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On the properties of zero-divisor graphs of posets

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Abstract. We determine the cut vertices in the zero-divisor graphs of posets and study the posets with end-regular zero-divisor graph. Also, we investigate the zero-divisor graph of the product of two posets. In particular, we determine all posets with planar and outerplanar zero-divisor graphs.

1. Introduction

The investigation of graphs related to various algebraic structures is a very large and growing area of research. In particular, Cayley graphs have attracted serious attention in the literature, since they have many useful applications, see [13], [16], [17], [20], [21], [24] for examples of recent results and further references. Several other classes of graphs associated with algebraic structures have been also actively investigated. For example, power graphs and divisibility graphs have been considered in [14], [15], zero-divisor graphs have been studied in [3], [4], [5], [8], [9], and cozero-divisor graphs and annihilating-ideal graphs have been considered in [1] and [2], respectively.

Recently, the zero-divisor graph of a poset was defined and studied in [11], [12], [19] and [23]. In this paper, we deal with the zero-divisor graphs of posets based on terminology of [19]. In [19], Lu and Wu defined the zero-divisor graph for an arbitrary partially ordered set $(P, \leqslant)$ (poset, briefly) with a least element 0, as an undirected graph whose vertices consists of all nonzero zero-divisors of P, and two distinct vertices x and y are adjacent if and only if $\{x, y\}^\ell = \{0\}$, where for a subset S of P, $\{S\}^\ell$ denotes the set of lower bounds of S. In this paper, we denote this graph by $\Gamma(P)$. In Section 2, we study the cut vertices in $\Gamma(P)$. Also, we investigate some basic properties of $\Gamma(P_1 \times P_2)$, where P_1 and P_2 are two finite posets. In Section 3, we study the planarity of $\Gamma(P_1 \times P_2)$. In Section 4, we investigate the outerplanarity in the zero-divisor graphs of posets. In the last section, we study the posets with end-regular zero-divisor graphs.

Now we recall some definitions and notations on graphs and partially ordered sets. We use the standard terminology of graphs in [6] and partially ordered sets in [7]. Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. In a graph G, the *distance* between two distinct vertices a and b, denoted by $\mathrm{d}(a, b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise,

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we set $\mathrm{d}(a, b) := \infty$. The *diameter* of a graph G is $\mathrm{diam}(G) = \sup\{\mathrm{d}(a, b) : a$ and b are distinct vertices of $G\}$. A graph G is said to be *connected* if there exists a path between any two distinct vertices, and it is *complete* if it is connected with diameter one. We use K_n to denote the complete graph with n vertices. Also, we say that G is *totally disconnected* if no two vertices of G are adjacent. The *valency* of a vertex a is the number of the edges of the graph G incident with a. A *clique* of a graph is a maximal complete subgraph of it and the number of vertices in a largest clique of G is called *clique number* of G and is denoted by $\omega(G)$. In the graph theory, a *unicycle graph* is a graph that has exactly one cycle. The graph with no vertices and no edges is the *null graph*.

In a partially ordered set $(P, \leqslant)$ with a least element 0, an element a is called an *atom* if $a \neq 0$, and, for an element x in P, the relation $0 \leqslant x \leqslant a$ implies either $x = 0$ or $x = a$. Also, for $a, b \in P$, we say that $a < b$, whenever $a \leqslant b$ and $a \neq b$. Assume that S is a subset of P. Then an element x in P is a *lower bound* of S if $x \leqslant s$ for all $s \in S$. An *upper bound* is defined in a dual manner. The set of all lower bounds of S is denoted by S^ℓ and the set of all upper bounds of S by S^u, that is,

$$S^\ell := \{x \in P \mid x \leqslant s, \text{ for all } s \in S\}$$

and

$$S^u := \{x \in P \mid s \leqslant x, \text{ for all } s \in S\}.$$

We say that a non-empty subset I of P is an *ideal* of P if, for arbitrary elements x and y in P, the relations $x \in I$ and $y \leqslant x$ imply that $y \in I$. Also the ideal I is *prime* if $x, y \in P$ with $\{x, y\}^\ell \subseteq I$, then $x \in I$ or $y \in I$. A *maximal* ideal of P is a proper ideal of P which is maximal among all ideals of P.

2. Cut vertices in the zero-divisor graph of a poset

Throughout the paper, P is a finite poset and $A(P) = \{a_1, a_2, ..., a_n\}$ is the set of all atoms of P. Also, we denote the set of zero-divisors of the poset P by $Z(P)$, that is,

$$Z(P) = \{x \in P \mid \{x, y\}^\ell = 0, \text{ for some } y \in P\}.$$

Clearly, if $|A(P)| = 1$, then $\Gamma(P)$ is a null graph. Therefore, we suppose that $|A(P)| \geqslant 2$.

A vertex a of a graph G is called a *cut vertex* if the removal of a and any edges incident on a creates a graph with more connected components than G.

Theorem 2.1. *If a is a cut vertex in $\Gamma(P)$, then $\{0, a\}$ is an ideal of P.*

Proof. One can easily see that $\{0, a\}$ is an ideal of P if and only if a is an atom of P. Hence it is sufficient to show that $a = a_i$, for some $i = 1, 2, \ldots, n$. Assume that a is not an atom. Since a is a cut vertex, $\Gamma(P)\backslash\{a\}$ has at least two components X and Y. We claim that $A(P) \subseteq X$ or $A(P) \subseteq Y$. Otherwise there are atoms

a_i and a_j, where $1 \leqslant i \neq j \leqslant n$, such that $a_i \in X$ and $a_j \in Y$. Now we have that a_i is adjacent to a_j, which is impossible. Without loss of generality, we may assume that $A(P) \subseteq X$. Then, for all $y \in Y$, we have $y \in \{a_i\}^u$, for $i = 1, 2, \ldots, n$. Thus $y \in \cap_{i=1}^{n} \{a_i\}^u$. This implies that $y \notin Z(P)$, which is impossible. Therefore $a \in A(P)$, and so $\{0, a\}$ is an ideal of P. □

The following example shows that the converse of Theorem 2.1 is not true in general.

Example 2.2. Suppose that P is a poset in Figure 1. Then, it is easy to see that a_1 is an atom, but it is not a cut vertex in $\Gamma(P)$.

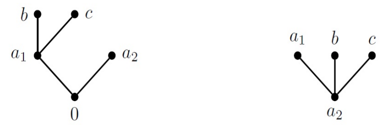

Figure 1. P and $\Gamma(P)$

Notation. Let $i_1, i_2, \ldots, i_n$ be integers with $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$. The notation $U^P_{i_1 i_2 \ldots i_k}$ stands for the following set:

$$\{x \in P; \quad x \in \cap_{s=1}^{k} \{a_{i_s}\}^u \setminus \cup_{j \neq i_1, i_2, \ldots, i_k} \{a_j\}^u\}$$

Note that no two distinct elements in $U_{i_1 i_2 \ldots i_k}$ are adjacent in $\Gamma(P)$. Also if the index sets $\{i_1, i_2, \ldots, i_k\}$ and $\{j_1, j_2, \ldots, j_{k'}\}$ of $U_{i_1 i_2 \ldots i_k}$ and $U_{j_1 j_2 \ldots j_{k'}}$, respectively, are distinct, then one can easily check that $U_{i_1 i_2 \ldots i_k} \cap U_{j_1 j_2 \ldots j_{k'}} = \emptyset$. Moreover $P \setminus \{0\} = \cup_{k=1,\ 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n}^{n} U_{i_1 i_2 \cdots i_k}$. Also, if there is no ambiguity, we denote $U^P_{i_1 i_2 \ldots i_k}$ by $U_{i_1 i_2 \ldots i_k}$. Also by $1 \cdots \hat{i} \cdots n$ we means that $1 \cdots i-1\ i+1 \cdots n$.

In the next theorem, we provide some conditions under which the converse of Theorem 2.1 holds.

Theorem 2.3. *Let $|P| \geqslant 4$. Then there exists i with $1 \leqslant i \leqslant n$ such that a_i is a cut vertex in $\Gamma(P)$, if $|U_i| = 1$ and $U_{1 \cdots \hat{i} \cdots n} \neq \emptyset$, for some $1 \leqslant i \leqslant n$.*

Proof. It is enough to show that there exist vertices b and c in P such that a_i is in every path from b to c in $\Gamma(P)$. Since $U_{1 \cdots \hat{i} \cdots n} \neq \emptyset$, there is an element b in $U_{1 \cdots \hat{i} \cdots n}$. Now, for some $j \neq i$, consider $c \in U_j$. Thus a_i is in every path from b to c in $\Gamma(P)$, and so it is a cut vertex in $\Gamma(P)$. □

Proposition 2.4. *Let a be a cut vertex in $\Gamma(P)$ and X be connected component of $\Gamma(P) \setminus \{a\}$. Also suppose that X is complete with at least two vertices. Then $V(X) \cup \{0\}$ is an ideal of P.*

Proof. Since a is a cut vertex in $\Gamma(P)$, by Theorem 2.1, a is an atom of P. Suppose that $a = a_1$. Now, we have the following cases:

Case 1. $A(P)\backslash\{a\} \subseteq X$. If X contains an element b such that b is not an atom, then since X is complete, we have that $b \in U_1$. Now, let $Y \neq X$ be another connected components of $\Gamma(P)\backslash\{a\}$ and let $c \in Y$. Clearly, $c \in U_{23\ldots n}$. Thus b and c are adjacent which is impossible. So we have that $X = A(P)\backslash\{a\}$, and thus $V(X) \cup \{0\}$ is an ideal of P.

Case 2. $A(P)\backslash\{a\} \nsubseteq X$. It is easy to see that in this situation X does not contain any atom. Now, let x and y be distinct elements in X. Then we have $x, y \in U_{23\ldots n}$, and so x is not adjacent to y, which is impossible. Therefore this case does not happen. □

The next example shows that the converse of Proposition 2.4 is not true in general.

Example 2.5. Suppose that P is a poset of Figure 2. Clearly a_1 is the cut vertex in $\Gamma(P)$. Let $V(X) = \{a_2, a_3, c\}$. Then, by Figure 2, it is easy to see that $V(X) \cup \{0\}$ is an ideal of P, but X is not a complete subgraph of $\Gamma(P)$.

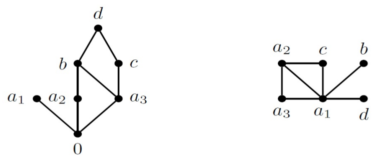

Figure 2. P and $\Gamma(P)$

Definition 2.6. Suppose that x is a vertex in $\Gamma(P)$. Set

$$Z_x := \{y \in P \mid \{x, y\}^\ell = \{0\}\}.$$

We say that Z_x is properly maximal if $Z_x \subseteq Z_b$, for some $b \in P\backslash\{0, x\}$, then we have $Z_x = Z_b$.

Theorem 2.7. *If a is a cut vertex in $\Gamma(P)$, then Z_a is properly maximal.*

Proof. Assume on the contrary that $Z_a \subsetneqq Z_b$, for some vertices b in $\Gamma(P)$ with $b \neq a$. Then clearly all vertices adjacent to a are also adjacent to b. This is a contradiction with the fact that a is a cut vertex. □

Let $(P_1, \leqslant_1)$ and $(P_2, \leqslant_2)$ be two posets with the least elements. Then the cartesian product $P_1 \times P_2$ is also a poset with the following relation. For two distinct elements $(x, y), (x', y') \in P_1 \times P_2$ we say that $(x, y) \leqslant (x', y')$ if and only if $x \leqslant_1 x'$ and $y \leqslant_2 y'$. Clearly $(P_1 \times P_2, \leqslant)$ has the minimum element $(0, 0)$. Suppose that P_1 and P_2 are two finite posets such that $A(P_1) = \{a_1, a_2, \ldots, a_n\}$ and $A(P_2) = \{b_1, b_2, \ldots, b_m\}$. In the following we study some properties of the zero-divisor graph $\Gamma(P_1 \times P_2)$.

Lemma 2.8. *In the poset $P_1 \times P_2$, we have $A(P_1 \times P_2) = (A(P_1) \times \{0\}) \cup (\{0\} \times A(P_2))$, and so $|A(P_1 \times P_2)| = |A(P_1)| + |A(P_2)|$.*

Proof. Suppose that (a, b) belongs to the set $A(P_1 \times P_2)$. If $a, b \neq 0$, then we have $(0, 0) < (a, 0) < (a, b)$ which is impossible. Then we have $a = 0$ or $b = 0$. Without loss of generally, we may assume that $b = 0$. If $a \notin A(P_1)$, then there exists an atom $a_i \in A(P_1)$, for some $1 \leqslant i \leqslant n$, such that $a_i < a$. Hence we have that $(0, 0) < (a_i, 0) < (a, 0)$ which is impossible. Thus $a \in A(P_1)$, and so the result holds. $\square$

We can extend the concept of $P_1 \times P_2$ for a product of finite number of posets.

Corollary 2.9. *Let $P = P_1 \times P_2 \times \cdots \times P_n$, where $(P_i, \leqslant_i)$'s are partially ordered sets for $i = 1, 2, \ldots, n$. Then $A(P)$ consists of elements $(a_1, a_2, \ldots, a_n)$ such that there exists $1 \leqslant j \leqslant n$ with $a_j \in A(P_j)$, and, for all i with $1 \leqslant i \neq j \leqslant n$, $a_i = 0$.*

Proposition 2.10. *Let $P = P_1 \times P_2 \times \cdots \times P_n$ be a poset such that $P \neq P_1 \times P_2$, with $|P_1| = |P_2| = 2$. If $a = (0, 0, \ldots, u_i, 0, \ldots, 0) \in Z(P)$ is a cut vertex with nonzero component u_i such that $u_i \notin Z(P_i)$, then $|P_i| = 2$.*

Proof. Assume on the contrary that P_i has at least three elements and so there exists v_i in $P_i \backslash \{0, u_i\}$. It is easy to see that $Z_a \subseteq Z_{(0,0,\ldots,v_i,0,\ldots,0)}$. Since a is a cut vertex, by Theorem 2.7, we have that $Z_a = Z_{(0,0,\ldots,v_i,0,\ldots,0)}$, which implies that $a = (0, 0, \ldots, v_i, 0, \ldots, 0)$. Hence $u_i = v_i$, which is a contradiction. $\square$

3. Planarity of $\Gamma(P_1 \times P_2)$

Recall that a graph is said to be *planar* if it can be drown in the plane, so that its edges intersect only at their ends. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

Theorem 3.1. *If $\Gamma(P_1 \times P_2)$ is planar, then $|A(P_1)| + |A(P_2)| \leqslant 4$.*

Proof. Suppose on the contrary that $|A(P_1)| + |A(P_2)| \geqslant 5$. Since the induced subgraph of $\Gamma(P_1 \times P_2)$ on the vertex-set $A(P_1 \times P_2)$ is a complete graph, one can find a subgraph of $\Gamma(P_1 \times P_2)$ isomorphic to K_5, and so, by Kuratowski's Theorem, $\Gamma(P_1 \times P_2)$ is not planar. Hence we have $|A(P_1)| + |A(P_2)| \leqslant 4$. $\square$

By Theorem 3.1, we must study the cases that $|A(P_1)| + |A(P_2)|$ is equal to 2, 3 and 4. In the following proposition, we state the necessary and sufficient condition for planarity of $\Gamma(P_1 \times P_2)$, when $|A(P_1)| + |A(P_2)| = 2$.

Proposition 3.2. *Suppose that $|A(P_1)| + |A(P_2)| = 2$ such that $|A(P_1)| = 1 = |A(P_2)|$. Then $\Gamma(P_1 \times P_2)$ is planar if and only if $|P_1| \leqslant 3$ or $|P_2| \leqslant 3$.*

Proof. Since $|A(P_1)| + |A(P_2)| = 2$, we have that $\Gamma(P_1 \times P_2)$ is a complete bipartite graph. Now one can easily see that $\Gamma(P_1 \times P_2)$ is planar if and only if $|P_1| \leqslant 3$ or $|P_2| \leqslant 3$. $\square$

Now, suppose that P_1 and P_2 are posets such that $|A(P_1)| + |A(P_2)| = 3$. Let $|A(P_1)| = 1$ and $|A(P_2)| = 2$. If $|P_1|, |P_2| \geqslant 4$, then we can find a copy of $K_{3,3}$ in the graph $\Gamma(P_1 \times P_2)$. Thus, by Kuratowski's Theorem, $\Gamma(P_1 \times P_2)$ is not planar. Therefore, if $\Gamma(P_1 \times P_2)$ is planar, then $|P_1| \leqslant 3$ or $|P_2| \leqslant 3$. Now, we have the following cases:

Case 1. Suppose that $|P_1| = 2$ and $|U_i^{P_2}| \geqslant 2$, for all $1 \leqslant i \leqslant 2$. In this situation we can find a subdivision of K_5 as in Figure 3, where $y_i \in U_i^{P_2} \backslash \{b_i\}$, for all $1 \leqslant i \leqslant 2$, and so $\Gamma(P_1 \times P_2)$ is not planar.

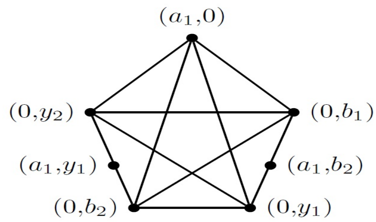

Figure 3.

If $|U_i^{P_2}| = 1$ and $|U_j^{P_2}| \geqslant 3$, for some $1 \leqslant i \neq j \leqslant 2$, then one can find a copy of $K_{3,3}$ with vertex-set $\{(a_1, 0), (0, b_1), (a_1, b_1)\} \cup \{(0, b_2), (0, y_2), (0, y_2')\}$, where $y_i, y_i' \in U_i^{P_2} \backslash \{b_i\}$, for all $1 \leqslant i \leqslant 2$, and so $\Gamma(P_1 \times P_2)$ is not planar.

Now, if $|U_i^{P_2}| = 1$ and $|U_j^{P_2}| \leqslant 2$, for all $1 \leqslant i \neq j \leqslant 2$, then $\Gamma(P_1 \times P_2)$ is pictured in Figure 4, and so $\Gamma(P_1 \times P_2)$ is planar.

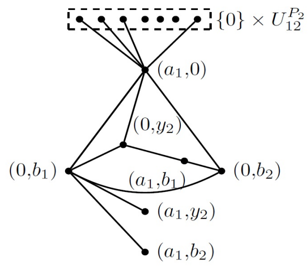

Figure 4.

Case 2. Suppose that $|P_1| = 3$ and $|U_i^{P_2}| \geqslant 3$, for some $1 \leqslant i \leqslant 2$. In this situation one can find a copy of $K_{3,3}$ with vertex-set $\{(a_1, 0), (0, b_2), (x, b_2)\} \cup \{(0, b_1), (0, y_1), (0, y_1')\}$, where $x \in P_1 \backslash \{0, a_1\}$ and $y_i, y_i' \in U_i^{P_2} \backslash \{b_i\}$ for some $1 \leqslant i \leqslant 2$, and so $\Gamma(P_1 \times P_2)$ is not planar.

Now, if $|U_i^{P_2}| \leqslant 2$, for all $1 \leqslant i \leqslant 2$, then one of the following situations happen:

(i) If $|U_i^{P_2}| = 2$, for all $1 \leqslant i \leqslant 2$, then we can find a subdivision of K_5 as in Figure 3, where $y_i \in U_i^{P_2} \backslash \{b_i\}$ for all $1 \leqslant i \leqslant 2$, and so $\Gamma(P_1 \times P_2)$ is not planar.

(ii) If $|U_i^{P_2}| = 2$, $|U_j^{P_2}| = 1$, for all $1 \leqslant i \neq j \leqslant 2$ and $U_{12}^{P_2} \neq \emptyset$, then we can find a subdivision of K_5 as in Figure 5, where $y_i \in U_i^{P_2} \backslash \{b_i\}$ for some $1 \leqslant i \leqslant 2$ and $c_{12} \in U_{12}^{P_2}$. So $\Gamma(P_1 \times P_2)$ is not planar.

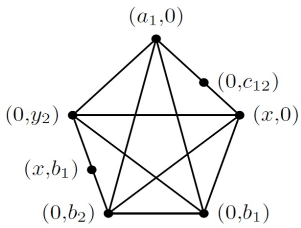

Figure 5.

If $|U_i^{P_2}| = 2$, $|U_j^{P_2}| = 1$, for all $1 \leqslant i \neq j \leqslant 2$ and $U_{12}^{P_2} = \emptyset$, then $\Gamma(P_1 \times P_2)$ is pictured in Figure 6, and so $\Gamma(P_1 \times P_2)$ is planar.

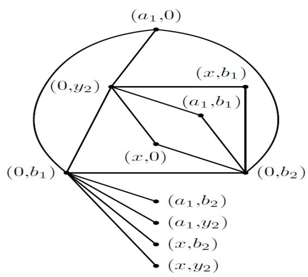

Figure 6.

(iii) If $|U_i^{P_2}| = 1$, for all $1 \leqslant i \leqslant 2$, then $\Gamma(P_1 \times P_2)$ is pictured in Figure 7, and so $\Gamma(P_1 \times P_2)$ is planar.

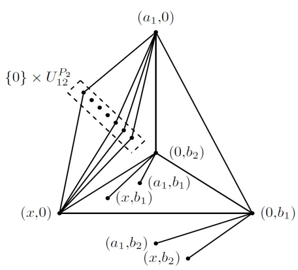

Figure 7.

Case 3. Suppose that $|P_2| = 3$. In this situation $\Gamma(P_1 \times P_2)$ is pictured in Figure 8, and hence $\Gamma(P_1 \times P_2)$ is planar.

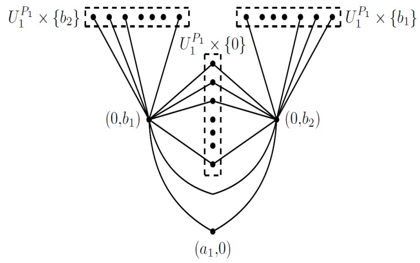

Figure 8.

Thus we have the following theorem.

Theorem 3.3. *Suppose that* $|A(P_1)| + |A(P_2)| = 3$ *such that* $|A(P_1)| = 1$ *and* $|A(P_2)| = 2$. *Then* $\Gamma(P_1 \times P_2)$ *is planar if and only if one of the following conditions hold.*

(*i*) $|P_1| = 2$, $|U_i^{P_2}| = 1$ *and* $|U_j^{P_2}| \leqslant 2$, *for all* $1 \leqslant i \neq j \leqslant 2$.

(*ii*) $|P_1| = 3$ *and* $|U_i^{P_2}| = 1$, *for all* $1 \leqslant i \leqslant 2$.

(*iii*) $|P_1| = 3$, $|U_i^{P_2}| = 2$ *and* $|U_j^{P_2}| = 1$, *for some* $1 \leqslant i \neq j \leqslant 2$ *and* $U_{12}^{P_2} = \emptyset$.

(*iv*) $|P_2| = 3$.

Finally, in order to complete the study of planarity of $\Gamma(P_1 \times P_2)$, we assume that $|A(P_1)| + |A(P_2)| = 4$. Now, we have the following cases:

Case 1. Suppose that $|A(P_1)| = 1$ and $|A(P_2)| = 3$. In this situation if $\Gamma(P_1 \times P_2)$ is planar, then $|P_1| \leqslant 3$. Note that if $\Gamma(P_1 \times P_2)$ is planar and $|P_1| \geqslant 4$, then one can find a copy of $K_{3,3}$ with vertex-set $\{(a_1, 0), (x, 0), (x', 0)\} \cup \{(0, b_1), (0, b_2), (0, b_3)\}$, where $x, x' \in P_1 \backslash \{0, a_1\}$. Thus $\Gamma(P_1 \times P_2)$ is not planar. Therefore $|P_1| \leqslant 3$.

Now, we investigate the planarity of $\Gamma(P_1 \times P_2)$ whenever, $|P_1| \leqslant 3$. To this end, we consider the following situations:

(i) Suppose that $|P_1| = 2$. If $|U_i^{P_2}| \geqslant 2$, for some $1 \leqslant i \leqslant 3$, then we can find a subdivision of K_5 as in Figure 9, where $y_i \in U_i^{P_2} \backslash \{b_i\}$ for some $1 \leqslant i \leqslant 3$.

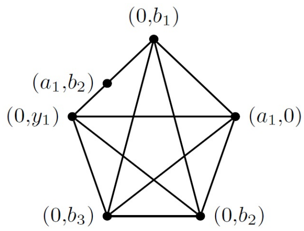

Figure 9.

If $|U_{ij}^{P_2}| \geqslant 1$, for some $1 \leqslant i \neq j \leqslant 3$, then one can find a copy of $K_{3,3}$ with vertex-set $\{(a_1,0),(0,b_3),(a_1,b_3)\} \cup \{(0,b_1),(0,b_2),(0,c_{12})\}$, where $c_{ij} \in U_{ij}^{P_2}$ for some $1 \leqslant i \neq j \leqslant 3$. So $\Gamma(P_1 \times P_2)$ is not planar.

Now, if $|U_i^{P_2}| = 1$, for all $1 \leqslant i \leqslant 3$ and $U_{ij}^{P_2} = \emptyset$, for all $1 \leqslant i \neq j \leqslant 3$, then $\Gamma(P_1 \times P_2)$ is pictured in Figure 10, and so $\Gamma(P_1 \times P_2)$ is planar.

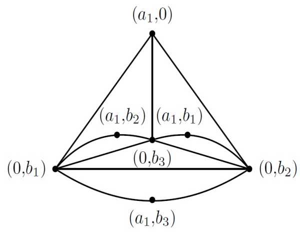

Figure 10.

(ii) Assume that $|P_1| = 3$. If $|U_i^{P_2}| \geqslant 2$, for some $1 \leqslant i \leqslant 3$, then we can find a subdivision of K_5 as in Figure 9, where $y_i \in U_i^{P_2} \backslash \{b_i\}$ for some $1 \leqslant i \leqslant 3$.

If $|U_{ij}^{P_2}| \geqslant 1$, for some $1 \leqslant i \neq j \leqslant 3$, then one can find a copy of $K_{3,3}$ with vertex-set $\{(a_1,0),(0,b_3),(a_1,b_3)\} \cup \{(0,b_1),(0,b_2),(0,c_{12})\}$, where $c_{ij} \in U_{ij}^{P_2}$, for some $1 \leqslant i \neq j \leqslant 3$. So $\Gamma(P_1 \times P_2)$ is not planar.

If $U_{123}^{P_2} \neq \emptyset$, then we can find a subdivition of K_5 as in Figure 11, where $c_{123} \in U_{123}^{P_2}$. So $\Gamma(P_1 \times P_2)$ is not planar.

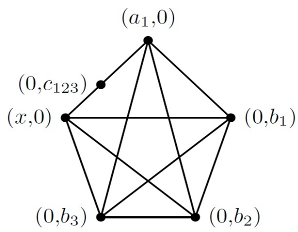

Figure 11.

Now, if $|U_i^{P_2}| = 1$, for all $1 \leqslant i \leqslant 3$ and $U_{i\dots j}^{P_2} = \emptyset$, for all $1 \leqslant i \neq j \leqslant 3$, then $\Gamma(P_1 \times P_2)$ is pictured in Figure 12, and so $\Gamma(P_1 \times P_2)$ is planar.

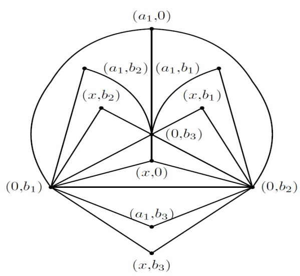

Figure 12.

Case 2. Assume that $|A(P_1)| = 2 = |A(P_2)|$. In this situation we can find a subdivision of $K_{3,3}$ as in Figure 13, and so $\Gamma(P_1 \times P_2)$ is not planar.

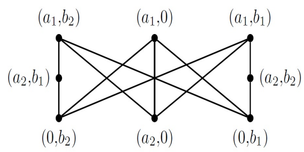

Figure 13.

Hence we have the following theorem.

Theorem 3.4. *Suppose that* $|A(P_1)| + |A(P_2)| = 4$ *such that* $|A(P_1)| = 1$ *and* $|A(P_2)| = 3$. *Then* $\Gamma(P_1 \times P_2)$ *is planar if and only if one of the following conditions hold.*

(*i*) $|P_1| = 2$ *and* $|U_i^{P_2}| = 1$ *for all* $1 \leqslant i \leqslant 3$ *and* $U_{ij}^{P_2} = \emptyset$ *for all* $1 \leqslant i \neq j \leqslant 3$.

(*ii*) $|P_1| = 3$, $|U_i^{P_2}| = 1$ *for all* $1 \leqslant i \leqslant 3$ *and* $U_{i\ldots j}^{P_2} = \emptyset$ *for all* $1 \leqslant i \neq j \leqslant 3$.

4. Outerplanarity of $\Gamma(P)$ and $\Gamma(P_1 \times P_2)$

A directed graph is *outerplanar* if it can be drawn in the plane without crossing in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization for outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$.

In the following, we characterize all posets P such that $\Gamma(P)$ is outerplanar.

Lemma 4.1. *If* $\Gamma(P)$ *is outerplanar, then* $|A(P)| \leqslant 3$.

Proof. Assume to the contrary that $|A(P)| \geqslant 4$. Since the induced subgraph of $\Gamma(P)$ on vertex-set $A(P)$ is a complete subgraph, one can find a copy of K_4 in $\Gamma(P)$, and so $\Gamma(P)$ is not outerplanar. Hence we have $|A(P)| \leqslant 3$. □

By Lemma 4.1, we must study the cases that $|A(P)|$ is equal to 2 and 3. In the following proposition, we state the necessary and sufficient condition for outerplanarity of $\Gamma(P)$, when $|A(P)| = 2$.

Proposition 4.2. *Suppose that* $|A(P)| = 2$. *Then* $\Gamma(P)$ *is outerplanar if and only if* $|U_i| = 1$, *for some* $1 \leqslant i \leqslant 2$, *or* $|U_i| \leqslant 2$, *for all* $1 \leqslant i \leqslant 2$.

Proof. Since $|A(P)| = 2$, we have that $\Gamma(P)$ is a complete bipartite graph. Now one can easily see that $\Gamma(P)$ is outerplanar if and only if $|U_i| = 1$, for some $1 \leqslant i \leqslant 2$, or $|U_i| \leqslant 2$, for all $1 \leqslant i \leqslant 2$. □

In the sequel of this section, we investigate the outerplanarity of $\Gamma(P)$, when $|A(P)| = 3$. If $|\cup_{i=1}^{3} U_i| \geqslant 5$, then we can find a copy of $K_{2,3}$ in the structure of $\Gamma(P)$, and so $\Gamma(P)$ is not outerplanar. Therefore, if $\Gamma(P)$ is outerplanar, then $|\cup_{i=1}^{3} U_i| \leqslant 4$. Now, we have the following cases:

Case 1. Suppose that $|\cup_{i=1}^{3} U_i| = 3$. In this situation $\Gamma(P)$ is a unicyclic graph which is in pictured in Figure 14, and so it is outerplanar.

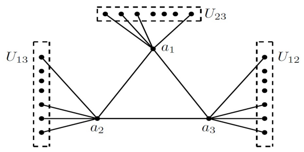

Figure 14.

Case 2. Suppose that $|\cup_{i=1}^{3} U_i| = 4$. Suppose that $|U_i| = 2$. If $|U_{jk}| \geqslant 1$, for some $1 \leqslant i \neq j \neq k \leqslant 3$, then we can find a copy of $K_{2,3}$ with vertex-set $\{a_1, a_1'\} \cup \{a_2, a_3, c_{23}\}$, where $a_i' \in U_i \backslash \{a_i\}$ and $c_{jk} \in U_{jk}$, for some $1 \leqslant i \neq j \neq k \leqslant 3$, and so $\Gamma(P)$ is not outerplanar.

Now, if $U_{jk} = \emptyset$, for all $1 \leqslant i \neq j \neq k \leqslant 3$, then $\Gamma(P)$ is isomorphic to the graph which is pictured in Figure 15, and so $\Gamma(P)$ is outerplanar.

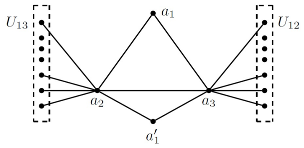

Figure 15.

Theorem 4.3. *Suppose that $|A(P)| = 3$. Then $\Gamma(P)$ is outerplanar if and only if one of the following conditions holds:*

(*i*) $|\cup_{i=1}^{3} U_i| = 3$.

(*ii*) $|\cup_{i=1}^{3} U_i| = 4$ *and if* $|U_i| = 2$, *for some* $1 \leqslant i \leqslant 3$, *then* $U_{jk} = \emptyset$, *for all* $1 \leqslant i \neq j \neq k \leqslant 3$.

In the following, we characterize all posets P_1 and P_2 such that $\Gamma(P_1 \times P_2)$ is outerplanar. Clearly, if $\Gamma(P_1 \times P_2)$ is outerplanar, then, by Lemmas 2.8 and 4.1, $|A(P_1)| + |A(P_2)| \leqslant 3$. In the next two Theorems, we investigate the cases $|A(P_1)| + |A(P_2)| = 2$ and $|A(P_1)| + |A(P_2)| = 3$.

Theorem 4.4. *Suppose that $|A(P_1)| + |A(P_2)| = 2$ such that $|A(P_1)| = 1 = |A(P_2)|$. Then $\Gamma(P_1 \times P_2)$ is outerplanar if and only if $|P_i| \leqslant 2$ or, $|P_j| \leqslant 3$ with $|P_i| \leqslant 2$, for some $1 \leqslant i \neq j \leqslant 2$.*

Proof. Since $|A(P_1)| + |A(P_2)| = 2$, we have that $\Gamma(P_1 \times P_2)$ is a complete bipartite graph. Now one can easily see that $\Gamma(P_1 \times P_2)$ is an outerplanar graph if and only if $|P_i| \leqslant 2$ or, $|P_j| \leqslant 3$ and $|P_i| \leqslant 2$, for some $1 \leqslant i \neq j \leqslant 2$. □

Now, suppose that P_1 and P_2 are posets such that $|A(P_1)| = 1$ and $|A(P_2)| = 2$. If $|P_i| \geqslant 3$ and $|P_j| \geqslant 4$, for all $1 \leqslant i \neq j \leqslant 2$, then we can find a copy of $K_{2,3}$ in the graph $\Gamma(P_1 \times P_2)$. Thus $\Gamma(P_1 \times P_2)$ is not outerplanar. Therefore, if $\Gamma(P_1 \times P_2)$ is outerplanar, then $|P_1| = 2$, or $|P_2| = 3$ with $|P_1| \leqslant 3$. Now, in the following two cases, we study the outerplanarity of $\Gamma(P_1 \times P_2)$ whenever $|P_1| = 2$, or $|P_1| \leqslant 3$ with $|P_2| = 3$.

Case 1. Suppose that $|P_1| = 2$ and $|U_i^{P_2}| \geqslant 2$, for some $1 \leqslant i \leqslant 2$. In this case we can find a subdivision of K_4 as in Figure 16, where $y_i \in U_i^{P_2} \setminus \{b_i\}$, and so $\Gamma(P_1 \times P_2)$ is not outerplanar.

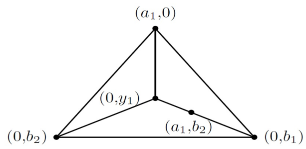

Figure 16.

Now, if $|U_i^{P_2}| = 1$, for all $1 \leqslant i \leqslant 2$, then $\Gamma(P_1 \times P_2)$ is pictured in Figure 17, and so $\Gamma(P_1 \times P_2)$ is outerplanar.

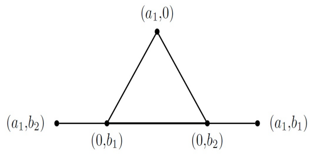

Figure 17.

Case 2. Suppose that $|P_2| = 3$ and $|P_1| \leqslant 3$. If $|P_1| = 3$. Then $\Gamma(P_1 \times P_2)$ is pictured in Figure 18, where $x \in P_1 \setminus \{0, a_1\}$, and so it is outerplanar.

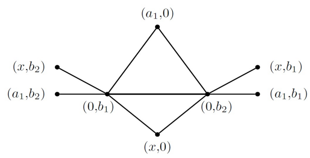

Figure 18.

If $|P_1| = 2$, then $\Gamma(P_1 \times P_2)$ is pictured in Figure 17, and so it is outerplanar.

Theorem 4.5. *Suppose that* $|A(P_1)| + |A(P_2)| = 3$ *such that* $|A(P_1)| = 1$ *and* $|A(P_2)| = 2$. *Then* $\Gamma(P_1 \times P_2)$ *is outerplanar if and only if one of the following conditions hold.*

(*i*) $|P_1| = 2$ *and* $|U_i^{P_2}| = 1$, *for all* $1 \leqslant i \leqslant 2$.

(*ii*) $|P_2| = 3$ *and* $|P_1| \leqslant 3$.

Let G be a graph with n vertices and q edges. We recall that a *chord* is any edge of G joining two nonadjacent vertices in a cycle of G. Let C be a cycle of G. We say C is a *primitive cycle* if it has no chords. Also, a graph G has the *primitive cycle property* (PCP) if any two primitive cycles intersect in at most one edge. The number $\mathrm{frank}(G)$ is called the *free rank* of G and it is the number of primitive cycles of G. Also, the number $\mathrm{rank}(G){=}q-n+r$ is called the *cycle rank* of G, where r is the number of connected components of G. The cycle rank of G can be expressed as the dimension of the cycle space of G. By [10, Proposition 2.2], we have $\mathrm{rank}(G) \leqslant \mathrm{frank}\ (G)$. A graph G is called a *ring graph* if it satisfies in one of the following equivalent conditions (see [10]).

(i) $\mathrm{rank}(G)= \mathrm{frank}(G)$,

(ii) G satisfies the PCP and G does not contain a subdivision of K_4 as a subgraph.

Clearly, every outerplanar graph is a ring graph and every ring graph is a planar graph.

Now, in view of the proofs of Proposition 4.2 and Theorem 4.3 we have the following result.

Theorem 4.6. *The zero-divisor graph* $\Gamma(P)$ *is a ring graph if and only if it is an outerplanar graph.*

5. End-regularity of zero-divisor graphs of posets

Let G and H be graphs. A *homomorphism* f from G to H is a map from $V(G)$ to $V(H)$ such that for any $a, b \in V(G)$, a is adjacent to b implies that $f(a)$ is adjacent to $f(b)$. Moreover, if f is bijective and its inverse mapping is also a homomorphism, then we call f an *isomorphism* from G to H, and in this case we say G is isomorphic to H, denoted by $G \cong H$. A homomorphism (resp, an isomorphism) from G to itself is called an *endomorphism* (resp, *automorphism*) of G. An endomorphism f is said to be *half-strong* if $f(a)$ is adjacent to $f(b)$ implies that there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that c is adjacent to d. By $\mathrm{End}(G)$, we denote the set of all the endomorphisms of G. It is well-known that $\mathrm{End}(G)$ is a monoid with respect to the composition of mappings. Let S be a semigroup. An element a in S is called *regular* if $a = aba$ for some $b \in S$ and S is called regular if every element in S is regular. Also, a graph G is called *end-regular* if $\mathrm{End}(G)$ is regular.

Now, we recall the following Lemma from [18].

Lemma 5.1. [18, Lemma 2.1] *Let G be a graph. If there are pairwise distinct vertices a, b, c in G satisfying* $\mathrm{N}(c) \subsetneqq \mathrm{N}(a) \subseteq \mathrm{N}(b)$*, then G is not end-regular.*

Lemma 5.2. *Suppose that $|A(P)| \geqslant 3$. If $U_{i\ldots j}, U_{i\ldots j\ldots k} \neq \emptyset$, such that $|U_{i\ldots j}| > 1$, for some $1 \leqslant i < j < k < n$, or $U_{i\ldots j}$, $U_{i\ldots j\ldots k}, U_{i\ldots j\ldots k\ldots t} \neq \emptyset$, for some $1 \leqslant i < j < k < t < n$, then $\Gamma(P)$ is not end-regular.*

Proof. First suppose that $U_{i\dots j}, U_{i\dots j\dots k} \neq \emptyset$ and $|U_{i\dots j}| > 1$, for some $1 \leqslant i < j < k < n$. Let $a, b \in U_{i\dots j}$ and $c \in U_{i\dots j\dots k}$. Then $N(c) \subsetneqq \mathrm{N}(a)$, since $a_k \in \mathrm{N}(a) \backslash \mathrm{N}(c)$. Now, we have $\mathrm{N}(c) \subsetneqq \mathrm{N}(a) \subseteq \mathrm{N}(b)$, and so, by Lemma 5.1, $\Gamma(P)$ is not end-regular. If $U_{i\dots j}$, $U_{i\dots j\dots k}, U_{i\dots j\dots k\dots t} \neq \emptyset$, for some $1 \leqslant i < j < k < t < n$, then consider the elements $a \in U_{i\dots j}$, $b \in U_{i\dots j\dots k}$ and $c \in U_{i\dots j\dots k\dots t}$. Now, we have $\mathrm{N}(c) \subsetneqq \mathrm{N}(b) \subseteq \mathrm{N}(a)$. Hence $\Gamma(P)$ is not end-regular. □

Proposition 5.3. *Suppose that $|A(P)| = 2$. Then $\Gamma(P)$ is end-regular.*

Proof. Clearly $\Gamma(P)$ is a complete bipartite graph. Now, by [22, Theorem 3.4], we have that $\Gamma(P)$ is end-regular. □

Lemma 5.4. *Suppose that $x, y \in Z(P)$. Then $N(x) \subseteq N(y)$ if and only if $Z_x \subseteq Z_y$ and $\{x, y\}^\ell \neq \{0\}$.*

Proof. First assume that $\mathrm{N}(x) \subseteq \mathrm{N}(y)$. Then $Z_x \subseteq Z_y$. Also, suppose to the contrary that $\{x, y\}^\ell = \{0\}$. Then x is adjacent to y. This means that $y \in \mathrm{N}(x) \subseteq \mathrm{N}(y)$, and so $y \in \mathrm{N}(y)$, which is a contradiction.

Conversly, one can easy to see that result holds. □

Proposition 5.5. *Suppose that $P = P_1 \times P_2 \times \cdots \times P_n$. Then we have the following statements.*

(*i*) *If $n \geqslant 3$, then $\Gamma(P_1 \times P_2 \times \cdots \times P_n)$ is not end-regular.*

(*ii*) *If $|A(P_1)| = 1 = |A(P_2)|$, then $\Gamma(P_1 \times P_2)$ is end-regular.*

Proof. (*i*) Suppose that $A(P_1) = \{a_1, a_2, \dots, a_n\}$, $A(P_2) = \{b_1, b_2, \dots, b_m\}$ and $A(P_3) = \{c_1, c_2, \dots, c_t\}$, where $m, n, t \geqslant 1$.

Set $x := (a_i, 0, \dots, 0)$, $y := (a_i, b_j, 0, \dots, 0)$ and $z := (a_i, b_j, c_k, 0, \dots, 0)$, for some $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$ and $1 \leqslant k \leqslant t$. Then $\mathrm{N}(z) \subsetneqq \mathrm{N}(y) \subsetneqq \mathrm{N}(x)$. Now, by Lemmas 5.1 and 5.4, $\Gamma(P)$ is not end-regular.

(*ii*) Note that in this case, $\Gamma(P_1 \times P_2)$ is a complete bipartite graph and, by [22, Theorem 3.4], every complete bipartite graph is end-regular. □

Lemma 5.6. *Assume that $\Gamma(P_2)$ has distinct vertices x and y such that $x, y \notin A(P_2)$ and $\mathrm{N}(x) \subseteq \mathrm{N}(y)$. Then $\Gamma(P_1 \times P_2)$ is not end-regular.*

Proof. Suppose that $b \in A(P_2)$. Then it follows from the fact that $\mathrm{N}(0, b) \subsetneqq \mathrm{N}(0, x) \subsetneqq \mathrm{N}(0, y)$. □

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Continuous homomorphisms, the left-gyroaddition action and topological quotient gyrogroups

Watchareepan Atiponrat and Rasimate Maungchang

Abstract. Recently, many properties of gyrogroups have been discovered. In this work, we investigate some properties of topological gyrogroups, specifically, the continuity of some homomorphisms, the canonical decomposition, and the continuity of the left-gyroaddition action.

1. Introduction

A gyrogroup is a generalization of a group of which the associative law is replaced by a more generalized version called, the left gyroassociative law and an additional property called, the left loop property, see Section 2 for more details and examples. Its structures were discovered by A. A. Ungar from the study of the Einstein velocity addition, see [13] and the references therein. Since then, many properties of gyrogroups have been discovered by active researchers in the field, see [3], [4], [7], [8], [9], [11], [12], [14]. A large portion of its algebraic properties was studied by T. Suksumran, for example, the isomorphism theorems, Cayley's Theorem, Lagrange's Theorem, gyrogroup actions, etc., see [7], [8], [11]. He is now extending his study to metric aspect of the gyrogroups, see [10].

From the topological aspect, W. Atiponrat, R. Maungchang, and T. Suksumran have been studying the separation axioms of the topological gyrogroups, see [1], [2], [15]. In this work, we continue the study of topological gyrogroups, in particular, we investigate the continuity of some homomorphisms, the canonical decomposition, and the continuity of the left-gyroaddition action.

2. Definitions and background

In this section, we include basic definitions, examples, and theorems involving the topological gyrogroups. Readers are recommended to see [1], [8], [11], and [14] for further details and examples.

Let $(G_1, \oplus_1)$ and $(G_2, \oplus_2)$ be groupoids. A function $f : G_1 \to G_2$ is a called a *homomorphism* if $f(x \oplus_1 y) = f(x) \oplus_2 f(y)$ for any $x, y \in G_1$. A bijective homomorphism is called an *isomorphism*. An isomorphism of a groupoid $(G, \oplus)$

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to itself is called a *groupoid automorphism* and we denote the set of all groupoid automorphisms of a groupoid $(G, \oplus)$ by $\mathrm{Aut}(G, \oplus)$.

Definition 2.1 (Definition 2.7 of [14])**.** Let $(G, \oplus)$ be a nonempty groupoid. We say that $(G, \oplus)$ or just G (when it is clear from the context) is a *gyrogroup* if the following hold:

1. There is a unique identity element $0_G \in G$ such that
$$0_G \oplus x = x = x \oplus 0_G \qquad \text{for all } x \in G;$$
2. For each $x \in G$, there exists a unique *inverse* element $\ominus x \in G$ such that
$$\ominus x \oplus x = 0_G = x \oplus (\ominus x);$$
3. For any $x, y \in G$, there exists $\mathrm{gyr}[x, y] \in \mathrm{Aut}(G, \oplus)$ such that
$$x \oplus (y \oplus z) = (x \oplus y) \oplus \mathrm{gyr}[x, y](z)$$
for all $z \in G$; (left gyroassociative law)
4. For any $x, y \in G$, $\mathrm{gyr}[x \oplus y, y] = \mathrm{gyr}[x, y]$. (left loop property)

We give an example of a gyrogroup which is not a group. It is called a *Möbius gyrogroup*.

Example 2.2. Let $\mathbb{D}$ be the complex open unit disk $\{z \in \mathbb{C} : |z| < 1\}$. Define a *Möbius addition* $\oplus_M : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ by
$$a \oplus_M b = \frac{a + b}{1 + \bar{a}b},$$
for all $a, b \in \mathbb{D}$. This map is well defined and its image lies in $\mathbb{D}$, see Theorem 5.5.2 of [5] for the proof. It is obvious that 0 is the identity and $-a$ is the inverse of a, for any $a \in \mathbb{D}$. $(\mathbb{D}, \oplus_M)$ is not a group because the associative property does not hold. For example, if $a = 1/2, b = i/2$, and $c = -1/2$, then $a \oplus_M (b \oplus_M c) = (10+15i)/26$ but $(a \oplus_M b) \oplus_M c = (8 + 15i)/34$. However, $(\mathbb{D}, \oplus_M)$ is a gyrogroup with
$$\mathrm{gyr}[a, b](c) = \frac{1 + a\bar{b}}{1 + \bar{a}b} c \qquad \text{for any } a, b, c \in \mathbb{D},$$
as proved in section 3.4 of [14].

Adding a topology to a gyrogroup motivates the following definition.

Definition 2.3 (Definition 1 of [1])**.** A triple $(G, \mathcal{T}, \oplus)$ is called a *topological gyrogroup* if and only if

1. $(G, \mathcal{T})$ is a topological space;

2. $(G, \oplus)$ is a gyrogroup; and

3. The binary operation $\oplus : G \times G \to G$ is continuous, where $G \times G$ is endowed with the product topology, and the operation of taking the inverse, i.e., $\ominus(\,\cdot\,)\colon G \to G,\ \ x \mapsto \ominus x$, is continuous.

Sometimes we will just say that G is a topological gyrogroup if the binary operation and the topology are clear from the context.

From the previous example, if we consider $\mathbb{D}$ as a subspace of $\mathbb{C}$ endowed with the standard topology, then it can be shown that $\oplus_M$ and $\ominus_M$ are continuous. So $\mathbb{D}$ is a topological gyrogroup.

The following are some basic algebraic and topological properties of gyrogroups and topological gyrogroups which will be needed later in our work.

Proposition 2.4 (Proposition 6 of [11]). *Suppose $(G, \oplus)$ is a gyrogroup and $A \subseteq G$. Then the following are equivalent:*

1. $\mathrm{gyr}[x, y](A) \subseteq A$ *for all* $x, y \in G$.

2. $\mathrm{gyr}[x, y](A) = A$ *for all* $x, y \in G$.

Lemma 2.5 (Proposition 32 of [7]). *Let $(G_1, \oplus_1)$ and $(G_2, \oplus_2)$ be gyrogroups, and let $f : G_1 \to G_2$ be a homomorphism. Then the following are true:*

1. $f(0_{G_1}) = 0_{G_2}$.

2. *For any* $x \in G_1$, $f(\ominus_1 x) = \ominus_2 f(x)$.

Following the notation in [14], for any pair of elements x, y in a gyrogroup $(G, \oplus)$, we let $x \boxplus y$ denote $x \oplus \mathrm{gyr}[x, \ominus y](y)$, and let $x \boxminus y$ denote $x \oplus \mathrm{gyr}[x, y](\ominus y)$.

Theorem 2.6 (Theorem 2.10, 2.22 and 2.35 of [14]). *Let $(G, \oplus)$ be a gyrogroup. For any $x, y, z \in G$, the following are true:*

1. $(\ominus x) \oplus (x \oplus y) = y$. (left cancellation law)

2. $(x \boxminus y) \oplus y = x$. (right cancellation law)

3. $\mathrm{gyr}[x, y](z) = \ominus(x \oplus y) \oplus (x \oplus (y \oplus z))$. (gyrator identity)

4. $(x \oplus y) \oplus z = x \oplus (y \oplus \mathrm{gyr}[y, x](z))$. (right gyroassociative law)

Akin to the case of topological groups, topological gyrogroups admit the following properties.

Proposition 2.7 (Proposition 3 of [1]). *Let $(G, \mathcal{T}, \oplus)$ be a topological gyrogroup. Then, for each $a \in G$, the maps $x \mapsto x \oplus a$ and $x \mapsto a \oplus x$, where $x \in G$, are homeomorphisms.*

Proposition 2.8 (Corollary 5 of [1]). *Suppose that $(G, \mathcal{T}, \oplus)$ is a topological gyrogroup, $x \in G$, and $A, B \subseteq G$. Then the following are true:*

1. *A is open if and only if $\ominus A$, $x \oplus A$ and $A \oplus x$ are open.*
2. *If A is open, then $A \oplus B$ and $B \oplus A$ are open.*

Next we introduce subgyrogroups and necessary concepts. This also leads us to the definition of quotient gyrogroups and the left-gyroaddition action.

Definition 2.9 (Section 4 of [11]). Let H be a nonempty subset of a gyrogroup $(G, \oplus)$. Then H is called a *subgyrogroup* of G and denoted by $H \leqslant G$ if $(H, \oplus_{|H \times H})$ is a gyrogroup and $\mathrm{gyr}[a, b]|_H \in \mathrm{Aut}(H, \oplus_{|H \times H})$ for all $a, b \in H$.

A subgyrogroup H is called an *L-subgyrogroup* and denoted by $H \leqslant_L G$ if

$$\mathrm{gyr}[a, h](H) = H,$$

for all $a \in G, h \in H$.

It is easy to see that $\{0\}$ is trivially an L-subgyrogroup. For a nontrivial example, see Example 18 of [11].

Proposition 2.10 (Proposition 14 of [11]). *Let H be a nonempty subset of a gyrogroup $(G, \oplus)$. Then $H \leqslant G$ if and only if $\ominus h \in H$ and $h \oplus k \in H$ for all $h, k \in H$.*

Lemma 2.11. *Let H be a subgyrogroup of a gyrogroup $(G, \oplus)$. Then $h \oplus H = H$ for each $h \in H$.*

Proof. Let $h \in H$. By Proposition 2.10, $h \oplus H \subseteq H$. On the other hand, if $k \in H$, then $k = h \oplus (\ominus h \oplus k)$ by the left cancellation law. Again, by Proposition 2.10, $\ominus h \oplus k \in H$ so $k = h \oplus (\ominus h \oplus k) \in h \oplus H$ which implies $H \subseteq h \oplus H$. As a result, $h \oplus H = H$. □

When H is a subgyrogroup of a gyrogroup $(G, \oplus)$, we use the notation G/H to stand for the set of all left cosets of H, i.e. $G/H = \{x \oplus H : x \in G\}$. The notion of L-subgyrogroups enables us to work with the set of all left cosets easily.

Proposition 2.12 (Proposition 19 of [11]). *Let H be an L-subgyrogroup of a gyrogroup $(G, \oplus)$. Then, for any $a, b \in G$, $a \oplus H = b \oplus H$ if and only if $\ominus a \oplus b \in H$.*

Proposition 2.13 (Proposition 20 of [11]). *Let H be an L-subgyrogroup of a gyrogroup $(G, \oplus)$. Then the set $G/H = \{x \oplus H : x \in G\}$ forms a partition of G.*

Being a subgyrogroup and an L-subgyrogroup are preserved by homomorphisms in the following sense.

Proposition 2.14 (Proposition 24 of [11]). *Let $f : G \to H$ be a homomorphism between gyrogroups, and let $K \leqslant G$. Then $f(K) \leqslant H$. Moreover, if $K \leqslant_L G$ and f is surjective, then $f(K) \leqslant_L H$.*

Proposition 2.15 (Proposition 25 of [11]). *Let $f : G \to H$ be a homomorphism between gyrogroups, and let $K \leqslant H$. Then $f^{-1}(K) \leqslant G$. Moreover, if $K \leqslant_L H$, then $f^{-1}(K) \leqslant_L G$. In particular,* $\ker f \leqslant_L G$.

Upcoming, trying to obtain a nice object like normal subgroups, we define normal subgyrogroups which allow a familiar binary operation on the set of all left cosets.

Definition 2.16 (Section 5 of [11]). Let H be a nonempty subset of a gyrogroup $(G, \oplus)$. Then H is called a *normal subgyrogroup* of G and denoted by $H \trianglelefteq G$ if $H = \ker f$ for some homomorphism $f : G \to K$ where K is a gyrogroup.

Lemma 2.17 (the paragraph after Proposition 25 of [11]). *Let $(G, \oplus)$ be a gyrogroup. If $H \trianglelefteq G$, then $\mathrm{gyr}[x, y](H) = H$ for all $x, y \in G$. In particular, H is an L-subgyrogroup of G.*

Theorem 2.18 (Theorem 27 of [11]). *Let $(G, \oplus)$ be a gyrogroup, and let $H \trianglelefteq G$. Then the function $\bigoplus : G/H \times G/H \to G/H$ defined by $(x \oplus H, y \oplus H) \mapsto (x \oplus y) \oplus H$ is a binary operation. Furthermore, $(G/H, \bigoplus)$ becomes a gyrogroup such that H is the identity element and $\ominus x \oplus H$ is the inverse of $x \oplus H$ for each $x \oplus H \in G/H$.*

Definition 2.19 (Section 5 of [11]). Let $(G, \oplus)$ be a gyrogroup, and let $H \trianglelefteq G$. The gyrogroup $(G/H, \bigoplus)$ in Theorem 2.18 is called the *quotient gyrogroup*, and the function $q : G \to G/H$ such that $x \mapsto x \oplus H$ is called a *canonical projection.*

Theorem 2.20 (Theorem 28 of [11] (The first isomorphism theorem)). *Let $(G_1, \oplus_1)$ and $(G_2, \oplus_2)$ be gyrogroups, and let $f : G \to H$ be a homomorphism. Then the map $g \oplus \ker f \mapsto f(g)$ gives rise to an isomorphism between $G/\ker f$ and $f(G)$.*

We end this section with the definition of the left-gyroaddition action.

Definition 2.21 (Definition 3.1 of [8]). Let $(G, \oplus)$ be a gyrogroup, and let X be a set. A function $\cdot : G \times X \to X$, written $\cdot((a, x)) = a \cdot x$, is a (gyrogroup) *action* of G on X if

1. $0_G \cdot x = x$ for all $x \in X$, and
2. $a \cdot (b \cdot x) = (a \oplus b) \cdot x$ for all $a, b \in G, x \in X$.

Theorem 2.22 (Theorem 4.5 of [8]). *Let H be a subgyrogroup of $(G, \oplus)$. Then the function $\cdot : G \times G/H \to G/H$ such that for all $g \in G, x \oplus H \in G/H$,*

$$g \cdot (x \oplus H) = (g \oplus x) \oplus H$$

defines a gyrogroup action of G on G/H if and only if

$$\mathrm{gyr}[a, b](x \oplus H) \subseteq x \oplus H$$

for all $a, b \in G, x \oplus H \in G/H$.

Definition 2.23 (Definition 4.4 of [8]). Following the language of Theorem 2.22, the function $\cdot : G \times G/H \to G/H$ is called the *left-gyroaddition action* if it is a gyrogroup action.

3. Continuous homomorphisms

In this section, we prove the continuity of some homomorphisms and the canonical decomposition of topological gyrogroups.

Proposition 3.24. *Let* $(G_1, \mathcal{T}_1, \oplus_1)$ *and* $(G_2, \mathcal{T}_2, \oplus_2)$ *be topological gyrogroups. Let* $f : G_1 \to G_2$ *be a homomorphism. Then* f *is continuous if and only if it is continuous at* 0_{G_1}.

Proof. ($\Rightarrow$) Obvious.

($\Leftarrow$) Let $x \in G_1$. If U is a neighborhood of $f(x)$, then $\ominus_2 f(x) \oplus_2 U$ is a neighborhood of 0_{G_2} by Proposition 2.8. So there is a neighborhood W of 0_{G_1} such that $f(W) \subseteq \ominus_2 f(x) \oplus_2 U$. As a result, $x \oplus_1 W$ is a neighborhood of x such that $f(x \oplus_1 W) = \{f(x \oplus_1 w) : w \in W\} = \{f(x) \oplus_2 f(w) : w \in W\} = f(x) \oplus_2 f(W) \subseteq f(x) \oplus_2 (\ominus_2 f(x) \oplus_2 U) = \{f(x) \oplus_2 (\ominus_2 f(x) \oplus_2 u) : u \in U\} = U$ by the left cancellation law (see Theorem 1). Hence f is continuous at x. Since x is arbitrary, f is continuous. $\square$

Lemma 3.25. *Let* H *be a subgyrogroup of a topological gyrogroup* $(G, \mathcal{T}, \oplus)$ *such that* $\mathrm{gyr}[a,b](x \oplus H) \subseteq x \oplus H$ *for all* $a, b \in G, x \oplus H \in G/H$ *[or let* $H \trianglelefteq G$*]. Suppose* G/H *is equipped with the quotient topology induced by* q. *Then the canonical projection* $q : G \to G/H$ *is a continuous open map.*

Proof. Since G/H is endowed with the quotient topology induced by q, q is continuous. Next, let $U \subseteq G$ be an open set. Then $q(U) = \{u \oplus H : u \in U\}$. We will show that $q^{-1}(q(U)) = U \oplus H$. If $a \in q^{-1}(q(U))$, then $q(a) = a \oplus H = u \oplus H$ for some $u \in U$. As a result, $\ominus u \oplus a \in H$ by Proposition 2.12. Thus $\ominus u \oplus a = h$ for some $h \in H$ so $a = u \oplus h \in U \oplus H$ by the left cancellation law. On the other hand, if $x \in U \oplus H$, then $x = v \oplus k$ for some $v \in U$, $k \in H$. We obtain that $q(x) = x \oplus H = (v \oplus k) \oplus H = v \oplus (k \oplus \mathrm{gyr}[k,v](H)) = v \oplus (k \oplus H) = v \oplus H \in q(U)$; the fourth and fifth equalities come from our assumption together with Proposition 2.4 [or come from Lemma 2.17 for the case $H \trianglelefteq G$] and Lemma 2.11. So $x \in q^{-1}(q(U))$ and we can conclude that $q^{-1}(q(U)) = U \oplus H$ which is an open set by Proposition 2.8. Hence q is an open map. $\square$

Theorem 3.26. (Canonical decomposition) *Let* $(G_1, \mathcal{T}_1, \oplus_1)$ *and* $(G_2, \mathcal{T}_2, \oplus_2)$ *be topological gyrogroups. Let* $f : G_1 \to G_2$ *be a continuous homomorphism. Then the following are true:*

$$\begin{tikzcd} G_1 \arrow[r, "f"] \arrow[d, "q"'] & G_2 \arrow[d, "i"] \\ G_1/\ker f \arrow[r, "\widetilde{f}"'] & f(G_1) \end{tikzcd}$$

(1) *The above diagram commutes where* $q : G_1 \to G_1/\ker f$ *is the canonical projection,* $\widetilde{f} : G_1/\ker f \to f(G_1)$ *is a function defined by* $g \oplus_1 \ker f \mapsto f(g)$ *for all* $g \in G_1$*, and* $i : f(G_1) \to G_2$ *is the inclusion map.*

(2) $i : f(G_1) \to G_2$ *is an injective continuous homomorphism, and* $\widetilde{f}$ *is a continuous isomorphism.*

(3) f *is an open map if and only if* $f(G_1)$ *is open in* G_2 *and* $\widetilde{f}$ *is an open map.*

(4) $\widetilde{f}$ *is an open map if and only if* $f(U)$ *is open in* $f(G_1)$ *for all open subset* U *of* G_1*.*

Proof. To see (1), we first show that $\widetilde{f}$ is well defined. If $a, b \in G$ are so that $a \oplus_1 \ker f = b \oplus_1 \ker f$, then $\ominus_1 b \oplus_1 a \in \ker f$ by Proposition 2.12. Thus $f(\ominus_1 b) \oplus_2 f(a) = f(\ominus_1 b \oplus_1 a) = 0_{G_2}$ so $\ominus_2 f(\ominus_1 b) = f(a)$ by the left cancellation law. Hence $f(b) = f(a)$ by Lemma 2.5. Next, the diagram commutes because for any $a \in G_1$, $f(a) = i(f(a)) = i(\widetilde{f}(a \oplus_1 \ker f)) = i(\widetilde{f}(q(a)))$.

To prove (2), i is injective and continuous because it is a restriction of the identity map. Moreover, it is a homomorphism since $f(G_1)$ is a gyrogroup by Proposition 2.14. On the other hand, $\widetilde{f}$ is an isomorphism by the first isomorphism theorem. Next, we show that $\widetilde{f}$ is continuous. Let U be an open subset of $f(G_1)$. Then there is an open subset W of G_2 such that $U = W \cap f(G_1)$. Since f is continuous, $f^{-1}(W)$ is open in G_1. Then $q(f^{-1}(W))$ is an open subset of $G_1/\ker f$ by Lemma 3.25. Now observe that

$$\begin{aligned}\widetilde{f}^{-1}(U) &= \widetilde{f}^{-1}(W \cap f(G_1)) = \widetilde{f}^{-1}(i^{-1}(W \cap f(G_1))) = \widetilde{f}^{-1}(i^{-1}(f(f^{-1}(W)))) \\ &= \widetilde{f}^{-1}(i^{-1}((i \circ \widetilde{f} \circ q)(f^{-1}(W)))) = q(f^{-1}(W)).\end{aligned}$$

So $\widetilde{f}^{-1}(U)$ is open in $G_1/\ker f$, and hence $\widetilde{f}$ is continuous.

Now we prove (3). ($\Rightarrow$): Suppose that f is an open map. Then $f(G_1)$ is open in G_2. To see that $\widetilde{f}$ is an open map, let U be an open subset of $G_1/\ker f$. Since q is continuous, $q^{-1}(U)$ is open. Moreover, $f(q^{-1}(U))$ is open because f is an open map. Then $\widetilde{f}(U) = (i^{-1} \circ f \circ q^{-1})(U)$ is open because i, q are continuous.
($\Leftarrow$): Let $f(G_1)$ be open in G_2, and let $\widetilde{f}$ be an open map. We will show that f is an open map. Let U be an open subset of G_1. Then $(\widetilde{f} \circ q)(U)$ is open in $f(G_1)$ because q and $\widetilde{f}$ are open maps. Since $f(G_1)$ is open in G_2, $(\widetilde{f} \circ q)(U)$ is open in G_2. Notice that $f(U) = (i \circ \widetilde{f} \circ q)(U) = i((\widetilde{f} \circ q)(U)) = (\widetilde{f} \circ q)(U)$. Hence $f(U)$ is open in G_2 which implies that f is an open map.

Finally, we prove (4). ($\Rightarrow$): Assume that $\widetilde{f}$ is an open map. Let U be an open subset of G_1. Then $(\widetilde{f} \circ q)(U)$ is open in $f(G_1)$ because q and $\widetilde{f}$ are open maps. Observe that $f(U) = i((\widetilde{f} \circ q)(U)) = (\widetilde{f} \circ q)(U)$. So $f(U)$ is open in $f(G_1)$.
($\Leftarrow$): Suppose that $f(U)$ is open in $f(G_1)$ for all open subset U of G_1. To see that $\widetilde{f}$ is an open map, let W be an open subset of $G_1/\ker f$. Then $(i^{-1} \circ f \circ q^{-1})(W) =$

$(f \circ q^{-1})(W) = f(q^{-1}(W))$ is open in $f(G_1)$ by the assumption and the fact that q is continuous. Since $\widetilde{f}(W) = (i^{-1} \circ f \circ q^{-1})(W)$, $\widetilde{f}$ is an open map. $\square$

4. Action and topological quotient gyrogroups

In our last section, we consider the set of all left cosets of an L-subgyrogroup H in a topological gyrogroup $(G, \mathcal{T}, \oplus)$. According to Proposition 2.13, we can assign the quotient topology induced by canonical projection to G/H and study the continuity of the left-gyroaddition action $\cdot : G \times G/H \to G/H$ where $G \times G/H$ is endowed with the product topology. In addition, if $H \trianglelefteq G$, then $(G/H, \bigoplus)$ is a gyrogroup so we can examine the continuity of $\bigoplus$.

From now on, let $\mathfrak{T}$ denote the quotient topology induced by the canonical projection $q : G \to G/H$. In addition, we will assume that G/H is endowed with $\mathfrak{T}$ in our proof when the topology is needed to be specify. We begin this section by providing some basic facts of G/H in the following proposition which the proof in topological group version can be adopted.

Proposition 4.1. *Let $(G, \mathcal{T}, \oplus)$ be a topological gyrogroup, and let $H \leqslant G$ be such that $\mathrm{gyr}[a, b](x \oplus H) \subseteq x \oplus H$ for all $a, b \in G, x \oplus H \in G/H$. Then the following are equivalent:*

1. *$(G/H, \mathfrak{T})$ is T_2.*
2. *$(G/H, \mathfrak{T})$ is T_1.*
3. *H is a closed subset of G.*

Proof. $(1 \Rightarrow 2)$: Trivial.

$(2 \Rightarrow 3)$: Observe that $H = q^{-1}(\{H\})$ because of Lemma 2.11 and Proposition 2.12. Since q is continuous and $\{H\}$ is closed because $(G/H, \mathfrak{T})$ is T_1, we gain the result.

