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Transversals in loops. 3. Loop transversals

Eugene Kuznetsov

Abstract. The investigation of the new notion of a transversal in a loop to its subloop (begun in [10]) is continued in the present article. This notion generalized the well-known notion of a transversal in a group to its subgroup and can be correctly defined only in the case when some specific condition (Condition A) for a loop and its subloop holds. The connections between loop transversals in some loop to its subloop and loop transversals in multiplicative group of this loop to some suitable subgroup are investigated in this work.

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1 Introduction

In group theory, in group representations theory and in quasigroup theory the following notion is well-known – the notion of a left (right) transversal in a group to its subgroup [1, 5, 6, 11].

Definition 1. Let G be a group and H be a subgroup in G. A complete set $T = \{t_i\}_{i \in E}$ of representatives of the left (right) cosets H_i in G to H ($e = t_1 \in H$, $t_i \in H_i$) is called a **left (right) transversal in G to H**.

In the present work we continue to study a variant of natural generalization of the notion of transversal to the class of loops, begun in [10]. As the elements of a left (right) transversal in a group to its subgroup are representatives of every left (right) coset to the subgroup, the notion of a left (right) transversal in a loop to its subloop can be correctly defined only in the case when this loop admits a left (right) coset decomposition by its subloop (see the Condition A below).

In Section 2 of this article we remember the most important notions and theorems from the first part of this investigation [10].

In Section 3 different structural theorems are proved. They demonstrate a connection between transversals in a loop to its subloop and transversals in a multiplicative group of this loop to its suitable subgroup.

In Section 4 one of the most important particular cases of transversals in a loop to its subloop is investigated – the case of a loop transversal. Some criteria of the existence of a loop transversal in a given loop to its subloop are proved.

Further we shall use the following notations:
$\langle L, \cdot, e\rangle$ is an initial loop with the unit e;
$\langle R, \cdot, e\rangle$ is its proper subloop;
E is a set of indexes ($1 \in E$) of the left (right) cosets R_i in L to R (moreover, $R_1 = R$).

2 Preliminaries

Definition 2. A system $\langle E, \cdot\rangle$ is called [2] a **right (left) quasigroup** if for arbitrary $a, b \in E$ the equation $x \cdot a = b$ ($a \cdot y = b$) has a unique solution in the set E. If the system $\langle E, \cdot\rangle$ is both a right and left quasigroup, then it is called a **quasigroup**. If in a right (left) quasigroup $\langle E, \cdot\rangle$ there exists an element $e \in E$ such that $x \cdot e = e \cdot x = x$ for every $x \in E$, then the system $\langle E, \cdot\rangle$ is called a **right (left) loop** (the element e is called a **unit** or **identity element**). If a system $\langle E, \cdot\rangle$ is both a right and left loop, then it is called a **loop**.

At the beginning let us define a partition of a loop by left (right) cosets to its proper subloop.

Definition 3 (see [12])**.** Let $\langle L, \cdot\rangle$ be a loop and $\langle R, \cdot\rangle$ be its proper subloop. Then a **left coset** in L to R is a set of the form $xR = \{xr \mid r \in R\}$, and a **right coset** is a set of the form $Rx = \{rx \mid r \in R\}$.

In a general case the cosets in a loop to its subloop do not necessarily form a partition of the loop. This leads us to the following definition.

Definition 4 (see [12])**.** A loop L has a **left (right) coset decomposition by its proper subloop** R, if the left (right) cosets form a partition of the loop L, i.e. for some set of indexes E:

1. $\bigcup\limits_{i \in E} (a_i R) = L$;
2. For every $i, j \in E, \;\; i \neq j, \qquad (a_i R) \cap (a_j R) = \varnothing$.

In order to define correctly the notion of a left (right) transversal in a loop to its proper subloop, the following condition must be necessarily fulfilled.

Definition 5 (see [10])**.** (**Left Condition A**) Let R be a subloop of a loop L. For all $a, b \in L$ there exists an element $c \in L$ such that

$$a(bR) = cR. \tag{1}$$

The **right Condition A** is defined analogously.
Let us denote (see [13]) $\forall a, b \in L$: a **left inner mapping**

$$l_{a,b}\left(x\right) = \left(a \cdot b\right) \setminus \left(a \cdot \left(b \cdot x\right)\right), \qquad x \in L, \tag{2}$$

where "\" is a left division in the loop $\langle L, \cdot, e\rangle$, and a **right inner mapping**

$$r_{a,b}(x) = ((x \cdot b) \cdot a) / (b \cdot a), \qquad x \in L, \tag{3}$$

where "/" is a right division in the loop $\langle L, \cdot, e\rangle$.

Lemma 1 (see [10])**.** *Let the left* ***Condition A*** *be fulfilled. Then* $\forall a, b \in L$: $l_{a,b}(R) = R$.

Lemma 2 (see [10])**.** *Let the right* ***Condition A*** *be fulfilled. Then* $\forall a, b \in L$: $r_{a,b}(R) = R$.

Definition 6 (see [9])**.** Let $\langle L, \cdot, e\rangle$ be a loop, $\langle R, \cdot, e\rangle$ be its subloop and the left **Condition A** be fulfilled. Let $\{R_x\}_{x\in E}$ be the set of all left cosets in L to R that form a left coset decomposition of the loop L. A set $T = \{t_x\}_{x\in E} \subset L$ is called a **left transversal** in L to R if T is a complete set of representatives of the left cosets R_x in L to R, i.e. there exists a unique element $t_x \in T$ such that $t_x \in R_x$ for every $x \in E$.

A right and two-sided transversal in L to R is defined analogously.

On a set E it is possible to define correctly the following operations:

$$x \overset{(T)}{\cdot} y = z \quad \overset{def}{\Leftrightarrow} \quad t_x \cdot t_y = t_z \cdot r, \text{ where } t_x, t_y, t_z \in T, \quad r \in R, \tag{4}$$

if T is a left transversal in L to R, and

$$x \overset{(T)}{\circ} y = z \quad \overset{def}{\Leftrightarrow} \quad t_x \cdot t_y = r \cdot t_z, \text{ where } \quad t_x, t_y, t_z \in T, \quad r \in R, \tag{5}$$

if T is a right transversal in L to R.

Definition 7. Let T be a left (right) transversal in L to R. If the transversal operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$ ($\langle E, \overset{(T)}{\circ}, 1\rangle$) is a loop then the transversal T is called a **left (right) loop transversal** in L to R.

Let still $\langle L, \cdot, e\rangle$ be a loop, $\langle R, \cdot, e\rangle$ be its subloop, and the left **Condition A** be fulfilled. Let $T = \{t_x\}_{x\in E}$ be a left transversal in L to R. Define the following map:

$$\begin{aligned} f &: L \times E \to E, \\ f &: (g, x) \to y = \hat{g}(x), \\ \hat{g}(x) &= y \quad \overset{def}{\Leftrightarrow} \quad g \cdot (t_x \cdot R) = t_y \cdot R. \end{aligned} \tag{6}$$

By virtue of the left **Condition A** this definition (a *left action* of the loop L on a set E) is correct.

Lemma 3 (see [10])**.** *A map* $\hat{g}$ *is a permutation on a set* E *for every element* $g \in L$.

Lemma 4 (see [10]). *For an arbitrary left transversal $T = \{t_x\}_{x\in E}$ in a loop $L = \langle L, \cdot, e\rangle$ to its subloop $R = \langle R, \cdot, e\rangle$ the following propositions are true:*

1. $\forall r \in R$: $\hat{r}(1) = 1$;

2. $\forall x, y \in E$: $\hat{t}_x(y) = x \overset{(T)}{\cdot} y$, $\hat{t}_x^{-1}(y) = x\backslash y$,

where $\hat{t}_x^{-1}$ is an inverse permutation to a permutation $\hat{t}_x$ in S_E, and "$\backslash$" is a left division in a left loop $\langle E, \overset{(T)}{\cdot}, 1\rangle$. Moreover,

$$\hat{t}_x(1) = x, \quad \hat{t}_1(x) = x, \quad \hat{t}_x^{-1}(1) = x\backslash 1, \quad \hat{t}_x^{-1}(x) = 1.$$

Lemma 5 (see [10]). *The following conditions are equivalent:*

1. *A set $T = \{t_x\}_{x\in E}$ is a left loop transversal in a loop L to its subloop R;*

2. *A set $\hat{T} = \{\hat{t}_x\}_{x\in E}$ is a sharply transitive set of permutations in the group S_E.*

3 Semidirect products of loops and suitable subgroups

Remind the definition (see [8, 13]) of the semidirect product of a left loop $L = \langle E, \cdot, 1\rangle$ with two-sided unit 1 on a suitable permutation group H on the set E ($H \subseteq St_1(S_E)$). Let the following conditions hold:

1. $\forall a, b \in E:\ l_{a,b}^{-1} L_a L_b \in H$;

2. $\forall u \in E$ and $\forall h \in H:\quad \varphi(u, h) L_{h(u)}^{-1} h\, L_u h^{-1} \in H$,

where L_a is the left translation by an element a in $\langle E, \cdot, 1\rangle$ (i.e. $L_{a(x)} = a \cdot x$).

Then on the set
$$E \times H = \{(u, h) | u \in E, h \in H\}$$
it is possible to define correctly the operation
$$(u, h_1) * (v, h_2) \overset{def}{=} (u \cdot h_1(v),\ l_{u,h_1(v)} \varphi(v, h_1) h_1 h_2).$$

The system $G = \langle E \times H, *, (1, id)\rangle$ is a group, which is called the **semidirect product** $G = L \leftthreetimes H$ of the left loop L on the group H. This product satisfies the following properties:

1. The map $\widehat{(u, h)} : E \to E$:
$$(\widehat{u, h}))\,(x) \overset{def}{=} u \cdot h(x)$$
is an **action**, i.e.

   (a) It is a permutation on E;

   (b) If $(\widehat{u, h_1})(x) = (\widehat{v, h_2})(x) \quad \forall x \in E$, then $u = v$ and $h_1 = h_2$;

(c) If $\widehat{(u,h)}(x) = (x) \quad \forall x \in E$, then $(u,h) \equiv (1, id)$.

2. $\forall x \in E$ it is true that

$$((u,\widehat{h_1) * (v},h_2))(x) = \widehat{(u,h_1)}(\widehat{(v,h_2)}(x)) = L_u h_1 L_v h_2(x).$$

3. $(u,h)^{-1} = (h^{-1}(u\backslash 1), L^{-1}_{h^{-1}(u\backslash 1)} h^{-1} L^{-1}_u)$,

   and, in particular $(u,id)^{-1} = (u\backslash 1, L^{-1}_{u\backslash 1} L^{-1}_u)$.

4. The system $\hat{H} = \langle H^*, *, (1, id)\rangle$ (where $H^* = \{(1,h)|\, h \in H\}$) is a subgroup in G, isomorphic to the group H.

5. The set $\hat{T} = \{(u,id)|\, u \in E\}$ is a left transversal in G to $\hat{H}$, and the operation $\left\langle E, \overset{(T)}{\cdot}, 1 \right\rangle$ coincides with the operation $\langle E, \cdot, 1\rangle$.

We remind the definitions of the **left multiplicative group** of a left loop L:

$$LM(L) \overset{def}{=} \langle L_x | x \in L, L_x(u) = x \cdot u\rangle\,,$$

and the **left inner permutation group** of a left loop L:

$$LI(L) \overset{def}{=} \langle l_{a,b} | a, b \in L\rangle\,.$$

It was shown in [8] that:

$$LI(L) = St_1(LM(L)) \subset LM(L), \qquad LM(L) = L \leftthreetimes LI(L).$$

Lemma 6. *Let $L = \langle E, \cdot, 1\rangle$ be a loop, $R = \langle E_1, \cdot, 1\rangle$ be its subloop, and the left* ***Condition A*** *be fulfilled for them. Let $T_0 = \{t_x\}_{x \in E_0}$ be a left transversal in L to R.*

Assume

$$G = LM(L) = L \leftthreetimes LI(L), \qquad H = LI(L).$$

Then:

1. The set $K = \{(r,h)|\, r \in R,\ h \in H\}$ is a subgroup in G, and $H \subseteq K \subset G$;

2. The set $T^*_0 = \{(t_x, id)|\, t_x \in T_0,\ x \in E_0\}$ is a left transversal in G to K, and

$$\left\langle E_0, \overset{(T_0)}{\cdot}, 1 \right\rangle \equiv \left\langle E_0, \overset{(T^*_0)}{\cdot}, 1 \right\rangle.$$

Proof. **1**. Let the conditions of the lemma hold. According to properties of semidirect product we have

$$H = \{(1,h)|h \in H = LI(L)\} \subset \{(u,h)|u \in L, h \in H\} = G.$$

Since $R \subseteq L$, then

$$\begin{gathered}\forall a, b \in R: \quad l_{a,b} \in LI(L) = H, \\ \forall u \in R \ \forall h \in H: \quad \varphi(u,h) \in \{\varphi(u,h) |\, u \in R,\, h \in H\} \subseteq \\ \subseteq \{\varphi(u,h) |\, u \in L,\, h \in H\} \subseteq LI(L) = H.\end{gathered}$$

Then it is possible to define correctly a semidirect product on the set

$$K = R \times H = \{(r,h) |\, r \in R, h \in H\} \subseteq G.$$

It is obvious that $H \subseteq K$.

Besides for any two elements (r_1, h_1) and (r_2, h_2) from K we have:

$$(r_1, h_1) * (r_2, h_2) = (r_1 \overset{(R)}{\cdot} h_1(r_2), l_{r_1, h_1(r_2)} \varphi(r_2, h_1) h_1 h_2).$$

In order that the group K be a subgroup in G it is necessary and sufficient that the following condition be fulfilled:

$$\forall r_1, r_2 \in R \quad \forall h \in H: \quad (r_1 \overset{(R)}{\cdot} h(r_2)) \in R.$$

But it is equivalent to the following: $\forall h \in H: \quad h(R) \subseteq R$, i.e. $\forall a, b \in L: l_{a,b}(R) \subseteq R$. According to Lemma 1 the last conditions are equivalent to the left Condition A for loops L and R.

2. Let $T_0 = \{t_x\}_{x \in E_0}$ be a left transversal in L to R. Then we consider the set

$$T_0^* = \{(t_x, id) |\; t_x \in T_0,\, x \in E_0\}.$$

For an arbitrary $x \in E_0$ we consider the set:

$$\begin{aligned}(t_x, id) * K &= \{(t_x, id) * (r,h) | r \in R, h \in H\} = \\ &= \{(t_x \overset{(L)}{\cdot} r, l_{t_x, r} h) | r \in R, h \in H\}.\end{aligned} \tag{7}$$

Let us show that this set is a left coset in G to K. Since the set

$$\{t_x \overset{(L)}{\cdot} r |\; r \in R\} = t_x \overset{(L)}{\cdot} R$$

is a left coset in L to R, if $x_1 \neq x_2$ then by (7) we have:

$$((t_{x_1}, id) * K) \cap ((t_{x_2}, id) * K) = \varnothing.$$

Further, let g_0 be an arbitrary element from G; by virtue of the representation $G = L \times H$ we have that $g_0 = (u_0, h_0)$, where $u_0 \in L,\ h_0 \in H$. Since $T_0 = \{t_x\}_{x \in E_0}$ is the left transversal in L to R, then $u_0 = t_{x_0} \overset{(L)}{\cdot} r_0$, where $t_{x_0} \in T_0$, $r_0 \in R$. Therefore supposing $h_1 = l^{-1}_{t_{x_0}, r_0} h_0 \in H$, we obtain

$$(t_{x_0}, id)_{\in T_0^*} * (r_0, h_1)_{\in K} = (t_{x_0} \overset{(L)}{\cdot} r_0, l_{t_{x_0}, r_0} h_1) = (u_0, h_0) = g_0.$$

So, sets of the form $(t_x, id) * K$, $x \in E_0$ are left cosets in G to K. Therefore the set

$$T_0^* = \{(t_x, id) |\, t_x \in T_0,\, x \in E_0\}$$

is a left transversal in G to K. The corresponding transversal operation is $\left\langle E_0, \overset{(T_0^*)}{\cdot}, 1 \right\rangle$, for which we have:

$$\begin{gathered} x \overset{(T_0^*)}{\cdot} y = z \quad \Leftrightarrow \quad (t_x, id) * (t_y, id) = (t_z, id) * (r, h), \quad (r, h \in K), \\ (t_x \overset{(L)}{\cdot} t_y, l_{t_x, t_y}) = (t_z \overset{(L)}{\cdot} r, l_{t_z, r} h), \\ t_x \overset{(L)}{\cdot} t_y = t_z \overset{(L)}{\cdot} r; \quad r \in R, \\ x \overset{(T_0)}{\cdot} y = z, \end{gathered}$$

i.e.

$$x \overset{(T_0^*)}{\cdot} y = x \overset{(T_0)}{\cdot} y, \quad \forall x, y \in E_0,$$

as required. □

Let us prove one additional lemma.

Lemma 7. *Let $T_0 = \{t_x\}_{x \in E_0}$ be a left transversal in L to R. Then $\forall t_u, t_x \in T_0$ and $\forall r \in R$ it is true that:*

$$(t_u \cdot r) \cdot t_x = t_u \cdot (r \cdot l^{-1}_{t_u, r}(t_x)),$$

where $l_{a,b} \in LI(L)$.

Proof. Really, by virtue of the definition of $l_{a,b}$,

$$l^{(z)}_{a,b} = (a \cdot b) \backslash (a \cdot (b \cdot z)).$$

Then

$$(t_u \cdot r) \backslash (t_u \cdot (r \cdot l^{-1}_{t_u, r}(r_x))) = l_{t_u, r} l^{-1}_{t_u, r}(t_x) = t_x,$$

i.e.

$$(t_u \cdot r) \cdot t_x = t_u \cdot (r \cdot l^{-1}_{t_u, r}(t_x)),$$

as required. □

Let us consider the permutation representations of loop L by left cosets to a subloop R and group G by left cosets to a subgroup K.

Lemma 8. *Let $\hat{L}$ be the permutation representation of a loop L by left cosets to a subloop R, i.e. $\forall g \in L$:*

$$\hat{g}(x) = y \Leftrightarrow g \overset{(L)}{\cdot} (t_x \overset{(L)}{\cdot} R) = t_y \overset{(L)}{\cdot} R,$$

where $T_0 = \{t_x\}_{x\in E_0}$ *is a left transversal in* L *to* R. *Then in the group* G *to its subgroup* K *(see Lemma 6) there exists such a left transversal* $T_0^* = \{t_x^*\}_{x\in E_0}$ *that for a suitable permutation representation* $\breve{G}$ *of the group* G *by left cosets to its subgroup* K *the following is true:*

$$\forall g\in L \;\; \exists \;\; g'\in G \quad such \quad that \quad \hat{g}(x) = \breve{g}'(x) \quad \forall x\in E_0.$$

Proof. Let the conditions of the lemma hold. According to Lemma 6, we can consider the following left transversal

$$T_0^* = \{(t_x, id) |\, t_x \in T_0\}.$$

We have in the loop L: if $g = t_u\cdot r$ (where $t_u\in T_0$, $r\in R$), then

$$\begin{gathered}\hat{g}(x) = y\,,\\ g\cdot(t_x\cdot R) = t_y\cdot R\,,\\ g\cdot t_x = t_y\cdot r'; \quad r'\in R;\\ (t_u\cdot r)\cdot t_x = t_y\cdot r'.\end{gathered}$$

By virtue of Lemma 7 we obtain:

$$t_u\cdot(r\cdot l^{-1}_{t_u,r}(t_x)) = t_y\cdot r'. \tag{8}$$

Now pass to the group G. As an element g' we take

$$g' = (t_u, k') = (t_u, id)\cdot(r, l^{-1}_{t_u,r}),$$

where $k'\in K$, $k' = (r, l^{-1}_{t_u,r})$. Then we have:

$$\breve{g}'(x) = z \quad \Leftrightarrow \quad g't_x^*K = t_z^*K \quad \Leftrightarrow \quad g't_x^* = t_z^*k', \quad k'\in K. \tag{9}$$

And so

$$\begin{gathered}(t_u, id)*(r, l^{-1}_{t_u,r})*(t_x, id) = \\ (t_u, id)*(r\cdot l^{-1}_{t_u,r}(t_x),\; l_{r,l^{-1}_{t_u,r}(t_*)}\varphi(t_x, l^{-1}_{t_u,r})l^{-1}_{t_u,r}) = \\ (t_u\cdot(r\cdot l^{-1}_{t_u,r}(t_x)),\; l_{t_u,r\cdot l^{-1}_{t_u,r}(t_x)}l_{r,l^{-1}_{t_u,r}(t_x)}\varphi(t_x, l^{-1}_{t_u,r})l^{-1}_{t_u,r}) = \\ \overset{(8)}{=} (t_y\cdot r', \underbrace{l_{t_u,r\cdot l^{-1}_{t_u,r}(t_x)}l_{r,l^{-1}_{t_u,r}(t_x)}\varphi(t_x, l^{-1}_{t_u,r})l^{-1}_{t_u,r}}_{h'}) = (t_y, id)*\underbrace{(r', h'')}_{\in K},\end{gathered}$$

where $h',\; h''\in LI(L)$.

Since $(r', h'')\in K$, then from (9) we obtain

$$t_z^*k' = g't_x^* = \underbrace{(t_y, id)}_{t_y^*}*\underbrace{(r', h'')}_{\in K}.$$

Since $T_0^* = \{t_x^*\}_{x\in E_0}$ is a left transversal in G to K then

$$t_z^*\equiv t_y^* \Leftrightarrow t_z = t_y; \quad \Leftrightarrow \quad z = y,$$

i. e. $\breve{g}'(x) = \hat{g}(x)$, as required. $\square$

4 Loop transversal in loop by its subloop

Let again L be a loop, R be its subloop, and **Condition A** be fulfilled for them. Define under what conditions a left transversal $T_0 = \{t_x\}_{x \in E_0}$ will be a left loop transversal in a loop L by its subloop R.

First prove one preliminary lemma.

Lemma 9. *Let L be a loop, R be its subloop and **Condition A** be fulfilled for them. Then*

1. $\forall a, b, c \in L$:
$$c \backslash (a \cdot (b \cdot R)) = (c \backslash (a \cdot b)) \cdot R; \tag{10}$$

2. $\forall a, b, c \in L$:
$$a \cdot (b \cdot (c \backslash R)) = (a \cdot (b \cdot (c \backslash 1))) \cdot R.$$

3. $\forall h \in LI(L)$:
$$h(a \cdot R) = h(a) \cdot R, \quad \forall a \in L. \tag{11}$$

Proof. **1**. $\forall a, b, c \in L$ by virtue of **Condition** A we have:

$$c \cdot [(c \backslash (a \cdot b)) \cdot R] = (c \cdot (c \backslash (a \cdot b)) \cdot R = (a \cdot b) \cdot R = a \cdot (b \cdot R),$$

i.e.

$$c \backslash (a \cdot (b \cdot R) = (c \backslash (a \cdot b) \cdot R.$$

2. Using **1** we have for $a \cdot b = 1$:

$$c \backslash R = c \backslash (1 \cdot R) = c \backslash ((a \cdot b) \cdot R) = (c \backslash (a \cdot b)) \cdot R = (c \backslash 1) \cdot R. \tag{12}$$

Then by virtue of **Condition** A and (12) we have:

$$a \cdot (b \cdot (c \backslash R)) = a \cdot (b \cdot ((c \backslash 1) \cdot R)) = a \cdot ((b \cdot (c \backslash 1)) \cdot R) =$$

$$= (a \cdot (b \cdot (c \backslash 1))) \cdot R.$$

3. For arbitrary $l_{a,b} \in LI(L)$ using **1** and **Condition A** we have: $\forall c \in L$

$$l_{a,b}(c \cdot R) = (a \cdot b) \backslash (a \cdot (b \cdot (c \cdot R))) = (a \cdot b) \backslash ((a \cdot (b \cdot c)) \cdot R) =$$

$$= ((a \cdot b) \backslash (a \cdot (b \cdot c))) \cdot R = l_{a,b}(c) \cdot R.$$

Besides $\forall a, b \in L$ we have: $\forall c \in L$

$$l^{-1}_{a,b}(c \cdot R) = b \backslash (a \backslash ((a \cdot b) \cdot (c \cdot R))) =$$

$$= b \backslash (a \backslash (((a \cdot b) \cdot c) \cdot R)) = b \backslash ((a \backslash ((a \cdot b) \cdot c)) \cdot R) =$$

$$= (b \backslash ((a \cdot b) \cdot c))) \cdot R = l^{-1}_{a,b}(c) \cdot R.$$

Since any $h \in LI(L)$ may be represented in the form

$$h = l^{\pm 1}_{a_1,b_1} \cdot \ldots \cdot l^{\pm 1}_{a_k,b_k},$$

then $\forall h \in LI(L)$ we have: $\forall a \in L$

$$h(a \cdot R) = l^{\pm 1}_{a_1,b_1} \cdot \ldots \cdot l^{\pm 1}_{a_k,b_k}(a \cdot R) = l^{\pm 1}_{a_1,b_1} \cdot \ldots \cdot l^{\pm 1}_{a_k,b_k}(a) \cdot R = h(a) \cdot R.$$

□

Lemma 10. *Let L be an arbitrary loop, R be its subloop, and* ***Condition A*** *be fulfilled for them. Then the following conditions for an arbitrary left transversal $T_0 = \{t_x\}_{x \in E_0}$ in L to R are equivalent:*

1. *T_0 is a left transversal in L to R;*
2. *$\forall u \in L$ and $\forall h \in LI(L)$ the set $T_{u,h}\{u \cdot h(t_x \cdot h^{-1}(u\backslash 1))\}_{x \in E_0}$ is a left transversal in L to R;*
3. *$\forall v \in E_0$ the set $T_v\{t_v \cdot (t_x(t_v\backslash 1))\}_{x \in E_0}$ is a left transversal in L to R;*
4. *$\forall u \in L$ the set $T^*_u\{(u\backslash(t_x \cdot u))\}_{x \in E_0}$ is a left transversal in L to R;*
5. *$\forall v \in E_0$ the set $T^*_v\{t_v\backslash(t_x \cdot t_v)\}_{x \in E_0}$ is a left transversal in L to R.*

Proof. Let conditions of the lemma hold. Using the results of the previous section we have the following sequence of equivalent statements (according to Lemma 6):

– a left transversal $T_0 = \{t_x\}_{x \in E_0}$ in L to R is a left loop by a transversal in L to R

$$\Leftrightarrow \quad \text{the operation } \left\langle E_0, \overset{(T_0)}{\cdot}, 1 \right\rangle \text{ is a loop} \quad \Leftrightarrow$$

– the left transversal $T^*_0 = \{\underbrace{(t_x, id)}_{t^*_x}\}_{x \in E_0}$ in a group G to its subgroup K is a loop transversal (where $G = L \rightthreetimes LI(L)$, $K = R \rightthreetimes LI(L)$), and $\left\langle E_0, \overset{(T^*_0)}{\cdot}, 1 \right\rangle$ is a loop, coincides with the loop $\left\langle E_0, \overset{(T_0)}{\cdot}, 1 \right\rangle$.

The last statement is equivalent to every of the following statements (see [1,6,11]):

1. $\forall g \in G$ the set $gT^*_0 g^{-1}$ is a left transversal in G to K;
2. $\forall x \in E_0$ the set $t^*_x T^*_0 {t^*_x}^{-1}$ is a left transversal in G to K;
3. $\forall g \in G$ the set $g^{-1}T^*_0 g$ is a left transversal in G to K;
4. $\forall x \in E_0$ the set ${t^*_x}^{-1} T^*_0 t^*_x$ is a left transversal in G to K.

Further we have: if $g \in G$, $g = (u, h)$, where $u \in L$, $h \in H = LI(L)$, therefore $\forall x \in E_0$:

$$((u,h) * (t_x, id) * (u,h)^{-1}(z) = (L_u h) * L_{t_x} * (L_u h)^{-1}(z) =$$
$$= L_u h L_{t_x} h^{-1} L_u^{-1}(z). \tag{13}$$

The set $gT_0^* g^{-1}$ is a left transversal in G to K if and only if

1) $\bigcup\limits_{x \in E_0} (g t_x^* g^{-1}) * K = G$;

2) $\forall x_1 \neq x_2$ from E_0:

$$(g t_{x_1}^* g^{-1}) * K \cap (g t_{x_2}^* g^{-1}) * K = \emptyset. \tag{14}$$

So $\forall v \in L$ and $h \in H$ we have

$$(v,h) * K = \bigcup_{r \in R,\ h_1 \in H} ((v,h) * (r, h_1)) =$$
$$= \bigcup_{r \in R,\ h_1 \in H} (v \cdot h(r), l_{v,h(v)} \varphi(r,h) h h_1) =$$
$$= (v \cdot h(R), H) = (L_v h(R), H).$$

Then the conditions (14) (using (13)) are equivalent to the following:

1) $\bigcup\limits_{x \in E_0} (L_u h L_{t_x} h^{-1} L_u^{-1}(R)) = L$;

2) $\forall x_1, x_2 \in E_0$, $x_1 \neq x_2$:

$$(L_u h L_{t_{x_1}}(R)) \cap (L_u h L_{t_{x_2}} h^{-1} L_u^{-1}(R)) = \emptyset. \tag{15}$$

By virtue of item **2** from Lemma 9 we obtain that the conditions (15) are equivalent to the following:

1) $\bigcup\limits_{x \in E_0} [(u \cdot h(t_x \cdot h^{-1}(u \backslash 1))) \cdot R] = L$;

2) $\forall x_1, x_2 \in E_0$, $x_1 \neq x_2$:

$$[(u \cdot h(t_{x_1} \cdot h^{-1}(u \backslash 1))) \cdot R] \cap [(u \cdot h(t_{x_2} \cdot h^{-1}(u \backslash 1))) \cdot R] = \emptyset. \tag{16}$$

The conditions (16) are equivalent to that the set $T_{u,h}\{u \cdot h(t_x \cdot h^{-1}(u \backslash 1))\}$ is a left transversal in L by R. Remembering that the reasoning was carried out $\forall g \in G$, i.e. $\forall u \in L$ and $\forall h \in H = LI(L)$, we obtain item 2 of the present lemma.

The items **3**, **4** and **5** are proved similarly to the previous reasoning, using the corresponding statements and Lemma 9. □

Corollary 1. *Let L be a loop, R be its subloop, and **Condition A** be fulfilled for them. Let $T_0 = \{t_x\}_{x \in E_0}$ be a left loop transversal in L to R. Then $\forall u \in L$ the set $T_u\{u \cdot (t_x \cdot (u \backslash 1))\}_{x \in E_0}$ is a left transversal in L to R.*

Proof. The proof easily follows from Lemma 10, **2**, when $h = id$. □

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Eugene Kuznetsov
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
5 Academiei str.
Chishinau MD-2028, Moldova
E-mail: *kuznet1964@mail.ru*

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Spectra of semimodules

Reza Ebrahimi Atani, Shahabaddin Ebrahimi Atani

Abstract. The purpose of this paper is to investigate possible structures and useful properties of prime subsemimodules of a semimodule M over a semiring R and show various applications of the properties. The main part of this work is to introduce a new class of semimodules over R called strong primeful R-semimodules. It is shown that every non-zero strong primeful semimodule possesses the non-empty prime spectrum with the surjective natural map. Also, it is proved that this class contains the family of finitely generated R-semimodules properly.

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1 Introduction

Semimodules over semirings also appear naturally in many areas of mathematics. For example, semimodules are useful in the area of theoretical computer science as well as in the solution of problems in the graph theory and cryptography [13, 18]. This paper generalizes some well know results on prime submodules in commutative rings to commutative semirings. The main difficulty is figuring out what additional hypotheses the ideal or subsemimodule must satisfy to get similar results. The two new key notions are that of a "strong ideal" and a "strong subsemimodule". Moreover, quotient semimodules are determined by equivalence relations rather than by subsemimodules as in the module case. Allen [1] has presented the notion of a partitioning ideal ($= Q$-ideal) I in the semiring R and constructed the quotient semiring R/I. Quotient semimodules over a semiring R have already been introduced and studied by present authors in [10]. Chaudhari and Bonde extended the definition of Q_M-subsemimodule of a semimodule and some results given in Section 2 in [10] to a more general quotient semimodules case in [3]. Of course "quotient semimodule" is a natural extension of "quotient semiring" and, hence, ought to be in the literature. So quotient semimodules are particularly important in the study of the representation theory of semimodules over semiring. The representation theory of semimodules over semirings has developed greatly in the recent years. One of the aims of the modern representation theory of semimodules is to generalize the properties of modules over rings to semimodules over semirings. The aim of present paper is to extend some basic results of C. P. Lu [15, 16, 17] to semimodules over semirings. We know (at least as far as we are aware) of no systematic study of the topological space Spec(M) in the semimodule over semiring context. Our results is particularly important in the topological space Spec(M) equipped with a topology called the Zariski topology in the semimodule context and, we hope to address in a later paper.

2 Preliminaries

In order to make this paper easier to follow, we recall in this section various notions from semimodule theory which will be used in the sequel. For the definitions of monoid, semirings, semimodules and subsemimodules we refer to [4, 9, 10, 13, 14]. All semiring in this paper are commutative with non-zero identity.

Definition 1. (a) A semiring R is said to be semidomain whenever $a, b \in R$ with $ab = 0$ implies that either $a = 0$ or $b = 0$.

(b) A semifield is a semiring in which non-zero elements form a group under multiplication.

(c) An R-semimodule M is said to be semivector space if R is a semifield.

(d) Let M be a semimodule over a semiring R. A subtractive subsemimodule (= k-subsemimodule) N is a subsemimodule of M such that if $x, x + y \in N$, then $y \in N$ (so $\{0_M\}$ is a k-subsemimodule of M).

(e) A prime subsemimodule (resp. primary subsemimodule) of M is a proper subsemimodule N of M in which $x \in N$ or $rM \subseteq N$ (resp. $x \in N$ or $r^n M \subseteq N$ for some positive integer n) whenever $rx \in N$. The collection of all prime (resp. maximal) subsemimodules of M is called the spectrum (resp. the maximal spectrum) of M and denoted by Spec(M) (resp. Max(M)). Similarly, the collection of all P-prime subsemimodules of M for any prime k-ideal P of R is designated by $\text{Spec}_P(M)$. We define k-ideals and prime ideals of a semiring R in a similar fashion.

(f) We say that $r \in R$ is a zero-divisor for a semimodule M if $rm = 0$ for some non-zero element m of M. The set of zero-divisors of M is written $Z_R(M)$.

(g) An R-semimodule M is called multiplication semimodule provided that for every subsemimodule N of M there exists an ideal I of R such that $N = IM$.

(h) We say that M is a torsion-free R-semimodule whenever $r \in R$ and $m \in M$ with $rm = 0$ implies that either $m = 0$ or $r = 0$ (so every semivector space over a semifield R is a torsion-free R-semimodule).

(i) A proper ideal I of a semiring R is said to be strong ideal (or strongly zero-sum ideal) if for each $a \in I$ there exists $b \in I$ such that $a + b = 0$ (see [11, Example 2.3] and [8]).

A subsemimodule N of a semimodule M over a semiring R is called a partitioning subsemimodule (= Q_M-subsemimodule) if there exists a subset Q_M of M such that $M = \cup\{q + N : q \in Q_M\}$ and if $q_1, q_2 \in Q_M$ then $(q_1 + N) \cap (q_2 + N) \neq \emptyset$ if and only if $q_1 = q_2$. Let N be a Q_M-subsemimodule of M and let $M/N = \{q + N : q \in Q_M\}$. Then M/N forms an R-semimodule under the operations $\oplus$ and $\odot$ defined as follows: $(q_1 + N) \oplus (q_2 + N) = q_3 + N$, where $q_3 \in Q_M$ is the unique element such that $q_1 + q_2 + N \subseteq q_3 + N$ and $r \odot (q_1 + N) = q_4 + I$, where $r \in R$ and $q_4 \in Q_M$ is the unique element such that $rq_1 + N \subseteq q_4 + N$. This R-semimodule M/N is called the quotient semimodule of M by N [3]. By [3, Lemma 2.3], there exists a unique element $q_0 \in Q_M$ such that $q_0 + N = N$. Thus $q_0 + N$ is the zero element of M/N. Also, [3, Theorem 2.4] show that the structure $(M/N, \oplus, \odot)$ is essentially independent of Q_M (see [3, Example 2.6]).

3 Spec(M)

In this section we extend some results of C. P. Lu [15] to semimodules over semirings.

Remark 1. (Change of semirings.) Assume that I is a Q-ideal of a semiring R and let N be a Q_M-subsemimodule of an R-semimodule M. We show now how M/N can be given a natural structure as a semimodule over R/I. Let $q_1, q_2 \in Q$ be such that $q_1 + I = q_2 + I$, and let $m_1, m_2 \in Q_M$ be such that $m_1 + N = m_2 + N$. Then $q_1m_1 + N = q_2m_2 + N$. By assumption, there exist the unique elements $t_1, t_2 \in Q_M$ such that $q_1m_1 + N \subseteq t_1 + N$ and $q_2m_2 + N \subseteq t_2 + N$; so $t_1 = t_2$. Hence we can unambiguously define a mapping $R/I \times M/N$ into M/N (sending $(q + I, m + N)$ to $t + N$), where $qm + N \subseteq t + N$ for some unique element $t \in Q_M$, and it is routine to check that this turns the commutative additive semigroup with a zero element M/N into an R/I-semimodule.

Definition 2. A proper subsemimodule N of a semimodule M over a semiring R is said to be strong subsemimodule if for each $x \in N$ there exists $y \in N$ such that $x + y = 0$.

Example 1. Let that E_0^+ be the set of all non-negative integers. The monoid $M = (Z_6, +_6)$ is a semimodule over $(E_0^+, +, .)$ (see [13, p. 151]). An inspection will show that $N = \{\bar{0}, \bar{2}, \bar{4}\}$ is a strong Q_M-subsemimodule of M, where $Q_M = \{\bar{0}, \bar{1}\}$.

Lemma 1. *Let N be a strong Q_M-subsemimodule of a module M over a semiring R. Then the following hold:*

(i) If $q_0 \in Q_M$ and $q_0 + N$ is the zero in M/N, then $q_0 \in N$.

(ii) If $q \in N \cap Q_M$ and $q_0 + N$ is the zero in M/N, then $q = q_0$.

(iii) If $q_0 + N$ is the zero in M/N, then $m \in N$ if and only if $m + N = \{m + a : a \in N\}$ and $N + m = N$ are equal as sets.

Proof. (i) By [3, Lemma 2.3], $q_0 + N = N$; hence $q_0 \in N$ since every Q_M-subsemimodule is a k-subsemimodule of M by [3, Theorem 3.2].

(ii) Since $q + q_0 \in (q + N) \cap (q_0 + N)$, we must have $q = q_0$. (iii) follows from (i) and (ii).

(iii) Let $m \in N$. Since the inclusion $m + N \subseteq N$ is clear, we will prove the reverse inclusion. Assume that $x \in N$. There exist $a, b, b' \in N$ such that $x = q_0 + a$, $m = q_0 + b$ and $b + b' = 0$; so $x = m + a + b' \in m + N$, and so we have equality. The other implication is obvious. □

Theorem 1. *Let N be a proper strong Q_M-subsemimodule of a semimodule M over a semiring R with $(N : M) = P$ a Q-ideal of R. Then the following statements are equivalent:*

(i) N is a prime subsemimodule of M;
(ii) M/N is a torsion-free R/P-semimodule;
(iii) $(N :_M < r >) = N$ for every $r \in R - P$;
(iv) $(N :_M J) = N$ for every ideal $J \nsubseteq P$;
(v) $(N :_R < m >) = P$ for every $m \in M - N$;
(vi) $(N :_R L) = P$ for every subsemimodule L of M properly containing N;
(vii) $Z_R(M/N) = P$.

Proof. $(i) \Rightarrow (ii)$ Note that M/N is an R/P-semimodule by Remark 1. Assume that q_0 is the unique element in Q_M such that $q_0 + N$ is the zero in M/N and let $(q + P)(m + N) = q_0 + N$, where $qm + N \subseteq q_0 + N$ for some $q \in Q$ and $m \in Q_M$, so $qm \in N$ since N is a k-subsemimodule of M. Therefore, N prime gives either $q \in P$ or $m \in N$. If $q \in P$, then $q + P$ is the zero in R/P by [6, Lemma 2.3]. If $m \in N$, then $m + N$ is the zero in M/N by Lemma 1. Thus M/N is torsion-free semimodule as an R/P-semimodule.

$(ii) \Rightarrow (iii)$ Assume that $q_0 + P$ is the zero element in R/P. It suffices to show that $(N :_M < r >) \subseteq N$. Let $m \in (N :_M < r >)$. Then $rm \in N$, $r = q + a$ and $m = t + x$ for some $q \in Q$, $a \in P$, $t \in Q_M$ and $x \in N$ (so $q \notin P$); hence $qt \in N$ since N is a k-subsemimodule. Since $(q + P)(t + N) = q_0 + N$ by Lemma 1 and $q + P \neq q_0 + P$, we must have $t + N = q_0 + N$; hence $t = q_0 \in N$. Therefore, $m = t + x \in N$, and so we have equality.

$(iii) \Rightarrow (iv)$ Clearly, $N \subseteq (N :_M J)$. For the reverse inclusion, assume that $m \in (N :_M J)$. By assumption, there exists $r \in J$ such that $r \in R - P$ and $rm \in N$; so $(N :_M < r >) = N$ by (iii). This completes the proof.

$(iv) \Rightarrow (v)$ Since $PM \subseteq N$, we conclude that $P \subseteq (N :_R < m >)$ for every $m \in M - N$. For the other containment, assume that $m \in M - N$ and $r \in (N :_R < m >)$; we show that $r \in P$. Suppose not. Then $J = < r > \nsubseteq P$, and so $m \in (N :_M J) = N$ by (iv), which is a contradiction, as required.

$(v) \Rightarrow (vi)$ If $a \in P$, then $aL \subseteq aM \subseteq N$; so $P \subseteq (N :_R L)$. Now suppose that $b \in (N :_R L)$. By assumption, there exists $m \in L$ such that $m \in M - N$. Then $b \in (N :_R < m >) = P$ by (v), as needed.

