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On generalized associativity in groupoids

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Abstract. Following an approach developed by Niemenmäa and Kepka, we prove that if a division groupoid G satisfies the identity $R_{x_1}L_{x_2}\dots R_{x_{2n-1}}L_{x_{2n}}y = L_{x_{2n}}R_{x_{2n-1}}\dots L_{x_2}R_{x_1}y$ for some $n \neq 2$, then G is an abelian group. Using equational reasoning, we also give a new proof of a result of Niemenmäa and Kepka that a division groupoid in which the generalized associative law $x_1(x_2(\dots x_{n-1}x_n)\dots)) = ((\dots(x_1x_2)\dots)x_{n-1})x_n$ holds must be a group.

1. Introduction

Let G be a groupoid, with composition written as juxtaposition. For $a \in G$, we define the left multiplication map $L_a : G \to G$ by $x \mapsto ax$ and the right multiplication map $R_a : G \to G$ by $x \mapsto xa$. If these maps are surjective for all $a \in G$, we call G a *division groupoid*. If these maps are bijective for all $a \in G$, we call G a *quasigroup*. For more on quaisgroups, we refer the reader to [2].

Since the operation on a groupoid is not in general associative, direct study of such objects is usually rather difficult. One way around this problem is to construct an auxiliary group whose properties reflect those of the groupoid operation, and then use this group to study the original structure. This approach was exploited successfully by Niemenmäa and Kepka in [1], in which they showed that any division groupoid satisfying the identity

$$\mathcal{I}_n : x_1(x_2(\dots(x_{n-1}x_n)\dots)) = ((\dots(x_1x_2)\dots)x_{n-1})x_n$$

is in fact a group. While it is clear that $\mathcal{I}_n$ constitutes a generalization of associativity, it is far from obvious that it *implies* associativity.

At the end of [1], the authors define a groupoid identity $M = N$ to be *linear* if M and N contain the same set of indeterminates, and each such indeterminate occurs exactly once on each side. They then pose the question of determining which linear groupoid identities imply associativity. Considering that the associative law $(xy)z = x(yz)$ may be written (in terms of the multiplication maps) as $R_zL_xy = L_xR_zy$, it is perhaps natural to consider the following family of linear groupoid identities:

$$\mathcal{J}_n : R_{x_1}L_{x_2}\dots R_{x_{2n-1}}L_{x_{2n}}y = L_{x_{2n}}R_{x_{2n-1}}\dots L_{x_2}R_{x_1}y$$

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as generalizations of the associative law, which is the case $n = 1$. Modifying the techniques of [1], we show, in the first part of this article, that if $n \geqslant 2$, then a division groupoid satisfies $\mathcal{J}_n$ if and only if it is an abelian group. In the second part of the article, we use equational reasoning to give a much shorter proof of the original Niemenmäa-Kepka result that $\mathcal{I}_n$ implies associativity. The first part of that proof is a relatively straightforward argument that any division groupoid satisfying $\mathcal{I}_n$ is in fact a quasigroup; this is essentially the same as the reasoning in the original paper [1]. In the second part of the proof, however, we use an inductive argument to show that any quasigroup satisfying $\mathcal{I}_n$ implies associativity, thereby circumventing the need to introduce an auxiliary group structure. The ideas in this part of the proof were inspired by output from `Prover9` for the implication $I_4 \Longrightarrow I_3$; however, our proof follows a different path from that outlined in the `Prover9` output.

Following [1], we make the following definitions. Let G be a groupoid and $P(G)$ the set of permutations of G.

$$AL(G) = \{f \in P(G) : f(xy) = g(x)y \text{ for some } g \in P(G) \text{ and all } x, y \in G\}$$
$$AR(G) = \{g \in P(G) : f(xy) = g(x)y \text{ for some } f \in P(G) \text{ and all } x, y \in G\}$$
$$BL(G) = \{f \in P(G) : f(xy) = xg(y) \text{ for some } g \in P(G) \text{ and all } x, y \in G\}$$
$$BR(G) = \{g \in P(G) : f(xy) = xg(y) \text{ for some } f \in P(G) \text{ and all } x, y \in G\}$$

We say that G is *AL-transitive* if for all $x, y \in G$ there exists $f \in AL(G)$ such that $f(x) = y$. The notions of $AR-$, $BL-$ and $BR-$transitivity are defined similarly.

A key property undergirding both parts of this paper is a rigidity principle which appears in [1] as Lemma 2.5. We give a slightly modified version of this below.

Lemma 1.1. [1, Lemma 2.5] *Suppose a division groupoid G is BL-transitive. If $f, f' \in AL(G)$ and $f(a) = f'(a)$ for some $a \in G$, then $f = f'$. The same is true if G is assumed to be AL-transitive and $f, f' \in BL(G)$.*

Proof. Suppose first that G is BL-transitive and $f, f' \in AL(G)$, $a \in G$ are such that $f(a) = f'(a)$. Select $c \in G$ arbitrarily, and use surjectivity of L_c to find $d \in G$ such that $a = cd$. Next, given $z \in G$, use BL-transitivity to find $h \in BL(G)$ such that $h(a) = z$. Let $g, g', k \in P(G)$ witness that the formulas $f(xy) = g(x)y$, $f'(xy) = g'(x)y$, and $h(xy) = xk(y)$ hold for $x, y \in G$. Now

$$\begin{aligned} f(z) &= f(h(a)) = f(h(cd)) = f(ck(d)) = g(c)k(d) = h(g(c)d) = h(f(cd)) \\ &= hf(a) = hf'(a) = hf'(cd) = h(g'(c)d) = g'(c)k(d) = f'(ck(d)) \\ &= f'(h(cd)) = f'(h(a)) = f'(z). \end{aligned}$$

The proof of the second statement is similar. □

We will also need the following key result:

Proposition 1.2. [1, Proposition 3.4] *Let G be a quasigroup which is both AL- and BL-transitive and satisfies $AL(G) \subseteq AR(G)$, $BL(G) \subseteq BR(G)$. Then there exists a binary operation $*$ such that $(G, *)$ is a group and $xy = A(x) * c * B(y)$ for some $c \in G$ and some automorphisms A, B of $(G, *)$.*

2. A generalized form of associativity

In this section, we consider the identity

$$\mathcal{J}_n : R_{x_1} L_{x_2} \dots R_{x_{2n-1}} L_{x_{2n}} y = L_{x_{2n}} R_{x_{2n-1}} \dots L_{x_2} R_{x_1} y$$

as another generalization of the associative law. We may rewrite $\mathcal{J}_n$ in two different ways:

$$L_{x_2} \dots R_{x_{2n-1}} L_{x_{2n}} y \cdot x_1 = L_{x_{2n}} R_{x_{2n-1}} \dots L_{x_2}(yx_1), \tag{1}$$

$$R_{x_1} L_{x_2} \dots R_{x_{2n-1}}(x_{2n} y) = x_{2n} \cdot R_{x_{2n-1}} \dots L_{x_2} R_{x_1} y. \tag{2}$$

These formulas witness that if G is a groupoid in which $\mathcal{J}_n$ is satisfied, then $L_{x_{2n}} R_{x_{2n-1}} \dots L_{x_2} \in AL(G) \cap AR(G)$ and $R_{x_1} L_{x_2} \dots R_{x_{2n-1}} \in BL(G) \cap BR(G)$. In particular, if G is a division groupoid, then G is both AL-transitive and BL-transitive.

We are now ready to prove our main result.

Theorem 2.1. *Let G be a division groupoid and $n \geqslant 2$. Then G satisfies $\mathcal{J}_n$ if and only if G is an abelian group.*

Proof. Suppose first that G is a division groupoid satisfying $\mathcal{J}_n$. We argue first that G must be a quasigroup. Given $a \in G$, fix $b \in G$, and use surjectivity of the multiplication maps to select $c_1, \dots, c_{2n-2}$ such that $L_{c_1} R_{c_2} \dots R_{c_{2n-2}} L_a b = b$. Now $L_{c_1} R_{c_2} \dots R_{c_{2n-2}} L_a \in AL(G)$, so by Lemma 1.1, $L_{c_1} R_{c_2} \dots R_{c_{2n-2}} L_a = 1_G$. Therefore, L_a is injective.

Next, we show that G satisfies the remaining hypotheses of Proposition 1.2. Given $f \in AL(G)$, fix $a \in G$ and use surjectivity of the multiplication maps to select $d_1, \dots, d_{2n-1} \in G$ such that $L_{d_1} R_{d_2} \dots L_{d_{2n-1}} a = f(a)$. Because we have $L_{d_1} R_{d_2} \dots L_{d_{2n-1}}$ and f are both members of $AL(G)$, Lemma 1.1 implies that $f = L_{d_1} R_{d_2} \dots L_{d_{2n-1}}$, so $f \in AR(G)$ also. Thus, $AL(G) \subseteq AR(G)$. The proof of the inclusion $BL(G) \subseteq BR(G)$ is similar.

Now we use Proposition 1.2 to deduce the existence of a binary operation $+$ on G such that $(G, +)$ is a group and $xy = A(x) + c + B(y)$ for some automorphisms A and B of $(G, +)$. (Even though $(G, +)$ is not assumed to be an abelian group, we will still use additive notation to avoid confusion with the groupoid operation on G.) The identity

$$\mathcal{J}_n : R_{x_1} L_{x_2} \dots R_{x_{2n-1}} L_{x_{2n}} y = L_{x_{2n}} R_{x_{2n-1}} \dots L_{x_2} R_{x_1} y$$

implies an identity in $(G,+)$; when this is written out, each of the indeterminates $x_1,\dots,x_n$ occurs in exactly one term on each side, with some automorphism of $(G,+)$ applied to it. For example, in the case $n=2$ we have:

$$\begin{aligned}A^2x_2 + Ac + ABA^2x_4 + ABAc + ABABy + ABc + AB^2x_3 + c + Bx_1 \\ = Ax_4+c+BA^2x_2+BAc+BABAy+BABc+BAB^2x_1+Bc+B^2x_3.\end{aligned}$$

In general, the automorphisms applied to the indeterminates $x_1,\dots,x_{2n}$ on the left are (respectively, in order):

$$B, A^2, (AB)B, (AB)A^2, (AB)^2B, (AB)^2A^2, \dots, (AB)^{n-1}B, (AB)^{n-1}A^2$$

and on the right the automorphisms are:

$$(BA)^{n-1}B^2, (BA)^{n-1}A, \dots, (BA)^2B^2, (BA)^2A, (BA)B^2, (BA)A, B^2, A.$$

For $2 \leqslant i \leqslant 2n$, set $x_i = \begin{cases} B^{-1}(c) & \text{if } i \text{ is odd,} \\ A^{-1}c & \text{if } i \text{ is even.}\end{cases}$

Next, set $y = A^{-1}(c)$ and substitute these values into the identity to obtain: $d + Bx_1 = (BA)^{n-1}B^2x_1$ for some $d \in G$. Evaluating at $x_1 = 0$, the fact that B and $(BA)^{n-1}B^2$ are automorphisms of G forces $d = 0$, so $Bx_1 = (BA)^{n-1}B^2x_1$ and hence $(BA)^{n-1}B = 1_G$.

Now for $i \neq 2$, $1 \leqslant i \leqslant n$, set $y = (BA)^{-1}(c)$; then, substitute this and the same values for x_i (as above) into the identity to obtain $A^2x_2 + d' = (BA)^{n-1}Ax_2$ for some $d' \in G$. Reasoning as before, we have $d' = 0$, so $(BA)^{n-1}A^{-1} = 1_G$. Thus, $A^{-1} = B$, so $B = (BA)^{n-1}B = 1_G$, which in turn implies $A = 1_G$.

Therefore, $xy = x + c + y$, and so we compute:

$$(xy)z = (x+c+y)z = (x+c+y)+c+z = x+c+(y+c+z) = x(y+c+z) = x(yz).$$

This shows that the quasigroup G is, in fact, a group. Now that we know that G has a neutral element e, simply set all x_i, $i \neq 1,3$ equal to e in the identity $\mathcal{J}_n$ to obtain $R_{x_1}R_{x_3} = R_{x_3}R_{x_1}$. Applying this equality of functions to e, we have $x_1x_3 = x_3x_1$ for all $x_1, x_3 \in G$, so G is abelian.

Conversely, if G is an abelian group, then the identities $R_xL_y = L_yR_x$, $R_xR_y = R_yR_x$ and $L_xL_y = L_yL_x$ hold in G. Now all left and right multiplication maps commute with each other, so $\mathcal{J}_n$ must hold. □

3. The Niemenmäa-Kepka Theorem

We conclude by giving a new proof of the main result of [1]. The first part of the proof (Proposition 3.1 below) follows the reasoning of [1, Theorem 4.1].

Proposition 3.1. *Let $n \geqslant 3$. Then a division groupoid satisfying $\mathcal{I}_n$ is a quasigroup.*

Proof. Note that $\mathcal{I}_n$ can be interpreted in two ways:

$$L_{x_1} \dots L_{x_{n-2}}(x_{n-1}x_n) = R_{x_{n-1}} \dots R_{x_3} R_{x_2} x_1 \cdot x_n, \tag{3}$$
$$R_{x_n} \dots R_{x_3}(x_1x_2) = x_1 \cdot L_{x_2} \dots L_{x_{n-1}} x_n. \tag{4}$$

In particular, for any division groupoid G satisfying $\mathcal{I}_n$, the first formula shows that $L_{x_1} \dots L_{x_{n-2}} \in AL(G)$, and the second formula that $R_{x_n} \dots R_{x_3} \in BL(G)$. Since all left and right multiplication maps are surjective, it follows that G is both AL-transitive and BL-transitive.

We now show that for $a \in G$, the map L_a is injective. To this end, fix $b \in G$ and use surjectivity of the left multiplication maps to select $y_1, \dots, y_{n-3} \in G$ such that $L_{y_1} \dots L_{y_{n-3}} L_a b = b$. By the rigidity principle (Lemma 1.1), $L_{y_1} \dots L_{y_{n-3}} L_a = 1_G$; so L_a has a left inverse and is hence injective. The proof of the injectivity of R_a is similar. □

We are now ready to give a new proof of [1, Theorem 4.1]. To prepare, define

$$\lambda(x_1, \dots, x_n) = ((x_1x_2) \cdots x_{n-1})x_n,$$
$$\rho(x_1, \dots, x_n) = x_1(x_2 \cdots (x_{n-1}x_n)).$$

Then $\mathcal{I}_n$ is simply the statement $\lambda(x_1, \dots, x_n) = \rho(x_1, \dots, x_n)$. All of the identities in the list below can be proved by direct calculation.

Lemma 3.2. *The following formulas hold for any $m \geqslant 1$:*

- *(HL)* $\lambda(x_1, \dots, x_m, y) = \lambda(x_1, \dots, x_m)y$,
- *(HR)* $\rho(y, x_1, \dots, x_m) = y\rho(x_1, \dots, x_m)$,
- *(CL)* $\lambda(\lambda(x_1, \dots, x_\ell), x_{\ell+1}, \dots, x_m) = \lambda(x_1, \dots, x_m)$,
- *(CR)* $\rho(x_1, \dots, x_\ell, \rho(x_{\ell+1}, \dots, x_m)) = \rho(x_1, \dots, x_m)$,
- *(DL)* $\lambda(yx_1, x_2, \dots, x_m) = \lambda(y, x_1, x_2, \dots, x_m)$,
- *(DR)* $\rho(x_1, \dots, x_{m-1}, x_my) = \rho(x_1, \dots, x_{m-1}, x_m, y)$.

Theorem 3.3. *A quasigroup satisfying $\mathcal{I}_n$ is a group.*

Proof. We will argue that when $n \geqslant 4$, $\mathcal{I}_n$ implies $\mathcal{I}_{n-1}$, and then apply induction. The designation at the end of each line shows which statement from Lemma 3.2 was used to deduce it from the previous line.

$$
\begin{aligned}
&y(\lambda(x_1,\ldots,x_{n-1})\rho(z_1,\ldots,z_{n-2}))\\
&\qquad = y(\lambda(x_1,\ldots,x_{n-1},\rho(z_1,\ldots,z_{n-2}))) && (HL)\\
&\qquad = y(\rho(x_1,\ldots,x_{n-1},\rho(z_1,\ldots,z_{n-2}))) && (\mathcal{I}_n)\\
&\qquad = \rho(y,x_1,\ldots,x_{n-1},\rho(z_1,\ldots,z_{n-2}))) && (HR)\\
&\qquad = \rho(y,x_1,\ldots,x_{n-2},x_{n-1},z_1,\ldots,z_{n-2}) && (CR)\\
&\qquad = \rho(y,x_1,\ldots,x_{n-2},\rho(x_{n-1},z_1,\ldots,z_{n-2})) && (CR)\\
&\qquad = \lambda(y,x_1,\ldots,x_{n-2},\rho(x_{n-1},z_1,\ldots,z_{n-2})) && (\mathcal{I}_n)\\
&\qquad = \lambda(y,x_1,\ldots,x_{n-2})\rho(x_{n-1},z_1,\ldots,z_{n-2}) && (HL)\\
&\qquad = \rho(\lambda(y,x_1,\ldots,x_{n-2}),x_{n-1},z_1,\ldots,z_{n-2}) && (HR)\\
&\qquad = \lambda(\lambda(y,x_1,\ldots,x_{n-2}),x_{n-1},z_1,\ldots,z_{n-2}) && (\mathcal{I}_n)\\
&\qquad = \lambda(y,x_1,\ldots,x_{n-1},z_1,\ldots,z_{n-2}) && (CL)\\
&\qquad = \lambda(\lambda(y,x_1,\ldots,x_{n-1}),z_1,\ldots,z_{n-2}) && (CL)\\
&\qquad = \lambda(\rho(y,x_1,\ldots,x_{n-1}),z_1,\ldots,z_{n-2}) && (\mathcal{I}_n)\\
&\qquad = \lambda(y\rho(x_1,\ldots,x_{n-1}),z_1,\ldots,z_{n-2}) && (HR)\\
&\qquad = \lambda(y,\rho(x_1,\ldots,x_{n-1}),z_1,\ldots,z_{n-2}) && (DR)\\
&\qquad = \rho(y,\rho(x_1,\ldots,x_{n-1}),z_1,\ldots,z_{n-2}) && (\mathcal{I}_n)\\
&\qquad = y\rho(\rho(x_1,\ldots,x_{n-1}),z_1,\ldots,z_{n-2}) && (HR)\\
&\qquad = y(\rho(x_1,\ldots,x_{n-1})\rho(z_1,\ldots,z_{n-2})) && (HR).
\end{aligned}
$$

Now cancel y from the left, and then cancel $\rho(z_1,\ldots,z_{n-2})$ from the right to obtain $\lambda(x_1,\ldots,x_{n-1}) = \rho(x_1,\ldots,x_{n-1})$, which is $\mathcal{I}_{n-1}$. □

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Characterizations of Clifford semigroups and t-Putcha semigroups by their quasi-ideals

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Abstract. There are bi-ideals of semigroups which are not quasi-ideals. In spite of this fact, here we show that a semigroup S is quasi-simple if and only if it is bi-simple, equivalently t-simple. Main results of this article are several equivalent characterizations of the Clifford semigroups and the semigroups which are semilattices of t-Archimedean semigroups by their quasi-ideals. A semigroup S is a Clifford semigroup if and only if every quasi-ideal of S is a semiprime ideal, whereas S is a semilattice of t-Archimedean semigroups if and only if $\sqrt{Q}$ is an ideal for every quasi-ideal Q of S.

1. Introduction

In 1952, R.A. Good and D.R. Hughes [3] first defined the notion of bi-ideals of a semigroup. The notion of quasi-ideals in rings and semigroups was introduced and developed by Otto Steinfeld [12], [13], [14], [15]. Different classes of semigroups has been characterized by using bi-ideals and quasi-ideals by many authors [7], [8], [9], [10]. Later on different classes of semigroups has been characterized by using minimal and maximal left-ideals, bi-ideals and quasi-ideals by many authors [1], [17], [4], [2], [9], [6]. Here we characterize the Clifford semigroups and the semigroups which are semilattices of t-Archimedean semigroups by their quasi-ideals.

There are several characterizations for a semigroup S equivalent to be a Clifford semigroup and t-Putcha semigroup by their bi-ideals. Every quasi-ideal of a semigroup is a bi-ideal but the converse is not true. So if a semigroup S is bi-simple or equivalently t-simple then it is quasi-simple. Here we have a strange observation that every quasi-simple semigroup is also t-simple and thus quasi-simplicity and t-simplicity becomes equivalent in semigroups. Therefore we hope that it may turns out to be the case that the semigroups which are semilattices of groups or t-Archimedean semigroups will be characterized by their quasi-ideals. We show that a semigroup S is a semilattice of t-Archimedean semigroups.

Some elementary results together with prerequisites have been discussed in Section 2. In Section 3 we have studied semilattice of quasi-simple semigroups.

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2. Preliminaries

A nonempty subset L of a semigroup S is called a *left ideal* of S if $SL \subseteq L$. The *right ideals* are defined dually. A subset I of S is called an *ideal* of S if it is both a left and a right ideal of S. For an element $a \in S$ the *principal left ideal* (*right ideal*) of S generated by $\{a\}$ is given by $Sa \cup \{a\}$ $(aS \cup \{a\})$ and are denoted by $L(a)$ and $R(a)$ respectively. A semigroup S is called *simple* (*left-simple, right-simple*) if it does not contain any proper ideal (left-ideal, right-ideal), and S is called *t-simple* if it is both left simple and right simple.

A nonempty subset Q is called a *quasi-ideal* of S if $QS \cap SQ \subseteq Q$. It follows that every quasi-ideal Q of S is a subsemigroup. Every nonempty intersection of a left ideal and a right ideal is a quasi ideal of S. Suppose Q is a quasi-ideal of S. Then $L = SQ \cup Q$ is a left ideal and $R = QS \cup Q$ is a right ideal of S such that $Q = L \cap R$. Thus a nonempty subset Q of S is a quasi-ideal if and only if it is an intersection of a left ideal and a right ideal. For $a \in S$, let $Q(a)$ be the quasi-ideal generated by $\{a\}$.

A semigroup S is called *quasi-simple* if it has no proper quasi-ideal.

The *Green's relations* $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ on a semigroup S are defined by, for $a, b \in S$,

$$a\mathcal{L}b \text{ if } L(a) = L(b), \quad a\mathcal{R}b \text{ if } R(a) = R(b) \quad \text{and} \quad \mathcal{H} = \mathcal{L} \cap \mathcal{R}.$$

Now we have the following theorem (cf. [9]).

Theorem 2.1. *Let S be a semigroup. Then $\mathcal{H}$ can be given as follows: for $a, b \in S$,*

$$a\mathcal{H}b \iff Q(a) = Q(b).$$

A nonempty subset A of S is called *semiprime* if for all $x \in S$ such that $x^2 \in A$ one has $x \in A$, and *completely prime* (resp. *semiprimary*) if for all $x, y \in S$ such that $xy \in A$ one has $x \in A$ or $y \in A$ (resp. $x^n \in A$ or $y^n \in A$ for some $n \in \mathbb{N}$). A subsemigroup F of S is called a *filter* of S if for any $a, b \in S$, $ab \in F \Rightarrow a, b \in F$. Let $N(a)$ be the filter generated by $\{a\}$. Define an equivalence relation $\mathcal{N}$ on S by: for $a, b \in S$,

$$a\mathcal{N}b \text{ if } N(a) = N(b).$$

The following lemma (proved in [9]) plays a crucial role in the main theorems of this article.

Lemma 2.2. *Let S be a semigroup. Then $\mathcal{N}$ is the least semilattice congruence on S.*

3. Semilattice of groups

In this section we characterize the semigroups which are semilattices (chains) of groups.

Theorem 3.1. *The following conditions are equivalent on a semigroup S:*

(1) *S is a semilattice of groups;*

(2) *for all $a, b \in S$, $ab, ba \in Q(a)$ and $a \in Q(a^2)$;*

(3) *for all $a \in S$, $Q(a)$ is a semiprime ideal of S;*

(4) *every quasi-ideal of S is a semiprime ideal of S;*

(5) *for all $a, b \in S$, $Q(ab) = Q(a) \cap Q(b)$;*

(6) *for all $a \in S$, $N(a) = \{x \in S | a \in Q(x)\}$;*

(7) *for every nonempty family $\{Q_\lambda | \lambda \in \Delta\}$ of quasi-ideals of S, $\bigcap_{\lambda \in \Delta} Q_\lambda$ is a semiprime ideal of S;*

(8) *$\mathcal{H} = \mathcal{N}$ is the least semilattice congruence of S such that each of its congruence classes is a group.*

Proof. (1) $\Rightarrow$ (2). Let S be a semilattice L of groups G_α, $(\alpha \in L)$. Consider $a, b \in S$. Then there are $\alpha, \beta \in L$ such that $a \in G_\alpha$, $b \in G_\beta$ and so aba, ab, ba are in $G_\alpha G_\beta \subseteq G_{\alpha\beta}$. Since $G_{\alpha\beta}$ is a group, $ab \in Q(aba) \subseteq Q(a)$. Similarly, $ba \in Q(a)$. Also $a, a^2 \in G_\alpha$ implies that $a \in Q(a^2)$.

(2) $\Rightarrow$ (3). Let $a \in S$. Consider $q \in Q(a)$ and $s \in S$. Then $sq, qs \in Q(q) \subseteq Q(a)$ implies that $Q(a)$ is an ideal of S. Let $u \in S$ be such that $u^2 \in Q(a)$. Then $u \in Q(u^2) \subseteq Q(a)$. Thus $Q(a)$ is a semiprime ideal of S.

(3) $\Rightarrow$ (4). Follows similarly.

(4) $\Rightarrow$ (5). Let $a, b \in S$. Since $a \in Q(a)$ is an ideal of S, so $ab \in Q(a)$ and similarly, $ab \in Q(b)$. Then $ab \in Q(a) \cap Q(b)$ implies that $Q(ab) \subseteq Q(a) \cap Q(b)$. Let $x \in Q(a) \cap Q(b)$. Then $x \in R(a)$ implies that there exists $s_1 \in S$ such that $x = as_1$. Then $x^2 = (as_1)x = a(s_1x)$. Since $Q(a) \cap Q(b)$ is an ideal of S, so $s_1x \in Q(a) \cap Q(b)$ and hence $s_1x \in R(b)$. Then $s_1x = bs_2$ for some $s_2 \in S$. Then $x^2 = abs_2$ which implies that $x^2 \in R(ab)$. Similarly, $x^2 \in L(ab)$. Thus $x^2 \in Q(ab)$ which yields $x \in Q(ab)$. Then $Q(a) \cap Q(b) \subseteq Q(ab)$ and hence $Q(a) \cap Q(b) = Q(ab)$.

(5) $\Rightarrow$ (6). Let $F = \{x \in S \,|\, a \in Q(x)\}$. Consider $x, y \in F$. Then $a \in Q(x) \cap Q(y) = Q(xy)$ implies that $xy \in F$. Thus F is a subsemigroup of S. If for $x, y \in S$, $xy \in F$, then $a \in Q(xy) = Q(x) \cap Q(y)$ implies that $x, y \in F$. Thus F is a filter of S.

Let T be a filter of S containing a and $u \in F$. Then there exists $s \in S$ such that $a = s_1u$. Then $s_1u \in T$ implies that $u \in T$. Hence $F = N(a)$.

(6) $\Rightarrow$ (7). Let $Q = \bigcap_{\lambda \in \Delta} Q_\lambda$. Then Q is a quasi-ideal of S. Let $q \in Q$ and $s \in S$. Now $q \in N(qs)$ implies that $qs \in Q(q) \subseteq Q$. Similarly, $sq \in Q$. Let $a^2 \in Q$. Then $a^2 \in N(a)$ implies that $a \in Q(a^2) \subseteq Q$. Thus Q is a semiprime ideal of S.

(7) $\Rightarrow$ (4). Obvious.

(6) $\Rightarrow$ (8). Let $a, b \in S$. Then $a\mathcal{H}b$ implies that $Q(a) = Q(b)$ and so $a \in N(b)$ and $b \in N(a)$. This implies that $N(a) = N(b)$, i.e., $a\mathcal{N}b$. Thus $\mathcal{H} \subseteq \mathcal{N}$. Similarly,

$\mathcal{N} \subseteq \mathcal{H}$. Hence $\mathcal{H} = \mathcal{N}$ is the least semilattice congruence on S. Then every $\mathcal{H}$-class is a group.

(8) $\Rightarrow$ (1). Obvious. $\square$

In the following theorem we characterize the semigroups which are chains of groups.

Theorem 3.2. *The following conditions are equivalent on a semigroup S:*

(1) *S is a chain of groups;*

(2) *for all $a, b \in S$, $ab, ba \in Q(a)$; and $a \in Q(ab)$ or $b \in Q(ab)$;*

(3) *for all $a \in S$, $Q(a)$ is a completely prime ideal of S;*

(4) *every quasi-ideal of S is a completely prime ideal of S;*

(5) *for all $a, b \in S$, $Q(ab) = Q(a) \cap Q(b)$; and $Q(a) \subseteq Q(b)$ or $Q(b) \subseteq Q(a)$;*

(6) *for all $a, b \in S$, $N(a) = \{x \in S \mid a \in Q(x)\}$ and $N(ab) = N(a) \cup N(b)$;*

(7) *for every nonempty family $\{Q_\lambda \mid \lambda \in \Delta\}$ of quasi-ideals of S, $\bigcap_{\lambda \in \Delta} Q_\lambda$ is a completely prime ideal of S;*

(8) *$\mathcal{H} = \mathcal{N}$ is the least chain congruence on S such that each of its congruence classes is a group.*

Proof. (1) $\Rightarrow$ (2). Let S be a chain $\mathcal{C}$ of groups $G_\alpha (\alpha \in \mathcal{C})$. Then the first part follows from Theorem 3.1. For the second part, let $a \in G_\alpha, b \in G_\beta$, $\alpha, \beta \in \mathcal{C}$. Since $\mathcal{C}$ is a chain, either $\alpha\beta = \alpha$ or $\alpha\beta = \beta$. If $\alpha\beta = \alpha$, then a, $ab \in G_\alpha$ implies that $a\mathcal{H}ab$ and hence $a \in Q(ab)$. Similarly, $\alpha\beta = \beta$ implies that $b \in Q(ab)$. Thus either $a \in Q(ab)$ or $b \in Q(ab)$.

(2) $\Rightarrow$ (3). Let $a \in S$. Then $Q(a)$ is an ideal of S by Theorem 3.1. Consider $x, y \in S$ such that $xy \in Q(a)$. Now $x \in Q(xy)$ or $y \in Q(xy)$ implies that $x \in Q(a)$ or $y \in Q(a)$. Thus $Q(a)$ is a semiprime ideal of S.

(3) $\Rightarrow$ (4). Follows similarly.

(4) $\Rightarrow$ (5). Let $a, b \in S$. Then $Q(ab) = Q(a) \cap Q(b)$, by Theorem 3.1.

Again $a \in Q(ab)$ or $b \in Q(ab)$ implies that $Q(a) \subseteq Q(ab) \subseteq Q(b)$ or $Q(b) \subseteq Q(ab) \subseteq Q(a)$. Thus $Q(a) \subseteq Q(b)$ or $Q(b) \subseteq Q(a)$.

(5) $\Rightarrow$ (6). Let $a \in S$. Then $N(a) = \{x \in S \mid a \in Q(x)\}$, by Theorem 3.1. Let $a, b \in S$. Then, $N(a) \cap N(b) \subseteq N(ab)$. Let $x \in N(ab)$. Then $ab \in Q(x)$. Now we have $Q(ab) = Q(a)$ or $Q(ab) = Q(b)$ which implies that $Q(a) \subseteq Q(x)$ or $Q(b) \subseteq Q(x)$. Then $x \in N(a)$ or $x \in N(b)$. Thus $N(ab) \subseteq N(a)$ or $N(ab) \subseteq N(b)$. Then $N(ab) \subseteq N(a) \cup N(b)$. Hence $N(ab) = N(a) \cup N(b)$.

(6) $\Rightarrow$ (7). Let $Q = \bigcap_{\lambda \in \Delta} Q_\lambda$. In view of Theorem 3.1, we are only to show that Q is completely prime. For $a, b \in S$, if $ab \in Q$, then $ab \in N(ab) = N(a) \cup N(b)$ implies that $a \in Q(ab) \subseteq Q$ or $b \in Q(ab) \subseteq Q$, i.e., $a \in Q$ or $b \in Q$. Thus Q is a completely prime ideal of S.

(7) $\Rightarrow$ (4). Obvious.

(6) $\Rightarrow$ (8). In view of Theorem 3.1, we are only to show that $\mathcal{N}$ is a chain congruence on S. Let $a, b \in S$. Then $ab \in N(ab) = N(a) \cup N(b)$. Thus $ab \in N(a)$ or $ab \in N(b)$, i.e., $N(ab) \subseteq N(a) \subseteq N(a) \cup N(b) = N(ab)$ or $N(ab) \subseteq N(b) \subseteq N(a) \cup N(b) = N(ab)$. Then $N(ab) = N(a)$ or $N(ab) = N(b)$. Then $ab\mathcal{N}a$ or $ab\mathcal{N}b$.

(8) $\Rightarrow$ (1). Obvious. $\square$

4. Semilattice of t-Archimedean semigroups

In this section we characterize the semigroups which are semilattices of t-Archimedean semigroups by their quasi-ideals. Also in this section the semigroups which are chains of t-Archimedean semigroups are characterized.

Let A be a nonempty subset of a semigroup S. Then the *radical of* A in S is given by

$$\sqrt{A} = \{x \in S \mid (\exists n \in \mathbb{N})\ x^n \in A\}.$$

A semigroup S is called *left* (*right*) *Archimedean* if for each $a \in S$, $S = \sqrt{Sa}$, ($S = \sqrt{aS}$) and *t-Archimedean semigroup* if it is both a left Archimedean semigroup and a right Archimedean semigroup. Thus a semigroup S is t-Archimedean if and only if for $a, b \in S$ there exist $n \in \mathbb{N}$ and $x_1, x_2 \in S$ such that $b^n = x_1 a$ and $b^n = a x_2$.

A semigroup S is called a *semilattice* (*chain*) of t-Archimedean semigroups if there exists a congruence ρ on S such that S/ρ is a semilattice (chain) and each ρ-class is a t-Archimedean semigroup.

Let S be a semigroup. Define a binary relation σ on S by : for $a, b \in S$,

$$a\sigma b \Longleftrightarrow b \in \sqrt{SaS} \Longleftrightarrow b^n \in SaS, \text{ for some } n \in \mathbb{N}.$$

Then $a^3 \in SaS$ shows that $a \in \sqrt{SaS}$, i.e., σ is reflexive. So the transitive closure $\rho = \sigma^*$ is reflexive and transitive and therefore the symmetric relation $\eta = \rho \cap \rho^{-1}$ is an equivalence relation. Thus the equivalence relation η is the least semilattice congruence on S.

Recall that for every $a \in S$, $Q(a) = L(a) \cap R(a)$. In general neither $L(a) = Sa$ nor $R(a) = aS$. Also, $Sa \cap aS$ is a quasi-ideal of S which may not contain a. But we have the following lemma.

Lemma 4.1. *Let S be a semigroup. Then $\sqrt{Q(a)} = \sqrt{Sa \cap aS} = \sqrt{Sa} \cap \sqrt{aS}$ for all $a \in S$.*

Lemma 4.2. *Let S be a semigroup such that for all $a, b \in S, ab \in \sqrt{Sa} \cap \sqrt{bS}$. Then*

(1) *for all $a, b \in S, a \in Sb \cap bS \Rightarrow$ for every $r \in \mathbb{N}$ there are $n \in \mathbb{N}, x \in S$ such that $a^n = b^{2^r} x b^{2^r}$ and hence $a \in \sqrt{Q(b^{2^r})}$;*

(2) *for all* $a, b \in S, a \in \sqrt{Q(b)}$ *implies that* $\sqrt{Q(a)} \subseteq \sqrt{Q(b)}$*;*

(3) *the least semilattice congruence* η *on* S *is given by: for all* $a, b \in S$*,*

$$a\eta b \text{ if } b \in \sqrt{Q(a)} \text{ and } a \in \sqrt{Q(b)}.$$

Proof. (1). Let $a, b \in S$ with $a \in Sb \cap bS$. Then there exist $s_1, s_2 \in S$ such that $a = s_1 b = bs_2$. Also, there exist $n \in \mathbb{N}$ and $u_1, u_2 \in S$ such that $(bs_1)^n = u_1 b$ and $(s_2 b)^n = bu_2$. Then $a^{n+1} = s_1(bs_1)^n b = s_1 u_1 b^2$ and $a^{n+1} = b(s_2 b)^n s_2 = b^2 u_2 s_2$. Then $a^{2(n+1)} = b^2 u_2 s_2 s_1 u_1 b^2$ implies that the result is true for $r = 1$. Let for $k \in \mathbb{N}$, there is $p \in \mathbb{N}$ and $x \in S$ such that $a^p = b^{2^k} x b^{2^k}$. Then proceeding as above, we have $q \in \mathbb{N}$ and $y \in S$ such that $a^q = b^{2^{k+1}} y b^{2^{k+1}}$. Thus the result follows by the principle of Mathematical induction.

The last part follows by Lemma 4.1.

(2). For $a \in \sqrt{Q(b)}$, there are $n \in \mathbb{N}$ and $s_1, s_2 \in S$ such that $a^n = s_1 b = bs_2$. Let $x \in \sqrt{Q(a)}$. Then there exists $m \in \mathbb{N}$ such that $x^m \in Sa \cap aS$. Let $r \in \mathbb{N}$ be such that $2^r > n$. Then, by (1), we find $p \in \mathbb{N}$ and $u \in S$ such that $x^p = a^{2^r} u a^{2^r}$ which implies that $x^p = a^n a^{2^r - n} u a^{2^r - n} a^n = bs_2 a^{2^r - n} u a^{2^r - n} s_1 b$. Then $x \in \sqrt{Q(b)}$, by the Lemma 4.1.

(3). Consider $a \in S$. Then $x \in \sqrt{Q(a)}$ implies that $x^n = s_1 a = as_2$ for some $n \in \mathbb{N}$ and $s_1, s_2 \in S$. Then $x^{n+n} = s_1 a^2 s_2$ implies that $x \in \sqrt{SaS}$. Thus $\sqrt{Q(a)} \subseteq \sqrt{SaS}$. Let $y \in \sqrt{SaS}$. Then there are $m \in \mathbb{N}$ and $t_1, t_2 \in S$ such that $y^m = t_1 a t_2$. Again $t_1 a t_2 \in \sqrt{St_1 a} \subseteq \sqrt{Sa}$ and $t_1 a t_2 \in \sqrt{at_2 S} \subseteq \sqrt{aS}$ implies that $y^m \in \sqrt{aS} \cap \sqrt{Sa} = \sqrt{Q(a)}$ and so $y \in \sqrt{Q(a)}$, by the Lemma 4.1. Thus $\sqrt{SaS} \subseteq \sqrt{Q(a)}$ and hence $\sqrt{Q(a)} = \sqrt{SaS}$.

Now for $a, b \in S$, $a\eta b$ implies that there are $c_1, c_2, \ldots, c_n, d_1, d_2, \ldots, d_m \in S$ such that $a\sigma c_1$, $c_1 \sigma c_2, \ldots, c_{n-1}\sigma c_n$, $c_n \sigma b$ and $b\sigma d_1$, $d_1 \sigma d_2, \ldots, d_{m-1}\sigma d_m$, $d_m \sigma a$. These give $c_1 \in \sqrt{Q(a)}$, $c_2 \in \sqrt{Q(c_1)}, \ldots, b \in \sqrt{Q(c_n)}$ and $d_1 \in \sqrt{Q(b)}$, $d_2 \in \sqrt{Q(d_1)}, \ldots, a \in \sqrt{Q(d_m)}$ so that $b \in \sqrt{Q(a)}$ and $a \in \sqrt{Q(b)}$, by (2). □

Recall that for $a, b \in S$,

$$a\mathcal{H}b \iff Q(a) = Q(b).$$

Let us define $\sqrt{\mathcal{H}}$, the radical of $\mathcal{H}$ on S by: for $a, b \in S$,

$$a\sqrt{\mathcal{H}}b \iff \sqrt{Q(a)} = \sqrt{Q(b)}.$$

Now we have the main theorem of this section:

Theorem 4.3. *The following conditions are equivalent on a semigroup* S*:*

(1) S *is a t-Putcha semigroup;*

(2) *for all* $a, b \in S$, $b \in SaS$ *implies* $b \in \sqrt{Q(a)}$*;*

(3) *for all* $a, b \in S$, $ab \in \sqrt{Sa} \cap \sqrt{bS}$;

(4) $\sqrt{Q}$ *is an ideal of* S *for every quasi-ideal* Q *of* S;

(5) $\sqrt{Q(a)}$ *is an ideal of* S, *for all* $a \in S$;

(6) $N(a) = \{x \in S \mid a \in \sqrt{Q(x)}\}$ *for all* $a \in S$;

(7) $\mathcal{N} = \sqrt{\mathcal{H}}$ *is the least semilattice congruence and the congruence classes are* t*-Archimedean semigroups.*

Proof. (1) $\Rightarrow$ (2). Let ρ be a semilattice congruence on S such that the ρ-classes $T_\alpha, \alpha \in S/\rho$ are t-Archimedean semigroups. Let $a, b \in S$ be such that $b \in SaS$. Then there are $s_1, s_2 \in$ such that $b = s_1 a s_2$. Now $s_1 a s_2 \rho a s_1 s_2 \rho s_1 s_2 a$ implies that $b, as_1s_2, s_1s_2a \in T_\alpha$ for some $\alpha \in S/\rho$. Since T_α is a t-Archimedean semigroup, there exist $n \in \mathbb{N}$ and $u_1, u_2 \in T_\alpha$ such that $b^n = as_1s_2u_1$ and $b^n = u_2s_1s_2a$. Thus $b \in \sqrt{Q(a)}$, by Lemma 4.1.

(2) $\Rightarrow$ (3). Let $a, b \in S$. Now $(ab)^2 = abab$ implies $(ab)^2 \in SaS \cap SbS$. Then $(ab)^2 \in \sqrt{Q(a)} \cap \sqrt{Q(b)} \subseteq \sqrt{Sa} \cap \sqrt{bS}$ and hence $ab \in \sqrt{Sa} \cap \sqrt{bS}$.

(3) $\Rightarrow$ (4). Let Q be a quasi-ideal of S and let $u \in \sqrt{Q}$ and $c \in S$. Then $u^n = q$ for some $n \in \mathbb{N}, q \in Q$. Also by (3), there is $m \in \mathbb{N}$ such that $(uc)^m \in Su$ and $(uc)^{m+1} \in uSu$. Consider $r \in \mathbb{N}$ such that $2^r > n$. Then by Lemma 4.2, there are $m_1 \in \mathbb{N}$ and $x \in S$ such that $(uc)^{m_1} = u^{2^r} x u^{2^r} = q u^{2^r - n} x u^{2^r - n} q$ which implies that $uc \in \sqrt{qS \cap Sq} = \sqrt{Q(q)} \subseteq \sqrt{Q}$, by Lemma 4.1. Similarly, $cu \in \sqrt{Q}$. Thus $\sqrt{Q}$ is an ideal of S.

(4) $\Rightarrow$ (5). Trivial.

(5) $\Rightarrow$ (3). Let $a, b \in S$. Then $\sqrt{Q(a)}$ and $\sqrt{Q(b)}$ are ideals of S. Then $ab \in \sqrt{Q(a)} \cap \sqrt{Q(b)}$ and hence $ab \in \sqrt{Sa} \cap \sqrt{bS}$.

(3) $\Rightarrow$ (6). Let $a \in S$ and $F = \{x \in S | a \in \sqrt{Q(x)}\}$. Consider $y, z \in F$. Then there exist $n \in \mathbb{N}, u_1, u_2 \in S$ such that $a^n = u_1 z$ and $a^n = y u_2$. Also, by (3), there are $m_1, m_2 \in \mathbb{N}, w_1, w_2 \in S$ such that $(u_2u_1zy)^{m_1} = zyw_1$ and $(zyu_2u_1)^{m_2} = w_1zy$. Now $a^{2n} = yu_2u_1z$ implies $a^{2n(m_1+1)} = (yu_2u_1z)^{m_1+1} = y(u_2u_1zy)^{m_1}u_2u_1z = (yz)yw_1u_2u_1z$. Also, $a^{2n(m_2+1)} = yu_2u_1zw_2z(yz)$. Thus $yz \in F$, by Lemma 4.1; and hence F is a subsemigroup of S.

Let $y, z \in S$ be such that $yz \in F$. Then $a \in \sqrt{Q(yz)} = \sqrt{yzS} \cap \sqrt{Syz} \subseteq \sqrt{yS} \cap \sqrt{Sz}$. Now, by (3), $yz \in \sqrt{Sy}$, and so $yz \in \sqrt{yS} \cap \sqrt{Sy} = \sqrt{Q(y)}$, by Lemma 4.1. Then $\sqrt{Q(yz)} \subseteq \sqrt{Q(y)}$, by Lemma 4.2. Thus $a \in \sqrt{Q(y)}$ and hence $y \in F$. Similarly, $z \in F$. Thus F is a filter that contains a. Let T be a filter of S containing a and $y \in F$. Then $a^m = sy$ for some $m \in \mathbb{N}, s \in S$. Now $a^m \in T$ implies $sy \in T$ and hence $y \in T$. Thus $F = N(a)$.

(6) $\Rightarrow$ (7). Consider $a, b \in S$. Then $ab \in N(ab)$ implies that $a, b \in N(ab)$. Then, by (6), $ab \in \sqrt{Q(a)} \cap \sqrt{Q(b)} \subseteq \sqrt{Sa} \cap \sqrt{bS}$. If $a\mathcal{N}b$ then $N(a) = N(b)$ implies that $b \in \sqrt{Q(a)}$ and $a \in \sqrt{Q(b)}$. So, $\sqrt{Q(b)} \subseteq \sqrt{Q(a)}$ and $\sqrt{Q(a)} \subseteq \sqrt{Q(b)}$, by Lemma 4.2. Thus $a\sqrt{\mathcal{H}}b$ and hence $\mathcal{N} \subseteq \sqrt{\mathcal{H}}$. Similarly, $\sqrt{\mathcal{H}} \subseteq \mathcal{N}$. Hence $\mathcal{N} = \sqrt{\mathcal{H}}$ is the least semilattice congruence.

Let T be an $\mathcal{N}$-class in S. Since $\mathcal{N}$ is a semilattice congruence, T is a subsemigroup. Consider $a, b \in T$. Then $a^2 \mathcal{N} b$ implies that $N(a^2) = N(b)$; and by (6) we have $b \in \sqrt{Q(a^2)}$. Thus there are $n \in \mathbb{N}$ and $s_1, s_2 \in S$ such that $b^n = s_1 a^2$ and $b^n = a^2 s_2$ which implies that $b^{n+1} = bs_1 a^2$ and $b^{n+1} = a^2 s_2 b$. Since $\mathcal{N}$ is a semilattice congruence, $t_1 = bs_1 a \mathcal{N} bs_1 a^2 \mathcal{N} b^{n+1} \mathcal{N} b$ and $t_2 = as_2 b \mathcal{N} b$ which implies that $t_1 = bs_1 a \in T$ and $t_2 = as_2 b \in T$. Thus $b \in \sqrt{Ta} \cap \sqrt{aT}$ and hence T is a t-Archimedean semigroup.

(7) $\Rightarrow$ (1). Follows directly. $\square$

Theorem 4.4. *The following conditions on a semigroup S are equivalent:*

(1) *S is a chain of t-Archimedean semigroups.*

(2) *S is a t-Putcha semigroup and for all $a, b \in S$, $b \in \sqrt{Q(a)}$ or $a \in \sqrt{Q(b)}$.*

(3) *For all $a, b \in S$, $N(a) = \{x \in S \mid a \in \sqrt{Q(x)}\}$ and $N(ab) = N(a) \cup N(b)$.*

(4) *$\mathcal{N} = \sqrt{\mathcal{H}}$ is the least chain congruence on S such that each of its congruence classes is t-Archimedean.*

Proof. (1) $\Rightarrow$ (2). Let S be a chain $\mathcal{C}$ of t-Archimedean semigroups $S_\alpha (\alpha \in \mathcal{C})$. Let $a, b \in S$. Then $a \in S_\alpha$ and $a \in S_\beta$ for some $\alpha, \beta \in \mathcal{C}$. Since $\mathcal{C}$ is a chain, either $\alpha\beta = \alpha$ or $\alpha\beta = \beta$. If $\alpha\beta = \alpha$, then $a, ab \in S_\alpha$; and since S_α is a t-Archimedean semigroup, there exist $n \in \mathbb{N}$ and $x_1, x_2 \in S_\alpha$ such that $a^n = x_1 ab$ and $a^n = abx_2$. Now, by Theorem 4.3, since S is a semilattice of t-Archimedean semigroup, there are $m \in \mathbb{N}$ and $s \in S$ such that $(abx_2)^m = bx_2 s$. Then we have $a^{nm} = s_1 b$ and $a^{nm} = bx_2 s$ for some $s_1 \in S$ and hence $a \in \sqrt{Q(b)}$, by Lemma 4.1. If $\alpha\beta = \beta$, then $b, ab \in S_\beta$ and similarly as above we have $b \in \sqrt{Q(a)}$.

(2) $\Rightarrow$ (3). By Theorem 4.3, we have $N(a) = \{x \in S \mid a \in \sqrt{Q(x)}\}$, since S is a t-Putcha semigroup. Let $a, b \in S$. Then $ab \in N(ab)$ implies that $a \in N(ab)$ and $b \in N(ab)$, and hence $N(a) \cup N(b) \subseteq N(ab)$. Again, either $a \in \sqrt{Q(b)}$ or $b \in \sqrt{Q(a)}$. If $a \in \sqrt{Q(b)}$, then there are $n \in \mathbb{N}$ and $s \in S$ such that $a^n = bs$ and so $a^{n+1} = abs$. Since S is a semilattice of t-Archimedean semigroups, there exist $m \in \mathbb{N}$ and $t \in S$ such that $(abs)^m = tab$, by Theorem 4.3. Then we have $a^{(n+1)m} = tab$ and $a^{(n+1)m} = abt_1$ for some $t_1 \in S$. Then $a \in \sqrt{Q(ab)}$ which implies that $ab \in N(a)$. Thus $N(ab) \subseteq N(a)$. If $b \in \sqrt{Q(a)}$, then similarly we have $N(ab) \subseteq N(b)$, which shows that $N(ab) \subseteq N(a) \cup N(b)$. Hence $N(ab) = N(a) \cup N(b)$.

(3) $\Rightarrow$ (4). It follows by Theorem 4.3 that $\mathcal{N} = \sqrt{\mathcal{H}}$ is the least semilattice congruence on S and each $\mathcal{N}$-class is a t-Archimedean semigroup.

Now consider $a, b \in S$. Then $ab \in N(a) \cup N(b)$ shows that $ab \in N(a)$ or $ab \in N(b)$. Again $N(a) \subseteq N(ab)$ and $N(b) \subseteq N(ab)$. Thus either $N(ab) \subseteq N(a) \subseteq N(ab)$ or $N(ab) \subseteq N(b) \subseteq N(ab)$. i.e., either $a \mathcal{N} ab$ or $b \mathcal{N} ab$. Hence $\mathcal{N}$ is a chain congruence on S. Since every chain is a semilattice and $\mathcal{N}$ is the least semilattice congruence, it is the least chain congruence on S.

(4) $\Rightarrow$ (1). Trivial. $\square$

Theorem 4.5. *The following conditions on a semigroup S are equivalent:*

(1) *S is a chain of t-Archimedean semigroups;*

(2) *$\sqrt{Q}$ is a completely prime ideal of S for every quasi-ideal Q of S;*

(3) *$\sqrt{Q(a)}$ is a completely prime ideal of S for every $a \in S$;*

(4) *$\sqrt{Q(ab)} = \sqrt{Q(a)} \cap \sqrt{Q(b)}$ for all $a, b \in S$ and every quasi-ideal of S is semiprimary .*

Proof. (1) $\Rightarrow$ (2). Let S be a chain $\mathcal{C}$ of t-archimedean semigroups $\{S_\alpha \mid \alpha \in \mathcal{C}\}$. We take a quasi-ideal Q of S. Then $\sqrt{Q}$ is an ideal of S, by Theorem 4.3. Let $x, y \in S$ be such that $xy \in \sqrt{Q}$. Then there is $n \in \mathbb{N}$ such that $(xy)^n = u \in Q$. Suppose $x \in S_\alpha$ and $y \in S_\beta$,$\alpha, \beta \in \mathcal{C}$. Since $\mathcal{C}$ is a chain, either $\alpha\beta = \alpha$ or $\alpha\beta = \beta$. If $\alpha\beta = \alpha$, then $x, u \in S_\alpha$. Since S_α is t-Archimedean, so $x \in \sqrt{Q(u)} \subseteq \sqrt{Q}$. Similarly, if $\alpha\beta = \beta$, then $y \in \sqrt{Q}$. Hence $\sqrt{Q}$ is a completely prime ideal of S.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (4). Let $a, b \in S$. Then $\sqrt{Q(a)}$ and $\sqrt{Q(b)}$ are ideals of S and hence $ab \in \sqrt{Q(a)} \cap \sqrt{Q(b)}$. This implies $\sqrt{Q(ab)} \subseteq \sqrt{Q(a)} \cap \sqrt{Q(b)}$, by Lemma 4.2 and Theorem 4.3. Since $\sqrt{Q(ab)}$ is completely prime, so $a, b \in \sqrt{Q(ab)}$ which implies $\sqrt{Q(a)} \cap \sqrt{Q(b)} \subseteq \sqrt{Q(ab)}$. Thus $\sqrt{Q(ab)} = \sqrt{Q(a)} \cap \sqrt{Q(b)}$.

Let Q be a quasi-ideal of S and $x, y \in S$ be such that $xy \in Q$. Then $xy \in \sqrt{Q(xy)}$ implies that $x \in \sqrt{Q(xy)}$ or $y \in \sqrt{Q(xy)}$. Thus $x^n \in \sqrt{Q(xy)} \subseteq Q$ or $y^n \in \sqrt{Q(xy)} \subseteq Q$ for some $n \in \mathbb{N}$. Hence Q is semiprimary.

(4) $\Rightarrow$ (1). Let $a, b \in S$. Then $ab \in \sqrt{Q(a)} \cap \sqrt{Q(b)} \subseteq \sqrt{Sa} \cap \sqrt{bS}$. Then by Theorem 4.3, S is a t-Putcha semigroup. Since $\sqrt{Q(ab)}$ is a semiprimary, $ab \in Q(ab)$ implies that $a \in \sqrt{Q(ab)} = \sqrt{Q(a)} \cap \sqrt{Q(b)} \subseteq \sqrt{Q(b)}$ or $b \in \sqrt{Q(ab)} \subseteq \sqrt{Q(a)}$. Thus S is a chain of t-Archimedean semigroups by Theorem 4.4. $\square$

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Normal edge-transitive Cayley graphs on certain groups of orders 4n and 8n

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Abstract. Normal edge-transitive Cayley graph $Cay(G, S)$ where G is the generalized quaternion group Q_{4n} of order $4n$ or a certain group V_{8n} of order $8n$ is investigated. It is shown that up to isomorphism there is only one tetravalent normal edge-transitive Cayley graph when $G \cong Q_{4n}$ is the generalized quaternion group and its automorphism group is found. In the case of V_{8n} we show that there is no normal edge-transitive Cayley graph on V_{8n}.

