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Continuous Extensions On Euclidean Combinatorial Configurations

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Abstract. In this paper, we introduce a concept of the Euclidean combinatorial configuration as a mapping of a set of certain objects into a point of Euclidean space. We classify Euclidean combinatorial configurations sets based on their structure and constraints. The proposed typology forms the basis for studying continuous functional representations of combinatorial configurations. Special classes of functional extensions are introduced, their properties are described, and corresponding examples are given.

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1 Introduction

Review of modern methods of Combinatorial Optimization (CO) allows us to specify the following important interrelated areas of research. On the one hand, this refers to deriving approaches to solving problems in their general formulation, e.g., branch and bound algorithms, cutting algorithms, branch-and-cut algorithms, relaxation approaches, dual methods, etc. On the other hand, there are special methods focused on solving specific problems of CO, its structure, and properties of objective function and constraints.

Of great interest are problems in which the domain of feasible solutions is the combinatorial space generated by combinatorial objects when they are mapped into the Euclidean space $\mathbb{R}^n$. Thus, combinatorial optimization problems can be equivalently formulated as discrete optimization (DO) problems, which in this case have a number of special properties. Classical approaches to solving discrete optimization problems include continuous approaches that can be roughly grouped in two directions: continuous formulations and continuous relaxations [**?**, 12, 14, 32]. In the first case, the problem is equivalently formulated in $\mathbb{R}^{\mathcal{N}}$ in terms of continuous variables. As a result of this formulation, we have a problem of nonlinear programming whose properties and methods of solving are determined by the class of objective function and functional constraints. In the second case, some of the constraints are relaxed, and a search of solutions for relaxation problems is performed. What is interesting is that, with the proper organization of the search, we can guarantee obtaining an exact solution of the original combinatorial problem.

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In both cases, the geometric peculiarities of the feasible domain are represented in algebraic form, i.e., allow it to be presented analytically. Thus, Polyhedral Combinatorics focuses on the construction and investigation of H-representations of convex hulls of combinatorial sets [1, 28, 59], which are applied to solving polyhedral relaxation problems. In nonlinear case, to improve methods of their solutions, one should try to make the relaxation problem convex. Therefore, it is of great relevance to single out classes of CO-problems that come with this property.

From the perspective of structural properties of representations of the simplest combinatorial objects, the most challenging are the ordered finite samples. This led to the introduction of a concept and selection of a class of Euclidean combinatorial sets (e-sets) [**?**, 39]. According to the definition provided, the elements of e-sets have the same number of components and differ by their composition or order. This is what gives an opportunity to consider the representations of e-sets as real-valued tuples, i.e., vectors in $\mathbb{R}^n$.

The purpose of this paper is to identify a class of e-sets connected with the configurations determined by C. Berge [2]. The research focuses on the study of images of e-sets in $\mathbb{R}^n$. The proposed typology of Euclidean combinatorial configurations will be the basis for algorithms of their continuous representation and functional extensions. The beginning of such research was laid in the papers [19–22, 24–26, 40, 43, 46].

2 Euclidean combinatorial configurations

Let us introduce the following notations. Suppose $\mathcal{P}$ is an e-set, whose representation $\mathcal{E}$ in $\mathbb{R}^n$ has the property as follows:

$$\exists\, \phi,\ \exists\, \mathcal{N} \in \mathbb{N}:\ \mathcal{E} = \phi(\mathcal{P}) \subset \mathbb{R}^{\mathcal{N}},\ \mathcal{P} = \phi^{-1}(\mathcal{E}). \tag{1}$$

Experts call the process of the representation of $\mathcal{P}$ in $\mathbb{R}^{\mathcal{N}}$ an immersion of the set $\mathcal{P}$ into Euclidean space. As a result of immersion, we obtain the correspondent image $\mathcal{E}$ of the set $\mathcal{P}$ in $\mathbb{R}^{\mathcal{N}}$ further referred to as s-sets.

We consider configurations according to [2]. This implies that any configuration is a mapping ψ of some initial set B of certain elements into a resulting finite abstract set $A = \{a_1, ..., a_k\}$ of certain elements and specific structure, where the given set of constrains Λ holds, i.e.,

$$\psi : B \to A. \tag{2}$$

Suppose B is a finite set $B = \{b_1, ..., b_n\}$, then the result of mapping (2) is an ordered set π of elements from A:

$$\pi = \begin{pmatrix} b_1 & \dots & b_n \\ a_{j_1} & \dots & a_{j_n} \end{pmatrix} = \langle a_{j_1} a_{j_2} ... a_{j_n} \rangle\,, \tag{3}$$

where $\{j_1, ..., j_n\} \subseteq J_k = \{1, ..., k\}$, further referred to as a combinatorial configuration or a c-configuration.

Now let us introduce a set Π of c-configurations, generated by various tuples (3) for the given A, B, and constraints Λ (further referred to as $\mathcal{E}_c$-set).

Then we single out the following class of c-configurations. Let set $\mathbf{A} = \{\mathbf{a}_1, ..., \mathbf{a}_k\}$ be a collection of vectors of the same dimension m, i.e.

$$\mathbf{a}_l = (a_{1l}, ..., a_{ml})^T \in \mathbb{R}^m, \quad l \in J_k. \tag{4}$$

Let us form a multiset of coordinates for vectors (4) and single out its ground set $\mathcal{A} = \{e_1, ..., e_K\} = S(\{a_{ij}\}_{i\in J_m,\, j\in J_k})$, where $S(M)$ is a ground set of a multiset M. Set $\mathcal{A}$ will be called a generated set (further referred to as $(E.\mathrm{GS})$) of $\mathcal{C}$-set E.

Let us consider $\mathbf{A}$ as a resulting set for configurations formation, i.e., set $A = \mathbf{A}$.