$(vi) \Rightarrow (vii)$ Let $r \in Z_R(M/N)$. Then there exists $t \in Q_M - N$ such that $r(t + N) = q_0 + N$, where $rt + N \subseteq q_0 + N$, so $rt \in N$ since N is a k-subsemimodule; hence $r \in (N :_R Rt + N) = P$ by (vi). Thus $Z_R(M/N) \subseteq P$. For the reverse conclusion, assume that $a \in P$. By assumption, there is an element $m \in M - N$ such that $am \in N$. There exist $s \in Q_M - N$ and $y \in N$ such that $m = s + y$ (so

$s \notin N$) such that $as \in N$; hence $a(s+N) = q_0+N$ by Lemma 1. Thus $a \in Z_R(M/N)$. This completes the proof.

$(vii) \Rightarrow (i)$ Let $rm \in N$ for some $r \in R$ and $m \in M - N$; we show that $r \in P$. By assumption, there are elements $t \in Q_M - N$ and $z \in N$ such that $m = t + z$, so $rt \in N$. Then $r(t + N) = q_0 + N$ by Lemma 1; hence $r \in Z_R(M/N) = P$ by (vii), as required. □

Proposition 1. *Let N be a proper strong Q_M-subsemimodule of a semimodule M over a semiring R with $(N : M) = P$ a maximal Q-ideal of R. Then N is a prime subsemimodule. In particular, $P'M$ is a prime subsemimodule of an R-semimodule M for every maximal Q-ideal P' of R such that $P'M \neq M$.*

Proof. By [4, Theorem 2.10], R/P is a semifield, so M/N is a semivector space over the semifield R/P by Remark 1; hence it is a torsion-free R/P-semimodule. Thus N is prime by Theorem 1. Finally, suppose that $(P'M : M) = J \neq R$. Then $P' \subseteq J$, so $J = P'$ since P' is maximal, as required. □

Theorem 2. *Let N be a proper strong Q_M-subsemimodule of a semimodule M over a semiring R with $(N : M) = P$ a Q-ideal of R and let P be a maximal ideal of R. Then N is P-prime if and only if $PM \subseteq N$. In particular, if N is a P-prime subsemimodule of M, then so is every proper subsemimodule of M containing N.*

Proof. It suffices to show that if $PM \subseteq M$, then N is P-prime. Let $p \in P$. Then $p \in (N : M)$, so $P = (N : M)$ by maximality of P. Now apply Proposition 1. □

Proposition 2. *Let M be a finitely generated semimodule over a semiring R and let I be a strong k-ideal of R such that $I = \mathrm{rad}(I)$. Then $(IM : M) = I$ if and only if $\mathrm{ann}(M) \subseteq I$.*

Proof. The necessity is clear. Assume that $\mathrm{ann}(M) \subseteq I$ and let $x \in (IM : M)$. First we show that if M is generated by n elements, then there exists a $y \in I$ such that $x^n+y \in \mathrm{ann}(M)$. To see that, we use induction on n. Consider first the case in which $n = 1$. Here we have $x < m > \subseteq I < m >$. So $xm = sm$ for some $s \in I$; hence there is an element $s' \in I$ such that $(x+s')m = sm+s'm = 0$. It follows that $(x+s')M = 0$. We now turn to the inductive step. Assume, inductively, that $n = k + 1$, where $k \geq 1$, and that the result has been proved in the case where $n = k$. Then we must have $(x + a)(x^k + b)M = (x^{k+1} + ax^k + bx + ab)(< m_1, ..., m_k > + < m_{k+1} >) = 0$ for some $a, b \in I$, so $(x^{k+1}+c)M = 0$, where $ax^k+bx+ab = c \in J$. Thus $x^n+y \in I$. Since I is a k-ideal, we must have $x^n \in I$ and, therefore, $(IM : M) \subseteq \mathrm{rad}(I) = I$. Now we can see easily that $(IM : M) = I$. □

Theorem 3. *If M is a finitely generated semimodule over a semiring R and P is a strong maximal Q-ideal of R containing $\mathrm{ann}(M)$, then $PM \neq M$ so that PM is a prime subsemimodule of M. In particular, if M is a finitely generated faithful R-semimodule, then PM is a prime subsemimodule of M for every strong maximal Q-ideal P of R.*

Proof. Apply Proposition 1 and Proposition 2 (note that every Q-ideal is a k-ideal). □

4 Spec(M_S)

Assume that S is a multiplicatively closed subset of the commutative semiring R and let M be an R-semimodule. We introduce a useful relationship between Spec(M) and Spec(M_S) (Theorem 6) and exhibit its application through the remaining of the paper.

Lemma 2. *Let R be a semiring. If N is a primary subsemimodule of an R-semimodule M, then $(N : M)$ (or equivalently* $\text{ann}(M/N)$*) is a primary ideal.*

Proof. Since $M \nsubseteq N$, the ideal $(N; M)$ is a proper ideal. Now suppose that $a, b \in R$ such that $ab \in (N : M)$, $b \notin (N : M)$. Since $b \notin (N : M)$, there exists $m \in M$ such that $bm \notin N$. But N is a primary submodule, consequently $a^s M \subseteq N$ for some integer s. This completes the proof. □

If N is a primary subsemimodule of an R-semomodule M, then Lemma 2 shows that $P' = (N : M)$ is a primary ideal. Consequently, $P = \text{rad}(P')$ is a prime ideal. In this case, we shall say that N is P-primary.

Lemma 3. *Let R be a semiring. A primary subsemimodule N of any R-semimodule M is prime if and only if $(N : M)$ is a prime ideal. In particular, if K is a P-primary subsemimodule of M containing a P-prime subsemimodule, then K is prime.*

Proof. The proof is straightforward. □

Definition 3. Let S be a multiplicatively closed subset of the commutative semiring R, and let M be an R-semimodule. M is called a S-cancellative semimodule whenever $am = an$ for some $0 \neq a \in S$ and $m, n \in M$, then $m = n$. A semiring is called a S-cancellative semiring if it is a S-cancellative semimodule over itself.

Example 2. Assume that E_0^+ is the set of all non-negative integers and let $S = E_0^+ - \{0\}$. Then $(E_0^+, +, .)$ is a S-cancellative semiring. Let $M = (E_0^+, \text{gcd})$. Clearly, M is a commutative monoid in which every element is idempotent. Moreover, M is a S-cancellative semimodule over E_0^+ with scalar multiplication defined by $rm = 0$ if $r = 0$ and $rm = m$ if $r > 0$ for all $r \in E_0^+$ and $m \in M$ [13, p. 151].

Let R be a S-cancellative semiring. Define a relation $\sim$ on $R \times S$ as follows: for $(a, s), (b, t) \in R \times S$, we write $(a, s) \sim (b, t)$ if and only if $ad = bc$. Then $\sim$ is an equivalence relation on $R \times S$. For $(a, s) \in R \times S$, denote the equivalence class of $\sim$ which contains (a, s) by a/s, and denote the set of all equivalence classes of $\sim$ by R_S. Then R_S can be given the structure of a commutative semiring under operations for which $a/s + b/t = (ta + sb)/st$, $(a/s)(b/t) = (ab)/st$ for all $a, b \in r$ and $s, t \in S$.

This new semiring R_S is called the semiring of fractions of R with respect to S; its zero element is $0/1$, its multiplicative identity element is $1/1$ and each element of S has a multiplicative inverse in R_S (see [9, 13, 19]). Assume that R is a semidomain and let $S = R - \{0_R\}$. Then R_S is a semifield. The semifield F constructed from the semidomain R is referred to as the semifield of fractions of the semidomain R. Moreover, assume that P is a prime ideal of R. Then $S = R - P$ is a multiplicatively closed subset of R. In this case we set $R_S = R_P$ and $I_S = IR_P$, where I is an ideal of R.

Let M be a S-cancellative semimodule over a S-cancellative semiring R. The relation $\sim'$ on $M \times S$ defined by, for $(m,s),(n,t) \in M \times S$, $(m,s) \sim' (n,t)$ if and only if $tm = sn$ is an equivalence relation on $M \times S$; for $(m,s) \in M \times S$, the equivalence class of $\sim'$ which contains (m,s) is denoted by m/s. Similarly, a simple argument will show that the set M_S of all equivalence classes of $\sim'$ has the structure of a semimodule over the semiring R_S of fractions of R with respect to S under operations for which $m/s + n/t = (tm + sn)/st$, $(r/s)(n/t) = (rn)/st$ for all $m, n \in M$, $s, t \in S$ and $r \in R$. The R_S-semimodule M_S is called the semimodule of fractions of M with respect to S; its zero element is $0_M/1$, and this is equal to $0_M/s$ for all $s \in S$.

Convention. Throughout this section we shall assume unless otherwise stated, that R denotes a commutative S-cancellative semiring with an identity element and S a non-empty multiplicatively closed subset of R. M will designate a fixed S-cancellative semimodule over R. If N is a subsemimodule of M, then N_S will be regarded as an R_S-subsemimodule of M_S.

Proposition 3. *Let N be a P-primary subsemimodule of M. If $P \cap S \neq \emptyset$, then $N_S = M_S$. On the other hand if $P \cap S = \emptyset$, then N_S is a P_S-primary subsemimodule of M_S and $N = N_S \cap M = \{m \in M : m/1 \in N_S\}$.*

Proof. First suppose that there is an element s which is common to P and S. Since $P = \mathrm{rad}(N : M)$, there is an integer n such that $s^n M \subseteq N$. Suppose now that $m/s' \in M_S$. Then $m/s' = (s^n m)/(s^n s') \in N_S$. This shows that $M_S = N_S$ and the first assertion follows. From here on we assume that P does not meet S. Since the inclusion $N \subseteq M \cap N_S$ is clear, we will prove the reverse inclusion. Let $m \in N_S \cap M$. Then there are elements $n \in N$ and $t \in S$ such that $m/1 = n/t$; hence $tm \in N$. Using the facts that N is P-primary in M and $t \notin P$, we conclude that $m \in N$, and so we have equality. This shows, in particular, that N_S is a proper subsemimodule of M_S. Assume next that $(r/s)(m/t) \in N_S$ and $m/t \notin N_S$. Then $m \notin N$. Multiplying $(rm)/(st)$ by $(s^2t^2)/(st)$, we obtain $(rm)/1 = (s^2t^2rm)/(s^2t^2) \in N_S$. Thus $rm \in N$. It now follows that $r^v M \subseteq N$ for some integer v, which in turn implies that $(r/s)^v M_S \subseteq N_S$. This establishes that N_S is a primary subsemimodule of M_S. By [5, Lemma 2.3] it must be P'_S-primary, where P' is a prime ideal of R with $P' \cap S = \emptyset$. Let $a \in P$. Then $a^n M \subseteq N$ for some integer n; hence if $s \in S$, then $((sa)/s)^n M_S \subseteq N_S$. It follows that $(sa)/s \in P'_S$ and therefore $a \in P'_S \cap R = P'$ by

[5, Lemma 2.3] again; thus $P \subseteq P'$. For the other containment, assume that $b \in P'$. If now $t \in S$, then there exists an integer u such that $((tb)/t)^u M_S \subseteq N_S$. Select $m \in M$ so that $m \notin N$. Then $(t^{u-1}b^u m)/t^{u+1} = ((tb)/t)^u (tm)/t \in N_S$ and therefore $t^u m \in N_S \cap M = N$; hence $t \in P$, and so we have equality, as required. □

Let N be a subsemimodule of a semimodule M. An inspection will show that $N_S \cap M = \{m \in M : sm \in N \text{ for some } s \in S\}$. Let P be a prime ideal of R. The saturation $S_P(N) = N_P \cap M$ of N with respect to P is known in the literature as the S-component of N in M for the multiplicatively closed subset $S = R - P$, but designated in various way. A subsemimodule K of M is said to be saturated with respect to P if $S_P(K) = K$.

Theorem 4. *Every P-primary subsemimodule of a semimodule M is a saturated subsemimodule.*

Proof. Apply Proposition 3. □

Theorem 5. *Let P be a prime ideal of R with $P \cap S = \emptyset$, and let M be an R-semimodule. Then there is a one-to-one correspondence between the P-primary subsemimodules N of M and the P_S-primary subsemimodules L of M_S. This is such that, when N and L correspond, $L = N_S$ and $N = L \cap M$.*

Proof. Let L be a P_S-primary subsemimodule of M_S. By Proposition 3, it is enough to show that there is a P-primary subsemimodule N of M such that $N = L \cap M$. Suppose that $L \cap M = N$; we show that $N_S = L$. Since the inclusion $N_S \subseteq L$ is clear, we will prove the reverse inclusion. Let $x \in L$. Then $x = m/s$ for some $m \in M$ and $s \in S$, so $(s^2/s)(m/s) = m/1 \in L$ and therefore $m \in N$. It follows that $m/s \in N_S$. This shows that $L = N_S$ and N is a proper subsemimodule of M. Now assume that $rm \in N$, where $r \in R$, $m \in M$ and $m \notin N$. If s is an arbitrary element of S, then $(rs)/s)((sm)/s^2) = (rs^2m)/s^2 \in N_S = L$. On the other hand, $(sm)/s \notin L$ for the contrary assumption would imply that $m \in N$. So there exists an integer w such that $((rs)/s)^w M_S \subseteq L$ since L is primary. Let $m' \in M$ then $(s^{w+1}r^w m')/s^{w+1} = ((rs)/s)^w (sm')/s \in L$, whence $r^w m' \in N$. As this holds for every $m' \in M$, we may conclude that $r^w M \subseteq N$. This proves that N is a primary subsemimodule of M. Let it be P'-primary. Since $N_S = L$ and $L \neq M_S$, Proposition 3 shows that $P' \cap S \neq \emptyset$. The same proposition shows that $N_S = L$ is P'_S-primary. Thus $P'_S = P_S$ and therefore $P = P'$ by [4, Lemma 2.3]. This completes the proof. □

Lemma 4. *Let M be an R-semimodule. Then the following hold:*

(i) If $N_1, N_2, ..., N_k$ are subsemimodules of M, then $(N_1 + N_2 + ... + N_k)_S = (N_1)_S + (N_2)_S + ... + (N_k)_S$ and $(N_1 \cap N_2 \cap ... \cap N_k)_S = (N_1)_S \cap (N_2)_S \cap ... \cap (N_k)_S$.

(ii) If $m \in M$ and N is a subsemimodule of M, then $(N : m)_S = (N_S : m/1)$.

(iii) If $m_1, m_2, ..., m_n$ are elements which generate M, then the R_S-semimodule generated by $m_1/1, m_2/2, ..., m_n/1$ is just M_S.

(iv) If I is an ideal of R, then $I_S = R_S$ if and only if $I \cap S \neq \emptyset$.

(v) If M is finitely generated and N a subsemimodule of M, then $(N : M)_S = (N_S : M_S)$. In particular, $(\mathrm{ann}(M))_S = \mathrm{ann}(M_S)$.

(vi) If M is finitely generated and N a subsemimodule of M, then $N_S = M_S$ if and only if $(N : M) \cap S \neq \emptyset$.

Proof. The proofs of (i), (ii), (iii) and (iv) are straightforward. To see that (v), let $m_1, m_2, ..., m_k$ be elements which generate M. Then $(N : M) = (N : m_1) \cap ... \cap (N : m_k)$ and therefore, by (i) and (ii), $(N : M)_S = (N : m_1)_S \cap ... \cap (N : m_k)_S = (N_S : R_S m/1 + ... + R_S m_k/1)$. This completes the proof by (iii).

(vi) We have $N_S = M_S$ if and only if $(N_S : M_S) = R_S$. By (v), $(N : M)_S = (N_S : M_S)$ and, by (iv), this equals R_S if and only if S meets $(N : M)$, as required. □

Theorem 6. *Let P be a prime ideal of R with $P \cap S = \emptyset$, and let M be an R-semimodule. Then there is a one-to-one correspondence between the P-prime subsemimodules N of M and the P_S-prime subsemimodules L of M_S. This is such that, when N and L correspond, $L = N_S$ and $N = L \cap M$.*

Proof. By Theorem 5, we need to show that, under this correspondence of primary subsemimodules, N is prime if and only if $L = N_S$ is prime. By Lemma 4, it suffices to show that $(N :_R M) = P$ if and only if $(N_S :_{R_S} M_S) = P_S$ provided that $P = \mathrm{rad}(N : M)$ and $P_S = \mathrm{rad}(N_S : M_S)$ as N and N_S are, respectively, P-primary and P_S-primary. If $P = (N : M)$, then $P_S = (N : M)_S \subseteq (N_S : M_S) \subseteq \mathrm{rad}(N_S : M_S) = P_S$ whence $(N_S : M_S) = P_S$. Conversely, if $(N_S : M_S) = P_S$, then $P_S M_S \subseteq N_S$ so that $(p/s)(m/t) \in P_S$ for every $p \in P$, $m \in M$, and $s, t \in S$. Since $(pm/st)(s^2t^2/st) \in N_S$, $pm \in N$ for every $m \in M$. Thus $p \in (N : M)$ for every $p \in P = \mathrm{rad}(N : M)$. Therefore, $(N : M) = \mathrm{rad}(N : M) = P$. □

Corollary 1. *If N is a prime subsemimodule of an R-semimodule M, then $(N : M)_S = (N_S : M_S)$.*

Proof. In the proof of Theorem 6 we have seen that if $(N : M) \cap S = \emptyset$, then $(N : M)_S = (N_S : M_S)$. On the other hand if $(N : M) \cap S \neq \emptyset$, then $N_S = M_S$ by Proposition 9 so that $(N : M)_S = (N_S : M_S) = R_S$. □

Corollary 2. *Let M be an R-semimodule and P a prime ideal of R. Then the prime subsemimodules of the R_P-semimodule M_P are in a one-to-one correspondence with those prime subsemimodules N of M with $(N : M) \subseteq P$.*

Proof. Set $S = R - P$ and apply Theorem 6. □

Proposition 4. *Let R be a semiring and N a subsemimodule of an R-semimodule M. If $N_S \neq M_S$, then $(N : M) \cap S = \emptyset$. Conversely, if $(N : M) \cap S = \emptyset$, then $N_S \neq M_S$ provided that either i) M is finitely generated or ii) N is a primary subsemimodule.*

Proof. Assume that $(N : M) \cap S \neq \emptyset$ and $r \in (N : M) \cap S$. Let $m/s \in M_S$. Then $rm \in N$ so that $m/s = (rm)/(rs) \in N_S$, which proves that $N_S = M_S$, a contradiction. Thus $(N : M) \cap S = \emptyset$. Conversely, assume that $(N : M) \cap S = \emptyset$. Note that $(N : M) \cap S = \emptyset$ if and only if $\mathrm{rad}(N : M) \cap S = \emptyset$. Now the assertion follows from Lemma 4 (vi) and Proposition 3. □

Proposition 5. *Let R be a semiring and N a subsemimodule of an R-semimodule M such that $(N : M)$ is a primary ideal (resp. $(N : M) = P$) for some prime ideal P of R. Then N is a P-primary (resp. P-prime) subsemimodule of M if and only if $N_P \cap M = N$.*

Proof. The necessity is due to Proposition 3 (resp. Theorem 6). To see the sufficiency, suppose that $rm \in N$ such that $m \in M - N$ and $r \in R$; we show that $r \in P$. Suppose not. Then $m/1 = (rm)/r \in N_P$, so $m \in N_P \cap M = N$, which is a contradiction. Thus, $r \in P$ so that N is a P-primary (resp. P-prime) subsemimodule of M. □

Theorem 7. *Let R be a semidomain which is not a semifield and F the field of fractions of R. Then the R-semimodule F has* $\mathrm{Spec}(F) = \{0\}$.

Proof. Let N be a proper subsemimodule of F. Then $(N : F) = 0$ since $aF = F$ for every non-zero element a of R. Let $r.a/b = (ra)/b \in \{0/1\}$ such that $r \in R$ and $a/b \neq 0/1$. Then $a \neq 0_R$ and $ra = 0_R$; so $r = 0$ since R is a semidomain. It follows that $\{0/1\}$ is a 0_F-prime subsemimodule of F. To show that $\{0/1\}$ is the only prime subsemimodule of F, we assume the contrary and let L be a non-zero prime subsemimodule of F. Since L is a non-zero subsemimodule, there exists $0/1 \neq x = c/d \in L$, where $c, d \in R$, such that $(d/1)x = c/1 \in L$. On the other hand, there exists $0 \neq y \in R$ such that $1/y \notin L$ since $L \neq F$. Now we have $(c/1)(y/1) = (cy)/1 \notin (L : F)$ and $1/y \notin L$, but $(cy/1)(1/y) = c/1 \in L$, which is a contradiction. Thus $\mathrm{Spec}(F) = \{0\}$. □

5 Strong primeful semimodules

In this section we extend some definitions and results of C. P. Lu [16, 17] to semimodules over semirings. Let M be a semimodule over a semiring R with $\mathrm{ann}(M)$ a Q-ideal of R. The map $\psi : \mathrm{Spec}(M) \to \mathrm{Spec}(R/\mathrm{ann}(M))$ defined by $\psi(N) = (N : M)/\mathrm{ann}(M)$ for every $N \in \mathrm{Spec}(M)$ will be called the natural map of $\mathrm{Spec}(M)$. The surjectivity of the natural map ψ is particularly important in the topological space $\mathrm{Spec}(M)$ equipped with a topology called the Zariski topology. An R-semimodule M is called primeful if either $M = 0$ or the natural map of $\mathrm{Spec}(M)$ is surjective [17].

We continue to use the notation already established, so R denotes a commutative S-cancellative semiring with an identity element and S a non-empty multiplicatively closed subset of R. M will designate a fixed S-cancellative semimodule over R.

Moreover, assume that P is a prime ideal of R. Then $S = R - P$ is a multiplicatively closed subset of R. In this case we set $R_S = R_P$ and $I_S = IR_P$, where I is an ideal of R.

Lemma 5. *Assume that P is a prime k-ideal of a semiring R and let R be a S-cancellative semiring, where $S = R - P$. Then R_P is a local semiring with unique maximal k-ideal of PR_P.*

Proof. By [8, Lemma 5 and Theorem 2], it suffices to show that PR_P is exactly the set of non-semi-units of R_P. Let $y \in R_P - PR_P$, and take any representation $y = a/s$ with $a \in R$, $s \in S$. We must have $a \notin P$, so that a/s is a unit of R_P with inverse s/a (so a/s is a semi-unit by [11, Remark 2.4]. On the other hand, if y is a semi-unit of R_P, and $y = b/t$ for some $b \in R$, $t \in S$, then there exist $c, d \in R$ and $u, w \in S$ such that $1/1 + (bc)/(tu) = (bd)/(tw)$. It follows that $t^2uw + bctw = tubc$; hence $b \notin P$ since P is a k-ideal, and since this reasoning applies to every representation $y = b/t$ with $b \in R$, $t \in S$, of y as a formal fraction, it follows that $y \notin PR_S$, and so the proof is complete (see [9, Theorem 3]). $\square$

Example 3. The monoid $M = (Z_6, +_6)$ is a semimodule over $(E_0^+, +, .)$ (see [13, p. 151]) with $\operatorname{ann}(M) = \{60k : k \in E_0^+\}$. It is easy to see that $\operatorname{ann}(M)$ is a Q-ideal of E_0^+ with respect to $Q = \{1, 2, ..., 59\}$.

Proposition 6. *Let M be a non-zero semimodule over a semiring R with* $\operatorname{ann}(M)$ *a Q-ideal of R. Then the following hold:*

(i) M is a primeful semimodule if and only if for every prime k-ideal P with $\operatorname{ann}(M) \subseteq P$*, there exists a prime subsemimodule N of M such that $(N : M) = P$.*

(ii) If M is a primeful semimodule, then $PM_P \neq M_P$ for every prime k-ideal P with $\operatorname{ann}(M) \subseteq P$.

Proof. (i) Assume that M is primeful and let P be a prime k-ideal of R with $\operatorname{ann}(M) \subseteq P$. Then $P/\operatorname{ann}(M)$ is a prime ideal of $R/\operatorname{ann}(M)$ by [4, Theorem 2.5]. By assumption, there exists a prime subsemimodule N of M such that $\psi(N) = (N : M)/\operatorname{ann}(M) = P/\operatorname{ann}(M)$; hence $(N : M) = P$ by [4, Lemma 2.13]. The reverse implication is clear.

(ii) For any prime k-ideal P of R with $\operatorname{ann}(M) \subseteq P$, let N be a P-prime subsemimodule of M. Then $PM \subseteq N$ with $N \neq M$ so that N_P is a PR_P-prime subsemimodule of M_P by Theorem 6. Since $PM_P \subseteq N_P$ with $N_P \neq M_P$, $M_P \neq PM_P$. $\square$

We begin this section by proving the following fundamental theorems of this paper:

Theorem 8. *Let M be a non-zero semimodule over a semiring R with* $\operatorname{ann}(M)$ *a Q-ideal of R. Then the following are equivalent:*

(i) M is primeful;

(ii) Let P be a prime partitioning ideal of R such that $\operatorname{ann}(M) \subseteq P$ *and PR_P is a partitioning ideal of R_P. Then there exists a prime subsemimodule N of M such that $(N : M) = P$;*

(iii) Let P be a prime partitioning ideal of R such that $\operatorname{ann}(M) \subseteq P$ *and* PR_P *is a partitioning ideal of* R_P*. Then* $PM_P \neq M_P$*;*

(iv) Let P be a prime partitioning ideal of R such that $\operatorname{ann}(M) \subseteq P$ *and* PR_P *is a partitioning ideal of* R_P*. Then* $S_P(PM)$ *is a P-prime subsimimodule;*

(v) Let P be a prime partitioning ideal of R such that $\operatorname{ann}(M) \subseteq P$ *and* PR_P *is a partitioning ideal of* R_P*. Then* $\operatorname{Spec}_P(M) \neq \emptyset$*.*

Proof. By Proposition 6, we prove $(iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (i)$. $(iii) \Rightarrow (iv)$: Since by assumption and Lemma 5, PR_P is a maximal partitioning ideal of R_P and $PM_P \neq M_P$, $(PR_P)M_P$ is a PR_P-prime subsemimodule of M_P by Proposition 1. Hence $S_P(PM) = PM_P \cap M$ is a P-prime subsemimodule of M by Theorem 6. Thus (iv) follows. $(iv) \Rightarrow (v)$ and $(v) \Rightarrow (ii)$ are clear. □

Let M be a semimodule over a semiring R with $\operatorname{ann}(M)$ a Q-ideal of R. The collection of all prime (resp. maximal) k-subsemimodules of M with $(N : M)$ a strong ideal of R (resp. with $(N : M)$ a strong Q-ideal of R) is called the k-spectrum (resp. the maximal k-spectrum) of M and denoted by $\operatorname{Spec}_k(M)$ (resp. $\operatorname{Max}_k(M)$). Set

$$\operatorname{Spec}_k(R/\operatorname{ann}(M)) = \{P/\operatorname{ann}(M) \in \operatorname{Spec}(R/\operatorname{ann}(M)) : P \text{ is a strong } k\text{-ideal of } R\}.$$

Definition 4. Let M be a semimodule over a semiring R with $\operatorname{ann}(M)$ a Q-ideal of R:

(i) M is called strong primeful if either $M = 0$ or the natural map $\psi : \operatorname{Spec}_k(M) \to \operatorname{Spec}_k(R/\operatorname{ann}(M))$ defined by $\psi(N) = (N : M)/\operatorname{ann}(M)$ for every $N \in \operatorname{Spec}_k(M)$ is surjective.

(ii) M is called strong fulmaximal if either $M = 0$ or the natural map $\psi : \operatorname{Max}_k(M) \to \operatorname{Max}_k(R/\operatorname{ann}(M))$ defined by $\psi(N) = (N : M)/\operatorname{ann}(M)$ for every $N \in \operatorname{Max}_k(M)$ is surjective.

Theorem 9. *Let M be a non-zero semimodule over a semiring R with* $\operatorname{ann}(M)$ *a Q-ideal of $R \neq \{0\}$. Then the following hold:*

(i) If M is finitely generated, then M is a strong primeful semimodule and, similarly, M is a strong fulmaximal semimodule. Consequently, $\operatorname{Spec}_k(M) \neq \emptyset$ *and* $\operatorname{Max}_k(M) \neq \emptyset$*.*

(ii) If M is multiplication, then M is a strong primeful semimodule. Consequently, $\operatorname{Spec}_k(M) \neq \emptyset$*.*

Proof. (i) Let $P/\operatorname{ann}(M) \in \operatorname{Spec}_k(R/\operatorname{ann}(M))$. Then by assumption and [4, Theorem 2.5], P is a strong prime k-ideal containing $\operatorname{ann}(M)$. Since M is a non-zero finitely generated R-semimodule, M_P is a non-zero finitely generated R_P-semimodule with $\operatorname{ann}(M_P) = (\operatorname{ann}(M))_P$ and $\operatorname{ann}(M_P) \subseteq PR_P$ by Lemma 4. If $a/s \in PR_P$ for some $a \in P$ and $s \notin P$, then $a + b = 0$ for some $b \in P$, and so $a/s + b/s = 0/1$; hence

PR_P is a strong prime k-ideal of R_P (see [9, Lemma 5]). According to Proposition 2 and Theorem 3, $PM_P \neq M_P$ so that PM_P is a PR_P-prime subsemimodule of M_P. Applying Theorem 6, we can conclude that $N = PM_P \cap M$ is a prime subsemimodule of M; hence $\psi(N) = (N : M)/\mathrm{ann}(M) = P/\mathrm{ann}(M)$. This proves that ψ is surjective. Finally, assume that $P/\mathrm{ann}(M) \in \mathrm{Max}_k(R/\mathrm{ann}(M))$. Then by assumption and [4, Theorem 2.14], P is a strong maximal k-ideal containing $\mathrm{ann}(M)$. Let $T(P)$ be the set of all P-prime k-subsemimodules N of M with $(N : M)$ a strong Q-ideal. In the proof above, we have seen that $T(P) \neq \emptyset$. With the aid of Zorn's lemma, we can see that there exists a maximal element L in $T(P)$. Since $(L : M) = P$ is a maximal Q-ideal, L is a maximal subsemimodule. (ii) follows from [12, Theorem 3.8]. $\square$

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Reza Ebrahimi Atani
Department of Computer Science
and Engineering University of Guilan
P.O. Box. 3756, Rasht, Iran
E-mail: *rebrahimi@guilan.ac.ir*

Shahabaddin Ebrahimi Atani
Department of Pure Mathematics
Faculty of Mathematical Science
University of Guilan
P.O. Box. 1914, Rasht, Iran
E-mail: *ebrahimi@guilan.ac.ir*

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A Generalization of Hardy-Hilbert's Inequality for Non-homogeneous Kernel

Namita Das, Srinibas Sahoo

Abstract. This paper deals with a generalization of Hardy-Hilbert's inequality for non-homogeneous kernel by considering sequences (s_n), (t_n), the functions ϕ_p, ϕ_q and parameter λ. This inequality generalizes both Hardy-Hilbert's inequality and Mulholland's inequality, which includes most of the recent results of this type. As applications, the equivalent form, some particular results and a generalized Hardy-Littlewood inequality are established.

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1 Introduction

If $a_n, b_n \geq 0$ satisfy $\sum_{n=1}^{\infty} a_n^2 < \infty$ and $\sum_{n=1}^{\infty} b_n^2 < \infty$, then the well known Hilbert's inequality (see [1]) is given by

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left\{\sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2\right\}^{\frac{1}{2}} \tag{1}$$

and an equivalent form is given by

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^2 < \pi^2 \sum_{n=1}^{\infty} a_n^2, \tag{2}$$

where the constant factors π and π^2 are the best possible. In 1925, Hardy [2] gave some extensions of (1) and (2) by introducing the (p,q)-parameters as: if $p > 1, \frac{1}{p}+\frac{1}{q} = 1$, $a_n, b_n \geq 0$ satisfy $\sum_{n=1}^{\infty} a_n^p < \infty$ and $\sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=1}^{\infty} a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{1/q} \tag{3}$$

and an equivalent form is given by

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^p < \left[\frac{\pi}{\sin(\pi/p)}\right]^p \sum_{n=1}^{\infty} a_n^p, \tag{4}$$

where the constant factors $\frac{\pi}{\sin(\pi/p)}$ and $[\frac{\pi}{\sin(\pi/p)}]^p$ are the best possible. Inequality (3) is called Hardy-Hilbert's inequality and is important in analysis and its applications (cf. Mintrinovic et al. [4]). Recently many generalizations and refinements of these inequalities were also obtained. Some of them are given in [5–15].

If $p > 1, \frac{1}{p}+\frac{1}{q} = 1, a_n, b_n \geq 0$ satisfy $0 < \sum_{m=2}^{\infty} \frac{1}{m}\ a_m^p < \infty$ and $0 < \sum_{n=2}^{\infty} \frac{1}{n}\ b_n^q < \infty$, then the Mulholland's inequality (cf. [1,3]) is given by

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{a_m b_n}{mn \ln mn} < \frac{\pi}{\sin(\pi/p)} \left\{\sum_{n=2}^{\infty} \frac{1}{n}\ a_n^p\right\}^{1/p} \left\{\sum_{n=2}^{\infty} \frac{1}{n}\ b_n^q\right\}^{1/q}; \tag{5}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Replacing a_m with ma_m and b_n with nb_n we have the following inequality:

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geq 0$ satisfy $0 < \sum_{n=2}^{\infty} n^{p-1} a_n^p < \infty$ and $0 < \sum_{n=2}^{\infty} n^{q-1} b_n^q < \infty$, then the inequality

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\pi/p)} \left\{\sum_{n=2}^{\infty} n^{p-1} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=2}^{\infty} n^{q-1} b_n^q\right\}^{\frac{1}{q}} \tag{6}$$

holds, where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. The inequality (6) is also referred to as Mulholland's inequality. Some generalizations of these inequalities are given in [16, 17].

Most of the recent generalizations of inequalities (1) and (3) (cf. [5–15]) estimate the upper bounds of the double sum of the form $\sum\sum K(m,n) a_m b_n$, where the kernel $K(m,n)$ is homogeneous in m and n. In this paper, we give a generalization of Hardy-Hilbert's inequality for non-homogeneous kernel $K(m,n) = (s_m + t_n)^{-1}$ by considering the sequences (s_n), (t_n), the functions ϕ_p, ϕ_q and parameter λ. This inequality generalizes both Hardy-Hilbert's inequality and Mulholland's inequality, from which all the inequalities given in [5–17] are obtained as particular cases. As applications, the equivalent form, some particular results and a generalized Hardy-Littlewood inequality are established.

2 Some Lemmas

We first set the following notations. Suppose $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1$ and $\phi_r\ (r = p, q)$ is a function of r such that $0 < \phi_r < \lambda\ (r = p, q)$. Let $m_0, n_0 \in \mathbb{N}$ and $s(x), t(x)$ are differentiable strictly increasing functions in (m_0-1, ∞) and (n_0-1, ∞), respectively, such that $s((m_0 - 1)+) = t((n_0 - 1)+) = 0$ and $s(\infty) = t(\infty) = \infty$, $\frac{s'(x)}{(s(x))^{1-\phi_q}}$ and $\frac{t'(x)}{(t(x))^{1-\phi_p}}$ are decreasing in $(m_0 - 1, \infty)$ and $(n_0 - 1, \infty)$, respectively. We write $s(m) = s_m$, $s'(m) = s'_m$, $t(n) = t_n$ and $t'(n) = t'_n$.

We need the formula of the $\beta-$function as (cf. Wang et al. [18]):

$$B(p,q) = \int_0^{\infty} \frac{1}{(1+u)^{p+q}} u^{p-1} du = B(q,p)\ \ (p, q > 0). \tag{7}$$

Lemma 1. *Define the weight functions $\omega_\lambda(s,t,p,m)$ and $\omega_\lambda(t,s,q,n)$ as*

$$\omega_\lambda(s,t,p,m) = \sum_{n=n_0}^{\infty} \frac{1}{(s_m+t_n)^\lambda} \frac{t_n'}{(t_n)^{1-\phi_p}},\ (m \geq m_0); \tag{8}$$

$$\omega_\lambda(t,s,q,n) = \sum_{m=m_0}^{\infty} \frac{1}{(s_m+t_n)^\lambda} \frac{s_m'}{(s_m)^{1-\phi_q}},\ (n \geq n_0). \tag{9}$$

Then

$$\omega_\lambda(s,t,p,m) < B(\phi_p, \lambda-\phi_p)(s_m)^{\phi_p-\lambda},\ (m \geq m_0); \tag{10}$$

$$\omega_\lambda(t,s,q,n) < B(\phi_q, \lambda-\phi_q)(t_n)^{\phi_q-\lambda},\ (n \geq n_0). \tag{11}$$

Proof. Since $\lambda > 0,\ s(x),\ t(x)$ are differentiable, strictly increasing functions and $\frac{s'(x)}{(s(x))^{1-\phi_q}}$ and $\frac{t'(x)}{(t(x))^{1-\phi_p}}$ are decreasing in (m_0-1,∞) and (n_0-1,∞), respectively. So

$$\begin{aligned}
\omega_\lambda(s,t,p,m) &< \sum_{n=n_0}^{\infty} \int_{n-1}^{n} \frac{1}{(s_m+t(y))^\lambda} \frac{t'(y)}{(t(y))^{1-\phi_p}} dy \\
&= \int_{n_0-1}^{\infty} \frac{(t(y))^{\phi_p-1} t'(y)}{(s_m+t(y))^\lambda} dy \\
&= (s_m)^{\phi_p-\lambda} \int_0^\infty \frac{1}{(1+u)^\lambda} u^{\phi_p-1} du \qquad \left(\text{setting } \ u = \frac{t(y)}{s_m}\right).
\end{aligned}$$

Then by (7), we get (10). Similarly, (11) can be proved. The lemma is proved. □

Lemma 2. *If $\phi_p + \phi_q = \lambda$ and $0 < \varepsilon < q\phi_p$, then*

$$\begin{aligned}
\sum\nolimits_1 := \sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} \frac{1}{(s_m+t_n)^\lambda} \times \frac{s_m'}{(s_m)^{1-\phi_q+\frac{\varepsilon}{p}}} \times \frac{t_n'}{(t_n)^{1-\phi_p+\frac{\varepsilon}{q}}} \\
> \frac{1}{\varepsilon (s_{m_0})^\varepsilon} B\left(\phi_p - \frac{\varepsilon}{q}, \phi_q + \frac{\varepsilon}{q}\right) - \bigcirc(1).
\end{aligned} \tag{12}$$

Proof. Since $\lambda > 0,\ s(x),\ t(x)$ are differentiable strictly increasing functions and $\frac{s'(x)}{(s(x))^{1-\phi_q}}$ and $\frac{t'(x)}{(t(x))^{1-\phi_p}}$ are decreasing in (m_0-1,∞) and (n_0-1,∞), respectively, we have

$$\begin{aligned}\sum\nolimits_1 &> \int_{m_0}^{\infty}\int_{n_0}^{\infty} \frac{1}{(s(x)+t(y))^{\lambda}} \times \frac{s'(x)}{(s(x))^{1-\phi_q+\frac{\varepsilon}{p}}} \times \frac{t'(y)}{(t(y))^{1-\phi_p+\frac{\varepsilon}{q}}} dxdy \\ &= \int_{m_0}^{\infty} \frac{s'(x)}{(s(x))^{1+\varepsilon}} \left[\int_{\frac{t(n_0)}{s(x)}}^{\infty} \frac{1}{(1+u)^{\lambda}} u^{\phi_p-\frac{\varepsilon}{q}-1} du\right] dx \qquad \left(\text{setting } \ u = \frac{t(y)}{s(x)}\right) \\ &= \int_{m_0}^{\infty} \frac{s'(x)}{(s(x))^{1+\varepsilon}} dx \int_0^{\infty} \frac{u^{\phi_p-\frac{\varepsilon}{q}-1}}{(1+u)^{\lambda}} du - \int_{m_0}^{\infty} \frac{s'(x)}{(s(x))^{1+\varepsilon}} \left[\int_0^{\frac{t(n_0)}{s(x)}} \frac{u^{\phi_p-\frac{\varepsilon}{q}-1}}{(1+u)^{\lambda}} du\right] dx \\ &> \frac{1}{\varepsilon(s(m_0))^{\varepsilon}} \int_0^{\infty} \frac{u^{\phi_p-\frac{\varepsilon}{q}-1}}{(1+u)^{\lambda}} du - \int_{m_0}^{\infty} \frac{s'(x)}{(s(x))^{1+\varepsilon}} \left[\int_0^{\frac{t(n_0)}{s(x)}} u^{\phi_p-\frac{\varepsilon}{q}-1} du\right] dx \\ &= \frac{1}{\varepsilon(s_{m_0})^{\varepsilon}} \int_0^{\infty} \frac{u^{\phi_p-\frac{\varepsilon}{q}-1}}{(1+u)^{\lambda}} du - \frac{(t_{n_0})^{\phi_p-\frac{\varepsilon}{q}}}{(u_{m_0})^{\phi_p-\frac{\varepsilon}{q}+\varepsilon}} \left(\phi_p - \frac{\varepsilon}{q}\right)^{-1} \left(\phi_p - \frac{\varepsilon}{q} + \varepsilon\right)^{-1}.\end{aligned}$$

Then by (7), (12) is valid. The lemma is proved. □

3 Main Result

Theorem 1. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \phi_r < \lambda$ $(r = p, q)$ *and* $a_n, b_n \geq 0$ *satisfy* $0 < \sum_{m=m_0}^{\infty} \frac{(s_m)^{\phi_p-\lambda+(p-1)(1-\phi_q)}}{(s'_m)^{p-1}} a_m^p < \infty$ *and* $0 < \sum_{n=n_0}^{\infty} \frac{(t_n)^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(t'_n)^{q-1}} b_n^q < \infty$, *then*

$$\begin{aligned}\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty} \frac{a_m b_n}{(s_m+t_n)^{\lambda}} < H_{\lambda}(\phi_p,\phi_q) &\left\{\sum_{m=m_0}^{\infty} \frac{(s_m)^{\phi_p-\lambda+(p-1)(1-\phi_q)}}{(s'_m)^{p-1}} a_m^p\right\}^{\frac{1}{p}} \\ &\times \left\{\sum_{n=n_0}^{\infty} \frac{(t_n)^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(t'_n)^{q-1}} b_n^q\right\}^{\frac{1}{q}}\end{aligned} \tag{13}$$

where $H_{\lambda}(\phi_p,\phi_q) = B^{\frac{1}{p}}(\phi_p, \lambda-\phi_p) B^{\frac{1}{q}}(\phi_q, \lambda-\phi_q)$.