$(3 \Rightarrow 1)$: We will show that the set $\{(x \oplus H, y \oplus H) : x \oplus H = y \oplus H\}$ is closed in $G/H \times G/H$ together with the product topology. Notice that $\{(x \oplus H, y \oplus H) : x \oplus H = y \oplus H\} \subseteq \{(x \oplus H, y \oplus H) : \ominus x \oplus y \in H\}$ by Proposition 2.12. On the other hand, $\{(x \oplus H, y \oplus H) : \ominus x \oplus y \in H\} \subseteq \{(x \oplus H, y \oplus H) : x \oplus H = y \oplus H\}$ by the fact that $\mathrm{gyr}[a, b](x \oplus H) \subseteq x \oplus H$ for all $a, b \in G, x \oplus H \in G/H$, Proposition 2.4 and, again, Proposition 2.12. So $\{(x \oplus H, y \oplus H) : x \oplus H = y \oplus H\} = \{(x \oplus H, y \oplus H) : \ominus x \oplus y \in H\}$. Next, observe that $G/H \times G/H - \{(x \oplus H, y \oplus H) : x \oplus H = y \oplus H\} = G/H \times G/H - \{(x \oplus H, y \oplus H) : \ominus x \oplus y \in H\} = \{(x \oplus H, y \oplus H) : \ominus x \oplus y \notin H\} = (q \times q) \circ (\ominus(\cdot) \times Id) \circ (\oplus^{-1})(G - H)$, where $q \times q$ is the product of two open quotient maps and $Id : G \to G$ is the identity function. Since $\oplus$ is continuous and H is closed, $\oplus^{-1}(G - H)$ is open. Moreover, $\ominus(\cdot) \times Id : G \times G \to G \times G$ is a homeomorphism so $(\ominus(\cdot) \times Id) \circ (\oplus^{-1})(G - H)$ is open. Finally, it is a well-known fact in topology that the product of two open maps is an open map. Hence $(q \times q) \circ (\ominus(\cdot) \times Id) \circ (\oplus^{-1})(G - H)$ is open. This implies that $\{(x \oplus H, y \oplus H) : x \oplus H = y \oplus H\}$ is closed. $\square$

Lemma 4.2. *Let H be a subgyrogroup of a gyrogroup $(G, \oplus)$,* $\text{gyr}[a,b](x \oplus H) \subseteq x \oplus H$ *for all $a, b \in G, x \oplus H \in G/H$. Then, for all $a \in G$ and $x \oplus H, y \oplus H \in G/H$, $(a \oplus x) \oplus H = (a \oplus y) \oplus H$ if and only if $x \oplus H = y \oplus H$.*

Proof. ($\Leftarrow$): Use Theorem 2.22.

($\Rightarrow$): Suppose $(a \oplus x) \oplus H = (a \oplus y) \oplus H$. We will show that $\ominus y \oplus x \in H$ which implies $x \oplus H = y \oplus H$. Let $(a \oplus x) \oplus h_1 \in (a \oplus x) \oplus H$. By assumption, $\text{gyr}[a,b](H) \subseteq H$, for all $a, b \in G$. So $\text{gyr}[a,b](H) = H$, for all $a, b \in G$, by Proposition 2.4. Then, for some $h_2, h_3, h_4, h_5 \in H$,

$$\begin{aligned}
(a \oplus x) \oplus h_1 &= (a \oplus y) \oplus h_2, \\
a \oplus (x \oplus h_3) &= a \oplus (y \oplus h_4), \\
x \oplus h_3 &= y \oplus h_4, \\
\ominus y \oplus (x \oplus h_3) &= h_4, \\
(\ominus y \oplus x) \oplus h_5 &= h_4, \\
\ominus y \oplus x &= h_4 \boxminus h_5.
\end{aligned}$$

Moreover, $h_4 \boxminus h_5 = h_4 \oplus \text{gyr}[h_4, h_5](\ominus h_5) \in H$. Hence $\ominus y \oplus x \in H$. □

Theorem 4.3. *Let H be a subgyrogroup of a topological gyrogroup $(G, \mathcal{T}, \oplus)$ such that* $\text{gyr}[a,b](x \oplus H) \subseteq x \oplus H$ *for all $a, b \in G, x \oplus H \in G/H$. Then the left-gyroaddition action $\cdot : G \times G/H \to G/H$ is transitive. Furthermore, for each $a \in G$, the function $f_a : G/H \to G/H$ defined by $f_a(x \oplus H) = a \cdot (x \oplus H) = (a \oplus x) \oplus H$ for all $x \oplus H \in G/H$ is a homeomorphism.*

Proof. To begin with, we show that the action is transitive. Let $x \oplus H, y \oplus H \in G/H$. Then $(y \boxminus x) \cdot (x \oplus H) = ((y \boxminus x) \oplus x) \oplus H = y \oplus H$, by the right cancellation law.

Next, we prove the last sentence of the theorem. Let $a \in G$. We first show that the function $f_a : G/H \to G/H$ defined by $f_a(x \oplus H) = a \cdot (x \oplus H) = (a \oplus x) \oplus H$ for each $x \oplus H \in G/H$ is a continuous bijection. Lemma 4.2 shows that f_a is injective. Moreover, for any $x \oplus H \in G/H$, $f_a((\ominus a \oplus x) \oplus H) = (a \oplus (\ominus a \oplus x)) \oplus H = x \oplus H$. So f_a is bijective. To see the continuity of f_a, let $L_a : G \to G$ be such that $L_a(x) = a \oplus x$ for all $x \in G$. Then L_a is a homeomorphism by Proposition 2.7. Observe that $q \circ L_a = f_a \circ q$ where $q : G \to G/H$ is the canonical projection. So, for each open set $U \subseteq G/H$, we have $f_a^{-1}(U) = q(L_a^{-1}(q^{-1}(U)))$ which is open by Lemma 3.25. We conclude that f_a is a continuous bijection. It is not hard to check that $f_a^{-1} = f_{\ominus a}$ which is a continuous bijection by similar proof. Thus f_a is a homeomorphism. □

In some special occasion, the continuity of the left-gyroaddition action is established.

Theorem 4.4. *Suppose that H is a compact subgyrogroup of a topological gyrogroup $(G, \mathcal{T}, \oplus)$ such that* $\text{gyr}[a,b](x \oplus H) \subseteq x \oplus H$ *for all $a, b \in G, x \oplus H \in G/H$.*

Then the left-gyroaddition action of G on G/H is transitive. Moreover, it is continuous when $G \times G/H$ is endowed with the product topology.

Proof. The action is transitive by Theorem 4.3. Next, we show that the map $\cdot : G \times G/H \to G/H$ defined by $\cdot((a, x \oplus H)) = a \cdot (x \oplus H) = (a \oplus x) \oplus H$ for all $a \in G, x \oplus H \in G/H$ is continuous when the topology on $G \times G/H$ is the product topology. Suppose $(a, x \oplus H) \in G \times G/H$. Let $U \subseteq G/H$ be an open set containing $\cdot((a, x \oplus H)) = (a \oplus x) \oplus H$. Observe that $a \oplus (x \oplus H) = (a \oplus x) \oplus \mathrm{gyr}[a, x](H) = (a \oplus x) \oplus H$ by our assumption and Proposition 2.4. Moreover, $q((a \oplus x) \oplus H) = q(\{(a \oplus x) \oplus h : h \in H\}) = \{((a \oplus x) \oplus h) \oplus H : h \in H\} = \{(a \oplus x) \oplus (h \oplus \mathrm{gyr}[h, a \oplus x](H)) : h \in H\} = \{(a \oplus x) \oplus (h \oplus H) : h \in H\} = \{(a \oplus x) \oplus H : h \in H\} \subseteq U$; the fourth and fifth equalities come from our assumption together with Proposition 2.4 and Lemma 2.11. So $a \oplus (x \oplus H) = (a \oplus x) \oplus H \subseteq q^{-1}(U)$ which is an open set because q is continuous. Thus, for each $h \in H$, there are open sets U_h, V_h of G such that $a \in U_h$, $x \oplus h \in V_h$, and $U_h \oplus V_h \subseteq q^{-1}(U)$ because $\oplus$ is continuous. It is clear that $x \oplus H \subseteq \bigcup_{h \in H} V_h$. Since H is compact, $x \oplus H$ is compact by Proposition 2.7. Hence $x \oplus H \subseteq V_{h_1} \cup ... \cup V_{h_l}$ for some $h_1, ..., h_l \in H, l \in \mathbb{N}$. Let $\widetilde{U} = U_{h_1} \cap ... \cap U_{h_l}$ and $\widetilde{V} = V_{h_1} \cup ... \cup V_{h_l}$. Then $\widetilde{U} \oplus \widetilde{V} \subseteq q^{-1}(U)$, $a \in \widetilde{U}$ and $x \oplus H \subseteq \widetilde{V}$ where $\widetilde{U}, \widetilde{V}$ are open in G. Notice that $x \in x \oplus H \subseteq \widetilde{V}$ which implies $x \oplus H = q(x) \in q(\widetilde{V})$. Moreover, $q(\widetilde{V})$ is open by Lemma 3.25. Hence $\widetilde{U} \times q(\widetilde{V})$ is a neighborhood of $(a, x \oplus H)$ such that

$$\begin{aligned}
\cdot(\widetilde{U} \times q(\widetilde{V})) &= \{u \cdot q(v) : u \in \widetilde{U} \text{ and } v \in \widetilde{V}\} \\
&= \{u \cdot (v \oplus H) : u \in \widetilde{U} \text{ and } v \in \widetilde{V}\} \\
&= \{(u \oplus v) \oplus H : u \in \widetilde{U} \text{ and } v \in \widetilde{V}\} \\
&= \{q(u \oplus v) : u \in \widetilde{U} \text{ and } v \in \widetilde{V}\} \\
&= q(\widetilde{U} \oplus \widetilde{V}) \subseteq q(q^{-1}(U)) = U.
\end{aligned}$$

We conclude that the action is continuous. □

Next, we will explore the continuity of $\bigoplus$ when $H \trianglelefteq G$. Let us start with the following theorem.

Theorem 4.5. *Let H be a subgyrogroup of a topological gyrogroup $(G, \mathcal{T}, \oplus)$ such that $\mathrm{gyr}[a, b](x \oplus H) \subseteq x \oplus H$ for all $a, b \in G, x \oplus H \in G/H$ [or let $H \trianglelefteq G$]. Then $\mathfrak{T}$ is a discrete topology if and only if H is an open subset of G.*

Proof. ($\Rightarrow$) Suppose $\mathfrak{T}$ is a discrete topology. We obtain that $\{H\}$ is an open subset of G/H. Since q is continuous, $q^{-1}(\{H\})$ is open. It is not hard to prove that $q^{-1}(\{H\}) = H$ by using Lemma 2.11 and Proposition 2.12. The result follows.

($\Leftarrow$) We will show that for each $x \in G$, the singleton set $\{x \oplus H\}$ is open. Since H is an open subgyrogroup of G, $x \oplus H$ is open in G by Proposition 2.8. Observe that $q(x \oplus H) = \{(x \oplus h) \oplus H : h \in H\} = \{x \oplus (h \oplus \mathrm{gyr}[h, x](H)) : h \in H\} = \{x \oplus (h \oplus H) : h \in H\} = \{x \oplus H\}$; again, the third and fourth equalities come from our assumption together with Proposition 2.4 [or come from Lemma 2.17 for the case $H \trianglelefteq G$] and Lemma 2.11. Since q is an open map, $\{x \oplus H\} = q(x \oplus H)$ is open in G/H. ☐

When H is a normal subgyrogroup of a topological gyrogroup $(G, \mathcal{T}, \oplus)$, it is possible that $(G/H, \mathfrak{T}, \bigoplus)$ turns into a topological gyrogroup. Fortunately, we can show that this is the case.

Definition 4.6. *Let $(G, \mathcal{T}, \oplus)$ be a topological gyrogroup, and let $H \trianglelefteq G$. Then the quotient gyrogroup $(G/H, \bigoplus)$ is called the topological quotient gyrogroup if $(G/H, \mathfrak{T}, \bigoplus)$ is a topological gyrogroup.*

Theorem 4.7. *Let $(G, \mathcal{T}, \oplus)$ be a topological gyrogroup, and let $H \trianglelefteq G$. Then $(G/H, \mathfrak{T}, \bigoplus)$ is a topological quotient gyrogroup.*

Proof. It is a well-known result in topology that the product of two open quotient maps is also a quotient map. So $q \times q : G \times G \to G/H \times G/H$ is a quotient map. To prove that $\bigoplus$ is continuous, it is enough to show that $\bigoplus \circ (q \times q)$ is continuous by Theorem 22.2 of [6]. Notice that $(\bigoplus \circ (q \times q))((x, y)) = q(x) \bigoplus q(y) = (x \oplus H) \bigoplus (y \oplus H) = (x \oplus y) \oplus H = (q \circ \oplus)((x, y))$ for all $x, y \in G$. Since q and $\oplus$ are continuous, we have that $\bigoplus \circ (q \times q)$ is continuous which implies the continuity of $\bigoplus$. Next, for each $x \oplus H \in G/H$, $\ominus x \oplus H$ is its inverse element by Theorem 2.18. As a result, the inverse operation $x \oplus H \mapsto \ominus x \oplus H$ is continuous since it is equal to q composed with $\ominus(\,\cdot\,)$. ☐

A careful reader might ask for the continuity of the left-gyroaddition action in general settings. On one hand, this problem is still open for us. On the other hand, we provide an easy example of occasion that the action is continuous without employing compactness of the subgroup H.

Remark 4.8. *Consider $(\mathbb{D}, \mathcal{T}, \oplus_M)$ where $\mathcal{T}$ is the discrete topology on $\mathbb{D}$ or the subspace topology of $\mathbb{C}$ endowed with the standard topology. It is clear that $(\mathbb{D}, \mathcal{T}, \oplus_M)$ is a topological gyrogroup which is not compact. Let $H = \mathbb{D}$. Then H is not compact, and H is a normal subgyrogroup of $\mathbb{D}$ such that $\mathrm{gyr}[a, b](x \oplus H) \subseteq x \oplus H$ for all $a, b \in \mathbb{D}, x \oplus H \in \mathbb{D}/H$. Since $\mathbb{D}/H$ is a singleton set, the left-gyroaddition action is continuous when $\mathbb{D} \times \mathbb{D}/H$ is equipped with the product topology.*

Finally, we would like to end our work with the succeeding question.

Question 1. *Is the left-gyroaddition action continuous in general?*

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From quotient trigroups to groups

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Abstract. In this paper, we study the notion of normality in the category of trigroups, and construct quotient trigroups. This allows us to establish analogues for trigroups of some useful results on groups, namely, the first, second and third isomorphism theorems as well as some of their related corollaries. Our construction provides a new functorial link between the categories of groups and trigroups.

1. Introduction

The concept of digroups originated from the work of J. L. Loday on dialgebras [9], and were formally axiomatized by M. Kinyon in his contribution to the Coquecigrue problem; an analogue of Lie's third theorem which consists to associate a grouplike object to a given Leibniz algebra by "antidifferentiation". More precisely, Kinyon showed in [4] that conjugating digroups equipped with a manifold structure differentiate to Leibniz algebras [7]. Digroups was also independently introduced by K. Liu [5] and R. Felipe [3], and further studied in [10].

In their study of trialgebras and families of polytopes [8], Loday and Ronco provided an axiomatic definition of associative trioids. This led the authors to introduce the category of trigroups as associative trioid – also called trisemigroups– equipped with bar-units and in which each element has a bar-inverse. Trigroups are generalizations of digroups to algebraic structures with three operations, since forgetting one operation of a trigroup yields a digroup structure. Analogue to the relationship between digroups and Leibniz algebras provided by Kinyon in [4], it is shown in [2] that conjugating linear trigroups yields Lie 3-racks [1], which produce Leibniz 3-algebras [6] when differentiated with respect to the distinguish bar-unit.

At the beginning of the last century, Evarist Galois introduced in the classical theory of groups the notion of normal subgroups which played a fundamental role in defining quotient groups and in the so-called isomorphism theorems which are very important in the general development of Group Theory (see [12]). In 2016, Ongay, Velasquez and Wills-Toro defined normal subdigroups [11] and studied a construction of quotient digroups and the corresponding analogues of Isomorphism Theorems. Our aim in this paper is to conduct a similar study on trigroups using a different approach. Our study produces a different quotient on the underlying digroup associated to a trigroup. More precisely, we use the notion of conjugation

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of trigoups provided in [2] to define the concept of normality on trigroups. This allows us to define a congruence for which the quotient set has a group structure, i.e. a trivial trigroup structure. It is worth mentioning that our construction of quotient trigroup produces a functor from the category of trigroups to the category of groups, other than the functor provided in [2].

2. Trigroups

Recall from [2] that a *trisemigroup* $(A, \vdash, \perp, \dashv)$ is a set A equipped with three binary associative operations $\vdash$, $\perp$ and $\dashv$ respectively called left, middle and right, and satisfying the following conditions:

$$\begin{cases} x \vdash (y \vdash z) = (x \dashv y) \vdash z & (p_1) \\ x \vdash (y \vdash z) = (x \perp y) \vdash z & (p_2) \\ x \vdash (y \dashv z) = (x \vdash y) \dashv z & (p_3) \\ x \vdash (y \perp z) = (x \vdash y) \perp z & (p_4) \\ x \dashv (y \dashv z) = x \dashv (y \vdash z) & (p_5) \\ x \dashv (y \dashv z) = x \dashv (y \perp z) & (p_6) \\ (x \perp y) \dashv z = x \perp (y \dashv z) & (p_7) \\ (x \dashv y) \perp z = x \perp (y \vdash z) & (p_8) \end{cases}$$

for all $x, y, z \in A$.

A trisemigroup A is a *trigroup* if there exists an element $1 \in A$ satisfying

$$1 \vdash x = x = x \dashv 1 \ \text{ for all } \ x \in A \tag{I}$$

and for all $x \in A$, there exists $x^{-1} \in A$ (called *inverse* of x) such that

$$x \vdash x^{-1} = 1 = x^{-1} \dashv x \ \text{ and } \ x \perp x^{-1} = 1 = x^{-1} \perp x.$$

Let $\mathfrak{U}_A := \{e \in A : e \vdash x = x = x \dashv e \text{ for all } x, y \in A\}$ be the set of *bar-units* of A.

Recall also that a morphism between two trigroups is a map that preserves the three binary operations and is compatible with bar-units and inverses.

Remark 2.1. [2, Lemma 4.5]

(a) The set $J_A = \{x^{-1} : x \in A\}$ is a group in which $\vdash = \perp = \dashv$.

(b) The mapping $\phi : A \to J_A$ defined by $x \mapsto (x^{-1})^{-1}$ is an epimorphism of trigroups that fixes J_A, and $\mathsf{Ker}\,\phi = \mathfrak{U}_A$.

(c) $x \vdash 1 = 1 \perp x = x \perp 1 = 1 \dashv x = (x^{-1})^{-1}$ for all $x \in A$.

(d) $(x \perp y)^{-1} = y^{-1} \perp x^{-1}$ for all $x, y \in A$.

(e) $(x \vdash y)^{-1} = y^{-1} \vdash x^{-1} = y^{-1} \dashv x^{-1} = (x \dashv y)^{-1}$ for all $x, y \in A$. Consequently, $((x^{-1})^{-1})^{-1} = x^{-1}$.

(f) $x^{-1} \vdash x \vdash y = x \vdash x^{-1} \vdash y = y$ for all $x, y \in A$.

The following results are consequences of Remark 2.1 and will be heavily used without reference throughout the paper to simplify proofs.

Remark 2.2.

(a) $x^{-1} \vdash 1 = x^{-1} = 1 \dashv x^{-1}$ for all $x \in A$.

(b) $x \vdash y = (x^{-1})^{-1} \vdash y$ for all $x, y \in A$.

(c) $x \dashv y = x \dashv (y^{-1})^{-1}$ for all $x, y \in A$.

Proof. The assertion (a) follows by Remark 2.1(c). For (b) and (c), we have again by Remark 2.1(c), $(x^{-1})^{-1} \vdash y = (x \vdash 1) \vdash y = x \vdash (1 \vdash y) = x \vdash y$ and $x \dashv (y^{-1})^{-1} = x \dashv (1 \dashv y) = (x \dashv 1) \dashv y = x \dashv y$. □

3. Subtrigroups

In this section we define sub-objects in the category of trigroups, and study the concept of normality on these sub-objects.

Definition 3.1. We say that a trigroup A is *trivial* if $A = J_A$.

Proposition 3.2. *A trigroup $(A, \vdash, \perp, \dashv)$ is trivial if and only if $(x^{-1})^{-1} = x$ for all $x \in A$.*

Proof. The proof is straightforward by Definition 3.1. □

For the rest of the paper, all trigroups are assumed to be non-trivial unless otherwise stated.

Definition 3.3. Let $(A, \vdash, \perp, \dashv)$ be a trigroup with distinguish bar-unit 1. A subset S of A is said to be a *subtrigroup* of A if $(S, \vdash, \perp, \dashv)$ is a trigroup with distinguish bar-unit 1.

Proposition 3.4. *Let $(A, \vdash, \perp, \dashv)$ be a trigroup with distinguish bar-unit 1, and H a nonempty subset of A. H is a subtrigroup of A if and only H is closed under the operations $\vdash, \perp, \dashv$, and $x^{-1} \in H$ for all $x \in H$.*

Proof. The proof of the forward direction is obvious. For the converse, it is enough to verify that $1 \in H$. Indeed, since H is nonempty there is some $x_0 \in H$, which yields $x_0^{-1} \in H$, and thus $1 = x_0 \vdash x_0^{-1} \in H$. □

Proposition 3.5. *Let A be a trigroup. Then $(J_A, \vdash = \dashv = \perp)$ and $(\mathfrak{U}_A, \vdash, \dashv, \perp)$ are subtrigroups of A.*

Proof. J_A is a subtrigroup of A since by Remark 2.1(a), $J_A \subseteq A$ and J_A is a group in which $\vdash = \perp = \dashv$. To show that $\mathfrak{U}_A$ is a subtrigroup of A, notice that for all $e, e' \in \mathfrak{U}_A$, $e \vdash e' = e'$, $\quad e \dashv e' = e$, $\quad (e \perp e') \vdash x \overset{p2}{=} e \vdash (e' \vdash x) = e \vdash x = x$ and $x \dashv (e \perp e') \overset{p6}{=} x \dashv (e \dashv e') = (x \dashv e) \dashv e' = x \dashv e = x$ for all $x \in A$. So $\mathfrak{U}_A$ is closed under the operations $\vdash, \perp, \dashv$. In addition, $e^{-1} \in \mathfrak{U}_A$ by [2, Lemma 4.6]. The result follows by Proposition 3.4. $\square$

Proposition 3.6. *Let $\phi : A \to A'$ be a morphism of trigroups. Then:*

(a) $\mathsf{Ker}\,\phi$ *is a subtrigroup of A.*

(b) *If S is a subtrigroup of A, then $\phi(S)$ is a subtrigroup of A'.*

(c) *If S' is a subtrigroup of A', then $\phi^{-1}(S')$ is a subtrigroup of A.*

Proof. To prove (a), first notice that $\phi(1_A) = 1_{A'}$, so $\mathsf{Ker}\,\phi \neq \emptyset$. Now Let $x, y \in \mathsf{Ker}\,\phi$. Then $\phi(x \vdash y) = \phi(x) \vdash \phi(y) = 1_{A'} \vdash 1_{A'} = 1_{A'}$, $\phi(x \dashv y) = \phi(x) \dashv \phi(y) = 1_{A'} \dashv 1_{A'} = 1_{A'}$, $\phi(x \perp y) = \phi(x) \perp \phi(y) = 1_{A'} \perp 1_{A'} = 1$ and $\phi(x^{-1}) = (\phi(x))^{-1} = 1_{A'}$. Thus by proposition 3.4, $\mathsf{Ker}\,\phi$ is a subtrigroup of A. The proofs of (b) and (c) are similar. $\square$

Consider the following sets: $x \star S = \{x \star s,\ s \in S\}$ and $S \star x = \{s \star x,\ s \in S\}$, where $\star \in \{\vdash, \perp, \dashv\}$. In [2], the operation $[-,-,-] : A \times A \times A \to A$ given by $[x, y, z] = (x \perp y) \vdash z \dashv (y^{-1} \perp x^{-1})$, was defined as a generalization of the conjugation on digroups [4, Equation (13)] to trigroups. Using this operation, we define normality of subtrigroups as follows:

Definition 3.7. Let $(A, \vdash, \perp, \dashv)$ be a trigroup with distinguish bar-unit 1. A subtrigroup S of A is said to be *normal* if $(x \perp y) \vdash S \dashv (y^{-1} \perp x^{-1}) \subseteq S$ for all $x, y \in A$.

This definition extends the following definition of normality in digroups to trigroups.

Definition 3.8. [11, Definition 4] A subdigroup S of a digroup $(A, \vdash, \dashv)$ is said to be *normal* if $x \vdash S \dashv x^{-1} \subseteq S$ for all $x \in A$.

It turns out that normality in a trigroup is completely determined by its underlying digroup structure, as proven in the following Lemma.

Lemma 3.9. *Let $(A, \vdash, \perp, \dashv)$ be a trigroup with distinguish bar-unit 1 and S a subtrigroup of A. Then S is a normal subtrigroup of A iff S is a normal subdigroup of the underlying digroup $(A, \vdash, \dashv)$.*

Proof. Let $(A, \vdash, \perp, \dashv)$ be a trigroup and S a normal subtrigroup of A. Then for all $x \in A$,

$$\begin{aligned} x \vdash S \dashv x^{-1} &= x \vdash (1 \vdash S) \dashv x^{-1} \overset{p_2}{=} (x \perp 1) \vdash S \dashv x^{-1} \\ &= (x \perp 1) \vdash (S \dashv 1) \dashv x^{-1} = (x \perp 1) \vdash S \dashv (1 \dashv x^{-1}) \\ &\overset{p_6}{=} (x \perp 1) \vdash S \dashv (1 \perp x^{-1}) \subseteq S. \end{aligned}$$

The converse is obvious since for all $x, y \in A$ we have by setting $z = x \perp y$, $(x \perp y) \vdash S \dashv (x \perp y)^{-1} = z \vdash S \dashv z^{-1} \subseteq S$ □

Lemma 3.10. *Let $(A, \vdash, \perp, \dashv)$ be a trigroup with distinguish bar-unit 1. A subtrigroup S of A is said to be normal if and only if $(x \perp y) \vdash S = S \dashv (x \perp y)$ for all $x, y \in A$.*

Proof. Assume that S is a normal subtrigroup of A. Let $x, y \in A$ and set $z = x \perp y$. For all $s \in S$, we have: $z \vdash s \dashv z^{-1} = s'$ for some $s' \in S$, i.e., $z \vdash s = z \vdash (s \dashv 1)$ $= z \vdash (s \dashv (z^{-1} \dashv z)) = z \vdash ((s \dashv z^{-1}) \dashv z) \overset{p_3}{=} (z \vdash s \dashv z^{-1}) \dashv z = s' \dashv z$. So $(x \perp y) \vdash S \subseteq S \dashv (x \perp y)$.

For the reverse inclusion,

$$\begin{aligned} S \dashv z &= ((z \vdash z^{-1}) \vdash S \dashv 1) \dashv z = (z \vdash (z^{-1} \vdash S) \dashv 1) \dashv z \\ &= z \vdash ((z^{-1} \vdash S) \dashv 1) \dashv z)) = z \vdash (z^{-1} \vdash S \dashv (1 \dashv z)) \\ &= z \vdash (z^{-1} \vdash S \dashv (z^{-1})^{-1}) \subseteq z \vdash S \quad \text{since} \quad S \quad \text{is normal.} \end{aligned}$$

Conversely, assume that $(x \perp y) \vdash S = S \dashv (x \perp y)$ for all $x, y \in A$. Then,

$$\begin{aligned} (x \perp y) \vdash S \dashv (y^{-1} \perp x^{-1}) &= ((x \perp y) \vdash S) \dashv (x \perp y)^{-1} \\ &= (S \dashv (x \perp y)) \dashv (x \perp y)^{-1} \\ &= S \dashv ((x \perp y) \dashv (x \perp y)^{-1}) \\ &= S \quad \text{since} \quad (x \perp y) \dashv (x \perp y)^{-1} \in \mathfrak{U}_A. \end{aligned}$$

Therefore S is a normal subtrigroup of A. □

The following Lemma is the normality transfer condition for trigroups.

Lemma 3.11. *Let $(A, \vdash, \perp, \dashv)$ be a trigroup. If S is a subtrigroup of A and R is a normal subtrigroup of A, then $S \cap R$ is a normal subtrigroup of S.*

Proof. The proof is obvious since for all $s \in S$, we have $s \vdash S \cap R \dashv s^{-1} \subseteq S$ due to closure under the operations $\vdash, \dashv$, and $s \vdash S \cap R \dashv s^{-1} \subseteq R$ since R is normal in A. The result follows by Lemma 3.9. □

Remark 3.12. Let $(A, \vdash, \perp, \dashv)$ be a trigroup and S a normal subtrigroup of A. Then $S \perp x^{-1} = x^{-1} \vdash S$ for all $x \in A$.

Proof. Let $x \in A$. Since $x^{-1} \vdash x \in \mathfrak{U}_A$, we have

$$\begin{aligned} S \perp x^{-1} &= (x^{-1} \vdash x) \vdash (S \perp x^{-1}) = x^{-1} \vdash (x \vdash (S \perp x^{-1})) \\ &\overset{p_7}{=} x^{-1} \vdash ((x \vdash S) \perp x^{-1}) = x^{-1} \vdash ((S \dashv x) \perp x^{-1}) \\ &= x^{-1} \vdash (S \perp (x \vdash x^{-1})) = x^{-1} \vdash (S \perp 1) = x^{-1} \vdash (1 \dashv S) \\ &\overset{p_3}{=} (x^{-1} \vdash 1) \dashv S = x^{-1} \dashv S. \end{aligned}$$

This completes the proof. □

Lemma 3.13. *Let $\phi : A \to A'$ be a morphism of trigroups. Then $\mathsf{Ker}\,\phi$ is a normal subtrigroup of A. Consequently, the set $\mathfrak{U}_A$ of bar-units of A is a normal subtrigroup of A.*

Proof. By Proposition 3.5, Proposition 3.6 and Lemma 3.9, it remains to show that for all $x \in A$, $x \vdash \mathsf{Ker}\,\phi \dashv x^{-1} \subseteq \mathsf{Ker}\,\phi$. Indeed, let $z \in \mathsf{Ker}\,\phi$,

$$\begin{aligned}\phi(x \vdash z \dashv x^{-1}) &= \phi(x) \vdash \phi(z) \dashv \phi(x^{-1}) = \phi(x) \vdash 1 \dashv (\phi(x))^{-1} \\ &= (\phi(x) \vdash 1) \dashv (\phi(x))^{-1} = ((\phi(x))^{-1})^{-1} \dashv (\phi(x))^{-1} = 1.\end{aligned}$$

So $x \vdash z \dashv x^{-1} \in \mathsf{Ker}\,\phi$. Consequently, $\mathfrak{U}_A$ is a normal subtrigroup by Remark 2.1. □

Lemma 3.14. *Let A be a trigroup. Then the group J_A of inverses of elements in A is a normal subtrigroup of A.*

Proof. By Proposition 3.5 and Lemma 3.9, it is enough to show that if $x \in A$, then $x \vdash J_A \dashv x^{-1} \subseteq J_A$. Notice that for all $y \in A$,

$$x \vdash y = x \vdash (1 \vdash y) = (x \vdash 1) \vdash y = (x^{-1})^{-1} \vdash y.$$

So $x \vdash J_A \dashv x^{-1} = (x^{-1})^{-1} \vdash J_A \dashv x^{-1} \subseteq J_A$ since $x^{-1}, (x^{-1})^{-1} \in J_A$. □

Lemma 3.15. *Let $\phi : A \to A'$ be a morphism of trigroups. Then,*

- (a) *If S is a normal subtrigroup of A and ϕ is surjective, then $\phi(S)$ is a normal subtrigroup of A'.*
- (b) *If S' is a normal subtrigroup of A', then $\phi^{-1}(S')$ is a normal subtrigroup of A.*

Proof. To prove (a), assume that S is a normal subtrigroup of A and ϕ is surjective. By Proposition 3.6 and Lemma 3.9, it remains to show that $y \vdash \phi(S) \dashv y^{-1} \subseteq \phi(S)$ for all $y \in A'$. let $y \in A'$ and $s \in S$. Then, $y = \phi(x)$ for some $x, \in A$. We have

$$\begin{aligned}y \vdash \phi(s) \dashv y^{-1} &= \phi(x) \vdash \phi(s) \dashv (\phi(x))^{-1} = \phi(x) \vdash \phi(s) \dashv \phi(x^{-1}) \\ &= \phi(x \vdash s \dashv x^{-1}) \in \phi(S) \ \text{ since } S \text{ is normal in } A.\end{aligned}$$

The proof of (b) is similar. □

4. Quotient trigroups

4.1. From quotient trigroups to groups

In an effort to study the notion of quotient of a given trigroup by a normal subtrigroup, we define an equivalence relation for which the equivalence classes are the cosets of the normal subtrigroup, and the equivalence class of the identity element is the normal subtrigroup.

Lemma 4.1. *Let $(A, \vdash, \perp, \dashv)$ be a trigroup, and S a subtrigroup of A. Then the following assertions are true:*

(*a*) $g \vdash S = S \iff g^{-1} \in S \iff S \dashv g = S$ *for all* $g \in A$.

(*b*) $g \vdash S = h \vdash S \iff g^{-1} \dashv h \in S$.

(*c*) $S \dashv g = S \dashv h, \iff g \vdash h^{-1} \in S$

Proof. For (a), it is clear that for all $g \in A$, $(g^{-1})^{-1} = g \vdash 1 \in g \vdash S$. So if $g \vdash S = S$, then $(g^{-1})^{-1} \in S$ which implies $g^{-1} \in S$. Conversely, let $g \in A$ such that $g^{-1} \in S$. So $g \vdash 1 = (g^{-1})^{-1} \in S$. Then $g \vdash S = g \vdash (1 \vdash S) = (g \vdash 1) \vdash S \subseteq S$ since S is closed under the operation $\vdash$. For the reverse inclusion, we have for all $s \in S$, that $s = 1 \vdash s = (g \vdash g^{-1}) \vdash s = g \vdash (g^{-1} \vdash s) \in g \vdash S$. This proves that $g \vdash S = S \iff g^{-1} \in S$. The proof of the other equivalence is similar.

To prove (b), let $g, h \in A$ such that $g \vdash S = h \vdash S$, then there exists $s \in S$ such that $h \vdash 1 = g \vdash s$. So

$$\begin{aligned} g^{-1} \dashv h &= g^{-1} \dashv (h \dashv 1) \overset{p_5)}{=} g^{-1} \dashv (h \vdash 1) = g^{-1} \dashv (g \vdash s) \\ &\overset{p_5}{=} g^{-1} \dashv (g \dashv s) = (g^{-1} \dashv g) \dashv s = 1 \dashv s \in S. \end{aligned}$$

Conversely, let $g, h \in A$ such that $g^{-1} \dashv h \in S$. Then

$$\begin{aligned} h \vdash S &= ((g \vdash g^{-1}) \vdash h) \vdash S = (g \vdash (g^{-1} \vdash h)) \vdash S \\ &= g \vdash ((g^{-1} \vdash h) \vdash S) = g \vdash (g^{-1} \vdash (h \vdash S)) \\ &\overset{p_1}{=} g \vdash ((g^{-1} \dashv h) \vdash S) \subseteq g \vdash S. \end{aligned}$$

The reverse inclusion holds also since $h^{-1} \dashv g = h^{-1} \dashv (g^{-1})^{-1} = (g^{-1} \dashv h)^{-1} \in S$.

The proof of (c) is similar to the proof of (b). $\square$

Proposition 4.2. *Let $(A, \vdash, \perp, \dashv)$ be a trigroup and S a subtrigroup of A. Define the relation: For $x, y \in A$,*

$$x \sim y \iff x^{-1} \dashv y \quad \in S.$$

Then $\sim$ is an equivalence relation and the equivalence classes are the left cosets $x \vdash S$, $x \in A$ (orbits of the action of S on A.)

Proof. For all $x, y, z \in A$, we have

i) $x^{-1} \dashv x = 1 \in S$,

ii) if $x^{-1} \dashv y \in S$ then $y^{-1} \dashv x = y^{-1} \dashv (x^{-1})^{-1} = (x^{-1} \dashv y)^{-1} \in S$,

iii) if $x^{-1} \dashv y \in S$ and $y^{-1} \dashv z \in S$, then

$$\begin{aligned} x^{-1} \dashv z &= (x^{-1} \vdash 1) \dashv z = (x^{-1} \vdash (y \vdash y^{-1})) \dashv z \overset{p_1}{=} ((x^{-1} \dashv y) \vdash y^{-1}) \dashv z) \\ &\overset{p_3}{=} (x^{-1} \dashv y) \vdash (y^{-1} \dashv z) \in S. \end{aligned}$$

These prove that $\sim$ is respectively reflexive, symmetric and transitive, and by Lemma 4.1(b), the equivalence classes are left cosets $x \vdash S$ □

By the fundamental theorem of equivalence relations, the relation $\sim$ partitions A into the left cosets $x \vdash S$, $x \in A$. Let A/S be the set of left cosets. Define the following binary operations $\rhd, \vartriangle, \lhd$: $A/S \times A/S \to A/S$ by:

$$(g \vdash S) \rhd (h \vdash S) = (h \vdash g) \vdash S$$

$$(g \vdash S) \lhd (h \vdash S) = (h \dashv g) \vdash S$$

$$(g \vdash S) \vartriangle (h \vdash S) = (h \perp g) \vdash S.$$

We have the following result.

Lemma 4.3. *Let $(A, \vdash, \perp, \dashv)$ be a trigroup and S a normal subtrigroup of A. Then for all $x, y \in A$, $x \sim y \iff x^{-1} \vdash S \dashv y \subseteq S$.*

Proof. Let $x, y \in A$ such that $x \sim y$ i.e. $x^{-1} \dashv y \in S$. Since $y \dashv y^{-1} \in \mathfrak{U}_A$, it follows that for all $s \in S$,

$$\begin{aligned} (x^{-1} \vdash s) \dashv y &= (x^{-1} \vdash ((y \dashv y^{-1}) \vdash s)) \dashv y \overset{p_1}{=} (x^{-1} \vdash (y \vdash (y^{-1} \vdash s))) \dashv y \\ &\overset{p_1}{=} ((x^{-1} \dashv y) \vdash (y^{-1} \vdash s)) \dashv y \overset{p_3}{=} (x^{-1} \dashv y) \vdash (y^{-1} \vdash (s \dashv y)) \in S \end{aligned}$$

since S is normal and S is closed under $\vdash$. For the converse, if $x, y \in A$ such that $x^{-1} \vdash S \dashv y \subseteq S$, then $x^{-1} \dashv y = (x^{-1} \vdash 1) \dashv y \in (x^{-1} \vdash S) \dashv y \subseteq S$. □

Proposition 4.4. *Let $(A, \vdash, \perp, \dashv)$ be a trigroup and S a normal subtrigroup of A. Then the binary operations $\rhd, \vartriangle, \lhd$ are well-defined and equip A/S with a structure of a group with unit S and the inverse of the class $g \vdash S$ is the class $g^{-1} \vdash S$.*

Proof. First we verify that the operations $\rhd$, $\vartriangle$, and $\lhd$ are equal, then we verify their well-definition and their compatibility with the equivalence relation $\sim$. Indeed, let $x, y \in A$. Then, since $y^{-1} \vdash x^{-1} = y^{-1} \dashv x^{-1} = y^{-1} \perp x^{-1}$ as $\vdash = \dashv = \perp$ in J_A, It follows that

$$(x \vdash y)^{-1} \dashv (x \dashv y) = (x \perp y)^{-1} \dashv (x \dashv y) = (x \dashv y)^{-1} \dashv (x \dashv y) = 1 \in S.$$

So $(x \vdash y) \sim (x \perp y) \sim (x \dashv y)$. Therefore,

$$(x \vdash S) \rhd (y \vdash S) = (x \vdash S) \vartriangle (y \vdash S) = (x \vdash S) \lhd (y \vdash S).$$

To show the well-definition, let $x, y, a, b \in A$ such that $x \sim y, a \sim b$.

So $z := a^{-1} \dashv b \in S$ and thus $x^{-1} \vdash z \dashv y \in S$ by Lemma 4.3. Then

$$\begin{aligned}(a \vdash x)^{-1} \dashv (b \vdash y) &= (x^{-1} \vdash a^{-1}) \dashv (b \vdash y) \overset{p_3}{=} x^{-1} \vdash (a^{-1} \dashv (b \vdash y))\\ &\overset{p_5}{=} x^{-1} \vdash (a^{-1} \dashv (b \dashv y)) = x^{-1} \vdash ((a^{-1} \dashv b) \dashv y)\\ &= x^{-1} \vdash (z \dashv y) \in S.\end{aligned}$$

So $(a \vdash x) \sim (b \vdash y)$.

To show that S is the unique bar-unit , we prove that $\mathfrak{U}_{A/S} = \{S\}$. Indeed, notice that for all $a, x \in A$,

$$(x \dashv a)^{-1} \dashv x = (a^{-1} \dashv x^{-1}) \dashv x = a^{-1} \dashv (x^{-1} \dashv x) = a^{-1} \dashv 1 = a^{-1}$$

and

$$(a \vdash x)^{-1} \dashv x = (x^{-1} \vdash a^{-1}) \dashv x = (a^{-1} \vdash x^{-1}) \dashv x \overset{p_3}{=} a^{-1} \vdash (x^{-1} \dashv x) = a^{-1}.$$

So $x \dashv a \ \sim x \iff a^{-1} \in S \iff a \vdash x \sim x$. Therefore,

$$\mathfrak{U}_{A/S} = \{a \vdash S : a^{-1} \in S\} = \{S\}$$

by the first property of Lemma 4.1. That the inverse of the class $g \vdash S$ is the class $g^{-1} \vdash S$ is straighforward. We can now conclude that if $(A, \vdash, \perp, \dashv)$ is a trigroup, then $(A/S, \rhd{=}\vartriangle{=}\lhd)$ is a group. □

Remark 4.5. Proposition 4.4 provides another functor from the category of trigroups to the category of groups.

Remark 4.6. Note that every normal subtrigroup is the kernel of some trigroup homomorphism. More precisely, if S is a normal subtrigroup of a trigroup A, then the natural projection $A \to A/S$ is a homomorphism with kernel equal to S.

4.2. A First Isomorphism Theorem for trigroups

Lemma 4.7. *Let $\phi : A \to A'$ be a morphism of trigroups and S a normal subtrigroup of A containing $\mathsf{Ker}\,\phi$. If $t \in A$ such that $\phi(t) \in \phi(S)$, then $t^{-1} \in S$.*

Proof. Under the hypothesis, we have $\phi(t) = \phi(s)$ for some $s \in S$. So $\phi(t \vdash s^{-1}) = \phi(t) \vdash \phi(s^{-1}) = 1$. Thus $t \vdash s^{-1} \in \mathsf{Ker}\,\phi \subseteq S$. Therefore $t^{-1} = ((t^{-1})^{-1})^{-1} \in S$ since $(t^{-1})^{-1} = t \vdash 1 = t \vdash (s^{-1} \dashv s)) = (t \vdash s^{-1}) \dashv s \quad \in S$. □

Proposition 4.8. *Let A and A' be two trigroups and S a normal subtrigroup of A. Let $\phi : A \to A'$ be a morphism of trigroups such that $\mathsf{Ker}\,(\phi) \subseteq S$. Then there is an isomorphism of groups $\hat{\phi} : A/S \to Im\phi/\phi(S)$. In particular, if $S = ker(\phi)$ then this isomorphism becomes $\hat{\phi} : A/ker(\phi) \to Im\phi/\{1\}$.*

Proof. Since S is a normal subtrigroup of A and $\phi : A \to A$ a morphism of trigroups, then $\phi(S)$ is normal subtrigroup of $Im\phi$ by Lemma 3.15. Moreover

$$\begin{aligned} x \sim y &\Longleftrightarrow x^{-1} \dashv y \in S \Longleftrightarrow \phi(x^{-1} \dashv y) \in \phi(S) \Longleftrightarrow \phi(x^{-1}) \dashv \phi(y) \in \phi(S) \\ &\Longleftrightarrow (\phi(x))^{-1} \dashv \phi(y) \in \phi(S) \Longleftrightarrow \phi(x) \sim \phi(y). \end{aligned}$$

Note that the implication $x^{-1} \dashv y \in S \Longleftarrow \phi(x^{-1} \dashv y) \in \phi(S)$ above is due to Lemma 4.7 since $y^{-1} \dashv x = y^{-1} \dashv (x^{-1})^{-1} = (x^{-1} \dashv y)^{-1} \in S$ and the relation $\sim$ is symmetric. Therefore ϕ induces the isomorphism: $\hat{\phi} : A/S \longrightarrow Im\phi/\phi(S)$ such that $x \vdash S \longmapsto \hat{\phi}(x \vdash S) = \phi(x) \vdash \phi(S)$. □

Corollary 4.9. *Let A be a trigroup. Then there is a group isomorphism*

$$A/\mathfrak{U}_A \cong J_A.$$

Proof. By the assertion (b) of Remark 2.1, the mapping $A \to J_A$ defined by $x \mapsto (x^{-1})^{-1}$ is an epimorphism of trigroups with kernel $\mathfrak{U}_A$. Moreover $J_A/\{1\} = J_A$ since J_A is a group. We conclude the proof using Proposition 4.8. □

Corollary 4.10. *Let A be a trigroup. Then there is a group isomorphism*

$$A/\{1\} \cong A/\mathfrak{U}_A.$$

Proof. Clearly, the map $A \xrightarrow{\pi} A/\mathfrak{U}_A\ ,\quad a \longmapsto a \vdash \mathfrak{U}_A$ is a trigroup epimorphism whose kernel is $ker(\pi) = \{1\}$ since by the first property of Lemma 4.1, we have $a \vdash \mathfrak{U}_A = \mathfrak{U}_A \iff a^{-1} \in \mathfrak{U}_A \cap J_A = \{1\} \iff a = 1$. By proposition 4.8, there is a group isomorphism $A/\{1\} \cong A/\mathfrak{U}_A$. □

Corollary 4.11. *Let A and B be two trigroups. Then A can be identified with a normal subtrigroup $A \times \mathfrak{U}_B$ of $A \times B$ and there is a group isomorphism $\frac{A\times B}{A\times \mathfrak{U}_B} \cong B/\{1\}$.*

Proof. Assume that $(A, \vdash, \dashv, \perp)$ and $(B, \vdash', \dashv', \perp')$ are two trigroups. Then clearly $(A \times B, \rhd, \lhd, \unrhd)$ is a trigroup with operations given by

$(a_1, b_1) \rhd (a_2, b_2) = (a_1 \vdash a_2, b_1 \vdash' b_2)$,
$(a_1, b_1) \lhd (a_2, b_2) = (a_1 \dashv a_2, b_1 \dashv' b_2)$,
$(a_1, b_1) \unrhd (a_2, b_2) = (a_1 \perp a_2, b_1 \perp' b_2)$.

It is easy to verify that the map $A \times B \xrightarrow{\theta} B/\mathfrak{U}_B\ ,\quad (a, b) \longmapsto b \vdash \mathfrak{U}_B$ is a trigroup epimorphism whose kernel is $ker(\theta) = A \times \mathfrak{U}_B$ by the first property of Lemma 4.1 and since $e \in \mathfrak{U}_B \iff e^{-1} \in \mathfrak{U}_B$. By proposition 4.8, there is a group isomorphism $\frac{A\times B}{A\times \mathfrak{U}_B} \cong B/\mathfrak{U}_B$. Now since $B/\mathfrak{U}_B \cong B/\{1\}$ thanks to Corollary 4.10, the proof is complete. □

4.3. A Second Isomorphism Theorem for trigroups

In this section, we use our construction of quotients on trigroups to prove an analogue of the second isomorphism theorem for trigroups. Consider the following set:

$S \star S' = \{x \star x',\ x \in S \text{ and } x' \in S'\}$ where $\star \in \{\vdash, \dashv\}$.

Lemma 4.12. *Let A be a trigroup, and S, R two subtrigroups of A such that $s \vdash R = R \dashv s$ for all $s \in S$. Then the following hold:*

(*a*) *The set $\widehat{R} =: \{x \in A\ :\ x^{-1} \in R\}$ is a subtrigroup of A containing R.*

(*b*) *$S \vdash R$ is a subtrigroup of A.*

(*c*) *R is a normal subtrigroup of $S \vdash R$.*

(*d*) *$S \cap \widehat{R}$ is a normal subtrigroup of S.*

Proof. The proof of (*a*) is straightforward since R is a subtrigroup of A.

To show (*b*), we verify the properties of Proposition 3.4. Indeed, Let $s, s_1 \in S$ and $r, r_1 \in R$. Since $R \dashv s_1 = s_1 \vdash R$, it follows that $r \dashv s_1 = s_1 \vdash r_2$ for some $r_2 \in R$.

1) $$\begin{aligned}(s \vdash r) \vdash (s_1 \vdash r_1) &\overset{p_1}{=} ((s \vdash r) \dashv s_1) \vdash r_1 \overset{p_3}{=} (s \vdash (r \dashv s_1)) \vdash r_1\\ &= (s \vdash (s_1 \vdash r_2)) \vdash r_1 = (s \vdash s_1) \vdash (r_2 \vdash r_1) \in S \vdash R.\end{aligned}$$

2) $$\begin{aligned}(s \vdash r) \dashv (s_1 \vdash r_1) &\overset{p_3}{=} s \vdash (r \dashv (s_1 \vdash r_1)) \overset{p_5}{=} s \vdash (r \dashv (s_1 \dashv r_1))\\ &= s \vdash ((r \dashv s_1) \dashv r_1) = s \vdash ((s_1 \vdash r_2) \dashv r_1)\\ &\overset{p_3}{=} (s \vdash (s_1 \vdash r_2)) \dashv r_1 = ((s \vdash s_1) \vdash r_2) \dashv r_1\\ &\overset{p_3}{=} (s \vdash s_1) \vdash (r_2 \dashv r_1) \in S \vdash R.\end{aligned}$$

3) $$\begin{aligned}(s \vdash r) \perp (s_1 \vdash r_1) &\overset{p_8}{=} ((s \vdash r) \dashv s_1) \perp r_1 \overset{p_3}{=} (s \vdash (r \dashv s_1)) \perp r_1\\ &= (s \vdash (s_1 \vdash r_2)) \perp r_1 = ((s \vdash s_1) \vdash r_2) \perp r_1\\ &\overset{p_4}{=} (s \vdash s_1) \vdash (r_2 \perp r_1) \in S \vdash R.\end{aligned}$$

4) Since $R \dashv s^{-1} = s^{-1} \vdash R$, then $r^{-1} \dashv s^{-1} = s^{-1} \vdash r_0$ for some $r_0 \in R$. So $(s \vdash r)^{-1} = r^{-1} \vdash s^{-1} = r^{-1} \dashv s^{-1} = s^{-1} \vdash r_0 \in\ S \vdash R$.

To show (*c*), we first notice that $R \subseteq S \vdash R$ since $r = 1 \vdash r$ for all $r \in R$. Now let $s \in S$ and $r, r_0 \in R$. Then

$$\begin{aligned}(s \vdash r) \vdash r_0 \dashv (s \vdash r)^{-1} &= (s \vdash r) \vdash r_0 \dashv (r^{-1} \vdash s^{-1})\\ &\overset{p_5}{=} (s \vdash r) \vdash r_0 \dashv (r^{-1} \dashv s^{-1})\\ &= s \vdash (r \vdash r_0 \dashv r^{-1}) \dashv s^{-1} \in s \vdash R \dashv s^{-1} \subseteq R.\end{aligned}$$

To show (*d*), we first notice that $S \cap \widehat{R} \neq \emptyset$ as $1 \in S \cap \widehat{R}$. Also it is clear that $S \cap \widehat{R} \subseteq S$. Now for all $s \in S$ and $t \in S \cap \widehat{R}$, we have $s \vdash t \dashv s^{-1} \in S$ since

S is a subtrigroup of A. Also, since $s \vdash R = R \dashv s$, $s \vdash t^{-1} = t' \dashv s$ for some $t' \in R$. So $s \vdash t^{-1} \dashv s^{-1} = (t' \dashv s) \dashv s^{-1} = t' \dashv (s \dashv s^{-1}) = t' \in R$. We now have $(s \vdash t \dashv s^{-1})^{-1} = (s^{-1})^{-1} \vdash t^{-1} \dashv s^{-1} = s \vdash t^{-1} \dashv s^{-1} \in R$, and thus $s \vdash t \dashv s^{-1} \in \widehat{R}$. Therefore, $s \vdash t \dashv s^{-1} \in S \cap \widehat{R}$. $\square$

Corollary 4.13. *Let A be a trigroup, and S and R two subtrigroups of A such that $s \vdash R = R \dashv s$ for all $s \in S$. Then there is a group isomorphism*

$$(S \vdash R)/R \cong S/(S \cap \widehat{R}).$$

Proof. By Lemma 4.12, $S \vdash R$ is a subtrigroup of A having R as a normal subtrigroup, and that $S \cap R$ is a normal subtrigroup of S. The map

$$S \longrightarrow (S \vdash R)/R, \quad s \mapsto s \vdash R$$

is clearly a surjective homomorphism. Its kernel is $S \cap \widehat{R}$ by the first property of Lemma 4.1. The result now follows using Proposition 4.8. $\square$

Corollary 4.14. *Let A be a trigroup, R a normal subtrigroup of A and S a subtrigroup of A such that $A = S \vdash R$. Then*

$$A/R \cong S/(S \cap \widehat{R}).$$

Proof. The proof is straightforward as a direct consequence of Corollary 4.13. $\square$

Corollary 4.15. *Let A be a trigroup. Then there are group isomorphisms*

$$(J_A \vdash \mathfrak{U}_A)/\mathfrak{U}_A \cong J_A \quad \textit{and} \quad (\mathfrak{U}_A \vdash J_A)/J_A \cong \{\mathfrak{U}_A\}$$

Proof. By Lemma 3.13 and Lemma 3.14, J_A and $\mathfrak{U}_A$ are normal subtrigroups of A. This implies that $e \vdash J_A = J_A \dashv e$ and $j \vdash \mathfrak{U}_A = \mathfrak{U}_A \dashv j$ for all $e \in \mathfrak{U}_A$ and $j \in J_A$. So, $\mathfrak{U}_A$ and J_A are respectively normal subgroups of $J_A \vdash \mathfrak{U}_A$ and $\mathfrak{U}_A \vdash J_A$ by Lemma 4.12. Note that $\widehat{J_A} = A$, thus $\mathfrak{U}_A \cap \widehat{J_A} = \mathfrak{U}_A$. Also, since J_A is a group, $J_A \cap \widehat{\mathfrak{U}}_A = \{1\}$. We now have $(J_A \vdash \mathfrak{U}_A)/\mathfrak{U}_A \cong J_A/\{1\} \cong J_A$ and $(\mathfrak{U}_A \vdash J_A)/J_A \cong \mathfrak{U}_A/\mathfrak{U}_A \cong \{\mathfrak{U}_A\}$ by Corollary 4.13. $\square$

4.4. A Third Isomorphism Theorem for trigroups

Lemma 4.16. *Let A be a trigroup, and S, R two normal subtrigroups of A such that S is a subtrigroup of R. Then $\widehat{R}/S$ is a normal subgroup of A/S.*

Proof. By Lemma 3.11, S is a normal subtrigroup of $\widehat{R}$, and $\widehat{R}/S$ is a subtrigroup of A/S. Now, let $a \in A$. Then for all $r \in R$, $r^{-1} \in R$. So, $(a \vdash r \dashv a^{-1})^{-1} = (a^{-1})^{-1} \vdash r^{-1} \dashv a^{-1} \in R$ since R is a normal subtrigroup of A. Hence $a \vdash r \dashv a^{-1} \in \widehat{R}$. We now have

$$\begin{aligned}(a \vdash S) \rhd (r \vdash S) \lhd (a^{-1} \vdash S) &= ((a \vdash r) \vdash S) \lhd (a^{-1} \vdash S)\\ &= (a \vdash r \dashv a^{-1}) \vdash S \in \widehat{R}/S.\end{aligned}$$

Hence $\widehat{R}/S$ is a normal subtrigroup of A/S. $\square$

Proposition 4.17. *Let A be a trigroup, and S and R two normal subtrigroups of A such that S is a normal subgroup of R. Then, there is a group ismorphism*

$$(A/S)/(\hat{R}/S) \cong A/R.$$

Proof. Under the hypothesis of the proposition, S is also a normal subtrigroup of $\hat{R}$. Now consider the map: $A/S \xrightarrow{\tau} A/R, (a \vdash S) \longmapsto (a \vdash R)$. Then τ is obviously a surjective morphism of groups whose kernel is $ker(\tau) = \hat{R}/S$, by the first property of Lemma 4.1. We now conclude by proposition 4.8 that $(A/S)/(\hat{R}/S) \cong A/R$. □

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The transitivity of primary conjugacy in a class of semigroups

Maria Borralho

Abstract. Elements a, b of a semigroup S are said to be *primarily conjugate* or just *p-conjugate*, if there exist $x, y \in S^1$ such that $a = xy$ and $b = yx$. The p-conjugacy relation generalizes conjugacy in groups, but for general semigroups, it is not transitive. Finding the classes of semigroups in which this notion is transitive is an open problem. The aim of this note is to show that for semigroups satisfying $xy \in \{yx, (xy)^n\}$ for some $n > 1$, primary conjugacy is transitive.

By a notion of conjugacy for a class of semigroups, we mean an equivalence relation defined in the language of that class of semigroups such that when restricted to groups, it coincides with the usual notion of conjugacy.

Before introducing the notion of conjugacy that will occupy us, we recall some standard definitions and notation (we generally follow [4]). For a semigroup S, we denote by S^1 the semigroup S if S is a monoid; otherwise S^1 denotes the monoid obtained from S by adjoining an identity element 1.

Any reasonable notion of semigroup conjugacy should coincide in groups with the usual notion. Elements a, b of a group G are conjugate if there exists $g \in G$ such that $a = g^{-1}bg$. Conjugacy in groups has several equivalent formulations that avoid inverses, and hence generalize syntactically to any semigroup. For many of these notions including the one we focus on here, we refer the reader to [2, 5, 8].

For example, if G is a group, then $a, b \in G$ are conjugate if and only if $a = uv$ and $b = vu$ for some $u, v \in G$. Indeed, if $a = g^{-1}bg$, then setting $u = g^{-1}b$ and $v = g$ gives $uv = a$ and $vu = b$; conversely, if $a = uv$ and $b = vu$ for some $u, v \in G$, then setting $g = v$ gives $g^{-1}bg = v^{-1}vuv = uv = a$.

This last formulation was used to define the following relation on a free semigroup S (see [9]):

$$a \sim_{\mathrm{p}} b \qquad \Longleftrightarrow \qquad \exists_{u,v \in S^1} \quad a = uv \text{ and } b = vu.$$

If S is a free semigroup, then $\sim_{\mathrm{p}}$ is an equivalence relation on S [9, Cor.5.2], and so it can be considered as a notion of conjugacy in S. In a general semigroup S, the relation $\sim_{\mathrm{p}}$ is reflexive and symmetric, but not transitive. If $a \sim_{\mathrm{p}} b$ in a semigroup, we say that a and b are *primarily conjugate* or just p-conjugate for short (hence the subscript in $\sim_{\mathrm{p}}$); a and b were said to be "primarily related" in [8].

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Lallement [9] credited the idea of the relation $\sim_{\rm p}$ to Lyndon and Schützenberger [10].

In spite of its name, $\sim_{\rm p}$ is a valid notion of conjugacy only in the class of semigroups in which it is transitive. Otherwise, the transitive closure ${\sim_{\rm p}}^*$ of $\sim_{\rm p}$ has been defined as a conjugacy relation in a general semigroup [3, 7, 8]. Finding classes of semigroups in which $\sim_{\rm p}$ itself is transitive, that is, $\sim_{\rm p}={\sim_{\rm p}}^*$, is an open problem. The aim of this note is to prove the following theorem.

Theorem. *Let $n > 1$ be an integer and let S be a semigroup satisfying the following: for all $x, y \in S$,*

$$xy \in \{yx, (xy)^n\} \ .$$

Then primary conjugacy $\sim_p$ is transitive in S.

There are various motivations for studying this particular class of semigroups. First, it naturally generalizes two classes of semigroups in which $\sim_{\rm p}$ is transitive.

Proposition. *Let S be a semigroup.*

(1) *If S is commutative, then $\sim_p$ is transitive.*

(2) *If S satisfies $xy = (xy)^2$ for all $x, y \in S$, then $\sim_p$ is transitive.*

Proof. (1). In a commutative semigroup, $\sim_{\rm p}$ is the identity relation and hence it is trivially transitive.

(2). If $a \sim_{\rm p} b$, then $a = uv$ and $b = vu$ for some $u, v \in S^1$. Thus $a^2 = (uv)^2 = uv = a$ and $b^2 = (vu)^2 = vu = b$ so that a, b are idempotents. In particular, a, b are completely regular elements of S. The restriction of $\sim_{\rm p}$ to the set of completely regular elements is a transitive relation [6]. $\square$

The other motivation for studying this class of semigroups is that it has been of recent interest in other contexts. In particular, J. P. Araújo and M. Kinyon [1] showed that a semigroup satisfying $x^3 = x$ and $xy \in \{yx, (xy)^2\}$ for all x, y is a semilattice of rectangular bands and groups of exponent 2.

The proof of Theorem was found by first proving the special cases $n = 2, 3, 4$ using the automated theorem prover `Prover9` developed by McCune [11]. After studying these proofs, the pattern became apparent, leading to the proof of the general case. Note that `Prover9` and other automated theorem provers usually cannot handle statements like our theorem directly because there is not a way to specify that n is a fixed positive integer. Thus the approach of examining a few special cases and then extracting a human proof of the general case is the most efficient way to use an automated theorem prover in these circumstances.

Proof of Theorem. Suppose $a, b, c \in S$ satisfy $a \sim_{\rm p} b$ and $b \sim_{\rm p} c$. Since $a \sim_{\rm p} b$, there exist $a_1, a_2 \in S^1$ such that $a = a_1a_2$ and $b = a_2a_1$. Similarly, since $b \sim_{\rm p} c$, there exist $b_1, b_2 \in S^1$ such that $b = b_1b_2$ and $c = b_2b_1$. We want to prove there exist $x, y \in S^1$ such that $a = xy$ and $c = yx$. If $a = b$ or if $b = c$, then there is nothing to prove. Thus we may assume without loss of generality that $a_1a_2 \neq a_2a_1$ and $b_2b_1 \neq b_1b_2$.

Assume first that $n = 2$. Then

$$a = a_1a_2 = (a_1a_2)(a_1a_2) = a_1(a_2a_1)a_2 = a_1ba_2 = (a_1b_1)(b_2a_2)\,,$$

and

$$c = b_2b_1 = (b_2b_1)(b_2b_1) = b_2(b_1b_2)b_1 = b_2bb_1 = (b_2a_2)(a_1b_1)\,.$$

Thus setting $x = a_1b_1$ and $y = b_2a_2$, we have $a \sim_{\mathrm{p}} c$ in this case.

Now assume $n > 2$. We have

$$\begin{aligned} a = a_1a_2 = (a_1a_2)^n &= \underbrace{(a_1a_2)\cdots(a_1a_2)}_{n} \\ &= a_1\underbrace{(a_2a_1)\cdots(a_2a_1)}_{n-1}a_2 \\ &= a_1b^{n-1}a_2 \\ &= a_1bb^{n-2}a_2 \\ &= a_1(b_1b_2)b^{n-2}a_2 \\ &= (a_1b_1)(b_2b^{n-2}a_2) \end{aligned}$$

and

$$\begin{aligned} c = b_2b_1 = (b_2b_1)^n &= \underbrace{(b_2b_1)\cdots(b_2b_1)}_{n} \\ &= b_2\underbrace{(b_1b_2)\cdots(b_1b_2)}_{n-1}b_1 \\ &= b_2b^{n-1}b_1 \\ &= b_2b^{n-2}bb_1 \\ &= b_2b^{n-2}(a_2a_1)b_1 \\ &= (b_2b^{n-2}a_2)(a_1b_1)\,. \end{aligned}$$

Thus setting $x = a_1b_1$ and $y = b_2b^{n-2}a_2$, we have that $a \sim_{\mathrm{p}} c$. $\square$

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Hyperidentities with permutations leading to the isotopy of invertible binary algebras to a group

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Abstract. Using the second-order formulas we obtained characterizations of binary invertible algebras principally isotopic to a group or to an abelian group.

1. Introduction

A binary algebra $(Q;\Sigma)$ is called an *invertible algebra* or *system of quasigroups* if each operation in Σ is a quasigroup operation. Invertible algebras with second order formulas first were considered by Shaufler [12, 13] in connection with coding theory. He pointed out that the resulting message would be more difficult to decode by unauthorized receiver than in the case when a single operation is used for calculation. Later such algebras were investigated by Aczel [1], Belousov [3, 4], Sade [11], Movsisyan [8, 9, 10] and others.

It is well known [5] that with each quasigroup A the next five quasigroups are connected:

$$A^{-1},\ {}^{-1}A,\ {}^{-1}(A^{-1}),\ ({}^{-1}A)^{-1},\ A^*,$$

where $A^*(x,y) = A(y,x)$. These quasigroups are called *inverse quasigroups* or *parastrophes*. Like this, with each invertible algebra $(Q;\Sigma)$ the next five invertible algebras are connected:

$$(Q;\Sigma^{-1}),\ (Q;{}^{-1}\Sigma),\ (Q;{}^{-1}(\Sigma^{-1})),\ (Q;({}^{-1}\Sigma)^{-1}),\ (Q;\Sigma^*),$$

where

$$\begin{aligned}\Sigma^{-1} &= \{A^{-1}\,|\,A\in\Sigma\},\\ {}^{-1}\Sigma &= \{{}^{-1}A\,|\,A\in\Sigma\},\\ {}^{-1}(\Sigma^{-1}) &= \{{}^{-1}(A^{-1})\,|\,A\in\Sigma\},\\ ({}^{-1}\Sigma)^{-1} &= \{({}^{-1}A)^{-1}\,|\,A\in\Sigma\},\\ \Sigma^* &= \{A^*\,|\,A\in\Sigma\}.\end{aligned}$$

Each of these invertible algebras is called a *parastrophe of the algebra* $(Q;\Sigma)$.

Let us recall that the following absolutely closed second-order formula:

$$\forall X_1,\dots,X_m\forall x_1,\dots,x_n\quad(\omega_1=\omega_2),$$
$$\forall X_1,\dots,X_k\exists X_{k+1}\dots,X_m\forall x_1,\dots,x_n\quad(\omega_1=\omega_2),$$

where ω_1, ω_2 are words written in the functional variables, $X_1,\dots,X_m$, and in the objective variables, $x_1,\dots,x_n$, are called $\forall(\forall)$-*identity* or *hyperidentity* and $\forall\exists(\forall)$-*identity.* For see [8].

The groupoid $Q(A)$ is *isotopic* to the groupoid $Q(B)$ if exist three permutations α,β,γ of Q such that $\gamma B(x,y) = A(\alpha x,\beta y)$ for all $x,y\in Q$. The isotopy of the form $T=(\alpha,\beta,\varepsilon)$, where ε is the identity map, is called a *principal isotopy.*

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The class of quasigroups isotopic to groups first were considered by Belousov [4]. Varieties of quasigroups isotopic to groups have been considered by Glukhov, Gvaramia, Sokhatsky and others. In [6] the concept of identities with permutations was introduced and isotopies of quasigroups to groups was characterized by these identities.

We introduce the notion of the hyperidentity with permutations and using these hyperidentities we obtain characterizations of binary invertible algebras principally isotopic to a group.

2. Auxiliary concepts and results

We start with some concepts and results, which are necessary for further considerations.

Definition 2.1. The triplet $T = (\alpha, \beta, \gamma)$ of permutations of the set Q is called an *autotopy* of the groupoid $Q(\cdot)$, if the identity $\gamma(x \cdot y) = \alpha x \cdot \beta y$ is true for for all $x, y \in Q$. If $T = (\alpha, \beta, \gamma)$ is an autotopy of the groupoid $Q(A)$, then we write $A^T = A$.

In the case $\alpha = \beta = \gamma$ the triplet $T = (\alpha, \alpha, \alpha)$ is an automorphism. It is easy to see that the set of autotopies of $Q(\cdot)$ forms a group.

