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Generalized Moufang G–loops

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Abstract

In this note some relations among generalized Moufang loops and G-loops are considered.

A loop $\mathcal{Q}(\cdot)$ is called:

(a) a *G-loop* [1], if every loop which is isotopic to $\mathcal{Q}(\cdot)$ is also isomorphic to it;

(b) a *generalized Moufang loop* [2], if one of the following identities holds:

$$x\cdot(yz\cdot x)=I(I^{-1}y\cdot I^{-1}x)\cdot zx\,,\qquad (x\cdot yz)\cdot x=xy\cdot I^{-1}(Ix\cdot Iz)\,;$$

(c) an *Osborn loop* [2], if the identity

$$xy\cdot(\Theta_x z\cdot x)=(x\cdot yz)\cdot x$$

holds, where Θ_x is a permutation which depends on x;

(d) a *K-loop* [3], if the following identities hold:

$$(x\cdot yIx)\cdot xz=x\cdot yz \quad\text{and}\quad (y\cdot x)\cdot((I^{-1}xz)\cdot x)=yz\cdot x\,, \tag{1}$$

where $Ix=x^{-1}$ and $I^{-1}x={}^{-1}x$;

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(e) a *VD-loop*, if we have the equalities

$$(\cdot)_x = (\cdot)_x^{L_x^{-1}R_x} \quad \text{and} \quad {}_x(\cdot) = (\cdot)^{R_x^{-1}L_x}, \tag{2}$$

which are true for any $x \in \mathcal{Q}$, where

$$(\cdot)_x = (\cdot)^{(L_x,\,1,\,L_x)}, \qquad {}_x(\cdot) = (\cdot)^{(1,\,R_x,\,R_x)}.$$

Any K-loop (any VD-loop) is a G-loop and any VD-loop is an Osborn loop.

Theorem 1. *A generalized Moufang loop $\mathcal{Q}(\cdot)$ is a K-loop if and only if $x^2 \in N$ for any $x \in \mathcal{Q}$.*

Proof. For a generalized Moufang loop $\mathcal{Q}(\cdot)$ the property *WIP* is universal [2]. Therefore, by a result from [4], one has the autotopy $T = (R_y^{-1}L_z, R_x^{-1}L_y, L_zR_x^{-1})$, where $z = I^{-1}(y \cdot x)$. By identifying x and y in the autotopy T, one obtains

$$T_1 = (R_x^{-1}L_{I^{-1}(x\cdot x)}\,, R_x^{-1}L_x\,, L_{I^{-1}(x\cdot x)}R_x^{-1}\,.$$

In any generalized Moufang loop $I^{-1}(x \cdot x) = I(x \cdot x)$ holds, hence T_1 provides the equality

$$R_x^{-1}L_{I(x\cdot x)}u \cdot R_x^{-1}L_xv = L_{I(x\cdot x)}R_x^{-1}(u \cdot v). \tag{3}$$

Let $v = 1$ in (3), then

$$R_x^{-1}L_{I(x\cdot x)} = L_{I(x\cdot x}R_x^{-1}. \tag{4}$$

Identity (4) implies $T_1 = (L_{I(x\cdot x)}R_x^{-1}\,, R_x^{-1}L_x\,, L_{I(x\cdot x}R_x^{-1})$. Hence T_1^{-1} implies $L_x^{-1}R_x$ is a pseudoautomorphism with the right companion $\alpha_1 = R_xL_{I(x\cdot x)}^{-1}1 = (x \cdot x) \cdot x$. Thus

$$(\cdot)_{x^2\cdot x} = (\cdot)^{R_x^{-1}L_x} \tag{5}$$

In any generalized Moufang loop the equalities

$$_{I_x}(\cdot) = (\cdot)_x \tag{6}$$

$$R_x L_{I^{-1}x} = L_x^{-1} R_x \,, \qquad L_x R_{I_x} = R_x^{-1} L_x \tag{7}$$

hold.

Let $x^2 \in N$, where N is the nucleus of the generalized Moufang loop $\mathcal{Q}(\cdot)$. Since $(\cdot)_{x^2} = (\cdot)$ (5) implies $(\cdot)_x = (\cdot)^{R_x^{-1} L_x}$, or, in other words,

$$T_2 = (R_x \,, L_x^{-1} R_x \,, R_x) = (R_x \,, R_x L_{I^{-1}x} \,, R_x)$$

(by (7)) is an autotopy, that yields $yx \cdot (I^{-1}xz \cdot x) = yz \cdot x$, which coincides with the second equality from (1). By using (6) in the equality $(\cdot)_x = (\cdot)^{R_x^{-1} L_x}$ we obtain $I_x(\cdot) = (\cdot)^{R_x^{-1} L_x}$, and consequently, have the autotopy

$$T_3 = (L_x^{-1} R_x \,, R_{Ix} L_x^{-1} R_x \,, R_{Ix} L_x^{-1} R_x) = (L_x^{-1} R_x \,, L_x^{-1}, L_x^{-1})$$

or

$$T_3^{-1} = (R_x^{-1} L_x \, L_x \,, L_x) = (L_x R_{Ix} \,, L_x \,, L_x) \,,$$

hence $(x \cdot yI_x) \cdot xz = x \cdot yz$, meaning that the first equality from (1) is true. Thus, if in the generalized Moufang loop $\mathcal{Q}(\cdot)$, $x^2 \in N$ for any $x \in Q$, then $\mathcal{Q}(\cdot)$ is a K-loop.

Now, let the generalized Moufang loop $\mathcal{Q}(\cdot)$ be a K-loop, then the equality (5) and the equality $(\cdot)_x = (\cdot)^{R_x^{-1} L_x}$ are true and they imply $(\cdot)_{x^2 \cdot x} = (\cdot)_x$, or $(\cdot)_{x^2} = (\cdot)$, or $x^2 \in N$. □

Theorem 2. *A generalized Moufang loop $\mathcal{Q}(\cdot)$ is a VD-loop, if $x^4 \in N$ whichever $x \in \mathcal{Q}$.*

Proof. If $\mathcal{Q}(\cdot)$ is a generalized Moufang loop, then (5) holds, so: $((\cdot)_{x^2 \cdot x})_x = ((\cdot)^{R_x^{-1} L_x})_x$ or $(\cdot)_{x^4} = (\cdot)_x^{R_x^{-1} L_x}$. Suppose $x^4 \in N$, then $(\cdot)_{x^4} = (\cdot)$, hence $(\cdot)_x^{R_x^{-1} L_x} = (\cdot)$, and $(\cdot)_x = (\cdot)^{L_x^{-1} R_x}$. The equalities ${}_{I_x}(\cdot) = (\cdot)_x = (\cdot)^{L_x^{-1} R_x}$ supply ${}_x({}_{I_x}(\cdot)) = {}_x((\cdot)^{L_x^{-1} R_x})$ or $(\cdot) = {}_x(\cdot)^{L_x^{-1} R_x}$. If in the generalized Moufang loop $\mathcal{Q}(\cdot)$ one has $x^4 \in N$ for every $x \in \mathcal{Q}$, then $\mathcal{Q}(\cdot)$ is a *VD*-loop. In each generalized Moufang loop the equality

$$(\cdot)_{x^4} = (\cdot)_x^{R_x^{-1} L_x} \tag{8}$$

holds. If the generalized Moufang loop is a *VD*-loop then (2) implies

$$(\cdot)_x^{R_x^{-1} L_x} = (\cdot) \,. \tag{9}$$

(8) and (9) provide $(\cdot)_{x^4} = (\cdot)$, therefore $x^4 \in N$. $\square$

Theorem 3. *Each VD-loop is an Osborn loop.*

Proof. Let $\mathcal{Q}(\cdot)$ be a *VD*-loop, then the equalities (2) are true, that is $(\cdot)_x = (.)^{L_x^{-1}L_x}$. They imply the autotopies

$$S = (L_x R_x^{-1} L_x \, , R_x^{-1} L_x \, , L_x R_x^{-1} L_x)$$

and

$$S_1 = (L_x^{-1} R_x \, , R_x L_x^{-1} R_x \, , R_x L_x^{-1} R_x)\,.$$

By multiplying the autotopies S and S_1 we obtain

$$SS_1 = (L_x \, , R_x^{-1} L_x R_x L_x^{-1} R_x \, , L_x R_x^{-1} L_x R_x L_x^{-1} R_x)\,,$$

which carries out the equality

$$L_x u \cdot R_x^{-1} L_x R_x L_x^{-1} R_x v = L_x R_x^{-1} L_x R_x L_x^{-1} R_x (u \cdot v)\,. \tag{10}$$

Let $v = 1$ in (10) (1 is the unit of the loop $\mathcal{Q}(\cdot)$), then

$$R_x L_x = L_x R_x^{-1} L_x R_x L_x^{-1} R_x \tag{11}$$

and

$$L_x^{-1} R_x L_x = R_x^{-1} L_x R_x L_x^{-1} R_x)\,. \tag{12}$$

Applying (11) and (12) to (10), we obtain

$$L_x u \cdot (R_x^{-1} L_x^{-1} R_x L_x v \cdot x) = R_x L_x (u \cdot v) \quad \text{or} \quad \mathrm{xu} \cdot \Theta_{\mathrm{x}} \mathrm{vx} = (\mathrm{x} \cdot \mathrm{uv}) \cdot \mathrm{x}\,,$$

that is $\mathcal{Q}(\cdot)$ is an Osborn loop. $\square$

Theorem 3 implies

Corollary 1. *The three nuclei of each VD-loop $\mathcal{Q}(\cdot)$ coincide, i.e. $N_r = N_m = N_l = N$. Moreover, N is a normal subloop in $\mathcal{Q}(\cdot)$.* $\square$

Proposition 1. *A K-loop $\mathcal{Q}(\cdot)$ is a VD-loop if $x^2 \in N$ for any $x \in \mathcal{Q}$.*

Proof. The following equalities are consequences of the definition of K-loops:

$$(\cdot)_x = (\cdot)^{R_x^{-1}L_x} \quad \text{and} \quad {}_x(\cdot) = (\cdot)^{L_x^{-1}R_x}.$$

By means of the properties of the derived operations, one has

$$((\cdot)_x)_x = ((\cdot)^{R_x^{-1}L_x})_x \quad \text{and} \quad {}_x({}_x(\cdot)) = {}_x((\cdot)^{L_x^{-1}R_x})$$

or

$$(\cdot)_{x^2} = (\cdot)_x^{R_x^{-1}L_x} \quad \text{and} \quad {}_{x^2}(\cdot) = {}_x(\cdot)^{L_x^{-1}R_x} \tag{13}$$

If $x^2 \in N$ then

$$(\cdot)_{x^2} = (\cdot) \quad \text{and} \quad {}_{x^2}(\cdot) = (\cdot)\,. \tag{14}$$

By using (14) in (13), we obtain $(\cdot) = (\cdot)_x^{R_x^{-1}L_x}$ and $(\cdot) = {}_x(\cdot)L_x^{-1}R_x$, or $(\cdot)_x = (\cdot)^{L_x^{-1}R_x}$ and ${}_x(\cdot) = (\cdot)^{R_x^{-1}L_x}$ however, this means $\mathcal{Q}(\cdot)$ is a *VD*-loop. □

Proposition 2. *A VD-loop $\mathcal{Q}(\cdot)$ is a K-loop if $x^2 \in N$ for any $x \in \mathcal{Q}$.*

The proof is analogous to that of Proposition 1. □

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Monoquasigroups isotopic to groups

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Abstract

In this work the quasigroups isotopic to groups are considered. The necessary and sufficient conditions are found which the isotopy must satisfy so that the corresponding group isotope be: monogenic quasigroup, momoquasigroup.

1. Introduction

The algebraic systems generated by one element (monogenic systems) are the simplest in the lattice-theoretic sense in every class of algebraic systems (in some cases, such as quasigroups, semigroups, in order to be able always to talk about lattice, we need to consider the empty set as a subsystem). These systems are contained as subsystems in some other systems. More precisely, every non-empty system of some class of algebraic systems includes some monogenic systems of this class as its subsystems. Hence the structure of algebraic systems depends on the structure of their monogenic systems.

In such classes of algebraic systems as groups and semigroups the monogenic systems are the cyclic groups and semigroups, respectively, which are completely described, as it is well known. In other classes it is very difficult to describe monogenic systems.

Definition 1. A quasigroup generated by one its element is called a *monogenic quasigroup*.

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Definition 2. A quasigroup generated by every its element is called a *monoquasigroup.*

From the definition it is clear that monoquasigroups have no nontrivial subquasigroups and, hence, its lattice of subquasigroups consists of one element. A nontrivial (or proper) subquasigroup is a subquasigroup different from the empty subquasigroup and the quasigroup itself [1].

Researching different kinds of functional completeness of universal algebras, A. V. Kuznetzov and A. F.Danilichenko A. F. announced during *The First All-Union Symposium on the Theory of Quasigroups and its Applications* (Suchumi, 1968) that for every positive integer n there exists a monoquasigroup with $|\mathcal{Q}| = n$, where by $|\mathcal{Q}|$ we denote the order of $\mathcal{Q}$.

A quasigroup $(\mathcal{Q}, \cdot)$ is said to be *without congruences* if it has no congruences except $\varepsilon = \mathcal{Q} \times \mathcal{Q} = \mathcal{Q}^2$ (the complete relation on $\mathcal{Q}$) and $\omega = \{(a, a) : a \in \mathcal{Q}\}$ (the equality relation on $\mathcal{Q}$, sometimes called the *diagonal* of $\mathcal{Q}^2$).

T. Kepka has proved in [4] that a quasigroup $(\mathcal{Q}, \cdot)$ such that $3 \leq |\mathcal{Q}| \leq \aleph_0$ is isotopic to a monoquasigroup. In [3] is proved

Theorem 1.
a) Every quasigroup $(\mathcal{Q}, \cdot)$ *such that* $3 \leq |\mathcal{Q}| \leq \aleph_0$ *is isotopic to a monoquasigroup without congruences.*
b) Every quasigroup $(\mathcal{Q}, \cdot)$ *such that* $5 \leq |\mathcal{Q}| \leq \aleph_0$ *is isotopic to a monoquasigroup without congruences and automorphisms.* □

A quasigroup without automorphisms is a quasigroup with the unitary group of automorphisms. There exist no monoquasigroups with more than the countable order, because a finitely generated free algebra with a finite set of operations has at most the countable order [5]. A quasigroup $(\mathcal{Q}, \cdot)$ with $|\mathcal{Q}| = 1$ satisfies Theorem 1a and a quasigroup $(\mathcal{Q}, \cdot)$ with $|\mathcal{Q}| = 2$ is a group and does not satisfies Theorem 1a.

In [3] a class of order $2^{\aleph_0}$ of pairwise non-isomorphic monoquasigroups without congruences and without automorphisms is given. This fact allows us to assert that it is not very probably to describe monoquasigroups or monogenic quasigroups. Therefore to describe monoquasigroups we shall restrict ourselves to particular classes. In the class of all idempotent quasigroups and the class of all loops there are no monoquasigroups except single-element groups. In other classes it is rather difficult to give examples of monoquasigroups.

Below we consider only quasigroups with the order greater than 2 and smaller than $\aleph_0$.

2. Preliminaries

A groupoid (i.e. a set $(\mathcal{Q}, \cdot)$ with a binary operation "$\exists$" on $\mathcal{Q}$) is called a *quasigroup* if equations $ax = b$, $ya = b$ have unique solutions for any elements $a, b \in \mathcal{Q}$. In a quasigroup $(\mathcal{Q}, \cdot)$ a mapping $x \to ax$ is called the *left translation by* a and is denoted by L_a. The *right translation by* a is the mapping $x \to xa$ that is denoted by R_a. For any $a \in \mathcal{Q}$ the translations R_a and L_a are permutations on the set $\mathcal{Q}$ and belong to the permutation group $\mathcal{S}(\mathcal{Q})$.