Proof. By Hölder's inequality with weight (cf. Kuang [19]), we have

$$\begin{aligned}&\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty} \frac{a_m b_n}{(s_m+t_n)^{\lambda}} \\ &= \sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty} \frac{1}{(s_m+t_n)^{\lambda}} \left\{\frac{(t_n)^{(\phi_p-1)/p}(t'_n)^{1/p}}{(s_m)^{(\phi_q-1)/q}(s'_m)^{1/q}} a_m\right\} \left\{\frac{(s_m)^{(\phi_q-1)/q}(s'_m)^{1/q}}{(t_n)^{(\phi_p-1)/p}(t'_n)^{1/p}} b_n\right\} \\ &\leq \left\{\sum_{m=m_0}^{\infty} \left[\sum_{n=n_0}^{\infty} \frac{1}{(s_m+t_n)^{\lambda}} \frac{t'_n}{(t_n)^{1-\phi_p}}\right] \frac{(s_m)^{(p-1)(1-\phi_q)}}{(s'_m)^{p-1}} \ a_m^p\right\}^{\frac{1}{p}} \\ &\quad \times \left\{\sum_{n=n_0}^{\infty} \left[\sum_{m=m_0}^{\infty} \frac{1}{(s_m+t_n)^{\lambda}} \frac{s'_m}{(s_m)^{1-\phi_q}}\right] \frac{(t_n)^{(q-1)(1-\phi_p)}}{(t'_n)^{q-1}} \ b_n^q\right\}^{\frac{1}{q}}.\end{aligned}$$

Then by (8) and (9), we have

$$\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{a_m b_n}{(s_m+t_n)^{\lambda}} \le \left\{\sum_{m=m_0}^{\infty}\omega_{\lambda}(s,t,p,m)\frac{(s_m)^{(p-1)(1-\phi_q)}}{(s'_m)^{p-1}}\, a_m^p\right\}^{\frac{1}{p}} \times \left\{\sum_{n=n_0}^{\infty}\omega_{\lambda}(t,s,q,n)\frac{(t_n)^{(q-1)(1-\phi_p)}}{(t'_n)^{q-1}}\, b_n^q\right\}^{\frac{1}{q}}$$

and in view of (10) and (11), it follows that (13) is valid. The theorem is proved. □

Theorem 2. *If* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $0 < \phi_r < \lambda$ $(r = p, q)$ *and* $a_n \ge 0$ *satisfy* $0 < \sum_{m=m_0}^{\infty}\frac{(s_m)^{\phi_p-\lambda+(p-1)(1-\phi_q)}}{(s'_m)^{p-1}}a_m^p < \infty$, *then we obtain an equivalent inequality of* (13) *as follows:*

$$\sum_{n=n_0}^{\infty}\frac{t'_n}{(t_n)^{1-\phi_p+(p-1)(\phi_q-\lambda)}}\left[\sum_{m=m_0}^{\infty}\frac{a_m}{(s_m+t_n)^{\lambda}}\right]^p < [H_{\lambda}(\phi_p,\phi_q)]^p\sum_{m=m_0}^{\infty}\frac{(s_m)^{\phi_p-\lambda+(p-1)(1-\phi_q)}}{(s'_m)^{p-1}}a_m^p. \tag{14}$$

Proof. Setting $b_n = \frac{t'_n}{(t_n)^{1-\phi_p+(p-1)(\phi_q-\lambda)}}\left[\sum_{m=m_0}^{\infty}\frac{a_m}{(s_m+t_n)^{\lambda}}\right]^{p-1}$ and using (13) we obtain

$$\begin{aligned}
0 &< \sum_{n=n_0}^{\infty}\frac{(t_n)^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(t'_n)^{q-1}}b_n^q \\
&= \sum_{n=n_0}^{\infty}\frac{t'_n}{(t_n)^{1-\phi_p+(p-1)(\phi_q-\lambda)}}\left[\sum_{m=m_0}^{\infty}\frac{a_m}{(s_m+t_n)^{\lambda}}\right]^p \\
&= \sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{a_m b_n}{(s_m+t_n)^{\lambda}} \\
&\le H_{\lambda}(\phi_p,\phi_q)\left\{\sum_{m=m_0}^{\infty}\frac{(s_m)^{\phi_p-\lambda+(p-1)(1-\phi_q)}}{(s'_m)^{p-1}}a_m^p\right\}^{\frac{1}{p}} \\
&\times\left\{\sum_{n=n_0}^{\infty}\frac{(t_n)^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(t'_n)^{q-1}}b_n^q\right\}^{\frac{1}{q}}.
\end{aligned} \tag{15}$$

Hence

$$
\begin{aligned}
0 &< \left[\sum_{n=n_0}^{\infty} \frac{(t_n)^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(t'_n)^{q-1}} b_n^q\right]^{\frac{1}{p}} \\
&= \left\{\sum_{n=n_0}^{\infty} \frac{t'_n}{(t_n)^{1-\phi_p+(p-1)(\phi_q-\lambda)}} \left[\sum_{m=m_0}^{\infty} \frac{a_m}{(s_m+t_n)^{\lambda}}\right]^p\right\}^{\frac{1}{p}} \qquad (16) \\
&\le H_\lambda(\phi_p,\phi_q)\left\{\sum_{m=m_0}^{\infty} \frac{(s_m)^{\phi_p-\lambda+(p-1)(1-\phi_q)}}{(s'_m)^{p-1}} a_m^p\right\}^{\frac{1}{p}} < \infty.
\end{aligned}
$$

By using (13) it follows that (15) takes the form of strict inequality; so does (16). Hence we get (14).

On the other hand, if (14) holds, then by Hölder's inequality, we have

$$
\begin{aligned}
&\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty} \frac{a_m b_n}{(s_m+t_n)^{\lambda}} \\
&= \sum_{n=n_0}^{\infty}\left[\frac{(t'_n)^{1/p}}{(t_n)^{(1-\phi_p+(p-1)(\phi_q-\lambda))/p}} \sum_{m=m_0}^{\infty} \frac{a_m}{(s_m+t_n)^{\lambda}}\right]\left[\frac{(t_n)^{(1-\phi_p+(p-1)(\phi_q-\lambda))/p}}{(t'_n)^{1/p}} b_n\right] \\
&\le \left\{\sum_{n=n_0}^{\infty} \frac{t'_n}{(t_n)^{1-\phi_p+(p-1)(\phi_q-\lambda)}} \left[\sum_{m=m_0}^{\infty} \frac{a_m}{(s_m+t_n)^{\lambda}}\right]^p\right\}^{\frac{1}{p}} \left\{\sum_{n=n_0}^{\infty} \frac{(t_n)^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(t'_n)^{q-1}} b_n^q\right\}^{\frac{1}{q}}.
\end{aligned}
$$

Hence by (14), (13) yields. Thus it follows that (13) and (14) are equivalent. The theorem is proved. □

Theorem 3. *If $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $\phi_r > 0$ $(r = p, q)$, $\phi_p + \phi_q = \lambda$, $a_n, b_n \ge 0$ satisfy $0 < \sum_{m=m_0}^{\infty} \frac{(s_m)^{p(1-\phi_q)-1}}{(s'_m)^{p-1}} a_m^p < \infty$ and $0 < \sum_{n=n_0}^{\infty} \frac{(t_n)^{q(1-\phi_p)-1}}{(t'_n)^{q-1}} b_n^q < \infty$, then*

$$
\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty} \frac{a_m b_n}{(s_m+t_n)^{\lambda}} < B(\phi_p,\phi_q)\left\{\sum_{m=m_0}^{\infty} \frac{(s_m)^{p(1-\phi_q)-1}}{(s'_m)^{p-1}} a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=n_0}^{\infty} \frac{(t_n)^{q(1-\phi_p)-1}}{(t'_n)^{q-1}} b_n^q\right\}^{\frac{1}{q}} \tag{17}
$$

where the constant factor $B(\phi_p, \phi_q)$ is the best possible.

Proof. Since $\phi_p + \phi_q = \lambda$, then by Theorem 1, (17) is valid. For $0 < \varepsilon < q\phi_p$, we take

$$
\begin{aligned}
\tilde{a}_m &= (s_m)^{-1+\phi_q-\varepsilon/p}\, s'_m \quad (m \ge m_0), \\
\tilde{b}_n &= (t_n)^{-1+\phi_p-\varepsilon/q}\, t'_n \quad (n \ge n_0).
\end{aligned}
$$

Since $\frac{s'(x)}{(s(x))^{1+\varepsilon}} = \frac{s'(x)}{(s(x))^{1-\phi_q}} \frac{1}{(s(x))^{\phi_q+\varepsilon}}$ is decreasing in $(m_0 - 1, \infty)$, we have

$$\begin{aligned}\sum_{m=m_0}^{\infty} \frac{(s_m)^{p(1-\phi_q)-1}}{(s'_m)^{p-1}} \tilde{a}_m^p &= \frac{s'_{m_0}}{(s_{m_0})^{1+\varepsilon}} + \sum_{m=m_0+1}^{\infty} \frac{s'_m}{(s_m)^{1+\varepsilon}} \\ &\leq \frac{s'_{m_0}}{(s_{m_0})^{1+\varepsilon}} + \int_{m_0}^{\infty} \frac{s'(x)}{(s(x))^{1+\varepsilon}} dx \\ &= \frac{1}{\varepsilon}\left[\varepsilon \frac{s'_{m_0}}{(s_{m_0})^{1+\varepsilon}} + \frac{1}{(s_{m_0})^{\varepsilon}}\right].\end{aligned} \tag{18}$$

Similarly,

$$\sum_{n=n_0}^{\infty} \frac{(t_n)^{q(1-\phi_p)-1}}{(t'_n)^{q-1}} \tilde{b}_n^q \leq \frac{1}{\varepsilon}\left[\varepsilon \frac{t'_{n_0}}{(t_{n_0})^{1+\varepsilon}} + \frac{1}{(t_{n_0})^{\varepsilon}}\right]. \tag{19}$$

If the constant factor $B(\phi_p, \phi_q)$ in (17) is not the best possible, then there exists a positive constant $K < B(\phi_p, \phi_q)$ such that (17) is still valid if we replace $B(\phi_p, \phi_q)$ by K. In particular by (12), (18) and (19), we have

$$\begin{aligned}&\frac{1}{(s_{m_0})^{\varepsilon}} B\left(\phi_p - \frac{\varepsilon}{q}, \phi_q + \frac{\varepsilon}{q}\right) - \varepsilon \bigcirc(1) \\ &< \varepsilon \sum\nolimits_1 = \varepsilon \sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} \frac{\tilde{a}_m \tilde{b}_n}{(s_m + t_n)^{\lambda}} \\ &< \varepsilon K \left\{\sum_{m=m_0}^{\infty} \frac{(s_m)^{p(1-\phi_q)-1}}{(s'_m)^{p-1}} \tilde{a}_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=n_0}^{\infty} \frac{(t_n)^{q(1-\phi_p)-1}}{(t'_n)^{q-1}} \tilde{b}_n^q\right\}^{\frac{1}{q}} \\ &< K \left\{\varepsilon \frac{s'_{m_0}}{(s_{m_0})^{1+\varepsilon}} + \frac{1}{(s_{m_0})^{\varepsilon}}\right\}^{\frac{1}{p}} \left\{\varepsilon \frac{t'_{n_0}}{(t_{n_0})^{1+\varepsilon}} + \frac{1}{(t_{n_0})^{\varepsilon}}\right\}^{\frac{1}{q}}\end{aligned}$$

and taking $\varepsilon \to 0^+$, we get $B(\phi_p, \phi_q) \leq K$. This contradiction leads to the conclusion that the constant factor $B(\phi_p, \phi_q)$ in (17) is the best possible. The theorem is proved. □

Corollary 1. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $a_m \geq 0$ *satisfy* $0 < \sum_{m=m_0}^{\infty} (s'_m)^{1-r} a_m^r < \infty$ $(r = p, q)$, *then*

$$\sum_{m=m_0}^{\infty} \sum_{n=m_0}^{\infty} \frac{a_m a_n}{s_m + s_n} < \frac{\pi}{\sin \frac{\pi}{p}} \left(\sum_{m=m_0}^{\infty} (s'_m)^{1-p} a_m^p\right)^{\frac{1}{p}} \left(\sum_{m=m_0}^{\infty} (s'_m)^{1-q} a_m^q\right)^{\frac{1}{q}} \tag{20}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ *is the best possible.*

Proof. Taking $a_n = b_n, s_n = t_n, \lambda = 1, \phi_r = \frac{1}{r}$ $(r = p,\ q)$ in (17), we get (20). The corollary is proved. □

Corollary 2. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $a_m \geq 0$ *satisfy* $0 < \sum_{m=m_0}^{\infty} \frac{(s_m)^{\frac{r}{2}-1}}{(s'_m)^{r-1}} a_m^r < \infty$ $(r = p, q)$, *then*

$$\sum_{m=m_0}^{\infty} \sum_{n=m_0}^{\infty} \frac{a_m a_n}{s_m + s_n} < \pi \left(\sum_{m=m_0}^{\infty} \frac{(s_m)^{\frac{p}{2}-1}}{(s'_m)^{p-1}} a_m^p \right)^{\frac{1}{p}} \left(\sum_{m=m_0}^{\infty} \frac{(s_m)^{\frac{q}{2}-1}}{(s'_m)^{q-1}} a_m^q \right)^{\frac{1}{q}} \tag{21}$$

where the constant factor π *is the best possible.*

Proof. Taking $a_n = b_n$, $s_n = t_n$, $\lambda = 1$, $\phi_r = \frac{1}{2}$ $(r = p, q)$ in (17), we get (21). The corollary is proved. □

Theorem 4. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\phi_r > 0$ $(r = p, q), \phi_p + \phi_q = \lambda$, $a_n \geq 0$ *satisfy* $0 < \sum_{m=m_0}^{\infty} \frac{(s_m)^{p(1-\phi_q)-1}}{(s'_m)^{p-1}} a_m^p < \infty$, *then we obtain an equivalent inequality of* (17) *as follows:*

$$\sum_{n=n_0}^{\infty} \frac{t'_n}{(t_n)^{1-p\phi_p}} \left[\sum_{m=m_0}^{\infty} \frac{a_m}{(s_m + t_n)^{\lambda}} \right]^p < [B(\phi_p, \phi_q)]^p \sum_{m=m_0}^{\infty} \frac{(s_m)^{p(1-\phi_q)-1}}{(s'_m)^{p-1}} a_m^p \tag{22}$$

where the constant factor $[B(\phi_p, \phi_q)]^p$ *is the best possible.*

Proof. Since $\phi_p + \phi_q = \lambda$, then by Theorem 2, we get inequalities (17) and (22) are equivalent. By Theorem 3, the constant factor in (17) is best possible, hence the constant factor in (22) is best possible. The theorem is proved. □

4 Generalization of Hardy-Hilbert's Inequality

Theorem 5. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \phi_r < \lambda(r = p, q)$, $A, B > 0$, $0 < \alpha \leq \frac{1}{\phi_q}$, $0 < \beta \leq \frac{1}{\phi_p}$, $a_n, b_n \geq 0$ *satisfy* $0 < \sum_{m=1}^{\infty} m^{\alpha(\phi_p - \lambda + (1-p)\phi_q) + p - 1} a_m^p < \infty$ *and* $0 < \sum_{n=1}^{\infty} n^{\beta(\phi_q - \lambda + (1-q)\phi_p) + q - 1} b_n^q < \infty$, *then the following two equivalent inequalities hold:*

$$\begin{aligned} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(Am^{\alpha} + Bn^{\beta})^{\lambda}} &< \mu H_{\lambda}(\phi_p, \phi_q) \left\{ \sum_{m=1}^{\infty} m^{\alpha(\phi_p - \lambda + (1-p)\phi_q) + p - 1} a_m^p \right\}^{\frac{1}{p}} \\ &\times \left\{ \sum_{n=1}^{\infty} n^{\beta(\phi_q - \lambda + (1-q)\phi_p) + q - 1} b_n^q \right\}^{\frac{1}{q}}; \end{aligned} \tag{23}$$

$$\begin{aligned} \sum_{n=1}^{\infty} n^{\beta(\phi_p + (1-p)(\phi_q - \lambda)) - 1} &\left[\sum_{m=1}^{\infty} \frac{a_m}{(Am^{\alpha} + Bn^{\beta})^{\lambda}} \right]^p \\ &< [\mu H_{\lambda}(\phi_p, \phi_q)]^p \sum_{m=1}^{\infty} m^{\alpha(\phi_p - \lambda + (1-p)\phi_q) + p - 1} a_m^p \end{aligned} \tag{24}$$

where $\mu = \left(\frac{A^{\phi_p - \lambda}}{\beta B^{\phi_p}} \right)^{\frac{1}{p}} \left(\frac{B^{\phi_q - \lambda}}{\alpha A^{\phi_q}} \right)^{\frac{1}{q}}$ *and* $H_{\lambda}(\phi_p, \phi_q) = B^{\frac{1}{p}}(\phi_p, \lambda - \phi_p) B^{\frac{1}{q}}(\phi_q, \lambda - \phi_q)$. *The constant factors* $\mu H_{\lambda}(\phi_p, \phi_q)$ *and* $[\mu H_{\lambda}(\phi_p, \phi_q)]^p$ *are the best possible if* $\phi_p + \phi_q = \lambda$.

Proof. Setting $s_m = Am^{\alpha}$, $t_n = Bn^{\beta}$ in Theorem 1 and Theorem 2, we get both the inequalities (23) and (24) are valid and equivalent. From Theorem 3 and Theorem 4, it follows that the constant factors are the best possible. This completes the proof. $\square$

We discuss a number of special cases of inequality (23). Similar inequalities can also be derived from inequality (24).

Example 1. Setting $\phi_p = 1 - A_2p$, $\phi_q = 1 - A_1q$ in Theorem 5, we have the following inequality: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $A, B > 0$, $A_1 < \frac{1}{q}$, $A_2 < \frac{1}{p}$, $0 < \alpha \leq \frac{1}{1-A_1q}$, $0 < \beta \leq \frac{1}{1-A_2p}$, $\lambda > \max\{1 - A_2p, 1 - A_1q\}$, $a_m, b_n \geq 0$ satisfy $0 < \sum_{m=1}^{\infty} m^{\alpha(2-p-\lambda+p(A_1-A_2))+p-1} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{\beta(2-q-\lambda+q(A_2-A_1))+q-1} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(Am^{\alpha} + Bn^{\beta})^{\lambda}} < L \left\{\sum_{m=1}^{\infty} m^{\alpha(2-p-\lambda+p(A_1-A_2))+p-1} a_m^p\right\}^{\frac{1}{p}} \times \left\{\sum_{n=1}^{\infty} n^{\beta(2-q-\lambda+q(A_2-A_1))+q-1} b_n^q\right\}^{\frac{1}{q}} \tag{25}$$

where $L = \left(\frac{A^{1-A_2p-\lambda}}{\beta B^{1-A_2p}}\right)^{\frac{1}{p}} \left(\frac{B^{1-A_1q-\lambda}}{\alpha A^{1-A_1q}}\right)^{\frac{1}{q}} H_{\lambda}(1 - A_2p, 1 - A_1q)$. For $A = B = \alpha = \beta = 1$, we get the result of Brnetic and Pecaric [5, Theorem 2].

Example 2. Setting $\phi_r = \frac{\lambda}{r}$ $(r = p, q)$ in Theorem 5, we have the following inequality: If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $A, B > 0$, $0 < \alpha \leq \frac{q}{\lambda}$, $0 < \beta \leq \frac{p}{\lambda}$, $\lambda > 0$, $a_m, b_n \geq 0$ satisfy $0 < \sum_{m=1}^{\infty} m^{(p-1)(1-\alpha\lambda)} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{(q-1)(1-\beta\lambda)} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(Am^{\alpha}+Bn^{\beta})^{\lambda}} < \mu B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left\{\sum_{m=1}^{\infty} m^{(p-1)(1-\alpha\lambda)} a_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{(q-1)(1-\beta\lambda)} b_n^q\right\}^{\frac{1}{q}} \tag{26}$$

where $\mu = \left(A^{\frac{\lambda}{q}} B^{\frac{\lambda}{p}} \alpha^{\frac{1}{q}} \beta^{\frac{1}{p}}\right)^{-1}$ and the constant factor $\mu B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$ is the best possible. For $A = B = \lambda = 1$, $\alpha = \beta$, we get the result of Yang [7]. Setting $\alpha = \beta = 1$, $p = q = 2$, we get the result of Yang [13] and setting $\alpha = \beta = 1$, we get the result of Yang [15].

Example 3. Setting $\phi_r = \lambda(1 - \frac{1}{r})$ $(r = p, q)$ in Theorem 5, we have the following inequality: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $A, B > 0$, $0 < \alpha \leq \frac{p}{\lambda}$, $0 < \beta \leq \frac{q}{\lambda}$, $\lambda > 0$ and $a_m, b_n \geq 0$ satisfy $0 < \sum_{m=1}^{\infty} m^{p-\alpha\lambda-1} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{q-\beta\lambda-1} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(Am^{\alpha}+Bn^{\beta})^{\lambda}} < \mu B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left\{\sum_{m=1}^{\infty} m^{p-\alpha\lambda-1} a_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{q-\beta\lambda-1} b_n^q\right\}^{\frac{1}{q}} \tag{27}$$

where $\mu = \left(A^{\frac{\lambda}{p}} B^{\frac{\lambda}{q}} \alpha^{\frac{1}{q}} \beta^{\frac{1}{p}}\right)^{-1}$ and the constant factor $\mu B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$ is the best possible. For $\lambda = 1, \alpha = \beta$, we get the result of Yang [8]. Setting $A = B = \alpha = \beta = 1$, we recover the result of Yang [9].

Example 4. Setting $\phi_r = 1 + \frac{\lambda-2}{r}$ $(r = p, q)$ in Theorem 5, we have the following inequality: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $A, B > 0$, $0 < \alpha \leq \frac{q}{q+\lambda-2}$, $0 < \beta \leq \frac{p}{p+\lambda-2}$, $\lambda > 2 - \min\{p, q\}$, a_m, $b_n \geq 0$ satisfy $0 < \sum_{m=1}^{\infty} m^{(p-1)(1-\alpha(q+\lambda-2))} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{(q-1)(1-\beta(p+\lambda-2))} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(Am^{\alpha}+Bn^{\beta})^{\lambda}} < L_1\left\{\sum_{m=1}^{\infty} m^{(p-1)(1-\alpha(q+\lambda-2))} a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{(q-1)(1-\beta(p+\lambda-2))} b_n^q\right\}^{\frac{1}{q}} \tag{28}$$

where the constant factor $L_1 = \left(A^{\frac{q+\lambda-2}{q}} B^{\frac{p+\lambda-2}{p}} \alpha^{\frac{1}{q}} \beta^{\frac{1}{p}}\right)^{-1} \times B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ is the best possible. In particular for $\alpha = \beta = 1$, $p = q = 2$, we get the result of Yang [13].

Example 5. Setting $\phi_r = 1 + (1 - \frac{1}{r})(\lambda - 2)$ $(r = p, q)$ in Theorem 5, we have the following inequality: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $A, B > 0$, $0 < \alpha \leq \frac{p}{p+\lambda-2}$, $0 < \beta \leq \frac{q}{q+\lambda-2}$, $\lambda > 2 - \min\{p, q\}$, $a_m, b_n \geq 0$ satisfy $0 < \sum_{m=1}^{\infty} m^{\alpha(2-\lambda-p)+p-1} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{\beta(2-\lambda-q)+q-1} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(Am^{\alpha}+Bn^{\beta})^{\lambda}} < L_2\left\{\sum_{m=1}^{\infty} m^{\alpha(2-\lambda-p)+p-1} a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{\beta(2-\lambda-q)+q-1} b_n^q\right\}^{\frac{1}{q}} \tag{29}$$

where the constant factor $L_2 = \left(A^{\frac{p+\lambda-2}{p}} B^{\frac{q+\lambda-2}{q}} \alpha^{\frac{1}{q}} \beta^{\frac{1}{p}}\right)^{-1} \times B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ is the best possible. For $\alpha = \beta = 1$, we get the result of Yang and Debnath [6]. Setting $A = B = \lambda = 1$, $\alpha = \beta$, we recover the result of Yang [7].

Example 6. Setting $\phi_r = \frac{\lambda-1}{2} + \frac{1}{r}$ $(r = p, q)$, in Theorem 5, we have the following inequality: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $A, B > 0$, $0 < \alpha \leq (\frac{\lambda-1}{2} + \frac{1}{q})^{-1}$, $0 < \beta \leq (\frac{\lambda-1}{2} + \frac{1}{p})^{-1}$, $\lambda > 1 - 2\min\{\frac{1}{p}, \frac{1}{q}\}$, $a_m, b_n \geq 0$ satisfy $0 < \sum_{m=1}^{\infty} m^{p-1+\alpha(2-p\lambda-p)/2} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{q-1+\beta(2-q\lambda-q)/2} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(Am^{\alpha}+Bn^{\beta})^{\lambda}} < L_3\left\{\sum_{m=1}^{\infty} m^{p-1+\alpha(2-p\lambda-p)/2} a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{q-1+\beta(2-q\lambda-q)/2} b_n^q\right\}^{\frac{1}{q}} \tag{30}$$

where the constant factor $L_3 = \left(A^{\frac{\lambda-1}{2}+\frac{1}{q}} B^{\frac{\lambda-1}{2}+\frac{1}{p}} \alpha^{\frac{1}{q}} \beta^{\frac{1}{p}}\right)^{-1} \times B\left(\frac{\lambda-1}{2} + \frac{1}{p}, \frac{\lambda-1}{2} + \frac{1}{q}\right)$ is the best possible. Setting $A = B = \lambda = 1$, $\alpha = \beta$, we recover the result of Yang [7].

Example 7. Setting $\phi_p = \lambda(\frac{1}{\alpha} + (1-\frac{1}{p})(\alpha-2))$, $\phi_q = \lambda(\frac{1}{\beta} + (1-\frac{1}{q})(\beta-2))$ in Theorem 5, we have the following inequality: If $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, A, B, $\lambda > 0$, $2-p < \alpha \leq 2 + p(\frac{\lambda-1}{\lambda})$, $2-q < \beta \leq 2 + q(\frac{\lambda-1}{\lambda})$, a_m, $b_n \geq 0$ satisfy $0 < \sum_{m=1}^{\infty} m^{\lambda(2-\alpha-p)+p-1} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{\lambda(2-\beta-q)+q-1} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(Am^{\alpha}+Bn^{\beta})^{\lambda}} < L_4 \left\{\sum_{m=1}^{\infty} m^{\lambda(2-\alpha-p)+p-1} a_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{\lambda(2-\beta-q)+q-1} b_n^q\right\}^{\frac{1}{q}} \tag{31}$$

where the constant factor $L_4 = \left(A^{\frac{\lambda(p+\alpha-2)}{p\alpha}} B^{\frac{\lambda(q+\beta-2)}{q\beta}} \alpha^{\frac{1}{q}} \beta^{\frac{1}{p}}\right)^{-1} \times B\left(\frac{\lambda(p+\alpha-2)}{p\alpha}, \frac{\lambda(q+\beta-2)}{q\beta}\right)$ is the best possible. For $A = B = \lambda = 1, \alpha = \beta$, we get the result of Yang [12].

Remark 1. Setting **(i)** $\phi_r = \frac{1}{r}$ $(r = p, q)$, **(ii)** $\phi_r = 1 - \frac{1}{r}$ $(r = p, q)$, **(iii)** $\phi_r = \frac{\lambda+1}{2} - \frac{1}{r}$ $(r = p, q)$ in Theorem 5, we get new inequalities.

Remark 2. Taking $\alpha = \beta$, $A = B = 1$, $\phi_r = \frac{\varphi_r}{\alpha}$ $(r = p, q)$ in (23), we get the result of Yang [10].

Remark 3. Taking $s_m = t_m = u(m)$ in Theorem 3, we get the result of Yang [14].

For other appropriate values of λ, ϕ_p, ϕ_q and suitably choosing sequences s_m and t_n in Theorem 1 and Theorem 3, one can obtain many new inequalities.

5 Generalization of Mulholland's Inequality

Theorem 6. *If $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $0 < \phi_r \leq 1$ $(r = p, q)$, $\lambda > \max\{\phi_p, \phi_q\}$, $\alpha, \beta > 0$ and $a_n, b_n \geq 0$ satisfy $0 < \sum_{m=2}^{\infty} m^{p-1}(\ln m)^{\phi_p-\lambda+(p-1)(1-\phi_q)} a_m^p < \infty$ and $0 < \sum_{n=2}^{\infty} n^{q-1}(\ln n)^{\phi_q-\lambda+(q-1)(1-\phi_p)} b_n^q < \infty$, then the following two equivalent inequalities holds:*

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{a_m b_n}{(\ln m^{\alpha} n^{\beta})^{\lambda}} < \eta H_{\lambda}(\phi_p, \phi_q) \left\{\sum_{m=2}^{\infty} m^{p-1}(\ln m)^{\phi_p-\lambda+(p-1)(1-\phi_q)} a_m^p\right\}^{\frac{1}{p}} \times \left\{\sum_{n=2}^{\infty} n^{q-1}(\ln n)^{\phi_q-\lambda+(q-1)(1-\phi_p)} b_n^q\right\}^{\frac{1}{q}}; \tag{32}$$

$$\sum_{n=2}^{\infty} \frac{(\ln n)^{\phi_p-1+(p-1)(\lambda-\phi_q)}}{n} \left[\sum_{m=2}^{\infty} \frac{a_m}{(\ln m^{\alpha} n^{\beta})^{\lambda}}\right]^p < [\eta H_{\lambda}(\phi_p, \phi_q)]^p \sum_{m=2}^{\infty} m^{p-1}(\ln m)^{\phi_p-\lambda+(p-1)(1-\phi_q)} a_m^p \tag{33}$$

where $\eta = \left(\frac{\alpha^{\phi_p-\lambda}}{\beta^{\phi_p}}\right)^{\frac{1}{p}} \left(\frac{\beta^{\phi_q-\lambda}}{\alpha^{\phi_q}}\right)^{\frac{1}{q}}$ and $H_{\lambda}(\phi_p, \phi_q) = B^{\frac{1}{p}}(\phi_p, \lambda-\phi_p) B^{\frac{1}{q}}(\phi_q, \lambda-\phi_q)$. The constant factors $\eta H_{\lambda}(\phi_p, \phi_q)$ and $[\eta H_{\lambda}(\phi_p, \phi_q)]^p$ are the best possible if $\phi_p + \phi_q = \lambda$.

Proof. Setting $s_m = \ln m^\alpha$, $t_n = \ln n^\beta$ in Theorem 1 and Theorem 2, we get both the inequalities (32) and (33) are valid and equivalent. The constant factors are the best possible obtained from Theorem 3 and Theorem 4. This completes the proof. $\square$

Example 8. Setting $\phi_r = \frac{1}{r}$ $(r = p, q)$ in Theorem 6, we obtain the following inequality: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > 0$, $\beta > 0, \lambda > \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$, $a_m, b_n \geq 0$ satisfy $0 < \sum_{m=2}^{\infty} m^{p-1}(\ln m)^{1-\lambda} a_m^p < \infty$ and $0 < \sum_{n=2}^{\infty} n^{q-1}(\ln n)^{1-\lambda} b_n^q < \infty$, then

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{a_m b_n}{(\ln m^\alpha n^\beta)^\lambda} < \eta H_\lambda\left(\frac{1}{p}, \frac{1}{q}\right)\left\{\sum_{m=2}^{\infty} m^{p-1}(\ln m)^{1-\lambda} a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=2}^{\infty} n^{q-1}(\ln n)^{1-\lambda} b_n^q\right\}^{\frac{1}{q}}. \tag{34}$$

In particular for $\alpha = \beta = \lambda = 1$, we get the result of Yang [16, Theorem 2.1].

Example 9. Setting $\phi_r = \frac{\lambda}{r}$ $(r = p, q)$ in Theorem 6, we have the following inequality: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > 0$, $\beta > 0, 0 < \lambda \leq \min\{p, q\}$, $a_m, b_n \geq 0$ satisfy $0 < \sum_{m=2}^{\infty} m^{p-1}(\ln m)^{(p-1)(1-\lambda)} a_m^p < \infty$ and $0 < \sum_{n=2}^{\infty} n^{q-1}(\ln n)^{(q-1)(1-\lambda)} b_n^q < \infty$, then

$$\begin{aligned}\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{a_m b_n}{(\ln m^\alpha n^\beta)^\lambda} &< \frac{1}{\alpha^{\frac{\lambda}{q}}\beta^{\frac{\lambda}{p}}} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)\left\{\sum_{m=2}^{\infty} m^{p-1}(\ln m)^{(p-1)(1-\lambda)} a_m^p\right\}^{\frac{1}{p}} \\ &\quad \times \left\{\sum_{n=2}^{\infty} n^{q-1}(\ln n)^{(q-1)(1-\lambda)} b_n^q\right\}^{\frac{1}{q}}\end{aligned} \tag{35}$$

where the constant factor $\frac{1}{\alpha^{\lambda/q}\beta^{\lambda/p}} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$ is the best possible. In particular for $\alpha = \beta = \lambda = 1$, we get the result of Yang [16, Theorem 2.1].

Example 10. Setting $\phi_r = \lambda(1 - \frac{1}{r})$ $(r = p, q)$ in Theorem 6, we have the following inequality: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > 0$, $\beta > 0$, $0 < \lambda \leq \min\{p, q\}$, $a_m, b_n \geq 0$ satisfy $0 < \sum_{m=2}^{\infty} m^{p-1}(\ln m)^{p-\lambda-1} a_m^p < \infty$ and $0 < \sum_{n=2}^{\infty} n^{q-1}(\ln n)^{q-\lambda-1} b_n^q < \infty$, then

$$\begin{aligned}\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{a_m b_n}{(\ln m^\alpha n^\beta)^\lambda} &< \frac{1}{\alpha^{\frac{\lambda}{p}}\beta^{\frac{\lambda}{q}}} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)\left\{\sum_{m=2}^{\infty} m^{p-1}(\ln m)^{p-\lambda-1} a_m^p\right\}^{\frac{1}{p}} \\ &\quad \times \left\{\sum_{n=2}^{\infty} n^{q-1}(\ln n)^{q-\lambda-1} b_n^q\right\}^{\frac{1}{q}}\end{aligned} \tag{36}$$

where the constant factor $\frac{1}{\alpha^{\lambda/p}\beta^{\lambda/q}} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$ is the best possible.

In particular for $\alpha = \beta = \lambda = 1$, it reduces to

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{a_m b_n}{(\ln mn)^\lambda} < \frac{\pi}{\sin\frac{\pi}{p}}\left\{\sum_{m=2}^{\infty} m^{p-1}(\ln m)^{p-2} a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=2}^{\infty} n^{q-1}(\ln n)^{q-2} b_n^q\right\}^{\frac{1}{q}} \tag{37}$$

and we obtain a new inequality in (p,q)-parameter form other than (6), with the same best constant factor.

Example 11. Setting $\phi_r = 1+(1-\frac{1}{r})(\lambda-2)\,(r=p,q)$ in Theorem 6, we have the following inequality: If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\alpha>0$, $\beta>0$, $2-\min\{p,q\}<\lambda\leq 2$, $a_m, b_n \geq 0$ satisfy $0<\sum_{m=2}^{\infty} m^{p-1}(\ln m)^{1-\lambda}a_m^p<\infty$ and $0<\sum_{n=2}^{\infty} n^{q-1}(\ln n)^{1-\lambda}b_n^q<\infty$, then

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{(\ln m^{\alpha}n^{\beta})^{\lambda}}<\eta k_{\lambda}(p)\left\{\sum_{m=2}^{\infty} m^{p-1}(\ln m)^{1-\lambda}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=2}^{\infty} n^{q-1}(\ln n)^{1-\lambda}b_n^q\right\}^{\frac{1}{q}} \tag{38}$$

where $\eta=\alpha^{\frac{2-\lambda-p}{p}}\beta^{\frac{2-\lambda-q}{q}}$, $k_{\lambda}(p)=B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)$ and the constant factor $\eta k_{\lambda}(p)$ is the best possible.

In particular for $\alpha=\beta=1$, we get

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{(\ln mn)^{\lambda}}<k_{\lambda}(p)\left\{\sum_{m=2}^{\infty} m^{p-1}(\ln m)^{1-\lambda}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=2}^{\infty} n^{q-1}(\ln n)^{1-\lambda}b_n^q\right\}^{\frac{1}{q}} \tag{39}$$

where the constant factor $k_{\lambda}(p)$ is the best possible. For $\lambda=1$, it reduces to the result of Yang [16, Theorem 2.1]. Replacing a_m, b_n by $\frac{a_m}{m^r}, \frac{b_n}{n^s}$ respectively, we get the result of Yang and Debnath [17, Theorem 1]).

Example 12. Setting $\phi_r=\frac{\lambda-1}{2}+\frac{1}{r}\ (r=p,q)$ in Theorem 6, we have the following inequality: If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\alpha>0$, $\beta>0$, $1-2\min\{\frac{1}{p},\frac{1}{q}\}<\lambda<1+2\min\{\frac{1}{p},\frac{1}{q}\}$, $a_m, b_n\geq 0$ satisfy $0<\sum_{m=2}^{\infty} m^{p-1}(\ln m)^{p(1-\lambda)/2}a_m^p<\infty$ and $0<\sum_{n=2}^{\infty} n^{q-1}(\ln n)^{q(1-\lambda)/2}b_n^q<\infty$, then

$$\begin{aligned}&\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{(\ln m^{\alpha}n^{\beta})^{\lambda}}\\ &<\eta\tilde{k}_{\lambda}(p)\left\{\sum_{m=2}^{\infty} m^{p-1}(\ln m)^{p(1-\lambda)/2}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=2}^{\infty} n^{q-1}(\ln n)^{q(1-\lambda)/2}b_n^q\right\}^{\frac{1}{q}}\end{aligned} \tag{40}$$

where $\eta=\alpha^{\frac{1-\lambda}{2}-\frac{1}{q}}\beta^{\frac{1-\lambda}{2}-\frac{1}{p}}$, $\tilde{k}_{\lambda}(p)=B\left(\frac{\lambda-1}{2}+\frac{1}{p},\frac{\lambda-1}{2}+\frac{1}{q}\right)$ and the constant factor $\eta\tilde{k}_{\lambda}(p)$ is the best possible. In particular for $\alpha=\beta=\lambda=1$, it reduces to (6).

Remark 4. Setting **(i)** $\phi_r=1-\frac{1}{r}\ (r=p,q)$, **(ii)** $\phi_r=1+\frac{\lambda-2}{r}\ (r=p,q)$, **(iii)** $\phi_r=\frac{\lambda+1}{2}-\frac{1}{r}\ (r=p,q)$ in Theorem 6, we get new inequalities.

6 Applications

In this section, we will give the generalizations of Hardy-Littlewood's inequality. Let $f \in L^2(0,1)$ and $f(x) \neq 0$. If

$$a_n = \int_0^1 x^n f(x)dx, \qquad n = 0, 1, 2, 3, \dots$$

then we have the Hardy-Littlewood's inequality (see [1]) of the form

$$\sum_{n=0}^{\infty} a_n^2 < \pi \int_0^1 f^2(x)dx \tag{41}$$

where the constant factor π is the best possible. Yang [11] gave a generalization of (41) for $p \geq 2$ as

$$\left(\sum_{n=0}^{\infty} a_n^p\right)^{1+\frac{1}{p}} < \frac{\pi}{\sin\frac{\pi}{p}} \left(\sum_{n=0}^{\infty} a_n^{p(p-1)}\right)^{\frac{1}{p}} \int_0^1 f^2(x)dx. \tag{42}$$

Theorem 7. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^2(0,1)$, $f(x) \neq 0$ and*

$$a_n = (s_n')^{\frac{1}{p}} \int_0^1 x^{s_n - \frac{1}{2}} f(x)dx, \qquad n \geq m_0.$$

If $0 < \sum_{n=m_0}^{\infty} (s_n')^{2-p} a_n^{p(p-1)} < \infty$, then

$$\left(\sum_{n=m_0}^{\infty} a_n^p\right)^{1+\frac{1}{p}} < \frac{\pi}{\sin\frac{\pi}{p}} \left(\sum_{n=m_0}^{\infty} (s_n')^{2-p} a_n^{p(p-1)}\right)^{\frac{1}{p}} \int_0^1 f^2(x)dx. \tag{43}$$

Proof. Applying Schwartz inequality, we have

$$\begin{aligned}
\left(\sum_{n=m_0}^{\infty} a_n^p\right)^2 &= \left(\sum_{n=m_0}^{\infty} a_n^{p-1}(s_n')^{\frac{1}{p}} \int_0^1 x^{s_n-\frac{1}{2}} f(x)dx\right)^2 \\
&= \left\{\int_0^1 \left(\sum_{n=m_0}^{\infty} a_n^{p-1}(s_n')^{\frac{1}{p}} x^{s_n-\frac{1}{2}}\right) f(x)dx\right\}^2 \\
&\leq \int_0^1 \left(\sum_{n=m_0}^{\infty} a_n^{p-1}(s_n')^{\frac{1}{p}} x^{s_n-\frac{1}{2}}\right)^2 dx \int_0^1 f^2(x)dx \\
&= \left\{\sum_{n=m_0}^{\infty}\sum_{m=m_0}^{\infty} \frac{a_n^{p-1}(s_n')^{\frac{1}{p}} a_m^{p-1}(s_m')^{\frac{1}{p}}}{s_n+s_m}\right\} \int_0^1 f^2(x)dx.
\end{aligned} \tag{44}$$

Since $f(x) \neq 0,\ s'_n > 0$. So, $a_n \neq 0$. Hence it is impossible to get equality in (44). Again by Corollary 1, we have

$$\sum_{n=m_0}^{\infty} \sum_{m=m_0}^{\infty} \frac{a_n^{p-1}(s'_n)^{\frac{1}{p}} a_m^{p-1}(s'_m)^{\frac{1}{p}}}{s_n + s_m}$$

$$\leq \frac{\pi}{\sin \frac{\pi}{p}} \left(\sum_{n=m_0}^{\infty} (s'_n)^{1-p} a_n^{p(p-1)} s'_n \right)^{\frac{1}{p}} \left(\sum_{n=m_0}^{\infty} (s'_n)^{1-q} a_n^{q(p-1)} (s'_n)^{\frac{q}{p}} \right)^{\frac{1}{q}}$$

$$= \frac{\pi}{\sin \frac{\pi}{p}} \left(\sum_{n=m_0}^{\infty} (s'_n)^{2-p} a_n^{p(p-1)} \right)^{\frac{1}{p}} \left(\sum_{n=m_0}^{\infty} a_n^p \right)^{\frac{1}{q}}.$$

Hence we obtain the inequality (43). This complete the proof of the theorem. □

Theorem 8. *Let* $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ f \in L^2(0,1),\ f(x) \neq 0$ *and*

$$a_n = \frac{(s'_n)^{\frac{1}{p}}}{(s_n)^{\frac{1}{p}-\frac{1}{2}}} \int_0^1 x^{s_n - \frac{1}{2}} f(x)dx, \qquad n \geq m_0.$$

If

$$0 < \sum_{n=m_0}^{\infty} \left(\frac{s'_n}{s_n} \right)^{2-p} a_n^{p(p-1)} < \infty$$

then

$$\left(\sum_{n=m_0}^{\infty} a_n^p \right)^{1+\frac{1}{p}} < \pi \left(\sum_{n=m_0}^{\infty} \left(\frac{s'_n}{s_n} \right)^{2-p} a_n^{p(p-1)} \right)^{\frac{1}{p}} \int_0^1 f^2(x)dx. \tag{45}$$

Proof. Proceeding as in Theorem 7 and using Corollary 2, the proof of the theorem follows. □

Remark 5. For $s_n = n$, (43) becomes (42). Taking $p = 2$ in Theorem 7 and Theorem 8, we get

$$\sum_{n=m_0}^{\infty} a_n^2 < \pi \int_0^1 f^2(x)dx \tag{46}$$

which reduces to (41) for $s_n = n$.

