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A parastrophic equivalence in quasigroups

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Abstract

In this paper there are found of "lowest" representants of classes of a parastrophic equivalence in quasigroups satisfying identities of the type

$$w_1\Box_1(w_2\Box_2\dots(w_n\Box_n x))) \doteq x, \quad 1<n,$$

where $\Box_i$ is a parastrophe of $\Box_1$ for all $i \leqslant n$ and $w_1,\dots,w_n$ are terms in $Q(\cdot)$ and its parastrophes that not contain variable x. These representants are listed for $1<n<5$ by a personal computer.

1. Introduction

With any given quasigroup $(Q,\cdot)$ there are associated five operations $*$, $/$, $\backslash$, $\triangle$, $\triangledown$ (see the following part 1) that we shall call *conjugates* of $(\cdot)$ (see [1], [4]) or *parastrophes* of $(\cdot)$ (see [3]). If a quasigroup $(Q,\cdot)$ satisfies a given identity, say I, then in general, for example $(Q,/)$ will satisfy a different conjugate identity, say II. Therefore it is in some sense true that the theory of quasigroups that satisfy the identity I is equivalent to the theory of quasigroups which satisfy the identity II, as has been remarked by Stein in [4].

In [3], Sade has given some general rules for determining the identities satisfied by the parastrophes of a quasigroup $(Q,\cdot)$ when $(Q,\cdot)$ satisfies a given identity involving some elements of the set $\sum(\cdot) = \{\cdot,*,/,\triangledown,\backslash,\triangle\}$. In [4], Stein has listed the conjugate identities for a

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number well-known identities. More extensive list is given in Belousov [1]. With respect to a parastrophic equivalence, Belousov in [2] has given a classification of all quasigroups identities which are of the type $x\square_1\big(x\square_2(x\square_3 y)\big) \doteq y$, where $\square_i \in \sum(\cdot)$ for all $i = 1, 2, 3$.

In this paper we give a generalization and a simplification of methods used in [2].

2. Preliminaries

Let $(Q, \cdot)$ be a fixed quasigroup, $\mathcal{T} = \{L, R, T, L^{-1}, R^{-1}, T^{-1}\}$, and $\sum(\cdot) = \{\cdot, *, /, \bigtriangledown, \backslash, \bigtriangleup\}$, where $x \cdot y = z \Leftrightarrow y * x = z \Leftrightarrow z/y = x \Leftrightarrow y \bigtriangledown z = x \Leftrightarrow x \backslash z = y \Leftrightarrow z \bigtriangleup x = y$.

Further, let $L_a x = a \cdot x$, $R_a x = x \cdot a$, $L^{/}_a x = a/x$, $R^{\bigtriangledown}_a x = x \bigtriangledown a$, $T_a x = x \backslash a$, $L_a L_a^{-1} x = x$, ... Then it holds the relations given by Table 1. This table we read like this: $L^{\bigtriangledown} = R^{-1}$, $(R^{-1})^{\backslash} = T^{-1}$, ..., $\varphi_2(R^{-1}) = \varphi_2 R^{-1} = (R^{-1})^{/} = R$, $\varphi_5 R = L^{-1}$, $R_a^{-1} = (R_a)^{-1}$, $T_a^{-1} = (T_a)^{-1}$.

Table 1

	$\cdot$	$*$	$/$	$\bigtriangledown$	$\backslash$	$\bigtriangleup$
L	L	R	T^{-1}	R^{-1}	L^{-1}	T
R	R	L	R^{-1}	T^{-1}	T	L^{-1}
T	T	T^{-1}	L^{-1}	L	R	R^{-1}
L^{-1}	L^{-1}	R^{-1}	T	R	L	T^{-1}
R^{-1}	R^{-1}	L^{-1}	R	T	T^{-1}	L
T^{-1}	T^{-1}	T	L	L^{-1}	R^{-1}	R
	φ_0	φ_1	φ_2	φ_3	φ_4	φ_5

From this table directly follows that $(\varphi_i x)^{-1} = \varphi_i(x^{-1})$ for all $i \in \{0, 1, ..., 5\}$ and for all $x \in \mathcal{T}$. If $(Q, \square)$ is a quasigroup, then the mappings $L^{\square}_a$, $R^{\square}_a$, ..., $(T_a^{-1})^{\square}$ are called *translations* of $\square$. Every operation in $\sum(\cdot)$ is named a *parastrophe* of $(\cdot)$.

If a quasigroup $(Q,\cdot)$ satisfies a given identity, for example

$$y \doteq (x \backslash yz)/zx\,, \tag{1}$$

then in general each of its parastrophes will satisfy a different conjugate identities. Thus, for example, (1) is equivalent to $y\cdot zx \doteq x\backslash yz$; if denote $zx = u$, $yz = v$ and $yu = t$ (i.e. $x = z\backslash u$, $z = y\backslash v$, $u = y\backslash t$), then

$$t \doteq \big((y\backslash v)\backslash(y\backslash t)\big)\backslash v\,. \tag{2}$$

Hence, $(Q,\cdot)$ satisfies (1) iff $(Q,\backslash)$ satisfies (2). If (2) is written with terms of $(Q,\cdot)$, then obtain

$$c \doteq (ab\cdot ac)b\,. \tag{3}$$

Thus (3) is a conjugate identity to (1). Further, from (3) we have $R_bL_{ab}L_a \doteq 1$, i.e. $L_aR_bL_{ab} \doteq 1$. Whence $c \doteq a\cdot(ab\cdot c)b$ and if denote $a = y$, $ab = z$, $c = z\backslash x$, then

$$y \doteq \big(x\cdot(y\backslash z)\big) \bigtriangledown (z\backslash x) \tag{4}$$

is a conjugate identity to (1). (4) we get from (1) if all operations in (1) are substituted by Table 2, i.e. $(\cdot)$ is substituted by $\backslash = \varphi_4(\cdot)$, $*$ by $\bigtriangleup = \varphi_4(*)$, . . . , $\bigtriangleup$ by $* = \varphi_4(\bigtriangleup)$ (see Sade [3]).

Table 2

	$\cdot$	$*$	$/$	$\bigtriangledown$	$\backslash$	$\bigtriangleup$
$\cdot$	$\cdot$	$*$	$/$	$\bigtriangledown$	$\backslash$	$\bigtriangleup$
$*$	$*$	$\cdot$	$\bigtriangledown$	$/$	$\bigtriangleup$	$\backslash$
$/$	$/$	$\bigtriangleup$	$\cdot$	$\backslash$	$\bigtriangledown$	$*$
$\bigtriangledown$	$\bigtriangledown$	$\backslash$	$*$	$\bigtriangleup$	$/$	$\cdot$
$\backslash$	$\backslash$	$\bigtriangledown$	$\bigtriangleup$	$*$	$\cdot$	$/$
$\bigtriangleup$	$\bigtriangleup$	$/$	$\backslash$	$\cdot$	$*$	$\bigtriangledown$
	φ_0	φ_1	φ_2	φ_3	φ_4	φ_5

Table 3

	φ_0	φ_1	φ_2	φ_3	φ_4	φ_5
φ_0	φ_0	φ_1	φ_2	φ_3	φ_4	φ_5
φ_1	φ_1	φ_0	φ_5	φ_4	φ_3	φ_2
φ_2	φ_2	φ_3	φ_0	φ_1	φ_5	φ_4
φ_3	φ_3	φ_2	φ_4	φ_5	φ_1	φ_0
φ_4	φ_4	φ_5	φ_3	φ_2	φ_0	φ_1
φ_5	φ_5	φ_4	φ_1	φ_0	φ_2	φ_3

The identities (1) and (4) may be written by the way as

$$R^{/}_{zx}L^{\backslash}_{x}R_z \doteq 1, \qquad R^{\bigtriangledown}_{z\backslash x}L_xR^{\backslash}_z \doteq 1$$

and with respect to Table 1 and Table 2

$$R_{zx}^{-1}L_x^{-1}R_z \doteqdot 1, \qquad T_{z\backslash x}^{-1}L_xT_z \doteqdot 1\,.$$

The ordered tripletes $R^{-1}L^{-1}R$, $T^{-1}LT$ may be assigned to the identities (2), (3). Therefore the triple $R^{-1}L^{-1}R$ will be called *conjugate* to the triple $T^{-1}LT$.

In what follows we shall denote:

$$\mathbf{N} = \{0,1,2,3,\dots\},$$

$$\mathcal{T} = \{L, R, T, L^{-1}, R^{-1}, T^{-1}\},$$

$$[0,n) = \{0,1,2,3,\dots,n-1\},\ n \in \mathbf{N},\ n > 0,$$

$$\mathcal{T}^n = \{\alpha : \alpha \text{ is a map } [0,n) \to \mathcal{T}\} \text{ for all } n \in \mathbf{N},\ n > 0,$$

if $\alpha \in \mathcal{T}^n$ then $\alpha = A_{n-1}\dots A_2A_1A_0$ and $\alpha(i) = A_i$ for all $i \in [0,n)$,

$$\mathcal{T}^\infty = \mathcal{T} \cup (\mathcal{T}\times\mathcal{T}) \cup (\mathcal{T}\times\mathcal{T}\times\mathcal{T}) \cup \dots,$$

$$l(\alpha) = n \iff \alpha \in \mathcal{T}^n,$$

$\omega : \mathcal{T}^\infty \to \mathcal{T}^\infty$, $(\omega\alpha)(i) = \alpha((i+1)(mod\, n))$ for all $i \in [0,n)$,
it holds $\alpha \in \mathcal{T}^n \Rightarrow \omega\alpha \in \mathcal{T}^n$,

$\sigma : \mathcal{T}^\infty \to \mathcal{T}^\infty$, $(\sigma\alpha)(j) = \alpha(n-1-j)$ for $n = l(\alpha)$ and for all $j \in [0,n)$,

$\rho : \mathcal{T} \to \mathcal{T}$, $L \mapsto L^{-1} \mapsto L$, $R \mapsto R^{-1} \mapsto R$, $T \mapsto T^{-1} \mapsto T$,
i.e. $\rho(A) = A^{-1}$ for all $A \in \mathcal{T}$,

$\rho : \mathcal{T}^\infty \to \mathcal{T}^\infty$, $(\rho\alpha)(i) = \rho\big(\alpha(i)\big)$ for all $j \in [0,n)$, $n = l(\alpha)$,

$\kappa : \mathcal{T} \to [0,6)$, $L \mapsto 0$, $R \mapsto 1$, $T \mapsto 2$, $L^{-1} \mapsto 3$, $R^{-1} \mapsto 4$, $T^{-1} \mapsto 5$,

$\kappa : \mathcal{T}^\infty \to \mathbf{N}$, $\kappa\alpha = \sum_{i=0}^{n-1} 10^i\kappa(\alpha(i))$, $n = l(\alpha)$,

$\alpha < \beta$ for $\alpha, \beta \in \mathcal{T}^\infty \iff \kappa\alpha < \kappa\beta$,

$\varphi_i : \mathcal{T}^\infty \to \mathcal{T}^\infty$, $(\varphi_i\alpha)(j) = \varphi_i(\alpha(j))$ for all $i \in [0,6)$ and $j \in [0,n)$, $n = l(\alpha)$, where $\varphi_i : \mathcal{T} \to \mathcal{T}$ is given in Table 1,

$\mathcal{P}_1 -$ be the group generated by $\{\varphi_i : \mathcal{T}^\infty \to \mathcal{T}^\infty,\ i \in [0,6)\}$,

$\mathcal{P}-$ the group generated by the set $\mathcal{P}_1 \cup \{\rho\sigma, \omega\}$,
(these maps are defined upon $\mathcal{T}^\infty$),

$\varphi_{i+6} : \mathcal{T}^\infty \to \mathcal{T}^\infty$, $\varphi_{i+6} = \sigma\rho\varphi_i$ for all $i \in [0,6)$, where
$\varphi_i : \mathcal{T} \to \mathcal{T}$ is given in Table 1,

$C(i,j,k)(\alpha) = \kappa\big((\omega^k\varphi_i\alpha)(j)\big)$ for all $i \in [0,12)$, $j,k \in [0,n)$, $n = l(\alpha)$.

