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Quasigroup power sets and cyclic S–systems

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Abstract

We give new constructions of power sets of quasigroups (latin squares) based on pairwise balanced block designs and complete cyclic S-systems of quasigroups.

1. Introduction

Let L be a fixed latin square of order n with elements of the set $Q = \{0, 1, \dots, n-1\}$ and $(\alpha_0, \alpha_1, \dots, \alpha_{n-1})$ be an ordered set of permutations $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$, where row i of L is the image of $(0, 1, \dots, n-1)$ under the permutation α_i, $0 \leqslant i \leqslant n-1$. We write $L = (\alpha_0, \alpha_1, \dots, \alpha_{n-1})$. If $R = (\beta_0, \beta_1, \dots, \beta_{n-1})$ is another latin square of order n, then the product square LR is defined as $(\alpha_0\beta_0, \alpha_1\beta_1, \dots \dots, \alpha_{n-1}\beta_{n-1})$, where $\alpha_i\beta_i$ denotes the usual product of the permutation α_i on β_i.

Let L be a latin square of order n and m a positive integer greater than one. If $L^2, L^3, \dots, L^m$ are all latin squares, then $\{L, L^2, \dots, L^m\}$ is called a *latin power set of size m*. This concept was introduced explicitly in [7] and implicitly in [13]. In this case the latin squares $L, L^2, \dots, L^m$ are pairwise orthogonal [15], Theorem 1.

The authors of [7] conjectured that for all $n \neq 2, 6$ there exists a latin power set consisting of at least two $n \times n$ latin squares. This problem was also put by J. Dénes in [5]. A proof of this conjecture would provide

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a new disproof of the Euler conjecture (if $n = 4k + 2$, then there is no pair of orthogonal latin squares of order n). A construction in [7], based on Mendelsohn designs, gives infinitely many counterexamples to the Euler conjecture but unfortunately the construction does not work when $n \equiv 2 \pmod 6$. In [7] it was proved that for $7 \leqslant n \leqslant 50$ and for all larger n except possibly those of the form $6k + 2$ there exists a latin power set containing at least two latin squares of order n.

In [8] J. Dénes and P.J. Owens gave a new construction of power sets of $p \times p$ latin squares for all primes $p \geqslant 11$ not based on group tables. Such latin power sets of prime order can be used to obtain latin power sets of a composite order by the known methods.

The main construction of [8] is based on a circular Tuscan square.

As is noted in [8], for both theoretical and practical reasons it is important to find latin power sets that are not based on group tables (the sets given in [8] are constructed by using rearrangements of rows of a group table). It is important, for example, for a ciphering device, whose algorithm is based on latin power sets [9]. It is obvious that latin power sets based on non-group tables are preferable to those based on group tables because the greater irregularity makes the cipher safer.

In this article we use an algebraic approach to latin power sets. In Section 1 some necessary information from [1, 2, 3] concerning S-systems of quasigroups is given. In Section 2 we use cyclic S-systems (they are a particular case of latin power sets) and pairwise balanced block designs of index one ($BIB(v, b, r, k, 1)$) for the construction of quasigroup power sets of different sizes.

The suggested construction, in particular, is used to obtain power sets of quasigroups of all orders $n = 12t + 8 = 6(2l + 1) + 2$, $t, l \geqslant 1$, i.e. for any $n = 6k + 2$ where k is an odd number, $k \geqslant 3$.

In Section 3, there is described a composite method of constructing quasigroup (latin) power sets based on pairwise balanced block designs of index one of type $(v; k_1, k_2, \ldots, k_m)$ $(BIB(v; k_1, k_2, \ldots, k_m))$. At the end of this section the sizes of quasigroup power sets are given that can be constructed using some known block designs and cyclic S-systems by means of the suggested methods.

2. Cyclic S-systems as quasigroup power sets

Let $Q(A)$ and $Q(B)$ be groupoids. *Mann's (right) multiplication* $B \cdot A$ of the operation B on A is defined in the following way [14]:

$$B \cdot A(x,y) = B(x, A(x,y)), \quad x, y \in Q.$$

The operation $(\cdot)$ on the set of all operations defined on the set Q is associative, i.e. $(A \cdot B) \cdot C = A \cdot (B \cdot C)$. If $Q(A)$ and $Q(B)$ are quasigroups and L, R are the latin squares corresponding to them, then

$$B \cdot A(x,y) = \beta_x \alpha_x y,$$

where $\beta_x y = B(x,y)$, $\alpha_x y = A(x,y)$ and β_x (α_x) is row x of R (L).

Thus, Mann's (right) multiplication of quasigroups corresponds to the product of the respective latin squares and conversely.

Let $A = B$, then we get

$$A \cdot A = A^2, \quad A \cdot A \cdot A = A^3, \quad \dots, \quad \underbrace{A \cdot A \cdot \dots \cdot A}_{m} = A^m.$$

If $A, A^2, \dots, A^m$ are quasigroups, then $\{A, A^2, \dots, A^m\}$ is called a *quasigroup power set* (briefly QPS), $\{L, L^2, \dots, L^m\}$ is the latin power set corresponding to this QPS.

Let $\Sigma = \{A, B, C, \dots\}$ be a system of binary operations given on Q.

Definition 1. [1] A system of operations $Q(\Sigma)$ is called an *S–system* if

1. Σ contains the unit operations F and E $(F(x,y) = x,\ E(x,y) = y$ $\forall x, y \in Q)$ and the remaining operations define quasigroups,
2. $A \cdot B \in \Sigma'$ for all $A, B \in \Sigma'$, where $\Sigma' = \Sigma \setminus F$,
3. $A^* \in \Sigma$ for all $A \in \Sigma$, where $A^*(x,y) = A(y,x)$.

An S-system $Q(\Sigma)$ is finite if Q is a finite set. In finite $Q(\Sigma)$ for any $A \in \Sigma$ is defined A^{-1} as the solution of the equation $A(a,x) = b$, i.e. $A^{-1}(a,b) = x$. Then $A^{-1} = A^k \in \Sigma$ for some natural k, because the set of all invertible to the right operations on Q forms a finite group with respect to the right multiplication of operations. In this group E is the unit and $A^{-1} \cdot A = A \cdot A^{-1} = E$.

We remind the reader that two binary operations A and B defined on Q are said to be orthogonal if the pair of equations $A(x,y) = a$ and $B(x,y) = b$ has a unique solution for any elements $a, b \in Q$.

All operations of an S-system $Q(\Sigma)$ are pairwise orthogonal and the following properties of finite S-systems are also important:

1. Σ' is a group with respect to the (right) multiplication of operations, E is the unit of this group and A^{-1} is the inverse element of A.

2. All the quasigroups of $Q(\Sigma)$ are idempotent, if $|\Sigma| \geqslant 4$, where $|\Sigma|$ is the number of operations of $Q(\Sigma)$.

Theorem 1. (Theorem 4.3 in [1]) *Let $Q(\Sigma)$ be an $S-$system, $|Q| = n$, $|\Sigma| = k$, then $k-1$ divides $n-1$ and $r = \frac{n-1}{k-1} \geqslant k$ or $r = 1$.*

In [1] the number r is called an *index* of the S-system $Q(\Sigma)$. The number k is called *order* of $Q(\Sigma)$.

An S-system is called *complete* if $r = 1$ (in this case $n = k$). It then contains $n-2$ quasigroups.

A characterization for a finite complete S-system was given in [1], Theorem 4.6.

Definition 2. [3] An S-system $Q(\Sigma)$ of order k is called *cyclic* if $\Sigma'(\cdot)$, where $\Sigma' = \Sigma \setminus F$ and $(\cdot)$ represents composition of operations (called Mann's multiplication above), is a cyclic group.

By Corollary 1 of [3] a complete S-system $Q(\Sigma)$ is cyclic iff it is an S-system over a field $Q(\cdot,+)$, i.e. iff every operation of Σ has the form

$$A_a(x,y) = (1-a)x + ay, \quad a,x,y \in Q, \tag{1}$$

where 1 is the unit of the multiplicative group of the field.

Remark 1. *If $Q(\Sigma)$, $\Sigma = \{F, E, A, A^2, \ldots, A^{k-2}\}$, is a complete cyclic $S-$system of order k, then*

$$A(x,y) = (1-a)x + ay,$$

where the element a is a generating element of the multiplicative (cyclic) group of a field. Indeed, it is easy to prove that

$$A^l(x,y) = (1-a^l)x + a^l y, \quad l = 1,2,\ldots,k-2,$$

and $A^{k-1} = E$ iff $a^{k-1} = 1$.

Conversely, if an element a is a generating element of the multiplicative group of a field, then the quasigroup A_a of (1) generates a complete cyclic $S-$system.

Every cyclic S-system of order k and index r corresponds to a quasigroup power set of size $k-2$ and consists of quasigroups of order $n = rk - r + 1$.

From the results of [2, 3], the description of an arbitrary cyclic S-system by means of a field and an incomplete balanced block design can be obtained. First we need the following definitions.

Definition 3. [6] A *balanced incomplete block design* (or $BIB(v, b, r, k, \lambda)$) is an arrangement of v elements $a_1, a_2, \ldots, a_v$ by b blocks such that

1. every block contains exactly k different elements;
2. every element appears in exactly r different blocks;
3. every pair of different elements (a_i, a_j) appears in exactly λ blocks.

Definition 4. [2] A $BIB(v, b, r, k, 1)$ is called an $S(r, k)-$*configuration* if k is a prime power, i.e. $k = p^\alpha$.

It is known that the parameters of a $BIB(v, b, r, k, 1)$ satisfy the following equalities

$$v = rk - r + 1, \qquad b = \frac{rk - r + 1}{k} r \ .$$

In accordance with Theorem 1 of [2] a cyclic S-system of index r and order k exists iff there exists an $S(r, k)$-configuration.

Let us give a construction of an S-system of order k and index r for the case of a cyclic S-system.

Let an $S(r, k)$-configuration be given on a set Q, where $|Q| = v = rk - r + 1$, and let $Q_1, Q_2, \ldots, Q_b$ be its blocks. Let $H(+, \cdot)$ be a field of order k (such a field exists as k is a prime power) and let $H(\tilde{\Sigma})$, $\tilde{\Sigma} = \{F, E, A_1, A_2, \ldots, A_{k-2}\}$, be a complete cyclic S-system over this field.

1. On the block Q_i $(i = 1, 2, \ldots, b)$ we define a quasigroup $Q_i(A_j^{(i)})$, $j = 1, 2, \ldots, k-2$, isomorphic to the quasigroup $H(A_j)$ of the S-system $H(\tilde{\Sigma})$:

$$A_j^{(i)}(x, y) = \alpha_i^{-1} A_j(\alpha_i x, \alpha_i y) = A_j^{\alpha_i}(x, y),$$

where α_i is an arbitrary one-to-one mapping of the set Q_i upon H, $i = 1, 2, \ldots, b$.

2. Then, on the set Q, we define the operations B_j, $j = 1, 2, \dots, k-2$, in the following way:

$$B_j(x,y) = \begin{cases} A_j^{(i)}(x,y), & \text{if} \quad x,y \in Q_i,\ x \neq y, \\ x, & \text{if} \quad x = y. \end{cases}$$

By Theorem 1 of [2] the system $Q(\Sigma)$, $\Sigma = \{F, E, B_1, \dots, B_{k-2}\}$, is an S-system of index r and order k. It is called an *$S-$system over the field $H(+,\cdot)$ and the $S(r,k)-$configuration.* Moreover, by Theorem 3 of [3] such an S-system is cyclic and any S-system over a field and an $S(r,k)$-configuration is cyclic.

If $\tilde{\Sigma} = \{F, E, A, A^2, \dots, A^{k-2}\}$, then by (1) and Remark 1

$$A_j(u,v) = A^j(u,v) = (1-a^j)u + a^j v, \quad j = 1, 2, \dots, k-2,$$

$u, v \in H$, where the element a is a generating element of the multiplicative group of the field $H(+,\cdot)$.

Hence,

$$\begin{aligned} A_j^{(i)}(x,y) &= \alpha_i^{-1}((1-a^j)\alpha_i x + (a^j \cdot \alpha_i y)) = \alpha_i^{-1} A^j(\alpha_i x, \alpha_i y) \\ &= (A^j)^{\alpha_i}(x,y) = (A^{\alpha_i})^j(x,y), \qquad x, y \in Q_i, \end{aligned}$$

since it is easy to see that $(A \cdot B)^\alpha = A^\alpha \cdot B^\alpha$ if α is an isomorphism, $(A \cdot B)^\alpha(x,y) = \alpha^{-1}[(A \cdot B)(\alpha x, \alpha y)]$. Then

$$B^j(x,y) = B_j(x,y) = \begin{cases} (A^{\alpha_i})^j(x,y), & \text{if} \quad x,y \in Q_i,\ x \neq y, \\ x, & \text{if} \quad x = y \end{cases}$$

and $\Sigma = \{F, E, B, B^2, \dots, B^{k-2}\}$.

In an Appendix we give an illustrative example of this construction.

3. Direct product of quasigroup power sets

Let $Q_1(A_1)$, $Q_2(A_2)$ be two binary groupoids. On the set $Q_1 \times Q_2$ which consists of all pairs (a_1, a_2), where $a_i \in Q_i$, $i = 1, 2$, define the direct product $A_1 \times A_2$ of the operations A_1 and A_2:

$$(A_1 \times A_2)((x_1, x_2), (y_1, y_2)) = (A_1(x_1, y_1), A_2(x_2, y_2)).$$

If A_1, A_2 are quasigroup operations, then $A_1 \times A_2$ also is a (binary) quasigroup operation.

Proposition 1. $(A_1 \times A_2)^m = A_1^m \times A_2^m$ *for any natural number* m.

Proof. Let $m = 2$, then

$$\begin{aligned}
&(A_1 \times A_2)^2((x_1, x_2), (y_1, y_2)) = \\
&\qquad = (A_1 \times A_2)((x_1, x_2), (A_1 \times A_2)((x_1, x_2), (y_1, y_2))) = \\
&\qquad = (A_1 \times A_2)((x_1, x_2), (A_1(x_1, y_1), A_2(x_2, y_2))) = \\
&\qquad = (A_1(x_1, A_1(x_1, y_1)), A_2(x_2, A_2(x_2, y_2))) = \\
&\qquad = (A_1^2(x_1, y_1), A_2^2(x_2, y_2)) = (A_1^2 \times A_2^2)((x_1, x_2), (y_1, y_2)) \, .
\end{aligned}$$

Hence,

$$(A_1 \times A_2)^2 = A_1^2 \times A_2^2 \, .$$

But then

$$(A_1 \times A_2)^3 = (A_1 \times A_2)(A_1 \times A_2)^2 = (A_1 \times A_2)(A_1^2 \times A_2^2)$$

since the Mann's multiplication $(\cdot)$ of operations is associative. Using that we can similarly show that

$$(A_1 \times A_2)^3 = A_1^3 \times A_2^3 \, .$$

Hence, by induction on the integer m, we may deduce that Proposition 1 is true. □

Let $Q_1(A_1), Q_2(A_2), \ldots, Q_n(A_n)$ be binary groupoids. On the set $Q_1 \times Q_2 \times \cdots \times Q_n$ define the direct product of the operations $A_1, \ldots, A_n$

$$\begin{aligned}
&(A_1 \times A_2 \times \cdots \times A_n)((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \\
&\qquad = (A_1(x_1, y_1), A_2(x_2, y_2), \ldots, A_n(x_n, y_n)).
\end{aligned}$$

Proposition 1 at once implies

Corollary 1. $(A_1 \times A_2 \times \cdots \times A_n)^m = A_1^m \times A_2^m \times \cdots \times A_n^m$.

Now let us consider the following n QPSs:

$$Q_i(\Sigma_i), \qquad \Sigma_i = \{A_i, A_i^2, \ldots, A_i^{m_i}\}, \qquad i = 1, 2, \ldots, n,$$

and on the set $Q_1 \times Q_2 \times \cdots \times Q_n$ define the set

$$\begin{aligned}
&\Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_n = \{(A_1 \times A_2 \times \cdots \times A_n), \\
&(A_1^2 \times A_2^2 \times \cdots \times A_n^2), \ldots, (A_1^m \times A_2^m \times \cdots \times A_n^m)\},
\end{aligned}$$

where $m = \min\{m_1, m_2, \dots m_n\}$. By Corollary 1

$$\Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_n = \{(A_1 \times A_2 \times \cdots \times A_n),\\ (A_1 \times A_2 \times \cdots \times A_n)^2, \dots, (A_1 \times A_2 \times \cdots \times A_n)^m\},$$

and $(Q_1 \times Q_2 \times \cdots \times Q_n)(\Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_n)$ is a QPS of size m which consists of quasigroups of order $|Q_1| \cdot |Q_2| \cdot \ldots \cdot |Q_n|$. We call this QPS the direct product of QPSs $Q_i(\Sigma_i)$, $i = 1, 2, \dots, n$.

Theorem 2. *Let $n = p_1^{u_1} p_2^{u_2} \dots p_s^{u_s}$, where for all $i = 1, \dots, s$, the p_i are prime numbers, the u_i are natural numbers and $m = \min\{p_1^{u_1}, \dots, p_s^{u_s}\} \geqslant 4$. Then there exists a quasigroup power set containing $m-2$ quasigroups of order n.*

Proof. Let

$$p_1^{u_1} \leqslant p_2^{u_2} \leqslant \cdots \leqslant p_s^{u_s}, \quad \text{where} \quad p_i^{u_i} \neq 2, 3,$$

$$\text{and} \quad Q_i(\Sigma_i) = \left\{F, E, A_i, A_i^2, \dots, A_i^{p_i^{u_i}-2}\right\}$$

be a complete cyclic S-system of order $p_i^{u_i} = |Q_i|$, $i = 1, 2, \dots, s$. By Corollary 1 of [3] such an S-system is an S-system over a field of order $p_i^{u_i}$ and its binary operations have the form (1). Using the direct product of QPSs, we deduce that $(Q_1 \times Q_2 \times \cdots \times Q_s)(\Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_s)$ is a QPS of size $p_1^{u_1} - 2$ containing quasigroups of order n. □

Note that, under different representations of a number n by powers of prime numbers, the quasigroup power sets obtained by Theorem 2 are different. For example, if $n = 7 \cdot 5^2$ we can construct a QPS of 5 quasigroups, whereas for $n = 5 \cdot 5 \cdot 7$ we obtain a QPS of 3 quasigroups of order n.

As has been noted, numbers of the form $6k+2$ present definite difficulties for the construction of latin power sets (or QPSs). As an application of Theorem 2 let us consider numbers of this form when k is odd, i.e.

$$\begin{aligned} n &= 6(2t+1)+2 = 12t+8 = 2^2(3t+2), \quad t \geqslant 1 \\ (\, n &= 20, 32, 44, 56, \dots, 92, 104, \dots, 140, 152, \dots). \end{aligned}$$

Corollary 2. *Let $n = 12t + 8$, $t \geqslant 1$. Then there exists a QPS containing at least two quasigroups of order n. If $t = 4k$, $k \geqslant 1$ then there exists a QPS containing at least three quasigroups. Moreover, if $k = 1$, then there exists a QPS of five quasigroups. For $2 \leqslant k \leqslant 9$ there exists a QPS of six quasigroups.*

Proof. The number $n = 2^2(3t+2)$ is not divisible by three. This implies that, in the factorization of n into prime powers, all $p_i^{\alpha_i} \geqslant 4$ and so, according to Theorem 2, there exists a QPS consisting of at least two quasigroups of order n.

Let $t = 4k$, $k \geqslant 1$, then $n = 2^3(6k+1)$ where $6k+1$ is an odd number $\geqslant 7$ that is not divisible by 2 and 3. Thus, the number 5 is the least possible divisor of $6k+1$ and by Theorem 2, there exists a QPS of three quasigroups of order n.

By Theorem 2 there exist QPSs of at least five quasigroups of order $n = 56$ ($t = 4$, $k = 1$). If $t = 4k$, $2 \leqslant k \leqslant 8$, then $6k+1 = 13, 19, 25, 31, 37, 43, 49\ldots$ and there exist QPSs of at least six quasigroups of order $n = 104 = (2^3 \cdot 13), 152 = (2^3 \cdot 19), 200, 248, \ldots, 392 = (2^3 \cdot 7^2)$. □

4. Quasigroup power sets and $\mathbf{BIB(v; k_1, k_2, \ldots, k_s)}$

To obtain a further construction of QPSs, we use a generalization of the concept of a balanced incomplete block design called by R.C. Bose and S.S. Shrikhande a *pairwise balanced block design of index unity and type* $(v; k_1, k_2, \ldots, k_s)$ (for brevity, we shall write $BIB(v; k_1^s)$) (see [6], page 400; [12], page 271). Such a design comprises a set of v elements arranged in $b = \sum_{i=1}^{s} b_i$ blocks such that there are b_1 blocks each of which contains k_1 elements; b_2 blocks each of which contains k_2 elements, ... b_s blocks each of which contains k_s elements ($k_i \leqslant v$ for $i = 1, 2, \ldots, s$), and in which each pair of the v distinct elements occurs together in exactly one of the b blocks.

The latter condition implies that

$$v(v-1) = \sum_{i=1}^{s} b_i k_i (k_i - 1).$$

If $k_1 = k_2 = \cdots = k_s = k$, then we obtain the (pairwise) balanced incomplete block design ($BIB(v, b, r, k, 1)$).

By Theorem 11.2.2 [6] if a pairwise balanced block design of index unity and type $(v; k_1^s)$ exists and for each k_i there exists a set of $q_i - 1$ mutually orthogonal latin squares of that order then it is possible to construct a set of $q-2$ mutually orthogonal latin squares of order v, where $q = \min\{q_1, q_2, \ldots, q_s\}$.

We prove that an analogous statement is true for latin power sets (that is for QPSs) using a constructing of idempotent quasigroups by means of

$BIB(v;k_1^s)$ given in [4] (see also [10]). First, we describe briefly the construction of such quasigroups from [4].

Let $Q_1, Q_2, \dots, Q_b$ be blocks of $BIB(v;k_1^s)$, given on a set Q, and $Q_1(A_1), Q_2(A_2), \dots, Q_b(A_b)$ be idempotent quasigroups. Note that, in contrast to [4], we assume for the sake of simplicity that these quasigroups are given on the blocks of the BIB.

Define the operation $(\cdot)$ on the set Q in the following way:

$$x \cdot y = \begin{cases} A_i(x,y), & \text{if } \ x,y \in Q_i, \ x \neq y; \\ x, & \text{if } \ x = y. \end{cases} \tag{2}$$

The groupoid $Q(\cdot)$ is an idempotent quasigroup and the operation $(\cdot)$ will be denoted by

$$(\cdot) = A = [A_i]_{i=1}^{b}(v;k_1^s). \tag{3}$$

The quasigroup $Q(\cdot)$ consists of quasigroups defined on the blocks of the $BIB(v;k_1^s)$.

Now we prove the following

Proposition 2. *In* (3)*, let A_i be an idempotent quasigroup for any $i = 1, 2, \dots, b$. Then*

$$A^k = [A_i^k]_{i=1}^{b}(v;k_1^s) \tag{4}$$

for any natural number k.

Proof. First notice that $x, A(x,y) \in Q_i$ where $x \neq y$, iff $x, y \in Q_i$. Granted this and the idempotency of A and A_i for any $i = 1, 2, \dots, b$, by (2) we have

$$\begin{aligned} A^2(x,y) &= A(x, A(x,y)) = \\ &= \begin{cases} A_i(x, A(x,y)), & \text{if } \ x, A(x,y) \in Q_i, \ A(x,y) \neq x; \\ x, & \text{if } \ A(x,y) = x; \end{cases} \\ &= \begin{cases} A_i(x, A_i(x,y)), & \text{if } \ x,y \in Q_i, \ x \neq y; \\ x, & \text{if } \ x = y; \end{cases} \\ &= \begin{cases} A_i^2(x,y), & \text{if } \ x,y \in Q_i, \ x \neq y; \\ x, & \text{if } \ x = y. \end{cases} \end{aligned}$$

Thus,

$$A^2 = [A_i^2]_{i=1}^{b}(v;k_1^s). \tag{5}$$

Further, since A, A_i, A_i^2 are idempotent quasigroups for all $i = 1, 2, \dots, b$, then using (2) and (5) we have

$$A^3(x,y) = A^2(x, A(x,y)) =$$

$$= \begin{cases} A_i^2(x, A(x,y)), & \text{if} \quad x, A(x,y) \in Q_i, \ A(x,y) \neq x; \\ x, & \text{if} \quad A(x,y) = x; \end{cases}$$
$$= \begin{cases} A_i^2(x, A_i(x,y)), & \text{if} \quad x, y \in Q_i, \quad x \neq y; \\ x, & \text{if} \quad x = y; \end{cases}$$
$$= \begin{cases} A_i^3(x,y), & \text{if} \quad x, y \in Q_i, \quad x \neq y; \\ x, & \text{if} \quad x = y. \end{cases}$$

Hence,

$$A^3 = [A_i^3]_{i=1}^b (v; k_1^s).$$

Extending this argument (that is, using induction on the index l) and taking into account that A_i^l, $i = 1, 2, \ldots, b$, $l = 1, 2, \ldots, k-1$, and A^l, $l = 1, 2, \ldots, k-1$, are all idempotent quasigroups we obtain equality (4). □

Now it is easy to prove the following

Theorem 3. *Suppose that there exists a BIB of index unity and type $(v; k_1, k_2, \ldots, k_s)$ and that, for every k_i, $i = 1, 2, \ldots, s$, there exists a QPS of a size m with idempotent quasigroups of order k_i. Then there exists a QPS of m quasigroups of order v.*

Proof. Let a $BIB(v; k_1^s)$ be given on a set Q and have the blocks $, Q_1, Q_2, \ldots, Q_b$, $|Q_i| \in \{k_1, k_2, \ldots, k_s\}$. Let the following quasigroup power sets of size m on these blocks be given:

$$\begin{array}{lcl} Q_1(\Sigma_1) & : & \Sigma_1 = \{A_1, A_1^2, \ldots, A_1^m\}, \\ Q_2(\Sigma_2) & : & \Sigma_2 = \{A_2, A_2^2, \ldots, A_2^m\}, \\ \cdots\cdots & & \cdots\cdots\cdots\cdots\cdots\cdots \\ Q_b(\Sigma_b) & : & \Sigma_b = \{A_b, A_b^2, \ldots, A_b^m\}, \end{array}$$

where $Q_1(A_1), Q_2(A_2), \ldots, Q_b(A_b)$ are idempotent quasigroups (then all their powers in the power sets are also idempotent).

Consider the following quasigroups on the set Q:

$$C_1 = [A_i]_{i=1}^b (v; k_1^s), \quad C_2 = [A_i^2]_{i=1}^b (v; k_1^s), \quad \ldots, \quad C_m = [A_i^m]_{i=1}^b (v; k_1^s).$$

Using (4) we obtain that $C_2 = C_1^2$, $C_3 = C_1^3$, $\ldots$, $C_m = C_1^m$. Hence, $Q(\Sigma) : \Sigma = \{C_1, C_1^2, \ldots, C_1^m\}$ is a QPS of size m containing quasigroups of order v. □

Corollary 3. *If there exists a $BIB(v; k_1^s)$ where k_i, $i = 1, 2, \ldots, s$ are powers of primes and $t = \min\{k_1, k_2, \ldots, k_s\} \geqslant 4$, then there exists a QPS containing $t-2$ quasigroups of order v.*

Proof. As k_i, $i = 1, 2, \ldots, s$, are prime powers then, by Corollary 1 of [3], for every k_i there exists a complete cyclic S-system (over a field of order k_i). This S-system contains $k_i - 2$ (idempotent) quasigroups. Now, applying Theorem 3 completes the proof. □

Corollary 4. *Let* $k, k+1, m, x$ *be prime powers,* $4 \leqslant k \leqslant m$, $4 \leqslant x \leqslant m$, $t = \min\{k, x\}$. *Then there exists a QPS which contains* $t - 2$ *quasigroups of order* $v = km + x$.

Proof. Let $N(m)$ denote the largest possible number of mutually orthogonal latin squares of order m which can exist in a single mutually orthogonal set and $k \leqslant N(m) + 1 \leqslant m$, $x \leqslant m$. Then (see [6], p. 412–413) there exists a $BIB(km + x; k, k+1, x, m)$ of index unity.

Since m is a prime power then there exists a complete set of mutually orthogonal latin squares (i.e. $N(m) = m - 1$) of order m. In this case the equalities $k \leqslant m$ and $k \leqslant N(m) + 1$ are equivalent. Finally use Corollary 3 taking into account that under our conditions $\min\{k, k+1, x, m\} = \min\{k, x\}$. □

Next we apply Theorem 3, Corollary 3 and Corollary 4 to construct a number of QPSs using some known BIBs $(v; b, r, k, 1)$ and BIBs $(v; k_1^s)$.

Let a $BIB(v, b, r, k, 1)$ be given on a set Q, $|Q| = v$. By removing one element from this BIB, we can obtain a $BIB(v-1; k-1, k)$, that contains r blocks of $k-1$ elements and $b-r$ blocks of k elements. In the table presented below we give initial BIBs $(v, b, r, k, 1)$ (with the numbers assigned to them in the Table of Appendix I of [12]), the corresponding $BIB(v-1; k-1, k)$, the size of QPS obtained by Corollary 3 and also that obtained by Theorem 2 (for comparison) for the same values of v.

