The varieties of Bol-Moufang quasigroups defined by a single operation

Reza Akhtar, Ashley Arp, Michael Kaminski, Jasmine Van Exel, Davian Vernon and Cory Washington

Abstract. A quasigroup identity is said to be of *Bol-Moufang* type if it involves three variables, two of which occur once on each side and one of which appears twice; moreover, the order in which the variables appear is the same on both sides, and there is only one binary operation in the identity. Answering a question of Drapál, we classify all varieties of quasigroups of Bol-Moufang type where the operation involved is $*$, $/$, or $\backslash$, determining all inclusions among these and providing all necessary counterexamples. This work extends that of Phillips and Vojtěchovský, who described the relationships among the 26 varieties obtained when the operation is $*$. We find that 52 varieties, distinct from each other and from the aforementioned 26, are obtained when one allows $/$ or $\backslash$ as the operation. We determine all inclusions among these varieties, furnishing all necessary counterexamples to complete the classification.

1. Introduction

A *quasigroup* is a set G together with a binary operation $*$ such that the maps $L(a) : G \to G$ and $R(a) : G \to G$ defined by $[L(a)](x) = a*x$ and $[R(a)](x) = x*a$ are bijective for all $a \in G$. As such, there are operations $\backslash : G \to G$ and $/ : G \to G$ defined by $a\backslash c = b$ and $c/b = a$ if only if $a * b = c$. We often refer to $*$ as the *principal operation* in the quasigroup. A quasigroup is called a *loop* if it has a two-sided neutral element, i.e., an element $e \in G$ such that $e * x = x = x * e$ for all $x \in G$. From the viewpoint of universal algebra, one may view the *variety of quasigroups* as consisting of universal algebras $(G, *, \backslash, /)$ satisfying the four identities:

$$a * (a\backslash b) = b,\ \ (b/a) * a = b,\ \ a\backslash(a * b) = b,\ \ (b * a)/a = b.$$

In this article, we classify varieties of quasigroups satisfying an additional identity, an identity of so-called *Bol-Moufang type.* Such identities involve three variables, two of which appear once on both sides of the equation and one of which appears twice on both sides. We also require that the variables appear in the same

2010 Mathematics Subject Classification: 20N05
Keywords: quasigroup, Bol-Moufang identity, variety
This research was funded through the SUMSRI program at Miami University.

order on both sides, and that only one operation (either $*$, $\backslash$, or $/$) appears in the identity. For example, $x*((y*x)*z) = (x*y)*(x*z)$ is an identity of Bol-Moufang type.

The equational perspective is useful in that it lends itself particularly well to automated theorem proving. Indeed, we made considerable use of the automated theorem prover `Prover9` [3] to deduce which implications among identities were valid; virtually all counterexamples were found using the finite model builder `Mace4` [3]. In hindsight, we realized that all the proofs could be written out by hand, only one of them being somewhat long. Therefore, all proofs that appear in this paper are "human" proofs, although some of them would have been difficult to find without the assistance of `Prover9`.

Our work builds upon that of Phillips and Vojtěchovský [5] who carried out this classification for varieties of quasigroups defined by identities of Bol-Moufang type involving only the operation $*$. Using the action of the group S_3 on the conjugates of a quasigroup, we argue that an analogous classification holds for varieties defined solely by $\backslash$ and for varieties defined by solely by $/$; hence, the problem is reduced to an understanding of how a variety defined by an identity involving one of the three operations is related (if at all) to a variety defined by an identity involving another operation. By using the Phillips-Vojtěchovský classification and the S_3-action, we reduce the problem to checking a much smaller number of implications. We then provide necessary counterexamples to complete our classification.

2. Notation and background

For simplicity of reference, we adopt and extend notation introduced by Phillips and Vojtěchovský in [4] and [5] for labeling identities of Bol-Moufang type.

A	$xxyz$
B	$xyxz$
C	$xyyz$
D	$xyzx$
E	$xyzy$
F	$xyzz$

1	0(0(00))
2	0((00)0)
3	(00)(00)
4	(0(00))0
5	((00)0)0

In labeling an identity, the first letter (S, L, or R) refers to the operation used (**s**tar ($*$), **l**eft division ($\backslash$) or **r**ight division ($/$)); the next letter, selected from A through F, refers to the variable ordering as labeled in the above chart, and the two numbers at the end refer to the parenthesization patterns on the two sides of the identity. For example, $LA25$ is the identity $x\backslash((x\backslash y)\backslash z) = ((x\backslash x)\backslash y)\backslash z$, while $SD34$ is the identity $(x*y)*(z*x) = (x*(y*z))*x$. Note also that an identity employing a variable ordering in which x, y, and z are not revealed in alphabetical order (e.g. $zxyz$) is equivalent to one described by the above notation by appropriate permutation of x, y, and z. Thus, there are 180 identities of Bol-Moufang type to consider, 60 for each operation.

If I is an identity of Bol-Moufang type, its *dual* is the identity $I^\vee$ obtained from I by reading from right to left; for example, $(SD34)^\vee$ is $x * ((z * y) * x) = (x * z) * (y * x)$; after switching y and z, we identify this as $SD24$. Thus the variable orders A and F are duals of each other, as are B and E, while C and D are self-dual. Similarly, patterns 1 and 5 are dual to each other, as are 2 and 4, whereas 3 is self-dual. Since the other three operations defined on G ($\circ$, $//$, and $\backslash\backslash$) are defined by

$$x \circ y = y * x, \quad x//y = y\backslash x, \quad \text{and} \quad x\backslash\backslash y = y/x$$

an identity of Bol-Moufang type involving any one of these operations is equivalent to an identity involving one of $*$, $\backslash$, or $/$. This explains our restriction to identities of the latter sort.

We say that an identity I *implies* another identity J and write $I \Rightarrow J$ if J holds in every quasigroup satisfying I – in other words, if the variety of quasigroups defined by I is contained in the variety of quasigroups defined by J. We say that I and J are *equivalent* if $I \Rightarrow J$ and $J \Rightarrow I$, or equivalently if I and J define the same variety of quasigroups.

Let G be a quasigroup with principal operation $*$. We refer to the operations in $\mathcal{O} = \{*, \backslash, /\circ, \backslash\backslash, //\}$ as *conjugates* of the principal operation $*$. If $\Box \in \mathcal{O}$ is any operation, we may consider the quasigroup $(G, \Box)$ whose underlying set is G and whose principal operation $*^\Box$ is defined by $a *^\Box b = a\Box b$. We call these quasigroups *conjugates* of the original quasigroup $(G, *)$. There is a natural action of the symmetric group S_3 on $\mathcal{O}$, summarized in Table 1; this extends to an action of S_3 on the conjugates of $(G, *)$ by setting $\sigma \cdot (G, \Box) = (G, \sigma \cdot \Box)$. The table also tells one how to interpret each of the conjugate operations in the various conjugate quasigroups. In particular, given $\sigma \in S_3$, let $\Box$ be the operation in the first column and in the row corresponding to σ. The entries of this row identify each of the six operations $*^\Box$, $\backslash^\Box$, $/^\Box$, $\circ^\Box$, $\backslash\backslash^\Box$, and $//^\Box$ with a corresponding operation in $\mathcal{O}$. For example, if $\sigma = (13)$, we have $\sigma \cdot (G, \cdot) = (G, \backslash)$. The entry in the third row and third column of the table tells us $/^\backslash = \backslash\backslash$; that is, for any $a, b \in G$, $a/^\backslash b = a\backslash\backslash b$.

	$*$	$\backslash$	$/$	$\circ$	$\backslash\backslash$	$//$
1	$*$	$\backslash$	$/$	$\circ$	$\backslash\backslash$	$//$
(1 2)	$\circ$	$\backslash\backslash$	$//$	$*$	$\backslash$	$/$
(1 3)	$\backslash$	$*$	$\backslash\backslash$	$//$	$/$	$\circ$
(2 3)	$/$	$//$	$*$	$\backslash\backslash$	$\circ$	$\backslash$
(1 2 3)	$//$	$/$	$\circ$	$\backslash$	$*$	$\backslash\backslash$
(1 3 2)	$\backslash\backslash$	$\circ$	$\backslash$	$/$	$//$	$*$

Table 1. Action of S_3 on $\mathcal{O}$

Conjugacy is particularly important in that it allows us to reduce further the number of implications among Bol-Moufang identities we need to consider. Extending the action of S_3 on $\mathcal{O}$ to an action on the set of all Bol-Moufang identities involving a single operation, we have the following:

Lemma 2.1. *Let I be an identity involving (only) one operation and J an identity involving a single (potentially different) operation. Then*

$$(I \Rightarrow J) \iff (\sigma \cdot I \Rightarrow \sigma \cdot J) \text{ for any } \sigma \in S_3.$$

Proof. Suppose $I \Rightarrow J$. If $\sigma \cdot I$ holds in some quasigroup $(G, *)$, then I holds in $\sigma^{-1}(G, *)$. Thus, J holds in $\sigma^{-1}(G, *)$, so $\sigma \cdot J$ holds in $(G, *)$. The proof of the reverse implication is similar. □

Corollary 2.2. *Any implication among identities of Bol-Moufang type is equivalent to one of the form $SUvw \Rightarrow LXab$.*

Proof. By Lemma 2.1, any implication whose premise $LUvw$ is equivalent, by application of the permutation $\sigma = (1\ 3)$, to an implication with premise $SUvw$. Similarly, any implication whose premise is $RUvw$ is equivalent, by application of (2 3), to an implication with premise $SUvw$. Now all implications of the form $SUvw \Rightarrow SXab$ have been determined by Phillips and Vojtěchovský [5], so it remains only to consider implications of the form $SUvw \Rightarrow LXab$ or $SUvw \Rightarrow RXab$. However, by applying (1 2), we see that the latter is equivalent to $S(Uvw)^\vee \Rightarrow L(Xab)^\vee$. □

A convenient summary of rules for converting implications is given in Table 2.

Before	After
$LUvw \Rightarrow SXab$	$SUvw \Rightarrow LXab$
$LUvw \Rightarrow LXab$	$SUvw \Rightarrow SXab$
$LUvw \Rightarrow RXab$	$SUvw \Rightarrow R(Xab)^\vee$
$RUvw \Rightarrow SXab$	$SUvw \Rightarrow RXab$
$RUvw \Rightarrow RXab$	$SUvw \Rightarrow SXab$
$RUvw \Rightarrow LXab$	$SUvw \Rightarrow L(Xab)^\vee$
$SUvw \Rightarrow RXab$	$S(Uvw)^\vee \Rightarrow L(Xab)^\vee$

Table 2. Conversion of implications

3. The main result

In this section we classify all valid implications among identities of Bol-Moufang type. By Corollary 2.2, we may restrict attention to implications of the form $SUvw \Rightarrow LXab$.

We will make heavy use of the Hasse diagram in Figure 1 which summarizes the results of [5]. Each node corresponds to a distinct variety of quasigroups defined by a single Bol-Moufang identity involving (only) the operation $*$. Inside the node is the abbreviated name of the variety, together with one identity which defines it. The full name of the variety corresponding to each abbreviation, together with the complete statement of the defining identity and what type of neutral element (2-sided, left, right, or none) exists, may be found in Table 5. The Hasse diagram is to be interpreted as follows: if there is a path from some variety to another variety on a lower level, then the upper variety is contained in the lower variety; that is, the identity defining the upper variety implies the one defining the lower variety. Note that by Proposition 2.1, there is a corresponding Hasse diagram for each of the other operations $\backslash$ and $/$.

For convenience, we say that an implication $SUvw \Rightarrow LXab$ is *irreducible* if whenever Vxy is an identity such that $SUvw \Rightarrow SVxy \Rightarrow LXab$, we must have $SUvw \Leftrightarrow SVxy$, and whenever Vxy is an identity such that $SUvw \Rightarrow LVxy \Rightarrow LXab$, we must have $LVxy \Leftrightarrow LXab$. It is clear that all valid implications may be constructed from a list of valid irreducible implications and the relevant Hasse diagram.

Theorem 3.1. *The only valid irreducible implications of the form* $SUvw \Rightarrow LXab$ *are* $SA25 \Rightarrow LB25$, $SB15 \Rightarrow LA35$, *and* $SC24 \Rightarrow LA35$.

Proof. We begin by arguing that all the implications described above are valid. Note first that in a loop both sides of the identity $LA35$: $(x\backslash x)\backslash(y\backslash z) = ((x\backslash x)\backslash y)\backslash z$ are equal to $y\backslash z$. Since $SB15$ and $SC24$ define varieties of loops, each of these implies $LA35$. From Table 2, $SA25 \Rightarrow LB25$ is equivalent to $SF14 \Rightarrow RE14$. The proof of the latter is rather lengthy and is deferred to Section 4.

We now show that no other irreducible implications hold. We begin by giving examples showing that the maximal identity $SA12$ in the Hasse diagram does not imply any minimal identity $LUvw$ when Uvw is equivalent to neither $A35$ nor $B25$. Observe that a quasigroup satisfying $SA12$ is necessarily a group. If $G = \mathbb{Z}_3 = \{e, a, b\}$ is a cyclic group of order 3 in which e denotes the neutral element and some identity $LUvw$ holds in G, then both sides of $LUvw$ must be equal when the element a is substituted for each of the variables x, y, and z. Now if $v = 1$, the left hand side of $LUvw$ is $a\backslash(a\backslash(a\backslash a)) = a\backslash(a\backslash e) = a\backslash b = a$. Similar computations show that if $v = 2, 3$, or 5 we obtain e and if $v = 4$ we obtain b. All this implies that the only identities $LUvw$ which could possibly hold in G are of form $LU23$, $LU25$ or $LU35$. Referencing Figure **??**, we are reduced to showing $SA12 \not\Rightarrow LUvw$ where $Uvw \in \{A23, E25, F25\}$. In fact, none of these three identities holds in S_3, the symmetric group on three letters: to show that $LA23$ does not hold, we take $x = z = (1\ 2)$, $y = (1\ 2\ 3)$, and to show that $LE25$ and $LF25$ do not hold we take $x = y = (1\ 2)$, $z = (1\ 2\ 3)$.

To show that $SB23$ does not imply $LB25$, we consider a nonassociative extra loop (i.e., a loop satisfying $SB23$) defined by Goodaire et. al. in [2]. We describe here a construction of this loop due to Chein [1]: given a group G, define $M(G, 2) = G\times\{0, 1\}$, where $(g, 0)(h, 0) = (gh, 0)$, $(g, 0)(h, 1) = (hg, 1)$, $(g, 1)(h, 0) = (gh^{-1}, 1)$

and $(g,1)(h,1) = (h^{-1}g, 0)$. For our counterexample, we consider $M(D_4, 2)$, where D_4 is the dihedral group of order 8 defined by generators R and F satsifying $R^4 = F^2 = 1$ and $RF = FR^{-1}$. Now define elements of $M(D_4, 2)$ by $x = (R, 1)$, $y = (R, 0)$ and $z = (F, 1)$; direct computation then shows that $LB25$ does not hold. The counterexamples associated to each of the remaining (potential) implications are described in Table 3. The entries in every third column correspond to quasigroups whose multiplication tables are catalogued in Section ; in each case below the counterexample is obtained by taking $x = y = z = 0$. □

Uvw	Xab	No.	Uvw	Xab	No.	Uvw	Xab	No.	Uvw	Xab	No.
A13	A35	3	F13	A35	1	A35	A35	10	A23	B25	6
A15	A35	5	F14	A35	1	B45	A35	2	B25	B25	9
A23	A35	6	F15	A35	8	C15	A35	2	F14	B25	1
A25	A35	7	F34	A35	1	C45	A35	4	F34	B25	1

Table 3. Table of counterexamples

By converting the implications of Theorem 3.1 using Table 2, one obtains a complete list of valid irreducible implications. The results are summarized below in Table 4; each box consists of logically equivalent implications.

$SA25 \Rightarrow LB25$	$LA25 \Rightarrow SB25$	$RA25 \Rightarrow LE14$
$SF14 \Rightarrow RE14$	$LF14 \Rightarrow RB25$	$RF14 \Rightarrow SE14$
$SB15 \Rightarrow LA35$	$LB15 \Rightarrow SA35$	$RB15 \Rightarrow LF13$
$SB15 \Rightarrow RF13$	$LB15 \Rightarrow RA35$	$RB15 \Rightarrow SF13$
$SC24 \Rightarrow LA35$	$LC24 \Rightarrow SA35$	$RC24 \Rightarrow LF13$
$SC24 \Rightarrow RF13$	$LC24 \Rightarrow RA35$	$RC24 \Rightarrow SF13$

Table 4. Valid irreducible implications

4. Proof of $SF14 \Rightarrow RE14$

In this section we give a proof that $SF14$ implies $RE14$, based on output from `Prover9`. Since $SF14$ has been shown to be equivalent to $SD14$ [5], we prove instead $SD14 \Rightarrow RE14$, as the output from `Prover9` is easier to parse. Although the proof is not particularly intuitive, it is short enough to be written out, and doing so ensures that all proofs in this article are “human" proofs.

For convenience, we write xy in place of $x * y$ and use juxtaposition notation to save parentheses. The notation $a \mapsto b$ (where a and b are formal expressions involving quasigroup elements and operations) means “substitute b for a".

We begin with the identity $SD14$:

$$(x \cdot yz)x = x(y \cdot zx).$$

This readily implies
$$(x \cdot yz)\backslash(x(y \cdot zx)) = x \tag{1}$$
and
$$[x(y \cdot zx)]/x = x \cdot yz. \tag{2}$$

On the other hand, substituting $y \mapsto y/z$ in SD14 gives
$$xy \cdot x = x(y/z \cdot zx). \tag{3}$$

By replacing $y \mapsto y/(zx)$ in (2), we have $(xy)/x = x[y/(zx) \cdot z]$. Substituting $y \mapsto x$ and $z \mapsto y$, we obtain
$$x = x \cdot (x/(yx))y \tag{4}$$
and dividing by x on the left yields
$$x\backslash x = (x/yx)y. \tag{5}$$

Returning to (1) and replacing $z \mapsto z/x$ we have $x = [x \cdot y(z/x)]\backslash[x \cdot y(z/x \cdot x)]$, which simplifies to
$$x = [x \cdot y(z/x)]\backslash[x \cdot yz]. \tag{6}$$

Replacing $y \mapsto x\backslash y$ in (3), we have
$$yx = x[(x\backslash y)/z \cdot zx]. \tag{7}$$

Putting $x \mapsto y/zy \cdot z$, $y \mapsto x$, and $z \mapsto y$ in (7), we have
$$x(y/zy \cdot z) = (y/zy \cdot z)[((y/zy \cdot z)\backslash x)/y \cdot y(y/zy \cdot z)]$$
which by (4) simplifies to $(y/zy \cdot z)[((y/zy \cdot z)\backslash x)/y \cdot y]] = x$. Thus $x = x(y/zy \cdot z) = x(y\backslash y)$ by (5), which establishes the existence of a right neutral element.

Using this we argue
$$[x/(y/z \cdot x)]y = z\backslash[z \cdot [x/(y/z \cdot x)]y] = [z \cdot (x\backslash x)]\backslash[z \cdot [x/(y/z \cdot x)]y].$$

Now using (5), the above may be written as
$$[z \cdot [x/(y/z \cdot x)](y/z)]\backslash[z \cdot [x/(y/z \cdot x)]y]$$
which by (6) reduces to z. Summarizing, we have
$$[x/(y/z \cdot x)]y = z. \tag{8}$$

Dividing this equation on the right by y on the right yields
$$x/(y/z \cdot x) = z/y, \tag{9}$$

and if instead we substitute $y \mapsto yz$, we obtain

$$x/yx \cdot yz = z. \tag{10}$$

Returning to (3) and substituting $z \mapsto z/(xz)$, we have $xy \cdot x = x(y/(z/xz) \cdot (z/xz)x)$. By (5), the right hand side reduces to $x(y/(z/xz) \cdot z\backslash z) = x(y/(z/xz))$. Thus, we have

$$x(y/(z/xz)) = xy \cdot x. \tag{11}$$

Using (11), (2), and (10) we reason

$$(y/zy)(zx \cdot z) = (y/zy)(z(x/(y/zy)) = (y/zy \cdot zx)/(y/zy) = x/(y/zy).$$

Thus we have

$$x/(y/zy) = (y/zy)(zx \cdot z). \tag{12}$$

We are finally ready to prove $RE14$. Applying (9), we have $(x/(y/z))/y = (x/[x/((z/y)x)])/y$, which by (12) equals $[(x/((z/y)x)) \cdot ((z/y)x) \cdot (z/y)]/y$. Using (3) we may rewrite this as $[(x/((z/y)x)) \cdot ((z/y) \cdot (x/w)(w \cdot (z/y)))]/y$, where for convenience we write $w = y/(z/y)$. By (10), the above expression reduces to $[(x/w) \cdot (w \cdot (z/y))]/y = [x/(y/(z/y)) \cdot y]/y = x/(y/(z/y)$, which establishes $RE14$. □

5. Counterexamples

1.

*	0	1	2
0	1	0	2
1	2	1	0
2	0	2	1

2.

*	0	1	2
0	1	0	2
1	0	2	1
2	2	1	0

3.

*	0	1	2	3	4	5	6	7	8
0	1	2	4	0	6	3	8	5	7
1	2	4	6	1	8	0	7	3	5
2	0	1	2	3	4	5	6	7	8
3	7	5	3	8	0	6	1	4	2
4	6	8	7	4	5	2	3	1	0
5	3	0	1	5	2	7	4	8	6
6	8	7	5	6	3	4	0	2	1
7	5	3	0	7	1	8	2	6	4
8	4	6	8	2	7	1	5	0	3

4.

*	0	1	2	3	4	5
0	1	2	4	0	5	3
1	2	0	5	1	3	4
2	0	1	3	2	4	5
3	4	5	2	3	0	1
4	5	3	0	4	1	2
5	3	4	1	5	2	0

5.

*	0	1	2	3	4
0	1	4	3	0	2
1	3	0	4	2	1
2	0	1	2	3	4
3	2	3	1	4	0
4	4	2	0	1	3

6.

*	0	1	2	3
0	1	0	3	2
1	2	3	0	1
2	0	1	2	3
3	3	2	1	0

7.

$*$	0	1	2	3	4	5
0	1	0	4	5	2	3
1	3	2	5	4	0	1
2	0	1	2	3	4	5
3	5	4	3	2	1	0
4	2	3	0	1	5	4
5	4	5	1	0	3	2

8.

$*$	0	1	2	3	4
0	1	2	4	3	0
1	3	0	2	4	1
2	0	4	3	1	2
3	4	1	0	2	3
4	2	3	1	0	4

9.

$*$	0	1	2	3	4	5
0	1	3	0	5	2	4
1	0	1	2	3	4	5
2	4	2	5	0	3	1
3	5	4	3	2	1	0
4	2	0	4	1	5	3
5	3	5	1	4	0	2

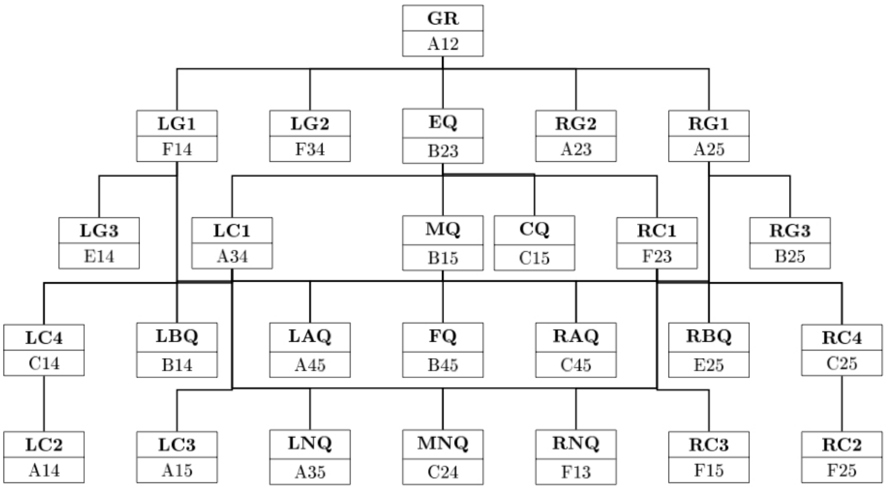
10.

$*$	0	1	2	3	4
0	1	3	0	4	2
1	0	1	2	3	4
2	4	0	1	2	3
3	2	4	3	1	0
4	3	2	4	0	1

Variety	Abbrev.	Defining identity	Name	Neutral elt.
Groups	GR	$x(yz) = (xy)z$	$A12$	2
RG1-quasigroups	RG1	$x((xy)z) = ((xx)y)z$	$A25$	L
LG1-quasigroups	LG1	$x(y(zz)) = (x(yz))z$	$F14$	R
RG2-quasigroups	RG2	$x(x(yz)) = (xx)(yz)$	$A23$	L
LG2-quasigroups	LG2	$(xy)(zz) = (x(yz))z$	$F34$	R
RG3-quasigroups	RG3	$x((yx)z) = ((xy)x)z$	$B25$	L
LG3-quasigroups	LG3	$x(y(zy)) = (x(yz))y$	$E14$	R
Extra q.	EQ	$x((yx)z) = (xy)(xz)$	$B23$	2
Moufang q.	MQ	$x(y(xz)) = ((xy)x)z$	$B15$	2
Left Bol q.	LBQ	$x(y(xz)) = (x(yx))z$	$B14$	R
Right Bol q.	RBQ	$x((yz)y) = ((xy)z)y$	$E25$	L
C-quasigroups	CQ	$x(y(yz)) = ((xy)y)z$	$C15$	0
LC1-quasigroups	LC1	$(xx)(yz) = (x(xy))z$	$A34$	2
LC2-quasigroups	LC2	$x(x(yz)) = (x(xy))z$	$A14$	0
LC3-quasigroups	LC3	$x(x(yz)) = ((xx)y)z$	$A15$	L
LC4-quasigroups	LC4	$x(y(yz)) = (x(yy))z$	$C14$	R
RC1-quasigroups	RC1	$x((yz)z) = (xy)(zz)$	$F23$	2
RC2-quasigroups	RC2	$x((yz)z) = ((xy)z)z$	$F25$	0
RC3-quasigroups	RC3	$x(y(zz)) = ((xy)z)z$	$F15$	R
RC4-quasigroups	RC4	$x((yy)z) = ((xy)y)z$	$C25$	L
Left alternative q.	LAQ	$x(xy) = (xx)y$	$A45$	L
Right alternative q.	RAQ	$x(yy) = (xy)y$	$C45$	R
Flexible q.	FQ	$x(yx) = (xy)x$	$B45$	0
Left nuclear q.	LNQ	$(xx)(yz) = ((xx)y)z$	$A35$	L
Middle nuclear q.	MNQ	$x((yy)z) = (x(yy))z$	$C24$	2
Right nuclear q.	RNQ	$x(y(zz)) = (xy)(zz)$	$F13$	R

Table 5. Definitions of varieties of quasigroups

Figure 1. Varieties of Bol-Moufang type under $*$



Acknowledgements. The authors would like to thank the NSA, the NSF, and Miami University for their financial support and also J.D. Phillips for several helpful suggestions. The authors also thank the referee for numerous improvements to this article.

References

[1] **O. Chein**, *Moufang loops of small order*, Mem. AMS **13** (1978), no. 197.

[2] **E. G. Goodaire, S. May, M. Raman**, *The Moufang loops of order less than 64.* Nova Science Publishers, 1999.

[3] **W. McCune**, *Prover9, equational reasoning tool and Mace4, finite model builder.* Available at `http://www.cs.unm.edu/~mccune/mace4/`.

[4] **J. D. Phillips and P. Vojtěchovský**, *The varieties of loops of Bol-Moufang type.* Algebra Universalis **54** (2005), no. 3, 259 – 271.

[5] **J. D. Phillips and P. Vojtěchovský**, *The varieties of quasigroups of Bol-Moufang type: An equational reasoning approach.* J. Algebra **293** (2005), 17 – 33.

[6] **J. D. H. Smith**, *An Introduction to quasigroups and their representations*, Studies in Advanced Mathematics, Chapman and Hall, 2007.

Received January 31, 2012.

Department of Mathematics, Miami University, Oxford, OH 45056, USA
E-mail: reza@calico.mth.muohio.edu

Quasigroups and Related Systems **20** (2012), 11 − 20

On WIP loops

Asif Ali and *Hasib Khan*

Abstract. A weak inverse property loop (WIP loop) is a loop L that satisfies $x(yx)^\rho = y^\rho$ or $(xy)^\lambda x = y^\lambda$ for all $x, y \in L$. In this paper we prove some necessary and sufficient conditions for a WIP loop to be LC, RC, left alternative, right alternative, and C-loop. We also construct infinite families of WIP loops of various orders.

1. Introduction

Let L be a loop with identity element 1, then L will be said to satisfy the *weak inverse property* if whenever three elements x, y, z of L satisfy the relation $xy{\cdot}z = 1$, they also satisfy the relation $x \cdot yz = 1$. The study of weak inverse property loops (WIP loops) was initiated by J. M. Osborn [4] as a class of loops which contains both IP loops and CIP loops. He proved that a WIP loop is a loop which satisfies one of the following equivalent identities

$$x(yx)^\rho = y \quad \text{or} \quad (xy)^\lambda x = y^\lambda.$$

He further proved that the left, middle and right nuclei of a WIP loop coincide. If L is a loop all of whose isotopes have the WIP and N is its nucleus, then N is normal and L/N is a Moufang loop. Isotopy-isomorphy conditions of WIP loops were considered in [2]. We prove some necessary and sufficient conditions for a WIP loop to be LC, RC, left alternative, right alternative, and C-loop in section 3 and construct infinite families of WIP loops of various orders in section 4.

2. Preliminaries

Let L be a loop. Then the sets

$$N_\lambda = \{x \in L \,:\, x(yz) = (xy)z \text{ for every } y, z \in L\},$$
$$N_\mu = \{x \in L \,:\, y(xz) = (yx)z \text{ for every } y, z \in L\},$$
$$N_\rho = \{x \in L \,:\, y(zx) = (yz)x \text{ for every } y, z \in L\}$$

are called the *left nucleus*, *middle nucleus* and *right nucleus* respectively. $N = N_\lambda \cap N_\mu \cap N_\rho$ is called the *nucleus.*

2010 Mathematics Subject Classification: 20N99
Keywords: WIP loop, C-loop, ARIF loop, LC-loop, RC-loop

A loop L is called *left alternative* if $xx \cdot y = x \cdot xy \; \forall x, y \in L$, *right alternative* if $x \cdot yy = xy \cdot y \; \forall x, y \in L$, and *alternative* if it is both left alternative and right alternative.

C-loops are loops satisfying the identity $x(y(yz)) = ((xy)y)z$. Loops satisfying the identity $(xx)(yz)) = (x(xy))z$ are called *LC-loops* and loops satisfying the identity $(xy)(yz) = x(y(zz))z$ are called *RC-loops*. Loops which are both LC-loops and RC-loops are C-loops. *ARIF* loops are defined to be flexible loops satisfying $(zx)(yxy) = (z(xyx))y$.

3. Necessary and sufficient conditions

LC-loops, RC-loops, C-loops, ARIF loops are subclasses of WIP loop. We prove here necessary and sufficient conditions for a WIP loop to satisfy these loops which are its subclasses. We define $L_x : a \longrightarrow xa$, $R_x : a \longrightarrow ax$, $J : x \longrightarrow x^{-1}$ and $P = R_x \circ L_x \; \forall \; x \in L$.

Theorem 3.1. *Let L be a WIP loop of unique inverses. Then $(JP)^n = I$ for any $n \in 2\mathbb{Z}^+$, where $\mathbb{Z}^+$ denotes the set of positive integers.*

Proof. Let $y \in L$. Since $P = R_x \circ L_x$, then for $(JP)^n = I$, where $n \in 2Z^+$. Consider $n = 2$. Then

$$y(JP)^2 = yJPJP = x((x(y^{-1}x))^{-1}x) = x(y^{-1}x)^{-1} = y.$$

Thus $(JP)^2 = I$. Now if any $n \in 2\mathbb{Z}^+$, then $n = 2m$ for some $m \in \mathbb{Z}^+$, so $(JP)^n = (JP)^{2m} = ((JP)^2)^m = (I)^m = I$. $\square$

Corollary 3.2. *$(JP)^n = I$ for all $n \in \mathbb{Z}^+$ if the loop is a WIP loop of exponent 2.*

Proof. Let L be a WIP loop of exponent 2. Then

$$\begin{aligned} y(JP) &= y^{-1}R_x \circ L_x = x(y^{-1}x) \\ &= x(y^{-1}x)^{-1} \qquad \text{since } L \text{ is of exponent } 2 \\ &= y^{-1} \qquad \text{since } L \text{ is a WIP loop} \\ &= y. \end{aligned}$$

Thus $JP = I$ and hence $(JP)^n = I$ for all $n \in \mathbb{Z}^+$ if the loop is a WIP loop of exponent 2. $\square$

Next we prove necessary and sufficient conditions for a WIP loop to be left alternative, and right alternative.

Theorem 3.3. *Let L be a WIP loop. Then L is left alternative if and only if $L_x = R_x J L_{x^2} JP$.*

Proof. Let L be a WIP loop satisfying $L_x = R_x J L_{x^2} JP$. Then

$$\begin{aligned} L_x &= R_x J L_{x^2} JP \\ JR_x^{-1}J &= R_x J L_{x^2} JP \quad \text{since } L_x = JR_x^{-1}J \\ R_x^{-1}J &= L_x^{-1} L_{x^2} JP \quad \text{since } L_x^{-1} = JR_xJ \\ L_x R_x^{-1} P &= L_{x^2}(JP)^2 \\ L_x R_x^{-1} R_x L_x &= L_{x^2} I \quad \text{by Theorem 3.1} \\ L_x L_x &= L_{x^2} \end{aligned}$$

Conversely, if is $x(xy) = x^2y$ for all $x, y \in L$, then $L_xL_x = L_{x^2}$ for all $x \in$L. Thus $L_xL_xP^{-1} = L_{x^2}P^{-1}$. From this, by Theorem 3.1, we obtain $L_xR_x^{-1} = L_{x^2}(JP)^2P^{-1}$, i.e., $R_x^{-1} = L_x^{-1}L_{x^2}JPJ$. The last, by left and right cancellation of J, implies $L_x = R_xJL_{x^2}JP$. □

Theorem 3.4. *Let L be a WIP loop. Then L is right alternative if and only if* $R_x = PJR_{x^2}JL_x$.

Proof. If L satisfies $R_x = PJR_{x^2}JL_x$, then

$$\begin{aligned} &JR_xJ = JPJR_{x^2}JL_xJ \quad \text{by multiplication of both sides by J} \\ &PL_x^{-1} = PJPJR_{x^2}R_x^{-1} \quad \text{by multiplication of both sides by P} \\ &R_xR_x = R_{x^2}. \end{aligned}$$

Conversely, let L be right alternative. Then $R_xR_x = R_{x^2}$. Hence $P^{-1}R_xR_x = P^{-1}R_{x^2}$. Thus $L_x^{-1}IR_x = P^{-1}R_{x^2}$, which implies $L_x^{-1}R_x = IP^{-1}R_{x^2}$, and consequently $R_x = PJR_{x^2}JL_x$. □

Theorem 3.5. *A WIP loop L is an LC loop if and only if it satisfies the identity* $JL_{x^2}T_z = L_zT_xJPL_z$, *where* $T_x = R_x^{-1}L_x$.

Proof. Let L be an LC loop. Then $xx \cdot yz = (x \cdot xy)z$, which implies $R_zL_{x^2} = L_xL_xR_z$. Thus $R_zL_{x^2}T_z = L_xL_xR_zT_z$, whence, putting $L_x^{-1} = JR_xJ$, we obtain $JL_{x^2}T_z = L_zR_x^{-1}L_xJR_xJJL_xL_z$. Thus $JL_{x^2}T_z = L_zT_xJPL_z$.

Conversely, if L satisfies $JL_{x^2}T_z = L_zT_xJPL_z$, then also $JR_zL_{x^2}R_z^{-1} = T_xJP$, which implies $R_zL_{x^2} = L_xL_xR_z$. Hence, L is an LC loop. □

Theorem 3.6. [2, Theorem 4.2]
A loop L (WIP loop) is a C-Loop if and only if $R_x = PJR_{x^2}JL_x$ *and* $JL_{x^2}T_z = L_zT_xJPL_z$. □

4. Various constructions of WIP loops

Here we give the construction of infinite families of non-associative WIP loops by extensions of loops.

Lemma 4.1. *Let $\mu : G \times G \to A$ be a factor set. Then (G, A, μ) is a WIP loop if and only if*

$$\mu(h, h^{-1}) + \mu(g, g^{-1}h^{-1}) = \mu(h, g) + \mu(hg, g^{-1}h^{-1}) \tag{D}$$

for all $g, h \in G$.

Proof. The loop (G, A, μ) is a WIP loop iff $(g, a)[(h, b)(g, a)]^{-1} = (h, b)^{-1}$ hold for every $g, h \in G$ and every $a, b \in A$. Straight forward calculation with (A) shows that this happens iff (D) holds. □

We call a factor set μ satisfies (A) and (D) a *W-factor set.* We now use a particular W-factor set to construct the above-mentioned families of WIP loops.

Proposition 4.2. *Let $n \geqslant 2$ be an integer and let A be an abelian group of order n with respect to addition with neutral element 0 and $\alpha \in A$ be an element of order bigger than 2. Let $G = \{1, x, x^2\}$ be the cyclic group of order 3 with respect to multiplication with neutral element 1. Define $\mu : G \times G \to A$ by*

$$\mu(h, g) = \begin{cases} \alpha & \textit{if} \ \ (h, g) = (x, x), \\ 0 & \textit{otherwise.} \end{cases}$$

Then (G, A, μ) is a non-alternative (hence non-associative) commutative WIP loop with $N = \{(1, a) : a \in A\}$.

Proof. The map μ is clearly a factor set. To show that (G, A, μ) is a WIP loop, we verify (D). Since μ is a factor set, there is nothing to prove when $g = 1$. Assume that $g = x$. Then (D) becomes $\mu(h, h^{-1}) + \mu(x, x^2h^{-1}) = \mu(h, x) + \mu(x, x^2h^{-1})$. If $h = 1$, then $\mu(1, 1) + \mu(x, x^2) = \mu(1, x) + \mu(x, x^2)$ and both sides of this equation are equal to 0. If $h = x$, then$\mu(x, x^2) + \mu(x, x) = \mu(x, x) + \mu(x, x)$ and both sides of this equation are equal to α. Assume $h = x^2$, then $\mu(x^2, x) + \mu(x, 1) = \mu(x^2, x) + \mu(1, xx)$ and both sides of this equation are equal to 0. Next assume that $g = x^2$, then (D) becomes $\mu(h, h^{-1}) + \mu(x^2, xh^{-1}) = \mu(h, x^2) + \mu(hx^2, xh^{-1})$. If $h = 1$, then both sides of this equation are equal to 0. Assume $h = x$, then both sides of this equation are equal to 0, Assume $h = x^2$, then $\mu(x^2, x) + \mu(x^2, x^2) = \mu(x^2, x^2) + \mu(x, x^2)$ and both sides of this equation are equal to 0. Since $\alpha \neq 0$, we have that, $(x, a)(x, a) \cdot (x^2, a) \neq (x, a) \cdot (x, a)(x^2, a)$. Thus (G, A, μ) is non-alternative and hence non-associative. Also neither $(x, a) \in N$ nor $(x^2, a) \in N$ for all $a \in A$. Also we have that $(1, a)((h, b)(g, c)) = ((1, a)(h, b))(g, c)$ for all $h, g \in G$ and $a, b, c \in A$. Which implies that $(1, a)$ belongs to nucleus. Thus $\{(1, a); a \in A\}$ is the nucleus of the loop (G, A, μ). □

Corollary 4.3. *For each natural number n there exists a non-alternative commutative WIP loop having nucleus of order n.*

Proof. It remains to show that there exist non-alternative commutative WIP loop having nucleus of order 1. This requirement is fulfilled by the following example. □

Example 4.4. A commutative, non-alternative WIP loop of order 10 having trivial nucleus.

$\cdot$	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	0	3	2	5	4	8	9	6	7
2	2	3	0	1	6	7	4	5	9	8
3	3	2	1	0	8	9	7	6	4	5
4	4	5	6	8	1	0	9	2	7	3
5	5	4	7	9	0	1	2	8	3	6
6	6	8	4	7	9	2	3	0	5	1
7	7	9	5	6	2	8	0	3	1	4
8	8	6	9	4	7	3	5	1	2	0
9	9	7	8	5	3	6	1	4	0	2

Example 4.5. The smallest group A satisfying the assumption of Proposition 4.2 is the cyclic group $\{0,1\}$ of order 2. The construction of Proposition 4.2 with $\alpha = 1$ yields the smallest non-alternative commutative WIP loop of order 6.

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	3	2	5	4
2	2	3	5	4	0	1
3	3	2	4	5	1	0
4	4	5	0	1	2	3
5	5	4	1	0	3	2

Proposition 4.6. *Let $n \geqslant 3$ be an integer and let A be an abelian group of order n with respect to addition with neutral element 0 and $\alpha \in A$ be an element of order bigger than 2. Let $G = \{1, u, v, w\}$ denotes the Klein group with respect to multiplication with neutral element 1. Define $\mu : G \times G \to A$ by*

$$\mu(x,y) = \begin{cases} \alpha & \text{if } \ (x,y) \in \{(u,v),(v,w),(w,u)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then (G, A, μ) is a non-alternative, non-commutative WIP loop with nucleus $N = \{(1,a) \,:\, a \in A\}$.

Proof. The map μ is clearly a factor set. To show that (G, A, μ) is a WIP loop, we verify (D). Since μ is a factor set, there is nothing to prove when $g = 1$. Assume that $g = u$, then (D) becomes $\mu(h, h^{-1}) + \mu(u, uh^{-1}) = \mu(h, u) + \mu(hu, uh^{-1})$. If $h = 1$, then both sides of this equation are equal to 0. Assume $h = v$, then $\mu(v,v) + \mu(u,w) = \mu(v,u) + \mu(w,w)$ and both sides of this equation are equal to 0. Assume $h = w$, then $\mu(w,w) + \mu(u,v) = \mu(w,u) + \mu(v,v)$ and both sides of this equation are equal to α. Next assume that $g = v$, then (D) becomes $\mu(h, h^{-1}) + \mu(v, vh^{-1}) = \mu(h, v) + \mu(hv, vh^{-1})$. If $h = 1$, then both sides of this equation are equal to 0. Assume $h = u$, $\mu(u,u) + \mu(v,w) = \mu(u,v) + \mu(w,w)$ and both sides of this equation are equal to α. Assume $h = v$, then $\mu(v,v) + \mu(v,1) =$

$\mu(v,v)+\mu(1,1)$ both sides of this equation are equal to 0. Assume $h=w$, then $\mu(w,w)+\mu(v,u)=\mu(w,v)+\mu(u,u)$ and both sides of this equation are equal to 0. Next assume that $g=w$, then (D) becomes $\mu(h,h^{-1})+\mu(w,wh^{-1})=\mu(h,w)+\mu(hw,wh^{-1})$. If $h=1$, then both sides of this equation are equal to 0. Assume $h=u$, then this equation is equal to $\mu(u,u)+\mu(w,v)=\mu(u,w)+\mu(v,v)$ and both sides of this equation are equal to 0. Assume $h=v$, then $\mu(v,v)+\mu(w,u)=\mu(v,w)+\mu(u,u)$ and both sides of this equation are equal to α. Assume $h=w$, then $\mu(w,w)+\mu(w,1)=\mu(w,w)+\mu(1,1)$ and both sides of this equation are equal to 0. Since $\alpha\neq 0$, and we have that, $(u,a)(u,a)\cdot(v,a)\neq(u,a)\cdot(u,a)(v,a)$ also we have that, $(w,a)(u,a)\cdot(u,a)\neq(w,a)\cdot(u,a)(u,a)$. Thus (G,A,μ) is non-alternative and hence non-associative. Also $(u,a),(v,a),(w,a)\notin N$ for all $a\in A$. Also we have that $(1,a)((h,b)(g,c))=((1,a)(h,b))(g,c)$ for all $h,g\in G$ and $a,b,c\in A$. Which implies that $(1,a)$ belongs to the nucleus. Thus $\{(1,a):a\in A\}$ is the nucleus of the loop (G,A,μ). □

Corollary 4.7. *For each $n\geqslant 1$ there exists a non-alternative non-commutative WIP loop having nucleus of order n.*

Proof. It remains to show that there exist a non-alternative non-commutative WIP loop having nuclei of order 1 and 2. The first requirement follows by Example 4.8 while the second requirement follows by Example 4.9. □

Example 4.8. A non-alternative non-commutative WIP loop having nucleus of order 1.

$\cdot$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	4	6	2	7	3	5
2	2	7	5	0	3	1	4	6
3	3	5	0	4	6	2	7	1
4	4	6	3	1	7	0	5	2
5	5	3	7	2	0	6	1	4
6	6	4	1	7	5	3	2	0
7	7	2	6	5	1	4	0	3

Example 4.9. A non-alternative non-commutative WIP loop having nucleus of order 2.

$\cdot$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	0	5	6	4	3
2	2	0	1	6	5	3	4
3	3	6	5	4	0	1	2
4	4	5	6	0	3	2	1
5	5	3	4	2	1	6	0
6	6	4	3	1	2	0	5

Example 4.10. The smallest group A satisfying the assumption of Proposition 4.6 is the cyclic group $\{0, 1, 2\}$. The construction of Proposition 4.6 with $\alpha = 1$ yields the smallest non-alternative commutative WIP loop of order 12.

$\cdot$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	0	4	5	3	7	8	6	10	11	9
2	2	0	1	5	3	4	8	6	7	11	9	10
3	3	4	5	0	1	2	11	9	10	6	7	8
4	4	5	3	1	2	0	9	10	11	7	8	6
5	5	3	4	2	0	1	10	11	9	8	6	7
6	6	7	8	9	10	11	0	1	2	5	3	4
7	7	8	6	10	11	9	1	2	0	3	4	5
8	8	6	7	11	9	10	2	0	1	4	5	3
9	9	10	11	8	6	7	3	4	5	0	1	2
10	10	11	9	6	7	8	4	5	3	1	2	0
11	11	9	10	7	8	6	5	3	4	2	0	1

GAP gives these extra informations about the above WIP loop. It is (1) power associative, (2) not Moufang, (3) neither automorphic nor anti-automorphic, (4) neither left nor right Bol.

Proposition 4.11. *Let $n \geqslant 3$ be an integer and let A be an abelian group of order n with respect to addition with neutral element 0 and $\alpha \in A$ be an element of order bigger than 2. Let $G = \{1, u, v, w\}$ denotes the Klein group with respect to multiplication with neutral element 1. Define $\mu : G \times G \to A$ by*

$$\mu(x, y) = \begin{cases} \alpha & \textit{if } (x, y) \in \{(u, v), (v, u), (u, w), (w, u), (v, w), (w, v)\}, \\ 0 & \textit{otherwise.} \end{cases}$$

Then (G, A, μ) is a non-alternative, commutative WIP loop with nucleus $N = \{(1, a) : a \in A\}$.

Proof. The map μ is clearly a factor set. To show that (G, A, μ) is a WIP loop, we verify (D). Since μ is a factor set, there is nothing to prove when $g = 1$. Assume that $g = u$, then (D) becomes $\mu(h, h^{-1}) + \mu(u, uh^{-1}) = \mu(h, u) + \mu(hu, uh^{-1})$. If $h = 1$, then $\mu(h, h^{-1}) + \mu(u, u) = \mu(1, u) + \mu(u, u)$ both sides of this equation are equal to 0. Assume $h = u$ then $\mu(u, u) + \mu(u, 1) = \mu(u, u) + \mu(1, 1)$ both sides of this equation are equal to 0. Assume $h = v$, then $\mu(v, v) + \mu(u, w) = \mu(v, u) + \mu(w, w)$ and both sides of this equation are equal to α. Assume $h = w$, then $\mu(w, w) + \mu(u, v) = \mu(w, u) + \mu(v, v)$ and both sides of this equation are equal to α. Next assume that $g = v$, then (D) becomes $\mu(h, h^{-1}) + \mu(v, vh^{-1}) = \mu(h, v) + \mu(hv, vh^{-1})$. If $h = 1$, then $\mu(1, 1) + \mu(v, v) = \mu(1, v) + \mu(v, v)$ and both sides of this equation are equal to 0. Assume $h = u$, then $\mu(u, u) + \mu(v, w) = \mu(u, v) + \mu(w, w)$ and both sides of this equation are equal to α. Assume $h = v$, then $\mu(v, v) + \mu(v, 1) = \mu(v, v) + \mu(1, 1)$ both sides of this equation are equal to 0. Assume $h = w$, then $\mu(w, w) + \mu(v, u) = \mu(w, v) + \mu(u, u)$ and both sides of this equation are equal to α. Next assume that $g = w$, then (D) becomes $\mu(h, h^{-1}) + \mu(w, wh^{-1}) = \mu(h, w) + \mu(hw, wh^{-1})$. If $h = 1$, then $\mu(1, 1) + \mu(w, w) = \mu(1, w) + \mu(w, w)$ both sides of this equation are equal to 0. Assume $h = u$, then $\mu(u, u) + \mu(w, v) = \mu(u, w) + \mu(v, v)$ and both sides of this equation are equal to α. Assume $h = v$, then $\mu(v, v) + \mu(w, u) = \mu(v, w) + \mu(u, u)$ and both sides of this equation are equal to α. Assume $h = w$, then $\mu(w, w) + \mu(w, 1) = \mu(w, w) + \mu(1, 1)$ and both sides of this equation are equal to 0. Since $\alpha \neq 0$, and we have that, $(u, a)(u, a) \cdot (v, a) \neq (u, a) \cdot (u, a)(v, a)$. Also $(w, a)(u, a) \cdot (u, a) \neq (w, a) \cdot (u, a)(u, a)$. Thus (G, A, μ) is non-alternative and hence non-associative. Also $(u, a), (v, a), (w, a) \notin N$ for all $a \in A$. Also we have that $(1, a)((h, b)(g, c)) = ((1, a)(h, b))(g, c)$ for all $h, g \in G$ and $a, b, c \in A$. Which implies that $(1, a)$ belongs to the nucleus. Thus $\{(1, a) : a \in A\}$ is the nucleus of the loop (G, A, μ). $\square$

Example 4.12. The smallest group A satisfying the assumption of Proposition 4.11 is the cyclic group $\{0, 1, 2\}$. The construction of Proposition 4.11 with $\alpha = 1$ then yields the smallest non-alternative commutative WIP loop of order 12.

$\cdot$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	0	4	5	3	7	8	6	10	11	9
2	2	0	1	5	3	4	8	6	7	11	9	10
3	3	4	5	0	1	2	11	9	10	8	6	7
4	4	5	3	1	2	0	9	10	11	6	7	8
5	5	3	4	2	0	1	10	11	9	7	8	6
6	6	7	8	11	9	10	0	1	2	5	3	4
7	7	8	6	9	10	11	1	2	0	3	4	5
8	8	6	7	10	11	9	2	0	1	4	5	3
9	9	10	11	8	6	7	5	3	4	0	1	2
10	10	11	9	6	7	8	3	4	5	1	2	0
11	11	9	10	7	8	6	4	5	3	2	0	1

GAP [3] gives these extra informations about the above WIP loop. It is (1) power associative, (2) not automorphic inverse property loop, (2) neither LC-loop nor RC-loop.

Proposition 4.13. *Let $n \geqslant 2$ be an integer and let A be an abelian group of order n with respect to addition with neutral element 0 and $\alpha \in A$ be an element of order bigger than 2. Let $G = \{1, x, x^2, x^3, x^4\}$ be the Cyclic group of order 5 with respect to multiplication with neutral element 1. Define $\mu : G \times G \to A$ by*

$$\mu(h, g) = \begin{cases} \alpha & \text{if } (h, g) \in \{(x^2, x^2), (x, x^2), (x^2, x)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then (G, A, μ) is a non-alternative commutative WIP loop with nucleus $N = \{(1, a) : a \in A\}$.

Proof. The map μ is clearly a factor set. To show that (G, A, μ) is a WIP loop, we verify (D). Since μ is a factor set, there is nothing to prove when $g = 1$. Assume that $g = x$, then (D) becomes $\mu(h, h^{-1}) + \mu(x, x^4h^{-1}) = \mu(h, x) + \mu(hx, x^4h^{-1})$. If $h = 1$, then $\mu(h, h^{-1}) + \mu(x, x^4h^{-1}) = \mu(h, x) + \mu(hx, x^4h^{-1})$ and both sides of this equation equals to 0. $h = x$, then $\mu(x, x^4) + \mu(x, x^3) = \mu(x, x) + \mu(x^2, x^3)$ then both sides of this equation are equal to 0,Assume $h = x^2$, then $\mu(x^2, x^3) + \mu(x, x^2) = \mu(x^2, x) + \mu(x^3, x^2)$ and both sides of this equation are equal to α. Assume $h = x^3$, then $\mu(x^3, x^2) + \mu(x, x) = \mu(x^3, x) + \mu(x^4, x)$ and both sides of this equation are equal to 0 Assume $h = x^4$, then $\mu(x^4, x) + \mu(x, 1) = \mu(x^4, x) + \mu(1, 1)$and both sides of this equation are equal to 0 assume that $g = x^2$, then (D) becomes $\mu(h, h^{-1}) + \mu(x^2, x^3h^{-1}) = \mu(h, x^2) + \mu(hx^2, x^3h^{-1})$. If $h = 1$, then $\mu(1, 1) + \mu(x^2, x^3) = \mu(1, x^2) + \mu(x^2, x^3)$ and both sides of this equation equals to 0. Assume $h = x$, then $\mu(x, x^4) + \mu(x^2, x^2) = \mu(x, x^2) + \mu(x^3, x^2)$ then both sides of this equation are equal to α, Assume $h = x^2$, then $\mu(x^2, x^3) + \mu(x^2, x) = \mu(x^2, x^2) + \mu(x^4, x)$ and both sides of this equation are equal to α. Assume $h = x^3$, then $\mu(x^3, x^2) + \mu(x^2, 1) = \mu(x^3, x^2) + \mu(1, 1)$ and both sides of this equation are equal to 0. Assume $h = x^4$, then $\mu(x^4, x) + \mu(x^2, x^4) = \mu(x^4, x^2) + \mu(x, x^4)$ and both sides of this equation are equal to 0. Assume that $g = x^3$, then $\mu(h, h^{-1}) + \mu(x^3, x^2h^{-1}) = \mu(h, x^3) + \mu(hx^3, x^2h^{-1})$. If $h = 1$, then $\mu(1, 1) + \mu(x^3, x^2) = \mu(1, x^3) + \mu(x^3, x^2)$ and both sides of this equation equals to 0. Assume $h = x$, then this equation equals to $\mu(x, x^4) + \mu(x^3, x) = \mu(x, x^3) + \mu(x^4, x)$ then both sides of this equation are equal to 0, Assume $h = x^2$, then $\mu(x^2, x^3) + \mu(x^3, 1) = \mu(x^2, x^3) + \mu(1, 1)$ and both sides of this equation are equal to 0. Assume $h = x^3$, then $\mu(x^3, x^2) + \mu(x^3, x^4) = \mu(x^3, x^3) + \mu(x, x^4)$ and both sides of this equation are equal to 0. Assume $h = x^4$, then $\mu(x^4, x) + \mu(x^3, x^3) = \mu(x^4, x^3) + \mu(x^2, x^3)$ and both sides of this equation are equal to 0, Assume that $g = x^4$, then (D) becomes $\mu(h, h^{-1}) + \mu(x^4, xh^{-1}) = \mu(h, x^4) + \mu(hx^4, xh^{-1})$. If $h = 1$, then $\mu(1, 1) + \mu(x^4, x) = \mu(1, x^4) + \mu(x^4, x)$ both sides of this equation equals to 0. Assume $h = x$, then $\mu(x, x4) + \mu(x^4, 1) = \mu(x, x^4) + \mu(1, 1)$ and both sides of this equation are equal to 0, Assume $h = x^2$, then $\mu(x^2, x^3) + \mu(x^4, x^4) = \mu(x^2, x^4) + \mu(x, x^4)$ and both sides of this equation are equal to 0. Assume $h = x^3$, then $\mu(x^3, x^2) + \mu(x^3, x^4) = \mu(x^3, x^3) + \mu(x, x^4)$ and both sides of this equation are equal to 0. Assume $h = x^4$, then $\mu(x^4, x) + \mu(x^4, x^2) = \mu(x^4, x^4) + \mu(x^3, x^2)$ and both sides of this equation are equal to 0. Since $\alpha \neq 0$, we have that, $(x^3, a) \cdot (x^2, a)(x^2, a) \neq (x3, a)(x^2, a) \cdot (x^2, a)$. Also $(x^2, a) \cdot (x, a)(x^3, a) \neq (x, 3a + \alpha) = (x^2, a)(x, a) \cdot (x^3, a)$. Thus (G, A, μ) is non-alternative and hence non-associative WIP loop. Also neither $(x, a), (x^2, a), (x^3, a) \in N$ for all $a \in A$. Similarly $(x^4, a) \notin A$. Also we have that $(1, a)((h, b)(g, c)) = ((1, a)(h, b))(g, c)$ for all $h, g \in G$ and $a, b, c \in A$. Which implies that $(1, a)$ belongs to the nucleus. Thus $\{(1, a);\, a \in A\}$ is the nucleus of the loop (G, A, μ). □

Example 4.14. The smallest group A satisfying the assumption of Proposition 4.13 is the cyclic group $\{0, 1, 2\}$ of order 3. The construction of Proposition 4.13 with $\alpha = 1$ yields the smallest non-alternative commutative WIP loop of order 10.