According to (3), e-configuration π is an ordered set of vectors from $\mathbf{A}$, i.e.,

$$\pi = \langle \mathbf{a}_{j_1}, \mathbf{a}_{j_2}, ..., \mathbf{a}_{j_n} \rangle . \tag{5}$$

With every π configuration (5), let us associate a multiset

$$\tilde{A}(\pi) = \{\alpha_i(\pi)\}_{i\in J_N} = \{a_{1j_1}, ..., a_{1j_n}, a_{2j_1}, ..., a_{2j_n}, ..., a_{mj_1}, ..., a_{mj_n}\}, \tag{6}$$

and a point

$$x = (x_1, ..., x_N) \in \mathbb{R}^N, \; N{=}m \cdot n, \tag{7}$$

according to the rule below.

First, we need to specify a bijective mapping φ_1 between $\mathcal{A}$ and $\mathcal{A}'$ sets of the same cardinality:

$$\varphi_1 : \; \mathcal{A}' = \varphi_1(\mathcal{A}) = \{\varphi_1(e_i)\}_{i\in J_K} \text{ such that } \mathcal{A}' \text{ is a set, i.e., } \; \mathcal{A}' = S(\mathcal{A}'). \tag{8}$$

Second, we need to fully order the multiset $\tilde{A}(\pi)$ (further mapping φ_2) and consider the result as a vector. Thus, the vector x is formed according to the rule as follows:

$$x_i = \varphi_1(\alpha_{\beta_i}(\pi)), \; i \in J_N, \tag{9}$$

where $\beta = (\beta_i)_{i\in J_n}$ is some permutation of J_n.

Point x is a result of mapping φ_1, φ_2 (further φ) such that:

$$x = \varphi(\pi), \; \pi = \varphi^{-1}(x), \tag{10}$$

where φ provides a bijection between any e-configurations (5), that satisfy $S(\tilde{A}(\pi)) \subseteq \mathcal{A}$, and point $x \in \mathbb{R}^N$ satisfying $S(\{x\}) \subseteq \mathcal{A}'$.

Definition 1. Euclidean combinatorial configuration (e-configuration) is a mapping as follows:

$$\varphi : (\psi, \mathbf{A}, \; \Theta) \to \mathbb{R}^N, \tag{11}$$

where ψ is a mapping $\psi : J_n \to \mathbf{A}$, and Θ is a system of constraints on φ, ψ.

The Euclidean combinatorial configuration is fully determined by a tuple $\langle \varphi, \psi, \mathbf{A}, \Theta \rangle$

It is an image of combinatorial configuration (4) in Euclidean space $\mathbb{R}^N$ with the given φ, ψ, being also the x-vector (10) of the dimension N.

By selecting the mapping φ and a way of multiset $\tilde{A}(\pi)$ ordering, we can obtain various e-configurations.

For example,

$$x = vec((\pi)) = (a_{1j_1}, ..., a_{1j_n}, a_{2j_1}, ..., a_{2j_n}, ..., a_{mj_1}, ..., a_{mj_n})^T, \tag{12}$$

where $vec((\pi))$ is a vectorization of matrix $(\pi) = (a_{lj_i})_{l,i} \in \mathbb{R}^{m\times n}$ that corresponds to c-configuration π.

Further we shall use mapping φ with reference to (12) with no other constraints to its form. As a result, we shall have $\Theta = \Lambda$ that fashions formula (11) to $\varphi : (\psi, \mathbf{A},\ \Lambda) \to \mathbb{R}^N$.

The mapping of $\mathcal{E}_c$-set Π into $\mathbb{R}^N$ brings about set

$$E = \varphi(\Pi) \subset \mathbb{R}^N \tag{13}$$

of all e-configurations $\langle \varphi, \psi, \mathbf{A},\ \Lambda\rangle$ (further reffered to as $\mathcal{C}$-set).

On the other hand, (10) implies that:

$$\Pi = \varphi^{-1}(E). \tag{14}$$

When choosing $\mathcal{P} = \Pi$, $\mathcal{E} = E$, $\mathcal{N} = N$, we see that condition (1) holds for Π, i.e., E is s-set that corresponds to $\mathcal{E}_c$-set Π.

Hence, $\mathcal{C}$-sets form a subclass of s-sets that are the resultant images of $\mathcal{E}_c$-sets in $\mathbb{R}^{nm}$, with the condition below to be satisfied that there exists $m \in \mathbb{N}$ such that $A = \mathbf{A}$ holds. This fact leads us to assert that $\mathcal{E}_c$-sets are interrelated with the class of e-sets.

We can single out a special class of $\mathcal{C}$-sets when vectors (4) have a single coordinate, i.e., $m = 1$, with the resultant set expected to be numerical. Thus, $K = k$, $N = n$, $\mathbf{a}_j = a_j \in \mathbb{R}^1$, $j \in J_k$. Hence, (12) can be expressed as $x = vec((\pi)) = (a_{j_1}, ..., a_{j_n})^T$.

Example 1. If $A = \{0, 1\}$, then c-configuration π is supposed to be an ordered Boolean sequence that determines the composition of the subset $B(\pi) \subseteq B$, where $a_{j_i} = 1$ means that b_i is included in $B(\pi)$, whereas $a_{j_i} = 0$ is not. The correspondent e-configuration $x = \varphi(\pi)$ is a characteristic vector of the set $B(\pi)$.

Example 2. Suppose A is the standard basis $\{\mathbf{e}_i\}_{i\in J_n}$ and ψ is a bijective mapping. Then c-configuration $\pi = \langle \mathbf{e}_{j_1}, ..., \mathbf{e}_{j_n}\rangle$ is to be a permutation of vectors of this basis. The configuration will correspond to the permutational matrix (π) of the order n, and $x = \varphi(\pi)$ will be the result of its vectorization in the form of a vector of the dimension n^2.