Lemma 1.1. *Let* $\alpha \in \mathcal{T}^\infty$, $n = l(\alpha)$ *and let* $j,k \in [0,n)$. *Then the following relations hold*

(i) $\sigma^2 = \rho^2 = 1$, $\omega\sigma\omega = \sigma$, $\omega^{-1}(\alpha) = \omega^{n-1}(\alpha)$, $\omega^k(\alpha) = \omega^t(\alpha)$ *if* $k \equiv t (mod\, n)$,

(ii) *every two elements of the set* $\{\omega, \sigma, \rho, \varphi_2, \varphi_4\}$ *commute, besides* ω, σ *and* φ_2, φ_4,

(iii) $\mathcal{P}_1 = \{\varphi_i : i \in [0,12)\}$; $\mathcal{P}_1$ *is generated by* $\{\varphi_2,\ \varphi_4\}$,

(iv) $\mathcal{P} = \{\omega^k\varphi_i : k \in \mathbf{N},\ i \in [0,12)\}$,

(v) $C(i,j,0)(\alpha) = i + (-1)^i\kappa\alpha(j) (mod\, 6)$ *for* $i = 0,1$,

(vi) $C(i,j,0))(\alpha) = 1 - i + (-1)^{i+1}\kappa\alpha(j)(mod\, 6)$ *for* $i = 2,3,4,5$,

(vii) $C(i+6,j,0)(\alpha) = C(i,n-1-j,0) + 3 (mod\, 6)$ *for* $i \in [0,6)$,

$(viii)$ $C(i,j,k)(\alpha) = C(i,(j-k)(mod\, 6),0)(\kappa\alpha)$ *for* $i \in [0,12)$.

Proof. (i) $\omega\sigma\omega\alpha(j) = \omega\sigma\big(\alpha(j+1)\big) = \omega\alpha(n-1-j-1) = \alpha(n-1-j) = \sigma\alpha(j)$. The rest of the proof is straightforward when we use Table 1 – Table 3 . □

Definition 1.2. $\alpha, \beta \in \mathcal{T}^\infty$ are called *parastrophic equivalent* if there exists $\varphi \in \mathcal{P}$ such that $\varphi(\alpha) = \beta$.

Obviously the parastrophic equivalence is an equivalence relation; by $[\alpha]$ it will be denoted the class of the relation that comprises α. With respect to (iv) we have

$$[\alpha] = \{\, \omega^k\varphi_i(\alpha) : i \in [0,12),\ k \in [0,n),\ n = l(\alpha)\,\}\,,$$

and by (v) – $(viii)$

$$[\alpha] = \{\, C(i,n-1,k)(\alpha)C(i,n-2,k)(\alpha)\ldots C(i,0,k)(\alpha) : i \in [0,12)\,\}\,,$$

where $k \in [0, n)$, $n = l(\alpha)$.

In the following (by a personal computer) it will be found the lowest element of a class $[\alpha]$ for all $\alpha \in \mathcal{T}^n$ and $1 < n < 5$.

2. The parastrophic equivalence in $\mathcal{T}^2 - \mathcal{T}^4$

Theorem 2.1. *Let $n \in \{2, 3, 4\}$. Then every α in T^n is parastrophic equivalent to exactly one of the following elements*

LL	LR	LT	LL^{-1}	$(PE2)$
LLL	LLR	LLT	LLL^{-1}	$(PE3)$
LRT	LRL^{-1}	LRR^{-1}	LRT^{-1}	
LTR^{-1}	$LR^{-1}T$			
$LLLL$	$LLTR$	$LRLR$	$LTLT$	$(PE4)$
$LLLR$	$LLTT$	$LRLT$	$LTLL^{-1}$	
$LLLT$	$LLTL^{-1}$	$LRLL^{-1}$	$LTLR^{-1}$	
$LLLL^{-1}$	$LLTR^{-1}$	$LRLR^{-1}$	$LTT^{-1}L^{-1}$	
$LLRR$	$LLL^{-1}R$	$LRLT^{-1}$	$LL^{-1}LL^{-1}$	
$LLRT$	$LLL^{-1}T$	$LRTL^{-1}$	$LL^{-1}T^{-1}T$	
$LLRL^{-1}$	$LLL^{-1}L^{-1}$	$LRTR^{-1}$		
$LLRR^{-1}$	$LLR^{-1}R$	$LRL^{-1}T$		
$LLRT^{-1}$	$LLR^{-1}T$	$LRL^{-1}R^{-1}$		
	$LLT^{-1}R$	$LRL^{-1}T^{-1}$		
		$LRR^{-1}T$		
		$LRR^{-1}L^{-1}$		
		$LRT^{-1}T$		
		$LRT^{-1}L^{-1}$		
		$LRT^{-1}R^{-1}$		

In [1] V.D. Belousov defines: A primitive quasigroup $(Q, \cdot, \backslash, /)$ is a $\Pi-$*quasigroup of type* (α, β, γ) if α, β, $\gamma \in \sum(\cdot)$ and the quasigroup satisfies the identity

$$L_x^\alpha L_x^\beta L_x^\gamma \doteq 1.$$

This identity is equivalent to the identity $A_xB_xC_x \doteq 1$ for some $A, B, C \in \mathcal{T}$, $A^{-1} \neq B$, $C \neq B^{-1}$, $A \neq C^{-1}$. Therefore we can say that Q is a *quasigroup of type* ABC.

By Belousov [2], two $\Pi-$quasigroups of types ABC, DEF, respectively, are called *parastrophic equivalent* if $ABC = \varphi(DEF)$ for some $\varphi \in \mathcal{P}$; it is in the view of the definition of the parastrophic equivalence given in this paper. Thus if from 10 elements of the set $PE3$ delete LLL^{-1}, LRR^{-1}, LRL^{-1} then obtain 7 elements that determine 7 equivalence classes of the parastrophic equivalence relation listed in [2, Table 1].

If we want to determine the equivalence class of the parastrophic equivalency (for example) of the identity

$$(x/y) \setminus (y \setminus x) \doteq x \tag{5}$$

(see [2, p. 16]), then proceed like this: (5) is equivalent to

$$y \setminus x \doteq (x/y)x,$$

i.e.

$$R_x \doteq R_x L_x^{/}$$

whence by Table 1

$$T_z \doteq R_z T_z^{-1}$$

and also

$$R_z T_z^{-1} T_z^{-1} \doteq 1, \qquad \varphi_3(RT^{-1}T^{-1}) = R^{-1}LL.$$

Hence (5) is parastrophic equivalent to

$$L_x L_x R_x^{-1} \doteq 1,$$

i.e. $x \cdot xy \doteq yx$ in $(Q, /)$.

The lowest element of the set $[RT^{-1}T^{-1}]$ we can determine by a computer. Similarly we can proceed for arbitrary $ABC \in \mathcal{T}^3$; more generally, for arbitrary $x \in \mathcal{T}^n$, $n > 1$.

By a computer we can get $card(PE5) = 148$, $card(PE6) = 718$, $card(PE7) = 3441$.

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On intuitionistic fuzzy subquasigroups of quasigroups

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Abstract

In this paper, we introduce the notion of an intuitionistic fuzzy subquasigroup of a quasigroup G, and then some related properties are investigated. Characterizations of intuitionistic fuzzy subquasigroup of a quasigroup G are given.

1. Introduction

After the introduction of the concept of fuzzy sets by Zadeh [11], several researches were conducted on the generalizations of the notion of fuzzy set. The idea of "intuitionistic fuzzy set" was first published by Atanassov [1, 2], as a generalization of the notion of fuzzy set. Jun and Kim considered the intuitionistic fuzzification of near-rings [8]. In [6], Dudek introduced the notion of fuzzy subquasigroup of a quasigroup G. Fuzzy subquasigroups with respect to a norm are considered by Dudek and Jun in [7]. In this paper, we apply the concepts of intuitionistic fuzzy sets to subquasigroups of a quasigroup and introduce the notion of an intuitionistic fuzzy subquasigroup of a quasigroup, and then some related properties are investigated. Also, we discuss equivalence relations on the family of all intuitionistic fuzzy subquasigroups of a quasigroup.

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2. Preliminaries

A groupoid $(G,\cdot)$ is called a *quasigroup* if each of the equations $ax=b$, $xa=b$ has a unique solution for any $a,b\in G$. A quasigroup $(G,\cdot)$ may be also defined as an algebra $(G,\cdot,\backslash,/)$ with the three binary operations $\cdot$, $\backslash$, $/$ satisfying the identies

$$(xy)/y=x,\quad x\backslash(xy)=y,\quad (x/y)y=x \quad\text{and}\quad x(x\backslash y)=y.$$

We say also that $(G,\cdot,\backslash,/)$ is an *equasigroup* (i.e., *equationally definable quasigroup*) [9] or a *primitive quasigroup* [3]. The quasigroup $(G,\cdot,\backslash,/)$ corresponds to quasigroup $(G,\cdot)$, where

$$x\backslash y=z \Longleftrightarrow xz=y \quad\text{and}\quad x/y=z\Longleftrightarrow zy=x.$$

A quasigroup is called *unipotent* if $xx=yy$ for all $x,y\in G$. These quasigroups are connected with Latin squares which have one fixed element in the diagonal (cf. [5]). Such quasigroups may be defined as quasigroups $(G,\cdot)$ with the special element θ satisfying the identity $xx=\theta$. In this case also $x\backslash\theta=x$ and $\theta/x=x$ for all $x\in G$.

A nonempty subset S of a quasigroup $\mathcal{G}=(G,\cdot,\backslash,/)$ is called a *subquasigroup* of $\mathcal{G}$ if it is closed under these three operations $\cdot$, $\backslash$, $/$, i.e., if $x*y\in S$ for all $*\in\{\cdot,\backslash,/\}$ and $x,y\in S$.

By a *fuzzy set* μ in a set G we mean a function $\mu:G\to[0,1]$. The *complement* of μ, denoted by $\overline{\mu}$, is the fuzzy set in G given by $\overline{\mu}(x)=1-\mu(x)$ for all $x\in G$.

For a unipotent quasigroup $\mathcal{G}=(G,\cdot,\backslash,/)$ and a fuzzy set μ in G, let G_μ denote the set of all elements of G such that $\mu(x)=\mu(\theta)$, i.e.,

$$G_\mu=\{x\in G:\mu(x)=\mu(\theta)\}.$$

$\mathrm{Im}(\mu)$ denote the image set of μ, $a\wedge b=\min\{a,\,b\}$, $a\vee b=\max\{a,\,b\}$.

An *intuitionistic fuzzy set* (IFS for short) of a nonempty set X is defined by Atanassov (cf. [2]) in the following way.

Definition 2.1. An *intuitionistic fuzzy set* A of a nonempty set X is an object having the form

$$A=\{(x,\mu_A(x),\gamma_A(x)):x\in X\},$$

where the functions $\mu_A:X\to[0,1]$ and $\gamma_A:X\to[0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leqslant \mu_A(x) + \gamma_A(x) \leqslant 1$ for all $x \in X$.

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$.

The concept of fuzzy subquasigroups was introduced in [6].

Definition 2.2. A fuzzy set μ in a quasigroup $\mathcal{G} = (G, \cdot, \backslash, /)$ is called a *fuzzy subquasigroup* of $\mathcal{G}$ if

$$\mu(xy) \wedge \mu(x\backslash y) \wedge \mu(x/y) \geqslant \mu(x) \wedge \mu(y)$$

for all $x, y \in G$.