BIB No. from [12]	BIB $(v; b, r, k, 1)$	BIB $(v-1; k-1, k)$	Size QPS by Cor. 3	Size QPS by Th. 2
7	$(21, 21, 5, 5, 1)$	$(20; 4, 5)$	2	2
11	$(25, 30, 6, 5, 1)$	$(24; 4, 5)$	2	–
25	$(57, 57, 8, 8, 1)$	$(56; 7, 8)$	5	5
36	$(64, 72, 9, 8, 1)$	$(63; 7, 8)$	5	5
37	$(73, 73, 9, 9, 1)$	$(72; 8, 9)$	6	6
42	$(41, 82, 10, 5, 1)$	$(40; 4, 5)$	2	3
45	$(81, 90, 10, 9, 1)$	$(80; 8, 9)$	6	3
51	$(45, 99, 11, 5, 1)$	$(44; 4, 5)$	2	2
108	$(61, 183, 15, 5, 1)$	$(60; 4, 5)$	2	–
[6], p.403		$(22; 4, 7)$	2	–

Now we use Corollary 4 to obtain new QPSs with quasigroups of order $v = km + x$, where the numbers k, m, x satisfy the conditions of the corollary. In the table given below, we present some $BIB(km+x; k, k+1, x, m)$ with such values of k, m, x and also the sizes of the QPSs (with quasigroups of order $v = km + x$) constructed by Corollary 4 and Theorem 2 corresponding to them.

BIB $(km+x; k, k+1, x, m)$	$v = km + x$	Size of QPS by Cor. 4	Size of QPS by Th. 2
$(60; 7, 8, 4, 8)$	$60 = 2^2 \cdot 3 \cdot 5$	2	–
$(63; 7, 8, 7, 8)$	$63 = 3^2 \cdot 7$	5	5
$(69; 8, 9, 5, 8)$	$69 = 3 \cdot 23$	3	–
$(76; 8, 9, 4, 9)$	$76 = 2^2 \cdot 19$	2	2
$(80; 8, 9, 8, 9)$	$80 = 2^4 \cdot 5$	6	3
$(92; 8, 9, 4, 11)$	$92 = 2^2 \cdot 23$	2	2
$(93; 8, 9, 5, 11)$	$93 = 3 \cdot 31$	3	–
$(95; 8, 9, 7, 11)$	$95 = 5 \cdot 19$	5	3
$(96; 8, 9, 8, 11)$	$96 = 2^5 \cdot 3$	6	–
$(99; 8, 9, 11, 11)$	$99 = 3^2 \cdot 11$	6	7
$(108; 8, 9, 4, 13)$	$108 = 2^2 \cdot 3^3$	2	2
$(111; 8, 9, 7, 13)$	$111 = 3 \cdot 37$	5	–
$(112; 8, 9, 8, 13)$	$112 = 2^4 \cdot 7$	6	5
$(115; 8, 9, 11, 13)$	$115 = 5 \cdot 23$	6	3
$(132; 8, 9, 4, 16)$	$132 = 2^2 \cdot 3 \cdot 11$	2	–
$(133; 8, 9, 5, 16)$	$133 = 7 \cdot 19$	3	5
$(135; 8, 9, 7, 16)$	$135 = 3^3 \cdot 5$	5	3
$(136; 8, 9, 8, 16)$	$136 = 2^3 \cdot 17$	6	6
$(140; 8, 9, 4, 17)$	$140 = 2^2 \cdot 5 \cdot 7$	2	2
$(141; 8, 9, 5, 17)$	$141 = 3 \cdot 47$	3	–
$(141; 8, 9, 13, 16)$	$141 = 3 \cdot 47$	6	–
$(143; 8, 9, 7, 17)$	$143 = 11 \cdot 13$	5	9
$(144; 8, 9, 8, 17)$	$144 = 2^4 \cdot 3^2$	6	7
$(145; 8, 9, 9, 17)$	$145 = 5 \cdot 29$	6	3
$(147; 8, 9, 11, 17)$	$147 = 3 \cdot 7^2$	6	–
$(152; 8, 9, 16, 17)$	$152 = 2^3 \cdot 19$	6	6
$(153; 8, 9, 17, 17)$	$153 = 3^2 \cdot 17$	6	7

The parameters of the following BIBs $(km+x; k, k+1, x, m)$:

$$(20;4,5,4,4),\ (24;4,5,4,5),\ (25;4,5,5,5),\ (32;4,5,4,7),$$
$$(33;4,5,5,7),\ (35;4,5,7,7),\ (36;4,5,4,8),\ (37;4,5,5,8),$$
$$(39;4,5,7,8),\ (40;4,5,8,8),\ (40;4,5,4,9),\ (41;4,5,5,9),$$
$$(43;4,5,7,9),\ (44;4,5,8,9), (45;4,5,9,9)$$

also satisfy the conditions of Corollary 4. Using these BIBs one can construct QPSs containing at least two quasigroups of order v.

Appendix

Now we give an illustrative example using the construction of QPSs from Section 2.

Let $H(\oplus,\cdot)$, where $H=\{0,1,2,3,4,\}$, be the finite field formed by the residues modulo 5. The element 2 is a generating element of the (cyclic) multiplicative group of this field, so we take the quasigroup $A(x,y) = (1-2)x+2y = 4x+2y$ as the defining quasigroup for a complete (cyclic) $S-$system $H(\tilde{\Sigma})$, $\tilde{\Sigma}=\{F,E,A,A^2,A^3\}$. The Cayley table of the quasigroup A is as follows:

A	0	1	2	3	4
0	0	2	4	1	3
1	4	1	3	0	2
2	3	0	2	4	1
3	2	4	1	3	0
4	1	3	0	2	4

As a block design we use the following

$$BIB(v,b,r,k,1) = BIB(21,21,5,5,1) = S(5,5)$$

on the set $Q=\{1,2,3,4,5,6,7,8,9,a,b,c,d,e,f,g,h,k,m,n,p\}$ of 21 elements and 21 blocks:

B_1 : $1,2,3,4,5,$
B_2 : $2,6,a,e,k,$
B_3 : $3,6,b,g,p,$
B_4 : $4,6,c,h,m,$
B_5 : $5,6,d,f,n,$
B_6 : $1,a,b,c,d,$
B_7 : $2,7,b,f,m,$
B_8 : $3,7,a,h,n,$
B_9 : $4,7,d,g,k,$
B_{10} : $5,7,c,e,p,$
B_{11} : $1,e,f,g,h,$
B_{12} : $2,8,c,g,n,$
B_{13} : $3,8,d,e,m,$
B_{14} : $4,8,a,f,p,$
B_{15} : $5,8,b,h,k,$
B_{16} : $1,k,m,n,p,$
B_{17} : $2,9,d,h,p,$
B_{18} : $3,9,c,f,k,$
B_{19} : $4,9,b,e,n,$
B_{20} : $5,9,a,g,m,$
B_{21} : $1,6,7,8,9.$

This block design is isomorphic to the finite projective plane of order 4 and corresponds to a complete set of orthogonal latin squares of order 4.

According to the results of Section 2 it is sufficient to construct the quasigroup $Q(B)$:

$$B(x,y) = \begin{cases} A^{\alpha_i}, & \text{if } x, y \in Q_i,\ x \neq y, \\ x, & \text{if } x = y. \end{cases}$$

Then $Q(\Sigma)$, $\Sigma = \{B, B^2, B^3\}$ is a QPS. The Cayley table for the quasigroup $Q(B)$ we fill out by subquasigroups given on the blocks of the BIB. These subquasigroups are isomorphic to the quasigroup $H(A)$:

$B(x,y) = \alpha_i^{-1} A(\alpha_i x, \alpha_i y),\ x, y \in Q_i$ or $B(\beta_i x, \beta_i y) = \beta_i A(x,y),\ x, y \in H,$

$$\beta_i = \alpha_i^{-1},\quad i = 1, 2, ..., 21,\ \ \alpha_i : Q_i \to H,\quad \alpha_i = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix},$$

if $Q_i = \{a_0, a_1, a_2, a_3, a_4\}$ (in the order of listing). For example

$$\alpha_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix},\quad \alpha_{15} = \begin{pmatrix} 5 & 8 & b & h & k \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}.$$

The quasigroup $Q(B)$ is defined by the following table.

B	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f	g	h	k	m	n	p
1	1	3	5	2	4	7	9	6	8	b	d	a	c	f	h	e	g	m	p	k	n
2	5	2	4	1	3	a	b	c	d	k	m	n	p	6	7	8	9	e	f	g	h
3	4	1	3	5	2	b	a	d	c	n	p	k	m	8	9	6	7	f	e	h	g
4	3	5	2	4	1	c	d	a	b	p	n	m	k	9	8	7	6	g	h	e	f
5	2	4	1	3	5	d	c	b	a	m	k	p	n	7	6	9	8	h	g	f	e
6	9	k	p	m	n	6	8	1	7	e	g	h	f	2	5	3	4	a	c	d	b
7	8	m	n	k	p	1	7	9	6	h	f	e	g	5	2	4	3	d	b	a	c
8	7	n	m	p	k	9	6	8	1	f	h	g	e	3	4	2	5	b	d	c	a
9	6	p	k	n	m	8	1	7	9	g	e	f	h	4	3	5	2	c	a	b	d
a	d	e	h	f	g	2	3	4	5	a	c	1	b	k	p	m	n	6	9	7	8
b	c	f	g	e	h	3	2	5	4	1	b	d	a	n	m	p	k	8	7	9	6
c	b	g	f	h	e	4	5	2	3	d	a	c	1	p	k	n	m	9	6	8	7
d	a	h	e	g	f	5	4	3	2	c	1	b	d	m	n	k	p	7	8	6	9
e	h	a	d	b	c	k	p	m	n	6	9	7	8	e	g	1	f	2	3	4	5
f	g	b	c	a	d	n	m	p	k	8	7	9	6	1	f	h	e	3	2	5	4
g	f	c	b	d	a	p	k	n	m	9	6	8	7	h	e	g	1	4	5	2	3
h	e	d	a	c	b	m	n	k	p	7	8	6	9	g	1	f	h	5	4	3	2
k	p	6	9	7	8	e	g	h	f	2	5	3	4	a	c	d	b	k	n	1	m
m	n	7	8	6	9	h	f	e	g	5	2	4	3	d	b	a	c	1	m	p	k
n	m	8	7	9	6	f	h	g	e	3	4	2	5	b	d	c	a	p	k	n	1
p	k	9	6	8	7	g	e	f	h	4	3	5	2	c	a	b	d	n	1	m	p

The subquasigroup on the block B_{15} has the following Cayley table:

	5	8	b	h	k
5	5	b	k	8	h
8	k	8	h	5	b
b	h	5	b	k	8
h	b	k	8	h	5
k	8	h	5	b	k

The subquasigroup on B_1 is in the left top corner of the Cayley table of the quasigroup $Q(B)$.

From the quasigroup $Q(B)$ it is easy to obtain the quasigroups B^2 and B^3. Thus we obtain a QPS $\{B, B^2, B^3\}$.

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The topological quasigroups with multiple identities

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Abstract

In this article we describe the topological quasigroups with (n, m)-identities, which are obtained by using isotopies of topological groups. Such quasigroups are called the (n, m)-homogeneous quasigroups. Our main goal is to extend some affirmations of the theory of topological groups on the class of topological (n, m)-homogeneous quasigroups.

1. General notes

A non-empty set G is said to be a *groupoid* relative to a binary operation denoted by $\cdot$ or by juxtaposition, if for every ordered pair a, b of elements of G, is defined a unique element $ab \in G$.

If the groupoid G is a topological space and the multiplication operation $(a, b) \rightarrow a \cdot b$ is continuous, then G is called a *topological groupoid.*

A groupoid G is called a *groupoid with division*, if for every $a, b \in G$ the equations $ax = b$ and $ya = b$ have solutions, not necessarily unique.

A groupoid G is called *reducible* or *cancellative*, if for each equality $xy = uv$ the equality $x = u$ is equivalent to the equality $y = v$.

A groupoid G is called a *primitive groupoid with the divisions*, if there exist two binary operations $l : G \times G \rightarrow G$, $r : G \times G \rightarrow G$ such that $l(a, b) \cdot a = b$, $a \cdot r(a, b) = b$ for all $a, b \in G$. Thus a primitive groupoid with divisions is a universal algebra with three binary operations.

If in a topological groupoid G the primitive divisions l and r are continuous, then we can say that G is a *topological primitive groupoid with continuous divisions.*

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A primitive groupoid G with divisions is called a *quasigroup* if every of the equations $ax = b$ and $ya = b$ has unique solution. In the quasigroup G the divisions l, r are uniques.

An element $e \in G$ is called an *identity* if $ex = xe = x$ for every $x \in X$. A quasigroup with an identity is called a *loop*.

If a multiplication operation in a quasigroup $(G, \cdot)$ with a topology is continuous, then G is called a *semitopological quasigroup*.

If in a semitopological quasigroup G the divisions l and r are continuous, then G is called a *topological quasigroup*.

A quasigroup G is called *medial* if it satisfies the law $xy \cdot zt = xz \cdot yt$ for all $x, y, z, t \in G$.

If a medial quasigroup G contains an element e such that $e \cdot x = x$ $(x \cdot e = x)$ for all x in G, then e is called a *left* (*right*) *identity element* of G and G is called a *left* (*right*) *medial loop*.

Let $N = \{1, 2, ...\}$ and $Z = \{..., -2, -1, 0, 1, 2, ...\}$. We shall use the terminology from [3, 5].

2. Multiple identities

We consider a groupoid $(G, +)$. For every two elements a, b from $(G, +)$ we denote

$$\begin{aligned} 1\,(a, b, +) &= (a, b, +)\,1 = a + b\,, \\ n\,(a, b, +) &= a + (n-1)\,(a, b, +)\,, \\ (a, b, +)\,n &= (a, b, +)\,(n-1) + b \end{aligned}$$

for all $n \geqslant 2$.

If a binary operation $(+)$ is given on a set G, then we shall use the symbols $n(a, b)$ and $(a, b)n$ instead of $n\,(a, b, +)$ and $(a, b, +)\,n$.

Definition 1. Let $(G, +)$ be a groupoid, $n \geqslant 1$ and $m \geqslant 1$. The element e of a groupoid $(G, +)$ is called an (n, m)-*zero* of G if $e + e = e$ and $n\,(e, x) = (x, e)\,m = x$ for every $x \in G$. If $e + e = e$ and $n\,(e, x) = x$ for every $x \in G$, then e is called an (n, ∞)-*zero*. If $e + e = e$ and $(x, e)\,m = x$ for every $x \in G$, then e is called an (∞, m)-*zero*. It is clear that $e \in G$ is an (n, m)-zero, if it is an (n, ∞)-zero and an (∞, m)-zero.

Remark 1. In the multiplicative groupoid $(G, \cdot)$ the element e is called an (n, m)-*identity*. The notion of the (n, m)-identity was introduced in [4].

Theorem 1. *Let $(G, \cdot)$ be a multiplicative groupoid, $e \in G$ and the following conditions hold:*

1. $ex = x$ *for every* $x \in G$;
2. $x^2 = x \cdot x = e$ *for every* $x \in G$;
3. $x \cdot yz = y \cdot xz$ *for all* $x, y, z \in G$;
4. *For every* $a, b \in G$ *there exists a unique point* $y \in G$ *such that* $ay = b$.

Then e *is a* $(1, 2)$*-identity in* G.

Proof. Fix $x \in G$. Pick $y \in G$ such that $xe \cdot y = x$. By virtue of the condition 2 we have $x \cdot (xe \cdot y) = x \cdot x = e$, i.e. $x \cdot (xe \cdot y) = e$. From the condition 3 it follows that $xe \cdot xy = e$. It is clear that $xe \cdot xe = e$. Thus $xe \cdot xy = xe \cdot xe$, $xy = xe$ and $y = e$. Therefore $(x \cdot e) \cdot e = (x \cdot e) \cdot y = x$ and e is a $(1, 2)$-identity. The proof is complete. □

Example 1. Let $(G, +)$ be a commutative additive group with a zero 0. Consider a new binary operation $x \cdot y = y - x$. Then $(G, \cdot)$ is a medial quasigroup with a $(1, 2)$-identity 0. If $x + x \neq 0$ for some $x \in G$, then 0 is not an identity in $(G, \cdot)$.

Theorem 2. *Let $(G, \cdot)$ be a multiplicative groupoid, $e \in G$ and the following conditions hold:*

1. $ex = x$ *for every* $x \in G$;
2. $x \cdot x = e$ *for every* $x \in G$;
3. $xy \cdot uv = xu \cdot yv$ *for all* $x, y, u, v \in G$;
4. *If* $xa = ya$, *then* $x = y$.

Then G *is a medial quasigroup with a* $(1, 2)$*-identity* e.

Proof. If $x \in G$, then $xe \cdot e = xe \cdot xx = xx \cdot ex = e \cdot ex = x$. Thus e is a $(1, 2)$-identity.

Consider the equation $xa = b$. Then $xa \cdot e = b \cdot e$, $xa \cdot ee = be$ and $xe \cdot ae = be$. Thus $(xe \cdot ae) \cdot (be) = e$, $(xe \cdot b) \cdot (ae \cdot e) = e$, $(xe \cdot b)\, a = e$, $(xe \cdot b) \cdot (ea) = e$, $(xe \cdot e) \cdot (ba) = e$ and $x \cdot ba = e$. Therefore $x \cdot ba = ba \cdot ba$ and $x = ba$. Since $ba \cdot a = ba \cdot ea = be \cdot aa = be \cdot e = b$, the element $x = ba$ is a unique solution of the equation $xa = e$. Now we consider the equation $ay = b$. In this case $be = ay \cdot e = ay \cdot aa = aa \cdot ya = e \cdot ya = ya$. Thus $y = be \cdot a$ is a unique solution of the equation $ay = b$. The proof is complete. □

Corollary 1. *Let* $(G, \cdot)$ *be a left medial loop,* $e \in G$ *and* $x^2 = e$ *for every* $x \in G$*. Then* e *is a* $(1, 2)$*-identity.*

3. Homogeneous isotopes

Definition 2. Let $(G, +)$ be a topological groupoid. A groupoid $(G, \cdot)$ is called a *homogeneous isotope* of the topological groupoid $(G, +)$ if there exist two topological automorphisms $\varphi, \psi : (G, +) \to (G, +)$ such that $x \cdot y = \varphi(x) + \psi(y)$ for all $x, y \in G$.

If $h : X \to X$ is a mapping, then $h^1(x) = h(x)$ and $h^n(x) = h\left(h^{n-1}(x)\right)$ for all $x \in X$ and $n \geqslant 2$.

Definition 3. Let $n, m \leqslant \infty$. A groupoid $(G, \cdot)$ is called an (n, m)*-homogeneous isotope* of a topological groupoid $(G, +)$ if there exist two topological automorphisms $\varphi, \psi : (G, +) \to (G, +)$ such that:

1. $x \cdot y = \varphi(x) + \psi(y)$ for all $x, y \in G$;
2. $\varphi\psi = \psi\varphi$;
3. If $n < +\infty$, then $\varphi^n(x) = x$ for every $x \in G$.
4. If $m < +\infty$, then $\psi^m(x) = x$ for every $x \in G$.

Definition 4. A groupoid $(G, \cdot)$ is called an *isotope* of a topological groupoid $(G, +)$, if there exist two homeomorphisms $\varphi, \psi : (G, +) \to (G, +)$ such that $x \cdot y = \varphi(x) + \psi(y)$ for all $x, y \in G$.

Under the conditions of Definition 4 we shall say that the isotope $(G, \cdot)$ is generated by the homeomorphisms φ, ψ of the topological groupoids $(G, +)$ and denote $(G, \cdot) = g(G, +, \varphi, \psi)$.

Theorem 3. *Let* $(G, +)$ *be a topological groupoid,* $\varphi, \psi : G \to G$ *be homeomorphisms and* $(G, \cdot) = g(G, +, \varphi, \psi)$*. Then:*

1. $(G, +) = \left(G, \cdot, \varphi^{-1}, \psi^{-1}\right)$*;*
2. $(G, \cdot)$ *is a topological groupoid;*
3. *If* $(G, +)$ *is a reducible groupoid, then* $(G, \cdot)$ *is a reducible groupoid too;*
4. *If* $(G, +)$ *is a groupoid with a division, then* $(G, \cdot)$ *is a groupoid with a division too;*
5. *If* $(G, +)$ *is a topological primitive groupoid with a division, then* $(G, \cdot)$

is a topological primitive groupoid with a division too;

6. *If* $(G,+)$ *is a topological quasigroup, then* $(G,\cdot)$ *is a topological quasigroup too;*
7. *If* $n,m,p,k \in N$ *and* $(G,\cdot)$ *is an* (n,m)*-homogeneous isotop of the groupoid* $(G,+)$ *and* e *is a* (k,p)*-zero in* $(G,+)$*, then* e *is an* (mk,np)*-identity in* $(G,\cdot)$.

Proof. We have $x \cdot y = \varphi(x) + \psi(y)$. Therefore

$$\varphi^{-1}(x) \cdot \psi^{-1}(y) = \varphi\left(\varphi^{-1}(x)\right) + \psi\left(\psi^{-1}(y)\right) = x + y$$

and $(G,+) = g\left(G,\cdot,\varphi^{-1},\psi^{-1}\right)$. The assertion 1 is proved. The assertion 2 and 3 are obvious.

Let $(G,+,r,l)$ be a topological primitive groupoid with the divisions, where $l : G \times G \to G$ and $r : G \times G \to G$ be continuous primitive divisions. Then the mappings $l_1(a,b) = \varphi^{-1}(l(\psi(a),b))$ and $r_1(a,b) = \psi^{-1}(r(\varphi(a),b))$ are the divisions of the groupoid $(G,\cdot)$. The divisions l_1, r_1 are continuous if and only if the divisions l, r are continuous. The assertions 4, 5 and 6 are proved.

Let $(G,\cdot)$ be an (n,m)-homogeneous isotope of the groupoid $(G,+)$ and e be a (k,p)-zero in $(G,+)$. We mention that $\varphi^q(e) = \psi^q(e) = e$ for every $q \in N$. If $k < +\infty$, then in $(G,+)$ we have $qk\,(e,x,+) = x$ for each $x \in G$ and for every $q \in N$.

Let $m < +\infty$ and $\psi^m(x) = x$ for all $x \in G$.

Then $1\,(e,x,\cdot) = 1\,(e,\psi(x),+)$ and $q\,(e,x,\cdot) = q\,(e,\psi^q(x),+)$ for every $q \geqslant 1$. Therefore

$$mk\,(e,x,\cdot) = mk\left(e,\psi^{mk}(x),+\right) = mk\,(e,x,+) = x.$$

Analogously we obtain that

$$(e,x,\cdot)\,np = (e,\varphi^{np}(x),+)\,np = (e,x,+)\,np = x.$$

Hence e is an (mk,np)-identity in $(G,\cdot)$. The statement 7 is proved. The proof of Theorem 3 is complete. □

Remark 2. Let $(G,+)$ be a topological quasigroup, $a,b \in G$ and φ,ψ be two automorphisms of $(G,+)$. If $x \cdot y = (a + \varphi(x)) + (\psi(y) + b)$, then we denote $(G,\cdot) = g\,(G,+,\varphi,\psi,a,b)$. It is clear that $(G,\cdot)$ is a topological quasigroup too. If $\varphi_1(x) = a + \varphi(x)$ and $\psi_1(x) = \psi(x) + b$, then φ_1, ψ_1 are homeomorphism of $(G,+)$ and $(G,+,\varphi,\psi,a,b) = (G,+,\varphi_1,\psi_1)$.

4. The homogeneous isotopes and congruences

We consider a topological groupoid $(G,+)$. If α is a relation on G, then $\alpha(x) = \{y \in G : x\alpha y\}$ for every $x \in G$.

An equivalence relation α on G is called a *congruence* on $(G,+)$ if from $x\alpha u$ and $y\alpha v$ it follows $(x+y)\,\alpha\,(u+v)$. If $(G,+)$ is a primitive groupoid with divisions l and r, then we consider that $l\,(x,y)\,\alpha l\,(u,v)$ and $r\,(x,y)\,\alpha r\,(u,v)$ provided $x\alpha u$ and $y\alpha v$.

Two congruences α and β on G are called *conjugate* if there exists a topological automorphism $\varphi : G \to G$ such that the relation $x\alpha y$ is equivalent to the relation $\varphi(x)\beta\varphi(y)$.

Let α, β be two conjugate congruences on G and φ be the topological automorphism for which the relation $x\alpha y$ is equivalent to the relation $\varphi(x)\beta\varphi(y)$. Let $\alpha\,(x) = \{y \in G : x\alpha y\}$. Then $\varphi\,(\alpha(x)) = \beta\,(\varphi(x))$. If $\{\beta_\mu : \mu \in M\}$ is a family of congruences on $(G,+)$, then there exists the intersection $\beta = \cap\{\beta_\mu : \mu \in M\}$, where $\beta\,(x) = \cap\{\beta_\mu\,(x) : \mu \in M\}$. The relation $x\beta y$ is hold, if and only if $x\beta_\mu y$ is hold for every $\mu \in M$.

Theorem 4. *Let $(G,\cdot) = g\,(G,+,\varphi,\psi)$ be an isotope of the topological primitive groupoid $(G,+)$ with the divisions $\{r,l\}$, φ,ψ be topological automorphisms of $(G,+)$, and α be a congruence on the groupoid $(G,+,l,r)$. Then:*

1. *If $(G,\cdot)$ is a homogeneous isotope, then there exists a countable set of congruences $\{\beta_n : n \in N\}$ of the groupoid $(G,+)$, conjugate to α, such that $\alpha \in \{\beta_n : n \in N\}$ and $\beta = \cap\{\beta_n : n \in N\}$ is a common congruence of the groupoids $(G,+)$ and $(G,\cdot)$.*
2. *If $(G,\cdot)$ is an (n,m)-homogeneous isotope of the groupoid $(G,+)$, and $n,m < +\infty$, then there exists a finite set of congruences $\{\beta_i : i \leqslant nm\}$ of the groupoid $(G,+)$, conjugate to α, such that $\beta = \cap\{\beta_i : i \leqslant nm\}$ is a common congruence of the groupoids $(G,+)$ and $(G,\cdot)$.*

Proof. Let Z be the set of all integer numbers. If $n = 0$, then $\varphi^0(x) = x$ for all $x \in G$. If $n \in Z$ and $n < 0$, then $\varphi^n = \left(\varphi^{-1}\right)^{-n}$. Denote by $\{h_n : n \in Z\}$ the set of the all automorphisms

$$\left\{\varphi^{k_1} \circ \psi^{m_1} \circ \varphi^{k_2} \circ \psi^{m_2} \circ ... \circ \varphi^{k_n} \circ \psi^{m_n} \;:\; n \in N,\;\; k_1, m_1, ..., k_n, m_n \in Z\right\}.$$

If $\varphi\psi = \psi\varphi$, then

$$\{h_n : n \in Z\} = \left\{\varphi^k \circ \psi^m \,:\, k, m \in Z\right\}.$$

For each $n \in N$ we define the congruence $\beta_n(x) = h_n(\alpha(x))$ for all $x \in G$.

Denote $\beta = \cap\{\beta_k : k \in N\}$. Then $\varphi(\beta(x)) = \psi(\beta(x)) = \beta(x)$ for each $x \in G$. Hence β is a common congruence of groupoids $(G,+)$ and $(G,\cdot)$. Suppose that automorphisms φ and ψ satisfy the Definition 3 and $(G,\cdot)$ is an (n,m)-isotope of groupoid $(G,+)$. In this case we have

$$\varphi^{k_1} \cdot \psi^{q_1} \cdot \varphi^{k_2} \cdot \psi^{q_2} \cdot \ldots \cdot \varphi^{k_n} \cdot \psi^{q_n} = \left(\varphi^{k_1+\ldots+k_n}\right) \cdot \left(\psi^{q_1+\ldots+q_n}\right)$$

Therefore

$$\{h_k : k \in N\} = \left\{\varphi^i \cdot \psi^j \;:\; i = 1,\ldots,n,\;\; j = 1,\ldots,m\right\} = \{h_k \;:\; k \leqslant nm\}$$

and the set $\{\beta_n : n \in N\}$ is finite and contains no more than nm distinct elements. The proof is complete. $\square$

Remark 3. Let α and β be two conjugate congruences on a topological groupoid G. Then:

1. The sets $\alpha(x)$ are G_δ-sets iff the sets $\beta(x)$ are G_δ-sets in G.
2. The sets $\alpha(x)$ are closed in G iff the sets $\beta(x)$ are closed in G.
3. The sets $\alpha(x)$ are open in G iff the sets $\beta(x)$ are open in G.

Remark 4. Let $\{\beta_n : n \in N' \subset N\}$ be a family of congruences on a topological goupoid G and $\beta = \cap\{\beta_n : n \in N\}$. Then:

1. If the sets $\beta_n(x)$ are G_δ-sets in G, then the sets $\beta(x)$ are G_δ-sets in G too.
2. If the set N' is finite and the sets $\beta_n(x)$ are open, then the sets $\beta(x)$ are open in G.

5. General properties of medial quasigroups

Let $(G,\cdot)$ be a topological medial quasigroup. By virtue of Toyoda's Theorem [7] there exist a binary operation $(+)$ on G, two elements $0, c \in G$ and two topological automorphisms $\varphi, \psi : (G,+) \to (G,+)$ such that $(G,+)$ is a topological commutative group, 0 is the zero of $(G,+)$ and $(G,\cdot) = g(G,+,\varphi,\psi,0,c)$ is a homogeneous isotope of $(G,+)$. In particular, $\varphi\psi = \psi\varphi$.

In [2] G.B. Beleavskaya has proved a generalization of Toyoda's Theorem.

Theorem 5. *Let $(G,+)$ be a topological quasigroup, $0 \in G, 0+0=0, \varphi, \psi$ be two automorphisms of $(G,+)$ and $(G,\cdot) = (G,+,\varphi,\psi)$. Then:*

1. *$\{0\}$ is a subquasigroup of the quasigroups $(G,+)$ and $(G,\cdot)$.*
2. *If $n < +\infty$, then 0 is an (n,∞)-identity of $(G,\cdot)$ iff $\varphi^n(x) = x$ for every $x \in G$.*
3. *If $m < +\infty$, then 0 is an (∞,m)-identity of $(G,\cdot)$ iff $\psi^m(x) = x$ for every $x \in G$.*
4. *If $n, m < +\infty$, then 0 is an (n,m)-identity of $(G,\cdot)$ iff $\varphi^n(x) = \psi^m(x) = x$ for every $x \in G$.*

Proof. Let n $< +\infty$. If $\varphi^n(x) = x$ for every $x \in G$, then from Theorem 3 it follows that 0 is an $(n,+\infty)$-identity in $(G,\cdot)$.