Example 3. If $k = n$ and ψ is a bijective mapping, then c-configuration π will be a permutation of A such as $\pi = \langle a_{j_1}, ..., a_{j_n}\rangle$, and the correspondent e-configuration x will be the vector $x = (a_{j_1}, ..., a_{j_n})$.

3 Typology of $\mathcal{C}$-sets

Suppose $\mathcal{C}$-set E is a finite point configuration in $\mathbb{R}^n$:

$$E = \{x^i\}_{i\in J_{n_E}} \subseteq \mathbb{R}^N,\ x^i = (x_{ij})^T_{j\in J_N},\ i \in J_{n_E},\ n_E > 1. \tag{15}$$

Then multiset $G = \bigcup\limits_{x\in E} \{x\}$ is a multiset that induces E (an induced multiset, (E.IM)). Its ground set $S(G)$ coincides with (E.GS), i.e., $S(G) = \mathcal{A}$. We can express (E.IM) and (E.GS) as follows:

$$G = \{g_1, ..., g_\eta\},\ g_i \le g_{i+1},\ i \in J_{\eta-1},$$
$$\mathcal{A} = \{e_1, ..., e_K\},\ e_i < e_{i+1},\ i \in J_{K-1},$$

or $G = \{e_i^{\eta_i}\}_{i\in J_K}$, where $[G] = (\eta_i)_{i\in J_K}$ is a primary specification of G, and $\eta_i = \mu_G(e_i)$ is a multiplicity of e_i in G, $i \in J_K$.

Further in the paper we shall solve posed problems using $\mathcal{C}$-sets, analyzing them in two independent ways as follows:
– analysis of $\mathcal{A}, G, n$ (a constructive analysis, C-A);
– analysis of algebraic and topological properties of E (a geometric analysis, G-A).

Further we shall consider the case when $m = 1$, that together with (15) means that $K = k$, wherefrom

$$E = \{x^i\}_{i\in J_{n_E}} \subseteq \mathbb{R}^n,\ x^i = (x_{ij})^T_{j\in J_n},\ i \in J_{n_E},$$
$$\mathcal{A} = \{e_1, ..., e_k\},\ e_i < e_{i+1},\ i \in J_{k-1},\ k, n, n_E > 1.$$

3.1 C-A-typology of $\mathcal{C}$-sets

The given classification can refer to all of three elements $\mathcal{A}, G, n$ or to some of them. Specifically, this can be exemplified as follows.

$\mathcal{A}$-A-typology: – $\mathcal{A} \subset \mathbb{Z}$ – E is an integer-valued $\mathcal{C}$-set ($\mathbb{Z}$S);
– $\mathcal{A} \subset \mathbb{Q}$ – E is a rational-valued $\mathcal{C}$-set ($\mathbb{Q}$S);
– $\mathcal{A} \subset \mathbb{R}^1_{>0}$ – E is a positive-valued $\mathcal{C}$-set ($\mathbb{R}_{>0}$S);
– $\mathcal{A} \subset \mathbb{R}^1_{+}$ – E is a non-negative-valued $\mathcal{C}$-set ($\mathbb{R}_{+}$S);
– if $\exists\ \Delta > 0 :\ e_{i+1} - e_i = \Delta,\ i \in J_{k-1}$, then E is a uniformly distributed $\mathcal{C}$-set ($\mathbb{U}(\Delta)$S).

With features of $\mathbb{Z}$-$\mathbb{U}(\Delta)$Ss combined, other classes of $\mathcal{C}$-sets are formed, including
– Boolean $\mathcal{C}$-sets ($\mathcal{B}$S): $\mathcal{A} = \{0, 1\}$;
– binary $\mathcal{C}$-sets ($\mathcal{B}'$S): $\mathcal{A} = \{-1, 1\}$;
– ternary $\mathcal{C}$-sets ($\mathcal{T}$S): $\mathcal{A} = \{-1, 0, 1\}$.

E is called a set of special e-configurations (special $\mathcal{C}$-set, $\mathcal{S}$S) if:

$$\max_j k_j = 2, \text{ where } k_j = |\mathcal{A}_j|,\ \mathcal{A}_j = S(\{x_{ij}\}_{i\in J_{n_E}}),\ j \in J_n. \tag{16}$$

Thus, $\mathcal{B}$S, $\mathcal{B}'$S are special classes of $\mathbb{Z}\mathcal{S}$S.

G/n-A-typology:
– E is called a set of e-configurations of permutations ($\mathcal{C}$-set of permutations, $\mathcal{P}$S) if $\forall\, x \in E \quad \{x\} = G$;
– E is called a set of e-configurations of partial permutations ($\mathcal{C}$-set of partial permutations, $\mathcal{PP}$S) if $\forall x \in E\ \{x\} \subset G$.

These classes can be also determined as $\mathcal{P}$S $: \eta = n$ and $\mathcal{PP}$S $: \eta > n$.

$G/\mathcal{A}$-A-typology: E is called a set of e-configurations:
– without repetitions ($\mathcal{C}$-set without repetitions, $\mathcal{R}^-$S) if $\forall\, x \in E\ |[\{x\}]| = n$;
– with repetitions ($\mathcal{C}$-set with repetitions, $\mathcal{R}^+$S) if $\exists\, x \in E\ |[\{x\}]| < n$.

These features are expressed below as $\mathcal{R}^-$S $: \eta = k$ and $\mathcal{R}^+$S $: \eta > k$.

C-A-typology: Combinations of the above classes produce various $\mathcal{C}$-sets, such as:
– $\mathcal{C}$-set of permutations without repetitions ($\mathcal{PR}^-$S): $\eta = k = n$;
– $\mathcal{C}$-set of permutations with repetitions ($\mathcal{PR}^+$S): $k < \eta = n$;
– $\mathcal{C}$-set of partial permutations without repetitions ($\mathcal{PPR}^-$S): $n < k = \eta$;
– $\mathcal{C}$- set of partial permutations with repetitions ($\mathcal{PPR}^+$S): $k, n \le \eta$;
and special, Boolean, binary, triple $\mathcal{C}$-sets of permutations and partial permutations, etc.