It is clear that this definition is equivalent to the following.

Definition 2.3. A fuzzy set μ in a quasigroup $\mathcal{G} = (G, \cdot, \backslash, /)$ is a *fuzzy subquasigroup* of $\mathcal{G}$ if

$$\mu(x * y) \geqslant \mu(x) \wedge \mu(y)$$

for all $* \in \{\cdot, \backslash, /\}$ and all $x, y \in G$.

3. Intuitionistic fuzzy subquasigroups

In what follows let $\mathcal{G} = (G, \cdot, \backslash, /)$ denote a quasigroup, and we start by defining the notion of intuitionistic fuzzy subquasigroups.

Definition 3.1. An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in $\mathcal{G}$ is called an *intuitionistic fuzzy subquasigroup* of $\mathcal{G}$ if

(IF1) $\mu_A(x * y) \geqslant \mu_A(x) \wedge \mu_A(y)$ and $\gamma_A(x * y) \leqslant \gamma_A(x) \vee \gamma_A(y)$

hold for all $x, y \in G$.

Proposition 3.2. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subquasigroup of a quasigroup $\mathcal{G}$, then*

(i) $\mu_A(x * y) \wedge \mu_A(x) = \mu_A(x * y) \wedge \mu_A(y) = \mu_A(x) \wedge \mu_A(y)$,

(ii) $\gamma_A(x * y) \vee \gamma_A(x) = \gamma_A(x * y) \vee \gamma_A(y) = \gamma_A(x) \vee \gamma_A(y)$

for all $x, y \in G$.

Proof. (i) is by Proposition 3.4 from [6].

(ii) We first consider the case $x*y=xy$. Since $(xy)/y=x$ for all $x,y\in G$, we have

$$\begin{aligned}\gamma_A(xy)\vee\gamma_A(y)&\leqslant\big[\,\gamma_A(x)\vee\gamma_A(y)\,\big]\vee\gamma_A(y)\\&=\gamma_A(x)\vee\gamma_A(y)=\gamma_A((xy)/y)\vee\gamma_A(y)\\&\leqslant\big[\,\gamma_A(xy)\vee\gamma_A(y)\,\big]\vee\gamma_A(y)\\&=\gamma_A(xy)\vee\gamma_A(y),\end{aligned}$$

which proves that $\gamma_A(xy)\vee\gamma_A(y)=\gamma_A(x)\vee\gamma_A(y)$.

In the similar way, using the identity $x\backslash(xy)=y$, we can show that $\gamma_A(xy)\vee\gamma_A(x)=\gamma(x)\vee\gamma_A(y)$.

Next we prove that the result for the case $x*y=x\backslash y$. Since $x(x\backslash y)=y$ for all $x,y\in G$, we get

$$\begin{aligned}\gamma_A(x)\vee\gamma_A(y)&=\gamma_A(x)\vee\gamma_A\big(\,x(x\backslash y)\,\big)\\&\leqslant\gamma_A(x)\vee\big[\,\gamma_A(x)\vee\gamma_A(x\backslash y)\,\big]=\gamma_A(x)\vee\gamma_A(x\backslash y)\\&\leqslant\gamma_A(x)\vee\big[\,\gamma_A(x)\vee\gamma_A(y)\,\big]=\gamma_A(x)\vee\gamma_A(y).\end{aligned}$$

Thus $\gamma_A(x)\vee\gamma_A(x\backslash y)=\gamma_A(x)\vee\gamma_A(y)$.

Noticing that $x\backslash y=z\Longleftrightarrow xz=y$, we obtain

$$\begin{aligned}\gamma_A(x\backslash y)\vee\gamma_A(y)&=\gamma_A(z)\vee\gamma_A(xz)=\gamma_A(z)\vee\gamma_A(x)\\&=\gamma_A(x\backslash y)\vee\gamma_A(x)=\gamma_A(x)\vee\gamma_A(y).\end{aligned}$$

Finally, we should prove the result for the case $x*y=x/y$. Using the equality $(x/y)y=x$, we have

$$\begin{aligned}\gamma_A(x)\vee\gamma_A(y)&=\gamma_A\big(\,(x/y)y\,\big)\vee\gamma_A(y)\\&\geqslant\big[\,\gamma_A(x/y)\vee\gamma_A(y)\,\big]\vee\gamma_A(y)=\gamma_A(x/y)\vee\gamma_A(y)\\&\geqslant\big[\,\gamma_A(x)\vee\gamma_A(y)\,\big]\vee\gamma_A(y)=\gamma_A(x)\vee\gamma_A(y).\end{aligned}$$

It follows that $\gamma_A(x/y)\vee\gamma_A(y)=\gamma_A(x)\vee\gamma_A(y)$.

Since $x/y=z\Longleftrightarrow zy=x$ for all $x,y,z\in G$, we get

$$\begin{aligned}\gamma_A(x/y)\vee\gamma_A(x)&=\gamma_A(z)\vee\gamma_A(zy)=\gamma_A(z)\vee\gamma_A(y)\\&=\gamma_A(x/y)\vee\gamma_A(y)=\gamma_A(x)\vee\gamma_A(y).\end{aligned}$$

This completes the proof. □

Corollary 3.3. *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy subquasigroup of $\mathcal{G}$ and let $* \in \{\cdot, \backslash, /\}$. Then $\mu_A(x*y) = \mu_A(x) \wedge \mu_A(y)$ (resp. $\gamma_A(x*y) = \gamma_A(x) \vee \gamma_A(y)$) whenever $\mu_A(x) \neq \mu_A(y)$ (resp. $\gamma_A(x) \neq \gamma_A(y)$).*

Proof. Straightforward. □

Lemma 3.4. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subquasigroup of $\mathcal{G}$ and e is a left (right) neutral element of $(G, \cdot)$, then $\mu_A(e) \geqslant \mu_A(x)$ and $\gamma_A(e) \leqslant \gamma_A(x)$ for all $x \in G$.*

Proof. Indeed, if $ex = x$, then also $x/x = e$ and $\mu_A(e) = \mu_A(x/x) \geqslant \mu_A(x) \wedge \mu_A(x) = \mu_A(x)$. Similarly $\gamma_A(e) = \gamma_A(x/x) \leqslant \gamma_A(x)$. □

Lemma 3.5. *If $A = (\mu_A, \gamma_A)$ is an intuitionictic fuzzy subquasigroup of a unipotent quasigroup $\mathcal{G}$, then $\mu_A(\theta) \geqslant \mu_A(x)$ and $\gamma_A(\theta) \leqslant \gamma_A(x)$ for all $x \in G$.*

Proof. Since $xx = \theta$ for all $x \in G$, we have

$$\mu_A(\theta) = \mu_A(xx) \geqslant \mu_A(x) \wedge \mu_A(x) = \mu_A(x)$$

and

$$\gamma_A(\theta) = \gamma_A(xx) \leqslant \gamma_A(x) \vee \gamma_A(x) = \gamma_A(x)$$

for all $x \in G$. □

Theorem 3.6. *If $A = (\mu_A, \gamma)$ is an intuitionistic fuzzy subquasigroup of $\mathcal{G}$, then so is $\Box A$, where $\Box A = \{(x, \mu_A(x), 1 - \mu_A(x)) : x \in G\}$.*

Proof. It is sufficient to show that $\overline{\mu_A}$ satisfies the second condition of (IF1). For any $x, y \in G$, we have

$$\begin{aligned} \overline{\mu_A}(x*y) &= 1 - \mu_A(x*y) \leqslant 1 - \big[\, \mu_A(x) \wedge \mu_A(y) \,\big] \\ &= \big[\, 1 - \mu_A(x) \,\big] \vee \big[\, 1 - \mu_A(y) \,\big] = \overline{\mu_A}(x) \vee \overline{\mu_A}(y). \end{aligned}$$

Therefore $\Box A$ is an intuitionistic fuzzy subquasigroup of $\mathcal{G}$. □

Theorem 3.7. *Let $\mathcal{G} = (G, \cdot, \backslash, /)$ be a unipotent quasigroup. If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subquasigroup of $\mathcal{G}$, then*

$$G_\mu = \{x \in G : \mu_A(x) = \mu_A(\theta)\} \quad \text{and} \quad G_\gamma = \{x \in G : \gamma_A(x) = \gamma_A(\theta)\}$$

are subquasigroups of $\mathcal{G}$.

Proof. Obviously $G_\mu \neq \emptyset \neq G_\gamma$. Let $x, y \in G_\mu$ and $* \in \{\cdot, \backslash, /\}$. Then $\mu_A(x * y) \geqslant \mu_A(x) \wedge \mu_A(y) = \mu_A(\theta)$. Since $\mu_A(\theta) \geqslant \mu_A(z)$ for all $z \in G$, it follows that $\mu_A(x * y) = \mu_A(\theta)$, i.e., $x * y \in G_\mu$.

Similarly $x, y \in G_\gamma$ implies $\gamma_A(x*y) \leqslant \gamma_A(x) \vee \gamma_A(y) = \gamma_A(\theta)$ and so $\gamma_A(x * y) = \gamma_A(\theta)$, i.e., $x * y \in G_\gamma$. This completes the proof. □

Corollary 3.8. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subquasigroup of $\mathcal{G}$ and e is a left (right) neutral element of $(G, \cdot)$, then*

$$G_\mu = \{x \in G : \mu_A(x) = \mu_A(e)\} \quad \text{and} \quad G_\gamma = \{x \in G : \gamma_A(x) = \gamma_A(e)\}$$

are subquasigroups of $\mathcal{G}$. □

For any $\alpha \in [0, 1]$ and fuzzy set μ of G, the set

$$U(\mu; \alpha) = \{x \in G : \mu(x) \geqslant \alpha\} \quad (\text{resp. } L(\mu; \alpha) = \{x \in G : \mu(x) \leqslant \alpha\})$$

is called an *upper* (resp. *lower*) *α-level cut* of μ.

Theorem 3.9. *If $A = (\mu_A, \gamma)$ is an intuitionistic fuzzy subquasigroup of $\mathcal{G}$, then the sets $U(\mu_A; \alpha)$ and $L(\gamma_A; \alpha)$ are subquasigroups of $\mathcal{G}$ for every $\alpha \in \mathrm{Im}(\mu_A) \cap \mathrm{Im}(\gamma_A)$.*

Proof. Let $\alpha \in \mathrm{Im}(\mu_A) \cap \mathrm{Im}(\gamma_A) \subseteq [0, 1]$ and $* \in \{\cdot, \backslash, /\}$ and let $x, y \in U(\mu_A; \alpha)$. Then $\mu_A(x) \geqslant \alpha$ and $\mu_A(y) \geqslant \alpha$. It follows from the first condition of (IF1) that

$$\mu_A(x * y) \geqslant \mu_A(x) \wedge \mu_A(y) \geqslant \alpha \quad \text{so that} \quad x * y \in U(\mu_A; \alpha).$$

If $x, y \in L(\gamma_A; \alpha)$, then $\gamma_A(x) \leqslant \alpha$ and $\gamma_A(y) \leqslant \alpha$, and so

$$\gamma_A(x * y) \leqslant \gamma_A(x) \vee \gamma_A(y) \leqslant \alpha.$$

Hence we have $x * y \in L(\gamma_A; \alpha)$. Therefore $U(\mu_A; \alpha)$ and $L(\gamma_A; \alpha)$ are subquasigroups of $\mathcal{G}$. □

Theorem 3.10. *Let $A = (\mu_A, \gamma_A)$ be an* IFS *in $\mathcal{G}$ such that the nonempty sets $U(\mu_A; \alpha)$ and $L(\gamma_A; \alpha)$ are subquasigroups of $\mathcal{G}$ for all $\alpha \in [0, 1]$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subquasigroup of $\mathcal{G}$.*

Proof. Let $\alpha \in [0,1]$. Assume that $U(\mu_A;\alpha) \neq \emptyset$ and $L(\gamma_A;\alpha) \neq \emptyset$ are subquasigroups of $\mathcal{G}$. We must show that $A = (\mu_A, \gamma_A)$ satisfies the condition (IF1).