A non-empty subset H of the quasigroup $(\mathcal{Q}, \cdot)$ is a subquasigroup of $(\mathcal{Q}, \cdot)$ provided $(H, \cdot)$ is a quasigroup with respect to the operation "$\exists$". The empty set $\emptyset$ will be considered as a subquasigroup iff the intersection of all non-empty subquasigroups of $(\mathcal{Q}, \cdot)$ is $\emptyset$. The set-theoretic intersection of all subquasigroups of $(\mathcal{Q}, \cdot)$ containing a subset M of $\mathcal{Q}$ is a subquasigroup that will be denoted by $\langle M \rangle$ and will be called the subquasigroup generated by M. The class $L(\mathcal{Q}, \cdot)$ of all subquasigroups of a quasigroup $(\mathcal{Q}, \cdot)$ is a complete lattice with respect to the set-theoretic intersection and "generate" operation. The last element in $L(\mathcal{Q}, \cdot)$ is the intersection of all non-empty subquasigroups and the greatest element is $(\mathcal{Q}, \cdot)$.

Let " $*$ " and " $\circ$ " be two operations defined on $\mathcal{Q}$. The operation " $*$ " is said *to be isotopic* to " $\circ$ " if there exist three permutations $\alpha, \beta, \gamma \in \mathcal{S}(\mathcal{Q})$ such that

$$x * y = \gamma^{-1}(\alpha x \circ \beta y) \tag{1}$$

for all $x, y \in \mathcal{Q}$.

We also say that $(\mathcal{Q}, *)$ and $(\mathcal{Q}, \circ)$ are *isotopic*, or that $(\mathcal{Q}, *)$

is an *isotop* of $(\mathcal{Q}, \circ)$ of the form (1). Shortly we write this as

$$(\mathcal{Q}, *) : x * y = \gamma^{-1}(\alpha x \circ \beta y), \quad \alpha, \beta, \gamma \in \mathcal{S}(\mathcal{Q}), \quad x, y \in \mathcal{Q}.$$

Then triple (α, β, γ) of permutations such that the relation (1) holds is called the *isotopy* of $(\mathcal{Q}, \circ)$.

If in (1) γ is the identical permutation ϵ, then $(\mathcal{Q}, *)$ is said to be a *principal isotope* of $(\mathcal{Q}, \circ)$.

If in (1) $\alpha = \beta = \gamma$, then

$$x * y = \gamma^{-1}(\gamma x \circ \gamma y),$$

which means that γ is an isomorphism between $(\mathcal{Q}, *)$ and $(\mathcal{Q}, \circ)$. The equality (1) is equivalent to

$$x * y = \gamma^{-1}(\alpha\gamma^{-1}\gamma\, x \circ \beta\gamma^{-1}\gamma\, y).$$

Whence we have proved the following:

Theorem 2 ([1] Theorem 1.2)**.** *An isotope $(\mathcal{Q}, *)$ such that $x * y = \gamma^{-1}(\alpha x \circ \beta y)$, $\alpha, \beta, \gamma \in \mathcal{S}(\mathcal{Q})$, $x, y \in \mathcal{Q}$ is isomorphic to the principal isotope $(\mathcal{Q}, \otimes)$, where $x \otimes y = \alpha\gamma^{-1}x \circ \beta\gamma^{-1}y$, $\alpha, \beta, \gamma \in \mathcal{S}(\mathcal{Q})$, $x, y \in \mathcal{Q}$, and γ is the isomorphism between them.* $\square$

3. Group isotopes

Let $(\mathcal{Q}, \cdot)$ be a group with the unit element e. We will find the necessary and sufficient conditions which the isotopy must satisfy in order that the corresponding isotope of $(\mathcal{Q}, \cdot)$ be a monogenic group; monoquasigroup (Theorem 3). Since an isomorphism keeps the number of generators, then taking into consideration Theorem 2 it is sufficient to find these conditions for principal isotopes of a group $(\mathcal{Q}, \cdot)$.

Lemma 1. *For a principal isotope*

$$(\mathcal{Q}, *) \ : \ x * y = \varphi x \cdot \psi y, \quad \varphi, \psi \in \mathcal{S}(\mathcal{Q}), \quad x, y \in \mathcal{Q}$$

of a group $(\mathcal{Q}, \cdot)$ with the unit e there exist permutations $\alpha, \beta \in \mathcal{S}(\mathcal{Q})$ such that $\beta e = e$ and $x * y = \alpha x \cdot \beta y$, i.e.

$$(\mathcal{Q}, *) \ : \ x * y = \alpha x \cdot \beta y, \quad \alpha, \beta \in \mathcal{S}(\mathcal{Q}), \quad \beta e = e, \quad x, y \in \mathcal{Q}. \quad (2)$$

Proof. For every $x, y \in \mathcal{Q}$ we have

$$x * y = \varphi x \cdot \psi y = \varphi x \cdot \psi e \cdot (\psi e)^{-1} \psi y = R_{\psi e} \varphi x \cdot L_{(\psi e)^{-1}} \psi y = \alpha x \cdot \beta y \,,$$

where $\alpha = R_{\psi e} \varphi$ and $\beta = L_{(\psi e)^{-1}} \psi$. Moreover,

$$\beta e = L_{(\psi e)^{-1}} \psi e = (\psi e)^{-1} \psi e = e \,,$$

which completes the proof. □

For all $\alpha \in \mathcal{S}(\mathcal{Q})$, $H \subseteq \mathcal{Q}$ put

$$\alpha H = \{ \, \alpha h \, : \, h \in H \, \} \,.$$

Lemma 2. *For an isotope*

$$(\mathcal{Q} \,, *) \, : \, x * y = \alpha x \cdot \beta y, \quad \alpha, \beta \in \mathcal{S}(\mathcal{Q}), \quad \beta e = e, \quad x, y \in \mathcal{Q}$$

of a group $(\mathcal{Q} \,, \cdot)$ *with the unit* e *the following conditions are equivalent*

a) $e \in (H \,, *) \in L(\mathcal{Q} \,, *)$,

b) $\alpha H = H = \beta H$ *and* $(H \,, \cdot) \in L(\mathcal{Q} \,, \cdot)$.

Proof. Let $e \in (H \,, *) \in L(\mathcal{Q} \,, *)$. Then for any $x \in H$, we have $x * e \in (H \,, *)$ and $x * e = \alpha x \cdot \beta e = \alpha x \cdot e = \alpha x$. Hence $\alpha x \in H$ and, as x is an arbitrary element, we have $\alpha H \subseteq H$. For any $x \in H$ there exists $y \in H$ such that $x = y * e$, since $(H \,, *)$ is a subquasigroup of $(\mathcal{Q} \,, *)$ and $e \in H$. From the last equality we get $y = \alpha^{-1} x$ and $\alpha^{-1} H \in H$ since x is an arbitrary element of H. Therefore $H \subseteq \alpha H$ and we have $H = \alpha H$. Let $h \in H$ be such that $\alpha h = e$. For any $x \in H$ we have $h * x \in H$ and $h * x = \alpha h \cdot \beta x = e \cdot \beta x = \beta x$. Therefore $\beta x \in H$ for any $x \in H$, so $\beta H \subseteq H$. There exists $y \in H$ such that $h * y = x$ for any $x \in H$. Then

$$h * y = \alpha h \cdot \beta y = e \cdot \beta y = \beta y = x$$

and $y = \beta^{-1}(x)$. Hence $\beta^{-1} H \subseteq H$, $H \subseteq \beta H$ and finally we have $\beta H = H$. So, the restrictions of α and β to H are permutations on H, and $(H, \cdot)$ is an associative quasigroup isotopic to the quasigroup $(H, *)$ since $x \cdot y = \alpha^{-1} x * \beta^{-1} y$, i.e. $(H, \cdot) \in L(\mathcal{Q}, *)$. Therefore we have proved that $e \in (H,*) \in L(\mathcal{Q},*)$ implies $\alpha H = H = \beta H$ and $(H,\cdot) \in L(\mathcal{Q},\cdot)$.

The converse implication is trivial. □

For any $\varphi \in \mathcal{S}(\mathcal{Q})$ put

$$Stab_{L(\mathcal{Q},\cdot)}\varphi = \{\, H \subseteq \mathcal{Q} \,:\, (H,\cdot) \in L(\mathcal{Q},\cdot) \quad \text{and} \quad \varphi H = H \,\}.$$

Lemma 3. *A quasigroup*

$$(\mathcal{Q},*) \,:\, x * y = \alpha x \cdot \beta y, \quad \alpha,\beta \in \mathcal{S}(\mathcal{Q}), \quad \beta e = e, \quad x,y \in \mathcal{Q}$$

which is isotopic to a group $(\mathcal{Q},\cdot)$ with the unit e is generated by e if and only if

$$Stab_{L(\mathcal{Q},\cdot)}\alpha \cap Stab_{L(\mathcal{Q},\cdot)}\beta = \{\mathcal{Q}\}. \tag{3}$$

Proof. Let $(\mathcal{Q},*)$ be generated by the unit e and

$$H \in Stab_{L(\mathcal{Q},\cdot)}\alpha \cap Stab_{L(\mathcal{Q},\cdot)}\beta.$$

Then $(H,\cdot) \in L(\mathcal{Q})$ and $\alpha H = H = \beta H$. By Lemma 2 we get $(H,*) \in L(\mathcal{Q},*)$ and $(H,*) = (\mathcal{Q},*)$ as $e \in (H,*)$. Hence

$$Stab_{L(\mathcal{Q},\cdot)}\alpha \cap Stab_{L(\mathcal{Q},\cdot)}\beta = \{\mathcal{Q}\}.$$

Conversely, let the relation (3) holds and $(H,*) \in L(\mathcal{Q},*)$, where $e \in (H,*)$. By Lemma 2 we have $\alpha H = H = \beta H$ and from (3) we get $H = \mathcal{Q}$. Therefore $(\mathcal{Q},*)$ is generated by the unit e. □

Directly from Lemmas 1 and 3 we get

Corollary 1. *A quasigroup*

$$(\mathcal{Q},*) \,:\, x * y = \alpha x \cdot \beta y, \quad \alpha,\beta \in \mathcal{S}(\mathcal{Q}), \quad x,y \in \mathcal{Q},$$

which is isotopic to a group $(\mathcal{Q},\cdot)$ with the unit e is generated by e if and only if

$$Stab_{L(\mathcal{Q},\cdot)}R_{\beta e}\alpha \cap Stab_{L(\mathcal{Q},\cdot)}L_{\beta e}^{-1}\beta = \{\mathcal{Q}\}. \tag{4}$$

Proof. By Lemma 1 we get the equalities $x * y = R_{\beta e}\alpha x \cdot L^{-1}_{\beta e}\beta y$ and $L^{-1}_{\beta e}\beta e = e$. The equality (4) follows from Lemma 3.

Remark that relation (4) gives $\alpha e \cdot \beta e \neq e$, otherwise the set $\{e\}$ is a subquasigroup of $(\mathcal{Q}\,, *)$, contrary to (4). $\square$

Now we will find the condition for a quasigroup

$$(\mathcal{Q}\,, *)\,:\, x * y = \alpha x \cdot \beta y, \quad \alpha, \beta \in \mathcal{S}(\mathcal{Q}), \quad x, y \in \mathcal{Q}$$

to be generated by any its element $a \in \mathcal{Q}\,,\ \ a \neq e$.

Let us consider the isotope

$$(\mathcal{Q}\,, \circ)\,:\, x \circ y = x \cdot a^{-1}y, \quad x, y \in \mathcal{Q}$$

for any fixed element $a \in \mathcal{Q}$. This isotope is a group with the unit a and the left translation L_a of a group $(\mathcal{Q}\,, \cdot)$ is an isomorphism between groups $(\mathcal{Q}\,, \cdot)$ and $(\mathcal{Q}\,, \circ)$, i.e. we have $L_a(x \cdot y) = L_a x \circ L_a y$. Then the equality

$$L_a^{-1}(x \circ y) = L_a^{-1}x \cdot L_a^{-1}y$$

and implications

$$(H, \cdot) \in L(\mathcal{Q}\,, \cdot) \ \Rightarrow\ (L_a H, \circ) \in L(\mathcal{Q}\,, \circ)\,,$$

$$(H, \circ) \in L(\mathcal{Q}\,, \circ) \ \Rightarrow\ (L_a^{-1}H, \cdot) \in L(\mathcal{Q}\,, \cdot) \tag{5}$$

hold. The quasigroup $(\mathcal{Q}\,, *)$ is an isotope of a group $(\mathcal{Q}\,, \circ)$ with the unit a since we have $x * y = \alpha x \cdot \beta y = \alpha x \circ L_a \beta y$. By Corollary 1 the quasigroup $(\mathcal{Q}\,, *)$ is generated by a if and only if the equality

$$Stab_{L(\mathcal{Q}\,,\circ)}\hat{R}^{-1}_{L_a\beta a}\alpha \,\cap\, Stab_{L(\mathcal{Q}\,,\circ)}\hat{L}^{-1}_{L_a\beta a}L_a\beta = \{\mathcal{Q}\} \tag{6}$$

holds, where by $\hat{R}_x\,, \hat{L}_x$ we denote the translations by x on the group $(\mathcal{Q}\,, \circ)$.

Remark. If the equality (6) holds, then we have $\alpha a \cdot \beta a \neq a$. The following equalities hold for any $a, u \in \mathcal{Q}$:

$$L_u^{-1} = L_{u^{-1}}\,,$$
$$\hat{L}_u = L_u L_{a^{-1}} = L_{ua^{-1}}\,, \quad \hat{L}_u^{-1} = L_a L_{u^{-1}} = L_{au^{-1}}\,,$$
$$\hat{R}_u = R_u R_{a^{-1}} = R_{a^{-1}u}\,, \quad \hat{R}_u^{-1} = R_a R_{u^{-1}} = R_{u^{-1}a}\,,$$

where a^{-1}, u^{-1} are the inverses of a, u in $(\mathcal{Q}\,,\cdot)$. Then

$$\hat{R}_{L_a\beta a}\alpha = \hat{R}_{a\beta a}\alpha = R_{a^{-1}\cdot a\beta a}\alpha = R_{\beta a}\alpha\,,$$

$$\hat{L}^{-1}_{L_a\beta a}L_a\beta = \hat{L}^{-1}_{a\beta a}L_a\beta = L_{a(\beta a)^{-1}a^{-1}}L_a\beta = L_{a(\beta a)^{-1}}\beta\,.$$

Now the equality (6) can be rewritten in the following way:

$$Stab_{L(\mathcal{Q}\,,\circ)}R_{\beta a}\alpha \cap Stab_{L(\mathcal{Q}\,,\circ)}L_{a(\beta a)^{-1}}\beta = \{\mathcal{Q}\}\,. \tag{7}$$

Lemma 4. *For any* $\varphi \in \mathcal{S}(\mathcal{Q})$ *we have*

$$Stab_{L(\mathcal{Q}\,,\circ)}\varphi = L_a(Stab_{L(\mathcal{Q}\,,\cdot)}L_{a^{-1}}\varphi L_a)$$

and

$$Stab_{L(\mathcal{Q}\,,\circ)}\varphi = \{\, L_aH \,:\, (H,\cdot) \in L(\mathcal{Q}\,,\cdot) \text{ and } L_{a^{-1}}\varphi L_a H = H\}\,.$$

Proof. If $(H,\cdot) \in L(\mathcal{Q}\,,\cdot)$ and $L_{a^{-1}}\varphi L_a H = H$, then $\varphi L_a H = L_a H$ and $(L_aH,\circ) \in Stab_{L(\mathcal{Q}\,,\circ)}\varphi$ since $(L_aH,\circ) \in L(\mathcal{Q}\,,\circ)$. Hence we have

$$Stab_{L(\mathcal{Q}\,,\circ)}\varphi \supseteq L_a(Stab_{L(\mathcal{Q}\,,\cdot)}L_{a^{-1}}\varphi L_a)\,.$$

Conversely, let $(\hat{H},\circ) \in Stab_{L(\mathcal{Q},\circ)}\varphi$. Then $\varphi\hat{H} = \hat{H}$ and we can write

$$\varphi L_a {L_a}^{-1}\hat{H} = L_a {L_a}^{-1}\hat{H}$$

which get

$$L_{a^{-1}}\varphi L_a L_a^{-1}\hat{H} = L_a^{-1}\hat{H}\,.$$

Hence,

$$(L_a^{-1}\hat{H},\cdot) \in Stab_{L(\mathcal{Q}\,,\cdot)}L_{a^{-1}}\varphi L_a$$

as $(L_a^{-1}\hat{H},\,\cdot) \in L(\mathcal{Q}\,,\,\cdot)$ by (5). Therefore

$$Stab_{L(\mathcal{Q},\circ)}\varphi \subseteq L_a(Stab_{L(\mathcal{Q},\cdot)}L_{a^{-1}}\varphi L_a)$$

and the statement of the lemma is proved. □

Now the equality (7) can be rewritten as

$$L_a(Stab_{L(\mathcal{Q},\cdot)}L_{a^{-1}}R_{\beta a}\alpha L_a) \cap L_a(Stab_{L(\mathcal{Q},\cdot)}L_{(\beta a)^{-1}}\beta L_a) = \{\mathcal{Q}\}\,,$$

from which we have

$$Stab_{L(\mathcal{Q},\cdot)}L_{a^{-1}}R_{\beta a}\alpha L_a \cap Stab_{L(\mathcal{Q},\cdot)}L_{(\beta a)^{-1}}\beta L_a = \{\mathcal{Q}\}\,. \tag{8}$$

So, we have proved the following:

Theorem 3. *A quasigroup*

$$(\mathcal{Q}\,,*) \;\; : \;\; x*y=\alpha x\cdot\beta y, \quad \alpha,\beta\in\mathcal{S}(\mathcal{Q}), \quad x,y\in\mathcal{Q}$$

isotopic to a group $(\mathcal{Q}\,,\,\cdot)$ *with the unit* e *is:*
a) generated by an element $a\in\mathcal{Q}$ *iff the equality (8) holds,*
b) a monoquasigroup iff the equality (8) holds for any $a\in\mathcal{Q}$ *.* □

Corollary 2. *If a quasigroup* $(\mathcal{Q}\,,*)$ *isotopic to a group* $(\mathcal{Q}\,,\cdot)$ *with the unit* e *is a monoquasigroup, then* $\alpha x\cdot\beta x\neq x$ *for all* $x\in\mathcal{Q}$.