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NAMITA DAS
P.G. Department of Mathematics
Utkal University, Vani Vihar
Bhubaneswar, Orissa, India 751004
E-mail: *namitadas440@yahoo.co.in*

SRINIBAS SAHOO
Department of Mathematics
Banki Autonomous College, Banki
Orissa, India 754008
E-mail: *sahoosrinibas@yahoo.co.in*

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Topologies on $Spec_g(M)$

Ahmad Yousefian Darani

Abstract. Let R be a G-graded commutative ring with identity and let M be a graded R-module. We endow $Spec_g(M)$, the collection of all graded prime submodules of M, analogous to that for $Spec(R)$, the spectrum of prime ideals of R, by two topologies: quasi-Zariski topology and Zariski topology. Then we study some properties of these topological spaces.

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1 Introduction

Throughout this paper all rings are commutative with a nonzero identity, and all modules are unitary. Let R be a commutative ring and consider $Spec(R)$, the spectrum of all prime ideals of R. The Zariski topology on $Spec(R)$ is a useful implement in algebraic geometry. For each ideal I of R, the variety of I is the set $V(I) = \{P \in Spec(R)|I \subseteq P\}$. Then the set $\{V(I)|I \trianglerighteq R\}$ satisfies the axioms for the closed sets of a topology on $Spec(R)$, called the Zariski topology on $Spec(R)$ [1].

Let M be an R-module and let N be a submodule of M. We denote the annihilator of M/N by $(N :_R M)$, i.e. $(N :_R M) = \{r \in R|rM \subseteq N\}$. We recall that a proper submodule N of M is called a prime submodule of M if, for every $a \in R$ and $m \in M$, $am \in N$ implies that either $m \in N$ or $a \in (N :_R M)$. The notion of prime submodules was first introduced and studied in [2] and recently it has received a good deal of attention from several authors. We denote the set of all prime submodules of M by $Spec(M)$. In [7], the $Spec(M)$ topologized with the Zariski topology (quasi-Zariski topology by the notions of [6]) in a similar way to that of $Spec(R)$. For any submodule $N \leq M$, denote by $V^*(N)$ the variety of N, which is the set $V^*(N) = \{P \in Spec(M)|N \subseteq P\}$. Then the set $\tau^*(M) = \{V^*(N)|N \leq M\}$ is not closed under finite unions. The R-module M is called a Top-module provided that $\tau^*(M)$ is closed under finite unions, whence $\tau^*(M)$ constitute the closed sets in a Zariski topology on $Spec(M)$.

A grading on a ring and its modules usually aids computations by allowing one to focus on the homogeneous elements, which are presumably simpler or more controllable than random elements. However, for this to work one needs to know that the constructions being studied are graded. One approach to this issue is to redefine the constructions entirely in terms of the category of graded modules and thus avoid any consideration of non-graded modules or non-homogeneous elements; Sharp gives such a treatment of attached primes in [10]. Unfortunately, while such an approach helps to understand the graded modules themselves, it will only help to understand the original construction if the graded version of the concept happens to coincide with the original one. Therefore, notably, the study of graded modules is very important.

For the sake of completeness, we recall some definitions and notations used throughout. Let G be an arbitrary group. A commutative ring R with a non-zero identity is G-graded if it has a direct sum decomposition $R = \bigoplus_{g\in G} R_g$ such that for all $g, h \in G$, $R_gR_h \subseteq R_{gh}$. The G-graded ring R is called a graded integral domain provided that $ab = 0$ implies that either $a = 0$ or $b = 0$ where $a, b \in h(R) := \bigcup_{g\in G} R_g$. If R is G-graded, then an R-module M is said to be G-graded if it has a direct sum decomposition $M = \bigoplus_{g\in G} M_g$ such that for all $g, h \in G$, $R_gM_h \subseteq M_{gh}$. For every $g \in G$, an element of R_g or M_g is said to be a homogeneous element. We denote by $h(M)$ the set of all homogeneous elements of M, that is $h(M) = \bigcup_{g\in G} M_g$. Let M be a G-graded R-module. A submodule N of M is called graded (or homogeneous) if $N = \bigoplus_{g\in G}(N \cap M_g)$ or equivalently N is generated by homogeneous elements. Moreover, M/N becomes a G-graded R-module with g-component $(M/N)_g = (M_g + N)/N$ for each $g \in G$. An ideal I of R is called a graded ideal if it is a graded submodule of R and a graded R-module.

Let R be a G-graded ring. A proper graded ideal I of R is said to be a graded prime ideal if whenever $ab \in I$, we have $a \in I$ or $b \in I$, where $a, b \in h(R)$. The graded radical of I , denoted by $Gr(I)$, is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x^{n_g} \in I$. A graded R-module M is called graded finitely generated if $M = \sum_{i=1}^{n} Rx_{g_i}$, where $x_{g_i} \in h(M)$ for every $1 \leq i \leq n$. It is clear that a graded module is finitely generated if and only if it is graded finitely generated. For M, consider the subset $T^g(M) = \{m \in M : rm = 0 \text{ for some nonzero } r \in h(R)\}$. If R is a graded integral domain, then $T^g(M)$ is a graded submodule of M. M is called graded torsion-free (g-torsion-free for short) if $T^g(M) = 0$, and it is called graded torsion (g-torsion for short) if $T^g(M) = M$. It is clear that if M is torsion-free, then it is g-torsion-free. Moreover, if M is g-torsion, then it is torsion.

Most of our results are related to the references [6, 7] which have been proved for the graded case.

2 Results

Let R be a G-graded R-module and consider $Spec_g(R)$, the spectrum of all graded prime ideals of R. The Zariski topology on $Spec_g(R)$ is defined in a similar way to that of $Spec(R)$. For each graded ideal I of R, the graded variety of I is the set $V_R^g(I) = \{P \in Spec_g(R) | I \subseteq P\}$. Then the set $\{V_R^g(I) | I \text{ is a graded ideal of } R\}$ satisfies the axioms for the closed sets of a topology on $Spec_g(R)$, called the Zariski topology on $Spec_g(R)$ (see [9]).

Let R be a G-graded ring and M a graded R-module. We recall from [3] that a proper graded submodule N of M is called graded prime if $rm \in N$, then $m \in N$

or $r \in (N :_R M) = \{r \in R | rM \subseteq N\}$, where $r \in h(R)$, $m \in h(M)$. It is shown in [3, Proposition 2.7] that if N is a graded prime submodule of M, then $P := (N :_R M)$ is a graded prime ideal of R. Let N be a graded submodule of M. Then N is a graded prime submodule of M if and only if $P := (N :_R M)$ is a graded prime ideal of R and M/N is a g-torsion-free R/P-module. Note that some graded R-modules M have no graded prime submodules. We call such graded modules g-primeless. For example, the zero module is clearly g-primeless. A submodule S of M will be called graded semiprime if S is an intersection of graded prime submodules of M. Let $Spec_g(M)$ denote the set of all graded prime submodules of M. Our goal is to endow $Spec_g(M)$ with some topologies. To this end, for each subset $E \subseteq h(M)$, let

$$V_*^g(E) = \{P \in Spec_g(M) | E \subseteq P\}.$$

Let N be a graded submodule of M. The graded radical of N in M, denoted by $Gr_M(N)$ is defined to be the intersection of all graded prime submodules of M containing N [5]. In the other words, $Gr_M(N) = \bigcap_{P \in V_*^g(N)} P$, and it is equal to M if $V_*^g(M) = \emptyset$. It is obvious that $N \subseteq Gr_M(N)$ and that $Gr_M(N) = M$ or $Gr_M(N)$ is a graded semiprime submodule of M.

Assume that N is the graded submodule generated by $E \subseteq h(M)$. Then from $E \subseteq N \subseteq Gr_M(N)$ we clearly have $V_*^g(Gr_M(N)) \subseteq V_*^g(N) \subseteq V_*^g(E)$. On the other hand, N is the smallest graded submodule of M containing E, so that if $P \in V_*^g(E)$, then $P \in V_*^g(N)$. Therefore $V_*^g(E) = V_*^g(N)$. Moreover $Gr_M(N)$ is the intersection of all graded prime submodules of M containing N; so $V_*^g(N) = V_*^g(Gr_M(N))$. Therefore $V_*^g(E) = V_*^g(N) = V_*^g(Gr_M(N))$. Consider the cases when $E = \{0\}$ or $E = M$. Then $V_*^g(0) = Spec_g(M)$ and $V_*^g(M) = \emptyset$. Now let $\{N_\lambda\}_{\lambda \in \Lambda}$ be a family of graded submodules of M. Then $\bigcap_{\lambda \in \Lambda} V_*^g(N_\lambda) = V_*^g(\sum_{\lambda \in \Lambda} N_\lambda)$. Moreover, for every pair N and K of graded submodules of M, we have $V_*^g(N) \cup V_*^g(K) \subseteq V_*^g(N \cap K)$. Summarizing, we have proved:

Proposition 1. *Let M be a graded R-module. Then*

(1) For each subset $E \subseteq h(M)$, $V_*^g(E) = V_*^g(N) = V_*^g(Gr_M(N))$, where N is the graded submodule of M generated by E.

(2) $V_*^g(0) = Spec_g(M)$, and $V_*^g(M) = \emptyset$.

(3) If $\{N_\lambda\}_{\lambda \in \Lambda}$ is a family of graded submodules of M, then $\bigcap_{\lambda \in \Lambda} V_*^g(N_\lambda) = V_*^g(\sum_{\lambda \in \Lambda} N_\lambda)$.

(4) For every pair N and K of graded submodules of M, we have $V_*^g(N) \cup V_*^g(K) \subseteq V_*^g(N \cap K)$.

Therefore if we set

$$\zeta_*^g(M) = \{V_*^g(N) | \text{N is a graded submodule of M}\}$$

then $\zeta_*^g(M)$ contains the empty set and $Spec_g(M)$, and $\tau_*^g(M)$ is closed under arbitrary intersections, but it is not necessarily closed under finite unions.

Definition 1. Let M be a graded R-module.

(1) We shall say that M is a $g-Top$ module if $\zeta^g_*(M)$ is closed under finite unions, i.e. for any graded submodules N and L of M there exists a graded submodule K of M such that $V^g_*(N) \cup V^g_*(L) = V^g_*(K)$.

(2) A graded prime submodule N of M will be called graded extraordinary, or g-extraordinary for short, if whenever K and L are graded semiprime submodules of M with $K \cap L \subseteq N$ then $K \subseteq N$ or $L \subseteq N$.

Assume that M is a $g-Top$ module. In this case $\zeta^g_*(M)$ satisfies the axioms for the closed sets of a unique topology τ^g_* on $Spec_g(M)$. Then the topology $\tau^g_*(M)$ on $Spec_g(M)$ is called the quasi-Zariski topology. Note that we are not excluding the trivial case where $Spec_g(M)$ is empty; i.e. g-primeless graded R-modules are $g-Top$ modules. Also any graded prime ideal of the graded ring R is an extraordinary graded prime submodule of the graded R-module R.

Theorem 1. *Let M be a graded R-module. Then, the following statements are equivalent:*

(i) M is a $g-Top$ module.

(ii) Every graded prime submodule of M is g-extraordinary.

(iii) $V^g_*(N) \cup V^g_*(L) = V^g_*(N \cap L)$ for any graded semiprime submodules N and L of M.

Proof. The result is clear when $Spec_g(M) = \emptyset$. So assume that $Spec_g(M) \neq \emptyset$.

$(i) \Rightarrow (ii)$ Let M be a $g-Top$ module. Assume that P is a graded prime submodule of M and that N, L are graded semiprime submodules of M with $N \cap L \subseteq P$. By assumption, there exists a graded submodule K of M with $V^g_*(N) \cup V^g_*(L) = V^g_*(K)$. Since N is a graded semiprime submodule, $N = \bigcap_{i\in I} P_i$ in which $\{P_i\}_{i\in I}$ is a collection of graded prime submodules of M. For every $i \in I$, we have

$$P_i \in V^g_*(N) \subseteq V^g_*(K) \Rightarrow K \subseteq P_i \Rightarrow K \subseteq \textstyle\bigcap_{i\in I} P_i = N.$$

Similarly, $K \subseteq L$. So $K \subseteq N \cap L$. Now we have

$$V^g_*(N) \cup V^g_*(L) \subseteq V^g_*(N \cap L) \subseteq V^g_*(K) = V^g_*(N) \cup V^g_*(L).$$

Consequently, $V^g_*(N) \cup V^g_*(L) = V^g_*(N \cap L)$. Now from $N \cap L \subseteq P$ we have $P \in V^g_*(N \cap L) = V^g_*(N) \cup V^g_*(L)$. Hence either $P \in V^g_*(N)$ or $P \in V^g_*(L)$, that is either $N \subseteq P$ or $L \subseteq P$. So P is g-extraordinary.

$(ii) \Rightarrow (iii)$ Suppose that every graded prime submodule of M is g-extraordinary. Assume that N and L are two graded semiprime submodules of M. Clearly $V^g_*(N) \cup V^g_*(L) \subseteq V^g_*(N \cap L)$. For the other containment, choose $P \in V^g_*(N \cap L)$. Then $N \cap L \subseteq P$. By assumption, P is g-extraordinary. So $N \subseteq P$ or $L \subseteq P$, that is either $P \in V^g_*(N)$ or $P \in V^g_*(L)$. Therefore $V^g_*(N \cap L) \subseteq V^g_*(N) \cup V^g_*(L)$, and so $V^g_*(N) \cup V^g_*(L) = V^g_*(N \cap L)$.

$(iii) \Rightarrow (i)$ Let N, L be two graded submodules of M. We can assume that $V^g_*(N)$ and $V^g_*(L)$ are both nonempty, for otherwise $V^g_*(N) \cup V^g_*(L) = V^g_*(N)$ or $V^g_*(N) \cup V^g_*(L) = V^g_*(L)$. We know that $Gr_M(N)$ and $Gr_M(L)$ are both graded semiprime submodules of M. Setting $K = Gr_M(N) \cap Gr_M(L)$ we have:

$V_*^g(N) \cup V_*^g(L) = V_*^g(Gr_M(N)) \cup V_*^g(Gr_M(L)) = V_*^g(Gr_M(N) \cap Gr_M(L)) = V_*^g(K)$ by (iii). Hence M is a $g-Top$ module. $\square$

Proposition 2. *Let M be a graded R-module with the property that for every graded prime submodule N of M, $(K :_R M) \subseteq (N :_R M)$ implies that $K \subseteq N$ for each graded semiprime submodule K of M. Then M is a $g-Top$ module.*

Proof. Let N be a graded prime submodule of M and assume that $S_1 \cap S_2 \subseteq N$, where S_1, S_2 are graded semiprime submodules of M. It follows from $(S_1 :_R M) \cap (S_2 :_R M) = (S_1 \cap S_2 :_R M) \subseteq (N :_R M)$ that either $(S_1 :_R M) \subseteq (N :_R M)$ or $(S_2 :_R M) \subseteq (N :_R M)$ since $(N :_R M)$ is a graded prime ideal of R. Now by assumption we have $S_1 \subseteq N$ or $S_2 \subseteq N$, that is N is g-extraordinary. Hence M is a $g-Top$ module by Theorem 1. $\square$

Theorem 2. *Let M be a $g-Top$ R-module.*

(1) If K is a graded submodule of M, then M/K is a $g-Top$ R-module.

(2) The graded R_P-module M_P is a $g-Top$ module for every graded prime ideal P of R.

(3) If $Gr_M(N) = N$ for every graded submodule N of M, then M is a graded distributive module.

Proof. There will be nothing to prove if M has no graded prime submodules. So assume that $Spec_g(M) \neq \emptyset$.

(1) By [3, Lemma 2.8], the graded prime submodules of M/K are just the submodules N/K where N is a graded prime submodule of M with $K \subseteq N$. Consequently, any graded semiprime submodule of M/K is of the form S/K in which S is a graded semiprime submodule of M with $K \subseteq S$. Assume that S_1/K and S_2/K are two graded semiprime submodules of M/K. Then, by Theorem 1, $V_*^g(S_1) \cup V_*^g(S_2) = V_*^g(S_1 \cap S_2)$ since M is a $g-Top$ module. Thus $V_*^g(S_1/K) \cup V_*^g(S_2/K) = V_*^g(S_1/K \cap S_2/K)$. It follows from Theorem 1 that M/K is a $g-Top$ module.

(2) By Theorem 1, it is enough to show that every graded prime submodule of M_P is g-extraordinary. Let N be a graded prime submodule of M_P, and let $S_1 \cap S_2 \subseteq N$ for some graded semiprime submodules S_1, S_2 of M_P. Clearly, $N \cap M$ is a proper graded submodule of M. Assume that $r \in h(R)$ and $m \in h(M)$ are such that $rm \in N \cap M$. Then, $r/1 \in h(R_P)$ and $m/1 \in h(M_P)$ with $(r/1)(m/1) = (rm)/1 \in N$. It follows that either $(r/1)M_P \subseteq N$ or $m/1 \in N$ since N is graded prime. Therefore, either $r \in (N \cap M :_R M)$ or $m \in N \cap M$. This implies that $N \cap M$ is a graded prime submodule of M. Hence N is g-extraordinary by Theorem 1. As another consequence, $S_1 \cap M$ and $S_2 \cap M$ are graded semiprime submodules of M with $(S_1 \cap M) \cap (S_2 \cap M) \subseteq N \cap M$. Therefore, $S_1 \cap M \subseteq N \cap M$ or $S_2 \cap M \subseteq N \cap M$. It follows that either $S_1 = (S_1 \cap M)R_P \subseteq (N \cap M)R_P$ or $S_2 = (S_2 \cap M)R_P \subseteq (N \cap M)R_P$. Hence N is a g-extraordinary submodule of M_P.

(3) For every graded submodules N, K and L of M we have:

$$
\begin{aligned}
(K+L)\cap N &= Gr_M((K+L)\cap N)\\
&= \bigcap\{P|P\in V^g_*((K+L)\cap N)\}\\
&= \bigcap\{P|P\in V^g_*(K+L)\cup V^g_*(N)\}\\
&= \bigcap\{P|P\in (V^g_*(K)\cap V^g_*(L))\cup V^g_*(N)\}\\
&= \bigcap\{P|P\in (V^g_*(K)\cup V^g_*(N))\cap (V^g_*(L)\cup V^g_*(N))\}\\
&= \bigcap\{P|P\in (V^g_*(K\cap N))\cap (V^g_*(L\cap N))\}\\
&= \bigcap\{P|P\in V^g_*((K\cap N)+(L\cap N))\}\\
&= Gr_M((K\cap N)+(L\cap N)) = (K\cap N)+(L\cap N).
\end{aligned}
$$

Thus M is graded distributive. □

Let M be a $g-Top$ module and let $X = Spec_g(M)$. We know that any closed subset of X is of the form $V^g_*(N)$ for some graded prime submodule N of M. But now the question arises as to what open subsets of X look like. To say that any open subset of X is of the form $X - V^g_*(N)$ for some graded prime submodule N of M, though true, is not very helpful. For every subset S of $h(M)$, define

$$X_S = X - V^g_*(S)$$

In particular, if $S=\{f\}$, we denote X_S be X_f.

Proposition 3. *The set $\{X_f|f\in h(M)\}$ is a basis for the quasi-Zariski topology on X.*

Proof. Let U be a non-void open subset in X. Then $U = X - V^g_*(N)$ for some graded submodule N of M. Assume that N is generated by some subset $E\subseteq h(M)$. Then we have

$$U = X - V^g_*(N) = X - V^g_*(E) = X - V^g_*(\bigcup_{f\in E}\{f\}) = X - \bigcap_{f\in E} V^g_*(f) = \bigcup_{f\in E}(X - V^g_*(f)) = \bigcup_{f\in E} X_f$$

Therefore the set $\{X_f|f\in h(M)\}$ is a basis for X. □

Let R be a G-graded ring. A graded R-module M is said to be a graded multiplication module if for each graded submodule N of M, $N = IM$ for some graded ideal I of R [4]. One can easily show that if N is a graded submodule of a graded multiplication module M, then $N = (N :_R M)M$. A graded multiplication module need not be multiplication. We first recall some results concerning graded prime submodules and graded multiplication modules.

Theorem 3 (see [8])**.** *Let M be a graded multiplication R-module, and N a proper graded submodule of M. Then, the following statements are equivalent:*

(1) N is a graded prime submodule;

(2) $(N :_R M)$ is a graded prime ideal of R;

(3) $N = PM$ for some graded prime ideal P of R with $Ann(M) \subseteq P$.

Suppose that M is a graded multiplication R-module, $N = IM$ and $K = JM$ are graded submodules of M, where I and J are graded ideals of R. The product of N and K, denoted by NK, is defined by $NK = (IJ)M$. It is proved in [8, Theorem 4] that this product is independent of the choice of I and J. For each pair m, m' of elements of $h(M)$, we define $mm' = (IJ)M$, where $Rm = IM$ and $Rm' = JM$.

Theorem 4. *Let N be a proper graded submodule of the graded multiplication R-module M. Then, the following statements are equivalent.*

(1) N is a graded prime submodule;

(2) $AB \subseteq N$ implies that $A \subseteq N$ or $B \subseteq N$ for each graded submodules A and B of M;

(3) $m.m' \subseteq N$ implies that $m \in N$ or $m' \in N$ for every $m, m' \in h(M)$.

Proof. It is a direct consequence of [8, Theorem 4 and Corollary 2]. □

Theorem 5. *Every graded multiplication module is a $g - Top$ module.*

Proof. Let M be a graded multiplication R-module. Assume that N and L are two graded semiprime submodules of M. Clearly $V_*^g(N) \cup V_*^g(L) \subseteq V_*^g(NL)$. For the converse containment, pick $P \in V_*^g(NL)$. Then from $NL \subseteq P$ we get either $N \subseteq P$ or $L \subseteq P$ by Theorem 4. Therefore $P \in V_*^g(N) \cup V_*^g(L)$, that is $V_*^g(NL) \subseteq V_*^g(N) \cup V_*^g(L)$. Consequently $V_*^g(N) \cup V_*^g(L) = V_*^g(NL)$. It follows from Theorem 1 that M is a $g - Top$ module. □

Corollary 1. *Let M be a graded multiplication R-module. Then $V_*^g(N) \cup V_*^g(L) = V_*^g(NL) = V_*^g(N \cap L)$ for each pair N and L of graded submodules of M.*

We end this paper by endowing $Spec_g(M)$ by another topology, called the Zariski topology on M. Let M be a graded module over the G-graded ring R. For every graded submodule N of M, set

$$V^g(N) = \{P \in Spec_g(M) | (P :_R M) \supseteq (N :_R M)\}$$

and

$$\zeta^g(M) = \{V^g(N) | N \text{ is a graded submodule of } M\}.$$

Then

Proposition 4. *(1) $V^g(0) = Spec_g(M)$, and $V^g(M) = \emptyset$.*

(2) If $\{N_\lambda\}_{\lambda \in \Lambda}$ is a family of graded submodules of M, then $\bigcap_{\lambda \in \Lambda} V^g(N_\lambda) = V^g(\sum_{\lambda \in \Lambda} N_\lambda)$.

(3) For each pair N and K of graded submodules of M, we have $V^g(N) \cup V^g(K) = V^g(N \cap K)$.

Therefore for any graded R-module M there always exists a topology τ^g on $Spec_g(M)$ in which $\zeta^g(M)$ is the family of all closed sets. τ^g is called the Zariski topology on $Spec_g(M)$. Now consider the set

$$\zeta^g_{**}(M) = \{V^g_*(IM) | I \text{ is a graded ideal of } R\}.$$

In contrast with $\zeta^g_*(M)$, $\zeta^g_{**}(M)$ is always closed under finite unions, and so it always induces a topology τ^g_{**} on $Spec_g(M)$. It is easy to verify that, for every $g - Top$ module, the topology τ^g_{**} is coarser than the topology τ^g_*.

Lemma 1. *Let R be a G-graded ring and let M be a graded R-module. For every graded prime ideal p of R, denote by $Spec^p_g(M)$, the set $\{P \in Spec_g(M) | (P :_R M) = p\}$. Then, for every graded submodules N and L of M, the following statements are satisfied.*

(1) If $(N :_R M) = (L :_R M)$, then $V^g(N) = V^g(L)$.

(2) Let both N and L be graded prime. Then $(N :_R M) = (L :_R M)$ if and only if $V^g(N) = V^g(L)$.

(3) $V^g(N) = \bigcup_{p \in V^g_R(N :_R M)} Spec^p_g(M)$.

Theorem 6. *Let R be a G-graded ring and let M be a graded R-module.*

(1) The Zariski topology τ^g on $Spec_g(M)$ and the topology τ^g_{**} are identical.

(2) If M is $g - Top$ module, then the quasi-Zariski topology τ^g_* on $Spec_g(M)$ is finer than the Zariski topology τ^g.

Proof. It is easy to show that $V^g(N) = V^g((N :_R M)M) = V^g_*((N :_R M)M)$ and $V^g(IM) = V^g_*(IM)$ for every graded submodule N of M and every graded ideal I of R. Therefore $\zeta^g(M) = \zeta^g_{**}(M) \subseteq \zeta^g_*(M)$. So the result follows. $\square$

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Ahmad Yousefian Darani
Department of Mathematics
University of Mohaghegh Ardabili
P. O. Box. 179, Ardabil, Iran
E-mail: *yousefian@uma.ac.ir, youseffian@gmail.com*

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Vague Lie Ideals of Lie Algebras

D. R. Prince Williams, Arsham Borumand Saeid

Abstract. In this paper, we have introduced the notion of vague Lie ideal and have studied their related properties. The cartesian products of vague Lie ideals are discussed. In particular, the Lie homomorphisms between the vague Lie ideals of a Lie algebra and the relationship between the domains and the co-domains of the vague Lie ideals under these Lie homomorphisms are investigated.

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1 Introduction

Lie algebras were first discovered by Sophus Lie (1842–1899) when he attempted to classify certain smooth subgroups of general linear groups. The groups he considered are now called Lie groups. By taking the tangent space at the identity element of such a group, he obtained the Lie algebra and hence the problems on groups can be reduced to problems on Lie algebras so that it becomes more tractable. To study more about Lie algebras see [12]. There are many applications of Lie algebras in many branches of mathematics and physics [9].

The notion of fuzzy sets was first introduced by Zadeh [18]. Fuzzy set theory has been developed in many directions by many scholars and has evoked great interest among mathematicians working in different fields of mathematics [15, 16]. Later many authors applied fuzzy set theory in Lie algebras [2–6, 10, 13, 14, 17].

The notion of vague theory was first introduced by Gau and Buechrer [11] in 1993. Later vague theory of the "group" concept into "vague group" was made by Biswas [7]. This work was the first vagueness of any algebraic structure and thus opened a new direction, new exploration, new path of thinking to mathematicians, engineers, computer scientists and many others in various tests. Further, in [1] Akram and Shum have studied vague Lie subalgebras over a vague field. Recently, Borumand Saeid applied vague set theory in $BCI/BCK-$algebras in [8]. The theory of vague sets started with the aim of interpreting the real life problems in a better way than the fuzzy sets do.

In this paper, we have introduced the notion of vague Lie ideals of Lie algebras and have studied their related properties. Characterization of vague Lie ideals on Lie homomorphisms is also presented.

2 Preliminaries

In this section, we first review some elementary aspects that are necessary for this paper.

Definition 2.1. A *Lie algebra* is a vector space L over a field F (equal to $\mathbb{R}$ or $\mathbb{C}$) on which $L \times L \longrightarrow L$ denoted by $(x, y) \longrightarrow [x, y]$ is defined satisfying the following axioms:

(L1) $[x, y]$ is bilinear,

(L2) $[x, x] = 0$ for all $x \in L$,

(L3) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in L$ (Jacobi identity).

In what follows, we denote L for Lie algebra, unless otherwise specified.

We note that the multiplication in a Lie algebra is not associative, i.e., it is not true in general that $[[x, y], z] = [x, [y, z]]$. But it is *anticommutative*, i.e., $[x, y] = -[y, x]$. We call a subspace H of L closed under $[\cdot, \cdot]$ a *Lie subalgebra*. A subspace I of L with the property $[I, L] \subseteq I$ is called a *Lie ideal* of L. Obviously, any Lie ideal is a subalgebra.

Definition 2.1 [13]. A fuzzy set $\mu : L \to [0, 1]$ is said to be a *fuzzy Lie ideal* of L if the following conditions are satisfied:

(F1) $((\forall x, y \in L), \mu(x + y) \geq \min\{\mu(x), \mu(y)\})$,

(F2) $((\forall x, y \in L$ and $\alpha \in F), \mu(\alpha x) \geq \mu(x))$,

(F3) $((\forall x, y \in L), \mu([x, y]) \geq \max\{\mu(x), \mu(y)\})$.

Definition 2.2 [3]. Let μ be a fuzzy set on L, i.e., a map $\mu : L \to [0, 1]$. Then, μ is said to be an *anti fuzzy Lie ideal* of L if the following conditions are satisfied:

(AF1) $((\forall x, y \in L), \mu(x + y) \leq \max\{\mu(x), \mu(y)\})$,

(AF2) $((\forall x, y \in L$ and $\alpha \in F), \mu(\alpha x) \leq \mu(x))$,

(AF3) $((\forall x, y \in L), \mu([x, y]) \leq \mu(x))$.

Definition 2.3 [11]. A *vague set* A in the universe of discourse U is characterized by two membership functions given by:

(V1) A true membership function $t_A : U \to [0, 1]$, and

(V2) A false membership function $f_A : U \to [0, 1]$,

where $t_A(u)$ is a lower bound on the grade of membership of u derived from the "evidence for u", $f_A(u)$ is a lower bound on the negation of u derived from the "evidence against u", and $t_A(u) + f_A(u) \leq 1$.

Thus the grade of membership of u in the vague set A is bounded by a subinterval $[t_A(u), 1 - f_A(u)]$ of $[0, 1]$. This indicates that if the actual grade of membership u is $\mu(u)$, then

$$t_A(u) \leq \mu(u) \leq 1 - f_A(u).$$

The vague set A is written as

$$A = \{\langle u, [t_A(u), f_A(u)]\rangle | u \in U\},$$

where the interval $[t_A(u), 1 - f_A(u)]$ is called the *vague value* of u in A, denoted by $V_A(u)$.

3 Vague Lie Ideals

In this section, we define the notion of vague Lie ideals.

For our discussion, we shall use the following notations on interval arithmetic:

Let $I[0,1]$ denote the family of all closed subintervals of $[0,1]$. We define the term "imax" to mean the maximum of two intervals as:

$$\operatorname{imax}(I_1, I_2) \simeq [\max(a_1, a_2), \max(b_1, b_2)],$$

where $I_1 = [a_1, b_1], I_2 = [a_2, b_2]$. Similarly, we define "imin". The concept of "imax" and "imin" could be extended to define "isup" and "iinf" of infinite number of elements of $[0,1]$.

It is obvious that $L = \{I[0,1], \text{isup}, \text{iinf}, \succeq\}$ is a lattice with universal bounds $[0,0]$ and $[1,1]$.

Also, if $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ are two subintervals of $[0,1]$, we can define a relation between I_1 and I_2 by $I_1 \succeq I_2$ if and only if $a_1 \geq a_2$ and $b_1 \geq b_2$.

Definition 3.1. Let L be a Lie algebra. A vague set A of L is called a *vague Lie subalgebra* of L if the following axioms hold:

(VLI1) $(\forall x, y \in L), (V(x+y) \succeq \operatorname{imin}\{V(x), V(y)\})$,

(VLI2) $(\forall x \in L, a \in F), (V(ax)) \succeq V(x))$.

(VLI3) $(\forall x, y \in L), (V([x,y]) \succeq \operatorname{imin}\{V(x), V(y)\})$.

That is,

$$\begin{aligned}
t_A(x+y) &\geq \min\{t_A(x), t_A(y)\}) \\
1 - f_A(x+y) &\geq \min\{1 - f_A(x), 1 - f_A(y)\} \\
t_A(ax) &\geq t_A(x) \\
1 - f_A(ax) &\geq 1 - f_A(x)) \\
t_A([x,y]) &\geq \min\{t_A(x), t_A(y)\}) \\
1 - f_A([x,y]) &\geq \min\{1 - f_A(x), 1 - f_A(y)\}.
\end{aligned}$$

Definition 3.2. Let L be a Lie algebra. A vague set A of L is called a *vague Lie ideal* of L if the following axioms hold:

It satisfies (VLI1), (VLI2) and (VLI4)$(\forall x, y \in L), (V([x,y]) \succeq \operatorname{imax}\{V(x), V(y)\})$.

That is,

$$\begin{aligned}
t_A([x,y]) &\geq \max\{t_A(x), t_A(y)\}) \\
1 - f_A([x,y]) &\geq \max\{1 - f_A(x), 1 - f_A(y)\}.
\end{aligned}$$

Example 3.3. Let $\mathbb{R}^3 = \{(x,y,z) : x, y, z \in \mathbb{R}\}$ be the set of all 3-dimensional real vectors. Then $\mathbb{R}^3$ with the bracket $[\cdot,\cdot]$ defined as the usual cross product, i.e.,

$$[x,y] = x \times y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1),$$

forms a real Lie algebra over the field $\mathbb{R}$.

(1) Let A be the vague set in $\mathbb{R}^3$ defined as follows:

$$A = \begin{cases} [0.8, 0.1] & if \ \ x = y = z = 0, \\ [0.7, 0.2] & if \ \ x \neq 0, y = z = 0, \\ [0.5, 0.3] & \quad otherwise. \end{cases}$$

By routine calculations, it is clear that A is a vague Lie subalgebra of $\mathbb{R}^3$, but not a vague ideal of $\mathbb{R}^3$, since

$$t_A([(1, 0, 0), (1, 1, 1)]) = t_A(0, -1, 1) = 0.5$$

and

$$\max\{t_A(1, 0, 0), t_A(1, 1, 1)\} = \max\{0.7, 0.5\} = 0.7.$$

Also,

$$1 - f_A([(1, 0, 0), (1, 1, 1)]) = 1 - f_A(0, -1, 1) = 0.7$$

and

$$\max\{1 - f_A(1, 0, 0), 1 - f_A(1, 1, 1)\} = \max\{0.8, 0.7\} = 0.8.$$

(2) Let A be the vague set in $\mathbb{R}^3$ defined as follows:

$$A = \begin{cases} [0.8, 0.1] & \text{if } \ (x, y, z) = (0, 0, 0), \\ [0.6, 0.2] & \text{otherwise.} \end{cases}$$

By routine calculations, it is clear that A is a vague Lie ideal of $\mathbb{R}^3$.

Proposition 3.4. *Let A be a vague set of L. Then A is a vague ideal of L if and only if t_A is a fuzzy ideal of L and f_A is an anti fuzzy ideal of L.*

Proof. The proof is obvious. □

For $\alpha, \beta \in [0, 1]$, now we define $(\alpha, \beta) - cut$ and $\alpha - cut$ of a vague set.

Definition 3.5. Let A be a vague set in L with true membership function t_A and the false membership function f_A. The $(\alpha, \beta) - cut$ of the vague set A is a crisp subset $A_{(\alpha,\beta)}$ of the set L given by

$$A_{(\alpha,\beta)} = \{x \in L \,|V_A(x) \succeq [\alpha, \beta]\}.$$

Clearly, $A_{(0,0)} = L$. The $(\alpha, \beta)-$cuts of the vague set A are also called *vague sets* of A.

Definition 3.6. The $\alpha - cut$ of the vague set A is a crisp subset A_α of the set L given by $A_\alpha = A_{(\alpha,\alpha)}$.

Note that $A_0 = L$, and if $\alpha \geq \beta$ then $A_\alpha \subseteq A_\beta$ and $A_{(\alpha,\beta)} = A_\alpha$. Equivalently, we can define the $\alpha-$cut as

$$A_{(\alpha)} = \{x \in L \,|t_A(x) \geq \alpha\}.$$

Theorem 3.7. *Let A be a vague set of L. Then A is a vague Lie ideal of L if and only if $A_{(\alpha,\beta)}$ is a Lie ideal of L for every $\alpha, \beta \in (0,1]$.*

Proof. Let A be a vague set of L. Suppose A is a vague Lie ideal of L.

For all $x, y \in A_{(\alpha,\beta)}$ and $\alpha, \beta \in (0,1]$, then

$$t_A(x), t_A(y) \geq \alpha \text{ and } 1 - f_A(x), 1 - f_A(y) \geq \beta.$$

Then we have

(i)

$$t_A(x + y) \geq t_A(x) \geq \min\{t_A(x), t_A(y)\} \geq \alpha$$

and

$$1 - f_A(x + y) \geq 1 - f_A(x) \geq \min\{1 - f_A(x), 1 - f_A(y)\} \geq \beta.$$

Thus $x + y \in A_{(\alpha,\beta)}$.

(ii)

$$t_A(ax) \geq t_A(x) \geq \alpha \text{ and } 1 - f_A(ax) \geq 1 - f_A(x) \geq \beta.$$

Thus $ax \in A_{(\alpha,\beta)}$.

(iii)

$$t_A([x, y]) \geq \max\{t_A(x), t_A(y)\} \geq \alpha,$$

and

$$1 - f_A([x, y]) \geq \max\{1 - f_A(x), 1 - f_A(y)\} \geq \beta,$$

which implies $[x, y] \in A_{(\alpha,\beta)}$. Thus $A_{(\alpha,\beta)}$ is a Lie ideal of L.

Conversely, assume that $A_{(\alpha,\beta)} \neq \emptyset$ is a Lie ideal of L for every $\alpha, \beta \in (0,1]$. Assume that

$$V(x + y) \prec \text{imin}\{V(x), V(y)\}$$

for some $x, y \in L$. Taking

$$\alpha_1 = \frac{1}{2}\{t_A(x + y) + \min\{t_A(x), t_A(y)\}\}$$

and

$$\beta_2 = \frac{1}{2}\{1 - f_A(x + y) + \min\{1 - f_A(x), 1 - f_A(y)\}\}$$

for some $x, y \in L$, we have

$$t_A(x + y) < \alpha_1 < \min\{t_A(x), t_A(y)\}$$

and

$$1 - f_A(x+y) < \beta_2 < \min\{1 - f_A(x), 1 - f_A(y)\}.$$

So, we have $x+y \notin A_{(\alpha_1,\beta_2)}$, for all $x, y \in A_{(\alpha_1,\beta_2)}$. This is a contradiction. Thus

$$V(x+y) \succeq \text{imin}\{V(x), V(y)\}.$$

Similarly, we can prove (VLI2), (VLI3) and (VLI4). Hence A is a vague ideal of L. This completes the proof. □

Theorem 3.8. *If $\{A_i | i \in I\}$ is an arbitrary family of vague Lie ideals of L then $\bigcap A_i$ is a vague Lie ideals of L, where $\bigcap A_i(x) = inf\{A_i(x) | i \in I)\}$, for all $x \in L$.*

Proof. The proof is trivial. □

However, the union of two vague Lie ideals cannot be a vague ideal. Let A and B be two vague Lie ideals of L. Define

$$(A \cup B)(x) = \max\{A(x), B(x)\}, \text{ for all } x \in L.$$

The following example shows that $A \cup B$, cannot be a vague Lie ideal of L.