·	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	0	3	2	5	4	7	6	9	8
2	2	3	4	5	7	6	8	9	0	1
3	3	2	5	4	6	7	9	8	1	0
4	4	5	7	6	9	8	0	1	2	3
5	5	4	6	7	8	9	1	0	3	2
6	6	7	8	9	0	1	2	3	4	5
7	7	6	9	8	1	0	3	2	5	4
8	8	9	0	1	2	3	4	5	6	7
9	9	8	1	0	3	2	5	4	7	6

GAP shows that the following properties do not hold in this WIP loop: (1) automorphic inverse property, (2) anti-automorphic inverse property, (3) LC, (4) RC, (5) left Bol, (6) right Bol, (7) Moufang, (8) power alternative, (9) power associative, (10) left nuclear square, (13) right nuclear square, (14) left inverse and (15) right inverse property.

References

[1] **T. G. Jaiyeola and J. O. Adeniran**, *Weak Inverse Property Loops and some Isotopy-Isomorphy Properties*, Acta Univ. Apulensis Math. Inform. **18** (2009), 19−33.

[2] **T. G. Jaiyeola and J. O. Adeniran**, *On central loops and central square property*, Quasigroups and Related Systems **15** (2007), 191 − 200.

[3] **G. P. Nagy and P. Vojtechovsky**, *LOOPS: Computing with quasigroups and loops in GAP, version* 1.0.0, *computational package for GAP*, http://www.math.du.edu/loop.

[4] **J. M. Osborn**, *Loops with the weak inverse property*, Pac. J. Math. **10** (1961), 295 − 304.

[5] **J. D. Philips and P. Vojtěchovský**, *C-loops: an introduction.*, Publ. Math. (Debrecen) **68** (2006), 115 − 137.

Received February 4, 2011

Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan.
E-mail: dr_asif_ali@hotmail.com (A.Ali), hasibkhan13@yahoo.com (H.Khan)

Quasigroups and Related Systems **20** (2012), 21 − 28

Parametrization of actions of a subgroup of the modular group

Muhammad Ashiq, Qaiser Mushtaq and Tariq Maqsood

Abstract. Graham Higman proposed the problem of parametrization of actions of the extended Modular Group $PGL(2,Z)$ on the projective line over F_q. The problem was solved by Q. Mushtaq. In this paper, we take up the problem and parametrize the actions of $\langle u,v,t : u^3 = v^3 = t^2 = (ut)^2 = (vt)^2 = 1\rangle$ on the projective line over finite Galois fields.

1. Introduction

It is well known [3, 4, 6] that the modular group $PSL(2,Z)$, where Z is the ring of integers, is generated by the linear-fractional transformations $x : z \longrightarrow \frac{-1}{z}$ and $y : z \longrightarrow \frac{z-1}{z}$ and has the presentation $\langle x,y : x^2 = y^3 = 1\rangle$.

Let $v = xyx$, and $u = y$. Then $(z)v = \frac{-1}{z+1}$ and thus $u^3 = v^3 = 1$. So, the group $G(2,Z) = \langle u,v\rangle$ is a proper subgroup of the modular group $PSL(2,Z)$ and the linear-fractional transformation $t{:}z \rightarrow \frac{1}{z}$ inverts u and v, that is, $t^2 = (ut)^2 = (vt)^2 = 1$ and so extends the group $G(2,Z)$ to $G^*(2,Z) = \langle u,v,t : u^3 = v^3 = t^2 = (ut)^2 = (vt)^2 = 1\rangle$.

As u and v have the same orders, there exists an automorphism which interchanges u and v yielding the split extension $G^*(2,Z)$.

Let $PL(F_q)$ denote the projective line over the Galois field F_q , where q is a prime, that is, $PL(F_q) = F_q \cup \{\infty\}$. The group $G^*(2,q)$ is then the group of linear-fractional transformations of the form $z \rightarrow \frac{az+b}{cz+d}$, where $a,b,c,d \in F_q$ and $ad-bc \neq 0$, while $G(2,q)$ is its subgroup consisting of all those linear-fractional transformations of the form $z \rightarrow \frac{az+b}{cz+d}$, where $a,b,c,d \in F_q$ and $ad-bc$ is a non-zero square in F_q.

We use coset diagrams for the group and study its action on $PL(F_q)$. Our coset diagrams consist of triangles; they are called coset diagrams because the vertices of the triangles are identified with cosets of the group. These diagrams are defined for a particular group which has a presentation with three generators. The coset diagrams defined for the actions of $G^*(2,Z)$ on $PL(F_q)$ are special in a number of ways [3]. First, they are defined for a particular group, namely, $G^*(2,Z)$, which has a presentation in terms of three generators t, u and v. Since there are only three

2010 Mathematics Subject Classification: 20F05; 20G40.
Keywords: Linear-fractional transformation, non-degenerate homomorphism, conjugacy classe, parametrization, projective line.

generators, it is possible to avoid using colors as well as the orientation of edges associated with the involution t. For u, and v both have order 3, there is a need to distinguish u from u^2 and v from v^2. The three cycles of the transformation u are denoted by three (blue) unbroken edges of a triangle permuted anti-clockwise by u and the three cycles of the transformation v are denoted by three (red) broken edges of a triangle permuted anti-clockwise by v. The action of t is depicted by the symmetry about vertical axis. Fixed points of u and v, if they exist, are denoted by heavy dots. The method is well explained in [1, 2].

G. Higman proposed the problem of parametrization of actions of $PGL(2, Z)$ on $PL(F_q)$. The problem was solved by Q. Mushtaq in [5]. In this paper, we take up the problem and parametrize the actions of $G^*(2, Z)$ on $PL(F_q)$. We have shown here that any non-degenerate homomorphism α from $G(2, Z)$ into $G(2, q)$ can be extended to a non-degenerate homomorphism α from $G^*(2, Z)$ into $G^*(2, q)$. It has been shown also that every element in $G^*(2, q)$, not of order 1 or 3, is the image of uv under α. It is also proved that the conjugacy classes of $\alpha : G^*(2, Z) \to G^*(2, q)$ are in one-to-one correspondence with the conjugacy classes of non-trivial elements of $G^*(2, q)$, under a correspondence which assigns to the homomorphism α the class containing $(uv)\alpha$.

2. Conjugacy classes

A homomorphism $\alpha : G^*(2, Z) \to G^*(2, q)$ amounts to choosing $\overline{u} = u\alpha$, $\overline{v} = v\alpha$ and $\overline{t} = t\alpha$, in $G^*(2, q)$ such that

$$\overline{u}^3 = \overline{v}^3 = \overline{t}^2 = (\overline{u}\overline{t})^2 = (\overline{v}\overline{t})^2 = 1. \tag{1}$$

We call α to be a *non-degenerate homomorphism* if neither of the generators u, v of $G^*(2, Z)$ lies in the kernel of α. Two homomorphisms α and β from $G^*(2, Z)$ to $G^*(2, q)$ are called *conjugate* if there exists an inner automorphism ρ of $G^*(2, q)$ such that $\beta = \rho\alpha$. Let δ be the automorphism on $G^*(2, Z)$ defined by $u\delta = tut$, $v\delta = v$, and $t\delta = t$. Then the homomorphism $\alpha' = \delta\alpha$ is called the *dual homomorphism* of α. This, of course, means that if α maps u, v, t to $\overline{u}$,$\overline{v}$, $\overline{t}$, then α' maps u, v, t to $\overline{t}\overline{u}\overline{t}$, $\overline{v}$, $\overline{t}$ respectively. Since the elements $\overline{u}$,$\overline{v}$,$\overline{t}$ as well as $\overline{t}\overline{u}\overline{t}$, $\overline{v}$, $\overline{t}$ satisfying the above relations, therefore the solutions of these relations occur in dual pairs. Of course, if α is conjugate to β then α' is conjugate to β'.

3. Parametrization

If the natural mapping $GL(2, q) \to G^*(2, q)$ maps a matrix M to the element of g of $G^*(2, q)$ then $\theta = (tr(M))^2 / \det(M)$ is an invariant of the conjugacy class of g. We refer to it as the parameter of g or of the conjugacy class. Of course, every element in F_q is the parameter of some conjugacy class in $G^*(2, q)$. For instance, the class represented by a matrix with characteristic polynomial $z^2 - \theta z + \theta$ if $\theta \neq 0$ or $z^2 - 1$ if $\theta = 0$.

If q is odd. There are two classes with parameter 0. Of course a matrix M in $GL(2,q)$ represents an involution in $G^*(2,q)$ if and only if its trace is zero. This means that the two classes with parameter 0 contain involutions. One of the classes is contained in $G(2,q)$ and the other not. In any case, there are two classes with parameter 4; the class containing the identity element and the class containing the element $z \to z+1$. Thus apart from these two exceptions, the correspondence between classes and parameters is one-to-one.

If q is odd and g is not an involution, then g belongs to $G(2,q)$ if and only if θ is a square in F_q. On the other hand, $g : z \to \frac{az+b}{cz+d}$, where $a,b,c,d \in F_q$, has a fixed point k in the representation of $G^*(2,q)$ on $PL(F_q)$ if and only if the discriminant, $a^2+d^2-2ad+4bc$, of the quadratic equation $k^2c+k(d-a)-b=0$ is a square in F_q. Since the determinant $ad-bc$ is 1 and the trace $a+d$ is r, the discriminant, $a^2+d^2-2ad+4bc = (a+d)^2-4(ad-bc) = r^2-4 = \theta-4$. Thus, g has fixed point in the representation of $G^*(2,q)$ on $PL(F_q)$ if and only if $(\theta-4)$ is a square in F_q.

If U and V are two non-singular 2×2 matrices corresponding to the generators $\overline{u}$ and $\overline{v}$ of $G^*(2,q)$ with $\det(UV)=1$ and trace r, then for a positive integer k

$$\begin{aligned}(UV)^k &= \{\binom{k-1}{0}r^{k-1} - \binom{k-2}{1}r^{k-3} + \ldots\}UV \\ &\quad -\{\binom{k-2}{0}r^{k-2} - \binom{k-3}{1}r^{k-4} + \ldots\}I. \end{aligned} \tag{2}$$

Furthermore, suppose

$$f(r) = \binom{k-1}{0}r^{k-1} - \binom{k-2}{1}r^{k-3} + \ldots \tag{3}$$

The replacement of θ for r^2 in $f(r)$ yields a polynomial $f(\theta)$ in θ. Thus, one can find a minimal polynomial for positive integer k such that $q \equiv \pm 1 (\bmod\ k)$ by the equation:

$$g_k(\theta) = \frac{f_k(\theta)}{g_{d_1}(\theta)g_{d_2}(\theta)...g_{d_n}(\theta)} \tag{4}$$

where $d_1, d_2, \ldots d_n$, are the divisors of k such that $1 < d_i < k$, $i = 1,2,...,n$ and $f_k(\theta)$ is obtained by the equation (3).

The degree of the minimal polynomial is obtained as:

$$\deg[g_k(\theta)] = \deg[f_k(\theta)] - \sum \deg[g_{d_i}(\theta)] \tag{5}$$

where $\deg[f_k(\theta)] = \left\{ \begin{array}{ll} \frac{k-1}{2}, & \text{if } k \text{ is odd} \\ \frac{k}{2}, & \text{if } k \text{ is even} \end{array} \right\}$. Also, $\deg[g_{2^n}(\theta)] = \frac{2^n}{2} - \frac{2^{n-1}}{2}$, and $\deg[g_{p^n}(\theta)] = \frac{p^n}{2} - \frac{p^{n-1}}{2}$, if p is an odd prime. Thus:

k	**Minimal equation satisfied by θ**
1	$\theta - 4 = 0$
2	$\theta = 0$
3	$\theta - 1 = 0$
4	$\theta - 2 = 0$
5	$\theta^2 - 3\theta + 1 = 0$
6	$\theta - 3 = 0$
7	$\theta^3 - 5\theta^2 + 6\theta - 1 = 0$
8	$\theta^2 - 4\theta + 2 = 0$
9	$\theta^3 - 6\theta^2 + 9\theta - 1 = 0$
10	$\theta^2 - 5\theta + 5 = 0$

Table 1.

and so on.

Let $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an element of $GL(2,q)$ corresponding to $\overline{u}$. Then, since $\overline{u}^3 = 1$, U^3 is a scalar matrix, and hence the $\det(U)$ is a square in F_q. Thus, replacing U by a suitable scalar multiple, we assume that $\det(U) = 1$.

Since, for any matrix M, $M^3 = \lambda I$ if and only if $(tr(M))^2 = \det(M)$, we may assume that $tr(U) = a + d = -1$ and $\det(U) = 1$. Thus $U = \begin{bmatrix} a & b \\ c & -a-1 \end{bmatrix}$. Similarly, $V = \begin{bmatrix} e & f \\ g & -e-1 \end{bmatrix}$. Since $\overline{u}^3 = 1$ also implies that the $tr(\overline{u}) = -1$, every element of $GL(2,q)$ of trace equal to -1 has up to scalar multiplication, a conjugate of the form $\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$. Therefore U will be of the form $\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$.

Now let $\overline{t}$ be represented by $T = \begin{bmatrix} l & m \\ n & j \end{bmatrix}$. Since $\overline{t}^2 = 1$, the trace of T is zero. So, up to scalar multiplication, the matrix representing $\overline{t}$ will be of the form $\begin{bmatrix} 0 & -k \\ 1 & 0 \end{bmatrix}$. Because $(\overline{u}\overline{t})^2 = (\overline{v}\overline{t})^2 = 1$, the $tr(\overline{u}\overline{t}) = tr(\overline{v}\overline{t}) = 0$ and so $b = kc$ and $f = gk$.

Thus the matrices corresponding to generators $\overline{u}$, $\overline{v}$ and $\overline{t}$ of $G^*(2,q)$ will be: $U = \begin{bmatrix} a & kc \\ c & -a-1 \end{bmatrix}$, $V = \begin{bmatrix} e & gk \\ g & -e-1 \end{bmatrix}$, and $T = \begin{bmatrix} 0 & -k \\ 1 & 0 \end{bmatrix}$ respectively, where $a, c, e, g, k \in F_q$. Then,

$$1 + a + a^2 + kc^2 = 0 \tag{6}$$

and

$$1 + e + e^2 + kg^2 = 0 \tag{7}$$

because the determinants of U and V are 1.

This certainly evolves elements satisfying the relations $U^3 = V^3 = \lambda I$, where λ is a scalar and I is the identity matrix. The non-degenerate homomorphism

α is determined by $\overline{u}, \overline{v}$ because one-to-one correspondence assigns to α the class containing $\overline{u}\,\overline{v}$. So it is sufficient to check on the conjugacy class of $\overline{u}\,\overline{v}$. The matrix UV has the trace

$$r = a(2e+1) + 2kgc + (e+1) \tag{8}$$

If $tr(UVT) = ks$, then

$$s = 2ag - c(2e+1) + g \tag{9}$$

So the relationship between (8) and (9) is

$$r^2 + ks^2 = r + 2. \tag{10}$$

We set

$$\theta = r^2 \tag{11}$$

4. Main results

Lemma 4.1. *Either $\overline{uv}$ is of order 3 or there exists an involution $\overline{t}$ in $G^*(2,q)$ such that $\overline{t}^2 = (\overline{u}\overline{t})^2 = (\overline{v}\overline{t})^2 = 1$.*

Proof. Let $tr(UV) = r = gk - g + e + 1$. Then, $gk - g = r - e - 1$. Also $\det(UV) = -g^2k - e^2 - e = -(g^2k + e^2 + e) = 1$. Because, $\overline{t}^2 = (\overline{u}\overline{t})^2 = (\overline{v}\overline{t})^2 = 1$, $m = n - l$ and so

$$(2e - g + 1)l + (gk + g)n = 0 \tag{12}$$

Now for T to be a non-singular matrix, we should have $det(T) \neq 0$, that is

$$nl - l^2 - n^2 \neq 0. \tag{13}$$

Thus the necessary and sufficient conditions for the existence of $\overline{t}$ in $G^*(2,q)$ are the equations (12), and (13). Hence $\overline{t}$ exists in $G^*(2,q)$ unless $nl - l^2 - n^2 = 0$. Of course, if both $2e-g+1$ and $gk+g$ are equal to zero, then the existence of $\overline{t}$ is trivial. If not, then $l/n = -(gk+g)/(2e-g+1)$, and so equation (13) is equivalent to $(gk+g)^2 + (2e-g+1)^2 + (2e-g+1)(gk+g) \neq 0$. Thus there exists $\overline{t}$ in $G^*(2,q)$ such that $\overline{t}^2 = (\overline{u}\overline{t})^2 = (\overline{v}\overline{t})^2 = 1$ unless $(gk+g)^2 + (2e-g+1)(gk+g) = -(2e-g+1)^2$. But if $(gk+g)^2 + (2e-g+1)(gk+g) = -(2e-g+1)^2$, then, $g^2k^2 + g^2 + 2g^2k + 2egk + 2eg - g^2k - g^2 + gk + g = -(4e^2 + g^2 + 1 + 4e - 2g - 4eg) = -\{4e^2 + 4e + 1 + g^2 - 2g - 4eg\} = -\{-4g^2k - 3 + g^2 - 2g - 4eg\}$. So, after simplification

$$(gk - g)^2 + (gk - g) + 2e(gk - g) - g^2k = 3 \tag{14}$$

Since $gk - g = r - e - 1$, equation (14) can be further simplified as

$$r^2 - 2 = r \tag{15}$$

Square both sides of equation (15), and substitute $r^2 = \theta$ in the equation $\theta^2 - 5\theta + 4 = 0$ giving $\theta = 1, 4$.

By Table 1, $\theta = 1$ implies that the order of $\overline{u}\,\overline{v}$ is 3 and $\theta = 4$ implies that the order of $\overline{u}\,\overline{v}$ is 1. □

It can happen that both $\overline{u}\,\overline{v}$ is of order 3 and the pair $(\overline{u}, \overline{v})$ is invertible if $\overline{u}\,\overline{v} = \overline{v}\,\overline{u}$. For example, if $U = \begin{bmatrix} 2 & 2 \\ 2 & -3 \end{bmatrix}$, $V = \begin{bmatrix} 2 & 2 \\ 2 & -3 \end{bmatrix}$, and $T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. In fact, because of the following result this is the only case in which $\overline{t}$ exists and $\overline{u}\,\overline{v}$ is of order 3.

Lemma 4.2. *One and only one of the following holds:*
(*i*) *The pair* $(\overline{u}, \overline{v})$ *is invertible.*
(*ii*) $\overline{u}\,\overline{v}$ *has order* 3 *and* $\overline{u}\,\overline{v} \neq \overline{v}\,\overline{u}$. □

In what follows we shall find a relationship between the parameters of the dual homomorphisms. We first prove the following.

Lemma 4.3. *Any non trivial element* $\overline{g}$ *of* $G^*(2, q)$ *whose order is not equal to* 2 *or* 6 *is the image of* uv *under some non-degenerate homomorphism* α *of* $G^*(2, Z)$ *into* $G^*(2, q)$.

Proof. Using Lemma4.1, we show that every non-trivial element of $G^*(2, q)$ is a product of two elements of orders 3. So we find elements $\overline{u}, \overline{v}$ and, $\overline{t}$ of $G^*(2, q)$ satisfying the equation (1) with $\overline{u}\,\overline{v}$ in a given conjugacy class.

The class to which we want $\overline{u}\,\overline{v}$ to belong do not consist of involutions because $\overline{g} = \overline{u}\,\overline{v}$ is not of order 2. Thus the traces of the matrices UV and UVT are not equal to zero. Hence $r \neq 0$, and $s \neq 0$, so that we have $\theta = r^2 \neq 0$; and it is sufficient to show that we can choose a, c, e, g, k, in F_q so that r^2 is indeed equal to θ. The solution of θ is therefore arbitrarily in F_q. We can choose r to satisfy $\theta = r^2$, equation (10), yields $ks^2 = 2 + r - r^2$. If $r^2 \neq 2 + r$, we select k as above.

Any quadratic polynomial $\lambda z^2 + \mu z + \nu$, with coefficients in F_q takes at least $(q+1)/2$ distinct values, as z runs through F_q; since the equation $\lambda z^2 + \mu z + \nu = k$ has at most two roots for fixed k; and there are q elements in F_q, where q is odd. In particular, $e^2 + e$ and $-kg^2 - 1$ each take at least $(q + 1)/2$ distinct values as e and g run through F_q. Hence we can find e and g so that $e^2 + e = -kg^2 - 1$ (equation 7).

Finally by substituting the values of r, s, e, g, k in equations (8) and (9) we obtain the values of a and c. □

It is clear from (10) and (11) that $\theta = 0$ when $r = 0$ and $\theta = 1$ or 4 when $s = 0$. The possibility that $\theta = 0$ gives rise to the situation where $\overline{uv}$ is of order 2. Similarly, the possibility $\theta = 1$ leads to the situation where $\overline{u}\,\overline{v}$ is of order 3, and similarly $\theta = 4$ yields $\overline{u}\,\overline{v}$ of order 1.

Lemma 4.4. *Any two non-degenerate homomorphisms* α, β *of* $G^*(2, Z)$ *into* $G^*(2, q)$ *are conjugate if* $(uv)\alpha = (uv)\beta$.

Proof. Let α: $G^*(2, Z) \to G^*(2, q)$ be such that $\overline{u}\,\overline{v}$ has parameter θ constructed as in the proof of lemma 4.3. We also suppose that β:$G^*(2, Z) \to G^*(2, q)$ has the same parameter θ.

First, since there are just two classes of elements of order 2 in $G^*(2,Z)$, one in $G^*(2,Z)$ and the other not, we can pass to a conjugate of β in which $t\beta$ is represented by $\begin{bmatrix} 0 & -k' \\ 1 & 0 \end{bmatrix}$ for some $k' \neq 0$ in F_q. Then because $u\beta$ and $v\beta$ are both of orders 3, $u\beta$ must be represented by a matrix $\begin{bmatrix} a' & k'c' \\ c' & -a'-1 \end{bmatrix}$ and $v\beta$ must be represented by a matrix $\begin{bmatrix} e' & k'g' \\ g' & -e'-1 \end{bmatrix}$, with a', c', e', g', k' satisfying the equations from (6) to (9). Then $\theta = r'^2 = r^2$ and $(2+r) - \theta = k's'^2 = ks^2$. Here since θ and $(2+r)-\theta$ are non-zero, so it follows that k'/k is a square in F_q.

Now $v\alpha$ and $v\beta$ are both of orders 3 and so are conjugate in $G^*(2,q)$. So we can pass to a conjugate of β (which we still call β) with $v\alpha = v\beta$. As $t\alpha$ and $t\beta$ are involutions which invert $v\alpha$, and so belong to $N(\langle v\alpha\rangle)$ there are two classes of such involutions, one in $G^*(2,q)$ and the other not. Because $t\alpha$ is $\begin{bmatrix} 0 & -k \\ 1 & 0 \end{bmatrix}$ and $t\beta$ is conjugate to $\begin{bmatrix} 0 & -k' \\ 1 & 0 \end{bmatrix}$ and k'/k is a square, $t\alpha$ and $t\beta$ either both belong to $G^*(2,q)$ or neither. Hence they are conjugate in $N(\langle v\alpha\rangle)$. That is, passing to a new conjugate (still called β) we can assume $v\alpha = v\beta, t\alpha = t\beta$. This means that in the notations above, we can assume $k' = k$, $g = g'$ and $e = e'$. We can also, by multiplying the matrix representing $u\beta$ by a scalar, assume $r = r'$ and $s = s'$. Then the equations from (6) to (9) with a, c, e, g, k and then with a', c', e', g', k' and ensure that $a = a'$and $c = c'$. That is $\alpha = \beta$. □

Theorem 4.5. *The conjugacy classes of non-degenerate homomorphisms of* $G^*(2,Z)$ *into* $G^*(2,q)$ *are in one-to-one correspondence with the non-trivial conjugacy classes of elements of* $G^*(2,q)$ *under a correspondence which assigns to any non-degenerate homomorphism* α *the class containing* $(uv)\alpha$.

Proof. Let $\alpha : G^*(2,Z) \to G^*(2,q)$ be such that it maps u, v to $\overline{u}$,$\overline{v}$. Let θ be the parameter of the class represented by $\overline{u}\ \overline{v}$. Now α is determined by $\overline{u}$,$\overline{v}$ and each θ evolves a pair $\overline{u}$,$\overline{v}$, so that α is associated with θ. We shall call the parameter θ of the class containing $\overline{u}\ \overline{v}$, the parameter of $G^*(2,Z) \to G^*(2,q)$. Now

$$UT = \begin{bmatrix} ck & -ak \\ -a-1 & -ck \end{bmatrix}$$

implies that $\det(UT) = -k(a^2 + a + kc^2) = k$ (equation 6). Also,

$$(UT)V = \begin{bmatrix} kec - akg & k^2gc + ak(e+1) \\ -ae - e - kgc & -akg - kg + ck(e+1) \end{bmatrix}$$

implies that the $tr((UT)V) = 2kec - 2akg - kg + kc = -1(2akg - 2kec + kg - kc) = -ks$. If $\overline{u}, \overline{v}, \overline{t}$ satisfy equation (1), then so do $\overline{t}\overline{u}\overline{t}, \overline{v}, \overline{t}$. So that the solution of equation (1) occur in dual pairs. Hence replacing the solutions in lemma-4.3 by

$\overline{t}\overline{u}\overline{t}, \overline{v}, \overline{t}$, we obtain $\theta = \frac{[tr((UT)V]^2}{\det(UT)} = \frac{k^2s^2}{k} = ks^2$. We then find a relationship between the parameters of the dual non-degenerate homomorphisms. □

There is an interesting relationship between the parameters of the dual non-degenerate homomorphisms.

Corollary 4.6. *If $\alpha : G^*(2,Z) \to G^*(2,q)$ is a non-degenerate homomorphism, α' is its dual and θ, φ are their respective parameters then $\theta + \varphi = r + 2$.*

Proof. Let $\alpha : G^*(2,Z) \to G^*(2,q)$ satisfy the relations $u\alpha = \overline{u}$, $v\alpha = \overline{v}$ and $t\alpha = \overline{t}$. Let α' be the dual of α. As, we choose the matrices $U = \begin{bmatrix} a & ck \\ a & -a-1 \end{bmatrix}$, $V = \begin{bmatrix} e & g\,k \\ g & -e-1 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & -k \\ 1 & 0 \end{bmatrix}$, representing $\overline{u}, \overline{v}$ and $\overline{t}$, respectively such that they satisfy the equations from (6) to (10). Now, $(\overline{u}\,\overline{v})^2 = 1$ implies that $tr(UV) = 0$. Also, we have $\{tr(UVT)\}/k = s = 0$ if and only if $(\overline{u}\,\overline{v}\overline{t})^2 = 1$. Now $\det(UV) = 1$, thus giving the parameter of $\overline{u}\ \overline{v}$ equal to $r^2 = \theta$, say. Also since $tr(UVT) = ks$ and $\det(UVT) = k$ (since $\det(U) = 1$, $\det(V) = 1$ and $\det(T) = k$), we obtain the parameter of $\overline{u}\ \overline{v}\overline{t}$ equal to ks^2, which we denote by φ. Thus $\theta + \varphi = r^2 + ks^2$. Substituting the values from equation (10), we thus obtain $\theta + \varphi = r + 2$. Hence if θ is the parameter of the non-degenerate homomorphism α, then $\varphi = r + 2 - \theta$ is the parameter of the dual α' of α. □

Theorem 4.5, of course, means that we can actually parametrize the non-degenerate homomorphisms of $G^*(2,Z)$ to $G^*(2,q)$ except for a few uninteresting ones, by the elements of F_q. Since $G^*(2,q)$ has a natural permutation representation on $PL(F_q)$, any homomorphism $\alpha : G^*(2,Z) \to G^*(2,q)$ gives rise to an action of $G^*(2,Z)$ on $PL(F_q)$.

References

[1] **M. Ashiq and Q. Mushtaq**, *Finite presentation of a linear-fractional group*, Algebra Colloquium **12** (2005), 585 – 589.

[2] **M. Ashiq and Q. Mushtaq**, *Coset diagrams for a homomorvarphic image of* $\triangle(3,3,k)$, Acta Math. Sci. **28B** (2008), 363 – 370.

[3] **M. H. Millington**, *Subgroups of the classical modular group*, J. Lond. Math. Soc. **1** (1970), 351 – 357.

[4] **Q. Mushtaq**, *Coset diagrams for Hurwitz groups*, Commun. Algebra **18** (1990), 3857 – 3888.

[5] **Q. Mushtaq**, *Parametrization of all homomorvarphisms from $PGL(2,Z)$ into $PGL(2,q)$*, Commun. Algebra **20** (1992), 1023 – 1040.

[6] **W. W. Stothers**, *Subgroups of the modular group*, Proc. Camb. Philos. Soc. **75** (1974), 4139 – 154.

Received February 14, 2011

Quasigroups and Related Systems **20** (2012), 29 – 36

A Zariski topology for k-semirings

Shahabaddin Ebrahimi Atani and Reza Ebrahimi Atani

Abstract. The prime k-spectrum $\mathrm{Spec}_k(R)$ of a k-semiring R will be introduced. It will be proven that it is a topological space, and some properties of this space will be investigated. Connections between the topological properties of $\mathrm{Spec}_k(R)$ and possible algebraic properties of the k-semiring R will be established.

1. Introduction

Semirings which are regarded as a generalization of rings have been found useful in solving problems in different disciplines of applied mathematics and information sciences because semirings provides an algebraic framework for modeling. Ideals of semirings play a central role in the structure theory and are useful for many purposes. However, they do not in general coincide with the usual ring ideals and, for this reason; their use is somewhat limited in trying to obtain analogues of ring theorems for semirings. Indeed, many results in rings apparently have no analogues in semirings using only ideals. Let R be a commutative ring with identity. The prime spectrum $\mathrm{Spec}(R)$ and the topological space obtained by introducing Zariski topology on the set of prime ideals of R play an important role in the fields of commutative algebra, algebraic geometry and lattice theory. Also, recently the notion of prime submodules and Zariski topology on $\mathrm{Spec}(M)$, the set of all prime submodules of a module M over R, are studied by many authors (for example see [11]). In this paper, we concentrate on Zariski topology of semirings and generalize the some well known results of Zariski topology on the sets of prime ideals of a commutative ring to prime ideals of a commutative semiring and investigate the basic properties of this topology. For example, we prove that if R is a k-semiring, then $\mathrm{Spec}_k(R)$ is a T_0-space and it is a compact space.

Throughout this paper R is a commutative semiring with identity. For the definitions of monoid, semirings, semimodules and subsemimodules we refer [1, 6, 8, 10, 11]. All semiring in this paper are commutative with non-zero identity. Allen [1] has presented the notion of Q-ideal I in the semiring R and constructed the quotient semiring R/I (also see [3, 5, 7]). Let R be a semiring. A *subtractive ideal* (= *k-ideal*) I is a ideal of R such that if $x, x+y \in I$, then $y \in I$ (so $\{0_R\}$ is a k-ideal of R). A *prime ideal* of R is a proper ideal P of R in which $x \in P$ or $y \in P$ whenever $xy \in P$. So P is prime if and only if whenever $IJ \subseteq P$ for some

2010 Mathematics Subject Classification: 16Y60
Keywords: k-prime spectrum, Zariski topology, k-semiring, top semiring.

ideals I, J of R implies that $I \subseteq P$ or $J \subseteq P$. Furthermore, the collection of all prime k-ideals of R is called the *spectrum* of R and denoted by $\mathrm{Spec}_k(R)$. An ideal I of R is said to be *semiprime* if I is an intersection of prime k-ideals of R. If I is a proper ideal of R, then the *radical* $\mathrm{rad}(I)$ of I (in R) is the intersection of all prime k-ideals of R containing I (see [4]). Note that $I \subseteq \mathrm{rad}(I)$ and that $\mathrm{rad}(I)$ is a semiprime k-ideal of R. An ideal I of R is called *extraordinary* if whenever A and B are semiprime k-ideals of R with $A \cap B \subseteq I$, then $A \subseteq I$ or $B \subseteq I$. A semiring is called a *partitioning semiring*, if every proper principal ideal of R is a partitioning ideal (= a Q-ideal) (see [7]). A non-zero element a of a semiring R with identity is said to be a *semiunit* in R if $1 + ra = sa$ for some $r, s \in R$.

Lemma 1.1. *Let R be a semiring. If $\{I_i\}_{i\in\Lambda}$ is a collection of k-ideals of R, then $\sum_{i\in\Lambda} I_i$ and $\bigcap_{i\in\Lambda} I_i$ are k-ideals of R.* □

2. Properties of top semirings

Let R be a semiring with $1 \neq 0$. Then R has at least one maximal k-ideal and if I is a proper Q-ideal of R, then $I \subseteq P$ for some maximal k-ideal P of R (see [5]). Now by [3], R/P is a semifield and hence it is a semidomain. Thus P is prime and $\mathrm{Spec}_k(R) \neq \emptyset$ (see [3]). Then we have the following

Lemma 2.1. *If P is a maximal Q-ideal of a semiring R, then P is a prime k-ideal of R. In particular,* $\mathrm{Spec}_k(R) \neq \emptyset$. □

Let R be a semiring R with non-zero identity. For any k-ideal I of R by $V(I)$ we mean the set of all prime k-ideals of R containing I. Clearly, $V(R) = \emptyset$ and $V(\{0\}) = \mathrm{Spec}(R)$.

Definition 2.2. A semiring is called a *k-semiring*, if every ideal of R is a k-ideal.

Example 2.3. Assume that E_+ be the set of all non-negative integers and let $R = E_+ \cup \{\infty\}$. Define $a + b = \max\{a, b\}$ and $ab = \min\{a, b\}$ for all $a, b \in R$. Then R is a commutative semiring with $1_R = \infty$ and $0_R = 0$. An inspection will show that the list of ideals of R are: R, E_+ and for every non-negative integer n

$$I_n = \{0, 1, \ldots, n\}.$$

It is clear that every ideal of R is a k-ideal; so R is a k-semiring. Moreover, every proper ideal of R is a prime k-ideal; so $\mathrm{Spec}(R) = \{E_+, I_0, \ldots\}$. □

Lemma 2.4. *Let R be a k-semiring. Then the following statements hold:*

(*i*) *If S is a subset of R, then $V(S) = V(\langle S\rangle)$.*

(*ii*) $V(I) \cup V(J) = V(IJ) = V(I \cap J)$ *for every k-ideals I and J of R.*

(*iii*) *If I is a k-ideal of R, then $V(I) = V(\mathrm{rad}(I))$.*

(*iv*) *If* $V(I) \subseteq V(J)$*, then* $J \subseteq \mathrm{rad}(I)$ *for every deals* I, J *of* R*.*

(*v*) $V(I){=}V(J)$ *if and only if* $\mathrm{rad}(I){=}\mathrm{rad}(J)$ *for every ideals* I, J *of* R*.*

(*vi*) *If* $\{I_i\}_{i\in\Lambda}$ *is a family of ideals of* R*, then* $V(\sum_{i\in\Lambda} I_i) = \bigcap_{i\in\Lambda} V(I_i)$*.*

Proof. (*i*) and (*iv*) are obvious.
(*ii*) It is clear that $V(I) \cup V(J) \subseteq V(I \cap J) \subseteq V(IJ)$. Let $P \in V(IJ)$. Then $IJ \subseteq P$, and hence $I \subseteq P$ or $J \subseteq P$. Thus $P \in V(I)$ or $P \in V(J)$, i.e., $P \in V(I) \cup V(J)$. Hence $V(IJ) \subseteq V(I) \cup V(J)$.

(*iii*) Since $I \subseteq \mathrm{rad}(I)$, we have $V(\mathrm{rad}(I)) \subseteq V(I)$. For the reverse inclusion, assume that $P \in V(I)$. Then $I \subseteq P$. Hence $\mathrm{rad}(I) \subseteq P$, and so we have the equality.

(*v*) Let $V(I) = V(J)$. By (*iii*), we have $V(I) \subseteq V(\mathrm{rad}(J)$; hence $\mathrm{rad}(J) \subseteq \mathrm{rad}(I)$ by (*iv*). Similarly, $\mathrm{rad}(I) \subseteq \mathrm{rad}(J)$, and so we have the equality. The other implication is similar.

(*vi*) Let $P \in \bigcap_{i\in\Lambda} V(I_i)$. Then $I_i \subseteq P$ for every $i \in \Lambda$, so $\sum_{i\in\Lambda} I_i \subseteq P$, which implies that $\bigcap_{i\in\Lambda} V(I_i) \subseteq V(\sum_{i\in\Lambda} I_i)$. The reverse inclusion is similar. □

Let R be a k-semiring. If $\zeta(R)$ denotes the collection of all subsets $V(I)$ of $\mathrm{Spec}_k(R)$, then $\zeta(R)$ contains the empty set and $\mathrm{Spec}(R) = X$ and is closed under arbitrary intersection by Lemma 2.4 (*vi*). If also $\zeta(R)$ is closed under finite union, that is, for every ideals I and J of R such that $V(I) \cup V(J) = V(L)$ for some ideal L of R, for in this case $\zeta(R)$ satisfies the axioms of closed subsetes of a topological spaces, which is called *Zariski topology.* The following definition is the same as that introduced by MacCasland, Moore, and Smith in [11].

Definition 2.5. Let R be a k-semiring. An R-semimodule M equipped with Zariski topology is called *top semimodule.* A k-semiring R which is a top semimodule as an R-semimodule is called a *top semiring.*

Proposition 2.6. *Every* k*-semiring with a non-zero identity is a top semiring.*

Proof. Apply Lemma 2.4. □

Theorem 2.7. *Every ideal of a* k*-semiring with a non-zero identity is extraordinary.*

Proof. Note that $\mathrm{Spec}_k(R) \neq \emptyset$ by Lemma 2.1. Let P be any ideal of R and let I and J be semiprime ideals of R such that $I \cap J \subseteq P$. By Proposition 2.6, there exists an ideal U of R such that $V(I) \cup V(J) = V(U)$. Since $I = \bigcap_{i\in\Lambda} P_i$, where P_i are prime k-ideals of R ($i \in \Lambda$), for each $i \in \Lambda$, $P_i \in V(I) \subseteq V(U)$, so that $U \subseteq P_i$. Thus $U \subseteq I$. Similarly, $U \subseteq J$. Thus $U \subseteq I \cap J$. Now we have $V(I) \cup V(J) \subseteq V(I \cap J) \subseteq V(U) = V(I) \cup V(J)$, that is, $V(I) \cup V(J) = V(I \cap J)$. Hence $P \in V(I \cap J)$ gives $I \subseteq P$ or $J \subseteq P$. □

Definition 2.8. A semiring is called a *strong partitioning semiring*, if every proper finitely generated ideal of R is a partitioning ideal (= a Q-ideal).

Proposition 2.9. *Assume that R is a strong partitioning semiring and let I be the proper ideal of R generated by a family $\{a_t\}_{t\in\Lambda}$ of elements R. Then I is a Q-ideal of R.*

Proof. Since $R = \bigcup\{q + Ra_t : q \in Q\}$ for some $t \in \Lambda$, we must have $R = \bigcup\{q + I : q \in Q\}$. Let $X \in (q_1 + I) \cap (q_2 + I) \neq \emptyset$. Then $X = q_1 + r_{i_1}a_{i_1} + \ldots + r_{i_n}a_{i_n} = q_2 + s_{j_1}a_{j_1} + \ldots + s_{j_m}a_{j_m}$ for some $a_{j_k}, a_{i_t} \in I$ and $r_{i_t}, s_{j_k} \in R$ ($1 \leqslant t \leqslant n$, $1 \leqslant k \leqslant m$). Let J be the ideal of R generated by $r_{i_1}a_{i_1}, ..., r_{i_n}a_{i_n}, s_{j_1}a_{j_1}, \ldots, s_{j_m}a_{j_m}$. By assumption, J is a Q-ideal of R and $X \in (q_1 + J) \cap (q_2 + J)$; hence $q_1 = q_2$. Thus I is a Q-ideal of R. □

Remark 2.10. Let $X = \mathrm{Spec}_k(R)$. For each subset S of R, by X_S we mean $X - V(S) = \{P \in X : S \nsubseteq P\}$. If $S = \{f\}$, then by X_f we denote the set $\{P \in X : f \notin P\}$. Clearly, the sets X_f are open, and they are called *basic open sets*. □

Theorem 2.11. *Let R be a strong partitioning semiring and $X = \bigcup_{i\in\Lambda} X_{a_i}$. If I is the ideal of R generated by $\{a_i\}_{i\in\Lambda}$, then $I = R$.*

Proof. Suppose not. Since I is a proper Q-ideal of R by Proposition 2.9, we have $I \subseteq P$ for some maximal k-ideal P of R. By assumption, $P \notin X_{a_i}$ for every $i \in \Lambda$, which is a contradiction. □

Theorem 2.12. *Let R be a strong partitioning semiring. Then the following statements hold:*

(i) $X_f \cap X_e = X_{fe}$ *for all* $f, e \in R$.

(ii) $X_f = \emptyset$ *if and only if f is nilpotent.*

(iii) $X_f = X$ *if and only if f is a semiunit in R.*

Proof. (i) If $P \in X_f \cap X_e$, then $e, f \notin P$, so $ef \notin P$, which implies that $P \in X_{fe}$. Thus $X_f \cap X_e \subseteq X_{ef}$. The other inclusion is similar.

(ii) Assume that an element f is nilpotent and let P be any element of X. Then $f^s = 0 \in P$ for some positive integer s. Thus P prime k-ideal gives $f \in P$; hence $P \notin X_f$ for every $P \in X$. Thus $X_f = \emptyset$. Conversely, assume that $X_f = \emptyset$. Then for each $P \in X$, we have $f \in P$; whence $f \in \bigcap_{P\in X} P = \mathrm{rad}(0)$ (see [4]). Thus f is nilpotent.

(iii) Let f be a semiunit. Since the inclusion $X_f \subseteq X$ is trivial, we will prove the reverse inclusion. Let P be any element of X. If $Rf \subseteq P$, then $R = P$ by [5], which is a contradiction. Thus $f \notin P$; hence $P \in X_f$, and so we have equality. Conversely, assume that $X = X_f$. Then for any $P \in X$, we must have $f \notin P$. If f is not a semiunit in R, then Rf is a Q-ideal of R and hence it is contained in a maximal k-ideal of R which is a prime k-ideal by Lemma 2.1, a contradiction. Thus f is semiunit. □

Theorem 2.13. *Let R be a k-semiring. Then the set $\mathcal{A} = \{X_f : f \in R\}$ forms a base for the Zariski topology on X.*

Proof. Suppose that U is an open set in X. Then $U = X - V(I)$ for some k-ideal I of R. Let $I = \langle \{f_i : i \in \Lambda\}\rangle$, where $\{f_i : i \in \Lambda\}$ is a generator set of I. Then $V(I) = V(\sum_{i\in\Lambda} Rf_i) = \bigcap_{i\in\lambda} V(Rf_i)$ by Lemma 2.4(vi). It follows that $U = X - V(I) = X - \bigcap_{i\in\Lambda} V(Rf_i) = \bigcup_{i\in\Lambda} X_{f_i}$. Thus $\mathcal{A}$ is a base for the Zariski topology on X. □

Proposition 2.14. *Let I be an ideal of a k-semiring R. Then*

(i) *$X_I = \bigcup_{a\in I} X_a$. Moreover, if $I = \langle a_1, a_2, \ldots, a_n\rangle$, then $X_I = \bigcup_{i=1}^{n} X_{a_i}$.*

(ii) *Let $\{a_i\}_{i\in\Lambda}$ be the collection of elements of R and $a \in R$. Then $X_a \subseteq \bigcup_{i\in\Lambda} X_{a_i}$ if and only if there are elements $a_{i_1}, \ldots, a_{i_n} \in \{a_i\}_{i\in\Lambda}$ such that $a \in \mathrm{rad}(\langle a_{i_1}, \ldots, a_{i_n}\rangle)$.*

Proof. (i) Assume that $a \in I$ and let $P \in X_a$. Then $a \notin P$ which implies $P \in X_I$. Thus $\bigcup_{a\in I} X_a \subseteq X_I$. For the reverse inclusion, assume that $P \in X_I$. Then $P \in X_b$ for some $b \in I - P$, and so we have the equality. Finally, since the inclusion $\bigcup_{i=1}^{n} X_{a_i} \subseteq X_I$ is clear, we will prove the reverse inclusion. Let $P \in X_I$. Then there exist $a \in I - P$ and $r_i \in R$ $(1 \leqslant i \leqslant n)$ such that $P \in X_a$ and $a = \sum_{i=1}^{n} r_i a_i$. It follows that there exists a positive integer j $(1 \leqslant j \leqslant n)$ such that $a_j \notin P$; hence $P \in X_{a_j}$, as needed.

(ii) Let $a \in \mathrm{rad}(\langle a_{i_1}, \ldots, a_{i_n}\rangle)$. Then there exists a positive integer m and $r_i \in R$ $(1 \leqslant i \leqslant n)$ such that $a^m = \sum_{j=1}^{n} r_j a_{i_j}$. Now, let $P \in X_a$. So $a \notin P$ gives $a^m \notin P$; hence $P \in X_{a_{i_k}}$ for some k. Thus $X_a \subseteq \bigcup_{i\in\Lambda} X_{a_i}$.

Conversely, assume that $X_a \subseteq \bigcup_{i\in\Lambda} X_{a_i}$ and let I be the ideal of R generated by $\{a_i : i \in \Lambda\}$. It is clear that if $P \in X$ and $P \notin \bigcup_{i\in\Lambda} X_{a_i}$, then $a_i \in P$ implies that $a \in P$. Therefore we have $V(I) \subseteq V(\langle a\rangle)$. It follows that $a \in \bigcap_{P\in V(<a>)} P \subseteq \bigcap_{P\in V(I)} P = \mathrm{rad}(I)$. So, there exist $i_1, i_2, \ldots, i_s \in \Lambda$ and $t_1, t_2, \ldots, t_s \in R$ such that $a^m = t_1 a_{i_1} + \ldots + t_s a_{i_s}$ for some positive integer m; thus $a \in \mathrm{rad}(\langle a_{i_1}, \ldots, a_{i_n}\rangle)$. □

Theorem 2.15. *Let R be a k-semiring. For every $a \in R$, the set X_a is compact. Specifically the whole space $X_1 = X$ is compact.*

Proof. By Theorem 2.13, it suffices to show that every cover of basic open sets has a finite subcover. Suppose that $X_a \subseteq \bigcup_{i\in\Lambda} X_{a_i}$. By Proposition 2.14 ($ii$), there are $a_{i_1}, \ldots, a_{i_n} \in R$ such that $a \in \mathrm{rad}(\langle a_{i_1}, \ldots, a_{i_n}\rangle)$. Since $V(\mathrm{rad}(\langle a_{i_1}, \ldots, a_{i_n}\rangle)) = V(\langle a_{i_1}, \ldots, a_{i_n}\rangle)$ by Lemma 2.4 (iii), we must have $X_a \subseteq \bigcup_{i=1}^{n} X_{a_i}$ by Proposition 2.14 (i). This completes the proof. □

From Theorem 2.13 and Theorem 2.15 the next result is immediate.

Corollary 2.16. *Let R be a k-semiring. Then an open set of X is compact if and only if it is a finite union of basic open sets.* □

Let R be a k-semiring. The topological space $X = \mathrm{Spec}_k(R)$ is said to be a T_0*-space* if for every $P, P' \in X$, $P \neq P'$ there is either a neighborhood X_a of P such that $X_a \cap P' = \emptyset$ or a neighborhood X_b of P' such that $X_b \cap P = \emptyset$.

Theorem 2.17. *Let R be a k-semiring. Then the topological space $X = \mathrm{Spec}_k(R)$ is a T_0-space.*

Proof. Let $P, P' \in X$ with $P \neq P'$. We note that the set X_a is a neighborhood of P if and only if $a \notin P$. Assume that $P' \in X_a$ for all $a \notin P$. Then we conclude that $a \in P'$ implies that $a \in P$; hence $P' \subset P$. Now let $b \in P - P'$. Then $b \notin P'$ gives X_b is a neighborhood of P', but $b \in P$, so $P \notin X_b$. This completes the proof. □

Quotient semimodules over a semiring R have already been introduced and studied by present authors in [6]. Chaudhari and Bonde extended the definition of Q_M-subsemimodule of a semimodule and some results given in the Section 2 in [6] to a more general quotient semimodules case in [8] (for the structure of quotient semimodules we refer [8]).

Convention. For each Q_R-subsemimodule I of the R-semimodule R, we mean I is a Q_R-ideal of R. Now If I is a Q_R-ideal of a semiring R, then R/I is a quotient semimodule of R by I. Now we give an example of semimodules over a semiring that are top semimodules.

Lemma 2.18. *Let I be a Q_R-ideal (or a Q_R-subsemimodule) of a semiring R. If J is a k-ideal of R containing I, then $(J :_R R) = (J/I :_R R/I)$.*

Proof. Let $r \in (J : R)$. If $q + I \in R/I$, then there exists a unique element q' of Q_R such that $r(q + I) = q' + I$, where $rq + I \subseteq q' + I$; so $q' \in J \cap Q_R$ since $rq \in J$ and J is a k-ideal. Thus $(J : R) \subseteq (J/I : R/I)$.

Conversely, assume that $a \in (J/I : R/I)$ and $s \in R$. Then $s = q_1 + t$ for some $q_1 \in Q_R$ and $t \in I$; so there is a unique element q_2 of Q_R with $a(q_1 + I) = q_2 + I \in J/I$, where $aq_1 + I \subseteq q_2 + I$. Thus J k-ideal gives $aq_1 \in J$. As $as = aq_1 + at \in J$, we have $a \in (J : R)$. □

Proposition 2.19. *Let I be a Q_R-ideal of a semiring R. Then there is a one-to-one correspondence between prime k-subsemimodules of R-semimodule R/I and prime k-ideals of R containing I.*

Proof. Let J be a prime k-ideal of R containing I. Then it follows from [3] that J/I is a proper k-subsemimodule of R/I. Let $a(q_1 + I) = q_2 + I \in J/I$, where $q_2 \in Q_R \cap J$ and $aq_1 + I \subseteq q_2 + I$, so $aq_1 \in J$ since J is a k-ideal of R. But J is prime, hence either $q_1 \in J$ (so $q_1 + I \in J/I$) or $a \in (J : R) = (J/I : R/I)$ by Lemma 2.18. Thus, J/I is a prime k-subsemimodule of R/I.

Conversely, assume that J/I is a prime k-subsemimodule of R/I. To show that J is a prime k-ideal of R, suppose that $rx \in J$, where $r, x \in R$. We may assume that $r \neq 0$. There are elements $q \in Q_R$ and $n \in I$ such that $x = q + n$, so $rx = rq + rn \in J$; hence $rq \in J$ since J is a k-ideal. Therefore, there exists a unique element $q' \in Q_R$ such that $r(q+I) = q' + I$, where $rq + I \subseteq q' + I$; hence $q' \in J$. Thus $r(q+I) \in J/I$. Then J/I prime gives either $q + I \in J/I$ (so $x \in J$) or $r \in (J/I : R/I) = (J : R)$, and the proof is complete. □

Corollary 2.20. *Let I be a Q_R-ideal of a semiring R. Then there is a one-to-one correspondence between semiprime k-subsemimodules of R/I and semiprime k-ideals of R containing I.*

Proof. Apply Theorem 2.19 (note that $(\bigcap_{i\in J} P_i)/I = \bigcap_{i\in J}(P_i/I)$, where P_i is a prime k-ideal for all $i \in J$). □

Theorem 2.21. *Let I be an Q_R-ideal of a semiring R with a non-zero ideantity. Then the following statements hold:*

(i) *Every k-subsemimodule of R/I is extraordinary.*

(ii) *R/I is a top R-semimodule.*

Proof. (i) We may assume that $\mathrm{Spec}(R/I) \neq \emptyset$. Then any semiprime k-subsemimodule of R/I has the form A/I where A is a semiprime k-ideal of R containing I by Corollary 2.20. Let B/I be any k-subsemimodule of R/I and let U/I and L/I be semiprime k-subsemimodules of R/I such that $(L/I) \cap (U/I() \subseteq B/N$. Then $(L \cap U)/I \subseteq (L/I) \cap (U/I) \subseteq B/I$, so $U \cap L \subseteq B$; hence either $U \subseteq B$ or $L \subseteq B$ since T is extraordinary by Theorem 2.7. Thus either $U/I \subseteq B/I$ or $L/I \subseteq B/I$, as needed.

(ii) First we show that $V(U/I) \cup V(L/I) = V(U/I \cap L/I)$ for any semiprime subsemimodules U/I and L/I of R/I.