1. Introduction

We will be concerned with simple graphs, which mean graphs with no multiple edges and loops. Let $\Gamma = (V, E)$ be a graph with vertex set V and edge set E. The edge joining the vertices u and v is denoted by $e = \{u, v\}$. The group of the automorphisms of the graph is denoted by $A = \mathbb{A}ut\,(\Gamma)$, and Γ is called *vertex* or *edge transitive* if A acts transitively on V or E respectively. Let G be a finite group and S be a subset of G such that $S = S^{-1}$ and $1 \notin S$. The Cayley graph of G on S is denoted by $\Gamma = Cay(G, S)$ and has its vertex set G and edge set $e = \{x, sx\}$ where $x \in G$ and $s \in S$. Therefore Γ is a regular graph of valency $|S|$, and it is connected if and only if S generates G. For $g \in G$ the mapping defined by $\rho_g : G \to G$, $\rho_g(x) = xg$, $x \in G$ is a permutation of G preserving the edges of Γ, hence it is an automorphism of Γ. It can be verified that $R(G) = \{\rho_g \mid g \in G\}$ is a subgroup of $Aut(\Gamma)$ isomorphic to G which acts regularly on the vertices of Γ, hence Γ is a vertex transitive graph.

For the Cayley graph $\Gamma = Cay(G, S)$ we define the group $Aut(G, S)$ by putting $Aut(G, S) = \{\sigma \in Aut(G) \mid \sigma(S) = S\}$. It can be verified that $Aut(G, S)$ is a subgroup of $A = Aut(\Gamma)$ which acts on $R(G)$ by $\rho_g^{\sigma} := \rho_{\sigma^{-1}(g)}$, where $\sigma \in Aut(G, S)$ and $\rho_g \in R(G)$. Therefore the semi-direct product $R(G) \rtimes Aut(G, S)$ is a subgroup of A.

It is proved in [3] that $N_A(R(G)) = R(G) \rtimes Aut(G, S)$, where $N_A(R(G))$ denotes the normalizer of $R(G)$ in A. In [7] the graph Γ is called *normal* if $R(G)$ is a normal subgroup of A and obviously in this case we have $A = R(G) \rtimes Aut(G, S)$.

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The normality of Cayley graphs has been studied by various authors from different point of views. If one is interested to study the normality of the Cayley graphs it suffices to consider the connected normal Cayley graphs, because in [5] all the disconnected normal Cayley graphs are determined. The research on edge-transitive Cayley graphs of small valency is of interest to many authors. In [6] the authors determined all the tetravalent edge-transitive Cayley graphs on the group $PSL_2(p)$ and Brian P. Corr et al. in [1] determined normal edge-transitive Cayley graphs of Frobenius group of order a product of two different primes. In [8] tetravalent non-normal Cayley graphs of order $4p$, p a prime number, are determined. In [2] the authors studied normal edge-transitive Cayley graphs on group of order $4p$ where p is an odd prime. Motivated by [2] we are interested to investigate normal edge-transitive Cayley graphs on the generalized quaternion group of order 4n and a certain group of order $8n$, where n is an arbitrary natural number. In particular we obtain:

Main result 1. *Let $Q_{4n} = \langle a, b \,|\, a^{2n} = b^4 = 1,\ a^n = b^2,\ b^{-1}ab = a^{-1}\rangle$ be the generalized quaternion group of order $4n$. Then up to isomorphism there is only one normal edge-transitive tetravalent Cayley graph of G and its automorphism group is isomorphic to $G \rtimes D_8$ if n is even and isomorphic to $G \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ if n is odd.*

Main result 2. *Let $V_{8n} = \langle a, b \,|\, a^{2n} = b^4 = 1,\ (ab)^2 = (a^{-1}b)^2 = 1\rangle$ be a group of order $8n$. Then there is no normal edge-transitive Cayley graph on V_{8n}.*

2. Preliminary results

Let G be a group and S be a subset of G such that $1 \notin S$. The *Cayley di-graph* (*directed graph*) $Cay(G, S)$ of G relative to S has G as its vertex set and (x, sx) as its edge set, where $x \in G$ and $s \in S$. If S is an inverse closed subset of G, i.e., $S = S^{-1}$, then $Cay(G, S)$ is an undirected graph that is simply called a Cayley graph. The following result can be found for example in [4].

Lemma 2.1. *Let $\Gamma = Cay(G, S)$ be the Cayley graph of G with respect to S. Then the following hold:*

(*i*) $N_A(R(G)) = R(G) \rtimes \mathbb{A}ut(G, S)$.

(*ii*) $R(G) \trianglelefteq A$ *if and only if* $A = R(G) \rtimes \mathbb{A}ut(G, S)$.

(*iii*) Γ *is normal iff* $A_1 = \mathbb{A}ut(G, S)$, *where* A_1 *denotes the stabilizer of the vertex* 1 *under* A.

We set $N = N_A(R(G)) = R(G) \rtimes \mathbb{A}ut(G, S)$ and we remark that for the normal edge-transitivity of $Cay(G, S)$ the group N need only be transitive on undirected edges, and may or may not be transitive on ordered pairs of adjacent vertices. From [4] we have the following result which is useful in our investigation.

Lemma 2.2. *Let $\Gamma = Cay(G,S)$ be an undirected Cayley graph of the group G on S and let $N = N_A(R(G)) = R(G) \rtimes \mathbb{A}ut(G,S)$. Then the following are equivalent:*

(i) *Γ is normal edge-transitive.*

(ii) *$S = T \cup T^{-1}$ where T is an orbit of $\mathbb{A}ut(G,S)$ on S.*

(iii) *There exist a subgroup H of $\mathbb{A}ut(G)$ and $g \in G$ such that $S = g^H \cup (g^{-1})^H$, where $g^H = \{gh \mid h \in H\}$.*

3. Cayley graphs on a certain group of order 4n

First we consider the generalized quaternion group. The generalized quaternion group of order $4n$ has the following presentation:

$$Q_{4n} = \langle a, b \mid a^{2n} = b^4 = 1,\ a^n = b^2,\ b^{-1}ab = a^{-1}\rangle.$$

It is easy to verify that the center Z of Q_{4n} has order 2 generated by $a^n = b^2$ and $\frac{Q_{4n}}{Z} \cong D_{2n}$. The elements of Q_{4n} are of the form a^ib^j, $0 \leqslant i \leqslant 2n-1$, $j = 0, 1$. Element orders of Q_{4n} is as follows:

$$O(a^k) = \tfrac{2n}{(k,2n)},\ \ 0 \leqslant k \leqslant 2n-1,\ \ (0,2n) = 2n,$$

$$O(a^kb) = 4,\ \ 0 \leqslant k \leqslant 2n-1.$$

Proposition 3.1. *The automorphism group of Q_{4n} is of order $2n\varphi(2n)$ and is isomorphic to the semi-direct product $\mathbb{Z}_{2n} \rtimes \Phi_{2n}$, where Φ_{2n} is the group of units of $\mathbb{Z}_{2n}$.*

Proof. Let $\varphi \in \mathbb{A}ut(Q_{4n})$. Then φ is completely determined by defining $\varphi(a)$ and $\varphi(b)$. Since φ preserves order of elements we have $O(\varphi(a)) = 2n$ and $O(\varphi(b)) = 4$. Therefore $\varphi(a) = a^k$, where $1 \leqslant k < 2n$, $(k,2n) = 1$. If $\varphi(b) = a^l$has order 4, then $\varphi(\langle a,b\rangle) \subseteq \langle a\rangle$ or $G \subseteq \langle a\rangle$ which is a contradiction. Therefore $\varphi(b) = a^lb$, $0 \leqslant l < 2n$. It can be verified that φ in fact defines an automorphism of Q_{4n} and if we set $\varphi_{k,l}(a) = a^k$, $\varphi_{k,l}(b) = a^lb$ with k, l satisfying the above conditions, then $\varphi_{k,l}\varphi_{k',l'} = \varphi_{kk',l+kl'}$, hence:

$$\begin{aligned} \mathbb{A}ut(Q_{4n}) &= \{\varphi_{k,l} \mid k \in \Phi_{2n},\ l \in \mathbb{Z}_{2n}\} \\ &\cong \left\{ \begin{bmatrix} k & l \\ 0 & 1 \end{bmatrix} : k \in \Phi_{2n}, l \in \mathbb{Z}_{2n} \right\} \end{aligned}$$

But if we set

$$N = \left\{ \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix} : l \in \mathbb{Z}_{2n} \right\}$$

and

$$H = \left\{ \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} : k \in \Phi_{2n} \right\},$$

then $\mathbb{A}ut(Q_{4n}) = N \rtimes H \cong \mathbb{Z}_{2n} \rtimes \Phi_{2n}$, where the group Φ_{2n} has order $\varphi(2n)$. The proof is completed now. $\square$

Now let S be a subset of Q_{4n} such that $1 \notin S$, $S = S^{-1}$ and $\langle S \rangle = Q_{4n}$. Our aim is to consider normal edge-transitive Cayley graphs Q_{4n} on S. By Lemma 2.2, elements of S have the same order d and $S = T \cup T^{-1}$ where T is an orbit of $\mathbb{A}ut(G, S)$. If S contains an element of order 2 this element must be b^2 which is a central element and invariant under $\mathbb{A}ut(G, S)$ and S can not break as $S = T \cup T^{-1}$. This implies that $|S|$ should be even. Since $\langle a \rangle$is a cyclic group of order $2n$, for each divisor d of $2n$ there is a unique subgroup of $\langle a \rangle$with order d and elements of order d of $\langle a \rangle$ lie in this subgroup. If $d \neq 4$, elements of order d of Q_{4n} lie in $\langle a \rangle$ and obviously can not generate Q_{4n}.

Next we assume elements of S are of order $d = 4$. Keeping fixed the above notations we state the following:

Proposition 3.2. *S can not contain elements of order 4 contained in $\langle a \rangle$.*

Proof. On the contrary suppose $a^k \in \langle a \rangle \cap S$ has order 4. Then$\frac{2n}{(k,2n)} = 4$ implying $n = 2(k, 2n)$. Hence n must be even and we set $n = 2t$ which implies k is an odd multiple of t, i.e., $k = (2l + 1)t = \frac{(2l+1)n}{2}$. Then from $0 \leqslant k < 2n$ we obtain $l = 0$ or 1, hence $k = \frac{n}{2}$ or$\frac{3n}{2}$. This implies that the only elements of order4 in $\langle a \rangle$ are $a^{\frac{n}{2}}$ and $a^{\frac{3n}{2}}$.

But in this case if we apply the automorphisms φ of Q_{4n} obtained in Proposition 3.1 we see that $\{a^{\frac{n}{2}}, a^{\frac{3n}{2}}\}$ is invariant under $\mathbb{A}ut(Q_{4n})$. Again S can not break as $S = T \cup T^{-1}$ with T as an $\mathbb{A}ut(G, S)$ orbit and this completes the proof. $\square$

By the above proposition if $Cay(G, S)$ is normal edge-transitive, then we will have $S \subseteq \{a^i b \,|\, 0 \leqslant i < 2n\}$.

Proposition 3.3. *Let $0 \leqslant i \neq j < 2n$. Then $\langle a^i b, a^j b \rangle = Q_{4n}$ if and only if $(i - j, 2n) = 1$.*

Proof. Suppose $j < i$, $(i - j, 2n) = d$ and $H = \langle a^i b, a^j b \rangle$. Then using the defining relations for Q_{4n} we deduce $(a^i b)^2 = b^2 \in H$. Therefore $a^{i-j} \in H$. Since $(i - j, 2n) = d$ we obtain $a^d \in H$ and d is the least power of a belonging to H. Now elements of H can be organized as a^{id}, $a^{id}b^2$, $0 \leqslant i < \dfrac{2n}{d}$. Hence$|H| = \frac{4n}{d}$ and $H = Q_{4n}$ if and only if $d = 1$ and the proof is complete. $\square$

Next we turn on tetravalent Cayley graphs of Q_{4n}. By what we proved earlier we have $S = \{a^i b, a^j b, a^i b^{-1}, a^j b^{-1}\}$, where $(i - j, 2n) = 1$. We define the following concept which is needed in the next result.

If G is a group with two subsets S and T such that $1 \notin S$, $1 \notin T$, and if there is an automorphism φ of G such that $\varphi(S) = T$, then $Cay(G, S)$ is isomorphic to $Cay(G, T)$. In this case S and T are called equivalent.

Proposition 3.4. *If* $(i - j, 2n) = 1$, *then* $\{b, ab, b^{-1}, ab^{-1}\}$ *is equivalent to* $\{a^i b, a^j b, a^i b^{-1}, a^j b^{-1}\}$.

Proof. It is enough to apply the automorphism $\varphi_{j-i,i}$ of Q_{4n} to one of the above sets. □

Theorem 3.5. *There is only one tetravalent normal edge-transitive Cayley graph of Q_{4n} and the automorphism group of this graph is isomorphic to $Q_{4n} \rtimes D_8$ if n is even and isomorphic to $Q_{4n} \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ if n is odd.*

Proof. By Proposition 3.3 we have $S \subseteq \{a^i b \,|\, 0 \leqslant i < 2n\}$ and $|S| = 4$, $S = S^{-1}$, $\langle S \rangle = Q_{4n}$ forces $S = \{a^i b, a^j b, a^i b^{-1}, a^j b^{-1}\}$ for some i, j where $(i - j, 2n) = 1$. Now by Proposition 3.4 we may take $S = \{b, ab, b^{-1}, ab^{-1}\}$. This proves that up to equivalence there is a unique tetravalent normal edge-transitive Cayley graph of Q_{4n}. Next we determine $\mathbb{A}ut(Q_{4n}, S)$.

Since $\langle S \rangle = Q_{4n}$ the group $\mathbb{A}ut(Q_{4n}, S)$ acts on S faithfully, from which we deduce $\mathbb{A}ut(Q_{4n}, S) \leqslant \mathbb{S}_4$. If $\mathbb{A}ut(Q_{4n}, S)$ contains an element σ of order 3, then σ would fix an element say $\alpha \in S$, but in this case $\sigma(\alpha^{-1}) = \alpha^{-1}$ and σ can not be a 3-cycle. Therefore $|\mathbb{A}ut(Q_{4n}, S)|$ is a divisor of 8. It is easy to verify that the elements $\varphi_{1,n}$ and $\varphi_{2n-1,1}$ belong to $\mathbb{A}ut(Q_{4n}, S)$ and $\langle \varphi_{1,n}, \varphi_{2n-1,1} \rangle \cong V_4$ the Klein's four group. We distinguish two cases:

Case (*i*). n is even. In this case $\varphi_{n-1,1}$ is also an element of $\mathbb{A}ut(G, S)$ of order 4 and $\langle \varphi_{n-1,1}, \varphi_{2n-1,1}, \varphi_{1,n} \rangle \cong D_8$ is a subgroup of $\mathbb{A}ut(Q_{4n}, S)$, hence $\mathbb{A}ut(Q_{4n}, S) \cong D_8$ therefore the automorphism group of $Cay(Q_{4n}, S)$ is isomorphic to $Q_{4n} \rtimes D_8$.

Case (*ii*). n is odd. In this case we will prove that $\mathbb{A}ut(Q_{4n}, S)$ does not contain an element of order 4. On the contrary suppose $\varphi_{k,l} \in \mathbb{A}ut(Q_{4n}, S)$ is an element of order 4. Therefore we have one of the cases $\varphi_{k,l}(b) = ab$, $\varphi_{k,l}(ab) = b^{-1}$ or $\varphi_{k,l}(b) = ab^{-1}$, $\varphi_{k,l}(ab) = b$. In the first case we obtain $a^l b = ab$ and $a^{k+l} b = b^{-1}$, hence $a^{l-1} = 1$, $a^{k+l+n} = 1$. Since a is of order $2n$ we obtain $k = n - 1$, and because n is odd, $2|(n - 1, 2n) = (k, 2n) = 1$, a contradiction. In the second case we obtain $a^l b = ab^{-1}$, $a^{k+l} b = b$, hence $a^{l+n-1} = 1$ and $a^{l+k} = 1$. Again from these relations we obtain $k = n - 1$, a contradiction.

Since $\mathbb{A}ut(Q_{4n}, S)$ does not contain elements of order 4 we obtain $\mathbb{A}ut(Q_{4n}, S) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, hence the automorphism group of $Cay(Q_{4n}, S)$ is isomorphic to $Q_{4n} \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ and the proof is complete. □

4. Cayley graph of a group of order 8n

Next we are going to study the normal edge-transitive Cayley graphs of a certain group of order $8n$ whose presentation is given as follows:

$$V_{8n} = \langle a, b \mid a^{2n} = b^4 = 1, (ab)^2 = (a^{-1}b)^2 = 1\rangle$$

where n is a natural number. Using similar techniques as used in the previous section in finding the automorphism group of Q_{4n} one can prove the following:

Lemma 4.1. *$\mathbb{A}ut(V_{8n})$ is a group of order $4n\varphi(2n)$ if $n > 1$ and it is a group of order 8 if $n = 1$.*

Proof. In fact if $n = 1$, the group $V_1 = D_8$ is the dihedral group of order 8. To define an automorphism f of V_{8n} it is enough to define $f(a)$ and $f(b)$ which can be verified they are of the form:

$$f_{i,r,s,t}(a) = a^i b^r$$
$$f_{i,r,s,t}(b) = a^{2t} b^s,$$

where $(i, 2n) = 1$, $r = 0, 2$, $s = \pm 1$, $1 \leqslant t \leqslant n$. □

Lemma 4.2. *For V_{8n} we have*

$$\langle a^2, b^2, ab\rangle = \{a^{2k}, a^{2k+1}b^{\pm 1}, a^{2k}b^2 \mid 1 \leqslant k \leqslant n\}$$

Proof. If we set $X = \{a^{2k}, a^{2k+1}b^{\pm 1}, a^{2k}b^2 \mid 1 \leqslant k \leqslant n\}$ since $\{a^2, b^2, ab\} \subseteq X$ it is sufficient to show that X is a subgroup of V_{8n} and it is obviously true. □

Theorem 4.3. *There is no normal edge-transitive Cayley graph $Cay(G, S)$ for $G = V_{8n}$ if S has an element of order 2.*

Proof. Suppose $Cay(G, S)$ is a normal edge-transitive Cayley graph and S has an element of order 2.

Elements of order 2 in V_{8n} are $Y = \{a^n, b^2, a^n b^2, a^{2k+1}b^{\pm 1} \mid 1 \leqslant k \leqslant n\}$. Since all elements of S have the same order we have $S \subseteq Y$. If n is even then $\langle S\rangle \subseteq \langle Y\rangle \subseteq \langle a^2, b^2, ab\rangle \neq V_{8n}$, a contradiction. Hence n is odd.

If $S \cap \{a^n, a^n b^2\} = \emptyset$ then $\langle S\rangle \subseteq \langle a^2, b^2, ab\rangle \neq V_{8n}$ a contradiction, hence $S \cap \{a^n, a^n b^2\} \neq \emptyset$. For all $f \in \mathbb{A}ut(G, S)$ we have $f(\{a^n, a^n b^2\}) = \{a^n, a^n b^2\}$, therefore $S \cap \{a^n, a^n b^2\}$ is an orbit of $f \in \mathbb{A}ut(G, S)$ on S and it is a contradiction by Lemma 2.2. □

Theorem 4.4. *There is no normal edge-transitive Cayley graph $Cay(G, S)$ for $G = V_{8n}$ if S has an element of order 4.*

Proof. Suppose $Cay(G, S)$ is a normal edge-transitive Cayley graph and S has an element of order 4. Elements of order 4 in V_{8n} are $a^{2t}b^{\pm 1}$ for odd n and are $\{a^{\frac{n}{2}}, a^{\frac{n}{2}}b^2, a^{2t}b^{\pm 1} \mid 1 \leqslant t \leqslant n\}$.

Since $Cay(G, S)$ is a normal edge transitive Cayley graph all elements of S have order 4. If $(n, 4) = 1$ or $(n, 4) = 4$ then $\langle S\rangle \subseteq \langle a^2, b\rangle \neq V_{8n}$, a contradiction. Hence $(n, 4) = 2$ or equivalently $\frac{n}{2}$ is odd.

If $S \cap \{a^{\frac{n}{2}}, a^{\frac{n}{2}}b^2\} = \emptyset$ then $\langle S\rangle \subseteq \langle a^2, b\rangle \neq V_{8n}$ a contradiction, hence $S \cap \{a^{\frac{n}{2}}, a^{\frac{n}{2}}b^2\} \neq \emptyset$. For all $f \in \mathbb{A}ut(G, S)$ we have $f(\{a^{\frac{n}{2}}, a^{\frac{n}{2}}b^2\}) = \{a^{\frac{n}{2}}, a^{\frac{n}{2}}b^2\}$

therefore $S \cap \{a^{\frac{n}{2}}, a^{\frac{n}{2}}b^2\}$ is an orbit of $\mathbb{A}ut(G,S)$ on S and it is a contradiction by Lemma 2.2. unless $|S| = 4$ and $S = \{a^{\frac{n}{2}}, a^{\frac{n}{2}}b^2, a^{-\frac{n}{2}}, a^{-\frac{n}{2}}b^2\}$ and in these case we also have $\langle S\rangle \neq V_{8n}$. □

Theorem 4.5. *There is no normal edge-transitive Cayley graph on V_{8n}.*

Proof. Suppose $Cay(G,S)$ is a normal edge-transitive Cayley graph. By Theorems 3.3 and 3.4 we know that S can not have elements of order 2 or 4, Hence we have $S \subseteq \{a^i, a^ib^2 \mid 1 \leqslant i \leqslant 2n\}$ consequently $\langle S\rangle \subseteq \langle a, b^2\rangle \neq V_{8n}$, a contradiction. □

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Filter theory in EQ-algebras based on soft sets

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Abstract. Int-soft prefilters (filters) of EQ-algebras are introduced, and related properties are investigated. Characterizations of int-soft prefilters (filters) of EQ-algebras are provided.

1. Introduction

Many-valued logics are uniquely determined by the algebraic properties of the structure of its truth values. As a precise logic to deal with uncertainty and approximate reasoning, one can consider fuzzy logics. As well-known fuzzy logics, one can also take residuated lattices based on fuzzy logics such as Łukasiewicz logic, BL-logic, R_0-logic, MTL-logic, and so forth. In fuzzy logics, it is generally accepted that the algebraic structure should be a residuated lattice. MV-algebras, BL-algebras, R_0-algebras, MTL-algebras, and so forth are well-known classes of residuated lattices. A new class of algebras called EQ-algebras has been recently introduced by V. Novák and B. De Baets [9] with the intent to develop an algebraic structure of truth values for fuzzy type theory. From the point of view of logic, the main difference between residuated lattices and EQ-algebras lies in the way the implication operation is obtained. It is obtained from a (strong) conjuction in residuated lattices, but it is obtained from equivalence in EQ-algebras. Consequently, the two types of algebras differ in several essential points, despite their many similar or identical properties. An EQ-algebra has three binary operations: meet ($\wedge$), multiplication ($\otimes$), and fuzzy equality ($\sim$), and a unit element, whereas the implication ($\rightarrow$) is derived from the fuzzy equality ($\sim$). Filter theory plays a vital role in studying several algebraic structures such as residuated lattices, MV-algebras, BL-algebras, R_0-algebras, MTL-algebras, BCK/BCI-algebras, lattice implication algebras, and so forth. M. El-Zekey et al. [2] have introduced and studied the prefilters and filters of EQ-algebras. Liu and Zhang [5] have introduced and studied the implicative and positive implicative prefilters (filters) of EQ-algebras.

Soft set theory [8] has been firstly proposed by a Russian researcher Molodtsov in 1999. This is a general mathematical tool for dealing with uncertain, fuzzy, not clearly defined objects. Generally, the soft set theory is different from traditional tools for dealing with uncertainties, such as the theory of probability, the theory

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of fuzzy sets and the theory of rough sets. Nowadays, work on the soft set theory is progressing rapidly. Maji et al. [7] has been firstly defined some operations on soft sets. They also have been introduced the soft set into the decision-making problem [6] that is based on the concept of knowledge reduction in the rough set theory [10]. Jun et al. [4] has been introduced and studied int-soft filters, int-soft G-filters, regular int-soft filters, and MV-int-soft filters in residuated lattices. Jun et al. has been studied (implicative) int-soft filters of R_0-algebras (see [3]).

The aim of this paper is to study prefilters (filters) and positive implicative prefilters (filters) of EQ-algebras based on soft set theory. We study characterizations of positive implicative int-soft prefilters (filters) of EQ-algebras, and establish the extension property for positive implicative int-soft filters.

2. Preliminaries

We display basic definitions and properties of EQ-algebras that will be used in this paper. For more details of EQ-algebras, we refer the reader to [1], [2], and [5].

By an *EQ-algebra* we mean an algebra $E := (E, \wedge, \otimes, \sim, 1)$ of type $(2, 2, 2, 0)$ in which the following axioms are valid:

(E1) $(E, \wedge, 1)$ is a commutative idempotent monoid (i.e., $\wedge$-semilattice with top element 1),

(E2) $(E, \otimes, 1)$ is a monoid and $\otimes$ is isotone with respect to $\leqslant$ (with $x \leqslant y$ defined as $x \wedge y = x$),

(E3) $x \sim x = 1$,

(E4) $((x \wedge y) \sim z) \otimes (a \sim x) \leqslant z \sim (a \wedge y)$,

(E5) $(x \sim y) \otimes (a \sim b) \leqslant (x \sim a) \sim (y \sim b)$,

(E6) $(x \wedge y \wedge z) \sim x \leqslant (x \wedge y) \sim x$,

(E7) $x \otimes y \leqslant x \sim y$

for all $x, y, z, a, b \in E$.

The operation "$\wedge$" is called *meet* (*infimum*) and "$\otimes$" is called *multiplication*. If the multiplication is commutative in an EQ-algebra E, then we say that E is a commutative EQ-algebra.

Let E be an EQ-algebra. For all $x \in L$, we put $\tilde{x} = x \sim 1$. We also define the derived operation, so called *implication* and denoted by $\rightarrow$, as follows:

$$(\forall x, y \in E)\, (x \rightarrow y = (x \wedge y) \sim x)\,. \tag{1}$$

An EQ-algebra E is said to be *residuated* if $(x \otimes y) \wedge z = x \otimes y$ if and only if $x \wedge ((y \wedge z) \sim y) = x$ for all $x, y, z \in E$.

Proposition 2.1. *Every* (*commutative*) *EQ-algebra E satisfies the following conditions for all $a, b, c, d \in E$:*

(1) *If* $a \leqslant b$, *then* $a \to b = 1$, $a \sim b = b \to a$, $\tilde{a} \leqslant \tilde{b}$, $c \to a \leqslant c \to b$ *and* $b \to c \leqslant a \to c$,

(2) $a \otimes b \leqslant a \wedge b \leqslant a, b$ *and* $b \otimes a \leqslant a \wedge b \leqslant a, b$,

(3) $a \to b = a \to (a \wedge b)$,

(4) $(a \to b) \otimes (b \to c) \leqslant a \to c$,

(5) $a \to b \leqslant (a \wedge c) \to (b \wedge c)$.

A subset F of an EQ-algebra E is called a *prefilter* of E if it satisfies the following conditions:

$$1 \in F, \tag{2}$$
$$(\forall a, b \in E)\,(a \to b \in F,\ a \in F \ \Rightarrow\ b \in F). \tag{3}$$

A subset F of an EQ-algebra E is called a *filter* of E if it is a prefilter of E with the following additional condition:

$$(\forall a, b, c \in E)\,(a \to b \in F \ \Rightarrow\ (a \otimes c) \to (b \otimes c) \in F,\ (c \otimes a) \to (c \otimes b) \in F). \tag{4}$$

A prefilter (resp. filter) F of an EQ-algebra E is said to be *positive implicative* if the following assertion is valid:

$$(\forall x, y, z \in E)\,(x \to (y \to z) \in F,\ x \to y \in F \ \Rightarrow\ x \to z \in F). \tag{5}$$

A soft set theory is introduced by Molodtsov [8]. In what follows, let U be an initial universe set and X be a set of parameters. Let $P(U)$ denotes the power set of U and $A, B, C, \ldots \subseteq X$.

A *soft set* $(\tilde{f}, A)$ of X over U is defined to be the set of ordered pairs

$$(\tilde{f}, A) := \{(x, \tilde{f}(x)) : x \in X,\ \tilde{f}(x) \in P(U)\},$$

where $\tilde{f} : X \to P(U)$ such that $\tilde{f}(x) = \emptyset$ if $x \notin A$.

3. Int-soft prefilters (filters)

In what follows, let E denote a commutative EQ-algebra unless otherwise specified.

Definition 3.1. A soft set $(\tilde{f}, E)$ on E over U is called an *int-soft prefilter* (resp. *int-soft filter*) of E if the set

$$i_E(\tilde{f}; \gamma) := \{x \in E \mid \gamma \subseteq \tilde{f}(x)\}$$

is a prefilter (resp. filter) of E for all $\gamma \in P(U)$ with $i_E(\tilde{f}; \gamma) \neq \emptyset$.

We say that $i_E(\tilde{f}; \gamma)$ is the γ-*inclusive set* of $(\tilde{f}, E)$.

Example 3.2. Let $E = \{0, a, b, 1\}$ be a chain. We define two binary operations '$\otimes$' and '$\sim$' by the following tables:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
1	0	a	b	1

$\sim$	0	a	b	1
0	1	a	a	a
a	a	1	b	b
b	a	b	1	1
1	a	b	1	1

Then $E := (E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra (see [5]). The derived operation "$\rightarrow$" is described as the following table:

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	a	b	1	1
1	a	b	1	1

Then a soft set $(\tilde{f}, E)$ on E over $U = \mathbb{Z}$ defined by

$$\tilde{f}(x) := \begin{cases} 4\mathbb{N} & \text{if } x \in \{0, a\}, \\ 4\mathbb{Z} & \text{if } x = b, \\ 2\mathbb{Z} & \text{if } x = 1 \end{cases}$$

is an int-soft prefilter of E.

Example 3.3. Let E by as in the previous example an let

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

$\sim$	0	a	b	1
0	1	0	0	0
a	0	1	a	a
b	0	a	1	1
1	0	a	1	1

Then $E := (E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra (see [5]). The derived operation "$\rightarrow$" is described by table:

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	1	1

Then a soft set $(\tilde{f}, E)$ on E over $U = \mathbb{Z}$ defined as follows:

$$\tilde{f}(x) := \begin{cases} 4\mathbb{Z} & \text{if } x \in \{0, a\}, \\ 2\mathbb{Z} & \text{if } x \in \{b, 1\} \end{cases}$$

is an int-soft filter of E.

Theorem 3.4. *A soft set $(\tilde{f}, E)$ on E over U is an int-soft prefilter of E if and only if the following assertions are valid.*

$$(\forall x \in E)\left(\tilde{f}(x) \subseteq \tilde{f}(1)\right), \tag{6}$$

$$(\forall x, y \in E)\left(\tilde{f}(x) \cap \tilde{f}(x \to y) \subseteq \tilde{f}(y)\right). \tag{7}$$

Proof. Assume that $(\tilde{f}, E)$ is an int-soft prefilter of E. For any $x \in E$, let $\tilde{f}(x) = \gamma$. Then $x \in i_E(\tilde{f}; \gamma)$, and so $i_E(\tilde{f}; \gamma) \neq \emptyset$. Thus $i_E(\tilde{f}; \gamma)$ is a prefilter of E, and therefore $1 \in i_E(\tilde{f}; \gamma)$. Hence $\tilde{f}(1) \supseteq \gamma = \tilde{f}(x)$ for all $x \in E$. For any $x, y \in E$, let $\tilde{f}(x) \cap \tilde{f}(x \to y) = \delta$. Then $\tilde{f}(x) \supseteq \delta$ and $\tilde{f}(x \to y) \supseteq \delta$, that is, $x \in i_E(\tilde{f}; \delta)$ and $x \to y \in i_E(\tilde{f}; \delta)$. It follows from (3) that $y \in i_E(\tilde{f}; \delta)$ and that $\tilde{f}(y) \supseteq \delta = \tilde{f}(x) \cap \tilde{f}(x \to y)$.

Conversely, let $(\tilde{f}, E)$ be a soft set on E over U that satisfies two conditions (6) and (7). Let $\varepsilon \in P(U)$ be such that $i_E(\tilde{f}; \varepsilon) \neq \emptyset$. Then $\tilde{f}(a) \supseteq \varepsilon$ for some $a \in i_E(\tilde{f}; \varepsilon)$. Using (6), we have $\tilde{f}(1) \supseteq \tilde{f}(a) \supseteq \varepsilon$, and so $1 \in i_E(\tilde{f}; \varepsilon)$. Let $x, y \in E$ be such that $x \in i_E(\tilde{f}; \varepsilon)$ and $x \to y \in i_E(\tilde{f}; \varepsilon)$. Then $\varepsilon \subseteq \tilde{f}(x)$ and $\varepsilon \subseteq \tilde{f}(x \to y)$. It follows from (7) that $\varepsilon \subseteq \tilde{f}(x) \cap \tilde{f}(x \to y) \subseteq \tilde{f}(y)$ and that $y \in i_E(\tilde{f}; \varepsilon)$. Hence $i_E(\tilde{f}; \varepsilon)$ is a prefilter of E for all $\varepsilon \in P(U)$ with $i_E(\tilde{f}; \varepsilon) \neq \emptyset$, and therefore $(\tilde{f}, E)$ is an int-soft prefilter of E. □

Theorem 3.5. *A soft set $(\tilde{f}, E)$ on E over U is an int-soft filter of E if and only if it satisfies* (6), (7) *and*

$$(\forall x, y, z \in E)\left(\tilde{f}(x \to y) \subseteq \tilde{f}((x \otimes z) \to (y \otimes z))\right). \tag{8}$$

Proof. Let $(\tilde{f}, E)$ be an int-soft filter of E. Then $(\tilde{f}, E)$ is an int-soft prefilter of E, and so two conditions (6) and (7) are valid by Theorem 3.4. Let $x, y \in E$ and $\tau \in P(U)$ be such that $\tilde{f}(x \to y) = \tau$. Then $x \to y \in i_E(\tilde{f}; \tau)$. Since $i_E(\tilde{f}; \tau)$ is a filter of E, we have $(x \otimes z) \to (y \otimes z) \in i_E(\tilde{f}; \tau)$ for all $x, y, z \in E$. It follows that

$$\tilde{f}((x \otimes z) \to (y \otimes z)) \supseteq \tau = \tilde{f}(x \to y)$$

for all $x, y, z \in E$.

Conversely, let $(\tilde{f}, E)$ be a soft set on E over U that satisfies (6), (7) and (8). Then $(\tilde{f}, E)$ is an int-soft prefilter of E by Theorem 3.4, and thus $i_E(\tilde{f}; \gamma)$ is a prefilter of E for all $\gamma \in P(U)$ with $i_E(\tilde{f}; \gamma) \neq \emptyset$. Let $x, y \in E$ be such that $x \to y \in i_E(\tilde{f}; \gamma)$. Then

$$\tilde{f}((x \otimes z) \to (y \otimes z)) \supseteq \tilde{f}(x \to y) \supseteq \gamma$$

by (8), and so $(x \otimes z) \to (y \otimes z) \in i_E(\tilde{f}; \gamma)$. Hence $i_E(\tilde{f}; \gamma)$ is a filter of E, and therefore $(\tilde{f}, E)$ is an int-soft filter of E. □

Proposition 3.6. *Every int-soft prefilter $(\tilde{f}, E)$ of E for all $x, y \in E$ satisfies the following assertions:*

(1) *if* $x \leqslant y$, *then* $\tilde{f}(x) \subseteq \tilde{f}(y)$,

(2) $\tilde{f}(x \otimes y) \subseteq \tilde{f}(x) \cap \tilde{f}(y)$.

Proof. (1). Let $x, y \in E$ be such that $x \leqslant y$. Then $x \to y = 1$ by Proposition 2.1. It follows from (6) and (7) that $\tilde{f}(y) \supseteq \tilde{f}(x) \cap \tilde{f}(x \to y) = \tilde{f}(x) \cap \tilde{f}(1) = \tilde{f}(x)$.

(2). Using Proposition 2.1(2) and item (1), we have $\tilde{f}(x \otimes y) \subseteq \tilde{f}(x) \cap \tilde{f}(y)$. □

Theorem 3.7. *For a soft set* $(\tilde{f}, E)$ *on* E *over* U, *the following are equivalent.*

(1) $(\tilde{f}, E)$ *is an int-soft prefilter of* E.

(2) $(\forall x, y, z \in E)\Big(x \leqslant y \to z \ \Rightarrow\ \tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(z)\Big)$.

(3) $(\forall x, y, z \in E)\Big(x \to (y \to z) = 1 \ \Rightarrow\ \tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(z)\Big)$.

Proof. (1) $\Rightarrow$ (2). Let $x, y, z \in E$ be such that $x \leqslant y \to z$. Then $\tilde{f}(x) \subseteq \tilde{f}(y \to z)$ by Proposition 3.6(1). Using (7), we get

$$\tilde{f}(z) \supseteq \tilde{f}(y) \cap \tilde{f}(y \to z) \supseteq \tilde{f}(x) \cap \tilde{f}(y).$$

(2) $\Rightarrow$ (3). Let $x, y, z \in E$ be such that $x \to (y \to z) = 1$. Then

$$x \leqslant 1 = x \to (y \to z),$$

and so $\tilde{f}(x) \subseteq \tilde{f}(y \to z)$ by (2). Since $y \to z \leqslant y \to z$, it follows from (2) that

$$\tilde{f}(z) \supseteq \tilde{f}(y \to z) \cap \tilde{f}(y) \supseteq \tilde{f}(x) \cap \tilde{f}(y).$$

(3) $\Rightarrow$ (1). Since $x \to (x \to 1) = 1$ for all $x \in E$, it follows from (3) that $\tilde{f}(x) \subseteq \tilde{f}(1)$ for all $x \in E$. Note that $(x \to y) \to (x \to y) = 1$ for all $x, y \in E$. Thus $\tilde{f}(x) \cap \tilde{f}(x \to y) \subseteq \tilde{f}(y)$ for all $x, y \in E$ by (3). Therefore $(\tilde{f}, E)$ is an int-soft prefilter of E by Theorem 3.4. □

Proposition 3.8. *For any int-soft filter* $(\tilde{f}, E)$ *of* E, *for all* $x, y, z \in E$ *the following assertions are valid.*

(1) $\tilde{f}(x \otimes y) = \tilde{f}(x) \cap \tilde{f}(y)$,

(2) $\tilde{f}(x \to z) \supseteq \tilde{f}(x \to y) \cap \tilde{f}(y \to z)$.

Proof. (1). The inclusion $\tilde{f}(x \otimes y) \subseteq \tilde{f}(x) \cap \tilde{f}(y)$ follows from Proposition 3.6(2). Note that $y \leqslant 1 \to y$ for all $y \in E$. It follows from Proposition 3.6(1) and (8) that

$$\tilde{f}(y) \subseteq \tilde{f}(1 \to y) \subseteq \tilde{f}((x \otimes 1) \to (x \otimes y)) = \tilde{f}(x \to (x \otimes y))$$

and from (7) that $\tilde{f}(x \otimes y) \supseteq \tilde{f}(x) \cap \tilde{f}(x \to (x \otimes y)) \supseteq \tilde{f}(x) \cap \tilde{f}(y)$ for all $x, y \in E$.

(2). Combining Proposition 2.1(4), Proposition 3.6(1) and item (1) induces

$$\tilde{f}(x \to z) \supseteq \tilde{f}((x \to y) \otimes (y \to z)) = \tilde{f}(x \to y) \cap \tilde{f}(y \to z)$$

for all $x, y, z \in E$. □

4. Int-soft prefilters (filters)

Definition 4.1. A soft set $(\tilde{f}, E)$ on E over U is called a *positive implicative int-soft prefilter* (*filter*) of E if the nonempty γ-inclusive set of $(\tilde{f}, E)$ is a positive implicative prefilter (filter) of E for all $\gamma \in P(U)$.

Example 4.2. The int-soft filter $(\tilde{f}, E)$ in Example 3.3 is positive implicative, but the int-soft prefilter $(\tilde{f}, E)$ in Example 3.2 is not positive implicative because if we take $\tau \in P(U)$ with $4\mathbb{N} \subsetneq \tau \subseteq 4\mathbb{Z}$, then $i_E(\tilde{f}; \tau) = \{b, 1\}$ is not a positive implicative prefilter of E.

Theorem 4.3. *A soft set $(\tilde{f}, E)$ on E over U is a positive implicative int-soft prefilter* (*filter*) *of E if and only if it is an int-soft prefilter* (*filter*) *of E that satisfies an additional condition:*

$$(\forall x, y, z \in E) \left(\tilde{f}(x \to (y \to z)) \cap \tilde{f}(x \to y) \subseteq \tilde{f}(x \to z) \right). \tag{9}$$

Proof. Assume that $(\tilde{f}, E)$ is a positive implicative int-soft prefilter (filter) of E. Then $i_E(\tilde{f}; \tau)$ is a positive implicative prefilter (filter) of E for all $\tau \in P(U)$ with $i_E(\tilde{f}; \tau) \neq \emptyset$, and therefore $i_E(\tilde{f}; \tau)$ is a prefilter (filter) of E. Hence $(\tilde{f}, E)$ is an int-soft prefilter (filter) of E. Let $x, y, z \in E$ be such that $\tilde{f}(x \to (y \to z)) \cap \tilde{f}(x \to y) = \varepsilon$. Then $x \to (y \to z) \in i_E(\tilde{f}; \varepsilon)$ and $x \to y \in i_E(\tilde{f}; \varepsilon)$, which implies from (5) that $x \to z \in i_E(\tilde{f}; \varepsilon)$. Thus

$$\tilde{f}(x \to z) \supseteq \varepsilon = \tilde{f}(x \to (y \to z)) \cap \tilde{f}(x \to y).$$

Conversely, let $(\tilde{f}, E)$ be an int-soft prefilter (filter) of E that satisfies (9). Then $i_E(\tilde{f}; \varepsilon)$ is a prefilter (filter) of E for all $\varepsilon \in P(U)$ with $i_E(\tilde{f}; \varepsilon) \neq \emptyset$. Let $x, y, z \in E$ be such that $x \to (y \to z) \in i_E(\tilde{f}; \varepsilon)$ and $x \to y \in i_E(\tilde{f}; \varepsilon)$. Then $\varepsilon \subseteq \tilde{f}(x \to (y \to z))$ and $\varepsilon \subseteq \tilde{f}(x \to y)$. It follows from (9) that

$$\varepsilon \subseteq \tilde{f}(x \to (y \to z)) \cap \tilde{f}(x \to y) \subseteq \tilde{f}(x \to z)$$

and that $x \to z \in i_E(\tilde{f}; \varepsilon)$. Hence $i_E(\tilde{f}; \varepsilon)$ is a positive implicative prefilter (filter) of E for all $\varepsilon \in P(U)$ with $i_E(\tilde{f}; \varepsilon) \neq \emptyset$, and therefore $(\tilde{f}, E)$ is a positive implicative int-soft prefilter (filter) of E. □

Theorem 4.4. *If an int-soft filter of E satisfies the following assertion*

$$(\forall x, y \in E) \left(\tilde{f}(((x \to y) \wedge x) \to y) = \tilde{f}(1) \right), \tag{10}$$

then it is a positive implicative int-soft filter of E.

Proof. Let $(\tilde{f}, E)$ be an int-soft filter of E that satisfies the condition (10). Using Proposition 2.1(5) and Proposition 2.1(3), we have

$$x \to (y \to z) \leqslant (x \wedge y) \to ((y \to z) \wedge y) \quad \text{and} \quad x \to y = x \to (x \wedge y).$$

It follows from (6), Proposition 3.6, Proposition 3.8(2) and (10) that

$$\begin{aligned}
&\tilde{f}(x \to y) \cap \tilde{f}(x \to (y \to z)) = \tilde{f}(x \to y) \cap \tilde{f}(x \to (y \to z)) \cap \tilde{f}(1) \\
&\subseteq \tilde{f}(x \to (x \wedge y)) \cap \tilde{f}((x \wedge y) \to ((y \to z) \wedge y)) \cap \tilde{f}(1) \\
&\subseteq \tilde{f}(x \to ((y \to z) \wedge y)) \cap \tilde{f}(((y \to z) \wedge y) \to z) \subseteq \tilde{f}(x \to z).
\end{aligned}$$

Therefore $(\tilde{f}, E)$ is a positive implicative int-soft filter of E by Theorem 4.3. □

Theorem 4.5. *Let $(\tilde{f}, E)$ and $(\tilde{g}, E)$ be int-soft filters of E such that $\tilde{f}(1) = \tilde{g}(1)$ and $\tilde{f}(x) \subseteq \tilde{g}(x)$ for all $x \in E$. If $(\tilde{f}, E)$ is positive implicative, then so is $(\tilde{g}, E)$.*

Proof. Indeed, $\tilde{g}(((x \to y) \wedge x) \to y) \supseteq \tilde{f}(((x \to y) \wedge x) \to y) = \tilde{f}(1) = \tilde{g}(1)$, and thus $\tilde{g}(((x \to y) \wedge x) \to y)) = \tilde{g}(1)$ for all $x, y \in E$. Therefore $(\tilde{g}, E)$ is a positive implicative int-soft filter of E by Theorem 4.4. □

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Minimal ideals of Abel-Grassmann's groupoids

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Abstract. We study minimal (0-minimal) ideals, simple (0-simple) Abel-Grassmann's groupoids and zeroids of an Abel-Grassmann's groupoid S. We consider S containing a minimal ideal which is the union of all minimal left ideals of S. The completely simple Abel-Grassmann's groupoid which is equal to the union of all its nonzero minimal left ideals is investigated. In addition, we discuss a universally minimal left ideal of S which is a right ideal and is the kernel of S. Finally, we prove that S contains a left zeroid if and only if it contains a universally minimal left ideal.

1. Introduction and preliminaries

The concept of an *Abel-Grassmann's groupoid* (abbreviated as *AG-groupoid*) was first introduced by M. A. Kazim and M. Naseeruddin in 1972 which they called a *left almost semigroup* [7]. P. Holgate [6] called the same structure a *left invertive groupoid*. P. V. Protić and N. Stevanović later called such a groupoid an *Abel-Grassmann's groupoid* [12]. An *AG-groupoid* is in fact a groupoid S satisfying the *left invertive law* $(ab)c = (cb)a$. The left invertive law can be stated by introducing the braces on the left of ternary commutative law $abc = cba$. An AG-groupoid satisfies the *medial law* $(ab)(cd) = (ac)(bd)$. Since AG-groupoids satisfy the medial law, they belong to the class of entropic groupoids. If an AG-groupoid S contains a left identity, then it satisfies the *paramedial law* $(ab)(cd) = (dc)(ba)$ and the identity $a(bc) = b(ac)$ [11]. An AG-groupoid is an algebraic structure which is midway between a groupoid and a commutative semigroup. Consequently, an AG-groupoid has many properties similar to to the properties of semigroups (cf. for example [3], [4] and [5]), but AG-groupoids (also AG-groupoids with a left identity) are non-associative and non-commutative in general.

The minimal ideals are interesting not only in itself but it also influences the other properties of semigroups. In the literature, some interesting articles on minimal ideals and their properties can be found, for instance, see [1, 2, 8] and [9].

In this paper, we investigate minimal ideals in a non-associative and non-commutative AG-groupoid. We also discuss zeroids and divisibility in an AG-groupoid and relate them with minimal ideals.

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By a *unitary AG-groupoid*, we mean a AG-groupoid S with a left identity e. It is worth noticing that if S is a unitary AG-groupoid then $Se = eS = S$ and $S = S^2$. A groupoid with the property $S = S^2$ is called *surjective*.

If $I \subseteq S$ and $SI \subseteq I$ $(IS \subseteq I)$, then I is called a *left* (*right*) *ideal* of S. If I is both a left and right ideal of S, then I is called a *two-sided ideal* or simply an *ideal* of S. A left ideal L of an AG-groupoid S is *minimal* if every left ideal M of S included in L coincides with L. A similar statement holds for the right ideal. Let S^* be an AG-groupoid and $S^* \supseteq S \supseteq A$ such that A is a left ideal of S and S is a left ideal of S^* with the assumption that A is idempotent. Then A is a left ideal of S^*. In fact, the following equalities always hold.

$$S^*A = S^* \cdot AA \subseteq S^* \cdot AS = A \cdot S^*S \subseteq AS = AA \cdot SS = SA \cdot SA \subseteq A.$$

Notice that the property of being left ideal is transitive only if we impose an extra condition on a left ideal A. In general, being a left ideal is not transitive. If S is an AG-groupoid and A and B are ideals of the same type, then $A \cap B$ is either empty or an ideal of the same type as A and B. Also if S is an AG-groupoid, then the union of any collection of ideals of the same type is an ideal of the same type.

If there is an element 0 of an AG-groupoid $(S, \cdot)$ such that $x0 = 0x = 0$ for all $x \in S$, then 0 is the *zero element* of S.

2. Minimal and 0-minimal ideals

In [8], the authors studied minimal ideals of an AG-groupoid. They have shown that if L is a minimal left ideal of a unitary AG-groupoid, then Lc forms a minimal left ideal of S for all $c \in S$ which is a consequence of the following lemma.

Lemma 2.1. *Let L be a left ideal of a unitary AG-groupoid S. Then the following conditions are equivalent:*

(*i*) *L is a minimal left ideal of S;*

(*ii*) *$Lx = L$ for every $x \in L$;*

(*iii*) *$Sx = L$ holds for every $x \in L$.*

Proof. (*i*) $\Rightarrow$ (*ii*). Let L be a minimal left ideal of S and $x \in L$. Then $Lx \subseteq L$. Moreover, $S \cdot Lx = Se \cdot Lx = SL \cdot ex \subseteq Lx$. Thus, Lx is a left ideal of S and, by the minimality of L, we have $Lx = L$ for every $x \in L$.

(*ii*) $\Rightarrow$ (*iii*) is simple.

(*iii*) $\Rightarrow$ (*i*). Let L be a left ideal of S such that $Sx = L$ holds for every $x \in L$. Assume that M is a left ideal of S which is contained in L and let $x \in M$. Then $x \in L$ and therefore, $L = Sx \subseteq SM \subseteq M$. Hence $L = M$. □

Lemma 2.2. *A left ideal L (a right ideal R of a unitary AG-groupoid S is a minimal left (right) ideal of S if and only if $L = Sa$ for all $a \in L$ (respectively, $R = Sa^2$ for every $a \in R$).* □

Theorem 2.3. *If a unitary AG-groupoid S contains a minimal left ideal L such that L is idempotent, then S contains a minimal ideal which is the union of all the minimal left ideals of S.*

Proof. Assume that L is a minimal left ideal of S. Then, as a consequence of Lemma 2.1, LS is the union of all the minimal left ideals of S, that is $LS = \bigcup_{s \in S} Ls$. Now we have the following equalities:

$$LS \cdot S = SS \cdot L = SL = SS \cdot LL = LL \cdot SS = LS,$$

and

$$S \cdot LS = SS \cdot LS = SL \cdot SS \subseteq LS.$$

Hence, we can easily show that LS is an ideal of S. Further, we may suppose that I is an ideal of S such that $I \subseteq LS$. Then $S(I \cdot LS) \subseteq LS$. Therefore by the minimality of L, we have $I \cdot LS = L$. Thus $LS = (I \cdot LS)S \subseteq IS \cdot S \subseteq I$. Hence, we can see that S contains a minimal two sided ideal which is a union of all the minimal left ideals of S. $\square$

Corollary 2.4. *A unitary AG-groupoid S will have no proper ideals if and only if S is the union of all its minimal left ideals.* $\square$

Corollary 2.5. *If a unitary AG-groupoid S contains a minimal left ideal L and an ideal I such that L is idempotent then $L \subseteq I$.* $\square$

Theorem 2.6. *Let L, R and I be the minimal left, minimal right and minimal ideal of a unitary AG-groupoid S respectively such that L is idempotent and $R \subseteq I$. Then $I = LR = LS \cdot R = LS = SR = LI = IR$.*

Proof. Since $L^2 = L$, and $R \subseteq I$ we have $S \cdot LR = L \cdot SR = L(SS \cdot R) = L(RS \cdot S) \subseteq LR$ and $LR \cdot S = SR \cdot L = SR \cdot LL = SL \cdot RL \subseteq LR$. So, LR is an ideal of S and therefore by minimality of I again, we have $I \subseteq LR$. Also it is easy to see that $LR \subseteq I$, which shows hat $I = LR$. Thus,

$$\begin{aligned} S(LS \cdot R) &= (SS)(LS \cdot R) = (S \cdot LS)(SR) \\ &= (SS \cdot LS)(SR) = (SL \cdot SS)(SR) \\ &\subseteq (L \cdot SS)(SR) \subseteq LS \cdot R, \end{aligned}$$

and

$$\begin{aligned} (LS \cdot R)S &= (LS \cdot R)(SS) = (LS \cdot S)(RS) \subseteq SL \cdot R \\ &= (SS \cdot LL)R = (SL \cdot SL)R \subseteq LS \cdot R. \end{aligned}$$

Hence, $LS \cdot R$ is an ideal of S and, by the minimality of I, we obtain $I \subseteq LS \cdot R$. Also it is easy to see that $LS \cdot R \subseteq I$, which implies that $I = LS \cdot R$. The remaining results can be proved in the similar manner. $\square$

Corollary 2.7. *If L, L', R, R' are minimal left and minimal right ideals of a unitary AG-groupoid S respectively, then $LR = L'R'$.* □

Lemma 2.8. *If L is a minimal left ideal of a unitary AG-groupoid S, then L is an AG-groupoid without proper left ideal.*

Proof. Let L' be a left ideal of L, then $LL' \subseteq L$. As L is a left ideal of S, we have $S \cdot LL' = Se \cdot LL' = SL \cdot eL' \subseteq LL'$, The above result shows that $LL^{'}$ is a left ideal of S contained in L and therefore by the minimality of L, we have $L = LL^{'} \subseteq L^{'}$. This equality shows that $L = L^{'}$ and thus L contains no proper left ideal. □

Definition 2.9. A left (right) ideal M of an AG-groupoid S with zero is called 0-*minimal* if $M \neq \{0\}$ and $\{0\}$ is the only left (right) ideal of S properly contained in M.