Proof. In fact, if $\alpha a\cdot\beta a=a$ for some $a\in\mathcal{Q}$, then a is an idempotent. Thus $(\mathcal{Q}\,,*)$ is not generated by a, contrary to the assertion of the corollary. □

Proposition 1. *The order of a subquasigroup of a group isotope* $(\mathcal{Q}\,,\oplus)$ *divides the order of the quasigroup* $(\mathcal{Q},\oplus)$.

Proof. Let $(H,\oplus)$ be a subquasigroup of a group isotope $(\mathcal{Q}\,,\oplus)$ and $\emptyset\neq H\neq\mathcal{Q}$. By Albert's Theorem the isotope

$$(\mathcal{Q}\,,\bullet) \;\; : \;\; x\bullet y=R_a^{-1}x\oplus L_a^{-1}y$$

is a group for every element $a\in H$ and $(H,\bullet)$ is a subgroup of $(\mathcal{Q}\,,\bullet)$ as $a\in H$. In a group the order of subgroup divides the order of the group. □

Corollary 3. *Every proper subquasigroup of a group isotope of prime order is a single-element set.* □

Corollary 4. *The isotope*

$$(\mathcal{Q}\,,*) \;\;:\;\; x*y = \alpha x\cdot\beta y, \quad \alpha,\beta\in\mathcal{S}(\mathcal{Q}), \quad x,y\in\mathcal{Q}$$

of a group $(\mathcal{Q}\,,*)$ *of a prime order is a monoquasigroup if and only if* $\alpha x\cdot\beta x\neq x$ *for all* $x\in\mathcal{Q}$.

Proof. Let $\alpha x\cdot\beta x\neq x$ for all $x\in\mathcal{Q}$ and $|\mathcal{Q}|$ be a prime number. The isotope $(\mathcal{Q}\,,*)$ has not single-element subquasigroups since $x* = \alpha x\cdot\beta x\neq x$ for all $x\in\mathcal{Q}$. From Corollary 3 we obtain that $(\mathcal{Q}\,,*)$ is generated by any its element, i.e. is a monoquasigroup. □

By Corollary 2 the relation $\alpha x\cdot\beta x\neq x$ holds in $(\mathcal{Q}\,,*)$ for all $x\in\mathcal{Q}$. We will adopt some of the above results for right loops principally isotopic to groups, i.e. we will find the necessary and sufficient conditions for a right loop isotopic to a group to be generated by any its non-unit element, therefore to have no proper subloops. A results can be obtained for left loops isotopic to groups.

Recall that a *right (left) loop* is a quasigroup $(\mathcal{Q}\,,*)$ with the right (left) unit f (respectively - (e)) i.e. such elements that $x*f=x$ $(e*x=x)$ for all $x\in\mathcal{Q}$. The sets $\{f\}$, $\{e\}$ and $\mathcal{Q}$ are right (left) subloops of $(\mathcal{Q}\,,*)$ called *improper subloops.* All other subloops are called *proper subloops.*

For all $a\in\mathcal{Q}$ put

$$\mathcal{S}_a(\mathcal{Q}) = \{\psi\in\mathcal{S}(\mathcal{Q}) \;\;:\;\; \psi a = a\}\,.$$

Proposition 2. *A right loop isotopic to a group* $(\mathcal{Q}\,,\cdot)$ *with the unit* e *is isomorphic to some right loop*

$$(\mathcal{Q}\,,\circ) \;\;:\;\; x\circ y = x\cdot\varphi y, \quad \varphi\in\mathcal{S}_e(\mathcal{Q}), \quad x,y\in\mathcal{Q}$$

with the unit e.

Proof. To prove that, by Theorem 2 it is sufficient to consider right loops principally isotopic to groups. Let

$$(\mathcal{Q}\,,*) \;\;:\;\; x*y = \alpha x\cdot\beta y, \quad \alpha,\beta\in\mathcal{S}(\mathcal{Q})$$

be a right loop isotopic to a group $(\mathcal{Q}\,,\cdot)$ and f be the right unit of $(\mathcal{Q}\,,*)$. For every $x\in\mathcal{Q}$ we have $x=x*f=\alpha x\cdot\beta f$ and we obtain

$$\alpha = R^{-1}_{(\beta f)} = R_{(\beta f)^{-1}}\,.$$

Therefore

$$x*y = R_{(\beta f)^{-1}}x\cdot\beta y = x(\beta f)^{-1}\beta y = x\cdot L_{(\beta f)^{-1}}\beta y$$

for all $x,y\in\mathcal{Q}$. Let us consider the isotope

$$(\mathcal{Q}\,,\circ)\quad:\quad x\circ y = x\cdot L_{(\beta f)^{-1}}\beta L_f y,\qquad x,y\in\mathcal{Q}\,.$$

Remark that $L_{(\beta f)^{-1}}\beta L_f e = e$, i.e. $(\mathcal{Q}\,,\circ)$ is a right loop with the unit e. The loop $(\mathcal{Q}\,,\circ)$ is isomorphic to $(\mathcal{Q}\,,*)$ since

$$L_f(x\circ y) = L_f(x\cdot L_{(\beta f)^{-1}}\beta L_f y) = L_f x\cdot L_{(\beta f)^{-1}}\beta L_f y = L_f x * L_f y$$

for all $x,y\in\mathcal{Q}$. $\square$

The following assertion can be proved in a way analogous to that used in Theorem 3.

Lemma 5. *A right loop*

$$(\mathcal{Q}\,,*)\quad:\quad x*y = x\cdot\alpha y,\qquad \alpha\in\mathcal{S}_e(\mathcal{Q}),\qquad x,y\in\mathcal{Q}$$

isotopic to a group $(\mathcal{Q}\,,\cdot)$ *with the unit* e *has no proper subloops if and only if*

$$Stab_{L(\mathcal{Q},\cdot)}\alpha = \{\{e\},\ \mathcal{Q}\}.$$ $\square$

Lemma 6. *A right loop*

$$(\mathcal{Q}\,,*)\quad:\quad x*y = x\cdot\alpha y,\qquad \alpha\in\mathcal{S}(\mathcal{Q}),\qquad \alpha f = e,\qquad x,y\in\mathcal{Q}$$

isotopic to a group $(\mathcal{Q}\,,\cdot)$ *with the unit* e *has no proper subloops if and only if*

$$Stab_{L(\mathcal{Q},\cdot)}\alpha L_e = \{\{e\},\mathcal{Q}\}$$

Proof. The right loop $(\mathcal{Q}\,,*)$ is isomorphic to the right loop

$$(\mathcal{Q}\,,\circ) \;\;:\;\; x\circ y = x\cdot \alpha L_f y, \quad x,y\in \mathcal{Q}\,.$$

In fact

$$L_f(x\circ y) = L_f(x\cdot \alpha L_f y) = L_f x\cdot \alpha L_f y = L_f x * L_f y$$

for all $x,y\in\mathcal{Q}$. Then the right loop $(\mathcal{Q}\,,*)$ has no proper subloops provided that $(\mathcal{Q}\,,\circ)$ has no ones. Now we apply Lemma 5 to the right loop $(\mathcal{Q}\,,\circ)$ and this completes the proof of our Lemma. □

Corollary 5. *The loop isotopic to a group of prime order has no proper right subloops.*

Proof. The assertion follows from Lemma 6 taking into account that a group of a prime order has no proper subgroups. □

Theorem 4. *Let* $(\mathcal{Q}\,,\cdot)$ *be a group with the unit* e. *The right loop*

$$(\mathcal{Q}\,,*) \;\;:\;\; x*y = \gamma^{-1}(\alpha x\cdot \beta y), \quad \alpha,\beta,\gamma\in \mathcal{S}(\mathcal{Q}), \quad x,y\in\mathcal{Q}$$

with the right unit f *has no proper right subloops if and only if*

$$Stab_{L(\mathcal{Q},\cdot)} L_{(\beta f)^{-1}}\beta\gamma^{-1}L_{\gamma f} = \{\{e\},\,\mathcal{Q}\}$$

Proof. By Theorem 2 the right loop $(\mathcal{Q}\,,*)$ is isomorphic to the right loop

$$(\mathcal{Q}\,,*) \;\;:\;\; x*y = \gamma^{-1}(\alpha x\cdot \beta y), \quad \alpha,\beta,\gamma\in \mathcal{S}(\mathcal{Q}), \quad x,y\in\mathcal{Q}$$

with the unit γf. For every $x\in\mathcal{Q}$ we have

$$x = x\circ \gamma f = \alpha\gamma^{-1}x\cdot \beta f\,,$$

hence $\alpha\gamma^{-1} = R_{(\beta f)^{-1}}$. Therefore

$$x\circ y = R_{(\beta f)^{-1}}x\cdot \beta\gamma^{-1}y = x\cdot(\beta f)^{-1}\beta\gamma^{-1}y = x\cdot L_{(\beta f)^{-1}}\beta\gamma^{-1}y\,.$$

Let us consider the isotope

$$(\mathcal{Q}\,,\times) \;\;:\;\; x\times y = x\cdot L_{(\beta f)^{-1}}\beta\gamma^{-1}L_{(\gamma f)}y, \quad x,y\in\mathcal{Q}\,.$$

This isotope is a right loop with the right unit e and the translation $L_{\gamma f}$ is an isomorphism between $(\mathcal{Q}\,,\times)$ and $(\mathcal{Q}\,,\circ)$. Now we apply Lemma 5 to the right loop $(\mathcal{Q}\,,\times)$ and this completes the proof. □

4. Examples

1. Let $(\mathcal{Q}\,,\,\cdot) = \langle h\rangle$ be a cyclic group generated by $h \in \mathcal{Q}$ and $|\mathcal{Q}| = r$, $3 \le r \le \aleph_0$. Let e be the unit of the group $(\mathcal{Q}\,,\,\cdot)$, I be the permutation of $(\mathcal{Q}\,,\,\cdot)$ defined by $Ix = x^{-1}$, $\alpha = (e\,h)$ be the transposition of elements e and h. Then the isotope

$$(\mathcal{Q}\,,*) \quad : x * y = \alpha x \cdot Iy, \quad x, y \in \mathcal{Q}$$

is a monoquasigroup which satisfies the identity $x * (x * y) = y$. In fact

$$x * (x * y) = \alpha x \cdot I(\alpha x \cdot Iy) = \alpha x \cdot I\alpha x \cdot y = y\,.$$

Recall that every element of the group $(\mathcal{Q}\,,\,\cdot) = \langle h\rangle$ is a power of the element h and every its subgroup is the cyclic subgroup generated by some power h^k of h. To prove that $(\mathcal{Q}\,,*)$ is a monoquasigroup, by Lemma 3, it is sufficient to prove the equality

$$Stab_{L(\mathcal{Q},\cdot)}\alpha \cap Stab_{L(\mathcal{Q},\cdot)}I = \{\mathcal{Q}\}\,.$$

This equality holds since we have $\alpha e = h$ and the group $(\mathcal{Q}\,,\,\cdot)$ is the unique subgroup of $(\mathcal{Q}\,,\,\cdot)$ which contains h.

Now we will prove that every subquasigroup $(H, *)$ of $(\mathcal{Q}\,,*)$ contains e. Really, if $h^k \in (H, *)$ for $k \neq 1$, then $e = h^k \cdot h^{-k} \in (H, *)$. If $h \in (H, *)$, then $h^{-1} = e \cdot h^{-1} = \alpha h \cdot Ih = h * h \in (H, *)$ and $e \in (H, *)$ as it is proved above.

2. Let $(\mathcal{Q}\,,\cdot) = \langle h\rangle$ be a cyclic group with the unit e generated by $h \in \mathcal{Q}$, $|\mathcal{Q}| = n$, $3 \le n \le \aleph_0$ and α be a cyclic permutation $(hh^2h^3...h^{n-1})$. Then the isotope

$$(\mathcal{Q}\,,*) \quad : \quad x * y = x \cdot \alpha y, \quad x, y \in \mathcal{Q}$$

is a right loop with the unit e and has no proper right subloops.

Really, we have $x * e = x \cdot \alpha e = x \cdot e = x$ for all $x \in \mathcal{Q}$ and thus $(\mathcal{Q}\,,*)$ is a right loop with the unit e. If $1 \le k \le n-2$, then $\alpha h^k = h^{k+1}$ and $\alpha h^{-1} = \alpha h^{n-1} = h$. Now, if a subgroup of the cyclic group $\langle h\rangle$ contains elements h^k and h^{k+1}, then it contains h as a solution of the equation $h^k \cdot x = h^{k+1}$, and thus it coincides with $\langle h\rangle$. Hence, we have

$$Stab_{L(\mathcal{Q},\cdot)}\alpha = \{\{e\}, \mathcal{Q}\},$$

and thus $(\mathcal{Q}, *)$ has no proper right subloops.

3. Let $(\mathcal{Q}, \cdot) = \langle h \rangle$ be the infinite cyclic group generated by the element $h \in \mathcal{Q}$. Let e be the unit of $(\mathcal{Q}, \cdot)$ and α be the following permutation: $\alpha e = e$, $\alpha h^{-1} = h$, $\alpha h^k = h^{k+1}$ for all $k \neq 1$. Like that in the example 2 we can prove that the right loop

$$(\mathcal{Q}, *) \quad : \quad x * y = x \cdot \alpha y, \quad x, y \in \mathcal{Q}$$

has no proper right subloops.

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Frobenius groups and one-sided S-systems

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Abstract

Frobenius groups are studied by the means of systems of orthogonal operations, naturally being built over these groups.

1. Introduction

Definition 1. [4,8] The transitive irregular permutation group G acting on a set E is called a *Frobenius group*, if $St_{ab}(G) = \langle id \rangle$ for any $a, b \in E,\ a \neq b$.