Let $* \in \{\cdot, \backslash, /\}$. If the first condition of (IF1) is false, then there exist $x_0, y_0 \in G$ such that $\mu_A(x_0 * y_0) < \mu_A(x_0) \wedge \mu_A(y_0)$. Taking

$$\alpha_0 = \frac{1}{2}\left[\, \mu_A(x_0 * y_0) + \left[\, \mu_A(x_0) \wedge \mu_A(y_0)\right]\right],$$

we have $\mu_A(x_0 * y_0) < \alpha_0 < \mu_A(x_0) \wedge \mu_A(y_0)$. It follows that x_0, y_0 are in $U(\mu_A;\alpha_0)$ but $x_0 * y_0 \notin U(\mu_A;\alpha_0)$, which is a contradiction.

Assume that the second condition of (IF1) does not hold. Then $\gamma_A(x_0 * y_0) > \gamma_A(x_0) \vee \gamma_A(y_0)$ for some $x_0, y_0 \in G$. Let

$$\beta_0 = \frac{1}{2}\left[\, \gamma_A(x_0 * y_0) + \left[\, \gamma_A(x_0) \vee \gamma_A(y_0)\,\right]\right].$$

Then $\gamma_A(x_0 * y_0) > \beta_0 > \gamma_A(x_0) \vee \gamma_A(y_0)$ and so $x_0, y_0 \in L(\gamma_A;\beta_0)$ but $x_0 * y_0 \notin L(\gamma_A;\beta_0)$. This is a contradiction.

Thus (IF1) must be satisfied. □

Theorem 3.11. *Let $\mathcal{H}$ be a subquasigroup of $\mathcal{G}$ and let $A = (\mu_A, \gamma_A)$ be an* IFS *in $\mathcal{G}$ defined by*

$$\mu_A(x) = \begin{cases} \alpha_0 & \text{if } \ x \in H, \\ \alpha_1 & \text{otherwise}, \end{cases} \qquad \gamma_A(x) = \begin{cases} \beta_0 & \text{if } \ x \in H, \\ \beta_1 & \text{otherwise}, \end{cases}$$

for all $x \in G$ and $\alpha_i, \beta_i \in [0,1]$ such that $\alpha_0 > \alpha_1$, $\beta_0 < \beta_1$ and $\alpha_i + \beta_i \leqslant 1$ for $i = 0, 1$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subquasigroup of $\mathcal{G}$ and $U(\mu_A;\alpha_0) = H = L(\gamma_A;\beta_0)$.

Proof. Let $x, y \in G$ and let $* \in \{\cdot, \backslash, /\}$. If any one of x and y does not belong to H, then

$$\mu_A(x * y) \geqslant \alpha_1 = \mu_A(x) \wedge \mu_A(y)$$

and

$$\gamma_A(x * y) \leqslant \beta_1 = \gamma_A(x) \vee \gamma_A(y).$$

Therefore $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subquasigroup of a quasigroup $\mathcal{G}$. Obviously $U(\mu_A;\alpha_0) = H = L(\gamma_A;\beta_0)$. □

Corollary 3.12. *Let χ_H be the characteristic function of a subquasigroup $\mathcal{H}$ of $\mathcal{G}$. Then $H = (\chi_H, \overline{\chi_H})$ is an intuitionistic fuzzy subquasigroup of $\mathcal{G}$.* □

Theorem 3.13. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subquasigroup of $\mathcal{G}$, then*

$$\mu_A(x) = \sup\{\alpha \in [0,1] : x \in U(\mu_A; \alpha)\}$$

and

$$\gamma_A(x) = \inf\{\alpha \in [0,1] : x \in L(\gamma_A; \alpha)\}$$

for all $x \in G$.

Proof. Let $\delta = \sup\{\alpha \in [0,1] : x \in U(\mu_A; \alpha)\}$ and let $\varepsilon > 0$ be given. Then $\delta - \varepsilon < \alpha$ for some $\alpha \in [0,1]$ such that $x \in U(\mu_A; \alpha)$. This means that $\delta - \varepsilon < \mu_A(x)$ so that $\delta \leqslant \mu_A(x)$ since ε is arbitrary.

We now show that $\mu_A(x) \leqslant \delta$. If $\mu_A(x) = \beta$, then $x \in U(\mu_A; \beta)$ and so

$$\beta \in \{\alpha \in [0,1] : x \in U(\mu_A; \alpha)\}.$$

Hence

$$\mu_A(x) = \beta \leqslant \sup\{\alpha \in [0,1] : x \in U(\mu_A; \alpha)\} = \delta.$$

Therefore

$$\mu_A(x) = \delta = \sup\{\alpha \in [0,1] : x \in U(\mu_A; \alpha)\}.$$

Now let $\eta = \inf\{\alpha \in [0,1] : x \in L(\gamma_A; \alpha)\}$. Then

$$\inf\{\alpha \in [0,1] : x \in L(\gamma_A; \alpha)\} < \eta + \varepsilon$$

for any $\varepsilon > 0$, and so $\alpha < \eta + \varepsilon$ for some $\alpha \in [0,1]$ with $x \in L(\gamma_A; \alpha)$. Since $\gamma_A(x) \leqslant \alpha$ and ε is arbitrary, it follows that $\gamma_A(x) \leqslant \eta$.

To prove $\gamma_A(x) \geqslant \eta$, let $\gamma_A(x) = \zeta$. Then $x \in L(\gamma_A; \zeta)$ and thus $\zeta \in \{\alpha \in [0,1] : x \in L(\gamma_A; \alpha)\}$. Hence

$$\inf\{\alpha \in [0,1] : x \in L(\gamma_A; \alpha)\} \leqslant \zeta,$$

i.e., $\eta \leqslant \zeta = \gamma_A(x)$. Consequently

$$\gamma_A(x) = \eta = \inf\{\alpha \in [0,1] : x \in L(\gamma_A; \alpha)\},$$

which completes the proof. □

Theorem 3.14. *Let* $\{\mathcal{H}_\alpha : \alpha \in \Lambda\}$, *where* Λ *is a nonempty subset of* $[0,1]$, *be a collection of subquasigroups of* $\mathcal{G}$ *such that*

(i) $G = \bigcup\limits_{\alpha\in\Lambda} H_\alpha$,

(ii) $\alpha > \beta \iff H_\alpha \subset H_\beta$ *for all* $\alpha, \beta \in \Lambda$.

Then an intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ *defined by*

$$\mu_A(x) = \sup\{\alpha \in \Lambda : x \in H_\alpha\} \text{ and } \gamma_A(x) = \inf\{\alpha \in \Lambda : x \in H_\alpha\}$$

for all $x \in G$ *is an intuitionistic fuzzy subguasigroup of* $\mathcal{G}$.

Proof. According to Theorem 3.10, it is sufficient to show that the nonempty sets $U(\mu_A;\alpha)$ and $L(\gamma_A;\beta)$ are subquasigroups of $\mathcal{G}$.

In order to prove that $U(\mu_A;\alpha) \neq \emptyset$ is a subquasigroup of $\mathcal{G}$, we consider the following two cases:

$$\text{(i)} \quad \alpha = \sup\{\delta \in \Lambda : \delta < \alpha\} \quad \text{and} \quad \text{(ii)} \quad \alpha \neq \sup\{\delta \in \Lambda : \delta < \alpha\}.$$

Case (i) implies that

$$x \in U(\mu_A;\alpha) \iff \big(x \in H_\delta \ for\ all\ \delta < \alpha\big) \iff x \in \bigcap_{\delta<\alpha} H_\delta\,,$$

so that $U(\mu_A;\alpha) = \bigcap\limits_{\delta<\alpha} H_\delta$ which is a subquasigroup of $\mathcal{G}$.

For the case (ii), we claim that $U(\mu_A;\alpha) = \bigcup\limits_{\delta\geqslant\alpha} H_\delta$. If $x \in \bigcup\limits_{\delta\geqslant\alpha} H_\delta$ then $x \in H_\delta$ for some $\delta \geqslant \alpha$. It follows that $\mu_A(x) \geqslant \delta \geqslant \alpha$, so that $x \in U(\mu_A;\alpha)$. This shows that $\bigcup\limits_{\delta\geqslant\alpha} H_\delta \subseteq U(\mu_A;\alpha)$.

Now assume that $x \notin \bigcup\limits_{\delta\geqslant\alpha} H_\delta$. Then $x \notin H_\delta$ for all $\delta \geqslant \alpha$. Since $\alpha \neq \sup\{\delta \in \Lambda : \delta < \alpha\}$, there exists $\varepsilon > 0$ such that $(\alpha-\varepsilon, \alpha) \cap \Lambda = \emptyset$. Hence $x \notin H_\delta$ for all $\delta > \alpha - \varepsilon$, which means that if $x \in H_\delta$ then $\delta \leqslant \alpha - \varepsilon$. Thus $\mu_A(x) \leqslant \alpha - \varepsilon < \alpha$, and so $x \notin U(\mu_A;\alpha)$. Therefore $U(\mu_A;\alpha) \subseteq \bigcup\limits_{\delta\geqslant\alpha} H_\delta$, and thus $U(\mu_A;\alpha) = \bigcup\limits_{\delta\geqslant\alpha} H_\delta$, which is a subquasigroup of $\mathcal{G}$.

Now we prove that $L(\gamma_A;\beta)$ is a subquasigroup of $\mathcal{G}$. We consider the following two cases:

$$\text{(iii)} \quad \beta = \inf\{\eta \in \Lambda : \beta < \eta\} \quad \text{and} \quad \text{(iv)} \quad \beta \neq \inf\{\eta \in \Lambda : \beta < \eta\}.$$

For the case (iii) we have

$$x \in L(\gamma_A;\beta) \Longleftrightarrow \big(x \in H_\eta \ \ for\ all\ \eta > \beta\big) \Longleftrightarrow x \in \bigcap_{\eta>\beta} H_\eta$$

and hence $L(\gamma_A;\beta) = \bigcap\limits_{\eta>\beta} H_\eta$ which is a subquasigroup of $\mathcal{G}$.

For the case (iv), there exists $\varepsilon > 0$ such that $(\beta, \beta+\varepsilon) \cap \Lambda = \emptyset$. We will show that $L(\gamma_A;\beta) = \bigcup\limits_{\eta\leqslant\beta} H_\eta$. If $x \in \bigcup\limits_{\eta\leqslant\beta} H_\eta$ then $x \in H_\eta$ for some $\eta \leqslant \beta$. It follows that $\gamma_A(x) \leqslant \eta \leqslant \beta$ so that $x \in L(\gamma_A;\beta)$. Hence $\bigcup\limits_{\eta\leqslant\beta} H_\eta \subseteq L(\gamma_A;\beta)$.

Conversely, if $x \notin \bigcup\limits_{\eta\leqslant\beta} H_\eta$ then $x \notin H_\eta$ for all $\eta \leqslant \beta$, which implies that $x \notin H_\eta$ for all $\eta < \beta+\varepsilon$, i.e., if $x \in H_\eta$ then $\eta \geqslant \beta+\varepsilon$. Thus $\gamma_A(x) \geqslant \beta+\varepsilon > \beta$, i.e., $x \notin L(\gamma_A;\beta)$. Therefore $L(\gamma_A;\beta) \subseteq \bigcup\limits_{\eta\leqslant\beta} H_\eta$ and consequently $L(\gamma_A;\beta) = \bigcup\limits_{\eta\leqslant\beta} H_\eta$ which is a subquasigroup of $\mathcal{G}$. This completes the proof. □

Theorem 3.15. $A = (\mu_A, \gamma_A)$ *is an intuitionistic fuzzy subquasigroup of* $\mathcal{G}$ *iff* μ_A *and* $\overline{\gamma_A}$ *are fuzzy subquasigroups of* $\mathcal{G}$.