Theorem 2.10. *Let M be a 0-minimal ideal of a unitary AG-groupoid S with zero such that $M^2 \neq \{0\}$ and $S \neq \{0\}$. If $R \neq \{0\}$ is a right ideal of S contained in M, then $R^2 \neq \{0\}$.*

Proof. Let R be right ideal of S, then it is easy to show that RS is an ideal of S. Therefore by the 0-minimality of M, either $RS = \{0\}$ or $RS = M$. Let $RS = \{0\}$. Since R is nonzero and would appear as an ideal of S contained in M, therefore $R = M$. Thus, $M^2 \subseteq MS = RS = \{0\}$. This contradicts the hypothesis of M. Thus $RS = M$ and therefore $M^2 = RS \cdot RS = R^2S$, which shows that $R^2 \neq \{0\}$. □

Lemma 2.11. *Let S be a unitary AG-groupoid with zero and $S \neq \{0\}$. Then $Sa \cdot S = S$ for every $0 \neq a \in S$ if $\{0\}$ is the only left ideal of S.*

Proof. Assume that $S^2 \neq \{0\}$ and $\{0\}$ is the only left ideal of S. Further, suppose that $C = \{c \in S : Sc \cdot S = \{0\}\} \neq \emptyset$. If $x \in C$ and $y \in S$, then

$$(S \cdot yx)S = (y \cdot Sx)(SS) = (yS)(Sx \cdot S) = (Sx)(yS \cdot S) \subseteq Sx \cdot S = \{0\}\,.$$

The above equality implies $yx \in C$. Thus $yx \in SC \subseteq C$ which means that C is a left ideal of S. Therefore, either $C = \{0\}$ or $C = S$. For the last case, we have

$$SC \cdot S = S^2S = SS = S = \{0\},$$

which contradicts our assumption. Hence, we have $C = \{0\}$ and $Sa \cdot S \neq \{0\}$ for all $0 \neq a \in S$. Since $Sa \cdot S$ is a left ideal of S, we have $Sa \cdot S = S$. □

Theorem 2.12. *If a 0-minimal ideal A of a unitary AG-groupoid S with zero contains at least one 0-minimal left ideal of S and $A^2 \neq \{0\}$, then every left ideal of A is also a left ideal of S.*

Proof. Assume that $L \neq \{0\}$ is a left ideal of A and $a \in L \setminus \{0\}$. By Lemmas 2.2, 2.11 and the fact that $A^2 \neq \{0\}$, we obtain $Aa \cdot A = A$ and $Aa \neq \{0\}$. By Lemma 6.8 [8], S contains a left ideal L_1 such that $a \in L_1 \subseteq A$. Since Aa is a nonzero left ideal of S contained in L_1, we have $Aa = L_1$. Thus, $a \in Aa$. Therefore $L \subseteq \cup\{Aa : a \in L\}$. To show the converse statement let $x \in \cup\{Aa : a \in L\}$. Then there exist elements $b \in A$ and $c \in L$ such that $x = bc$. Since $AL \in L$, it is evident that $x \in L$. Thus $L = \cup\{Aa : a \in L\}$. By the union of a set of ideals, $\cup\{Aa : a \in L\}$ is a left ideal of S. $\square$

3. Simple and completely 0-simple AG-groupoids

In this section, we consider an AG-groupoid which contains a zero but contains no proper ideal except zero. If zero is the only element of an AG-groupoid, then it would be a proper ideal. The fact that the intersection of two nonzero minimal ideals might contain a zero element of an AG-groupoid differentiates it from the class of non-zero ideals.

Theorem 3.1. *If an AG-groupoid S without zero has at least one minimal left ideal, then the sum of all its minimal left ideals is a two-sided ideal of S.*

Proof. Let A_α be the minimal left ideals of S and $B = \sum_\alpha A_\alpha$. Then B is a left ideal. In fact: $SB = S\sum_\alpha A_\alpha = \sum_\alpha SA_\alpha \subseteq \sum_\alpha A_\alpha = B$. Also let $a \in S$, then $Ba = \sum_\alpha A_\alpha a$. But since $A_\alpha a$ is a minimal left ideal of S is contained in the sum of all minimal left ideals, *i.e* $A_\alpha a \subseteq B$ holds for all $a \in S$. It shows that $Ba \in BS \subseteq B$. Hence B is a two sided ideal of S. $\square$

Theorem 3.2. *An AG-groupoid without zero having at least one minimal left ideal is the sum of all its minimal left ideals if and only if it is simple.*

Proof. Let S be simple and has at least one minimal left ideal L. By Theorem 3.1 the sum B of all the minimal left ideals is a two sided ideal of S. Thus $B = S$. As $B \subset S$ is contrary to the definition of simplicity of S.

Conversely, suppose that $S = \sum_\alpha L_\alpha$. Suppose that S has a two-sided subideal A distinct from S, i.e., $AS \subseteq A \subset S$ and $SA \subseteq A \subset S$. Then AL_α is a left ideal of S contained in L_α. In fact: $S(AL_\alpha) = A(SL_\alpha) \subseteq AL_\alpha \subset L_\alpha$. Since every L_α is a left ideal of S, according to the minimality, $AL_\alpha = L_\alpha$. Therefore, $AS = A\sum_\alpha L_\alpha = \sum_\alpha AL_\alpha = \sum L_\alpha = S$, which contradicts our supposition. Thus S has no proper two sided ideal and hence is simple. $\square$

In a unitary AG-groupoid S the situation $Sa \neq S$ $(Sa^2 \neq S)$ for every $a \in S$ is possible. Indeed, such situation take place in a unitary AG-groupoid S with the following multiplication table:

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	c	d	e
c	a	e	b	c	d
d	a	d	e	b	c
e	a	c	d	e	b

Definition 3.3. An AG-groupoid S is called *left* (*right*) *simple* if S is the only left (right) ideal of S. It is called *simple* if it contains no proper ideal.

Theorem 3.4. *A unitary AG-groupoid S is left* (*right*) *simple if and only if $Sa = S$* ($Sa^2 = S$) *for every $a \in S$.*

Proof. Suppose that S is a left simple AG-groupoid. Let $a \in S$, Then

$$S \cdot Sa = SS \cdot Sa = aS \cdot SS = aS \cdot S = SS \cdot a = Sa.$$

Thus Sa is a proper left ideal of S, but this contradicts our assumption. So, $Sa = S$.

Conversely, suppose that $Sa = S$ for all $a \in S$. Let L be a left ideal and $b \in L$. Then $S = Sb \subseteq SL \subseteq L$ and hence $S = L$.

Let S be right simple and $a \in S$. Then

$$Sa^2 \cdot S = SS \cdot a^2S = S \cdot a^2S = a^2 \cdot SS = SS \cdot aa = Sa^2.$$

The above shows that Sa^2 is a proper right ideal of S, which is a contradiction to the fact that S is right simple and therefore $Sa^2 = S$.

The converse statement is obvious. □

Definition 3.5. An AG-groupoid S with zero is called 0*-simple* (*left* 0*-simple, right* 0*-simple*) if $S^2 \neq \{0\}$ and $\{0\}$ is the only ideal (left ideal, right ideal) of S.

Theorem 3.6. *Let S be a unitary AG-groupoid with zero and $S \neq \{0\}$. Then S is left* (*right*) 0*-simple if and only if $Sa \cdot S = S$* ($Sa^2 \cdot S = S$) *for every $0 \neq a \in S$.*

Proof. The first part of the proof is a consequence of Lemma 2.11. To prove the second part assume that $Sa \cdot S = S$. Then $S^2 \neq \{0\}$ because $S = Sa \cdot S \subseteq S^2$. Let $A \neq \{0\}$ be a left ideal of S and $a \in A$, then $S = Sa \cdot S \subseteq SA \cdot S \subseteq A$. Hence S is left 0-simple. Similarly it can be proved for a right 0-simple AG-groupoid. □

Corollary 3.7. *A unitary AG-groupoid S without zero is left* (*right*) *simple if and only if $Sa \cdot S = S$* ($Sa^2 \cdot S = S$) *for all $a \in S$.* □

Lemma 3.8. *Let $\{0\}$ be the only ideal properly contained in a unitary AG-groupoid S with* 0*. Then S is* 0*-simple.*

Proof. Since S^2 is an ideal of S. We have either $S^2 = \{0\}$ or $S^2 = S$. If $S^2 = \{0\}$, then $S = \{0\}$ and $\{0\}$ is not the proper ideal of S, a clear contradiction. Now if $S^2 \neq \{0\}$, then by definition, S is 0-simple. □

Lemma 3.9. *Let L (R) be a 0-minimal left (right) ideal of a 0-simple unitary AG-groupoid S with zero. Then $Sa = L$ $(Sa^2 = R)$ for $a \in L\backslash 0$ $(a \in R\backslash 0)$.*

Proof. Let L be a 0-minimal left ideal of S and $a \in L\backslash 0$. Then Sa is a left ideal of S contained in L. By minimality of L, either $Sa =\{0\}$ or $Sa = L$. The case $Sa = 0$ is impossible because $a \neq \{0\}$ and therefore $Sa = L$. Similarly in the case for a right ideal. □

Definition 3.10. If S is an AG-groupoid with zero such that $S^2 \neq \{0\}$ and has no proper nonzero ideal and has minimal left and minimal right nonzero ideals, then S is said to *completely simple AG-groupoid with zero.*

Theorem 3.11. *Let L be a minimal left ideal of a completely simple unitary AG-groupoid S with zero such that L is idempotent. Then $LS = S = A$, where A is a nonzero left ideal of S contained in LS.*

Proof. Let S be an AG-groupoid and L be a nonzero minimal left ideal such that $L^2 = L$. Since we have $LS \cdot S = (LL \cdot SS)S = (LS \cdot LS)S \subseteq LL \cdot S \subseteq LS$, and $S \cdot LS = SS \cdot LS = SL \cdot SS \subseteq LS$, we see that LS is an ideal of S. If $LS = \{0\}$, then there exists only one minimal left ideal L, i.e., the zero ideal and S reduces to L. Therefore $LS = SS = S^2 = \{0\}$, which the contradicts the argument of S. Hence our assumption is false and hence $LS = S$. Let A be a nonzero left ideal of S contained in LS. Let $a \in LS$. Then there exists $b \in L$ and $y \in S$ such that $a = by$. Since $A \subseteq LS$, therefore $0 \neq f \in A$ has the form $f = ty$ for $t \in L$ and $y \in S$. According to Lemma 2.1, every $b \in L$ has the form $b = st$ where $s \in S$. Therefore, $a = by = st \cdot y = se \cdot ty = se \cdot f \in SA \subseteq A$. It follows that $LS \subseteq A$ and hence $LS = A$. □

Corollary 3.12. *Let L be an idempotent minimal left ideal of a completely simple unitary AG-groupoid S with zero. Then LS is a minimal left ideal of S.* □

Theorem 3.13. *If S is a completely simple unitary AG-groupoid with a zero and L and R are nonzero minimal left and right ideals of S respectively such that L and R are idempotents. Then $RL \neq \{0\}$. If $LR \neq \{0\}$, then $LR = S$.*

Proof. Similarly as in the proof of Theorem 3.11 we can prove that $LS = S$ and $SR = S$. Hence,

$$\begin{aligned} S &= SS = SR \cdot LS = (SS \cdot RR)(LS) = (RS \cdot L)S = (LS \cdot RR)S \\ &= (LR \cdot SR)S = S(LR \cdot R) \cdot S = (S \cdot RL)S. \end{aligned}$$

The above equality implies that $RL \neq \{0\}$. If $LR \neq \{0\}$, then

$$S \cdot LR = SS \cdot LR = L(SS \cdot R) = L(RS \cdot S) \subseteq LR$$

and $LR \cdot S = (LL \cdot R)S \subseteq LR$, which shows that LR is a two sided ideal of S and therefore $LR = S$. □

Corollary 3.14. *If L is a nonzero minimal left ideal of a completely simple unitary AG-groupoid S, then $LR = S$ for some nonzero minimal right ideal R of S.* □

4. Zeroids and divisibility in AG-groupoids

The concept of zeroids in an AG-groupoid was given by Q. Mushtaq in [10], where it is shown that every AG-groupoid has a left zeroid and characterized an AG-groupoid in terms of zeroids.

Definition 4.1. An element u of an AG-groupoid S is said to be a *left* (*right*) *zeroid* of S if for every element $a \in S$, there exists $x \in S$ such that $u = xa$ ($u = ax$), that is $u \in Sa$ ($u \in aS$). An element is called *zeroid* if it is both a left and a right zeroid.

Definition 4.2. A left (right) ideal of an AG-groupoid S is called an *universally minimal left ideal* of S if it is contained in every left (right) ideal of S. If an AG-groupoid S has a minimal ideal K, then K is called the *kernel* of S.

Lemma 4.3. *A unitary AG-groupoid S contains a left zeroid if and only if it contains a universally minimal left ideal L and L contains all the left zeroids of S.*

Proof. Assume that S contains a left zeroid and L consist of all left zeroids of S. Then for $a \in SL$ there exists $x \in S$ and $y \in L$ such that $a = xy$. Since L is the set of all left zeroids, $y = bc$ for some $b \in S$. Thus

$$a = xy = x \cdot bc = ex \cdot bc = cb \cdot xe = (xe \cdot b)c.$$

So, a is a left zeroid belonging to L. Hence $SL \subseteq L$ and L is a left ideal of S. Let L_1 be a left ideal of S. Then for $b \in L_1$, $Sb \subseteq SL_1 \subseteq L_1$. Let $z \in L$, then since z is a left zeroid, $z \in Sb \subseteq L_1$ and therefore $L \subseteq L_1$.

Conversely, if S contains a universally minimal left ideal L, then for any $x \in S$, Sx is a left ideal of S and $L \subseteq Sx$. Hence for every $a \in L$ we have $a = yx$ for some $y \in S$. Thus we a is a left zeroid of S. □

Lemma 4.4. *An universally minimal left ideal of a unitary AG-groupoid S is a right ideal of S and is the kernel of S.*

Proof. Assume that L is an universally minimal left ideal of S. Let $p \in LS$. Then $p = xy$ for $x \in L$ and $y \in S$. By Theorem 2.3, Ly is a minimal left ideal of S and by definition of L, $L \subseteq Ly$ and hence $L = Ly$. Thus $p \in Ly = L$ and therefore $LS \subseteq L$, which shows that L is a right ideal of S. By definition, L contains no proper ideal and hence is the kernel of S. □

Theorem 4.5. *In a unitary AG-groupoid S with zeroids, every left zeroid is a right zeroid and vice versa. The set of all zeroids of S is the kernel of S.*

Proof. The proof follows from Lemmas 4.7 and 4.4. □

An element $a \in S$ is said to be *divisible on the left (right) by* $b \in S$ if there exist $x, y \in S$ such that $a = ax$ ($a = yb$).

Theorem 4.6. *Let a and b be two distinct elements of a unitary AG-groupoid S. Then a is divisible by b on the right if and only if the left ideal of a is contained in the left ideal of b.*

Proof. Suppose that a is divisible by b on the right. Then for some $x \in S$, $a = xb$. Thus

$$\begin{aligned} Sa \cup a &= S \cdot xb \cup xb \subseteq S \cdot Sb \cup Sb = SS \cdot Sb \cup Sb \\ &= bS \cdot SS \cup Sb = Sb \cup Sb = Sb \subseteq Sb \cup b. \end{aligned}$$

Conversely, let $Sa \cup a \subseteq Sb \cup b$. Since a and b are distinct elements, therefore we have $a \in Sb$, this means that there exists some $y \in S$ such that $a = yb$. □

Corollary 4.7. *If some elements of a unitary AG-groupoid S are divisible by all the elements of S, then the collection of such elements is a universally minimal left ideal of S.*

Proof. Let B be a non-empty collection of all such elements which are divisible by all the elements of S on the right, then B is a left ideal of S. Indeed, for $a_1, a_2 \in S$, there exists $x \in S$ such that $b = xa_1$ for $b \in S$. Thus

$$a_2b = a_2 \cdot xa_1 = ea_2 \cdot xa_1 = a_1x \cdot a_2e = (a_2e \cdot x)a_1.$$

So, a_2b is divisible on the right by $a_1 \in S$ and hence $a_2b \in B$.

Let L be any arbitrary left ideal of S. Then for $l \in L$ and $b \in B$, there exists $x \in S$ such that $b = xl \in B$. Hence, $B \subseteq L$ and it is an universally minimal left ideal of S. □

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Flocks, groups and heaps, joined with semilattices

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Abstract. This article describes the lattice of varieties generated by those of flocks and near heaps. Flocks and heaps are two ways of presenting groups by a ternary operation rather than a binary one. Their varieties joined with that of ternary semilattices create the varieties of near flocks and near heaps. This is done by finding normal forms for words that make up free algebras. Simple sets of identities define these varieties. Identities in general are decidable. Each near flock is a Płonka sum of flocks, and each near heap is a Płonka sum of heaps. An algorithm translates any binary group identity to one in a ternary operation satisfied by near heaps.

1. Introduction

This article merges groups, which arise from composing permutations, with semilattices, which are partial orders with least upper bounds. This is not done by imposing an order on groups, of which there is an extensive literature, but by joining their varieties. The varietal join of groups with semilattices is achieved seamlessly with operations having three arguments instead of the usual two. There are several ways to do this. We look at flocks and heaps.

Figure 1 shows the lattice of these varieties. In each box are the algebras, the name of their variety, and the identities defining it. The top box, containing all of the varieties, is the new variety of near flocks. We prove that these varieties are related as depicted, and decompose algebras higher up into those lower down, wherever possible.

The traditional binary operation $\times$ of a group has five desirable properties: associativity, unique solvability in each argument, and hence the existence of a unity, from which follow inverses and cancellation. There are several ways to change the binary operation to a ternary one $[,,]$, where these properties diverge. Our way defines it by:

$$[x, y, z] = (x \times y^{-1}) \times z.$$

This satisfies (1) and (2), which together are called the *para-associative law.* This operation is also uniquely solvable in each argument, which implies cancellation. However there is neither a unity nor an inverse operation.

A set with a ternary operation satisfying the para-associative law and being solvable in each argument is a flock in the original sense. But solvability is not

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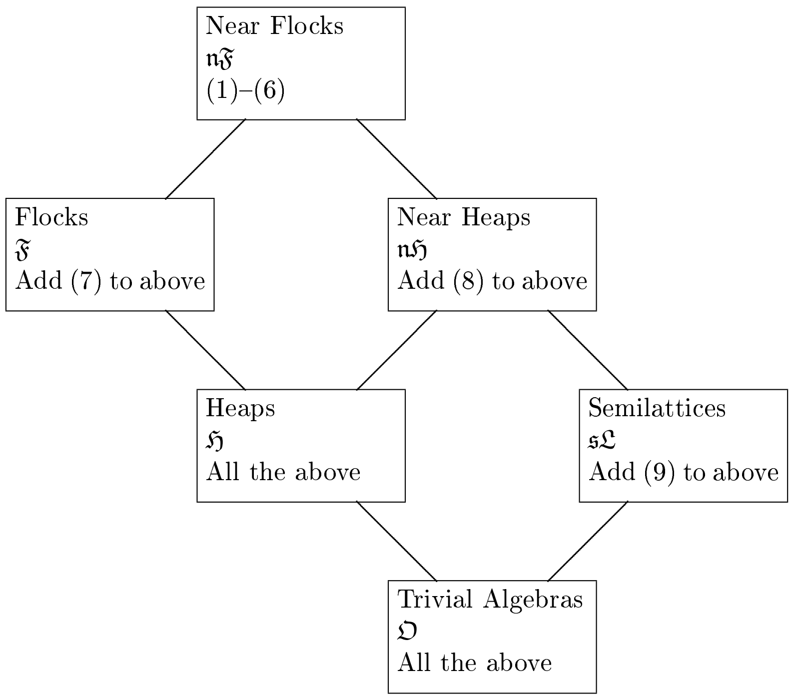
definable by identities. To do so requires adding a unary operation $\overline{x}$ that captures regularity in the sense of von Neumann: for each x, the element $\overline{x}$ satisfies $[x, \overline{x}, x] = x$. This is not necessarily the inverse, although it may be is some cases, and in other cases it may be the identity function. When $\overline{x}$ is the identity function we have heaps.



$$[v, w, [x, y, z]] \approx [[v, w, x], y, z], \tag{1}$$
$$[v, w, [x, y, z]] \approx [v, [y, x, w], z], \tag{2}$$
$$\overline{[x, y, z]} \approx [\overline{x}, \overline{y}, \overline{z}], \tag{3}$$
$$\overline{\overline{x}} \approx x, \tag{4}$$
$$[x, \overline{x}, x] \approx x, \tag{5}$$
$$[x, \overline{x}, y] \approx [\overline{x}, x, y] \approx [y, x, \overline{x}] \approx [y, \overline{x}, x], \tag{6}$$
$$[x, \overline{x}, y] \approx y, \tag{7}$$
$$\overline{x} \approx x, \tag{8}$$
$$[x, x, y] \approx [x, y, y]. \tag{9}$$

Figure 1. Lattice of varieties of flocks, heaps and semilattices, and their defining identities.

The binary operation $\wedge$ of semilattices may be turned into a ternary operation:

$$[x, y, z] = (x \wedge y) \wedge z.$$

Then the equation $[x, y, x] = x$ always has the solution $y = x$ although it may not be unique. Nevertheless we set $\overline{x} = x$, the identity function. Although it could be dropped in semilattices and heaps, for uniformity in comparing varieties we type algebras with a ternary operation and a unary operation, that is the type is $< 3, 1 >$, except where otherwise noted later in this article.

For groups there is also the triple composition:

$$[x, y, z] = (x \times y) \times z,$$

without a middle inverse y^{-1}. This satisfies associative laws and solvability. The inverse operation $\overline{x} = x^{-1}$ may be again added to the type as the solution to $[x, \overline{x}, x] = x$. But this is not the main line of investigation, and will be passed over.

The lack of a unity is no loss and may be an advantage. In the study of vector spaces, where a base-free presentation favors no particular axes, just as in the physical world no particular directions are preferred, so a presentation with no origin should be applauded, as it goes along with the universe having no designated center. Still, the ternary operation has a physical meaning, at least for vector spaces, it is the completion of a parallelogram, that is, it is the fourth vertex, $d = [a, b, c]$, of a parallelogram when the other three vertices are a, b and c.

With the definition of the varieties by ever increasing sets of identities as we go downward in Figure 1, it is clear that the lines represent set-theoretical inclusion as we go upwards. It remains to be proven that the joins and meets are varietal: for example, that $\mathfrak{nF}$ is the smallest variety that includes both $\mathfrak{F}$ and $\mathfrak{nH}$, and that $\mathfrak{H}$ is the largest variety included in both $\mathfrak{F}$ and $\mathfrak{nH}$.

To do this for joins, we find for each variety a normal form for its terms. These constitute the free algebras. The identities in each variety are decidable.

Algebras in the joins are built from algebras below them. A near flock is a Płonka sum of flocks, which is a special kind of extension of flocks by a semilattice. A near heap is a Płonka sum of heaps.

Through the next four sections we descend from the top of the lattice of Figure 1. Since the operation $\overline{x}$ has no effect on the variety $\mathfrak{nH}$, because of identity (8), it will eventually be left out in the treatment of the varieties lower in the lattice.

The next to last section spells out the close connection between heaps and groups as an adjoint situation that is almost a categorical equivalence. The last section translates any group identity to its counterpart in heaps.

2. Near flocks

The variety $\mathfrak{nF}$ of near flocks is defined by the set nF of identities (1)–(6), and is at the top of the lattice of Figure 1. Free algebras are built with normal words.

With these it will be proven in the next section that the variety of near flocks is the join of those of flocks and near heaps.

First, we derive some consequences of the identities defining near flocks. Only some of what is needed is written out here. More identities may be manufactured by their reflection. The *reflection* of an identity is it written backwards, literally. For instance, the reflection of (2), $[vw[xyz]] \approx [v[yxw]z]$, is $[z[wxy]v] \approx [[zyx]wv]$. Since the reflections of (1)–(6) are consequences of these axioms, a reflection of any consequence of (1)–(6) is also a consequence of them.

Proposition 2.1. *These identities for near flocks follow from* $(1)-(6)$.

$$[[w,x,\overline{x}],y,z] \approx [[w,y,z],x,\overline{x}] \tag{10}$$

$$[[w,y,x],\overline{x},z] \approx [[w,y,z],x,\overline{x}], \tag{11}$$

$$[[x,y,z],\overline{[x,y,z]},w] \approx [x,\overline{x},[y,\overline{y},[z,\overline{z},w]]]. \tag{12}$$

$$[[x,v,w],[y,\overline{v},\overline{w}],[z,v,w]] \approx [[x,y,z],v,w]. \tag{13}$$

Proof.

(10).
$$\begin{aligned}[][[w,x,\overline{x}],y,z] &\approx [[x,\overline{x},w],y,z] && (6)\\ &\approx [x,\overline{x},[w,y,z]] && (1)\\ &\approx [[w,y,z],x,\overline{x}] && (6).\end{aligned}$$

(11).
$$\begin{aligned}[][[w,y,x],\overline{x},z] &\approx [w,y,[x,\overline{x},z]] && (1)\\ &\approx [w,y,[z,x,\overline{x}]] && (6)\\ &\approx [[w,y,z],x,\overline{x}] && (1).\end{aligned}$$

(12).
$$\begin{aligned}[][[x,y,z],\overline{[x,y,z]},w] &\approx [[x,y,z],[\overline{x},\overline{y},\overline{z}],w] && (3)\\ &\approx [[[x,y,z],\overline{z},\overline{y}],\overline{x},w] && (2)\\ &\approx [[[x,y,\overline{y}],\overline{x},w],z,\overline{z}] && (11),(10)\\ &\approx [[x,y,\overline{y}],\overline{x},[z,\overline{z},w]] && (1),(6)\\ &\approx [x,\overline{x},[y,\overline{y},[z,\overline{z},w]]] && (1),(11).\end{aligned}$$

(13).
$$\begin{aligned}[][[x,v,w],[y,\overline{v},\overline{w}],[z,v,w]] &\approx [[[x,v,w],\overline{w},\overline{v}],y,[z,v,w]] && (2)\\ &\approx [[[[x,v,w],\overline{w},\overline{v}],y,z],v,w] && (1)\\ &\approx [[[[x,y,z],v,\overline{v}],v,w],\overline{w},w] && (10),(11)\\ &\approx [[x,y,z],v,w] && (1),(5).\ \square\end{aligned}$$

Identities (10) and (11) of this proposition suggest isolating pairs of adjacent variables when one is barred and the other is not.

Normal near flock words, introduced in the next definition, will serve as the elements of free near flocks. A distinction is made between terms and words.

Definition 2.2. In contrast to a term, built with variables and operations symbols, a *word* is simply a finite string of these letters with no ternary operation symbols but with single bars over some of the letters. For example, $\left[x_2, \overline{\overline{x}}_1, \overline{[x_4, x_1, \overline{x}_4]}\right]$ is a term, and $x_2 x_1 \overline{x}_4 \overline{x}_1 x_4$ is a word. A letter adjacent to itself barred, $x_i \overline{x}_i$ or $\overline{x}_i x_i$, is called a *skew pair*. Let $|w|$ be the *length* of a word, that is, the number of occurrences of letters in it. A *normal near flock word*, or simply normal word in this section, is a word, $w = w^\phi w^\sigma$, with two parts, namely a flock part w^ϕ and a semilattice part w^σ — their names will be motivated later. The *flock part* w^ϕ is of odd length in which no variable x_i and its bar $\overline{x}_i$ are adjacent, in either order. The *semilattice* or *skew part* w^σ is of even length and is a sequence, $x_{i_1}\overline{x}_{i_1} x_{i_2}\overline{x}_{i_2} \dots x_{i_k}\overline{x}_{i_k}$, of skew pairs $x_i \overline{x}_i$ with the indices in increasing order: $i_1 < i_2 < \cdots < i_k$. All letters occurring in the flock part w^ϕ must occur in the semilattice part w^σ, but not all the letters in w^σ need be in w^ϕ. For example, here are the parts of a normal word:

$$\begin{aligned} w &= x_5 x_5 x_2 \overline{x}_5 x_6 \overline{x}_2 \overline{x}_2 \; x_2 \overline{x}_2 x_5 \overline{x}_5 x_6 \overline{x}_6 x_9 \overline{x}_9, \\ w^\phi &= x_5 x_5 x_2 \overline{x}_5 x_6 \overline{x}_2 \overline{x}_2, \\ w^\sigma &= x_2 \overline{x}_2 x_5 \overline{x}_5 x_6 \overline{x}_6 x_9 \overline{x}_9. \end{aligned}$$

Definition 2.3. To define free near flocks we need to manipulate words with some operators: the first operator w^ρ reverses the order of the variables; for example, $(x_1 x_2 \overline{x}_3)^\rho$ is $\overline{x}_3 x_2 x_1$. Note that $(uvw)^\rho = w^\rho v^\rho u^\rho$ for words u, v and w. The second operator $\Cup$ joins semilattice parts: $v^\sigma \Cup w^\sigma$ is the string of all skew pairs $x_i \overline{x_i}$ for all variables x_i in v or w, put in order of increasing index with no skew pair occurring more than once. The third transforms a word w of odd length back into a term w^β by appropriately inserting pairs of brackets to form ternary operations all associated to the left; for example, the word $w = x_5 x_5 x_2 x_2 \overline{x}_5 x_6 \overline{x}_2$ becomes the term $w^\beta = [[[x_5, x_5, x_2], x_2, \overline{x}_5], x_6, \overline{x}_2]$. The fourth is an algorithm, given in the next definition, that normalizes any term.

Definition 2.4. Here is how to turn any near flock term t into a normal near flock word t^ν by using (1)–(6). Let $x_{i_1}, x_{i_2}, \dots, x_{i_n}$ be the variables of t with $i_1 < i_2 < \ldots < i_n$. As a running example, consider $t = [[x_2, \overline{\overline{x_2}}, x_1], \overline{[\overline{x}_1, x_2, x_1]}, \overline{x}_1]$. Use (3) to push all bars of t onto individual variables, and (4) to eliminate more than one bar on a variable. The example becomes $[[x_2, x_2, x_1], [x_1, \overline{x}_2, \overline{x}_1], \overline{x}_1]$.

With (5) create a skew pair $x_i \overline{x}_i$ for each variable x_i in t not already in such a pair. Use (10) and (11) to move these skew pairs, one at a time, to the extreme right, in order of increasing index. The example has now become

$$[[[[x_2, x_2, x_1], [x_1, \overline{x}_2, \overline{x}_1], \overline{x}_1], \; x_1, \overline{x}_1], x_2, \overline{x}_2].$$

No skew pair will now be across a bracket. Then use (1) and (2) to associate all occurrences of $[,,]$ to the far left. So we have

$$[[[[[x_2, x_2, x_1], \overline{x}_1, \overline{x}_2], x_1, \overline{x}_1], \; x_1, \overline{x}_1], x_2, \overline{x}_2].$$

At this point we may as well remove the brackets and work with the resulting word $w = w^{\upsilon} w^{\sigma}$. Here w^{σ} is the semilattice part — the word $x_{i_1}\overline{x}_{i_1}x_{i_2}\overline{x}_{i_2}\ldots x_{i_n}\overline{x}_{i_n}$ of all skew pairs that have been moved. The remainder w^{υ} of w on the right will be reworked to give the flock part. The example turns into a word with parts

$$w^{\upsilon} = x_2x_2x_1\overline{x}_1\overline{x}_2x_1\overline{x}_1 \text{ and } w^{\sigma} = x_1\overline{x}_1x_2\overline{x}_2.$$

Remember that at any time the operator ${}^{\beta}$ can return brackets, associated to the left.

Up to now the algorithm has been deterministic; there have been no choices that might make a difference. But new skew pairs may appear in w^{υ} as a result of having moved old ones to the right; for example, when $x_1\overline{x}_1$ is removed from the middle of w^{υ}, then $x_2\overline{x}_2$ is a new skew pair. Now there will be choices as to which order to eliminate these unnecessary skew pairs. The next proposition will show that these choices make no difference in the final outcome.

With (10) and (11) move skew pairs as they appear over to their corresponding pairs in w^{σ} on the right, and eliminate duplicates with (5). The remaining part of the word on the left, with no skew pairs, is the desired flock part w^{ϕ}. Our term, $t = [[x_2, \overline{\overline{x_2}}, x_1], \overline{[\overline{x}_1, x_2, x_1]}, \overline{x}_1]$ has become the normal word $t^{\nu} = x_2\ x_1\overline{x}_1x_2\overline{x}_2$.

Proposition 2.5. *The outcome of the algorithm of Definition* 2.4 *does not depend on the order of eliminating skew pairs.*

Proof. By induction on the length $|w^{\upsilon}|$ of what is to become the flock part of a word w. Suppose the proposition is true when the length is less than n. Assume a particular w^{υ} has length n. Consider sequences, $p = \langle p_1, p_2, \ldots\rangle$, of occurrences p_i of skew pairs that appear in it and that are being successively eliminated; call them *paths*. Think of two different paths, $p = \langle p_1, p_2, \ldots\rangle$ and $q = \langle q_1, q_2, \ldots\rangle$. We will show that, after removing the skew pairs in them, we arrive at the same flock part w^{ϕ}. There are three possibilities for the first pairs: p_1 and q_1 are the same; p_1 and q_1 are not the same but overlap; p_1 and q_1 do not overlap, that is, they are disjoint. We dispose of these possibilities in order.

Suppose that p_1 is q_1, and this skew pair is eliminated from both paths, Then the remaining words will be the same and have length less than n. By the induction hypothesis, after all the remaining skew pairs are eliminated from the two paths, we end up with the same word.

Next suppose that p_1 and q_1 overlap, that is, we have for example $x_i\overline{x}_ix_i$ in w^{υ} with p_1 being $x_i\overline{x}_i$ and q_1 being $\overline{x}_ix_i$. Eliminating either skew pair leaves the same word of lesser length, and the induction hypothesis applies again.

Now assume the two paths start out with disjoint skew pairs, that is, a path can start out at either p_1 or q_1, which are not the same. In particular new paths can start as $p' = \langle p_1, q_1, \ldots\rangle$ and $q' = \langle q_1, p_1, \ldots\rangle$. Now, by the induction hypothesis, the elimination of the same first skew pairs of p and p' will end up with the same word, since they start out the same; and so will q and q'. As p_1 and q_1 are disjoint, what is left after they are both removed is the same word r. Its length $|r|$ is less than n, and hence we are led to the same normal word no matter in what order skew pairs of r are thrown out. So p' and q' will terminate the algorithm at the same flock part, and hence so will p and q. $\square$

We now extend the use of $^\phi$ from designating the flock part of a normal word to its use as an operator that creates the normal flock part from any word of odd length. Similarly $^\sigma$ becomes the operator that creates the semilattice part

Definition 2.6. Write $F_{\mathfrak{n}\mathfrak{F}}(\alpha)$ for the set of all normal words for near flocks, each with a finite number of letters from the set $\{x_i \mid i < \alpha\}$. Turn this into an algebra of type $\langle 3, 1\rangle$, soon to be proven a near flock. For normal words u, v, w define the flock part $[u, v, w]^\phi$ of the operation $[u, v, w]$ to be the string $(u^\phi v^{\phi\rho} w^\phi)^\phi$, and the semilattice part $[u, v, w]^\sigma$ to be $u^\sigma \Cup v^\sigma \Cup w^\sigma$. (The operation $\Cup$ is associative; see Definition 2.3.) The operation $\overline{w}$ adds bars to those variables in the flock part that have none and removes bars from those that do, it leaves the semilattice part alone. Write $\boldsymbol{F}_{\mathfrak{n}\mathfrak{F}}(\alpha)$ for the algebra $\langle F_{\mathfrak{n}\mathfrak{F}}(\alpha); [\,,\,], {}^-\rangle$.

Proposition 2.7. *For each nonzero cardinal α, $\boldsymbol{F}_{\mathfrak{n}\mathfrak{F}}(\alpha)$ is a near flock.*

Proof. We prove that the identities (2), (3) and (4) are satisfied in $\boldsymbol{F}_{\mathfrak{n}\mathfrak{F}}(\alpha)$; the others are proven similarly.

(2). Let u, v, w, x, y and z be normal words. It suffices to prove (2) separately on the flock parts and the semilattice parts of words.

For the flock parts on each side of (2), we expand them to a common word:

$$[v, w, [x, y, z]]^\phi = (v^\phi w^{\phi\rho}[x, y, z]^\phi)^\phi = (v^\phi w^{\phi\rho}(x^\phi y^{\phi\rho} z^\phi)^\phi)^\phi = (v^\phi w^{\phi\rho} x^\phi y^{\phi\rho} z^\phi)^\phi,$$

$$[v, [y, x, w], z]^\phi = (v^\phi [yxw]^{\phi\rho} z^\phi)^\phi = (v^\phi (y^\phi x^{\phi\rho} w^\phi)^{\phi\rho} z^\phi)^\phi = (v^\phi w^{\phi\rho} x^\phi y^{\phi\rho} z^\phi)^\phi.$$

In the first line, Proposition **??** tells us that $(uvw^\phi)^\phi = (uvw)^\phi$ for words u, v, w. In the second line, we also use the fact that $w^{\phi\rho} = w^{\rho\phi}$.

The semilattice parts are equal since $\Cup$ is associative and commutative.

(3) $\overline{[u, v, w]} = [\overline{u}, \overline{v}, \overline{w}]$. For the flock part this follows from the fact that $\overline{w}^\phi = \overline{w^\phi}$. So each path eliminating skew pairs from w has a corresponding path in $\overline{w}$. For the semilattice parts of u, v, w, the bar has no effect.

(4) $\overline{\overline{w}} = w$. The operation $^-$ toggles the bar operation on the flock part, leaving the semilattice part alone. $\square$

Theorem 2.8. *For each nonzero cardinal α, $\boldsymbol{F}_{\mathfrak{n}\mathfrak{F}}(\alpha)$ is the free near flock on α generators.*

Proof. We verify the universal property that characterizes free algebras: for any near flock $\boldsymbol{A}$ generated by α elements a_i $(i < \alpha)$, there is a unique homomorphism h from $\boldsymbol{F}_{\mathfrak{n}\mathfrak{F}}(\alpha)$ to $\boldsymbol{A}$ such that $h(x_i) = a_i$. To that end define $h : F_{\mathfrak{n}\mathfrak{F}}(\alpha) \to A$ by $h(w) = w^\beta(a_{i_1}, \ldots, a_{i_n})$ where $x_{i_1}, \ldots, x_{i_n}$ are the letters of the normal word w.

First we prove that h is a homomorphism, that is, it preserves the operations; then we prove that it is unique. Preserving the operation $[,,]$ means that

$$h([u,v,w]) = [h(u), h(v), h(w)] \qquad (u, v, w \in F_{\mathfrak{n}\mathfrak{F}}(\alpha)),$$

which is equivalent to

$$[u,v,w]^\beta \approx [u^\beta, v^\beta, w^\beta]. \tag{14}$$

This last requires a proof by induction on the length $|v|$ of the middle argument v.

If v is a single variable y, then identity (1) only, when applied to the left side of (14), will move all brackets of the normal word w to the left without any need of reversals by (2). Now suppose $v = [x, y, z]$ with x, y, z normal near flock words of length less than v. We calculate that

$$\begin{aligned} [u,v,w]^\beta &= [u,[x,y,z],w]^\beta \\ &= [[u,z,y],x,w]^\beta && (2) \\ &= [[u^\beta, z^\beta, y^\beta], x^\beta, w^\beta] && \text{(induction hypothesis twice)} \\ &= [u^\beta, [x^\beta, y^\beta, z^\beta], w^\beta] && (2) \\ &= [[u^\beta, [x,y,z]^\beta, w^\beta] && \text{(induction hypothesis)} \\ &= [u^\beta, v^\beta, w^\beta]. \end{aligned}$$

Since the bar $^-$ does not change the order of the variables, h preserves it:

$$h(\overline{w}) = \overline{w}^\beta = \overline{w^\beta} = \overline{h(w)}.$$

To show ϕ is unique, let $g : F_{\mathfrak{n}\mathfrak{F}}(\alpha) \to A$ be another homomorphism such that $g(x_i) = a_i$ $(i < \alpha)$. Then, if $x_{i_1}, \ldots, x_{i_n}$ are the letters of a normal word w, we have that $g(w) = w^\beta(a_{i_1}, \ldots, a_{i_n}) = h(w)$, since a homomorphism preserves terms. □

Write nF for the set of identities (1) – (6) defining near flocks.

Proposition 2.9.

(a) *For any near flock term t there exists a unique normal near flock word w such that $\mathsf{nF} \vdash t \approx w^\beta$.*

(b) *For any normal near flock words v and w, $\mathsf{nF} \vdash v^\beta \approx w^\beta$ iff $v = w$.*

Proof. Existence falls out of Definition 2.4. Uniqueness follows from Theorem 2.8: as normal words, like v and w, make up $\boldsymbol{F}_{\mathfrak{n}\mathfrak{F}}(\alpha)$, a free near flock of the variety defined by the identities of nF, we conclude (b), from which follows uniqueness. □

Corollary 2.10. *The equational theory of near flocks is decidable.*

Proof. From (b) of Proposition 2.9, for terms t_1 and t_2 of type $\langle 3, 1\rangle$,

$$\mathsf{nF} \vdash t_1 \approx t_2 \text{ iff } t_1^\nu = t_2^\nu.$$

This is true since $\mathsf{nF} \vdash t^{\nu\beta} \approx t$. Here t^ν is the normal near flock word obtained from a term t by the algorithm of Definition 2.4. □

3. Flocks

The set F of identities (1)–(7) define the variety $\mathfrak{F}$ of flocks. Originally flocks were defined by Dudek [6] as a nonempty set A with a ternary operation $[,,]$ that satisfies (1)–(2) and is uniquely solvable in each argument: for all a, b and c in A there are unique x, y, z such that $[x, a, b] = c$, $[a, y, b] = c$, and $[a, b, z] = c$. It is not possible to define unique solvability by identities in $[,,]$ alone without additional operations (see the end of this section for why not). However, by [6, Proposition 3.2], unique solvability does allow us to define a unary operation $^{-}$:

$$\overline{x} \text{ is the unique } y \text{ such that } [x, y, x] = x.$$

Adding (7) to those identities defining near flocks simplifies the theory since (7) allows all skew pairs to be removed from normal flock words. With them free algebras are defined and used to prove that the variety of near flocks is the join of flocks and near heaps. Finally, it is shown that each near flock is a Płonka sum of flocks by a semilattice.

Definition 3.1. As before, *words* are strings of letters, some with bars. But now, a *normal word* for flocks is a word of odd length in which no skew pairs occur, that is, neither $x_i\overline{x}_i$ nor $\overline{x}_ix_i$ occur. They are merely the flock parts of normal near flock words.

We pass between terms and words similarly to what was done in the last section. A normal flock word is obtained from any term by using the identities (1)–(6) to push all bars onto individual letters and eliminate multiple bars. Identities (10), (11) and (7) eliminate skew pairs. Identities (1) and (2) associate brackets to the left. With brackets removed, this is a normal flock word.

Definition 3.2. To define the *free flock* on α generators, let $F_{\mathfrak{F}}(\alpha)$ be the set of normal flock words on the set of α letters $\{x_i \mid i < \alpha\}$. Then $\boldsymbol{F}_{\mathfrak{F}}(\alpha)$ is the algebra $\langle F_{\mathfrak{F}}(\alpha);\ [,,],\ ^{-}\rangle$ of type $\langle 3, 1\rangle$. Here, for normal flock words, u, v and w, the ternary operation $[u, v, w]$ is the catenation of them, $(uv^{\rho}w)^{\phi}$, with the order of the letters in the middle argument v reversed to v^{ρ}. This is followed by erasing any skew pairs that arise. The unary operation $\overline{w}$ removes bars from letters in w that have them and adds them otherwise.

The next proposition is on the way to showing that $\boldsymbol{F}_{\mathfrak{F}}(\alpha)$ is a free flock.

Proposition 3.3. *For any non-zero cardinal α, $\boldsymbol{F}_{\mathfrak{F}}(\alpha)$ is a flock.*

Proof. Axiom (7) is satisfied since, for normal flock words w and $v = x_{i_1} \dots x_{i_n}$,

$$[v, \overline{v}, w] = (v\overline{v}^{\rho}w)^{\phi} = (x_{i_1} \dots x_{i_n}\overline{x}_{i_n} \dots \overline{x}_{i_1}w)^{\phi} = w^{\phi} = w.$$

Cancelling inner letters by the operator ϕ also works when some of the letters of v are barred.

It was proven in Proposition 2.7 that $\boldsymbol{F}_{\mathfrak{n}\mathfrak{F}}(\alpha)$ satisfies (1)–(6). With the help of (7), these proofs may be extended to $\boldsymbol{F}_{\mathfrak{F}}(\alpha)$. For example, to prove (1), let v, w, x, y, z be normal flock words. For all x_i appearing in any of $v, \ldots, z$, use (7) to add the skew pair $x_i\overline{x}_i$ at the right side of each word $v, \ldots, z$. Then (1) holds for these words since their modifications are normal near flock words. Use (7) again to wipe out all skew pairs, returning $v, \ldots, z$ to satisfy (1). □

The proof of the next theorem builds on that for free near flocks.

Theorem 3.4. *For any non-zero cardinal α, $\boldsymbol{F}_{\mathfrak{F}}(\alpha)$ is the free flock on α generators.*

Proof. It was just proven that $\boldsymbol{F}_{\mathfrak{F}}(\alpha)$ is a flock. The argument that $\boldsymbol{F}_{\mathfrak{F}}(\alpha)$ satisfies the universal property for freedom is like that for Theorem 2.8. □

Proposition 3.5.

(*a*) *For each term t of type $\langle 3, 1\rangle$, there is a unique normal flock word w such that $\mathsf{F} \vdash t \approx w^\beta$.*

(*b*) *For any normal flock words v and w, $\mathsf{F} \vdash v^\beta \approx w^\beta$ iff $v = w$.*

Proof. By Proposition 2.9 there is a unique normal free near flock word w such that $\mathsf{nF} \vdash t \approx v^\beta$. Eliminating the semilattice part of v give w. □

Corollary 3.6. *The equational theory of flocks is decidable.*

Proof. Like that of Corollary 2.10. □

The proofs and structure of normal forms suggest building any free near flock as a subalgebra of a product of a free flock and a free semilattice. To set the stage, here is a sort review of semilattices. They are traditionally binary algebras with one operation $\vee$ that is idempotent, commutative and associative. A term-equivalent variety with a ternary operation $[,,]$ and a unary operation $^-$ is obtained by 'stammering' the binary operation, and making the unary a dummy:

$$[x, y, z] = (x \vee y) \vee z,$$
$$\overline{x} = x.$$

An example of a semilattice lies in the semilattice parts w^σ of normal near flock words w. They make up the free semilattice $\boldsymbol{F}_{\mathfrak{s}\mathfrak{L}}(\alpha)$ on α generators $x_i\overline{x}_i$ ($i < \alpha$). The ternary operation $[u^\sigma, v^\sigma, w^\sigma]$ is the word consisting of all skew pairs $x_i\overline{x}_i$ occurring in any of u^σ, v^σ or w^σ, arranged in order of ascending index, that is, $[u^\sigma, v^\sigma, w^\sigma] = u^\sigma \Cup v^\sigma \Cup w^\sigma$. Bar does nothing. $\boldsymbol{F}_{\mathfrak{s}\mathfrak{L}}(\alpha)$ is term-equivalent to the semilattice of all nonempty finite subsets of a set with α elements. It is almost a distributive lattice in that every interval of it is a distributive lattice with the operations of union and intersection. All that is missing to make it distributive is the empty set, a bottom element.

Theorem 3.7. *For any nonzero cardinal α, the free near flock $\boldsymbol{F}_{\mathfrak{nF}}(\alpha)$ is isomorphic to a subalgebra of the product, $\boldsymbol{F}_{\mathfrak{F}}(\alpha) \times \boldsymbol{F}_{\mathfrak{sL}}(\alpha)$, of a corresponding free flock and free semilattice. The carrier of the subalgebra is $\{\langle w^{\phi}, w^{\sigma}\rangle \mid w \in F_{\mathfrak{nF}}(\alpha)\}$.*

Proof. Define a function $h : F_{\mathfrak{nF}}(\alpha) \to F_{\mathfrak{F}}(\alpha) \times F_{\mathfrak{sL}}(\alpha)$ by $h(w) = \langle w^{\phi}, w^{\sigma}\rangle$. That h is an injection follows from the definition of normal words. It is a homomorphism since it preserves the operations:

$$h(\overline{w}) = h\left(\overline{w^{\phi}w^{\sigma}}\right) = h\left(\overline{w^{\phi}}w^{\sigma}\right) = h\left(\overline{w}^{\phi}w^{\sigma}\right) = \langle \overline{w^{\phi}}, w^{\sigma}\rangle = \overline{\langle w^{\phi}, w^{\sigma}\rangle} = \overline{h(w)};$$

and $h([u, v, w] = [h(u), h(v), h(w)]$ similarly. $\square$

With this theorem we may check the top part of Figure 1.

Theorem 3.8.

(a) *The variety of near flocks is the join of those of flocks and semilattices:* $\mathfrak{nF} = \mathfrak{F} \vee \mathfrak{sL}$.

(b) *The variety of near flocks is the join of those of flocks and near heaps:* $\mathfrak{nF} = \mathfrak{F} \vee \mathfrak{nH}$.

Proof. (a). The inclusions of their defining identities are passed to the varieties themselves, and hence $\mathfrak{F} \vee \mathfrak{sL} \subseteq \mathfrak{nF}$. As each free near flock $\boldsymbol{F}_{\mathfrak{nF}}(\alpha)$ is a subalgebra of a product of a flock and a semilattice (Theorem 3.7), we have that $\boldsymbol{F}_{\mathfrak{nF}}(\alpha)$ belongs to the join $\mathfrak{F} \vee \mathfrak{sL}$. As any near flock $\boldsymbol{A}$ is a homomorphic image of a free near flock, it follows that $\boldsymbol{A}$ is in the join. Therefore, $\mathfrak{nF} \subseteq \mathfrak{F} \vee \mathfrak{sL}$.

(b). This follows from (a) by the inclusion of varieties: $\mathfrak{sL} \subseteq \mathfrak{nH} \subseteq \mathfrak{nF}$. $\square$

In the language of extensions and Płonka sums more can be said about the structure of near flocks. We first define extensions and prove Theorem 3.10. Then we define connecting homomorphisms that turn this extension into a Płonka sum.

Definition 3.9. An *extension* (or *union* or *sum*) of a nonempty set $\mathcal{A}$ of algebras by another algebra $\boldsymbol{B}$ (all of the same type) is an algebra $\boldsymbol{E}$ and a congruence θ of $\boldsymbol{E}$ such that:

1. each congruence class of θ that is an algebra is isomorphic to a member of $\mathcal{A}$;

2. each member of $\mathcal{A}$ is isomorphic to some congruence class of θ; and

3. $\boldsymbol{E}/\theta$ is isomorphic to $\boldsymbol{B}$.

This definition came from specializing Mal'cev's definition for classes of algebras to individual algebras [11]. In turn, his definition grew out of classical extensions in group theory, where not every coset is a subgroup. However, when $\mathbf{B}$ is idempotent, say a semilattice, then all the congruence classes of $\mathbf{E}/\theta$ will be subalgebras.

Theorem 3.10. *Each near flock $\boldsymbol{A}$ is an extension of flocks by a semilattice.*

The proof of this theorem proceeds by a series of lemmas and interspersed definitions. Easy proofs are omitted without mention.

Now assume that $\boldsymbol{A}$ is a near flock. A congruence θ of $\boldsymbol{A}$ is found such that $\boldsymbol{A}$ is an extension of its congruence classes a/θ by its quotient $\boldsymbol{A}/\theta$.

Definition 3.11. On A define the binary relation:

$$a \leqslant b \quad \text{if} \quad [a, \overline{a}, b] = b. \tag{15}$$

Lemma 3.12.
(*a*) *The relation $\leqslant$ is a quasi-order.*
(*b*) *The operations $^-$ and $[,,]$ preserve $\leqslant$.*

Proof. (*a*). Reflexivity is clear from (5). To prove transitivity, suppose that $a \leqslant b$ and $b \leqslant c$, that is, $[a, \overline{a}, b] = b$ and $[b, \overline{b}, c] = c$. Then by (1),

$$[a, \overline{a}, c] = [a, \overline{a}, [b, \overline{b}, c]] = [[a, \overline{a}, b], \overline{b}, c] = [b, \overline{b}, c] = c,$$

and hence $a \leqslant c$.

(*b*). Bar is preserved by (3). We prove that $\leqslant$ preserves $[,,]$ in its middle argument; the other arguments are simpler.

We assume that $b \leqslant d$, that is $[b, \overline{b}, d] = d$, and prove that $[a, b, c] \leqslant [a, d, c]$, with the help of (1), (2), (5), (10) and (12):

$$\begin{aligned}
[[a, b, c], \overline{[a, b, c]}, [a, d, c]] &= [a, \overline{a}, [b, \overline{b}, [c, \overline{c}, [a, d, c]]]] = [a, \overline{a}, [b, \overline{b}, [a, d, [c, \overline{c}, c]]]] \\
&= [a, \overline{a}, [b, \overline{b}, [a, d, c]]] = [b, \overline{b}, [a, d, c]] \\
&= [a, d, [b, \overline{b}, c] = [a, [b, \overline{b}, d], c] \\
&= [[a, d, c].
\end{aligned}$$
□

Definition 3.13. On A define the binary relation θ by:

$$a \,\theta\, b \quad \text{if} \quad a \leqslant b \text{ and } b \leqslant a.$$

Lemma 3.14. *The relation θ is a congruence of $\boldsymbol{A}$.*

Proof. By Lemma 3.12, $\leqslant$ is a quasi-order preserving $^-$ and $[,,]$. Therefore, θ is an equivalence relation preserving the operations. □

Lemma 3.15. *Each coset of θ is a flock.*

Proof. First one must prove that, for any element of e of A, the coset e/θ is an algebra, that is, it is closed to the operations $^-$ and $[,,]$. To prove closure to $^-$, suppose $a \in e/\theta$. Then $a \,\theta\, e$, and hence from the definition of θ,

$$[a, \overline{a}, e] = e \quad \text{and} \quad [e, \overline{e}, a] = a.$$

From the first equation, with the identities for near flocks we get that $[\overline{a}, \overline{\overline{a}}, e] = [\overline{a}, a, e] = [a, \overline{a}, e] = e$, and so $\overline{a} \leqslant e$. From the second, similarly $e \leqslant \overline{a}$, and thus $\overline{a} \in e/\theta$.

To prove closure to $[,,]$, suppose $a, b, c \in e/\theta$. As before, $[a, \overline{a}, e] = e$ and $[e, \overline{e}, a] = a$, and likewise for b and c. From these equations, Proposition 2.1, and the axioms for near flocks, we deduce that

$$\begin{aligned} [[a,b,c], \overline{[a,b,c]}, e] &= [[a,b,c], [\overline{a}, \overline{b}, \overline{c}], e] = [[[a,b,c], \overline{c}, \overline{b}], \overline{a}, e] \\ &= [[a,b,\overline{b}], \overline{a}, [c, \overline{c}, e]] = [[a,b,\overline{b}], \overline{a}, e] \\ &= [a, \overline{a}, [b, \overline{b}, e]] = [a, \overline{a}, e] = e. \end{aligned}$$

Hence $[a, b, c] \leqslant e$, and with less work $e \leqslant [a, b, c]$; therefore, $[a, b, c] \in e/\theta$.