A special class ($\mathcal{PPR}^+$S) is $\mathcal{C}$-set of partial permutations with unbound repetitions if $\eta = k \cdot n$.

Other constraints imposed on $\mathcal{C}$-set elements form new classes, e.g., $\mathcal{C}$-sets of even and odd permutations [29], even and odd Boolean vectors [11], signed permutations [58], etc.

3.2 G-A-typology of $\mathcal{C}$-sets

G-A-classification of $\mathcal{C}$-sets consists in analyzing the mutual position of their points. The given classification can be only provided when the initial combinatorial set is mapped into Euclidean space that allows considering a polytope

$$P = convE, \tag{17}$$

as well as hypersurfaces containing E.

Definition 2. E is called a vertex-located $\mathcal{C}$-set (VLS) if

$$E = vert\, conv\, E. \tag{18}$$

Definition 3. E is a surface-located $\mathcal{C}$-set (SLS) if there is function $f(x)$ that is strictly convex on the convex set $\mathcal{K} \supseteq E$, and if

$$f(x) \underset{E}{=} 0. \tag{19}$$

Definition 4. E is a polyhedral-surfaced $\mathcal{C}$-set (PSS) if there exists a hypersurface S for which the following is true:

$$E = P \cap S, \tag{20}$$

where P is a polytope (17).

Representation (20) is called a polyhedral-surfaced one of E (PSR) if, in an expression

$$S = \{x \in \mathbb{R}^n : \ f(x) = 0\}, \tag{21}$$

function $f(x)$ is strictly convex on a convex set $\mathcal{K} \supseteq C = conv\ S$.This implies that $C = \{x \in \mathbb{R}^n : \ f(x) \leq 0\}$ is a convex body bounded by a strictly convex surface S that itself is a full surface of this body [27] .

Typology of SLSs: SLS E is
– spherically-located (SSpS) if $\exists r > 0,\ a \in \mathbb{R}^n$:

$$S = S_r(a) = \{x \in \mathbb{R}^n : \ \|x - a\|_2 = r\};$$

– superspherically-located (SSsS) if $\exists r > 0,\ a \in \mathbb{R}^n,\ \alpha \in (1, \infty)$:

$$S = S_r(a, \alpha) = \{x \in \mathbb{R}^n : \ \|x - a\|_\alpha = r\};$$

– ellipsoidally-located (SES) if $\exists A \in \mathbb{R}^{n \times n},\ A = A^T,\ A \succ 0,\ a \in \mathbb{R}^n$:

$$S = El(a, A) = \{x \in \mathbb{R}^n : \ \|x - a\|_A = 1\}, \ \text{ where } \|x\|_A = x^T A x.$$

Typology of PSSs: PSS E is
– polyhedral-spherical (PSpS) if S in (20) is a hypersphere;
– polyhedral-superspherical (PSsS) if S is a supersphere;
– polyhedral-ellipsoidal (PES) if S is an ellipsoid.

SSpS, SSsS, and SES enable PSR, called polyhedral-spherical (PSpR), polyhedral-superspherical (PSsR), and polyhedral-ellipsoidal representations (PER), respectively.

PSS E is a polyhedral-spherical $\mathcal{C}$-set (PSpS) if S is a hypersphere; polyhedral-ellipsoidal $\mathcal{C}$-set (PES) if S is an ellipsoid; polyhedral-superspherical $\mathcal{C}$-set (PSsS) if S is a supersphere. Clearly, if E is a PSpS, a PSsS or a PES, it enables PSR. In view of this, we shall focus on the three classes of $\mathcal{C}$-sets, PSpSs in particular, since it is formed by an intersection of PSsS- and PES-classes AS a result, PSpSs posses features of both PSsSs and PESs in combination with its specific properties.

4 Continuous functional representations of $\mathcal{C}$-sets

Suppose

$$\mathcal{F} = \{f_j(x)\}_{j \in J_m}, \tag{22}$$

where $f_j : \ E \to \mathbb{R}^1$ are continuous functions for $j \in J_m$.

Definition 5. The representation of $\mathcal{C}$-set E with the help of functional dependencies expressed as follows:

$$f_j(x) = 0,\ j \in J_{m'}, \tag{23}$$
$$f_j(x) \leq 0,\ j \in J_m \backslash J_{m'} \tag{24}$$

will be called a continuous functional representation (a f-representation) of E.

In f-representation (23), (24): a) (23) is a strict part; b) (24) is a nonstrict part; c) m is an order; d) m', $m'' = m - m'$ is an order of the strict and nonstrict parts, respectively.

When we introduce a notation for a geometric locus, determined by expressions (23), (24):

$$S_j = \{x \in \mathbb{R}^n : f_j(x) = 0\},\ j \in J_{m'}, \tag{25}$$
$$C_j = \{x \in \mathbb{R}^n : f_{j+m'}(x) \leq 0\},\ j \in J_{m''}, \tag{26}$$

then $E = \left(\underset{j \in J_{m'}}{\cap} S_j\right) \bigcap \left(\underset{i \in J_{m''}}{\cap} C_j\right)$.

Thus, if dimension of varieties (25) is $n-1$, and $f_j(x),\ j \in J_m \backslash J_{m'}$ are convex, $\mathcal{C}$-set E is formed as an intersection of hypersurfaces (25) with convex bodies (26).

We shall provide the classification of f-representations in several ways according to: a) the type of functions (22) including linear, nonlinear, differentiable, smooth, convex, polynomial, trigonometrical, etc.; b) the correlation of parameters m, m', m''.

