# Some structures of Hom-Poisson color algebras 

Ibrahima Bakayoko and Sylvain Attan


#### Abstract

In many previous papers, the authors used an algebra endomorphism to twist the original algebraic structures in order to produce the corresponding Hom-algebraic structures. In this work, we use a bijective linear map, an element of centroid, an averaging operator, a Rota-Baxter operator or a multiplier to produce a Hom-Poisson color algebra from a given one.


## 1. Introduction

Poisson algebras are algebras which has simultaneously a Lie and a commutative associative algebra structures satisfying the Leibniz identity. They naturally appear in very different forms and contexts. Many examples coming from geometry and mathematical physics lead to a certain type of Poisson structures. These are always a key element coming along with interesting problems in the fields of classical/quantum mechanics, differential geometry and algebraic geometry.

The first motivation to study nonassociative Hom-algebras comes from quasideformations of Lie algebras of vector fields, in particular q-deformations of Witt and Virasoro algebras $[1,5,6,8,9]$. Hom-Lie algebras were first introduced by Hartwig, Larsson and Silvestrov in order to describe q-deformations of Witt and Virasoro algebras using $\sigma$-derivations [7]. The corresponding associative type objects, called Hom-associative algebras were introduced by Makhlouf and Silvestrov in [10]. Next, generalizations of Hom-type algebras are introduced and discussed in the framework of color algebras. In particular, Hom-associative color algebras [11] has been introduced as a generalization of both Hom-associative algebras and associative color algebras. Furthermore, relying on the well-known relationship between (Hom-) associative and (Hom-)Lie algebras, Hom-Lie color algebras were also introduced in [11] as a natural generalization of Hom-Lie algebras and as a special case of quasi-hom-Lie algebras. It is proved that the commutator of any Hom-associative color algebras gives rise to Hom-Lie color algebras and a way to obtain Hom-Lie color algebras from classical Lie color algebras along with even color algebra endomorphisms is presented. Also, we have introduced a multiplier $\sigma$ on an abelian group and constructions of new Hom-Lie color algebras from given ones by the $\sigma$-twists are obtained. Furthermore, Hom-Poisson color algebras are

[^0]introduced in [3] as the color version of Hom-Poisson algebras [4]. Some constructions of Hom-Poisson color algebras from Hom-associative color algebras which twisting map is an averaging operator or from a given Hom-Poisson color algebra together with an averaging operator or from a Hom-post-Poisson color algebra are given in [4]. In particular, it is shown that any Hom-pre-Poisson color algebra leads to a Hom-Poisson color algebra. Moreover, in [2] is obtained a description of HomPoisson color algebras by using only one operation of its two binary operations via the polarisation-depolarisation process.

The goal of this paper is to give a continuation of constructions of Hom-Poisson color algebras [4]. While many authors working on Hom-algebras use a morphism of Hom-algebras to build another one, we ask ourselves if there are others kinds of twists which are not morphisms such that we can get Hom-algebraic structures from others one. To give a positive answer to the above question, we organize this paper as follows. In Section 2, we recall some basic definitions about Rota-Baxter Hom-associative color algebras and Rota-Baxter Hom-Lie color algebras as well as averaging operators and centroids. In Section 3, we give the main results of the paper. The proceeding is by twisting the original multiplications of Hom-Poisson color algebras by a bijective linear map, an element of centroid, an averaging operator, a Rota-Baxter operator or a multiplier.

Throughout this paper, all graded vector spaces are assumed to be over a field $\mathbb{K}$ of characteristic different from 2.

## 2. Definitions

In this section, we recall some relevant definitions about $G$-graded vetor space and color Hom-algebras. In particular, we recall the notion of a color Hom-associative algebra as well as the one of a color Hom-Lie algebra. Some examples are given and some results are also proved.

First, let recall that if $G$ is an abelian group, a vector space $L$ is said to be $G$-graded if, there exists a family $\left(L_{a}\right)_{a \in G}$ of vector subspaces of $L$ such that $L=\oplus_{a \in G} L_{a}$. An element $u \in L$ is said to be homogeneous of degree $a \in G$ if $u \in L_{a}$. The set of all homogeneous elements in $L$ is denoted by $\mathcal{H}(L)$.

Definition 2.1. Let $G$ be an abelian group. A map $\varepsilon: G \times G \rightarrow \mathbb{K}^{*}$ is called a skew-symmetric bicharacter on $G$ if the following identities hold:
(1) $\varepsilon(a, b) \varepsilon(b, a)=1$,
(2) $\varepsilon(a, b+c)=\varepsilon(a, b) \varepsilon(a, c)$,
(3) $\varepsilon(a+b, c)=\varepsilon(a, c) \varepsilon(b, c)$,
for all $a, b, c \in G$.
Remark 2.2. (1) Observe that $\varepsilon(a, 0)=\varepsilon(0, a)=1, \varepsilon(a, a)= \pm 1$ for all $a \in G$.
(2) If $x$ and $y$ are two homogeneous elements of degree $a$ and $b$ respectively and $\varepsilon$ is a bicharacter, then we shorten the notation by writing $\varepsilon(x, y)$ instead of $\varepsilon(a, b)$. Also unless stated, in the sequel all the graded space are over the same abelian group $G$ and the bicharacter will be the same for all the structures.

Example 2.3. For $G=\mathbb{Z}_{2}^{n}=\left\{\left(\alpha_{1}, \ldots, \alpha_{n}\right) \mid \alpha_{i} \in \mathbb{Z}_{2}\right\}$,

$$
\varepsilon\left(\left(\alpha_{1}, \ldots, \alpha_{n}\right),\left(\beta_{1}, \ldots, \beta_{n}\right)\right):=(-1)^{\alpha_{1} \beta_{1}+\cdots+\alpha_{n} \beta_{n}}
$$

is a skew-symmetric bicharacter.
Definition 2.4. Let $G$ be an abelian group. A bicharacter on $G$ is a map $\delta: G \times G \rightarrow \mathbb{K}^{*}$ defined by

$$
\delta(x, y):=\sigma(x, y) \sigma(y, x)^{-1} \text { for all } x, y, z \in G
$$

where $\sigma: G \times G \rightarrow \mathbb{K}^{*}$ is any mapping such that

$$
\sigma(x, y+z) \sigma(y, z)=\sigma(x, y) \sigma(x+y, z), \text { for all } x, y, z \in G
$$

In this case, $\sigma$ is called a multiplier on $G$ and $\delta$ the bicharacter associated with $\sigma$.
Example 2.5. If we define the mapping $\sigma: G \times G \rightarrow \mathbb{R}^{*}$ by

$$
\sigma\left(\left(i_{1}, i_{2}\right),\left(j_{1}, j_{2}\right)\right):=(-1)^{i_{1} j_{2}}, \text { for all } i_{k}, j_{k} \in \mathbb{Z}_{2}, k=1,2
$$

it is easy to verify that $\sigma$ is a multiplier on $G$ and

$$
\delta\left(\left(i_{1}, i_{2}\right),\left(j_{1}, j_{2}\right)\right):=(-1)^{i_{1} j_{2}-i_{2} j_{1}}, \text { for all } i_{k}, j_{k} \in \mathbb{Z}_{2}, i=1,2
$$

is a bicharacter on $G$.
Definition 2.6. A color Hom-algebra is a quadruple $(A, \mu, \varepsilon, \alpha)$ in which
(1) $A$ is a G-graded vector space i.e., $A=\bigoplus_{a \in G} A_{a}$,
(2) $\mu: A \times A \rightarrow A$ is an even bilinear map i.e., $\mu\left(A_{a}, A_{b}\right) \subset A_{a+b}$, for all $a, b \in G$,
(3) $\alpha: A \rightarrow A$ is an even linear map i.e., $\alpha\left(A_{a}\right) \subset A_{a}$ for all $a \in G$,
(4) $\varepsilon: G \times G \rightarrow \mathbb{K}^{*}$ is a bicharacter.

Definition 2.7. A Hom-associative color algebra is a color Hom-algebra ( $A, \mu, \varepsilon, \alpha$ ) satisfying the Hom-associativity condition:

$$
a s_{\mu}(x, y, z):=\mu(\alpha(x), \mu(y, z))-\mu(\mu(x, y), \alpha(z))=0
$$

for all $x, y, z \in \mathcal{H}(A)$.
If, in addition, $\mu$ satisfies $\mu=\varepsilon(\cdot, \cdot) \mu^{o p}$ i.e., $\mu(x, y)=\varepsilon(x, y) \mu(y, x)$ for all $x, y \in \mathcal{H}(A)$ ( $\varepsilon$-commutativity), the Hom-associative color algebra $(A, \mu, \varepsilon, \alpha)$ is said to be a $\varepsilon$-commutative Hom-associative color algebra.

Whenever, $\alpha=I d_{A}$ we recover associative color algebra.
Proposition 2.8. Let $(A, \mu, \varepsilon)$ be an associative color algebra and $\alpha: A \rightarrow A$ be an even linear map such that $(A, \mu, \varepsilon, \alpha)$ is a Hom-associative color algebra. Then, for any fixed element $\xi \in A$, the quadruple $\left(A, \mu_{\xi}, \varepsilon, \alpha\right)$ is a Hom-associative color algebra with

$$
\mu_{\xi}(x, y):=x \xi y
$$

for all $x, y \in \mathcal{H}(A)$.
Proof. For any $x, y, z \in \mathcal{H}(A)$ we have

$$
\begin{aligned}
a s_{\mu_{\xi}}(x, y, z) & =\mu_{\xi}\left(\mu_{\xi}(x, y), \alpha(z)\right)-\mu_{\xi}\left(\alpha(x), \mu_{\xi}(y, z)\right) \\
& =(x \xi y) \xi \alpha(z)-\alpha(x) \xi(y \xi z) \\
& =x(\xi y \xi) \alpha(z)-\alpha(x)(\xi y \xi) z \text { (associativity) } \\
& =(x \xi y \xi) \alpha(z)-(x \xi y \xi) \alpha(z) \text { (Hom-associativity) } \\
& =0
\end{aligned}
$$

Now, we recall the definition of Hom-Lie color algebra.
Definition 2.9. A Hom-Lie color algebra is a color Hom-algebra ( $A,[],, \varepsilon, \alpha$ ) satisfying
(1) $[x, y]=-\varepsilon(x, y)[y, x]$ ( $\varepsilon$-skew-symmetry),
(2) $\varepsilon(z, x)[\alpha(x),[y, z]]+\varepsilon(x, y)[\alpha(y),[z, x]]+\varepsilon(y, z)[\alpha(z),[x, y]]=0$ (color HomJacobi identity)
for any $x, y, z \in \mathcal{H}(A)$.
Example 2.10. It is clear that Lie color algebras are examples of Hom-Lie color algebras by setting $\alpha=i d$. If, in addition, $\varepsilon(x, y)=1$ (resp. $\varepsilon(x, y)=(-1)^{|x||y|}$ ) then, the Hom-Lie color algebra is a classical Lie algebra (resp. Lie superalgebra). Moreover, Hom-Lie algebras (resp. Hom-Lie superalgebras) are also obtained when $\varepsilon(x, y)=1$ (resp. $\left.\varepsilon(x, y)=(-1)^{|x||y|}\right)$. See [11] for other examples as Hom-Lie color $s l(2, \mathbb{K})$, Heisenberg Hom-Lie color algebra and Hom-Lie color algebra of Witt type.

Definition 2.11. i) A Rota-Baxter Hom-associative color algebra of weight $\lambda \in \mathbb{K}$ is a Hom-associative color algebra $(A, \cdot, \varepsilon, \alpha)$ together with an even linear map $R: A \rightarrow A$ that satisfies the identities

$$
\begin{align*}
R \circ \alpha & =\alpha \circ R  \tag{1}\\
R(x) \cdot R(y) & =R(R(x) \cdot y+x \cdot R(y)+\lambda x \cdot y) \tag{2}
\end{align*}
$$

for all $x, y \in \mathcal{H}(A)$.
ii) A Rota-Baxter Hom-Lie color algebra of weight $\lambda \in \mathbb{K}$ is a Hom-Lie color algebra $(L,[],, \varepsilon, \alpha)$ together with an even linear map $R: L \rightarrow L$ that satisfies the identities

$$
\begin{align*}
R \circ \alpha & =\alpha \circ R \\
{[R(x), R(y)] } & =R([R(x), y]+[x, R(y)]+\lambda[x, y]), \tag{3}
\end{align*}
$$

for all $x, y \in \mathcal{H}(L)$.
Example 2.12. Consider the abelian multiplicative group $G=\{-1,+1\}$ and the $G$-graded 2-dimensional vector space $A=A_{(-1)} \oplus A_{(1)}=\left\langle e_{2}\right\rangle \oplus\left\langle e_{1}\right\rangle$. Then the quintuple $(A, \cdot, \varepsilon, \alpha, R)$ is a Rota-Baxter Hom-associative color algebra of weight $\lambda$ with

- the multiplication: $e_{1} \cdot e_{1}:=-e_{1}, \quad e_{1} \cdot e_{2}:=e_{2}, \quad e_{2} \cdot e_{1}:=e_{2}, \quad e_{2} \cdot e_{2}:=e_{1}$,
- the bicharacter: $\varepsilon(i, j):=(-1)^{(i-1)(j-1) / 4}$,
- the even linear map $\alpha: A \rightarrow A$ defined by $\alpha\left(e_{1}\right):=e_{1}, \quad \alpha\left(e_{2}\right):=-e_{2}$,
- the Rota-Baxter operator $R: A \rightarrow A$ given by $R\left(e_{1}\right):=-\lambda e_{1}, R\left(e_{2}\right):=$ $-\lambda e_{2}$.

Definition 2.13. Let $k \geqslant 0$ be an integer.
i) An $\alpha^{k}$-averaging operator over a Hom-associative color algebra $(A, \mu, \varepsilon, \alpha)$, is an even linear map $\beta: A \rightarrow A$ such that

$$
\begin{align*}
\alpha \circ \beta & =\beta \circ \alpha,  \tag{4}\\
\beta\left(\mu\left(\beta(x), \alpha^{k}(y)\right)\right. & =\mu(\beta(x), \beta(y))=\beta\left(\mu\left(\alpha^{k}(x), \beta(y)\right)\right), \tag{5}
\end{align*}
$$

for all $x, y \in \mathcal{H}(A)$.
ii) An $\alpha^{k}$-averaging operator over a Hom-Lie color algebra ( $L,[],, \varepsilon, \alpha$ ), is an even linear map $\beta: L \rightarrow L$ such that

$$
\begin{align*}
\alpha \circ \beta & =\beta \circ \alpha, \\
{[\beta(x), \beta(y)] } & =\beta\left(\left[\beta(x), \alpha^{k}(y)\right]\right), \tag{6}
\end{align*}
$$

for all $x, y \in \mathcal{H}(L)$.
Definition 2.14. Let $k \geqslant 0$ be an integer.
An element of $\alpha^{k}$-centroid of a Hom-associative color algebra $(A, \cdot, \varepsilon, \alpha)$, is an even linear map $\beta: A \rightarrow A$ such that

$$
\begin{align*}
\beta \circ \alpha & =\alpha \circ \beta  \tag{7}\\
\beta(x \cdot y) & =\beta(x) \cdot \alpha^{k}(y)=\alpha^{k}(x) \cdot \beta(y) \tag{8}
\end{align*}
$$

for all $x, y \in \mathcal{H}(A)$.

In the case of a Hom-Lie color algebra $(L,[],, \varepsilon, \alpha)$, an element of $\alpha^{k}$-centroid is an even linear map $\beta: L \rightarrow L$ such that

$$
\begin{align*}
\beta \circ \alpha & =\alpha \circ \beta, \\
\beta([x, y]) & =\left[\beta(x), \alpha^{k}(y)\right], \tag{9}
\end{align*}
$$

for all $x, y \in \mathcal{H}(L)$.
Observe that $\beta([x, y])=\left[\alpha^{k}(x), \beta(y)\right]$ thanks to the $\varepsilon$-skew-symmetry.

## 3. Hom-Poisson Color Algebras

This section is devoted to various constructions of Hom-Poisson color algebras. It contains relevant results of this paper. In the most proofs, we don't establish the $\varepsilon$-skew-symmetry condition as well as the color Hom-Jacobi identity.

Definition 3.1. A Hom-Poisson color algebra consists of a $G$-graded vector space $A$, a multiplication $\mu: A \times A \rightarrow A$, an even bilinear bracket $\{\}:, A \times A \rightarrow A$ and an even linear map $\alpha: A \rightarrow A$ such that :
(1) $(A, \mu, \varepsilon, \alpha)$ is a Hom-associative color algebra,
(2) $(A,\{\},, \varepsilon, \alpha)$ is a Hom-Lie color algebra,
(3) the Hom-Leibniz color identity

$$
\{\alpha(x), \mu(y, z)\}=\mu(\{x, y\}, \alpha(z))+\varepsilon(x, y) \mu(\alpha(y),\{x, z\}),
$$

is satisfied for any $x, y, z \in \mathcal{H}(A)$.
A Hom-Poisson color algebra $(A, \mu,\{\},, \varepsilon, \alpha)$ in which $\mu$ is $\varepsilon$-commutative is said to be a commutative Hom-Poisson color algebra.

Example 3.2. Let $A=A_{(0)} \oplus A_{(1)}=\left\langle e_{1}, e_{2}\right\rangle \oplus\left\langle e_{3}\right\rangle$ be a 3-dimensional graded vector space and $\cdot: A \times A \rightarrow A$ and [, ] : $A \times A \rightarrow A$ the multiplications defined by

$$
\begin{aligned}
& e_{1} \cdot e_{1}:=e_{1}, e_{1} \cdot e_{2}:=e_{2}, e_{1} \cdot e_{3}:=a e_{3}, e_{2} \cdot e_{1}:=e_{2} \\
& e_{2} \cdot e_{1}:=\frac{1}{a} e_{2}, e_{2} \cdot e_{3}:=e_{3}, e_{3} \cdot e_{1}:=a e_{3},\left[e_{2}, e_{3}\right]:=e_{3}
\end{aligned}
$$

and the omitted products being zero. Then, the quintuple $(A, \cdot,[],, \varepsilon, \alpha)$ is a Hom-Poisson color algebra with

$$
\alpha\left(e_{1}\right):=e_{1}, \quad \alpha\left(e_{2}\right):=e_{2}, \quad \alpha\left(e_{3}\right):=a e_{3},
$$

and any bicharacter $\varepsilon$.
Theorem 3.3. Let $(P, \cdot,[],, \varepsilon, \alpha)$ be a Hom-Poisson color algebra and a map $\sigma: G \times G \rightarrow \mathbb{K}^{*}$ be a symmetric multiplier on $G$ i.e.,
(1) $\sigma(x, y)=\sigma(y, x), \forall x, y \in G$,
(2) $\sigma(x, y) \sigma(z, x+y)$ is invariant under cyclic permutation of $x, y, z \in G$.

Then, $P^{\sigma}=\left(P, \cdot^{\sigma},[,]^{\sigma}, \varepsilon, \alpha\right)$ is also a Hom-Poisson color algebra with

$$
x \cdot \cdot^{\sigma} y:=\sigma(x, y) x \cdot y \quad \text { and } \quad[x, y]^{\sigma}:=\sigma(x, y)[x, y]
$$

for any $x, y \in \mathcal{H}(P)$.
Proof. For any homogeneous elements $x, y, z \in P$,

$$
\begin{aligned}
\text { as. }(x, y, z) & =\left(x \cdot{ }^{\sigma} y\right) \cdot \cdot^{\sigma} \alpha(z)-\alpha(x) \cdot{ }^{\sigma}\left(y \cdot{ }^{\sigma} z\right) \\
& =\sigma(x, y) \sigma(x+y, z)(x \cdot y) \cdot z-\sigma(x, y+z) \sigma(y, z) \alpha(x) \cdot(y \cdot z) \\
& =\sigma(x, y) \sigma(x+y, z) \text { as. }(x, y, z) \\
& =0
\end{aligned}
$$

Thus the Hom-associativity condition holds in $P^{\sigma}$. Next, the color Hom-Jacobi identity follows from [11]. Finally, for verifying the Hom- Leibniz color identity consider any homogeneous elements $x, y, z \in P$,

$$
\begin{aligned}
& {\left[\alpha(x), y \cdot{ }^{\sigma} z\right]^{\sigma}} \\
& \quad=[\alpha(x), \sigma(y, z) y \cdot z]^{\sigma} \\
& \quad=\sigma(y, z) \sigma(x, y+z)[\alpha(x), y \cdot z] \\
& \quad=\sigma(y, z) \sigma(x, y+z)[x, y] \cdot \alpha(z)+\sigma(y, z) \sigma(x, y+z) \varepsilon(x, y) \alpha(y) \cdot[x, z] \\
& \quad=\sigma(x, y) \sigma(z, x+y)[x, y] \cdot \alpha(z)+\sigma(z, x) \sigma(y, z+x) \varepsilon(x, y) \alpha(y) \cdot[x, z] \\
& \quad=\sigma(z, x+y)[x, y] \cdot{ }^{\sigma} \alpha(z)+\sigma(y, x+z) \varepsilon(x, y) \alpha(y) \cdot{ }^{\sigma}[x, z] \\
& \quad=[x, y]^{\sigma} \cdot{ }^{\sigma} \alpha(z)+\varepsilon(x, y) \alpha(y) \cdot{ }^{\sigma}[x, z]^{\sigma} .
\end{aligned}
$$

The following theorem can be proved as the previous one.
Theorem 3.4. Let $(P, \cdot,[],, \varepsilon, \alpha)$ be a Hom-Poisson color algebra and a map $\delta: G \times G \rightarrow \mathbb{K}^{*}$ be the bicharacter associated with the multiplier $\sigma$ on $G$. Then, $\left(P, \cdot{ }^{\sigma},[,]^{\sigma}, \varepsilon \delta, \alpha\right)$ is also a Hom-Poisson color algebra with
$x \cdot{ }^{\sigma} y:=\sigma(x, y) x \cdot y,[x, y]^{\sigma}:=\sigma(x, y)[x, y]$ and $\varepsilon \delta(x, y):=\varepsilon(x, y) \sigma(x, y) \sigma(y, x)^{-1}$,
for any $x, y \in \mathcal{H}(P)$. Moreover, an endomorphism of $(P, \cdot,[],, \varepsilon, \alpha)$ is also an endomorphism of $\left(P, \sigma^{\sigma},[,]^{\sigma}, \varepsilon \delta, \alpha\right)$.
Theorem 3.5. Let $\left(P^{\prime}, .^{\prime},[,]^{\prime}, \varepsilon, \alpha^{\prime}\right)$ be a Hom-Poisson color algebra and $P$ a graded vector space with an even bilinear map ". ", a $\varepsilon$-skew-symmetric even bilinear bracket"[,]" and an even linear map $\alpha$. Let $f: P \rightarrow P^{\prime}$ be an even bijective linear map such that $f \circ \alpha=\alpha^{\prime} \circ f$,

$$
f(x \cdot y)=f(x) \cdot^{\prime} f(y) \quad \text { and } \quad f([x, y])=[f(x), f(y)]^{\prime}, \forall x, y \in \mathcal{H}(P)
$$

Then $(P, \cdot,[],, \varepsilon, \alpha)$ is a Hom-Poisson color algebra.

Proof. First, we obtain for all $x, y, z \in \mathcal{H}(P)$,

$$
\begin{aligned}
& (x \cdot y) \cdot \alpha(z)-\alpha(x) \cdot(y \cdot z) \\
= & f^{-1}\left(\left(f(x) \cdot \cdot^{\prime} f(y)\right) \cdot^{\prime} f(\alpha(z))\right)-f^{-1}\left(f(\alpha(x)) \cdot \cdot^{\prime}\left(f(y) \cdot^{\prime} f(z)\right)\right) \\
= & f^{-1}\left(\left(f(x) \cdot \cdot^{\prime} f(y)\right) \cdot^{\prime} \alpha^{\prime}\left(f(z)-\alpha^{\prime}(f(x)) \cdot \cdot^{\prime}\left(f(y) \cdot^{\prime} f(z)\right)\right) .\right.
\end{aligned}
$$

Thus, the Hom-associativity identity follows from the one in $P^{\prime}$. Similarly, we get the color Hom-Jacobi identity. Finally, for any $x, y, z \in \mathcal{H}(P)$, the Hom-Leibniz color identity is proved as follows

$$
\begin{aligned}
{[\alpha(x), y \cdot z]=} & f^{-1}[f(\alpha(x)), f(y \cdot z)]^{\prime} \\
= & f^{-1}\left[f(\alpha(x)), f\left(f^{-1}\left(f(y) \cdot^{\prime} f(z)\right)\right)\right]^{\prime} \\
= & f^{-1}\left[\alpha^{\prime}(f(x)), f(y) \cdot^{\prime} f(z)\right]^{\prime} \\
= & f^{-1}\left([f(x), f(y)]^{\prime} \cdot^{\prime} \alpha^{\prime}(f(z))+\varepsilon(x, y) \alpha^{\prime}(f(y)) \cdot^{\prime}\left[f(x) \cdot^{\prime} f(z)\right]^{\prime}\right) \\
= & f^{-1}\left(f\left(f^{-1}[f(x), f(y)]^{\prime}\right) \cdot^{\prime} \alpha^{\prime}(f(z))\right) \\
& +\varepsilon(x, y) f^{-1}\left(\alpha^{\prime}(f(y)) \cdot^{\prime} f\left(f^{-1}\left[f(x) \cdot^{\prime} f(z)\right]^{\prime}\right)\right) \\
= & f^{-1}\left(f([x, y]) \cdot^{\prime} f(\alpha(z))\right)+\varepsilon(x, y) f^{-1}\left(f(\alpha(y)) \cdot^{\prime} f([x \cdot z])\right) \\
= & {[x, y] \cdot \alpha(z)+\varepsilon(x, y) \alpha(y) \cdot[x \cdot z] . }
\end{aligned}
$$

Definition 3.6. Let $(P, \cdot,[],, \varepsilon, \alpha)$ be a Hom-Poisson color algebra. An even linear map $\beta: P \rightarrow P$ is said to be
(1) an element of $\alpha^{k}$-centroid of $P$ if (7), (8) and (9) hold.
(2) an $\alpha^{k}$-averaging operator of $P$ if (4), (5) and (6) hold.
(3) a Rota-Baxter operator over $P$ if (1), (2) and (3) hold.

Example 3.7. The even linear map $R: P \rightarrow P$ defined on the Hom-Poisson color algebra of Example 3.2, by

$$
R\left(e_{1}\right):=-\lambda e_{1}, \quad R\left(e_{2}\right):=-\lambda e_{2}, \quad R\left(e_{3}\right):=-\lambda e_{3},
$$

is a Rota-Baxter operator of weight $\lambda$ on $P$.
Example 3.8. If $(A, \mu, \varepsilon, \alpha, R)$ is a Rota-Baxter Hom-associative color algebra, then

$$
\left(A, \mu,\{,\}:=\mu-\varepsilon(\cdot, \cdot) \mu^{o p}, \varepsilon, \alpha, R\right),
$$

is a Rota-Baxter Hom-Poisson color algebra.

Now, we have the following result whose proof is similar to the one of the previous.

Theorem 3.9. Let $P^{\bullet}:=(P, \cdot[],, \varepsilon, \alpha)$ be a Hom-Poisson color algebra and $\beta: P \rightarrow P$ be an element of $\alpha^{0}$-centroid of $P$. If we define the multiplications $*: P \times P \rightarrow P$ and $\{\}:, P \times P \rightarrow P$ by

$$
\begin{equation*}
x * y:=x \cdot y \quad \text { and } \quad\{x, y\}:=[\beta(x), y], \forall x, y \in \mathcal{H}(P), \tag{10}
\end{equation*}
$$

then $P^{*}:=(P, *,\{\},, \varepsilon, \alpha)$ is also a Hom-Poisson color algebra. Moreover, the map $\beta:(P, *,\{\},, \varepsilon, \alpha) \longrightarrow(P, \cdot,[],, \varepsilon, \alpha)$ becomes a morphism of Hom-Poisson color algebras.

Proof. It is clear that the Hom-associativity identity in $P^{*}$ follows from the one in $P^{\bullet}$. Next, the color Hom-Jacobi identity is proved as follows

$$
\begin{aligned}
& \varepsilon(z, x)\{\alpha(x),\{y, z\}\}+\varepsilon(x, y)\{\alpha(y),\{z, x\}\}+\varepsilon(y, z)\{\alpha(z),\{x, y\}\} \\
& =\varepsilon(z, x)[\beta(\alpha(x)),[\beta(y), z]]+\varepsilon(x, y)[\beta(\alpha(y)),[\beta(z), x]]+\varepsilon(y, z)[\beta(\alpha(z)),[\beta(x), y]] \\
& =\varepsilon(z, x)[\beta(\alpha(x)), \beta([y, z])]+\varepsilon(x, y)[\beta(\alpha(y)), \beta([z, x])]+\varepsilon(y, z)[\beta(\alpha(z)), \beta([x, y])] \\
& =\beta^{2}(\varepsilon(z, x)[\alpha(x),[y, z]]+\varepsilon(x, y)[\alpha(y),[z, x]]+\varepsilon(y, z)[\alpha(z),[x, y]]) \\
& =\beta^{2}(0)=0 .
\end{aligned}
$$

In order to prove the Hom-Leibniz color identity we consider $x, y, z \in \mathcal{H}(P)$. Then

$$
\begin{aligned}
\{\alpha(x), y * z\} & =[\beta(\alpha(x)), y \cdot z]=[\alpha(\beta(x)), y \cdot z] \\
& =[\beta(x), y] \cdot \alpha(z)+\varepsilon(x, y) \alpha(y)) \cdot[\beta(x), z] \\
& =\{x, y\} * \alpha(z)+\varepsilon(x, y) \alpha(y) *\{x, y\} .
\end{aligned}
$$

Theorem 3.10. Let $(P, \cdot,[],, \varepsilon, \alpha)$ be a Hom-Poisson color algebra and $\beta: P \rightarrow P$ be an $\alpha^{0}$-averaging operator. Then with the products defined as

$$
\begin{equation*}
x * y:=\beta(x) \cdot \beta(y) \quad \text { and } \quad\{x, y\}:=[\beta(x), \beta(y)], \forall x, y \in \mathcal{H}(P), \tag{11}
\end{equation*}
$$

$(P, *,\{\},, \varepsilon, \alpha)$ is a Hom-Poisson color algebra.
Proof. First, the $\varepsilon$-skew-symmetry is obvious to obtain. Next, let $x, y, z \in \mathcal{H}(P)$, then

$$
\begin{aligned}
& (x * y) * \alpha(z)-\alpha(x) *(y * z) \\
& =\beta(\beta(x) \cdot \beta(y)) \cdot \beta(\alpha(z))-\beta(\alpha(x)) \cdot \beta(\beta(y) \cdot \beta(z)) \\
& =\beta((\beta(x) \cdot \beta(y)) \cdot \alpha(\beta(z))-\alpha(\beta(x)) \cdot(\beta(y) \cdot \beta(z))) \\
& =\beta(0)=0,
\end{aligned}
$$

which is the Hom-associativity. Similarly, we get the color Hom-Jacobi identity as follows

$$
\begin{aligned}
& \varepsilon(z, x)\{\alpha(x),\{y, z\}\}+\varepsilon(x, y)\{\alpha(y),\{z, x\}\}+\varepsilon(y, z)\{\alpha(z),\{x, y\}\} \\
= & \varepsilon(z, x)[\beta(\alpha(x)), \beta([\beta(y), \beta(z)])]+\varepsilon(x, y)[\beta(\alpha(y)), \beta([\beta(z), \beta(x)])] \\
& +\varepsilon(y, z)[\beta(\alpha(z)), \beta([\beta(x), \beta(y)])] \\
= & \beta(\varepsilon(z, x)[\alpha(\beta(x)),[\beta(y), \beta(z)]]+\varepsilon(x, y)[\alpha(\beta(y)),[\beta(z), \beta(x)]] \\
& +\varepsilon(y, z)[\alpha(\beta(z)),[\beta(x), \beta(y)]]) \\
= & \beta(0)=0 .
\end{aligned}
$$

Finally, let us prove the Hom-Leibniz color identity as follows

$$
\begin{aligned}
\{\alpha(x), y * z\} & =[\beta(x), \beta(y * z)] \\
& =[\beta \alpha(x), \beta(\beta(y) \cdot \beta(z))] \\
& =\left[\alpha \beta(x), \beta^{2}(y) \cdot \beta(z)\right] \\
& =\left[\beta(x), \beta^{2}(y)\right] \cdot \alpha \beta(z)+\varepsilon(x, y) \alpha \beta^{2}(y) \cdot[\beta(x), \beta(z)] \\
& =\beta[\beta(x), \beta(y)] \cdot \beta \alpha(z)+\varepsilon(x, y) \alpha \beta^{2}(y) \cdot[\beta(x), \beta(z)] \\
& =\beta[\beta(x), \beta(y)] \cdot \beta \alpha(z)+\varepsilon(x, y) \beta^{2} \alpha(y) \cdot \beta([x, \beta(z)]) \\
& =\beta[\beta(x), \beta(y)] \cdot \beta \alpha(z)+\varepsilon(x, y) \beta(\beta \alpha(y) \cdot \beta([x, \beta(z)])) \\
& =\beta[\beta(x), \beta(y)] \cdot \beta \alpha(z)+\varepsilon(x, y) \beta(\beta \alpha(y) \cdot[\beta(x), \beta(z)])) \\
& =\beta[\beta(x), \beta(y)] \cdot \beta \alpha(z)+\varepsilon(x, y) \beta \alpha(y) \cdot \beta[\beta(x), \beta(z)] \\
& =\{x, y\} * \alpha(z)+\varepsilon(x, y) \alpha(y) *\{x, z\} .
\end{aligned}
$$

The following theorem is proved by a straighforward calculation.
Theorem 3.11. Let $(P, \cdot,[],, \varepsilon)$ be a Poisson color algebra and $\beta: P \rightarrow P$ an $\alpha^{0}$-averaging operator. Then with the products

$$
\begin{equation*}
x * y:=\beta(x) \cdot y \quad \text { and } \quad\{x, y\}:=[\beta(x), y], \forall x, y \in \mathcal{H}(P), \tag{12}
\end{equation*}
$$

$(P, *,\{\},, \varepsilon, \beta)$ becomes a Hom-Poisson color algebra.
Theorem 3.12. Let $(P, \cdot,[],, \varepsilon, \alpha)$ be a Hom-Poisson color algebra and $\beta: P \rightarrow P$ be an injective $\alpha^{k}$-averaging operator. Then with the products

$$
\begin{equation*}
x * y:=\beta(x) \cdot \alpha^{k}(y) \quad \text { and } \quad\{x, y\}:=\left[\beta(x), \alpha^{k}(y)\right], \forall x, y \in \mathcal{H}(P) \tag{13}
\end{equation*}
$$

$(P, *,\{\},, \varepsilon, \alpha)$ is a Hom-Poisson color algebra. Moreover, $\beta:(P, *,\{\},, \varepsilon, \alpha) \rightarrow$ ( $P, \cdot,[],, \varepsilon, \alpha)$ is a morphism of Hom-Poisson color algebras.

Proof. Note that the $\varepsilon$-skew-symmetry is obvious to prove. Next, to prove the Hom-associativity, pick $x, y, z \in \mathcal{H}(P)$ then

$$
\begin{aligned}
& \beta((x * y) * \alpha(z)-\alpha(x) *(y * z)) \\
& =\beta\left(\beta\left(\beta(x) \cdot \alpha^{k}(y)\right) \cdot \alpha^{k+1}(z)-\beta \alpha(x) \cdot \alpha^{k}(\beta(y) \cdot \beta(z))\right) \\
& =\beta\left(\left(\beta(x) \cdot \alpha^{k}(y)\right) \cdot \alpha \beta(z)-\alpha \beta(x) \cdot \beta\left(\beta(y) \cdot \alpha^{k}(z)\right)\right. \\
& =(\beta(x) \cdot \beta(y)) \cdot \alpha(\beta(z))-\alpha(\beta(x)) \cdot(\beta(y) \cdot \beta(z)) \\
& =0 .
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& \beta(\varepsilon(z, x)\{\alpha(x),\{y, z\}\}+\varepsilon(x, y)\{\alpha(y),\{z, x\}\}+\varepsilon(y, z)\{\alpha(z),\{x, y\}\}) \\
&= \beta\left(\varepsilon(z, x)\left[\beta \alpha(x), \alpha^{k}\left(\left[\beta(y), \alpha^{k}(z)\right]\right)\right]+\varepsilon(x, y)\left[\beta \alpha(y), \alpha^{k}\left(\left[\beta(z), \alpha^{k}(x)\right]\right)\right]\right. \\
&\left.+\varepsilon(y, z)\left[\beta \alpha(z), \alpha^{k}\left(\left[\beta(x), \alpha^{k}(y)\right]\right)\right]\right)=\varepsilon(z, x)\left[\beta \alpha(x), \beta\left(\left[\beta(y), \alpha^{k}(z)\right]\right)\right] \\
&+\varepsilon(x, y)\left[\beta \alpha(y), \beta\left(\left[\beta(z), \alpha^{k}(x)\right]\right)\right]+\varepsilon(y, z)\left[\beta \alpha(z), \beta\left(\left[\beta(x), \alpha^{k}(y)\right]\right)\right] \\
&= \varepsilon(z, x)[\alpha(\beta(x)),[\beta(y), \beta(z)]]+\varepsilon(x, y)[\alpha(\beta(y)),[\beta(z), \beta(x)]] \\
&+\varepsilon(y, z)[\alpha(\beta(z)),[\beta(x), \beta(y)]] \\
&= \beta(0)=0
\end{aligned}
$$

which is the color Hom-Jacobi identity. Finally, let us prove the Hom-Leibniz color identity as follows

$$
\begin{aligned}
\beta(\{\alpha(x), y * z\}) & =\left[\beta \alpha(x), \alpha^{k}(y * z)\right]=\left[\beta \alpha(x), \alpha^{k}\left(\beta(y) \cdot \alpha^{k}(z)\right)\right] \\
& =\left[\beta \alpha(x), \beta\left(\beta(y) \cdot \alpha^{k}(z)\right)\right]=[\alpha \beta(x), \beta(y) \cdot \beta(z)] \\
& =[\beta(x), \beta(y)] \cdot \alpha \beta(z)+\varepsilon(x, y) \alpha \beta(y) \cdot[\beta(x), \beta(z))] \\
& \left.=\beta\left[\beta(x), \alpha^{k}(y)\right] \cdot \alpha \beta(z)+\varepsilon(x, y) \alpha \beta(y) \cdot \beta\left[\beta(x), \alpha^{k}(z)\right)\right] \\
& \left.=\beta\left(\beta\left[\beta(x), \alpha^{k}(y)\right] \cdot \alpha^{k+1}(z)+\varepsilon(x, y) \beta \alpha(y) \cdot \alpha^{k}\left[\beta(x), \alpha^{k}(z)\right)\right]\right) \\
& =\beta\left(\beta\{x, y\} \cdot \alpha^{k+1}(z)+\varepsilon(x, y) \beta \alpha(y) \cdot \alpha^{k}\{x, z\}\right) \\
& =\beta(\{x, y\} * \alpha(z)+\varepsilon(x, y) \alpha(y) *\{x, z\}) .
\end{aligned}
$$

Theorem 3.13. Let $(P, \cdot,[],, \varepsilon, \alpha)$ be a Hom-Poisson color algebra and the map $R: P \rightarrow P$ be a Rota-Baxter operator of weight $\lambda \in \mathbb{K}$ on $P$. Then $P$ is a Hom-Poisson color algebra with the multiplications:

$$
\begin{aligned}
x * y & :=R(x) \cdot y+x \cdot R(y)+\lambda x \cdot y, \\
\{x, y\} & :=[R(x), y]+[x, R(y)]+\lambda[x, y],
\end{aligned}
$$

for all $x, y \in \mathcal{H}(P)$. Moreover, $R$ is a morphism of Hom-Poisson color algebra $(P, *,\{\},, \varepsilon, \alpha)$ onto ( $P, \cdot,[],, \varepsilon, \alpha$ ).

Proof. First, let $x, y, z \in \mathcal{H}(P)$, then

$$
\begin{aligned}
& (x * y) * \alpha(z)=R(x * y) \cdot \alpha(z)-(x * y) \cdot R \alpha(z)+\lambda(x * y) \cdot \alpha(z) \\
& =(R(x) \cdot R(y)) \cdot \alpha(z)+(R(x) \cdot y) \alpha R(z)+(x \cdot R(y)) \cdot \alpha R(z)+\lambda(x * y) \cdot \alpha R(z) \\
& +\lambda(R(x) \cdot y) \cdot \alpha(z)+\lambda(x \cdot R(y)) \cdot \alpha(z)+\lambda(x \cdot y) \cdot \alpha(z) \\
& =\alpha R(x) \cdot(R(y) \cdot z)+\alpha R(x) \cdot(y \cdot R(z))+\alpha(x) \cdot(R(y) \cdot R(z))+\lambda \alpha(x) \cdot(y \cdot R(z)) \\
& +\lambda \alpha R(x) \cdot(y \cdot z)+\lambda \alpha(x) \cdot(R(y) \cdot z)+\lambda(x \cdot y) \cdot \alpha(z),
\end{aligned}
$$

and also,

$$
\begin{aligned}
& \alpha(x) *(y * z)=R \alpha(x) \cdot(y * z)+\alpha(x) \cdot(y * z)+\lambda \alpha(x) \cdot(y * z) \\
& =\alpha R(x) \cdot(R(y) \cdot z)+\alpha R(x) \cdot(y \cdot R(z))+\alpha(x) \cdot(R(y) \cdot R(z))+\lambda \alpha(x) \cdot(y \cdot R(z)) \\
& +\lambda \alpha R(x) \cdot(y \cdot z)+\lambda \alpha(x) \cdot(R(y) \cdot z)+\lambda(x \cdot y) \cdot \alpha(z) \\
& \quad \text { using (1), (2) and rearranging terms) }
\end{aligned}
$$

therefore, the Hom-associativity holds. Next, we get by using Equation (3) that

$$
\begin{aligned}
& \varepsilon(z, x)\{\alpha(x),\{y, z\}\}=\varepsilon(z, x)([R \alpha(x),\{y, z\}]+[\alpha(x), R(\{y, z\})]+\lambda[\alpha(x),\{y, z\}]) \\
& =\varepsilon(z, x)([\alpha R(x),[R(y), z]]+[\alpha R(x),[y, R(z)]]+\lambda[\alpha R(x),[y, z]]+[\alpha(x),[R(y), R(z)]] \\
& \left.\lambda[\alpha(x),[R(y), z]]+\lambda[\alpha(x),[y, R(z)]]+\lambda^{2}[\alpha(x),[y, z]]\right)
\end{aligned}
$$

and similarly, after rearranging terms

$$
\begin{aligned}
& \varepsilon(x, y)\{\alpha(y),\{z, x\}\} \\
& =\varepsilon(x, y)([\alpha R(y),[z, R(x)]]+[\alpha(y),[R(z), R(x)]]+\lambda[\alpha(y),[z, R(x)]] \\
& \left.+[\alpha R(y),[R(z), x]]+\lambda[\alpha R(y),[z, x]]+\lambda[\alpha(y),[R(z), x]]+\lambda^{2}[\alpha(y),[z, x]]\right), \\
& \varepsilon(y, z)\{\alpha(z),\{x, y\}\} \\
& =\varepsilon(y, z)([\alpha R(z),[x, R(y)]]+[\alpha(z),[R(x), R(y)]]+\lambda[\alpha(z),[x, R(y)]] \\
& \left.+[\alpha R(z),[R(x), y]]+\lambda[\alpha R(z),[x, y]]+\lambda[\alpha(z),[R(x), y]]+\lambda^{2}[\alpha(z),[x, y]]\right) .
\end{aligned}
$$

Then adding memberwise these two previous equalities, we observe that the color Hom-Jacobi identity in ( $P, *,\{\},, \varepsilon, \alpha$ ) follows from the one in ( $P, \cdot,[],, \varepsilon, \alpha$ ).

Finally, let us prove the Hom-Leibniz color identity as follows

$$
\begin{aligned}
& \{\alpha(x), y * z\} \\
= & \{\alpha(x), R(y) z+y R(z)+\lambda y z\} \\
= & {[R \alpha(x), R(y) z+y R(z)+\lambda y z]+[\alpha(x), R(R(y) z+y R(z)+\lambda y z)] } \\
& +\lambda[\alpha(x), R(y) z+y R(z)+\lambda y z] \\
= & {[R \alpha(x), R(y) z]+[R \alpha(x), y R(z)]+\lambda[R \alpha(x), y z]+[\alpha(x), R(y) R(z)] } \\
& +\lambda[\alpha(x), R(y) z]+\lambda[\alpha(x), y R(z)]+\lambda^{2}[\alpha(x), y z] .
\end{aligned}
$$

By the Hom-Leibniz the color identity in $(P, \cdot,[],, \varepsilon, \alpha)$,

$$
\begin{aligned}
& \{\alpha(x), y * z\} \\
= & {[R(x), R(y)] \alpha(z)+\varepsilon(x, y) \alpha(R(y))[R(x), z]+[R(x), y] \alpha(R(z)) } \\
& +\varepsilon(x, y) \alpha(y) \cdot[R(x), R(z)]+\lambda[R(x), y] \alpha(z)+\lambda \varepsilon(x, y) \alpha(y) \cdot[R(x), z] \\
& +[x, R(y)] \alpha(R(z))+\varepsilon(x, y) \alpha(R(y))[x, R(z)]+\lambda[x, R(y)] \alpha(z) \\
& +\varepsilon(x, y) \lambda \alpha R(y)[x, z]+\lambda[x, y] \alpha(R(z))+\lambda \varepsilon(x, y) \alpha(y)[x, R(z)] \\
& +\lambda^{2}[x, y] \alpha(z)+\lambda^{2} \varepsilon(x, y) \alpha(y)[x, z] .
\end{aligned}
$$

By reorganizing the terms, we have

$$
\begin{aligned}
& \{\alpha(x), y * z\} \\
= & {[R(x), R(y)] \alpha(z)+[R(x), y] \alpha(R(z))+\lambda[x, y] \alpha(R(z))+[x, R(y)] \alpha(R(z)) } \\
& +\lambda[R(x), y] \alpha(z)+\lambda[x, R(y)] \alpha(z)+\lambda^{2}[x, y] \alpha(z) \\
& +\varepsilon(x, y) \alpha(R(y))[R(x), z]+\varepsilon(x, y) \alpha(R(y))[x, R(z)]+\varepsilon(x, y) \lambda \alpha R(y)[x, z] \\
& +\varepsilon(x, y) \alpha(y)[R(x), R(z)]+\lambda \varepsilon(x, y) \alpha(y)[R(x), z]+\lambda \varepsilon(x, y) \alpha(y)[x, R(z)] \\
& +\lambda^{2} \varepsilon(x, y) \alpha(y)[x, z] \\
= & R([R(x), y] \alpha(z)+[x, R(y)]+\lambda[x, y]) \alpha(z) \\
& +([R(x), y]+[x, R(y)]+\lambda[x, y]) \alpha(R(z)) \\
& +\lambda([R(x), y]+[x, R(y)]+\lambda[x, y]) \alpha(z) \\
& +\varepsilon(x, y) \alpha(R(y))([R(x), z]+[x, R(z)]+\lambda[x, z])+\varepsilon(x, y) \alpha(y)[R(x), R(z)] \\
& +\lambda \varepsilon(x, y) \alpha(y)([R(x), z]+[x, R(z)]+\lambda[x, z]) . \\
= & ([R(x), y]+[x, R(y)]+\lambda[x, y]) * \alpha(z) \\
& +\varepsilon(x, y) \alpha(y) *([R(x), z]+[x, R(z)]+\lambda[x, z]) . \\
= & \{x, y\} * \alpha(z)+\varepsilon(x, y) \alpha(y) *\{x, z\} .
\end{aligned}
$$

The following result can be proved easily.

Theorem 3.14. Let $(P, \cdot,[],, \varepsilon, \alpha)$ be a Hom-Poisson color algebra over a field $\mathbb{K}$ and suppose $\mathbb{K}$ an extension of $\mathbb{K}$. Then, the graded $\mathbb{K}$-vector space

$$
\hat{\mathbb{K}} \otimes P=\sum_{g \in G}(\mathbb{K} \otimes P)_{g}=\sum_{g \in G} \mathbb{K} \otimes P_{g}
$$

is a Hom-Poisson color algebra with
(1) the associative product $(\xi \otimes x) \cdot^{\prime}(\eta \otimes y):=\xi \eta \otimes(x \cdot y)$,
(2) the bracket $[\xi \otimes x, \eta \otimes y]^{\prime}:=\xi \eta \otimes[x, y]$,
(3) the even linear map $\alpha^{\prime}(\xi \otimes x):=\xi \otimes \alpha(x)$,
(4) the bicharacter $\varepsilon(\xi+x, \eta+y):=\varepsilon(x, y)$,
for all $\xi, \eta \in \hat{\mathbb{K}}$ and $x, y \in \mathcal{H}(P)$.
Theorem 3.15. Let $\left(A, \cdot, \varepsilon, \alpha_{A}\right)$ be a commutative Hom-associative color algebra and $\left(P, *,[],, \varepsilon, \alpha_{P}\right)$ be a Hom-Poisson color algebra. Then the tensor product $A \otimes P$ endowed with the even linear map $\alpha=\alpha_{A} \otimes \alpha_{P}: A \otimes P \rightarrow A \otimes P$, the even bilinear maps $\diamond:(A \otimes P) \times(A \otimes P) \rightarrow A \otimes P$ and $\{\}:,(A \otimes P) \times(A \otimes P) \rightarrow A \otimes P$ defined, for any $a, b \in \mathcal{H}(A), x, y \in \mathcal{H}(P)$, by
(1) $\alpha(a \otimes x):=\alpha_{A}(a) \otimes \alpha_{P}(x)$,
(2) $(a \otimes x) \diamond(b \otimes y):=\varepsilon(x, b)(a \cdot b) \otimes(x * y)$,
(3) $\{a \otimes x, b \otimes y\}:=\varepsilon(x, b)(a \cdot b) \otimes[x, y]$,
is a Hom-Poisson color algebra.
Proof. First, let $a, b, c \in \mathcal{H}(A)$ and $x, y, z \in \mathcal{H}(P)$. By the Hom-associativity of . and $*$, we get:

$$
\begin{aligned}
& ((a \otimes x) \diamond(b \otimes y)) \diamond \alpha(c \otimes z) \\
& =\varepsilon(x, b) \varepsilon(x+y, c)(a \cdot b) \cdot \alpha_{A}(c) \otimes(x * y) * \alpha_{P}(z) \\
& =\varepsilon(x, b) \varepsilon(x, c) \varepsilon(y, c)(a \cdot b) \cdot \alpha_{A}(c) \otimes(x * y) * \alpha_{P}(z) \\
& =\alpha(a \otimes x) \diamond((b \otimes y) \diamond(c \otimes z))
\end{aligned}
$$

Hence the Hom-associativity of $\diamond$ holds. Next, we get

$$
\begin{aligned}
& \varepsilon(c+z, a+x)\{\alpha(a \otimes x),\{b \otimes y, c \otimes z\}\} \\
& =\varepsilon(c, a) \varepsilon(x, b) \varepsilon(y, c) \varepsilon(z, a) \varepsilon(z, x)\left(\alpha_{A}(a) \cdot(b \cdot c) \otimes\left[\alpha_{P}(x),[y, z]\right]\right)
\end{aligned}
$$

and similarly, by the $\varepsilon$-commutativity and the Hom-associativity of $\cdot$, we get

$$
\begin{aligned}
& \varepsilon(a+x, b+y)\{\alpha(b \otimes y),\{c \otimes z, a \otimes x\}\} \\
& =\varepsilon(c, a) \varepsilon(x, b) \varepsilon(y, c) \varepsilon(z, a) \varepsilon(x, y)\left(\alpha_{A}(a) \cdot(b \cdot c) \otimes\left[\alpha_{P}(y),[z, x]\right]\right) \\
& \varepsilon(b+y, c+z)\{\alpha(c \otimes z),\{a \otimes x, b \otimes y\}\} \\
& =\varepsilon(c, a) \varepsilon(x, b) \varepsilon(y, c) \varepsilon(z, a) \varepsilon(y, z)\left(\alpha_{A}(a) \cdot(b \cdot c) \otimes\left[\alpha_{P}(z),[x, y]\right]\right) .
\end{aligned}
$$

Thus the color Hom-Jacobi identity in $(P, \diamond,\{\},, \varepsilon, \alpha)$ follows from the one in $(P, *,[],, \varepsilon, \alpha)$. Finally, let us prove the Hom-Leibniz color identity as follows

$$
\begin{aligned}
& \{\alpha(a \otimes x),(b \otimes y) \diamond(c \otimes z)\} \\
= & \varepsilon(y, c)\left\{\alpha_{A}(a) \otimes \alpha_{P}(x),(b \cdot c) \otimes(y * z)\right\} \\
= & \varepsilon(y, c) \varepsilon(x, b+c) \alpha_{A}(a) \cdot(b \cdot c) \otimes\left[\alpha_{P}(x), y * z\right] \\
= & \varepsilon(y, c) \varepsilon(x, b+c) \alpha_{A}(a) \cdot(b \cdot c) \otimes\left([x, y] * \alpha_{P}(z)+\varepsilon(x, y) \alpha_{P}(y) *[x, z]\right. \\
= & \varepsilon(y, c) \varepsilon(x, b) \varepsilon(x, c) \alpha_{A}(a) \cdot(b \cdot c) \otimes[x, y] * \alpha_{P}(z)+ \\
& \varepsilon(y, c) \varepsilon(x, b+c) \varepsilon(x, y)(a \cdot b) \cdot \alpha_{A}(c) \otimes \alpha_{P}(y) *[x, z] \\
= & \varepsilon(x, b) \varepsilon(x+y, c)(a \cdot b) \cdot \alpha_{A}(c) \otimes[x, y] * \alpha_{P}(z)+ \\
& \varepsilon(y, c) \varepsilon(x, b+c) \varepsilon(x, y) \varepsilon(a, b)(b \cdot a) \cdot \alpha_{A}(c) \otimes \alpha_{P}(y) *[x, z] \\
= & \varepsilon(x, b)(a \cdot b \otimes[x, y]) \diamond\left(\alpha_{A}(c) \otimes \alpha_{P}(z)\right)+ \\
& \varepsilon(y, c) \varepsilon(x, b) \varepsilon(x, y) \varepsilon(a, b)\left(\alpha_{A}(b) \otimes \alpha_{P}(y)\right) \diamond(a \cdot c \otimes[x, z]) \\
= & \{a \otimes x, b \otimes y\} \diamond \alpha(c \otimes z)+\varepsilon(a+x, b+y) \alpha(b \otimes y) \diamond\{a \otimes x, c \otimes z\} .
\end{aligned}
$$

## References

[1] N. Aizawa and H. Sato, $q$-deformation of the Virasoro algebra with central extension, Physics Letters B, Phys. Lett. B, 256 (1991), no. 1, 185-190.
[2] S. Attan, Some characterizations of color Hom-Poisson algebras, Hacettepe J. Math. Statist., 47 (2018), no. 6, 1552-1563.
[3] I. Bakayoko, Modules over color Hom-Poisson algebras, J. Gen. Lie Theory Appl., 8 (2014), no. 1, Art. ID ID 1000226, 7 pp.
[4] I. Bakayoko and B.M. Touré, Constructing Hom-Poisson color algebras, Intern. J. Algebra, 13 (2019), no. 1, 1-16
[5] M. Chaichian, P. Kulish and J. Lukierski, q-Deformed Jacobi identity, $q$ oscillators and $q$-deformed infinite-dimensional algebras, Phys. Lett. B, 237 (1990), no. 3-4, 401-406.
[6] T.L. Curtright and C.K. Zachos, Deforming maps for quantum algebras, Phys. Lett. B, 243 (1990), no. 3, 237-244.
[7] J.T. Hartwig, D. Larsson and S.D. Silvestrov, Deformations of Lie algebras using $\sigma$-derivations, J. Algebra 295 (2006), 314-361.
[8] N. Hu, $q$-Witt algebras, $q$-Lie algebras, $q$-holomorph structure and representations, Algebra Colloq. 6(1999), no. 1, 51-70.
[9] C. Kassel, Cyclic homology of differential operators, the Virasoro algebra and a q-analogue, Commun. Math. Phys. 146 (1992), 343-351.
[10] A. Makhlouf and S. Silvestrov, Notes on formal deformations of Hom-Associative and Hom-Lie algebras, Forum Math., 22 (2010), no. 4, 715-759.
[11] L. Yuan, Hom-Lie color algebras, Commun. Algebra, 40 (2012), 575-592.

Received November 06, 2020
I. Bakayoko

D épartement de Mathématiques, Université de N'Zérékoré, BP:50, N'Zérékoré, Guinée
E-mail: ibrahimabakayoko27@gmail.com
S. Attan

Département de Mathématiques, Université d'Abomey Calavi, 01 BP 4521, Cotonou 01, Bénin. E-mail: syltane2010@yahoo.fr

# Biquasigroups linear over a group 

Wieslaw A. Dudek and Robert A. R. Monzo


#### Abstract

We determine the structure of biquasigroups $(Q, \circ, *)$ satisfying variations of Polonijo's Ward double quasigroup identity $(x \circ z) *(y \circ z)=x * y$, including those that are linear over a group.


## 1. Introduction

J.M. Cardoso and C.P. da Silva, inspired by Ward's paper [11] on postulating the inverse operations in groups, introduced in [1] the notion of Ward quasigroups as quasigroups $(Q, \circ)$ containing an element $e$ such that $x \circ x=e$ for all $x \in Q$, and satisfying the identity $(x \circ y) \circ z=x \circ(z \circ(e \circ y))$. Polonijo [8] proved that these two conditions can be replaced by the identity:

$$
\begin{equation*}
(x \circ z) \circ(y \circ z)=x \circ y \tag{1}
\end{equation*}
$$

In [1] it is proved that if $(Q, \circ)$ is a Ward quasigroup, then $(Q, \cdot)$, where $x \cdot y=$ $x \circ(e \circ y)$, is a group in which $e=x \circ x$ and $x^{-1}=e \circ x$ for all $x \in Q$. Also, $x \circ e=x$, $e \circ(e \circ x)=x$ and $e \circ(x \circ y)=y \circ x$. Conversely, if $(Q, \cdot)$ is a group, then $Q$ with the operation $x \circ y=x \cdot y^{-1}$ is a Ward quasigroup (cf. [11]). Other characterizations of Ward quasigroups can be found in [2] and [10], some applications in [5]. Note that the Ward quasigroups corresponding to commutative groups sometimes are called subtractive quasigroups (cf. [6] and [12]). A Ward quasigroup $(Q, \circ)$ is subtractive if and only if it is medial (that is, it satisfies the identity $(x \circ y) \circ(z \circ w)=(x \circ z) \circ(y \circ w))$ if and only if it is left modular (that is, it satisfies the identity $x \circ(y \circ z)=z \circ(y \circ x))$ (cf. Lemma 2.4, [3]).

A biquasigroup, i.e. an algebra of the form $(Q, \circ, *)$ where $(Q, \circ)$ and $(Q, *)$ are quasigroups, is called a Ward double quasigroup if it satisfies the identity

$$
\begin{equation*}
(x \circ z) *(y \circ z)=x * y . \tag{2}
\end{equation*}
$$

Obviously each Ward quasigroup ( $Q, \circ$ ) can be considered as a Ward double quasigroup of the form $(Q, \circ, \circ)$. Ward double quasigroups have a similar characterization as Ward quasigroups.

Theorem 1.1. (cf. [7]) A biquasigroup $(Q, \circ, *)$ is a Ward double quasigroup if and only if there is a group $(Q,+)$ and bijections $\alpha, \beta$ on $Q$ such that $x \circ y=x-\beta y$ and $x * y=\alpha(x-y)$.

2010 Mathematics Subject Classification: 20M15, 20N02
Keywords: Biquasigroups, linear quasigroups, Ward quasigroups.

Note that Ward double quasigroups are distinct from the double Ward quasigroups considered by Fiala (cf. [4]).

Let us consider the identity (2). Keeping the variables $x, y$ and $z$ the same and varying only the quasigroup operations $\circ$ and $*$, there are sixteen possible identities. Eight of these have reversible versions obtained by replacing the operation - with the operation $*$ and, simultaneously, replacing the operation $*$ with the operation 0 .

For example, the identity $(x \circ z) *(y * z)=x \circ y$ has the reversible version $(x * z) \circ(y \circ z)=x * y$. So, if we are to consider all possible versions of Theorem 1.1, we need to explore the following identities:

$$
\begin{align*}
& (x \circ z) \circ(y \circ z)=x \circ y,  \tag{3}\\
& (x \circ z) \circ(y \circ z)=x * y,  \tag{4}\\
& (x \circ z) \circ(y * z)=x \circ y,  \tag{5}\\
& (x \circ z) *(y \circ z)=x \circ y,  \tag{6}\\
& (x \circ z) \circ(y * z)=x * y,  \tag{7}\\
& (x \circ z) *(y * z)=x \circ y,  \tag{8}\\
& (x \circ z) *(y * z)=x * y . \tag{9}
\end{align*}
$$

The biquasigroup ( $Q, \circ, \circ$ ) satisfies identity (3) if and only if ( $Q, \circ$ ) is a Ward quasigroup. Our interest is in finding non-trivial models of the other six identities, where 'non-trivial' means that the set $Q$ has more than one element. In particular, since Ward quasigroups are unipotent, we will be interested in biquasigroups $(Q, \circ, *)$ where $(Q, \circ)$ or $(Q, *)$ is unipotent, both are unipotent or when one or both are Ward quasigroups.

Note that a biquasigroup $(Q, \circ, *)$, where $(Q, *)$ is a commutative group and $x \circ y=x * y^{-1}$ satisfies identities (1) through (9) if and only if $(Q, *)$ is a Boolean group.

## 2. Main Results

We will now characterize the biquasigroups satisfying the identities (2) to (9). First we will describe their general properties then we will characterize biquasigroups linear over a group and satisfying identities (2) to (9).

1. Recall that a quasigroup $(Q, \cdot)$ is linear over a group (cf. [9]) if there exists a group $(Q,+)$, its automorphisms $\varphi, \psi$ and $a \in Q$ such that $x \cdot y=\varphi x+a+\psi y$ for all $x, y \in Q$. Consequently, a biquasigroup $(Q, \circ, *)$ will be called linear over a group if both its quasigroups $(Q, \circ)$ and $(Q, *)$ are linear over the same group, i.e. if there is a group $(Q,+)$, its automorphisms $\varphi, \psi, \alpha, \beta$ and elements $a, b \in Q$ such that

$$
x \circ y=\varphi x+a+\psi y \quad \text { and } \quad x * y=\alpha x+b+\beta y .
$$

According to the Toyoda Theorem (cf. [9]), a quasigroup $(Q, \cdot)$ is medial if and only if it is linear over a commutative group with commuting automorphisms $\varphi, \psi$. In an analogous way we can shows that a quasigroup $(Q, \cdot)$ is paramedial, that is it satisfies the identity $(x \cdot y) \cdot(z \cdot u)=(u \cdot y) \cdot(z \cdot x)$ if and only if it is linear over a commutative group with automorphisms $\varphi, \psi$ such that $\varphi^{2}=\psi^{2}$. Based on these facts we say that a biquasigroup $(Q, \circ, *)$ is medial (paramedial) if both its quasigroups $(Q, \circ)$ and $(Q, *)$ are medial (paramedial) and linear over the same commutative group.

A biquasigroup $(Q, \circ, *)$ is unipotent if there is $q \in Q$ such that $x \circ x=q=x * x$ for and all $x \in Q$. If both quasigroups $(Q, \circ)$ and $(Q, *)$ are idempotent then we say that $(Q, \circ, *)$ is an idempotent biquasigroup.
2. We will start with biquasigroups satisfying the identity (2).

A general characterization of such biquasigroups is given by Theorem 1.1. Now we describe a biquasigroup linear over a group $(Q,+)$ and satisfying identity (2).

From (2) for $x=y=z=0$ we obtain $\alpha a+b+\beta a=b$, This together with (2) implies $\varphi=\varepsilon$ (the identity map). Thus

$$
\alpha x+\alpha a+\alpha \psi z+b+\beta y+\beta a+\beta \psi z=\alpha x+b+\beta y
$$

This for $z=0$ gives

$$
\alpha a+b+\beta y+\beta a=b+\beta y=\alpha a+b+\beta a+\beta y .
$$

So $\beta y+\beta a=\beta a+\beta y$, i.e. $a$ is in the center $Z(Q,+)$ of the group $(Q,+)$. Thus using (2) and the above facts we obtain $\alpha \psi z+b+\beta y+\beta \psi z=b+\beta y$. Hence $\alpha v+u+\beta v=u$ for all $u, v \in Q$. Thus $\beta=-\alpha$ and consequently $\alpha v+u=u+\alpha v$ for all $u, v \in Q$, which means that $(Q,+)$ is a commutative group.

In this way we have proved the "only if" part of the following Theorem. The second part is trivial.

Theorem 2.1. A biquasigroup $(Q, \circ, *)$ linear over a group $(Q,+)$ is a Ward double quasigroup (that is, it satisfies (2)) if and only if $(Q,+)$ is a commutative group, $x \circ y=x+\psi y+a$ and $x * y=\alpha x-\alpha y+b$.

Obviously such a biquasigroup is medial. The quasigroup $(Q, \circ)$ has a right neutral element and the quasigroup $(Q, *)$ is unipotent. Moreover, a biquasigroup $(Q, \circ, *)$ satisfying (2) is paramedial if and only if $\psi^{2}=\varepsilon$.
3. Now consider biquasigroups satisfying the identity (3).

Since this identity contains only one operation, it is enough to examine the quasigroup $(Q, \circ)$. Quasigroups satisfying (3) were characterized at the beginning of this paper. If a quasigroup $(Q, \circ)$ linear over a group $(Q,+)$ satisfies $(3)$, then $\varphi=\varepsilon$ and $a+\psi a=0$. So (3) for $y=0$, can be reduced to $\psi z+\psi^{2} z=0$. This means that $\psi z=-z$ and $(Q,+)$ is a commutative group. Consequently $x \circ y=x-y+a$.

Theorem 2.2. A quasigroup $(Q, \circ)$ linear over a group $(Q,+)$ satisfies (3) if and only if $(Q,+)$ is a commutative group and $x \circ y=x-y+a$ for some fixed $a \in Q$.

This quasigroup is medial, paramedial, unipotent and has a right neutral element.

Note that $(Q, \circ)$ is a Ward quasigroup if and only if there is a group $(Q,+)$ and an element $a \in Q$ such that $x \circ y=x-y+a$. The group $(Q,+)$ need not be commutative.
4. Will now consider a biquasigroup $(Q, \circ, *)$ satisfying the identity (4), i.e.

$$
(x \circ z) \circ(y \circ z)=x * y
$$

Theorem 2.3. If a biquasigroup $(Q, \circ, *)$ satisfies the identity (4), then both quasigroups $(Q, \circ)$ and $(Q, *)$ are unipotent with $q \in Q$ such that $x \circ x=q=x * x$ and $x * y=(x \circ y) \circ q=q \circ(y \circ x)$ for all $x, y \in Q$.
Proof. If $(Q, \circ)$ is idempotent, then $x=(x \circ x) \circ(x \circ x)=x * x$. So $(Q, *)$ is idempotent too. If $(Q, *)$ is idempotent, then $x=x * x=(x \circ z) \circ(x \circ z)$ for all $x, z \in Q$. In particular, for $z=x^{\prime} \in Q$ such that $x=x \circ x^{\prime}$ we obtain $x=x \circ x$. This shows that both these quasigroups are idempotent or none of them are idempotent.

If both are idempotent, then $x \circ z=(x \circ z) \circ(x \circ z)=x * x=x=x \circ x$ for all $x, z \in Q$, which implies $x=z$. Hence $Q$ has only one element. So it is unipotent.

Now suppose both quasigroups $(Q, \circ)$ and $(Q, *)$ are not idempotent. Then there exists $b \in Q$ such that $b * b=q \neq b$ and for any $x \in Q$ there exist $x^{\prime}, x^{\prime \prime} \in Q$ such that $b \circ x^{\prime}=x$ and $x \circ x^{\prime \prime}=x$. Then $x \circ x=\left(b \circ x^{\prime}\right) \circ\left(b \circ x^{\prime}\right)=b * b=q$ and $x * x=\left(x \circ x^{\prime \prime}\right) \circ\left(x \circ x^{\prime \prime}\right)=x \circ x=q$. Hence, $(Q, \circ)$ and $(Q, *)$ are unipotent, with $q=x \circ x=x * x$ for all $x \in Q$. Also, $x * y=(x \circ x) \circ(y \circ x)=q \circ(y \circ x)$ and $x * y=(x \circ y) \circ(y \circ y)=(x \circ y) \circ q$.

Corollary 2.4. If a biquasigroup $(Q, \circ, *)$ satisfies (4) and $(Q, \circ)$ has a right neutral element, then $(Q, \circ)=(Q, *)$ is a Ward quasigroup. If $(Q, \circ)$ has a left neutral element, then $x * y=y \circ x$. If $(Q, \circ)$ has a neutral element, then $(Q, \circ)=$ $(Q, *)$ is a commutative Ward quasigroup.

Any medial unipotent quasigroup ( $Q, \circ$ ) can be 'extended' to a medial unipotent biquasigroup ( $Q, \circ, *$ ) satisfying the identity (4), as follows.

Proposition 2.5. If $(Q, \circ)$ is a medial unipotent quasigroup, then $(Q, \circ, *)$, where $x \circ x=q$ and $x * y=(x \circ y) \circ q$ for all $x, y \in Q$, is a biquasigroup satisfying (4). Moreover, if $q$ is a left neutral element of $(Q, \circ)$, then $x * y=y \circ x$.

Proof. Indeed, $(Q, *)$ is a quasigroup and $x * y=(x \circ y) \circ q=(x \circ y) \circ(z \circ z)=$ $(x \circ z) \circ(y \circ z)$. Also, if $q$ is a left neutral element of $(Q, \circ)$, then $x * y=(x \circ y) \circ q=$ $(x \circ y) \circ(x \circ x)=(x \circ x) \circ(y \circ x)=y \circ x$.

Let $(Q, \circ, *)$ be a biquasigroup linear over a group $(Q,+)$. If it satisfies (4), then $\varphi a+a+\psi a=b$ and $\alpha=\varphi^{2}$. So (4) for $x=y=0$ and $\psi z=a$ gives $2 \varphi a+a+2 \psi a=b=\varphi a+a+\psi a$ which implies $a=b$. Consequently $a \circ a=a$. Thus, by Theorem 2.3, $a=q$ and $x * y=a \circ(y \circ x)$. Hence

$$
x * 0=a \circ(0 \circ x)=\alpha x+a=\varphi a+a+\psi a+\psi^{2} x=a+\psi^{2} x
$$

and

$$
a=x \circ x=\alpha x+a+\beta x=a+\psi^{2} x+\beta x .
$$

This gives $\psi^{2} x+\beta x=0$, i.e. $\beta=-\psi^{2}$. Hence $x * y=\varphi^{2} x+a-\psi^{2} y$. Since $x \circ x=a=z * z$ we also have $\varphi x+a=a-\psi x$ and $\varphi^{2} z+a=a+\psi^{2} z$. This for $x=\varphi z$ gives $\varphi^{2} z+a=a-\psi \varphi z$. Hence $a+\psi^{2} z=a-\psi \varphi z$. Consequently, $\psi=-\varphi$ and $x \circ y=\varphi x+a-\varphi y$. So $a=x \circ x=\varphi x+a-\varphi x$. Thus $a \in Z(Q,+)$. Also $\varphi^{2}=\psi^{2}$.

Therefore, $x \circ y=\varphi x+a-\varphi y$ and $x * y=\varphi^{2} x+a-\varphi^{2} y$. Inserting these operations to (4) we obtain $-\varphi^{2} z-\varphi^{2} y+\varphi^{2} z=-\varphi^{2} y$ for all $y, z \in Q$. Hence $(Q,+)$ is a commutative group. Consequently $(Q, \circ, *)$ is medial and unipotent. This proves the "only if" part of the Theorem 2.6 below. The proof of the "if" part follows from a direct calculation and is omitted

Theorem 2.6. A biquasigroup $(Q, \circ, *)$ linear over a group $(Q,+)$ satisfies the identity (4) if and only if $(Q,+)$ is a commutative group, $x \circ y=\varphi x+a-\varphi y$ and $x * y=\varphi^{2} x+a-\varphi^{2} y$.

It is clear that such a biquasigroup is medial and paramedial. If $a=0$ then it is unipotent.
5. Will now consider a biquasigroup $(Q, \circ, *)$ satisfying the identity (5), i.e.

$$
(x \circ z) \circ(y * z)=x \circ y .
$$

Theorem 2.7. If a biquasigroup $(Q, \circ, *)$ satisfies the identity (5), then both quasigroups $(Q, \circ)$ and $(Q, *)$ have only one idempotent. This idempotent is a right neutral element of these quasigroups. Moreover, $(Q, *)$ is unipotent.

Proof. For each $x \in Q$ there is uniquely determined $\bar{x} \in Q$ such that $x \circ \bar{x}=x$. Then for $x, y \in Q$, by (5), we have

$$
x \circ y=(x \circ \bar{x}) \circ(y * \bar{x})=x \circ(y * \bar{x}) .
$$

So, $y=y * \bar{x}$ for each $y \in Q$. Also $y \circ y=(y \circ \bar{x}) \circ(y * \bar{x})=(y \circ \bar{x}) \circ y$, hence $y=y \circ \bar{x}$ for all $y \in Q$. Thus $e=\bar{x}$ is a right neutral element of $(Q, \circ)$ and $(Q, *)$. There are no other idempotents in $(Q, \circ)$ and $(Q, *)$. Indeed, if $a * a=a$, then for each $x \in Q$

$$
x \circ a=(x \circ a) \circ(a * a)=(x \circ a) \circ a,
$$

so $x \circ a=x=x \circ e$. Hence $a=e$. Similarly for $a \circ a=a$ we have

$$
a \circ x=(a \circ a) \circ(x * a)=a \circ(x * a),
$$

which implies $x * a=x=x * e$, so also in this case $a=e$.
For each $x \in Q$ there exists $x^{\prime} \in Q$ such that $x * x=x \circ x^{\prime}$. Thus, by (5),

$$
(x \circ x) \circ e=x \circ x=(x \circ x) \circ(x * x)=(x \circ x) \circ\left(x \circ x^{\prime}\right),
$$

which implies $e=x \circ x^{\prime}=x * x$. So, $(Q, *)$ is unipotent.
The following example shows that $(Q, \circ)$ may not be unipotent.
Example 2.8. Let $(Q, \circ)$ be a group. Then $(Q, \circ, *)$, where $x * y=y^{-1} \circ x$ is an example of a biquasigroup satisfying (5) in which only one of quasigroups $(Q, \circ)$ and $(Q, *)$ has a left neutral element. Moreover, $(Q, *)$ is unipotent but $(Q, \circ)$ is unipotent only in the case when it is a Boolean group.

Corollary 2.9. If in a biquasigroup $(Q, \circ, *)$ satisfying (5) one of quasigroups $(Q, \circ)$ or $(Q, *)$ is idempotent, then $Q$ has only one element.