Let 0 be an (n,∞)-identity in $(G,\cdot)$. By construction, $\varphi(0) = \psi(0) = 0$ and $x \cdot y = \varphi(x) + \psi(y)$. Then $(x,0)\,k = \varphi^k(x)$ and $(0,x)\,k = \varphi^k(x)$ for every $k \in N$. Since $(x,0)\,n = x$ we obtain that $\varphi^n(x) = x$. The proof is complete. □

Consider on G some equivalence relation α. Denote by G/α the collection of classes of equivalence $\alpha(x)$ and $\pi_\alpha : G \to G/\alpha$ is the natural projection. On G/α we consider the quotient topology. The mapping π_α is continuous. If α is a congruence on $(G,\cdot)$ (or on $(G,+)$), then the mapping π_α is open.

An equivalence relation α on G is called compact if the sets $\alpha(x)$ are compact.

Theorem 6. *Let $(G,+)$ be a commutative topological group, 0 be a zero of $(G,+)$, $c \in G$, φ and ψ be two automorphisms of the topological group $(G,+)$ and $(G,\cdot) = g(G,+,\varphi,\psi,0,c)$. If the space G contains a non-empty compact subset F of countable character, then for every open subset U of G containing 0 there exists a compact equivalence relation α_U on G such that:*

1. *$\alpha_U(0) \subseteq U$.*
2. *α_U is a congruence on $(G,\cdot)$.*
3. *α_U is a congruence on $(G,+)$.*
4. *The natural projection $\pi_U = \pi_{\alpha_U} : G \to G/\alpha_U$ is an open perfect mapping.*
5. *The space G/α_U is metrizable.*

Proof. We consider that $0 \in F \subseteq U$. Fix a sequence $\{U_n : n \in N\}$ of open subsets of G such that for every open set V containing F there exists $n \in N$ such that $F \subseteq U_n \subseteq V$. Suppose that $F \subseteq U_n$ and $U_{n+1} \subseteq U_n$ for every $n \in N$.

Then there exists a sequence $\{V_n : n \in N\}$ of open sets of G such that for every $n \in N$ we have:

- $V_{n+1} + V_{n+1} \subseteq V_n \subseteq U_n, \quad cl_G V_{n+1} \subseteq V_n$ and $V_n = -V_n$,
- $\varphi(V_{n+1}) \cup \psi(V_{n+1}) \subseteq V_n$.

We put $H = \cap \{V_n : n \in N\}$. By construction, H is a compact subgroup and the natural projection $\pi : G \to G/H$ is open and perfect. Let $\alpha(x) = x + H$ for every $x \in G$. Then α is a congruence on $(G, +)$. Suppose that $x\alpha z$ and $y\alpha v$. Then

$$\begin{array}{rcl} x \cdot y & = & \varphi(x) + \psi(y) + c, \\ z \cdot v & = & \varphi(z) + \psi(v) + c, \\ \varphi(x) - \varphi(z) & \in & H, \ \psi(y) - \psi(v) \in H. \end{array}$$

Thus

$$\begin{gathered} (x \cdot y) - (z \cdot v) = \\ = (\varphi(x) + \psi(y)) - (\varphi(z) + \psi(v)) = \\ = (\varphi(x) - \varphi(z)) + (\psi(y) - \psi(v)) \in H. \end{gathered}$$

Therefore α is a congruence on $(G, \cdot)$ too.

It is clear that the space G/H is metrizable. The proof is complete. □

Corollary 2. *A first countable topological medial quasigroup is metrizable.*

A space X is called a *paracompact p-space* if there exists a perfect mapping $g : X \to Y$ onto some metrizable space Y (see [1]).

Corollary 3. *If a topological medial quasigroup contains a non-empty compact subset of countable character then it is a paracompact space p-space and admits an open perfect homomorphism onto a medial metrizable quasigroup.*

Corollary 4. *A Čech complete topological medial quasigroup is paracompact and admits an open perfect homomorphism onto a complete metrizable medial quasigroup.*

Corollary 5. *A locally compact medial quasigroup is paracompact and admits an open perfect homomorphism onto a metrizable locally compact medial quasigroup.*

6. On Haar measures on medial quasigroups

By $B(X)$ denote the family of all Borel subsets of the space X.

A non-negative real-valued function μ defined on the family $B(X)$ of Borel subsets of a space X is said to be a *Radon measure* on X if it has the following properties:

- $\mu(H) = \sup\{\mu(F) : F \subseteq H, F$ is a compact subset of $H\}$ for every $H \in B(X)$;
- for every point $x \in X$ there exists an open subset V_x containing x such that $\mu(V_x) < \infty$.

Definition 5. Let $(A, \cdot)$ be a topological quasigroup with the divisions $\{r, l\}$. A Radon measure μ on A is called:

- a *left invariant Haar measure*, if $\mu(U) > 0$ and $\mu(xH) = \mu(H)$ for every non-empty open set $U \subseteq A$, a point $x \in A$ and a Borel set $H \in B(A)$;
- a *right invariant Haar measure*, if $\mu(U) > 0$ and $\mu(Hx) = \mu(H)$ for every non-empty open set $U \subseteq A$, a point $x \in A$ and Borel set $H \in B(A)$;
- an *invariant Haar measure* if $\mu(U) > 0$ and $\mu(xH) = \mu(Hx) = \mu(l(x, H)) = \mu(r(H, x)) = \mu(H)$ for every non-empty open set $U \subseteq A$, a point $x \in A$ and a Borel set $H \in B(A)$;

Definition 6. We say that on a topological quasigroup $(A, \cdot)$ *there exists a unique left (right) invariant Haar measure*, if for every two left (right) invarinat Haar measures μ_1, μ_2 on A there exists a constant $c > 0$ such that $\mu_2(H) = c \cdot \mu_1(H)$ for every Borel set $H \in B(A)$.

If $(G, +)$ is a locally compact commutative group, then on G there exists a unique invariant Haar measure μ_G (see [6]).

Theorem 7. *Let $(G, \cdot)$ be a locally compact medial quasigroup, $(G, +)$ be a commutative topological group, $\varphi, \psi : G \to G$ be automorphisms of $(G, +), b \in G$ and $(G, \cdot) = g(G, +, \varphi, \psi, 0, b)$. On the group $(G, +)$ consider the invariant Haar measure μ_G. Then :*

1. *On $(G, \cdot)$ the right (left) invariant Haar measure is unique.*
2. *If μ is a left (right) invariant Haar measure on $(G, \cdot)$, then μ is a left (right) invariant Haar measure on $(G, +)$ too.*
3. *On $(G, \cdot)$ there exists some right invariant Haar measure if and only*

if $\mu_G(\varphi(H)) = \mu_G(H)$ *for every* $H \in B(A)$.

4. *If* $n < +\infty$, *and on* G *there exists some* $(n, +\infty)-$*identity, then on* $(G, \cdot)$ *the measure* μ_G *is a unique right invariant Haar measure.*
5. *If* $m < +\infty$, *and on* G *there exists some* $(+\infty, m)-$*identity, then on* $(G, \cdot)$ *the measure* μ_G *is a unique left invariant Haar measure.*
6. *If* $n, m < +\infty$, *and on* G *there exists some* $(n, m)-$*identity, then on* $(G, \cdot)$ *the measure* μ_G *is a unique invariant Haar measure.*

Proof. Let μ be a right invariant Haar measure on $(G, \cdot)$. Since $x \cdot y = \varphi(x) + \psi(y) + b$ for all $x, y \in G$, then $Hx = \varphi(H) + \psi(H) + b$. Thus μ is an invariant Haar measure on $(G, +)$ and there exists a constant $c > 0$ such that $\mu(H) = c \cdot \mu_G(H)$. Thus μ_G is a right invariant Haar measure on $(G, \cdot)$. The assertions 1,2 and 3 are proved.

Consider some topological automorphism h of $(G, +)$. Then $\mu_h(H) = \mu_G(h(H))$ is an invariant Haar measure on $(G, +)$. There exists a constant $c_h > 0$ such that $\mu_h(H) = \mu_G(h(H)) = c_h \cdot \mu_G(H)$ for every Borel subset $H \in B(G)$. In particular, $\mu_G\left(h^k(H)\right) = c_h^k \mu_G(H)$ for every $k \in N$. If $n < +\infty$ and 0 is an $(n, +\infty)-$identity, then $\varphi^n(x) = x$ for every $x \in G$ and $c_\varphi^n = 1$. Thus $c_\varphi = 1$, $\mu_G(H) = \mu_G(h(H))$ and μ_G is a right invariant Haar measure on $(G, \cdot)$. The assertions 4, 5 and 6 are proved. The proof is complete. $\square$

In this way we can prove the following results.

Theorem 8. *Let* $(G, +)$ *be a topological quasigroup and* $(G, \cdot)$ *be an* (n, m)*-homogeneous isotope of* $(G, +)$. *Then:*

1. *On* $(G, +)$ *there exists a left (right) invariant Haar measure if and only if on* $(G, \cdot)$ *there exists a left (right) invariant Haar measure.*
2. *If on* $(G, +)$ *the a left (right) invariant Haar measure is unique, then on* $(G, \cdot)$ *the a left (right) invariant Haar measure is unique too.*

Theorem 9. *On a compact medial quasigroup* G *there exists a unique Haar measure* μ *for which* $\mu(G) = 1$.

Theorem 10. *Let* $(G, +)$ *be a locally compact group,* μ_G *be the left invariant Haar measure on* $(G, +)$ *and* $\varphi, \psi : G \to G$ *be the topological automorphism of* $(G, +)$. *Fix* $c \in G$ *and consider the binary operation* $x \cdot y = \varphi(x) + \psi(y) + c$. *Then:*

1. $(G, \cdot)$ *is a topological quasigroup.*
2. *If* $\mu_G(\psi(H)) = \mu_G(H)$ *for every Borel subset* $H \in B(G)$, *then* μ_G

is a left invariant Haare measure on $(G, \cdot)$.

3. *If* $m \in N$ *and* $\psi^m(x) = x$ *for every* $x \in G$*, then* μ_G *is a left invariant Haar measure on* $(G, \cdot)$.
4. *If* $(G, +)$ *is a compact group, then* μ_G *is an invariant Haar measure on* $(G, \cdot)$.

7. Examples

Example 2. Let $(R, +)$ be a topological commutative group of real numbers, $a > 0$, $b > 0$, $\varphi(x) = ax$, $\psi(y) = bx$ and $x \cdot y = \varphi(x) + \psi(y)$. Then $(R, \cdot)$ is a commutative locally compact medial quasigroup. If $H = [c, d]$, then $0 \cdot H = [ac, ad]$ and $H \cdot 0 = [bc, bd]$. Thus:

- on $(G, \cdot)$ there exists some right invariant Haar measure if and only if $a = 1$;
- on $(G, \cdot)$ there exists some left invariant Haar measure if and only if $b = 1$;
- if $a \neq 1$ and $b \neq 1$, then on $(G, \cdot)$ does not exist any left or right invariant Haar measure.

Example 3. Denote by $Z_p = Z/pZ = \{0, 1, ..., p-1\}$ the cyclic Abelian group of order n. Consider the Abelian group $(G, +) = (Z_5, +)$ and $\varphi(x) = 2x$, $\psi(x) = 4x$. Then $(G, \cdot) = g(G, +; \varphi, \psi)$ is a medial quasigroup and each element from $(G, \cdot)$ is $(2, 4)$-identity in G.

Example 4. Consider the Abelian group $(G, +) = (Z_5, +)$ and $\varphi(x) = \psi(x) = 3x$. Then $(G, \cdot) = g(G, +; \varphi, \psi)$ is medial quasigroup and all elements from $(G, \cdot)$ are the $(4, 4)$-identities in G.

Example 5. Consider the commutative group $(G, +) = (Z_5, +)$, $\varphi(x) = 2x, \psi(x) = 2x + 1$ and $x \cdot y = 2x + 2y + 1$. Then $(G, \cdot) = g(G, +; \varphi, \psi, 0, 1)$ is a commutative medial quasigroup and $(G, \cdot)$ does not contain (n, m)-identities.

Example 6. Consider the commutative group $(G, +) = (Z, +)$, $\varphi(x) = x, \psi(x) = x + 1$ and $x \cdot y = x + y + 1$. Then $(G, \cdot) = g(G, +; \varphi, \psi)$ is a medial quasigroup and $(G, \cdot)$ does not contain (n, m)-identities. On $(G, \cdot)$ there exists an invariant Haar measure.

Example 7. Let $(G,+)$ be an Abelian group and $x+x\neq 0$ for each $x\in G$. For example $(G,+)\in\{(Z_p,+):p\in N,p\geqslant 2\}$. Denote $\varphi(x)=x$ and $\psi(x)=-x$ for each $x\in G$. Then $(G,\cdot)=g(G,+;\varphi,\psi)$ is a medial quasigroup and $(G,\cdot)$ contains the unique $(1,2)$-identity, which coincide with the zero element in $(G,+)$.

Example 8. Let $(G,+)=(Z_7,+)$, and $\varphi(x)=3x$ and $\psi(x)=5x$. Then $(G,\cdot)=g(G,+;\varphi,\psi)$ is a medial quasigroup. In this case 0 and 3 are $(12,6)$-identities.

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Product of the symmetric group with the alternating group on seven letters

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Abstract

We will find the structure of groups $G = AB$ where A and B are subgroups of G with A isomorphic to the alternating group on 7 letters and B isomorphic to the symmetric group on $n \geqslant 5$ letters.

1. Introduction

If A and B are subgroups of the group G and $G = AB$, then G is called a *factorizable group* and A and B are called *factors of the factorization*. We also say that G is the *product of its subgroups* A and B. If any of A or B is a non-proper subgroup of G, then $G = AB$ is called a *trivial factorization* of G, and by a *non-trivial* or a *proper factorization* we mean $G = AB$ with both A and B are proper subgroups of G. It is an interesting problem to know the groups with proper factorization. Of course not every group has a proper factorization, for example an infinite group with all proper subgroups finite has no proper factorization, also the Janko simple group J_1 of order 175 560 has no proper factorization. In what follows G is assumed to be a finite group. Now we recall some research papers towards the problem of factorization of groups under additional conditions on A and B. In [8] factorization of the simple group $L_2(q)$ are obtained and in [1] simple groups G with proper factorizations $G = AB$ such that $(|A|, |B|) = 1$ are given. Factorizations $G = AB$ with $A \cap B = 1$ are called *exact* and in [18] such factorizations for the alternating and symmetric groups are investigated. If A and B are maximal subgroups of G and $G = AB$, then this is called a *maximal factorization* of G. In [11] all

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maximal factorizations of the simple groups and their automorphism groups are obtained. Factorizations of sporadic simple groups and simple groups of Lie type with rank 1 or 2 as the product of two simple subgroups are obtained in [6] and [7] respectively. In [12] all groups with factorization $G = AB$, with A and B simple subgroups of G such that a Sylow 2-subgroup of A has rank 2 and a Sylow 2-subgroup of B is elementary abelian are completely classified.

Factorizations of groups involving alternating or symmetric groups have been investigated in some papers. In [10] groups G with factorization $G = AB$, where $A \cong B \cong \mathbf{A}_5$, are classified and in [13] groups $G = AB$ where A is a non-abelian simple group and B is isomorphic to to the alternating group on 5 letters are determined. In a series of papers G.L. Walls considered factorizations $G = AB$ of a group G with both factors simple [14], [15]. In [16] he began the study of factorizations when one factor is simple and the other is almost simple. To begin this study it is natural to start with the case where one factor is isomorphic to a simple alternating group and the other is isomorphic to a symmetric group. In [3] we classified all groups G with factorization $G = AB$ where $A \cong \mathbf{A}_6$ and B is isomorphic to a symmetric group on $n \geqslant 5$ letter, and in [4] we determined all groups G with factorization $G = AB$ where A is a simple group and $B \cong \mathbf{S}_6$. Motivated by the above results and to get a picture for the general case, in this paper we will study groups G with factorization $G = AB$, where $A \cong \mathbf{A}_7$ and B is isomorphic to a symmetric group on $n \geqslant 5$.

2. Preliminary results

In this section we obtain results which are needed in the proof of our main Theorem. Suppose Ω is a set of cardinality m and G is a k-homogeneous, $1 \leqslant k \leqslant m$, group on Ω. If H is a k-homogeneous subgroup of G, then it is easy to see that $G = G_{(\Delta)}H$ where Δ is a subset of cardinality k in Ω. We can give some factorization of groups using the previous observation. It is easy to verify that the order of a subgroup of $\mathbf{A}_7$ is one of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 18, 20, 21, 24, 36, 60, 72, 120, 168, 360 or 2520 and therefore the index of a proper subgroup of $\mathbf{A}_7$ is one of the numbers 7,15, 21, 35, 42, 70, 105, 120, 126, 140, 210, 252, 280, 315, 360, 420, 504, 630, 840, 1260 or 2520. Therefore $\mathbf{A}_7$ has transitive action on sets of cardinality equal to any of the latter numbers. Therefore we always have the factorization $\mathbf{S}_{n+1} = \mathbf{S}_n\mathbf{A}_7$ where $n = 6$, 14, 20, 34, 41, 69, 104, 119, 125, 139, 209, 251, 279, 314, 359, 419, 503, 629, 839, 1259 or 2519. It is well-know that $\mathbf{A}_7$ has a 2-transitive action on 15 points and hence we have the factorization $\mathbf{S}_{15} = \mathbf{S}_{13}\mathbf{A}_7$. If we consider the 7-transitive action of $\mathbf{A}_7$ on 7 points we also have the factorizations $\mathbf{S}_7 = \mathbf{S}_n\mathbf{A}_7$, $2 \leqslant n \leqslant 7$. Therefore we have the following Lemma.

Lemma 1.

(*a*) $\mathbf{S}_{n+1} = \mathbf{A}_7\mathbf{S}_n$ *and* $\mathbf{A}_{n+1} = \mathbf{A}_7\mathbf{A}_n$ *for* $n = 6$, 14, 20, 34, 41, 69, 104, 119, 125, 139, 209, 251, 279, 314, 359, 419, 503, 629, 839, 1259 *or* 2519.

(*b*) $\mathbf{S}_{15} = \mathbf{A}_7\mathbf{S}_{13}$ *and* $\mathbf{A}_{15} = \mathbf{A}_7\mathbf{S}_{13}$.

(*c*) $\mathbf{S}_7 = \mathbf{A}_7\mathbf{S}_n$ *for* $2 \leqslant n \leqslant 7$.

In the following Lemmas we will find a special kind of factorizations for the alternating and symmetric groups.

Lemma 2. *Let* $m, r, n \geqslant 5$ *be natural numbers. If* $\mathbf{A}_m = \mathbf{A}_r\mathbf{A}_n$ *or* $\mathbf{A}_m = \mathbf{A}_r\mathbf{S}_n$ *are proper factorizations, then either* $r = m-1$ *and* $\mathbf{A}_m$ *has a transitive subgroup isomorphic to* $\mathbf{A}_n$ *or* $\mathbf{S}_n$ *which gives the factorizations* $\mathbf{A}_m = \mathbf{A}_{m-1}\mathbf{A}_n$ *and* $\mathbf{A}_m = \mathbf{A}_{m-1}\mathbf{S}_n$, *or* $(m, r, n) = (10, 6, 8)$, $(15, 7, 13)$, $(15, 8, 13)$, $(10, 8, 6)$ *giving the factorizations* $\mathbf{A}_{10} = \mathbf{A}_6\mathbf{A}_8 = \mathbf{A}_6\mathbf{S}_8$, $\mathbf{A}_{15} = \mathbf{A}_7\mathbf{A}_{13} = \mathbf{A}_7\mathbf{S}_{13}$, $\mathbf{A}_{15} = \mathbf{A}_8\mathbf{A}_{13} = \mathbf{A}_8\mathbf{S}_{13}$ *and* $\mathbf{A}_{10} = \mathbf{A}_8\mathbf{S}_6$. *Moreover all the above factorizations occurs.*

Proof. We use Theorem D of [11], but note that the case (ii) of this Theorem can not happen for these special factorizations of $\mathbf{A}_m$ stated in our Theorem. First we assume $\mathbf{A}_m = \mathbf{A}_r\mathbf{A}_n$. In this case without loss of generality we may assume $\mathbf{A}_{m-k} \trianglelefteq \mathbf{A}_r \leqslant \mathbf{S}_{m-k} \times \mathbf{S}_k$ for some $k, 1 \leqslant k \leqslant 5$, and $\mathbf{A}_n$ is k-homogeneous on m letter. Since the factorization is proper hence $m > r$ and $m > n$. If $m - k = 1$, then from $1 \leqslant k \leqslant 5$ we get $m = 6$ and we have the factorization $\mathbf{A}_6 = \mathbf{A}_5\mathbf{A}_5$. Hence $m - k = r$. If $k = 1$, then $\mathbf{A}_m = \mathbf{A}_{m-1}\mathbf{A}_n$ and $\mathbf{A}_n$ has a transitive action on m letters. If $k \geqslant 2$, then since $m > n$, from [9] and [5] we get $k = 2$ and the 2-transitive actions of $\mathbf{A}_r$ occurs if and only if $(r, m) = (5, 6), (6, 10), (7, 15), (8, 15)$. since we have assumed $m, r, n \geqslant 5$, so we obtain the triples listed in the Lemma.

Next we assume $\mathbf{A}_m = \mathbf{A}_r\mathbf{S}_n$. Again we use Theorem D in [11], but we must consider two cases

Case (*i*). $\mathbf{A}_{m-k} \trianglelefteq \mathbf{A}_r \leqslant \mathbf{S}_{m-k} \times \mathbf{S}_k$ for some $k, 1 \leqslant k \leqslant 5$, and $\mathbf{S}_n$ is k-homogeneous on m letters. Reasoning as above we must have $m - k = r$. If $k = 1$, then $\mathbf{A}_m = \mathbf{A}_{m-1}\mathbf{S}_n$ and $\mathbf{S}_n$ must have a transitive permutation representation on m points. Otherwise since $\mathbf{S}_n$ has no $k > 2$ transitive permutation representations except the trivial ones we don't get a possibility. However $\mathbf{S}_6$ has a 2-transitive permutation representations on 10 points giving the factorization $\mathbf{A}_{10} = \mathbf{A}_8\mathbf{S}_6$.

CASE (*ii*). $\mathbf{A}_{m-k} \trianglelefteq \mathbf{S}_n \leqslant \mathbf{S}_{m-k} \times \mathbf{S}_k$ for some $k, 1 \leqslant k \leqslant 5$, and $\mathbf{A}_r$ is k-homogeneous on m points. In this case $m - k = 1$ is not possible, hence $m - k = n$. Since we must have $m \geqslant n - 2$, so $k = 1$ is not possible. Natural k-homogeneous permutation representation of $\mathbf{A}_r$ don't give proper factorizations, therefore we must have $\mathbf{A}_r = \mathbf{A}_5, \mathbf{A}_6, \mathbf{A}_7$ or $\mathbf{A}_8$ acting 2-transitively on sets of cardinality 6, 10, 15 and 15 respectively. In this case we obtain $(m, r, n) = (6, 5, 4), (10, 6, 8), (15, 7, 13), (15, 8, 13), (10, 8, 6)$ and the admissible triples are the ones listed in the Lemma. □

Lemma 3. *Let $m, r, n \geqslant 5$ be integers and $\mathbf{S}_m = \mathbf{A}_r\mathbf{S}_n$ be a non-trivial factorization of $\mathbf{S}_m$. Then we have one of the following possibilities:*

(*a*) $n = m - 1$ *and $\mathbf{A}_r$ has a transitive action on m points and the factorization $\mathbf{S}_m = \mathbf{A}_r\mathbf{S}_{m-1}$ occurs.*

(*b*) $r = m - 1$ *and $\mathbf{S}_n$ has a transitive action on r points and moreover* $2m \mid n!$.

(*c*) $(m, r, n) = (10, 6, 8), (15, 7, 13), (15, 8, 13)$ *and the factorizations* $\mathbf{S}_{10} = \mathbf{A}_6\mathbf{S}_8$, $\mathbf{S}_{15} = \mathbf{A}_7\mathbf{S}_{13}$, $\mathbf{S}_{15} = \mathbf{A}_8\mathbf{S}_{13}$ *all occurs.*

(*d*) $(m, n, r) = (10, 8, 6)$ *and* $\mathbf{S}_{10} = \mathbf{A}_8\mathbf{S}_6$.

Proof. Again we use Theorem D of [11], knowing that case (ii) of the Theorem does not hold in this special case. We consider two cases. Note that $m > r$ and $m > n$.

CASE (*i*). $\mathbf{A}_{m-k} \trianglelefteq \mathbf{S}_n \leqslant \mathbf{S}_{m-k} \times \mathbf{S}_k$ and $\mathbf{A}_r$ has a k-homogeneous action on m letters. If $k = 1$, then $n = m - 1$ and we have the factorization $\mathbf{S}_m = \mathbf{S}_{m-1}\mathbf{A}_r$ where $\mathbf{A}_r$ acts transitively on m letters. If $k \geqslant 2$, then by [9] and [5] the only non-trivial k-homogeneous representation of $\mathbf{A}_r$ on m letters occurs if and only if $k = 2$ and $(m, r) = (6, 5), (10, 6), (15, 7), (15, 8)$ and for these pairs we have $n =$ 4,8,13,13 respectively. Therefore cases (a) and (c) are proved and it is clear that the appropriate factorizations exists.

CASE (*ii*). $\mathbf{A}_{m-k} \trianglelefteq \mathbf{A}_r \leqslant \mathbf{S}_{m-k} \times \mathbf{S}_k$ and $\mathbf{S}_n$ has a k-homogeneous action on m letters. In this case $\mathbf{S}_n$ does not have a k-homogeneous action on m letters except the trivial ones if $k > 2$. In the case of $k = 2$, $\mathbf{S}_6$ has a non-trivial 2-transitive action on 10 letters. Therefore $k = 1$ which forces $r = m - 1$ and if we have the factorization $\mathbf{S}_m = \mathbf{A}_{m-1}\mathbf{S}_n$, then $\mathbf{S}_n$ must act on m letters transitively and order consideration yields $2m \mid n!$. In this way we obtain cases (b) and (d) and the Lemma is proved. □

3. Factorizations involving $\mathbf{A}_7$

To obtain our main result concerning groups with factorizations $G = \mathbf{A}_7\mathbf{S}_n$, $n \geqslant 5$ we need to know about simple primitive groups of certain degrees, and these degrees are indices of subgroups of $\mathbf{A}_7$ which are greater than 1. In section 2 we listed the 21 possible numbers, and we see that except 1260 and 2520 the rest of them are less than 1000. Simple primitive groups of degree up to 1000 are listed in [5] and we can obtain the simple primitive groups with the degree we want. These are listed in Table I. But we don't know about the simple primitive groups of degree 1260 and 2520 in the existing literature. The following Lemma deals with these cases.

Lemma 4. *Suppose G is not an alternating simple group but G is a simple permutation group of degree* 1260 *or* 2520*. Then it is not possible to decompose G as $G = \mathbf{A}_7\mathbf{A}_n$, for any n.*

Proof. According to the classification of finite simple groups any finite non-abelian simple group is isomorphic to either an alternating group, a sporadic group or a simple group of Lie type. Since G is written as the product of two simple groups results of [6] show that G can not be a sporadic simple group. If G is a simple group of Lie type, then by [7] the only possibility is $L_2(9) = \mathbf{A}_5\mathbf{A}_5$ which is not possible because $L_2(9)$ is not a permutation group of degree 1260 or 2520. Therefore we assume that G is a simple group of Lie type with Lie rank at least 3. Here we use results about the minimum index of a subgroup of a group of Lie type and consider the following cases.

(i). For $L_n(q), n \geqslant 4$, the proper subgroups have index at least $\frac{q^n-1}{q-1}$ and form $\frac{q^n-1}{q-1} \leqslant 2520$ we get the following possibilities: $L_4(2)$, $L_4(3)$, $L_4(4)$, $L_4(5)$, $L_4(7)$, $L_4(8)$, $L_4(9)$, $L_4(11)$, $L_4(13)$, $L_5(2)$, $L_5(3)$, $L_5(4)$, $L_5(5)$, $L_5(7)$, $L_6(2)$, $L_6(3)$, $L_6(4)$, $L_7(2)$, $L_8(2)$, $L_9(2)$, $L_{10}(2)$, $L_{11}(2)$. $L_4(2) \cong \mathbf{A}_8$ is not the case. If $L_4(3) = \mathbf{A}_7\mathbf{A}_n$, then by [2] we see that$L_4(3)$ does not contain a subgroup isomorphic to $\mathbf{A}_7$. If $L_4(4) = \mathbf{A}_7\mathbf{A}_n$, then since $17||L_4(4)|$ we must have $n \geqslant 17$, but in this case we must have $13||L_4(4)|$ which is not the case. Now using similar argument as above we rule out all the above possibilities.

(ii). For $U_n(q), n \geqslant 6$, the proper subgroups have index at least $(q^n - (-1)^n)(q^{n-1} - (-1)^{n-1})/(q^2 - 1)$ and for this number to be at least 2520 we get only $U_6(2)$. If $U_6(2) = \mathbf{A}_7\mathbf{A}_n$, then since $11||U_6(2)|$ we must have $n \geqslant 11$, but then from [2] we see that $U_6(2)$ does not have a subgroup isomorphic to A_{11}.