Definition 6. System (23), (24) is called:
– a strict f-representation of E(further referred to as (E.SR)) if it contains only a strict part $m' = m$, $m'' = 0$;
– a nonstrict such a representation (further referred to as (E.NR)) if an f-representation contains only a nonstrict part $m' = 0$, $m'' = m$;
– a mixed f-representation of a set E (further referred to as (E.MR)) if it contains both strict and non-strict parts, i.e., $m'\,(m - m') > 0$.

System of constraints (23), (24) will be called an irredundant f-representation of E-set (E.IR) if the exclusion some of its constraints results in a formation of its proper superset E:

$$\begin{aligned} &\forall\ j \in J_{m'}\ E \backslash S_j \supset E; \\ &\forall\ i \in J_{m''}\ E \backslash C_i \supset E. \end{aligned} \tag{27}$$

Finally, a bi-component strict f-representation of E will be called tangential (E.TR) if set E coincides with a set of tangential points between S_1 and S_2 surfaces that are tangent to each other; a n-component irredundant strict f-representation will be called intersected (E.IIR).

Among the mixed f-representations we can single out a class of polyhedral-surfaced representations (further referred to as E.PSR) consisting of an equation of a strictly convex surface S, circumscribed around E, and H-representation of P. Classification of f-representations can be done with respect to a type of surface S. Particularly, polyhedral-spherical, polyhedral-superspherical and polyhedral-ellipsoidal f-representations can be singled out. If there is a constraint that the family of functions (22) consists only of polynomials, one can use instruments of Real Algebraic Geometry [4,9,41]. For instance, (E.SR) can be formed in the way of finding a base for an ideal of an algebraic set E. Determining (E.NR) or (E.MR), we bear in mind that E is a semi-algebraic set.

4.1 Approaches to construction of nonstrict and mixed f-representations of $\mathcal{C}$-sets

We can construct nonstrict and mixed f-representations of $\mathcal{C}$-sets in the following ways:
– to find the equation of strictly convex surface S and H-representation of P if the existence of (E.PSR) is substantiated;
– to extract a family of functions, for which the range of value changes over E is known, form (23) and (24) in view of the above and check $x \in E$ iff x satisfies (23), (24).

We can construct a strict f-representation of $\mathcal{C}$-set $E' \supset E$ and define E as E' subject to some constraints including some inequalities. To exemplify, we can use as E' the grid $E' = \mathcal{A}^n$.

Example 4. Taking into account that $E_{nk}(G)$, B'_n are PSpSs, and their convex hulls are a generalized permutohedron and a hypercube correspondingly [59], we obtain: (B'_n.PSR) : $-1 \leq x_i \leq 1, i \in J_n;\ \sum\limits_{i=1}^{n} x_i^2 = n$;

$$(E_{nk}(G).\mathrm{PSR}): \quad \sum_{j\in\omega} x_j \geq \sum_{j=1}^{|\omega|} g_j,\ \omega \subset J_n;\ \sum_{i=1}^{n} x_i^l = \sum_{i=1}^{n} g_i^l,\ l = 1, 2.$$

4.2 Approaches to construction of strict f-representations of $\mathcal{C}$-sets

System (23) is (E.SR) if and only if $x \in E \Longleftrightarrow x$ satisfies (23).

Algorithm 1. We can single out a family $\Phi(E)$ that takes constant values on E and use it to form (23). Selecting subfamily (23) from $\Phi(E)$, we are to justify that: a) it yields a finite point configuration (FPC) [8]; b) the FPC contains no other points except for E.

In order to single out a specific set E among other $\mathcal{C}$-sets of the same combinatorial type that are induced by $G = (E.\mathrm{IM})$, we shall associate it with a basic $\mathcal{C}$-set (further referred to as $\mathcal{C}_b$-set) of the same combinatorial type as E, that has the same parameters $\mathcal{A}$, G, n and unites $\mathcal{C}$-sets of the type and parameters. Hence, the general $\mathcal{C}_b$-set of permutations induced by n-element multiset (3) will be expressed as follows:

$$E_{nk}(G) = \{x \in \mathbb{R}^n : \{x\} = \{x_1, ..., x_n\} = G\}; \tag{28}$$

$\mathcal{C}_b$-set of permutations without repetitions induced by G is

$$E_n(G) = \{x \in \mathbb{R}^n : \{x\} = \{x_1, ..., x_n\} = \mathcal{A}\}; \qquad (29)$$

a binary $\mathcal{C}_b$-set – $B'_n = \{x \in \mathbb{R}^n : x_i \in \{-1, 1\}, i \in J_n\}$, etc.

Theorem 1. *$\Phi(E_{nk}(G)) = \Phi^{sym}(G)$, where $\Phi^{sym}(G)$ is the set of symmetric functions that are zero-valued at a point $g = (g_1, ..., g_n)$.*

Theorem 2. *$\Phi(B'_n) = \Phi^{even}(\{-1^n, 1^n\})$, where $\Phi(B'_n)$ is the set of functions even on every coordinate that are zero-valued at a point $e = (1, ..., 1)$.*

Theorems 1 and 2 imply that strict f-representations of $E_{nk}(G)$, B'_n are constructed only by functions of families $\Phi^{sym}(G)$ and $\Phi(B'_n)$, respectively, whereas the construction of their proper subsets requires the application of other functions as well. In this relation, we need to find such functions that are zero-valued on $\mathcal{C}$-set E and nonzero-valued on $E' \backslash E$, where E' is the correspondent $\mathcal{C}_b$-set.

Example 5. (B'_n.SR): $x_i^2 - 1 = 0$, $i \in J_n$.

Algorithm 1 was used to construct strict f-representations of the general $\mathcal{C}_b$-set of permutations and its special classes [22].