Definition 2.2. The third component γ of the autotopy $T = (\alpha, \beta, \gamma)$ of the groupoid $Q(\cdot)$ is called a *quasi-automorphism* of $Q(\cdot)$.

Lemma 2.3. (cf. [3]) *Any quasi-automorphism γ of a group $Q(\cdot)$ has the form:*

$$\gamma = \widetilde{R}_s \gamma_0, \quad (\gamma = \widetilde{L}_s \delta_0) \tag{1}$$

where γ_0 (δ_0) is an automorphism of the group $Q(\cdot)$, $\widetilde{R}_s x = x \cdot s$ ($\widetilde{L}_s x = s \cdot x$), $s \in Q$ and, conversely, the map γ defined by the equality (1) *is a quasi-automorphism of the group $Q(\cdot)$.*

Lemma 2.4. (cf. [3]) *Let γ be a quasi-automorphism of the group $Q(\cdot)$. Then γ is an automorphism if and only if $\gamma 1 = 1$, where 1 is the identity element of the group $Q(\cdot)$.*

Lemma 2.5. (cf. [3]) *Let $\alpha, \beta, \gamma, \delta, \sigma, \tau$ be permutations of the set Q, such that the equality*

$$\beta(\alpha(x \cdot y) \cdot z) = \gamma x \cdot \delta(\sigma y \cdot \tau z)$$

is valid in the group $Q(\cdot)$ for all $x, y, z \in Q$. Then the permutations $\alpha, \beta, \gamma, \delta, \sigma, \tau$ are quasi-automorphisms of the group $Q(\cdot)$.

Lemma 2.6. (cf. [3]) *A permutation α of Q is a quasi-automorphism of the group $Q(\cdot)$ if and only if for all $x, y \in Q$ the equality*

$$\alpha(xy) = \alpha x \cdot (\alpha 1)^{-1} \cdot \alpha y,$$

where 1 is the identity of $Q(\cdot)$, is valid.

Theorem 2.7. (cf. [3]) *If a non-empty set Q is a quasigroup under each of four operations A_1, A_2, A_3, A_4 satisfying the identity:*

$$A_1(A_2(x, y), z) = A_3(x, A_4(y, z)), \tag{2}$$

then there exists the operation $(\cdot)$ such $Q(\cdot)$ is a group isotopic to all these four quasigroups.

Theorem 2.8. (cf. [2]) *if a non-empty set Q is a quasigroup under each of six operations $A_1, A_2, A_3, A_4, A_5, A_6$ satisfying the identity:*

$$A_1(A_2(x, y), A_3(z, u)) = A_4(A_5(x, z), A_6(y, u)), \tag{3}$$

then there exists the operation $(\cdot)$ such that $Q(\cdot)$ is an abelian group isotopic to all these six quasigroups, i.e.,

$$\begin{aligned} A_1(x, y) &= \alpha x \cdot \beta y, & A_4(x, y) &= \chi x \cdot \varphi y, \\ A_2(x, y) &= \alpha^{-1}(\gamma x \cdot \delta y), & A_5(x, y) &= \chi^{-1}(\gamma x \cdot \theta y), \\ A_3(x, y) &= \beta^{-1}(\theta x \cdot \psi y), & A_6(x, y) &= \varphi^{-1}(\delta x \cdot \psi y), \end{aligned}$$

where $\alpha, \beta, \gamma, \delta, \chi, \varphi, \psi, \theta$ are permutations of Q.

Definition 2.9. We say that a binary algebra $(Q;\Sigma)$ is *isotopic* to the groupoid $Q(\cdot)$, if each operation in Σ is isotopic to the groupoid $Q(\cdot)$, i.e., for every operation $A \in \Sigma$ there exists permutations $\alpha_A, \beta_A, \gamma_A$ of Q such that:

$$\gamma_A A(x,y) = \alpha_A x \cdot \beta_A y,$$

for every $x, y \in Q$.

Theorem 2.10. (cf. [7]) *The invertible algebra $(Q;\Sigma)$ is principally isotopic to a group if and only if for all $A, B \in \Sigma$ the following second-order formula*

$$A({}^{-1}A(B(x, B^{-1}(y,z)),u),v) = B(x, B^{-1}(y, A({}^{-1}A(z,u),v))),$$

is valid in the algebra $(Q;\Sigma \cup \Sigma^{-1} \cup {}^{-1}\Sigma)$.

3. Main results

We denote by $L_{A,a}$ and $R_{A,a}$ the left and right translations of the binary algebra $(Q;\Sigma)$:

$$L_{A,a} : x \mapsto A(a,x) \;\; (R_{A,a} : x \mapsto A(x,a)).$$

If $(Q;\Sigma)$ is an invertible algebra, then these translations are bijections for all $a \in Q$.

We will consider second order formulas (called *hyperidentities with permutations* or *hyperidentities in* $(Q;\Sigma)$) of the following form:

$$\beta_1^{A,B} A(\beta_2^{A,B} B(\beta_3^{A,B} x, \beta_4^{A,B} y), \beta_5^{A,B} z) = B(\beta_6^{A,B} x, \beta_7^{A,B} A(\beta_8^{A,B} y, \beta_9^{A,B} z)),$$

where x, y, z are objective variables, $\beta_i^{A,B} (i = 1, \ldots, 9)$ are permutations on Q dependent on $A, B \in \Sigma$. By doing paremeter replacement those formulas may be transformed into second order formulas with less number of paramaters:

$$\alpha_1^{A,B} A(\alpha_2^{A,B} B(x,y), z) = B(\alpha_3^{A,B} x, \alpha_4^{A,B} A(\alpha_5^{A,B} y, \alpha_6^{A,B} z)). \tag{4}$$

Theorem 3.11. *If the second order formula* (4) *is valid in the algebra $(Q;\Sigma)$ for all $A, B \in \Sigma$ and for some permutations $\alpha_i^{A,B} (i = 1, \ldots, 6)$, then the algebra $(Q;\Sigma)$ is principally isotopic to a group.*

Conversely, if the invertible algebra $(Q;\Sigma)$ is principally isotopic to a group $Q(\cdot)$, then for all $A, B \in \Sigma$ there exist permutations $\alpha_i^{A,B} (i = 1, \ldots, 6)$ such that the second order formula (4) *is valid in the algebra $(Q;\Sigma)$.*

Proof. Let (4) hold in $(Q;\Sigma)$ for all $A, B \in \Sigma$ and for some permutations $\alpha_i^{A,B} (i = 1, \ldots, 6)$. The second order formula (4) is a particular case of (2), where

$$A_1(x,y) = \alpha_1^{A,B} A(x,y), \qquad A_2(x,y) = \alpha_2^{A,B} B(x,y),$$

$$A_3(x,y) = B(\alpha_3^{A,B} x, y), \qquad A_4(x,y) = \alpha_4^{A,B} A(\alpha_5^{A,B} x, \alpha_6^{A,B} y).$$

According to Theorem 2.7, the quasigroups A_1, A_2, A_3, A_4 are isotopic to the same group $Q(\cdot)$:

$$A_1(x,y) = \alpha^{-1}(\beta x \cdot \gamma y), \qquad A_2(x,y) = \alpha_1^{-1}(\beta_1 x \cdot \gamma_1 y),$$

$$A_3(x,y) = \lambda^{-1}(\mu x \cdot \nu y), \qquad A_4(x,y) = \lambda_1^{-1}(\mu_1 x \cdot \nu_1 y).$$

Having in consideration the last equalities and (2) we get:

$$\alpha^{-1}(\beta \alpha_1^{-1}(\beta_1 x \cdot \gamma_1 y) \cdot \gamma z) = \lambda^{-1}(\mu x \cdot \nu \lambda_1^{-1}(\mu_1 y \cdot \nu_1 z))$$

or

$$\lambda \alpha^{-1}(\beta \alpha_1^{-1}(x \cdot y) \cdot z) = \mu \beta_1^{-1} x \cdot \nu \lambda_1^{-1}(\mu_1 \gamma_1^{-1} y \cdot \nu_1 \gamma^{-1} z).$$

According to Lemma 2.5, $\lambda\alpha^{-1} = \theta$ is a quasi-automorphism of the group $Q(\cdot)$. Fixing the operation A, we fix the permutation α, too. Then, every operation $B \in \Sigma$ has the form:

$$B(x, y) = A_3((\alpha_3^{A,B})^{-1}x, y) = A_3(\phi x, y) = \lambda^{-1}(\mu\phi x \cdot \nu y)$$

or

$$B(x, y) = \alpha^{-1}\theta^{-1}(\phi' x \cdot \nu y).$$

Since the permutation θ^{-1} is a quasi-automorphism of the group $Q(\cdot)$, then

$$B(x, y) = \alpha^{-1}(\theta^{-1}\phi' x \cdot (\theta^{-1}1)^{-1} \cdot \theta^{-1}\nu y) = \alpha^{-1}(\phi'' x \cdot \psi y),$$

where $\phi'' x = \theta^{-1}\phi' x(\theta^{-1}1)^{-1}, \psi x = \theta^{-1}\nu x$ and 1 is the identity element of the group $Q(\cdot)$.

Consider the operation:

$$x \circ y = \alpha^{-1}(\alpha x \cdot \alpha y).$$

$Q(\circ)$ is isomorphic to the group $Q(\cdot)$. Thus, $(Q(\circ)$ is a group and

$$B(x, y) = \alpha^{-1}\phi'' x \circ \alpha^{-1}\psi y$$

or

$$B(x, y) = fx \circ gy.$$

Hence, $Q(B)$ is principally isotopic to the group $Q(\circ)$ and since B is an arbitrary operation from Σ, this proves the statement.

Conversely, if an invertible algebra is principally isotopic to a group, then according to Theorem 2.10 the following formula is valid:

$$A({}^{-1}A(B(x, B^{-1}(y, z)), u), v) = B(x, B^{-1}(y, A({}^{-1}A(z, u), v))).$$

Taking into account that

$$A^{-1}(x, u) = R_{A^{-1},u}x = L_{A^{-1},x}u \quad \text{and} \quad {}^{-1}A(v, x) = L_{{}^{-1}A,v}x = R_{{}^{-1}A,x}v$$

the above formula may be re-written in the form:

$$A[R_{{}^{-1}A,u}B(x, z), v] = B[x, L_{B^{-1},y}A(R_{{}^{-1}A,u}L^{-1}_{B^{-1},y}z, v)].$$

This for $u = a, y = b$, where $a, b \in Q$ are fixed, gives (4), where

$$\alpha_1^{A,B} = \alpha_3^{A,B} = \alpha_6^{A,B} = \epsilon, \quad \alpha_2^{A,B} = R_{{}^{-1}A,a}, \quad \alpha_4^{A,B} = L_{B^{-1},b}, \quad \alpha_5^{A,B} = R_{{}^{-1}A,a}L^{-1}_{B^{-1},b},$$

and completes the proof. □

Corollary 3.12. (cf. [6]) *The class of quasigroups isotopic to a group is characterized by the identity:*

$$x(b\backslash((z/a)v)) = ((x(b\backslash z))/a)v,$$

where a and b are fixed.

Theorem 3.13. *The invertible algebra $(Q; \Sigma)$ is principally isotopic to an abelian group if and only if for all $A, B \in \Sigma$ the second-order formula*

$$A({}^{-1}A(B(x, z), y), A^{-1}(y, B(w, u))) = A({}^{-1}A(B(w, z), y), A^{-1}(y, B(x, u))). \quad (5)$$

Proof. Let $(Q; \Sigma)$ be an invertible algebra principally isotopic to an abelian group $Q(\cdot)$, i.e., every operation $A \in \Sigma$ has the form:

$$A(x, y) = \alpha_A x \cdot \beta_A y, \quad (6)$$

where α_A, β_A are permutations of the set Q. Then from (6) we obtain:

$$A^{-1}(x, y) = \beta_A^{-1}(\overline{\alpha_A x} \cdot y) \quad \text{and} \quad {}^{-1}A(x, y) = \alpha_A^{-1}(x \cdot \overline{\beta_A y}), \quad (7)$$

where $\overline{x}$ is the inverse element of x in the group $Q(\cdot)$.

Using the identities (6) and (7) we can prove that left and right sides of (5) are the same.

Conversely, let (5) be satisfied in $(Q;\Sigma\cup\Sigma^{-1}\cup^{-1}\Sigma)$ for all $A,B\in\Sigma$. For $y=a$ it has the form:

$$A(C(x,z),D(w,u))=A(C(w,z),D(x,u)), \tag{8}$$

where $C(x,y)={}^{-1}A(B(x,y),a)$ and $D(x,y)=A^{-1}(a,B(x,y))$.

Let's write (8) in the form:

$$A(C^*(z,x),D(w,u))=A(C^*(z,w),D(x,u)). \tag{9}$$

Obviously, the operations C, C^* and D are inverse operations. According to Theorem 2.8, the quasigroups $Q(A)$, $Q(C^*)$ and $Q(D)$ are isotopic to the same abelian group $Q(\cdot)$. Hence,

$$A(x,y)=\alpha x\cdot\beta y,\qquad C^*(x,y)=\alpha^{-1}(\gamma x\cdot\delta y),\qquad D(x,y)=\beta^{-1}(\theta x\cdot\psi y),$$

for some permutations $\alpha,\beta,\gamma,\delta,\theta,\psi$ of Q.

Fixing the operation A, we also fix the permutation α. Then:

$$C^*(y,x)=C(x,y)={}^{-1}A(B(x,y),a)=R_{{}^{-1}A,a}B(x,y)=\alpha^{-1}(\gamma y\cdot\delta x),$$

or

$$B(x,y)=R^{-1}_{{}^{-1}A,a}\alpha^{-1}(\gamma y\cdot\delta x),\qquad B(x,y)=R^{-1}_{{}^{-1}A,a}\alpha^{-1}I(I\delta x\cdot I\gamma y),$$

where $I(x)=\overline{x}$ assigns to x its inverse $\overline{x}$ calculated in the group $Q(\cdot)$. Then the permutation $\phi=I\alpha R_{{}^{-1}A,a}$ depends only on A. Thus, $Q(\circ)$, where $x\circ y=\varphi^{-1}(\varphi x\cdot\varphi y)$, is an abelian group is isomorphic to the group $Q(\cdot)$. In the group $Q(\circ)$ the operation B has the form:

$$B(x,y)=fx\circ gy,$$

where $f=\varphi^{-1}I\delta$, $g=\varphi^{-1}I\gamma$ are permutations of Q. Thus, $Q(B)$ is principally isotopic to the group $Q(\circ)$ and since B is an arbitrary operation from Σ, this proves the theorem. □

Theorem 3.14. *If the second order formula*

$$\alpha_1^{A,B}A[\alpha_2^{A,B}B(\alpha_3^{A,B}x,\alpha_4^{A,B}z),\alpha_5^{A,B}B(\alpha_6^{A,B}w,\alpha_7^{A,B}v)]=A[\alpha_8^{A,B}B(w,z),\alpha_9^{A,B}B(x,v)] \tag{10}$$

is valid in the invertible algebra $(Q;\Sigma)$ for all $A,B\in\Sigma$ and for some permutations $\alpha_i^{A,B}$ where $i=1,2,\dots,9$, then the algebra $(Q;\Sigma)$ is principally isotopic to an abelian group.

Conversely, if the invertible algebra $(Q;\Sigma)$ is principally isotopic to an abelian group $Q(\cdot)$, then for all $A,B\in\Sigma$ there are permutations $\alpha_i^{A,B}$, $i=1,2,\dots,9$, such that the second order formula (10) is valid in the algebra $(Q;\Sigma)$.

Proof. Let (10) holds in $(Q;\Sigma)$ for all $A,B\in\Sigma$ and for some permutations $\alpha_i^{A,B}$, $i=1,2,\dots,9$. Then (10) is a particular case of (3), where

$$A_1(x,y)=\alpha_1^{A,B}A(x,y),\quad A_2(x,y)=\alpha_2^{A,B}B(\alpha_3^{A,B}x,\alpha_4^{A,B}y),\quad A_3(x,y)=\alpha_5^{A,B}B(\alpha_6^{A,B}x,\alpha_7^{A,B}y),$$

$$A_4(x,y)=A(x,y),\qquad A_5(x,y)=\alpha_8^{A,B}B(x,y),\qquad A_6(x,y)=\alpha_9^{A,B}B(x,y).$$

According to Theorem 2.8, the quasigroups A_1, A_2, A_3, A_4, A_5, A_6 are isotopic to the same abelian group $Q(\cdot)$:

$$A_1(x,y)=\alpha x\cdot\phi y,\qquad A_2(x,y)=\alpha^{-1}(\gamma x\cdot\delta y),\qquad A_3(x,y)=\phi^{-1}(\lambda x\cdot\beta y),$$

$$A_4(x,y)=\psi x\cdot\sigma y,\qquad A_5(x,y)=\psi^{-1}(\gamma x\cdot\lambda y),\qquad A_6(x,y)=\sigma^{-1}(\delta x\cdot\beta y).$$

Fixing B, we obtain $A_5(x,y)=\alpha_8^{A,B}B(x,y)=\psi^{-1}(\gamma x\cdot\lambda y)$. Thus ψ is fixed too. Then $Q(\circ)$, where

$$x\cdot y=\psi^{-1}x\circ\psi^{-1}y.$$

is an abelian group and $A(x,y)=A_4(x,y)=\psi x\cdot\sigma y=x\circ\psi^{-1}\sigma y$. Thus, $Q(A)$ is principally isotopic to the group $Q(\circ)$ and as $A\in\Sigma$ is an arbitrary operation, this proves the statement.

Conversely, if the invertible algebra $(Q;\Sigma)$ is principally isotopic to an abelian group, then according to Theorem 3.13 the formula is valid:

$$A({}^{-1}A(B(x,z),y),A^{-1}(y,B(w,u)))=A({}^{-1}A(B(w,z),y),A^{-1}(y,B(x,u))).$$

Then,

$$A[R_{{}^{-1}A,y}B(x,z),L_{A^{-1},y}B(w,u)]=A[R_{{}^{-1}A,y}B(w,z),L_{A^{-1},y}B(x,u)].$$

This for fixed $y=a\in Q$ gives (10) with

$$\alpha_1=\alpha_3=\alpha_4=\alpha_6=\alpha_7=\epsilon,\quad \alpha_8=\alpha_2=R_{{}^{-1}A,a},\quad \alpha_5=\alpha_9=L_{A^{-1},a}.$$ □

Corollary 3.15. *The class of quasigroups isotopic to an abelian group is characterized by the identity:*

$$(xz/y)(y\backslash wu)=(wz/y)(y\backslash xu),$$

where y is fixed.

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Translatability determines the structure of certain types of idempotent quasigroups

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Abstract. We prove that in certain types of k-translatable idempotent quasigroups, the value of k determines all possible orders of k-translatable idempotent quasigroups of a particular type. From this, all k-translatable idempotent quesigroups of that type can be calculated, as well as their parastrophe types. Four operators on the collection of all idempotent, translatable quasigroups are defined and formulae determining relationships amongst them are given. Necessary and sufficient conditions are given for particular types of idempotent, translatable quasigroups to be perpendicular to their dual quasigroup.

1. Introduction

The notion of a k-translatable groupoid was an outcrop of the observation that certain quadratical quasigroups are translatable [6]. This led to the determination of the structure of idempotent, translatable quasigroups in general and of types of idempotent, translatable quasigroups in particular (Theorems 4.2 and 4.27 [5]). These results and Theorem 4.2 [7] inspired the work in this paper.

To say that an idempotent quasigroup $(Q,\cdot)$ of order n is k-translatable is a powerful statement. It implies that $x\cdot y=[ax+by]_n$ for some $a\in\{2,3,\ldots,n\}$ and odd $n>1$, where $[a+b]_n=1$, $[a+kb]_n=0$ and $[t]_n$ equals t calculated modulo n (cf. [5]). In addition, the greatest common divisor of a and n is 1, as is that of b and n and k and n. Also, there exist unique values a', b' and k' such that $[aa']_n=[bb']_n=[kk']_n=1$, where k' is the value of the translatability of the dual quasigroup $(Q,*)$ and $x*y=[bx+ay]_n$. Therefore, $[b+k'a]_n=0$. The products of the parastrophes of $(Q,\cdot)$ and their translatability can also be determined (cf. [5]). We note that idempotent k-translatable quasigroups are medial, that is they satisfy the identity $xy\cdot zw=xz\cdot yw$, and therefore they are what is called in the literature IM-quasigroups (cf. [9]). We denote the collection of all idempotent, medial quasigroups as **IMQ**. We define **IKQ** as the collection of all idempotent, k-translatable quasigroups. By Corollary 4.5 [5], **IKQ** $\subset$ **IMQ**.

To simplify the size of some of the tables we will sometimes let (a,b) denote the idempotent k-translatable quasigroup $x\cdot y=[ax+by]_n$, where $[a+b]_n=1$. For

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example, $(3,3)$ denotes the idempotent 4-translatable quasigroup $x \cdot y = [3x+3y]_5$, and $(2,10)$ denotes the idempotent 2-translatable quasigroup $x \cdot y = [2x+10y]_{11}$.

In this paper we examine certain types of idempotent k-translatable quasigroups. Each type $\mathbf{T}$ in Table 3.1 satisfies a single identity $u_T = v_T$, with $u_T = u_T(x,y)$ and $v_T = v_T(x,y)$. Each identity yields a function $F_T(a)$ such that $[F_T(a)]_n = 0$. This formula allows us to calculate the possible values of n; that is, for each value of a, the formula determines the possible orders of the members of $\mathbf{T}$. Also, the value of a' and k' are determined by the value of k.

The function H_T denotes the function $H_T = H_T(k)$, where $[H_T(k)]_n = 0$. The products of the parastrophes of a given $(Q,\cdot) \in \mathbf{T}$ and the value of their translatability can also be determined by k, the value of the translatability of $(Q,\cdot)$. Also, in any type $\mathbf{T}$ we can calculate all k-translatable quasigroup members of $\mathbf{T}$, for any value of k. We give tables of such quasigroup members for each type $\mathbf{T}$ and each value of k, for $k \in \{2,3,\ldots,10\}$. The main results are given in Tables 3.1, 3.2, 3.3 and 3.4, from which most other results and tables follow.

We examine, for each $\mathbf{T}$, the dual collection $\mathbf{T}^*$ and the inverse collection $-\mathbf{T}$ and prove that the above analysis also applies to these collections of quasigroups. Some interrelationships between different types of idempotent k-translatable quasigroups, their dual collections, their inverse collections and the collections $\mathbf{T}^{+1}$ and $\mathbf{T}^{-1}$ are also given.

We will show how these results link with the work of Belousov. He proved that any minimal non-trivial identity in a quasigroup is parastrophically equivalent to one of seven identity types [1]. We prove that five of those identities determine types of idempotent k-translatable quasigroups and that the remaining two identities do not. We prove in Corollary 6.4 that if $\mathbf{T}$ is the collection of quadratical quasigroups or the collection of affine regular octagonal quasigroups, then any quasigroup member of $\mathbf{T}$ is perpendicular to its dual quasigroup.

2. Preliminary definitions, examples and results

A *groupoid* (in other terminology: a *magma*) is a non-empty set Q with a binary operation (called a *multiplication*) defined on Q and denoted by dot or juxtaposition. For clarity of record we will limit the number of parentheses. Instead of $(x \cdot y) \cdot z$, we will write $xy \cdot z$.

Let us recall that a groupoid $(,\cdot)$ is a *quasigroup* if for every $a,b \in Q$ there exist unique elements $x,y \in Q$ such that $ax = b$ and $ya = b$. An element x of a groupoid $(Q,\cdot)$ is idempotent if $x \cdot x = x$. A finite groupoid $Q = \{1,2,\ldots,n\}$ is called *k-translatable*, where $1 \leqslant k < n$, if the second row of its multiplication table is obtained from the first row by inserting the last k entries of the first row into the first k entries of the second row and the first $n-k$ entries of the first row into the last $n-k$ entries of the second row. This operation is repeated from the second row, to obtain the entries of the third row, and so on until the table is filled (cf. [5]).

The following are the Cayley tables of a 2-translatable idempotent quasigroup of order 3, a 3-translatable idempotent quasigroup of order 5 and a 4-translatable idempotent quasigroup of order 7.

	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

	1	2	3	4	5
1	1	3	5	2	4
2	5	2	4	1	3
3	4	1	3	5	2
4	3	5	2	4	1
5	2	4	1	3	5

	1	2	3	4	5	6	7
1	1	3	5	7	2	4	6
2	7	2	4	6	1	3	5
3	6	1	3	5	7	2	4
4	5	7	2	4	6	1	3
5	4	6	1	3	5	7	2
6	3	5	7	2	4	6	1
7	2	4	6	1	3	5	7

It is known that an idempotent k-translatable quasigroup of order n is induced by the additive group of integers modulo n, where, for simplicity of our calculations, 0 is identified with n, i.e., instead of $Q = \{0, 1, \ldots, n-1\}$ we consider the set $Q = \{1, 2, \ldots, n\}$. In this convention, an idempotent k-translatable quasigroup of order n has the form

$$x \cdot y = [ax + (1-a)y]_n, \quad \text{where} \quad [a + k(1-a)]_n = 0$$

and the greatest common divisor of k and n is 1. Obviously, the greatest common divisor of a and n (also $a-1$ and n) must be 1. The value n must be odd and greater than or equal to 3, while $k \geqslant 2$ (cf. [5, Lemma 4.1]).

It follows that idempotent k-translatable quasigroups satisfy particular identity types if and only if $[F_T(a)]_n = 0$ for some function $F_T(a)$ that is determined by the identity that defines the type T.

The identity types here explored determine well-known types of quasigroups, such as *quadratical* ($\mathbf{Q}$: $xy \cdot x = zx \cdot yz$), *hexagonal* ($\mathbf{H}$: $xy \cdot x = y$), *golden square* ($\mathbf{GS}$: $(xy \cdot z) \cdot z = y$), *right modular* ($\mathbf{RM}$: $xy \cdot z = zy \cdot x$) and *left modular* ($\mathbf{LM}$: $x \cdot yz = z \cdot yx$), *affine regular octagonal* ($\mathbf{ARO}$: $xy \cdot y = yx \cdot x$) and *pentagonal* ($\mathbf{P}$: $(xy \cdot x)y \cdot x = y$). In addition we examine the identities $(yx \cdot x)x = y$ (denoted as $\mathbf{C3}$) and $x(y \cdot yx) = y$ (denoted as $\mathbf{U}$).

For a given collection $\mathbf{T}$ of idempotent k-translatable quasigroups we define the following collection of quasigroups

$$\begin{aligned}
\mathbf{T}^* &= \{(1-a, a) \in \mathbf{IMQ}\} \mid (a, 1-a) \in \mathbf{T}\},\\
-\mathbf{T} &= \{(-a, 1+a) \in \mathbf{IMQ} \mid (a, 1-a) \in \mathbf{T}\},\\
\mathbf{T}^{+\mathbf{t}} &= \{(a+t, 1-a-t) \in \mathbf{IMQ} \mid (a, 1-a) \in \mathbf{T}\},\\
\mathbf{T}^{-\mathbf{t}} &= \{(a-t, 1+t-a) \in \mathbf{IMQ} \mid (a, 1-a) \in \mathbf{T}\},
\end{aligned}$$

where $t \in \{1, 2, \ldots\}$.

These two theorems, that are a modification of Theorems 4.26 and 4.27 from [5], will be used later.

Theorem 2.1. *A k-translatable, naturally ordered quasigroup $(Q,\cdot)$ of order n with the multiplication defined by $x\cdot y=[ax+(1-a)y]_n$, where $a\in\mathbb{Z}_n$ and $[a+(1-a)k]_n=0$ is*

(1) *quadratical if and only if $[2a^2-2a+1]_n=0$,*

(2) *hexagonal if and only if $[a^2-a+1]_n=0$,*

(3) *GS-quasigroup if and only if $[a^2-a-1]_n=0$,*

(4) *right modular quasigroup if and only if $[a^2+a-1]_n=0$,*

(5) *left modular quasigroup if and only if $[a^2-3a+1]_n=0$,*

(6) *ARO-quasigroup if and only if $[2a^2]_n=1$,*

(7) *$C3$ quasigroup if and only if $[a^3]_n=1$.*

Theorem 2.2. *A naturally ordered quasigroup $(Q,\cdot)$ of order n with the multiplication defined by $x\cdot y=[ax+(1-a)y]_n$, where $a\in\mathbb{Z}_n$ and $[a+(1-a)k]_n=0$ is a k-translatable*

(1) *quadratical quasigroup if and only if $k=[1-2a]_n$,*

(2) *hexagonal quasigroup if and only if $k=[1-a]_n$,*

(3) *GS-quasigroup if and only if $k=[a+1]_n$,*

(4) *right modular quasigroup if and only if $k=[-1-a]_n$,*

(5) *left modular quasigroup if and only if $k=[a-1]_n$,*

(6) *ARO-quasigroup if and only if $k=[-1-2a]_n$,*

(7) *$C3$ quasigroup if and only if $[(1-a^2)k]_n=1$.*

We will also need the following characterization of a pentagonal quasigroup proved in [7].

Theorem 2.3. *A groupoid $(Q,\cdot)$ of order $n>2$ is a pentagonal quasigroup induced by the group $\mathbb{Z}_n$ if and only if $x\cdot y=[ax+(1-a)y]_n$ and $[a^4-a^3+a^2-a+1]_n=0$ for some $a\in\mathbb{Z}_n$ such that a and n, also $a-1$ and n, are relatively prime. Such a quasigroup is k-translatable for $k=[1-a-a^3]_n$.*

3. The main theorem

In this section we find identities amongst various types of idempotent, k-translatable quasigroup types $\mathbf{T}$ and their dual and inverse collections $\mathbf{T}^*$ and $-\mathbf{T}$. We then find the values of $H_T(k)$, a, a' and k' as functions of k.

Theorem 3.1. *The following identities between classes of idempotent quasigroups induced by the additive groups $\mathbb{Z}_n$ are valid:*

(1) $\mathbf{Q} = \mathbf{Q}^*$,

(2) $\mathbf{H} = \mathbf{H}^* = -\mathbf{C3}$,

(3) $\mathbf{GS} = \mathbf{GS}^* = -\mathbf{RM}$,

(4) $\mathbf{RM} = -(\mathbf{GS}^*)$,

(5) $\mathbf{LM} = \mathbf{RM}^*$,

(6) $\mathbf{ARO} = -\mathbf{ARO}$,

(7) $\mathbf{C3} = -\mathbf{H} = -(\mathbf{H}^*)$.

Proof. In the proof we use Theorem 2.1.

(1): $(a, 1-a) \in \mathbf{Q} \Leftrightarrow [2a^2 - 2a + 1]_n = 0 \Leftrightarrow [2(1-a)^2 - 2(1-a) + 1]_n = 0 \Leftrightarrow$
$(1-a, a) \in \mathbf{Q} \Leftrightarrow (a, 1-a) \in \mathbf{Q}^*$.

(2): $(a, 1-a) \in \mathbf{H} \Leftrightarrow [a^2 - a + 1]_n = 0 \Leftrightarrow [(1-a)^2 - (1-a) + 1]_n = 0 \Leftrightarrow$
$(1-a, a) \in \mathbf{H} \Leftrightarrow (a, 1-a) \in \mathbf{H}^*$
and
$(a, 1-a) \in \mathbf{C3} \Leftrightarrow [a^2 + a + 1]_n = 0 \Leftrightarrow [(-a)^2 - (-a) + 1]_n = 0 \Leftrightarrow$
$(-a, a+1) \in \mathbf{H} \Leftrightarrow (a, 1-a) \in -\mathbf{H}$. So, $\mathbf{H}^* = \mathbf{H} = -(-\mathbf{H}) = -\mathbf{C3}$.

(3): $\mathbf{GS} = \mathbf{GS}^* = -\mathbf{RM}$ and $\mathbf{RM} = -\mathbf{GS}$.
$(a, 1-a) \in \mathbf{GS} \Leftrightarrow [a^2 - a - 1]_n = 0 \Leftrightarrow [(1-a)^2 - (1-a) - 1]_n = 0 \Leftrightarrow$
$(1-a, a) \in \mathbf{GS} \Leftrightarrow (a, 1-a) \in \mathbf{GS}^*$

(4): $(a, 1-a) \in \mathbf{RM} \Leftrightarrow [a^2 + a - 1]_n = 0 \Leftrightarrow [(-a)^2 - (-a) - 1]_n = 0 \Leftrightarrow$
$(-a, a+1) \in \mathbf{GS} \Leftrightarrow (a, 1-a) \in -\mathbf{GS}$. So, $-\mathbf{RM} = -(-\mathbf{GS}) = \mathbf{GS}$.

(5): $\mathbf{RM} = \mathbf{LM}^*$ and $\mathbf{LM} = \mathbf{RM}^*$.
$(a, 1-a) \in \mathbf{RM} \Leftrightarrow [a^2 + a - 1]_n = 0 \Leftrightarrow [(1-a)^2 - 3(1-a) + 1]_n = 0 \Leftrightarrow$
$(1-a, a) \in \mathbf{LM} \Leftrightarrow (a, 1-a) \in \mathbf{LM}^*$.

(6): $(a, 1-a) \in \mathbf{ARO} \Leftrightarrow [2a^2 - 1]_n = 0 \Leftrightarrow [2(-a)^2 - 1]_n = 0 \Leftrightarrow$
$(-a, a+1) \in \mathbf{ARO} \Leftrightarrow (a, 1-a) \in -\mathbf{ARO}$.

(7) is a consequence of the above facts. $\square$

Theorem 3.2. *If* **T** *is any one of the following types:* **Q**, **H**, **GS**, **RM**, **LM**, **ARO**, **ARO***, **C3**, **C3***, **P**, **P***, **U**, **U***, $-$**LM**, $-$(**C3***), $-$**U**, $-$(**U***), $-$(**ARO***), $-$**P** *or* $-$(**P***), *then the values of* $F_T(a)$, $H_T(k)$, k, a, a' *and* k' *are as indicated in the tables below, where all entries are calculated modulo* n.

Table 3.1.

T	$F_T(a)$	k	$H_T(k)$
Q	$2a^2-2a+1$	$1-2a$	k^2+1
H	a^2-a+1	$1-a$	k^2-k+1
GS	a^2-a-1	$a+1$	k^2-3k+1
RM	a^2+a-1	$-1-a$	k^2+k-1
LM	a^2-3a+1	$a-1$	k^2-k-1
ARO	$2a^2-1$	$-1-2a$	k^2+2k-1
ARO*	$2a^2-4a+1$	$2a-1$	k^2-2k-1
C3	a^2+a+1	$ta-t$	$3k^2-3k+1$
C3*	a^2-3a+3	$3-a$	k^2-3k+3
P	$a^4-a^3+a^2-a+1$	$1-a^3-a$	$k^4-2k^3+4k^2-3k+1$
P*	$a^4-3a^3+4a^2-2a+1$	$1-a^3+2a^2-2a$	$k^4-3k^3+4k^2-2k+1$
U	a^3-3a^2+2a-1	a^2-2a+1	k^3-2k^2+k-1
U*	a^3-a+1	$1-a^2-a$	k^3-k^2+2k-1

Table 3.2.

T	a	a'	k'
Q	$2a=1-k$	$k+1$	$-k$
H	$1-k$	k	$1-k$
GS	$k-1$	$k-2$	$3-k$
RM	$-1-k$	$-k$	$k+1$
LM	$k+1$	$2-k$	$k-1$
ARO	$2a=-1-k$	$-k-1$	$k+2$
ARO*	$2a=k+1$	$3-k$	$k-2$
C3	$1-3k$	$3k-2$	$3-3k$
C3*	$3-k$	$-tk$	$tk+1$
P	$-k^3+k^2-3k+1$	k^3-2k^2+4k-2	$-k^3+2k^2-4k+3$
P*	$-k^3+2k^2-2k+1$	k^3-3k^2+4k-1	$-k^3+3k^2-4k+2$
U	k^3-k^2	$2k-k^2$	k^2-2k+1
U*	$-1-k^2$	$-k^2+k-1$	k^2-k+2

Table 3.3.

T	$F_T(a)$	k	$H_T(k)$
$-$**LM**	a^2+3a+1	$5k=1-a$	$5k^2-5k+1$
$-$(**C3***)	a^2+3a+3	$7k=3-a$	$7k^2-9k+3$
$-$**U**	a^3+3a^2+2a+1	$7k=-a^2-4a+1$	$7k^3-10k^2+5k-1$
$-$(**U***)	a^3-a-1	a^2+a+1	k^3-5k^2+4k-1
$-$(**ARO***)	$2a^2+4a+1$	$7k=1-2a$	$7k^2-6k+1$
$-$**P**	$a^4+a^3+a^2+a+1$	$5k=a^4-a^2-2a+2$	$5k^4-10k^3+10k^2-5k+1$
$-$(**P***)	$a^4+3a^3+4a^2+2a+1$	$11k=-a^3-4a^2-8a+1$	$11k^4-21k^3+16k^2-6k+1$

Table 3.4.

T	a	a'	k'
$-$**LM**	$1-5k$	$5k-4$	$5-5k$
$-$(**C3***)	$3-7k$	$3a'=7k-6$	$3k'=9-7k$
$-$**U**	$-7k^2+3k-1$	$-7k^2+10k-4$	$7k^2-10k+5$
$-$(**U***)	k^2-4k+1	$-k^2+5k-3$	k^2-5k+4
$-$(**ARO***)	$2a=1-7k$	$7k-5$	$6-7k$
$-$**P**	$-5k^3+5k^2-5k+1$	$5k^3-10k^2+10k-4$	$-5k^3+10k^2-10k+5$
$-$(**P***)	$-11k^3+10k^2-6k+1$	$11k^3-21k^2+16k-5$	$-11k^3+21k^2-16k+6$

Proof. The values of k listed in Table 3.1, column 3, can be checked using the fact that $[a+k(1-a)]_n=0$. In the case of **P**, $[a+(1-a-a^3)(1-a)]_n = [a^4-a^3+a^2-a+1]_n=0$. Note that $C3$ quasigroups have order $n=3t+1$ (cf. [2]) and so $[2t]_n=[-1-t]_n$. Therefore, $[a+(ta-t)(1-a)]_n=[-ta^2+2ta-t+a]_n = [-t(a^2+a+1)]_n=0$, which proves that $k=[ta-t]_n$ in $C3$ quasigroups with order $n=3t+1$.

Once the values of k in Table 3.1 have been verified, these can be used to check the values of a, listed in Table 3.2, as a function of k, using also the value of $F_T(a)$. For example, in the case of **P*** since $k=[1-a^3+2a^2-2a]_n$, using the fact that $[a^4-3a^3+4a^2-2a+1]_n=0$ it follows that $k^2=[-a^3+a^2-a]_n$ and $k^3=[-2a^2+a-1]_n$. Then, we get $[-k^3+2k^2-2k+1]_n = [(2a^2-a+1)+(-2a^3+2a^2-2a)+(-2+2a^3-4a^2+4a)+1]_n=a$. Similarly, for **U** we can calculate that $k^2=[a^2-a]_n$ and $k^3=[a^2]_n$. Hence, $a=[k^3-k^2]_n$. Using these values of a as a function of k, substituting them into the formula $0=[F_T(a)]_n$ gives the value of $H_T(k)$ listed in column 3 of Table 3.1. Alternatively, we can substitute the value of a as a function of k into the formula $[a+k(1-a)]_n=0$. So, with **P** for example, $[a+k(1-a)]_n=0$ and $a=[-k^3+k^2-3k+1]$. Therefore, $0=[-k^3+k^2-3k+1+k(k^3-k^2+3k)]_n=[k^4-2k^3+4k^2-3k+1]_n$.

The listings of the values of a' in Table 3.2 can be checked using the fact that $[ka]_n = [k+a]_n$. For example, in $\mathbf{Q}$, $[2a^2-2a+1]_n = 0$ and $k = [1-2a]_n$. Then $[a(k+1)]_n = [(1-2a)+2a]_n = 1$ and so $a' = k+1$. In the case of $\mathbf{C3}^*$, $[(-tk)a]_n = [-t(k+a)]_n = [-t((3-a)+a)]_n = [-3t]_n = 1$ and so $a' = [-tk]_n$ in a $\mathbf{C3}^*$ quasigroup of order $n = 3t+1$.

The values of k' in Table 3.2 follow from the fact that $k' = [1-a']_n$, which in turn follows from the fact that $0 = [b+k'a]_n = [k'a+(1-a)]_n = [k'+(1-a)a']_n$.

$-\mathbf{LM}$: If $(a, 1-a) \in -\mathbf{LM}$, then $(-a, a+1) \in \mathbf{LM}$ and, by Theorem 2.1, $0 = [(-a)^2-3(-a)+1]_n = [a^2+3a+1]_n$. Now $1 = [a(-a-3)]_n$ and so, $a' = [-a-3]_n$. But $k' = [1-a']_n = [a+4]_n$. Then, $1 = [kk']_n = [k(a+4)]_n = [5k+a]_n$ and so, $[5k]_n = [1-a]_n$ and $a = [1-5k]_n$. Therefore, $a' = [-a-3]_n = [5k-4]_n$ and $k' = [a+4]_n = [5-5k]_n$. Finally, $1 = [kk']_n = [5k-5k^2]_n$ and so, $0 = [5k^2-5k+1]_n$.

$-(\mathbf{C3}^*)$: If $(a, 1-a) \in -(\mathbf{C3}^*)$, then $(-a, 1+a) \in \mathbf{C3}^*$ and, by Theorem 2.1, $0 = [(-a)^3-3(-a)+3]_n = [a^2+3a+3]_n$. But $k = [a(k-1)]_n$ and so, $0 = [(k-1)a^2+3(k-1)a+3(k-1)]_n$ which, using the fact that $[ka]_n = [k+a]_n$, implies $0 = [7k+a-3]_n$. Therefore, $[7k]_n = [3-a]_n$ and $a = [3-7k]_n$. Now, $1 = [kk']_n = [a(k-1)k']_n = [(3-7k)(k-1)k']_n = [10-7k-3k']_n$ and so $[3k']_n = [9-7k]_n$. The last gives $3 = [9k-7k^2]_n$ and so, $0 = [7k^2-9k+3]_n$. Moreover, $k' = [1-a']_n$ implies $[3k']_n = [3-3a']_n$ and $[3a']_n = [3-3k']_n = [7k-6]_n$.

$-\mathbf{U}$: If $(a, 1-a) \in -\mathbf{U}$, then $(-a, a+1) \in \mathbf{U}$ and, according to Table 3.1, $0 = [(-a)^3-3(-a)+2(-a)-1]_n = [a^3+3a^2+2a+1]_n$. Using this fact and the fact that $k = [a(k-1)]_n$, the identity $0 = [(k-1)^3(a^3+3a^2+2a+1)]_n$ implies $0 = [7k^3-10k^2+5k-1]_n$. Then, $1 = [7k^3-10k^2+5k]_n = [k(7k^2-10k+5)]_n$ implies $k' = [7k^2-10k+5]_n$. Consequently, $a' = [1-k']_n = [-7k^2+10k-4]_n$.

Using the fact that $[ka]_n = [k+a]_n$, the identity $0 = [k(a^3+3a^2+2a+1)]_n$ implies $[7k]_n = [-a^2-4a+1]_n$. Also, since $1 = [7k+a^2+4a]_n$, $a' = [7ka'+a+4]_n$ we obtain $a = [a'-4-7ka']_n = [(-7k^2+10k-4)-4-7k(-7k^2+10k-4)]_n = [49k^3-77k^2+38k-8]_n = [7(7k^3-10k^2+5k-1)+(-7k^2+3k-1)]_n$. Thus, $a = [-7k^2+3k-1]_n$.

$-(\mathbf{U}^*)$: If $(a, 1-a) \in -(\mathbf{U}^*)$, then $(-a, 1+a) \in \mathbf{U}^*$. Hence, by Table 3.1, $0 = [(-a)^3-(-a)+1]_n = [a^3-a-1]_n$. Then, $[a+(a^2+a+1)(1-a)]_n = [-a^3+a+1]_n = 0$ implies $k = [a^2+a+1]_n$. But $k = [a(k-1)]_n$, so $[(k-1)k]_n = [(k-1)(a^2+a+1)]_n = [3k+a-1]_n$. Hence, $a = [k^2-4k+1]_n$. Also, $k = [a(k-1)]_n = [(k^2-4k+1)(k-1)]_n = [k^3-5k^2+5k-1]_n$ and so, $[k^3-5k+4k-1]_n = 0$. Then, $[k(k^2-5k+4)]_n = 1$. Thus, $k' = [k^2-5k+4]_n$ and $a' = [1-k']_n = [-k^2+5k-3]_n$.

$-(\mathbf{ARO}^*)$: If $(a, 1-a) \in -(\mathbf{ARO}^*)$, then $(-a, 1+a) \in \mathbf{ARO}^*$ and, by Table 3.1, $0 = [2(-a)^2-4(-a)+1]_n = [2a^2+4a+1]_n$. Since $k = [a(k-1)]_n$ we also have $0 = [(k-1)^2(2a^2+4a+1)]_n = [7k^2-6k+1]_n$. So, $1 = [6k-7k^2]_n = [k(6-7k)]_n$ and therefore, $k' = [6-7k]_n$ and $a' = [7k-5]_n$. Now, $0 = [k(2a^2+4a+1)]_n$ together with $[ka]_n = [k+a]_n$ imply $0 = [2a+7k-1]_n$. So, $[2a]_n = [1-7k]_n$ and $[7k]_n = [1-2a]_n$.

$-\mathbf{P}$: If $(a, 1-a) \in -\mathbf{P}$, then $(-a, 1+a) \in \mathbf{P}$. Hence, by Table 3.1, we have $0 = [(-a)^4 - (-a)^3 + (-a)^2 - (-a) + 1]_n = [a^4 + a^3 + a^2 + a + 1]_n$. Using the fact that $[ka]_n = [k+a]_n$, the identity $0 = [k(a^4 + a^3 + a^2 + a + 1)]_n$ implies $0 = [5k + a^3 + 2a^2 + 3a - 1]_n$. Applying $k = [a(k-1)]_n$ to the identity $0 = [(k-1)^4(a^4+a^3+a^2+a+1)]_n$ we obtain $0 = [5k^4 - 10k^3 + 10k^2 - 5k + 1]_n$. Thus, $1 = [k(-5k^3 + 10k^2 - 10k + 5)]_n$. Consequently, $k' = [-5k^3 + 10k^2 - 10k + 5]_n$ and $a' = [5k^3 - 10k^2 + 10k - 4]_n$. Now, from $[(-5k^3 + 5k^2 - 5k + 1)a']_n = [-25k^6 + 75k^5 - 125k^4 + 125k^3 - 80k^2 + 30k - 4]_n = [-5k^2(5k^4 - 10k^3 + 10k^2 - 5k + 1) + 5k(5k^4 - 10k^3 + 10k^2 - 5k + 1) - 5(5k^4 - 10k^3 + 10k^2 - 5k + 1) + 1]_n = 1$ we conclude that $a = [-5k^3 + 5k^2 - 5k + 1]_n$.

$-(\mathbf{P}^*)$: If $(a, 1-a) \in -(\mathbf{P}^*)$, then $(-a, 1+a) \in \mathbf{P}^*$. Hence, by Table 3.1, we have $0 = [(-a)^4 - 3(-a)^3 + 4(-a)^2 - 2(-a) + 1]_n = [a^4 + 3a^3 + 4a^2 + 2a + 1]_n$. Using the fact that $[ka]_n = [k+a]_n$, the identity $0 = [k(a^4 + 3a^3 + 4a^2 + 2a + 1)]_n$ implies $0 = [11k + a^3 + 4a^2 + 8a - 1]_n$. Then, using the fact that $k = [a(k-1)]_n$, the identisty $0 = [(k-1)^4(a^4 + 3a^3 + 4a^2 + 2a + 1)]_n$ implies $0 = [11k^4 - 21k^3 + 16k^2 - 6k + 1]_n$. This means that $1 = [-11k^3 + 21k^2 - 16k + 6]_n$. So, $k' = [-11k^4 - 21k^3 + 16k^2 + 6]_n$ and $a' = [11k^3 - 21k^2 + 16k - 5]_n$. Finally, using $0 = [11k^4 - 21k^3 + 16k^2 - 6k + 1]_n$, we can calculate that $[aa']_n = 1$ for $a = [-11k^3 + 10k^2 - 6k + 1]_n$.

This completes the proof of Theorem 3.2 □

Theorem 3.3. *Let $(Q, \cdot)$ be an idempotent k-translatable quasigroup of order n. If m divides n, then $(Q, \cdot)$ has an idempotent k'-translatable subquasigroup of order m, where $k' = [k]_m$.*

Proof. An idempotent k-translatable quasigroup $(Q, \cdot)$ of order n is induced by the group $\mathbb{Z}_n$ and its automorphism $\varphi(x) = [ax]_n$, where a and n are relatively prime. If m divides n, then $\mathbb{Z}_n$ has a subgroup $(H, +)$ of order m. It is isomorphic to the group $\mathbb{Z}_m$. Since a and m are relatively prime too, φ calculated modulo m, is an automorphism of the group $\mathbb{Z}_m$ and $[a + (1-a)k']_m = 0$ for $k' = [k]_m$. So, $(H, \cdot)$ is an idempotent k'-translatable quasigroup induced by $\mathbb{Z}_m$ and consequently by the subgroup $(H, +)$. □

4. Idempotent k-translatable quasigroups for $k \leqslant 10$

Using our Theorem 3.2 for each value of k we can calculate all idempotent k-translatable quasigroups for the types of quasigroups discussed in the previous section. To calculate the orders of these quasigroups we bear in mind that the order n is odd and that the values of $F_T(a)$ and $H_T(k)$ calculated in Tables 3.1 to 3.4 are equivalent to 0 modulo n. For example, for $k = 5$ in $\mathbf{H}$, we have $0 = [k^2 - k + 1]_n = [21]_n = [3 \cdot 7]_n$. This means that for $k = 5$ the possible orders $n > k$ are 7 or 21. Using Table 3.2 we see that for $n = 7$, $a = [1-k]_7 = [-4]_7 = 3$; for $n = 21$, $a = [-4]_{21} = 17$. Thus, $(3, 5)$ and $(17, 5)$ are members of $\mathbf{H}$. Similarly for $\mathbf{C3}^*$ and $k = 6$ we have $H_T(6) = 21$, so possible order n of a 6-translatable $C3^*$

quasigroup is 3, 7 or 21. But, in this case should be $n > 6$ and $n = 3t+1$. Thus a 6-translatable $C3^*$ quasigroup has order 7. Then, by Table 3.2, $a = [-3]_7 = 4$ and $[F_T(4)]_7 = 0$. Hence a multiplication of a 6-translatable $C3^*$ quasigroup of order 7 is given by $x \cdot y = [4x+4y]_7$. Therefore $(4,4) \in \mathbf{C3}^*$.

Calculations for other cases are similar and we skip them. Obtained results are presented in Tables 4.1 and 4.2.

Table 4.1.

T	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
Q	$(2,4)$	$(4,2)$	$(7,11)$	$(11,3)$	$(16,22)$
H	$(2,2)$	$(5,3)$	$(10,4)$	$(3,5),(17,5)$	$(26,6)$
GS	–	–	$(3,3)$	$(4,8)$	$(5,15)$
RM	$(2,4)$	$(7,5)$	$(14,6)$	$(23,7)$	$(34,8)$
LM	–	$(4,2)$	$(5,7)$	$(6,14)$	$(7,23)$
ARO	$(2,6)$	$(5,3)$	$(9,15)$	$(14,4)$	$(20,28)$
ARO*	–	–	$(6,2)$	$(3,5)$	$(15,9)$
C3	$(2,6)$	$(11,9)$	$(26,12)$	$(47,15)$	$(4,4),(9,5)$ $(74,18)$
C3*	–	–	$(6,2)$	$(11,3)$	$(4,4)$
P	$(2,10)$	$(4,2),(7,5)$ $(29,27)$	$(122,60)$	$(347,115)$	$(794,198)$
P*	$(2,4)$	$(17,15)$	$(5,7),(82,40)$	$(4,8),(9,23)$ $(257,85)$	(10,2), (58,14) $(626,156)$
U	–	$(7,5)$	$(3,3),(6,2)$ $(13,23)$	$(21,59)$	$(31,119)$
U*	$(2,6)$	$(13,11)$	$(3,3),(5,7)$ $(38,18)$	$(83,27)$	$(154,38)$
−**LM**	$(2,10)$	$(17,15)$	$(42,20)$	$(77,25)$	$(122,30)$
−(**C3***)	$(2,12)$	$(8,6),(21,19)$	$(54,26)$	$(3,5),(6,14)$ $(101,33)$	$(28,40)$
−(**ARO***)	$(2,16)$	$(13,11)$	$(31,59)$	$(56,18)$	(26,6), (88,130)
−**U**	(2,4) $(2,24)$	$(58,56)$	$(206,102)$	$(4,8),(16,44)$ $(488,162)$	$(946,236)$
−(**U***)	–	–	–	$(6,14)$	$(13,47)$
−**P**	$(2,30)$	$(107,105)$	$(5,7),(25,47)$ $(522,260)$	$(4,8),(49,143)$ $(1577,525)$	$(3722,930)$
−(**P***)	$(2,60)$	(7,5),(22,20) $(227,225)$	$(3,3),(5,7)$ $(22,10),(38,18)$ (53,103),(115,327) $(1138,568)$	$(3467,1155)$	(26,6), (266,66) $(8210,2052)$

Table 4.2.

T	$k=7$	$k=8$	$k=9$	$k=10$
Q	$(22,4)$	$(3,11),(29,37)$	$(37,5)$	$(46,56)$
H	$(37,7)$	$(12,8),(50,8)$	$(65,9)$	$(4,10),(82,10)$
GS	$(6,24)$	$(7,35)$	$(8,4),(8,48)$	$(9,63)$
RM	$(3,9),(47,9)$	$(62,10)$	$(79,11)$	$(98,12)$
LM	$(8,34)$	$(9,3),(9,47)$	$(10,62)$	$(11,79)$
ARO	$(27,5)$	$(35,45)$	$(44,6)$	$(3,15)$
ARO*	$(4,14)$	$(28,20)$	$(5,27)$	$(45,35)$
C3	$(107,21)$	$(3,11),(146,24)$	$(5,27)$	$(242,30)$
C3*	$(27,5)$	$(38,6)$	$(13,7),(51,7)$	$(66,8)$
P	$(27,5),(52,10)$ $(1577,315)$	$(190,472)$ $(2834,472)$	$(8,4),(308,184)$ $(4727,675)$	$(6,6),(593,169)$ $(7442,930)$
P*	$(53,259)$ $(1297,259)$	$(2402,400)$	$(5,27),(20,132)$ $(4097,585)$	$(6,6),(35,27)$ $(28,94),(523,148)$ $(6562,820)$
U	$(43,209)$	$(6,12),(11,13)$ $(57,335)$	$(4,20),(23,3)$ $(73,43),(73,503)$	$(91,719)$
U*	$(257,51)$	$(398,66)$	$(13,7),(23,13))$ $(13,83),(51,83)$ $(583,83)$	$(818,102)$
$-$**LM**	$(177,35)$	$(242,40)$	$(13,7),(317,45)$	$(6,6),(33,9)$ $(402,50)$
$-$(**C3***)	$(237,47)$	$(326,54)$	$(103,61),(429,61)$	$(546,68)$
$-$(**ARO***)	$(127,25)$	$(173,229)$	$(226,32)$	$(286,356)$
$-$**U**	$(66,324)$ $(1622,324)$	$(12,8),(46,112)$ $(2558,426)$	$(3796,542)$	$(19,5),(118146)$ $(5378,672)$
$-$(**U***)	$(22,4)$	$(33,191)$	$(46,314)$	$(6,6),(12,38)$ $(61,17),(61,479)$
$-$**P**	$(3,9),(138,684)$ $(7527,1505)$	$(9,3),(623,829)$ $(13682,2280)$	$(37,5),(562,80)$ $(22997,3285)$	$(8,24),(735,587)$ $(36402,4550)$
$-$(**P***)	$(13,59),(48,234)$ $(16627,3325)$	$(30242,5040)$	$(4359,7263)$ $(50843,7263)$	$(6,6),(6403,1829)$ $(80482,10060)$

Note that similar results can be obtained for negative values of k. Obtained quasigroups will be $[k]_n$-translatable quasigroups of order $n > 2$, where n is a divisor of $H_T(k)$.

For example, for $\mathbf{U}^*$, where $0 = [k^3 - k^2 + 2k - 1]_n$, substituting $k = -5$ gives $0 = [-161]_n = [161]_n = [7 \cdot 23]_n$. If $n = 7$, then $a = [-1 - k^3]_n = [-26]_7 = 2$, which gives $x \cdot y = [2x + 6y]_7$. Since $[2^3 - 2 + 1]_7 = 0$, $(2,6) \in \mathbf{U}^*$. If $n = 23$, then $a = [-26]_{23} = [-3]_{23} = 20$ and $[(-3)^3 - (-3) + 1]_{23} = 0$. So, $(20,4) \in \mathbf{U}^*$.

In this case, $k = [-5]_{23} = 18$. Finally, if $n = 161$, then $a = [-26]_{161} = 135$, $(135, 27) \in \mathbf{U}^*$ and $k = [-5]_{161} = 156$.

In a similar way we can calculate analogous results for quasigroups other types $\mathbf{T}$ mentioned in the previous sections.

Below we present obtained results for $\mathbf{U}^*$, where $k \in \{-1, -2, \dots, -10\}$, once again omitting the detailed calculations.

Table 4.3.

k	n	a	$\mathbf{U}^*$	$[k]_n$
-1	5	3	$(3,3)$	4
-2	17	12	$(16,6)$	15
-3	43	33	$(33,11)$	40
-4	89	72	$(72,18)$	85
-5	$161 = 7 \cdot 23$	$[-26]_n$	$(2,6),(20,4),(135,27)$	$2,18,156$
-6	$265 = 5 \cdot 53$	$[-37]_n$	$(3,3),(16,38),(228,38)$	$4,47,259$
-7	$407 = 11 \cdot 37$	$[-50]_n$	$(5,7),(24,14),(357,51)$	$4,30,400$
-8	593	528	$(528,66)$	585
-9	829	747	$(747,83)$	820
-10	$1121 = 19 \cdot 59$	$[-101]_n$	$(13,7),(17,43),(1020,102)$	$9,49,1111$

In [10] Vidak proved that if $(Q, \cdot)$ is a pentagonal quasigroup then $(Q, \circ)$, defined as $x \circ y = (yx \cdot x)x \cdot y$, is a golden square quasigroup. If the pentagonal quasigroup $(Q, \cdot)$ is also translatable and of order n then, as we have seen, $x \cdot y = [ax+(1-a)y]_n$, with $[a^4-a^3+a^2-a+1]_n = 0$ and $x \circ y = [(a-a^4)x+(1+a^4-a)y]_n$. We can easily check that $(Q, \circ) \in \mathbf{GS}$ using Table 3.1. Since $[a^5+1]_n = 0$, we have also $[(a-a^4)+(1-a^4+a)(1+a^4-a)]_n = 0$. Therefore, $(Q, \circ)$ is $[1-a^4+a]_n$-translatable. So, for every translatable pentagonal quasigroup of order n there is a translatable golden square quasigroup of order n. Note that by [7] a finite pentagonal quasigroup has order $5s$ or $5s+1$. By Table 4.1, a 6-translatable GS-quasigroup has order 19. Hence, it is not pentagonal.

Notice that $\{(3,9),(9,3)\} \subseteq -\mathbf{P}$. Accordingly, we have the following definition.

Definition 4.1. The set $dp(\mathbf{T}) = \{(a, 1-a) \,|\, (a, 1-a), (1-a, a) \in \mathbf{T}\}$ is called the *set of* $\mathbf{T}$ *dual pairs*.

If $\mathbf{T} \in \{\mathbf{Q}, \mathbf{H}, \mathbf{GS}\}$ then, by Theorem 3.1, $\mathbf{T} = \mathbf{T}^*$ and $dp(\mathbf{T}) = \mathbf{T} = dp(\mathbf{T}^*)$. From Table 3.1, it follows that if $(a, 1-a) \in \mathbf{RM} \cap \mathbf{RM}^* = \mathbf{RM} \cap \mathbf{LM}$, then $0 = [a^2+a-1]_n = [a^2-3a+1]_n$ and so $[4a]_n = 2$. Thus $0 = [4(a^2+a-1)]_n = [2a-2]_n$ gives $[2a]_n = 2$. Hence, $2 = [4a]_n = [2(2a)]_n = 4$ and so $[2]_n = 0$. This is impossible because $2 < a < n$. Similarly, $\mathbf{LM} \cap \mathbf{LM}^* = \emptyset$. In this way we have proved:

Proposition 4.2. $dp(\mathbf{LM}) = \emptyset = dp(\mathbf{RM})$.

Proposition 4.3. $dp(\mathbf{C3}) = \{(4,4)\} = dp(\mathbf{C3}^*)$.

Proof. $C3$ and $C3^*$-quasigroups have order $n = 3t + 1$.

If $(a, 1-a) \in dp(\mathbf{C3})$, then, by Table 3.1, we have $0 = [(1-a)^2 + (1-a) + 1]_n = [a^2 - 3a + 3]_n$ which together with $0 = [a^2 + a + 1]_n$ gives $[4a]_n = 2$. Consequently, $0 = [4(a^2 + a + 1)]_n = [2a + 6]_n$, i.e., $[2a]_n = [-6]_n$. So, $2 = [4a]_n = [2(2a)]_n = [-12]_n$ which means that $0 = [14]_n$. But $n = 3t + 1$, so $n = 7$. Therefore, $[2a]_7 = 1$ and $a = 4$.

If $(a, 1-a) \in dp(\mathbf{C3}^*)$, then, by Table 3.1, we have $0 = [a^2 - 3a + 3]_n$. Also, $0 = [(1-a)^2 - 3(1-a) + 3]_n = [a^2 + a + 1]_n$ and consequently, $0 = [a^2 - 3a + 3]_n = [(a^2 + a + 1) - 4a + 2]_n = [-4a + 2]_n$. So, $[4a]_n = 2$. Thus $0 = [4(a^2 - 3a + 3)]_n = [2a + 6]_n$, i.e., $[2a]_n = [-6]_n$. Hence $2 = [2(2a)]_n = [-12]_n$. So, $[14]_n = 0$ and, as in the previous case, $n = 7$, $a = 4$. □

Proposition 4.4. $dp(\mathbf{ARO}) = \emptyset = dp(\mathbf{ARO}^*)$.

Proof. $0 = [2a^2 - 1]_n$ and $0 = [2(1-a)^2 - 1]_n = [2a^2 - 4a + 1]_n$. So, $[4a - 2]_n = 0$ and $2 = [4a^2]_n = [2a]_n$. Hence, $1 = [2a^2]_n = [2a]_n = 2$, contradiction. □

Proposition 4.5. $dp(\mathbf{U}) = \{(3,3)\} = dp(\mathbf{U}^*)$.

Proof. If $(a, 1-a) \in dp(\mathbf{U})$, then $0 = [(1-a)^3 - 3(1-a)^2 + 2(1-a) - 1]_n = [-a^3 + a - 1]_n$, which gives $[a^3]_n = [a-1]_n$. Therefore, $0 = [a^3 - 3a^2 + 2a - 1]_n = [-3a^2 + 3a - 2]_n$, i.e., $[3a^2]_n = [3a - 2]_n$. Hence, $[3(a-1)]_n = [3a^3]_n = [3a^2 - 2a]_n = [a-2]_n$. So, $[2a]_n = 1$. Thus $[a^2]_n = [2a(a^2)]_n = [2(a-1)]_n = [2a-2]_n = [-1]_n$. Consequently, $a = [(2a)a]_n = [-2]_n$. This together with $[a^3]_n = [a-1]_n$ implies $n = 5$ and $a = 3$.

Now, if $(a, 1-a) \in dp(\mathbf{U}^*)$, then $0 = [a^3 - a + 1]_n$ and $0 = [(1-a)^3 - (1-a) + 1]_n = [-(a^3 - a + 1) + 3a^2 - 3a + 2]_n = [3a^2 - 3a + 2]_n$, by Table 3.1. Thus, $[3a^2]_n = [3a-2]_n$ and $0 = [3a^2 - 2a + 3]_n = [(3a-2)a - 3a + 3]_n = [-2a + 1]_n$. Hence, $[2a]_n = 1 = [4a^2]_n$. So, $[a+1]_n = [(2a)a + 4a^2]_n = [6a^2]_n = [6a - 4]_n = [3 - 4]_n = [-1]_n$. So, $a = [-2]_n$ and $1 = [2a]_n = [-4]_n$. Thus, $0 = [5]_n$ and $a = 3$. □

Proposition 4.6. $dp(-\mathbf{LM}) = \{(6,6)\}$.

Proof. If $(a, 1-a) \in dp(-\mathbf{LM})$, then $0 = [a^2 + 3a + 1]_n$ and $0 = [(1-a)^2 + 3(1-a) + 1]_n = [a^2 - 5a + 5]_n = [(a^2 + 3a + 1) - 8a + 4]_n = [-8a + 4]_n$. Hence, $[8a]_n = 4$ and $0 = [8(a^2 + 3a + 1)]_n = [4a + 20]_n$. Thus, $4 = [2(4a)]_n = [-40]_n$ and so $[44]_n = 0$. Since n must be odd (cf. [5, Lemma 4.1]), $n = 11$ and $[8a]_{11} = 4$. This equation has only one solution $a = 6$. □

The proofs of the next two propositions are very similar to the proof of Proposition 4.6.

Proposition 4.7. $dp(-(\mathbf{C3}^*) = \{(10,10)\}$.

Proposition 4.8. $dp(-(\mathbf{U}^*)) = \{(6,6)\}$.

Proposition 4.9. $dp(-(\mathbf{ARO}^*)) = \{(4,4)\}$.

Proof. For $(a, 1-a) \in dp(-(\mathbf{ARO}^*))$ we have $0 = [2a^2+4a+1]_n$. Also $0 = [2(1-a)^2+4(1-a)+1]_n = [2a^2-8a+7]_n = [(2a^2+4a+1)-12a+6]_n = [-12a+6]_n$. Hence, $[12a]_n = 6$ and $0 = [6(2a^2+4a+1)]_n = [6a+18]_n$. Thus, $6 = [2(6a)]_n = [-36]_n$ and so $[42]_n = 0$. Since, n must be odd, n is equal to 3, 7 or 21. For $n = 3$ the possible values of a are 1 or 2. These values do not satisfy the condition $[2a^2+4a+1]_3 = 0$, so the case $n = 3$ is impossible. For $n = 21$ the equation $[12a]_{21} = 6$ is solved only by $a = 4$, but then $[2a^2+4a+1]_{21} \neq 0$, This also is impossible. The equation $[12a]_7 = 6$ has only one solution $a = 4$. It satisfies the equation $[2a^2+4a+1]_7 = 0$. Hence $dp(-(\mathbf{ARO}^*)) = \{(4,4)\}$. □

Proposition 4.10. $dp(-\mathbf{U}) = \{(12,12)\})$.

Proof. For the pair $(a, 1-a) \in dp(-\mathbf{U})$ we have $0 = [a^3+3a^2+2a+1]_n$ and $0 = [(1-a)^3+3(1-a)^2+2(1-a)+1]_n = [-a^3+6a^2-11a+7]_n = [9a^2-9a+8]_n$. Hence, $[9a^2]_n = [9a-8]_n$ which together with $0 = [9(a^3+3a^2+2a+1)]_n$ gives $[46a]_n = [23]_n$. Consequently, $[a^2]_n = [-22a+40]_n$ and $[207a]_n = [368]_n$. So, $[23a]_n = [230a-207a]_n = [115-368]_n = [-253]_n$. Thus, $[23]_n = [46a]_n = [-506]_n$. Therefore $n = 529$ or $n = 23$.

For $n = 529$ we have $[23a]_{529} = [-253]_{529} = 276$ and $a = 12$. But such a does not satisfy $[a^3+3a^2+2a+1]_{529} = 0$. If $n = 23$, then from $[a^2]_{23} = [-22a+40]_{23}$ it follows that $a = 12$. Such a satisfies $[a^3+3a^2+2a+1]_{23} = 0$. □

Proposition 4.11. $dp(\mathbf{P}) = \{(6,6)\} = dp(\mathbf{P}^*)$.