(iii). For $S_{2m}(q), m \geqslant 3$, the proper subgroup have index at least $\frac{q^{2m}-1}{q-1}$ when $q > 2$ and at least $2^m(2^m-1)$ when $q = 2$ and $m > 2$. In this case the following symplectic groups are the possibilities $S_6(2)$, $S_6(3)$, $S_6(4)$, $S_8(2)$, $S_{10}(2)$, $S_{12}(2)$, and again using [2] and order consideration we rule out the possibility $G = \mathbf{A}_7\mathbf{A}_n$.

(iv). For $O^{\epsilon}_{2m}(q), m \geqslant 4, \epsilon \equiv \pm$, the proper subgroups have index at least $(q^m-1)(q^{m-1}+1)/(q-1)$ when $\epsilon \equiv +$ and at least $(q^m+1)(q^{m-1}-1)/(q-1)$ when $\epsilon \equiv -$ except for the case $(q,\epsilon) = (2,+)$ when a proper subgroup has index at least $2^{m-1}(2^m-1)$. For $O_{2m+1}(q), m \geqslant 3, q$ odd, the proper subgroups have index at least $(q^{2m}-1)/(q-1)$ except when $q = 3$ and in this latter case the minimum index is $(q^{2m}-q^m)/2$. Now for this index to be at most 2520 we obtain the following orthogonal groups $O_7(3)$, $O^{\pm}_8(2)$, $O^{\pm}_8(3)$, $O^{\pm}_{10}(2)$, $O^{\pm}_{12}(2)$. Again order consideration rules out the possibility $G = \mathbf{A}_7\mathbf{A}_n$.

(v). For G to be an exceptional simple group of Lie type we use the argument in the proof of Theorem 9 in [5] from which only the possibilities $E_6(q)$ or $F_4(q)$ can arise and both of them are ruled out by order consideration. The Lemma is now proved. □

Theorem 5. *If $M = \mathbf{A}_7\mathbf{A}_n$ is a simple group, then*

(a) $M = \mathbf{A}_n$ *for* $n \geqslant 7$,

(b) $M = \mathbf{A}_{15} = \mathbf{A}_7\mathbf{A}_{13}$,

(c) $M = \mathbf{A}_{n+1} = \mathbf{A}_7\mathbf{A}_n$ *for* $n = 14, 20, 34, 41, 69, 104, 119, 125, 139,$ 219, 251, 279, 314, 359, 419, 503, 629, 839, 1259 *or* 2519.

Proof. Case (a) corresponds to the trivial factorization of M. Now suppose $M = \mathbf{A}_7\mathbf{A}_n$ is a non-trivial factorization of a simple group M. If C is a maximal subgroup of M containing $\mathbf{A}_n$, then we have $[M : C]|[\mathbf{A}_7 : \mathbf{A}_7 \cap C]$. Therefore M is a simple primitive group of degree equal to the index of a proper subgroup of $\mathbf{A}_7$. If M is an alternating group, then by Lemma 2 we get cases (b) and (c). Therefore we assume M is not an altrnating group. By Lemma 4 M can not be a primitive group of degree 1260 or 2520. Therefore we may assume M is a simple primitive group of degree less than 1000. Simple primitive groups of these special degrees are listed in Table I. Now using [6] and [7] the only cases that need to be considered are $S_6(2), S_8(2), O^+_8(2)$ or J_2. If $S_6(2) = \mathbf{A}_7\mathbf{A}_n$, then since $2^9||S_6(2)|$ we must have $n \geqslant 8$ and by [2] we get $n = 8$. But in this case if $S_6(2) = \mathbf{A}_7\mathbf{A}_8$, then $|\mathbf{A}_7 \cap \mathbf{A}_8| = 35$ which is a contradiction because $\mathbf{A}_7$ does not contain a subgroup of order 35. Order consideration rules out the possibilities $S_8(2)$

or $O_8^+(2)$ to be factorized as $\mathbf{A}_7\mathbf{A}_n$, for any n. By [2] the group J_2 does not contain a subgroup isomorphic to $\mathbf{A}_7$, and the Lemma is proved now. □

Table I. Simple primitive groups of certain degrees

degree	Groups
7	$\mathbf{A}_7, L_2(7)$
15	$\mathbf{A}_{15}, \mathbf{A}_6, \mathbf{A}_7, \mathbf{A}_8$
21	$\mathbf{A}_{21}, \mathbf{A}_7, L_2(7), L_3(4)$
35	$\mathbf{A}_{35}, \mathbf{A}_7, \mathbf{A}_8$
42	$\mathbf{A}_{42}$
70	$\mathbf{A}_{70}$
105	$\mathbf{A}_{105}, \mathbf{A}_{15}, L_3(4)$
120	$\mathbf{A}_{120}, \mathbf{A}_9, \mathbf{A}_{10}, L_2(16), L_3(4), S_4(4), S_6(2), S_8(2), O_8^+(2)$
126	$\mathbf{A}_{126}, \mathbf{A}_9, \mathbf{A}_{10}, L_2(125), U_3(5), U_4(3)$
140	$\mathbf{A}_{140}, L_2(139)$
210	$\mathbf{A}_{210}, \mathbf{A}_{10}, \mathbf{A}_{21}$
252	$\mathbf{A}_{252}, L_2(251)$
280	$\mathbf{A}_{280}, \mathbf{A}_9, L_3(4), U_4(3), J_2$
315	$\mathbf{A}_{315}, S_6(2), J_2$
360	$\mathbf{A}_{360}, L_2(359)$
420	$\mathbf{A}_{420}, L_2(419)$
504	$\mathbf{A}_{504}, L_2(503)$
630	$\mathbf{A}_{630}, \mathbf{A}_{36}$
840	$\mathbf{A}_{840}, \mathbf{A}_9, L_2(839), J_2$

Lemma 6. *There is no non-trivial factorization $G = \mathbf{A}_7\mathbf{S}_n$ with G simple except $G = \mathbf{A}_{15} = \mathbf{A}_7\mathbf{S}_{13}$.*

Proof. Let G be a simple group with a non-trivial factorization $G = \mathbf{A}_7\mathbf{S}_n$ for some natural number n. If G is isomorphic to an alternating group, then by Lemma 2 the only possibility is $G = \mathbf{A}_{15} = \mathbf{A}_7\mathbf{S}_{13}$. Hence we assume G is not an alternating group. As in the proof of Lemmas if C is a maximal subgroup of G containing $\mathbf{S}_n$, then $[M : C]|[\mathbf{A}_7 : \mathbf{A}_7 \cap C] = d$ and so G is represented as a simple primitive group of degree d where d is the index of a proper subgroup of $\mathbf{A}_7$. First we consider simple primitive groups of degree $d \leqslant 1000$ which are listed in Table I. We can exclude the linear groups $L_2(q)$ and the groups $L_3(4), S_4(4)$ and J_2 as by [2] they don't contain the alternating group of degree 7. Therefore we have to examine the groups $S_6(2), S_8(2), O_8^+(2), U_3(5)$ or $U_4(3)$ for appropriate decomposition. If $S_8(2) = \mathbf{A}_7\mathbf{S}_n$, then since $17||S_8(2)|$ we must have $n \geqslant 17$, but in this case we must have $13||S_n|$ which is a contradiction. If $O_8^+(2) = \mathbf{A}_7\mathbf{S}_n$. then order consideration implies $n \geqslant 12$ which is a contradiction because by [2] $O_8^+(2)$ does not contain a subgroup isomorphic to $\mathbf{S}_{12}$. For $S_6(2) = \mathbf{A}_7\mathbf{S}_n$ order consideration yields $n = 8$, but then $|\mathbf{A}_7 \cap \mathbf{S}_8| = 70$ contradicting the fact that $\mathbf{A}_7$ does not have a subgroup of order 70. If $U_3(5) = \mathbf{A}_7\mathbf{S}_n$, then since $5^3||U_5(3)|$ we must have $n \geqslant 10$ which is not possible because, by [2], the group $U_3(5)$ does not contain a subgroup isomorphic to $\mathbf{S}_{10}$. If $U_4(3) = \mathbf{A}_7\mathbf{S}_n$, then order consideration will imply $n \geqslant 9$, which is not possible because, by [2], $U_4(3)$ does not contain a subgroup is isomorphic to $\mathbf{S}_9$.

Secondly we should consider simple primitive groups G of degree $d > 1000$ which can be written as $G = \mathbf{A}_7\mathbf{S}_n$, and these degrees are 1260 and 2520. But by Lemma 4 we know a list of simple groups which possibly have this property. Now case by case examination of these groups, with the same method as used in the proof of Lemma 4, will end to a contradiction. The Lemma is proved now. □

Lemma 7. *Let $H = AB, A \cong \mathbf{A}_7, B \cong \mathbf{A}_n$, be a proper factorization of a group H and $H \ncong A \times B$. Then H is isomorphic to an alternating group $\mathbf{A}_m$ for possible m.*

Proof. Let N be a normal subgroup of H. Since A is a simple group therefore $N \cap A = A$ or 1. If $N \cap A = A$, then $A \subseteq N$ and we will have $H = AB = NB$ and by [15] we must have $H = B$, $H \cong A \times B$ or $H = \text{Hol}\,(\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2)$, and non of them is the case. Therefore $N \cap A = 1$ and similarly $N \cap B = 1$.

Now assume N is a maximal normal subgroup of G. We have $\frac{H}{N} =$

$(\frac{AN}{N})(\frac{BN}{N})$ and $\frac{AN}{N} \cong A \cong \mathbf{A}_7$ and $\frac{BN}{N} \cong B \cong \mathbf{A}_n$, and hence the simple group $\frac{H}{N}$ is the product of $\mathbf{A}_7$ and $\mathbf{A}_n$ and by Theorem 5 we must have $\frac{H}{N} \cong \mathbf{A}_m$ for suitable m. Now by [17] we must have $N = 1$ and $H \cong \mathbf{A}_m$. This completes the proof. $\square$

Now we state and prove our final result.

Theorem 8. *Let G be a group such that $G = AB$, $A \cong \mathbf{A}_7$ and $B \cong \mathbf{S}_n$, $n \geqslant 5$, then one of the following cases occurs:*

(a) $G \cong \mathbf{A}_7$,

(b) $G \cong \mathbf{S}_n, n \geqslant 7$,

(c) $G \cong \mathbf{A}_7 \times \mathbf{S}_n$,

(d) $G \cong \mathbf{A}_{15} = \mathbf{A}_7\mathbf{S}_n$,

(e) $G \cong \mathbf{S}_{n+1} = \mathbf{A}_7\mathbf{S}_n$, $n = 14, 20, 34, 41, 69, 104, 119, 125, 139, 219,$ 251, 279, 314, 359, 419, 503, 629, 839, 1259 *or* 2519,

(f) $G \cong \mathbf{S}_{15} = \mathbf{A}_7\mathbf{S}_{13}$ *or* $G \cong \mathbf{A}_{15} \times \mathbf{Z}_2 = \mathbf{A}_7\mathbf{S}_{13}$,

(g) $G \cong (\mathbf{A}_7 \times \mathbf{A}_7)\langle\tau\rangle$, τ *an automorphism of order* 2 *of* $\mathbf{A}_7$ *and* $\mathbf{A}_7 \times \mathbf{A}_7$ *is the minimal normal subgroup of* G,

(h) $G \cong (\mathbf{A}_7 \times \mathbf{A}_n)\langle\tau\rangle$, $n \neq 7$, *where* τ *acts as an automorphism of order* 2 *on both factors (in this case* $\mathbf{A}_7$ *or* $\mathbf{A}_n$ *is the minimal normal subgroup of* G*).*

Proof. We will use Lemma 4 of [3]. Let M be a minimal normal subgroup of $G = AB$, $G \ncong A \times B$, where $A \cong \mathbf{A}_7$ and $B \cong \mathbf{S}_n$, $n \geqslant 5$. Then we have the following possibilities.

(i). $M = G = AB$ is simple . In this case by Lemma 6 only $G \cong \mathbf{A}_{15} = \mathbf{A}_7\mathbf{S}_{13}$ is possible which is case (d) of our Theorem.

(ii). $G = MB, M = AB'$ is simple, where $B' \cong \mathbf{A}_n$ denotes the commutator subgroup of B. If $M = A$ or B', then we get trivial factorizations which are case (a) and case (b) of our Theorem. Therefore we assume $M = AB', A \cong \mathbf{A}_7, B' \cong \mathbf{A}_n$, is a simple group with non-trivial factorization. By Lemma 5 we must have either $M \cong \mathbf{A}_{15} = \mathbf{A}_7\mathbf{A}_{13}$ or $M \cong \mathbf{A}_{n+1} = \mathbf{A}_7\mathbf{A}_n$ for the n's specified in the Theorem. In the latter case $[G : M] = 2$, hence $G = M\langle\tau\rangle$ where τ is an element of order 2 in $\mathbf{S}_n \backslash \mathbf{A}_n$. Now in the latter case the same reasoning as used in the proof of Theorem 4 in [16] yields $G = \mathbf{S}_{n+1}$, which is the case (e) of the Theorem. In the case of $M \cong \mathbf{A}_{15} = \mathbf{A}_7\mathbf{A}_{13}$ if τ acts as an inner automorphism on M we obtain $\mathbf{A}_{15} \times \mathbf{Z}_2 \cong \mathbf{A}_7\mathbf{S}_{13}$ and if τ acts as an outer automorphism on M we obtain $\mathbf{S}_{15} \cong \mathbf{A}_7\mathbf{S}_{13}$ which are included in case (f) of the Theorem.

(iii). $G = MB, B \cong \mathbf{S}_n, M \cong \mathbf{A}_7 \times \mathbf{A}_7$. In this case $n = 7$ and therefore $G \cong (\mathbf{A}_7 \times \mathbf{A}_7)\langle\tau\rangle$, with τ an automorphism of order 2 and $\mathbf{A}_7 \times \mathbf{A}_7$ a minimal normal subgroup of G, which is the case (g) of the Theorem.

(iv). $M = A$ or B', $AB' \cong A \times B' \cong \mathbf{A}_7 \times \mathbf{A}_n$, $[G : AB'] = 2$. In this case $G = (\mathbf{A}_7 \times \mathbf{A}_n)\langle\tau\rangle$ where τ acts as an automorphism of order 2 on both factors with $\mathbf{A}_7$ or $\mathbf{A}_n$ as the minimal normal subgroup of G. This is the case (h) in our Theorem.

(v). Finally we must have $M \cap A = M \cap B = 1$, $|M||[A : A \cap B]$ and $|M||[B : A \cap B]$, furthermore $|M|.|A \cap B| = |\frac{AM}{M} \cap \frac{BM}{M}|$. We will show that no new possibilities arise in this case and the proof of our Theorem will be completed. M is isomorphic to the direct product of isomorphic simple groups. From $|M||[A : A \cap B]$ it follows that if M is abelian, then $|M| = 2$, 3, 4, 5, 7, 8, 9 and if M is non-abelian, then $M \cong \mathbf{A}_5, \mathbf{A}_6, L_2(7)$ or $\mathbf{A}_7$.

Now the groups A, B and G act on M by conjugation with the kernels $C_A(M) \trianglelefteq A, C_B(M) \trianglelefteq B$ and $C = C_G(M)$ respectively. If $C_A(M) = 1$, then A would be isomorphic to a subgroup of $\mathrm{Aut}(M)$ and by the structure of M the only possibility is $M = A$ which has been considered above. Therefore $C_A(M) = A$ which implies $A \leqslant C_G(M) = C$. Now $C_B(M) = 1, B'$ or B because $B \cong \mathbf{S}_n, n \geqslant 5$. Since $A \leqslant C$ we must have $G = AB = CB$ and hence $|A||C \cap B| = |C||A \cap B|$. We have $C \cap B = C_B(M)$, and if $C_B(M) = 1$, then $A = C$ and $A \cap B = 1$. Since $C \trianglelefteq G$ we consider the group $H = AB' = CB'$ where A and B' are simple alternating groups and $[G : H] = 2$. Now by Lemma 6 either this factorization is not proper or $H \cong A \times B'$ or H isomorphic to an alternating group. If the factorization is not proper, then either $A \subseteq B'$ or $B' \subseteq A$ contradicting $A \cap B = 1$. The other cases force G to be a symmetric group which is considered above. If $C_B(M) = B$, then $B \leqslant C_G(M)$ and we will get $M \leqslant Z(G)$.

Finally we will assume $C_B(M) = B \cap C = B'$. Now from $G = AB = CB$ we get $[G : C] = 2$. We know that $[A \cap B : A \cap B'] = 1$ or 2. If $[A \cap B : A \cap B'] = 2$, then $|AB'| = |AB| = |G|$ implying $G = AB' \subseteq C$ or $G = C$ which is a contradiction. Therefore $A \cap B = A \cap B'$ from which we obtain $|AB'| = \frac{1}{2}|G| = |C|$, hence $C = AB'$.

Our arguments so far show that either $M \leqslant Z(G)$ or $C = AB'$ where $A \cong \mathbf{A}_7$ and $B' \cong \mathbf{A}_n$. If $C = AB'$ then the factorization must be proper because $M \trianglelefteq C$ and A and B' are simple groups, therefore by Lemma 7 either $C \cong A \times B'$ or $C \cong \mathbf{A}_m$ for suitable m. If $C \cong A \times B'$, then as $[G : C] = 2$ we will obtain case (f) again. The case $C \cong \mathbf{A}_m$ can not happen because $M \trianglelefteq C$. Now we will deal with the case $M \leqslant Z(G)$. We have $\frac{G}{M} \cong (\frac{AM}{M})(\frac{BM}{M})$ with $\frac{AM}{M} \cong A \cong \mathbf{A}_7$ and $\frac{BM}{M} \cong B \cong \mathbf{S}_n$ and by induction either $G = MB \cong M \times B$ or $\frac{G}{M} = \mathbf{S}_{n+1}$ for n's as in case (e) of the Theorem. Now the same reasoning as used in the proof of Theorem 4 in [16] leads to a contradiction. The Theorem is proved now. $\square$

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Rough set theory applied to BCI-algebras

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Abstract

As a generalization of subalgebras/ideals in BCI-algebras, the notion of rough subalgebras/ideals is introduced, and some of their properties are discussed.

1. Introduction

In 1982, Pawlak introduced the concept of a rough set (see [13]). This concept is fundamental for the examination of granularity in knowledge. It is a concept which has many applications in data analysis (see [14]). An algebraic approach to rough sets has been given by Iwiński [7]. Rough set theory is applied to semigroups and groups (see [10, 11]). In 1994, Biswas and Nanda [2] introduced and discussed the concept of rough groups and rough subgroups. Recently, Jun [8] applied rough set theory to BCK-algebras. In this paper, we apply the rough set theory to BCI-algebras, and we introduce the notion of upper/lower rough subalgebras/ideals in BCI-algebras, and discuss some of their properties.

Note that BCI-algebras are an algebraic characterization of some types of non-classical logics. Moreover, BCI-algebras are also a generalization of BCK-algebras. On the other side, BCI-algebras are a generalization of T-quasigroups, too. Namely, as it is proved in [3] and [4], a BCI-algebra is a quasigroup if and only if it is medial. Such BCI-algebra is uniquely determined by some abelian group. In fact, such BCI-algebra is isotopic to this group. The class of associative BCI-algebras coincides with the class of Boolean groups.

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2. Preliminaries

Recall that a *BCI-algebra* is an algebra $(G, *, 0)$ of type $(2, 0)$ satisfying the following axioms: for every $x, y, z \in G$,

- $((x * y) * (x * z)) * (z * y) = 0$,
- $(x * (x * y)) * y = 0$,
- $x * x = 0$,
- $x * y = 0$ and $y * x = 0$ imply $x = y$.

For any BCI-algebra G, the relation $\leqslant$ defined by $x \leqslant y$ if and only if $x * y = 0$ is a partial order on G. In any BCI-algebra the following two identities hold:

(P_1) $x * 0 = x$,
(P_2) $(x * y) * z = (x * z) * y$.

A non-empty subset S of a BCI-algebra G is said to be a *subalgebra* of G if $x * y \in S$ whenever $x, y \in S$. A non-empty subset A of a BCI-algebra G is called an *ideal* of G, denoted by $A \sqsubseteq G$, if

- $0 \in A$,
- $x * y \in A$ and $y \in A$ imply $x \in A$.

An ideal A of a BCI-algebra G is said to be *closed* if $0 * x \in A$ for all $x \in A$. Note that an ideal of a BCI-algebra may not be a subalgebra in general, but every closed ideal is closed with respect to a BCI-operation, i.e. it is a subalgebra (cf. [5]).

A non-empty subset A of a BCI-algebra G is called a *p-ideal* of G if it satisfies the following two conditions

- $0 \in A$,
- $(x * z) * (y * z) \in A$ and $y \in A$ imply $x \in A$.

Note that in BCI-algebras every p-ideal is an ideal, but not converse (see [15]).

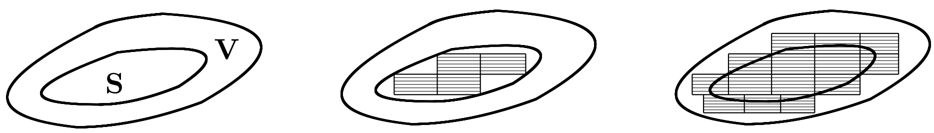
Let ρ be a congruence relation on G, that is, ρ is an equivalence relation on G such that $(x, y) \in \rho$ implies $(x * z, y * z) \in \rho$ and $(z * x, z * y) \in \rho$ for all $z \in G$. The set of all equivalence classes of G with respect to ρ will be denoted by G/ρ. On G/ρ we define an operation $*$ putting $[x]_\rho * [y]_\rho = [x * y]_\rho$ for all $[x]_\rho, [y]_\rho \in G/\rho$. It is clear that such operation is well-defined, but $(G/\rho, *, [0]_\rho)$ may not be a BCI-algebra, because G/ρ does not satisfy the fourth condition of a BCI-algebra.

For any non-empty subsets A and B of a BCI-algebra G we define the *complex multiplication* putting $A * B = \{x * y \mid x \in A,\ y \in B\}$.

3. Roughness of some ideals

Let V be a set and ρ an equivalence relation on V and let $\mathcal{P}(V)$ denote the power set of V. For all $x \in V$, let $[x]_\rho$ denote the equivalence class of G with respect to ρ. Define the functions ρ_- and ρ^- from $\mathcal{P}(V)$ to $\mathcal{P}(V)$ putting for every $S \in \mathcal{P}(V)$

$$\rho_-(S) = \{x \in V \mid [x]_\rho \subseteq S\},$$
$$\rho^-(S) = \{x \in V \mid [x]_\rho \cap S \neq \emptyset\}.$$



$S \subset V$ $\qquad$ $\rho_-(S) \subseteq S$ $\qquad$ $S \subseteq \rho^-(S)$

$\rho_-(S)$ is called the *lower approximation* of S while $\rho^-(S)$ is called the *upper approximation*. The set S is called *definable* if $\rho_-(S) = \rho^-(S)$ and *rough* otherwise. The pair (V, ρ) is called an *approximation space*.

Directly from the definition by simple calculations we can see that the following proposition holds.

Proposition 1. *Let A and B be non-empty subsets of a BCI-algebra G. If ρ is a congruence relation on G, then the following hold:*

(1) $\rho_-(A) \subseteq A \subseteq \rho^-(A)$,
(2) $\rho^-(A \cup B) = \rho^-(A) \cup \rho^-(B)$,
(3) $\rho_-(A \cap B) = \rho_-(A) \cap \rho_-(B)$,
(4) $A \subseteq B$ *implies* $\rho_-(A) \subseteq \rho_-(B)$ *and* $\rho^-(A) \subseteq \rho^-(B)$,
(5) $\rho_-(A \cup B) \supseteq \rho_-(A) \cup \rho_-(B)$,
(6) $\rho^-(A \cap B) \subseteq \rho^-(A) \cap \rho^-(B)$,
(7) $\rho^-(A) * \rho^-(B) \subseteq \rho^-(A * B)$,
(8) $\rho_-(A) * \rho_-(B) \subseteq \rho_-(A * B)$ *whenever* $\rho_-(A) * \rho_-(B) \neq \emptyset$ *and* $\rho_-(A * B) \neq \emptyset$.

Proposition 2. *If* ρ *is a congruence relation on a* BCI*-algebra* G*, then the following are equivalent:*

(1) $x * y \in [0]_\rho$ *and* $y * x \in [0]_\rho$ *imply* $(x, y) \in \rho$,

(2) ρ *is regular, i.e.* $[x]_\rho * [y]_\rho = [0]_\rho = [y]_\rho * [x]_\rho$ *implies* $[x]_\rho = [y]_\rho$,

(3) $(G/\rho, *, [0]_\rho)$ *is a* BCI*-algebra.*

Proof. (1) $\Rightarrow$ (2) Suppose $[x]_\rho * [y]_\rho = [0]_\rho = [y]_\rho * [x]_\rho$. Then $[x * y]_\rho = [0]_\rho = [y * x]_\rho$, and so $(x * y, 0) \in \rho$ and $(y * x, 0) \in \rho$. It follows from (1) that $(x, y) \in \rho$. Hence $[x]_\rho = [y]_\rho$.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Let $x, y \in G$ be such that $x * y \in [0]_\rho$ and $y * x \in [0]_\rho$. Then

$$[x]_\rho * [y]_\rho = [x * y]_\rho = [0]_\rho = [y * x]_\rho = [y]_\rho * [x]_\rho \,.$$

It follows from the fourth condition of the definition of a BCI-algebra that $[x]_\rho = [y]_\rho$. Thus $(x, y) \in \rho$. This completes the proof. □

Theorem 3. *If* ρ *is a congruence relation on* G*, then* $[0]_\rho$ *is a closed ideal, and hence a subalgebra of* G*.*

Proof. Obviously, $0 \in [0]_\rho$. Let $x, y \in G$ be such that $x * y \in [0]_\rho$ and $y \in [0]_\rho$. Then $(x * y, 0) \in \rho$ and $(y, 0) \in \rho$. Since ρ is a congruence relation on G, it follows from (P_1) that $(x * y, x) = (x * y, x * 0) \in \rho$ so that $(x, 0) \in \rho$, that is, $x \in [0]_\rho$. If $x \in [0]_\rho$, then $(x, 0) \in \rho$ and hence $(0 * x, 0) = (0 * x, 0 * 0) \in \rho$, that is, $0 * x \in [0]_\rho$. Hence $[0]_\rho$ is a closed ideal of G. □

Definition 4. A non-empty subset S of a BCI-algebra G is called an *upper* (resp. a *lower*) *rough subalgebra* (or, (*closed*) *ideal*) of G if the upper (resp. nonempty lower) approximation of S is a subalgebra (or, (closed) ideal) of G. If S is both an upper and a lower rough subalgebra (or, (closed) ideal) of G, we say that S is a *rough subalgebra* (or (*closed*) *ideal*) of G.

Theorem 5. *Every subalgebra is a rough subalgebra.*

Proof. Let S be a subalgebra of a BCI-algebra G. Taking $A = B = S$ in Proposition 1(8), we have

$$\rho_-(S) * \rho_-(S) \subseteq \rho_-(S * S) \subseteq \rho_-(S)$$

because S is a subalgebra of G. Hence $\rho_-(S)$ is a subalgebra of G, that is, S is a lower rough subalgebra of G. We now show that $\rho^-(S)$ is a subalgebra of G. Let $x, y \in \rho^{-}(S)$. Then $[x]_\rho \cap S \neq \emptyset$ and $[y]_\rho \cap S \neq \emptyset$. Thus there exist $a_x, a_y \in S$ such that $a_x \in [x]_\rho$ and $a_y \in [y]_\rho$. It follows that $(a_x, x) \in \rho$ and $(a_y, y) \in \rho$ so that $(a_x * a_y, x * y) \in \rho$, that is, $a_x * a_y \in [x * y]_\rho$. On the other hand, since S is a subalgebra of G, we have $a_x * a_y \in S$. Hence $a_x * a_y \in [x * y]_\rho \cap S$, that is, $[x * y]_\rho \cap S \neq \emptyset$. This shows that $x * y \in \rho^{-}(S)$. Therefore S is an upper rough subalgebra of G. This completes the proof. $\square$

For any subset I of a BCI-algebra G, define a relation $\rho(I)$ on G induced by I in the following way:

$$(x, y) \in \rho(I) \iff x * y,\ y * x \in I.$$

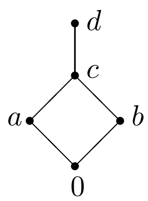
$\rho(I)_{-}(S)$ is called the *lower approximation* of S by I, while $\rho(I)^{-}(S)$ is called the *upper approximation* by I. In the case $\rho(I)_{-}(S) = \rho(I)^{-}(S)$ we say that S is called *definable* with respect to I. In otherwise S is *rough* with respect to I. Obviously $\rho(I)_{-}(G) = G = \rho(I)^{-}(G)$ for any $I \sqsubseteq G$. This means that any BCI-algebra is definable with respect to any its ideal.

If I is an ideal of G, then $\rho(I)$ is a regular congruence relation on G (see [9]). Note that in the case of BCI-quasigroups every subalgebra is an ideal. The converse is not true (see [4]), but a finite subset of such quasigroup is an ideal if and only if it is a subalgebra. Thus in BCI-algebras all relations $\rho(I)$ induced by a finite set I are regular congruences.

The following example shows that there exists non-empty subset S of G which is not an ideal, but for which S is an upper rough subalgebra of G. Hence we know that the notion of an upper rough subalgebra is an extended notion of a subalgebra.

Example 6. Let $G = \{0, a, b, c, d\}$ be a BCI-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	d	c	0



Then $I = \{0, a\} \sqsubseteq G$, and thus $[0]_{\rho(I)} = [a]_{\rho(I)} = I$, $[b]_{\rho(I)} = \{b\}$, $[c]_{\rho(I)} = \{c\}$, and $[d]_{\rho(I)} = \{d\}$. Consider a subset $S = \{a, b\}$ of G which is not a subalgebra of G. Then $\rho(I)^{-}(S) = \{0, a, b\}$ which is a subalgebra.