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy subquasigroup of $\mathcal{G}$. Then clearly μ_A is a fuzzy subquasigroup of $\mathcal{G}$. Let $x, y \in G$ and $* \in \{\cdot, \backslash, /\}$. Then

$$\begin{aligned}\overline{\gamma_A}(x*y) &= 1-\gamma_A(x*y) \geqslant 1-\big[\,\gamma_A(x)\vee\gamma_A(y)\,\big]\\ &= \big[\,1-\gamma_A(x)\,\big]\wedge\big[\,1-\gamma_A(y)\,\big] = \overline{\gamma_A}(x)\wedge\overline{\gamma_A}(y)\,.\end{aligned}$$

Hence $\overline{\gamma_A}$ is a fuzzy subquasigroup of $\mathcal{G}$.

Conversely suppose that μ_A and $\overline{\gamma_A}$ are fuzzy subquasigroups of $\mathcal{G}$. If $x, y \in G$ and $* \in \{\cdot, \backslash, /\}$, then

$$\begin{aligned}1-\gamma_A(x*y) &= \overline{\gamma_A}(x*y) \geqslant \overline{\gamma_A}(x)\wedge\overline{\gamma_A}(y)\\ &= \big[\,1-\gamma_A(x)\,\big]\wedge\big[\,1-\gamma_A(y)\,\big]\\ &= 1-\big[\,\gamma_A(x)\vee\gamma_A(y)\,\big]\,,\end{aligned}$$

which proves $\gamma_A(x*y) \leqslant \gamma_A(x)\vee\gamma_A(y)$. This completes the proof. □

If $\mathcal{H}$ is a subquasigroup of $\mathcal{G}$, then $H = (\chi_H, \overline{\chi_H})$ is an intuitionistic fuzzy subquasigroup of $\mathcal{G}$ from Corollary 3.12, where χ_H is the characteristic function of H.

Let $IFS(\mathcal{G})$ be the family of all intuitionistic fuzzy subquasigroups of $\mathcal{G}$ and $\alpha \in [0,1]$ be a fixed real number. For any $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ from $IFS(\mathcal{G})$ we define two binary relations $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$ on $IFS(\mathcal{G})$ as follows:

$$(A,B) \in \mathfrak{U}^\alpha \Longleftrightarrow U(\mu_A;\alpha) = U(\mu_B;\alpha)$$

and

$$(A,B) \in \mathfrak{L}^\alpha \Longleftrightarrow L(\gamma_A;\alpha) = L(\gamma_B;\alpha)\,.$$

These two relations $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$ are equivalence relations, give rise to partitions of $IFS(\mathcal{G})$ into the equivalence classes of $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$, denoted by $[A]_{\mathfrak{U}^\alpha}$ and $[A]_{\mathfrak{L}^\alpha}$ for any $A = (\mu_A, \gamma_A) \in IFS(\mathcal{G})$, respectively. And we will denote the quotient sets of $IFS(\mathcal{G})$ by $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$ as $IFS(\mathcal{G})/\mathfrak{U}^\alpha$ and $IFS(\mathcal{G})/\mathfrak{L}^\alpha$, respectively.

If $\mathcal{S}(\mathcal{G})$ is the family of all subquasigroups of $\mathcal{G}$ and $\alpha \in [0,1]$, then we define two maps U_α and L_α from $IFS(\mathcal{G})$ to $\mathcal{S}(\mathcal{G}) \cup \{\emptyset\}$ as follows:

$$U_\alpha(A) = U(\mu_A;\alpha) \quad \text{and} \quad L_\alpha(A) = L(\gamma_A;\alpha),$$

respectively, for each $A = (\mu_A, \gamma_A) \in IFS(\mathcal{G})$. Then the maps U_α and L_α are well-defined.

Theorem 3.16. *For any $\alpha \in (0,1)$, the maps U_α and L_α are surjective from $IFS(\mathcal{G})$ onto $\mathcal{S}(\mathcal{G}) \cup \{\emptyset\}$.*

Proof. Let $\alpha \in (0,1)$. Note that $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1})$ is in $IFS(\mathcal{G})$, where $\mathbf{0}$ and $\mathbf{1}$ are fuzzy sets in $\mathcal{G}$ defined by $\mathbf{0}(x) = 0$ and $\mathbf{1}(x) = 1$ for all $x \in G$. Obviously, $U_\alpha(\mathbf{0}_\sim) = L_\alpha(\mathbf{0}_\sim) = \emptyset$. If $\mathcal{H}$ is a subquasigroup of $\mathcal{G}$, then for the intuitionistic fuzzy subquasigroup $H = (\chi_H, \overline{\chi_H})$, $U_\alpha(H) = U(\chi_H;\alpha) = H$ and $L_\alpha(H) = L(\overline{\chi_H};\alpha) = H$. Hence U_α and L_α are surjective. □

Theorem 3.17. *The quotient sets $IFS(\mathcal{G})/\mathfrak{U}^\alpha$ and $IFS(\mathcal{G})/\mathfrak{L}^\alpha$ are equipotent to $\mathcal{S}(\mathcal{G}) \cup \{\emptyset\}$ for any $\alpha \in (0,1)$.*

Proof. Let $\alpha \in (0,1)$ and let $\overline{U_\alpha} : IFS(\mathcal{G})/\mathfrak{U}^\alpha \longrightarrow \mathcal{S}(\mathcal{G}) \cup \{\emptyset\}$ and $\overline{L_\alpha} : IFS(\mathcal{G})/\mathfrak{L}^\alpha \longrightarrow \mathcal{S}(\mathcal{G}) \cup \{\emptyset\}$ be the maps defined by

$$\overline{U_\alpha}([A]_{\mathfrak{U}^\alpha}) = U_\alpha(A) \quad \text{and} \quad \overline{L_\alpha}([A]_{\mathfrak{L}^\alpha}) = L_\alpha(A),$$

respectively, for each $A = (\mu_A, \gamma_A) \in IFS(\mathcal{G})$.

If $U(\mu_A;\alpha) = U(\mu_B;\alpha)$ and $L(\gamma_A;\alpha) = L(\gamma_B;\alpha)$ for $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ from $IFS(\mathcal{G})$, then $(A,B) \in \mathfrak{U}^\alpha$ and $(A,B) \in \mathfrak{L}^\alpha$, whence $[A]_{\mathfrak{U}^\alpha} = [B]_{\mathfrak{U}^\alpha}$ and $[A]_{\mathfrak{L}^\alpha} = [B]_{\mathfrak{L}^\alpha}$. Hence the maps $\overline{U_\alpha}$ and $\overline{L_\alpha}$ are injective.

To show that the maps $\overline{U_\alpha}$ and $\overline{L_\alpha}$ are surjective, let $\mathcal{H}$ be a subquasigroup of $\mathcal{G}$. Then for $H = (\chi_H, \overline{\chi_H}) \in IFS(\mathcal{G})$ we have $\overline{U_\alpha}([H]_{\mathfrak{U}^\alpha}) = U(\chi_H;\alpha) = H$ and $\overline{L_\alpha}([H]_{\mathfrak{L}^\alpha}) = L(\overline{\chi_H};\alpha) = H$. Also $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in IFS(\mathcal{G})$. Moreover $\overline{U_\alpha}([\mathbf{0}_\sim]_{\mathfrak{U}^\alpha}) = U(\mathbf{0};\alpha) = \emptyset$ and $\overline{L_\alpha}([\mathbf{0}_\sim]_{\mathfrak{L}^\alpha}) = L(\mathbf{1};\alpha) = \emptyset$. Hence $\overline{U_\alpha}$ and $\overline{L_\alpha}$ are surjective. □

For any $\alpha \in [0,1]$, we define another relation $\mathfrak{R}^\alpha$ on $IFS(\mathcal{G})$ as following:

$$(A,B) \in \mathfrak{R}^\alpha \Longleftrightarrow U(\mu_A;\alpha) \cap L(\gamma_A;\alpha) = U(\mu_B;\alpha) \cap L(\gamma_B;\alpha)$$

for any $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ from $IFS(\mathcal{G})$. Then the relation $\mathfrak{R}^\alpha$ is also an equivalence relation on $IFS(\mathcal{G})$.

Theorem 3.18. *For any $\alpha \in (0,1)$ and any $A = (\mu_A, \gamma_A) \in IFS(\mathcal{G})$ the map $I_\alpha : IFS(\mathcal{G}) \longrightarrow \mathcal{S}(\mathcal{G}) \cup \{\emptyset\}$ defined by*

$$I_\alpha(A) = U_\alpha(A) \cap L_\alpha(A)$$

is suriective.

Proof. Indeed, if $\alpha \in (0,1)$ is fixed, then for $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in IFS(\mathcal{G})$ we have

$$I_\alpha(\mathbf{0}_\sim) = U_\alpha(\mathbf{0}_\sim) \cap L_\alpha(\mathbf{0}_\sim) = U(\mathbf{0};\alpha) \cap L(\mathbf{1};\alpha) = \emptyset\,,$$

and for any $\mathcal{H} \in \mathcal{S}(\mathcal{G})$, there exists $H = (\chi_H, \overline{\chi_H}) \in IFS(\mathcal{G})$ such that $I_\alpha(H) = U(\chi_H;\alpha) \cap L(\overline{\chi_H};\alpha) = H$. □

Theorem 3.19. *For any* $\alpha \in (0,1)$*, the quotient set* $IFS(\mathcal{G})/\mathfrak{R}^{\alpha}$ *is equipotent to* $\mathcal{S}(\mathcal{G}) \cup \{\emptyset\}$*.*

Proof. Let $\alpha \in (0,1)$ and let $\overline{I_\alpha} : IFS(\mathcal{G})/\mathfrak{R}^{\alpha} \longrightarrow \mathcal{S}(\mathcal{G}) \cup \{\emptyset\}$ be a map defined by

$$\overline{I_\alpha}([A]_{\mathfrak{R}^\alpha}) = I_\alpha(A) \quad \text{ for each } \quad [A]_{\mathfrak{R}^\alpha} \in IFS(\mathcal{G})/\mathfrak{R}^{\alpha}.$$

If $\overline{I_\alpha}([A]_{\mathfrak{R}^\alpha}) = \overline{I_\alpha}([B]_{\mathfrak{R}^\alpha})$ for any $[A]_{\mathfrak{R}^\alpha}, [B]_{\mathfrak{R}^\alpha} \in IFS(\mathcal{G})/\mathfrak{R}^{\alpha}$, then

$$U(\mu_A;\alpha) \cap L(\gamma_A;\alpha) = U(\mu_B;\alpha) \cap L(\gamma_B;\alpha),$$

hence $(A,B) \in \mathfrak{R}^\alpha$ and $[A]_{\mathfrak{R}^\alpha} = [B]_{\mathfrak{R}^\alpha}$. It follows that $\overline{I_\alpha}$ is injective.

For $\mathbf{0}_\sim = (\mathbf{0},\mathbf{1}) \in IFS(\mathcal{G})$ we have $\overline{I_\alpha}(\mathbf{0}_\sim) = I_\alpha(\mathbf{0}_\sim) = \emptyset$. If $H \in \mathcal{S}(\mathcal{G})$, then for $H = (\chi_H, \overline{\chi_H}) \in IFS(\mathcal{G})$, $\overline{I_\alpha}(H) = I_\alpha(H) = H$. Hence $\overline{I_\alpha}$ is a bijective map. □

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On the classes of algebras reciprocally closed under direct products

O. U. Kirnasovsky

Abstract

The class K of algebras with the property that two algebras belongs to K iff their direct product belongs to K is studied.