Proof. If $(a, 1-a) \in dp(\mathbf{P})$, then $0 = [a^4-a^3+a^2-a+1]_n$, i.e., $[a^5]_n = [-1]_n$. In this case also $0 = [(1-a)^4-(1-a)^3+(1-a)^2-(1-a)+1]_n = [-2a^3+3a^2-a]_n$. So, $[2a^3]_n = [3a^2-a]_n$, whence, multiplying by a^3, a^2 and a we obtain, respectively, $[a^4]_n = [2a-3]_n$, $[a^3]_n = [3a^4+2]_n = [6a-7]_n$ and $[a^2]_n = [3a^3-2a^4]_n = [14a-15]_n$, which together with $[a^4-a^3+a^2-a+1]_n = 0$ gives $[9a]_n = 10$. Thus, $[10a]_n = [9a^2]_n = 5$. So, $a = [-5]_n$ and $[55]_n = 0$. Hence n is equal to 5, 11 or 55. The case $n = 5$ is impossible because in this case $a = 0$, Also the case $n = 55$ is impossible since a and n should be relatively prime. For $n = 11$, $a = 6$ satisfies these conditions.

If $(a, 1-a) \in dp(\mathbf{P}^*)$, then $0 = [a^4-3a^3+4a^2-2a+1]_n$. In this case also $0 = [(1-a)^4-3(1-a)^3+4(1-a)^2-2(1-a)+1]_n = [a^4-a^3+a^2-a+1]_n$. So, $(a, 1-a) \in \mathbf{P} \cap \mathbf{P}^*$. Also $(1-a, a) \in \mathbf{P} \cap \mathbf{P}^*$. Thus, $dp(\mathbf{P}^*) \subseteq dp(\mathbf{P}) = \{(6,6)\}$. Direct computation shows that $(6,6) \in dp(\mathbf{P}^*)$. Therefore $dp(\mathbf{P}) = dp(\mathbf{P}^*)$. □

Proposition 4.12. $dp(-\mathbf{P}) = \{(3,9), (9,3), (16,16), (47,295), (295,47)\}$.

Proof. If $(a, 1-a) \in dp(-\mathbf{P})$, then, by Table 3.3,

$$[a^4+a^3+a^2+a+1]_n = 0, \tag{1}$$

which implies $[a^5]_n = 1$. Then also, $0 = [(1-a)^4+(1-a)^3+(1-a)^2+(1-a)+1]_n = [a^4 - 5a^3 + 10a^2 - 10a + 5]_n$, i.e.,

$$[a^4]_n = [5a^3 - 10a^2 + 10a - 5]_n. \tag{2}$$

From this, multiplying by a and 4, we obtain $[5a^4]_n = [10a^3 - 10a^2 + 5a + 1]_n$ and $[4a^4]_n = [20a^3 - 40a^2 + 40a - 20]_n$. So,

$$[a^4]_n = [-10a^3 + 30a^2 - 35a + 21]_n.$$

Therefore, $[-50a^3 + 150a^2 - 175a + 105]_n = [5a^4]_n = [10a^3 - 10a^2 + 5a + 1]_n$, whence, as a consequence, we get

$$[60a^3]_n = [160a^2 - 180a + 104]_n.$$

On the other hand, (1) together with (2) imply $[6a^3]_n = [9a^2 - 11a + 4]_n$. Thus, $[90a^2 - 110a + 40]_n = [60a^3]_n = [160a^2 - 180a + 104]_n$. So,

$$[70a^2]_n = [70a - 64]_n. \tag{3}$$

From this, multiply successively by a^4, a and a^2 we get $[70a]_n = [70 - 64a^4]_n$, $[70a^3]_n = [6a - 64]_n$ and $[70a^4]_n = [6a^2 - 64a]_n$, which, together with (1), gives $0 = [70(a^4 + a^3 + a^2 + a + 1)]_n = [6a^2 + 82a - 58]_n$, i.e.,

$$[6a^2]_n = [58 - 82a]_n. \tag{4}$$

Since $[64a^2]_n = [70a - 6a^2 - 64]_n$, by (3), we also have $[4a^2]_n = [64a^2 - 60a^2]_n = [(70a - 6a^2 - 64) - (580 - 820a)]_n = [890 - 6a^2 - 644]_n$ and so, $[890a - 644]_n = [4a^2 + 6a^2]_n = [4a^2 + 58 - 82a]_n$. Hence, $[4a^2]_n = [972a - 702]_n$. Then $[2a^2]_n = [6a^2 - 4a^2]_n = [760 - 1054a]_n$. Thus, $[972a - 702]_n = [2(2a^2]_n = [1520 - 2108a]_n$. So, $[3080a]_n = [2222]_n$ and $[3080a^2]_n = [2222a]_n$. Now, using this equation and (3), we obtain $0 = [44(70a^2 - 70a + 64)]_n = [3080a^2 - 3080a + 2816]_n = [-858a + 2816]_n$. Thus, $[858a]_n = [2816]_n$, which implies, $[2574a]_n = [3(858a)]_n = [8448]_n$. Hence, $[506a]_n = [3080a - 2574a] = [-6226]_n$, $[352a]_n = [858a - 506a] = [9042]_n$ and $[308a]_n = [2(858a) - 4(352a)]_n = [-30536]_n$. Consequently,

$$[44a]_n = [352a - 308a]_n = [39578]_n. \tag{5}$$

But $[39578]_n = [44a]_n = [308a - 6(44a)]_n = [-30536 - 237468]_n = [-268004]_n$. So, $[307582]_n = 0$. Since $307582 = 2 \times 11^2 \times 31 \times 41$ and n must be an odd number, the possible values of n are 11, 31, 41, 121, 341, 451, 1 271, 3 751, 4 961, 13 981 and 153 791.

We will consider each case separately. Note first that $(a, b) \in dp(-\mathbf{P})$ if and only if both a and b satisfy (1) and $[a+b]_n = 1$. Then a and b satisfy the congruence (5) too.

($n = 11$). Since $k < n = 11$, from Tables 4.1 and 4.2 it follows that in this case only pairs $(3, 9)$ and $(9, 3)$ are dual.

($n = 31$). Then (5) reduces to the congruence $13a \equiv 22 (\mathrm{mod}\ 31)$. Since the greatest common divisor of 13 and 31 is 1, this congruence has only one solution $a = 16$. This solution satisfies (1). Obviously, $(16, 16) \in dp(-\mathbf{P})$.

($n = 41$). Then (5) has the form $3a \equiv 13 (\mathrm{mod}\ 41)$ and has only one solution $a = 18$. The pair $(18, 24) \in -\mathbf{P}$, but 24 does not satisfies the above congruence. Thus for $n = 41$ the set $dp(-\mathbf{P})$ is empty.

($n = 121$). Then $44a \equiv 11 (\mathrm{mod}\ 121)$. Since the greatest common divisor of 44 and 11 is 11, this congruence has 11 solutions. Any a satisfying the congruence $44a \equiv 11 (\mathrm{mod}\ 121)$ satisfies also the congruence $4a \equiv 1 (\mathrm{mod}\ 11)$, which has only one solution $a = 3$. Thus the set S of solutions of $44a \equiv 11 (\mathrm{mod}\ 121)$ consists of the numbers the form $3 + 11k$, $k = 0, 1, 2, \ldots, 10$. Since for any $a, b \in S$ we have $[a + b]_{11} = 6$, so $[a + b]_{121} \neq 1$. This means that for $n = 121$ the set $dp(-\mathbf{P})$ is empty.

($n = 341$). Then $44a \equiv 22 (\mathrm{mod}\ 341)$. This congruence has 11 solutions. Any a satisfying this congruence satisfies also the congruence $4a \equiv 2 (\mathrm{mod}\ 31)$, which has only one solution $a = 16$. Thus the solutions of $44a \equiv 22 (\mathrm{mod}\ 341)$ have the form $x = 16 + 31k$, $k = 0, 1, 2, \ldots, 10$. Direct calculations shows that only pairs $(47, 295)$ and $(295, 47)$ are dual.

($n = 451$). Then $44a \equiv 341 (\mathrm{mod}\ 451)$. This congruence has 11 solutions. Any a satisfying this congruence also satisfies the congruence $4a \equiv 31 (\mathrm{mod}\ 41)$, which has only one solution $a = 18$. Thus $S = \{18 + 41k \mid k = 0, 1, \ldots, 10\}$ is the set of solutions of $44a \equiv 341 (\mathrm{mod}\ 451)$. Since $[a + b]_{41} = 36$ for all $a, b \in S$, in the case $n = 451$ there no dual pairs.

($n = 1271$). Then $44a \equiv 177 (\mathrm{mod}\ 1271)$. This congruence is satisfied only by $a = 264$. The pair $(264, 1008) \in -\mathbf{P}$, but 1008 does not satisfy this congruence. So, for $n = 1271$ the set $dp(-\mathbf{P})$ is empty.

($n = 3751$). Then $44a \equiv 2068 (\mathrm{mod}\ 3751)$. This congruence has 11 solutions. Any a satisfying this congruence satisfies also the congruence $4a \equiv 188 (\mathrm{mod}\ 341)$, which has only one solution $x = 47$. Thus $S = \{47 + 341k \mid k = 0, 1, \ldots, 10\}$ contains all solutions of the congruence $44a \equiv 2068 (\mathrm{mod}\ 3751)$. Since $[a + b]_{341} = 94$ for all $a, b \in S$, in this case there no dual pairs.

($n = 4961$). Then $44a \equiv 4851 (\mathrm{mod}\ 4961)$. This congruence has 11 solutions. Any a satisfying this congruence satisfies also the congruence $4a \equiv 441 (\mathrm{mod}\ 451)$, which has only one solution $a = 223$. Thus $S = \{223 + 451k \mid k = 0, 1, \ldots, 10\}$ contains all solutions of the congruence $44a \equiv 441 (\mathrm{mod}\ 4851)$. Since $[a + b]_{451} = 446$ for all $a, b \in S$, also in this case there no dual pairs.

($n = 13\,981$). Then $44a \equiv 11616 (\mathrm{mod}\ 13981)$. This congruence has 11 solutions. Proceeding as in previous cases we can see that $S = \{264 + 1271k \mid k = 0, 1, \ldots, 10\}$ contains all solutions of this congruence. Since $[a + b]_{1271} = 528$ for all $a, b \in S$, in this case there no dual pairs too.

($n = 153\,791$). Then $44a \equiv 39578 (\mathrm{mod}\ 153791)$. Analogously as in previous cases we can see that the set $S = \{7890 + 13981k \mid k = 0, 1, \ldots, 10\}$ contains all solutions of this congruence and $[a+b]_{13981} \neq 1$ for $a, b \in S$. So, in this case there no dual pairs.

This completes the proof. $\square$

Proposition 4.13. $dp(-(\mathbf{P}^*)) = \{(3,3), (5,7), (6,6), (7,5)\}$.

Proof. If $(a, 1-a) \in dp((-\mathbf{P})^*)$, then, by Table 3.3,

$$[a^4 + 3a^3 + 4a^2 + 2a + 1]_n = 0 \tag{6}$$

and $0 = [(1-a)^4 + 3(1-a)^3 + 4(1-a)^2 + 2(1-a) + 1]_n = [a^4 - 7a^3 + 19a^2 - 23a + 11]_n$, i.e.,

$$[a^4]_n = [7a^3 - 19a^2 + 23a - 11]_n. \tag{7}$$

Comparing (6) with (7) we obtain

$$[10a^3]_n = [15a^2 - 25a + 10]_n. \tag{8}$$

Multiplying this equation by 11 and a we obtain $[110a^3]_n = [165a^2 - 275a + 110]_n$ and $[10a^4]_n = [15a^3 - 25a^2 + 10a]_n$.

From (6) we have $[10a^4]_n = [-30a^3 - 40a^2 - 20a - 10]_n$, which together with the last equation implies $[45a^3]_n = [-15a^2 - 30a - 10]_n$. Comparing this equation with (8) multiplied by 4 we obtain

$$[5a^3]_n = [-75a^2 + 70a - 50]_n. \tag{9}$$

Consequently, $[-150a^2 + 140a - 100]_n = [10a^3]_n = [15a^2 - 25a + 10]_n$. So, $[165a^2]_n = [165a - 110]_n$. Thus,

$$[110a^3]_n = [165a^2 - 275a + 110]_n = [-110a]_n \tag{10}$$

and $[110a^4]_n = [-110a^2]_n$. Now, multiplying (6) by 110 and applying the last two expressions we obtain $[330a^2]_n = [110a - 110]_n$. This and (10) imply $[-330a]_n = [330a^3]_n = [110a^2 - 110a]_n$. So, $[110a^2]_n = [-220a]_n$ and $[110a^3]_n = [-220a^2]_n$. Hence $[-110a]_n = [110a^3]_n = [-220a^2]_n$. Thus $[110a]_n = [220a^2]_n$. Consequently, $[110a - 110]_n = [330a^2]_n = [220a^2 + 110a^2]_n = [110a + 110a^2]_n$. Hence $[110a^2]_n = [-110]_n$. Therefore, $[110a]_n = [220a^2]_n = [-220]_n$ and $[-110]_n = [110a^2] = [-220a]_n = [440]_n$, i.e., $[550]_n = 0$. Since n must be odd, the possible values of n are 5, 11, 25, 55 and 275.

($n = 5$). Direct calculation shows that in this case only $(3,3) \in dp(-(\mathbf{P})^*)$.

($n = 11$). In this case only $(5,7), (6,6), (7,5) \in dp(-(\mathbf{P}^*))$.

($n = 25$). Any a satisfying (6) and (7) satisfies also (9), which for $n = 25$ has the form $[5a^3]_{25} = [20a]_{25}$. Solutions of this equation also satisfy the equation $[a^3]_5 = [4a]_5$. This equation has two solutions that are relatively prime to 5, namely $a = 2$ and $a = 3$. Thus the solutions of $[5a^3]_{25} = [20a]_{25}$ should be in one of the following sets: $S' = \{2+5k \mid k = 0,1,2,3,4\}$ or $S'' = \{3+5k \mid k = 0,1,2,3,4\}$. For $(a,b) \in dp(-(\mathbf{P}^*))$, $[a+b]_{25} = 1$. This is possible only for $a, b \in S''$. But it is easy to check that none of $a \in S''$ satisfies (6). (Also none of $a \in S'$ satisfies (6).) Hence for $n = 25$ the set $dp(-(\mathbf{P}^*))$ is empty.

($n = 275$). The number of solutions of the congruence (9) calculated modulo $275 = 11 \times 25$ is equal to $t_1 \times t_2$, where t_1 is the number of the solutions of (9) calculated modulo 11 and t_2 is the number of the solutions of (9) calculated modulo 25 (cf. [11]). Since $t_2 = 0$, for $n = 275$ the set $dp(-(\mathbf{P}^*))$ is empty. $\Box$

5. Moving from one type to another

The mappings $\mathbf{T} \mapsto \mathbf{T}^*$, $\mathbf{T} \mapsto -\mathbf{T}$, $\mathbf{T} \mapsto \mathbf{T}^{+\mathbf{t}}$ and $\mathbf{T} \mapsto \mathbf{T}^{-\mathbf{t}}$ transform one type of idempotent k-translatable quasigroups to another. We already know that $\mathbf{H} = \mathbf{H}^* = -\mathbf{C3}$, $\mathbf{GS} = \mathbf{GS}^* = -\mathbf{RM}$, $\mathbf{RM} = \mathbf{LM}^* = -\mathbf{GS}$, $\mathbf{LM} = \mathbf{RM}^*$, $\mathbf{ARO} = -\mathbf{ARO}$ and $\mathbf{C3} = -\mathbf{H}$. These formulae allow us to move from certain types to others. For example, to move from $\mathbf{GS}$ to $\mathbf{RM}$ we convert any $(a, 1-a) \in \mathbf{GS}$ to $(-a, 1+a)$ and then $(-a, 1+a) \in \mathbf{RM}$. Similarly, to move from $\mathbf{C3}$ to $\mathbf{H}$ we convert any $(a, 1-a) \in \mathbf{C3}$ to $(-a, 1+a)$ and then $(-a, 1+a) \in \mathbf{H}$. To move from $\mathbf{RM}$ to $\mathbf{LM}$ we convert any $(a, 1-a) \in \mathbf{RM}$ to $(1-a.a)$ and then $(1-a, a) \in \mathbf{LM}$. Also, $(\mathbf{GS})^{+\mathbf{1}} = \mathbf{LM}$, $(\mathbf{LM})^{-\mathbf{1}} = \mathbf{GS}$ and $\mathbf{LM} = (\mathbf{GS})^{+\mathbf{1}} = (-\mathbf{RM})^{+\mathbf{1}} = (-(\mathbf{LM}^*))^{+\mathbf{1}}$. We prove below that $\mathbf{T} = (-(\mathbf{T}^*))^{+\mathbf{1}}$ for any type $\mathbf{T} \subseteq \mathbf{IKQ}$.

Notice that $\mathbf{T} = \mathbf{T}^*$ does not imply $-(\mathbf{T}^*) = (-\mathbf{T})^*$ because, $\mathbf{H} = \mathbf{H}^*$ and $-\mathbf{H} = \mathbf{C3}$ and so, $(-\mathbf{H})^* = \mathbf{C3}^* \neq \mathbf{C3} = -(\mathbf{H}^*)$. This proves the following proposition.

Proposition 5.1. *In general,* $(-\mathbf{T})^* \neq -(\mathbf{T}^*)$.

Theorem 5.2. *For any type* $\mathbf{T}$ *of idempotent k-translatable quasigroups*

$$-\mathbf{T} = (\mathbf{T}^*)^{-\mathbf{1}} \quad \textit{and} \quad \mathbf{T} = -((\mathbf{T}^*)^{-\mathbf{1}}) = (-(\mathbf{T}^*))^{+\mathbf{1}}.$$

Proof. We have $(a, 1-a) \in (\mathbf{T}^*)^{-\mathbf{1}} \Leftrightarrow (a+1, -a) \in \mathbf{T}^* \Leftrightarrow (-a, a+1) \in \mathbf{T} \Leftrightarrow (a, 1-a) \in -\mathbf{T}$. Since $-(-\mathbf{T}) = \mathbf{T}$, from $-\mathbf{T} = (\mathbf{T}^*)^{-\mathbf{1}}$ it follows $\mathbf{T} = -((\mathbf{T}^*)^{-\mathbf{1}})$. Also, $(a, 1-a) \in \mathbf{T} \Leftrightarrow (1-a, a) \in \mathbf{T}^* \Leftrightarrow (a-1, 2-a) \in -(\mathbf{T}^*) \Leftrightarrow (a, 1-a) \in (-(\mathbf{T}^*))^{+\mathbf{1}}$. $\Box$

Corollary 5.3. $\mathbf{T}^* = -(\mathbf{T}^{-\mathbf{1}}) = (-\mathbf{T})^{+\mathbf{1}} = ((\mathbf{T}^{-\mathbf{1}})^*)^{-\mathbf{1}} = ((-(\mathbf{T}^*))^*)^{-\mathbf{1}}$.

Proof. As a consequence of Theorem 5.2 we get, $\mathbf{T}^* = -(-(\mathbf{T}^*)) = ((-(\mathbf{T}^*))^*)^{-\mathbf{1}}$. Also, $-(\mathbf{T}^*) = ((\mathbf{T}^*)^*)^{-\mathbf{1}} = \mathbf{T}^{-\mathbf{1}}$ implies $\mathbf{T}^* = -(\mathbf{T}^{-\mathbf{1}}) = ((\mathbf{T}^{-\mathbf{1}})^*)^{-\mathbf{1}}$. Finally, $\mathbf{T}^* = (-\mathbf{T})^{+\mathbf{1}}$. $\Box$

Corollary 5.4. $\mathbf{T} = (-(\mathbf{T}^*))^{+\mathbf{1}} = -((\mathbf{T}^*)^{-\mathbf{1}})$.

Proof. Observe that $(a, 1-a) \in \mathbf{T} \Leftrightarrow (1-a, a) \in \mathbf{T}^* \Leftrightarrow (a-1, 2-a) \in -(\mathbf{T}^*) \Leftrightarrow (a, 1-a) \in (-(\mathbf{T}^*))^{+\mathbf{1}}$. So, $\mathbf{T} = (-(\mathbf{T}^*))^{+\mathbf{1}}$. Also, $(\mathbf{T}^*)^{-\mathbf{1}} = ((-\mathbf{T})^{+\mathbf{1}})^{-\mathbf{1}} = -\mathbf{T}$, by Corollary 5.3. Hence, $\mathbf{T} = -((\mathbf{T}^*)^{-\mathbf{1}})$. □

Corollary 5.5. $-(\mathbf{T}^*) = \mathbf{T}^{-\mathbf{1}}$ *and* $(-\mathbf{T})^* = \mathbf{T}^{+\mathbf{1}}$.

Proof. From Corollary 5.3 it follows that $-(\mathbf{T}^*) = \mathbf{T}^{-\mathbf{1}}$. Then, $(a, 1-a) \in (-\mathbf{T})^* \Leftrightarrow (1-a, a) \in -\mathbf{T} \Leftrightarrow (a-1, 2-a) \in \mathbf{T} \Leftrightarrow (a, 1-a) \in \mathbf{T}^{+\mathbf{1}}$. □

We can now answer the question, *when does* $-(\mathbf{T}^*) = (-\mathbf{T})^*$?

Theorem 5.6. $(-\mathbf{T})^* = -(\mathbf{T}^*) \Leftrightarrow \mathbf{T}^{+\mathbf{1}} = \mathbf{T}^{-\mathbf{1}} \Leftrightarrow \mathbf{T} = \mathbf{T}^{+\mathbf{2}} \Leftrightarrow \mathbf{T} = \mathbf{T}^{-\mathbf{2}}$.

Proof. Indeed, by Corollary 5.5, $(-\mathbf{T})^* = -(\mathbf{T}^*) \Leftrightarrow \mathbf{T}^{+\mathbf{1}} = \mathbf{T}^{-\mathbf{1}}$. We also have, $\mathbf{T}^{+\mathbf{1}} = \mathbf{T}^{-\mathbf{1}} \Leftrightarrow \mathbf{T} = \mathbf{T}^{+\mathbf{2}} \Leftrightarrow \mathbf{T} = \mathbf{T}^{-\mathbf{2}}$. □

Theorem 5.7. $-(\mathbf{T}^{+\mathbf{1}}) = (-\mathbf{T})^{+\mathbf{1}} \Leftrightarrow -(\mathbf{T}^{-\mathbf{1}}) = (-\mathbf{T})^{-\mathbf{1}} \Leftrightarrow (\mathbf{T}^*)^{+\mathbf{1}} = (\mathbf{T}^{-\mathbf{1}})^* \Leftrightarrow (\mathbf{T}^*)^{-\mathbf{1}} = (\mathbf{T}^{-\mathbf{1}})^* \Leftrightarrow (-\mathbf{T})^* = -(\mathbf{T}^*)$.

Proof. We have $(a, 1-a) \in -(\mathbf{T}^{+\mathbf{1}}) \Leftrightarrow (-a, 1+a) \in \mathbf{T}^{+\mathbf{1}} \Leftrightarrow (-1-a, 2+a) \in \mathbf{T} \Leftrightarrow (2+a, -1-a) \in \mathbf{T}^* \Leftrightarrow (a, 1-a) \in (\mathbf{T}^*)^{-\mathbf{2}} = (-\mathbf{T})^{-\mathbf{1}}$ and $(-\mathbf{T})^{+\mathbf{1}} = \mathbf{T}^*$, by Corollary 5.3. Therefore, $-(\mathbf{T}^{+\mathbf{1}}) = (-\mathbf{T})^{+\mathbf{1}} \Leftrightarrow \mathbf{T}^* = (-\mathbf{T})^{-\mathbf{1}} \Leftrightarrow -(\mathbf{T}^{-\mathbf{1}}) = (-\mathbf{T})^{-\mathbf{1}}$. But by Corollary 5.3 we also have $(-\mathbf{T})^* = -((-\mathbf{T})^{-\mathbf{1}})$, so $\mathbf{T}^* = (-\mathbf{T})^{-\mathbf{1}} \Leftrightarrow (-\mathbf{T})^* = -(\mathbf{T}^*)$. □

6. Orthogonality

Definition 6.1. Two quasigroups $(Q, \cdot)$ and $(Q, \circ)$ are called *orthogonal* if, for every $s, t \in Q$, the equations $x \cdot y = s$ and $x \circ y = t$ have unique solutions $x, y \in Q$.

Not every pair of idempotent translatable quasigroups of the same order are orthogonal. The criterion of orthogonality of such quasigroups is given by the following theorem that also can be deduced from results obtained in [8].

Theorem 6.2. *The quasigroups* $(Q, \cdot)$ *and* $(Q, \circ)$*, where* $x \cdot y = [ax + (1-a)y]_n$ *and* $x \circ y = [cx + (1-c)y]_n$ *are orthogonal if* $a - c$ *and* n *are relatively prime.*

Proof. Since $x \cdot y = [ax + (1-a)y]_n$ and $x \circ y = [cx + (1-c)y]_n$ are quasigroup operations, a and n (also c and n) are relatively prime. So, there are $a', c' \in Q$ such that $[aa']_n = [cc']_n = 1$.

Let $s, t \in Q$. Suppose that

$$\begin{cases} x \cdot y = [ax + (1-a)y]_n = s, \\ x \circ y = [cx + (1-c)y]_n = t. \end{cases}$$

Multiply the first equation by a' and the second by c', we obtain the following system of equations

$$\begin{cases} [x+(a'-1)y]_n = sa', \\ [x+(c'-1)y]_n = tc', \end{cases}$$

that will be written as

$$\begin{cases} [(a'-c')y]_n = sa'-tc', \\ [x+(c'-1)y]_n = tc'. \end{cases}$$

This system has a unique solution if and only if the mapping $\varphi(y) = [(a'-c')y]_n$ transforms Q onto Q. This is possible only in the case when $a'-c'$ and n are relatively prime. Since p divides $a'-c'$ if and only if p divides $a-c$, $a'-c'$ and n are relatively prime if and only if $a-c$ and n are relatively prime. This observation completes the proof. □

Corollary 6.3. *A quasigroup $(Q,\cdot)$, where $x\cdot y = [ax+(1-a)y]_n$, and its dual quasigroup $(Q,*)$ are orthogonal if and only if $2a-1$ and n are relatively prime.*

Applying this corollary to Table 3.1 we obtain

Corollary 6.4. *Quasigroups from* **Q** *and* **ARO** *are orthogonal to their dual quasigroups.*

7.Belousov's identities

Belousov in [1] proved the following Theorem.

Theorem 7.1. *Any minimal nontrivial identity in a quasigroup is parastrophically equivalent to one of the following identity types: $x(x\cdot xy) = y$, $x(y\cdot yx) = y$, $x\cdot xy = yx$, $xy\cdot x = y\cdot xy$, $xy\cdot yx = y$, $xy\cdot y = x\cdot xy$ and $yx\cdot xy = y$.*

We now explore these identities within **IKQ**. Observe first that the identity $x(x\cdot xy) = y$ defines the type **C3**, the identity $x(y\cdot yx) = y$ defines the type **U** and the identity $x\cdot xy = yx$ defines the type **LM**.

Proposition 7.2. *In* **IKQ** *each of the identities $xy\cdot x = y\cdot xy$ and $xy\cdot yx = y$ define a quadratical quasigroup.*

Proof. Since $x\cdot y = [ax+(1-a)y]_n$, each of these identities implies the identity $2a^2-2a+1_n = 0$. So, by Theorem 2.1, $(G,\cdot)$ is quadratical. □

Proposition 7.3. *There are no quasigroups in* **IKQ** *that satisfy either of the identities $xy\cdot y = x\cdot xy$ or $yx\cdot xy = y$.*

Proof. In **IKQ** each of these identities imply the identity $[2a^2-2a]_n = 0$. This implies $0 = [k(2a^2-2a)]_n = [2(ka)a-2ka]_n = [2(k+a)a-2(k+a)]_n = [2a^2]_n = [2a]_n$. So, $0 = [2ak]_n = [2(k+a)]_n = [2k]_n$, and consequently $2 = [2kk']_n = 0$, a contradiction. □

8. Parastrophes

Each quasigroup $Q = (Q, \cdot)$ determines five new quasigroups $Q_i = (Q, \circ_i)$ (called *parastrophes* or *conjugate quasigroups*), where the operation $\circ_i$ is defined as follows:

$$\begin{aligned} x \circ_1 y = z &\Leftrightarrow x \cdot z = y, \\ x \circ_2 y = z &\Leftrightarrow z \cdot y = x, \\ x \circ_3 y = z &\Leftrightarrow z \cdot x = y, \\ x \circ_4 y = z &\Leftrightarrow y \cdot z = x, \\ x \circ_5 y = z &\Leftrightarrow y \cdot x = z. \end{aligned}$$

It is not difficult to observe that these parastrophes are pairwise dual. Namely, $Q^* = Q_5$, $Q_1^* = Q_4$ and $Q_2^* = Q_3$.

In general, such defined parastrophes are not isotopic, but if $(Q, \cdot)$ is an idempotent k-translatable quasigroup of order n, then all its parastrophes are isotopic (cf. [5]) and have simple form.

Theorem 8.1. *Parastrophes of a k-translatable idempotent quasigroup $(Q, \cdot)$ with the multiplication defined by $x \cdot y = [ax + by]_n$ are t-translatable idempotent quasigroups of the form:*

$$\begin{aligned} x \circ_1 y &= [(1 - b')x + b'y]_n, \\ x \circ_2 y &= [a'x + (1 - a')y]_n, \\ x \circ_3 y &= [(1 - a')x + a'y]_n, \\ x \circ_4 y &= [b'x + (1 - b')y]_n, \\ x \circ_5 y &= [(1 - a)x + ay]_n. \end{aligned}$$

Q_1 is t-translatable for $t = a$, Q_2 for $t = b'$, Q_3 for $t = b$, Q_4 for $t = a'$, Q_5 for $t = k'$.

Proof. By simple computations we can see that the parastrophes of $(Q, \cdot)$ have the above form. So they are idempotent quasigroups. Their t-translatability follows from the fact that $[a + b]_n = 1$ and $[a' + b']_n = [a'b']_n$. □

Corollary 8.2. *Parastrophes of a k-translatable quadratical quasigroup $(Q, \cdot)$ with the multiplication $x \cdot y = [ax + by]_n$, have the form:*

$$\begin{aligned} x \circ_1 y &= [kx + (1 - k)y]_n, \\ x \circ_2 y &= [(k + 1)x - ky]_n, \\ x \circ_3 y &= [-kx + (k + 1)y]_n, \\ x \circ_4 y &= [(1 - k)x + ky]_n, \\ x \circ_5 y &= [(1 - a)x + ay]_n. \end{aligned}$$

Theorem 8.3. *If $(Q, \cdot)$ with $x \cdot y = [ax + by]_n$ is a k-translatable quadratical quasigroup, then its parastrophe types are as in the table below, where (u, v) in the column $x \circ_i y$ and the row* **T** *means that the parastrophe $x \circ_i y$ of $(Q, \cdot)$ is of type* **T** *only for $a = u$ and $b = v$.*

	$x \cdot y$	$x \circ_1 y$	$x \circ_2 y$	$x \circ_3 y$	$x \circ_4 y$	$x \circ_5 y$
$\mathbf{Q}$	*always*	$(2,4)$	$(4,2)$	$(2,4)$	$(4,2)$	*always*
$\mathbf{H}$	*never*	*never*	*never*	*never*	*never*	*never*
$\mathbf{GS}$	*never*	$(4,2)$	$(2,4)$	$(4,2)$	$(2,4)$	*never*
$\mathbf{RM}$	$(2,4)$	$(2,4)$	*never*	*never*	$(4,2)$	$(4,2)$
$\mathbf{LM}$	$(4,2)$	*never*	$(4,2)$	$(2,4)$	*never*	$(2,4)$
$\mathbf{ARO}$	*never*	*never*	$(11,7)$	$(7,11)$	*never*	*never*
$\mathbf{ARO}^*$	*never*	$(7,11)$	*never*	*never*	$(11,7)$	*never*
$\mathbf{C3}$	$(3,11)$	*never*	$(3,11)$	$(11,3)$	*never*	$(11,3)$
$\mathbf{C3}^*$	$(11,3)$	$(11,3)$	*never*	*never*	$(3,11)$	$(3,11)$
$\mathbf{P}$	$(4,2)$	*never*	$(4,2)$	$(2,4)$	*never*	$(2,4)$
$\mathbf{P}^*$	$(2,4)$	$(2,4)$	*never*	*never*	$(4,2)$	$(4,2)$
$\mathbf{U}$	*never*	$(4,2)$	$(2,4)$	$(4,2)$	$(2,4)$	*never*
$\mathbf{U}^*$	*never*	$(4,2)$	$(2,4)$	$(4,2)$	$(2,4)$	*never*
$-\mathbf{LM}$	$(5,37)$	*never*	$(5,37)$	$(37,5)$	*never*	$(37,5)$
$-(\mathbf{C3}^*)$	$(60,38)$	$(3,11)$	$(56,6)$	$(6,56)$	$(11,3)$	$(38,60)$
$-(\mathbf{ARO}^*)$	*never*	$(11,7)$	$(11,3),(62,28)$ $(596,562)$	$(3,11),(28,62)$ $(562,596)$	$(7,11)$	*never*
$-\mathbf{U}$	*never*	$(2,4)$	$(46,56)$	$(56,46)$	$(4,2)$	*never*
$-(\mathbf{U}^*)$	*never*	$(2,4)$	$(7,11)$	$(11,7)$	$(4,2)$	*never*
$-\mathbf{P}$	$(37,5)$	*never*	$(37,5)$	$(5,37)$	*never*	$(5,37)$
$-(\mathbf{P}^*)$	$(153,89)$	$(4,2)$	*never*	*never*	$(2,4)$	$(89,153)$

Proof. In the proof we will use conditions given in Table 3.1 and the fact that an idempotent k-translatable quasigroup $(Q,\cdot)$ is quadratical if and only if $x\cdot y = [ax+(1-a)y]_n$, where $n>1$ is odd, $[2a^2-2a+1]_n=0$, $k=[1-2a]_n$ and $[k^2]_n=-1$. Moreover, since $Q^*=Q_5$, $Q_1^*=Q_4$ and $Q_2^*=Q_3$, it is sufficient verity only when Q, Q_1 and Q_2 are fixed type $\mathbf{T}$, i.e., for which values of (a,b) $Q,Q_1,Q_2\in\mathbf{T}$.

$\mathbf{T}=\mathbf{Q}$.

• Since, $\mathbf{Q}=\mathbf{Q}^*$ (Theorem 3.1), the quasigroup Q_5 always is quadratical.

• $x\circ_1 y=[kx+(1-k)y]_n$. Thus, $0=[2k^2-2k+1]_n=[4a-3]_n=[-2k-1]_n$. So, $0=[(-2k-1)k]_n=[2-k]_n$. Hence $k=2$, $n=k^2+1=5$ and $[2a]_5=4$, which gives $(2,4)$.

• $x\circ_2 y=[(k+1)x-ky]_n$. Then $0=[2(k_1)^2-2(k+1)=1]_n=[2k-1]_n$. So, $0=[(2k-1)k]_n=[-2-k]_n$. Hence, $n=k^2+1=5$, $[2a]_5=3$ and $a=4$, which gives $(4,2)$.

$\mathbf{T}=\mathbf{H}$.

If $Q \in \mathbf{H}$, then $0 = [2a^2 - 2a + 1]_n = [a^2 - a + 1]_n$, which is impossible. So, $Q \notin \mathbf{H}$.
• If $Q_1 \in \mathbf{H}$, then $0 = [k^2 - k + 1]_n = [-1 - k + 1]_n = [-k]_n$, a contradiction.
• If $Q_2 \in \mathbf{H}$, then $0 = [(k+1)^2 - (k+1) + 1]_n = [k^2 + k + 1]_n = [k]_n$, a contradiction.
$\mathbf{T} = -(\mathbf{ARO}^*)$. If $Q \in -(\mathbf{ARO}^*)$, then $0 = [2a^2 - 2a + 1]_n = [a^2 + 4a + 1]_n$. This gives $[6a]_n = 0$. But then $0 = [3(2a^2 - 2a + 1)]_n = 3$. So, must be $n = 3$ and $a = 2$, which is impossible
• If $Q_1 \in -(\mathbf{ARO}^*)$, then $0 = [2k^2 + 4k + 1]_n$ implies $[4k]_n = 1$. Thus $[-4]_n = k$, $n = k^2 + 1 = 17$ and $[2a]_{17} = [1 - k]_{17} = 5$. So, $a = 11$, which gives the pair $(11, 7)$.
• If $Q_2 \in -(\mathbf{ARO}^*)$, then $0 = [2(k+1)^2 + 4(k+1) + 1]_n = [8k + 5]_n$. Hence, $[8k]_n = [-5]_n$, $[-5k]_n = [-8]_n$ and $k = [16k - 15k]_n = [-34]_n$. Thus $0 = [k^2 + 1]_n = 1157$ means that the possible values of n are 13, 89 and 1157. For $n = 13$ we obtain $k = [-13]_{13} = 5$ and $2a = [1 - k]_{13} = 9$. So, $a = 11$, which gives the pair $(11, 3)$. By similar calculations, for $n = 89$ we get $k = 55$ and $(62, 28)$, for $n = 1157$ we obtain $k = 1123$ and $(598, 562)$.

For other types the proof is analogous, so we omit it. □

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Characterization of inverse ordered semigroups by their ordered idempotents and bi-ideals

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Abstract. We prove that an ordered semigroup is complete semilattice of group-like ordered semigroups if and only if it is completely regular and inverse. The relation between principal bi-ideals generated by two inverses of an element in an inverse ordered semigroup has been presented here. Furthermore we bring the opportunity to study complete regularity on an inverse ordered semigroups by their bi-ideals.

1. Introduction

Inverse semigroups have a natural ordering which has deep impact on their structure. The study of behavior of inverses of an element in ordered semigroups had been an area of interest among the semigroup theorists since last fifty years. Bhuniya and Hansda [1] have deal with ordered semigroups in which any two inverses of an element are $\mathcal{H}$-related. Class of these ordered semigroups are natural generalization of class of inverse semigroups (without order). We call these ordered semigroups as inverse ordered semigroups.

We characterize inverse ordered semigroups by their ordered idempotents. We study complete regularity in an inverse ordered semigroup and explore the look of resulting ordered semigroup. Keeping in mind that bi-ideals have been studied more, we give several characterizations of inverse ordered semigroups by their bi-ideals.

2. Preliminaries

An ordered semigroup is a partially ordered set $(S, \leqslant)$, and at the same time a semigroup $(S, \cdot)$ such that for all $a, b, x \in S$ $a \leqslant b$ implies $xa \leqslant xb$ and $ax \leqslant bx$. It is denoted by $(S, \cdot, \leqslant)$.

For every $H \subseteq S$, we define $(H] = \{t \in S : t \leqslant h,$ for some $h \in H\}$.

Throughout this paper unless otherwise stated S stands for an ordered semigroup. An equivalence relation ρ is called a *left* (*right*) *congruence* on S if for $a, b, c \in S$ $a\rho b$ implies $ca\rho cb$ $(ac\rho bc)$. By a *congruence* we mean both left and

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right congruence. A congruence ρ is called a *semilattice congruence* on S if for all $a, b \in S$ $a\rho a^2$ and $ab\rho ba$. By a *complete semilattice congruence* on S we mean a semilattice congruence σ on S such that for $a, b \in S$ $a \leqslant b$ implies that $a\sigma ab$. An ordered semigroup S is called a *complete semilattice of subsemigroups* of type τ if there exists a complete semilattice congruence ρ such that $(x)_\rho$ is a type τ subsemigroup of S.

Let I be a nonempty subset of an ordered semigroup S. I is a *left* (*right*) *ideal* of S, if $SI \subseteq I$ ($IS \subseteq I$) and $(I] = I$. I is an *ideal* of S if it is both a left and a right ideal of S.

Following Kehayopulu [4], a nonempty subset B of an ordered semigroup S is called a *bi-ideal* of S if $BSB \subseteq B$ and $(B] = B$. Here our aim is to study completely regular and inverse ordered semigroups by their bi-ideals.

The principal [5] left ideal, right ideal, ideal and bi-ideal [4] generated by $a \in S$ are denoted by $L(a)$, $R(a)$, $I(a)$and $B(a)$ respectively and have form

$$L(a) = (a \cup Sa],\ R(a) = (a \cup aS],\ I(a) = (a \cup Sa \cup aS \cup SaS] \text{ and } B(a) = (a \cup aSa].$$

Kehayopulu [5] defined Greens relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$ and $\mathcal{H}$ on an ordered semigroup S as follows:

$$a\mathcal{L}b \text{ if } L(a) = L(b),$$
$$a\mathcal{R}b \text{ if } R(a) = R(b),$$
$$a\mathcal{J}b \text{ if } I(a) = I(b),$$
$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}.$$

These four relations are equivalence relations on S.

A regular ordered semigroup S is said to be a *group-like* (resp. *left group-like*) [1] *ordered semigroup* if for every $a, b \in S$, $a \in (Sb]$ and $b \in (aS]$ (resp. $a \in (Sb]$). A right group-like ordered semigroup can be defined dually. Two elements $a, b \in S$ are said to *$\mathcal{H}$-related* if $a\mathcal{H}b$. An ordered semigroup S is called an *regular* (*completely regular*) [3] if for every $a \in S$, $a \in (aSa]$ $(a \in (a^2Sa^2])$. An element $b \in S$ is *inverse* of a if $a \leqslant aba$ and $b \leqslant bab$. The set of all inverses of an element $a \in S$ is denoted by $V_\leqslant(a)$. Two elements $a, b \in S$ are said to *$\mathcal{H}$-commutative* [1] if $ab \leqslant bxa$ for some $x \in S$. A regular ordered semigroup S is called *inverse* [1] if for every $a \in S$ and $a', a'' \in V_\leqslant(a)$, $a'\mathcal{H}a''$, that is, any two inverses of a are $\mathcal{H}-$related.

By an *ordered idempotent* [1] in an ordered semigroup S, we shall mean an element $e \in S$ such that $e \leqslant e^2$. We denote the set of all ordered idempotents of S by $E_\leqslant(S)$.

For the convenience of readers we state the following three results from [1].

Lemma 2.1. *Let S be completely regular ordered semigroup. Then for every $a \in S$ there is $x \in S$ such that $a \leqslant axa^2$ and $a \leqslant a^2xa$.*

Theorem 2.2. *An ordered semigroup S is completely regular if and only if for all $a \in S$ there exists $a' \in V_\leqslant(a)$ such that $aa' \leqslant a'ua$ and $a'a \leqslant ava'$ for some $u, v \in S$.*

Lemma 2.3. *Let S be a completely regular ordered semigroup. Then following statements hold in S:*

1. *$\mathcal{J}$ is the least complete semilattice congruence on S.*

2. *S is a complete semilattice of completely simple ordered semigroups.*

3. Inverse ordered semigroup

Let S be an ordered semigroup and ρ be an equivalence on S. We say thatl an ideal I of S is generated by a ρ-unique element $b \in S$ if $b\rho x$ for any generator x of I.

Definition 3.1. A regular ordered semigroup S is called *inverse* if for every $a \in S$, any two inverses of a are $\mathcal{H}$-related.

Example 3.2. The ordered semigroup $S = \{a, e, f\}$ with the multiplication defined below and with the discrete order is an inverse ordered semigroup.

$\cdot$	a	e	f
a	a	e	f
e	f	e	a
f	e	a	f

We present a role of ordered idempotents in an inverse ordered semigroup in the next theorem.

Theorem 3.3. *An ordered semigroup S is inverse if and only if every principal left ideal and every principal right ideal of S are generated by an $\mathcal{H}$-unique ordered idempotent.*

Proof. Suppose that S is inverse. Let I be a principal left ideal of S. Then there exists $e \in E_{\leqslant}(S)$ such that $I = (Se]$. If possible let $I = (Sf]$ for some $f \in E_{\leqslant}(S)$. Then $e\mathcal{L}f$ and thus $e \leqslant xf$ and $f \leqslant ye$ for some $x, y \in S$. Now $e \leqslant ee \leqslant eee \leqslant exfe$. Therefore $exf \leqslant exfexf$ so that $exf \in E_{\leqslant}(S)$. Also $exf \leqslant exfexf \leqslant exf(fe)exf$ and $fe \leqslant feee \leqslant fexfe \leqslant fe(exf)fe$. Therefore $fe \in V_{\leqslant}(exf)$. Also $exf \in V_{\leqslant}(exf)$. Since S is inverse, we have $fe\mathcal{H}exf$. Then $e \leqslant ee \leqslant exffe \leqslant fezexf$ for some $z \in S$, and so $e \leqslant fz_1$, where $z_1 = ezexf$. Similarly $f \leqslant ez_2$ for some $z_2 \in S$. So $e\mathcal{R}f$. Hence $e\mathcal{H}f$. Likewise every principal right ideal of S generated by certain $\mathcal{H}$-unique ordered idempotent.

Conversely assume that given condition holds in S. Let $a \in S$ and $a', a'' \in V_{\leqslant}(a)$. Clearly $(Sa] = (Sa'a] = (Sa''a]$. Since $a'a, a''a \in E_{\leqslant}(S)$ we have that $a'a\mathcal{H}a''a$, by given condition. Then there are $s, t \in S$ such that $a' \leqslant a''asa'$ and $a'' \leqslant a'ata''$. Thus $a'\mathcal{R}a''$. Likewise $a'\mathcal{L}a''$, that is $a'\mathcal{H}a''$. Hence S is an inverse ordered semigroup. $\square$

In the following we show that an ordered semigroup S is inverse if and only if any two ordered idempotents of S are $\mathcal{H}$-commutative.

Theorem 3.4. *The following conditions are equivalent on an ordered semigroup S.*

(1) *S is an inverse semigroup;*

(2) *S is regular and its idempotents are $\mathcal{H}$-commutative;*

(3) *For every $e, f \in E_{\leqslant}(S)$, $e\mathcal{L}f(e\mathcal{R}f)$ implies $e\mathcal{H}f$.*

Proof. (1) $\Rightarrow$ (2): Obviously S is regular. Let us assume that $a \in S$ and $a', a'' \in V_{\leqslant}(a)$.

Consider $e, f \in E_{\leqslant}(S)$. Since S is regular, so there is $x \in S$ such that $x \in V_{\leqslant}(ef)$. Now $x \leqslant xefx$ implies that $fxe \leqslant fxe(ef)fxe$ and $ef \leqslant efxef$ implies $ef \leqslant ef(fxe)ef$. Thus $ef \in V_{\leqslant}(fxe)$. Also $fxe \leqslant fxefxe$ that is $fxe \in E_{\leqslant}(S)$. So $fxe \in V_{\leqslant}(fxe)$. Since S is inverse, so $fxe\mathcal{H}ef$. Then there are $s_1, s_2 \in S$ such that $ef \leqslant fxes_1$ and $ef \leqslant s_2fxe$. Now $ef \leqslant efxef$ implies that $ef \leqslant f(xes_1xs_2fx)e$. Therefore $ef \leqslant fye$, where $y = xes_1xs_2fx$. Similarly there is $z \in S$ such that $fe \leqslant ezf$. Hence any two idempotents are $\mathcal{H}$-commutative.

(2) $\Rightarrow$ (3): Let $e, f \in E_{\leqslant}(S)$ be such that $e\mathcal{L}f$. Then $e \leqslant xf$ and $f \leqslant ye$ for some $x, y \in S$. Now $e \leqslant xf$ implies $e \leqslant exf$, and so $e \leqslant ee \leqslant exfe$ which implies that $exf \leqslant exfexf$. So $exf \in E_{\leqslant}(S)$. Similarly $fye \in E_{\leqslant}(S)$. Now $e \leqslant exf \leqslant exff \leqslant exffye$. Since $exf, fye \in E_{\leqslant}(S)$, by condition (2) we have $exffye \leqslant (fye)z(exf)$ for some $z \in S$. Hence $e \leqslant ft$, where $t = yezexf$. Similarly $f \leqslant ew$ for some $w \in S$, so that $e\mathcal{R}f$. Hence $e\mathcal{H}f$. If $e\mathcal{R}f$ then $e\mathcal{H}f$ can be done dually.

(3) $\Rightarrow$ (1): Let $a \in S$ and $a', a'' \in V_{\leqslant}(a)$. Now $aa' \leqslant aa''aa'$ and $aa'' \leqslant aa'aa''$. So $aa'\mathcal{R}aa''$ which implies that $aa'\mathcal{H}aa''$, by the condition (3). Also $a'a\mathcal{H}a''a$. Then $a' \leqslant a'aa'$ gives that $a' \leqslant a''axa$ for some $x \in S$. Therefore $a' \leqslant a''t$ where $t = axa$. In similar way it is possible to obtained $u, v, w \in S$ such that $a' \leqslant ua''$, $a'' \leqslant a'v$ and $a'' \leqslant wa'$. So $a'\mathcal{H}a''$. Hence S is an inverse ordered semigroup. $\square$

Lemma 3.5. *Let S be an inverse ordered semigroup. Then following statements hold in S.*

(1) *$a\mathcal{L}b$ if and only if $a'a\mathcal{H}b'b$ for some $a, b \in S$ and $a' \in V_{\leqslant}(a)$ $b' \in V_{\leqslant}(b)$;*

(2) *$a\mathcal{R}b$ if and only if $aa'\mathcal{H}bb'$ for some $a, b \in S$ and $a' \in V_{\leqslant}(a)$ $b' \in V_{\leqslant}(b)$;*

(3) *for any $a \in S$ and $e \in E_{\leqslant}(S)$ there are $x, y \in S$ such that $aexa', a'eya \in E_{\leqslant}(S)$; where $a' \in V_{\leqslant}(a)$.*

(4) *for any $a, b \in S$ there are $x, y \in S$ such that $ab \leqslant abb'xa'ab$ and $b'a' \leqslant b'a'aybb'a'$, where $a' \in V_{\leqslant}(a)$ and $b' \in V_{\leqslant}(b)$.*

Proof. (1): Let $a, b \in S$ be such that $a\mathcal{L}b$. Let $a' \in V_{\leqslant}(a)$, $b' \in V_{\leqslant}(b)$. Since $a \leqslant aa'a$ and $a'a \leqslant a'aa'a$, we have $a\mathcal{L}a'a$ which implies that $b\mathcal{L}a'a$. Also $b\mathcal{L}b'b$. Hence $a'a\mathcal{L}b'b$. Since $a'a, b'b \in E_{\leqslant}(S)$ and S is inverse we have $a'a\mathcal{H}b'b$, by Theorem 3.4(3).

Conversely suppose that given condition holds in S. Let $a, b \in S$ with $a' \in V_{\leqslant}(a)$ and $b' \in V_{\leqslant}(b)$. Then by given condition $aa'\mathcal{H}bb'$. Also we have $a\mathcal{L}a'a$ and $b\mathcal{L}b'b$ so that $a\mathcal{L}b$.

(2): This is similar to (1).

(3): Let $a \in S$ and $e \in E_{\leqslant}(S)$. Also $a'a \in E_{\leqslant}(S)$. Since S is an inverse, there is an $x \in S$ such that $a'ae \leqslant exa'a$, by Theorem 3.4(2). Now $aexa' \leqslant aa'aeexa' \leqslant aexa'aexa'$. So $aexa' \in E_{\leqslant}(S)$. Likewise $a'eya \in E_{\leqslant}(S)$; for some $y \in S$.

(4): Let $a, b \in S$ with $a' \in V_{\leqslant}(a)$, $b' \in V_{\leqslant}(b)$. So $a'a, b'b \in E_{\leqslant}(S)$. Now $ab \leqslant aa'abb'b \leqslant$ and $a'abb' \leqslant b'bxa'a$, by Theorem 3.4(2). Thus $ab \leqslant abb'xa'ab$. Likewise $b'a' \leqslant b'a'aybb'a'$; for some $y \in S$. □

In the following theorem an inverse ordered semigroup has been characterized by the inverse of an element of the set $(eSf]$.

Theorem 3.6. *Let S be an ordered semigroup and $e, f \in E_{\leqslant}(S)$. Then S is inverse if and only if for every $x \in (eSf]$ implies $x' \in (fSe]$, where $x' \in V_{\leqslant}(x)$.*

Proof. First suppose that S is an inverse ordered semigroup and $x \in (eSf]$. Then $x \leqslant es_1f$ for some $s_1 \in S$. Let $x' \in V_{\leqslant}(x)$. Now $x' \leqslant x'xx' \leqslant x'es_1fx'$, and so $es_1fx' \leqslant es_1fx'es_1fx'$. Hence $es_1fx' \in E_{\leqslant}(S)$. Similarly $x'es_1f \in E_{\leqslant}(S)$. Now there is $s_2 \in S$ such that $x'es_1fx' \leqslant x'es_1ffx' \leqslant fs_2x'es_1fx'$, by Theorem 3.4(2) . Also $fs_2x'es_1fx' \leqslant fs_2x'ees_1fx' \leqslant fs_2x'es_1fx's_3e$, for some $s_3 \in S$. Then $x' \leqslant x'xx'$ implies that $x' \leqslant fs_2x'es_1fx' \leqslant fs_2x'es_1fx's_3e$. Hence $x' \in (fSe]$.

Conversely assume that the given conditions hold in S. First consider a left ideal L of S such that $L = (Se] = (Sf]$ for $e, f \in E_{\leqslant}(S)$. Then $e\mathcal{L}f$, so that $e \leqslant ee \leqslant ezf$ for some $z \in S$. Therefore $e \in (eSf]$. Since $e \in V_{\leqslant}(e)$ we have $e \in (fSe]$, by given condition. Likewise $f \in (eSf]$. This implies that $e\mathcal{R}f$ and so $e\mathcal{H}f$. Similarly it can be shown that every principal right ideal of S generated by $\mathcal{H}$-unique ordered idempotent. Thus by Theorem 3.3, S is an inverse ordered semigroup. □

Corollary 3.7. *The following conditions are equivalent on a regular ordered semigroup S.*

(1) *S is an inverse ordered semigroup;*

(2) *for any $a \in S$ and for any $a' \in V_{\leqslant}(a)$, $aa', a'a$ are $\mathcal{H}$-commutative;*

(3) *for any $e \in E_{\leqslant}(S)$, any two inverses of e are $\mathcal{H}$-related;*

(4) *for any $e \in E_{\leqslant}(S)$ and all its inverses are $\mathcal{H}$-commutative;*

5) *for any* $e \in E_{\leqslant}(S)$ *and* $e' \in V_{\leqslant}(e)$, ee' *and* $e'e$ *are* $\mathcal{H}$*-commutative.*

Proof. (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4), and (4) $\Rightarrow$ (5): These are obvious.

(5) $\Rightarrow$ (1): Let $e, f \in E_{\leqslant}(S)$ and $x \in V_{\leqslant}(ef)$. So $ef \leqslant efxef \leqslant effxeef$ and $x \leqslant xefx$ implies that $fxe \leqslant fxeeffxe$. So $ef \in V_{\leqslant}(fxe)$. Also $fxe \in E_{\leqslant}(S)$. Now $ef \leqslant efxef \leqslant effxeef \leqslant effxefxeef \leqslant fxez_1efz_2fxe$, for some $z_1, z_2 \in S$, by the given condition. So $ef \leqslant fz_3e$ where $z_3 = xemefnfx$. Similarly $fe \leqslant ez_4f$, for some $z_4 \in S$. So e, f are $\mathcal{H}$-commutative. Hence by Theorem 3.4 S is inverse ordered semigroup. □

We study inverse ordered semigroup together with complete regularity in the following theorem.

Theorem 3.8. *The following conditions are equivalent on a regular ordered semigroup* S.

(1) S *is inverse and completely regular;*

(2) S *is a complete semilattice of group like ordered semigroups;*

(3) $ab\mathcal{H}ba$ *whenever* $ab, ba \in E_{\leqslant}(S)$*;*

(4) *any ordered idempotent of* S *is* $\mathcal{H}$*-commutative to any element of* S*;*

(5) *for any* $e, f \in E_{\leqslant}(S)$ $e\mathcal{J}f$ *implies* $e\mathcal{H}f$*;*

(6) $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{J}$.

Proof. (1) $\Rightarrow$ (2): Let S be a completely regular and inverse ordered semigroup. Then by Lemma 2.3, $\mathcal{J}$ is the complete semilattice congruence on S and every $\mathcal{H}$-class is a group-like ordered semigroup. We now prove $\mathcal{H} = \mathcal{J}$. Let $a, b \in S$ be such that $a\mathcal{J}b$. So there are $x, y, u, v \in S$ such that $a \leqslant xby$ and $b \leqslant uav$. Since S is completely regular, so there are $h, g, f \in S$ such that $x \leqslant x^2hx$, $b \leqslant b^2gb$, $b \leqslant bgb^2$, $y \leqslant yfy^2$, by Lemma 2.3. Now $a \leqslant x^2hxb^2gbyfy^2 \leqslant x^2hxb^2gbgb^2yfy^2$.

Let $p \in V_{\leqslant}(x^2hxb^2g)$. So

$$x^2hxb^2g \leqslant x^2hxb^2gpx^2hxb^2g \leqslant x^2hxb^2g(b^2gpx^2h)x^2hxb^2g$$

and

$$b^2gpx^2h \leqslant b^2gpx^2hxb^2gpx^2h \leqslant b^2gpx^2h(x^2hxb^2g)b^2gpx^2h.$$

This shows that $b^2gpx^2h \in V_{\leqslant}(x^2hxb^2g)$. Also

$$x^2hxb^2g \leqslant x^2hxb^2gpx^2hxb^2g \leqslant x^2hxb^2g(x^2hxb^2gp^2)x^2hxb^2g$$

and

$$x^2hxb^2gp^2 \leqslant x^2hxb^2gpx^2hxb^2gp^2 \leqslant x^2hxb^2gp^2(x^2hxb^2g)x^2hxb^2gp^2,$$

which implies that $x^2hxb^2gp^2 \in V_{\leqslant}(x^2hxb^2g)$. Similarly $p^2x^2hxb^2g \in V_{\leqslant}(x^2hxb^2g)$. Since $b^2gpx^2h, x^2hxb^2gp^2 \in V_{\leqslant}(x^2hxb^2g)$ and S is inverse, so there is $t \in S$ such that $x^2hxb^2gp^2 \leqslant b^2gpx^2ht$. Thus

$x^2hxb^2g \leqslant x^2hxb^2gpx^2hxb^2g \leqslant x^2hxb^2gp^2(x^2hxb^2g)^2$

implies that $x^2hxb^2g \leqslant b^2gpx^2hxt(x^2hxb^2g)^2 = bs$ where $s = bgpx^2ht(x^2hxb^2g)^2$.

Similarly there is $s_1 \in S$ such that $b^2gyfy^2 \leqslant s_1b$. Hence $a \leqslant x^2hxb^2gbyfy^2 \leqslant bsbyfy^2 = bs_2$, where $s_2 = sbyfy^2$. Similarly $a \leqslant s_3b$ for some $s_3 \in S$. Likewise $b \leqslant s_4a$ and $b \leqslant as_5$, for some $s_4, s_5 \in S$. So $a\mathcal{H}b$. Thus $\mathcal{J} \subseteq \mathcal{H}$. Also $\mathcal{H} \subseteq \mathcal{J}$, and Hence $\mathcal{J} = \mathcal{H}$. Therefore S is complete semilattice of group-like ordered semigroups.

(2) $\Rightarrow$ (3): Suppose that S is a complete semilattice Y of group like ordered semigroups $\{S_\alpha\}_{\alpha\in Y}$. Let $a, b \in S$ such that $ab, ba \in E_{\leqslant}(S)$. Let ρ be the corresponding semilattice congruence on S. Then there is $\alpha \in Y$ such that $ab, ba \in S_\alpha$. Since S_α is group like ordered semigroups, so $ab\mathcal{H}ba$.

(3) $\Rightarrow$ (4): Let $a \in S$ and $e \in E_{\leqslant}(S)$. Since S is regular there is an $x \in S$ such that $a \leqslant axa$. Clearly $ax, xa \in E_{\leqslant}(S)$. Thus by condition (3) $ax\mathcal{H}xa$. So $xa \leqslant axu$ and $ax \leqslant vxa$, for some $u, v \in S$. Then we have $a \leqslant axa \leqslant axaxa \leqslant axaxaxa \leqslant aaxuxvxaa = a^2ta^2$, where $t = xuxvx$. Now $a \leqslant a^2ta^2 \leqslant a(a^2ta^2ta^2ta^2)a \leqslant a^2(a^2ta^2ta^2ta^2ta^2)a$, that is $a \leqslant a^2ya$, where $y = a^2ta^2ta^2ta^2ta^2$. Similarly $a \leqslant aya^2$. Clearly $a^2y, ya^2 \in E_{\leqslant}(S)$.

Let $e, f \in E_{\leqslant}(S)$ and $x \in V_{\leqslant}(ef)$. Then we have $x \leqslant xefx$. So $fxe \leqslant fxefxe \leqslant fxeeffxe$ and $ef \leqslant efxef \leqslant effxeef$. So $ef \in V_{\leqslant}(fxe)$. Also $ef \leqslant effxeef$ implies that $effxe \leqslant effxeeffxe$, and $fxeef \leqslant fxeeffxeef$. So $effxe, fxeef \in E_{\leqslant}(S)$ and thus $effxe\mathcal{H}fxeef$, by the condition (3). Then there are $u, v \in S$ such that $effxe \leqslant fxeefu$ and $fxeef \leqslant veffxe$. Now $ef \leqslant effxefxeef \leqslant fxeefuveffxe = fce$; where $c = xe^2fuvef^2x$. Likewise $fe \leqslant edf$, for some $d \in S$.

Now $ae \leqslant a^2yae$. Let $z \in V_{\leqslant}(a^2yae)$. So

$$a^2yae \leqslant a^2yaeza^2yae \leqslant a^2yae(eza^2y)a^2yae.$$

Clearly $a^2yaeeza^2y, eza^2ya^2yae \in E_{\leqslant}(S)$ and thus $a^2yaeeza^2y\mathcal{H}eza^2ya^2yae$, by condition (3). Now $ae \leqslant a^2yae \leqslant a^2yaeeza^2ya^2yae \leqslant eza^2ys_1a^2yaea^2yae$, for some $s_1 \in S$. So $ae \leqslant es_2ae$, where $s_2 = za^2ys_1a^2yaea^2y$. Again $ae \leqslant es_2aya^2e \leqslant es_2aes_3ya^2$, for some $s_3 \in S$, since $ya^2, e \in E_{\leqslant}(S)$. That is $ae \leqslant es_4a$, for some $s_4 \in S$. Similarly $ea \leqslant as_5e$, for some $s_5 \in S$. So a, e are $\mathcal{H}$-commutative.

(4) $\Rightarrow$ (5): Let $e, f \in E_{\leqslant}(S)$ such that $e\mathcal{J}f$. Then there are $x, y, z, u \in S$ such that $e \leqslant xfy$ and $f \leqslant zeu$. Now $e \leqslant xfy$ implies that $e \leqslant fhxy$ and $e \leqslant xykf$ by the given condition for some $h, k \in S$. Similarly $f \leqslant zeu$ gives $f \leqslant es_1zu$ and $f \leqslant zus_2e$ for some $s_1, s_2 \in S$. Hence $e\mathcal{H}f$.

(5) $\Rightarrow$ (6): Let $a, b \in S$ such that $a\mathcal{J}b$. Then there are $s, t, u, v \in S$ such that $a \leqslant sbt$ and $b \leqslant uav$. Since S is regular so $a \leqslant axa$ and $b \leqslant byb$ for some $x, y \in S$ so that $ax \leqslant axax$ and $by \leqslant byby$. Now $axax \leqslant axsbtx \leqslant axsbybtx$ that is $ax \leqslant axsbybtx$. Likewise $by \leqslant byuaxavy$. Thus $ax\mathcal{J}by$, and so from given condition $ax\mathcal{H}by$. Similarly $xa\mathcal{H}yb$. So there is $c \in S$ such that $ax \leqslant byc$, that is $a \leqslant byca = bd$, for some $d = yca \in S$. Likewise $a \leqslant pb$, $b \leqslant qa$ for some $p, q \in S$.

Thus $a\mathcal{H}b$. So $\mathcal{H} = \mathcal{J}$. Now $\mathcal{J} = \mathcal{H} = \mathcal{L} \cap \mathcal{R}$ gives $\mathcal{J} \subseteq \mathcal{L}$ and $\mathcal{J} \subseteq \mathcal{R}$. Therefore $\mathcal{L} = \mathcal{J} = \mathcal{R}$.

(6) $\Rightarrow$ (1): Let $a \in S$. Since S is regular so there exists $a' \in V_{\leqslant}(a)$. Clearly $a\mathcal{L}a'a$ and $a\mathcal{R}aa'$. So by the given condition $a\mathcal{R}a'a$ and $a\mathcal{L}aa'$. Now $a \leqslant aa'a \leqslant aa'aa'a \leqslant aa'aa'aa'a \leqslant aas_1a's_2aa$ for some $s_1, s_2 \in S$. So $a \leqslant a^2pa^2$ where $p = s_1a's_2$. So S is completely regular.

Also let $a', a'' \in V_{\leqslant}(a)$. Now $a\mathcal{L}a'a\mathcal{L}a''a$ implies that $a\mathcal{R}a'a\mathcal{R}a''a$. Also by the given condition we can show that $a\mathcal{L}aa'\mathcal{L}a''a$. So it is to check that $a'\mathcal{R}a''$ and $a'\mathcal{L}a''$. So $a'\mathcal{H}a''$. Hence S is inverse ordered semigroup. □

4. Bi-ideals in inverse ordered semigroups

Following Hansda [2] an ordered semigroup S is *completely regular* if and only if for every $a \in S$ there is some $e \in E_{\leqslant}(S)$ such that $a \leqslant ae$, $a \leqslant ea$ and $B(a) = B(e)$. Here our approach allows one to see the role of principal bi-ideal generated by an inverse of an element in an inverse ordered semigroup.

Lemma 4.1. *Let S be a regular ordered semigroup. Then the following conditions are equivalent.*

1. *S is a completely regular ordered semigroup;*
2. *for any $a \in S$ there is $a' \in V_{\leqslant}(a)$ such that $B(a) = B(a')$;*
3. *for any $a \in S$ there is $a' \in V_{\leqslant}(a)$ such that $B(aa') = B(a) \cap B(a') = B(a') \cap B(a) = B(a'a)$;*
4. *$B(a) = B(a^2)$ for any $a \in S$.*

Proof. (1) $\Rightarrow$ (2): First suppose that S is completely regular ordered semigroup. Let $a \in S$. Then by Theorem 2.2 there is $a' \in V_{\leqslant}(a)$ such that $aa' \leqslant a'ua$ and $a'a \leqslant ava'$ for some $u, v \in S$. Let $x \in B(a)$. Therefore $x \leqslant a$ or $x \leqslant as_1a$ for some $s_1 \in S$. If $x \leqslant a$ then $x \leqslant aa'a \leqslant aa'aa'a \leqslant a'uaaava' = a'za'$ where $z = uaaav$. Again if $x \leqslant as_1a$ then there is $t \in S$ such that $x \leqslant a'ta'$. Therefore in either case $x \in B(a')$. Also $a \in B(a')$. So $B(a) \subseteq B(a')$. Similarly $B(a') \subseteq B(a)$. Hence $B(a) = B(a')$.

(2) $\Rightarrow$ (3): Suppose that condition (2) holds. Let $a \in S$. Then there is $a' \in V_{\leqslant}(a)$ such that $a \leqslant aa'a$. Let $x \in B(aa')$. Then $x \leqslant aa'$ or $x \leqslant aa'saa'$ for some $s \in S$. By given condition $a' \in B(a)$. So $a' \leqslant a$ or there is $y \in S$ such that $a' \leqslant aya$. If $x \leqslant aa'saa'$ and $a' \leqslant aya$ then $x \leqslant aa'saaya$. If $x \leqslant aa'saa'$ and $a' \leqslant a$ then $x \leqslant aa'saa$. If $x \leqslant aa'$ and $a' \leqslant a$ then $x \leqslant aa$. Also if $x \leqslant aa'$ and $a' \leqslant aya$ then $x \leqslant aaya$. Therefore in either case $x \in B(a)$. Hence $B(aa') \subseteq B(a)$. Likewise $B(aa') \subseteq B(a')$ and hence $B(aa') \subseteq B(a) \cap B(a')$.