On the other hand, for $M = \{0, a, c\}$ which is a subalgebra but not an ideal, we have $\rho(I)^{-}(M) = \rho(I)_{-}(M) = M$. Hence M is definable with respect to I. It is not to difficult to see that M is not definable with respect to $J = \{0, b\} \sqsubseteq G$.

Proposition 7. *Every non-empty subset of a BCI-algebra is definable with respect to the trivial ideal* $\{0\}$.

Proof. If $a \in [x]_{\rho(\{0\})}$ then $(a, x) \in \rho(\{0\})$ and so $a * x \in \{0\}$ and $x * a \in \{0\}$. It follows that $a = x$ so that $[x]_{\rho(\{0\})} = \{x\}$ for all $x \in G$. Hence

$$\rho(\{0\})_-(S) = \big\{x \in G \mid [x]_{\rho(\{0\})} \subseteq S\big\} = S$$

and

$$\rho(\{0\})^-(S) = \big\{x \in G \mid [x]_{\rho(\{0\})} \cap S \neq \emptyset\big\} = S.$$

This completes the proof. $\square$

Lemma 8. *If I and J are ideals of a BCI-algebra G such that $I \subseteq J$, then $\rho(I) \subseteq \rho(J)$.*

Proof. If $(x, y) \in \rho(I)$, then $x * y \in I \subseteq J$ and $y * x \in I \subseteq J$. Hence $(x, y) \in \rho(J)$, completing the proof. $\square$

Remark 9. Let I and J be ideals of G such that $I \neq J$. Then $\rho(I)_-(J)$ is not an ideal of G in general. Indeed, it is easy to see that $I = \{0, a\}$ and $J = \{0, b\}$ are ideals of a BCI-algebra G defined in Example 6. But

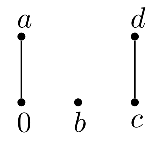
$$\rho(I)_-(J) = \{x \in G \mid [x]_{\rho(I)} \subseteq J\} = \{b\}$$

is not an ideal of G.

The following example shows that there exists a non-ideal J of G for which J is an upper rough ideal of G with respect to an ideal of G. Hence we know that the notion of an upper rough ideal is an extended notion of an ideal.

Example 10. Consider a BCI-algebra $G = \{0, a, b, c, d\}$ with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	c	b	c
a	a	0	c	b	b
b	b	b	0	c	c
c	c	c	b	0	0
d	d	c	b	a	0



Then for $I = \{0, a\} \sqsubseteq G$ we have $[0]_{\rho(I)} = [a]_{\rho(I)} = I$, $[b]_{\rho(I)} = \{b\}$, and $[c]_{\rho(I)} = [d]_{\rho(I)} = \{c, d\}$. Thus for $J = \{0, b, c\}$, which is not an ideal of G, we obtain

$$\rho(I)^-(J) = \{x \in G \mid [x]_{\rho(I)} \cap J \neq \emptyset\} = \{0, a, b, c, d\} \sqsubseteq G.$$

Theorem 11. *Let $I \subseteq J$ be two ideals of a BCI-algebra G. Then*

(1) *$\rho(I)_-(J)$ $(\neq \emptyset)$ is an ideal of G, that is, J is a lower rough ideal of G with respect to I.*

(2) *$\rho(I)^-(J)$ is an ideal of G, that is, J is an upper rough ideal of G with respect to I. Moreover if J is closed, then so is $\rho(I)^-(J)$.*

Proof. (1) Let $x \in [0]_{\rho(I)}$. Then $x = x*0 \in I \subseteq J$ and so $[0]_{\rho(I)} \subseteq J$. Hence $0 \in \rho(I)_-(J)$. Let $x, y \in G$ be such that $x*y \in \rho(I)_-(J)$ and $y \in \rho(I)_-(J)$. Then $[y]_{\rho(I)} \subseteq J$ and

$$[x]_{\rho(I)} * [y]_{\rho(I)} = [x * y]_{\rho(I)} \subseteq J.$$

Let $a_x \in [x]_{\rho(I)}$ and $a_y \in [y]_{\rho(I)}$. Then $(a_x, x) \in \rho(I)$ and $(a_y, y) \in \rho(I)$, which imply $(a_x * a_y, x * y) \in \rho(I)$. Hence $a_x * a_y \in [x * y]_{\rho(I)} \subseteq J$. Since $a_y \in [y]_{\rho(I)} \subseteq J$, it follows that $a_x \in J$. Therefore $[x]_{\rho(I)} \subseteq J$, or equivalently, $x \in \rho(I)_-(J)$. This shows that $\rho(I)_-(J)$ is an ideal of G.

(2) Obviously, $0 \in \rho(I)^-(J)$. Let $x, y \in G$ be such that $y \in \rho(I)^-(J)$ and $x*y \in \rho(I)^-(J)$. Then $[y]_{\rho(I)} \cap J \neq \emptyset$ and $[x*y]_{\rho(I)} \cap J \neq \emptyset$, and so there exist $u, v \in J$ such that $u \in [y]_{\rho(I)}$ and $v \in [x * y]_{\rho(I)}$. Hence $(u, y) \in \rho(I)$ and $(v, x * y) \in \rho(I)$ which imply $y * u \in I \subseteq J$ and $(x * y) * v \in I \subseteq J$. Since $u, v \in J$ and J is an ideal, it follows that $y \in J$ and $x * y \in J$ so that $x \in J$. Note that $x \in [x]_{\rho(I)}$, thus $x \in [x]_{\rho(I)} \cap J$, that is, $[x]_{\rho(I)} \cap J \neq \emptyset$. Therefore $x \in \rho(I)^-(J)$, and consequently J is an upper rough ideal of G with respect to I. Now let $x \in \rho(I)^-(J)$. Then $[x]_{\rho(I)} \cap J \neq \emptyset$, and so there exists $a_x \in J$ such that $a_x \in [x]_{\rho(I)}$. Since J is closed, it follows that $0 * a_x \in J$ and hence

$$0 * a_x \in \left([0]_{\rho(I)} * [x]_{\rho(I)}\right) \cap J = [0 * x]_{\rho(I)} \cap J,$$

that is, $[0 * x]_{\rho(I)} \cap J \neq \emptyset$. Hence $0 * x \in \rho(I)^-(J)$. This completes the proof. □

Lemma 12. ([15, Theorem 4.1]) *An ideal I of a BCI-algebra G is a p-ideal if and only if for each $x, y, z \in G$,*

$$(x * z) * (y * z) \in I \text{ implies } x * y \in I.$$

It is not difficult to see that in the case of BCI-quasigroups every ideal is a p-ideal and conversely.

Theorem 13. *Let $I \sqsubseteq G$ and let J be a p-ideal of a BCI-algebra G containing I. Then $\rho(I)_-(J)$ $(\neq \emptyset)$ and $\rho(I)^-(J)$ are p-ideals of G.*

Proof. Let $x, y, z \in G$ be such that $(x * z) * (y * z) \in \rho(I)_-(J)$. Then

$$\big([x]_{\rho(I)} * [z]_{\rho(I)}\big) * \big([y]_{\rho(I)} * [z]_{\rho(I)}\big) = [(x * z) * (y * z)]_{\rho(I)} \subseteq J.$$

Let $w \in [x * y]_{\rho(I)} = [x]_{\rho(I)} * [y]_{\rho(I)}$. Then $w = a_x * a_y$ for some $a_x \in [x]_{\rho(I)}$ and $a_y \in [y]_{\rho(I)}$. From $a_x \in [x]_{\rho(I)}$ and $a_y \in [y]_{\rho(I)}$, we have $(a_x, x) \in \rho(I)$ and $(a_y, y) \in \rho(I)$. Taking $a_z \in [z]_{\rho(I)}$, then $(a_z, z) \in \rho(I)$. Since $\rho(I)$ is a congruence relation, we get $(a_x * a_z, x * z) \in \rho(I)$ and $(a_y * a_z, y * z) \in \rho(I)$, and thus

$$\big((a_x * a_z) * (a_y * a_z),\ (x * z) * (y * z)\big) \in \rho(I).$$

This means that

$$(a_x * a_z) * (a_y * a_z) \in [(x * z) * (y * z)]_{\rho(I)} \subseteq J.$$

Since J is a p-ideal, it follows from Lemma 12 that $w = a_x * a_y \in J$ so that $[x * y]_{\rho(I)} \subseteq J$, or equivalently, $x * y \in \rho(I)_-(J)$. Combining Theorem 11(1) and Lemma 12, $\rho(I)_-(J)$ is a p-ideal of G.

Now let $x, y, z \in G$ be such that $(x * z) * (y * z) \in \rho(I)^-(J)$ and $y \in \rho(I)^-(J)$. Then $[y]_{\rho(I)} \cap J \neq \emptyset$ and $[(x * z) * (y * z)]_{\rho(I)} \cap J \neq \emptyset$, and thus there are $a, b \in J$ such that $a \in [y]_{\rho(I)}$ and $b \in [(x * z) * (y * z)]_{\rho(I)}$. Hence $(a, y) \in \rho(I)$ and $\big(b, (x * z) * (y * z)\big) \in \rho(I)$, which imply that $y * a \in I \subseteq J$ and $\big((x * z) * (y * z)\big) * b \in I \subseteq J$. Since J is an ideal and since $a, b \in J$, we have $y \in J$ and $(x * z) * (y * z) \in J$. Since J is a p-ideal, it follows that $x \in J$. Note that $x \in [x]_{\rho(I)}$, and thus $x \in [x]_{\rho(I)} \cap J$, that is, $[x]_{\rho(I)} \cap J \neq \emptyset$. Therefore $x \in \rho(I)^-(J)$. This completes the proof. □

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A note on the Akivis algebra of a smooth hyporeductive loop

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Abstract

Using the fundamental tensors of a smooth loop and the differential geometric characterization of smooth hyporeductive loops, the Akivis operations of a local smooth hyporeductive loop are expressed through the two binary and the one ternary operations of the hyporeductive triple algebra (h.t.a.) associated with the given hyporeductive loop. Those Akivis operations are also given in terms of Lie brackets of a Lie algebra of vector fields with the hyporeductive decomposition which generalizes the reductive decomposition of Lie algebras. A nontrivial real two-dimensional h.t.a. is presented.

1. Introduction

A *quasigroup* is a set Q with a binary operation of multiplication denoted by $\circ$ or juxtaposition such that the knowledge of any two of x, y, z in the equation $x \circ y = z$ uniquely specifies the third. A *loop* is a quasigroup $(Q, \circ)$ with a two-sided identity e. In the case when Q is a neighborhood of the fixed point e in a smooth (real finite-dimensional) manifold and the operation $\circ$ is a smooth function $Q \times Q \to Q$, then $(Q, \circ)$ is called a *local smooth* loop.

As for Lie groups, an infinitesimal theory for smooth quasigroups is considered by M. A. Akivis (see [1], [2], [3]). If $(Q, \circ)$ is a smooth loop then in a sufficiently small neighborhood of e, the binary operation $\circ$ has the following Taylor expansion [1]:

$$(x \circ y)^i = x^i + y^i + \tau^i_{jk} x^j y^k + \mu^i_{jkl} x^j x^k y^l + \nu^i_{jkl} x^j y^k y^l + \dots$$

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where the quantities μ^i_{jkl} and ν^i_{jkl} have the properties $\mu^i_{jkl} = \mu^i_{kjl}$ and $\nu^i_{jkl} = \nu^i_{jlk}$. The so-called *fundamental tensors* α^i_{jk}, β^i_{ljk} of the given smooth loop $(Q, \circ, e)$ are defined as follows:

$$\alpha^i_{jk} = \frac{1}{2}\left(\tau^i_{jk} - \tau^i_{kj}\right), \qquad \beta^i_{ljk} = 2\mu^i_{jkl} - 2\nu^i_{jkl} + \alpha^s_{jk}\alpha^i_{sl} - \alpha^i_{js}\alpha^s_{kl}\ .$$

The *commutator* and the *associator* at the identity e of $(Q, \circ, e)$ are expressed in terms of the fundamental tensors α^i_{jk} and β^i_{ljk} as follows:

$$(x \circ y)^i - (y \circ x)^i = 2\alpha^i_{jk} x^i y^k + o(\rho^2),$$

$$[(x \circ y) \circ z]^i - [x \circ (y \circ z)]^i = \beta^i_{ljk} x^l y^j z^k + o(\rho^3),$$

where $\rho = max(|x^i|, |y^i|)$.

Therefore the tensor α^i_{jk} (respectively β^i_{ljk}) characterizes the principal part of the deviation degree from commutativity (respectively associativity) of the loop $(Q, \circ, e)$. It should be noted that these expressions of the commutator and the associator hold in any smooth loop (more precisely, in a sufficiently small neighborhood of any element of that loop) and the tensors α^i_{jk} and β^i_{ljk} are defined at any point of the manifold Q (cf. [1]). For $\alpha^i_{jk} = 0$ and $\beta^i_{ljk} = 0$, the loop $(Q, \circ, e)$ becomes locally an abelian group and for $\beta^i_{ljk} = 0$ it is a local Lie group.

Using the fundamental tensors, the tangent space T_eQ may be provided with a structure of a binary-ternary algebra (the *tangent algebra of the smooth loop*) if define

$$(X \diamond Y)^i = 2\alpha^i_{jk} X^j Y^k\,, \qquad [X, Y, Z]^i = \beta^i_{ljk} X^l Y^j Z^k\,, \tag{1}$$

for all $X, Y, Z \in T_eQ$. It is shown [2] that $\diamond$ and $[-,-,-]$ satisfy the following identities

$$X \diamond X = 0, \tag{2}$$

$$[X, X, X] = 0, \tag{3}$$

$$\sigma\{XY \diamond Z\} = \sigma\{[X, Y, Z]\} - \sigma\{[Y, X, Z]\}, \tag{4}$$

where σ denotes the cyclic sum with respect to X, Y, Z and juxtaposition is used to reduce the number of brackets, that is $XY \diamond Z$ means $(X \diamond Y) \diamond Z$. Following [4], a (real finite-dimensional) vector space is called an *Akivis algebra* if it carries a bilinear operation $\diamond$ and a trilinear operation $[-,-,-]$ satisfying the identities $(2) - (4)$. The identity (4) is known as the *Akivis* *identity*. Hereafter we shall refer to the operations $\diamond$ and $[-,-,-]$ as defined in (1) as to the Akivis operations.

We will be interested in the situation when a smooth loop $(Q,\circ,e)$ is related to an affine connection space (Q,∇). In [8], [11] a construction of a loop centered at a fixed point e of (Q,∇) is given. Such a loop is called the *geodesic loop* of (Q,∇) at the point e (it turns out that e is the two-sided identity element of that loop). Moreover the geodesic loop operation $\circ$ is supplemented by an unary multiplication $(t,x)\mapsto tx$ of any element $x \in (Q,\circ,e)$ by a real scalar t, giving rise to the concept of a *geodesic odule* (see [11]). The identity

$$((t+u)x)\circ y = tx\circ(ux\circ y) \tag{5}$$

is called the *left monoalternative property*, where t and u are real numbers; likewise is defined the *right* monoalternative property. The right monoalternative property plays a key role in the differential geometric theory of some classes of loops. It turns out that (see [3]) for a geodesic loop $(Q,\circ,e)$ of an affine connection space (Q,∇), its fundamental tensors are expressed in terms of the torsion and curvature of the space (Q,∇) as follows:

$$\alpha^i_{jk} = -\frac{1}{2}\,T^i_{jk}(e), \qquad \beta^i_{ljk} = \frac{1}{2}\left(R^i_{l,jk} - \nabla_k T^i_{lj}\right)(e)\,. \tag{6}$$

Accordingly the Akivis operations of $(Q,\circ,e)$ are also expressed in terms of the torsion and curvature of (Q,∇).

For the general theory of specific classes of smooth loops it is sometimes convenient to give the explicit form of their Akivis operations. This is easy, according to (6), whenever a suitable differential geometric theory is built for a given class of smooth loops. The tangent algebra to a smooth *Bol loop* is called a *Bol algebra* (see [10], [15]) while the tangent algebra to a smooth *homogeneous loop* is called a *Lie triple algebra* (see [9], [12]). One observes that a Bol algebra (resp. a Lie triple algebra) is an Akivis algebra of a smooth Bol loop (resp. a smooth homogeneous loop) with additional conditions.

In [5] the Lie triple algebra of a smooth homogeneous loop was related to its Akivis algebra. It is our purpose in this note to do the same for hyporeductive loops since they are a generalization both of Bol loops and homogeneous loops ([13], [14]). Here the approach is geometric in the sense of (6) (see Section 2) and algebraic meaning that the Akivis operations of a smooth hyporeductive loop are expressed in terms of the Lie brackets of a Lie algebra satisfying some specific conditions (see Section 3). We wonder whether the method of the algebraic calculus of formal power series, developed in [5] for the case of smooth homogeneous loops, could be applied to smooth hyporeductive loops.

2. Tangent algebras to smooth hyporeductive loops: hyporeductive triple algebras (h.t.a.)

A loop $(Q, \circ, e)$ is said *left hypospecial* if there exists $b(x,y) \in Q$ with $x, y \in Q$ such that $b(x,e) = e = b(e,x)$ and the mapping $\phi(x,y) = L_{b(x,y)} l_{x,y}$ has the property

$$\phi L_z \phi^{-1} = L_{(\phi z)/b(x,y)}$$

where $L_u v = u \circ v$, $l_{u,v} = L^{-1}_{u \circ v} L_u L_v$ and $/$ denotes the right division in $(Q, \circ, e)$. A *left hyporeductive loop* is a left hypospecial loop with the left monoalternative property (5). Similarly is defined a *right* hyporeductive loop. An infinitesimal theory for smooth hyporeductive loops is initiated by L.V. Sabinin in [13], [14], where he constructed a tangent algebra for such loops that is called a *hyporeductive algebra*. It should be noted that there is a one-to-one correspondance between hyporeductive algebras and smooth hyporeductive loops. In [6] (see also [7]) a differential geometric study for smooth hyporeductive loops is suggested. In particular it is shown that a smooth hyporeductive loop $(Q, \circ, e)$ can locally be seen as an affine connection space (Q, ∇) with zero curvature satisfying the following structure equations

$$d\omega^i = \frac{1}{2} \; T^i_{jk} \;\; \omega^j \; \wedge \; \omega^k, \tag{7}$$

$$dT^i_{jk} = \left(T^i_{ls}(T^s_{jk} + a^s_{jk}) - r^i_{l,jk}\right) \omega^l, \tag{8}$$

where a^s_{jk} and $r^i_{l,jk}$ are constants and $a^i_{jk} = -a^i_{kj}$, $r^i_{l,jk} = -r^i_{l,kj}$. Moreover, the geodesic loop at a fixed point of an affine connection space with structure equations (7), (8) is a (right) hyporeductive loop. Using the known differential geometric techniques we obtained [6] that the integrability criteria of (7), (8) constitute the determining identities of a hyporeductive algebra if we set

$$\begin{aligned} (X.Y)^i = a^i_{jk} X^j Y^k \,, \qquad (X * Y)^i \;&=\; (-T^i_{jk}(e) - a^i_{jk}) X^j Y^k \,, \\ < Z; X, Y >^i \;&=\; -r^i_{l,jk} X^j Y^k Z^k \,, \end{aligned} \tag{9}$$

for $X, Y, Z \in T_e Q$. The operations $*$, . and $< -; -, - >$ are linked by a certain set of identities ([6], [7], [14]). They are as follows:

$$\sigma\{\ \xi\,.\,(\eta\,.\,\zeta) - <\xi;\eta,\zeta>\} = 0\,,$$

$$\sigma\{\ \zeta * (\xi\,.\,\eta)\,\} = 0\,,$$

$$\sigma\{<\theta;\zeta,\ \xi\,.\,\eta>\} = 0\,,$$

$$\begin{aligned}&\kappa.<\zeta;\xi,\eta> -\zeta.<\kappa;\xi,\eta> + <\zeta.\kappa;\xi,\eta> =\\ &\quad = <\xi*\eta;\zeta,\kappa> - <\zeta*\kappa;\xi,\eta> + \zeta*<\kappa;\xi,\eta> - \kappa*<\zeta;\xi,\eta> +\\ &\quad + (\xi*\eta)*(\zeta*\kappa) + (\xi*\eta).(\zeta*\kappa)\,,\end{aligned}$$

$$\begin{aligned}&\chi.(\kappa.<\zeta;\xi,\eta> -\zeta.<\kappa;\xi,\eta> + <\zeta.\kappa;\xi,\eta>) +\\ &\quad + <<\chi;\xi,\eta>;\zeta,\kappa> - <<\chi;\zeta,\kappa>;\xi,\eta> +\\ &\quad + <\chi;\zeta,<\kappa;\xi,\eta>> - <\chi;\kappa,<\zeta;\xi,\eta>> = 0\,,\end{aligned}$$

$$\chi * (\kappa\,.<\zeta;\xi,\eta> -\zeta\,.<\kappa;\xi,\eta> + <\zeta\,.\,\kappa;\xi,\eta>) = 0\,,$$

$$<\theta;\chi,\ \kappa\,.<\zeta;\xi,\eta> -\zeta\,.<\kappa;\xi,\eta> + <\zeta\,.\,\kappa;\xi,\eta>> = 0\,,$$

$$\begin{aligned}&\kappa\,.<\zeta;\xi,\eta> -\zeta\,.<\kappa;\xi,\eta> + <\zeta\,.\,\kappa;\xi,\eta> +\\ &\quad + \eta\,.<\xi;\zeta,\kappa> -\xi\,.<\eta;\zeta,\kappa> + <\xi\,.\,\eta;\zeta,\kappa> = 0\,,\end{aligned}$$

$$\zeta*<\kappa;\xi,\eta> -\kappa*<\zeta;\xi,\eta> +\xi*<\eta;\zeta,\kappa> +\eta*<\xi;\zeta,\kappa> = 0\,,$$

$$\begin{aligned}&\Sigma\{<(<\xi\,.\,\eta;\zeta,\kappa> +\eta\,.<\xi;\zeta,\kappa> -\xi\,.<\eta;\zeta,\kappa>);\lambda,\mu> +\\ &\quad + <\lambda.\mu;<\eta;\zeta,\kappa>,\xi> +\mu\,.<\lambda;<\eta;\zeta,\kappa>,\xi> -\\ &\quad -\lambda\,.<\mu;<\eta;\zeta,\kappa>,\xi> -(<\lambda.\mu;<\xi;\zeta,\kappa>,\eta> +\\ &\quad +\mu\,.<\lambda;<\xi;\zeta,\kappa>,\eta> -\lambda<\mu;<\xi;\zeta,\kappa>,\eta>)\,\} = 0\,,\end{aligned}$$

$$\begin{aligned}&\Sigma\{\,(<\mu;<\eta;\zeta,\kappa>,\xi> - <\mu;<\xi;\zeta,\kappa>,\eta>)*\lambda +\\ &\quad +(<\lambda;<\xi;\zeta,\kappa>,\eta> - <\lambda;<\eta;\zeta,\kappa>,\xi>)*\mu\} = 0\,,\end{aligned}$$

$$\begin{aligned}&\Sigma\{<\theta;(<\mu;<\eta;\zeta,\kappa>,\xi> - <\mu;<\xi;\zeta,\kappa>,\eta>),\lambda> +\\ &\quad + <\theta;(<\lambda;<\xi;\zeta,\kappa>,\eta> - <\lambda;<\eta;\zeta,\kappa>,\xi>),\mu>\,\} = 0\,,\end{aligned}$$

where σ denotes the cyclic sum with respect to ξ,η,ζ and Σ the one with respect to pairs (ξ,η), (ζ,κ), (λ,μ). Any (real finite-dimensional) vector space with two anticommutative bilinear operations and one trilinear, skew-symmetric with respect to the two last variables, operation satisfying those identities is called an *abstract hyporeductive triple algebra* (h.t.a. for short).

It is worthy of note that such identities are obtained [14] if work out the Jacobi identities of the Lie algebra of vector fields enveloping the given hyporeductive algebra and satisfying some specific conditions.

We give an example of a nontrivial real 2-dimensional h.t.a.

Example. Let m be a 2-dimensional algebra over the field of real numbers with basis $\{u, v\}$. Define on m the following operations:

$$u * v = u, \quad u.v = v, \quad < u; u, v >= v, \quad < v; u, v >= 0$$

with the symmetries $u * u = 0 = u.u$, $< t; u, u >= 0$, where $t = u$ or v. Then it could be checked that m is a nontrivial h.t.a. that is not a Bol algebra nor a Lie triple algebra. □

We have the following theorem whose proof is somewhat elementary in view of structure equations (7), (8) above.

Theorem 1. *Let* $(Q, \circ, e)$ *be a given smooth local hyporeductive loop and* $(T_eQ, ., *, < -; -, - >)$ *be the corresponding (up to an isomorphism) h.t.a. Then the Akivis operations* $\diamond$ *and* $[-, -, -]$ *of* $(Q, \circ, e)$ *are linked with* $.$, $*$, $< -; -, - >$ *as follows:*

(i) $\quad X \diamond Y = X.Y + X * Y,$

(ii) $\quad [X, Y, Z] = -\frac{1}{2}\big(< Z; X, Y > + Z \diamond (X * Y)\big)$

for all $X, Y, Z \in T_eQ$.

Proof. Let $(X * Y)^i = b^i_{jk} X^j Y^k$, that is $b^i_{jk} = -T^i_{jk}(e) - a^i_{jk}$. Then from (1), (6) and (9) we get (i).

Next, from (8) we know that $-r^i_{l,jk} = (\nabla_l T^i_{jk} + T^i_{ls} b^s_{jk})(e)$. Therefore, since $< Z; X, Y >^i = -r^i_{l,jk} X^j Y^k Z^l = ((\nabla_l T^i_{jk} + T^i_{ls} b^s_{jk})(e)) X^j Y^k Z^l$, from (1), (6) we get (ii) (recall that $R^i_{l,jk} = 0$). □

Remark 1. (a) Using (i) the Akivis operation $[X, Y, Z]$ in (ii) can also be expressed by $\diamond$ and $.$ as follows:

(iii) $\quad [X, Y, Z] = -\frac{1}{2}\big(< Z; X, Y > + Z \diamond (X \diamond Y) - Z \diamond (X.Y)\big).$

(b) From (i) and (ii) we see that if $X.Y = 0$ for all $X, Y \in T_eQ$, then $X \diamond Y = X * Y$ and $[X, Y, Z] = (-1/2)(< Z; X, Y > + Z \diamond (X \diamond Y))$ and we are in the situation of Bol algebras (see [10], [15]). Likewise for $X * Y = 0$ for all $X, Y \in T_eQ$ we get $X \diamond Y = X.Y$ and $[X, Y, Z] = (-1/2) < Z; X, Y >$

and we have the case of Lie triple algebras [5].

With the remarks above one could think of the operation . (resp. $*$) as of a deviation degree of a h.t.a. from a Bol algebra (resp. a Lie triple algebra). Although the transformations are somewhat tedious and lengthy, one could write down the determining identities of a h.t.a. in terms of the Akivis operations $\diamond$, $[-,-,-]$ and the operation . (or $*$).

3. An alternative approach

Let m be a (real finite-dimensional) vector space of covariantly constant vector fields of an affine connection space with zero curvature (Q, ∇) and $e \in Q$ a fixed point. Let g be the Lie algebra of vector fields generated by m and such that $g = m + [m, m]$ (here $[m, m]$ denotes the subset of g generated by all $[X, Y]$ with $X, Y \in m$) and let h be the subalgebra of g defined by $h = \{X \in g : X(e) = o\}$. Then

$$g = m \dot{+} h \tag{10}$$

(direct sum of vector spaces; see [16]). Additionally let assume that there exists in g a subspace n such that

$$g = m \dot{+} n \ \ \text{(direct sum of subspaces)}, \tag{11}$$

$$[n, m] \subset m. \tag{12}$$

A pair (g, h) with the decomposition (10) such that (11), (12) hold is said *hyporeductive* ([13], [14]).

Proposition 2. *The hyporeductive pair* (g, h) *with conditions* (10) – (12) *induces on* m *a structure of a h.t.a.*

Proof. If $X, Y \in m$ then $[X, Y] \in g$ and the decomposition (11) induces a binary operation, say ., on m

$$X_i.X_j = [X_i, X_j]^n_m \tag{13}$$

(here and in the sequel $[X, Y]^w_v$ denotes the projection on v parallely w), where X_s ($s = 1, ..., l,\ l = dim\ m$) constitute a basis of m. We denote by $D(X_i, X_j) = [X_i, X_j] - X_i.X_j$ ($i \neq j$) the basis elements of n. Further, using (10) and (12), we define on m a binary operation

$$X_i * X_j = [X_i, X_j]^h_m - X_i.X_j \tag{14}$$

and a ternary operation

$$< X_k; X_i, X_j >= -[X_k, D(X_i, X_j)]. \tag{15}$$

Now using the procedure described in [13], [14] one could write down the Jacobi identities in g with respect to the set $\{X_\alpha,\ D(X_\beta, X_\gamma)\}$ of basis elements. This in turn leads to the set of determining identities of a h.t.a. so that $(m, ., *, < -; -, - >)$ becomes a h.t.a. of vector fields. □

Above we considered m as the linear space of covariantly constant vector fields on an affine connection manifold (Q, ∇) with zero curvature; this is intended for a relation with local smooth loops with the right monoalternative property and, further, with local smooth hyporeductive loops. Specifically we mean the following. If e is a fixed point on (Q, ∇), then m may be identified with the tangent space T_eQ and therefore, in the case when m is a h.t.a., T_eQ is a h.t.a. Moreover, since (Q, ∇) has zero curvature, the geodesic loop $(Q, ., e)$ of (Q, ∇) centered at the point e has the right monoalternative property [15] and, if T_eQ is a h.t.a., $(Q, ., e)$ has the (right) hypospecial property ([6], [7]). Thus we get a (right) hyporeductive geodesic loop $(Q, ., e)$ with T_eQ as its tangent algebra. But then from (6), (8), (9), (13), (14) and (15) we see that its Akivis operations have the following expressions through the Lie brackets of g:

$$X \diamond Y = [X, Y]^h_m\,, \tag{16}$$

$$[X, Y, Z] = \frac{1}{2}\big([Z, [X, Y]^m_n\,] - [Z, [X, Y]^h_m\,]^h_m + [Z, [X, Y]^n_m\,]^h_m \big). \tag{17}$$

Thus we have the following

Theorem 3. *Let g be a real finite-dimensional Lie algebra generated by a subspace of vector fields and let (g, h) be the hyporeductive pair with the hyporeductive decomposition* $(10) - (12)$*. Then the Akivis operations of the local smooth hyporeductive loop corresponding (up to an isomorphism) to the h.t.a. in g are expressed as in* (16), (17). □

One observes that we have worked with an h.t.a. of covariantly constant vector fields in a smooth affine connection space with zero curvature. But one can also start from a structure of abstract h.t.a. given on the tangent space W to a fixed point e of that connection space and then extend this structure to the one of a h.t.a. of covariantly constant vector fields V through the identification of W with $V = \{X_\xi : X_\xi(e) = \xi \in W\}$.