Frobenius groups are one of the classical group classes in permutation group theory. The studying of these groups was begun in the Frobenius article [3] at the beginning of 20th century and was continued by M.Hall [4], H.Wielandt [8] etc. Frobenius proved in [3] by means of character group theory that there exists an invariant regular subgroup consisting of all fixed-point free permutations and the identity permutation in a finite Frobenius group (Frobenius theorem). It is not known any other proof of this theorem (without using of character group theory) till now.

In present article a 1-1 correspondence between Frobenius groups and one-sided S-systems of orthogonal operations [1] (on the same set of symbols E), whose cell permutations form a group, is built.

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In the section 2 the incident system of (left) cosets in an arbitrary finite Frobenius group G by stabilizer $St_a(G)$ $(a \in E)$ is investigated. It is proved that this incident system is an algebraic m-net [2], where $m = |St_a(G)|$.

In the section 3 the construction of two systems of the orthogonal operations over an arbitrary finite Frobenius group G is given. It is proved that they form a left and a right S-systems [1]. Some other properties of these one-sided S-systems are studied too. A numbering correlation between permutations degree n and $m = |St_a(G)|$ is obtained. As a corollary of this correlation it is proved that finite Frobenius p-groups doesn't exist (the negative answer on the problem 6.55 from [5] in a finite case). At the end of the part 2 the right (left) cell permutations of right (left) S-system are defined and it is shown that the set of all cell permutations forms a group coinciding with the group G.

In the section 3 an arbitrary right (left) S-systems of binary idempotent quasigroups on some set E (finite or infinite) are investigated. In any right (left) S-system of operations the cell functions are introduced, and it is proved that all these functions are permutations on the set E. If the set of all cell permutations forms a group (with respect to natural operation of composition), then this group is a Frobenius group. As a corollary, the proof of Frobenius theorem is obtained (when the set E is finite). Another construction of one-sided S-systems of operations on E over the Frobenius group G, no depending from the cardinality of the set E, are given in order to demonstrate preserving of the correspondence between Frobenius groups and one-sided S-systems of operations on E with the property mentioned above in the case when the set E is infinite.

We will use the following notations:

$H_a = St_a(G)$ is the stabilizer of the element $a \in E$ in the group G,

$0,\ 1$ are two distinguished elements in the set E,

$E^* = \{0\} \cup \{h(1)\ :\ h \in H_0 = St_0(G)\} \subseteq E$.

2. Incident system of cosets

In this paragraph we suppose that the set E is finite, i.e. the permutations from the Frobenius group G have the finite degree $n = |E|$.

In a Frobenius group all subgroups H_a $(a \in E)$ are conjugate and so we can denote

$$m = |H_0| = |H_a|\,.$$

At last, we can suppose the elements from E are renamed so that

$$E^* = \{0, 1, ..., m\}.$$

Let's consider all (left) cosets $H_a^b = \{\alpha \in G \ : \ \alpha(a) = b\}$ in G by the subgroup H_a and define the following incident system $\mathcal{R} =< X, \mathcal{L}, I >$:

points from X are (left) cosets H_a^b,

lines from $\mathcal{L}$ are permutations $\alpha \in G$,

incidence I is a belonging relation, i.e.

$$(a,b)I[\alpha] \ \Leftrightarrow \ (\,point\ H_a^b)I(\,line\ \alpha) \overset{def}{\Leftrightarrow} (\,\alpha \in H_a^b\,). \qquad (1)$$

Definition 2. By an *algebraic k-net* [2] we mean an incidence system $\mathcal{R} =< X, \mathcal{L}, L_1, ..., L_k, I >$ consisting of the point set X, the line set $\mathcal{L}$ which is separated on k distinct classes of "parallel" lines $l_1, L_2, ..., L_k$, and the incidence relation I between elements from X and $\mathcal{L}$, which satisfy the following two conditions:

1) any two lines from the different classes L_i and L_j are incident to one and only one point from X,
2) every point from X is incident to one and only one line from each class L_i.

Lemma 1. *The system $\mathcal{R} =< X, \mathcal{L}, I >$ defined in (1) is an algebraic m-net.*

Proof. According to Frobenius theorem [8], in a finite Frobenius group G of permutations of degree n all fixed-point-free permutations with the identity permutation form a transitive invariant subgroup A, moreover, $|A| = n$. It is easy to see that A is a group transversal (see [6]) in G to H_a $\forall a \in E$.

Let's define the classes L_i of "parallel" lines in $\mathcal{L}$ by the following:

$$L_i = \{\alpha h_i \,:\, \alpha \in A,\ h_i \in H_0, \quad h_i(1) = i\}, \quad i = 1, ..., n\,.$$

Note that $h_i = id$ and $L_i = A$.

Lemma A. *Let $\alpha, \beta \in \mathcal{L}$ and $\alpha \neq \beta$. The following conditions are equivalent:*

1) *both of lines α and β are in the class L_i for some i,*
2) *$\alpha(t) \neq \beta(t) \quad \forall t \in E$.*

Proof of Lemma A. 1) $\Rightarrow$ 2). Let $\alpha, \beta \in L_i$ and $\alpha \neq \beta$. Let's assume there exists $t_0 \in E$ such that $\alpha(t_0) = \beta(t_0)$. Then we have

$$\alpha_0 h_i(t_0) = \beta_0 h_i(t_0),$$
$$\alpha_0(t_1) = \beta_0(t_1)$$

where $\alpha_0, \beta_0 \in A,\ t_1 = h_i(t_0)$. The last equality contradicts the regularity of the group A. So

$$\alpha(t) \neq \beta(t) \quad \forall t \in E.$$

2) $\Rightarrow$ 1). Let $\alpha, \beta \in \mathcal{L}$, $\alpha \neq \beta$ and

$$\alpha(t) \neq \beta(t) \quad \forall t \in E.$$

The set A is a (left) transversal in G to H_0, so we have

$$\alpha = \alpha_0 h_i, \qquad \beta = \beta_0 h_j,$$

where $\alpha_0, \beta_0 \in A,\ h_i, h_j \in H_0$. It is necessary to prove that $h_i = h_j$. We have

$$\alpha_0 h_i(t) \neq \beta_0 h_j(t) \quad \forall t \in E,$$
$$\alpha_0^{-1} \beta_0 h_j h_i^{-1}(t') \neq t' \quad \forall t' = h_i(t) \in E,$$

i.e. $\gamma_0 = \alpha_0^{-1} \beta_0 h_j h_i^{-1}$ is a fixed-point-free permutation. Then $\gamma_0 \in A$ and we obtain

$$h_j h_i^{-1} = h_k = \beta_0^{-1} \alpha_0 \gamma_0 \in H_0 \cap A = \{id\},$$

i.e. $h_i = h_j$. The proof of Lemma A is completed. □

Let's return to the proof of Lemma 1. It is necessary to check the realization of the conditions 1) and 2) from Definition 2 for the incidence system $\mathcal{R} =< X, \mathcal{L}, I >$.

a) Let α and β be two different lines from the different classes L_i and L_j. Then there exist element $t_0 \in E$ such that

$$\alpha(t_0) = \beta(t_0) = d, \tag{2}$$

(in a contrary case we would have $\alpha(t) \neq \beta(t) \quad \forall t \in E$, and so $\alpha, \beta \in L_k$ for some k according to Lemma A. Moreover, there exist an unique element $t_0 \in E$ satisfying (2), because in a opposite case the permutation $\alpha^{-1}\beta$ would fix two different elements from E and so $\alpha^{-1}\beta = id$ according to Definition 1. So we have:

$$\alpha, \beta \in H^d_{t_0},$$

i.e. the lines α and β are incident to the unique point $H^d_{t_0}$. The condition 1) is proved.

b) Let H^b_a be an arbitrary point from X. This point is incident to all lines $\alpha_i \in G$ such that $\alpha_i \in H^b_a$, i.e. $\alpha_i(a) = b$. By means of Lemma A we obtain that different such lines α_i lie in different classes L_i. As $|H^b_a| = |H_a| = m$, then the point H^b_a is incident to m different lines α_i from different classes L_i; moreover, it is incident to an unique line in each class L_i. The number of classes L_i is equal to m, so every of these classes consists of a line being incident to the point H^b_a. The condition 2) is proved.

The proof of Lemma 1 is completed. $\square$

3. One-sided S-systems being constructed over a Frobenius group

In this paragraph we will suppose that Frobenius group G is finite.

Let's define the following two binary operations $(\cdot)$ and $(*)$:

$$(\cdot) \; : \; E \times E \to E,$$
$$x \cdot y = z \overset{def}{\Longleftrightarrow} z = \varphi_x(y),$$

where $\varphi_x \in A,\ \varphi_x(0) = x$,

$$(*)\ :\ E^* \times E \to E,$$

$$0 * v \overset{def}{\Longleftrightarrow} 0,$$

$$u \neq 0\ :\ u * v = w \overset{def}{\Longleftrightarrow} w = h_u(v),$$

where $h_u \in H_0,\ h_u(1) = u$. Note that $(*)$ is a partial operation.

Lemma 2. *The following statements are true:*

1) $< E, \cdot, 0 > \cong\ A$,
2) $< E^* - \{0\}, *, 1 > \cong\ H_0$,
3) $x * (y \cdot z) = (x * y) \cdot (x * z) \quad \forall x \in E^*, \quad \forall y, z \in E$,
4) *every permutation* $h \in H_0$ *is an automorphism of the subgroup* A,
5) $G = \{\alpha_{a,b}\ :\ \alpha_{a,b}(x) = a \cdot (b * x),\ \ a \in E\ \ b \in E^* - \{0\}\}$.

Proof. 1) Let's consider the following mapping

$$\alpha\ :< E, \cdot, 0 > \to\ A\,, \qquad \alpha(x) = \varphi_x$$

where $x \in E$ and the permutation φ_x is defined above. Then α is a bijection, because the group A is regular on the set E. Further we have

$$(\alpha(x \cdot y))(0) = \varphi_{x \cdot y}(0) = x \cdot y.$$

On the other hand we have

$$(\alpha(x)\alpha(y))(0) = \varphi_x \varphi_y(0) = \varphi_x(y) = x \cdot y\,.$$

So we obtain

$$(\alpha(x \cdot y))(0) = x \cdot y = (\alpha(x)\alpha(y))(0),$$

and

$$\alpha(x \cdot y) = \alpha(x)\alpha(y),$$

because the group A is regular (i.e. sharply transitive) on the set E. We obtain that the mapping α is an isomorphism.

2) can be proved analogously, and the isomorphism is determined by the mapping

$$\beta\ :< E^* - \{0\}, *, 1 > \to\ H_0\,, \quad \beta(u) \overset{def}{=} h_u,$$

where $u \in E^* - \{0\}$ and the permutation h_u is defined above.

3) and 4) can be proved analogously to 3) of Lemma 8 from [7].

5) can be proved analogously to Lemma 9 from [7]. $\square$

Note that $\alpha_{0,1} \equiv id$ and $\alpha_{a,1}(x) = a \cdot x$ is a fixed-point-free permutation if $a \neq 0$; moreover $\{\alpha_{a,1}\}_{a \in E} \equiv A$.

Now let's define the following partial ternary operation

$$(\, , \, , \,) \, : \, E \times E^* \times E \to E,$$

$$(x, a, y) \overset{def}{=} x \cdot (a * (x^{-1} \cdot y)), \tag{3}$$

where $a \in E^*$, $x, y \in E$, and x^{-1} is the inverse element to x in $< E, \cdot, 0 >$.

Lemma 3. *The following statements are true:*

1) $(x, 0, y) = x, \quad (x, 1, y) = y,$
   $(x, a, x) = x, \quad (0, a, 1) = a, \quad \forall a \in E^*, \quad x, y \in E.$
2) *The system of operations* $A_a(x, y) = (x, a, y)$ $(a \in E^* - \{0\})$ *is a right* S*-system.*
3) *The operations* (x, a, y) *and* (x, b, y) *are orthogonal for any* $a \neq b$*, and they are quasigroups for* $a \neq 0, 1$*.*
4) *The operations* (x, a, y) *and* $x \circ y = x^{-1} \cdot y$ *are orthogonal for any* $a \in E^*$*.*

Proof. 1)
$$\begin{gathered}(x, 0, y) = x \cdot (0 * (x^{-1} \cdot y)) = x \cdot 0 = x,\\ (x, 1, y) = x \cdot (1 * (x^{-1} \cdot y)) = x \cdot x^{-1} \cdot y = y,\\ (x, a, x) = x \cdot (a * (x^{-1} \cdot x)) = x \cdot (a * 0) = x \cdot 0 = x,\\ (0, a, 1) = 0 \cdot (a * (0^{-1} \cdot 1)) = a * 1 = a.\end{gathered}$$

2) According to the definition from [1], a system of operations $A_a(x, y)$ $(a \in E^* \subseteq E)$ on some set E is a *right (left)* S*-system*, if for any $a, b \in E^*$ and $x, y \in E$ there exists $c = c(a, b) \in E^*$ such that the following equality

$$(A_a \circ A_b)(x, y) = A_a(x, A_b(x, y)) = A_c(x, y)$$

holds, and moreover, the system $< A_u, \circ, A_1 >$, where $u \neq 0$, is a group (correspondingly, if for any $a, b \in E^*$ and $x, y \in E$ there exist such $c = c(a, b) \in E^*$ that the following equality

$$(A_a \bullet A_b)(x,y) = A_a(A_b(x,y),y) = A_c(x,y)$$

holds, and moreover, the system $< A_u(u \neq 1), \bullet, A_0 >$ is a group).

According to the equality (3) we obtain for the operations $A_a(x,y) = (x,a,y)$ and $A_b(x,y) = (x,b,y)$ (where $a,b \in E^* - \{0\}$):

$$\begin{gathered}(A_a \circ A_b)(x,y) = A_a(x, A_b(x,y)) = (x,a,(x,b,y)) = \\ = x \cdot (a * (x^{-1} \cdot (x \cdot (b * (x^{-1} \cdot y))))) = \\ = x \cdot (a * b * (x^{-1} \cdot y)) = A_{a*b}(x,y).\end{gathered}$$

With the help of Lemma 2, we obtain that the system $< A_u, \circ, A_1 >$, where $u \neq 0$ is a group (this group is isomorphic to the group H_0), i.e. the system of operations $A_a(x,y) = (x,a,y)$ is a right S-system.

3) We notice that for any $a \in E^*$, $z \in E$,

$$(a * z)^{-1} = a * x^{-1}.$$

Really, with the help of Lemma 2, we obtain

$$\begin{gathered}(a * z) \cdot (a * z^{-1}) = a * (z \cdot z^{-1}) = a * 0 = 0, \\ (a * z^{-1}) \cdot (a * z) = a * (z^{-1} \cdot z) = a * 0 = 0.\end{gathered}$$

Further, let we have the following system

$$\begin{cases} (x,a,y) = c, \\ (x,b,y) = d, \end{cases}$$

where $a,b \in E^*$, $a \neq b, c, d \in E$ are arbitrary given elements.

If $a = 0$ then $x = c$, $y = c \cdot (b^{-1} * (c^{-1} \cdot d))$, i.e. this system has an unique solution in $E \times E$; so the operations (x,a,y) and (x,b,y) are orthogonal. If $b = 0$ then we obtain the same result.

Let $a,b \neq 0$. Then we have

$$\begin{cases} (x,a,y) = c \\ (x,b,y) = d \end{cases} \Longleftrightarrow \begin{cases} x \cdot (a * (x^{-1} \cdot y)) = c \\ x \cdot (b * (x^{-1} \cdot y)) = d \end{cases} \Longleftrightarrow$$

$$\begin{cases} a * (x^{-1} \cdot y) = x^{-1} \cdot c \\ b * (x^{-1} \cdot y) = x^{-1} \cdot d \end{cases} \Longleftrightarrow \begin{cases} (a * (x^{-1} \cdot y))^{-1} = c^{-1} \cdot x \\ (b * (x^{-1} \cdot y))^{-1} = d^{-1} \cdot x \end{cases} \Longleftrightarrow$$

$$\begin{cases} a * (y^{-1} \cdot x) = c^{-1} \cdot x \\ b * (y^{-1} \cdot x) = d^{-1} \cdot x \end{cases} \Leftrightarrow a^{(-1)} * (c^{-1} \cdot x) = y^{-1} \cdot x = b^{(-1)} * (d^{-1} \cdot x),$$

where a^{-1} is the inverse element to a in $< E^* - \{0\}, *, 1 >$. From the last equality we obtain

$$c^{-1} \cdot x = (a * b^{-1}) * ((d^{-1} \cdot c) \cdot (c^{-1} \cdot x)),$$

i.e. (see Lemma 2)

$$c^{-1} \cdot x = \alpha_{d^{-1} \cdot c, a * b^{(-1)}}(c^{-1} \cdot x)$$

As $a \neq b$ then $a * b^{(-1)} \neq 1$; so the permutation $\alpha_{d^{-1} \cdot c, a * b^{(-1)}}$ has an unique fixed-point element p_0. So we obtain that $x = c \cdot p_0$, $y = c \cdot p_0 \cdot (a^{(-1)} * p_0)$, i.e. the operations (x, a, y) and (x, b, y) are orthogonal.