Proposition 2.10. Let $(Q, \circ, *)$ be a biquasigroup satisfying (5). If $(Q, \circ)$ is a Ward quasigroup, then $(Q, \circ)=(Q, *)$.
Proof. Since $(Q, \circ)$ is a Ward quasigroup, there exists a group $(Q, \cdot)$ such that $x \circ y=x \cdot y^{-1}$ and $e \circ(x \circ y)=y \circ x$, where $e$ is the neutral element of the group $(Q, \cdot)(\mathrm{cf}$. [1]). Then $x \circ y=(x \circ x) \circ(y * x)=e \circ(y * x)$ and so $x \circ y=e \circ(y \circ x)=$ $e \circ(e \circ(x * y))=x * y$. Hence $(Q, \circ)=(Q, *)$.

Proposition 2.11. Let $(Q, \circ, *)$ be a biquasigroup satisfying (5). If $(Q, \circ)$ is medial and unipotent, then $(Q, \circ)=(Q, *)$.
Proof. For every $x, y \in Q$ there exists $z \in Q$ such that $x * y=x \circ z$. Since $(Q, \circ)$ is medial,

$$
(x \circ x) \circ e=x \circ x=(x \circ y) \circ(x * y)=(x \circ y) \circ(x \circ z)=(x \circ x) \circ(y \circ z) .
$$

Thus, $y \circ z=e=y \circ y$, where $e$ is the right neutral element of $(Q, \circ)$. Therefore $y=z$ and consequently, $x * y=x \circ y$.

Proposition 2.12. A biquasigroup $(Q, \circ, *)$ linear over a group $(Q,+)$ satisfies (5) if and only if a group $(Q,+)$ is a commutative group, $x \circ y=x+\psi y-\psi b$ and $x * y=x-y+b$.

Proof. If a biquasigroup $(Q, \circ, *)$ linear over a group $(Q,+)$ satisfies (5), then $\varphi a+a+\psi b=a$ and $\varphi=\varepsilon$. Thus $a+\psi b=0$. This together with (5) for $y=0$ gives $\psi z+\psi \beta z=0$. So, $\beta z=-z$ for all $z \in Q$. Thus $(Q,+)$ is a commutative group. Consequently, $\alpha=\varepsilon$. Therefore, $x \circ y=x+\psi y-\psi b, x * y=x-y+b$.

The proof of the converse follows from a direct calculation and is omitted.

A biquasigroup ( $Q, \circ, *$ ) linear over a group and satisfying (5) is medial and both its quasigroups $(Q, \circ)$ and $(Q, *)$ have the same right neutral element. If $\psi^{2}=\varepsilon$ then this biquasigroup is also paramedial.
6. Will now consider a biquasigroup $(Q, \circ, *)$ satisfying the identity (6), i.e.

$$
(x \circ z) *(y \circ z)=x \circ y .
$$

Theorem 2.13. A biquasigroup $(Q, \circ, *)$ satisfies the identity (6) if and only if there is a group $(G, \cdot)$ and a bijection $\alpha$ on $Q$ such that $x \circ y=(\alpha x)^{-1} \cdot(\alpha y)$ and $x * y=x \cdot y^{-1}$.

Proof. $\Rightarrow$ : Let $x, y, z \in Q$. Then for fixed $q \in Q$ there are $x^{\prime}, y^{\prime}, z^{\prime} \in Q$ such that and $x=x^{\prime} \circ q, y=y^{\prime} \circ q$ and $z=z^{\prime} \circ q$. Then, $x * z=\left(x^{\prime} \circ q\right) *\left(z^{\prime} \circ q\right)=x^{\prime} \circ z^{\prime}$ and $y * z=\left(y^{\prime} \circ q\right) *\left(z^{\prime} \circ q\right)=y^{\prime} \circ z^{\prime}$. So,

$$
(x * z) *(y * z)=\left(x^{\prime} \circ z^{\prime}\right) *\left(y^{\prime} \circ z^{\prime}\right)=x^{\prime} \circ y^{\prime}=\left(x^{\prime} \circ q\right) *\left(y^{\prime} \circ q\right)=x * y .
$$

Therefore, $(Q, *)$ is a Ward quasigroup and there exists a group $(Q, \cdot)$ such that $x * y=x \cdot y^{-1}$ and $x \cdot y=x *(e * y)$, where $e=w * w$ for any $w \in Q$ and $x^{-1}=e * x$.

Let $x \in Q$. Then, $e=(x \circ z) *(x \circ z)=x \circ x$. So, $e *(x \circ y)=(y \circ y) *(x \circ y)=y \circ x$, for any $y \in Q$. Let $\alpha x=e \circ x$. Then, $(\alpha x)^{-1}=e *(e \circ x)=x \circ e$. Thus, $(\alpha x)^{-1} \cdot(\alpha y)=(x \circ e) \cdot(e \circ y)=(x \circ e) *(e *(e \circ y))=(x \circ e) *(y \circ e)=x \circ y$.
$\Leftarrow:$ Let $x, y, z \in Q$. Then, $(x \circ z) *(y \circ z)=\left[(\alpha x)^{-1} \cdot(\alpha z)\right] *\left[(\alpha y)^{-1} \cdot(\alpha z)\right]=$ $(\alpha x)^{-1} \cdot(\alpha z) \cdot(\alpha z)^{-1} \cdot(\alpha y)=(\alpha x)^{-1} \cdot(\alpha y)=x \circ y$.

Corollary 2.14. If a biquasigroup $(Q, \circ, *)$ satisfies the identity ( 6 ), then it is unipotent.

Corollary 2.15. If in a biquasigroup $(Q, \circ, *)$ satisfying the identity (6) one of quasigroups $(Q, \circ)$ and $(Q, *)$ is commutative, then also the second is commutative. In this case both quasigroups are induced by the same Boolean group.

Proposition 2.16. A biquasigroup $(Q, \circ, *)$ linear over a group $(Q,+)$ satisfies the identity (6) if and only if a group $(Q,+)$ is commutative, $x \circ y=\varphi x-\varphi y+a$ and $x * y=x-y+a$.

Proof. If a biquasigroup $(Q, \circ, *)$ linear over a group $(Q,+)$ satisfies the identity (6) then $\alpha a+b+\beta a=a$ and $\alpha=\varepsilon$. So, $b+\beta a=0$. Thus (6) for $x=y=0$ gives $\psi z+\beta \psi z=0$ which means that $\beta v=-v$ for each $v \in Q$. Hence $(Q,+)$ is commutative and $a=b$. Therefore $x * y=x-y+a$. Substituting this operation to (6) we obtain $x \circ y=\varphi x-\varphi y+a$.

The converse statement is obvious.
Corollary 2.17. A linear biquasigroup satisfying the identity (6) is unipotent.
7. Will now consider a biquasigroup $(Q, \circ, *)$ satisfying the identity (7), i.e.

$$
(x \circ z) \circ(y * z)=x * y
$$

A simple example of a biquasigroup $(Q, \circ, *)$ satisfying the identity (7) is a commutative group $(Q,+)$ with the operations $x \circ y=y-x$ and $x * y=x+y$. This biquasigroup is medial, both quasigroups $(Q, \circ)$ and $(Q, *)$ have left neutral element but only the first is unipotent.

Suppose now that in a biquasigroup $(Q, \circ, *)$ satisfying the identity (7) the first quasigroup is medial and the second is idempotent. Then, by Toyoda Theorem (cf. [9]), there exists a commutative group $(Q,+)$ and its commuting automorphisms $\varphi, \psi$ such that $x \circ y=\varphi x+\psi y+a$ for some fixed $a \in Q$. Then $x * y=(x \circ y) \circ(y * y)=$ $(x \circ y) \circ y=\varphi^{2} x+\varphi \psi y+\psi y+\varphi a+a$. This, by (7), implies $\varphi^{2}-\varphi=\varepsilon, \varphi+\varphi \psi+\psi=0$ and $\varphi a=-a$. Thus $a=0$ and $x * y=\varphi^{2} x+\varphi \psi y+\psi y$. Since $(Q, *)$ is idempotent, $\varphi^{2}+\varphi \psi+\psi=\varepsilon$. Hence $x * y=\varphi^{2} x+y-\varphi^{2} y=\varphi^{2} x-\varphi y$. Consequently $(Q, *)$ is medial. Therefore $(Q, \circ, *)$ is medial too.

In this way we have proved
Proposition 2.18. If in a biquasigroup $(Q, \circ, *)$ satisfying the identity (7) the first quasigroup is medial and the second is idempotent, then the second is medial too and there exists a commutative group $(Q,+)$ and its commuting automorphisms $\varphi, \psi$ such that $\varphi+\varphi \psi+\psi=0, \varphi^{2}=\varphi+\varepsilon, x \circ y=\varphi x+\psi y$ and $x * y=\varphi^{2} x-\varphi y$.

Conversely we have:
Proposition 2.19. Let $(Q,+)$ be a commutative group and $\varphi, \psi$ be its commuting automorphisms such that $\varphi+\varphi \psi+\psi=0$ and $\varphi^{2}=\varphi+\varepsilon$. Then $(Q, \circ, *)$, where $x \circ y=\varphi x+\psi y$ and $x * y=\varphi^{2} x-\varphi y$, is a medial biquasigroup satisfying the identity (7).
Proof. This is a straightforward calculation.
As a consequence of the above results we obtain
Corollary 2.20. An idempotent medial biquasigroup $(Q, \circ, *)$ satisfies the identity (7) if and only if there exist a commutative group $(Q,+)$ and its automorphism $\varphi$ such that $x \circ y=\varphi x+y-\varphi y, x * y=\varphi^{2} x-\varphi y$ and $\varphi^{2}=\varphi+\varepsilon$.

In the case of quasigroups induced by the group $\mathbb{Z}_{n}$ we have stronger result. For simplicity the value of the integer $t \geqslant 0$ modulo $n$ will be denoted by $[t]_{n}$.

Corollary 2.21. An idempotent medial biquasigroup induced by the group $\mathbb{Z}_{n}$ satisfies the identity (7) if and only if has the form $\left(\mathbb{Z}_{n}, \circ, *\right)$, where $x \circ y=$ $[a x+(1-a) y]_{n}, x * y=\left[a^{2} x+\left(1-a^{2}\right) y\right]_{n}$ and $\left[a^{2}-a\right]_{n}=1$.

Corollary 2.22. For every $a \geqslant 3$ there is an idempotent medial biquasigroup of order $n=a^{2}-a-1$ satisfying (7). It has the form $\left(\mathbb{Z}_{n}, \circ, *\right)$, where $x \circ y=$ $[a x+(1-a) y]_{n}$ and $x * y=[(a+1) x-a y]_{n}$, or $x \circ y=[(1-a) x+a y]_{n}$ and $x * y=[(2-a) x+(a-1) y]_{n}$.

Proposition 2.23. A medial biquasigroup $(Q, \circ, *)$ satisfies the identity (7) if and only if there exist a commutative group $(Q,+)$ and its commuting automorphisms $\varphi, \psi$ such that $x \circ y=\varphi x+\psi y+c, x * y=\varphi^{2} x-\varphi y+d, \varphi \psi+\varepsilon=0$ and $\varphi c+\psi d+c=d$ for some fixed $c, d \in Q$.

Proposition 2.24. A medial biquasigroup $\left(\mathbb{Z}_{n}, \circ, *\right)$ satisfies the identity (7) if and only if there exists $a, b, c, d \in \mathbb{Z}_{n}$ such that $[a b+1]_{n}=0,[a c+b d+c]_{n}=d$, $x \circ y=[a x+b y+c]_{n}$ and $\left.x * y=\left[a^{2} x-a y+d\right]\right)_{n}$.

For linear biquasigroup we have the following result.
Theorem 2.25. A biquasigroup $(Q, \circ, *)$ linear over a group $(Q,+)$ satisfies the identity (7) if and only if $(Q,+)$ is a commutative group, $x \circ y=\varphi x+\psi y+a$, $x * y=\varphi^{2} x+\psi \varphi^{2} y+b, \varphi \psi+\psi^{2} \varphi^{2}=0$ and $\varphi a+a+\psi b=b$.

Proof. If a biquasigroup $(Q, \circ, *)$ linear over a group $(Q,+)$ satisfies (7), then $\varphi a+a+\psi b=b$ and $\varphi^{2}=\alpha$. Thus (7) can be reduced to

$$
\varphi a+\varphi \psi z+a+\psi \alpha y+\psi b+\psi \beta z=b+\beta y
$$

which for $z=0$ gives $\varphi a+a+\psi \alpha y+\psi b=b+\beta y=(\varphi a+a+\psi b)+\beta y$. Hence $\psi \alpha y+\psi b=\psi b+\beta y$. So $\psi \alpha y=\psi b+\beta y-\psi b$. Therefore the previous identity implies $\varphi \psi z+a+\psi b+\beta y+\psi \beta z=a+\psi b+\beta y$. Since every element $v \in Q$ can be presented in the form $v=a+\psi b+\beta y$, the last identity means that $\varphi \psi z+v+\psi \beta z=v$ for all $v, z \in Q$. This implies $\varphi \psi=-\psi \beta$. Hence $\varphi \psi z+v=v-\psi \beta z=v+\varphi \psi z$. So, $(Q,+)$ is commutative. Applying these facts to (7) we can see that $\beta=\psi \alpha$. Hence $x \circ y=\varphi x+\psi y+a$ and $x * y=\varphi^{2} x+\psi \varphi^{2} y+b$.

The proof of the converse follows from a direct calculation and is omitted.
8. Will now consider a biquasigroup $(Q, \circ, *)$ satisfying the identity (8), i.e.

$$
(x \circ z) *(y * z)=x \circ y .
$$

Proposition 2.26. If a biquasigroup $(Q, \circ, *)$ satisfies ( 8 ), then $(Q, *)$ has no more than one idempotent. If such idempotent exists then it is a right neutral element of a quasigroup $(Q, *)$. Moreover, if $x \circ x=u$ for some $u \in Q$ and all $x \in Q$, then $x \circ y=x *(y * x), x * x=w$ and $w \circ u=w$ for all $x, y \in Q$.

Proof. Let $e * e=e$. Since $(Q, \circ)$ is a quasigroup, each $z \in Q$ can be expressed in the form $z=x \circ e$. Thus $z=x \circ e=(x \circ e) *(e * e)=z * e$, so $e$ is a right neutral element of $(Q, *)$. If $\bar{e}$ is the second idempotent of $(Q, *)$, then $\bar{e} * e=\bar{e}=\bar{e} * \bar{e}$. Therefore $\bar{e}=e$, so $(Q, *)$ has no more than one idempotent. If $x \circ x=u$ for all $x \in Q$, then, by ( 8 ), $u *(x * x)=(x \circ x) *(x * x)=x \circ x=u$. Analogously $u *(y * y)=u$. Thus, $x * x=y * y=w$ for some $w \in Q$, i.e. $(Q, *)$ is unipotent and $w$ is its right neutral element. Then, $x \circ y=(x \circ x) *(y * x)=u *(y * x)$ and $w \circ u=(w \circ w) *(u * w)=u * u=w$.

Let ( $Q, \circ, *$ ) be linear over a group $(Q,+)$. If it satisfies (8), then $\alpha a+b+\beta b=a$, $\alpha=\varepsilon$ and $\beta b=-b$. Thus (8) can be reduced to

$$
\begin{equation*}
\psi z+b+\beta y+\beta b+\beta^{2} z=\psi y \tag{10}
\end{equation*}
$$

This for $y=0$ gives $\psi z+\beta^{2} z=0$. So, $\psi=-\beta^{2}$ and $\psi b=-b=\beta b$.
Now putting $y=b$ in (10) we obtain $\psi z+b+\beta b+\beta b+\beta^{2} z=\psi$ b, i.e. $-\beta^{2} z+\beta b+\beta^{2} z=\psi b=\beta b$. So, $\beta b+\beta^{2} z=\beta^{2} z+\beta b$. This means that $b$ is in the center of $(Q,+)$. Thus putting $z=0$ in (10) and using the above facts, we obtain $\beta=\psi=-\beta^{2}$. Hence $\beta=-\varepsilon$. So $(Q,+)$ is commutative, $x \circ y=\varphi x-y+a$ and $x * y=x-y+b$.

In this way, we have proved the "only if" part of Theorem 2.27 below. The proof of the converse part of Theorem 2.27 follows from a direct calculation and is omitted.

Theorem 2.27. A biquasigroup $(Q, \circ, *)$ linear over a group $(Q,+)$ satisfies (8) if and only if $(Q,+)$ is commutative, $x \circ y=\varphi x-y+a$ and $x * y=x-y+b$.

Corollary 2.28. A linear biquasigroup satisfying (8) is medial.
Corollary 2.29. A medial biquasigroup induced by the group $\mathbb{Z}_{n}$ satisfies (8) if and only if $x \circ y=[a x-y+c]_{n}$ and $x * y=[x-y+d]_{n}$ for some $a, c, d \in \mathbb{Z}_{n}$ such that $(a, n)=1$.
9. Finally, let us consider a biquasigroup $(Q, \circ, *)$ satisfying the identity (9), i.e.

$$
(x \circ z) *(y * z)=x * y
$$

Theorem 2.30. In a biquasigroup $(Q, \circ, *)$ satisfying the identity (9) the quasigroups $(Q, \circ)$ and $(Q, *)$ have no more than one idempotent. If such idempotent exists then it is a common right neutral element of these quasigroups.
Proof. Assume $(Q, \circ)$ has an idempotent $a$. Then $a * a=(a \circ a) *(a * a)=a *(a * a)$ and so $a * a=a$. Analogously, for $a * a=a$ we have $a * a=(a \circ a) *(a * a)=(a \circ a) * a$, which implies $a \circ a=a$. So, $(Q, \circ)$ and $(Q, *)$ have the same idempotent. Then for each $x \in Q x * a=(x \circ a) *(a * a)=(x \circ a) * a$, which implies $x=x \circ a$. Thus $a$ is a right neutral element of ( $Q, \circ$ ). On the other hand, $x \circ a=x$ gives $x * x=(x \circ a) *(x * a)=x *(x * a)$, and consequently $x=x * a$. Thus, $a$ is a right neutral element of $(Q, \circ)$ and $(Q, *)$.
Corollary 2.31. If in a biquasigroup $(Q, \circ, *)$ satisfying (9) the quasigroup $(Q, *)$ is unipotent, then $(Q, \circ)=(Q, *)$ and $(Q, \circ)$ is a Ward quasigroup.
Proof. Let $x * x=a$ for all $x \in Q$ and some $a \in Q$. Then $a * a=x * x=(x \circ x) *$ $(x * x)=(x \circ x) * a$. Therefore, $x \circ x=a$, i.e. $(Q, \circ)$ is unipotent. Consequently, $a *(x * y)=(y \circ y) *(x * y)=y * x$, which implies $x \circ y=(x \circ y) * a=(x \circ y) *(y * y)=x * y$. Hence $(Q, \circ)=(Q, *)$ and (9) coincides with (1). This means that $(Q, \circ)$ is a Ward quasigroup.

Theorem 2.32. A biquasigroup $(Q, \circ, *)$ linear over a group $(Q,+)$ satisfying the identity (9) is medial and can be presented in the form $x \circ y=x-\beta^{2} y-\beta b$ and $x * y=x+\beta y+b$, where $(Q,+)$ is a commutative group, $\beta \in \operatorname{Aut}(Q,+)$ and $b \in Q$. This biquasigroup has a right neutral element $e=-\varphi^{-1} b$.

Conversely, if $(Q,+)$ is a commutative group, $\beta \in \operatorname{Aut}(Q,+), b \in Q, x \circ y=$ $x-\beta^{2} y-\beta b$ and $x * y=x+\beta y+b$, then the biquasigroup $(Q, \circ, *)$ satisfies (9).

Proof. If a biquasigroup $(Q, \circ, *)$ linear over a group $(Q,+)$ satisfies the identity (9), then $\alpha a+b+\beta b=b$ and $\varphi=\varepsilon$. So, (9) can be reduced to

$$
\begin{equation*}
\alpha a+\alpha \psi z+b+\beta \alpha y+\beta b+\beta^{2} z=b+\beta y \tag{11}
\end{equation*}
$$

which for $z=0$ gives $\alpha a+b+\beta \alpha y+\beta b=b+\beta y=\alpha a+b+\beta b+\beta y$. Since $\alpha a=b=b-\beta b$, the last implies $\beta \alpha y=\beta b+\beta y-\beta b$. This together with (11) (for $y=v$ ) gives

$$
\begin{equation*}
\alpha a+\alpha \psi z+b+\beta b+\beta v+\beta^{2} z=b+\beta v \tag{12}
\end{equation*}
$$

Now adding $\beta v$ on the right side to (11) and putting $y=0$ we get

$$
\alpha a+\alpha \psi z+b+\beta b+\beta^{2} z+\beta v=b+\beta v
$$

Comparing this identity with (12) we obtain $\beta v+\beta^{2} z=\beta^{2} z+\beta v$ for all $v, z \in Q$. This shows that $(Q,+)$ is a commutative group. Consequently, $\beta \alpha y=\beta y$, so $\alpha=\varepsilon$. This by $\alpha a+b+\beta b=b$ gives $a=-\beta b$. Again putting $y=0$ in (11) and using the above facts we obtain $\psi=-\beta^{2}$. Therefore, $x \circ y=x-\beta^{2} y-\beta b$ and $x * y=x+\beta y+b$.

The proof of the converse part of the Theorem follows from a direct calculation and is omitted.

Proposition 2.33. A medial biquasigroup $\left(\mathbb{Z}_{n}, \circ, *\right)$ satisfies the identity (9) if and only if $x \circ y=\left[x-a^{2} y-a b\right]_{n}, x * y=[x+a y+b]_{n}$, where $a, b \in \mathbb{Z}_{n}$ are fixed and $(a, n)=1$.

Example 2.34. Let $n=a^{2}+1>4$. Then $\left(\mathbb{Z}_{n}, \circ, *\right)$, where $x \circ y=[x+y]_{n}$ and $x * y=[x+a y]_{n}$ is an example of a biquasigroup satisfying (9).
10. Many authors study linear quasigroups of the second type, namely quasigroups $(Q, \cdot)$ where, in the definition of the operation, the constant element is not placed in the middle of the formula but at its end, i.e. $x \cdot y=\varphi x+\psi y+a$.

Biquasigroups of this type satisfying the identities $(2)-(9)$ coincide with the quasigroups of the previous type. Namely, if a biquasigroup $\widehat{Q}=(Q, \circ, *)$ with the operations $x \circ y=\varphi x+\psi y+a$ and $x * y=\alpha x+\beta y+b$, where $\alpha, \beta, \varphi, \psi$ are automorphisms of a group $(Q,+)$, satisfies (2) then $\alpha a+\beta a=0$ and $\varphi=\varepsilon$. Thus $\alpha \psi z+\alpha a+\beta y+\beta \psi z+\beta a=\beta y$. This for $y=0$ and $\psi z=v$ gives $\alpha v=-\beta a-\beta v-\alpha a$. Since $\alpha$ and $\beta$ are automorphisms of $(Q,+)$ the last
expression for $v=u+w$ implies $\beta(u+w)=\beta w+\beta u$. Thus $\beta u+\beta w=\beta u+\beta w$ for all $u, w \in Q$. Hence $(Q,+)$ is a commutative group. Such biquasigroups are described in subsection 2.

If a biquasigroup $(Q, \circ, \circ)$ with $x \circ y=\varphi x+\psi y+a$ satisfies (3), then $\varphi a+\psi a=0$, $\varphi=\varepsilon$ and $\psi z+a+\psi y+\psi^{2} z+\psi a=\psi y$, which for $y=a$ gives $\psi=-\varepsilon$. This shows that $(Q,+)$ is a commutative group and $x \circ y=x-y+a$. Also in the case when $(Q, \circ)$ with $x \circ y=\varphi x+a+\psi y$ satisfies (1), the group $(Q,+)$ must be commutative and $x \circ y=x-y+a$. This means that these two cases coincide.

If a biquasigroup $\widehat{Q}$ satisfies (4), then $\varphi a+\psi a+a=b$ and $\alpha=\varphi^{2}$. Because by Theorem 2.3 we have $q=\varphi 0+\psi 0+a=\alpha 0+\beta 0+b$, must be $q=a=b$. Consequently, $a=\varphi x+\psi x+a$. This implies $\varphi=-\psi$, which together with $(x \circ 0) \circ(x \circ 0)=a$ implies $\varphi^{2} x+\varphi a-\varphi^{2} x-\varphi a=0$. Hence $\varphi x+a=a+\varphi x$ for all $x \in Q$. So, $a$ is in the center of $(Q,+)$. Therefore this case is reduced to the case described in subsection 4 .

If a biquasigroup $\widehat{Q}$ satisfies (5), then by Theorem 2.7 the quasigroup $(Q, *)$ has a right neutral element $e$. Thus $x=x * e=\alpha x+\beta e+b$ for all $x \in Q$. In particular $0=0 * e=\beta e+b$. Consequently, $x=x * e=\alpha x$ and $x * y=x+\beta y+b$. Applying this formula to (5) we can see that $\varphi=\varepsilon$ and $\varphi b=-a$. Therefore the identity (5) can be written in the form

$$
\psi z+a+\psi y+\psi \beta z=\psi y+a .
$$

This for $z=0$ implies $a+\psi y=\psi y+a$. Hence $a$ is in the center of $(Q,+)$. Also $b$ is in the center of $(Q,+)$ because $\varphi b=-a$. Thus this case reduces to the case from subsection 5 .

By Corollary 2.14 any quasigroup satisfying (6) is unipotent. Thus if $\widehat{Q}$ satisfies (6), then $\alpha 0+\beta 0+b=b$ implies $b=x * x=\alpha x+\beta x+b$, i.e. $\beta x=-\alpha x$ for all $x \in Q$. From (6) it follows $\alpha=\varepsilon$. Thus $\beta x=-x$. Since $\beta$ is an automorphism of $(Q,+),(Q,+)$ is commutative. Hence this case reduces to subsection 6.

If a biquasigroup $\widehat{Q}$ satisfies (7), then $\varphi a+\psi b+a=b$ which together with (7) fort $x=y=0$ implies

$$
\varphi \psi z+\varphi a+\psi \beta z+\psi b+a=b=\varphi a+\psi b+a
$$

Thus $\varphi \psi z=\varphi a-\psi \beta z-\varphi a$. Since $\varphi \psi$ and $\psi \beta$ are automorphisms of $(Q,+)$ the last for $z=u+v$ gives

$$
\varphi \psi(u+v)=\varphi a-\psi \beta(u+v)-\varphi a .
$$

On the other side,

$$
\varphi \psi u+\varphi \psi v=\varphi a-\psi \beta u-\varphi a+\varphi a-\psi \beta v-\varphi a=\varphi a-\psi \beta(v+u)-\varphi a
$$

Comparing these two expression we obtain $\psi \beta(u+v)=\psi \beta(v+u)$. Hence $(Q,+)$ is a commutative group and this case reduces to 7 .

If a biquasigroup $\widehat{Q}$ satisfies (8), then $\alpha a+\beta b+b=a$ and $\alpha=\varepsilon$. This together with (8) for $x=y=0$ implies $\psi z+a+\beta^{2} z=a$. Hence $\beta^{2} z=-a-\psi z+a$.

From this for $z=u+v$, in a similar way as in the previous case, we obtain $\psi(u+v)=\psi(v+u)$. Therefore $(Q,+)$ is a commutative group and this case reduces to 8 .

The case when $\widehat{Q}$ satisfies (9) reduces to 9 . Indeed, in this case $\alpha a+\beta b=0$, which together (9) for $x=y=0$ shows that $\beta^{2} z=-\alpha a-\alpha \psi z-\beta b$. From this we compute $\alpha \psi(u+v)=\alpha \psi(v+u)$. Hence $(Q,+)$ is commutative.

## References

[1] J.M. Cardoso, C.P. da Silva, On Ward quasigroups, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Secţ. I a Mat., 24 (1978), 231-233.
[2] S.K. Chatterjea, On Ward quasigroups, Pure Math. Manuscript, 6 (1987), 31-34.
[3] W.A. Dudek, R.A.R. Monzo, Double magma associated with Ward and double Ward quasigroups, Quasigroups and Related Systems, 27 (2019), 33-52.
[4] N.C. Fiala, Double Ward quasigroups, Quasigroups and Related Systems, 15 (2007), 261-262.
[5] K.W. Johnson, P. Vojtěchovský, Right division in groups, Dedekind-Frobenius group matrices, and Ward quasigroups, Abh. Math. Sem. Univ. Hamburg, 75 (2005), 121-136.
[6] J. Morgado, Definição de quasigrupo subtractivo por único axioma, Gaz. Mat. (Lisboa) 92-93 (1963), 17-18.
[7] M. Polonijo, Ward double quasigroups, in Proc. Conf. "Algebra and Logic". Cetinje, 1986, 153-156, Univ. Novi Sad, 1987.
[8] M. Polonijo, A note on Ward quasigroups, An. Ştiinţ. Univ. Al.I. Cuza Iaşi Secţ. I a Mat., 32 (1986), 5-10.
[9] V. Shcherbacov, Elements of quasigroup theory and applications, Champan and Hall Book (2017).
[10] C.P. da Silva, F.K. Miyaoka, Relations among some classes of quasigroups, Rev. Colombia Mat. 13 (1979), 311-321.
[11] M. Ward, Postulates for the inverse operations in a group, Trans. Amer. Math. Soc., 32 (1930), 520-526.
[12] J.V. Whittaker, On the postulates defining a group, Amer. Math. Monthly, 62 (1955), 636-640.

Received February 03, 2021
W.A. Dudek

Faculty of Pure and Applied Mathematics, Wroclaw University of Science and Technology
50-370 Wroclaw, Poland
Email: wieslaw.dudek@pwr.edu.pl
R.A.R. Monzo

Flat 10, Albert Mansions, Crouch Hill, London N8 9RE, United Kingdom
E-mail: bobmonzo@talktalk.net

# $w$-supplemented property in the lattices 

Shahabaddin Ebrahimi Atani


#### Abstract

Let $L$ be a lattice with the greatest element 1 . In this paper, we introduce and investigate the latticial counterpart of the filter-theoretical concepts of $w$-supplemented. The basic properties and possible structures of such filters are studied.


## 1. Introduction

The notion of an order plays an important role not only throughout mathematics but also in adjacent such as logic and computer science, hence, ought to be in the literature. The beauty of lattice theory derives in part from the extreme simplicity of its basic concepts: (partial) ordering, least upper and greatest lower bounds. In structure, lattices lie between semigroups and rings. In this respect, it closely resembles group theory. Thus lattices and groups provide two of the most basic tools of universal algebra, and in particular the structure of algebraic systems is usually most clearly revealed through the analysis of appropriate lattices. In this paper, we extend several concepts from module theory to lattice theory. The main difficulty is figuring out what additional hypotheses the lattice or filter must satisfy to get similar results. Nevertheless, growing interest in developing the algebraic theory of lattices can be found in several papers and books (see for example [2, 4, $5,6,7]$ ).

The notion of a supplement submodule was introduced in [10] in order to characterize semiperfect modules, that is projective modules whose factor modules have projective cover. For submodules $U$ and $V$ of a module $M, V$ is said to be a supplement of $U$ in $M$ or $U$ is said to have a supplement $V$ in $M$ if $U+V=M$ and $U \cap V \ll V$. The module M is called supplemented if every submodule of $M$ has a supplement in $M$. See [4] and [12] for results and the definitions related to supplements and supplemented modules. In a series of papers, Zöschinger has obtained detailed information about supplemented and related modules [14]. Supplemented modules are also discussed in [11]. Recently, several authors have studied different generalizations of supplemented modules. Rad-supplemented modules have been studied in [13] and [3]. See [13]; these modules are called generalized supplemented modules. For submodules $U$ and $V$ of a module $M, V$ is said to be a rad-supplement of $U$ in $M$ or $U$ is said to have a rad-supplement $V$ in $M$ if

2000 MSC: 06B35; 05C25.
Keywords: Lattice; filter; semisimple; w-supplemented.
$U+V=M$ and $U \cap V \subseteq \operatorname{rad}(V) . \quad M$ is called a rad-supplemented module if every submodule of $M$ has a rad-supplement in $M$. We shall say that a module $M$ is $w$-supplemented if every semisimple submodule of $M$ has a supplement in $M$ [1]. Recently, the study of the supplemented property in the rings, modules, and lattices has become quite popular (see for example [1, 3, 4, 8, 9, 12, 13]. Supplemented property in the lattices have already been investigated in [8]. This paper is based on another variation of supplemented filters. In fact, in the present paper, we are interested in investigating (amply) $w$-supplemented filters in a distributive lattice with 1 to use other notions of $w$-supplemented, and associate which exist in the literature as laid forth in [1] (see Sections 2 and Section 3).

Let us briefly review some definitions and tools that will be used later [2]. By a lattice we mean a poset $(L, \leqslant)$ in which every couple elements $x, y$ has a g.l.b. (called the meet of $x$ and $y$, and written $x \wedge y$ ) and a l.u.b. (called the join of $x$ and $y$, and written $x \vee y$ ). A lattice $L$ is complete when each of its subsets $X$ has a l.u.b. and a g.l.b. in $L$. Setting $X=L$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that $L$ is a lattice with 0 and 1). A lattice $L$ is called a distributive lattice if $(a \vee b) \wedge c=(a \wedge c) \vee(b \wedge c)$ for all $a, b, c$ in $L$ (equivalently, $L$ is distributive if $(a \wedge b) \vee c=(a \vee c) \wedge(b \vee c)$ for all $a, b, c$ in $L$ ). A non-empty subset $F$ of a lattice $L$ is called a filter, if for $a \in F$, $b \in L, a \leqslant b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if $L$ is a lattice with 1, then $1 \in F$ and $\{1\}$ is a filter of $L$ ). A proper filter $P$ of $L$ is said to be maximal if $E$ is a filter in $L$ with $P \varsubsetneqq E$, then $E=L$. If $F$ is a filter of a lattice $L$, then the radical of $F$, denoted by $\operatorname{rad}(F)$, is the intersection of all maximal subfilters of $F$.

Let $L$ be a lattice. If $A$ is a subset of $L$, then the filter generated by $A$, denoted by $T(A)$, is the intersection of all filters that is containing $A$. A filter $F$ is called finitely generated if there is a finite subset $A$ of $F$ such that $F=T(A)$. A subfilter $G$ of a filter $F$ of $L$ is called small in $F$, written $G \ll F$, if, for every subfilter $H$ of $F$, the equality $T(G \cup H)=F$ implies $H=F$ [8]. A subfilter $G$ of $F$ is called essential in $F$, written $G \unlhd F$, if $G \cap H \neq\{1\}$ for any subfilter $H \neq\{1\}$ of $F$. Let $G$ be a subfilter of a filter $F$ of $L$. A subfilter $H$ of $F$ is called a supplement of $G$ in $F$ if $F=T(G \cup H)$ and $H$ is minimal with respect to this property, or equivalently, $F=T(G \cup H)$ and $G \cap H \ll H . H$ is said to be a supplement subfilter of $F$ if $H$ is a supplement of some subfilter of $F . F$ is called a supplemented filter if every subfilter of $F$ has a supplemented in $F$. A subfilter $G$ of a filter $F$ of $L$ has ample supplements in $F$ if, for every subfilter $H$ of $F$ with $F=T(H \cup G)$, there is a supplement $H^{\prime}$ of $G$ with $H^{\prime} \subseteq H$. If every subfilter of a filter $F$ has ample supplements in $F$, then we call $F$ amply supplemented. Let $G, H$ be subfilters of a filter $F$ of $L$. If $F=T(G \cup H)$ and $G \cap H \ll F$, then $H$ is called a weak supplement of $G$ in $F$. If every subfilter of $F$ has a weak supplement in $F$, then $F$ is called a weak supplemented filter. If $F=T(G \cup H)$ and $G \cap H \subseteq \operatorname{rad}(H)$, then $H$ is called a rad-supplement of $G$ in $F$. If every subfilter of $F$ has a rad-supplement in $F$, then $F$ is called a rad-supplemented filter.

A lattice $L$ is called semisimple, if for each proper filter $F$ of $L$, there exists a filter $G$ of $L$ such that $L=T(F \cup G)$ and $F \cap G=\{1\})$. In this case, we say that
$F$ is a direct summand of $L$, and we write $L=F \oplus G$. A filter $F$ of $L$ is called a semisimple filter, if every subfilter of $F$ is a direct summand. A simple lattice (resp. filter), is a lattice (resp. filter) that has no filters besides the $\{1\}$ and itself. For a filter $F, \operatorname{Soc}(F)=T\left(\cup_{i \in \Lambda} F_{i}\right)$, where $\left\{F_{i}\right\}_{i \in \Lambda}$ is the set of all simple filters of $L$ contained in $F$.

Proposition 1.1. $[6,7]$ A non-empty subset $F$ of a lattice $L$ is a filter if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F, z \in L$. Moreover, since $x=x \vee(x \wedge y)$, $y=y \vee(x \wedge y)$ and $F$ is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in L$.

Proposition 1.2. [8, Lemma 2.4, Theorem 2.6 and Theorem 2.9] Let $F$ be a filter of a distributive lattice $L$ with 1 .
(1) If $A \ll F$ and $C \subseteq A$, then $C \ll F$.
(2) If $A, B$ are subfilters of $F$ with $A \ll B$, then then $A$ is a small subfilter in subfilters of $F$ that contains the subfilter of $B$. In particular, $A \ll F$.
(3) If $F_{1}, F_{2}, \ldots, F_{n}$ are small subfilters of $F$, then $T\left(F_{1} \cup F_{2} \cup \cdots \cup F_{n}\right)$ is also small in $F$.
(4) If $A, B, C$ and $D$ are subfilters of $F$ with $A \ll B$ and $C \ll D$, then $T(A \cup C) \ll T(B \cup D)$.
(5) Let $G, H$ be subfilters of $F$ such that $H$ is a supplement of $G$ in $F$. If $F=T(U \cup H)$ for some subfilter $U$ of $G$, then $H$ is a supplement of $U$ in $F$.
(6) $\operatorname{rad}(F)=T\left(\cup_{G \ll F} G\right)$.
(7) Every finitely generated subfilter of $\operatorname{rad}(F)$ is small in $\operatorname{rad}(F)$.
(8) $x \in \operatorname{rad}(F)$ if and only if $T(\{x\}) \ll \operatorname{rad}(F)$.

Lemma 1.3. [8, Proposition 2.1]
(1) Let $A$ be an arbitrary non-empty subset of $L$. Then $T(A)=\left\{x \in L: a_{1} \wedge a_{2} \wedge \cdots \wedge a_{n} \leqslant x\right.$ for some $\left.a_{i} \in A(1 \leqslant i \leqslant n)\right\}$. Moreover, if $F$ is a filter and $A$ is a subset of $L$ with $A \subseteq F$, then $T(A) \subseteq F$, $T(F)=F$ and $T(T(A))=T(A)$.
(2) Let $A, B$ and $C$ be subfilters of a filter $F$ of $L$. Then $T(T(A \cup B) \cup C) \subseteq$ $T(A \cup T(B \cup C))$. In particular, if $F=T(T(A \cup B) \cup C)$, then $F=$ $T(T(C \cup B) \cup A)=T(T(A \cup C) \cup B)$.
(3) (Modular law) If $F_{1}, F_{2}, F_{3}$ are filters of $L$ with $F_{2} \subseteq F_{1}$, then
$F_{1} \cap T\left(F_{2} \cup F_{3}\right)=T\left(F_{2} \cup\left(F_{1} \cap F_{3}\right)\right)$.

## 2. Basic Properties of $w$-supplemented Filters

Throughout this paper, we shall assume unless otherwise stated, that $L$ is a distributive lattice with 1 . In this section we collect some basic properties concerning $w$-supplemented filters of $L$. Our starting point is the following lemma.

Lemma 2.1. Every subfilter of a semisimple filter of $L$ is semisimple.

Proof. Assume that $K$ is a subfilter of a semisimple filter $F$ of $L$ and let $G$ be a subfilter of $K$. By assumption, $F=T(G \cup H)$ and $H \cap G=\{1\}$ for some subfilter $H$ of $F$. Then by modular law, $K=K \cap T(G \cup H)=T(G \cup(H \cap K))$ and $G \cap(H \cap K)=\{1\}$, as required.

Lemma 2.2. Let $F$ be a filter of $L$ such that $F=T(G \cup H)$, where $H$ is a subfilter of $F$ and $G$ is a semisimple subfilter of $F$. Then $F=G^{\prime} \oplus H$ for some subfilter $G^{\prime}$ of $G$.

Proof. By assumption, $G=(G \cap H) \oplus G^{\prime}$ for some subfilter $G^{\prime}$ of $G$. Then by Lemma 1.3, $F=T\left(H \cup T\left((G \cap H) \cup G^{\prime}\right)\right)=T\left(G^{\prime} \cup T(H \cup(G \cap H))\right)=T\left(G^{\prime} \cup H\right)$ and $G^{\prime} \cap H=G \cap G^{\prime} \cap H=\{1\}$. So $F=G^{\prime} \oplus H$.

Lemma 2.3. Let $U, V$ be subfilters of a filter $F$ of $L$ such that $V$ is a direct summand of $F$ with $U \subseteq V$. Then $U \ll F$ if and only if $U \ll V$.

Proof. If $U \ll V$, then $U \ll F$ by Proposition 1.2 (2). Conversely, assume that $U \ll F$ and $F=T\left(V \cup V^{\prime}\right)$ with $V \cap V^{\prime}=\{1\}$. Let $V=T(U \cup K)$ for some subfilter $K$ of $V$. It follows from Lemma 1.3 that

$$
F=T\left(V^{\prime} \cup T(U \cup K)\right)=T\left(U \cup T\left(V^{\prime} \cup K\right)\right) ;
$$

hence $F=T\left(V^{\prime} \cup K\right)$ since $U \ll F$. Now it is enough to show that $V \subseteq K$. Let $x \in V$. Then $x \in T\left(V^{\prime} \cup K\right)$ gives $x=x \vee\left(v^{\prime} \wedge k\right)=\left(x \vee v^{\prime}\right) \wedge(x \vee k)$ for some $v^{\prime} \in V^{\prime}$ and $k \in K$. Since $x \vee v^{\prime} \in V \cap V^{\prime}=\{1\}$, we get $x=x \vee k \in K$, as required.

Lemma 2.4. Let $F$ be a filter of $L$. Then the following hold:
(1) $\operatorname{Soc}(\operatorname{rad}(F)) \ll F$.
(2) If $G$ is a semisimple subfilter of $F$ such that $G \subseteq \operatorname{rad}(F)$, then $G \ll F$.

Proof. (1). Put $G=\operatorname{Soc}(\operatorname{rad}(F))$ and suppose that $F=T(G \cup K)$ for some subfilter $K$ of $F$. Set $H=G \cap K$. Then we have $G=H \oplus H^{\prime}$ for some subfilter $H^{\prime}$ of $G \subseteq \operatorname{rad}(F), F=T\left(K \cup T\left(H \cup H^{\prime}\right)\right)=T\left(H^{\prime} \cup T(H \cup K)\right)=T\left(H^{\prime} \cup K\right)$ and $\{1\}=H \cap H^{\prime}=(G \cap K) \cap H^{\prime}=H^{\prime} \cap K$; hence $F=H^{\prime} \oplus K$. We claim that $H^{\prime}=\{1\}$. To see this, it suffices to show that every simple subfilter of $H^{\prime}$ is $\{1\}$. If $S$ is any simple subfilter of $H^{\prime} \subseteq \operatorname{rad}(F)$, then $S$ is a direct summand of $H^{\prime}$; hence it is a direct summand of $F$. By Proposition $1.2(7), S \ll \operatorname{rad}(F)$ which implies that $S \ll F$ by Proposition 1.2 (2). Thus $S$ is a direct summand of $F$ and is small in $F$ and hence $S=\{1\}$. Thus $F=T(K)=K$. This completes the proof.
(2). By assumption, $G \subseteq \operatorname{rad}(F)$ gives $G=\operatorname{Soc}(G) \subseteq \operatorname{Soc}(\operatorname{rad}(F))$. Now the assertion follows from (1) and Proposition 1.2 (1).

Definition 2.5. A filter $F$ of $L$ is called $w$-supplemented, if every semisimple subfilter of $F$ has a supplement in $F$.

We next give three other characterizations of $w$-supplemented filters.
Theorem 2.6. Let $F$ be a filter of $L$. Then the following statements are equivalent.
(1) $F$ is w-supplemented.
(2) Every semisimple subfilter of $F$ has a supplement that is a direct summand.
(3) Every semisimple subfilter of $F$ has a weak supplement.
(4) Every semisimple submodule of $F$ has a rad-supplement.

Proof. (1) $\Rightarrow(2)$ Let $G$ be a semisimple subfilter of $F$. Then $F=T(G \cup H)$ and $G \cap H \ll H$ for some subfilter $H$ of $F$. By Lemma 2.2, there exists a subfilter $G^{\prime}$ of $G$ such that $F=G^{\prime} \oplus H$.
$(2) \Rightarrow(3)$. Let $G$ be a semisimple subfilter of $F$. Then By $(2), F=T(G \cup H)$ and $G \cap H \ll H$ for some subfilter $H$ of $F$. By Proposition 1.2 (2), $G \cap H \ll H$ gives $G \cap H \ll F$; hence $G$ has a weak supplement.
$(3) \Rightarrow(4)$. Let $G$ be a semisimple subfilter of $F$. By assumption, $F=T(G \cup H)$ and $G \cap H \ll F$ for some subfilter $H$ of $F$. By Lemma 2.2, there exists a subfilter $G^{\prime}$ of $G$ such that $F=G^{\prime} \oplus H$. Since $G \cap H \subseteq H$ and $G \cap H \ll F$, we get $G \cap H \ll H$ by lemma 2.3. This implies $G \cap H \subseteq \operatorname{rad}(H)$ by Proposition 2.1 (6). Thus $H$ is rad-supplement of $G$ in $F$.
$(4) \Rightarrow(1)$. Let $G$ be a semisimple subfilter of $F$. By (4), $F=T(G \cup H)$ and $G \cap H \subseteq \operatorname{rad}(H)$ for some subfilter $H$ of $F$. Since $G \cap H \subseteq \operatorname{rad}(H) \subseteq \operatorname{rad}(F)$, Lemma 2.4 and Lemma 2.1 gives $G \cap H \ll F$. Since $G$ is semisimple, by Lemma 2.2, $F=G^{\prime} \oplus H$ for some subfilter $G^{\prime}$ of $G$. So we get $G \cap H \ll H$ by Lemma 2.3. This completes the proof.

Corollary 2.7. Let $F$ be a filter of $L$. Then $F$ is $w$-supplemented if and only if for each semisimple submodule $G$ of $F$, there exists a decomposition $F=F_{1} \oplus F_{2}$ such that $F_{1}$ is a subfilter of $G$ and $G \cap F_{2} \ll F_{2}$.

Proof. Apply Theorem 2.6
Proposition 2.8. If $F$ is a w-supplemented filter of $L$, then $F=X \oplus S$ for some semisimple subfilter $X$ and a subfilter $S$ of $F$.

Proof. Let $G$ be a semisimple subfilter of $F$. If there is no $G \neq\{1\}$, then $F=$ $F \oplus\{1\}$ and result follows. Otherwise, by assumption, $F=T(G \cup S)$ and $G \cap S \ll S$ for some subfilter $S$ of $F$. By Lemma 2.1, there exists a semisimple subfilter $X$ of $G$ such that $G=(G \cap S) \oplus X$; hence by Lemma 1.3, $F=T(G \cup S)=$ $T(S \cup T((G \cap S) \cup X))=T(X \cup T(S \cup(G \cap S)))=T(X \cup T(S))=T(X \cup S)$ and $X \cap S=(G \cap S) \cap X=\{1\}$. So $F=X \oplus S$.

Theorem 2.9. Every direct summand of a w-supplemented filter $F$ of $L$ is wsupplemented.

Proof. Let $G$ be a direct summand of $F$. Then $F=T(G \cup H)$ and $G \cap H=\{1\}$ for some subfilter $H$ of $F$. Let $S$ be a semisimple subfilter of $G$. If $S=\{1\}$, then $G$ trivially $w$-supplemented. So we may assume that $S \neq\{1\}$. Since $S$ is a semisimple subfilter of $F$, we have $F=T(S \cup K)$ and $S \cap K \ll K$ for some subfilter $K$ of $F$. Then by modular law, $G=G \cap T(S \cup K)=T(S \cup(K \cap G))$. It is enough to show that $S \cap(K \cap G)=K \cap S \ll K \cap G$. By Lemma 2.2, $G=T\left(S^{\prime} \cup(K \cap G)\right)$ and $S^{\prime} \cap(K \cap G)=S^{\prime} \cap K=\{1\}$ for some subfilter $S^{\prime}$ of $S$. That is, $K \cap G$ is a direct summand of $G$. By Proposition 1.2 (2), $S \cap K \ll K$ gives $S \cap K \ll F$. Since $S \cap K \subseteq G$ and $G$ is a direct summand of $F$, we get $K \cap S \ll G$ by Lemma 2.3. As $K \cap G$ is a direct summand of $G, K \cap S \subseteq K \cap G$ and $K \cap S \ll G$, we obtain $K \cap S \ll K \cap G$ by Lemma 2.3. This completes the proof.

Lemma 2.10. Let $H, G$ be subfilters of $F$ such that $T(H \cup G)$ has a supplement of $U$ in $F$ and $H \cap T(U \cup G)$ has a supplement $V$ in $H$. Then $T(U \cup V)$ is a supplement of $G$ in $F$.

Proof. To simplify our notation let $B=T(U \cup G) \cap H \subseteq T(U \cup G)$. By hypothesis, $T(U \cup T(H \cup G))=F, U \cap T(H \cup G) \ll U, T(V \cup B)=H$ and $V \cap B=$ $V \cap T(U \cup G)=A \ll V$. By Lemma 1.3, we have $F=T(U \cup T(H \cup G))=$ $T(H \cup T(U \cup G))=T(T(B \cup V) \cup T(U \cup G))=$

$$
T(V \cup T(B \cup T(U \cup G)))=T(V \cup T(U \cup G)) \subseteq T(G \cup T(U \cup V)) \subseteq F
$$

hence $F=T(G \cup T(U \cup V))$. It is enough to show that $T(U \cup V) \cap G \ll T(U \cup V)$. Since $T(G \cup V) \subseteq T(H \cup G)$ and $F=T(G \cup T(U \cup V))=T(U \cup T(G \cup V))$, Proposition $1.2(5)$ gives $U$ also is a supplement of $T(G \cup V)$ in $F$ which implies that $C=T(G \cup V) \cap U \ll U$. Now by Proposition 1.2 (4), $T(U \cup V) \cap G \subseteq$ $T(A \cup C)) \ll T(U \cup V)$; hence $T(U \cup V) \cap G \ll T(U \cup V)$ by Proposition 1.2 (1).

Theorem 2.11. Let $F_{1}, F_{2}$ and $F$ be filters of $L$ such that $F=F_{1} \oplus F_{2}$. If $F_{1}$ and $F_{2}$ are $w$-supplemented, then $F$ is $w$-supplemented.

Proof. Let $K$ be a semisimple subfilter of $F$. At the first we show that $F_{1} \cap T(K \cup$ $\left.F_{2}\right)$ is a semisimple subfilter of $F_{1}$. Assume that $G$ is a subfilter of $F_{1} \cap T\left(K \cup F_{2}\right)$ and let $x \in G$. Then there are elements $h \in K$ and $f_{2} \in F_{2}$ such that $x=$ $x \vee\left(h \wedge f_{2}\right)=(x \vee h) \wedge\left(x \vee f_{2}\right)$. Then $x \vee f_{2} \in G \cap F_{2} \subseteq F_{1} \cap F_{2}=\{1\}$ which implies that $x=x \vee h \in K$; hence $G \subseteq K$. By Lemma 2.1, $G$ is semisimple. If $G=F_{1} \cap T\left(K \cup F_{2}\right)$, we are done. So we may assume that $G \neq F_{1} \cap T\left(K \cup F_{2}\right)$. There exists a subfilter $G^{\prime}$ of $K$ such that $K=G \oplus G^{\prime}$. Then by Lemma 1.3,

$$
\begin{aligned}
& F_{1} \cap T\left(K \cup F_{2}\right)=F_{1} \cap T\left(T\left(G \cup G^{\prime}\right) \cup F_{2}\right) \subseteq T\left(G \cup T\left(G^{\prime} \cup F_{2}\right)\right) \cap F_{1} \\
& =T\left(G \cup\left(F_{1} \cap T\left(G^{\prime} \cup F_{2}\right)\right)\right) \text { with } F_{1} \cap T\left(G^{\prime} \cup F_{2}\right) \neq\{1\} .
\end{aligned}
$$

As

$$
G \cup\left(F_{1} \cap T\left(G^{\prime} \cup F_{2}\right)\right) \subseteq F_{1} \cap T\left(K \cup F_{2}\right)
$$

we get $T\left(G \cup\left(F_{1} \cap T\left(G^{\prime} \cup F_{2}\right)\right)\right) \subseteq F_{1} \cap T\left(K \cup F_{2}\right)$; hence

$$
F_{1} \cap T\left(K \cup F_{2}\right)=T\left(G \cup\left(F_{1} \cap T\left(G^{\prime} \cup F_{2}\right)\right)\right.
$$

It is enough to show that $G \cap\left(F_{1} \cap T\left(G^{\prime} \cup F_{2}\right)\right)=G \cap T\left(G^{\prime} \cup F_{2}\right)=\{1\}$. Let $z \in G$ and $z \in T\left(G^{\prime} \cup F_{2}\right)$. Thus $z=z \vee\left(g^{\prime} \wedge f\right)=\left(z \vee g^{\prime}\right) \wedge(z \vee f)$ for some $g^{\prime} \in G^{\prime}$ and $f \in F_{2}$. Since $z \vee g^{\prime} \in G \cap G^{\prime}=\{1\}$ and $z \vee f \in G \cap F_{2}=\{1\}$, we get $z=1$. Thus $A=F_{1} \cap T\left(K \cup F_{2}\right)$ is a semisimple subfilter of $F_{1}$. Similarly, $B=F_{2} \cap T\left(K \cup F_{1}\right)$ is a semisimple subfilter of $F_{2}$. Then $A$ and $B$ have supplements $V_{1}$ and $V_{2}$ in $F_{1}$ and $F_{2}$, respectively. Clearly, $F=T(F \cup\{1\})=T\left(T\left(F_{1} \cup F_{2}\right) \cup K\right)=T\left(F_{2} \cup T\left(F_{1} \cup K\right)\right)$ has a supplement $\{1\}$ in $F$. If $G=T\left(F_{1} \cup K\right)$ and $H=F_{2}$, then $V_{2}$ is a supplement $T\left(F_{1} \cup K\right)$ in $F$ by Lemma 2.10. Also $F_{1} \cap T\left(K \cup V_{2}\right) \subseteq F_{1} \cap T\left(F_{2} \cup K\right)$ gives $F_{1} \cap T\left(V_{2} \cup K\right)$ is semisimple by Lemma 2.1 which implies that it has a supplement $S$ in $F_{1}$. Again applying Lemma 2.10, $T\left(V_{2} \cup S\right)$ is a supplement of $K$ in $F$. Thus $F$ is $w$-supplemented.

Corollary 2.12. Every finite direct sum of $w$-supplemented filters of $L$ is $w$ supplemented.

Proof. Apply Theorem 2.11.
Proposition 2.13. Let $G$ be a subfilter of a filter $F$ of $L$. Then the following hold:
(1) If $A$ is the intersection of filters of $L$ which are essential in $F$, then $A=\operatorname{Soc}(F)$.
(2) $\operatorname{Soc}(G)=G \cap \operatorname{Soc}(F)$ and $\operatorname{Soc}(\operatorname{Soc}(F))=\operatorname{Soc}(F)$.

Proof. (1). Let $\operatorname{Soc}(F)=T\left(\cup_{i \in I} F_{i}\right)$, where $\left\{F_{i}\right\}_{i \in I}$ is the set of all simple filters of $L$ contained in $F$. Let $G \unlhd F$. For each $i \in I, F_{i} \cap G \neq 1$ which implies that $F_{i} \subseteq G$; hence $\operatorname{Soc}(F) \subseteq A$. For the reverse inclusion, it is enough to show that $A$ is semisimple. Let $G$ be a filter of $L$ such that $G \subseteq A$. If $G \unlhd F$, then $A \subseteq G$; so $G=A$. So we may assume that $G$ is not essential in $F$. Let $G^{\prime}$ be a complement of $G$ in $F$; so $T\left(G \cup G^{\prime}\right) \unlhd F$ by [8, Lemma 2.15 (3)]. It follows that $G \subseteq A \subseteq T\left(G \cup G^{\prime}\right) ;$ thus $A=A \cap T\left(G \cup G^{\prime}\right)=T\left(G \cup\left(A \cap G^{\prime}\right)\right)$ by Lemma 1.3 which implies that $A=G \oplus\left(A \cap G^{\prime}\right)$; hence $A$ is semisimple. Thus $A \subseteq \operatorname{Soc}(F)$, and so we have equality.
(2). Let $\operatorname{Soc}(F)=T\left(\cup_{i \in I} F_{i}\right)$, where $\left\{F_{i}\right\}_{i \in I}$ is the set of all simple filters of $L$ contained in $F$. Since the inclusion $\operatorname{Soc}(G) \subseteq G \cap \operatorname{Soc}(F)$ is clear, we will prove the reverse inclusion. Let $x \in G \cap \operatorname{Soc}(F)$. So $x=x \vee\left(f_{1} \wedge f_{2} \wedge \cdots \wedge f_{t}\right)=$ $\left(x \vee f_{1}\right) \wedge \cdots \wedge\left(x \vee f_{t}\right)$ for some $f_{j} \in F_{i_{j}}(1 \leqslant j \leqslant t)$. If for each $1 \leqslant j \leqslant$ $t, F_{i_{j}} \subseteq G$, then we are done. Therefore, without loss of generality, we can assume that $F_{i_{1}}, F_{i_{2}}, \cdots, F_{i_{m}} \nsubseteq G$ (so $G \cap F_{i_{1}}=\{1\}, \cdots G \cap F_{i_{m}}=\{1\}$ ) and $F_{i_{m+1}}, \cdots, F_{i_{t}} \subseteq G$. As for each $1 \leqslant j \leqslant m, F_{i_{j}}, G$ are filters, we get $x \vee f_{i_{j}}=1$; hence $x=\left(x \vee f_{m+1}\right) \wedge \cdots \wedge\left(x \vee f_{t}\right) \in T\left(F_{m+1} \cup \cdots \cup F_{t}\right) \subseteq \operatorname{Soc}(G)$, and so we have equality. Finally, if $G=\operatorname{Soc}(F)$, then $\operatorname{Soc}(\operatorname{Soc}(F))=\operatorname{Soc}(F)$.

Lemma 2.14. If $F$ is a filter of $L$ with $\operatorname{Soc}(F) \ll F$, then $F$ is $w$-supplemented.
Proof. It is clear that if $\operatorname{Soc}(F)=\{1\}$, then $F$ is $w$-supplemented. Let $G$ be a semisimple subfilter of $F$. Since $\operatorname{Soc}(F)$ is the largest semisimple subfilter of $F$, then $G \subseteq \operatorname{Soc}(F) \ll F$ which implies that $G \ll F$ by Proposition 1.2 (1). Now $F=T(F \cup G)$ and $G \cap F=G \ll F$ gives $G$ has a supplement in $F$. Thus $F$ is $w$-supplemented.

Theorem 2.15. Let $F$ be a filter of $L$. Then $F$ is $w$-supplemented if and only if $\operatorname{Soc}(F)$ has a supplement in $F$.

Proof. If $F$ is $w$-supplemented, then $\operatorname{Soc}(F)$ has a supplement in $F$ since it is semisimple. Conversely, let $H$ be a supplement of $\operatorname{Soc}(F)$ in $F$. Then by Proposition 2.13, $F=T(H \cup \operatorname{Soc}(F))$ and $\operatorname{Soc}(H)=H \cap \operatorname{Soc}(F) \ll H$; hence $H$ is $w$-supplemented by Lemma 2.14. By Lemma 2.2, $F=H \oplus S$, where $S$ is a semisimple subfilter of $F$ (so it is $w$-supplemented). Thus $F$ is $w$-supplemented by Theorem 2.11.

A subfilter $G$ of a filter $F$ of $L$ is said to be radical if $\operatorname{rad}(G)=G$.
Proposition 2.16. Every radical filter $F$ of $L$ is $w$-supplemented.
Proof. Since $\operatorname{Soc}(F)=\operatorname{Soc}(\operatorname{rad}(F)) \ll F$ by Lemma 2.4, we get $F$ is $w$-supplemented by Lemma 2.14.

Definition 2.17. A filter $F$ of $L$ is called amply $w$-supplemented, if $F=T(A \cup B)$, where $A$ is a semisimple subfilter of $F$, then $B$ contains a supplement of $A$.
Theorem 2.18. Let $F$ be a filter of $L$. Then $F$ is $w$-supplemented if and only if $F$ is amply w-supplemented.

Proof. Clearly, if $F$ is amply $w$-supplemented, then it is $w$-supplemented. Conversely, let $A$ be a semisimple subfilter of $F$ such that $F=T(A \cup B)$. It suffices to show that $B$ contains a supplement of $A$ in $F$. Since $A \cap B$ is semisimple, we have $F=T(H \cup(A \cap B))$ and $A \cap B \cap H \ll H$ for some subfilter $H$ of $F$. By Lemma 2.2, $F=H \oplus F_{1}$ for some subfilter $F_{1}$ of $A \cap B$. Then by Lemma 1.3, $B=B \cap T\left(H \cup F_{1}\right)=T\left(F_{1} \cup(H \cap B)\right)$ and

$$
F=T\left(A \cup T\left(F_{1} \cup(B \cap H)\right)\right)=T\left((B \cap H) \cup T\left(A \cup F_{1}\right)\right)=T((B \cap H) \cup A)
$$

with $B \cap H \subseteq B$. It follows that $H=H \cap T((B \cap H) \cup A)=T((B \cap H) \cup(H \cap A))$. Since $H \cap A$ is semisimple, by Lemma 2.2, $H=(B \cap H) \oplus K$ for some subfilter $K$ of $H \cap A$. Now $B \cap H$ is a direct summand of $H$ and $A \cap B \cap H \ll H$ gives $A \cap B \cap H \ll H \cap B$ by Lemma 2.3, as required.

Lemma 2.19. Let $F$ be a filter of $L$ such that $\operatorname{rad}(F) \unlhd F$. Let $K \subseteq G \subseteq F$ be subfilters of $F$ and assume $K$ to be a direct summand of $F$. Then $\operatorname{rad}(K)=\operatorname{rad}(G)$ if and only if $G=K$.

Proof. Let $\operatorname{rad}(K)=\operatorname{rad}(G)$. By assumption, $F=K \oplus K^{\prime}$ for some subfilter $K^{\prime}$ of $F$. Then by modular law, $G=G \cap T\left(K \cup K^{\prime}\right)=T\left(K \cup\left(K^{\prime} \cap G\right)\right)$ with $K \cap\left(K^{\prime} \cap G\right)=\{1\}$. Then $\operatorname{rad}(G)=T\left(\operatorname{rad}(K) \cup \operatorname{rad}\left(G \cap K^{\prime}\right)\right)$ with $\operatorname{rad}(G) \cap$ $\operatorname{rad}\left(G \cap K^{\prime}\right)=\{1\}$ by [8, Proposition 2.16] $\left(\operatorname{so} \operatorname{rad}\left(G \cap K^{\prime}\right)=\{1\}\right)$. Clearly, $G \cap K^{\prime}$ is a supplement of $K$ in $G$. If $\operatorname{rad}(G)=\operatorname{rad}(K)$, then by [8, Theorem 2.9 (3)], $\{1\}=\operatorname{rad}\left(G \cap K^{\prime}\right)=\left(G \cap K^{\prime}\right) \cap \operatorname{rad}(F)$ which implies that $G \cap K^{\prime}=\{1\}$ since $\operatorname{rad}(F) \unlhd F$, and so $G=K$. The other implication is clear.

Theorem 2.20. Let $F$ be a filter of $L$ such that $\operatorname{rad}(F) \unlhd F$. Then the following statements are equivalent:
(1) $F$ is $w$-supplemented;
(2) Every semisimple submodule of $F$ is a direct summand;
(3) $\operatorname{Soc}(F)$ is a direct summand of $F$.

Proof. (1) $\Rightarrow$ (2). Let $G$ be a semisimple subfilter of $F$. By (1), there is a subfilter $K$ of $F$ such that $F=T(G \cup K)$ and $G \cap K \ll K$. By Lemma 2.2, $F=K \oplus G^{\prime}$ for some subfilter $G^{\prime}$ of $G$. By $\left[8\right.$, Proposition 2.16], we have $\operatorname{rad}(G)=\operatorname{rad}\left(G^{\prime}\right)=\{1\}$ which implies that $G=G^{\prime}$ by Lemma 2.19. Thus $F=K \oplus G$.
$(2) \Rightarrow(3)$. Since $\operatorname{Soc}(F)$ is semisimple subfilter of $F$, we get it is a direct summand of $F$ by (2).
$(3) \Rightarrow(1)$. Let $G$ be a semisimple subfilter of $F$. So $G$ is a subfilter and a direct summand of $\operatorname{Soc}(F)$; hence $G$ is a direct summand of $F$ by (3). So $F=G \oplus H$ and $G \cap H=\{1\} \ll H$ for some subfilter $H$ of $F$. Thus $F$ is $w$-supplemented.

Corollary 2.21. Let $F$ be a filter of $L$ such that $\operatorname{rad}(F) \unlhd F$. If $F$ is $w$ supplemented, then every subfilter of $F$ is w-supplemented.

Proof. Assume that $G$ is a subfilter of $F$ and let $K$ be a semisimple subfilter of $G$. By Theorem 2.20, there exists a subfilter $H$ of $F$ such that $F=K \oplus H$. By modularity, $G=G \cap T(K \cup H)=T(K \cup(G \cap H))$ with $K \cap(G \cap H)=\{1\}$, that is, $G=K \oplus(G \cap H)$. Therefore $G$ is $w$-supplemented.

Definition 2.22. We say that a filter $F$ of $L$ is totally w-supplemented, if every subfilter of $F$ is $w$-supplemented. A lattice $L$ is called a $V$-lattice if $\operatorname{rad}(F)=\{1\}$ for every filter $F$ of $L$.

Proposition 2.23. For $a V$-lattice $L$ and a filter $F$ of $L$, the following statements are equivalent:
(1) $F$ is w-supplemented;
(2) $F$ is amply w-supplemented;
(3) $F$ is totally $w$-supplemented.

Proof. (1) $\Leftrightarrow(2)$. The proof is followed from Theorem 2.18.
$(1) \Leftrightarrow(3)$. Let $F$ be totally $w$-supplemented. Since $F \subseteq F, F$ is also $w$ supplemented. Conversely, assume that $F$ is $w$-supplemented and $G$ be a subfilter of $F$. We will show that $G$ is $w$-supplemented. Let $K$ be a semisimple subfilter of $G$. By assumption, there is a subfilter $H$ of $F$ such that $F=T(H \cup K)$ and $H \cap K \ll H$. So $H \cap K \subseteq \operatorname{rad}(H) \subseteq \operatorname{rad}(F)=\{1\}$; hence $F=H \oplus K$. By modularity, $G=G \cap T(K \cup H)=T(K \cup(G \cap H))$ with $K \cap(G \cap H)=\{1\}$; so $G=K \oplus(G \cap H)$. Thus $G$ is $w$-supplemented.

Proposition 2.24. Let $F=F_{1} \oplus F_{2}$ be a filter of $L$ such that $\operatorname{rad}(F) \unlhd F$, where $F_{1}$ and $F_{2}$ are totally w-supplemented filters. Then $F$ is totally $w$-supplemented.

Proof. Let $G$ be a subfilter of $F$ and $K$ be a semisimple subfilter of $G$. Clearly, $F_{1}$ and $F_{2}$ are $w$-supplemented; so $F$ is $w$-supplemented by Theorem 2.11. Then $F=T(K \cup H)$ and $K \cap H \ll H$ for some subfilter $H$ of $F$. By Lemma 2.2, $F=K^{\prime} \oplus H$ for some subfilter $K^{\prime}$ of $K$. By Lemma $2.19, K=K^{\prime}$ which implies that $F=K \oplus H$. So by modular law, $G=G \cap T(H \cup K)=T(K \cup(G \cap H))$ and $K \cap(G \cap H)=K \cap H \ll H$. Hence $G$ is $w$-supplemented.

Theorem 2.25. Let $F=F_{1} \oplus F_{2}$ be a filter of $L$ such that $F_{2}$ is semisimple. Then $F$ is totally $w$-supplemented if and only if $F_{1}$ is totally $w$-supplemented.

Proof. It suffices to show that if $F_{1}$ is totally $w$-supplemented, then $F$ is totally $w$-supplemented. Let $G$ be a subfilter of $F$. Since $F_{2}$ is semisimple, there is a subfilter $H$ of $F_{2}$ such that $F_{2}=\left(G \cap F_{2}\right) \oplus H$ (so $G \cap H=\{1\}$ and $\left.H \cap F_{1}=\{1\}\right)$. By Lemma 1.3, since

$$
F=T\left(F_{1} \cup T\left(\left(G \cap F_{2}\right) \cup H\right)\right)=T\left(\left(G \cap F_{2}\right) \cup\left(F_{1} \cup H\right)\right),
$$

we get $G=T\left(\left(G \cap F_{2}\right) \cup(G \cap T(F!\cup H))\right)$ with $\left(G \cap F_{2}\right) \cap\left(G \cap T\left(F_{1} \cup H\right)\right)=\{1\}$, that is, $G=\left(G \cap F_{2}\right) \oplus\left(G \cap\left(F_{1} \oplus H\right)\right)$. If $x \in G \cap T\left(F_{1} \cup H\right)$, then $x=\left(x \vee f_{1}\right) \wedge(x \vee h)$ for some $f_{1} \in F_{1}$ and $h \in H$. As $x \vee h \in G \cap H=\{1\}$, we get $x \in F_{1}$, and so $G \cap\left(F_{1} \oplus H\right)$ is a subfilter of $F_{1}$; hence it is $w$-supplemented. Also, $G \cap F_{2}$ is $w$-supplemented since it is semisimple. Now the assertion follows from Theorem 2.11.

## 3. $W$-supplemented Quotient Filters

Quotient lattices are determined by equivalence relations rather than by ideals as in the ring case. If $F$ is a filter of a lattice $(L, \leqslant)$, we define a relation on $L$, given by $x \sim y$ if and only if there exist $a, b \in F$ satisfying $x \wedge a=y \wedge b$. Then $\sim$ is an equivalence relation on $L$, and we denote the equivalence class of $a$ by $a \wedge F$ and these collection of all equivalence classes by $\frac{L}{F}$. We set up a partial order $\leqslant_{Q}$ on $\frac{L}{F}$ as follows: for each $a \wedge F, b \wedge F \in \frac{L}{F}$, we write $a \wedge F \leqslant_{Q} b \wedge F$ if and only if $a \leqslant b$. It is straightforward to check that $\left(\frac{L}{F}, \leqslant_{Q}\right)$ is a poset. The following notation below
will be kept in this section: Let $a \wedge F, b \wedge F \in \frac{L}{F}$ and set $X=\{a \wedge F, b \wedge F\}$. By definition of $\leqslant_{Q},(a \vee b) \wedge F$ is an upper bound for the set $X$. If $c \wedge F$ is any upper bound of $X$, then we can easily show that $(a \vee b) \wedge F \leqslant_{Q} c \wedge F$. Thus $(a \wedge F) \vee_{Q}(b \wedge F)=(a \vee b) \wedge F$. Similarly, $(a \wedge F) \wedge_{Q}(b \wedge F)=(a \wedge b) \wedge F$. Thus $\left(\frac{L}{F}, \leqslant_{Q}\right)$ is a lattice.
Remark 3.1. Let $G$ be a subfilter of a filter $F$ of $L$.
(1) If $a \in F$, then $a \wedge F=F$. By the definition of $\leqslant_{Q}$, it is easy to see that $1 \wedge F=F$ is the greatest element of $\frac{L}{F}$.
(2) If $a \in F$, then $a \wedge F=b \wedge F$ (for every $b \in L$ ) if and only if $b \in F$. In particular, $c \wedge F=F$ if and only if $c \in F$. Moreover, if $a \in F$, then $a \wedge F=F=1 \wedge F$.
(3) By the definition $\leqslant_{Q}$, we can easily show that if $L$ is distributive, then $\frac{L}{F}$ is distributive.
(4) $\frac{F}{G}=\{a \wedge G: a \in F\}$ is a filter of $\frac{L}{G}$.
(5) If $K$ is a filter of $\frac{L}{G}$, then $K=\frac{F}{G}$ for some filter $F$ of $L$.
(6) If $H$ is a filter of $L$ such that $G \subseteq H$ and $\frac{F}{G}=\frac{H}{G}$, then $F=H$.
(7) If $H$ and $V$ are filters of $L$ containing $G$, then $\frac{F}{G} \cap \frac{H}{G}=\frac{V}{G}$ if and only if $V=H \cap F$.
(8) If $H$ is a filter of $L$ containing $G$, then $\frac{T(F \cup H)}{G}=T\left(\frac{H}{G} \cup \frac{F}{G}\right)$.

Proposition 3.2. Every quotient of a semisimple filter of $L$ is semisimple.
Proof. Let $K$ be a subfilter of a semisimple filter $F$. We show that $\frac{F}{K}$ is semisimple. Let $\frac{G}{K}$ be a subfilter of $\frac{F}{K}$. Since $F$ is semisimple, $F=T(G \cup H)$ with $G \cap H=\{1\}$ for some subfilter $H$ of $F$. Then we have $\frac{F}{K}=\frac{T(G \cup H)}{K}=$

$$
\frac{T(G \cup T(H \cup K))}{K}=T\left(\frac{G}{K} \cup \frac{T(H \cup K)}{K}\right)
$$

and $\frac{G}{K} \cap \frac{T(H \cup K)}{K}=\frac{G \cap T(H \cup K)}{K}$. It is enough to show that $G \cap T(H \cup K)=K$. Clearly, $K \subseteq G \cap T(H \cup K)$. For the reverse inclusion, suppose that $x \in G \cap$ $T(H \cup K)$. Then $x=x \vee(h \wedge k)=(x \vee h) \wedge(x \vee k)$ for some $h \in H$ and $k \in K$. As $x \vee h \in G \cap H=\{1\}$, we get $x=x \vee k \in K$, and so we have equality. Thus $\frac{F}{K}=\frac{G}{K} \oplus \frac{T(H \cup K)}{K}$.