Example 3.9. Let $\{e_1, e_2, ..., e_8\}$ be a basis of a vector space V over a field F. Then, it is not difficult to see that, by putting: $[e_1, e_2] = e_5$, $[e_1, e_3] = e_6$, $[e_1, e_4] = e_7$, $[e_1; e_5] = -e_8$, $[e_2, e_3] = e_8$, $[e_2; e_4] = e_6$, $[e_2, e_6] = -e_7$, $[e_3, e_4] = -e_5$, $[e_3, e_5] = -e_7$, $[e_4; e_6] = -e_8$, $[e_i; e_j] = -[e_j, e_i]$ and $[e_i, e_j] = 0$ for all $i \leq j$, we can obtain a Lie algebra over the field F. Define the vague sets A and B for all $x \in V$ as follows:

$$A = \begin{cases} [0.8, 0.1] & if\ \ x = 0, e_8, \\ [0.6, 0.2] & if\ \ x = e_7, \\ [0.2, 0.6] & otherwise. \end{cases}$$

and

$$B = \begin{cases} [0.8, 0.1] & if\ \ x = 0, e_7, \\ [0.5, 0.3] & if\ \ x = e_8, \\ [0.2, 0.6] & otherwise. \end{cases}$$

Then A and B are vague Lie ideal of V, since by Theorem 3.7, the vague-cut sets, $A_{(0.8,0.1)} = \langle e_8 \rangle$, $B_{(0.8,0.1)} = \langle e_7 \rangle$ and $A_{(0.6,0.2)} = B_{(0.5,0.3)} = \langle e_7, e_8 \rangle$ are vague Lie ideals of V, but

$$(t_A \cup t_B)(e_7 + e_8) = \max\{t_A(e_7 + e_8), t_B(e_7 + e_8)\} \geq$$

$$\geq \max\{\min\{t_A(e_7), t_A(e_8)\}, \min\{t_B(e_7), t_B(e_8)\}\} = \max\{0.6, 0.5\} = 0.6.$$

and

$$(1 - f_A \cup f_B)(e_7 + e_8) = \max\{1 - f_A(e_7 + e_8), 1 - f_B(e_7 + e_8)\} \geq$$

$$\geq \max\{\min\{1 - f_A(e_7), 1 - f_A(e_8)\}, \min\{1 - f_B(e_7), 1 - f_B(e_8)\}\} =$$

$$= \max\{0.8, 0.7\} = 0.8.$$

On the other hand

$$\min\{(t_A \cup t_B)(e_7), (t_A \cup t_B)(e_8)\} =$$
$$= \min\{\max\{t_A(e_7), t_B(e_7)\}, \max\{t_A(e_8), t_B(e_8)\}\} =$$
$$= \min\{0.8, 0.8\} = 0.8.$$

and

$$\min\{1 - (f_A \cup f_B)(e_7), 1 - (f_A \cup f_B)(e_8)\} =$$
$$= \min\{\max\{1 - f_A(e_7), 1 - f_B(e_7)\}, \max\{1 - f_A(e_8), 1 - f_B(e_8)\}\} =$$
$$= \min\{0.9, 0.9\} = 0.9.$$

Thus we have

$$(t_A \cup t_B)(e_7 + e_8) = 0.6 \ngeq 0.8 = \min\{(t_A \cup t_B)(e_7), (t_A \cup t_B)(e_8)\}$$

and

$$1 - (f_A \cup f_B)(e_7 + e_8) = 0.8 \ngeq 0.9 = \min\{1 - (f_A \cup f_B)(e_7), 1 - (f_A \cup f_B)(e_8)\}.$$

Therefore, $(A \bigcup B)$ is not a vague Lie ideal.

Definition 3.10. Let A and B be two vague Lie ideals of L. We define the *sup − min product* $[AB]$ of A and B by

$$[t_A t_B](x) = \begin{cases} \sup\limits_{x=[yz]} \min\{t_A(y), t_B(z)\}, \\ \\ 0, \qquad x \neq yz \end{cases}$$

and

$$[1 - f_A f_B](x) = \begin{cases} \sup\limits_{x=[yz]} \min\{1 - f_A(y), 1 - f_B(z)\}, \\ \\ 0, \qquad x \neq yz. \end{cases}$$

Let A and B be vague Lie ideals of the Lie algebra L. Then $[AB]$ may not be a vague Lie ideal of L as this can be seen in the following counter-example:

Example 3.11. Let $\{e_1, e_2, ..., e_8\}$ be a basis of a vector space over a field F. Then, it is not difficult to see that, by putting: $[e_1, e_2] = e_5$, $[e_1, e_3] = e_6$, $[e_1, e_4] = e_7$, $[e_1; e_5] = -e_8$, $[e_2, e_3] = e_8$, $[e_2; e_4] = e_6$, $[e_2, e_6] = -e_7$, $[e_3, e_4] = -e_5$, $[e_3, e_5] = -e_7$, $[e_4; e_6] = -e_8$, $[e_i; e_j] = -[e_j, e_i]$ and $[e_i, e_j] = 0$ for all $i \leq j$, we can obtain a Lie algebra over the field F. The following vague sets

$$A = \begin{cases} [0.7, 0.1] \quad if \ x = 0, e_1, e_5, e_6, e_7, e_8, \\ \\ [0.2, 0.6] \quad otherwise. \end{cases}$$

and

$$B = \begin{cases} [0.7, 0.1] & if \ \ x = 0, \\ \\ [0.5, 0.2] & if \ \ x = e_2, e_5, e_6, e_7, e_8, \\ \\ [0.2, 0.6] & otherwise. \end{cases}$$

Thus A and B are vague Lie ideals of L because the cut Lie ideals of L $A_{(0.7,0.1)} =< e_1, e_5, e_6, e_7, e_8 >$ and $B_{(0.5,0.2)} =< e_2, e_5, e_6, e_7, e_8 >$ are vague-cut Lie ideals of L. But $[AB]$ is not a vague Lie ideal because the following condition does not hold:

$$[V_A V_B](e_7 + e_8) \succeq \text{imin}\{[V_{AB}](e_7), [V_{AB}](e_8)\},$$

$$t_A t_B(e_7) = \sup \begin{cases} \min\{t_A(e_1), t_B(e_4)\} = \min\{0.7, 0.2\} = 0.2, e_7 = [e_1, e_4], \\ \min\{t_A(e_2), t_B(e_6)\} = \min\{0.2, 0.5\} = 0.2, e_7 = -[e_2, e_6], \\ \min\{t_A(e_3), t_B(e_5)\} = \min\{0.2, 0.5\} = 0.2, e_7 = -[e_3, e_5], \\ \min\{t_A(e_4), t_B(e_1)\} = \min\{0.2, 0.2\} = 0.2, e_7 = -[e_4, e_1], \\ \min\{t_A(e_6), t_B(e_2)\} = \min\{0.7, 0.5\} = 0.5, e_7 = [e_6, e_2], \\ \min\{t_A(e_5), t_B(e_3)\} = \min\{0.7, 0.2\} = 0.2, e_7 = [e_5, e_3] \end{cases}$$

and

$1 - f_A f_B(e_7) =$

$$= \sup \begin{cases} \min\{1 - f_A(e_1), 1 - f_B(e_4)\} = \min\{0.9, 0.4\} = 0.4, e_7 = [e_1, e_4], \\ \min\{1 - f_A(e_2), 1 - f_B(e_6)\} = \min\{0.4, 0.8\} = 0.4, e_7 = -[e_2, e_6], \\ \min\{1 - f_A(e_3), 1 - f_B(e_5)\} = \min\{0.4, 0.8\} = 0.4, e_7 = -[e_3, e_5], \\ \min\{1 - f_A(e_4), 1 - f_B(e_1)\} = \min\{0.9, 0.4\} = 0.4, e_7 = -[e_4, e_1], \\ \min\{1 - f_A(e_6), 1 - f_B(e_2)\} = \min\{0.9, 0.8\} = 0.8, e_7 = [e_6, e_2], \\ \min\{1 - f_A(e_5), 1 - f_B(e_3)\} = \min\{0.9, 0.4\} = 0.4, e_7 = [e_5, e_3]. \end{cases}$$

Thus $t_A t_B(e_7) = 0.5$ and $1 - f_A f_B(e_7) = 0.8$.

Similarly, we can get $t_A t_B(e_8) = 0.5$ and $1 - f_A f_B(e_8) = 0.8$.

On the other hand, we have

$$t_A t_B(e_7 + e_8) = sup\{i - vi\}$$

and

$$1 - f_A f_B(e_7 + e_8) = sup\{i - vi\}$$

(i) if $e_7 + e_8 = [e_1(e_4 - e_5)]$, then

$$\min\{t_A(e_1), t_B(e_4 - e_5)\} = \min\{t_A(e_1), t_B(e_4), t_B(e_5)\} = 0.2,$$

since $t_B(e_4) = 0.2$, and if $e_7 + e_8 = [(e_5 - e_4)e_1]$, then

$$\min\{t_A(e_5 - e_4), t_B(e_1)\} = \min\{t_A(e_5), t_A(e_4), t_B(e_1)\} = 0.2,$$

since $t_A(e_4) = 0.2$.

Similarly,
(ii) the cases $e_7 + e_8 = [e_2(e_3 - e_6)]$; then the value is 0.2,
(iii) the cases $e_7 + e_8 = [e_3(-e_2 - e_5)]$; then the value is 0.2,
(iv) the cases $e_7 + e_8 = [e_4(-e_1 - e_6)]$; then the value is 0.2,
(v) the cases $e_7 + e_8 = [e_5(-e_3 - e_1)]$; then the value is 0.2,
(vi) the cases $e_7 + e_8 = [e_6(-e_2 - e_4)]$; then the value is 0.2.
Thus,

$$t_A t_B(e_7 + e_8) = \min\{0.2, 0.2, 0.2, 0.2, 0.2, 0.2\} = 0.2.$$

Hence, we have proved that

$$t_A t_B(e_7 + e_8) = 0.2 \ngeq 0.5 = \min\{t_A t_B(e_7), t_A t_B(e_8)\}.$$

On the other hand, we can prove
$1 - f_A f_B(e_7 + e_8) = 0.4 \ngeq 0.8 = \min\{1 - f_A f_B(e_7), 1 - f_A f_B(e_8)\}$.
Now we redefine the product of two vague Lie ideals A and B of L to an extended form.

Definition 3.12. Let A and B be two vague sets of L. Then, we define the *sup − min product* $[\![AB]\!]$ of A and B, as follows, for all $x, y, z \in L$:

$$[\![t_A t_B]\!](x) = \begin{cases} \sup\limits_{x=\sum\limits_{i=1}^{n}[x_i y_i]} \left\{ \min\limits_{i \in N} \{\min(t_A(x_i), t_B(y_i))\} \right\}, \\ 0, \qquad x \neq \sum\limits_{i=1}^{n}[x_i y_i] \end{cases}$$

and

$$[\![1 - f_A f_B]\!](x) = \begin{cases} \sup\limits_{x=\sum\limits_{i=1}^{n}[x_i y_i]} \left\{ \min\limits_{i \in N} \{\min(1 - f_A(x_i), 1 - f_B(y_i))\} \right\}, \\ 0, \qquad x \neq \sum\limits_{i=1}^{n}[x_i y_i]. \end{cases}$$

From the definitions of $[AB]$ and $[\![AB]\!]$, we can easily see that $[AB] \subseteq [\![AB]\!]$ and $[AB] \neq [\![AB]\!]$.

The following theorem proves $[\![AB]\!]$ is a vague Lie ideal of L if A and B are vague Lie ideals of L.

Theorem 3.13. *Let A and B be any two vague Lie ideals of L. Then $[\![AB]\!]$ is also a vague Lie ideal of L.*

Proof. It is easy to prove $[\![AB]\!]$ is a vague Lie subalgebra of L.

(iv) Suppose $x, y \in L$. Let if possible,

$$[\![V_A V_B]\!]([x, y]) \prec \operatorname{imax}\{[\![V_A V_B]\!](x), [\![V_A V_B]\!](y)\}.$$

Then we have

$$[\![V_A V_B]\!]([x,y]) \prec [\![V_A V_B]\!](x) \text{ or } [\![V_A V_B]\!]([x,y]) \prec [\![V_A V_B]\!](y).$$

Choose a number $t < s \in [0,1]$ such that

$$[\![t_A t_B]\!]([x,y]) < t < [\![t_A t_B]\!](x), [\![t_A t_B]\!]([x,y]) < t < [\![t_A t_B]\!](y).$$

and

$$\begin{aligned}
&[\![1 - f_A f_B]\!]([x,y]) < s < [\![1 - f_A f_B]\!](x),\\
&[\![1 - f_A f_B]\!]([x,y]) < s < [\![1 - f_A f_B]\!](y).
\end{aligned}$$

There exist $x_i, y_i \in L$ such that $x = \sum\limits_{i=1}^{n} [x_i y_i]$.
For all i, j we have,

$$t_A(x_i) > t, t_B(y_i) > t$$

and

$$1 - f_A(x_i) > s, 1 - f_B(y_i) > s.$$

Since $[x,y] = \left[\sum\limits_{i=1}^{n} [x_i, y_i], y\right]$, we have

$$\begin{aligned}
[\![t_A t_B]\!]([x,y]) &= [\![t_A t_B]\!]\left(\left[\sum_{i=1}^{n} [x_i, y_i], y\right]\right)\\
&= [\![t_A t_B]\!]\left(\sum_{i=1}^{n} [[x_i, y_i], y]\right)\\
&\geq [\![t_A t_B]\!]([[x_i, y_i], y]), \text{for all } i\\
&= [\![t_A t_B]\!]([[x_i, y], y_i] - [[y_i, y], x_i])\\
&\geq [\![t_A t_B]\!]([[x_i, y], y_i])\\
&\geq \max\{t_A [x_i, y], t_B(y_i)\}\\
&\geq \max\{\max\{t_A(x_i), t_A(y)\}, t_B(y_i)\}\\
&> t.
\end{aligned}$$

and

$$\begin{aligned}
[\![1 - f_A f_B]\!]([x,y]) &= [\![1 - f_A f_B]\!]\left(\left[\sum_{i=1}^{n} [x_i, y_i], y\right]\right)\\
&= [\![1 - f_A f_B]\!]\left(\sum_{i=1}^{n} [[x_i, y_i], y]\right)\\
&\geq [\![1 - f_A f_B]\!]([[x_i, y_i], y]), \text{for all } i\\
&= [\![1 - f_A f_B]\!]([[x_i, y], y_i] - [[y_i, y], x_i])\\
&\geq [\![1 - f_A f_B]\!]([[x_i, y], y_i])
\end{aligned}$$

$$\begin{aligned} &\geq \max\{1 - f_A[x_i, y], 1 - f_B(y_i)\} \\ &\geq \max\{\max\{1 - f_A(x_i), 1 - f_A(y)\}, 1 - f_B(y_i)\} \\ &> s. \end{aligned}$$

Thus, we have

$$[\![1 - f_A f_B]\!]([x, y]) > t \text{ and } [\![1 - f_A f_B]\!]([x, y]) > s,$$

which is a contradiction. Thus $[\![AB]\!]$ satisfies (VLI4). Hence $[\![AB]\!]$ is a vague Lie ideal of L. □

The following theorem characterized congruence relation on L.

Theorem 3.14. *Let A be a vague Lie ideal of L. Define a binary relation $\sim$ on L by $x \sim y$ if and only if $t_A(x - y) = t_A(0), 1 - f_A(x - y) = 1 - f_A(0)$ for all $x, y \in L$. Then $\sim$ is a congruence relation on L.*

Proof. To prove $\sim$ is an equivalent relation, it is enough to show the transitivity of $\sim$ because the reflectivity and symmetricity of $\sim$ hold trivially. Let $x, y, z \in L$. If $x \sim y$ and $y \sim z$, then $t_A(x - y) = t_A(0)$, $t_A(y - z) = t_A(0)$ and $1 - f_A(x - y) = 1 - f_A(0)$, $1 - f_A(y - z) = 1 - f_A(0)$. Hence it follows that

$$t_A(x - z) = t_A(x - y + y - z) \geq \min\{t_A(x - y), t_A(y - z)\} = t_A(0)$$

and

$$1 - f_A(x - z) = 1 - f_A(x - y + y - z) \geq \min\{1 - f_A(x - y), 1 - f_A(y - z)\} = 1 - f_A(0).$$

Consequently $x \sim z$. We now verify that " $\sim$ " is a congruence relation on L. For this purpose, we let $x \sim y$ and $y \sim z$. Then

$$t_A(x - y) = t_A(0), t_A(y - z) = t_A(0)$$

and

$$1 - f_A(x - y) = 1 - f_A(0), 1 - f_A(y - z) = 1 - f_A(0).$$

Now, for $x_1, x_2, y_1, y_2 \in L$, we have

(i)

$$\begin{aligned} t_A((x_1 + x_2) - (y_1 + y_2)) &= t_A((x_1 - y_1) + (x_2 - y_2)) \\ &\geq \min\{t_A(x_1 - y_1), t_A(x_2 - y_2) = t_A(0) \end{aligned}$$

and

$$\begin{aligned} 1 - f_A((x_1 + x_2) - (y_1 + y_2)) &= 1 - f_A((x_1 - y_1) + (x_2 - y_2)) \\ &\geq \min\{1 - f_A(x_1 - y_1), 1 - f_A(x_2 - y_2)\} \\ &= 1 - f_A(0), \end{aligned}$$

(ii)

$$t_A((ax_1 - ay_1)) = t_A(a(x_1 - y_1)) \geq t_A(x_1, y_1) = t_A(0)$$

and

$$1 - f_A((ax_1 - ay_1)) = 1 - f_A(a(x_1 - y_1)) \geq 1 - f_A(x_1, y_1) = 1 - f_A(0),$$

(iii)

$$\begin{aligned} t_A([x_1, x_2] - [y_1, y_2]) &= t_A([(x_1 - y_1), (x_2 - y_2)]) \\ &\geq \max\{t_A(x_1 - y_1), t_A(x_2 - y_2)\} = t_A(0) \end{aligned}$$

and

$$\begin{aligned} 1 - f_A([x_1, x_2] - [y_1, y_2]) &= 1 - f_A([(x_1 - y_1), (x_2 - y_2)]) \\ &\geq \max\{1 - f_A(x_1 - y_1), 1 - f_A(x_2 - y_2)\} \\ &= 1 - f_A(0). \end{aligned}$$

That is, $x_1 + x_2 \sim y_1 + y_2$, $ax_1 \sim ay_1$ and $[x_1, x_2] \sim [y_1, y_2]$. Thus, " $\sim$ " is indeed a congruence relation on L. □

4 Characterization of vague Lie ideals on Lie Homomorphisms

Definition 4.1. Let L and L' be two Lie algebras over a field F. Then a linear transformation $f : A \to B$ is called a *Lie homomorphism* if $g[x, y] = [g(x), g(y)]$ holds, for all $x, y \in L$.

Let $g : L \longrightarrow L'$ be a Lie homomorphism. For any vague set A in L', we define the *preimage* of A under g, denoted by $g^{-1}(A)$, is a vague set in L defined by

$$g^{-1}(t_A) = t_{A_{g^{-1}}}(x) = t_A(g(x))$$

and

$$1 - g^{-1}(f_A) = 1 - f_{A_{g^{-1}}}(x) = 1 - f_A(g(x)), \forall x \in L.$$

For any vague set A in G, we define the *image* of A under a linear transformation g, denoted by $g(A)$, is a vague set in G' defined by

$$g(t_A)(y) = \begin{cases} \sup\limits_{x \in g^{-1}(y)} \quad t_A(x) & if \quad g^{-1}(y) \neq \phi, \\ 0 & otherwise. \end{cases}$$

and

$$g(f_A)(y) = \begin{cases} \inf\limits_{x \in g^{-1}(y)} \quad f_A(x) & if \quad g^{-1}(y) \neq \phi, \\ 0 & otherwise. \end{cases}$$

for all $x \in L$ and $y \in L'$.

Theorem 4.2. *Let g be a surjective Lie homomorphism from L into L'.*

(i) If A and B are two vague Lie ideals of L, then

$$g(A + B) = g(A) + g(B).$$

(ii) *If* $\{A_i | i \in I\}$ *is a set of* $g-$*invariant vague Lie ideal of* L*, then*

$$g\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} g\left(A_i\right).$$

(iii) *If* A *and* B *are two vague Lie ideals of* L*, then*

$$g([\![V_A V_B]\!]) \simeq [\![g(V_A)g(V_B)]\!].$$

Proof. The proofs of (i) and (ii) are trivial. To prove (iii), let $x \in L$. Suppose $g([\![V_A V_B]\!])(x) \prec [\![g(V_A)g(V_B)]\!](x)$. Now, we can choose a number $t < s \in [0,1]$ such that

$$g([\![t_A t_B]\!])(x) < t < [\![g(t_A)g(t_B)]\!](x)$$

and

$$g([\![1 - f_A f_B]\!])(x) < s < [\![g(1 - f_A)g(1 - f_B)]\!](x).$$

Then, there exist $y_i, z_i \in L'$ such that $x = \sum\limits_{i=1}^{n} [y_i, z_i]$ with $g(t_A) > t$, $g(t_B) > t$ and $g(1 - f_A) > s$, $g(1 - f_B) > s$. Since g is surjective, there exists a $y \in L$ such that $g(y) = x$ and $y = \sum\limits_{i=1}^{n} [a_i, b_i]$, for some $a_i \in g^{-1}(y_i)$ and $b_i \in g^{-1}(z_i)$ with $g(a_i) = y_i$ and $g(a_i) = y_i$, $t_A(a_i) > t$, $t_B(b_i) > t$ and $1 - f_A(a_i) > s$, $1 - f_B(b_i) > s$. Since

$$\begin{aligned} g\left(\sum_{i=1}^{n} [a_i, b_i]\right) &= \sum_{i=1}^{n} g\left([a_i, b_i]\right) = \sum_{i=1}^{n} [g(a_i), g(b_i)] \\ &= \sum_{i=1}^{n} [y_i, z_i] = x, \end{aligned}$$

we have $g([\![t_A t_B]\!])(x) > t$ and $g([\![1 - f_A f_B]\!])(x) > s$. This is a contradiction.

Similarly, for the case $g([\![V_A V_B]\!])(x) \succ [\![g(V_A)g(V_B)]\!](x)$ we get the contradiction. Hence, $g([\![V_A V_B]\!])(x) \simeq [\![g(V_A)g(V_B)]\!](x)$. □

Definition 4.3. Let A and B be two vague Lie ideals of L. Then A is said to be of the *same type* as B if there exists $g \in Aut(L)$ such that $A = B \circ g$, i.e., $V_A(x) \simeq V_B(g(x))$, for all $x \in L$.

Theorem 4.4. *Let* A *and* B *be two vague Lie ideals of* L*. Then* A *is a vague Lie ideal having the same type as* B *if and only if* A *is isomorphic to* B*.*

Proof. We only need to prove the necessity because the sufficiency part is trivial. Let A be a vague Lie ideal having the same type as B. Then there exists $g \in Aut(L)$ such that $V_A(x) \simeq V_B(g(x))$, for all $x \in L$. Let $\phi : A(L) \longrightarrow B(L)$ be a mapping defined by $\phi(A(x)) = B(g(x))$, for all $x \in L$, that is $\phi(V_A(x)) \simeq V_B(g(x))$, for all $x \in L$. Then it is clear that f is surjective. For all $x, y \in L$, if $\phi(t_A(x)) =$

$\phi(t_A(y))$, then $t_B(g(x)) = t_B(g(y))$ and hence $t_A(x) = t_A(y)$. Similarly, we can prove $\phi(1 - f_A(x)) = \phi(1 - f_A(y))$, for all $x \in L$ implies $1 - f_B(g(x)) = 1 - f_B(g(y))$. Thus ϕ is one-to-one. Now we need to prove ϕ is a homomorphism. Let all $x, y \in L$, we have

$$\begin{aligned}\phi(t_A(x+y)) &= t_B(g(x+y)) = t_B(g(x)+g(y)) = t_B(g(x)) + t_B(g(y)) \\ &= \phi(t_A(x)) + \phi(t_A(y))\end{aligned}$$

and

$$\begin{aligned}\phi(1 - f_A(x+y)) &= 1 - f_B(g(x+y)) = 1 - f_B(g(x)+g(y)) \\ &= 1 - f_B(g(x)) + 1 - f_B(g(y)) \\ &= \phi(1 - f_A(x)) + \phi(1 - f_A(y)).\end{aligned}$$

Let all $x \in L$ and $a \in F$, we have

$$\phi(t_A(ax)) = t_B(g(ax)) = t_B(ag(x)) = at_B(g(x)) = a\phi(t_A(x)).$$

and

$$\begin{aligned}\phi(1 - f_A(ax)) &= 1 - f_B(g(ax)) = 1 - f_B(ag(x)) = a(1 - f_B(g(x))) \\ &= a\phi(1 - f_A(x)).\end{aligned}$$

Let all $x, y \in L$, we have

$$\begin{aligned}\phi(t_A([x,y])) &= t_B(g([x,y])) = t_B([g(x), g(y)]) = [t_B(g(x)), t_B(g(y))] \\ &= [\phi(t_A(x)), \phi(t_A(y)]\end{aligned}$$

and

$$\begin{aligned}\phi(1 - f_A([x,y])) &= 1 - f_B(g([x,y])) = 1 - f_B([g(x), g(y)]) \\ &= [1 - f_B(g(x)), 1 - f_B(g(y))] \\ &= [\phi(1 - f_A(x)), \phi(1 - f_A(y))].\end{aligned}$$

This completes the proof. □

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D. R. Prince Williams
Department of Information Technology
College of Applied Sciences (Ministry of Higher Education)
Post Box. 135, Sohar-311, Sultanate of Oman.
E-mail: *princeshree1@gmail.com*

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Arsham Borumand Saeid
Department of Mathematics
Shahid Bahonar University of Kerman
Kerman, Iran.
E-mail: *arsham@mail.uk.ac.ir*

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On conjugate sets of quasigroups

Tatiana Popovich

Abstract. It is known that the set of conjugates (the conjugate set) of a binary quasigroup can contain 1, 2, 3 or 6 elements. We establish a connection between different pairs of conjugates and describe all six possible conjugate sets, with regard to the equality ("assembling") of conjugates. Four identities which correspond to the equality of a quasigroup to its conjugates are pointed out. Every conjugate set is characterized with the help of these identities. The conditions of the equality of a T-quasigroup to conjugates are established and some examples of T-quasigroups with distinct conjugate sets are given.

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1 Introduction

A quasigroup is an ordered pair (Q, A) where Q is a set and A is a binary operation defined on Q such that each of the equations $A(a, y) = b$ and $A(x, a) = b$ is uniquely solvable for any pair of elements a, b in Q. It is known that the multiplication table of a finite quasigroup defines a Latin square and six (not necessarily distinct) conjugates (or parastrophes) are associated with each quasigroup (Latin square) [1, 3].

In [5] a connection between five identities of two variables and the equality of a quasigroup to some of the rest five its conjugates was established. It was also proved that the number of distinct conjugates in a finite quasigroup can be 1, 2, 3 or 6 and for any $m = 1, 2, 3, 6$ and any $n \geq 4$ there exists a quasigroup of order n with m distinct conjugates (see Theorem 6 of [5]).

We divide all pairs of conjugates of a quasigroup into four classes and consider six possible conjugate sets, with regard to the equality ("assembling") of conjugates. Four identities which correspond to these four classes (or to the equality of a quasigroup to its conjugates) are pointed out. It is proved that each of six conjugate sets can be described with the help of these identities and any two of these identities imply the rest two identities. The conditions of the equality of a T-quasigroup to its conjugates are established and some examples of T-quasigroups with distinct conjugate sets are given.

2 Preliminaries

Remind some necessary notions and results.

With any quasigroup (Q,A) the system $\Sigma(A)$ of six (not necessarily distinct) *conjugates (parastrophes)* is connected:

$$\Sigma(A) = (A, A^{-1}, {}^{-1}A, {}^{-1}\left(A^{-1}\right), ({}^{-1}A)^{-1}, A^*),$$

where $A(x,y) = z \Leftrightarrow A^{-1}(x,z) = y \Leftrightarrow {}^{-1}A(z,y) = x \Leftrightarrow A^*(y,x) = z$.

Using the suitable Belousov's designation of conjugates of a quasigroup (Q,A) of [2] we have the following conjugate system $\Sigma(A)$:

$$\Sigma(A) = (A,\ {}^{r}\!A,\ {}^{l}\!A,\ {}^{lr}\!A,\ {}^{rl}\!A,\ {}^{s}\!A),$$

where ${}^{r}\!A = A^{-1}$, ${}^{l}\!A = {}^{-1}A$, ${}^{lr}\!A = {}^{-1}(A^{-1})$, ${}^{rl}\!A = ({}^{-1}\!A)^{-1}$, ${}^{s}\!A = A^*$.

Note that

$$\left({}^{-1}(A^{-1})\right)^{-1} = {}^{rlr}\!A = {}^{-1}\left(({}^{-1}\!A)^{-1}\right) = {}^{lrl}\!A = {}^{s}\!A$$

and ${}^{rr}\!A = {}^{ll}\!A = A$, ${}^{\sigma\tau}\!A = {}^{\sigma}({}^{\tau}\!A)$.

Let $\overline{\Sigma}(A)$ be the set of conjugates (*the conjugate set*) of a quasigroup (Q,A).

It is known from [5] that $\mid \overline{\Sigma}(A) \mid = 1,2,3$ or 6.

A quasigroup is a totally-symmetric quasigroup (a TS-quasigroup) if $\mid \overline{\Sigma}(A) \mid = 1$.

3 Conjugates of quasigroups

We start with the following useful result concerning the (unordered) pairs of conjugates of a quasigroup.

Proposition 1. *All pairs of conjugates of the conjugate system $\Sigma(A)$ of a quasigroup (Q,A) can be divided into four disjoint classes:*

I. $(A,{}^{r}\!A)$, $({}^{l}\!A,{}^{lr}\!A)$, $({}^{rl}\!A,{}^{s}\!A)$;

II. $(A,{}^{l}\!A)$, $({}^{r}\!A,{}^{rl}\!A)$, $({}^{s}\!A,{}^{lr}\!A)$;

III. $(A,{}^{s}\!A)$, $({}^{r}\!A,{}^{lr}\!A)$, $({}^{l}\!A,{}^{rl}\!A)$;

IV. $({}^{l}\!A,{}^{r}\!A)$, $(A,{}^{lr}\!A)$, $({}^{r}\!A,{}^{s}\!A)$, $({}^{lr}\!A,{}^{rl}\!A)$, $(A,{}^{rl}\!A)$, $({}^{l}\!A,{}^{s}\!A)$

such that the equality (inequality) of components of one pair in a class implies the equality (inequality) of components of any pair in this class.

Proof. There are 15 unordered pairs of conjugates of a quasigroup. It is easy to check that if we take any conjugate of the operations in a pair of any class of I, II, III or IV , then we obtain some pair from the same class. For example, if we apply the conjugations r, l, rl, lr and s to the operations of the pair $({}^{l}\!A,{}^{lr}\!A)$ of class I, we obtain, respectively, the pairs of conjugates $({}^{rl}\!A,{}^{s}\!A)$, $(A,{}^{r}\!A)$, $({}^{r}\!A, A)$, $({}^{s}\!A,{}^{rl}\!A)$, $({}^{lr}\!A,{}^{l}\!A)$ of class I. Here we take into account that ${}^{rllr}\!A = A$, ${}^{lrl}\!A = {}^{s}\!A$, ${}^{lrlr}\!A = {}^{rlrr}\!A = {}^{rl}\!A$ and ${}^{slr}\!A = {}^{lrllr}\!A = {}^{l}\!A$. Analogously, the conjugations can be applied to the rest two pairs of class I and to the every pair of other classes.

Thus, any class pointed out in the proposition is closed with respect to taking the same conjugate of both operations in a pair from this class. □

Proposition 2. *If the components of pairs from any two classes of I, II, III, IV coincide for a quasigroup (Q, A), then the components of every pair of all classes coincide and (Q, A) is a TS-quasigroup.*

Proof. According to Proposition 1 for the proof we can take any pair of a class.

I, II: Let $A = {}^rA$ and $A = {}^lA$, then ${}^lA = {}^rA$ (it gives a pair of IV) and ${}^sA = {}^{rlr}A = {}^rA = A$ where $(A, {}^sA)$ is a pair of III.

I, III: If $A = {}^rA$ and $A = {}^sA$, then ${}^rA = {}^sA$ (it corresponds to a pair of IV); the last and the second equalities imply $A = {}^{lr}A = {}^sA$ (we obtain a pair $({}^sA, {}^{lr}A)$ of II).

I, IV: Let $A = {}^rA$ and ${}^lA = {}^rA$, then $A = {}^lA$ and we have a pair of II, so ${}^rA = {}^{rl}A$, and ${}^lA = {}^{rl}A$ (a pair of III).

III, IV: If $A = {}^sA$ and ${}^lA = {}^sA$, then $A = {}^lA$ (a pair of II). Let $A = {}^sA$ and ${}^rA = {}^sA$, then $A = {}^rA$ (a pair of I).

Analogously the rest cases can be considered. In every case all conjugates coincide, so (Q, A) is a TS-quasigroup. □

The following theorem describes all possible conjugate sets for quasigroups and points out the only possible variants of the equality ("assembling") of conjugates in every case.

Theorem 1. *The following conjugate sets of a quasigroup (Q, A) are only possible:*

$\overline{\Sigma}_1(A) = \{A\}$;

$\overline{\Sigma}_2(A) = \{A, {}^sA\} = \{A = {}^{lr}A = {}^{rl}A,\ {}^lA = {}^rA = {}^sA\}$;

$\overline{\Sigma}_6(A) = \{A,\ {}^rA,\ {}^lA,\ {}^{lr}A,\ {}^{rl}A,\ {}^sA\}$;

$\overline{\Sigma}_3(A) = \{A, {}^{lr}A, {}^{rl}A\}$ *and three cases are possible:*

$\overline{\Sigma}_3^1(A) = \{A = {}^rA,\ {}^lA = {}^{lr}A,\ {}^{rl}A = {}^sA\}$;

$\overline{\Sigma}_3^2(A) = \{A = {}^lA,\ {}^rA = {}^{rl}A,\ {}^{lr}A = {}^sA\}$;

$\overline{\Sigma}_3^3(A) = \{A = {}^sA,\ {}^rA = {}^{lr}A,\ {}^lA = {}^{rl}A\}$.

Proof. It follows from Proposition 1 that if the components of pairs of all classes I, II, III, IV (or by Proposition 2 at least any of two classes) coincide then all conjugates coincide and (Q, A) is a TS-quasigroup.

If the components of pairs from all classes do not coincide, then all conjugates of (Q, A) are different and $\overline{\Sigma}(A) = \overline{\Sigma}_6(A)$. In the rest cases by Proposition 2 we have exactly one of the groups of conjugate equalities:

I'. $A = {}^rA$, ${}^lA = {}^{lr}A$, ${}^{rl}A = {}^sA$;

II'. $A = {}^lA$, ${}^rA = {}^{rl}A$, ${}^{lr}A = {}^sA$;

III'. $A = {}^sA$, ${}^rA = {}^{lr}A$, ${}^lA = {}^{rl}A$;

IV'. $A = {}^{lr}A = {}^{rl}A$, ${}^lA = {}^rA = {}^sA$.

Moreover, different equalities in a group do not "assemble": if some conjugate of one equality from a group coincides with a conjugate from another equality of this group, then new equalities arise from the group of equalities corresponding to another class of pairs. So by Proposition 2 all six conjugates of the quasigroup coincide. Thus, in each of these cases there are exactly two (see equality group IV') or three (every of equality groups I', II' and III') distinct conjugates. Note that every equality group of I', II', III' contains conjugates A, ${}^{lr}A$ and ${}^{rl}A$, but there are distinct variants of their "assembling" with the rest conjugates which give the conjugate sets $\overline{\Sigma}_3^1(A)$, $\overline{\Sigma}_3^2(A)$ and $\overline{\Sigma}_3^3(A)$ respectively. $\square$

From Proposition 1 and Theorem 1 follows immediately

Corollary 1. *Let $\overline{\Sigma}(A)$ be the conjugate set of a quasigroup (Q, A), then $\mid \overline{\Sigma}(A) \mid= 1$ or 3 in the case of the coincidence of the components of a pair in any of classes I, II, III and $\mid \overline{\Sigma}(A) \mid= 1$ or 2 by coincidence of the components of a pair in class IV. For a commutative quasigroup $\mid \overline{\Sigma}(A) \mid= 1$ or 3.*

Corollary 2. *If (Q, A) is a commutative quasigroup, then $\overline{\Sigma}(A) = \overline{\Sigma}_1(A)$ or $\overline{\Sigma}_3^3(A)$. For a noncommutative quasigroup $\overline{\Sigma}(A) = \overline{\Sigma}_2(A), \overline{\Sigma}_6(A), \overline{\Sigma}_3^1(A)$ or $\overline{\Sigma}_3^2(A)$.*

Proof. Indeed, for a commutative quasigroup we have $A =^s A$, it corresponds to the equality group III'. If it corresponds only to the equality group III', then $\overline{\Sigma}(A) = \overline{\Sigma}_3^3(A)$. If there are equalities of another group, then by Proposition 2 $\overline{\Sigma}(A) = \overline{\Sigma}_1(A)$. The rest conjugate sets are possible only for a noncommutative quasigroup. $\square$

Let $\overline{\Sigma}(A) = \{A_1, A_2, ..., A_i\}$, $i = 1, 2, 3$ or 6, be the conjugate set of a quasigroup (Q, A) and ${}^{\sigma}\overline{\Sigma}(A) = \{{}^{\sigma}A_1, {}^{\sigma}A_2, ..., {}^{\sigma}A_i\}$ where σ is some conjugation of a quasigroup (Q, A).

Proposition 3. *Let σ be any conjugation of (Q, A). Then ${}^{\sigma}\overline{\Sigma}(A) = \overline{\Sigma}(A)$. If $\overline{\Sigma}(A) = \overline{\Sigma}_1, \overline{\Sigma}_2(A)$ or $\overline{\Sigma}_6(A)$, then $\overline{\Sigma}(A) = \overline{\Sigma}({}^{\sigma}A)$. If $\overline{\Sigma}(A) = \overline{\Sigma}_3(A)$, then $\mid \overline{\Sigma}(A) \mid= \mid \overline{\Sigma}({}^{\sigma}A) \mid$.*

Proof. The equality ${}^{\sigma}\overline{\Sigma}(A) = \overline{\Sigma}(A)$ follows from Proposition 1 as any class pointed out in this proposition is closed with respect to taking the same conjugate of both operations in a pair from this class.

If $\overline{\Sigma}(A) = \overline{\Sigma}_1(A) = \{A\}$, then $\overline{\Sigma}({}^{\sigma}A) = \overline{\Sigma}(A)$ since ${}^{\sigma}A = A$ for any σ. Let $\overline{\Sigma}(A) = \overline{\Sigma}_2(A) = \{A = {}^{lr}A = {}^{rl}A,\ {}^{l}A = {}^{r}A = {}^{s}A\} = \{A, {}^{s}A\}$, then these equalities can be written via the operation ${}^{r}A$ in the following way: $\overline{\Sigma}(A) = \{{}^{r}({}^{r}A) = {}^{l}({}^{r}A) = {}^{s}({}^{r}A),\ {}^{lr}({}^{r}A) = {}^{r}A = {}^{rl}({}^{r}A)\} = \overline{\Sigma}_2({}^{r}A)$. Analogously each of the rest conjugates can be used.

It is evident that if $\overline{\Sigma}(A) = \overline{\Sigma}_6(A)$, then analogous passage from A to ${}^{\sigma}A$ gives all six distinct conjugates.

In the case $\overline{\Sigma}(A) = \overline{\Sigma}_3(A) = \overline{\Sigma}_3^i(A)$ for some $i = 1, 2, 3$ writing corresponding equalities via ${}^{\sigma}A$ we obtain also three pairs of equal conjugates. But these pairs can correspond to $\overline{\Sigma}_3^j({}^{\sigma}A)$ where $i \neq j$. For example, let $\overline{\Sigma}(A) = \overline{\Sigma}_3^2(A) = \{A = {}^{l}A,\ {}^{r}A = {}^{rl}A,\ {}^{lr}A = {}^{s}A\}$ or using the conjugate ${}^{r}A$ we obtain $\overline{\Sigma}(A) = \{{}^{r}({}^{r}A) = {}^{lr}({}^{r}A),\ {}^{r}A = {}^{s}({}^{r}A),\ {}^{l}({}^{r}A) = {}^{rl}({}^{r}A)\} = \overline{\Sigma}({}^{r}A) = \{{}^{r}B = {}^{lr}B,\ B = {}^{s}B,\ {}^{l}B = {}^{rl}B\}$ where ${}^{r}A = B$. Thus, $\overline{\Sigma}(A) = \overline{\Sigma}_3^2(A) = \overline{\Sigma}_3^3({}^{r}A)$. $\square$

Using Proposition 1 we obtain (see also Theorem 4 of [5]) the following

Proposition 4. *The components of any pair of a class of Proposition 1 coincide if and only if a quasigroup (Q, A) satisfies the identity*

$A(x, A(x, y)) = y$ *for class I;*

$A(A(y,x),x) = y$ *for class II;*
$A(x,y) = A(y,x)$ *for class III;*
$A(A(x,y),x) = y$ *for class IV.*

Proof. It is known that the identities $A(x, A(y,x)) = y$ and $A(A(x,y),x) = y$ are equivalent (see, for example, [3], p. 61). This fact follows also from Proposition 1 since the components of the pairs $({}^rA, {}^sA)$ and $({}^lA, {}^sA)$, giving equivalence of these identities, coincide simultaneously (see class IV).

I. By the definition of the conjugate ${}^rA(x,y)$ we have that $A(x,y) = {}^rA(x,y)$ if and only if $A(x, A(x,y)) = y$.

II. Analogously, $A(x,y) = {}^lA(x,y)$ if and only if $A(A(x,y),y) = x$ or $A(A(y,x),x) = y$.

III. $A = {}^sA$ means that $A(x,y) = A(y,x)$.

IV. ${}^rA = {}^sA$ (taking into account Proposition 1 we can take the last pair of class IV) if and only if $A(x, A(y,x)) = y$. But this identity is equivalent to the identity $A(A(x,y),x) = y$, as it was noted above. □

From Propositions 2 and 4 it follows immediately

Corollary 3. *Any two identities of four identities of Proposition 4 imply the rest two identities.*

In Theorem 4 of [5] it was shown that $|\ \overline{\Sigma}(A)\ | = 1$ if and only if in a quasigroup (Q, A) all five identities of the set $T = \{A(x, A(x,y)) = y, A(A(y,x),x) = y, A(x,y) = A(y,x), A(x, A(y,x)) = y, A(A(x,y),x) = y\}$ (corresponding to the equality of a quasigroup to one of its conjugates) are fulfilled,

it is 2 if and only if (Q, A) satisfies exactly 2 of the identities,

it is 3 if and only if (Q, A) satisfies exactly one of the identities,

it is 6 if and only if (Q, A) satisfies none of the identities.

In this case (and below) we assume that a quasigroup satisfies exactly k identities of a set of identities if it satisfies k identities and does not satisfy the rest identities of this set. Let $\overline{T} = \{A(x, A(x,y)) = y,\ A(A(y,x),x) = y,\ A(x,y) = A(y,x),\ A(A(x,y),x) = y\}$ be the set of identities of Proposition 4.