Clearly $V(U/I) \cup V(L/I) \subseteq V(U/I \cap L/I)$. Let $P/I \in V(U/I \cap L/I)$, where P is a semiprime by Corollary 2.20. Then $U \cap L \subseteq P$ and hence $L \subseteq P$ or $U \subseteq P$ (see Theorem 2.7), i.e., $P/I \in V(U/I)$ or $P/I \in V(L/I)$. This proves that $V(U/I \cap L/I) \subseteq V(U/I) \cup V(L/I)$ ans hence $V(U/I) \cup V(L/I) = V(U/I \cap L/I)$. Next, let A/I and B/I be any subsemimodules of R/I. If $V(A/I)$ is empty then $V(A/I) \cup V(B/I) = V(B/I)$. Suppose that $V(A/I)$ and $V(B/I)$ are both non-empty. Then $V(A/I) \cap V(B/I) = V(\mathrm{rad}(A/I)) \cap V(\mathrm{rad}(B/I)) = V(\mathrm{rad}(A/I) \cap \mathrm{rad}(B/I))$. This proves (ii). □

Example 2.22. Let R be the k-semiring as described in Example 2.3. Then $\mathrm{Spec}(R)$ is compact and it is a T_0-space by Theorems 2.15 and 2.17. □

References

[1] **P. J. Allen**, *A fundamental theorem of homomorphisms for simirings*, Proc. Amer. Math. Soc. **21** (1969), 412 – 416.

[2] **M. F. Atiyah and I. G. Macdonald**, *Introduction to Commutative Algebra*, Addison Wesley Publishing Company, 1969.

[3] **S. E. Atani**, *The ideal theory in quotients of commutative semirings*, Glasnik Mat. **42** (2007), 301 – 308.

[4] **S. E. Atani**, *An ideal-based zero-divisor graph of a commutative semiring*, Glasnik Mat. **44** (2009), 141-153.

[5] **R. E. Atani and S. E. Atani**, *Ideal theory in commutative semirings*, Bul. Acad. Ştiinţe Repub. Mold. Mat. **2** (2008), 14 – 23.

[6] **R. E. Atani and S. E. Atani**, *Subsemimodules of semimodules*, Bul. Acad. Ştiinţe Repub. Mold. Mat. **63** (2010), 20 – 30.

[7] **S. E. Atani and R. E. Atani**, *Some remarks on partitioning semirings*, An. St. Univ. Ovidius Constanta **18** (2010), 49 – 62.

[8] **J. N. Chaudhari and D. Bonde**, *On partitioning and subtractive subsemimodules of semimodules over semirings*, Kyungpook Math. J. **50** (2010), 329 – 336.

[9] **J. S. Golan**, *Semirings and their Applications*, Kluwer Acad. Pub. Dodrecht, 1999.

[10] **U. Hebisch and U. J. Weinert**, Halbringe - Algebraische Theorie und Anwendungen in der Informatik, Teubner, Stuttgart 1993.

[11] **R. Y. McCasland, M. E. Moore and P. F. Smith**, *On the spectrum of a module over a commutative ring*, Comm. Algebra **25** (1997), 79 – 103.

Received February 1, 2011

S.E.Atani
Faculty of Mathematics, University of Guilan, P.O.Box 1914, Rasht, Iran
E-mail: ebrahimi@guilan.ac.ir

R.E.Atani
Department of Computer Engineering, University of Guilan, P.O.Box 3756, Rasht, Iran
E-mail: rebrahimi@guilan.ac.ir

Quasigroups and Related Systems **20** (2012), 37 − 52

Some enumerational results relating the numbers of latin and frequency squares of order n

Francis N. Castro, Gary L. Mullen and Ivelisse Rubio

Abstract We discuss some enumerational results relating the numbers of $F(n;\lambda_1,...,\lambda_m)$ and $F(n;\lambda'_1,...,\lambda'_k)$ frequency squares of order n. In particular, for any frequency vector $(\lambda_1,...,\lambda_m)$ of n, we discuss some enumerational results relating the number of $F(n;\lambda_1,...,\lambda_m)$ frequency squares and the number of latin squares of order n. In Section 4 we also discuss some enumerational results for latin rectangles.

1. Introduction

A *latin square of order* n is an $n\times n$ array in which each of the numbers $1,2,\ldots,n$ appears exactly once in each row and each column. By an $F(n;\lambda_1,\ldots,\lambda_m)$ *frequency square* is meant an $n\times n$ array in which each of the numbers i with $1\leqslant i\leqslant m$ appears exactly λ_i times in each row and each column. Thus we have $n=\lambda_1+\cdots+\lambda_m$ and an $F(n;1,\ldots,1)$ frequency square is a latin square of order n.

Let $\mathcal{F}(n;\lambda_1,\ldots,\lambda_m)$ denote the total number of distinct $F(n;\lambda_1,\ldots,\lambda_m)$ frequency squares and let $f(n;\lambda_1,\ldots,\lambda_m)$ represent the number of reduced squares where a frequency square as above is reduced if the first row and first column are both in standard order with λ_1 1's, λ_2 2's, and continuing, λ_m m's.

It is known from [1] that

Theorem 1.1. *For any frequency vector* $(\lambda_1,\ldots,\lambda_m)$ *of* n

$$\mathcal{F}(n;\lambda_1,\ldots,\lambda_m)=\binom{n}{\lambda_1,\ldots,\lambda_m}\binom{n-1}{\lambda_1-1,\ldots,\lambda_m}f(n;\lambda_1,\ldots,\lambda_m). \qquad \square$$

See [9] for some enumerational and classification results concerning latin squares. Let L_n denote the total number of latin squares of order n and let l_n denote the number of reduced latin squares of order n. It is known ([2], page 142) and easy to prove that

Corollary 1.2. *For* $n\geqslant 2$, $L_n=n!(n-1)!l_n$. $\square$

2010 Mathematics Subject Classification: 05B15.
Keywords: Latin square, frequency square, latin rectangle.

In this paper we prove several results relating the total number L_n of distinct latin squares of order n and the number of frequency squares with a fixed frequency vector. We also prove results relating the numbers of frequency squares of order n with two different frequency vectors.

It is known (see for example [8], Thm. 7.1) that a latin square of order n is equivalent to a 1-factorization of $K_{n,n}$, a bipartite graph in which each vertex of U is joined to each vertex of W, where U, W represent the rows and columns of a latin square of order n so that both U and W contain exactly n elements. If the symbol in position (i, j) is k, then we color the edge from i to j with color k. See page 107 of [8] for more details.

Now let $\vec{K_n}$ (see page 111 of [8]) be the complete directed graph with loops on n vertices. Then in Cor. 7.10 of [8] it is shown that the number of latin squares of order n with first row in standard order is the same as the number of 1-factorizations of $\vec{K_n}$. Also see [5] for connections between enumerating certain frequency squares and 1-factorizations of certain graphs.

Thus one can certainly show that counting latin squares can be done by counting 1-factorizations of an appropriate graph. In our paper we are not just counting or enumerating frequency squares, rather we are showing how to enumerate frequency squares with one frequency vector relative to the number of frequency squares with a different frequency vector. This is the main point of the current paper.

In [10] Wanless considers k-plexes for latin squares. Such objects are generalizations of transversals in latin squares. Many of our results could be stated using the terminlogy of k-plexes, but we prefer to use terminology involving i-transversals that is defined in the next section.

In [6] it was shown in Theorem 3.1 that one could relate the number of latin squares of order n to the number of 1-factorizations of frequency squares with frequency vector $\lambda_1, ..., \lambda_m$ via the use of isotopy classes. While the result in that paper is valid, the proof was incomplete in that it assumed (without proof) that each frequency square in an isotopy class had the same number of 1-factorizations. While this fact turns out to be true, it does require some proof. This proof is now given in Lemma 2.1 of the current paper.

In this paper we also extend the result from equation (2) in [6] dealing with latin and frequency squares, to the case where we relate the number of frequency squares with one frequency vector to the number of frequency squares with a different frequency vector.

2. Numbers of frequency and latin squares

Let $F(n; \lambda_1, \ldots, \lambda_m)$ be a frequency square of order n with frequency vector $(\lambda_1, \ldots, \lambda_m)$. For $i = 1, \ldots, m$, by an *i-transversal* is meant a set of n cells, one in each row and one in each column, each containing the symbol i. A set of n transversals containing λ_i, i-transversals for each $i = 1, \ldots, m$, forms a *partition* of the frequency square if for each i, the i-transversals disjointly partition the set of $n\lambda_i$ cells containing i. We define an *i-partition* to be the subset of a partition consisting of all i-transversals in the partition.

As in [1] two frequency squares F_1 and F_2 of the same order and frequency vector, are said to be *isotopic* if there exist permutations $\sigma_r, \sigma_c, \sigma_\#$ so that F_2 can be obtained from F_1 by applying σ_r to the rows of F_1, and then successively applying σ_c to the columns and $\sigma_\#$ to the numbers of each resulting square, respectively.

We now prove that frequency squares from the same isotopy class yield exactly the same number of partitions. This will greatly reduce our calculations which will of course be very helpful for larger values of n.

Lemma 2.1. *Assume that two frequency squares F_1 and F_2 (of the same order n and frequency vector) are isotopic. Then the number of partitions of F_1 is the same as the number of partitions of F_2.*

Proof. Let F_1 and F_2 be frequency squares of order n with the same frequency vector. Suppose that F_1 and F_2 are *isotopic*. Fix permutations σ_r, σ_c and $\sigma_\#$ and define a function from the set of partitions of F_1 to the set of partitions of F_2 by applying $\sigma_r, \sigma_c, \sigma_\#$ to the transversals of the partitions. Let F_1^r be the frequency square obtained after we apply σ_r to F_1. Given an i-transversal $\{(1, i_1), (2, i_2), \ldots, (n, i_n)\}$ of F_1 and applying σ_r to the i-transversal we obtain

$$\{(\sigma_r(1), i_1), \ldots, (\sigma_r(n), i_n)\},$$

an i-transversal of F_1^r. Let F_1^c be the frequency square obtained after we apply σ_c to F_1^r. Given an i-transversal $\{(1, i_1), (2, i_2), \ldots, (n, i_n)\}$ of F_1^r and applying σ_c to the i-transversal, we obtain $\{(1, \sigma_c(i_1)), \ldots, (n, \sigma_c(i_n))\}$, an i-transversal of F_1^c. Let $F_1^\#$ be the frequency square obtained after we apply $\sigma_\#$ to F_1^c. Note that $F_2 = F_1^\#$ for some $r, c, \#$. Given an i-transversal $\{(1, i_1), (2, i_2), \ldots, (n, i_n)\}$ of F_1^c we obtain the $\sigma_\#(i)$-transversal $\{(1, i_1), \ldots, (n, i_n)\}$ of F_2. Hence $\sigma_r, \sigma_c, \sigma_\#$ take a transversal of F_1 to a transversal of F_2.

Let $A = \{(1, i_1), \ldots, (n, i_n)\} \neq B = \{(1, j_1), \ldots, (n, j_n)\}$ be two distinct i-transversals of F_1. We claim that applying σ_r, σ_c, or $\sigma_\#$ to A and B we obtain distinct transversals. Suppose that $\sigma_c(A) = \{(1, \sigma_c(i_1)), \ldots, \ (n, \sigma_c(i_n))\} = \sigma_c(B) = \{(1, \sigma_c(j_1)), \cdots, (n, \sigma_c(j_n))\}$. Then $\sigma_c(i_k) = \sigma_c(j_k)$ for $k = 1, \ldots, n$. This implies that $i_k = j_k$ for $k = 1, \ldots, n$, contradicting the fact that $A \neq B$. The same can be proved for σ_r and $\sigma_\#$. We also claim that if $A \cap B = \emptyset$, then $\sigma_c(A) \cap \sigma_c(B) = \emptyset$. Suppose not. Then $(k, \sigma_c(i_k)) = (k, \sigma_c(j_k))$ for some $k = 1, \ldots, n$. Then $i_k = j_k$, contradicting that $A \cap B = \emptyset$. The same can be proved for σ_r and $\sigma_\#$. Hence, applying $\sigma_r, \sigma_c, \sigma_\#$ to a partition of F_1 we obtain a partition of F_2.

The above shows that $\sigma_\# \circ \sigma_c \circ \sigma_r$ is a well defined function between the sets of partitions of F_1 and F_2. This implies that the number of partitions of F_1 is less than or equal to the number of partitions of F_2. But we can repeat the same process starting with F_2 and we obtain that the number of partitions of F_2 is less than or equal to the number of partitions of F_1. Therefore, the number of partitions of F_1 and F_2 are equal. $\square$

It is clear from the previous proof that permutations of rows and columns take an i-transversal to another i-transversal. These permutations also take different i-transversals into different i-transversals; hence the number of i-transversals is preserved by permutations of rows and columns as the next lemma states.

Lemma 2.2. *Let F_1 and F_2 be frequency squares of the same order and frequency vector. Suppose that F_2 can be obtained from F_1 by successively applying permutations of rows and columns. Then, F_1 and F_2 have the same number of i-transversals.* □

Remark 1. Note that permutations $\sigma_{\#}$ of symbols of a frequency square take i-transversals to $\sigma_{\#}(i)$-transversals and therefore it is false in general that the number of i-transversals of frequency squares belonging to the same isotopy class is fixed, as it is shown in the next example.

Example 2.3. Considere the following reduced frequency squares with vector $(5; 2, 2, 1)$:

$$F_1 = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 2 & 3 & 2 \\ 2 & 2 & 3 & 1 & 1 \\ 2 & 3 & 1 & 1 & 2 \\ 3 & 2 & 1 & 2 & 1 \end{pmatrix}, \quad F_1' = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 \\ 1 & 3 & 1 & 2 & 2 \\ 2 & 2 & 3 & 1 & 1 \\ 2 & 2 & 1 & 3 & 1 \\ 3 & 1 & 2 & 1 & 2 \end{pmatrix}.$$

The square F_1' can be obtained from square F_1 by interchanging entries $1 \leftrightarrow 2$ and permuting the rows and columns to convert it into a reduced square and hence the two squares are isotopic. It can be checked that F_1 has 2, 1-transversals and 4, 2-transversals, and F_1' has 4, 1-transversals and 2, 2-transversals. Note that $\sigma_{\#}(1) = 2$ and the number of 1-transversals of F_1 is the number of 2-transversals of F_1'. □

Let $\Lambda(n; \lambda_1, \dots, \lambda_m)$ denote the number of distinct isotopy classes of frequency squares $F(n; \lambda_1, \dots, \lambda_m)$. For a fixed frequency vector, from Theorem 1.1, we know that the number of isotopy classes of frequency squares is the same as the number of isotopy classes of reduced frequency squares. Assume that the j-th class contains n_j reduced squares so that

$$\sum_{j=1}^{\Lambda(n;\lambda_1,\dots,\lambda_m)} n_j = f(n; \lambda_1, \dots, \lambda_m). \tag{1}$$

We now prove

Theorem 2.4. *For any frequency vector $(\lambda_1, \dots, \lambda_m)$ of n*

$$\binom{n}{\lambda_1, \dots, \lambda_m} \binom{n-1}{\lambda_1 - 1, \dots, \lambda_m} \sum_{j=1}^{\Lambda(n;\lambda_1,\dots,\lambda_m)} n_j \delta^{(j)} \lambda_1! \cdots \lambda_m! \tag{2}$$

$$= n!(n-1)!l_n = L_n,$$

where $\delta^{(j)}$ denotes the number of distinct partitions of any reduced frequency square $F(n; \lambda_1, \dots, \lambda_m)$ in the j-th isotopy class of reduced squares which contains n_j reduced squares.

Proof. How many distinct latin squares of order n does the left hand side of (2) generate? Consider the j-th isotopy class. By Lemma 2.1 each frequency square in this class has the same number $\delta^{(j)}$ of partitions so consider a fixed reduced frequency square $F = F(n; \lambda_1, \dots, \lambda_m)$ in this class. Using this reduced frequency square one can construct different latin squares in the following way.

Fix a partition P of F. For each 1-transversal in P, replace each value 1 in the cells given by the 1-transversal by a number k, $k = 1, \cdots, \lambda_1$, one number for each of the λ_1 1-transversals. Since the 1-transversals are disjoint, this gives $\lambda_1!$ different latin squares of order n. Similarly, for each 2-transversal of F, replace the number 2 by $\lambda_1 + 1, \cdots, \lambda_1 + \lambda_2$. Doing the same for each $i = 1, \cdots, m$, the partition P generates $\lambda_1! \times \cdots \times \lambda_m!$ distinct latin squares of order n. Each of the $\binom{n}{\lambda_1, \dots, \lambda_m}\binom{n-1}{\lambda_1 - 1, \dots, \lambda_m}$ distinct frequency squares obtained by permuting rows and columns of F will also produce $\lambda_1! \times \cdots \times \lambda_m!$ latin squares.

Continuing, this can be repeated for each of the n_j reduced squares in the j-th isotopy class. Finally, we doing this for each class we get that the number of latin squares of order n generated from the left hand side will be at most L_n.

Conversely, given a latin square L_1 of order n, construct a frequency square $FS_1 = F_1(n; \lambda_1, \dots, \lambda_m)$ in the following way: replace the numbers $1, 2, \dots, \lambda_1$ in the latin square by 1, the numbers $\lambda_1 + 1, \dots, \lambda_1 + \lambda_2$ by 2 and continuing, until the numbers $\lambda_1 + \cdots + \lambda_{m-1} + 1, \dots, n$ by m.

Consider the $a_1, \dots, a_{\lambda_1}$, 1-transversals forming a 1-partition of FS_1. Note that any latin square with the numbers $\lambda_1 + 1, \dots, n$ in the same positions as L_1 and with a value i_1, $1 \leqslant i_1 \leqslant \lambda_1$ in the positions of a_1, a value $i_2 \neq i_1$, $1 \leqslant i_2 \leqslant \lambda_1$ in the positions of a_2 and so on gives FS_1 if we apply the above construction. There are $\delta_1(FS_1)\lambda_1!$ latin squares that give FS_1 under this construction, where $\delta_1(FS_1)$ is the number of 1-partitions of FS_1 and there are no other latin squares that give FS_1 under this construction. Something similar happens for all the other i-partitions. Let C_1 be the set of all these latin squares; this is, C_1 is the set of all the latin squares that give FS_1 under this construction. There are exactly $\delta_1(FS_1) \cdots \delta_m(FS_1)\lambda_1! \cdots \lambda_m!$ different latin squares in C_1, where $\delta_i(FS_1)$ is the number of i-partitions of FS_1.

Take another latin square of order n that it is not in C_1 and construct a frequency square FS_2 with the above construction. This gives another set C_2 of latin squares associated to FS_2. Repeat until we have a set $\{C_1, \cdots, C_k\}$ such that any latin square of order n belongs to a C_s and each C_s corresponds to a unique FS_s. We then have that

$$L_n = \sum_{s=1}^{k} |C_s| = \sum_{s=1}^{k} \delta^{(s)} \lambda_1! \cdots \lambda_m!$$

$$\leqslant \sum_{s=1}^{\mathcal{F}} \delta^{(s)} \lambda_1! \cdots \lambda_m! = \binom{n}{\lambda_1, \ldots, \lambda_m} \binom{n-1}{\lambda_1 - 1, \ldots, \lambda_m} \sum_{s=1}^{f} \delta^{(s)} \lambda_1! \cdots \lambda_m!,$$

where $\mathcal{F}$ is the total number of frequency squares $F(n; \lambda_1, \ldots, \lambda_m)$, f is the total number of reduced frequency squares with the same frequency vector and $\delta^{(s)} = \delta_1(FS_s) \cdots \delta_m(FS_s)$ is the number of partitions of the frequency square FS_s.

Using (1) one can now sum over the isotopy classes of reduced frequency squares to see that $\delta^{(s)}$ coincides with $\delta^{(j)}$ in equation (2) and get that

$$L_n \leqslant \binom{n}{\lambda_1, \ldots, \lambda_m} \binom{n-1}{\lambda_1 - 1, \ldots, \lambda_m} \sum_{j=1}^{\Lambda(n;\lambda_1,\ldots,\lambda_m)} n_j \delta^{(j)} \lambda_1! \cdots \lambda_m!. \qquad \square$$

One can easily simplify the result of the theorem to obtain

Corollary 2.5. *For any frequency vector* $(\lambda_1, \ldots, \lambda_m)$ *of* n

$$n! \binom{n-1}{\lambda_1 - 1, \ldots, \lambda_m} \sum_{j=1}^{\Lambda(n;\lambda_1,\ldots,\lambda_m)} n_j \delta^{(j)} = n!(n-1)! l_n = L_n,$$

where $\delta^{(j)}$ *denotes the number of distinct partitions of any reduced frequency square* $F(n; \lambda_1, \ldots, \lambda_m)$ *in the* j*-th isotopy class which contains* n_j *reduced squares.* $\square$

We note that results for the number of isotopy classes of frequency squares of order $n \leqslant 6$ can be found in [1] while results for orders 7 and 8 can be found in [7].

Example 2.6. For $n = 4$, from [1] there are five reduced $F(4; 2, 2)$ frequency squares and these are given by

$$F_1 = \begin{matrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{matrix}, \quad F_2 = \begin{matrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \\ 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 \end{matrix}, \quad F_3 = \begin{matrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 \\ 2 & 1 & 1 & 2 \end{matrix}$$

$$F_4 = \begin{matrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \end{matrix}, \quad F_5 = \begin{matrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 1 & 1 \end{matrix}$$

Square	$\#1-trans.$	$\#2-trans.$	δ_j
F_1	4	4	4
F_2	2	2	1
F_3	2	2	1
F_4	2	2	1
F_5	2	2	1

Note that from [1], there are just two distinct isotopy classes; the first containing just the square F_1 while the second class contains the four squares $F_2, \dots, F_5$. Hence our theorem yields

$$\binom{4}{2,2}\binom{3}{2,1}[4(2!)(2!)+4(2!)(2!)] = 6(3)(16+16) = 576 = 4!3!(4) = L_4. \qquad \square$$

Remark 2. The above results simplify considerably when there is only one isotopy class. This is the case for frequency squares $F(n; n-1, 1)$.

The next argument shows that there is only one isotopy class for $F(n; n-1, 1)$ frequency squares. Since each row and column contains only one 2 and the rest 1's, we can easily interchange rows and columns to show that every $F(n; n-1, 1)$ frequency square is isotopic to the square

$$\begin{array}{ccccc}
1 & 1 & \cdots & 1 & 2 \\
1 & 1 & \cdots & 2 & 1 \\
. & . & . & . & . \\
. & . & . & . & . \\
. & . & . & . & . \\
2 & 1 & \cdots & 1 & 1
\end{array}$$

which has 2's on the back diagonal. It is easy to see that there are $(n-2)!$ reduced frequency squares of this type.

3. Enumerating frequency squares

In this section we enumerate frequency squares of certain frequency vectors using the number of i-transversals of frequency squares of a related frequency vector. We also give a formula to compute the number of 1-transversals of frequency squares $F(n; n-1, 1)$. As a consequence we can compute the number of frequency squares $F(n; n-2, 1, 1)$ for any $n \geqslant 3$. Let $F(n)$ be a frequency square of order n and let $T_i(F(n))$ be the number of i-transversals of $F(n)$.

Lemma 3.1. *Let* $(\lambda_1, \dots, \lambda_m, \underbrace{1, \dots, 1}_{s})$ *be a frequency vector of* n *where* $\lambda_m \neq \lambda_j$ *for all* $j \neq m$, *and let* $\Lambda = \Lambda(n; \lambda_1, \dots, \lambda_m, \underbrace{1, \dots, 1}_{s})$ *be the number of distinct isotopy classes of frequency squares associated to it. Then*

$$\binom{n}{\lambda_1, \dots, \lambda_m}\binom{n-1}{\lambda_1 - 1, \dots, \lambda_m}\sum_{j=1}^{\Lambda} n_j T_m(F_j(n)) \tag{3}$$
$$= \mathcal{F}(n; \lambda_1, \dots, \lambda_{m-1}, \lambda_m - 1, \underbrace{1, \dots, 1}_{s+1})$$

where $\lambda_m \geqslant 2$, $s \geqslant 0$, and $T_m(F_j(n))$ denotes the number of distinct m-transversals of any reduced frequency square $F(n;\lambda_1,\ldots,\lambda_m,1,\ldots,1)$ in the j-th isotopy class of reduced frequency squares which contains n_j reduced squares.

Proof. Assume that $\lambda_m \neq \lambda_j$ for all $j \neq m$. This implies that the permutations used to construct the isotopy classes of the frequency vector $(n;\lambda_1,\ldots,\lambda_m,1,\ldots,1)$ do not include permutations $\sigma_{\#}$ of the symbol m because, if one apply the permutation $\sigma_{\#}(m)$, the resulting frequency square will have a different frequency vector and all the vectors in the isotopy class must have the same frequency vector. Hence, by Lemma 2.2 the number of m-transversals within an isotopy class is fixed.

Given a frequency square $FS^m = F(n;\lambda_1,\ldots,\lambda_m,1,\ldots,1)$ we construct another frequency square $FS^{m-1} = F(n;\lambda_1,\ldots,\lambda_{m-1},\lambda_m-1,1,1,\ldots,1)$ in the following way: consider an m-transversal of FS^m and replace the m's in the entries given by the m-transversal by the number $l = m+s+1$. Each of the $T_m(FS^m)$ different m-transversals of FS^m gives a different frequency square FS^{m-1}. The same can be done with each of the $T_m(F_j(n))$ m-transversals of the $\binom{n}{\lambda_1,\ldots,\lambda_m}\binom{n-1}{\lambda_1-1,\ldots,\lambda_m}$ different frequency squares FS^m given by each of the n_j reduced frequency squares in the j-th isotopy class of FS^m. Hence,

$$\binom{n}{\lambda_1,\ldots,\lambda_m}\binom{n-1}{\lambda_1-1,\ldots,\lambda_m}\sum_{j=1}^{\Lambda} n_j T_m(F_j(n))$$

$$\leqslant \mathcal{F}(n;\lambda_1,\ldots,\lambda_{m-1},\lambda_m-1,\underbrace{1,\ldots,1}_{s+1})$$

Conversely, given a frequency square FS_1^{m-1} construct a frequency square FS_1^m by replacing the number $l = m+s+1$ by the number m. Any frequency square with the number i in the λ_i positions of FS_1^{m-1} for $i \neq m, l$ will produce the same frequency square FS_1^m. Let C_1 be the set of all the frequency squares FS^{m-1} that produce FS_1^m under the above construction. The number of squares FS^{m-1} in C_1 is the number of m-transversals of FS_1^m. Take another frequency square FS_2^{m-1} that it is not in C_1 and construct FS_2^m. This gives another set C_2, and, repeating the construction, we get a set $\{C_1,\cdots,C_k\}$, where each frequency square FS^{m-1} belongs to a C_i and each C_s corresponds to a unique FS^m. This gives

$$\begin{aligned}\mathcal{F}(n;\lambda_1,\ldots,\lambda_{m-1},\lambda_m-1,\underbrace{1,\ldots,1}_{s+1}) &= \sum_{i=1}^{k} |C_i| \\ &= \sum_{i=1}^{k} T_m(FS_i^m) \leqslant \sum_{i=1}^{\mathcal{F}} T_m(FS_i^m),\end{aligned}$$

where $\mathcal{F}$ is the total number of frequency squares $F(n;\lambda_1,\ldots,\lambda_m,1,\ldots,1)$. Since the number of m-transversals do not change with row and column permutations and the number of m-transversals does not change within the isotopy classes we have that

$$\begin{aligned}&\mathcal{F}(n;\lambda_1,\ldots,\lambda_{m-1},\lambda_m-1,\underbrace{1,\ldots,1}_{s+1})\\ &\leqslant \binom{n}{\lambda_1,\ldots,\lambda_m}\binom{n-1}{\lambda_1-1,\ldots,\lambda_m}\sum_{j=1}^{f}T_m(F_j(n))\\ &=\binom{n}{\lambda_1,\ldots,\lambda_m}\binom{n-1}{\lambda_1-1,\ldots,\lambda_m}\sum_{j=1}^{\Lambda}n_jT_m(F_j(n)),\end{aligned}$$

where f is the number of reduced frequency squares with frequency vector of the form $(n;\lambda_1,\ldots,\lambda_m,1,\ldots,1)$ and n_j is the number of reduced squares in the j-th isotopy class. □

Example 3.2. The above lemma gives a way to compute $\mathcal{F}(8;6,1,1)$ using reduced frequency squares with frequency vector $(7,1)$. Namely, it is known that $f(n;n-1,1)=(n-2)!$ and, by Remark 2, there is only one isotopy class of frequency squares with frequency vector $(n-1,1)$. Hence

$$\mathcal{F}(8;6,1,1)=8\times 7\times 6!\times T_1(8;7,1)=598,066,560,$$

as reported in [7]. □

Example 3.3. In general, to compute $\mathcal{F}(n;n-2,1,1)$ using reduced frequency squares with frequency vector $(n-1,1)$, we need to compute $T_1(F(n;n-1,1))$, and then

$$\mathcal{F}(n;n-2,1,1)=n!\times T_1(F(n;n-1,1)).$$

Theorem 3.8 gives a formula to compute $\mathcal{F}(n;n-2,1,1)$ for any n. □

Remark 3. If $\lambda_m=\lambda_i$ for some i, then Lemma 3.1 is false. The reason is that one can interchange the numbers m and i in a frequency square to obtain another frequency square in the same isotopy class but both having different numbers of m-transversals. In fact, two reduced frequency squares in the same isotopy class can have have different m-transversals as we saw in Example 2.3. Therefore, in this case one cannot group the reduced squares in the isotopy class to get n_j in equation (3). However, if instead of summing over the isotopy classes, one sums over all the reduced frequency squares, one obtains a formula that works for any frequency vector as we see in Lemma 3.5.

Remark 4. Note that, since one can relabel $i\leftrightarrow m$, and interchange the positions of λ_m,λ_i, it is enough to have any λ_i be such that $\lambda_i\neq\lambda_j$ for all $j\neq i$.

Lemma 3.1 can be applied successively to obtain the following result.

Theorem 3.4. *Let* $(\lambda_1, \ldots, \lambda_l, \cdots, \lambda_m, \underbrace{1, \ldots, 1}_{s})$ *be a frequency vector of* n *where* $\lambda_i \neq \lambda_j$ *for* $i = l, \cdots, m,\ j = 1, \cdots, m,$ *and let* Λ *be the number of distinct isotopy classes of reduced frequency squares associated to it. Then*

$$\binom{n}{\lambda_1, \ldots, \lambda_m}\binom{n-1}{\lambda_1 - 1, \ldots, \lambda_m}\sum_{j=1}^{\Lambda} n_j T_{l+1}(F_j(n)) \cdots T_m(F_j(n))$$
$$= \mathcal{F}(n; \lambda_1, \ldots, \lambda_l, \lambda_{l+1} - 1, \ldots, \lambda_{m-1} - 1, \lambda_m - 1, \underbrace{1, \ldots, 1}_{s+m-l+1}),$$

where $\lambda_l \geqslant 2, \ldots, \lambda_m \geqslant 2,\ s \geqslant 0,$ *and* $T_l(F_j(n))$ *denote the number of distinct* l*-transversals of any reduced frequency square* $F_j(n; \lambda_1, \ldots, \lambda_m, 1, \ldots, 1)$ *in the* j*-th isotopy class of reduced squares which contains* n_j *reduced squares.* □

Note that Lemma 3.1 requires $\lambda_m \neq \lambda_i$ for all $i \neq m$. Alternatively, one can sum over all the reduced frequency squares and then this assumption is not needed:

Lemma 3.5. *For any frequency vector* $(\lambda_1, \ldots, \lambda_m, \underbrace{1, \ldots, 1}_{s})$ *of* n*, let* f *be the number of distinct reduced frequency squares with this frequency vector. Then*

$$\binom{n}{\lambda_1, \ldots, \lambda_m}\binom{n-1}{\lambda_1 - 1, \ldots, \lambda_m}\sum_{j=1}^{f} T_m(F_j(n))$$
$$= \mathcal{F}(n; \lambda_1, \ldots, \lambda_{m-1}, \lambda_m - 1, \underbrace{1, \ldots, 1}_{s+1})$$

where $\lambda_m \geqslant 2,\ s \geqslant 0,$ *and* $T_m(F_j(n))$ *denotes the number of distinct* m*-transversals of the reduced frequency square* $F_j(n; \lambda_1, \ldots, \lambda_m, 1, \ldots, 1)$ *and the sum is over the* f *different reduced frequency squares.* □

Theorem 3.6. *For any frequency vector* $(\lambda_1, \ldots, \lambda_m, \underbrace{1, \ldots, 1}_{s})$ *of* n*, let* f *be the number of distinct reduced frequency squares with this frequency vector. Then*

$$\binom{n}{\lambda_1, \ldots, \lambda_m}\binom{n-1}{\lambda_1 - 1, \ldots, \lambda_m}\sum_{j=1}^{f} T_{l+1}(F_j(n)) \cdots T_m(F_j(n))$$
$$= \mathcal{F}(n; \lambda_1, \ldots, \lambda_l, \lambda_{l+1} - 1, \ldots, \lambda_{m-1} - 1, \lambda_m - 1, \underbrace{1, \ldots, 1}_{s+m-l+1}),$$

where $\lambda_l \geqslant 2, \ldots, \lambda_m \geqslant 2,\ s \geqslant 0,$ *and* $T_l(F_j(n))$ *denote the number of distinct* l*-transversals of the reduced frequency square* $F_j(n; \lambda_1, \ldots, \lambda_m, 1, \ldots, 1)$ *and the sum is over the* f *different reduced frequency squares.* □

The following is a well known result for derangements. When it is reinterpreted for frequency squares, it gives a formula to compute the number of 1-transversals of a frequency square with frequency vector $(n-1,1)$.

Lemma 3.7. *Let $T_1(F(n;n-1,1))$ be the number of 1-transversals of an $F(n;n-1,1)$ frequency square. Then*

$$T_1(F(n;n-1,1)) = (n-1)\left(T_1(F(n-1;n-2,1)) + T_1(F(n-2;n-3,1))\right)$$

$$= n!\sum_{i=2}^{n}\frac{(-1)^i}{i!}. \qquad \square$$

Note that this is the number of derangements of n symbols. The above result, together with Lemma 3.1, and the fact that there is only one isotopy class for frequency squares $F(n;n-1,1)$ with $(n-2)!$ reduced frequency squares is used to obtain a formula for the number of frequency squares $\mathcal{F}(n;n-2,1,1)$ for any $n \geqslant 3$.

Theorem 3.8. *Let $\mathcal{F}(n;n-2,1,1)$ be the number of frequency squares with frequency vector $(n-2,1,1)$. Then,*

$$\mathcal{F}(n;n-2,1,1) = n!n!\sum_{i=2}^{n}\frac{(-1)^i}{i!}. \qquad \square$$

The number of reduced frequency squares $f(n;n-2,1,1)$ for $n \leqslant 8$ where given in [1] and [7]. Theorem 3.8 gives a formula for the value of $f(n;n-2,1,1)$ for any $n \geqslant 3$.

Corollary 3.9. *Let $f(n;n-2,1,1)$ be the number of reduced frequency squares with frequency vector $(n-2,1,1)$. Then,*

$$f(n;n-2,1,1) = (n-3)!(n-2)!n\sum_{i=2}^{n}\frac{(-1)^i}{i!}.$$

n	$f(n,n-2,1,1)$
7	7416
8	254280
9	12014640
10	747578160
11	59329146240
12	5814256049280

4. Transversals and latin rectangles

Let $T_1(n; n-1, 1)$ be the number of 1-transversals of an $F(n; n-1, 1)$ frequency square. Consider the two line latin rectangles with first row 1,2,3:

$$R_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

We can associate 1-transversals to the above two line latin rectangles as follows. Consider the frequency square

$$F_d(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

with 2's on the main diagonal. The 1-transversal of $F_d(3)$ associated to R_1 is

$$\{(1,2),(2,3),(3,1)\},$$

and the 1-transversal associated to R_2 is

$$\{(1,3),(2,1),(3,2)\}.$$

Note that there are correspondences $\{(1,2),(2,3),(3,1)\} \mapsto (2\ 3\ 1)$ and $\{(1,3),(2,1),(3,2)\} \mapsto (3\ 1\ 2)$.

We can generalize this construction for any n since no 1-transversal of the frequency square $F_d(n)$ with 2's in the diagonal will contain the pair (i,i) for $i = 1, \ldots, n$. In general, consider the "diagonal" frequency square of order n

$$F_d(n) = \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ & \vdots & & \\ 1 & 1 & \cdots & 2 \end{pmatrix}. \tag{4}$$

Note that the set of 1-transversals of $F_d(n)$ is

$$A = \{\{(1,i_1),(2,i_2),\cdots,(n,i_n)\} \mid i_l \neq l, i_k \neq i_l \text{ for } k \neq l\},$$

and

$$\{(1,i_1),(2,i_2),\cdots,(n,i_n)\} \mapsto (i_1\ i_2\ \cdots\ i_n)$$

defines a 1-1 correspondence between the set of 1-transversals A and the set of two line latin rectangles whose first row is in the natural order $1, 2, \ldots, n$ and second row is $(i_1\ i_2\ \cdots\ i_n)$.

For $m \leqslant n$, let $R(m,n)$ be the number of m line latin rectangles of order n whose first row is in standard order $1, 2, \ldots, n$.

Corollary 4.1. *For each* $n \geqslant 2$, $R(2,n) = T_1(n; n-1, 1)$. □

The correspondence of pairs of disjoint 1-transversals of $F_d(n)$ and 3 line latin rectangles is similar. Consider the diagonal frequency square (4) and note that the set of pairs of disjoint 1-transversals of this square is

$$A = \{\{\{(1,i_1),(2,i_2),\cdots,(n,i_n)\},\{(1,j_1),(2,j_2),\cdots,(n,j_n)\}\} \mid$$
$$i_l, j_l \neq l, i_k \neq i_l \text{ and } j_k \neq j_l \text{ for } k \neq l, \text{ and } i_k \neq j_k\}.$$

Now each element in A (a pair) defines the last two rows

$$(i_1\ i_2\ \cdots\ i_n),(j_1\ j_2\ \cdots\ j_n)$$

of a three line latin rectangle with first row in the natural order. Since we can interchange the order of the last 2 rows, we have 2 different three line latin rectangles with first row in the natural order for each element in A. Let $T_1^{(m)}(n;n-1,1)$ be the number of sets of m disjoint 1-transversals of the frequency square (2). Hence $T_1^{(1)}(n;n-1,1) = T_1(n;n-1,1)$.

Corollary 4.2. *For each $n \geqslant 3$, $R(3,n) = 2T_1^{(2)}(n;n-1,1)$.* □

The construction for m line latin rectangles is similar: the set A is the set of all sets of $m-1$ disjoint 1-transversals of (4). Each element in A gives $m-1$ rows of the m line latin rectangle. There are $(m-1)!$, m line latin rectangles for each element in A.

Corollary 4.3. *For $1 \leqslant m \leqslant n$, $R(m,n) = (m-1)!T_1^{(m-1)}(n;n-1,1)$.* □

See page 142 of [2] for the number of m line latin rectangles of order $n \leqslant 11$.

Corollary 4.4. *For each $n \geqslant 2$, $T_1^{(n-1)}(n;n-1,1) = l_n$, the number of reduced latin squares of order n.* □

5. Relating the numbers of frequency squares with two different frequency vectors

In this section we extend our results from Section 2 in order to be able to go from one frequency vector to another, not just from a given frequency vector to the vector $(1,\ldots,1)$ involving latin squares.

Let $\lambda_1 + \cdots + \lambda_m$ be a partition of n. Another partition

$$\lambda'_{11} + \cdots + \lambda'_{1e_1} + \cdots + \lambda'_{m1} + \cdots + \lambda'_{me_m}$$

of n is a *refinement*, if for each $i = 1,\ldots,m$, $\lambda_i = \lambda'_{i1} + \cdots + \lambda'_{ie_i}$. In this case, will call $(\lambda'_{11},\ldots,\lambda'_{me_m})$ a *refinement vector* of $(\lambda_1,\ldots,\lambda_m)$

For each $i = 1,\ldots,m$, we have $\lambda_i n$ cells (λ_i in each row and column) in the $F(n;\lambda_1,\ldots,\lambda_m)$ frequency square containing the symbol i. For each $i = 1,\ldots,m$, we now form an $(\lambda'_{i1}, \ldots, \lambda'_{ie_i})$-array containing e_i disjoint blocks. The first block has $\lambda'_{i1}n$ cells with λ'_{i1} cells in each row and column. Continuing, the e_i-th block has $\lambda'_{ie_i}n$ cells with λ'_{ie_i} cells occurring in each row and column.

In Section 2, to construct latin squares from frequency squares, we replaced the values of the cells given by each of the i-transversals of an i-partition by a symbol, one symbol for each transversal, hence λ_i symbols for each i-partition. Now, to construct frequency squares with frequency vector $(n; \lambda'_{11}, \ldots, \lambda'_{me_m})$, we will replace the values of the cells given in each block of a $(\lambda'_{i1}, \ldots, \lambda'_{ie_i})$-array by a symbol, one symbol for each block, hence e_i symbols for each $(\lambda'_{i1}, \ldots, \lambda'_{ie_i})$-array.

Let $\delta_i(F)$ be the number of such arrays arising from the symbol i which occurs in the reduced frequency square $F = F(n; \lambda_1, \ldots, \lambda_m)$. Following the proof of Lemma 2.1, one can prove that the product $\delta = \delta_1(F) \cdots \delta_m(F)$ is invariant in an isotopy class:

Lemma 5.1. *Assume that two frequency squares F_1 and F_2 (of the same order n and frequency vector) are isotopic. Then the number of arrays from F_1 is the same as the number of arrays from F_2; that is $\delta_1(F_1) \cdots \delta_m(F_1) = \delta_1(F_2) \cdots \delta_m(F_2)$.* □

Remark 5. As in Example 2.3, for a fixed i, $\delta_i(F_1)$ might not be equal to $\delta_i(F_2)$, but, since we are considering all the symbols in the product, we get that we have $\delta_1(F_1) \cdots \delta_m(F_1) = \delta_1(F_2) \cdots \delta_m(F_2)$.

We now obtain a theorem that extends the result in Theorem 2.4:

Theorem 5.2. *If $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a frequency vector of n and $(\lambda'_{11}, \ldots, \lambda'_{me_m})$ is a fixed refinement vector of λ, then*

$$\binom{n}{\lambda_1, \ldots, \lambda_m}\binom{n-1}{\lambda_1 - 1, \ldots, \lambda_m} \sum_{j=1}^{\Lambda(n;\lambda_1,\ldots,\lambda_m)} n_j \delta^{(j)} e_1! \cdots e_m!$$

$$= \binom{n}{\lambda'_{11}, \ldots, \lambda'_{me_m}}\binom{n-1}{\lambda'_{11} - 1, \ldots, \lambda'_{me_m}} f(n; \lambda'_{11}, \ldots, \lambda'_{me_m})$$

$$= \mathcal{F}(n; \lambda'_{11}, \ldots, \lambda'_{me_m})$$

where $\delta^{(j)}$ denotes the number of distinct arrays (as defined above) of any reduced frequency square $F(n; \lambda_1, \ldots, \lambda_m)$ in the j-th isotopy class of reduced squares which contains n_j reduced squares. □

As the proof of this theorem is similar to the proof of Theorem 2.4 in Section 2 for determining the total number of latin squares from reduced $F(n; \lambda_1, \ldots, \lambda_m)$ frequency squares, we omit the proof and instead, provide the reader with the following illustrative example.

We start with reduced $F(5; 4, 1)$ frequency squares and determine the total number of $F(5; 2, 2, 1)$ frequency squares. There is only one isotopy class and

$(5-2)!$ reduced frequency squares with the frequency vector $(4,1)$. Consider

$$F = \begin{matrix} 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \end{matrix}\ .$$

There are (4)(5)=20 cells containing the symbol 1. Form a (2,2)-array containing 2 blocks with 10 cells each, 2 per row and column. This is the same as considering a partition and selecting 2, 1-transversals to construct one block and 2 other 1-transversals to construct the other block. For example, from the partition

$$P = \{\{(1,1),(2,2),(3,4),(4,3),(5,5)\},\{(1,2),(2,3),(3,5),(4,1),(5,4)\}\ ,$$

$$\{(1,3),(2,5),(3,1),(4,4),(5,2)\},\{(1,4),(2,1),(3,2),(4,5),(5,3)\},$$

$$\{(1,5),(2,4),(3,3),(4,2),(5,1)\}\},$$

one can form an array $\{B_1, B_2\}$ with the two blocks

$$B_1 = \{(1,1),(2,2),(3,4),(4,3),(5,5),(1,2),(2,3),(3,5),(4,1),(5,4)\},$$

$$B_2 = \{(1,3),(2,5),(3,1),(4,4),(5,2),(1,4),(2,1),(3,2),(4,5),(5,3)\}.$$

The 1's in B_1 can be changed to 3's to obtain

$$F' = \begin{matrix} 3 & 3 & 1 & 1 & 2 \\ 1 & 3 & 3 & 2 & 1 \\ 1 & 1 & 2 & 3 & 3 \\ 3 & 2 & 3 & 1 & 1 \\ 2 & 1 & 1 & 3 & 3 \end{matrix}\ .$$

Note that there are $e_1! = 2!$ ways to replace the symbol 1 using this array. There are a total of $\delta_1 = 108$ distinct arrays containing the symbol 1. Theorem 5.2 implies that there are 72 reduced frequency squares $F(5;2,2,1)$, which agrees with the results from [1].

Acknowledgements: The authors appreciate the careful review and comments made by the referees which helped to correct and improve this final version. The second author would like to sincerely thank the Departments of Computer Science and Mathematics at the University of Puerto Rico, Río Piedras for their hospitality during his two visits in 2008 and 2009 when most of this work was completed. The third author appreciates the hospitality of the Department of Mathematics of the Pennsylvania State University during her visit in 2011.

References

[1] **L.J. Brant and G.L. Mullen**, *Some results on enumeration and isotopic classification of frequency squares*, Utilitas Math. **29**(1986), 231 – 244.

[2] **C.J. Colbourn and J.H. Dinitz, Editors**, *Handbook of Combinatorial Designs*, Sec. Ed., Chapman and Hall/CRC, Boca Raton, FL, 2007.

[3] **J. Dénes and A.D. Keedwell**, *Latin Squares and their Applications*, Academic Press, New York, 1974.

[4] **J. Dénes and A.D. Keedwell**, *Latin Squares*, Annals of Disc. Math., Vol. **46** (1991), North-Holland, Amsterdam.

[5] **J. Dénes, A.D. Keedwell, and G.L. Mullen**, *Connections between r-factors of labelled graphs and frequency squares*, Ars Combinatoria **28**(1989), 201 – 202.

[6] **J. Dénes and G.L. Mullen**, *Enumeration formulas for latin and frequency squares*, Discrete Math. **111**(1993), 157 – 163.

[7] **V. Krčadinac**, *Frequency squares of orders 7 and 8*, Utilitas Math. **72**(2007), 89 – 95.

[8] **C.F. Laywine and G.L. Mullen**, *Discrete Mathematics Using Latin Squares*, Wiley-Interscience Series in Discrete Mathematics and Optimization, New York, 1998.

[9] **B.D. McKay and I.M. Wanless**, On the number of latin squares, Ann. Comb. **9**(2005), 335 – 344.

[10] **I.M. Wanless**, A generalization of transversals for latin squares, Electron J. Combinatorics **9**(2002), #R12.

Received December 10, 2010
Revised January 10, 2012

F. N. Castro
Department of Mathematics, University of Puerto Rico, Río Piedras. PO Box 70377, San Juan, PR 00936-8377
E-mail: franciscastr@gmail.com

G. L. Mullen
Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA
E-mail: mullen@math.psu.edu

I. Rubio
Department of Computer Science, University of Puerto Rico, Río Piedras, PO Box 70377, San Juan, PR 00936-8377
E-mail: iverubio@gmail.com

Quasigroups and Related Systems **20** (2012), 53 − 60

Some results on E-inversive semigroups

Roman S. Gigoń

Abstract. In the paper we study E-inversive semigroups. We show that E-inversive semigroups are M-semigroups and we prove that M-biordered sets arise from E-inversive semigroups. Moreover, some connections between bi-ideals of an E-inversive semigroup S and bi-order ideals, order bi-ideals of the biordered set E_S of S are given. Further, some results of Janet Mills concerning matrix congruences on orthodox semigroups are generalized to E-inversive E-semigroups. Also, we prove that the class of all E-inversive semigroups is structurally closed.

1. Introduction and preliminaries

In the paper we present some results on E-inversive semigroups. The main result of this article is Theorem 2.18 i.e. we show that every M-biordered set arises from some E-inversive semigroup. Our proof of this result is quite simple. Proving this result we used the characterization of the M-set of a semigroup (see Prop. 2.12) and an important Easdown's result (that is, every biordered set comes from some semigroup). Moreover, we can show in a similar way Nambooripad's Theorem (i.e., each regular biordered set comes from some regular semigroup). The proofs of this result were more complicated, see [2, 13]. Also, some equivalent conditions for a semigroup to be E-inversive are given (Corollaries 2.4, 2.11). Further, some connections between bi-ideals of an E-inversive semigroup S and order bi-ideals, bi-order ideals of the biordered set E_S are presented in this work (see Prop. 2.14 and Th. 2.16). Moreover, we give some remarks concerning matrix congruences on E-inversive (E-)semigroups (see Cor. 2.7 and Th. 2.10). Finally, we prove that the class of E-inversive semigroups is structurally closed (Cor. 2.6).

Let S be a semigroup, $a \in S$. The set $W(a) = \{x \in S : x = xax\}$ is called the set of all *weak inverses* of a, and so the elements of $W(a)$ will be called *weak inverse elements* of a. A semigroup S is called *E-inversive* iff for every $a \in S$ there exists $x \in S$ such that $ax \in E_S$, where E_S (or briefly E) is the set of idempotents of S (more generally, if $A \subseteq S$, then E_A denotes the set of all idempotents of A). It is easy to see that a semigroup S is E-inversive if and only if $W(a)$ is nonempty for all $a \in S$. Hence if S is E-inversive, then for every $a \in S$ there is $x \in S$ such that $ax, xa \in E_S$ (see [10, 11]).

Further, by $Reg(S)$ we shall mean the set of *regular elements* of S (an element a of S is called *regular* if $a \in aSa$) and by $V(a) = \{x \in S : a = axa, x = xax\}$ the

2010 Mathematics Subject Classification: 20M10, 20M12
Keywords: E-inversive semigroup, M-semigroup, bi-ideal, matrix congruence.

set of all *inverse elements* of a. It is well known that an element a of S is regular iff $V(a) \neq \emptyset$, so a semigroup S is regular iff $V(a) \neq \emptyset$ for every $a \in S$ [6]. Finally, a semigroup S is said to be *eventually regular* if every element of S has a regular power [4]. Clearly, eventually regular semigroups are E-inversive.

In [5] Hall observed that the set $Reg(S)$ of a semigroup S with $E_S \neq \emptyset$ forms a regular subsemigroup of S iff the product of any two idempotents of S is regular. In that case, S is said to be an *R-semigroup.* Also, we say that S is an *E-semigroup* if ${E_S}^2 \subseteq E_S$.

A subsemigroup B of a semigroup S is said to be a *bi-ideal* of S if $BSB \subseteq B$. It is clear that there exists the least bi-ideal (X) containing a nonempty subset X of S. One can easily seen that (X) is of the form: $X \cup X^2 \cup XSX$ [1].

A nonempty subset A of a semigroup S is called a *quasi ideal* iff $AS \cap SA \subseteq A$. Note that every quasi ideal A of S is a bi-ideal of S and each one-sided ideal of S is a quasi ideal of S, so it is a bi-ideal of S. If $\emptyset \neq C \subseteq S$, then $(C \cup SC) \cap (C \cup CS)$ is the smallest quasi ideal of S containing C.

Each subsemigroup eSe of a semigroup S, where $e \in E_S$, will be called a *local subsemigroup* of S. Furthermore, we say that a semigroup S with $E_S \neq \emptyset$ is *locally E-inversive* iff *every* local subsemigroup of S is E-inversive.

By a *rectangular band* we shell mean a semigroup M with the property $aba = a$ for all $a, b \in M$. Note that in that case, $M = E_M$. Also, we say that a congruence ρ on a semigroup S is a *matrix* congruence if S/ρ is a rectangular band [9].

Some background material on *biordered sets* will be useful. For a definition of a *biordered set*, its related axioms and concepts see [13, 3, 2]. Let S be a semigroup with $E_S = E \neq \emptyset$. Define

$$\omega^l = \{(e,f) \in E \times E : \ ef = e\}, \quad \omega^r = \{(e,f) \in E \times E : \ fe = e\},$$

$$\leqslant = \omega^l \cap \omega^r, \quad L = \omega^l \cap (\omega^l)^{-1}, \quad R = \omega^r \cap (\omega^r)^{-1},$$

$$D_E = \{(e,f) \in E \times E : ef = e \text{ or } ef = f \text{ or } fe = e \text{ or } fe = f\}.$$

Then the partial algebra E with domain D_E is a biordered set, Th. 1.1 (a1) [13]. It is easy to see that the relation $\leqslant$ is the natural partial order on the set E, and if $e, f \in E$, then $(e,f) \in L$ $[R]$ iff $(e,f) \in \mathcal{L}$ $[\mathcal{R}]$ (in a semigroup S), where $\mathcal{L}, \mathcal{R}$ are Green's relations on S. Furthermore, the relations ω^l and ω^r are quasi-orders on E. For $\rho = \omega^l$ or $\rho = \omega^r$ and any $e \in E$, we put $\rho(e) = \{g \in E : (g,e) \in \rho\}$.

Let E be a biordered set and $e, f \in E$. We define the *M-set* $M(e,f)$ of e, f by $M(e,f) = \omega^l(e) \cap \omega^r(f) = \{g \in E : g = ge = fg\}$. Also, define the *sandwich-set* $S(e,f)$ of e, f [13] by

$$S(e,f) = \{g \in M(e,f) : (\forall h \in M(e,f)) \ (eh, eg) \in \omega^r, (hf, gf) \in \omega^l\}.$$

Moreover, we define E to be an *M-biordered set* iff $M(e,f) \neq \emptyset$ for all $e, f \in E$. Let S be a semigroup with $E_S \neq \emptyset$. We say that S is an *M-semigroup* if E_S is an M-biordered set. Finally, a subset F of E_S is called an *order bi-ideal* of E_S iff $M(e,f) \subseteq F$ for all $e, f \in F$.

The following result is probably known:

Lemma 1.1. *Let S be an R-semigroup, $e, f \in E_S$. Then*:

$$S(e,f) = \{g \in M(e,f) : egf = ef\} = \{g \in M(e,f) : g \in V(ef)\} = fV(ef)e.$$

Proof. Denote the above four sets by A, B, C and D, respectively.

If $g \in B$, then $fge = g$, so $efgef = egf = ef, gefg = gg = g$ i.e., $g \in V(ef)$. Thus $B \subset C$.

If $g \in C$, then $g = fge$ and $g \in V(ef)$. Hence $g \in fV(ef)e$. Thus $C \subset D$.

Let $g = fxe$ for some $x \in V(ef)$. Then clearly $g \in M(e,f)$. If $h \in M(e,f)$ (i.e. $fh = h = he$), then $(eg)(eh) = efxeeh = efxe(fh) = (efxef)h = efh = eh$. Thus $(eh, eg) \in \omega^r$, and similarly $(hf, gf) \in \omega^l$, so $g \in A$. Consequently, $D \subset A$.

Finally, let $g \in A, x \in V(ef)$. Then $fxe \in D \subset A$. In particular, $eg \mathrel{\mathcal{R}} efxe$ (by the definition of A). Hence

$$egf = e(ge)f = (eg)(ef) = eg(efxef) = (eg \cdot efxe)f = efxef = ef.$$

Thus $g \in B$, as exactly required. □

Let S be an R-semigroup. A subset F of E_S is called a *biorder ideal* if and only if the following two conditions hold:

(i) $(\forall e \in E_S, f \in F)\ e \leqslant f \Longrightarrow e \in F$;

(ii) $(\forall e, f \in F)\ S(e,f) \cap F \neq \emptyset$.