Proposition 3.3. Let $H$ and $G$ be subfilters of a filter $F$ of $L$. IF $H$ is semisimple, then $\frac{T(H \cup G)}{G}$ is a semisimple subfilter in $\frac{F}{G}$.
Proof. Let $\frac{U}{G}$ be a subfilter of $\frac{T(H \cup G)}{G}$ (so $U \subseteq T(H \cup G)$ ). By assumption, $H=(H \cap U) \oplus K$ for some subfilter $K$ of $H$ (so $U \cap K=\{1\}$ ). At first we show that $T(U \cup K)=T(H \cup G)$. Since $U \subseteq T(H \cup G)$ and $K \subseteq H$, we get $T(U \cup K) \subseteq T(H \cup G)$. For the reverse inclusion, by Lemma 1.3, we have
$T(H \cup G)=T(G \cup T(K \cup(H \cap U))) \subseteq T(G \cup T(U \cup K)) \subseteq T(K \cup T(U \cup G))=T(U \cup K)$,
and so we have equality. Next we show that $T(U \cup K)=T(U \cup T(G \cup K))$. Since the inclusion $T(U \cup K) \subseteq T(U \cup T(G \cup K))$ is clear, we will prove the reverse containment. Let $x \in T(U \cup T(G \cup K))$. Then $x=(x \vee u) \wedge(x \vee t)$ for some $u \in U$ and $t \in T(G \cup K)$ which implies that $x=(x \vee u) \wedge(x \vee t \vee g) \wedge(x \vee t \vee k) \in T(K \cup U)$ for some $g \in G$ and $k \in K$. Thus $T(U \cup T(G \cup K))=T(U \cup K)=T(G \cup H))$. Clearly, $G \subseteq U \cap T(G \cup K)$. If $z \in U \cap T(G \cup K)$, then $z=(z \vee g) \wedge(z \vee k)$ for some $g \in G$ and $k \in K$. As $z \vee k \in U \cap K=\{1\}$, we get $z=z \vee g \in G$; hence $G=U \cap T(G \cup U)$. Now we have

$$
T\left(\frac{U}{G} \cup \frac{T(G \cup K}{G}\right)=\frac{T(U \cup T(G \cup K))}{G}=\frac{T(H \cup G)}{G}
$$

and $\frac{U}{G} \cap \frac{T(G \cup K)}{G}=\frac{U \cap T(G \cup K)}{G}=\frac{G}{G}=\{G\}$. Thus $\frac{T(H \cup G)}{G}=\frac{U}{G} \oplus \frac{T(G \cup K)}{G}$.
Theorem 3.4. Let $G$ be a subfilter a filter $F$ of $L$ such that $G \ll F$. If $G$ and $\frac{F}{G}$ are $w$-supplemented, then $F$ is w-supplemented.
Proof. If $H$ is any semisimple subfilter of $F$, then $\frac{T(H \cup G)}{G}$ is a semisimple subfilter in $\frac{F}{G}$ by Proposition 3.3. If $\frac{F}{G}=\frac{T(H \cup G)}{G}$, then $F=T(H \cup G)$. By Lemma 2.2, $F=H^{\prime} \oplus G$ for some subfilter $H^{\prime}$ of $H$ which implies that $F$ is $w$-supplemented as a finite direct sum of $w$-supplemented filters. So we may assume that $\frac{F}{G} \neq \frac{T(H \cup G)}{G}$. By assumption, there exists a subfilter $\frac{K}{G}$ of $\frac{F}{G}$ such that $\frac{F}{G}=T\left(\frac{T(H \cup G)}{G} \cup \frac{K}{G}\right)=$ $\frac{T(K \cup T(H \cup G))}{G}=\frac{T(K \cup H)}{G}$ and $\frac{T(H \cup G)}{G} \cap \frac{K}{G}=\frac{T(H \cup G) \cap K}{G}=\frac{T(G \cup(H \cap K)}{G} \ll \frac{K}{G}$. Since $F=T(K \cup H)$, it is enough to show that $H \cap K \ll K$. Let $K=T(X \cup(H \cap K))$ for some subfilter $X$ of $K$. Then $\frac{K}{G}=T\left(\frac{T(G \cup(H \cap K)}{G} \cup \frac{T(X \cup G)}{G}\right)$. Since $\frac{T(G \cup(H \cap K)}{G} \ll$ $\frac{K}{G}$, then $\frac{K}{G}=\frac{T(X \cup G)}{G}$; hence $K=T(X \cup G)$. As $F=T(K \cup H)$, there is a subfilter $U$ of $H$ such that $F=K \oplus U$ by Lemma 2.2. As $K$ is a direct summand of $F$ and $G \subseteq K, G \ll F$ gives $G \ll K$ by Lemma 2.3; hence $K=X$. Thus $H \cap K \ll K$. This completes the proof.
Theorem 3.5. Let $F$ be a filter of L. If every semisimple subfilter of $\frac{F}{\operatorname{rad}(F)}$ is a direct summand, then $F$ is (amply) w-supplemented. In particular, if $\frac{F}{\operatorname{rad}(F)}$ is semisimple, then $F$ is (amply) w-supplemented.

Proof. Let $G$ be a semisimple subfilter of $F$. Then by Proposition 3.3, $\frac{T(G \cup \operatorname{rad}(F))}{\operatorname{rad}(F)}$ is a semisimple subfilter of $\frac{F}{\operatorname{rad}(F)}$. If $\frac{T(G \cup \operatorname{rad}(F))}{\operatorname{rad}(F)}=\frac{F}{\operatorname{rad}(F)}$, then $T(G \cup \operatorname{rad}(F))=$ $F$. Thus $F=\operatorname{rad}(F) \oplus G^{\prime}$ for some subfilter $G^{\prime}$ of $G$ by Lemma 2.2. Since $G \cap \operatorname{rad}(F)$ is semisimple and $G \cap \operatorname{rad}(F) \subseteq \operatorname{rad}(F), G \cap \operatorname{rad}(F) \ll F$ by Lemma 2.4 and also by Lemma 2.3, $G \cap \operatorname{rad}(F) \ll \operatorname{rad}(F)$ since $\operatorname{rad}(F)$ is a direct summand of $F$. Thus $F$ is $w$-supplemented. So we may assume that $\frac{T(G \cup \operatorname{rad}(F))}{\operatorname{rad}(F)} \neq \frac{F}{\operatorname{rad}(F)}$. By assumption and Lemma 1.3, there is a subfilter $\frac{H}{\operatorname{rad}(F)}$ of $\frac{F}{\operatorname{rad}(F)}$ such that

$$
\frac{F}{\operatorname{rad}(F)}=T\left(\frac{T(G \cup \operatorname{rad}(F))}{\operatorname{rad}(F)} \cup \frac{H}{\operatorname{rad}(F)}\right)=\frac{T(G \cup H)}{\operatorname{rad}(F)}
$$

and $\frac{T(G \cup \operatorname{rad}(F))}{\operatorname{rad}(F)} \cap \frac{H}{\operatorname{rad}(F)}=\frac{T(\operatorname{rad}(F) \cup(G \cap H))}{\operatorname{rad}(F)}=\{\operatorname{rad}(F)\}$; so $F=T(G \cup H)$ and $T(\operatorname{rad}(F) \cup(G \cap H))=\operatorname{rad}(F)$ (so $G \cap H \subseteq \operatorname{rad}(F))$. By Lemma 2.2, $F=H \oplus K$ for some subfilter $K$ of $G$. Since $G \cap H$ is semisimple, by Lemma 2.4, $G \cap H \ll F$. By Lemma 2.3, since $H$ is a direct summand of $F$ and $G \cap H \ll F$, we get $G \cap H \ll H$. Therefore, $F$ is $w$-supplemented. The in particular statement is clear.

Definition 3.6. A lattice $L$ is called a semilocal lattice if $\frac{F}{\operatorname{rad}(F)}$ is semisimple for every filter $F$ of $L$.
Corollary 3.7. If $L$ is a semilocal lattice. Then the following hold:
(1) Every filter of $L$ is (amply) w-supplemented.
(2) Every filter of $L$ is totally $w$-supplemented.

Proof. Apply Theorem 3.5.

## References

[1] G. Bilhan and A.T. Güroglu, A variation of supplemented modules, Turkish J. Math., 37 (2013), 418 - 426.
[2] G. Birkhoff, Lattice theory, Amer. Math. Soc., 1973.
[3] E. Büyükasik, E. Mermut and S. Özdemir, Rad-supplemented modules, Rend. Semin. Mat. Univ. Padova, 124 (2010), 157 - 177.
[4] J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, Lifting modules. Supplements and projectivity in module theory, Frontiers Math. (Birkhäuser, Boston, 2006).
[5] G. Calugareanu, Lattice Concepts of Module Theory, Kluwer Academic Publishers, 2000.
[6] S. Ebrahimi Atani and M. Sedghi Shanbeh Bazari, On 2-absorbing filters of lattices, Discuss. Math. Gen. Algebra Appl, 36 (2016), 157 - 168.
[7] S. Ebrahimi Atani, S. Dolati Pish Hesari, M. Khoramdel and M. Sedghi Shanbeh Bazari, A semiprime filter-based identity-summand graph of a lattice, Le Matematiche, 70 (2018), no. 2, 297 - 318.
[8] S. Ebrahimi Atani and M. Chenari, Supplemented property in the lattices, Serdica Math. J., 46 (2020), no. 1, $73-88$.
[9] A. Harmanci, D. Keskin and P.F. Smith, On $\oplus$-supplemented modules, Acta Math. Hungar., 83 (1999), no. 1-2, 161 - 169.
[10] F. Kasch and E.A. Mares, Eine Kennzeichnung semi-perfekter Moduln, Nagoya Math. J., 27 (1966), $525-529$.
[11] S.H. Mohamed and B.J. Müller, Continuous and discrete modules, Cambridge University Press, London, 1990.
[12] R. Wisbauer, Foundations of Module and Ring Theory, Gordon and Breach, Reading, 1991.
[13] Y. Wang and D. Ding, Generalized supplemented modules, Taiwanese J. Math., 10 (2006), $1589-1601$.
[14] H. Zöschinger, Komplementierte Moduln über Dedekindringen, J. Algebra, 29 (1974), $42-56$.

Faculty of Mathematical Sciences
University of Guilan
P.O.Box 1914, Rasht, Iran

E-mail: ebrahimi@guilan.ac.ir

# $S S$-supplemented property in the lattices 

Shahabaddin Ebrahimi Atani, Mehdi Khoramdel, Saboura Dolati Pish Hesari and Mahsa Nikmard Rostam Alipour


#### Abstract

Let $L$ be a lattice with the greatest element 1 . We introduce and investigate the latticial counterpart of the filter-theoretical concepts of $s s$-supplemented. The basic properties and possible structures of such filters are studied.


## 1. Introduction

Since Kasch and Mares [13] have defined the notions of perfect and semiperfect for modules, the notion of a supplemented module has been used extensively by many authors. For submodules $U$ and $V$ of a module $M, V$ is said to be a supplement of $U$ in $M$ or $U$ is said to have a supplement $V$ in $M$ if $U+V=M$ and $U \cap V \ll V$. The module M is called supplemented if every submodule of $M$ has a supplement in $M$. In a series of papers, Zöschinger has obtained detailed information about supplemented and related modules [17]. Supplemented modules are also discussed in [14]. Recently, several authors have studied different generalizations of supplemented modules. Rad-supplemented modules have been studied in [15] and [3]. See [15]; these modules are called generalized supplemented modules. For submodules $U$ and $V$ of a module $M, V$ is said to be a rad-supplement of $U$ in $M$ or $U$ is said to have a rad-supplement $V$ in $M$ if $U+V=M$ and $U \cap V \subseteq \operatorname{rad}(V) . M$ is called a rad-supplemented module if every submodule of $M$ has a rad-supplement in $M$. We shall say that a module $M$ is $w$-supplemented if every semisimple submodule of $M$ has a supplement in $M$ [1]. We say that $V$ is an $s s$-supplement $U$ in $M$ if $M=U+V$ and $U \cap V \ll V$ and $V \cap U$ is semisimple. We call a module $M$ is $s s$-supplemented if every submodule of $M$ has an $s s$-supplement in $M$ [12]. Recently, the study of the supplemented property in the rings, modules, and lattices has become quite popular (see for example [2, $3,4,10,11,12,13]$. Supplemented property (resp. $w$-supplemented property) in the lattices have already been investigated in [7] (resp. [6]). This paper is based on another variation of supplemented filters. In fact, in the present paper, we are interested in investigating strongly local filters and (amply) $s s$-supplemented filters in a distributive lattice with 1 to use other notions of $s s$-supplemented, and associate which exist in the literature as laid forth in [12] (see Sections 2, 3, 4).

2010 Mathematics Subject Classification: 06B05.
Keywords: Lattice; Filter; Small; semisimple; $S S$-supplemented.

Let us briefly review some definitions and tools that will be used later [2]. By a lattice we mean a poset $(L, \leqslant)$ in which every couple elements $x, y$ has a g.l.b. (called the meet of $x$ and $y$, and written $x \wedge y$ ) and a l.u.b. (called the join of $x$ and $y$, and written $x \vee y$ ). A lattice $L$ is complete when each of its subsets $X$ has a l.u.b. and a g.l.b. in $L$. Setting $X=L$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that $L$ is a lattice with 0 and 1). A lattice $L$ is called a distributive lattice if $(a \vee b) \wedge c=(a \wedge c) \vee(b \wedge c)$ for all $a, b, c$ in $L$ (equivalently, $L$ is distributive if $(a \wedge b) \vee c=(a \vee c) \wedge(b \vee c)$ for all $a, b, c$ in $L)$. A non-empty subset $F$ of a lattice $L$ is called a filter, if for $a \in F, b \in L, a \leqslant b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if $L$ is a lattice with 1 , then $1 \in F$ and $\{1\}$ is a filter of $L$ ). A proper filter $P$ of $L$ is said to be maximal if $E$ is a filter in $L$ with $P \varsubsetneqq E$, then $E=L$. If $F$ is a filter of a lattice $L$, then the radical of $F$, denoted by $\operatorname{rad}(F)$, is the intersection of all maximal subfilters of $F$.

Let $L$ be a lattice. If $A$ is a subset of $L$, then the filter generated by $A$, denoted by $T(A)$, is the intersection of all filters that is containing $A$. A filter $F$ is called finitely generated if there is a finite subset $A$ of $F$ such that $F=T(A)$. A subfilter $G$ of a filter $F$ of $L$ is called small in $F$, written $G \ll F$, if, for every subfilter $H$ of $F$, the equality $T(G \cup H)=F$ implies $H=F$ [7]. A subfilter $G$ of $F$ is called essential in $F$, written $G \unlhd F$, if $G \cap H \neq\{1\}$ for any subfilter $H \neq\{1\}$ of $F$. Let $G$ be a subfilter of a filter $F$ of $L$. A subfilter $H$ of $F$ is called a supplement of $G$ in $F$ if $F=T(G \cup H)$ and $H$ is minimal with respect to this property, or equivalently, $F=T(G \cup H)$ and $G \cap H \ll H . H$ is said to be a supplement subfilter of $F$ if $H$ is a supplement of some subfilter of $F . F$ is called a supplemented filter if every subfilter of $F$ has a supplemented in $F$. A subfilter $G$ of a filter $F$ of $L$ has ample supplements in $F$ if, for every subfilter $H$ of $F$ with $F=T(H \cup G)$, there is a supplement $H^{\prime}$ of $G$ with $H^{\prime} \subseteq H$. If every subfilter of a filter $F$ has ample supplements in $F$, then we call $F$ amply supplemented. Let $G, H$ be subfilters of a filter $F$ of $L$. If $F=T(G \cup H)$ and $G \cap H \subseteq \operatorname{rad}(H)$, then $H$ is called a rad-supplement of $G$ in $F$. If every subfilter of $F$ has a rad-supplement in $F$, then $F$ is called a rad-supplemented filter.

A lattice $L$ is called semisimple, if for each proper filter $F$ of $L$, there exists a filter $G$ of $L$ such that $L=T(F \cup G)$ and $F \cap G=\{1\})$. In this case, we say that $F$ is a direct summand of $L$, and we write $L=F \oplus G$. A filter $F$ of $L$ is called a semisimple filter, if every subfilter of $F$ is a direct summand. A simple lattice (resp. filter), is a lattice (resp. filter) that has no filters besides the $\{1\}$ and itself. For a filter $F, \operatorname{Soc}(F)=T\left(\cup_{i \in \Lambda} F_{i}\right)$, where $\left\{F_{i}\right\}_{i \in \Lambda}$ is the set of all simple filters of $L$ contained in $F$. In [17], Zhou and Zhang generalized the concept of socle a module $M$ to that of $\operatorname{Soc}_{g}(M)$ by considering of all simple submodules of $M$ that are small in $M$ in place of the class of all simple submodules of $M$, that is, $\operatorname{Soc}_{g}(M)=\sum\{N \ll M: N$ is simple $\}$. For a filter $F$, we define $\operatorname{Soc}_{g}(F)=T\left(\cup_{i \in \Lambda} F_{i}\right)$, where $\left\{F_{i}\right\}_{i \in \Lambda}$ is the set of all simple filters of $L$ contained in $F$ and $F_{i} \ll F$ for each $i \in \Lambda$. Clearly, $\operatorname{Soc}_{g}(F) \subseteq \operatorname{Soc}(F)$ and $\operatorname{Soc}_{g}(F) \subseteq \operatorname{rad}(F)$. Let $F$ be a filter of a lattice $L . F$ is called hollow if $F \neq\{1\}$ and every proper
subfilter $G$ of $F$ is small in $F . F$ is called local if it has exactly one maximal subfilter that contains all proper subfilters.

Proposition 1.1. (cf. [9], [8]) A non-empty subset $F$ of a lattice $L$ is a filter if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F, z \in L$. Moreover, since $x=x \vee(x \wedge y), y=y \vee(x \wedge y)$ and $F$ is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in L$.

Proposition 1.2. (cf. [6]) Let $F$ be a filter of a distributive lattice $L$ with 1.
(1) If $A \ll F$ and $C \subseteq A$, then $C \ll F$.
(2) If $A, B$ are subfilters of $F$ with $A \ll B$, then $A$ is a small subfilter in subfilters of $F$ that contains the subfilter of $B$. In particular, $A \ll F$.
(3) $\operatorname{rad}(F)=T\left(\cup_{G \ll F} G\right)$.
(4) Every finitely generated subfilter of $\operatorname{rad}(F)$ is small in $\operatorname{rad}(F)$.
(5) $x \in \operatorname{rad}(F)$ if and only if $T(\{x\}) \ll F$.
(6) If $F_{1}, F_{2}, \ldots, F_{n}$ are small subfilters of $F$, then $T\left(F_{1} \cup F_{2} \cup \cdots \cup F_{n}\right)$ is also small in $F$.

Lemma 1.3. (cf. [6])
(1) $T(A)=\left\{x \in L: a_{1} \wedge a_{2} \wedge \cdots \wedge a_{n} \leqslant x\right.$ for some $\left.a_{i} \in A(1 \leqslant i \leqslant n)\right\}$ an arbitrary non-empty subset $A$ of $L$. Moreover, if $F$ is a filter and $A$ is a subset of $L$ with $A \subseteq F$, then $T(A) \subseteq F, T(F)=F$ and $T(T(A))=T(A)$.
(2) $T(T(A \cup B) \cup C) \subseteq T(A \cup T(B \cup C))$ for subfilters $A, B, C$ of a filter $F$ of L. In particular, $F=T(T(C \cup B) \cup A)=T(T(A \cup C) \cup B)$ for all $F=T(T(A \cup B) \cup C)$.
(3) (Modular law) $F_{1} \cap T\left(F_{2} \cup F_{3}\right)=T\left(F_{2} \cup\left(F_{1} \cap F_{3}\right)\right)$ for filters $F_{1}, F_{2}, F_{3}$ of $L$ such that $F_{2} \subseteq F_{1}$.

Proposition 1.4. (cf. [11])
(a) Let $G$ be a semisimple subfilter of a filter $F$ of $L$ such that $G \subseteq \operatorname{rad}(F)$. Then $G \ll F$.
(b) Let $H$ and $G$ be subfilters of a filter $F$ of $L$. Then the following hold:
(1) If $H$ is semisimple, then $\frac{T(H \cup G)}{G}$ is a semisimple subfilter in $\frac{F}{G}$.
(2) If $\operatorname{Soc}(F)=\cap_{K \unlhd F} K$.
(3) $\operatorname{Soc}(G)=G \cap \operatorname{Soc}(F)$.
(c) Let $U, V$ be subfilters of a filter $F$ of $L$ such that $V$ is a direct summand of $F$ with $U \subseteq V$. Then $U \ll F$ if and only if $U \ll V$.

## 2. Strongly Local Filters

Throughout this paper, we shall assume unless otherwise stated, that $L$ is a distributive lattice with 1 . In this section we collect some properties concerning strongly local filters of $L$. Our starting point is the following lemma.

Lemma 2.5. Let $F$ be a filter of $L$. Then the following hold:
(1) If $E$ is a simple subfilter of $F$, then $E=T(\{a\})$ for some $1 \neq a \in E$.
(2) If $f_{1}, f_{2}, \ldots, f_{n} \in F$, then $T\left(T\left(\left\{f_{1}\right\}\right) \cup \ldots \cup T\left(\left\{f_{n}\right\}\right)\right)=T\left(\left\{f_{1}, \ldots, f_{n}\right\}\right)$.
(3) If $F$ is semisimple, then $F$ is a direct sum of a finite family of simple subfilters if and only if $F$ is finitely generated.

Proof. (1). Since $E$ is simple, there is an element $1 \neq a \in E$ such that $T(\{a\}) \neq$ $\{1\}$ is a subfilter of $E$; hence $E=T(\{a\})$.
(2). Since the inclusion $A=T\left(\left\{f_{1}, \ldots, f_{n}\right\}\right) \subseteq T\left(T\left(\left\{f_{1}\right\}\right) \cup \ldots \cup T\left(\left\{f_{n}\right\}\right)\right)=B$ is clear we will prove the reverse inclusion. Let $x \in B$. Then $a_{1} \wedge a_{2} \wedge \cdots \wedge a_{n} \leqslant x$ for some $a_{i} \in T\left(\left\{f_{i}\right\}\right)(1 \leqslant i \leqslant n)$. By assumption, there exist $s_{1}, s_{2}, \ldots, s_{n} \in L$ such that $a_{i}=f_{i} \vee s_{i}(1 \leqslant i \leqslant n)$. Then $\left(f_{1} \vee s_{1}\right) \wedge \ldots \wedge\left(f_{n} \vee s_{n}\right) \leqslant x$. Since for each $i, f_{i} \leqslant f_{i} \vee s_{i}$ and $f_{i} \in A$, we get $f_{i} \vee s_{i} \in A(1 \leqslant i \leqslant n)$; so $x \in A$, and so we have equality.
(3). Let $F=F_{1} \oplus \cdots \oplus F_{n}$, where for each $i(1 \leqslant i \leqslant n), F_{i}$ is a simple subfilter of $F$, so by (1), $F_{i}=T\left(\left\{f_{i}\right\}\right)$ for some $1 \neq f_{i} \in F_{i}$. Then by (2), $F=T\left(T\left(\left\{f_{1}\right\}\right) \cup \cdots \cup T\left(\left\{f_{n}\right\}\right)\right)=T\left(\left\{f_{1}, \ldots, f_{n}\right\}\right)$. Thus $F$ is finitely generated. Conversely, suppose that $F=T(A)$, where $A=\left\{e_{1}, \ldots, e_{m}\right\}$. As $F$ is semisimple, we can write $F=T\left(\cup_{i \in I} F_{i}\right)$, where for each $i \in I, F_{i}$ is simple. We can now pick out a finite collection $i_{1}, i_{2}, \ldots, i_{r}$ of elements of $I$ such that $e_{i} \in T\left(F_{i_{1}} \cup \cdots \cup F_{i_{r}}\right)$ for $1 \leqslant i \leqslant m$. But then $F=T\left(F_{i_{1}} \cup \cdots \cup F_{i_{r}}\right)$, that is, $F=F_{i_{1}} \oplus \cdots \oplus F_{i_{r}}$.

Proposition 2.6. If $F$ is a filter of $L$, then $\operatorname{Soc}_{g}(F)=\operatorname{rad}(F) \cap \operatorname{Soc}(F)$.
Proof. It suffices to show that $\operatorname{rad}(F) \cap \operatorname{Soc}(F) \subseteq \operatorname{Soc}_{g}(F)$. Let $a \in \operatorname{rad}(F) \cap$ $\operatorname{Soc}(F)$. Then $T(\{a\})$ is semisimple and so there exist simple subfilters $F_{i}$ of $F$ such that $T(\{a\})=F_{1} \oplus \cdots \oplus F_{n}$ by Lemma 2.5 (3). By Proposition 1.2 (5), $T(\{a\}) \ll \operatorname{rad}(F)$; hence it is small in $F$ by Proposition 1.2 (2). Since for each $i$, $F_{i} \subseteq T(\{a\})$, we get $F_{i} \ll F$ by Proposition 1.2 (1). Thus $a \in T(\{a\}) \subseteq \operatorname{Soc}_{g}(F)$, and so we have equality.

A filter $F$ is called indecomposable if $F \neq\{1\}$ and $F=T(G \cup H)$ with $G \cap H=$ $\{1\}$, then either $G=\{1\}$ or $H=\{1\}$.

Lemma 2.7. Let $F$ be an indecomposable filter of $L$. Then $F$ is either simple or $\operatorname{Soc}(F) \subseteq \operatorname{rad}(F)$.

Proof. If $F$ is simple, we are done. Thus we may assume that $F$ is not simple. It suffices to show that $\operatorname{Soc}(F) \ll F$ by Proposition 1.2 (3). Let $F=T(K \cup \operatorname{Soc}(F))$ for some subfilter $K$ of $F$. By assumption, there is a semisimple subfilter $H$ of $\operatorname{Soc}(F)$ such that $\operatorname{Soc}(F)=(\operatorname{Soc}(F) \cap K) \oplus H$, and so by Lemma 1.3 (2), $F=T(K \cup T(H \cup(\operatorname{Soc}(F) \cap K)))=T(K \cup H)$ and $K \cap H=H \cap(\operatorname{Soc}(F) \cap K)=\{1\}$. Since $F$ is indecomposable and not simple, we get $H=\{1\}$; hence $F=K$. Thus $\operatorname{Soc}(F) \ll F$, as required.

By [6, Remark 2.19 (2)], every local filter is hollow and by [6, Remark 2.19 (1)], every hollow filter is indecomposable. Using Proposition 2.6 and Lemma 2.7 we have the following Corollary:

Corollary 2.8. Let $F$ be a local filter of $L$ such that it is not simple. Then $\operatorname{Soc}_{g}(F)=\operatorname{Soc}(F)$.
Definition 2.9. A filter $F$ of $L$ is called strongly local if it is local and $\operatorname{rad}(F)$ is semisimple. A filter $F$ of $L$ is called radical if $F$ has no maximal subfilters, that is, $F=\operatorname{rad}(F)$.

Assume that $F$ is a filter of $L$ and let $P(F)$ be the filter generated by $\cup_{G \subseteq F} G$, where for each subfilter $G$ of $F, G=\operatorname{rad}(G)$, that is, $P(F)=T\left(\cup_{G \subseteq F} G\right)$, where $G=\operatorname{rad}(G)$. It is easy to see that $P(F) \subseteq \operatorname{rad}(F)$.

Lemma 2.10. If $F$ is a filter of $L$, then $P(F)$ is the largest radical subfilter of $F$.
Proof. It suffices to show that $P(F) \subseteq \operatorname{rad}(P(F))$. If $x \in P(F)$, then there exist radical subfilters $G_{1}, \ldots, G_{n}$ of $F$ and $g_{1} \in G_{1}, . ., g_{n} \in G_{n}$ such that $g_{1} \wedge \cdots \wedge g_{n} \leqslant x$. Since $g_{1} \in G_{1}=\operatorname{rad}\left(G_{1}\right), \ldots, g_{n} \in G_{n}=\operatorname{rad}\left(G_{n}\right)$, by Proposition 1.2 (5) we have $T\left(\left\{g_{i}\right\}\right) \ll G_{i}$, for each $1 \leqslant i \leqslant n$. By Proposition $1.2(2), T\left(\left\{g_{i}\right\}\right) \ll P(F)$, for each $1 \leqslant i \leqslant n$. Therefore $g_{i} \in \operatorname{rad}(P(F))$, for each $1 \leqslant i \leqslant n$. This implies that $x \in \operatorname{rad}(P(F))$.

Proposition 2.11. If a filter $F$ of $L$ is strongly local, then $F$ is reduced (that is, $P(F)=\{1\}$ ).

Proof. Since $\operatorname{rad}(F)$ is semisimple and $P(F) \subseteq \operatorname{rad}(F) \subseteq \operatorname{Soc}(F)$, we get $P(F)$ is semisimple and so $P(F)=\operatorname{rad}(P(F))=\{1\}$ by [6, Proposition 2.16 (2)] and Lemma 2.10, as required.

Example 2.12. The collection of ideals of $Z$, the ring of integers, form a lattice under set inclusion which we shall denote by $L(Z)$ with respect to the following definitions: $m Z \vee n Z=(m, n) Z$ and $m Z \wedge n Z=[m, n] Z$ for all ideals $m Z$ and $n Z$ of $Z$, where ( $m, n$ ) and $[m, n]$ are greatest common divisor and least common multiple of $m, n$, respectively. Note that $L(Z)$ is a distributive complete lattice with least element the zero ideal and the greatest element $Z$. Then by $[7$, Proposition 2.1 (iii) and Theorem 3.1 (ii)], every simple filter of $L(Z)$ is of the form $F=\{Z, p Z\}$ for some prime number $p$. Let $\mathbf{P}$ be the set of all prime numbers. For each $p \in \mathbf{P}$, set $F_{p}=\{Z, p Z\}$. Then $\left\{F_{p}\right\}_{p \in \mathbf{P}}$ is the set of all simple filters of $L(Z)$. Moreover, by [7, Lemma 3.1], $\mathrm{m}=L(Z) \backslash\{0\}$ is the only maximal filter of $L(Z)$; so $L(Z)$ is a local filter of $L(Z)$ (so it is hollow). If $G$ is a proper subfilter of $L(Z)$ with $G \neq \operatorname{rad}(G)$, then $G$ has a maximal subfilter, say $H$. There exists $x \in G$ such that $x \notin H$; hence $T(H \cup T(\{x\}))=G$. By [6, Remark 2.19 (4)], $G$ has a supplement $K$ in $L(Z)$; so by Lemma 1.3,

$$
L(Z)=T(T(H \cup T(\{x\})) \cup K)=T(H \cup T(K \cup T(\{x\}))) ;
$$

hence $L(Z)=H$ which is impossible since $T(K \cup T(\{x\})) \ll L(Z)$. Thus $P(L(Z))=\mathbf{m} \neq\{1\}$. If $L(Z)=T\left(\cup_{p \in \mathbf{P}} F_{p}\right)$, then $\{0\}=p_{i_{1}} Z \wedge \cdots \wedge p_{i_{k}} Z=$ $p_{i_{1}} \cdots p_{i_{k}} Z$, a contradiction. So $L(Z)$ is not semisimple. Similarly, $\operatorname{rad}(L(Z))=\mathbf{m}$ is not semisimple. Therefore the condition "strongly" in the Proposition 2.11 is necessary.

## 3. $S S$-supplemented Filters

In this section, the basic properties and possible structures of $s s$-supplemented filters are investigated. Our starting point is the following lemma.

Lemma 3.1. Let $G$ and $H$ be subfilters of a filter $F$ of $L$. If $G$ is a maximal subfilter of $F$, then $H$ is a supplement of $G$ in $F$ if and only if $F=T(G \cup H)$ and $H$ is local.

Proof. Let $H$ be a supplement of $G$ in $F$. By [6, Theorem 2.9 (4)], $H$ is cyclic, and $G \cap H=\operatorname{rad}(H)$ is a (the unique) maximal subfilter of $H$; so $H$ is local. Conversely, let $H$ be local (so it is hollow) and $F=T(G \cup H)$. If $H \cap G=H$, then $F=G$ which is impossible. Thus $H \cap G \neq H$. Now $H$ is hollow gives $H \cap G \ll H$. Thus $H$ is a supplement of $G$ in $F$.

Definition 3.2. Let $G$ be any subfilter of a filter $F$ of $L$. We say that $H$ is an ss-supplement $G$ in $F$ if $F=T(G \cup H)$ and $G \cap H \ll H$ and $G \cap H$ is semisimple. We call a filter $F$ ss-supplemented if every its subfilter has an ss-supplement in $F$.

A subfilter $G$ of $F$ has ample ss-supplements in $F$ if every subfilter $K$ of $F$ such that $F=T(K \cup G)$ contains an $s s$-supplement of $G$ in $F$. We call a filter $F$ amply ss-supplemented if every subfilter of $F$ has ample $s s$-supplements in $F$.

We next give two other characterizations of $s s$-supplements filters.
Proposition 3.3. Let $G, H$ be subfilters of a filter $F$ of $L$. Then the following statements are equivalent:
(1) $F=T(G \cup H)$ and $G \cap H \subseteq \operatorname{Soc}_{g}(H)$;
(2) $F=T(G \cup H)$ and $G \cap H \subseteq \operatorname{rad}(H)$ and $G \cap H$ is semisimple;
(3) $F=T(G \cup H)$ and $G \cap H \ll H$ and $G \cap H$ is semisimple.

Proof. (1) $\Rightarrow$ (2). By (1) and Proposition 2.6, $G \cap H$ is semisimple and $G \cap H \subseteq$ $\operatorname{rad}(H) \cap \operatorname{Soc}(H) \subseteq \operatorname{rad}(H)$.
$(2) \Rightarrow(3)$. It is clear by (2) and Proposition 1.4 (a).
$(3) \Rightarrow(1)$. It is clear by (3) and Proposition 2.6.
Analogous to that Lemma 3.1 we have the following proposition:
Proposition 3.4. Let $G$ and $H$ be subfilters of a filter $F$ of $L$. If $G$ is a maximal subfilter of $F$, then $H$ is a ss-supplement of $G$ in $F$ if and only if $F=T(G \cup H)$ and $H$ is strongly local.

Proof. Let $H$ be an $s s$-supplement of $G$ in $F$. By [6, Theorem 2.9 (4)], $H$ is local with the unique maximal subfilter $G \cap H=\operatorname{rad}(H)$; so $H$ is strongly local since $G \cap H$ is semisimple. Conversely, since $H$ is local and $F=T(G \cup H)$, we can write $G \cap H \subseteq \operatorname{rad}(H)$. Now $\operatorname{rad}(H)$ is semisimple gives $G \cap H$ is semisimple. Hence, $H$ is an $s s$-supplement of $G$ in $F$.

Lemma 3.5. Let $G$ be a subfilter of a ss-supplemented filter $F$ of $L$. If $G \ll F$, then $G \subseteq \operatorname{Soc}_{g}(F)$. In particular, if $\operatorname{rad}(F) \ll F$, then $\operatorname{rad}(F) \subseteq \operatorname{Soc}(F)$.

Proof. Let $H$ be an $s s$-supplement of $G$ in $F$. Then $F=T(G \cup H)$ and $G \ll F$ gives $H=F$ and $G=G \cap H$ is semisimple; so $G \subseteq \operatorname{rad}(F) \cap \operatorname{Soc}(F)=\operatorname{Soc}_{g}(F)$ by Proposition 2.6. The in particular statement is clear.

Let $F$ be a local filter of $L$ (so it is hollow). It is easy to see that $F$ has no supplement subfilter except for $\{1\}$ and $F$. Thus every local filter is amply supplemented. Analogous to that we have:

Proposition 3.6. Every strongly local filter of $L$ is amply ss-supplemented.
Proof. Let $F$ be a strongly local filter (so $\operatorname{rad}(F)$ is semisimple). Then $F$ is local and so it is amply supplemented. If $G$ is a proper subfilter of $F$, then $F=T(F \cup G)$ and $G=G \cap F \ll F$; so $G \subseteq \operatorname{rad}(F)$; hence $G$ is semisimple. Thus $F$ is amply $s s$-supplemented.

Proposition 3.7. If $F$ is a hollow filter of $L$, then $F$ is (amply) ss-supplemented if and only if it is strongly local.

Proof. Assume that $F$ is $s s$-supplemented and let $x \in \operatorname{rad}(F)$. By Proposition $1.2(5), T(\{x\}) \ll \operatorname{rad}(F)$, and so it is small in $F$ by Proposition 1.2 (2). As $F$ is $s s$-supplemented, it follows from Lemma 3.5 that $x \in T(\{x\}) \subseteq \operatorname{Soc}_{g}(F)=$ $\operatorname{rad}(F) \cap \operatorname{Soc}(F)$; hence $x \in \operatorname{Soc}(F)$, and so $\operatorname{rad}(F) \subseteq \operatorname{Soc}(F)$. Suppose that $F=\operatorname{rad}(F)$. Then $F=\operatorname{rad}(F)=\operatorname{Soc}(F)$, and so $F$ is semisimple. Thus $F=\{1\}$ by [6, Proposition 2.16 (2)]. This is a contradiction because $F$ is hollow. So we may assume that $F \neq \operatorname{rad}(F)$, that is, $F$ is local by [6, Theorem 2.21]. Hence $F$ is strongly local. The other implication follows from Proposition 3.6.

The following example shows in general a (amply) supplemented filter need not be (amply) $s s$-supplemented.

Example 3.8. Assume that $R$ is a local Dedekind domain with unique maximal ideal $P=R p$ and let $E=E(R / P)$, the $R$-injective hull of $R / P$. For each positive integer $n$, set $A_{n}=\left(0:_{E} P^{n}\right)$. Then by [9, Lemma 2.6], every non-zero proper submodule of $E$ is equal to $A_{m}$ for some $m$ with a strictly increasing sequence of submodules $A_{1} \subset A_{2} \subset \cdots \subset A_{n} \subset A_{n+1} \subset \cdots$. The collection of submodules of $E$ form a complete lattice which is a chain under set inclusion which we shall denote by $L(E)$ with respect to the following definitions: $A_{n} \vee A_{m}=A_{n}+A_{m}$ and $A_{n} \wedge A_{m}=A_{n} \cap A_{m}$ for all submodules $A_{n}$ and $A_{m}$ of $E$. Then by [7, Example 2.3
(b)], every proper filter of $L(E)$ is of the form $\left[A_{n}, E\right]$ for some $n$. Clearly, $L(E)$ is a hollow filter which is not local. As hollow filters are (amply) supplemented, $L(E)$ is (amply) supplemented. However, $L(E)$ is not (amply) ss-supplemented filter by Proposition 3.7.

Theorem 3.9. If $F$ is a filter of $L$ with $\operatorname{rad}(F) \ll F$, then the following statements are equivalent:
(1) $F$ is ss-supplemented;
(2) $F$ is supplemented and $\operatorname{rad}(F)$ has an ss-supplement in $F$;
(3) $F$ is supplemented and $\operatorname{rad}(F) \subseteq \operatorname{Soc}(F)$.

Proof. (1) $\Rightarrow(2)$. It is clear.
$(2) \Rightarrow(3)$. Since $\operatorname{rad}(F) \ll F$ and $\operatorname{rad}(F)$ has $s s$-supplement in $F$, we get $F$ is a supplement of $\operatorname{rad}(F)$; hence $\operatorname{rad}(F)=\operatorname{rad}(F) \cap F$ is semisimple. Thus $\operatorname{rad}(F) \subseteq \operatorname{Soc}(F)$.
$(3) \Rightarrow(1)$. Let $G$ be a subfilter of $F$. By assumption, there exists a subfilter $H$ of $F$ such that $F=T(G \cup H)$ and $G \cap H \ll H$. Then $G \cap H \subseteq \operatorname{rad}(H) \subseteq \operatorname{rad}(F) \subseteq$ $\operatorname{Soc}(F)$; so $G \cap H$ is semisimple. It means that $F$ is $s s$-supplemented.

Corollary 3.10. If $F$ is a finitely generated filter of $L$, then $F$ is ss-supplemented if and only if it is supplemented and $\operatorname{rad}(F) \subseteq \operatorname{Soc}(F)$.

Proof. By Theorem 3.9, it suffices to show that $\operatorname{rad}(F) \ll F$. Assume that $F=$ $T(A)$, where $A=\left\{a_{1}, \ldots, a_{n}\right\}$ and let $F=T(H \cup \operatorname{rad}(F))$ for some subfilter $H$ of $F$. Since $\operatorname{rad}(F)=T\left(\cup_{G \ll F} G\right)$, there exists a finite subfilters $F_{i_{1}} \ll F, F_{i_{2}} \ll F$, $\ldots, F_{i_{r}} \ll F$ such that $a_{i} \in T\left(T\left(F_{i_{1}} \cup \cdots \cup F_{i_{r}}\right) \cup H\right)$ for $1 \leqslant i \leqslant r$ which implies that $F=T\left(T\left(F_{i_{1}} \cup \cdots \cup F_{i_{r}}\right) \cup H\right)$; hence $H=F$ by Proposition 1.2 (6)..

Lemma 3.11. If $K$ and $H$ are semisimple filters of $L$, then $T(K \cup H)$ is semisimple.

Proof. Let $G$ be a subfilter of $T(K \cup H)$. There exist a subfilter $K^{\prime}$ of $K$ and a subfilter $H^{\prime}$ of $H$ such that $K=(G \cap K) \oplus K^{\prime}$ (so $K^{\prime} \cap G=\{1\}$ ) and $H=$ $(H \cap G) \oplus H^{\prime}\left(\right.$ so $\left.H^{\prime} \cap G=\{1\}\right)$. If $x \in G \cap T\left(K^{\prime} \cup H^{\prime}\right)$, then $a \wedge b \leqslant x$ for some $a \in K^{\prime}$ and $b \in H^{\prime}$; so $x=(x \vee a) \wedge(x \vee b)=1$. Thus $G \cap T\left(K^{\prime} \cup H^{\prime}\right)=\{1\}$. It enough to show that $T(H \cup K)=T\left(G \cup T\left(K^{\prime} \cup H^{\prime}\right)\right)$. Since the inclusion $T\left(G \cup T\left(K^{\prime} \cup H^{\prime}\right)\right) \subseteq$ $T(K \cup H)$ is clear, we will prove the reverse inclusion. Let $z \in T(K \cup H)$. Then $c \wedge d \leqslant z$ for some $c \in K=T\left((G \cap K) \cup K^{\prime}\right)$ and $d \in H=T\left((H \cap G) \cup H^{\prime}\right)$. It follows that there are elements $c_{1} \in G \cap K, c_{2} \in K^{\prime}, d_{1} \in G \cap H$ and $d_{2} \in H^{\prime}$ such that $c_{1} \wedge c_{2} \leqslant c$ and $d_{1} \wedge d_{2} \leqslant d$; hence $\left(c_{1} \wedge d_{1}\right) \wedge\left(c_{2} \wedge d_{2}\right) \leqslant z$, where $c_{1} \wedge d_{1} \in G$ and $c_{2} \wedge d_{2} \in T\left(H^{\prime} \cup K^{\prime}\right)$ which implies that $z \in T\left(G \cup T\left(K^{\prime} \cup H^{\prime}\right)\right)$, and so we have equality. Thus $T(K \cup H)=G \oplus T\left(K^{\prime} \cup H^{\prime}\right)$.

Proposition 3.12. Let $F_{1}$ and $G$ be subfilters of a filter $F$ of $L$ with $F_{1}$ sssupplemented. If there is a ss-supplement for $T\left(F_{1} \cup G\right)$ in $F$, then the same is true for $G$.

Proof. Let $X$ be an $s s$-supplement of $T\left(F_{1} \cup G\right)$ in $F$ and $Y$ is an $s s$-supplement $T(X \cup G) \cap F_{1}$ in $F_{1}$. Then by an argument like that in [6, Proposition 2.10], we get $F=T(G \cup T(X \cup Y))$ and $T(X \cup Y) \cap G \ll T(X \cup Y)$. Moreover, $A=X \cap T(Y \cup G)$ is semisimple as a subfilter of the semisimple filter $X \cap T\left(F_{1} \cup G\right)$. Also, $Y \cap\left(F_{1} \cap T(X \cup G)\right)=Y \cap T(X \cup G)=B$ is semisimple; so $T(A \cup B)$ is semisimple by Lemma 3.11. Since $T(A \cup B)=G \cap T(X \cup Y)$, we get $T(X \cup Y)$ is an $s s$-supplement of $G$ in $F$.

Theorem 3.13. Let $F_{1}$ and $F_{2}$ be subfilters of $F$ such that $F=T\left(F_{1} \cup F_{2}\right)$. If $F_{1}$ and $F_{2}$ are ss-supplemented, then $F$ is ss-supplemented.

Proof. Let $G$ be a subfilter of $F$. The subfilter $\{1\}$ is $s s$-supplement of $F=$ $T\left(F_{1} \cup T\left(F_{2} \cup G\right)\right)$ in $F$. Since $F_{1}$ is $s s$-supplemented, $T\left(F_{2} \cup G\right)$ has an $s s$ supplement in $F$ by Proposition 3.12. Again applying Proposition 3.12, $G$ has an ss-supplement in $F$. This completes the proof.

Corollary 3.14. If $F_{1}, \ldots, F_{n}$ are ss-supplemented filters of $L$, then $T\left(U_{i=1}^{n} F_{i}\right)$ is an ss-supplemented filter.

Proof. Apply Theorem 3.13.
Lemma 3.15. Let $F$ be a filter of $L$. If every subfilter of $F$ is ss-supplemented, then $F$ is amply ss-supplemented.

Proof. Let $G$ and $H$ be subfilters of $F$ such that $F=T(G \cup H)$. By the assumption, $H=T\left((G \cap H) \cup H^{\prime}\right),(G \cap H) \cap H^{\prime}=G \cap H^{\prime} \ll H^{\prime}$ and $G \cap H^{\prime}$ is semisimple for some subfilter $H^{\prime}$ of $H$. Since $F=T\left(G \cup T\left((G \cap H) \cup H^{\prime}\right)\right)=T\left(G \cup H^{\prime}\right)$, we get $G$ has ample $s s$-supplements in $F$. Thus $F$ is amply $s s$-supplemented.

Lemma 3.16. Assume that $F$ is a amply ss-supplemented filter of $L$ and let $H$ be an ss-supplement subfilter in $F$. Then $H$ is amply ss-supplemented.

Proof. Let $H$ be an $s s$-supplement of a subfilter $G$ of $F$. Let $X$ and $Y$ be subfilters of $H$ such that $H=T(X \cup Y)$. Then

$$
F=T(H \cup G)=T(G \cup T(X \cup Y))=T(Y \cup T(G \cup X)) .
$$

As $F$ is amply $s s$-supplemented, $T(X \cup G)$ has an $s s$-supplement $Y^{\prime} \subseteq Y$ in $F$; so $F=T\left(Y^{\prime} \cup T(X \cup G)\right)=T\left(G \cup T\left(X \cup Y^{\prime}\right)\right)$. Since $X \cup Y^{\prime} \subseteq X \cup Y$, we obtain $T\left(X \cup Y^{\prime}\right) \subseteq T(X \cup Y)=H$. Then $H$ is an $s s$-supplement of $G$ in $F$ gives $H=T\left(X \cup Y^{\prime}\right)$ by minimality of $H$. Moreover, $X \cap Y^{\prime} \subseteq T(G \cup X) \cap Y^{\prime} \ll Y^{\prime}$, and so $X \cap Y^{\prime} \ll Y^{\prime}$ by Proposition 1.2 (1). As $T(G \cup X) \cap Y^{\prime}$ is semisimple, $X \cap Y^{\prime} \subseteq T(G \cup X) \cap Y^{\prime}$ is semisimple. Thus $H$ is amply ss-supplemented.

The next theorem gives a more explicit description of amply $s s$-supplemented filters.

Theorem 3.17. For a filter $F$ of $L$, the following statements are equivalent:
(1) $F$ is amply ss-supplemented;
(2) Every subfilter $G$ of $F$ is of the form $G=T(X \cup Y)$, where $X$ is ss-supp lemented and $Y \subseteq \operatorname{Soc}_{g}(F)$.
Proof. (1) $\Rightarrow(2)$. Assume that $F$ is amply $s s$-supplemented and let $G$ be a subfilter of $F$. Since $F$ is $s s$-supplemented, $G$ has an $s s$-supplements $H$ in $F$; so $F=$ $T(H \cup G)$. By the assumption, there exists a subfilter $X$ of $G$ such that $X$ is an $s s$-supplement of $H$ in $F$; so $F=T(X \cup H)$. Set $Y=G \cap H$. Since $H$ is an $s s$-supplement of $G$ in $F$, we have $Y=G \cap H \subseteq \operatorname{Soc}_{g}(H) \subseteq \operatorname{Soc}(F)$ by Proposition 3.3. By the modular law, $G=G \cap T(X \cup H)=T(X \cup(G \cap H))=T(X \cup Y)$, where $Y \subseteq \operatorname{Soc}_{g}(F)$ and $X$ is $s s$-supplemented by Lemma 3.16.
$(2) \Rightarrow(1)$. By the assumption, if $G$ is a subfilter of $F$, then $G=T(X \cup Y)$ with $X$ is $s s$-supplemented and $Y \subseteq \operatorname{Soc}_{g}(F) \subseteq \operatorname{Soc}(F)$ (so $Y$ is $s s$-supplemented). By Theorem 3.13, $G$ is $s s$-supplemented. Therefore $F$ is amply $s s$-supplemented by Lemma 3.15.

Corollary 3.18. For a filter $F$ of $L$, the following statements are equivalent:
(1) $F$ is amply ss-supplemented;
(2) Every subfilter of $F$ is ss-supplemented;
(3) Every subfilter of $F$ is amply ss-supplemented.

Proof. Apply Theorem 3.17.

## 4. $S S$-supplemented Quotient Filters

Quotient lattices are determined by equivalence relations rather than by ideals as in the ring case. If $F$ is a filter of a lattice $(L, \leqslant)$, we define a relation on $L$, given by $x \sim y$ if and only if there exist $a, b \in F$ satisfying $x \wedge a=y \wedge b$. Then $\sim$ is an equivalence relation on $L$, and we denote the equivalence class of $a$ by $a \wedge F$ and these collection of all equivalence classes by $\frac{L}{F}$. We set up a partial order $\leqslant_{Q}$ on $\frac{L}{F}$ as follows: for each $a \wedge F, b \wedge F \in \frac{L}{F}$, we write $a \wedge F \leqslant_{Q} b \wedge F$ if and only if $a \leqslant b$. It is straightforward to check that $\left(\frac{L}{F}, \leqslant_{Q}\right)$ is a poset. The following notation below will be kept in this section: Let $a \wedge F, b \wedge F \in \frac{L}{F}$ and set $X=\{a \wedge F, b \wedge F\}$. By definition of $\leqslant_{Q},(a \vee b) \wedge F$ is an upper bound for the set $X$. If $c \wedge F$ is any upper bound of $X$, then we can easily show that $(a \vee b) \wedge F \leqslant_{Q} c \wedge F$. Thus $(a \wedge F) \vee_{Q}(b \wedge F)=(a \vee b) \wedge F$. Similarly, $(a \wedge F) \wedge_{Q}(b \wedge F)=(a \wedge b) \wedge F$. Thus $\left(\frac{L}{F}, \leqslant_{Q}\right)$ is a lattice.
Remark 4.1. Let $G$ be a subfilter of a filter $F$ of $L$.
(1). If $a \in F$, then $a \wedge F=F$. By the definition of $\leqslant_{Q}$, it is easy to see that $1 \wedge F=F$ is the greatest element of $\frac{L}{F}$.
(2). If $a \in F$, then $a \wedge F=b \wedge F$ (for every $b \in L$ ) if and only if $b \in F$. In particular, $c \wedge F=F$ if and only if $c \in F$. Also, if $a \in F$, then $a \wedge F=F=1 \wedge F$.
(3). By the definition $\leqslant_{Q}$, we can easily show that if $L$ is distributive, then $\frac{L}{F}$ is
distributive.
(4). $\frac{F}{G}=\{a \wedge G: a \in F\}$ is a filter of $\frac{L}{G}$.
(5). If $K$ is a filter of $\frac{L}{G}$, then $K=\frac{F}{G}$ for some filter $F$ of $L$.
(6). If $H$ is a filter of $L$ such that $G \subseteq H$ and $\frac{F}{G}=\frac{H}{G}$, then $F=H$.
(7). If $H$ and $V$ are filters of $L$ containing $G$, then $\frac{F}{G} \cap \frac{H}{G}=\frac{V}{G}$ if and only if $V=H \cap F$.
(8). If $H$ is a filter of $L$ containing $G$, then $\frac{T(F \cup H)}{G}=T\left(\frac{H}{G} \cup \frac{F}{G}\right)$.

Proposition 4.2. Every quotient filter of a strongly local filter of $L$ is strongly local.

Proof. Let $G$ be a subfilter of a strongly local filter $F$ of $L$. Clearly, if $H$ is a subfilter of $F$ with $G \subseteq H$, then $H$ is a maximal subfilter of $F$ if and only if $\frac{H}{G}$ is a maximal subfilter of $\frac{F}{G}$; so the quotient filter $\frac{F}{G}$ is local. By assumption, $\operatorname{rad}\left(\frac{F}{G}\right)=$ $\frac{\operatorname{rad}(F)}{G} \subseteq \frac{\operatorname{Soc}(F)}{G}=\frac{\cap_{K \unlhd F} K}{G} \subseteq \cap_{\frac{K}{G} \unlhd \frac{F}{G}} \frac{K}{G} \subseteq \operatorname{Soc}\left(\frac{F}{G}\right) ;$ so $\operatorname{rad}\left(\frac{F}{G}\right)$ is semisimple. Thus $\frac{F}{G}$ is strongly local.

Lemma 4.3. Let $G, H, K$ be filters of $L$ such that $H \ll K$. Then $\frac{T(H \cup G)}{G} \ll$ $\frac{T(K \cup G)}{G}$.

Proof. Let $\frac{T(K \cup G)}{G}=T\left(\frac{U}{G} \cup \frac{T(H \cup G)}{G}\right)=\frac{T(U \cup T(H \cup G))}{G}$ for some subfilter $\frac{U}{G}$ of $\frac{T(K \cup G)}{G}($ so $U \subseteq T(K \cup G)) ;$ hence $T(K \cup G)=T(U \cup H)$. As $K=K \cap T(U \cup H)=$ $T(H \cup(U \cap K))$, we get $U \cap K=K$ since $H \ll K$. It follows that $T(K \cup G) \subseteq U$, and so $\frac{T(K \cup G)}{G}=\frac{U}{G}$.

Theorem 4.4. If $F$ is an ss-supplemented filter, then every quotient filter of $F$ is ss-supplemented.

Proof. Assume that $F$ is an $s s$-supplemented filter and let $\frac{F}{G}$ be a quotient filter of $F$. Let $\frac{H}{G}$ be a subfilter of $\frac{F}{G}$. By the assumption, there exists a subfilter $K$ of $F$ such that $F=T(K \cup H), K \cap H \ll H$ and $H \cap K$ is semisimple. Then $\frac{F}{G}=T\left(\frac{H}{G} \cup \frac{T(K \cup G)}{G}\right)$ and

$$
\frac{H}{G} \cap \frac{T(K \cup G)}{G}=\frac{H \cap T(K \cup G)}{G}=\frac{T((H \cap K) \cup G)}{G} \ll \frac{T(K \cup G)}{G}
$$

by Lemma 4.3 and Lemma 1.3. Since $H \cap K$ is semisimple, it follows from Proposition 1.4 that $\frac{H}{G} \cap \frac{T(K \cup G)}{G}=\frac{T((H \cap K) \cup G)}{G}$ is semisimple; so $\frac{T(K \cup G)}{G}$ is an sssupplement of $\frac{H}{G}$ in $\frac{F}{G}$. This completes the proof.

Corollary 4.5. If $F$ is an amply ss-supplemented filter of $L$, then every quotient filter of $F$ is amply ss-supplemented.

Proof. Let $\frac{V}{X}$ be a subfilter of $\frac{F}{X}$ such that $\frac{F}{X}=T\left(\frac{V}{X} \cup \frac{U}{X}\right)$ for some subfilter $\frac{U}{X}$ of $\frac{F}{X}$; so $F=T(V \cup U)$. Since $F$ is amply $s s$-supplemented, there is a subfilter $H \subseteq U$ such that $H$ is a $s s$-supplement of $V$ in $F$. By a similar argument like that in Theorem 4.4, $\frac{T(H \cup X)}{X} \subseteq \frac{U}{X}$ is a $s s$-supplement $\frac{V}{X}$ in $\frac{F}{X}$. Thus $\frac{F}{X}$ is amply ss-supplemented.

Lemma 4.6. Let $G$ and $H$ be subfilters of a filter $F$ of $L$ such that $F=T(G \cup H)$. If $K$ is a proper subfilter of $F$ such that $G \varsubsetneqq K$, then $K \cap H$ is a proper subfilter of $H$.

Proof. If $H \subseteq K$, then $F=T(G \cup H)$ gives $F=K$, a contradiction. Thus $H \nsubseteq K$ and $K \cap H \neq H$. By the relations, $K=K \cap T(G \cup H)=T(G \cup(H \cap K))$ and $K \neq G$, we obtain $K \cap H \neq\{1\}$. Therefore, $K \cap H$ is a proper subfilter of $H$.

Lemma 4.7. Let $G$ and $H$ be proper subfilters of a filter $F$ of $L$. If $F=T(G \cup H)$ and $H$ is simple, then $G$ is a maximal subfilter of $F$.

Proof. If $K$ is a subfilter of $F$ such that $G \varsubsetneqq K \varsubsetneqq F$, then $K \cap H$ is a proper subfilter of $H$ by Lemma 4.6 which is impossible since $H$ is simple. This completes the proof.

Proposition 4.8. Let $G$ and $H$ be subfilters of a filter $F$ of $L$. Assume $H$ to be a supplement of $G$ in $F$. Then the following hold:
(1). If $K$ is a maximal subfilter of $H$, then $T(G \cup K)$ is a maximal subfilter of $F$. In this case, $K=T(G \cup K) \cap H$.
(2). If $\operatorname{rad}(F) \ll F$, then $G$ is contained in a maximal subfilter of $F$.

Proof. (1). Since $K$ is a maximal subfilter of $H$, we find $K \neq H$. Since $H$ is a supplement of $G$ in $F$, we get $F \neq T(G \cup K)$. As $G \cap H \ll H$ and $K$ is a maximal subfilter of $H$, we conclude that $H \cap G \subseteq K$; hence $K=T(K \cup(G \cap H))=$ $H \cap T(G \cup K)$. Since $\frac{H}{K}$ is simple and $\frac{F}{K}=T\left(\frac{H}{K} \cup \frac{T(G \cup K)}{K}\right)$, we conclude that $\frac{T(G \cup K)}{K}$ is a maximal filter of $\frac{F}{K}$ by Lemma 4.7 which implies that $T(G \cup K)$ is a maximal subfilter of $F$ which contains $G$.
(2). If $G \subseteq \operatorname{rad}(F)$, then the assertion is clear. If $G \nsubseteq \operatorname{rad}(F)$, then by $[6$, Theorem $2.9(3)], \operatorname{rad}(H)=H \cap \operatorname{rad}(F) \neq H$, i.e. there is a maximal subfilter $K$ of $H$. Now the assertion follows from (1).

Definition 4.9. Let $F$ be a filter of $L . F$ is called the internal direct sum of the set $\left\{F_{i}: i \in I\right\}$ of subfilters of $F: F=\oplus_{i \in I} F_{i}$ if and only if $F=T\left(\cup_{i \in I} F_{i}\right)$ and for each $j \in I, F_{j} \cap T\left(\cup_{i \in I_{i \neq j}} F_{i}\right)=\{1\}$.

Lemma 4.10. If $\left\{F_{i}\right\}_{i \in I}$ is an indexed set of subfilters of a filter $F$ of $L$ with $F=\oplus_{i \in I} F_{i}$, then $\operatorname{rad}(F)=\oplus_{i \in I} \operatorname{rad}\left(F_{i}\right)$ and $\operatorname{Soc}(F)=\oplus_{i \in I} \operatorname{Soc}\left(F_{i}\right)$.

Proof. By the assumption, for each $i \in I, \operatorname{rad}\left(F_{i}\right)=F_{i} \cap \operatorname{rad}(F)$ by [6, Theorem 2.9 (3)]. It suffices to show that $\operatorname{rad}(F) \subseteq \oplus_{i \in I} \operatorname{rad}\left(F_{i}\right)$. Let $x \in \operatorname{rad}(F)$. Then $\left(x_{i_{1}} \wedge x_{i_{2}} \wedge \cdots \wedge x_{i_{k}}\right) \leqslant x$, where $x_{i_{1}} \in F_{i_{1}} \subseteq \operatorname{rad}(F), \ldots, x_{i_{k}} \in F_{i_{k}} \subseteq$ $\operatorname{rad}(F)$. Therefore, $F=\oplus_{i \in I} F_{i}$ gives there exist subfilters $F_{t_{1}}, \cdots, F_{t_{s}}$ of $F$ such that $x_{i_{1}} \in F_{t_{1}} \cap \operatorname{rad}(F)=\operatorname{rad}\left(F_{t_{1}}\right), \ldots, x_{i_{k}} \in F_{t_{s}} \cap \operatorname{rad}(F)=\operatorname{rad}\left(F_{t_{s}}\right)$; so $x \in T\left(\operatorname{rad}\left(F_{t_{1}}\right) \cup \cdots \cup \operatorname{rad}\left(F_{t_{s}}\right)\right) \subseteq \oplus_{i \in I} \operatorname{rad}\left(F_{i}\right)$, and so we have equality. Since the inclusion $\oplus_{i \in I} \operatorname{Soc}\left(F_{i}\right) \subseteq \operatorname{Soc}(F)$ is clear, we will prove the reverse inclusion. Let $z \in \operatorname{Soc}(F)$. Then

$$
z=\left(z \vee a_{1}\right) \wedge \cdots \wedge\left(z \vee a_{n}\right)
$$

for some $a_{1} \in F_{j_{1}} \subseteq F, \ldots, a_{n} \in F_{j_{n}} \subseteq F$; hence $z \vee a_{1} \in F_{j_{1}} \cap \operatorname{Soc}(F)=$ $\operatorname{Soc}\left(F_{j_{1}}\right), \ldots, z \vee a_{n} \in F_{j_{n}} \cap \operatorname{Soc}(F)=\operatorname{Soc}\left(F_{j_{1}}\right)$. It follows that $z \in T\left(\operatorname{Soc}\left(F_{j_{1}}\right) \cup\right.$ $\left.\cdots \cup \operatorname{Soc}\left(F_{j_{n}}\right)\right) \subseteq \oplus_{i \in I} \operatorname{Soc}\left(F_{i}\right)$. This completes the proof.

Let $L, L^{\prime}$ be two lattice. Then a lattice homomorphism $f: L \rightarrow L^{\prime}$ is a map from $L$ to $L^{\prime}$ satisfying $f(x \vee y)=f(x) \vee f(y)$ and $f(x \wedge y)=f(x) \wedge f(y)$ for all $x, y \in L$. A bijective lattice homomorphism $f$ is called a lattice isomorphism (in this case we write $L \cong L^{\prime}$ ).

Lemma 4.11. If $A$ and $B$ are filters of $L$, then $\frac{T(A \cup B)}{A} \cong \frac{B}{A \cap B}$.
Proof. Define $f: \frac{B}{A \cap B} \rightarrow \frac{T(A \cup B)}{A}$ by $f(b \wedge(A \cap B))=b \wedge A$. It is clear that $f$ is well-defined. We will show $f$ is one-to-one: Let $f\left(b_{1} \wedge(A \cap B)\right)=f\left(b_{2} \wedge(A \cap B)\right)$, where $b_{1}, b_{2} \in B$. Then $b_{1} \wedge A=b_{2} \wedge A$; and so $b_{1} \wedge c_{1}=b_{2} \wedge c_{2}$ for some $c_{1}, c_{2} \in A$. Hence

$$
\left(b_{1} \wedge c_{1}\right) \vee\left(b_{1} \wedge b_{2}\right)=\left(b_{2} \wedge c_{2}\right) \vee\left(b_{2} \wedge b_{1}\right)
$$

The left side is equal to $\left[b_{1} \vee\left(b_{1} \wedge b_{2}\right)\right] \wedge\left[c_{1} \vee\left(b_{1} \wedge b_{2}\right)\right]=b_{1} \wedge\left[c_{1} \vee\left(b_{1} \wedge b_{2}\right)\right]$. Similarly, the right side is equal to $b_{1} \wedge\left[c_{1} \vee\left(b_{1} \wedge b_{2}\right)\right]$. Thus $b_{1} \wedge(A \cap B)=b_{2} \wedge(A \cap B)$. We claim $f$ is surjective: Let $z \wedge A \in \frac{T(A \cup B)}{A}$, where $z \in T(A \cup B)$. Hence there exist $a \in A, b \in B$ such that $a \wedge b \leqslant z$. Thus $(z \vee b) \wedge a=(z \wedge a) \vee(b \wedge a)=z \wedge a$. Therefore $(z \vee b) \wedge A=z \wedge A$. Thus $f((z \vee b) \wedge(A \cap B))=(z \vee b) \wedge A=z \wedge A$ and $(z \vee b) \wedge(A \cap B) \in \frac{B}{A \cap B}$. Now, we show that $f$ is a lattice homomorphism. Let $b_{1} \wedge(A \cap B), b_{2} \wedge(A \cap B) \in \frac{B}{A \cap B}$. Then $f\left(\left(b_{1} \wedge(A \cap B)\right) \wedge_{Q}\left(b_{2} \wedge(A \cap B)\right)\right)=f\left(\left(b_{1} \wedge\right.\right.$ $\left.\left.b_{2}\right) \wedge(A \cap B)\right)=\left(b_{1} \wedge b_{2}\right) \wedge A=\left(b_{1} \wedge A\right) \wedge_{Q}\left(b_{2} \wedge A\right)=f\left(b_{1} \wedge(A \cap B)\right) \wedge_{Q} f\left(b_{2} \wedge(A \cap B)\right)$.

Similarly, $f\left(\left(b_{1} \wedge(A \cap B)\right) \vee_{Q}\left(b_{2} \wedge(A \cap B)\right)\right)=f\left(b_{1} \wedge(A \cap B)\right) \vee_{Q} f\left(b_{2} \wedge(A \cap B)\right)$. This completes the proof.

Lemma 4.12. Assume that $\left\{F_{i}\right\}_{i \in I}$ is an indexed set of subfilters of a filter $F$ of $L$ such that $F=\oplus_{i \in I} F_{i}$ and let $G$ be a subfilter of $F$. Then $\frac{F}{G}=\oplus_{i \in I} \frac{T\left(F_{i} \cup G\right)}{G}$.

Proof. For each $j \in I$, let $x \wedge G \in \frac{T\left(F_{j} \cup G\right)}{G} \cap T\left(\cup_{i \in I_{i \neq j}} \frac{T\left(F_{i} \cup G\right)}{G}\right)$. Then $x \in T\left(F_{j} \cup G\right)$ gives there exist $f_{j} \in F_{j}$ and $g_{j} \in G$ such that $x \wedge G=\left(\left(x \vee f_{j}\right) \wedge\left(x \vee g_{j}\right)\right) \wedge G=$ $\left(x \vee f_{j}\right) \wedge G$; so $x=f_{j} \vee x \in F_{j}$. Similarly, there are subfilters $F_{i_{1}}, \ldots, F_{i_{s}}$ such that $x \in T\left(\cup_{k=1_{k \neq j}}^{s} F_{i_{k}}\right) ;$ hence $x=1$. Thus $\frac{T\left(F_{j} \cup G\right)}{G} \cap T\left(\cup_{i \in I_{i \neq j}} \frac{T\left(F_{i} \cup G\right)}{G}\right)=\{1 \wedge G\}$.

It is enough to show that $\frac{F}{G} \subseteq \oplus_{i \in I} \frac{T\left(F_{i} \cup G\right)}{G}$. Let $y \wedge G \in \frac{F}{G}$. Then there exist $f_{i_{1}} \in F_{i_{1}}, \ldots, f_{i_{t}} \in F_{i_{t}}$ such that $f_{i_{1}} \wedge \cdots \wedge f_{i_{t}} \leqslant y$; so $\left(f_{i_{1}} \wedge G\right) \wedge_{Q} \cdots \wedge_{Q}\left(f_{i_{t}} \wedge G\right) \leqslant$ $y \wedge G$. It follows that $\left.y \wedge G \in T\left(\frac{T\left(F_{i_{1}} \cup G\right)}{G} \cup \cdots \cup \frac{T\left(F_{i_{t}} \cup G\right.}{G}\right) \subseteq T\left(\cup_{i \in I} \frac{T\left(F_{i} \cup G\right.}{G}\right)\right)$, as required.

Remark 4.13. Let $F$ be a filter of $F$.
(1). If $G$ is a hollow subfilter of a filter $F$ of $L$ that is not small in $F$. Then there exists a proper subfilter $K$ of $F$ such that $F=T(G \cup K)$. Since $G$ is hollow, we get $G \cap K \ll G$. Thus $G$ is a supplement in $F$. Thus $\operatorname{rad}(G)=G \cap \operatorname{rad}(F)$ by [6, Theorem 2.9 (3)].
(2). If $G$ is a direct summand of $F$ such that $G \ll F$, then $G=\{1\}$.
(3). A filter $F$ of $L$ is said to be coatomic if every proper subfilter of $F$ is contained in a maximal subfilter of $F$. It is easy to see that $\operatorname{rad}(F) \ll F$.
Lemma 4.14. Let $\left\{H_{\alpha}\right\}_{\alpha \in A}$ be an indexed set of simple subfilters of the filter $F$ of a lattice L. If $F=T\left(\cup_{\alpha \in A} H_{\alpha}\right)$, then for each subfilter $K$ of $F$ there is a subset $B$ of $A$ such that $\left\{H_{\alpha}\right\}_{\alpha \in B}$ is independent and $F=K \oplus\left(T\left(\cup_{\alpha \in B} H_{\alpha}\right)\right)$.
Proof. Let $K$ be a subfilter of $F$. Then there is a subset $B$ of $A$ maximal with respect to conditions that $\left\{H_{\alpha}\right\}_{\alpha \in B}$ is independent and $K \cap\left(T\left(\cup_{\alpha \in B} H_{\alpha}\right)\right)=\{1\}$. Let $M=T\left(K \cup\left(T\left(\cup_{\alpha \in B} H_{\alpha}\right)\right)\right)$. For each $\alpha \in A$, we have either $H_{\alpha} \cap M=\{1\}$ or $H_{\alpha} \cap M=H_{\alpha}$. If $H_{\alpha} \cap M=\{1\}$, then we have a contradiction with the maximality of $B$. Thus $H_{\alpha} \subset M$ for each $\alpha \in A$, hence $F=K \oplus\left(T\left(\cup_{\alpha \in B} H_{\alpha}\right)\right)$.

Proposition 4.15. Let $F=\oplus_{i \in I} F_{i}$ be a filter of $L$, where each $F_{i}$ is a local filter. If $\operatorname{rad}(F) \ll F$, then $F$ is supplemented.
Proof. By [6, Theorem 2.21] and Remark 4.13, for each $i \in I, F_{i}$ is not small in $F\left(\right.$ so $\left.\operatorname{rad}\left(F_{i}\right)=F_{i} \cap \operatorname{rad}(F) \neq F_{i}\right)$ and $\frac{F_{i}}{\operatorname{rad}\left(F_{i}\right)}$ is simple. Let $U$ be a subfilter of $F$. By Lemma 4.11 and Lemma 4.12, we have $\bar{F}=\frac{F}{\operatorname{rad}(F)}=\oplus_{i \in I} \frac{T\left(F_{i} \cup \operatorname{rad}(F)\right)}{r^{\operatorname{rad}(F)}} \cong$ $\oplus_{i \in I} \frac{F_{i}}{\operatorname{rad}\left(F_{i}\right)}$ is a direct sum of simple filters, and so $\bar{F}=\bar{U} \oplus\left(\oplus_{i \in J} \frac{F_{i}}{\operatorname{rad}\left(F_{i}\right)}\right)$ for some $J \subseteq I$, where $\bar{U}=\frac{T(U \cup \operatorname{rad}(F))}{\operatorname{rad}(\bar{F})}$, by Lemma 4.15. Now we set $\left.\bar{V}=\oplus_{i \in J} \frac{F_{i}}{\operatorname{rad}\left(F_{i}\right)}\right)$ (so $\left.V=\oplus_{i \in J} F_{i}\right)$. Since $\bar{F}=\bar{U} \oplus \bar{V}$, we get that $F=T(\operatorname{rad}(F) \cup T(U \cup V)$ ) which implies $F=T(U \cup V)$ since $\operatorname{rad}(F) \ll F$. Moreover, $\bar{U} \cap \bar{V}=\{\operatorname{rad}(F)\}$ gives $U \cap V \subseteq \operatorname{rad}(F)$; so $U \cap V \ll F$ by Proposition 1.2 (1). Since $V$ is a direct summand of $F, U \cap V \ll V$ by Proposition 1.4 (c). Thus $F$ is supplemented.

Theorem 4.16. Let $F=\oplus_{i \in I} F_{i}$ be a filter of $L$, where each $F_{i}$ is a strongly local filter. Then $F$ is ss-supplemented and coatomic.
Proof. Since $F_{i}$ is strongly local for every $i \in I$, it is local and $\operatorname{rad}\left(F_{i}\right) \subseteq \operatorname{Soc}\left(F_{i}\right)$ $(i \in I)$. It then follows from Lemma 4.10 that $\operatorname{rad}(F)=\oplus_{i \in I} \operatorname{rad}\left(F_{i}\right) \subseteq \oplus_{i \in I} \operatorname{Soc}\left(F_{i}\right)$ $=\operatorname{Soc}(F)$; hence $\operatorname{rad}(F) \ll F$ by Proposition 1.4 (a). As strongly local filters are local, Proposition 4.16 gives $F$ is supplemented. Therefore, $F$ is $s s$-supplemented by Theorem 3.9. Let $H$ be a proper subfilter of $F$. By Proposition 4.8 (2), $H$ is contained in a maximal subfilter of $F$, that is, $F$ is coatomic.

Acknowledgement. We would like to thank the referees for valuable comments.

## References

[1] G. Bilhan and A.T. Güroglu, A variation of supplemented modules, Turkish J. Math., 37 (2013), 418 - 426.
[2] G. Birkhoff, Lattice theory, Amer. Math. Soc., 1973.
[3] E. Büyükasik, E. Mermut and S. Özdemir, Rad-supplemented modules, Rend. Semin. Mat. Univ. Padova, 124 (2010), 157 - 177.
[4] G. Calugareanu, Lattice Concepts of Module Theory. Kluwer Academic Publishers, 2000.
[5] S. Ebrahimi-Atani, On secondary modules over Dedekind domains, Southeast Asian Bull. Math. 25 (2001), $1-6$.
[6] S. Ebrahimi Atani, w-Supplemented property in the lattices, Quasigroups and Related Systems, 29 (2021), $31-44$.
[7] S. Ebrahimi Atani and M. Chenari, Supplemented property in the lattices, Serdica Math. J., 46 (2020), $73-88$.
[8] S. Ebrahimi Atani, S. Dolati Pish Hesari, M. Khoramdel and M. Sedghi Shanbeh Bazari, A simiprime filter-based identity-summand graph of a lattice, Le Matematiche, 73 (2018), no.2, 297-318.
[9] S. Ebrahimi Atani and M. Sedghi Shanbeh Bazari, On 2-absorbing filters of lattices, Discuss. Math. Gen. Algebra Appl. 36 (2016), 157 - 168.
[10] S. Ebrahimi Atani and M. Sedghi Shanbeh Bazari, Decomposable filters of lattices, Kragujevac J. Math., 43 (1) (2019), $59-73$.
[11] A. Harmanci, D. Keskin and P.F. Smith, On $\oplus$-supplemented modules, Acta Math. Hungar., 83 (1999), $161-169$.
[12] E. Kaynar, H. Calisici and E. Türkmen, SS-supplemented modules, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., 69 (2020), $473-485$.
[13] F. Kasch and E.A. Mares, Eine Kennzeichnung semi-perfekter Moduln. Nagoya Math. J., 27 (1966), $525-529$.
[14] S.H. Mohamed and B.J. Müller, Continuous and discrete modules. Cambridge University Press, London, 1990.
[15] Y. Wang and D. Ding, Generalized supplemented modules. Taiwanese J. Math., 10 (2006), $1589-1601$.
[16] H. Zömpschinger, Komplementierte Moduln über Dedekindringen. J. Algebra, 29 (1974), $42-56$.
[17] D.X. Zhou and X.R. Zhang, Small-essential submodules and Morita duality, Southeast Asian Bull. Math., 35 (2011), 1051 - 1062.