Let $w \in B(a) \cap B(a')$. So $w \in B(a)$ and $w \in B(a')$. Therefore $w \leqslant a$ or $w \leqslant as_2a$ and $w \leqslant a'$ or $w \leqslant a's_3a'$ for some $s_2, s_3 \in S$. Since S is regular, there is $d \in S$

such that $w \leqslant wdw$. If $w \leqslant a$ and $w \leqslant a'$ then $w \leqslant wdw \leqslant ada' \leqslant aa'ada'aa'$. If $w \leqslant as_2a$ and $w \leqslant a'$ then $w \leqslant wdw \leqslant as_2ada' \leqslant aa'as_2ada'aa'$. If $w \leqslant as_2a$ and $w \leqslant a's_3a'$ then $w \leqslant wdw \leqslant as_2ada's_3a' \leqslant aa'as_2ada's_3a'aa'$. If $w \leqslant a$ and $w \leqslant a's_3a'$ then $w \leqslant wdw \leqslant ada's_3a' \leqslant aa'ada's_3a'aa'$. Therefore in either case $w \in B(aa')$. Hence $B(a) \cap B(a') \subseteq B(aa')$. Thus $B(aa') = B(a) \cap B(a')$.

(3) $\Rightarrow$ (4): Suppose that condition (3) holds. Let $a \in S$. Then there exists $a' \in V_{\leqslant}(a)$ such that $B(aa') = B(a'a)$. Now $a \leqslant aa'a \leqslant aa'aa'a = a(a'a)a'(aa')a$. Now by condition (3) $a'a \leqslant aa'zaa'$ and $aa' \leqslant a'awa'a$ for some $z, w \in S$. Then $a \leqslant a(a'a)a'(aa')a$ implies that $a \leqslant a(aa'zaa')a'(a'awa'a)a = a^2(a'zaa'a'a'awa')a^2$. Thus $B(a) \subseteq B(a^2)$. It is evident that $B(a^2) \subseteq B(a)$ and hence $B(a) = B(a^2)$.

(4) $\Rightarrow$ (1): Suppose condition (4) holds. Therefore $a \leqslant a^2$ or $a \leqslant a^2s_2a^2$ and $a^2 \leqslant a$ or $a^2 \leqslant as_3a$ for some $s_2, s_3 \in S$. Therefore in either case $a\mathcal{H}a^2$. Since S is regular, so $a \leqslant aza$ for some $z \in S$. So $a \leqslant aza \leqslant a^2s_4zs_5a^2$ for some $s_4, s_5 \in S$. Hence S is completely regular ordered semigroup. □

Corollary 4.2. *A regular ordered semigroup S is completely regular if and only if for any $a \in S$ there is $a' \in V_{\leqslant}(a)$ such that $B(aa') = B(a) \cap B(a') = B(a'a) = B(a) = B(a')$.*

Proof. This follows from Lemma 4.1 . □

Theorem 4.3. *Let S be a regular ordered semigroup. Then the following conditions are equivalent.*

(1) *S is an inverse ordered semigroup;*

(2) *for any $a \in S$, $B(a') = B(a'')$ for every $a', a'' \in V_{\leqslant}(a)$;*

(3) *for any $e, f \in E_{\leqslant}(S)$, $B(ef) = B(e) \cap B(f)$;*

(4) *for any $e \in E_{\leqslant}(S)$ and $x \in V_{\leqslant}(e)$, $B(ex) = B(xe)$.*

Proof. (1) $\Rightarrow$ (2): First suppose that S is an inverse ordered semigroup. Let $a \in S$ and $a', a'' \in V_{\leqslant}(a)$. Suppose $x \in B(a')$. Therefore $x \leqslant a'$ or $x \leqslant a'ya'$ for some $y \in S$. Since S is inverse, so $a'\mathcal{H}a''$. If $x \leqslant a'$ then $x \leqslant a'aa' \leqslant a''s_1as_2a''$ for some $s_1, s_2 \in S$. Therefore $x \leqslant a''sa''$ where $s = s_1as_2$. Again if $x \leqslant a'ya'$ then there is $s_3 \in S$ such that $x \leqslant a''s_3a''$. Therefore in either case $x \in B(a'')$. Also $a' \in B(a'')$. So $B(a') \subseteq B(a'')$. Similarly $B(a'') \subseteq B(a')$. Hence $B(a') = B(a'')$.

(2) $\Rightarrow$ (3): First suppose that condition (2) holds and let $e, f \in E_{\leqslant}(S)$. Let $x \in V_{\leqslant}(ef)$. Therefore $ef \leqslant efxef$ and $x \leqslant xefx$. So $fxe \leqslant fxefxe$. Therefore $fxe \in E_{\leqslant}(S)$. Also $ef \leqslant ef(fxe)ef$ and $fxe \leqslant fxe(ef)fxe$. Therefore $ef, fxe \in V_{\leqslant}(fxe)$. So by the condition $B(ef) = B(fxe)$. Clearly $ef\mathcal{H}fxe$.

Let $w \in B(ef)$. Therefore $w \leqslant ef$ or $w \leqslant efs_1ef$ for some $s_1 \in S$. If $w \leqslant ef$ then $w \leqslant ef \leqslant efxef \leqslant efxs_2fxe$ for some $s_2 \in S$. Again if $w \leqslant efs_1ef$ then $w \leqslant efs_1ef \leqslant efs_1s_2fxe$. So in either case $w \in B(e)$. Similarly $w \in B(f)$. Hence $w \in B(e) \cap B(f)$. Therefore $B(ef) \subseteq B(e) \cap B(f)$.

Again let $y \in B(e) \cap B(f)$. So $y \leqslant e$ or $y \leqslant es_4e$ and $y \leqslant f$ or $y \leqslant fs_5f$, for some $s_4, s_5 \in S$. Since S is inverse, there exists $z \in V_{\leqslant}(y)$ such that $z \leqslant zyz$ and $y \leqslant yzy$. If $y \leqslant es_4e$ and $y \leqslant fs_5f$ then $z \leqslant zyz \leqslant zes_4ez$. Therefore $es_4ez \leqslant es_4ezes_4ez$. So $es_4ez \in E_{\leqslant}(S)$. Similarly $zfs_5f \in E_{\leqslant}(S)$. Now $es_4ez \leqslant es_4ezes_4ez \leqslant es_4ez(yz)es_4ez$ and $yz \leqslant yzyz \leqslant yz(es_4ez)yz$. Therefore $es_4ez, yz \in V_{\leqslant}(es_4ez)$. So condition (2) $B(es_4ez) = B(yz)$. Similarly $B(zfs_5f) = B(zy)$. Clearly $es_4ez\mathcal{H}yz$ and $zfs_5f\mathcal{H}zy$. Now $y \leqslant yzy \leqslant es_4ezfs_5f \leqslant es_4ezyzfs_5f \leqslant ees_4ezyzfs_5ff \leqslant eyzs_6ys_7zyf \leqslant efs_5fzs_6ys_7zes_4ef$ for some $s_6, s_7 \in S$. If $y \leqslant e$ and $y \leqslant f$ then clearly $B(ez) = B(yz)$ and $B(zf) = B(zy)$. Now $y \leqslant yzy \leqslant ezf \leqslant eezyzff \leqslant eyzs_8ys_9zyf \leqslant efzs_8ys_9zef$ for some $s_8, s_9 \in S$. If $y \leqslant e$ and $y \leqslant fs_5f$ then $zfs_5f \in E_{\leqslant}(S)$. Now $y \leqslant yzy \leqslant ezfs_5f \leqslant eezfs_5ff \leqslant ezfs_5fs_{10}ef \leqslant ezfs_5ffs_{10}ef \leqslant efs_{11}zfs_5fs_{10}ef$ for some $s_{10}, s_{11} \in S$. Again if $y \leqslant es_4e$ and $y \leqslant f$ then $es_4ez \in E_{\leqslant}(S)$. Now $y \leqslant yzy \leqslant es_4ezf \leqslant ees_4ezff \leqslant efs_{12}es_4ezf \leqslant efs_{12}ees_4ezf \leqslant efs_{12}es_4ezs_{13}ef$ for some $s_{12}, s_{13} \in S$. Therefore in either case $y \in B(ef)$ and so $B(e) \cap B(f) \subseteq B(ef)$. Hence $B(e) \cap B(f) = B(ef)$.

(3) $\Rightarrow$ (4): First suppose that condition (3) holds in S. Let $e \in E_{\leqslant}(S)$ and $x \in V_{\leqslant}(e)$ so $e, xe, ex \in E_{\leqslant}(S)$. By condition (3) $B(exe) = B(e) \cap B(xe)$,that is, $B(e) = B(e) \cap B(xe)$. Therefore $B(e) \subseteq B(xe)$. Again $B(xee) = B(e) \cap B(xe)$ that is $B(xe) = B(e) \cap B(xe)$. So $B(xe) \subseteq B(e)$. Therefore $B(e) = B(xe)$. Similarly $B(e) = B(ex)$. Therefore $B(xe) = B(ex)$.

(4) $\Rightarrow$ (1): Suppose that condition (4) holds in S. Now $ex \in B(e)$ and $ex \in B(x)$. So $ex \leqslant e$ or $ex \leqslant eb_1e$, and $ex \leqslant x$ or $ex \leqslant xb_2x$ for some $b_1, b_2 \in S$. If $ex \leqslant e$ and $ex \leqslant x$ then $ex \leqslant exex \leqslant xe \leqslant xexe = xae$ where $a = ex$. If $ex \leqslant e$ and $ex \leqslant xb_2x$ then $ex \leqslant exex \leqslant xb_2xe = xbe$ where $b = b_2x$. If $ex \leqslant eb_1e$ and $ex \leqslant x$ then $ex \leqslant exex \leqslant xeb_1e = xce$ where $c = eb_1$. Again if $ex \leqslant eb_1e$ and $ex \leqslant xb_2x$ then $ex \leqslant exex \leqslant xb_2xeb_1e = xde$ where $d = b_2xeb_1$. Therefore in either case $ex \leqslant xse$ for some $s \in S$. Similarly $xe \leqslant etx$ for some $t \in S$. Hence e, x are $\mathcal{H}$-commutative. So S is an inverse ordered semigroup, by Corollary 3.7. $\square$

Corollary 4.4. *A regular ordered semigroup S is inverse if and only if for any $e \in E_{\leqslant}(S)$ and $x \in V_{\leqslant}(e)$, $B(ex) = B(e) \cap B(x) = B(xe) = B(e) = B(x)$.*

Corollary 4.5. *A regular ordered semigroup S is inverse if and only if for any $e, f \in E_{\leqslant}(S)$, $e\mathcal{L}f(e\mathcal{R}f)$ implies $B(e) = B(f)$.*

Proof. Let S be an inverse ordered semigroup. Since S is inverse, $e\mathcal{L}f(e\mathcal{R}f)$ implies $e\mathcal{H}f$ by Theorem 3.4. So it is easy to check that $B(e) = B(f)$.

Conversely suppose that the condition holds in S. Now $B(e) = B(f)$ gives that $e \in B(f)$ and $f \in B(e)$. Therefore $e \leqslant f$ or $e \leqslant fxf$ and $f \leqslant e$ or $f \leqslant eye$. In either case $e\mathcal{R}f$. So $e\mathcal{L}f$ implies $e\mathcal{H}f$. Hence S is inverse ordered semigroup by Theorem 3.4. $\square$

Lemma 4.6. *Let S be an inverse ordered semigroup and $a' \in V_{\leqslant}(a)$, $b' \in V_{\leqslant}(b)$, where $a, b \in S$. Then following conditions hold on S:*

(1) $a\mathcal{L}b$ *if and only if* $B(a'a) = B(b'b)$.

(2) $a\mathcal{R}b$ *if and only if* $B(aa') = B(bb')$.

Proof. (1): Let S be an inverse ordered semigroup and $a' \in V_{\leqslant}(a)$, $b' \in V_{\leqslant}(b)$ where $a, b \in S$. So by Lemma 3.5 $a'a\mathcal{H}b'b$. Let $x \in B(a'a)$. Therefore $x \leqslant a'a$ or $x \leqslant a'as_1a'a$ for some $s_1 \in S$. So it is easy to verify that $x \in B(b'b)$. Also $a'a \in B(b'b)$. So $B(a'a) \subseteq B(b'b)$. Similarly $B(b'b) \subseteq B(a'a)$. So $B(a'a) = B(b'b)$.

The converse statement is obvious.

(2): Analogously as (1). □

Characterization of ordered semigroups which are both completely regular and inverse have been presented in the next theorem.

Theorem 4.7. *Let S be a regular ordered semigroup. Then the following conditions are equivalent.*

(1) *S is completely regular and inverse ordered semigroup;*

(2) *for any $a, b \in S$, $B(ab) = B(ba) = B(a) \cap B(b)$;*

(3) *$B(ab) = B(ba)$ where $a, b \in S$ and $ab, ba \in E_{\leqslant}(S)$;*

(4) *for any $a, b \in S$, $a\mathcal{L}b$ implies $B(a) = B(b)$.*

Proof. (1) $\Rightarrow$ (2): First suppose that S is completely regular and inverse ordered semigroup. Then any ordered idempotent of S is $\mathcal{H}$ commutative to any element of S by Theorem 3.8 . Let $a, b \in S$. Since S is regular, so there are $p, q, r \in S$ such that $a \leqslant apa$, $b \leqslant bqb$ and $ab \leqslant abrab$. Clearly $bq, pa \in E_{\leqslant}(S)$. Now $ab \leqslant abrab \leqslant abqbrapab \leqslant bqp_1abrabp_2pa = bs_2a$ where $s_2 = qp_1abrabp_2pa$. Let $x \in B(ab)$. Therefore $x \leqslant ab$ or $x \leqslant abs_1ab$ for some $s_1 \in S$. If $x \leqslant abs_1ab$, then $x \leqslant abs_1bs_2a$. So $x \leqslant aya$ where $y = bs_1bs_2$. Again if $x \leqslant ab$, then $x \leqslant abrab \leqslant abrbs_2a$. So in either case $x \in B(a)$. Also $ab \in B(a)$. Similarly $x \in B(b)$ and $ab \in B(b)$. Hence $B(ab) \subseteq B(a) \cap B(b)$.

Again let $y \in B(a) \cap B(b)$. So $y \leqslant a$ or $y \leqslant as_4a$ and $y \leqslant b$ or $y \leqslant bs_5b$ for some $s_4, s_5 \in S$. Since S is regular, So there is $z \in S$ such that $y \leqslant yzy$. Now if $y \leqslant as_4a$ and $y \leqslant bs_5b$ then $y \leqslant yzy \leqslant as_4azbs_5b \leqslant as_4azbqbs_5b \leqslant abqs_6s_4azbs_5b \leqslant abqs_6s_4apazbs_5b \leqslant abqs_6s_4azbs_5s_7pab$ for some $s_6, s_7 \in S$. Again if $y \leqslant a$ and $y \leqslant b$ then $y \leqslant yzy \leqslant azb \leqslant apazbqb \leqslant abqs_8pazb \leqslant abqs_8zs_9pab$ for some $s_8, s_9 \in S$. Again if $y \leqslant a$ and $y \leqslant bs_5b$ then $y \leqslant yzy \leqslant azbs_5b \leqslant apazbqbs_5b \leqslant abqs_{10}pazbs_5b \leqslant abqs_{10}zbs_5s_{11}pab$ for some $s_{10}, s_{11} \in S$. Also if $y \leqslant as_4a$ and $y \leqslant b$ then $y \leqslant yzy \leqslant as_4azb \leqslant as_4apazbqb \leqslant abqs_{12}s_4apazb \leqslant abqs_{12}s_4azs_{13}pab$ for some $s_{12}, s_{13} \in S$. Therefore in either case $y \in B(ab)$. Hence $B(a) \cap B(b) \subseteq B(ab)$. Therefore $B(ab) = B(a) \cap B(b) = B(b) \cap B(a) = B(ba)$.

(2) $\Rightarrow$ (3): Suppose that the given condition (2) holds. Therefore $B(ab) = B(a) \cap B(b) = B(b) \cap B(a) = B(ba)$.

(3) $\Rightarrow$ (4): First suppose that condition (3) holds and let $a\mathcal{L}b$. So there exists $s,t \in S$ such that $a \leqslant sb$ and $b \leqslant ta$. Since S is regular, $a \leqslant aza$ and $z \leqslant zaz$ for some $z \in V_{\leqslant}(a)$. Clearly $az, za \in E_{\leqslant}(S)$. Now $z \leqslant zaz \leqslant zsbz$. So $zsb \leqslant zsbzsb$. Therefore $zsb \in E_{\leqslant}(S)$. Similarly $bzs \in E_{\leqslant}(S)$. So by the condition (3) $B(zsb) = B(bzs)$. Clearly $zsb\mathcal{H}bzs$. Similarly $za\mathcal{H}az$. Let $x \in B(a)$. Therefore $x \leqslant a$ or $x \leqslant as_1a$ for some $s_1 \in S$. If $x \leqslant a$ then $x \leqslant a \leqslant aza \leqslant azsb \leqslant zas_2sb \leqslant zsbs_2sb \leqslant bzss_3s_2sb$ for some $s_2, s_3 \in S$. Similarly if $x \in as_1a$ then $x \leqslant bs_4b$ for some $s_4 \in S$. So in either case $x \in B(b)$. Therefore $B(a) \subseteq B(b)$. Similarly $B(b) \subseteq b(a)$. Therefore $B(a) = B(b)$.

Conversely suppose that the given condition holds, that is $a\mathcal{L}b$ implies $B(a) = B(b)$ for any $a, b \in S$. Now $B(a) = B(b)$ implies that $a\mathcal{R}b$. So $a\mathcal{L}b$ implies that $a\mathcal{R}b$. Therefore $\mathcal{L} \subseteq \mathcal{H}$. Also $\mathcal{H} \subseteq \mathcal{L}$. Hence $\mathcal{L} = \mathcal{H}$. So S is completely regular and an inverse ordered semigroup by Theorem 3.8. □

Corollary 4.8. *A regular ordered semigroup S is completely regular and inverse if and only if for any $e \in E_{\leqslant}(S)$ and for any $a \in S$, $B(ea) = B(e) \cap B(a)$.*

Proof. For $b = e$ we obtain the result. □

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The equivalence graph of the comaximal graph of a group

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Abstract. Let G be a finite group. The comaximal graph of G, denoted by $\Gamma_m(G)$, is a graph whose vertices are the proper subgroups of G that are not contained in the Frattini subgroup of G and join two distinct vertices H and K, whenever $G = \langle H, K\rangle$. In this paper, we define an equivalence relation $\sim$ on $V(\Gamma_m(G))$ by taking $H \sim K$ if and only if their open neighborhoods are the same. We introduce a new graph determined by equivalence classes of $V(\Gamma_m(G))$, denoted $\Gamma_E(G)$, as follows. The vertices are $V(\Gamma_E(G)) = \{[H]|H \in V(\Gamma_m(G))\}$ and two equivalence classes $[H]$ and $[K]$ are adjacent in $\Gamma_E(G)$ if and only if H and K are adjacent in $\Gamma_m(G)$. We will state some basic graph theoretic properties of $\Gamma_E(G)$ and study the relations between some properties of graph $\Gamma_m(G)$ and $\Gamma_E(G)$, such as the chromatic number, clique number, girth and diameter. Moreover, we classify the groups for which $\Gamma_E(G)$ is complete, regular or planar. Among other results, we show that if the number of maximal subgroups of the group G is less or equal than 4, then $\Gamma_m(G)$ and $\Gamma_E(G)$ are perfect graphs.

1. Introduction

The study of algebraic structures using the properties of graphs has been an exciting research topic, leading to many fascinating results and questions. Associating a graph to a group or a ring and using information on one of the two objects to solve a problem for the other is an interesting research topic, for instance, see [?, ?, ?]. For example, in [?] Sharma and Bhatwadekar defined a graph on a non-zero commutative ring with identity R, $\Gamma(R)$, with vertices as elements of R, where two distinct vertices a and b are adjacent if and only if $Ra + Rb = R$. In [?] the authors introduced and studied the comaximal graph of a finite bounded lattice, denoted by $\Gamma(R)$. They investigated some graph-theoretic properties of $\Gamma(R)$. It is shown that for two finite semi-local rings R and S, if R is reduced, then $\Gamma(R) \cong \Gamma(S)$ if and only if $R \cong S$.

Let G be a group and $L(G)$ be the set of all subgroups of G. We can associate a graph to G in many different ways (see, for example, [1, 2, 3, 14]). Here we consider the following way: Let $\Phi(G)$ be the Frattini subgroup of G. Associate a graph $\Gamma_m(G)$ to G, the comaximal graph of G, as follows: The vertex set is all proper subgroups of G that are not contained in $\Phi(G)$ and two distinct vertices H

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and K joined by an edge if and only if $G = \langle H, K\rangle$. Note that if $G \cong C_{p^n}$, a cyclic group of order p^n, then $\Phi(G) \cong C_{p^{n-1}}$ and so $\Gamma_m(G)$ is a null graph. Recently, this graph was investigated by H. Ahmadi and B. Taeri in [?, ?, ?], in which it is referred to as *the graph related to the join of subgroups*.

For a simple graph Γ, two vertices H, K are equivalent if and only if their open neighborhoods are the same, i.e., $H \sim K$ if and only if $N(H) = N(K)$ where $N(H) = \{L \in V(\Gamma) \mid H$ and L are adjacent in $\Gamma\}$. It is clear that $\sim$ is an equivalence relation on $V(\Gamma)$ and we denote the class of H by $[H]$. The graph of equivalence classes of Γ, denoted by Γ_E, is the simple graph whose vertex set is $V(\Gamma_E) = \{[H] | H \in V(\Gamma)\}$ and two distinct equivalence classes $[H]$ and $[K]$ are adjacent in Γ_E, denoted $[H] - [K]$, if H and K are adjacent in Γ. The remarkable thing is that Γ_E can be considered as a subgraph of Γ, and it can inherit many properties of Γ. In particular, in many cases, some graph theoretic properties of Γ and Γ_E are the same, such as the chromatic number, clique number and diameter. For example, in [?] the authors considered the graph of equivalence classes of the non-commuting graph of a group G and investigated some graph-theoretic properties of this graph.

In this paper, we will introduce the graph of equivalence classes of $\Gamma_m(G)$ and we will state some of basic graph theoretical properties of $\Gamma_E(G)$, for instance determining diameter, girth, dominating set, planarity of the graph and we give some relation between the graph properties of $\Gamma_m(G)$ and $\Gamma_E(G)$. We will classify all solvable groups G for which $\Gamma_E(G)$ is a complete graph. Furthermore, we show that for a non-nilpotent group G, $\Gamma_E(G)$ is planner if and only if $|G| = 2^n 3^m$ and $G/\Phi(G) \cong S_3$. In Section 3, some results on groups whose equivalence graph of comaximal graphs are complete are given. In Section 4, we will state some results on planarity of $\Gamma_E(G)$. Finally, in Section 5 we will study on the perfection of $\Gamma_E(G)$ and we will show that if $|\mathrm{Max}(G)| \leqslant 4$, then $\Gamma_E(G)$ is a perfect graph and conclude if $|\mathrm{Max}(G)| \leqslant 4$, then $\Gamma_m(G)$ is a perfect graph, too, where $\mathrm{Max}(G)$ is the set of all maximal subgroups of the group G.

2. Definitions and basic results

For a simple graph Γ, we denote the sets of the vertices and the edges of Γ by $V(\Gamma)$ and $E(\Gamma)$, respectively. A graph Γ is said to be connected if there exists a path between any two distinct vertices. The distance between two distinct vertices H and K, denoted by $d(H, K)$, is the length of the shortest path connecting H and K, if such a path exists; otherwise, we set $d(H, K) := \infty$. The degree of H, denoted by $\deg(H)$, is the number of edges incident with H. The graph Γ is regular if and only if for any two distinct vertices of graph have a same degree. Moreover, the diameter of a connected graph Γ, denoted by $\mathrm{diam}(\Gamma)$, is $\sup\{d(H, K) : H, K \in V(\Gamma)\}$. A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. We use K_n for the complete graph with n vertices. For a positive integer r, an r-partite graph is one whose vertex-set can be partitioned into r subsets so that no edge has both ends in any one subset. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. The girth of Γ, denoted by girth(Γ), is the length of the shortest cycle in Γ, if Γ contains a cycle; otherwise, we set girth$(\Gamma) := \infty$. A subset X of $V(\Gamma)$ is called a clique if the induced subgraph on X is a complete graph. The maximum size of a clique in a graph Γ is called the clique number of Γ and denoted by $\omega(\Gamma)$. The chromatic number of a graph Γ, denoted by $\chi(\Gamma)$, is the minimal number of colors which can be assigned to the vertices of Γ in such a way that every two adjacent vertices have different colors. A subset X of the vertices of Γ is called an independent set if the induced subgraph on X has no edges. The maximum size of an independent set in a graph Γ is called the independence number of Γ and denoted by $\alpha(\Gamma)$. A subset D of $V(\Gamma)$ is a dominating set of Γ if every vertex in $V(\Gamma)\setminus D$ is adjacent to some vertex in D. The domination number $\lambda(\Gamma)$ of Γ is the minimum cardinality of a dominating set. The complement of a graph Γ, denoted by $\overline{\Gamma}$, is the graph with the same vertex set such that two distinct vertices H and K are adjacent in $\overline{\Gamma}$ if and only if they are not adjacent in Γ.

Let $\Gamma_m(G)$ be the comaximal graph of a group G and

$$N(H) = \{L \in V(\Gamma_m(G)) \mid H \text{ and } L \text{ are adjacent in } \Gamma_m(G)\}$$

be the open neighborhood of the vertex H in $\Gamma_m(G)$. Two vertices H and K are equivalent in $\Gamma_m(G)$ if and only if their open neighborhoods are the same, i.e., $H \sim K$ if and only if $N(H) = N(K)$. One can see that $\sim$ is an equivalence relation on $V(\Gamma_m(G))$ and we denote the class of H by $[H]$.

Definition 2.1. Let G be a group and $\Gamma_m(G)$ be its comaximal graph. The *graph of equivalence classes of* $\Gamma_m(G)$, denoted by $\Gamma_E(G)$, is the graph whose vertex set is $V(\Gamma_E(G)) = \{[H] : H \in V(\Gamma_m(G))\}$, and two distinct equivalence classes $[H]$ and $[K]$ are adjacent in $\Gamma_E(G)$ if and only if H and K are adjacent in $\Gamma_m(G)$.

Proposition 2.2. *Let C_n be a cyclic group of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$, where $\alpha_i \in \mathbb{N}$ and $m \geqslant 2$. Then $\Gamma_E(C_n) \cong \Gamma_E(C_{p_1 \dots p_m})$.*

Proof. Assume that $C_n = \langle a \rangle$. It is easy to check that

$$N(\langle a^{p_{i_1}^{\beta_1} p_{i_2}^{\beta_2} \cdots p_{i_k}^{\beta_k}} \rangle) = N(\langle a^{p_{i_1} p_{i_2} \cdots p_{i_k}} \rangle)$$

where $\{i_1, i_2, \dots i_k\} \subset \{1, 2, \dots, m\}$ and $1 \leqslant \beta_i \leqslant \alpha_i$. Therefore $[\langle a^{p_{i_1}^{\beta_1} p_{i_2}^{\beta_2} \cdots p_{i_k}^{\beta_k}} \rangle] = [\langle a^{p_{i_1} p_{i_2} \cdots p_{i_k}} \rangle]$ and so the result follows. $\square$

Let $\pi(G)$ be the set of all prime divisors of $|G|$. By Proposition 2.2 we have the following result.

Proposition 2.3. *Let C_n and C_m be two cyclic groups of order n, m. If $\pi(C_n) = \pi(C_m) = \{p_1, \dots, p_k\}$, then $\Gamma_E(C_n) \cong \Gamma_E(C_m) \cong \Gamma_E(C_{p_1 \dots p_k})$.*

Let H be a proper subgroup of G. Set $M(H) = \{M \in \mathrm{Max}(G) | H \subseteq M\}$.

Lemma 2.4. *Let H and K be proper subgroups of G. Then*

(*i*) *$[H]$ and $[K]$ are adjacent in $\Gamma_E(G)$ if and only if $M(H) \cap M(K) = \emptyset$.*

(*ii*) *$[H] = [K]$ if and only if $M(H) = M(K)$.*

In particular, if H is only contained in a single maximal subgroup M, then $[H] = [M]$.

Proof. (i). Assume that H and K are adjacent in $\Gamma_m(G)$. If M is a maximal subgroup of G that contains both of them, then $\langle H, K\rangle \neq G$, a contradiction. Conversely, assume that the intersection of $M(H)$ and $M(K)$ is the empty set and $[H]$ and $[K]$ are not adjacent in $\Gamma_E(G)$. Then $\langle H, K\rangle$ is a proper subgroup of G and so H and K lie in a maximal subgroup of G which is a contradiction.

(ii). Let $[H] = [K]$ and M be a maximal subgroup of G such that $M \in N(H) = N(K)$. Then M is adjacent to both of H and K, which implies that for any maximal subgroup N of G, $H \subseteq N$ if and only if $K \subseteq N$. Therefore $M(H) = M(K)$. Conversely, assume that $M(H) = M(K)$ and $[H] \neq [K]$. Then $H \nsim K$ and so $N(H) \neq N(K)$. Therefore there is a vertex L in $\Gamma_m(G)$ such that $G = \langle L, H\rangle$ and $\langle L, K\rangle$ lies in a maximal subgroup of G, which is a contradiction. □

Remark 2.5. Let G be a group and $\mathrm{Max}(G) = \{M_1, \ldots, M_n\}$. For $I_n = \{1, \ldots, n\}$ we put

$$V_{i_1 i_2 \ldots i_r} = \{H \in V(\Gamma_m(G)) | M(H) = \{M_{i_1}, M_{i_2}, \ldots, M_{i_r}\}\}$$

where $i_1, i_2, \ldots, i_r \in I_n$ and $r \leqslant n-1$. By Lemma 2.4 we have $H, H' \in V_{i_1 i_2 \ldots i_r}$ if and only if $[H] = [H']$. Now if $V_{i_1 i_2 \ldots i_r} \neq \emptyset$, we may denote the vertex $V_{i_1 i_2 \ldots i_r}$ in $\Gamma_E(G)$ by $[v_{i_1 i_2 \ldots i_r}]$. Furthermore, for $1 \leqslant i \leqslant n$ we denote the class of V_i by $[M_i]$. Then we have

$$V(\Gamma_E(G)) = \{[M_i] : 1 \leqslant i \leqslant n\} \cup_{r=2}^{n-1} \{[v_{i_1 i_2 \cdots i_r}] : 1 \leqslant i_1, \cdots, i_r \leqslant n, V_{i_1 i_2 \ldots i_r} \neq \emptyset\}.$$

Furthermore, It is clear that $[v_{i_1 i_2 \ldots i_r}]$ and $[v_{j_1 j_2 \ldots j_s}]$ are adjacent in $\Gamma_E(G)$ if and only if $\{i_1, i_2, \ldots, i_r\} \cap \{j_1, j_2, \ldots, j_s\} = \emptyset$ where $1 \leqslant r, s \leqslant n-1$.

Proposition 2.6. *Assume that G is a finite group. Then*

(*i*) $\omega(\Gamma_E(G)) = \omega(\Gamma_m(G)) = \chi(\Gamma_m(G)) = \chi(\Gamma_E(G)) = |\mathrm{Max}(G)|$.

(*ii*) $\mathrm{diam}(\Gamma_E(G)) = \mathrm{diam}(\Gamma_m(G)) \leqslant slant3$. *In particular, $\Gamma_E(G)$ is connected.*

(*iii*) *If* $|\mathrm{Max}(G)| \geqslant 3$, *then* $\mathrm{girth}(\Gamma_E(G)) = 3$.

(*iv*) $\alpha(\Gamma_E(G)) \leqslant \alpha(\Gamma_m(G))$.

Proof. (i). Let $|\mathrm{Max}(G)| = n$. We claim that $\{[M_1], \dots, [M_n]\}$ is a maximum clique in $\Gamma_E(G)$. Let $\{[H_1], \dots, [H_r]\}$ be a clique in graph $\Gamma_E(G)$. Since $[H_i]$ and $[H_j]$ are adjacent, by Lemma 2.4, $M(H_i) \cap M(H_j) = \emptyset$, thus every subgroup H_i is contained in a maximal subgroup of G and so $r \leqslant n$. By the same way we have $\{M_1, \dots, M_n\}$ is a maximum clique in $\Gamma_m(G)$. Therefore $\omega(\Gamma_E(G)) = \omega(\Gamma_m(G)) = |\mathrm{Max}(G)|$. Moreover, it is clear that for any graph Γ, $\omega(\Gamma) \leqslant \chi(\Gamma)$. Now assume that $\omega(\Gamma_m(G)) = t$ and $\mathrm{Max}(G) = \{M_1, \cdots, M_t\}$. Then for $1 \leqslant i \leqslant t$, $S_i = L(M_i) \setminus L(\Phi(G))$ is an independent set and $V(\Gamma_m(G)) = \cup_{i=1}^{t} S_i$, where $L(X)$ is the set of all subgroups of a group X. Hence $\chi(\Gamma) \leqslant \omega(\Gamma)$ and the proof is complete.

(ii). Assume that $[H]$ and $[K]$ are two distinct vertices in $\Gamma_E(G)$. If $H \cap K \nsubseteq \Phi(G)$, then there is a maximal subgroup M of G such that $G = \langle M, H \rangle = \langle M, K \rangle$ and so $d([H], [K]) \leqslant 2$. Now assume that $H \cap K \subseteq \Phi(G)$. Then there are maximal subgroups M_1 and M_2 of G such that

$$G = \langle M_1, H \rangle = \langle M_2, K \rangle = \langle M_1, M_2 \rangle$$

and so $d([H], [K]) \leqslant 3$. Therefore $\mathrm{diam}(\Gamma_E(G)) \leqslant slant3$. By the same way one may have $\mathrm{diam}(\Gamma_m(G)) \leqslant slant3$, as required.

(iii). Suppose that a group G contains at least three maximal subgroups M_1, M_2 and M_3. Then $\{M_1, M_2, M_3\}$ and $\{[M_1], [M_2], [M_3]\}$ are triangles in $\Gamma_m(G)$ and $\Gamma_E(G)$ respectively and so $\mathrm{girth}(\Gamma_m(G)) = \mathrm{girth}(\Gamma_E(G)) = 3$.

(iv). It is clear that if $\{H_1, \dots, H_r\}$ is an independent set in the graph $\Gamma_m(G)$, then $\{[H_1], \dots, [H_r]\}$ is an independent set in $\Gamma_E(G)$. Thus $\alpha(\Gamma_E(G)) \leqslant \alpha(\Gamma_m(G))$. □

3. On the completeness of $\Gamma_E(G)$

Let G be a finite group. In [14], the authors have introduced the concept of *maximal graph*, denoted by $\Gamma M(G)$, as the graph whose vertices are the maximal subgroups of G and join two distinct vertices M_1 and M_2, whenever $M_1 \cap M_2 \neq 1$. If the intersection of every pair of distinct maximal subgroups of G is trivial, then the graph $\Gamma M(G)$ has no edges. Now we may recall the following theorem.

Theorem 3.1. [14, Proposition 1.3] *Let G be a finite group. The intersection of every pair of distinct maximal subgroups of G is trivial if and only if G is solvable and one of the following holds:*

(i) $G \cong C_{p^n}$ *(p is prime).*

(ii) $G \cong C_{pq}$ *(p, q different primes).*

(iii) $G \cong C_p \times C_p$ *(p is prime).*

(iv) $G = P \rtimes Q$, *where P is an elementary abelian p-group of order p^n (p a prime), $|Q| = q$, where q is a prime different from p, and Q acts irreducibly and fixed point freely on P.*

In the following theorem, we characterize all groups whose graph of equivalence classes of comaximal graph of G are complete.

Theorem 3.2. *The equivalence graph of the comaximal graph of G is complete if and only if G is solvable and one of the following holds.*

(i) $G \cong C_{p^n}$ *(p is prime).*

(ii) $G \cong C_{p^r q^s}$ *(p, q different primes).*

(iii) *G is a p-group, where $G/\Phi(G) \cong C_p \times C_p$ (p a prime). In particular, if G is an abelian p-group then $G \cong C_{p^r} \times C_{p^s}$ and $\Gamma_E(G) \cong K_{p+1}$.*

(iv) *$G/\Phi(G) \cong P \rtimes Q$, where P is an elementary abelian p-group of order p^n (p is prime), $|Q| = q$, where q is a prime different from p, and Q acts irreducibly and fixed point freely on P. Moreover, in this case, G is not nilpotent.*

Proof. Let $\Gamma_E(G)$ be a complete graph and $\mathrm{Max}(G) = \{M_1, \ldots, M_k\}$. Since M_i and $M_i \cap M_j$ are not joined by an edge, then $[M_i \cap M_j]$ is not one of the vertices of $\Gamma_E(G)$. Hence $M_i \cap M_j = \Phi(G)$ and so $V(\Gamma_E(G)) = \{[M_1], \ldots, [M_k]\}$. Moreover, the intersection of every pair of distinct maximal subgroups of $G/\Phi(G)$ is trivial. Now by Theorem 3.1 we have the following cases:

(i). If $G/\Phi(G) \cong C_{p^n}$, then $n = 1$ and $G \cong C_{p^m}$, for some integer m. Thus in this case $\Gamma_E(G)$ is an empty graph.

(ii). If $G/\Phi(G) \cong C_{pq}$, then G is a cyclic group with two maximal subgroups. Therefore $G \cong C_{p^r q^s}$.

(iii). If $G/\Phi(G) \cong C_p \times C_p$, then G is nilpotent. Therefore

$$G \cong S(p_1) \times \ldots \times S(p_k),$$

where $S(p_i)$ is the Sylow p_i-subgroup of G and $\pi(G) = \{p_1, \ldots, p_k\}$ is the set of all prime divisors of $|G|$. Assume that $k \geqslant 2$. We know $\Phi(G) \cong \Phi(S(p_1)) \times \ldots \times \Phi(S(p_k))$ and $\Phi(S(p_i)) \neq 1$. Therefore

$$C_p \times C_p \cong \frac{G}{\Phi(G)} \cong \frac{S(p_1)}{\Phi(S(p_1))} \times \ldots \times \frac{S(p_k)}{\Phi(S(p_k))},$$

which contradicts $\pi(G) = \pi(G/\Phi(G))$. Hence $k = 1$ and so G is a p-group, where $G/\Phi(G) \cong C_p \times C_p$. In particular, if G is an abelian p-group, $G/\Phi(G) \cong C_p \times C_p$ follows that $G \cong C_{p^r} \times C_{p^s}$ and so $\Gamma_E(G) \cong K_{p+1}$.

(iv). If $G/\Phi(G) = P \rtimes Q$, Since Q is a non-normal maximal subgroup of G, then G is non-nilpotent.

Conversely, If $G \cong C_{p^n}$ or $C_{p^r q^s}$, then it is clear that $\Gamma_E(G)$ is complete. Now assume that G is a p-group of order p^n, where $G/\Phi(G) \cong C_p \times C_p$. Then $|\Phi(G)| = p^n$ and for all M_i and M_j in $\mathrm{Max}(G)$, $|M_i \cap M_j| = |\Phi(G)|$. Therefore $M_i \cap M_j = \Phi(G)$ for all M_i and M_j in $\mathrm{Max}(G)$ and so $V(\Gamma_E(G)) = \{[M_1], \ldots, [M_k]\}$. Thus $\Gamma_E(G)$ is a complete graph.

For the last case there is a bijection between $\mathrm{Max}(G)$ and $\mathrm{Max}(G/\Phi(G))$ and we may assume that $G/\Phi(G) \cong P'/\Phi(G) \rtimes Q'/\Phi(G)$, where $P = P'/\Phi(G)$ and $Q = Q'/\Phi(G)$. Then $V(\Gamma_E(G)) = \{[P'], [Q'], [Q'^g] |\ g \in G\}$ and so $\Gamma_E(G)$ is a complete graph. $\square$

Proposition 3.3. $\lambda(\Gamma_E(G)) = 1$ *if and only if* $\Gamma_E(G)$ *is a complete graph.*

Proof. Let $D = \{[H]\}$ be a dominating set. It is easy to show that H is only contained in a single maximal subgroup M and so $[H] = [M]$ by Lemma 2.4. On the other hand, one can see that $M \cap N = \Phi(G)$ for all $N \in \mathrm{Max}(G) \setminus \{M\}$. Therefore $M/\Phi(G) \cap N/\Phi(G) = \{\Phi(G)\}$ and so the maximal graph of $G/\Phi(G)$, $\Gamma M(G/\Phi(G))$, is nonconnected. Thanks to Theorem 1.2 in [14], $G/\Phi(G)$ is isomorphic to one of the groups $C_p \times C_p$, C_{pq} or $P \rtimes Q$, where P is an elementary abelian p-group of order p^n (p a prime), $|Q| = q$, where q is a prime different from p, and Q acts irreducibly and fixed point freely on P. Now the result follows by Theorem 3.2. $\square$

Proposition 3.4. $\Gamma_E(G)$ *is a regular graph if and only if* $\Gamma_E(G)$ *is a complete graph.*

Proof. Let $\Gamma_E(G)$ be a regular graph and let, for a contradiction, there is maximal subgroups M_i and M_j of G such that $\Phi(G) \subsetneq M_i \cap M_j$. Then $[M_i \cap M_j]$ is one of the vertices of $\Gamma_E(G)$. But $\deg([M_i \cap M_j]) < \deg([M_i])$, which contradicts the regularity of $\Gamma_E(G)$. Therefore $\Phi(G) = M_i \cap M_j$ and so $V(\Gamma_E(G)) = \mathrm{Max}(G)$ and the result follows. $\square$

Proposition 3.5. *If* G *is a finite* p*-group which has a maximal cyclic subgroup, then* $\Gamma_E(G)$ *is a complete graph.*

Proof. Thanks to Theorem 5.3.4 in [?], G is one of the following groups:

(*i*) C_{p^n}

(*ii*) $C_{p^n} \times C_{p^{n-1}}$

(*iii*) $D_{2^n} =< x, y | x^{2^{n-1}} = y^2 = (xy)^2 = 1 >,\ n \geqslant 3.$

iv) $Q_{2^n} =< x, y | x^{2^{n-1}} = 1, y^2 = x^{2^{n-2}}, x^y = x^{-1} >,\ n \geqslant 3.$

(*v*) $SD_{2^n} =< x, a | x^2 = 1 = a^{2^{n-1}}, a^x = a^{2^{n-2}-1} >,\ n \geqslant 3.$

(*vi*) $M_n(p) =< x, a | x^p = 1 = a^{p^{n-1}}, a^x = a^{1+p^{n-2}} >,\ n \geqslant 3.$

Now by using the parts (i) and (iii) of Theorem 3.2, $\Gamma_E(G)$ is a complete graph. $\square$

Proposition 3.6. $\Gamma_E(G) \cong K_4$ *if and only if one of the following holds.*

(*i*) *G is a 3-group and $G/\Phi(G) \cong C_3 \times C_3$. In particular, if G is an abelian 3-group then $G \cong C_{3^r} \times C_{3^s},\ r, s \geqslant 1$.*

(*ii*) *$G/\Phi(G) \cong S_3$.*

Proof. Assume that $\Gamma_E(G) \cong K_4$. Since $\Gamma_E(G)$ is complete graph, then $|V(\Gamma_E(G))| = |\mathrm{Max}(G)| = 4$. Then we have the following cases:
(*i*). By part (*iii*) of Theorem 3.2, G is a 3-group and $G/\Phi(G) \cong C_3 \times C_3$. In particular, if G is an abelian 3-group then $G \cong C_{3^r} \times C_{3^s}, r, s \geqslant 1$.
(*ii*). By part (*iv*) of Theorem 3.2, assume that $G/\Phi(G) \cong P \rtimes Q$, where P is an elementary abelian p-group of order p^n (p a prime), $|Q| = q$, where q is a prime different from p. One can see that the number of Sylow q-subgroups and Sylow p-subgroup of $G/\Phi(G)$ are $q + 1 = 3$ and 1 respectively. Therefore $G/\Phi(G) \cong C_3 \rtimes C_2 \cong S_3$.

□

Proposition 3.7. $\Gamma_E(G) \cong K_5$ *if and only of $G/\Phi(G) \cong A_4$.*

Proof. Assume that $\Gamma_E(G) \cong K_5$. Since $\Gamma_E(G)$ is complete graph, then $|V(\Gamma_E(G))| = |\mathrm{Max}(G)| = 5$ and so by the last part of Theorem 3.2 the number of Sylow q-subgroups and of $G/\Phi(G)$ are $q+1 = 4$ and so $G/\Phi(G) \cong (C_2 \times C_2) \rtimes C_3 \cong A_4$. □

4. On the planarity of $\Gamma_E(G)$

In this section, we will investigate the planarity of the equivalence graph $\Gamma_E(G)$. First we recall the following well-known theorem of Kuratowski.

Theorem 4.1. [13, Theorem 4.4.6] *A graph is planar if and only if it has no subdivisions of K_5 or $K_{3,3}$.*

In the following theorem, we characterize all cyclic groups whose equivalence graph are planar.

Theorem 4.2. *Let C_n be a cyclic group of order n. $\Gamma_E(C_n)$ is planar if and only if $|\pi(C_n)| = 2$ or 3.*

Proof. Since $|\mathrm{Max(C_n)}| = |\pi(C_n)|$, then $|\mathrm{Max(C_n)}| \leqslant 4$, otherwise $\Gamma_E(G)$ will have a subgraph isomorphic to K_5 which is a contradiction. First we assume that $|\mathrm{Max(C_n)}| = 4$. According to Proposition 2.3 if $\pi(C_n) = \{p_1, \ldots, p_4\}$, we have $\Gamma_E(C_n) = \Gamma_E(C_m) = \Gamma_E(C_{p_1 \ldots p_4})$. Hence the induced subgraph on vertices

$$\{< a^{p_1} >, < a^{p_2} >, < a^{p_3} >, < a^{p_4} >, < a^{p_1 p_3} >, < a^{p_2 p_4} >\}$$

contains a subgraph isomorphic to $K_{3,3}$ and so $\Gamma_E(C_n)$ is not planar. Now, one can check that if $|\pi(C_n)| = 2$ or 3, then $\Gamma_E(C_n)$ is planar. □

Theorem 4.3. *Assume that G is a p-group of order p^n where p is a prime and $n \geqslant 2$. Then $\Gamma_E(G)$ is planar if and only if $G/\Phi(G) \cong C_2 \times C_2$ or $C_3 \times C_3$. In particular, if G is an abelian non-cyclic p-group of order p^n and $n \geqslant 2$, then $\Gamma_E(G)$ is planar if and only if $G \cong C_{3^r} \times C_{3^s}$ or $G \cong C_{2^r} \times C_{2^s}$, where $r, s \geqslant 1$.*

Proof. Let G be a p-group of order p^n and $\Gamma_E(G)$ be planar. Then $G/\Phi(G) \cong C_p \times \cdots \times C_p$ with rank r, $|\mathrm{Max}(G)| = (p^r - 1)/(p-1)$ and $|\mathrm{Max}(G)| \leqslant 4$. Hence we must have $p = 2$ or $p = 3$ and $r = 2$ and so by Theorem 3.2 $\Gamma_E(G) \cong K_3$ or K_4, which they are planar.

Assume that G is a group isomorphic to D_{2^n}, Q_{2^n} or SD_{2^n}, $n \geqslant 3$. Then $G/\Phi(G) \cong C_2 \times C_2$. Furthermore, $M_n(p)/\Phi(M_n(p)) \cong C_p \times C_p$ for $p = 2$ or 3. Thanks to Theorem 4.3 we have the following result. $\square$

Corollary 4.4. *Let G be a group isomorphic to one of the group $D_{2^n}, Q_{2^n}, SD_{2^n}$, $n \geqslant 3$ or $M_n(p)$, $p = 2$ or 3. Then $\Gamma_E(G)$ is planar.*

Theorem 4.5. *Let G be a non-nilpotent group. $\Gamma_E(G)$ is planar if and only if $|G| = 2^n 3^m$ and $G/\Phi(G) \cong S_3$, where $n, m \geqslant 1$.*

Proof. Assume that $\Gamma_E(G)$ is planar. Then $|\mathrm{Max}(G)| \leqslant 4$. On the other hand, since G is not nilpotent by Lemma 3, in [9], we have $|\mathrm{Max}(G)| \geqslant 4$. So $\mathrm{Max}(G) = 4$ and by theorem 3 in [9], G is a supersolvable group of order $2^n 3^m$ $n, m \geqslant 1$ and $G/\Phi(G) \cong S_3$ and the result follows. $\square$

5. On the perfection of $\Gamma_E(G)$

In this section, we will study the perfection of the equivalence graph. We show that if $|\mathrm{Max}(G)| \leqslant 4$ then $\Gamma_E(G)$ and $\Gamma_m(G)$ are perfect. First, we recall the following definitions and theorems.

Definition 5.1. A graph Γ is *perfect* whenever $\omega(\Gamma') = \chi(\Gamma')$, for all induced subgraphs Γ' of Γ.

Definition 5.2. A graph is *chordal* (or *triangulated*) if each of its cycles of length at least 4 has a chord, i.e., if it contains no induced cycles other than triangles.

Proposition 5.3. [13, Proposition 5.5.1] *Every chordal graph is perfect. In particular, complete graphs, empty graphs and k-partite graphs are perfect.*

Theorem 5.4. [?, Theorem 1.2] *A graph Γ is perfect if and only if neither Γ nor $\overline{\Gamma}$ contains an odd cycle of length at least 5 as an induced subgraph.*

Theorem 5.5. *If $|\mathrm{Max}(G)| \leqslant 3$, then $\Gamma_E(G)$ is chordal.*

Proof. If $|\mathrm{Max}(G)| = 1$, then $\Phi(G)$ is the maximal subgroup of G and so $\Gamma_E(G)$ is empty. Furthermore, if $|\mathrm{Max}(G)| = \{M_1, M_2\}$, then $V(\Gamma_E(G)) = \{[M_1], [M_2]\}$

and so $\Gamma_E(G) \cong K_2$. Hence by Proposition 5.3 they are perfect. Now assume that $\mathrm{Max}(G) = \{M_1, M_2, M_3\}$ and

$$[H_1] - [H_2] - \cdots - [H_n] - [H_1]$$

be a cycle of length n in $\Gamma_E(G)$. Since for all $1 \leqslant i \leqslant 3$, $\deg([M_i]) = 2$ or 3 and by Remark 2.5 $\deg([v_{ij}]) = 1$, then $n \leqslant 3$ and so $\Gamma_E(G)$ is chordal. $\square$

Corollary 5.6. *If* $|\mathrm{Max}(G)| \leqslant 3$*, then* $\Gamma_E(G)$ *is perfect.*

It must be noted that if $|\mathrm{Max}(G)| \geqslant 4$, then there exists a finite group like G such that $\Gamma_E(G)$ is not chordal. For example, assume that $G = \langle a \rangle \cong C_{p_1 \ldots p_4}$, where $p_1, \ldots, p_4$ are primes, then

$$C_4 : [a^{p_1}] - [a^{p_2}] - [a^{p_1 p_3}] - [a^{p_2 p_4}] - [a^{p_1}]$$

is a cycle of length 4 without a chord.

Theorem 5.7. *If* $|\mathrm{Max}(G)| = 4$ *then* $\Gamma_E(G)$ *is perfect.*

Proof. We use Theorem 5.4 and show that $\Gamma_E(G)$ and $\overline{\Gamma_E(G)}$ do not contain an odd cycle of length at least 5 as an induced subgraph. For $\Gamma_E(G)$, by Remark 2.5 we have

$$V(\Gamma_E(G)) = \{[M_1], [M_2], [M_3], [M_4], [v_{ij}], [v_{ijk}] | i, j, k \in \{1, 2, 3, 4\}\}.$$

In the general case, we may assume that all of $[v_{ij}]$'s and $[v_{ijk}]$'s are not empty. It must be noted that there is not a cycle of length at least 5 which contains $[v_{ijk}]$, because each $[v_{ijk}]$ has degree 1 and cannot be part of a cycle. Therefore, if $n \geqslant 5$ and $C_n : [H_1] - [H_2] - \cdots - [H_n] - [H_1]$ is an odd cycle in $V(\Gamma_E(G))$, then for $1 \leqslant i \leqslant n$, $[H_i]$ is equal to either $[M_i]$ or $[v_{ij}]$. Without loss of generality, we may assume that $[H_1] = [M_1]$ or $[H_1] = [v_{12}]$.
If $[H_1] = [M_1]$, there are two choices for $[H_2]$.

Case 1: $[H_2] = [M_2]$, $[M_3]$ or $[M_4]$. If for example $[H_2] = [M_2]$, then we can choose just $[v_{13}]$ or $[v_{14}]$ for $[H_3]$. If $[H_3] = [v_{13}]$, then $[H_4] = [v_{24}]$ and so $[H_1], [H_4]$ are adjacent. Hence $n = 4$, a contradiction. On the other hand, if $[H_3] = [v_{14}]$, then there is no choice for $[H_4]$, a contradiction too.

Case 2: $[H_2] = [v_{23}]$ or $[v_{24}]$. Then $[H_3] = [v_{14}]$ or $[v_{13}]$ respectively and we have no choice for $[H_4]$ which is a contradiction.

Now assume that $[H_1] = [v_{12}]$. We have two choices for $[H_2]$.

Case 1: $[H_2] = [M_3]$ or $[M_4]$. Let for example $[H_2] = [M_3]$. If $[H_3] = [M_1]$ or $[M_2]$, then $[H_4] = [v_{23}]$ or $[v_{13}]$ respectively and there exists no choice for $[H_5]$, a contradiction. Similarly, if $[H_3] = [v_{14}]$ or $[v_{24}]$, then $[H_4] = [v_{23}]$ or $[v_{13}]$ respectively and there exists no choice for $[H_5]$, a contradiction too.

Case 2: $[H_2] = [v_{34}]$. Then $[H_3] = [M_1]$ or $[M_2]$. If for example $[H_3] = [M_1]$, then $[H_4] = [v_{23}]$ or $[v_{24}]$ and so $[H_5] = [v_{14}]$ or $[v_{13}]$ respectively. Now there exists no choice for $[H_6]$ and so this case does not hold. Consequently, $\Gamma_E(G)$ does not contain an odd cycle of length at least 5 as an induced subgraph.

Now, we prove the same result for $\overline{\Gamma_E(G)}$. First we note that since $[v_{ijk}]$ has degree 1 in $\Gamma_E(G)$, all but one vertex of the complement are neighbors of $[v_{ijk}]$, and so it cannot be contained in a chordless cycle of length at least 3. Let $n \geqslant 5$ and $C_n : [H_1] - [H_2] - \cdots - [H_n] - [H_1]$ be an odd cycle in $\overline{\Gamma_E(G)}$. Then for $1 \leqslant i \leqslant n$, $[H_i]$ is equal to either $[M_i]$ or $[v_{ij}]$.

Without loss of generality, we may assume that $[H_1] = [M_1]$ or $[H_1] = [v_{12}]$. First assume that $[H_1] = [M_1]$. Then $[H_2] = [v_{12}], [v_{13}]$ or $[v_{14}]$. If for example $[H_2] = [v_{12}]$, then $[H_3] = [v_{23}], [v_{24}]$ or $[M_2]$. If $[H_3] = [M_2]$, then we have no choice for $[H_4]$. Let $[H_3] = [v_{23}]$ (or $[H_3] = [v_{24}]$), then $[H_4] = [M_3]$ or $[v_{34}]$. If $[H_4] = [M_3]$, then there is no choice for $[H_5]$ and if $[H_4] = [v_{34}]$, then $[H_5] = [M_4]$ and we have no choice for $[H_6]$. Therefore in this case we have a contradiction.

Now assume that $[H_1] = [v_{12}]$. We have the following cases for $[H_2]$:

Case 1: If $[H_2] = [M_1]$ or $[M_2]$, then $[H_3] = [v_{13}]$ or $[v_{23}]$ respectively and so we have a cycle of length at most 3, a contradiction.

Case 2: $[H_2] = [v_{13}]$, $[v_{14}]$, $[v_{23}]$ or $[v_{24}]$. If for example $[H_2] = [v_{13}]$, then $[H_3] = [M_3]$ or $[v_{34}]$ and finally we have the paths $[v_{12}] - [v_{13}] - [M_3]$ or $[v_{12}] - [v_{13}] - [v_{34}] - [M_4]$ respectively, which they are not cycles in $\overline{\Gamma_E(G)}$. Then we get a contradiction in this case too.

Therefore $\overline{\Gamma_E(G)}$ does not contain an odd cycle of length at least 5 and so $\Gamma_E(G)$ is a perfect graph. □

One can easily check that if $C_n : H_1 - H_2 - \cdots - H_n - H_1$ is a cycle of length n in $\Gamma_m(G)$, then $\overline{C_n} : [H_1] - [H_2] - \cdots - [H_n] - [H_1]$ is a cycle of length n in $\Gamma_E(G)$. Then by Corollary 5.6 and Theorem 5.7 we have the following result for $\Gamma_m(G)$.

Corollary 5.8. *If* $|\mathrm{Max}(G)| \leqslant 4$, *then* $\Gamma_m(G)$ *is a perfect graph.*

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Matched pairs of m-invertible Hopf quasigroups

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Abstract. The matched pair theory (of groups) is studied for a class of quasigroups; namely, the m-inverse property loops. The theory is upgraded to the Hopf level, and the *m-invertible Hopf quasigroups* are introduced.

1. Introduction

One of the main motivations of the theory of quasigroups may be considered to be the extension of the representation theoretical properties of the groups on the level of quasigroups; such as the character theory [27, 58], module theory [57], or homogeneous spaces [59, 60]. See also [19, 22, 23].

Not much later, it was discovered that there are a plethora of areas for quasigroups to apply. Among others, an incomplete list may consists of the coding theory (see, for instance, [20] for the quasigroup-based MDS codes, and [42, 43] for the quasigroup point of view towards the codes with one check symbol, as well as [21]), cryptology [18, 24, 54], and combinatorics [9, 25, 30, 37].

In order to shed further light on the well deserved analysis of the quasigroups, we shall develop in the present paper the matched pair construction for these non-associative structures. The matched pair theory was introduced, initially, for groups in order to recover the structure of a group in terms of two subgroups with mutual actions, [12, 36, 38, 39, 61, 64]. More precisely, given a pair of groups (G, H) with mutual actions

$$\triangleright : H \times G \to G, \qquad \triangleleft : H \times G \to H$$

satisfying

$$\begin{aligned} y \triangleright (xx') &= (y \triangleright x)((y \triangleleft x) \triangleright x'), \qquad y \triangleright 1 = 1, \\ (yy') \triangleleft x &= (y \triangleleft (y' \triangleright x))(y' \triangleleft x), \qquad 1 \triangleleft x = 1, \end{aligned}$$

for any $x, x' \in G$, and any $y, y' \in H$, the cartesian product $G \bowtie H := G \times H$ is a group with the multiplication

$$(x, y)(x', y') = (x(y \triangleright x'), (y \triangleleft x)y'),$$

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and the unit $(1,1) \in G \times H$. In this case, the pair (G,H) of groups is called a matched pair of groups.

As for the quasigroups, there are many such constructions. To begin with, there are of course the direct product construction [17, 56, 44, 26, 6, 7], and the semi-direct product construction [55, 49, 15]. There is also the crossed product construction [14, 13, 5], which is referred as quasi-direct product in [63]. Considering these as the binary crossed products, there are, on top of these, the n-ary crossed products [11]. The other generalizations goes under the titles of the generalized crossed product [8], and the generalized singular direct product [52, 53]. Finally, there is the Sabinin's product [48, 51] and its generalization [15, 50]. We refer the reader also to [16].

More recently, a bicrossed product construction for quasigroups (based on the mutual interaction of the quasigroups through permutations) is developed in [1, Sect. 5]. The structure of the resulting bipoduct quasigroup of [1, Thm. 5.1] encompases to that of the bicrossedproduct group of [40], and is closest to the one developed in the present manuscript.

The matched pair construction that we shall develop here is also based on the "mutual actions" of two objects through certain maps, though this time the objects being m-inverse property loops, and not merely quasigroups. We shall, furthermore, be able to relate our construction to the matched pair of groups; which will enable us to produce an ample amount of examples motivated from the matched pairs of groups.

Let us note that the matched pair theory of groups suit also to Hopf algebras, the quantum analogues of groups, [40, 41, 62]. Just as well, there will be a Hopf analogue of the theory we shall develop here.

In [35], see also [34], the authors managed to develop successfully a not-necessarily associative (but coassociative, counital, and unital) (co)algebra H, that they call a Hopf quasigroup, with a map $S : H \to H$ satisfying compatibility conditions more general then those satisfied by the antipode of a Hopf algebra. It is further shown that kQ is a Hopf quasigroup if Q is an inverse property (IP) loop, and that for any Hopf quasigrop H, the set $G(H)$ of group-like elements form an IP-loop.

Considering the Hopf algebras as linearizations of groups, one sees the antipode of a Hopf algebra as the manifestation of the inversion on a group. This point of view is precisely what has been studied in [34, 35], where the authors succesfully developed the correct axioms for the *antipode* of the quantum analogue of an IP-loop. Looking from a similar perspective, in the present paper we shall develop the quantum analogue of a strictly larger family; the m-inverse property loops, which is general enough to encompass the weak-inverse-property (WIP) of [3], as well as the crossed-inverse (CI) property of [2]. The resulting quantum objects shall be addressed as m-inverse property Hopf quasigroups, and their matched pair theory (the quantum analogue of the matcehd pair theory developed for the m-inverse property loops) will be developed.

Finally, it deserves to be mentioned that in the level of Hopf objects there are constructions that fell beyond the matched pair construction; most notably, the Radford's biproduct construction, [45]. However, the Radford's biproduct construction uses the category of Yetter-Drinfeld objects; that we intend to explore for the m-inverse property Hopf quasigroups in a separate paper. As such, we expect also to penetrate into a Hopf-cyclic type (co)homology theory for the quantum objects constructed here.

The paper is organized as follows.

Section 2 below is about the inverse properties on quasigroups, and serves to fix the basic definitions of the main objects of study. To this end, in Subsection 2.1 we collect the definitions of quasigroups and loops, while in Subsection 2.2 we recall briefly the various inverse properties on quasigroups (with a special emphasis on the m-inverse property).

Section 3 is where we develop the matched pair theory for the m-inverse property loops. Based on the lack of literature on semi-direct product of quasigroups (in the sense that one quasigroup acts on the other, see for instance Proposition 3.3 and Proposition 3.4 below), and for the convenience of the reader, we begin with a recollection of the basic results (Theorem 3.1 and Theorem 3.2) on the direct products of quasigroups in Subsection 3.1, and then extend it to the semi-direct products of m-inverse property loops (Proposition 3.6 and Proposition 3.7). Finally, we achieve the full generality (proving our main results on the quasigroup level) in Subsection 3.3, and succeed the matched pair construction for the m-inverse property loops (Proposition 3.8 and Proposition 3.9). We also discuss the universal property of this construction in Proposition 3.12, as an analogue of [41, Prop. 6.2.15] for the m-inverse property loops.

Section 4, finally, is reserved for the quantum counterparts of the main results of Subsection 3.3. Accordingly, in Subsection 4.1 we introduce the notion of m-invertible Hopf quasigroup in Definition 4.1. Then, in Subsection 4.2 we establish the matched pair theory for the m-invertible Hopf quasigroups (Proposition 4.6), along with a suitable version (Proposition 4.9) of [41, Thm. 7.2.3].

Notation and Conventions

We shall adopt the Sweedler's notation (suppressing the summation) to denote a comultiplication; $\Delta : A \to A\otimes A$, $\Delta(a) := a_{<1>} \otimes a_{<2>}$. For the sake of simplicity, we shall also denote, occasionally, an element in the cartesian product $A \times B$, or tensor product $A \otimes B$ as (a, b), rather than $a \otimes b$.

2. Quasigroups with inverse properties

In this section we shall discuss the semi-direct product, and then the matched pair constructions on two large classes of semigroups; namely the m-inverse loops, and the Hom-groups. To this end, we review the basics of the quasigroup theory first. We shall then focus on the inverse-properties (IP) over quasigroups, in order to be able to recall the (r,s,t)-inverse quasigroups, as well as the m-inverse loops. Finally, on the other extreme, we shall recall/review the basics of the Hom-groups.

2.1. Quasigroups

A *quasigroup* is a set Q with a multiplication such that for all $a,b \in Q$, there exist unique elements $x,y \in Q$ such that $ax = b,\ \ ya = b$. In this case, $x = a\backslash b$ is called the *left division*, and $y = b/a$ the *right division*.

Given two quasigroups Q and Q', a *quasigroup homotopy* from Q to Q' is a triple (α,β,γ) of maps $Q \to Q'$ such that $\alpha(x)\beta(y) = \gamma(xy)$ for all $x,y \in Q$. In case $\alpha = \beta = \gamma$, then we arrive at the notion of a *quasigroup homomorphism*. On the other hand, a *quasigroup isotopy* is a quasigroup homotopy (α,β,γ) such that all three maps are bijective.

A quasigroup Q with a distinguished idempotent element $\delta \in Q$ is called a *pointed idempotent quasigroup*, or in short, a *pique*, [16]. A pique (Q,δ) is called a *loop* if the idempotent element $\delta \in Q$ acts like an identity, *i.e.* $x\delta = \delta x = x$ for any $x \in Q$. It, then, follows that the idempotent element $\delta \in Q$ is unique, and that any $x \in Q$ has a unique *left inverse* $x^\lambda := \delta/x,\ \ x^\lambda x = \delta$ as well as a unique *right inverse* $x^\sigma := x\backslash\delta,\ \ xx^\sigma = \delta$. A loop Q is said to have *two-sided inverses* if $x^\lambda = x^\sigma$ for all $x \in Q$. Furthermore, a loop Q is said to have the *left inverse property* if $x^\lambda(xy) = y$ for all $x,y \in Q$, and similarly Q is said to have the *right inverse property* if $(yx)x^\sigma = y$, for all $x,y \in Q$. Finally, a loop is said to have the *inverse property* if it has both the left inverse property and the right inverse property. Such loops are also called the *IP-loops*.

Given a pique (Q,δ), there corresponds a loop $B(Q)$ - called the *corresponding loop* or *cloop* - with the multiplication $x*y := (x/\delta)(\delta\backslash y)$ for any $x,y \in Q$, and the identity element $\delta \in Q$. We note that it is possible to recover the multiplication on a pique from the one on the cloop as $xy := (x\delta)*(\delta y)$, see, for instance, [47].

Finally, a pique (Q,δ) is called *central* if $B(Q)$ is an abelian group, and the set of all left and right multiplications of Q that fix the idempotent element $\delta \in Q$ is the group $\operatorname{Aut}\big(B(Q)\big)$.

A convenient way to construct quasigroups, out of groups, is the cocycle-type group extensions, [4], see also [55, Subsect. 1.6.2].

Example 2.1. Let G be a group, $(V,+)$ an abelian group with a right action

$\triangleleft : V \times G \to V$, $(v,x) \mapsto v \triangleleft x$. Then, given any $\varphi : G \times G \to V$, the operation

$$(x,v)(x',v') := (xx', \varphi(x,x') + v \triangleleft x' + v') \tag{2.1}$$

is associative on $G \ltimes_\varphi V := G \times V$ if and only if

$$d\varphi(x,x',x'') := \varphi(x',x'') - \varphi(xx',x'') + \varphi(x,x'x'') - \varphi(x,x') \triangleleft x'' = 0, \tag{2.2}$$

that is, $\varphi : G \times G \to V$ is 2-cocycle in the group cohomology of G, with coefficients in V; in other words, $\varphi \in H^2(G,V)$. As such, giving up the cocycle condition (2.2) we arrive at the quasigroup $G \ltimes_\varphi V$ with the multiplication (2.1).

Similarly, we may construct a loop.

Example 2.2. Considering the quasigroup $G \ltimes_\varphi V$ of Example 2.1, we see at once that $(1,0) \in G \ltimes_\varphi V$ acts as unit, with respect to (2.1), if and only if

$$\varphi(1,x) = 0 = \varphi(x,1) \tag{2.3}$$

for any $x \in G$. Hence, given a group G, an abelian group $(V,+)$ with a right action $\triangleleft : V \times G \to V$, and a mapping $\varphi : G \times G \to V$ satisfying (2.3) is a loop.

We shall, for the sake of simplicity, drop the right action (that is, we shall assume the right action to be trivial) on the sequel, and consider the examples of the form $G \times_\varphi V$, with the multiplication

$$(x,v)(x',v') := (xx', \varphi(x,x') + v + v'). \tag{2.4}$$

2.2. Inverse properties on quasigroups

In the present subsection we shall recall the inverse properties on quasigroups, and in particular, on loops.