2. The main results

Proposition 2.1. *Let S be a semigroup. The following conditions are equivalent*:

(i) *S is E-inversive;*

(ii) *every bi-ideal of S contains some idempotent of S;*

(iii) *every quasi ideal of S contains some idempotent of S;*

(iv) *every ideal of S contains some idempotent of S.*

Proof. (i) $\Longrightarrow$ (ii). Let B be a bi-ideal of S, $b \in B$ and $x \in W(b^2)$. Then $x = xbbx$. Hence $(bxb)^2 = b(xbbx)b = bxb \in BSB \subseteq B$. Thus $bxb \in E_B$.

(ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv). This is evident.

(iv) $\Longrightarrow$ (i). Let $a \in S$. By assumption SaS has at least one idempotent, that is, $xay = e$ for some $x, y \in S$, $e \in E_S$, so $exaye = e$. Hence $yexayex = yex$. Thus $yex \in W(a)$. □

Lemma 2.2. *Every E-inversive semigroup S is locally E-inversive.*

Proof. Let $a \in eSe$, where $e \in E_S$, $x \in W(a)$. Then $x = xax = x(eae)x$. It follows that $exe = (exe)a(exe)$. Thus $exe \in W(a)$ in eSe, as exactly required. □

Corollary 2.3. *Every bi-ideal of an E-inversive semigroup S is E-inversive. Hence a semigroup S is E-inversive if and only if every bi-ideal of S is E-inversive.*

Proof. Let B be a bi-ideal of S and $b \in B$. By Proposition 2.1, B contains some idempotent of S, say e. By Lemma 2.2, $eSe \in BSB \subseteq B$ is E-inversive and so $(ebe)y \in E_{eSe}$ for some $y \in eSe$. Hence $(eb)(ey) \in E_{eSe}$, say $(eb)(ey) = f$, where $ey \in e(eSe) = eSe$. Therefore $f(eb)eyf = f$, so $eyf(eb)eyf = eyf$. We conclude that there exists $x \in W(eb)$ in B (for example: $x = (ey)f \in (eSe)(eSe) \subseteq B$), so $x = xebx$. Thus $(xe)b(xe) = xe$ and $xe \in Be \subseteq B$. Consequently, B is E-inversive (remark that even $xe = eyfe \in eSe$). □

Let a semigroup S (with $E_S \neq \emptyset$) be locally E-inversive, $b \in S$ and $e \in E_S$. Consider the least bi-ideal, say B, of S containing the set $\{e, b\}$. Note that $(e) \subseteq B$ i.e., $eSe \subseteq B$. From the proof of Corollary 2.3 and from Lemma 2.2 we obtain:

Corollary 2.4. *A semigroup is E-inversive if and only if it is locally E-inversive.*

In [7] S. Kopamu defined a countable family of congruences on a semigroup S, as follows: for each ordered pair of non-negative integers (m, n), he put:

$$\theta_{m,n} = \{(a, b) \in S \times S : (\forall x \in S^m, y \in S^n)\ xay = xby\},$$

and he made the convention that $S^1 = S$ and S^0 denotes the set containing the empty word. In particular, $\theta_{0,0}$ is the identity relation on S. Let $\mathcal{C}$ be a class of semigroups of the same type $\mathcal{T}$ (for example: the class of E-inversive semigroups); call its elements *$\mathcal{C}$-semigroups*. A semigroup S is called a *structurally $\mathcal{C}$-semigroup* if $S/\theta_{m,n} \in \mathcal{C}$ for some integers $m, n \geq 0$. Further, denote by $\mathcal{SC}$ the class of all structurally $\mathcal{C}$-semigroups. It is clear that $\mathcal{C} \subseteq \mathcal{SC}$. Finally, we say that the class $\mathcal{C}$ is *structurally closed* if $\mathcal{C} = \mathcal{SC}$ [8].

Lemma 2.5. *Every structurally E-inversive semigroup is locally E-inversive.*

Proof. Let S be a structurally E-inversive semigroup, say $S/\theta_{m,n}$ is E-inversive; $a \in eSe$, where $e \in E_S$. Since the class of E-inversive semigroups is closed under homomorphic images, then we may suppose that m, n are both positive integers. Moreover, $a = eae$, $(x, xax) \in \theta_{m,n}$ for some $x \in S$. Hence $e^m x e^n = e^m xaxe^n$, that is, $exe = exaxe = ex(eae)xe$ and so $exe = (exe)a(exe)$. Therefore $exe \in W(a)$ in the semigroup eSe. Consequently, S is locally E-inversive. □

Combining the above lemma with Corollary 2.4 we obtain the following:

Corollary 2.6. *The class of all E-inversive semigroups is structurally closed.* □

By the *trace* $\operatorname{tr}\rho$ of a congruence ρ on a semigroup S we mean $\rho \cap (E_S \times E_S)$.

Corollary 2.7. *If ρ is a matrix congruence on an E-inversive semigroup S, then every ρ-class of S is E-inversive.*

Moreover, every matrix congruence on an E-inversive semigroup is uniquely determined by its trace.

Proof. The first part follows from Corollary 2.3 and the following easy observation: if A is any ρ-class of S, where ρ is a matrix congruence on S, then A is a bi-ideal.

We show the second part. Let ρ_1, ρ_2 be matrix congruences on an E-inversive semigroup S, $\mathrm{tr}\rho_1 \subset \mathrm{tr}\rho_2$, $e \in E_S$. If $a \in e\rho_1$, then there exists $x \in W(a)$ in $e\rho_1$. Hence $ax(\mathrm{tr}\rho_1)e(\mathrm{tr}\rho_1)xa$ and so $ax(\mathrm{tr}\rho_2)e(\mathrm{tr}\rho_2)xa$. Therefore we get $a\,\rho_2\,axxa\,\rho_2\,e$ i.e., $a \in e\rho_2$. Thus $\rho_1 \subset \rho_2$. Consequently, if $\mathrm{tr}\rho_1 = \mathrm{tr}\rho_2$, then $\rho_1 = \rho_2$. □

Remark 2.8. The second part of the above corollary generalizes Theorem 2.1 [9]. One can modify all results of J. Mills in Section 2 of [9] for E-inversive E-semigroups. Denote by ψ the least matrix congruence on a semigroup S. It is clear that the interval $[\psi,\ S \times S]$ consists of all matrix congruences on S and it is a complete sublattice of the lattice of all congruences on S. Denote it by $\mathcal{MC}(S)$. Moreover, if S is an E-semigroup, then the symbol $\mathcal{MC}(E_S)$ means the complete lattice of matrix congruences on E_S.

For terminology and elementary facts about lattices the reader is referred to the book [14] (Section I.2). The following result will be useful (see Lemma I.2.8 and Exercise I.2.15 (iii) in [14]):

Lemma 2.9. *If φ is an order isomorphism of a lattice L onto a lattice M, then φ is a lattice isomorphism. Moreover, every lattice ismomorhism of complete lattices is a complete lattice isomorphism.* □

In particular, the following theorem is valid (see Theorems 2.5, 2.6 and Corollary 2.7 in [9]):

Theorem 2.10. *Let S be an E-inversive E-semigroup. Suppose also that the least matrix congruence on E_S can be extended to a matrix congruence on S. Then each matrix congruence on E_S can be extended uniquely to a matrix congruence on S. In fact, if it is the case, then for any matrix congruence ρ_E on E_S, the relation ρ defined on S by:*

$$(a,b) \in \rho \iff (\exists\ e,f \in E_S)\ (a\psi e)\rho_E(f\psi b)$$

is the unique matrix congruence on S which extends ρ_E. Thus there is an inclusion-preserving bijection θ between the lattice $\mathcal{MC}(S)$ and the lattice $\mathcal{MC}(E_S)$. In fact, θ is defined by:

$$\theta : \rho \to tr\rho$$

for every $\rho \in \mathcal{MC}(S)$. Furthermore, θ^{-1} is an inclusion-preserving bijection, too (by the proof of the second part of Corollary 2.7), so θ is an order isomorphism of the lattice $\mathcal{MC}(S)$ onto the lattice $\mathcal{MC}(E_S)$. Consequently, θ is a complete lattice isomorphism between the complete lattices $\mathcal{MC}(S)$ and $\mathcal{MC}(E_S)$, respectively.

Also, ρ is a matrix congruence on an E-inversive E-semigroup S if and only if $tr\rho$ is a matrix congruence on E_S and every ρ-class of S contains some idempotent of S. □

Clearly, every semigroup S is an ideal (of S) and so S is a bi-ideal. Also, if A is a left [right or bi-] ideal of S, $a \in A$, then the principle left [right or bi-] ideal of S containing a is contained in A. Thus by Proposition 2.1 and Corollary 2.3 we obtain the following:

Corollary 2.11. *Let S be a semigroup. The following conditions are equivalent*:

(i) *S is E-inversive*;

(ii) *every left $[$right$]$ $($principle$)$ ideal of S contains some idempotent of S*;

(iii) *every $($principle$)$ ideal of S contains some idempotent of S*;

(iv) *every $($principle$)$ quasi ideal of S contains some idempotent of S*;

(v) *every $($principle$)$ bi-ideal of S contains some idempotent of S*;

(vi) *every $($principle$)$ bi-deal of S is E-inversive*;

(vii) *every $($principle$)$ quasi ideal of S is E-inversive*;

$(viii)$ *every $($principle$)$ left $[$right$]$ ideal of S is E-inversive*;

(ix) *every $($principle$)$ ideal of S is E-inversive.* □

Proposition 2.12. *Every E-inversive semigroup S is an M-semigroup. In fact,*

$$M(e,f) = fW(ef)e$$

for all $e, f \in E_S$.

Proof. Let $g \in M(e,f)$, where $e, f \in E_S$. Then $g = fge$. Also, $gefg = gg = g$ and so $g \in W(ef)$. Consequently, $g \in fW(ef)e$.

Conversely, if $g = fxe$ for some $x \in W(ef)$, then $gg = f(xefx)e = fxe = g$. Hence $g \in E_S$. Clearly, $g = ge = fg$. Thus $g \in M(e,f)$, as required. □

Remark 2.13. The free monoids are M-semigroups but they are not E-inversive. Note that in [4] Edwards shows that eventually regular semigroups are M-semigroups and gives an example of an M-biordered set which does not arise from eventually regular semigroups.

In the following three results are presented some connections between bi-ideals of an E-inversive semigroup S and order bi-ideals, bi-order ideals of the biordered set E_S.

Proposition 2.14. *Let S be an R-semigroup. Then F is an order bi-ideal of E_S if and only if F is a biorder ideal of E_S.*

Proof. Let F be an order bi-ideal of E_S. Then $S(g,h) \subseteq M(g,h) \subseteq F$ for every $g, h \in F$, so $S(g,h) \cap F = S(g,h) \neq \emptyset$, since S is an R-semigroup (Lemma 1.1). Also, if $e \in E_S$, then for every $f \in F$ such that $e \leqslant f$ (i.e., $e = ef = fe$) we have $e \in W(f)$. Consequently, $e = fef \in fW(ff)f = M(f,f) \subseteq F$. Therefore F is a biorder ideal of E_S.

The proof of the opposite implication is similar to the proof of Theorem 1 [1] and is omitted. □

Lemma 2.15. *Let B be a bi-ideal of an E-inversive semigroup S. Then E_B is an order bi-ideal of E_S.*

Proof. Let B be a bi-ideal of $S, g, h \in E_B, e \in M(g,h)$. Then $e = hxg$ for some $x \in W(gh)$ (Proposition 2.12), so $e \in BSB \subseteq B$ i.e., $e \in E_B$. Thus $M(g,h) \subseteq E_B$ for all $g, h \in E_B$. Consequently, E_B is an order bi-ideal of E_S. □

The following theorem generalizes Theorem 2 [1].

Theorem 2.16. *Let S be an E-inversive semigroup and B be a bi-ideal of S. Then E_B is an order bi-ideal of E_S. Also, $A = E_B S E_B$ is an E-inversive bi-ideal of S such that $E_A = E_B$.*

Conversely, if F is an order bi-ideal of E_S, then $B = FSF$ is an E-inversive bi-ideal of S such that $E_B = F$.

Proof. Indeed, E_B is an order bi-ideal of E_S. It is clear that A is a bi-ideal of S and so A is E-inversive (Corollary 2.3). Also, $E_A = E_B$, since $BSB \subseteq B$.

We may show in a similar way the second part of the theorem. □

Finally, we show that every M-biordered set E arises from some E-inversive semigroup. Firstly, we have need the following important Easdown's Theorem:

Theorem 2.17. (Corollary from Theorem 3.3 [3]) *Every biordered set comes from some semigroup.* □

We say that an element a of a semigroup is *E-inversive* if $W(a) \neq \emptyset$.

The following theorem is the main result of the paper.

Theorem 2.18. *Each M-biordered set E arises from some E-inversive semigroup.*

Proof. Let E be an M-biordered set. By Easdown's Theorem there exists some semigroup S with $E_S = E$. Since E_S is M-biordered, then $M(e,f)$ is nonempty for all $e, f \in E_S$, so by Proposition 2.12, $W(ef) \neq \emptyset$ for all $e, f \in E_S$. We show that the set T (say) of all E-inversive elements of S forms an E-inversive subsemigroup of S. Clearly, $E_S \subset T$ and so $T \neq \emptyset$. Moreover, if $W(a), W(b)$ are nonempty, then $xa, by \in E_S$ for some $x, y \in S$. Thus $W(xaby) \neq \emptyset$ and so $s = sxabys$ for some $s \in S$. It follows that $ysx = ysx(ab)ysx$. Therefore $W(ab) \neq \emptyset$. We conclude that E is the set of idempotents of an E-inversive semigroup T (since if $t \in T$ and $x \in W(t)$ in S, then $x \in Reg(S) \subset T$, so $x \in W(t)$ in T). □

Remark 2.19. A biordered set E is called *regular* if $S(e,f) \neq \emptyset$ for all $e, f \in E$. By Hall's result, Easdown's Theorem and Lemma 1.1 we obtain Nambooripad's Theorem [13]:

Theorem 2.20. *Every regular biordered set comes from some regular semigroup.*

References

[1] **F. Catino**, *On bi-ideals in eventually regular semigroups*, Riv. Mat. Pura Appl. **4** (1989), 89 – 92.

[2] **D. Easdown**, *A new proof that regular biordered sets come from regular semigroups*, Proc. Roy. Soc. Edinburgh (Sec. A) **96** (1984), 109 – 116.

[3] **D. Easdown**, *Biordered sets come from semigroups*, J. Algebra **96** (1985), 581 – 591.

[4] **P. M. Edwards**, *Eventually regular semigroups*, Bull. Austral. Math. Soc. **28** (1983), 23 – 38.

[5] **T. E. Hall**, *On regular semigroups*, J. Algebra **24** (1973), 1 – 24.

[6] **J. M. Howie**, *An introduction to semigroup theory*, Academic Press, London (1984).

[7] **S. J. L. Kopamu**, *On semigroup species*, Comm. Algebra **23** (1995), 5513 – 5537.

[8] **S. J. L. Kopamu**, *The concept of structural regularity*, Portugaliae Math. **53** (1996), 435 – 456.

[9] **J. E. Mills**, *Matrix congruences on orthodox semigroups*, Semigroup Forum **31** (1985), 87 – 97.

[10] **H. Mitsch and M. Petrich**, *Basic properties on E-inversive semigroups*, Comm. Algebra **28** (2000), 5169 – 5182.

[11] **H. Mitsch and M. Petrich**, *Restricting idempotents in E-inversive semigroups*, Acta Sci. Math. (Szeged) **67** (2001), 555 – 570.

[12] **K. S. S. Nambooripad**, *Structure of regular semigroups I: fundamental regular semigroups*, Semigroup Forum **9** (1975), 354 – 363.

[13] **K. S. S. Nambooripad**, *Structure of regular semigroups I*, Memoirs Amer. Math. Soc. **22** (1979).

[14] **M. Petrich**, *Inverse semigroups*, Wiley, New York (1984).

Received August 28, 2011

Institute of Mathematics and Computer Science
Wroclaw University of Technology
Wyb. Wyspiańskiego 27
50-370 Wroclaw
Poland
E-mail: romekgigon@tlen.pl

Quasigroups and Related Systems **20** (2012), 61 − 70

On fuzzy ordered semigroups

Niovi Kehayopulu and Michael Tsingelis

Abstract. There are two equivalent definitions of a fuzzy right ideal, fuzzy left ideal, fuzzy bi-ideal or fuzzy quasi-ideal f of an ordered semigroup (or a semigroup) S in the bibliography. The first one is based on the fuzzy subset f itself, the other on the multiplication of fuzzy sets and the greatest fuzzy subset of S. Investigations in the existing bibliography are based on the first definition. The present paper serves as an example to show that using the second definition the proofs of the results can be simplified, drastically in some cases, using only the definitions themselves.

1. Introduction and prerequisites

As we have seen in [6], there are two equivalent definitions for each of the following: Fuzzy right ideal, fuzzy left ideal, fuzzy bi-ideal and fuzzy quasi-ideal. These are the following:

Definition 1.1. Let $(S, ., \leqslant)$ be an ordered groupoid. A fuzzy subset f of S is called a *fuzzy right ideal* of $(S, ., \leqslant)$ (or just a *fuzzy right ideal* of S) if

(1) $f(xy) \geqslant f(x)$ for all $x, y \in S$ and
(2) if $x \leqslant y$, then $f(x) \geqslant f(y)$.

Definition 1.2. Let $(S, ., \leqslant)$ be an ordered groupoid. A fuzzy subset f of S is called a *fuzzy right ideal* of S if

(1) $f \circ 1 \preceq f$ and
(2) if $x \leqslant y$, then $f(x) \geqslant f(y)$.

Definition 1.3. Let $(S, ., \leqslant)$ be an ordered groupoid. A fuzzy subset f of S is called a *fuzzy left ideal* of S if

(1) $f(xy) \geqslant f(y)$ for all $x, y \in S$ and
(2) if $x \leqslant y$, then $f(x) \geqslant f(y)$.

Definition 1.4. Let $(S, ., \leqslant)$ be an ordered groupoid. A fuzzy subset f of S is called a *fuzzy left ideal* of S if

(1) $1 \circ f \preceq f$ and
(2) if $x \leqslant y$, then $f(x) \geqslant f(y)$.

2010 Mathematics Subject Classification: 06F05, 08A72.
Keywords: Ordered semigroup, regular, intra-regular, left (right) ideal, bi-ideal, fuzzy subset, fuzzy right (left) ideal, fuzzy bi-ideal.

Definition 1.5. Let $(S, ., \leqslant)$ be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy bi-ideal* of S if

(1) $f(xyz) \geqslant \min\{f(x), f(z)\}$ for all $x, y, z \in S$ and

(2) if $x \leqslant y$, then $f(x) \geqslant f(y)$.

Definition 1.6. Let S be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy bi-ideal* of S if

(1) $f \circ 1 \circ f \preceq f$ and

(2) if $x \leqslant y$, then $f(x) \geqslant f(y)$.

Definition 1.7. Let $(S, ., \leqslant)$ be an ordered groupoid. A fuzzy subset f of S is called a *fuzzy quasi-ideal* of S if

(1) if $x \leqslant bs$ and $x \leqslant tc$ for some x, b, s, t, c in S, then $f(x) \geqslant \min\{f(b), f(c)\}$ and

(2) if $x \leqslant y$, then $f(x) \geqslant f(y)$.

Definition 1.8. Let $(S, ., \leqslant)$ be an ordered groupoid. A fuzzy subset f of S is called a *fuzzy quasi-ideal* of S if

(1) $(f \circ 1) \wedge (1 \circ f) \preceq f$ and

(2) if $x \leqslant y$, then $f(x) \geqslant f(y)$.

A fuzzy subset f of $(S, ., \leqslant)$ is said to be a fuzzy right (resp. left) ideal, fuzzy bi-ideal or fuzzy quasi-ideal of $(S, .)$ if the following assertions, respectively hold in $(S, ., \leqslant)$: $f(xy) \geqslant f(x)$ (resp. $f(xy) \geqslant f(y)$); $f(xyz) \geqslant \min\{f(x), f(z)\}$; $x \leqslant bs$ and $x \leqslant tc$ imply $f(x) \geqslant \min\{f(b), f(c)\}$.

Definitions 1.1, 1.3, 1.5 and 1.7 are based on the fuzzy subset f itself while in 1.2, 1.4, 1.6, 1.8 the greatest fuzzy subset 1 of S and the multiplication of fuzzy subsets play an essential role. Investigations in the existing bibliography are based on Definitions 1.1, 1.3, 1.5 and 1.8. Definition 1.7 has been first introduced by Kehayopulu and Tsingelis in [6]. The present paper serves as an example to show that with Definitions 1.2, 1.4, 1.6, 1.8 the proofs of the results can be simplified, drastically is some cases, using only the definitions themselves.

It has been announced without proof in [7] that an ordered semigroup $(S, ., \leqslant)$ is intra-regular if and only if for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of $(S, ., \leqslant)$, we have $f \wedge h \wedge g \preceq g \circ h \circ f$ and that it is both regular and intra-regular if and only if for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of $(S, ., \leqslant)$, we have $f \wedge h \wedge g \preceq h \circ f \circ g$. Some more general situations are given in the present paper. According to the present paper, if an ordered semigroup $(S, ., \leqslant)$ is intra-regular, then for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of $(S, .)$, we have $f \wedge h \wedge g \preceq g \circ h \circ f$. If an ordered semigroup $(S, ., \leqslant)$ is both regular and intra-regular, then for every fuzzy right ideal f, every fuzzy subset g and every fuzzy bi-ideal h of $(S, .)$, we have $f \wedge h \wedge g \preceq h \circ f \circ g$. We also prove that if an ordered semigroup $(S, ., \leqslant)$ is regular, then for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of $(S, .)$ we have $f \wedge h \wedge g \preceq f \circ h \circ g$. "Conversely", if for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of $(S,.,\leqslant)$ we have $f \wedge h \wedge g \preceq f \circ h \circ g$, then S is regular. Characterizations of regular and both regular and intra-regular ordered semigroups in terms of fuzzy sets have been also given by Xie in [8].

Let $(S,.,\leqslant)$ be an ordered semigroup. For a subset A of S, denote by $(A]$ the subset of S defined by

$$(A] := \{t \in S \mid t \leqslant a \text{ for some } a \in A\}.$$

A nonempty subset A of $(S,.,\leqslant)$ is called a *left* (resp. *right*) *ideal* of $(S,.,\leqslant)$ (or just of S) if (1) $SA \subseteq A$ (resp. $AS \subseteq A$) and (2) if $a \in A$ and $S \ni b \leqslant a$, then $b \in A$. A is called a *bi-ideal* of S if (1) $ASA \subseteq A$ and (2) if $a \in A$ and $S \ni b \leqslant a$, then $b \in A$. It is called a *quasi-ideal* of S if (1) $(SA] \cap (AS] \subseteq A$ and (2) if $a \in A$ and $S \ni b \leqslant a$, then $b \in A$. A nonempty subset A of $(S,.,\leqslant)$ is said to be a left ideal, right ideal, bi-ideal or quasi-ideal of $(S,.)$ if the relations $SA \subseteq A$, $AS \subseteq A$, $SAS \subseteq A$ or $(AS] \cap (SA] \subseteq A$, respectively hold in S. An ordered semigroup $(S,.,\leqslant)$ is called *regular* if for every $a \in S$ there exists $x \in S$ such that $a \leqslant axa$. Equivalently, if $A \subseteq (ASA]$ for every $A \subseteq S$. It is called *intra-regular* if for every $a \in S$ there exist $x, y \in S$ such that $a \leqslant xa^2y$. Equivalently, if $A \subseteq (SA^2S]$ for every $A \subseteq S$.

Denote by 1 the fuzzy subset of S defined by $1 : S \to [0,1] \mid a \to 1$. The fuzzy set 1 is the greatest element in the set of fuzzy subsets of S, that is, $f \preceq 1$ for every fuzzy subset f of S. If S is a regular or an intra-regular ordered semigroup, then we have $1 \circ 1 = 1$. It is well known that an ordered semigroup S is regular if and only if for every fuzzy right ideal f and every fuzzy left ideal g of $(S,.,\leqslant)$, we have $f \wedge g = f \circ g$ equivalently $f \wedge g \preceq f \circ g$ [4]. It is intra-regular if and only if for every fuzzy right ideal f and every fuzzy left ideal g of $(S,.,\leqslant)$, we have $f \wedge g \preceq g \circ f$ [7]. Moreover, an ordered semigroup S is regular if and only if for every fuzzy subset f of S, we have $f \preceq f \circ 1 \circ f$ [6]. It is intra-regular if and only if for every fuzzy subset f of S, we have $f \preceq 1 \circ f^2 \circ 1$ [5]. If $(S,.,\leqslant)$ is an ordered groupoid, f, g fuzzy subsets of $(S,.)$ and $f \preceq g$ then, for any fuzzy subset h of $(S,.)$, we have $f \circ h \preceq g \circ h$ and $h \circ f \preceq h \circ g$ (cf. also [4]). It is also well known that if S is a semigroup or an ordered semigroup, then the multiplication of fuzzy subsets of S is associative (cf. [3]). For the definitions and notations not given in the present paper we refer to [4].

2. Main results

The first theorem characterizes the ordered semigroups which are intra-regular in terms of fuzzy sets. Let us prove it using first the first and then the second definitions.

Theorem 2.1. *Let $(S,.,\leqslant)$ be an ordered semigroup. If S is intra-regular, then for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of*

$(S,.)$ we have

$$f \wedge h \wedge g \preceq g \circ h \circ f.$$

"Conversely", if for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of $(S,.,\leqslant)$ we have $f \wedge h \wedge g \preceq g \circ h \circ f$, then $(S,.,\leqslant)$ is intra-regular.

Proof of Theorem 2.1 using the Definitions 1.1, 1.3, 1.5

We need the following lemmas. As our aim is to compare the two definitions, we would like to mention everything we use in the proofs. In that sense, for the sake of completeness, it is no harm to mention the next lemma related to the real numbers, as well.

Lemma 2.1. *If a, b, c, d, e, f are real numbers, then*
(1) *If $a \geqslant b$ and $c \geqslant d$, then $\min\{a, c\} \geqslant \min\{b, d\}$.*
(2) $\min\{\min\{a, b\}, c\} = \min\{a, b, c\}$.
(3) *If $a \geqslant b$, $c \geqslant d$ and $e \geqslant f$, then $\min\{a, c, e\} \geqslant \min\{b, d, f\}$.*

Lemma 2.2. (cf. also [2; Proposition 2]) *Let $(S,.,\leqslant)$ be an ordered groupoid. If A is a left (resp. right) ideal of $(S,.,\leqslant)$, then the characteristic function f_A is a fuzzy left (resp. fuzzy right) ideal of $(S,.,\leqslant)$. "Conversely", if A is a nonempty set and f_A a fuzzy left (resp. right) ideal of $(S,.,\leqslant)$, then A is a left (resp. right) ideal of $(S,.,\leqslant)$.*

Lemma 2.3. (cf. also [7; Lemma 4]) *Let $(S,.,\leqslant)$ be an ordered semigroup. If B is a bi-ideal of $(S,.,\leqslant)$, then the characteristic function f_B is a fuzzy bi-ideal of $(S,.,\leqslant)$. "Conversely", if B is a nonempty set and f_B a fuzzy bi-ideal of $(S,.,\leqslant)$, then B is a bi-ideal of $(S,.,\leqslant)$.*

Lemma 2.4. [4; Proposition 7] *If S is an ordered groupoid (or groupoid) and $\{A_i \mid i \in I\}$ a nonempty family of subsets of S, then we have*

$$\bigwedge_{i\in I} f_{A_i} = f_{\bigcap\limits_{i\in I} A_i}.$$

Lemma 2.5. *Let S be an ordered semigroup, n a natural number, $n \geqslant 2$ and $\{A_1, A_2, \ldots, A_n\}$ a set of nonempty subsets of S. Then we have*

$$f_{A_1} \circ f_{A_2} \circ \ldots \circ f_{A_n} = f_{(A_1 A_2 \ldots A_n]}.$$

Proof. For $n = 2$ it is true [4; Proposition 8]. Suppose $f_{A_1} \circ f_{A_2} \circ \ldots \circ f_{A_m} = f_{(A_1 A_2 \ldots A_m]}$ for a natural number m, $m \geqslant 2$. Then we have

$$\begin{aligned} f_{A_1} \circ f_{A_2} \circ \ldots \circ f_{A_{m+1}} &= f_{(A_1 A_2 \ldots A_m]} \circ f_{A_{m+1}} = f_{((A_1 A_2 \ldots A_m] A_{m+1}]} \\ &= f_{((A_1 A_2 \ldots A_m) A_{m+1}]} = f_{(A_1 A_2 \ldots A_{m+1}]}. \end{aligned}$$

□

Lemma 2.6. [4; Proposition 5] *If S is an ordered groupoid (or groupoid) and A, B subsets of S, then we have*

$$A \subseteq B \Longleftrightarrow f_A \preceq f_B.$$

Taking into account the Proposition 2 and Lemma 2 in [1], one can easily see that the following lemma is satisfied:

Lemma 2.7. *Let $(S, ., \leqslant)$ be an ordered semigroup. If $(S, ., \leqslant)$ is intra-regular, then for every right ideal X, every left ideal Y and every bi-ideal B of $(S, .)$ we have*

$$X \cap B \cap Y \subseteq (YBX].$$

"Conversely", if for every right ideal X, every left ideal Y and every bi-ideal B of $(S, ., \leqslant)$ we have $X \cap B \cap Y \subseteq (YBX]$, then S is intra-regular.

Proof of Theorem 2.1

$\Longrightarrow$. Let f be a fuzzy right ideal, g a fuzzy left ideal, h a fuzzy bi-ideal of $(S, .)$, and $a \in S$. Since $(S, ., \leqslant)$ is intra-regular, there exist $x, y \in S$ such that $a \leqslant xa^2y$. Then we have

$$a \leqslant x(xa^2y)(xa^2y)y = x^2a^2yxa^2y^2,$$

which implies $(x^2a^2yxa, ay^2) \in A_a$ .... $(*)$ and $A_a \neq \emptyset$. Then we have

$$\begin{aligned}((g \circ h) \circ f)(a) :&= \bigvee_{(u,v) \in A_a} \min\{(g \circ h)(u), f(v)\} \text{ (since } A_a \neq \emptyset) \\ &\geqslant \min\{(g \circ h)(x^2a^2yxa), f(ay^2)\} \text{ (by } (*)).\end{aligned}$$

Since $(x^2a, ayxa) \in A_{x^2a^2yxa}$, we have $A_{x^2a^2yxa} \neq \emptyset$, hence

$$\begin{aligned}(g \circ h)(x^2a^2yxa) :&= \bigvee_{(w,t) \in A_a} \min\{(g(w), h(t)\} \\ &\geqslant \min\{g(x^2a), h(ayxa)\}.\end{aligned}$$

Then, by Lemma 2.1(1) and (2), we have

$$\begin{aligned}((g \circ h) \circ f)(a) &\geqslant \min\{\min\{g(x^2a), h(ayxa)\}, f(ay^2)\} \\ &= \min\{g(x^2a), h(ayxa), f(ay^2)\} \\ &= \min\{f(ay^2), h(ayxa), g(x^2a)\}\end{aligned}$$

Since f is a fuzzy right ideal of $(S, .)$, we have $f(ay^2) \geqslant f(a)$. Since h is a fuzzy bi-ideal of $(S, .)$, we have $h(ayxa) \geqslant h(a)$. Since g is a fuzzy left ideal of $(S, .)$, we have $g(x^2a) \geqslant g(a)$. Then, by Lemma 2.1(3), we have

$$((g \circ h) \circ f)(a) \geqslant \min\{f(a), h(a), g(a)\} = (f \wedge h \wedge g)(a).$$

As the multiplication of fuzzy subsets is associative, we obtain $f \wedge h \wedge g \preceq g \circ h \circ f$.

$\Longleftarrow$. Let X be a right ideal, Y a left ideal, and B a bi-ideal of $(S,.,\leqslant)$. By Lemma 2.7 it is enough to prove that $X \cap B \cap Y \subseteq (YBX]$. By Lemmas 2.2 and 2.3, f_X is a fuzzy right ideal, f_Y a fuzzy left ideal and f_B a fuzzy bi-ideal of $(S,.,\leqslant)$. By hypothesis, we have $f_X \wedge f_B \wedge f_Y \preceq f_Y \circ f_B \circ f_X$. By Lemma 2.4, we have $f_X \wedge f_B \wedge f_Y = f_{X\cap B\cap Y}$. By Lemma 2.5, $f_Y \circ f_B \circ f_X = f_{(YBX]}$. Hence we have $f_{X\cap B\cap Y} \preceq f_{(YBX]}$. Then, by Lemma 2.6, $X \cap B \cap Y \subseteq (YBX]$. $\square$

Proof of Theorem 2.1 using the Definitions 1.2, 1.4, 1.6

$\Longrightarrow$. Let f be a fuzzy right ideal, g a fuzzy left ideal, h a fuzzy bi-ideal of $(S,.)$. Since $f \wedge h \wedge g$ is a fuzzy subset of S and S is intra-regular, we have

$$\begin{aligned}
f \wedge h \wedge g &\preceq 1 \circ (f \wedge h \wedge g)^2 \circ 1 = 1 \circ (f \wedge h \wedge g) \circ (f \wedge h \wedge g) \circ 1\\
&\preceq 1 \circ 1 \circ (f \wedge h \wedge g)^2 \circ 1 \circ 1 \circ (f \wedge h \wedge g)^2 \circ 1 \circ 1\\
&= 1 \circ (f \wedge h \wedge g) \circ (f \wedge h \wedge g) \circ 1 \circ (f \wedge h \wedge g) \circ (f \wedge h \wedge g) \circ 1\\
&\preceq (1 \circ g) \circ (h \circ 1 \circ h) \circ (f \circ 1)\\
&\preceq g \circ h \circ f.
\end{aligned}$$

$\Longleftarrow$. Let f be a fuzzy right ideal and g a fuzzy left ideal of $(S,.,\leqslant)$. Since 1 is a fuzzy right ideal and f a fuzzy bi-ideal of $(S,.,\leqslant)$, by hypothesis, we have $f \wedge g = 1 \wedge f \wedge g \preceq g \circ f \circ 1 \preceq g \circ f$, so S is intra-regular. $\square$

The next theorem characterizes the ordered semigroups which are both regular and intra-regular using fuzzy sets.

Theorem 2.2. *Let $(S,.,\leqslant)$ be an ordered semigroup. If S is both regular and intra-regular, then for every fuzzy right ideal f, every fuzzy subset g and every fuzzy bi-ideal h of $(S,.)$ we have*

$$f \wedge h \wedge g \preceq h \circ f \circ g.$$

"Conversely", if for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of $(S,.,\leqslant)$ we have $f \wedge h \wedge g \preceq h \circ f \circ g$, then S is both regular and intra-regular.

Proof of Theorem 2.2 using the Definitions 1.1, 1.3, 1.5

In addition to Lemmas 2.1–2.6 mentioned above, we need the following lemma.

Lemma 2.8. (cf. also [1; Proposition 3]) *Let $(S,.,\leqslant)$ be an ordered semigroup. If $(S,.,\leqslant)$ is both regular and intra-regular, then for every right ideal X, every subset Y and every bi-ideal B of $(S,.)$ we have*

$$X \cap B \cap Y \subseteq (BXY].$$

"Conversely", if for every right ideal X, every left ideal Y and every bi-ideal B of $(S,.,\leqslant)$ we have $X\cap B\cap Y\subseteq (BXY]$, then $(S,.,\leqslant)$ is both regular and intra-regular.

Proof of Theorem 2.2

$\Longrightarrow$. Let f be a fuzzy right ideal of $(S,.)$, g a fuzzy subset of S, h a fuzzy bi-ideal of $(S,.)$, and $a\in S$. Since S is regular, there exists $x\in S$ such that $a\leqslant axa$. Since S is intra-regular, there exist $z,y\in S$ such that $a\leqslant za^2y$. Then we have

$$a\leqslant ax(axa)\leqslant ax(za^2y)xa=axza^2yxa,$$

$(axza^2yx,a)\in A_a$, $A_a\neq\emptyset$, and

$$\begin{aligned}((h\circ f)\circ g)(a):&=\bigvee_{(u,v)\in A_a}\min\{(h\circ f)(u),g(v)\}\\ &\geqslant\min\{(h\circ f)(axza^2yx),g(a)\}.\end{aligned}$$

Since $(axza,ayx)\in A_{axza^2yx}$, we have $A_{axza^2yx}\neq\emptyset$, and

$$\begin{aligned}(h\circ f)(axza^2yx):&=\bigvee_{(w,t)\in A_{xza^2yx}}\min\{h(w),f(t)\}\\ &\geqslant\min\{h(axza),f(ayx)\}.\end{aligned}$$

Hence we obtain

$$\begin{aligned}((h\circ f)\circ g)(a)&\geqslant\min\{\min\{h(axza),f(ayx)\},g(a)\}\\ &=\min\{h(axza),f(ayx),g(a)\}\end{aligned}$$

Since h is a fuzzy bi-ideal, f a fuzzy right ideal and g a fuzzy subset of S, we obtain

$$((h\circ f)\circ g)(a)\geqslant\min\{h(a),f(a),g(a)\}=(f\wedge h\wedge g)(a).$$

$\Longleftarrow$. Let X be a right ideal, Y a left ideal and B a bi-ideal of $(S,.,\leqslant)$. Since f_X is a fuzzy right ideal, f_Y a fuzzy left ideal and f_B a fuzzy bi-ideal of $(S,.,\leqslant)$, by hypothesis, we have $f_X\wedge f_B\wedge f_Y\preceq f_B\circ f_X\circ f_Y$. Then $f_{X\cap B\cap Y}\preceq f_{(BXY]}$, and $X\cap B\cap Y\subseteq (BXY]$. By Lemma 2.8, S is both regular and intra-regular. $\square$

Proof of Theorem 2.2 using the Definitions 1.2, 1.4, 1.6

$\Longrightarrow$. Since S is both regular and intra-regular, for any fuzzy subset f of S, we have $f\preceq f\circ 1\circ f^2\circ 1\circ f$. Indeed: Since S is regular, we have $f\preceq f\circ 1\circ f$. Since S is intra-regular, we have $f\preceq 1\circ f^2\circ 1$. Thus we have

$$\begin{aligned}f&\preceq(f\circ 1\circ f)\circ 1\circ f\preceq f\circ 1\circ(1\circ f^2\circ 1)\circ 1\circ f\\ &=f\circ 1\circ f^2\circ 1\circ f.\end{aligned}$$

Let now f be a fuzzy right ideal, g a fuzzy subset and h a fuzzy bi-ideal of $(S,.)$. Since $f \wedge h \wedge g$ is a fuzzy subset of S, we have

$$\begin{aligned} f \wedge h \wedge g &\preceq (f \wedge h \wedge g) \circ 1 \circ (f \wedge h \wedge g) \circ (f \wedge h \wedge g) \circ 1 \circ (f \wedge h \wedge g) \\ &\preceq (h \circ 1 \circ h) \circ (f \circ 1) \circ g \\ &\preceq h \circ f \circ g. \end{aligned}$$

$\Longleftarrow$. Let f be a fuzzy right and g a fuzzy left ideal of $(S,.,\leqslant)$. Since 1 is a fuzzy right ideal of $(S,.,\leqslant)$ and f a fuzzy bi-ideal of $(S,.,\leqslant)$, by hypothesis, we have $f \wedge g = 1 \wedge f \wedge g \preceq f \circ 1 \circ g \preceq f \circ g$, and S is regular. Since g is a fuzzy bi-ideal and 1 a fuzzy left ideal of $(S,.,\leqslant)$, by hypothesis, we have $f \wedge g = f \wedge g \wedge 1 \preceq g \circ f \circ 1 \preceq g \circ f$, and S is intra-regular. □

We finally characterize the ordered semigroups which are regular in terms of fuzzy sets.

Theorem 2.3. *Let $(S,.,\leqslant)$ be an ordered semigroup. If S is regular, then for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of $(S,.)$ we have*

$$f \wedge h \wedge g \preceq f \circ h \circ g.$$

"Conversely", if for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of $(S,.,\leqslant)$ we have $f \wedge h \wedge g \preceq f \circ h \circ g$, then S is regular.

Proof of Theorem 2.3 using the Definitions 1.1, 1.3, 1.5

In addition to Lemmas 2.1–2.6, we need the following lemma.

Lemma 2.9. (cf. also [1; Proposition 1]) *Let $(S,.,\leqslant)$ be an ordered semigroup. If S is regular, then for every right ideal X, every left ideal Y and every bi-ideal B of $(S,.)$ we have*

$$X \cap B \cap Y \subseteq (XBY].$$

"Conversely", if for every right ideal X, every left ideal Y and every bi-ideal B of $(S,.,\leqslant)$ we have $X \cap B \cap Y \subseteq (XBY]$, then S is regular.

Proof of Theorem 2.3

$\Longrightarrow$. Let f be a fuzzy right ideal, g a fuzzy left ideal, h a fuzzy bi-ideal of $(S,.)$, and $a \in S$. Then $a \leqslant axa \leqslant (axa)x(axa)$ for some $x \in S$. Then $(axaxa, xa) \in A_a$, and

$$\begin{aligned} ((f \circ h) \circ g)(a) :&= \bigvee_{(u,v) \in A_a} \min\{(f \circ h)(u), g(v)\} \\ &\geqslant \min\{(f \circ h)(axaxa), g(xa)\}. \end{aligned}$$

Since $(ax, axa) \in A_{axaxa}$, we have

$$(f \circ h)(axaxa) := \bigvee_{(w,t) \in A_{axaxa}} \min\{(f(w), h(t)\} \geqslant \min\{f(ax), h(axa)\}.$$

Then we have

$$\begin{aligned}((f \circ h) \circ g)(a) &\geqslant \min\{\min\{f(ax), h(axa)\}, g(xa)\}\\ &= \min\{f(ax), h(axa), g(xa)\}\\ &\geqslant \min\{f(a), h(a), g(a)\}\\ &= (f \wedge h \wedge g)(a).\end{aligned}$$

Hence we obtain $f \wedge h \wedge g \preceq f \circ h \circ g$.

$\Longleftarrow$. Let X be a right ideal, Y a left ideal, and B a bi-ideal of $(S, ., \leqslant)$. Then f_X is a fuzzy right ideal, f_Y a fuzzy left ideal and f_B a fuzzy bi-ideal of $(S, ., \leqslant)$. By hypothesis, we have $f_X \wedge f_B \wedge f_Y \preceq f_X \circ f_B \circ f_Y$. Since $f_X \wedge f_B \wedge f_Y = f_{X \cap B \cap Y}$ and $f_Y \circ f_B \circ f_X = f_{(YBX]}$, we have $f_{X \cap B \cap Y} \preceq f_{(YBX]}$. Then, by Lemma 2.9, $X \cap B \cap Y \subseteq (XBY]$, and S is regular. □

Proof of Theorem 2.3 using the Definitions 1.2, 1.4, 1.6

$\Longrightarrow$. Let f be a fuzzy right ideal, g a fuzzy left ideal, h a fuzzy bi-ideal of $(S, .)$. Since S is regular and $f \wedge h \wedge g$ a fuzzy subset of S, we have

$$\begin{aligned}f \wedge h \wedge g &\preceq (f \wedge h \wedge g) \circ 1 \circ (f \wedge h \wedge g)\\ &\preceq (f \wedge h \wedge g) \circ 1 \circ (f \wedge h \wedge g) \circ 1 \circ (f \wedge h \wedge g) \circ 1 \circ (f \wedge h \wedge g)\\ &\preceq (f \circ 1) \circ (h \circ 1 \circ h) \circ (1 \circ g)\\ &\preceq f \circ h \circ g\end{aligned}$$

$\Longleftarrow$. Let f be a fuzzy right ideal and g a fuzzy left ideal of $(S, ., \leqslant)$. Since 1 is a fuzzy bi-ideal of $(S, ., \leqslant)$, by hypothesis, we have $f \wedge g = f \wedge 1 \wedge g \preceq f \circ (1 \circ g) \preceq f \circ g$, and S is regular. □

As a conclusion, we have the following

Theorem. *An ordered semigroup S is intra-regular (resp. regular) if and only if for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of S we have $f \wedge h \wedge g \preceq g \circ h \circ f$ (resp. $f \wedge h \wedge g \preceq f \circ h \circ g$). It is both regular and intra-regular if and only if for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of S, we have $f \wedge h \wedge g \preceq h \circ f \circ g$.*

References

[1] **N. Kehayopulu**, *On regular, intra-regular ordered semigroups*, Pure Math. Appl. (PU.M.A.) **4** (1993), 447 – 461.

[2] **N. Kehayopulu, M. Tsingelis**, *Fuzzy sets in ordered groupoids*, Semigroup Forum **65** (2002), 128 – 132.

[3] **N. Kehayopulu, M. Tsingelis**, *The embedding of an ordered groupoid into a poe-groupoid in terms of fuzzy sets*, Inform. Sci. **152** (2003), 231 – 236.

[4] **N. Kehayopulu, M. Tsingelis**, *Regular ordered semigroups in terms of fuzzy subsets*, Inform. Sci. **176** (2006), 3675 – 3693.

[5] **N. Kehayopulu, M. Tsingelis**, *Characterization of some types of ordered semigroups in terms of fuzzy sets*, Lobachevskii J. Math. **29** (2008), 14 – 20.

[6] **N. Kehayopulu, M. Tsingelis**, *Fuzzy right, left, quasi-ideals, bi-ideals in ordered semigroups*, Lobachevskii J. Math. **30** (2009), 17 – 22.

[7] **N. Kehayopulu, M. Tsingelis**, *Intra-regular ordered semigroups in terms of fuzzy sets*, Lobachevskii J. Math. **30** (2009), 23 – 29.

[8] **X. Y. Xie, J. Tang**, *Regular ordered semigroup and intra-regular ordered semigroups in terms of fuzzy subsets*, Iran. J. Fuzzy Syst. **7** (2010), 121 – 140.

Received September 16, 2011
Revised December 20, 2011

University of Athens
Department of Mathematics
157 84 Panepistimiopolis
Athens, Greece

E-mail: nkehayop@math.uoa.gr

Quasigroups and Related Systems **20** (2012), 71 − 80

Varieties of rectangular quasigroups

Aleksandar Krapež

Abstract. For the given variety $\mathbb{V}$ of quaisgroups, the class of all *rectangular* $\mathbb{V}$*–quasigroups* is defined as the class of all groupoids isomorphic to $L \times Q \times R$, where $Q \in \mathbb{V}$ and $L(R)$ is a left (right) zero semigroup. The identities axiomatizing the new class are given, proving that it is a variety in the language of the original variety.

1. Introduction

In the papers [6], [7] and [8], the so called rectangular loops and rectangular quasigroups were defined.

Definition 1.1. Groupoid is a *rectangular quasigroup* (*loop*) iff it is isomorphic to the direct product of a left zero semigroup, a quasigroup (loop) and a right zero semigroup.

Several different axiomatizations for both these structures were given and the problems of the axiomatization by independent systems of axioms were posed.

In their paper [5] M. Kinyon and J. D. Phillips solved these problems by giving the following axioms:

$(RQ1)$ $$x\backslash xx = x$$

$(RQ2)$ $$xx/x = x$$

$(RQ3)$ $$x(x\backslash y) = x\backslash xy$$

$(RQ4)$ $$(x/y)y = xy/y$$

$(RQ5)$ $$(x\backslash y)\backslash((x\backslash y) \cdot zu) = (x\backslash xz)u$$

$(RQ6)$ $$(xy \cdot (z/u))/(z/u) = x(yu/u)$$

2010 Mathematics Subject Classification: 20N02, 20N05, 08B25, 08A5016Y60
Keywords: rectangular $\mathbb{V}$–quasigroup, axiomatization, axiom independence.
Supported by the Ministry of Education and Science of Serbia, grants 174008 and 174026

$$(RL) \qquad x\backslash x(y\backslash y) = (x/x)y/y$$

The system (RQ1)–(RQ6) axiomatizes rectangular quasigroups and, if we add (RL) to it, we get axioms for rectangular loops.

In this paper we give some new axiomatizations of rectangular loops. More importantly, if $\mathbb{V}$ is a quasigroup variety, we give an axiomatization of the variety of rectangular $\mathbb{V}$–quasigroups.

2. Axioms for rectangular $\mathbb{V}$–quasigroups

We need to adjust the types of (equational) quasigroups and left (right) zero semigroups. To achieve this we extend the language of groupoids with further operations.

Definition 2.1. Let $L = \{\cdot, \backslash, /\}$ be the language of quasigroups and M a further (possibly empty) set of operation symbols disjoint from L. The language $\hat{L} = L \cup M$ is *an extended language of quasigroups.*

The language $L_1 = \{\cdot, \backslash, /, e\}$, obtained from L by the addition of a single constant, is *the language of loops.*

Definition 2.2. A left (right) zero semigroup is an algebra in $\hat{L}$ satisfying identities $x\backslash y = x/y = xy$ and $xy = x(xy = y)$.

Definition 2.3. Let $\mathbb{V}$ be a variety of quasigroups in an extended language $\hat{L}$. An algebra in the language $\hat{L}$ is a *rectangular* $\mathbb{V}$*–quasigroup* if it is isomorphic to the direct product of a left zero semigroup, a quasigroup from the variety $\mathbb{V}$ and a right zero semigroup.

There are three exceptions to the definition above. In the Section 3 (4) we consider rectangular left (right) symmetric quasigroups which have only two binary operations. But in that case one of the division operations coincide with multiplication, so this algebra is equivalent to the (proper) rectangular left (right) symmetric quasigroup with three binary operations. Similarly, for TS–quasigroups in which both division operations are equal to multiplication, rectangular TS–quasigroups are just special groupoids.

Theorem 2.4. *Let $\mathbb{V}$ be a variety of quasigroups satisfying additional identities $s_i = t_i$ $(i \in I)$ in an extended language $\hat{L}$ and let x be a variable which does not occur in either s_i or t_i. Then the variety $\square\mathbb{V}$ of rectangular $\mathbb{V}$–quasigroups can be axiomatized by (RQ1)–(RQ6) together with (for all $i \in I$):*

$$(V_i) \qquad x \cdot s_i x = x \cdot t_i x \ .$$

Proof. Left (right) zero semigroups as well as all $\mathbb{V}$–quasigroups satisfy (RQ1)–(RQ6) and all (V_i) $(i \in I)$. So do their direct products i.e. rectangular $\mathbb{V}$–quasigroups.

If an algebra satisfies (RQ1)–(RQ6) then it is a rectangular quasigroup. Since all (V_i) are satisfied, the quasigroup factor has to satisfy them too. But in quasigroups identities (V_i) are equivalent to $s_i = t_i$ and these define $\mathbb{V}$. □

Theorem 2.5. *Theorem* 2.4 *remains valid if we replace* (V_i) *by any of the following identities:*

$$x \circ (s_i \diamond x) = x \circ (t_i \diamond x)$$

$$(x \circ s_i) \diamond x = (x \circ t_i) \diamond x$$

$$x/(s_i \backslash x) = (x/t_i) \backslash x$$

$$x \circ (s_i \diamond y) = x \circ (t_i \diamond y)$$

$$(x \circ s_i) \diamond y = (x \circ t_i) \diamond y$$

where x, y *do not occur in* s_i, t_i *and* $\circ, \diamond \in \{\cdot, \backslash, /\}$.

Proof. In the proof of Theorem 2.4 we can replace any (V_i) by some of the above identities which are, in quasigroups, equivalent to $s_i = t_i$. The line of reasoning remains the same. □

Definition 2.6. $head(t)(tail(t))$ is the first (last) variable of the term t.

Theorem 2.7. *The equality* $u = v$ *is true in all rectangular* $\mathbb{V}$*–quasigroups iff* $head(u) = head(v)$, $tail(u) = tail(v)$ *and* $u = v$ *is true in all* $\mathbb{V}$*–quasigroups.*

Proof. In one direction the theorem is true because projections are epimorphisms and so preserve identities. The converse is true because direct products also preserve identities. □

Theorem 2.8. *Theorem* 2.4 *remains valid if we replace* (V_i) *by any of the following identities:*

$$s_i \circ x = t_i \circ x \quad (\textit{if } head(s_i) = head(t_i))$$

$$x \circ s_i = x \circ t_i \quad (\textit{if } tail(s_i) = tail(t_i))$$

$$s_i = t_i \quad (\textit{provided both } head(s_i) = head(t_i) \textit{ and } tail(s_i) = tail(t_i))$$

where x *does not occur in* s_i, t_i *and* $\circ \in \{\cdot, \backslash, /\}$. □

Example 2.9. Adding associativity $x \cdot yz = xy \cdot z$ to identities $(RQ1)$–$(RQ6)$ gives yet another axiomatization of *rectangular groups.* □

Example 2.10. Adding identity $x \cdot yx = x \cdot zx$ to (RQ1)–(RQ6) gives a (way too complicated) axiomatization of *rectangular bands.* □

Example 2.11. Rectangular commutative quasigroups have identities (RQ1) – (RQ6) and $x(yz \cdot x) = x(zy \cdot x)$ as axioms. □

However, note that commutative rectangular quasigroups are just commutative quasigroups.

Example 2.12. Rectangular medial quasigroups are axiomatized by $(RQ1)$–$(RQ6)$ and $xy \cdot uv = xu \cdot yv$. □

Example 2.13. Commutative medial quasigroups are characterized by the axiom $xy \cdot uv = uy \cdot xv$ (among others). *Rectangular commutative medial quasigroups* are rectangular quasigroups satisfying $x(yz \cdot uv) = x(uz \cdot yv)$. □

Example 2.14. Paramedial quasigroups are characterized by the identity $xy \cdot uv = vy \cdot ux$. *Rectangular paramedial quasigroups* are axiomatized by adding identity $x \cdot (yz \cdot uv)x = x \cdot (vz \cdot uy)x$ to (RQ1)–(RQ6). □

It is rather obvious that the following corollaries are true:

Corollary 2.15. *If the variety* $\mathbb{V}$ *of quasigroups is defined by the identities* $s_i = t_i$ $(i \in I)$ *such that* $head(s_i) = head(t_i)$, $tail(s_i) = tail(t_i)$ *for all* $i \in I$, *then the class of rectangular quasigroups satisfying all identities* $s_i = t_i$ $(i \in I)$ *is the class of all rectangular* $\mathbb{V}$*–quasigroups.* □

Corollary 2.16. *If the variety* $\mathbb{V}$ *of quasigroups is defined by the identities* $s_i = t_i$ $(i \in I)$ *such that* $head(s_i) \neq head(t_i)$ *and* $tail(s_j) \neq tail(t_j)$ *for some* $i, j \in I$, *then the class of rectangular quasigroups satisfying all identities* $s_i = t_i$ $(i \in I)$ *is just the class of all* $\mathbb{V}$*–quasigroups.* □

Example 2.17. Moufang loops are defined as loops satisfying any of the four equivalent identities:

$$xy \cdot zx = (x \cdot yz)x$$

$$x(yz \cdot x) = xy \cdot zx$$

$$x(y \cdot xz) = (xy \cdot x)z$$

$$x(y \cdot zy) = (xy \cdot z)y.$$

K. Kunen recently proved in [9] that the existence of the neutral element follows from any of these identities. Therefore, *rectangular Moufang loops* are axiomatized by $(RQ1)$–$(RQ6)$ and for example $xy \cdot zx = (x \cdot yz)x$. □

Example 2.18. Let $(S; \cdot)$ and $(T; \circ)$ be groupoids and $f, g, h : S \longrightarrow T$ three bijections. If $f(xy) = g(x) \circ h(y)$ we say that $(T; \circ)$ is an *isotope* of $(S; \cdot)$. Isotopy is an important invariant of quasigroups which generalizes isomorphism. □

The result that every quasigroup is an isotope of some loop is a classical one in quasigroup theory. The class of all isotopes of groups is also significant and constitutes a variety of quasigroups as proved by V. D. Belousov in [1]. The defining identity of group isotopes is

$$x(y\backslash(z/u)v) = (x(y\backslash z)/u)v. \tag{2.1}$$

By the theorem 2.8 the axioms for the class of all *rectangular group isotopes* are $(RQ1)$–$(RQ6)$ and (2.1).