To show that e/θ is a flock one need only show that (7) is an identity in e/θ; that is, show $[a, \overline{a}, b] = b$ for all a and b related by θ; but this last implies $a \leqslant b$, that is, $[a, \overline{a}, b] = b$. □

Lemma 3.16. *The quotient $\boldsymbol{A}/\,\theta$ is a semilattice.*

Proof. We need only prove (8) and (9) in $\boldsymbol{A}/\,\theta$. For the latter, this amounts to showing that $[a, a, b]\,\theta\,[a, b, b]$ for any a and b in A. Arguing with the axioms and Proposition 2.1 as before, we show that $[[a, a, b], \overline{[a, a, b]}, [a, b, b]] = \cdots = [a, b, b]$, which implies $[a, a, b] \leqslant [a, b, b]$. The converse of this relation is proven similarly, and so the two sides are related by θ. This completes the proof of Theorem 3.10. □

This extension is refined with Płonka sums [13], which are defined here only for near flocks. In [7, Theorem 11] a Płonka sum of heaps is also called a 'strong semilattice of heaps'. We need the partial order found in any semilattice $\boldsymbol{S}$:

$$r \leqslant s \text{ if } [r, \overline{r}, s] = s \quad (r, s \in S).$$

Definition 3.17. A near flock $\boldsymbol{A}$ is a *Płonka sum* of flocks if it is the union of a family $\{\boldsymbol{A}_s \mid s \in S\}$ of disjoint flocks indexed by a semilattice $\boldsymbol{S}$ together with a family of homomorphisms, $\{h_{rs} : \boldsymbol{A}_r \to \boldsymbol{A}_s \,|\, r \leqslant s \text{ in } \boldsymbol{S}\}$, that evaluate the ternary and unary operations of $\boldsymbol{A}$:

$$[a,b,c]^{\boldsymbol{A}} = [(h_{\pi(a),s}(a), h_{\pi(b),s}(b), h_{\pi(c),s}(c)]^{\boldsymbol{A}_s}, \text{ where } s = [\pi(a), \pi(b), \pi(c)]^{\boldsymbol{S}}; \tag{16}$$

$$\overline{(a)}^{\boldsymbol{A}} = \overline{h_{\pi(a),s}(a)}^{\boldsymbol{A}_s}, \qquad \text{where } s = \overline{\pi(a)}^{\boldsymbol{S}}. \tag{17}$$

Here the homomorphisms are assumed to be functorial in that $h_{st} \circ h_{rs} = h_{rt}$ when $r \leqslant s \leqslant t$; and π is the projection map from the disjoint flocks to their indices: $\pi(a) = s$ if $a \in A_s$. The class of Płonka sums of flocks is denoted $s_P\mathfrak{F}$.

The next theorem follows from the more general theory of Płonka [15, Theorem 7.1]. It is also proven in [7, Section 4]; but we sketch another proof that depends in part on Theorem 3.10.

Theorem 3.18. *Every Płonka sum of flocks is a near flock, and every near flock is a Płonka sum of flocks. In short,* $\mathfrak{n}\mathfrak{F} = s_P\mathfrak{F}$.

Proof. That a Płonka sum of flocks is a near flock follows from proving that the identities satisfied by a Płonka sum $\boldsymbol{A}$ are precisely those common to the stalks $\boldsymbol{A}_s$ and semilattices. This follows from verifying by induction on terms that for any term t with n variables,

$$t^{\boldsymbol{A}}(a_1, \ldots, a_n) = t^{\boldsymbol{A}_s}(h_{s_1,s}(a_1), \ldots, h_{s_n,s}(a_n)) \qquad (a_i \in A_{s_i});$$

here s is the semilattice join of the s_i.

For the other direction, by Theorem 3.10, $\boldsymbol{A}$ is the extension of flocks by a semilattice. Let θ be the congruence in Definition 3.13. To create a Płonka sum take the index set S to be the set A/θ of congruence classes a/θ of θ, and define the projection, $\pi(a) = a/\theta$. Define the connecting homomorphisms by

$$h_{\frac{a}{\theta},\frac{b}{\theta}}(x) = [x, \overline{b}, b] \qquad (a, b \in A \text{ and } x \in a/\theta). \tag{18}$$

It remains to be proven that these homomorphisms are well-defined and functorial, and that the operations are evaluated correctly.

These connecting homomorphisms are well-defined since different choices of related a's and related b's yield the same answer for (18). In detail, supposing $x \in a_1/\theta$, $a_1\theta a_2$ and $b_1\theta b_2$, we know that $b_1 = [b_1, \overline{b}_2, b_2]$ and find that

$$h_{\frac{a_1}{\theta},\frac{b_1}{\theta}}(x) = [x, \overline{b}_1, b_1] = [x, \overline{b}_1, [b_1, \overline{b}_2, b_2]] = [x, [\overline{b}_2, b_1, \overline{b}_1], b_2] = [x, \overline{b}_2, b_2] = h_{\frac{a_2}{\theta},\frac{b_2}{\theta}}(x).$$

They are functorial since, if $a \leqslant b \leqslant c$ and $x \in a/\theta$, then

$$h_{\frac{b}{\theta},\frac{c}{\theta}}(h_{\frac{a}{\theta},\frac{b}{\theta}}(x)) = h_{\frac{b}{\theta},\frac{c}{\theta}}([x, \overline{b}, b]) = [[x, \overline{b}, b], \overline{c}, c] = [x, [\overline{c}, b, \overline{b}], c] = [x, \overline{c}, c] = h_{\frac{a}{\theta},\frac{c}{\theta}}(x).$$

They evaluate correctly according to (16) and (17) since for a, b, c in A and $d = [a, b, c]$, we have in $\boldsymbol{A}/\theta$ that

$$s = [\pi(a), \pi(b), \pi(c)] = \left[\frac{a}{\theta}, \frac{b}{\theta}, \frac{c}{\theta}\right] = \frac{[a, b, c]}{\theta} = \frac{d}{\theta},$$

and hence, with the help of (13) and (5),

$$\begin{aligned}
[(h_{\pi(a),s}(a), h_{\pi(b),s}(b), h_{\pi(c),s}(c)] &= [h_{\frac{a}{\theta},\frac{d}{\theta}}(a)]h_{\frac{b}{\theta},\frac{d}{\theta}}(b)]h_{\frac{c}{\theta},\frac{d}{\theta}}(c)] \\
&= [[a, \overline{d}, d], [b, \overline{d}, d], [c, \overline{d}, d]] \\
&= [[a, b, c], \overline{d}, d] \\
&= [[a, b, c], \overline{[a, b, c]}, [a, b, c]] \\
&= [a, b, c].
\end{aligned}$$

For the unary operation, since $s = \overline{\pi(a)}^{\boldsymbol{S}} = \pi(a) = \frac{a}{\theta}$, we check that $\overline{h_{\pi(a),s}(a)} = \overline{h_{\frac{a}{\theta},\frac{a}{\theta}}(a)} = \overline{[a, \overline{a}, a]} = \overline{a}$. $\square$

A free near flock may also be described as a Płonka sum; this is a refinement of Theorem 3.7. We realize this by looking closely at the definition of $F_{\mathfrak{n}\mathfrak{F}}(\alpha)$.

Theorem 3.19. *For a nonzero cardinal α, the free near flock $F_{\mathfrak{n}\mathfrak{F}}(\alpha)$ is the Płonka sum of the free flocks $F_{\mathfrak{F}}(w)$ indexed by elements w of the free semilattice $F_{\mathfrak{s}\mathfrak{L}}(\alpha)$. Here the $F_{\mathfrak{F}}(w)$ are free flocks on generators that are the skew pairs $x_i\overline{x}_i$ in w.*

The article [17] also describes free near flocks as Płonka sums of flocks, but assumes the free flocks $F_{\mathfrak{F}}(w)$ are already known.

The lattice of subvarieties of $\mathfrak{n}\mathfrak{F}$ has been described in [5] and [18, Section 4.3].

Theorem 3.20. *The lattice of varieties of near flocks is isomorphic to the product of the lattice of varieties of flocks and the two-element lattice.A subvariety of $\mathfrak{n}\mathfrak{F}$ is either a subvariety $\mathfrak{K}$ of $\mathfrak{F}$, or a join, $\mathfrak{K} \vee \mathfrak{s}\mathfrak{L}$, of it with the variety of semilattices.*

There is a curiosity about the flocks $\langle A; [,,]\rangle$ originally defined by (1), (2) and the unique solvability of $[,,]$. The class $\mathfrak{F}_3$ of all such is categorically isomorphic to $\mathfrak{F}$. However, $\mathfrak{F}$ is a variety, but $\mathfrak{F}_3$ is not. To understand this, let A be the set $\{x_0^n \mid n \geq 1, n \text{ odd}\}$ of words in $F_{\mathfrak{F}}(1)$. This set is closed to $[,,]$ and thus it is a subalgebra of $\langle F_{\mathfrak{F}}(1); [,,]\rangle$. Because there is no bar, the operation $[,,]$ is not solvable in it. Hence $\mathfrak{F}_3$ is not closed to taking subalgebras, and so fails to be a variety. Therefore, $\mathfrak{F}_3$ is not definable by identities.

4. Near heaps

The variety $\mathfrak{n}\mathfrak{H}$ of near heaps is the class of algebras satisfying the identities: (1)–(6) and (8). The last identity means that the bar operation may be omitted, and we will do so for the remainder of this article, changing the type of near heaps, heaps and semilattices from $\langle 3, 1\rangle$ to $\langle 3\rangle$, only retaining the ternary operation $[,,]$. With that understanding, the defining identities are equivalent to

$$[v, w, [x, y, z]] \approx [[v, w, x], y, z], \tag{1}$$
$$[v, w, [x, y, z]] \approx [v, [y, x, w], z], \tag{2}$$
$$[x, x, x] \approx x, \tag{19}$$
$$[x, x, y] \approx [y, x, x], \tag{20}$$

which is the way Hawthorn and Stokes [7] introduced near heaps.

These identities hold in any group when $[x, y, z]$ is interpreted as $x(y^{-1}z)$, and in any semilattice when $[x, y, z]$ is interpreted as $x(yz)$. In this section a new normal form describes the elements of free near heaps. In the next, the variety of near heaps is proven to be the join of the varieties of heaps and semilattices, in fact, Płonka sums.

Definition 4.1. *Near heap words* w now have no bars, they are simply finite sequences of letters x_i. With bars no more, twin pairs take the place of skew pairs; a *twin pair* is a double occurrence $x_i x_i$ of the same letter adjacent to itself. A *normal near heap word* is in two parts: on the left will be the heap part $w^{\widehat{\phi}}$, a string of odd length of isolated occurrences of individual letters; on the right will be the semilattice part $w^{\widehat{\sigma}}$, a string of twin pairs, one for each letter occurring in w, and ordered by increasing indices, with no twin pair duplicated. Hats over operators indicate their adjustment to there no longer being any bars. Recall that w^{β} restores brackets, associated to the left.

A normal near heap word w is derived from nH for any term t built from the ternary operation and letters alone. To convert t use (1) and (2) to associate all the brackets to the left; (2) may change the order of the letters. Move to the right side any twin pair by using (10) and (11). (They have no bars now.) Reorder these pairs by increasing indices, removing duplicate pairs with (5). If some letter is isolated on the left side and does not occur on the right, triplicate it with (19) to create a twin pair, and move the twin pair to the side, absorbing it among the twin pairs already ordered there.

For example, if t is $[x_3, x_2, [x_2, x_1, x_3]]$, then, with the algorithm in the proof of the next lemma, w is $x_3x_1x_3\,x_1x_1x_2x_2x_3x_3$ with $w^{\widehat{\phi}} = x_3x_1x_3$ and $w^{\widehat{\sigma}} = x_1x_1x_2x_2x_3x_3$.

Definition 4.2. For any cardinal α, the algebra $\boldsymbol{F}_{\mathfrak{n}\mathfrak{H}}(\alpha)$ has as carrier $F_{\mathfrak{n}\mathfrak{H}}(\alpha)$ of all normal near heap words with letters from $\{x_i \mid i < \alpha\}$ and a ternary operation defined by,

$$[u, v, w] = (u^{\widehat{\phi}} v^{\widehat{\phi}\rho} w^{\widehat{\phi}})^{\widehat{\phi}} (u^{\widehat{\sigma}} \Cup v^{\widehat{\sigma}} \Cup w^{\widehat{\sigma}}),$$

where $u^{\widehat{\sigma}} \Cup v^{\widehat{\sigma}} \Cup w^{\widehat{\sigma}}$ is the sequence of all twin pairs in order of increasing index without repetition.

Proposition 4.3. *For any cardinal α, $\boldsymbol{F}_{\mathfrak{n}\mathfrak{H}}(\alpha)$ is a near heap.*

Proof. When the operation, $\overline{x} = x$, is introduced into the type, the identities of nH follow readily from those of nF. □

Theorem 4.4. *For any cardinal α, $\boldsymbol{F}_{\mathfrak{n}\mathfrak{H}}(\alpha)$ is the free near heap on α generators.*

Proof. We use the universal mapping property of free algebras. Let $\boldsymbol{A}$ be a near heap generated by $\{a_i \mid i < \alpha\}$. Define h on a w of $F_{\mathfrak{n}\mathfrak{H}}(\alpha)$ in the letters $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$ by $h(w) = w^{\beta}(a_{i_1}, a_{i_2}, \ldots, a_{i_n})$. Proving that h is the unique homomorphism of $\boldsymbol{F}_{\mathfrak{n}\mathfrak{H}}(\alpha)$ to $\boldsymbol{A}$ taking x_i to a_i parallels the proof of Theorem 2.8. □

Proposition 4.5.

(*a*) *For any near heap term t, there is a unique normal near heap word w such that $\mathsf{nH} \vdash t \approx w^{\beta}$.*

(*b*) *For any normal near heap words v and w, $\mathsf{nF} \vdash v^{\beta} \approx w^{\beta}$ iff $v = w$.*

Proof. Existence comes from the algorithm of Definition 4.1, which uses only the identities of nH and their consequences.

Uniqueness and part (b) parallel the proof of Proposition 2.9. □

Corollary 4.6. *The equational theory of near heaps is decidable.*

Proof. Like that of Corollary 2.10. □

5. Heaps

These were defined in Section as algebras satisfying the identities (1)–(8). As the last identity makes the bar $^-$ pointless, these identities are equivalent to (1), (2) and

$$[x, x, y] \approx y \approx [y, x, x] \tag{21}$$

in algebras with only a ternary operation. Let H be this last set of identities. It is proven that the variety $\mathfrak{nH}$ of near heaps is the smallest variety containing heaps and semilattices. Even better, any near heap is a Płonka sum of heaps over a semilattice. From results in the literature, the lattice of subvarieties of $\mathfrak{nH}$ is sketched, and their subdirectly irreducibles are determined modulo those of heaps. The equivalence of these varieties with the traditional ones for ordinary groups and semilattices with binary operations will be addressed in Section 6. Heaps were first studied by Prüfer [16] in the context of commutative groups where $[x, y, z] = x - y + z$.

Of all the free algebras in this article, free heaps are the simplest to describe; their elements are just the left part, the heap part, of normal near heap words.

Definition 5.1. A *normal heap word* is a string of letters of odd length in which no letter occurs next to itself. The set $F_{\mathfrak{H}}(\alpha)$ of normal heap words on the alphabet $\{_i |\ i < \alpha\}$ is the carrier of the algebra $\boldsymbol{F}_{\mathfrak{H}}(\alpha)$ with the ternary operation

$$[u, v, w] = (uv^\rho w)^{\widehat{\phi}}.$$

Proposition 5.2. *For α a nonzero cardinal, $\boldsymbol{F}_{\mathfrak{H}}(\alpha)$ is a heap.*

Proof. As a heap is a near heap, only axiom (21) needs be proven:

$$[v, v, w] = (vv^\rho w)^{\widehat{\phi}} = (x_{i_1} x_{i_2} \dots x_{i_n} x_{i_n} \dots x_{i_2} x_{i_1} w)^{\widehat{\phi}} = w,$$

when $v = x_{i_1} x_{i_2} \dots x_{i_n}$; here (7) cancels duplicate pairs successively. □

Theorem 5.3. *For α a nonzero cardinal, $\boldsymbol{F}_{\mathfrak{H}}(\alpha)$ is the free heap on α generators.*

Proof. This parallels the proof for Theorem 2.8. □

Proposition 5.4.

(a) *For any heap term t, there is a unique normal heap word such that $\mathsf{H} \vdash t \approx w^\beta$.*

(b) *For any normal heap words v and w, $\mathsf{nF} \vdash v^\beta \approx w^\beta$ iff $v = w$.*

Proof. This is like that of Proposition 3.5. □

Corollary 5.5. *The equational theory of heaps is decidable.*

Proof. Similar to that for Corollary 2.10. □

Theorem 5.6. *For any nonzero cardinal α, the free near heap on α generators is isomorphic to a subalgebra of the product of the free heap and the free semilattice, both on α generators. Symbolically, $\boldsymbol{F}_{\mathfrak{n}\mathfrak{H}}(\alpha) \hookrightarrow \boldsymbol{F}_{\mathfrak{H}}(\alpha) \times \boldsymbol{F}_{\mathfrak{s}\mathfrak{L}}(\alpha)$.*

Proof. Define the function $F_{\mathfrak{n}\mathfrak{H}}(\alpha) \hookrightarrow F_{\mathfrak{H}}(\alpha) \times F_{\mathfrak{s}\mathfrak{L}}(\alpha)$ by $h(w) = \langle w^{\widehat{\phi}}, w^{\widehat{\sigma}} \rangle$. It is a homomorphism since it preserves the ternary operation:

$$\begin{aligned}
h([u,v,w]) &= \langle [u,v,w]^{\widehat{\phi}}, [u,v,w]^{\widehat{\sigma}} \rangle \\
&= \langle (u^{\widehat{\phi}} v^{\widehat{\phi}\rho} w^{\widehat{\phi}})^{\widehat{\phi}}, (u^{\widehat{\sigma}} v^{\widehat{\sigma}} w^{\widehat{\sigma}})^{\widehat{\sigma}} \rangle \\
&= \langle [u^{\widehat{\phi}}, v^{\widehat{\phi}}, w^{\widehat{\phi}}], [u^{\widehat{\sigma}}, v^{\widehat{\sigma}}, w^{\widehat{\sigma}}] \rangle \\
&= [\langle u^{\widehat{\phi}}, u^{\widehat{\sigma}} \rangle, \langle v^{\widehat{\phi}}, v^{\widehat{\sigma}} \rangle, \langle w^{\widehat{\phi}}, w^{\widehat{\sigma}} \rangle] \\
&= [h(u), h(v), h(w)].
\end{aligned}$$ □

Theorem 5.7. *The variety of hear heaps is the join of the varieties of heaps and semilattices: $\mathfrak{n}\mathfrak{H} = \mathfrak{H} \vee \mathfrak{s}\mathfrak{L}$; that is, it is the smallest variety containing them.*

Proof. It is the proof of Theorem 3.8 mutatis mutandis. □

Theorem 5.8. *The lattice of Figure 1 is a sublattice of the lattice of all varieties of algebra with one ternary operation and one unary operation.*

Proof. Note that the free algebras of the different varieties in Figure 1 being non-isomorphic shows that the inclusions in it are proper. That each join of Figure 1 is the smallest variety including those below it is covered by Theorems 3.8 and 5.7. For the each meet of the figure, recall that the meet of two varieties is their intersection, and that the inclusions of the varieties in Figure 1 correspond to that of their generating sets, for example, that $\mathsf{F} \cap \mathsf{nH} = \mathsf{H}$. □

The next theorem follows immediately from Theorem 3.10. It is also proven in [7, Section 4] in a different language, and also follows from [18, Theorem 4.3.2].

Theorem 5.9. *Every Płonka sum of heaps is a near heap, and every near heap is a Płonka sum of heaps. In short, $\mathfrak{n}\mathfrak{H} = s_P\mathfrak{H}$.*

In parallel with Theorem 3.19, a free near heap may also be described as a Płonka sum of free heaps over a free semilattice (see also [17] and [18, Theorem 4.3.8]).

6. Types for groups and heaps

This section clarifies the relationship between groups with a binary operation and heaps with a ternary one. We view their varieties as categories. Two functors pass back and forth between them, giving almost a categorical equivalence. To make clear what is preserved, the intermediary of pointed heaps is introduced. The point serves as an identity element and can be chosen arbitrarily in a heap. Some of these ideas and results were presented noncategorically by Baer [1] and Certaine [3], where there are many references to their origins. See [6] for related concepts.

Definition 6.1. The variety $\mathfrak{G}$ of *groups* is the class of all algebras $\langle G; \times, ^{-1}, e\rangle$ of type $\langle 2, 1, 0\rangle$ satisfying these identities:

$$x \times (y \times z) \approx (x \times y) \times z,$$
$$x \times x^{-1} \approx 1 \approx x^{-1} \times x,$$
$$1 \times x \approx x \approx x \times 1.$$

The variety $\mathfrak{pH}$ of *pointed heaps* consists of all algebras $\langle G; [\,,\,,\,], e\rangle$ of type $\langle 3, 0\rangle$ satisfying identities (1)–(2) and (21).

Surprisingly, no additional identities beyond these defining heaps are needed to define pointed heaps. Identities cannot nail the constant e — its choice is arbitrary!

The varieties $\mathfrak{G}$ and $\mathfrak{pH}$ are term-equivalent and hence categorically isomorphic. To see this, replace the three operations $x \times y$, x^{-1} and 1 in a group by the two operations $[x, y, z] = x \times y^{-1} \times z$ and $e = 1$ in a pointed heap, and replace the operations $[x, y, z]$ and e in a pointed heap by $x \times y = [x, e, y]$, $x^{-1} = [e, x, e]$, and $1 = e$ in a group. Now we define an adjoint situation between $\mathfrak{pH}$ and $\mathfrak{G}$.

Definition 6.2. The function $D : \mathfrak{pH} \to \mathfrak{H}$ drops the constant e as an operation from any pointed heap $\langle A; [\,,\,,\,], e\rangle$. Homomorphisms are left alone by $D(h) = h$, although there may be more of them in $\mathfrak{H}$. The function $E : \mathfrak{H} \to \mathfrak{pH}$ uses the axiom of choice to add to each heap $\langle A; [\,,\,,\,]\rangle$ an arbitrary element e of A. A homomorphism $h : \boldsymbol{A} \to \boldsymbol{A}'$ in $\mathfrak{H}$ is mapped by E to one in $\mathfrak{pH}$ by the formula:

$$E(h)(a) = [h(a), h(e), e'] \qquad (a \in A),$$

where e and e' are the constants chosen by E.

Theorem 6.3. *The functions $D : \mathfrak{pH} \to \mathfrak{H}$ and $E : \mathfrak{H} \to \mathfrak{pH}$ are functors, and D is both a right and left adjoint of E.*

Proof. That D and E are indeed functors is straightforward to verify.

To show that D is a left adjoint of E, it is easiest to prove an equivalent universal situation: for all $\boldsymbol{A}$ in $\mathfrak{H}$, there exists an $\boldsymbol{B}$ in $\mathfrak{pH}$ and a homomorphism

$f : \boldsymbol{A} \to D(\boldsymbol{B})$ such that for all $\boldsymbol{B}'$ in $\mathfrak{p}\mathfrak{H}$ and all homomorphisms $h : \boldsymbol{A} \to D(\boldsymbol{B}')$ there is a unique homomorphism $\bar{h} : \boldsymbol{B} \to \boldsymbol{B}'$ such that this diagram commutes:

$$\begin{tikzcd} \boldsymbol{A} \arrow[r, "f"] \arrow[d, "h"'] & D(\boldsymbol{B}) \arrow[d, "D(\bar{h})"] & \boldsymbol{B} \arrow[d, "\bar{h}"] \\ D(\boldsymbol{B}') \arrow[r, equal] & D(\boldsymbol{B}') & \boldsymbol{B}' \end{tikzcd}$$

It follows that D is a left adjoint of E by Theorem 27.3 of [8].

It is proven similarly that D is a right adjoint of E. □

How far this adjunction falls short of a categorical equivalence is seen in a proposition that traces how common concepts pass across. Its proof is routine. But its statement needs the Cayley representation of elements of a group as permutations.

Definition 6.4. For any pointed heap $\boldsymbol{A}$, $\operatorname{Cay} \boldsymbol{A} = \{f_{ab} \mid a, b \in A\}$, where f_{ab} is the function given by $f_{ab}(c) = [a, b, c]$.

In the following, $\operatorname{Sub} \boldsymbol{A}$, $\operatorname{Con} \boldsymbol{A}$ and $\operatorname{Aut} \boldsymbol{A}$ mean respectively the sets of all subalgebras, congruences and automorphisms of an algebra $\boldsymbol{A}$. For sets $\mathcal{F}_1$ and $\mathcal{F}_2$ of functions, $\mathcal{F}_1 \circ \mathcal{F}_2$ means complex composition: $\{f_1 \circ f_2 \mid f_i \in \mathcal{F}_i\}$.

Proposition 6.5. *For $\boldsymbol{A}, \boldsymbol{A}_1, \boldsymbol{A}_2$ in $\mathfrak{p}\mathfrak{H}$ and their images $D\boldsymbol{A}, D\boldsymbol{A}_1, D\boldsymbol{A}_2$ in $\mathfrak{H}$:*

(1) $\operatorname{Sub}(D\boldsymbol{A}) =$ *the set of congruence classes of* $\boldsymbol{A}$;

(2) $\operatorname{Con}(D\boldsymbol{A}) = \operatorname{Con} \boldsymbol{A}$;

(3) $\operatorname{Aut}(D\boldsymbol{A}) = \operatorname{Aut} \boldsymbol{A} \circ \operatorname{Cay} \boldsymbol{A} = \operatorname{Cay} \boldsymbol{A} \circ \operatorname{Aut} \boldsymbol{A}$;

(4) $\operatorname{Hom}(D\boldsymbol{A}_1, D\boldsymbol{A}_2) = \operatorname{Cay} \boldsymbol{A}_2 \circ \operatorname{Hom}(\boldsymbol{A}_1, \boldsymbol{A}_2)$;

(5) $D\boldsymbol{A}_1 \times D\boldsymbol{A}_2 = D(\boldsymbol{A}_1 \times \boldsymbol{A}_2)$.

Since the congruences are the same under D, so are the simple algebras and subdirectly irreducibles.

Dudek [6, Section 4] approaches the groups in flocks by looking at the binary operations, $x \cdot_a y = [x, a, y]$, which are isomorphic groups in a given flock.

Płonka sums of groups are developed in [14].

7. Transfer of identities

What do the identities defining a variety of ordinary groups become when transfered to a corresponding variety of near heaps? This section starts with algorithms for modifying identities to define varieties of a new type, then a theorem justifies them, and two examples follow. There are two or three steps, depending on whether the subvariety is only a variety of heaps or it is a join of one with semilattices (Theorem 3.20). Notation is from Section 6. Regularity is needed.

Definition 7.1. An identity is *regular* if each variable occurring in a term on one side of it occurs also in the term on the other side. A variety is *regular* if it can be defined by regular identities. The *regularization* of a variety $\mathfrak{K}$ is the variety defined by the regular identities satisfied by $\mathfrak{K}$.

Step 1 — from $\mathfrak{G}$ to $\mathfrak{pH}$. Here is the recipe for the first step to translate a term t of type $\langle 2,1,0\rangle$ to one of type $\langle 3,0\rangle$; it follows the scheme in Section 6.

- Replace each product $t_1 \times t_2$ of subterms t_1 and t_2 of t by $[t_1, e, t_2]$.
- Replace each inverse t_1^{-1} of a subterm t_1 of t by $[e, t_1, e]$.
- Replace the constant 1 by e.

Write $\bar{t}$ for the translated term, and $\bar{t}_1 \approx \bar{t}_2$ for the translation of an identity $t_1 \approx t_2$. For a set K of identities defining a variety of groups, let $\overline{\mathsf{K}}$ be the set of translations.

Step 2 — from $\mathfrak{pH}$ to $\mathfrak{H}$. Assume w is a variable not in any of the identities defining $\mathcal{H}$, a subvariety of $\mathfrak{pH}$. Replace the constant e by w in all the identities of K. Write $\overline{\mathsf{K}}$ for the set modified identities.

Step 3 — from $\mathfrak{H}$ to $\mathfrak{sH}$. If a subvariety of $\mathfrak{nH}$ is the join of a subvariety $\mathfrak{K}$ of $\mathfrak{H}$ with $\mathfrak{sL}$, then the identities K defining $\mathfrak{K}$ must be regularized. One can do this for an identity, $t_1 \approx t_2$, by adding to the right side of the term t_1 the pair xx for any variable x that appears only in t_2 to get $[t_1, x, x] \approx t_2$, and likewise for t_2 .

Theorem 7.2.

(1) *For $\mathfrak{K}$ a subvariety of $\mathfrak{G}$ defined by a set K of identities, the set $\overline{\mathsf{K}}$ of term-translated identities of* Step 1 *defines $\overline{\mathfrak{K}}$, the subvariety of $\mathfrak{pH}$ of pointed heaps term-equivalent to the groups of $\mathfrak{G}$.*

(2) *For a subvariety $\mathfrak{K}$ of $\mathfrak{pH}$ defined by a set K of identities, the set $\overline{\mathsf{K}}$ of translated identities of* Step 2 *defines the subvariety, $\overline{\mathfrak{K}} = D(\mathfrak{K})$, of $\mathfrak{H}$.*

(3) *For a subvariety $\mathfrak{K}$ of $\mathfrak{nH}$ with a defining set K of identities, a defining set of identities for its join, $\mathfrak{K} \vee \mathfrak{sL}$, with the variety of semilattices is given by the regularization $\overline{\mathsf{K}}$ of K, as done in* Step 3.

Proof. (1). This follows from term-equivalence of $\mathfrak{G}$ of $\mathfrak{pH}$.

(2). We must show that, if an identity $t_1 \approx t_2$ is satisfied by an algebra $\boldsymbol{A}$ of $\mathfrak{K}$, then its translation $\bar{t}_1 \approx \bar{t}_2$ is satisfied by $D(\boldsymbol{A})$, and conversely. Write the terms as $t_i(x_1, \ldots, x_n, e)$ where the t_i are of type $\langle 3\rangle$. Then the translated terms will be $t_i(x_1, \ldots, x_n, w)$ with w replacing e. We will show that $t_1(a_1, \ldots, a_n, b) = t_2(a_1, \ldots, a_n, b)$ for all $a_1, \ldots, a_n, b$ in A. Define an automorphism α of $\boldsymbol{A}$ by $\alpha(x) = [x, b, e]$. Then $\alpha(b) = e$. So

$$\begin{aligned}\alpha(t_1(a_1,\ldots,a_n,b)) &= t_1(\alpha(a_1),\ldots,\alpha(a_n),\alpha(b))\\ &= t_1(\alpha(a_1),\ldots,\alpha(a_n),e)\\ &= t_2(\alpha(a_1),\ldots,\alpha(a_n),e)\\ &= t_2(\alpha(a_1),\ldots,\alpha(a_n),\alpha(b))\\ &= \alpha(t_2(a_1,\ldots,a_n,b)).\end{aligned}$$

Therefore, $t_1(a_1,\ldots,a_n,b) = t_2(a_1,\ldots,a_n,b)$. The converse is proven by replacing w by e.

We have shown for any identity $t_1 \approx t_2$ and any algebra $\boldsymbol{A}$ of $\mathfrak{K}$ that

$$\boldsymbol{A} \models t_1 \approx t_2 \text{ iff } D(\boldsymbol{A}) \models \bar{t}_1 \approx \bar{t}_2.$$

This equivalence also applies to the sets of identities:

$$\mathfrak{K} \models \mathsf{K} \text{ iff } \overline{\mathfrak{K}} \models \overline{\mathsf{K}}.$$

Therefore, $\overline{\mathsf{K}}$ defines $\overline{\mathfrak{K}}$ since K defines $\mathfrak{K}$.

(3). Let $t_1 \approx t_2$ be an identity of $\mathfrak{K} \vee \mathfrak{sL}$. We must show that it is derivable from the regularization $\overline{\mathsf{K}}$ of K. As an identity satisfied by $\mathfrak{K} \vee \mathfrak{sL}$, $t_1 \approx t_2$ is satisfied by $\mathfrak{K}$, and so is derivable from K alone. Such a derivation is a sequence of identities, each one of which is either in K or derivable from previous ones using the rules of equational logic. Now regularize each identity in this derivation. This is a derivation from $\overline{\mathsf{K}}$ of the regularization $\bar{t}_1 \approx \bar{t}_2$ of the original identity, with the proviso that some new identities must be interpolated to accommodate instances of substitution in equation logic. □

Two examples illustrate this process. The identities of Definition 6.1, which define groups, translate to identities that are seen to be equivalent to (1), (2) and (21), which define heaps. If the binary commutative law, $x \times y \approx y \times x$, is added, it becomes

$$[x,w,y] \approx [y,w,x] \tag{22}$$

in the first two steps. This is already regular, and so Step 3 is not needed. Hence the join of semilattices and commutative groups is defined by (1),(2), (21) and (22).

Elementary 2-groups are defined by the identity, $x \times x \approx 1$. The first and second steps give $[x,w,x] \approx w$, and the third regularizes it:

$$[x,w,x] \approx [w,x,x]. \tag{23}$$

So the join of semilattices and 2-groups is defined by (1), (2), (21) and (23).

As 2-groups are commutative, it is an elementary exercise to show directly in the language of heaps that (22) follows from (23).

A note on the references. Some of the notions in this paper have an extensive literature reaching back more than a century. A sampling is included here, from which the reader may find more, as well as related concepts.

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On 2-absorbing ideals in commutative semirings

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Abstract. We study 2-absorbing ideals in a commutative semiring S with $1 \neq 0$ and prove some important results analogous to ring theory. More general form of the Prime Avoidance Theorem is also given. We also prove that if $I = \langle a_1, a_2, \dots, a_r \rangle$ is a finitely generated ideal of a semiring S and $P_1, P_2, \dots, P_n$ are subtractive prime ideals of S such that $I \nsubseteq P_i$ for each $1 \leqslant i \leqslant n$, then there exist $b_2, \dots, b_r \in S$ such that $c = a_1 + b_2a_2 + \dots + b_ra_r \notin \bigcup\limits_{i=1}^{n} P_i$.

1. Introduction

The semiring is an important algebraic structure which plays a prominent role in various branches of mathematics like combinatorics, functional analysis, topology, graph theory, optimization theory, cryptography etc. as well as in diverse areas of applied science such as theoretical physics, computer science, control engineering, information science, coding theory etc. The concept of semiring was first introduced by H. S. Vandiver [14] in 1934. After that several authors have apllied this concept in various disciplines in many ways.

A *commutative semiring* is a commutative semigroup $(S, \cdot)$ and a commutative monoid $(S, +, 0_S)$ in which 0_S is the additive identity and $0_S \cdot x = x \cdot 0_S = 0_S$ for all $x \in S$, both are connected by ring like distributivity. A subset I of a semiring S is called an *ideal* of S if $a, b \in I$ and $r \in S$, $a + b \in I$ and $ra, ar \in I$. An ideal I of a semiring S is called *subtractive* if $a, a+b \in I$, $b \in S$ then $b \in I$. A proper ideal P of a semiring S is said to be *prime* (resp. *weakly prime*) if for some $a, b \in S$ such that $ab \in P$ (resp. $0 \neq ab \in P$), then either $a \in P$ or $b \in P$.

Throughout this paper, semiring S will be considered as commutative with identity $1 \neq 0$.

2. Prime ideals

The concept of prime ideal plays an important role in ring and semiring theory. we refer ([8], [10], [13]), for more understanding about prime ideals. In this section, we give the more general form of The Prime Avoidance Theorem for semirings. We start this section with the statement of the following lemma.

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Lemma 2.1 ([15], Lemma 2.5). *Let P_1, P_2 be subtractive ideals of a commutative semiring S and I be an ideal of S such that $I \subseteq P_1 \cup P_2$. Then $I \subseteq P_1$ or $I \subseteq P_2$.*

Theorem 2.2 ([15], Theorem 2.6). (The Prime Avoidance Theorem)
Let S be a semiring and $P_1, \dots, P_n$ ($n \geqslant 2$) be subtractive ideals of S such that almost two of $P_1, \dots, P_n$ are not prime. Let I be an ideal of S such that $I \subseteq \bigcup_{i=1}^{n} P_i$. Then $I \subseteq P_j$ for some $1 \leqslant j \leqslant n$.

The next theorem is the more general form of the Prime Avoidance Theorem of semirings.

Theorem 2.3. (Extented version of the Prime Avoidance Theorem)
Let S be a semiring and $P_1, \dots, P_n$ be subtractive prime ideals of S. Let I be an ideal of S and $a \in S$ such that $aS + I \nsubseteq \bigcup_{i=1}^{n} P_i$. Then there exists $c \in I$ such that $a + c \notin \bigcup_{i=1}^{n} P_i$.

Proof. Assume that $P_i \nsubseteq P_j$ and $P_j \nsubseteq P_i$ for all $i, j \in \{1, 2, \dots, n\}$ and $i \neq j$. Suppose that a lies in all of $P_1, P_2, \dots, P_k$ but none of $P_{k+1}, \dots, P_n$. If $k = 0$, then $a = a + 0 \notin \bigcup_{i=1}^{n} P_i$, which is required. So, let $k \geqslant 1$. Now, $I \nsubseteq \bigcup_{i=1}^{k} P_i$, for otherwise, by the Prime Avoidance Theorem, we would get $I \subseteq P_j$ for some $1 \leqslant j \leqslant k$, which gives $aS + I \subseteq P_j \subseteq \bigcup_{i=1}^{n} P_i$, which contradicts to the hypothesis. Thus, there exists $d \in I \setminus \bigcup_{i=1}^{k} P_i$. Also, $P_{k+1} \cap \ldots \cap P_n \nsubseteq P_1 \cup \ldots \cup P_k$. Otherwise, if $P_{k+1} \cap \ldots \cap P_n \subseteq P_1 \cup \ldots \cup P_k$, by the Prime Avoidance Theorem, we would get a contradiction. Therefore there exists $b \in P_{k+1} \cap \ldots . \cap P_n \setminus (P_1 \cup \ldots \cup P_k)$. Now, define $c = db \in I$ and note that $c \in P_{k+1} \cap \ldots \cap P_n \setminus (P_1 \cup \ldots \cup P_k)$. Since $a \in P_1 \cap \ldots \cap P_k \setminus (P_{k+1} \cup \ldots \cup P_n)$, it follows that $a + c \notin \bigcup_{i=1}^{n} P_i$ (since $P_i's$ are subtractive). $\square$

Next theorem says that if I is a finitely generated ideal of S satisfying the assumption of the Prime Avoidance Theorem for semirings, then the linear combination of the generators of I also avoids $\bigcup_{i=1}^{n} P_i$, where $P_i's$, ($1 \leqslant i \leqslant n$) are subtractive prime ideals of S.

Theorem 2.4. *Let S be a semiring and $I = \langle a_1, a_2, \dots, a_r \rangle$ be a finitely generated ideal of S. Let $P_1, P_2, \dots, P_n$ be subtractive prime ideals of S such that $I \nsubseteq P_i$ for each i, $1 \leqslant i \leqslant n$. Then there exist $b_2, \dots, b_r \in S$ such that $c = a_1 + b_2 a_2 + \ldots + b_r a_r \notin \bigcup_{i=1}^{n} P_i$.*

Proof. We prove it by induction on n. Without loss of generality, assume that $P_i \nsubseteq P_j$ for all $i \neq j$. If $n = 1$, then clearly $c = a_1 + b_2a_2 + \ldots + b_ra_r \notin P_1$. Assume that the result is true for $(n-1)$ subtractive prime ideals of S. Then, there exist $c_2, c_3, \ldots, c_r \in S$ such that $d = a_1 + c_2a_2 + \ldots + c_ra_r \notin \bigcup_{i=1}^{n-1} P_i$. If $d \notin P_n$, then we are through. So assume that $d \in P_n$. If $a_2, \ldots, a_r \in P_n$, then from the expression for d, we have $a_1 \in P_n$, (since $d = a_1 + c_2a_2 + \ldots + c_ra_r$ and $d \in P_n$ implies $a_1 \in P_n$, since P_n is subtractive), which is a contradiction to $I \nsubseteq P_n$ (since, if $a_1 \in P_n$ and we have already assumed that $a_2, \ldots, a_r \in P_n$, we get $a_1, \ldots, a_r \in P_n$, this implies that $I \subseteq P_n$). So for some i, $a_i \notin P_n$. Without loss of generality, let $i = 2$. Since $P_i \nsubseteq P_j$ for all $i \neq j$, we can find $x \in \bigcap_{i=1}^{n-1} P_i$ such that $x \notin P_n$. Thus, $c = a_1 + (c_2 + x)a_2 + \ldots + c_ra_r \notin \bigcup_{i=1}^{n} P_i$. □

3. 2-absorbing ideals

The concept of 2-absorbing and weakly 2-absorbing ideals of a commutative ring with non-zero unity was first introduced by Badawi and Darani in [3], [4] which are generalizations of prime and weakly prime ideals in commutative ring, see [1]. After that Darani [7] and Kumar et. al [11], explored these concepts in commutative semiring and characterized many results in terms of 2-absorbing and weakly 2-absorbing ideals in commutative semiring. Most of the results of this section are inspired from [5] and [6].

Definition 3.1. A proper ideal I of a semiring S is said to be a *2-absorbing ideal* of S if $abc \in I$ implies $ab \in I$ or $bc \in I$ or $ac \in I$ for some $a, b, c \in S$.

Definition 3.2. A proper ideal I of a semiring S is said to be a *weakly 2-absorbing ideal* if whenever $a, b, c \in S$ such that $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$.

Clearly, one can see that every 2-absorbing ideal of a semiring S is a weakly 2-absorbing ideal of S but converse need not be true. For more details of 2-absorbing and weakly 2-absorbing ideals in commutative semirings, we refer [7], [11].

Lemma 3.3. *Let I be a subtractive 2-absorbing ideal of S. Suppose that $abJ \subseteq I$ for some $a, b \in S$ and an ideal J of S. If $ab \notin I$, then either $aJ \subseteq I$ or $bJ \subseteq I$.*

Proof. Suppose that $aJ \nsubseteq I$ and $bJ \nsubseteq I$. Therefore, there are some $x, y \in J$ such that $ax \notin I$ and $by \notin I$. Since $abx \in I$ and $ab \notin I$ and $ax \notin I$, we have $bx \in I$. Since $aby \in I$ and $ab \notin I$ and $by \notin I$, we have $ay \in I$. Now, since $ab(x+y) \in I$ and $ab \notin I$, we have $a(x+y) \in I$ or $b(x+y) \in I$, since I is a 2-absorbing ideal of S. If $a(x+y) \in I$ and $ay \in I$, then $ax \in I$ (since I is subtractive), which is a contradiction. Similarly, if $b(x+y) \in I$ and $bx \in I$, we get $by \in I$ (since I is subtractive), which is again a contradiction. Hence, either $aJ \subseteq I$ or $bJ \subseteq I$. □

Theorem 3.4. *Let I be a proper subtractive ideal of S. Then I is a 2-absorbing ideal of S if and only if whenever $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of S, then either $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq I$ or $I_3I_1 \subseteq I$.*

Proof. Let $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of S, then $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq I$ or $I_1I_3 \subseteq I$. Then by definition, I is a 2-absorbing ideal of S. Conversely, let I be a 2-absorbing ideal of S and $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of S, such that $I_1I_2 \nsubseteq I$. We show that $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$. If possible, suppose that $I_1I_3 \nsubseteq I$ and $I_2I_3 \nsubseteq I$. Then there exist $a_1 \in I_1$ and $a_2 \in I_2$ such that $a_1I_3 \nsubseteq I$ and $a_2I_3 \nsubseteq I$. Also, $a_1a_2I_3 \subseteq I$ and $a_1I_3 \nsubseteq I$ and $a_2I_3 \nsubseteq I$, we have $a_1a_2 \in I$ by above lemma. Since $I_1I_2 \nsubseteq I$, therefore for some $a \in I_1$, $b \in I_2$, $ab \notin I$. Since $abI_3 \subseteq I$ and $ab \notin I$, we have $aI_3 \subseteq I$ or $bI_3 \subseteq I$ by above lemma. Here three cases arise.

Case I: Suppose that $aI_3 \subseteq I$, but $bI_3 \nsubseteq I$. Since $a_1bI_3 \subseteq I$ and $bI_3 \nsubseteq I$ and $a_1I_3 \nsubseteq I$, by above lemma, we have $a_1b \in I$. Since $(a + a_1)bI_3 \subseteq I$ and $aI_3 \subseteq I$, but $a_1I_3 \nsubseteq I$, therefore $(a + a_1)I_3 \nsubseteq I$. Since $bI_3 \nsubseteq I$ and $(a + a_1)I_3 \nsubseteq I$, we have $(a + a_1)b \in I$ by above lemma. Again, $(a + a_1)b = ab + a_1b \in I$ and $a_1b \in I$, we conclude that $ab \in I$ (since I is subtractive), which is a contradiction.

Case II: Suppose that $bI_3 \subseteq I$, but $aI_3 \nsubseteq I$. Since $aa_2I_3 \subseteq I$ and $aI_3 \nsubseteq I$ and $a_2I_3 \nsubseteq I$, by above lemma, we have $aa_2 \in I$. Again, $a(b + a_2)I_3 \subseteq I$ and $bI_3 \subseteq I$, but $a_2I_3 \nsubseteq I$, we have $(b + a_2)I_3 \nsubseteq I$. Since $aI_3 \nsubseteq I$ and $(b + a_2)I_3 \nsubseteq I$, we have $a(b + a_2) \in I$ by above lemma. Since $a(b + a_2) = ab + aa_2 \in I$ and $aa_2 \in I$, we conclude that $ab \in I$ (since I is subtractive), which is a contradiction.

Case III: Suppose that $aI_3 \subseteq I$ and $bI_3 \subseteq I$. Since $bI_3 \subseteq I$ and $a_2I_3 \nsubseteq I$, we have $(b+a_2)I_3 \nsubseteq I$. Since $a_1(b+a_2)I_3 \subseteq I$ and $a_1I_3 \nsubseteq I$ and $(b+a_2)I_3 \nsubseteq I$, we have $a_1(b + a_2) = a_1b + a_1a_2 \in I$ by lemma above. Since $a_1b + a_1a_2 \in I$ and $a_1a_2 \in I$, we have $ba_1 \in I$ (since I is subtractive). Since $aI_3 \subseteq I$ and $a_1I_3 \nsubseteq I$, we have $(a + a_1)I_3 \nsubseteq I$. Since $(a + a_1)a_2I_3 \subseteq I$ and $a_2I_3 \nsubseteq I$ and $(a + a_1)I_3 \nsubseteq I$, we have $(a + a_1)a_2 = aa_2 + a_1a_2 \in I$ by above lemma. Since $a_1a_2 \in I$ and $aa_2 + a_1a_2 \in I$, we have $aa_2 \in I$ (since I is subtractive). Now, since $(a + a_1)(b + a_2)I_3 \subseteq I$ and $(a+a_1)I_3 \nsubseteq I$ and $(b+a_2)I_3 \nsubseteq I$, we have $(a+a_1)(b+a_2) = ab+aa_2+ba_1+a_1a_2 \in I$ by above lemma. Since $aa_2, ba_1, a_1a_2 \in I$, we have $aa_2 + ba_1 + a_1a_2 \in I$. Since $ab + aa_2 + ba_1 + a_1a_2 \in I$ and $aa_2 + ba_1 + a_1a_2 \in I$, we conclude that $ab \in I$ (since I is subtractive), which is a contradiction. Hence $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$. □

Result 3.5 ([2], Lemma 2.1 (ii))**.** *If I is a subtractive ideal of S, then $(I : a)$ is a subtractive ideal of S, where $(I : a) = \{s \in S : sa \in I\}$.*

Proof. It is straight forward. □

Next theorem gives some characterizations of 2-absorbing ideals of semiring. Mostafanasab and Darani in [12], proved it for 2-absorbing primary ideals of rings.

Theorem 3.6. *Let S be a semiring and I be a proper subtractive ideal of S. Then the following are equivalent:*

(1) *I is a 2-absorbing ideal of S;*

(2) *For all* $a, b \in S$ *such that* $ab \notin I$, $(I : ab) \subseteq (I : a)$ *or* $(I : ab) \subseteq (I : b)$*;*

(3) *For all* $a \in S$ *and for all ideal* J *of* S *such that* $aJ \nsubseteq I$, $(I : aJ) \subseteq (I : J)$ *or* $(I : aJ) \subseteq (I : a)$*;*

(4) *For all ideals* J, K *of* S *such that* $JK \nsubseteq I$, $(I : JK) \subseteq (I : J)$ *or* $(I : JK) \subseteq (I : K)$*;*

(5) *For all ideals* J, K, L *of* S *such that* $JKL \subseteq I$, *either* $JK \subseteq I$ *or* $KL \subseteq I$ *or* $JL \subseteq I$.

Proof. (1) $\Rightarrow$ (2). Let $ab \notin I$ where $a, b \in S$ and $x \in (I : ab)$. Then $xab \in I$. Therefore, either $xa \in I$ or $xb \in I$ and hence either $x \in (I : a)$ or $x \in (I : b)$. Thus, $(I : ab) \subseteq (I : a) \cup (I : b)$. Then we have $(I : ab) \subseteq (I : a)$ or $(I : ab) \subseteq (I : b)$ (since if A, B are subtractive ideals of a semiring S such that $C \subseteq A \cup B$ where C is an ideal of S, then either $C \subseteq A$ or $C \subseteq B$).
(2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4), (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (1) is similar as the proof of ([12], Theorem 2.1), by using the result (if A, B are subtractive ideals of a semiring S such that $C \subseteq A \cup B$ where C is an ideal of S, then either $C \subseteq A$ or $C \subseteq B$). □

Theorem 3.7. *Let* I *be a* 2*-absorbing ideal of* S *and* A *be a multiplicatively closed subset of* S *such that* $I \cap A = \Phi$. *Then* $A^{-1}I$ *is also a* 2*-absorbing ideal of* $A^{-1}S$.

Proof. Let $(a/s)(b/t)(c/k) \in A^{-1}I$ for some $a, b, c \in S$ and $s, t, k \in A$. Then there exists $u \in A$ such that $uabc \in I$. Therefore, we have $uab \in I$ or $bc \in I$ or $uac \in I$, since I is a 2-absorbing ideal of S. If $uab \in I$, then $(a/s)(b/t) = (uab/ust) \in A^{-1}I$. If $bc \in I$, then $(b/t)(c/k) \in A^{-1}I$. If $uac \in I$, then $(a/s)(c/k) = (uac/usk) \in A^{-1}I$. □

Lemma 3.8. *Let* S *be a semiring and* P_1 *and* P_2 *be distinct weakly prime ideals of* S. *Then* $P_1 \cap P_2$ *is also a weakly* 2*-absorbing ideal of* S.

Proof. Let $0 \neq abc \in P_1 \cap P_2$ for some $a, b, c \in S$. Suppose that $ab \notin P_1 \cap P_2$ and $ac \notin P_1 \cap P_2$. Assume that $ab \notin P_1$ and $ac \notin P_1$. Since $0 \neq abc \in P_1$ and P_1 is weakly prime, we get $c \in P_1$ and hence $ac \in P_1$, a contradiction. Similarly, if $ab \notin P_2$ and $ac \notin P_2$, we would get a contradiction. Therefore, either $ab \notin P_1$ and $ac \notin P_2$ or $ab \notin P_2$ and $ac \notin P_1$. First assume that, $ab \notin P_1$ and $ac \notin P_2$. Since $0 \neq abc \in P_1$, we get $c \in P_1$ and hence $bc \in P_1$. Similarly, since $0 \neq abc \in P_2$, we get $b \in P_2$ and hence $bc \in P_2$. Thus, $bc \in P_1 \cap P_2$. Hence $P_1 \cap P_2$ is a weakly 2-absorbing ideal of S. Likewise, we can prove for the second case when $ab \notin P_2$ and $ac \notin P_1$, we have $bc \in P_1 \cap P_2$. □

Definition 3.9. Let I be a weakly 2-absorbing ideal of S. We say that (a, b, c), where $a, b, c \in S$ is a *triple zero* of I if $abc = 0$, $ab \notin I$, $bc \notin I$ and $ac \notin I$.

Theorem 3.10. *Let* I *be a subtractive weakly* 2*-absorbing ideal of* S *and* (a, b, c) *be a triple zero of* I *for some* $a, b, c \in S$. *Then*

(1) $abI = bcI = acI = \{0\}$.

(2) $aI^2 = bI^2 = cI^2 = \{0\}$.

Proof. (1). Let $abI \neq 0$. Then there exists $x \in I$ such that $abx \neq 0$. Therefore, $ab(c+x) \neq 0$. Since I is a weakly 2-absorbing ideal of S and $ab \notin I$, we have $a(c+x) \in I$ or $b(c+x) \in I$ and hence $ac \in I$ or $bc \in I$ (since I is subtractive), which is a contradiction. Thus, $abI = 0$. Similarly, $bcI = acI = 0$.

(2). Let $aI^2 \neq 0$. Then there exist $x, y \in I$ such that $axy \neq 0$. Therefore (1) gives, $a(b+x)(c+y) = axy \neq 0$. Since I is a weakly 2-absorbing ideal of S, we have either $a(b+x) \in I$ or $a(c+y) \in I$ or $(b+x)(c+y) \in I$. Thus, $ab \in I$ or $ac \in I$ or $bc \in I$ (since I is subtractive), which is a contradiction. Hence $aI^2 = 0$. Similarly, $bI^2 = cI^2 = 0$. □

Definition 3.11. Let I be a weakly 2-absorbing ideal of S and let $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of S. We say that I is a *free triple zero with respect to* $I_1I_2I_3$ if (a, b, c) is not a triple zero of I for every $a \in I_1$, $b \in I_2$, and $c \in I_3$.

Conjecture 3.12. *If I is a weakly 2-absorbing ideal of S with $0 \neq I_1I_2I_3 \subseteq I$ for some ideals $I_1, I_2, I_3 \in S$, then I is a free triple zero with respect to $I_1I_2I_3$.*

Lemma 3.13. *Let I be a subtractive weakly 2-absorbing ideal of S. Let $abJ \subseteq I$ for some $a, b \in S$ and some ideal J of S such that (a, b, c) is not a triple zero of I for every $c \in J$. If $ab \notin I$, then either $aJ \subseteq I$ or $bJ \subseteq I$.*

Proof. Suppose that $aJ \nsubseteq I$ and $bJ \nsubseteq I$. Then, there are some $x, y \in J$ such that $ax \notin I$ and $by \notin I$. Since (a, b, x) is not a triple zero of I and $abx \in I$ and $ab \notin I$ and $ax \notin I$, we have $bx \in I$. Since (a, b, y) is not a triple zero of I and $aby \in I$ and $ab \notin I$ and $by \notin I$, we have $ay \in I$. Again, $(a, b, x+y)$ is not a triple zero of I and $ab(x+y) \in I$ and $ab \notin I$, we have $a(x+y) \in I$ or $b(x+y) \in I$, since I is a weakly 2-absorbing ideal of S. If $a(x+y) \in I$ and $ay \in I$, then $ax \in I$ (since I is subtractive), which is a contradiction. Similarly, if $b(x+y) \in I$ and $bx \in I$, we get $by \in I$ (since I is subtractive), which is a contradiction. Hence, either $aJ \subseteq I$ or $bJ \subseteq I$. □

Remark 3.14. If I is a weakly 2-absorbing ideal of S and $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of S such that I is a free triple zero with respect to $I_1I_2I_3$. Then $ab \in I$ or $ac \in I$ or $bc \in I$ for all $a \in I_1$, $b \in I_2$ and $c \in I_3$.