We conclude with the following remarks in full analogy with the ones we done in Section 2.

Remark 2. (a) We get the Bol theory ([10], [15]) if $n = [m, m]$, i.e. $[X, Y]^n_m = 0$ in which case we have $g = m \dotplus h$, and $[[m, m], m] \subset m$ so that (17) reads

$$[X, Y, Z] = \frac{1}{2}\big([Z, [X, Y]] - [Z, [X, Y]^h_m]^h_m \big)$$

((16) remains the same).

(b) The hyporeductive pair (g, h) (see (10) – (12)) becomes reductive when n coincides with h, i.e. $g = m \dotplus h$ and $[h, m] \subset m$. Therefore the Akivis operation (17) reduces to the following

$$[X, Y, Z] = \frac{1}{2}\, [Z, [X, Y]^m_h]$$

(again (16) remains the same) and one observes that we get precisely the Akivis operations of the local smooth loop associated with the corresponding reductive decomposition ([5]).

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A note on trimedial quasigroups

Michael K. Kinyon and J. D. Phillips

Abstract

The purpose of this brief note is to sharpen a result of Kepka [2] [3] about the axiomatization of the variety of trimedial quasigroups.

A groupoid is *medial* if it satisfies the identity $wx \cdot yz = wy \cdot xz$. A groupoid is *trimedial* if every subgroupoid generated by 3 elements is medial. Medial groupoids and quasigroups have also been called abelian, entropic, and other names, while trimedial quasigroups have also been called triabelian, terentropic, etc. (See [1], especially p. 120, for further background.)

In [2] [3], Kepka showed that a quasigroup satisfying the following three identities must be trimedial.

$$xx \cdot yz = xy \cdot xz \tag{1}$$
$$yz \cdot xx = yx \cdot zx \tag{2}$$
$$(x \cdot xx) \cdot uv = xu \cdot (xx \cdot v) \tag{3}$$

The converse is trivial, and so these three identities characterize trimedial quasigroups. Here, we show that, in fact, (2) and (3) are sufficient to characterize this variety (as a subvariety of the variety of quasigroups). Note that in the theorem we only assume left cancellation, not the full strength of the quasigroup axioms.

Theorem. *A groupoid with left cancellation which satisfies* (2) *and* (3) *must also satisfy* (1).

Proof. $(x{\cdot}xz)(xx{\cdot}yz) = (x{\cdot}xx)(xz{\cdot}yz) = (x{\cdot}xx)(xy{\cdot}zz) = (x{\cdot}xy)(xx{\cdot}zz) = (x \cdot xy)(xz \cdot xz) = (x \cdot xz)(xy \cdot xz)$. Now cancel. □

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In [2] [3], Kepka showed that the following single identity characterizes trimedial quasigroups:

$$[(xx \cdot yz)]\{[xy \cdot uu][(w \cdot ww) \cdot zv]\} = [(xy \cdot xz)]\{[xu \cdot yu][wz \cdot (ww \cdot v)]\}.$$

Using the theorem we can sharpen this.

Corollary. *The following identity characterizes trimedial quasigroups*:

$$[(xy \cdot uu)][(w \cdot ww) \cdot zv] = [(xu \cdot yu)][wz \cdot (ww \cdot v)].$$

Proof. To obtain (2) set $z = ww$ and use right cancellation. To obtain (3) set $y = u$ and use left cancellation. □

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Transversals in groups. 4. Derivation construction

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Abstract

In the present work the derivation construction is studied by means of transversals in a group to a proper subgroup. It is shown that the method of derivation may be understood as a connection between different transversals in a group to a subgroup.

1. Introduction

The method of derivation has appeared in Dickson's works at the first time. It has been used to construct nearfields and quasifields from fields and skew-fields (see [14], [15]). Karzel [6] axiomatized and generalized this method for groups. Kiechle [7] gave a generalization of method of derivation which applied to construct loops with determined conditions by the help of groups.

In a present work the derivation construction is studied by means of transversals in a group to a proper subgroup. It is shown that the method of derivation may be understood as a connection between different transversals in a group to a subgroup. It give us a possibility to generalize the derivation construction for loops, i.e. to construct loops with some determined conditions by the help of some "good" loops.

2. Necessary definitions and notations

Definition 1. [2] A system $\langle E, \cdot \rangle$ is called a *right* (*left*) *quasigroup*, if for arbitrary $a, b \in E$ the equation $x \cdot a = b$ $(a \cdot y = b)$ has a unique solution in the set E. If in quasigroup $\langle E, \cdot \rangle$ there exists element $e \in E$ such that

$$x \cdot e = e \cdot x = e$$

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for every $x \in E$, then system $\langle E, \cdot \rangle$ is called a *loop*.

Definition 2. [1] Let G be a group and H be a subgroup in G. A complete system $T = \{t_i\}_{i \in E}$ of representatives of the left (right) cosets of H in G ($e = t_1 \in H$) is called a *left* (*right*) *transversal in* G *to* H.

Let $T = \{t_i\}_{i \in E}$ be a left transversal in G to H. We can define correctly (see [1, 9]) the following operation on the set E (E is an index set; left cosets of H in G are numbered by indexes from E):

$$x \overset{(T)}{\cdot} y = z \quad \overset{def}{\Longleftrightarrow} \quad t_x t_y = t_z h, \quad h \in H. \tag{1}$$

In [1] (and [9]) it is proved that $\langle E, \overset{(T)}{\cdot} \rangle$ is a left quasigroup with two-sided unit 1.

Below we shall consider (for simplicity) that $Core_G(H) = e$ (where

$$Core_G(H) = \bigcap_{g \in G} gHg^{-1}$$

is the maximal normal subgroup of the group G contained in the subgroup H) and shall study a permutation representation $\hat{G}$ of group G by left cosets on the subgroup H. According to [5], we have $\hat{G} \cong G$, where

$$\hat{g}(x) = y \quad \overset{def}{\Longleftrightarrow} \quad g t_x H = t_y H.$$

Note that $\hat{H} = St_1(\hat{G})$.

Lemma 1. ([9], Lemma 4) *Let T be an arbitrary left transversal in G to H. Then the following statements are true:*

1. $\hat{h}(1) = 1$ *for all* $h \in H$.
2. *For every $x, y \in E$ we have:* $\hat{t}_x(y) = x \overset{(T)}{\cdot} y, \quad \hat{t}_1(x) = \hat{t}_x(1) = x,$
$$\hat{t}_x^{-1}(y) = x \overset{(T)}{\backslash} y, \quad \hat{t}_x^{-1}(1) = x \overset{(T)}{\backslash} 1, \quad \hat{t}_x^{-1}(x) = 1$$

where $\overset{(T)}{\backslash}$ *is the left division in the system* $\langle E, \overset{(T)}{\cdot}, 1 \rangle$

(*i.e.* $x \overset{(T)}{\backslash} y = z \;\Leftrightarrow\; x \overset{(T)}{\cdot} z = y$). □

Let us denote
$$l_{a,b}^{(T)} = L^{-1}_{a \overset{(T)}{\cdot} b} L_a L_b$$

where $L_a(x) = a \overset{(T)}{\cdot} x$ is a left translation in the left loop $\langle E, \overset{(T)}{\cdot}, 1\rangle$. The group

$$LI(\,\langle E, \overset{(T)}{\cdot}, 1\rangle\,) = \langle\, l^{(T)}_{a,b} \mid a, b \in E\,\rangle$$

is called a *left inner permutation group*. It is easy to see that

$$l^{(T)}_{a,b}(x) = (a \overset{(T)}{\cdot} b) \overset{(T)}{\backslash} (a \overset{(T)}{\cdot} (b \overset{(T)}{\cdot} x)) = \hat{t}^{-1}_{a \overset{(T)}{\cdot} b}\hat{t}_a\hat{t}_b(x)\,. \tag{2}$$

Note that for every $a, b \in E \quad l^{(T)}_{a,b}(1) = 1$.

Definition 3. A left loop $\langle E, \cdot, 1\rangle$ is called

1. *left alternative*, if for every $x, y \in E: \quad x \cdot (x \cdot y) = (x \cdot x) \cdot y$,
2. *left IP-loop* (or *LIP-loop*), if for every $x \in E$ there exists the element $x' \in E$ such that $x' \cdot (x \cdot y) = y$ for every $y \in E$,
3. *left A_l-loop*, if for every $a, b \in E \;\; l_{a,b} \in Aut(\langle E, \cdot, 1\rangle)$.

Lemma 2. *Let a set $T = \{t_i\}_{i\in E}$ be a left transversal in the group G to its subgroup H. Then the following conditions are equivalent:*

1. *The system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a left A_l-loop,*
2. *For every $\alpha \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ we have $\alpha\hat{T}\alpha^{-1} \subseteq \hat{T}$.*

Proof. $1 \Longrightarrow 2$. Let the system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ be a left A_l-loop, i.e. let

$$l^{(T)}_{a,b} = \hat{t}^{-1}_{a\cdot b}\hat{t}_a\hat{t}_b \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$$

for every $a, b \in E$. Then

$$LI(\langle E, \overset{(T)}{\cdot}, 1\rangle) \subseteq Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle). \tag{3}$$

Let $\alpha \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$. Because $\alpha\hat{t}_x\alpha^{-1} \in \hat{G}$ for every $x \in E$, then

$$\alpha\hat{t}_x\alpha^{-1} = \hat{t}_u\hat{h}_1 \tag{4}$$

for some $u \in E$ and $h_1 \in H$. In view of Lemma 1 we have

$$u = \hat{t}_u(1) = \hat{t}_u\hat{h}_1(1) = \alpha\,\hat{t}_x\alpha^{-1}(1) = \alpha\,\hat{t}_x(1) = \alpha(x),$$

i.e. the equation (4) may be rewritten in the following form

$$\alpha \hat{t}_x \alpha^{-1} = \hat{t}_{\alpha(x)} \hat{h}_1 \,. \tag{5}$$

On the other hand, for every $x, y \in E$

$$\hat{t}_x \hat{t}_y = \hat{t}_{x \overset{(T)}{\cdot} y} l^{(T)}_{x,y},$$

$$\alpha \hat{t}_x \alpha^{-1} \alpha \hat{t}_y \alpha^{-1} = \alpha \hat{t}_{x \overset{(T)}{\cdot} y} \alpha^{-1} \alpha l^{(T)}_{x,y} \alpha^{-1},$$

which, by (5), gives

$$\hat{t}_{\alpha(x)} \hat{h}_1 \hat{t}_{\alpha(y)} \hat{h}_2 = \hat{t}_{\alpha(x \overset{(T)}{\cdot} y)} \hat{h}_3 \, \alpha l^{(T)}_{x,y} \alpha^{-1}$$

for some $h_2, h_3 \in H$. In view of Lemma 1 we have also

$$\hat{t}_{\alpha(x)} \hat{h}_1 \hat{t}_{\alpha(y)} \hat{h}_2(1) = \hat{t}_{\alpha(x \overset{(T)}{\cdot} y)} \hat{h}_3 \, \alpha l^{(T)}_{x,y} \alpha^{-1}(1),$$

$$\alpha(x) \overset{(T)}{\cdot} \hat{h}_1(\alpha(y)) = \alpha(x \overset{(T)}{\cdot} y). \tag{6}$$

But $\alpha \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$. Thus, by (3), we have $\alpha(x \overset{(T)}{\cdot} y) = \alpha(x) \overset{(T)}{\cdot} \alpha(y)$. So, by (6), we obtain:

$$\alpha(x) \overset{(T)}{\cdot} \hat{h}_1(\alpha(y)) = \alpha(x) \overset{(T)}{\cdot} \alpha(y)$$

for every $x, y \in E$. Since the system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a left quasigroup with two-sided unit 1, for every $y \in E$ we get

$$\hat{h}_1(\alpha(y)) = \alpha(y).$$

Function α is a permutation on the set E, so for every $z \in E$ $\hat{h}_1(z) = z$, i.e. $h_1 = e$. Then in view of (5) we obtain:

$$\alpha \hat{t}_x \alpha^{-1} = \hat{t}_{\alpha(x)} \in \hat{T} \tag{7}$$

for every $\alpha \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ and $x \in E$.

$2 \Longrightarrow 1$. (See [8]) Let for every $\alpha \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ and $x \in E$ exists an element $u \in E$ such that

$$\alpha \hat{t}_x \alpha^{-1} = \hat{t}_u \in \hat{T}. \tag{8}$$

Then, in view of Lemma 1, we have

$$u = \hat{t}_u(1) = \alpha\,\hat{t}_x\alpha^{-1}(1) = \alpha\,\hat{t}_x(1) = \alpha(x),$$

i.e. the equation (8) may be rewritten in the following way

$$\alpha\hat{t}_x\alpha^{-1} = \hat{t}_{\alpha(x)}. \tag{9}$$

But every $x, y \in E$ we have

$$\hat{t}_x\hat{t}_y = \hat{t}_{x \overset{(T)}{\cdot} y}\, l^{(T)}_{x,y}\,.$$

Then

$$\alpha\hat{t}_x\alpha^{-1}\alpha\hat{t}_y\alpha^{-1} = \alpha\hat{t}_{x \overset{(T)}{\cdot} y}\alpha^{-1}\alpha l^{(T)}_{x,y}\alpha^{-1},$$

which, by (9), gives

$$\hat{t}_{\alpha(x)}\hat{t}_{\alpha(y)} = \hat{t}_{\alpha(x \overset{(T)}{\cdot} y)}\alpha l^{(T)}_{x,y}\alpha^{-1}$$

and, in the consequence,

$$\alpha l^{(T)}_{x,y}\alpha^{-1} \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle) \subseteq \hat{H}\,,$$

because $\alpha, l^{(T)}_{x,y} \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$. Now, applying the definition of the left transversal T, we obtain

$$\alpha(x) \overset{(T)}{\cdot} \alpha(y) = \alpha(x \overset{(T)}{\cdot} y).$$

This means that $\alpha \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ and $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a left A_l-loop. $\square$

Lemma 3. *Let $\langle E, \cdot, 1\rangle$ be a left loop. Then the following statements are true:*

1. *System $\langle E, \cdot, 1\rangle$ is LIP-loop if and only if for every $a \in E$: if $a \cdot a' = 1$ for some $a' \in E$ then $l_{a,a'} = id$.*

2. *System $\langle E, \cdot, 1\rangle$ is left alternative if and only if $l_{a,a} = id$ for every $a \in E$.*

Proof. See [7]. $\square$

3. Derivation as a connection between transversals in a group by the same subgroup

Let us remind the general method of derivation used for construction of loops [7], section 7.

Let $\langle A, \cdot, 1\rangle$ be a group. The function $\varphi: A \to S_A$ such that $\varphi(a) \rightleftharpoons \varphi_a$, $\varphi_1 = id$ and $\varphi_a(1) = 1$ for any $a \in A$ is called a *weak derivation*. It is called a *derivation* if furthemore for all $a, b \in A$ there exists a unique $x \in A$ such that $x \cdot \varphi_x(a) = b$.

In [7], section 7, it was proved the following

Lemma 4. *Let $\langle A, \cdot, 1\rangle$ be a group with a weak derivation φ. Let us define the operation*

$$x \circ y \stackrel{def}{=} x \cdot \varphi_x(y). \tag{10}$$

Then:

1. *The system $\langle A, \circ, 1\rangle$ is a left loop with two-sided unit 1 (the identity elements of $\langle A, \cdot, 1\rangle$ and $\langle A, \circ, 1\rangle$ coincide). Moreover, for all $a \in A$ if $a \circ a' = 1$, then $a' = \varphi_a^{-1}(a^{-1})$.*
2. *If φ is a derivation, then system $\langle A, \circ, 1\rangle$ is a loop.*

The system $\langle A, \circ, 1\rangle$ is called a *derived (left) loop*. If $\varphi_a \in Aut\, A$ for every $a \in A$, then derivation is called *automorphic derivation*.

For the connection between two different left transversals in a group G by the same subgroup H see [9, 10].

Let $T = \{t_x\}_{x\in E}$ and $P = \{p_x\}_{x\in E}$ are two left transversals in a group G to its subgroup H. It is evident that for every $x \in E$ $p_x = t_x h_{(x)}$ for some collection $\{h_{(x)}\}_{x\in E}$, $h_{(x)} \in H$. As it was proved in [9], for systems $\langle E, \overset{(T)}{\cdot}, 1\rangle$ and $\langle E, \overset{(P)}{\cdot}, 1\rangle$ corresponding to the transversals T and P, by formula (1), we have:

$$x \overset{(P)}{\cdot} y = x \overset{(T)}{\cdot} \hat{h}_{(x)}(y). \tag{11}$$

It is easy to see that formulas (10) and (11) almost coincide; moreover, $\hat{h}_{(1)} = id$ and $\hat{h}_{(x)}(1) = 1$ for every $x \in E$. Note that unlike the derivation construction described above the system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is not a group; in a general case it is a left loop with two-sided unit. This means that the construction of new operations by the help of the connection between different left transversals in a group G to its subgroup H (formula (11)) generalize the derivation method (formula (10)). So the construction of weak derivation from formula (10) may be generalized up to the class of left loops which are corresponding to the left transversals in a some group G to its subgroup H. The system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ from formula (11) will be called *derived left loop* and the set of permutations $\{\hat{h}_{(x)}\}_{x\in E}$ will be called a *deriving set.*

Lemma 5. *Let $\langle E, \overset{(P)}{\cdot}, 1\rangle$ be a derived left loop obtained by the method of weak derivation (formula (11)) from the left loop $\langle E, \overset{(T)}{\cdot}, 1\rangle$ by the help of derived set $\{\hat{h}_{(x)}\}_{x\in E}$. Then the following sentences are true:*

1. *Two-sided units of the left loops $\langle E, \overset{(P)}{\cdot}, 1\rangle$ and $\langle E, \overset{(T)}{\cdot}, 1\rangle$ coincide.*
2. *If $\overset{(P)}{a^{-1}}$ is a right inverse to the element a in $\langle E, \overset{(P)}{\cdot}, 1\rangle$, then $\overset{(P)}{a^{-1}} = \hat{h}^{-1}_{(a)}(\overset{(T)}{a^{-1}})$, where $\overset{(T)}{a^{-1}}$ is a right inverse to the element a in $\langle E, \overset{(T)}{\cdot}, 1\rangle$.*
3. *The left loop $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is a loop (i.e. weak derivation is a derivation) if and only if the operations $\langle E, \overset{(T)}{\cdot}, 1\rangle$ and $B(x,y) = \hat{h}^{-1}_{(x)}(y)$ are orthogonal.*

Proof. 1. It is easy to see that if 1 is a unit in $\langle E, \overset{(T)}{\cdot}, 1\rangle$, then

$$1 \overset{(P)}{\cdot} x = 1 \overset{(T)}{\cdot} \hat{h}_{(1)}(x) = 1 \overset{(T)}{\cdot} x = x,$$

$$x \overset{(P)}{\cdot} 1 = x \overset{(T)}{\cdot} \hat{h}_{(x)}(1) = x \overset{(T)}{\cdot} 1 = x.$$

2. If $\overset{(P)}{a^{-1}}$ is a right inverse to a in $\langle E, \overset{(P)}{\cdot}, 1\rangle$, then $a \overset{(P)}{\cdot} \overset{(P)}{a^{-1}} = 1$. Thus $a \overset{(T)}{\cdot} \hat{h}_{(a)}(\overset{(P)}{a^{-1}}) = 1$ and $\overset{(P)}{a^{-1}} = \hat{h}^{-1}_{(a)}(\overset{(T)}{a^{-1}})$.

3. (See also [3], [10]) It is enough to prove that the equation $x \overset{(P)}{\cdot} a = b$ has a unique solution in E for any fixed $a, b \in E$ if and only if the operations $\langle E, \overset{(T)}{\cdot}, 1\rangle$ and $B(x,y) = \hat{h}^{-1}_{(x)}(y)$ are orthogonal, i.e. if and only if the system

$$\begin{cases} x \overset{(T)}{\cdot} y = a \\ B(x,y) = b \end{cases}$$

has a unique solution in the set $E \times E$ for any $a, b \in E$.

We have

$$\begin{cases} x \overset{(P)}{\cdot} a = b \\ x \overset{(T)}{\cdot} \hat{h}_{(x)}(a) = b \end{cases} \Longleftrightarrow \begin{cases} \hat{h}_{(x)}(a) = z \\ x \overset{(T)}{\cdot} z = b \end{cases} \Longleftrightarrow$$

$$\Longleftrightarrow \begin{cases} \hat{h}^{-1}_{(x)}(z) = a \\ x \overset{(T)}{\cdot} z = b \end{cases} \Longleftrightarrow \begin{cases} B(x,z) = a \\ x \overset{(T)}{\cdot} z = b \end{cases}$$

Last system has a unique solution if and only if the operations $\langle E, \overset{(T)}{\cdot}, 1\rangle$ and $B(x,y) = \hat{h}^{-1}_{(x)}(y)$ are orthogonal. $\square$

Remark 1. According to Cayley's Theorem (see [5], theorem 12.1.1, 12.1.3) every group K may be represented as a permutation group on the set K; this representation is regular. So any group K may be represented as a group transversal in S_K to $St_1(S_K)$. Then the construction of weak derivation of an arbitrary group $\langle A, \cdot, 1\rangle$ to the derived left loop $\langle A, \circ, 1\rangle$ may be represented as a construction of the left transversal $P = \{p_x\}_{x\in E}$ in the group S_A to $St_1(S_A)$ by the help of the group transversal $A^* = \{t_x\}_{x\in E}$ in the group S_A to $St_1(S_A)$. The corresponding system $\langle E, \overset{(A)}{\cdot}, 1\rangle$ is isomorphic to the group $\langle A, \cdot, 1\rangle$ and the system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is isomorphic to the derived left loop $\langle A, \circ, 1\rangle$.

Remark 2. The construction of weak derivation may also take place when there exists a group transversal T in the group G to its subgroup H. Then any other left transversal P in the group G to its subgroup H may be represented as a weak derivation of the group transversal T by the help of the deriving set $\{\hat{h}_{(x)}\}_{x\in E} \subset \hat{H}$.

Remark 3. The construction of automorphic derivation may be naturally represented as a connection between left transversals in the group G to its subgroup H, where G is a semidirect product (see [13], [11]) of a left loop $\langle E, \cdot, 1\rangle$ and group H, and $LI(\langle E, \cdot, 1\rangle) \subseteq H \subseteq Aut(\langle E, \cdot, 1\rangle)$.

4. Automorphic derivations

Let us investigate the case of weak automorphic derivation of left loops, i.e. the case of weak derivation with the condition

$$\{\hat{h}_{(x)}\}_{x\in E} \subseteq Aut(\langle E, \cdot, 1\rangle).$$

Lemma 6. *Let $\langle E, \overset{(P)}{\cdot}, 1\rangle$ be a derived left loop, which is obtained from a left loop $\langle E, \overset{(T)}{\cdot}, 1\rangle$ by means of weak automorphic derivation by the help of the deriving set $\{\hat{h}_{(x)}\}_{x\in E}$. We have*

1. *The following conditions are equivalent:*

(a) $\alpha \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ *is an automorphism of* $\langle E, \overset{(P)}{\cdot}, 1\rangle$,
(b) $\alpha\hat{h}_{(x)}\alpha^{-1} = \hat{h}_{(\alpha(x))}$ *for every* $x \in E$,
(c) $\alpha \in Aut(\langle E, \overset{(P)}{\cdot}, 1\rangle)$ *is an automorphism of* $\langle E, \overset{(T)}{\cdot}, 1\rangle$.

2. $l^{(P)}_{a,b} = \hat{h}^{-1}_{(a \overset{(P)}{\cdot} b)} l^{(T)}_{a,\hat{h}_{(a)}(b)} \hat{h}_{(a)}\hat{h}_{(b)}$ *for every* $a, b \in E$,

3. *The system* $\langle E, \overset{(P)}{\cdot}, 1\rangle$ *is left alternative if and only if for every* $a \in E$
$$\hat{h}_{(a \overset{(P)}{\cdot} a)} = l^{(T)}_{a,\hat{h}_{(a)}(a)} \hat{h}^2_{(a)}.$$

4. *The system* $\langle E, \overset{(P)}{\cdot}, 1\rangle$ *is a LIP-loop if and only if for every* $a \in E$
$$\hat{h}^{-1}_{(a)} = \hat{h}_{(\hat{h}^{-1}_{(a)}(a'))} l^{(T)}_{a,a'}, \quad \text{where } a' = \overset{(T)}{a^{-1}}.$$

5. *The system* $\langle E, \overset{(P)}{\cdot}, 1\rangle$ *is a left Bol loop if and only if for every* $a, b \in E$
$$\hat{h}_{(a \overset{(P)}{\cdot} (b \overset{(P)}{\cdot} a))} = l^{(T)}_{a,\hat{h}_{(a)}(b \overset{(P)}{\cdot} a)} \hat{h}_{(a)} l^{(T)}_{b,h_{(b)}(a)} \hat{h}_{(b)}\hat{h}_{(a)}.$$

6. *The system* $\langle E, \overset{(P)}{\cdot}, 1\rangle$ *is a group if and only if for every* $a, b \in E$
$$\hat{h}_{(a \overset{(P)}{\cdot} b)} = l^{(T)}_{a,\hat{h}_{(a)}(b)} \hat{h}_{(a)}\hat{h}_{(b)}.$$

Proof. 1. $(a) \Longleftrightarrow (b)$. If $\alpha \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$, we have for every $x, y \in E$
$$\alpha(x) \overset{(P)}{\cdot} \alpha(y) = \alpha(x \overset{(P)}{\cdot} y),$$
$$\alpha(x) \overset{(T)}{\cdot} \hat{h}_{(\alpha(x))}\alpha(y) = \alpha(x \overset{(T)}{\cdot} \hat{h}_{(x)}(y)) = \alpha(x) \overset{(T)}{\cdot} \alpha\,\hat{h}_{(x)}(y),$$
$$\hat{h}_{(\alpha(x))}\alpha(y) = \alpha\,\hat{h}_{(x)}(y),$$
$$\hat{h}_{(\alpha(x))} = \alpha\hat{h}_{(x)}\alpha^{-1}.$$

$(c) \Longleftrightarrow (b)$. For every $x, y \in E$ we have $x \overset{(T)}{\cdot} y = x \overset{(P)}{\cdot} \hat{h}^{-1}_{(x)}(y)$. So the result follows as before.

2. Because of $\hat{h}_{(a)} \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ for every $a \in E$, so we have for every $a, b, x \in E$
$$a \overset{(P)}{\cdot} (b \overset{(P)}{\cdot} x) = (a \overset{(P)}{\cdot} b) \overset{(P)}{\cdot} l^{(P)}_{a,b}(x),$$
$$a \overset{(T)}{\cdot} \hat{h}_{(a)}(b \overset{(T)}{\cdot} \hat{h}_{(b)}(x)) = (a \overset{(T)}{\cdot} \hat{h}_{(a)}(b)) \overset{(T)}{\cdot} \hat{h}_{(a \overset{(P)}{\cdot} b)} l^{(P)}_{a,b}(x),$$
$$a \overset{(T)}{\cdot} (\hat{h}_{(a)}(b) \overset{(T)}{\cdot} \hat{h}_{(a)}\hat{h}_{(b)}(x)) = (a \overset{(T)}{\cdot} \hat{h}_{(a)}(b)) \overset{(T)}{\cdot} \hat{h}_{(a \overset{(P)}{\cdot} b)} l^{(P)}_{a,b}(x),$$

$$(a \overset{(T)}{\cdot} \hat{h}_{(a)}(b)) \overset{(T)}{\backslash} (a \overset{(T)}{\cdot} (\hat{h}_{(a)}(b) \overset{(T)}{\cdot} \hat{h}_{(a)}\hat{h}_{(b)}(x))) = \hat{h}_{(a \overset{(P)}{\cdot} b)} l^{(P)}_{a,b}(x).$$

In view of formula (2) we obtain for every $x \in E$

$$l^{(T)}_{a,\hat{h}_{(a)}(b)} \hat{h}_{(a)}\hat{h}_{(b)}(x) = \hat{h}_{(a \overset{(P)}{\cdot} b)} l^{(P)}_{a,b}(x),$$
$$l^{(P)}_{a,b} = \hat{h}^{-1}_{(a \overset{(P)}{\cdot} b)} l^{(T)}_{a,\hat{h}_{(a)}(b)} \hat{h}_{(a)}\hat{h}_{(b)}.$$

3. According to Lemma 3 the system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is left alternative if and only if $l^{(P)}_{a,a} = id$ for every $a \in E$. So by the condition 2) the result follows.