Received November 2, 2020
Department of Mathematics, University of Guilan, P.O.Box 1914, Rasht, Iran
E-mails: ebrahimi@guilan.ac.ir, mehdikhoramdel@gmail.com, saboura_dolati@yahoo.com, mhs.nikmard@gmail.com

# Hom-Jacobi-Jordan and Hom-antiassociative algebras with symmetric invariant nondegenerate bilinear forms 

Cyrille Essossolim Haliya and Gbêvèwou Damien Houndedji


#### Abstract

The aim of this paper is first to introduce and study quadratic Hom-Jacobi-Jordan algebras, which are Hom-Jacobi-Jordan algebras with symmetric invariant nondegenerate bilinear forms. We provide several constructions leading to examples. We reduce the case where the twist map is invertible to the study of involutive quadratic Jacobi-Jordan algebras. Also elements of a representation theory for Hom-Jacobi-Jordan algebras, including adjoint and coadjoint representations are supplied with application to quadratic Hom-Jacobi-Jordan algebras.

Secondly, introduce a hom-antiassociative algebra built as a direct sum of a given hom- antiassociative algebra $(\mathcal{A}, \cdot, \alpha)$ and its dual $\left(\mathcal{A}^{*}, \circ, \alpha^{*}\right)$, endowed with a non-degenerate symmetric bilinear form $\mathcal{B}$, where $\cdot$ and $\circ$ are the products defined on $\mathcal{A}$ and $\mathcal{A}^{*}$, respectively, $\alpha$ and $\alpha^{*}$ stand for the corresponding algebra homomorphisms.


## 1. Introduction

The Hom-algebra structures arose first in quasi-deformation of Lie algebras of vector fields. Discrete modifications of vector fields via twisted derivations lead to Hom-Lie and quasi-Hom-Lie structures in which the Jacobi condition is twisted. The first examples of $q$-deformations, in which the derivations are replaced by $\sigma$-derivations, concerned the Witt and Virasoro algebras, see for example [2, 9, $10,11,12,14,16]$. A general study and construction of Hom-Lie algebras are considered in $[13,17,18]$ and a more general framework bordering color and super Lie algebras was introduced in $[13,17,18,19]$. In the subclass of Hom-Lie algebras skew-symmetry is untwisted, whereas the Jacobi identity is twisted by a single linear map and contains three terms as in Lie algebras, reducing to ordinary Lie algebras when the twisting linear map is the identity map.

In [21] and [22], the theory of Hom-coalgebras and related structures are developed. Further development could be found in [3, 4, 15].

The quadratic Lie algebras, also called metrizable or orthogonal, are intensively studied, one of the fundamental results of constructing and characterizing quadratic Lie algebras is due to Medina and Revoy (see [23]) using double extension, while the concept of $T^{*}$-extension is due to Bordemann (see [7]). The $T^{*}$ -

[^1]extension concerns nonassociative algebras with nondegenerate associative symmetric bilinear form, such algebras are called metrizable algebras. In [7], the metrizable nilpotent associative algebras and metrizable solvable Lie algebras are described. The study of graded quadratic Lie algebras could be found in [5]. Jacobi-Jordan algebras (JJ algebras for short) were introduced in [8] in 2014 as vector spaces $\mathcal{A}$ over a field k , equipped with a bilinear map $\cdot: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ satisfying the Jacobi identity and instead of the skew-symmetry condition valid for Lie algebras the commutativity $x \cdot y=y \cdot x$, for all $x, y \in \mathcal{A}$ is imposed. This class of algebras appear under different names in the literature reflecting, perhaps, the fact that it was considered from different viewpoints by different communities, sometimes not aware of each other's results (see [27] for more details). Wörz-Busekros in [26] relates these type of algebras with Bernstein algebras. One crucial remark is that JJ algebras are examples of the more popular and well-referenced Jordan algebras [1, 24] introduced in order to achieve an axiomatization for the algebra of observables in quantum mechanics. In [8] the authors achieved the classification of these algebras up to dimension 6 over an algebraically closed field of characteristic different from 2 and 3. There's two entertaining facts on Jacobi-Jordan algebras. The first one is that in [1] prove that a finite dimensional JJ algebras is Frobenius if and only if there exists an invariant non degenerate bilinear form (Proposition 1.8). The other entertaining fact (noted in [25]) is that Jacobi-Jordan algebras can be produced from antiassociative algebras the same way as they are produced from associative ones. Hence there's a strong link in between antiassociative algebras and Jacobi-Jordan algebras. By antiassociative algebras, we mean algebras subject to operation $(a, b) \rightarrow a b$ satisfying $(a b) c+a(b c)=0$ for each $a, b$ and $c$. This class of algebras first arise in the literature specially in [25] where the authors gave their main properties. The purpose of this paper on the first hand is to study and construct quadratic Hom-Jacobi-Jordan algebras as S. Benayadi and A. Makhlouf did for the case of Lie algebra structures in [6]. On the other hands to establish a double construction of hom-antiassociative algebra equipped with a non degenerate symmetric invariant bilinear form.

In the first Section, we define the notions of Hom-Jacobi-Jordan algebras, Homantiassociative algebras and their related propreties. Some key constructions of Hom-Jacobi-Jordan algebras are derived. Section 2 is dedicated to a theory of representations of Hom-Jacobi-Jordan algebras including adjoint and coadjoint representations. In Section 3, we introduce the notion of quadratic Hom-Jacobi-Jordan algebra and give some properties. Several procedures of construction leading to some examples are provided in Section 4. We show in Section 5 that there exists biunivoque correspondence between some classes of Jacobi-Jordan algebras and classes of Hom-Jacobi-Jordan algebras. In Section 6, we introduce the concepts of matched pairs of hom-antiassociative algebras and establish some properties. In Section 7, we give and discuss of double constructions of multiplicative homantiassociative algebras. In section 8, we end with some concluding remarks.

## 2. Preliminaries

In the following we give the definitions of Hom-Jacobi-Jordan and Hom- antiassociative algebraic structures generalizing the well known Jacobi-Jordan and antiassociative algebras. Also we define in this case the notion of modules over Hom-algebras.

Throughout the article we let $\mathbb{K}$ be an algebraically closed field of characteristic 0 . We mean by a Hom-algebra a triple $(A, \mu, \alpha)$ consisting of a vector space $A$, a bilinear map $\mu$ and a linear map $\alpha$. In all the examples involving the unspecified products are either symmetric or zero.

The notion of Hom-Lie algebra was introduced by Hartwig, Larsson and Silvestrov in $[13,17,18]$ motivated initially by examples of deformed Lie algebras coming from twisted discretizations of vector fields. In this article, we follow notations from [20]. In this part, we analogously define the Hom-Jacobi-Jordan algebras which is a kind of deformation of Jacobi-Jordan algebras. But first let's recall the notions antiassociative and Jacobi-Jordan algebras.

Definition 2.1. [25] Let "." be a bilinear product in a vector space $\mathcal{A}$. Suppose that it satisfies the following law:

$$
\begin{equation*}
(x \cdot y) \cdot z=-x \cdot(y \cdot z) \tag{2.1}
\end{equation*}
$$

Then, we call the pair $(\mathcal{A}, \cdot)$ an antiassociative algebra.
Definition 2.2. [27] An algebra ( $\mathfrak{g},[$,$] ) over K$ is called Jacobi-Jordan if it is commutative:

$$
\begin{equation*}
[x, y]=[y, x] \tag{2.2}
\end{equation*}
$$

and satisfies the Jacobi identity:

$$
\begin{equation*}
[[x, y], z]+[[z, x], y]+[[y, z], x]=0 \tag{2.3}
\end{equation*}
$$

for any $x, y, z \in \mathfrak{g}$.
Theorem 2.3. [27] Given an antiassociative algebra $(\mathcal{A}, \cdot)$, the new algebra $\mathcal{A}^{\dagger}$ with multiplication give by the "anticommutator"

$$
\begin{equation*}
[a, b]=\frac{1}{2}(a \cdot b+b \cdot a) \tag{2.4}
\end{equation*}
$$

is a Jacobi-Jordan algebra.
Since Jacobi-Jordan algebras are commutative, the left and right actions of an algebra coincide, so we can speak about just modules.
Definition 2.4. [27] A vector space $V$ is a module over a Jacobi-Jordan algebra $\mathfrak{g}$, if there is a linear map (a representation) $\rho: \mathfrak{g} \rightarrow \operatorname{End}(V)$ such that

$$
\begin{equation*}
\rho([x, y])(v)=-\rho(x)(\rho(y) v)-\rho(y)(\rho(x) v) \tag{2.5}
\end{equation*}
$$

for any $x, y \in \mathfrak{g}$ and $v \in V$.

Definition 2.5. A Hom-Jacobi-Jordan algebra is a triple ( $\mathfrak{g},[],, \alpha$ ) consisting of a linear space $\mathfrak{g}$ on which $[]:, \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ is a bilinear map and $\alpha: \mathfrak{g} \rightarrow \mathfrak{g}$ a linear map satisfying

$$
\begin{gather*}
{[x, y]=[y, x], \quad(\text { symmetry })}  \tag{2.6}\\
\circlearrowleft_{x, y, z}[\alpha(x),[y, z]]=0 \quad \text { (Hom-Jacobi condition) } \tag{2.7}
\end{gather*}
$$

for all $x, y, z$ from $\mathfrak{g}$, where $\circlearrowleft_{x, y, z}$ denotes summation over the cyclic permutation on $x, y, z$.

We recover classical Jacobi-Jordan algebra when $\alpha=i d_{\mathfrak{g}}$ and the identity (2.7) is the Jacobi identity in this case.

Proposition 2.6. Every symmetric bilinear map on a 2-dimensional linear space defines a Hom-Jacobi-Jordan algebra.

Proof. The Hom-Jacobi identity (2.7) is satisfied for any triple $(x, x, y)$.
Let $(\mathfrak{g}, \mu, \alpha)$ and $\mathfrak{g}^{\prime}=\left(\mathfrak{g}^{\prime}, \mu^{\prime}, \alpha^{\prime}\right)$ be two Hom-Jacobi-Jordan algebras. A linear map $f: \mathfrak{g} \rightarrow \mathfrak{g}^{\prime}$ is a morphism of Hom-Jacobi-Jordan algebras if

$$
\mu^{\prime} \circ(f \otimes f)=f \circ \mu \quad \text { and } \quad f \circ \alpha=\alpha^{\prime} \circ f
$$

In particular, Hom-Jacobi-Jordan algebras $(\mathfrak{g}, \mu, \alpha)$ and $\left(\mathfrak{g}, \mu^{\prime}, \alpha^{\prime}\right)$ are isomorphic if there exists a bijective linear map $f$ such that

$$
\mu=f^{-1} \circ \mu^{\prime} \circ(f \otimes f) \quad \text { and } \quad \alpha=f^{-1} \circ \alpha^{\prime} \circ f
$$

A subspace $I$ of $\mathfrak{g}$ is said to be an ideal if for $x \in I$ and $y \in \mathfrak{g}$ we have $[x, y] \in I$ and $\alpha(x) \in I$. A Hom-Jacobi-Jordan algebra in which the anticommutator is not identically zero and which has no proper ideals is called simple.

Example 2.7. Let $\left\{x_{1}, x_{2}, x_{3}\right\}$ be a basis of a 3-dimensional linear space $\mathfrak{g}$ over $\mathbb{K}$. The following bracket and linear map $\alpha$ on $\mathfrak{g}=\mathbb{K}^{3}$ define a Hom-Jacobi-Jordan algebra over $\mathbb{K}$ :

$$
\begin{array}{lll}
{\left[x_{1}, x_{1}\right]=-b x_{3},} & {\left[x_{1}, x_{2}\right]=b\left(-x_{1}+\frac{1}{2} x_{3}\right),} & \alpha\left(x_{1}\right)=x_{1}, \\
{\left[x_{2}, x_{2}\right]=a x_{3},} & {\left[x_{1}, x_{3}\right]=\frac{b}{2} x_{2},} & \alpha\left(x_{2}\right)=2 x_{2}, \\
{\left[x_{3}, x_{3}\right]=a x_{3},} & {\left[x_{2}, x_{3}\right]=2\left(a x_{1}+b x_{3}\right),} & \alpha\left(x_{3}\right)=2 x_{3}
\end{array}
$$

with $\left[x_{2}, x_{1}\right],\left[x_{3}, x_{1}\right]$ and $\left[x_{3}, x_{2}\right]$ defined via symmetry. It's a Jacobi-Jordan algebra only in case $b=0$ and $a=0$ or $b=0$ and $a \neq 0$, since

$$
\left[x_{1},\left[x_{2}, x_{3}\right]\right]+\left[x_{3},\left[x_{1}, x_{2}\right]\right]+\left[x_{2},\left[x_{3}, x_{1}\right]\right]=\frac{b^{2}}{2} x_{2}+a b x_{3}
$$

For simplicity we will use in the sequel the following terminology and notations.

Definition 2.8. Let $(\mathfrak{g},[\cdot, \cdot], \alpha)$ be a Hom-Jacobi-Jordan algebra. The Homalgebra is called

- a multiplicative Hom-Jacobi-Jordan algebra if for all $x, y \in \mathfrak{g}$ we have

$$
\alpha([x, y])=[\alpha(x), \alpha(y)] ;
$$

- a regular Hom-Jacobi-Jordan algebra if $\alpha$ is an automorphism;
- an involutive Hom-Jacobi-Jordan algebra if $\alpha$ is an involution, that is $\alpha^{2}=i d$.

The center of the Hom-Jacobi-Jordan algebra is denoted $\mathcal{Z}(\mathfrak{g})$ and defined by

$$
\mathcal{Z}(\mathfrak{g})=\{x \in \mathfrak{g}:[x, y]=0 \forall y \in \mathfrak{g}\} .
$$

We give in the following the definition of Hom-antiassociative algebra which provide a different way for constructing Hom-Jacobi-Jordan algebras by extending the fundamental construction of Jacobi-Jordan algebras from antiassociative algebras via anticommutator bracket multiplication.

Definition 2.9. A Hom-antiassociative algebra is a triple $(A, \mu, \alpha)$ consisting of a linear space $A, \mu: A \times A \rightarrow A$ is a bilinear map and $\alpha: A \rightarrow A$ is a linear map, satisfying

$$
\begin{equation*}
\mu(\alpha(x), \mu(y, z))=-\mu(\mu(x, y), \alpha(z)) . \tag{2.8}
\end{equation*}
$$

We can talk about functor from the category of Hom-antiassociative algebras in the category of Hom-Jacobi-Jordan algebras.

Proposition 2.10. Let $(A, \mu, \alpha)$ be a Hom-antiassociative algebra defined on the linear space $A$ by the multiplication $\mu$ and a homomorphism $\alpha$. Then the triple $(A,[],, \alpha)$, where the bracket is defined for $x, y \in A$ by $[x, y]=\mu(x, y)+\mu(y, x)$, is a Hom-Jacobi-Jordan algebra.

Proof. The bracket is obviously symmetric and with a direct computation we have

$$
\begin{aligned}
& {[\alpha(x),[y, z]]+[\alpha(z),[x, y]]+[\alpha(y),[z, x]]} \\
& \quad=\mu(\alpha(x), \mu(y, z))+\mu(\alpha(x), \mu(z, y))+\mu(\mu(y, z), \alpha(x))+\mu(\mu(z, y), \alpha(x)) \\
& \quad+\mu(\alpha(z), \mu(x, y))+\mu(\alpha(z), \mu(y, x))+\mu(\mu(x, y), \alpha(z))+\mu(\mu(y, x), \alpha(z)) \\
& \quad+\mu(\alpha(y), \mu(z, x))+\mu(\alpha(y), \mu(x, z))+\mu(\mu(z, x), \alpha(y))+\mu(\mu(x, z), \alpha(y))=0 .
\end{aligned}
$$

A structure of module over Hom-associative algebras is defined in [21] and [22]. Here we define the analogous notion over Hom-antiassociative algebras as follows.
Definition 2.11. Let $(\mathcal{A}, \mu, \alpha)$ be a Hom-antiassociative algebra. A (left) $\mathcal{A}$ module is a triple $(M, f, \gamma)$ where $M$ is a $\mathbb{K}$-vector space and $f, \gamma$ are $\mathbb{K}$-linear maps, $f: M \rightarrow M$ and $\gamma: \mathcal{A} \otimes M \rightarrow M$, such that the following diagram commutes:


Remark 2.12. A Hom-antiassociative algebra $(\mathcal{A}, \mu, \alpha)$ is a left $\mathcal{A}$-module with $M=\mathcal{A}, f=\alpha$ and $\gamma=\mu$.

The following result shows that Jacobi-Jordan algebras deform into Hom-Jacobi-Jordan algebras via endomorphisms.

Theorem 2.13. Let $(\mathfrak{g},[]$,$) be a Jacobi-Jordan algebra and \alpha: \mathfrak{g} \rightarrow \mathfrak{g}$ be a Jacobi-Jordan algebra endomorphism. Then $\mathfrak{g}_{\alpha}=\left(\mathfrak{g},[,]_{\alpha}, \alpha\right)$ is a Hom-JacobiJordan algebra, where $[,]_{\alpha}=\alpha \circ[$,$] . Moreover, suppose that \left(\mathfrak{g}^{\prime},[,]^{\prime}\right)$ is another Jacobi-Jordan algebra and $\alpha^{\prime}: \mathfrak{g}^{\prime} \rightarrow \mathfrak{g}^{\prime}$ is a Jacobi-Jordan algebra endomorphism. If $f: \mathfrak{g} \rightarrow \mathfrak{g}^{\prime}$ is a Jacobi-Jordan algebra morphism that satisfies $f \circ \alpha=\alpha^{\prime} \circ f$ then

$$
f:\left(\mathfrak{g},[,]_{\alpha}, \alpha\right) \longrightarrow\left(\mathfrak{g}^{\prime},[,]_{\alpha^{\prime}}^{\prime}, \alpha^{\prime}\right)
$$

is a morphism of Hom-Jacobi-Jordan algebras.
Proof. Observe that $\left[\alpha(x),[y, z]_{\alpha}\right]_{\alpha}=\alpha[\alpha(x), \alpha[y, z]]=\alpha^{2}[x,[y, z]]$. Therefore the Hom-Jacobi identity for $\mathfrak{g}_{\alpha}=\left(\mathfrak{g},[,]_{\alpha}, \alpha\right)$ follows obviously from the Jacobi identity of $(\mathfrak{g},[]$,$) . The symmetry and the second assertion are proved similarly.$

In the sequel we denote by $\mathfrak{g}_{\alpha}$ the Hom-Jacobi-Jordan algebra ( $\mathfrak{g}, \alpha \circ[],, \alpha$ ) corresponding to a given Jacobi-Jordan algebra ( $\mathfrak{g},[$,$] ) and an endomorphism \alpha$. We say that the Hom-Jacobi-Jordan algebra is obtained by composition.

Proposition 2.14. Let $(\mathfrak{g},[],, \alpha)$ be a regular Hom-Jacobi-Jordan algebra. Then $\left(\mathfrak{g},[,]_{\alpha^{-1}}=\alpha^{-1} \circ[],\right)$ is a Jacobi-Jordan algebra.

Proof. It follows from
$\circlearrowleft_{x, y, z}\left[x,[y, z]_{\alpha^{-1}}\right]_{\alpha^{-1}}=\circlearrowleft_{x, y, z} \alpha^{-1}\left(\left[x, \alpha^{-1}([y, z])\right]\right)=\circlearrowleft_{x, y, z} \alpha^{-2}[\alpha(x),[y, z]]=0$.
Remark 2.15. In particular the proposition is valid when $\alpha$ is an involution.
We may also derive new Hom-Jacobi-Jordan algebras from a given multiplicative Hom-Jacobi-Jordan algebra using the following procedure.

Definition 2.16. Let $(\mathfrak{g},[],, \alpha)$ be a multiplicative Hom-Jacobi-Jordan algebra and $n \geqslant 0$. The $n$th derived Hom-algebra of $\mathfrak{g}$ is defined by

$$
\begin{equation*}
\mathfrak{g}_{(n)}=\left(\mathfrak{g},[,]^{(n)}=\alpha^{n} \circ[,], \alpha^{n+1}\right) \tag{2.9}
\end{equation*}
$$

Note that $\mathfrak{g}_{(0)}=\mathfrak{g}$ and $\mathfrak{g}_{(1)}=\left(\mathfrak{g},[,]^{(1)}=\alpha \circ[],, \alpha^{2}\right)$.
Observe that for $n \geqslant 1$ and $x, y, z \in \mathfrak{g}$ we have

$$
\left[[x, y]^{(n)}, \alpha^{n+1}(z)\right]^{(n)}=\alpha^{n}\left(\left[\alpha^{n}([x, y]), \alpha^{n+1}(z)\right]\right)=\alpha^{2 n}([[x, y], \alpha(z)])
$$

Hence, one obtains the following result.

Theorem 2.17. Let $(\mathfrak{g},[],, \alpha)$ be a multiplicative Hom-Jacobi-Jordan algebra. Then its $n$th derived Hom-algebra is a Hom-Jacobi-Jordan algebra.

In the following we construct Hom-Jacobi-Jordan algebras involving elements of the centroid of Jacobi-Jordan algebras. Let $(\mathfrak{g},[\cdot, \cdot])$ be a Jacobi-Jordan algebra. An endomorphism $\theta \in \operatorname{End}(\mathfrak{g})$ is said to be an element of the centroid if $\theta[x, y]=[\theta(x), y]$ for any $x, y \in \mathfrak{g}$. The centroid is defined by

$$
\operatorname{Cent}(\mathfrak{g})=\{\theta \in \operatorname{End}(\mathfrak{g}): \theta[x, y]=[\theta(x), y], \forall x, y \in \mathfrak{g}\} .
$$

The same definition is assumed for Hom-Jacobi-Jordan algebra.
Proposition 2.18. Let $(\mathfrak{g},[\cdot, \cdot])$ be a Jacobi-Jordan algebra and $\theta \in \operatorname{Cent}(\mathfrak{g})$. Set for $x, y \in \mathfrak{g}$

$$
\begin{aligned}
& \{x, y\}=[\theta(x), y] \\
& {[x, y]_{\theta}=[\theta(x), \theta(y)] .}
\end{aligned}
$$

Then $(\mathfrak{g},\{\cdot, \cdot\}, \theta)$ and $\left(\mathfrak{g},[\cdot, \cdot]_{\theta}, \theta\right)$ are Hom-Jacobi-Jordan algebras.
Proof. For $\theta \in \operatorname{Cent}(\mathfrak{g})$ we have $[\theta(x), y]=\theta([x, y])=\theta([y, x])=[\theta(y), x]=$ $[x, \theta(y)]$. Then

$$
\{x, y\}=[\theta(x), y]=[\theta(y), x]=\theta[y, x]=\{y, x\}
$$

Also we have

$$
\begin{aligned}
\{\theta(x),\{y, z\}\} & =\left[\theta^{2}(x),\{y, z\}\right]=\left[\theta^{2}(x),[\theta(y), z]\right] \\
& =\theta([\theta(x),[\theta(y), z]])=[\theta(x), \theta([\theta(y), z])] \\
& =[\theta(x),[\theta(y), \theta(z)]]
\end{aligned}
$$

It follows $\circlearrowleft_{x, y, z}\{\theta(x),\{y, z\}\}=\circlearrowleft_{x, y, z}[\theta(x),[\theta(y), \theta(z)]]=0$ since $(\mathfrak{g},[]$,$) is a Lie$ algebra. Therefore the Hom-Jacobi is satisfied. Thus $(\mathfrak{g},\{\cdot, \cdot\}, \theta)$ is a Hom-JacobiJordan algebra.

Similarly we have the symmetry and the Hom-Jacobi identity satisfied for $\left(\mathfrak{g},[\cdot, \cdot]_{\theta}, \theta\right)$. Indeed

$$
[x, y]_{\theta}=[\theta(x), \theta(y)]=[\theta(y), \theta(x)]=[y, x]_{\theta}
$$

and

$$
\left[\theta(x),[y, z]_{\theta}\right]_{\theta}=\left[\theta^{2}(x), \theta\left([y, z]_{\theta}\right)\right]=\left[\theta^{2}(x), \theta([\theta(y), \theta(z)]]=\theta^{2}([\theta(x),[\theta(y), \theta(z)]]\right.
$$

which leads to $\circlearrowleft_{x, y, z}\left[\theta(x),[y, z]_{\theta}\right]_{\theta}=\theta^{2}\left(\circlearrowleft_{x, y, z}[\theta(x),[\theta(y), \theta(z)]]\right)=0$.

## 3. Representations of Hom-Jacobi-Jordan Algebras

In this section we introduce a representation theory of Hom-Jacobi-Jordan algebras and discuss the cases of adjoint and coadjoint representations for Hom-JacobiJordan algebras.

Definition 3.1. Let $(\mathfrak{g},[],, \alpha)$ be a Hom-Jacobi-Jordan algebra. A representation of $\mathfrak{g}$ is a triple $(V, \rho, \beta)$, where $V$ is a $\mathbb{K}$-vector space, $\beta \in \operatorname{End}(V)$ and $\rho: \mathfrak{g} \rightarrow \operatorname{End}(V)$ is a linear map satisfying

$$
\begin{equation*}
\rho([x, y]) \circ \beta=-\rho(\alpha(x)) \circ \rho(y)-\rho(\alpha(y)) \circ \rho(x) \quad \forall x, y \in \mathfrak{g} \tag{3.1}
\end{equation*}
$$

One recovers the definition of a representation in the case of Jacobi-Jordan algebras by setting $\alpha=I d_{\mathfrak{g}}$ and $\beta=I d_{V}$.
Definition 3.2. Let $(\mathfrak{g},[],, \alpha)$ be a Hom-Jacobi-Jordan algebra. Two representations $(V, \rho, \beta)$ and $\left(V^{\prime}, \rho^{\prime}, \beta^{\prime}\right)$ of $\mathfrak{g}$ are said to be isomorphic if there exists a linear map $\phi: V \rightarrow V^{\prime}$ such that

$$
\forall x \in \mathfrak{g} \quad \rho^{\prime}(x) \circ \phi=\phi \circ \rho(x) \quad \text { and } \quad \phi \circ \beta=\beta^{\prime} \circ \phi .
$$

Proposition 3.3. Let $\left(\mathfrak{g},[,]_{\mathfrak{g}}, \alpha\right)$ be a Hom-Jacobi-Jordan algebra and $(V, \rho, \beta)$ be a representation of $\mathfrak{g}$. The direct summand $\mathfrak{g} \oplus V$ with a bracket defined by

$$
\begin{equation*}
[x+u, y+w]:=[x, y]_{\mathfrak{g}}+\rho(x)(w)+\rho(y)(u) \quad \forall x, y \in \mathfrak{g} \forall u, w \in V \tag{3.2}
\end{equation*}
$$

and the twisted map $\gamma: \mathfrak{g} \oplus V \rightarrow \mathfrak{g} \oplus V$ defined by

$$
\begin{equation*}
\gamma(x+w)=\alpha(x)+\beta(u) \quad \forall x \in \mathfrak{g} \forall u \in V \tag{3.3}
\end{equation*}
$$

is a Hom-Jacobi-Jordan algebra.
Proof. The symmetry of the bracket is obvious. We show that the Hom-Jacobi identity is satisfied:

Let $x, y, z \in \mathfrak{g}$ and $\forall u, v, w \in V$.

$$
\begin{aligned}
& \circlearrowleft_{(x, u),(y, v),(z, w)}[\gamma(x+u),[y+v, z+w]] \\
= & \circlearrowleft_{(x, u),(y, v),(z, w)}\left[\alpha(x)+\beta(u),[y, z]_{\mathfrak{g}}+\rho(y)(w)-\rho(z)(v)\right] \\
= & \circlearrowleft_{(x, u),(y, v),(z, w)}\left[\alpha(x),[y, z]_{\mathfrak{g}}\right]_{\mathfrak{g}}+\rho\left(\alpha(x)(\rho(y)(w)-\rho(z)(v))+\rho\left([y, z]_{\mathfrak{g}}\right)(\beta(u))\right. \\
= & \circlearrowleft_{(x, u),(y, v),(z, w)} \rho(\alpha(x)(\rho(y)(w))+\rho(\alpha(x)(\rho(z)(v))+\rho(\alpha(y)(\rho(z)(u)) \\
& +\rho(\alpha(z)(\rho(y)(u)) \\
= & \rho(\alpha(x)(\rho(y)(w))+\rho(\alpha(x)(\rho(z)(v))+\rho(\alpha(y)(\rho(z)(u)) \rho(\alpha(z)(\rho(y)(u)) \\
& +\rho(\alpha(y)(\rho(z)(u))+\rho(\alpha(y)(\rho(x)(w))+\rho(\alpha(z)(\rho(x)(v))+\rho(\alpha(x)(\rho(z)(v)) \\
& +\rho(\alpha(z)(\rho(x)(v))+\rho(\alpha(z)(\rho(y)(u))+\rho(\alpha(x)(\rho(y)(w))+\rho(\alpha(y)(\rho(x)(w)) \\
= & 0,
\end{aligned}
$$

where $\circlearrowleft_{(x, u),(y, v),(z, w)}$ denotes summation over the cyclic permutation on $(x, u),(y, v),(z, w)$.

Now, we discuss the adjoint representations of a Hom-Jacobi-Jordan algebra.
Proposition 3.4. Let $(\mathfrak{g},[],, \alpha)$ be a Hom-Jacobi-Jordan algebra and $\operatorname{ad}: \mathfrak{g} \rightarrow \operatorname{End}(\mathfrak{g})$ be an operator defined for $x \in \mathfrak{g}$ by $\operatorname{ad}(x)(y)=[x, y]$. Then $(\mathfrak{g}, \mathrm{ad}, \alpha)$ is a representation of $\mathfrak{g}$.
Proof. Since $\mathfrak{g}$ is a Hom-Jacobi-Jordan algebra, the Hom-Jacobi condition on $x, y, z \in \mathfrak{g}$ is

$$
[\alpha(x),[y, z]]+[\alpha(y),[z, x]]+[\alpha(z),[x, y]]=0
$$

and may be written as

$$
a d[x, y](\alpha(z))=-a d(\alpha(x))(a d(y)(z))-a d(\alpha(y))(a d(x)(z))
$$

Then the operator ad satisfies

$$
a d[x, y] \circ \alpha=-a d(\alpha(x)) \circ a d(y)-a d(\alpha(y)) \circ(a d(x) .
$$

Therefore, it determines a representation of the Hom-Jacobi-Jordan algebra $\mathfrak{g}$.
We call the representation defined in the previous proposition the adjoint representation of the Hom-Jacobi-Jordan algebra.

In the following, we explore the dual representations and coadjoint representations of Hom-Jacobi-Jordan algebras.

Let $(\mathfrak{g},[],, \alpha)$ be a Hom-Jacobi-Jordan algebra and $(V, \rho, \beta)$ be a representation of $\mathfrak{g}$. Let $V^{*}$ be the dual vector space of $V$.

We define a linear map $\widetilde{\rho}: \mathfrak{g} \rightarrow \operatorname{End}\left(V^{*}\right)$ by $\widetilde{\rho}(x)=-{ }^{t} \rho(x)$.
Let $f \in V^{*}, x, y \in \mathfrak{g}$ and $u \in V$. We compute the right hand side of the identity (3.1)

$$
\begin{aligned}
-(\widetilde{\rho}(\alpha(x)) \circ \widetilde{\rho}(y)-\widetilde{\rho}(\alpha(y)) \circ \widetilde{\rho}(x))(f)(u) & =-(\widetilde{\rho}(\alpha(x))(\widetilde{\rho}(y)(f))-\widetilde{\rho}(\alpha(y))(\widetilde{\rho}(x)(f)))(u) \\
& =\widetilde{\rho}(y)(f)(\rho(\alpha(x))(u))+\widetilde{\rho}(x)(f)(\rho(\alpha(y))(u)) \\
& =-f(\rho(y) \rho(\alpha(x))(u))-f(\rho(x) \rho(\alpha(y))(u)) \\
& =-f(\rho(y) \rho(\alpha(x))-\rho(x) \rho(\alpha(y))(u)) .
\end{aligned}
$$

On the other hand, we set that the twisted map for $\widetilde{\rho}$ is $\widetilde{\beta}={ }^{t} \beta$, then the left hand side of (3.1) writes

$$
((\widetilde{\rho}([x, y]) \widetilde{\beta})(f))(u)=(\widetilde{\rho}([x, y])(f \circ \beta)(u)=-f \circ \beta(\rho([x, y])(u)) .
$$

Therefore, we have the following proposition:
Proposition 3.5. Let $(\mathfrak{g},[\cdot, \cdot], \alpha)$ be a Hom-Jacobi-Jordan algebra and $(V, \rho, \beta)$ be a representation of $\mathfrak{g}$. The triple $\left(V^{*}, \widetilde{\rho}, \widetilde{\beta}\right)$, where $\widetilde{\rho}: \mathfrak{g} \rightarrow \operatorname{End}\left(V^{*}\right)$ is given by $\widetilde{\rho}(x)=-{ }^{t} \rho(x)$, defines a representation of the Hom-Jacobi-Jordan algebra $(\mathfrak{g},[\cdot, \cdot], \alpha)$ if and only if

$$
\begin{equation*}
\beta \circ \rho([x, y])=-\rho(x) \rho(\alpha(y))-\rho(y) \rho(\alpha(x)) . \tag{3.4}
\end{equation*}
$$

We obtain the following characterization in the case of adjoint representation.
Corollary 3.6. Let $(\mathfrak{g},[],, \alpha)$ be a Hom-Jacobi-Jordan algebra and ( $\mathfrak{g}$, ad, $\alpha$ ) be the adjoint representation of $\mathfrak{g}$, where $\operatorname{ad}: \mathfrak{g} \rightarrow \operatorname{End}(\mathfrak{g})$. We set $\widetilde{\mathrm{ad}}: \mathfrak{g} \rightarrow \operatorname{End}\left(\mathfrak{g}^{*}\right)$ and $\widetilde{\operatorname{ad}}(x)(f)=-f \circ \operatorname{ad}(x)$. Then $\left(\mathfrak{g}^{*}, \widetilde{\operatorname{ad}}, \widetilde{\alpha}\right)$ is a representation of $\mathfrak{g}$ if and only if

$$
\begin{equation*}
\alpha([[x, y], z])=[x,[\alpha(y), z]]+[y,[\alpha(x), z]] \quad \forall x, y, z \in \mathfrak{g} . \tag{3.5}
\end{equation*}
$$

## 4. Quadratic Hom-Jacobi-Jordan Algebras

In this section we extend the notion of quadratic Jacobi-Jordan algebra to Hom-Jacobi-Jordan algebras and provide some properties. But let's first define quadratic Jacobi-Jordan algebra.

Definition 4.1. Let $(\mathfrak{g},[]$,$) be a Jacobi-Jordan algebra and B: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{K}$ a symmetric nondegenerate bilinear form satisfying

$$
\begin{equation*}
B([x, y], z)=B(x,[y, z]) \quad \forall x, y, z \in \mathfrak{g} \tag{4.1}
\end{equation*}
$$

The identity (4.1) may be written $B([x, y], z)=-B(y,[x, z])$ and is called an invariance of $B$. The triple ( $\mathfrak{g},[],$,$B ) is called the quadratic Jacobi-Jordan algebra.$

More generally, for nonassociative algebras $(A, \cdot)$, a triple $(A, \cdot, B)$ where $B$ is a symmetric nondegenerate bilinear form satisfying

$$
\begin{equation*}
B(x \cdot y, z)=B(x, y \cdot z) \quad \forall x, y, z \in A \tag{4.2}
\end{equation*}
$$

defines a quadratic algebra, called also metrizable algebra. A bilinear form $B$ satisfying (4.2) is said to be invariant form.

Definition 4.2. Let $(\mathfrak{g},[],, \alpha)$ be a Hom-Jacobi-Jordan algebra and $B: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{K}$ be an invariant symmetric nondegenerate bilinear form satisfying

$$
\begin{equation*}
B(\alpha(x), y)=B(x, \alpha(y)) \quad \forall x, y \in \mathfrak{g} . \tag{4.3}
\end{equation*}
$$

The quadruple $(\mathfrak{g},[],, \alpha, B)$ is called a quadratic Hom-Jacobi-Jordan algebra.
If $\alpha$ is an involution (resp. invertible), the quadratic Hom-Jacobi-Jordan algebra is said to be involutive (resp. regular) quadratic Hom-Jacobi-Jordan algebra and we write for shortness IQH-Jacobi-Jordan algebra (resp. RQH-Jacobi-Jordan algebra).

We recover the notion of quadratic Jacobi-Jordan algebra when $\alpha$ is the identity map. One may consider a larger class with a definition without condition (4.3). We may also introduce in the following a generalized quadratic Hom-Jacobi-Jordan algebra notion where the invariance is twisted by a linear map.

Definition 4.3. A Hom-Jacobi-Jordan algebra ( $\mathfrak{g},[],, \alpha$ ) is called Hom-quadratic if there exist a pair $(B, \gamma)$ where $B: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{K}$ is a symmetric nondegenerate bilinear form and $\gamma: \mathfrak{g} \rightarrow \mathfrak{g}$ is a linear map satisfying

$$
\begin{equation*}
B([x, y], \gamma(z))=-B(\gamma(y),[x, z]) \quad \forall x, y, z \in \mathfrak{g} \tag{4.4}
\end{equation*}
$$

We call the identity (4.4) the $\gamma$-invariance of $B$. We recover the quadratic Hom-Jacobi-Jordan algebras when $\gamma=i d$.

Proposition 4.4. Let $(\mathfrak{g},[\cdot, \cdot], \alpha)$ be a Hom-Jacobi-Jordan algebra. If there exists $B: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{K}$ a bilinear form such that the quadruple $(\mathfrak{g},[\cdot, \cdot], \alpha, B)$ is a quadratic Hom-Jacobi-Jordan algebra then

1. $\left(\mathfrak{g}^{*}, \widetilde{\text { ad }}, \widetilde{\alpha}\right)$ is a representation of $\mathfrak{g}$.
2. The representations ( $\mathfrak{g}$, ad, $\alpha$ ) and ( $\mathfrak{g}^{*}, \widetilde{\text { ad }}, \widetilde{\alpha}$ ) are isomorphic.

Proof. To prove the first assertion, we should show that for any $z$ we have

$$
\begin{equation*}
\alpha \circ a d([x, y])(z)+\rho(x) a d(\alpha(y))(z)+a d(y) a d(\alpha(x))(z)=0 \tag{4.5}
\end{equation*}
$$

that is

$$
\alpha[[x, y], z]+[x,[\alpha(y), z]]+[y,[\alpha(x), z]]=0
$$

Let $u \in \mathfrak{g}$

$$
\begin{aligned}
B(\alpha[[x, y], z]+ & {[x,[\alpha(y), z]]+[y,[\alpha(x), z]], u) } \\
& =B(\alpha[[x, y], z], u)+B([x,[\alpha(y), z]], u)+B([y,[\alpha(x), z]], u) \\
& =B([[x, y], z], \alpha(u))-B([\alpha(y), z],[x, u])-B([\alpha(x), z],[y, u]) \\
& =-(B(z,[[x, y], \alpha(u)])+B(z,[\alpha(y),[x, u]])+B(z,[\alpha(x),[y, u]])) \\
& =-(B(z,[[x, y], \alpha(u)]+[\alpha(y),[x, u]])+[\alpha(x),[y, u]]) \\
& =-(B(z,[\alpha(u),[y, x]])+[\alpha(y),[x, u]])+[\alpha(x),[u, y]])) \\
& =0 .
\end{aligned}
$$

This proves (4.5) since $B$ is nondegenerate.
For the second assertion let's consider the map $\phi: \mathfrak{g} \rightarrow \mathfrak{g}^{\star}$ defined by $x \rightarrow B(x, \cdot)$ which is bijective since $B$ is nondegenerate. It's obvious to prove that $\phi$ is also a module morphism.

Definition 4.5. Let $(\mathfrak{g},[\cdot, \cdot], \alpha, B)$ be a quadratic Hom-Jacobi-Jordan algebra.

1. An ideal $I$ of $\mathfrak{g}$ is said to be nondegenerate if $B_{\mid I \times I}$ is nondegenerate.
2. The quadratic Hom-Jacobi-Jordan algebra is said to be irreducible (or $B$ irreducible) if $\mathfrak{g}$ doesn't contain any nondegenerate ideal $I$ such that $I \neq\{0\}$ and $I \neq \mathfrak{g}$.
3. Let $I$ be an ideal of $\mathfrak{g}$. The orthogonal $I^{\perp}$ of $I$ with respect to $B$ is defined by $\{x \in \mathfrak{g}: B(x, y)=0 \forall y \in I\}$.

Remark 4.6. Let $I$ be a nondegenerate ideal of a quadratic Hom-Jacobi-Jordan algebra $(\mathfrak{g},[\cdot, \cdot], \alpha, B)$. Then $\left(I,[,]_{\mid I \times I}, \alpha_{\mid} I, B_{\mid I \times I}\right)$ is a quadratic Hom-JacobiJordan algebra.

Lemma 4.7. Let $(\mathfrak{g},[\cdot, \cdot], \alpha)$ be a multiplicative Hom-Jacobi-Jordan algebra. Then the center $\mathcal{Z}(\mathfrak{g})$ is an ideal of $\mathfrak{g}$.

Proof. We have $[\mathfrak{g}, \mathcal{Z}(\mathfrak{g})]=\{0\} \subseteq \mathcal{Z}(\mathfrak{g})$. Let $x \in \mathcal{Z}(\mathfrak{g})$ and $y \in \mathfrak{g}$. For any $z \in \mathfrak{g}$ the invariance and the symmetry of $B$ leads to $B([\alpha(x), y], z)=B(\alpha(x),[y, z])=$ $B(x, \alpha([y, z]))=B(x,[\alpha(y), \alpha(z)])=B([x, \alpha(y)], \alpha(z)])=0($ since $x \in \mathcal{Z}(\mathfrak{g}))$.

Then for any $y \in \mathfrak{g}$ we have $[\alpha(x), y]=0$ since $B$ is nondegenerate. Thus $\alpha(x) \in \mathcal{Z}(\mathfrak{g})$.

Lemma 4.8. Let $(\mathfrak{g},[\cdot, \cdot], \alpha, B)$ be a quadratic Hom-Jacobi-Jordan algebra and $I$ be an ideal of $\mathfrak{g}$. Then the orthogonal $I^{\perp}$ of $I$ with respect to $B$ is an ideal of $\mathfrak{g}$.

Proof. It is clear that $\left[\mathfrak{g}, I^{\perp}\right] \subseteq I^{\perp}$. Let $y \in I$ and $z \in I^{\perp}$, then $B(\alpha(y), z)=$ $B(y, \alpha(z))=0$ since $\alpha(I) \subseteq I$. We conclude that $I^{\perp}$ is an ideal of $\mathfrak{g}$.

Proposition 4.9. Let $(\mathfrak{g},[\cdot, \cdot], \alpha, B)$ be a quadratic Hom-Jacobi-Jordan algebra. Then $\mathfrak{g}=\mathfrak{g}_{1} \oplus \cdots \oplus \mathfrak{g}_{n}$ such that

1. $\mathfrak{g}_{i}$ is an irreducible ideal of $\mathfrak{g}$, for any $i \in\{1, \cdots, n\}$,
2. $B\left(\mathfrak{g}_{i}, \mathfrak{g}_{j}\right)=\{0\}$, for any $i, j \in\{1, \cdots, n\}$ such that $i \neq j$,
3. $\left(\mathfrak{g}_{i},[\cdot, \cdot]_{\mathfrak{g}_{i} \times \mathfrak{g}_{i}}, \alpha_{\mid \mathfrak{g}_{i}}, B_{\mid \mathfrak{g}_{i} \times \mathfrak{g}_{i}}\right)$ is an irreducible quadratic Hom-Jacobi-Jordan algebra.

Proof. By induction on the dimension of $\mathfrak{g}$.
Now, let $\mathfrak{g}=(\mathfrak{g},[],, \alpha, B)$ be a quadratic multiplicative Hom-Jacobi-Jordan algebra. We provide in the following some observations.

Proposition 4.10. If the linear map $\alpha$ is an automorphism and the center $\mathcal{Z}(\mathfrak{g})=\{0\}$ then $\alpha$ is an involution i.e. $\alpha^{2}=i d$.

Proof. For $x, y, z \in \mathfrak{g}$ we have

$$
\begin{aligned}
B([\alpha(x), y], z) & =B(\alpha(x),[y, z])=B(x, \alpha([y, z]) \\
& =B(x,[\alpha(y), \alpha(z)])=B([x, \alpha(y)], \alpha(z)) \\
& =B(\alpha([x, \alpha(y)]), z)=B\left(\left[\alpha(x), \alpha^{2}(y)\right], z\right) .
\end{aligned}
$$

Then $B\left([\alpha(x), y]-\left[\alpha(x), \alpha^{2}(y)\right], z\right)=0$ which may be written $B\left(\left[\alpha(x), y-\alpha^{2}(y)\right], z\right)$ $=0$. Hence, for any $x, y \in \mathfrak{g}$ we have $\left[\alpha(x),\left(i d-\alpha^{2}\right)(y)\right]=0$. Since $\alpha$ is bijective and $\mathcal{Z}(\mathfrak{g})=\{0\}$ then $\alpha^{2}=i d$.

Proposition 4.11. There exist two nondegenerate ideals $I, J$ of $\mathfrak{g}=(\mathfrak{g},[],, \alpha, B)$ such that

1. $B(I, J)=\{0\}$,
2. $\mathfrak{g}=I \oplus J$,
3. $\alpha_{\mid I}$ is nilpotent and $\alpha_{\mid J}$ is invertible.

Proof. The fitting decomposition with respect to the linear map $\alpha$ leads to the existence of an integer $n$ such that $\mathfrak{g}=I \oplus J$, where $I=\operatorname{Ker}\left(\alpha^{n}\right)$ and $J=\operatorname{Im}\left(\alpha^{n}\right)$, such that $\alpha(I) \subseteq I, \alpha(J) \subseteq I, \alpha_{\mid I}$ is nilpotent and $\alpha_{\mid J}$ is invertible.

Let $x \in \mathfrak{g}, y \in I$. We have $\alpha^{n}([x, y])=\left[\alpha^{n}(x), \alpha^{n}(y)\right]=0$ since $\alpha^{n}(y)=0$, and $[x, y] \in I$. Then $[\mathfrak{g}, I] \subseteq I$. In addition $\alpha^{n}(\alpha(y))=\alpha^{n+1}(y)=0$ which implies that $\alpha(y) \in \operatorname{Ker}\left(\alpha^{n}\right)$. Therefore $I$ is an ideal of $\mathfrak{g}$.

Let $x, y \in J$ then there exist $x^{\prime}, y^{\prime} \in \mathfrak{g}$ such that $x=\alpha^{n}\left(x^{\prime}\right)$ and $y=\alpha^{n}\left(y^{\prime}\right)$. We have $[x, y]=\left[\alpha^{n}\left(x^{\prime}\right), \alpha^{n}\left(y^{\prime}\right)\right]=\alpha^{n}\left(\left[x^{\prime}, y^{\prime}\right]\right) \in J$. In addition $\alpha(J) \subseteq J$. Therefore $J$ is a subalgebra.

Let $x \in I$ and $y \in J$. There exists $y^{\prime} \in \mathfrak{g}$ such that $y=\alpha^{n}\left(y^{\prime}\right)$. For any $z \in \mathfrak{g}$, we have $B([x, y], z)=-B([y, x], z)=-B(y,[x, z])=-B\left(\alpha^{n}\left(y^{\prime}\right),[x, z]\right)=$ $-B\left(y^{\prime}, \alpha^{n}([x, z])=-B\left(y^{\prime},\left[\alpha^{n}(x), \alpha^{n}(z)\right]\right)=0\right.$. Then $[x, y]=0$, since $B$ is a nondegenerate bilinear form. We conclude that $I=\operatorname{Im}\left(\alpha^{n}\right)$ is an ideal of $\mathfrak{g}$ and $[I, J]=0$.

Now let $x \in I$ and $y=\alpha^{n}\left(y^{\prime}\right) \in J$, where $y^{\prime} \in \mathfrak{g}$. We have $B(x, y)=$ $B\left(x, \alpha^{n}\left(y^{\prime}\right)\right)=B\left(\alpha^{n}(x), y^{\prime}\right)=0$ since $\alpha^{n}(x)=0$. Therefore $B(I, J)=0$.
Corollary 4.12. Let $(\mathfrak{g},[\cdot, \cdot], \alpha, B)$ be a quadratic Hom-Jacobi-Jordan algebra which is $B$-irreducible. Then either $\alpha$ is nilpotent or $\alpha$ is an automorphism of $\mathfrak{g}$.

## 5. Constructions and Examples

We show in the following some constructions leading to some examples of quadratic Hom-Jacobi-Jordan algebras. We use Theorem 2.13 and Theorem 2.17 to provide some classes of quadratic Hom-Jacobi-Jordan algebras starting from an ordinary quadratic Jacobi-Jordan algebras, respectively from any multiplicative quadratic Hom-Jacobi-Jordan algebra. Also we provide constructions using elements in the centroid of a Jacobi-Jordan algebras and constructions of $T^{*}$-extension type.

Let $(\mathfrak{g},[], B$,$) be a quadratic Jacobi-Jordan algebra. We denote by A u t_{S}(\mathfrak{g}, B)$ the set of symmetric automorphisms of $\mathfrak{g}$ with respect of $B$, that is automorphisms $f: \mathfrak{g} \rightarrow \mathfrak{g}$ such that $B(f(x), y)=B(x, f(y)), \forall x, y \in \mathfrak{g}$.

Proposition 5.1. Let $(\mathfrak{g},[], B$,$) be a quadratic Jacobi-Jordan algebra and \alpha \in$ Aut ${ }_{S}(\mathfrak{g}, B)$. Then $\mathfrak{g}_{\alpha}=\left(\mathfrak{g},[,]_{\alpha}, \alpha, B_{\alpha}\right)$, where for any $x, y \in \mathfrak{g}$

$$
\begin{align*}
{[x, y]_{\alpha} } & =[\alpha(x), \alpha(y)]  \tag{5.1}\\
B_{\alpha}(x, y) & =B(\alpha(x), y), \tag{5.2}
\end{align*}
$$

is a quadratic Hom-Jacobi-Jordan algebra.

Proof. The triple $\left(\mathfrak{g},[,]_{\alpha}, \alpha\right)$ is a Hom-Jacobi-Jordan algebra by Theorem 2.13.
The linear form $B_{\alpha}$ is nondegenerate since $B$ is nondegenerate and $\alpha$ bijective.
We show that the identity (4.1) is satisfied by $\mathfrak{g}_{\alpha}=\left(\mathfrak{g},[,]_{\alpha}, \alpha, B_{\alpha}\right)$. Let $x, y, z \in \mathfrak{g}$, then

$$
\begin{aligned}
B_{\alpha}\left([x, y]_{\alpha}, z\right) & =B(\alpha([\alpha(x), \alpha(y)]), z)=B([\alpha(x), \alpha(y)], \alpha(z)) \\
& =B(\alpha(x),[\alpha(y), \alpha(z)]) \quad \quad(\text { Invariance of } B) \\
& =B\left(\alpha(x),[y, z]_{\alpha}\right)=B_{\alpha}\left(x,[y, z]_{\alpha}\right)
\end{aligned}
$$

Therefore $B_{\alpha}$ is invariant.
We have $\alpha \in A u t_{S}\left(\mathfrak{g}_{\alpha}, B_{\alpha}\right)$. Indeed

$$
\alpha\left([x, y]_{\alpha}\right)=\alpha([\alpha(x), \alpha(y)])=\left[\alpha^{2}(x), \alpha^{2}(y)\right]=[\alpha(x), \alpha(y)]_{\alpha},
$$

and

$$
B_{\alpha}(\alpha(x), y)=B(\alpha(\alpha(x)), y)=B(\alpha(x), \alpha(y))=B_{\alpha}(x, \alpha(y))
$$

The following theorem permits to obtain new quadratic Hom-Jacobi-Jordan algebras starting from a multiplicative quadratic Hom-Jacobi-Jordan algebra.
Proposition 5.2. Let $(\mathfrak{g},[],, \alpha, B)$ be a multiplicative quadratic Hom-JacobiJordan algebra. For any $n \geqslant 0$, the quadruple

$$
\begin{equation*}
\mathfrak{g}_{(n)}=\left(\mathfrak{g},[,]^{(n)}=\alpha^{n} \circ[,], \alpha^{n+1}, B_{\alpha^{n}}\right), \tag{5.3}
\end{equation*}
$$

where $B_{\alpha^{n}}$ is defined for $x, y \in \mathfrak{g}$ by $B_{\alpha^{n}}(x, y)=B\left(\alpha^{n}(x), y\right)$, determine a multiplicative quadratic Hom-Jacobi-Jordan algebra.

Proof. The triple $\mathfrak{g}_{(n)}=\left(\mathfrak{g},[,]^{(n)}=\alpha^{n} \circ[],, \alpha^{n+1}\right)$ is a Hom-Jacobi-Jordan algebra by Theorem 2.17.

Since $\alpha \in \operatorname{Aut}(\mathfrak{g})$ by induction we have $\alpha^{n} \in \operatorname{Aut}(\mathfrak{g})$. The bilinear form $B_{\alpha^{n}}$ is nondegenerate because $B$ is nondegenerate and $\alpha^{n}$ is bijective. It is is symmetric. Indeed

$$
B_{\alpha^{n}}(x, y)=B\left(\alpha^{n}(x), y\right)=B\left(x, \alpha^{n}(y)\right)=B\left(\alpha^{n}(y), x\right)=B_{\alpha^{n}}(y, x) .
$$

The invariance of $B_{\alpha^{n}}$ is given by

$$
\begin{aligned}
B_{\alpha^{n}}\left([x, y]^{n}, z\right) & =B\left(\alpha^{n} \circ \alpha^{n}([x, y]), z\right)=B\left(\alpha^{n}([x, y]), \alpha^{n}(z)\right)=B\left(\left[\alpha^{n}(x), \alpha^{n}(y)\right], \alpha^{n}(z)\right) \\
& =B\left(\alpha^{n}(x),\left[\alpha^{n}(y), \alpha^{n}(z)\right]\right)=B\left(\alpha^{n}(x), \alpha^{n}([y, z])\right)=B_{\alpha^{n}}\left(x,[y, z]^{n}\right) .
\end{aligned}
$$

We have also $B_{\alpha^{n}}\left(\alpha^{n}(x), y\right)=B_{\alpha^{n}}\left(x, \alpha^{n}(y)\right)$, indeed

$$
B_{\alpha^{n}}\left(\alpha^{n}(x), y\right)=B\left(\alpha^{2 n}(x), y\right)=B\left(\alpha^{n}(x), \alpha^{n}(y)\right)=B_{\alpha^{n}}\left(x, \alpha^{n}(y)\right)
$$

We provide here a construction a Hom-Jacobi-Jordan algebra $\mathcal{L}$ and also the double extension of $\{0\}$ by $\mathcal{L}$ see [23].

Proposition 5.3. Let $\left(\mathfrak{g},[,]_{\mathfrak{g}}\right)$ be a Jacobi-Jordan algebra and $\mathfrak{g}^{*}$ be the underlying dual vector space. The vector space $\mathcal{L}=\mathfrak{g} \oplus \mathfrak{g}^{*}$ equipped with the following product

$$
\begin{equation*}
[,]: \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}, \quad(x+f, y+h) \mapsto[x, y]_{\mathfrak{g}}+f \circ a d y+h \circ a d x \tag{5.4}
\end{equation*}
$$

and a bilinear form

$$
\begin{equation*}
B: \mathcal{L} \times \mathcal{L} \rightarrow \mathbb{K}, \quad(x+f, y+h) \mapsto f(y)+h(x) \tag{5.5}
\end{equation*}
$$

is a quadratic Jacobi-Jordan algebra, which we denote by $\mathcal{L}$.
In the sequel we denote $\mathcal{L}$ by $T^{*}(\mathfrak{g})$ and $B$ by $B_{0}$.
Theorem 5.4. Let $(\mathfrak{g},[]$,$) be a Jacobi-Jordan algebra and \alpha \in \operatorname{Aut}(\mathfrak{g})$. Then the endomorphism $\Omega:=\alpha+{ }^{t} \alpha$ of $T^{*}(\mathfrak{g})$ is a symmetric automorphism of $T^{*}(\mathfrak{g})$ with respect to $B_{0}$ if and only if $\operatorname{Im}\left(\alpha^{2}-i d\right) \subseteq \mathcal{Z}(\mathfrak{g})$, where $\mathcal{Z}(\mathfrak{g})$ is the center of $\mathfrak{g}$. Hence, if $\operatorname{Im}\left(\alpha^{2}-i d\right) \subseteq \mathcal{Z}(\mathfrak{g})$ then $\left(T_{0}^{*}(\mathfrak{g})_{\Omega},[,]_{\Omega}, \Omega, B_{\Omega}\right)$ is a RQH-Jacobi-Jordan algebra where $\Omega=\alpha+{ }^{t} \alpha$.
Proof. Let $x, y \in \mathfrak{g}$ and $f, h \in \mathfrak{g}^{*}$.

$$
\begin{aligned}
\Omega([x+f, y+h]) & =\Omega\left([x, y]_{\mathfrak{g}}+f \circ a d y+h \circ a d x\right) \\
& =\alpha\left([x, y]_{\mathfrak{g}}\right)+f \circ a d y \circ \alpha+h \circ a d x \circ \alpha,
\end{aligned}
$$

and

$$
\begin{aligned}
{[\Omega(x+f), \Omega(y+h)] } & =[\alpha(x)+f \circ \alpha, \alpha(y)+h \circ \alpha] \\
& =[\alpha(x), \alpha(y)]_{\mathfrak{g}}+f \circ \alpha \circ \operatorname{ad} \alpha(y)+h \circ \alpha \circ \operatorname{ad\alpha }(x) .
\end{aligned}
$$

Then $\Omega([x+f, y+h])=[\Omega(x+f), \Omega(y+h)]$ if and only if

$$
\forall x, y \in \mathfrak{g}, \quad f \circ a d y \circ \alpha+h \circ a d x \circ \alpha=f \circ \alpha \circ a d \alpha(y)+h \circ \alpha \circ a d \alpha(x) .
$$

That is for all $z \in \mathfrak{g}$

$$
f([y, \alpha(z)])+h([x, \alpha(z)])=f(\alpha[\alpha(y), z])+h(\alpha[\alpha(x), z]) .
$$

Hence, $\Omega$ is an automorphism of $T^{*}(\mathfrak{g})$ if and only if $f([x, \alpha(y)])=f(\alpha[\alpha(x), y])$, $\forall f \in \mathfrak{g}^{*} \forall x, y \in \mathfrak{g}$, which is equivalent to $[x, \alpha(y)]=\alpha[\alpha(x), y] \forall x, y \in \mathfrak{g}$.

As a consequence, $\Omega \in \operatorname{Aut}\left(T_{0}^{*}(\mathfrak{g})\right)$ if and only if $\left[\alpha^{2}(x)-x, \alpha(y)\right]_{\mathfrak{g}}=0 \forall x, y \in \mathfrak{g}$, i.e. $\operatorname{Im}\left(\alpha^{2}-i d\right) \subseteq \mathcal{Z}(\mathfrak{g})$, since $\alpha \in \operatorname{Aut}(\mathfrak{g})$.

In the following we show that $\Omega$ is symmetric with respect to $B_{0}$. Indeed, let $x, y \in \mathfrak{g}$ and $f, h \in \mathfrak{g}^{*}$

$$
\begin{aligned}
B_{0}(\Omega(x+f), y+h) & =B_{0}(\alpha(x)+f \circ \alpha, y+h)=f \circ \alpha(y)+h(\alpha(x)) \\
& =f \circ \alpha(y)+h \circ \alpha(x)=B_{0}(x+f, \alpha(y)+h \circ \alpha) \\
& =B_{0}(x+f, \Omega(y+h)) .
\end{aligned}
$$

The last assertion is a consequence of the previous calculations and Proposition 3.3.

In the following we provide examples which show that the class of Jacobi-Jordan algebras with automorphisms satisfying the condition $\operatorname{Im}\left(\alpha^{2}(x)-x\right) \in \mathcal{Z}(\mathfrak{g})$ is large. We consider first Jacobi-Jordan algebras with involutions.
Corollary 5.5. Let $\left(\mathfrak{g},[,]_{\mathfrak{g}}\right)$ be a Jacobi-Jordan algebra and $\theta \in \operatorname{Aut}(\mathfrak{g})$ such that $\theta^{2}=i d$ ( $\theta$ is an involution), then $\theta^{2}(x)-x=0 \in \mathcal{Z}(\mathfrak{g})$ for all $x \in \mathfrak{g}$. Thus $\left(T_{0}^{*}(\mathfrak{g})_{\Omega},[,]_{\Omega}, \Omega, B_{\Omega}\right)$ is a RQH-Jacobi-Jordan algebra where $\Omega=\alpha+{ }^{t} \alpha$.
Example 5.6. Considering an involution on a Jacobi-Jordan algebra $\mathfrak{g}$ is equivalent to have a $\mathbb{Z}_{2}$-graduation on $\mathfrak{g}$. From the above he Jacobi-Jordan algebras with involutions are symmetric.

Starting from a Jacobi-Jordan algebra one may construct a symmetric JacobiJordan algebra in the following way :

Let $(\mathfrak{g},[\cdot, \cdot])$ be a Jacobi-Jordan algebra, we consider the Jacobi-Jordan algebra $\left(\mathfrak{L},[\cdot, \cdot]_{\mathfrak{L}}\right)$ where $\mathfrak{L}=\mathfrak{g} \times \mathfrak{g}$ and the bracket defined by for all $x, y, x^{\prime}, y^{\prime} \in \mathfrak{g}$ by $\left[(x, y),\left(x^{\prime}, y^{\prime}\right)\right]_{\mathfrak{L}}:=\left(\left[x, x^{\prime}\right],\left[y, y^{\prime}\right]\right)$.

It is easy to check that the map $\theta: \mathfrak{L} \rightarrow \mathfrak{L}, \quad(x, y) \mapsto(y, x)$ is an automorphism of $\mathfrak{L}$. Then the trivial $T^{*}$-extension of $\mathfrak{L}$ has $\Omega=\theta+{ }^{t} \theta$ as a symmetric automorphism with respect to $B_{0}$. Moreover, $\Omega$ is an involution. According to Corollary 5.5, we have $\left(T_{0}^{*}(\mathfrak{L})_{\Omega},[,]_{\Omega}, \Omega,\left(B_{0}\right)_{\Omega}\right)$ is a quadratic Hom-Jacobi-Jordan algebra.

Example 5.7. Let $\mathfrak{g}=V \oplus \mathcal{Z}(\mathfrak{g})$, where $V \neq\{0\}$ is a subspace of the vector space $\mathfrak{g}$ with $[V, V]=[\mathfrak{g}, \mathfrak{g}] \subseteq \mathcal{Z}(\mathfrak{g})$. Let $\lambda: \mathfrak{g} \rightarrow \mathcal{Z}(\mathfrak{g})$ be a nontrivial linear map and $\alpha: \mathfrak{g} \rightarrow \mathfrak{g}$ is an endomorphism of $\mathfrak{g}$ defined by

$$
\alpha(v+z):=v+\lambda(v)+z \quad \forall v \in V \quad \forall z \in \mathcal{Z}(\mathfrak{g}) .
$$

We have $\alpha\left(\left[v+z, v^{\prime}+z^{\prime}\right]\right)=\alpha\left(\left[v, v^{\prime}\right]\right)=\left[v, v^{\prime}\right]$ since $\left[v, v^{\prime}\right] \in \mathcal{Z}(\mathfrak{g})$.
Also $\left.\left[\alpha(v+z), \alpha\left(v^{\prime}+z^{\prime}\right)\right]\right)=\left[v, v^{\prime}\right]$. Therefore, the map $\alpha$ is an injective Jacobi-Jordan algebra morphism. Thus $\alpha$ is an automorphism of $\mathfrak{g}$.

Moreover, if $v \in \mathfrak{g}$ and $z \in \mathcal{Z}(\mathfrak{g})$, we have

$$
\begin{aligned}
\left(\alpha^{2}-i d\right)(v+z) & =\alpha^{2}(v+z)-(v+z)=\alpha(v+\lambda(v)+z)-(v+z) \\
& =v+2 \lambda(v)+z-v-z=2 \lambda(v)
\end{aligned}
$$

Then $\alpha^{2}-i d \neq 0$ and $\operatorname{Im}\left(\alpha^{2}-i d\right) \subseteq \mathcal{Z}(\mathfrak{g})$. It follows that $\left(T_{0}^{*}(\mathfrak{g})_{\Omega},[,]_{\Omega}, \Omega,\left(B_{0}\right)_{\Omega}\right)$, where $\Omega=\alpha+{ }^{t} \alpha$, is a RQH-Jacobi-Jordan algebra.

It is clear that $T_{0}^{*}(\mathfrak{g})_{\Omega}$ is 2-nilpotente. It 's also a quadratic Jacobi-Jordan algebra.
Proposition 5.8. Let $\mathcal{A}$ be an anticommutative antiassociative algebra and $\mathfrak{g}$ be a Jacobi-Jordan algebra. If $\mathcal{A}$ has an automorphism $\theta$ such that
$\operatorname{Im}\left(\theta^{2}-i d\right) \subseteq \operatorname{Ann}(\mathcal{A})$, where $\operatorname{Ann}(\mathcal{A})$ denotes the annihilator of $\mathcal{A}$, then the endomorphism $\widetilde{\theta}:=i d_{\mathfrak{g}} \otimes \theta$ of $\mathfrak{g} \otimes \mathcal{A}$ is an automorphism of the Jacobi-Jordan algebra $(\mathfrak{g} \otimes \mathcal{A},[]$,$) , where [x \otimes a, y \otimes b]:=[x, y]_{\mathfrak{g}} \otimes a b$ for all $x, y \in \mathfrak{g}$ and $a, b \in \mathcal{A}$. In addition, $\operatorname{Im}\left(\widetilde{\theta}^{2}-i d_{\mathfrak{g} \otimes \mathcal{A}}\right) \subseteq \mathcal{Z}(\mathfrak{g} \otimes \mathcal{A})$. Then $\left(T_{0}^{*}(\mathfrak{g} \otimes \mathcal{A})_{\Omega},[,]_{\Omega}, \Omega,\left(B_{0}\right)_{\Omega}\right)$ is a RQH-Jacobi-Jordan algebra. Moreover, if $\theta^{2} \neq i d_{\mathcal{A}}$ then $\widetilde{\theta}^{2} \neq i d_{\mathfrak{g} \otimes \mathcal{A}}$.
Proof. It follows from direct calculation and Theorem 5.4.

## 6. Connection Between Algebras

We establish a connection between some classes of Jacobi-Jordan algebras (resp. quadratic Jacobi-Jordan algebras) and classes of Hom-Jacobi-Jordan algebras (resp. quadratic Hom-Jacobi-Jordan algebras).

Theorem 6.1. There exists a biunivoque correspondence between the class of Jacobi-Jordan algebras (quadratic Jacobi-Jordan algebras) admitting involutive automorphisms (symmetric involutive automorphisms) and the class of Hom-JacobiJordan algebras (quadratic Hom-Jacobi-Jordan algebras) where twist maps are involutive automorphisms (symmetric involutive automorphisms).

Proof. Let $(\mathfrak{g},[]$,$) be a symmetric Jacobi-Jordan algebra with \theta$ an involutive automorphism of $\mathfrak{g}$. Then, according to Theorem 2.13, $\left(g_{\theta},[,]_{\theta}, \theta\right)$ is a Hom-Jacobi-Jordan algebra where $\theta$ is an involutive automorphism of $\mathfrak{g}_{\theta}$. Moreover, if $\mathfrak{g}$ has an invariant scalar product $B$ such that $\theta$ is symmetric with respect to $B$, we have seen that

$$
\begin{equation*}
B_{\theta}: \mathfrak{g}_{\theta} \times \mathfrak{g}_{\theta} \rightarrow \mathbb{K}, \quad(x, y) \mapsto B_{\theta}(x, y):=B(\theta(x), y) \tag{6.1}
\end{equation*}
$$

defines a quadratic structure on $\mathfrak{g}_{\theta}$.
Conversely, let $\left(H,[,]_{H}, \theta\right)$ be a Hom-Jacobi-Jordan algebra where $\theta$ is an involutive automorphism of $H$.

We will untwist the Hom-Jacobi-Jordan algebra structure by considering the vector space $H$ and the bracket

$$
\begin{equation*}
[,]: H \times H \rightarrow H \quad(x, y) \mapsto[x, y]:=[\theta(x), \theta(y)]_{H} . \tag{6.2}
\end{equation*}
$$

Obviously the new bracket is bilinear and symmetric. We show that it satisfies the Jacobi identity.

Indeed, for $x, y, z \in H$ we have

$$
\begin{aligned}
{[x,[y, z]] } & =[\theta(x), \theta([y, z])]_{H}=\left[\theta(x), \theta\left([\theta(y), \theta(z)]_{H}\right)\right]_{H} \\
& \left.\left.=\left[\theta(x),\left[\theta^{2}(y), \theta^{2}(z)\right]_{H}\right)\right]_{H}=\left[\theta(x),[y, z]_{H}\right)\right]_{H} .
\end{aligned}
$$

Thus

$$
\left.\circlearrowleft_{x, y, z}[x,[y, z]]=\circlearrowleft_{x, y, z}\left[\theta(x),[y, z]_{H}\right)\right]_{H}=0
$$

Thus $(H,[]$,$) is a Jacobi-Jordan algebra.$
Furthermore, for $x, y \in H$

$$
\theta([x, y])=\theta\left([\theta(x), \theta(y)]_{H}\right)=\left[\theta^{2}(x), \theta^{2}(y)\right]_{H}=[x, y]_{H}
$$

and

$$
[\theta(x), \theta(y)]=\left[\theta^{2}(x), \theta^{2}(y)\right]_{H}=[x, y]_{H}
$$

Then $\theta([x, y])=[\theta(x), \theta(y)]$. Therefore $\theta$ is an involutive automorphism of the Jacobi-Jordan algebra $(H,[]$,$) .$

Also for $x, y \in H$

$$
[x, y]_{\theta}:=[\theta(x), \theta(y)]=\left[\theta^{2}(x), \theta^{2}(y)\right]_{H}=[x, y]_{H}
$$

Then $\left(H,[,]_{\theta}, \theta\right)$ is the Hom-Jacobi-Jordan algebra $\left(H,[,]_{H}, \theta\right)$.
Now, let $\left(H,[,]_{H}, \theta, B\right)$ be a quadratic Hom-Jacobi-Jordan algebra.
The bilinear form

$$
\begin{equation*}
T: H \times H \rightarrow \mathbb{K}, \quad(x, y) \mapsto T(x, y)=B(\theta(x), y) \tag{6.3}
\end{equation*}
$$

is symmetric and nondegenerate.
Indeed, for Let $x, y, z \in H$, we have

$$
\begin{aligned}
T([x, y], z) & =B(\theta([x, y]), z)=B\left(\theta[\theta(x), \theta(y)]_{H}, z\right)=B\left([x, y]_{H}, z\right) \\
& =B\left(x,[y, z]_{H}\right)=B\left(\theta(x), \theta\left([y, z]_{H}\right)\right) \quad \theta \text { is } B \text {-symmetric } \\
& \left.\left.=B\left(\theta(x),[\theta(y), \theta(z)]_{H}\right)\right)=B(\theta(x),[y, z])\right)=T(x,[y, z])
\end{aligned}
$$

Then $T$ is invariant. In the other hand,

$$
T(\theta(x), y)=B(x, y)=B(\theta(x), \theta(y))=T(x, \theta(y))
$$

That is $\theta$ is symmetric with respect to $T$. Therefore $(H,[], T$,$) is a quadratic$ Jacobi-Jordan algebra and $\left(H,[,]_{\theta}, \theta, T_{\theta}\right)$ is an IQH-Jacobi-Jordan algebra.

Now we discuss the connection between Hom-Jacobi-Jordan algebras where the twist map is in the centroid and quadratic Jacobi-Jordan algebras. Let ( $\mathfrak{g},[],$,$B )$ be a quadratic Jacobi-Jordan algebra and $\theta \in \operatorname{Cent}(\mathfrak{g})$ such that $\theta$ is invertible and symmetric with respect to $B$. We set

$$
\operatorname{Cent}_{S}(\mathfrak{g})=\{\theta \in \operatorname{Cent}(\mathfrak{g}): \theta \text { symmetric with respect to } B\} .
$$

We consider

$$
\begin{equation*}
B_{\theta}: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{K} \quad(x, y) \mapsto B_{\theta}(x, y):=B(\theta(x), y) \tag{6.4}
\end{equation*}
$$

Then $B_{\theta}$ is symmetric, nondegenerate and invariant. Indeed,

$$
\begin{aligned}
B_{\theta}(\{x, y\}, z) & =B_{\theta}([\theta(x), y], z)=B(\theta([\theta(x), y]), z) \\
& =B([\theta(x), y], \theta(z))=B(\theta(x),[y, \theta(z)]) \\
& =B(\theta(x),[\theta(y), z])=B(\theta(x),\{y, z\}) \\
& =B_{\theta}(x,\{y, z\}) .
\end{aligned}
$$

Also,

$$
B_{\theta}(\theta(x), y)=B\left(\theta^{2}(x), y\right)=B(\theta(x), \theta(y))=B_{\theta}(x, \theta(y))
$$

Then $\left(\mathfrak{g},\{\},, \theta, B_{\theta}\right)$ is a quadratic Hom-Jacobi-Jordan algebra.
Notice that $B_{\theta}$ is an invariant scalar product of the Jacobi-Jordan algebra $\mathfrak{g}$.

We have also that $\left(\mathfrak{g},[,]_{\theta}, \theta, B_{\theta}\right)$ is a quadratic Hom-Jacobi-Jordan algebra. Indeed,

$$
\begin{aligned}
B_{\theta}\left([x, y]_{\theta}, z\right) & =B_{\theta}([\theta(x), \theta(y)], z)=B(\theta([\theta(x), \theta(y)]), z) \\
& =B([\theta(x), \theta(y)], \theta(z))=B(\theta(x),[\theta(y), \theta(z)]) \\
& =B\left(\theta(x),[y, z]_{\theta}\right)=B_{\theta}\left(x,[y, z]_{\theta}\right)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\theta\left([x, y]_{\theta}\right)=\theta[\theta(x), \theta(y)]=\left[\theta^{2}(x), \theta(y)\right] & =[\theta(x), y]_{\theta} . \\
\theta(\{x, y\})=\theta[\theta(x), y]=\left[\theta^{2}(x), y\right] & =\{\theta(x), y\} .
\end{aligned}
$$

We may say that $\theta \in \operatorname{Cent}(\mathfrak{g},\{\}$,$) and \theta \in \operatorname{Cent}\left(\mathfrak{g},[,]_{\theta}\right)$.
Conversely, let $(\mathfrak{g},[],, \alpha)$ be a Hom-Jacobi-Jordan algebra such that $\alpha \in \operatorname{Cent}(\mathfrak{g},[],, \alpha)$.