Let I be a weakly 2-absorbing ideal of S. According to the following result, we see that Conjecture 3.12 is valid if and only if whenever $0 \neq I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of S, then either $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq I$ or $I_1I_3 \subseteq I$.

Theorem 3.15. *Let I be a subtractive weakly 2-absorbing ideal of S. If $0 \neq I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of S such that I is a free triple zero with respect to $I_1I_2I_3$, then either $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq I$ or $I_3I_1 \subseteq I$.*

Proof. Let I be a subtractive weakly 2-absorbing ideal of S and $0 \neq I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of S such that I is a free triple zero with respect to $I_1I_2I_3$. Let $I_1I_2 \nsubseteq I$. We show that $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$. By using above remark 1 and lemma 3.13, it will proceed as the proof of theorem 3.4. If possible, suppose that $I_1I_3 \nsubseteq I$ and $I_2I_3 \nsubseteq I$. Then there exist $a_1 \in I_1$ and $a_2 \in I_2$ such that $a_1I_3 \nsubseteq I$ and $a_2I_3 \nsubseteq I$. Also, $a_1a_2I_3 \subseteq I$ and $a_1I_3 \nsubseteq I$ and $a_2I_3 \nsubseteq I$, we have $a_1a_2 \in I$ by lemma 3.13. Since $I_1I_2 \nsubseteq I$, therefore for some $a \in I_1$, $b \in I_2$, $ab \notin I$. Since $abI_3 \subseteq I$ and $ab \notin I$, we have $aI_3 \subseteq I$ or $bI_3 \subseteq I$ by lemma 3.13 . Here three cases arise.

Case I: Suppose that $aI_3 \subseteq I$, but $bI_3 \nsubseteq I$. Since $a_1bI_3 \subseteq I$ and $bI_3 \nsubseteq I$ and $a_1I_3 \nsubseteq I$, by lemma 3.13, we have $a_1b \in I$. Since $(a + a_1)bI_3 \subseteq I$ and $aI_3 \subseteq I$, but $a_1I_3 \nsubseteq I$, therefore $(a + a_1)I_3 \nsubseteq I$. Since $bI_3 \nsubseteq I$ and $(a + a_1)I_3 \nsubseteq I$, we have $(a + a_1)b \in I$ by lemma 3.13. Again, $(a + a_1)b = ab + a_1b \in I$ and $a_1b \in I$, we conclude that $ab \in I$ (since I is subtractive), which is a contradiction.

Case II: Suppose that $bI_3 \subseteq I$, but $aI_3 \nsubseteq I$. Since $aa_2I_3 \subseteq I$ and $aI_3 \nsubseteq I$ and $a_2I_3 \nsubseteq I$, by lemma 3.13, we have $aa_2 \in I$. Again, $a(b + a_2)I_3 \subseteq I$ and $bI_3 \subseteq I$, but $a_2I_3 \nsubseteq I$, we have $(b + a_2)I_3 \nsubseteq I$. Since $aI_3 \nsubseteq I$ and $(b + a_2)I_3 \nsubseteq I$, we have $a(b + a_2) \in I$ by lemma 3.13. Since $a(b + a_2) = ab + aa_2 \in I$ and $aa_2 \in I$, we conclude that $ab \in I$ (since I is subtractive), which is a contradiction.

Case III: Suppose that $aI_3 \subseteq I$ and $bI_3 \subseteq I$. Since $bI_3 \subseteq I$ and $a_2I_3 \nsubseteq I$, we have $(b + a_2)I_3 \nsubseteq I$. Since $a_1(b + a_2)I_3 \subseteq I$ and $a_1I_3 \nsubseteq I$ and $(b + a_2)I_3 \nsubseteq I$, we have $a_1(b+a_2) = a_1b+a_1a_2 \in I$ by lemma 3.13. Since $a_1b+a_1a_2 \in I$ and $a_1a_2 \in I$, we have $ba_1 \in I$ (since I is subtractive). Since $aI_3 \subseteq I$ and $a_1I_3 \nsubseteq I$, we have $(a + a_1)I_3 \nsubseteq I$. Since $(a + a_1)a_2I_3 \subseteq I$ and $a_2I_3 \nsubseteq I$ and $(a + a_1)I_3 \nsubseteq I$, we have $(a + a_1)a_2 = aa_2 + a_1a_2 \in I$ by lemma 3.13. Since $a_1a_2 \in I$ and $aa_2 + a_1a_2 \in I$, we have $aa_2 \in I$ (since I is subtractive). Now, since $(a + a_1)(b + a_2)I_3 \subseteq I$ and $(a+a_1)I_3 \nsubseteq I$ and $(b+a_2)I_3 \nsubseteq I$, we have $(a+a_1)(b+a_2) = ab+aa_2+ba_1+a_1a_2 \in I$ by lemma 3.13. Since $aa_2, ba_1, a_1a_2 \in I$, we have $aa_2 + ba_1 + a_1a_2 \in I$. Since $ab + aa_2 + ba_1 + a_1a_2 \in I$ and $aa_2 + ba_1 + a_1a_2 \in I$, we conclude that $ab \in I$ (since I is subtractive), which is a contradiction. Hence $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$. □

Proposition 3.16. *Let S be a semiring and I be a proper subtractive ideal of S. Then the following statements are equivalent:*

(1) *For any ideals I_1, I_2, I_3 of S, $0 \neq I_1I_2I_3 \subseteq I$ implies either $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$;*

(2) *For any ideals I_1, I_2, I_3 of S such that $I \subseteq I_1$, $0 \neq I_1I_2I_3 \subseteq I$ implies either $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1). Let $0 \neq JI_2I_3 \subseteq I$ for some ideals J, I_2, I_3 of S. Then obviously $0 \neq (J + I)I_2I_3 = (JI_2I_3) + (II_2I_3) \subseteq I$. Let $I_1 = J + I$. Then, either $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$ by given hypothesis. Therefore, $(J + I)I_2 \subseteq I$ or $(J + I)I_3 \subseteq I$ or $I_2I_3 \subseteq I$. Thus, either $JI_2 \subseteq I$ or $JI_3 \subseteq I$ or $I_2I_3 \subseteq I$ (since I is subtractive). □

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On the principal (m,n)-ideals in the direct product of two semigroups

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Abstract. We characterize properties of the equivalence relation determined by two (m, n)-ideals of a semigroup S and describe properties of this relation in the direct product of two semigroups.

1. Preliminaries

Let m, n be non-negative integers. A subsemigroup A of a semigroup S is called an (m, n)-*ideal* of S if

$$A^m S A^n \subseteq A$$

(Here, $A^0 S = S A^0 = S$). This notion was first introduced by S. Lajos [3] in 1961. The principal (m, n)-ideal of S generated by $a \in S$ will be denoted by $[a]_{(m,n)}$, and it is of the form

$$[a]_{(m,n)} = \bigcup_{i=1}^{m+n} \{a^i\} \bigcup a^m S a^n$$

(see [2]).

Now, let T be a semigroup, and thus the direct product $S \times T$ is a semigroup under the coordinate wise multiplications. In this paper we introduce the equivalence relation $\mathcal{J}_{(m,n)}$ on S by, for any $a, b \in S$,

$$a \mathcal{J}_{(m,n)} b \iff [a]_{(m,n)} = [b]_{(m,n)}.$$

2. Main results

Throughout this section, let m, n be non-negative integers and S be a semigroup. We begin this section with the following lemmas:

Lemma 2.1. *Let S and T be any two semigroups, and let $s \in S$, $t \in T$. Then*

$$(s, t)^m (S \times T)(s, t)^n = s^m S s^n \times t^m T t^n.$$

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Hence,

$$[(s,t)]_{(m,n)} = \bigcup_{i=1}^{m+n}\{(s,t)^i\} \cup s^mSs^n \times t^mTt^n.$$

Proof. This follows by

$$(s,t)^m(S\times T)(s,t)^n = (s^m,t^n)(S\times T)(s^n,t^n) = s^mSs^n \times t^mTt^n. \qquad \square$$

Lemma 2.2. *Let s be an element of a semigroup S. Then*

$$[s]_{(m,n)} = s^mSs^n \;\longleftrightarrow\; s \in s^mSs^n.$$

Proof. It is clear that $[s]_{(m,n)} = s^mSs^n$ implies $s \in s^mSs^n$.
Conversely, if $s \in s^mSs^n$, then

$$[s]_{(m,n)} = \bigcup_{i=1}^{m+n}\{s^i\} \cup s^mSs^n \subseteq s^mSs^n.$$

By $s^mSs^n \subseteq [s]_{(m,n)}$, $[s]_{(m,n)} = s^mSs^n$. $\square$

Lemma 2.3. *Let S and T be any two semigroups, and let $s \in S$, $t \in T$. Then*

$$[(s,t)]_{(m,n)} \subseteq [s]_{(m,n)} \times [t]_{(m,n)}.$$

Proof. This follows by Lemma 2.1. $\square$

We now prove the first main purpose of this paper.

Theorem 2.4. *Let S and T be any two semigroups, and let $s \in S$, $t \in T$. Then $[(s,t)]_{(m,n)} = [s]_{(m,n)} \times [t]_{(m,n)}$ if and only if at least one of the following conditions holds:*

(1) $s^mSs^n = \{s\}$,

(2) $t^mTt^n = \{t\}$,

(3) $s \in s^mSs^n$ *and* $t \in t^mTt^n$.

Proof. Assume first that $[(s,t)]_{(m,n)} = [s]_{(m,n)} \times [t]_{(m,n)}$. Suppose that $s \notin s^mSs^n$ or $t \notin t^mTt^n$. If $s \notin s^mSs^n$, then $s \neq s^k$ for all $k \in \{2,3,\ldots\}$; hence

$$\{s\} \times t^mTt^n = \{(s,t)\}.$$

This implies that $t^mTt^n = \{t\}$. Similarly, if $t \notin t^mTt^n$, then $s^mSs^n = \{s\}$.

Conversely, we assume that (1), (2) or (3) holds. If $s^mSs^n = \{s\}$, then, by Lemma 2.2, it follows that

$$\begin{aligned}[s]_{(m,n)} \times [t]_{(m,n)} &= \{s\} \times [t]_{(m,n)} = \bigcup_{i=1}^{m+n}\{(s,t^i)\} \bigcup \{s\} \times t^mTt^n \\ &= \bigcup_{i=1}^{m+n}\{(s^i,t^i)\} \bigcup s^mSs^n \times t^mTt^n = [(s,t)]_{(m,n)}.\end{aligned}$$

Thus $[(s,t)]_{(m,n)} = [s]_{(m,n)} \times [t]_{(m,n)}$.

Similarly, if $t^mTt^n = \{t\}$, then $[(s,t)]_{(m,n)} = [s]_{(m,n)} \times [t]_{(m,n)}$.

Finally, we assume that $s \in s^mSs^n$ and $t \in t^mTt^n$. By Lemmas $2.2 - 2.3$, we have

$$[(s,t)]_{(m,n)} \subseteq [s]_{(m,n)} \times [t]_{(m,n)} = s^mSs^n \times t^mTt^n \subseteq [(s,t)]_{(m,n)}.$$

Therefore $[(s,t)]_{(m,n)} = [s]_{(m,n)} \times [t]_{(m,n)}$, as required. □

Now, we consider the second aim of this paper.

Lemma 2.5. *For any $s \in S$, if $J_{(m,n),s} \cap s^mSs^n \neq \emptyset$, then $J_{(m,n),s} \subseteq s^mSs^n$.*

Proof. For $J_{(m,n),s} \cap s^mSs^n \neq \emptyset$ there exists $u \in J_{(m,n),s} \cap s^mSs^n$. Thus

$$s \in [s]_{(m,n)} = [u]_{(m,n)}.$$

We have $s \in s^mSs^n$. Indeed, if $s = u^i$ for some $i \in \{1, 2, \ldots, m+n\}$, then $s \in s^mSs^n$. And, if $s \in u^mSu^n$, then $s \in u^mSu^n \subseteq (s^mSs^n)S(s^mSs^n) \subseteq s^mSs^n$, and so $s \in s^mSs^n$.

Now, if $v \in J_{(m,n),s}$, then $[v]_{(m,n)} = [s]_{(m,n)}$; hence $v \in [s]_{(m,n)}$. This implies $v \in [s]_{(m,n)} = s^mSs^n$ by Lemma 2.2. Therefore $J_{(m,n),s} \subseteq s^mSs^n$. □

Lemma 2.6. *If for $s \in S$ the cardinality $|J_{(m,n),s}| > 1$, then $J_{(m,n),s} \subseteq s^mSs^n$.*

Proof. For $|J_{(m,n),s}| > 1$ there exists $u \in J_{(m,n),s}$ such that $u \neq s$. We have

$$u \in [u]_{(m,n)} = [s]_{(m,n)}.$$

If $u \in s^mSs^n$, then $J_{(m,n),s} \cap s^mSs^n \neq \emptyset$. So, $J_{(m,n),s} \subseteq s^mSs^n$, by Lemma 2.5. Let $u = s^i$ for some $i \in \{2, 3, \ldots, m+n\}$. If $s = u^j$ for some $j \in \{2, 3, \ldots, m+n\}$, then $s \in s^mSs^n$. Therefore, by Lemma 2.5, it follows that $J_{(m,n),s} \subseteq s^mSs^n$. If $s \in u^mSu^n$, then $J_{(m,n),s} = J_{(m,n),u} \subseteq u^mSu^n = s^{mi}Ss^{ni} \subseteq s^mSs^n$. This completes the proof. □

Let S and T be any two semigroups. Define $\pi_S : S \times T \to S$ and $\pi_T : S \times T \to T$, respectively, by:

$$(s,t)\pi_S = s \text{ for all } s \in S \quad \text{and} \quad (s,t)\pi_T = t \text{ for all } t \in T.$$

Then π_S (resp. π_T) is a projection from $S \times T$ onto S (resp. T). Moreover, for any $(s,t) \in S \times T$ we have $[(s,t)]_{(m,n)}\pi_S = [s]_{(m,n)}$ and $[(s,t)]_{(m,n)}\pi_T = [t]_{(m,n)}$.

Theorem 2.7. *Let S and T be any two semigroups, and let $(s,t) \in S \times T$. Then*

(1) $J_{(m,n),(s,t)} \subseteq J_{(m,n),s} \times J_{(m,n),t}$, *and*

(2) *if $J_{(m,n),(s,t)}$ is a proper subset of $J_{(m,n),s} \times J_{(m,n),t}$, then $J_{(m,n),s} \times J_{(m,n),t}$ is the union of at least two $\mathcal{J}_{(m,n)}$-classes in $S \times T$.*

Proof. To prove (1), let $(u,v) \in J_{(m,n),(s,t)}$. Then $[(s,t)]_{(m,n)} = [(u,v)]_{(m,n)}$,

$$[s]_{(m,n)} = [(s,t)]_{(m,n)}\pi_S = [(u,v)]_{(m,n)}\pi_S = [u]_{(m,n)}$$

and

$$[t]_{(m,n)} = [(s,t)]_{(m,n)}\pi_T = [(u,v)]_{(m,n)}\pi_T = [v]_{(m,n)}.$$

Thus $(u,v) \in J_{(m,n),s} \times J_{(m,n),t}$.

(2). Let $(u,v) \in J_{(m,n),s} \times J_{(m,n),t} \setminus J_{(m,n),(s,t)}$. Then $[u]_{(m,n)} = [s]_{(m,n)}$ and $[v]_{(m,n)} = [t]_{(m,n)}$. Thus

$$J_{(m,n),(u,v)} \subseteq J_{(m,n),u} \times J_{(m,n),v} = J_{(m,n),s} \times J_{(m,n),t}. \qquad \square$$

Corollary 2.8. *Let S and T be any two semigroups, and let $(s,t) \in S \times T$. If $J_{(m,n),s} = \{s\}$ and $J_{(m,n),t} = \{t\}$, then*

$$J_{(m,n),(s,t)} = J_{(m,n),s} \times J_{(m,n),t} = \{(s,t)\}.$$

Theorem 2.9. *Let S and T be any two semigroups, and let $s \in S, t \in T$. Then $J_{(m,n),s} \times J_{(m,n),t} = J_{(m,n),(s,t)}$ if and only if at least one of the following conditions holds:*

(1) $J_{(m,n),s} = \{s\}$ *and* $J_{(m,n),t} = \{t\}$,

(2) $s \in s^mSs^n$ *and* $t \in t^mTt^n$.

Proof. Assume that $J_{(m,n),s} \times J_{(m,n),t} = J_{(m,n),(s,t)}$. If $|J_{(m,n),(s,t)}| = 1$, then

$$J_{(m,n),s} \times J_{(m,n),t} = J_{(m,n),(s,t)} = \{(s,t)\}.$$

That is, $J_{(m,n),s} = \{s\}$ and $J_{(m,n),t} = \{t\}$.

If $|J_{(m,n),(s,t)}| > 1$, then, $(s,t) \in J_{(m,n),(s,t)} \subseteq s^mSs^n \times t^mTt^n$, by Lemma 2.2.

Conversely, if (1) holds, then $J_{(m,n),s} \times J_{(m,n),t} = J_{(m,n),(s,t)}$, by Corollary 2.8. By (2) and Theorem 2.4, we get $[(s,t)]_{(m,n)} = [s]_{(m,n)} \times [t]_{(m,n)}$. By Theorem 2.7, $J_{(m,n),(s,t)} \subseteq J_{(m,n),s} \times J_{(m,n),t}$.

To prove the reverse inclusion let $(u,v) \in J_{(m,n),s} \times J_{(m,n),t}$.

Case 1: $(u,v) = (s,t)$. Then $(u,v) \in J_{(m,n),(s,t)}$, and so $J_{(m,n),s} \times J_{(m,n),t} \subseteq J_{(m,n),(s,t)}$.

Case 2: $u \neq s$. By Lemma 2.6, we have $u \in J_{(m,n),u} \subseteq u^mSu^n$, because $s,u \in J_{(m,n),s} = J_{(m,n),u}$.

Case 2.1: $v = t$. We have $v \in v^mTv^n$. By Theorem 2.4,

$$[(u,v)]_{(m,n)} = [u]_{(m,n)} \times [v]_{(m,n)} = [s]_{(m,n)} \times [t]_{(m,n)} = [(s,t)]_{(m,n)}.$$

Thus $(u,v) \in J_{(m,n),(s,t)}$. Therefore $J_{(m,n),s} \times J_{(m,n),t} \subseteq J_{(m,n),(s,t)}$.

Case 2.2: $v \neq t$. We have $v \in J_{(m,n),v} \subseteq v^mTv^n$. As Case 2.1 we have $(u,v) \in J_{(m,n),(s,t)}$, and thus

$$J_{(m,n),s} \times J_{(m,n),t} \subseteq J_{(m,n),(s,t)}.$$

Case 3: $u = t, v \neq t$. Analogously as Case 2.1.

This completes the proof. $\square$

Using the thereom above we have the following.

Corollary 2.10. *Let S and T be any two semigroups, and let $s \in S, t \in T$. If $|J_{(m,n),s}| > 1$ and $|J_{(m,n),t}| > 1$, then $J_{(m,n),(s,t)} = J_{(m,n),s} \times J_{(m,n),t}$.*

Corollary 2.11. *Let S and T be any two semigroups, and let $s \in S, t \in T$. If $J_{(m,n),s} \times J_{(m,n),t}$ is the union of at least two $\mathcal{J}_{(m,n)}$-classes, then necessarily either $|J_{(m,n),s}| > 1, J_{(m,n),t} = \{t\}$ or $|J_{(m,n),t}| > 1, J_{(m,n),s} = \{s\}$.*

Theorem 2.12. *Let S and T be any two semigroups, and let $s \in S, t \in T$. Then $J_{(m,n),s} \times J_{(m,n),t}$ is the union of at least two $\mathcal{J}_{(m,n)}$-classes if and only if either*

$$|J_{(m,n),s}| > 1, \quad J_{(m,n),t} = \{t\}, \quad t \notin t^m T t^n$$

or

$$|J_{(m,n),t}| > 1, \quad J_{(m,n),s} = \{s\}, \quad s \notin s^m S s^n.$$

Proof. Assume that $J_{(m,n),s} \times J_{(m,n),t}$ is the union of at least two $\mathcal{J}_{(m,n)}$-classes. By Corollary 2.11,

$$|J_{(m,n),s}| > 1, \quad J_{(m,n),t} = \{t\}$$

or

$$|J_{(m,n),t}| > 1, \quad J_{(m,n),s} = \{s\}.$$

Case 1: $|J_{(m,n),s}| > 1$, $J_{(m,n),t} = \{t\}$. Then $t \notin t^m T t^n$ because otherwise, $s \in J_{(m,n),s} \subseteq s^m S s^n$ and $t \in t^m T t^n$ imply that $J_{(m,n),s} \times J_{(m,n),t} = J_{(m,n),(s,t)}$.

Case 2: $|J_{(m,n),t}| > 1$, $J_{(m,n),s} = \{s\}$. This can be proceed analogously, and hence $s \notin s^m S s^n$.

For the opposite direction, it suffices to consider the case $|J_{(m,n),s}| > 1$, $J_{(m,n),t} = \{t\}$, $t \notin t^m T t^n$. Let $u \in J_{(m,n),s}$ such that $u \neq s$. Then $(u,t) \in J_{(m,n),s} \times J_{(m,n),t}$. Since $t \notin t^m T t^n$, we have $(s,t) \notin s^m S s^n \times t^m T t^n$. Thus, by Lemma 2.6, we have $(u,t) \notin \{(s,t)\} = J_{(m,n),(s,t)}$. □

The rest of this paper, relationships between maximal $\mathcal{J}_{(m,n)}$-classes in $S \times T$ and maximal $\mathcal{J}_{(m,n)}$-classes in S and in T will be investigated.

Theorem 2.13. *Let S and T be any two semigroups, and let $s \in S$, $t \in T$ be such that $(s,t) \in s^m S s^n \times t^m T t^n$. Then, for any $u \in S, v \in T$, $[(s,t)]_{(m,n)} \subseteq [(u,v)]_{(m,n)}$ if and only if $[s]_{(m,n)} \subseteq [u]_{(m,n)}$ and $[t]_{(m,n)} \subseteq [v]_{(m,n)}$.*

Proof. Assume that $[(s,t)]_{(m,n)} \subseteq [(u,v)]_{(m,n)}$. Then

$$[s]_{(m,n)} = [(s,t)]_{(m,n)}\pi_S \subseteq [(u,v)]_{(m,n)}\pi_S = [u]_{(m,n)},$$

$$[t]_{(m,n)} = [(s,t)]_{(m,n)}\pi_T \subseteq [(u,v)]_{(m,n)}\pi_T = [v]_{(m,n)}.$$

Hence $[s]_{(m,n)} \subseteq [u]_{(m,n)}$ and $[t]_{(m,n)} \subseteq [v]_{(m,n)}$.

Assume that $[s]_{(m,n)} \subseteq [u]_{(m,n)}$ and $[t]_{(m,n)} \subseteq [v]_{(m,n)}$. Since $s \in s^m S s^n$ and $t \in t^m T t^n$, it follows by Theorem 2.4 and Lemma 2.2 that

$$[(s,t)]_{(m,n)} = [s]_{(m,n)} \times [t]_{(m,n)} = s^m S s^n \times t^m T t^n.$$

If $(x, y) \in [(s, t)]_{(m,n)}$, then

$$(x, y) \in s^m S s^n \times t^m T t^n \subseteq u^m S u^n \times v^m T v^n \subseteq [(u, v)]_{(m,n)}.$$

Thus $[(s, t)]_{(m,n)} \subseteq [(u, v)]_{(m,n)}$. □

Theorem 2.14. *Let S and T be any two semigroups, and let $s \in S$, $t \in T$ be such that $(s, t) \in s^m S s^n \times t^m T t^n$. Then $J_{(m,n),(s,t)}$ is a maximal $\mathcal{J}_{(m,n)}$-class in $S \times T$ if and only if $J_{(m,n),s}$ and $J_{(m,n),t}$ are maximal $\mathcal{J}_{(m,n)}$-classes in S and in T, respectively.*

Proof. Assume first that $J_{(m,n),(s,t)}$ is a maximal $\mathcal{J}_{(m,n)}$-class in $S \times T$. Suppose that $J_{(m,n),s}$ is not a maximal $\mathcal{J}_{(m,n)}$-class in S. Then there exists $u \in S$ such that $[s]_{(m,n)} \subset [u]_{(m,n)}$. By Theorem 2.13, $[(s, t)]_{(m,n)} \subseteq [(u, t)]_{(m,n)}$. We have

$$(u, t) \notin [s]_{(m,n)} \times [t]_{(m,n)} = [(s, t)]_{(m,n)}.$$

Thus $[(s, t)]_{(m,n)} \subset [(u, t)]_{(m,n)}$. This contradicts to the maximality of $J_{(m,n),(s,t)}$. In the same manner, if $J_{(m,n),t}$ is not a maximal $\mathcal{J}_{(m,n)}$-class in T, then we get a contradiction.

Conversely, we assume that $J_{(m,n),s}$ and $J_{(m,n),t}$ are maximal $\mathcal{J}_{(m,n)}$-classes in S and in T, respectively. Suppose that $J_{(m,n),(s,t)}$ is not maximal. Then there exists $(u, v) \in S \times T$ such that $[(s, t)]_{(m,n)} \subset [(u, v)]_{(m,n)}$. Thus

$$[s]_{(m,n)} \times [t]_{(m,n)} = [(s, t)]_{(m,n)} \subset [(u, v)]_{(m,n)} \subseteq [u]_{(m,n)} \times [v]_{(m,n)}.$$

This is a contradiction. □

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Characterizing monomorphisms of actions on directed complete posets (S-dcpo)

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Abstract. Domain Theory is a branch of mathematics that studies special kinds of partially ordered sets (posets) commonly called *domains*. It was introduced in the 1970s by Scott as a foundation for programming semantics and provides an abstract model of computation, and has grown into a respected field on the borderline between Mathematics and Computer Science.

In this paper we take domains as ordered algebraic structures and consider the actions of a partially ordered monoid which is itself a domain, on them. To study algebraic notions, in particular injectivity and flatness, in the categories so obtained, one needs to know the different kinds of monomorphisms, their properties and the relations between them. This is what we are going to discuss in this paper.

1. Introduction and preliminaries

Domain theory is a branch of mathematics that studies special kinds of partially ordered sets (posets) commonly called domains. It was introduced in the 1970s by Scott as a foundation for programming semantics and provides an abstract model of computation using order structures and topology, and has grown into a respected field on the borderline between Mathematics and Computer Science [1].

Relationships between domain theory and logic were noted early on by Scott [10], and subsequently developed by many authors, including Smyth [11], Abramsky [2], and Zhang [12]. There has been much work on the use of domain logics as logics of types and of program correctness, with a focus on functional and imperative languages.

In this paper we take domains as ordered algebraic structures and consider the actions of a pomonoid which is itself a domain, on them. To study algebraic notions, in particular injectivity and flatness, in the categories so obtained, one needs to know the properties of different kinds of monomorphisms and the relations between them. This is what we are trying to do in the following.

First we recall some preliminaries needed in the sequel. The reader can find more details in [2, 4, 5, 6]. Let **Pos** denote the category of all partially ordered sets (posets) with order-preserving (monotone) maps between them. A non-empty subset D of a partially ordered set is called *directed*, denoted by $D \subseteq^d P$, if for every $a, b \in D$ there exists $c \in D$ such that $a, b \leqslant c$; and P is called *directed complete*,

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or briefly a *dcpo*, if for every $D \subseteq^d P$, the directed join $\bigvee^d D$ exists in P. A dcpo which has a bottom element $\perp$ is said to be a *cpo*.

A *dcpo map* or a *continuous map* $f : P \to Q$ between dcpo's is a map with the property that for every $D \subseteq^d P$, $f(D)$ is a directed subset of Q and $f(\bigvee^d D) = \bigvee^d f(D)$. A dcpo map $f : P \to Q$ between cpo's is called *strict* if $f(\perp) =\perp$. Thus we have the category **Dcpo** (**Cpo**) of all dcpo's (cpo's) with (strict) continuous maps between them.

A *po-monoid* S is a monoid with a partial order $\leqslant$ which is compatible with the binary operation (that is, for $s, t, s', t' \in S$, $s \leqslant t$ and $s' \leqslant t'$ imply $ss' \leqslant tt'$). Similarly, a *dcpo* (*cpo*)*-monoid* is a monoid which is also a dcpo (cpo) whose binary operation is a (strict) continuous map.

Recall that an (*right*) *S-act* or an *S-set* for a monoid S is a set A equipped with an *action* $A \times S \to A$, $(a, s) \rightsquigarrow as$, such that $ae = a$ (e is the identity element of S) and $a(st) = (as)t$, for all $a \in A$ and $s, t \in S$. Let **Act**-S denote the category of all S-acts with *action preserving* maps ($f : A \to B$ with $f(as) = f(a)s$, for all $a \in A, s \in S$). Let A be an S-act. An element $a \in A$ is called a *zero, fixed,* or *a trap element* if $as = a$, for all $s \in S$.

For a po-monoid S, an (*right*) *S-poset* is a poset A which is also an S-act whose action $\lambda : A \times S \to A$ is order-preserving, where $A \times S$ is considered as a poset with componentwise order. The category of all S-posets with action preserving monotone maps between them is denoted by **Pos**-S.

Also, for a dcpo (cpo)-monoid S, an (*right*) *S-dcpo* (*S-cpo*) is a dcpo (cpo) A which is also an S-act whose action $\lambda : A \times S \to A$ is a (strict) continuous map.

Notice that in the definition of an S-cpo, the continuity of the action implies that it is also strict. This is because, since $\perp_S \leqslant e$ and the action is continuous, we have $\perp_A \perp_S \leqslant \perp_A e = \perp_A$ and so $\perp_A \perp_S \leqslant \perp_A$. Also, $\perp_A \leqslant \perp_A \perp_S$. Therefore, $\perp_A \perp_S = \perp_A$ and the action is strict. Also, note that the bottom element of an S-cpo in not necessarily a zero element. For example, consider the cpo-monoid $S = \{s, e\}$ where e is the identity element of S, $e \leqslant s$, and $ss = s$. Take the S-cpo $A = \{\perp_A, a\}$, where $\perp_A \leqslant a$, with the action $\perp_A s = a = as$. We see that $\perp_A$ is not a zero element.

A (possibly empty) subset B of an S-dcpo (S-cpo) A is called a *sub S-dcpo* (*sub S-cpo*) of A if B is both a sub dcpo (sub cpo) and a subact of A.

By an *S-dcpo map* (*S-cpo map*) between S-dcpo's (S-cpo's), we mean a map $f : A \to B$ which is both (strict) continuous and action preserving. We denote the category of all S-dcpo's (S-cpo's) and S-dcpo (S-cpo) maps between them by **Dcpo**-S (**Cpo**-S).

A separately (or semi-)cpo-monoid is a monoid which is also a cpo whose right and left translations $R_s : S \to S$, $t \rightsquigarrow ts$ and $L_s : S \to S$, $t \rightsquigarrow st$ are strict continuous.

Now, let S be a separately cpo-monoid. A separately S-cpo is a cpo A which is also an S-act with the action $A \times S \to A$ such that every $R_s : A \to A$, $a \rightsquigarrow as$ and $L_a : S \to A$, $s \rightsquigarrow as$, are strict continuous. The category of all separately

S-cpo's with action preserving strict continuous maps between them is denoted by **Sep-Cpo**-S.

Finally, let S be a monoid with identity e. By a cpo S-act, we mean an S-act in the category **Cpo**. In other words, a pair $(A; (\lambda_s)_{s\in S})$ is called a cpo S-act if A is a cpo, and each $\lambda_s : A \to A$, $a \rightsquigarrow as$, is a cpo map, called an action, such that for all $s, t \in S$, and $a \in A$, denoting $\lambda_s(a)$ by as we have:

(1) $a(st) = (as)t$;

(2) $ae = a$.

By a cpo S-act map between cpo S-acts, we mean a cpo map which is also action preserving. The category of all cpo S-acts with cpo S-act maps between them is denoted by $\mathbf{Cpo_{Act\text{-}S}}$.

Definition 1.1. A morphism $h : A \to B$ in **Dcpo**-S (**Cpo**-S, **Sep-Cpo**-S, $\mathbf{Cpo_{Act\text{-}S}}$) is called *order-embedding* provided that for all $x, y \in A$, $h(x) \leqslant h(y)$ if and only if $x \leqslant y$.

In this paper, first we characterize different kinds of monomorphisms namely regular, strict, strong and extremal in [3], in the categories **Dcpo**-S, **Cpo**-S, **Sep-Cpo**-S and $\mathbf{Cpo_{Act\text{-}S}}$ and see that they are the same as order-embeddings. Then, we study the relation of monomorphisms with one-one morphisms and see that in the categories **Dcpo**-S, **Sep-Cpo**-S, $\mathbf{Cpo_{Act\text{-}S}}$, **Dcpo** and **Cpo**, monomorphisms are exactly one-one morphisms. Also, we show that under some conditions the same result is true for the category **Cpo**-S. In the last section we consider some categorical properties of monomorphisms and regular monomorphisms, in the mentioned categories, the properties such as factorization properties of morphisms and some categorical properties related to limits and colimits.

2. Characterization of monomorphisms

In this section we characterize different kinds of monomorphisms in categories **Dcpo**-S, **Cpo**-S, **Sep-Cpo**-S and $\mathbf{Cpo_{Act\text{-}S}}$, also we study their relation with one-one morphisms. First, we recall some related definitions from [3].

Definition 2.1. Let $\mathcal{E}$ and $\mathcal{M}$ be classes of morphisms in a category $\mathcal{C}$. Then, the pair $(\mathcal{E}, \mathcal{M})$ is called a *factorization structure for morphisms* in $\mathcal{C}$ and $\mathcal{C}$ is called $(\mathcal{E}, \mathcal{M})$-structured provided that:

(1) each of $\mathcal{E}$ and $\mathcal{M}$ is closed under composition with isomorphisms,

(2) $\mathcal{C}$ has $(\mathcal{E}, \mathcal{M})$-*factorizations* (*of morphisms*); that is, each morphism f in $\mathcal{C}$ has a factorization $f = he$, with $e \in \mathcal{E}$ and $h \in \mathcal{M}$, and

(3) $\mathcal{C}$ has the unique $(\mathcal{E}, \mathcal{M})$-*diagonalization property*; that is, for each commutative square

$$\begin{array}{ccc} \mathrm{A} & \xrightarrow{e} & B \\ f\big\downarrow & & \big\downarrow g \\ C & \xrightarrow{h} & D \end{array}$$

with $e \in \mathcal{E}$ and $h \in \mathcal{M}$, there exists a unique *diagonal*; that is, a morphism $d : B \to C$ such that $de = f$ and $hd = g$.

Definition 2.2. A monomorphism $h : A \to B$ in a category $\mathcal{C}$ is called:

(1) *regular* if it is an equalizer of a pair of morphisms;

(2) *strict* if it has the universal property that given any morphism $h' : A' \to B$ such that $rh = sh$ implies $rh' = sh'$, for all $r, s : B \to C$, there exists a unique morphism $\bar{h} : A' \to A$ with $h' = h\bar{h}$;

(3) *strong* provided that $\mathcal{C}$ has the unique (Epi,$\{h\}$)-diagonalization property (Epi is the class of all epimorphisms);

(4) *extremal* provided that if $h = me$, where e is an epimorphism, then e is an isomorphism.

2.1. Monomorphisms and order-embeddings

In this subsection, we characterize different kinds of monomorphisms such as regular, strict, strong and extremal in **Dcpo**-S, **Cpo**-S, **Sep**-**Cpo**-S and $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$.

Remark 2.3. Notice that order-embeddings are one-one, and hence monomorphisms in the categories **Dcpo**-S, **Cpo**-S, **Sep**-**Cpo**-S and $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$. But the converse is not necessarily true. For example, take $S = \{e, s\}$ where $s \leqslant e$ and $s^2 = s$. Then S is a dcpo (cpo, separately cpo)-monoid. Now, take $A = \{\perp, a, a'\}$ with the order $\perp \leqslant a, a'$, $a \parallel a'$, and define the action on A as follows: $\perp$ is a zero element and $as = a's = \perp$. Also, take B to be the three element chain $\mathbf{3} = \{0, 1, 2\}$ with $0 \leqslant 1 \leqslant 2$, and define the action on B as follows: 0 is a zero element and $1s = 2s = 0$. Now, define $h : A \to B$ as $h(\perp) = 0$, $h(a) = 1$, $h(a') = 2$. Then h is one-one and hence a monomorphism in these categories, but it is not an order-embedding.

Theorem 2.4. *A monomorphism $h : A \to B$ in* **Dcpo**-S*,* **Cpo**-S*,* **Sep**-**Cpo**-S *and* $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ *is regular if and only if it is order-embedding.*

Proof. Let $h : A \to B$ be a regular monomorphism in **Dcpo**-S (**Cpo**-S, **Sep**-**Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$). Then h is the equalizer of morphisms $g_1, g_2 : B \to C$. Note that, the equalizer of g_1 and g_2 in these categories is $E = \{b \in B : g_1(b) = g_2(b)\}$ with order and action inherited from B (see also [7], [8], [9]). Hence there exists an isomorphism between E and A, so h is an order-embedding.

Conversely, let $h : A \to B$ be an order-embedding in one of the categories **Dcpo**-S, **Cpo**-S, **Sep**-**Cpo**-S or $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$. In each category, we define two morphisms whose equalizer is h.

(i). In **Dcpo**-S, consider the disjoint union $(B \times \{1\}) \cup (B \times \{2\})$ of B with itself, which is the coproduct $B \sqcup B$ by Theorem 2.4 of [7]. Take B' to be the quotient $(B \sqcup B)/\theta(H)$, where $\theta(H)$ is the congruence generated by $H = \{((h(a), 1), (h(a), 2)) : a \in A\}$. Now, consider the natural epimorphism $q : B \sqcup B \to B'$ and the coproduct maps $g_1, g_2 : B \to B \sqcup B$. We prove later on that h is the equalizer of qg_1 and qg_2.

(ii). In **Cpo**-S, we consider the same S-dcpo B' as defined in (i). Since in this case h is strict, $h(\perp_A) = \perp_B$, and then $[(\perp_B, 1)]_{\theta(H)} = [(\perp_B, 2)]_{\theta(H)}$ is the bottom element of B'. So, B' is an S-cpo. Also qg_1 and qg_2 introduced in part (i) are strict, because $qg_1(\perp_B) = [(\perp_B, 1)]_{\theta(H)}$ and $qg_2(\perp_B) = [(\perp_B, 2)]_{\theta(H)}$. We will see that h is the equalizer of qg_1 and qg_2 in **Cpo**-S.

(iii). In **Sep**-**Cpo**-S, B is a separately S-cpo, and hence by Remark 3.3 of [8], B is also an S-cpo. So, from the discussion given in (ii), B' which introduced in part (i), is an S-cpo. Now again by applying Remark 3.3 of [8], we get that B' is a separately S-cpo. This is because, B is a separately S-cpo, and so for every $b \in B$ and $s \in S$ we have $b\perp_S = \perp_B$ and $\perp_B s = \perp_B$, therefore for every $b \in B$, $s \in S$, and $i = 1, 2$, we have $[(b, i)]\perp_S = [(b\perp_S, i)] = [(\perp_B, i)]$ and $[(\perp_B, i)]s = [(\perp_B s, i)] = [(\perp_B, i)]$. Also, similar to part (ii), qg_1 and qg_2 are strict continuous maps. We will see later on that h is the equalizer of qg_1 and qg_2.

(iv). In $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$, similar to (i), take the coporoduct of B with itself (which is called the coalesced sum, see [9]), and apply the same argument to define $q,\ g_1,\ g_2$. We show that h is the equalizer of qg_1 and qg_2.

Now, we prove that h is the equalizer of qg_1 and qg_2 in all the above cases.

It is clear that $(qg_1)h = (qg_2)h$. Consider an S-dcpo (an S-cpo, a separately S-cpo, a cpo S-act) map $k : C \to B$ with $(qg_1)k = (qg_2)k$. Notice that $k(C) \subseteq h(A)$. This because, on the contrary if $x \in k(C) \setminus h(A)$, then since $x \notin h(A)$, we get $qg_1(x) \neq qg_2(x)$ but since $x \in k(C)$ and $(qg_1)k = (qg_2)k$, we have $qg_1(x) = qg_2(x)$ which is a contradiction. On the other hand, since h is an order-embedding, it is one-one, and so there exists a map $h' : B \to A$ such that $h'h = id_A$. Now we see that $k' = h'k : C \to A$ is the unique S-dcpo (S-cpo, separately S-cpo, cpo S-act) map with $hk' = k$. First, we prove that k' preserves the order. To see this, let $x, x' \in C$, $x \leqslant x'$. Then $k(x) \leqslant k(x')$. Since $k(C) \subseteq h(A)$, there exist $a, a' \in A$, $k(x) = h(a)$ and $k(x') = h(a')$. Therefore, $h(a) \leqslant h(a')$, and so $a \leqslant a'$ (since h is an order-embedding). Now, $h'h(a) \leqslant h'h(a')$ (since $h'h = id_A$) and hence $k'(x) = h'k(x) = h'h(a) \leqslant h'h(a') = h'k(x') = k'(x')$. Also, k' preserves the action. To show this, let $x \in C$ and $s \in S$, then $k'(xs) = h'k(xs) = h'(k(x)s) = h'(h(a)s) = h'(h(as)) = as$ where $k(x) = h(a), a \in A$. On the other hand, $k'(x)s = h'k(x)s = h'h(a)s = as$. To see that k' is continuous, let $D \subseteq^d C$. Then $k'(D) \subseteq^d A$, since k' is order-preserving. Also for each $d \in D$, there exists $a_d \in A$ with $k(d) = h(a_d)$ and $T = \{a_d : d \in D, h(a_d) = k(d)\} \subseteq^d A$. This is because, if $a_{d_1}, a_{d_2} \in T$, then $d_1, d_2 \in D \subseteq^d C$. Therefore, there exists $d_3 \in D$ with $d_1, d_2 \leqslant d_3$. Now, $k(d_1), k(d_2) \leqslant k(d_3)$ and so $h(a_{d_1}), h(a_{d_2}) \leqslant h(a_{d_3})$ for some $a_{d_3} \in A$, and hence $a_{d_1}, a_{d_2} \leqslant a_{d_3}$, since h is an order-embedding. Now, $k'(\bigvee^d D) = h'k(\bigvee^d D) = h'h(a) = a$ where $k(\bigvee^d D) = h(a), a \in A$. On the other hand, $\bigvee^d_{d\in D} k'(d) = \bigvee^d_{d\in D} h'k(d) = \bigvee^d_{d\in D} h'h(a_d) = \bigvee^d_{d\in D} a_d$. It is enough to prove that $\bigvee^d T = \bigvee^d_{d\in D} a_d = a$. For every $d \in D$, $a_d \leqslant a$, since $h(a_d) = k(d) \leqslant k(\bigvee^d D) = h(a)$ and h is an order-embedding. If $a' \in A$ is also an upper bound of T in A, then for every $d \in D$, $h(a_d) \leqslant h(a')$ and so $k(d) = h(a_d) \leqslant h(a')$ which implies $h(a) = k(\bigvee^d D) = \bigvee^d_{d\in D} k(d) \leqslant h(a')$. Thus $a \leqslant a'$, since h is an order-embedding. Therefore, $\bigvee^d T = a$. Notice that $hk' = k$ and k' is unique with this property. Also, in the case where h and k are strict, then so is k'. □

Definition 2.5. Recall from [4] that considering **Dcpo**-S (**Cpo**-S, **Sep**-**Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$) as a concrete category over **Set**, a monomorphism h is said to be an *embedding over* **Set** if whenever g is a map between S-dcpo's (S-cpo's, separately S-cpo's, cpo S-acts) such that hg is an S-dcpo (an S-cpo, a separately S-cpo, a cpo S-act) map, then g itself is an S-dcpo (an S-cpo, a separately S-cpo, a cpo S-act) map.

As a consequence of Theorem 2.4 we have:

Corollary 2.6. *If $h : A \to B$ is a regular monomorphism in* **Dcpo**-S (**Cpo**-S, **Sep**-**Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$) *then h is an S-dcpo (an S-cpo, a separately S-cpo, a cpo S-act) embedding over* **Set**.

Proof. Suppose that $h : A \to B$ is a regular monomorphism in **Dcpo**-S (**Cpo**-S, **Sep**-**Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$). By Theorem 2.4, h is an order-embedding. Now, let $g : C \to A$ be a function between S-dcpo's (S-cpo's, separately S-cpo's, cpo S-acts) such that hg is an S-dcpo (an S-cpo, a separately S-cpo, a cpo S-act) map. Then we prove that g is an S-dcpo (an S-cpo, a separately S-cpo, a cpo S-act) map. First, we show that g preserves the action. This is because, for $x \in C$ and $s \in S$,

$$h(g(xs)) = (hg)(xs) = ((hg)(x))s = (h(g(x)))s = h(g(x)s)$$

and so $g(xs) = g(x)s$, since h is one-one. Also, g preserves the order. To see this, let $x, x' \in C$ with $x \leqslant x'$. Then, $h(g(x)) \leqslant h(g(x'))$. Now, since h is an order-embedding we have $g(x) \leqslant g(x')$. Finally, g is continuous. To show this, let $D \subseteq^d C$. Then $g(D) \subseteq^d A$, since g preserves the order. Further,

$$h(g(\bigvee^d D)) = (hg)(\bigvee^d D) = \bigvee_{d\in D}^{d} (hg)(d) = \bigvee_{d\in D}^{d} h(g(d)) = h(\bigvee_{d\in D}^{d} g(d))$$

and so $g(\bigvee^d_{d\in D} D) = \bigvee^d_{d\in D} g(d)$. Also, $h(g(\perp_C)) = hg(\perp_C) = \perp_B = h(\perp_A)$ and $g(\perp_C) = \perp_A$. □

Now, we will study the relation of different kinds of monomorphisms. First recall the following proposition.

Proposition 2.7. [3] *If the category $\mathcal{C}$ has equalizers and pushouts, also regular monomorphisms in $\mathcal{C}$ are closed under composition, then a monomorphism is regular if and only if it is extremal.* □

Theorem 2.8. *For a monomorphism $h : A \to B$ in* **Dcpo**-S (**Sep**-**Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$) *the following are equivalent:*

(1) *h is regular,*
(2) *h is strict,*
(3) *h is strong,*
(4) *h is extremal.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are general category-theoretic results. For implication (4) $\Rightarrow$ (1), since these categories are complete and cocomplete (see [7], [8], [9]), and by Theorem 2.4, regular monomorphisms are exactly order-embeddings and hence they are closed under composition, applying Proposition 2.7, we get that any extremal monomorphism is regular. $\square$

Lemma 2.9. *If* $h : A \to B$ *is a morphism in* **Dcpo**-S (**Cpo**-S) *then* $h' : A \to\ <h(A)>$*, to the sub* S*-dcpo (sub* S*-cpo) of* B *generated by* $h(A)$*, with* $h'(a) = h(a)$ *for all* $a \in A$*, is an epimorphism in* **Dcpo**-S (**Cpo**-S).

Proof. Let $h : A \to B$ be a morphism in **Dcpo**-S (**Cpo**-S). Then take $h' : A \to\ <h(A)>$, to the sub S-dcpo (sub S-cpo) of B generated by $h(A)$, with $h'(a) = h(a)$ for all $a \in A$. To show that h' is an epimorphism, consider $g_1, g_2 : <h(A)> \to C$ such that $g_1h' = g_2h'$. Since for all $D \subseteq h(A)$, $g_1(D) = g_2(D)$ and g_1 and g_2 are continuous, it is straightforward to show that $g_1(<h(A)>) = g_2(<h(A)>)$. Therefore, h' is an epimorphism in **Dcpo**-S (**Cpo**-S). $\square$

Remark 2.10. Notice that, if $h : A \to B$ is a morphism in the category **Dcpo**-S (**Cpo**-S), then $h(A)$ is not necessarily an S-dcpo (S-cpo). To see this, consider $A = (\mathbb{N})_\perp$ where the natural numbers $\mathbb{N}$ is considered with the discrete order and $\perp \leqslant n$, for all $n \in \mathbb{N}$. Also consider $B = (\mathbb{N}^\infty)_\perp$ where $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$ and the order on $\mathbb{N}$ is the usual one and $\perp \leqslant n \leqslant \infty$, for all $n \in \mathbb{N}$. It is straightforward to show that A and B with the identity action are S-dcpo's (S-cpo's). Now, define the map $h : A \to B$ by $h(\perp) = \perp$ and $h(n) = n$, for all $n \in \mathbb{N}$. We get h is a (strict) continuous map and $h(A) = (\mathbb{N})_\perp$ is not an S-dcpo (S-cpo). This is because $D = \mathbb{N}$ is a directed subset of $h(A)$ and $\bigvee^d D = \bigvee^d \mathbb{N} = \infty \notin h(A)$.

Lemma 2.11. *A monomorphism* $h : A \to B$ *in* **Cpo**-S *is order-embedding if it is extremal.*

Proof. Suppose that $h : A \to B$ is an extremal mono in **Cpo**-S and consider $h' : A \to\ <h(A)>$, $h'(a) = h(a)$ for all $a \in A$. It is clear that $h = ih'$, where $i : <h(A)> \hookrightarrow B$. Also by Lemma 2.9, h' is an epimorphism in **Cpo**-S. Hence, by the definition of extremal monomorphisms, h' is an isomorphism in **Cpo**-S, and consequently h is an order-embedding. $\square$

As a consequence of Lemma 2.11 and Theorem 2.4, we have:

Corollary 2.12. *For a monomorphism* $h : A \to B$ *in* **Cpo**-S*, the following are equivalent:*

(1) *h is regular,*
(2) *h is strict,*
(3) *h is strong,*
(4) *h is extremal.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are general category-theoretic results. For implication (4) $\Rightarrow$ (1), by Lemma 2.11 and Theorem 2.4, we get the result. $\square$

2.2. Monomorphisms and one-one morphisms

In this subsection, we study the relation between monomorphisms and one-one morphisms in the categories **Dcpo**-S, **Dcpo**, **Cpo**, **Sep**-**Cpo**-S, and $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$.

Remark 2.13. Notice that in **Dcpo**-S, monomorphisms are exactly one-one morphisms (see [7]). Furthermore, in **Dcpo**, **Cpo**, **Sep**-**Cpo**-S, and $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ by applying the adjoint pairs given in Corollary 2.5 and Theorem 3.4 of [6], Corollary 4.4 of [8] and Corollary 4.2 of [9] and the fact that right adjoints preserves limits, we get that monomorphisms are exactly one-one morphisms. In the category **Cpo**-S, whenever $\perp_S = e$ or $\top_S = e$, monomorphisms are exactly one-one morphisms (by the adjoint pairs given in Corollaries 3.2 and 3.7 of [6]).

Remark 2.14. In Remark 2.3, we see that in the categories **Dcpo**-S, **Cpo**-S, **Sep**-**Cpo**-S, and $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$, order-embeddings are monomorphisms, but the converse is not necessarily true. But it is clearly shown that in the ordered structures, if $h : A \to B$ is a monomorphism and A is a chain then we have h is an order-embedding.

Lemma 2.15. *If $h : A \to B$ is a monomorphism in* **Cpo**-S *such that for every $a, a' \in A$ with $h(a) = h(a')$, we have $a \perp_S = a' \perp_S = \perp_A$, then h is one-one.*

Proof. Let $h : A \to B$ be a monomorphism in **Cpo**-S with the property mentioned in the hypothesis and $h(a) = h(a')$ for some $a, a' \in A$. Then $a = a'$. This is because, on the contrary if $a \neq a'$, then there exist S-cpo maps $g, k : S \to A$ given by $g(s) = as$ and $k(s) = a's$, for $s \in S$ where $hg = hk$ while $g \neq k$, which is a contradiction. Therefore, h is one-one. □

As a corollary of Lemma 2.15, we have:

Theorem 2.16. *If $h : A \to B$ is a monomorphism in* **Cpo**-S *and for every $a \in A$, $a\perp_S = \perp_A$, then h is one-one.* □

Theorem 2.17. *If $h : A \to B$ is a monomorphism in* **Cpo**-S *and $\perp_A$ is a zero element then h is one-one.*

Proof. Let $h : A \to B$ be a monomorphism in **Cpo**-S such that $\perp_A$ is a zero element. To see that h is a monomorphism in **Dcpo**-S, let $g_1, g_2 : D \to A$ be S-dcpo maps such that $hg_1 = hg_2$. Then, consider $D_\perp$ the S-cpo where $\perp$ is a zero element, and define $g'_i : D_\perp \to A$ for $i = 1, 2$ by

$$g'_i(d) = \begin{cases} g_i(d) & \text{if} \quad d \neq \perp \\ \perp_A & \text{if} \quad d = \perp \end{cases}$$

It is clear that g'_1 and g'_2 are S-cpo maps and $hg'_1 = hg'_2$. So $g'_1 = g'_2$, and hence $g_1 = g_2$. Therefore h is a monomorphism in **Dcpo**-S, and so h is one-one by Remark 2.13. □

As a consequence of Theorem 2.17 we get the following corollary.

Corollary 2.18. *If S is a cpo-monoid whose bottom element is a zero element or S is left zero as a semigroup, then in* **Cpo**-S *monomorphisms are exactly one-one morphisms.*

Proof. Let S be a cpo-monoid whose bottom element is a zero element and A be an S-cpo. Then $\perp_A$ is a zero element (for all $s \in S$, $\perp_A s = (\perp_A \perp_S)s = \perp_A(\perp_S s) = \perp_A \perp_S = \perp_A$). So by Theorem 2.17, in **Cpo**-S, monomorphisms are exactly one-one morphisms. In the case where S is left zero as a semigroup, since $\perp_S$ is zero element, the result follows similarly. □

3. Monomorphisms and regular monomorphisms

We have divided this section into two subsections as follows:

3.1. Factorization properties of morphisms

Let $\mathcal{E}'$ be the class of order-embeddings in **Dcpo**-S, **Cpo**-S, **Sep**-**Cpo**-S, and $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$. Then, in the following theorem we show that **Dcpo**-S, **Cpo**-S, **Sep**-**Cpo**-S, and $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ have unique (Epi, $\mathcal{E}'$)-diagonalization property.