Theorem 3. *If (23) is a strict f-representation of $\mathcal{C}$-set $E' \supset E$,*

$$f \in \overline{\Phi}(E') = \Phi(E) \backslash \Phi(E'), \qquad (30)$$

$$f(x) \underset{E' \backslash E}{\neq} 0, \qquad (31)$$

then (23), $f(x) = 0$ is (E.SR).

The given theorem specifies the conditions necessary to single out one $\mathcal{C}$-set from another using the only equality constraint.

Corollary 1. *If (23) and (24) are f-representations of $\mathcal{C}$-set $E' \supset E$, and function f satisfies (30) and (31), then (23), (24), and $f(x) = 0$ form (E.FR).*

Example 6. Function

$$f(x) = \prod_{i=1}^{n} x_i \underset{B'_n}{\in} \{-1, 1\}, \qquad (32)$$

that enables splitting of B'_n into two subsets – $B_n'^-$ and $B_n'^+$, where $f(x)$ takes the value of -1 and 1 correspondingly. According to Theorem 3, (B'_n.SR) with $f(x) = 1$ is ($B_n'^+$.SR), whilst (B'_n.SR) and $f(x) = -1$ form ($B_n'^-$.SR). Taking into account that $B'_n = 2B_n - 1$, these f-representations allow obtaining strict f-representations of $\mathcal{C}_b$-sets of even and odd Boolean vectors.

According to Corollary 1, (B'_n.PSpR) together with $f(x) = 1$ forms a mixed f-representation $B_n'^+$ (further referred to as ($B_n'^+$.MR1)) of the order of $2n + 2$, and (B'_n.PSpR) together with $f(x) = 1$ forms f-representation $B_n'^-$ (further referred to as ($B_n'^-$.MR1)).

Algorithm 2. We can express constraints on $\mathcal{A}, G, \psi, \Omega$ in terms of coordinates of e-configurations. This is much more convenient for constructing the above-mentioned strict f-representations of $\mathcal{C}_b$-sets as for the way of their constructions. We shall illustrate this with a $\mathcal{C}_b$-set of permutations.

Example 7. Suppose $E = E_n(G)$. The constraint $k = n$ is expressed in Cartesian variables in (29). The constraint $\{x_1, ..., x_n\} = \mathcal{A}$ can be expressed in its turn in two ways as follows: 1) $x_i \in \mathcal{A},\ i \in J_n$; 2) $x_i \neq x_j,\ i, j \in J_n, i \neq j$. Consequently, we obtain the following $(E_n(G).SR)$: $\prod\limits_{j=1}^{n} (x_i - e_j) = 0,\ i \in J_n$; $(x_i - x_j)^2 \geq \delta^2,\ 1 \leq i < j \leq n$, where $\delta = \min\limits_{i \in J_{n-1}} \{e_{i+1} - e_i\}$.

To construct a strict f-representation of $E_{nk}(G)$, we can simply express the condition below in Cartesian variables:

$$\{x_1, ..., x_n\} = \{g_1, ..., g_n\} \tag{33}$$

given in (28). It can be expressed as shown below:

$$(x - g_1) \cdot ... \cdot (x - g_n) = 0,\ x \in \mathbb{R}. \tag{34}$$

Indeed, to solve equation (34) with respect to x, we need to find the collection $x_1, ..., x_n$ of its roots, that exactly coincides with the multiset G, thus making condition (33) true.

Now we shall rewrite (34) in terms of its roots according to Viete formula:
$x^n - (g_1 + ... + g_n)\, x^{n-1} + (g_1 g_2 + ... + g_{n-1} g_n)\, x^{n-2} + ... + (-1)^n g_1 \cdot g_2 \cdot ... g_n = 0.$

As a result we obtain a system of n equations:

$$\sum_{\omega \subseteq J_n, |\omega| = j} \prod_{i \in \omega} x_i = \sum_{\omega \subseteq J_n, |\omega| = j} \prod_{i \in \omega} g_i,\ j \in J_n, \tag{35}$$

whose solution is no other set except for the set of n real numbers $x_1, ..., x_n$ the same as in (33). On the other hand, dealing with every equation of the system (35) as with the equation of some variety in $\mathbb{R}^n$, we come up with the fact that a complete solution to this nonlinear system is exactly $\mathcal{C}$-set of permutations $E_{nk}\,(G)$. Thus, (35) is a strict polynomial representation of this set (further referred to as $(E_{nk}\,(G).\text{SR1})$) whose degree and order coincide with the dimension of the Euclidean space and are equal to n.

Let us denote elementary symmetric polynomials as follows:

$$u_j\,(x) = \sum_{\omega \subseteq J_n, |\omega| = j} \prod_{i \in \omega} x_i,\ j \in J_n, \tag{36}$$

and rewrite $(E_{nk}\,(G).\text{SR1})$ as:

$$u_j\,(x) = u_j\,(g)\,,\ j \in J_n. \tag{37}$$

We should note that the use of $(E_{nk}(G).\mathrm{SR1})$ for large dimensions is problematic because it is quite difficult to evaluate functions (36).

We shall construct another functional representation of a $\mathcal{C}$-set $E_{nk}(G)$ based on $(E_{nk}(G).\mathrm{SR1})$ that relies on the given interrelations between elementary symmetric polynomials (36) with power sums:

$$q_j(x) = \sum_{i=1}^{n} x_i^j, \ j \in J_n^0, \tag{38}$$

that are reflected in the Newton-Girard identities [3] :

$$q_j(x) = j \cdot (-1)^{-j+1} h_j(x) + \sum_{i=1}^{j-1} (-1)^{i-j+1} q_i(x) \cdot h_{j-i}(x), \ j \in J_n. \tag{39}$$

Applying the recurrent formula (39) to both parts of the equation (37), we obtain $q_j(x) = q_j(g), \ j \in J_n$, or, considering (38),

$$\sum_{i=1}^{n} x_i^j = \sum_{i=1}^{n} g_i^j, \ j \in J_n. \tag{40}$$

Similar to (35), the system of equations (40) can be considered from two points of view: first, as a system used to determine a set of solutions of equation (34); second, as a system that defines a set of varieties in $\mathbb{R}^n$ whose intersection is exactly the set $E_{nk}(G)$. Thus, we found another f-representation (40) of $E_{nk}(G)$ (further referred to as $(E_{nk}(G).\mathrm{SR2})$). Similar to $(E_{nk}(G).\mathrm{SR1})$, it is strict, polynomial, with its degree and order being equal to n. At the same time, it has apparent benefits over $(E_{nk}(G).\mathrm{SR1})$, namely, the simplicity of the functions involved, and its convexity in $\mathbb{R}^n_{\geq e_1}$.