The class K of algebras with the property that two algebras belongs to K iff their direct product belongs to K is called *reciprocally closed under direct products.* The formula Φ is *reciprocally preserved under direct products* if the class of algebras satisfying Φ is reciprocally closed under direct products (cf. [1]).

Three following assertions are evident.

Proposition 1. *A class of algebras closed under direct products and homomorphisms is reciprocally closed under direct products.*

Proposition 2. *A class of idempotent algebras closed under direct products and subalgebras is reciprocally closed under direct products.*

Proposition 3. *The conjunction of formulas of a fixed signature, which are reciprocally preserved under direct products, is reciprocally preserved under direct products. Similarly, the intersection of classes of algebras reciprocally closed under direct products is a class of algebras reciprocally closed under direct products.*

In this paper by a *groupoid* we mean an algebra (Q, f) with one (binary or n-ary) operation f. A groupoid (Q, f) in which for all

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$1 \leqslant i \leqslant n$ and $a_i \in Q$ the equation

$$f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n) = a_i$$

has a unique solution $x_i \in Q$ (denoted by $f^i(a_1, \ldots, a_n)$) is called a *quasigroup*. A *loop* is a quasigroup with a neutral element; a *semigroup* - an associative groupoid; a *group* - an associative quasigroup.

A formula Φ of the signature Ω is called *conjunctively-positive* iff its record has no predicate letter, except the symbols of equality, no logical connective, except the symbols of conjunction, and no term, except terms of the signature Ω.

A formula Φ is *prenex almost conjunctively-positive formula of a signature* Ω, iff all quantifiers and symbols "$\exists!$" in its shortened record, obtained only by reductions to the symbols "$\exists!$", precede the quantifier-free part, and the shortened record has no predicate letter, except the symbols of equality, no logical connective, except the symbols of conjunction, and no term, except terms of the signature Ω. Obviously, prenex normal form of a conjunctively-positive formula of a signature Ω is a prenex almost conjunctively-positive formula of the signature Ω.

Lemma 4. *Every prenex almost conjunctively-positive formula is reciprocally preserved under direct products.*

Proof. The given formula is equivalent to a closed formula of the form

$$(Q_1 x_1) \ldots (Q_k x_k)(w_1 = w_2 \,\&\, \ldots \,\&\, w_{2m-1} = w_{2m}), \tag{1}$$

where $Q_1, ..., Q_k$ are quantifiers $\forall$, $\exists$ and symbols "$\exists!$", and $w_1, ..., w_{2m}$ are terms of the signature of the given formula. The formula (1) has the signature of algebras of some type. Fix arbitrary algebras $\langle G, \Omega_1 \rangle$ and $\langle H, \Omega_2 \rangle$ of the type. Denote the direct product of the first of them by the second of them by $\langle M, \Omega \rangle$. Validity of the formula (1) in the algebra $\langle M, \Omega \rangle$ is equivalent to the formula

$$(Q_1 \langle y_1, z_1 \rangle \in M) \ldots (Q_k \langle y_k, z_k \rangle \in M)\, P(\langle y_1, z_1 \rangle, \ldots, \langle y_k, z_k \rangle), \tag{2}$$

where $P(x_1, \ldots, x_k)$ is the quantifier-free part of the formula (1). Next, the formula (2) is equivalent to the formula

$$\begin{aligned}(Q_1 y_1 \in G, z_1 \in H) \ldots (Q_k y_k \in G, z_k \in H)\, (P'(y_1, \ldots, y_k) \,\&\, \\ \&\, P''(z_1, \ldots, z_k)),\end{aligned} \tag{3}$$

where P' and P'' are formulas obtained from P by the way of the replacement of every propositional variable x_i respectively with y_i and z_i and of every functional variable f of the signature Ω with the respective functional variable (f_1 of the signature Ω_1 and f_2 of the signature Ω_2 respectively). At last, formula (3) and, therefore, formula (2), are equivalent to the formula

$$((Q_1 y_1 \in G) \ldots (Q_k y_k \in G)\, P'(y_1, \ldots, y_k))\,\& \\ \&\,((Q_1 z_1 \in H) \ldots (Q_k z_k \in H)\, P''(z_1, \ldots, z_k)),$$

that is equivalent to simultaneous validity of the formula (1) in both $\langle G, \Omega_1 \rangle$ and $\langle H, \Omega_2 \rangle$ algebras. □

Corollary 5. *Every conjunctively-positive formula is reciprocally preserved under direct products.*

Corollary 6. *The class of all quasigroups (of all groups, of all semigroups, of all monoids, of all loops) is reciprocally closed under direct products.*

As it is well known, the *direct product* $\rho \times \tau$ *of binary relations* ρ *and* τ is defined as the relation

$$\langle a, b \rangle (\rho \times \tau) \langle c, d \rangle \Longleftrightarrow (a\,\rho\,c)\,\&\,(b\,\tau\,d).$$

It is clear, that for mappings f and g the relation $f \times g$ is a mapping with the domain equal to the Cartesian product of the domains of the mappings f and g and $(f \times g)(\langle x, y \rangle) = \langle f(x), g(y) \rangle$.

A groupoid (G, g) is called an *isotope of a binary semigroup* $(Q, +)$ iff there exists a collection $\langle \alpha_1, \ldots, \alpha_n, \alpha \rangle$ of bijections from the set G onto the set Q satisfying the identity

$$\alpha g(x_1, \ldots, x_n) = \alpha_1 x_1 + \cdots + \alpha_n x_n. \qquad (4)$$

An isotope of a group is called also a *group isotope.* It is easy to see that an isotope of a group is a quasigroup. A transformation α of a set Q is called a *linear transformation of a group* $(Q, +)$ if there exist an endomorphism θ and a right translation R_c of this group such that $\alpha = R_c \theta$. An isotope of a group $(Q, +)$ defined by (4) is called *i-linear* if the bijections α_i and α are linear transformations of

$(Q, +)$. An isotope is *linear* if it is i-linear for all i. Obviously, every groupoid isomorphic to a linear or i-linear group isotope is a linear or, respectively, i-linear group isotope.

Lemma 7. *The direct product of an isotope (A, g) of a semigroup $(G, +)$ by an isotope (B, h) of a semigroup $(H, \cdot)$ defined by (4) and*

$$\beta h(x_1, \dots, x_n) = \beta_1 x_1 \cdot \ldots \cdot \beta_n x_n$$

is an isotope (C, f) of the semigroup $(M, \circ)$ determined by

$$(M, \circ) = (G \times H, \circ) = (G, +) \times (H, \cdot)$$

and by

$$(\alpha \times \beta) f(x_1, \dots, x_n) = (\alpha_1 \times \beta_1) x_1 \circ \ldots \circ (\alpha_n \times \beta_n) x_n,$$

where $\alpha_1, \dots, \alpha_n$ and α are bijections from A onto G, and $\beta_1, \dots, \beta_n$ and β are bijections from B onto H.

Proof. Indeed, let f be the operation of the given direct product of the isotopes of the semigroups. Then

$$\begin{aligned}
&(\alpha \times \beta) f(\langle x_1, y_1 \rangle, \dots, \langle x_n, y_n \rangle) \\
&\qquad = (\alpha \times \beta)(\langle g(x_1, \dots, x_n), h(y_1, \dots, y_n) \rangle) \\
&\qquad = \langle \alpha g(x_1, \dots, x_n), \beta h(y_1, \dots, y_n) \rangle \\
&\qquad = \langle \alpha_1 x_1 + \ldots + \alpha_n x_n, \beta_1 y_1 \cdot \ldots \cdot \beta_n y_n \rangle \\
&\qquad = \langle \alpha_1 x_1, \beta_1 y_1 \rangle \circ \ldots \circ \langle \alpha_n x_n, \beta_n y_n \rangle \\
&\qquad = (\alpha_1 \times \beta_1) \langle x_1, y_1 \rangle \circ \ldots \circ (\alpha_n \times \beta_n) \langle x_n, y_n \rangle,
\end{aligned}$$

which completes the proof. □

If (Q, f) is a quasigroup of an arity $n \geqslant 2$ then $(Q, f, f^1, \dots, f^n)$ is called the *primitive quasigroup* which corresponds to the quasigroup (Q, f). Such quasigroup maybe defined as an algebra $(Q, f, f^1, \dots, f^n)$ with $n+1$ n-ary operations satisfying $2n$ identities:

$$\begin{aligned}
f(x_1, \dots, x_{i-1}, f^i(x_1, \dots, x_n), x_{i+1}, \dots, x_n) &= x_i, \\
f^i(x_1, \dots, x_{i-1}, f(x_1, \dots, x_n), x_{i+1}, \dots, x_n) &= x_i.
\end{aligned}$$

A congruence on a quasigroup (Q, f) is called *normal* if it is a congruence on the corresponding primitive quasigroup.

Lemma 8. *The homomorphic image of a group isotope, where the congruence which corresponds to the respective homomorphism is normal, is a group isotope.*

Proof. Let (Q, f) be the given group isotope, φ the given homomorphism of the group isotope (Q, f) onto a groupoid (G, g), and π the respective normal congruence on (Q, f). Then π is a congruence on the primitive quasigroup $(Q, f, f^1, \ldots, f^n)$. Denote the respective natural homomorphism by ψ. From [2] it follows that the class of all n-ary group isotopes is a variety of quasigroups. Therefore, the class of all primitive quasigroups which correspond to n-ary group isotopes is closed under homomorphisms, whence ψ is a homomorphism of the group isotope (Q, f) onto some group isotope $(Q/\pi, h)$. Hence (G, g) is a group isotope. □

Lemma 9. *A homomorphism of a quasigroup (Q, f) into a quasigroup (G, g) is a homomorphism of a quasigroup $(Q, f, f^1, \ldots, f^n)$ into a quasigroup $(G, g, g^1, \ldots, g^n)$.*

Proof. Denote the given homomorphism by φ. Let $a_1, \ldots, a_n$ be arbitrary elements from Q, and i be a natural number not greater than n. If $b_i = f^i(a_1, \ldots, a_n)$, then

$$\varphi a_i = \varphi f(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_n) = g(\varphi a_1, \ldots, \varphi a_{i-1}, \varphi b_i, \varphi a_{i+1}, \ldots, \varphi a_n),$$

whence, it follows that

$$\varphi f^i(a_1, \ldots, a_n) = \varphi b_i = g^i(\varphi a_1, \ldots, \varphi a_n)$$

for all $a_1, \ldots, a_n \in Q$ and all $1 \leqslant i \leqslant n$. Thus we have the identity

$$\varphi f^y(x_1, \ldots, x_n) = g^y(\varphi x_1, \ldots, \varphi x_n).$$

This completes the proof. □

Corollary 10. *The congruence which corresponds to a homomorphism of a quasigroup into a quasigroup is normal.*

Corollary 11. *If there exists a homomorphism φ of a group isotope into a quasigroup (Q, f), then the groupoid $(Im\varphi, f)$ is a group isotope.*

Proof. It is enough to add the statement of Lemma 8 to the statement of Corollary 10. □

Example 14. Let $(Q,+)$ be an arbitrary infinite group. Since the sets Q and Q^2 have the same cardinal number, then there exists a bijection f of Q^2 onto Q. Let $(Q^3,*)$ be the direct product $(Q,+)\times(Q,+)\times(Q,+)$ and let (Q^3,g) be the isotope of the group $(Q^3,*)$ defined by the identity

$$g(x_1,\ldots,x_n)=\alpha x_1*\ldots*\alpha x_n,$$

where $n\geqslant 2$ is an arbitrary fixed number and α is a substitution of Q^3 defined by the identity

$$\alpha(\langle x,y,z\rangle)=\langle f^{-1}(x),f(y,z)\rangle.$$

Let φ be a mapping $\varphi: Q^3\to Q^2$ such that

$$\varphi:\langle x,y,z\rangle\mapsto\langle x,y\rangle,$$

and let h be the operation of the arity $n\geqslant 2$ defined on Q^2 by the formula

$$h(\langle x_1,y_1\rangle,\ldots,\langle x_n,y_n\rangle)=\varphi g(\langle x_1,y_1,z_1\rangle,\ldots,\langle x_n,y_n,z_n\rangle),$$

where $z_1,\ldots,z_n\in Q$ are arbitrary.