We define a new bracket as $\{x, y\}:=[\alpha(x), y]$. Then $(\mathfrak{g},\{\})$,$) is a Jacobi-$ Jordan algebra. Indeed, the bracket is symmetric and

$$
\begin{aligned}
& \{x,\{y, z\}\}=[\alpha(x),[\alpha(y), z]] \\
& \{y,\{z, x\}\}=\left[\alpha(y),[\alpha(z), x]=\left[\alpha^{2}(y),[z, x]\right]\right. \\
& \{z,\{x, y\}\}=[\alpha(z),[\alpha(x), y]]=[\alpha(z),[x, \alpha(y)]] .
\end{aligned}
$$

Then

$$
\circlearrowleft_{x, y, z}\{x,\{y, z\}\}=[\alpha(x),[\alpha(y), z]]+\left[\alpha^{2}(y),[z, x]\right]+[\alpha(z),[x, \alpha(y)]]=0 .
$$

We may define another bracket which gives rise to also a Jacobi-Jordan algebra by $[x, y]_{\alpha}:=[\alpha(x), \alpha(y)]$. Indeed, the bracket is symmetric and

$$
\begin{aligned}
& {\left[x,[y, z]_{\alpha}\right]_{\alpha}=[\alpha(x), \alpha([\alpha(y), \alpha(z)])]=\left[\alpha(x),\left[\alpha^{2}(y), \alpha(z)\right]\right]=\left[\alpha^{2}(x),[\alpha(y), \alpha(z)]\right],} \\
& {\left[y,[z, x]_{\alpha}\right]_{\alpha}=[\alpha(y), \alpha([\alpha(z), \alpha(x)])]=\left[\alpha(y),\left[\alpha^{2}(z), \alpha(x)\right]\right]=\left[\alpha^{2}(y),[\alpha(z), \alpha(x)]\right],} \\
& {\left[z,[x, y]_{\alpha}\right]_{\alpha}=[\alpha(z), \alpha([\alpha(x), \alpha(y)])]=\left[\alpha(z),\left[\alpha^{2}(x), \alpha(y)\right]\right]=\left[\alpha^{2}(z),[\alpha(x), \alpha(y)]\right] .}
\end{aligned}
$$

Therefore

$$
\left[\alpha^{2}(x),[\alpha(y), \alpha(z)]\right]+\left[\alpha^{2}(y),[\alpha(z), \alpha(x)]\right]+\left[\alpha^{2}(z),[\alpha(x), \alpha(y)]\right]=0
$$

Now if there is an invariant scalar product $B$ on $(\mathfrak{g},[]$,$) and assume that \alpha$ is invertible and symmetric with respect to $B$. Consider the bilinear form $B_{\alpha}$ defined by $B_{\alpha}(x, y)=B(\alpha(x), y)$. We have

$$
\begin{aligned}
B_{\alpha}(\{x, y\}, z) & =B(\alpha(\{x, y\})]), z)=B(\alpha([\alpha(x), y], z)=B(\alpha(x),[y, \alpha(z)]) \\
& =B(\alpha(x),[\alpha(y), z])=B_{\alpha}(x,\{y, z\})
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
B_{\alpha}\left([x, y]_{\alpha}, z\right) & =B(\alpha([\alpha(x), \alpha(y)]), z)=B([\alpha(x), \alpha(y)], \alpha(z)) \\
& =B(\alpha(x),[\alpha(y), \alpha(z)])=B\left(\alpha(x),[y, z]_{\alpha}\right) \\
& =B_{\alpha}\left(x,[y, z]_{\alpha}\right) .
\end{aligned}
$$

Therefore $\left(\mathfrak{g},\{\},, B_{\alpha}\right)$ and $\left(\mathfrak{g},[,]_{\alpha}, B_{\alpha}\right)$ are quadratic Jacobi-Jordan algebras. Hence, we have the following theorem:

Theorem 6.2. There exists a biunivoque correspondence between the class of Jacobi-Jordan algebras (quadratic Jacobi-Jordan algebras) admitting an element in the centroid (symmetric invertible element in the centroid) and the class of Hom-Jacobi-Jordan algebras (quadratic Hom-Jacobi-Jordan algebras) where twist map is in the centroid (symmetric invertible element in the centroid).

## 7. Bimodules of Hom-antiassociative Algebras

Definition 7.1. A hom-antiassociative algebra is said to be multiplicative if the triple $(\mathcal{A}, \cdot, \alpha)$ consisting of a linear space $\mathcal{A}, \mathcal{K}$-bilinear map $: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ and a linear space map $\alpha: \mathcal{A} \rightarrow \mathcal{A}$ satisfies

$$
\begin{equation*}
\alpha(x \cdot y)=\alpha(x) \cdot \alpha(y) \quad(\text { multiplicativity }) \tag{7.1}
\end{equation*}
$$

Example 7.2. Let $\left\{e_{1}, e_{2}\right\}$ be a basis of a 2-dimensional vector space $\mathcal{A}$ over $\mathcal{K}$. The following multiplication $\cdot$ and map on $\mathcal{A}$ define a hom-antiassociative algebra:

$$
\begin{align*}
& e_{1} \cdot e_{1}=e_{2} \\
& \alpha\left(e_{1}\right)=a_{1} e_{1}+a_{2} e_{2}, \quad \alpha\left(e_{2}\right)=0, \tag{7.2}
\end{align*}
$$

where $a_{1}, a_{2} \in \mathcal{K}$.
Definition 7.3. A hom-module is a pair $(V, \beta)$ where $V$ is a $\mathcal{K}$-vector space and $\beta: V \rightarrow V$ is a linear map.

Definition 7.4. Let $(\mathcal{A}, \cdot, \alpha)$ be a hom-antiassociative algebra and let $(V, \beta)$ be a hom-module. Let $l, r: \mathcal{A} \rightarrow g l(V)$ be two linear maps. The quadruple $(l, r, \beta, V)$ is called a bimodule of $\mathcal{A}$ if

$$
\begin{gather*}
l(x \cdot y) \beta(v)=-l(\alpha(x)) l(y) v, \quad r(x \cdot y) \beta(v)=-r(\alpha(y)) r(x) v, \\
 \tag{7.3}\\
l(\alpha(x)) r(y) v=-r(\alpha(y)) l(x) v,  \tag{7.4}\\
\beta(l(x) v)=l(\alpha(x)) \beta(v),  \tag{7.5}\\
\beta(r(x) v)=r(\alpha(x)) \beta(v),
\end{gather*}
$$

for all $x, y \in \mathcal{A}, v \in V$.

Proposition 7.5. Let $(l, r, \beta, V)$ a hom-bimodule of a hom-antiassociative algebra $(\mathcal{A}, \cdot, \alpha)$. Then the direct sum $\mathcal{A} \oplus V$ of vectors spaces is turned into a homantiassociative algebra by defining multiplication in $\mathcal{A} \oplus V$ by

$$
\begin{aligned}
\left(x_{1}+v_{1}\right) *\left(x_{2}+v_{2}\right) & =x_{1} \cdot x_{2}+\left(l\left(x_{1}\right) v_{2}+r\left(x_{2}\right) v_{1}\right) \\
(\alpha \oplus \beta)\left(x_{1}+v_{1}\right) & =\alpha\left(x_{1}\right)+\beta\left(v_{1}\right)
\end{aligned}
$$

for all $x_{1}, x_{2} \in \mathcal{A}, v_{1}, v_{2} \in V$.
Proof. Let $v_{1}, v_{2}, v_{3} \in V$ and $x_{1}, x_{2}, x_{3} \in \mathcal{A}$. Set

$$
\begin{equation*}
\left[\left(x_{1}+v_{1}\right) *\left(x_{2}+v_{2}\right)\right] *\left(\alpha\left(x_{3}\right)+\beta\left(v_{3}\right)\right)=-\left(\alpha\left(x_{1}\right)+\beta\left(v_{1}\right)\right) *\left[\left(x_{2}+v_{2}\right) *\left(x_{3}+v_{3}\right)\right] . \tag{7.6}
\end{equation*}
$$

After computation of (7.6), one easily obtains the conditions of (7.3). Hence the proposition is established.

We denote such a hom-antiassociative algebra $(\mathcal{A} \oplus V, *, \alpha+\beta)$ or $\mathcal{A} \ltimes_{l, r, \alpha, \beta}^{-1} V$.
Example 7.6. Let $(\mathcal{A}, \cdot, \alpha)$ be a multiplicative hom-antiassociative algebra. Let $L .(x)$ and $R .(x)$ denote the left and right multiplication operators, respectively, that is, $L \cdot(x)(y)=x \cdot y, R .(x)(y)=y \cdot x$ for any $x, y \in \mathcal{A}$. Let $L:: \mathcal{A} \rightarrow g l(\mathcal{A})$ with $x \mapsto L .(x)$ and $R .: \mathcal{A} \rightarrow g l(\mathcal{A})$ with $x \mapsto R .(x)$ (for every $x \in \mathcal{A})$ be two linear maps. Then $(L ., 0, \alpha),\left(0, R_{.}, \alpha\right)$ and $\left(L ., R_{.}, \alpha\right)$ are bimodules of $(\mathcal{A}, \cdot, \alpha)$.

Proposition 7.7. Let $(l, r, \beta, V)$ be a bimodule of a multiplicative hom-antiassociative algebra $(\mathcal{A}, \cdot, \alpha)$. Then $\left(l \circ \alpha^{n}, r \circ \alpha^{n}, \beta, V\right)$ is a bimodule of $\mathcal{A}$ for any entiger $n \geqslant 1$.

Proof. We have:

$$
\begin{gathered}
l \circ \alpha^{n}(x \cdot y) \beta(v)=l\left(\alpha^{n}(x) \cdot \alpha^{n}(y)\right) \beta(v)=-l\left(\alpha\left(\alpha^{n}(x)\right)\right) l\left(\alpha^{n}(y)\right) v \\
-l\left(\alpha^{n+1}(x)\right) l\left(\alpha^{n}(y)\right) v=-l \circ \alpha^{n}(\alpha(x)) l \circ \alpha^{n}(y) v .
\end{gathered}
$$

Similarly, the other relations are established.
Example 7.8. Let $(\mathcal{A}, \cdot, \alpha)$ be a multiplicative hom-antiassociative algebra. Then $\left(L . \circ \alpha^{n}, R . \circ \alpha^{n}, \alpha, \mathcal{A}\right)$ is a bimodule of $\mathcal{A}$ for any entiger $n \geqslant 1$.

Example 7.9. Let $(\mathcal{A}, \cdot, \alpha)$ be a multiplicative antiassociative algebra. Also let $\beta: \mathcal{A} \rightarrow \mathcal{A}$ be a morphism. Then $\mathcal{A}_{\beta}=\left(\mathcal{A}, \cdot \beta=\beta \circ \cdot, \alpha_{\beta}=\beta \circ \alpha\right)$ is also a multiplicative hom-antiassociative algebra. Hence $\left(L_{._{\beta}} \circ \alpha_{\beta}^{n}, R_{\cdot} \circ \alpha_{\beta}^{n}, \alpha_{\beta}, \mathcal{A}\right)$ is a bimodule of $\mathcal{A}$ for any integer $n \geqslant 0$.

Theorem 7.10. Let $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{B}, \circ, \beta)$ be two hom-antiassociative algebras. Suppose that there are linear maps $l_{\mathcal{A}}, r_{\mathcal{A}}: \mathcal{A} \rightarrow \operatorname{gl}(\mathcal{B})$ and $l_{\mathcal{B}}, r_{\mathcal{B}}: \mathcal{B} \rightarrow \operatorname{gl}(\mathcal{A})$ such that $\left(l_{\mathcal{A}}, r_{\mathcal{A}}, \beta, \mathcal{B}\right)$ is a bimodule of $\mathcal{A}$ and $\left(l_{\mathcal{B}}, r_{\mathcal{B}}, \alpha, \mathcal{A}\right)$ is a bimodule of $\mathcal{B}$, satisfying the following conditions:

$$
\begin{align*}
& l_{\mathcal{A}}(\alpha(x))(a \circ b)=-l_{\mathcal{A}}\left(r_{\mathcal{B}}(a) x\right) \beta(b)-\left(l_{\mathcal{A}}(x) a\right) \circ \beta(b),  \tag{7.7}\\
& r_{\mathcal{A}}(\alpha(x))(a \circ b)=-r_{\mathcal{A}}\left(l_{\mathcal{B}}(b) x\right) \beta(a)-\beta(a) \circ\left(r_{\mathcal{A}}(x) b\right), \tag{7.8}
\end{align*}
$$

$$
\begin{gather*}
l_{\mathcal{B}}(\beta(a))(x \cdot y)=-l_{\mathcal{B}}\left(r_{\mathcal{A}}(x) a\right) \alpha(y)-\left(l_{\mathcal{B}}(a) x\right) \cdot \alpha(y),  \tag{7.9}\\
r_{\mathcal{B}}(\beta(a))(x \cdot y)=-r_{\mathcal{B}}\left(l_{\mathcal{A}}(y) a\right) \alpha(x)-\alpha(x) \cdot\left(r_{\mathcal{B}}(a) y\right),  \tag{7.10}\\
l_{\mathcal{A}}\left(l_{\mathcal{B}}(a) x\right) \beta(b)+\left(r_{\mathcal{A}}(x) a\right) \circ \beta(b)+r_{\mathcal{A}}\left(r_{\mathcal{B}}(b) x\right) \beta(a)+\beta(a) \circ\left(l_{\mathcal{A}}(x) b\right)=0,  \tag{7.11}\\
l_{\mathcal{B}}\left(l_{\mathcal{A}}(x) a\right) \alpha(y)+\left(r_{\mathcal{B}}(a) x\right) \cdot \alpha(y)+r_{\mathcal{B}}\left(r_{\mathcal{A}}(y) a\right) \alpha(x)+\alpha(x) \cdot\left(l_{\mathcal{B}}(a) y\right)=0, \tag{7.12}
\end{gather*}
$$

for any $x, y \in \mathcal{A}, a, b \in \mathcal{B}$. Then, there is a hom-antiassociative algebra structure on the direct sum $\mathcal{A} \oplus \mathcal{B}$ of the underlying vector spaces of $\mathcal{A}$ and $\mathcal{B}$ given by

$$
\begin{align*}
& (x+a) *(y+b)=\left(x \cdot y+l_{\mathcal{B}}(a) y+r_{\mathcal{B}}(b) x\right)+\left(a \circ b+l_{\mathcal{A}}(x) b+r_{\mathcal{A}}(y) a\right),  \tag{7.13}\\
& (\alpha \oplus \beta)(x+a)=\alpha(x)+\beta(a) \tag{7.14}
\end{align*}
$$

for all $x, y \in \mathcal{A}, a, b \in \mathcal{B}$.
Proof. Let $v_{1}, v_{2}, v_{3} \in V$ and $x_{1}, x_{2}, x_{3} \in \mathcal{A}$. Set

$$
\begin{equation*}
\left[\left(x_{1}+v_{1}\right) *\left(x_{2}+v_{2}\right)\right] *\left(\alpha\left(x_{3}\right)+\beta\left(v_{3}\right)\right)=-\left(\alpha\left(x_{1}\right)+\beta\left(v_{1}\right)\right) *\left[\left(x_{2}+v_{2}\right) *\left(x_{3}+v_{3}\right)\right] . \tag{7.15}
\end{equation*}
$$

After computation of (7.15), we obtain (7.7) - (7.12). Hence the theorem is proved.

This hom-antiassociative algebra will be denoted by $\left(\mathcal{A} \bowtie_{-1} \mathcal{B}, *, \alpha+\beta\right)$ or by $\mathcal{A} \bowtie_{l_{\mathcal{B}}, r_{\mathcal{B}}, \alpha}^{-1, l_{\mathcal{A}}, r_{\mathcal{A}}, \beta} \mathcal{B}$.

Definition 7.11. Let $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{B}, \circ, \beta)$ be two hom-antiassociative algebras. Suppose that there are linear maps $l_{\mathcal{A}}, r_{\mathcal{A}}: \mathcal{A} \rightarrow \operatorname{gl}(\mathcal{B})$ and $l_{\mathcal{B}}, r_{\mathcal{B}}: \mathcal{B} \rightarrow \operatorname{gl}(\mathcal{A})$ such that $\left(l_{\mathcal{A}}, r_{\mathcal{A}}, \beta\right)$ is a bimodule of $\mathcal{A}$ and $\left(l_{\mathcal{B}}, r_{\mathcal{B}}, \alpha\right)$ is a bimodule of $\mathcal{B}$. If the equations (7.7) - (7.12) are satisfied, then $\left(\mathcal{A}, \mathcal{B}, l_{\mathcal{A}}, r_{\mathcal{A}}, \beta, l_{\mathcal{B}}, r_{\mathcal{B}}, \alpha\right)$ is called a matched pair of hom-antiassociative algebras.

## 8. Quadratique Hom-antiassociative Algebras

In this section, we consider the multiplicative hom-antiassociative algebra $(\mathcal{A}, \cdot, \alpha)$ such that $\alpha$ involutive, i.e., $\alpha^{2}=\operatorname{id}_{\mathcal{A}}$.

Definition 8.1. Let $V_{1}, V_{2}$ be two vector spaces. For a linear map $\phi: V_{1} \rightarrow V_{2}$, we denote the dual (linear) map by $\phi^{*}: V_{2}^{*} \rightarrow V_{1}^{*}$ given by

$$
\left\langle v, \phi^{*}\left(u^{*}\right)\right\rangle=\left\langle\phi(v), u^{*}\right\rangle \text { for all } v \in V_{1}, u^{*} \in V_{2}^{*} .
$$

Lemma 8.2. Let $(l, r, \beta, V)$ be a bimodule of a multiplicative hom-antiassociative algebra $(\mathcal{A}, \cdot, \alpha)$.
(i) Let $l^{*}, r^{*}: \mathcal{A} \rightarrow g l\left(V^{*}\right)$ be the linear maps given by

$$
\begin{equation*}
\left\langle l^{*}(x) u^{*}, v\right\rangle=\left\langle l(x) v, u^{*}\right\rangle,\left\langle r^{*}(x) u^{*}, v\right\rangle=\left\langle r(x) v, u^{*}\right\rangle \tag{8.1}
\end{equation*}
$$

for all $x \in \mathcal{A}, u^{*} \in V^{*}, v \in V$. Then, $\left(r^{*}, l^{*}, \beta^{*}, V^{*}\right)$ is a bimodule of $(\mathcal{A}, \cdot, \alpha)$.
(ii) $\left(r^{*}, 0, \beta^{*}, V^{*}\right)$ and $\left(0, l^{*}, \beta^{*}, V^{*}\right)$ are also bimodules of $\mathcal{A}$.

Proof. ( $i$ ): Let $(l, r, \beta, V)$ be a bimodule of a multiplicative hom-antiassociative algebra $(\mathcal{A}, \cdot, \alpha)$. Show that $\left(r^{*}, l^{*}, \beta^{*}, V^{*}\right)$ is a bimodule of $\mathcal{A}$.

Let $x, y \in \mathcal{A}, u^{*} \in V^{*}, v \in V$, we have

$$
\begin{aligned}
\left\langle r^{*}(x \cdot y) \beta^{*}\left(u^{*}\right), v\right\rangle & =\left\langle\beta(r(x \cdot y) v), u^{*}\right\rangle=\left\langle r(\alpha(x \cdot y)) \beta(v), u^{*}\right\rangle \\
& =\left\langle r(\alpha(x) \cdot \alpha(y)) \beta(v), u^{*}\right\rangle=\left\langle-r\left(\alpha^{2}(y)\right) r(\alpha(x)) v, u^{*}\right\rangle \\
& =\left\langle-(r(y) r(\alpha(x)))^{*} u^{*}, v\right\rangle=\left\langle-r^{*}(\alpha(x)) r^{*}(y) u^{*}, v\right\rangle
\end{aligned}
$$

leading to $r^{*}(x \cdot y) \beta^{*}\left(u^{*}\right)=-r^{*}(\alpha(x)) r^{*}(y) u^{*}$.

$$
\begin{aligned}
\left\langle l^{*}(x \cdot y) \beta^{*}\left(u^{*}\right), v\right\rangle & =\left\langle\beta(l(x \cdot y)(v)), u^{*}\right\rangle=\left\langle l(\alpha(x \cdot y)) \beta(v), u^{*}\right\rangle \\
& =\left\langle l(\alpha(x) \cdot \alpha(y)) \beta(v), u^{*}\right\rangle=\left\langle-l\left(\alpha^{2}(x)\right) l(\alpha(y)) \beta(v), u^{*}\right\rangle \\
& =\left\langle-(l(x) l(\alpha(y)))^{*} u^{*}, v\right\rangle=\left\langle-l^{*}(\alpha(y)) l^{*}(x) u^{*}, v\right\rangle
\end{aligned}
$$

giving $l^{*}(x \cdot y) \beta^{*}\left(u^{*}\right)=-l^{*}(\alpha(y)) l^{*}(x) u^{*}$.

$$
\begin{aligned}
\left\langle r^{*}(\alpha(x)) l^{*}(y) u^{*}, v\right\rangle & =\left\langle l(y) r(\alpha(x)) v, u^{*}\right\rangle \\
& \left.=\left\langle l\left(\alpha^{2}(y)\right) r(\alpha(x)) v, u^{*}\right\rangle=\langle(l \circ \alpha)(\alpha(y))(r \circ \alpha)(x)) v, u^{*}\right\rangle \\
& =\left\langle-r\left(\alpha^{2}(x)\right) l(\alpha(y)) v, u^{*}\right\rangle \\
& =\left\langle-r(x) l(\alpha(y)) v, u^{*}\right\rangle=\left\langle-l^{*}(\alpha(y)) r^{*}(x) u^{*}, v\right\rangle
\end{aligned}
$$

providing that $r^{*}(\alpha(x)) l^{*}(y) u^{*}=-l^{*}(\alpha(y)) r^{*}(x) u^{*}$.

$$
\begin{aligned}
\left\langle\beta^{*}\left(r^{*}(x)\right) u^{*}, v\right\rangle & =\left\langle r(x)(\beta(v)), u^{*}\right\rangle=\left\langle r\left(\alpha^{2}(x)\right)(\beta(v)), u^{*}\right\rangle \\
& =\left\langle(r \circ \alpha)(\alpha(x))(\beta(v)), u^{*}\right\rangle \\
& =\left\langle\beta(r(\alpha(x))) v, u^{*}\right\rangle=\left\langle r^{*}(\alpha(x)) \beta^{*}\left(u^{*}\right), v\right\rangle .
\end{aligned}
$$

Then $\beta^{*}\left(r^{*}(x)\right) u^{*}=r^{*}(\alpha(x)) \beta^{*}\left(u^{*}\right)$.
Similarly, we show that $\beta^{*}\left(l^{*}(x)\right) u^{*}=l^{*}(\alpha(x)) \beta^{*}\left(u^{*}\right)$. Hence, $\left(r^{*}, l^{*}, \beta^{*}, V^{*}\right)$ is a bimodule of $\mathcal{A}$.
(ii): Similarly, we can show also that $\left(r^{*}, 0, \beta^{*}, V^{*}\right)$ and $\left(0, l^{*}, \beta^{*}, V^{*}\right)$ are well bimodules of $\mathcal{A}$.

Definition 8.3. Let $(\mathcal{A}, \cdot, \alpha)$ be a hom-antiassociative algebra and $B: \mathcal{A} \times \mathcal{A} \rightarrow K$ be a non degenerate symmetric bilinear form on $\mathcal{A}$. $B$ is said $\alpha$-invariant if

$$
B(\alpha(x) \cdot \alpha(y), \alpha(z))=B(\alpha(x), \alpha(y) \cdot \alpha(z))
$$

Definition 8.4. We call $(\mathcal{A}, \alpha, B)$ a double construction of an involutive quadratic hom-antiassociative algebra associated to $\left(\mathcal{A}_{1}, \alpha_{1}\right)$ and $\left(\mathcal{A}_{1}^{*}, \alpha_{1}^{*}\right)$ if
(1) $\mathcal{A}=\mathcal{A}_{1} \oplus \mathcal{A}_{1}^{*}$ as the direct sum of vector spaces;
(2) $\left(\mathcal{A}_{1}, \alpha_{1}\right)$ and $\left(\mathcal{A}_{1}^{*}, \alpha_{1}^{*}\right)$ are hom-antiassociative subalgebras of $(\mathcal{A}, \alpha)$ with $\alpha=\alpha_{1} \oplus \alpha_{1}^{*} ;$
(3) $B$ is the natural non-degenerate $\left(\alpha_{1} \oplus \alpha_{1}^{*}\right)$-invariant symmetric bilinear form on $\mathcal{A}_{1} \oplus \mathcal{A}_{1}^{*}$ given by

$$
\begin{align*}
B\left(x+a^{*}, y+b^{*}\right) & =\left\langle x, b^{*}\right\rangle+\left\langle a^{*}, y\right\rangle  \tag{8.2}\\
B\left(\left(\alpha+\alpha^{*}\right)\left(x+a^{*}\right), y+b^{*}\right) & =B\left(x+a^{*},\left(\alpha+\alpha^{*}\right)\left(y+b^{*}\right)\right) \tag{8.3}
\end{align*}
$$

for all $x, y \in \mathcal{A}_{1}, a^{*}, b^{*} \in \mathcal{A}_{1}^{*}$ where $\langle$,$\rangle is the natural pair between the vector$ space $\mathcal{A}_{1}$ and its dual space $\mathcal{A}_{1}^{*}$.

Let $(\mathcal{A}, \cdot, \alpha)$ be an involutive hom-antiassociative algebra. Suppose that there is an involutive hom-antiassociative algebra structure " $\circ$ " on its dual space $\mathcal{A}^{*}$. We construct an involutive hom-antiassociative algebra structure on the direct sum $\mathcal{A} \oplus \mathcal{A}^{*}$ of the underlying vector spaces of $\mathcal{A}$ and $\mathcal{A}^{*}$ such that $(\mathcal{A}, \cdot, \alpha)$ and $\left(\mathcal{A}^{*}, \circ, \alpha^{*}\right)$ are hom-subalgebras and equipped with the non-degenerate $\left(\alpha_{1} \oplus \alpha_{1}^{*}\right)$ invariant symmetric bilinear form on $\mathcal{A} \oplus \mathcal{A}^{*}$ given by the equation (8.2). That is, $\left(\mathcal{A} \oplus \mathcal{A}^{*}, \alpha \oplus \alpha^{*}, B\right)$ is an involutive quadratic hom- antiassociative algebra. Such a construction is called a double construction of an involutive quadratic hom-antiassociative algebra associated to $(\mathcal{A}, \cdot, \alpha)$ and $\left(\mathcal{A}^{*}, \circ, \alpha^{*}\right)$.

Theorem 8.5. Let $(\mathcal{A}, \cdot, \alpha)$ be an involutive hom-antiassociative algebra. Suppose that there is an involutive hom-antiassociative algebra structure " ○" on its dual space $\mathcal{A}^{*}$. Then, there is a double construction of an involutive quadratic hom-antiassociative algebra associated to $(\mathcal{A}, \cdot, \alpha)$ and $\left(\mathcal{A}^{*}, \circ, \alpha^{*}\right)$ if and only if $\left(\mathcal{A}, \mathcal{A}^{*}, R_{.}^{*}, L_{.}^{*}, \alpha^{*}, R_{\circ}^{*}, L_{\circ}^{*}, \alpha\right)$ is a matched pair of involutive hom-antiassociative algebras.

Proof. Let us consider the four maps

$$
\begin{aligned}
& L_{.}^{*}: \mathcal{A} \rightarrow g l\left(\mathcal{A}^{*}\right),\left\langle L_{.}^{*}(x) u^{*}, v\right\rangle=\left\langle L \cdot(x) v, u^{*}\right\rangle=\left\langle x \cdot v, u^{*}\right\rangle, \\
& R_{\cdot}^{*}: \mathcal{A} \rightarrow g l\left(\mathcal{A}^{*}\right),\left\langle R_{\cdot}^{*}(x) u^{*}, v\right\rangle=\left\langle R .(x) v, u^{*}\right\rangle=\left\langle v \cdot x, u^{*}\right\rangle, \\
& R_{\circ}^{*}: \mathcal{A}^{*} \rightarrow g l(\mathcal{A}),\left\langle R_{\circ}^{*}\left(x^{*}\right) u, v^{*}\right\rangle=\left\langle R_{\circ}\left(x^{*}\right) v^{*}, u\right\rangle=\left\langle v^{*} \circ x^{*}, u\right\rangle, \\
& L_{\circ}^{*}: \mathcal{A}^{*} \rightarrow g l(\mathcal{A}),\left\langle L_{\circ}^{*}\left(x^{*}\right) u, v^{*}\right\rangle=\left\langle L_{\circ}\left(x^{*}\right) v^{*}, u\right\rangle=\left\langle x^{*} \circ v^{*}, u\right\rangle,
\end{aligned}
$$

for all $x, v, u \in \mathcal{A}, x^{*}, v^{*}, u^{*} \in \mathcal{A}^{*}$.
If $\left(\mathcal{A}, \mathcal{A}^{*}, R_{.}^{*}, L_{.}^{*}, \alpha^{*}, R_{\circ}^{*}, L_{\circ}^{*}, \alpha\right)$ is a matched pair of multiplicative hom-antiassociative algebras, then $\left(\mathcal{A} \bowtie_{-1} \mathcal{A}^{*}, *, \alpha+\alpha^{*}\right)$ is a multiplicative hom-antiassociative algebra with its product $*$ given by the equation (7.13) and the bilinear form $\mathcal{B}(\cdot, \cdot)$ defined by the equation (8.2) is $\left(\alpha \oplus \alpha^{*}\right)$-invariant, that is

$$
\begin{aligned}
\mathcal{B}\left[\left(\alpha(x)+\alpha^{*}\left(a^{*}\right)\right) *\right. & \left.\left(\alpha(y)+\alpha^{*}\left(b^{*}\right)\right),\left(\alpha(z)+\alpha^{*}\left(c^{*}\right)\right)\right] \\
& =\mathcal{B}\left[\alpha(x)+\alpha^{*}\left(a^{*}\right),\left(\alpha(y)+\alpha^{*}\left(b^{*}\right)\right) *\left(\alpha(z)+\alpha^{*}\left(c^{*}\right)\right)\right]
\end{aligned}
$$

for all $x, y \in \mathcal{A}^{*}, a^{*}, b^{*} \in \mathcal{A}^{*}$ and

$$
\left(x+a^{*}\right) *\left(y+b^{*}\right)=\left(x \cdot y+l_{\mathcal{B}}(a) y+r_{\mathcal{B}}(b) x\right)+\left(a \circ b+l_{\mathcal{A}}(x) b+r_{\mathcal{A}}(y) a\right)
$$

with $l_{\mathcal{A}}=R_{.}^{*}, r_{\mathcal{A}}=L_{.}^{*}, l_{\mathcal{B}}=R_{\circ}^{*}, r_{\mathcal{B}}=L_{\circ}^{*}$.

## Indeed,

$$
\begin{aligned}
& \mathcal{B}\left[\left(\alpha(x)+\alpha^{*}\left(a^{*}\right)\right) *\left(\alpha(y)+\alpha^{*}\left(b^{*}\right)\right),\left(\alpha(z)+\alpha^{*}\left(c^{*}\right)\right)\right] \\
& \begin{aligned}
&= \mathcal{B}\left[\left(\alpha(x) \cdot \alpha(y)+l_{A^{*}}\left(\alpha^{*}\left(a^{*}\right)\right) \alpha(y)+r_{A^{*}}\left(\alpha^{*}\left(b^{*}\right)\right) \alpha(x)\right)+\left(\alpha^{*}\left(a^{*}\right) \circ \alpha^{*}\left(b^{*}\right)\right.\right. \\
&\left.\left.\quad+l_{\mathcal{A}}(\alpha(x)) \alpha^{*}\left(b^{*}\right)+r_{\mathcal{A}}(\alpha(y)) \alpha^{*}\left(a^{*}\right)\right), \alpha(z)+\alpha^{*}\left(c^{*}\right)\right] \\
&=\left\langle\alpha(x) \cdot \alpha(y), \alpha^{*}\left(c^{*}\right)\right\rangle+\left\langle\alpha^{*}\left(c^{*}\right) \circ \alpha^{*}\left(a^{*}\right), \alpha(y)\right\rangle+\left\langle\alpha^{*}\left(b^{*}\right) \circ \alpha^{*}\left(c^{*}\right), \alpha(x)\right\rangle \\
& \quad+\left\langle\alpha^{*}\left(a^{*}\right) \circ \alpha^{*}\left(b^{*}\right), \alpha(z)\right\rangle+\left\langle\alpha(z) \cdot \alpha(x), \alpha^{*}\left(b^{*}\right)\right\rangle+\left\langle\alpha(y) \cdot \alpha(z), \alpha^{*}\left(a^{*}\right)\right\rangle
\end{aligned}
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathcal{B}\left[\alpha(x)+\alpha^{*}\left(a^{*}\right),\left(\alpha(y)+\alpha^{*}\left(b^{*}\right)\right) *\left(\alpha(z)+\alpha^{*}\left(c^{*}\right)\right)\right] \\
& =\mathcal{B}\left[\alpha(x)+\alpha^{*}\left(a^{*}\right),\left(\alpha(y) \cdot \alpha(z)+l_{\mathcal{A}^{*}}\left(\alpha^{*}\left(b^{*}\right)\right) \alpha(z)+r_{\mathcal{A}^{*}}\left(\alpha^{*}\left(c^{*}\right)\right) \alpha(y)\right)\right. \\
& \left.\quad \quad+\left(\alpha^{*}\left(b^{*}\right) \circ \alpha^{*}\left(c^{*}\right)+l_{A}(\alpha(y)) \alpha^{*}\left(c^{*}\right)+r_{\mathcal{A}}(\alpha(z)) \alpha^{*}\left(b^{*}\right)\right)\right] \\
& =\left\langle\alpha(x), \alpha^{*}\left(b^{*}\right) \circ \alpha^{*}\left(c^{*}\right)\right\rangle+\left\langle\alpha^{*}\left(c^{*}\right), \alpha(x) \cdot \alpha(y)\right\rangle+\left\langle\alpha^{*}\left(b^{*}\right), \alpha(z) \cdot \alpha(x)\right\rangle \\
& \quad+\left\langle\alpha(y) \cdot \alpha(z), \alpha^{*}\left(a^{*}\right)\right\rangle+\left\langle\alpha^{*}\left(a^{*}\right) \circ \alpha^{*}\left(b^{*}\right), \alpha(z)\right\rangle+\left\langle\alpha\left(c^{*}\right) \circ \alpha^{*}\left(a^{*}\right), \alpha(y)\right\rangle .
\end{aligned}
$$

Thus, $\mathcal{B}$ is well $\left(\alpha \oplus \alpha^{*}\right)$-invariant. Conversely, let

$$
x * a^{*}=l_{\mathcal{A}}(x) a^{*}+r_{\mathcal{A}^{*}}\left(a^{*}\right) x, a^{*} * x=l_{\mathcal{A}^{*}}\left(a^{*}\right) x+r_{\mathcal{A}}(x) a^{\star},
$$

for $x \in \mathcal{A}, a^{*} \in \mathcal{A}^{*}$. Then, $\left(\mathcal{A}, \mathcal{A}^{*}, R_{.}^{*}, L_{.}^{*}, \alpha^{*}, R_{\circ}^{*}, L_{\circ}^{*}, \alpha\right)$ is a matched pair of multiplicative hom-antiassociative algebras, since the double construction of the involutive quadratic hom-antiassociative algebra associated to $(\mathcal{A}, \cdot, \alpha)$ and $\left(\mathcal{A}^{*}, \circ, \alpha^{*}\right)$ produces the equations (7.7) - (7.12).

Theorem 8.6. Let $(\mathcal{A}, \cdot, \alpha)$ be an involutive hom-associative algebra. Suppose that there is an involutive hom-associative algebra structure " $\circ$ " on its dual space $\left(\mathcal{A}^{*}, \alpha^{*}\right)$. Then, $\left(\mathcal{A}, \mathcal{A}^{*}, R_{.}^{*}, L_{.}^{*}, \alpha^{*}, R_{\circ}^{*}, L_{0}^{*}, \alpha\right)$ is a matched pair of involutive homassociative algebras if and only if for any $x \in \mathcal{A}$ and $a^{*}, b^{*} \in \mathcal{A}^{*}$,

$$
\begin{gather*}
R_{.}^{*}(\alpha(x))\left(a^{*} \circ b^{*}\right)=-R_{.}^{*}\left(L_{\circ}^{*}\left(a^{*}\right) x\right) \alpha^{*}\left(b^{*}\right)-\left(R_{.}^{*}(x) a^{*}\right) \circ \alpha^{*}\left(b^{*}\right),  \tag{8.4}\\
R_{.}^{*}\left(R_{\circ}^{*}\left(a^{*}\right) x\right) \alpha^{*}\left(b^{*}\right)+L_{.}^{*}(x) a^{*} \circ \alpha^{*}\left(b^{*}\right)=-L_{.}^{*}\left(L_{\circ}^{*}\left(b^{*}\right) x\right) \alpha^{*}\left(a^{*}\right)-\alpha^{*}\left(a^{*}\right) \circ\left(R_{.}^{*}(x) b^{*}\right) . \tag{8.5}
\end{gather*}
$$

Proof. Obviously, (8.4) gives (7.7) and (8.5) reduces to (7.11) when $l_{\mathcal{A}}=R_{\text {. }}^{*}$, $r_{\mathcal{A}}=L_{.}^{*}, l_{\mathcal{B}}=l_{\mathcal{A}^{*}}=R_{0}^{*}, r_{\mathcal{B}}=r_{\mathcal{A}^{*}}=L_{\circ}^{*}$. Now, show that

$$
(7.7) \Leftrightarrow(7.8) \Leftrightarrow(7.9) \Leftrightarrow(7.10) \quad \text { and } \quad(7.11) \Leftrightarrow(7.12) .
$$

Suppose (8.4) and (8.5) are satisfied and show that one has:

$$
\begin{array}{r}
L_{.}^{*}(\alpha(x))\left(a^{*} \circ b^{*}\right)=-L_{.}^{*}\left(R_{\circ}^{*}\left(b^{*}\right) x\right) \alpha^{*}\left(a^{*}\right)-\alpha^{*}\left(a^{*}\right) \circ\left(L_{.}^{*}(x) b^{*}\right), \\
R_{\circ}^{*}\left(\alpha^{*}\left(a^{*}\right)\right)(x \cdot y)=-R_{\circ}^{*}\left(L_{.}^{*}(x) a^{*}\right) \alpha(y)-\left(R_{\circ}^{*}(a) x\right) \cdot \alpha(y), \\
L_{\circ}^{*}\left(\alpha^{*}\left(a^{*}\right)\right)(x \cdot y)=L_{\circ}^{*}\left(R_{.}^{*}(y) a^{*}\right) \alpha(x)+\alpha(x) \cdot\left(L_{\circ}^{*}\left(a^{*}\right) y\right), \\
R_{\circ}^{*}\left(R_{.}^{*}(x) a^{*}\right) \alpha(y)+\left(L_{\circ}^{*}\left(a^{*}\right) x\right) \cdot \alpha(y)+L_{\circ}^{*}\left(L .(y) a^{*}\right) \alpha(x)+\alpha(x) \cdot\left(R_{\circ}^{*}(a) y\right)=0 .
\end{array}
$$

We have:

$$
\begin{gather*}
\left\langle R_{.}^{*}(x) a^{*}, y\right\rangle=\left\langle L_{.}^{*}(y) a^{*}, x\right\rangle=\left\langle y \cdot x, a^{*}\right\rangle, \\
\left\langle R_{\circ}^{*}\left(b^{*}\right) x, a^{*}\right\rangle=\left\langle L_{\circ}^{*}\left(a^{*}\right) x, b^{*}\right\rangle=\left\langle a^{*} \circ b^{*}, x\right\rangle, \\
\alpha^{*}\left(R_{\cdot}^{*}(x) a^{*}\right)=R_{.}^{*}(\alpha(x)) \alpha^{*}\left(a^{*}\right), \alpha^{*}\left(L_{.}^{*}(x) a^{*}\right)=L_{.}^{*}(\alpha(x)) \alpha^{*}\left(a^{*}\right),  \tag{8.6}\\
\alpha\left(R_{\circ}^{*}\left(a^{*}\right) x\right)=R_{\circ}^{*}\left(\alpha^{*}\left(a^{*}\right)\right) \alpha(x), \alpha\left(L_{\circ}^{*}\left(a^{*}\right) x\right)=L_{\circ}^{*}\left(\alpha^{*}\left(a^{*}\right)\right) \alpha(x), \tag{8.7}
\end{gather*}
$$

for all $x, y \in \mathcal{A}, a^{*}, b^{*} \in \mathcal{A}^{*}$. Set $\alpha(x)=z, \alpha(y)=t, \alpha^{*}\left(a^{*}\right)=c^{*}$ and $\alpha^{*}\left(b^{*}\right)=d^{*}$. Then

$$
\begin{aligned}
\left\langle-R_{.}^{*}(\alpha(x))\left(a^{*} \circ b^{*}\right), y\right\rangle & =\left\langle-y \cdot \alpha(x), a^{*} \circ b^{*}\right\rangle=\left\langle-(L .(y) \circ \alpha) x, a^{*} \circ b^{*}\right\rangle \\
& =\left\langle-x, \alpha^{*}\left(L_{.}^{*}(y)\left(a^{*} \circ b^{*}\right)\right)\right\rangle=\left\langle-L_{.}^{*}(\alpha(y)) \alpha^{*}\left(a^{*} \circ b^{*}\right), x\right\rangle \\
& =\left\langle-L_{.}^{*}(\alpha(y))\left(\alpha^{*}\left(a^{*}\right) \circ \alpha^{*}\left(b^{*}\right)\right), x\right\rangle=\left\langle-L_{.}^{*}(\alpha(y))\left(c^{*} \circ d^{*}\right), x\right\rangle, \\
\left\langle-R_{\cdot}^{*}\left(L_{\circ}^{*}\left(a^{*}\right) x\right) \alpha\left(b^{*}\right), y\right\rangle & =\left\langle-y \cdot L_{\circ}^{*}\left(a^{*}\right) x, \alpha^{*}\left(b^{*}\right)\right\rangle=\left\langle-L_{.}^{*}(y)\left(\alpha^{*}\left(b^{*}\right)\right), L_{\circ}^{*}\left(a^{*}\right) x\right\rangle \\
& =\left\langle-L_{\circ}^{*}\left(a^{*}\right) x, L_{.}^{*}(y)\left(\alpha^{*}\left(b^{*}\right)\right)\right\rangle=\left\langle-a^{*} \circ\left(L_{.}^{*}(y)\left(\alpha^{*}\left(b^{*}\right)\right)\right), x\right\rangle \\
& =\left\langle-\alpha^{*}\left(c^{*}\right) \circ\left(L_{.}^{*}(y)\left(d^{*}\right)\right), x\right\rangle, \\
\left\langle\left(R_{.}^{*}(x) a^{*}\right) \circ \alpha^{*}\left(b^{*}\right), y\right\rangle & =\left\langle-R_{\circ}^{*}\left(\alpha^{*}\left(b^{*}\right)\right) y, R_{.}^{*}(x) a^{*}\right\rangle=\left\langle-a^{*},\left(R_{\circ}^{*}\left(\alpha^{*}\left(b^{*}\right)\right) y\right) \cdot x\right\rangle \\
& =\left\langle-L_{.}^{*}\left[R_{\circ}^{*}\left(\alpha^{*}\left(b^{*}\right)\right) y\right] a^{*}, x\right\rangle=\left\langle-L_{.}^{*}\left(R_{\circ}^{*}\left(d^{*}\right) y\right) \alpha^{*}\left(c^{*}\right), x\right\rangle
\end{aligned}
$$

leading to $(7.7) \Leftrightarrow(7.8)$.

$$
\begin{gathered}
\left\langle L^{*}(\alpha(x))\left(a^{*} \circ b^{*}\right), y\right\rangle=\left\langle-a^{*} \circ b^{*}, \alpha(x) \cdot y\right\rangle=\left\langle-R_{\circ}^{*}\left(b^{*}\right)(\alpha(x) \cdot y), a^{*}\right\rangle \\
=\left\langle-R_{\circ}^{*}\left(\alpha^{*}\left(d^{*}\right)\right)(z \cdot y), a^{*}\right\rangle, \\
\left\langle\alpha^{*}\left(a^{*}\right) \circ\left(L_{.}^{*}(x) b^{*}\right), y\right\rangle=\left\langle\alpha^{*}\left(a^{*}\right), R_{\circ}^{*}\left(L_{.}^{*}(x) b^{*}\right) y\right\rangle=\left\langle a^{*}, \alpha\left[R_{\circ}^{*}\left(L_{.}^{*}(x) b^{*}\right) y\right]\right\rangle \\
=\left\langle a^{*}, R_{\circ}^{*}\left[\alpha^{*}\left(L_{.}^{*}(x) b^{*}\right)\right] \alpha(y)\right\rangle=\left\langle a^{*}, R_{\circ}^{*}\left[L_{.}^{*}(\alpha(x)) \alpha^{*}\left(b^{*}\right)\right] \alpha(y)\right\rangle \\
=\left\langle a^{*}, R_{\circ}^{*}\left(L_{.}^{*}(z) d^{*}\right) \alpha(y)\right\rangle, \\
\left\langle L_{.}^{*}\left(R_{\circ}^{*}\left(b^{*}\right) x\right) \alpha^{*}\left(a^{*}\right), y\right\rangle=\left\langle\left(R_{\circ}^{*}\left(b^{*}\right) x\right) \circ y, \alpha^{*}\left(a^{*}\right)\right\rangle=\left\langle\alpha\left[\left(R_{\circ}^{*}\left(b^{*}\right) x\right) \circ y\right], a^{*}\right\rangle \\
\\
=\left\langle\left(R_{\circ}^{*}\left(\alpha^{*}\left(b^{*}\right)\right) \alpha(x)\right) \circ \alpha(y), a^{*}\right\rangle=\left\langle R_{\circ}^{*}\left(d^{*}\right) z \cdot \alpha(y), a^{*}\right\rangle
\end{gathered}
$$

giving (7.8) $\Longleftrightarrow(7.9)$.

$$
\begin{aligned}
\left\langle R^{*}(\alpha(x))\left(a^{*} \circ b^{*}\right), y\right\rangle & =\left\langle a^{*} \circ b^{*}, y \cdot \alpha(x)\right\rangle=\left\langle L \cdot\left(a^{*}\right) b^{*}, y \cdot z\right\rangle \\
& =\left\langle L_{\circ}^{*}\left(a^{*}\right)(y \cdot z)\right\rangle=\left\langle L_{\circ}^{*}\left(\alpha^{*}\left(c^{*}\right)\right)(y \cdot z)\right\rangle \\
\left\langle\left(R_{.}^{*}(x) a^{*}\right) \circ \alpha^{*}\left(b^{*}\right), y\right\rangle & =\left\langle\alpha^{*}\left(b^{*}\right), L_{.}^{*}\left(R_{.}^{*}(x) a^{*}\right) y\right\rangle=\left\langle b^{*}, \alpha^{*}\left[L_{.}^{*}\left(R_{.}^{*}(x) a^{*}\right) y\right]\right\rangle \\
& =\left\langle b^{*}, L_{.}^{*}\left(R_{.}^{*}(\alpha(x)) \alpha^{*}\left(a^{*}\right)\right) \alpha(y)\right\rangle \\
& =\left\langle b^{*}, L_{.}^{*}\left(R_{\cdot}^{*}(z) c^{*}\right) \alpha(y)\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
\left\langle R_{:}^{*}\left(L_{\circ}^{*}\left(a^{*}\right) x\right) \alpha^{*}\left(b^{*}\right), y\right\rangle & =\left\langle y \cdot L_{\circ}^{*}\left(a^{*}\right) x, \alpha^{*}\left(b^{*}\right)\right\rangle=\left\langle\alpha(y) \cdot \alpha\left(L_{\circ}^{*}\left(a^{*}\right) x\right), b^{*}\right\rangle \\
& =\left\langle\alpha(y) \cdot L_{\circ}^{*}\left(\alpha^{*}\left(a^{*}\right)\right) \alpha(x), b^{*}\right\rangle=\left\langle\alpha(y) \cdot L_{\circ}^{*}\left(c^{*}\right) z, b^{*}\right\rangle
\end{aligned}
$$

providing that $(7.7) \Longleftrightarrow$ (7.10).

$$
\begin{aligned}
\left\langle L_{.}^{*}\left(L_{\circ}^{*}\left(b^{*}\right) x\right) \alpha^{*}\left(a^{*}\right), y\right\rangle & =\left\langle\left(L_{\circ}^{*}\left(b^{*}\right) x\right) \cdot y, \alpha^{*}\left(a^{*}\right)\right\rangle=\left\langle a^{*}, \alpha\left(L_{\circ}^{*}\left(b^{*}\right) x\right) \cdot \alpha(y)\right\rangle \\
& =\left\langle a^{*}, L_{\circ}^{*}\left(\alpha^{*}\left(b^{*}\right)\right) \alpha(x) \cdot \alpha(y)\right\rangle=\left\langle a^{*}, L_{\circ}^{*}\left(d^{*}\right) z \cdot \alpha(y)\right\rangle, \\
\left\langle\alpha^{*}\left(a^{*}\right) \circ\left(R_{.}^{*}(x) b^{*}\right), y\right\rangle & =\left\langle R_{\circ}^{*}\left(R_{\circ}^{*}(x) b^{*}\right) y, \alpha^{*}\left(a^{*}\right)\right\rangle=\left\langle\alpha^{*}\left(a^{*}\right) \circ\left(R_{.}^{*}(x) b^{*}\right), y\right\rangle \\
& =\left\langle\alpha\left[R_{\circ}^{*}\left(R_{\circ}^{*}(x) b^{*}\right) y\right], a^{*}\right\rangle=\left\langle R_{\circ}^{*}\left[R_{\circ}^{*}(\alpha(x)) \alpha^{*}\left(b^{*}\right)\right] \alpha(y), a^{*}\right\rangle \\
& =\left\langle R_{\circ}^{*}\left(R_{.}^{*}(z) d^{*}\right) \alpha(y), a^{*}\right\rangle \\
\left\langle\left(L_{.}^{*}(x) a^{*}\right) \circ \alpha^{*}\left(b^{*}\right), y\right\rangle & =\left\langle R_{\circ}^{*}\left(\alpha^{*}\left(b^{*}\right)\right) y, L_{.}^{*}(x) a^{*}\right\rangle=\left\langle x \cdot\left(R_{\circ}^{*}\left(d^{*}\right) y\right), a^{*}\right\rangle \\
& =\left\langle\alpha(z) \cdot\left(R_{\circ}^{*}\left(d^{*}\right) y\right), a^{*}\right\rangle, \\
\left\langle R_{.}^{*}\left(R_{\circ}^{*}\left(a^{*}\right) x\right) \alpha^{*}\left(b^{*}\right), y\right\rangle & =\left\langle y \cdot R_{\circ}^{*}\left(a^{*}\right) x, \alpha^{*}\left(b^{*}\right)\right\rangle=\left\langle\alpha^{*}\left(b^{*}\right), L .(y)\left(R_{\circ}^{*}\left(a^{*}\right) x\right)\right\rangle \\
& =\left\langle\left(L_{.}^{*}(y)\left(d^{*}\right), R_{\circ}^{*}\left(a^{*}\right) x\right\rangle=\left\langle L_{.}^{*}(y) d^{*} \circ a^{*}, x\right\rangle\right. \\
& =\left\langle L_{\circ}^{*}\left(L_{.}^{*}(y) d^{*}\right) x, a^{*}\right\rangle=\left\langle L_{\circ}^{*}\left(L_{.}^{*}(y) d^{*}\right) \alpha(z), a^{*}\right\rangle
\end{aligned}
$$

implying that $(7.11) \Longleftrightarrow$ (7.12).

## References

[1] A.L. Agore and G. Militaru, On a type of commutative algebras, Lin. Algebra Appl., 485 (2015), 222 - 249.
[2] N. Aizawa and H. Sato, q-Deformation of the Virasoro algebra with central extension, Physics Letters B, 256 (1991), no. 2, 185 - 190.
[3] F. Ammar and A. Makhlouf, Hom-Lie algebras and Hom-Lie admissible superalgebras, J. Algebra, 324 (2010), 1513 - 1528.
[4] H. Ataguema, A. Makhlouf and S. Silvestrov, Generalization of n-ary Nambu algebras and beyond, J. Math. Phys., 50 (2009), no. 8, 083501.
[5] H. Benamor and S. Benayadi, Double extension of quadratic Lie superalgebras, Commun. Algebra, 27 (1999), $67-88$.
[6] S. Beneyadi and A. Makhlouf, Hom-Lie algebras with symmetric invariant bilinear forms, J. Geom. Phys., 76 (2014), $38-60$.
[7] M. Bordemann, Nondegenerate invariant bilinear forms in nonassociative algebras, Acta Math. Univ. Comenian. 66 (1997), no. 2, 151 - 201.
[8] D. Burde and A. Fialowski, Jacobi-Jordan algebras, Lin. Algebra Appl., 459 (2014), 586 - 594.
[9] M. Chaichian, D. Ellinas and Z. Popowicz, Quantum conformal algebra with central extension, Phys. Lett. B, 248 (1990), no. 1-2, $95-99$.
[10] M. Chaichian, A.P. Isaev, J. Lukierski, Z. Popowicz and P. Prešnajder, $q$-Deformations of Virasoro algebra and conformal dimensions, Phys. Lett. B, 262 (1991), no. 1, $32-38$.
[11] M. Chaichian, Z. Popowicz, and P. Prešnajder, $q$-Virasoro algebra and its relation to the $q$-deformed KdV system, Phys. Lett. B, 249 (1990), no. 1, $63-65$.
[12] T. L. Curtright, C. K. Zachos, Deforming maps for quantum algebras, Phys. Lett. B, 243 (1990), no. 3, $237-244$.
[13] J. T. Hartwig, D. Larsson and S.D. Silvestrov, Deformations of Lie algebras using $\sigma$-derivations, J. Algebra, 295 (2006), 314 - 361.
[14] N. Hu, $q$-Witt algebras, $q$-Lie algebras, $q$-holomorph structure and representations, Algebra Colloq., 6 (1999), no. 1, 51 - 70.
[15] Q. Jin and X. Li, Hom-Lie algebra structures on semi-simple Lie algebras, J. Algebra, 319 (2008), 1398 - 1408.
[16] C. Kassel, Cyclic homology of differential operators, the Virasoro algebra and a q-analogue, Commun. Math. Phys., 146 (1992), 343 - 351.
[17] D. Larsson and S.D. Silvestrov, Quasi-Hom-Lie algebras, central extensions and 2-cocycle-like identities, J. Algebra, 288 (2005), 321 - 344.
[18] D. Larsson and S.D. Silvestrov, Quasi-Lie algebras in noncommutative geometry and representation theory in mathematical physics, Contemp. Math. 391 (2005), $241-248$.
[19] D. Larsson and S.D. Silvestrov, Quasi-deformations of $s l_{2}(\mathbb{F})$ using twisted derivations, Commun. Algebra, 35 (2007), $4303-4318$.
[20] A. Makhlouf and S.D. Silvestrov, Hom-algebra structures, J. Gen. Lie Theory Appl. 2 (2008), no. 2, $51-64$.
[21] A. Makhloufand S.D. Silvestrov, Hom-Lie admissible Hom-coalgebras and HomHopf algebras, in S. Silvestrov, E. Paal, V. Abramov, A. Stolin, (Eds.), Generalized Lie theory in Mathematics, Physics and Beyond, 2 (2008), 189 - 206.
[22] A. Makhlouf and S.D. Silvestrov, Hom-Algebras and Hom-Coalgebras, J. Algebra Appl., 9 (2010), $553-589$.
[23] A. Medina and Ph. Revoy, Algèbres de Lie et produit scalaire invariant, Ann. Sci. Ecole Norm. Sup., 4 (1985), no. 18, 553 - 561.
[24] J. Neumann, J. von Neumann and E. Wigner, On an algebraic generalization of the quantum mechanical formalism, Ann. Math., $\mathbf{3 5}$ (1934), $29-64$.
[25] S. Okubo and N. Kamiya, Jordan-Lie super algebra and Jordan-Lie triple system, J. Algebra, 198 (1997), 388 - 411.
[26] A. Wörz-Busekros, Bernstein algebras, Arch. Math., 48 (1987), $388-398$.
[27] P. Zumanovich, Special and exceptional mock-Lie algebras, Linear Algebra Appl., 518 (2017), 79 - 96.

Received July 26, 2020
International Chair in Mathematical Physics and Applications, ICMPA-UNESCO Chair, University of Abomey-Calavi, 072 BP 50, Cotonou, Rep. of Benin
E-mails: cyzille19@gmail.com, houndedjid@gmail.com

# On completely regular 2-duo semigroups 

Panuwat Luangchaisri and Thawhat Changphas


#### Abstract

We present characterizations of completely regular 2-duo semigroups using (2,0)ideals, $(0,2)$-ideals, $(2,2)$-ideals and (2,2)-quasi-ideals of semigroups. We then consider 2-duo semigroups when every (2,2)-ideal is quasi-prime.


## 1. Introduction

Let $S$ be a semigroup. An element $a \in S$ is said to be regular if there exits $x \in S$ such that $a=a x a$, and $S$ is said to be regular if every element of $S$ is regular. Let $A$ be a nonempty subset of $S$. We say that $A$ is a left ideal (respectively, right ideal) of $S$ if $S A \subseteq A$ (respectively, $A S \subseteq A$ ). $A$ is called a two-sided ideal of $S$ if it is both a left and a right ideal of $S . S$ is called a duo semigroup if its left ideals and right ideals are two-sided. In [6], the author characterized regular duo semigroups by left ideals and right ideals.

Let $m$ and $n$ be non-negative integers. A subsemigroup $A$ of a semigroup $S$ is said to be an $(m, n)$-ideal of $S$ if $A^{m} S A^{n} \subseteq A$. Here, $A^{0} S=S A^{0}=S$. An $(m, n)$ ideal was firstly introduced by S. Lajos in [4]; the author considered ( $m, n$ )-ideals on regular duo semigroups in [5]. The results were extended to ordered semigroups by L. Bussaban and T. Changphas in [1].

In this paper, we define an $n$-duo semigroup extending the concept of duo semigroups. We then characterize completely regular 2 -duo semigroups by $(2,2)$ ideals. Moreover, we consider when (2,2)-ideals of 2-duo semigroups are all quasiprime.

## 2. Main Results

Definition 2.1. (cf. [2],[3],[8]) Let $S$ be a semigroup and let $a \in S$. We say that $a$ is completely regular if $a \in a^{2} S a^{2}$. A semigroup $S$ is completely regular if every element of $S$ is completely regular.

Definition 2.2. Let $S$ be any semigroup and let $n$ be a non-negative integer. We say that $S$ is an $n$-duo semigroup if it satisfies the following conditions:
(i) Every $(n, 0)$-ideal of $S$ is a $(0, n)$-ideal of $S$;

2010 Mathematics Subject Classification: 20M17
Keywords: 2-duo semigroup, completely regular, $(m, n)$-ideal, quasi-prime
(ii) Every $(0, n)$-ideal of $S$ is an $(n, 0)$-ideal of $S$.

Let $A$ be a nonempty subset of a semigroup $S$. The set $L(A)$ (respectively, $R(A)$ ) is a left (respectively, right) ideal of $S$ generated by $A$. It is well known that $L(A)=A \cup S A$ and $R(A)=A \cup A S$. Moreover, the set $L(A)$ coincide the set $R(A)$ on duo semigroups. By Theorem 2.4 and Example 2.3, we show that every duo semigroup is an $n$-duo semigroup ( $n \geqslant 2$ ), but the converse is not generally true.

Example 2.3. Let $S=\{a, b, c, d\}$. Consider a semigroup $S$ with an associative operation defined by:

| $\cdot$ | $a$ | $b$ | $c$ | $d$ |
| :---: | :---: | :---: | :---: | :---: |
| $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $a$ | $a$ | $a$ |
| $c$ | $a$ | $a$ | $a$ | $a$ |
| $d$ | $a$ | $a$ | $b$ | $a$ |

Then $(S, \cdot)$ is a 2 -duo semigroup, but it is not a duo semigroup.
Theorem 2.4. Let $S$ be a semigroup. If $S$ is a duo semigroup, then $S$ is an $n$-duo semigroup where $n \geqslant 2$.
Proof. Assume that $S$ is a duo semigroup. Let $A$ be an $(n, 0)$-ideal of $S$. Then

$$
\begin{aligned}
A^{n} S & \subseteq A^{n-1}(A \cup A S) \\
& =A^{n-1} R(A) \\
& =A^{n-1} L(A) \\
& =A^{n-1}(A \cup S A) \\
& =A^{n} \cup A^{n-1} S A \\
& \subseteq A \cup A^{n-1} S A
\end{aligned}
$$

Continue in the same manner, we obtain that

$$
A^{n} S \subseteq A \cup S A^{n} \subseteq A
$$

Thus, $A$ is a $(0, n)$-ideal of $S$. Similarly, we have that every $(0, n)$-ideal of $S$ is an ( $n, 0$ )-ideal of $S$. Therefore, $S$ is an $n$-duo semigroup.

Let $S$ be a semigroup. For each $a \in S$, the symbol $J_{(m, n)}(a)$ stands for the $(m, n)$-ideal of $S$ generated by $a$. S. Lajos proved in [4] that

$$
J(a)_{(m, n)}=\bigcup_{i=1}^{m+n} a^{i} \bigcup a^{m} S a^{n}
$$

It is observed that $J_{(0,2)}(a)=J_{(2,0)}(a)$ for all $a \in S$ if $S$ is a 2-duo semigroup.

Theorem 2.5. Let $S$ be a semigroup. Then $S$ is a completely regular 2-duo semigroup if and only if the following conditions hold:
(1) $\left(A^{2} \cup A^{2} S\right)^{2}=A$ for all $(0,2)$-ideals $A$ of $S$;
(2) $\left(B^{2} \cup S B^{2}\right)^{2}=B$ for all $(2,0)$-ideals $B$ of $S$.

Proof. Assume that $S$ is a completely regular 2-duo semigroup. Let $A$ be a ( 0,2 )ideal of $S$. Then $A=A^{2}$ because

$$
A \subseteq A^{2} S A^{2} \subseteq A^{3} \subseteq A^{2} \subseteq A
$$

Next, we prove the main equation of this theorem. Consider

$$
\begin{aligned}
A & =A^{2} \\
& =(A \cup A)^{2} \\
& \subseteq\left(A^{2} \cup A^{2} S A^{2}\right)^{2} \\
& \subseteq\left(A^{2} \cup A^{2} S\right)^{2} \\
& \subseteq A^{2} \\
& =A
\end{aligned}
$$

Therefore, $\left(A^{2} \cup A^{2} S\right)^{2}=A$. If $B$ is a $(2,0)$-ideal of $S$, we can proceed similarly and then we obtain $\left(B^{2} \cup S B^{2}\right)^{2}=B$.

Conversely, assume that (1) and (2) hold. Let $A$ be a ( 0,2 )-ideal of $S$. Then

$$
\begin{aligned}
A^{2} S & =\left(A^{2} \cup A^{2} S\right)^{2}\left(A^{2} \cup A^{2} S\right)^{2} S \\
& \subseteq\left(A^{2} \cup A^{2} S\right)^{2} S \\
& \subseteq\left(A^{2} \cup A^{2} S\right)\left(A^{2} S\right) \\
& \subseteq\left(A^{2} \cup A^{2} S\right)\left(A^{2} \cup A^{2} S\right) \\
& =A
\end{aligned}
$$

Thus, $A$ is a (2,0)-ideal of $S$. Similarly, if $B$ is a (2,0)-ideal of $S$, then by (2) we obtain $B$ is a $(0,2)$-ideal of $S$. Therefore, $S$ is 2 -duo.

To prove that $S$ is completely regular, let $a \in S$. Consider

$$
\begin{aligned}
a \in J(a)_{(2,0)} & =\left(\left(J(a)_{(2,0)}\right)^{2} \cup\left(J(a)_{(2,0)}\right)^{2} S\right)^{2} \\
& =\left(\left(J(a)_{(2,0)}\right)^{2} \cup\left(J(a)_{(2,0)}\right)^{2} S\right)\left(\left(J(a)_{(0,2)}\right)^{2} \cup\left(J(a)_{(2,0)}\right)^{2} S\right) \\
& \subseteq\left(a^{2} \cup a^{2} S\right)\left(\left(J(a)_{(0,2)}\right)^{2} \cup J(a)_{(2,0)}\right) \\
& \subseteq\left(a^{2} \cup a^{2} S\right)\left(a^{2} \cup S a^{2} \cup J(a)_{(0,2)}\right) \\
& =\left(a^{2} \cup a^{2} S\right)\left(a \cup a^{2} \cup S a^{2}\right) \\
& \subseteq a^{3} \cup a^{4} \cup a^{2} S a^{2} .
\end{aligned}
$$

Thus, $a$ is completely regular.

Theorem 2.6. Let $S$ be any semigroup. Then $S$ is a completely regular 2-duo semigroup if and only if

$$
\left(B^{2} \cup B^{2} S\right)^{2}=B=\left(B^{2} \cup S B^{2}\right)^{2}
$$

for all (2,2)-ideal B of $S$.
Proof. Assume that $S$ is a completely regular 2-duo semigroup. Let $B$ be a (2, 2)ideal of $S$. Then

$$
\begin{aligned}
\left(B^{2} \cup B^{2} S\right)^{2} & \subseteq B^{4} \cup B^{4} S \cup B^{2} S B^{2} \cup B^{2} S B^{2} S \\
& \subseteq B \cup B^{4} S \\
& \subseteq B \cup B S \\
& \subseteq B \cup B^{2} S B^{2} S \\
& \subseteq B \cup B\left(B^{2} S B^{2}\right) S B^{2} S
\end{aligned}
$$

Since $S B^{2}$ is a $(0,2)$-ideal of $S$ and $S$ is a 2-duo semigroup, it follows that

$$
\begin{aligned}
B \cup B\left(B^{2} S B^{2}\right) S B^{2} S & =B \cup B^{3}\left(S B^{2} S B^{2}\right) S \\
& \subseteq B \cup B^{3} S B^{2} \\
& \subseteq B \cup B^{2} S B^{2} \\
& \subseteq B
\end{aligned}
$$

According to the proof of Theorem 2.5, we have that $B=B^{2}$. Thus,

$$
B=B^{4} \subseteq\left(B^{2} \cup B^{2} S\right)^{2}
$$

Therefore, $B=\left(B^{2} \cup B^{S}\right)^{2}$. Similarly, $B=\left(B^{2} \cup B^{2} S\right)$. Hence,

$$
\left(B^{2} \cup B^{2} S\right)^{2}=B=\left(B^{2} \cup B^{2} S\right)
$$

Conversely, let $A$ be a $(0,2)$-ideal of $S$. Then $A$ is a $(2,2)$ ideal of $S$. By assumption,

$$
A=\left(A^{2} \cup A^{2} S\right)^{2}
$$

On the same way, we obtain that

$$
B=\left(B^{2} \cup S B^{2}\right)
$$

for every (2,0)-ideal $B$ of $S$. By Theorem $2.5, S$ is a completely regular 2-duo semigroup.
Definition 2.7. Let $S$ be a semigroup and let $m, n$ be non-negative integers. A subsemigroup $Q$ of $S$ is said to be an ( $m, n$ )-quasi-ideal of $S$ if $S Q^{m} \cap Q^{n} S \subseteq Q$. Here, $Q^{0} S=S Q^{0}=S$.

Theorem 2.8. Let $S$ be a semigroup. Then $S$ is a completely regular 2-duo semigroup if and only if

$$
\left(Q^{2} \cup Q^{2} S\right)^{2}=Q=\left(Q^{2} \cup S Q^{2}\right)^{2}
$$

for all (2,2)-quasi-ideal $Q$ of $S$.
Proof. Assume that $S$ is a completely regular 2-duo semigroup. Let $Q$ be a $(2,2)$ -quasi-ideal of $S$. Then

$$
\left(Q^{2} \cup Q^{2} S\right)^{2}=Q^{2} \cup Q^{4} S \cup Q^{2} S Q^{2} \cup Q^{2} S Q^{2} S \subseteq Q^{2} S
$$

and

$$
\begin{aligned}
\left(Q^{2} \cup Q^{2} S\right)^{2} & =Q^{4} \cup Q^{4} S \cup Q^{2} S Q^{2} \cup Q^{2} S Q^{2} S \\
& \subseteq Q^{2} S Q^{2} \cup S Q^{2} S \\
& \subseteq S Q^{2} \cup S Q^{2} S \\
& \subseteq S Q^{2} \cup S Q\left(Q^{2} S Q^{2}\right) S \\
& \subseteq S Q^{2} \cup S Q^{2} S Q^{2} S \\
& \subseteq S Q^{2} \cup S Q^{2} \\
& =S Q^{2}
\end{aligned}
$$

Thus, $\left(Q^{2} \cup Q^{2} S\right)^{2} \subseteq Q S^{2} \cap S Q^{2} \subseteq Q$. The opposite inclusion is obtained by the following equation:

$$
Q \subseteq Q^{2} S Q^{2} \subseteq\left(Q^{2} \cup Q^{2} S\right)^{2}
$$

Similarly, we have

$$
Q=\left(Q^{2} \cup S Q^{2}\right)^{2}
$$

This implement has been proven.
Conversely, let $A$ and $B$ be a ( 0,2 )-ideal of $S$ and a (2,0)-ideal of $S$, respectively. Then $A$ and $B$ are (2,2)-quasi-ideals as well. By assumption, we have

$$
A=\left(A^{2} \cup A^{2} S\right)^{2}
$$

and

$$
B=\left(B^{2} \cup S B^{2}\right)^{2}
$$

By Theorem 2.5, we have that $S$ is a completely regular 2-duo semigroup.
Example 2.9. Let $S=\{0,1,2,3\}$ and defined a binary operation on $S$ by

| $\cdot$ | $a$ | $b$ | $c$ | $d$ |
| :---: | :---: | :---: | :---: | :---: |
| $a$ | $a$ | $b$ | $a$ | $d$ |
| $b$ | $b$ | $a$ | $b$ | $d$ |
| $c$ | $a$ | $b$ | $c$ | $d$ |
| $d$ | $d$ | $d$ | $d$ | $d$. |

Then $(S, \cdot)$ is a semigroup. We have that $\{d\},\{a, b, d\}$ and $S$ are only (2,2)-ideals of $S$. Moreover every (2,2)-ideal $B$ of $S$ satisfies the equation

$$
\left(B^{2} \cup B^{2} S\right)^{2}=B=\left(B^{2} \cup S B^{2}\right)^{2}
$$

Thus, $S$ is a completely regular 2-duo semigroup.
Lemma 2.10. Let $S$ be a semigroup. Then $S$ is completely regular if and only if $A=A^{2}$ for every $(2,2)$-ideal $A$ of $S$.
Proof. Assume that $S$ is completely regular. Let $A$ be a $(2,2)$-ideal of $S$. Then

$$
A \subseteq A^{2} S A^{2} \subseteq A^{2} S\left(A^{2} S A^{2}\right)\left(A^{2} S A^{2}\right) \subseteq\left(A^{2} S A^{2}\right)\left(A^{2} S A^{2}\right) \subseteq A^{2} \subseteq A
$$

Thus, $A=A^{2}$.
Conversely, assume that $A=A^{2}$ for every (2,2)-ideal $A$ of $S$. Let $a \in S$. Then

$$
\begin{aligned}
a & \in J_{(2,2)}(a) \\
& =\left(J_{(2,2)}(a)\right)^{2} \\
& \subseteq a^{2} \cup a^{3} \cup a^{4} \cup a^{2} S a^{2} .
\end{aligned}
$$

Thus, $a \in a^{2} S a^{2}$. This implies that $S$ is completely regular.
Remark 2.11. If $S$ is completely regular, then $A B$ is a (2,2)-ideal of $S$ for all $(2,2)$-ideals $A, B$ of $S$.

Definition 2.12. Let $S$ be a semigroup. A (2,2)-ideal $P$ of $S$ is said to be quasiprime if

$$
A B \subseteq P \Rightarrow A \subseteq P \text { or } B \subseteq P
$$

for all (2,2)-ideals $A, B$ of $S$.
Definition 2.13. Let $S$ be a semigroup. A (2,2)-ideal $P$ of $S$ is said to be quasisemiprime if

$$
A^{2} \subseteq P \Rightarrow A \subseteq P
$$

for every (2,2)-ideal $A$ of $S$.
Recall and apply Lemma 2.11 in [7], we have the following lemma:
Lemma 2.14. Let $S$ be a semigroup. Then $A=A^{2}$ for every $(2,2)$-ideal $A$ of $S$ if and only if every $(2,2)$-ideal of $S$ is quasi-semiprime.

Theorem 2.15. Let $S$ be a-duo semigroup. Then every (2,2)-ideal of $S$ is quasiprime if and only if $S$ is completely regular and (2,2)-ideals of $S$ form a chain by inclusion.

Proof. Assume that every ( 2,2 )-ideal of $S$ is quasi-prime. Then they are quasisemiprime as well. By Lemma 2.14, we have that $A=A^{2}$ for every ( 2,2 )-ideal $A$ of $S$. By Theorem 2.10, $S$ is completely regular. Next, we show that ( 2,2 )-ideals of $S$ form a chain by inclusion. Let $A, B$ be $(2,2)$-ideals of $S$. By Remark2.11, we obtain that $A B$ is also a ( 2,2 )-ideal of $S$. By assumption, $A B$ is quasi-prime. Then we have two cases to consider:

Case 1: $A \subseteq A B$. Then

$$
A \subseteq A B \subseteq A\left(B^{2} S B^{2}\right) \subseteq A B^{2} S B\left(B^{2} S B^{2}\right) \subseteq A B^{2} S B^{2} S B^{2}
$$

Since $B^{2} S$ is a $(2,0)$-ideal of $S$ and $S$ is a 2 -duo semigroup, it follows that $B^{2} S$ is a ( 0,2 )-ideal of $S$. Thus,

$$
A B^{2} S B^{2} S B^{2} \subseteq B^{2} S B^{2} \subseteq B .
$$

These imply that $A \subseteq B$.
Case 2: $B \subseteq A$. Then

$$
B \subseteq A B \subseteq\left(A^{2} S A^{2}\right) B \subseteq\left(A^{2} S A^{2}\right) A S A^{2} B \subseteq A S A^{2} S A^{2} B
$$

Since $S A^{2}$ is a $(0,2)$-ideal of $S$ and $S$ is a 2 -duo semigroup, it follows that $S A^{2}$ is a (2,0)-ideal of $S$. Thus,

$$
A S A^{2} S A^{2} B \subseteq A^{2} S A^{2} \subseteq A
$$

These imply that $B \subseteq A$. From Case 1 and Case 2, we conclude that (2,2)-ideals of $S$ form a chain by inclusion.

To prove the opposite direction, let $P$ be a $(2,2)$-ideal of $S$. Assume that $A, B$ are (2,2)-ideals of $S$ such that $A B \subseteq P$. If $A \subseteq B$, then

$$
A=A^{2} \subseteq A B \subseteq P
$$

Otherwise, $B \subseteq A$ implies that

$$
B=B^{2} \subseteq A B \subseteq P
$$

Thus, $P$ is quasi-prime.
Acknowledgments: This work has received scholarship under the Post-Doctoral Training Program from Khon Kaen University, Thailand.

## References

[1] L. Bussaban and T. Changphas, $A$ note on $(m, n)$-ideals in regular duo semigroups, Quasigroups and Related Systems, 23 (2015), 211-216.
[2] A.H. Clifford, Semigroups admitting relative inverses, Annals of Math., 42 (1941), 1037-1049.
[3] R. Croisot, Demi-groupes inversifs et demi-groupes réunions de demi-groupes simples, Ann. Sci. Ecole Norm. Sup. 70 (1953), 361-379.
[4] S. Lajos, Generalize ideals in semigroups, Acta Sci. Math., 22 (1961), 217-222.
[5] S. Lajos, On ( $m, n$ )-ideals in regular duo semigroups, Acta Sci. Math., 31 (1970), 179-180.
[6] S. Lajos, A new characterization of regular duo semigroups, Proc. Japan Acad., 47 (1971), 394-395.
[7] P. Luangchaisri and T. Changphas, On $(m, n)$-regular and intra-regular ordered semigroups, Quasigroups and Related Systems, 27 (2019), 267-272.
[8] M. Petrich, Introduction to Semigroups, Charles E. Merrill Publishing Company, (1973).