Along the lines of [33], see also [3], a loop Q is said to have the *weak-inverse property* (WIP) if there is a permutation $J : Q \to Q$ such that

$$xJ(x) = \delta, \tag{2.5}$$

and that

$$xJ(yx) = J(y), \tag{2.6}$$

for any $x,y \in Q$. Dropping the condition (2.5), a quasigroup with 2.6 is called a *WIP quasigroup.*

Similarly, a loop/quasigroup Q is said to have the *crossed-inverse property* (*CI property*) if (2.6) is replaced by

$$(xy)J(x) = y. \tag{2.7}$$

We refer the reader to [31] for the applications of the CI quasigroups in cryptography.

On the other hand, the loop/quasigroup Q has the *m-inverse property* if (2.6), or (2.7), is now substituted with

$$J^m(xy)J^{m+1}(x) = J^m(y), \tag{2.8}$$

where $m \in \mathbb{Z}$, [29].

Finally, we recall that the loop/quasigroup Q is said to have the *(r,s,t)-inverse property* if (2.6), (2.7), or (2.8), is exchanged with

$$J^r(xy)J^s(x) = J^t(y), \tag{2.9}$$

where $r, s, t \in \mathbb{Z}$, [33].

Remark 2.3. The condition (2.9) generalizes those given by (2.6), (2.7), or (2.8). More precisely, the weak-inverse property is a $(-1, 0, -1)$-inverse property, [33], and a crossed-inverse property is nothing but a 0-inverse property; where, in general an m-inverse property is an $(m, m+1, m)$-inverse property.

On the other hand, it is observed in [32] that every (r,s,t)-inverse loop is an $(r, r+1, r)$-inverse loop, that is, an r-inverse loop. Though, on the level of quasigroups, there are proper (r,s,t)-inverse quasigroups, [33].

Remark 2.4. It is critical to recall from [33, Rk. 2.2] that if Q is an (r,s,t)-inverse quasigroup with the permutation $J : Q \to Q$ so that $J^h \in \operatorname{Aut}(Q)$ for some $h \in \mathbb{Z}$, then Q is an $(r+uh, s+uh, t+uh)$-inverse quasigroup for any $u \in \mathbb{Z}$.

Let us finally discuss an odd-invertible loop.

Example 2.5. Let us consider the loop $G \times_\varphi V$ of Example 2.2. Let also

$$J : G \times_\varphi V \to G \times_\varphi V, \qquad J(x, v) := (x^{-1}, -v). \tag{2.10}$$

It is quite clear then that $J^2 = \operatorname{Id}_{G\times V} \in \operatorname{Aut}(G \times_\varphi V)$. Accordingly, we see at once that

$$(x, v)J(x, v) = (1, 0)$$

if and only if

$$\varphi(x, x^{-1}) = 0 \tag{2.11}$$

for any $x \in G$, and that for any $m = 2\ell + 1$,

$$J^m((x, v)(x', v'))J^{m+1}(x, v) = J^m(x', v')$$

if and only if

$$J((x, v)(x', v'))(x, v) = J(x', v'),$$

if and only if

$$\varphi(x'^{-1}x^{-1}, x) = \varphi(x, x') \tag{2.12}$$

for any $x, x' \in G$.

To sum up, we may say that given any group G, an abelian group $(V, +)$, and any $\varphi : G \times G \to V$ satisfying (2.3), (2.11), and (2.12), $G \times_\varphi V$ is an $(2\ell + 1)$-invertible loop with (2.10) for any $\ell \in \mathbb{Z}$.

3. Matched pairs of m-invertible loops

In this section we shall introduce the matched pair theory for the quasigroups with the m-inverse property. The theory that we shall develop here will thus generalize the direct product theory in [33, Sect. 5], and the semi-direct product theory in [55, Sect. 1.6.2] for quasigroups.

3.1. Direct products of quasigroups

To this end we shall first recall the direct product theory from [33, Sect. 5]. In the utmost generality, let Q_1 be an (r_1, s_1, t_1)-inverse quasigroup with the permutation $J_1 : Q_1 \to Q_1$, and let Q_2 an (r_2, s_2, t_2)-inverse quasigroup with $J_2 : Q_2 \to Q_2$. Then the direct product $Q_1 \times Q_2$ is defined to be the quasigroup with the permutation $J_1 \times J_2 : Q_1 \times Q_2 \to Q_1 \times Q_2$, and the multiplication given by $(q_1, q_2)(q_1', q_2') := (q_1q_1', q_2q_2')$.

Along the lines of [33, Sect. 5], let $J_1^{h_1} \in \mathrm{Aut}(Q_1)$ and $J_2^{h_2} \in \mathrm{Aut}(Q_2)$. In the case that Q_1 is an m_1-inverse quasigroup and Q_2 is an m_2-inverse quasigroup, the structure of $Q_1 \times Q_2$ is given in [33, Thm. 5.1], that we recall below.

Theorem 3.1. *Assume that Q_1 is an m_1-inverse quasigroup with the permutation $J_1 : Q_1 \to Q_1$ so that $J_1^{h_1} \in \mathrm{Aut}(Q_1)$, and Q_2 is an m_2-inverse quasigroup with $J_2 : Q_2 \to Q_2$ such that $J_2^{h_2} \in \mathrm{Aut}(Q_2)$. Then $Q_1 \times Q_2$ is an m-inverse quasigroup with $J_1 \times J_2 : Q_1 \times Q_2 \to Q_1 \times Q_2$, for any $m \in \mathbb{Z}$ that satisfies*

$$\begin{aligned} m &\equiv m_1 \pmod{h_1}, \\ m &\equiv m_2 \pmod{h_2}. \end{aligned} \tag{3.1}$$

As is noted in the proof of [33, Thm. 5.1], a solution to (3.1) exists if and only if there is $\ell \in \mathbb{N}$ such that $m_1 - m_2 = (h_1, h_2)\ell$. Here (h_1, h_2) refers to the greatest common divisor of $h_1 \in \mathbb{Z}$ and $h_2 \in \mathbb{Z}$.

If, on the other hand, Q_1 is an (r_1, s_1, t_1)-inverse quasigroup, and Q_2 is an (r_2, s_2, t_2)-inverse quasigroup, the structure of the direct product is given by [33, Thm. 5.2], which we recall now.

Theorem 3.2. *Let Q_1 is an (r_1, s_1, t_1)-inverse quasigroup with the permutation $J_1 : Q_1 \to Q_1$ so that $J_1^{h_1} \in \mathrm{Aut}(Q_1)$, and Q_2 is an (r_2, s_2, t_2)-inverse quasigroup with $J_2 : Q_2 \to Q_2$ such that $J_2^{h_2} \in \mathrm{Aut}(Q_2)$. Then $Q_1 \times Q_2$ is an (r, s, t)-inverse quasigroup with $J_1 \times J_2 : Q_1 \times Q_2 \to Q_1 \times Q_2$, if there are $u_1, u_2 \in \mathbb{Z}$ such that*

$$r - r_1 = s - s_1 = t - t_1 = u_1 h_1, \qquad r - r_2 = s - s_2 = t - t_2 = u_2 h_2.$$

3.2. Semi-direct products of m-invertible loops

As for the semi-direct products of quasigroups, there seems to be no approach involving the notion of an action of a quasigroup on another. A semi-direct product construction, using groups, is the one given in [46, 28], see also [55, Sect. 1.6.2] which we recall below.

Proposition 3.3. *Let $(G, +)$ and $(H, \cdot)$ be two groups, and: $G \to \mathrm{Aut}(H)$. Then, $G \times H$ is a quasigroup with the multiplication*

$$(g, h)(g', h') := (g + g'(g')(h) \cdot h').$$

The construction given in [51] uses a quasigroup, and its transassociant.

Proposition 3.4. *Let Q be a quasigroup, and H be the group generated by $\{\ell(q, q') \mid q, q' \in Q\}$, where $\ell(q, q') := L_{qq'}^{-1} \circ L_q \circ L_{q'}$, and $L_q : Q \to Q$, $L_q(r) := qr$, is the left translation. Then, $Q \times H$ is a quasigroup with the multiplication given by*

$$(q, h)(q', h') := (qh(q'), \ell(q, h(q')) \circ m_{q'}(h) \circ h \circ h'),$$

where, for any $q \in Q$ and any $h \in H$,

$$m_q(h) := L_{h(q)}^{-1} \circ h \circ L_q \circ h^{-1}.$$

Let us note also that this was the point of view considered in [34, 35].

None of these constructions lead to a possible discussion on the matched pairs of quasigroups. We thus adopt the following (more general, given in terms of quasigroup homomorphisms) definition given in [55, Def. 1.318].

Definition 3.5. A quasigroup Q is called the *semi-direct product* of two quasigroups R and S, if there is a (quasigroup) homomorphism $h : Q \to S$, such that the kernel $\ker(h) = R$, and that $h|_S = \mathrm{Id}_S$. In this case, Q is denoted by $R \rtimes S$.

The motivating examples are the ones discussed within the following propositions below, on the level of (m-inverse) loops, and Hom-groups.

Proposition 3.6. *Let R and S be two loops, and let $\varphi : S \times R \to R$ be a map satisfying $\varphi(s,\delta) = \delta$ and $\varphi(\delta, r) = r$. Then a loop Q is isomorphic to the loop $R \rtimes S := R \times S$ with the multiplication given by*

$$(r,s)(r',s') := (r\varphi(s,r'), ss'), \tag{3.2}$$

if and only if there are quasigroup homomorphisms $i_S : S \to Q$, $i_R : R \to Q$, $p_S : Q \to S$, and a map $p_R : Q \to R$ satisfying the Moufang-type identities

$$p_R\big(\big(i_R(r)i_S(s)\big)\big(i_R(r')i_S(s')\big)\big) = p_R(i_R(r))\Big(\big(p_R(i_S(s)i_R(r'))\big)p_R(i_S(s'))\Big) = \Big(p_R(i_R(r))\big(p_R(i_S(s)i_R(r'))\big)\Big)p_R(i_S(s')), \tag{3.3}$$

as well as $p_R \circ i_R = \mathrm{Id}_R$ and $p_S \circ i_S = \mathrm{Id}_S$, such that $R \rtimes S \to Q$, $(r,s) \mapsto i_R(r)i_S(s)$ and $Q \to R \rtimes S$, $q \mapsto (p_R(q), p_S(q))$ are inverse to each other.

Proof. Letting $\Phi : Q \to R \rtimes S$ to be the (quasigroup) isomorphism, we consider the mappings

$$i_R : R \to Q, \quad i_R(r) := \Phi^{-1}(r,\delta), \qquad i_S : S \to Q, \quad i_S(s) := \Phi^{-1}(\delta, s)$$

and

$$p_R : Q \to R, \quad p_R(q) := \pi_1(\Phi(q)), \qquad p_S : Q \to S, \quad p_S(q) := \pi_2(\Phi(q)),$$

where π_i's denote the projection onto the ith component. It is evident that

$$(p_R \circ i_R)(r) = \pi_1(r,\delta) = r,$$

for any $r \in R$, as such $p_R \circ i_R = \mathrm{Id}_R$. Similarly, $p_S \circ i_S = \mathrm{Id}_S$. We further see that

$$i_S(ss') = \Phi^{-1}(\delta, ss') = \Phi^{-1}\Big((\delta,s)(\delta,s')\Big) = \Phi^{-1}(\delta,s)\Phi^{-1}(\delta,s') = i_S(s)i_S(s'),$$

that

$$i_R(rr') = \Phi^{-1}(rr',\delta) = \Phi^{-1}\Big((r,\delta)(r',\delta)\Big) = \Phi^{-1}(r,\delta)\Phi^{-1}(r',\delta) = i_R(r)i_R(r'),$$

and that

$$p_S(qq') = p_2(\Phi(qq')) = \pi_2(\Phi(q)\Phi(q')) = \pi_2(\Phi(q))\pi_2(\Phi(q')) = p_S(q)p_S(q').$$

On the other hand, the mapping $R \rtimes S \to Q$, $(r,s) \mapsto i_R(r)i_S(s)$, becomes $\Phi^{-1} : R \rtimes S \to Q$, whereas the map $Q \to R \rtimes S$, $q \mapsto (p_R(q), p_S(q))$ becomes $\Phi : Q \to R \rtimes S$. Finally, we note also that

$$p_R\big(\big(i_R(r)i_S(s)\big)\big(i_R(r')i_S(s')\big)\big) = p_R\big(\Phi^{-1}(r,s)\Phi^{-1}(r',s')\big) = p_R\big(\Phi^{-1}(r\varphi(s,r'),ss')\big) = r\varphi(s,r') = p_R(i_R(r))\Big(\big(p_R(i_S(s)i_R(r'))\big)p_R(i_S(s'))\Big) = \Big(p_R(i_R(r))\big(p_R(i_S(s)i_R(r'))\big)\Big)p_R(i_S(s')).$$

Conversely, let $i_S : S \to Q$, $i_R : R \to Q$, and $p_S : Q \to S$ the quasigroup homomorphisms, together with the map $p_R : Q \to R$ satisfying (3.3), such that $\Psi : R \rtimes S \to Q$, $\Psi(r,s) := i_R(r)i_S(s)$, and $\Phi : Q \to R \rtimes S$, $\Phi(q) := (p_R(q), p_S(q))$ are inverse to each other. Thus, the loop structure on Q induces a loop structure on $R \times S$. We shall, furthermore, see that this induced loop structure is in fact one of the form (3.6). Indeed,

$$(\delta, s)(r', \delta) = \Phi(\Psi(\delta, s)\Psi(r', \delta)) = \Phi\Big(\big(i_R(\delta)i_S(s)\big)\big(i_R(r')i_S(\delta)\big)\Big) = \Phi\Big(i_S(s)i_R(r')\Big)$$
$$= \Big(p_R\big(i_S(s)i_R(r')\big), p_S\big(i_S(s)i_R(r')\big)\Big) = \Big(p_R\big(i_S(s)i_R(r')\big), p_S\big(i_S(s)\big)p_S\big(i_R(r')\big)\Big) =$$
$$\Big(p_R\big(i_S(s)i_R(r')\big), s\Big) = \Big(\varphi(s, r'), s\Big),$$

where $\varphi : S \times R \to R$, $\varphi(s, r') := p_R\big(i_S(s)i_R(r')\big)$. On the third equality we used the assumption that i_R, i_S are quasigroup homomorphisms, while on the fifth equality we used that of $p_S : Q \to S$ being a quasigroup homomorphism. Finally, on the sixth equality we used $p_S \circ i_S = \mathrm{Id}_S$. Accordingly,

$$(r, s)(r', s') = \Phi(\Psi(r, s)\Psi(r', s')) = \Phi\Big(\big(i_R(r)i_S(s)\big)\big(i_R(r')i_S(s')\big)\Big) =$$
$$\Big(p_R\Big(\big(i_R(r)i_S(s)\big)\big(i_R(r')i_S(s')\big)\Big), p_S\Big(\big(i_R(r)i_S(s)\big)\big(i_R(r')i_S(s')\big)\Big)\Big) =$$
$$\Big(p_R(i_R(r))\Big(\big(p_R(i_S(s)i_R(r'))\big)p_R(i_S(s'))\Big), ss'\Big) = \Big(r\varphi(s, r'), ss'\Big). \qquad \square$$

If we ask the semi-direct product loop to have the m-inverse property, then we have the following more precise result.

Proposition 3.7. *Let (R, δ) be an m_1-inverse loop with the permutation $J_R : R \to R$ so that $J_R(\delta) = \delta$, and that $J_R^{h_1} \in \mathrm{Aut}(R)$, and (S, δ) is an m_2-inverse loop with $J_S : S \to S$ such that $J_S(\delta) = \delta$, and that $J_S^{h_2} \in \mathrm{Aut}(S)$. Furthermore, let there be a map $\varphi : S \times R \to R$ satisfying*

$$\begin{aligned} &\varphi(\delta, r) = r, \qquad \varphi(s, \delta) = \delta, \\ &\varphi(J_S^m(ss'), \varphi(J_S^{m+1}(s), r)) = \varphi(J_S^m(s'), r), \\ &\varphi(s, J_R^m(rr'))\varphi(s, J_R^{m+1}(r)) = \varphi(s, J_R^m(r')), \end{aligned} \tag{3.4}$$

for any $r, r' \in R$, any $s, s' \in S$, and any $m \in \mathbb{Z}$ that satisfies

$$\begin{aligned} m &\equiv m_1 \ (\mathrm{mod}\, h_1), \\ m &\equiv m_2 \ (\mathrm{mod}\, h_2). \end{aligned} \tag{3.5}$$

Then, $\Big(R \rtimes S := R \times S, (\delta, \delta)\Big)$ is an m-invertible loop with the multiplication

$$(r, s)(r', s') := \Big(r\varphi(s, r'), ss'\Big) \tag{3.6}$$

and the permutation $J : R \rtimes S \to R \rtimes S$,

$$J(r,s) := (\delta, J_S(s))(J_R(r),\delta) = \Big(\varphi\big(J_S(s), J_R(r)\big), J_S(s)\Big), \tag{3.7}$$

if and only if

$$\begin{cases} \varphi(s,r) = r & \text{if } m = 2\ell, \\ \varphi(J_S^m(ss'), \varphi(s,r)) = \varphi(J_S^m(s'), r) & \text{if } m = 2\ell+1, \end{cases} \tag{3.8}$$

for any $s, s' \in S$, *and any* $r \in R$.

Proof. Assuming the conditions are met, we see at once that

$$\begin{aligned}
&(r,s)J(r,s) = \big[(r,\delta)(\delta,s)\big]\big[(\delta,J_S(s))(J_R(r),\delta)\big] = \\
&\big[(r,\delta)(\delta,s)\big](\varphi(J_S(s),J_R(r)),J_S(s)) = \\
&(r,\delta)\big[(\delta,s)(\varphi(J_S(s),J_R(r)),J_S(s))\big] = \\
&(r,\delta)(\varphi(s,\varphi(J_S(s),J_R(r))),sJ_S(s)) = \\
&(r,\delta)(J_R(r),\delta) = (rJ_R(r),\delta) = (\delta,\delta).
\end{aligned}$$

On the other hand, since

$$\varphi(s,r)J_R(\varphi(s,r)) = \delta = \varphi(s,r)\varphi(s,J_R(r)),$$

we conclude

$$J_R(\varphi(s,r)) = \varphi(s,J_R(r)),$$

which, in turn, implies that

$$\begin{aligned}
&J\big((\delta,s)(r,\delta)\big) = J(\varphi(s,r),s) = (\delta,J_S(s))(J_R(\varphi(s,r)),\delta) = \\
&(\delta,J_S(s))(\varphi(s,J_R(r)),\delta) = (\varphi(J_S(s),\varphi(s,J_R(r))),J_S(s)) = (J_R(r),J_S(s)),
\end{aligned}$$

and then that

$$J^m(r,s) = \begin{cases} (J_R^m(r), J_S^m(s)), & \text{if } m = 2\ell, \\ (\delta, J_S^m(s))(J_R^m(r),\delta), & \text{if } m = 2\ell+1. \end{cases}$$

Accordingly, in the case $m = 2\ell$,

$$\begin{aligned}
&J^m\big((r,s)(r',s')\big)J^{m+1}(r,s) = J^m(r\varphi(s,r'),ss')J^{m+1}(r,s) = \\
&\big[(J_R^m(r\varphi(s,r')),\delta)(\delta,J_S^m(ss'))\big]\big[(\delta,J_S^{m+1}(s))(J_R^{m+1}(r),\delta)\big] = \\
&(J_R^m(r\varphi(s,r')),\delta)\Big\{(\delta,J_S^m(ss'))\big[(\delta,J_S^{m+1}(s))(J_R^{m+1}(r),\delta)\big]\Big\} = \\
&(J_R^m(r\varphi(s,r')),\delta)\big[((\delta,J_S^m(ss')))(\varphi(J_S^{m+1}(s),J_R^{m+1}(r)),J_S^{m+1}(s))\big] =
\end{aligned}$$

$$
\begin{aligned}
&(J_R^m(r\varphi(s,r')),\delta)(\varphi(J_S^m(ss'),\varphi(J_S^{m+1}(s),J_R^{m+1}(r))),J_S^m(ss')J_S^{m+1}(s)) = \\
&(J_R^m(r\varphi(s,r')),\delta)(\varphi(J_S^m(ss'),\varphi(J_S^{m+1}(s),J_R^{m+1}(r))),J_S^m(s')) = \\
&(J_R^m(r\varphi(s,r')),\delta)(\varphi(J_S^m(s'),J_R^{m+1}(r)),J_S^m(s')) = \\
&\Big(J_R^m(r\varphi(s,r'))\varphi(J_S^m(s'),J_R^{m+1}(r)),J_S^m(s')\Big) = \\
&\Big(J_R^m(r\varphi(s,r'))J_R^{m+1}(\varphi(J_S^m(s'),r)),J_S^m(s')\Big) = \big(J_R^m(r'),J_S^m(s')\big) = J^m(r',s')
\end{aligned}
\tag{3.9}
$$

where; on the sixth equality we used Remark 2.4, and that $m \in \mathbb{Z}$ is a solution of the system (3.5), on the tenth equality we used (3.8), in addition to Remark 2.4 and (3.5). In the case $m = 2\ell + 1$,

$$
\begin{aligned}
&J^m\big((r,s)(r',s')\big)J^{m+1}(r,s) = J^m(r\varphi(s,r'),ss')J^{m+1}(r,s) = \\
&\big[(\delta,J_S^m(ss'))(J_R^m(r\varphi(s,r')),\delta)\big]\big[(J_R^{m+1}(r),\delta)(\delta,J_S^{m+1}(s))\big] = \\
&(\varphi(J_S^m(ss'),J_R^m(r\varphi(s,r'))),J_S^m(ss'))\big[(J_R^{m+1}(r),\delta)(\delta,J_S^{m+1}(s))\big] = \\
&\big[(\varphi(J_S^m(ss'),J_R^m(r\varphi(s,r'))),J_S^m(ss'))(J_R^{m+1}(r),\delta)\big](\delta,J_S^{m+1}(s)) = \\
&(\varphi(J_S^m(ss'),J_R^m(r\varphi(s,r')))\varphi(J_S^m(ss'),J_R^{m+1}(r)),J_S^m(ss'))(\delta,J_S^{m+1}(s)) = \\
&(\varphi(J_S^m(ss'),J_R^m(\varphi(s,r'))),J_S^m(ss'))(\delta,J_S^{m+1}(s)) = \\
&(\varphi(J_S^m(ss'),J_R^m(\varphi(s,r'))),J_S^m(s')) = (J_R^m(\varphi(J_S^m(ss'),\varphi(s,r'))),J_S^m(s')) = \\
&(J_R^m(\varphi(J_S^m(s'),r')),J_S^m(s')) = (\varphi(J_S^m(s'),J_R^m(r')),J_S^m(s')) = \\
&(\delta,J_S^m(s'))(J_R^m(r'),\delta) = J^m(r',s')
\end{aligned}
\tag{3.10}
$$

where; in the sixth equation we used (3.4), in the seventh equation we used Remark 2.4 and (3.5), and in the ninth equation we used (3.8).

Let, conversely, $R \rtimes S$ be an m-inverse loop with the multiplication (3.6) and the permutation (3.7).

In the case $m = 2\ell$, the tenth equation of (3.9) holds, and we have

$$J_R^m(r\varphi(s,r'))J_R^{m+1}(\varphi(J_S^m(s'),r)) = J_R^m(r')$$

for any $r, r' \in R$, and any $s, s' \in S$. In particular, for $r = \delta$, we see that

$$J_R^m(\varphi(s,r')) = J_R^m(r'),$$

and that $\varphi(s,r') = r'$, for any $r' \in R$, and any $s \in S$.

In the case $m = 2\ell + 1$, however, we have the ninth equation of (3.10), that is,

$$J_R^m(\varphi(J_S^m(ss'),\varphi(s,r'))) = J_R^m(\varphi(J_S^m(s'),r')).$$

But then, since $J_R : R \to R$ is a permutation, we obtain

$$\varphi(J_S^m(ss'),\varphi(s,r')) = \varphi(J_S^m(s'),r')$$

for any $r' \in R$, and any $s, s' \in S$. □

3.3. Matched pairs of m-invertible loops

In order to be able to generalize Definition 3.5 in the presence of two quasigroups, none of which is necessarily the kernel of a quasigroup homomorphism, we adopt the point of view of [10, 40, 45].

Proposition 3.8. *Let R and S be two loops, with the maps $\varphi : S \times R \to R$ and $\psi : S \times R \to S$ satisfying*

$$\varphi(s,\delta) = \delta, \qquad \varphi(\delta, r) = r, \qquad \psi(s,\delta) = s, \qquad \psi(\delta, r) = \delta.$$

Then a loop Q is isomorphic to the loop $R \bowtie S := R \times S$ with the multiplication given by

$$(r,s)(r',s') := (r\varphi(s,r'), \psi(s,r')s'), \tag{3.11}$$

if and only if there are quasigroup homomorphisms $i_S : S \to Q$, $i_R : R \to Q$, together with the maps $p_R : Q \to R$ and $p_S : Q \to S$ satisfying the Moufang-type identities

$$p_R\big(\big(i_R(r)i_S(s)\big)\big(i_R(r')i_S(s')\big)\big) = p_R(i_R(r))\Big(\big(p_R(i_S(s)i_R(r'))\big)p_R(i_S(s'))\Big) = \\ \Big(p_R(i_R(r))\big(p_R(i_S(s)i_R(r'))\big)\Big)p_R(i_S(s')) \tag{3.12}$$

and

$$p_S\big(\big(i_R(r)i_S(s)\big)\big(i_R(r')i_S(s')\big)\big) = p_S(i_R(r))\Big(\big(p_S(i_S(s)i_R(r'))\big)p_S(i_S(s'))\Big) = \\ \Big(p_S(i_R(r))\big(p_S(i_S(s)i_R(r'))\big)\Big)p_S(i_S(s')), \tag{3.13}$$

as well as $p_R \circ i_R = \mathrm{Id}_R$ and $p_S \circ i_S = \mathrm{Id}_S$, such that $R \bowtie S \to Q$, $(r,s) \mapsto i_R(r)i_S(s)$ and $Q \to R \bowtie S$, $q \mapsto (p_R(q), p_S(q))$ are inverse to each other.

Proof. Letting $\Phi : Q \to R \bowtie S$ to be the (quasigroup) isomorphism, we consider the mappings

$$i_R : R \to Q, \quad i_R(r) := \Phi^{-1}(r,\delta), \qquad i_S : S \to Q, \quad i_S(s) := \Phi^{-1}(\delta, s)$$

and

$$p_R : Q \to R, \quad p_R(q) := \pi_1(\Phi(q)), \qquad p_S : Q \to S, \quad p_S(q) := \pi_2(\Phi(q)),$$

where π_i's denote the projection onto the ith component. It is evident that

$$(p_R \circ i_R)(r) = \pi_1(r,\delta) = r,$$

for any $r \in R$, as such $p_R \circ i_R = \mathrm{Id}_R$. Similarly, $p_S \circ i_S = \mathrm{Id}_S$. We further see that

$$i_S(ss') = \Phi^{-1}(\delta, ss') = \Phi^{-1}\Big((\delta,s)(\delta,s')\Big) = \Phi^{-1}(\delta,s)\Phi^{-1}(\delta,s') = i_S(s)i_S(s'),$$

and that

$$i_R(rr') = \Phi^{-1}(rr',\delta) = \Phi^{-1}\Big((r,\delta)(r',\delta)\Big) = \Phi^{-1}(r,\delta)\Phi^{-1}(r',\delta) = i_R(r)i_R(r').$$

On the other hand, the mapping $R \bowtie S \to Q$, $(r,s) \mapsto i_R(r)i_S(s)$, becomes $\Phi^{-1} : R \bowtie S \to Q$, whereas the map $Q \to R \bowtie S$, $q \mapsto (p_R(q), p_S(q))$ becomes $\Phi : Q \to R \bowtie S$. Finally, we note also that

$$\begin{aligned}
&p_R\big(\big(i_R(r)i_S(s)\big)\big(i_R(r')i_S(s')\big)\big) = p_R\big(\Phi^{-1}(r,s)\Phi^{-1}(r',s')\big) = \\
&p_R\big(\Phi^{-1}(r\varphi(s,r'),\psi(s,r')s')\big) = r\varphi(s,r') = p_R(i_R(r))\Big(\big(p_R(i_S(s)i_R(r'))\big)p_R(i_S(s'))\Big) \\
&= \Big(p_R(i_R(r))\big(p_R(i_S(s)i_R(r'))\big)\Big)p_R(i_S(s')),
\end{aligned}$$

and that, similarly,

$$\begin{aligned}
&p_S\big(\big(i_R(r)i_S(s)\big)\big(i_R(r')i_S(s')\big)\big) = p_S(i_R(r))\Big(\big(p_S(i_S(s)i_R(r'))\big)p_S(i_S(s'))\Big) = \\
&\Big(p_S(i_R(r))\big(p_S(i_S(s)i_R(r'))\big)\Big)p_S(i_S(s')).
\end{aligned}$$

Conversely, let $i_S : S \to Q$ and $i_R : R \to Q$ be quasigroup homomorphisms, together with the maps $p_R : Q \to R$ and $p_S : Q \to S$ satisfying (3.12) and (3.13), such that $\Psi : R \bowtie S \to Q$, $\Psi(r,s) := i_R(r)i_S(s)$, and $\Phi : Q \to R \bowtie S$, $\Phi(q) := (p_R(q), p_S(q))$ are inverse to each other. Thus, the loop structure on Q induces a loop structure on $R \times S$. We shall, furthermore, see that this induced loop structure is in fact of the form (3.20). Indeed,

$$\begin{aligned}
&(\delta,s)(r',\delta) = \Phi(\Psi(\delta,s)\Psi(r',\delta)) = \Phi\Big(\big(i_R(\delta)i_S(s)\big)\big(i_R(r')i_S(\delta)\big)\Big) = \Phi\Big(i_S(s)i_R(r')\Big) \\
&= \Big(p_R\big(i_S(s)i_R(r')\big), p_S\big(i_S(s)i_R(r')\big)\Big) = \Big(\varphi(s,r'),\psi(s,r')\Big),
\end{aligned}$$

where $\varphi : S \times R \to R$, $\varphi(s,r') := p_R\big(i_S(s)i_R(r')\big)$, and $\psi : S \times R \to S$, $\psi(s,r') := p_S\big(i_S(s)i_R(r')\big)$. On the third equality we used the assumption that i_R, i_S are quasigroup homomorphisms. Accordingly,

$$\begin{aligned}
&(r,s)(r',s') = \Phi(\Psi(r,s)\Psi(r',s')) = \Phi\Big(\big(i_R(r)i_S(s)\big)\big(i_R(r')i_S(s')\big)\Big) = \\
&\Big(p_R\big(\big(i_R(r)i_S(s)\big)\big(i_R(r')i_S(s')\big)\big), p_S\big(\big(i_R(r)i_S(s)\big)\big(i_R(r')i_S(s')\big)\big)\Big) = \\
&\Big(p_R(i_R(r))\Big(\big(p_R(i_S(s)i_R(r'))\big)p_R(i_S(s'))\Big), \Big(p_S(i_R(r))\big(p_S(i_S(s)i_R(r'))\big)\Big)p_S(i_S(s'))\Big) \\
&= \Big(r\varphi(s,r'),\psi(s,r')s'\Big). \qquad \square
\end{aligned}$$

Next, we discuss the matched pair construction for the m-inverse property loops.

Proposition 3.9. *Let (R,δ) be an m_1-inverse loop with the permutation $J_R : R \to R$ so that $J_R(\delta) = \delta$, and that $J_R^{h_1} \in \mathrm{Aut}(R)$, and (S,δ) is an m_2-inverse loop with $J_S : S \to S$ such that $J_S(\delta) = \delta$, and that $J_S^{h_2} \in \mathrm{Aut}(S)$. Furthermore, let there be two maps $\phi : S \times R \to R$ and $\psi : S \times R \to S$ satisfying*

$$\phi(\delta,r) = r, \quad \phi(s,\delta) = \delta, \qquad \psi(\delta,r) = \delta, \quad \psi(s,\delta) = s, \tag{3.14}$$
$$\phi(s,\phi(J_S(s),r)) = r, \tag{3.15}$$
$$\psi(\psi(s,J_R^m(rr')),J_R^{m+1}(r)) = \psi(s,J_R^m(r')), \tag{3.16}$$
$$\phi(s,J_R^m(rr'))\phi(\psi(s,J_R^m(rr')),J_R^{m+1}(r)) = \phi(s,J_R^m(r')), \tag{3.17}$$
$$\psi(s,\phi(J_S(s),r))\psi(J_S(s),r) = \delta, \tag{3.18}$$

for any $r,r' \in R$, any $s,s' \in S$, and any $m \in \mathbb{Z}$ that satisfies

$$\begin{aligned} m &\equiv m_1 \ (\mathrm{mod}\, h_1), \\ m &\equiv m_2 \ (\mathrm{mod}\, h_2). \end{aligned} \tag{3.19}$$

Then, $\Big(R \bowtie S := R \times S, (\delta,\delta)\Big)$ is an m-invertible loop with the multiplication

$$(r,s)(r',s') := \Big(r\phi(s,r'), \psi(s,r')s'\Big) \tag{3.20}$$

and the permutation

$$\begin{aligned} &J : R \bowtie S \to R \bowtie S, \\ &J(r,s) := (\delta, J_S(s))(J_R(r),\delta) = \Big(\phi\big(J_S(s),J_R(r)\big), \psi\big(J_S(s),J_R(r)\big)\Big), \end{aligned} \tag{3.21}$$

if and only if

$$\left\{\begin{array}{ll} \left.\begin{array}{l} \phi(s,r) = r, \\ \psi(s,r) = s, \end{array}\right\} & \text{if } m = 2\ell, \\ \left.\begin{array}{l} \phi(J_S^m(\psi(s,J_R^{-m}(r))s'), \phi(\psi(s,J_R^{-1}(r)),r)) = \phi(J_S^m(s'),r), \\ \Big[\psi(J_S^m(\psi(s,r)s'),J_R^m(\phi(s,r)))\Big]J_S^{m+1}(s) = \psi(J_S^m(s'),J_R^m(r)), \end{array}\right\} & \text{if } m = 2\ell+1, \end{array}\right. \tag{3.22}$$

for any $s,s' \in S$, and any $r,r' \in R$.

Proof. Assuming the conditions (3.22) are met, we see at once that

$$\begin{aligned} &(r,s)J(r,s) = \big[(r,\delta)(\delta,s)\big]\big[(\delta,J_S(s))(J_R(r),\delta)\big] = \\ &\big[(r,\delta)(\delta,s)\big](\phi(J_S(s),J_R(r)),\psi(J_S(s),J_R(r))) = \\ &(r,\delta)\big[(\delta,s)(\phi(J_S(s),J_R(r)),\psi(J_S(s),J_R(r)))\big] = \\ &(r,\delta)\Big(\phi(s,\phi(J_S(s),J_R(r))),\psi(s,\phi(J_S(s),J_R(r)))\psi(J_S(s),J_R(r))\Big) = \\ &(r,\delta)(J_R(r),\delta) = (rJ_R(r),\delta) = (\delta,\delta), \end{aligned}$$

where on the fifth equality we used (3.15), and (3.18). Next, in view of (3.17) and (3.16), we have

$$\big((\delta,s)(r,\delta)\big)(J_R(r),J_S(s)) = (\phi(s,r),\psi(s,r))(J_R(r),J_S(s)) =$$
$$\big(\phi(s,r)\phi(\psi(s,r),J_R(r)),\psi(\psi(s,r),J_R(r))J_S(s)\big) = (\delta,\delta),$$

which implies that

$$J\big((\delta,s)(r,\delta)\big) = (J_R(r),J_S(s)).$$

On the other hand, in view of (3.17) we have

$$\phi(s,r)J_R(\phi(s,r)) = \delta = \phi(s,r)\phi(\psi(s,r),J_R(r)),$$

and hence we conclude

$$J_R(\phi(s,r)) = \phi(\psi(s,r),J_R(r)). \tag{3.23}$$

Let us note further that (3.23), together with (3.16), implies

$$J_R^m(\phi(s,r)) = \begin{cases}\phi(s,J_R^m(r)) & \text{if } m = 2\ell,\\ \phi(\psi(s,J_R^{m-1}(r)),J_R^m(r)) & \text{if } m = 2\ell+1.\end{cases}$$

Accordingly, in the case $m = 2\ell$,

$$\begin{aligned}
&J^m\big((r,s)(r',s')\big)J^{m+1}(r,s) = J^m(r\phi(s,r'),\psi(s,r')s')J^{m+1}(r,s) =\\
&\big[(J_R^m(r\phi(s,r')),\delta)(\delta,J_S^m(\psi(s,r')s'))\big]\big[(\delta,J_S^{m+1}(s))(J_R^{m+1}(r),\delta)\big] =\\
&(J_R^m(r\phi(s,r')),\delta)\Big\{(\delta,J_S^m(\psi(s,r')s'))\big[(\delta,J_S^{m+1}(s))(J_R^{m+1}(r),\delta)\big]\Big\} =\\
&(J_R^m(r\phi(s,r')),\delta)\big[((\delta,J_S^m(\psi(s,r')s')))(\phi(J_S^{m+1}(s),J_R^{m+1}(r)),\psi(J_S^{m+1}(s),J_R^{m+1}(r)))\big]\\
&= (J_R^m(r\phi(s,r')),\delta)\Big(\phi(J_S^m(\psi(s,r')s'),\phi(J_S^{m+1}(s),J_R^{m+1}(r))),\\
&\qquad\qquad \psi(J_S^m(\psi(s,r')s'),\phi(J_S^{m+1}(s),J_R^{m+1}(r)))\psi(J_S^{m+1}(s),J_R^{m+1}(r))\Big) =\\
&\Big(J_R^m(r\phi(s,r'))\phi(J_S^m(\psi(s,r')s'),\phi(J_S^{m+1}(s),J_R^{m+1}(r))),\\
&\qquad\qquad \psi(J_S^m(\psi(s,r')s'),\phi(J_S^{m+1}(s),J_R^{m+1}(r)))\psi(J_S^{m+1}(s),J_R^{m+1}(r))\Big) =\\
&\big(J_R^m(r'),J_S^m(s')\big) = J^m(r',s'),
\end{aligned} \tag{3.24}$$

where; on the seventh equality we used (3.22). In the case $m = 2\ell+1$,

$$\begin{aligned}
&J^m\big((r,s)(r',s')\big)J^{m+1}(r,s) = J^m(r\phi(s,r'),\psi(s,r')s')J^{m+1}(r,s) =\\
&\big[(\delta,J_S^m(\psi(s,r')s'))(J_R^m(r\phi(s,r')),\delta)\big]\big[(J_R^{m+1}(r),\delta)(\delta,J_S^{m+1}(s))\big] =\\
&(\phi(J_S^m(\psi(s,r')s'),J_R^m(r\phi(s,r'))),\psi(J_S^m(\psi(s,r')s'),J_R^m(r\phi(s,r'))))\big[(J_R^{m+1}(r),\delta)(\delta,J_S^{m+1}(s))\big] =\\
&\big[(\phi(J_S^m(\psi(s,r')s'),J_R^m(r\phi(s,r'))),\psi(J_S^m(\psi(s,r')s'),J_R^m(r\phi(s,r'))))(J_R^{m+1}(r),\delta)\big](\delta,J_S^{m+1}(s)) =
\end{aligned}$$

$$
\begin{aligned}
&\Big[\Big(\phi(J_S^m(\psi(s,r')s'),J_R^m(r\phi(s,r')))\phi(\psi(J_S^m(\psi(s,r')s'),J_R^m(r\phi(s,r'))),J_R^{m+1}(r)),\\
&\qquad\qquad \psi(\psi(J_S^m(\psi(s,r')s'),J_R^m(r\phi(s,r'))),J_R^{m+1}(r))\Big)\Big](\delta,J_S^{m+1}(s))=\\
&\Big(\phi(J_S^m(\psi(s,r')s'),J_R^m(\phi(s,r'))),\psi(J_S^m(\psi(s,r')s'),J_R^m(\phi(s,r')))\Big)(\delta,J_S^{m+1}(s))=\\
&\Big(\phi(J_S^m(\psi(s,r')s'),J_R^m(\phi(s,r'))),\psi(J_S^m(\psi(s,r')s'),J_R^m(\phi(s,r')))J_S^{m+1}(s)\Big)=\\
&\Big(\phi(J_S^m(\psi(s,r')s'),[\phi(\psi(s,J_R^{m-1}(r')),J_R^m(r'))]),\psi(J_S^m(\psi(s,r')s'),J_R^m(\phi(s,r')))J_S^{m+1}(s)\Big)=\\
&\Big(\phi(J_S^m(s'),J_R^m(r')),\psi(J_S^m(s'),J_R^m(r'))\Big)=(\delta,J_S^m(s'))(J_R^m(r'),\delta)=J^m(r',s') \qquad (3.25)
\end{aligned}
$$

where; in the sixth equation we used (3.17) and the second identity of (3.16), on the eighth equation we used (3.23), and finally on the ninth equation we used (both identities of) (3.22), in addition to Remark 2.4 and (3.19).

Let, conversely, $R \bowtie S$ be an m-inverse loop with the multiplication (3.20) and the permutation (3.21).

In the case $m = 2\ell$, the seventh equation of (3.24) holds, and we have

$$J_R^m(r\phi(s,r'))\phi(J_S^m(\psi(s,r')s'),\phi(J_S^{m+1}(s),J_R^{m+1}(r))) = J_R^m(r')$$

together with

$$\psi(J_S^m(\psi(s,r')s'),\phi(J_S^{m+1}(s),J_R^{m+1}(r)))\psi(J_S^{m+1}(s),J_R^{m+1}(r)) = J_S^m(s')$$

for any $r, r' \in R$, and any $s, s' \in S$. In particular, for $r = \delta$, the former equality yields

$$J_R^m(\varphi(s,r')) = J_R^m(r'),$$

hence $\varphi(s,r') = r'$, for any $r' \in R$, and any $s \in S$. For, on the other hand, $s = \delta$, the latter results in

$$\psi(J_S^m(s),J_R^{m+1}(r)) = J_S^m(s).$$

Once again, in view of the fact that $J_R : R \to R$ and $J_S : S \to S$ are both permutations, we deduce that $\psi(s,r) = s$ for any $r \in R$ and any $s \in S$.

In the case $m = 2\ell + 1$, however, the ninth equation of (3.25) holds, that is,

$$\phi(J_S^m(\psi(s,r')s'),[\phi(\psi(s,J_R^{m-1}(r')),J_R^m(r'))]) = \phi(J_S^m(s'),J_R^m(r')),$$

and

$$\psi(J_S^m(\psi(s,r')s'),J_R^m(\phi(s,r')))J_S^{m+1}(s) = \psi(J_S^m(s),J_R^m(r')).$$

The latter is nothing but the second identity of (3.22), whereas the first identity of (3.22) is obtained by taking $r' = J_R^{-m}(r)$ in the former. $\square$

Definition 3.10. Assume that (R, J_R, δ_R) is an m_1-inverse property loop such that $J_R(\delta_R) = \delta_R$, and that $J_R^{h_1} \in \mathrm{Aut}(R)$, and (S, J_S, δ_S) be an m_2-inverse property loop such that $J_S(\delta_S) = \delta_S$, and that $J_S^{h_2} \in \mathrm{Aut}(S)$. Let also $m \in \mathbb{Z}$ be a solution of

$$\begin{aligned} m &\equiv m_1 \pmod{h_1}, \\ m &\equiv m_2 \pmod{h_2}. \end{aligned}$$

Then, (R, S) is called a *matched pair of m-inverse property loops* if (R, J_R, δ_R) and (S, J_S, δ_S) satisfy the conditions (3.14) – (3.18).

Remark 3.11. We see that if (R, S) is a matched pair of m-inverse property quasigroups, then $R \bowtie S := R \times S$ is an m-inverse property quasigroup if and only if (3.22) holds. From the point of view of the generalization of groups, this is a manifestation of the fact that any group may be considered as an odd-inverse property quasigroup, while only commuttative groups fall into the category of even-inverse property quasigroups. Furthermore, we already know from the theory of matched pairs (of groups) that the matched pair group is commutative if and only if the mutual actions are trivial.

The following is an analogue of [41, Prop. 6.2.15].

Proposition 3.12. *Let (R, δ) be an m_1-inverse loop with the permutation $J_R : R \to R$ so that $J_R(\delta) = \delta$, and that $J_R^{h_1} \in \mathrm{Aut}(R)$, and (S, δ) is an m_2-inverse loop with $J_S : S \to S$ such that $J_S(\delta) = \delta$, and that $J_S^{h_2} \in \mathrm{Aut}(S)$. Let also $m \in \mathbb{Z}$ be a solution of*

$$\begin{aligned} m &\equiv m_1 \pmod{h_1}, \\ m &\equiv m_2 \pmod{h_2}. \end{aligned}$$

and (Q, δ) be an m-inverse loop so that (R, δ) is an m_1-inverse subloop of (Q, δ), and (S, δ) is an m_2-inverse subloop of (Q, δ);

$$(R, \delta) \hookrightarrow (Q, \delta) \hookleftarrow (S, \delta),$$

that the multiplication in Q yields an isomorphism

$$\Theta : R \times S \to Q, \qquad (r, s) \mapsto rs, \tag{3.26}$$

under which the multiplications are compatible as

$$(rs)q = r(sq), \qquad q(rs) = (qr)s, \tag{3.27}$$

and the inversions as

$$J_Q(rs) = J_S(s)J_R(r), \qquad J_Q(sr) = J_R(r)J_S(s) \tag{3.28}$$

for any $r \in R$, any $s \in S$, and any $q \in Q$. Then, (R, S) is a matched pair of m-inverse loops, and $Q \cong R \bowtie S$ as quasigroups.

Proof. Let us begin with the mappings

$$\phi : S \times R \to R, \qquad \psi : S \times R \to S \tag{3.29}$$

given by

$$\phi(s,r) := (\pi_1 \circ \Theta^{-1})(sr), \qquad \psi(s,r) := (\pi_2 \circ \Theta^{-1})(sr),$$

where $\pi_1 : R \times S \to R$, $\pi_2 : R \times S \to S$ are the projections onto the first and the second component respectively. It then follows at once that

$$sr = \Theta\big(\phi(s,r), \psi(s,r)\big) = \Theta((\delta,s)(r,\delta)), \tag{3.30}$$

that is, the isomorphism (3.26) respect the multiplications in Q and $R \bowtie S$.

It remains to show that the mappings (3.29) have the properties (3.14) $-$ (3.18).

The first one, (3.14), follows from the consideration of $r = \delta$ and $s = \delta$ in (3.30), respectively.

Next, in view of (3.28) the property $qJ_Q(q) = \delta$ implies $(rs)J_Q(rs) = \delta$ for any $r \in R$ and any $s \in S$, which in turn implies (3.15) and (3.18).

On the other hand, (3.27), and $J_Q^m(qq')J_Q^{m+1}(q) = J_Q^m(q')$ for any $q, q' \in Q$ yields, along the lines of (3.25),

$$\Big[\Big(\phi(J_S^m(\psi(s,r')s'), J_R^m(r\phi(s,r')))\phi(\psi(J_S^m(\psi(s,r')s'), J_R^m(r\phi(s,r'))), J_R^{m+1}(r)),$$
$$\psi(\psi(J_S^m(\psi(s,r')s'), J_R^m(r\phi(s,r'))), J_R^{m+1}(r))\Big)\Big](\delta, J_S^{m+1}(s)) =$$
$$\Big(\phi(J_S^m(s'), J_R^m(r')), \psi(J_S^m(s'), J_R^m(r'))\Big).$$

In particular, for $s = \delta$ then we see that

$$\Big[\Big(\phi(J_S^m(s'),J_R^m(rr'))\phi(\psi(J_S^m(s'),J_R^m(rr')),J_R^{m+1}(r)), \psi(\psi(J_S^m(s'),J_R^m(rr')),J_R^{m+1}(r))\Big)\Big]$$
$$= \Big(\phi(J_S^m(s'), J_R^m(r')), \psi(J_S^m(s'), J_R^m(r'))\Big),$$

which is equivalent to (3.16) and (3.17).

Finally, having obtained (3.14) – (3.18), the condition (3.22) follows from the ninth equality of (3.25) in the odd case, while it is a result of the seventh equality of (3.24) in the even case. $\square$

Let us illustrate with an example.

Example 3.13. Given a matched pair of groups (G, H), and two abelian groups V and W, let

$$\Lambda : (G \bowtie H) \times (G \bowtie H) \to V \times W, \quad \Lambda\Big((x,y),(x',y')\Big) := \Big(\varphi(x,x'), \chi(y,y')\Big), \tag{3.31}$$

for such $\varphi : G \times G \to V$ and $\chi : H \times H \to W$ that

$$\varphi(x, x') = \varphi(x, y \triangleright x') \tag{3.32}$$

and

$$\chi(y, y') = \varphi(y \triangleleft x, y') \tag{3.33}$$

for any $x, x' \in G$, and any $y, y' \in H$. Then let $(G \bowtie H) \times_\Lambda (V \times W)$ be the $(2\ell + 1)$-invertible loop of Example 2.5. As such, (3.31) satisfies (2.3), and we obtain $\varphi(1, x) = 0 = \varphi(x, 1)$, $\chi(1, y) = 0 = \chi(y, 1)$ for any $(x, y) \in G \times H$.

Similarly, imposing (2.11) onto (3.31),

$$\Lambda\Big((x, y), (x, y)^{-1}\Big) = \Lambda\Big((x, y), (y^{-1} \triangleright x^{-1}, y^{-1} \triangleleft x^{-1})\Big) = 0$$

we obtain $\varphi(x, y^{-1} \triangleright x^{-1}) = 0$, $\chi(y, y^{-1} \triangleleft x^{-1}) = 0$ for any $(x, y) \in G \times H$.

In particular, for $y = 1 \in H$ we obtain $\varphi(x, x^{-1}) = 0$, for any $x \in G$, and setting $x = 1 \in G$ we arrive at $\chi(y, y^{-1}) = 0$, for any $y \in H$.

Finally, since (3.31) is bound to satisfy (2.12), that is,

$$\Lambda\Big((x', y')^{-1}(x, y)^{-1}, (x, y)\Big) = \Lambda\Big((x, y), (x', y')\Big),$$

for any $(x, y), (x', y') \in G \times H$, or equivalently

$$\Lambda\Big((x', y')(x, y), (x, y)^{-1}\Big) = \Lambda\Big((x, y)^{-1}, (x', y')^{-1}\Big),$$

we have

$$\Lambda\Big((x'(y' \triangleright x), (y' \triangleleft x)y), (y^{-1} \triangleright x^{-1}, y^{-1} \triangleleft x^{-1})\Big) = \\ \Lambda\Big((y^{-1} \triangleright x^{-1}, y^{-1} \triangleleft x^{-1}), (y'^{-1} \triangleright x'^{-1}, y'^{-1} \triangleleft x'^{-1})\Big),$$

that is,

$$\varphi(x'(y' \triangleright x), y^{-1} \triangleright x^{-1}) = \varphi(y^{-1} \triangleright x^{-1}, y'^{-1} \triangleright x'^{-1})$$

and

$$\chi((y' \triangleleft x)y, y^{-1} \triangleleft x^{-1}) = \chi(y^{-1} \triangleleft x^{-1}, y'^{-1} \triangleleft x'^{-1})$$

for any $x, x' \in G$, and any $y, y' \in H$. Now $y = y' = 1 \in H$ (resp. $x = x' = 1 \in G$) leads to $\varphi(x'x, x^{-1}) = \varphi(x^{-1}, x'^{-1})$ (resp. $\chi(y'y, y^{-1}) = \chi(y^{-1}, y'^{-1})$). As a result, we have the $(2\ell_1 + 1)$-invertible loop $G \times_\varphi V$, and the $(2\ell_2 + 1)$-invertible loop $H \times_\chi W$, for any $\ell_1, \ell_2 \in \mathbb{Z}$, in such a way that

$$G \times_\varphi V \to (G \bowtie H) \times_\Lambda (V \times W), \qquad (x, v) \mapsto \Big((x, 1), (v, 0)\Big)$$

and

$$H \times_\chi W \to (G \bowtie H) \times_\Lambda (V \times W), \qquad (y, w) \mapsto \Big((1, y), (0, w)\Big)$$

are quasigroup homomorphisms.

Moreover, the multiplication in $(G \bowtie H) \times_\Lambda (V \times W)$ yields the isomorphism

$$\begin{aligned}
&\Theta : (G \times_\varphi V) \times (H \times_\chi W) \to (G \bowtie H) \times_\Lambda (V \times W), \\
&\Big((x, v), (y, w)\Big) \mapsto \Big((x, y), (v, w)\Big).
\end{aligned}$$

Let us finally show that (3.27) and (3.28) are satisfied. As for the former, we simply observe for any $(x, v) \in G \times_\varphi V$, any $(y, w) \in H \times_\chi W$, and any $((x', y'), (v', w')) \in (G \bowtie H) \times_\Lambda (V \times W)$,

$$\begin{aligned}
&\Big[(x, v)(y, w)\Big]\Big((x', y'), (v', w')\Big) = \\
&\Big[\Big((x, 1), (v, 0)\Big)\Big((1, y), (0, w)\Big)\Big]\Big((x', y'), (v', w')\Big) = \\
&\Big((x, y), (v, w)\Big)\Big((x', y'), (v', w')\Big) = \\
&\Big((x(y \triangleright x'), (y \triangleleft x')y'), (\varphi(x, x') + v + v', \chi(y, y') + w + w')\Big) = \\
&\Big((x(y \triangleright x'), (y \triangleleft x')y'), (\varphi(x, y \triangleright x') + v + v', \chi(y, y') + w + w')\Big) = \\
&\Big((x, 1), (v, 0)\Big)\Big((y \triangleright x', (y \triangleright x')y'), (v', \chi(y, y') + w + w')\Big) = \\
&\Big((x, 1), (v, 0)\Big)\Big[\Big((1, y), (0, w)\Big)\Big((x', y'), (v', w')\Big)\Big] = \\
&(x, v)\Big[(y, w)\Big((x', y'), (v', w')\Big)\Big],
\end{aligned}$$

where we used (3.32) in the fourth equality. Similarly, (3.33) yields

$$\Big((x, y), (v, w)\Big)\Big[(x', v')(y', w')\Big] = \Big[\Big((x, y), (v, w)\Big)(x', v')\Big](y', w').$$

Accordingly, (3.27) holds. As for (3.28), we do note that

$$\begin{aligned}
&J\Big((x, v)(y, w)\Big) = J\Big((x, y), (v, w)\Big) = \Big((x, y)^{-1}, (-v, -w)\Big) = \\
&\Big((y^{-1} \triangleright x^{-1}, y^{-1} \triangleleft x^{-1}), (v, w)\Big) = \Big((1, y^{-1}), (0, -w)\Big)\Big((x^{-1}, 1), (-v, 0)\Big) = \\
&(y^{-1}, -w)(x^{-1}, -v) = J_{H \times_\chi W}(y, w) J_{G \times_\varphi V}(x, v),
\end{aligned}$$

and that

$$\begin{aligned}
&J\Big((y, w)(x, v)\Big) = J\Big((y \triangleright x, y \triangleleft x), (v, w)\Big) = \Big((y \triangleright x, y \triangleleft x)^{-1}, (-v, -w)\Big) = \\
&\Big((x^{-1}, y^{-1}), (-v, -w)\Big) = \Big((x^{-1}, 1), (-v, 0)\Big)\Big((1, y^{-1}), (0, -w)\Big) = \\
&(x^{-1}, -v)(y^{-1}, -w) = J_{G \times_\varphi V}(x, v) J_{H \times_\chi W}(y, w).
\end{aligned}$$

We may now say that the hypotheses of Proposition 3.12 hold with $R := G \times_{\varphi} V$, $S := H \times_{\chi} W$, $Q := (G \bowtie H) \times_{\Lambda} (V \times W)$, $m := 2\ell+1$, $m_1 := 2\ell_1+1$, $m_2 := 2\ell_2+1$, for any $\ell, \ell_1, \ell_2 \in \mathbb{Z}$, and $h_1 = 2 = h_2$, that $(G \times_{\varphi} V, H \times_{\chi} W)$ is a matched pair of $(2\ell+1)$-invertible loops, and that

$$(G \bowtie H) \times_{\Lambda} (V \times W) \cong (G \times_{\varphi} V) \bowtie (H \times_{\chi} W).$$

Indeed, the mutual *actions*

$$\phi : (H \times_{\chi} W) \times (G \times_{\varphi} V) \to (G \times_{\varphi} V), \qquad \Big((y, w), (x, v)\Big) \mapsto (y \triangleright x, v)$$

and

$$\psi : (H \times_{\chi} W) \times (G \times_{\varphi} V) \to (H \times_{\chi} W), \qquad \Big((y, w), (x, v)\Big) \mapsto (y \triangleleft x, w)$$

which fit (in view of (3.32) and (3.33)) into

$$\begin{aligned}&\Big((x, v); (y, w)\Big)\Big((x', v'); (y', w')\Big) = \Big((x, y), (v, w)\Big)\Big((x', y'), (v', w')\Big) = \\ &\Big((x, v)\phi\Big((y, w), (x', v')\Big); \psi\Big((y, w), (x', v')\Big)(y', w')\Big)\end{aligned}$$

satisfy the compatibilities (3.14) – (3.18), as well as (3.22), merely from the matched pair compatibilities for groups.

4. Linearizations

Following the terminology and the point of view of [34, 35], we shall consider the Hopf analogues of the m-inverse property loops, under the name *m-invertible Hopf quasigroup.*

4.1. m-invertible Hopf quasigroups

Along the lines of [35, Def. 4.1], see also [34, Def. 2.1], we now introduce what we call an *m-inverse property Hopf quasigroup.*

Definition 4.1. Let $\mathcal{H}$ be a k-linear space equipped with the linear maps

$$\mu : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H},\ \eta : k \to \mathcal{H},\ \Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H},\ \varepsilon : \mathcal{H} \to k, \text{ and } S : \mathcal{H} \to \mathcal{H}.$$

Then, the six-tuple $(\mathcal{H}, \mu, \eta, \Delta, \varepsilon, S)$ is called an *m-inverse property Hopf quasigroup* if

(i) $(\mathcal{H}, \mu, \eta)$ is a unital, not-necessarily associative algebra,

(ii) $(\mathcal{H}, \Delta, \varepsilon)$ is a coassociative and counital coalgebra,

(iii) $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ and $\varepsilon : \mathcal{H} \to k$ are multiplicative,

(iv) $S : \mathcal{H} \to \mathcal{H}$ is the unique coalgebra anti-automorphism satisfying

$$h_{<1>} S(h_{<2>}) = \varepsilon(h)\delta = S(h_{<1>}) h_{<2>}, \tag{4.1}$$

so that

$$S^m(h_{<2>} g) S^{m+1}(h_{<1>}) = \varepsilon(h) S^m(g) \tag{4.2}$$

holds for any $h, g \in \mathcal{H}$.

Example 4.2. Let (Q, δ, J) be an m-inverse property loop. Then the linear space kQ is a m-inverse property Hopf quasigroup via

(i) the multiplication $\mu : kQ \otimes kQ \to kQ,\ \ \mu(q, q') := qq'$, defined as a linear extension of the multiplication on Q, the unit $\eta : k \to kQ,\ \ \eta(\alpha) := \alpha\delta$,

(ii) the comultiplication $\Delta : kQ \to kQ \otimes kQ,\ \ \Delta(q) := q \otimes q$ as the linear extension of the diagonal map, the counit $\varepsilon : kQ \to k,\ \ \varepsilon(q) = 1$,

(iii) and the *antipode* $S : kQ \to kQ,\ \ S(q) := J(q)$.

The following adaptation of [33, Rk. 2.2] will be instrumental in the construction of the products of Hopf quasigroups.

Remark 4.3. Let $(\mathcal{H}, \mu, \eta, \Delta, \varepsilon, S)$ be an m-inverse property Hopf quasigroup such that $S^r \in \operatorname{Aut}(\mathcal{H})$, *i.e.* $S^r(hg) = S^r(h) S^r(g)$, and $\Delta(S^r(h)) = S^r(h_{<1>}) \otimes S^r(h_{<2>})$, for any $h \in \mathcal{H}$. Then, $(\mathcal{H}, \mu, \eta, \Delta, \varepsilon, S)$ be an $(m + ur)$-inverse property Hopf quasigroup for any $u \in \mathbb{Z}$.

Indeed,

$$\begin{aligned} &S^{m+ur}(h_{<2>} g) S^{m+1+ur}(h_{<1>}) = S^m(S^{ur}(h_{<2>}) S^{ur}(g)) S^{m+1}(S^{ur}(h_{<1>})) = \\ &S^m(S^{ur}(h)_{<2>} S^{ur}(g)) S^{m+1}(S^{ur}(h)_{<1>}) = S^m(S^{ur}(g)) = S^{m+ur}(g). \end{aligned}$$

4.2. Matched pairs of m-inverse property Hopf quasigroups

For convenience, let us begin with the tensor product Hopf quasigroups. More precisely, the following result is the Hopf counterpart of [33, Thm. 5.1], that is, Theorem 3.1 above.

Theorem 4.4. *Let $(\mathcal{H}_1, \mu_1, \eta_1, \Delta_1, \varepsilon_1, S_1)$ be an m_1-inverse Hopf quasigroup so that $S_1^{h_1} \in \operatorname{Aut}(\mathcal{H}_1)$, and $(\mathcal{H}_2, \mu_2, \eta_2, \Delta_2, \varepsilon_2, S_2)$ be an m_2-inverse Hopf quasigroup such that $S_2^{h_2} \in \operatorname{Aut}(\mathcal{H}_2)$. Then $\mathcal{H}_1 \otimes \mathcal{H}_2$ is an m-inverse quasigroup with the tensor product structure maps, for any $m \in \mathbb{Z}$ that satisfies*

$$\begin{aligned} m &\equiv m_1 \pmod{h_1}, \\ m &\equiv m_2 \pmod{h_2}. \end{aligned} \tag{4.3}$$

Proof. It follows at once that

(i) $(\mathcal{H}_1 \otimes \mathcal{H}_2, \mu_\otimes, \eta_\otimes)$ is a (not necessarily associative) unital algebra via

$$\mu_\otimes := (\mu_1 \otimes \mu_2) \circ (\mathrm{Id} \otimes \tau \otimes \mathrm{Id}) : (\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes (\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{H}_1 \otimes \mathcal{H}_2,$$
$$\mu_\otimes\Big((h \otimes h') \otimes (g \otimes g')\Big) := \mu_1(h \otimes g) \otimes \mu_2(h' \otimes g')$$

and $\eta_\otimes := \eta_1 \otimes \eta_2 : k \to \mathcal{H}_1 \otimes \mathcal{H}_2$, $\eta_\otimes(\alpha) := \alpha\eta_1(1) \otimes \eta_2(1)$,

(ii) $(\mathcal{H}_1 \otimes \mathcal{H}_2, (\mathrm{Id} \otimes \tau \otimes \mathrm{Id}) \circ (\Delta_1 \otimes \Delta_2), \varepsilon_1 \otimes \varepsilon_2)$ is a coassociative counital coalgebra, such that

(iii) the coalgebra structure maps

$$\Delta_\otimes := (\mathrm{Id} \otimes \tau \otimes \mathrm{Id}) \circ (\Delta_1 \otimes \Delta_2) : \mathcal{H}_1 \otimes \mathcal{H}_2 \to (\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes (\mathcal{H}_1 \otimes \mathcal{H}_2),$$
$$\Delta_\otimes(h \otimes h') = (h_{<1>} \otimes h'_{<1>}) \otimes (h_{<2>} \otimes h'_{<2>})$$

and $\varepsilon_\otimes := \varepsilon_1 \otimes \varepsilon_2 : \mathcal{H}_1 \otimes \mathcal{H}_2 \to k$, $\varepsilon_\otimes(h \otimes h') = \varepsilon_1(h)\varepsilon_2(h')$ are multiplicative.

(iv) Finally, in view of Remark 4.3 above, for any solution $m \in \mathbb{Z}$ of (4.3)

$$(S_1 \otimes S_2)^m((h_{<2>} \otimes h'_{<2>})(g \otimes g'))(S_1 \otimes S_2)^{m+1}(h_{<1>} \otimes h'_{<1>}) =$$
$$S_1^m(h_{<2>}g)S_1^{m+1}(h_{<1>}) \otimes S_2^m(h'_{<2>}g')S_2^{m+1}(h'_{<1>}) =$$
$$S_1^m(g) \otimes S_2^m(g') = (S_1 \otimes S_2)^m(g \otimes g'). \qquad \square$$

As for the matched pair construction, Proposition 3.9 upgrades to the following proposition. However, we shall first need a technical lemma.