Corollary 3.1. **Dcpo**-S*,* **Cpo**-S*,* **Sep**-**Cpo**-S*, and* $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ *have unique* (*Epi*, $\mathcal{E}'$)*-diagonalization property.*

Proof. By Theorem 2.4, every order-embedding is a regular monomorphism in the mentioned categories and by Theorem 2.8 and Corollary 2.12 every regular monomorphism is a strong monomorphism. Now, by the definition of a strong monomorphism we get the result. □

Theorem 3.2. **Dcpo**-S *and* **Cpo**-S *have* (*Epi, Mono*)*-factorization.*

Proof. Let $f : A \to B$ be a morphism in **Dcpo**-S (**Cpo**-S). Then, take $f' : A \to <f(A)>$ by $f'(a) = f(a)$. So by Lemma 2.9, f' is an S-dcpo (S-cpo) epimorphism and $f = if'$, where $i : <f(A)> \hookrightarrow B$ is an S-dcpo (S-cpo) monomorphism. □

Remark 3.3. The factorization mentioned in Theorem 3.2, is not necessarily unique. To see this, consider $A = (\mathbb{N}^\infty)_\perp$, where $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$ has been considered with the discrete order, $\perp \leqslant n$ for all $n \in \mathbb{N}^\infty$ and the action on A is the identity action. Also consider $B = \perp \oplus \mathbb{N}^\infty \oplus \top$ where the order on $\mathbb{N}$ is the usual one, $\infty \parallel n$ for all $n \in \mathbb{N}$, and the action on B is the identity action. Define the map $f : A \to B$ as $f(\perp) = \perp$ and $f(n) = n$, for all $n \in \mathbb{N}^\infty$. It is straightforward to show that A and B are S-dcpo's (S-cpo's) and f is an S-dcpo (S-cpo) map. Furthermore, f is an epimorphism in **Dcpo**-S (**Cpo**-S). To prove this, let $g_1, g_2 : B \to D$ be S-dcpo (S-cpo) maps with $g_1 f = g_2 f$. Then, $g_1(n) = g_1(f(n)) = g_2(f(n)) = g_2(n)$, for all $n \in \mathbb{N}^\infty \cup \{\perp\}$. Also $g_1(\top) = g_1(\bigvee^d \mathbb{N}) =$

$\bigvee^d_{n\in\mathbb{N}} g_1(n) = \bigvee^d_{n\in\mathbb{N}} g_2(n) = g_2(\bigvee^d \mathbb{N}) = g_2(\top)$, since $g_1(n) = g_2(n)$, for all $n \in \mathbb{N}$. Therefore, $g_1(\top) = g_2(\top)$ and so $g_1 = g_2$. Hence, f is an epimorphism and it has the factorization $f = id_B f$. Now, let $C = \perp \oplus ((\mathbb{N} \oplus \top) \cup \{\infty\})$ where the order on $\mathbb{N}$ is the natural one, $n \leqslant \top$ for all $n \in \mathbb{N}$, $\infty \parallel n$ for all $n \in (\mathbb{N} \oplus \top)$, and the action on C is the identity action. Then define $f' : A \to C$ by $f'(\perp) = \perp$ and $f'(n) = n$, for all $n \in \mathbb{N} \cup \{\infty\}$. It is clear that f' is an S-dcpo (S-cpo) map. Also f' is an epimorphism in **Dcpo**-S (**Cpo**-S) (the proof of the fact that f' is an epimorphism is similar to proof of the fact that f is an epimorphism) and $f = if'$ where i is an inclusion map from C to B. Hence, we have two factorizations for f, which are not equal.

Theorem 3.4. *The category* **Dcpo**-S (**Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$) *has neither* (*Onto,Mono*)*-diagonalization property nor* (*Epi,Mono*)*-diagonalization property.*

Proof. Suppose that $A = \{\perp_A, a_1, a_2, a_3\}$ where $\perp_A$ is the bottom element, $a_2 \leqslant a_3$ and $a_1 \parallel a_2, a_3$, $B = \{\perp_B, b_1, b_2\}$ where the order on B is $\perp_B \leqslant b_1 \leqslant b_2$, $C = \{\perp_C, c_1, c_2\}$ where $\perp_C$ is the bottom element and $c_1 \parallel c_2$ and $D = \{\perp_D, d_1, d_2, d_3\}$ where $\perp_D$ is the bottom element, $d_1 \parallel d_2$ and $d_1, d_2 \leqslant d_3$. It is clear that A, B, C and D with the identity action are S-dcpo's (S-cpo's, cpo S-acts). Now, define $e : A \to B$ as $e(\perp_A) = \perp_B$, $e(a_1) = b_1$ and $e(a_3) = e(a_2) = b_2$, $f : A \to C$ as $f(\perp_A) = \perp_C$, $f(a_1) = c_1$ and $f(a_2) = f(a_3) = c_2$, $h : C \to D$ as $h(\perp_C) = \perp_D$, $h(c_1) = d_1$ and $h(c_2) = d_3$, $g : B \to D$ as $g(\perp_B) = \perp_D$, $g(b_1) = d_1$ and $g(b_2) = d_3$. It is straightforward to show that e, g, f and h are S-dcpo (S-cpo, cpo S-act) maps and $ge = hf$, but if there exists an S-dcpo (an S-cpo, a cpo S-act) map $k : B \to C$, such that $ke = f$ and $hk = g$, then $k(b_1) = k(e(a_1)) = f(a_1) = c_1$ and $k(b_2) = k(e(a_2)) = f(a_2) = c_2$ but $c_1 \nleqslant c_2$, which is a contradiction (because k is an order-preserving and $b_1 \leqslant b_2$). So **Dcpo**-S (**Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$) does not have (Onto,Mono)-diagonalization property. Also **Dcpo**-S (**Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$) does not have (Epi,Mono)-diagonalization property. □

3.2. Limits and colimits

The following theorem is easily proved, and it is in fact a corollary of the next result.

Theorem 3.5. *In* **Dcpo**-S (**Cpo**-S, **Sep**-**Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$) *we have:*

(1) *The class of monomorphisms is closed under products;*
(2) *Let* $\{f_\alpha : A \to B_\alpha | \alpha \in I\}$ *be a family of monomorphisms. Then their product morphism* $f : A \to \prod B_\alpha$ *is also a monomorphism.* □

Theorem 3.6. *Let* $\{f_\alpha : A \to B_\alpha | \alpha \in I\}$ *be a source of monomorphisms in the categories* **Dcpo**-S (**Cpo**-S, **Sep**-**Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$)*. Then the morphism* $f : A \to limB_\alpha$ (*existing by the universal property of limits*) *is also a monomorphism.*

Proof. Let $\{f_\alpha : A \to B_\alpha | \alpha \in I\}$ be a source of monomorphisms in one of the categories mentioned in the hypothesis. To prove that $f : A \to limB_\alpha$ is a monomorphism, let $g_1, g_2 : C \to A$ be such that $fg_1 = fg_2$. Then, $fg_1(c) = fg_2(c)$ for all $c \in C$. Also for all $c \in C$ and $\alpha \in I$, $\pi_\alpha(fg_1(c)) = f_\alpha(g_1(c)) = f_\alpha(g_2(c)) = \pi_\alpha(fg_2(c))$, where $\pi_\alpha : limB_\alpha \to B_\alpha$ is a limit morphism. Hence, $f_\alpha g_1 = f_\alpha g_2$ for all $\alpha \in I$, and since f_α is a monomorphism, we have $g_1 = g_2$. □

Proposition 3.7. *In* **Dcpo**-S (**Cpo**-S, **Sep-Cpo**-S, $\mathbf{Cpo_{Act\text{-}S}}$) *we have:*

(1) *The class of regular monomorphisms is closed under products;*

(2) *Let* $\{f_\alpha : A \to B_\alpha | \alpha \in I\}$ *be a family of regular monomorphisms. Then their product morphism* $f : A \to \prod B_\alpha$ *is also a regular monomorphism.*

Proof. We just prove (1) in **Dcpo**-S and the rest are proved similarly.

Let $\{f_\alpha : A_\alpha \to B_\alpha | \alpha \in I\}$ be a family of regular monomorphisms in **Dcpo**-S. We show that $f = \prod f_\alpha : \prod A_\alpha \to \prod B_\alpha$ where $f((a_\alpha)_{\alpha\in I}) = (f_\alpha(a_\alpha))_{\alpha\in I}$ is an order-embedding and so by Theorem 2.4, it is a regular monomorphism. Suppose that $f((a_\alpha)_{\alpha\in I}) \leqslant f((a'_\alpha)_{\alpha\in I})$ for $(a_\alpha)_{\alpha\in I}$, $(a'_\alpha)_{\alpha\in I} \in \prod A_\alpha$. We have $f((a_\alpha)_{\alpha\in I}) \leqslant f((a'_\alpha)_{\alpha\in I})$ if and only if $(f_\alpha(a_\alpha))_{\alpha\in I} \leqslant (f_\alpha(a'_\alpha))_{\alpha\in I}$ if and only if $f_\alpha(a_\alpha) \leqslant f_\alpha(a'_\alpha)$, for all $\alpha \in I$ if and only if $a_\alpha \leqslant a'_\alpha$, for all $\alpha \in I$ (since each f_α is an order-embedding) if and only if $(a_\alpha)_{\alpha\in I} \leqslant (a'_\alpha)_{\alpha\in I}$. So f is a regular monomorphism. □

Theorem 3.8. *Let* $\{f_\alpha : A \to B_\alpha | \alpha \in I\}$ *be a source of regular monomorphisms in* **Dcpo**-S (**Cpo**-S, **Sep-Cpo**-S, $\mathbf{Cpo_{Act\text{-}S}}$). *Then the morphism* $f : A \to limB_\alpha$ (*existing by the universal property of limits*) *is also a regular monomorphism.*

Proof. Let $\{f_\alpha : A \to B_\alpha | \alpha \in I\}$ be a source of regular monomorphisms in one of the categories mentioned in the hypothesis. To prove that $f : A \to limB_\alpha$ is a regular monomorphism, by Theorem 2.4 it is enough to show that f is an order-embedding. To see this, let $f(a) \leqslant f(a')$ where $a, a' \in A$. We have $f_\alpha(a) = \pi_\alpha(f(a)) \leqslant \pi_\alpha(f(a')) = f_\alpha(a')$, for all $\alpha \in I$ ($\pi_\alpha : limB_\alpha \to B_\alpha$ is a limit morphism). So $a \leqslant a'$, because by Theorem 2.4, f_α is an order-embedding, for every $\alpha \in I$ and hence f is an order-embedding and also it is a regular monomorphism. □

Proposition 3.9. *In* **Dcpo**-S, **Sep-Cpo**-S *and* $\mathbf{Cpo_{Act-S}}$, *the class of monomorphisms and regular monomorphisms are closed under coproducts.*

Proof. Assume that $\{f_\alpha : A_\alpha \to B_\alpha | \alpha \in I\}$ is a family of monomorphisms and $\coprod f_\alpha : \coprod A_\alpha \to \coprod B_\alpha$ is the coproduct morphism. We show that $\coprod f_\alpha$ defined by $(\coprod f_\alpha)(a, \alpha) = (f_\alpha(a), \alpha)$, $a \in A_\alpha$, $\alpha \in I$, is a monomorphism. By Remark 2.13, it is enough to show that $\coprod f_\alpha$ is one-one. To see this, let $(\coprod f_\alpha)(a, \alpha) = (\coprod f_\alpha)(a', \alpha')$ where $a \in A_\alpha, a' \in A_{\alpha'}, \alpha, \alpha' \in I$. Therefore, $(f_\alpha(a), \alpha) = (f_\alpha(a'), \alpha')$ and so $\alpha = \alpha'$ and $f_\alpha(a) = f_\alpha(a')$. Since f_α is one-one we have $a = a'$. Consequently, $(a, \alpha) = (a', \alpha) = (a', \alpha')$. Now, suppose that $\{f_\alpha : A_\alpha \to B_\alpha | \alpha \in I\}$ is a family of regular monomorphisms. We show that $\coprod f_\alpha$ is a regular monomorphism. By Theorem 2.4, it is enough to show that $\coprod f_\alpha$ is an order-embedding. To prove this, let $(\coprod f_\alpha)(a, \alpha) \leqslant (\coprod f_\alpha)(a', \alpha')$

where $a \in A_\alpha, a' \in A_{\alpha'}, \alpha, \alpha' \in I$. Therefore, $(f_\alpha(a), \alpha) \leqslant (f_\alpha(a'), \alpha')$. But this is impossible except $\alpha = \alpha'$ and then $f_\alpha(a) \leqslant f_\alpha(a')$. Since f_α is order-embedding, we have $a \leqslant a'$. Consequently, $(a, \alpha) \leqslant (a', \alpha) = (a', \alpha')$. □

Recall that a class of morphisms of a category is called *pullback stable* if pullbacks transfer those morphisms. In the final theorem, we see that the class of order-embeddings satisfying this property.

Theorem 3.10. *The class of order-embeddings in* **Dcpo**-S (**Cpo**-S, **Sep-Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$) *is pullback stable.*

Proof. By Proposition 11.18 of [3], the class of regular monomorphisms is pullback stable. Therefore by Theorem 2.4, we get the result. □

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Some properties of a graph associated to a lattice

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Abstract. Some properties of the graph $\Gamma_S(L)$, where L is a lattice and S is a $\wedge$-closed subset of L, are obtained. Moreover, the graph structure of $\Gamma_S(L)$ under graph operations union, join, lexicographic product and tensor product are determined. The graph associated to quotient lattice is also studied.

1. Introduction

Making connection between various algebraic structures and graph theory by assigning graphs to an algebraic structure and investigating the properties of one from the another is an exciting research methods in the last decade. Barati et al. [2] associated a simple graph $\Gamma_S(R)$ to a multiplicatively closed subset S of a commutative ring R with all elements of R as vertices, and two distinct vertices x, y are adjacent if and only if $x + y \in S$. Afkhami et al. [1] introduced the same graph structure on a lattice. They considered a lattice L and defined a graph $\Gamma_S(L)$ with all elements of L as vertices and two distinct vertices $x, y \in L$ are adjacent if and only if $x \vee y \in S$ where S is a subset of L which is closed under $\wedge$ operation.

Throughout this paper L means a finite bounded lattice. Let x, y be two distinct elements of L, whenever $x < y$ and there is no element z in L such that $x < z < y$, we say that y *covers* x. In bounded lattice L an element $p \in L$ is said to be an *atom* if it covers 0, also an element $m \in L$ is a *coatom* of L if 1 covers it. We denote the set of all coatoms of L by $Coatom(L)$ and the set of atoms of L by $Atom(L)$. The set of all lower bounds of a subset A of L is denoted by A^ℓ and the set of all upper bounds of A is denoted by A^u i.e.,

$$A^\ell = \{x \in L : x \leqslant a \ \ for\ all\ a \in A\},$$

$$A^u = \{x \in L : a \leqslant x \ \ for\ all\ a \in A\},$$

$\{x\}^\ell$ and $\{x\}^u$ (or simply x^ℓ and x^u) are also denoted by $(x]$ and $[x)$ respectively.

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Let L and L' be lattices. A mapping $\theta : L \longrightarrow L'$ is called a *homomorphism* if for all $a, b \in L$, $\theta(a \vee b) = \theta(a) \vee \theta(b)$ and $\theta(a \wedge b) = \theta(a) \wedge \theta(b)$. If the map θ is also bijective, we call θ to be an *isomorphism*.

A mapping $\theta : L \longrightarrow L'$ is called an *anti-homomorphism* if $\theta(a \vee b) = \theta(a) \wedge \theta(b)$ and $\theta(a \wedge b) = \theta(a) \vee \theta(b)$ for all $a, b \in L$. A bijective anti-homomorphism is called an *anti-isomorphism*. An equivalence relation R on a lattice L is called a *congruence* if $a_1 R b_1$ and $a_2 R b_2$ imply $(a_1 \wedge a_2) R (b_1 \wedge b_2)$ and $(a_1 \vee a_2) R (b_1 \vee b_2)$. The set of all such relations is denoted by $Con(L)$ or L/R. It is well-known that the set of all congruence relations, under inclusion constitutes a complete lattice. The (ordinal) sum $P + Q$ of P and Q can be defined on the (disjoint) union $P \cup Q$ ordered as follows: for the elements $x, y \in P \cup Q$, define $x \leqslant y$ if one of the following conditions holds:

i) $x, y \in P$ and $x \leqslant_P y$,

ii) $x, y \in Q$ and $x \leqslant_Q y$,

iii) $x \in P$ and $y \in Q$.

For an order set P with unit 1_P, and an order set Q with zero, 0_Q, the *glued sum*, $P \dot{+} Q$, is obtained from $P + Q$ by identifying 1_P and 0_Q[5, p. 8]. We refer to [4, 5] for a complete description of these notions.

Let G be an undirected graph with the vertex set $V(G)$. The notation $ab \in E$ means that vertices a and b are adjacent in G. The degree of a vertex v is denoted by $deg(v)$ and the notations P_n, C_n, S_n and K_n are used for the path, cycle, star and complete graphs with n vertices, respectively. Recall that a subgraph H of a graph G is a graph whose the set of vertices and the set of edges are both subsets of G. A *vertex-induced subgraph* of graph G is one that consists of some of the vertices of G and all of the edges that connect them in G. An edge-induced subgraph of graph G consists of some of the edges of G and the vertices that are at their endpoints. The *complement* of G is a graph denoted by $\overline{G}$ with the same vertex set as G and two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in G. The complement of the complete graph K_n is called the *null graph* on n vertices, see [3] for more details.

We now recall some graph operations [6]. Suppose G and H are graphs with disjoint vertex sets. The *disjoint union* $G + H$ is a graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H)$. The *join* $G \oplus H$ defined as $\overline{\bar{G} + \bar{H}}$. The *tensor product* (or *direct product*) $G \times H$ of graphs G and H is the graph whose vertex set is $V(G) \times V(H)$ in such a way that vertices (g, h) and (g', h') are adjacent if and only if $gg' \in E(G)$ and $hh' \in E(H)$.

2. Main results

The aim of this section is to compute $\Gamma_S(L)$, for some special lattice L and a subset S of L. We start by an example:

Example 2.1. Let L be a chain with n elements and S be any nonempty subset of L. Then $deg_{\Gamma_S(L)}(x) = |x^l| + |S \cap x^u| - 2$, $x \in S$, and for any $x \in S^c$, $deg(x) = |x^u \cap S|$. In some special cases, we have:

- If $S = y^u$ for some $y \in L$, then $deg(x) = |L| - 1$, for all $x \in S$ and $deg(x) = |S|$, for every $x \in S^c$.
- If $S = y^l$ for some $y \in L$, then $deg(x) = |S| - 1$, for all $x \in S$ and $deg(x) = 0$, for every $x \in S^c$.

Proposition 2.2. *We have:*

(i) $\Gamma_S(L)$ is a cycle if and only if $|L| = 3$ and $\Gamma_S(L)$ is complete. On the other word $\Gamma_S(L) \neq C_n$ for all subset S of L, unless $n = 3$.

(ii) $\Gamma_S(L)$ is a tree if and only if it is a star.

Proof. (i). Since the cycle is two regular, if $\Gamma_S(L)$ is a cycle, then $deg(1) = deg(0) = 2$. Hence by [1, Lemma 2.2], $1 \in S$ and $deg(1) = |L| - 1 = 2$ i.e., $|L| = 3$. On the other hand, if $0 \in S$, then $deg(0) = |S| - 1 = 2$ and $|S| = 3$ i.e., $S = L$, and if $0 \notin S$, then $deg(0) = |S| = 2$ i.e., $S = L \setminus \{0\}$ [1, Lemma 2.2]. So, $\Gamma_S(L)$ is a complete graph [1, Proposition 2.4].

(ii). If $\Gamma_S(L)$ is a tree, then it is connected, so, $1 \in S$ [1, Theorem 2.3]. Thus $deg(1) = |L| - 1$ [1, Lemma 2.2]. Since $\Gamma_S(L)$ is a tree, it has no other edge, so, $|Coatom(L)| = 1$ and by [1, Lemma 2.2], $S = \{1\}$ or $S = \{0, 1\}$. The result follows from [1, Theorem 2.5]. $\square$

Lemma 2.3. *Let L be a bounded lattice. Then*

(1) *$\Gamma_S(L)$ is null graph if and only if $S = \{0\}$ or $S = \emptyset$.*

(2) *$\Gamma_S(L) = P_2 + \overline{K}_{|L|-2}$ if and only if $S = \{p\}$ or $S = \{0, p\}$ that $p \in Atom(L)$, in fact in this case, $deg(p) = deg(0) = 1$ and $deg(x) = 0$, for every $x \neq 0, p$.*

(3) *$\Gamma_S(L) = P_3 + \overline{K}_{|L|-3}$ if and only if $S = \{p_1, p_2\}$ or $S = \{0, p_1, p_2\}$ for some $p_1, p_2 \in Atom(L)$, in this case, $deg(p_1) = deg(p_2) = 1$, $deg(0) = 2$ and for every $x \neq 0, p_1, p_2$, $deg(x) = 0$.*

(4) *$\Gamma_S(L) = C_3 + \overline{K}_{|L|-3}$ if and only if $S = \{0, p, x\}$ such that $x^\ell = \{0, p\}$ and $p \in Atom(L)$.*

(5) *$\Gamma_S(L) = S_\alpha + \overline{K}_{|L|-\alpha}$ (where $\alpha = |S| - 1$ or $\alpha = |S^l|$) if and only if $S \subseteq \{0\} \cup AtomL$ or $S = \{x\}$, for some nonzero element of lattice L.*

Proof. The proof is straightforward and so it is omitted. $\square$

Remark 2.4. Suppose that S is a $\wedge$-closed subset of a lattice L and $a, b, x \in L$, we know that $a \vee (a \vee b) = b \vee (a \vee b) = (a \wedge b) \vee (a \vee b) = (a \wedge x) \vee (a \vee b) = (b \wedge x) \vee (a \vee b) = a \vee b$. So if in a graph $\Gamma_S(L)$ a, b are adjacent i.e., $a \vee b \in S$, then $a \vee (a \vee b) \in S$, $b \vee (a \vee b) \in S$, $(a \wedge b) \vee (a \vee b) \in S$ and $(a \wedge x) \vee (a \vee b) \in S$. Hence, summarizing, we have:

If $n \geqslant 3$, then $\Gamma_S(L) \neq P_n + \overline{K}_{|L|-(n)}$ for all $\wedge$-closed subsets S of L; and if $n \geqslant 4$, then $\Gamma_S(L) \neq C_n + \overline{K}_{|L|-n}$ for all subset S of L.

Remark 2.5. If S is a sublattice of L, then the subgraph $\Gamma_S(L)$ on S is complete. Since for all $a, b \in S$ we have $a \vee b \in S$, every two elements of subset S in $\Gamma_S(L)$ are adjacent.

Remark 2.6. It is easy to show that $\Gamma_{S'}(L)$ is a subgraph of $\Gamma_S(L)$, when S, S' are two $\wedge$-closed subsets of L and $S' \subseteq S$. But in general $\Gamma_{S'}(L)$ is neither edge-induced nor vertex-induced subgraph of $\Gamma_S(L)$. For example, let L be the modular lattice M_3 containing $0, 1$ and three incomparable elements a, b, c. Define $S = \{0, b, c, 1\}$ and $S' = \{0, b\}$. Then it is clear to see that $\Gamma_{S'}(L)$ is not edge-induced and vertex-induced subgraph of $\Gamma_S(L)$.

Theorem 2.7. *A $\wedge$-closed subset S of L is an ideal if and only if*

$$\Gamma_S(L) = K_{|S|} + \overline{K}_{|S^c|}.$$

Proof. Suppose $\Gamma_S(L) = K_{|S|} + \overline{K}_{|S^c|}$. Then by definition of $\Gamma_S(L)$, we have $a \vee b \in S$ if and only if $a, b \in S$ which implies that S is an ideal.

Conversely, if S is an ideal of L. Then, S is closed under taking join of elements, consequently all vertices of S are adjacent in graph $\Gamma_S(L)$. Moreover, since S is a lower set, for all $a, b \in S^c$, $a \vee b \notin S$. In fact, if in contrary $a \vee b \in S$ then $a \wedge (a \vee b) = a \in S$ which is a contraction. So, all vertices of S^c aren't adjacent in $\Gamma_S(L)$. Moreover, since S is a lower set, it follows that all $a \in S$ and $b \in S^c$ aren't adjacent in $\Gamma_S(L)$. Therefore, $\Gamma_S(L) = K_{|S|} + \overline{K}_{|S^c|}$. □

Clearly we have:

Lemma 2.8. *Let $\alpha : L \longrightarrow L'$ be a lattice isomorphism and S be a $\wedge$-closed subset of L. Then*

$$\Gamma_S(L) \cong \Gamma_{\alpha(S)}(L').$$

Theorem 2.9. *A $\wedge$-closed subset S of L is a prime filter if and only if*

$$\Gamma_S(L) = K_{|S|} \oplus \overline{K}_{|S^c|}.$$

Proof. Assume that S is a prime filter. Then for any $x, y \in S$, we have $x \vee y \in S$, i.e., $xy \in E(\Gamma_S(L))$. Since S is an upper subset of L, $x \vee y \in S$ for each $x \in S$ and $y \in S^c$. This means that x and y are adjacent. In addition, since S is a prime filter, S^c is an ideal. Hence for any $x, y \in S^c$, $x \vee y \in S^c$ and so $x \vee y \notin S$. This implies that x, y aren't adjacent in $\Gamma_S(L)$. On the other hand, if $\Gamma_S(L) = K_{|S|} \oplus \overline{K}_{|S^c|}$, then obviously for any $x \in S$ and $y \in L$, $x \vee y \in S$ and if $x \vee y \in S$, then $x \in S$ or $y \in S$. This completes the proof. □

A *semiregular graph* is a graph in which the set of degree of vertices includes only two elements. The following corollary immediately follows from Theorem 2.9.

Corollary 2.10. *If S is a prime filter of L, then $\Gamma_S(L)$ is a semiregular graph.*

Proof. Suppose S is a prime filter of L. Then by Theorem 2.9, we conclude that $deg(x) = |L| - 1$, for all $x \in S$ and $deg(y) = |S|$, for all $y \in S^c$, and the proof is completed. □

Proposition 2.11. *Assume that $\alpha : L \longrightarrow L'$ is a lattice isomorphism and S is a prime ideal or a filter of L, then*

$$\overline{\Gamma_S(L)} \cong \Gamma_{\alpha(S)^c}(L').$$

Proof. It is easy to show that if S is a prime ideal or a filter of L, then $\alpha(S)^c$ is a $\wedge$-closed subset of L'. The details are left to the readers. □

Corollary 2.12. *If S is a filter or a prime ideal of L, then $\overline{\Gamma_S(L)} = \Gamma_{S^c}(L)$.*

Proof. The proof by Proposition 2.11 and $\alpha = IdL$ (the identity map) is done. □

The *disjunction graph* $G \vee H$ of graphs G and H is the graph whose vertex set is $V(G) \times V(H)$ in such a way that vertices (g, h) and (g', h') are adjacent if and only if $gg' \in E(G)$ or $hh' \in E(H)$.

Theorem 2.13. *Let L, L' be two lattices and $L \times L'$ be its direct product. If S and T are $\wedge$-closed subset of L, L', respectively, Then*

(1) $\Gamma_{S \times T}(L \times L') = \Gamma_S(L) \times \Gamma_T(L')$,

(2) $\Gamma_S(L) + \Gamma_T(L') = \Gamma_{S \cup T}(L + L')$,

(3) *Let $S_0 = S \times L'$ and $T_0 = L \times T$. If S or T is a lower set, then we have* $\Gamma_{S_0 \cup T_0}(L \times L') = \Gamma_S(L) \vee \Gamma_T(L')$.

Proof. (1). At first, we notice that $S \times T$ is a $\wedge$-closed subset of $L \times L'$. Two distinct vertices (a, b) and (c, d) of $\Gamma_{S \times T}(L \times L')$ are adjacent if and only if $(a, b) \vee (c, d) = (a \vee c, b \vee d) \in S \times T$, which is equivalent to $a \vee c \in S$ and $b \vee d \in T$. This means that a, c are adjacent in $\Gamma_S(L)$ and b, d are adjacent in $\Gamma_T(L')$. Therefore, (a, b) and (c, d) are adjacent in $\Gamma_S(L) \times \Gamma_T(L')$.

(2). If a, b are adjacent in $\Gamma_{S \cup T}(L + L')$, then $a \vee b \in S \cup T$. So, $a \vee b \in S$ or $a \vee b \in T$, i.e., a, b are adjacent in $\Gamma_S(L)$ or a, b are adjacent in $\Gamma_T(L')$ which implies that a, b are adjacent in $\Gamma_S(L) + \Gamma_T(L')$. On the other hand, if a, b are adjacent in $\Gamma_S(L) + \Gamma_T(L')$, then a, b are adjacent in $\Gamma_S(L)$ or a, b are adjacent in $\Gamma_T(L')$. So, $a \vee b \in S$ or $a \vee b \in T$, i.e., $a \vee b \in S \cup T$. Hence a, b are adjacent in $\Gamma_{S \cup T}(L + L')$,

(3). Since S or T is a lower set, $S_0 \cup T_0$ is a $\wedge$-closed subset of $L \times L'$. Two distinct vertices (a, b) and (c, d) are adjacent in $\Gamma_{S_0 \cup T_0}(L \times L')$ if and only if $(a \vee c, b \vee d) \in (S \times L') \cup (L \times T)$ if and only if $a \vee c \in S$ or $b \vee d \in T$ and this means that a, c are adjacent in $\Gamma_S(L)$ or b, d are adjacent in $\Gamma_T(L')$. The later is equivalent to (a, b) and (c, d) are adjacent in $\Gamma_S(L) \vee \Gamma_T(L')$. □

Recall that the *lexicographic product* of two graph G and H, denoted by $G[H]$, is defined as $V(G[H]) = V(G) \times V(H)$ where two vertices $(a,b),(c,d)$ of $G[H]$ are adjacent whenever $ac \in E(G)$, or $a = c$ and $bd \in E(H)$ [6, p. 43].

If P and Q are two partially ordered sets, then $P \times Q$, by ordering $(a,b) \leqslant (c,d)$ if $a <_P c$, or $a = c$ and $b \leqslant_Q d$ will be a partially ordered set again. We use the notation $P \oslash Q$ to denote $(P \times Q, \leqslant)$. Notice that if P and Q are totally ordered sets, then $P \oslash Q$ is a totally ordered set too. One can check at once that if L and L' are two lattices and L' is bounded, then $L \oslash L'$ is a lattice [5, p. 260] with join and meet operations as follows:

$$(a,b) \wedge (c,d) = \begin{cases} (a, b \wedge d) & \text{if } a = c, \\ (a,b)(\text{or } (c,d)) & \text{if } a < c \text{ (or } c < a), \\ (a \wedge c, 1) & \text{if } a \parallel c, \end{cases}$$

$$(a,b) \vee (c,d) = \begin{cases} (a, b \vee d) & \text{if } a = c, \\ (c,d)(\text{or } (a,b)) & \text{if } a < c \text{ (or } c < a), \\ (a \vee c, 0) & \text{if } a \parallel c. \end{cases}$$

Theorem 2.14. *Let L, L' be two totally ordered lattices and L' be bounded. If S and T are subsets of L, L', respectively, then*

$$\Gamma_{S \times T}(L \oslash L') = \Gamma_S(L)[\Gamma_T(L')].$$

Proof. Since L, L' are totally ordered, $L \oslash L'$ is totally ordered and so $S \times T$ is a $\wedge$-closed subset of $L \oslash L'$ and $\Gamma_{S\times T}(L \oslash L')$ is well defined. We now assume that (a,b) and (c,d) are two distinct vertices of $\Gamma_{S\times T}(L \oslash L')$. These two vertices are adjacent if and only if $(a,b) \vee (c,d) \in S \oslash T$ if and only if $(a,b) \in S \times T$ or $(c,d) \in S \times T$ if and only if $(a > c \text{ or } a = c, b > d)$ or $(a < c \text{ or } a = c, b < d)$, equivalently $a \vee c \in S$ or $(a = c, b \vee d \in T)$. This is equivalent to $ac \in E(\Gamma_S(L))$ or $a = c$, $bd \in E(\Gamma_T(L'))$. So, (a,b) and (c,d) are adjacent in $\Gamma_S(L)[\Gamma_T(L')]$. □

Proposition 2.15. *Let L and L' be lattices and L' be bounded. Suppose that T is a $\wedge$-closed subset of L' and S is a lower set of L. We also assume that $S_0 = S \times L'$ and $T_0 = L \times T$, then $\Gamma_S(L)[\Gamma_T(L')]$ is a subgraph of $\Gamma_{S_0 \cup T_0}(L \oslash L')$.*

Proof. At first, since T is a $\wedge$-closed subset of L' and S is a lower set of L, $S_0 \cup T_0$ is a $\wedge$-closed subset of $L \oslash L'$, so $\Gamma_{S_0\cup T_0}(L \oslash L')$ can be defined. On the other hand, $V(\Gamma_S(L)[\Gamma_T(L')]) = V(\Gamma_{S_0\cup T_0}(L \oslash L')) = L \times L'$. Also, if two distinct vertices (a,b) and (c,d) are adjacent in $\Gamma_S(L)[\Gamma_T(L')]$, by definition of lexicographic product of graphs, one of the following two cases are occurred:

1. a and c are adjacent in graph $\Gamma_S(L)$,

2. $a = c$ and b and d are adjacent in graph $\Gamma_T(L')$.

Thus we have $a \vee c \in S$ or $(a = c$ and $b \vee d \in T)$. Hence, according to join operation in a lattice $L \oslash L'$, we conclude that $(a,b) \vee (c,d) \in S_0 \cup T_0$, so two vertices (a,b) and (c,d) are adjacent in $\Gamma_{S_0\cup T_0}(L \oslash L')$. This completes the proof. □

Corollary 2.16. *Let L, L' be two totally ordered lattices and L' be bounded. If S and T are ($\wedge$-closed) subsets of L, L', respectively, then $\Gamma_{S\times T}(L \oslash L')$ is a subgraph of $\Gamma_{S_0\cup T_0}(L \oslash L')$.*

The *Cartesian product* of two graph G and H is a graph, denoted by $G\Box H$, whose vertex set is $V(G) \times V(H)$ and two vertices (a, b) and (c, d) are adjacent if $a = c$ and $bd \in E(G)$, or $ac \in E(H)$ and $b = d$ [6, p. 35].

Proposition 2.17. *Let L and L' be lattices and T, S are $\wedge$-closed subsets of L, L', respectively. We also assume that $S_0 = S\times L'$ and $T_0 = L\times T$. Then $\Gamma_S(L)\Box\Gamma_T(L')$ is a subgraph of $\Gamma_{S_0\cup T_0}(L \times L')$.*

Proof. Assume that (a, b) and (c, d) are two distinct vertices of $\Gamma_S(L)\Box\Gamma_T(L')$. These two vertices are adjacent if and only if $(a = c, bd \in E(\Gamma_T(L')))$ or $(ac \in E(\Gamma_S(L)), b = d)$, if and only if $(a = c, b\vee d \in T)$ or $(a\vee c \in S, b = d)$, equivalently $(a, b) \vee (c, d) = (a \vee c, b \vee d) \in S_0 \cup T_0$. So, (a, b) and (c, d) are adjacent in $\Gamma_{S_0\cup T_0}(L \times L')$. □

The *strong product* of two graph G and H is the graph denoted as $G \boxtimes H$, whose vertex set is $V(G) \times V(H)$ and $E(G \boxtimes H) = E(G\Box H) \cup E(G \times H)$ [6, p. 36].

Corollary 2.18. *Let L and L' be lattices and T, S are $\wedge$-closed subsets of L, L' respectively. We also assume that $S_0 = S\times L'$ and $T_0 = L\times T$. Then $\Gamma_S(L)\boxtimes\Gamma_T(L')$ is a subgraph of $\Gamma_{S\times T}(L \times L') \cup \Gamma_{S_0\cup T_0}(L \times L')$.*

Proof. The result follows from definition of $G \boxtimes H$, part (1) of Theorem 2.13 and previous preposition. □

Suppose Π is a partition of the vertices of a graph G. The *quotient graph* G/Π is a graph with vertex set Π, and for which distinct classes $C_1, C_2 \in \Pi$ are adjacent if some vertex in C_1 is adjacent to a vertex of C_2 [6, p. 159]. In the following, we let $\varphi : L \longrightarrow K$ be an onto lattice homomorphism and α be the congruence relation of L defined by $x \equiv_\alpha y$ if and only if $\varphi(x) = \varphi(y)$. Therefore, $L/\alpha \cong K$. In other words, a homomorphic image of L is isomorphic to some quotient lattice of L. Obviously, if S is a $\wedge$-closed subset of L, then S_1, the set of all equivalence classes of α on S, is a $\wedge$-closed subset of L/α. So, we can define graph $\Gamma_{S_1}(L/\alpha)$. We have the following description for the graph associated to L/α.

Theorem 2.19. *Suppose that $\varphi : L \longrightarrow K$ is an onto lattice homomorphism and α is corresponding congruence relation with it. If S is an ideal of L and S_1 is the set of all equivalence classes of α on S, then*

$$\Gamma_{S_1}(L/\alpha) = \Gamma_S(L)/\alpha.$$

Proof. Consider $\alpha = \{[x]_\alpha : x \in L\}$ to be a partition for the vertex set of $\Gamma_S(L)$. So, the vertices of $\Gamma_S(L)/\alpha$ and $\Gamma_{S_1}(L/\alpha)$ are equal. On the other hand, according to definition of a quotient graph, if two distinct vertices $[x]$ and $[y]$ are adjacent in $\Gamma_S(L)/\alpha$, there exists $a \in [x]$ and $b \in [y]$ which are adjacent in $\Gamma_S(L)$ i.e. $a \vee b \in S$. So, $[a] \vee [b] = [a \vee b] \in S_1$. Thus $[a], [b]$ are adjacent in $\Gamma_{S_1}(L/\alpha)$, which is equivalent to $[x]$ and $[y]$ are adjacent in $\Gamma_{S_1}(L/\alpha)$.

Moreover, if $[x]$ and $[y]$ are adjacent in $\Gamma_{S_1}(L/\alpha)$, then $[x \vee y] = [x] \vee [y] \in S_1$. So, there exists a $s \in S$ such that $x \vee y \equiv_\alpha s$. According to the properties of congruence relations we have:

$$x = x \wedge (x \vee y) \equiv_\alpha x \wedge s, \qquad y = y \wedge (x \vee y) \equiv_\alpha y \wedge s.$$

So, $s \wedge x \in [x]$ and $s \wedge y \in [y]$. Since S is an ideal, $s \wedge x, s \wedge y \in S$ and $(s \wedge x) \vee (s \wedge y) \in S$. Thus $s \wedge x$ and $s \wedge y$ are adjacent in $\Gamma_S(L)$. This follows that $[x]$ and $[y]$ are adjacent in $\Gamma_S(L)/\alpha$ and the proof is complete. □

Corollary 2.20. *Suppose that $\varphi : L \longrightarrow K$ is an onto lattice anti-homomorphism and α is corresponding congruence relation with it. If S is a filter of L and S_1 is the set of all equivalence classes of α on S and $(L', \vee', \wedge')$ is dual of a lattice L, then*

$$\Gamma_{S_1}(L/\alpha) = \Gamma_S(L')/\alpha.$$

Proof. At first the vertex set of $\Gamma_S(L')/\alpha$ and $\Gamma_{S_1}(L/\alpha)$ are equal. On the other hand, if two distinct vertices $[x]$ and $[y]$ are adjacent in $\Gamma_S(L')/\alpha$, there exists $a \in [x]$ and $b \in [y]$ which are adjacent in $\Gamma_S(L')$ i.e. $a \vee' b \in S$. So, by definition of S_1, $[a \vee' b] \in S_1$ i.e., $[x] \vee [y] = [a] \vee [b] = [a \wedge b] = [a \vee' b] \in S_1$, so $[x]$ and $[y]$ are adjacent in $\Gamma_{S_1}(L/\alpha)$. Moreover, if $[x]$ and $[y]$ are adjacent in $\Gamma_{S_1}(L/\alpha)$, then $[x \wedge y] = [x] \vee [y] \in S_1$. So, there exist some $s \in S$ such that $x \wedge y \equiv_\alpha s$. By the properties of congruence relations, we have:

$$x = x \vee (x \wedge y) \equiv_\alpha x \vee s, \qquad y = y \vee (x \wedge y) \equiv_\alpha y \vee s.$$

So, $s \vee x \in [x]$ and $s \vee y \in [y]$. Since S is a filter, $s \vee x, s \vee y \in S$ and $(s \vee x) \wedge (s \vee y) \in S$. Thus $(s \vee x) \vee' (s \vee y) \in S$, i.e., $s \vee x$ and $s \vee y$ are adjacent in $\Gamma_S(L')$. This follows that $[x]$ and $[y]$ are adjacent in $\Gamma_S(L')/\alpha$. □

From now on L is a distributive lattice and S is a filter of L. We state here an important result of Stone [5, Theorem 115] as follows:

Theorem 2.21. *Let L be a distributive lattice, let I be an ideal, let D be a filter of L, and let $I \cap D = \emptyset$. Then there exists a prime ideal P of L such that $P \supseteq I$ and $P \cap D = \emptyset$.*

For a filter S of L and arbitrary element $x \in S^c$, by Stone theorem, there exists a prime ideal P_x such that $P_x \cap S = \emptyset$ and $(x] \subseteq P_x$. This means that S^c is a union of some prime ideals. Hence $S^c = \bigcup_{x \in S^c} P_x$. Set $I = \bigcap_{x \in S^c} P_x$ and define a congruence relation θ_0 on L as follows;

$$\theta_0 = \bigwedge \{\theta \in Con(L) \ : \ I^2 \subseteq \theta\}.$$

We consider $\tilde{S} = \{[x]_{\theta_0} \ : \ x \in S\}$.

Example 2.22. Suppose $L = \{0, x_1, x_2, x_3, x_4, x_5, \ldots\}$. Define an order $\leqslant$ on L as follows: for each $i \geqslant 1$, $0 \leqslant x_i$. Moreover, $x_1, x_2 \leqslant x_3$ and for $i, j \geq 3$ that $i \leqslant j$, $x_i \leqslant x_j$. Define $S = [x_5)$. So, $I = (x_3]$ and by definition of θ_0, we have $L/\theta_0 = \{I\} \cup \{\{x\} : x \notin I\}$.

Lemma 2.23. *$\tilde{S} = \{[x]_{\theta_0} : x \in S\}$ is a filter of L/θ_0.*

Proof. S is a $\wedge$-closed subset of L and in L/θ_0, we have $[a \wedge b] = [a] \wedge [b]$. So, $\tilde{S}$ is a $\wedge$-closed subset of L/θ_0. It is now enough to show that if $[a] \wedge [b] \in \tilde{S}$ then $[a], [b] \in \tilde{S}$. To do this, suppose that $[a \wedge b] = [a] \wedge [b] \in \tilde{S}$. Hence, there exist some element $s \in S$ such that $a \wedge b \equiv_{\theta_0} s$. According to properties of congruence relations, we have $a = a \vee (a \wedge b) \equiv_{\theta_0} a \vee s$, $b = b \vee (a \wedge b) \equiv_{\theta_0} b \vee s$. This means that $[a] = [a \vee s], [b] = [b \vee s]$. Since S is a filter of L, $b \vee s, a \vee s \in S$ which implies that $[a], [b] \in \tilde{S}$. □

Theorem 2.24. *$\Gamma_{\tilde{S}}(L/\theta_0)$ is connected.*

Proof. By [1, Theorem 2.3] the graph $\Gamma_S(L)$ is connected if and only if $1 \in S$. Now the result follows from Lemma 2.23. □

Theorem 2.25. *If $\Gamma_S(L)$ is complete, then $\Gamma_{\tilde{S}}(L/\theta_0)$ is complete.*

Proof. Suppose that $\Gamma_S(L)$ is complete. Thus $S = L$ or $S = L \setminus \{0\}$ [1, Theorem 4.2] and we have the following two cases:

- If $S = L$, then $I = \emptyset$, so $\theta_0 = \bigwedge\{\theta : I^2 \subseteq \theta\} = L \times L$. Thus $\tilde{S} = \{[x]_{\theta_0} : x \in S\}$ $= L/\theta_0$ and therefore $\Gamma_{\tilde{S}}(L/\theta_0)$ is complete.
- If $S = L \setminus \{0\}$ then $I = \{0\}$. So, $\theta_0 = \bigwedge\{\theta : I^2 \subseteq \theta\} = \triangle$. Hence, $\tilde{S} = (L/\theta_0) \setminus \{[0]\}$ and so $\Gamma_{\tilde{S}}(L/\theta_0)$ is complete. □

Notice that the converse of previous theorem is not true in general. Suppose $L = \{0, x_1, x_2, x_3, x_4, x_5, x_6, \ldots\}$. Define an order $\leqslant$ on L as follows: for each $i \geqslant 1$, $0 \leqslant x_i$. Moreover, $x_1 \leqslant x_2, x_3$ and $x_2, x_3 \leqslant x_4$ and for $i, j \geqslant 4$ that $i \leqslant j$, $x_i \leqslant x_j$. Define $S = [x_5)$, so $I = (x_4]$ and $\theta_0 = \{I\} \cup \{\{x\} : x \notin I\}$. Therefore, $\{I\}$ is zero element of a lattice L/θ_0 and so $\tilde{S} = (L/\theta_0) \setminus \{[0]\}$. By [1, Proposition 2.4] the graph $\Gamma_{\tilde{S}}(L/\theta_0)$ is complete. But by Theorem 2.9, $\Gamma_S(L)$ is not complete.

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Steiner loops satisfying the statement of Moufang's theorem

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Abstract. Andrew Rajah posed at the Loops'11 Conference in Trest, Czech Republic, the following conjecture: *Is every variety of loops that satisfies Moufang's theorem contained in the variety of Moufang loops?* This paper is motivated by that problem. We give a partial answer to this question and present two types of Steiner loops, one that satisfies Moufang's theorem and another that does not, and neither is Moufang loop.

1. Introduction

A nonempty set L with a binary operation is a *loop* if there exists an identity element 1 with $1x = x = x1$ for every $x \in L$ and both left and right multiplication by any fixed element of L permutes every element of L.

A loop L has the *inverse property* (and is an *IP loop*), if and only if there is a bijection $L \longrightarrow L : x \mapsto x^{-1}$ such whenever $x, y \in L$, $x^{-1}(xy) = y = (yx)x^{-1}$. It can be seen that *IP* loops also satisfy $(xy)^{-1} = y^{-1}x^{-1}$. A *Steiner loop* is an IP loop of exponent 2. A loop M is a *Moufang loop* if it satisfies any of the following equivalent identities:

$$\begin{aligned} x(y \cdot xz) &= (xy \cdot x)z, \\ y(x \cdot zx) &= (yx \cdot z)x, \\ xy \cdot zx &= x(yz \cdot x). \end{aligned}$$

Such loops were introduced by Moufang [3] in 1934. The *associator* of elements $a, b, c \in L$ is the unique element (a, b, c) of L satisfying the equation: $ab \cdot c = (a \cdot bc)(a, b, c)$.

Theorem 1.1. [Moufang's Theorem [4]] *Let M be a Moufang loop. If $a, b, c \in M$ such that $(a, b, c) = 1$, then a, b, c generate a subgroup of M.*

In view of Theorem 1.1, every Moufang loop is *diassociative*, that is, any two of its elements generate a group. However, Theorem 1.1 was formulated for Moufang

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loops. We consider its statement for another class of loops, namely, for the variety of Steiner loops.

Our motivation arises from the question posed by Andrew Rajah at the Loops'11 Conference concerning the relationship between Moufang loops and loops that satisfy Moufang's theorem. The results in this paper were first presented at the Third Mile High Conference on Nonassociative Mathematics in Denver, 2013. Later, Stuhl [7] explored solutions based on Steiner Oriented Hall Loops, and a combinatorial characterization of Steiner loops satisfying Moufang's theorem in terms of configurations has been established [1]. Despite the combinatorial characterization in [1], the algebraic treatment here remains useful for two reasons. First, these results provide the foundational work for [1]; and second, they provide an algebraic framework to understand such loops, which complements the combinatorial framework.

2. Steiner loops and Moufang's theorem

Definition 2.1. A loop L *satisfies Moufang's Property*, $\mathcal{MP}$, if L is not Moufang loop, but it satisfies the statement of Moufang's theorem, i.e., if $a, b, c \in L$ such that $(a, b, c) = 1$, then a, b, c generate a subgroup of L.

It is known that there exists only one Steiner loop S of order 10. We prove that this Steiner loop S satisfies Moufang's Property $\mathcal{MP}$. Its Cayley table can be found, for example, using the GAP Library [9], as seen below:

$\cdot$	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	1	4	3	8	10	9	5	7	6
3	3	4	1	2	10	9	8	7	6	5
4	4	3	2	1	9	8	10	6	5	7
5	5	8	10	9	1	7	6	2	4	3
6	6	10	9	8	7	1	5	4	3	2
7	7	9	8	10	6	5	1	3	2	4
8	8	5	7	6	2	4	3	1	10	9
9	9	7	6	5	4	3	2	10	1	8
10	10	6	5	7	3	2	4	9	8	1

For any $x, y, z \in S$, such that $x \neq y$; $y \neq z$; $z \neq x$, $x \neq 1, y \neq 1, z \neq 1$, $x \cdot yz = xy \cdot z$ implies that $z = xy$. So $< x, y, z > = < x, y >$, and hence x, y, z generate a group.

A *Steiner triple system* $(Q, \mathcal{B})$, or $STS(n)$, is a non-empty set Q with n elements and a set $\mathcal{B}$ of unordered triples $\{a, b, c\}$ such that

(i) a, b, c are distinct elements of Q;

(ii) when $a, b \in Q$ and $a \neq b$, there exists a unique triple $\{a, b, c\} \in \mathcal{B}$.

A Steiner triple system $(Q,\mathcal{B})$ with $|Q| = n$ elements exists if and only if $n \geqslant 1$ and $n \equiv 1$ *or* $3 \pmod 6$ [8]. Because there is a one-to-one correspondence between the variety of Steiner triple systems and the variety of all Steiner Loops [2], Steiner loops have order $m \equiv 2$ *or* $4 \pmod 6$. This underlies the study of Steiner triple systems from an algebraic point of view as in [4], [5] and [6].

We use the following standard construction of Steiner triple systems [8], sometimes called the *Bose construction*. Let $n = 2t + 1$ and define $Q := \mathbb{Z}_n \times \mathbb{Z}_3$. A Steiner triple system $(Q,\mathcal{B})$ can be formed with $\mathcal{B}$ consisting of the following triples

$$\begin{array}{ll} \{(x,0),(x,1),(x,2)\} & \text{where} \quad x \in \mathbb{Z}_n, \text{ and} \\ \{(x,i),(y,i),(\frac{x+y}{2},i+1)\} & \text{where} \quad x \neq y; x,y \in \mathbb{Z}_n, i \in \mathbb{Z}_3 \end{array}$$

The corresponding Steiner loops can be defined directly. Let $S = Q \cup \{1\}$. Define a binary operation $*$ with identity element 1 as follows:

$$\begin{aligned} (x,i)*(x,j) &= (x,k) & i \neq j,\ i \neq k,\ j \neq k, \\ (x,i)*(y,i) &= (\tfrac{x+y}{2}, i+1) & x \neq y, \\ (x,i)*(y,i+1) &= (2y-x,i) & x \neq y, \\ (x,i)*(y,i-1) &= (2x-y,i-1) & x \neq y, \\ (x,i)*(x,i) &= 1 \end{aligned}$$

Then $(S,*)$ is commutative loop. However, $(S,*)$ is not a Moufang loop. If we take the elements $x = (0,0)$ $y = (1,0)$ and $z = (0,1)$ then $(xy)(zx) = (-1/2,1)$. On the other hand, $x((yz)x) = (-1,0)$, so $(S,*)$ does not satisfy one of the Moufang identities.

Analyzing Steiner loops from the Bose construction, there are two types: one that satisfies $\mathcal{MP}$, and another that does not. Using computer calculations and the `Loops` package in GAP [9], first we studied the Steiner loops of order k with $k \in M_1$ where

$$M_1 = \{16,28,34,40,46,52,58,79,76,82,88,94,100,112,118,124,130,136,142,154\}$$

from the Bose construction. Each of these Steiner loops satisfies $\mathcal{MP}$. However, none of the Steiner loops of order $k \in \{22,64,106,148\}$ from the Bose construction satisfies $\mathcal{MP}$. The explanation for this follows.

Theorem 2.2. *Let S be a Steiner loop from the Bose construction. Then S has the property $\mathcal{MP}$ if and only if 7 is an invertible element in $\mathbb{Z}_n$.*

Proof. Suppose S has property $\mathcal{MP}$. If 7 is not invertible in $\mathbb{Z}_n$, then exists an element $a \in \mathbb{Z}_n$, $a \neq 0$ such that $7a = 0$. Hence $8a = a$. Because n is odd, $2a = a/4$. The associator $((0,1),(0,0),(a,0)) = 1$ while $((0,1),(a,0),(0,0)) \neq 1$, thus the elements $(0,1),(0,0),(a,0)$ associate in some order, but not in every order, a contradiction.

Now, suppose that 7 is invertible in $\mathbb{Z}_n$. We consider all possible triples of elements of S. Our strategy is to show that if the associator $(a, b, c) = 1$, then a, b, c are in the same triple. There are 25 generic triple elements of S; here $x, y, z \in \mathbb{Z}_n$ are distinct and $i, j, k \in \mathbb{Z}_3$ are distinct:

$\{(x,i),(x,i),(x,i)\}$, $\{(x,i),(x,i),(x,j)\}$, $\{(x,i),(x,i),(y,i)\}$, $\{(x,i),(x,i),(y,j)\}$, $\{(x,i),(x,j),(x,i)\}$, $\{(x,i),(x,j),(x,j)\}$, $\{(x,i),(x,j),(y,i)\}$, $\{(x,i),(x,j),(y,j)\}$, $\{(x,i),(x,j),(x,k)\}$, $\{(x,i),(x,j),(y,k)\}$, $\{(x,i),(y,i),(x,i)\}$, $\{(x,i),(y,i),(x,j)\}$, $\{(x,i),(y,i),(y,i)\}$, $\{(x,i),(y,i),(y,j)\}$, $\{(x,i),(y,i),(z,i)\}$, $\{(x,i),(y,i),(z,j)\}$, $\{(x,i),(y,j),(x,i)\}$, $\{(x,i),(y,j),(x,j)\}$, $\{(x,i),(y,j),(y,i)\}$, $\{(x,i),(y,j),(y,j)\}$, $\{(x,i),(y,j),(z,i)\}$, $\{(x,i),(y,j),(z,j)\}$, $\{(x,i),(y,j),(x,k)\}$, $\{(x,i),(y,j),(y,k)\}$, $\{(x,i),(y,j),(z,k)\}$.