Note that the class of all *isotopes of rectangular groups* is strictly greater than the class of all rectangular group isotopes. Namely, if $S = \{0, 1\}, f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $xy = f(x)$, then $(S; \cdot)$ is an isotope of the left zero semigroup with two elements but is not a rectangular quasigroup. □

Example 2.19. The variety of *rectangular quasigroups with an idempotent* may be axiomatized by $(RQ1)$–$(RQ6)$ and $ee = e$. □

Example 2.20. The variety of *rectangular left loops* is axiomatized by $(RQ1)$–$(RQ6)$ and any of the following 37 identities:

$$x \circ ((y/y) \diamond x) = x \circ ((z/z) \diamond x)$$

$$(x \circ (y/y)) \diamond x = (x \circ (z/z)) \diamond x$$

$$x/((y/y)\backslash x) = (x/(z/z))\backslash x$$

$$x \circ ((y/y) \diamond u) = x \circ ((z/z) \diamond u)$$

$$(x \circ (y/y)) \diamond u = (x \circ (z/z)) \diamond u$$

where $\circ, \diamond \in \{\cdot, \backslash, /\}$. □

Example 2.21. If the variety of left loops is defined in the language of loops i.e. by the identity $ex = x$, then the variety of rectangular left loops is axiomatized by $(RQ1)$–$(RQ6)$ and

$$x \cdot ey = xy. \tag{2.2}$$

Example 2.22. The variety of rectangular loops is axiomatized by $(RQ1)$–$(RQ6)$ and any of the identities from the Example 2.20, together with the dual of one of them (to ensure the existence of a right neutral in quasigroup). However, we can apply the Theorem 2.5 to the single identity $y\backslash y = z/z$ which axiomatizes loops within quasigroups, and add any of the following identities to $(RQ1)$–$(RQ6)$ to obtain axioms for rectangular loops.

$$x \circ ((y\backslash y) \diamond x) = x \circ ((z/z) \diamond x)$$

$$(x \circ (y\backslash y)) \diamond x = (x \circ (z/z)) \diamond x$$

$$x/((y\backslash y)\backslash x) = (x/(z/z))\backslash x$$

$$x/((y/y)\backslash x) = (x/(z\backslash z))\backslash x$$

$$x \circ ((y\backslash y) \diamond u) = x \circ ((z/z) \diamond u)$$

$$(x \circ (y\backslash y)) \diamond u = (x \circ (z/z)) \diamond u$$

where $\circ, \diamond \in \{\cdot, \backslash, /\}$. This gives us a total of 1407 axiom systems for rectangular loops. □

Example 2.23. In the language of loops, the variety of rectangular loops can be axiomatized by ($RQ1$)–($RQ6$), (2.2) and

$$xe \cdot y = xy \tag{2.3}$$

The identity (2.2) may be replaced by any of identities from the Example 2.20. Likewise, the identity (2.3) may be replaced by the dual of some of these identities. This gives us 75 further axiomatizations of rectangular loops.

However, it should be admitted that the axiom system of Kinyon and Phillips is shorter (smaller language and/or less identities and/or less variables and/or less symbols) and more appealing then any of the above 1482 systems. The only exception is perhaps the system with identities (2.2) and (2.3). □

3. Rectangular left symmetric quasigroups

The important class of *left symmetric quasigroups* is characterized by the identity $x \cdot xy = y$. Just as in numerous examples in the previous section, we can axiomatize rectangular left symmetric quasigroups by identities ($RQ1$)–($RQ6$) and the identity

$$x(y \cdot yz) = xz \tag{LS}$$

as prescribed by the Theorem 2.8.

However, in this case we can do more. Note that by the Definition 2.2 $x\backslash y = xy$ in both left and right zero semigroups. In left symmetric quasigroups this is also true. Therefore, the identity $x\backslash y = xy$ is true in rectangular left symmetric quasigroups as well. But then the operation $\backslash$ can be eliminated from axioms and from the language itself. We have:

Theorem 3.1. *An algebra $(S; \cdot, /)$ is a rectangular left symmetric quasigroup iff it satisfies:*

$$x \cdot xx = x \tag{LS1}$$

$$xx/x = x \tag{LS2}$$

$$(x/y)y = xy/y \tag{LS3}$$

$$xy \cdot (xy \cdot uv) = (x \cdot xu)v \tag{LS4}$$

$$(xy \cdot (u/v))/(u/v) = x(yv/v). \tag{LS5}$$

Proof. Axiom ($RQ3$) transforms into trivial identity and may be eliminated. Axioms ($RQ1$) and ($RQ5$) become axioms ($LS1$) and ($LS4$) respectively.

Only (LS) remains to be proved. We do it by the series of lemmas below. □

Lemma 3.2. $(x \cdot xy)z = x(x \cdot yz)$

Proof.
$$\begin{aligned} (x \cdot xy)z &= (x \cdot xx) \cdot ((x \cdot xx) \cdot yz) && \text{(by (LS4))} \\ &= x(x \cdot yz) && \text{(by (LS1))} \end{aligned}$$
□

Lemma 3.3. $xy \cdot (xy \cdot z) = x \cdot xz$

Proof.
$$\begin{aligned} xy \cdot (xy \cdot z) &= xy \cdot (xy \cdot (z \cdot zz)) && \text{(by (LS1))} \\ &= (x \cdot xz) \cdot zz && \text{(by (LS4))} \\ &= (x \cdot xx) \cdot ((x \cdot xx) \cdot (z \cdot zz)) && \text{(by (LS4))} \\ &= x \cdot xz && \text{(by (LS1))} \end{aligned}$$
□

Lemma 3.4. $x(x \cdot xy) = xy$

Proof.
$$\begin{aligned} x(x \cdot xy) &= (x \cdot xx)y && \text{(by Lemma 3.2)} \\ &= xy && \text{(by (LS1))} \end{aligned}$$
□

Lemma 3.5. $xy \cdot x(x \cdot zu) = xy \cdot zu$

Proof.
$$\begin{aligned} xy \cdot x(x \cdot zu) &= xy \cdot (x \cdot xz)u && \text{(by Lemma 3.2)} \\ &= xy \cdot (xy \cdot (xy \cdot zu)) && \text{(by (LS4))} \\ &= xy \cdot zu && \text{(by Lemma 3.4)} \end{aligned}$$
□

Lemma 3.6. $x \cdot x(y \cdot yz) = x \cdot xz$

Proof.
$$\begin{aligned} x \cdot x(y \cdot yz) &= (x \cdot xy) \cdot yz && \text{(by Lemma 3.2)} \\ &= (x \cdot xy) \cdot x(x \cdot yz) && \text{(by Lemma 3.5)} \\ &= (x \cdot xy) \cdot (x \cdot xy)z && \text{(by Lemma 3.2)} \\ &= x \cdot xz && \text{(by Lemma 3.3)} \end{aligned}$$
□

Lemma 3.7. $x(y \cdot yz) = xz$

Proof.
$$\begin{aligned} x(y \cdot yz) &= x(x \cdot x(y \cdot yz)) && \text{(by Lemma 3.4)} \\ &= x(x \cdot xz) && \text{(by Lemma 3.6)} \\ &= xz && \text{(by Lemma 3.4)} \end{aligned}$$
□

The proof above is an adaptation of the proof found by the automated reasoning program `Prover9`. `Prover9` is the first order logic theorem prover developed by W. W. McCune [11] which is capable of solving difficult mathematical problems. For instance, McCune in [10] solved the so called Robbins conjecture using `Otter` (an earlier version of `Prover9`). See [12] for the gentle introduction to `Otter` with the leaning to quasigroup theory.

McCune also wrote the model builder program `Mace4` [11], which is used in the following examples to verify the independence of the axioms $(LS1)$–$(LS5)$.

Example 3.8. Table 1 is the smallest model that satisfies (LS2), (LS3), (LS4), and (LS5), but not (LS1).

$\bullet$	0	1
0	1	1
1	0	0

$\backslash$	0	1
0	1	1
1	0	0

Table 1. (LS2), (LS3), (LS4) and (LS5), but not (LS1). □

Example 3.9. Table 2 is the smallest model that satisfies (LS1), (LS3), (LS4), and (LS5), but not (LS2).

$\bullet$	0	1
0	0	1
1	1	0

$\backslash$	0	1
0	1	0
1	0	1

Table 2. (LS1), (LS3), (LS4) and (LS5), but not (LS2). $\square$

Example 3.10. Table 3 is the smallest model that satisfies (LS1), (LS2), (LS4), and (LS5), but not (LS3).

$\bullet$	0	1	2
0	1	0	2
1	2	1	0
2	0	2	1

$\backslash$	0	1	2
0	2	0	1
1	0	1	2
2	1	2	0

Table 3. (LS1), (LS2), (LS4) and (LS5), but not (LS3). $\square$

Example 3.11. Table 4 is the smallest model that satisfies (LS1), (LS2), (LS3), and (LS5), but not (LS4).

$\bullet$	0	1	2
0	0	1	0
1	1	2	1
2	2	0	2

$\backslash$	0	1	2
0	0	2	0
1	1	0	1
2	2	1	2

Table 4. (LS1), (LS2), (LS3) and (LS5), but not (LS4). $\square$

Example 3.12. Table 5 is the smallest model that satisfies (LS1), (LS2), (LS3), and (LS4), but not (LS5).

$\bullet$	0	1
0	0	0
1	1	1

$\backslash$	0	1
0	0	0
1	0	1

Table 5. (LS1), (LS2), (LS3) and (LS4), but not (LS5). $\square$

4. Right symmetric quasigroups

Right symmetric quasigroups are defined by the identity $xy \cdot y = x$. From the Theorem 3.1 it follows, by the duality principle for groupoids (see [2]), that the class of all *rectangular right symmetric quasigroups* can be axiomatized by the identities:

$$x\backslash xx = x \tag{RS1}$$

$$xx \cdot x = x \tag{RS2}$$

$$x(x\backslash y) = x\backslash xy \tag{RS3}$$

$$(x\backslash y)\backslash((x\backslash y)\cdot uv) = (x\backslash xu)v \tag{RS4}$$

$$(xy\cdot uv)\cdot uv = x(yv\cdot v) \tag{RS5}$$

in the language $\{\cdot, \backslash\}$. Moreover, the axioms are mutually independent.

If a quasigroup satisfies both left and right symmetry identities, i.e. if both $x \cdot xy = y$ and $xy \cdot y = x$ are true, then such a quasigroup is called a *totally symmetric* or *TS–quasigroup*. TS–quasigroups are commutative and both division operations in them coincide with multiplication. Applying Theorem 3.1 and its dual we get:

Theorem 4.1. *A groupoid $(S;\cdot)$ is a rectangular TS–quasigroup iff*

$$x\cdot xx = x \tag{TS1}$$

$$xx\cdot x = x \tag{TS2}$$

$$xy\cdot(xy\cdot uv) = (x\cdot xu)v \tag{TS3}$$

$$(xy\cdot uv)\cdot uv = x(yv\cdot v). \tag{TS4}$$

Example 4.2. Table 6 is the smallest model that satisfies (TS2), (TS3) and (TS4), but not (TS1).

$\bullet$	0	1
0	1	1
1	0	0

Table 6. (TS2), (TS3) and (TS4), but not (TS1).

$\bullet$	0	1	2
0	0	2	0
1	1	1	1
2	2	0	2

Table 7. (TS1), (TS2) and (TS4), but not (TS3).

Example 4.3. Table 7 is the smallest model that satisfies (TS1), (TS2), and (TS4), but not (TS3). □

Independence of (TS2) and (TS4) is proved by models dual to those in Examples 4.2 and 4.3 respectively.

Acknowledgement. The author would like to thank M. K. Kinyon for his help and comments on the early version of this paper.

References

[1] **V. D. Belousov**, *Balanced identities in quasigroups* (Russian), Mat. Sb. **70(112)** (1966), 55 – 97.

[2] **A. H. Cliffod and G. B. Preston**, *The algebraic theory of semigroups*, vol. 1, AMS, Providence, (1964).

[3] **Z. Daroóczy and Zs. Páles (eds.)**, *Functional equations – Results and advances*, Kluwer Academic Publishers, Dordrecht, Boston, London, (2002).

[4] **Z. Kadelburg (ed.)**, Proc. 10th Congress Yugoslav Math., Univ. Belgrade Fac. Math., Belgrade, (2001).

[5] **M. K. Kinyon and J. D. Phillips**, *Rectangular quasigroups and loops*, Comput. Math. Appl. **49** (2005), 1679 – 1685.

[6] **A. Krapež**, *Rectangular loops*, Publ. Inst. Math (Belgrade) (N.S.) **68(82)** (2000), 59 – 66.

[7] **A. Krapež**, *Rectangular quasigroups*, in the book [4], 169 – 171.

[8] **A. Krapež**, *Generalized associativity on rectangular quasigroups*, in the book [3], 335 – 349.

[9] **K. Kunen**, *Moufang quasigroups*, J. Algebra **183** (1996), 231 – 234.

[10] **W. W. McCune**, *Solution of the Robbins problem*, *J. Automated Reasoning* **19** (1997), 263 – 276.

[11] **W. W. McCune**, `Prover9` version 2008–06A, (http://www.cs.unm.edu/ mccune/prover9/).

[12] **J. D. Phillips**, *See Otter digging for algebraic pearls*, *Quasigroups and Related Systems* **10** (2003), 95 – 114.

Received September 4, 2011

Matematički institut
Kneza Mihaila 36
11001 Beograd, p.p. 367
Serbia
E-mail: sasa@mi.sanu.ac.rs

Quasigroups and Related Systems **20** (2012), 81 − 85

Essential operations of clones

Adam W. Marczak

Abstract. Clones of algebras consist not only of essential operations but also of operations not depending on every variable. However, the sets of all essential operations of clones uniquely determine the clones. In this note we present a short precise proof of this fact and indicate these essential operations that are equal to inessential elements of clones.

1. Introduction

In the last century research in the theory of finite automata and deterministic operators led to problems concerning essential variables of functions. From that time the theory of essential variables of finite operations became a quite frequent research direction. The study of essential variables in functions defined on finite sets, initiated by A. Salomaa in [11], goes with multiple-valued logic and currently plays an important role in computer sciences. Essential variables of functions and essential term operations of algebras were widely studied under different aspects, see e.g. [1]–[6], [8],[9], [12],[13].

The clone of a given algebra consists of all its term operations – it contains both essentially n-ary term operations as well as term operations not depending on every variable. But the clone is uniquely determined by the set of all its constants and essential operations. This fact is sometimes assumed as intuitive, since every term operation not depending on every its variable can be obtained by adding inessential variables to an essential operation. However, this argumentation is imprecise and it cannot be regarded as sufficient, especially when the essential operation equal to a given inessential one has to be indicated, as e.g. in [10]. Therefore we decided to give in this note a short precise argument that clones of algebras are determined only by constants and essentially n-ary term operations. We indicate these essential elements of clones that are equal to the elements not depending on every variable.

By an *algebra* we mean a pair $\mathfrak{A} = (A; F^{\mathfrak{A}})$, where A is a nonempty set and $F^{\mathfrak{A}}$ is a family of mappings $f^{\mathfrak{A}}\colon A^n \to A$ called fundamental operations of $\mathfrak{A}$. The number n is called the *arity* of $f^{\mathfrak{A}}$. A *type* of algebras we define as a mapping $\tau\colon F \to \mathbb{N} \cup \{0\}$, where F is a nonempty set of fundamental operation symbols and $\mathbb{N}$ is the set of positive integers. An algebra is said to be of type τ if it is of

2010 Mathematics Subject Classification: 20N15
Keywords: algebra, clone of an algebra, essentially n-ary term operation

the form $\mathfrak{A} = (A;\, F^{\mathfrak{A}})$, where $F^{\mathfrak{A}} = \{f^{\mathfrak{A}} \colon\, f \in F\}$, and the arity of $f^{\mathfrak{A}}$ equals $\tau(f)$ for every $f \in F$.

Let an algebra $\mathfrak{A} = (A;\, F^{\mathfrak{A}})$ of type τ be given. Recall that for every $1 \leqslant i \leqslant n$, the i-th n-ary *projection* is the mapping $(a_1, \ldots, a_n) \mapsto a_i$. It is usually denoted by $e_i^n(x_1, \ldots, x_n) = x_i$. The smallest set containing all projections and all elements of $F^{\mathfrak{A}}$ that is closed under superpositions is called the set of *term operations* of $\mathfrak{A}$, or the *clone* of $\mathfrak{A}$. We denote it by $Cl(\mathfrak{A})$. An n-ary term operation $f^{\mathfrak{A}} \in Cl(\mathfrak{A})$ *depends* on the variable x_i, if there exist elements $a_1, \ldots, a_n, b \in A$ such that

$$f^{\mathfrak{A}}(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) \neq f^{\mathfrak{A}}(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n).$$

The number of essential variables in $f^{\mathfrak{A}}$ is called the *essential arity* of $f^{\mathfrak{A}}$. If the term operation $f^{\mathfrak{A}}$ depends on every of its variable, then it is said to be *essentially* n-ary, or an *essential operation* of $\mathfrak{A}$. Otherwise $f^{\mathfrak{A}}$ is called *inessential*.

Following [6], for an algebra $\mathfrak{A}$ and every positive integer n, $\mathbb{P}_n(\mathfrak{A})$ denotes the set of all essentially n-ary term operations of $\mathfrak{A}$. $\mathbb{P}_0(\mathfrak{A})$ denotes the set of all constant non-nullary term operations of $\mathfrak{A}$ and all its nullary operations.

2. The result

Let an algebra $\mathfrak{A} = (A;\, F^{\mathfrak{A}})$ of type τ be given. For an n-ary term operation $f^{\mathfrak{A}}(x_1, \ldots, x_n) \in Cl(\mathfrak{A})$ and a permutation σ of $1, \ldots, n$, define

$$f^{\mathfrak{A}}_{\sigma}(x_1, \ldots, x_n) = f^{\mathfrak{A}}(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Recall the following two simple observations. They are both easily provable by induction on the complexity of term operation, see also [7], § 8.

(2.i) *Let $n > 1$. For every n-ary term operation $f^{\mathfrak{A}} \in Cl(\mathfrak{A})$, there exists an $(n-1)$-ary term operation $g^{\mathfrak{A}} \in Cl(\mathfrak{A})$ such that*

$$f^{\mathfrak{A}}(a_1, \ldots, a_{n-1}, a_{n-1}) = g^{\mathfrak{A}}(a_1, \ldots, a_{n-1})$$

for all $a_1, \ldots, a_{n-1} \in A$.

(2.ii) *If an n-ary term operation $f^{\mathfrak{A}} \in \mathbb{P}_n(\mathfrak{A})$, then also $f^{\mathfrak{A}}_{\sigma} \in \mathbb{P}_n(\mathfrak{A})$ for every permutation σ of $1, \ldots, n$.*

Then we have the following.

Lemma. *For a given algebra $\mathfrak{A}$, if a term operation $f^{\mathfrak{A}}(x_1, \ldots, x_n)$ depends only on the variables $x_1, \ldots, x_k$ for some $k < n$, then there exists a term operation $(f^*)^{\mathfrak{A}}(x_1, \ldots, x_k) \in \mathbb{P}_k(\mathfrak{A})$ such that*

$$f^{\mathfrak{A}}(x_1, \ldots, x_n) = (f^*)^{\mathfrak{A}}\bigl(e_1^n(x_1, \ldots, x_n), \ldots, e_k^n(x_1, \ldots, x_n)\bigr),$$

where $e_i^n(x_1, \ldots, x_n) = x_i$ for every $i = 1, \ldots, k$.

Proof. Consider a term operation $f^{\mathfrak{A}}(x_1, \ldots, x_n) \in Cl(\mathfrak{A})$ that depends on $x_1, ..., x_k$ for some $k < n$. From (2.i), there exists a k-ary term operation $(f^*)^{\mathfrak{A}} \in Cl(\mathfrak{A})$ such that

$$(f^*)^{\mathfrak{A}}(a_1, \ldots, a_k) = f^{\mathfrak{A}}(a_1, \ldots, a_k, \ldots, a_k)$$

for every $a_1, \ldots, a_k \in A$. We shall prove that $(f^*)^{\mathfrak{A}}$ is essentially k-ary. Indeed, since $f^{\mathfrak{A}}(x_1, \ldots, x_n)$ depends on x_i for every $i = 1, \ldots, k-1$, there exist elements $a_1, \ldots, a_i, \ldots, a_n, b_i \in A$ such that

$$f^{\mathfrak{A}}(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) \neq f^{\mathfrak{A}}(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_n).$$

Since $f^{\mathfrak{A}}$ does not depend on x_j for $j > k$, so we have

$$f^{\mathfrak{A}}(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_k, a_{k+1}, \ldots, a_n) = f^{\mathfrak{A}}(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_k, a_k, \ldots, a_k)$$

and

$$f^{\mathfrak{A}}(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_k, a_{k+1}, \ldots, a_n) = f^{\mathfrak{A}}(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_k, a_k, \ldots, a_k),$$

and consequently

$$(f^*)^{\mathfrak{A}}(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_k) \neq (f^*)^{\mathfrak{A}}(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_k)$$

for every $i = 1, \ldots, k-1$. Therefore the term operation $(f^*)^{\mathfrak{A}}$ depends on x_i for every $i < k$. Moreover, since $f^{\mathfrak{A}}$ depends also on x_k, we have

$$f^{\mathfrak{A}}(c_1, \ldots, c_{k-1}, c_k, c_{k+1}, \ldots, c_n) \neq f^{\mathfrak{A}}(c_1, \ldots, c_{k-1}, d_k, c_{k+1}, \ldots, c_n)$$

for some elements $c_1, \ldots, c_n, d_k \in A$. But $f^{\mathfrak{A}}$ does not depend on x_j for every $j > k$, so we have

$$f^{\mathfrak{A}}(c_1, \ldots, c_{k-1}, c_k, c_{k+1}, \ldots, c_n) = f^{\mathfrak{A}}(c_1, \ldots, c_{k-1}, c_k, c_k, \ldots, c_k)$$

and

$$f^{\mathfrak{A}}(c_1, \ldots, c_{k-1}, d_k, c_{k+1}, \ldots, c_n) = f^{\mathfrak{A}}(c_1, \ldots, c_{k-1}, d_k, d_k, \ldots, d_k)$$

and consequently

$$(f^*)^{\mathfrak{A}}(c_1, \ldots, c_{k-1}, c_k) \neq (f^*)^{\mathfrak{A}}(c_1, \ldots, c_{k-1}, d_k).$$

Thus $(f^*)^{\mathfrak{A}}(x_1, \ldots, x_k) \in \mathbb{P}_k(\mathfrak{A}) \subset Cl(\mathfrak{A})$. Finally, let $(f^{**})^{\mathfrak{A}}$ denote the term operation obtained from $(f^*)^{\mathfrak{A}}$ by substituting every its variable x_i for the n-ary projection $e_i^n(x_1, \ldots, x_n)$ for every $i = 1, \ldots, k$. We have

$$(f^{**})^{\mathfrak{A}}(x_1, \ldots, x_n) = (f^*)^{\mathfrak{A}}\big(e_1^n(x_1, \ldots, x_n), \ldots, e_k^n(x_1, \ldots, x_n)\big).$$

Note that for every $a_1, \ldots, a_n \in A$ we have

$$(f^{**})^{\mathfrak{A}}(a_1, \ldots, a_n) = (f^*)^{\mathfrak{A}}\big(e_1^n(a_1, \ldots, a_n), \ldots, e_k^n(a_1, \ldots, a_n)\big) = (f^*)^{\mathfrak{A}}(a_1, \ldots, a_k) = f^{\mathfrak{A}}(a_1, \ldots, a_k, a_k, \ldots, a_k)$$

and since $f^{\mathfrak{A}}$ does not depend on x_j for any $j > k$, we obtain

$$f^{\mathfrak{A}}(a_1, \ldots, a_k, a_k, \ldots, a_k) = f^{\mathfrak{A}}(a_1, \ldots, a_k, a_{k+1}, \ldots, a_n)$$

and consequently

$$(f^{**})^{\mathfrak{A}}(x_1, \ldots, x_n) = f^{\mathfrak{A}}(x_1, \ldots, x_n),$$

completing the proof. □

Theorem. *Let* $\mathfrak{A}_1 = (A; F^{\mathfrak{A}_1})$ *and* $\mathfrak{A}_2 = (A; G^{\mathfrak{A}_2})$ *be algebras of types* τ_1 *and* τ_2*, respectively. Then* $Cl(\mathfrak{A}_1) = Cl(\mathfrak{A}_2)$ *if and only if* $\mathbb{P}_n(\mathfrak{A}_1) = \mathbb{P}_n(\mathfrak{A}_2)$ *for every* $n \in \mathbb{N} \cup \{0\}$*.*

In another words, the clone $Cl(\mathfrak{A})$ of a given algebra $\mathfrak{A}$ is uniquely determined by the subset of $Cl(\mathfrak{A})$ consisting of all term operations depending on every variable and all constant operations.

Proof. The necessity of the theorem is obvious. For the proof of sufficiency assume that $\mathbb{P}_n(\mathfrak{A}_1) = \mathbb{P}_n(\mathfrak{A}_2)$ for every nonnegative integer n. Let a mapping f be a nullary, constant non-nullary or essentially n-ary term operation of $\mathfrak{A}_1$. Then, by the assumption, $f \in \mathbb{P}_n(\mathfrak{A}_1)$ if and only if $f \in \mathbb{P}_n(\mathfrak{A}_2)$ for some $n \in \mathbb{N} \cup \{0\}$. Let $f^{\mathfrak{A}_1}(x_1, \ldots, x_n) \in Cl(\mathfrak{A}_1)$ be a term operation depending only on k, $k < n$, its variables. Consider a term operation $f_\sigma^{\mathfrak{A}_1}(x_1, \ldots, x_n) = f^{\mathfrak{A}_1}(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for a permutation $\sigma \in \mathbb{S}_n$ such that $f_\sigma^{\mathfrak{A}_1}$ depends on $x_1, \ldots, x_k$. From (2.ii), $f^{\mathfrak{A}_1} \in Cl(\mathfrak{A}_1)$ implies that $f_\sigma^{\mathfrak{A}_1} \in Cl(\mathfrak{A}_1)$. Then, from Lemma, there exists a term operation $(f_\sigma^*)^{\mathfrak{A}_1} \in \mathbb{P}_k(\mathfrak{A}_1)$ such that

$$(f_\sigma^*)^{\mathfrak{A}_1}(a_1, \ldots, a_k) = f_\sigma^{\mathfrak{A}_1}(a_1, \ldots, a_k, a_{k+1}, \ldots, a_n)$$

for every $a_1, \ldots, a_n \in A$. But since $(f_\sigma^*)^{\mathfrak{A}_1}$ is essentially k-ary, so – by the assumption – $(f_\sigma^*)^{\mathfrak{A}_1}$ belongs also to the set $\mathbb{P}_k(\mathfrak{A}_2) \subseteq Cl(\mathfrak{A}_2)$ and hence $f_\sigma^{\mathfrak{A}_1} \in Cl(\mathfrak{A}_2)$. Now, from (2.ii) again, $f^{\mathfrak{A}_1} \in Cl(\mathfrak{A}_2)$ and consequently the inclusion $Cl(\mathfrak{A}_1) \subseteq Cl(\mathfrak{A}_2)$ holds. The proof of the opposite inclusion is analogous. So, $Cl(\mathfrak{A}_1) = Cl(\mathfrak{A}_2)$, completing the proof. □

References

[1] **J. Berman and A. Kisielewicz**, On the number of operations on a clone, Proc. Amer. Math. Soc. **122** (1994), 359 – 369.

[2] **M. Couceiro and E. Lehtonen**, *On the arity gap of finite functions: results and applications*, Proc. of the First Int. Conf. on Relations, Orders and Graphs: Interaction with Computer Science (2008), 65 – 72.

[3] **M. Couceiro, E. Lehtonen and T. Waldhauser**, *Decompositions of functions based on arity gap*, arXiv:1003.1294v1 [math.CO] (2010).

[4] **R. O. Davies**, *Two theorems on essential variables*, J. London Math. Soc. **41** (1966), 333 – 335.

[5] **K. Denecke and J. Koppitz**, *Essential variables in hypersubstitutions*, Algebra Univers. **46** (2001), 443 – 45.

[6] **G. Grätzer**, *Composition of functions*, in: "Proceedings of the Conference on Universal Algebra", 1969, Queen's University, Kingston, Ont., (1970), 1 – 106.

[7] **G. Grätzer**, "Universal Algebra", 2nd ed., Springer-Verlag, New York-Heidelberg-Berlin 1979.

[8] **G. Grätzer and A. Kisielewicz**, *A survey of some open problems on p_n-sequences and free spectra of algebras and varieties*, 57 – 88, in: Universal Algebra and Quasigroup Theory, Romanowska, A., Smith, J.D.H. (eds.), Helderman Verlag, Berlin 1992.

[9] **A. I. Mal'cev**, *Iterative algebras and Post varieties*, Studies in Logic and the Foundations of Math. **66** (1971), 396 – 415.

[10] **A. W. Marczak and J. Płonka**, *Mapping extension of an algebra*, Algebra Colloq. **16** (2009), 479 – 494.

[11] **A. Salomaa**, *On essential variables of functions, especially in the algebra of logic*, Ann. Acad. Scient. Fennicæ Ser. A. I. Math. **339** (1963), 3 – 11.

[12] **Sl. Shtrakov and K. Denecke**, *Essential variables and separable sets in universal algebra*, Multiple Valued Logic: An International Journal **8** (2002), 165 – 181.

[13] **R. Willard**, *Essential arities of term operations in finite algebras*, Discrete Math. **149** (1996), 239 – 259.

Received November 11, 2011

Institute of Mathematics and CS
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27
50-370 Wrocław
Poland
e-mail: amarczak@pwr.wroc.pl

Quasigroups and Related Systems **20** (2012), 87 − 96

The spectrum of a variety of modular groupoids

Robert A. R. Monzo

Abstract. We prove that the spectrum of the variety of idempotent, right modular and anti-rectangular groupoids consists of all powers of four. We also prove that any finite or countable groupoid anti-isomorphic to a groupoid in that variety is isomorphic to it. Finally, it is proved that, to within isomorphism, there is only one countable groupoid in that variety and that it is isomorphic to a proper subgroupoid of itself.

1. Introduction

Kazim and Naseeruddin studied a groupoid variety consisting of what they called *left almost semigroups*, groupoids satisfying the equation $xy \cdot z = zy \cdot x$ [9]. Such groupoids have also been referred to as *left invertive* [5], *Abel-Grassmann's* [8, 10, 11, 12, 14, 15, 16] and *right modular* [7]. Various aspects of these groupoids have been studied over the years, such as partial ordering and congruences [6], inflations [15], idempotent structure [14], zeroids and idempoids [12], structure of unions of groups [10], power groupoids and inclusion classes [11] simplicity [7] and combinatorial chacterization [1].

In this paper we study the variety $I \cap RM \cap AR$ of idempotent, right modular, anti-rectangular groupoids, the collection of groupoids that satisfy the equations $x = x^2$, $xy \cdot z = zy \cdot x$ and $xy \cdot x = y$. These groupoids also satisfy the equation $x \cdot yz = z \cdot yx$ and are therefore modular. They were called *anti-rectangular* AG-*bands* in [14] and are also known, perhaps more commonly, as *affine spaces over* $GF(4)$ [1, 4]. The main result of this paper is that there is, up to isomorphism, exactly one groupoid of order 4^n in $I \cap RM \cap AR$ for each $n \in \{0, 1, 2, \dots\}$ and that there are no finite groupoids in $I \cap RM \cap AR$ of any other orders. We also prove that, up to isomorphism, there is only one countable groupoid in $I \cap RM \cap AR$ and that it is isomorphic to a proper subgroupoid of itself.

2. Preliminary definitions, notation and results

We use $G, H, J, \dots$ to denote groupoids, xy or $x \cdot y$ to denote the product of x on the left with y on the right. For example, $(xy \cdot z) \cdot yz = [(x \cdot y) \cdot z] \cdot (y \cdot z)$. The varieties of *idempotent* and *anti-rectangular* groupoids are denoted by I and AR

2010 Mathematics Subject Classification: 20N02, 14R10
Keywords: anti-rectangular, spectrum, right modular.

and are the collection of groupoids satisfying the equations $x = x^2$ and $xy \cdot x = y$ respectively.

The set of orders of the finite algebras in a groupoid variety V is called the *spectrum of* V. We will denote this by $sp(V)$. T. Evans [3] showed that the spectrum of the groupoid variety defined by the equation $xy \cdot yz = y$ is the set $\{n^2 : n \in N\}$. Evans generalised this result and obtained, for each positive integer $n \in N$, a variety of groupoids having as spectrum all n^{th} powers [2]. The main result in this paper, referred to in the introduction above, is that the spectrum of $I \cap RM \cap AR$ is $\{4^n : n \in N \cup \{0\}\}$.

There is another reason to study the structure of groupoids in $I \cap RM \cap AR$. Let RM denote the variety of *right modular* groupoids determined by the equation $xy \cdot z = zy \cdot x$. Protić and Stepanović [14] proved that any idempotent, right modular groupoid G is an idempotent, right modular groupoid Y_G of members of $I \cap RM \cap AR$. In other words,

Lemma 2.1. [14, Theorem 2.1]
If $G \in I \cap RM$, then there exists a groupoid $Y_G \in I \cap RM$ such that G is a disjoint union of groupoids G_α $(\alpha \in Y_G)$, $G_\alpha G_\beta \subseteq G_{\alpha\beta}$ $(\alpha, \beta \in Y_G)$ and $G_\alpha \in I \cap RM \cap AR$ $(\alpha \in Y_G)$.

So, the finite members of $I \cap RM \cap AR$ are basic building blocks of the finite members of $I \cap RM$. As we shall see, the basic building block of the finite members of $I \cap RM \cap AR$ is the following groupoid T_4 of order 4, called *Traka* 4 in [14]. It is isomorphic to any groupoid generated by any two distinct elements, a and b say, of any member of $I \cap RM \cap AR$ and, therefore, $\mathrm{T}_4 \in I \cap RM \cap AR$ (see Lemma 2.4 below). The multiplication table of T_4 is:

T_4	a	b	ab	ba
a	a	ab	ba	b
b	ba	b	a	ab
ab	b	ba	ab	a
ba	ab	a	b	ba

We will also show that if $G \in I \cap RM \cap AR$ and $|G| = 4^n$ then G consists of 4^{n-1} disjoint copies of T_4 (see Corollary 3.8). Some of the following results will be used throughout this paper. Several of the proofs are straightforward and are omitted.

Lemma 2.2. [13] *If $G \in RM$, then G satisfies the identity $xu \cdot vy = xv \cdot uy$.*

Lemma 2.3. *If $G \in I \cap RM \cap AR$. then G satisfies the identity $x \cdot yz = z \cdot yx$.*

Proof. $z \cdot yx = (yx \cdot z) \cdot z = (zx \cdot y) \cdot z = [zx \cdot (zy \cdot z)] \cdot z =$
$= [(z \cdot zy) \cdot (xz)] \cdot z = (z \cdot xz) \cdot (z \cdot zy) = x \cdot [(zy \cdot z) \cdot z] = x \cdot yz.$ □

Lemma 2.4. *Let $G \in I \cap RM \cap AR$ with $\{c, d\} \subseteq G$ and $c \neq d$. Then the subgroupoid $\langle c, d\rangle$ of G generated by c and d is isomorphic to T_4. One isomorphism is given by the mapping $c \to a$, $d \to b$, $cd \to ab$ and $dc \to ba$.*

Lemma 2.5. *Any two distinct elements of* T_4 *generate* T_4.

Lemma 2.6. *Any bijection on* T_4 *is either an isomorphism or an anti-isomorphism. Four-cycles and two-cycles are anti-isomorphisms and the identity mapping, three-cycles and products of two-cycles are isomorphisms.*

Lemma 2.7. *Any groupoid anti-isomorphic to* T_4 *is isomorphic to* T_4. *In particular, if* $\Phi : \mathrm{T}_4 \to G$ *is an anti-isomorphism, then the mapping* $a \to \Phi a$, $b \to \Phi b$, $ab \to \Phi(ba)$ *and* $ba \to \Phi(ab)$ *is an isomorphism.*

Lemma 2.8. *Suppose that* H *and* K *are subgroupoids of* $G \in I \cap RM \cap AR$ *and that* $H \cong \mathrm{T}_4 \cong K$. *Then either* $H = K$, $H \cap K = \emptyset$ *or* $H \cap K = \{c\}$.

Notation 2.9. $G \cong H$ $[G \underset{\leftarrow}{\cong} H]$ will denote that G and H are isomorphic [anti-isomorphic].

Lemma 2.10. *If* $G \in I \cap RM \cap AR$ *and* $G \underset{\leftarrow}{\cong} H$, *then* $H \in I \cap RM \cap AR$.

Proof. Let $\Phi : G \to H$ be an anti-isomorphism. Then it is straightforward to show that H is an idempotent groupoid that satisfies the equation $xy \cdot x = y$. Let $\{h_1, h_2, h_3\} \subseteq H$. Then there exists $\{g_1, g_2, g_3\} \subseteq G$ such that $h_i = \Phi g_i$, $i \in \{1, 2, 3\}$. Using Lemma 2.3, $h_1 h_2 \cdot h_3 = (\Phi g_1)(\Phi g_2) \cdot (\Phi g_3) = \Phi(g_2 g_1) \cdot (\Phi g_3) = \Phi(g_3 \cdot g_2 g_1) = \Phi(g_1 \cdot g_2 g_3) = \Phi(g_2 g_3) \cdot (\Phi g_1) = (\Phi g_3)(\Phi g_2) \cdot (\Phi g_1) = h_3 h_2 \cdot h_1$ and so H satisfies the equation $xy \cdot z = zy \cdot x$. Hence, $H \in I \cap RM \cap AR$. $\square$

3. The structure of finite members of $\mathbf{I} \cap \mathbf{RM} \cap \mathbf{AR}$

We use $G \leq H$ $[G \prec H]$ to denote that G is a subgroupoid [proper subgroupoid] of the groupoid H. Recall that $a \in \mathrm{T}_4$.

Theorem 3.1. *If* $\mathrm{T}_4 \leq H \prec R$, $R \in I \cap RM \cap AR$ *and* $r \in R - H$, *then* $H_r = H \cup \{rh\}_{h \in H} \cup \{hr\}_{h \in H} \cup \{ar \cdot h\}_{h \in H}$ *is a subgroupoid of* R *and, therefore,* $H_r \in I \cap RM \cap AR$. *If* H *has* n *elements then* H_r *has* $4n$ *elements.*

Proof. We will prove that H_r is closed under the multiplication inherited from R and that its multiplication table is as follows:

H_r	k	rk	kr	$ar \cdot k$
h	hk	$ar \cdot (ka \cdot h)$	$r \cdot kh$	$(hk \cdot ah)\, r$
rh	$kh \cdot r$	$r \cdot hk$	$ar \cdot (k \cdot ah)$	$a \cdot hk$
hr	$ar \cdot (ha \cdot kh)$	kh	$hk \cdot r$	$r\,(ah \cdot k)$
$ar \cdot h$	$r\,(h \cdot ka)$	$(hk \cdot a)\, r$	$ak \cdot ha$	$ar \cdot hk$

Table 1. The multiplication table for $\{h, k\} \subseteq H$.

We will use Lemma 2.2 and Lemma 2.3, together with the fact that R is in $I \cap RM \cap AR$ to calculate the products in rows 2, 3, 4 and 5 of the table.

Row 2: The product in column 2 follows from the fact that H is a subgroupoid of R. The product in column 4 follows from Lemma 2.3. For column 3, $h \cdot rk = h \cdot (ar \cdot a)\, k = h \cdot (ka \cdot ar) = ar \cdot (ka \cdot h)$. For column 5, $h \cdot (ar \cdot k) = (h \cdot ar) \cdot hk = (r \cdot ah) \cdot hk = (hk \cdot ah) \cdot r$.

Row 3: The product in column 2 follows from the right modularity of R. The product in column 3 follows from Lemma 2.2 and the fact that R is an idempotent groupoid. For the product in column 4, $rh \cdot kr = rk \cdot hr = rk \cdot (ah \cdot a)\, r = rk \cdot (ra \cdot ah) = (r \cdot ra)\,(k \cdot ah) = ar \cdot (k \cdot ah)$. For the product in column 5, $rh \cdot (ar \cdot k) = (r \cdot ar) \cdot hk = a \cdot hk$.

Row 4: For the product in column 2, $hr \cdot k = [h\,(ar \cdot a)]\, k = [k\,(ar \cdot a)]\, h = kh \cdot [(ar \cdot a)\, h] = kh \cdot (ha \cdot ar) = ar\,(ha \cdot kh)$. For the product in column 3, $hr \cdot rk = (rk \cdot r)\, h = kh$. For the product in column 4, $hr \cdot kr = hk \cdot r$. For column 5, $hr \cdot (ar \cdot k) = (h \cdot ar) \cdot rk = (r \cdot ah) \cdot rk = r\,(ah \cdot k)$.

Row 5: For the product in column 2, $(ar \cdot h)\, k = (ar \cdot h)\,(a \cdot ka) = r\,(h \cdot ka)$. For column 3, $(ar \cdot h) \cdot rk = (ar \cdot r) \cdot hk = ra \cdot hk = (hk \cdot a)\, r$. For column 4, $(ar \cdot h) \cdot kr = (hr \cdot a) \cdot kr = (ha \cdot ra) \cdot kr = (ha \cdot k) \cdot a = ak \cdot ha$. The product in column 5 follows from Lemma 2.2 and the fact that R is an idempotent groupoid.

Thus, H_r is closed under the groupoid operation and hence H_r belongs to $I \cap RM \cap AR$.

It is straightforward to show that the sets H, $\{rh\}_{h \in H}$, $\{hr\}_{h \in H}$ and $\{ar \cdot h\}_{h \in H}$ are pairwise disjoint sets. Furthermore, it is easy to show that, for $\{h, k\} \subseteq H$, two elements rh and rk [hr and kr; $ar \cdot h$ and $ar \cdot k$] are equal if and only if $h = k$. Therefore, if H contains n elements then H_r contains $4n$ elements. $\square$

Definition 3.2. We will call H_r the *extension of* H *by* r.

Corollary 3.3. $sp\,(I \cap RM \cap AR) = \{4^n : n \in N \cup \{0\}\}$.

Corollary 3.4. *A groupoid* $G \in I \cap RM \cap AR$ *of order* 4^n *has* $(n+1)$ *generators,* $n \in \{0, 1, \dots\}$.

Theorem 3.5. *Suppose that* $\mathrm{T}_4 \leq H \in I \cap RM \cap AR$ *and* $r \notin H$. *We define pairwise disjoint sets* $A = \{rh\}_{h \in H}$, $B = \{hr\}_{h \in H}$ *and* $C = \{ar \circ h\}_{h \in H}$ *such that* $A \cap H = B \cap H = C \cap H = \emptyset$. *Define* $H^r = H \cup A \cup B \cup C$ *with a product* $\circ$ *defined as in Table 2 below. Then* $H^r \cong H_r$ *and therefore* $H^r \in I \cap RM \cap AR$.

H^r	k	rk	kr	$ar \circ k$
h	hk	$ar \circ (ka \cdot h)$	$r(kh)$	$(hk \cdot ah)r$
rh	$(kh)r$	$r(hk)$	$ar \circ (k \cdot ah)$	$a \cdot hk$
r	$ar \circ (ha \cdot kh)$	kh	$(hk)r$	$r(ah \cdot k)$
$ar \circ h$	$r(h \cdot ka)$	$(hk \cdot a)r$	$ak \cdot ha$	$ar \circ hk$

Table 2. The multiplication table for $\circ$ with $\{h, k\} \subseteq H$.

Proof. The product $\circ$ is well defined and closed and so H^r is a groupoid. We define a mapping $\Phi : H^r \to H_r$ as follows: for any $h \in H$, $\Phi h = h$, $\Phi(rh) = rh$,

$\Phi(hr) = hr$ and $\Phi(ar \circ h) = ar \cdot h$. It is clear that Φ is one-to-one and onto H_r. We now show that Φ is a homomorphism. Let $\{x, y\} \subseteq H^r$. There are 16 possible forms $x \circ y$ can take.

Let $\{h, k\} \subseteq H$.

Case 1. $x = h$, $y = k$. Then $\Phi(x \circ y) = \Phi(hk) = hk = \Phi h \cdot \Phi k = \Phi x \cdot \Phi y$.

Case 2. $x = h$, $y = rk$. Then $\Phi(x \circ y) = \Phi(h \circ rk) = \Phi(ar \circ ka \cdot h)) = ar(ka \cdot h) = h \cdot rk = \Phi h \cdot \Phi(rk) = \Phi x \cdot \Phi y$.

Case 3. $x = h$, $y = kr$. Then $\Phi(x \circ y) = \Phi(h \circ kr) = \Phi(r \circ kh) = r \cdot kh = h \cdot kr = \Phi h \cdot \Phi(kr) = \Phi x \cdot \Phi y$.

Case 4. $x = h$, $y = ar \circ k$. Then we have $\Phi(x \circ y) = \Phi(h \circ (ar \circ k)) = \Phi((hk \cdot ah)r) = (hk \cdot ah)r = h(ar \cdot k) = \Phi h \cdot \Phi(ar \circ k) = \Phi x \cdot \Phi y$.

Case 5. $x = rh$, $y = k$. Then $\Phi(x \circ y) = \Phi(rh \circ k) = \Phi((kh)r) = kh \cdot r = rh \cdot k = \Phi(rh) \cdot \Phi k = \Phi x \cdot \Phi y$.

Case 6. $x = rh$, $y = rk$. Then $\Phi(x \circ y) = \Phi(rh \circ rk) = \Phi(r(hk)) = r \cdot hk = rh \cdot rk = \Phi(rh) \cdot \Phi(rk) = \Phi x \cdot \Phi y$.

Case 7. $x = rh$, $y = kr$. Then $\Phi(x \circ y) = \Phi(rh \circ kr) = \Phi(ar \circ (k \cdot ah)) = ar \cdot (k \cdot ah) = rh \cdot kr = \Phi(rh) \cdot \Phi(kr) = \Phi x \cdot \Phi y$.

Case 8. $x = rh$, $y = ar \circ k$. Then $\Phi(x \circ y) = \Phi(rh \circ (ar \circ k)) = a \cdot hk = rh \cdot (ar \cdot k) = \Phi(rh) \cdot \Phi(ar \cdot k) = \Phi x \cdot \Phi y$.

Case 9. $x = hr$, $y = k$. Then $\Phi(x \circ y) = \Phi(hr \circ k) = \Phi(ar \circ (ha \cdot kh)) = ar \cdot (ha \cdot kh) = hr \cdot k = \Phi(hr) \cdot \Phi k = \Phi x \cdot \Phi y$.

Case10. $x = hr$, $y = rk$. Then $\Phi(x \circ y) = \Phi(hr \circ rk) = \Phi(kh) = kh = hr \cdot rk = \Phi(hr) \cdot \Phi(rk) = \Phi x \cdot \Phi y$.

Case 11. $x = hr$, $y = kr$. Then $\Phi(x \circ y) = \Phi(hr \circ kr) = \Phi((hk)r) = hk \cdot r = hr \cdot kr = \Phi(hr) \cdot \Phi(kr) = \Phi x \cdot \Phi y$.

Case 12. $x = hr$, $y = ar \circ k$. Then $\Phi(x \circ y) = \Phi(hr \circ (ar \circ k)) = \Phi(r(ah \cdot k)) = r(ah \cdot k) = hr \cdot (ar \cdot k) = \Phi(hr) \cdot \Phi(ar \cdot k) = \Phi x \cdot \Phi y$.

Case 13. $x = ar \cdot h$, $y = k$. Then $\Phi(x \circ y) = \Phi((ar \circ h) \circ k) = \Phi(r(h \cdot ka)) = r(h \cdot ka) = (ar \cdot h) \cdot k = \Phi(ar \circ h) \cdot \Phi k = \Phi x \cdot \Phi y$.

Case 14. $x = ar \circ h$, $y = rk$. Then $\Phi(x \circ y) = \Phi((ar \circ h) \circ rk) = \Phi((hk \cdot a)r) = (hk \cdot a)r = (ar \cdot h) \cdot rk = \Phi(ar \cdot h) \cdot \Phi(rk) = \Phi x \cdot \Phi y$.

Case 15. $x = ar \circ h$, $y = kr$. Then $\Phi(x \circ y) = \Phi((ar \circ h) \circ kr) = ak \cdot ha = (ar \cdot h) \cdot kr = \Phi(ar \cdot h) \cdot \Phi(kr) = \Phi x \cdot \Phi y$.

Case 16. $x = ar \circ h$, $y = ar \circ k$. Then $\Phi(x \circ y) = \Phi((ar \circ h) \circ (ar \circ k)) = \Phi(ar(hk)) = ar \cdot hk = (ar \cdot h) \cdot (ar \cdot k) = \Phi(ar \cdot h) \cdot \Phi(ar \cdot k) = \Phi x \cdot \Phi y$.

Hence, Φ is an isomorphism and $H^r \cong H_r$. □

Definition 3.6. We define G_0 as the trivial groupoid, $G_1 = \mathrm{T}_4$ and by induction, $G_n = G_{n-1}^{r_{n-1}}$, $n \geqslant 2$, where $r_n \notin G_n$, $n \geqslant 1$.

Corollary 3.7. *Any finite member of $I \cap RM \cap AR$ is isomorphic to G_n for some $n \in \{0, 1, 2 \ldots\}$. If $G \in I \cap RM \cap AR$ and $|G| = 4^n$, then $G \cong G_n$.*

Corollary 3.8. *For $n \in N$, G_n is a disjoint union of groupoids G_α with $G_\alpha G_\beta \subseteq G_{\alpha\beta}$ and $G_\alpha \cong G_{n-1}$, $\alpha, \beta \in \mathrm{T}_4$. Therefore, G_n is a disjoint union of 4^{n-1} copies of T_4.*

4. The countable member of $\mathbf{I} \cap \mathbf{RM} \cap \mathbf{AR}$

In this section we show that, to within isomorphism, there is precisely one countable member of $I \cap RM \cap AR$. This result will follow from the following construction of such a groupoid.

Construction 4.1. Let $H = \bigcup_{n=1}^{\infty} G_n$, with the G_n's as in Definition 3.6. Define a product $*$ on H as follows. If $\{u, v\} \subseteq H$ with $u \in G_{n_u} - G_{n_u - 1}$ and $v \in G_{n_v} - G_{n_v - 1}$ then $u * v$ is defined as the product of u and v in $G_{\max\{n_u, n_v\}}$.

Theorem 4.2. *H in Construction 4.1 is countable and $H \in I \cap RM \cap AR$.*

Proof. Clearly $*$ is well defined and H is closed with respect to $*$. By Theorem 3.5, $G_n \in I \cap RM \cap AR$, $n \in N$, and since $\max\{\max\{n_u, n_v\}, n_w\} = \max\{\max\{n_w, n_v\}, n_u\}$, it follows easily that $H \in I \cap RM \cap AR$. Since each G_n, $n \in N$, is countable, so is H. $\square$

Theorem 4.3. *A countable $K \in I \cap RM \cap AR$ is isomorphic to H in Construction 4.1.*

Proof. Let $K = \bigcup_{n=1}^{\infty} \{y_n\}$, with $y_i = y_j$ if and only if $i = j$. Define $K_0 = \emptyset$, $K_1 = \{y_1, y_2, y_1 y_2, y_2 y_1\}$ and $R_1 = K - K_1$. Define $K_2 = K_1^{y_{t_1}}$, where t_1 is the minimum of the subscripts of the y_n's in R_1. Define $R_2 = K - K_2$ and $K_3 = K_2^{y_{t_2}}$, where t_2 is the minimum subscript of the y_n's in R_2. In general, by induction we define $R_n = K - K_n$ and $K_{n+1} = K_n^{y_{t_n}}$, where t_n is the minimum subscript of the y_n's in R_n. Then every y_n must eventually appear in some K_t and therefore $K = \bigcup_{n=0}^{\infty} K_n$. Note that if $\{h, k\} \subseteq K$, with $h \in K_n - K_{n-1}$ and $k \in K_m - K_{m-1}$, then the product hk in K equals the product hk in K_M, where $M = \max\{n, m\}$.

By Lemma 2.4, $K_1 \cong G_1 = \mathrm{T}_4$. Call this isomorphism Φ_1. Note that $\Phi_1(y_1) = a$, $\Phi_1(y_2) = b$, $\Phi_1(y_1 y_2) = ab$ and $\Phi_1(y_2 y_1) = ba$.

Now by induction we define $\Phi_n : K_n \to G_n$, $n \geqslant 2$, as follows. Firstly, $\Phi_n = \Phi_{n-1}$ on K_{n-1}. Then for $k \in K_n - K_{n-1}$ we define

$$\Phi_n(y_{t_{n-1}} k) = r_{n-1} * (\Phi_{n-1} k), \quad \Phi_n(k y_{t_{n-1}}) = (\Phi_{n-1} k) * r_{n-1} \ \text{ and}$$
$$\Phi_n((y_1 y_{t_{n-1}})k) = ((\Phi_{n-1} y_1) * r_{n-1}) * (\Phi_{n-1} k).$$

We now prove by induction on n that Φ_n is an isomorphism ($n \geqslant 2$). Assume that for $1 \leqslant t \prec n$, Φ_t is an isomorphism and $\Phi_t y_1 = a$. Then the fact that Φ_n is one-to-one and onto G_n follows from the definition of Φ_n and the fact that Φ_{n-1} is one-to-one and onto G_{n-1}. The fact that $\Phi_n(xy) = (\Phi_n x)(\Phi_n y)$ for any $\{x, y\} \subseteq K_n$ follows from the definition of product in K_n and G_n (see Tables 3 and 4 below) and the facts that Φ_{n-1} is an isomorphism and $\Phi_{n-1}y_1 = a$. We leave the straightforward details of these calculations to the reader.

$K_n{=}K_{n-1}^{y_{t_{n-1}}}$	m	$y_{t_{n-1}}m$	$my_{t_{n-1}}$	$y_1y_{t_{n-1}} \cdot m$
l	lm	$y_1y_{t_{n-1}}{\cdot}(my_1{\cdot}l)$	$y_{t_{n-1}} \cdot ml$	$(lm{\cdot}y_1l){\cdot}y_{t_{n-1}}$
$y_{t_{n-1}}l$	$ml \cdot y_{t_{n-1}}$	$y_{t_{n-1}} \cdot lm$	$y_1y_{t_{n-1}}{\cdot}(m{\cdot}y_1l)$	$y_1 \cdot lm$
$ly_{t_{n-1}}$	$y_1y_{t_{n-1}}{\cdot}(ly_1{\cdot}ml)$	ml	$lm \cdot y_{t_{n-1}}$	$y_{t_{n-1}}{\cdot}(y_1l{\cdot}m)$
$y_1y_{t_{n-1}} \cdot l$	$y_{t_{n-1}} \cdot (l \cdot my_1)$	$(lm \cdot y_1) \cdot y_{t_{n-1}}$	$y_1l \cdot my_1$	$y_1y_{t_{n-1}} \cdot lm$

Table 3. The multiplication table for $\{l, m\} \subseteq K_{n-1}$.