Lemma 4.5. *Let $(\mathcal{H}_1, \mu_1, \eta_1, \Delta_1, \varepsilon_1, S_1)$ be an m_1-inverse Hopf quasigroup, and $(\mathcal{H}_2, \mu_2, \eta_2, \Delta_2, \varepsilon_2, S_2)$ be an m_2-inverse Hopf quasigroup. Moreover, let there be two maps $\phi : \mathcal{H}_2 \otimes \mathcal{H}_1 \to \mathcal{H}_1$ and $\psi : \mathcal{H}_2 \otimes \mathcal{H}_1 \to \mathcal{H}_2$ satisfying*

$$\phi(S(h'_{<1>}), \phi(h'_{<2>}, h)) = \varepsilon_2(h')h = \phi(h'_{<1>}, \phi(S(h'_{<2>}), h)), \tag{4.4}$$
$$\psi(\psi(h', S(h_{<1>})), h_{<2>}) = \varepsilon_1(h)h' = \psi(\psi(h', h_{<1>}), S(h_{<2>})) \tag{4.5}$$
$$\Delta_1(\phi(h', h)) = \phi(h'_{<1>}, h_{<1>}) \otimes \phi(h'_{<2>}, h_{<2>}), \quad \varepsilon_1(\phi(h', h)) = \varepsilon_1(h)\varepsilon_2(h'), \tag{4.6}$$
$$\Delta_2(\psi(h', h)) = \psi(h'_{<1>}, h_{<1>}) \otimes \psi(h'_{<2>}, h_{<2>}), \quad \varepsilon_2(\psi(h', h)) = \varepsilon_1(h)\varepsilon_2(h'), \tag{4.7}$$
$$\phi(h'_{<1>}, S(h_{<2>}))\big[\phi(\psi(h'_{<2>}, S(h_{<1>})), h_{<3>})\big] = \varepsilon_1(h)\varepsilon_2(h') = \phi(h'_{<1>}, h_{<1>})\big[\phi(\psi(h'_{<2>}, h_{<2>}), S(h_{<3>}))\big], \tag{4.8}$$
$$\big[\psi(S(h'_{<1>}), \phi(h'_{<2>}, h_{<1>}))\big]\psi(h'_{<3>}, h_{<2>}) = \varepsilon_1(h)\varepsilon_2(h') = \big[\psi(h'_{<1>}, \phi(S(h'_{<3>}), h_{<1>}))\big]\psi(S(h'_{<2>}), h_{<2>}), \tag{4.9}$$
$$\psi(h'_{<1>}, h_{<1>}) \otimes \phi(h'_{<2>}, h_{<2>}) = \psi(h'_{<2>}, h_{<2>}) \otimes \phi(h'_{<1>}, h_{<1>}) \tag{4.10}$$

for any $h, g \in \mathcal{H}_1$*, any* $h', g' \in \mathcal{H}_2$*. Then the mapping*

$$\begin{aligned} &S_{\bowtie} : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2, \\ &S_{\bowtie}(h \otimes h') := (\delta_1 \otimes S_2(h'))(S_1(h) \otimes \delta_2) = \\ &\qquad \Big(\phi\big(S_2(h'_{<2>}), S_1(h_{<2>})\big) \otimes \psi\big(S_2(h'_{<1>}), S_1(h_{<1>})\big)\Big), \end{aligned} \tag{4.11}$$

satisfies

$$S_{\bowtie}((\delta_1, h')(h, \delta_2)) = (S_1(h), S_2(h'))$$

for any $h \in \mathcal{H}_1$*, and any* $h' \in \mathcal{H}_2$*.*

Proof. For any $h \in \mathcal{H}_1$, and any $h' \in \mathcal{H}_2$ we have

$$\begin{aligned}
&S_{\bowtie}((\delta_1, h')(h, \delta_2)) = S_{\bowtie}(\phi(h'_{<1>}, h_{<1>}), \psi(h'_{<2>}, h_{<2>})) = \\
&S_{\bowtie}(\phi(h'_{<2>}, h_{<2>}), \psi(h'_{<1>}, h_{<1>})) = \Big(\delta_1, S_2\big(\psi(h'_{<1>}, h_{<1>})\big)\Big)\Big(S_1\big(\phi(h'_{<2>}, h_{<2>})\big), \delta_2\Big) = \\
&\Big(\phi\big(S_2(\psi(h'_{<1>}, h_{<1>}))_{<1>}, S_1(\phi(h'_{<2>}, h_{<2>}))_{<1>}\big), \\
&\qquad \psi\big(S_2(\psi(h'_{<1>}, h_{<1>}))_{<2>}, S_1(\phi(h'_{<2>}, h_{<2>}))_{<2>}\big)\Big) = \\
&\Big(\phi\big(S_2(\psi(h'_{<1>}, h_{<1>}))_{<2>}, S_1(\phi(h'_{<2>}, h_{<2>}))_{<2>}\big), \\
&\qquad \psi\big(S_2(\psi(h'_{<1>}, h_{<1>}))_{<1>}, S_1(\phi(h'_{<2>}, h_{<2>}))_{<1>}\big)\Big) = \\
&\Big(\phi\big(S_2(\psi(h'_{<1><1>}, h_{<1><1>})), S_1(\phi(h'_{<2><1>}, h_{<2><1>}))\big), \\
&\qquad \psi\big(S_2(\psi(h'_{<1><2>}, h_{<1><2>})), S_1(\phi(h'_{<2><2>}, h_{<2><2>}))\big)\Big) = \\
&\Big(\phi\big(S_2(\psi(h'_{<1>}, h_{<1>})), S_1(\phi(h'_{<3>}, h_{<3>}))\big), \\
&\qquad \psi\big(S_2(\psi(h'_{<2>}, h_{<2>})), S_1(\phi(h'_{<4>}, h_{<4>}))\big)\Big) = \\
&\Big(\phi\big(S_2(\psi(h'_{<1>}, h_{<1>})), S_1(\phi(h'_{<2><2>}, h_{<2><2>}))\big), \\
&\qquad \psi\big(S_2(\psi(h'_{<2><1>}, h_{<2><1>})), S_1(\phi(h'_{<3>}, h_{<3>}))\big)\Big) = \\
&\Big(\phi\big(S_2(\psi(h'_{<1>}, h_{<1>})), S_1(\phi(h'_{<2><1>}, h_{<2><1>}))\big), \\
&\qquad \psi\big(S_2(\psi(h'_{<2><2>}, h_{<2><2>})), S_1(\phi(h'_{<3>}, h_{<3>}))\big)\Big) = \\
&\Big(\phi\big(S_2(\psi(h'_{<1>}, h_{<1>})), S_1(\phi(h'_{<2>}, h_{<2>}))\big), \\
&\qquad \psi\big(S_2(\psi(h'_{<3>}, h_{<3>})), S_1(\phi(h'_{<4>}, h_{<4>}))\big)\Big) =
\end{aligned}$$

$$\Big(\phi\big(S_2(\psi(h'_{<1>}, h_{<1>})), [\phi(\psi(h'_{<2>}, h_{<2><1>}), S_1(h_{<2><2>}))]\big),$$
$$\psi\big(S_2(\psi(h'_{<3>}, h_{<3>})), S_1(\phi(h'_{<4>}, h_{<4>}))\big)\Big) =$$
$$\Big(S_1(h_{<1>}), \psi\big(S_2(\psi(h'_{<1>}, h_{<2>})), S_1(\phi(h'_{<2>}, h_{<2>}))\big)\Big) =$$
$$\Big(S_1(h_{<1>}), \psi\big(\psi(S_2(h'_{<1><1>}), \phi(h'_{<1><2>}, h_{<2>})), S_1(\phi(h'_{<2>}, h_{<2>}))\big)\Big) = \Big(S_1(h), S_2(h')\Big),$$

where on the second, fifth, and ninth equations we used (4.10), on the sixth equation we used (4.6) and (4.7), on the eleventh equation we used the fact that

$$S_1(\phi(h', h)) = \phi(\psi(h', h_{<1>}), S_1(h_{<2>})),$$

which follows from (4.8), and on the twelfth equation we used (4.4). Finally, on the thirteenth equation we used

$$S_2(\psi(h', h)) = \psi(S_2(h'_{<1>}), \phi(h'_{<2>}, h)),$$

which is a consequence of (4.9), and on the fourteenth we used (4.5). $\square$

We are now ready for the main result.

Proposition 4.6. *Let $(\mathcal{H}_1, \mu_1, \eta_1, \Delta_1, \varepsilon_1, S_1)$ be an m_1-inverse Hopf quasigroups such that $S_1(\delta_1) = \delta_1$, and that $S_1^{h_1} \in \mathrm{Aut}(\mathcal{H}_1)$, and $(\mathcal{H}_2, \mu_2, \eta_2, \Delta_2, \varepsilon_2, S_2)$ be an m_2-inverse Hopf quasigroup such that $S_2(\delta_2) = \delta_2$, and that $S_2^{h_2} \in \mathrm{Aut}(\mathcal{H}_2)$. Furthermore, let there be two maps $\phi : \mathcal{H}_2 \otimes \mathcal{H}_1 \to \mathcal{H}_1$ and $\psi : \mathcal{H}_2 \otimes \mathcal{H}_1 \to \mathcal{H}_2$ satisfying*

$$\phi(\delta_2, h) = h, \quad \phi(h', \delta_1) = \delta_1, \qquad \psi(\delta_2, h) = \delta_2, \quad \psi(h', \delta_1) = h', \tag{4.12}$$
$$\phi(S(h'_{<1>}), \phi(h'_{<2>}, h)) = \varepsilon_2(h')h = \phi(h'_{<1>}, \phi(S(h'_{<2>}), h)), \tag{4.13}$$
$$\psi(\psi(h', S_1^m(h_{<2>}g)), S_1^{m+1}(h_{<1>})) = \psi(h', S_1^m(g)), \tag{4.14}$$
$$\psi(\psi(h', S(h_{<1>})), h_{<2>}) = \varepsilon_1(h)h' = \psi(\psi(h', h_{<1>}), S(h_{<2>})) \tag{4.15}$$
$$\Delta_1(\phi(h', h)) = \phi(h'_{<1>}, h_{<1>}) \otimes \phi(h'_{<2>}, h_{<2>}), \qquad \varepsilon_1(\phi(h', h)) = \varepsilon_1(h)\varepsilon_2(h'), \tag{4.16}$$
$$\Delta_2(\psi(h', h)) = \psi(h'_{<1>}, h_{<1>}) \otimes \psi(h'_{<2>}, h_{<2>}), \qquad \varepsilon_2(\psi(h', h)) = \varepsilon_1(h)\varepsilon_2(h'), \tag{4.17}$$
$$\phi(h', S_1^m(g)) =$$
$$\begin{cases} \phi(h'_{<1>}, S_1^m(h_{<3>}g_{<2>}))\phi(\psi(h'_{<2>}, S_1^m(h_{<2>}g_{<1>})), S_1^{m+1}(h_{<1>})) & \text{if } m = 2\ell+1, \\ \phi(h'_{<1>}, S_1^m(h_{<2>}g_{<1>}))\phi(\psi(h'_{<2>}, S_1^m(h_{<3>}g_{<2>})), S_1^{m+1}(h_{<1>})) & \text{if } m = 2\ell, \end{cases} \tag{4.18}$$
$$\phi(h'_{<1>}, S(h_{<2>}))\big[\phi(\psi(h'_{<2>}, S(h_{<1>})), h_{<3>})\big] = \varepsilon_1(h)\varepsilon_2(h') =$$
$$\phi(h'_{<1>}, h_{<1>})\big[\phi(\psi(h'_{<2>}, h_{<2>}), S(h_{<3>}))\big], \tag{4.19}$$
$$\big[\psi(S(h'_{<1>}), \phi(h'_{<2>}, h_{<1>}))\big]\psi(h'_{<3>}, h_{<2>}) = \varepsilon_1(h)\varepsilon_2(h') =$$
$$\big[\psi(h'_{<1>}, \phi(S(h'_{<3>}), h_{<1>}))\big]\psi(S(h'_{<2>}), h_{<2>}), \tag{4.20}$$

$$\psi(h'_{<1>}, h_{<1>}) \otimes \phi(h'_{<2>}, h_{<2>}) = \psi(h'_{<2>}, h_{<2>}) \otimes \phi(h'_{<1>}, h_{<1>}) \tag{4.21}$$

for any $h, g \in \mathcal{H}_1$, any $h', g' \in \mathcal{H}_2$, and any $m \in \mathbb{Z}$ that satisfies

$$\begin{aligned} m &\equiv m_1 \pmod{h_1}, \\ m &\equiv m_2 \pmod{h_2}. \end{aligned} \tag{4.22}$$

Then $\Big(\mathcal{H}_1 \bowtie \mathcal{H}_2 := \mathcal{H}_1 \otimes \mathcal{H}_2, \mu_{\bowtie}, \eta_{\otimes}, \Delta_{\otimes}, \varepsilon_{\otimes}, S_{\bowtie}\Big)$ *is an m-invertible Hopf quasigroup with the multiplication*

$$\mu_{\bowtie}((h\otimes h')\otimes(g\otimes g')) =: (h\otimes h')(g\otimes g') := \Big(h\phi(h'_{<1>}, g_{<1>}), \psi(h'_{<2>}, g_{<2>})g'\Big), \tag{4.23}$$

and the antipode

$$\begin{aligned} &S_{\bowtie} : \mathcal{H}_1 \bowtie \mathcal{H}_2 \to \mathcal{H}_1 \bowtie \mathcal{H}_2, \\ &S_{\bowtie}(h \otimes h') := (\delta_1 \otimes S_2(h'))(S_1(h) \otimes \delta_2) = \\ &\qquad \Big(\phi\big(S_2(h'_{<2>}), S_1(h_{<2>})\big) \otimes \psi\big(S_2(h'_{<1>}), S_1(h_{<1>})\big)\Big), \end{aligned} \tag{4.24}$$

if and only if

$$\begin{cases} \left.\begin{array}{l} \phi(h', h) = h, \\ \psi(h', h) = h', \end{array}\right\} & \text{if } m = 2\ell, \\ \left.\begin{array}{l} \phi(S_2^m(\psi(h'_{<2>}, g_{<2>})g'), S_1^m(\phi(h'_{<1>}, g_{<1>}))) = \\ \varepsilon_2(h')\phi(S_2^m(g'), S_1^m(g)), \\ \psi(S_2^m(\psi(h'_{<3>}, g_{<2>})g'), S_1^m(\phi(h'_{<2>}, g_{<1>})))S_2^{m+1}(h'_{<1>}) = \\ \varepsilon_2(h')\psi(S_2^m(g'), S_1^m(g)), \end{array}\right\} & \text{if } m = 2\ell+1, \end{cases} \tag{4.25}$$

for any $h, g \in \mathcal{H}_1$, and any $h', g' \in \mathcal{H}_2$.

Proof. Let us first assume that the conditions (4.25) are met. We shall begin with the observation that

$$\begin{aligned} &(h_{<1>}, h'_{<1>})S_{\bowtie}(h_{<2>}, h'_{<2>}) = \big[(h_{<1>}, \delta_2)(\delta_1, h'_{<1>})\big]\big[(\delta_1, S_2(h'_{<2>}))(S_1(h_{<2>}), \delta_2)\big] = \\ &\big[(h_{<1>}, \delta_2)(\delta_1, h'_{<1>})\big]\Big(\phi(S_2(h'_{<3>}), S_1(h_{<3>})), \psi(S_2(h'_{<2>}), S_1(h_{<2>}))\Big) = \\ &(h_{<1>}, \delta_2)\Big[(\delta_1, h'_{<1>})\big(\phi(S_2(h'_{<3>}), S_1(h_{<3>})), \psi(S_2(h'_{<2>}), S_1(h_{<2>}))\big)\Big] = \\ &(h_{<1>}, \delta_2)\Big(\phi(h'_{<1><1>}, \phi(S_2(h'_{<3>}), S_1(h_{<3>}))_{<1>}), \\ &\qquad \psi(h'_{<1><2>}, \phi(S_2(h'_{<3>}), S_1(h_{<3>}))_{<2>})\psi(S_2(h'_{<2>}), S_1(h_{<2>}))\Big) = \\ &(h_{<1>}, \delta_2)\Big(\phi(h'_{<1>}, \phi(S_2(h'_{<5>}), S_1(h_{<4>}))), \\ &\qquad \psi(h'_{<2>}, \phi(S_2(h'_{<4>}), S_1(h_{<3>})))\psi(S_2(h'_{<3>}), S_1(h_{<2>}))\Big) = \end{aligned}$$

$$(h_{<1>},\delta_2)\Big(\phi(h'_{<1>},\phi(S_2(h'_{<4>}),S_1(h_{<3>}))),\psi(h'_{<2>}S_2(h'_{<3>}),S_1(h_{<2>}))\Big) =$$
$$(h_{<1>},\delta_2)\Big(\phi(h'_{<1>},\phi(S_2(h'_{<2>}),S_1(h_{<2>}))),\delta_2\Big) =$$
$$(h_{<1>},\delta_2)(S_1(h_{<2>}),\varepsilon_2(h')\delta_2) = (h_{<1>}S_1(h_{<2>}),\delta_2) = (\varepsilon_1(h)\delta_1,\varepsilon_2(h')\delta_2),$$

where on the fifth equality we used (4.16) and (4.17), on the sixth equality (4.20), and on the eighth equality we use (4.13). Similarly,

$$S_{\bowtie}(h_{<1>},h'_{<1>})(h_{<2>},h'_{<2>}) = \big[(\delta_1,S_2(h'_{<1>}))(S_1(h_{<1>}),\delta_2)\big]\big[(h_{<2>},\delta_2)(\delta_1,h'_{<2>})\big] =$$
$$\Big(\phi(S_2(h'_{<2>}),S_1(h_{<2>})),\psi(S_2(h'_{<1>}),S_1(h_{<1>}))\Big)\big[(h_{<3>},\delta_2)(\delta_1,h'_{<3>})\big] =$$
$$\Big[\Big(\phi(S_2(h'_{<2>}),S_1(h_{<2>})),\psi(S_2(h'_{<1>}),S_1(h_{<1>}))\Big)(h_{<3>},\delta_2)\Big](\delta_1,h'_{<3>}) =$$
$$\Big(\phi(S_2(h'_{<2>}),S_1(h_{<2>}))\phi(\psi(S_2(h'_{<1>}),S_1(h_{<1>}))_{<1>},h_{<3><1>}),$$
$$\psi(\psi(S_2(h'_{<1>}),S_1(h_{<1>}))_{<2>},h_{<3><2>})\Big)(\delta_1,h'_{<3>}) =$$
$$\Big(\phi(S_2(h'_{<2>}),S_1(h_{<2>}))\phi(\psi(S_2(h'_{<1><2>}),S_1(h_{<1><2>})),h_{<3><1>}),$$
$$\psi(\psi(S_2(h'_{<1><1>}),S_1(h_{<1><1>})),h_{<3><2>})\Big)(\delta_1,h'_{<3>}) =$$
$$\Big(\phi(S_2(h'_{<3>}),S_1(h_{<3>}))\phi(\psi(S_2(h'_{<2>}),S_1(h_{<2>})),h_{<4>}),$$
$$\psi(\psi(S_2(h'_{<1>}),S_1(h_{<1>})),h_{<5>})\Big)(\delta_1,h'_{<4>}) =$$
$$\Big(\phi\big(S_2(h'_{<2>}),S_1(h_{<2>})h_{<3>}\big),\psi\big(\psi(S_2(h'_{<1>}),S_1(h_{<1>})),h_{<4>}\big)\Big)(\delta_1,h'_{<3>}) =$$
$$\Big(\delta_1,\psi\big(\psi(S_2(h'_{<1>}),S_1(h_{<1>})),h_{<2>}\big)\Big)(\delta_1,h'_{<2>}) = \Big(\varepsilon_1(h)\delta_1,S_2(h'_{<1>})\Big)(\delta_1,h'_{<2>}) =$$
$$(\varepsilon_1(h)\delta_1,S_2(h'_{<1>})h'_{<2>}) = (\varepsilon_1(h)\delta_1,\varepsilon_2(h')\delta_2),$$

using (4.19) on the seventh equality, and (4.15) on the tenth. Furthermore, (4.24) is unique with the property (4.1). Indeed, if $T:\mathcal{H}_1\otimes\mathcal{H}_2\to\mathcal{H}_1\otimes\mathcal{H}_2$, say $T(h,h')=(T_1(h,h'),T_2(h,h'))$, is a coalgebra anti-automorphism so that

$$(h_{<1>},h'_{<1>})T(h_{<2>},h'_{<2>}) = (\varepsilon_1(h)\delta_1,\varepsilon_2(h')\delta_2) = T(h_{<1>},h'_{<1>})(h_{<2>},h'_{<2>}), \tag{4.26}$$

then on one hand (from the first equality of (4.26))

$$(\varepsilon_1(h)\delta_1,\varepsilon_2(h')\delta_2) = (h_{<1>},h'_{<1>})T(h_{<2>},h'_{<2>}) =$$
$$(h_{<1>},h'_{<1>})\Big(T_1(h_{<2>},h'_{<2>}),T_2(h_{<2>},h'_{<2>})\Big) =$$
$$\Big(h_{<1>}\phi\big(h'_{<1><1>},T_1(h_{<2>},h'_{<2>})_{<1>}\big),\psi\big(h'_{<1><2>},T_1(h_{<2>},h'_{<2>})_{<2>}\big)T_2(h_{<2>},h'_{<2>})\Big), \tag{4.27}$$

while on the other hand (this time from the second equality of (4.26)),

$$
\begin{aligned}
&(\varepsilon_1(h)\delta_1, \varepsilon_2(h')\delta_2) = T(h_{<1>}, h'_{<1>})(h_{<2>}, h'_{<2>}) = \\
&\Big(T_1(h_{<1>}, h'_{<1>}), T_2(h_{<1>}, h'_{<1>})\Big)(h_{<2>}, h'_{<2>}) = \\
&\Big(T_1(h_{<1>}, h'_{<1>})\phi\big(T_2(h_{<1>}, h'_{<1>})_{<1>}, h_{<2><1>}\big), \psi\big(T_2(h_{<1>}, h'_{<1>})_{<2>}, h_{<2><2>}\big)h'_{<2>}\Big).
\end{aligned} \tag{4.28}
$$

Application of $\mathrm{Id}\otimes\varepsilon_2 : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1$ to (4.27) yields

$$h_{<1>}\phi\big(h'_{<1>}, T_1(h_{<2>}, h'_{<2>})\big) = \varepsilon_1(h)\varepsilon_2(h')\delta_1,$$

which, in turn, leads to

$$\phi\big(h'_{<1>}, T_1(h, h'_{<2>})\big) = \varepsilon_2(h')S_1(h).$$

But then,

$$T_1(h, h') = \phi\Big(S_2(h'_{<1>}), \phi\big(h'_{<2>}, T_1(h, h'_{<3>})\big)\Big) = \phi\big(S_2(h'), S_1(h)\big). \tag{4.29}$$

Similarly, applying $\varepsilon_1 \otimes \mathrm{Id} : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_2$ to (4.28) we derive

$$T_2(h, h') = \psi\big(S_2(h'), S_1(h)\big). \tag{4.30}$$

Now, from (4.29) and (4.30) we conclude $T = S_{\bowtie}$.

We next proceed to show that (4.24) satisfies (4.2). In case of $m = 2\ell + 1$, we have

$$
\begin{aligned}
&S_{\bowtie}^m\big((h_{<2>}, h'_{<2>})(g, g')\big)S_{\bowtie}^{m+1}(h_{<1>}, h'_{<1>}) = \\
&\qquad S_{\bowtie}^m\big(h_{<2>}\phi(h'_{<2>}, g_{<1>}), \psi(h'_{<3>}, g_{<2>})g'\big)S_{\bowtie}^{m+1}(h_{<1>}, h'_{<1>}) = \\
&\Big[\big(\delta_1, S_2^m(\psi(h'_{<3>}, g_{<2>})g')\big)\big(S_1^m(h_{<2>}\phi(h'_{<2>}, g_{<1>})), \delta_2\big)\Big]\big(S_1^{m+1}(h_{<1>}), S_2^{m+1}(h'_{<1>})\big) = \\
&\Big(\phi\big(S_2^m(\psi(h'_{<3>}, g_{<2>})g')_{<1>}, S_1^m(h_{<2>}\phi(h'_{<2>}, g_{<1>}))_{<1>}\big), \\
&\quad \psi\big(S_2^m(\psi(h'_{<3>}, g_{<2>})g')_{<2>}, S_1^m(h_{<2>}\phi(h'_{<2>}, g_{<1>}))_{<2>}\big)\Big)\big(S_1^{m+1}(h_{<1>}), S_2^{m+1}(h'_{<1>})\big) = \\
&\Big(\phi\big(S_2^m(\psi(h'_{<3>}, g_{<2>})g')_{<1>}, S_1^m(h_{<2>}\phi(h'_{<2>}, g_{<1>}))_{<1>}\big)\times \\
&\quad \phi\big(\psi\big(S_2^m(\psi(h'_{<3>}, g_{<2>})g')_{<2><1>}, S_1^m(h_{<3>}\phi(h'_{<2>}, g_{<1>}))_{<2><1>}, S_1^{m+1}(h_{<1>})_{<1>}\big)\big), \\
&\psi\big(\psi\big(S_2^m(\psi(h'_{<3>}, g_{<2>})g')_{<2><2>}, S_1^m(h_{<2>}\phi(h'_{<2>}, g_{<1>}))_{<2><2>}, S_1^{m+1}(h_{<1>})_{<2>}\big)\big)S_2^{m+1}(h'_{<1>})\Big) \\
&= \Big(\phi\big(S_2^m(\psi(h'_{<3>}, g_{<2>})g')_{<1>}, \big[S_1^m(h_{<2>}\phi(h'_{<2>}, g_{<1>}))_{<1>}S_1^{m+1}(h_{<1>})_{<1>}\big]\big), \\
&\psi\big(S_2^m(\psi(h'_{<3>}, g_{<2>})g')_{<2>}, \big[S_1^m(h_{<2>}\phi(h'_{<2>}, g_{<1>}))_{<2>}S_1^{m+1}(h_{<1>})_{<2>}\big]\big)S_2^{m+1}(h'_{<1>})\Big) = \\
&\Big(\varepsilon_1(h)\phi\big(S_2^m(g')_{<1>}, S_1^m(g)_{<1>}\big), \varepsilon_2(h')\psi\big(S_2^m(g')_{<2>}, S_1^m(g)_{<2>}\big)\Big) = \\
&(\delta_1, \varepsilon_2(h')S_2^m(g'))(\varepsilon_1(h)S_1^m(g), \delta_2) = \varepsilon_1(h)\varepsilon_2(h')S_{\bowtie}^m(g, g'),
\end{aligned} \tag{4.31}
$$

where on the second and the eighth equalities we used Lemma 4.5, on the fifth equality we used (4.18) and (4.14), and on the sixth equality we used (4.25). If, on the other hand, $m = 2\ell$

$$
\begin{aligned}
&S_{\bowtie}^{m}\big((h_{<2>}, h'_{<2>})(g, g')\big)S_{\bowtie}^{m+1}(h_{<1>}, h'_{<1>}) = \\
&S_{\bowtie}^{m}\big(h_{<2>}\phi(h'_{<2>}, g_{<1>}), \psi(h'_{<3>}, g_{<2>})g'\big)S_{\bowtie}^{m+1}(h_{<1>}, h'_{<1>}) = \\
&\Big(S_1^m(h_{<2>}\phi(h'_{<2>}, g_{<1>})), S_2^m(\psi(h'_{<3>}, g_{<2>})g')\Big)\Big[(\delta_1, S_2^{m+1}(h'_{<1>}))\,(S_1^{m+1}(h_{<1>}), \delta_2)\Big] = \\
&\Big(S_1^m(h_{<2>}\phi(h'_{<2>}, g_{<1>})), S_2^m(\psi(h'_{<3>}, g_{<2>})g')\Big)\times \\
&\qquad\Big(\phi\big(S_2^{m+1}(h'_{<1>})_{<1>}, S_1^{m+1}(h_{<1>})_{<1>}\big), \psi\big(S_2^{m+1}(h'_{<1>})_{<2>}, S_1^{m+1}(h_{<1>})_{<2>}\big)\Big) = \\
&\Big(S_1^m(h_{<2>}\phi(h'_{<2>}, g_{<1>}))\times \\
&\qquad\Big[\phi\big(S_2^m(\psi(h'_{<3>}, g_{<2>})g')_{<1>}, \phi\big(S_2^{m+1}(h'_{<1>})_{<1><1>}, S_1^{m+1}(h_{<1>})_{<1><1>}\big)\big)\Big], \\
&\Big[\psi\big(S_2^m(\psi(h'_{<3>}, g_{<2>})g')_{<2>}, \phi\big(S_2^{m+1}(h'_{<1>})_{<1><2>}, S_1^{m+1}(h_{<1>})_{<1><2>}\big)\big)\Big]\times \\
&\qquad\qquad\psi\big(S_2^{m+1}(h'_{<1>})_{<2>}, S_1^{m+1}(h_{<1>})_{<2>}\big)\Big) = \\
&\big(\varepsilon_1(h)S_1^m(g), \varepsilon_2(h')S_2^m(g')\big),
\end{aligned}
\tag{4.32}
$$

where on the second equality we used Lemma 4.5, and on the fifth equality we used (4.25).

Conversely, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be subject to the hypothesis of the theorem. Then, in the case of $m = 2\ell + 1$, the application of $\mathrm{Id}\otimes\varepsilon_2 : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1$ to the sixth equality

$$
\begin{aligned}
&\Big(\phi\big(S_2^m(\psi(h'_{<3>}, g_{<2>})g')_{<1>}, S_1^m(\phi(h'_{<2>}, g_{<1>}))_{<1>}\big), \\
&\qquad\psi\big(S_2^m(\psi(h'_{<3>}, g_{<2>})g')_{<2>}, S_1^m(\phi(h'_{<2>}, g_{<1>}))_{<2>}\big)S_2^{m+1}(h'_{<1>})\Big) = \\
&\Big(\phi\big(S_2^m(g')_{<1>}, S_1^m(g)_{<1>}\big), \varepsilon_2(h')\psi\big(S_2^m(g')_{<2>}, S_1^m(g)_{<2>}\big)\Big)
\end{aligned}
$$

of (4.31) yields

$$
\phi\big(S_2^m(\psi(h'_{<2>}, g_{<2>})g'), S_1^m(\phi(h'_{<1>}, g_{<1>}))\big) = \phi\big(S_2^m(g'), S_1^m(g)\big)\varepsilon_2(h')
$$

for any $g \in \mathcal{H}_1$, and any $g', h' \in \mathcal{H}_2$.

Similarly, the application of $\varepsilon_1 \otimes \mathrm{Id} : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_2$ on the other hand (to the sixth equality of (4.31)) this times yields

$$
\psi\big(S_2^m(\psi(h'_{<3>}, g_{<2>})g'), S_1^m(\phi(h'_{<2>}, g_{<1>}))\big)S_2^{m+1}(h'_{<1>}) = \varepsilon_2(h')\psi\big(S_2^m(g'), S_1^m(g)\big).
$$

Next, if $m = 2\ell$, then we apply $\mathrm{Id} \otimes \varepsilon_2 : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1$ to the fifth equality

$$\Big(S_1^m(h_{<2>}\phi(h'_{<2>}, g_{<1>}))\times \Big[\phi\big(S_2^m(\psi(h'_{<3>}, g_{<2>})g')_{<1>}, \phi\big(S_2^{m+1}(h'_{<1>})_{<1><1>}, S_1^{m+1}(h_{<1>})_{<1><1>}\big)\big)\Big],$$
$$\Big[\psi\big(S_2^m(\psi(h'_{<3>}, g_{<2>})g')_{<2>}, \phi\big(S_2^{m+1}(h'_{<1>})_{<1><2>}, S_1^{m+1}(h_{<1>})_{<1><2>}\big)\big)\Big]\times$$
$$\psi\big(S_2^{m+1}(h'_{<1>})_{<2>}, S_1^{m+1}(h_{<1>})_{<2>}\big)\Big) = \big(\varepsilon_1(h)S_1^m(g), \varepsilon_2(h')S_2^m(g')\big)$$

of (4.32) to get

$$S_1^m(h_{<2>}\phi(h'_{<2>}, g_{<1>}))\Big[\phi\big(S_2^m(\psi(h'_{<3>}, g_{<2>})g'), \phi\big(S_2^{m+1}(h'_{<1>}), S_1^{m+1}(h_{<1>})\big)\big)\Big] =$$
$$\varepsilon_1(h)S_1^m(g)\varepsilon_2(h')\varepsilon_2(g')$$

for any $g, h \in \mathcal{H}_1$, and any $g', h' \in \mathcal{H}_2$. In particular, for $h = 1$ and $g' = 1$ we arrive at

$$S_1^m(\phi(h', g)) = \varepsilon_2(h')S_1^m(g),$$

from which we conclude that

$$\phi(h', g) = \varepsilon_2(h')g. \tag{4.33}$$

Similarly, the application of $\varepsilon_1 \otimes \mathrm{Id} : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_2$ to the fifth equality of (4.32) yields

$$\Big[\psi\big(S_2^m(\psi(h'_{<3>}, g_{<2>})g'), \phi\big(S_2^{m+1}(h'_{<1>})_{<1>}, S_1^{m+1}(h_{<1>})_{<1>}\big)\big)\Big]\times$$
$$\psi\big(S_2^{m+1}(h'_{<1>})_{<2>}, S_1^{m+1}(h_{<1>})_{<2>}\big)\Big) =$$
$$\varepsilon_1(h)\varepsilon_1(g)\varepsilon_2(h')S_2^m(g').$$

Now, invoking (4.33), and setting $g = 1$ and $h' = 1$, we obtain (in view of (4.12))

$$\psi\big(S_2^m(g'), S_1^{m+1}(h)\big) = \varepsilon_1(h)S_2^m(g'),$$

from which the the triviality of the left action follows. □

Definition 4.7. Let $(\mathcal{H}_1, \mu_1, \eta_1, \Delta_1, \varepsilon_1, S_1)$ be an m_1-inverse Hopf quasigroup such that $S_1(\delta_1) = \delta_1$, and that $S_1^{h_1} \in \mathrm{Aut}(\mathcal{H}_1)$, and $(\mathcal{H}_2, \mu_2, \eta_2, \Delta_2, \varepsilon_2, S_2)$ be an m_2-inverse Hopf quasigroup such that $S_2(\delta_2) = \delta_2$, and that $S_2^{h_2} \in \mathrm{Aut}(\mathcal{H}_2)$. Let also $m \in \mathbb{Z}$ be a solution of

$$m \equiv m_1 \pmod{h_1},$$
$$m \equiv m_2 \pmod{h_2}.$$

Then, $(\mathcal{H}_1, \mathcal{H}_2)$ is called a *matched pair of m-inverse property Hopf quasigroups* if the Hopf quasigroups $(\mathcal{H}_1, \mu_1, \eta_1, \Delta_1, \varepsilon_1, S_1)$ and $(\mathcal{H}_2, \mu_2, \eta_2, \Delta_2, \varepsilon_2, S_2)$ satisfy the conditions (4.12) – (4.21).

A remark is in order.

Remark 4.8. Given an m_1-inverse property quasigroup Q_1, an m_2-inverse property quasigroup Q_2, and a solution $m \in \mathbb{Z}$ of

$$\begin{aligned} m &\equiv m_1 \pmod{h_1}, \\ m &\equiv m_2 \pmod{h_2}. \end{aligned}$$

Let $\Big((Q_1, J_1, \delta_1), (Q_2, J_2, \delta_2)\Big)$ be a matched pair of m-inverse property quasigroups such that $J_1(q)q = \delta_1$ for any $q \in Q_1$ and $J_2(q')q' = \delta_2$ for any $q' \in Q_2$. Then (kQ_1, kQ_2) is a matched pair of m-inverse property Hopf quasigroups.

The following result is the universal property of the matched pair construction for m-inverse property Hopf quasigroups, that is, the analogue of [41, Thm. 7.2.3].

Proposition 4.9. *Let $(\mathcal{H}_1, \mu_1, \eta_1, \Delta_1, \varepsilon_1, S_1)$ be an m_1-inverse Hopf quasigroups such that $S_1(\delta_1) = \delta_1$, and that $S_1^{h_1} \in \mathrm{Aut}(\mathcal{H}_1)$, and $(\mathcal{H}_2, \mu_2, \eta_2, \Delta_2, \varepsilon_2, S_2)$ be an m_2-inverse Hopf quasigroup such that $S_2(\delta_2) = \delta_2$, and that $S_2^{h_2} \in \mathrm{Aut}(\mathcal{H}_2)$. Let also $m \in \mathbb{Z}$ be a solution of*

$$\begin{aligned} m &\equiv m_1 \pmod{h_1}, \\ m &\equiv m_2 \pmod{h_2}. \end{aligned}$$

and $\mathcal{G}$ be an m-inverse Hopf quasigroup so that $\mathcal{H}_1$ and $\mathcal{H}_2$ are m-inverse Hopf quasi-subgroups of $\mathcal{G}$;

$$\mathcal{H}_1 \hookrightarrow \mathcal{G} \hookleftarrow \mathcal{H}_2,$$

such that the multiplication on $\mathcal{G}$ yields an isomorphism

$$\Theta : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{G}, \qquad h \otimes h' \mapsto hh', \tag{4.34}$$

of vector spaces, under which the multiplications are compatible as

$$(hh')g = h(h'g), \qquad g(hh') = (gh)h',$$

for any $h \in \mathcal{H}_1$, any $h' \in \mathcal{H}_2$, and any $g \in \mathcal{G}$, while the antipodes are compatible as

$$S(hh') = S_2(h')S_1(h), \qquad S(h'h) = S_1(h)S_2(h') \tag{4.35}$$

for any $h \in \mathcal{H}_1$, any $h' \in \mathcal{H}_2$, and any $g \in \mathcal{G}$. Then, $(\mathcal{H}_1, \mathcal{H}_2)$ is a matched pair of m-inverse Hopf quasigroups, and $\mathcal{G} \cong \mathcal{H}_1 \bowtie \mathcal{H}_2$ as Hopf quasigroups.

Proof. Let us begin with the mappings

$$\phi : \mathcal{H}_2 \otimes \mathcal{H}_1 \to \mathcal{H}_1, \qquad \psi : \mathcal{H}_2 \otimes \mathcal{H}_1 \to \mathcal{H}_2 \tag{4.36}$$

given by

$$\phi(h', h) := ((\mathrm{Id} \otimes \varepsilon_2) \circ \Theta^{-1})(h'h), \qquad \psi(h', h) := ((\varepsilon_1 \otimes \mathrm{Id}) \circ \Theta^{-1})(h'h),$$

through

$$hh' = \Theta\big(\phi(h'_{<1>}, h_{<1>}), \psi(h'_{<2>}, h_{<2>})\big). \tag{4.37}$$

It then follows at once that the isomorphism (4.34) respect the multiplications in $\mathcal{G}$ and $\mathcal{H}_1 \bowtie \mathcal{H}_2$.

It remains to show that the mappings (4.36) have the properties (4.12)−(4.21).

The first one, (4.12), follows from the consideration of $h = \delta_1$ and $h' = \delta_2$ in (4.37), respectively.

Next, the linear map $\Psi : \mathcal{H}_2 \otimes \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ given by

$$\Psi(h' \otimes h) := \Theta^{-1}(hh') = \phi(h'_{<1>}, h_{<1>}) \otimes \psi(h'_{<2>}, h_{<2>})$$

being a coalgebra homomorphism, we have

$$\Delta_{\otimes} \circ \Psi = (\Psi \otimes \Psi) \circ \Delta_{\otimes}, \qquad \big((\varepsilon_1 \otimes \varepsilon_2) \circ \Psi\big)(h \otimes h') = \varepsilon_1(h)\varepsilon_2(h'),$$

for any $h \in \mathcal{H}_1$, and any $h' \in \mathcal{H}_2$. Applying on an arbitrary $h' \otimes h \in \mathcal{H}_2 \otimes \mathcal{H}_1$, we arrive at

$$\Big[\phi(h'_{<1>}, h_{<1>})_{<1>} \otimes \psi(h'_{<2>}, h_{<2>})_{<1>}\Big] \otimes \Big[\phi(h'_{<1>}, h_{<1>})_{<2>} \otimes \psi(h'_{<2>}, h_{<2>})_{<2>}\Big] =$$
$$\Big(\phi(h'_{<1><1>}, h_{<1><1>}) \otimes \psi(h'_{<1><2>}, h_{<1><2>})\Big) \otimes$$
$$\Big(\phi(h'_{<2><1>}, h_{<2><1>}) \otimes \psi(h'_{<2><2>}, h_{<2><2>})\Big).$$

Now, $\mathrm{Id} \otimes \varepsilon_2 \otimes \mathrm{Id} \otimes \varepsilon_2$ yields (4.16), and $\varepsilon_1 \otimes \mathrm{Id} \otimes \varepsilon_1 \otimes \mathrm{Id}$ results in (4.17). Furthermore, $\varepsilon_1 \otimes \mathrm{Id} \otimes \mathrm{Id} \otimes \varepsilon_2$ leads to (4.21).

On the other hand, in view of (4.35) the property $g_{<1>}S(g_{<2>}) = \varepsilon(g)\delta$ implies $(h_{<1>}h'_{<1>})S(h_{<2>}h'_{<2>}) = \varepsilon_1(h)\varepsilon_2(h')\delta$ for any $h \in \mathcal{H}_1$ and any $h' \in \mathcal{H}_2$, which in turn implies

$$\Big(h_{<1>}\phi(h'_{<1>}, \phi(S_2(h'_{<5>}), S_1(h_{<4>}))),$$
$$\psi(h'_{<2>}, \phi(S_2(h'_{<4>}), S_1(h_{<3>})))\psi(S_2(h'_{<3>}), S_1(h_{<2>}))\Big) =$$
$$(h_{<1>}S_1(h_{<2>}), \delta_2) = (\varepsilon_1(h)\delta_1, \varepsilon_2(h')\delta_2).$$

We then obtain the second equality of (4.13) by applying $\mathrm{Id} \otimes \varepsilon_2$, as well as the second equality of (4.20) via $\varepsilon_1 \otimes \mathrm{Id}$. Similarly, $S(g_{<1>})g_{<2>} = \varepsilon(g)\delta$ yields

$$\Big(\phi(S_2(h'_{<3>}), S_1(h_{<3>}))\phi(\psi(S_2(h'_{<2>}), S_1(h_{<2>})), h_{<4>}),$$
$$\phi(\psi(S_2(h'_{<1>}), S_1(h_{<1>})), h_{<5>})h'_{<4>}\Big) =$$
$$(\varepsilon_1(h)\delta_1, S_2(h'_{<1>})h'_{<2>}) = (\varepsilon_1(h)\delta_1, \varepsilon_2(h')\delta_2),$$

which in turn implies the first equality of (4.19) by $\mathrm{Id}\otimes\varepsilon_2$, and the first equality of (4.15) by $\varepsilon_1\otimes\mathrm{Id}$.

On the next step, $J^m{}_Q(qq')J^{m+1}{}_Q(q) = J^m{}_Q(q')$ for any $q, q' \in Q$ provides, along the lines of (4.31),

$$\Big(\phi\big(S_2^m(\psi(h'_{<3>}, g_{<2>})g')_{<1>}, S_1^m(h_{<2>}\phi(h'_{<2>}, g_{<1>}))_{<1>}\big)\times$$
$$\phi\big(\psi\big(S_2^m(\psi(h'_{<3>}, g_{<2>})g')_{<2><1>}, S_1^m(h_{<3>}\phi(h'_{<2>}, g_{<1>}))_{<2><1>}, S_1^{m+1}(h_{<1>})_{<1>}\big)\big),$$
$$\psi\big(\psi\big(S_2^m(\psi(h'_{<3>},g_{<2>})g')_{<2><2>},S_1^m(h_{<2>}\phi(h'_{<2>},g_{<1>}))_{<2><2>},S_1^{m+1}(h_{<1>})_{<2>}\big)\big)S_2^{m+1}(h'_{<1>})\Big)$$
$$= \Big(\varepsilon_1(h)\phi\big(S_2^m(g')_{<1>}, S_1^m(g)_{<1>}\big), \varepsilon_2(h')\psi\big(S_2^m(g')_{<2>}, S_1^m(g)_{<2>}\big)\Big) =$$
$$(\delta_1, \varepsilon_2(h')S_2^m(g'))(\varepsilon_1(h)S_1^m(g), \delta_2) = \varepsilon_1(h)\varepsilon_2(h')S_{\bowtie}^m(g, g').$$

In particular, for $h' = \delta_2$ we see that

$$\Big(\phi\big(S_2^m(g')_{<1>}, S_1^m(h_{<2>}g_{<1>})_{<1>}\big)\phi\big(\psi\big(S_2^m(g')_{<2><1>}, S_1^m(h_{<3>}g_{<1>})_{<2><1>},$$
$$S_1^{m+1}(h_{<1>})_{<1>}\big)\big), \psi\big(\psi\big(S_2^m(g')_{<2><2>}, S_1^m(h_{<2>}g_{<1>})_{<2><2>}, S_1^{m+1}(h_{<1>})_{<2>}\big)\big)\Big) =$$
$$\Big(\varepsilon_1(h)\phi\big(S_2^m(g')_{<1>}, S_1^m(g)_{<1>}\big), \psi\big(S_2^m(g')_{<2>}, S_1^m(g)_{<2>}\big)\Big),$$

which implies (4.18) by $\mathrm{Id}\otimes\varepsilon_2$, and (4.14) by $\varepsilon_1\otimes\mathrm{Id}$. Let us also remark that (4.18) implies the second equality of (4.19), and that (4.14) implies the second equation of (4.15).

Equipped with these now, (4.35) gives

$$S_{\bowtie}((\delta_1, h')(h, \delta_2)) =$$
$$\Big(\phi\big(S_2(\psi(h'_{<1>}, h_{<1>})), S_1(\phi(h'_{<2>}, h_{<2>}))\big), \psi\big(S_2(\psi(h'_{<3>}, h_{<3>})), S_1(\phi(h'_{<4>}, h_{<4>}))\big)\Big)$$
$$= \Big(\phi\big(S_2(\psi(h'_{<1>}, h_{<1>})), [\phi(\psi(h'_{<2>}, h_{<2><1>}), S_1(h_{<2><2>}))]\big),$$
$$\psi\big(S_2(\psi(h'_{<3>}, h_{<3>})), S_1(\phi(h'_{<4>}, h_{<4>}))\big)\Big) =$$
$$\Big(S_1(h), S_2(h')\Big).$$

Then, the application of $\mathrm{Id}\otimes\varepsilon_2$ yields

$$\phi\big(S_2(\psi(h'_{<1>}, h_{<1>})), [\phi(\psi(h'_{<2>}, h_{<2>}), S_1(h_{<3>}))]\big) = \varepsilon_2(h')S_1(h),$$

in particular,

$$\phi\big(S_2(\psi(\psi(h', S_1(h_{<1>}))_{<1>}, h_{<2><1>})), [\phi(\psi(\psi(h', S_1(h_{<1>}))_{<2>}, h_{<2><2>}),$$
$$S_1(h_{<2><3>}))]\big) = \varepsilon_2(\psi(h', S_1(h_{<1>})))S_1(h_{<2>}),$$

that is,

$$\phi\big(S_2(h'_{<1>}), \phi(h'_{<2>}, S_1(h))\big) = \varepsilon_2(h')S_1(h),$$

the first equality of (4.13). Similarly, the application of $\varepsilon_1 \otimes \mathrm{Id}$ onto

$$S_{\bowtie}((\delta_1,h')(h,\delta_2)) = \Big(S_1(h_{<1>}), \psi\big(S_2(\psi(h'_{<1>}, h_{<2>})), S_1(\phi(h'_{<2>}, h_{<2>}))\big)\Big) = \Big(S_1(h), S_2(h')\Big),$$

implies

$$\psi\big(S_2(\psi(h'_{<1>}, h_{<2>})), S_1(\phi(h'_{<2>}, h_{<2>}))\big) = \varepsilon_1(h)S_2(h').$$

Hence, we see that

$$\psi\Big(\psi\big(S_2(\psi(h'_{<1><1>}, h_{<1><2>})), S_1(\phi(h'_{<1><2>}, h_{<1><2>}))\big), \phi(h'_{<2>}, h_{<2>})\Big) =$$
$$\varepsilon_1(h_{<1>})\psi\big(S_2(h'_{<1>}), \phi(h'_{<2>}, h_{<2>})\big),$$

that is,

$$S_2(\psi(h', h)) = \psi\big(S_2(h'_{<1>}), \phi(h'_{<2>}, h)\big).$$

But then,

$$\big[\psi(S(h'_{<1>}), \phi(h'_{<2>}, h_{<1>}))\big]\psi(h'_{<3>}, h_{<2>}) = S_2(\psi(h'_{<1>}, h_{<1>}))\psi(h'_{<2>}, h_{<2>}) = \varepsilon_1(h)\varepsilon_2(h'),$$

the first equality of (4.20) is satisfied.

Finally, having obtained (4.12) – (4.21), it is possible to derive (4.25) from (4.31) in the case $m = 2\ell + 1$, and from (4.32) in the case $m = 2\ell$. □

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Computational approach for intransitive action of $\Delta(2,4,k)$ on $PL(F_q)$

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Abstract In this paper, we have investigated actions of triangle group $\Delta(2,4,k)$ defined by $< r,s : r^2 = s^4 = (rs)^k = 1 >$, on projective line over the finite field $PL(F_q)$ by using the concept of coset diagrams. We will parameterize this action and prove that actions of $\Delta(2,4,4)$ is intransitive on $PL(F_q)$, where q is such a prime that $q+2$ gives a perfect square. We have also developed a useful computational technique to parameterize this action and also to draw coset diagrams. Throughout -1 represents ∞ ,in diagrams as these are computer generated.

1. Introduction

The *linear-fractional group* $\Delta(2,4,k)$ is defined by the transformations $r : z \to \frac{-1}{z}$ and $s : z \to \frac{-1}{2(z+1)}$ that satisfies the relations $r^2 = s^4 = 1$. This group can be extended by adjoining an involution $t : z \to \frac{1}{2z}$ such that $(rt)^2 = (st)^2 = 1$. This extended group is denoted by $\Delta^*(2,4,k)$ [1, 2, 6].

Let $\alpha : PGL\,(2,Z) \longrightarrow PGL\,(2,q)$ be a non-degenerate homomorphism. We know that every non-degenerate homomorphism gives rise to an action. So, this non-degenerate homomorphism gives rise to an action of $PGL\,(2,Z)$ on $PL\,(F_q)$. The action which arises from this non-degenerate homomorphism not only corresponds to the non-degenerate homomorphism but to a conjugacy class of the homomorphisms [3, 5].

Since, there is one-to-one correspondence between the conjugacy classes of elements of order greater than 2 in $PGL\,(2,q)$ and the non-zero elements of F_q, such that the class corresponding to an element θ in F_q consists of all the elements represented by matrices A [6]. It follows that we can actually parameterize the non-degenerate homomorphisms of $PGL\,(2,Z)$ into $PGL\,(2,q)$, except for a few uninteresting ones, by the elements of F_q. If α is any such non-degenerate homomorphism, and R, S and T are in $GL\,(2,q)$, which yield the elements $\overline{r}$, $\overline{s}$, $\overline{t}$ then letting $\theta = m_2^2/\Delta$ (where $m_2 = trace(RS)$, $\triangle = det(RS)$), we associate the parameter θ with the homomorphism α. This non-zero element θ of F_q provides a permutation representation of the action corresponding to the homomorphism α. We draw a coset diagram corresponding to this action which is a diagram corresponding to not only one action but to a class of actions whose parameter is θ.

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We are looking for a condition on θ and q which ensures action of $PGL\,(2, Z)$ on $PL\,(F_q)$ evolving the required coset diagrams [4, 6, 7].

2. Conjugacy classes and coset diagrams

In this section, construction of coset diagrams for the generalized triangle group $< r, s, t : r^2 = s^4 = t^2 = (rt)^2 = (st)^2 = (rs)^k = 1 >$ are considered along-with certain observations about this case. The coset diagrams for action of $\Delta^*(2, 4, k)$ on finite space are defined as follows.

The four cycles of s are represented by squares whose vertices are permuted anti-clock wise by S. Any two vertices which are interchanged by involution r is represented by an edge. The action of t is represented by reflection about a vertical axis of symmetry. For example, action of $\Delta^*(2, 4, k)$ on $PL(F_{31})$ gives us the following permutation representations:

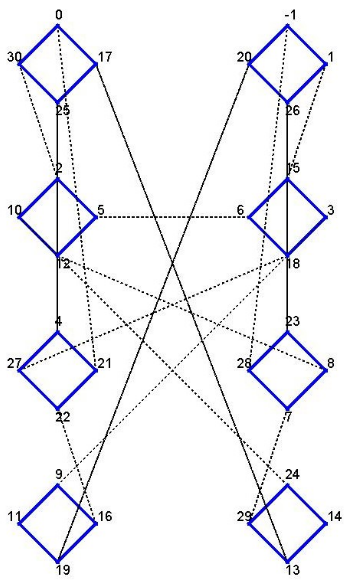

Figure 1: Action of $\Delta^*(2, 4, k)$ on $PL(F_{31})$

Theorem 2.1. *Corresponding to each $\theta = m_4 \in F_q$ there exists a conjugacy class of non-degenerate homomorphism $\alpha : PGL(2, Z) \to PGL(2, q)$ which yields the homomorphic image of $< r, s : r^2 = s^4 = (rs)^4 = 1 >$ under α.*

Proof. Define a homomorphism $\alpha : PGL(2, Z) \longrightarrow PGL(2, q)$ such that $\overline{r} = r\alpha$, $\overline{s} = s\alpha$ and $\overline{t} = t\alpha$ satisfying the relations:

$$\overline{r}^2 = \overline{s}^4 = \overline{t}^2 = \left(\overline{r}\overline{t}\right)^2 = \left(\overline{s}\overline{t}\right)^2 = 1. \tag{1}$$

So, there is requirement to see for elements $\overline{r}$, $\overline{s}$, $\overline{t} \in PGL(2,q)$ satisfying the relations 1 with $\overline{r}\,\overline{s}$ in given conjugacy class. Let $\overline{r}$, $\overline{s}$ and $\overline{t}$ be represented by matrices,
$R = \begin{bmatrix} r_1 & kr_3 \\ r_3 & -r_1 \end{bmatrix}$, $S = \begin{bmatrix} s_1 & ks_3 \\ s_3 & -s_1 - \sqrt{2} \end{bmatrix}$ and $T = \begin{bmatrix} 0 & -k \\ 1 & 0 \end{bmatrix}$ respectively, as defined in [4], where r_1, r_3, s_1, s_3, $k \in F_q$. Let $det(R) = \Delta$ and $det(S) = 1$, then

$$det(R) = \Delta = -r_1^2 - kr_3^2 = r_1^2 + kr_3^2 \neq 0 \tag{2}$$

and,

$$\begin{aligned} det(S) = 1 = -s_1^2 - \sqrt{2}s_1 - ks_3^2 \\ s_1^2 + \sqrt{2}s_1 + ks_3^2 + 1 = 0. \end{aligned} \tag{3}$$

This surely, yields such elements that satisfy the relations (1). Now the product of matrices R and S is given by,

$$RS = \begin{bmatrix} r_1 & kr_3 \\ r_e & -r_1 \end{bmatrix} \begin{bmatrix} s_1 & ks_3 \\ s_3 & -s_1 - 1 \end{bmatrix} = \begin{bmatrix} r_1s_1 + kr_3s_3 & kr_1s_3 - kr_3s_1 - \sqrt{2}kr_3 \\ r_3s_1 - r_1s_3 & kr_3s_3 + r_1s_1 + \sqrt{2}r_1 \end{bmatrix}$$

As already supposed that $tr(RS) = m_2$, therefore

$$m_2 \quad = \quad 2r_1s_1 + 2kr_3s_3 + \sqrt{2}r_1. \tag{4}$$

The matrix RST is given by

$$\begin{aligned} RST &= \begin{bmatrix} r_1s_1 + kr_3s_3 & kr_1s_3 - kr_3s_1 - \sqrt{2}kr_3 \\ r_3s_1 - r_1s_3 & kr_3s_3 + r_1s_1 + \sqrt{2}r_1 \end{bmatrix} \begin{bmatrix} 0 & -k \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} kr_1s_3 - kr_3s_1 - \sqrt{2}kr_3 & -k(r_1s_1 + kr_3s_3) \\ kr_3s_3 + r_1s_1 + \sqrt{2}r_1 & -k(r_3s_1 - r_1s_3) \end{bmatrix} \end{aligned}$$

and so the trace of RST is given by

$$tr(RST) = kr_1s_3 - kr_3s_1 - \sqrt{2}kr_3 - k(r_3s_1 - r_1s_3) = 2kr_1s_3 - kr_3\left(2s_1 + \sqrt{2}\right)$$

and as already considered, $m_3k = trace(RST)$ so

$$\begin{aligned} m_3k &= 2kr_1s_3 - kr_3\left(2s_1 + \sqrt{2}\right) \\ m_3 &= 2r_1s_3 - r_3\left(2s_1 + \sqrt{2}\right). \end{aligned} \tag{5}$$

Now squaring equations (4) and (5) we get,

$$\begin{aligned} m_2^2 = [2r_1s_1 + 2kr_3s_3 + \sqrt{2}r_1]^2 = 4r_1^2s_1^2 + 4k^2r_3^2s_3^2 + 2r_1^2 + 8kr_1s_1r_3s_3 \\ +4\sqrt{2}r_1r_3s_3 + 4\sqrt{2}r_1^2s_1 \end{aligned}$$

and

$$m_3^2 = [2r_1s_3 - r_3(2s_1+\sqrt{2})]^2 = 4r_1^2s_3^2 + r_3^2(4s_1^2+2+4\sqrt{2}s_1) - 4r_1r_3s_3(2s_1+\sqrt{2}))$$
$$= 4r_1^2s_3^2 + 4r_3^2s_1^2 + 2r_3^2 + 4\sqrt{2}r_3^2s_1 - 8r_1r_3s_1s_3 - 4\sqrt{2}r_1r_3s_3.$$

Multiplying m_3^2 by k and then adding in m_2^2, we get

$$\begin{aligned} m_2^2+km_3^2 =& 4r_1^2s_1^2 + 4k^2r_3^2s_3^2 + 2r_1^2 + 8kr_1s_1r_3s_3 + 4\sqrt{2}r_1r_3s_3 + 4\sqrt{2}r_1^2s_1 \\ &+ 4kr_1^2s_3^2 + 4kr_3^2s_1^2 + 2kr_3^2 + 4\sqrt{2}kr_3^2s_1 - 8kr_1r_3s_1s_3 - 4\sqrt{2}kr_1r_3s_3 \\ =& 4r_1^2s_1^2+4k^2r_3^2s_3^2 + 2r_1^2+4\sqrt{2}r_1^2s_1 + 4kr_1^2s_3^2+4kr_3^2s_1^2 + 2kr_3^2+4\sqrt{2}kr_3^2s_1 \\ =& 2(r_1^2+kr_3^2) + 4s_1^2(r_1^2+kr_3^2) + 4\sqrt{2}s_1(r_1^2+kr_3^2) + 4ks_3^2(r_1^2+kr_3^2) \\ =& (r_1^2+kr_3^2)(2+4s_1^2+4\sqrt{2}s_1+4ks_3^2) \\ =& [r_1^2+kr_3^2][2+4(s_1^2+\sqrt{2}s_1+ks_3^2)]. \end{aligned}$$

By using equations (3), we obtain

$$m_2^2 + km_3^2 = [r_1^2+kr_3^2][2+4(-1)] = (-\Delta)(-2) = 2\Delta.$$

That is,

$$2\Delta = m_2^2 + km_3^2. \tag{6}$$

We have

$$R^{-1}S^{-1} = \frac{1}{\Delta}\begin{bmatrix} r_1s_1+\sqrt{2}r_1+kr_3s_3 & kr_1s_3-kr_3s_1 \\ r_3s_1+\sqrt{2}r_3-r_1s_3 & kr_3s_1+r_1s_1 \end{bmatrix}.$$

The product $RSR^{-1}S^{-1}$ is

$$\frac{1}{\Delta}\begin{bmatrix} r_1s_1+kr_3s_3 & kr_1s_3-kr_3s_1-\sqrt{2}kr_3 \\ r_3s_1-r_1s_3 & kr_3s_3+r_1s_1+\sqrt{2}r_1 \end{bmatrix}\begin{bmatrix} r_1s_1+\sqrt{2}r_1+kr_3s_3 & kr_1s_3-kr_3s_1 \\ r_3s_1+\sqrt{2}r_3-r_1s_3 & kr_3s_1+r_1s_1 \end{bmatrix}.$$

Now further as considered in previous section $trace(RSR^{-1}S^{-1}) = m_4$, then $m_4 = \frac{1}{\Delta}[\Delta - km_2^2 - r_1^2 - kr_3^2]$ and consequently, $m_4\Delta = \Delta - km_3^2 - r_1^2 - kr_3^2 = \Delta - km_3^2 - (r_1^2+kr_3^2) = \Delta - km_3^2 - (-\Delta) = 2\Delta - km_3^2$, which together with (6) implies $m_2^2 = m_4\Delta$. This together with $m_2^2 = \Delta\theta$ gives $\theta = m_4 \in F_q$. Hence θ is the permutation representation of the action corresponding to the homomorphism α. □

Theorem 2.2. *The transformation $\bar{t}$ has fixed vertices in $D(\theta,q)$ if and only if $\theta(\theta-2)$ is a square in F_q.*

Proof. Let $\alpha\colon \Gamma^* \to G^{*3,4}(2,q)$ be a non-degenerate homomorphism that satisfies the relations $r\alpha = \bar{r}, s\alpha = \bar{s}$ and $t\alpha = \bar{t}$ and α' be its dual. Choose the matrices, $R = \begin{bmatrix} r_1 & kr_3 \\ r_3 & -r_1 \end{bmatrix}$, $S = \begin{bmatrix} s_1 & ks_3 \\ s_3 & -\sqrt{2}-s_1 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & -k \\ 1 & 0 \end{bmatrix}$, representing $\bar{r}, \bar{s}$

and $\bar{t}$ respectively, where r_1, r_3, s_1, s_3, $k \in F_q$ and satisfies the equations (2) to (6). As we know that, $tr(RS) = 0$ if and only if $(\bar{r}\,\bar{s})^2 = 1$. Also, $\frac{tra(RST)}{k} = m_3 = 0$ if and only if $(\bar{r}\,\bar{s}\bar{t})^2 = 1$. Now $det(RS) = 1$, gives parameter of $\bar{r}\,\bar{s}$ as $m_2^2 = \theta$. Also $tr(RST) = km_3$ and $det(RST) = k$ [Since $det(R) = 1$, $det(S) = 1$ and $det(T) = k \Rightarrow det(RST) = k$], gives parameter of $\bar{r}\,\bar{s}\bar{t}$ as km_3^2. Let this parameter be denoted by ϕ. Therefore, $\theta + \phi = \frac{m_2^2+km_3^2}{\Delta}$. Putting values from equation (6), $\theta + \phi = 2$. Hence, $\phi = \theta - 2$.

Since change from α to α' interchanges both $\bar{r}$ and $\bar{r}\bar{t}$ and θ and $\theta - 2$, so $\bar{r}\bar{t}$ maps to an element $\Delta^*(2,4,k)$ if and only if $\theta(\theta - 2)$ is a square in F_q. Since $\bar{t}$ lies in $\Delta^*(2,4,k)$ if both of $\bar{r}$ and $\bar{r}\bar{t}$, so $\bar{t}$ belongs to $G^*(2,4,k)$ if and only if $\theta(\theta - 2)$ is a square in F_q. Now $\bar{t}$ has fixed points in $PL(F_q)$ if either $\bar{t}$ belongs to $\Delta^*(2,4,k)$ and $q \equiv -1(mod4)$ or $\bar{t}$ does not belong to $\Delta^*(2,4,k)$ and $q \equiv 1(mod4)$, which means that -1 is a square in F_q. Hence it can be concluded that $\bar{t}$ has fixed vertices in $D(\theta, q)$ if and only if $-\theta(2-\theta) = \theta(\theta-2)$ is a square in F_q. □

3. Action of $\Delta(2,4,k)$ on $PL(F_q)$ for $\theta = 2$

Following computer coding scheme generate parameterizations and coset diagrams for actions of $\Delta(2,4,k)$ over $PL(F_q)$, wherein q is a prime number $q+2$ gives perfect square.

3.1. Computer program to parameterize action

```
m4 = Input["m4"];
delta = Input["Delta"];
m2sq = delta*m4;
While[! (Element[Sqrt[m2sq], Integers]), m2sq += q];
m2 = Sqrt[m2sq];
m3sq = ((2*delta ) - (m2sq))/k;
While[m3sq < 0, m3sq += q;];
m3 = Sqrt[m3sq];
s3sq = (-1 - s1^2 - (Sqrt[2 + q]*s1))/k;
While[s3sq < 0, s3sq += q;];
While[! (Element[Sqrt[s3sq], Integers]), s3sq += q];
s3 = Sqrt[s3sq];
{c, d} = {a, b} /.
   First@Solve[{2*a* s1 + 2*k*b*s3 + (Sqrt[2 + q])*a == m2,
      2*a*s3 - 2*b*s1 - (Sqrt[2 + q])*b == m3}, {a, b}];
nom = Numerator[c];
denom = Denominator[c];
While[! (Element[nom/denom, Integers]), nom += q];
r1 = nom/denom;
```

```
nom = Numerator[d];
denom = Denominator[d];
While[! (Element[nom/denom, Integers]), nom += q];
r3 = nom/denom;
r11 = r1;
r12 = k*r3;
r13 = r3;
r14 = -r1;
s11 = s1;
s12 = k*s3;
s13 = s3;
s14 = -s1 - (Sqrt[2 + q]);
t2 = -k;
While[t2 < 0, t2 += q];
matrix_X = MatrixForm[{{r11, r12}, {r13, r14}}]
matrix_Y = MatrixForm[{{s11, s12}, {s13, s14}}]
matrix_T = MatrixForm[{{0, t2}, {1, 0}}]
```

3.2. Computer program to draw coset diagrams

Following coding scheme using java programming language to draw coset diagrams with respect to the primes q for the action of $\Delta(2,4,k)$ has been developed. The code given below will generate the permutations for R. Similar code is used for generating the permutations for S and T.

```
List<Integer> tmp=new ArrayList<Integer>();
     int count=R_values.get(0);
     tmp.add(count);
     while(cycle==true)
     {
     int permut_temp=(int) calculateFunc_R(count,a,b,c,d);
     count=permut_temp;
     if(!(tmp.contains(permut_temp))&& tmp.size()<2)
     {
     tmp.add((int) permut_temp);
     }
     else
     {
     Permutation_R.add(tmp);
     cycle=false;
     }
     }
```

Following code separates the fix points from permutation of S.

```
for(List<Integer> innerList : Permutation_S) {
        if(innerList.size()<4)
        {
         fixPointS.add(innerList);
        }
        }
```

The code given below will make the nodes symmetrical basing on the permutations of T.

```
for(List<Integer> innerList : Permutation_T) {
        if(innerList.size()==1)
        {
         fix=(Integer) Permutation_T.get(Permutation_T.indexOf(innerList)).get(0);
         for(List<Integer> innerSList : Permutation_S)
         {
         if(innerSList.contains(fix))
         {
         if(!PermutationS_toDrawCenter.contains(innerSList))
         {
         PermutationS_toDrawCenter.add(innerSList);
         toremove_S.add(innerSList);
         }
         toremove_T.add(innerList);
         }
         }
        }
        }
```

The symmetrical nodes will then be drawn by using the code given below:

```
public Node(Point p,int n_v, int r, Color color, Kind kind,int pos) {
            this.p = p;
            this.r = r;
            this.node_value=n_v;
            this.color = color;
            this.kind = kind;
            this.pos=pos;
            setBoundary(b);
        }
public void draw(Graphics g) {
         int x,y,r=5;
         if(this.pos==0)
         {
         x=b.x;
```

```
        y=b.y-r;
        }
        else if(this.pos==1)
        {x=b.x-r-8;
    y=b.y;}
        else if(this.pos==2)
        {x=b.x;
    y=b.y+r+15;}
        else
        {
        x=b.x+r;
        y=b.y;
        }
           g.setColor(this.color);
           if (this.kind == Kind.Circular) {
               g.fillOval(b.x, b.y, b.width, b.height);
           } else if (this.kind == Kind.Rounded) {
               g.fillRoundRect(b.x, b.y, b.width, b.height, r, r);
           } else if (this.kind == Kind.Square) {
               g.fillRect(b.x, b.y, b.width, b.height);
           }
           g.setColor(Color.BLACK);
           g.setFont(g.getFont().deriveFont(18.0f));
           g.drawString(Integer.toString(this.node_value), x, y);
       }
```

Example 3.1. Consider $q = 7$. Then $m_2^2 = m_4\Delta$. Also, $m_4 = \theta = 2$, $m_2^2 = 2\triangle$. Considering $\triangle = k = s_1 = 1$, and then by using the code given in section 2.3, corresponding matrices R, S, and T thus obtained are:

$R = \begin{bmatrix} 3 & 5 \\ 5 & 4 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 3 \\ 3 & 3 \end{bmatrix}$, $T = \begin{bmatrix} 0 & 6 \\ 1 & 0 \end{bmatrix}$.

Therefore, linear-fractional transformations are,

$r : z \mapsto \dfrac{3z+5}{5z+4}$, $s : z \mapsto \dfrac{z+3}{3z+3}$, $t : z \mapsto \dfrac{6}{z}$.

Applying r, s and t transformations on the elements of $PL(F_7)$, the permutations will be: r act as: $(0\ 3)(1\ 4)(2\ \infty)(5\ 6)$, s act as: $(0\ 1\ 3\ 4)(2\ 6\ \infty\ 5)$, t act as: $(0\ \infty)(1\ 6)(2\ 3)(4\ 5)$.

Obtained coset diagram is as follows.

This diagram is disconnected and consisting of two diagrams each having 4 vertices. Also note that each vertex of these diagrams is fixed by $(rs)^4$ and the group $\Delta(2,4,4) =< r, s : r^2 = s^4 = (rs)^4 = 1 >$. So G is is abelian and cyclic.

Example 3.2. Consider $q = 23$. Then $m_2^2 = m_4\Delta$. Also, $m_4 = \theta = 2$, $m_2^2 = 2\triangle$. Considering $\triangle = k = s_1 = 1$, and then by using the code given in sections 3.1 and 3.2, corresponding matrices R, S, and T thus obtained are:

$R = \begin{bmatrix} 17 & 3 \\ 3 & 6 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 4 \\ 4 & 17 \end{bmatrix}$, $T = \begin{bmatrix} 0 & 22 \\ 1 & 0 \end{bmatrix}$.

Therefore, linear-fractional transformations are $r : z \mapsto \dfrac{17z+3}{3z+6}$, $s : z \mapsto \dfrac{z+4}{4z+17}$, $t : z \mapsto \dfrac{22}{z}$.

Applying r, s and t transformations on the elements of $PL(F_{23})$, the permutations will be,
r act as: $(0\ 21)(1\ 3)(2\ 9)(4\ 14)(5\ 11)(6\ 7)(8\ 16)(10\ 20)(12\ \infty)(13\ 19)(15\ 22)(17\ 18)$
s act as: $(0\ 7\ 12\ 19)(1\ 9\ 15\ 11)(2\ 3\ 5\ 22)(4\ 10\ 18\ 8)(8\ 13)(8\ 20)(10\ 16)(12\ 21)(14\ 18)$
t act as: $(0\ \infty)(1\ 22)(2\ 11)(3\ 15)(4\ 17)(5\ 9)(6\ 19)(7\ 13)(8\ 20)(10\ 16)(12\ 21)(14\ 18)$.