Received January 13, 2021
P. Luangchaisri

Department of Mathematics,
Faculty of Science,
Khon Kaen University
Khon Kaen 40002, Thailand
E-mail: desparadoskku@hotmail.com
T. Changphas

Centre of Excellence in Mathematics,
CHE, Si Ayuttaya Rd.,
Bangkok 10400, Thailand
E-mail: thacha@kku.ac.th

# A new design of the signature schemes based on the hidden discrete logarithm problem 

Dmitriy N. Moldovyan, Alexandr A. Moldovyan, Nikolay A. Moldovyan


#### Abstract

A new design of the signature scheme based on the computational complexity of the hidden discrete logarithm problem, which meets the criterion of elimination of periodicity associated with the value of the discrete logarithm, is introduced as a candidate for post-quantum public-key cryptoscheme. The used design criterion is oriented to provide security to the known and potential future quantum attacks. Three different 6 -dimensional finite non-commutative associative algebras sets over the field $G F(p)$ are considered as the algebraic support of the developed signature have algorithm that is characterized in using a commutative finite group possessing 2-dimensional cyclicity as a hidden group. Besides, the following two different types of masking operations are applied: i) operations that are mutual commutative with the exponentiation operation and ii) operations that are free of this property.


## 1. Introduction

Development of practical post-quantum (PQ) public-key (PK) cryptosystems is a current challenge in the area of cryptography, which attracts considerable attention from the research community [15, 16]. The most widely used in practice, PK cryptographic algorithms and protocols are not resistant to quantum attacks (attacks on computations on a quantum computer), since they are based on the computational difficulty of the factoring problem (FP) and the discrete logarithm problem (DLP) each of which can be solved in polynomial time on a quantum computer [2, 18]. Quantum algorithms for solving the FP and DLP exploit the extremely high efficiency of quantum computers to perform a discrete Fourier transform [3] which is used to calculate the period length of periodic functions. In particular, to solve DLP, one constructs a periodic function containing a period with the length depending on the value of the logarithm.

Among the computationally difficult problems used as a basic primitive of PQ PK cryptoschemes the hidden discrete logarithm problem (HDLP) [4, 6, 8] is of particular interest for the development of PQ signature schemes [13, 7] having high performance and comparetively small size of the PK and signature.

[^2]Recently [10], an enhanced design criterion has been proposed to provide the resistance of the HDLP-based signature schemes to quantum attacks. That criterion consists in the requirement to eliminate periodicities depending on the value of the discrete logarithm when defining periodic functions on the base of public parameters of the signature scheme. The signature scheme proposed in [10] meets the said design criterion, however, that scheme uses a doubled verification equation reducing the rate and increasing the signature size.

The present paper consideres another design of HDP-based signature schemes meeting the advanced criterion of PQ resistance. The introduced new signature scheme has significantly smaller size of signature and PK.

## 2. Preliminaries

### 2.1. Masking operations and hidden logarithm problem

Usually the HDLP is defined in finite non-commutative associative algebras (FNAAs) [6, 7, 13]. The HDLP can be briefly described as follows. It is a selected a random cyclic group having sufficiently large prime order, which is represented by its generator $G$. Then one computes the PK in the form of the pair of vectors $Z=\psi_{1}(G)$ and $Y=\psi_{2}\left(G^{x}\right)$, where $x$ is private key; $\psi_{1}$ and $\psi_{2}$ are masking operations representing two different homomorphism-map (or automorphism-map) operations which are mutually commutative with the exponentiation operation.

Due to using the masking operations $\psi_{1}$ and $\psi_{2}$ the vectors $Z$ and $Y$ are contained in different cyclic groups. Each of the masking operations is mutually comutative with the exponentiation operation, therefore, one can use a DLP-based signature (for example, well-known Schnorr signature algorithm [17]) and replace in it the signature verification procedure using the values $G$ and $G^{x}$ by a signature verification procedure using the values $Z$ and $Y$. To compute a signature, a potential forger needs to know only the value $x$ that is a discrete logarithm value in a hidden cyclic group, no element of which is known to the forger. The rationale of the security of the HDLP-based signature scheme is connected with the fact that every set of periodic functions constructed using the public parameters of the signature scheme takes on values in many different cyclic groups contained in FNAA used as algebraic support. Therefore, the Shor quantum algorithm is not directly applicable to compute the value $x$, even in the case when a periodic function contains a period depending on the value $x$ although.

For example, in the case of the signature scheme [13] the function $F(i, j)=$ $Y^{i} \circ Z^{j}$, where $\circ$ denotes the multiplication operation in the FNAA, contains a period of the length $(-1, x)$, however one cannot select a fixed cyclic group such that the function $F(i, j)$ take on with sufficiently high probability the values in the fixed cyclic group.

Thus, for the development of the HDLP-based signature schemes, one can formulate the following design criterion:

Criterion 1. The periodic functions constructed on the base of public parameters of the signature scheme and containing a period with the length depending on the discrete logarithm value should take on values in different finite cyclic groups contained in the FNAA used as algebraic support. Besides, no cyclic group can be pointed out as a preferable finite group for the values of the function $F(i, j)$.

However, the future progress in quantum computations can lead to developing new quantum algorithms that will allow one to compute the period length for periodic functions that take on values in algebraic sets that are not groups. Possible emergence of such quantum algorithms will mean breaking the known HDLP-based signature schemes.

In the paper [10] the following strengthened criterion for ensuring the security of the HDLP-based cryptoschemes to hypothetic quantum attacks is proposed:

Criterion 2. Based on the public parameters of the signature scheme, the construction of a periodic function containing a period with the length depending on the discrete logarithm value should be a computationally intractable task.

Using Criterion 2, in the present paper, a new HDLP-base signature scheme is developed which has smaller sizes of signature and PK.

### 2.2. The used 6 -dimensional FNAAs

Suppose a finite $m$-dimensional vector space is defined over the ground finite field $G F(p)$. Then defining additionally the vector multiplication that is distributive at the right and at the left relatively the addition operation one gets a finite $m$-dimensional algebra. Some algebra element ( $m$-dimensional vector) $A$ can be denoted in the following two forms: $A=\left(a_{0}, a_{1}, \ldots, a_{m-1}\right)$ and $A=\sum_{i=0}^{m-1} a_{i} \mathbf{e}_{i}$, where $a_{0}, a_{1}, \ldots, a_{m-1} \in G F(p)$ are called coordinates; $\mathbf{e}_{0}, \mathbf{e}_{1}, \ldots \mathbf{e}_{m-1}$ are basis vectors.

The vector multiplication operation ( $\circ$ ) of two $m$-dimensional vectors $A$ and $B$ is defined as follows:

$$
A \circ B=\sum_{i=0}^{m-1} \sum_{j=0}^{m-1} a_{i} b_{j}\left(\mathbf{e}_{i} \circ \mathbf{e}_{j}\right)
$$

where every of the products $\mathbf{e}_{i} \circ \mathbf{e}_{j}$ is to be replaced by a single-component vector $\lambda \mathbf{e}_{k}$, where $\lambda \in G F(p)$, indicated in the cell at the intersection of the $i$ th row and $j$ th column of so called basis vector multiplication table (BVMT) like Tables 1, 2, and 3. To define the associative vector multiplication operation, the BVMT should define the associative multiplication of all possible triples of the basis vectors $\left(\mathbf{e}_{i}, \mathbf{e}_{j}, \mathbf{e}_{k}\right)$ :

$$
\left(\mathbf{e}_{i} \circ \mathbf{e}_{j}\right) \circ \mathbf{e}_{k}=\mathbf{e}_{i} \circ\left(\mathbf{e}_{j} \circ \mathbf{e}_{k}\right) .
$$

Three different 6 -dimensional FNAAs defined by Tables 1 , 2 , and 3 with the structural constant $\lambda \neq 0$ are considered as the algebraic support of the HDLPbased signature scheme described in the next Section 3. The BVMT shown as

Tablea 1 is constructed using a unified method [12] for setting FNAA of arbitrary even dimensions. Other two BVMTs are presented as alternative variants of setting the 6 -dimensional FNAAs which also suit well for applying them as an algebraic support of the proposed signature scheme.

Every of these FNAAs contains a global two-sided unit. The unit in the algebra defined by Tables 1 and 3 represents the vector $E=(1,0,0,0,0,0)$. The unit in the algebras defined by Table 2 is the vector $E=(0,0,0,1,0,0)$. Invertible vectors having prime order of sufficiently large size are used as parameters of the signature scheme. In every of the said FNAAs the maximum order of the elements is equal to $\omega_{\max }=p\left(p^{2}-1\right)$ and the algebras are set over the field $G F(p)$ with characteristic equal to prime $p=2 q+1$, where $q$ is a 255 -bit prime number.

It is easy to see that every of the considered FNAAs contains a large number of different commutative groups possessing 2-dimensional cyclicity. The notion of $\mu$-dimensional cyclicity was proposed in [11, 14] in order to highlight the finite groups generated by a minimum generator system including $\mu$ elements of the same order.

Consider the vector $V_{d}=(d, 0,0,0,0,0)$, where $d$ is primitive element in $G F(p)$. Evidently, the vector $V_{d}$ is generator of the cyclic group $\Gamma_{d}$ including all vectors of the form $(i, 0,0,0,0,0)$, where $i \neq 0$, and every vector $V \in \Gamma_{d}$ satisfies the condition $A \circ V=V \circ A$, since multiplication by $V$ represents the scalar multiplication.

Suppose the vector $J \notin \Gamma_{d}$ has order equal to $p-1$. Then the minimum generator system $<J, V_{d}>$ defines the finite commutative group possessing 2dimensional cyclicity and having the order $\Omega=(p-1)^{2}$. Every of the considered 6 -dimensional FNAAs contains a large number of different commutative groups of the said type and the cyclic group $\Gamma_{d}$ is contained in every of these commutative groups.

Table 1
The BVMT [12] setting the 6-dimensional FNAA used as algebraic support

| $\circ$ | $\mathbf{e}_{0}$ | $\mathbf{e}_{1}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{3}$ | $\mathbf{e}_{4}$ | $\mathbf{e}_{5}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{e}_{0}$ | $\mathbf{e}_{0}$ | $\mathbf{e}_{1}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{3}$ | $\mathbf{e}_{4}$ | $\mathbf{e}_{5}$ |
| $\mathbf{e}_{1}$ | $\mathbf{e}_{1}$ | $\lambda \mathbf{e}_{0}$ | $\mathbf{e}_{5}$ | $\lambda \mathbf{e}_{4}$ | $\mathbf{e}_{3}$ | $\lambda \mathbf{e}_{2}$ |
| $\mathbf{e}_{2}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{3}$ | $\mathbf{e}_{4}$ | $\mathbf{e}_{5}$ | $\mathbf{e}_{0}$ | $\mathbf{e}_{1}$ |
| $\mathbf{e}_{3}$ | $\mathbf{e}_{3}$ | $\lambda \mathbf{e}_{2}$ | $\mathbf{e}_{1}$ | $\lambda \mathbf{e}_{0}$ | $\mathbf{e}_{5}$ | $\lambda \mathbf{e}_{4}$ |
| $\mathbf{e}_{4}$ | $\mathbf{e}_{4}$ | $\mathbf{e}_{5}$ | $\mathbf{e}_{0}$ | $\mathbf{e}_{1}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{3}$ |
| $\mathbf{e}_{5}$ | $\mathbf{e}_{5}$ | $\lambda \mathbf{e}_{4}$ | $\mathbf{e}_{3}$ | $\lambda \mathbf{e}_{2}$ | $\mathbf{e}_{1}$ | $\lambda \mathbf{e}_{0}$ |

TABLE 2
The BVMT setting the first alternative 6 -dimensional FNAA; $\lambda \neq 0$.

| $\circ$ | $\mathbf{e}_{0}$ | $\mathbf{e}_{1}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{3}$ | $\mathbf{e}_{4}$ | $\mathbf{e}_{5}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{e}_{0}$ | $\lambda \mathbf{e}_{3}$ | $\mathbf{e}_{2}$ | $\lambda \mathbf{e}_{1}$ | $\mathbf{e}_{0}$ | $\lambda \mathbf{e}_{5}$ | $\mathbf{e}_{4}$ |
| $\mathbf{e}_{1}$ | $\mathbf{e}_{4}$ | $\mathbf{e}_{5}$ | $\mathbf{e}_{0}$ | $\mathbf{e}_{1}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{3}$ |
| $\mathbf{e}_{2}$ | $\lambda \mathbf{e}_{5}$ | $\mathbf{e}_{4}$ | $\lambda \mathbf{e}_{3}$ | $\mathbf{e}_{2}$ | $\lambda \mathbf{e}_{1}$ | $\mathbf{e}_{0}$ |
| $\mathbf{e}_{3}$ | $\mathbf{e}_{0}$ | $\mathbf{e}_{1}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{3}$ | $\mathbf{e}_{4}$ | $\mathbf{e}_{5}$ |
| $\mathbf{e}_{4}$ | $\lambda \mathbf{e}_{1}$ | $\mathbf{e}_{0}$ | $\lambda \mathbf{e}_{5}$ | $\mathbf{e}_{4}$ | $\lambda \mathbf{e}_{3}$ | $\mathbf{e}_{2}$ |
| $\mathbf{e}_{5}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{3}$ | $\mathbf{e}_{4}$ | $\mathbf{e}_{5}$ | $\mathbf{e}_{0}$ | $\mathbf{e}_{1}$ |

Table 3
The BVMT setting the second alternative 6 -dimensional FNAA; $\lambda \neq 0$.

| $\circ$ | $\mathbf{e}_{0}$ | $\mathbf{e}_{1}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{3}$ | $\mathbf{e}_{4}$ | $\mathbf{e}_{5}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{e}_{0}$ | $\mathbf{e}_{0}$ | $\mathbf{e}_{1}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{3}$ | $\mathbf{e}_{4}$ | $\mathbf{e}_{5}$ |
| $\mathbf{e}_{1}$ | $\mathbf{e}_{1}$ | $\lambda \mathbf{e}_{0}$ | $\lambda \mathbf{e}_{4}$ | $\lambda \mathbf{e}_{5}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{3}$ |
| $\mathbf{e}_{2}$ | $\mathbf{e}_{2}$ | $\lambda \mathbf{e}_{5}$ | $\lambda \mathbf{e}_{0}$ | $\lambda \mathbf{e}_{4}$ | $\mathbf{e}_{3}$ | $\mathbf{e}_{1}$ |
| $\mathbf{e}_{3}$ | $\mathbf{e}_{3}$ | $\lambda \mathbf{e}_{4}$ | $\lambda \mathbf{e}_{5}$ | $\lambda \mathbf{e}_{0}$ | $\mathbf{e}_{1}$ | $\mathbf{e}_{2}$ |
| $\mathbf{e}_{4}$ | $\mathbf{e}_{4}$ | $\mathbf{e}_{3}$ | $\mathbf{e}_{1}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{5}$ | $\mathbf{e}_{0}$ |
| $\mathbf{e}_{5}$ | $\mathbf{e}_{5}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{3}$ | $\mathbf{e}_{1}$ | $\mathbf{e}_{0}$ | $\mathbf{e}_{4}$ |

## 3. The proposed signature scheme

### 3.1. Setting the hidden commutative group

One can use different values of the structural constant $\lambda \neq 0$ in the BVMTs defining the 6-dimensional FNAAs used as an algebraic support of the developed signature scheme. For any fixed value $\lambda$ every of the said algebras contains sufficiently large number of commutative groups with 2-dimensional cyclicity. Computation of the private and public parameters of the signature scheme begins with setting a private hidden finite commutative group $\Gamma_{<G, Q>}$ that is generated by the minimum generator system $<G, Q>$ that includes two vectors $G$ and $Q$ each of which has order equal to the prime $q$. Actually, the group $\Gamma_{<G, Q>}$ of the order $q^{2}$ is set as computation of the vectors $G$ and $Q$ of the order $q$, which is performed as follows:

1. Select a random invertible vector $R_{1}$ and compute $G_{1}=R_{1}^{2 p(p+1)} \neq E$.
2. Select a random invertible vector $R_{2}$ and compute $G_{2}=R_{2}^{2 p(p+1)} \neq E$.
3. If $G_{1} \circ G_{2}=G_{2} \circ G_{1}$, then go to step 1. Otherwise, take $G=G_{1}$.
4. Select a random integer $r$ and compute $b=r^{2} \bmod p \neq 1$.
5. Performing scalar multiplication, compute the vector $Q=b G$.

One can easily see that the order of each of the vectors $G$ and $Q$ is equal
to the prime $q$, therefore we have the minimum generator system $<G, Q>$ of the commutative group with 2-dimensional cyclicity, which has order equal to the value $q^{2}$.

### 3.2. Masking operations and computation of the public key.

Two different types of masking operations are used:
i) the automorphism map operation $\psi_{B}(X)=B \circ X \circ B^{-1}$, where $B$ is an invertible vector (private value), which is a mutually commutative the exponentiation operation;
ii) map operations that are not mutually commutative with the exponentiation operation, which are defined as $F_{A B}(X)=A \circ X \circ B^{-1}$ and $F_{B A}(X)=B \circ X \circ A^{-1}$.

Computation of the PK in the form of the triple of vectors $(U, Y, Z)$ is performed as follows:

1. Generate at random the minimum generator system $<G, Q>$ of the hidden commutative group $\Gamma_{<G, Q>}$ possessing the 2-dimensional cyclicity.
2. Generate at random the invertible vector $B$ of the order $p^{2}-1$, which satisfies the conditions $G \circ B \neq B \circ G$, and compute the vector $Y=B \circ G \circ B^{-1}$.
3. Generate at random the integers $x(1<x<q)$ and $w(1<w<q)$ and the invertible vector $A$ of the order $p^{2}-1$, which satisfies the conditions $A \circ B \neq B \circ A$ and $A \circ G \neq G \circ A$. Then compute the vectors $U=A \circ G^{x} \circ Q \circ B^{-1}$ and $Z=B \circ Q^{w} \circ A^{-1}$.

The integers $x, w$ and the vectors $G, Q, A$, and $B$ are the private parameters of the signature scheme. The private key represents the subset $\{x, w, G, Q, A\}$ of private elements that are used when computing a signature. The size of the PK $(U, Y, Z)$ is equal to 576 bytes.

### 3.3. Signature generation algorithm:

1. Generate at random the integers $k(1<k<q)$ and $t(1<t<q)$. Then compute $V=A \circ G^{k} Q^{t} \circ A^{-1}$.
2. Using a specified hash function $f_{H}$, compute the first signature element $e$ : $e=f_{H}(M, V)$, where $M$ is a document to be signed.
3. Compute the second $s$ and third $\sigma$ signature elements as one of the two solutions of the following system of two congruences

$$
\left\{\begin{array}{l}
e s^{2}+x s+x \sigma=k \bmod q \\
s+w s+\sigma+w \sigma=t \bmod q
\end{array}\right.
$$

If this system has no solution, then go to step 1.
On average, computation of one 96 -byte signature $(e, s, \sigma)$ requires performing the signature generation procedure two times. On the whole, the computational difficulty of the signature computation procedure is roughly equal to four exponentiation operations in the 6-dimensional FNAA selected as the algebraic support of the signature scheme.

### 3.4. Verification and correctness of the signature scheme

Signature verification procedure includes the following steps:

1. Using the signature $(e, s, \sigma)$ and the $\mathrm{PK}(U, Y, Z)$, compute the vector

$$
V^{\prime}=\left(U \circ Y^{e s} \circ Z\right)^{s} \circ(U \circ Z)^{\sigma} .
$$

2. Compute the hash function value $e^{\prime}=f_{H}\left(M, V^{\prime}\right)$.
3. If $e^{\prime}=e$, then the signature is genuine. Otherwise, the signature is rejected.

The computational difficulty of the signature verification procedure is roughly equal to three exponentiation operations in the 6 -dimensional FNAA. Correctness proof of the signature scheme consists in proving that the signature $(e, s, \sigma)$ computed correctly will pass the verification procedure as a genuine signature.

Correctness proof:

$$
\begin{aligned}
& V_{1}^{\prime}=\left(U \circ Y^{e s} \circ Z\right)^{s} \circ(U \circ Z)^{\sigma}= \\
= & \left(A \circ G^{x} \circ Q \circ B^{-1} \circ\left(B \circ G \circ B^{-1}\right)^{e s} \circ B \circ Q^{w} \circ A^{-1}\right)^{s} \circ \\
& \circ\left(A \circ G^{x} \circ Q \circ B^{-1} \circ B \circ Q^{w} \circ A^{-1}\right)^{\sigma}= \\
= & \left(A \circ G^{x} \circ Q \circ G^{e s} \circ Q^{w} \circ A^{-1}\right)^{s} \circ A \circ G^{x \sigma} \circ Q^{\sigma} \circ Q^{w \sigma} \circ A^{-1}= \\
= & A \circ G^{x s} \circ Q^{s} \circ G^{e s^{2}} \circ Q^{w s} \circ G^{x \sigma} \circ Q^{\sigma+w \sigma} \circ A^{-1}= \\
= & A \circ G^{e s^{2}+x s+x \sigma} \circ Q^{s+w s+\sigma+w \sigma} \circ A^{-1}=A \circ G^{k} \circ Q^{t} \circ A^{-1}=V .
\end{aligned}
$$

Since $V^{\prime}=V$, the equality $e^{\prime}=e$ holds true, i. e. the signature is accepted as a genuine one.

## 4. Discussion

Consider some periodic functions composed on the base of public parameters of the introduced signature scheme.

1. Suppose the function $F_{1}(i, j)=(Z \circ U)^{i} \circ Y^{j}=B \circ G^{x i+j} \circ Q^{w i+i} \circ B^{-1}$ includes a period with the length $\left(\delta_{i}, \delta_{j}\right)$. Then, we have

$$
\left\{\begin{array}{l}
x \delta_{i}+\delta_{j} \equiv 0 \bmod q \\
(w+1) \delta_{i} \equiv 0 \bmod q
\end{array}\right.
$$

From the last system, one gets $\delta_{i} \equiv \delta_{j} \equiv 0 \bmod q$. The last means the function $F_{1}(i, j)$ possesses only the periodicity connected with the value $q$ that is the order of cyclic groups contained in the hidden commutative group with 2-dimensional cyclicity.
2. Suppose the function $F_{2}(i, j)=(U \circ Y \circ Z)^{i} \circ(U \circ Z)^{j}=A \circ G^{x i+i+x j} \circ$ $Q^{i+w i+j+w j} \circ A^{-1}$ contains a period with the length $\left(\delta_{i}, \delta_{j}\right)$. Then, taking into
account that $G$ and $Q$ are generators of different cyclic groups of the same order $q$, we have

$$
\left\{\begin{array}{l}
(x+1) \delta_{i}+x \delta_{j} \equiv 0 \bmod q \\
(w+1) \delta_{i}+(w+1) \delta_{j} \equiv 0 \bmod q
\end{array}\right.
$$

The main determinant of this system of two linear equations is not equal to zero, therefore, $\delta_{i} \equiv \delta_{j} \equiv 0 \bmod q$, i. e., the function $F_{2}(i, j)$ also possesses only the periodicity connected with the value $q$.
3. Suppose the function $F_{3}(i, j, k)=(U \circ Z)^{i} \circ\left(U \circ Y^{j} \circ Z\right)^{k}=B \circ G^{x i+x k+j k} \circ$ $Q^{w i+i+k+w k} \circ B^{-1}$ contains a period with the length $\left(\delta_{i}, \delta_{j}\right)$. Then we have

$$
\left\{\begin{array}{l}
x \delta_{i}+x \delta_{k}+j \delta_{k}+k \delta_{j}+\delta_{j} \delta_{k} \equiv 0 \bmod q \\
(w+1) \delta_{i}+(w+1) \delta_{k} \equiv 0 \bmod q
\end{array}\right.
$$

When solving the last system of two linear congruencies relatively, the unknowns $\delta_{i}, \delta_{j}$, and $\delta_{k}$, one obtains solutions that depend on the values $j$ and $k$, except the solution $\left(\delta_{i}, \delta_{j}, \delta_{k}\right)=(0,0,0)$. This means that the function $F_{3}(i, j, k)$ possesses only the periodicity with the length $(q, q, q)$, i. e., the function $F_{3}(i, j, k)$ also possesses only the periodicity connected with the order of the vectors $G$ and $Q$.

Thus, the proposed signature scheme meets the advanced design criterion of PQ resistance.

Among the nine signature algorithms developed in framework of the NIST competition as candidates for PQ signature standard the algorithms Falcon [https:// falcon-sign.info/], Dilithium [https://pq-crystals.org/dilithium/index.shtml], Rainbow [1], and qTESLA [https://qtesla.org/] attracts attention from the view point of the trade off between rate and size of the PK and the signature. Table 4 presents a rough comparison of the proposed signature algorithm with Falcon-512, Dilithium-1024x768, Rainbow, and qTESLA-p-I (versions related to the 128-bit security level).

The signature algorithm proposed in this article has a significant advantage in the size of the signature, but it is inferior in performance than Falcon-512. However, for potential versions of the proposed signature scheme, which will be implemented using a 4-dimensional FNAA with two-sided global unit as the algebraic support, the rate can be increased by 2.25 times (with simultaneous reducing the PK size to the value 384 bytes). Suitable 4 -dimensional FNAAs are presented, for example, in papers [5, 9]. When using a 256 -bit prime integer as the value $q$, one can expect the 128-bit security is provided for the both cases of the algebra dimension $m=6$ and $m=4$. However consideration of the security of the proposed signature scheme represents a task of individual study.

## 5. Conclusion

This paper introduces a HDLP-based signature scheme that meets the advanced design criterion of PQ resistance, significant merit of which is the significantly

TABLE 4
Comparison with the NIST candidates for PQ signature standard

| Signature <br> scheme | signature <br> size, bytes | publi-key <br> size, bytes | sign. gener. <br> rate, arb. un. | sign. verific. <br> rate, arb. un. |
| :---: | :---: | :---: | :---: | :---: |
| Falcon-512 | 657 | 897 | 50 | 25 |
| Dilithium | 2044 | 1184 | 15 | 10 |
| Rainbow | 64 | 150000 | - | - |
| qTESLA-p-I | 2592 | 15000 | 20 | 40 |
| Proposed <br> $m=6$ | 96 | 576 | 30 | 40 |
| Proposed <br> $m=4$ | 96 | 384 | 65 | 90 |
| $[10]$ | 192 | 768 | 85 | 65 |

smaller size of both the signature and the PK in comparison with the earlier proposed analog [10]. The used design method is characterized in applying both the masking operations that are mutually commutative with the exponentiation operation and the masking operations that are free of such properties. Another feature of the introduced cryptoscheme is the use of the signature verification equation with cascade exponentiation.

In comparison with the PQ signature schemes that are currently considered as candidates for PQ signature standards, the propose scheme is significantly more practical. Besides, implementation of the last one on the base of one of the 4dimensional FNAA with two-sided global units, which are described in [5, 9], will supposedly also provide 128-bit security, but will have 2.25 times higher performance rate.

## References

[1] J. Ding, D. Schmidt, Rainbow, a new multivariable polynomial signature scheme, Lecture Notes Computer Sci., 3531 (2005), $164-175$.
[2] A. Ekert, R. Jozsa, Quantum computation and Shor's factoring algorithm, Rev. Mod. Phys., 68 (1996), 733.
[3] R. Jozsa, Quantum algorithms and the fourier transform, Proc. Roy. Soc. London Ser A, 454 (1998), 323 - 337.
[4] A.S. Kuzmin, V.T. Markov, A.A. Mikhalev, A.V. Mikhalev, A.A. Nechaev, Cryptographic algorithms on groups and algebras, J. Math. Sci., 223 (2017), no. 5, $629-641$.
[5] A.A. Moldovyan, N.A. Moldovyan, Post-quantum signature algorithms based on the hidden discrete logarithm problem, Computer Sci. J. Moldova, 26 (2018), no. 3(78), $301-313$.
[6] A.A. Moldovyan, N.A. Moldovyan, Finite non-commutative associative algebras as carriers of hidden discrete logarithm problem, Bull. South Ural State Univ., Ser. Mathematical Modelling, Programming \& Computer Software, 12 (2019), 66-81.
[7] A.A. Moldovyan, N.A. Moldovyan, New forms of defining the hidden discrete logarithm problem, SPIIRAS Proceedings, 18 (2019), no 2, 504-529.
[8] D.N. Moldovyan, Non-commutative finite groups as primitive of public-key cryptoschemes, Quasigroups and Related Systems, 18 (2010), $165-176$.
[9] D.N. Moldovyan, A unified method for setting finite none-commutative associative algebras and their properties, Quasigroups and Related Systems, 27 (2019), 293 308.
[10] D.N. Moldovyan, A.A. Moldovyan, N.A. Moldovyan, An enhanced version of the hidden discrete logarithm problem and its algebraic support, Quasigroups and Related Systems, 28 (2020), $269-284$.
[11] N.A. Moldovyan, Fast signatures based on non-cyclic finite groups, Quasigroups and Related Systems, 18 (2010), $83-94$.
[12] N.A. Moldovyan, Unified method for defining finite associative algebras of arbitrary even dimensions, Quasigroups and Related Systems, 26 (2018), 263 - 270.
[13] N.A. Moldovyan, Finite non-commutative associative algebras for setting the hidden discrete logarithm problem and post-quantum cryptoschemes on its base, Bul. Acad.e Stiinte Republ. Moldova. Matematica, 1(89) (2019), 71 - 78.
[14] N.A. Moldovyan, P.A. Moldovyanu, New primitives for digital signature algorithms, Quasigroups and Related Systems, 17 (2009), 271 - 282.
[15] Post-Quantum Cryptography, Lecture Notes Computer Sci., 10786, (2018).
[16] Post-Quantum Cryptography, Lecture Notes Computer Sci., 11505 (2019).
[17] C.P. Schnorr Efficient signature generation by smart cards, Cryptology, 4 (1991), 161-174.
[18] P.W. Shor, Polynomial-time algorithms for prime factorization and discrete logarithms on quantum computer, SIAM J. Computing, 26 (1997), 1484-1509.

Received March 25, 2020
St. Petersburg Federal Research Center of the Russian Academy of Sciences (SPC RAS),
St. Petersburg Institute for Informatics and Automation of Russian Academy of Sciences, 14-th line 39, 199178, St. Petersburg, Russia
E-mails: mdn.spectr@mail.ru, nmold@mail.ru

# Characterizations of ordered $k$-regularities on ordered semirings 

Pakorn Palakawong na Ayutthaya and Bundit Pibaljommee


#### Abstract

We investigate the connections among some types of ordered $k$-regularities of ordered semirings and give some of their characterizations using their ordered $k$-ideals, prime ordered $k$ ideals, semiprime ordered $k$-ideals and pure ordered $k$-ideals.


## 1. Introduction

Regularities are important and interesting properties to research on algebraic structures, especially, semigroups and semirings. Some notable types of regularities defined by Kehayopulu [7,8] and Kehayopulu and Tsingelis [9] on semigroups and ordered semigroups are the bases of many works about regularities on semirings and ordered semirings. A semiring, a well-known generalization of a ring, is an algebraic system $(S,+, \cdot)$ such that $(S,+)$ and $(S, \cdot)$ are semigroups connected by a distributive law. Originally, the regular property of a semiring $(S,+, \cdot)$ is defined on ( $S, \cdot \cdot$ ) as a similar way of a regular ring defined by von Neumann [11]. He called a semiring $(S,+, \cdot)$ to be regular if the semigroup $(S, \cdot)$ is regular, i.e., for each $a \in S, a=a x a$ for some $x \in S$. However, in the sense of Bourne [3], a semiring $(S,+, \cdot)$ is regular if for each $a \in S, a+a x a=a y a$ for some $x, y \in S$. Later, Adhikari, Sen and Weinert [1] renamed Bourne regular semirings to be $k$ regular semirings. It is easy to obtain that a $k$-regular semiring is a generalization of a regular semiring. In 1958, Henriksen [5] defined a more restricted class of ideals in a semiring, which he called $k$-ideals, a considerably useful kind of ideals to characterize $k$-regular semirings. Afterwards, Bhuniya and Jana $[2,6]$ defined the notions of quasi- $k$-ideals and $k$-bi-ideals of semirings and use them to characterize $k$-regular and intra $k$-regular semirings.

A notable generalization of semirings is an ordered semiring. In the sense of Gan and Jiang [4], an ordered semiring $(S,+, \cdot, \leqslant)$ is a semiring $(S,+, \cdot)$ together with a partially ordered relation $\leqslant$ on $S$ satisfying the compatibility property. In 2014, Mandal [10] defined an ordered semiring $(S,+, \cdot, \leqslant)$ to be regular and $k$ regular if for each $a \in S, a \leqslant a x a$ and $a+a x a \leqslant a y a$ for some $x, y \in S$, respectively. In 2016, we gave some characterizations of regular, left regular, right regular, and intra-regular ordered semirings using many kinds of their ordered ideals in [12].

[^3]Later, Patchakheio and Pibaljommee [16] defined an ordered semiring ( $S,+, \cdot, \leqslant$ ) to be ordered $k$-regular if $a \in \overline{(a S a]}$ for all $a \in S$. This notion is a generalization of $k$-regular ordered semirings defined by Mandal. Moreover, in [16] they gave the notions of left ordered $k$-regular, right ordered $k$-regular, left weakly ordered $k$ regular and right weakly ordered $k$-regular semirings and characterize them using their ordered $k$-ideals. In 2017, Senarat and Pibaljommee [18] used prime and irreducible ordered $k$-bi-ideals to characterize left and right weakly ordered $k$ regular semirings.

In our previous works [13-15, 17], we characterized ordered $k$-regular, left ordered $k$-regular, right ordered $k$-regular, ordered intra $k$-regular, completely ordered $k$-regular, left weakly ordered $k$-regular, right weakly ordered $k$-regular and fully ordered $k$-idempotent semirings in terms of many kinds of their ordered $k$-ideals. In this work, we recollect all types of mentioned kinds of ordered $k$-regularities, investigate connections among them and left generalized ordered $k$-regular, right generalized ordered $k$-regular and generalized ordered $k$-regular semirings and give some more their characterizations. Furthermore, we use the concepts of prime ordered $k$-ideals, semiprime ordered $k$-ideals and pure ordered $k$-ideals of ordered semirings to characterize some kinds of ordered $k$-regularities.

## 2. Preliminaries

An ordered semiring [4] is a system $(S,+, \cdot, \leqslant)$ consisting of the semiring $(S,+, \cdot)$ and the partially ordered set $(S, \leqslant)$ connected by the compatibility property. If $(S,+)$ is commutative, $(S,+, \cdot, \leqslant)$ is called additively commutative [1]. Throughout this work, we simple write $S$ instead of an ordered semiring ( $S,+, \cdot, \leqslant$ ) and always assume that it is additively commutative.

For any $\emptyset \neq A, B \subseteq S$, we denote $A+B=\{a+b \in S \mid a \in A, b \in B\}$, $A B=\{a b \in S \mid a \in A, b \in B\},(A]=\{x \in S \mid x \leqslant a$ for some $a \in A\}$ and

$$
\Sigma A=\left\{\sum_{i \in I} a_{i} \mid a_{i} \in A \text { and } I \text { is a finite nonempty set }\right\}
$$

The $k$-closure [16] of $\emptyset \neq A \subseteq S$ is denoted by $\bar{A}=\{x \in S \mid x+a \leqslant b$ for some $a, b \in A\}$. By the elementary properties of the finite sums $\Sigma$, the operator ( ] and the $k$-closure of a nonempty subset of an ordered semiring, we refer to [13-16]. Nevertheless, we give the following lemma to be useful accessories for reaching the main results.

Lemma 2.1. Let $A$ and $B$ be nonempty subsets of an ordered semiring $S$. The following statements hold:
(i) $\Sigma \overline{(A]} \subseteq \overline{(\Sigma A]}$;
(ii) $\overline{(A]}=\overline{(\overline{(A)}]}$;
(iii) $\overline{A(B]} \subseteq \overline{(A]} \overline{(B]} \subseteq \overline{(\Sigma A B]}$ and $\overline{(A]} B \subseteq \overline{(A]} \overline{(B]} \subseteq \overline{(\Sigma A B]}$;
(iv) $A+\overline{(B]} \subseteq \overline{(A]}+\overline{(B]} \subseteq \overline{(A+B]}$;
(v) $\overline{(A \overline{(B]})} \subseteq \overline{(\overline{(A)} \overline{(B]})} \subseteq \overline{(\Sigma A B]}$ and $\overline{(\overline{(A]} B]} \subseteq \overline{(\overline{(A]} \overline{(B)]}} \subseteq \overline{(\Sigma A B]}$;
(vi) $\overline{(A+\overline{(B]}]} \subseteq \overline{(\overline{(A]}+\overline{(B]}]} \subseteq \overline{(A+B]}$.

A nonempty subset $A$ of an ordered semiring $S$ such that $A+A \subseteq A$ is called a left (resp. right) ordered $k$-ideal of $S$ if $S A \subseteq A$ (resp. $A S \subseteq A$ ) and $A=\bar{A}$. If $A$ is both a left and a right ordered $k$-ideal of $S$, then $A$ is called an ordered $k$-ideal [16] of $S$. A nonempty subset $Q$ of $S$ is called an ordered quasi-k-ideal [13] of $S$ if $\overline{(\Sigma Q S]} \cap \overline{(\Sigma S Q]} \subseteq Q$ and $Q=\bar{Q}$. A nonempty subset $B$ of $S$ such that $B+B \subseteq B, B^{2} \subseteq B$ and $B=\bar{B}$ is said to be an ordered $k$-bi-ideal [18] (resp. ordered $k$-interior ideal) [14] of $S$ if $B S B \subseteq B$ (resp. $S B S \subseteq B$ ).

For $a \in S$, by the notations $L(a), R(a), J(a), Q(a), B(a)$ and $I(a)$, we mean the intersection of all left ordered $k$-ideals, right ordered $k$-ideals, ordered $k$-ideals, ordered quasi- $k$-ideals, ordered $k$-bi-ideals and ordered $k$-interior ideals of $S$ containing $a$, respectively. Now, we recollect their constructions which occur in [13-16] as follows.

Lemma 2.2. For $\emptyset \neq A \subseteq S$, the following statements hold:
(i) $L(a)=\overline{(\Sigma a+S a]}$;
(iv) $Q(a)=\overline{(\Sigma a+(\overline{(a S]} \cap \overline{(S a]})}]$;
(ii) $R(a)=\overline{(\Sigma a+a S]}$;
(v) $B(a)=\overline{\left(\Sigma a+\Sigma a^{2}+a S a\right]}$;
(iii) $J(a)=\overline{(\Sigma a+S a+a S+\Sigma S a S]}$;
(vi) $I(a)=\overline{\left(\Sigma a+\Sigma a^{2}+\Sigma S a S\right]}$.

We define the relations $\mathcal{L}$ and $\mathcal{R}$ on an ordered semiring $S$ by

$$
\mathcal{L}:=\{(x, y) \in S \times S \mid L(x)=L(y)\} \text { and } \mathcal{R}:=\{(x, y) \in S \times S \mid R(x)=R(y)\}
$$

## 3. Ordered $k$-regularities of Ordered Semirings

We recall the notions of some types of ordered $k$-regularities of ordered semirings as the following definition.

Definition 3.1. An ordered semiring $S$ is called:
(i) ordered $k$-regular if $a \in \overline{(a S a]}$ for all $a \in S$ (cf. [16]);
(ii) left ordered $k$-regular if $a \in \overline{\left(S a^{2}\right]}$ for all $a \in S$ (cf. [16]);
(iii) right ordered $k$-regular if $a \in \overline{\left(a^{2} S\right]}$ for all $a \in S$ (cf. [16]);
(iv) completely ordered $k$-regular if $S$ is ordered $k$-regular, left ordered $k$-regular and right ordered $k$-regular (cf. [15]);
(v) ordered intra $k$-regular if $a \in \overline{\left(\Sigma S a^{2} S\right]}$ for all $a \in S$ (cf. [14]);
(vi) left weakly ordered $k$-regular if $a \in \overline{(\Sigma S a S a]}$ for all $a \in S$ (cf. [16]);
(vii) right weakly ordered $k$-regular $\mathrm{f} a \in \overline{(\Sigma a S a S]}$ for all $a \in S$ (cf. [16]);
(viii) fully ordered $k$-idempotent if $I=\overline{\left(\Sigma I^{2}\right]}$ for each ordered $k$-ideal $I$ of $S$ (cf. [15]).

According to Definition 3.1 (viii), we note that an ordered semiring $S$ is fully ordered $k$-idempotent if and only if $a \in \overline{(\Sigma S a S a S]}$ for all $a \in S$ [15].

Here, we give two lemmas which will be significantly used later.
Lemma 3.2. An ordered semiring $S$ is ordered intra $k$-regular if $a \in \overline{\left(\Sigma a^{2}+\Sigma S a^{2} S\right]}$ for all $a \in S$.

Proof. Let $a \in S$. Assume that

$$
\begin{equation*}
a \in \overline{\left(\Sigma a^{2}+\Sigma S a^{2} S\right]} . \tag{1}
\end{equation*}
$$

Using (1), we get

$$
\begin{align*}
a^{2}=a a & \in \overline{\left(\Sigma a^{2}+\Sigma S a^{2} S\right]} \overline{\left(\Sigma a^{2}+\Sigma S a^{2} S\right]} \subseteq \overline{\left(\Sigma\left(\Sigma a^{2}+\Sigma S a^{2} S\right)\left(\Sigma a^{2}+\Sigma S a^{2} S\right)\right]} \\
& \subseteq \overline{\left(\Sigma\left(\Sigma a^{4}+\Sigma S a^{2} S\right]\right]} \subseteq \overline{\left(\Sigma\left(\Sigma S a^{2} S\right]\right]}=\overline{\left(\Sigma S a^{2} S\right]} . \tag{2}
\end{align*}
$$

Using (1) and (2), we obtain

$$
\begin{aligned}
a & \in \overline{\left(\Sigma a^{2}+\Sigma S a^{2} S\right]} \subseteq \overline{\left(\Sigma \overline{\left(\Sigma S a^{2} S\right]}+\Sigma S a^{2} S\right]} \subseteq \overline{\left(\overline{\left(\Sigma S a^{2} S\right]}+\overline{\left(\Sigma S a^{2} S\right]}\right)} \\
& \subseteq \overline{\left(\overline{\left(\Sigma S a^{2} S+\Sigma S a^{2} S\right]}\right]}=\overline{\left(\Sigma S a^{2} S\right]}
\end{aligned}
$$

Therefore, $S$ is ordered intra $k$-regular.
Lemma 3.3. If an ordered semiring $S$ is ordered intra $k$-regular, then $J(a)=$ $\overline{(\Sigma S a S]}$ for all $a \in S$.

Proof. Let $a \in S$. Assume that $S$ is ordered intra $k$-regular. Then

$$
\begin{aligned}
J(a) & =\overline{(\Sigma a+S a+a S+\Sigma S a S]} \\
& \subseteq \overline{\left(\Sigma \overline{\left(\Sigma S a^{2} S\right]}+S \overline{\left(\Sigma S a^{2} S\right]}+\overline{\left(\Sigma S a^{2} S\right]} S+\Sigma S \overline{\left(\Sigma S a^{2} S\right]} S\right]} \\
& \left.\subseteq \overline{\left(\overline{\left(\Sigma S a^{2} S\right]}+\overline{\left(\Sigma S a^{2} S\right]}+\overline{\left(\Sigma S a^{2} S\right]}+\overline{\left(\Sigma S a^{2} S\right]}\right.}\right) \\
& \subseteq \overline{\left(\overline{\left(\Sigma S a^{2} S+\Sigma S a^{2} S+\Sigma S a^{2} S+\Sigma S a^{2} S\right]}\right)} \\
& =\overline{\left(\overline{\left(\Sigma S a^{2} S\right]}\right)}=\overline{\left(\Sigma S a^{2} S\right]} \subseteq \overline{(\Sigma S a S]} .
\end{aligned}
$$

On the other hand, we show that $\overline{(\Sigma S a S]} \subseteq J(a)$. Let $s \in \Sigma S a S$ and $t \in \Sigma a+$ $S a+a S$. Then $s+(t+s) \leqslant t+s+s$ such that $t+s, t+s+s \in \Sigma a+S a+a S+\Sigma S a S$ and so $s \in \overline{\Sigma a+S a+a S+\Sigma S a S} \subseteq \overline{(\Sigma a+S a+a S+\Sigma S a S]}=J(a)$. This means


Theorem 3.4. [16] An ordered semiring $S$ is ordered $k$-regular if and only if $R \cap L=\overline{(R L]}$ for every right ordered $k$-ideal $R$ and left ordered $k$-ideal $L$ of $S$.

Corollary 3.5. [13] An ordered semiring $S$ is ordered $k$-regular if and only if $a \in \overline{(R(a) L(a)]}$ for all $a \in S$.

Now, we give more characterizations of an ordered $k$-regular semiring in terms of many kinds of their ordered $k$-ideals.

Theorem 3.6. The following conditions are equivalent:
(i) $S$ is ordered $k$-regular;
(ii) $B \cap L \subseteq \overline{(B L]}$ for every ordered $k$-bi-ideal $B$ and left ordered $k$-ideal $L$ of $S$;
(iii) $R \cap B \subseteq \overline{(R B]}$ for every right ordered $k$-ideal $R$ and ordered $k$-bi-ideal $B$ of $S$;
(iv) $R \cap B \cap L \subseteq \overline{(R B L]}$ for every right ordered $k$-ideal $R$, ordered $k$-bi-ideal $B$ and left ordered $k$-ideal $L$ of $S$;
(v) $B \cap I=\overline{(B I B]}$ for every ordered $k$-bi-ideal $B$ and ordered $k$-interior ideal $I$ of $S$;
(vi) $B \cap J=\overline{(B J B]}$ for every ordered $k$-bi-ideal $B$ and ordered $k$-ideal $J$ of $S$;
(vii) $B \cap I \cap L \subseteq \overline{(B I L]}$ for every ordered $k$-bi-ideal $B$, ordered $k$-interior ideal $I$ and left ordered $k$-ideal $L$ of $S$;
(viii) $Q \cap I \cap L \subseteq \overline{(Q I L]}$ for every ordered quasi-k-ideal $Q$, ordered $k$-interior ideal $I$ and left ordered $k$-ideal $L$ of $S$;
(ix) $R \cap I \cap L \subseteq \overline{(R I L]}$ for every right ordered $k$-ideal $R$, ordered $k$-interior ideal $I$ and left ordered $k$-ideal $L$ of $S$;
(x) $B \cap J \cap L \subseteq \overline{(B J L]}$ for every ordered $k$-bi-ideal $B$, ordered $k$-ideal $J$ and left ordered $k$-ideal $L$ of $S$;
(xi) $Q \cap J \cap L \subseteq \overline{(Q J L]}$ for every ordered quasi-k-ideal $Q$, ordered $k$-ideal $J$ and left ordered $k$-ideal $L$ of $S$;
(xii) $R \cap J \cap L \subseteq \overline{(R J L]}$ for every right ordered $k$-ideal $R$, ordered $k$-ideal $J$ and left ordered $k$-ideal $L$ of $S$;
(xiii) $R \cap I \cap B \subseteq \overline{(R I B]}$ for every right ordered $k$-ideal $R$, ordered $k$-interior ideal $I$ and ordered $k$-bi-ideal $B$ of $S$;
(xiv) $R \cap I \cap Q \subseteq \overline{(R I Q]}$ for every right ordered $k$-ideal $R$, ordered $k$-interior ideal $I$ and ordered quasi-k-ideal $Q$ of $S$;
(xv) $R \cap J \cap B \subseteq \overline{(R J B]}$ for every right ordered $k$-ideal $R$, ordered $k$-ideal $J$ and ordered $k$-bi-ideal $B$ of $S$;
(xvi) $R \cap J \cap Q \subseteq \overline{(R J Q]}$ for every right ordered $k$-ideal $R$, ordered $k$-ideal $J$ and ordered quasi-k-ideal $Q$ of $S$.
Proof. $(i) \Rightarrow(i i)$. Let $B$ and $L$ be an ordered $k$-bi-ideal and a left ordered $k$-ideal of $S$, respectively. If $x \in B \cap L$ then by $(i), x \in \overline{(x S x]} \subseteq \overline{(B S L]} \subseteq \overline{(B L]}$.
$(i i) \Rightarrow(i)$. Let $a \in S . \operatorname{By}(i i), a \in B(a) \cap L(a) \subseteq \overline{(B(a) L(a)]}$. Since every right ordered $k$-ideal is an ordered $k$-bi-ideal [13], we get $a \in \overline{(B(a) L(a)]} \subseteq \overline{(R(a) L(a)]}$. By Corollary 3.5, $S$ is ordered $k$-regular.
$(i) \Rightarrow(i i i)$. and $(i i i) \Rightarrow(i)$ can be proved in a similar way of $(i) \Rightarrow(i i)$ and (ii) $\Rightarrow(i)$, respectively.
$(i) \Rightarrow(i v)$. Let $R, B$ and $L$ be a right ordered $k$-ideal, an ordered $k$-biideal and a left ordered $k$-ideal of $S$, respectively. If $x \in R \cap B \cap L$ then by ( $i$ ), $x \in \overline{(x S x]} \subseteq \overline{(\overline{(x S x]} S x]} \subseteq \overline{(x S x S x]} \subseteq \overline{(R S B S L]} \subseteq \overline{(R B L]}$.
$(i v) \Rightarrow(i)$. Let $a \in S$. By (iv), $a \in R(a) \cap B(a) \cap L(a) \subseteq \overline{(R(a) B(a) L(a)]} \subseteq$ $\overline{(R(a) L(a)]}$. Using Corollary 3.5, $S$ is ordered $k$-regular.
$(i) \Rightarrow(v)$. Let $B$ and $I$ be an ordered $k$-bi-ideal and an ordered $k$-interior ideal of $S$, respectively. If $x \in B \cap I$ then by $(i), x \in \overline{(x S x]} \subseteq \overline{(\overline{(x S x]} S x]} \subseteq \overline{(x S x S x]} \subseteq$ $\overline{(B S I S B]} \subseteq \overline{(B I B]}$. Clearly, $\overline{(B I B]} \subseteq B \cap I$. Hence, $B \cap I=\overline{(B I B]}$.
$(v) \Rightarrow(v i)$. It follows from the fact that every ordered $k$-ideal is an ordered $k$-interior ideal [14].
$(v i) \Rightarrow(i)$. Let $a \in S . \quad$ By $(v i), a \in B(a) \cap J(a)=\overline{(B(a) J(a) B(a)]}$. Since every one-sided ordered $k$-ideal is an ordered $k$-bi-ideal [13], $a \in \overline{(B(a) J(a) B(a)]} \subseteq$ $\overline{(R(a) J(a) L(a)]} \subseteq \overline{(R(a) L(a)]}$. Using Corollary 3.5, $S$ is ordered $k$-regular.
$(i) \Rightarrow(v i i)$. Let $B, I$ and $L$ be an ordered $k$-bi-ideal, an ordered $k$-interior ideal and a left ordered $k$-ideal of $S$, respectively. If $x \in B \cap I \cap L$ then by $(i)$, $x \in \overline{(x S x]} \subseteq \overline{(\overline{(x S x]} S x]} \subseteq \overline{(x S x S x]} \subseteq \overline{(B S I S L]} \subseteq \overline{(B I L]}$.
$(v i i) \Rightarrow(v i i i)$. It follows from the fact that every ordered quasi- $k$-ideal is an ordered $k$-bi-ideal [12].
$(v i i i) \Rightarrow(i x)$. It follows from the fact that every right ordered $k$-ideal is an ordered quasi- $k$-ideal [12].
$(i x) \Rightarrow(i)$. Let $a \in S$. By $(i x), a \in R(a) \cap I(a) \cap L(a) \subseteq \overline{(R(a) I(a) L(a)]} \subseteq$ $\overline{(R(a) L(a)]}$. Using Corollary 3.5, $S$ is ordered $k$-regular.
$(i) \Rightarrow(x) \Rightarrow(x i) \Rightarrow(x i i) \Rightarrow(i)$ can be proved in a similar way of $(i) \Rightarrow(v i i) \Rightarrow$ $(v i i i) \Rightarrow(i x) \Rightarrow(i)$.
$(i) \Rightarrow(x i i i)$. Let $R, I$ and $B$ be a right ordered ideal, an ordered $k$-interior ideal and an ordered $k$-bi-ideal of $S$, respectively. If $x \in R \cap I \cap B$ then by $(i)$, $x \in \overline{(x S x]} \subseteq \overline{(\overline{(x S x]} S x]} \subseteq \overline{(x S x S x]} \subseteq \overline{(R S I S B]} \subseteq \overline{(R I B]}$.
(xiii) $\Rightarrow$ (xiv). It follows from the fact that every ordered quasi- $k$-ideal is an ordered $k$-bi-ideal [13].
$(x i v) \Rightarrow(i)$. Let $a \in S . \operatorname{By}(x i v), a \in R(a) \cap I(a) \cap Q(a) \subseteq \overline{(R(a) I(a) Q(a)]} \subseteq$ $\overline{(R(a) Q(a)]}$. Using the fact that every left ordered $k$-ideal is an ordered quasi- $k$ ideal $[13], a \in \overline{(R(a) Q(a)]} \subseteq \overline{(R(a) L(a)]}$. By Corollary 3.5, $S$ is ordered $k$-regular.
$(i) \Rightarrow(x v) \Rightarrow(x v i) \Rightarrow(i)$ can be proved in a similar way of $(i) \Rightarrow(x i i i) \Rightarrow$ (xiv) $\Rightarrow(i)$.

Definition 3.7. Let $a$ be an element of an ordered semiring $S$. Then $a$ is called: left generalized ordered $k$-regular (resp. right generalized ordered $k$-regular, generalized ordered $k$-regular) if $a \in \overline{(S a]}$ (resp. $a \in \overline{(a S]}, a \in \overline{(\Sigma S a S]})$.

If $a$ is left generalized ordered $k$-regular (resp. right generalized ordered $k$ regular, generalized ordered $k$-regular) for all $a \in S$, then $S$ is called left generalized ordered $k$-regular (resp. right generalized ordered $k$-regular, generalized ordered $k$ regular).
Remark 3.8. Let $a$ and $b$ be elements of an ordered semiring $S$. If $a$ is left (resp. right) generalized ordered $k$-regular and $a \mathcal{L} b(a \mathcal{R} b)$, then $b$ is also left (resp. right) generalized ordered $k$-regular.
Proof. Let $a, b \in S$. If $a$ is left generalized ordered $k$-regular and $a \mathcal{L} b$, then

$$
\begin{aligned}
b \in L(a) & =\overline{(\Sigma a+S a]} \subseteq \overline{(\Sigma \overline{(S a]}+S a]} \subseteq \overline{(S a]} \subseteq \overline{(S L(b)]} \\
& \subseteq \overline{(S \overline{(\Sigma b+S b]}]} \subseteq \overline{(\Sigma S b+S b]}=\overline{(S b+S b]}=\overline{(S b]}
\end{aligned}
$$

Hence, $b$ is also left generalized ordered $k$-regular.
Remark 3.9. Let $a$ and $b$ be elements of an ordered semiring $S$ such that $a$ is generalized ordered $k$-regular. If $a \mathcal{L} b$ or $a \mathcal{R} b$, then $b$ is also generalized ordered $k$-regular.

Proof. Let $a, b \in S$. Assume that $a$ is generalized ordered $k$-regular and $a \mathcal{L} b$. Then

$$
\begin{aligned}
b \in L(a) & =\overline{(\Sigma a+S a]} \subseteq \overline{(\Sigma \overline{(\Sigma S a S]}+S \overline{(\Sigma S a S]}]} \subseteq \overline{(\overline{(\Sigma S a S]}+\overline{(\Sigma S a S]}]} \subseteq \overline{(\Sigma S a S]} \\
& \subseteq \overline{(\Sigma S L(b) S]} \subseteq \overline{(\Sigma S \overline{(\Sigma b+S b]} S]} \subseteq \overline{(\Sigma S b S+\Sigma S b S]}=\overline{(\Sigma S b S]}
\end{aligned}
$$

Hence, $b$ is generalized ordered $k$-regular. The case of $a \mathcal{R} b$ can be proved similarly.

Connections among eleven types of ordered $k$-regularities can be summarized by the following diagram. Each arrow represents the implication between two regularities and its converse is not generally true.


Example 3.10. Let $S=\{a, b, c, d\}$. Define two binary operations + and $\cdot$ on $S$ by the following tables:

| + | $a$ | $b$ | $c$ | $d$ |
| :---: | :---: | :---: | :---: | :---: |
| $a$ | $a$ | $b$ | $c$ | $d$ |
| $b$ | $b$ | $b$ | $c$ | $d$ |
| $c$ | $c$ | $c$ | $c$ | $d$ |
| $d$ | $d$ | $d$ | $d$ | $d$ |


| $\cdot$ | $a$ | $b$ | $c$ | $d$ |
| :---: | :---: | :---: | :---: | :---: |
| $a$ | $a$ | $a$ | $c$ | $d$ |
| $b$ | $a$ | $a$ | $c$ | $d$ |
| $c$ | $a$ | $a$ | $c$ | $d$ |
| $d$ | $a$ | $a$ | $c$ | $d$ |

Define a binary relation $\leqslant$ on $S$ by $\leqslant:=\{(a, a),(b, b),(c, c),(d, d),(a, d),(b, d),(c, d)\}$.
Then $(S,+, \cdot, \leqslant)$ is an ordered semiring.
Since $x \in \overline{\left(\Sigma S x^{2} S\right]}=S$ for all $x \in S$, we have that $S$ is ordered intra $k$-regular and hence $S$ is fully ordered $k$-idempotent and generalized ordered $k$-regular.

Since $x \in \overline{\left(\Sigma x^{2} S\right]}=S$ for all $x \in S$, we have that $S$ is right ordered $k$-regular and hence $S$ is right weakly ordered $k$-regular and right generalized ordered $k$ regular.

However, $b \notin \overline{(S b]}=\{a\}$ and so $S$ is not left generalized ordered $k$-regular. Consequently, $S$ is not left weakly ordered $k$-regular and also neither left ordered $k$-regular nor ordered $k$-regular.

Example 3.11. Consider the set $S=\{a, b, c, d\}$ together with the operation + and the relation $\leqslant$ of Example 3.10. Define a binary operation $\cdot$ on $S$ by the following table:

| $\cdot$ | $a$ | $b$ | $c$ | $d$ |
| :---: | :---: | :---: | :---: | :---: |
| $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $a$ | $a$ | $a$ |
| $c$ | $c$ | $c$ | $c$ | $c$ |
| $d$ | $d$ | $d$ | $d$ | $d$ |

Then $(S,+, \cdot, \leqslant)$ is an ordered semiring.
Since $x \in \overline{\left(\Sigma S x^{2} S\right]}=S$ for all $x \in S$, we have that $S$ is ordered intra $k$-regular and hence $S$ is fully ordered $k$-idempotent and generalized ordered $k$-regular.

Since $x \in \overline{\left(\Sigma S x^{2}\right]}=S$ for all $x \in S$, we have that $S$ is left ordered $k$-regular and hence $S$ is left weakly ordered $k$-regular and left generalized ordered $k$-regular.

However, $b \notin \overline{(b S]}=\{a\}$ and so $S$ is not right generalized ordered $k$-regular. Consequently, $S$ is not right weakly ordered $k$-regular and also neither right ordered $k$-regular nor ordered $k$-regular.

Example 3.12. Let $S=\{a, b, c\}$. Define two binary operations + and $\cdot$ on $S$ by the following tables:

| + | $a$ | $b$ | $c$ |
| :---: | :---: | :---: | :---: |
| $a$ | $a$ | $b$ | $c$ |
| $b$ | $b$ | $b$ | $c$ |
| $c$ | $c$ | $c$ | $c$ |

and

| $\cdot$ | $a$ | $b$ | $c$ |
| :---: | :---: | :---: | :---: |
| $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $a$ | $a$ |
| $c$ | $a$ | $b$ | $c$ |

Define a binary relation $\leqslant$ on $S$ by $\leqslant:=\{(a, a),(b, b),(c, c),(a, b),(a, c),(b, c)\}$. Then $(S,+, \cdot, \leqslant)$ is an ordered semiring. Since $a \in \overline{(S a]}=\{a\}, b \in \overline{(S b]}=\{a, b\}$ and $c \in \overline{(S c]}=S$, we get that $S$ is left generalized ordered $k$-regular. However, $b \notin \overline{(\Sigma S b S]}=\{a\}$ and so $S$ is not generalized ordered $k$-regular. Consequently, $S$ is not fully ordered $k$-idempotent and also not left weakly ordered $k$-regular.

Example 3.13. Consider the set $S=\{a, b, c\}$ together with the operation + and the relation $\leqslant$ of Example 3.12. Define a binary operation • on $S$ by the following table;

| $\cdot$ | $a$ | $b$ | $c$ |
| :---: | :---: | :---: | :---: |
| $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $a$ | $b$ |
| $c$ | $a$ | $a$ | $c$ |

Then $(S,+, \cdot, \leqslant)$ is an ordered semiring. Since $a \in \overline{(a S]}=\{a\}, b \in \overline{(b S]}=\{a, b\}$ and $c \in \overline{(c S]}=S$, we get that $S$ is right generalized ordered $k$-regular. However, $b \notin \overline{(\Sigma S b S]}=\{a\}$ and so $S$ is not generalized ordered $k$-regular. Consequently, $S$ is not fully ordered $k$-idempotent and also not left weakly ordered $k$-regular.

Example 3.14. Consider the set $S=\{a, b, c\}$ together with the operation + and the relation $\leqslant$ of Example 3.12. Define a binary operation $\cdot$ on $S$ by the following table;

| $\cdot$ | $a$ | $b$ | $c$ |
| :---: | :---: | :---: | :---: |
| $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $a$ | $b$ |
| $c$ | $a$ | $b$ | $c$ |

Then $(S,+, \cdot, \leqslant)$ is an ordered semiring. Since $a \in \overline{(\Sigma S a S]}=\{a\}, b \in \overline{(\Sigma S b S]}=$ $\{a, b\}$ and $c \in \overline{(\Sigma S c S]}=S$, we get that $S$ is generalized ordered $k$-regular. However, $S$ is not fully ordered $k$-idempotent because $b \notin \overline{(\Sigma S b S b S]}=\{a\}$.

## 4. Prime and Semiprime Ordered $k$-ideals

Now, we use the concepts of prime and semiprime ordered $k$-ideals to characterize several kinds of ordered $k$-regularities on ordered semirings.

Definition 4.1. A nonempty subset $T$ of an ordered semiring $S$ is said to be prime if for any $a, b \in S, a b \in T$ implies $a \in T$ or $b \in T$.

Definition 4.2. A nonempty subset $T$ of an ordered semiring $S$ is said to be semiprime if for any $a \in S, a^{2} \in T$ implies $a \in T$.

It is clear that every prime subset of an ordered semiring is semiprime but not conversely.

Example 4.3. Consider the ordered semiring $(\mathbb{N},+, \cdot, \leqslant)$ such that $\mathbb{N}$ is the set of all natural numbers, + is the usual addition, $\cdot$ is the usual multiplication and $\leqslant$ is the natural order. We easily get that $2 \mathbb{N}$ is a prime subset and $6 \mathbb{N}$ is a semiprime subset of $(\mathbb{N},+, \cdot, \leqslant)$. However, $6 \mathbb{N}$ is not prime because $2 \cdot 3 \in 6 \mathbb{N}$ but $2,3 \notin 6 \mathbb{N}$.

Theorem 4.4. An ordered semiring $S$ is left (right) ordered $k$-regular if and only if every left (right) ordered $k$-ideal of $S$ is semiprime.
Proof. Assume that $S$ is left ordered $k$-regular. Let $L$ be a left ordered $k$-ideal of $S$ and $x \in S$. If $x^{2} \in L$ then by assumption, $x \in \overline{\left(S x^{2}\right]} \subseteq \overline{(S L]} \subseteq \overline{(L]}=L$. Hence, $L$ is semiprime.

Conversely, assume that every left ordered $k$-ideal of $S$ is semiprime. Let $a \in S$. Since $a^{2}$ belongs to $L\left(a^{2}\right)$ a semiprime left ordered $k$-ideal, we get

$$
\begin{equation*}
a \in L\left(a^{2}\right)=\overline{\left(\Sigma a^{2}+S a^{2}\right]} \tag{3}
\end{equation*}
$$

Using (3), we obtain

$$
\begin{equation*}
a^{2}=a a \in a \overline{\left(\Sigma a^{2}+S a^{2}\right]} \subseteq \overline{\left(\Sigma a^{3}+S a^{2}\right)} \subseteq \overline{\left(S a^{2}\right)} \tag{4}
\end{equation*}
$$

Using (3) and (4), we obtain

$$
a \in \overline{\left(\Sigma a^{2}+S a^{2}\right]} \subseteq \overline{\left(\Sigma \overline{\left(S a^{2}\right]}+S a^{2}\right]} \subseteq \overline{\left(S a^{2}\right]}
$$

Therefore, $S$ is left ordered $k$-regular.
Theorem 4.5. [15] An ordered semiring $S$ is completely ordered $k$-regular if and only if every ordered $k$-bi-ideal of $S$ is semiprime.

Theorem 4.6. [15] An ordered semiring $S$ is both left and right ordered $k$-regular if and only if every ordered quasi-k-ideal of $S$ is semiprime.

Theorem 4.7. An ordered semiring $S$ is ordered intra $k$-regular if and only if every ordered $k$-interior ideal of $S$ is semiprime.

Proof. Assume that $S$ is ordered intra $k$-regular. Let $I$ be an ordered $k$-interior ideal of $S$ and $x \in S$. If $x^{2} \in I$ then by assumption, $x \in \overline{\left(\Sigma S x^{2} S\right]} \subseteq \overline{(\Sigma S I S]} \subseteq$ $\overline{(\Sigma I]}=I$. Hence, $I$ is semiprime.

Conversely, assume that every ordered $k$-interior ideal $I$ of $S$ is semiprime. Let $a \in S$. Since $a^{2}$ belongs to $I\left(a^{2}\right)$ a semiprime ordered $k$-interior ideal, we get $a \in I\left(a^{2}\right)=\overline{\left(\Sigma a^{2}+\Sigma a^{4}+\Sigma S a^{2} S\right]} \subseteq \overline{\left(\Sigma a^{2}+\Sigma S a^{2} S\right]}$. By Lemma 3.2, $S$ is ordered intra $k$-regular.

We note that every ordered $k$-ideal of an ordered semiring is an ordered $k$ interior ideal $[13,14]$ and they coincide in ordered intra $k$-regular semirings [14]. As a consequence of Theorem 4.7 and using the above fact, we obtain the following corollary.

Corollary 4.8. An ordered semiring $S$ is ordered intra $k$-regular if and only if every ordered $k$-ideal of $S$ is semiprime.

Theorem 4.9. An ordered semiring $S$ is ordered intra $k$-regular and the set of all ordered $k$-ideals of $S$ forms a chain if and only if every ordered $k$-ideal of $S$ is prime.

Proof. Let $T$ be an ordered $k$-ideal of $S$ and let $a, b \in S$ be such that $a b \in T$. Using Lemma 3.3, we have $J(a)=\overline{(\Sigma S a S]}, J(b)=\overline{(\Sigma S b S]}$ and $J(a b)=\overline{(\Sigma S a b S]}$. We show that $J(a) \cap J(b) \subseteq J(a b)$. Let $z \in J(a) \cap J(b)$. Then

$$
\begin{equation*}
z^{2} \in J(b) J(a)=\overline{(\Sigma S b S]} \overline{(\Sigma S a S]} \subseteq \overline{(\Sigma S b S a S]} \tag{5}
\end{equation*}
$$

If $w \in b S a$, then $w^{2} \in b S a b S a \subseteq S a b S \subseteq \overline{(\Sigma S a b S]}=J(a b)$. By assumption and Theorem 4.7, $J(a b)$ is semiprime and so $w \in J(a b)$. Thus, $b S a \subseteq J(a b)$. By (5), it turns out that $z^{2} \in \overline{(\Sigma S(b S a) S]} \subseteq \overline{(\Sigma S J(a b) S]} \subseteq \overline{(\Sigma J(a b)]}=J(a b)$. Since $J(a b)$ is semiprime, $z \in J(a b)$. Hence, $J(a) \cap J(b) \subseteq J(a b)$. Since the set of all ordered $k$-ideals of $S$ is a chain, $J(a) \subseteq J(b)$ or $J(b) \subseteq J(a)$. If $J(a) \subseteq J(b)$, then $a \in J(a)=J(a) \cap J(b) \subseteq J(a b)=\overline{(\Sigma S a b S]} \subseteq \overline{(\Sigma S T S]} \subseteq T$. If $J(b) \subseteq J(a)$, then $b \in J(b)=J(a) \cap J(b) \subseteq J(a b)=\overline{(\Sigma S a b S]} \subseteq \overline{(\Sigma S T S]} \subseteq T$. Therefore, $T$ is prime.

Conversely, assume that every ordered $k$-ideal of $S$ is prime. Let $A$ and $B$ be ordered $k$-ideals of $S$. We want to show that $A \subseteq \overline{(\Sigma A B]}$ or $B \subseteq \overline{(\Sigma A B]}$. Suppose that $B \nsubseteq \overline{(\Sigma A B]}$. There exists $b \in B$ such that $b \notin \overline{(\Sigma A B]}$. Then for any $a \in A$, we have that $a b \in A B \subseteq \overline{(\Sigma A B]}$. Since $\overline{(\Sigma A B]}$ is prime, $a \in \overline{(\Sigma A B]}$ and so $A \subseteq \overline{(\Sigma A B]}$. Hence, $A \subseteq \overline{(\Sigma A B]} \subseteq \overline{(\Sigma B]}=B$ or $B \subseteq \overline{(\Sigma A B]} \subseteq \overline{(\Sigma A]}=A$. It follows that the set of all ordered $k$-ideals of $S$ forms a chain. By assumption, every ordered $k$-ideal of $S$ is also semiprime. Hence, by Theorem 4.7, $S$ is ordered intra $k$-regular.

Using the fact that every ordered $k$-ideal is an ordered $k$-interior ideal, together with Theorem 4.9, we directly obtain the following corollary.

Corollary 4.10. An ordered semiring $S$ is ordered intra $k$-regular and the set of all ordered $k$-ideals of $S$ forms a chain if and only if every ordered $k$-interior ideal of $S$ is prime.

## 5. Pure Ordered $k$-ideals

In this section, we present the notions of left pure, right pure, quasi-pure, bi-pure, left weakly pure and right weakly pure ordered $k$-ideals of ordered semirings and use them to characterize ordered $k$-regular, left weakly ordered $k$-regular, right weakly ordered $k$-regular and fully ordered $k$-idempotent semirings.

Definition 5.1. An ordered $k$-ideal $A$ of an ordered semiring $S$ is called left pure (resp. right pure) if $x \in \overline{(A x]}$ (resp. $x \in \overline{(x A]})$ for all $x \in A$.

Theorem 5.2. Let $A$ be an ordered $k$-ideal of an ordered semiring $S$. Then $A$ is left pure (resp. right pure) if and only if $A \cap L=\overline{(A L]}$ for every left ordered $k$-ideal $L$ (resp. $R \cap A=\overline{(R A]}$ for every right ordered $k$-ideal $R$ ) of $S$.

Proof. (i) Assume that $A$ is left pure. Let $L$ be a left ordered $k$-ideal of $S$. If $x \in A \cap L$, then $x \in \overline{(A x]} \subseteq \overline{(A L]}$. Therefore, $A \cap L \subseteq \overline{(A L]}$. Clearly, $\overline{(A L]} \subseteq A \cap L$. Hence, $A \cap L=\overline{(A L]}$.

Conversely, let $x \in A$. Using assumption and Lemmas 2.1 and 2.2, we get

$$
\begin{aligned}
x \in A \cap L(x) & =\overline{(A L(x)]}=\overline{(A \overline{(\Sigma x+S x]}]} \subseteq \overline{(\overline{(\Sigma A x+A S x]}]} \\
& \subseteq \overline{(A x+A x]} \subseteq \overline{(A x]}
\end{aligned}
$$

Hence, $A$ is a left pure ordered $k$-ideal of $S$.
(ii) It can be proved similarly.

Definition 5.3. An ordered $k$-ideal $A$ of an ordered semiring $S$ is called quasi-pure if $x \in \overline{(x A]} \cap \overline{(A x]}$ for all $x \in A$.

It is clear that every quasi-pure ordered $k$-ideal of an ordered semiring is both left pure and right pure.

Theorem 5.4. An ordered $k$-ideal $A$ of an ordered semiring $S$ is quasi-pure if and only if $A \cap Q=\overline{(Q A]} \cap \overline{(A Q]}$ for every ordered quasi-k-ideal $Q$ of $S$.

Proof. Assume that $A$ is quasi-pure. Let $Q$ be an ordered quasi- $k$-ideal of $S$. If $x \in A \cap Q$, then $x \in \overline{(x A]} \cap \overline{(A x]} \subseteq \overline{(Q A]} \cap \overline{(A Q]}$. Thus, $A \cap Q \subseteq \overline{(Q A]} \cap \overline{(A Q]}$. Clearly, $\overline{(Q A]} \cap \overline{(A Q]} \subseteq A \cap Q$. Hence, $A \cap Q=\overline{(Q A]} \cap \overline{(A Q]}$.

Conversely, let $x \in A$. Using assumption and Lemmas 2.1 and 2.2, we get

$$
\begin{aligned}
x \in A \cap Q(x) & =\overline{(Q(x) A]} \cap \overline{(A Q(x)]} \\
& =\overline{\overline{((\Sigma x+(\overline{(x S]} \cap \overline{(S x]})} A]} \cap \overline{\overline{(A(\Sigma x+(\overline{(x S]} \cap \overline{(S x]})}]}] \\
& \subseteq \overline{(\overline{(\Sigma x+\overline{(x S]}]} A]} \cap \overline{(A(\overline{(\Sigma x+\overline{(S x]})}]} \\
& \subseteq \overline{(\overline{(\overline{(\Sigma x+x S]} A]} \cap \overline{(A \overline{(\Sigma x+S x)})}} \\
& \subseteq \overline{(\Sigma x A+x S A]} \cap \overline{(\Sigma A x+A S x]} \\
& \subseteq \overline{(x A+x A]} \cap \overline{(A x+A x]} \subseteq \overline{(x A]} \cap \overline{(A x]} .
\end{aligned}
$$

Hence, $A$ is a quasi-pure ordered $k$-ideal of $S$.
Definition 5.5. An ordered $k$-ideal $A$ of an ordered semiring $S$ is called bi-pure if $x \in \overline{(x A x]}$ for all $x \in A$.

It is easy to obtain that every bi-pure ordered $k$-ideal of an ordered semiring is quasi-pure.

Theorem 5.6. An ordered $k$-ideal $A$ of an ordered semiring $S$ is bi-pure if and only if $A \cap B=\overline{(B A B]}$ for every ordered $k$-bi-ideal $B$ of $S$.

Proof. Assume that $A$ is bi-pure. Let $B$ be an ordered $k$-bi-ideal of $S$. If $x \in A \cap B$, then $x \in \overline{\overline{(x A x]}} \subseteq \overline{(B A B]}$. Thus, $A \cap B \subseteq \overline{(B A B]}$. Clearly, $\overline{(B A B]} \subseteq A \cap B$. Hence, $A \cap B=\overline{(B A B]}$.