$G_n = G_{n-1}^{r_{n-1}}$	k	$r_{n-1}k$	kr_{n-1}	$ar_{n-1} \cdot k$
h	hk	$ar_{n-1}{\cdot}(ka{\cdot}h)$	$r_{n-1}(kh)$	$(hk{\cdot}ah)r_{n-1}$
$r_{n-1}h$	$(kh)r_{n-1}$	$r_{n-1}(hk)$	$ar_{n-1}{\cdot}(k{\cdot}ah)$	$a \cdot hk$
hr_{n-1}	$ar_{n-1}{\cdot}(ha{\cdot}kh)$	kh	$(hk)r_{n-1}$	$r_{n-1}(ah{\cdot}k)$
$ar_{n-1}{\cdot}h$	$r_{n-1}(h \cdot ka)$	$(hk \cdot a)r_{n-1}$	$ak \cdot ha$	$ar_{n-1}{\cdot}hk$

Table 4. The multiplication table for $\{h, k\} \subseteq G_{n-1}$.

So every $\Phi_n : K_n \to G_n$ is an isomorphism.

We now define $\Phi : K \to H$ as follows: for $x \in K_n - K_{n-1}$, $\Phi x = \Phi_n x$. Note that if $x \in K_n - K_{n-1}$ and $M \geqslant n$ then, since $K_n \subseteq K_{n+1} \subseteq \ldots \subseteq K_{M-1}$ and $\Phi_t = \Phi_{t-1}$ on K_{t-1}, $t \in N - \{1\}$, $\Phi_M = \Phi_n$ on K_n. Then for any $\{x, y\} \subseteq K$, with $x \in K_n - K_{n-1}$ and $y \in K_m - K_{m-1}$, $\Phi(xy) = \Phi_M(xy) = (\Phi_M x)(\Phi_M y) = (\Phi_n x)(\Phi_m y) = (\Phi x)(\Phi y)$, where $M = \max\{n, m\}$. Using the definition of the Φ_n's it is straightforward to prove that Φ is one-to-one and onto H. So, $H \cong K$. □

Corollary 4.4. *A countable member of $I \cap RM \cap AR$ is a union of a countable number of disjoint, isomorphic copies of* T_4.

Corollary 4.5. *A countable member of $I \cap RM \cap AR$ is isomorphic to a proper subgroupoid of itself.*

Proof. Consider H in Construction 4.1. Let $J_1 = \{a, ar_1, r_1a, r_1\}$. For $1 \prec n$ define J_n by induction as $J_n = J_{n-1}^{r_n}$. Then $J = \bigcup_{n=1}^{\infty} J_n$, with the multiplication inherited from H, is a proper, countable subgroupoid of H. By Theorem 4.3, J and H are isomorphic. □

It follows from Lemma 2.10, Corollary 3.7 and Theorem 4.3 that:

Corollary 4.6. *If $G \in I \cap RM \cap AR$, G is finite or countable and $G \underset{\leftarrow}{\cong} H$, then $G \cong H$.*

5. Smallest $(\mathbf{W}, \mathbf{W})$ groupoids in RM – AR

Definition 5.1. A groupoid G is called a *groupoid* Y_G of groupoids G_α, $\alpha \in Y_G$ if G is a disjoint union of the groupoids G_α and $G_\alpha G_\beta \subseteq G_{\alpha\beta}$, $\alpha, \beta \in Y_G$. If $a \in G_\alpha$, then G_a will denote G_α.

In Definition 5.1, if $Y_G \in U$ and $G_\alpha \in V$ $(\alpha \in Y_G)$ for some groupoid varieties U and V, then G is called a (U, V)*-groupoid*.

In this section W will denote the variety $I \cap RM \cap AR$.

Looking closely at Lemma 2.1, it is natural to wonder whether a right modular (W, W)-groupoid is anti-rectangular and, hence, a member of W. The converse statement is trivial, since any $G \in W$ is a groupoid $Y_G = G$ of trivial members of W. However, there is a (W, W)-groupoid $G \in RM - AR$. In fact we find a right modular (W, W)-groupoid G of order 16, which is the minimal order for a right modular (W, W)-groupoid that is <u>not</u> anti-rectangular, as we proceed to prove. We also prove that G is unique up to isomorphism and that any right modular (W, W)-groupoid $K \notin AR$ contains an isomorphic copy of G.

Lemma 5.2. *If $K \in RM$ is a groupoid Y_K of groupoids K_α, $\alpha \in Y_K$, with $Y_K \in W$ and $K_\alpha \in W$ $(\alpha \in Y_K)$, then*

1) *K is cancellative,*
2) *for any $\{a, b\} \subseteq K$, $|K_a| = |K_b|$,*
3) *for any $\{a, b\} \subseteq K$, $ab \cdot a = b$ if and only if $ba \cdot b = a$.*

Proof. 1) Suppose that $a \in K_\alpha = K_a$, $b \in K_\beta = K_b$ and $c \in K_\gamma = K_c$. If $ca = cb$, then $\gamma\alpha = \gamma\beta$ and, since Y_K is cancellative, $\alpha = \beta$. Then $ab \cdot a = b$ and $bc = (ab \cdot a)c = ca \cdot ab = cb \cdot ab = (ab \cdot b)c = ba \cdot c$.

Hence, $(ca \cdot c)b = bc \cdot ca = (ba \cdot c) \cdot ca = (ca \cdot c) \cdot ba$. But since $\{b, ba, ca \cdot c\} \subseteq K_\beta$, and K_β is cancellative, $b = ba$. Therefore $b = ba = bb$. So $a = b$. Dually, if $ac = bc$, then $a = b$. Therefore K is cancellative.

2) Now let $c \in K_\alpha = K_a$. Then $ab \cdot c \in K_\beta$. Since K is cancellative $|K_\alpha| \leqslant |K_\beta|$. Dually $|K_\beta| \leqslant |K_\alpha|$ and so $|K_\alpha| = |K_\beta|$.

3) Note that $ab \cdot a = a \cdot ba$ and so we can write aba to denote $ab \cdot a$. If $aba = b$, then $ba \cdot b = a((bab)a) = a((ba)(aba)) = a((ba)b)$. But $\{a, bab\} \subset K_a$ and K_a is cancellative. Hence $a = bab$. Dually, $bab = a$ implies $aba = b$. $\square$

Now suppose that $K \in RM$ is a groupoid Y_K of groupoids K_α $(\alpha \in Y_K)$, with $Y_K \in W$ and $K_\alpha \in W$, $(\alpha \in Y_K)$. If K is <u>not</u> anti-rectangular, then it follows from Lemma 5.2 that there is a set $\{a, b, c, d\} \subseteq K$ with $aba = d \neq b$, $bab = c \neq a$, $\{a, c, ac, ca\} \subseteq K_a$, $\{b, d, bd, db\} \subseteq K_b$, $ab \neq cd$ and $ba \neq dc$.

It follows from Lemma 2.4 and the fact that K is a groupoid Y_K of groupoids K_α, $(\alpha \in Y_K)$, with $Y_K \in W$ and $K_\alpha \in W$ that $\{a, c, ac, ca\} = G_a$, $\{b, d, bd, db\} = G_b$, $\{ab, cd, ab \cdot cd, cd \cdot ab\} = G_{ab}$ and $\{ba, dc, ba \cdot dc, dc \cdot ba\} = G_{ba}$ are disjoint, isomorphic copies of T_4 contained in K_a, K_b, K_{ab} and K_{ba} respectively. We proceed to demonstrate that the union $G = \bigcup G_g$, $g \in \{a, b, ab, ba\}$, of these four copies of T_4 is a subgroupoid of K and is a groupoid T_4 of groupoids G_g.

Recall that $K \in I \cap RM$ is cancellative. We have $ab \cdot a = d$. Then $ab \cdot c = cb \cdot a = (bab \cdot b)a = (b \cdot ba)a = aba \cdot b = db$, $ab \cdot ac = (aba)(ab \cdot c) = aba \cdot (cb \cdot a) = d \cdot db = bd$ and $ab \cdot ca = (ab \cdot c) \cdot aba = db \cdot d = b$. We have shown that $G_b = (ab)G_a$.

Similarly we can calculate that $G_{ab} = G_a b$ and $G_{ba} = bG_a$.

We can then calculate the Cayley table consisting of the 256 products of pairs of elements of G. In order to have sufficient space to show the Cayley table we define the following two ordered 16–tuples as equal:

$$(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16) = (a, c, ac, ca, b, d, bd, db, ab, cd, ab \cdot cd, cd \cdot ab, ba, dc, ba \cdot dc, dc \cdot ba).$$

G	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	3	4	2	9	11	12	10	16	14	13	15	6	8	7	5
2	4	2	1	3	12	10	9	11	13	15	16	14	7	5	6	8
3	2	4	3	1	10	12	11	9	15	13	14	16	5	7	8	6
4	3	1	2	4	11	9	10	12	14	16	12	13	8	6	5	7
5	13	15	16	14	5	7	8	6	2	4	3	1	12	10	9	11
6	16	14	13	15	8	6	5	7	3	1	2	4	9	11	12	10
7	14	16	15	13	6	8	7	5	1	3	4	2	11	9	10	12
8	15	13	14	16	7	5	6	8	4	2	1	3	10	12	11	9
9	6	8	7	5	13	15	16	14	9	11	12	10	4	2	1	3
10	7	5	6	8	16	14	13	15	12	10	9	11	1	3	4	2
11	5	7	8	6	14	16	15	13	10	12	11	9	3	1	2	4
12	8	6	5	7	15	13	14	16	11	9	10	12	2	4	3	1
13	9	11	12	10	2	4	3	1	8	6	5	7	13	15	16	14
14	12	10	9	11	3	1	2	4	5	7	8	6	16	14	13	15
15	10	12	11	9	1	3	4	2	7	5	6	8	14	16	15	13
16	11	9	10	12	4	2	1	3	6	8	7	5	15	13	14	16

Table 5.

G	h	$(ab) \cdot h$	hb	bh
g	gh	$[c(g \cdot ah)]\, b$	$b\,[(a \cdot hg)c]$	$(ab) \cdot (ga \cdot h)$
$(ab) \cdot g$	$b(ca \cdot hg)$	$(ab) \cdot (gh)$	$cg \cdot ha$	$(gh \cdot a)b$
gb	$(ab) \cdot (ha \cdot gh)$	$b(hg \cdot ca)$	$(gh)b$	$h \cdot (ag \cdot c)$
bg	$(hg)b$	$h \cdot gc$	$(ab) \cdot (g \cdot ch)$	$b(gh)$

Table 6. The multiplication table for $\{g, h\} \subseteq G_a = \{a, c, ac, ca\}$.

Table 6 is derived using calculations obtained from Table 5. Notice that Table 6 yields the following Cayley table in set theoretic notation:

G	G_a	$G_b = (ab)G_a$	$G_{ab} = G_a b$	$G_{ba} = bG_a$
G_a	G_a	G_{ab}	G_{ba}	G_b
$G_b = (ab)G_a$	G_{ba}	G_b	G_a	G_{ab}
$G_{ab} = G_a b$	G_b	G_{ba}	G_{ab}	G_a
$G_{ba} = bG_a$	G_{ab}	G_a	G_b	G_{ba}

Table 7.

Note that the subscripts of the $G_g's$, $g \in \{a, b, ab, ba\}$, multiply in exactly the same way as the elements of T_4. The fact that $G \in RM$ follows from the fact that $G \leq K$ and $K \in I \cap RM \subseteq RM$. This proves that G is a right modular groupoid

T_4 of groupoids G_g, where each $G_g \cong \mathrm{T}_4$. Note however that $\{a, b, ab, ba\}$ is not even a subgroupoid of G! We have therefore proved:

Theorem 5.3. *$G \in I \cap RM$ and G is a groupoid T_4 of (four) isomorphic copies of T_4. However $G \notin W$. Also, if (W, W)-groupoid $K \in RM - AR$, then K contains an isomorphic copy of G.*

References

[1] **J. Dudek**, *Medial idempotent groupoids I*, Czechoslovak Math. J. **41** (1991), 249 – 259.

[2] **T. Evans**, *The spectrum of a variety*, Zeitchr. J. Math. Logic und Grandlagen **13** (1967), 213 – 218.

[3] **T. Evans**, *Products of points – some simple algebras and their identities*, Amer. Math. Monthly **74** (1967), 362 – 372.

[4] **B. Ganter and H. Werner**, *Equational classes of Steiner systems*, Algebra Universalis **5** (1975), 125 – 140.

[5] **P. Holgate**, *Groupoids satisfying a simple invertive law*, Math. Student **61** (1992), 101 – 106.

[6] **M. Iqbal and Q. Mushtaq**, *Partial ordering and congruences on LA-semigroups.* Indian J. Pure Appl. Math. **22** (1991), 331 – 336.

[7] **J. Ježek and T. Kepka**, *Modular groupoids*, Czechoslovak Math. J. **34** (1984), 477 – 487.

[8] **M. S. Kamran and Q. Mushtaq**, *Finite AG-groupoid with left identity and left zero*, Internat. J. Math. Math. Sci. **27** (2001), 387 – 389.

[9] **M. A. Kazim and M. Naseeruddin**, *On almost semigroups*, Portugaliae Math. **36** (1977), 41 – 47.

[10] **R. A. R. Monzo**, *On the structure of Abel-Grassmann unions of groups*, Int. J. Algebra **4** (2010), 1261 – 1275.

[11] **R. A. R. Monzo**, *Power groupoids and inclusion classes of Abel-Grassmann groupoids*, Int. J. Algebra **5** (2011), 91 – 105.

[12] **Q. Mushtaq**, *Zeroids and idempoids in AG-groupoids*, Quasigroups and Related Systems **11** (2004), 79 – 84.

[13] **Q. Mushtaq and S. M. Yusuf**, *On LA-semigroups*, Alig. Bull. Math. **8** (1978), 65 – 70.

[14] **P. V. Protić and N. Stevanović**, *Abel-Grassmann's bands.* Quasigroups and Related Systems **11**(2004), 95 – 101.

[15] **P. V. Protić and N. Stevanović**, *Inflations of the AG-groupoids*, Novi Sad J. Math. **29** (1999), 19 – 26.

[16] **P. V. Protić and N. Stevanović**, *Composition of Abel-Grassman's 3-bands.* Novi Sad J. Math. **34** (2004), 175 – 182.

Received March 11, 2011

Flat 10 Albert Mansions, Crouch Hill, London N8 9RE, United Kingdom
E-mali: bobmonzo@talktalk.net

Quasigroups and Related Systems **20** (2012), 97 − 112

Right k-weakly regular hemirings

Muhammad Shabir and Rukhshanda Anjum

Abstract. In this paper we define right k-weakly regular hemirings, which are generalization of k-regular hemirings. We characterize these hemirings by the properties of their right k-ideals and also by the properties of their fuzzy right k-ideals.

1. Introduction

There are many concepts of universal algebra generalizing an associative ring $(R, +, \cdot)$. Some of them, nearrings and several kinds of semirings, have been proven very useful. The notion of semiring was introduced by H. S. Vandiver in 1934 [12]. Semirings provide a common generalization of rings and distributive lattices, appear in a natural manner in some applications to the theory of automata, formal languages, optimization theory and other branches of applied mathematics. Hemirings, semirings with commutative addition and zero element, have also proved to be an important algebraic tool in theoretical computer science. The concept of a fuzzy set, introduced by Zadeh [14], was applied by many researchers to generalize some of the basic concepts of algebra. The notions of automata and formal languages have been generalized and extensively studied in a fuzzy frame work.

Ideals of semirings play a central role in the structure theory and are useful for many purposes. However in general, they do not coincide with usual ring ideals. For this, their use is somewhat limited in trying to obtain analogues of ring theorems for semirings. Henriksen defined in [6] a more restricted class of ideals in semirings, which is called the class of k-ideals. These ideals have the property that if the semiring R is a ring then a complex in R is a k-ideal if and only if it is a ring ideal.

Investigations of fuzzy semirings were initiated in [2]. Fuzzy k-ideals are studied in [3, 5, 7, 11]. In this paper we characterize hemirings in which each right k-ideal is idempotent and those hemirings for which each fuzzy right k-ideal is idempotent. We also study right pure and purely prime k-ideals and fuzzy right pure and fuzzy purely prime k-ideals in hemirings.

2010 Mathematics Subject Classification: 20N05
Keywords: hemiring, right weakly regular hemiring, k-ideal, right pure k-ideal, purely prime k-ideal, right pure fuzzy k-ideal, purely prime fuzzy k-ideal.

2. Preliminaries

For the definitions of semiring, hemiring, left (right) ideal we refer to [4].

A left (right) ideal A of a hemiring R is called a *left* (*right*) *k-ideal* of R if for any $a, b \in A$ and $x \in R$ from $x + a = b$ it follows $x \in A$.

The *k-closure* of a non-empty subset A of a hemiring R is defined as

$$\overline{A} = \{x \in R \mid x + a = b \ \text{ for some } a, b \in A\}.$$

It is clear that if A is a left (right) ideal of R, then $\overline{A}$ is the smallest left (right) k-ideal of R containing A. Also, $\overline{A} = A$ for all left (right) k-ideals of R. Obviously $\overline{\overline{A}} = \overline{A}$ for each non-empty $A \subseteq R$. Also $\overline{A} \subseteq \overline{B}$ for all $A \subseteq B \subseteq R$. A right k-ideal A with the property $\overline{A^2} = A$ is called *k-idempotent*.

Lemma 2.1. *$\overline{AB} = \overline{\overline{A}\,\overline{B}}$ for any subsets A, B of a hemiring R.* □

Lemma 2.2. [10] *If A and B are right and left k-ideals of a hemiring R respectively, then $\overline{AB} \subseteq A \cap B$.* □

An element a of a hemiring R is called *regular* if there exists $x \in R$ such that $a = axa$. A hemiring R is called *regular* if each element of R is regular. Generalizing the concept of regularity, in [1, 9] k-regular hemirings are defined as a hemiring in which for each $a \in R$, there exist $x, y \in R$ such that $a + axa = aya$.

Obviously, every regular hemiring is a k-regular but the converse is not true. If R is a ring, then the regular and k-regular coincide.

Theorem 2.3. [9] *A hemiring R is k-regular if and only if for any fuzzy right k-ideal A and any fuzzy left k-ideal B, we have $\overline{AB} = A \cap B$.* □

For any fuzzy subsets λ and μ of X we define

$$\begin{aligned}
&\lambda \leqslant \mu \iff \lambda(x) \leqslant \mu(x), \\
&(\lambda \wedge \mu)(x) = \lambda(x) \wedge \mu(x) = \min\{\lambda(x), \mu(x)\}, \\
&(\lambda \vee \mu)(x) = \lambda(x) \vee \mu(x) = \max\{\lambda(x), \mu(x)\}
\end{aligned}$$

for all $x \in X$.

More generally, if $\{\lambda_i : i \in I\}$ is a collection of fuzzy subsets of X, then by the *intersection* and the *union* of this collection we mean the fuzzy subsets

$$\begin{aligned}
\Big(\bigwedge_{i\in I} \lambda_i\Big)(x) &= \bigwedge_{i\in I} \lambda_i(x) = \inf_{i\in I}\{\lambda_i(x)\}, \\
\Big(\bigvee_{i\in I} \lambda_i\Big)(x) &= \bigvee_{i\in I} \lambda_i(x) = \sup_{i\in I}\{\lambda_i(x)\},
\end{aligned}$$

respectively.

A fuzzy subset λ of a hemiring R is called a *fuzzy left* (*right*) *ideal* of R if for all $a, b \in R$ we have

(1) $\lambda(a+b) \geqslant \lambda(a) \wedge \lambda(b)$,

(2) $\lambda(ab) \geqslant \lambda(b)$, $(\lambda(ab) \geqslant \lambda(a))$.

Note that $\lambda(0) \geqslant \lambda(x)$ for all $x \in R$.

A fuzzy left (right) ideal λ of a hemiring R is called a *fuzzy left (right) k-ideal* if $x + y = z \Longrightarrow \lambda(x) \geqslant \lambda(y) \wedge \lambda(z)$ holds for all $x, y, z \in R$.

A fuzzy right k-ideal is defined analogously. The basic properties of fuzzy k-ideals in semirings are described in [3].

Let λ be a fuzzy subset of a universe X and $t \in [0,1]$. Then the subset $U(\lambda; t) = \{x \in X \ : \ \lambda(x) \geqslant t\}$ is called *level subset* of λ.

The following Proposition is a consequence of transfer principle [8].

Proposition 2.4. *Let A be a non-empty subset of a hemiring R. Then a fuzzy set λ_A defined by*

$$\lambda_A(x) = \begin{cases} t & if\ x \in A \\ s & otherwise \end{cases}$$

where $0 \leqslant s < t \leqslant 1$, is a fuzzy left (right) k-ideal of R if and only if A is a left (right) k-ideal of R. □

Corollary 2.5. *Let A be a non-empty subset of a hemiring R. Then the characteristic function χ_A of A is a fuzzy right k-ideal of R if and only if A is a right k-ideal of R.* □

Proposition 2.6. *If A, B are subsets of a hemiring R such that $Im\lambda_A = Im\lambda_B$ then*

(1) $A \subseteq B \Longleftrightarrow \lambda_A \leqslant \lambda_B$,

(2) $\lambda_A \wedge \lambda_B = \lambda_{A\cap B}$. □

Definition 2.7. [11] The *k-product* of two fuzzy subsets μ and ν on R is defined by

$$(\mu \odot_k \nu)(x) = \bigvee_{x+\sum\limits_{i=1}^{m} a_i b_i = \sum\limits_{j=1}^{n} a'_j b'_j} \left[\bigwedge_{i=1}^{m} \left[\mu(a_i) \wedge \nu(b_i)\right] \wedge \bigwedge_{j=1}^{n} \left[\mu(a'_j) \wedge \nu(b'_j)\right] \right]$$

and $(\mu \odot_k \nu)(x) = 0$ if x cannot be expressed as $x + \sum\limits_{i=1}^{m} a_i b_i = \sum\limits_{j=1}^{n} a'_j b'_j$.

A fuzzy subset λ such that $\lambda \odot_k \lambda = \lambda$ is called *k-idempotent*.

Proposition 2.8. *Let μ, ν, ω, λ be fuzzy subsets on R. Then*

(1) $\mu \leqslant \omega$ *and* $\nu \leqslant \lambda \Longrightarrow \mu \odot_k \nu \leqslant \omega \odot_k \lambda$.

(2) $\chi_A \odot_k \chi_B = \chi_{\overline{AB}}$ *for characteristic functions of $A, B \subset R$.* □

Lemma 2.9. *If μ, ν are fuzzy left (right) k-ideals of a hemiring R, then $\mu \wedge \nu$ is also a fuzzy left (right) k-ideal of R.* □

Theorem 2.10. [11]
(*i*) *If λ and μ are fuzzy k-ideals of R, then so is $\lambda \odot_k \mu$. Moreover, $\lambda \odot_k \mu \leqslant \lambda \wedge \mu$.*
(*ii*) *If λ is fuzzy right k-ideal of R and μ a fuzzy left k-ideals of R, then $\lambda \odot_k \mu \leqslant \lambda \wedge \mu$.* □

Theorem 2.11. [11] *A hemiring R is k-regular if and only if for any fuzzy right k-ideal μ and any fuzzy left k-ideal ν of R we have $\mu \odot_k \nu = \mu \wedge \nu$.* □

3. Right k-weakly regular hemirings

Definition 3.1. A hemiring R is called *right* (*left*) *k-weakly regular* if for each $x \in R$, $x \in \overline{(xR)^2}$ $\left(\text{res. } x \in \overline{(Rx)^2}\right)$.

That is for each $x \in R$ we have $r_i, s_i, t_j, p_j \in R$ such that $x + \sum_{i=1}^{n} x r_i x s_i = \sum_{j=1}^{m} x t_j x p_j$ $\left(x + \sum_{i=1}^{n} r_i x s_i x = \sum_{j=1}^{m} t_j x p_j x\right)$. Thus each k-regular hemiring with identity is right k-weakly regular but the converse is not true. However for a commutative hemiring both the concept coincide.

Proposition 3.2. *The following statements are equivalent for a hemiring R with identity:*

1. *R is right k-weakly regular hemiring,*
2. *all right k-ideals of R are k-idempotent,*
3. *$\overline{BA} = B \cap A$ for all right k-ideals B and two-sided k-ideals A of R.*

Proof. (1) $\Longrightarrow$ (2) Let R be a right k-weakly regular hemiring and B be a right k-ideal of R. Clearly $\overline{B^2} \subseteq B$.

Let $x \in B$. Since R is right k-weakly regular, so $x \in \overline{(xR)^2}$ where xR is the right ideal of R generated by x and so $\overline{xR}$ is the right k-ideal of R generated by x. Thus $xR \subseteq B$, this implies $x \in \overline{(xR)(xR)} \subseteq \overline{BB} = \overline{B^2}$. Thus $B \subseteq \overline{B^2}$. So, $\overline{B^2} = B$.

(2) $\Longrightarrow$ (3) Let B be a right k-ideal of R and A a two-sided k-ideal of R, then by Lemma 2.2, $\overline{BA} \subseteq B \cap A$. To prove the reverse inclusion, let $x \in B \cap A$ and xR and RxR are right ideal and two-sided ideal of R generated by x, respectively. Thus $xR \subseteq B$ and $RxR \subseteq A$.

$$x \in xR \subseteq \overline{xR} = \overline{\overline{xR}\,\overline{xR}} = \overline{xRxR} = \overline{(xR)(xR)} = \overline{x(RxR)} \subseteq \overline{xA} \subseteq \overline{BA}$$

Hence $B \cap A \subseteq \overline{BA}$ and so $B \cap A = \overline{BA}$.

(3) $\Longrightarrow$ (1) Let $x \in R$ and RxR and xR be the two-sided ideal and right ideal of R generated by x, respectively. Then

$$x \in xR \cap RxR \subseteq \overline{xR} \cap \overline{RxR} = \overline{\overline{xR}\ \overline{RxR}} = \overline{xRRxR} = \overline{xR^2xR} = \overline{(xR)^2}.$$

Hence R is right k-weakly regular hemiring. □

Theorem 3.3. *For a hemiring R with identity, the following statements are equivalent:*

1. *R is right k-weakly regular hemiring,*
2. *all fuzzy right k-ideals of R are k-idempotent,*
3. *$\lambda \odot_k \mu = \lambda \wedge \mu$ for all fuzzy right k-ideals λ and all fuzzy two-sided k-ideals μ of R.*

Proof. (1) $\Longrightarrow$ (2) Let λ be a fuzzy right k-ideal of R, then $\lambda \odot_k \lambda \leqslant \lambda$.

For the reverse inclusion, let $x \in R$. Since R is right k-weakly regular so there exist $s_i, t_i, s'_j, t'_j \in R$ such that $x + \sum_{i=1}^{m} xs_ixt_i = \sum_{j=1}^{n} xs'_jxt'_j$. Hence

$$\lambda(x) = \lambda(x) \wedge \lambda(x) \leqslant \bigwedge_{i=1}^{m} (\lambda(xs_i) \wedge \lambda(xt_i)).$$

Also

$$\lambda(x) = \lambda(x) \wedge \lambda(x) \leqslant \bigwedge_{j=1}^{n} \left(\lambda(xs'_j) \wedge \lambda(xt'_j)\right).$$

Therefore

$$\begin{aligned}
\lambda(x) &\leqslant \bigwedge_{i=1}^{m} (\lambda(xs_i) \wedge \lambda(xt_i)) \wedge \bigwedge_{j=1}^{n} \left(\lambda(xs'_j) \wedge \lambda(xt'_j)\right) \\
&\leqslant \bigvee_{x+\sum\limits_{i=1}^{m} xs_ixt_i = \sum\limits_{j=1}^{n} xs'_jxt'_j} \left[\bigwedge_{i=1}^{m} (\lambda(xs_i) \wedge \lambda(xt_i)) \wedge \bigwedge_{j=1}^{n} \left(\lambda(xs'_j) \wedge \lambda(xt'_j)\right)\right] \\
&= (\lambda \odot_k \lambda)(x).
\end{aligned}$$

Hence $\lambda \leqslant \lambda \odot_k \lambda$, which proves $\lambda \odot_k \lambda = \lambda$.

(2) $\Longrightarrow$ (3) Let λ and μ be fuzzy right and two sided k-ideal of R, respectively. Then $\lambda \wedge \mu$ is a fuzzy right k-ideal of R. By Theorem 2.10, $\lambda \odot_k \mu \leqslant \lambda \wedge \mu$. By hypothesis,$(\lambda \wedge \mu) = (\lambda \wedge \mu) \odot_k (\lambda \wedge \mu) \leqslant \lambda \odot_k \mu$. Hence $\lambda \odot_k \mu = \lambda \wedge \mu$.

(3) $\Longrightarrow$ (1) Let B be a right k-ideal of R and A be a two-sided k-ideal of R, then the characteristic functions χ_B and χ_A of B and A are fuzzy right and fuzzy two-sided k-ideal of R, respectively. Hence by the hypothesis and Propositions 2.6 and 2.8, we have $\chi_B \odot_k \chi_A = \chi_B \wedge \chi_A$, i.e., $\chi_{\overline{BA}} = \chi_{B\cap A}$, which implies $\overline{BA} = B \cap A$. Thus, by Proposition 3.2, R is right k-weakly regular hemiring. □

Theorem 3.4. *For a hemiring R with identity, the following statements are equivalent:*

1. *R is right k-weakly regular hemiring,*
2. *all right k-ideals of R are k-idempotent,*
3. *$\overline{BA} = B \cap A$ for all right k-ideals B and two-sided k-ideals A of R,*
4. *all fuzzy right k-ideals of R are k-idempotent,*
5. *$\lambda \odot_k \mu = \lambda \wedge \mu$ for all fuzzy right k-ideals λ and all fuzzy two-sided k-ideals μ of R.*

If R is commutative, then the above statements are equivalent to

6. *R is k-regular.*

Proof. $1, 2, 3$ are equivalent by Proposition 3.2. $1, 4, 5$ are equivalent by Theorem 3.3. Finally, if R is commutative, then by Theorem 2.3, also 1 and 6 are equivalent. □

Definition 3.5. [11] The *k-sum* $\lambda +_k \mu$ of fuzzy subsets λ and μ of R is defined by

$$(\lambda +_k \mu)(x) = \bigvee_{x+(a_1+b_1)=(a_2+b_2)} [\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)],$$

where $x, a_1, b_1, a_2, b_2 \in R$.

Theorem 3.6. [11] *The k-sum of fuzzy k-ideals of R is also a fuzzy k-ideal of R.* □

Theorem 3.7. *The collection of all k-ideals of a right k-weakly regular hemiring R forms a complete distributive lattice.*

Proof. The collection $\mathcal{L}_R$ of all k-ideals of a right k-weakly regular hemiring R is a partially ordered set under the inclusion of sets and is a complete lattice under the operations $\sqcup$, $\sqcap$ defined as $A \sqcup B = \overline{A + B}$ and $A \sqcap B = A \cap B$.

Let $A, B, C \in \mathcal{L}_R$, then obviously $\overline{(A \cap B) + (A \cap C)} \subseteq A \cap (\overline{B + C})$. For the reverse inclusion, let $x \in A \cap (\overline{B + C}) = \overline{A(\overline{B + C})}$. Then $x + a = b$ for some $a, b \in A(\overline{B + C})$. Hence $a = a_1 y_1$ and $b = a_2 y_2$ for some $a_1, a_2 \in A$ and $y_1, y_2 \in (\overline{B + C})$. Then $y_1 + b_1 + c_1 = b_2 + c_2$ and $y_2 + b_3 + c_3 = b_4 + c_4$ for some $b_1, b_2, b_3, b_4 \in B$ and $c_1, c_2, c_3, c_4 \in C$. Thus $a_1 y_1 + a_1 b_1 + a_1 c_1 = a_1 b_2 + a_1 c_2$ yields $a + a_1 b_1 + a_1 c_1 = a_1 b_2 + a_1 c_2$ which implies $a \in \overline{AB + AC}$. Similarly $b \in \overline{AB + AC}$ and thus $x \in \overline{AB + AC}$. Hence $A \cap (\overline{B + C}) = \overline{A(\overline{B + C})} \subseteq \overline{AB + AC} \subseteq \overline{\overline{AB} + \overline{AC}} = \overline{(A \cap B) + (A \cap C)}$. Thus $\overline{(A \cap B) + (A \cap C)} = A \cap (\overline{B + C})$. □

The following example shows that if the collection of all k-ideals of a hemiring R is a complete distributive lattice then R is not necessarily a right k-weakly regular hemiring.

Example 3.8. Consider the hemiring $R = \{0, a, b\}$ with $+$ and $\cdot$ defined by $x + y = \max\{x, y\}$, where $0 < a < b$ and $x \cdot y = b$ for $x = y = b$ and $x \cdot y = 0$ otherwise.

The k-ideals of R are $\{0\}, \{0, a\}$ and R. Since $\{0\} \subseteq \{0, a\} \subseteq R$. So the collection of k-ideals is a complete distributive lattice but R is not right k-weakly regular hemiring. □

Theorem 3.9. *If R is a right k-weakly regular hemiring, then the set $\mathcal{L}_R$ of all fuzzy k-ideals of R (ordered by $\leqslant$) is a distributive lattice.*

Proof. The set $\mathcal{L}_R$ of all fuzzy k-ideals of R (ordered by $\leqslant$) is clearly a lattice under the k-sum and intersection of fuzzy k-ideals. Now we show that $\mathcal{L}_R$ is a distributive lattice, that is for any fuzzy k-ideals λ, μ, δ of R we have $(\lambda \wedge \delta) + \mu = (\lambda + \mu) \wedge (\delta + \mu)$.

For any $x \in R$

$$\begin{aligned}
[(\lambda \wedge \delta) + \mu](x) &= \bigvee_{x+(a_1+b_1)=(a_2+b_2)} \begin{bmatrix} (\lambda \wedge \delta)(a_1) \wedge (\lambda \wedge \delta)(a_2) \wedge \\ (\mu)(b_1) \wedge (\mu)(b_2) \end{bmatrix} \\
&= \bigvee_{x+(a_1+b_1)=(a_2+b_2)} \begin{bmatrix} \lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \\ \mu(b_2) \wedge \delta(a_1) \wedge \delta(a_2) \end{bmatrix} \\
&= \bigvee_{x+(a_1+b_1)=(a_2+b_2)} \begin{bmatrix} [\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \wedge \\ [\delta(a_1) \wedge \delta(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \end{bmatrix} \\
&= \Big(\bigvee_{x+(a_1+b_1)=(a_2+b_2)} [\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \Big) \\
&\quad \wedge \Big(\bigvee_{x+(a_1+b_1)=(a_2+b_2)} [\delta(a_1) \wedge \delta(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \Big) \\
&= (\lambda + \mu)(x) \wedge (\delta + \mu)(x) = [(\lambda + \mu) \wedge (\delta + \mu)](x).
\end{aligned}$$ □

4. Prime and Fuzzy prime right k-ideals

Definition 4.1. A right k-ideal P of a hemiring R is called *k-prime* (*k-semiprime*) if for any right k-ideals A, B of R,

$$AB \subseteq P \Longrightarrow A \subseteq P \text{ or } B \subseteq P \quad \big(A^2 \subseteq P \Longrightarrow A \subseteq P\big).$$

P is *k-irreducible* (*k-strongly irreducible*) if for any right k-ideals A, B of R

$$A \cap B = P \Longrightarrow A = P \text{ or } B = P \quad (A \cap B \subseteq P \Longrightarrow A \subseteq P \text{ or } B \subseteq P).$$

A fuzzy right k-ideal μ of a hemiring R is called a *fuzzy k-prime* (*k-semiprime*) *right k-ideal* of R if for any fuzzy k-right ideals λ, δ of R,

$$\lambda \odot_k \delta \leqslant \mu \Longrightarrow \lambda \leqslant \mu \text{ or } \delta \leqslant \mu \ (\lambda \odot_k \lambda \leqslant \mu \Longrightarrow \lambda \leqslant \mu).$$

μ is called a *fuzzy k-irreducible* (*k-strongly irreducible*) if for any fuzzy right k-ideals λ, δ of R,

$$\lambda \wedge \delta = \mu \Longrightarrow \lambda = \mu \text{ or } \delta = \mu \ \ (\lambda \wedge \delta \leqslant \mu \Longrightarrow \lambda \leqslant \mu \text{ or } \delta \leqslant \mu).$$

Lemma 4.2. *In any hemiring R*

(a) *a (fuzzy) k-prime right k-ideal is a (fuzzy) k-semiprime right k-ideal,*

(b) *an intersection of (fuzzy) k-prime right k-ideals is a (fuzzy) k-semi prime right k-ideal.* □

Theorem 4.3. *Each proper right k-ideal of a right k-weakly regular hemiring R is the intersection of right k-irreducible k-ideals which contain it.*

Proof. Let I be a proper right k-ideal of R and let $\{I_\alpha : \alpha \in \Lambda\}$ be a family of right k-irreducible k-ideals of R which contain I. Clearly $I \subseteq \cap_{\alpha\in\Lambda} I_\alpha$. Suppose $a \notin I$. Then by Zorn's Lemma there exists a right k-ideal I_β such that I_β is maximal with respect to the property $I \subseteq I_\beta$ and $a \notin I_\beta$. We will show that I_β is k-irreducible. Let A, B be right k-ideals of R such that $I_\beta = B \cap A$. Suppose $I_\beta \subset B$ and $I_\beta \subset A$. Then by the maximality of I_β, we have $a \in B$ and $a \in A$. But this implies $a \in B \cap A = I_\beta$, which is a contradiction. Hence either $I_\beta = B$ or $I_\beta = A$. So there exists a k-irreducible k-ideal I_β such that $a \notin I_\beta$ and $I \subseteq I_\beta$. Hence $\cap I_\alpha \subseteq I$. Thus $I = \cap I_\alpha$. □

Proposition 4.4. *Let R be a right k-weakly regular hemiring. If λ is a fuzzy right k-ideal of R with $\lambda(a) = \alpha$, where a is any element of R and $\alpha \in (0,1]$, then there exists a fuzzy k-irreducible right k-ideal δ of R such that $\lambda \leqslant \delta$ and $\delta(a) = \alpha$.*

Proof. Let $X = \{\mu : \mu \text{ is a fuzzy right } k\text{-ideal of } R,\ \mu(a) = \alpha \text{ and } \lambda \leqslant \mu\}$. Then $X \neq \emptyset$, since $\lambda \in X$. Let F be a totally ordered subset of X, say $F = \{\lambda_i : i \in I\}$. We claim that $\bigvee_{i\in I} \lambda_i$ is a fuzzy right k-ideal of R. For any $x, r \in R$, we have

$$\Big(\bigvee_i \lambda_{i\in I}\Big)(x) = \bigvee_{i\in I}(\lambda_i(x)) \leqslant \bigvee_{i\in I}(\lambda_i(xr)) = \Big(\bigvee_{i\in I}\lambda_i\Big)(xr).$$

Let $x, y \in R$, consider

$$\begin{aligned}
\Big(\bigvee_{i\in I}\lambda_i\Big)(x) \wedge \Big(\bigvee_{i\in I}\lambda_i\Big)(y) &= \Big(\bigvee_{i\in I}\lambda_i(x)\Big) \wedge \Big(\bigvee_{j\in I}\lambda_j(y))\Big) \\
&= \bigvee_{j\in I}\Big(\bigvee_{i\in I}(\lambda_i(x) \wedge \lambda_j(y))\Big) \\
&\leqslant \bigvee_{j\in I}\Big(\bigvee_{i\in I}(\max\{\lambda_i(x), \lambda_j(x)\} \wedge \max\{\lambda_i(y), \lambda_j(y)\})\Big) \\
&\leqslant \bigvee_{j\in I}\Big(\bigvee_{i\in I}\max\{\lambda_i(x+y), \lambda_j(x+y)\}\Big) \\
&\leqslant \bigvee_{i\in I}\max\{\lambda_i(x+y), \lambda_j(x+y)\} = \Big(\bigvee_{i\in I}\lambda_i\Big)(x+y).
\end{aligned}$$

Now, let $x+a=b$, where $a,b\in R$. Then

$$\begin{aligned}\Big(\bigvee_{i\in I}\lambda_i\Big)(a)\wedge\Big(\bigvee_{i\in I}\lambda_i\Big)(b) &= \Big(\bigvee_{i\in I}\lambda_i(a)\Big)\wedge\Big(\bigvee_{j\in I}\lambda_j(b)\Big)\\ &= \bigvee_{j\in I}\Big(\bigvee_{i\in I}\lambda_i(a)\wedge\lambda_j(b)\Big)\\ &\leqslant \bigvee_{j\in I}\Big(\bigvee_{i\in I}\max\{\lambda_i(a),\lambda_j(a)\}\wedge\max\{\lambda_i(b),\lambda_j(b)\}\Big)\\ &= \bigvee_{i,j\in I}\max\{\lambda_i(x),\lambda_j(x)\}\leqslant\bigvee_{i\in I}\lambda_i(x).\end{aligned}$$

Thus $\bigvee\limits_{i\in I}\lambda_i$ is a fuzzy right k-ideal of R. Clearly $\lambda\leqslant\bigvee\limits_i\lambda_i$ and $\bigvee\limits_i\lambda_i(a)=\alpha$. Thus $\bigvee\limits_i\lambda_i$ is the l.u.b of F. Hence by Zorn's lemma there exists a fuzzy right k-ideal δ of R which is maximal with respect to the property that $\lambda\leqslant\delta$ and $\delta(a)=\alpha$.

We will show that δ is fuzzy k-irreducible right k-ideal of R. Let $\delta=\delta_1\wedge\delta_2$, where δ_1,δ_2 are fuzzy right k-ideals of R. Thus $\delta\leqslant\delta_1$ and $\delta\leqslant\delta_2$. We claim that either $\delta=\delta_1$ or $\delta=\delta_2$. Suppose $\delta\neq\delta_1$ and $\delta\neq\delta_2$. Since δ is maximal with respect to the property that $\delta(a)=\alpha$ and since $\delta\lneqq\delta_1$ and $\delta\lneqq\delta_2$, so $\delta_1(a)\neq\alpha$ and $\delta_2(a)\neq\alpha$. Hence $\alpha=\delta(a)=(\delta_1\wedge\delta_2)(a)=(\delta_1)(a)\wedge(\delta_2)(a)\neq\alpha$, which is impossible. Hence $\delta=\delta_1$ or $\delta=\delta_2$. Thus δ is a fuzzy k-irreducible right k-ideal of R. □

Theorem 4.5. *Every fuzzy right k-ideal of a hemiring R is the intersection of all fuzzy k-irreducible right k-ideals of R which contain it.*

Proof. Let λ be the fuzzy right k-ideal of R and let $\{\lambda_\alpha:\alpha\in\Lambda\}$ be the family of all fuzzy k-irreducible right k-ideals of R which contain λ. Obviously $\lambda\leqslant\bigwedge\limits_{\alpha\in\Lambda}\lambda_\alpha$. We show that $\bigwedge\limits_{\alpha\in\Lambda}\lambda_\alpha\leqslant\lambda$. Let a be any element of R, then by Proposition 4.4, there exists a fuzzy k-irreducible right k-ideal λ_β such that $\lambda\leqslant\lambda_\beta$ and $\lambda(a)=\lambda_\beta(a)$. Hence $\lambda_\beta\in\{\lambda_\alpha:\alpha\in\Lambda\}$. Hence $\bigwedge\limits_{\alpha\in\Lambda}\lambda_\alpha\leqslant\lambda_\beta$, so $\bigwedge\limits_{\alpha\in\Lambda}\lambda_\alpha(a)\leqslant\lambda_\beta(a)=\lambda(a)$, i.e., $\bigwedge\limits_{\alpha\in\Lambda}\lambda_\alpha\leqslant\lambda$. Hence $\bigwedge\limits_{\alpha\in\Lambda}\lambda_\alpha=\lambda$. □

Theorem 4.6. *A hemiring with identity is right k-weakly regular if and only if each its right k-ideal is k-semiprime.*

Proof. Suppose every right k-ideal is idempotent. Let I,J be right k-ideals of R, such that $J^2\subseteq I$. Thus $\overline{J^2}\subseteq\overline{I}$. By Theorem 3.4, $J=\overline{J^2}$, so $J\subseteq I$. Hence I is a k-semiprime right k-ideal of R.

Conversely, if each each right k-ideal I of R is k-semiprime, then $\overline{I^2}$ is also a right k-ideal of R and $I^2\subseteq\overline{I^2}$. Hence by hypothesis $I\subseteq\overline{I^2}$. But $\overline{I^2}\subseteq I$ always. Hence $I=\overline{I^2}$. Thus by Theorem 3.4, R is right k-weakly regular. □

Theorem 4.7. *For a hemiring R with identity the following statements are equivalent:*

1. *R is right k-weakly regular hemiring,*
2. *all fuzzy right k-ideals of R are k-idempotent,*
3. *$\lambda \odot_k \mu = \lambda \wedge \mu$ for all fuzzy right k-ideals λ and all fuzzy two-sided k-ideals μ of R,*
4. *each fuzzy right k-ideal of R is also fuzzy k-semiprime.*

Proof. $1, 2, 3$ are equivalent by Theorem 3.3.

If δ is a fuzzy right k-ideal of R, then $\lambda \odot_k \lambda \leqslant \delta$, where λ is a fuzzy right k-ideal of R. By (2) $\lambda \odot_k \lambda = \lambda$, so $\lambda \leqslant \delta$. Thus δ is a fuzzy k-semiprime right k-ideal of R.

Conversely, if δ is a fuzzy right k-ideal of R, then also $\delta \odot_k \delta$ is a fuzzy right k-ideal of R and so by (4) $\delta \odot_k \delta$ is a fuzzy k-semiprime right k-ideal of R. As $\delta \odot_k \delta \leqslant \delta \odot_k \delta$ we have $\delta \leqslant \delta \odot_k \delta$. But $\delta \odot_k \delta \leqslant \delta$ always. So $\delta \odot_k \delta = \delta$. □

Theorem 4.8. *If every right k-ideal of a hemiring R is k-prime , then R is a right k-weakly regular hemiring and the set of k-ideals of R is totally ordered.*

Proof. Suppose R is a hemiring in which each right k-ideal is prime right k-ideal. Let A be a right k-ideal of R then $\overline{A^2}$ is a right k-ideal of R. As $A^2 \subseteq \overline{A^2}$ $\Longrightarrow A \subseteq \overline{A^2}$. But $\overline{A^2} \subseteq A$ always. Hence $A = \overline{A^2}$. Thus R is right k-weakly regular.

Let A, B be any k-ideals of R then $AB \subseteq A \cap B$. As $A \cap B$ is a k-ideal of R, so a k-prime right k-ideal. Thus either $A \subseteq A \cap B$ or $B \subseteq A \cap B$. That is either $A \subseteq B$ or $B \subseteq A$. □

Theorem 4.9. *If R is a right k-weakly regular hemiring and the set of all right k-ideals of R is totally ordered, then every right k-ideal of R is k-prime.*

Proof. Let A, B, C be right k-ideals of R such that $AB \subseteq C$. Since the set of all right k-ideals of R is totally ordered, so we have $A \subseteq B$ or $B \subseteq A$. If $A \subseteq B$ then $A = \overline{AA} \subseteq \overline{AB} \subseteq C$. If $B \subseteq A$ then $B = \overline{BB} \subseteq \overline{AB} \subseteq C$. Thus C is a k-prime right k-ideal. □

Theorem 4.10. *If every fuzzy right k-ideal of a hemiring R is a fuzzy k-prime right k-ideal, then R is a right k-weakly regular hemiring and the set of fuzzy k-ideals of R is totally ordered.*

Proof. Suppose R is a hemiring in which each fuzzy right k-ideal is fuzzy prime. Let λ be a fuzzy right k-ideal of R. Then $\lambda \odot_k \lambda$ is also a fuzzy right k-ideal of R. As $\lambda \odot_k \lambda \leqslant \lambda \odot_k \lambda \Longrightarrow \lambda \leqslant \lambda \odot_k \lambda$. But $\lambda \odot_k \lambda \leqslant \lambda$ always. Hence $\lambda = \lambda \odot_k \lambda$. Thus R is a right k-weakly regular hemiring.

Let λ, μ be any fuzzy k-ideals of R. Then $\lambda \odot_k \mu \leqslant \lambda \wedge \mu$. As $\lambda \wedge \mu$ is a fuzzy k-ideal of R so it is fuzzy k-prime. Thus either $\lambda \leqslant \lambda \wedge \mu$ or $\mu \leqslant \lambda \wedge \mu$. That is either $\lambda \leqslant \mu$ or $\mu \leqslant \lambda$. □

Theorem 4.11. *If the set of all fuzzy right k-ideals of a right k-weakly regular hemiring R is totally ordered, then every fuzzy right k-ideal of R is a fuzzy k-prime right k-ideal of R.*

Proof. Let λ, μ, ν be fuzzy right k-ideals of R such that $\lambda \odot_k \mu \leqslant \nu$. Since the set of all fuzzy right k-ideals of R is totally ordered, so we have $\lambda \leqslant \mu$ or $\mu \leqslant \lambda$. If $\lambda \leqslant \mu$ then $\lambda = \lambda \odot_k \lambda \leqslant \lambda \odot_k \mu \leqslant \nu$. If $\mu \leqslant \lambda$ then $\mu = \mu \odot_k \mu \leqslant \lambda \odot_k \mu \leqslant \nu$. Thus ν is a fuzzy k-prime right k-ideal. □

Example 4.12. Consider the set $R = \{0, x, 1\}$ in which $a + b = \max\{a, b\}$ and $ab = \min\{a, b\}$ are defined by the chains $0 < 1 < x$ and $0 < x < 1$. Then $(R, +, \cdot)$ is a hemiring.

The right k-ideals of R are $\{0\}, \{0, x\}, \{0, x, 1\}$. The k-ideals $\{0\}$ $\{0, x, 1\}$ are idempotent.

In order to examine the right fuzzy k-ideals of R, we observe the following facts.

Fact 1. A fuzzy subset λ of R is a fuzzy right ideal if and only if $\lambda(0) \geqslant \lambda(x) \geqslant \lambda(1)$.

Indeed, since $0 = x \cdot 0 = 1 \cdot 0$ so $\lambda(0) \geqslant \lambda(x)$ and $\lambda(0) \geqslant \lambda(1)$. Also $\lambda(x) = \lambda(1 \cdot x) \geqslant \lambda(1)$. Thus $\lambda(0) \geqslant \lambda(x) \geqslant \lambda(1)$.

Conversely, If λ is a fuzzy subset of R such that $\lambda(0) \geqslant \lambda(x) \geqslant \lambda(1)$, then by the definition of $+$ in R, we have $m + m' = m$ or m' for every $m, m' \in R$, and certainly $\lambda(m) \wedge \lambda(m') \leqslant \lambda(m)$ and $\lambda(m) \wedge \lambda(m') \leqslant \lambda(m')$. Thus $\lambda(m + m') \geqslant \lambda(m) \wedge \lambda(m')$. By the definition of $\cdot$ defined on R, it is easy to verify that $\lambda(ma) \geqslant \lambda(m)$ for all m, a in R. Hence λ is a fuzzy right ideal of R.

Fact 2. λ is a fuzzy right k-ideal of R if and only if $\lambda(0) \geqslant \lambda(x) = \lambda(1)$.

Indeed, by the Fact 1 we have $\lambda(0) \geqslant \lambda(x) \geqslant \lambda(1)$. Since $1 + x = x$, so $\lambda(1) \geqslant \lambda(x) \wedge \lambda(x) = \lambda(x)$. Thus $\lambda(0) \geqslant \lambda(x) = \lambda(1)$. Conversely, if $\lambda(0) \geqslant \lambda(x) = \lambda(1)$, then by the Fact 1, λ is a fuzzy right ideal of R.

If $x + a = b$ for $a, b, x \in R$ then $\lambda(x) \geqslant \lambda(a) \wedge \lambda(b)$. So λ is a fuzzy right k-ideal of R.

Obviously R is a right k-weakly regular hemiring. But each fuzzy right k-ideal of R is not k-prime. Because λ, μ, ν defined by $\lambda(0) = 0.8, \lambda(x) = \lambda(1) = 0.6$, $\mu(0) = 0.9$, $\mu(x) = \mu(1) = 0.5$ and $\nu(0) = 0.85$, $\nu(x) = \nu(1) = 0.55$ are fuzzy k-ideals of R such that $\lambda \odot_k \mu \leqslant \nu$ but neither $\lambda \leqslant \nu$ nor $\mu \leqslant \nu$. □

5. Right pure k-ideals

In this section we define right pure k-ideals of a hemiring R and also right pure fuzzy k-ideals of R. We prove that a two-sided k-ideal I of a hemiring R is right pure if and only if for every right k-ideal A of R, we have $A \cap I = \overline{AI}$.

Definition 5.1. A k-ideal I of a hemiring R is called *right pure* if for each $x \in I$, $x \in \overline{xI}$, i.e., if for each $x \in I$ there exist $y, z \in I$ such that $x + xy = xz$.

Lemma 5.2. *A k-ideal I of a hemiring R is right pure if and only if $A \cap I = \overline{AI}$ for every right k-ideal A of R.*

Proof. Suppose that I is a right pure k-ideal of R and A is a right k-ideal of R. Then $\overline{AI} \subseteq A \cap I$. Clearly, $a \in A \cap I$ implies $a \in A$ and $a \in I$. Since I is right pure, so $a \in \overline{aI} \subseteq \overline{AI}$. Thus $A \cap I \subseteq \overline{AI}$. Hence $A \cap I = \overline{AI}$.

Conversely, assume that $A \cap I = \overline{AI}$ for every right k-ideal A of R. Let $x \in I$. Take A, the principal right k-ideal generated by x, that is, $A = \overline{xR + \mathbb{N}_\circ x}$, where $\mathbb{N}_\circ = \{0, 1, 2, ...\}$. By hypothesis $A \cap I = \overline{AI} = \overline{\overline{(xR + \mathbb{N}_\circ x)}I} = \overline{(xR + \mathbb{N}_\circ x)I} \subseteq \overline{xI}$. So $x \in \overline{xI}$. Hence I is a right pure k-ideal of R. □

Definition 5.3. A fuzzy k-ideal λ of a hemiring R is called *right pure* if and only if $\mu \wedge \lambda = \mu \odot_k \lambda$ for every fuzzy right k-ideal μ of R.