The operation h is not dependent on that choice of $z_1,\ldots,z_n\in Q$, since for the direct product $(Q^2,\star)$ of the group $(Q,+)$ we have

$$\begin{aligned}\varphi g(\langle x_1,y_1,z_1\rangle,\ldots,\langle x_n,y_n,z_n\rangle)\\ &=\varphi(\alpha(\langle x_1,y_1,z_1\rangle)*\ldots*\alpha(\langle x_n,y_n,z_n\rangle))\\ &=\varphi(\langle f^{-1}(x_1),f(y_1,z_1)\rangle*\ldots*\langle f^{-1}(x_n),f(y_n,z_n)\rangle)\\ &=\varphi(\langle f^{-1}(x_1)\star\ldots\star f^{-1}(x_n),f(y_1,z_1)+\ldots+f(y_n,z_n)\rangle)\\ &=f^{-1}(x_1)\star\ldots\star f^{-1}(x_n).\end{aligned}$$

Moreover, from these equalities it follows that the operation h is not a quasigroup one, since all divisions are multivalued. But the identity

$$h(\varphi x_1,\ldots,\varphi x_n)=\varphi g(x_1,\ldots,x_n)$$

holds. Thus, φ is a homomorphism of the group isotope (Q^3, g) onto the groupoid (Q^2, h), which is not even a quasigroup. The congruence corresponding to it by Lemma 8 is not normal. □

Theorem 13. *The class of all group isotopes is reciprocally closed under direct products.*

Proof. By Lemma 7 the direct product of two group isotopes of the same arity is a group isotope. Let the direct product (M, f) of a groupoid (G, g) by a groupoid (H, h) be a group isotope. By Corollary 6 the groupoids (G, g) and (H, h) are quasigroups. It is easy to see that the mappings φ_1 and φ_2 from the group isotope (M, f) into the quasigroups (G, g) and (H, h) respectively, for which

$$(\forall x \in G)(\forall y \in H)(\varphi_1(\langle x, y\rangle) = x \;\&\; \varphi_2(\langle x, y\rangle) = y),$$

are homomorphisms of the group isotope (M, f) onto the quasigroups (G, g) and (H, h), respectively. By Corollary 11 these two quasigroups are group isotopes. □

Theorem 14. *The class of all i-linear n-ary group isotopes, where i and n are fixed numbers, is reciprocally closed under direct products.*

Proof. By Lemma 7 the direct product of two i-linear n-ary group isotopes is an i-linear group isotope. Let the direct product (M, f) of a groupoid (G, g) by a groupoid (H, h) be i-linear n-ary group isotope. By Theorem 13 (G, g) and (H, h) are group isotopes. The repeated application of Lemma 7 gives i-linearity of these group isotopes. □

Corollary 15. *The class of all linear group isotopes is reciprocally closed under direct products.*

In spite of the collection of the above results and the results of Horn from [1] which describe the structure of the classes of algebras reciprocally closed under direct products, the question about criterion for a class of algebras to be reciprocally closed under direct products, or, at least, for a formula to be reciprocally preserved under direct products, remains open.

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Squares in quadratical quasigroups

Vladimir Volenec

Abstract

"Geometrical" concept of square is defined and investigated in any quadratical quasigroup.

A groupoid $(Q,\cdot)$ is said to be *quadratical* if the identity

$$ab \cdot a = ca \cdot bc \tag{1}$$

holds and the equation $ax = b$ has a unique solution $x \in Q$ for any $a, b \in Q$ (cf. [10] and [3]). Every quadratical groupoid $(Q,\cdot)$ is a quasigroup, i.e. the equation $xa = b$ has a unique solution $x \in Q$ for any $a, b \in Q$. In a quadratical quasigroup $(Q,\cdot)$ the identities

$$\begin{aligned} aa &= a \qquad \text{(idempotency)}, &(2)\\ a \cdot ba &= ab \cdot a \qquad \text{(elasticity)}, &(3)\\ ab \cdot a &= ba \cdot b, &(4)\\ ab \cdot cd &= ac \cdot bd \qquad \text{(mediality)} &(5) \end{aligned}$$

and the equivalency

$$ab = c \iff bc = ca \tag{6}$$

hold (cf. [10]).

If C is the set of all points of an Euclidean plane and if a groupoid $(C,\cdot)$ is defined so that $aa = a$ for any $a \in C$ and for any two

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different points $a, b \in C$ the point ab is the centre of the positively oriented square with two adjacent vertices a and b (Fig. 1), then $(C, \cdot)$ is a quadratical quasigroup. The figures in this quasigroup $(C, \cdot)$ can be used for illustration of "geometrical" relations in any quadratical quasigroup $(Q, \cdot)$ and for motivation of the study of this quasigroup.

From now on let $(Q, \cdot)$ be any quadratical quasigroup. The elements of the set Q are said to be *points*.

If an operation $\bullet$ is defined on the set Q by

$$a \bullet b = ab \cdot a = ca \cdot bc, \tag{7}$$

then $(Q, \bullet)$ is an idempotent medial commutative quasigroup (cf. [2]), i.e. the identities

$$\begin{aligned} a \bullet a &= a, & (8)\\ (a \bullet b) \bullet (c \bullet d) &= (a \bullet c) \bullet (b \bullet d), & (9)\\ a \bullet b &= b \bullet a \end{aligned}$$

hold, and the operations $\cdot$ and $\bullet$ are mutually medial, i.e. the identity

$$ab \bullet cd = (a \bullet c)(b \bullet d) \tag{10}$$

holds. For any two points a and b the point $a \bullet b$ is said to be the *midpoint* of a and b (cf. Fig. 1).

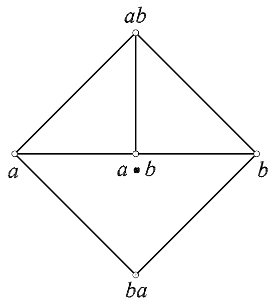

Fig. 1.

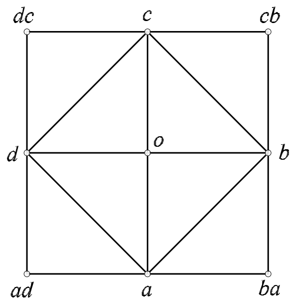

Fig. 2.

Theorem 1. *If any three of four products ab, bc, cd, da are equal, then all four products are equal (cf. Fig. 2).*

Proof. Let $ab = bc = cd$. The equality $bc = cd$ implies by (6) $db = c$. Therefore, by (4), we obtain

$$bd \cdot b = db \cdot d = cd = ab,$$

where from it follows $bd = a$ and then by (6) finally $da = ab$. □

Corollary 1. *Any three of four equalities*

$$ab = o, \quad bc = o, \quad cd = o, \quad da = o \tag{11}$$

imply the remaining equality. □

A quadrangle (a, b, c, d) is said to be a *square* and is denoted by $S(a, b, c, d)$ if any three of four products ab, bc, cd, da (and then all four products) are equal. More exactly, a quadrangle (a, b, c, d) is said to be a square with the *centre* o and is denoted by $S_o(a, b, c, d)$ if any three of four equalities (11) (and then all four equalities) hold.

If (e, f, g, h) is a cyclic permutation of (a, b, c, d), then $S(a, b, c, d)$ implies $S(e, f, g, h)$ and $S_o(a, b, c, d)$ implies $S_o(e, f, g, h)$.

The point o is said to be the *centre of a square on a segment* (a, b) if $S_o(a, b, c, d)$ holds for some points c and d.

Let us prove some simple results about squares.

Theorem 2. $S(a, b, c, d)$ *implies* $S_o(a, b, c, d)$, *where* $o = a \bullet c = b \bullet d$. (*cf. Fig.* 2)

Proof. Let $S_o(a, b, c, d)$ holds. From (11) we obtain

$$o \overset{(2)}{=} oo = da \cdot cd \overset{(7)}{=} a \bullet c,$$

and analogously $o = b \bullet d$. □

Theorem 3. *The statement* $S(a, b, c, d)$ *is equivalent with any of four* (*and then all four*) *equalities*

$$ac = d, \quad bd = a, \quad ca = b, \quad db = c. \tag{12}$$

Proof. According to the proof of Theorem 1 $S(a,b,c,d)$ implies $bd = a$, $db = c$ and analogously $ac = d$, $ca = b$. Conversely, because of cyclical permutations of (a,b,c,d), it suffices to prove the implications

$$ac = d,\ bd = a \quad \Longrightarrow \quad S(a,b,c,d),$$
$$ac = d,\ ca = b \quad \Longrightarrow \quad S(a,b,c,d).$$

From $ac = d$ and $bd = a$ by (6) it follows $cd = da$ and $da = ab$ and then Theorem 1 implies $S(a,b,c,d)$.

If $ac = d$ and $ca = b$, then we obtain

$$ab = a \cdot ca \overset{(3)}{=} ac \cdot a = da = ac \cdot a \overset{(4)}{=} ca \cdot c = bc$$

and Theorem 1 implies $S(a,b,c,d)$ again. □

Corollary 2. *For any two points a and b it holds $S_{a \bullet b}(a, ba, b, ab)$ and $ba \bullet ab = a \bullet b$ (cf. Fig. 1).* □

Theorem 4. *Let $S_{o'}(a', b', c', d')$ holds. The statements $S_o(a,b,c,d)$, $S_{oo'}(aa', bb', cc', dd')$, $S_{o'o}(a'a, b'b, c'c, d'd)$ are equivalent.*

Proof. It is sufficient to prove that the equalities $ab = o$ and $aa' \cdot bb' = oo'$ are equivalent if $a'b' = o'$ holds. But, this is obvious, because of

$$ab \cdot o' = ab \cdot a'b' \overset{(5)}{=} aa' \cdot bb'. \qquad □$$

For any point p we obviously have $S_p(p,p,p,p)$. Therefore:

Corollary 3. *The following three statements:*

$$S_o(a,b,c,d), \quad S_{po}(pa, pb, pc, pd), \quad S_{op}(ap, bp, cp, dp)$$

are mutually equivalent. □

Theorem 5. *$S_o(a,b,c,d)$ implies $S_o(ba, cb, dc, ad)$ and $ad \bullet ba = a$, $ba \bullet cb = b$, $cb \bullet dc = c$, $dc \bullet ad = d$ (cf. Fig. 2).*

Proof. $S_o(a,b,c,d)$ obviously implies $S_o(b,c,d,a)$ and according to Theorem 4 it follows $S_o(ba,cb,dc,ad)$ because of $oo \overset{(2)}{=} o$. Further we obtain

$$ad \bullet ba \overset{(10)}{=} (a \bullet b)(d \bullet a) \overset{(9)}{=} (a \bullet b)(a \bullet d) =$$

$$\overset{(10)}{=} aa \bullet bd \overset{(2)}{=} a \bullet bd \overset{(12)}{=} a \bullet a \overset{(8)}{=} a. \qquad \square$$

Theorem 6. *Let $S_{o'}(a',b',c',d')$ holds. The statements $S_o(a,b,c,d)$ and $S_{o\bullet o'}(a \bullet a', b \bullet b', c \bullet c', d \bullet d')$ are equivalent.*

Proof. It suffices to prove the equivalency of the equalities $ab = o$ and $(a \bullet a')(b \bullet b') = o \bullet o'$ if the equality $a'b' = o'$ holds. This is obvious because of

$$ab \bullet o' = ab \bullet a'b' \overset{(10)}{=} (a \bullet a')(b \bullet b'). \qquad \square$$

Corollary 4. $S_o(a,b,c,d) \iff S_{p\bullet o}(p \bullet a,\ p \bullet b,\ p \bullet c,\ p \bullet d)$. $\square$

Corollary 5. $S_o(a,b,c,d) \implies S_o(a \bullet b,\ b \bullet c,\ c \bullet d,\ d \bullet a)$. $\square$

Theorem 7. *If $ab = c$, $b \bullet c = d$, $c \bullet a = e$, $a \bullet b = f$, then $bc = ca = f$, $af = e$, $fb = d$ and $S_{c\bullet f}(e,f,d,c)$ (cf. Fig. 3).*

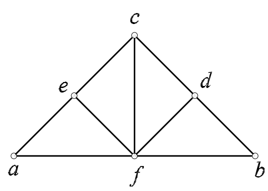

Fig. 3.