When we consider $j \neq i$, $j \neq k$, $k \neq i$, we assume that $j = i+1$ and $k = i-1$ or $j = i-1$ and $k = i+1$. We identify 59 different sets of triples of elements and calculate the associators of each set. We found that in the first 37 triples the associator is different from 1, as listed below:

$\{(x,i),(x,i+1),(y,i)\}$, $\{(x,i),(x,i-1),(y,i)\}$, $\{(x,i),(x,i+1),(y,i+1)\}$,
$\{(x,i),(y,i),(x,i+1)\}$, $\{(x,i),(y,i),(x,i-1)\}$, $\{(x,i),(y,i),(y,i-1)\}$,
$\{(x,i),(y,i),(z,i)$, where $z \neq \frac{x+y}{2}$ and $x \neq \frac{y+z}{2}\}$,
$\{(x,i),(y,i),(z,i)$, where $z \neq \frac{x+y}{2}$ and $x = \frac{y+z}{2}\}$,
$\{(x,i),(y,i),(z,i)$, where $z = \frac{x+y}{2}$ and $x \neq \frac{y+z}{2}\}$,
$\{(x,i),(y,i),(z,i)$, where $z = \frac{x+y}{2}$ and $x = \frac{y+z}{2}\}$,
$\{(x,i),(y,i),(z,i+1)$, where $z \neq \frac{x+y}{2}\}$, $\{(x,i),(y,i+1),(x,i+1)\}$,
$\{(x,i),(y,i),(z,i-1)$, where $z \neq \frac{x+y}{2}$ and $x \neq 2y-z\}$,
$\{(x,i),(y,i),(z,i-1)$, where $z = \frac{x+y}{2}$ and $x \neq 2y-z\}$,
$\{(x,i),(y,i),(z,i-1)$, where $z = \frac{x+y}{2}$ and $x = 2y-z\}$,
$\{(x,i),(y,i-1),(x,i-1)\}$, $\{(x,i),(y,i+1),(y,i)\}$, $\{(x,i),(y,i-1),(y,i)\}$,
$\{(x,i),(y,i+1),(z,i)$, where $z \neq 2y-x\}$,
$\{(x,i),(y,i-1),(z,i)$, where $z \neq 2x-y$ and $x \neq 2z-y\}$,
$\{(x,i),(y,i-1),(z,i)$, where $z \neq 2x-y$ and $x = 2z-y\}$,
$\{(x,i),(y,i-1),(z,i)$, where $z = 2x-y$ and $x \neq 2z-y\}$,
$\{(x,i),(y,i-1),(z,i)$, where $z = 2x-y$ and $x = 2z-y\}$,
$\{(x,i),(y,i+1),(z,i+1)$, where $z \neq 2y-x$ and $x \neq \frac{y+z}{2}\}$,
$\{(x,i),(y,i+1),(z,i+1)$, where $z \neq 2y-x$ and $x = \frac{y+z}{2}\}$,
$\{(x,i),(y,i+1),(z,i+1)$, where $z = 2y-x$ and $x = \frac{y+z}{2}\}$,
$\{(x,i),(y,i-1),(z,i-1)$, where $z \neq 2x-y\}$, $\{(x,i),(y,i+1),(x,i-1)\}$,
$\{(x,i),(y,i-1),(x,i+1)\}$, $\{(x,i),(y,i+1),(y,i-1)\}$,
$\{(x,i),(y,i+1),(z,i-1)$, where $z \neq 2y-x$ and $x \neq 2z-y\}$,
$\{(x,i),(y,i+1),(z,i-1)$, where $z = 2y-x$ and $x \neq 2z-y\}$,
$\{(x,i),(y,i+1),(z,i-1)$, where $z = 2y-x$ and $x = 2z-y\}$,
$\{(x,i),(y,i-1),(z,i+1)$, where $z \neq 2x-y$ and $x \neq 2y-z\}$,
$\{(x,i),(y,i-1),(z,i+1)$, where $z \neq 2x-y$ and $x = 2y-z\}$,

$\{(x,i),(y,i-1),(z,i+1)$, where $z=2x-y$ and $x=2y-z\}$,
$\{(x,i),(x,i-1),(y,i+1)\}$.

Next, there are 14 triples for which the associator is 1 and they are in the same triple of the STS; consequently, they are in a Klein group (and so generate a subgroup).

$\{(x,i),(x,i),(x,i)\}$, $\{(x,i),(x,i),(x,j)\}$, $\{(x,i),(x,i),(y,i)\}$, $\{(x,i),(x,i),(y,j)\}$,
$\{(x,i),(x,j),(x,i)\}$, $\{(x,i),(x,j),(x,j)\}$, $\{(x,i),(y,i),(x,i)\}$, $\{(x,i),(y,i),(y,i)\}$,
$\{(x,i),(y,i),(z,i+1)$, where $z=\frac{x+y}{2}\}$, $\{(x,i),(y,j),(x,i)\}$, $\{(x,i),(y,j),(y,j)\}$,
$\{(x,i),(y,i+1),(z,i)$, where $z=2y-x\}$,
$\{(x,i),(y,i-1),(z,i-1)$, where $z=2x-y\}$, $\{(x,i),(x,j),(x,k)\}$

There remain 8 cases to consider:

$\{(x,i),(x,i-1),(y,i-1)\}$, $\{(x,i),(y,i),(y,i+1)\}$,
$\{(x,i),(y,i-1),(y,i+1)\}$, $\{(x,i),(x,i+1),(y,i-1)\}$,
$\{(x,i),(y,i),(z,i-1)$, where $x=2y-z, z\neq\frac{x+y}{2}\}$,
$\{(x,i),(y,i+1),(z,i+1)$ where $z=2y-x, x\neq\frac{y+z}{2}\}$,
$\{(x,i),(y,i+1),(z,i-1)$ where $z\neq 2y-x, x=2z-y\}$,
$\{(x,i),(y,i-1),(z,i+1)$ where $z=2x-y, x\neq 2y-z\}$

Each has associator different from 1 because 7 is invertible in $\mathbb{Z}_n$. Take for instance the triple $\{(x,i),(x,i+1),(y,i-1)\}$ with $x\neq y$ of the STS. Now $(x,i)*((x,i+1)*(y,i-1))=(4y-3x,i)$ and $((x,i)*(x,i+1))*(y,i-1)=(\frac{x+y}{2},i)$. The associator $((x,i),(x,i+1),(y,i-1))$ is 1 if and only if $7x=7y$. Because 7 is invertible in $\mathbb{Z}_n$, we obtain $x=y$, a contradiction. □

3. Beyond Steiner loops

We have seen that certain Steiner loops from the Bose construction provide examples of loops satisfying $\mathcal{MP}$. Further examples can be obtained by the direct product of loops, the proof of which is straightforward:

Lemma 3.1. *Let S and M be loops that satisfy Moufang's theorem. Then $S\times M$ satisfies Moufang's theorem, and $S\times M$ satisfies $\mathcal{MP}$ if one or both of S and M satisfy $\mathcal{MP}$.*

Taking S to satisfy $\mathcal{MP}$ and M to be a group or a Moufang loop provides numerous examples of loops that satisfy $\mathcal{MP}$ but are neither Steiner nor Moufang loops. A characterization of loops that satisfy Moufang's theorem must therefore consider loops beyond the varieties examined here.

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On injective and subdirectly irreducible S-posets over left zero posemigroups

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Abstract. The notion of a Cauchy sequence in an S-poset is a useful tool to study algebraic concepts, specially the concept of injectivity. This paper is concerned with the relations between injectivity and Cauchy sequences in the category of S-posets in which S is a left zero posemigroup. We characterize subdirectly irreducible S-posets over this posemigroup and by Birkhof's Representation Theorem we get a description of such S-posets.

1. Introduction and preliminaries

The category of S-posets, as the ordered version of the category of S-acts, recently have captured the interest of some mathematicians [4, 5]. And it is always interesting to verify the counterpart results of S-acts in the category of S-posets (see [1, 4, 8]). Cauchy sequences in an S-act first introduced by E. Giuli in [3] for a particular class of acts, then generalized to S-acts, in [2]. Recently we generalized this concept to S-posets, [4, 5].

Left zero semigroups, all of whose elements are left zero, are an important class of semigroups, since every non-empty set S can be turned into a left zero semigroup by defining $st = s$ for all $s, t \in S$ also this semigroup is applied in automata theory, theory of computations, Boolean algebras.

Here we are going to use the notion of Cauchy sequences to study the dc-regular injectivity of S-posets over a left zero posemigroup, as we did in [7] for injectivity of S-acts. But the order here plays an important role and to get the counterpart results here we need to modify (some times strongly) the S-act version of the proofs. The aim of this paper is to determine the structure of dc-injective in the category of S-posets and characterize the subdiretly irreducible S-posets over a left zero semigroup. Therefore, throughout this article, we assume S to be a left zero posemigroup. Now let us briefly recall some necessary concepts.

A *partially ordered semigroup* (or simply, a *posemigoup*) is a semigroup which is also a poset whose partial order is compatible with its binary operation (that is $s \leqslant s'$ implies $st \leqslant s't$, for every $s, s', t \in S$).

For a posemigoup S, a (*right*) *S-poset* is a poset A equipped with a function $\alpha : A \times S \to A$, called the action of S on A, such that for $a, b \in A$, $s, t \in S$ (denoting $\alpha(a, s)$ by as): (1) $a(st) = (as)t$, (2) $a \leqslant b \Rightarrow as \leqslant bs$, (3) $s \leqslant t \Rightarrow as \leqslant at$.

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By an *S-poset morphism* $f : A \to B$, we mean a monotone map between S-posets which preserves the action (that is $f(as) = f(a)s$).

An element a of an S-poset A is called a *fixed* or *zero element* if $as = a$ for all $s \in S$. We denote the set of all fixed elements of an S-poset A by $FixA$, which is in fact a sub-S-poset of A that is $as \in FixA$ for all $a \in FixA$ and $s \in S$.

We define an S-poset A to be *separated* if it is separated as an S-act, that is any two points $a \neq b$ in A can be separated by at least one $s \in S$, by $sa \neq sb$.

We say that an S-poset A is *subseparated* if $a \leqslant b$ in A whenever $as \leqslant bs$ for all $s \in S$. It is clear that every subseparated S-poset is a separated one.

A *regular monomorphism* or an *embedding* is an S-poset morphism (that is, a monoton and action preserving map) $f : A \to B$ such that $a \leqslant b$ if and only if $f(a) \leqslant f(b)$, for each $a, b \in A$.

2. Cauchy sequences

Our central object of study in this paper is the notion of Cauchy sequences in S-posets [2, 3, 4].

First of all it is easy to check that:

- *If S is a left zero semigroup, then for every S-poset A, $AS \subseteq FixA$.*

Definition 2.1. A *Cauchy sequence* in an S-poset A is an S-poset morphism $f : S \to A$. More explicitly, $f : S \to A$ is a Cauchy sequence when it is order preserving and $f(st) = f(s)t$.

We denote a Cauchy sequence by $(a_s)_{s\in S}$, which expresses the fact that the element $s \in S$ is mapped to the element a_s in A. Since S is a left zero posemigroup, with this notation we have $a_s t = a_{st} = a_s$ and for $s, t \in S$ if $s \leqslant t$ then $a_s \leqslant a_t$.

It is worth noting that in an S-poset A (over the left zero posemigroup S) the terms of a Cauchy sequence are fixed elements of A. So if we denote the set of Cauchy sequences of A by $\mathcal{C}(A)$ then $\mathcal{C}(A) = (FixA)^S$ in which $(FixA)^S$ is the set of monotone mappings from S to $FixA$.

Definition 2.2. Let $(a_s)_{s\in S}$ be a Cauchy sequence in an S-poset A. An element b in an extension B of A is called a *limit* of $(a_s)_{s\in S}$ whenever $bs = a_s$ for each $s \in S$.

Lemma 2.3. *Given an S-poset A over a left zero posemigroup S, the set $\mathcal{C}(A)$ of all Cauchy sequences in A, is a subseparated S-poset.*

Proof. First we note that $\mathcal{C}(A)$ is an S-poset, by the action $\mathcal{C}(A) \times S \to \mathcal{C}(A)$ mapping each $((a_s)_{s\in S}, t) \in \mathcal{C}(A) \times S$ to $(a_s)_{s\in S} \cdot t = (a_{ts})_{s\in S}$ which is obviously in $\mathcal{C}(A)$, for every $t \in S$. We should note that $\mathcal{C}(A)$ is a poset with point-wise order and $((a_s)_{s\in S} \cdot t) \cdot r = (a_s)_{s\in S} \cdot (tr)$. Indeed, $(a_s)_{s\in S} \cdot (tr) = (a_s)_{s\in S} \cdot t = (a_{ts})_{s\in S} = (a_t)_{s\in S}$, namely $(a_s)_{s\in S} \cdot (tr)$ is the constant sequence $(a_t)_{s\in S}$, also we have $((a_s)_{s\in S} \cdot t) \cdot r = (a_{ts})_{s\in S} \cdot r = (a_t)_{s\in S} \cdot r = (a_t)_{s\in S}$; the last equality is true because $(a_t)_{s\in S}$ is a constant sequence. Now if $r \leqslant t$ in S, then $rs = r \leqslant t = ts$, for every $s \in S$ and since $(a_s)_{s\in S}$ is a Cauchy sequence, $a_{rs} = a_r \leqslant a_t = a_{ts}$. That is $(a_s)_{s\in S}\cdot r \leqslant (a_s)_{s\in S}\cdot t$. Finally if $(a_s)_{s\in S}\leqslant (b_s)_{s\in S}$, then $a_s\leqslant b_s$, for every $s\in S$. Hence $a_{ts}\leqslant b_{ts}$ for every $s,t\in S$. That is $(a_s)_{s\in S}\cdot t\leqslant (b_s)_{s\in S}\cdot t$, for every $t\in S$. To prove subseparatedness, let $(a_s)_{s\in S}\cdot t\leqslant (b_s)_{s\in S}\cdot t$, for every $t\in S$. Then $a_{ts}\leqslant b_{ts}$, for every $t,s\in S$. Now, since S is a left zero posemigroup, $a_t\leqslant b_t$, for every $t\in S$. That is $(a_s)_{s\in S}\leqslant (b_s)_{s\in S}$. □

Lemma 2.4. *Let A be an S-poset over a left zero posemigroup S and $(a_s)_{s\in S}$ be a sequence (indexed family of elements of A by $s\in S$). Then $(a_s)_{s\in S}$ has a limit in some extension B of A if and only if it is a Cauchy sequence.*

Proof. One way is clear. In fact the limit of the sequence $(a_s)_{s\in S}$ makes it to have the Cauchy property in Definition 2.1. For the converse, let $(a_s)_{s\in S}$ be a Cauchy sequence in A. Then take the extension B of A to be $A\dot{\cup}\{(a_s)_{s\in S}\}$ with the action $(a_s)_{s\in S}\cdot t=a_t$ for $t\in S$ and no order between $(a_s)_{s\in S}$ and the elements of A. The constructed B is an S-poset. This is because, for all $t,r\in S$, we have $((a_s)_{s\in S}\cdot t)\cdot r=a_t\cdot r=a_{tr}=(a_s)_{s\in S}\cdot(tr)$, and if $t\leqslant r$ then $a_t\leqslant a_r$ follows from this fact that $(a_s)_{s\in S}$ is a Cauchy sequence, and hence $(a_s)_{s\in S}\cdot t\leqslant (a_s)_{s\in S}\cdot r$. Now, by the defined action, we have that $(a_s)_{s\in S}$ is a limit of $(a_s)_{s\in S}$. □

Definition 2.5. An S-poset A is said to be *complete* if every Cauchy sequence over A has a limit in A.

For a given left zero posemigroup S and an S-poset A Lemma 2.3 shows that $\mathcal{C}(A)$ is an S-poset. In fact, $\mathcal{C}(A)$ is a complete S-poset.

Theorem 2.6. *Let A be an S-poset over a left zero posemigroup S. The S-poset $\mathcal{C}(A)$ is complete.*

Proof. Let $(f_s)_{s\in S}$ be a Cauchy sequence in $\mathcal{C}(A)$, in which $f_s=(a^s_r)_{r\in S}$ for each $s\in S$. Hence for each $s,t\in S$ we have $f_{st}=f_st$. Since S is a left zero semigroup, $f_s=f_{st}=f_st$, i.e., for each $s\in S$, f_s is a fixed element in $\mathcal{C}(A)$. Now, by the defined action of S over $\mathcal{C}(A)$ in Lemma 2.3, we have $f_s=f_st=(a^s_{tr})_{r\in S}=(a^s_t)_{r\in S}$. So $(a^s_r)_{r\in S}=(a^s_t)_{r\in S}$ for each $r\in S$. Namely, for each $s\in S$, f_s is a constant sequence. Now we define the Cauchy sequence $(a_s)_{s\in S}$ to be $a_s=a^s_t$, for every $s\in S$ and claim that $(a_s)_{s\in S}$ is a limit of $(f_s)_{s\in S}$. This is because $(a_s)_{s\in S}\cdot r$ $=(a^s_t)_{s\in S}\cdot r=(a^s_{rt})_{s\in S}=(a^s_r)_{s\in S}=(a^s_t)_{r\in S}=f_s$. Indeed, the third equation follows from this fact that S is a left zero posemigroup. Also since f_s is a constant sequence and $(a^s_r)_{r\in S}=(a^s_t)_{r\in S}$, we have the fourth and fifth equations. □

3. dc-injective of S-posets

A sub-S-poset A of an S-poset B is called *down-closed* in B if $b\leqslant a$ for $a\in A$, $b\in B$ then $b\in A$. By a *down-closed embedding* or *dc-regular monomorphism*, we mean an embedding $f:A\to B$ such that $f(A)$ is a down-closed sub-S-poset of B.

An S-poset A is said to be *down-closed injective* or simply *dc-injective* if for every down-closed embedding $f : B \to C$ and each S-poset morphism $\varphi : B \to A$ there exists an S-poset morphism $\varphi^* : C \to A$ making the diagram

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "\varphi"'] & C \arrow[dl, dotted, "\varphi^*"] \\ A & \end{tikzcd}$$

commutative.

Theorem 3.1. *For a left zero posemigroup S every dc-injective S-poset is complete.*

Proof. Let $(a_s)_{s\in S}$ be a Cauchy sequence in a dc-injective S-poset A. Consider the extension $B = A\dot{\cup}\{(a_s)_{s\in S}\}$ of A with the action $(a_s)_{s\in S} \cdot t = a_t$ and no order relation between $(a_s)_{s\in S}$ and the elements of A as introduced in the proof of Lemma 2.4. It is clear that A is embedded in B, so the dc-injective property of A completes the diagram

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, equal] & B \arrow[dl, dotted, "\varphi"] \\ A & \end{tikzcd}$$

by an S-poset morphism φ. Now we claim that $\varphi((a_s)_{s\in S}) \in A$ is a limit of the Cauchy sequence $(a_s)_{s\in S}$. This is because $\varphi((a_s)_{s\in S})\cdot t = \varphi((a_s)_{s\in S}\cdot t) = \varphi(a_t) = a_t$, for every $t \in S$. $\square$

The converse of Theorem 3.1 is true if the S-poset has a "*good*" property. See the next theorem as the counterpart of Theorem 2.3 of [7] with the compeletly different method of proof.

Theorem 3.2. *If S is a left zero posemigroup S, then every complete subseparated S-poset A with a top fixed element is dc-injective.*

Proof. To prove, we show that A is a retract of each of its down-closed extensions (that is, to say A is an absolute down-closed retract) (see [8]). To do so, let B be a down-closed extension of A. Define $g : B \to A$ with $g|_A = id_A$ and for $b \in B \setminus A$ take $g(b) = a_b$ where a_b is a limit of the Cauchy sequence $(a_s)_{s\in S}$ with $a_s = bs$ for $bs \in A$, and $a_s = a_0$ for $bs \notin A$, where $a_0 \in FixA$ is the top fixed element in A mentioned in the hypotheses.

First we show that $(a_s)_{s\in S}$ is a Cauchy sequence. To do so, we note that $a_st = a_{st}$. This is because, if $a_s = bs$, then $a_st = (bs)t = b(st) = bs$ also $a_{st} = a_s = bs$, and if $a_s = a_0$, then $a_st = a_0t = a_0$ also $a_{st} = a_s = a_0$. Also if $s \leqslant t$, then $bs \leqslant bt$. This is because if $bt \in A$, then $bs \in A$, since A is down-closed in B, therefore $a_s \leqslant a_t$, and if $bt \notin A$, then $a_t = a_0$ but a_0 is a top fixed element and hence $bs \leqslant a_0$, that is $a_s \leqslant a_t$. Thus $(a_s)_{s\in S}$ is a Cauchy sequence.

Now we show that g is order preserving. To do so, let $b \leqslant b'$. Then $bs \leqslant b's$ for all $s \in S$. Therefore, by definition of $a_b, a_{b'}$, we have $a_b s \leqslant a_{b'} s$. But, since A is subseparated, $a_b \leqslant a_{b'}$. That is $g(b) \leqslant g(b')$. Finally g is equivariant on $B \setminus A$. Because $g(b)s = a_b s = a_s = bs = g(bs)$, if $bs \in A$, for every $b \in B \setminus A$ and $s \in S$. And if $bs \notin A$, then, since $(bs)t = bs$ for all $t \in S$, we get $g(bs) = a_{bs} = a_0 = a_0 s = a_b s = g(b)s$. □

As a corollary of Theorems 3.1 and 3.2 we get the following Theorem.

Theorem 3.3. *Let S be a left zero posemigroup S. Then a subseparated S-poset A with a top fixed element is dc-injective if and only if it is complete.*

Theorem 3.4. *For each S-poset A over a left zero posemigroup S with a top fixed element, $\mathcal{C}(A)$ is dc-injective.*

Proof. Let a_0 be a top fixed element in A. One can easily see that the constant sequence $(a_s = a_0)_{s \in S}$ is a Cauchy sequence and is a top fixed element in $\mathcal{C}(A)$. Now Theorems 3.3 and 2.6 give the result. □

Before the next definition it is worth noting that by a right down-closed ideal I of a posemigroup S we mean a non-empty subset I of S such that (i) $IS \subset I$ and (ii) $a \leqslant b \in I$ implies $a \in I$, for all $a, b \in S$.

Definition 3.5. An S-poset A is said to be

- *I-injective*, for a right down-closed ideal I of S, if each S-poset morphism $f : I \to A$ is of the form λ_a for some $a \in A$, that is $f(s) = as$ for $s \in I$.
- *S-injective*, if each S-poset morphism $f : S \to A$ is of the form λ_a for some $a \in A$, that is $f(s) = as$ for $s \in S$.

In the next theorem we compare the concept of completeness with the different types of injectivity for some special S-poset over a left zero posemigroup S, and we see that they are surprisingly equivalent to each other.

Theorem 3.6. *For a subseparated S-poset A with a top fixed element a_0, the following are equivalent:*

(1) *A is dc-injective;*
(2) *A is dc-absolutely retract;*
(3) *A is complete;*
(4) *A is I-injective, for each right down-closed ideal I of S;*
(5) *A is S-injective.*

Proof. (1)⇔(2). It is given in [8].

(1)⇔(3). See Theorem 3.3.

(3)⇒(4). Let A be complete and I be a right down-closed ideal of S and $f : I \to A$ be an S-poset morphism. Consider the sequence $(a_s)_{s \in S}$ to be $a_s = f(s)$ for $s \in I$, and $a_s = a_0$ for $s \in S - I$. The sequence $(a_s)_{s \in S}$ is a Cauchy sequence. This is because, if $s \leqslant t$ then four cases may occur:

∘ If $s, t \in I$ then $f(s) \leqslant f(t)$, since f is S-poset morphism, that is $a_s \leqslant a_t$.

∘ If $s, t \in S - I$, then $a_s = a_t = a_0$, that is $a_s \leqslant a_t$.

∘ It may $s \in S - I$, $t \in I$. But since $s \leqslant t$ and I is down-closed ideal, we must have $s \in I$ which is a contradiction. Hence this case is not possible.

∘ And finally if $s \in I$ and $t \in S - I$, then $f(s) = a_s$, $f(t) = a_0$. But a_0 is the top fixed element, hence $f(s) = a_s \leqslant a_0 = f(t)$.

Also let $s, t \in S$. Then if $s \in I$, we have $a_s t = f(s)t = f(st) = f(s) = a_s$ and if $s \in S - I$, then $f(s)t = a_0 t = a_0 = f(s)$.

Now since $(a_s)_{s \in S}$ is a Cauchy sequence, it has a limit a in A. So $a_s = as$, for all $s \in S$, which means $f(s) = a_s = \lambda_a(s)$. That is $f = \lambda_a$.

(4)⇒ (5). It is trivial.

(5)⇒ (3). Let A be S-injective and $(a_s)_{s \in S}$ be a Cauchy sequence over A. So $f : S \to A$ with $f(s) = a_s$ is an S-poset morphism. Now (5) gives $a \in A$ such that $f = \lambda_a$, hence $a_s = as$ for all $s \in S$, i.e., a is a limit of the given sequence. □

4. Subdirectly irreducible

By Birkhoff's Representation Theorem (see [6]) every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras. This theorem, by analogous proof is established in the category of S-posets. In [7], characterization of subdirectly irreducible acts, respectively over the monoid $(\mathbb{N} \cup \{\infty\}, \min, \infty)$, and left zero semigroups can be seen. In this section we give a characterization of subdirectly irreducible S-posets over a left zero posemigroup.

Definition 4.1. (see [6]) An equivalence relation ρ on an S-act A is called a *congruence* on A, if $a\rho a'$ implies $(as)\rho(a's)$, for all $s \in S$. We denote the set of all congruences on A by $Con(A)$.

A congruence on an S-poset A is a congruence θ on the S-act A with the property that the S-act A/θ can be made into an S-poset in such a way that the natural map $A \to A/\theta$ is an S-poset map (see [1]).

For any relation θ on A, define the relation $\leqslant_\theta$ on A by

$$a \leqslant_\theta a' \quad \text{if and only if} \quad a \leqslant a_1 \theta a'_1 \leqslant a_2 \theta a'_2 \leqslant \ldots \leqslant a_n \theta a'_n \leqslant a',$$

where $a_i, a'_i \in A$ (such a sequence of elements is called a *θ-chain*). Then an S-act congruence θ on an S-poset A is an S-poset congruence if and only if $a\theta a'$ whenever $a \leqslant_\theta a' \leqslant_\theta a$.

For $a, b \in A$, $\rho_{a,b}$ denotes the *smallest S- act congruence* on A containing (a, b). It is in fact, the equivalence relation generated by $\{(as, bs) : s \in S \cup \{1\}\}$. Its elements are as follows:

$$x\rho_{a,b}y \;\Leftrightarrow\; \exists s_1, s_2, ..., s_n \in S \cup \{1\}\ , p_1, p_2, ..., p_n, q_1, q_2, ..., q_n \in A,$$

$$\begin{array}{ccccccccc} x = p_1 s_1 & & & q_2 s_2 = p_3 s_3 & & \ldots & & q_n s_n = y \\ & q_1 s_1 = p_2 s_2 & & & q_3 s_3 = p_4 s_4 & \ldots & & \end{array}$$

where $(p_i, q_i) = (as, bs)$ or $(p_i, q_i) = (bs, as)$ for some $s \in S \cup \{1\}$.

Lemma 4.2. *Let A be an S-act over a posemigroup S. Then $\rho_{x,y}$, for every distinct $x, y \in$ FixA, is an S-poset congruence.*

Proof. To prove we show the equivalence condition of an S-poset congruence. Namely, we show that if $a \leqslant_{\rho_{x,y}} a' \leqslant_{\rho_{x,y}} a$ then $a\rho_{x,y}a'$. But first we note that $\rho_{x,y} = \Delta \bigcup\{(x, y), (y, x)\}$ since $x, y \in FixA$. Now if $a \leqslant_{\rho_{x,y}} a' \leqslant_{\rho_{x,y}} a$ then two:

1) $a \leqslant x\rho_{x,y}y \leqslant y\rho_{x,y}x \leqslant a' \leqslant x\rho_{x,y}y \leqslant y\rho_{x,y}x \leqslant a$. Therefore $a \leqslant x \leqslant a' \leqslant x \leqslant a$ and hence $a = a'$ thus $a\rho_{x,y}a'$; or

2) $a \leqslant x\rho_{x,y}y \leqslant y\rho_{x,y}x \leqslant a' \leqslant y\rho_{x,y}x \leqslant x\rho_{x,y}y \leqslant a$. Therefore $a \leqslant x \leqslant a' \leqslant y \leqslant a$ and hence $x = y$ which is a contradiction. Hence this case is not possible. Thus we have $a = a'$, that is $a\rho_{x,y}a'$. □

Definition 4.3. *(see [6])*An S-poset A is called *subdirectly irreducible* if $\bigcap_{i\in I} \rho_i \neq \Delta$ for all congruences ρ_i on A with $\rho_i \neq \Delta$. If A is not subdirectly irreducible, then it is called *subdirectly reducible.*

It is worth noting that for each posemigroup S and an S-poset A with $|A| = 2$ there exist only two congruences Δ and $\bigtriangledown$ on A and therefore these S-posets are subdirectly irreducible.

Lemma 4.4. *Every S-poset A over a left zero posemigroup S with $|FixA| = 1$ or $|FixA| \geqslant 3$ is subdirectly reducible.*

Proof. It is clear that for a left zero semigroup S, every S-poset with only one fixed element is subirectly reducible. Also, let A be an S-poset with at least three distinct fixed elements a, b, c. Then we consider the S-poset congruences $\rho_{a,b}$ and $\rho_{b,c}$, by Lemma 4.2. Since $a, b, c \in \mathrm{Fix}A$ we obviously have $\rho_{a,b} = \Delta \bigcup\{(a, b), (b, a)\}$ and $\rho_{b,c} = \Delta \bigcup\{(b, c), (c, b)\}$. Therefore $\rho_{a,b} \cap \rho_{a,c} = \triangle$, and we are done. □

We give the following theorm as the counterpart of Theorem 3.2 of [7] in the category of S-posets over a left zero posemigroup.

Theorem 4.5. *An S-poset A over a left zero posemigroup S is subdirectly irreducible if and only if it is separated and $|FixA| = 2$.*

Proof. Let A be subdirectly irreducible. Then Lemma 4.4 ensures that $|FixA| = 2$ such as $\{a_0, b_0\}$. To show that A is separated, we suppose that there exists $x \neq y \in A$ such that $xs = ys$, for all $s \in S$, and find a contradiction. To do so, consider the S-act congruence $\rho_{x,y}$. Since $xs = ys$, for all $s \in S$, $\rho_{x,y} = \Delta \bigcup\{(x, y), (y, x)\}$. By the analogous method of the proof of Lemma 4.2 one can see that $\rho_{x,y}$ is an S-poset congruence on A. Also since $a_0, b_0 \in FixA$, by Lemma 4.2, we have the S-posset congruence ρ_{a_0,b_0} on A. But $\rho_{a_0,b_0} \cap \rho_{x,y} = \Delta$ which is a contradiction, therefore A is separated.

For the converse, let A be separated, $FixA = \{a_0, b_0\}$, and $\theta(\neq \Delta)$ be an S-poset congruence on A. Then there exists $x \neq y \in A$ such that $(x, y) \in \theta$. Thus $(xs, ys) \in \theta$ for every $s \in S$. But since $xs, ys \in \mathrm{Fix}A = \{a_0, b_0\}$ and A is separated, there exists $s \in S$ such that $xs \neq ys$. This means $(a_0, b_0), (b_0, a_0) \in \theta$. Therefore $\bigcap_{\theta \neq \Delta} \theta$ contains $\Delta \cup \{(a_0, b_0), (b_0, a_0)\}$, hence A is subdirectly irreducible. □

Finally, by the above theorem, and Birkhoff's Representation Theorem we have:

Theorem 4.6. *Every S-poset over a left zero posemigroup S is isomorphic to a subdirect product of separated S-posets each of which has exactly two fixed elements.*

It is worth noting that every S-poset A over a left zero posemigroup S with one or two elements and $|FixA| = 1$ is dc-injective.

We close the paper by characterizing simple S-poset. Recall that an S-poset A is called *simple* if $ConA = \{\triangle, \nabla\}$. It is easy to check that every S-poset A with $|A| \leqslant 2$ is simple but no S-poset A with trivial action and $|A| > 2$ is simple.

Theorem 4.7. *For a left zero posemigroup S, there exists no simple S-poset A with $|A| > 2$.*

Proof. Let $a \neq b$ be elements of A. Then in the case where $a, b \in \mathrm{Fix}A$ we have $\rho_{a,b} \neq \nabla$, (where $\rho_{a,b}$ is an S-poset congruence that discussed in Lemma 4.2) since $|A| > 2$, hence there exists $(a, b \neq) x \in A$ and $(a, x) \notin \rho_{a,b}$. Therefore, $\rho_{a,b}$ is a nontrivial congruence on A. Also in the case that one of a, b is not fixed, taking $a \notin FixA$, then $\rho_{a,b} \neq \nabla$. Because otherwise, if $\rho_{a,b} = \nabla$ then for each $x \neq y \in A$, we have $(x, y) \in \rho_{a,b}$. Consequently there exist $s, t \in S$ such that $as = x,\ bt = y$. Hence $x, y \in FixA$. Thus $(a, x) \notin \rho_{x,y}$, and so $\rho_{x,y}$ is a nontrivial congruence. □

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Dynamic groups

Mohammad Reza Molaei

Abstract. In this essay we introduce a class of groups which any member of it has a dynamic product. We prove that any subgroup of a dynamic group is a dynamic group and the product of two dynamic groups is a dynamic group. We deduce a new equivalency on dynamical systems via Rees matrix semigroups.

1. Introduction

Groups with dynamic products are a class of groups which have important role in topological cocycles [5]. Cocycles [1, 2, 3] are time-dependent dynamical systems and they can describe by these kind of groups [5]. To present the definition of a dynamic group we first recall the definition of a dynamical system. We assume that $(T, +)$ is the group of real numbers or the group of integer numbers. The binary operation $+$ can be any group operation on this set. If Y is a non-empty set, then a family $\xi = \{\xi^t : t \in T\}$ of the maps $\xi^t : Y \to Y$ is called a *dynamical system* if

(i) $\xi^0 = id_Y$;

(ii) $\xi^{t+s} = \xi^t \circ \xi^s$ for all $t, s \in T$.

$(T, +)$ is called the *time group* of ξ, and T is called the *time set* of ξ. If $(T, +)$ is a semigroup, and ξ satisfies the condition (ii), then it is called a *semi-dynamical system.*

Definition 1.1. Suppose ξ is a dynamical system (semi-dynamical system) on Y, and G is a group (semigroup). (G, ξ) is called a *dynamic group* (*dynamic semigroup*) if there is a one-to-one map $h : G \to \xi$ such that $h(b) \circ h(c) = h(cb)$.

If G is a group with the identity e, then the above definition implies that $h(e) = \xi^0$. One must pay attention to this point that: in the above definition if h is an onto map, and T is a commutative group, then h is a group isomorphism.

Example 1.2. We define a self map η on the circle S^1 by $\eta(e^{2\pi i\theta}) = e^{2\pi i(\theta+\frac{1}{4})}$, and we take $\xi = \{\eta^n : n \in Z\}$, where

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$$\eta^n = \begin{cases} \underbrace{\eta \circ \eta \circ \eta \circ \cdots \circ \eta}_{n\ times} & if\ \ n \in N, \\ id & if\ \ n = 0, \\ \underbrace{\eta^{-1} \circ \eta^{-1} \circ \eta^{-1} \circ \cdots \circ \eta^{-1}}_{-n\ times} & if\ \ -n \in N. \end{cases}$$

Let G be the additive group modulo 4. Then (G, ξ) is a dynamic group. $\square$

In the next section we present an example of a non trivial dynamic group.

Example 1.3. The set of integer numbers with the product $m * n = n|m|$ is a semigroup. Z with the product $m \times n = m|n|$ is also a semigroup. If for given $n \in Z$, we define $\eta^n : Z \to Z$ by

$$\eta^n(m) = \begin{cases} n^3|m| & if\ \ m^{\frac{1}{3}} \in Z, \\ m & if\ \ m^{\frac{1}{3}} \notin Z, \end{cases}$$

then $\xi = \{\eta^n : n \in (Z, \times)\}$ is a semi-dynamical system, and $((Z, *), \xi)$ is a dynamic semigroup. In fact, if we define $h : Z \to \xi$ by $h(n) = \eta^n$, then $h(n) \circ h(m) = h(m*n)$ and h is one-to-one. $\square$

In the next section we present two methods for constructing dynamic groups (dynamic semigroups), and we show that a dynamic product is an algebraic property. We associate a completely simple semigroup to a dynamic group. By using of completely simple semigroups or Rees matrix semigroups we present an equivalence relation on dynamical systems.

2. Structural consideration

We begin this section by presenting a nontrivial example of a dynamic group.

Example 2.1. Let

$$Y = \{y : R^2 \to R : y(t, x) = 2x + g(t)\ where\ g(t)\ is\ a\ continuous\ function\},$$

and for given $s \in R$ let $\xi^t : Y \to Y$ be defined by $\xi^s(y)(t, x) = y(t + s, x)$. Then $\xi = \{\xi^s\ :\ s \in R\}$ is a dynamical system on Y. For given $x, t, s \in R\ and\ y \in Y$ we take $\varphi^t(y(t + s, x), x) = e^{2t}[x + \int_0^t e^{-2u} g(u + s)du]$. Suppose $G = \{\varphi^t(y(t, .), .) : t \in R\ and\ y \in Y\}$. We define a product on G by the following form

$$\varphi^t(y(t, .), .)\varphi^s(z(s, .), .) = \varphi^t(y(t + s, .), .) o \varphi^s(z(s, .), .).$$

Then G with this product is a group and (G, ξ) is a dynamic group. The map $h : G \to \xi$ defined by $h(\varphi^t(y(t, .), .)) = \xi^t$ has the properties of Definition 1.1. $\square$

Dynamic group is a kind of groups which it's product is look alike to an evolution operator up to a one-to-one map. To see this let (G,ξ) be a dynamic group with a one-to-one mapping $h : G \to \xi$. Any member of ξ is called an *evolution operator*. If $P : G \times G \to G$ is the product of G, and if

$$A = \{P_b = P(.,b) : G \to G : b \in G\},$$

then there is a bijection

$$\phi : G \longrightarrow A, \quad b \mapsto P_b.$$

Under the map $h \circ \phi^{-1}$ any given P_b is look alike to the evolution operator $(h \circ \phi^{-1})(P_b)$. So there is a dynamics on the product of G.

Theorem 2.2. *If H is a subgroup of a dynamic group (G,ξ), then H is a dynamic group.*

Proof. Suppose $h : G \to \xi$ is a one-to-one map with the properties of Definition 1.1, then $h|_H : H \to \xi$ has the properties of Definition 1.1 for (H,ξ). □

Theorem 2.3. *If (G_1,ξ_1) and (G_2,ξ_2) are two dynamic groups with a common time set T, then $G_1 \times G_2$ is a dynamic group.*

Proof. Suppose $h_1 : G_1 \longrightarrow \xi_1,\ g \mapsto \xi_1^{t_g}$ and $h_2 : G_2 \longrightarrow \xi_1,\ g \mapsto \xi_2^{t_g}$ are the one-to-one maps which satisfy the conditions of Definition 1.1. We know that $G_1 \times G_2$ with the multiplication $(g_1,g_2)(j_1,j_2) = (g_1j_1, g_2j_2)$ is a group, and we know that there is a bijection $\sigma : T \times T \to T$, where $T = R$ or $T = Z$. We define the following binary operation on T:

$$+_\sigma : T \times T \longrightarrow T, \quad (t,s) \mapsto \sigma(t+s).$$

Clearly $(T,+_\sigma)$ is a group. We assume that ξ_i is a dynamical system on Y_i for $i \in \{1,2\}$. If $t \in T$, then we define

$$\xi^t : Y_1 \times Y_2 \longrightarrow Y_1 \times Y_2, \quad (y_1,y_2) \mapsto (\xi_1^{t_1}(y_1), \xi_2^{t_2}(y_2)),$$

where $t = \sigma(t_1,t_2)$. The straightforward calculations imply that

$$\xi = \{\xi^t : t \in T \textit{ and the operation of } T \textit{ is } +_\sigma\}$$

is a dynamical system on $Y_1 \times Y_2$. Now we define $h : G_1 \times G_2 \to \xi$ by $h(g_1,g_2) = \xi^{\sigma(t_{g_1},t_{g_2})}$. Since σ, h_1, h_2 are one-to-one, then h is one-to-one.

For given $(g_1,g_2),(l_1,l_2) \in G_1 \times G_2$, we have

$$h(g_1,g_2) \circ h(l_1,l_2) = \xi^{\sigma(t_{g_1},t_{g_2})} \circ \xi^{\sigma(t_{l_1},t_{l_2})} = \xi^{\sigma(t_{g_1}+t_{l_1},t_{g_2}+t_{l_2})}$$

$$= (h_1(g_1) \circ h_1(l_1), h_2(g_2) \circ h_2(l_2)) = (h_1(l_1g_1), h_2(l_2g_2)) = (\xi_1^{t_{l_1}+t_{g_1}}, \xi_2^{t_{l_2}+t_{g_2}})$$

$$= \xi^{\sigma(t_{l_1}+t_{g_1},t_{l_2}+t_{g_2})} = h(l_1g_1, l_2g_2) = h((l_1,l_2)(g_1,g_2)).$$

Thus $(G_1 \times G_2,\xi)$ is a dynamic group. □

We say that a property is an *algebraic property* if it preserves under algebraic isomorphisms. The next theorem show that the concept of dynamic product is an algebraic concept.

Theorem 2.4. *If (G,ξ) is a dynamic group, and if $f : G \to H$ is a group isomorphism, then (H,ξ) is a dynamic group.*

Proof. Suppose $h : G \to \xi$ has the properties of Definition 1.1. We define $\widetilde{h} : H \to \xi$ by $\widetilde{h}(a) = h(f^{-1}(a))$. Clearly $\widetilde{h}$ is one-to-one. If $a, b \in H$, then

$$\widetilde{h}(ab) = h(f^{-1}(ab)) = h(f^{-1}(a)f^{-1}(b)) = h(f^{-1}(b)) \circ h(f^{-1}(a)) = \widetilde{h}(b) \circ \widetilde{h}(a).$$

We also have $\widetilde{h}(e_H) = h(e_G) = id$. Thus (H,ξ) is a dynamic group. □

3. Dynamic mappings

We begin this section by definition of a Rees matrix semigroup which is defined first in [6]. Suppose that G is a group and Λ and I are two sets. If $p : \Lambda \times I \to G$ is a mapping then $I \times G \times \Lambda$ with the product $(i, a, \lambda)(j, b, \mu) = (i, ap(\lambda, j)b, \mu)$ is a completely simple semigroup [4]. $I \times G \times \Lambda$ with this product is denoted by $M(G, I, \Lambda, p)$ and it is called a *Rees matrix semigroup.* Rees proved in [6] that any completely simple semigroup is isomorphic to a Rees matrix semigroup.

Now we are going to associate a Rees matrix semigroup to a dynamic group.

We assume that (G,ξ) is a dynamic group with the mapping $h : G \to \xi$, and ξ is a dynamical system on Y, then the mapping $p : Y \times Y \to G$ defined by

$$p(y,z) = \begin{cases} h^{-1}(\xi^{t_0}) & if\ A = \{|t| : \xi^t(y) = z\} \neq \emptyset\ and\ t_0 = inf A, \\ e & if\ A = \emptyset \end{cases}$$

is a well defined map. In this case the Rees matrix $M(G, Y, Y, p)$ is associated to (G,ξ).

Definition 3.1. If (G,ξ) and (H,η) are two dynamic groups, and ξ and η are dynamical systems on Y and X respectively, then we say that (G,ξ) and (H,η) *are equivalent* if their associated Rees matrices $M(G, Y, Y, p)$ and $M(H, X, X, q)$ are isomorphic semigroups.

Theorem 3.2. *Suppose (G,ξ) and (H,η) are two dynamic groups with the time set T and one-to-one maps $h : G \to \xi$ and $g : H \to \eta$. If there exists a bijection $f : Y \to X$ such that $f \circ \xi^t = \eta^t \circ f$ for all $t \in T$, then (G,ξ) is equivalent to (H,η).*

Proof. We define $w : h(G) \to g(H)$ by $w(\xi^t) = \eta^t$. The condition $f \circ \xi^t = \eta^t \circ f$ implies that w is a bijection. If $l = g^{-1} \circ w \circ h$, then $l : G \to H$ is an isomorphism. Because if $a, b \in G$, then

$$l(ab) = (g^{-1} \circ w)(h(ab)) = (g^{-1} \circ w)(h(b) \circ h(a)) = g^{-1}(w(h(b)) \circ w(h(a)))$$

$$= (g^{-1}(w(h(a))))(g^{-1}(w(h(b)))) = l(a)l(b).$$

Since l is a bijection, by similar method we can show that l^{-1} is a homomorphism. So it is an isomorphism.

Now we show that the mapping $\psi : M(G,Y,Y,p) \to M(H,X,X,q)$ defined by $\psi(y,s,z) = (f(y), l(s), f(z))$ is a semigroup isomorphism. If $(y_1,s_1,z_1), (y_2,s_2,z_2)$ are in $M(G,Y,Y,p)$, then

$$\psi((y_1,s_1,z_1),(y_2,s_2,z_2)) = \psi(y_1, s_1 p(z_1,y_2)s_2, z_2)$$

$$= (f(y_1, l(s_1)l(p(z_1),y_2))l(s_2), f(z_2))$$

and

$$(\psi(y_1,s_1,z_1))(\psi(y_2,s_2,z_2)) = (f(y_1), l(s_1), f(z_1))((f(y_2), l(s_2), f(z_2))$$

$$= (f(y_1), l(s_1)q(f(z_1), f(z_2))l(s_2), f(z_2)).$$

So ψ is a homomorphism if we prove that $l(p(z_1,y_2)) = q(f(z_1), f(y_2))$. To prove this we have the following two cases.

CASE 1. If $p(z_1,y_2) = e_G$, then $\xi^t(z_1) \neq y_2$ for all $t \in T$. So $f^{-1} \circ \eta^t \circ f(z_1) \neq y_2$ for all $t \in T$. Thus $\eta^t(f(z_1)) \neq f(y_2)$ for all $t \in T$. Hence $q(f(z_1), f(y_2)) = e_H$. Thus $l(p(z_1,y_2)) = l(e_G) = e_H = q(f(z_1), f(y_2))$.

CASE 2. If there is $t \in T$ such that $\xi^t(z_1) = y_2$, then $(f^{-1} \circ \eta^t \circ f)(z_1) = y_2$. So $\eta^t(f(z_1)) = f(y_2)$. Thus

$$A = \{|t| : \xi^t(z_1) = y_2\} = \{|t| : \eta^t(f(z_1)) = f(y_2)\}.$$

Hence $p(z_1,y_2) = h^{-1}(\xi^{t_0})$ and $q(f(z_1), f(y_2)) = g^{-1}(\eta^{t_0})$, where $t_0 = infA$. Thus $l(p(z_1,y_2)) = l(h^{-1}(\xi^{t_0})) = g^{-1}(\eta^{t_0}) = q(f(z_1), f(y_2))$. So ψ is a homomorphism.

Since ψ is one-to-one and onto, then by similar method we can show that ψ^{-1} is a homomorphism. Hence it is an isomorphism. □

If a finite set Y and a finite group G are given and if a is the cardinality of the set $\{p : p : Y \times Y \to G \textit{ is a mapping}\}$, then the number of non-equivalent dynamical systems on Y which can make G a dynamic group is at most a.

One must attention to this point that there exist completely simple semigroups which are not associated to any dynamic group. For example, if Y and G have more than two elements, and if $p : Y \times Y \to G$ is the constant mapping $p(y,z) = e$, then there is no any dynamical system on Y such that $M(G,Y,Y,p)$ can be associated to it. Because if there is a ξ and a one-to-one mapping $h : G \to \xi$, then the condition $p(y,z) = e$ implies that ξ can not have more than one element, and it's element is the identity mapping on Y. Since h is one-to-one, then the order of G is 1, and this is a contradiction.

To determine dynamical systems on Y which can prove a group G is a dynamic group is basically related to the number of $M(G,Y,Y,.)$. In fact when we determine $M(G,Y,Y,p)$, then we must check the existence of h.

4. Conclusion

We introduce dynamic groups, and we consider their properties. We show that if (G_1, ξ_1) and (G_2, ξ_2) are two dynamic groups, then there is a dynamical system ξ such that $(G_1 \times G_2, \xi)$ is a dynamic group. In Theorem 2.2 time sets of ξ_1 and ξ_2 can be different groups. Now let us to pose a problem.

Problem. *Suppose that the time groups of ξ_1 and ξ_2 are equal, and it is a group $(T, +)$. Is it possible to find a dynamical system ξ with the time group $(T, +)$ such that $(G_1 \times G_2, \xi)$ be a dynamic group?*

We present an equivalence relation on a set of dynamical systems. The characterization of dynamical systems via this kind of equivalency can be a topic for further research.

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On the generalization of Brešer theorems

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Abstract. If S is a prime semiring with char $S \neq 2$ and $f : S \to S$ is an additive mapping which is skew-commuting on an ideal I of S, then $f(I) = 0$. We also prove that zero is the only additive mapping which is skew-commuting on a 2-torsion free semiprime semiring. These statements are the generalization of Brešar's theorems.

1. Introduction

The notion of semiring was first introduced by H. S. Vandiver in 1934 [10]. An algebraic system $(S, +, \cdot)$ is called a *semiring* if $(S, \cdot)$ is a semigroup; $(S, +)$ is a commutative semigroup with 0 and distributive laws of multiplication over addition hold; furthermore, $0s = s0 = 0$ for all $s \in S$. A subsemiring I of S is called a *right ideal* of S if $s \in S$, $x \in I$ implies $xs \in I$. *Left ideals* are defined in a similar way. A subset which is both left and right ideal is called an *ideal*. An ideal I of a semiring S is called a *k-ideal* if $x + y \in I$, $x \in I$ implies $y \in I$. A proper ideal P of a semiring S is said to be *prime* if $AB \subset P$ implies $A \subset P$ or $B \subset P$ for any ideals A and B of S. A proper ideal P of a semiring S is called a *semiprime ideal* if $A^2 \subset P$ implies $A \subset P$ for every ideal A of S. A k-ideal I of a semiring S is semiprime ideal if and only if I is the intersection of all prime k-ideals of S containing it [9, Theorem 3.12]. A semiring S is prime if 0 is a prime ideal. A semiring S is semiprime if 0 is a semiprime ideal. For further details of semirings, we refer [2, 3, 4, 5, 6, 7]. An additive mapping $f : S \to S$ is said to be *skew-commuting* on a set $T \subseteq S$ if $f(s)s + sf(s) = 0$ for all $s \in T$.

In [1], M. Brešar proved that if S is a prime ring of characteristic not 2, and $f : S \to S$ is an additive mapping which is skew-commuting on an ideal I of S, then $f(I) = 0$. He also proved that zero is the only additive mapping which is skew-commuting on a 2-torsion free semiprime rings. In this paper, we observe that these results still hold in the wider spectrum of semirings.

2. Preliminaries

One can easily prove the statement of following lemma.

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Lemma 2.1. *Let S be a semiring. If S has a nonzero nilpotent right ideal R, then it has a nonzero nilpotent ideal I containing R.* □

Now, we extend Lemma 1.1 [7] and Lemma 1 [1] in the framework of semirings.

Lemma 2.2. *Let S be a semiring and $I \neq (0)$ a right ideal of S. If there exists a positive integer n such that $x^n = 0$ for all $x \in I$, then S has a nonzero nilpotent ideal.*

Proof. The proof is given by induction on n. For $n = 2$, we have $x^2 = 0$ for all $x \in I$. As $x + xs \in I$ for all $s \in S$, so we get $(x + xs)^2 = 0$. This implies $xsx = 0$. Multiply from right by $t \in S$ to get $xsxt = 0$, so we obtain $(xS)^2 = 0$. Now if $xS \neq 0$, then S has a nonzero nilpotent right ideal xS and hence, by Lemma 2.1, S has a nonzero nilpotent ideal. When $xS = 0$, then $I^2 \subseteq IS = 0$. So S has a nonzero nilpotent right ideal I and hence has a nonzero nilpotent ideal.

Now suppose that Lemma is true for all positive integers less than n. Since $x^n = 0$ for all $x \in I$ for a fixed integer n and n is least such integer, therefore $x^{n-1} \neq 0$ and $(x^{n-1})^2 = 0$. Take $b = x^{n-1}$, then $b^2 = 0$. Let $B = bI$, then two cases arise. In the first case, let $B \neq (0)$. As $b + bs \in I$ for all $s \in S$, so we have $(b + bs)^n = 0$. On expansion, we arrive $(bs)^{n-1}b = 0$. This results in $(bs)^{n-1}B = (0)$. Let $T = \{x \in B \,|\, xB = 0\}$. It is easy to see that T is a k-ideal of B. Moreover, $y \in B$ implies that $y^{n-1} \in T$. Now let $y + T \in B/T$, then $(y + T)^{n-1} = y^{n-1} + T = T$. Hence by induction hypothesis B/T has a nilpotent ideal $U/T \neq T$. This yields $U \not\subset T$ and $(U/T)^k = U^k/T = T$ for some positive integer k. Since T is a k-ideal of B, therefore $U^k \subset T$ and hence $U^{k+1} \subset TU \subseteq TB = (0)$. As $U \not\subset T$ and U is an ideal of B, so we have $(0) \neq UB \subset U$ and $(UB)^{k+1} \subset U^{k+1} = (0)$. This implies that UB is a nonzero nilpotent right ideal of S and hence, by Lemma 2.1, S has a nonzero nilpotent ideal. In the second case, when $B = x^{n-1}I = (0)$. Let $W = \{x \in I \,|\, xI = (0)\}$, then W is a k-ideal of I. If $W = I$, then $I^2 = (0)$ and so I is a nonzero nilpotent right ideal and hence, by Lemma 2.1, S has a nonzero nilpotent ideal. If $W \neq I$, then for each element $x \in I$, $x^{n-1} \in W$. Hence in I/W, each element $x + W$ satisfies $(x + W)^{n-1} = x^{n-1} + W = W$. So our induction hypothesis gives us a nilpotent ideal $V/W \neq W$, this means $V \not\subset W$ and $(V/W)^m = V^m/W = W$ for some positive integer m. Hence we have $V^m \subset W$ and $V^{m+1} \subset WV \subseteq WI = (0)$. Since $(0) \neq VI \subset V$, where V is ideal of I, so we have $(VI)^{m+1} \subset V^{m+1} = (0)$. This means that S has a nonzero nilpotent right ideal VI and hence again, in view of Lemma 2.1, S has a nonzero nilpotent ideal. □

Lemma 2.3. *Let I be a nonzero ideal of a prime semiring S. If $I_n = \{x^n \,|\, x \in I\}$, then $I_n a = 0$ (or $aI_n = 0$) implies $a = 0$.*

Proof. Let $I_n a = 0$ and suppose on contrary $a \neq 0$. If $at = 0$ for all $t \in I$, then replacing t by st, where $s \in S$, we get $ast = 0$. As S is prime semiring, so we get $t = 0$ for all $t \in I$. This implies $I = 0$, which is not possible, hence $av \neq 0$ for some $v \in I$. As $avx \in I$ for all $x \in I$, so $(avx)^n a = 0$, this implies that $(avx)^{n+1} = 0$.

So we get right ideal $(av)S$ in which each element r satisfies $r^{n+1} = 0$. Hence, by Lemma 2.2, S has a nonzero nilpotent ideal but this is not possible in prime semiring, so we conclude $a = 0$. Similarly, we can prove the case when $aI_n = 0$. $\square$

One can also observe the following statements.