Thus the use of the concept of Euclidean combinational configurations and property (33) of e-configuration of permutations allowed us to offer a new, much simpler proof of the following theorem.

Theorem 4. *[22] Each of the systems of equations* (35), (40) *defines a strict continuous functional representation of* $E_{nk}(G)$.

Here is a generalized Theorem 4.

Theorem 5. *If* ξ *is a bijective mapping between* $(E_{nk}(G).GS)$ $\mathcal{E}$ *and a real set* $\mathcal{E}' = \{e_1^{'}, ..., e_k^{'}\}$*, then each of the systems of equations:*

$$\sum_{i=1}^{n} \xi(x_i)^j = \sum_{i=1}^{n} \xi(g_i)^j, \ j \in J_n; \tag{41}$$

$$\sum_{\omega \subseteq J_n, |\omega|=j} \prod_{i \in \omega} \xi(x_i) = \sum_{\omega \subseteq J_n, |\omega|=j} \prod_{i \in \omega} \xi(g_i), \ j \in J_n, \tag{42}$$

defines a strict f-representation of $\mathcal{C}_b$*-set* $E_{nk}(G)$.

Corollary 2. $\forall a \in \mathbb{R}^1$, *each of the following systems of equations*

$$\sum_{i=1}^{n} (x_i - a)^j - \sum_{i=1}^{n} (g_i - a)^j = 0,\ j \in J_n;$$
$$\sum_{\omega \subseteq J_n, |\omega| = j} \prod_{i \in \omega} (x_i - a)^j - \sum_{\omega \subseteq J_n, |\omega| = j} \prod_{i \in \omega} (g_i - a)^j = 0,\ j \in J_n,$$

is a strict f-representation of $E_{nk}(G)$*-set.*

4.3 Approaches to construction of tangential f-representations of $\mathcal{C}$-sets

In the class of convex f-representations, tangential ones have a minimal number of components. The minimum degree of polynomial tangential f-representation is three. In view of the fact that the minimum degree of a polynomial f-representation is two, it becomes apparent that (E.TR) has benefits as for an order and a degree.

We shall describe some ways of constructing these f-representations. Let us outline a general construction scheme for tangential f-representations of $\mathcal{C}$-set E that rests on properties of differentiable functions over E (further referred to as (TR.Scheme1)).

The tangential representation will be constructed as shown below:

$$f_1(x) = 0, \tag{43}$$
$$f_2(x) = 0. \tag{44}$$

First, we should select the differentiable functions $f_1(x), f_2(x) \in \Phi(E)$, $f_1(x) \neq f_2(x)$ that have no singularities. This means that, in $\mathbb{R}^n$,

$$f_1(x) \underset{E}{=} 0,\ f_2(x) \underset{E}{=} 0 \tag{45}$$

determine surfaces

$$S_j = \{x \in \mathbb{R}^n :\ f_j(x) = 0\},\ j = 1, 2. \tag{46}$$

Second, we should check if the condition below is true:

$$\forall x \in E \quad \exists k(x) \neq 0 :\ \nabla f_2(x) \underset{E}{=} k(x) \cdot \nabla f_1(x). \tag{47}$$

Next, we should identify $j \in J_2$ and solve the optimization problem using the method of Lagrange multipliers:

$$f_j(x) \underset{S_{3-j}}{\to} \text{extr}. \tag{48}$$

Suppose

$$\begin{aligned} X^{j\,\min} &= \underset{S_{3-j}}{\operatorname{Argmin}}\ f_j(x),\ Z^{j\,\min} \underset{X^{j\,\min}}{=} f_j(x); \\ X^{j\,\max} &= \underset{S_{3-j}}{\operatorname{Argmax}}\ f_j(x),\ Z^{j\,\max} \underset{X^{j\,\max}}{=} f_j(x) \end{aligned} \tag{49}$$

is a complete solution of the problem (46).

Then the system of equations (43), (44) will be a tangential f-representation of E if one of the conditions below holds: $X^{j\min} = E$, $Z^{j\min} = 0$ or $X^{j\max} = E$, $Z^{j\max} = 0$.

(TR.Scheme1) can be applied to quite a narrow class of functions $f_1(x)$, $f_2(x)$ that allow solving problem (46), (48) explicitly (ref. e.g., [24]).

Next we shall introduce another method (further referred to as (TR.Scheme2)) of analytic foundation of a tangential f-representation of two-level sets [8], i.e., sets that can be decomposed exactly along two parallel hyperplanes towards the normal vectors to facets of the correspondent polytope (17).

Theorem 6. *If E is two-level, then*

$$S^2: \ f_0(x,2) = \sum_{F\in\mathbf{F}} (\overline{n}_F'^T x - a_F')^2 - |\mathbf{F}| = 0;$$
$$S^4: \ f_0(x,4) = \sum_{F\in\mathbf{F}} (\overline{n}_F'^T x - a_F')^4 - |\mathbf{F}| = 0 -$$

is its tangential representation (further referred to as (E(2-level).TR)), where $\mathbf{F}$ *is a set of P-facets;* $\overline{n}_F'^T \in \mathbb{R}^n$, $a_F' \in \mathbb{R}^1$, $\overline{n}_F'^T = \frac{\overline{n}_F}{\delta_F}$, $a_F' = \frac{a_F}{\delta_F}$ *for all* $F \in \mathbf{F}$.