4. According to Lemma 3 the system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is a LIP-loop if and only if for every $a \in E$: if $a \overset{(P)}{\cdot} a'' = 1$ for some $a'' \in E$, then $l^{(P)}_{a,a''} = id$. So in view of the proposition 2) of present Lemma we obtain

$$id = \hat{h}^{-1}_{(a \overset{(P)}{\cdot} a'')} l^{(T)}_{a,\hat{h}_{(a)}(a'')} \hat{h}_{(a)}\hat{h}_{(a'')} = l^{(T)}_{a,\hat{h}_{(a)}(a'')} \hat{h}_{(a)}\hat{h}_{(a'')}.$$

Because of $a'' = \hat{h}^{-1}_{(a)}(\overset{(T)}{a^{-1}})$, then $\hat{h}^{-1}_{(a)} = \hat{h}_{(\hat{h}^{-1}_{(a)}(a'''))} l^{(T)}_{a,a'''}$, where $a''' = \overset{(T)}{a^{-1}}$.

5. It is easy to prove (see [7]) that the left Bol identity

$$(a \overset{(P)}{\cdot} (b \overset{(P)}{\cdot} a) \overset{(P)}{\cdot} x = a \overset{(P)}{\cdot} (b \overset{(P)}{\cdot} (a \overset{(P)}{\cdot} x)))$$

for every $a, b, x \in E$ is equivalent to the identity $l^{(P)}_{a,b \overset{(P)}{\cdot} a} = (l^{(P)}_{b,a})^{-1}$ for every $a, b \in E$. So in view of the condition 2) of the present Lemma we obtain

$$\hat{h}^{-1}_{(a \overset{(P)}{\cdot} (b \overset{(P)}{\cdot} a))} l^{(T)}_{a,\hat{h}_{(a)}(b \overset{(P)}{\cdot} a)} \hat{h}_{(a)}\hat{h}_{(b \overset{(P)}{\cdot} a)} = \hat{h}^{-1}_{(a)}\hat{h}^{-1}_{(b)} l^{(T)\,-1}_{b,\hat{h}_{(b)}(a)} \hat{h}_{(b \overset{(P)}{\cdot} a)},$$
$$\hat{h}_{(a \overset{(P)}{\cdot} (b \overset{(P)}{\cdot} a))} = l^{(T)}_{a,\hat{h}_{(a)}(b \overset{(P)}{\cdot} a)} \hat{h}_{(a)} l^{(T)}_{b,\hat{h}_{(b)}(a)} \hat{h}_{(b)}\hat{h}_{(a)}.$$

6. It is easy to prove that system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is a group if and only if $l^{(P)}_{a,b} = id$ for every $a, b \in E$. So in view of the condition 2) of present Lemma we obtain the result. □

Corollary 1. *Let the general assumptions of Lemma 6 hold. If the system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is a group, then the system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is an A_l-loop.*

Proof. In view of proposition 6) of Lemma 6 if the system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is a group then $\hat{h}_{(a \overset{(P)}{\cdot} b)} = l^{(T)}_{a,\hat{h}_{(a)}(b)} \hat{h}_{(a)}\hat{h}_{(b)}$. Because of $\hat{h}_{(a)} \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ for every $a \in E$, then $l^{(T)}_{a,\hat{h}_{(a)}(b)} \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ for every $a, b \in E$. But $\hat{h}_{(a)}$ is a permutation on the set E for every $a \in E$, then $l^{(T)}_{a,c} \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ for every $a, c \in E$. So $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is an A_l-loop. □

4.1. An automorphic derivation of group

Let us apply the previous lemma to the case when $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a group.

Lemma 7. (See [7]) *Let $\langle E, \overset{(P)}{\cdot}, 1\rangle$ be a derived left loop, which is obtained from a group $\langle E, \overset{(T)}{\cdot}, 1\rangle$ by means of weak automorphic derivation by the help of deriving set $\{\hat{h}_{(x)}\}_{x\in E}$. Then the following statements are true:*

1. *The following conditions are equivalent:*
   (*a*) $\alpha \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ *is an automorphism of* $\langle E, \overset{(P)}{\cdot}, 1\rangle$,
   (*b*) $\alpha \hat{h}_{(x)} \alpha^{-1} = \hat{h}_{(\alpha(x))}$ *for every* $x \in E$,
   (*c*) $\alpha \in Aut(\langle E, \overset{(P)}{\cdot}, 1\rangle)$ *is an automorphism of the operation* $\langle E, \overset{(T)}{\cdot}, 1\rangle$.
2. $l^{(P)}_{a,b} = \hat{h}^{-1}_{(a \overset{(P)}{\cdot} b)} \hat{h}_{(a)}\hat{h}_{(b)} \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ *for every* $a, b \in E$,
3. *The system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is left alternative if and only if for every $a \in E$*
$$\hat{h}_{(a \overset{(P)}{\cdot} a)} = \hat{h}^2_{(a)}.$$
4. *The system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is LIP-loop if and only if for every $a \in E$*
$$\hat{h}^{-1}_{(a)} = \hat{h}_{(\hat{h}^{-1}_{(a)}(a'))}, \quad \text{where } a' = \overset{(T)}{a^{-1}}.$$
5. *The system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is a left Bol loop if and only if for every $a, b \in E$*
$$\hat{h}_{(a \overset{(P)}{\cdot} (b \overset{(P)}{\cdot} a))} = \hat{h}_{(a)}\hat{h}_{(b)}\hat{h}_{(a)}.$$
6. *The system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is a group if and only if for every $a, b \in E$*
$$\hat{h}_{(a \overset{(P)}{\cdot} b)} = \hat{h}_{(a)}\hat{h}_{(b)}.$$

Proof. It is the evident corollary of Lemma 6, because $l^{(T)}_{a,b} = id$ for every $a, b \in E$, if the system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a group. See also [7]. □

Corollary 2. *Let the conditions of Lemma 7 hold. If for every $h \in H$ we have* $hh_{(x)}h^{-1} = h_{(\hat{h}(x))}$, *then the system* $\langle E, \overset{(P)}{\cdot}, 1\rangle$ *is an* A_l*-loop.*

Proof. It is evident because $LI(\langle E, \overset{(P)}{\cdot}, 1\rangle) \subseteq \hat{H}$ for every left transversal P in a group G to a subgroup H. $\square$

Corollary 3. (See [15]) *Let the conditions of Lemma 7 hold. If* $\langle E, \overset{(P)}{\cdot}, 1\rangle$ *is a group, then* $\langle H_{(*)}, \cdot, h_{(1)}\rangle$, *where* $H_{(*)} \rightleftharpoons \{h_{(x)} | x \in E\}$, *is a subgroup of the group* H.

Proof. It is an evident corollary of 4) and 6) from Lemma 7. $\square$

4.2. An automorphic derivation of A$_l$-loop

Let us apply Lemma 6 in a case when $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is an A_l-loop.

Lemma 8. *Let* $\langle E, \overset{(P)}{\cdot}, 1\rangle$ *be a derived left loop, which is obtained from the* A_l*-loop* $\langle E, \overset{(T)}{\cdot}, 1\rangle$ *by means of weak automorphic derivation by the help of the deriving set* $\{\hat{h}_{(x)}\}_{x \in E}$. *Then:*

1. $l^{(P)}_{a,b} = \hat{h}^{-1}_{(a \overset{(P)}{\cdot} b)} l^{(T)}_{a,\hat{h}_{(a)}(b)} \hat{h}_{(a)} \hat{h}_{(b)} \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ *for every* $a, b \in E$,
2. $\langle E, \overset{(P)}{\cdot}, 1\rangle$ *is an* A_l*-loop if and only if for every* $\alpha \in LI(\langle E, \overset{(P)}{\cdot}, 1\rangle)$ *we have* $\alpha \hat{h}_{(x)} \alpha^{-1} = \hat{h}_{(\alpha(x))}$.

Proof. 1. Since the system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is an A_l-loop, $l^{(T)}_{a,b} \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ for every $a, b \in E$. But by the definition of a weak automorphic derivation $\hat{h}_{(x)} \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ for every $x \in E$. Thus, by the condition 2) of Lemma 6, we obtain our thesis.

2. If $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is a A_l-loop, then in view of 1) from Lemma 6 and 1) from the present Lemma it is equivalent to $l^{(P)}_{a,b} \hat{h}_{(x)} l^{(P)\,-1}_{a,b} = \hat{h}_{(l^{(P)}_{a,b}(x))}$ for every $a, b \in E$. Since $LI(\langle E, \overset{(P)}{\cdot}, 1\rangle) = \langle l^{(P)}_{a,b} | a, b \in E\rangle$, for every $\alpha \in LI(\langle E, \overset{(P)}{\cdot}, 1\rangle)$ we obtain $\alpha \hat{h}_{(x)} \alpha^{-1} = \hat{h}_{(\alpha(x))}$. But according to the condition 2) of the present Lemma we have $LI(\langle E, \overset{(P)}{\cdot}, 1\rangle) \subseteq Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$. $\square$

Corollary 4. *Let $\langle E, \overset{(T)}{\cdot}, 1\rangle$ be an A_l-loop and $G = LM(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ its left multiplication group. Then the system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is an A_l-loop if and only if $\alpha h_{(x)}\alpha^{-1} = h_{(\alpha(x))}$ for every $\alpha \in LI(\langle E, \overset{(P)}{\cdot}, 1\rangle)$.*

Proof. Since the system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is an A_l-loop and $H = LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$, every element $h \in H$ is an automorphism of $\langle E, \overset{(T)}{\cdot}, 1\rangle$. So in this case every weak derivation of the A_l-loop $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is an automorphic weak derivation. In view of the condition 2) of Lemma 8 we obtain the necessity. $\square$

Corollary 5. *Let the conditions of Corollary 4 hold. If for every $h \in H$ we have $hh_{(x)}h^{-1} = h_{(h(x))}$, then the system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is an A_l-loop.*

Proof. It is a consequence of our Corollary 4, because $LI(\langle E, \overset{(P)}{\cdot}, 1\rangle) \subseteq H$ for every left transversal P in a group G to a subgroup H. $\square$

Definition 4. A transversal T in the group G by its subgroup H is called a *gyrotransversal* if $T^{-1} = T$ and $hTh^{-1} \subseteq T$ for every $h \in H$.

Lemma 9. *Let the set $T = \{t_x\}_{x\in E}$ be a gyrotransversal in the group G by the subgroup H. Then $ht_xh^{-1} = t_{\hat{h}(x)}$ and $t_x^{-1} = t_{x\backslash 1}$ for every $h \in H$, $x \in E$.*

Proof. As the set $T = \{t_x\}_{x\in E}$ is a gyrotransversal in the group G by the subgroup H, so for every $h \in H$ we have $hTh^{-1} \subseteq T$, i.e. $ht_xh^{-1} = t_v$, where $v = \hat{t}_v(1) = \hat{h}\,\hat{t}_x\hat{h}^{-1}(1) = \hat{h}\,\hat{t}_x(1) = \hat{h}\,(x)$. Thus $ht_xh^{-1} = t_{\hat{h}(x)}$.

By the definition we have also $T^{-1} = T$, i.e. $t_x^{-1} = t_z$ for every $x \in T$, where $z = \hat{t}_z(1) = \hat{t}_x^{-1}(1) = x\backslash 1$. Thus $t_x^{-1} = t_{x\backslash 1}$. $\square$

Lemma 10. *Let the conditions of Lemma 8 hold. If T is a gyrotransversal in the group G by a subgroup H, then the following conditions are equivalent:*

1. *the set P is a gyrotransversal in the group G by a subgroup H,*
2. *$hh_{(x)}h^{-1} = h_{(\hat{h}(x))}$ and $h_{(x)}^{-1} = h_{(x\backslash 1)}$ for every $h \in H$, $x \in E$.*

Proof. 1. $\Longrightarrow$ 2. Let the conditions of the present Lemma hold and let P be a gyrotransversal in the group G by its subgroup H. Then, by the definition, for $h \in H$ we have $hPh^{-1} \subseteq P$ and $P^{-1} = P$, which implies $ht_xh_{(x)}h^{-1} = t_uh_{(u)}$ for all $h \in H$, $x \in E$ and some $u \in E$. Thus

$$u = \hat{t}_u(1) = \hat{t}_u\hat{h}_{(u)}(1) = \hat{h}\,\hat{t}_x\hat{h}_{(x)}\hat{h}^{-1}(1) = \hat{h}\,\hat{t}_x(1) = \hat{h}\,(x).$$

Thus the previous equation can be rewritten in the form $ht_xh_{(x)}h^{-1} = t_{\hat{h}\,(x)}h_{(\hat{h}\,(x))}$ and, in the consequence, in the form

$$t^{-1}_{\hat{h}\,(x)}ht_xh^{-1} = h_{(\hat{h}\,(x))}hh^{-1}_{(x)}h^{-1}\,. \tag{12}$$

As the set T is a gyrotransversal in the group G by the subgroup H, so, by Lemma 9, $ht_xh^{-1} = t_{\hat{h}\,(x)}$ for $h \in H$ and $x \in E$. Hence (12) has the form $e = h_{(\hat{h}\,(x))}hh^{-1}_{(x)}h^{-1}$, i.e. $hh_{(x)}h^{-1} = h_{(\hat{h}\,(x))}$, which proves the first condition of 2.

To prove the second, observe that $P^{-1} = P$ implies $(t_xh_{(x)})^{-1} = t_wh_{(w)}$ for $x \in E$, where

$$w = \hat{t}_w(1) = \hat{t}_w\hat{h}_{(w)}(1) = (\hat{t}_x\hat{h}_{(x)})^{-1}(1) = \hat{h}^{-1}_{(x)}\hat{t}^{-1}_x(1) = \hat{h}^{-1}_{(x)}(x\backslash 1).$$

This means that the equation $(t_xh_{(x)})^{-1} = t_wh_{(w)}$ can be written in the form $h^{-1}_{(x)}t^{-1}_x = t_{\hat{h}^{-1}_{(x)}(x\backslash 1)}h_{(\hat{h}^{-1}_{(x)}(x\backslash 1))}$, i.e. in the form

$$h_{(\hat{h}^{-1}_{(x)}(x\backslash 1))} = t^{-1}_{\hat{h}^{-1}_{(x)}(x\backslash 1)}h^{-1}_{(x)}t^{-1}_x = (t^{-1}_{\hat{h}^{-1}_{(x)}(x\backslash 1)}h^{-1}_{(x)}t_{x\backslash 1}h_{(x)})h^{-1}_{(x)}t^{-1}_{x\backslash 1}t^{-1}_x.$$

This together with $ht_xh^{-1} = t_{\hat{h}\,(x)}$ gives $t^{-1}_{\hat{h}^{-1}_{(x)}(x\backslash 1)}h^{-1}_{(x)}t_{x\backslash 1}h_{(x)} = e$, i.e. $h_{(\hat{h}^{-1}_{(x)}(x\backslash 1))} = h^{-1}_{(x)}t^{-1}_{x\backslash 1}t^{-1}_x$, which by $hh_{(x)}h^{-1} = h_{(\hat{h}\,(x))}$ implies the equation $h_{(\hat{h}^{-1}_{(x)}(x\backslash 1))} = h^{-1}_{(x)}h_{(x\backslash 1)}h_{(x)}$.

So we can write the equation $h_{(\hat{h}^{-1}_{(x)}(x\backslash 1))} = h^{-1}_{(x)}t^{-1}_{x\backslash 1}t^{-1}_x$ in the form $h^{-1}_{(x)}h_{(x\backslash 1)}h_{(x)} = h^{-1}_{(x)}t^{-1}_{x\backslash 1}t^{-1}_x$, i.e. $t_xt_{x\backslash 1}h_{(x\backslash 1)}h_{(x)} = e$.

Since the set T is a gyrotransversal in the group G by a subgroup H, by Lemma 9, for every $x \in E$ we have $t^{-1}_x = t_{x\backslash 1}$, which together with $t_xt_{x\backslash 1}h_{(x\backslash 1)}h_{(x)} = e$ implies $h_{(x\backslash 1)}h_{(x)} = e$. Hence $h^{-1}_{(x)} = h_{(x\backslash 1)}$. This proves the second condition of 2.

2. $\Longrightarrow$ 1. Let the conditions of Lemma 8 hold. If $hh_{(x)}h^{-1} = h_{(\hat{h}(x))}$ and $h^{-1}_{(x)} = h_{(x\backslash 1)}$ for all $h \in H$, $x \in E$, then, by Lemma 9, for every $h \in H$ and $x \in E$ we have

$$hp_xh^{-1} = ht_xh_{(x)}h^{-1} = (ht_xh^{-1})(hh_{(x)}h^{-1}) = t_{\hat{h}(x)}h_{(\hat{h}(x))} = p_{\hat{h}(x)},$$

i.e. $hPh^{-1} \subseteq P$. Moreover, by Lemma 9, for every $x \in E$ we have also

$$\begin{aligned} p_x^{-1} &= (t_x h_{(x)})^{-1} = h_{(x)}^{-1} t_x^{-1} = h_{(x\backslash 1)} t_{x\backslash 1} = \\ &= h_{(x\backslash 1)} t_{x\backslash 1} h_{(x\backslash 1)}^{-1} h_{(x\backslash 1)} = t_{\hat{h}_{(x\backslash 1)}(x\backslash 1)} h_{(x\backslash 1)} = \\ &= t_{\hat{h}_{(x\backslash 1)}(x\backslash 1)} h_{(\hat{h}_{(x\backslash 1)}(x\backslash 1))} h_{(\hat{h}_{(x\backslash 1)}(x\backslash 1))}^{-1} h_{(x\backslash 1)} = \\ &= t_{\hat{h}_{(x\backslash 1)}(x\backslash 1)} h_{(\hat{h}_{(x\backslash 1)}(x\backslash 1))} h_{(x\backslash 1)} h_{(x\backslash 1)}^{-1} h_{(x\backslash 1)}^{-1} h_{(x\backslash 1)} = \\ &= t_{\hat{h}_{(x\backslash 1)}(x\backslash 1)} h_{(\hat{h}_{(x\backslash 1)}(x\backslash 1))} = p_{\hat{h}_{(x\backslash 1)}(x\backslash 1)} \in P. \end{aligned}$$

This together with $hPh^{-1} \subseteq P$ proves that set P is a gyrotransversal in the group G by a subgroup H. □

5. Examples

Using propositions that are proved in the previous section, we will demonstrate some methods of construction of A_l-loops by the help of groups and A_l-loops.

Lemma 11. *If K is a group, $Inn\,(K)$ the group of its inner automorphisms, $M = K \times Inn\,(K)$ the semidirect product of K and $Inn\,(K)$, then the set $D = \{(x, \alpha_{a_x}) |\, x \in K\}$, where $a_x \in K$ are the indexes depended on $x \in K$, is a left transversal in the group M by a subgroup $H = Inn\,(K)$. Moreover, if $\alpha_u(a_x) = a_{\alpha_u(x)}$ for every $x, u \in K/Z(K)$, where $Z(K)$ is the center of K, then the system $\langle K, \overset{(D)}{\cdot}, 1\rangle$ is an A_l-loop.*

Proof. Let the conditions of the Lemma hold. Because $Inn\,(K) \subseteq Aut\,K$, then for the group transversal $K_0 = \{(x, id) |\, x \in K\}$ in the group M by subgroup $H = Inn\,(K)$ any weak derivation of group $\langle K, \overset{(K)}{\cdot}, 1\rangle$ is a weak automorphic derivation. According to Corollary 5, the system $\langle K, \overset{(D)}{\cdot}, 1\rangle$ is an A_l-loop if $\alpha_u \alpha_{a_x} \alpha_u^{-1} = \alpha_{a_{\alpha_u(x)}}$ for $x \in K$, $\alpha_u \in Inn\,(K)$.

This shows that for every $x, u, y \in K$ holds $\alpha_u \alpha_{a_x} \alpha_u^{-1}(y) = \alpha_{a_{\alpha_u(x)}}(y)$. Therefore

$$\begin{aligned} (ua_x u^{-1}) y (ua_x^{-1} u^{-1}) &= \alpha_u (a_x u^{-1} y u a_x^{-1}) = \alpha_u \alpha_{a_x} (u^{-1} y u) = \\ &= \alpha_u \alpha_{a_x} \alpha_u^{-1}(y) = \alpha_{a_{\alpha_u(x)}}(y) = a_{\alpha_u(x)} y\, a_{\alpha_u(x)}^{-1}. \end{aligned}$$

Hence $\alpha_u(a_x) y (\alpha_u(a_x))^{-1} = a_{\alpha_u(x)} y\, a_{\alpha_u(x)}^{-1}$ and $\alpha_{\alpha_u(a_x)} = \alpha_{a_{\alpha_u(x)}}$.

So, if $\alpha_u(a_x) = a_{\alpha_u(x)}$ holds, the system $\langle K, \overset{(D)}{\cdot}, 1\rangle$ is an A_l-loop. □

Corollary 6. *Let the conditions of Lemma* 11 *hold. If* $\alpha_u(a_x) = a_{\alpha_u(x)}$, *then* $a_x \in C_K(x)$ *for every* $x \in K$, *where* $C_K(x)$ *is the centralizer of* x *in the group* K.

Proof. It follows from $\alpha_u(a_x) = a_{\alpha_u(x)}$ for $u = x$. □

Remark 4. In the case $a_x = x$ we obtain so-called *diagonal transversal* $D = \{(x, \alpha_x) | \, x \in K\}$ investigated in [4].

Remark 5. For $a_x = x^m$, where $m \in Z - \{0, 1\}$ is fixed, we obtain the *generalized diagonal transversals* $D = \{(x, \alpha_x^m) | \, x \in K\}$ described in [12].

Now we will demonstrate the method of constructing of A_l-loops by the help of A_l-loop L with nontrivial right nucleus $N_r(L)$.

Lemma 12. *Let the conditions of Lemma* 8 *hold. If* $\hat{h}_{(x)} = l^{(T)}_{x, x \overset{(T)}{\cdot} c_0}$, *where* $c_0 \in N_r(\langle E, \overset{(T)}{\cdot}, 1\rangle)$, $c_0 \neq 1$, *then the system* $\langle E, \overset{(P)}{\cdot}, 1\rangle$ *is an* A_l*-loop.*

Proof. Since the system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is an A_l-loop, then

$$\hat{h}_{(x)} = l^{(T)}_{x, x \overset{(T)}{\cdot} c_0} \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle) \subseteq Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle), \tag{13}$$

i.e. the weak derivation of A_l-loop $\langle E, \overset{(T)}{\cdot}, 1\rangle$ onto left loop $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is a weak automorphic derivation. Moreover, in view of 1), Lemma 8 and (13) we obtain $l^{(P)}_{a,b} = \hat{h}^{-1}_{(a \overset{(P)}{\cdot} b)} l^{(T)}_{a, \hat{h}_{(a)}(b)} \hat{h}_{(a)} \hat{h}_{(b)} \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ for every $a, b \in E$, i.e.

$$LI(\langle E, \overset{(P)}{\cdot}, 1\rangle) \subseteq LI(\langle E, \overset{(T)}{\cdot}, 1\rangle). \tag{14}$$

But $c_0 \in N_r(\langle E, \overset{(T)}{\cdot}, 1\rangle)$. Then $l^{(T)}_{a,b}(c_0) = c_0$ for every $a, b \in E$. So $\alpha(c_0) = c_0$ for every $\alpha \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$. Then, by (7) and (14), for every $x \in E$ and $\alpha \in LI(\langle E, \overset{(P)}{\cdot}, 1\rangle)$ we obtain:

$$\begin{aligned}
\alpha \hat{h}_{(x)} \alpha^{-1} &= \alpha l^{(T)}_{x, x \overset{(T)}{\cdot} c_0} \alpha^{-1} = \alpha \hat{t}^{-1}_{x \overset{(T)}{\cdot} (x \overset{(T)}{\cdot} c_0)} \hat{t}_x \hat{t}_{x \overset{(T)}{\cdot} c_0} \alpha^{-1} = \\
&= (\alpha \hat{t}^{-1}_{x \overset{(T)}{\cdot} (x \overset{(T)}{\cdot} c_0)} \alpha^{-1})(\alpha \hat{t}_x \alpha^{-1})(\alpha \hat{t}_{x \overset{(T)}{\cdot} c_0} \alpha^{-1}) = \\
&= \hat{t}^{-1}_{\alpha(x \overset{(T)}{\cdot} (x \overset{(T)}{\cdot} c_0))} \hat{t}_{\alpha(x)} \hat{t}_{\alpha(x \overset{(T)}{\cdot} c_0)} = \hat{t}^{-1}_{\alpha(x) \overset{(T)}{\cdot} \alpha(x \overset{(T)}{\cdot} c_0)} \hat{t}_{\alpha(x)} \hat{t}_{\alpha(x \overset{(T)}{\cdot} c_0)} = \\
&= l^{(T)}_{\alpha(x), \alpha(x \overset{(T)}{\cdot} c_0)} = l^{(T)}_{\alpha(x), \alpha(x) \overset{(T)}{\cdot} \alpha(c_0)} = l^{(T)}_{\alpha(x), \alpha(x) \overset{(T)}{\cdot} c_0} = \hat{h}_{(\alpha(x))} \,.
\end{aligned}$$

This, by 2) of Lemma 8, proves that $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is an A_l-loop. □

Remark 6. The set $\{h_{(x)}\}_{x\in E}$ may be chosen in the another way. For example $\hat{h}_{(x)} = l^{(T)}_{x,c_0 \overset{(T)}{\cdot} x}$, or, in the general case $\hat{h}_{(x)} = l^{(T)}_{R_1(x,c_1),R_2(x,c_2)}$, where R_1, R_2 are terms of two variables on E. If $c_1, c_2 \in N_r(\langle E, \overset{(T)}{\cdot}, 1\rangle)$, then the system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is an A_l-loop.

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Dual positive implicative hyper K-ideals of type 3

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Abstract

In this note first we define the notion of *dual positive implicative hyper K-ideal of type* 3, where for simplicity is written by $DPIHKI-T3$. Then we determine all of the non-isomorphic hyper K-algebras of order 3, which have $D = \{0, 1\}$ as a $DPIHKI-T3$. To do this first we show that $D = \{1\}$ and $D = \{1, 2\}$ can not be $DPIHKI-T3$. Then we prove some lemmas which are needed for proving the main theorem. Finally we conclude that there are exactly 219 non-isomorphic hyper K-algebras of order 3 with the requested property.

1. Introduction

The hyperalgebraic structure theory was introduced by F. Marty [7] in 1934. Around the 40's, several authors worked on hypergroups, especially in France and in the United States, but also in Italy, Russia and Japan. Over the following decades, many important results appeared, but above all since 70's onwards the most luxuriant flourishing of hyperstructures has been seen (see for example [4]). Hyperstructures have many applications to several sectors of both pure and applied sciences.

Imai and Iéki [5] in 1966 introduced the notion of a BCK-algebra. Recently [2, 3, 9] Borzooei, Jun and Zahedi et .al. applied the hyperstructure to BCK-algebras and introduced the concept of hyper K-algebra which is a generalization of BCK-algebra. In [1], the authors have defined 8 types of positive implicative hyper K-ideals. Now in this note we define the notion of dual positive implicative hyper K-ideal of type 3, then we obtain some related results which have been mentioned in the abstract.

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2. Preliminaries

Definition 2.1. Let H be a non-empty set and " $\circ$ " be a *hyperoperation* on H, that is " $\circ$ " is a function from $H \times H$ to $\mathcal{P}^*(H) = \mathcal{P}(H) \backslash \{\emptyset\}$. Then H is called a *hyper K-algebra* if it contains a constant 0 and satisfies the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) < x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x < x$,
(HK4) $x < y,\ y < x \Longrightarrow x = y$,
(HK5) $0 < x$,

for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists\, a \in A$, $\exists\, b \in B$ such that $a < b$.

Note that if $A, B \subseteq H$, then by $A \circ B$ we mean the set-theoretic union of all $a \circ b$ such that $a \in A$, $b \in B$.

The main properties of hyper K-algebras are described in [2] and [3]. For example in [2] the following theorem is proved.

Theorem 2.2. *Let $(H, \circ, 0)$ be a hyper K-algebra. Then for all $x, y, z \in H$ and for all non-empty subsets A, B and C of H we have:*

(i)	$x \circ y < z \Longleftrightarrow x \circ z < y$,	(vi)	$x \in x \circ 0$,
(ii)	$(x \circ z) \circ (x \circ y) < y \circ z$,	(vii)	$(A \circ C) \circ (A \circ B) < B \circ C$,
(iii)	$x \circ (x \circ y) < y$,	$(viii)$	$(A \circ C) \circ (B \circ C) < A \circ B$,
(iv)	$x \circ y < x$,	(ix)	$A \circ B < C \Longleftrightarrow A \circ C < B$,
(v)	$A \subseteq B \Longrightarrow A < B$,	(x)	$A \circ B < A$. $\square$

Definition 2.3. Let $(H, \circ, 0)$ be a hyper K-algebra. If there exist an element $1 \in H$ such that $1 < x$ for all $x \in H$, then H is called a *bounded hyper K-algebra* and 1 is said to be the unit of H.

3. Dual positive implicative hyper K-algebras

From now H is a bounded hyper K-algebra with unit 1 and $1 \circ x = Nx$.

Definition 3.1. A non-empty subset D of H is called a *dual positive implicative hyper K-ideal type* 3 (shortly: DPIHKI-T3) if

(i) $1 \in D$,
(ii) $N((Nx \circ Ny) \circ Nz) < D$ and $N(Ny \circ Nz) < D$ imply
$N(Nx \circ Nz) \subseteq D$, $\forall\, x, y, z \in H$.