Lemma 2.4. *Let S be a semiring. If $a + b = 0$ and $a + c = 0$ for $a, b, c \in S$, then $b = c$.* $\square$

Lemma 2.5. *Let P be a prime ideal of semiring S and $ax \in P$ (or $xa \in P$) for all $x \in S$, then $a \in P$.* $\square$

3. Main results

Theorem 3.1. *Let S be a prime semiring of characteristic not 2. If an additive mapping $f : S \to S$ is skew-commuting on some ideal I of S, then $f(x) = 0$ for all $x \in I$.*

Proof. As f is skew-commuting on I, so we have

$$f(x)x + xf(x) = 0 \quad \forall x \in I. \tag{1}$$

Multiplying (1) from the right and left separately by x and applying Lemma 2.4, we get

$$f(x)x^2 = x^2 f(x). \tag{2}$$

Linearization of (1) yields

$$f(x)y + yf(x) + f(y)x + xf(y) = 0 \quad \forall x, y \in I. \tag{3}$$

Replacing y by x^2 in (3) and using (2), we get

$$2x^2 f(x) + f(x^2)x + xf(x^2) = 0. \tag{4}$$

After multiplying the last relation from right by x^2 and using (2), one can get the relation $2x^4 f(x) + f(x^2)x^3 + xf(x^2)x^2 = 0$. Now by adding $x^2 f(x^2)x + x^3 f(x^2)$ on both sides of this relation and using (1), we obtain

$$2x^4 f(x) = x^2 f(x^2)x + x^3 f(x^2). \tag{5}$$

Multiplying (4) by x^2 from left, the last relation reduces to $4x^4 f(x) = 0$. As S is of characteristic not 2, so we have

$$x^4 f(x) = 0. \tag{6}$$

Using (2), we obtain

$$f(x)x^4 = 0. \tag{7}$$

Now multiplying (1) from right by $2x$ and applying the Lemma 2.4 to (4), we have $2xf(x)x = f(x^2)x + xf(x^2)$. By multiplying this from left and right by x simultaneously and using (2) and (6), we reach $xf(x^2)x^2 + x^2f(x^2)x = 0$. This, along with (5) and (6), becomes

$$x^3f(x^2) = xf(x^2)x^2. \tag{8}$$

Now (1) can be written as $f(x^2)x^2 + x^2f(x^2) = 0$. Multiplying it from left by x and using (8), we get $2x^3f(x^2) = 0$. This becomes

$$x^3f(x^2) = 0. \tag{9}$$

Similarly, we can prove

$$f(x^2)x^3 = 0. \tag{10}$$

Replace x by x^2 in (3) to get $f(x^2)y+yf(x^2)+f(y)x^2+x^2f(y) = 0$ for all $x, y \in I$. By multiplying this from right and left by x^3 simultaneously and using (9) and (10), we obtain

$$x^3f(y)x^5 + x^5f(y)x^3 = 0 \quad \forall x, y \in I. \tag{11}$$

Replace x by x^2 in last relation to get

$$x^6f(y)x^{10} + x^{10}f(y)x^6 = 0. \tag{12}$$

First multiplying (11) from left by x^3 and right by x^5, then the last relation, in view of Lemma 2.4, becomes $x^{10}f(y)x^6 = x^8f(y)x^8$. Similarly, we get $x^6f(y)x^{10} = x^8f(y)x^8$. So (12) becomes $x^8f(y)x^8 = 0$ for all $x, y \in I$. This can be written as

$$zf(y)z = 0 \quad \forall y \in I, \forall z \in I_8. \tag{13}$$

Replace y by z in (3) to get

$$f(x)z + zf(x) + f(z)x + xf(z) = 0 \quad \forall x \in I, \forall z \in I_8. \tag{14}$$

Multiplying last relation from right by z and using (13), we obtain

$$f(x)z^2 + f(z)xz + xf(z)z = 0. \tag{15}$$

Suppose $x \in I_8$, then (13) can be written as $xf(x)x = 0$. Left multiplying (1) by x and using this relation, we get $x^2f(x) = 0$ for all $x \in I_8$. Now multiplying (15) from left by x^2, using this relation and (13), we arrive $x^3f(z)z = 0$ for all $x, z \in I_8$. By Lemma 2.3, this reduces to $f(z)z = 0$, hence we have $zf(z) = 0$. In view of this, (15) reduces to

$$f(x)z^2 + f(z)xz = 0 \quad \forall x \in I, \forall z \in I_8. \tag{16}$$

Now replacing x by xz in last relation, we obtain $f(xz)z^2 + f(z)xz^2 = 0$, then multiplying (16) from right by z and using Lemma 2.4, we arrive

$$f(x)z^3 = f(xz)z^2 \quad \forall x \in I, z \in I_8. \tag{17}$$

Left multiplying (14) by z, where $z \in I_8$, using $zf(z) = 0$ and (13), we get $z^2f(x) + zxf(z) = 0$. Replace x by xz in this relation and use $zf(z) = 0$ to have

$$z^2f(xz) = 0 \quad \forall x \in I, z \in I_8. \tag{18}$$

As a special case of (3), we have

$$f(x)yz + yzf(x) + f(yz)x + xf(yz) = 0 \ \ \forall x, y \in I, z \in I_8.$$

Multiplying the last relation from left and right by z^2 simultaneously and using (13), (17) and (18), we get $z^2f(x)yz^3 + z^2xf(y)z^3 = 0$. Multiplying this relation from left by z, one can see $tf(x)yt + txf(y)t = 0$ for all $x, y \in I$ and all $t \in I_{24}$. Now replacing y by $ytf(s)$, where $s \in I$ and $t \in I_{24}$, in this relation and using (13), one can arrive $txf(ytf(s))t = 0$ for all $x, y, s \in I$ and $t \in I_{24}$. As S is a prime semiring, we get

$$f(ytf(s))t = 0 \tag{19}$$

Replacing y by $ytf(s)$ in (3), where $s \in I$, we obtain

$$f(x)ytf(s) + ytf(s)f(x) + f(ytf(s))x + xf(ytf(s)) = 0.$$

Multiplying the last equation from left by t, using (13) and (19), we have

$$ytf(s)f(x)t + f(ytf(s))xt = 0 \quad \forall x, y, s \in I, \forall t \in I_{24} \tag{20}$$

Putting ry for y in last relation, where $r \in S$, leads to

$$rytf(s)f(x)t + f(rytf(s))xt = 0.$$

Multiplying (20) from left by r and using Lemma 2.4, we obtain $f(ryf(s))xt = rf(ytf(s))xt$. Again multiplying this from left by z, we obtain $zf(ryf(s))xt = zrf(ytf(s))xt$ for all $x, y, s \in I$, $z \in I_8$, $t \in I_{24}$, $r \in S$. Replace x by zx in this relation and use (13) to get $zrf(ytf(s))zxt = 0$. Due to primeness of S, this becomes $f(ytf(s))zxt = 0$. Again by primeness of S, we get $f(ytf(s))z = 0$. In view of Lemma 2.3, we have

$$f(ytf(s)) = 0. \tag{21}$$

Now suppose $f(s) \neq 0$ for some $s \in I$, otherwise theorem is proved. By Lemma 2.3, $tf(s) \neq 0$ for some $t \in I_{24}$. As $I \neq 0$, therefore for some $x \in I$, $a = xtf(s) \neq 0$. Thus $L = Sa$ is a nonzero left ideal of S contained in I. Hence using (21), we get $f(L) = 0$. Now, using (3), we have $f(x)t + tf(x) = 0$ for all $t \in L$ and $x \in I$. Substituting st for t, where $s \in S$, gives $f(x)st + stf(x) = 0$. Now by replacing s by x^4s and using (7), we have $x^4stf(x) = 0$. As S is a prime semiring, so we get $tf(x) = 0$. This implies that $f(x)t = 0$ and hence $f(x) = 0$ for all $x \in I$. This completes the proof. □

Theorem 3.2. *Let S be a 2-torsion free semiprime semiring. If an additive mapping $f : S \to S$ is skew-commuting on S, then $f = 0$.*

Proof. As S is a semiprime semiring, there exits a collection of prime k-ideals τ such that $\cap\tau = 0$. Let $\tau_1 = \{P \in \tau | charS/P \neq 2\}$ and $\tau_2 = \{P \in \tau | charS/P = 2\}$. Let $x \in \cap\tau_1$, then $2x \in (\cap\tau_1) \cap (\cap\tau_2) = \cap\tau = 0$, since S is 2-torsion free, so $x = 0$. Hence $\cap\tau_1 = 0$. The theorem will be complete if we prove $f(x) \in \cap\tau_1$ for all $x \in S$. Take a prime k-ideal $P \in \tau_1$. Linearize $f(x)x + xf(x) = 0$ to get $f(x)y + yf(x) + f(y)x + xf(y) = 0$ for all $x, y \in S$. This implies $f(p)x + xf(p) \in P$ for all $p \in P$, $x \in S$, so we get $xf(p)x + x^2f(p) \in P$ and $f(p)x^2 + xf(p)x \in P$. This gives $2xf(p)x + x^2f(p) + f(p)x^2 \in P$. As P is k-ideal and $x^2f(p) + f(p)x^2 \in P$, so we have $2xf(p)x \in P$. As char $S/P \neq 2$, so by Lemma 2.5, we obtain $xf(p)x \in P$ for all $p \in P, x \in S$. Since the k-ideal P is prime, therefore, in view of Lemma 2.6, $f(p) \in P$ for every $p \in P$. Now define a mapping F on S/P by $F(x + P) = f(x) + P$. It can be seen that F is additive and skew-commuting on prime semiring S/P. Hence $F = 0$ by Theorem 3.1. This gives $f(x) \in P$ for all $x \in S$. Hence $f(x) \in \cap\tau_1 = 0$. This completes the proof. $\square$

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Characterizations of ordered k-regular semirings by closure operations

Satyt Patchakhieo and Bundit Pibaljommee

Abstract. We introduce relations on the set of all closure operations on ordered semirings and then we characterize regular ordered semirings and ordered k-regular semirings using these relations.

1. Introduction

In 1936, J. von Neumann [14] called a ring $(S, +, \cdot)$ to be regular if $(S, \cdot)$ is regular. S. Bourne [3] has defined a semiring $(S, +, \cdot)$ to be regular if for all $a \in S$ there exist $x, y \in S$ such that $a + axa = aya$ which is different from Neumann regularity in general but both are equivalent in rings. In 1996, M. R. Adhikari, M. K. Sen and H. J. Weinert [1] have renamed the Bourne regularity of semirings to be a k-regularity.

In 1958, M. Henricksen [6] introduced the notion of k-ideals in a semiring. M. K. Sen and P. Mukhopadhyay [13] showed that k-regular semirings were characterized by k-ideals. A. K. Bhuniya and K. Jana [2] have shown that k-regular semirings and intra k-regular semirings can be characterized by k-bi-ideals where these semirings are additive semilattices. Subsequently, K. Jana [7, 8] continued to consider additive semilattice semirings and investigated some properties of quasi k-ideals in k-regular semirings and intra k-regular semirings, k-bi-ideals and quasi k-ideals in k-Clifford semirings. For more information about k-regular semirings and k-ideals in semirings, the reader may refer e.g., [2, 7, 8, 11].

A. P. Gan and Y. L. Jiang [5] investigated some properties of ordered ideals in ordered semirings. S. Patchakhieo and B. Pibaljommee [11] introduced the notions of an ordered k-regular semiring and an ordered k-ideal in an ordered semiring and characterized ordered k-regular semirings by their ordered k-ideals.

In 1970, B. Pondělíček [12] investigated a relation on the set of all closure operations on a semigroup and characterized a regular semigroup by this relation. After that T. Changphas [4] generalized Pondělíček's relation to an ordered semigroup.

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In this paper, we investigate a relationship between ordered semirings and closure operations on the ordered semirings. Moreover, we introduce relations on the set of all closure operations on ordered semirings and characterize regular ordered semirings and ordered k-regular semirings using these relations.

2. Preliminaries

In this section, we recall notions of an ordered semiring, an ordered ideal in an ordered semiring and notions of closure operations.

Let S be a nonempty set and $+$ and $\cdot$ be binary operations on S, named addition and multiplication, respectively. Then $(S, +, \cdot)$ is called a *semiring* if the following conditions are satisfied:

1. $(S, +)$ is a commutative semigroup;
2. $(S, \cdot)$ is a semigroup;
3. both operations are connected by the distributive laws, namely, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in S$.

A semiring $(S, +, \cdot)$ is said to have a *zero element* if there exists an element $0 \in S$ such that $0 + x = x = x + 0$ and $0 \cdot x = 0 = x \cdot 0$ for all $x \in S$. In particular, a semiring $(S, +, \cdot)$ is called *commutative* if $(S, \cdot)$ is a commutative semigroup, and called a *ring* if $(S, +)$ is a commutative group.

Definition 2.1. Let $(S, +, \cdot)$ be a semiring and $\emptyset \neq A \subseteq S$. Then A is called a *left* (*right*) *ideal* if the following conditions are satisfied:

1. $x + y \in A$ for all $x, y \in A$;
2. $SA \subseteq A$ $(AS \subseteq A)$.

We call A an *ideal* if it is both left ideal and right ideal of S.

Definition 2.2. Let $(S, \leqslant)$ be a partially ordered set. Then $(S, +, \cdot, \leqslant)$ is called an *ordered semiring* if the following conditions are satisfied:

1. $(S, +, \cdot)$ is a semiring;
2. if $a \leqslant b$ then $a + x \leqslant b + x$ and $x + a \leqslant x + b$;
3. if $a \leqslant b$ then $ax \leqslant bx$ and $xa \leqslant xb$

for all $a, b, x \in S$.

Instead of writing an ordered semiring $(S, +, \cdot, \leqslant)$, we denote S, for short, as an ordered semiring. Let A be a nonempty subset of S. We define

$$(A] = \{x \in S \mid x \leqslant a, \exists a \in A\}.$$

Definition 2.3. Let S be an ordered semiring and $\emptyset \neq A \subseteq S$. Then A is called a *left* (*right*) *ordered ideal* if the following conditions are satisfied:

1. A is a left (right) ideal of S;
2. if $x \leqslant a$ for some $a \in A$ then $x \in A$.

We call A an *ordered ideal* if it is both left ordered ideal and right ordered ideal of S.

It is known, a result in [5], that if A is a left (right, two-sided) ideal of an ordered semiring S then $(A]$ is the smallest left ordered ideal (right ordered ideal, two-sided ordered ideal) containing A.

Now we recall the notion of a C-closure operation and some of its properties proved in [12].

Let S be a nonempty set and $Sub(S)$ be the set of all subsets of S. A mapping $\mathbf{U} : Sub(S) \to Sub(S)$ is called a *C-closure operation* on S if

1. $\mathbf{U}(\emptyset) = \emptyset$;
2. $A \subseteq B \Rightarrow \mathbf{U}(A) \subseteq \mathbf{U}(B)$;
3. $A \subseteq \mathbf{U}(A)$;
4. $\mathbf{U}(\mathbf{U}(A)) = \mathbf{U}(A)$

for all $A, B \in Sub(S)$.

Let $x \in S$. We define $\mathbf{U}(x) = \mathbf{U}(\{x\})$. We denote by

$$\mathcal{F}(\mathbf{U}) = \{A \subseteq S \mid \mathbf{U}(A) = A\}$$

the set of all closed sets with respect to the operation $\mathbf{U}$ and by $\mathcal{C}(S)$ the set of all C-closure operations on S. Define a relation $\leqslant$ on $\mathcal{C}(S)$ by

$$\mathbf{U} \leqslant \mathbf{V} \iff \mathbf{U}(A) \subseteq \mathbf{V}(A) \text{ for all } A \in Sub(S).$$

We define a C-closure operation $\mathbf{I}$ on S by

$$\mathbf{I}(A) = \begin{cases} S, & \text{if } A \neq \emptyset, \\ \emptyset, & \text{if } A = \emptyset, \end{cases}$$

and for any $\mathbf{U}, \mathbf{V} \in \mathcal{C}(S)$ we denote by $\mathbf{U} \wedge \mathbf{V}$ and $\mathbf{U} \vee \mathbf{V}$ the infimum and the supremum, respectively, of $\mathbf{U}$ and $\mathbf{V}$ in $\mathcal{C}(S)$. It is known that for any $\mathbf{U}, \mathbf{V} \in \mathcal{C}(S)$,

1. $\mathbf{U} \leqslant \mathbf{I}$,
2. $\mathbf{U} \leqslant \mathbf{V} \iff \mathcal{F}(\mathbf{V}) \subseteq \mathcal{F}(\mathbf{U})$,
3. $\mathbf{U} \vee \mathbf{V}, \mathbf{U} \wedge \mathbf{V}$ exist and
   (a) $\mathcal{F}(\mathbf{U} \vee \mathbf{V}) = \mathcal{F}(\mathbf{U}) \cap \mathcal{F}(\mathbf{V})$,
   (b) $\mathcal{F}(\mathbf{U} \wedge \mathbf{V}) = \{A \cap B \mid A \in \mathcal{F}(\mathbf{U}), B \in \mathcal{F}(\mathbf{V})\}$.

3. Regular ordered semirings

In this section, we define a relation on the set of all closure operations on an ordered semiring and characterizes a regular ordered semiring by the relation.

Let S be an ordered semiring and $\emptyset \neq A \subseteq S$. We denote by $\Sigma_{fin}A$ the set of all finite sums of elements of A. We define a relation ρ on $\mathcal{C}(S)$ by letting $\mathbf{U}, \mathbf{V} \in \mathcal{C}(S)$,

$$\mathbf{U}\rho\mathbf{V} \iff A \cap B = (\Sigma_{fin}AB]$$

for all nonempty set $A \in \mathcal{F}(\mathbf{U})$ and for all nonempty set $B \in \mathcal{F}(\mathbf{V})$.

The following lemma is easy to prove using the definition of ρ.

Lemma 3.1. *Let S be an ordered semiring and $\mathbf{U}, \mathbf{U}', \mathbf{V}, \mathbf{V}' \in \mathcal{C}(S)$ such that $\mathbf{U}\rho\mathbf{V}$. If $\mathbf{U} \leqslant \mathbf{U}'$ and $\mathbf{V} \leqslant \mathbf{V}'$ then $\mathbf{U}'\rho\mathbf{V}'$.*

Let S be an ordered semiring. Then we define mappings $\mathbf{L}$ and $\mathbf{R}$ on $Sub(S)$ by letting $A \subseteq S$,

$$\mathbf{L}(A) = \begin{cases} (\Sigma_{fin}A + \Sigma_{fin}SA], & \text{if } A \neq \emptyset, \\ \emptyset, & \text{if } A = \emptyset, \end{cases}$$

and

$$\mathbf{R}(A) = \begin{cases} (\Sigma_{fin}A + \Sigma_{fin}AS], & \text{if } A \neq \emptyset, \\ \emptyset, & \text{if } A = \emptyset. \end{cases}$$

It is easy to show that $\mathbf{L}$ and $\mathbf{R}$ are closure operations on $Sub(S)$.

Now, we show that $\mathcal{F}(\mathbf{L})$ is the set of all left ordered ideals of S (including the empty set). Let A is a left ordered ideal of S. Then we obtain $A \subseteq \mathbf{L}(A) = (\Sigma_{fin}A + \Sigma_{fin}SA] \subseteq (A] = A$. Hence, $A \in \mathcal{F}(\mathbf{L})$. Conversely, let $\emptyset \neq A \in \mathcal{F}(\mathbf{L})$. Then $A = \mathbf{L}(A) = (\Sigma_{fin}A + \Sigma_{fin}SA]$. Hence, A is a left ordered ideal of S. Similarly, we have $\mathcal{F}(\mathbf{R})$ is the set of all right ordered ideals of S (including the empty set).

The following lemma can be proved straightforward.

Lemma 3.2. *Let S be an ordered semiring and A be a nonempty subset of S. Then $\Sigma_{fin}(AS] \subseteq (\Sigma_{fin}AS] = \Sigma_{fin}(\Sigma_{fin}AS]$ and $\Sigma_{fin}(SA] \subseteq (\Sigma_{fin}SA] = \Sigma_{fin}(\Sigma_{fin}SA]$.*

Theorem 3.3. *Let S be an ordered semiring and $\mathbf{U}, \mathbf{V} \in \mathcal{C}(S)$. Then $\mathbf{U}\rho\mathbf{V}$ if and only if $\mathbf{R} \leqslant \mathbf{U}, \mathbf{L} \leqslant \mathbf{V}$ and $x \in (\Sigma_{fin}\mathbf{U}(x)\mathbf{V}(x)]$ for all $x \in S$.*

Proof. ($\Rightarrow$). Assume that $\mathbf{U}\rho\mathbf{V}$. First, we show that $\mathbf{R} \leqslant \mathbf{U}$. Let $A \in \mathcal{F}(\mathbf{U})$. It is clear that $S \in \mathcal{F}(\mathbf{V})$. By assumption, we have $A = A \cap S = (\Sigma_{fin}AS]$. By Lemma 3.2, we have $A \subseteq \mathbf{R}(A) = (\Sigma_{fin}A + \Sigma_{fin}AS] = (\Sigma_{fin}(\Sigma_{fin}AS] + \Sigma_{fin}AS] = ((\Sigma_{fin}AS] + \Sigma_{fin}AS] \subseteq ((\Sigma_{fin}AS]] = (\Sigma_{fin}AS] = A$. Hence, $\mathbf{R}(A) = A$. Thus, $A \in \mathcal{F}(\mathbf{R})$. It follows that $\mathbf{R} \leqslant \mathbf{U}$. Similarly, $\mathbf{L} \leqslant \mathbf{V}$. Let $x \in S$. Since $\mathbf{U}, \mathbf{V} \in \mathcal{C}(S)$, we obtain $x \in \mathbf{U}(x) \cap \mathbf{V}(x)$. Since $\mathbf{U}(x) \in \mathcal{F}(\mathbf{U})$ and $\mathbf{V}(x) \in \mathcal{F}(\mathbf{V})$, we obtain $\mathbf{U}(x) \cap \mathbf{V}(x) = (\Sigma_{fin}\mathbf{U}(x)\mathbf{V}(x)]$. Thus, $x \in (\Sigma_{fin}\mathbf{U}(x)\mathbf{V}(x)]$.

($\Leftarrow$). Assume that $\mathbf{R} \leqslant \mathbf{U}, \mathbf{L} \leqslant \mathbf{V}$ and $x \in (\Sigma_{fin}\mathbf{U}(x)\mathbf{V}(x)]$ for all $x \in S$. We show that $\mathbf{U}\rho\mathbf{V}$. Let $A \in \mathcal{F}(\mathbf{U})\setminus\{\emptyset\}$ and $B \in \mathcal{F}(\mathbf{V})\setminus\{\emptyset\}$. By assumption, we have $A \in \mathcal{F}(\mathbf{R})$ and $B \in \mathcal{F}(\mathbf{L})$. Hence, $(\Sigma_{fin}AB] \subseteq (\Sigma_{fin}AS] \subseteq (\Sigma_{fin}A] \subseteq (A] = A$ and $(\Sigma_{fin}AB] \subseteq (\Sigma_{fin}SB] \subseteq (\Sigma_{fin}B] \subseteq (B] = B$. Thus, $(\Sigma_{fin}AB] \subseteq A \cap B$. Let $x \in A \cap B$. Then $\mathbf{U}(x) \subseteq \mathbf{U}(A) = A$ and $\mathbf{V}(x) \subseteq \mathbf{V}(B) = B$. By assumption, we have $x \in (\Sigma_{fin}\mathbf{U}(x)\mathbf{V}(x)] \subseteq (\Sigma_{fin}AB]$. Hence, $A \cap B \subseteq (\Sigma_{fin}AB]$. Thus, $A \cap B = (\Sigma_{fin}AB]$. Therefore, $\mathbf{U}\rho\mathbf{V}$. $\square$

As the notion of a regular ordered semigroup [9, 10], we define a notion of a regular ordered semiring as follows. An ordered semiring S is called *left* (*right*) *regular* if $a \in (Sa^2]$($a \in (a^2S]$) for all $a \in S$ and called *regular* if $a \in (aSa]$ for all $a \in S$. Similar to a result in ordered semigroups, we obtain the following theorem.

Theorem 3.4. *An ordered semiring S is ordered regular if and only if $A \cap B = (AB]$ for all right ordered ideal A and for all left ordered ideal B of S.*

Theorem 3.5. *An ordered semiring S is regular if and only if $\mathbf{R}\rho\mathbf{L}$.*

Proof. ($\Rightarrow$). Assume that S is regular. Let $a \in S$. By assumption, we have $a \in (aSa] \subseteq (\mathbf{R}(a)S\mathbf{L}(a)] \subseteq (\mathbf{R}(a)\mathbf{L}(a)] \subseteq (\Sigma_{fin}\mathbf{R}(a)\mathbf{L}(a)]$. By Theorem 3.3, we obtain $\mathbf{R}\rho\mathbf{L}$.

($\Leftarrow$). Assume that $\mathbf{R}\rho\mathbf{L}$. Let $a \in S$. By Theorem 3.3, $a \in (\Sigma_{fin}\mathbf{R}(a)\mathbf{L}(a)]$. Since $(\Sigma_{fin}\mathbf{R}(a)\mathbf{L}(a)] \subseteq (aS]$ and $(\Sigma_{fin}\mathbf{R}(a)\mathbf{L}(a)] \subseteq (Sa]$, we get $a \in (aS] \cap (Sa]$. Since $(aS] \in \mathcal{F}(\mathbf{R}), (Sa] \in \mathcal{F}(\mathbf{L})$ and $\mathbf{R}\rho\mathbf{L}$, we obtain $a \in (\Sigma_{fin}(aS](Sa]]$. Then there exist $x_1, x_2, \ldots, x_n \in (aS]$ and $y_1, y_2, \ldots, y_n \in (Sa]$ for some $n \in \mathbb{N}$ such that $x \leqslant x_1y_1 + x_2y_2 + \cdots + x_ny_n$. Since $x_i \in (aS]$ and $y_i \in (Sa]$ for all $i = 1, 2, \ldots, n$, there exist $s_i, r_i \in S$ such that $x_i \leqslant as_i$ and $y_i \leqslant r_ia$ for all $i = 1, 2, \ldots, n$. Hence, $x_iy_i \leqslant as_ir_ia$ for all $i = 1, 2, \ldots, n$. It follows that $a \leqslant as_1r_1a + as_2r_2a + \cdots + as_nr_na = a(s_1r_1 + s_2r_2 + \cdots + s_nr_n)a \in aSa$. Thus, $a \in (aSa]$. Therefore, S is regular. $\square$

As a consequence of Theorem 3.4 and Theorem 3.5, we obtain the following result.

Corollary 3.6. *Let S be an ordered semiring. Then $\mathbf{R}\rho\mathbf{L}$ if and only if $A \cap B = (AB]$ for all nonempty set $A \in \mathcal{F}(\mathbf{R})$ and for all nonempty set $B \in \mathcal{F}(\mathbf{L})$.*

Theorem 3.7. *Let S be a commutative ordered semiring, A be a nonempty subset of S and $\mathbf{R}\rho\mathbf{L}$. Then A is an ordered ideal of S if and only if there exist $H \in \mathcal{F}(\mathbf{R})$ and $K \in \mathcal{F}(\mathbf{L})$ such that $A = (HK]$.*

Let S be an ordered semiring. We denote the C-closure operation $\mathbf{R} \vee \mathbf{L}$ on S by $\mathbf{M}$. Note that $\mathcal{F}(\mathbf{M})$ is the set of all ordered ideals of S (including empty set).

Theorem 3.8. *Let S be an ordered semiring. Then the following statements are equivalent:*

(i) $\mathbf{L}\rho\mathbf{L}$;

(ii) $\mathbf{L}\rho\mathbf{M}$;

(iii) *S is left regular and* $\mathbf{R} \leqslant \mathbf{L}$.

Proof. $(i) \Rightarrow (ii)$. Since $\mathbf{L}\rho\mathbf{L}$ and by Lemma 3.1, we obtain $\mathbf{L}\rho\mathbf{M}$.

$(ii) \Rightarrow (iii)$. Assume that $\mathbf{L}\rho\mathbf{M}$. By Theorem 3.3, we have $\mathbf{R} \leqslant \mathbf{L}$. It follows that $\mathbf{L} = \mathbf{M}$. For any $x \in S$, we get

$$\begin{aligned} x \in (\Sigma_{fin}\mathbf{L}(x)\mathbf{M}(x)] &= (\Sigma_{fin}\mathbf{L}(x)\mathbf{L}(x)] \\ &\subseteq (\mathbb{N}x^2 + \Sigma_{fin}Sx^2 + \Sigma_{fin}xSx + \Sigma_{fin}SxSx] \\ &\subseteq (\mathbb{N}x^2 + \Sigma_{fin}Sx^2 + \Sigma_{fin}\mathbf{R}(x)x + \Sigma_{fin}S\mathbf{R}(x)x] \\ &\subseteq (\mathbb{N}x^2 + \Sigma_{fin}Sx^2 + \Sigma_{fin}\mathbf{L}(x)x + \Sigma_{fin}S\mathbf{L}(x)x] \\ &\subseteq (\mathbb{N}x^2 + \Sigma_{fin}Sx^2] \\ &= (\mathbb{N}x^2 + Sx^2]. \end{aligned}$$

Then there exist $k_1 \in \mathbb{N}, s \in S$ such that

$$x \leqslant k_1x^2 + sx^2. \tag{1}$$

Similarly, there exist $k_2 \in \mathbb{N}, r \in S$ such that $x^2 \leqslant k_2x^4 + rx^4$. Hence, $k_1x^2 \leqslant k_1k_2x^4 + k_1rx^4$. From (1), we have $x \leqslant k_1k_2x^4 + k_1rx^4 + sx^2$. This implies $x \in (Sx^2]$. Therefore, S is left regular.

$(iii) \Rightarrow (i)$. Assume that S is left regular and $\mathbf{R} \leqslant \mathbf{L}$. Then for any $x \in S$, we get $x \in (Sx^2] \subseteq (Sx\mathbf{L}(x)] \subseteq (\mathbf{L}(x)\mathbf{L}(x)] \subseteq (\Sigma_{fin}\mathbf{L}(x)\mathbf{L}(x)]$. By Theorem 3.3, it turns out $\mathbf{L}\rho\mathbf{L}$. □

Theorem 3.9. *Let S be an ordered semiring. Then the following statements are equivalent:*

(i) $\mathbf{R}\rho\mathbf{R}$;

(ii) $\mathbf{M}\rho\mathbf{R}$;

(iii) *S is right regular and* $\mathbf{L} \leqslant \mathbf{R}$.

Proof. The proof of this theorem is similar to Theorem 3.8. □

An ordered semiring S is called *left simple* (*right simple, simple*) if S has no proper left (right, two-sided) ordered ideal.

Now we give characterizations of left simple, right simple and simple as the following theorem which is easy to verify.

Theorem 3.10. *Let S be an ordered semiring. Then*

(i) *S is left simple if and only if* $\mathbf{L} = \mathbf{I}$*;*

(ii) *S is right simple if and only if* $\mathbf{R} = \mathbf{I}$*;*

(iii) *S is simple if and only if* $\mathbf{M} = \mathbf{I}$.

4. Ordered k-regular semirings

In this section, we define a relation on the set of all closure operations on an ordered semiring S and characterizes an ordered k-regular semiring by the relation.

The *k-closure* of a nonempty subset A of an ordered semiring S is defined by

$$\overline{A} = \{x \in S \mid \exists a, b \in A,\ x + a \leqslant b\}.$$

Lemma 4.1. [11] *Let S be an ordered semiring and A be a nonempty subset of S. If $A + A \subseteq A$ then $A \subseteq \overline{(A]} = \overline{\overline{(A]}}$.*

Let A be a nonempty subset of S. We note that if A is closed under addition then $\overline{(A]}$ is also closed.

Definition 4.2. [11] A left (right, two-sided) ordered ideal A of an ordered semiring S is called a *left ordered k-ideal* (*right ordered k-ideal, ordered k-ideal*) if $\overline{A} = A$.

In [11], it is known that if A is a left (right, two-sided) ideal of S then $\overline{(A]}$ is the smallest left ordered k-ideal (right ordered k-ideal, ordered k-ideal) containing A.

Definition 4.3. [11] An ordered semiring S is called *left* (*right*) *ordered k-regular* if $a \in \overline{(Sa^2]}$($a \in \overline{(a^2S]}$) for all $a \in S$ and called *ordered k-regular* if $a \in \overline{(aSa]}$ for all $a \in S$.

Theorem 4.4. [11] *An ordered semiring S is ordered k-regular if and only if $A \cap B = \overline{(AB]}$ for all right ordered k-ideal A and for all left ordered k-ideal B of S.*

Let S be an ordered semiring. We define a relation β on $\mathcal{C}(S)$ by letting $\mathbf{U}, \mathbf{V} \in \mathcal{C}(S)$,

$$\mathbf{U}\beta\mathbf{V} \iff A \cap B = \overline{(\Sigma_{fin} AB]}$$

for all nonempty set $A \in \mathcal{F}(\mathbf{U})$ and for all nonempty set $B \in \mathcal{F}(\mathbf{V})$.

By the definition of β, we have the following lemma.

Lemma 4.5. *Let S be an ordered semiring and $\mathbf{U}, \mathbf{U}', \mathbf{V}, \mathbf{V}' \in \mathcal{C}(S)$ such that $\mathbf{U}\beta\mathbf{V}$. If $\mathbf{U} \leqslant \mathbf{U}'$ and $\mathbf{V} \leqslant \mathbf{V}'$ then $\mathbf{U}'\beta\mathbf{V}'$.*

Lemma 4.6. [11] *Let S be an ordered semiring and A be a nonempty subset of S. Then*

(*i*) *$\overline{(\Sigma_{fin} A + \Sigma_{fin} SA]}$ is the smallest left ordered k-ideal of S containing A;*

(*ii*) *$\overline{(\Sigma_{fin} A + \Sigma_{fin} AS]}$ is the smallest right ordered k-ideal of S containing A.*

Let $(S,+,\cdot,\leqslant)$ be an ordered semiring. Then we define mappings $\mathbf{L}_k$ and $\mathbf{R}_k$ on $Sub(S)$ by letting $A \subseteq S$,

$$\mathbf{L}_k(A) = \begin{cases} \overline{(\Sigma_{fin}A + \Sigma_{fin}SA]}, & \text{if } A \neq \emptyset, \\ \emptyset, & \text{if } A = \emptyset, \end{cases}$$

and

$$\mathbf{R}_k(A) = \begin{cases} \overline{(\Sigma_{fin}A + \Sigma_{fin}AS]}, & \text{if } A \neq \emptyset, \\ \emptyset, & \text{if } A = \emptyset. \end{cases}$$

It is easy to show that $\mathbf{L}_k$ and $\mathbf{R}_k$ are closure operations on $Sub(S)$ and if $A \neq \emptyset$ then $\mathbf{L}_k(A)$ and $\mathbf{R}_k(A)$ are left ordered k-ideal and right ordered k-ideal of S, respectively.

Now, we show that $\mathcal{F}(\mathbf{L}_k)$ is the set of all left ordered k-ideals of S (including the empty set). Let A be a left ordered k-ideal of S. Then we obtain $A \subseteq \mathbf{L}_k(A) = \overline{(\Sigma_{fin}A + \Sigma_{fin}SA]} \subseteq \overline{(A]} = A$. Hence, $A \in \mathcal{F}(\mathbf{L}_k)$. Conversely, let $A \in \mathcal{F}(\mathbf{L}_k)\backslash\{\emptyset\}$. By Lemma 4.6, we get $A = \mathbf{L}_k(A) = \overline{(\Sigma_{fin}A + \Sigma_{fin}SA]}$ is a left ordered k-ideal of S. Similarly, we have $\mathcal{F}(\mathbf{R}_k)$ is the set of all right ordered k-ideals of S (including the empty set).

Lemma 4.7. *Let S be an ordered semiring and A be a nonempty subset of S. Then $\Sigma_{fin}\overline{(AS]} \subseteq \overline{(\Sigma_{fin}AS]} = \Sigma_{fin}\overline{(\Sigma_{fin}AS]}$ and $\Sigma_{fin}\overline{(SA]} \subseteq \overline{(\Sigma_{fin}SA]} = \Sigma_{fin}\overline{(\Sigma_{fin}SA]}$.*

Proof. Since $\Sigma_{fin}AS$ is closed under addition, then $\overline{(\Sigma_{fin}AS]}$ is also closed. Since $\overline{(AS]} \subseteq \overline{(\Sigma_{fin}AS]}$ and $\overline{(\Sigma_{fin}AS]}$ is closed under addition, we have $\Sigma_{fin}\overline{(AS]} \subseteq \overline{(\Sigma_{fin}AS]}$ and $\Sigma_{fin}\overline{(\Sigma_{fin}AS]} \subseteq \overline{(\Sigma_{fin}AS]}$. Hence, $\overline{(\Sigma_{fin}AS]} = \Sigma_{fin}\overline{(\Sigma_{fin}AS]}$. Similarly, we have $\Sigma_{fin}\overline{(SA]} \subseteq \overline{(\Sigma_{fin}SA]} = \Sigma_{fin}\overline{(\Sigma_{fin}SA]}$. $\square$

For any element a of an ordered semiring S, $\mathbb{N}a$ means $\{na \mid n \in \mathbb{N}\}$.

As a consequence of definitions of $\mathbf{L}_k$ and $\mathbf{R}_k$, we have the following lemma.

Lemma 4.8. *Let S be an ordered semiring and $a \in S$. Then $\mathbf{L}_k(a) = \overline{(\mathbb{N}a + Sa]}$ and $\mathbf{R}_k(a) = \overline{(\mathbb{N}a + aS]}$.*

Theorem 4.9. *Let S be an ordered semiring and $\mathbf{U},\mathbf{V} \in \mathcal{C}(S)$. Then $\mathbf{U}\beta\mathbf{V}$ if and only if $\mathbf{R}_k \leqslant \mathbf{U}$, $\mathbf{L}_k \leqslant \mathbf{V}$ and $x \in \overline{(\Sigma_{fin}\mathbf{U}(x)\mathbf{V}(x)]}$ for all $x \in S$.*

Proof. $(\Rightarrow)$. Assume that $\mathbf{U}\beta\mathbf{V}$. We first show that $\mathbf{R}_k \leqslant \mathbf{U}$. Let $A \in \mathcal{F}(\mathbf{U})\backslash\{\emptyset\}$. It is clear that $S \in \mathcal{F}(\mathbf{V})$. By assumption, $A = A \cap S = \overline{(\Sigma_{fin}AS]}$. By Lemma 4.7, we have $A \subseteq \mathbf{R}_k(A) = \overline{(\Sigma_{fin}A + \Sigma_{fin}AS]} = \overline{(\Sigma_{fin}\overline{(\Sigma_{fin}AS]} + \Sigma_{fin}AS]} = \overline{(\overline{(\Sigma_{fin}AS]} + \Sigma_{fin}AS]} \subseteq \overline{(\overline{(\Sigma_{fin}AS]}]} = \overline{(\Sigma_{fin}AS]} = A$. Hence, $\mathbf{R}_k(A) = A$. Thus, $A \in \mathcal{F}(\mathbf{R}_k)$. It follows that $\mathbf{R}_k \leqslant \mathbf{U}$. Similarly, $\mathbf{L}_k \leqslant \mathbf{V}$. Since $\mathbf{U},\mathbf{V} \in \mathcal{C}(S)$, we obtain $x \in \mathbf{U}(x) \cap \mathbf{V}(x)$ for all $x \in S$. Since $\mathbf{U}(x) \in \mathcal{F}(\mathbf{U})$ and $\mathbf{V}(x) \in \mathcal{F}(\mathbf{V})$, we obtain $\mathbf{U}(x) \cap \mathbf{V}(x) = \overline{(\Sigma_{fin}\mathbf{U}(x)\mathbf{V}(x)]}$. Thus, $x \in \overline{(\Sigma_{fin}\mathbf{U}(x)\mathbf{V}(x)]}$ for all $x \in S$.

($\Leftarrow$). Assume that $\mathbf{R}_k \leqslant \mathbf{U}, \mathbf{L}_k \leqslant \mathbf{V}$ and $x \in \overline{(\Sigma_{fin}\mathbf{U}(x)\mathbf{V}(x)]}$ for all $x \in S$. We show that $\mathbf{U}\beta\mathbf{V}$. Let $A \in \mathcal{F}(\mathbf{U}) \setminus \{\emptyset\}$ and $B \in \mathcal{F}(\mathbf{V}) \setminus \{\emptyset\}$. By assumption, $A \in \mathcal{F}(\mathbf{R}_k)$ and $B \in \mathcal{F}(\mathbf{L}_k)$. We obtain $\overline{(\Sigma_{fin}AB]} \subseteq \overline{(\Sigma_{fin}AS]} \subseteq \overline{(\Sigma_{fin}A]} \subseteq \overline{(A]} = A$ and $\overline{(\Sigma_{fin}AB]} \subseteq \overline{(\Sigma_{fin}SB]} \subseteq \overline{(\Sigma_{fin}B]} \subseteq \overline{(B]} = B$. Hence, $\overline{(\Sigma_{fin}AB]} \subseteq A \cap B$. Let $x \in A \cap B$. Then $\mathbf{U}(x) \subseteq A$ and $\mathbf{V}(x) \subseteq B$. By assumption, we have $x \in \overline{(\Sigma_{fin}\mathbf{U}(x)\mathbf{V}(x)]} \subseteq \overline{(\Sigma_{fin}AB]}$. Hence, $A \cap B \subseteq \overline{(\Sigma_{fin}AB]}$. Thus, $A \cap B = \overline{(\Sigma_{fin}AB]}$. Therefore, $\mathbf{U}\beta\mathbf{V}$. $\square$

The following theorem gives a characterization of an ordered k-regular semiring by closure operations.

Theorem 4.10. *An ordered semiring S is ordered k-regular if and only if $\mathbf{R}_k\beta\mathbf{L}_k$.*

Proof. ($\Rightarrow$). Assume that S is ordered k-regular. Let $a \in S$. Then we have $a \in \overline{(aSa]} \subseteq \overline{(\mathbf{R}_k(a)S\mathbf{L}_k(a)]} \subseteq \overline{(\mathbf{R}_k(a)\mathbf{L}_k(a)]} \subseteq \overline{(\Sigma_{fin}\mathbf{R}_k(a)\mathbf{L}_k(a)]}$. By Theorem 4.9, we obtain $\mathbf{R}_k\beta\mathbf{L}_k$.

($\Leftarrow$). Assume that $\mathbf{R}_k\beta\mathbf{L}_k$. Let $a \in S$. Then $a \in \overline{(\Sigma_{fin}\mathbf{R}_k(a)\mathbf{L}_k(a)]}$ by Theorem 4.9. Since $\overline{(\Sigma_{fin}\mathbf{R}_k(a)\mathbf{L}_k(a)]} \subseteq \overline{(aS]}$ and $\overline{(\Sigma_{fin}\mathbf{R}_k(a)\mathbf{L}_k(a)]} \subseteq \overline{(Sa]}$, we get $a \in \overline{(aS]} \cap \overline{(Sa]}$. Since $\overline{(aS]} \in \mathcal{F}(\mathbf{R}_k), \overline{(Sa]} \in \mathcal{F}(\mathbf{L}_k)$ and $\mathbf{R}_k\beta\mathbf{L}_k$, we obtain $a \in \overline{(\Sigma_{fin}\overline{(aS]}\ \overline{(Sa]}]}$. There exist $x, x' \in (\Sigma_{fin}\overline{(aS]}\ \overline{(Sa]}]$ such that $a + x \leqslant x'$. But $x, x' \in (\Sigma_{fin}\overline{(aS]}\ \overline{(Sa]}]$, so there exist $x_1, x_2, \ldots, x_n, x'_1, \ldots, x'_m \in \overline{(aS]}$, $y_1, y_2, \ldots, y_n, y'_1, \ldots, y'_m \in \overline{(Sa]}$ such that $x \leqslant \Sigma_{i=1}^n x_i y_i$ and $x' \leqslant \Sigma_{j=1}^m x'_j y'_j$. For each $1 \leqslant i \leqslant n$, we get

$$x_i + u_i \leqslant u'_i, \tag{2}$$

$$y_i + v_i \leqslant v'_i, \tag{3}$$

where $u_i \leqslant as_i, u'_i \leqslant as'_i, v_i \leqslant t_i a, v'_i \leqslant t'_i a$ for some $s_i, s'_i, t_i, t'_i \in S$. From (2), we have $x_i y_i + u_i y_i \leqslant u'_i y_i$. From (3), we have $u_i y_i + u_i v_i \leqslant u_i v'_i$ and $u'_i y_i + u'_i v_i \leqslant u'_i v'_i$. Hence, $x_i y_i + u_i y_i + u_i v_i + u'_i v_i \leqslant u'_i y_i + u_i v_i + u'_i v_i$. Then we get $u_i y_i + u_i v_i + u'_i v_i \leqslant u_i v'_i + u'_i v_i \leqslant a s_i t'_i a + a s'_i t_i a = a(s_i t'_i + s'_i t_i)a \in aSa$ and $u'_i y_i + u_i v_i + u'_i v_i \leqslant u_i v_i + u'_i v'_i \leqslant a s_i t_i a + a s'_i t'_i a = a(s_i t_i + s'_i t'_i)a \in aSa$. It follows that $x_i y_i \in \overline{(aSa]}$. Hence, $\Sigma_{i=1}^n x_i y_i \in \overline{(aSa]}$. Similarly, we obtain $\Sigma_{j=1}^m x'_j y'_j \in \overline{(aSa]}$. Since $x \leqslant \Sigma_{i=1}^n x_i y_i$ and $x' \leqslant \Sigma_{j=1}^m x'_j y'_j$, we have $x, x' \in (\overline{(aSa]}] = \overline{(aSa]}$. Then there exist $c, c'd, d' \in (aSa]$ such that $x + c \leqslant d$ and $x' + c' \leqslant d'$. It follows that $a + x + c + c' \leqslant x' + c + c' \leqslant c + d' \in (aSa]$ and $x + c + c' \leqslant d + c' \in (aSa]$. Thus, $a \in \overline{(aSa]}$. Therefore, S is ordered k-regular. $\square$

By Theorem 4.4 and Theorem 4.10, we have the following result.

Corollary 4.11. *Let S be an ordered semiring. Then $\mathbf{R}_k\beta\mathbf{L}_k$ if and only if $A \cap B = \overline{(AB]}$ for all nonempty set $A \in \mathcal{F}(\mathbf{R}_k)$ and for all nonempty set $B \in \mathcal{F}(\mathbf{L}_k)$.*

Example 4.12. Let $S = \{a, b, c\}$ with a partially ordered set $\leqslant$ be defined $a \leqslant b \leqslant c$. Define binary operations $+$ and $\cdot$ on S by the following tables.

$+$	a	b	c
a	a	a	a
b	a	b	c
c	a	c	c

and

$\cdot$	a	b	c
a	b	b	b
b	b	b	b
c	b	b	b

Then we have $(S,+,\cdot,\leqslant)$ is an ordered semiring. Moreover, $\overline{(\Sigma_{fin}\mathbf{R}(x)\mathbf{L}(x)]} = S$ for every $x \in S$. It follows that $x \in \overline{(\Sigma_{fin}\mathbf{R}(x)\mathbf{L}(x)]}$ for every $x \in S$. By Theorem 4.9 and Theorem 4.10, we obtain that S is an ordered k-regular semiring.

Theorem 4.13. *Let S be a commutative ordered semiring, A be a nonempty subset of S and $\mathbf{R}_k\beta\mathbf{L}_k$. Then A is an ordered k-ideal of S if and only if there exist $H \in \mathcal{F}(\mathbf{R}_k)$ and $K \in \mathcal{F}(\mathbf{L}_k)$ such that $A = \overline{(HK]}$.*

Proof. $(\Rightarrow)$. Assume that A is an ordered k-ideal of S. Let $H = \mathbf{R}_k(A)$ and $K = \mathbf{L}_k(A)$. Then we have $H \in \mathcal{F}(\mathbf{R}_k)$ and $K \in \mathcal{F}(\mathbf{L}_k)$. Since S is a commutative ordered semiring, $H = A = K$. Let $a \in A$. Since $\mathbf{R}_k\beta\mathbf{L}_k, a \in \overline{(\mathbf{R}_k(a)\mathbf{L}_k(a)]} \subseteq \overline{(\mathbf{R}_k(A)\mathbf{L}_k(A)]} = \overline{(HK]}$. Hence, $A \subseteq \overline{(HK]}$. Since $A^2 \subseteq A, \overline{(HK]} = \overline{(A^2]} \subseteq \overline{(A]} = A$. Therefore, $A = \overline{HK}$.

$(\Leftarrow)$. Assume that there exist $H \in \mathcal{F}(\mathbf{R}_k)$ and $K \in \mathcal{F}(\mathbf{L}_k)$ such that $A = \overline{(HK]}$. Since $\mathbf{R}_k\beta\mathbf{L}_k$, we have $H \cap K = \overline{(HK]}$. Since $A = \overline{(HK]} = H \cap K$ and S is commutative, A is an ordered ideal. Since $\overline{A} = \overline{\overline{(HK]}} = \overline{(HK]} = A$, A is an ordered k-ideal. □

Let S be an ordered semiring. We denote the C-closure operation $\mathbf{R}_k \vee \mathbf{L}_k$ on S by $\mathbf{M}_k$. Note that $\mathcal{F}(\mathbf{M}_k)$ is the set of all ordered k-ideals of S (including empty set).

Theorem 4.14. *Let S be an ordered semiring. Then the following statements are equivalent:*

(i) $\mathbf{L}_k\beta\mathbf{L}_k$;

(ii) $\mathbf{L}_k\beta\mathbf{M}_k$;

(iii) *S is left ordered k-regular and $\mathbf{R}_k \leqslant \mathbf{L}_k$.*

Proof. $(i) \Rightarrow (ii)$. Since $\mathbf{L}_k\beta\mathbf{L}_k$ and by Lemma 4.5, we obtain $\mathbf{L}_k\beta\mathbf{M}_k$.

$(ii) \Rightarrow (iii)$. Assume that $\mathbf{L}_k\beta\mathbf{M}_k$. By Theorem 4.9, we have $\mathbf{R}_k \leqslant \mathbf{L}_k$. It follows that $\mathbf{L}_k = \mathbf{M}_k$. For any $x \in S$, we get

$$\begin{aligned}
x \in \overline{(\Sigma_{fin}\mathbf{L}_k(x)\mathbf{M}_k(x)]} &= \overline{(\Sigma_{fin}\mathbf{L}_k(x)\mathbf{L}_k(x)]} \\
&\subseteq \overline{(\mathbb{N}x^2 + \Sigma_{fin}Sx^2 + \Sigma_{fin}xSx + \Sigma_{fin}SxSx]} \\
&\subseteq \overline{(\mathbb{N}x^2 + \Sigma_{fin}Sx^2 + \Sigma_{fin}\mathbf{R}_k(x)x + \Sigma_{fin}S\mathbf{R}_k(x)x]} \\
&\subseteq \overline{(\mathbb{N}x^2 + \Sigma_{fin}Sx^2 + \Sigma_{fin}\mathbf{L}_k(x)x + \Sigma_{fin}S\mathbf{L}_k(x)x]} \\
&\subseteq \overline{(\mathbb{N}x^2 + \Sigma_{fin}Sx^2]} \\
&= \overline{(\mathbb{N}x^2 + Sx^2]}.
\end{aligned}$$

Then there exist $y, z \in (\mathbb{N}x^2 + Sx^2]$ such that $x + y \leqslant z$. It follows that there exist $k_1, k_2 \in \mathbb{N}, s, t \in S$ such that $y \leqslant k_1x^2 + sx^2, z \leqslant k_2x^2 + tx^2$. Similarly, there exist $u, v \in (\mathbb{N}x^4 + Sx^4]$ such that $x^2 + u \leqslant v$. It follows that there exist $k_3, k_4 \in \mathbb{N}, q, r \in S$ such that $u \leqslant k_3x^4 + qx^4, v \leqslant k_4x^4 + rx^4$. Since $x^2 + u \leqslant v$, we obtain $k_1x^2 + k_1u \leqslant k_1v$ and $k_2x^2 + k_2u \leqslant k_2v$. Hence, $y + k_1u \leqslant k_1x^2 + sx^2 + k_1u \leqslant k_1v + sx^2 \leqslant k_1k_4x^4 + k_1rx^4 + sx^2$ and $z + k_2u \leqslant k_2x^2 + tx^2 + k_2u \leqslant k_2v + tx^2 \leqslant k_2k_4x^4 + k_2rx^4 + tx^2$. It turns out $y + k_1u + k_2u \leqslant k_1k_4x^4 + k_1rx^4 + sx^2 + k_2k_3x^4 + k_2qx^4 \in Sx^2$ and $z + k_1u + k_2u \leqslant k_2k_4x^4 + k_2rx^4 + tx^2 + k_1k_3x^4 + k_1qx^4 \in Sx^2$. Since $x + y \leqslant z$, we get $x + y + k_1u + k_2u \leqslant z + k_1u + k_2u$. This implies that $x \in \overline{(Sx^2]}$. Therefore, S is left ordered k-regular.

$(iii) \Rightarrow (i)$. Assume that S is left ordered k-regular and $\mathbf{R}_k \leqslant \mathbf{L}_k$. Then $x \in \overline{(Sx^2]} \subseteq \overline{(Sx\mathbf{L}_k(x)]} \subseteq \overline{(\mathbf{L}_k(x)\mathbf{L}_k(x)]} \subseteq \overline{(\Sigma_{fin}\mathbf{L}_k(x)\mathbf{L}_k(x)]}$ for all $x \in S$. By Theorem 4.9, it turns out $\mathbf{L}_k\beta\mathbf{L}_k$. □

Theorem 4.15. *Let S be an ordered semiring. Then the following statements are equivalent:*

(i) $\mathbf{R}_k\beta\mathbf{R}_k$;

(ii) $\mathbf{M}_k\beta\mathbf{R}_k$;

(iii) *S is right ordered k-regular and* $\mathbf{L}_k \leqslant \mathbf{R}_k$.

Proof. The proof of this theorem is similar to Theorem 4.14. □

An ordered semiring S is called *left k-simple* (*right k-simple, k-simple*) if S has no proper left (right, two-sided) ordered k-ideal.

Theorem 4.16. *Let S be an ordered semiring. Then*

(i) *S is left k-simple if and only if* $\mathbf{L}_k = \mathbf{I}$;

(ii) *S is right k-simple if and only if* $\mathbf{R}_k = \mathbf{I}$;

(iii) *S is k-simple if and only if* $\mathbf{M}_k = \mathbf{I}$.

Proof. (i). Assume that S is left k-simple. It is clear that $\mathbf{L}_k(\emptyset) = \emptyset = \mathbf{I}(\emptyset)$. Let A be a nonempty subset of S. Then we have $\mathbf{L}_k(A) = \overline{(\Sigma_{fin}A + \Sigma_{fin}SA]} = S = \mathbf{I}(A)$. Hence, $\mathbf{L}_k = \mathbf{I}$. Conversely, if A is a left ordered k-ideal, then we obtain $S = \mathbf{I}(A) = \mathbf{L}_k(A) = \overline{(\Sigma_{fin}A + \Sigma_{fin}SA]} \subseteq \overline{(A]} = A \subseteq S$. Hence, $A = S$. Thus, S is left k-simple.

The proof of (ii) and (iii) are similar to (i). □

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