Taking into account the previous results we obtain the following result making more precise Theorem 4 of [5]:

Corollary 4. *Let* (Q, A) *be a quasigroup, then*
$\overline{\Sigma}(A) = \overline{\Sigma}_1(A)$ *if and only if any two identities of* $\overline{T}$ *are fulfilled;*
$\overline{\Sigma}(A) = \overline{\Sigma}_2(A)$ *if and only if exactly the identity* $A(A(x,y),x) = y$ *of* $\overline{T}$ *is fulfilled;*
$\overline{\Sigma}(A) = \overline{\Sigma}_3^1(A)$ *if and only if exactly the identity* $A(x, A(x,y)) = y$ *of* $\overline{T}$ *is fulfilled;*
$\overline{\Sigma}(A) = \overline{\Sigma}_3^2(A)$ *if and only if exactly the identity* $A(A(y,x),x) = y$ *of* $\overline{T}$ *is fulfilled;*
$\overline{\Sigma}(A) = \overline{\Sigma}_3^3(A)$ *if and only if exactly the identity* $A(x,y) = A(y,x)$ *of* $\overline{T}$ *is fulfilled;*
$\overline{\Sigma}(A) = \overline{\Sigma}_6(A)$ *if and only if* (Q, A) *satisfies none of four identities of* $\overline{T}$.

Proof. Let $|\ \overline{\Sigma}(A)\ | = m$. By Corollary 3 any two identities of $\overline{T}$ imply the rest ones, so by Proposition 2 the components of any pair of each class of Proposition 1 coincide. In this case all conjugates coincide, thus, $m = 1$.

Conversely, in a TS-quasigroup all conjugates coincide, so by Proposition 4 this quasigroup satisfies all identities of $\overline{T}$.

By Theorem 1 $m = 2$ can be only for class IV of pairs. The identity $A(A(x, y), x) = y$ corresponds to this class by Proposition 4.

The case $m = 3$ by Theorem 1 can be for the every of classes I,II and III. The identities $A(x, A(x, y)) = y$, $A(A(y, x), x) = y$, $A(x, y) = A(y, x)$ correspond to these classes, respectively, according to Proposition 4.

At last, $m = 6$ if components of any pair of all pairs in Proposition 1 do not coincide. That means that the quasigroup satisfies none of the four identities of Proposition 4. □

4 Conjugates of T-quasigroups

A quasigroup (Q, A) is *a T-quasigroup* if there exist an abelian group $(Q, +)$, its automorphisms φ, ψ and an element $c \in Q$ such that $A(x, y) = \varphi x + \psi y + c$ for any $x, y \in Q$ [4].

The conjugates of a T-quasigroup $A(x, y) = \varphi x + \psi y + c$ (which are also T-quasigroups) have the following form: ${}^{s}\!A(x, y) = \psi x + \varphi y + c$, ${}^{r}\!A(x, y) = \psi^{-1}(y - \varphi x - c)$, ${}^{l}\!A(x, y) = \varphi^{-1}(x - \psi y - c)$, ${}^{rl}\!A(x, y) = \psi^{-1}(x - \varphi y - c)$, ${}^{lr}\!A(x, y) = \varphi^{-1}(y - \psi x - c)$ where $Ix = -x$ (see, for example, [6]). Note that $I\varphi = \varphi I$ for any automorphism φ of a group.

An operation A of the form $A(x, y) = ax + by \pmod{n}$, $n \geq 2$, is a T-quasigroup if and only if the numbers a, b modulo n are relatively prime to n. In this case $\varphi = L_a$, $\psi = L_b$, where $L_a x = ax \pmod{n}$, $x \in Q = \{0, 1, 2, \ldots, n - 1\}$, are permutations (automorphisms of the additive group modulo n). For these quasigroups the conjugates have the following form:

$${}^{s}\!A(x, y) = L_b x + L_a y \pmod{n},\ {}^{r}\!A(x, y) = L_b^{-1}(y - L_a x) \pmod{n},$$
$${}^{l}\!A(x, y) = L_a^{-1}(x - L_b y) \pmod{n},\ {}^{rl}\!A(x, y) = L_b^{-1}(x - L_a y) \pmod{n},$$
$${}^{lr}\!A(x, y) = L_a^{-1}(y - L_b x) \pmod{n}.$$

Theorem 2. *The components of any pair of a class I, II, III or IV for a T-quasigroup (Q, A): $A(x, y) = \varphi x + \psi y$ coincide if and only if*

$\psi = I$ *for class I;*

$\varphi = I$ *for class II;*

$\varphi = \psi$ *for class III;*

$\varphi^2 = I\psi$ *and* $\psi^2 = I\varphi$ *(or $\varphi = \psi^{-1}$ and $\varphi^3 = I$) for class IV.*

Proof. I. Let $A = {}^{r}\!A$, then

$$\varphi x + \psi y = I\psi^{-1}\varphi x + \psi^{-1} y = \psi^{-1}(y - \varphi x). \tag{1}$$

For $y = 0$ (0 is the identity element of the group $(Q, +)$) we have $\varphi x = I\psi^{-1}\varphi x$ and $\psi = I$.

II. Let $A = {}^{l}\!A$, then

$$\varphi x + \psi y = \varphi^{-1}(x - \psi y), \tag{2}$$

whence for $x = 0$ it follows $\varphi = I$.

III. If $A = {}^{s}\!A$, then $\psi x + \varphi y = \varphi x + \phi y$, whence $\varphi = \psi$ for $x = 0$.

IV. Let ${}^{l}\!A = {}^{r}\!A$, then

$$\psi^{-1}(y - \varphi x) = \varphi^{-1}(x - \psi y). \tag{3}$$

Taking $x = 0$ we have $\psi^{-1}y = I\varphi^{-1}\psi y$, $\psi^2 = I\varphi$. If $y = 0$, then $\varphi^{-1}x = I\psi^{-1}\varphi x$ and so $\varphi^2 = I\psi$.

Note that the second pair of equalities in the theorem for the pairs of IV is equivalent to the first one: from $\varphi^2 = I\psi$ and $\psi^2 = I\varphi$ it follows that $\varphi^4 = \psi^2 = I\varphi$, whence $\varphi^3 = I$ and so $\psi^{-1} = I\varphi^{-2} = \varphi$. Conversely, if $\varphi = \psi^{-1}$ and $\varphi^3 = I$, then $\varphi^2 = I\varphi^{-1} = I\psi$ and $\psi^2 = \varphi^4 = I\varphi$.

Now check sufficiency of the conditions. Let $\psi = I$, then $\varphi x + Iy = Iy + \varphi x$, that is we obtain (1) and $A = {}^{r}\!A$.

If $\varphi = I$, then $Ix + \psi y = I(x - \psi y)$. It is (2), so $A = {}^{l}\!A$. It is evident that if $\varphi = \psi$, then $A = {}^{s}\!A$.

At last, let $\varphi^2 = I\psi$ and $\psi^2 = I\varphi$. Show that ${}^{l}\!A = {}^{r}\!A$, that is (3) holds. Indeed, $\psi^{-1}(y - \varphi x) = I\varphi^{-2}(y - \varphi x) = I\varphi^{-2}y + \varphi^{-1}x$. But $\varphi^{-1}(x - \psi y) = \varphi^{-1}x + \varphi y = \varphi^{-1}x + I\varphi^{-2}y$ as $\varphi^3 = I$ (it was shown above) and (3) is true. □

From Proposition 2 and Theorem 2 it follows

Corollary 5. *The conditions of Theorem 2 for any two classes of I, II, III, IV define a TS-T-quasigroup.*

Corollary 6. *If in Theorem 2 $A(x, y) = ax + by \pmod n$, then the conjugates of the T-quasigroup in a class of I, II, III or IV have the form:*

I. $A(x, y) = {}^{r}\!A(x, y) = ax + (n-1)y \pmod n$, ${}^{l}\!A(x, y) = {}^{lr}\!A(x, y) = a^{-1}x + a^{-1}y \pmod n$ *and* ${}^{rl}\!A(x, y) = {}^{s}\!A(x, y) = (n-1)x + ay \pmod n$;

II. $A(x, y) = {}^{l}\!A(x, y) = (n-1)x + by \pmod n$, ${}^{r}\!A(x, y) = {}^{rl}\!A(x, y) = b^{-1}x + b^{-1}y \pmod n$ *and* ${}^{lr}\!A(x, y) = {}^{s}\!A(x, y) = bx + (n-1)y \pmod n$;

III. $A(x, y) = {}^{s}\!A(x, y) = ax + ay \pmod n$, ${}^{r}\!A(x, y) = {}^{lr}\!A(x, y) = (n-1)x + a^{-1}y \pmod n$ *and* ${}^{l}\!A(x, y) = {}^{rl}\!A(x, y) = a^{-1}x + (n-1)y \pmod n$;

IV. $A(x, y) = {}^{lr}\!A(x, y) = {}^{rl}\!A(x, y) = ax + a^{-1}y \pmod n$ *and* ${}^{s}\!A(x, y) = {}^{l}\!A(x, y) = {}^{r}\!A(x, y) = a^{-1}x + ay \pmod n$.

Proof. Follows from Proposition 1 and Theorem 2 if to take into account the form of a T-quasigroup, of its conjugates and that in this case $I = L_{n-1}$. For example, in class I: $A(x, y) = {}^{r}\!A(x, y) = ax + Iy = ax + (n-1)y \pmod n$, ${}^{l}\!A(x, y) = {}^{lr}\!A(x, y) = L_a^{-1}(y - Ix) = a^{-1}x + a^{-1}y \pmod n$. The rest cases are checked analogously. Note that in this case a, b modulo n are relatively prime to n, so they are invertible and belong to the multiplicative group of the residue-class ring (mod n). This multiplicative group consists of all numbers from 1 to $n-1$ relatively prime to n. In this case $L_a^{-1}x = L_{a^{-1}}x \pmod n$. □

Examples. Using Corollary 6 we obtain that the operations $A_1(x,y) = 2x+4y$ (mod 5), $A_2(x,y) = 6x+4y$ (mod 7), $A_3(x,y) = 4x+4y$ (mod 9) define quasigroups with three different conjugates: $\overline{\Sigma}(A_1) = \overline{\Sigma}_3^1(A_1)$, $\overline{\Sigma}(A_2) = \overline{\Sigma}_3^2(A_2)$, $\overline{\Sigma}(A_3) = \overline{\Sigma}_3^3(A_3)$. The operation $A_4(x,y) = 2x+5y$ (mod 9) defines a quasigroup with two different conjugates: $\overline{\Sigma}(A_4) = \overline{\Sigma}_2(A_4)$.

For the operation A_1 these conjugates have the following form:

$A_1(x,y) = {}^rA_1(x,y) = 2x+4y$ (mod 5), ${}^lA_1(x,y) = {}^{lr}A_1(x,y) = 3x+3y$ (mod 5), ${}^{rl}A_1(x,y) = {}^sA_1(x,y) = 4x+2y$ (mod 5) and are given by the following Cayley Tables:

A_1	0	1	2	3	4
0	0	4	3	2	1
1	2	1	0	4	3
2	4	3	2	1	0
3	1	0	4	3	2
4	3	2	1	0	4

Tab. 1

${}^lA_1$	0	1	2	3	4
0	0	2	4	1	3
1	4	1	3	0	2
2	3	0	2	4	1
3	2	4	1	3	0
4	1	3	0	2	4

Tab. 2

${}^sA_1$	0	1	2	3	4
0	0	3	1	4	2
1	3	1	4	2	0
2	1	4	2	0	3
3	4	2	0	3	1
4	2	0	3	1	4

Tab. 3

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Tatiana Popovich
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
Academiei str. 5, MD-2028 Chişinău
Moldova

E-mail: *tanea-popovici@mail.ru*

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Categorial aspects of the semireflexivity

Dumitru Botnaru, Olga Cerbu

Abstract. We examine the properties of semireflexive product, the relations between semireflexive subcategories, the right product of two subcategories and the factorization structures. We construct examples of semireflexive subcategories, also some problems are formulated.

Mathematics subject classification: 18A40, 46A03.
Keywords and phrases: Locally convex spaces, semireflexive spaces, the factorization structures, reflective subcategories, coreflective subcategories, semireflexive subcategories.

1 Introduction

For the theory of locally convex spaces we refer the reader to the monograph of Schaeffer (see [6]). Semireflexive and reflexive spaces are defined using the dual space. Many scientists have studied different classes of semireflexive spaces (see [1, 2, 5, 6]) by modifying this definition.

Definition 1 (see [6, Section IV 5.4])**.** *A locally convex space E is called semireflexive if the canonic inclusion $E \longrightarrow (E'_\beta)'$ is a surjective mapping: $E = (E'_\beta)'$.*

Definition 2 (see [6, Section IV 5.5])**.** *A locally convex space E is called reflexive if the canonic inclusion $E \longrightarrow (E'_\beta)'_\beta$ is a topological isomorphism of the space E on the second dual space with strong topology: $E = (E'_\beta)'_\beta$.*

Proposition 1 (see [6, Section IV 5.5])**.** *For a locally convex space E the following statements are equivalent:*

(a) the space E is semireflexive;

(b) every functional $\beta(E', E)$-continuous on E' is also continuous in the weak topology $\sigma(E', E)$;

(c) the space E'_τ (the space E' endowed with Mackey topology τ) is tunneled;

(d) every bounded set in E is compact in the weak topology $\sigma(E, E')$;

(e) the space E is quasicomplete in the weak topology $\sigma(E, E')$.

The criterium (e) permits a categorial formulation. It is used in the definition of the semireflexive product and of the semireflexive subcategories (see Definition 7).

We study the properties of semireflexive subcategories, the relations of the semireflexive product with the right product and we construct some examples.

Concerning the factorization structures (bicategory structures) see [4].

In Section 2 we examine the problem of factorization of one reflector functor within the factorization structures. In Section 3 we introduce the notion of $\varkappa$-functor (Definition 5), and Theorem 5 allows to construct examples of such functors. The property $(\mathcal{SR}t)$ generalizes the property $(\mathcal{SR})$ which often takes place in locally convex spaces. These conditions permit us to characterize and to construct examples of semireflexive subcategories in category $\mathcal{C}_2\mathcal{V}$ of locally convex topological Hausdorff vector spaces.

Definition 3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of morphisms of the category $\mathcal{C}$. The class $\mathcal{A}$ is $\mathcal{B}$-hereditary if $f \cdot g \in \mathcal{A}$ and $f \in \mathcal{B}$ imply that $g \in \mathcal{A}$.*

Dual notion: the class is $\mathcal{B}$-cohereditary.

Definition 4. *The class $\mathcal{A}$ of morphisms of the category $\mathcal{C}$ is called right stable if from the fact that $u' \cdot v = v' \cdot u$ is pullback and $u \in \mathcal{A}$ it follows that $u' \in \mathcal{A}$.*

Dual notion: the class of morphisms is left stable.

We denote by $\mathcal{M}_u$ the class of right stable monomorphisms.

In the category $\mathcal{C}_2\mathcal{V}$ the monomorphism $m : X \longrightarrow Y$ belongs to the class $\mathcal{M}_u$ iff any functional defined on X is expanded through m (see [4]).

2 The factorization of the reflector functor

Any factorization structure $(\mathcal{P}, \mathcal{I})$ of the category $\mathcal{C}_2\mathcal{V}$ divides the class $\mathbb{R}$ of the non-zero reflective subcategories into three classes:

a) The class $\mathbb{R}(\mathcal{P})$ of the $\mathcal{P}$-reflective subcategories.

b) The class $\mathbb{R}(\mathcal{I})$ of the $\mathcal{I}$-reflective subcategories.

c) The class $\mathbb{R}(\mathcal{P}, \mathcal{I}) = (\mathbb{R} \backslash (\mathbb{R}(\mathcal{P}) \cup \mathbb{R}(\mathcal{I}))) \cup \{\mathcal{C}_2\mathcal{V}\}$ consisting of the subcategories which are neither $\mathcal{P}$-reflective nor $\mathcal{I}$-reflective (with the exception of the element $\mathcal{C}_2\mathcal{V}$). All these classes have the last element $\mathcal{C}_2\mathcal{V}$.

Theorem 1. *1 (see [7, Theorem 1.3]). The class $\mathbb{R}(\mathcal{P})$ possesses the first element $\overline{\mathcal{S}}$.*

2 (see [7, Theorem 2.2]). Let $(\mathcal{I} \cap \mathcal{E}pi, (\mathcal{I} \cap \mathcal{E}pi)^{\perp})$ be a right factorization structure. Then $\mathbb{R}(\mathcal{I})$ possesses the first element $\overline{\mathcal{A}}$ and

$$\mathbb{R}(\mathcal{I}) = \{\mathcal{R} \in \mathbb{R} \mid \overline{\mathcal{A}} \subset \mathcal{R}\}.$$

We mention that in the category $\mathcal{C}_2\mathcal{V}$ a proper class of the factorization structures has been constructed which possesses the property indicated in the previous theorem.

In the case of the factorization structures $(\mathcal{E}_u, \mathcal{M}_p) =$ (the class of universal epimorphisms, the class of exact monomorphisms) $=$ (the class of surgective mappings, the class of topological embeddings) we have the following division of the lattice $\mathbb{R}$ in three complete sublattices:

a) The sublattice $\mathbb{R}(\mathcal{E}_u)$ of the $\mathcal{E}_u$-reflective subcategories. A $\mathcal{E}_u$-reflective subcategory $\mathcal{R}$ is characterized by the fact that the $\mathcal{R}$-replica of every object of the category $\mathcal{C}_2\mathcal{V}$ is a bijection. Another characteristic is:

$$\mathbb{R}(\mathcal{E}_u) = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{R} \supset \mathcal{S}\},$$

where $\mathcal{S}$ is the subcategory of spaces with the weak topology.

b) The sublattice $\mathbb{R}(\mathcal{M}_p)$ of the $\mathcal{M}_p$-reflective subcategories, that means the class of those reflective subcategories $\mathcal{R}$ for which the $\mathcal{R}$-replica for any object of the category $\mathcal{C}_2\mathcal{V}$ is a topological embedding:

$$\mathbb{R}(\mathcal{M}_p) = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{R} \supset \Gamma_0\},$$

where Γ_0 is the subcategory of complete spaces.

c) $\mathbb{R}(\mathcal{E}_u, \mathcal{M}_p) = (\mathbb{R} \backslash (\mathbb{R}(\mathcal{E}_u) \cup \mathbb{R}(\mathcal{M}_p))) \cup \{\mathcal{C}_2\mathcal{V}\}$.
$\mathbb{R}(\mathcal{E}_u, \mathcal{M}_p)$ is also a complete sublattice with the first element Π and the last element $\mathcal{C}_2\mathcal{V}$.

Let $(\mathcal{P}, \mathcal{I})$ be a factorization structure in the category $\mathcal{C}_2\mathcal{V}$, and $\mathcal{L}$ a non-zero reflective subcategory. For any object X of the category $\mathcal{C}_2\mathcal{V}$ let $l^X : X \to lX$ be the $\mathcal{L}$-replica, and

$$l^X = i^X \cdot p^X, \tag{1}$$

the $(\mathcal{P}, \mathcal{I})$-factorization of respective morphism. We denote by $\mathcal{B} = \mathcal{B}(\mathcal{L})$ the full subcategory of the category $\mathcal{C}_2\mathcal{V}$ formed from all objects isomorphic with the objects bX when $X \in \mid \mathcal{C}_2\mathcal{V} \mid$. The subcategory $\mathcal{B}$ is $\mathcal{P}$-reflective, and $b^X : X \to bX$ is the $\mathcal{B}$-replica of object X.

Let $\mathcal{A}'' = \mathcal{A}''(\mathcal{L})$ be the full subcategory of all objects A with the property:

For any object X of the category $\mathcal{C}_2\mathcal{V}$, every morphism $f : bX \to A$ is extended via the morphism i^X: $f = g \cdot i^X$ for some morphism g.

The subcategory $\mathcal{A}''$ is closed under products and $\mathcal{M}_f$-subobjects. So it is reflective, and $i^X : bX \longrightarrow lX$ is the $\mathcal{A}''$-replica of the object bX.

Let $l : \mathcal{C}_2\mathcal{V} \to \mathcal{L}$, $b : \mathcal{C}_2\mathcal{V} \to \mathcal{B}$ and $a'' : \mathcal{C}_2\mathcal{V} \to \mathcal{A}''$ be the respective reflector functors. Then

$$l = a'' \cdot b. \tag{2}$$

Starting from this remarks we will denote:

by $G(\mathcal{L})$ the class of all $\mathcal{I}$-reflective subcategories $\mathcal{A}$ of the category $\mathcal{C}_2\mathcal{V}$ for which the reflector functor $a : \mathcal{C}_2\mathcal{V} \to \mathcal{A}$ verifies the relation $l = a \cdot b$;

by $\overline{G}(\mathcal{L})$ the class of all reflective subcategories $\mathcal{A}$ for which $l = a \cdot b$.

It is possible that $G(\mathcal{L})$ be the empty class. Also we mention that $G(\mathcal{L}) = \overline{G}(\mathcal{L}) \cap \mathbb{R}(\mathcal{I})$.

Theorem 2 (see [7, Theorem 3.2]). *Let $(\mathcal{P}, \mathcal{I})$ be a factorization structure in the category $\mathcal{C}_2\mathcal{V}$ so that $(\mathcal{I} \cap \mathcal{E}pi, (\mathcal{I} \cap \mathcal{E}pi)^{\perp})$ is a right factorization structure. Then for every element $\mathcal{L} \in \mathbb{R}$ we have:*

1. *$\mathcal{A}''(\mathcal{L}) \in G(\mathcal{L})$.*
2. *The subcategory $\mathcal{A}' = \mathcal{A}'(\mathcal{L}) = \cap\{\mathcal{R} \mid \mathcal{R} \in G(\mathcal{L})\}$ belongs to class $\mathbb{R}(\mathcal{I})$.*
3. *$G(\mathcal{L}) = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{A}' \subset \mathcal{R} \subset \mathcal{A}''\}$.*

For the class $\overline{G}(\mathcal{L})$ things are easier.

Theorem 3 (see [5, Theorem 2.7])**.** *For any factorization structure $(\mathcal{P}, \mathcal{I})$ we have:*

$$\overline{G}(\mathcal{L}) = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{L} \subset \mathcal{R} \subset \mathcal{A}''\}.$$

3 $\varkappa$-Functors

Definition 5. *A functor $t\colon \mathcal{C}_2\mathcal{V} \to \mathcal{C}_2\mathcal{V}$ is called a $\varkappa$-functor if*

$$t(E, u) = (E, t(u)), \quad u \leq t(u)$$

for every object (E, u).

Any non-zero coreflector functor $t : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K}$ in composition with the embedding functor $i : \mathcal{K} \longrightarrow \mathcal{C}_2\mathcal{V}$ is a $\varkappa$-functor.

Let $\mathcal{R}$ and $\mathcal{K}$ be two non-zero subcategories of the category $\mathcal{C}_2\mathcal{V}$, where the $\mathcal{R}$ is reflective and the $\mathcal{K}$ is coreflective. For every object X of the category $\mathcal{C}_2\mathcal{V}$ let $r^X : X \longrightarrow rX$ and $k^{rX} : krX \longrightarrow rX$ be the $\mathcal{R}$-replica and the $\mathcal{K}$-coreplica of the respective objects. On these two morphisms we construct a pullback:

$$r^X \cdot t^X = k^{rX} \cdot u^X. \tag{1}$$

Theorem 4 (see [5, Theorem 3.4])**.** *The correspondence $X \longrightarrow tX$ defines a $\varkappa$-functor in the category $\mathcal{C}_2\mathcal{V}$.*

Definition 6. *The functor t defined in the previous section is called the $\varkappa$-functor generated by the reflective subcategory $\mathcal{R}$ and the coreflective one $\mathcal{K}$.*

Remark 1. We mention that a $\varkappa$-functor is not always a coreflector functor, since a $\varkappa$-functor is not necessarily idempotent.

Let $t\colon \mathcal{C}_2\mathcal{V} \to \mathcal{C}_2\mathcal{V}$ be a $\varkappa$-functor. For subcategory $\mathcal{R}$ we define the following condition:

($\mathcal{SR}t$) *Let $(E, u) \in\mid \mathcal{R} \mid$. Then, for every locally convex topology v on the vector space E*

$$u \leq v \leq t(u),$$

the space (E, v) belongs to subcategory $\mathcal{R}$.

Remark 2. 1. Let $\mathcal{M}$ be the coreflective subcategory of the spaces with Mackey topology, $m\colon \mathcal{C}_2\mathcal{V} \to \mathcal{M}$ be the coreflector functor. We denote the ($\mathcal{SR}m$) condition simply by ($\mathcal{SR}$).

2. Categorically, the condition ($\mathcal{SR}t$) can be formulated as follows:

($\mathcal{SR}t$) If $X \in\mid \mathcal{R} \mid$ and $f\colon Y \to X$ is a monomorphism such that $t^X = f \cdot g$ for some morphism g, then $Y \in |\mathcal{R}|$.

Since t^X is a bijective mapping, we deduce that so is f. In the given equality f and t^X are bijective mappings, so it follows that g is also a bijective mapping.

We will examine the property $(\mathcal{SR})$ for any elements of the classes $\mathbb{R}(\mathcal{E}_u), \mathbb{R}(\mathcal{M}_p)$ and $\mathbb{R}(\mathcal{E}_u, \mathcal{M}_p)$.

Theorem 5 (see [5, Theorem 3.8]). *1. Every element of the class $\mathbb{R}(\mathcal{M}_p)$ possesses the property $(\mathcal{SR})$.*

2. Let an element $\mathcal{L}$ of the class $\mathbb{R}(\mathcal{E}_u)$ possess the property $(\mathcal{SR})$. Then $\mathcal{L} = \mathcal{C}_2\mathcal{V}$.

4 Semireflexive product of two subcategories

Definition 7. *1. Let $\mathcal{R}$ be a reflective subcategory, and $\mathcal{A}$ be a subcategory of the category $\mathcal{C}$. The object X of the category C is called $(\mathcal{R}, \mathcal{A})$-semireflexive if its $\mathcal{R}$-replica belongs to the subcategory $\mathcal{A}$.*

2. The full subcategory of all $(\mathcal{R}, \mathcal{A})$-semireflexive objects is called the semireflexive product of the subcategories $\mathcal{R}$ and $\mathcal{A}$, and is denoted by $\mathcal{L} = \mathcal{R} \times_{sr} \mathcal{A}$.

3. The subcategory $\mathcal{L} \in \mathbb{R}(\mathcal{E}_u, \mathcal{M}_p), \mathcal{L} \neq \mathcal{C}_2\mathcal{V}$ of the category $\mathcal{C}_2\mathcal{V}$ is called semireflexive if these exists a reflective subcategory $\mathcal{R} \in \mathbb{R}(\mathcal{E}_u)$ and a reflective subcategory $\Gamma \in \mathbb{R}(\mathcal{M}_p)$ of the category $\mathcal{C}_2\mathcal{V}$ so that $\mathcal{L} = \mathcal{R} \times_{sr} \Gamma$.

Remark 3. The respective condition from the definition of the semireflexive subcategories has been imposed to exclude the trivial cases. So every subcategory $\mathcal{L}$ of the category $\mathcal{C}$ can be presented as $\mathcal{L} = \mathcal{C} \times_{sr} \mathcal{L}$.

Let $(\mathcal{P}, \mathcal{I})$ be a factorization structure in the category $\mathcal{C}_2\mathcal{V}$, and $\mathcal{L}$ be a non-zero reflective subcategory. The $(\mathcal{P}, \mathcal{I})$-factorization of the reflector functor $l\colon \mathcal{C}_2\mathcal{V} \to \mathcal{L}$ generates the $\mathcal{P}$-reflective subcategory $\mathcal{B} = \mathcal{B}(\mathcal{L})$ and the lattice $\overline{G}(\mathcal{L})$. Let $\Gamma \in \overline{G}(\mathcal{L})$. We examine the following conditions:

A. $\mathcal{L} = \mathcal{B} \times_{sr} \Gamma$, *where $\mathcal{B} = \mathcal{B}(\mathcal{L})$ and $\Gamma \in \overline{G}(\mathcal{L})$.*

B. *There exists a pair of reflective subcategories $\mathcal{R} \in \mathbb{R}(\mathcal{P})$ and $\Gamma \in \mathbb{R}(\mathcal{I})$ of the category $\mathcal{C}_2\mathcal{V}$ so that $\mathcal{L} = \mathcal{R} \times_{sr} \Gamma$.*

C. *The subcategory $\mathcal{L}$ is closed under $(\mathcal{P} \cap \mathcal{M}_u)$-subobjects.*

D. *The subcategory $\mathcal{L}$ verifies the condition $(\mathcal{SR})$ that means the subcategory $\mathcal{L}$ is closed under $(\mathcal{E}_u \cap \mathcal{M}_u)$-subobjects.*

E. *The subcategory $\mathcal{B} = \mathcal{B}(\mathcal{L})$ verifies the condition $(\mathcal{SR}t)$ for $\varkappa$-functor $t\colon \mathcal{C}_2\mathcal{V} \to \mathcal{C}_2\mathcal{V}$ generated by the reflective subcategory $\Gamma \in \overline{G}(\mathcal{L})$ and the coreflective subcategory $\mathcal{M}$ of the spaces with Mackey topology.*

Lemma 1. *1. In the previous conditions we have $\mathcal{L} \subset \mathcal{B} \times_{sr} \Gamma$, where $\mathcal{B} = \mathcal{B}(\mathcal{L})$ and $\Gamma \in \overline{G}(\mathcal{L})$.*

2. For the objects of the subcategories $\mathcal{L}(\mathcal{L} \subset \mathcal{B}(\mathcal{L}))$ the condition $(\mathcal{SR}t)$ coincides with the condition $(\mathcal{SR})$.

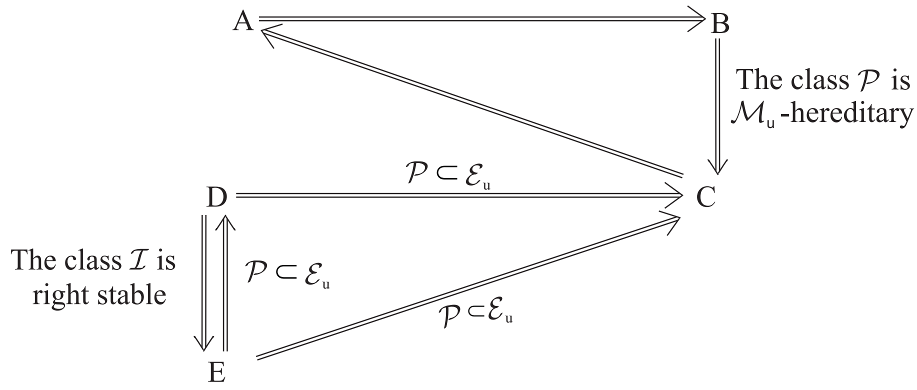
Theorem 6 (see [5, Theorem 4.5]). *The following implications are true:*

1. $C \Longrightarrow A \Longrightarrow B$.

2. Let $\mathcal{P}$ be an $\mathcal{M}_u$-hereditary class. Then $B \Longrightarrow C$.

3. Let $\mathcal{P} \subset \mathcal{E}_u$. Then $E \Longrightarrow D \Longrightarrow C$.

4. Let $\mathcal{I}$ be a right stable class. Then $D \Longrightarrow E$.



Theorem 7 (see [5, Theorem 4.6])**.** *In the case when $(\mathcal{P}, \mathcal{I}) = (\mathcal{E}_u, \mathcal{M}_p)$ the conditions $A - E$ are equivalent.*

5 Examples, conclusions, problems

Let $q\Gamma_0$ be a subcategory of the quasicomplete spaces, $s\mathcal{R}$ be a subcategory of semireflexive spaces [6], $\mathcal{S}$ be a subcategory of the spaces with weak topology, $\mathcal{N}$ be a subcategory of nuclear spaces. Then

$$\mathcal{R} \times_{sr} (q\Gamma_0) = s\mathcal{R},$$

for any reflective subcategory $\mathcal{R}$ with the property $\mathcal{S} \subset \mathcal{R} \subset \mathcal{N}$.

For the subcategory $\mathcal{S}c$ of the Schwartz spaces and the subcategory Γ_0 of the complete spaces we have

$$\mathcal{S}c \times_{sr} \Gamma_0 = \mathcal{K} \times_d (\mathcal{S}c \cap \Gamma_0) = i\mathcal{R},$$

where $i\mathcal{R}$ is the subcategory of semireflexive inductive spaces (see [1, Theorem 1.5]), and $\mathcal{K}$ is the coreflective subcategory of the category $\mathcal{C}_2\mathcal{V}$ which forms with the subcategory $\mathcal{S}c$ a pair of conjugate subcategories [4].

The subcategory Π of complete spaces with weak topology is semireflexive. For the case $(\mathcal{P}, \mathcal{I}) = (\mathcal{E}_u, \mathcal{M}_p)$ we have $\mathcal{B}(\Pi) = \mathcal{S}$, the subcategory of spaces with weak topology, $\mathcal{A}'(\Pi) = \Gamma_0$, and $\mathcal{A}''(\Pi)$ contains all normed spaces. From this, it follows that $G(\Pi)$ is a proper class.

The condition D from Theorem 6 indicates the fact that the property of any subcategory to be semireflexive does not depend on the factorization structure $(\mathcal{P}, \mathcal{I})$.

Definition 8. *The subcategory $\mathcal{A}$ of the category $\mathcal{C}_2\mathcal{V}$ is called closed under extensions if $f\colon A \to B \in \mathcal{E}pi \cap \mathcal{M}_p$ and $A \in \mid \mathcal{A} \mid$ implies also that $B \in \mid \mathcal{A} \mid$.*

Problem 1. Let $\mathcal{R}$ be a reflective subcategory closed under extensions, and $\mathcal{K}$ be a coreflective subcategory of the category $\mathcal{C}_2\mathcal{V}$. When the right product $\mathcal{K} \times_d \mathcal{R}$ of the subcategories $\mathcal{K}$ and $\mathcal{R}$ is closed under extensions?

Let $\mathcal{B} = \mathcal{K} \times_d \mathcal{R}$ and assume that $\mathcal{B}$ is a reflective subcategory (see [3, Theorem 2.5] and [2, Theorem 5.3]), and moreover $\mathcal{B}$ is closed under extensions. In this case for every $\Gamma \in \mathbb{R}(\mathcal{M}_p)$ we have

$$\mathcal{B} \cap \Gamma = \mathcal{B} \times_{sr} \Gamma_1, \quad \Gamma_1 \in \widetilde{G}(\mathcal{B} \cap \Gamma).$$

Based on Theorem 2.12 [5] the subcategory $\mathcal{B}$ verifies the condition $(\mathcal{SR}t)$, where $t\colon \mathcal{C}_2\mathcal{V} \to \mathcal{K}$ is the coreflector functor.

Problem 2. Is it true that $\mathcal{B} \cap \Gamma$ is a semireflexive subcategory?

Often, semireflexive subcategories can be presented as the right product of some subcategories [2, Theorem 5.4].

Problem 3. Is it true that every semireflexive subcategory is the right product of two subcategories?

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Dumitru Botnaru, Olga Cerbu
State University of Moldova
str. Al. Mateevici, 60
Chisinau, MD-2009 Moldova

E-mail: *dumitru.botnaru@gmail.com, olga.cerbu@gmail.com*

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Inclusion Radii for the Zeros of Special Polynomials

Matthias Dehmer

Abstract. To locate the zeros of complex-valued polynomials is a classical problem in algebra and function theory. For this, numerous inclusion radii have been established to estimate the moduli of the zeros of an underlying polynomial. In this note, we particularly state bounds for polynomials whose coefficients satisfy special conditions.

Mathematics subject classification: 30C15.
Keywords and phrases: Complex polynomials, zeros, inequalities.

1 Introduction

The analytic theory of polynomials [5] investigates properties of polynomials representing analytic functions. In particular the location of zeros of complex and real-valued polynomials has been extensively investigated [1–5]. To tackle this problem, we determine disks in the complex plane

$$K(z_0, r) := \{z \in \mathbb{C} \mid |z - z_0| \le r\},\ z_0 \in \mathbb{C},\ r \in \mathbb{R}_+,$$

containing all zeros of a complex valued polynomial

$$f(z) = \sum_{i=0}^{n} a_i z^i,\ a_i \in \mathbb{C}, a_n \neq 0.$$

r is called inclusion radius. Clearly, $r = r(a_0, a_1, \ldots, a_n)$.

In this paper, we examine the location of zeros of special complex-valued polynomials of the form

$$f(z) := f_{n_1}(z) g_{n_2}(z) = (b_{n_1} z^{n_1} + f_{n_1-1}(z))(c_{n_2} z^{n_2} + g_{n_2-1}(z)). \tag{1}$$

That means, we infer bounds for the moduli of their zeros given by an inclusion radius. It turns out that these bounds are more practicable for this class of polynomials rather than applying existing zero bounds for general polynomials, see [1–5].

2 Results

In [1], Dehmer proved the following theorem.

Theorem 2.1. *Let $f(z)$ be a complex polynomial, such that $f(z)$ is reducible in $\mathbb{C}[z]$, namely*

$$f(z) := f_{n_1}(z)g_{n_2}(z) = (b_{n_1}z^{n_1} + f_{n_1-1}(z))(c_{n_2}z^{n_2} + g_{n_2-1}(z))$$

where

$$|b_{n_1}| > |b_i|, 0 \leq i \leq n_1 - 1, |c_{n_2}| > |c_i|, 0 \leq i \leq n_2 - 1. \tag{2}$$

If $n_1 + n_2 > 1$, then all zeros of the polynomial $f(z)$ lie in the closed disk $K(0, \delta)$, where $\delta > 1$ is the positive root of the equation

$$z^{n_1+n_2+2} - 4z^{n_1+n_2+1} + 2z^{n_1+n_2} + z^{n_2+1} + z^{n_1+1} - 1 = 0. \tag{3}$$

It holds $1 < \delta < 2 + \sqrt{2}$.

In the following, we prove some related theorems for this class of polynomials (see Equation (1)). An improvement of Theorem 2.1 is

Theorem 2.2. *Let*

$$f(z) := f_{n_1}(z)g_{n_2}(z) = (b_{n_1}z^{n_1} + f_{n_1-1}(z))(c_{n_2}z^{n_2} + g_{n_2-1}(z)),$$

where

$$\phi_1 := \frac{|b_{n_1-1}|}{|b_{n_1}|} \quad \textit{and} \quad \phi_2 := \frac{|c_{n_2-1}|}{|c_{n_2}|}, \tag{4}$$

and

$$|b_{n_1}| > |b_i|, \quad 0 \leq i \leq n_1 - 1, \quad |c_{n_2}| > |c_i|, \quad 0 \leq i \leq n_2 - 1. \tag{5}$$

All zeros of the polynomial $f(z)$ lie in the closed disk

$$K\left(0, \max\left[\frac{1+\phi_1}{2} + \frac{\sqrt{(\phi_1 - 1)^2 + 4}}{2}, \frac{1+\phi_2}{2} + \frac{\sqrt{(\phi_2 - 1)^2 + 4}}{2}\right]\right).$$

Proof. We start the proof by obtaining the estimation

$$\begin{aligned} |f_{n_1}(z)| &= |b_{n_1}z^{n_1} + f_{n_1-1}(z)| = \\ &= |b_{n_1}z^{n_1} + b_{n_1-1}z^{n_1-1} + \cdots + b_1 z + b_0| \geq \\ &\geq |b_{n_1}||z|^{n_1} - \left[|b_{n_1-1}||z|^{n_1-1} + \cdots + |b_1||z| + |b_0|\right]. \end{aligned}$$

Using the relations $|b_{n_1}| > |b_i|, 0 \leq i \leq n_1 - 1$ (see Inequalities (5)), Equation (4) and $|z| > 1$, we further obtain

$$|f_{n_1}(z)| \geq |b_{n_1}| \left[|z|^{n_1} - \phi_1 |z|^{n_1-1} - \left[|z|^{n_1-2} + \cdots + |z| + 1 \right] \right] \tag{6}$$
$$= |b_{n_1}| \left[|z|^{n_1} - \phi_1 |z|^{n_1-1} - \frac{|z|^{n_1-1} - 1}{|z| - 1} \right]$$
$$> |b_{n_1}| \left[|z|^{n_1} - \phi_1 |z|^{n_1-1} - \frac{|z|^{n_1-1}}{|z| - 1} \right]$$
$$= \frac{|b_{n_1}||z|^{n_1-1}}{|z|-1} \left[|z|^2 - |z|(1+\phi_1) + (\phi_1 - 1) \right].$$

Clearly, applying this procedure to $f_{n_2}(z)$ also yields

$$|f_{n_2}(z)| > \frac{|c_{n_2}||z|^{n_2-1}}{|z|-1} \left[|z|^2 - |z|(1+\phi_2) + (\phi_2 - 1) \right].$$

By defining

$$H_{1,2}(z) := z^2 - z(1 + \phi_{1,2}) + (\phi_{1,2} - 1),$$

we get

$$|f_{n_1}(z) \cdot f_{n_2}(z)| > \frac{|b_{n_1}||z|^{n_1-1}}{|z|-1} \cdot \frac{|c_{n_2}||z|^{n_2-1}}{|z|-1} H_1(|z|) \cdot H_2(|z|),$$

and

$$|f_{n_1}(z) \cdot f_{n_2}(z)| > 0 \quad \text{if} \quad H_1(|z|) \cdot H_2(|z|) > 0.$$

Solving the last inequality requires to determine the zeros of $H_{1,2}(z)$. The zeros of $H_1(z)$ and $H_2(z)$ are

$$\frac{1+\phi_1}{2} \pm \frac{\sqrt{(\phi_1 - 1)^2 + 4}}{2},$$

and

$$\frac{1+\phi_2}{2} \pm \frac{\sqrt{(\phi_2 - 1)^2 + 4}}{2},$$

respectively. We easily see that

$$\alpha_1 := \frac{1+\phi_1}{2} + \frac{\sqrt{(\phi_1 - 1)^2 + 4}}{2} > 1,$$

and

$$\alpha_2 := \frac{1+\phi_2}{2} + \frac{\sqrt{(\phi_2 - 1)^2 + 4}}{2} > 1.$$

This finally implies

$$|f_{n_1}(z) \cdot f_{n_2}(z)| > 0,$$

if

$$|z| > \max\left(\frac{1+\phi_1}{2} + \frac{\sqrt{(\phi_1-1)^2+4}}{2}, \frac{1+\phi_2}{2} + \frac{\sqrt{(\phi_2-1)^2+4}}{2}\right),$$

and all zeros of $f(z)$ lie in $|z| \leq \max(\alpha_1, \alpha_2)$. $\square$

Remark 1. The bound given by Theorem 2.2 is an improvement of the upper bound of Equation (3) given in Theorem 2.1 since

$$\frac{1+\phi_{1,2}}{2} + \frac{\sqrt{(\phi_{1,2}-1)^2+4}}{2} < 2 + \sqrt{2}$$

if $\phi_{1,2} < 3$. But this is fulfilled by assumption, see Inequalities (5).