4) We have for any $a \in E^* - \{0\}$ and $c, d \in E$:

$$\begin{cases} (x, a, y) = c \\ x \circ y = d \end{cases} \Longleftrightarrow \begin{cases} x \cdot (a * (x^{-1} \cdot y)) = c \\ x^{-1} \cdot y = d \end{cases} \Longleftrightarrow$$

$$\begin{cases} x \cdot (a * d) = c \\ y = x \cdot d \end{cases} \Longleftrightarrow \begin{cases} x = c \cdot (a * d)^{-1} \\ y = c \cdot (a * d)^{-1} \cdot d, \end{cases}$$

i.e. the operations (x, a, y) and $x \circ y$ are orthogonal. $\square$

By an analogical way it can be defined one more partial ternary operation

$$[x, t, y] \ : \ E \times E^* \times E \to E$$
$$[x, a, y] \overset{def}{=} (a * (x \cdot y^{-1})) \cdot y\,, \quad a \in E^*.$$

Lemma 4. *The following statements are true:*

1) $[x, 0, y] = y, \quad [x, 1, y] = x,$
   $[x, a, x] = x, \quad [1, a, 0] = a, \quad \forall a \in E^*, \quad x, y \in E.$
2) *The system of operations* $A_a(x, y) = [x, a, y]$ $(a \in E^* - \{0\})$ *is a left S-system.*
3) *The operations* $[x, a, y]$ *and* $-[x, b, y]$ *are orthogonal for any* $a \neq b$ *and they are quasigroups for* $a \neq 0, 1$.
4) *The operations* $[x, a, y]$ *and* $x \bullet y = x \cdot y^{-1}$ *are orthogonal for any* $a \in E^*$.

Proof. 1). It is evident.

2)-4) can be proved analogously to the proof of Lemma 3. $\square$

Remark. There is a 1-1 correspondence between algebraic m-net from Lemma 1 and some system of m orthogonal operations on the set E. Defining the partial ternar $< x, t, y >$ by the following:

$$(a, b)I[c, d] \iff < a, c, b >= d,$$

where I is the incidence relation, we obtain

$$< x, a, y >= y \cdot (a * x)^{-1}, \ a \in E^*$$

This system of operations is not a left or a right S-system of operations.

Let's return back to the ternary operation (x, a, y).

Lemma 5. *The following statements are true:*
1) *The mapping* $\alpha_{b,a}(x) = b \cdot (a * x)$, $b \in E$, $a \in E^* - \{0\}$, $x \in E$ *is an isomorphism between operations* (x, c, y) *and* $(x, a * c * a^{-1}, y)$.
2) *The mapping* $\alpha_{b,I}$ *is an automorphism of the operation* (x, c, y) *for any* $b \in E$, $c \in E^*$.

Proof. 1) We have:

$$\alpha_{b,a}((x, c, y)) = b \cdot (a * (x \cdot (c * (x^{-1} \cdot y)))) =$$
$$= b \cdot (a * x) \cdot (a * c * (x^{-1} \cdot y)) =$$
$$= b \cdot (a * x) \cdot ((a * c * a^{(-1)}) * a * (x^{-1} \cdot y)) =$$
$$= b \cdot (a * x) \cdot ((a * c * a^{(-1)}) * ((a * x^{-1}) \cdot (a * y))) =$$
$$= b \cdot (a * x) \cdot ((a * c * a^{(-1)}) * ([(a * x)^{-1} \cdot b^{-1}] \cdot [b \cdot (a * y)])) =$$
$$= (\alpha_{b,a}(x)) \cdot ((a * c * a^{(-1)}) * ((\alpha_{b,a}(x))^{-1} \cdot (\alpha_{b,a}(y)))) =$$
$$= (\alpha_{b,a}(x), a * c * a^{(-1)}, \alpha_{b,a}(y)).$$

It means that the mapping $\alpha_{b,a}$ is an isomorphism between operations (x, c, y) and $(x, a * c * a^{-1}, y)$.

2) is an evident corollary of 1). $\square$

Lemma 6. *The following equality is true:*

$$n - 1 = km$$

for some $k \in \mathcal{N}$, *i.e.* $(|E^*| - 1) \,|\, (|E| - 1)$.

Proof. (see [1] too). Let $a_1 \in E$ and $a_1 \neq 0$. Let's consider the following equalities:

$$A_1(0, a_1) = (0, 1, a_1) = 1 * a_1 = a_1$$
$$A_2(0, a_1) = (0, 2, a_1) = 2 * a_1$$
$$\dots\dots\dots\dots\dots\dots$$
$$A_m(0, a_1) = (0, m, a_1) = m * a_1.$$

All values in the right sides of the equalities are different. Really, we have

$$a * a_1 \neq b * a_1, \quad a \neq b$$

because $< E^* - \{0\}, *, 1 >$ is a group.

Let's denote

$$M_1 = \{A_i(0, a_1) \,:\, i = 1, ..., m\}, \quad |M_1| = m.$$

Let $a_2 \neq 0$ and $a_2 \in E \backslash M_1$. By analogy with above we have:

$$A_1(0, a_2) = a_2$$
$$A_2(0, a_2) = 2 * a_2$$
$$\dots\dots\dots\dots$$
$$A_m(0, a_2) = m * a_2$$

and we obtain analogously that the right sides of these equalities are different. Let's denote

$$M_2 = \{A_i(0, a_2) \,:\, i = 1, ..., m\}, \quad |M_2| = m$$

If we assume that

$$M_1 \cap M_2 \neq \emptyset,$$

i.e. there exists $b \in M_1 \cap M_2$, then there exist such $k, r \in E^* - \{0\}$ that

$$b = A_k(o, a_1) = A_r(o, a_2),$$
$$b = k * a_1 = r * a_2$$
$$a_2 = r^{(-1)} * k * a_1 = k' * a_1\,, \quad k' = r^{(-1)} * k,$$

i.e. $a_2 \in M_1$, contradicting to the choosing of the element a_2. Continuing this process up to the complete exhaustion of the set E, we obtain

$$E \backslash \{0\} = \coprod_{i=1}^{k} M_i \,,$$

moreover, $M_i \cap M_j = \emptyset$ if $i \neq j$, and $|M_i| = m$ for every $i = 1, ..., k$. Then

$$|E - \{0\}| = k \cdot |M_1| \,,$$
$$n - 1 = k \cdot m$$

and Lemma 6 is proved. □

Corollary. *Finite nontrivial (i.e. $m > 1$ and $n > 1$) Frobenius p-group does not exist.*

Proof. Let's assume the contrary and let G be a Frobenius p-group with $m, n > 1$. Then we have

$$|G| = p' \,, \quad m = |H_0| \mid |G| \;\Rightarrow\; m = p^s, \quad s > 0,$$
$$n = |A| \mid |G| \;\Rightarrow\; n = p^t, \quad t > 0.$$

With the help of Lemma 6 we obtain

$$n - 1 = k \cdot m$$
$$p^t - 1 = k \cdot p^s$$
$$p^t - k \cdot p^s = 1$$

that is impossible because the left part of the last equality is divisible by number p, but the right one is not divisible. □

This corollary gives a negative solution of the problem 6.55 from Kourovskaya notebook [5] in the case of a finite Frobenius group.

Lemma 7. *The mappings*

$$\varphi_{b,a}(x) = (b, a, x),$$
$$\psi_{b,a}(x) = (b, a, (0, a^{(-1)}, x)),$$

where $b \in E$, $a \in E^* - \{0, 1\}$ and $a^{(-1)}$ is inverse to a in the group $< E^* - \{0\}, *, 1 >$, form a permutation group, which is isomorphic to *the group G.*

Proof. By means of Lemma 2 we have for $a \in E^*\backslash\{0,1\}$ and $b \in E$:
$\varphi_{b,a}(x) = (b,a,x) = b \cdot (a * (b^{-1} \cdot x)) = b \cdot (a * b^{-1}) \cdot (a * x) = \alpha_{c,a}(x)$,
where
$$c = b \cdot (a * b^{-1}) = (b,a,0).$$
In a such way it can be represented all the permutations $\alpha_{b,a}$ from G, except the permutations like $\alpha_{b,1}$. Further we obtain
$$\psi_{b,a}(x) = (b,a,(0,a^{(-1)},x)) \\ = b \cdot (a * (b^{-1} \cdot (a^{(-1)} * x))) = b \cdot (a * b^{-1}) \cdot x = \alpha_{c,1}(x)$$
where
$$c = b \cdot (a * b^{-1}) = (b,a,0).$$
In a such way it can be represented all the permutations from G like $\alpha_{b,1}$. It means that the set of permutations like $\varphi_{b,a}$ and $\psi_{b,a}$ ($a \in E^*\backslash\{0,1\}$ and $b \in E$) coincides with the set of permutations $\alpha_{b,a}$. By the help of Lemma 2 we obtain that this set of permutations forms a group, which is isomorphic to the group G. □

The mappings like $\varphi_{b,a}$ and $\psi_{b,a}$ are called *right cell permutations of the ternar* (x,t,y) (cf. [7]).

It is evident, that the analogous symmetric constructions can be done for the ternar $[x,t,y]$ too.

4. One sided S-systems of operations, whose cell permutations forms a group

In this paragraph the set E may be as finite as infinite.

Let $A_0(x,y), A_1(x,y), ..., A_m(x,y), ...$ be a collection of binary operations on some set E ($E^* = \{0,1,...,m,...\}$ is the set of indexes of th operations $A_i(x,y)$, and moreover, $E^* \subseteq E$), and let this collection forms a right S-system of indempotent quasigroups $A_1(x,y)$, $i \neq 0,1$, i.e.
$$A_0(x,y) = x\,, \quad A_1(x,y) = y\,, \quad A_i(x,x) = x,$$

$$(A_a \circ A_b)(x,y) = A_a(x, A_b(x,y)) = A_c(x,y) \tag{4}$$

for some $c \in E^* - \{0\}$, moreover, system $< A_u, \circ, A_1 >$, where $u \neq 0$, is a group. Rewrite (4) as a (partial) ternary operation (x,t,y):

$$(\ ,\ ,\ )\ :\ E \times E^* \times E \to E\,, \quad E^* \subseteq E, \quad 0, 1 \in E,$$
$$(x,t,y) = A_t(x,y),$$

i.e.

$$(x,0,y) = x\,, \quad (x,1,y) = y,$$

$$\forall\, a,b \in E^* - \{0\}\ :\ (x,a,(x,b,y)) = (x,c,y) \tag{5}$$

for some $c = c(a,b) \in E^* - \{0\}$,

$$(x,a,x) = x \quad \forall\, x \in E, \quad \forall\, a \in E^*. \tag{6}$$

It is easy to prove that all operations (x,a,y) are mutually orthogonal for different $a \in E^*$. As all the operations (x,a,y) are mutually orthogonal and the identity (6) is true, we obtain that all values $(0,a,1)$ are different for differenet $a \in E^*$. So renumerating the indexes from E^* it can obtain the following identity $(a \in E^*)$:

$$(0,a,1) = a. \tag{7}$$

Let's define the following operation $(*)$ on the set E^*:

$$(*)\ :\ E^* * E^* \to E^*,$$
$$0 * a = a * 0 = 0,$$
$$x,y,z \neq 0\,, \quad x * y = z \Longleftrightarrow (x,a,(x,b,y)) = (x,c,y).$$

As a corollary of (4) and (5) we obtain the system $< E^* - \{0\}, *, 1 >$ is a group. Further we have from (5) when $x = 0$ and $y \in E^* - \{0\}$:

$$(0,a,(0,b,y)) = (0,a*b,1),\ \forall\, a,b \in E - \{0\}. \tag{8}$$

If $y = 1$, then we obtain from (8) with the help of (7):

$$(0,a,b) = (0,a,(0,b,1)) = (0,a*b,1) = a*b.$$

So $\forall\, a,b,y \in E^* - \{0\}$ we obtain from (8):

$$(0,a,(0,b,y)) = (0,a,b*y) = a*(b*y) =$$

$$= (a * b) * y = (0, a * b, y) = (0, (0, a, b), y),$$

i.e. the operation $x \bullet y = (0, x, y)$ on the set $E^* - \{0\}$ is a group, and this operation coincide with the operation $(*)$ from the initial S-system. □

Lemma 8. *The mappings*

$$\varphi_{b,a}(x) = (b, a, x), \quad b \in E, \quad a \in E^* - \{0\},$$
$$\psi_{b,a,d}(x) = (b, a, (d, a^{(-1)}, x)), \quad b, d \in E, \quad a \in E^* - \{0\},$$

are permutations on the set E.

Proof. We have from (5):

$$\varphi_{b,1}(x) = (b, 1, x) = x,$$
$$\psi_{b,a,d}(x) = (b, a, (b, a^{(-1)}, x)) = (b, a * a^{(-1)}, x) = x,$$
$$\psi_{b,1,d}(x) = (b, 1, (d, 1, x)) = x.$$

If $a \in E^* - \{0\}$, then for any arbitrary b and a the mapping $\varphi_{b,a}(x) = L_b^{(a)}(x)$ is a left translation in the quasigroup (x, a, y) with respect to the element b, i.e. it is a permutation. If $a \in E^* - \{0, 1\}$, then for any arbitrary b, a and d we have:

$$\psi_{b,a,d}(x) = L_b^{(a)} L_d^{(a^{(-1)})}(x),$$

i.e. the mapping $\psi_{b,a,d}$ is a composition of two translations: $L_b^{(a)}$ in the quasigroup (x, a, y) and $L_d^{(a^{(-1)})}$ in the quasigroup $(x, a^{(-1)}, y)$; so it is a permutation too. □

Lemma 9. *The following statements are true:*

1) *permutation* $\varphi_{b,a}$ $(b \in E,\ a \in E^* - \{0, 1\})$ *has one and only one fixed element* b,
2) *permutation* $\psi_{b,a,d}$ $(b, d \in E, \quad a \in E^* - \{0, 1\})$ *is a fixed-point-free permutation, if* $b \neq d$,
3) *the set of permutations*
   $T = \{\psi_{b,a_0,0} \ :\ b \in E, \quad a_0$ *is a fixed element from* $E^* - \{0, 1\}\}$
   is transitive on the set E.

Proof. 1) We have

$$\varphi_{b,a}(b) = (b, a, b) = b.$$

Let's assume there exists an element $x_0 \in E,\ x_0 \neq b$ such that

$$\varphi_{b,a}(x_0) = x_0.$$

Then we obtain

$$(b, a, x_0) = x_0.$$

But it is evident that

$$(x_0, a, x_0) = x_0.$$

As the operation (x, a, y) is a quasigroup and $x_o \neq b$ we obtain a contradiction between the last two equalities. So

$$\varphi_{b,a}(x) \neq x$$

for any $x \in E - \{b\}$.