Here is one more method for tangential f-representation construction (further referred to as (TR.Scheme3)) that is used when equations (43), (44) can be expressed in terms of some norm:

$$\exists\, \|.\|_{(\alpha)},\ \exists\, \alpha_1, \alpha_2 \in \mathbb{R}^1_+ : \ f_1(x) = \|x\|_{(\alpha_1)} - 1; \ f_2(x) = \|x\|_{(\alpha_2)} - 1. \tag{50}$$

Correspondingly, surfaces (46) are spheres in a norm space equipped with a norm $\|.\|_{(\alpha_i)}$ (further referred to as $\|.\|_{(\alpha_i)}$-spheres, $i = 1, 2$).

The proof of the fact that this norm is strictly monotonous with respect to α, i.e., $\forall \alpha_1, \alpha_2 \in \mathbb{R}^1_+$, $\alpha_1 \neq \alpha_2$ one of the conditions is true:

$$\forall\, x \in \mathbb{R}^n : \ \|x\|_{(\alpha_1)} \geq \|x\|_{(\alpha_2)}, \tag{51}$$
$$\forall\, x \in \mathbb{R}^n : \ \|x\|_{(\alpha_1)} \leq \|x\|_{(\alpha_2)}, \tag{52}$$

substantiates that (43), (44) is (E.SR). Besides, if this norm is a differentiable function in domain $K \supset E$, then (43), (44) will be (E.TR).

In turn, it means that in case (51) true, then $S_1 \subseteq C_2$, with$E = S_1 \cap \partial C_2$. And in case (52) we have $E = S_2 \cap \partial C_1$, where $C_i = conv\ S_i$, $i = 1, 2$. In other words, in case (51), S_1 is inscribed into S_2, and in case (52), S_1 is circumscribed around S_2. At that, in terms of norm $\|.\|_{(\alpha)}$, (45), (50) are expressed as: $\|x\|_{(\alpha_1)} \underset{E}{=} \|x\|_{(\alpha_2)} \underset{E}{=} 1$.

Theorem 7. *If there is function $\|.\|_{(\alpha)}$ that is strictly monotonous with respect to α and such that functions $f_1(x), f_2(x)$ in* (50) *satisfy the conditions of* (45), (50), *then the pair of equations* (43), (44) *form (E.TR).*

(TR.Scheme3) can be used to substantiate the existence of (B'_n.TR) expressed as:

$$(B'_n.\mathrm{TR}(\alpha_1,\alpha_2)) : \sum_{i=1}^{n} x_i^{\alpha_1} = n,\ \sum_{i=1}^{n} x_i^{\alpha_2} = n,$$

where $1 \leq \alpha_1 < \alpha_2 < \infty$, because the scaled l_p norm $\|x\|_{\langle p\rangle} = \frac{1}{n}\|x\|_p$ is monotonous [7,15].

(TR.Scheme2) is applied to both B_n, B'_n, and further two-level $\mathcal{C}_b$-sets, that belong to class $\mathcal{S}$Ss, such as $E_{n2}(G)$, $E^n_{n+1,2}(G)$. When using Theorem 7 in this case, we should note that polytopes $convE_{n2}(G)$, $convE^n_{n+1,2}(G)$ are hypercubes with maximum two additional constraints.

Finally, (TR.Scheme1) can be used to substantiate the existence of (B'_n.TR(α_1, α_2)) for even α_1, α_2 and to substantiate the existence of cubic ($E_{n2}(G)$.TR) for the cases $G = \{e_1, e_2^{n-1}\}$ and $G = \{e_1^{n-1}, e_2\}$ [24].

5 Conclusion and Further Research

The presented results form the basis for the development of Euclidean combinatorial optimization methods [18,23,25,26,38,39,42,48,49,51,53–57].Theoretically, it is of interest to develop new approaches to the construction of convex extensions of functions defined on the corresponding $\mathcal{C}_b$-sets. At that, it is natural to single out various special classes of $\mathcal{C}$-sets, such as sets of e-configurations of permutation matrices, even, cyclic, or signed permutations, and so on. Considering these sets as new types of $\mathcal{C}_b$-sets, it is of considerable interest to single out their special subclasses and explore properties of the classes of $\mathcal{C}_b$-sets both in general and particular cases. Expectedly, this will provide new approaches to the construction of the required convex extensions. On the other hand, we need to conduct a comparative analysis of various continuous functional representations of $\mathcal{C}$-sets, since this greatly affects the efficiency of applied methods of Nonlinear Optimization that use the representations. Naturally, both of these directions should be considered integrally. Further, we intend to proceed to the study of genetic algorithms for optimization problems on $\mathcal{C}$-sets in view of previous research [50–52]. Of interest are methods of parametric and multicriteria optimization on $\mathcal{C}$-sets, taking into account the results of [5,30]. We also plan to focus further research on the development of methods for solving the problems of clustering, packing, layout, and covering [6,10,16,17,31,33–35,44,48]. Historically, it was this class of tasks, also called the geometric design problems [?] that laid the foundation for Euclidean Combinatorial Optimization. The problems of geometric design focus on spatial objects having a shape, metric and placement parameters that characterize their mutual position in space. Taken together, these characteristics determine geometric information that induces the configurational space of geometric objects [36,45]. The generalized variables of this space are considered as components of a vector represented a Euclidean combinatorial configuration. In turn, selecting the combinatorial structure in problems of placement of geometric objects [?, 39, 47] we can consider this class of optimization problems as CO problems on $\mathcal{C}$-sets.

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