Conversely, let $x \in A$. Using assumption and Lemmas 2.1 and 2.2, we get

$$
\begin{aligned}
x \in A \cap B(x) & =\overline{(B(x) A B(x)]}=\overline{\left(\overline{\left(\Sigma x+\Sigma x^{2}+x S x\right]} A \overline{\left(\Sigma x+\Sigma x^{2}+x S x\right]}\right]} \\
& \subseteq \overline{(\Sigma x A x]}=\overline{(x A x]} .
\end{aligned}
$$

Hence, $A$ is a bi-pure ordered $k$-ideal of $S$.
Definition 5.7. An ordered $k$-ideal $A$ of $S$ is called left weakly pure (resp. right weakly pure) if $A \cap I=\overline{(\Sigma A I]}$ (resp. $I \cap A=\overline{(\Sigma I A]})$ for every ordered $k$-ideal $I$ of $S$.

We note that every left (resp. right) pure ordered $k$-ideal of an ordered semiring is left (resp. right) weakly pure.

Now, we characterize some kinds of ordered $k$-regularities by pure and weakly pure ordered $k$-ideals of ordered semirings.

Lemma 5.8. [17] Let $S$ be an ordered semiring. Then the following statements hold:
(i) if $a \in \overline{\left(\Sigma a^{2}+a S a+S a^{2}+\Sigma S a S a\right]}$ for any $a \in S$, then $S$ is left weakly ordered $k$-regular;
(ii) if $a \in \overline{\left(\Sigma a^{2}+a S a+a^{2} S+\Sigma a S a S\right]}$ for any $a \in S$, then $S$ is right weakly ordered $k$-regular.

Theorem 5.9. An ordered semiring $S$ is left (resp. right) weakly ordered $k$-regular if and only if every ordered $k$-ideal of $S$ is left (resp. right) pure.

Proof. Assume that $S$ is left weakly ordered $k$-regular. Let $A$ be an ordered $k$-ideal of $S$ and let $x \in A$. By assumption, $x \in \overline{(\Sigma S x S x]} \subseteq \overline{(\Sigma S A S x]} \subseteq \overline{(\Sigma A x]}=\overline{(A x]}$. Hence, $A$ is left pure.

Conversely, let $a \in S$. By assumption, we obtain that $J(a)$ is left pure. Using Lemmas 2.1 and 2.2 and Theorem 5.2, we obtain that

$$
\begin{aligned}
a \in J(a) \cap L(a) & =\overline{(J(a) L(a)]}=\overline{(\overline{(\Sigma a+a S+S a+\Sigma S a S]} \overline{(\Sigma a+S a]}]} \\
& \subseteq \overline{\left(\Sigma a^{2}+a S a+S a^{2}+\Sigma S a S a\right]}
\end{aligned}
$$

By Lemma $5.8(i)$, we get that $S$ is left weakly ordered $k$-regular.
As a consequence of Theorem 5.9 and the fact that every quasi-pure ordered $k$-ideal is both left pure and right pure, we directly obtain the following corollary.

Corollary 5.10. An ordered semiring $S$ is both left and right weakly ordered $k$ regular if and only if every ordered $k$-ideal of $S$ is quasi-pure.

We note that an ordered $k$-ideal of an ordered semiring is bi-pure if and only if it is an ordered $k$-regular subsemiring. Accordingly, we obtain the following remark.
Remark 5.11. An ordered semiring $S$ is ordered $k$-regular if and only if every ordered $k$-ideal of $S$ is bi-pure.

Proof. Assume that $S$ is ordered $k$-regular. Let $A$ be an ordered $k$-ideal of $S$ and let $x \in A$. By the ordered $k$-regularity of $S$, we have that $x \in \overline{(x S x]} \subseteq \overline{(x S x S x]} \subseteq$ $\overline{(x S A S x]} \subseteq \overline{(x S A x]} \subseteq \overline{(x A x]}$. Hence, $A$ is bi-pure.

The converse is obvious since $S$ itself is a bi-pure ordered $k$-ideal and so $S$ is ordered $k$-regular.
Corollary 5.12. [15] Let $S$ be an ordered semiring. If

$$
a \in \overline{\left(\Sigma a^{2}+a S a+a^{2} S+\Sigma a S a S+S a^{2}+\Sigma S a S a+\Sigma S a^{2} S+\Sigma S a S a S\right]}
$$

for all $a \in S$, then $S$ is fully ordered $k$-idempotent.
Theorem 5.13. Let $S$ be an ordered semiring. Then
(i) if $S$ is fully ordered $k$-idempotent, then every ordered $k$-ideal of $S$ is both left and right weakly pure;
(ii) if every ordered $k$-ideal of $S$ is left weakly pure or right weakly pure, then $S$ is fully ordered $k$-idempotent.

Proof. (i). Assume that $S$ is fully ordered $k$-idempotent. Let $A$ and $I$ be any ordered $k$-ideals of $S$. By assumption, it turns out that if $x \in A \cap I$, then

$$
\begin{aligned}
& x \in \overline{(\Sigma S x S x S]} \subseteq \overline{(\Sigma S A S I S]} \subseteq \overline{(\Sigma A S I]} \subseteq \overline{(\Sigma A I)} \quad \text { and } \\
& x \in \overline{(\Sigma S x S x S]} \subseteq \overline{(\Sigma S I S A S]} \subseteq \overline{(\Sigma I S A]} \subseteq \overline{(\Sigma I A]} .
\end{aligned}
$$

So, $A \cap I \subseteq \overline{(\Sigma A I]}$ and $A \cap I \subseteq \overline{(\Sigma I A]}$. Clearly, $\overline{(\Sigma A I]} \subseteq A \cap I$ and $\overline{(\Sigma I A]} \subseteq A \cap I$. Hence, $A \cap I=\overline{(\Sigma A I]}=\overline{(\Sigma I A]}$ and thus $A$ is both left and right weakly pure.
(ii). Assume that every ordered $k$-ideal of $S$ is left weakly pure. Let $a \in S$. Then $J(a)$ is left weakly pure. It follows that $J(a)=\overline{(\Sigma J(a) J(a)]}$. By Lemmas 2.1 and 2.2 , we obtain that

$$
\begin{aligned}
a \in J(a) & =\overline{(\Sigma J(a) J(a)]}=\overline{(\Sigma \overline{(\Sigma a+S a+a S+\Sigma S a S]} \overline{(\Sigma a+S a+a S+\Sigma S a S]}]} \\
& =\overline{\left(\Sigma \overline{\left(\Sigma a^{2}+a S a+a^{2} S+\Sigma a S a S+S a^{2}+\Sigma S a S a+\Sigma S a^{2} S+\Sigma S a S a S\right]}\right]} \\
& =\overline{\left(\Sigma a^{2}+a S a+a^{2} S+\Sigma a S a S+S a^{2}+\Sigma S a S a+\Sigma S a^{2} S+\Sigma S a S a S\right]} .
\end{aligned}
$$

By Corollary 5.12 , we obtain that $S$ is fully ordered $k$-idempotent.
It can be proved analogously if every ordered $k$-ideal of $S$ is right weakly pure.

## References

[1] M.R. Adhikari, M.K.Sen and H.J. Weinert, On $k$-regular semirings, Bull. Calcutta Math. Soc. 88 (1996), $141-144$.
[2] A.K. Bhuniya and K. Jana, Bi-ideals in $k$-regular and intra $k$-regular semirings, Discuss. Math. Gen. Algebra Appl. 31 (2011), no. 1, 5-25.
[3] S. Bourne, The Jacobson radical of a semiring, Proc. Natl. Acad. Sci. USA 31 (1951), $163-170$.
[4] A.P. Gan and Y.L. Jiang, On ordered ideals in ordered semirings, J. Math. Res. Exposition 31 (2011), no. 6, 989 - 996.
[5] M. Henriksen, Ideals in semirings with commutative addition, Amer. Math. Soc. Notices 6 (1958), 321.
[6] K. Jana, Quasi $k$-ideals in $k$-regular and intra $k$-regular semirings, Pure Math. Appl. 22 (2011), no. 1, $65-74$.
[7] N. Kehayopulu, On intra-regular ordered semigroups, Semigroup Forum 46 (1993), 271-278.
[8] N. Kehayopulu, On completely regular ordered semigroups, Scinetiae Math. 1 (1998), $27-32$.
[9] N. Kehayopulu and M. Tsingelis, On left regular ordered semigroups, Southeast Asian Bull. Math. 25 (2002), no. 4, 609-615.
[10] D. Mandal, Fuzzy ideals and fuzzy interior ideals in ordered semirings, Fuzzy Inf. Eng. 6 (2014), no. 1, 101 - 114.
[11] J. von Neumann, On regular rings, Proc. Natl. Acad. Sci. USA 22 (1936),707-113.
[12] P. Palakawong na Ayuthaya and B. Pibaljommee, Characterizations of regular ordered semirings by ordered quasi-ideals, Int. J. Math. Math. Sci. 2016 (2016), Article ID. 4272451.
[13] P. Palakawong na Ayuthaya and B. Pibaljommee, Characterizations of ordered $k$-regular semirings by ordered quasi $k$-ideals, Quasigroups and Related Systems 25 (2017), 109 - 120.
[14] P. Palakawong na Ayuthaya and B. Pibaljommee, Characterizations of ordered intra $k$-regular semirings by ordered $k$-ideals, Commun. Korean Math. Soc. 33 (2018), no. 1, 1-12.
[15] P. Palakawong na Ayuthaya and B. Pibaljommee, Characterizations of completely ordered $k$-regular semirings, Songklanakarin J. Sci. Techn. 41(2019), 501-505.
[16] S. Patchakhieo and B. Pibaljommee, Characterizations of ordered $k$-regular semirings by ordered $k$-ideals, Asian-European J. Math. 10 (2017), Article ID. 4272451.
[17] B. Pibaljommee and P. Palakawong na Ayuthaya, Characterizations of weakly ordered $k$-regular hemirings by $k$-ideals, Discuss. Math. Gen. Algebra Appl. 39 (2019), no. 2, 289 - 302.
[18] P. Senarat and P. Pibaljommee, Prime ordered $k$-bi-ideals in ordered semirings, Quasigroups and Related Systems 25 (2017), 121 - 132.

Received July 26, 2020
Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen, Thailand
E-mail: pakorn1702@gmail.com banpib@kku.ac.th

# Magnifying elements of some semigroups of partial transformations 

Chadaphorn Punkumkerd and Preeyanuch Honyam


#### Abstract

Let $X$ be a nonempty set and let $P(X)$ denote the semigroup (under the composition) of partial transformations from a subset of $X$ to $X$ and $E(X)$ denote the subsemigroup of $P(X)$ containing surjective partial transformations on $X$. For a fixed nonempty subset $Y$ of $X$, let $\overline{P T}(X, Y)=\{\alpha \in P(X) \mid(\operatorname{dom} \alpha \cap Y) \alpha \subseteq Y\}$ and $P T_{(X, Y)}=\{\alpha \in P(X) \mid(\operatorname{dom} \alpha \cap Y) \alpha=Y\}$ We give necessary and sufficient conditions for elements in semigroups $\overline{P T}(X, Y)$ and $P T_{(X, Y)}$ to be left or right magnifying.


## 1. Introduction

Let $S$ be a semigroup. An element $a \in S$ is called a left (right) magnifying element if there exist a proper subset $M$ of $S$ such that $S=a M(S=M a)$. Such elements are mentioned in 1963 by E. S. Ljapin [5]. M. Gutan showed in [1] that there exists semigroups containing both strong and non-strong magnifying elements. In [2] he proved that every semigroup containing magnifying elements is factorizable. In [3] he proposed the method of construction of semigroups having good left magnifying elements.

Let $B(X)$ be the set of all binary relations on the set $X$. Then $P(X)$, where $P(X)=\{\alpha \in B(X) \mid \alpha: A \rightarrow B$ when $A, B \subseteq X\}$, is a semigroup called the semigroup of partial transformations on $X$. The semigroup of surjective partial transformations on $X$ is denoted by $E(X)$, i.e. $E(X)=\{\alpha \in P(X) \mid \operatorname{ran} \alpha=X\}$. The necessary and sufficient conditions for elements of $P(X)$ to be the left or right magnifying elements were found in [6].
$T(X)=\{\alpha \in P(X) \mid \operatorname{dom} \alpha=X\}$ is a semigroup called the full transformation semigroup on $X . E T(X)=E(X) \cap T(X)$ is a semigroup of surjective full transformations on $X$.

For a fixed nonempty subset $Y$ of $X$, let

$$
\bar{T}(X, Y)=\{\alpha \in T(X) \mid Y \alpha \subseteq Y\} \quad \text { and } \quad T_{(X, Y)}=\{\alpha \in T(X) \mid Y \alpha=Y\}
$$

where $Y \alpha=\{y \alpha \mid y \in Y\}$. Then $\bar{T}(X, Y)$ and $T_{(X, Y)}$ are subsemigroups of $T(X)$. $T_{(X, Y)}$ is also a subsemigroup of $\bar{T}(X, Y)$.

[^4]The semigroups $\overline{P T}(X, Y)$ and $P T_{(X, Y)}$ are defined similarly. Namely,

$$
\overline{P T}(X, Y)=\{\alpha \in P(X) \mid(\operatorname{dom} \alpha \cap Y) \alpha \subseteq Y\}
$$

and

$$
P T_{(X, Y)}=\{\alpha \in P(X) \mid(\operatorname{dom} \alpha \cap Y) \alpha=Y\},
$$

where $\operatorname{dom} \alpha$ is the domain of $\alpha$ and $(\operatorname{dom} \alpha \cap Y) \alpha=\{z \alpha \mid z \in \operatorname{dom} \alpha \cap Y\}$. Then $\overline{P T}(X, Y)$ and $P T_{(X, Y)}$ are subsemigroups of $P(X) . P T_{(X, Y)}$ also is a subsemigroup of $\overline{P T}(X, Y)$.

The purpose of this paper is providing the necessary and sufficient conditions for elements in semigroups $\overline{P T}(X, Y)$ and $P T_{(X, Y)}$ to be left or right magnifying.

## 2. Preliminaries

Throughout this paper, the cardinality of a set $X$ is denoted by $|X|$ and $X=A \dot{\cup} B$ means $X$ is a disjoint union of $A$ and $B$. The proper subset $B$ of a set $A$ is denoted by $B \subset A$.

For $\alpha, \beta \in P(X), \alpha \beta \in P(X)$ is defined by $x(\alpha \beta)=(x \alpha) \beta$ for all $x \in \operatorname{dom}(\alpha \beta)$. The identity map on $X$, i.e. $i d_{X}$, is the identity element of $\overline{P T}(X, Y)$ and $P T_{(X, Y)}$. The empty function on $X$, i.e. $\emptyset_{X}$ is a zero element of $\overline{P T}(X, Y)$ but $\emptyset_{X} \notin P T_{(X, Y)}$.

For $\alpha \in P(X)$, we write

$$
\alpha=\binom{X_{i}}{a_{i}}
$$

where the subscript $i$ belongs to some (unmentioned) index set $I$, the abbreviation $\left\{a_{i}\right\}$ denotes $\left\{a_{i} \mid i \in I\right\}$. Then $\operatorname{ran} \alpha=\left\{a_{i}\right\}$ and $a_{i} \alpha^{-1}=X_{i}$.

For $\alpha \in \overline{P T}(X, Y)$, we write

$$
\alpha=\left(\begin{array}{ccc}
A_{i} & B_{j} & C_{k} \\
a_{i} & b_{j} & c_{k}
\end{array}\right)
$$

where $A_{i} \cap Y \neq \emptyset ; B_{j}, C_{k} \subseteq X \backslash Y ;\left\{a_{i}\right\} \subseteq Y,\left\{b_{j}\right\} \subseteq Y \backslash\left\{a_{i}\right\}$ and $\left\{c_{k}\right\} \subseteq X \backslash Y$.
For $\alpha \in P T_{(X, Y)}$, we write

$$
\alpha=\left(\begin{array}{ll}
A_{i} & B_{j} \\
a_{i} & b_{j}
\end{array}\right)
$$

where $A_{i} \cap Y \neq \emptyset ; B_{j} \subseteq X \backslash Y ;\left\{a_{i}\right\}=Y,\left\{b_{j}\right\} \subseteq X \backslash Y$.
If $X$ is finite, then $Y$ is also finite. So we get $\overline{P T}(X, Y)$ and $P T_{(X, Y)}$ are finite semigroups. Since finite semigroups do not contain left and right magnifying elements (cf. [4]), we will consider only the case when $X$ is an infinite set.

## 3. Left Magnifying Elements in $\overline{P T}(X, Y)$

Lemma 3.1. If $\alpha \in \overline{P T}(X, Y)$ is a left magnifying element in $\overline{P T}(X, Y)$, then $\operatorname{dom} \alpha=X, \alpha$ is injective and $y \alpha^{-1} \subseteq Y$ for all $y \in Y \cap \operatorname{ran} \alpha$.

Proof. Assume that $\alpha$ is a left magnifying element in $\overline{P T}(X, Y)$. Then there exists a proper subset $M$ of $\overline{P T}(X, Y)$ such that $\alpha M=\overline{P T}(X, Y)$. Since $i d_{X} \in$ $\overline{P T}(X, Y)$, there exists $\beta \in M$ such that $\alpha \beta=i d_{X}$. Thus $X=\operatorname{dom} i d_{X} \subseteq \operatorname{dom} \alpha$ and hence $\operatorname{dom} \alpha=X$. Since $i d_{X}$ is injective, we also have $\alpha$ is injective. Since $\alpha$ is not an empty function, we have $Y \cap \operatorname{ran} \alpha \neq \emptyset$. Let $y \in Y \cap \operatorname{ran} \alpha$ and let $x \in y \alpha^{-1}$. Then $x \alpha=y$ and so $x=x i d_{X}=x \alpha \beta=y \beta \in Y$. So $y \alpha^{-1} \subseteq Y$ for all $y \in Y \cap \operatorname{ran} \alpha$.
Lemma 3.2. If $\alpha \in \overline{P T}(X, Y)$ is a left magnifying element in $\overline{P T}(X, Y)$, then $\alpha$ is not surjective.
Proof. Assume that $\alpha$ is a left magnifying element in $\overline{P T}(X, Y)$ and $\alpha$ is surjective. Then there exists $M \subset \overline{P T}(X, Y)$ such that $\alpha M=\overline{P T}(X, Y)$. By Lemma 3.1, we get $\operatorname{dom} \alpha=X, \alpha$ is injective and $y \alpha^{-1} \subseteq Y$ for all $y \in Y \cap \operatorname{ran} \alpha$. Then

$$
\alpha=\left(\begin{array}{ll}
a_{i} & b_{j} \\
y_{i} & z_{j}
\end{array}\right)
$$

where $\left\{a_{i}\right\}=Y=\left\{y_{i}\right\}$ and $\left\{a_{i}\right\} \dot{\cup}\left\{b_{j}\right\}=X=\left\{y_{i}\right\} \dot{\cup}\left\{z_{j}\right\}$. There is

$$
\alpha^{-1}=\left(\begin{array}{ll}
y_{i} & z_{j} \\
a_{i} & b_{j}
\end{array}\right) \in \overline{P T}(X, Y)
$$

such that $\alpha^{-1} \alpha=i d_{X}$. Let $\beta \in \overline{P T}(X, Y)$. Then $\alpha \beta \in \overline{P T}(X, Y)$. Since $\overline{P T}(X, Y)=\alpha M$, we get $\alpha \beta=\alpha \gamma$ for some $\gamma \in M$. So $\beta=i d_{X} \beta=\alpha^{-1}(\alpha \beta)=$ $\alpha^{-1}(\alpha \gamma)=i d_{X} \gamma=\gamma \in M$. Thus $\overline{P T}(X, Y) \subseteq M$ that contradicts with $M$ is a proper subset of $\overline{P T}(X, Y)$. Therefore, $\alpha$ is not surjective.
Theorem 3.3. $\alpha \in \overline{P T}(X, Y)$ is a left magnifying element in $\overline{P T}(X, Y)$ if and only if the following statements hold:

1. $\operatorname{dom} \alpha=X$,
2. $y \alpha^{-1} \subseteq Y$ for all $y \in Y \cap \operatorname{ran} \alpha$ and
3. $\alpha$ is injective but not surjective.

Proof. Assume that $\alpha$ is a left magnifying element in $\overline{P T}(X, Y)$. By the above lemmas, we have $\operatorname{dom} \alpha=X, y \alpha^{-1} \subseteq Y$ for all $y \in Y \cap \operatorname{ran} \alpha$ and $\alpha$ is injective but not surjective.

Conversely, choose $M=\{\delta \in \overline{P T}(X, Y) \mid \operatorname{dom} \delta \neq X\}$ and assume that the conditions 1-3 hold. Then we get $M \subset \overline{P T}(X, Y)$. Let $\beta \in \overline{P T}(X, Y)$. If $\beta=\emptyset_{X}$, then there is $\emptyset_{X} \in M$ such that $\beta=\alpha \emptyset_{X}$. If $\beta \neq \emptyset_{X}$, we let $Y=\left\{a_{i}\right\} \dot{\cup}\left\{b_{j}\right\}$ when $\operatorname{dom} \beta \cap Y=\left\{a_{i}\right\}$ and $X \backslash Y=\left\{s_{k}\right\} \dot{\cup}\left\{t_{l}\right\}$ when dom, $\beta \cap(X \backslash Y)=\left\{s_{k}\right\}$. Then

$$
\alpha=\left(\begin{array}{llll}
a_{i} & b_{j} & s_{k} & t_{l} \\
y_{i} & z_{j} & u_{k} & v_{l}
\end{array}\right)
$$

where $\left\{y_{i}\right\},\left\{z_{j}\right\} \subseteq Y$ and $\left\{u_{k}\right\},\left\{v_{l}\right\} \subseteq X \backslash Y$. Since $\alpha$ is not surjective, we have $\operatorname{ran} \alpha \neq X$. Define $\gamma:\left\{y_{i}\right\} \cup\left\{u_{k}\right\} \rightarrow X$ by

$$
\gamma=\left(\begin{array}{cc}
y_{i} & u_{k} \\
a_{i} \beta & s_{k} \beta
\end{array}\right) .
$$

Since $\alpha$ is injective, $\gamma$ is well-defined. Since $(\operatorname{dom} \gamma \cap Y) \gamma=\left\{y_{i}\right\} \gamma=\left\{a_{i} \beta\right\} \subseteq Y$, $\gamma \in \overline{P T}(X, Y)$. But dom $\gamma=\left\{y_{i}\right\} \cup\left\{u_{k}\right\} \subseteq \operatorname{ran} \alpha \neq X$, so $\gamma \in M$.

Let $x \in \operatorname{dom} \beta=\left\{a_{i}\right\} \cup\left\{s_{k}\right\}=\operatorname{dom}(\alpha \gamma)$.
If $x=a_{i}$ for some $i \in I$, then $x(\alpha \gamma)=a_{i}(\alpha \gamma)=\left(a_{i} \alpha\right) \gamma=y_{i} \gamma=a_{i} \beta=x \beta$.
If $x=s_{k}$ for some $k \in K$, then $x(\alpha \gamma)=s_{k}(\alpha \gamma)=\left(s_{k} \alpha\right) \gamma=u_{k} \gamma=s_{k} \beta=x \beta$.
Thus $\beta=\alpha \gamma$. Hence $\overline{P T}(X, Y)=\alpha M$. Therefore, $\alpha$ is a left magnifying element in $\overline{P T}(X, Y)$.

Taking $Y=X$ in Theorem 3.3 we obtain
Corollary 3.4. $\alpha \in P(X)$ is a left magnifying element in $P(X)$ if and only if $\operatorname{dom} \alpha=X$ and $\alpha$ is injective but not surjective.

Example 3.5. Let $X=\mathbb{N}$ and $Y=2 \mathbb{N}$. Define

$$
\alpha=\binom{n}{n+2}_{n \in \mathbb{N}}
$$

Then $(\operatorname{dom} \alpha \cap Y) \alpha=(2 \mathbb{N}) \alpha=2 \mathbb{N} \backslash\{2\} \subseteq Y$ and so $\alpha \in \overline{P T}(X, Y)$. Moreover, we get $\operatorname{dom} \alpha=\mathbb{N}=X, y \alpha^{-1} \subseteq Y$ for all $y \in Y \cap \operatorname{ran} \alpha$ and $\alpha$ is injective but $\alpha$ is not surjective. By Theorem 3.3, $\alpha$ is a left magnifying element in $\overline{P T}(X, Y)$. By the proof of Theorem 3.3, there exists $M=\{\delta \in \overline{P T}(X, Y) \mid \operatorname{dom} \delta \neq \mathbb{N}=X\} \subset$ $\overline{P T}(X, Y)$ such that $\alpha M=\overline{P T}(X, Y)$.

## 4. Right Magnifying Elements in $\overline{P T}(X, Y)$

Lemma 4.1. If $\alpha \in \overline{P T}(X, Y)$ is a right magnifying element in $\overline{P T}(X, Y)$, then $\alpha$ is surjective.
Proof. Assume that $\alpha$ is a right magnifying element in $\overline{P T}(X, Y)$. Then there is a proper subset $M$ of $\overline{P T}(X, Y)$ such that $M \alpha=\overline{P T}(X, Y)$. Since $i d_{X} \in \overline{P T}(X, Y)$, there exists $\beta \in M$ such that $\beta \alpha=i d_{X}$. From $i d_{X}$ is surjective, this implies $\alpha$ is surjective.

Lemma 4.2. If $\alpha \in \overline{P T}(X, Y)$ is a right magnifying element in $\overline{P T}(X, Y)$, then $y \alpha^{-1} \cap Y \neq \emptyset$ for all $y \in Y$.

Proof. Assume $\alpha$ is a right magnifying element in $\overline{P T}(X, Y)$. Then there exists a proper subset $M$ of $\overline{P T}(X, Y)$ such that $M \alpha=\overline{P T}(X, Y)$. By Lemma 4.1, $\alpha$ is surjective.

Suppose that $y_{0} \alpha^{-1} \cap Y=\emptyset$ for some $y_{0} \in Y$ and define

$$
\beta=\binom{Y}{y_{0}}
$$

Then $\beta \in \overline{P T}(X, Y)$. Since $M \alpha=\overline{P T}(X, Y)$, there is $\gamma \in M$ such that $\gamma \alpha=\beta$. But $\alpha$ is surjective and $y_{0} \alpha^{-1} \cap Y=\emptyset$, so $y_{0} \alpha^{-1} \subseteq X \backslash Y$. Thus for each $y \in Y$,
$y_{0}=y \beta=(y \gamma) \alpha$. So $y \gamma \in y_{0} \alpha^{-1} \subseteq X \backslash Y$ which is a contradiction. Therefore $y \alpha^{-1} \cap Y \neq \emptyset$ for all $y \in Y$.

Lemma 4.3. If $\alpha \in \overline{P T}(X, Y)$ is a right magnifying element in $\overline{P T}(X, Y)$, then $\operatorname{dom} \alpha \neq X$ or $\alpha$ is not injective.

Proof. Assume that $\alpha$ is a right magnifying element in $\overline{P T}(X, Y)$. By Lemmas 4.1 and 4.2, $\alpha$ is surjective and $y \alpha^{-1} \cap Y \neq \emptyset$ for all $y \in Y$. Suppose that $\operatorname{dom} \alpha=X$ and $\alpha$ is injective. Let $X=\left\{a_{i}\right\} \cup\left\{b_{j}\right\}$ be such that $Y=\left\{a_{i}\right\}$. Then

$$
\alpha=\left(\begin{array}{ll}
a_{i} & b_{j} \\
y_{i} & z_{j}
\end{array}\right)
$$

where $\left\{y_{i}\right\}=Y$ and $\left\{z_{j}\right\}=X \backslash Y$. There is $\alpha^{-1} \in \overline{P T}(X, Y)$ such that $\alpha \alpha^{-1}=$ $i d_{X}$. Let $\beta \in \overline{P T}(X, Y)$. Then $\beta \alpha \in \overline{P T}(X, Y)$. Since $\overline{P T}(X, Y)=M \alpha$, we have $\beta \alpha=\delta \alpha$ for some $\delta \in M$. Thus $\beta=(\beta \alpha) \alpha^{-1}=(\delta \alpha) \alpha^{-1}=\delta \in M$. Hence $\overline{P T}(X, Y) \subseteq M$. That yields $M=\overline{P T}(X, Y)$ which contradicts with $M \subset$ $\overline{P T}(X, Y)$. Therefore, $\operatorname{dom} \alpha \neq X$ or $\alpha$ is not injective.

Theorem 4.4. $\alpha \in \overline{P T}(X, Y)$ is a right magnifying element in $\overline{P T}(X, Y)$ if and only if the following statements hold:

1. $\alpha$ is surjective,
2. $y \alpha^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ and
3. $\operatorname{dom} \alpha \neq X$ or $\alpha$ is not injective.

Proof. Assume that $\alpha$ is a right magnifying element in $\overline{P T}(X, Y)$. Conditions 1-3 are a consequence of Lemmas 4.1, 4.2 and 4.3.

Conversely, assume that conditions 1-3 are satisfied. We have two cases.
Case 1: dom $\alpha \neq X$. Choose $M=\{\delta \in \overline{P T}(X, Y) \mid \delta$ is not surjective $\}$. Then $M \subset \overline{P T}(X, Y)$. Let $\beta \in \overline{P T}(X, Y)$. Then

$$
\beta=\left(\begin{array}{ccc}
A_{i} & B_{j} & C_{k} \\
a_{i} & b_{j} & c_{k}
\end{array}\right) .
$$

where $A_{i} \cap Y \neq \emptyset ; B_{j}, C_{k} \subseteq X \backslash Y ;\left\{a_{i}\right\} \subseteq Y,\left\{b_{j}\right\} \subseteq Y \backslash\left\{a_{i}\right\}$ and $\left\{c_{k}\right\} \subseteq X \backslash Y$. Since $\alpha$ is surjective, we have $\operatorname{ran} \beta \subseteq X=\operatorname{ran} \alpha$. From $y \alpha^{-1} \cap Y \neq \emptyset$ for all $y \in Y$, we have $a_{i} \alpha^{-1} \cap Y \neq \emptyset \neq b_{j} \alpha^{-1} \cap Y$. Choose $d_{a_{i}} \in a_{i} \alpha^{-1} \cap Y$ and $d_{b_{j}} \in b_{j} \alpha^{-1} \cap Y$. Then $d_{a_{i}} \alpha=a_{i}$ and $d_{b_{j}} \alpha=b_{j}$. Since $\operatorname{ran} \beta \subseteq \operatorname{ran} \alpha$, we have $c_{k} \in \operatorname{ran} \alpha$ and we can choose $c_{k}^{\prime} \in \operatorname{dom} \alpha$ such that $c_{k}^{\prime} \alpha=c_{k}$. Define

$$
\gamma=\left(\begin{array}{ccc}
A_{i} & B_{j} & C_{k} \\
d_{a_{i}} & d_{b_{j}} & c_{k}^{\prime}
\end{array}\right)
$$

Then $\gamma \in \overline{P T}(X, Y)$. Since $\operatorname{ran} \gamma \subseteq \operatorname{dom} \alpha \neq X, \gamma$ is not surjective. Thus $\gamma \in M$.
Let $\operatorname{dom}(\gamma \alpha)=(\operatorname{ran} \gamma \cap \operatorname{dom} \alpha) \gamma^{-1}=(\operatorname{ran} \gamma) \gamma^{-1}=\operatorname{dom} \gamma=\operatorname{dom} \beta$ and $x \in \operatorname{dom} \beta$.

If $x \in A_{i}$ for some $i \in I$, then $x(\gamma \alpha)=(x \gamma) \alpha=d_{a_{i}} \alpha=a_{i}=x \beta$.
If $x \in B_{j}$ for some $j \in J$, then $x(\gamma \alpha)=(x \gamma) \alpha=d_{b_{j}} \alpha=b_{j}=x \beta$.

If $x \in C_{k}$ for some $k \in K$, then $x(\gamma \alpha)=(x \gamma) \alpha=c_{k}^{\prime} \alpha=c_{k}=x \beta$.
Thus $\gamma \alpha=\beta$ and hence $\overline{P T}(X, Y) \subseteq M \alpha$ which implies that $M \alpha=\overline{P T}(X, Y)$. Case 2: $\alpha$ is not injective. Choose $M=\{\delta \in \overline{P T}(X, Y) \mid \delta$ is not surjective $\}$. Then $M \subset \overline{P T}(X, Y)$. Let $\beta \in \overline{P T}(X, Y)$. Then

$$
\beta=\left(\begin{array}{ccc}
A_{i} & B_{j} & C_{k} \\
a_{i} & b_{j} & c_{k}
\end{array}\right)
$$

where $A_{i} \cap Y \neq \emptyset ; B_{j}, C_{k} \subseteq X \backslash Y ;\left\{a_{i}\right\},\left\{b_{j}\right\} \subseteq Y$ and $\left\{c_{k}\right\} \subseteq X \backslash Y$.
Let $\gamma \in \overline{P T}(X, Y)$ be as in Case 1. Since $\alpha$ is not injective, there is $x_{0} \in \operatorname{ran} \alpha$ and distinct elements $x_{1}, x_{2} \in \operatorname{dom} \alpha$ such that $x_{1} \alpha=x_{0}=x_{2} \alpha$. Note that $\operatorname{ran} \beta \subseteq \operatorname{ran} \alpha$. If $x_{0} \in \operatorname{ran} \beta$, then there is exactly one (either $x_{1}$ or $x_{2}$ ) in ran $\gamma$. If $x_{0} \notin \operatorname{ran} \beta$, then $x_{1}, x_{2} \notin \operatorname{ran} \gamma$. Thus $\gamma$ is not surjective and so $\gamma \in M$. Analogously as in Case 1, we get $\gamma \alpha=\beta$ and hence $\overline{P T}(X, Y) \subseteq M \alpha$. This means that $M \alpha=\overline{P T}(X, Y)$.

Therefore, $\alpha$ is a right magnifying element in $\overline{P T}(X, Y)$.
For $Y=X$ we obtain the following corollary.
Corollary 4.5. $\alpha \in P(X) \alpha$ is a right magnifying element in $P(X)$ if and only if $\alpha$ is surjective and ( $\operatorname{dom} \alpha \neq X$ or $\alpha$ is not injective).

Example 4.6. Let $X=\mathbb{N}$ and $Y=2 \mathbb{N}$. Define

$$
\alpha=\left(\begin{array}{ccccc}
1 & \{2,3\} & 4 & \{5,6\} & n+2 \\
1 & 2 & 3 & 4 & n
\end{array}\right)_{n \geq 5} \text { and } \beta=\left(\begin{array}{ccccc}
1 & 4 & 5 & 8 & n+4 \\
1 & 2 & 3 & 4 & n
\end{array}\right)_{n \geq 5}
$$

Then $(\operatorname{dom} \alpha \cap Y) \alpha=(2 \mathbb{N}) \alpha=2 \mathbb{N} \subseteq Y$ and $(\operatorname{dom} \beta \cap Y) \beta=(2 \mathbb{N} \backslash\{2,6\}) \beta=2 \mathbb{N} \subseteq$ $Y$. So $\alpha, \beta \in \overline{P T}(X, Y)$. It is clear that $\alpha$ is surjective. Furthermore, $y \alpha^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ and $\alpha$ is not injective but $\operatorname{dom} \alpha=\mathbb{N}=X$. We can see that $\beta$ is a bijection and $y \beta^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ but $\operatorname{dom} \beta=\mathbb{N} \backslash\{2,3,6,7\} \neq X$. By Theorem 4.4, $\alpha, \beta$ are right magnifying elements in $\overline{P T}(X, Y)$. Then by the proof of Theorem 4.4, there is $M=\{\delta \in \overline{P T}(X, Y) \mid \delta$ is not surjective $\} \subset \overline{P T}(X, Y)$ such that $M \alpha=\overline{P T}(X, Y)$ and $M \beta=\overline{P T}(X, Y)$.

## 5. Left Magnifying Elements in $P T_{(X, Y)}$

Lemma 5.1. If $\alpha \in P T_{(X, Y)}$ is a left magnifying element in $P T_{(X, Y)}$, then $\operatorname{dom} \alpha=X$ and $\alpha$ is injective.

Proof. Assume that $\alpha$ is a left magnifying element in $P T_{(X, Y)}$. Then there exists a proper subset $M$ of $P T_{(X, Y)}$ such that $\alpha M=P T_{(X, Y)}$. Since $i d_{X} \in P T_{(X, Y)}$, there exists $\beta \in M$ such that $\alpha \beta=i d_{X}$. Thus $\operatorname{dom} \alpha=X$ and $\alpha$ is injective.

Lemma 5.2. If $\alpha \in P T_{(X, Y)}$, where $Y \neq X$, is a left magnifying element in $P T_{(X, Y)}$, then $\alpha$ is not surjective.

Proof. Given $Y \neq X$. Assume that $\alpha$ is a left magnifying element in $P T_{(X, Y)}$ and $\alpha$ is surjective. Then there exists $M \subset P T_{(X, Y)}$ such that $\alpha M=P T_{(X, Y)}$. By Lemma 5.1, we get $\operatorname{dom} \alpha=X$ and $\alpha$ is injective. Thus $\alpha$ is a bijection on $X$. Since $\alpha \beta \in P T_{(X, Y)}=\alpha M, \alpha \beta=\alpha \gamma$ for some $\gamma \in M$. So $\beta=\gamma$ and hence $\beta \in M$. Thus $P T_{(X, Y)} \subseteq M$. So $M=P T_{(X, Y)}$ which is a contradiction. Therefore, $\alpha$ is not surjective.

Theorem 5.3. If $Y \neq X$, then $\alpha \in P T_{(X, Y)}$ is a left magnifying element in $P T_{(X, Y)}$ if and only if $\operatorname{dom} \alpha=X$ and $\alpha$ is injective but not surjective.
Proof. Let $Y \neq X$. Assume that $\alpha$ is a left magnifying element in $P T_{(X, Y)}$. By Lemmas 5.1 and 5.2 , we have $\operatorname{dom} \alpha=X$ and $\alpha$ is injective but not surjective. Conversely, assume that $\operatorname{dom} \alpha=X$ and $\alpha$ is injective but not surjective. Choose $M=\left\{\delta \in P T_{(X, Y)} \mid \operatorname{dom} \delta \neq X\right\}$. Then $M \subset P T_{(X, Y)}$.

We prove that $\alpha M=P T_{(X, Y)}$. Let $\beta \in P T_{(X, Y)}$ and $Y=\left\{a_{i}\right\} \dot{\cup}\left\{b_{j}\right\}$ where $\operatorname{dom} \beta \cap Y=\left\{a_{i}\right\}$ and $X \backslash Y=\left\{s_{k}\right\} \dot{\cup}\left\{t_{l}\right\}$ when $\operatorname{dom} \beta \cap(X \backslash Y)=\left\{s_{k}\right\}$. Then

$$
\alpha=\left(\begin{array}{llll}
a_{i} & b_{j} & s_{k} & t_{l} \\
y_{i} & z_{j} & u_{k} & v_{l}
\end{array}\right)
$$

where $Y=\left\{y_{i}\right\} \cup\left\{z_{j}\right\}$ and $\left\{u_{k}\right\},\left\{v_{l}\right\} \subseteq X \backslash Y$. Define $\gamma:\left\{y_{i}\right\} \cup\left\{u_{k}\right\} \rightarrow X$ by

$$
\gamma=\left(\begin{array}{cc}
y_{i} & u_{k} \\
a_{i} \beta & s_{k} \beta
\end{array}\right) .
$$

Since $\alpha$ is injective, $\gamma$ is well-defined and $(\operatorname{dom} \gamma \cap Y) \gamma=\left\{y_{i}\right\} \gamma=\left\{a_{i} \beta\right\}=$ $(\operatorname{dom} \beta \cap Y) \beta=Y$, hence $\gamma \in P T_{(X, Y)}$. Since $\alpha$ is not surjective, from dom $\gamma=$ $\left\{y_{i}\right\} \cup\left\{u_{k}\right\} \subseteq \operatorname{ran} \alpha \neq X$ it follows $\gamma \in M$. But $x(\alpha \gamma)=(x \alpha) \gamma=x \beta$ for all $x \in \operatorname{dom} \beta=\left\{a_{i}\right\} \cup\left\{s_{k}\right\}=\operatorname{dom}(\alpha \gamma)$. Hence $\alpha \gamma=\beta$ and so $\alpha M=P T_{(X, Y)}$. So, $\alpha$ is a left magnifying element in $P T_{(X, Y)}$.

Theorem 5.4. $E(X)$ has no left magnifying elements.
Proof. Suppose that $\alpha$ is a left magnifying element in $E(X)$. Then $\alpha$ is a left magnifying element in $P T_{(X, Y)}$ when $Y=X$. By Lemma 5.1, $\operatorname{dom} \alpha=X$ and $\alpha$ is injective. Since $\alpha \in E(X), \alpha$ is surjective. Then there is $\alpha^{-1} \in E(X)$ such that $\alpha^{-1} \alpha=i d_{X}$. Since $\alpha$ is left magnifying, there is $M \subset E(X)$ such that $\alpha M=E(X)$. Let $\beta \in E(X)$. Analogously as in the proof of Lemma 5.2, we obtain $\beta \in M$. Thus $M=E(X)$. That is a contradiction. Hence, $E(X)$ has no left magnifying elements.

Example 5.5. Let $X=\mathbb{N}$ and $Y=2 \mathbb{N}$. Define

$$
\alpha=\left(\begin{array}{ll}
2 n-1 & 2 n \\
2 n+1 & 2 n
\end{array}\right)_{n \in \mathbb{N}} .
$$

Since $(\operatorname{dom} \alpha \cap Y) \alpha=(2 \mathbb{N}) \alpha=2 \mathbb{N}=Y, \alpha \in P T_{(X, Y)}, \operatorname{dom} \alpha=\mathbb{N}=X$ and $\alpha$ is injective. But $\operatorname{ran} \alpha=\mathbb{N} \backslash\{1\} \neq X$, then $\alpha$ is not surjective. By Theorem 5.3, $\alpha$
is a left magnifying element in $P T_{(X, Y)}$. Let $M=\left\{\delta \in P T_{(X, Y)} \mid \operatorname{dom} \delta \neq \mathbb{N}\right\}$. Then, analogously as in the proof of Theorem 5.3, for each $\beta \in P T_{(X, Y)}$, there exists $\gamma \in M$ such that $\alpha \gamma=\beta$. Thus $P T_{(X, Y)}=\alpha M$ for some $M \subset P T_{(X, Y)}$.

## 6. Right Magnifying Elements in $P T_{(X, Y)}$

Lemma 6.1. If $\alpha \in P T_{(X, Y)}$ is a right magnifying element in $P T_{(X, Y)}$, then $\alpha$ is surjective.

Proof. Assume that $\alpha$ is a right magnifying element in $P T_{(X, Y)}$. Then $M \alpha=$ $P T_{(X, Y)}$ for some proper subset $M$ of $P T_{(X, Y)}$. Since $i d_{X} \in P T_{(X, Y)}$, there exists $\beta \in M$ such that $\beta \alpha=i d_{X}$. So, $\alpha$ must be surjective.

Lemma 6.2. If $\alpha \in P T_{(X, Y)}$ is a right magnifying element in $P T_{(X, Y)}$, then $\operatorname{dom} \alpha \neq X$ or $\alpha$ is not injective.
Proof. Assume $\alpha$ is a right magnifying element in $P T_{(X, Y)}$. Then $M \alpha=P T_{(X, Y)}$ for some $M \subset P T_{(X, Y)}$. Suppose that dom $\alpha=X$ and $\alpha$ is injective. By Lemma 6.1, $\alpha$ is surjective. Let $\beta \in P T_{(X, Y)}$. Then $\beta \alpha \in P T_{(X, Y)}$. Since $P T_{(X, Y)}=M \alpha$, we have $\beta \alpha=\delta \alpha$ for some $\delta \in M$. Since $\alpha$ is a bijection on $X$ with $Y \alpha=Y$, we get $\beta=\delta \in M$. Hence $P T_{(X, Y)} \subseteq M$. That yields $M=P T_{(X, Y)}$ which contradicts with $M \subset P T_{(X, Y)}$. Therefore, $\operatorname{dom} \alpha \neq X$ or $\alpha$ is not injective.

Theorem 6.3. $\alpha \in P T_{(X, Y)}$ is a right magnifying element in $P T_{(X, Y)}$ if and only if $\alpha$ is surjective and ( $\operatorname{dom} \alpha \neq X$ or $\alpha$ is not injective).
Proof. Assume that $\alpha$ is a right magnifying element in $P T_{(X, Y)}$. By Lemmas 6.1 and $6.2, \alpha$ is surjective and ( $\operatorname{dom} \alpha \neq X$ or $\alpha$ is not injective).

Conversely, assume that $\alpha$ is surjective and ( $\operatorname{dom} \alpha \neq X$ or $\alpha$ is not injective). We have two cases:

Case 1: $\operatorname{dom} \alpha \neq X$. Choose $M=\left\{\delta \in P T_{(X, Y)} \mid \delta\right.$ is not surjective $\}$. Then $M \subset P T_{(X, Y)}$. Let $\beta \in P T_{(X, Y)}$. Then

$$
\beta=\left(\begin{array}{cc}
A_{i} & B_{j} \\
a_{i} & b_{j}
\end{array}\right)
$$

where $A_{i} \cap Y \neq \emptyset, B_{j} \subseteq X \backslash Y,\left\{a_{i}\right\}=Y$ and $\left\{b_{j}\right\} \subseteq X \backslash Y$. $(\operatorname{dom} \alpha \cap Y) \alpha=Y$ implies $y \alpha^{-1} \cap Y \neq \emptyset$ for all $y \in Y$. Then $a_{i} \alpha^{-1} \cap Y \neq \emptyset$ and $d_{a_{i}} \alpha=a_{i}$ for $d_{a_{i}} \in a_{i} \alpha^{-1} \cap Y$. Since $\operatorname{ran} \beta \subseteq \operatorname{ran} \alpha, b_{j} \in \operatorname{ran} \alpha$ and $b_{j}^{\prime} \alpha=b_{j}$ for somee $b_{j}^{\prime} \in \operatorname{dom} \alpha$. Define

$$
\gamma=\left(\begin{array}{cc}
A_{i} & B_{j} \\
d_{a_{i}} & b_{j}^{\prime}
\end{array}\right)
$$

Then $\gamma \in P T_{(X, Y)}$. Since $\operatorname{ran} \gamma \subseteq \operatorname{dom} \alpha \neq X, \gamma$ is not surjective. Thus $\gamma \in M$. Consequently, $x(\gamma \alpha)=(x \gamma) \alpha=x \beta$ for all $x \in \operatorname{dom} \beta=\operatorname{dom}(\gamma \alpha)$. Hence $\gamma \alpha=\beta$ and $P T_{(X, Y)} \subseteq M \alpha$ which gives $M \alpha=P T_{(X, Y)}$.

CASE 2: $\alpha$ is not injective. Choose $M=\left\{\delta \in P T_{(X, Y)} \mid \delta\right.$ is not surjective $\}$. Then $M \subset P T_{(X, Y)}$. Let $\beta \in P T_{(X, Y)}$. Then

$$
\beta=\left(\begin{array}{cc}
A_{i} & B_{j} \\
a_{i} & b_{j}
\end{array}\right)
$$

where $A_{i} \cap Y \neq \emptyset ; B_{j} \subseteq X \backslash Y ;\left\{a_{i}\right\}=Y$ and $\left\{b_{j}\right\} \subseteq X \backslash Y$. Let $\gamma \in P T_{(X, Y)}$ be as in Case 1. Since $\alpha$ is not injective, there is $x_{0} \in \operatorname{ran} \alpha$ and distinct elements $x_{1}, x_{2} \in \operatorname{dom} \alpha$ such that $x_{1} \alpha=x_{0}=x_{2} \alpha$. Obviously $\operatorname{ran} \beta \subseteq \operatorname{ran} \alpha$. If $x_{0} \in \operatorname{ran} \beta$, then there is exactly one (either $x_{1}$ or $x_{2}$ ) in $\operatorname{ran} \gamma$. If $x_{0} \notin \operatorname{ran} \beta$, then $x_{1}, x_{2} \notin$ $\operatorname{ran} \gamma$. Thus $\gamma$ is not surjective and so $\gamma \in M$. Analogously as in Case 1, we obtain $\gamma \alpha=\beta$. Hence $P T_{(X, Y)} \subseteq M \alpha$. This means that $M \alpha=P T_{(X, Y)}$. Therefore, $\alpha$ is a right magnifying element in $P T_{(X, Y)}$.

Corollary 6.4. $\alpha \in E(X)$ is a right magnifying element in $E(X)$ if and only if $\operatorname{dom} \alpha \neq X$ or $\alpha$ is not injective.

Example 6.5. Let $X=\mathbb{N}$ and $Y=2 \mathbb{N}$. Define

$$
\alpha=\left(\begin{array}{cc}
2 n & 2 n+1 \\
2 n & 2 n-1
\end{array}\right)_{n \in \mathbb{N}} \text { and } \beta=\left(\begin{array}{ccccc}
1 & 2 & \{3,4\} & \{5,6\} & n+2 \\
1 & 2 & 3 & 4 & n
\end{array}\right)_{n \geqslant 5} .
$$

Then $(\operatorname{dom} \alpha \cap Y) \alpha=2 \mathbb{N}=(\operatorname{dom} \beta \cap Y) \beta$ and so $\alpha, \beta \in P T_{(X, Y)}$. It is clear that $\alpha$ is injective. Since $\operatorname{ran} \alpha=\mathbb{N}=X, \alpha$ is surjective. but $\operatorname{dom} \alpha=\mathbb{N} \backslash\{1\} \neq X$, so $\operatorname{dom} \beta=\mathbb{N}=X$ and $\beta$ is surjective but not injective. By Theorem 6.3, $\alpha, \beta$ are right magnifying elements in $P T_{(X, Y)}$. Then there is $M=\left\{\delta \in P T_{(X, Y)} \mid \delta\right.$ is not surjective $\} \subset P T_{(X, Y)}$ such that $M \alpha=P T_{(X, Y)}$ and $M \beta=P T_{(X, Y)}$.

Added in proof (January 5, 2021). One of the Reviewers informed us that the results of our Sections 3 and 4 are similar to results obtained in the paper: R. Chinram, S. Buapradist, N. Yaqoob, P. Petchkaew, Left and right magnifying elements in some generalized partial transformation semigroups, submitted to Commun. Algebra, but the proofs are different.

## References

[1] M. Gutan, Semigroups with strong and non-trong magnifying elements, Semigroup Forum 53 (1966), no. 3, $384-386$.
[2] M. Gutan, Semigroups which contain magnifying elements are factorizable, Commun. Algebra 25 (1997), no. 12, 3953 - 3963.
[3] M. Gutan, Good and very good magnifiers, Bollettio dell' Unione Matematica Italiana 3 (2000), no. 3, $793-810$.
[4] E.S. Ljapin, Semigroups, Amer. Math. Soc.: Providence, R. I., USA, 1974.
[5] E. S. Ljapin, Translations of Mathematical Monographs Vol.3, Semigroups, Amer. Math. Soc.: Providence, R. I., USA, 1963.
[6] P. Luangchaisri, T. Changpas, C. Phanlert, Left (right) magnifying elements of a partial transformation semigroup, Asian-Eur. J. Math. 13 (2020), no. 1, 7 pp.

Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, Thailand E-mails: Chadaphorn_p@cmu.ac.th, preeyanuch.h@cmu.ac.th

# Characterizations of regularities on ordered semirings by idempotency of ordered ideals 

Kongpop Siribute, Pakorn Palakawong na Ayutthaya<br>and Jatuporn Eanborisoot


#### Abstract

We characterize regular, intra-regular, left weakly regular, right weakly regular and fully idempotent ordered semirings using idempotency of several kinds of ordered ideals including left ordered ideals, right ordered ideals, ordered ideals, ordered quasi-ideals, ordered bi-ideals and ordered interior ideals. Moreover, we characterize $(m, n)$-regular ordered semirings in terms of their ordered ( $m, n$ )-ideals.


## 1. Introduction

The notion of a regular semiring was defined as a similar way of a regular ring defined by von Neumann [6], i.e., for each element $a$ of a semiring $S, a=a x a$ for some $x \in S$ (equivalently, $a \in a S a$ for all $a \in S$ ). Later, in sense of Ahsan, Mordeson and Shabir [2], a semiring $S$ is called intra-regular if for each $a \in S$, $a=\sum_{i \in I} x_{i} a^{2} y_{i}$ for some $x_{i}, y_{i} \in S$ and finite index set $I$. This notion is equivalent to $a \in \Sigma S a^{2} S$ for all $a \in S$ where $\Sigma S a^{2} S$ is the set of all finite sums of elements in $S a^{2} S$. In 1993, Ahsan [1] called a semiring $S$ to be fully idempotent if every ideal of $S$ is idempotent. We are able to study the fully idempotency of a semiring as a kind of regularities due to the fact that a semiring $S$ is fully idempotent if and only if $a \in \Sigma S a S a S$ for all $a \in S$. Later, Shabir and Anjum [12] studied the concept of a right $k$-weakly regular hemiring in terms of its fuzzy ideals. They defined a hemiring to be right $k$-weakly regular if $a \in \overline{\Sigma a S a S}$ for all $a \in S$.

An ordered semiring is a notable generalization of a semiring, in other words, a semiring $S$ is an ordered semiring together with the relation $\{(x, x) \mid x \in S\}$. In sense of Gan and Jiang [4], an ordered semiring is a semiring $S$ together with a partial order on $S$ satisfying the compatibility property. In [4], the notion of an ordered ideal of an ordered semiring was defined. In 2012, Mandal [5] introduced the notion of a regular ordered semiring by for each $a \in S, a \leqslant a x a$ for some $x \in S$, i.e., $a \in(a S a]$ for all $a \in S$.

In this work, as generalizations of intra-regular semirings [2] and fully idempotent semirings [1], we give the notions of intra-regular ordered semirings and

[^5]fully idempotent ordered semirings. In addition, as a similar way of [12], we study the notion of left weakly regular and right weakly regular ordered semirings in the form $a \in(\Sigma S a S a]$ and $a \in(\Sigma a S a S]$ for all $a \in S$, respectively. Then, we use idempotency of left ordered ideals, right ordered ideals, ordered ideals, ordered quasi-ideals, ordered bi-ideals and ordered interior ideals to characterize mentioned kinds of regularities on ordered semirings. Moreover, we define an ordered $(m, n)$-ideals of an ordered semiring in a similar way of an $(m, n)$-ideals of an ordered semigroup defined by Sanborisoot and Changphas [11] and also study it on an ( $m, n$ )-regular ordered semiring as an analogous way on an $(m, n)$-regular ordered semigroup [3]. In conclusion, we have that the idempotency of each kind of ordered ideals of an ordered semiring is able to lead the ordered semiring to be different kinds of regularities.

## 2. Preliminaries

An ordered semiring [4] is a system $(S,+, \cdot, \leqslant)$ such that $(S,+, \cdot)$ is a semiring and $(S, \leqslant)$ is a poset satisfying the compatibility property, i.e., if $a \leqslant b$, then $a+c \leqslant b+c, c+a \leqslant c+b, a c \leqslant b c$ and $c a \leqslant c b$ for all $a, b, c \in S$. An element 0 of an ordered semiring $(S,+, \cdot, \leqslant)$ is called an absorbing zero if $x+0=x=0+x$ and $x 0=0=0 x$ for all $x \in S$.

Throughout this work, we simply write $S$ instead of an ordered semiring $(S,+, \cdot, \leqslant)$ and always assume that $S$ is additively commutative (i.e., $a+b=b+a$ for all $a, b \in S$ ) together with an absorbing zero 0 .

For $\emptyset \neq A, B \subseteq S$, we denote that $A+B=\{a+b \mid a \in A, b \in B\}$, $A B=\{a b \mid a \in A, b \in B\}$ and $(A]=\{x \in S \mid x \leqslant a$ for some $a \in S\}$. The set of all finite sums of elements in $\emptyset \neq A \subseteq S$ is denoted by $\Sigma A=$ $\left\{\sum_{i \in I} a_{i} \mid a_{i} \in A\right.$ and $I$ is a finitie set $\}$. If $I=\emptyset$, then we set $\sum_{i \in I} a_{i}=0$ for all $a_{i} \in S$.

For basic properties of the finite sums $\Sigma$ and the operator ( ], we refer to [8-10] However, we give the following useful remark which will be used in the main results.

Remark 2.1. Let $A$ and $B$ be nonempty subsets of an ordered semiring $S$. Then $(\Sigma(A](B]] \subseteq(\Sigma A B]$.

Definition 2.2. Let $A$ be a nonempty subset of an ordered semiring $S$ such that $A+A \subseteq A$ and $A=(A]$. Then $A$ is called:
(i) a left (right) ordered ideal [4] of $S$ if $S A \subseteq A(A S \subseteq A)$;
(ii) an ordered ideal [4] of $S$ if $A$ is both a left and a right ordered ideal of $S$;
(iii) an ordered quasi-ideal $[7]$ of $S$ if $(\Sigma A S] \cap(\Sigma S A] \subseteq A$;
(iv) an ordered bi-ideal of $S$ if $A^{2} \subseteq A$ and $A S A \subseteq A$;
(v) an ordered interior ideal of $S$ if $A^{2} \subseteq A$ and $S A S \subseteq A$.

Let $A$ be a nonempty subset of an ordered semiring $S$. We denote the notation $L(A), R(A), J(A), Q(A)$ and $I(A)$ to be the smallest left ordered ideals, right ordered ideals, ordered ideals, ordered quasi-ideals and ordered interior ideals of $S$ containing $A$, respectively. We recall constructions of $L(A), R(A) J(A)$ and $Q(A)$ which occur in [7] as follows.

Lemma 2.3. Let $A$ be a nonempty subset of an ordered semiring $S$. The following statements hold:
(i) $L(A)=(\Sigma A+\Sigma S A]$;
(ii) $R(A)=(\Sigma A+\Sigma A S]$;
(iii) $J(A)=(\Sigma A+\Sigma S A+\Sigma A S+\Sigma S A S]$;
(iv) $Q(A)=(\Sigma A+((\Sigma A S] \cap(\Sigma S A])]$.

Lemma 2.4. Let $A$ be a nonempty subset of an ordered semiring $S$. Then $I(A)=$ $\left(\Sigma A+\Sigma A^{2}+\Sigma S A S\right]$.

Proof. Let $\emptyset \neq A \subseteq S$. and $I=\left(\Sigma A+\Sigma A^{2}+\Sigma S A S\right]$. Clearly, $I+I \subseteq I, I=(I]$ and $A \subseteq I$. We have

$$
\begin{aligned}
I^{2} & =\left(\Sigma A+\Sigma A^{2}+\Sigma S A S\right]\left(\Sigma A+\Sigma A^{2}+\Sigma S A S\right] \\
& \subseteq\left(\left(\Sigma A+\Sigma A^{2}+\Sigma S A S\right)\left(\Sigma A+\Sigma A^{2}+\Sigma S A S\right)\right] \\
& \subseteq\left(\Sigma A^{2}+\Sigma S A S\right] \subseteq I \quad \text { and } \\
S I S & =S\left(\Sigma A+\Sigma A^{2}+\Sigma S A S\right] S \\
& \subseteq\left(S\left(\Sigma A+\Sigma A^{2}+\Sigma S A S\right) S\right] \\
& \subseteq\left(\Sigma S A S+\Sigma S A^{2} S+\Sigma S S A S S\right] \\
& \subseteq(\Sigma S A S+\Sigma S A S+\Sigma S A S]=(\Sigma S A S] \subseteq I
\end{aligned}
$$

So, $I$ is an ordered interior-ideal of $S$ containing $A$. If $J$ is an ordered interiorideal of $S$ containing $A$, then $I=\left(\Sigma A+\Sigma A^{2}+\Sigma S A S\right] \subseteq\left(\Sigma J+\Sigma J^{2}+\Sigma S J S\right] \subseteq$ $(\Sigma J+\Sigma J+\Sigma J]=(\Sigma J]=J$.

In a particular case of $A=\{a\}$ for some $a \in S$, we write $L(a), R(a), J(a)$, $Q(a)$ and $I(a)$ instead of $L(\{a\}), R(\{a\}), J(\{a\}), Q(\{a\})$ and $I(\{a\})$, respectively. The following corollary is directly obtained by Lemma 2.3 and 2.4.

Corollary 2.5. Let $a$ be an element of an ordered semiring $S$. The following statements hold:
(i) $L(a)=(\Sigma a+S a]$;
(ii) $R(a)=(\Sigma a+a S]$;
(iii) $J(a)=(\Sigma a+S a+a S+\Sigma S a S]$;
(iv) $Q(a)=(\Sigma a+((a S] \cap(S a])]$;
(v) $I(A)=\left(\Sigma a+\Sigma a^{2}+\Sigma S a S\right]$.

To define the notion of an ordered (m.n)-ideal of an ordered semiring $S$, for any $\emptyset \neq A, B \subseteq S$, we set $A^{m} B A^{0}=A^{m} B, A^{0} B A^{n}=B A^{n}$ and $A^{0} B A^{0}=B$ for all non-negative integers $m, n$.

Definition 2.6. Let $m$ and $n$ be non-negative integers. A subsemiring $A$ of an ordered semiring $S$ such that $A=(A]$ is called an ordered $(m, n)$-ideal of $S$ if $A^{m} S A^{n} \subseteq A$.

Clearly, $\emptyset \neq A \subseteq S$ is an ordered ( 0,0 )-ideal of $S$ if and only if $A=S$. It is easy to see that a left ordered ideal, a right ordered ideal and an ordered bi-ideal of an ordered semiring is an ordered ( 0,1 )-ideal, an ordered ( 1,0 )-ideal and an ordered ( 1,1 )-ideal, respectively.

For a nonempty subset $A$ of an ordered semiring $S$, we denote the notation $[A]_{(m, n)}$ to be the smallest ordered ( $m, n$ )-ideal of $S$ containing $A$.
Theorem 2.7. Let $A$ be a nonempty subset of an ordered semiring $S$. Then

$$
[A]_{(m, n)}=\left(\Sigma A+\Sigma A^{2}+\ldots+\Sigma A^{m+n}+\Sigma A^{m} S A^{n}\right]
$$

for all non-negative integers $m$ and $n$.
Proof. Let $\emptyset \neq A \subseteq S$ and $X=\left(\Sigma A+\Sigma A^{2}+\ldots+\Sigma A^{m+n}+\Sigma A^{m} S A^{n}\right]$. It is clear that $A \subseteq X \neq \emptyset, X=(X]$ and $X+X \subseteq X$. We obtain that

$$
\begin{aligned}
& X^{2}=\left(\Sigma A+\Sigma A^{2}+\ldots+\Sigma A^{m+n}+\Sigma A^{m} S A^{n}\right] \cdot\left(\Sigma A+\Sigma A^{2}+\ldots+\Sigma A^{m+n}+\Sigma A^{m} S A^{n}\right] \\
& \subseteq\left(\left(\Sigma A+\Sigma A^{2}+\ldots+\Sigma A^{m+n}+\Sigma A^{m} S A^{n}\right) \cdot\left(\Sigma A+\Sigma A^{2}+\ldots+\Sigma A^{m+n}+\Sigma A^{m} S A^{n}\right)\right] \\
& \subseteq\left(\Sigma A^{2}+\Sigma A^{3}+\ldots+\Sigma A^{m+n}+\Sigma A^{m+n+1}+\ldots+\Sigma A^{2 m+2 n}+\Sigma A^{m} S A^{n}\right] \\
& \subseteq\left(\Sigma A^{2}+\Sigma A^{3}+\ldots+\Sigma A^{m+n}+\Sigma A^{m} S A^{n}+\ldots+\Sigma A^{m} S A^{n}+\Sigma A^{m} S A^{n}\right] \\
&=\left(\Sigma A^{2}+\Sigma A^{3}+\ldots+\Sigma A^{m+n}+\Sigma A^{m} S A^{n}\right] \subseteq X \quad \text { and } \\
& X^{m} S X^{n}=\left(\left(\Sigma A+\Sigma A^{2}+\ldots+\Sigma A^{m+n}+\Sigma A^{m} S A^{n}\right]\right)^{m} S \\
& \cdot\left(\left(\Sigma A+\Sigma A^{2}+\ldots+\Sigma A^{m+n}+\Sigma A^{m} S A^{n}\right]\right)^{n} \\
& \subseteq\left(\left(\Sigma A+\Sigma A^{2}+\ldots+\Sigma A^{m+n}+\Sigma A^{m} S A^{n}\right)^{m}\right] S \\
& \cdot\left(\left(\Sigma A+\Sigma A^{2}+\ldots+\Sigma A^{m+n}+\Sigma A^{m} S A^{n}\right)^{n}\right] \\
& \subseteq\left(\Sigma A^{m}+\Sigma A^{m} S\right] S\left(\Sigma A^{n}+\Sigma S A^{n}\right] \\
& \subseteq\left(\Sigma A^{m} S+\Sigma A^{m} S\right]\left(\Sigma A^{n}+\Sigma S A^{n}\right] \\
&=\left(\Sigma A^{m} S\right]\left(\Sigma A^{n}+\Sigma S A^{n}\right] \subseteq\left(\left(\Sigma A^{m} S\right)\left(\Sigma A^{n}+\Sigma S A^{n}\right)\right] \\
& \subseteq\left(\Sigma A^{m} S A^{n}+\Sigma A^{m} S S A^{n}\right] \subseteq\left(\Sigma A^{m} S A^{n}+\Sigma A^{m} S A^{n}\right] \\
&=\left(\Sigma A^{m} S A^{n}+\Sigma A^{m} S A^{n}\right]=\left(\Sigma A^{m} S A^{n}\right] \subseteq X .
\end{aligned}
$$

Now, $X$ is an ordered $(m, n)$-ideal of $S$ containing $A$. Let $Y$ be an ordered $(m, n)$ ideal of $S$ containing $A$. Then $X=\left(\Sigma A+\Sigma A^{2}+\ldots+\Sigma A^{m+n}+\Sigma A^{m} S A^{n}\right] \subseteq(\Sigma Y+$ $\left.\Sigma Y^{2}+\ldots+\Sigma Y^{m+n}+\Sigma Y^{m} S Y^{n}\right] \subseteq(\Sigma Y+\Sigma Y+\ldots+\Sigma Y+\Sigma Y]=(\Sigma Y]=Y$.

In a particular case of $A=\{a\}$ for some $a \in S$, we write $[a]_{(m, n)}$ instead of $[\{a\}]_{(m, n)}$. The following corollary is directly obtained by Theorem 2.7.

Corollary 2.8. Let $a$ be an element of an ordered semiring $S$. Then

$$
[a]_{(m, n)}=\left(\Sigma a+\Sigma a^{2}+\ldots+\Sigma a^{m+n}+a^{m} S a^{n}\right]
$$

for all non-negative integers $m$ and $n$.
We immediately get that $[a]_{(1,1)}$ is the smallest ordered bi-ideal of $S$ containing $a$ for any elements $a$ of an ordered semiring $S$; accordingly, we use the notation $B(a)$ instead of $[a]_{(1,1)}$.
Corollary 2.9. Let $a$ be an element of an ordered semiring $S$. Then $B(a)=$ $\left(\Sigma a+\Sigma a^{2}+a S a\right]$.

## 3. Main Results

Definition 3.1. We call an ordered semiring $S$ to be:
(i) regular [5] if $a \in(a S a]$ for all $a \in S$;
(ii) intra-regular if $a \in\left(\Sigma S a^{2} S\right]$ for all $a \in S$;
(iii) left weakly regular if $a \in(\Sigma S a S a]$ for all $a \in S$;
(iv) right weakly regular if $a \in(\Sigma a S a S]$ for all $a \in S$.

Lemma 3.2. An ordered semiring $S$ is both regular and intra-regular if and only if $a \in\left(a S a^{2} S a\right]$ for all $a \in S$.

Proof. Assume that $S$ is both regular and intra-regular. Let $a \in S$. It follows that $a \in(a S a] \subseteq(a S(a S a]] \subseteq(a S a S a] \subseteq\left(a S\left(\Sigma S a^{2} S\right] S a\right] \subseteq\left(a\left(\Sigma S a^{2} S\right] a\right] \subseteq$ $\left(\left(\Sigma a S a^{2} S a\right]\right]=\left(a S a^{2} S a\right]$. Conversely, if $a \in\left(a S a^{2} S a\right]$ for all $a \in S$, then $a \in$ $\left(a S a^{2} S a\right] \subseteq(a S a]$ and $a \in\left(a S a^{2} S a\right] \subseteq\left(S a^{2} S\right] \subseteq\left(\Sigma S a^{2} S\right]$. Hence, $S$ is regular and intra-regular.

Definition 3.3. A nonempty subset $T$ of an ordered semiring $S$ is called idempotent if $T=\left(\Sigma T^{2}\right]$.

Definition 3.4. An ordered semiring $S$ is called fully idempotent if every ordered ideal of $S$ is idempotent.
Example 3.5. (i) ( $\mathbb{N},+, \cdot,=$ ) is an ordered semiring where $\mathbb{N}$ is the set of all natural numbers, + is the usual addition, $\cdot$ is the usual multiplication and $=$ is the equal relation. We have that the ordered ideal $2 \mathbb{N}$ is idempotent in ( $\mathbb{N},+, \cdot,=$ ).
(ii) $(\mathbb{N},+, \cdot, \leqslant)$ is an ordered semiring where $\leq$ is the natural ordered relation. Since $(\mathbb{N},+, \cdot, \leqslant)$ has no proper ordered ideal and $\mathbb{N}=\left(\Sigma \mathbb{N}^{2}\right]$, it is fully idempotent.

We characterize fully idempotent ordered semirings by idempotency of ordered ideals.

Theorem 3.6. The following statements are equivalent:
(i) $S$ is fully idempotent;
(ii) $J_{1} \cap J_{2}=\left(\Sigma J_{1} J_{2}\right]$ for all ordered ideals $J_{1}$ and $J_{2}$ of $S$;
(iii) $J(a) \cap J(b)=(\Sigma J(a) J(b)]$ for all $a, b \in S$;
(iv) $J(a)=\left(\Sigma(J(a))^{2}\right]$ for all $a \in S$;
(v) $a \in(\Sigma S a S a S]$ for all $a \in S$.

Proof. $(i) \Rightarrow\left(\right.$ ii). Let $J_{1}$ and $J_{2}$ be ordered ideals of $S$. Then $\left(\Sigma J_{1} J_{2}\right] \subseteq\left(\Sigma J_{1}\right]=J_{1}$ and $\left(\Sigma J_{1} J_{2}\right] \subseteq\left(\Sigma J_{2}\right]=J_{2}$. It follows that $\left(\Sigma J_{1} J_{2}\right] \subseteq J_{1} \cap J_{2}$. It is easy to show that $J_{1} \cap J_{2}$ is an ordered ideal of $S$. By ( $\left.i\right)$, we get $J_{1} \cap J_{2}=\left(\Sigma\left(J_{1} \cap J_{2}\right)^{2}\right]=$ $\left(\Sigma\left(J_{1} \cap J_{2}\right)\left(J_{1} \cap J_{2}\right)\right] \subseteq\left(\Sigma J_{1} J_{2}\right]$.

$$
\begin{aligned}
& (i i) \Rightarrow(i i i) . \text { and }(i i i) \Rightarrow(i v) \text { are obvious. } \\
& (i v) \Rightarrow(v) . \text { Let } a \in S . \text { By }(i v) \text {, we obtain that }
\end{aligned}
$$

$$
\begin{align*}
a & \in J(a)=\left(\Sigma(J(a))^{2}\right]=(\Sigma(\Sigma a+S a+a S+\Sigma S a S](\Sigma a+S a+a S+\Sigma S a S]] \\
& \subseteq(\Sigma(\Sigma a+S a+a S+\Sigma S a S)(\Sigma a+S a+a S+\Sigma S a S)] \\
& \subseteq\left(\Sigma\left(\Sigma a^{2}+a S a+a^{2} S+\Sigma a S a S+S a^{2}+\Sigma S a S a+\Sigma S a^{2} S+\Sigma S a S a S\right)\right] \\
& \subseteq\left(\Sigma a^{2}+a S a+a^{2} S+\Sigma a S a S+S a^{2}+\Sigma S a S a+\Sigma S a^{2} S+\Sigma S a S a S\right] \tag{1}
\end{align*}
$$

Using equation (1), we get that

$$
\begin{align*}
& a \in(a S+\Sigma S a S+\Sigma S a S a S]  \tag{2}\\
& a \in(S a+\Sigma S a S+\Sigma S a S a S] \tag{3}
\end{align*}
$$

Using equations (2) and (3), we get that

$$
\begin{align*}
a^{2}=a a & \in(a S+\Sigma S a S+\Sigma S a S a S](S a+\Sigma S a S+\Sigma S a S a S] \\
& \subseteq((a S+\Sigma S a S+\Sigma S a S a S)(S a+\Sigma S a S+\Sigma S a S a S)] \\
& \subseteq(a S a+\Sigma a S a S+\Sigma S a S a+\Sigma S a S a S] . \tag{4}
\end{align*}
$$

Using equations (2) and (3) again, we get that

$$
\begin{align*}
a S a & \subseteq(S a+\Sigma S a S+\Sigma S a S a S] S(a S+\Sigma S a S+\Sigma S a S a S] \\
& \subseteq((S a+\Sigma S a S+\Sigma S a S a S) S](a S+\Sigma S a S+\Sigma S a S a S] \\
& \subseteq(\Sigma S a S+\Sigma S a S a S](a S+\Sigma S a S+\Sigma S a S a S] \\
& \subseteq((\Sigma S a S+\Sigma S a S a S)(a S+\Sigma S a S+\Sigma S a S a S)] \\
& \subseteq(\Sigma S a S a S] . \tag{5}
\end{align*}
$$

Using equation (2), we get that

$$
\begin{equation*}
S a S a \subseteq S a S(a S+\Sigma S a S+\Sigma S a S a S] \subseteq(\Sigma S a S a S] \tag{6}
\end{equation*}
$$

Using equation (3), we get that

$$
\begin{equation*}
a S a S \subseteq(S a+\Sigma S a S+\Sigma S a S a S] S a S \subseteq(\Sigma S a S a S] \tag{7}
\end{equation*}
$$

Using equations (4), (5), (6) and (7) we get that

$$
\begin{align*}
a^{2} & \in(a S a+\Sigma a S a S+\Sigma S a S a+\Sigma S a S a S] \\
& \subseteq((\Sigma S a S a S]+\Sigma(\Sigma S a S a S]+\Sigma(\Sigma S a S a S]+\Sigma S a S a S] \\
& \subseteq((\Sigma S a S a S]+(\Sigma S a S a S]+(\Sigma S a S a S]+(\Sigma S a S a S]] \\
& \subseteq((\Sigma S a S a S+\Sigma S a S a S+\Sigma S a S a S+\Sigma S a S a S]] \\
& =(\Sigma S a S a S] . \tag{8}
\end{align*}
$$

Using equation (8), we get that

$$
\begin{align*}
& a^{2} S \subseteq(\Sigma S a S a S] S \subseteq(\Sigma S a S a S]  \tag{9}\\
& S a^{2} \subseteq S(\Sigma S a S a S] \subseteq(\Sigma S a S a S]  \tag{10}\\
& S a^{2} S \subseteq S(\Sigma S a S a S] S \subseteq(\Sigma S a S a S] . \tag{11}
\end{align*}
$$

Using equations (1) and (5)-(11), we obtain that

$$
\begin{aligned}
a \in & \left(\Sigma a+a S a+a^{2} S+\Sigma a S a S+S a^{2}+\Sigma S a S a+\Sigma S a^{2} S+\Sigma S a S a S\right] \\
\subseteq & (\Sigma(\Sigma S a S a S]+(\Sigma S a S a S]+(\Sigma S a S a S]+\Sigma(\Sigma S a S a S] \\
& +(\Sigma S a S a S]+\Sigma(\Sigma S a S a S]+\Sigma(\Sigma S a S a S]+\Sigma S a S a S] \\
\subseteq & ((\Sigma S a S a S]+(\Sigma S a S a S]+(\Sigma S a S a S]+(\Sigma S a S a S] \\
& +(\Sigma S a S a S]+(\Sigma S a S a S]+(\Sigma S a S a S]+(\Sigma S a S a S]] \\
\subseteq & ((\Sigma S a S a S+\Sigma S a S a S+\Sigma S a S a S+\Sigma S a S a S \\
& +\Sigma S a S a S+\Sigma S a S a S+\Sigma S a S a S+\Sigma S a S a S]] \\
= & (\Sigma S a S a S] .
\end{aligned}
$$

$(v) \Rightarrow(i)$. Let $J$ be an ordered ideal of $S$. Clearly, $\left(\Sigma J^{2}\right] \subseteq(\Sigma J]=J$. If $x \in J$, then by $(v)$, we get $x \in(\Sigma S x S x S] \subseteq(\Sigma S J S J S] \subseteq(\Sigma J J]=\left(\Sigma J^{2}\right]$ and so $J \subseteq\left(\Sigma J^{2}\right]$. Hence, $J=\left(\Sigma J^{2}\right]$ and thus $S$ is fully idempotent.