Proposition 5.4. *The characteristic function of a non-empty subset A of a hemiring R is its right pure fuzzy k-ideal if and only if A is a right pure k-ideal of R.*

Proof. Let A be a right pure k-ideal of R. Then χ_A is a fuzzy k-ideal of R. To prove that χ_A is right pure we have to show that for any fuzzy right k-ideal μ of R, $\mu \wedge \chi_A = \mu \odot_h \chi_A$. Now if $x \notin A$, then

$$(\mu \wedge \chi_A)(x) = \mu(x) \wedge \chi_A(x) = 0 \leqslant (\mu \odot_h \chi_A)(x).$$

For the case $x \in A$, as A is a right pure k-ideal of R, so there exist $a, b \in A$, such that $x + xa = xb$. As $x, a, b \in A$, this implies $\chi_A(x) = \chi_A(a) = \chi_A(b) = 1$. Now,

$$\begin{aligned}
(\mu \odot_k \chi_A)(x) &= \bigvee_{x + \sum\limits_{i=1}^{m} a_i b_i = \sum\limits_{j=1}^{n} a'_j b'_j} \left[\bigwedge_{i=1}^{m} [\mu(a_i) \wedge \chi_A(b_i)] \wedge \bigwedge_{j=1}^{n} [\mu(a'_j) \wedge \chi_A(b'_j)] \right] \\
&\geqslant \min [\mu(x) \wedge \chi_A(a) \wedge \mu(x) \wedge \chi_A(b)] \\
&\geqslant \min [\mu(x) \wedge \chi_A(x) \wedge \mu(x) \wedge \chi_A(x)] \\
&\geqslant \mu(x) \wedge \chi_A(x) = (\mu \wedge \chi_A)(x).
\end{aligned}$$

So, in both the cases $\mu \odot_k \chi_A \geqslant \mu \wedge \chi_A$. But $\mu \odot_k \chi_A \leqslant \mu \wedge \chi_A$ is always true. Thus, $\mu \wedge \chi_A = \mu \odot_k \chi_A$. So, χ_A is right pure fuzzy k-ideal of R.

Conversely, let χ_A be a right pure fuzzy k-ideal of R. Then A is a k-ideal of R. Let B be a right k-ideal of R, then χ_B is a fuzzy right k-ideal of R. Hence by hypothesis $\chi_B \odot_k \chi_A = \chi_B \wedge \chi_A = \chi_{B \cap A}$. By Proposition 2.8, $\chi_B \odot_k \chi_A = \chi_{\overline{BA}}$. This implies that $B \cap A = \overline{BA}$. Therefore A is a right pure k-ideal of R. □

Proposition 5.5. *Intersection of right pure k-ideals of R is a right pure k-ideal of R.*

Proof. Let A, B be right pure k-ideals of R and I be any right k-ideal of R. Then $I \cap (A \cap B) = (I \cap A) \cap B = (\overline{IA}) \cap B = \overline{(\overline{IA})B} = \overline{(IA)B} = \overline{I(AB)} = \overline{I(A \cap B)}$ because $(\overline{IA})$ is a right k-ideal. Hence $A \cap B$ is a right pure k-ideal of R. □

Proposition 5.6. *Let λ_1, λ_2 are right pure fuzzy k-ideals of R, then so is $\lambda_1 \wedge \lambda_2$.*

Proof. Indeed, $\lambda_1 \wedge \lambda_2$ is a fuzzy k-ideal of R. We have to show that, for any fuzzy right k-ideal μ of R, $\mu \odot_k (\lambda_1 \wedge \lambda_2) = \mu \wedge (\lambda_1 \wedge \lambda_2)$.

Since λ_2 is right pure fuzzy k-ideal of R so it follows that $\lambda_1 \odot_k \lambda_2 = \lambda_1 \wedge \lambda_2$. Hence $\mu \odot_k (\lambda_1 \odot_k \lambda_2) = \mu \odot_k (\lambda_1 \wedge \lambda_2)$.

Also, $\mu \wedge (\lambda_1 \wedge \lambda_2) = (\mu \wedge \lambda_1) \wedge \lambda_2 = (\mu \odot_k \lambda_1) \wedge \lambda_2 = (\mu \odot_k \lambda_1) \odot_k \lambda_2 = \mu \odot_k (\lambda_1 \odot_k \lambda_2)$ since $\mu \odot_k \lambda_1$ is a fuzzy right k-ideal of R.

Thus $\mu \wedge (\lambda_1 \wedge \lambda_2) = \mu \odot_k (\lambda_1 \wedge \lambda_2)$. □

Proposition 5.7. *For a hemiring R with identity the following statements are equivalent:*

1. *R is right k-weakly regular hemiring,*
2. *all right k-ideals of R are k-idempotent,*
3. *every k-ideal of R is right pure.*

Proof. 1 and 2 are equivalent by Proposition 3.2.

(1) $\Longrightarrow$ (3) Let I and A be k-ideal and right k-ideal of R, respectively. Then $A \cap I = \overline{AI}$. Thus by Lemma 5.2, A is right pure.

(3) $\Longrightarrow$ (1) Let I be a k-ideal of R and A a right k-idealof R, then by hypothesis, I is right pure and so $A \cap I = \overline{AI}$. Thus, by Proposition 3.2, R is right k-weakly regular. □

Proposition 5.8. *The following statements are equivalent for a hemiring R with identity:*

1. *R is right k-weakly regular hemiring,*
2. *all right k-ideals of R are k-idempotent,*
3. *every k-ideal of R is right pure,*
4. *all fuzzy right k-ideals of R are k-idempotent,*
5. *every fuzzy k-ideal of R is right pure.*

If R is commutative, then the above statements are equivalent to

6. *R is k-regular.*

Proof. $1, 2, 3$ are equivalent by Proposition 5.7, $1, 4$ by Theorem 3.3.

(4) $\Longrightarrow$ (5) Let λ and μ be fuzzy right and two sided k-ideals of R, respectively. Then $\lambda \wedge \mu$ is a fuzzy right k-ideal of R. By Theorem 2.10, $\lambda \odot_k \mu \leqslant \lambda \wedge \mu$. By hypothesis,$(\lambda \wedge \mu) = (\lambda \wedge \mu) \odot_k (\lambda \wedge \mu) \leqslant \lambda \odot_k \mu$. Hence $\lambda \odot_k \mu = \lambda \wedge \mu$. Thus μ is right pure.

(5) $\Longrightarrow$ (1) Let B be a right k-ideal of R and A be a two-sided k-ideal of R then the characteristic functions χ_B and χ_A are fuzzy right and fuzzy two-sided k-ideals of R, respectively. Hence $\chi_B \odot_h \chi_A = \chi_B \wedge \chi_A$ implies $\chi_{\overline{BA}} = \chi_{B\cap A}$, i.e., $\overline{BA} = B \cap A$. Thus by Proposition 3.2, R is right k-weakly regular.

Finally, for a commutative hemiring, by Theorem 2.11, 1 and 6 are equivalent. □

6. Purely prime k-ideals

Definition 6.1. A proper right pure k-ideal I of a hemiring R is called *purely prime* if for any right pure k-ideals A, B of R, $A \cap B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$, or equivalently, if $\overline{AB} \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$.

Definition 6.2. A proper right pure k-ideal μ of a hemiring R is called *purely prime* if for any right pure fuzzy k-ideals λ, δ of R, $\lambda \wedge \delta \leqslant \mu$ implies $\lambda \leqslant \mu$ or $\delta \leqslant \mu$, or equivalently, if $\lambda \odot_k \delta \leqslant \mu$ implies $\lambda \leqslant \mu$ or $\delta \leqslant \mu$.

Proposition 6.3. *For a k-ideal I of a right k-weakly regular hemiring R with identity the following statements are equivalent:*
1. $A \cap B = I \Longrightarrow A = I$ *or* $B = I$,
2. $A \cap B \subseteq I \Longrightarrow A \subseteq I$ *or* $B \subseteq I$,

where A, B are k-ideals of R.

Proof. (1) $\Longrightarrow$ (2) Suppose A, B are k-ideals of R such that $A \cap B \subseteq I$. Then by Theorem 3.4, $I = \overline{(A \cap B) + I} = \overline{(A + I)} \cap \overline{(B + I)}$. Hence by the hypothesis $I = \overline{(A + I)}$ or $I = \overline{(B + I)}$, i.e., $A \subseteq I$ or $B \subseteq I$.

(2) $\Longrightarrow$ (1) Suppose A, B are k-ideals of R such that $A \cap B = I$. Then $I \subseteq A$ and $I \subseteq B$. On the other hand by hypothesis $A \subseteq I$ or $B \subseteq I$. Thus $A = I$ or $B = I$. □

Proposition 6.4. *Let R be a right k-weakly regular hemiring. Then any proper right pure k-ideal of R is contained in a purely prime k-ideal of R.*

Proof. Let I be a proper right pure k-ideal of a weakly regular hemiring R and $a \in R$ such that $a \notin I$. Consider the set X of all proper right pure k-ideals J of R containing I and such that $a \notin J$. Then X is non-empty because $I \in X$. By Zorn's Lemma this family contains a maximal element, say M. This maximal element is purely prime. Indeed, let $A \cap B = M$ for some some right pure k-ideals A, B of R. If A, B both properly contains M, then by the maximality of M, $a \in A$ and $a \in B$. Thus $a \in A \cap B = M$, which is a contradiction. Hence either $A = M$ or $B = M$. □

Proposition 6.5. *Let R be a right k-weakly regular hemiring. Then each proper right pure k-ideal is the intersection of all purely prime k-ideals of R which contain it.*

Proof. The proof is similar to the proof of Theorem 4.3. □

Proposition 6.6. *Let R be a right k-weakly regular hemiring. If λ is a right pure fuzzy k-ideal of R with $\lambda(a) = t$ where $a \in R$ and $t \in [0, 1]$, then there exists a purely prime fuzzy k-ideal μ of R such that $\lambda \leqslant \mu$ and $\mu(a) = t$.*

Proof. The proof is similar to the proof of Proposition 4.4. □

Proposition 6.7. *Let R be a right k-weakly regular hemiring. Then each proper fuzzy right pure k-ideal is the intersection of all purely prime fuzzy k-ideals of R which contain it.*

Proof. The proof is similar to the proof of Theorem 4.5. □

References

[1] **M. R. Adhikari, M. K. Sen, H. J. Weinert**, *On k-regular semirings*, Bull. Cal. Math. Soc. **88** (1996), 141 – 144.

[2] **J. Ahsan, K. Saifullah, M. Farid Khan**, *Fuzzy semirings*, Fuzzy Sets Syst. **60** (1993), 309 – 320.

[3] **S. I. Baik, H. S. Kim**, *On Fuzzy k-ideals in semirings*, Kangweon-Kyungki Math. J. **8** (2000), 147 – 154.

[4] **W.A. Dudek, M. Shabir, R. Anjum**, *Characterizations of hemirings by their h-ideals*, Computer Math. Appl. **59** (2010), 3167 – 3179.

[5] **S. Ghosh**, *Fuzzy k-ideals of semirings*, Fuzzy Sets Syst. **95** (1998), 103 – 108.

[6] **M. Henriksen**, *Ideals in semirings with commutative addition*, Amer. Math. Soc. Notices **6** (1958), 321.

[7] **C. B. Kim, M. Park**, *k-fuzzy ideals in semirings*, Fuzzy Sets Syst. **81** (1996), 281 – 286.

[8] **M. Kondo, W. A. Dudek**, *On the transfer principle in fuzzy theory*, Mathware Soft Comput. **12** (2005), 41 – 55.

[9] **P. Mukhopadhyay**, *Characterizations of regular semirings*, Mat. Vesnik **48** (1996), 83 – 85.

[10] **M. K. Sen, P. Mukhopadhyay**, *Von Neumannian regularity in semirings*, Kyungpook Math. J. **35** (1995), 249 – 258.

[11] **M. Shabir, R. Anjum**, *Characterization of hemirings by their k-ideals*, (submitted)

[12] **H. S. Vandiver**, *Note on a simple type of algebra in which cancellation law of addition does not hold*, Bull. Amer. Math. Soc. **40** (1934), 914 – 920.

[13] **Y. Yin, H. Li**, *The characterizations of h-hemiregular hemirings and h-intra-hemiregular hemirings*, Inform. Sci. **178** (2008), 3451 – 3464.

[14] **L.A. Zadeh**, *Fuzzy Sets*, Inf. Contr. **8** (1965), 338 – 353.

[15] **J.Zhan, W. A. Dudek**, *Fuzzy h-ideals of hemirings*, Inform. Sci. **177** (2007), $676 - 886$.

Received July 23, 2010
Revised March 25, 2011

Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan
E-mails: mshabirbhatti@yahoo.co.uk (M.Shabir), rukh-28@hotmail.com (R.Anjum)

Quasigroups and Related Systems **20** (2012), 113 − 124

Congruences on ternary semigroups

*Sheeja G. and Sri Bala S.**

Abstract. We study congruences on ternary semigroups. We have extended Lallement's lemma for a regular ternary semigroups. We have characterized minimum group congruence and maximum idempotent pair separating congruence on a strongly regular ternary semigroups. We have also obtained a characterization for maximum idempotent pair separating congruence and smallest strongly regular congruence on an orthodox ternary semigroup.

1. Introduction

Ternary semigroups, i.e., algebras of the form $(T, [\ \ ])$, where $[\ \ ]$ is a *ternary operation* $T^3 \longrightarrow T : (x, y, z) \longrightarrow [xyz]$ satisfying the associative law

$$[xy[uvw]] = [x[yuv]w] = [[xyu]vw]$$

are studied by many authors. The study of ideals and radicals of ternary semigroups was initiated in [11]. The concept of regular ternary semigroups was introduced in [10]. In [6] regular ternary semigroups was characterized by ideals. In [8] regular ternary semigroups are characterized by idempotent pairs. Orthodox ternary semigroups are investigated in [9]. Congruences on ternary semigroups are described in [2].

In this paper we generalize to ternary semigroups some important results on congruences on binary semigroups such as the Lallement's Lemma for example. We also characterize the minimal congruence on ternary semigroup under which the quotient algebra is a ternary group and find a maximal congruence separating idempotent pairs.

2. Preliminaries

For simplicity a ternary semigroup $(T, [\ \ ])$ will be denoted by T and the symbol of an inner ternary operation $[\ \ ]$ will be deleted, i.e., instead of $[[xyz]uw]$ or $[x[yzu]w]$ or $[xy[zuw]]$ we will write $[xyzuw]$.

*According to the authors request we write their names in the form used in India.

2010 Mathematics Subject Classification: 20N10

Keywords: Idempotent pair, strongly regular ternary semigroup, congruence.

The second author is supported by University Grants Commission, India.

Recall that an element x of a ternary semigroup T is called *regular* if there exists $y \in T$ such that $[xyx] = x$. A ternary semigroup in which each element is regular is called *regular*. An element $x \in T$ is *inverse* to $y \in T$ if $[xyx] = x$ and $[yxy] = y$. Clearly, if x is inverse to y, then y is inverse to x. Thus every regular element has an inverse. The set of all inverses of x in T is denoted by $I(x)$.

Definition 2.1. A pair (a, b) of elements of T is an *idempotent pair* if $[ab[abt]] = [abt]$ and $[[tab]ab] = [tab]$ for all $t \in T$. An idempotent pair (a, b) in which an element a is inverse to b is called a *natural idempotent pair*.

According to Post [7] two pairs (a, b) and (c, d) are *equivalent* if $[abt] = [cdt]$ and $[tab] = [tcd]$ for all $t \in T$. Equivalent pairs are denoted by $(a, b) \sim (c, d)$. If (a, b) is an idempotent pair, then $([aba], [bab])$ is a natural idempotent pair and $(a, b) \sim ([aba], [bab])$. The equivalence class containing (a, b) will be denoted by $\langle a, b\rangle$. By E_T we denote the set of all equivalence classes of idempotent pairs in T.

For $a, b \in T$ consider the maps $\mathrm{L}_{a,b} : T \longrightarrow T : x \longrightarrow [abx]$ and $\mathrm{R}_{a,b} : x \longrightarrow [xab]$. On the set

$$M = \{m(a, b) \,|\, m(a, b) = (\mathrm{L}_{a,b}, \mathrm{R}_{a,b}), a, b \in T\},$$

which can be identified with $T \times T$, we introduce a binary product by putting

$$m(a, b)m(c, d) = m([abc], d) = m(a, [bcd]).$$

Then M is a semigroup. This semigroup can be extended to the semigroup $S_T = T \cup M$ as follows. For $A, B \in S_T$ we define

$$AB = \begin{cases} m(a,b) & \text{if} \quad A = a,\ B = b \in T, \\ [abx] & \text{if} \quad A = m(a,b) \in S_T,\ B = x \in T, \\ [xab] & \text{if} \quad A = x \in T,\ B = m(a,b) \in S_T, \\ m([abc],d) & \text{if} \quad A = m(a,b),\ B = m(c,d) \in S_T. \end{cases}$$

The semigroup S_T is a covering semigroup in the sense of Post [7] (see also [1]). The product $[abc]$ in T is equal to abc in S_T. The element $m(a, b)$ in S_T is usually denoted by ab.

It is shown in [8] that T is a regular (strongly regular) ternary semigroup if and only if S_T is a regular (inverse) semigroup. There is a bijective correspondence between E_T and the set E_{S_T} of idempotents of S_T. Note that (a, b) is an idempotent pair in T if and only if $m(a, b)$ is an idempotent in S_T and $\langle a, b\rangle$ corresponds to $m(a, b)$.

Definition 2.2. A ternary semigroup T is called a *ternary group* if for $a, b, c \in T$ the equations $[abx] = c$, $[ayb] = c$ and $[zab] = c$ have (unique) solutions in T.

Definition 2.3. An element a of a ternary semigroup T is said to be *invertible* if there exists an element $b \in T$ such that $[abx] = x = [bax] = [xab] = [xba]$ for all $x \in T$.

An invertible element is regular. In ternary group each element is invertible. Moreover, directly from the definition of a ternary group it follows that in ternary groups each element is regular and invertible. An element which is inverse to x is called it skew to x and is denoted by $\overline{x}$ (see [1] or [3]). Obviously it is uniquely determined and $\overline{\overline{x}} = x$.

In this paper we will denote the unique inverse of x (also in ternary semigroups) by x^{-1}.

As a simple consequence of results proved in [3] and [7] we can deduce

Theorem 2.4. *A ternary semigroup T is a ternary group if and only if one of the following equivalent conditions is satisfied.*

(i) *T is regular and cancellative.*

(ii) *T is regular and all idempotent pairs are equivalent.*

(iii) *All elements of T are invertible.*

(iv) *T contains no proper one sided ideals.* □

More information on ternary groups one can find in [4] and [5].

Definition 2.5. A regular ternary semigroup T is called *orthodox* if for any two idempotents pairs (a, b) and (c, d) the pair $([abc], d)$ is also an idempotent pair.

If T is an orthodox ternary semigroup, then E_T is a band. Hence E_T is a semilattice of rectangular bands. Clearly $E_T \simeq E_{S_T}$ as bands.

For $a, b \in T$ denote by $W(a, b)$ the set of all equivalence classes $\langle u, v\rangle$ such that $(u, v) \in T \times T$ and $[abuvabt] = [abt]$, $[tabuvab] = [tab]$, $[uvabuvt] = [uvt]$, $[tuvabuv] = [tuv]$.

Clearly, $\langle x, y\rangle \in W(a, b)$ if and only if $xy \in I(ab)$ in S_T. Since E_T is a semilattice of rectangular bands, from the fact that $\langle a, b\rangle$ and $\langle c, d\rangle$ are elements of E_T it follows that $\langle [abc], d\rangle$ and $\langle [cda], b\rangle$ are in the same component of E_T and consequently $W([abc], d) = W([cda], b)$.

Proposition 2.6. $[I(c)I(b)I(a)] \subset I([abc])$ *for all elements a, b, c of each orthodox ternary semigroup.* □

Proposition 2.7. *A regular ternary semigroup is orthodox if and only if for all its elements a, b from $I(a) \cap I(b) \neq \emptyset$ it follows $I(a) = I(b)$.* □

The proofs of the above two facts are found in [9].

3. Congruences on ternary semigroups

Lemma 3.1. *If (a, b) is an idempotent pair in an orthodox ternary semigroup T, then $([uab], u')$, $([abu], u')$, $([uu'a], b)$ and $([buu'], a)$ are idempotent pairs for any $u \in T$ and $u' \in I(u)$.*

Proof. Indeed, we have $[uabu'uabu't] = [uabu'uab[u'uu']t] = [u[abu'uabu'uu']t] = [u[abu'uu']t] = [uabu't]$ for all $t \in T$. Similarly, $[tuabu'uabu'] = [tuabu'uabu'uu'] = [tu[abu'uabu'uu']] = [tuabu']$. Therefore $([uab], u')$ is an idempotent pair. For $([abu], u')$, $([uu'a], b)$ and $([buu'], a)$ the proof is analogous. □

Corollary 3.2. *If (a, b) is an idempotent pair in a strongly regular ternary semigroup T, then $([uab], u^{-1})$,$([abu], u^{-1})$ $([uu^{-1}a], b)$ and $([buu^{-1}], a)$ are idempotent pairs for any $u \in T$.* □

Lemma 3.3. *If (a, b) is an idempotent pair in an orthodox ternary semigroup T, then $([uva], [bv'u'])$ is an idempotent pair for all $u' \in I(u)$, $v' \in I(v)$ and $u, v \in T$.*

Proof. By Lemma 3.1 $([vab], v')$ is an idempotent pair and for all $u' \in I(u)$ and $v' \in I(v)$ we obtain $[uvabv'u'uvabv'u't] = [u[vabv'u'uvabv'u'uu't]] = [uvabv'u'uu't] = [uvabv'u't]$ for $t \in T$. Similarly $[tuvabv'u'uvabv'u'] = [tuvabv'u'uvabv'u'uu'] = [tuvabv'u'uu'] = [tuvabv'u']$. □

Corollary 3.4. *If (a, b) is an idempotent pair in a strongly regular ternary semigroup T, then $([uva], [bv^{-1}u^{-1}])$ is an idempotent pair for all $u, v \in T$.*

Lemma 3.5. (Generalised Lallement's Lemma)
Let ρ be a congruence on a regular ternary semigroup T. If $(a\rho, b\rho)$ is an idempotent pair in T/ρ then there exists an idempotent pair (p, q) in T such that $(a\rho, b\rho) \sim (p\rho, q\rho)$. Moreover, (p, q) satisfies the property that $[Tpq] \subseteq [Tab]$ and $[pqT] \subseteq [abT]$

Proof. It is clear that T/ρ is a ternary semigroup. Let $(a\rho, b\rho)$ be an idempotent pair in T/ρ. If b' is an inverse of b and u be an inverse of $[[aba]bb']$, then for $p = [abb']$, $q = [uab]$ and $t \in T$ we have $[pq[pqt]] = [[abb'][uab][abb'][uab]t] = [abb'[uababb'u]abt] = [[abb'][uab]t] = [pqt]$. Similarly $[[tpq]pq] = [t[abb'][uababb'u]ab] = [tpq]$. Hence (p, q) is an idempotent pair. Moreover $[p\rho q\rho x\rho] = [[abb']\rho[uab]\rho x\rho] = [a\rho b\rho b'\rho u\rho a\rho b\rho x\rho] = [a\rho b\rho a\rho b\rho b'\rho u\rho a\rho b\rho a\rho b\rho b'\rho b\rho x\rho] = [[[[aba]bb']u[[aba]bb']]bx]\rho = [[aba]bx]\rho = [a\rho b\rho a\rho b\rho x\rho] = [a\rho b\rho x\rho]$ for $x \in T$. Analogously $[x\rho p\rho q\rho] = [x\rho a\rho b\rho]$ for all $x \in T$. Thus $(a\rho, b\rho) \sim (p\rho, q\rho)$ in T/ρ. From the choice of p and q it is clear that $[Tpq] \subseteq [Tab]$ and $[pqT] \subseteq [abT]$. □

Corollary 3.6. *If T is a regular ternary semigroup and ρ is a congruence on T, then T/ρ is a regular ternary semigroup.* □

Definition 3.7. A congruence ρ on a ternary semigroup T is said to be a *ternary group congruence* if T/ρ is a ternary group.

Definition 3.8. A congruence ρ on a regular ternary semigroup T is called *strongly regular* if T/ρ is a strongly regular ternary semigroup, and *idempotent pair separating* if (a, b) and (c, d) are equivalent in T for each idempotent pairs (a, b), (c, d) such that $(a\rho, b\rho)$ and $(c\rho, d\rho)$ are equivalent in T/ρ.

Lemma 3.9. *Let $\rho : T \longrightarrow T\rho$ be a ternary homomorphism of an orthodox ternary semigroup T. Then $T\rho$ is an orthodox ternary semigroup.* □

Lemma 3.10. *Let ρ be a ternary homomorphism of a strongly regular ternary semigroup T. Then $T\rho$ is a strongly regular ternary semigroup such that $(a\rho)^{-1} = a^{-1}\rho$ for all $t \in T$.*

Proof. For idempotent pairs $(a\rho, b\rho)$ and $(x\rho, y\rho)$ in $T\rho$, by Lemma 3.5, there exists idempotent pairs (p, q) and (u, v) such that $(p\rho, q\rho) \sim (a\rho, b\rho)$ and $(u\rho, v\rho) \sim (x\rho, y\rho)$. Thus $[a\rho b\rho x\rho y\rho t\rho] = [p\rho q\rho u\rho v\rho t\rho] = [pquvt]\rho = [uvpqt]\rho = [u\rho v\rho p\rho q\rho t\rho] = [x\rho y\rho a\rho b\rho t\rho]$ and $[t\rho a\rho b\rho x\rho y\rho] = [t\rho x\rho y\rho a\rho b\rho]$. Hence the idempotent pairs $(a\rho, b\rho)$ and $(x\rho, y\rho)$ commute in T/ρ. Thus $T\rho$ is strongly regular. Moreover, for any $a \in T$ we have $[a\rho a^{-1}\rho a\rho] = a\rho$ and $[a^{-1}\rho a\rho a^{-1}\rho] = a^{-1}\rho$. Thus $a^{-1}\rho = (a\rho)^{-1}$, by [9]. □

Any congruence ρ on a ternary semigroup T can be extended to the relation ρ^e defined on $S_T = T \cup M$ in the following way:

$$(x,y) \in \rho^e \Leftrightarrow \begin{cases} (x,y) \in \rho \text{ and } x, y \in T, \text{ or} \\ x{=}ab,\ y{=}cd{\in}M \text{ and } ([abt],[cdt]), ([tab],[tcd]){\in}\rho\ \ \forall t{\in}T. \end{cases}$$

Lemma 3.11. *ρ^e is a congruence on S_T.*

Proof. It is clear that ρ^e is an equivalence relation on S_T. To prove that it is a congruence suppose $x\rho^e y$ and $x, y \in S_T$.

(i) If $x, y \in T$ and $z \in T$, then $[zxt]\rho[zyt]$ and $[tzx]\rho[tzy]$ for any $t \in T$, so $zx\rho^e zy$. Similarly $[xzt]\rho[yzt]$ and $[txz]\rho[tyz]$. Hence $xz\rho^e yz$. If $z = uv$, then $zx = [uvx]$, $zy = [uvy]$ and $[uvx]\rho[uvy]$. Also $[xuv]\rho[yuv]$. Thus $zx\rho^e zy$ and $xz\rho^e yz$.

(ii) Suppose $x = ab$, $y = cd$ and $z = pq$. Then $xz = ([abp], q)$ and $yz = ([cdp], q)$. Since $x\rho^e y$, we have $[abt] = [cdt]$ and $[tab] = [tcd]$ for all $t \in T$. Therefore $[abpqt] = [cdpqt]$ and $[tabpq] = [tcdpq]$. Hence $xz\rho^e yz$. Similarly, $[pqabt] = [pqcdt]$ and $[tpqab] = [tpqcd]$. So, $zx\rho^e zy$.

(iii) If $x = ab, y = cd$, then for any $z \in T$ we have $[zab]\rho[zcd]$ and $[abz]\rho[cdz]$. Therefore $zx\rho^e zy$ and $xz\rho^e yz$. Hence ρ^e is a congruence. □

Lemma 3.12. *If T is a regular ternary semigroup, then ρ^e is an idempotent separating congruence in S_T if and only if ρ is an idempotent pair separating congruence in T.*

Proof. Let ρ^e be an idempotent separating congruence in S_T. If (a, b) and (c, d) are idempotent pairs in T such that $(a\rho, b\rho)$ and $(c\rho, d\rho)$ are equivalent in T/ρ, then $[abt]\rho[cdt]$ and $[tab]\rho[tcd]$ for all $t \in T$. Hence $ab\rho^e cd$ in S_T. Since ab and cd are idempotents in S_T and ρ^e is idempotent separating we have $ab = cd$. This means that $[abt] = [cdt]$ and $[tab] = [tcd]$ and so $(a, b) \sim (c, d)$. Conversely suppose ρ is an idempotent pair separating congruence in T. Let e, f be idempotents in S_T such that $e\rho^e f$. Let $e = ab$ and $f = cd$ for some idempotent pairs (a, b) and (c, d) in T. Then $e\rho^e f$ implies $[abt]\rho[cdt]$ and $[tab]\rho[tcd]$. Hence $(a\rho, b\rho) \sim (c\rho, d\rho)$ in T/ρ, which gives $(a, b) \sim (c, d)$ in T. So, $e = f$. Thus ρ^e is an idempotent separating congruence on S_T. □

4. Strongly regular ternary semigroups

In this section T denotes a strongly regular ternary semigroup. Below we will construct congruences on T which are analogous to the group congruence and maximum idempotent separating congruence on an ordinary inverse semigroup.

We start with the relation σ defined on T as follows:

$$(x, y) \in \sigma \iff [abx] = [aby] \text{ for some idempotent pair } (a, b) \in T.$$

Lemma 4.1. *σ is a congruence on T.*

Proof. Clearly σ is an equivalence relation on T. To prove that it is a congruence suppose $x\sigma y$ and $u, v \in T$. Then $[abx] = [aby]$ for some idempotent pair (a, b), and so $[abxuv] = [abyuv]$. Hence $([xuv], [yuv]) \in \sigma$. By Corollary 3.2, for any $u, v \in T$, $([v^{-1}u^{-1}u], v)$ is an idempotent pair and by Corollary 3.4, $([uva], [bv^{-1}u^{-1}])$ is also an idempotent pair. So,

$$\begin{aligned}
\big[[uva][bv^{-1}u^{-1}][uvx]\big] &= \big[uvabv^{-1}u^{-1}uvabv^{-1}u^{-1}uvx\big] \\
&= \big[uvabv^{-1}u^{-1}uvv^{-1}u^{-1}uvabx\big] \\
&= \big[uvabv^{-1}u^{-1}uvabx\big] = [uvabv^{-1}u^{-1}uvaby] \\
&= \big[uvababv^{-1}u^{-1}uvy\big] = [uvabv^{-1}u^{-1}uvy].
\end{aligned}$$

Therefore $([uvx], [uvy]) \in \sigma$. Similarly $([vab], v^{-1})$ and (u^{-1}, u) are idempotent pairs and they commute. Hence

$$\begin{aligned}
\big[[uva][bv^{-1}u^{-1}][uxv]\big] &= [uvabv^{-1}u^{-1}uvabv^{-1}u^{-1}uxv] \\
&= [u[vabv^{-1}u^{-1}uv]abv^{-1}u^{-1}uxv]] \\
&= \big[uu^{-1}u[vabv^{-1}vabv^{-1}u^{-1}]uxv\big] \\
&= \big[u[vabv^{-1}u^{-1}]uxv\big] = [uv[abv^{-1}u^{-1}u]xv] \\
&= [uvv^{-1}u^{-1}uabxv] = [uvv^{-1}u^{-1}uabyv] \\
&= [uvabv^{-1}u^{-1}uyv].
\end{aligned}$$

Therefore $([uxv], [uyv]) \in \sigma$. Hence σ is a congruence. $\square$

Proposition 4.2. *T/σ is a ternary group.*

Proof. By Theorem 2.4 and Lemma 3.9, it is enough to show that all idempotent pairs in T/σ are equivalent. If $(a\sigma, b\sigma)$, $(u\sigma, v\sigma)$ are two idempotent pairs in T/σ, then we have to prove $[abt]\sigma[uvt]$ and $[tab]\sigma[tuv]$ for all $t \in T$. By Lemma 3.5, without loss of generality we can assume that (a, b) and (u, v) are idempotent pairs of T. Then $([abu], v)$ and $([uva], b)$ are idempotent pairs. For any $t \in T$ we have $[[abu]v[abt]] = [ababuvt] = [abuvt] = [abuvuvt] = [[abu]v[uvt]]$ since idempotent pairs commute in T. Therefore $[abt]\sigma[uvt]$. Similarly $[[tab][uva]b] = [tabuvab] = [tuvab] = [[tuv][uva]b]$. Hence $[tab]\sigma[tuv]$. So, $(a\sigma, b\sigma)$ and $(u\sigma, v\sigma)$ are equivalent in T/σ. Thus in T/σ all idempotent pairs are equivalent and T/σ is a ternary group. $\square$

Theorem 4.3. *σ is the minimum ternary group congruence on a strongly regular ternary semigroup T.*

Proof. By Proposition 4.2, T/σ is a ternary group. Suppose θ is a congruence on T such that T/θ is a ternary group. We prove that $\sigma \subseteq \theta$. Suppose $(p, q) \in \sigma$, then $[abp] = [abq]$ for some idempotent pair (a, b) in T. Then $[a\theta b\theta p\theta] = [a\theta b\theta q\theta]$. Since T/θ is a ternary group cancellation law holds and so $p\theta = q\theta$. □

Now we consider the relation μ defined as follows:

$$(a, b) \in \mu \iff ([axx^{-1}], a^{-1}) \sim ([bxx^{-1}], b^{-1}) \ \ \forall (x, x^{-1}) \in T \times T.$$

In other words, $(a, b) \in \mu$ if $[axx^{-1}a^{-1}t] = [bxx^{-1}b^{-1}t]$ and $[taxx^{-1}a^{-1}] = [tbxx^{-1}b^{-1}]$ for every $t \in T$.

Lemma 4.4. *μ is a congruence on T.*

Proof. Clearly μ is an equivalence relation. Suppose $(a, b) \in \mu$ and $u, v \in T$. For every idempotent pair (x, x^{-1}), by Corollary 3.2 $([uvx], [x^{-1}v^{-1}u^{-1}])$ is an idempotent pair and so we obtain $[auvxx^{-1}v^{-1}u^{-1}a^{-1}t] = [buvxx^{-1}v^{-1}u^{-1}b^{-1}t]$, $[tauvxx^{-1}v^{-1}u^{-1}a^{-1}] = [tbuvxx^{-1}v^{-1}u^{-1}b^{-1}]$. Hence $([auv], [buv]) \in \mu$. Since $[axx^{-1}a^{-1}t] = [bxx^{-1}b^{-1}t]$ for all $t \in T$, we have $[uvaxx^{-1}a^{-1}t] = [uvbxx^{-1}b^{-1}t]$. Replacing t by $[v^{-1}u^{-1}t]$ we get $[uvaxx^{-1}a^{-1}v^{-1}u^{-1}t] = [uvbxx^{-1}b^{-1}v^{-1}u^{-1}t]$. In a similar way we obtain $[tuvaxx^{-1}a^{-1}v^{-1}u^{-1}] = [tuvbxx^{-1}b^{-1}v^{-1}u^{-1}]$. Thus $([uva], [uvb]) \in \mu$. Hence for every idempotent pair (x, x^{-1}) also $([vxx^{-1}], v^{-1})$ is an idempotent pair. Therefore for all $t \in T$ we have $[avxx^{-1}v^{-1}a^{-1}t] = [bvxx^{-1}v^{-1}b^{-1}t]$. In particular for $t = u^{-1}$ we obtain $[avxx^{-1}v^{-1}a^{-1}u^{-1}] = [bvxx^{-1}v^{-1}b^{-1}u^{-1}]$. Hence $[[uav]xx^{-1}[v^{-1}a^{-1}u^{-1}]t] = [[ubv]xx^{-1}[v^{-1}b^{-1}u^{-1}]t]$ for $t \in T$. Analogously we obtain $[tuavxx^{-1}v^{-1}a^{-1}u^{-1}] = [tubvxx^{-1}v^{-1}b^{-1}u^{-1}]$. Hence $([uav], [ubv]) \in \mu$. Thus μ is a congruence. □

Theorem 4.5. *μ is the maximum idempotent pair separating congruence on T.*

Proof. Let (a, a^{-1}) and (b, b^{-1}) be such that $(a\mu, a^{-1}\mu)$ and $(b\mu, b^{-1}\mu)$ are equivalent idempotent pairs in T/μ. We claim that (a, a^{-1}) and (b, b^{-1}) are equivalent idempotent pairs in T. From the hypothesis it follows that in T we have $[aa^{-1}t]\mu[bb^{-1}t]$ and $[taa^{-1}]\mu[tbb^{-1}]$ for all $t \in T$. The first relation for $t = a$ and $t = b$ gives $a\mu[bb^{-1}a]$ and $[aa^{-1}b]\mu b$. Putting in the second relation $t = a^{-1}$ and $t = b^{-1}$ we obtain $a^{-1}\mu[a^{-1}bb^{-1}]$ and $[b^{-1}aa^{-1}]\mu b^{-1}$. Therefore for all idempotent pairs (z, z^{-1}) and for all $t \in T$ we have

$$[azz^{-1}a^{-1}t] = [bb^{-1}azz^{-1}a^{-1}bb^{-1}t], \tag{4.1}$$

$$[bzz^{-1}b^{-1}t] = [aa^{-1}bzz^{-1}b^{-1}aa^{-1}t]. \tag{4.2}$$

From (4.1) for $z = a^{-1}$ and $t = a$ we get $[aa^{-1}aa^{-1}a] = [bb^{-1}aa^{-1}bb^{-1}a] = [bb^{-1}a]$. Therefore

$$a = [bb^{-1}a]. \tag{4.3}$$

Thus $a^{-1} = [a^{-1}bb^{-1}]$. From (4.2) putting $z = b^{-1}$ and $t = b$ we obtain $[bb^{-1}bb^{-1}b]$ $= [aa^{-1}bb^{-1}aa^{-1}b] = [aa^{-1}b]$. Therefore

$$b = [aa^{-1}b]. \tag{4.4}$$

Hence $b^{-1} = [b^{-1}aa^{-1}]$. Now using (4.3) and (4.4) we see that

$$[aa^{-1}t] = [bb^{-1}a[a^{-1}bb^{-1}]t] = [b[b^{-1}aa^{-1}]bb^{-1}t] = [bb^{-1}bb^{-1}t] = [bb^{-1}t]$$

for all $t \in T$. Similarly

$$[taa^{-1}] = [t[bb^{-1}a][a^{-1}bb^{-1}]] = [tb[b^{-1}aa^{-1}]bb^{-1}] = [tbb^{-1}bb^{-1}] = [tbb^{-1}].$$

Therefore $(a, a^{-1}) \sim (b, b^{-1})$. Hence μ is an idempotent pair separating congruence in T.

Suppose that ρ is another idempotent pair separating congruence on T. If $a\rho = b\rho$, then $a^{-1}\rho = b^{-1}\rho$ by Lemma 3.10. For any idempotent pair $(x, x^{-1}) \in T$ we have $[axx^{-1}a^{-1}t]\rho = [bxx^{-1}b^{-1}t]\rho$ and $[taxx^{-1}a^{-1}]\rho = [tbxx^{-1}b^{-1}]\rho$. Hence $([axx^{-1}]\rho, a^{-1}\rho)$ and $([bxx^{-1}]\rho, b^{-1}\rho)$ are equivalent idempotent pairs in T/ρ. Since $([axx^{-1}], a^{-1})$ and $([bxx^{-1}], b^{-1})$ are idempotent pairs in T we see that they are equivalent in T. Hence $a\mu b$. Therefore $\rho \subseteq \mu$. □

5. Congruences on orthodox ternary semigroups

In this section by T will denote an orthodox ternary semigroup. By γ we denote the relation on T such that

$$(a, b) \in \gamma \Longleftrightarrow I(a) = I(b).$$

Theorem 5.1. *The relation γ is a congruence on T.*

Proof. Clearly γ is an equivalence relation. Suppose $(a, b) \in \gamma$ and $x, y \in T$. Then for any $u \in I(a) = I(b)$ and for any $v \in I(x), w \in I(y)$ it follows from Proposition 2.6, that $[uwv] \in I([xya]) \cap I([xyb])$. Hence by Proposition 2.7 we get $I([xya]) = I([xyb])$ and so $([xya], [xyb]) \in \gamma$. Similarly $[wvu] \in I([axy]) \cap I([bxy])$. Therefore $([axy], [bxy]) \in \gamma$. Also $([xay], [xby]) \in \gamma$. Hence γ is a congruence. □

Theorem 5.2. *The relation γ is the smallest congruence on T for which T/γ is a strongly regular ternary semigroup.*

Proof. $E_T = \cup E_\alpha$ is a semilattice of rectangular bands. For any $\langle a, b\rangle$, $\langle c, d\rangle$ and $\langle e, f\rangle$ in E_T, elements $([abcde], f)$ and $([cdabe], f)$ belong to the same class E_α and so $I(\langle [abcde], f\rangle) = I(\langle [cdabe], f\rangle)$ in E_T. This can be interpreted in T as $W([abcde], f) = W([cdabe], f) = W(a, [bcdef])$. Let $(a\gamma, b\gamma)$ and $(c\gamma, d\gamma)$ be two idempotent pairs in T/γ. Fix $t \in T$. If $u \in I([abcdt])$, then $[abcdtuabcdt] = [abcdt]$ and $[uabcdtu] = u$. We first show that $(t, u) \in W([cdabt], t')$, for some

$t' \in I(t)$. For all $z \in T$ we have $[tuz] = [t[uabcdtu]z] = [tuabcdtt'tuz]$ and $[abcdtt'z] = [[abcdtuabcdt]t'z] = [abcdtt'tuabcdtt'z]$. Therefore we see that (t,u) is in $W([abcdt],t') = W([cdabt],t')$. Thus, for all $z \in T$

$$[cdabtt'tucdabtt'z] = [cdabtt'z], \tag{5.1}$$

$$[tucdabtt'tuz] = [tuz], \tag{5.2}$$

$$[zcdabtt'tucdabtt'] = [zcdabtt'], \tag{5.3}$$

$$[ztucdabtt'tu] = [ztu]. \tag{5.4}$$

(5.1) for $z = t$ gives $[cdabtt'tucdabtt't] = [cdabtt't]$. Therefore

$$[cdabtucdabt] = [cdabt]. \tag{5.5}$$

Multiplying (5.2) on the left by $[uabcd]$ and on the right by u we obtain the equation $[uabcdtucdabtt'tuzu] = [uabcdtuzu]$. Therefore $[ucdabtuzu] = [uzu]$, which for $z = [abcdt]$ gives $[ucdabt[uabcdtu]] = [uabcdtu]$. Hence

$$[ucdabtu] = u. \tag{5.6}$$

From (5.5) and (5.6) we get $u \in I([cdabt])$. Thus $u \in I([abcdt]) \cap I([cdabt])$, which implies $I([abcdt]) = I([cdabt])$ (cf. [9]). Hence

$$[abcdt]\gamma[cdabt]. \tag{5.7}$$

Now we show that $I([tabcd]) = I([tcdab])$. Indeed, if $u \in I([tabcd])$, then $[tabcdutabcd] = [tabccd]$ and $[utabcdu] = u$. Moreover, for every z from T we have $[utz] = [[utabcdu]tz] = [utt'tabcdutz]$, $[zut] = [zutabcdut] = [zutt'tabcdut]$. Similarly, $[t'tabcdz] = [t'[tabcd]z] = [t'[tabcdutabcd]z] = [t'tabcdutt'tabcdz]$, $[zt'tabcd] = [zt'[tabcdutabcd]] = [zt'tabcdutt'tabcd] = [zt'tabcdutt'tabcd]$. Therefore (u,t) is in $W([t',[tabcd]) = W(t',[tcdab])$. Hence for all $z \in T$,

$$[utt'tcdabutz] = [utz], \tag{5.8}$$

$$[t'tcdabutt'tcdabz] = [t'tcdabz], \tag{5.9}$$

$$[zutt'tcdabut] = [zut], \tag{5.10}$$

$$[zt'tcdabutt'tcdab] = [zt'tcdab]. \tag{5.11}$$

Multiplying (5.10) on the left by u and on the right by $[abcdu]$ we obtain the equation $[uzutcdab[utabcdu]] = [uz[utabcdu] = [uzu]$. This for $z = [tabcd]$ gives $[[utabcdu]tcdabu] = [utabcdu] = [utabcdu]$. Therefore

$$[utcdabu] = u. \tag{5.12}$$

(5.11) for $z = t$ gives $[tt'tcdabutt'tcdab] = [tt'tcdab]$. Therefore

$$[tcdabutcdab] = [tcdab]. \tag{5.13}$$

From (5.12) and (5.13) we get $u \in I([tcdab])$. Thus $I([tabcd]) = I([tcdab])$. Hence

$$[tabcd]\gamma[tcdab]. \tag{5.14}$$

Now, from (5.7) and (5.14) it follows that $(a\gamma, b\gamma)$ and $(c\gamma, d\gamma)$ commute in T/γ and so T/γ is strongly regular.

Suppose that ρ is a congruence on T such that T/ρ is a strongly regular ternary semigroup. If $(a, b) \in \gamma$, then for any $x \in I(a) = I(b)$, $a\rho$ and $b\rho$ are both inverses of $x\rho$ in T/ρ. Since T/ρ is strongly regular, the element $x\rho$ has a unique inverse and so $a\rho = b\rho$. Hence $\gamma \subseteq \rho$. Thus γ is the smallest strongly regular ternary semigroup congruence. $\square$

Theorem 5.3. *The relation μ defined by*

$$(a, b) \in \mu \Longleftrightarrow \begin{cases} \text{for every idempotent pair } (x, x') \ \exists a' \in I(a), \exists b' \in I(b) \\ ([axx'], a') \sim ([bxx'], b') \text{ and } ([a'xx'], a) \sim ([b'xx'], b). \end{cases}$$

is a congruence on T.

Proof. We first prove that μ is an equivalence relation. Clearly μ is reflexive and symmetric. For any $(a, b), (b, c) \in \mu$ there exists $a' \in I(a)$, b', $b'' \in I(b)$ and $c' \in I(c)$ such that for every idempotent pair (x, x') we have $[axx'a't] = [bxx'b't]$ and $[taxx'a'] = [tbxx'b']$, $[a'xx'at] = [b'xx'bt]$ and $[ta'xx'a] = [tb'xx'b]$, $[bxx'b''t] = [cxx'c't]$ and $[tbxx'b''] = [tcxx'c']$, $[b''xx'bt] = [c'xx'ct]$ and $[tb''xx'b] = [tc'xx'c]$. Put $a^* = [b''ba'bb']$. We see that $[bb'a] = [bb'aa'aa'a] = [bb'ba'ab'a] = [ba'ab'a] = [aa'aa'a] = [aa'a] = a$ and $[aa^*a] = [ab''ba'bb'a] = [bb''bb'bb'a] = [bb'bb'a] = [bb'a] = a$. Thus $[a^*aa^*] = [b''ba'bb'ab''ba'bb'] = [b''bb'bb'bb''ba'bb'] = [b''bb''ba'bb'] = [b''ba'bb'] = a^*$. Hence $a^* \in I(a)$. Similarly for $c^* = [b''bc'bb']$ we have $[cc^*c] = [cb''bc'bb'c] = [cb''bb''bb'b] = [cb''bb'b] = [cc'bb'c] = [cc'c] = c$, $[c^*cc^*] = [b''bc'bb'cb''bc'bb'] = [b''bb''bb'bb''bc'bb'] = [b''bb''bc'bb'] = [b''bc'bb'] = c^*$. Therefore $c^* \in I(c)$. Now for all idempotent pair (x, x') in T and all $t \in T$ we obtain $[a^*xx'at] = [b''ba'bb'xx'at] = [b''bb'bb'xx'bt] = [b''bb'xx'bt] = [b''bb''bb'xx'bt] = [b''bc'bb'xx'ct] = [c^*xx'ct]$ and $[ta^*xx'a] = [tc^*xx'c]$, $[axx'a^*t] = [axx'b''ba'bb't] = [bxx'b''bb'bb't] = [bxx'b''bb't] = [bxx'b''bb''bb't] = [cxx'b''bc'bb't] = [cxx'c^*t]$. Also we have $[taxx'a^*] = [tcxx'c^*]$. Hence $(a, c) \in \mu$, proving μ is a transitive relation. Thus μ is an equivalence relation.

Suppose $(a, b) \in \mu$ and $u, v \in T$ so that for every idempotent pair (x, x') in T and for all $t \in T$,

$$[axx'a't] = [bxx'b't], \tag{5.15}$$

$$[taxx'a'] = [tbxx'b'], \tag{5.16}$$

$$[a'xx'at] = [b'xx'bt], \tag{5.17}$$

$$[ta'xx'a] = [tb'xx'b]. \tag{5.18}$$

In (5.15), replacing (x, x') by $([uvx], [x'v'u'])$ we get $[auvxx'v'u'a't] = [buvxx'v'u'b't]$. Similarly, (5.16) becomes $[tauvxx'v'u'a'] = [tbuvxx'v'u'b']$. In (5.17) replacing

t by $[uvt]$ and multiplying on the left by v' and u' we get $[v'u'a'xx'auvt] = [v'u'b'xx'buvt]$ $\forall t \in T$. In (5.18) replacing t by $[tv'u']$ and multiplying on the right by u and v, we get $[tv'u'a'xx'auv] = [tv'u'b'xx'buv]$. Since $[v'u'a'] \in I([auv])$ and $[v'u'b'] \in I([buv])$ we have $([auv],[buv]) \in \mu$. Similarly we can show that $([uva],[uvb]) \in \mu$ and $([uav],[ubv]) \in \mu$. □

Theorem 5.4. *μ is the maximum idempotent pair separating congruence on T.*

Proof. Let $(a\mu, a'\mu)$ and $(b\mu, b'\mu)$ be two equivalent idempotent pairs in T/μ so that $[aa't]\mu[bb't]$, $[taa']\mu[tbb']$, $[a'at]\mu[b'bt]$ and $[ta'a]\mu[tb'b]$ $\forall t \in T$. Putting $t = a$ and $t = b$ in the first relation we get $a\mu[bb'a]$ and $[aa'b]\mu b$. Putting $t = a'$ and $t = b'$ in the second relation we get $a'\mu[a'bb']$ and $[b'aa']\mu b'$. Hence for every idempotent pair (x, x') and for all $t \in T$ we have

$$[axx'a't] = [bb'axx'[bb'a]'t], \tag{5.19}$$

$$[bxx'b't] = [aa'bxx'[aa'b]'t], \tag{5.20}$$

$$[ta''xx'a'] = [t[a'bb']'xx'a'bb'], \tag{5.21}$$

$$[tb''xx'b'] = [t[b'aa']'xx'b'aa'] \tag{5.22}$$

for some $[bb'a]' \in I([bb'a])$. From (5.19) for $(x, x') = (a', a)$ and $t = a$ we get $a = [bb'aa'a[bb'a]'t] = [[bb'a][bb'a]'t]$. Multiplying on the left by b and b' we have $[bb'a] = [bb'[bb'a][bb'a]'t] = a$. Therefore $[bb'a] = a$. Putting $(x, x') = (b', b)$ and $t = b$ in (5.20) we obtain $b = [aa'bb'b[aa'b]'b] = [aa'b[aa'b]'b]$. Multiplying on the left by a and a' we get $[aa'b] = [aa'aa'b[aa'b]'b] = [aa'b[aa'b]'b] = a$. Therefore $[aa'b] = a$. Replacing in (5.21) x by a' and x' by a'' we obtain $[ta''a'] = [t[a'bb']'a'a''a'bb'] = [t[a'bb']'a'bb']$ for every $t \in T$, which for $t = a'$ implies $a' = [a'[a'bb']'a'bb']$. Multiplying this on the right by b and b' we get $[a'bb'] = [a'[a'bb']'a'bb'bb'] = a'$. Therefore $[a'bb'] = a'$. (5.22) for $x = b'$ and $x' = b''$ gives $[tb''b'] = [t[b'aa']'b'b''b'aa'] = [t[b'aa']'b'aa']$, $\forall t \in T$. In particular, for $t = b'$ we get $b' = [b'[b'aa']'b'aa']$. Multiplying this on the right by a and a' we obtain $[b'aa'] = [b'[b'aa']'b'aa'aa'] = [b'[b'aa']'b'aa'] = b'$. Therefore $[b'aa'] = b'$ and $[aa't] = [[bb'a][a'bb']t] = [b[b'aa']bb't] = [bb'bb't] = [bb't]$, $\forall t \in T$, Similarly $[taa'] = [t[bb'a][a'bb']] = [tb[b'aa']bb'] = [tbb'bb'] = [tbb']$. Hence $(a, a') \sim (b, b')$. Thus μ is an idempotent pair separating congruence on T.

Suppose that θ is an idempotent pair separating congruences on T and θ_e is the congruence induced on S_T by θ. If $x\theta y$, then $x\theta_e y$ in S_T. S_T is orthodox and by Lemma 3.12, θ_e is an idempotent separating congruences on S_T. Hence $\theta_e \subset \mathcal{H}$, where $\mathcal{H}$ is the Green's equivalence on S_T. Hence $x\mathcal{H}y$ in S_T we can find inverse x' of x and y' of y such that $xx' = yy'$ and $x'x = y'y$ in S_T. Therefore for all $t \in T$, $[xx't] = [yy't]$ and $[txx'] = [tyy']$. Similarly, $[x'xt] = [y'yt]$ and $[tx'x] = [ty'y]$ in T. Therefore $x = [xx'x] = [yy'x]$, $x' = [x'xx'] = [x'yy']$, $y' = [y'yy']$. Thus $x' = [x'yy']\theta[x'xy'] = y'$. Hence for every idempotent pair (u, v) in T, $[x'uvxt]\theta[y'uvyt]; [xuvx't]\theta[yuvy't]$. $([x'uv]\theta, x\theta) \sim ([y'uv]\theta, y\theta)$ and $([xuv]\theta, x'\theta) \sim ([yuv]\theta, y'\theta)$ in T/θ. Since θ is idempotent pair separating we have

$([xuv], x') \sim ([yuv], y')$. In a similar way we can show that $([x'uv], x) \sim ([y'uv], y)$. Thus $x\mu y$. Hence $\theta \subseteq \mu$ and so μ is the maximum idempotent pair separating congruences on T. $\square$

Acknowledgements. The authors are grateful to Prof. V. Thangaraj and Prof. M. Loganathan for their valuable support. The authors express their gratitude to Prof. W. A. Dudek for his valuable suggestions.

References

[1] **A. Borowiec, W. A. Dudek and S. Duplij**, *Bi-element representations of ternary groups*, Commun. Algebra **34** (2006), 1651 – 1670.

[2] **A. Chronowski**, *Congruences on ternary semigroups*, Ukrainian Math. J. **56** (2004), 662 – 681.

[3] **W. Dörnte**, *Untersuchungen über einen verallgemeinerten Gruppenbegriff*, Math. Z. **29** (1928), 1 – 19.

[4] **W. A. Dudek**, *On some old and new problems in n-ary groups*, Quasigroups and Related Systems **8** (2001), 15 – 36.

[5] **W. A. Dudek and K. Głazek**, *Around the Hosszú-Gluskin Theorem for n-ary groups*, Discrete Math. **308** (2008), 4861 – 4876.

[6] **W. A. Dudek and I. M. Groździńska**, *On ideals in regular n-semigroups*, Mat. Bilten (Skopje) **3/4(29/30)** (1979-1980), 35 – 44.

[7] **E. L. Post**, *Polyadic groups*, Trans. Amer. Math. Soc. **48** (1920), 208 – 350.

[8] **M. L. Santiago and S. Sri Bala**, *Ternary semigroups*, Semigroup Forum **81** (2010), 380 – 388.

[9] **G. Sheeja and S. Sri Bala**, *Orthodox ternary semigroups*, Quasigroups and Related Systems **19** (2011), 339 – 348.

[10] **F. M. Sioson**, *On regular algebraic systems*, Proc. Japan Acad. **39** (1963), 283 – 286.