Assuming the special conditions for the polynomial's coefficients also leads to a bound whose value does not depend on any coefficients.

Theorem 2.3. *Let*

$$f(z) := f_{n_1}(z) g_{n_2}(z) = (b_{n_1} z^{n_1} + f_{n_1-1}(z))(c_{n_2} z^{n_2} + g_{n_2-1}(z)),$$

where

$$\phi_1 := \frac{|b_{n_1-1}|}{|b_{n_1}|} \quad \textit{and} \quad \phi_2 := \frac{|c_{n_2-1}|}{|c_{n_2}|},$$

and

$$|b_{n_1}| > |b_i|, \quad 0 \leq i \leq n_1 - 1, \quad |c_{n_2}| > |c_i|, \quad 0 \leq i \leq n_2 - 1. \tag{7}$$

All zeros of the polynomial $f(z)$ lie in the closed disk $K(0,2)$.

Proof. Using the Inequalities (7) and $|z| > 1$, we obtain

$$\begin{aligned}
|f_{n_1}(z)| &\geq |b_{n_1}| \left[|z|^{n_1} - \left[\frac{|b_{n_1-1}|}{|b_{n_1}|}|z|^{n_1-1} + \cdots + \frac{|b_1|}{|b_{n_1}|}|z| + \frac{|b_0|}{|b_{n_1}|}\right]\right] \\
&= |b_{n_1}| \left[|z|^{n_1} - \frac{|z|^{n_1} - 1}{|z| - 1}\right] > |b_{n_1}| \left[|z|^{n_1} - \frac{|z|^{n_1}}{|z| - 1}\right] \\
&= \frac{|b_{n_1}||z|^{n_1}}{|z| - 1} \left[|z| - 2\right].
\end{aligned}$$

Analogously, we also conclude ($|z| > 1$)

$$|f_{n_2}(z)| > \frac{|c_{n_2}||z|^{n_1}}{|z| - 1} \left[|z| - 2\right].$$

Finally,

$$|f_{n_1}(z) \cdot f_{n_2}(z)| > 0 \quad \text{if} \quad |z| > 2,$$

and, hence, all zeros of $f(z)$ lie in $|z| \leq 2$. □

A more general statement is

Theorem 2.4. *Let*

$$f(z) := f_{n_1}(z) g_{n_2}(z) = (b_{n_1} z^{n_1} + f_{n_1-1}(z))(c_{n_2} z^{n_2} + g_{n_2-1}(z)).$$

Define

$$\phi_1 := \frac{|b_{n_1-1}|}{|b_{n_1}|} \quad \textit{and} \quad \phi_2 := \frac{|c_{n_2-1}|}{|c_{n_2}|},$$

$$M_1 := \max_{0 \leq i \leq n_1-2} \left| \frac{b_i}{b_{n_1}} \right| \quad \textit{and} \quad M_2 := \max_{0 \leq i \leq n_2-2} \left| \frac{c_i}{c_{n_2}} \right|.$$

All zeros of the polynomial $f(z)$ lie in the closed disk

$$K\left(0, \max\left[\frac{1+\phi_1}{2} + \frac{\sqrt{(\phi_1-1)^2 + 4M_1}}{2}, \frac{1+\phi_2}{2} + \frac{\sqrt{(\phi_2-1)^2 + 4M_2}}{2}\right]\right).$$

Proof. Similar to Inequality (6) and by assuming $|z| > 1$, we infer

$$|f_{n_1}(z)| \geq |b_{n_1}| \left[|z|^{n_1} - \frac{|b_{n_1-1}|}{|b_{n_1}|} |z|^{n_1-1} - M_1 \left[|z|^{n_1-2} + \cdots + |z| + 1 \right] \right]$$

$$= |b_{n_1}| \left[|z|^{n_1} - \frac{|b_{n_1-1}|}{|b_{n_1}|} |z|^{n_1-1} - M_1 \frac{|z|^{n_1-1} - 1}{|z| - 1} \right]$$

$$> |b_{n_1}| \left[|z|^{n_1} - \frac{|b_{n_1-1}|}{|b_{n_1}|} |z|^{n_1-1} - M_1 \frac{|z|^{n_1-1}}{|z| - 1} \right]$$

$$= \frac{|b_{n_1}|}{|z| - 1} \left[|z|^{n_1+1} - |z|^{n_1} \left(1 + \frac{|b_{n_1-1}|}{|b_{n_1}|} \right) + |z|^{n_1-1} \left(\frac{|b_{n_1-1}|}{|b_{n_1}|} - M_1 \right) \right]$$

$$= \frac{|b_{n_1}||z|^{n_1-1}}{|z| - 1} \left[|z|^2 - |z| \left(1 + \phi_1 \right) + \left(\phi_1 - M_1 \right) \right],$$

and

$$|f_{n_1}(z)| \geq \frac{|c_{n_2}||z|^{n_1-1}}{|z| - 1} \left[|z|^2 - |z| \left(1 + \phi_2 \right) + \left(\phi_2 - M_2 \right) \right].$$

The rest of the proof is analogous to the proof steps of Theorem (2.2). □

3 Numerical Results

In this section, we evaluate the obtained bounds by using the following polynomials:

$$f_1(z) := (100z^3 - z^2 + iz + 50) \cdot (4z^4 + z^3 + 3z - 1),$$

$$f_2(z) := \left(2z^3 - z^2 + \frac{z}{2} + \frac{1}{10}\right) \cdot \left(\frac{z^3}{2} + \frac{z^2}{3} - \frac{z}{5} + \frac{1}{3}\right).$$

We start by evaluating the statements for $f_1(z)$ and first determine its zeros:

$$z_1 \doteq -1.3039,$$

$$z_2 \doteq -0.7903 + 0.0041i,$$

$$z_3 \doteq 0.1285 - 0.7933i,$$

$$z_4 \doteq 0.1285 + 0.7933i,$$

$$z_5 \doteq 0.2968,$$

$$z_6 \doteq 0.3965 + 0.6852i,$$

$$z_7 \doteq 0.4038 - 0.6894i.$$

$\max(|z_1|, |z_2|, \ldots, |z_7|) = 1.3039$. Then, we yield $K(0, 3.3734)$ (Theorem 2.1), $K(0, 1.8827)$ (Theorem 2.2), $K(0, 2)$ (Theorem 2.3) and $K(0, 1.75)$ (Theorem 2.4). We see that Theorem 2.2 – Theorem 2.4 clearly outperform Theorem 2.1. For polynomials for which the conditions of the Equations (2) are satisfied, the bound given by Theorem 2.4 is always an improvement of Theorem 2.2 as $M_1, M_2 < 1$.

For $f_2(z)$, we get

$$z_1 \doteq -1.3380,$$

$$z_2 \doteq -0.1454,$$

$$z_3 \doteq 0.3227 - 0.4896i,$$

$$z_4 \doteq 0.3227 + 0.4896i,$$

$$z_5 \doteq 0.3356 - 0.6209i,$$

$$z_6 \doteq 0.3356 + 0.6209i.$$

$\max(|z_1|, |z_2|, \ldots, |z_6|) = 1.3380$. This leads to the disks $K(0, 3.3499)$ (Theorem 2.1), $K(0, 1.847127)$ (Theorem 2.2), $K(0, 2)$ (Theorem 2.3) and $K(0, 1.6666)$ (Theorem 2.4). By inspecting the bound values for this polynomial, we see that we get the same situation as in the case of $f_1(z)$.

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Matthias Dehmer
UMIT, Institute for Bioinformatics
and Translational Research
Eduard Wallnoefer Zentrum 1
A-6060, Hall in Tyrol, Austria
E-mail: *matthias.dehmer@umit.at*

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On LCA groups whose rings of continuous endomorphisms have at most two non-trivial closed ideals. I

Valeriu Popa

Abstract. We describe the torsion, locally compact abelian (LCA) groups X for which the ring $E(X)$ of continuous endomorphisms of X, endowed with the compact-open topology, has no more than two non-trivial closed ideals.

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1 Introduction

Motivated by the work of F. Perticani [6], the author discussed in [10] general topological rings with identity having at most two non-trivial closed ideals. The main results of [10] characterize the mentioned rings in terms of ideal extensions of topological rings. In the present paper, we are interested in a more concrete class of topological rings of the mentioned type, namely, those which occur as rings of continuous endomorphisms of LCA groups. Precisely speaking, we are dealing with the problem of determining the LCA groups X with the property that the ring $E(X)$ of all continuous endomorphisms of X, taken with the compact-open topology, has no more than two non-trivial closed ideals.

In the following, we establish some bounds for the class of groups in question and solve completely the considered problem in the case of torsion LCA groups.

2 Notation

We use without explanations some terminology and notations introduced in [10]. In addition, we denote by $\mathbb{P}$ the set of primes and by $\mathcal{L}$ the class of LCA groups. For $p \in \mathbb{P}$, we denote by $\mathbb{Z}(p^\infty)$ the quasi-cyclic group corresponding to p and by $\mathbb{Z}(p^n)$, where n is a positive integer, the cyclic group of order p^n (both with the discrete topology). For $X \in \mathcal{L}$, we let 1_X, $t(X)$, X^*, and $E(X)$, denote, respectively, the identity map on X, the torsion subgroup of X, the character group of X, and the ring of continuous endomorphisms of X, endowed with the compact-open topology. Recall that the compact-open topology on $E(X)$ is generated by the sets

$$\Omega(K, U) = \{u \in E(X) \mid u(K) \subset U\},$$

where $K, U \subset X$, K is compact and U is open. For a positive integer n, we let $X[n] = \{x \in X \mid nx = 0\}$ and $nX = \{nx \mid x \in X\}$. Also, $o(a)$ denotes the order of a in X, $\langle S \rangle$ the subgroup of X generated by S, and (M) the ideal of $E(X)$ generated by M. Further, given a family $(X_i)_{i \in I}$ of groups in $\mathcal{L}$, we write $\prod_{i \in I} X_i$ for its topological direct product. In case each X_i coincides with a fixed $X \in \mathcal{L}$, we use X^I for $\prod_{i \in I} X_i$. For a discrete $X \in \mathcal{L}$, we let $X^{(I)}$ denote the discrete direct sum of I copies of X. Finally, $\oplus$ stands for topological direct sum and $\cong$ for topological isomorphism.

3 Some necessary conditions

In this section we shall reduce the study of groups $X \in \mathcal{L}$ with the property that the ring $E(X)$ has no more than two non-trivial closed ideals to the case of some more special groups.

We begin with the following preparatory lemma.

Lemma 1. *Let X be a group in $\mathcal{L}$ such that $\overline{mnE(X)} = \overline{nE(X)}$ for some positive integers m and n. Then $\overline{nX} = \overline{mnX}$ and $X[mn] = X[n]$.*

Proof. In view of our hypothesis, $n1_X \in \overline{mnE(X)}$, and hence there exists a net $(u_\lambda)_{\lambda \in L}$ of elements of $E(X)$ such that $n1_X = \lim_{\lambda \in L} mnu_\lambda$ [4, Proposition 1.6.3]. Pick any $x \in X$, and define $\delta_x : E(X) \to X$ by setting $\delta_x(u) = u(x)$ for all $u \in E(X)$. Then δ_x is a continuous group homomorphism, so

$$\begin{aligned} nx &= \delta_x(n1_X) = \delta_x(\lim_{\lambda \in L} mnu_\lambda) \\ &= \lim_{\lambda \in L} \delta_x(mnu_\lambda) = \lim_{\lambda \in L} \big((mnu_\lambda)(x)\big) \in \overline{mnX}. \end{aligned}$$

Since $x \in X$ was arbitrary, it follows that $nX \subset \overline{mnX}$, which gives $\overline{nX} = \overline{mnX}$. Further, since $E(X)$ and $E(X^*)$ are topologically anti-isomorphic [8, (1.1)], we also have $\overline{mnE(X^*)} = \overline{nE(X^*)}$, so as above $\overline{nX^*} = \overline{mnX^*}$, and hence $X[n] = X[mn]$ by [5, (24.22)]. $\square$

Next we recall two definitions.

Definition 1. Let n be a positive integer. A group $X \in \mathcal{L}$ is said to be of finite exponent n if n is the least positive integer satisfying $nX = \{0\}$.

Definition 2. A subgroup F of a group $X \in \mathcal{L}$ is said to be topologically fully invariant in X if $u(F) \subset F$ for all $u \in E(X)$.

Let $X \in \mathcal{L}$. Then X can be viewed as a left topological module over $E(X)$. It is clear that the topologically fully invariant subgroups of X are just the $E(X)$-submodules of X. Now, if F is a topologically fully invariant subgroup of X, then $ann_{E(X)}(F)$, the annihilator of F in $E(X)$, is a closed ideal of $E(X)$ because X is Hausdorff. Further, if F is in addition closed in X, then X/F is a Hausdorff topological $E(X)$-module, so that $ann_{E(X)}(X/F)$ is a closed ideal of $E(X)$ as well. In fact

$$ann_{E(X)}(X/F) = \{u \in E(X) \mid \mathrm{im}(u) \subset F\}.$$

We now state the main result of this section.

Theorem 1. *Let X be a non-zero group in $\mathcal{L}$ such that $E(X)$ has no more than two non-trivial closed ideals. Then exactly one of the following conditions holds:*

(i) $X \cong \mathbb{Z}(p)^{(\alpha)} \times \mathbb{Z}(p)^{\beta}$ *for some $p \in \mathbb{P}$ and some cardinal numbers α, β satisfying $\alpha + \beta \geq 1$.*

(ii) $X \cong \mathbb{Z}(p)^{(\alpha_p)} \times \mathbb{Z}(p)^{\beta_p} \times \mathbb{Z}(q)^{(\alpha_q)} \times \mathbb{Z}(q)^{\beta_q}$ *for some distinct $p, q \in \mathbb{P}$ and some cardinal numbers $\alpha_p, \beta_p, \alpha_q, \beta_q$ satisfying $\alpha_p + \beta_p \geq 1$ and $\alpha_q + \beta_q \geq 1$.*

(iii) *X is a group of finite exponent p^2 for some $p \in \mathbb{P}$.*

(iv) *X is a group of finite exponent p^3 for some $p \in \mathbb{P}$.*

(v) *X is densely divisible and torsion-free.*

(vi) *There exists $p \in \mathbb{P}$ such that $t(X) = X[p]$, $\overline{pX}$ is non-zero and densely divisible, and $\overline{pE(X)} \subsetneq E(X)$.*

(vii) *There exist $p, q \in \mathbb{P}$ such that $t(X) = X[pq]$, $\overline{pqX}$ is non-zero and densely divisible, and $\overline{pqE(X)} \subsetneq \overline{pE(X)} \subsetneq E(X)$.*

Proof. If there exists $p \in \mathbb{P}$ such that $pE(X) = \{0\}$, then $pX = (p1_X)(X) = 0$, and so, by [2, Ch. 2, §4, Theorem 2], $X \cong \mathbb{Z}(p)^{(\alpha)} \times \mathbb{Z}(p)^{\beta}$ for some cardinal numbers α and β satisfying $\alpha + \beta \geq 1$. Consequently, in this case we are led to (i).

Suppose $mE(X) \neq \{0\}$ for all $m \in \mathbb{P}$. If there exist $p, q \in \mathbb{P}$ such that $pqE(X) = \{0\}$, then $pqX = (pq1_X)(X) = \{0\}$, and hence $X = X[pq]$. Now, if $q \neq p$, we conclude from [1, Theorem 3.13] that $X = X[p] \oplus X[q]$, where the primary components $X[p]$ and $X[q]$ are non-zero. Thus, again appealing to [2, Ch. 2, §4, Theorem 2], in this case we get (ii). Further, in the remaining case when $q = p$, we have $X = X[p^2] \neq X[p]$, which gives us (iii).

Next suppose $lmE(X) \neq \{0\}$ for all $l, m \in \mathbb{P}$. If there exist $p, q, r \in \mathbb{P}$ such that $pqrE(X) = \{0\}$, then $pqrX = (pqr1_X)(X) = \{0\}$, and hence $X = X[pqr]$. We claim that $p = q = r$. Indeed, if the numbers p, q, and r were distinct, we could write $X = X[p] \oplus X[q] \oplus X[r]$. Since, in view of our assumption, the topologically fully invariant subgroups $X[p]$, $X[q]$, and $X[r]$ of X are non-zero, it would follow, as can be seen by considering the endomorphisms $p1_X$, $q1_X$, $r1_X$, and 1_X, that the annihilators $ann_{E(X)}(X[p])$, $ann_{E(X)}(X[q])$, and $ann_{E(X)}(X[r])$ are distinct, non-trivial, closed ideals of $E(X)$. This contradicts the hypothesis. Similarly, if only two of the numbers p, q, and r coincided, say $p \neq q = r$, we would have $X = X[p^2] \oplus X[q]$, where $X[p^2] \neq X[p]$ and $X[q] \neq \{0\}$. By invoking the endomorphisms $p^2 1_X$, $p1_X$, $q1_X$, and 1_X, we would then conclude that $ann_{E(X)}(X[p^2])$, $ann_{E(X)}(X[p])$, and

$ann_{E(X)}(X[q])$ are distinct, non-trivial, closed ideals of $E(X)$, again in contradiction with the hypothesis. Thus we must have $p = q = r$, so $X = X[p^3]$, getting (iv).

Further suppose $klmE(X) \neq \{0\}$ for all $k, l, m \in \mathbb{P}$. If $\overline{pE(X)} = E(X)$ for all $p \in \mathbb{P}$, it follows from Lemma 1 that $\overline{pX} = \overline{1X} = X$ and $X[p] = X[1] = \{0\}$ for all $p \in \mathbb{P}$, so X is densely divisible and torsion-free, and in this case we are led to (v).

Next assume there exists $p \in \mathbb{P}$ such that $\overline{pE(X)} \neq E(X)$. There are two possibilities: either (1) $\overline{qpE(X)} = \overline{pE(X)}$ for all $q \in \mathbb{P}$, or (2) there is $q \in \mathbb{P}$ such that $\overline{qpE(X)} \neq \overline{pE(X)}$. In the former case, it follows from Lemma 1 that $\overline{q\overline{pX}} = \overline{qpX} = \overline{pX}$ and $X[pq] = X[p]$ for all $q \in \mathbb{P}$, so that $\overline{pX}$ is non-zero, densely divisible and $t(X) = X[p]$, which gives us (vi). In the second case, $\overline{qpE(X)}$ and $\overline{pE(X)}$ are distinct non-trivial closed ideals of $E(X)$. Since, by our assumption, $\overline{pqrE(X)} \neq \{0\}$ for all $p, q, r \in \mathbb{P}$, it follows that $\overline{pqrE(X)} = \overline{pqE(X)}$ for all $r \in \mathbb{P}$, so $\overline{r\overline{pqX}} = \overline{pqrX} = \overline{pqX}$ and $X[pqr] = X[pq]$ for all $r \in \mathbb{P}$, and hence $\overline{pqX}$ is non-zero, densely divisible and $t(X) = X[pq]$, whence (vii). $\square$

Remark 1. We know from [8, (2.3)] that any group X appearing in item (i) of the preceding theorem has a topologically simple ring $E(X)$. It is also clear from [9, (2.2)] and [8, (2.3)] that for any group X appearing in item (ii), the ring $E(X)$ is a topological direct product of two topologically simple rings, and so it has exactly two non-trivial closed ideals. In particular, we see from [10, Theorem 3] that in this case every non-trivial closed ideal of $E(X)$ is strongly topologically maximal.

In the remaining part of this paper we handle the problem stated in Introduction for groups appearing in items (iii) and (iv). Since the groups appearing in items (v), (vi), and (vii) contain non-torsion elements, this furnishes a solution to the considered problem in the case of torsion LCA groups.

4 Groups of finite exponent p^2

Our aim in this section is to describe the groups $X \in \mathcal{L}$ of finite exponent p^2, where $p \in \mathbb{P}$, such that the ring $E(X)$ has no more than two non-trivial closed ideals. First, we note a lemma from [7, (3.8)], which will be frequently used in the sequel.

Lemma 2. *Let $X \in \mathcal{L}$ be a group of finite exponent p^n, where p is a prime and n is a positive integer. If $a \in X$ is an element of order p^n, then $\langle a \rangle$ splits topologically from X. Moreover, the complement of $\langle a \rangle$ can be chosen so as to contain a preassigned open subgroup V of X satisfying $\langle a \rangle \cap V = \{0\}$.*

We continue with five lemmas, that are needed for establishing the desired description.

Lemma 3. *Let $p \in \mathbb{P}$, and let $X \in \mathcal{L}$ be a group of finite exponent p^2. If $\overline{pX} \neq X[p]$, then $E(X)$ has more than two non-trivial closed ideals, which are comparable with respect to set-theoretic inclusion.*

Proof. Assume that $\overline{pX} \neq X[p]$. Clearly, $p1_X \neq 0$ and $p1_X \in ann_{E(X)}\big(X[p]\big)$. It is also clear that $\mathrm{im}(p1_X) \subset \overline{pX}$, so $p1_X \in ann_{E(X)}\big(X/\overline{pX}\big)$, and thus

$$ann_{E(X)}\big(X[p]\big) \cap ann_{E(X)}\big(X/\overline{pX}\big) \neq \{0\}.$$

Further, since $\overline{pX} \subset X[p]$, we have $ann_{E(X)}\big(X[p]\big) \subset ann_{E(X)}\big(\overline{pX}\big)$. Finally, since $pX \neq \{0\}$, it follows that $\overline{pX} \not\subset \ker(1_X)$, so $1_X \notin ann_{E(X)}\big(\overline{pX}\big)$, and hence $ann_{E(X)}\big(\overline{pX}\big) \neq E(X)$. We shall show that the inclusions

$$ann_{E(X)}\big(X[p]\big) \cap ann_{E(X)}\big(X/\overline{pX}\big) \subset ann_{E(X)}\big(X[p]\big) \subset ann_{E(X)}\big(\overline{pX}\big)$$

are strict. Let $\xi : X \to X/X[p]$ and $\eta : X \to X/\overline{pX}$ denote the canonical projections, and fix any $a \in X \backslash X[p]$ and $b \in X[p] \backslash \overline{pX}$. Then $o(a) = p^2$ and $o(\xi(a)) = p = o(\eta(b))$. By Lemma 2, we can write

$$X/X[p] = \langle \xi(a) \rangle \oplus A \qquad \text{and} \qquad X/\overline{pX} = \langle \eta(b) \rangle \oplus B,$$

where A and B are closed subgroups in $X/X[p]$ and $X/\overline{pX}$, respectively. Let $\lambda : \langle \xi(a) \rangle \to X$ and $\mu : \langle \eta(b) \rangle \to X$ be the group homomorphisms given by $\lambda(\xi(a)) = \mu(\eta(b)) = b$. Denoting by φ the canonical projection of $X/X[p]$ onto $\langle \xi(a) \rangle$ with kernel A, we see that $\lambda \circ \varphi \circ \xi \in ann_{E(X)}\big(X[p]\big)$, and $\lambda \circ \varphi \circ \xi \notin ann_{E(X)}\big(X/\overline{pX}\big)$ (because $(\lambda \circ \varphi \circ \xi)(a) = b \notin \overline{pX}$), so $ann_{E(X)}\big(X[p]\big)$ properly contains $ann_{E(X)}\big(X[p]\big) \cap ann_{E(X)}\big(X/\overline{pX}\big)$. Similarly, denoting by ψ the canonical projection of $X/\overline{pX}$ onto $\langle \eta(b) \rangle$ with kernel B, we see that $\mu \circ \psi \circ \eta \in ann_{E(X)}\big(\overline{pX}\big)$ and $\mu \circ \psi \circ \eta \notin ann_{E(X)}\big(X[p]\big)$ (because $b \in X[p]$ and $(\mu \circ \psi \circ \eta)(b) = b$), so $ann_{E(X)}\big(\overline{pX}\big)$ properly contains $ann_{E(X)}\big(X[p]\big)$. Consequently, the inclusions

$$ann_{E(X)}\big(X[p]\big) \cap ann_{E(X)}\big(X/\overline{pX}\big) \subset ann_{E(X)}\big(X[p]\big) \subset ann_{E(X)}\big(\overline{pX}\big)$$

are strict. □

Lemma 4. *Let $p \in \mathbb{P}$, let $X \in \mathcal{L}$ be a group of finite exponent p^2 such that $\overline{pX} = X[p]$, and let C be a non-zero closed ideal of $E(X)$. Further, let $\mathcal{P}$ be the set of all ordered pairs (a, G), where a is an element of order p^2 of X and G is a closed subgroup of X satisfying $X = \langle a \rangle \oplus G$, and for each $(a, G) \in \mathcal{P}$ let $\varepsilon_{a,G} \in E(X)$ denote the canonical projection of X onto $\langle a \rangle$ with kernel G. Then:*

(i) *If C contains elements of order p^2, then $C \supset (\varepsilon_{a,G} \mid (a, G) \in \mathcal{P})$.*

(ii) *If $pC = \{0\}$, then $C \supset (p\varepsilon_{a,G} \mid (a, G) \in \mathcal{P})$.*

Proof. For $(a, G) \in \mathcal{P}$ and $b \in X$, we define $f_{a,G,b} \in E(X)$ by the rule

$$f_{a,G,b}(t) = \begin{cases} b, & \text{if } t = a; \\ 0, & \text{if } t \in G. \end{cases}$$

(i) Pick any $u \in C$ with $o(u) = p^2$. Since $pu \neq 0$, there exists $x \in X$ such that $(pu)(x) \neq 0$, and so $o(u(x)) = p^2$. It then follows from Lemma 2 that there exists a closed subgroup Y of X such that $X = \langle u(x) \rangle \oplus Y$. Now, given any $(a, G) \in \mathcal{P}$, it is straightforward to check that $\varepsilon_{a,G} = f_{u(x),Y,a} \circ u \circ f_{a,G,x}$, so $\varepsilon_{a,G} \in C$.

(ii) Pick any non-zero $u \in C$ and any $x \in X$ such that $u(x) \neq 0$. Since $pu = 0$, we have $pX \subset \ker(u)$, so $X[p] \subset \ker(u)$, and therefore $o(x) = p^2$. In particular, by Lemma 2 we may write $X = \langle x \rangle \oplus Y$ for some closed subgroup Y of X. Now, fix an arbitrary open subgroup U of X such that $u(x) \notin U$. Since $X[p] = \overline{pX}$, there exists $z \in X$ satisfying $pz - u(x) \in U$. As $u(x) \notin U$, we cannot have $pz = 0$, and so $o(z) = p^2$. Let π denote the canonical projection of X onto the quotient group X/U. Clearly, $\pi(u(x)) \neq 0$ and $\pi(u(x)) = \pi(pz) = p\pi(z)$, so $o(\pi(z)) = p^2$. Hence we can write $X/U = \langle \pi(z) \rangle \oplus \Gamma$ for some subgroup Γ of X/U [3, Lemma 15.1]. Denoting by φ the canonical projection of X/U onto $\langle \pi(z) \rangle$ with kernel Γ and letting $h : \langle \pi(z) \rangle \to X$ be the group homomorphism defined by $h(\pi(z)) = x$, it is clear that $h \circ \varphi \circ \pi \in E(X)$ and $(h \circ \varphi \circ \pi) \circ u \circ \varepsilon_{x,Y} = p\varepsilon_{x,Y}$, so $p\varepsilon_{x,Y} \in C$. Finally, given any $(a, G) \in \mathcal{P}$, we have $p\varepsilon_{a,G} = f_{x,Y,a} \circ (p\varepsilon_{x,Y}) \circ f_{a,G,x} \in C$. □

Lemma 5. *Let $X \in \mathcal{L}$ be a group of finite exponent p^n, where p is a prime and n is a positive integer. If the subgroup A of X is a finite direct sum of cyclic groups of order p^n, then A splits topologically from X. Moreover, the complement of A can be chosen so as to contain a preassigned open subgroup V of X with property $A \cap V = \{0\}$.*

Proof. We induct on the number of summands, k, in the decomposition of A as a direct sum $A = A_1 \oplus \ldots \oplus A_k$ of cyclic groups of order p^n. If $k = 1$, the assertion holds trivially since this is just Lemma 2. Assume $k \geq 2$, and assume the result is true for any group of finite exponent p^n in $\mathcal{L}$ and any its subgroup written as a direct sum of $k-1$ cyclic subgroups of order p^n. Given an arbitrary open subgroup V of X satisfying $A \cap V = \{0\}$, it is clear that $V_1 = A_2 \oplus \ldots \oplus A_k \oplus V$ is an open subgroup of X and $A_1 \cap V_1 = \{0\}$. By Lemma 2, we can write $X = A_1 \oplus X_1$ for some subgroup X_1 of X containing V_1. Now, applying the inductive hypothesis to X_1, $A_2 \oplus \ldots \oplus A_k$, and V, we can find a subgroup X_k of X such that $X_k \cap V = \{0\}$ and $X_1 = A_2 \oplus \ldots \oplus A_k \oplus X_k$. Then $X = A_1 \oplus A_2 \oplus \ldots \oplus A_k \oplus X_k$. □

Lemma 6. *Let $p \in \mathbb{P}$, and let $X \in \mathcal{L}$ be a group of finite exponent p^2 satisfying $\overline{pX} = X[p]$. For any compact subset K of X and any neighbourhood U of zero in X, there exist two compact open subgroups K', U' of X such that $K \cup U' \subset K'$, $U' \subset U$, and $K' = \bigoplus_{i \in I} \langle a_i \rangle \oplus U'$ for some finite family $(a_i)_{i \in I}$ of elements of order p^2 of K'.*

Proof. Pick an arbitrary compact subset K of X and an arbitrary neighbourhood U of zero in X. Since X is totally disconnected, we can find a compact open subgroup U_0 of X such that $U_0 \subset U$ [5, (7.7)]. Let $K_0 = \langle K \cup U_0 \rangle$. Then K_0 is compact [5, (9.8)], and $U_0 \subset K_0$. In particular, K_0 is topologically isomorphic to a topological direct product of cyclic p-groups of order at most p^2 [5, (25.9)], and so there exist two disjoint sets I_1 and I_2 such that $K_0 \cong \prod_{i \in I_1 \cup I_2} C_i$, where $C_i = \mathbb{Z}(p)$ for $i \in I_1$ and $C_i = \mathbb{Z}(p^2)$ for $i \in I_2$. Fix a topological isomorphism f from K_0 onto

$\prod_{i\in I_1\cup I_2} C_i$. Given an arbitrary subset J of $I_1 \cup I_2$, we denote by C'_J the subgroup of all $(c_i)_{i\in I_1\cup I_2} \in \prod_{i\in I_1\cup I_2} C_i$ satisfying $c_i = 0$ for all $i \notin J$. Since U_0 is open in K_0, there exist finite subsets $J_1 \subset I_1$ and $J_2 \subset I_2$ such that $f(U_0) \supset C'_{(I_1\setminus J_1)\cup(I_2\setminus J_2)}$. We then have

$$\prod_{i\in I_1\cup I_2} C_i = \Big(\bigoplus_{i\in J_1\cup J_2} C'_i\Big) \oplus C'_{(I_1\setminus J_1)\cup(I_2\setminus J_2)},$$

so

$$K_0 = \Big(\bigoplus_{i\in J_1\cup J_2} f^{-1}(C'_i)\Big) \oplus f^{-1}(C'_{(I_1\setminus J_1)\cup(I_2\setminus J_2)}),$$

where C'_i stands for $C'_{\{i\}}$. Set $U' = f^{-1}(C_{(I_1\setminus J_1)\cup(I_2\setminus J_2)})$ and, for $i \in J_1 \cup J_2$, let a_i be a generator of $f^{-1}(C'_i)$. Then U' is an open subgroup of X contained in U_0 and

$$K_0 = \Big(\bigoplus_{i\in J_1\cup J_2} \langle a_i\rangle\Big) \oplus U'.$$

We also have $o(a_i) = p$ if $i \in J_1$, and $o(a_i) = p^2$ if $i \in J_2$. In the following, we shall construct a compact subgroup $K' \supset K_0$ which admits a decomposition similar to that of K_0, by replacing the elements a_i with $i \in J_1$ by elements of order p^2. If $J_1 = \varnothing$, we set $K' = K_0$. Suppose $J_1 \neq \varnothing$, and pick an arbitrary $j \in J_1$. Since $X[p] = \overline{pX}$, there exists $b_j \in X$ such that $a_j - pb_j \in U'$. As $a_j \notin U'$, we cannot have $pb_j = 0$, so $o(b_j) = p^2$. We claim that

$$\langle pb_j\rangle \cap \bigg(\Big(\bigoplus_{i\in(J_1\setminus\{j\})\cup J_2} \langle a_i\rangle\Big) \oplus U'\bigg) = \{0\}.$$

Indeed, given any $x \in \langle pb_j\rangle \cap \bigg(\Big(\bigoplus_{i\in(J_1\setminus\{j\})\cup J_2}\langle a_i\rangle\Big) \oplus U'\bigg)$, we can write

$$x = lpb_j = \Big(\sum_{i\in(J_1\setminus\{j\})\cup J_2} l_i a_i\Big) + y'$$

for some non-negative integers l, l_i and some $y' \in U'$. Since $y' + l(a_j - pb_j) \in U'$, it follows that

$$\begin{aligned} la_j &= \Big(\sum_{i\in(J_1\setminus\{j\})\cup J_2} l_i a_i\Big) + y' + l(a_j - pb_j) \\ &\in \langle a_j\rangle \cap \bigg(\Big(\bigoplus_{i\in(J_1\setminus\{j\})\cup J_2} \langle a_i\rangle\Big) \oplus U'\bigg) \;=\; \{0\}, \end{aligned}$$

so p divides l, and hence $x = 0$. This proves our claim that

$$\langle pb_j\rangle \cap \bigg(\Big(\bigoplus_{i\in(J_1\setminus\{j\})\cup J_2} \langle a_i\rangle\Big) \oplus U'\bigg) = \{0\}.$$

Clearly, we then also have

$$\langle b_j\rangle \cap \Big(\Big(\bigoplus_{i\in(J_1\setminus\{j\})\cup J_2}\langle a_i\rangle\Big)\oplus U'\Big)=\{0\}.$$

We replace K_0 by

$$K_1=\langle b_j\rangle\oplus\Big(\bigoplus_{i\in(J_1\setminus\{j\})\cup J_2}\langle a_i\rangle\Big)\oplus U'.$$

Now, if $J_1\setminus\{j\}\neq\varnothing$, we can apply the preceding procedure to K_1, and so, after a finite number of steps, we shall arrive at a compact subgroup K' of X having the following form:

$$K'=\Big(\bigoplus_{i\in J_1}\langle b_i\rangle\Big)\oplus\Big(\bigoplus_{i\in J_2}\langle a_i\rangle\Big)\oplus U',$$

where $o(b_i)=p^2$ for all $i\in J_1$. Since $a_i\in K'$ for all $i\in J_1$, we also have $K\cup U'\subset K_0\subset K'$, so K' and U' are those required. $\square$

Lemma 7. *Let $p\in\mathbb{P}$, let $X\in\mathcal{L}$ be a group of finite exponent p^2 satisfying $\overline{pX}=X[p]$, and let $\mathcal{P}$ be the set of all ordered pairs (a,G), where a is an element of order p^2 of X and G is a closed subgroup of X satisfying $X=\langle a\rangle\oplus G$. Then the ideal $(\varepsilon_{a,G}\mid(a,G)\in\mathcal{P})$, where $\varepsilon_{a,G}\in E(X)$ denotes the canonical projection of X onto $\langle a\rangle$ with kernel G, is dense in $E(X)$.*

Proof. Pick an arbitrary compact subset K of X and an arbitrary open neighbourhood U of zero in X. It suffices to show that

$$(\varepsilon_{a,G}\mid(a,G)\in\mathcal{P})\cap[1_X+\Omega(K,U)]\neq\varnothing.$$

By Lemma 6, we can find two compact open subgroups K', U' of X such that $K\cup U'\subset K'$, $U'\subset U$, and $K'=\bigoplus_{i\in I}\langle a_i\rangle\oplus U'$ for some finite family $(a_i)_{i\in I}$ of elements of order p^2 of K'. Further, by Lemma 5 there is a subgroup G of X such that $U'\subset G$ and $X=\bigoplus_{i\in I}\langle a_i\rangle\oplus G$. Then $(a_j,\bigoplus_{i\in I\setminus\{j\}}\langle a_i\rangle\oplus G)\in\mathcal{P}$ for all $j\in I$, and

$$\sum_{j\in I}\varepsilon_{(a_j,\bigoplus_{i\in I\setminus\{j\}}\langle a_i\rangle\oplus G)}-1_X\in\Omega(K',U')\subset\Omega(K,U).$$

$\square$

We now combine the preceding lemmas to obtain the main result of this section.

Theorem 2. *Let $p\in\mathbb{P}$, and let $X\in\mathcal{L}$ be a group of finite exponent p^2. The following statements are equivalent:*

(i) *$E(X)$ has only one non-trivial closed ideal.*

(ii) *Every non-trivial closed ideal of* $E(X)$ *is strongly topologically maximal.*

(iii) *Every non-trivial closed ideal of* $E(X)$ *is topologically maximal.*

(iv) $X[p] = \overline{pX}$.

Proof. Obviously, (i) implies (ii), and (ii) implies (iii). The fact that (iii) implies (iv) follows from Lemma 3.

Assume (iv), and let $\mathcal{P}$ be the set of all ordered pairs (a, G), where a is an element of order p^2 of X and G is a closed subgroup of X satisfying $X = \langle a \rangle \oplus G$. Further, for $(a, G) \in \mathcal{P}$, let $\varepsilon_{a,G} \in E(X)$ denote the canonical projection of X onto $\langle a \rangle$ with kernel G. Now, pick an arbitrary non-zero closed ideal C of $E(X)$. We distinguish cases when C contains elements of order p^2 and when $pC = \{0\}$.

First, suppose C contains elements of order p^2. Then $(\varepsilon_{a,G} \mid (a, G) \in \mathcal{P}) \subset C$ by Lemma 4, and hence $C = E(X)$ by Lemma 7.

Next suppose that $pC = \{0\}$. Then $\overline{(p\varepsilon_{a,G} \mid (a, G) \in \mathcal{P})} \subset C$ by Lemma 4. In order to establish the reverse inclusion, pick any $u \in C$, and let K be a compact subset of X and U an open neighbourhood of zero in X. By Lemma 6, we can find two compact open subgroups K', U' of X such that $K \cup U' \subset K'$, $U' \subset u^{-1}(U)$, and $K' = \bigoplus_{i \in I} \langle a_i \rangle \oplus U'$ for some finite family $(a_i)_{i \in I}$ of elements of order p^2 of K'. It then follows from Lemma 5 that $X = \bigoplus_{i \in I} \langle a_i \rangle \oplus Y$ for some closed subgroup Y of X containing U'. Since $\mathrm{im}(u) \subset X[p]$ and $X[p] = \overline{pX}$, for every $i \in I$ there exists $b_i \in X$ such that $pb_i - u(a_i) \in U'$. Define $v \in E(X)$ by setting $v(a_i) = b_i$ for all $i \in I$ and $v(y) = 0$ for all $y \in Y$. Then clearly $pv - u \in \Omega(K', U')$. Finally, by Lemma 7, there exists $w \in (\varepsilon_{a,G} \mid (a, G) \in \mathcal{P})$ such that $w - v \in \Omega(K', U')$. Since U' is a subgroup in X, we have $p(w - v) \in \Omega(K', U')$, and hence

$$pw - u = p(w - v) + (pv - u) \in \Omega(K', U') \subset \Omega(K, U).$$

As $pw \in (p\varepsilon_{a,G} \mid (a, G) \in \mathcal{P})$, we conclude that $u \in \overline{(p\varepsilon_{a,G} \mid (a, G) \in \mathcal{P})}$. $\square$

Remark 2. Lemma 3 and Theorem 2 give an answer to the considered question in the case of LCA groups of finite exponent p^2, where $p \in \mathbb{P}$.

5 Groups of finite exponent p^3

In this section, we determine the groups $X \in \mathcal{L}$ of finite exponent p^3, where $p \in \mathbb{P}$, such that the ring $E(X)$ has at most two non-trivial closed ideals. In preparation to this we establish four lemmas, which are similar to Lemmas 3, 4, 6, and 7 of the preceding section.

Lemma 8. *Let* $p \in \mathbb{P}$, *and let* $X \in \mathcal{L}$ *be a group of finite exponent* p^3. *If* $E(X)$ *has no more than two non-trivial closed ideals, then* $\overline{pX} = X[p^2]$ *and* $\overline{p^2X} = X[p]$.