Proof. By Corollary 2 we have $S_f(a,ba,b,c)$ and $ba \bullet c = f$. Therefore, Corollary 4 implies $S_{c\bullet f}(e,f,d,c)$ because of $c \bullet a = e$, $c \bullet ba = f$, $c \bullet b = d$, $c \bullet c = c$. Further, we obtain

$$bc = b \cdot ab \overset{(3)}{=} ba \cdot b \overset{(7)}{=} b \bullet a = f,$$

$$ca = ab \cdot a \overset{(7)}{=} a \bullet b = f,$$

$$af = a(a \bullet b) \overset{(8)}{=} (a \bullet a)(a \bullet b) \overset{(10)}{=} aa \bullet ab \overset{(2)}{=} a \bullet c = e,$$

$$fb = (a \bullet b)b \overset{(8)}{=} (a \bullet b)(b \bullet b) \overset{(10)}{=} ab \bullet bb \overset{(2)}{=} c \bullet b = d. \qquad \square$$

Theorem 8. *If b' and c' are the centres of squares on the segments (c, a) and (a, b), then $b \bullet c$ is the centre of a square on the segment (c', b') (cf. Fig. 4).*

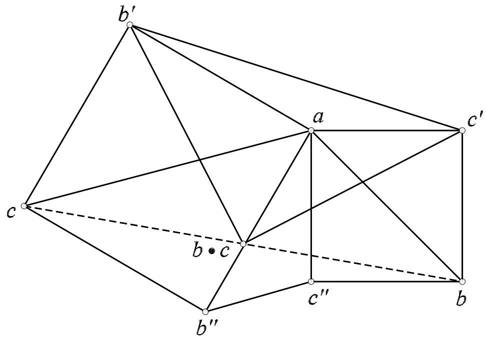

Fig. 4.

Proof. As $ca = b'$ and $ab = c'$, so we have $c'b' = ab \cdot ca \overset{(7)}{=} b \bullet c$. $\square$

In the case of the quasigroup $(C, \cdot)$ Theorem 8 proves a statement from [1], [7], [8], [9] and [11] which can be stated as a very famous problem of Captain Kidd burried treasure (cf. [6] and [4]).

The *rotation* about a point *a through a* (positively oriented) *right angle* is a transformation $x \longmapsto y$ of points such that $xy = a$.

Theorem 9. *If b', b'', c', c'' are the centres of squares on the segments (c, a), (a, c), (a, b), (b, a), then the rotation about the point $b \bullet c$ through a right angle maps the segment (c', b'') onto the segment (b', c'') (cf. Fig. 4).*

Proof. We have the equality from the above proof and analogously

$$b''c'' = ac \cdot ba \overset{(5)}{=} ab \cdot ca \overset{(7)}{=} b \bullet c. \qquad \square$$

Theorem 10. *Let* $S_o(p,a,u,b)$ *be fixed. If* (p,a',u',b') *is a square with the center* o*, then* $(o, b\bullet a', o', a\bullet b')$ *is a square with the centre* $o\bullet o'$ *and* $a\bullet b' = oo'$, $b\bullet a' = o'o$, $ba' = b'a = u\bullet u'$ *(cf. Fig.* 5*).*

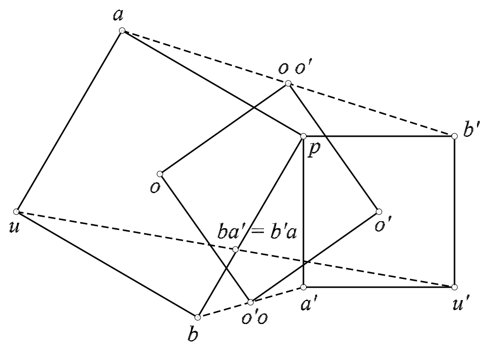

Fig. 5.

Proof. By Theorem 6 from $S_o(u,b,p,a)$ and $S_{o'}(p,a',u',b')$ it follows $S_{o\bullet o'}(u\bullet p, b\bullet a', p\bullet u', a\bullet b')$. But, $u\bullet p = o$ and $p\bullet u' = o'$ and we obtain $S_{o\bullet o'}(o, b\bullet a', o', a\bullet b')$, where from $oo' = a\bullet b'$, $o'o = b\bullet a'$ follows by Theorem 3. □

In the case of the quasigroup $(C,\cdot)$ Theorem 10 proves a result from [2] and [5].

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Remarks on polyadic groups

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Abstract

We prove that an n-ary semigroup $(G,[\ ])$ is an n-ary group $(n \geqslant 3)$ iff there exists $d \in G$ such that for every $a,b \in G$ and some fixed $i,j \in \{1,\dots,n-1\}$ the following two equations $[\overset{(i)}{a}, \overset{(n-i-1)}{b}, x] = d$ and $[y, \overset{(n-j-1)}{b}, \overset{(j)}{a}] = b$ are solvable.

Generalizing the group result from [4], V. I. Tyutin proved in [3] that an n-ary group $(G,[\ ])$ may be defined as an n-ary semigroup $(G,[\ ])$ in which for some fixed $d \in G$ and every $a,b \in G$ the following two equations

$$[\overset{(n-1)}{a}, x] = b, \qquad [y, \overset{(n-1)}{a}] = d$$

are solvable.

On the other hand, the author proved in [2] the following characterization of n-ary group

Theorem 1. *An n-ary semigroup $(G,[\ ])$ is an n-ary group iff for every $a,b \in G$ and some fixed $i,j \in \{1,\dots,n-1\}$ the following two equations*

$$[\overset{(i)}{a}, \overset{(n-i-1)}{b}, x] = b, \qquad [y, \overset{(n-j-1)}{b}, \overset{(j)}{a}] = b$$

are solvable. □

Note that this theorem was also obtained by W. A. Dudek as a consequence of some general results (cf. [1]).

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Theorem 2. *An n-ary semigroup $(G,[\ ])$ is an n-ary group ($n \geqslant 3$) iff there exists $d \in G$ such that for every $a, b \in G$ and some fixed $i, j \in \{1, \ldots, n-1\}$ the following equations*

$$[\overset{(i)}{a}, \overset{(n-i-1)}{b}, x] = d, \qquad [y, \overset{(n-j-1)}{b}, \overset{(j)}{a}] = b$$

are solvable.

Proof. It is clear that in an n-ary group $(G,[\ ])$ these equations have unique solutions for every $a, b, d \in G$ and every $i, j \in \{1, \ldots, n-1\}$.

On the other hand, if $(G,[\ ])$ is an n-ary semigroup in which there exists an element d such that for some fixed $i, j \in \{1, \ldots, n-1\}$ and every $a, b \in G$ these equations are solvable, then there are elements $u, v, w, z \in G$ such that

$$[\overset{(n-1)}{d}, u] = d, \quad [v, \overset{(n-j-1)}{a}, \overset{(j)}{d}] = a, \quad [w, \overset{(n-1)}{a}] = a, \quad [\overset{(i)}{a}, \overset{(n-i-1)}{b}, z] = d.$$

For such $u, v, w, z, d \in G$ and every $a \in G$ we have also

$$[a, \overset{(n-2)}{d}, u] = a, \quad [w, \overset{(n-2)}{a}, d] = d, \quad [\overset{(n-2)}{d}, u, a] = a,$$

because

$$[a, \overset{(n-2)}{d}, u] = [[v, \overset{(n-j-1)}{a}, \overset{(j)}{d}], \overset{(n-2)}{d}, u] = [v, \overset{(n-j-1)}{a}, \overset{(j-1)}{d}, [\overset{(n-1)}{d}, u]]$$
$$= [v, \overset{(n-j-1)}{a}, \overset{(j-1)}{d}, d] = [v, \overset{(n-j-1)}{a}, \overset{(j)}{d}] = a,$$

$$[w, \overset{(n-2)}{a}, d] = [w, \overset{(n-2)}{a}, [\overset{(i)}{a}, \overset{(n-i-1)}{b}, z]] = [[w, \overset{(n-1)}{a}], \overset{(i-1)}{a}, \overset{(n-i-1)}{b}, z]$$
$$= [a, \overset{(i-1)}{a}, \overset{(n-i-1)}{b}, z] = [\overset{(i)}{a}, \overset{(n-i-1)}{b}, z] = d$$

and

$$[\overset{(n-2)}{d}, u, a] = [[w, \overset{(n-2)}{a}, d], \overset{(n-3)}{d}, u, a] = [w, \overset{(n-3)}{a}, [a, \overset{(n-2)}{d}, u], a]$$
$$= [w, \overset{(n-3)}{a}, a, a] = [w, \overset{(n-1)}{a}] = a.$$

As a consequence we obtain

$$[\overset{(i)}{a}, \overset{(n-i-1)}{b}, [z, \overset{(n-3)}{d}, u, b]] = [[\overset{(i)}{a}, \overset{(n-i-1)}{b}, z], \overset{(n-3)}{d}, u, b]$$
$$= [d, \overset{(n-3)}{d}, u, b] = [\overset{(n-2)}{d}, u, b] = b,$$

which proves that $x = [\, z, \overset{(n-3)}{d}, u, b\,]$ is the solution of the first equation from our Theorem 1.

Since the second equation in Theorem 2 is the same as the second equation in Theorem 1 and has (by the assumption) a solution, Theorem 1 shows that $(G, [\ ])$ is an n-ary group.

This completes the proof. $\square$

Corollary. *An n-ary semigroup $(G, [\ ])$ is an n-ary group $(n \geqslant 3)$ iff there exists $d \in G$ such that for every $a, b \in G$ (at least) one of the following pairs of equations is solvable:*

a) $[\, a, \overset{(n-2)}{b}, x\,] = d, \qquad [\, y, \overset{(n-2)}{b}, a\,] = b;$

b) $[a, \overset{(n-2)}{b}, x\,] = d, \qquad [\, y, \overset{(n-1)}{a}\,] = b;$

c) $[\overset{(n-1)}{a}, x\,] = d, \qquad [\, y, \overset{(n-2)}{b}, a] = b;$

d) $[\overset{(n-1)}{a}, x\,] = d, \qquad [\, y, \overset{(n-1)}{a}\,] = b.$ $\square$

In the same manner as the above Theorem 2 we can prove

Theorem 3. *An n-ary semigroup $(G, [\ ])$ is an n-ary group $(n \geqslant 3)$ iff there exists $d \in G$ such that for every $a, b \in G$ and some fixed $i, j \in \{1, \ldots, n-1\}$ the following equations*

$$[\, \overset{(i)}{a}, \overset{(n-i-1)}{b}, x\,] = b, \qquad [\, y, \overset{(n-j-1)}{b}, \overset{(j)}{a}\,] = d$$

are solvable. $\square$

Putting in this Theorem $i = j = n-1$, we obtain Tyutin's definition of n-ary groups mentioned at the beginning of this paper.

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