2) Let's assume there exists an element $x_0 \in E$ such that $b \neq d$ and

$$\psi_{b,a,d}(x_0) = x_0,$$

i.e.

$$(b, a, (d, a^{(-1)}, x_0)) = x_0.$$

Then we obtain with the help of (5)

$$\begin{aligned}(b, a^{(-1)}, x_0) = (b, a^{(-1)}, (b, a, (d, a^{(-1)}, x_0))) = \\ = (b, a^{(-1)} * a, (d, a^{(-1)}, x_0)) = (b, 1, (d, a^{(-1)}, x_0)) = \\ = (d, a^{(-1)}, x_0),\end{aligned}$$

i.e. $b = d$ (because the operation $(x, a^{(-1)}, y)$ is a quasigroup when $a \neq 0, 1$). We obtain a contradiction; so if $b \neq d$, then $\varphi_{b,a,d}(x) \neq x$ for any $x \in E$.

3) Let a_0 be an arbitrary element from $E^* - \{0, 1\}$. We have for an arbitrary fixed element $c \in E$:

$$\varphi_{t,a_0,0}(c) = (t, a_0, (0, a_0^{(-1)}, c)) = (t, a_0, a_0^{(-1)} * c) = R^{(a_0)}_{a_0^{(-1)} * c}(t),$$

where $R^{(a)}_b$ denotes the right translation in the quasigroup (x, a, y) with respect to the element $b \in E$. As the mapping $R^{(a_0)}_{a_0^{(-1)} * c}$ is a permutation on the set E, so for any $c, d \in E$ there exists an element

$t_0 \in E$ such that we have

$$\psi_{t_0,a_0,0}(c) = R^{(a_0)}_{a_0^{(-1)}*c}(t_0) = d,$$

i.e. the set T is a transitive set of permutations on E. $\square$

Lemma 10. *Let permutations $\varphi_{b,a}$ and $\psi_{b,a,d}$ where $b, d \in E$, $a \in E^* - \{0\})$, form a group G under the natural product of permutations. Then G is a Frobenius group.*

Proof is an easy corollary of Lemma 9. $\square$

Lemma 11. *Let the set E be a finite one. If the conditions of Lemma 10 take place, then the set of fixed-point-free permutations*

$$T = \{\psi_{b,a_0,0} \; : \; b \in E, \;\; \text{is any fixed element from } \; E^* - \{0,1\}\}$$

with the identity permutation $id = \psi_{0,a_0,0}$ is a normal subgroup in the Frobenius group G.

Proof. Let the conditions of Lemma hold. Then G is a finite Frobenius group of permutations of degree n, where $n = |E|$. It is easy to show that the group G contains exactly $n-1$ fixed-point-free permutations (see [4]). As the set T contains exactly $n-1$ different fixed-point-free permutations (see Lemma 9), so T contains all the fixed-point-free permutations of group G.

Let's denote

$$H_a = St_a(G), \quad a \in E.$$

As the set T is a transitive set of permutations on E, so T is a left transversal in the group G to its subgroup H_a for any $a \in E$. So we have

$$t_i^{-1} t_j \notin H_a \quad \forall \, i \neq j.$$

Then we obtain that $t_i^{-1} t_j$ is a fixed-point-free permutation, i.e.

$$t_i^{-1} t_j = t_k$$

for some element $t_k \in T$, because the set T contains all the fixed-point-free permutations of the group G. So all fixed-point-free permutations of the group G with the identity permutation id form a group which is a normal subgroup of the group G. $\square$

By means of Lemmas 10 and 11 we obtain that there exist normal subgroups, consisting from fixed-point-free permutations and the identity permutation, in the finite Frobenius groups, which are groups of cell permutations of the right S-system.

Further we will demonstrate one more method of definition of the operations (x, a, y) by an arbitrary Frobenius group (that method is different from the method described in the part 3; moreover, that method does not use the fact of existing a normal subgroup in the Frobenius group and is independent for the cardinality of the set E.

Let G be an arbitrary Frobenius group of permutations on some set E and $0, 1$ be two distinct distinguished elements from E. As the group G is transitive on the set E, then there exists a set of n permutations $P = \{\sigma_x\}_{x \in E}$ such that $\sigma_x(0) = x \ \ \forall\, x \in E$, and $\sigma_0 = id$. Let's define the operation (x, a, y) as follows:

$$(x, 0, y) = x\,, \quad (x, 1, y) = y\,,$$

$$\begin{gathered} \forall\, a \in E^*,\ \ a \neq 0, 1,\ (x, a, y) = z \Longleftrightarrow z = \alpha(y)\,, \\ \text{where } \alpha \in G,\ \ \alpha(x) = x, \quad \beta(1) = (\sigma_x^{-1} \alpha \sigma_x)(1) = a. \end{gathered} \tag{9}$$

This definition is correct, because there exist an unique permutation $h = H_0 = St_0(G)$ satisfying the condition $h(1) = a$ and so there exist an unique permutation $\alpha \in G$ satisfying the condition (9).

Lemma 12. *The operation (x, a, y) (defined by (9)) satisfies the following properties:*

1) $(0, a, 1) = a, \quad (x, a, x) = x,$
2) $\forall\, a, b \in E^*\ :\ (x, a, (x, b, y)) = (x, c, y)$ *for some* $c = c(a, b) \in E^*$.

Proof. 1) We have

$$(0, a, 1) = u \Longleftrightarrow \begin{cases} u = \alpha(1) \\ \alpha(0) = 0 \\ \beta(1) = a \end{cases}$$

which implies $\sigma = id$, $\alpha = \beta$, and in the consequence $u = \alpha(1) = \beta(1) = a$, i.e. $(0, a, 1) = a$.

Similarly

$$(x,a,x)=u \Leftrightarrow \begin{cases} u=\alpha(x) \\ \alpha(x)=x \\ \beta(1)=a \end{cases} \Rightarrow u=\alpha(x)=x,$$

i.e. $(x,a,x)=x$.

2). If $a=0$, then we have

$$(x,a,(x,b,y))=x=(x,0,y) \Rightarrow c=c(0,b)=0.$$

If $b=0$, then we have

$$(x,a,(x,b,y))=(x,a,x)=x=(x,0,y) \Rightarrow c=c(a,0)=0.$$

Let $a,b\neq 0$. Then we obtain

$$(x,a,(x,b,y))=u \Longleftrightarrow \begin{cases} (x,a,v) &= u \\ (x,b,y) &= v \end{cases} \Longleftrightarrow$$

$$\Longleftrightarrow \begin{cases} \alpha(x)=x, & \alpha(v)=u \\ \sigma_x^{-1}\alpha\sigma_x=h_a, & h_a(1)=a \\ \alpha_1(x)=x, & \alpha_1(y)=v \\ \sigma_x^{-1}\alpha_1\sigma_x=h_b, & h_b(1)=b \end{cases} \Longleftrightarrow$$

$$\Longleftrightarrow \begin{cases} u=\alpha(v)=\alpha\alpha_1(y) \\ \alpha\alpha_1(x)=\alpha(x)=x \\ \sigma_x^{-1}\alpha\alpha_1\sigma_x=\sigma_x^{-1}\alpha\sigma_x\sigma_x^{-1}\alpha_1\sigma_x=h_ah_b=h_c \\ c=h_c(1)=h_ah_b(1)=h_a(b) \end{cases}$$

$$\Longleftrightarrow \begin{cases} u=\gamma(y)=(c,x,y)=(x,h_a(b),y) \\ \gamma=\alpha\alpha_1 \\ \gamma(x)=x \\ \sigma_x^{-1}\gamma\sigma_x=h_c \end{cases}$$

i.e.

$$(x,a,(x,b,y))=(x,c,y).$$

So we have demonstrated that it can define a right S-system of idempotent operations (x,a,y) over an arbitrary Frobenius group G with the help of equalities (9).

Moreover, it can define the left S-system of idempotent operations

$[x, a, y]$ over an arbitrary Frobenius group G changing symmetrically the definitive equalities (9).

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On loops with universal elasticity

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Abstract

A property P of a loop $Q(\cdot)$ is called universal for $Q(\cdot)$ if it holds in every loop isotopic to $Q(\cdot)$ ([1,2]). Loops with universal law of elasticity are considered in this article. Necessary and sufficient conditions for a commutative IP-loop with universal elasticity to be a Moufang loop are proved.

Loops with universal law of elasticity $x \cdot yx = xy \cdot x$ were mentioned in [2] and partially studied in [6]. It is our purpose in this paper to continue the study of algebraic properties of loops with universal elasticity. It was shown in [6] that the identity of elasticity $x \cdot yx = xy \cdot x$ is universal for a loop $Q(\cdot)$ if and only if one of identities

$$x\backslash[(xy/b)(a\backslash xb)] = a\backslash[(ay/b)(a\backslash xb)] \tag{1}$$

or

$$[(bx/a)(b\backslash yx)]/x = [(bx/a)(b\backslash ya)]/a\,, \tag{2}$$

holds in the primitive loop $Q(\cdot, /, \backslash)$. So, the identities (1) and (2) are equivalent in $Q(\cdot, /, \backslash)$. More, they are symmetric (dual). If $Q(\cdot)$ is an IP-loop then each of the identities (1) and (2) is equivalent in $Q(\cdot)$ to the following identity:

$$[x(az \cdot y) \cdot zx]a = x[az \cdot (y \cdot zx)a]\,, \tag{3}$$

which is universal for a loop $Q(\cdot)$ if and only if $Q(\cdot)$ is a Moufang loop.

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It is clear that each Moufang loop is a loop with universal elasticity. More, the class of Moufang loops is strictly contained in that of loops with universal elasticity. The following examples give loops with universal elasticity which are not Moufang.

Example 1. Let $R(+,\cdot)$ be the ring of integers modulo 2 and $Q = R^3$. Define on Q the operation $(\cdot)$ as follows:

$$(i\,,j\,,k)\cdot(p\,,q\,,r) = (i+p\,,j+q\,,k+r+jp+jpq+ijq)\,.$$

$Q(\cdot)$ is a loop with universal elasticity of order 8 and exponent 4 (with $(0,0,0)$ as a neutral element). $\square$

Example 2. Let $Q(+,\cdot)$ be the ring from Example 1 and $Q = R^3$. Define on Q the operation $(\cdot)$:

$$(i\,,j\,,k)\cdot(p\,,q\,,r) = (i+p\,,j+q\,,k+r+ijp+ipq).$$

Then $Q(\cdot)$ is a loop with universal elasticity (not Moufang). $\square$

In what follows will be useful the identity

$$(bx\cdot ay)x\cdot ab = bx\cdot a(yx\cdot ab)\,, \tag{4}$$

which is equivalent to (3). Indeed, making the substitutions:$a \to az^{-1}$, $x \to z^{-1}x$ and after this $z^{-1} \to b$ in (3), we find the identity (4), and analogously, from (4) follows (3).

I. As it was shown in [6], loops with universal elasticity are strong power-associative (i.e. each element generates an associative subloop).

Let $Q(\circ)$ be a strong power-associative loop. Define a new loop $Q(\cdot)$ as follows:

$$x\cdot y = x//y^{(-1)}\,,$$

where "//" is the left division in $Q(\circ)$, $y\cdot y^{(-1)} = 1$, 1 is the neutral element of $Q(\cdot)$. Then

$$x\circ y = x/y^{(-1)}\,,$$

where "/" is the left division in $Q(\cdot)$.

Proposition 1.
i) $1 = e$, *where* e *is the unit of* $Q(\circ)$,
ii) *if* $x \circ x^{-1} = e$, *then* $x^{-1} = x^{(-1)} = {}^{(-1)}x$,
where ${}^{-1}x \cdot x = x \cdot x^{(-1)} = 1$ *for every* x *in* $Q(\cdot)$.

Proof. *i)* Indeed, as

$$x = x \circ e = x/e^{(-1)},$$

we have $x = x \cdot e^{(-1)}$. So $e^{(-1)} = 1$, or $e = 1$.

ii) $Q(\circ)$ is strong power-associative and

$$e = x^{-1} \circ x = x^{-1}/x^{(-1)},$$

so

$$x^{-1} = ex^{(-1)} = x^{(-1)}.$$

Now

$${}^{(-1)}x \cdot x = e \implies {}^{(-1)}x = e/x = e \circ x^{-1} = x^{-1}. \qquad \square$$

Proposition 2. $Q(\cdot)$ *is a* LIP*-loop if and only if the permutation* $I : x \to x^{-1}$ *is an antiautomorphism of* $Q(\circ)$.

Proof. Let $Q(\cdot)$ be a LIP-loop, i.e. $x^{-1} \cdot xy = y$ for every x, y in Q hence $y/xy = x^{-1}$. Making the substitution $x \to x/y$ and using the strong power-associativity of $Q(\circ)$, we get $y/x = (x/y)^{-1}$, i.e.

$$y \circ x^{-1} = (x \circ y^{-1})^{-1} \quad \text{or} \quad y^{-1} \circ x^{-1} = (x \circ y)^{-1}.$$

The proof is reversible. $\square$

Remind that B_3-loops have been considered in [3, 4] by A.Gwaramija. A loop $Q(\cdot)$ is called a B_3*-loop* (or *a medial Bol loop*) if $Q(\cdot, /, \backslash)$ satisfies the identity

$$x(yz\backslash x) = (x/z)(y\backslash x).$$

Mention that the loops from Examples 1 and 2 are B_3-loops as well. We have below another proof of a Gwaramija's result.

Corollary. (A. Gwaramija [4]) *If* $Q(\cdot)$ *is a left Bol loop, then* $Q(\circ)$ *is a* B_3*-loop.*

Proof. It is known ([2]) that a B_3-loop is a loop for which the identity

$$(xy)^{-1} = y^{-1}x^{-1}$$

is universal and a left Bol loop is a loop with universal LIP-property. Let $Q(\circ)$ be a B_3-loop and consider an arbitrary principal isotope $(*) = (\circ)^{(\alpha,\beta,\epsilon)}$ of $Q(\circ)$. If $Q(\circ)$ is a B_3-loop, then $I(x*y) = Iy*Ix$, or

$$I(\alpha x/I\beta y) = I(\alpha x \circ \beta y) = I(x*y) = Iy*Ix = \alpha Iy \circ \beta Ix = \alpha Iy/I\beta x.$$

So we get

$$I(\alpha x/I\beta y) = \alpha Iy/I\beta Ix\,,$$

or, making the substitution $x \to \alpha^{-1}x$ and after that $x \to x \cdot I\beta y$ in the last equality, we obtain:

$$Ix = \alpha Iy/[I\beta I\alpha^{-1}(x \cdot I\beta y)]\,,$$
$$\alpha Iy = Ix \cdot [I\beta I\alpha^{-1}(x \cdot I\beta y)]\,,$$

or finally,

$$y = Ix \bullet (x \bullet y)\,,$$

where $(\bullet) = (\cdot)^{(\epsilon, I\beta I\alpha^{-1},\epsilon)}$. So, if $Q(\cdot)$ is a left Bol loop, then each its principal isotope $(\bullet)$ (defined above) has the LIP-property and this fact implies that every principal isotope $(*) = (\circ)^{(\alpha,\beta,\epsilon)}$ of $Q(\circ)$ satisfies the identity $I(x*y) = Iy*Ix$. □

Proposition 3. $Q(\circ)$ *is a* RIP*-loop if and only if* $Q(\cdot)$ *is a* RIP*-loop.*

Proof. Let $Q(\circ)$ be a RIP-loop. If by "$//$" is denoted the left division in $Q(\circ)$, then we have $x//y = xy^{-1}$. So

$$(y \circ x) \circ x^{-1} = y \implies y//x^{-1} = y \circ x \implies yx = y \circ x$$

for every x, y in Q. Conversely, if $yx \cdot x^{-1} = y$ for every x, y in Q then $y/x^{-1} = u \cdot x$ or $y \circ x = y \cdot x$. So the operations $(\cdot)$ and $(\circ)$ coincide in both cases. □

Let $Q(\circ)$ be a loop with universal elasticity and $Q(\cdot)$ be the loop defined in previous propositions: $x \circ y = x/y^{-1}$ for every $x, y \in Q$ (where $(/)$ is the left division for $(\cdot)$).

Proposition 4. *$Q(\cdot)$ is left alternative if and only if the identity*

$$(x \circ y \circ x^{-1})^2 = (x \circ y) \circ (y \circ x^{-1})$$

holds in $Q(\circ)$.