[11] **F. M. Sioson**, *Ideal theory in ternary semigroups*, Math. Japonica **10** (1965), 63 – 84.

Received January 24, 2011

Ramanujan Institute for Advanced Study in Mathematics, University of Madras,
Chennai 600 005, India
E-mail: sheejag.dhe@gmail.com (Sheeja G.), sribalamath@yahoo.co.in (Sri Bala S.)

Quasigroups and Related Systems **20** (2012), 125 − 138

Quotient hyper residuated lattices

Omid Zahiri, Rajabali A. Borzooei and Mahmud Bakhshi

Abstract. We define the concept of regular compatible congruence on hyper residuated lattices. Then we attempt to construct quotient hyper residuated lattices. Finally, we state and prove some theorem with appropriate results such as the isomorphism theorems.

1. Introduction

Residuated lattices, introduced by Ward and Dilworth [7], are a common structure among algebras associated with logical systems. In this definition to any bounded lattice $(\mathcal{L}, \vee, \wedge, 0, 1)$, a multiplication '$*$' and an operation '$\rightarrow$' are equipped such that $(\mathcal{L}, *, 1)$ is a commutative monoid and the pair $(*, \rightarrow)$ is an adjoint pair, i.e.,

$$x * y \leqslant z \text{ if and only if } x \leqslant y \rightarrow z, \quad \forall x, y, z \in \mathcal{L}.$$

The main examples of residuated lattices are MV-algebras introduced by Chang [2] and BL-algebras introduced by Hájek [4].

The hyperstructure theory was introduced by Marty [5], at the 8th Congress of Scandinavian Mathematicians. In his definition, a function $f : A \times A \longrightarrow P^*(A)$, of the set $A \times A$ into the set of all nonempty subsets of A, is called a binary *hyperoperation*, and the pair (A, f) is called a *hypergroupoid*. If f is associative, A is called a *semihypergroup*, and it is said to be *commutative* if f is commutative. Also, an element $1 \in A$ is called the *unit* or the *neutral element* if $a \in f(1, a)$, for all $a \in A$.

Recently, R. A. Borzooei et al. introduced and study hyper K-algebras and Sh. Ghorbani et al. applied the hyper structure to MV-algebras and introduced the concept of hyper MV-algebra, which is generalization of MV-algebra. In this paper, we want to introduced the concept of hyper residuated lattices and construct the quotient structure in hyper residuated lattices and give results as mentioned in the abstract.

2010 Mathematics Subject Classification: 03G10, 06B99, 06B75.
Keywords: Hyper residuated lattice, quotient, filter, isomorphism theorem.

2. Preliminaries

Definition 2.1. A *residuated lattice* is a structure $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 1)$ satisfying the following axioms:

(1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,
(2) $(L, \odot, 1)$ is a commutative monoid,
(3) $x \odot y \leqslant z$ if and only if $x \leqslant y \rightarrow z$, for all $x, y \in L$.

Let $(L', \vee', \wedge', \odot', \rightarrow', 0', 1')$ and $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ be two residuated lattices. The map $f : L \rightarrow L'$ is called a *homomorphism* if $f(x * y) = f(x) * f(y)$, for all $x, y \in L$, where $* \in \{\odot, \vee, \wedge, \rightarrow\}$

Definition 2.2. [6] A *super lattice* is a partially ordered set $(S; \leqslant)$ endowed with two binary hyperoperations $\vee$ and $\wedge$ satisfying the following properties: for all $a, b, c \in S$,

(SL1) $a \in (a \vee a) \cap (a \wedge a)$,
(SL2) $a \vee b = b \vee a$, $a \wedge b = b \wedge a$,
(SL3) $(a \vee b) \vee c = a \vee (b \vee c)$, $(a \wedge b) \wedge c = a \wedge (b \wedge c)$,
(SL4) $a \in ((a \vee b) \wedge a) \cap ((a \wedge b) \vee a)$,
(SL5) $a \leqslant b$ implies $b \in a \vee b$ and $a \in a \wedge b$,
(SL6) if $a \in a \wedge b$ or $b \in a \vee b$ then $a \leqslant b$.

Definition 2.3. Let A be a set, $\odot$ be a binary hyperoperation on A and $1 \in A$. $(A; \odot, 1)$ is called a *commutative semihypergroup* with 1 as an identity if it satisfies the following properties: for all $x, y, z \in A$,

(CSHG1) $x \odot (y \odot z) = (x \odot y) \odot z$,
(CSHG2) $x \odot y = y \odot x$,
(CSHG3) $x \in 1 \odot x$.

Proposition 2.4. *Let $(L, \leqslant)$ be a partially ordered set. Define the binary hyperoperations $\vee$ and $\wedge$ on L as follows: $a \vee b = \{c \,|\, a \leqslant c \text{ and } b \leqslant c\}$ and $a \wedge b = \{c \,|\, c \leqslant a \text{ and } c \leqslant b\}$, for all $a, b \in L$.Then $(L; \vee, \wedge)$ is a bounded super lattice.* □

Definition 2.5. Let $(P, \leqslant)$ be a partially ordered set and γ be an equivalence relation on P. Then γ is called *regular* if the set $P/\gamma = \{[x] \,|\, x \in P\}$ can be ordered in such a way that the natural map $\pi : P \rightarrow P/\gamma$ is order preserving.

Definition 2.6. Let γ be a regular equivalence relation on partially ordered set $(P, \leqslant)$.

(i) By a *γ-fence* we shall mean an ordered subset of P having the following diagram (Figure 1), where $a_i \leqslant b_{i+1}$ and three vertical lines indicate the equivalence modulo γ. We often denote this γ-fence by $\langle a_1, b_n \rangle_\gamma$ and say that a γ-fence

$\langle a_1, b_n\rangle_\gamma$ joins a_1 to b_n.

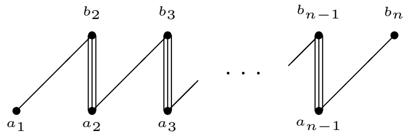


Figure 1. γ-fence

(ii) By a *γ-crown* we shall mean an ordered subset of P having the following diagram (Figure 2)

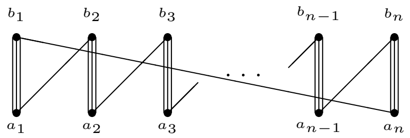


Figure 2. γ-crown

where $a_i \leqslant b_{i+1}$, $a_n \leqslant b_1$ and three vertical lines indicate the equivalence modulo γ. We often denote this γ-crown by $\langle\langle a_1, b_n\rangle\rangle_\gamma$.

(iii) A γ-crown $\langle a_1, b_n\rangle_\gamma$ is called *γ-closed*, when $a_i\gamma b_j$, for all $i, j \in \{1, 2, ..., n\}$.

Theorem 2.7. [1] *Let γ be an equivalence relation on ordered set $(P, \leqslant)$ and $\leqslant_\gamma$ be the relation on $P/\gamma = \{[x] \mid x \in P\}$ defined by $[x] \leqslant_\gamma [y]$ if and only if there is a γ-fence that joins x to y. Then the following statements are equivalent:*

(*i*) *$\leqslant_\gamma$ is an order on P/γ,*
(*ii*) *γ is regular,*
(*iii*) *every γ-crown is γ-closed.* □

3. Quotient hyper residuated lattices

Definition 3.1. By a *hyper residuated lattice* we mean a nonempty set L endowed with four binary hyperoperations $\vee$, $\wedge$, $\odot$, $\rightarrow$ and two constants 0 and 1 satisfying the following conditions:

(HRL1) $(L; \vee, \wedge, 0, 1)$ is a bounded super lattice,
(HRL2) $(L; \odot, 1)$ is commutative semihypergroup with 1 as an identity,
(HRL3) $a \odot c \ll b$ if and only if $c \ll a \rightarrow b$,

where $A \ll B$ means that there exist $a \in A$ and $b \in B$ such that $a \leqslant b$, for all nonempty subset A and B of L.

A hyper residuated lattice is called *nontrivial* if $0 \neq 1$. An element a of hyper residuated lattice L is called *scalar* if $|a \odot x| = 1$, for all $x \in L$.

Definition 3.2. Let $(L; \vee, \wedge, \odot, \rightarrow, 0, 1)$ and $(L'; \vee', \wedge', \odot', \rightarrow', 0', 1')$ be two hyper residuated lattices and $f : L \rightarrow L'$ be a function. f is called a *homomorphism* if it satisfies the following conditions: for all $x, y \in L$,

(i) $f(x \vee y) \subseteq f(x) \vee' f(y)$,

(ii) $f(x \wedge y) \subseteq f(x) \wedge' f(y)$,

(iii) $f(x \odot y) \subseteq f(x) \odot' f(y)$,

(iv) $f(x \rightarrow y) \subseteq f(x) \rightarrow' f(y)$,

(v) $f(1) = 1'$ and $f(0) = 0'$.

If f satisfies (v) and the conditions (i)–(iv) holds for the equality instead of the inclusion, f is said to be a *strong homomorphism*, briefly an *S-homomorphism*.

A homomorphism which is one to one, onto or both is called a *monomorphism*, *epimorphism* or an *isomorphism*, respectively. Similarly, an S-homomorphism which is one-to-one, onto or both is called an *S-monomorphism*, *S-epimorphism* or *S-isomorphism*, respectively.

Definition 3.3. A nonempty subset F of L satisfying

(F) $x \leqslant y$ and $x \in F$ imply $y \in F$

is called a

- *hyper filter* if $x \odot y \subseteq F$, for all $x, y \in F$,
- *weak hyper filter* if $F \ll x \odot y$, for all $x, y \in F$.

A filter F of L is called *proper* if $F \neq L$ and this is equivalent to that $0 \notin F$. Let F be a proper (weak) hyper filter of L. Then F is called a *maximal* if $F \subseteq J \subseteq L$ implies $F = J$ or $J = L$, for all (weak) hyper filters J of L. Moreover, hyper residuated lattice L is called *simple* if $\{\{1\}, L\}$ is the set of all weak hyper filters of L. Obviously, in any hyper residuated lattice L, $\{1\}$ is a weak hyper filter and L is a hyper filter of L.

Remark 3.4. Clearly, any hyper filter of L is a weak hyper filter of L. Moreover, $1 \in F$, for any (weak) hyper filter F of L.

From now on, in this section, L and L' will denote two hyper residuated lattices and for convenience, we use the same notations for the hyper operations of L and L', unless otherwise stated.

In the following, we introduced the concept of regular compatible congruence relations on a hyper residuated lattices and verify some useful properties of these relations. Then we attempted to fine the S-homomorphisms, whose *ker* are regular compatible congruence relations. Then we stated and proved isomorphism theorems on hyper residuated lattices.

Definition 3.5. Let θ be an equivalence relation on L and $A, B \subseteq L$. Then

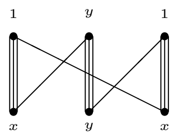
(i) $A\theta B$ means that there exist $a \in A$ and $b \in B$ such that $a\theta b$,

(ii) $A\overline{\theta}B$ means that for all $a \in A$, there exists $b \in B$ such that $a\theta b$ and for all $b \in B$, there exists $a \in A$ such that $a\theta b$,

Definition 3.6. An equivalence relation θ on L is called a *congruence relation* if for all $x, y, z, w \in L$, $x\theta y$ and $z\theta w$ imply $(x * z)\overline{\theta}(y * w)$, where $* \in \{\wedge, \vee, \odot, \rightarrow\}$.

Proposition 3.7. *Let θ be a regular equivalence on L. Then $[1] = \{x \in L \,|\, x\theta 1\}$ is a weak hyper filter of L.*

Proof. Clearly, $[1] \neq \emptyset$. Let $x, y \in [1]$. Since $(x \odot y)\overline{\theta}(1 \odot 1)$ and $1 \in 1 \odot 1$, then $(x \odot y)\theta 1$. Hence $(x \odot y) \cap [1] \neq \emptyset$ and so $[1] \ll x \odot y$. Now, let $x, y \in L$ be such that $x \in [1]$ and $x \leqslant y$. Then we have



and so $\{x, 1, y, y, x, 1\}$, forms a θ-crown on L. Since θ is regular, by Theorem 2.7, $x\theta y$ and so $y \in [1]$. Therefore, $[1]$ is a weak hyper filter of L. $\square$

Lemma 3.8. *Let θ be a regular congruence relation on L, $L/\theta = \{[x] \,|\, x \in L\}$ and $\leqslant_\theta$ be the relation on L/θ defined as in Theorem 2.7. For all $x, y \in L$, define $[x]\overline{\odot}[y] = [x \odot y]$, $[x]\overline{\vee}[y] = [x \vee y]$, $[x]\overline{\wedge}[y] = [x \wedge y]$ and $[x] \rightsquigarrow [y] = [x \to y]$, where $[A] = \{[a] \,|\, a \in A\}$, for all $A \subseteq L$. Then*

- (i) *$\overline{\odot}, \overline{\vee}, \overline{\wedge}$ and $\rightsquigarrow$ are well defined,*
- (ii) *$[x] \ll_\theta [y] \rightsquigarrow [z]$ if and only if $[x]\overline{\odot}[y] \ll_\theta [z]$, where $[A] \ll_\theta [B]$ if and only if $[a] \leqslant_\theta [b]$, for some $a \in A$ and $b \in B$.*

Proof. (i) Let $[x_1] = [x_2]$ and $[y_1] = [y_2]$, for some $x_1, x_2, y_1, y_2 \in L$. Since θ is a congruence relation on L, we have $(x_1 \odot y_1)\overline{\theta}(x_2 \odot y_2)$. Let $u \in [x_1]\overline{\odot}[y_1]$. Then $[u] = [a]$, for some $a \in x_1 \odot y_1$. By $(x_1 \odot y_1)\overline{\theta}(x_2 \odot y_2)$, we conclude that $a\theta b$, for some $b \in x_2 \odot y_2$ and so $[u] = [a] = [b] \in [x_2]\overline{\odot}[y_2]$. Hence $[x_1]\overline{\odot}[y_1] \subseteq [x_2]\overline{\odot}[y_2]$. By the similar way, we can prove that $[x_2]\overline{\odot}[y_2] \subseteq [x_1]\overline{\odot}[y_1]$. Therefore, $\overline{\odot}$ is well defined. Similarly, it is proved that $\overline{\vee}, \overline{\wedge}$ and $\rightsquigarrow$ are well defined.

(ii) Let $[x]\overline{\odot}[y] \ll_\theta [z]$. Then there exists $u \in x \odot y$ such that $[u] \leqslant_\theta [z]$ and so there exists a θ-fence that joins u to z. Let $\langle a_1, b_n\rangle$ be a θ-fence of L that joins u to z, where $u = a_1$ and $z = b_n$. Since $u \in x \odot y$ and $u \leqslant b_2$, then $x \odot y \ll b_2$ and so $x \leqslant c_2 \in y \to b_2$. By $b_2\theta a_2$, we get $(y \to b_2)\overline{\theta}(y \to a_2)$ whence $c_2\theta d_2$, for some $d_2 \in y \to a_2$. Now, from $d_2 \in y \to a_2$ it follows that $d_2 \ll y \to a_2$, and so $d_2 \odot y \ll a_2 \leqslant b_3$. Hence $d_2 \leqslant c_3 \in y \to b_3$. Since $(y \to b_3)\overline{\theta}(y \to a_3)$, then $c_3\theta d_3$, for some $d_3 \in y \to a_3$. Hence $x \leqslant c_2\theta d_2 \leqslant c_3\theta d_3$. By the similar way, there are $c_i \in y \to b_i$, for any $i \in \{2, 3, \ldots, n\}$ and $d_j \in y \to a_j$, for any $j \in \{2, 3, \ldots, n-1\}$ such that $x \leqslant c_2\theta d_2 \leqslant c_3\theta d_3 \leqslant \ldots \leqslant c_{n-1}\theta d_{n-1} \leqslant c_n$. Hence the set $\{x, d_2, \ldots, d_{n-1}, c_2, \ldots, c_n\}$ forms a θ-fence that joins x to c_n and so $[x] \leqslant_\theta [c_n]$. Since $c_n \in y \to b_n = y \to z$, we have $[x] \ll_\theta [y \to z] = [y] \rightsquigarrow [z]$. Conversely, let $[x] \ll_\theta [y] \rightsquigarrow [z]$. Then $[x] \leqslant_\theta [u]$, for some $u \in y \to z$. Hence there is a θ-fence, $\langle a_1, b_n\rangle_\theta$, that joins x to u, where $x = a_1$ and $u = b_n$. By $a_{n-1} \leqslant u \in y \to z$, we get $a_{n-1} \odot y \ll z$, whence $e_{n-1} \leqslant z$, for some $e_{n-1} \in a_{n-1} \odot y$. Since $a_{n-1}\theta b_{n-1}$, then $(a_{n-1} \odot y)\overline{\theta}(b_{n-1} \odot y)$ and so there exists $f_{n-1} \in b_{n-1} \odot y$ such that $f_{n-1}\theta e_{n-1}$. From $f_{n-1} \in b_{n-1} \odot y$ it follows that $b_{n-1} \odot y \ll f_{n-1}$, whence $a_{n-2} \leqslant b_{n-1} \ll y \to f_{n-1}$. Hence $a_{n-2} \odot y \ll f_{n-1}$ and so there is $e_{n-2} \in a_{n-2} \odot y$ such that $e_{n-2} \leqslant f_{n-1}$. From $(a_{n-2} \odot y)\overline{\theta}(b_{n-2} \odot y)$ it follows that $e_{n-2}\theta f_{n-2}$, for some $f_{n-2} \in b_{n-2} \odot y$. By a similar way, there are $e_i \in a_i \odot y$ and $f_i \in b_i \odot y$ such that $f_i\theta e_i$ and $e_j \leqslant f_{j+1}$, for all $i \in \{2, \ldots, n-1\}$ and $j \in \{1, 2, \ldots, n-2\}$. Therefore, $\{e_1, \ldots, e_{n-1}, f_2, \ldots, f_{n-1}, z\}$ forms a θ-fence that joins e_1 to z and so $[e_1] \leqslant_\theta [z]$. Since $e_1 \in a_1 \odot y = x \odot y$, then $[x]\overline{\odot}[y] = [x \odot y] \ll_\theta [z]$. □

Definition 3.9. Let θ be a regular congruence relation on L. We say that $\leqslant_\theta$, $\overline{\vee}$ and $\overline{\wedge}$ are *compatible* if they satisfy the following conditions: for all $x, y \in L$,

(*i*) $[x] \in [x]\overline{\vee}[y]$ if and only if $[x] \leqslant_\theta [y]$,

(*ii*) $[x] \in [x]\overline{\wedge}[y]$ if and only if $[x] \leqslant_\theta [y]$.

By a *regular compatible congruence relation* on L we mean a regular congruence relation on L such that $\leqslant_\theta$, $\overline{\vee}$ and $\overline{\wedge}$ are compatible.

Theorem 3.10. *Let θ be a regular compatible congruence relation on L. Then $(L/\theta, \overline{\vee}, \overline{\wedge}, \overline{\odot}, \rightsquigarrow, [0], [1])$ is a hyper residuated lattice.*

Proof. Since θ is regular, by Theorem 2.7, $\leqslant_\theta$ is a partially order on L. Clearly, $[0]$ and $[1]$ are the minimum and the maximum elements of $(L/\theta, \leqslant_\theta)$. Moreover, $[x]\overline{\odot}[y] = [x \odot y] = [y \odot x] = [y]\overline{\odot}[x]$, for any $x, y \in L$. By the similar way, we can show that $(L/\theta, \overline{\odot}, [1])$ is a commutative semihypergroup with $[1]$ as an identity. Hence by Lemma 3.8 and Definition 3.9, $(L/\theta, \overline{\vee}, \overline{\odot}, \rightsquigarrow, [0], [1])$ is a hyper residuated lattice. □

Example 3.11. Let $(\{0, a, b, c, 1\}, \leqslant)$ be a partially ordered set such that $0 < a < b < c < 1$, $L = \{0, a, b, c, 1\}$. Consider the following tables:

Table 1

$\vee$	0	a	b	c	1
0	{0,a,c,1}	{a,c,1}	{b,c,1}	{c,1}	{1}
a	{a,c,1}	{a,c,1}	{b,c,1}	{c,1}	{1}
b	{b,c,1}	{b,c,1}	{b,c,1}	{c,1}	{1}
c	{c,1}	{c,1}	{c,1}	{c,1}	{1}
1	{1}	{1}	{1}	{1}	{1}

Table 2

$\wedge$	0	a	b	c	1
0	{0}	{0}	{0}	{0}	{0}
a	{0}	{a,0}	{a,0}	{a,0}	{a,0}
b	{0}	{a,0}	{b,0}	{b,a,0}	{b,a,0}
c	{0}	{a,0}	{b,a,0}	{c,a,0}	{c,a,0}
1	{0}	{a,0}	{a,b,0}	{c,a,0}	{0,1,a,c}

Table 3

$\to$	0	a	b	c	1
0	{1}	{1}	{1}	{1}	{1}
a	{1,a,c}	{a,1}	{b,1}	{c,1}	{1}
b	{1,b,c}	{b,1}	{b,1}	{c,1}	{1}
c	{1,c}	{c}	{c}	{c,1}	{1}
1	{1,c}	{1,c}	{1,c}	{1,c}	{1}

Let $\odot = \wedge$. It is easy to verify that $(L; \vee, \wedge, \odot, \to, 0, 1)$ is a hyper residuated lattice. Let $\theta = \{(x, x) \mid x \in L\} \cup \{(a, b), (b, a)\}$. Routine calculations show that θ is a congruence relation on L, such that $\overline{\vee}$, $\overline{\wedge}$ and $\leqslant_\theta$ are compatible. Consider the partially order relation $[0] \prec [a] \prec [c] \prec [1]$ on L/θ. Since the mapping $\pi : L \to L/\theta$ defined by $\pi(x) = [x]$, for all $x \in L$ is an ordered preserving map, then θ is regular. Therefore, by Theorem 3.10, $(L; \overline{\vee}, \overline{\wedge}, \overline{\odot}, \rightsquigarrow, [0], [1])$ is a hyper residuated lattice. $\square$

Proposition 3.12. *Let θ be a regular compatible congruence relation on L. Then*

(*i*) $[1]$ *is a hyper filter of L if and only if $\{[1]\}$ is a hyper filter of L/θ.*

(*ii*) *if $[1]$ is a maximal weak hyper filter of L, then L/θ is simple.*

Proof. (*i*) Let $[1]$ be a hyper filter of L. Then $\{[1]\}$ is a weak hyper filter of L/θ. It suffices to show that $[1]\overline{\odot}[1] = [1]$. Since $1 \in [1]$ and $[1]$ is a hyper filter of L, then $1 \odot 1 \subseteq [1]$ and so $[1]\overline{\odot}[1] = [1 \odot 1] = [1]$. Hence $\{[1]\}$ is a hyper filter of L. Conversely, assume that $\{[1]\}$ is a hyper filter of L/θ. By Proposition 3.7, $[1]$ is a weak hyper filter of L. Let $a, b \in [1]$. Since $[1]\overline{\odot}[1] = [1]$ and $[a] = [b] = [1]$, then $[a \odot b] = [a]\overline{\odot}[b] = [1]\overline{\odot}[1] = [1]$. Hence $a \odot b \subseteq [1]$ and so $[1]$ is a hyper filter of L/θ.

(*ii*) By Proposition 3.7, $[1]$ is a weak hyper filter of L. Assume $[1]$ is a maximal weak hyper filter of L and F is a weak hyper filter of L. Let $M = \cup\{[x] \mid [x] \in F\}$. Then clearly, $M \neq \emptyset$. If $u, v \in M$, then $[u] \in F$ and $[v] \in F$ and so $[u \odot v] = [u]\overline{\odot}[v] \cap F \neq \emptyset$. Hence there exists $a \in u \odot v$ such that $[a] \in F$ and so $a \in M$. Hence $(u \odot v) \cap M \neq \emptyset$. Now, let $x \in M$ and $x \leqslant y$, for some $y \in L$. Then clearly, $\{x, y\}$ formes a θ-fence that joins x to y and so $[x] \leqslant_\theta [y]$. Since $[x] \in F$ and F is a weak hyper filter of L/θ, then $[y] \in F$ and so $y \in M$. Therefore, M is a weak hyper filter of L. Clearly, $[1] \subseteq M$. Since $[1]$ is a maximal weak hyper filter of L, then $[1] = M$ or $M = L$. If $M = L$, then $F = L/\theta$. Moreover, if $[1] = M$, then $F = \{[1]\}$. Therefore, $\{\{[1]\}, L/\theta\}$ is the set of all weak hyper filters of L/θ and so L/θ is simple. $\square$

The converse of Proposition 3.12(ii) may not be true.

Example 3.13. Let $L = \{0, a, b, c, 1\}$ and $(L, \leqslant)$ be a partially ordered set such that $0 < c < a < b < 1$. Define the binary hyperoperations $\vee$, $\odot$ and $\wedge$ on L as follows: $a \vee b = \{c \mid a \leqslant c \text{ and } b \leqslant c\}$ and $a \odot b = a \wedge b = \{c \mid c \leqslant a \text{ and } c \leqslant b\}$, for all $a, b \in L$. Now, let $\rightarrow$ be a hyperoperation on L defined by the following table.

Table 4

$\rightarrow$	0	a	b	c	1
0	{1}	{1}	{1}	{1}	{1}
a	{0,1}	{1}	{1}	{c,1}	{1}
b	{0,1}	{b,a,1}	{1}	{c,1}	{1}
c	{1}	{1}	{1}	{1}	{1}
1	{0,1}	{a,b,1}	{b,1}	{c,1}	{1}

It is not difficult to check that $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a hyper residuated lattice.

Let $\theta = \{(x, x)|x \in L\} \cup \{(1, a), (a, 1), (1, b), (b, 1), (a, b), (b, a), (c, 0), (0, c)\}$. Clearly, θ is an equivalence relation on L and $L/\theta = \{[1], [0]\}$. Define a relation $\prec$ on L/θ by $[0] \prec [1]$ and $[x] \prec [x]$, for all $x \in L/\theta$. Then $\prec$ is a partially order on L/θ. Moreover, the map $f : L \rightarrow L/\theta$ defined by $f(x) = [x]$, for all $x \in L$ is an ordered preserving map and so θ is regular. Hence By Theorem 2.7, $\leqslant_\theta$ is a partially order on L/θ. It is easy to check that $\leqslant_\theta = \prec$. Clearly, $[y] \in [x]\overline{\vee}[y]$ ($[x] \in [x]\overline{\wedge}[y]$) if and only if $[x] \leqslant_\theta [y]$, for all $[x], [y] \in L/\theta$. Hence θ is a regular compatible congruence relation of L and so by Theorem 3.10, $(L/\theta, \overline{\vee}, \overline{\wedge}, \overline{\odot}, \rightsquigarrow, [0], [1])$ is a hyper residuated lattice. Since $L/\theta = \{[0], [1]\}$, then L/θ is simple. Moreover, $F = \{1, a, b, c\}$ is a weak hyper filter of L and $[1] \subset F \subset L$ and so $[1] = \{1, a, b\}$ is not a maximal weak hyper filter of L. Therefore, the converse of Proposition 3.12 (ii) may not be true. $\square$

Let L and L' be two hyper residuated lattices and $f : L \rightarrow L'$ be a homomorphism. It is straightforward to check that $ker(f) = \{(x, y) \in L \times L \,|\, f(x) = f(y)\}$ is an equivalence relation on L. In Theorem 3.14, we want to verify this relation.

Theorem 3.14. *Let $f : L \rightarrow L'$ be an S-homomorphism and $\theta = ker(f)$. If $f(x) \leqslant f(y)$ implies there is a θ-fence that joins x to y, for all $x, y \in L$, then*

- (i) *θ is a regular compatible congruence relation on L and $L/ker(f)$ is a hyper residuated lattice,*
- (ii) *f induces a unique S-homomorphism $\overline{f} : L/ker(f) \rightarrow L'$ by $\overline{f}([x]) = f(x)$, for all $x \in L$ such that $Im(\overline{f}) = Im(f)$ and $\overline{f}$ is an S-monomorphism.*

Proof. (i) Let $x\theta y$ and $u\theta v$, for some $x, y, u, v \in L$. Then $f(x) = f(y)$ and $f(u) = f(v)$. Since f is an S-homomorphism, then $f(x \wedge u) = f(x) \wedge f(u) = f(y) \wedge f(v) = f(y \wedge v)$ and so $(x \wedge u)\overline{\theta}(y \wedge v)$. By the similar way we can prove the other cases. Now, we show that θ is regular. Let $\langle\langle a_1, b_n\rangle\rangle_\theta$ be a θ-crown of L. Then $f(a_i) = f(b_i)$, for all $i \in \{1, 2, ..., n\}$. Since $a_i \leqslant b_{i+1}$, then $a_i \in a_i \wedge b_{i+1}$ and so $f(a_i) \in f(a_i \wedge b_{i+1}) = f(a_i) \wedge f(b_{i+1}) = f(a_i) \wedge f(a_{i+1})$. Similarly, $a_n \leqslant b_1$ implies that $f(a_n) \leqslant f(b)_1$. Hence $f(a_i) \leqslant f(a_{i+1})$, for all $i \in \{1, 2, \ldots, n-1\}$ and so $f(x) = f(a_1) \leqslant f(a_2) \leqslant f(a_3) \leqslant \cdots \leqslant f(a_{n-1}) \leqslant f(a_n) \leqslant f(b_1) = f(a_1)$. Therefore, $f(a_i) = f(b_j)$, for all $i, j \in \{1, 2, \ldots, n\}$ and so $[a_i] = [a_j] = [b_k]$, for all $i, j, k \in \{1, 2, \ldots, n\}$. By Proposition 2.7, θ is regular. In the follow, we show that $[x] \in [x]\overline{\wedge}[y] \Leftrightarrow [x] \leqslant_\theta [y] \Leftrightarrow [y] \in [x]\overline{\vee}[y]$. Let $[x] \leqslant_\theta [y]$, for some $x, y \in L$. Then there exists a θ-fence, $\langle a_1, b_n\rangle$ that joins x to y, where $x = a_1$ and $y = b_n$. By $a_1 \leqslant b_2$, it follows that $f(a_1) \in f(a_1) \wedge f(b_2) = f(a_1) \wedge f(a_2)$ and so $f(a_1) \leqslant f(a_2)$. By a similar way, we can show that $f(a_i) \leqslant f(a_{i+1})$, for all $i \in \{1, 2, \ldots, n-1\}$. Since $f(a_{n-1}) \leqslant f(b_n) = f(y)$, then we conclude that $f(x) \leqslant f(y)$ and so $f(x) \in f(x) \wedge f(y) = f(x \wedge y)$ $(f(y) \in f(x) \vee f(y) = f(x \vee y))$. Hence $f(x) = f(a)$, for some $a \in x \wedge y$ $(a \in x \vee y)$, whence $[x] \in [x \wedge y] = [x]\overline{\wedge}[y]$ $([y] \in [x \vee y] = [x]\overline{\vee}[y])$. Conversely, let $[x] \in [x \wedge y] = [x]\overline{\wedge}[y]$ $([y] \in [x \vee y] = [x]\overline{\vee}[y])$, for some $x, y \in L$. Then there is $a \in x \wedge y$ $(a \in x \vee y)$ such that $[x] = [a]$ and so $f(x) = f(a) \in f(x \wedge y) = f(x) \wedge f(y)$ $(f(x) = f(a) \in f(x \vee y) = f(x) \vee f(y))$. Hence $f(x) \leqslant f(y)$, whence by hypothesis, there is a θ-fence that joins x to y. That is $[x] \leqslant_\theta [y]$. Therefore, θ is a regular congruence relation on L and θ, $\overline{\vee}$, $\overline{\wedge}$ are compatible and so by Theorem 3.10, $(L/ker(f), \overline{\vee}, \overline{\wedge}, \overline{\odot}, \rightsquigarrow, [0], [1])$ is a hyper residuated lattice.

(ii) Clearly, $\overline{f} : L/ker(f) \to L'$ is an S-homomorphism and $Im(\overline{f}) = Im(f)$. Let $\overline{f}([x]) = \overline{f}([y])$, for some $x, y \in L$. Then $f(x) = f(y)$ and so $[x] = [y]$. Therefore, $\overline{f}$ is a one to one S-homomorphism. □

Example 3.15. If L and L' are two residuated lattices and $f : L \to L'$ is a homomorphism, then $f(x) \leqslant f(y)$ implies $f(x) = f(x) \wedge f(y) = f(x \wedge y)$ and so the set $\{x, x, x \wedge y, y\}$, forms a $ker(f)$-fence that joins x to y. Therefore, f satisfies the conditions (i) and (ii) in Theorem 3.14. □

Example 3.16. Let $(L = \{0, a, b, c, 1\}, \leqslant)$ and $(L' = \{0, e, 1\}, \leqslant')$ be two partially ordered sets such that $0 < a < b < c < 1$ and $0 < e < 1$. Define the binary hyperoperations $\vee, \wedge, \vee'$ and $\wedge'$ by $x \vee y = \{u \in L \,|\, x \leqslant u, y \leqslant u\}$, $a \vee' b = \{u \in L' \,|\, a \leqslant' u, b \leqslant' u\}$, $x \wedge y = \{u \in L \,|\, u \leqslant x, u \leqslant y\}$ and $a \wedge' b = \{u \in L' \,|\, u \leqslant' a, u \leqslant' b\}$, for all $x, y \in L$ and $a, b \in L'$. Then by Proposition 2.4, $(L, \vee, \wedge, 0, 1)$ and $(L', \vee', \wedge', 0, 1)$ are two bounded super lattices. Let $\odot$ and $\odot'$ are defined by

$$a \odot b = \begin{cases} \{0\} & \text{if } a = 0 \text{ or } b = 0, \\ (a \wedge b) - \{0\} & \text{if } a, b \in L - \{0\}. \end{cases}$$

$$a \odot' b = \begin{cases} \{0\} & \text{if } a = 0 \text{ or } b = 0, \\ (a \wedge' b) - \{0\} & \text{if } a, b \in L' - \{0\}. \end{cases}$$

Now, consider the following tables:

Table 5

$\to$	0	a	b	c	1
0	{1}	{1}	{1}	{1}	{1}
a	{0}	{b,1}	{b,1}	{c,1}	{1}
b	{0}	{b,c}	{b,1}	{1}	{1}
c	{0}	{a,c}	{b,c}	{c,1}	{1}
1	{0}	{b,1}	{b,1}	{1}	{1}

Table 6

$\to'$	0	e	1
0	{1}	{1}	{1}
e	{0}	{e,1}	{1}
1	{0}	{1,e}	{1}

It is easy to verify that $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ and $(L', \vee', \wedge', \odot', \rightarrow', 0, 1)$ are hyper residuated lattices. Define $f : L \rightarrow L'$ by $f(0) = 0$, $f(a) = f(b) = e$ and $f(c) = f(1) = 1$. Then f is an S-homomorphism,

$ker(f) = \{(x, x) \mid x \in L\} \cup \{(a, b), (b, a), (1, c), (c, 1)\}$ and $L/ker(f) = \{[0], [a], [1]\}$.

Assume $\prec = \{(x, x) \mid x \in L/ker(f)\} \cup \{([0], [a]), ([a], [1]), ([0], [1])\}$. Then clearly, $\prec$ is a partially order on $L/ker(f)$. Since the map $\pi : L \rightarrow L/\theta$ defined by $\pi(x) = [x]$ is an order preserving map, then $ker(f)$ is regular. Easy calculations show that $f(x) \leqslant f(y)$ implies there exists a θ-fence on L that joins x to y, for any $x, y \in L$ and so by Theorem 3.14, $\overline{f} : L/\theta \rightarrow L'$ is a one to one homomorphism. $\square$

Lemma 3.17. *Let θ and χ be two regular compatible congruence relations on L such that $\theta \subseteq \chi$. Then χ/θ is a regular compatible congruence relation on L/θ, where $\chi/\theta = \{([x]_\theta, [y]_\theta) \in L/\theta \times L/\theta \mid (x, y) \in \chi\}$.*

Proof. By Theorem 3.10, $(L/\theta, \overline{\vee}, \overline{\wedge}, \overline{\odot}, \rightsquigarrow, [0], [1])$ is a hyper residueted lattice. Clearly, χ/θ is an equivalence relation on L/θ. Let $([x]_\theta, [y]_\theta)([a]_\theta, [b]_\theta) \in \chi/\theta$. Then $(x, y), (a, b) \in \chi$. Since χ is a congruence relation on L we have $(a \wedge x)\overline{\chi}(b \wedge y)$ and so by definition of $\overline{\wedge}$ we get $([a]_\theta \overline{\wedge}[x]_\theta)\overline{\chi/\theta}([b]_\theta \overline{\wedge}[y]_\theta)$. By the similar way, we can show that

$$([a]_\theta \overline{\vee}[x]_\theta)\overline{\chi/\theta}([b]_\theta \overline{\vee}[y]_\theta, \ ([a]_\theta \overline{\odot}[x]_\theta)\overline{\chi/\theta}([b]_\theta \overline{\odot}[y]_\theta), \ ([a]_\theta \rightsquigarrow [x]_\theta)\overline{\chi/\theta}([b]_\theta \rightsquigarrow [y]_\theta).$$

Hence χ/θ is a congruence relation on L/θ. Let $R = \chi/\theta$ and $(L/\theta)/R = \{[[x]_\theta]_R \mid [x]_\theta \in L/\theta\}$. Define the hyperoperations $\sqcup$, $\sqcap$, $\otimes$ and $\mapsto$ by

$$[[x]_\theta]_R \sqcup [[y]_\theta]_R = [[x]_\theta \overline{\vee}[y]_\theta]_R, \quad [[x]_\theta]_R \sqcap [[y]_\theta]_R = [[x]_\theta \overline{\wedge}[y]_\theta]_R,$$

$$[[x]_\theta]_R \otimes [[y]_\theta]_R = [[x]_\theta \overline{\odot}[y]_\theta]_R \text{ and } [[x]_\theta]_R \mapsto [[y]_\theta]_R = [[x]_\theta \rightsquigarrow [y]_\theta]_R$$

for all $[[x]_\theta]_R, [[y]_\theta]_R \in (L/\theta)/R$. Since R is a congruence relation on L/θ, then these hyperoperations are well defined. Now, we show that R is regular. Let $\langle\langle [a_1]_\theta, [b_n]_\theta \rangle\rangle_R$ be an R-crown in L/θ. Then $[a_n]_\theta \leqslant_\theta [b_1]_\theta$, $[a_i]_\theta \overline{R}[b_i]_\theta$ and $[a_j]_\theta \leqslant_\theta [b_{j+1}]_\theta$, for all $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, n-1\}$. Hence there are $n_i \in \mathbb{N}$ such that $a_{2,i}, a_{3,i}, \ldots, a_{n_i-1,i}, b_{2,i}, b_{3,i}, \ldots, b_{n_i-1,i} \in L/\theta$ such that

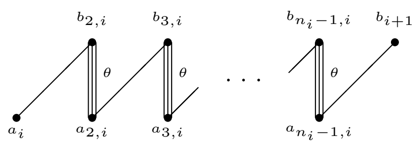


Figure 3. θ-fence joins a_i to b_{i+1}

for all $i \in \{1, 2, \ldots, n-1\}$. Moreover, there exists a θ-fence $\langle x_1, y_n \rangle_\theta$, that joins a_n to b_1. Since $[a_i]_\theta \overline{R}[b_i]_\theta$, for all $i \in \{1, 2, \ldots, n\}$ and $\theta \subseteq \chi$, then we can obtain the following χ-crown.

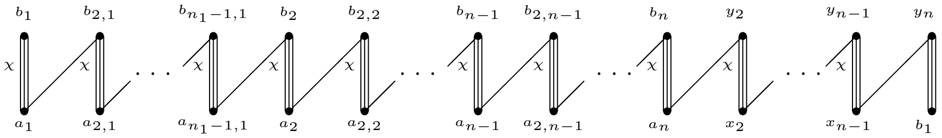


Figure 4. χ-crown

Since χ is regular, then by Theorem 2.7, $(a_i, b_j) \in \chi$ and so $[a_i]_\theta R[b_j]_\theta$, for all $i, j \in \{1, 2, \dots, n\}$, whence $\langle [a_1]_\theta, [b_n]_\theta \rangle_R$ is χ/θ closed. Now, by Theorem 2.7, R is regular. Finally, we show that R is compatible. Let $x, y \in L$ such that $[x]_\theta \leqslant_R [y]_\theta$. Then there is an R-fence $\langle [a_1]_\theta, [b_n]_\theta \rangle_R$ that joins $[x]_\theta$ to $[y]_\theta$, where $[x]_\theta = [a_1]_\theta$ and $[y]_\theta = [b_n]_\theta$. By $[a_j]_\theta R[b_j]_\theta$, we get $(a_j, b_j) \in \chi$, for all $j \in \{2, 3, \dots, n-1\}$. Since $[a_i]_\theta \leqslant_\theta [b_{i+1}]_\theta$, for all $i \in \{1, 2, \dots, n-1\}$, then there exists θ-fence $\langle a_{1,i}, b_{n_i,i} \rangle_\theta$ joins a_i to b_{i+1}, where $a_i = a_{1,i}$ and $b_{i+1} = b_{n_i,i}$, for all $i \in \{1, 2, \dots, n-1\}$. Hence by $\theta \subseteq \chi$, we can obtain the following χ-fence that joins x to y.

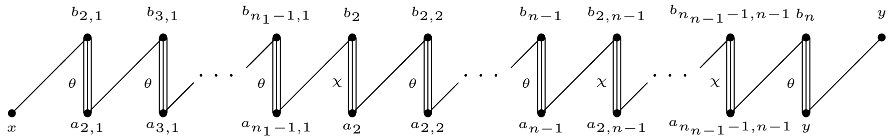


Figure 5. χ-fence joins x to y

Therefore, $[x]_\chi \leqslant_\chi [y]_\chi$. Since χ is a compatible regular congruence relation on L, then $[x]_\chi \in [x]_\chi \overline{\overline{\wedge}} [y]_\chi = [x \wedge y]_\chi$ and $[y]_\chi \in [y]_\chi \overline{\overline{\vee}} [y]_\chi = [x \vee y]_\chi$, where $\overline{\overline{\wedge}}$ and $\overline{\overline{\vee}}$ are hyper operation induced by χ in Lemma 3.8. Hence

$$[[y]_\theta]_R \in [[x \vee y]_\theta]_R = [[x]_\theta \overline{\vee} [y]_\theta]_R = [[x]_\theta]_R \sqcup [[y]_\theta]_R.$$

By the similar way, $[[x]_\theta]_R \in [[x]_\theta]_R \sqcap [[y]_\theta]_R$. Conversely, let $[[x]_\theta]_R \in [[x]_\theta]_R \sqcap [[y]_\theta]_R$. Then $[[x]_\theta]_R \in [[x \wedge y]_\theta]_R$ and so $[[x]_\theta]_R = [[u]_\theta]_\chi$, for some $u \in x \wedge y$. By definition of R, we conclude that $(x, u) \in \chi$ and so $[x]_\chi \in [x \wedge y]_\chi = [x]_\chi \overline{\overline{\wedge}} [y]_\chi$. Since χ is a compatible regular congruence relation on L, then $[x]_\chi \leqslant_\chi [y]_\chi$ and so there exists a χ-fence $\langle a_1, b_n \rangle_\chi$, that joins x to y, where $x = a_1$ and $y = b_n$. Clearly, $\langle [a_1]_\theta, [b_n]_\theta \rangle_R$ is a R-fence on L/θ and so $[[x]_\theta]_R \leqslant_R [[y]_\theta]_R$. By a similar way, $[[y]_\theta]_R \in [[x]_\theta]_R \sqcup [[y]_\theta]_R$ implies $[[x]_\theta]_R \leqslant_R [[y]_\theta]_R$. Therefore, R is a compatible regular congruence relation on L/θ. □

Theorem 3.18. *Let θ and χ be two regular compatible congruence relations on L such that $\theta \subseteq \chi$. Then $\frac{L/\theta}{\chi/\theta}$ and L/χ are S-isomorphic.*

Proof. By Theorem 3.10, $(L/\theta, \overline{\vee}, \overline{\wedge}, \overline{\odot}, \rightsquigarrow, [0]_\theta, [1]_\theta)$ and $(L/\chi, \overline{\vee}, \overline{\wedge}, \overline{\odot}, \rightsquigarrow, [0]_\chi, [1]_\chi)$ are two hyper residuated lattices. Let $\sqcup$, $\sqcap$, $\otimes$ and $\mapsto$ be the hyperoperations defined in Lemma 3.17. Then by Lemma 3.17 and Theorem 3.10, we see that $(\frac{L/\theta}{\chi/\theta}, \sqcup, \sqcap, \otimes, \mapsto, [[0]_\theta]_{\chi/\theta}, [[1]_\theta]_{\chi/\theta})$ is a hyper residuated lattice.

Define $f : \frac{L/\theta}{\chi/\theta} \to L/\chi$ by $f([[x]_\theta]_{\chi/\theta}) = [x]_\chi$. Let $[[x]_\theta]_{\chi/\theta} = [[y]_\theta]_{\chi/\theta}$, for some $x, y \in L$. Then by definition of χ/θ, we get $(x, y) \in \chi$ and so $[x]_\chi = [y]_\chi$. Hence f is well defined. Let $x, y \in L$. Then

$$\begin{aligned} f([[x]_\theta]_{\chi/\theta} \sqcap [[y]_\theta]_{\chi/\theta}) &= f([[x]_\theta \overline{\wedge} [y]_\theta]_{\chi/\theta}) \\ &= f([[x \wedge y]_\theta]_{\chi/\theta}) = \{f([[u]_\theta]_{\chi/\theta}) | u \in x \wedge y\} \\ &= \{[u]_\chi | u \in x \wedge y\} = [x \wedge y]_\chi = [x]_\chi \overline{\wedge} [y]_\chi \\ &= f([[x]_\theta]_{\chi/\theta}) \overline{\wedge} f([[y]_\theta]_{\chi/\theta}). \end{aligned}$$

By the similar way, we can show that

$$f([[x]_\theta]_{\chi/\theta} \sqcup [[y]_\theta]_{\chi/\theta}) = f([[x]_\theta]_{\chi/\theta}) \overline{\vee} f([[y]_\theta]_{\chi/\theta}),$$

$$f([[x]_\theta]_{\chi/\theta} \otimes [[y]_\theta]_{\chi/\theta}) = f([[x]_\theta]_{\chi/\theta}) \overline{\odot} f([[y]_\theta]_{\chi/\theta}),$$

$$f([[x]_\theta]_{\chi/\theta} \mapsto [[y]_\theta]_{\chi/\theta}) = f([[x]_\theta]_{\chi/\theta}) \rightsquigarrow f([[y]_\theta]_{\chi/\theta}).$$

Hence f is an S-homomorphism. Now, we show that f is one to one and onto. Clearly, f is an onto map. Let $f([[x]_\theta]_{\chi/\theta}) = f([[y]_\theta]_{\chi/\theta})$, for some $x, y \in L$. Then $[x]_\chi = [y]_\chi$ and so $(x, y) \in \chi$. Hence $[[x]_\theta]_{\chi/\theta} = [[y]_\theta]_{\chi/\theta}$ and so f is one to one. Therefore, f is an S-isomorphism. □

Remark 3.19. Let $(L_1; \vee_1, \wedge_1, \odot_1, \to_1, 0_1, 1_1)$ and $(L_2; \vee_2, \wedge_2, \odot_2, \to_2, 0_2, 1_2)$ be two hyper residuated lattices. We define the hyperoperations $\vee$, $\wedge$, $\to$ and $\odot$ on $L = L_1 \times L_2$ as follows:

$$\begin{aligned} (x_1, x_2) \vee (y_1, y_2) &= (x_1 \vee_1 y_1, x_2 \vee_2 y_2), \\ (x_1, x_2) \wedge (y_1, y_2) &= (x_1 \wedge_1 y_1, x_2 \wedge_2 y_2), \\ (x_1, x_2) \odot (y_1, y_2) &= (x_1 \odot_1 y_1, x_2 \odot_2 y_2), \\ (x_1, x_2) \to (y_1, y_2) &= (x_1 \to_1 y_1, x_2 \to_2 y_2). \end{aligned}$$

where $(A, B) = \{(a, b) \,|\, a \in A, b \in B\}$, for all subsets $A \subseteq L_1$ and $B \subseteq L_2$. Then $(L_1 \times L_2, \leqslant)$ satisfies (HRL1)-(HRL3) in which the order $\leq$ is given by

$$(a, b) \leqslant (c, d) \Leftrightarrow a \leqslant c, bd, \quad \forall a, c \in L_1, b, d \in L_2.$$

Hence $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a hyper residuated lattice, where $1 = (1, 1)$ and $0 = (0, 0)$.

Theorem 3.20. *If θ_1 and θ_2 are two regular compatible congruence relations on L_1 and L_2, respectively, and θ is a relation on $L_1 \times L_2$ defined by $(a, b)\theta(u, v)$ if and only if $(a, u) \in \theta_1$ and $(b, v) \in \theta_2$. Then θ is a regular compatible congruence relation on L and*

$$L/\theta \cong (L_1/\theta_1) \times (L_2/\theta_2).$$

Proof. Since θ_1 and θ_2 are regular compatible congruence relations on L_1 and L_2, respectively, then by Theorem 3.10, $(L_1/\theta_1, \leqslant_{\theta_1})$ and $(L_2/\theta_2, \leqslant_{\theta_2})$ are hyper residuated lattices. Let $\leqslant'$ be a partial order on $(L_1/\theta_1) \times (L_2/\theta_2)$, where $([x],[y]) \leqslant' ([a],[b])$ means that $[x] \leqslant_{\theta_1} [a]$ and $[y] \leqslant_{\theta_2} [b]$. Clearly, θ is a congruence relation on $L = L_1 \times L_2$. Let $\langle\langle (a_1,b_1),(c_1,d_1)\rangle\rangle_\theta$ be a θ-crown in L. Then by definition of $\leqslant$, we get $\langle\langle a_1, c_n\rangle\rangle$ is a θ_1-crown on L_1 and $\langle\langle b_1, d_n\rangle\rangle$ is a θ_2 crown on L_2. Since θ_1 is regular, then by Theorem 2.7, $a_i \cong c_j$, for all $i,j \in \{1,2,\ldots,n\}$. By a similar way, we can show that $b_i \cong d_j$, for all $i,j \in \{1,2,\ldots,n\}$. Hence $(a_i,b_i)\theta(c_i,d_i)$, for all $i,j \in \{1,2,\ldots,n\}$ and so by Theorem 2.7, θ is regular. Now, we show that θ is compatible. Let $[x]_i = \{a \in L_i \mid x\theta_i a\}$, for all $i \in \{1,2\}$. If $x,a \in L_1$, $y,b \in L_2$ and $\overline{\vee}$ $\overline{\wedge}$ are the hyperoperations on L induced by $\vee$ and $\wedge$, then we have

$$\begin{aligned}
[(x,y)] \in [(x,y)]\overline{\wedge}[(a,b)] &\Leftrightarrow [(x,y)] \in [(x \wedge_1 a, y \wedge_2 b)] \\
&\Leftrightarrow [x] \in [x \wedge_1 a]_1 \text{ and } [y] \in [y \wedge_1 b]_2 \\
&\Leftrightarrow x \leqslant_1 a, y \leqslant_2 b, \text{ since } \theta_1 \text{ and } \theta_2 \text{ are compatible} \\
&\Leftrightarrow (x,y) \leqslant (a,b).
\end{aligned}$$

By a similar way, we can show that $[(x,y)] \in [(x,y)]\overline{\vee}[(a,b)] \Leftrightarrow (x,y) \leqslant (a,b)$. Hence θ is compatible and so by Theorem 3.10, L/θ is a hyper residuated lattice. Define the map $f : L \to (L_1/\theta_1) \times (L_2/\theta_2)$, by $f((x,y)) = ([x]_1,[y]_2)$, for any $(x,y) \in L$. Let $* \in \{\vee, \wedge, \odot, \to\}$. Then

$$\begin{aligned}
f((x,y) * (a,b)) &= f(x * a, y * b) \\
&= ([x *_1 a]_1, [y *_2 b]_2) \\
&= ([x]_1 *_1 [a]_1, [y]_2 *_2 [b]_2) \\
&= ([x]_1,[y]_2) * ([a]_1,[b]_2) \\
&= f((x,y)) * f((a,b)).
\end{aligned}$$

Hence f is a S-homomorphism. Clearly, f is onto. Now, we show that $ker(f) = \theta$.

$$\begin{aligned}
ker(f) &= \{((x,y),(a,b)) \in L \times L \mid f((x,y)) = f((a,b))\} \\
&= \{((x,y),(a,b)) \in L \times L \mid ([x]_1,[y]_2) = ([a]_1,[b]_2)\} \\
&= \{((x,y),(a,b)) \in L \times L \mid [x]_1 = [a]_1, \quad [y]_1 = [b]_1\} \\
&= \theta.
\end{aligned}$$

Now, let $f((x,y)) \leqslant' f((a,b))$. Then $([x]_1,[y]_2) \leqslant' ([a]_1,[b]_2)$ and so $[x]_1 \leqslant_{\theta^1} [a]_1$ and $[y]_2 \leqslant_{\theta_2} [b]_2$. Hence by definition of $\leqslant_{\theta_1}$ and $\leqslant_{\theta_2}$, there are $\langle u_1, v_n\rangle_{\theta_1}$, that joins x to a and $\langle w_1, z_m\rangle_{\theta_2}$, that joins y to b. Without loss of generality, we assume that $n \leqslant m$. Then the set

$$\{(u_1,w_1),(v_2,z_2)\ldots,(v_n,z_n),(v_n,w_{n+1})(v_n,z_{n+1}),\ldots, \\ \ldots,(v_n,z_{m-1}),(v_n,w_{m-1}),(v_n,z_m)\}$$

is a θ-fence that joins (x,y) to (a,b). Hence by Theorem 3.14 we obtain $L/\theta = L/ker(f) \cong (L_1/\theta_1) \times (L_2/\theta_2)$, which completes the proof. □

References

[1] **T. S. Blyth**, *Lattices and ordered algebraic structures*, Springer-Verlag, 2005.

[2] **C. C. Chang**, *Algebraic analysis of many valued logics*, Trans. Am. Math. Soc. **88** (1958), 467 – 490.

[3] **B. A. Davey, H. A. Priestley**, *Introduction to lattices and order*, Cambridge Univ. Press, 2002.

[4] **P. Hájek**, *Metamathematics of fuzzy logic*, Kluwer Academic Publ., Dordrecht, 1998.

[5] **F. Marty**, *Sur une generalization de la notion de groups*, 8th Congress Math. Scandinaves, Stockholm, 1934, 45 – 49.

[6] **J. Mittas, M. Konstantinidou**, *Sur une nouvelle génération de la notion de treillis. Les supertreillis et certaines de leurs propriťés générales.* Ann. Sci. Univ. Blaise Pascal (Clermont II), Sér. Math. Fasc. **25** (1989), 61 – 83.

[7] **M. Ward, R. P. Dilworth**, *Residuated lattices.* Trans. Amer. Math. Soc. **45** (1939), 335 – 354.

Received November 29, 2011

O. Zahiri and R. Borzooei
Department of Mathematics, Shahid Beheshti University, Evin, Tehran, Iran
E-mail: o.zahiri@yahoo.com, borzooei@sbu.ac.ir

M. Bakhshi
Department of Mathematics, Bojnord University, Bojnord, Iran
E-mail: bakhshi@ub.ac.ir