Example 3.2. Let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H with unit 1.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

And $I = \{0, 1\}$ is a $DPIHKI - T3$. □

Theorem 3.3. *A non-empty subset D of H is a $DPIHKI - T3$ if and only if $N(Nx \circ Nz) \subseteq D$ for all $x, z \in H$.*

Proof. Let D be a $DPIHKI - T3$. Then by Definition 2.1 and Theorem 2.3 (x) we conclude that $N((Nx \circ Ny) \circ Nz) < D$ and $N(Ny \circ Nz) < D$ for all $x, y, z \in H$. So by hypothesis we get that $N(Nx \circ Nz) \subseteq D$ for all $x, z \in H$. The converse statement is obvious. □

To avoid repetitions let in the sequel $H = \{0, 1, 2\}$ be a bounded hyper K-algebra with unit 1.

Lemma 3.4. *In H we have* $1 \circ 0 = \{1\}$.

Proof. On the contrary let $1 \circ 0 \neq \{1\}$. Then we must have $1 \circ 0 = \{1, 2\}$. By $(HK2)$ we have $(1 \circ 0) \circ 2 = (1 \circ 2) \circ 0$, so $0 \in 2 \circ 2 \subseteq (1 \circ 0) \circ 2 = (1 \circ 2) \circ 0$. Thus there exists $x \in 1 \circ 2$ such that $0 \in x \circ 0$, which implies that $x < 0$, thus from (HK4) and $(HK5)$ we get that $x = 0$. Hence $0 \in 1 \circ 2$, that is $1 < 2$. Since $2 < 1$, thus $2 = 1$, which is a contradiction. □

Lemma 3.5. *For all $x \in H$ we have $NNx = x$ if and only if $1 \circ 1 = \{0\}$ and $1 \circ 2 = \{2\}$.*

Proof. Let $NNx = x$, i.e. $1 \circ (1 \circ x) = x$ for all x. Since $1 \circ (1 \circ 2) = 2$, we get that $0 \notin 1 \circ 2$ and $1 \notin 1 \circ 2$. So $1 \circ 2 = \{2\}$. Now since $1 \circ (1 \circ 1) = 1$, we conclude that $1 \notin 1 \circ 1$ and $2 \notin 1 \circ 1$. Thus $1 \circ 1 = \{0\}$.

The converse follows from Lemma 3.4 and hypothesis. □

Lemma 3.6. *Let $D_1 = \{1\}$ and $D_2 = \{1, 2\}$ in H. Then D_1 and D_2 are not $DPIHKI - T3$.*

Proof. Since $0 \in 1 \circ ((1 \circ 0) \circ (1 \circ 1)) = N(N0 \circ N1)$, $0 \notin D_1$ and $0 \notin D_2$, then by Theorem 3.3 D_1 and D_2 are not $DPIHKI-T3$. □

Lemma 3.7. *Let $D=\{0,1\}$ in H. Then the following hold:*
(i) *if* $2\in 1\circ 2$, *then D is not a $DPIHKI-T3$,*
(ii) *if* $2\in 1\circ 1$, *then D is not a $DPIHKI-T3$.*

Proof. (i) Since $2\in 1\circ 2\subseteq 1\circ((1\circ 2)\circ(1\circ 1))=N(N2\circ N1)$ and $2\notin D$, then, by Theorem 3.3, D is not $DPIHKI-T3$.
(ii) The proof is similar as (i). □

Theorem 3.8. *Let $D=\{0,1\}$ in H. Then D is a $DPIHKI-T3$ if and only if $2\notin 1\circ 2$ and $2\notin 1\circ 1$.*

Proof. Let $2\notin 1\circ 2$ and $2\notin 1\circ 1$. Thus $1\circ 2=\{1\}$ and $1\circ 1=\{0,1\}$ or $1\circ 1=\{0\}$. Now by some calculations we can get that $N(Nx\circ Nz)\subseteq D$, for all $x,z\in H$.

Conversely, on the contrary let $2\in 1\circ 2$ or $2\in 1\circ 1$. Then Lemma 3.7 $(i),(ii)$ gives a contradiction. Thus $2\notin 1\circ 2$ and $2\notin 1\circ 1$. □

Remark 3.9. From now let $D=\{0,1\}$ be a $DPIHKI-T3$. Thus:
(i) From Theorem 3.8 we conclude that $1\circ 2=\{1\}$ and $1\circ 1=\{0,1\}$ or $1\circ 1=\{0\}$.
(ii) By (HK2) we have $(1\circ 1)\circ 0=(1\circ 0)\circ 1$ and $(1\circ 1)\circ 2=(1\circ 2)\circ 1$. Thus by (i) and Lemma 3.4 we conclude that $0\circ 0\subseteq\{0,1\}$ and $0\circ 2\subseteq\{0,1\}$. □

Lemma 3.10. *If $2\circ 2=\{0\}$ and $0\circ 0=\{0\}$ in H, then $2\circ 0=\{2\}$.*

Proof. By (HK2) we have $(2\circ 0)\circ 2=(2\circ 2)\circ 0=0\circ 0=\{0\}$. If $1\in 2\circ 0$, then $1\circ 2\subseteq(2\circ 0)\circ 2=\{0\}$. Thus $1\circ 2=\{0\}$, which is a contradiction because $1\circ 2=\{1\}$, by Remark 3.9 (i). Thus $1\notin 2\circ 0$, hence $2\circ 0=\{2\}$. □

Lemma 3.11. *If $2\circ 2=\{0\}$ and $0\circ 1=\{0\}$ in H, then $1\notin 2\circ 1$.*

Proof. On the contrary let $1\in 2\circ 1$. By (HK2) we have $(2\circ 1)\circ 2=(2\circ 2)\circ 1=0\circ 1=\{0\}$. Thus by Remark 3.9 ($i$) and hypothesis we have $1\in 1\circ 2\subseteq(2\circ 1)\circ 2=\{0\}$, which is a contradiction. □

Lemma 3.12. If $2 \circ 1 = \{0, 2\}$ and $1 \circ 1 = \{0\}$ in H, then $2 \circ 0 = \{2\}$.

Proof. On the contrary let $2 \circ 0 \neq \{2\}$. Then we must have $2 \circ 0 = \{1, 2\}$. By (HK2) we have $(2 \circ 1) \circ 0 = (2 \circ 0) \circ 1$. By hypothesis we have $(2 \circ 1) \circ 0 = \{0, 1, 2\}$ and $(2 \circ 0) \circ 1 = \{0, 2\}$, which is a contradiction. $\square$

Lemma 3.13. *Let* $0 \circ 1 = \{0, 1\}$ *and* $0 \circ 2 = \{0\}$ *in* H.
(*i*) *If* $2 \circ 2 \subseteq \{0, 2\}$, *then* $2 \circ 1 \not\subseteq \{0, 2\}$.
(*ii*) *If* $2 \circ 2 = \{0, 1\}$ *or* $2 \circ 2 = \{0, 1, 2\}$, *then* $2 \circ 1 \neq \{0\}$.

Proof. (*i*) On the contrary let $2 \circ 1 \subseteq \{0, 2\}$. If $2 \circ 1 = \{0, 2\}$ by (HK2) we have $(2 \circ 2) \circ 1 = (2 \circ 1) \circ 2$. By hypothesis we have $(2 \circ 2) \circ 1 = \{0, 1\}$ if $2 \circ 2 = \{0\}$ and $(2 \circ 2) \circ 1 = \{0, 1, 2\}$ if $2 \circ 2 = \{0, 2\}$. On the other hand $(2 \circ 1) \circ 2 = \{0\}$ if $2 \circ 2 = \{0\}$ and $(2 \circ 1) \circ 2 = \{0, 2\}$ if $2 \circ 2 = \{0, 2\}$, which is a contradiction. If $2 \circ 1 = \{0\}$, then the proof is similar as the case of $2 \circ 1 = \{0, 2\}$.

The proof of (*ii*) is similar as (*i*). $\square$

Lemma 3.14. *Let* $0 \circ 1 = \{0, 2\}$ *in* H. *Then:*
(*i*) $2 \circ 2 \not\subseteq \{0, 1\}$,
(*ii*) $2 \circ 1 \not\subseteq \{0, 1\}$,
(*iii*) *if* $1 \circ 1 = \{0\}$, *then* $2 \circ 2 \neq \{0, 1, 2\}$,
(*iv*) *if* $0 \circ 0 = \{0\}$, *then* $2 \circ 0 = \{2\}$,
(*v*) *if* $2 \circ 2 = \{0, 1, 2\}$, *then* $0 \circ 2 = \{0, 1\}$.

Proof. (*i*) On the contrary let $2 \circ 2 \subseteq \{0, 1\}$. By (HK2) we have $(0 \circ 2) \circ 1 = (0 \circ 1) \circ 2$. If $2 \circ 2 = \{0\}$ by hypothesis and Remark 3.9 we get that $(0 \circ 1) \circ 2 \subseteq \{0, 1\}$ and $(0 \circ 2) \circ 1 = \{0, 2\}$ or $\{0, 1, 2\}$, which is a contradiction. If $2 \circ 2 = \{0, 2\}$, then the proof is similar as the case of $2 \circ 2 = \{0\}$.

The proof of the other cases are similar as above by considering the suitable modifications. $\square$

Lemma 3.15. *Let* $0 \circ 1 = \{0, 1, 2\}$ *in* H. *Then:*
(*i*) $2 \circ 2 \not\subseteq \{0, 1\}$,
(*ii*) $2 \circ 1 \not\subseteq \{0, 1\}$,
(*iii*) *if* $2 \circ 2 = \{0, 2\}$ *and* $0 \circ 2 = \{0\}$, *then* $2 \circ 1 \neq \{0, 2\}$,
(*iv*) *if* $2 \circ 1 = \{0, 2\}$ *and* $1 \notin 2 \circ 2$, *then* $0 \circ 2 = \{0, 1\}$.

Proof. (*i*) On the contrary let $2 \circ 2 \subseteq \{0, 1\}$. By (HK2) we have $(0 \circ 2) \circ 1 =$

$(0 \circ 1) \circ 2$. If $2 \circ 2 = \{0\}$, then by hypothesis and Remark 3.9 we get that $(0 \circ 1) \circ 2 = \{0, 1\}$ and $(0 \circ 2) \circ 1 = \{0, 1, 2\}$, which is a contradiction. If $2 \circ 2 = \{0, 2\}$, then the proof is similar as the case of $2 \circ 2 = \{0\}$.

The proof of the other cases are similar as above by considering the suitable modifications. □

Lemma 3.16. *If* $2 \circ 1$, $2 \circ 2$ *and* $0 \circ 1 \subseteq \{0, 2\}$, *then* $0 \circ 2 = \{0\}$.

Proof. By (HK2) we have $(2 \circ 1) \circ 2 = (2 \circ 2) \circ 1 \subseteq \{0, 2\}$. Since $0 \circ 2 \subseteq (2 \circ 1) \circ 2 \subseteq \{0, 2\}$, and by Remark 3.9 (ii) $2 \notin 0 \circ 2$, we get that $0 \circ 2 = \{0\}$. □

Lemma 3.17. *Let* $2 \circ 1 = \{0\}$ *in* H. *Then:*

(i) *if* $1 \circ 1 = \{0, 1\}$ *and* $2 \circ 2 = \{0, 1\}$ *or* $\{0, 1, 2\}$, *then* $0 \circ 2 = \{0, 1\}$,

(ii) *if* $0 \circ 0 = \{0\}$ *and* $1 \circ 1 = \{0, 1\}$, *then* $2 \circ 0 = \{2\}$,

(iii) *if* $0 \circ 1 = \{0, 1\}$, *then* $0 \circ 2 = \{0, 1\}$,

(iv) *if* $0 \circ 0 = \{0, 1\}$, *then* $2 \circ 0 = \{1, 2\}$.

Proof. (i) By (HK2) we have $(2 \circ 2) \circ 1 = (2 \circ 1) \circ 2$. Now $(2 \circ 2) \circ 1 = \{0, 1\}$ and $(2 \circ 1) \circ 2 = 0 \circ 2$, therefore $0 \circ 2 = \{0, 1\}$.

(ii) On the contrary let $2 \circ 0 \neq \{2\}$. Then we must have $2 \circ 0 = \{1, 2\}$. By (HK2) we have $(2 \circ 1) \circ 0 = (2 \circ 0) \circ 1$, which is contradiction, because $1 \in (2 \circ 0) \circ 1$, while $1 \notin (2 \circ 1) \circ 0 = \{0\}$.

The proofs of (iii) and (iv) are similar. □

Lemma 3.18. *Let* $2 \circ 1 = \{0, 2\}$ *in* H. *Then:*

(i) *if* $1 \in 0 \circ 1$ *and* $1 \notin 2 \circ 2$, *then* $0 \circ 2 = \{0, 1\}$,

(ii) *if* $0 \circ 0 = \{0, 1\}$, *then* $2 \circ 0 = \{1, 2\}$.

Proof. (i) On the contrary let $0 \circ 2 \neq \{0, 1\}$. Then we must have $0 \circ 2 = \{0\}$, by Remark 3.9 (ii). By (HK2) we have $(2 \circ 1) \circ 2 = (2 \circ 2) \circ 1$. Now by hypothesis we have $(2 \circ 1) \circ 2 \subseteq \{0, 2\}$ and $1 \in (2 \circ 2) \circ 1$, which is a contradiction.

The proof of (ii) is similar as (i). □

Lemma 3.19. *If* $2 \circ 2 \subseteq \{0, 2\}$ *and* $0 \circ 0 = \{0, 1\}$, *then* $2 \circ 0 = \{1, 2\}$.

Proof. On the contrary let $2 \circ 0 \neq \{1, 2\}$. Then we must have $2 \circ 0 = \{2\}$. By (HK2) we have $(2 \circ 2) \circ 0 = (2 \circ 0) \circ 2$. Now by hypothesis we have

$1 \in (2 \circ 2) \circ 0$ and $1 \notin (2 \circ 0) \circ 2$, which is a contradiction. $\square$

Now we are ready to determine all of hyper K-algebras of order 3, in which $D = \{0, 1\}$ is a $DPIHKI - T3$.

Theorem 3.20 (Main theorem) *There are* 219 *non-isomorphic bounded hyper K-algebras of order* 3, *to have $D = \{0, 1\}$ as a $DPIHKI - T3$.*

Proof. Let $H = \{0, 1, 2\}$ and 1 be its unit. The following table shows a probable hyper K-algebra structure on H, in which $D = \{0, 1\}$ is a $DPIHKI - T3$:

$\circ$	0	1	2
0	a_{11}	a_{12}	a_{13}
1	a_{21}	a_{22}	a_{23}
2	a_{31}	a_{32}	a_{33}

By Remark 3.9 we have $a_{21} = 1 \circ 0 = \{1\}$, $a_{22} = 1 \circ 1 = \{0\}$ or $\{0, 1\}$, $a_{23} = 1 \circ 2 = \{1\}$, $a_{11} = 0 \circ 0 \subseteq \{0, 1\}$ and $a_{13} = 0 \circ 2 \subseteq \{0, 1\}$. Also since H is bounded, then by $(HK3)$ and $(HK5)$ we have $0 \in a_{12} \bigcap a_{32} \bigcap a_{33}$. There are two cases for $a_{22} = 1 \circ 1$. Let $1 \circ 1 = \{0\}$. Then by (HK2) we have $(1 \circ 1) \circ 0 = (1 \circ 0) \circ 1$, so $0 \circ 0 = \{0\}$. Similarly $(1 \circ 1) \circ 2 = (1 \circ 2) \circ 1$ implies that $0 \circ 2 = \{0\}$. We will show that in this case there exist exactly 40 non-isomorphic hyper K-algebras. In the other hand if $1 \circ 1 = \{0, 1\}$, then by Remark 3.9 (ii) we get that $0 \circ 0 \subseteq \{0, 1\}$ and $0 \circ 2 \subseteq \{0, 1\}$ and in this situation we will obtain exactly 179 non-isomorphic hyper K-algebras other than the previous 40 ones. So totally we have 219 different non-isomorphic bounded hyper K-algebras of order 3, to have $D = \{0, 1\}$ as a $DPIHKI - T3$. Now we give the details. To do this we consider two main cases $1 \circ 1 = \{0\}$ and $1 \circ 1 = \{0, 1\}$, and many subcases of them.

1. $1 \circ 1 = \{0\}$

We consider some subcases as follows:

1.1. $0 \circ 1 = \{0\}$

In this case also we consider 4 states as follows:

1.1.1. $2 \circ 2 = \{0\}$

By Lemmas 3.10 and 3.11 we must have $2 \circ 0 = \{2\}$ and $2 \circ 1 \subseteq \{0, 2\}$. So there exist 2 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0, 2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0\}$

1.1.2. $2 \circ 2 = \{0,1\}$

By $(HK2)$ We have $(2\circ 2)\circ 1 = (2\circ 1)\circ 2$. We get that $(2\circ 2)\circ 1 = \{0\}$. If $1 \in 2\circ 1$ or $2 \in 2\circ 1$, then $1 \in (2\circ 1)\circ 2 = \{0\}$, which is a contradiction. Thus $1 \notin 2\circ 1$ and $2 \notin 2\circ 1$, hence $2\circ 1 = \{0\}$. So there exists two hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,1\}$

1.1.3. $2 \circ 2 = \{0,2\}$

If $2\circ 1 = \{0,2\}$, then by Lemma 3.12 we have $2\circ 0 = \{2\}$. So there exists seven hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

1.1.4. $2 \circ 2 = \{0,1,2\}$

We prove that $2\circ 1 \neq \{0,2\}$. On the contrary, let $2\circ 1 = \{0,2\}$. By $(HK2)$ we have $(2\circ 2)\circ 1 = (2\circ 1)\circ 2$, while $(2\circ 2)\circ 1 = \{0,2\}$ and $(2\circ 1)\circ 2 = \{0,1,2\}$, which is a contradiction. So there exist six hyper K-algebras:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

1.2. $0 \circ 1 = \{0, 1\}$

In this case also we consider four states as follows:

1.2.1. $2 \circ 2 = \{0\}$

By Lemmas 3.10 and 3.13(i) we have $2 \circ 0 = \{2\}$ and $2 \circ 1 \not\subseteq \{0, 2\}$. So there exist two hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0\}$

1.2.2. $2 \circ 2 = \{0, 1\}$

By Lemma 3.13 (ii) we have $2 \circ 1 \neq \{0\}$. If $2 \circ 1 = \{0, 2\}$, then by Lemma 3.12 we have $2 \circ 0 = \{2\}$. So there exist five hyper K-algebras:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

1.2.3. $2 \circ 2 = \{0, 2\}$

By Lemma 3.13 (i) we have $2 \circ 1 \not\subseteq \{0, 2\}$. So there exist four hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

1.2.4. $2 \circ 2 = \{0, 1, 2\}$

This case is similar to 1.2.2. So there exist five hyper K-algebras:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

1.3. $0 \circ 1 = \{0,2\}$

In this case we have only one state, since by Lemma 3.14 $(i),(ii)$ we have $2 \circ 2 \not\subseteq \{0,1\}$ and $2 \circ 2 \neq \{0,1,2\}$.

1.3.1. $2 \circ 2 = \{0,2\}$

By Lemma 3.14 $(i),(iv)$ we have $2 \circ 1 \not\subseteq \{0,1\}$ and $2 \circ 0 = \{2\}$. So there exist two hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

1.4. $0 \circ 1 = \{0,1,2\}$

We have two states, since $2 \circ 2 \not\subseteq \{0,1\}$, by Lemma 3.15 (i).

1.4.1. $2 \circ 2 = \{0,2\}$

By Lemma 3.15 $(ii),(iii)$ we have $2 \circ 1 \not\subseteq \{0,1\}$ and $2 \circ 1 \neq \{0,2\}$. So there exist two hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0\}$

1.4.2. $2 \circ 2 = \{0,1,2\}$

By Lemma 3.15 (ii) we have $2 \circ 1 \not\subseteq \{0,1\}$. If $2 \circ 1 = \{0,2\}$, then by Lemma 3.12 we have $2 \circ 0 = \{2\}$. So there exist three hyper K-algebras:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

Now we consider the following case:

2. $1 \circ 1 = \{0,1\}$

This case has two subcases $0 \circ 0 = \{0\}$ or $\{0,1\}$.

2.1. $0 \circ 0 = \{0\}$

We consider the following subcases as:

2.1.1. $0 \circ 1 = \{0\}$

In this case also we consider four states as follows:

2.1.1.1. $2 \circ 2 = \{0\}$

By Lemmas 3.10 and 3.11 we have $2 \circ 0 = \{2\}$ and $1 \notin 2 \circ 1$. If $2 \circ 1 \subseteq \{0, 2\}$, then by Lemma 3.16 we have $0 \circ 2 = \{0\}$. So there exist two hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0\}$

2.1.1.2. $2 \circ 2 = \{0, 1\}$

If $2 \circ 1 = \{0\}$, then by Lemma 3.17 $(i), (ii)$ we have $0 \circ 2 = \{0, 1\}$ and $2 \circ 0 = \{2\}$. So there exist 13 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

2.1.1.3. $2 \circ 2 = \{0, 2\}$

If $2 \circ 1 \subseteq \{0, 2\}$, then $0 \circ 2 = \{0\}$ by Lemma 3.16. If $2 \circ 1 = \{0\}$, then $2 \circ 0 = \{2\}$ by Lemma 3.17 (i). So there exist 11 hyper K-algebras:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,2\}$

2.1.1.4. $2 \circ 2 = \{0,1,2\}$

If $2 \circ 1 = \{0\}$, then by Lemma 3.17 $(i), (ii)$ we have $0 \circ 2 = \{0,1\}$ and $2 \circ 0 = \{2\}$. So there exist 13 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

2.1.2. $0 \circ 1 = \{0,1\}$

In this case also we consider four states as follows:

2.1.2.1. $2 \circ 2 = \{0\}$

By Lemma 3.10 we have $2 \circ 0 = \{2\}$. If $0 \circ 2 = \{0\}$, then by Lemma 3.13 (i) we have $2 \circ 1 \nsubseteq \{0, 2\}$. So there exist six hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0\}$

2.1.2.2. $2 \circ 2 = \{0, 1\}$

If $2 \circ 1 = \{0\}$, then by Lemma 3.17 $(i), (ii)$ we have $0 \circ 2 = \{0, 1\}$ and $2 \circ 0 = \{2\}$. So there exist 13 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

2.1.2.3. $2 \circ 2 = \{0, 2\}$

If $2 \circ 1 = \{0\}$, then by lemma 3.17 $(i), (ii)$ we have $0 \circ 2 = \{0, 1\}$ and $2 \circ 0 = \{2\}$. If $2 \circ 1 = \{0, 2\}$, then by Lemma 3.18 (i) we have $0 \circ 2 = \{0, 1\}$.

So there exist 11 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

2.1.2.4. $2 \circ 2 = \{0,1,2\}$

This case is similar as the case of 2.1.2.2. So there exist 13 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

2.1.3. $0 \circ 1 = \{0,2\}$

In this case we have two states since by Lemma 3.14 (i) we obtain $2 \circ 2 \not\subseteq \{0,1\}$.

2.1.3.1. $2 \circ 2 = \{0,2\}$

If $2 \circ 1 = \{0,2\}$, then by Lemma 3.16 we have $0 \circ 2 = \{0\}$. By Lemma 3.14 $(i),(iv)$ we have $2 \circ 1 \not\subseteq \{0,1\}$ and $2 \circ 0 = \{2\}$. So there exist three hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

2.1.3.2. $2 \circ 2 = \{0,1,2\}$

By Lemma 3.14 $(i),(iv),(v)$ we must have $2 \circ 1 \not\subseteq \{0,1\}$, $2 \circ 0 = \{2\}$ and $0 \circ 2 = \{0,1\}$. So there exist two hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

2.1.4. $0 \circ 1 = \{0,1,2\}$

By Lemma 3.15 (i) we have $2 \circ 2 \not\subseteq \{0,1\}$, thus in this case we have only two states as follows:

2.1.4.1. $2 \circ 2 = \{0,2\}$

By Lemma 3.15 (i) we have $2 \circ 1 \not\subseteq \{0,1\}$. If $2 \circ 1 = \{0,2\}$, then by Lemma 3.18 (i) we have $0 \circ 2 = \{0,1\}$. So there exist six hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

2.1.4.2. $2 \circ 2 = \{0, 1, 2\}$

By Lemma 3.15 (ii) we have $2 \circ 1 \not\subseteq \{0, 1\}$. So there exist eight hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

Now we consider the following case:

2.2. $0 \circ 0 = \{0, 1\}$

We consider some subcases as follows:

2.2.1. $0 \circ 1 = \{0\}$

In this case also we consider four states as follows:

2.2.1.1. $2 \circ 2 = \{0\}$

By Lemmas 3.19 and 3.11 we have $2 \circ 0 = \{1, 2\}$ and $1 \notin 2 \circ 1$. Since $1 \notin 2 \circ 1$, hence $2 \circ 1 \subseteq \{0, 2\}$ and by Lemma 3.16 we have $0 \circ 2 = \{0\}$. So there exist two hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0\}$

2.2.1.2. $2 \circ 2 = \{0, 1\}$

If $2 \circ 1 \subseteq \{0, 2\}$, then by Lemmas 3.17 (iv) and 3.18 (ii) we have $2 \circ 0 = \{1, 2\}$. If $2 \circ 1 = \{0\}$, then by Lemma 3.17 (i) we have $0 \circ 2 = \{0, 1\}$. So there exist 11 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

2.2.1.3. $2 \circ 2 = \{0, 2\}$

By Lemma 3.19 we have $2 \circ 0 = \{1, 2\}$. If $2 \circ 1 \subseteq \{0, 2\}$, then by Lemma 3.16 we have $0 \circ 2 = \{0\}$. So there exist six hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

2.2.1.4. $2 \circ 2 = \{0, 1, 2\}$

This case is similar to 2.2.1.2. So there exist 11 hyper K-algebras:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

2.2.2. $0 \circ 1 = \{0,1\}$

Consider the following four states:

2.2.2.1. $2 \circ 2 = \{0\}$

By Lemma 3.19 we have $2 \circ 0 = \{1,2\}$. If $2 \circ 1 \subseteq \{0,2\}$, then by Lemmas 3.17 (iii) and 3.18 (i) we have $0 \circ 2 = \{0,1\}$. So there exist six hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0\}$

2.2.2.2. $2 \circ 2 = \{0,1\}$

If $2 \circ 1 \subseteq \{0,2\}$, then by Lemmas 3.17 (iv) and 3.18 (ii) we have $2 \circ 0 = \{1,2\}$. If $2 \circ 1 = \{0\}$, then by Lemma 3.17 (iii) we have $0 \circ 2 = \{0,1\}$. So there exist 11 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

2.2.2.3. $2 \circ 2 = \{0,2\}$

By Lemma 3.19 we have $2 \circ 0 = \{1,2\}$. If $2 \circ 1 \subseteq \{0,2\}$, then by Lemmas 3.18 (i) and 3.17 (i) we have $0 \circ 2 = \{0,1\}$. So there exist six hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,2\}$

2.2.2.4. $2 \circ 2 = \{0,1,2\}$

If $2 \circ 1 \subseteq \{0,2\}$, then by Lemmas 3.17 (iv) and 3.18 (ii) we have $2 \circ 0 = \{1,2\}$. If $2 \circ 1 = \{0\}$, then by Lemma 3.17 (i) we have $0 \circ 2 = \{0,1\}$. So there exist 11 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

2.2.3. $0 \circ 1 = \{0, 2\}$

By Lemma 3.14 (i) we have $2 \circ 2 \not\subseteq \{0, 1\}$. Thus this case has only two stats as follows:

2.2.3.1. $2 \circ 2 = \{0, 2\}$

By Lemmas 3.14 (ii) and 3.19 we have $2 \circ 1 \not\subseteq \{0, 1\}$ and $2 \circ 0 = \{1, 2\}$. If $2 \circ 1 = \{0, 2\}$, then by Lemma 3.16 we have $0 \circ 2 = \{0\}$. So there exist three hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

Case 2.2.3.2. $2 \circ 2 = \{0, 1, 2\}$.

By Lemmas 3.14 $(ii), (v)$ we have $2 \circ 1 \not\subseteq \{0, 1\}$ and $0 \circ 2 = \{0, 1\}$. If $2 \circ 1 = \{0, 2\}$, then by Lemma 3.18 (ii) we have $2 \circ 0 = \{1, 2\}$. So there exist three hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

2.2.4. $0 \circ 1 = \{0, 1, 2\}$

By Lemma 3.15 (i) we have $2 \circ 2 \not\subseteq \{0, 1\}$. Thus this case has only two states as follows:

2.2.4.1. $2 \circ 2 = \{0, 2\}$

By Lemmas 3.15 (i) and 3.19 we have $2 \circ 1 \not\subseteq \{0, 1\}$ and $2 \circ 0 = \{1, 2\}$. If $2 \circ 1 = \{0, 2\}$, then by Lemma 3.18 (i) we have $0 \circ 2 = \{0, 1\}$. So there exist three hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

2.2.4.2. $2 \circ 2 = \{0, 1, 2\}$

By Lemma 3.15 (ii) we have $2 \circ 1 \not\subseteq \{0, 1\}$. If $2 \circ 1 = \{0, 2\}$, then by Lemma 3.18 (ii) we have $2 \circ 0 = \{1, 2\}$. So there exist six hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

Now we show that each pair of the above 219 hyper K-algebras are not isomorphic together. On the contrary let $(H_1, \circ_1, 0)$ and $(H_2, \circ_2, 0)$ be isomorphic. then there exists an isomorhpism $f : H_1 \to H_2$. So $f(x \circ_1 y) = f(x) \circ_2 f(y)$, for all $x, y \in H$, thus we have $f(0_1) = 0_2$, $f(1) = 2$, $f(2) = 1$. But $f(1 \circ_1 2) = f(\{1\}) = \{2\}$ and $f(1) \circ_2 f(2) = 2 \circ 1 \supseteq \{0\}$, which is a contradiction, since $0 \notin f(1 \circ_1 2) = \{2\}$. $\square$

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