Proof. If $Q(\circ)$ is left alternative, i.e. $x \cdot xy = x^2 \cdot y$, for every x, y in Q, then the equality

$$x//(x//y^{-1})^{-1} = (x//x^{-1})//y^{-1}$$

is true for every x, y in Q as $x//y = xy^{-1}$. Now using $x//x^{-1} = x^2$ in the previous equality ($Q(\circ)$ is strong power-associative) and replacing x by $x \circ y^{-1}$ we get:

$$(x \circ y^{-1})//x^{-1} = (x \circ y^{-1})^2//y^{-1}$$

or

$$[(x \circ y^{-1})//x^{-1}] \circ y^{-1} = (x \circ y^{-1})^2 \,. \tag{5}$$

But $Q(\circ)$ is a loop with universal elasticity, so it satisfies the identity

$$(x \circ y) \circ x^{-1} = x \circ (y \circ x^{-1})$$

(which is a corollary of (1) for $x = b = e$). From the last identity (using $y \to y^{-1}//x^{-1}$) we get

$$(x \circ y^{-1})//x^{-1} = x \circ (y^{-1}//x^{-1})$$

and using this identity in (5) it follows

$$[x \circ (y^{-1}//x^{-1})] \circ y^{-1} = (x \circ y^{-1})^2 \,,$$

or after replacing y^{-1} by $y \circ x^{-1}$ in the last equality:

$$(x \circ y) \circ (y \circ x^{-1}) = (x \circ y \circ x^{-1})^2 \,.$$

The proof is reversible. $\square$

II. An element a of a loop $Q(\cdot)$ is called *Moufang*, if for every $x, y \in Q$ we have $ax \cdot ya = a(xy \cdot a)$ ([5]). Denote

$$M = \{a \in Q \,:\, ax \cdot ya = a(xy \cdot a) \;\; \forall\, x, y \in Q\}\,.$$

The *Moufang center* of a loop is defined as the set of all elements c such that

$$c^2 \cdot xy = cx \cdot cy$$

for all $x, y \in Q$. It is known ([5]), that the Moufang center of a Moufang loop $Q(\cdot)$ is a commutative subloop of $Q(\cdot)$. More, if N is the nucleus, C is the Moufang center and Z is the center of a Moufang loop, then $N \cap C = Z$. For an arbitrary loop $Q(\cdot)$ the fact that $c \in C$ does not necessarily imply $cx = xc$.

Proposition 5. *The Moufang center $C(\cdot)$ of an IP-loop with universal elasticity $Q(\cdot)$ is a commutative subloop of $Q(\cdot)$.*

Proof. Let $a \in C$ and put $y = e$ in $a^2 \cdot xy = ax \cdot ay$, where e is the unit of $Q(\cdot)$. We get

$$a^2x = a^2(xe) = ax \cdot a = a \cdot xa\,.$$

So

$$a^2 \cdot x = a \cdot ax = a \cdot xa$$

and $ax = xa$ for every $x \in Q$. Here was used the law of left alternativity which holds in IP-loops with universal elasticity as was proved in [6]. We shall prove below that $C(\cdot)$ is a subloop of $Q(\cdot)$. If $a \in C$ then replacing x by au and y by av in the equality $a^2 \cdot xy = ax \cdot ay$ we get

$$a^2(au \cdot av) = a^2u \cdot a^2v\,, \quad \text{or} \quad a^4 \cdot uv = a^2u \cdot a^2v$$

for every $u, v \in Q$. Hence $a^2 \in C$. Let $a, b \in C$. Then

$$a^4(b^2 \cdot xy) = a^4(bx \cdot by) = (a^2 \cdot bx)(a^2 \cdot by) = (ab \cdot ax)(ab \cdot ay)\,,$$

for every $x, y \in Q$. But

$$a^4(b^2 \cdot xy) = a^2b^2 \cdot (a^2 \cdot xy) = a^2b^2 \cdot (ax \cdot ay)\,,$$

for every $x, y \in Q$ and

$$a^2b^2 = a^2 \cdot bb = ab \cdot ab = (ab)^2\,,$$

so we get

$$(ab)^2(ax \cdot ay) = (ab \cdot ax)(ab \cdot ay)\,,$$

for every $x, y \in Q$, i.e. $ab \in C$. If $a \in C$ then we get

$$a^{-2} \cdot uv = a^{-1}u \cdot a^{-1}v\,,$$

after replacing x by $a^{-1}u$ and y by $a^{-1}v$ in $a^2 \cdot xy = ax \cdot ay$. So, $a^{-1} \in C$ and $C(\cdot)$ is a commutative subloop of $Q(\cdot)$. □

Corollary. *If $Q(\cdot)$ is an IP-loop with universal elasticity then $C \leq M \leq Q$.* □

The following proposition contains some properties of the Moufang elements in IP-loops with universal elasticity.

Proposition 6. *Let $Q(\cdot)$ be an IP-loop with universal elasticity. Then $a \in M$ if and only if at least one of the equalities*

i)	$(ax \cdot y)x = a \cdot xyx,$	*ii)*	$axa \cdot y = a(x \cdot ay),$
iii)	$x(y \cdot xa) = xyx \cdot a,$	*iv)*	$x \cdot aya = (xa \cdot y)a,$
v)	$xy \cdot ax = x \cdot ya \cdot x,$	*vi)*	$xa \cdot yx = x \cdot ay \cdot x$

holds for every $x, y \in Q$.

Proof. To prove this proposition we need the following identities which are corollaries of (3) and (4):

(a) $\quad (x \cdot ay \cdot x)a = x(a \cdot yx \cdot a)\,,$

(b) $\quad (uv \cdot zu)v = u(vz \cdot uv)\,,$

(c) $\quad (v \cdot uy) \cdot uv = v[u(y \cdot uv)]\,,$

(d) $\quad vu \cdot (yu \cdot v) = [(vu \cdot y)u]v\,.$

Indeed, the identity (a) can be obtained from (3) taking $z = e$; the identity (b) is a corollary of (4) for $x \to b^{-1}u$, $a \to vb^{-1}$, $y = b$ and after this $b^{-1} = z$; the identity (c) was obtained replacing x by z^{-1}, a by uz^{-1} and after that z^{-1} by v in (3); (c) and (d) are symmetric. Consider $c \in M$ and substitute x by c in (a):

$$(c \cdot ay \cdot c)a = c(a \cdot yc \cdot a).$$

Hence

$$(ca \cdot yc)a = c(a \cdot yc \cdot a)$$

or, replacing yc by y:

$$(ca \cdot y)a = c \cdot aya$$

for every $a, y \in Q$, so i) is proved.

Analogously, taking $u = c$ in (b) we get: $czc \cdot v = c(x \cdot cv)$ for every $v, z \in Q$, i.e. ii) holds. For $a = c$ in (a) we have:

$$(x \cdot cy \cdot x)c = x(c \cdot yx \cdot c) = x(cy \cdot xc),$$

so

$$(x \cdot cy \cdot x)c = x(cy \cdot xc),$$

and after replacing cy by y: $xyx \cdot c = x(y \cdot xc)$ for every $x, y \in Q$, i.e. iii) is proved.

Taking $v = c$ in (b) we get

$$(uc \cdot zu)c = u(cz \cdot uc) = u(c \cdot zu \cdot c),$$

so

$$(uc \cdot zu)c = u(c \cdot zu \cdot c)$$

or, replacing zu by z: $(uc \cdot x)c = u \cdot czc$ for every $u, z \in Q$, i.e. iv) is proved.

Substitute now u by c in (c) and using ii) we get:

$$(v \cdot cy)cv = v[c(y \cdot cv)] = v(cyc \cdot v),$$

hence

$$(v \cdot cy) \cdot cv = v(cyc \cdot v),$$

or $vz \cdot cv = v(zc \cdot v)$, where by z was denoted cy, i.e. we get v). Analogously, taking $u = c$ in (d) we have:

$$vc \cdot (yc \cdot v) = [(vc \cdot y)c]v = (v \cdot cyc)v,$$

so

$$vc \cdot (yc \cdot v) = (v \cdot cyc)v,$$

or (putting $yc = z$) $vc \cdot zv = (v \cdot cz)v$, for every $v, z \in Q$, i.e. vi) is proved. In each of this cases the proof is reversible, so Proposition 6 is proved. $\square$

A bijection γ on Q is called a *right (left) pseudo-automorphism* of a quasigroup $Q(\cdot)$ if there exists at least one element $c \in Q$ such that

$$\gamma x \cdot (\gamma y \cdot c) = \gamma(xy) \cdot c\,, \qquad (c \cdot \gamma x)\gamma y = c \cdot \gamma(xy)\,,$$

for every $x, y \in Q$. The element c is called a *companion* of γ ([1,5]). It is known ([1]), that every companion of a pseudo-automorphism in IP-loops is a Moufang element. Let $Q(\cdot)$ be a loop with the law of elasticity. A bijection Θ on Q is called a *semiautomorphism* of $Q(\cdot)$, if

$$\Theta(xyx) = \Theta x \cdot \Theta y \cdot \Theta x$$

for every $x, y \in Q$ and $\Theta e = e$. It is known ([1]) that every pseudo-automorphism of a Moufang loop is its semiautomorphism.

Proposition 7. *Any pseudo-automorphism of an IP-loop $Q(\cdot)$ with universal elasticity is its semiautomorphism.*

Proof. Let φ be a pseudo-automorphism of a loop $Q(\cdot)$ with universal elasticity and c be a right companion of φ. Since

$$\varphi(xy) \cdot c = \varphi x \cdot (\varphi y \cdot c)$$

for every $x, y \in Q$, then

$$\varphi(xyx) \cdot c = \varphi x \cdot [\varphi(yx) \cdot c] = \varphi x \cdot [\varphi y \cdot (\varphi x \cdot c)] = (\varphi x \cdot \varphi y \cdot \varphi x)c$$

because c is a Moufang element and so we can apply here iii) from Proposition 6. Hence $\varphi(xyx) = \varphi x \cdot \varphi y \cdot \varphi x$ for every $x, y \in Q$. The proof is analogous in the case when c is a left companion of φ. Note that $\varphi e = e$ for every pseudo-automorphism of an IP-loop with universal elasticity. Indeed, such loops are left and right alternative, so taking $x = y = e$ in $\varphi(xy) \cdot c = \varphi x \cdot (\varphi y \cdot c)$, we get $\varphi e \cdot c = (\varphi e)^2 c$, thus $e = \varphi e$.

III. Let $Q(\cdot)$ be an IP-loop with universal elasticity and define on Q the operation $(+)$ by

$$x + y = xy^{-1}x$$

for every $x, y \in Q$. Then the groupoid $Q(+)$ is called the *core* of $Q(\cdot)$.

Remind ([1]) that the core of a Moufang loop $Q(\cdot)$ is a left-distributive groupoid and it is a quasigroup if and only if the mapping $x \to x^2$ is a permutation on Q. For our class of loops an analogous proposition is true.

Proposition 8. *The core $Q(+)$ of an IP-loop with universal elasticity is a quasigroup if and only if the mapping $x \to x^2$ is a permutation on Q.*

Proof. Let $Q(+)$ be a quasigroup. Then there exists for each $a \in Q$ an unic element $x \in Q$ such that $x + e = a$, where e is the unity of the loop $Q(\cdot)$. So the equation $xe^{-1}x = x^2 = a$ has an unique solution in $Q(\cdot)$ and consequently, $x \to x^2$ is a permutation on Q.

Conversely, suppose that the mapping $\psi : x \to x^2$ is a permutation on Q. The equation $a + x = b$ where $a, b \in Q$ or $ax^{-1}a = b$ has the unique solution $x = ab^{-1}a$. Consider now the equation $x + a = b$, i.e. $xa^{-1}x = b$, where $a, b \in Q$. The last equation is equivalent to $xa^{-1}x \cdot a^{-1} = ba^{-1}$. Returning to (3) and making the substitution $x = e,\ y = a$, we get

$$[(az \cdot a)z]a = az \cdot (az \cdot a)\,,$$

or, using the alternativity of $Q(\cdot)$,

$$(az \cdot a)z \cdot a = (az)^2 \cdot a\,,$$

so by (a) from Proposition 6:

$$aza \cdot z = a \cdot zaz = (az)^2$$

for every $a, z \in Q$. Now the considered equation $x \cdot a^{-1}xa^{-1} = ba^{-1}$ can be represented as follows:

$$(xa^{-1})^2 = ba^{-1}$$

and it has an unique solution because ψ is a permutation on Q. □

Note that the core of an IP-loop with universal elasticity is a groupoid with the law of elasticity. Indeed, using (c),

$$(x + y) + x = xy^{-1}x \cdot (x^{-1} \cdot xy^{-1}x) = xy^{-1}x \cdot y^{-1}x =$$
$$= x(y^{-1}x)^2 = x(y^{-1}xy^{-1} \cdot x) = x(yx^{-1}y)^{-1}x = x + (y + x)$$

for every $x, y \in Q$.

Proposition 9. *The core $Q(+)$ of an IP-loop$Q(\cdot)$ with universal elasticity is a left-distributive groupoid if and only if the following identity*

$$x(y \cdot xzx \cdot y)x = xyx \cdot z \cdot xyx \tag{6}$$

holds in $Q(\cdot)$.

Proof. Let $Q(+)$ be a left-distributive groupoid, i.e.

$$x + (y + z) = (x + y) + (x + z)\,,$$

or

$$x \cdot y^{-1}zy^{-1} \cdot x = xy^{-1}x \cdot x^{-1}zx^{-1} \cdot xy^{-1}x\,.$$

After replacing y^{-1} by y and z by xzx in the last identity we shall obtain (6). □

Corollary 1. ([1]) *The core of a Moufang loop is a left-distributive groupoid.* □

Corollary 2. *If $Q(\cdot)$ is a commutative IP-loop with universal elasticity for which the mapping $x \to x^2$ is a bijection, then $Q(\cdot)$ is a commutative Moufang loop if and only if its core $Q(+)$ is a left-distributive quasigroup.*

Proof. Let $Q(+)$ be a left-distributive quasigroup. The identities of alternativity and the identity (d) hold in $Q(\cdot)$ (see [6]). So, from (6) we have:

$$x^2(y^2 \cdot x^2z) = (x^2y)^2z = x^4y^2 \cdot z = x^2y^2x^2 \cdot z\,.$$

For $x^2 \to x$ and $y^2 \to y$ in the last identity we get

$$x(y \cdot xz) = xyx \cdot z\,,$$

thus $Q(\cdot)$ is a commutative Moufang loop. The converse statement is proved in [1]. □

Proposition 10. *The core $Q(+)$ of an IP-loop $Q(\cdot)$ with universal elasticity is right-distributive if and only if the identity*

$$xyx \cdot z \cdot xyx = xzx \cdot yz^{-1}y \cdot xzx \tag{7}$$

holds in $Q(\cdot)$.

Proof follows from the law of right-distributivity. □

Corollary. *Let a quasigroup $Q(+)$ be the left-distributive core of an IP-loop with universal elasticity $Q(\cdot)$. Then $Q(+)$ is right-distributive if and only if the identity*

$$xy^2x = yx^2y \tag{8}$$

holds in $Q(\cdot)$.