Proof. Suppose first that $\overline{pX} \neq X[p^2]$. To get a contradiction, it is enough to indicate three distinct, non-trivial, closed ideals of $E(X)$. Clearly, $p^2 1_X \neq 0$

and $p^2 1_X \in ann_{E(X)}\big(X[p^2]\big)$. It is also clear that $\mathrm{im}(p^2 1_X) = p^2 X \subset \overline{pX}$, so $p^2 1_X \in ann_{E(X)}\big(X/\overline{pX}\big)$, and thus

$$ann_{E(X)}\big(X[p^2]\big) \cap ann_{E(X)}\big(X/\overline{pX}\big) \neq \{0\}.$$

Further, since $\overline{pX} \subset X[p^2]$, we have $ann_{E(X)}\big(X[p^2]\big) \subset ann_{E(X)}\big(\overline{pX}\big)$. Now, given any $u \in ann_{E(X)}\big(\overline{pX}\big)$, we have $pu(X) = u(pX) = \{0\}$, so $\mathrm{im}(u) \subset X[p] \subset X[p^2]$, and hence $u \in ann_{E(X)}\big(X/X[p^2]\big)$. As $u \in ann_{E(X)}\big(\overline{pX}\big)$ was arbitrary, it follows that $ann_{E(X)}\big(\overline{pX}\big) \subset ann_{E(X)}\big(X/X[p^2]\big)$. Finally, since $p^2 X \neq \{0\}$, it follows that $\mathrm{im}(1_X) \not\subset X[p^2]$, so $1_X \notin ann_{E(X)}\big(X/X[p^2]\big)$, and hence $ann_{E(X)}\big(X/X[p^2]\big) \neq E(X)$. We shall show that at least two of the inclusions

$$\begin{aligned} ann_{E(X)}\big(X[p^2]\big) \cap ann_{E(X)}\big(X/\overline{pX}\big) &\subset ann_{E(X)}\big(X[p^2]\big) \\ &\subset ann_{E(X)}\big(\overline{pX}\big) \subset ann_{E(X)}\big(X/X[p^2]\big) \end{aligned}$$

are strict. Let $\xi : X \to X/X[p^2]$ and $\eta : X \to X/\overline{pX}$ be the canonical projections, and fix any $a \in X \setminus X[p^2]$ and $b \in X[p^2] \setminus \overline{pX}$. Then $o(a) = p^3$ and $o(\xi(a)) = p = o(\eta(b))$. By Lemma 2, we can write

$$X/X[p^2] = \langle \xi(a) \rangle \oplus A, \quad X/\overline{pX} = \langle \eta(b) \rangle \oplus B, \quad \text{and} \quad X = \langle a \rangle \oplus Y,$$

where A, B, and Y are closed subgroups in $X/X[p^2]$, $X/\overline{pX}$, and X, respectively. In the following, we distinguish the cases when $o(b) = p$ and when $o(b) = p^2$.

First assume that $o(b) = p$. Let $\lambda : \langle \xi(a) \rangle \to X$ and $\mu : \langle \eta(b) \rangle \to X$ be the group homomorphisms given by the rule $\lambda(\xi(a)) = \mu(\eta(b)) = b$. Denoting by φ the canonical projection of $X/X[p^2]$ onto $\langle \xi(a) \rangle$ with kernel A, we see that $\lambda \circ \varphi \circ \xi \in ann_{E(X)}\big(X[p^2]\big)$, and $\lambda \circ \varphi \circ \xi \notin ann_{E(X)}\big(X/\overline{pX}\big)$ (because $(\lambda \circ \varphi \circ \xi)(a) = b \notin \overline{pX}$), so $ann_{E(X)}\big(X[p^2]\big)$ properly contains $ann_{E(X)}\big(X[p^2]\big) \cap ann_{E(X)}\big(X/\overline{pX}\big)$. Similarly, denoting by ψ the canonical projection of $X/\overline{pX}$ onto $\langle \eta(b) \rangle$ with kernel B, we see that $\mu \circ \psi \circ \eta \in ann_{E(X)}\big(\overline{pX}\big)$, and $\mu \circ \psi \circ \eta \notin ann_{E(X)}\big(X[p^2]\big)$ (because $b \in X[p^2]$ and $(\mu \circ \psi \circ \eta)(b) = b$), so $ann_{E(X)}\big(\overline{pX}\big)$ properly contains $ann_{E(X)}\big(X[p^2]\big)$ as well.

Next we consider the case when $o(b) = p^2$. Let $\mu' : \langle \eta(b) \rangle \to X$ denote the group homomorphism given by the rule $\mu'(\eta(b)) = pb$. Then $\mu' \circ \psi \circ \eta \in ann_{E(X)}\big(\overline{pX}\big)$ and $\mu' \circ \psi \circ \eta \notin ann_{E(X)}\big(X[p^2]\big)$, so $ann_{E(X)}\big(\overline{pX}\big)$ properly contains $ann_{E(X)}\big(X[p^2]\big)$. Further, let $v \in E(X)$ be defined by $v(a) = b$ and $v(y) = 0$ for all $y \in Y$. Since $v(pa) = pb \neq 0$, we conclude that $v \notin ann_{E(X)}\big(\overline{pX}\big)$. On the other hand, since $p^2 v(a) = p^2 b = 0$, it is clear that $\mathrm{im}(v) \subset X[p^2]$, so $v \in ann_{E(X)}\big(X/X[p^2]\big)$, and hence $ann_{E(X)}\big(X/X[p^2]\big)$ properly contains $ann_{E(X)}\big(\overline{pX}\big)$.

We have shown that at least two of the inclusions

$$\begin{aligned} ann_{E(X)}\big(X[p^2]\big) \cap ann_{E(X)}\big(X/\overline{pX}\big) &\subset ann_{E(X)}\big(X[p^2]\big) \\ &\subset ann_{E(X)}\big(\overline{pX}\big) \subset ann_{E(X)}\big(X/X[p^2]\big) \end{aligned}$$

are strict, a contradiction. Consequently, we must have $\overline{pX} = X[p^2]$.

Now suppose that $\overline{p^2X} \neq X[p]$. As we already mentioned, $p^2 1_X \neq 0$ and $p^2 1_X \in ann_{E(X)}\big(X[p^2]\big)$. Since $\mathrm{im}(p^2 1_X) \subset \overline{p^2X}$, we also have $p^2 1_X \in ann_{E(X)}\big(X/\overline{p^2X}\big)$, so

$$ann_{E(X)}\big(X[p^2]\big) \cap ann_{E(X)}\big(X/\overline{p^2X}\big) \neq \{0\}.$$

Further, since $X[p] \subset X[p^2]$, we have $ann_{E(X)}\big(X[p^2]\big) \subset ann_{E(X)}\big(X[p]\big)$. Finally, since $X[p] \neq \{0\}$, it follows that $X[p] \not\subset \ker(1_X)$, so $1_X \notin ann_{E(X)}\big(X[p]\big)$, and hence $ann_{E(X)}\big(X[p]\big) \neq E(X)$. We shall show that the inclusions

$$ann_{E(X)}\big(X[p^2]\big) \cap ann_{E(X)}\big(X/\overline{p^2X}\big) \subset ann_{E(X)}\big(X[p^2]\big) \subset ann_{E(X)}\big(X[p]\big)$$

are strict. Let $\xi : X \to X/X[p^2]$ denote the canonical projection, and fix any $a \in X \setminus X[p^2]$ and $b \in X[p] \setminus \overline{p^2X}$. Then $o(a) = p^3$, so $o(\xi(a)) = p = o(b)$. By Lemma 2, we can write

$$X/X[p^2] = \langle \xi(a) \rangle \oplus A,$$

where A is a closed subgroup of $X/X[p^2]$. Let $\lambda : \langle \xi(a) \rangle \to X$ be the group homomorphism given by $\lambda(\xi(a)) = b$. Denoting by φ the canonical projection of $X/X[p^2]$ onto $\langle \xi(a) \rangle$ with kernel A, we see that $\lambda \circ \varphi \circ \xi \in ann_{E(X)}\big(X[p^2]\big)$, and $\lambda \circ \varphi \circ \xi \notin ann_{E(X)}\big(X/\overline{p^2X}\big)$ (because $(\lambda \circ \varphi \circ \xi)(a) = b \notin \overline{p^2X}$), so $ann_{E(X)}\big(X[p^2]\big)$ properly contains $ann_{E(X)}\big(X[p^2]\big) \cap ann_{E(X)}\big(X/\overline{p^2X}\big)$. Finally, since $p1_X \notin ann_{E(X)}\big(X[p^2]\big)$ and $p1_X \in ann_{E(X)}\big(X[p]\big)$, $ann_{E(X)}\big(X[p]\big)$ properly contains $ann_{E(X)}\big(X[p^2]\big)$. Consequently, the inclusions

$$ann_{E(X)}\big(X[p]\big) \cap ann_{E(X)}\big(X/\overline{p^2X}\big) \subset ann_{E(X)}\big(X[p^2]\big) \subset ann_{E(X)}\big(X[p]\big)$$

are strict. As this contradicts our hypothesis, we must have $\overline{p^2X} = X[p]$. $\square$

Lemma 9. *Let $p \in \mathbb{P}$, let $X \in \mathcal{L}$ be a group of finite exponent p^3 such that $\overline{pX} = X[p^2]$ and $\overline{p^2X} = X[p]$, and let C be a non-zero closed ideal of $E(X)$. Further, let $\mathcal{P}$ be the set of all ordered pairs (a, G), where a is an element of order p^3 of X and G is a closed subgroup of X satisfying $X = \langle a \rangle \oplus G$, and for each $(a, G) \in \mathcal{P}$ let $\varepsilon_{a,G} \in E(X)$ denote the canonical projection of X onto $\langle a \rangle$ with kernel G. Then:*

(i) *If C contains elements of order p^3, then $C \supset (\varepsilon_{a,G} \mid (a, G) \in \mathcal{P})$.*

(ii) *If $p^2C = \{0\}$ and $pC \neq \{0\}$, then $C \supset (p\varepsilon_{a,G} \mid (a, G) \in \mathcal{P})$.*

(iii) *If $pC = \{0\}$, then $C \supset (p^2\varepsilon_{a,G} \mid (a, G) \in \mathcal{P})$.*

Proof. As in the proof of Lemma 4, for any $(a, G) \in \mathcal{P}$ and $b \in X$, we let $f_{a,G,b} \in E(X)$ be defined by the rule

$$f_{a,G,b}(t) = \begin{cases} b, & \text{if } t = a; \\ 0, & \text{if } t \in G. \end{cases}$$

(i) Pick any $u \in C$ with $o(u) = p^3$. Since $p^2u \neq 0$, there exists $x \in X$ such that $(p^2u)(x) \neq 0$, and so $o(u(x)) = p^3$. By Lemma 2, there exists a closed subgroup Y of X such that $X = \langle u(x)\rangle \oplus Y$. Given any $(a, G) \in \mathcal{P}$, we then have $\varepsilon_{a,G} = f_{u(x),Y,a} \circ u \circ f_{a,G,x}$, so $\varepsilon_{a,G} \in C$.

(ii) Pick any $u \in C$ with $o(u) = p^2$. Since $pu \neq 0$, there exists $x \in X$ such that $(pu)(x) \neq 0$, so $px \notin \ker(u)$ and $o(u(x)) = p^2$. On the other hand, since $p^2u = 0$, we have $p^2X \subset \ker(u)$, so $X[p] \subset \ker(u)$. It follows that $px \notin X[p]$, so $p^2x \neq 0$, and hence $o(x) = p^3$. In particular, we can write $X = \langle x\rangle \oplus Y$ for some closed subgroup Y of X. Now, fix an arbitrary open subgroup U of X such that $pu(x) \notin U$. Since $u(x) \in X[p^2] = \overline{pX}$, there exists $z \in X$ such that $pz - u(x) \in U$. As $p(pz - u(x)) \in U$ and $pu(x) \notin U$, we cannot have $p^2z = 0$, so $o(z) = p^3$. Let π denote the canonical projection of X onto X/U. Clearly, $\pi(pu(x)) \neq 0$ and $\pi(pu(x)) = \pi(p^2z) = p^2\pi(z)$, so $o(\pi(z)) = p^3$. Hence we can write $X/U = \langle\pi(z)\rangle \oplus \Gamma$ for some subgroup Γ of X/U [3, Lemma 15.1]. Denoting by φ the canonical projection of X/U onto $\langle\pi(z)\rangle$ with kernel Γ and letting $h : \langle\pi(z)\rangle \to X$ be the group homomorphism defined by $h(\pi(z)) = x$, it is clear that $h \circ \varphi \circ \pi \in E(X)$ and $(h \circ \varphi \circ \pi) \circ u \circ \varepsilon_{x,Y} = p\varepsilon_{x,Y}$, so $p\varepsilon_{x,Y} \in C$. Now, given any $(a, G) \in \mathcal{P}$, we have $p\varepsilon_{a,G} = f_{x,Y,a} \circ (p\varepsilon_{x,Y}) \circ f_{a,G,x} \in C$.

(iii) Pick any non-zero $u \in C$ and any $x \in X$ such that $u(x) \neq 0$. Since $pu = 0$, we have $pX \subset \ker(u)$, so $X[p^2] \subset \ker(u)$, and therefore $o(x) = p^3$. In particular, $X = \langle x\rangle \oplus Y$ for some closed subgroup Y of X. Now, fix an arbitrary open subgroup U of X such that $u(x) \notin U$. Since $u(x) \in X[p] = \overline{p^2X}$, there exists $z \in X$ such that $p^2z - u(x) \in U$. As $u(x) \notin U$, we cannot have $p^2z = 0$, so $o(z) = p^3$. Let $\pi : X \to X/U$ be the canonical projection. Clearly, $\pi(u(x)) \neq 0$ and $\pi(u(x)) = \pi(p^2z) = p^2\pi(z)$, so $o(\pi(z)) = p^3$. Hence we can write $X/U = \langle\pi(z)\rangle \oplus \Gamma$ for some subgroup Γ of X/U [3, Lemma 15.1]. Denoting by φ the canonical projection of X/U onto $\langle\pi(z)\rangle$ with kernel Γ and letting $h : \langle\pi(z)\rangle \to X$ be the group homomorphism defined by $h(\pi(z)) = x$, it is clear that $h \circ \varphi \circ \pi \in E(X)$ and $(h \circ \varphi \circ \pi) \circ u \circ \varepsilon_{x,Y} = p^2\varepsilon_{x,Y}$, so $p^2\varepsilon_{x,Y} \in C$. Consequently, for any $(a, G) \in \mathcal{P}$, we have $p^2\varepsilon_{a,G} = f_{x,Y,a} \circ (p^2\varepsilon_{x,Y}) \circ f_{a,G,x} \in C$. $\square$

Lemma 10. *Let $p \in \mathbb{P}$, and let $X \in \mathcal{L}$ be a group of finite exponent p^3 satisfying $\overline{pX} = X[p^2]$ and $\overline{p^2X} = X[p]$. For any compact subset K of X and any neighbourhood U of zero in X, there exist two compact open subgroups K', U' of X such that $K \cup U' \subset K'$, $U' \subset U$, and $K' = \bigoplus_{i\in I}\langle a_i\rangle \oplus U'$ for some finite family $(a_i)_{i\in I}$ of elements of order p^3 of K'.*

Proof. Pick an arbitrary compact subset K of X and an arbitrary neighbourhood U of zero in X. Since X is totally disconnected, we can find a compact open subgroup U_0 of X such that $U_0 \subset U$ [5, (7.7)]. Let $K_0 = \langle K \cup U_0\rangle$. Then K_0 is compact [5, (9.8)], and $U_0 \subset K_0$. In particular, K_0 is topologically isomorphic to a topological direct product of cyclic p-groups of order at most p^3 [5, (25.9)], and so there exist three disjoint sets I_1, I_2, and I_3 such that $K_0 \cong \prod_{i\in I_1\cup I_2\cup I_3} C_i$, where $C_i = \mathbb{Z}(p)$ for $i \in I_1$, $C_i = \mathbb{Z}(p^2)$ for $i \in I_2$, and $C_i = \mathbb{Z}(p^3)$ for $i \in I_3$. Fix a topological isomorphism f from K_0 onto $\prod_{i\in I_1\cup I_2\cup I_3} C_i$. Given an arbitrary subset J of $I_1 \cup I_2 \cup I_3$, we denote by C'_J the subgroup of all $(c_i)_{i\in I_1\cup I_2\cup I_3} \in \prod_{i\in I_1\cup I_2\cup I_3} C_i$ satisfying $c_i = 0$ for all $i \notin J$. Since U_0 is open in K_0, there exist finite subsets $J_1 \subset I_1$, $J_2 \subset I_2$, and $J_3 \subset I_3$

such that $f(U_0) \supset C'_{(I_1\setminus J_1)\cup(I_2\setminus J_2)\cup(I_3\setminus J_3)}$. We then have

$$\prod_{i\in I_1\cup I_2\cup I_3} C_i = \Big(\bigoplus_{i\in J_1\cup J_2\cup J_3} C'_i\Big) \oplus C'_{(I_1\setminus J_1)\cup(I_2\setminus J_2)\cup(I_3\setminus J_3)},$$

so

$$K_0 = \Big(\bigoplus_{i\in J_1\cup J_2\cup J_3} f^{-1}(C'_i)\Big) \oplus f^{-1}(C'_{(I_1\setminus J_1)\cup(I_2\setminus J_2)\cup(I_3\setminus J_3)}),$$

where C'_i stands for $C'_{\{i\}}$. Set $U' = f^{-1}(C'_{(I_1\setminus J_1)\cup(I_2\setminus J_2)\cup(I_3\setminus J_3)})$ and, for $i \in J_1\cup J_2\cup J_3$, let a_i be a generator of $f^{-1}(C'_i)$. Then U' is an open subgroup of X contained in U_0 and

$$K_0 = \Big(\bigoplus_{i\in J_1\cup J_2\cup J_3} \langle a_i\rangle\Big) \oplus U'.$$

We also have $o(a_i) = p$ if $i \in J_1$, $o(a_i) = p^2$ if $i \in J_2$, and $o(a_i) = p^3$ if $i \in J_3$. In the following, we shall construct a compact subgroup $K' \supset K_0$ which admits a decomposition similar to that of K_0, by replacing the elements a_i with $i \in J_1\cup J_2$ by elements of order p^3. If $J_1\cup J_2 = \varnothing$, we set $K' = K_0$. Suppose $J_1\cup J_2 \neq \varnothing$, and fix an arbitrary $j \in J_1\cup J_2$. We distinguish the cases when $j \in J_1$ and when $j \in J_2$. In the former case we use the equality $X[p] = \overline{p^2X}$ to find an element $b_j \in X$ such that $a_j - p^2b_j \in U'$. As $a_j \notin U'$, we cannot have $p^2b_j = 0$, so $o(b_j) = p^3$. We claim that

$$\langle p^2b_j\rangle \cap \bigg(\Big(\bigoplus_{i\in(J_1\setminus\{j\})\cup J_2\cup J_3} \langle a_i\rangle\Big) \oplus U'\bigg) = \{0\}.$$

Indeed, given any $x \in \langle p^2b_j\rangle \cap \Big(\big(\bigoplus_{i\in(J_1\setminus\{j\})\cup J_2\cup J_3}\langle a_i\rangle\big) \oplus U'\Big)$, we can write

$$x = lp^2b_j = \Big(\sum_{i\in(J_1\setminus\{j\})\cup J_2\cup J_3} l_ia_i\Big) + y'$$

for some non-negative integers l, l_i and some $y' \in U'$. Since $y' + l(a_j - p^2b_j) \in U'$, it follows that

$$\begin{aligned} la_j &= \Big(\sum_{i\in(J_1\setminus\{j\})\cup J_2\cup J_3} l_ia_i\Big) + y' + l(a_j - p^2b_j) \\ &\in \langle a_j\rangle \cap \bigg(\Big(\bigoplus_{i\in(J_1\setminus\{j\})\cup J_2\cup J_3} \langle a_i\rangle\Big) \oplus U'\bigg) \;=\; \{0\}, \end{aligned}$$

so p divides l, and hence $x = 0$. This proves our claim that

$$\langle p^2b_j\rangle \cap \bigg(\Big(\bigoplus_{i\in(J_1\setminus\{j\})\cup J_2\cup J_3} \langle a_i\rangle\Big) \oplus U'\bigg) = \{0\}.$$

Clearly, we then also have

$$\langle b_j\rangle \cap \Big(\Big(\bigoplus_{i\in (J_1\setminus\{j\})\cup J_2\cup J_3} \langle a_i\rangle\Big) \oplus U'\Big) = \{0\}.$$

In this case, we replace K_0 by

$$K_1 = \langle b_j\rangle \oplus \Big(\bigoplus_{i\in (J_1\setminus\{j\})\cup J_2\cup J_3} \langle a_i\rangle\Big) \oplus U'.$$

Next we consider the second case when $j \in J_2$. We use the equality $X[p^2] = \overline{pX}$ to find an element $b_j \in X$ such that $a_j - pb_j \in U'$. Since then $pa_j - p^2 b_j \in U'$ and $pa_j \notin U'$, we cannot have $p^2 b_j = 0$, so $o(b_j) = p^3$. We claim that

$$\langle pb_j\rangle \cap \Big(\Big(\bigoplus_{i\in J_1\cup(J_2\setminus\{j\})\cup J_3} \langle a_i\rangle\Big) \oplus U'\Big) = \{0\}.$$

Indeed, given any $x \in \langle pb_j\rangle \cap \Big(\Big(\bigoplus_{i\in J_1\cup(J_2\setminus\{j\})\cup J_3}\langle a_i\rangle\Big) \oplus U'\Big)$, we can write

$$x = lpb_j = \Big(\sum_{i\in J_1\cup(J_2\setminus\{j\})\cup J_3} l_i a_i\Big) + y'$$

for some non-negative integers l, l_i and $y' \in U'$. Since $y' + l(a_j - pb_j) \in U'$, it follows that

$$\begin{aligned} la_j &= \sum_{i\in J_1\cup(J_2\setminus\{j\})\cup J_3} l_i a_i + y' + l(a_j - pb_j) \\ &\in \langle a_j\rangle \cap \Big(\Big(\bigoplus_{i\in J_1\cup(J_2\setminus\{j\})\cup J_3} \langle a_i\rangle\Big) \oplus U'\Big) \;=\; \{0\}, \end{aligned}$$

so p^2 divides l, and hence $x = 0$. This proves our claim that

$$\langle pb_j\rangle \cap \Big(\Big(\bigoplus_{i\in J_1\cup(J_2\setminus\{j\})\cup J_3} \langle a_i\rangle\Big) \oplus U'\Big) = \{0\}.$$

Clearly, we then also have

$$\langle b_j\rangle \cap \Big(\Big(\bigoplus_{i\in J_1\cup(J_2\setminus\{j\})\cup J_3} \langle a_i\rangle\Big) \oplus U'\Big) = \{0\}.$$

Consequently, in this case we can enlarge K_0 by considering

$$K_1 = \langle b_j\rangle \oplus \Big(\bigoplus_{i\in J_1\cup(J_2\setminus\{j\})\cup J_3} \langle a_i\rangle\Big) \oplus U'.$$

Now, if $(J_1 \cup J_2) \setminus \{j\} \neq \varnothing$, we can apply the preceding procedure to K_1, and so, after a finite number of steps, we shall arrive at a compact subgroup K' of X having the following form:

$$K' = \Big(\bigoplus_{i \in J_1 \cup J_2} \langle b_i \rangle \Big) \oplus \Big(\bigoplus_{i \in J_3} \langle a_i \rangle \Big) \oplus U',$$

where $o(b_i) = p^3$ for all $i \in J_1 \cup J_2$. Since $a_i \in K'$ for all $i \in J_1 \cup J_2$, we also have $K \cup U' \subset K_0 \subset K'$, so K' and U' are those required. □

Lemma 11. *Let $p \in \mathbb{P}$, let $X \in \mathcal{L}$ be a group of finite exponent p^3 satisfying $\overline{pX} = X[p^2]$ and $\overline{p^2 X} = X[p]$, and let $\mathcal{P}$ be the set of all ordered pairs (a, G), where a is an element of order p^3 of X and G is a closed subgroup of X satisfying $X = \langle a \rangle \oplus G$. Then the ideal $(\varepsilon_{a,G} \mid (a, G) \in \mathcal{P})$, where $\varepsilon_{a,G} \in E(X)$ denotes the canonical projection of X onto $\langle a \rangle$ with kernel G, is dense in $E(X)$.*

Proof. Pick an arbitrary compact subset K of X and an arbitrary open neighbourhood U of zero in X. It suffices to show that

$$(\varepsilon_{a,G} \mid (a, G) \in \mathcal{P}) \cap [1_X + \Omega(K, U)] \neq \varnothing.$$

By Lemma 10, we can find two compact open subgroups K', U' of X such that $K \cup U' \subset K'$, $U' \subset U$, and $K' = \bigoplus_{i \in I} \langle a_i \rangle \oplus U'$ for some finite family $(a_i)_{i \in I}$ of elements of order p^3 of K'. Further, by Lemma 5 there is a subgroup G of X such that $U' \subset G$ and $X = \bigoplus_{i \in I} \langle a_i \rangle \oplus G$. Then $(a_j, \bigoplus_{i \in I \setminus \{j\}} \langle a_i \rangle \oplus G) \in \mathcal{P}$ for all $j \in I$, and

$$\sum_{j \in I} \varepsilon_{(a_j, \bigoplus_{i \in I \setminus \{j\}} \langle a_i \rangle \oplus G)} - 1_X \in \Omega(K', U') \subset \Omega(K, U).$$

□

With this preparation, we can now state the main result of this section.

Theorem 3. *Let $p \in \mathbb{P}$, and let $X \in \mathcal{L}$ be a group of finite exponent p^3. The following statements are equivalent:*

(i) *$E(X)$ has exactly two non-trivial closed ideals.*

(ii) *$\overline{pX} = X[p^2]$ and $\overline{p^2 X} = X[p]$.*

Moreover, in case these conditions hold, the corresponding ideals are comparable with respect to set-theoretic inclusion.

Proof. The fact that (i) implies (ii) follows from Lemma 8. Assume (ii), and let $\mathcal{P}$ denote the set of all ordered pairs (a, G), where a is an element of order p^3 of X and G is a closed subgroup of X satisfying $X = \langle a \rangle \oplus G$. Further, for $(a, G) \in \mathcal{P}$, let $\varepsilon_{a,G} \in E(X)$ denote the canonical projection of X onto $\langle a \rangle$ with kernel G. Now, fix a non-zero closed ideal C of $E(X)$. We can have three possibilities for C.

First, suppose C contains elements of order p^3. Then $(\varepsilon_{a,G} \mid (a,G) \in \mathcal{P}) \subset C$ by Lemma 9, and hence $C = E(X)$ by Lemma 11.

Next suppose $p^2C = \{0\}$ and $pC \neq \{0\}$. Then $\overline{(p\varepsilon_{a,G} \mid (a,G) \in \mathcal{P})} \subset C$ by Lemma 9. To show the opposite inclusion, we pick any $u \in C$, and let K be a compact subset of X and U an open neighbourhood of zero in X. By Lemma 10, we can find two compact open subgroups K', U' of X such that $K \cup U' \subset K'$, $U' \subset U$, and $K' = \bigoplus_{i\in I}\langle a_i\rangle \oplus U'$ for some finite family $(a_i)_{i\in I}$ of elements of order p^3 of K'. It then follows from Lemma 5 that $X = \bigoplus_{i\in I}\langle a_i\rangle \oplus Y$ for some closed subgroup Y of X containing U'. Since $\mathrm{im}(u) \subset X[p^2]$ and $X[p^2] = \overline{pX}$, for every $i \in I$ there exists $b_i \in X$ such that $pb_i - u(a_i) \in U'$. Define $v \in E(X)$ by setting $v(a_i) = b_i$ for all $i \in I$ and $v(y) = 0$ for all $y \in Y$. Then clearly $pv - u \in \Omega(K', U')$. Finally, by Lemma 11, there exists $w \in (\varepsilon_{a,G} \mid (a,G) \in \mathcal{P})$ such that $w - v \in \Omega(K', U')$. Since U' is a subgroup in X, we have $p(w - v) \in \Omega(K', U')$, and hence

$$pw - u = p(w - v) + (pv - u) \in \Omega(K', U') \subset \Omega(K, U).$$

As $pw \in (p\varepsilon_{a,G} \mid (a,G) \in \mathcal{P})$, it follows that $u \in \overline{(p\varepsilon_{a,G} \mid (a,G) \in \mathcal{P})}$.

Lastly, suppose $pC = \{0\}$. Then $\overline{(p^2\varepsilon_{a,G} \mid (a,G) \in \mathcal{P})} \subset C$ by Lemma 9. In order to establish the reverse inclusion, fix any $u \in C$, and let K be a compact subset of X and U an open neighbourhood of zero in X. By Lemma 10, there exist two compact open subgroups K', U' of X such that $K \cup U' \subset K'$, $U' \subset U$, and $K' = \bigoplus_{i\in I}\langle a_i\rangle \oplus U'$ for some finite family $(a_i)_{i\in I}$ of elements of order p^3 of K'. Consequently, $X = \bigoplus_{i\in I}\langle a_i\rangle \oplus Y$ for some closed subgroup Y of X containing U'. Since $\mathrm{im}(u) \subset X[p]$ and $X[p] = \overline{p^2X}$, for every $i \in I$ there exists $b_i \in X$ such that $p^2b_i - u(a_i) \in U'$. Define $v \in E(X)$ by setting $v(a_i) = b_i$ for all $i \in I$ and $v(y) = 0$ for all $y \in Y$. It is then clear that $p^2v - u \in \Omega(K', U')$. By Lemma 11, there exists $w \in (\varepsilon_{a,G} \mid (a,G) \in \mathcal{P})$ such that $w - v \in \Omega(K', U')$. As U' is a subgroup in X, we have $p^2(w - v) \in \Omega(K', U')$, and hence

$$p^2w - u = p^2(w - v) + (p^2v - u) \in \Omega(K', U') \subset \Omega(K, U).$$

Since $p^2w \in (p^2\varepsilon_{a,G} \mid (a,G) \in \mathcal{P})$, it follows that $u \in \overline{(p^2\varepsilon_{a,G} \mid (a,G) \in \mathcal{P})}$. □

Remark 3. Lemma 8 and Theorem 3 give an answer to our question in the case of LCA groups of finite exponent p^3, where $p \in \mathbb{P}$.

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Valeriu Popa
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
Academiei str. 5, MD-2028 Chişinău
Moldova
E-mail: *vpopa@math.md*

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Semilattices of r-archimedean subdimonoids

Anatolii V. Zhuchok

Abstract. We characterize dimonoids which are semilattices of r-archimedean (ℓ-archimedean, $(t;r)$-archimedean) subdimonoids.

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1 Introduction

Dimonoids were introduced by J.-L. Loday [1] for the study of properties of Leibniz algebras. Dialgebras, which are based on the notion of a dimonoid, have been studied by many mathematicians (see, for example, [1-4]). It is well-known that the notion of a dimonoid generalizes the notion of a digroup [5]. Digroups play a prominent role in an important open problem from the theory of Leibniz algebras. Dimonoids were studied in the papers of the author (see, for example, [6–11]). Moreover, note that algebras with two associative operations (so-called bisemigroups) were considered earlier in some other aspects in the paper of B. M. Schein [12]. The study of connections between dimonoids and bisemigroups was started in [11].

In this work we characterize dimonoids which are bands of subdimonoids. In Section 2 we give necessary definitions, auxiliary results (Lemma 1 and Theorem 3) and some properties of dimonoids (Lemma 2, Theorem 4 and Corollary 1). Putcha [13] gave necessary and sufficient conditions under which an arbitrary semigroup is a semilattice of r-archimedean (ℓ-archimedean, t-archimedean) semigroups. In Section 3 we extend Putcha's results to the case of dimonoids (Theorem 5).

2 Preliminaries

A nonempty set D equipped with two binary associative operations $\prec$ and $\succ$ satisfying the following axioms:

$$(x \prec y) \prec z = x \prec (y \succ z),$$

$$(x \succ y) \prec z = x \succ (y \prec z),$$

$$(x \prec y) \succ z = x \succ (y \succ z)$$

for all $x, y, z \in D$, is called a dimonoid. If the operations of a dimonoid coincide, then the dimonoid becomes a semigroup.

Different examples of dimonoids can be found in [1, 6–11].

The notion of a diband of subdimonoids was introduced in [6] and investigated in [7]. Recall this definition.

A dimonoid $(D, \prec, \succ)$ is called an idempotent dimonoid or a diband if $x \prec x = x = x \succ x$ for all $x \in D$. If $\varphi : S \to T$ is a homomorphism of dimonoids, then the corresponding congruence on S will be denoted by Δ_φ.

Let S be an arbitrary dimonoid, J be some idempotent dimonoid. Let

$$\alpha : S \to J : x \mapsto x\alpha$$

be a homomorphism. Then every class of the congruence Δ_α is a subdimonoid of the dimonoid S, and the dimonoid S itself is a union of such dimonoids $S_\xi,\ \xi \in J$, that

$$x\alpha = \xi \Leftrightarrow x \in S_\xi = \Delta_\alpha^x = \{t \in S\,|(x;t) \in \Delta_\alpha\},$$

$$S_\xi \prec S_\varepsilon \subseteq S_{\xi \prec \varepsilon}, \quad S_\xi \succ S_\varepsilon \subseteq S_{\xi \succ \varepsilon},$$

$$\xi \neq \varepsilon \Rightarrow S_\xi \bigcap S_\varepsilon = \varnothing.$$

In this case we say that S is decomposable into a diband of subdimonoids (or S is a diband J of subdimonoids $S_\xi,\ \xi \in J$). If J is a band (=idempotent semigroup), then we say that S is a band J of subdimonoids $S_\xi,\ \xi \in J$. If J is a commutative band, then we say that S is a semilattice J of subdimonoids $S_\xi,\ \xi \in J$.

We denote the set of positive integers by N. Let $(D, \prec, \succ)$ be a dimonoid and $a \in D$, $n \in N$. Denote the power n of an element a with respect to the operation $\prec$ (respectively, $\succ$) by a^n (respectively, by $n\,a$).

Lemma 1 (see [8], Lemma 2.4)**.** *Let $(D, \prec, \succ)$ be an arbitrary dimonoid. For all $x \in D,\ n \in N$*

(i) $x^n \succ x = (n+1)x;$

(ii) $x \prec nx = x^{n+1}.$

A semigroup S is called r-archimedean (respectively, ℓ-archimedean) if for all $a, b \in S$ there exist $x \in S^1,\ n \in N$ such that $b^n = ax$ (respectively, $b^n = xa$). A semigroup S is called t-archimedean if for all $a, b \in S$ there exist $x, y \in S^1,\ n \in N$ such that $b^n = ax = ya$.

Let $(D, \prec, \succ)$ be a dimonoid. We denote the semigroup $(D, \prec)$ (respectively, $(D, \succ)$) with an identity by $D^1_\prec$ (respectively, by $D^1_\succ$).

Lemma 2. *Let $(D, \prec, \succ)$ be an arbitrary dimonoid.*

(i) If $(D, \prec)$ is an r-archimedean semigroup, then $(D, \succ)$ is an r-archimedean semigroup.

(ii) If $(D, \succ)$ is an ℓ-archimedean semigroup, then $(D, \prec)$ is an ℓ-archimedean semigroup.

(iii) If $(D, \prec)$ is a t-archimedean semigroup, then $(D, \succ)$ is an r-archimedean semigroup.

(iv) If $(D, \succ)$ is a t-archimedean semigroup, then $(D, \prec)$ is an ℓ-archimedean semigroup.

Proof. (i) Let $(D, \prec)$ be an r-archimedean semigroup. Then for all $a, b \in D$ there exist $x \in D^1_{\prec}$, $n \in N$ such that $a \prec x = b^n$. Multiply both parts of the last equality by b with respect to the operation $\succ$:

$$(a \prec x) \succ b = a \succ (x \succ b) = b^n \succ b = (n+1)b$$

according to the axiom of a dimonoid and Lemma 1 (i). So, $(D, \succ)$ is an r-archimedean semigroup.

(ii) Let $(D, \succ)$ be an ℓ-archimedean semigroup. Then for all $a, b \in D$ there exist $x \in D^1_{\succ}$, $n \in N$ such that $x \succ a = nb$. Multiply both parts of the last equality by b with respect to the operation $\prec$:

$$b \prec (x \succ a) = (b \prec x) \prec a = b \prec nb = b^{n+1}$$

according to the axiom of a dimonoid and Lemma 1 (ii). So, $(D, \prec)$ is an ℓ-archimedean semigroup.

The proofs of (iii) and (iv) are similar. □

A semigroup S is called archimedean if for all $a, b \in S$ there exist $x, y \in S^1$, $n \in N$ such that $b^n = xay$. A dimonoid is called archimedean if its both semigroups are archimedean.

Let $(D, \prec, \succ)$ be a dimonoid, $a, b \in D$. Introduce the following notations: $a_{\prec}|b$ if $b \in D^1_{\prec} \prec a \prec D^1_{\prec}$ and $a_{\succ}|b$ if $b \in D^1_{\succ} \succ a \succ D^1_{\succ}$.

Theorem 3 (see [8], Theorem 4.1). *A dimonoid $(D, \prec, \succ)$ is a semilattice of archimedean subdimonoids if and only if for all $a, b \in D$,*

$$a_{\prec}|b \Rightarrow a^2_{\prec}|b^n \quad \textit{for some} \quad n \in N. \tag{1}$$

Dually, the following theorem can be proved.

Theorem 4. *A dimonoid $(D, \prec, \succ)$ is a semilattice of archimedean subdimonoids if and only if for all $a, b \in D$,*

$$a_{\succ}|b \Rightarrow 2a_{\succ}|nb \quad \textit{for some} \quad n \in N. \tag{2}$$

From Theorem 4 we obtain

Corollary 1. *Let $(D, \prec, \succ)$ be a dimonoid. Then*

(i) $(D, \prec, \succ)$ with a medial semigroup $(D, \succ)$ is a semilattice of archimedean subdimonoids;

(ii) $(D, \prec, \succ)$ with a commutative operation $\succ$ is a semilattice of archimedean subdimonoids;

(iii) $(D, \prec, \succ)$ with an exponential semigroup $(D, \succ)$ is a semilattice of archimedean subdimonoids;

(iv) $(D, \prec, \succ)$ with a weakly exponential semigroup $(D, \succ)$ is a semilattice of archimedean subdimonoids.

Dually to Corollary 4.1 from [8], this corollary can be proved.

3 The main result

Observe that a commutative dimonoid was decomposed into a semilattice of archimedean subdimonoids in [6]. In [9] a free commutative dimonoid was constructed and this dimonoid was decomposed into a semilattice of archimedean subdimonoids. In [8] we gave necessary and sufficient conditions under which an arbitrary dimonoid is a semilattice of archimedean subdimonoids (see also Theorems 3 and 4).

In this section we give necessary and sufficient conditions under which an arbitrary dimonoid is a semilattice of r-archimedean (ℓ-archimedean, $(t;r)$-archimedean) subdimonoids.

Let $(D, \prec, \succ)$ be a dimonoid and $a, b \in D$. Introduce the following notations: $a_{\prec}|_r b$ if $a \prec x = b$ for some $x \in D^1_{\prec}$; $a_{\prec}|_{\ell} b$ if $x \prec a = b$ for some $x \in D^1_{\prec}$; $a_{\succ}|_{\ell} b$ if $x \succ a = b$ for some $x \in D^1_{\succ}$; $a_{\prec}|_t b$ if $a_{\prec}|_r b$ and $a_{\prec}|_{\ell} b$.

A dimonoid will be called r-archimedean (respectively, ℓ-archimedean) if both its semigroups are r-archimedean (respectively, ℓ-archimedean). A dimonoid $(D, \prec, \succ)$ will be called $(t;r)$-archimedean if $(D, \prec)$ is a t-archimedean semigroup and $(D, \succ)$ is an r-archimedean semigroup.

Theorem 5. *Let $(D, \prec, \succ)$ be an arbitrary dimonoid. Then*

(i) $(D, \prec, \succ)$ is a semilattice of r-archimedean subdimonoids if and only if for all $a, b \in D$,

$$a_{\prec}|b \Rightarrow a_{\prec}|_r b^n \quad \textit{for some} \quad n \in N. \tag{3}$$

(ii) $(D, \prec, \succ)$ is a semilattice of ℓ-archimedean subdimonoids if and only if for all $a, b \in D$,

$$a_{\succ}|b \Rightarrow a_{\succ}|_{\ell} nb \quad \textit{for some} \quad n \in N. \tag{4}$$

(iii) $(D, \prec, \succ)$ is a semilattice of $(t;r)$-archimedean subdimonoids if and only if for all $a, b \in D$,

$$a_{\prec}|b \Rightarrow a_{\prec}|_t b^n \quad \textit{for some} \quad n \in N. \tag{5}$$

Proof. (i) Let the condition (3) hold. By Theorem 3 (1) from [13] the condition (1) follows from (3). Hence according to Theorem 3 $(D, \prec, \succ)$ is a semilattice Y of archimedean subdimonoids $(D_i, \prec, \succ), i \in Y$. From Theorem 3 (1) [13] it follows that $(D_i, \prec)$, $i \in Y$, is an r-archimedean semigroup. Then by Lemma 2 (i) $(D_i, \succ)$, $i \in Y$, is an r-archimedean semigroup. Thus, $(D_i, \prec, \succ)$, $i \in Y$, is an r-archimedean subdimonoid of $(D, \prec, \succ)$.

The necessity follows from Theorem 3 (1) [13].

(ii) Let the condition (4) hold. By Theorem 3 (2) from [13] the condition (2) follows from (4). Hence according to Theorem 4 $(D, \prec, \succ)$ is a semilattice Y of archimedean subdimonoids $(D_i, \prec, \succ)$, $i \in Y$. From Theorem 3 (2) [13] it follows that $(D_i, \succ)$, $i \in Y$, is an ℓ-archimedean semigroup. Then by Lemma 2 (ii) $(D_i, \prec)$, $i \in Y$, is an ℓ-archimedean semigroup. Thus, $(D_i, \prec, \succ)$, $i \in Y$, is an ℓ-archimedean subdimonoid of $(D, \prec, \succ)$.

The necessity follows from Theorem 3 (2) [13].

(iii) Let the condition (5) hold. By Theorem 3 (3) from [13] the condition (1) follows from (5). Hence according to Theorem 3 $(D, \prec, \succ)$ is a semilattice Y of archimedean subdimonoids $(D_i, \prec, \succ)$, $i \in Y$. From Theorem 3 (3) [13] it follows that $(D_i, \prec)$, $i \in Y$, is a t-archimedean semigroup. Then by Lemma 2 (iii) $(D_i, \succ)$, $i \in Y$, is an r-archimedean semigroup. Thus, $(D_i, \prec, \succ)$, $i \in Y$, is a $(t; r)$-archimedean subdimonoid of $(D, \prec, \succ)$.

The necessity follows from Theorem 3 (3) [13]. $\square$

Theorem 5 extends Theorem 3 from [13] about necessary and sufficient conditions under which an arbitrary semigroup is a semilattice of r-archimedean (ℓ-archimedean, t-archimedean) semigroups.

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Anatolii V. Zhuchok
Department of Mechanics and Mathematics
Kyiv National Taras Shevchenko University
Volodymyrska str., 64, Kyiv, 01033, Ukraine
E-mail: *zhuchok_a@mail.ru*

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