In general, every ordered ideal of an ordered semiring is an ordered interior ideal but not conversely [7]. However, they are same in fully idempotent ordered semirings.

Proposition 3.7. Ordered ideals and ordered interior ideals coincide in fully idempotent ordered semirings.
Proof. Let $I$ be an ordered interior ideal of an ordered semiring $S$. Assume that $S$ is fully idempotent. If $x \in I S$, then by Theorem $3.6, x \in(\Sigma S x S x S] \subseteq(\Sigma S x S] \subseteq$ $(\Sigma S(I S) S] \subseteq(\Sigma S I S] \subseteq(\Sigma I]=I$. Similarly, we have that $S I \subseteq I$. Hence, $I$ is an ordered ideal of $S$.

Using Theorem 3.6 and Proposition 3.7, we directly obtain the following corollary as characterizations of fully idempotent ordered semirings by idempotency of ordered interior ideals.

Corollary 3.8. The following statements are equivalent:
(i) $S$ is fully idempotent;
(ii) $I_{1} \cap I_{2}=\left(\Sigma I_{1} I_{2}\right]$ for all ordered interior ideals $I_{1}$ and $I_{2}$ of $S$;
(iii) $I(a) \cap I(b)=(\Sigma I(a) I(b)]$ for all $a, b \in S$;
(iv) $I(a)=\left(\Sigma(I(a))^{2}\right]$ for all $a \in S$.

Now, we use idempotency of ordered quasi-ideals to characterize an ordered semiring which is both regular and intra-regular.

Theorem 3.9. The following statements are equivalent:
(i) $S$ is both regular and intra-regular;
(ii) every ordered quasi-ideal of $S$ is idempotent, i.e., $Q=\left(\Sigma Q^{2}\right]$ for all ordered quasi-ideals $Q$ of $S$;
(iii) $Q(a)=\left(\Sigma(Q(a))^{2}\right]$ for all $a \in S$.

Proof. $(i) \Rightarrow(i i)$. Assume that $S$ is both regular and intra-regular. Let $Q$ be an ordered quasi-ideal of $S$. Obviously, $\left(\Sigma Q^{2}\right] \subseteq(\Sigma Q]=Q$ (every ordered quasiideal is always a subsemiring [7]). If $x \in Q$, then using Remark 3.2, we get $x \in\left(x S x^{2} S x\right]=(x S x x S x] \subseteq(Q S Q Q S Q] \subseteq(Q Q]=\left(Q^{2}\right] \subseteq\left(\Sigma Q^{2}\right]$ (every ordered quasi-ideal is an ordered bi-ideal [7] and so $Q S Q \subseteq Q$ for every ordered quasi-ideal $Q$ of an ordered semiring $S)$. Hence, $Q=\left(\Sigma Q^{2}\right]$ and so $Q$ is idempotent.
$(i i) \Rightarrow(i i i)$. It is obvious.
$(i i i) \Rightarrow(i)$. Assume that (iii) holds and let $a \in S$. Since every left (right) ordered ideal is an ordered quasi-ideal [7], we obtain that

$$
\begin{align*}
& a \in Q(a)=(\Sigma Q(a) Q(a)] \subseteq(\Sigma R(a) L(a)]  \tag{12}\\
& a \in Q(a)=(\Sigma Q(a) Q(a)] \subseteq(\Sigma L(a) R(a)] \tag{13}
\end{align*}
$$

We consider equation (12). Using Corollary 2.5, it turns out that

$$
\begin{align*}
a & \in(\Sigma R(a) L(a)]=(\Sigma(\Sigma a+a S](\Sigma a+S a]] \\
& \subseteq(\Sigma(\Sigma a+a S)(\Sigma a+S a)] \subseteq\left(\Sigma\left(\Sigma a^{2}+a S a\right)\right] \\
& \subseteq\left(\Sigma a^{2}+a S a\right] \tag{14}
\end{align*}
$$

Using equation (14), we get that

$$
a^{2}=a a \in a\left(\Sigma a^{2}+a S a\right] \subseteq\left(\Sigma a^{3}+a S a\right] \subseteq(\Sigma a S a+a S a]=(a S a+a S a]=(a S a]
$$

Using equation (14) again, we obtain that

$$
\left.\begin{array}{rl}
a & \in\left(\Sigma a^{2}+a S a\right] \subseteq(\Sigma(a S a]+a S a] \subseteq((\Sigma a S a]+a S a] \\
& =((a S a]+a S a] \subseteq((a S a+a S a]]
\end{array}\right)=(a S a] . \quad .
$$

Now, $S$ is regular. We consider equation (13). Using Corollary 2.5. We get

$$
\begin{align*}
a & \in(\Sigma L(a) R(a)]=(\Sigma(\Sigma a+S a](\Sigma a+a S]] \\
& \subseteq(\Sigma(\Sigma a+S a)(\Sigma a+a S)] \subseteq\left(\Sigma\left(\Sigma a^{2}+\Sigma a^{2} S+\Sigma S a^{2}+\Sigma S a^{2} S\right)\right] \\
& \subseteq\left(\Sigma a^{2}+\Sigma a^{2} S+\Sigma S a^{2}+\Sigma S a^{2} S\right]=\left(\Sigma a^{2}+a^{2} S+S a^{2}+\Sigma S a^{2} S\right] \tag{15}
\end{align*}
$$

Using equation (15), we get that

$$
\begin{align*}
a^{2}=a a & \in\left(\Sigma a^{2}+a^{2} S+S a^{2}+\Sigma S a^{2} S\right]\left(\Sigma a^{2}+a^{2} S+S a^{2}+\Sigma S a^{2} S\right] \\
& \subseteq\left(\left(\Sigma a^{2}+a^{2} S+S a^{2}+\Sigma S a^{2} S\right)\left(\Sigma a^{2}+a^{2} S+S a^{2}+\Sigma S a^{2} S\right)\right] \\
& \subseteq\left(\Sigma a^{4}+\Sigma S a^{2} S\right] \subseteq\left(\Sigma S a^{2} S+\Sigma S a^{2} S\right]=\left(\Sigma S a^{2} S\right] \tag{16}
\end{align*}
$$

Using equation (16), we get that

$$
\begin{equation*}
a^{2} S \subseteq\left(\Sigma S a^{2} S\right] S \subseteq\left(\Sigma S a^{2} S\right] \text { and } S a^{2} \subseteq S\left(\Sigma S a^{2} S\right] \subseteq\left(\Sigma S a^{2} S\right] \tag{17}
\end{equation*}
$$

Using equations (15), (16) and (17), we have that

$$
\begin{aligned}
a & \in\left(\Sigma a^{2}+a^{2} S+S a^{2}+\Sigma S a^{2} S\right] \\
& \subseteq\left(\Sigma\left(\Sigma S a^{2} S\right]+\left(\Sigma S a^{2} S\right]+\left(\Sigma S a^{2} S\right]+\Sigma S a^{2} S\right] \\
& \subseteq\left(\left(\Sigma S a^{2} S\right]+\left(\Sigma S a^{2} S\right]+\left(\Sigma S a^{2} S\right]+\left(\Sigma S a^{2} S\right]\right] \\
& \subseteq\left(\Sigma S a^{2} S+\Sigma S a^{2} S+\Sigma S a^{2} S+\Sigma S a^{2} S\right]=\left(\Sigma S a^{2} S\right]
\end{aligned}
$$

Now, $S$ is intra-regular. Therefore, $S$ is both regular and intra-regular.
In general, every ordered quasi-ideal of an ordered semiring is an ordered biideal but not conversely [7]. However, they coincide in regular ordered semirings. Using this fact and Theorem 3.9, we obtain the following corollary as characterizations of an ordered semiring which is both regular and intra-regular by idempotency of ordered bi-ideals.

Corollary 3.10. The following statements are equivalent:
(i) $S$ is both regular and intra-regular;
(ii) every ordered bi-ideal of $S$ is idempotent, i.e., $B=\left(\Sigma B^{2}\right]$ for all ordered bi-ideals $B$ of $S$;
(iii) $B(a)=\left(\Sigma(B(a))^{2}\right]$ for all $a \in S$.

Now, we use idempotency of left ordered ideals to characterize a left weakly regular ordered semiring.

Theorem 3.11. The following statements are equivalent:
(i) $S$ is left weakly regular;
(ii) every left ordered ideal of $S$ is idempotent, i.e., $L=\left(\Sigma L^{2}\right]$ for all left ordered ideals $L$ of $S$;
(iii) $L(a)=\left(\Sigma(L(a))^{2}\right]$ for all $a \in S$.

Proof. $(i) \Rightarrow($ ii $)$. Let $L$ be a left ordered ideal of $S$. Clearly, $\left(\Sigma L^{2}\right] \subseteq(\Sigma L]=L$. If $x \in L$, then by $(i)$, we get that $x \in(\Sigma S a S a] \subseteq(\Sigma S L S L] \subseteq(\Sigma L L]=\left(\Sigma L^{2}\right]$. Hence, $L=\left(\Sigma L^{2}\right]$.
$(i i) \Rightarrow(i i i)$. It is obvious.
(iii) $\Rightarrow(i)$. Let $a \in S$. Using Corollary 2.5, we obtain that

$$
\begin{align*}
a & \in L(a)=(\Sigma L(a) L(a)]=(\Sigma(\Sigma a+S a](\Sigma a+S a]] \\
& \subseteq(\Sigma(\Sigma a+S a)(\Sigma a+S a)] \subseteq\left(\Sigma\left(\Sigma a^{2}+a S a+S a^{2}+\Sigma S a S a\right)\right] \\
& \subseteq\left(\Sigma a^{2}+a S a+S a^{2}+\Sigma S a S a\right] \tag{18}
\end{align*}
$$

Using equation (18), we get that

$$
\begin{equation*}
a \in\left(\Sigma a^{2}+a S a+S a^{2}+\Sigma S a S a\right] \subseteq(S a+\Sigma S a S a] . \tag{19}
\end{equation*}
$$

Using equation (19), we get that

$$
\begin{align*}
a^{2} & =a a \in(S a+\Sigma S a S a](S a+\Sigma S a S a] \\
& \subseteq((S a+\Sigma S a S a)(S a+\Sigma S a S a)] \subseteq(\Sigma S a S a] . \tag{20}
\end{align*}
$$

Using equation (19) again, we get that

$$
\begin{align*}
a S a & \subseteq(S a+\Sigma S a S a] S(S a+\Sigma S a S a] \subseteq((S a+\Sigma S a S a) S](S a+\Sigma S a S a] \\
& \subseteq(\Sigma S a S](S a+\Sigma S a S a] \subseteq((\Sigma S a S)(S a+\Sigma S a S a)] \subseteq(\Sigma S a S a] \tag{21}
\end{align*}
$$

Using equation (20), we get that

$$
\begin{equation*}
S a^{2} \subseteq S(\Sigma S a S a] \subseteq(S(\Sigma S a S a)] \subseteq(\Sigma S a S a] \tag{22}
\end{equation*}
$$

Using equations (18) and (20)-(22), it turns out that

$$
\begin{aligned}
a & \in\left(\Sigma a^{2}+a S a+S a^{2}+\Sigma S a S a\right] \\
& \subseteq(\Sigma(\Sigma S a S a]+(\Sigma S a S a]+(\Sigma S a S a]+\Sigma S a S a] \\
& \subseteq((\Sigma S a S a]+(\Sigma S a S a]+(\Sigma S a S a]+(\Sigma S a S a]] \\
& \subseteq((\Sigma S a S a+\Sigma S a S a+\Sigma S a S a+\Sigma S a S a]]=(\Sigma S a S a] .
\end{aligned}
$$

Therefore, $S$ is left weakly regular.
As a duality of Theorem 3.11, we obtain characterizations of right weakly regular ordered semirings in terms of idempotency of right ordered ideals analogously.

Theorem 3.12. The following statements are equivalent:
(i) $S$ is right weakly regular;
(ii) every right ordered ideal of $S$ is idempotent, i.e., $R=\left(\Sigma R^{2}\right]$ for all right ordered ideals $R$ of $S$;
(iii) $R(a)=\left(\Sigma(R(a))^{2}\right]$ for all $a \in S$.

To define the notion of an $(m, n)$-regular ordered semiring, for any elements $a$ and for any nonempty subsets $B$ of an ordered semiring $S$, we set $a^{m} B a^{0}=a^{m} B$, $a^{0} B a^{n}=B a^{n}$ and $a^{0} B a^{0}=B$ for all non-negative integers $m$ and $n$.

Definition 3.13. Let $m$ and $n$ be non-negative integers. An ordered semiring $S$ is called $(m, n)$-regular if $a \in\left(a^{m} S a^{n}\right]$ for all $a \in S$.

Theorem 3.14. Let $S$ be an ordered semiring and $m$, $n$ be positive integers. Then the following statements hold:
(i) $S$ is $(m, 0)$-regular if and only if $R=\left(R^{m} S\right]$ for each ordered $(m, 0)$-ideal $R$ of $S$;
(ii) $S$ is $(0, n)$-regular if and only if $L=\left(S L^{n}\right]$ for each ordered $(0, n)$-ideal $L$ of $S$.
Proof. (i). Assume that $S$ is ( $m, 0$ )-regular. Let $R$ be an ordered ( $m, 0$ )-ideal of $S$. If $a \in R$, then $a \in\left(a^{m} S\right] \subseteq\left(R^{m} S\right]$ implies $R \subseteq\left(R^{m} S\right]$. Clearly, $\left(R^{m} S\right] \subseteq R$. Hence, $R=\left(R^{m} S\right]$.

Conversely. let $a \in S$. By assumption and Corollary 2.8, we obtain that

$$
\begin{aligned}
a \in[a]_{(m, 0)} & =\left(\left([a]_{(m, 0)}\right)^{m} S\right]=\left(\left(\left(\Sigma a+\Sigma a^{2}+\ldots+\Sigma a^{m}+a^{m} S\right]\right)^{m} S\right] \\
& \subseteq\left(\left(\left(\Sigma a+\Sigma a^{2}+\ldots+\Sigma a^{m}+a^{m} S\right)^{m}\right] S\right] \\
& \subseteq\left(\left(\left(\Sigma a+\Sigma a^{2}+\ldots+\Sigma a^{m}+a^{m} S\right)^{m} S\right]\right] \\
& =\left(\left(\Sigma a+\Sigma a^{2}+\ldots+\Sigma a^{m}+a^{m} S\right)^{m} S\right] \\
& \subseteq\left(\left(\Sigma a^{m}+\Sigma a^{m} S\right) S\right] \subseteq\left(\Sigma a^{m} S+\Sigma a^{m} S\right]=\left(\Sigma a^{m} S\right]=\left(a^{m} S\right] .
\end{aligned}
$$

Therefore, $S$ is ( $m, 0$ )-regular.
(ii). It can be proved in a similar way of $(i)$.

It is not interesting to characterize a $(0,0)$-regular ordered semiring $S$ because $a \in\left(a^{0} S a^{0}\right]=(S]=S$ for all $a \in S$. Consequently, we obtain the following theorem as a characterization of an $(m, n)$-regular ordered semiring where $m$ and $n$ are not being zero at the same time.

Theorem 3.15. Let $m$ and $n$ be non-negative integers where $m$ and $n$ are not being zero at the same time. An ordered semigroup $S$ is $(m, n)$-regular if and only if $R \cap L=\left(R^{m} L^{n}\right]$ for every ordered ( $m, 0$ )-ideal $R$ and every ordered $(0, n)$-ideal $L$ of $S$.

Proof. The case of $m \neq 0, n=0$ and $m=0, n \neq 0$ is directly follows from Theorem 3.14(i) and (ii), respectively. Hence, we assume that $m \neq 0$ and $n \neq 0$.

Assume that $S$ is $(m, n)$-regular. Let $R$ and $L$ be an ordered ( $m, 0$ )-ideal and an ordered ( $0, n$ )-ideal of $S$, respectively. If $x \in R \cap L$, then by assumption, $x \in\left(a^{m} S a^{n}\right] \subseteq\left(R^{m} S L^{n}\right] \subseteq\left(R^{m} L^{n}\right]$ implies $R \cap L \subseteq\left(R^{m} L^{m}\right]$. Clearly, $\left(R^{m} L^{n}\right] \subseteq$ $R \cap L$. Hence, $R \cap L=\left(R^{m} L^{n}\right]$.

Conversely, let $a \in S$. Then by assumption and Corollary 2.8, we get

$$
\begin{aligned}
a & \in[a]_{(m, 0)} \cap[a]_{(0 . n)}=\left([a]_{(m, 0)}^{m}[a]_{(0 . n)}^{n}\right] \\
& =\left(\left(\Sigma a+\Sigma a^{2}+\ldots+\Sigma a^{m}+a^{m} S\right]^{m}\left(\Sigma a+\Sigma a^{2}+\ldots+\Sigma a^{n}+S a^{n}\right]^{n}\right] \\
& \left.\left.\subseteq\left(\left(\Sigma a+\Sigma a^{2}+\ldots+\Sigma a^{m}+a^{m} S\right)^{m}\right]\left(\Sigma a+\Sigma a^{2}+\ldots+\Sigma a^{n}+S a^{n}\right)^{n}\right]\right] \\
& \left.\left.\subseteq\left(\left(\left(\Sigma a+\Sigma a^{2}+\ldots+\Sigma a^{m}+a^{m} S\right)^{m}\right)\left(\Sigma a+\Sigma a^{2}+\ldots+\Sigma a^{n}+S a^{n}\right)^{n}\right)\right]\right] \\
& \left.\left.=\left(\left(\Sigma a+\Sigma a^{2}+\ldots+\Sigma a^{m}+a^{m} S\right)^{m}\right)\left(\Sigma a+\Sigma a^{2}+\ldots+\Sigma a^{n}+S a^{n}\right)^{n}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
& \subseteq\left(\left(\Sigma a^{m}+a^{m} S\right)\left(\Sigma a^{n}+S a^{n}\right)\right] \subseteq\left(\Sigma a^{m+n}+a^{m} S a^{n}\right] \\
& \subseteq\left(\Sigma a^{m+n-1}\left(\Sigma a^{m+n}+a^{m} S a^{n}\right]+a^{m} S a^{n}\right] \subseteq\left(\Sigma\left(a^{m} S a^{n}+a^{m} S a^{n}\right]+a^{m} S a^{n}\right] \\
& \subseteq\left(\left(a^{m} S a^{n}\right]+a^{m} S a^{n}\right] \subseteq\left(\left(a^{m} S a^{n}+a^{m} S a^{n}\right]\right]=\left(a^{m} S a^{n}\right] .
\end{aligned}
$$

Therefore, $S$ is $(m, n)$-regular.

## References

[1] J. Ahsan, Fully idempotent semirings, Proc. Japan Acad. 69 (1993), no. 6, 185-188.
[2] J. Ahsan, J.N. Mordeson and M. Shabir, Fuzzy semirings with applications to automata theory, Studies in Fuzziness and Soft Computing, Springer, 2012.
[3] L. Bussaban and T. Changphas, On $(m, n)$-ideals and $(m, n)$-regular ordered semigroups, Songklanakarin J. Sci. Technol. 38 (2016), no. 2, 199 - 206.
[4] A.P. Gan and Y.L. Jiang, On ordered ideals in ordered semirings, J. Math. Res. Exposition 31 (2011), no. 6, 989 - 996.
[5] D. Mandal, Fuzzy ideals and fuzzy interior ideals in ordered semirings, Fuzzy Inf. Eng. 6 (2014), no. 1, 101 - 114.
[6] J. von Neumann, On regular rings, Proc. Natl. Acad. Sci. USA 22 (1936), 707 113.
[7] P. Palakawong na Ayuthaya and B. Pibaljommee, Characterizations of regular ordered semirings by ordered quasi-ideals, Int. J. Math. Math. Sci. 2016 (2016), Article ID. 4272451.
[8] P. Palakawong na Ayuthaya and B. Pibaljommee, Characterizations of ordered $k$-regular semirings by ordered quasi $k$-ideals, Quasigroups Related Systems 25 (2017), $109-120$.
[9] P. Palakawong na Ayuthaya and B. Pibaljommee, Characterizations of ordered intra $k$-regular semirings by ordered $k$-ideals, Commun. Korean Math. Soc. 33 (2018), no. 1, 1-12.
[10] P. Palakawong na Ayuthaya and B. Pibaljommee, Characterizations of completely ordered $k$-regular semirings, Songklanakarin J. Sci. Technol. 41 (2019), no. 3, 501 - 505.
[11] J. Sanborisoot and T. Changphas, On characterizations of ( $m, n$ )-regular ordered semigroups, Far East J. Math. Sci. 65 (2012), no. 1, $75-86$.
[12] M. Shabir and R. Anjum, Right $k$-weakly regular hemirings, Quasigroups Related Systems 20 (2012), no. 1, $97-112$.

Received August 03, 2020
K. Siribute

Department of Curriculum and Instruction (Mathematics), Faculty of Education, Sakon Nakhon Rajabhat University, Sakon Nakhon, Thailand, 47000, E-ail: ozilthaipu@gmail.com
P. Palakawong na Ayutthaya

Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen, Thailand, E-mail:pakorn1702@gmail.com
J. Eanborisoot

Department of Mathematics, Faculty of Science, Mahasarakham University, Mahasarakham, Thailand, 44150, E-mail: jatuporn.san@msu.ac.th

# Half-isomorphisms whose inverses are also half-isomorphisms 

Giliard Souza dos Anjos


#### Abstract

Let $(G, *)$ and $\left(G^{\prime}, \cdot\right)$ be groupoids. A bijection $f: G \rightarrow G^{\prime}$ is called a halfisomorphism if $f(x * y) \in\{f(x) \cdot f(y), f(y) \cdot f(x)\}$, for any $x, y \in G$. A half-isomorphism of a groupoid onto itself is a half-automorphism. A half-isomorphism $f$ is called special if $f^{-1}$ is also a half-isomorphism. In this paper, necessary and sufficient conditions for the existence of special half-isomorphisms on groupoids and quasigroups are obtained. Furthermore, some examples of non-special half-automorphisms for loops of infinite order are provided.


## 1. Introduction

A groupoid consists of a nonempty set with a binary operation. A groupoid $(Q, *)$ is called a quasigroup if for each $a, b \in Q$ the equations $a * x=b$ and $y * a=b$ have unique solutions for $x, y \in Q$. A quasigroup $(L, *)$ is a loop if there exists an identity element $1 \in L$ such that $1 * x=x=x * 1$, for any $x \in L$. The fundamental definitions and facts from groupoids, quasigroups, and loops can be found in [1, 14].

Let $(G, *)$ and $\left(G^{\prime}, \cdot\right)$ be groupoids. A bijection $f: G \rightarrow G^{\prime}$ is called a half-isomorphism if $f(x * y) \in\{f(x) \cdot f(y), f(y) \cdot f(x)\}$, for any $x, y \in G$. A halfisomorphism of a groupoid onto itself is a half-automorphism. We say that a halfisomorphism is trivial when it is either an isomorphism or an anti-isomorphism.

In 1957, Scott [15] showed that every half-isomorphism on groups is trivial. In the same paper, the author provided an example of a loop of order 8 that has a nontrivial half-automorphism, then the result for groups can not be generalized to all loops. Recently, a similar version of Scott's result was proved for some subclasses of Moufang loops [3, 6, 8] and automorphic loops [10]. A Moufang loop is a loop that satisfies the identity $x(y(x z))=((x y) x) z$, and an automorphic loop is a loop in which every inner mapping is an automorphism [2]. We note that there are Moufang loops and automorphic loops that have nontrivial half-automorphisms [4, 9, 11].

In [10], the authors defined the concept of special half-isomorphism. A halfisomorphism $f: G \rightarrow G^{\prime}$ is called special if the inverse mapping $f^{-1}: G^{\prime} \rightarrow G$ is

[^6]also a half-isomorphism. It is easy to construct an example of a half-isomorphism that is not special, as we can see below.

Example 1.1. Let $G=\{1,2, \ldots, 6\}$ and consider the following Cayley tables of $(G, *)$ and $(G, \cdot)$ :

| $*$ | 1 | 2 | 3 | 4 | 5 | 6 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 2 | 3 | 4 | 5 | 6 | 1 |
| 3 | 3 | 4 | 5 | 6 | 1 | 2 |
| 4 | 4 | 5 | 6 | 1 | 2 | 3 |
| 5 | 5 | 6 | 1 | 2 | 3 | 4 |
| 6 | 6 | 1 | 2 | 3 | 4 | 5 |


| $\cdot$ | 1 | 2 | 3 | 4 | 5 | 6 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 2 | 3 | 4 | 5 | 6 | 1 |
| 3 | 3 | 1 | 5 | 6 | 4 | 2 |
| 4 | 4 | 5 | 6 | 1 | 2 | 3 |
| 5 | 5 | 6 | 1 | 2 | 3 | 4 |
| 6 | 6 | 4 | 2 | 3 | 1 | 5 |

Note that $(G, *)$ is isomorphic to $C_{6}$, the cyclic group of order 6, and $(G, \cdot)=L$ is a nonassociative loop. Consider the mapping $f: C_{6} \rightarrow L$ defined by $f(x)=x$, for all $x \in G$. For $x, y \in G$ such that $x \leqslant y$ and $(x, y) \neq(3,5)$, we have $y * x=x * y=$ $x \cdot y$. Furthermore, $3 * 5=5 * 3=5 \cdot 3$. Thus, $f$ is a half-isomorphism. From $3 \cdot 5=4$ and $3 * 5=5 * 3=1$, it follows that $f^{-1}(3 \cdot 5) \notin\left\{f^{-1}(3) * f^{-1}(5), f^{-1}(5) * f^{-1}(3)\right\}$, and hence $f^{-1}$ is not a half-isomorphism.

We note that providing some examples for the case of non-special half-automorphisms can be very complicated. For finite loops, every half-automorphism is special [10, Corollary 2.7], and in section 3 we show that the same is valid for finite groupoids.

As we can see in Example 1.1, in general, a half-isomorphism does not preserve the structure of the loop. For instance, $C_{6}$ is associative and commutative and has a subgroup $H=\{1,3,5\}$, while $L$ is nonassociative and noncommutative, and $f(H)$ is not a subloop of $L$. However, the inverse mapping of a half-isomorphism can preserve some structure, like the commutative property and subloops [10, Proposition 2.2]. The same naturally holds for special half-isomorphisms.

This paper is organized as follows: Section 2 presents the definitions and basic results about half-isomorphisms. In Section 3, some presented results in [10] on half-isomorphisms in loops are generalized to groupoids. In Section 4, the concept of principal h-groupoid of a groupoid is defined, and then a necessary and sufficient condition for the existence of special half-isomorphisms between groupoids is obtained. Furthermore, equations related to the number of special half-automorphisms, automorphisms and anti-automorphisms of a groupoid are obtained. In Section 5, the concept of principal h-quasigroup of a quasigroup is defined, and then the set of these quasigroups is described. Some examples of non-special half-automorphisms in loops are provided in Section 6.

## 2. Preliminaries

Here, the required definitions and basic results on half-isomorphisms are stated.

Definition 2.1. Let $G$ and $G^{\prime}$ be groupoids. We will say that $G$ is half-isomorphic to $G^{\prime}$, denoted by $G \stackrel{H}{\cong} G^{\prime}$, if there exists a special half-isomorphism between $G$ and $G^{\prime}$. Note that $\stackrel{H}{\cong}$ is an equivalence relation. If $G$ is isomorphic to $G^{\prime}$, we write $G \cong G^{\prime}$.

The next proposition assures that quasigroups half-isomorphic to loops are also loops.

Proposition 2.2. Let $(G, *)$ and $\left(G^{\prime}, \cdot\right)$ be groupoids and $f: G \rightarrow G^{\prime}$ be a halfisomorphism. If $G^{\prime}$ has an identity element 1 , then $f^{-1}(1)$ is the identity element of $G$.

Proof. Let $x=f^{-1}(1) \in G$. For $y \in G$, we have that $\{f(x * y), f(y * x)\} \subset$ $\{1 \cdot f(y), f(y) \cdot 1\}=\{f(y)\}$. Since $f$ is a bijection, we have $x * y=y * x=y$. Therefore, $x$ is an identity element of $G$.

Now, let $(G, *),\left(G^{\prime}, \cdot\right),\left(G^{\prime \prime}, \bullet\right)$ be groupoids, and $f: G \rightarrow G^{\prime}$ and $g: G^{\prime} \rightarrow G^{\prime \prime}$ be half-isomorphisms. For $x, y \in G$, we have

$$
g f(x * y) \in\{g(f(x) \cdot f(y)), g(f(y) \cdot f(x))\}=\{g f(x) \bullet g f(y), g f(y) \bullet g f(x))\}
$$

Thus, $g f$ is a half-isomorphism. If $f$ and $g$ are special half-isomorphisms, then $(g f)^{-1}=f^{-1} g^{-1}$ is also a special half-isomorphism.

We denote the sets of the half-automorphisms, special half-automorphisms, and trivial half-automorphisms of a groupoid $G$ by $\operatorname{Half}(G), \operatorname{Half}_{S}(G)$, and $\operatorname{Half}_{T}(G)$, respectively. Note that automorphisms and anti-automorphisms are always special half-automorphisms, and consequently $\operatorname{Half}_{T}(G) \subset \operatorname{Half}_{S}(G) \subset \operatorname{Half}(G)$.

For $f, g \in \operatorname{Half}(G)$, we already see that $f g \in \operatorname{Half}(G)$. The identity mapping $I_{d}$ of $G$ is the identity element of $\operatorname{Half}(G)$. Thus, $\operatorname{Half}(G)$ is a group if and only if it is closed under inverses, which is equivalent to $\operatorname{Half}(G)=\operatorname{Half}_{S}(G)$. In particular, $\operatorname{Half}_{S}(G)$ is always a group.

A composition of two automorphisms or two anti-automorphisms is an automorphism, and if $f$ is an automorphism and $g$ is an anti-automorphism, then $f g$ and $g f$ are anti-automorphisms and $g^{-1} f g$ is an automorphism. Thus, $H a l f_{T}(G)$ is a group and the automorphism group of $G$, denoted by $\operatorname{Aut}(G)$, is a normal subgroup of $\operatorname{Half}_{T}(G)$.

The following result summarizes the discussion above.
Proposition 2.3. Let $G$ be a groupoid. Then:
(a) $\operatorname{Half}_{S}(G)$ is a group and $\operatorname{Half}_{T}(G)$ is a subgroup of $\operatorname{Half}_{S}(G)$.
(b) $\operatorname{Half}(G)$ is a group if and only if $\operatorname{Half}(G)=\operatorname{Half}_{S}(G)$.
(c) $\operatorname{Aut}(G) \triangleleft \operatorname{Half}_{T}(G)$.

Remark 2.4. It is shown in Section 6 that in general $\operatorname{Half}(G)$ is not a group.

## 3. Special Half-isomorphisms on Groupoids

Considering $(G, *)$ and $\left(G^{\prime}, \cdot\right)$ as groupoids, define the following set:

$$
K(G)=\{(x, y) \in G \times G \mid x y=y x\}
$$

The next two results are respectively extensions of Proposition 2.3 and Theorem 2.5 of [10] to groupoids. We note that the proofs are similar to the ones for corresponding results given in [10].

Lemma 3.1. Let $f: G \rightarrow G^{\prime}$ be a half-isomorphism. Then

$$
\begin{aligned}
\psi_{\left(G, G^{\prime}\right)}: K\left(G^{\prime}\right) & \rightarrow K(G) \\
(x, y) & \mapsto\left(f^{-1}(x), f^{-1}(y)\right)
\end{aligned}
$$

is injective.
Proof. For $(x, y) \in K\left(G^{\prime}\right)$, we have

$$
\left\{f\left(f^{-1}(x) * f^{-1}(y)\right), f\left(f^{-1}(y) * f^{-1}(x)\right)\right\} \subseteq\{x \cdot y, y \cdot x\}=\{x \cdot y\}
$$

Then, $f\left(f^{-1}(x) * f^{-1}(y)\right)=f\left(f^{-1}(y) * f^{-1}(x)\right)$, so $f^{-1}(x) * f^{-1}(y)=f^{-1}(y) * f^{-1}(x)$. Thus, $\left(f^{-1}(x), f^{-1}(y)\right) \in K(Q)$ and the mapping $\psi_{\left(G, G^{\prime}\right)}$ is well-defined.

Now, let $(x, y),\left(x^{\prime}, y^{\prime}\right) \in K\left(G^{\prime}\right)$ such that $\psi_{\left(G, G^{\prime}\right)}((x, y))=\psi_{\left(G, G^{\prime}\right)}\left(\left(x^{\prime}, y^{\prime}\right)\right)$. Then, $f^{-1}(x)=f^{-1}\left(x^{\prime}\right)$ and $f^{-1}(y)=f^{-1}\left(y^{\prime}\right)$. Since $f$ is a bijection, the mapping $\psi_{\left(G, G^{\prime}\right)}$ is injective.

Theorem 3.2. Let $f: G \rightarrow G^{\prime}$ be a half-isomorphism. Then, the following statements are equivalent:
(a) $f$ is special.
(b) $\{f(x * y), f(y * x)\}=\{f(x) \cdot f(y), f(y) \cdot f(x)\}$ for any $x, y \in G$.
(c) For all $x, y \in G$ such that $x * y=y * x$, we have $f(x) \cdot f(y)=f(y) \cdot f(x)$.
(d) $\psi_{\left(G, G^{\prime}\right)}$ is a bijection.

Proof. $(a) \Rightarrow(b)$. Let $x, y \in G$. Since, by the assumption, $f$ is a half-isomorphism, we have $\{f(x * y), f(y * x)\} \subseteq\{f(x) \cdot f(y), f(y) \cdot f(x)\}$. Since $f^{-1}$ is a half-isomorphism, we have $\left\{f^{-1}(f(x) \cdot f(y)), f^{-1}(f(y) \cdot f(x))\right\} \subseteq\{x * y, y * x\}$, and hence $\{f(x) \cdot f(y), f(y) \cdot f(x)\} \subseteq\{f(x * y), f(y * x)\}$.
$(b) \Rightarrow(c)$. Let $x, y \in G$ such that $x * y=y * x$. Then, $f(x * y)=f(y * x)$. Using the hypothesis, we get $\{f(x) \cdot f(y), f(y) \cdot f(x)\}=\{f(x * y), f(y * x)\}=\{f(x * y)\}$, and therefore $f(x) \cdot f(y)=f(y) \cdot f(x)$.
$(c) \Rightarrow(d)$. From Lemma 3.1, we know that $\psi_{\left(G, G^{\prime}\right)}$ is injective. Let $(x, y) \in$ $K(G)$. By hypothesis, we have $f(x) \cdot f(y)=f(y) \cdot f(x)$, and then $(f(x), f(y)) \in$ $K\left(G^{\prime}\right)$. It is clear that $\psi_{\left(G, G^{\prime}\right)}((f(x), f(y)))=(x, y)$, and hence $\psi_{\left(G, G^{\prime}\right)}$ is a bijection.
$(d) \Rightarrow(a)$. Let $x, y \in G^{\prime}$. If $(x, y) \in K\left(G^{\prime}\right)$, then $\left(f^{-1}(x), f^{-1}(y)\right) \in K(G)$ since $\psi_{\left(G, G^{\prime}\right)}$ is a bijection. Thus, $f\left(f^{-1}(x) * f^{-1}(y)\right)=x \cdot y$, and therefore $f^{-1}(x \cdot y)=f^{-1}(x) * f^{-1}(y)$. If $(x, y) \notin K\left(G^{\prime}\right)$, then $\left(f^{-1}(x), f^{-1}(y)\right) \notin K(G)$ since $\psi_{\left(G, G^{\prime}\right)}$ is a bijection. Consequently, we have

$$
\left\{f\left(f^{-1}(x) * f^{-1}(y)\right), f\left(f^{-1}(y) * f^{-1}(x)\right)\right\}=\{x \cdot y, y \cdot x\}
$$

and hence $f^{-1}(x \cdot y) \in\left\{f^{-1}(x) * f^{-1}(y), f^{-1}(y) * f^{-1}(x)\right\}$.
As direct consequences of Lemma 3.1 and Theorem 3.2, we have the following corollaries.

Corollary 3.3. Let $f: G \rightarrow G^{\prime}$ be a half-isomorphism. If $|K(G)|=\left|K\left(G^{\prime}\right)\right|<\infty$, then $f$ is special.

Corollary 3.4. Let $G$ be a groupoid such that $|K(G)|<\infty$. Then, Half $(G)$ is a group.

Corollary 3.5. Let $G$ be a finite groupoid. Then, $\operatorname{Half}(G)$ is a group.
A loop is diassociative if any two of its elements generate an associative subloop. Moufang loops and groups are examples of diassociative loops. In [8, Lemma 2.1], the authors showed that the item (c) of Theorem 3.2 holds for any half-isomorphism on diassociative loops. Therefore, we have the next result.

Corollary 3.6. Let $(L, *)$ and $\left(L^{\prime}, \cdot\right)$ be diassociative loops. Then, every halfisomorphism between $L$ and $L^{\prime}$ is special.

Remark 3.7. The Corollary 3.6 cannot be extended for some important classes of loops. In Example 6.1, a non-special half-isomorphism between a right Bol loop and a group is introduced. A loop is called right Bol loop if it satisfies the identity $((x y) z) y=x((y z) y)$.

This section is finished with a property of half-isomorphic groupoids.
Proposition 3.8. If $G \stackrel{H}{\cong} G^{\prime}$, then:
(a) $\operatorname{Half}(G) \cong \operatorname{Half}\left(G^{\prime}\right)$
(b) $\operatorname{Half}_{S}(G) \cong \operatorname{Half}_{S}\left(G^{\prime}\right)$.

Proof. Let $\phi: G \rightarrow G^{\prime}$ be a special half-isomorphism and $\varphi: \operatorname{Half}(G) \rightarrow \operatorname{Half}\left(G^{\prime}\right)$ by $\varphi(f)=\phi f \phi^{-1}$. It is clear that $\varphi$ is a bijection. For $f, g \in \operatorname{Half}(G)$, we have $\varphi(f g)=\phi f g \phi^{-1}=\phi f \phi^{-1} \phi g \phi^{-1}=\varphi(f) \varphi(g)$. Thus, $\operatorname{Half}(G) \cong \operatorname{Half}\left(G^{\prime}\right)$. The rest of the claim is concluded from the fact that $\varphi\left(\operatorname{Half}_{S}(G)\right)=\operatorname{Half}_{S}\left(G^{\prime}\right)$.

Remark 3.9. If $G \stackrel{H}{\cong} G^{\prime}$, then $\operatorname{Aut}(G)$ is not isomorphic to $\operatorname{Aut}\left(G^{\prime}\right)$ in general (see Example 4.6).

## 4. Principal h-Groupoids of G

In this section, $G_{0}=(G, *)$ is considered as a noncommutative groupoid.
Let $\left(G^{\prime}, \bullet\right)$ be a groupoid such that $G_{0} \stackrel{H}{\cong}\left(G^{\prime}, \bullet\right)$. Then, there exists a special half-isomorphism $f$ of $G_{0}$ into $\left(G^{\prime}, \bullet\right)$. Define an operation $\cdot$ on $G$ by $x \cdot y=$ $f^{-1}(f(x) \bullet f(y))$. Thus, $f$ is an isomorphism of $(G, \cdot)$ into $\left(G^{\prime}, \bullet\right)$, and hence $I_{d}: G_{0} \rightarrow(G, \cdot)$ is a special half-isomorphism, where $I_{d}$ is the identity mapping of $G$.

A groupoid $(G, \cdot)$ for which $I_{d}: G_{0} \rightarrow(G, \cdot)$ is a special half-isomorphism is called a principal $h$-groupoid of $G_{0}$. Therefore, the following result is at hand.

Proposition 4.1. Let $G^{\prime}$ be a groupoid. Then, $G_{0} \stackrel{H}{\cong} G^{\prime}$ if and only if $G^{\prime}$ is isomorphic to a principal h-groupoid of $G_{0}$.

Denote by $\mathcal{M}\left(G_{0}\right)$ the set of the principal h-groupoids of $G_{0}$. Note that for $(G, \cdot),(G \cdot \bullet) \in \mathcal{M}\left(G_{0}\right)$, we have $(G, \cdot)=(G, \bullet)$ if $x \cdot y=x \bullet y$, for all $x, y \in G$, which is equivalent to $I_{d}$ being an isomorphism between $(G, \cdot)$ and $(G, \bullet)$.

Let $(G, \cdot) \in \mathcal{M}\left(G_{0}\right)$. Since $I_{d}: G_{0} \rightarrow(G, \cdot)$ is a special half-isomorphism, we have

$$
\begin{equation*}
\{x * y, y * x\}=\{x \cdot y, y \cdot x\}, \text { for all } x, y \in G . \tag{1}
\end{equation*}
$$

If $(x, y) \in K\left(G_{0}\right)$, then $x \cdot y=y \cdot x=x * y$. For each pair $(x, y),(y, x) \in$ $G \times G \backslash K\left(G_{0}\right)$, there are two possible values for $x \cdot y$ and $y \cdot x$ by (1). Thus, if $G$ is finite, we have $2^{\left|G \times G \backslash K\left(G_{0}\right)\right| / 2}$ possibilities for a principal h-groupoid of $G_{0}$. Hence, the following result is at hand.

Proposition 4.2. If $G$ is finite, then $\left|\mathcal{M}\left(G_{0}\right)\right|=2^{\left(|G|^{2}-\left|K\left(G_{0}\right)\right|\right) / 2}$.
Define $\mathcal{M}_{I}\left(G_{0}\right)=\left\{G^{\prime} \in \mathcal{M}\left(G_{0}\right) \mid G^{\prime} \cong G_{0}\right\}$ and let $S(G)$ be the set of permutations of $G$. For $G^{\prime}=(G, \cdot) \in \mathcal{M}_{I}(G)$, define

$$
\operatorname{Iso}\left(G^{\prime}, G_{0}\right)=\left\{f \in S(G) \mid f \text { is an isomorphism of } G^{\prime} \text { into } G_{0}\right\} .
$$

Note that $\operatorname{Iso}\left(G_{0}, G_{0}\right)=\operatorname{Aut}\left(G_{0}\right)$. In the next result, we determine a relationship between $\operatorname{Half}_{S}\left(G_{0}\right), \operatorname{Aut}\left(G_{0}\right)$ and $\mathcal{M}_{I}\left(G_{0}\right)$.

Proposition 4.3. We have:
(a) $\operatorname{Iso}\left(G^{\prime}, G_{0}\right) \subset \operatorname{Half}_{S}\left(G_{0}\right)$, for every $G^{\prime} \in \mathcal{M}_{I}\left(G_{0}\right)$.
(b) For each $G^{\prime} \in \mathcal{M}_{I}\left(G_{0}\right)$, Iso $\left(G^{\prime}, G_{0}\right)$ is a right coset of $\operatorname{Aut}\left(G_{0}\right)$ in Half $_{S}\left(G_{0}\right)$, that is, there exists $f \in \operatorname{Half}_{S}\left(G_{0}\right)$ such that $\operatorname{Iso}\left(G^{\prime}, G_{0}\right)=\operatorname{Aut}\left(G_{0}\right) f$.
(c) For $G_{1}, G_{2} \in \mathcal{M}_{I}\left(G_{0}\right)$, if $\operatorname{Iso}\left(G_{1}, G_{0}\right) \cap \operatorname{Iso}\left(G_{2}, G_{0}\right) \neq \emptyset$, then $G_{1}=G_{2}$.
(d) $\operatorname{Half}_{S}\left(G_{0}\right)=\bigcup_{G^{\prime} \in \mathcal{M}_{I}\left(G_{0}\right)} \operatorname{Iso}\left(G^{\prime}, G_{0}\right)$.
(e) $\left|\mathcal{M}_{I}\left(G_{0}\right)\right|=\left[\operatorname{Half}_{S}\left(G_{0}\right): \operatorname{Aut}\left(G_{0}\right)\right]$, which is the index of $\operatorname{Aut}\left(G_{0}\right)$ in Half $_{S}\left(G_{0}\right)$.

Proof. (a). For $G^{\prime}=(G, \cdot) \in \mathcal{M}_{I}\left(G_{0}\right)$, let $f \in \operatorname{Iso}\left(G^{\prime}, G_{0}\right)$. Then $f(x \cdot y)=$ $f(x) * f(y)$, for all $x, y \in G$. By (1), $\{f(x \cdot y), f(y \cdot x)\}=\{f(x) * f(y), f(y) * f(x)\}$, for all $x, y \in G$. By Theorem 3.2, $f \in \operatorname{Half}_{S}\left(G_{0}\right)$.
(b). Fix $f \in \operatorname{Iso}\left(G^{\prime}, G_{0}\right)$. It is clear that $g f^{-1} \in \operatorname{Aut}\left(G_{0}\right)$, for every $g \in$ $\operatorname{Iso}\left(G^{\prime}, G_{0}\right)$, and $\alpha f \in \operatorname{Iso}\left(G^{\prime}, G_{0}\right)$, for every $\alpha \in \operatorname{Aut}\left(G_{0}\right)$. Hence, we have the desired result.
(c). Let $f \in \operatorname{Iso}\left(G_{1}, G_{0}\right) \cap \operatorname{Iso}\left(G_{2}, G_{0}\right)$. Note that $I_{d}=f^{-1} f: G_{1} \rightarrow G_{2}$ is an isomorphism. From the definition of $\mathcal{M}\left(G_{0}\right)$, it follows that $G_{1}=G_{2}$.
$(d)$. Let $f \in \operatorname{Half}_{S}\left(G_{0}\right)$. Define the operation $\cdot$ on $G$ by $x \cdot y=f^{-1}(f(x) * f(y))$, for all $x, y \in G$. Note that $f:(G, \cdot) \rightarrow(G, *)$ is an isomorphism. Furthermore, since $f \in$ Half $_{S}\left(G_{0}\right)$, and $f(x \cdot y)=f(x) * f(y)$ and $f(y \cdot x)=f(y) * f(x)$, for all $x, y \in G$, we have $\{x \cdot y, y \cdot x\}=\{x * y, y * x\}$, for all $x, y \in G$. Thus, $G^{\prime}=(G, \cdot) \in \mathcal{M}_{I}\left(G_{0}\right)$, and hence $f \in \operatorname{Iso}\left(G^{\prime}, G_{0}\right)$.
$(e)$. It is a consequence of the previous items.
As a consequence of the Proposition 3.8 and the item (e) of Proposition 4.3, we have the following result.

Corollary 4.4. Let $G^{\prime}, G^{\prime \prime}$ be groupoids such that $G^{\prime} \stackrel{H}{\cong} G^{\prime \prime}$ and Half $\left(G^{\prime}\right)$ is finite. Then,

$$
\left|\mathcal{M}_{I}\left(G^{\prime}\right)\right| \cdot\left|\operatorname{Aut}\left(G^{\prime}\right)\right|=\left|\mathcal{M}_{I}\left(G^{\prime \prime}\right)\right| \cdot\left|\operatorname{Aut}\left(G^{\prime \prime}\right)\right|
$$

Define $G_{0}^{T}=(G, \cdot)$, where $x \cdot y=y * x$, for all $x, y \in G$, and denote the set of anti-automorphisms of $G_{0}$ by $\operatorname{Ant}\left(G_{0}\right)$. Since $G_{0}$ is noncommutative, we have $\operatorname{Aut}\left(G_{0}\right) \cap \operatorname{Ant}\left(G_{0}\right)=\emptyset$.

Proposition 4.5. $G_{0}$ has an anti-automorphism if and only if $G_{0}^{T} \in \mathcal{M}_{I}\left(G_{0}\right)$. In this case, $|\operatorname{Ant}(G)|=|A u t(G)|$.

Proof. Note that a bijection $f$ of $G$ is an anti-automorphism of $G_{0}$ if and only if $f$ is an isomorphism of $G_{0}$ into $G_{0}^{T}$. The rest of the claim is concluded from the item (b) of Proposition 4.3.

Example 4.6. Let $Q=\{1,2, \ldots, 8\}$ and consider the following Cayley tables of $(Q, *)$ and $(Q, \cdot)$ :

| $*$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 1 | 2 | 3 | 4 | 6 | 5 | 7 | 8 |
| 2 | 2 | 1 | 4 | 3 | 5 | 6 | 8 | 7 |
| 3 | 4 | 3 | 1 | 2 | 7 | 8 | 5 | 6 |
| 4 | 3 | 4 | 2 | 1 | 8 | 7 | 6 | 5 |
| 5 | 5 | 6 | 8 | 7 | 1 | 2 | 4 | 3 |
| 6 | 6 | 5 | 7 | 8 | 2 | 1 | 3 | 4 |
| 7 | 8 | 7 | 6 | 5 | 3 | 4 | 1 | 2 |
| 8 | 7 | 8 | 5 | 6 | 4 | 3 | 2 | 1 |


| $\cdot$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 2 | 4 | 3 | 6 | 5 | 7 | 8 |
| 2 | 2 | 1 | 3 | 4 | 5 | 6 | 8 | 7 |
| 3 | 3 | 4 | 1 | 2 | 7 | 8 | 5 | 6 |
| 4 | 4 | 3 | 2 | 1 | 8 | 7 | 6 | 5 |
| 5 | 5 | 6 | 8 | 7 | 1 | 2 | 4 | 3 |
| 6 | 6 | 5 | 7 | 8 | 2 | 1 | 3 | 4 |
| 7 | 8 | 7 | 6 | 5 | 3 | 4 | 1 | 2 |
| 8 | 7 | 8 | 5 | 6 | 4 | 3 | 2 | 1 |

We have $(Q, *)$ and $(Q, \cdot)$ being quasigroups. Note that, for $x, y \in Q$ : $x * y=\left\{\begin{array}{l}y \cdot x, \text { if }(x, y) \in\{(1,3),(1,4),(2,3),(2,4),(3,1),(3,2),(4,1),(4,2)\}, \\ x \cdot y, \text { otherwise } .\end{array}\right.$

Thus, $(Q, \cdot) \in \mathcal{M}((Q, *))$. Using the LOOPS package [13] for GAP [5] we get $|\operatorname{Aut}((Q, *))|=4$ and $|\operatorname{Aut}((Q, \cdot))|=8$. This illustrates Remark 3.9.

Note that $|K((Q, *))|=16$, and hence $|\mathcal{M}((Q, *))|=2^{24}=16777216$. Using a GAP computation with the LOOPS package, we get that there are 64 quasigroups in $\mathcal{M}((Q, *))$ and $\left|\mathcal{M}_{I}((Q, *))\right|=12$. By Proposition 4.3, we have $|\operatorname{Half}((Q, *))|=$ 48 and $\left|\mathcal{M}_{I}((Q, \cdot))\right|=6$.

It is observed that the number of quasigroups in $\mathcal{M}((Q, *))$ is much smaller than $|\mathcal{M}((Q, *))|$. In the next section, we will see that the same occurs for any finite noncommutative quasigroup.

## 5. Principal h-Quasigroups of Q

Here, $Q_{0}=(Q, *)$ is considered as a noncommutative quasigroup. A quasigroup $(Q, \cdot)$ is a principal $h$-quasigroup of $Q_{0}$ if $(Q, \cdot) \in \mathcal{M}\left(Q_{0}\right)$. Denote by $\mathcal{N}\left(Q_{0}\right)$ the set of the principal h-quasigroups of $Q_{0}$. It is clear that $\mathcal{M}_{I}\left(Q_{0}\right) \subset \mathcal{N}\left(Q_{0}\right) \subset \mathcal{M}\left(Q_{0}\right)$. The next result is concluded from Proposition 4.1.
Proposition 5.1. Let $Q^{\prime}$ be a quasigroup. Then $Q_{0} \stackrel{H}{\cong} Q^{\prime}$ if and only if $Q^{\prime}$ is isomorphic to a principal h-quasigroup of $Q_{0}$.

Now, we describe $\mathcal{N}\left(Q_{0}\right)$. For $(x, y),\left(x^{\prime}, y^{\prime}\right) \in Q \times Q \backslash K\left(Q_{0}\right)$, we say that $(x, y) \sim\left(x^{\prime}, y^{\prime}\right)$ if one of the following holds:
(i) $\left(x^{\prime}, y^{\prime}\right)=(y, x)$,
(ii) $x=x^{\prime}$ and $\{x * y, y * x\} \cap\left\{x * y^{\prime}, y^{\prime} * x\right\} \neq \emptyset$,
(iii) $y=y^{\prime}$ and $\{x * y, y * x\} \cap\left\{x^{\prime} * y, y * x^{\prime}\right\} \neq \emptyset$.

We say that $(x, y) \equiv\left(x^{\prime}, y^{\prime}\right)$ if there are $z_{1}, z_{2}, \ldots, z_{l} \in Q \times Q \backslash K\left(Q_{0}\right)$ such that $(x, y) \sim z_{1} \sim z_{2} \sim \ldots \sim z_{l} \sim\left(x^{\prime}, y^{\prime}\right)$.

The relation $\sim$ is reflexive and symmetric, and hence $\equiv$ is an equivalence relation. Denote by $r\left(Q_{0}\right)$ the number of equivalence classes of $\equiv$ on $Q \times Q \backslash K\left(Q_{0}\right)$.

Suppose that $Q$ is finite and let $\tau=\left\{\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right), \ldots,\left(x_{r\left(Q_{0}\right)}, y_{r\left(Q_{0}\right)}\right)\right\}$ be a set of representatives of the equivalence classes of $\equiv$ on $Q \times Q \backslash K\left(Q_{0}\right)$. Consider $\mathbb{Z}_{2}=\{0,1\}$, and for $\sigma=\left\{\sigma_{1}, \sigma_{2}, \ldots, \sigma_{r\left(Q_{0}\right)}\right\} \in \mathbb{Z}_{2}^{r\left(Q_{0}\right)}$, define the operation ${ }^{\sigma} \cdot$ on $Q$ by:

$$
x \bullet y=\left\{\begin{array}{l}
x * y, \text { if }(x, y) \in K\left(Q_{0}\right) \text { or }(x, y) \equiv\left(x_{i}, y_{i}\right), \text { where } \sigma_{i}=0, \\
y * x, \text { if }(x, y) \equiv\left(x_{i}, y_{i}\right), \text { where } \sigma_{i}=1 .
\end{array}\right.
$$

Denote $(Q, \stackrel{\sigma}{\bullet})$ by $Q_{\sigma}$ and let $\mathcal{N}_{\tau}\left(Q_{0}\right)=\left\{Q_{\sigma} \mid \sigma \in \mathbb{Z}_{2}^{r\left(Q_{0}\right)}\right\}$. Note that $\mathcal{N}_{\tau}\left(Q_{0}\right) \subset$ $\mathcal{M}\left(Q_{0}\right)$ and $\left|\mathcal{N}_{\tau}\left(Q_{0}\right)\right|=2^{r\left(Q_{0}\right)}$.
Theorem 5.2. If $Q$ is finite, then $\mathcal{N}\left(Q_{0}\right)=\mathcal{N}_{\tau}\left(Q_{0}\right)$. So, $\left|\mathcal{N}\left(Q_{0}\right)\right|=2^{r\left(Q_{0}\right)}$.

Proof. Let $Q_{\sigma} \in \mathcal{N}_{\tau}\left(Q_{0}\right)$. Since $Q$ is finite, in order to prove that $Q_{\sigma}$ is a quasigroup, we only need to show that the cancellation laws are satisfied, that is, $x \stackrel{\boldsymbol{\sigma}}{\bullet} y=x \stackrel{\boldsymbol{\sigma}}{\bullet} y^{\prime} \Rightarrow y=y^{\prime}$ and $x \stackrel{\boldsymbol{\sigma}}{\bullet} y=x^{\prime} \bullet y \Rightarrow x=x^{\prime}$.

Let $x, y, y^{\prime} \in Q$ be such that $x \stackrel{\sigma}{\bullet} y=x{ }^{\sigma} y^{\prime}$. If $(x, y) \in K\left(Q_{0}\right)$, then $x * y=y * x \in\left\{x * y^{\prime}, y^{\prime} * x\right\}$, and hence $y=y^{\prime}$. Now suppose that $(x, y) \notin K\left(Q_{0}\right)$. We have four possibilities:
(i) $x \stackrel{\sigma}{\bullet} y=x * y$ and $x \stackrel{\sigma}{\bullet} y^{\prime}=x * y^{\prime}$,
(ii) $x \stackrel{\sigma}{\bullet} y=y * x$ and $x \stackrel{\sigma}{\bullet} y^{\prime}=y^{\prime} * x$,
(iii) $x{ }^{\circ} \bullet y=x * y$ and $x \stackrel{\sigma}{\bullet} y^{\prime}=y^{\prime} * x$,
(iv) $x \stackrel{\boldsymbol{\sigma}}{\bullet} y=y * x$ and $x \stackrel{\sigma}{\bullet} y^{\prime}=x * y^{\prime}$.

In (i) and (ii), it is immediately seen that $y=y^{\prime}$.
For (iii) and (iv), we have $(x, y) \sim\left(x, y^{\prime}\right)$. Hence, there exists $\left(x_{i}, y_{i}\right) \in \tau$ such that $(x, y) \equiv\left(x_{i}, y_{i}\right)$ and $\left(x, y^{\prime}\right) \equiv\left(x_{i}, y_{i}\right)$. By definition of ${ }^{\boldsymbol{\sigma}}$, we have either $x \stackrel{\sigma}{\bullet} y=x * y$ and $x \stackrel{\sigma}{\bullet} y^{\prime}=x * y^{\prime}$, or $x \stackrel{\sigma}{\bullet} y=y * x$ and $x{ }^{\circ} \bullet y^{\prime}=y^{\prime} * x$. Since $(x, y) \notin K\left(Q_{0}\right)$, it follows that $\left(x, y^{\prime}\right) \in K\left(Q_{0}\right)$. Similarly to the case $(x, y) \in K\left(Q_{0}\right)$, one can conclude that $y=y^{\prime}$.

Thus, the cancellation law $x \stackrel{\sigma}{\bullet} y=x \stackrel{\sigma}{\bullet} y^{\prime} \Rightarrow y=y^{\prime}$ holds in $Q_{\sigma}$. The second cancellation law can be proven similarly. Therefore, $Q_{\sigma} \in \mathcal{N}\left(Q_{0}\right)$.

Conversely, let $Q^{\prime}=(Q, \cdot) \in \mathcal{N}\left(Q_{0}\right)$. Then, there exists $\sigma \in \mathbb{Z}_{2}^{r\left(Q_{0}\right)}$ such that $x_{i} \cdot y_{i}=x_{i} \bullet y_{i}$, for any $\left(x_{i}, y_{i}\right) \in \tau$. For $(x, y) \in K\left(Q_{0}\right)$, it is vividly deduced that $x \cdot y=x \stackrel{\sigma}{\bullet} y$.

Consider $\left(x_{i}, y_{i}\right) \in \tau$. Then, $y_{i} \cdot x_{i}=y_{i}{ }^{\sigma} x_{i}$. Let $(x, y) \in Q \times Q \backslash$ $\left\{\left(x_{i}, y_{i}\right),\left(y_{i}, x_{i}\right)\right\}$ such that $(x, y) \sim\left(x_{i}, y_{i}\right)$. By (1) and the definition of $\boldsymbol{\bullet}$, we have $x \cdot y \neq x_{i} \cdot y_{i}=x_{i} \bullet y_{i}$ and $x \bullet y \neq x_{i} \bullet y_{i}$, and therefore the only possibility is $x \cdot y=x \stackrel{\sigma}{\bullet} y$. For every $(x, y) \sim\left(x_{i}, y_{i}\right)$, one can use the previous arguments and result in $x^{\prime} \cdot y^{\prime}=x^{\prime} \bullet y^{\prime}$, for all $\left(x^{\prime}, y^{\prime}\right) \sim(x, y)$. Since $Q$ is finite, this procedure must end at some point, and hence $x \cdot y=x \stackrel{\sigma}{\bullet} y$, for all $(x, y) \equiv\left(x_{i}, y_{i}\right)$. As a result, we have $Q^{\prime}=Q_{\sigma}$.

By Proposition 4.2, if $Q$ is finite, then $r\left(Q_{0}\right) \leqslant\left(|Q|^{2}-\left|K\left(Q_{0}\right)\right|\right) / 2$. The next proposition provides a better estimate for $r\left(Q_{0}\right)$. According to this result, it is seen that $\left|\mathcal{N}\left(Q_{0}\right)\right|$ is much smaller that $\left|\mathcal{M}\left(Q_{0}\right)\right|$.

Proposition 5.3. If $Q$ is finite, then

$$
r\left(Q_{0}\right) \leqslant\left(|Q|^{2}-\left|K\left(Q_{0}\right)\right|\right) / 6 \quad \text { and } \quad\left|\mathcal{N}\left(Q_{0}\right)\right| \leqslant \sqrt[3]{\left|\mathcal{M}\left(Q_{0}\right)\right|}
$$

In particular, $\left|\mathcal{M}\left(Q_{0}\right)\right| \geqslant 8$.
Proof. Let $(x, y) \in Q \times Q \backslash K\left(Q_{0}\right)$ and $[(x, y)]$ be the equivalence class of $(x, y)$ with respect to $\equiv$. Since $Q_{0}$ is a quasigroup, there are $x^{\prime}, y^{\prime} \in Q$ such that $x^{\prime} \neq x$,
$y^{\prime} \neq y,\left(x^{\prime}, y\right) \sim(x, y)$, and $\left(x, y^{\prime}\right) \sim(x, y)$. We have $x \neq y, x^{\prime} \neq y$ and $x \neq y^{\prime}$ since $(x, y),\left(x^{\prime}, y\right),\left(x, y^{\prime}\right) \notin K\left(Q_{0}\right)$. Thus,

$$
|[(x, y)]| \geqslant\left|\left\{(x, y),\left(x^{\prime}, y\right),\left(x, y^{\prime}\right),(y, x),\left(y, x^{\prime}\right),\left(y^{\prime}, x\right)\right\}\right|=6
$$

Hence, $\left|Q \times Q \backslash K\left(Q_{0}\right)\right| \geqslant 6 r\left(Q_{0}\right)$. The rest of the claim follows from Proposition 4.2, Theorem 5.2 and the fact that $r\left(Q_{0}\right) \geqslant 1$.

If $Q$ is finite and $r\left(Q_{0}\right)$ is small, one can generate all quasigroups of $\mathcal{N}\left(Q_{0}\right)$ computationally. Then, by using Propositions 5.1 and 4.5 it can be verified if a quasigroup $Q^{\prime}$ is half-isomorphic to $Q_{0}$ and generated all elements of $\operatorname{Half}\left(Q_{0}\right)$. However, $r\left(Q_{0}\right)$ can be a large number even for groups of small order, and therefore generating all the quasigroups of $\mathcal{N}\left(Q_{0}\right)$ becomes computationally unviable. The next example illustrates both situations. In this example, $r\left(Q_{0}\right)$ and $\left|\mathcal{M}\left(Q_{0}\right)\right|$ are obtained by using GAP computing with the LOOPS package [5, 13].
Example 5.4. (a). Let $A_{5}$ be the alternating group of order 60 . We have that $r\left(A_{5}\right)=91$, and hence $\left|\mathcal{N}\left(A_{5}\right)\right|=2^{91}$. Furthermore, $\left|\mathcal{M}\left(A_{5}\right)\right|=2^{1650}$.
(b). The LOOPS package for GAP contains all nonassociative right Bol loops of order 141 (there are 23 such loops). The right Bol loops of this order were classified in [7]. If $L$ is one of these loops, then $3 \leqslant r(L) \leqslant 8$, and hence $|\mathcal{N}(L)| \leqslant 256$. Furthermore, $|\mathcal{M}(L)| \geqslant 2^{5405}$.

By Proposition 2.2, every quasigroup half-isomorphic to a loop is also a loop. Consequently, the same results as those presented for quasigroups in this section can be proven for loops. For more structured classes of loops, as it is seen in the following result, one can provide more information about the loops of $\mathcal{N}(L)$.
Proposition 5.5. Let $G$ be a finite noncommutative group. Then, $\left|\mathcal{M}_{I}(G)\right|=2$. Proof. From Scott's result [15], we have $\operatorname{Half}(G)=\operatorname{Half}_{T}(G)$. Since $G$ is noncommutative, the mapping $J: G \rightarrow G$, defined by $J(x)=x^{-1}$, is an antiautomorphism of $G$. By Proposition 4.5, we have $|\operatorname{Half}(G)|=2|\operatorname{Aut}(G)|$. Thus, the claim follows from Proposition 4.3.

In fact, the previous proposition can be extended to any noncommutative loop that has an anti-automorphism and where every half-automorphism is trivial, such as the noncommutative loops of the subclass of Moufang loops in [8, Thereom 1.4], which include the noncommutative Moufang loops of odd order [3]. Notice that this result cannot be extended even to all Moufang loops. In [16, Example 4.6], a noncommutative Moufang loop $L$ of order 16 is given for which $\left|\mathcal{M}_{I}(L)\right|=$ $[\operatorname{Half}(L): \operatorname{Aut}(L)]=16$.

## 6. A Construction of a Non-special Half-automorphism

Let $G$ be a nonempty set with binary operations $*$ and $\cdot$ such that there exists a non-special half-isomorphism $f:(G, *) \rightarrow(G, \cdot)$. Define $G_{\infty}=\prod_{i=1}^{\infty} G$. The
elements of $G_{\infty}$ will be denoted by $\left(x_{i}\right)=\left(x_{i}\right)_{i=1}^{\infty}$, where $x_{i} \in G$, for all $i$. For $\left(x_{i}\right),\left(y_{i}\right) \in G_{\infty}$, define the operation $\left(x_{i}\right) \bullet\left(y_{i}\right)=\left(z_{i}\right)$, where

$$
z_{j}=\left\{\begin{array}{l}
x_{j} * y_{j}, \text { if } j \text { is odd, } \\
x_{j} \cdot y_{j}, \text { if } j \text { is even. }
\end{array}\right.
$$

Then, $\left(G_{\infty}, \bullet\right)$ is a groupoid. It is easy to see that if $(G, *)$ and $(G, \cdot)$ are quasigroups (loops), then $\left(G_{\infty}, \bullet\right)$ is also a quasigroup (loop). Define the mapping $\phi: G_{\infty} \rightarrow G_{\infty}$ by $\phi\left(x_{i}\right)=\left(y_{i}\right)$, where

$$
y_{j}=\left\{\begin{array}{l}
f\left(x_{1}\right), \text { if } j=2, \\
x_{j+2}, \text { if } j \text { is odd, } \\
x_{j-2}, \text { if } j>2 \text { and } j \text { is even. }
\end{array}\right.
$$

Thus, $\phi$ is a bijection and in each entry of $\left(x_{i}\right)$ it behaves like an isomorphism or a half-isomorphism. Hence, $\phi$ is a half-automorphism of $G_{\infty}$. Since $f$ is a non-special half-isomorphism, there are $x, y \in G$ such that $f^{-1}(x \cdot y) \notin\left\{f^{-1}(x) *\right.$ $\left.f^{-1}(y), f^{-1}(y) * f^{-1}(x)\right\}$. Then,

$$
\phi^{-1}\left((x)_{i=1}^{\infty} \bullet(y)_{i=1}^{\infty}\right) \notin\left\{\phi^{-1}\left((x)_{i=1}^{\infty}\right) \bullet \phi^{-1}\left((y)_{i=1}^{\infty}\right), \phi^{-1}\left((y)_{i=1}^{\infty}\right) \bullet \phi^{-1}\left((x)_{i=1}^{\infty}\right)\right\} .
$$

Therefore, $\phi$ is a non-special half-automorphism of $G_{\infty}$.
In example 1.1, we have loops $C_{6}=(G, *)$ and $L=(G, \cdot)$ for the conditions above, hence the loop $G_{\infty}$ has a non-special half-automorphism. Note that $\operatorname{Half}\left(G_{\infty}\right)$ is not a group.

In the following example, a non-special half-isomorphism between a right Bol loop and a group is provided. This example is obtained by using MACE4 [12].

Example 6.1. Let $G=\{1,2, \ldots, 8\}$ and consider the following Cayley tables of $(G, *)$ and $(G, \cdot)$ :

| $*$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | 2 | 1 | 4 | 6 | 3 | 5 | 8 | 7 |
| 3 | 3 | 4 | 1 | 2 | 7 | 8 | 5 | 6 |
| 4 | 4 | 3 | 2 | 8 | 1 | 7 | 6 | 5 |
| 5 | 5 | 6 | 7 | 1 | 8 | 2 | 3 | 4 |
| 6 | 6 | 5 | 8 | 7 | 2 | 1 | 4 | 3 |
| 7 | 7 | 8 | 5 | 3 | 6 | 4 | 1 | 2 |
| 8 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |


| $\cdot$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | 2 | 1 | 4 | 3 | 6 | 5 | 8 | 7 |
| 3 | 3 | 4 | 1 | 2 | 7 | 8 | 5 | 6 |
| 4 | 4 | 3 | 2 | 1 | 8 | 7 | 6 | 5 |
| 5 | 5 | 7 | 6 | 8 | 1 | 3 | 2 | 4 |
| 6 | 6 | 8 | 5 | 7 | 2 | 4 | 1 | 3 |
| 7 | 7 | 5 | 8 | 6 | 3 | 1 | 4 | 2 |
| 8 | 8 | 6 | 7 | 5 | 4 | 2 | 3 | 1 |

We have $(G, *)=L$ as a right Bol loop and $(G, \cdot)$ being isomorphic to $D_{8}$, which is the dihedral group of order 8. The permutation $f=(357)(468)$ of $G$ is a half-isomorphism of $L$ into $D_{8}$. Since $|K(L)|=56$ and $\left|K\left(D_{8}\right)\right|=40, f$ is a non-special half-isomorphism by Theorem 3.2. Since $L$ and $D_{8}$ are right Bol loops, $G_{\infty}$ is also a right Bol loop, and from the previous construction we have a non-special half-automorphism in a right Bol loop of infinite order.

Acknowledgments. Some calculations in this work have been made by using the finite model builder MACE4, developed by McCune [12], and the LOOPS package [13] for GAP [5].

## References

[1] R.H. Bruck, A Survey of Binary Systems, Springer, 1971.
[2] R.H. Bruck and L.J. Paige, Loops whose inner mappings are automorphisms, Ann. Math. 63 (1956), 308-323.
[3] S. Gagola III and M.L. Merlini Giuliani, Half-isomorphisms of Moufang loops of odd order, J. Algebra Appl. 11 (2012), 194-199.
[4] S. Gagola III and M.L. Merlini Giuliani, On half-automorphisms of certain Moufang Loops with even order, J. Algebra 386 (2013), 131-141.
[5] The GAP Group, GAP - Groups, Algorithms, and Programming, Version 4.10.1, 2019. http://www.gap-system.org.
[6] A. Grishkov, M.L. Merlini Giuliani, M. Rasskazova and L. Sabinina, HalfIsomorphisms of finite Automorphic Moufang Loops, Comm. Algebra 44 (2016), 4252-4261.
[7] M.K. Kinyon, G.P. Nagy and P. Vojtěchovský, Bol loops and bruck loops of order pq, J. Algebra 473 (2017), 481-512.
[8] M. Kinyon, I. Stuhl and P. Vojtěchovský, Half-Isomorphisms of Moufang Loops, J. Algebra 450 (2016), 152-161.
[9] M.L. Merlini Giuliani and G. Souza dos Anjos, Half-isomorphisms of dihedral automorphic loops, Comm. Algebra 48 (2020), no. 3, 1150-1162.
[10] M.L. Merlini Giuliani and G. Souza dos Anjos, Lie automorphic loops under half-automorphisms, J. Algebra Appl., 19 (2020), no.11, 205 - 211.
[11] M.L. Merlini Giuliani, P. Plaumann and L. Sabinina, Half-automorphisms of Cayley-Dickson loops, In: Falcone, G., ed., Lie Groups, Differential Equations, and Geometry. Cham: Springer (2017), 109-125.
[12] W. McCune, Prover9 and Mace4, http://www.cs.unm.edu/~mccune/pover9, 2005-2010.
[13] G.P. Nagy and P. Vojtěchovský. LOOPS: Computing with quasigroups and loops in GAP, version 3.4.0, package for GAP, https://cs.du.edu/~petr/loops/
[14] H.O. Pflugfelder, Quasigroups and Loops: Introduction, Sigma Series in Pure Math., 7, Heldermann, 1990.
[15] W.R. Scott, Half-homomorphisms of groups, Proc. Amer. Math. Soc. 8 (1957), 1141-1144.
[16] G. Souza dos Anjos, Half-automorphism group of Chein loops, arXiv:2002.06853.

Instituto de Matemática e Estatística, Universidade de São Paulo
Rua do Matão, 1010, 05508-090, São Paulo - SP, Brazil
E-mail: giliards@ime.usp.br


[^0]:    2010 Mathematics Subject Classification: 17A30, 17B63, 17B70, 17B75.
    Keywords: Hom-Poisson color algebras, bijective even linear map, element of centroid, averaging operator, Rota-Baxter operator and multiplier.

[^1]:    2010 Mathematics Subject Classification: 16T25, 05C25, 16S99, 16 Z 05.
    Keywords: Hom-Jacobi-Jordan algebra, hom-antiassociative algebra, representation, quadratic form

[^2]:    2010 Mathematics Subject Classification: 94A60, 16Z05, 14G50, 11T71, 16S50
    Keywords: non-commutative algebra, finite associative algebra, single-sided units, postquantum cryptography, public-key cryptoscheme, signature scheme, discrete logarithm problem, hidden logarithm problem

[^3]:    2010 Mathematics Subject Classification: 16Y60, 06F25.
    Keywords: ordered $k$-ideal, prime ordered $k$-ideal, semiprime ordered $k$-ideal, pure ordered $k$-ideal, ordered semiring, ordered $k$-regular semiring.

[^4]:    2010 Mathematics Subject Classification: 20M10, 20M20
    Keywords: left magnifying element, right magnifying element, partial transformation semi-
    group, partial transformation semigroup with invariant set
    This research was supported by Chiang Mai University.

[^5]:    2010 Mathematics Subject Classification: 16Y60, 06F25.
    Keywords: ordered semiring, regular ordered semiring, ordered ideal, fully idempotent ordered semiring, $(m, n)$-regular ordered semiring, ordered $(m, n)$-ideal.

[^6]:    2010 Mathematics Subject Classification: 20N02, 20N05
    Keywords: half-isomorphism, half-automorphism, special half-isomorphism, groupoid, quasigroup, loop.