The coset diagram generated by using code in section 2.3 is shown in Figure 2,

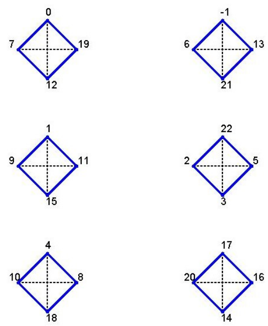

Figure 2: Intransitive action of $\Delta(2,4,k)$ on $PL(F_{23})$

This diagram is disconnected and has six diagrams each consisting of 4 vertices. Also note that each vertex of these diagrams is fixed by $(rs)^4$ and the group

$$\Delta(2,4,4) =< r, s : r^2 = s^4 = (rs)^4 = 1 > .$$

So G is an abelian and cyclic.

In Table 1, we have listed few primes and the number of diagrams corresponding to each prime. Here it can be observed that for each prime q, the coset diagram is disconnected. So the action of $\Delta(2,4,k)$ is intransitive on $PL(F_q)$.

Table 1: Number of disconnected diagrams

Primes	Diagrams of 4 Vertices
7	2
23	6
47	12
79	20

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On regularities in *po*-ternary semigroups

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Abstract. In this paper, we show the way to get into some results of partially ordered (in short, *po*-) ternary semigroup based on quasi-ideals, bi-ideals and semiprime ideals. We extend some results of *po*-semigroup into *po*-ternary semigroup under certain methodology. In particular, we characterize some properties of regular *po*-ternary semigroup, left (resp. right) regular *po*-ternary semigroup, completely regular *po*-ternary semigroup and intra-regular *po*-ternary semigroup by using quasi-ideal, bi-ideal and semiprime ideal of *po*-ternary semigroup.

1. Introduction

The ideal theory of ternary semigroup was introduced and studied by Sioson in [12]. Dixit and Dewan [2] studied the notion of quasi-ideals and bi-ideals in ternary semigroup. Later on Santiago, Sri Bala [11] developed the theory of ternary semigroup and semiheaps. Further Kar and Maity developed the ideal theory on ternary semigroup in [6]. Some properties of regular ternary semigroup were discussed by Dutta, Kar and Maity in [4]. Ternary semigroups were studied by many authors, semiheaps (and similar) by V. Vagner [13], W.A. Dudek [3], A. Knorbel [9] and many others.

Kehayapulu ([7], [8]) introduced and studied the notion of completely regular ordered semigroup. In 2012, Daddi and Power [1] studied the concept of ordered quasi-ideals and ordered bi-ideals in ordered ternary semigroup and also discussed about their properties. The result on the minimality and maximality theory of ordered quasi-ideal in odered ternary semigroup was developed by Jailoka and Iampan in [5].

In this paper, we study the notion of regular ordered ternary semigroups. We also introduce the notion of completely regular and intra-regular ordered ternary semigroups. Finally we characterize these classes of ordered ternary semigroups in terms of ideals, quasi-ideals, bi-ideals, semiprime ideals of ternary semigroup.

2. Preliminaries and Prerequisites

Here we provide some definitions and relevant results of *po*-ternary semigroup which will be required to develop our paper.

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A ternary semigroup S is called a *partially ordered ternary semigroup* (*po-ternary semigroup*) if there is a partial order " $\leq$ " on S such that for $x, y \in S$; $x \leqslant y \Longrightarrow xx_1x_2 \leqslant yx_1x_2,\ x_1xx_2 \leqslant x_1yx_2,\ x_1x_2x \leqslant x_1x_2y$ for all $x_1, x_2 \in S$.

For a *po*-ternary semigroup $(S, \cdot, \leqslant)$ and a subset H of S, we define

$$(H] := \{t \in S \mid t \leqslant h \text{ for some } h \in H\}.$$

A nonempty subset A of S is called a *left ideal* of S if (i) $SSA \subseteq A$ and (ii) $(A] = A$, a *right ideal* of S if (i) $ASS \subseteq A$ and (ii) $(A] = A$ and a *lateral ideal* of S if (i) $SAS \subseteq A$ and (ii) $(A] = A$. A nonempty subset A of S is called an *ideal* of S if it is a left ideal, right ideal and lateral ideal of S.

For a *po*-ternary semigroup S and $a \in S$, we denote by $R(a)$ (resp. $L(a)$, $M(a)$) the right (resp. left, lateral) ideal of S generated by a and $I(a)$ the ideal generated by a.

It can be easily proved that for an element a of S the right (resp. left, lateral) ideal and the ideal $I(a)$ of S generated by a have the form

$R(a) = (a \cup aSS], \quad L(a) = (a \cup SSa], \quad M(a) = (a \cup SaS \cup SSaSS],$

$I(a) = (a \cup SSa \cup SaS \cup SSaSS \cup aSS] = (a \cup S^2a \cup SaS \cup S^2aS^2 \cup aS^2].$

If A, B, C are subsets of a *po*-ternary semigroup $(S, \cdot, \leqslant)$, then (cf. [5])

(1) $A \subseteq (A]$.

(2) If $A \subseteq B$ then $(A] \subseteq (B]$.

(3) $((A]] = (A]$.

(4) $(A](B](C] \subseteq (ABC]$.

(5) $((A](B](C]] = ((A](B]C] = (AB(C]] = (ABC]$.

(6) $(A \cup B] = (A] \cup (B]$.

(7) $(A \cap B] \subseteq (A] \cap (B]$.

In particular, if A and B are some types of ideals in S, then $(A \cap B] = (A] \cap (B]$.

(8) $(SSA]$, $(ASS]$, $(SAS \cup SSASS]$ are left, right and lateral ideal in S.

A nonempty subset Q of S is called a *quasi-ideal* of S, if (i) $(SSQ] \cap (SQS] \cap (QSS] \subseteq Q$, (ii) $(SSQ] \cap (SSQSS] \cap (QSS] \subseteq Q$ and (iii) $(Q] = Q$.

Every left, right and lateral ideal of a *po*-ternary semigroup S is a quasi-ideal.

A subsemigroup B of S is called a *bi-ideal* of S, if (i) $BSBSB \subseteq B$ and (ii) $(B] = B$.

Every quasi-ideal is a bi-ideal. Since every left, right and lateral ideal is a quasi-ideal, it follows that every left, right and lateral ideal is a bi-ideal.

A proper ideal T of a *po*-ternary semigroup S is called *semiprime* if for any ideal A of S with $A^3 \subseteq T$, we have $A \subseteq T$.

3. Regular *po*-ternary semigroups

A *po*-ternary semigroup S is said to be *regular* (*left, right regular*) if $A \subseteq (ASA]$ (respectively, $A \subseteq (SA^2]$, $A \subseteq (A^2S]$) for every $A \subseteq S$.

Lemma 3.1. *A lateral ideal of a regular po-ternary semigroup is regular.*

Proof. Let I be a lateral ideal of a regular *po*-ternary semigroup S. Let $A \subseteq I$. Since S is regular, $A \subseteq (ASA]$. Now $A \subseteq (ASA] \subseteq (AS(ASA]] = (ASASA] = (A(SAS)A] \subseteq (A(SIS)A] \subseteq (AIA]$. Consequently, I is regular. $\square$

Corollary 3.2. *In a regular po-ternary semigroup S, every ideal is regular.*

Theorem 3.3. (cf. [10]) *In a po-ternary semigroup S, the following are equivalent:*

(*i*) *S is regular,*

(*ii*) *$(RML] = R \cap M \cap L$ where R, M, L are right ideal, lateral ideal and left ideal of S respectively,*

(*iii*) *for every bi-ideal B of S, $(BSBSB] = B$,*

(*iv*) *for every quasi-ideal Q of S, $(QSQSQ] = Q$.*

Theorem 3.4. *A po-ternary subsemigroup B of a regular po-ternary semigroup S is a bi-ideal of S if and only if $B = (BSB]$.*

Proof. Let S be a regular *po*-ternary semigroup and $B \subseteq S$. Let $B = (BSB]$. Then $B = (BSB] = (BS(BSB]] = (BS(BSB)] = (BSBSB]$. Thus $BSBSB \subseteq (BSBSB] = B$. It remains to show that $(B] = B$. Let $x \in (B]$. Then $x \in ((BSB]] = (BSB] = B$. Thus $(B] \subseteq B$. Hence B is a bi-ideal of S.

Conversely, let B be any bi-ideal of a regular *po*-ternary semigroup S. Since S is regular and $B \subseteq S$ we have $B \subseteq (BSB]$. Again $(BSB] \subseteq (BS(BSB]] = (BS(BSB)] = (BSBSB] \subseteq (B] = B$. Thus $B = (BSB]$. $\square$

Theorem 3.5. *In a regular po-ternary semigroup S, every bi-ideal of S is a quasi-ideal of S.*

Proof. Let B be a bi-ideal of a regular *po*-ternary semigroup S. Then $BSBSB \subseteq B$ and $(B] = B$. Now $S^2(S^2B] \subseteq (S](S](SSB] \subseteq (S^4B] \subseteq (SSB]$ and $((SSB]] = (SSB]$. Hence $(SSB]$ is a left ideal of S. Also $(BS^2]S^2 \subseteq (BS^2](S](S] \subseteq (BS^4] \subseteq (BS^2]$ and $((BS^2]] = (BS^2]$. Thus $(BSS]$ is a right ideal of S. Again $S(SBS \cup S^2BS^2]S \subseteq (S](SBS \cup S^2BS^2](S] \subseteq (S^2BS^2 \cup S^3BS^3] \subseteq (S^2BS^2 \cup SBS]$ and $((SBS \cup S^2BS^2]] = (SBS \cup S^2BS^2]$. So $(SBS \cup S^2BS^2]$ is a lateral ideal of S. From Theorem 3.3, we have $(BS^2] \cap (SBS \cup S^2BS^2] \cap (S^2B] = ((BS^2](SBS \cup S^2BS^2](S^2B]] = ((BS^2)(SBS \cup S^2BS^2)(S^2B)] = (BS^3BS^3B \cup BS^4BS^4B] \subseteq (BSBSB \cup BS^2BS^2B] \subseteq (BSBSB \cup BSB] = (BSBSB] \cup (BSB] = B \cup B = B$, by using Theorem 3.3 and Theorem 3.4. Consequently, B is a quasi-ideal of S. $\square$

Theorem 3.6. *Let S be a po-ternary semigroup. Then S is left* (*resp. right*) *regular if and only if every left* (*resp. right*) *ideal of S is semiprime.*

Proof. Let S be a left regular *po*-ternary semigroup and L be a left ideal of S. Let $A^3 \subseteq L$ for some left ideal A of S. Since S is left regular, we have $A \subseteq (SA^2] \subseteq (S(SA^2]A] = (S(SA^2)A] = (SSA^3] \subseteq (SSL] \subseteq (L] = L$. Thus L is semiprime.

Conversely, suppose that every left ideal of S is semiprime. Let $A \subseteq S$. Then $SS(SAA] \subseteq (S](S](SAA] \subseteq (S^3AA] \subseteq (SAA]$ and $((SAA]] = (SAA]$. Therefore,

$(SAA]$ is a left ideal of S. Now $A^3 \subseteq SA^2 \subseteq (SA^2]$. Since every left ideal of S is semiprime, we have $A \subseteq (SA^2]$. Thus S is a left regular *po*-ternary semigroup.

Similarly, we can also prove the same for right ideal of S. □

Theorem 3.7. *Let S be a commutative po-ternary semigroup. Then S is regular if and only if every ideal of S is semiprime.*

Proof. Let S be a commutative regular *po*-ternary semigroup and I be any ideal of S. Let $A^3 \subseteq I$ for $A \subset S$. Since S is regular and $A \subseteq S$ we have $A \subseteq (ASA] = (AAS] \subseteq (A(ASA]S] = (A(ASA)S] = (A(A^2S)S] = (A^3SS] \subseteq (ISS] \subseteq (I] = I$. Thus I is a semiprime ideal of S.

Conversely, we assume that every ideal of commutative *po*-ternary semigroup S is semiprime. Let $A \subseteq S$. Then $(ASA]$ is an ideal of S. If $(ASA] = (S] = S$, we get our conclusion. If $(ASA] \neq S$, then by hypothesis, $(ASA]$ is a semiprime ideal of S. Now $A^3 \subseteq ASA \subseteq (ASA]$ implies that $A \subseteq (ASA]$. Consequently, S is regular. □

Definition 3.8. Let S be a *po*-ternary semigroup. A nonempty subset B_w of S is called a *weak bi-ideal* of S, if (i) $bSbSb \subseteq B_w$ for all $b \in B_w$ and (ii) $(B_w] = B_w$.

Clearly, we have the following results:

Lemma 3.9. *Every bi-ideal of a po-ternary semigroup S is a weak bi-ideal of S.*

Lemma 3.10. *The intersection of arbitrary set of weak bi-ideals of a po-ternary semigroup S is either empty or a weak bi-ideal of S.*

Theorem 3.11. *Let S be a po-ternary semigroup. Then S is regular if and only if $B_w = (\bigcup_{b \in B_w} bSbSb]$ for any weak bi-ideal B_w of S.*

Proof. Let S be a regular *po*-ternary semigroup and B_w be any weak bi-ideal of S. Then $bSbSb \subseteq B_w$ for all $b \in B_w$. So $\bigcup_{b \in B_w} bSbSb \subseteq B_w$. This implies that $(\bigcup_{b \in B_w} bSbSb] \subseteq (B_w] = B_w$. Let $b \in B_w$. Since S is regular, there exists $x \in S$ such that $b \leqslant bxb$. So $b \leqslant bxb \leqslant bxbxb \in bSbSb \subseteq \bigcup_{b \in B_w} bSbSb$. Therefore, $b \in (\bigcup_{b \in B_w} bSbSb]$. Thus $B_w \subseteq (\bigcup_{b \in B_w} bSbSb]$. Hence $B_w = (\bigcup_{b \in B_w} bSbSb]$.

Conversely, let $B_w = (\bigcup_{b \in B_w} bSbSb]$, where B_w is a weak bi-ideal of S. Let R be a right ideal, M be a lateral ideal and L be a left ideal of S. Since every left, right and lateral ideal of a *po*-ternary semigroup S is a bi-ideal of S, it follows that every left, right and lateral ideal of a *po*-ternary semigroup S is a weak bi-ideal of S. So R, M, L are weak bi-ideals of S. Thus by Lemma 3.10, $R \cap M \cap L$ is a weak bi-ideal of S. Clearly, $(RML] \subseteq R \cap M \cap L$. Now let $a \in R \cap M \cap L$. Since $R \cap M \cap L$ is weak bi-ideal of S, by hypothesis we have $R \cap M \cap L = (\bigcup_{x \in R \cap M \cap L} xSxSx]$. Then $a \leqslant xs_1xs_2x$ for some $x \in R \cap M \cap L$ and $s_1, s_2 \in S$. So $a \leqslant xs_1xs_2ys_3ys_4y$ for some $x, y \in R \cap M \cap L$ and $s_1, s_2, s_3, s_4 \in S$. This implies that $a \in (RML]$. Thus $R \cap M \cap L \subseteq (RML]$ and hence $(RML] = R \cap M \cap L$. Consequently, S is a regular *po*-ternary semigroup by Theorem 3.3. □

4. Completely regular *po*-ternary semigroups

In this section, we characterize completely regular *po*-ternary semigroup by using quasi-ideals, bi-ideals and semiprime ideals.

Definition 4.1. A *po*-ternary semigroup S is said to be *completely regular* if it is regular, left regular and right regular i.e., $A \subseteq (ASA]$, $A \subseteq (SA^2]$ and $A \subseteq (A^2S]$ for every $A \subseteq S$.

Example 4.2. Let $S = \{a, b, c, d, e\}$ be a *po*-ternary semigroup with the ternary operation defined on S as $abc = a * (b * c)$, where the binary operation * is defined by

*	a	b	c	d	e
a	a	a	c	d	a
b	a	b	c	d	a
c	a	a	c	d	a
d	a	a	c	d	a
e	a	a	c	d	e

and the order defined as

$$\leqslant = \{(a,a),(a,c),(a,d),(b,b),(b,d),(b,a),(b,c),(c,c),(c,d),(d,d),(e,a),(e,c),(e,d),(e,e)\}.$$

Then S is a completely regular *po*-ternary semigroup.

Theorem 4.3. *In a po-ternary semigroup S, the following conditions are equivalent:*

(i) *S is completely regular;*

(ii) *$A \subseteq (A^2SA^2]$ for every $A \subseteq S$.*

Proof. $(i) \Rightarrow (ii)$. Then for any $A \subseteq S$, we have $A \subseteq (ASA] \subseteq ((A^2S]S(SA^2]] = ((A^2S)S(SA^2)] = (A^2S^3A^2] \subseteq (A^2SA^2]$.

$(ii) \Rightarrow (i)$. Let $A \subseteq S$. Then $A \subseteq (A^2SA^2] = (A(ASA)A] \subseteq (ASA]$, $A \subseteq (A^2SA^2] = ((A^2S)A^2] \subseteq (SA^2]$ and $A \subseteq (A^2SA^2] = (A^2(SA^2)] \subseteq (A^2S]$. This implies that S is regular, left regular and right regular. Consequently, S is completely regular. □

In the following result we provide another characterization of completely regular *po*-ternary semigroup in terms of quasi-ideal.

Theorem 4.4. *Let S be a po-ternary semigroup. Then S is completely regular if and only if every quasi-ideal of S is a completely regular subsemigroup of S.*

Proof. Let S be a completely regular *po*-ternary semigroup and Q be a quasi-ideal in S. Since $\phi \neq Q \subseteq S$ and $Q^3 \subseteq QSS \cap SQS \cap SSQ \subseteq (QSS] \cap (SQS] \cap (SSQ] \subseteq Q$, Q is a subsemigroup of S. Let $A \subseteq Q \subseteq S$. We have to show that Q is completely regular. Since S is completely regular and $A \subseteq S$, we have $A \subseteq (ASA] \subseteq ((A^2S]S(SA^2]] = ((A^2S)S(SA^2)] = (A^2SSSA^2] \subseteq (A^2SA^2] = (A(ASA)A] \subseteq (A(ASA]SAA] = (A(ASA)SAA] = (A(ASASA)A]$. Now $ASASA \subseteq SSASS \subseteq SSQSS$, $ASASA \subseteq SSA \subseteq SSQ$ and $ASASA \subseteq ASS \subseteq QSS$. Therefore, $ASASA \subseteq SSQ \cap SSQSS \cap QSS \subseteq (SSQ] \cap (SSQSS] \cap (QSS] \subseteq Q$. Hence $A \subseteq (AQA]$. Again $A \subseteq (ASA] \subseteq (AS(SA^2]] = (AS(SA^2)] \subseteq (ASS(SA^2]A] = (AS^2(SA^2)A] = ((AS^3A)A^2] \subseteq ((ASA)A^2] \subseteq (AS(ASA]A^2] = (AS(ASA)A^2] = ((ASASA)A^2] \subseteq (QA^2]$ and $A \subseteq (ASA] \subseteq ((A^2S]SA] = ((A^2S)SA] \subseteq (A(A^2S]SSA] = (A(A^2S)SSA] = (A^2(ASSSA)] \subseteq (A^2(ASA)] \subseteq (A^2(ASA]SA] = (A^2(ASA)SA] = (A^2(ASASA)] \subseteq (A^2Q]$. Thus Q is regular, left regular and right regular. Consequently, Q is completely regular subsemigroup.

Conversely, suppose that every quasi-ideal of S is a completely regular subsemigroup of S. Since S itself a quasi-ideal in S, S is completely regular. □

Theorem 4.5. *Let S be a po-ternary semigroup. Then S is left regular and right regular if and only if every quasi-ideal of S is semiprime.*

Proof. Let S be a left regular and right regular *po*-ternary semigroup and Q be a quasi-ideal of S. Let $A \subseteq S$ and $A^3 \subseteq Q$. Since S is left regular and right regular, $A \subseteq (SA^2]$ and $A \subseteq (A^2S]$. Now $A \subseteq (SA^2] \subseteq (S(SA^2]A] = (S(SA^2)A] = (SSA^3] \subseteq (SSQ]$, $A \subseteq (A^2S] \subseteq (A(A^2S]S] = (A(A^2S)S] = (A^3SS] \subseteq (QSS]$ and $A \subseteq (SA^2] \subseteq (SA(A^2S]] = (SA^3S] \subseteq (SQS]$. Therefore, $A \subseteq (SSQ] \cap (SQS] \cap (QSS] \subseteq Q$. Hence Q is semiprime.

Conversely, suppose that every quasi-ideal of S is semiprime. Since every right ideal and left ideal of S is a quasi-ideal of S, every right ideal and left ideal are semiprime. Now by using Theorem 3.6, we find that S is left regular and right regular. □

Corollary 4.6. *If S is a completely regular po-ternary semigroup then quasi-ideals of S are semiprime.*

The converse of the above result does not hold.

Example 4.7. Let $S = \{a, b, c, d, e\}$ be a *po*-ternary semigroup with ternary operation product defined on S by $abc = a * (b * c)$, where binary operation * is defined as

*	a	b	c	d	e
a	a	e	e	a	e
b	d	b	b	d	b
c	d	b	b	d	b
d	d	b	b	d	b
e	a	e	e	a	e

and the order defined by

$$\leqslant\ := \{(a,a),(b,a),(b,b),(b,d),(b,e),(c,a),(c,c),(c,d),(c,e),(d,d),(d,a),(e,a),(e,e)\}.$$

Then S is a left regular and right regular *po*-ternary semigroup. So every quasi-ideal of S is semiprime by Theorem 4.5 but S is not completely regular. In fact it is not regular since $c \in S$ is not regular.

Theorem 4.8. *A po-ternary semigroup S is completely regular if and only if every bi-ideal of S is semiprime.*

Proof. Let S be a completely regular *po*-ternary semigroup and B be any bi-ideal of S. Let $A \subseteq S$ and $A^3 \subseteq B$. Since S is completely regular *po*-ternary semigroup and $A \subseteq S$ we have $A \subseteq (A^2SA^2] \subseteq (A(A^2SA^2]S(A^2SA^2]A] = (A(A^2SA^2)S(A^2SA^2)A] = ((A^3SA^2S)(A^2S)A^3] \subseteq ((A^3SA^2S)(A^2SA^2](A^2SA^2]SA^3] = ((A^3SA^2S)(A^2SA^2)(A^2SA^2)SA^3] = (A^3(SA^2SA^2S)A^3(ASA^2S)A^3] \subseteq (BSBSB] \subseteq (B] = B$. Therefore B is semiprime.

Conversely, suppose that every bi-ideal of S is semiprime. Let $\phi \neq A \subseteq S$. Then we have $A^2SA^2 \subseteq S$ i.e. $(A^2SA^2] \subseteq S$. Now $(A^2SA^2]S(A^2SA^2]S(A^2SA^2] \subseteq (A^2SA^2](S](A^2SA^2](S](A^2SA^2] \subseteq (A^2SA^2SA^2SA^2SA^2SA^2] \subseteq (A^2SA^2]$ and also $((A^2SA^2]] = (A^2SA^2]$. Thus $(A^2SA^2]$ is a bi-ideal in S. Now $A^9 = A^2(A^5)A^2 \subseteq A^2SA^2 \subseteq (A^2SA^2]$. By hypothesis, since every bi-ideal is semiprime, $A^9 = (A^3)^3 \subseteq (A^2SA^2] \Longrightarrow A^3 \subseteq (A^2SA^2] \Longrightarrow A \subseteq (A^2SA^2]$. Since A is arbitrary, $A \subseteq (A^2SA^2]$ for every $A \subseteq S$. Hence S is completely regular. $\square$

5. Intra-regular *po*-ternary semigroups

In this section, we characterize intra-regular *po*-ternary semigroup by using properties of ideals.

Definition 5.1. A *po*-ternary semigroup S is called intra-regular if for every $a \in S$, there exists $x, y \in S$ such that $a \leqslant xa^3y$ or equivalently, $a \in (Sa^3S]$ for all $a \in S$.

In other words, a *po*-ternary semigroup S is intra-regular if $A \subseteq (SA^3S]$ for every $A \subseteq S$.

Lemma 5.2. *If S is a left (resp. right) regular po-ternary semigroup, then S is intra-regular.*

Proof. Let S be left regular *po*-ternary semigroup and $A \subseteq S$. Then $A \subseteq (SA^2] \subseteq (S(SA^2]A] = (S(SA^2)A] \subseteq (SS(SA^2]AA] = (SS(SA^2)AA] = (SSSA^3A] \subseteq (SSSA^3S] \subseteq (SA^3S]$. Thus S is intra-regular.

Similarly, we can prove the result for right regular *po*-ternary semigroup. □

But the converse of the above result is not true.

Example 5.3. Let $S = \{a, b, c, d, e\}$ be a *po*-ternary semigroup with ternary operation defined on S by $abc = a * (b * c)$, where the binary operation * is defined as

*	a	b	c	d	e
a	a	b	a	d	a
b	a	b	a	d	a
c	a	b	a	d	a
d	a	b	a	d	a
e	a	b	a	d	a

and the order defined by

$$\leqslant \ := \{(a, a), (a, b), (a, c), (a, e), (b, b), (c, c), (c, b), (c, e), (d, d), (e, b), (e, e)\}.$$

Then $(S, \cdot, \leqslant)$ is an intra-regular *po*-ternary semigroup but not left regular, since c and e are not left regular elements of S.

Now we can easily prove the following fact:

Theorem 5.4. *In an intra-regular po-ternary semigroup S, $L \cap M \cap R \subseteq (LMR]$, where L, M, R are left ideal, lateral ideal and right ideal of S respectively.*

Clearly, every ideal of a *po*-ternary semigroup S is also a lateral ideal of S. Certainly a lateral ideal of S is not necessarily an ideal of S. But in particular, for intra-regular *po*-ternary semigroup S we have the following result:

Theorem 5.5. *Let S be an intra-regular po-ternary semigroup. Then a non-empty subset I of S is an ideal of S if and only if I is a lateral ideal of S.*

Proof. Clearly, if I is an ideal of S, then I is a lateral ideal of S.

Conversely, assume that I is a lateral ideal of an intra-regular *po*-ternary semigroup S. Then $SIS \subseteq I$ and $(I] = I$. Since S is intra-regular and $I \subseteq S$ we have $I \subseteq (SI^3S]$. Now $SSI \subseteq (SSI] \subseteq (SS(SI^3S]] = (SS(SI^3S)] = (S^3I^3S] \subseteq (S^3(SI^3S]I^2S] = (S^3(SI^3S)I^2S] = ((S^4I)I(ISIIS)] \subseteq (SIS] \subseteq (I] = I$ and $ISS \subseteq (ISS] \subseteq ((SI^3S]SS] = ((SI^3S)SS] = (SI^3S^3] \subseteq (SI^2(SI^3S]S^3] = (SI^2(SI^3S)S^3] = ((SIISI)I(IS^4] \subseteq (SIS] \subseteq (I] = I$. Thus I is a left ideal as well as a right ideal of S. Consequently, I is an ideal of S. □

Lemma 5.6. *Let S be an intra-regular po-ternary semigroup and I be a lateral ideal of S. Then I is an intra-regular po-ternary semigroup.*

Proof. Let S be an intra-regular *po*-ternary semigroup and I be a lateral ideal of S. Let $A \subseteq I \subseteq S$. Since S is intra-regular, it follows that $A \subseteq (SA^3S]$. Now we have $A \subseteq (SA^3S] \subseteq (S(SA^3S](SA^3S](SA^3S]S] = (S(SA^3S)(SA^3S)(SA^3S)S] = ((SSA^3S^2)A^3(S^2A^3S^2)] \subseteq ((S^3AS^3)A^3(S^3AS^3)] \subseteq ((SAS)A^3(SAS)] \subseteq ((SIS)A^3(SIS)] \subseteq (IA^3I]$. Consequently, I is intra-regular. □

Corollary 5.7. *Let S be an intra-regular po-ternary semigroup and I be an ideal of S. Then I is an intra-regular po-ternary semigroup.*

Theorem 5.8. *Let S be an intra-regular po-ternary semigroup. Let I be an ideal of S and J be an ideal of I. Then J is an ideal of the entire po-ternary semigroup S.*

Proof. It is sufficient to show that J is a lateral ideal of S. Now $J \subseteq I \subseteq S$ and $SJS \subseteq SIS \subseteq I$. We have to show that $SJS \subseteq J$. From Corollary 5.7, it follows that I is an intra-regular *po*-ternary semigroup. Also $SJS \subseteq I$. So we have $(SJS) \subseteq (I(SJS)^3I] = (I(SJS)(SJS)(SJS)I] = ((ISJSS)J(SSJSI)] \subseteq ((ISISS)J(SSISI)] \subseteq ((IIS)J(SII)] \subseteq ((ISS)J(SSI)] \subseteq (IJI] \subseteq (J] = J$. Consequently, J is a lateral ideal of S. □

Theorem 5.9. *Let S be a po-ternary semigroup. Then S is intra-regular if and only if every ideal of S is semiprime.*

Proof. Let S be an intra-regular *po*-ternary semigroup and I be an ideal of S. Let $A^3 \subseteq I$ for $A \subseteq S$. Since S is intra-regular *po*-ternary semigroup, we have $A \subseteq (SA^3S] \subseteq (SIS] \subseteq (I] = I$. Hence I is a semiprime ideal of S.

Conversely, suppose that every ideal of S is semiprime. Let $A \subseteq S$. Since $A^3 \subseteq I(A^3)$, where $I(A^3)$ is the ideal generated by A^3 and by hypothesis $I(A^3)$ is a semiprime ideal of S, so $A \subseteq I(A^3)$.

Now $I(A^3) = (A^3 \cup SSA^3 \cup SA^3S \cup SSA^3SS \cup A^3SS] = (A^3] \cup (SSA^3] \cup (SA^3S] \cup (SSA^3SS] \cup (A^3SS]$.

1) If $A \subseteq (A^3]$. Then $A \subseteq (A(A^3]A] = (A(A^3)A] \subseteq (SA^3S]$.

2) If $A \subseteq (S^2A^3]$ then $A^3 \subseteq (S^2A^3]A^2$. Hence $A \subseteq (S^2(S^2A^3]A^2] = (S^2(S^2A^3)A^2] = (S^4A^5] \subseteq (S^5A^3S] \subseteq (SA^3S]$.

3) If $A \subseteq (SA^3S]$ we get our conclusion.

4) If $A \subseteq (SSA^3SS]$, then $A^3 \subseteq A(S^2A^3S^2]A$. Hence $A \subseteq (S^2A(S^2A^3S^2]AS^2] = (S^2A(S^2A^3S^2)AS^2] = (S^2AS^2A^3S^2AS^2] \subseteq (S^5A^3S^5] \subseteq (SA^3S]$.

5) If $A \subseteq (A^3SS]$, then $A^3 \subseteq A^2(A^3SS]$. Hence $A \subseteq (A^2(A^3SS]SS] = (A^2(A^3SS)SS] = (A^5S^4] \subseteq (SA^3S^5] = (SA^3S^5] \subseteq (SA^3S]$.

In each case, S is intra-regular. Consequently, S is an intra-regular. □

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Table of marks and markaracter table of certain finite groups

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Abstract. Let G be a finite group and $C(G)$ be a family of representatives of the conjugacy classes of subgroups in G. The table of marks of G is a matrix $TM(G) = (a_{HK})$, where $H, K \in C(G)$ and a_{HK} is the number of fixed points of the right cosets of H in G under the action of K. The markaracter table of G is a matrix obtained from the table of marks of G by selecting rows and columns corresponding to cyclic subgroups of G. In this paper, the table of marks and markaracter table of some classes of finite groups are computed.

1. Introduction

Throughout this paper all groups and sets are assumed to be finite. Our calculations are done with the aid of Gap [10] and we refer to the books [5, 6] for notions and notations not presented here.

Suppose G is a finite group containing subgroups H and K. Define $C(H)$ to be the set of all conjugates of H in G and $\mathcal{K}(G) = \{C(H_1), C(H_2), \ldots, C(H_s)\}$ to be a complete set of representatives of the conjugacy classes of subgroups in G. The right cosets of H in H is denoted by $G\backslash H$. It is well-known that the action of G on $G\backslash H$ is transitive and all transitive actions have such a form up to isomorphism. The mark $\beta_H(K) = \beta_{G\backslash H}(K)$ is defined as $|Fix_{G\backslash H}(K)| = |\{Hx \in G\backslash H \mid Hxk = Hx, \forall k \in K\}$. The table of marks of G, Table 1, is the square matrix $MT(G) = (\beta_{G\backslash G_i}(G_j))$, where $G_i, G_j \in \mathcal{X}$. The table $MT(G)$ was introduced in the second edition of the famous book of W. Burnside [2]. We refer the interested reader to consult an old but interesting paper by Pfeiffer [7] for more information on this topic.

The markaracter table of a finite group was introduced by a Japanese chemist Shinsaku Fujita in the context of stereochemistry and enumeration of molecules [3]. This table can be obtained from the table of marks by removing all rows and columns corresponding to non-cyclic subgroups. The markaracter table of dihedral, generalized quaternion and groups of order pqr, p, q, r are distinct primes, were computed in some earlier paper [1, 4, 8]. The aim of this paper is to continue these works by computing the table of marks and markaracter table of certain classes of groups.

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Table 1. The table of marks of group G

$*$	$C(H_1)$	$C(H_2)$	$\cdots$	$C(H_s)$
G/H_1	$\beta_{H_1}(K_1)$	$\beta_{H_1}(K_2)$	$\cdots$	$\beta_{H_1}(K_s)$
G/H_2	$\beta_{H_2}(K_1)$	$\beta_{H_2}(K_2)$	$\cdots$	$\beta_{H_2}(K_s)$
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$
G/H_s	$\beta_{H_s}(K_1)$	$\beta_{H_s}(K_2)$	$\cdots$	$\beta_{H_s}(K_s)$

where $K_i \in C(H_i)$ for all i.

2. Main Results

The aim of this section is to calculate the table of marks and markaracter table of the dicyclic group T_{4n}, the semi-dihedral group SD_{2^n}, and the group $H(n)$ that will be defined later. For the sake of completeness we mention here a known result about table of marks. The interested readers can be consulted an interesting paper of G. Pfeiffer [7].

Theorem 2.1. *Let G be a finite group, $\mathcal{K}(G) = \{C(H_1), C(H_2), \cdots, C(H_s)\}$ and $MT(G) = (m_{ij})$ in which $|K_i| \leqslant |K_j|$, when $K_i \in C(H_i), K_j \in C(H_j)$ and $i \leqslant j$. Then,*

1. *The matrix $M(G)$ is a lower triangular matrix,*
2. *m_{ij} divides m_{i1}, for all $1 \leqslant i, j \leqslant r$,*
3. *$m_{i1} = [G : H_i]$, for all $1 \leqslant i \leqslant s$,*
4. *$m_{ii} = [N_G(H_i) : H_i]$,*
5. *If $H_i \trianglelefteq G$, then $m_{ij} = m_{i1}$ whenever $K_j \nleq H_i$ and zero otherwise.*

2.1. Dicyclic group T_{4n}

The dicyclic group T_{4n} can be presented as $T_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1}\rangle$. The present authors [9], obtained the structure and the number of all subgroups of the dicyclic group T_{4n}. Based on the given information on subgroup lattice of dicyclic group, we know that it has two types of subgroups. The first type is cyclic subgroups of $\langle a\rangle$ and the second type is a subgroup H of index $2^l d$ conjugate to $C_{\frac{m}{d}} : Q_{\frac{2^{r+2}}{2^l}}$, where $n = 2^r m$. It is clear that $H = \langle a^n, a^j b\rangle$, $1 \leqslant j \leqslant n$, is a cyclic subgroup of order four. Thus, we have $\tau(2n)$ subgroups of the first type and the second type subgroups can be partitioned into two parts. The first part are subgroups in the form of $\langle a^d, a^j b\rangle$, where d is odd. These subgroups are all conjugate. If d is even then all subgroups in the form $\langle a^d, a^j b\rangle$, $2 \mid j$, are in a conjugacy class of subgroups and all subgroups in the form $\langle a^d, a^j b\rangle$, $2 \nmid j$, are in another conjugacy classes of subgroups. In Table 2 the table of marks are computed in two different cases that n is a prime number greater than or equal to five or $n = 3$.

Table 2. Table of marks when $n = p$ is odd prime.

$n = 3$	e	$\langle x^3\rangle$	$\langle x^2\rangle$	$\langle x^3, ab\rangle$	$\langle x\rangle$	G
G/e	12	0	0	0	0	0
$G/\langle x^3\rangle$	6	6	0	0	0	0
$G/\langle x^2\rangle$	4	0	4	0	0	0
$G/\langle x^3, ab\rangle$	3	3	0	1	0	0
$G/\langle x\rangle$	2	2	2	0	2	0
e	1	1	1	1	1	1

$n \geqslant 5$	e	$\langle x^p\rangle$	$\langle x^p, ab\rangle$	$\langle x^2\rangle$	$\langle x\rangle$	G
G/e	$4p$	0	0	0	0	0
$G/\langle x^p\rangle$	$2p$	$2p$	0	0	0	0
$G/\langle x^p, ab\rangle$	p	p	1	0	0	0
$G/\langle x^2\rangle$	4	0	0	4	0	0
$G/\langle x\rangle$	2	2	0	2	2	0
e	1	1	1	1	1	1

From calculations given in [9, Section 2.2], one can see that this group has exactly $\mid \mathcal{K}(G) \mid = \tau(2n) + 2r\tau(m) + \tau(m) = \tau(2n) + \tau(m)(r+1) + r\tau(m) = \tau(2n) + \tau(n) + r\tau(m)$ subgroups. This shows that we have the following lemma:

Lemma 2.2. *The order of the table of marks of the dicyclic group T_{4n}, $n = 2^r m$ and m is odd is $\tau(2n) + \tau(n) + r\tau(m)$.*

Proposition 2.3. *In the dicyclic group T_{4n}, $m_{i2} = [G : H_i]$, for any subgroup H_i if $\langle a^n\rangle \leqslant H$. In other case, $m_{i2} = 0$.*

Proof. To prove $m_{i2} = [G : H_i]$, we put $C_2 = \langle a^n\rangle$. If $C_2 \leqslant H_i$, then by definition

$$m_{i2} = [N_G(H_i) : H_i] \cdot \mid \{H^g \mid \langle a^n\rangle \leqslant H^g \ \&\ g \in T_{4n}\}|.$$

If H is a normal subgroup then $m_{i2} = [G : H]$. Suppose $H = \langle a^d, a^j b\rangle$, $1 \leqslant j \leqslant d$ and d is even. Then $H \cong T_{4\frac{n}{d}}$ and $N_G(\langle a^d, a^j b\rangle) = \langle a^{\frac{d}{2}}, a^j b\rangle$ which implies that $[N_G(\langle a^d, a^j b\rangle) : \langle a^d, a^j b\rangle] = 2$. On the other hand, $|\{(\langle a^d, a^j b\rangle)^g \mid \langle a^n\rangle \leqslant (\langle a^d, a^j b\rangle)^g \ \&\ g \in T_{4n}\}| = \frac{d}{2}$. Now, since $[T_{4n} : \langle a^d, a^j b\rangle] = d$, we have that $m_{i2} = [T_{4n} : \langle a^d, a^j b\rangle]$. Next we assume that d is odd which shows that $\langle a^d, a^j b\rangle$ is self-normalizer. Therefore, $[N_{T_{4n}}(\langle a^d, a^j b\rangle) : \langle a^d, a^j b\rangle] = 1$. This proves that the number $H-$conjugate classes is d. $\square$

In [6, Lemma 3.5.3(a)], it is proved that if $M(G) = [m_{ij}]$ is the table of marks of G then $m_{ij} = [N_G(H_i) : H_i]\cdot b_{ij}$, where b_{ij} is the number of subgroups conjugate to H_i which contain H_j. In particular, $m_{ii} = [N_G(H_i) : H_i]$. By this result, one can easily seen that if H_i is normal then $\beta_{G/H}(K) = [G : H]$.

Proposition 2.4. *Let d is an odd positive divisor and $H = \langle a^d, a^j b\rangle$. Then*

$$\beta_{T_{4n}/H}(K) = [T_{4n} : H] = d \text{ or } 1.$$

Proof. Since d is odd, H is a self-normalizing subgroup of T_{4n}. We first assume that $K \leqslant T_{4n}$ is normal. Then $\beta_{T_{4n}/H}(K) = |\{H^g \mid K \leqslant H^g \ \&\ g \in T_{4n}\}| = |\{H^g \mid K \leqslant H^g, g \in T_{4n}\}| = |\{H^g \mid K \leqslant H\}| = [T_{4n} : N_G(H)] = [T_{4n} : H]$. But $H \cong T_{4\frac{n}{d}}$ and so $\beta_{T_{4n}/H}(K) = d$, as desired. If K is not normal in T_{4n}, then $K = \langle a^h, a^j b\rangle$, where $h < d$. Thus $\beta_{T_{4n}/H}(K) = |\{H^g \mid \langle a^h, a^j b\rangle \leqslant H^g \ \&\ g \in T_{4n}\}| = 1$. □

By Lemma 2.2 and Propositions 2.3, 2.4 we have the following theorem:

Theorem 2.5. *The table of marks of the dicyclic group T_{4n} is given in Tables 3 and 4.*

Table 3. Table of marks of the dicyclic group T_{4n}, when $n = 2^r m$ and $3 \mid m$.

$*$	e	$\langle a^n\rangle$	$\langle a^{\frac{2n}{p_1}}\rangle$	$\langle a^{\frac{n}{2}}\rangle$	$\langle a^n, b\rangle$	$\langle a^n, ab\rangle$	$K_{j,\, 7\leqslant j\leqslant s}$
G/e	$4n$	0	0	0	0	0	$\cdots$
$G/\langle a^n\rangle$	$2n$	$2n$	0	0	0	0	$\cdots$
$G/\langle a^{\frac{2n}{p_1}}\rangle$	$\frac{4n}{p_1}$	-	$\frac{4n}{p_1}$	0	0	0	$\cdots$
$G/\langle a^{\frac{n}{2}}\rangle$	n	n	0	n	0	0	$\cdots$
$G/\langle a^n, b\rangle$	n	n	0	0	2	0	$\cdots$
$G/\langle a^n, ab\rangle$	n	n	0	0	0	2	$\cdots$
$G/H_i, 7 \leqslant i \leqslant s$				δ_{ij}			

Table 4. Table of marks of the dicyclic group T_{4n}, when $n = 2^r m$ and $3 \nmid m$.

$*$	e	$\langle a^n\rangle$	$\langle a^{\frac{n}{2}}\rangle$	$\langle a^n, b\rangle$	$\langle a^n, ab\rangle$	$K_{j,\, 6\leqslant j\leqslant s}$
G/e	$4n$	0	0	0	0	$\cdots$
$G/\langle a^n\rangle$	$2n$	$2n$	0	0	0	$\cdots$
$G/\langle a^{\frac{n}{2}}\rangle$	n	n	n	0	0	$\cdots$
$G/\langle a^n, b\rangle$	n	n	0	2	0	$\cdots$
$G/\langle a^n, ab\rangle$	n	n	0	0	2	$\cdots$
$G/H_i, 6 \leqslant i \leqslant s$				δ_{ij}		

In Tables 3 and 4, the quantity δ_{ij} can be computed by the following formula:

$$\delta_{ij} = \begin{cases} m_{i1} & \textit{if } K_j \leqslant H_i \trianglelefteq T_{4n} \\ 2 & \textit{if } K_j \leqslant H_i \leqslant T_{4n} \\ 1 & \textit{if } K_j \leqslant N_{T_{4n}}(H_i) = H_i \\ 0 & \textit{otherwise.} \end{cases}.$$

Suppose $\mathcal{K}(G)$ denotes the set of all conjugacy classes of a given group G. By definition of the markaracter table, one can easily seen that the markaracter table of G has exactly $\mathcal{K}(G)$ rows and columns.

We are now ready to calculate the markaracter table of the dicyclic group T_{4n}. The matrix $MC(T_{4n})$ can be obtained from $MT(T_{4n})$ in which we select rows and columns corresponding to cyclic subgroups of T_{4n}. By Lemma 2.2, the dicyclic group T_{4n}, $n = 2^r m$ and m is odd is $\tau(2n) + \tau(n) + r\tau(m)$.

Lemma 2.6. *The number of conjugacy classes of dicyclic group T_{4n} can be computed by the following formula:*

$$|\mathcal{K}(T_{4n})| = \begin{cases} \tau(2n) + 2 & 2 \mid n, \\ \tau(2n) + 1 & 2 \nmid n. \end{cases}$$

Proof. It is easy to see that for each i, i $2n$, $\langle a^i \rangle$ is a normal subgroup of T_{4n} and so there are $\tau(2n)$ conjugacy classes of cyclic subgroups of this type. Suppose n is even. Among two generator subgroups $\langle a^i, a^j b\rangle$ of T_{4n}, $\langle a^n, a^j b\rangle$ is a cyclic subgroup of order 4 and other subgroups of this form are not cyclic. On the other hand, all subgroup of the form $\langle a^n, a^j b\rangle$, j is odd, are conjugate in T_{4n}, and all subgroups of the form $\langle a^n, a^j b\rangle$, j is even, are conjugate in T_{4n}. This shows that in the case that n is even, we have exactly $\tau(2n) + 2$ conjugacy classes of cyclic subgroups. If n is even then all subgroups of the form $\langle a^n, a^j b\rangle$ (j can be odd or even) are conjugate in T_{4n} and so we have exactly $\tau(n) + 1$ conjugacy classes of cyclic subgroups in T_{4n}. This completes our argument. $\square$

By previous lemma the non-conjugate subgroups of T_{4n} are as follows:

- $C(H_1) = \langle e \rangle$,
- $C(H_2) = \langle a^n \rangle$,
- $C(H_3) = \langle a^{\frac{2n}{3}} \rangle$,
- $C(H_4) = \langle a^{\frac{n}{2}} \rangle$,
- $C(H_5) = \langle a^n, a^j b\rangle, 2 \mid j$,
- $C(H_6) = \langle a^n, a^j b\rangle, 2 \nmid j$,
- $C(H_i)_{7 \leqslant i \leqslant s} = \langle a^{\frac{2n}{d}} \rangle, d \neq 2, 3$, where $|\mathcal{K}(T_{4n})| = s$.

By Lemma 2.6, the markaracter table of the dicyclic group T_{4n} are recorded in Tables 5 and 6 in which

$$\delta_{ij} = \begin{cases} m_{i1} & \textit{if } K_j \leqslant H_i \trianglelefteq T_{4n} \\ 2 & \textit{if } K_j \leqslant H_i \leqslant T_{4n} \\ 1 & \textit{if } K_j \leqslant N_{T_{4n}}(H_i) = H_i \\ 0 & \textit{otherwise.} \end{cases}$$

Table 5. The markaracter table of T_{4n}, when $n = 2^r m$ and $3 \mid m$.

$*$	e	$\langle a^n \rangle$	$\langle a^{\frac{2n}{p_1}} \rangle$	$\langle a^{\frac{n}{2}} \rangle$	$\langle a^n, b \rangle$	$\langle a^n, ab \rangle$	$K_{j,\ 7 \leqslant i \leqslant s}$
G/e	$4n$	0	0	0	0	0	$\cdots$
$G/\langle a^n \rangle$	$2n$	$2n$	0	0	0	0	$\cdots$
$G/\langle a^{\frac{2n}{3}} \rangle$	$\frac{4n}{3}$	0	$\frac{4n}{3}$	0	0	0	$\cdots$
$G/\langle a^{\frac{n}{2}} \rangle$	n	n	0	n	0	0	$\cdots$
$G/\langle a^n, b \rangle$	n	n	0	0	**2**	0	$\cdots$
$G/\langle a^n, ab \rangle$	n	n	0	0	0	**2**	$\cdots$
$G/H_{i_{7 \leqslant i \leqslant s}}$				δ_{ij}			

Table 6. The Markaracter Table of T_{4n}, when $n = 2^r m$ and $3 \nmid m$.

$*$	e	$\langle a^n \rangle$	$\langle a^{\frac{n}{2}} \rangle$	$\langle a^n, b \rangle$	$\langle a^n, ab \rangle$	$K_{j,\ 5 \leqslant j \leqslant s}$
G/e	$4n$	0	0	0	0	$\cdots$
$G/\langle a^n \rangle$	$2n$	$2n$	0	0	0	$\cdots$
$G/\langle a^{\frac{n}{2}} \rangle$	n	n	n	0	0	$\cdots$
$G/\langle a^n, b \rangle$	n	n	0	**2**	0	$\cdots$
$G/\langle a^n, ab \rangle$	n	n	0	0	**2**	$\cdots$
$G/H_{i_{6 \leqslant i \leqslant s}}$				δ_{ij}		

2.2. Table of marks of the semi-dihedral group SD_{2^n}

In [9, Section 2.5], the present authors studied the structure of subgroups of the group SD_{2^n}. From the results given the mentioned paper, we can see that we have two types of cyclic subgroups in SD_{2^n}. The first type subgroups are in the form $\langle a^d \rangle$ of order $\frac{2^{n-1}}{d}$, where $d \mid 2^{n-1}$. The second type of subgroups have the form $\langle a^d, a^k b \rangle$, where $1 \leqslant k \leqslant d$. If $2 \mid k$ then $\langle a^d, a^k b \rangle \cong D_{\frac{2n}{d}}$, and if $2 \nmid k$ then $\langle a^d, a^k b \rangle \cong Q_{\frac{2n+1}{d}}$.

Since all subgroups of the first type are normal, there are $\tau(2^{n-1}) = n$ conjugacy classes of cyclic subgroups. Among subgroups of the second time, it is easy to see that all subgroups of the form $\langle a^j b \rangle$, $1 \leqslant j \leqslant 2^{n-1}$ and $2 \mid j$, are conjugate and so these subgroups constitute a conjugacy class of subgroups in SD_{2^n}. Choose the subgroups $\langle a^{2^k}, a^j b \rangle$, $1 \leqslant j \leqslant k$ and $k \mid 2^{n-3}$. Fix a positive integer k. Then all subgroups of the form $\langle a^{2^k}, a^j b \rangle$ with even positive integer j are conjugate and so we have $2(n-2)$ conjugacy classes of subgroups of this form. The same will be happened when j varies on the set of all odd integers with condition $1 \leqslant j \leqslant k$. Hence there are $2(n-2)+n+2 = 3n-2$ conjugacy classes of subgroups in SD_{2^n}. Therefore, the non-conjugate subgroups of SD_{2^n} are as follows:

- $C(H_1) = \{\langle e \rangle\}$;
- $C(H_2) = \{\langle a^{2^{n-2}} \rangle\}$;

- $C(H_3) = \{\langle a^{2^{n-1}}, a^j b\rangle \mid j \text{ is even}\}$;
- $C(H_{4+3i}) = \{\langle a^{2^{n-3-i}}\rangle\}$, $0 \leqslant i \leqslant n-3$;
- $C(H_{5+3i}) = \{\langle a^{2^{n-2-i}}, a^j b\rangle, j \text{ is even}\}$, $0 \leqslant i \leqslant n-3$;
- $C(H_{6+3i}) = \{\langle a^{2^{n-1-i}}, a^j b\rangle, j \text{ is odd}\}$, $0 \leqslant i \leqslant n-3$;
- $C(H_{3n-2}) = \{\langle a, b\rangle\}$.

Therefore, we proved the following proposition:

Proposition 2.7. *The semi-dihedral group SD_{2^n} has exactly $3n-2$ conjugacy classes of subgroups.*

Theorem 2.8. *The table of marks of the semi-dihedral group SD_{2^n} is given in Table 7.*

Table 7. Table of marks of the dicyclic group SD_{2^n}.

$*$	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}	K_{11}	K_{12}	$\cdots$	K_s
G/H_2	2^{n-1}	2^{n-1}	0	0	0	0	0	0	0	0	0	0	$\cdots$	0
G/H_3	2^{n-1}	0	**2**	0	0	0	0	0	0	0	0	0	$\cdots$	0
G/H_4	2^{n-2}	2^{n-2}	0	2^{n-2}	0	0	0	0	0	0	0	0	$\cdots$	0
G/H_5	2^{n-2}	2^{n-2}	0	0	**2**	0	0	0	0	0	0	0	$\cdots$	0
G/H_6	2^{n-2}	2^{n-2}	**2**	0	0	**2**	0	0	0	0	0	0	$\cdots$	0
G/H_7	2^{n-3}	2^{n-3}	0	2^{n-3}	0	0	2^{n-3}	0	0	0	0	0	$\cdots$	0
G/H_8	2^{n-3}	2^{n-3}	0	2^{n-3}	**2**	0	0	**2**	0	0	0	0	$\cdots$	0
G/H_9	2^{n-3}	2^{n-3}	**2**	2^{n-3}	0	**2**	0	0	**2**	0	0	0	$\cdots$	0
G/H_{10}	2^{n-4}	2^{n-4}	0	2^{n-4}	0	0	2^{n-4}	0	0	2^{n-4}	0	0	$\cdots$	0
G/H_{11}	2^{n-4}	2^{n-4}	0	2^{n-4}	**2**	0	2^{n-4}	**2**	0	0	**2**	0	$\cdots$	0
G/H_{12}	2^{n-4}	2^{n-4}	**2**	2^{n-4}	0	**2**	2^{n-4}	0	**2**	0	0	**2**	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
G/H_s	1	1	1	1	1	1	1	1	1	1	1	1	$\cdots$	1

where $s = 3n-2$.

Proof. We first calculate the entry m_{ij} in table of marks of semi-dihedral group SD_{2^n}. We claim that

$$m_{ij} = \beta_{(SD_{2^n}/H_i)}(k_j) = \begin{cases} [SD_{2^n} : H_i] & \textit{if } \ K_j \trianglelefteq H_i \trianglelefteq SD_{2^n} \text{or} K_j \leqslant H_i \trianglelefteq SD_{2^n} \\ 2 & \textit{if } \ K_j \leqslant H_i \leqslant SD_{2^n} \\ 0 & \textit{if } \ K_j \nleqslant H_i \end{cases}.$$

To prove, we assume that $K_j \trianglelefteq H_i \trianglelefteq SD_{2^n}$. Thus

$$\begin{aligned} [N_{SD_{2^n}}(H) : H] &= [SD_{2^n} : H] \\ |\{H^g \mid K \leqslant H^g \ \& \ g \in SD_{2^n}\}| &= 1. \end{aligned}$$

Since H_i is normal, $m_{ij} = \beta_{(SD_{2^n}/H_i)}(K_j) = [SD_{2^n} : H_i]$. Next we assume that $K_j \leqslant H_i \leqslant SD_{2^n}$ and H_i is not normal in SD_{2^n}. Then $[N_{SD_{2^n}}(H) : H] = 2$. We write $K_j = \langle a^r, a^j b\rangle$ and $H_i = \langle a^d, a^j b\rangle$. If $r \mid d$, then it easy see to that K_j is contained in a unique conjugate of H_i.

Since $H_i \ntrianglelefteq SD_{2^n}$ and $K_j \leqslant SD_{2^n}$,

$$\begin{aligned} N_{SD_{2^n}}(\langle a^d, a^j b\rangle) &= \langle a^{\frac{d}{2}}, a^j b\rangle, \\ |\{K_j \leqslant H_i^g \ \&\ g \in SD_{2^n}\}| &= 1. \end{aligned}$$

Finally, if $K_j \nleqslant H_i$ then $|\{H_i^g \mid K_j \leqslant H_i^g \ \&\ g \in SD_{2^n}\}| = 0$ and so $\beta_{SD_{2^n}/H_i}(K_j) = 0$. □

By the proof of the previous theorem, one can see that the number of cyclic subgroups of the semi-dihedral group SD_{2^n} are $n + 2^{n-3} + 2^{n-2}$. There are two conjugacy classes of subgroups of index 2^{n-1} with representatives $C_2 = \langle a^{2^{n-2}}\rangle$ and $D_2 = \langle a^2 b\rangle$. There are also two conjugacy classes of subgroup of index 2^{n-2} with representatives $C_4 = \langle a^{2^{n-3}}\rangle$ and $Q_4 = \langle a^{2^{n-2}}, ab\rangle$. For all other integers appeared as the index of a subgroup in SD_{2^n}, there exists a unique conjugacy classes of cyclic subgroups. In an exact phrase, there exists a unique subgroup of index 2^{n-3-k}, $0 \leqslant k \leqslant n-3$, generated by $a^{2^{n-4-k}}$. Therefore, there are $n+2$ conjugacy classes of cyclic subgroups. Hence we proved the following proposition:

Corollary 2.9. *The order of markaracter table in the group SD_{2^n} is equal to $s = n + 2$.*

Theorem 2.10. *The markaracter table of semi-dihedral group SD_{2^n} is given by Table 8.*

Table 8. Markaracter table of the semi-dihedral group SD_{2^n}.

$*$	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	$\cdots$	K_s
G/H_1	2^n	0	0	0	0	0	0	0	$\cdots$	0
G/H_2	2^{n-1}	2^{n-1}	0	0	0	0	0	0	$\cdots$	0
G/H_3	2^{n-1}	0	**2**	0	0	0	0	0	$\cdots$	0
G/H_4	2^{n-2}	2^{n-2}	0	2^{n-2}	0	0	0	0	$\cdots$	0
G/H_5	2^{n-2}	2^{n-2}	0	0	**2**	0	0	0	$\cdots$	0
G/H_6	2^{n-3}	2^{n-3}	0	2^{n-3}	0	2^{n-3}	0	0	$\cdots$	0
G/H_7	2^{n-4}	2^{n-4}	0	2^{n-4}	0	2^{n-4}	2^{n-4}	0	$\cdots$	0
G/H_8	2^{n-5}	2^{n-5}	0	2^{n-5}	0	2^{n-5}	2^{n-5}	2^{n-5}	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
G/H_s	2	2	0	2	0	2	2	2	$\cdots$	2

Proof. Apply Theorem 2.8. □

2.3. The group $H(n)$

Define $H(n) = \langle x, y, z \mid x^{2^{n-2}} = y^2 = z^2 = e, [x, y] = [y, z] = e, xz = xy\rangle$. The aim of this section is to calculate the table of marks and markaracter table of the group

$H(n)$. In [9, Section 2.6], the present authors studied the structure of subgroups of this group and proved that the normal subgroups of $H(n)$ have the following forms:

- $G_1 = \langle a^d \rangle$, where $d \mid 2^{n-2}$ and $d \neq 1$;
- $G_2 = \langle a^d, b \rangle$, where $d \mid 2^{n-2}$;
- $G_3 = \langle a^d b \rangle$, where $d \mid 2^{n-3}$ and $d \neq 1$;
- $G_4 = \langle a^d c, a^d bc \rangle$, where $d \mid 2^{n-3}$;
- $G_5 = \langle a^d, b, c \rangle$, where $d \mid 2^{n-2}$.

We now consider non-normal subgroups of $H(n)$. Suppose $d \mid 2^{n-2}$. Since $a^{-1}\langle a^d, c\rangle a = \langle a^d, bc\rangle$ and $a^{-1}\langle a^d b, a^d c\rangle a = \langle a^d b, a^d bc\rangle$, $\langle a^d, c\rangle$, $\langle a^d, bc\rangle$ and also $\langle a^d b, a^d c\rangle$, $\langle a^d b, a^d bc\rangle$ are conjugate subgroups of $H(n)$. Moreover, $c^{-1}\langle a\rangle c = \langle ab\rangle$ and so $\langle a\rangle$ and $\langle ab\rangle$ are conjugate. In what follows, we record the representatives of conjugacy classes of subgroups of $H(n)$. In the case that the conjugacy class has one or two elements, the complete conjugacy class of those subgroups are recorded.

1. $C(H_1) = \{\langle e\rangle\}$, $C(H_2) = \{\langle a^{2^{n-3}}\rangle\}$, $C(H_3) = \{\langle b\rangle\}$, $C(H_4) = \{\langle a^{2^{n-3}} b\rangle\}$, $C(H_5) = \{\langle c\rangle, \langle bc\rangle\}$, $C(H_6) = \{\langle a^{2^{n-3}} c\rangle, \langle a^{2^{n-3}} bc\rangle\}$;
2. $C(H_{7+8j}) = \{\langle a^{2^{n-4-j}}\rangle\}$, $0 \leqslant j \leqslant n-5$;
3. $C(H_{8+8j}) = \{\langle a^{2^{n-3-j}}, b\rangle\}$, $0 \leqslant j \leqslant n-5$;
4. $C(H_{9+8j}) = \{\langle a^{2^{n-4-j}} b\rangle\}$, $0 \leqslant j \leqslant n-5$;
5. $C(H_{10+8j}) = \{\langle a^{2^{n-3-j}} b, a^{2^{n-3-j}} bc\rangle\}$, $0 \leqslant j \leqslant n-5$;
6. $C(H_{11+8j}) = \{\langle a^{2^{n-2-j}}, b, c\rangle\}$, $0 \leqslant j \leqslant n-5$;
7. $C(H_{12+8j}) = \{\langle a^{2^{n-4-j}} c\rangle, \langle a^{2^{n-4-j}} bc\rangle\}$, $0 \leqslant j \leqslant n-5$;
8. $C(H_{13+8j}) = \{\langle a^{2^{n-3-j}}, c\rangle, \langle a^{2^{n-3-j}}, bc\rangle\}$, $0 \leqslant j \leqslant n-5$;
9. $C(H_{14+8j}) = \{\langle a^{2^{n-3-j}} c, a^{2^{n-3-j}} b\rangle, \langle a^{2^{n-3-j}} c, a^{2^{n-3-j}} bc\rangle\}$, $0 \leqslant j \leqslant n-5$;
10. $C(H_{8n-25}) = \{\langle a^2, b\rangle\}$, $C(H_{8n-24}) = \{\langle a^2 c, a^2 bc\rangle\}$, $C(H_{8n-23}) = \{\langle a^4, b, c\rangle\}$;
11. $C(H_{8n-22}) = \{\langle a\rangle, \langle ab\rangle\}$, $C(H_{8n-21}) = \{\langle ac\rangle, \langle abc\rangle\}$;
12. $C(H_{8n-20}) = \{\langle a^2 b, a^2 bc\rangle, \langle a^2 b, a^2 c\rangle\}$, $C(H_{8n-19}) = \{\langle a^2, c\rangle, \langle a^2, bc\rangle\}$, $C(H_{8n-18}) = \{\langle a, b\rangle\}$, $C(H_{8n-17}) = \{\langle a, c\rangle\}$, $C(H_{8n-16}) = \{\langle a^2, b, c\rangle\}$, $C(H_{8n-15}) = \{\langle a, b, c\rangle\}$.

Among these classes of subgroups, conjugacy classes recorded in the cases $1, 2, 4, 7$ and 11 are related to cyclic subgroups. We now record our calculations in the following lemma:

Lemma 2.11. *There are $8n-15$ conjugacy classes of subgroups in the group $H(n)$ and among them there are $3n-4$ conjugacy classes of cyclic subgroups. In particular, the order of table of marks and markaracter table of $H(n)$ are $8n-15$ and $3n-4$, respectively.*

To calculate the table of marks of $H(n)$, we have to calculate the values $m_{ij}(H(n))$.

Proposition 2.12.

$$\delta_{ij} = \beta_{H(n)/H_i}(K_j) = \begin{cases} [H(n):H_i] & K_j \trianglelefteq H_i \trianglelefteq H(n) \text{ or } K_j \leqslant H_i \trianglelefteq H(n), \\ [N_{H(n)}(H_i):H_i] & K_j \leqslant H_i \leqslant H(n), \\ 0 & K_j \nleq H_i. \end{cases}$$

Proof. Suppose $K_j \leqslant H_i$. It is easy to see that $|N_{H(n)}(H_i)| = 2^{n-1}$, when H_i is a non-normal subgroup of $H(n)$. On the other hand,

$$\begin{aligned} \beta_{H(n)/H_i}(K_j) &= [N_{H(n)}(H_i):H_i]|\{H_i^g \mid K_j \leqslant H_i^g \ \& \ g \in H(n)\}| \\ &= [N_{H(n)}(H_i):H_i], \end{aligned}$$

proving the result. □

Theorem 2.13. *The table of marks and markaracter table of the group $H(n)$ are given in Tables 9 and 10, respectively.*

Table 9. Table of marks of the group $H(n)$.

$*$	K_1	K_2	K_3	K_4	K_5	K_6	$K_{j,\ 7\leqslant j\leqslant 8n-15}$
$H(n)/e$	2^n	0	0	0	0	0	$\cdots$
$H(n)/\langle a^{2^{n-3}}\rangle$	2^{n-1}	2^{n-1}	0	0	0	0	$\cdots$
$H(n)/\langle b\rangle$	2^{n-1}	0	2^{n-1}	0	0	0	$\cdots$
$H(n)/\langle a^{2^{n-3}}b\rangle$	2^{n-1}	0	0	2^{n-1}	0	0	$\cdots$
$H(n)/\langle bc\rangle$	2^{n-1}	0	0	0	$\mathbf{2^{n-2}}$	0	$\cdots$
$H(n)/\langle a^{2^{n-3}}bc\rangle$	2^{n-1}	0	0	0	0	$\mathbf{2^{n-2}}$	$\cdots$
$H(n)/(H_i)_{7\leqslant i\leqslant 8n-15}$					δ_{ij}		

Table 10. The markaracter table of the group $H(n)$.

$*$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
$H(n)/e$	2^n	0	0	0	0	0	0
$\mathbf{H(n)/\langle a^{2^{n-3}}\rangle}$	2^{n-1}	$\mathbf{2^{n-1}}$	0	0	0	0	0
$H(n)/\langle b\rangle$	2^{n-1}	0	2^{n-1}	0	0	0	0
$H(n)/\langle a^{2^{n-3}}b\rangle$	2^{n-1}	0	0	2^{n-1}	0	0	0
$H(n)/\langle bc\rangle$	2^{n-1}	0	0	0	2^{n-2}	0	0
$H(n)/\langle a^{2^{n-3}}bc\rangle$	2^{n-1}	0	0	0	0	2^{n-2}	0
$\mathbf{H(n)/\langle a^{2^{n-4}}\rangle}$	2^{n-2}	2^{n-2}	0	0	0	0	$\mathbf{2^{n-2}}$
$H(n)/\langle a^{2^{n-4}}b\rangle$	2^{n-2}	2^{n-2}	0	0	0	0	0
$H(n)/\langle a^{2^{n-4}}bc\rangle$	2^{n-2}	2^{n-2}	0	0	0	0	0
$\mathbf{H(n)/\langle a^{2^{n-5}}\rangle}$	2^{n-3}	2^{n-3}	0	0	0	0	$\mathbf{2^{n-3}}$
$H(n)/\langle a^{2^{n-5}}b\rangle$	2^{n-3}	2^{n-3}	0	0	0	0	$\mathbf{2^{n-3}}$
$H(n)/\langle a^{2^{n-5}}bc\rangle$	2^{n-3}	2^{n-3}	0	0	0	0	$\mathbf{2^{n-3}}$
$H(n)/H_{i,\ 13\leqslant i\leqslant 3n-4}$							δ_{ij}

$*$	K_8	K_9	K_{10}	K_{11}	K_{12}	$K_{i,\ 13\leqslant i\leqslant 3n-4}$
$H(n)/e$	0	0	0	0	0	$\cdots$
$\mathbf{H(n)/\langle a^{2^{n-3}}\rangle}$	0	0	0	0	0	$\cdots$
$H(n)/\langle b\rangle$	0	0	0	0	0	$\cdots$
$H(n)/\langle a^{2^{n-3}}b\rangle$	0	0	0	0	0	$\cdots$
$H(n)/\langle bc\rangle$	0	0	0	0	0	$\cdots$
$H(n)/\langle a^{2^{n-3}}bc\rangle$	0	0	0	0	0	$\cdots$
$\mathbf{H(n)/\langle a^{2^{n-4}}\rangle}$	0	0	0	0	0	$\cdots$
$H(n)/\langle a^{2^{n-4}}b\rangle$	2^{n-2}	0	0	0	0	$\cdots$
$H(n)/\langle a^{2^{n-4}}bc\rangle$	0	2^{n-3}	0	0	0	$\cdots$
$\mathbf{H(n)/\langle a^{2^{n-5}}\rangle}$	0	0	$\mathbf{2^{n-3}}$	0	0	$\cdots$
$H(n)/\langle a^{2^{n-5}}b\rangle$	0	0	0	2^{n-3}	0	$\cdots$
$H(n)/\langle a^{2^{n-5}}bc\rangle$	0	0	0	0	2^{n-4}	$\cdots$
$H(n)/H_{i_{13\leqslant i\leqslant 3n-4}}$						

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