Proof. Let $Q(\cdot)$ be a loop with the identity (8). For $y \to zyz$ in (8) we get

$$x(zyz)^2x = zyz \cdot x^2 \cdot zyz$$

or, using the left-distributivity of $Q(+)$ and (8):

$$x(zyz)^2x = z(y \cdot zx^2z \cdot y)z\,.$$

Now we shall apply (8) to the last identity:

$$x(zyz)^2x = z(y \cdot xz^2x \cdot y)z\,,$$

or, after replacing $y \to z^{-1}yz^{-1}$:

$$xy^2x = z(z^{-1}yz^{-1} \cdot xz^2x \cdot z^{-1}yz^{-1})z\,.$$

So

$$z^{-1} \cdot xy^2x \cdot z^{-1} = z^{-1}yz^{-1} \cdot xz^2x \cdot z^{-1}yz^{-1}$$

and, making the substitutions $y \to y^2$, $z \to z^{-1}$ in the last identity and using (8),

$$z \cdot y^2x^2y^2 \cdot z = yz^2y \cdot xz^{-2}x \cdot yz^2y\,.$$

But $Q(+)$ is a left-distributive groupoid, hence taking $z = e$ in (6), we get

$$x \cdot yx^2y \cdot x = (xyx)^2\,.$$

Thus from (8) and the last identity it follows

$$x \cdot xy^2x \cdot x = x^2y^2x^2 = (xyx)^2$$

($(x^py)x^q = x^p(yx^q)$ in all IP-loops) and from these identities, using (8), we get

$$z(yxy)^2z = yz^2y \cdot xz^{-2}x \cdot yz^2y\,,$$

or

$$yxy \cdot z^2 \cdot yxy = yz^2y \cdot xz^{-2}x \cdot yz^2y\,.$$

Making the substitution $z^2 \to z$ in the last identity, we obtain

$$yxy \cdot z \cdot yxy = yzy \cdot xz^{-1}x \cdot yzy\,,$$

i.e. $Q(+)$ is right-distributive.

Conversely, let $Q(+)$ be a right-distributive and left-distributive groupoid. Then the identities (6) and (7) hold in $Q(\cdot)$. Hence

$$x(y \cdot xzx \cdot y)x = xzx \cdot yz^{-1}y \cdot xzx\,,$$

and for $z = e$

$$x \cdot yx^2y \cdot x = x^2y^2x^2 = x \cdot xy^2x \cdot x\,.$$

Thus $yz^2y = xy^2x$, which completes the proof. □

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On topological n-ary semigroups

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Abstract

In this note some we describe topologies on n-ary semigroups induced by families of deviations.

1. Introduction

Topological n–groups were investigated by many authors. For example, Čupona proved in [5] that each topological n–group can be embedded into a topological group. Žižović described topological medial n–groups (cf. [20]), topological n–groups with the Baire property (cf. [21]) and proved a topological analog of Hosszú theorem (cf. [19]). Crombez and Six described a fundamental system of open neighborhoods of a fixed element (cf. [4]). Endres proved that every topological n–group is homeomorphic to some canonical topological group (cf. [9]). Topologies induced by norms are considered by Boujuf and Mukhin (cf. [2]). Balci Dervis (cf. [1]) described free topological n–groups. In [12] is described a method of embedding topological abelian n–semigroups in topological n–group.

On the other hand, we known that topological n–semigroups have many properties which are not true for binary semigroups.

In this paper we investigate topologies on n–semigroups and n–groups determined by families of left invariant deviations. We describe

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the conditions under which such topology is compatible with the n–ary operation. We find also the necessary and sufficient conditions for the topologically embedding a semiabelian topological n–semigroup in a topological n–group.

2. Preliminaries

Traditionally in the theory of n-ary groups we use the following abbreviated notation: the sequence $x_i, ..., x_j$ is denoted by x_i^j (for $j < i$ this symbol is empty). If $x_{i+1} = ... = x_{i+k} = x$, then instead of x_{i+1}^{i+k} we write $\overset{(k)}{x}$. Obviously $\overset{(0)}{x}$ is the empty symbol. In this notation the formula

$$f(x_1, ..., x_i, x_{i+1}, ..., x_{i+k}, x_{i+k+1}, ..., x_n)\,,$$

where $x_{i+1} = ... = x_{i+k} = x$, will be written as $f(x_1^i, \overset{(k)}{x}, x_{i+k+1}^n)$.

If $m = k(n-1)+1$, then the m-ary operation g given by

$$g(x_1^{k(n-1)+1}) = \underbrace{f(f(..., f(f}_{k-times}(x_1^n), x_{n+1}^{2n-1}), ...), x_{(k-1)(n-1)+2}^{k(n-1)+1})$$

will be denoted by $f_{(k)}$. In certain situations, when the arity of g does not play a crucial role, or when it will differ depending on additional assumptions, we write $f_{(.)}$, to mean $f_{(k)}$ for some $k = 1, 2, ...$

An n–ary operation f defined on G is called *associative* if

$$f(f(x_1^n), x_{n+1}^{2n-1}) = f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1})$$

holds for all $x_1, x_2, ..., x_{2n-1} \in G$ and $i = 1, 2, ..., n$. The set G together with one associative operation f is called an *n–ary semigroup* (briefly: *n–semigroup*). An n–semigroup (G, f) in which for for all $a_1, a_2, ..., a_n, b \in G$ there exits an uniquely determined $x_i \in G$ such that $f(a_1^{i-1}, x_i, a_{i+1}^n) = b$ is called an *n–group*.

From this definition it follows that a group (a semigroup) is a 2-group (a 2–semigroup) in the above sense. Moreover, it is worthwhile to note that, under the assumption of the associativity of f, it suffices only to postulate the existence of a solution of the last equation at the places $i = 1$ and $i = n$ or at one place i other than 1 and n (cf. [13], p.213[17]). This means that an n–group may be considered as an algebra (G, f, f_1, f_n) with one associative n–ary operation f and two n–ary operations f_1, f_n such that

$$f(f_1(a_2^n, b), a_2^n) = f(a_a^n, f_n(a_2^n, b)) = b \tag{1}$$

for all $a_2^n, b \in G$.

Following E.L.Post ([13], p.282) the solution of the equation

$$f(x, a, ..., a, f(a, ..., a)) = a$$

is denoted by $a^{[-2]}$. An n–semigroup (G, f) with an unary operation ${}^{[-2]} : G \to G$ satisfying some natural identities is an n–group (cf. [16]).

The map $x \mapsto f(a_1^{j-1}, x, a_{j+1}^n)$ is called an *j–th n–ary translation* determined by $a_1, ..., a_n$. In an n–group each n–ary translation is a bijection.

In an n–group (G, f) for any sequence a_1^{n-2} there exists only one $a \in G$ such that

$$f(x, a_1^{n-2}, a) = f(a_1^{n-2}, a, x) = f(a, a_1^{n-2}, x) = f(x, a, a_1^{n-2}) = x$$

for all $x \in G$ (cf. [17]). An element a is called *inverse* for a_1^{n-2}. In the binary case, i.e. in the case $n = 2$, when the sequence a_1^{n-2} is empty by the inverse we mean the neutral element of a group (G, f) .

A sequence a_2^n is called a *left (right) neutral sequence* if $f(a_2^n, x) = x$ (respectively $f(x, a_2^n) = x$) holds for all $x \in G$. A left and right neutral sequence is called a *neutral sequence*. In an n–group for every sequence a_1^{n-2} may be extended to a neutral sequence, but there are n–semigroups without left (right) neutral sequences.

Let (G, f) be an n–semigroup and let a_2^{n-1} be fixed. Then $(G, *)$, where

$$x * y = f(x, a_2^{n-1}, y) \tag{2}$$

is a semigroup, which is called a *binary retract* of (G, f) and is denoted by $ret_{a_2^{n-1}}(G, f)$. A binary retract of an n–group is a group. Moreover, all binary retracts of a given n–group are isomorphic (cf. [7]), but n–groups with the same retract are not isomorphic, in general.

By so-called Hosszú theorem (cf. [11] or [7]), every n–group (G,f) has the form

$$f(x_1^n) = x_1 * \beta(x_2) * \beta^2(x_3) \ldots * \beta^{n-1}(x_n) * b\,, \tag{3}$$

where a_2^n is a fixed right neutral sequence of (G,f) , $(G,*) = ret_{a_2^{n-1}}(G,f)$, $b = f(\overset{(n)}{a_n})$ and $\beta(x) = f(a_n, x, a_2^{n-1})$.

The identical result holds for n–semigroups with a right neutral sequence.

3. Topology

An n–semigroup (G,f) defined on a topological space $(G,\mathcal{T})$ is called a *topological n–semigroup* if the operation f is continuous in all variables together.

A topological n–group is defined as a topological n–semigroup with two additional continuous operations f_1 and f_n satisfying (1) (cf. [5]). A topological n–group may be defined also a topological n–semigroup with additional continuous operation ${}^{[-2]}$. These definitions are equivalent (cf. [15]).

It is clear that retracts of a topological n–semigroup (n–group) are topological semigroups (groups). Obviously all translations of a topological n–semigroup (n–group) are continuous maps. On the other hand, every n–ary operation which may by written in the form (3), where $*$ and β are continuous, is continuous in all variables together. Thus the following lemma is true.

Lemma 3.1. *Assume that an n–semigroup (G,f) with a topology $\mathcal{T}$ has a right neutral sequence a_2^n. Then $(G,f,\mathcal{T})$ is a topological n–semigroup if and only if $ret_{a_2^{n-1}}(G,f)$ is a topological semigroup and $\beta(x) = f(a_n, x, a_2^{n-1})$ is continuous.* □

Corollary 3.2. *An n–group (G,f) defined on a topological space $(G,\mathcal{T})$ is a topological n–group if and only if there exists a right neutral sequence a_2^n such that $x * y = f(x, a_2^{n-1}, y)$, $\beta(x) = f(a_n, x, a_2^{n-1})$ and ${}^{[-2]} : x \mapsto x^{[-2]}$ are continuous.* □

Proposition 3.3. *An n–group (G, f) defined on a topological space $(G, \mathcal{T})$ is a topological n–group if and only if there exists a right neutral sequence a_2^n such that $ret_{a_2^{n-1}}(G, f)$ is a topological semigroup, $\beta(x) = f(a_n, x, a_2^{n-1})$ and $s : x \to s(x)$, where $f(s(x), a_2^{n-1}, x) = a_n$, are continuous.*

Proof. Let a_2^n be a fixed right neutral sequence on an n–group (G, f) . If $(G, *) = ret_{a_2^{n-1}}(G, f)$ is a topological semigroup and $\beta(x) = f(a_n, x, a_2^{n-1})$ is continuous, then (G, f) is a topological n–semigroup by Lemma 3.1.

Moreover, a_n is the neutral element of $(G, *)$ and $s(x)$ is the solution of $f(s(x), a_2^{n-1}, x) = a_n$, i.e. $s(x) * x = a_n$ in $(G, *)$. Thus $s(x)$ is the inverse of x in $(G, *)$. Hence $(G, *)$ is a topological group, because $s(x)$ is continuous, by the assumption.

Since $f(z, c_2^n) = f(f(z, a_2^n), c_2^n) = z * f(a_n, c_2^n)$ for all $c_j \in G$, then the solution z of $f(z, c_2^n) = b$ in (G, f) is the solution of $z * f(a_n, c_2^n) = b$ in $(G, *)$, then z continuously depends on b and $f(a_n, c_2^n)$. Thus z is a continuous function of variables $b, c_2, ..., c_n$. This, for $b = c_2 = ... = c_{n-1} = x$, $c_n = f(x, ..., x)$, implies that $z = x^{[-2]}$ is a continuous function of x. Thus (G, f) is a topological n–group.

The converse is obvious. □

Corollary 3.4. *Let $\mathcal{T}$ be a locally compact topology on an n–group (G, f) with a right neutral sequence a_2^n. If for every $b \in G$ translations $x \mapsto f(x, a_2^{n-1}, b)$, $x \mapsto f(b, a_2^{n-1}, x)$ and $x \mapsto f(a_n, x, a_2^{n-1})$ are continuous, then $(G, f, \mathcal{T})$ is a topological n–group.*

Proof. In the group $(G, *) = ret_{a_2^{n-1}}(G, f)$ translations $x \mapsto x * b$ and $x \mapsto b * x$ are continuous for every $b \in G$. Thus, by the theorem of Ellis (cf. Theorem 3 in [8]), $(G, *)$ is a topological group. In this group $s(x)$ defined in the previous Proposition is a continuous operation. Hence (G, f) is a topological n–group. □

4. Deviations

By a *deviation* defined on a nonempty set X we mean every map $\varphi : X \times X \to [0, +\infty)$ such that $\varphi(x,x) = 0$, $\varphi(x,y) = \varphi(y,x)$, and $\varphi(x,y) \leq \varphi(x,z) + \varphi(z,y)$ for all $x,y,z \in X$. A deviation φ defined on a semigroup (group) $(G,\cdot)$ is left invariant if $\varphi(cx,cy) = \varphi(x,y)$ for all $c,x,y \in G$. A deviation φ defined on an n–semigroup (G,f) is a *left invariant* if

$$\varphi(f(c_1^{n-1},x), f(c_1^{n-1},y)) = \varphi(x,y)$$

for all $x,y,c_1^{n-1} \in G$.

Theorem 4.1 ([2]) **.** *A binary semigroup (group) $(G,\cdot)$ with a topology $\mathcal{T}$ is a topological semigroup (group) if and only if there exists a family Φ of continuous left invariant deviations on G which induces $\mathcal{T}$ and $\varphi_z \in \Phi$ for every $z \in G$ and $\varphi \in \Phi$, where φ_z is defined by $\varphi_z(x,y) = \varphi(xz,yz)$.* □

In the case of an n–semigroup (G,f) every deviation φ on (G,f) induces a new deviation (φ,k,c_2^n) defined by

$$(\varphi,k,c_2^n)(x,y) = \varphi(f(c_2^k,x,c_{k+1}^n), f(c_2^k,y,c_{k+1}^n)),$$

where $c_2^n \in G$ and $k = 1,...,n$ are fixed.

Theorem 4.2. *Let a_2^n be a right neutral sequence of an n–semigroup (G,f) . If a topology $\mathcal{T}$ on G is induced by the family Φ of deviations such that for all $x,y,z \in G$ and $\varphi \in \Phi$*

(a) $\varphi(f(z,a_2^{n-1},x), f(z,a_2^{n-1},y)) = \varphi(x,y)$,

(b) $(\varphi,1,a_2^{n-1},z)$, $(\varphi,2,a_n,a_2^{n-1}) \in \Phi$,

then (G,f) is a topological n–semigroup.

Proof. Let Φ be as in the assumption. By (a) every $\varphi \in \Phi$ is a left invariant deviation on a semigroup $(G,*) = ret_{a_2^{n-1}}(G,f)$. From (b) we obtain

$$\varphi_z(x,y) = \varphi(x*z, y*z) = \varphi(f(x,a_2^{n-1},z), f(y,a_2^{n-1},z)) =$$

$$= (\varphi, 1, a_2^{n-1}, z)(x, y)$$

for every $z \in G$, which gives $\varphi_z \in \Phi$. By Theorem 4.1 $(G, *)$ is a topological semigroup.

Let $\varepsilon > 0$. If $x, x_0 \in G$ are such that $(\varphi, 2, a_n, a_2^{n-1})(x, x_0) < \varepsilon$, where $\varphi \in \Phi$, then

$$\varphi(\beta(x), \beta(x_0)) = \varphi(f(a_n, x, a_2^{n-1}), f(a_n, x_0, a_2^{n-1})) =$$
$$= (\varphi, 2, a_n, a_2^{n-1})(x, x_0) < \varepsilon\,,$$

which proves that β is continuous. Lemma 3.1 finish the proof. □

Theorem 4.3. *An n–group (G, f) with a topology $\mathcal{T}$ is a topological n–group if and only if there exists the family Φ of deviations such that a topology $\mathcal{T}$ is induced by Φ and for some right neutral sequence a_2^n of G and for all $x, y, z \in G$, $\varphi \in \Phi$ the conditions (a), (b) from the previous theorem are satisfied.*

Proof. Let $(G, f, \mathcal{T})$ be a topological n–group. Then the retract $(G, *) = ret_{a_2^{n-1}}(G, f)$ is a binary topological group for every choice of $a_2, ..., a_{n-1} \in G$. Thus, by Theorem 4.1, there exists the family Φ of continuous left invariant deviations of $(G, *)$ which induces the topology $\mathcal{T}$. Hence, for all $x, y, z \in G$ and $\varphi \in \Phi$, we have

$$\varphi(f(z, a_2^{n-1}, x), f(z, a_2^{n-1}, y)) = \varphi(z * x,\ z * y) = \varphi(x, y),$$

which proves (a).

Moreover, since for all $a_2, ..., a_{n-1} \in G$ there exista $a_n \in G$ such that a_2^n is a right neutral sequence, then from the above follows

$$\varphi(f(c_1^{n-1}, x), f(c_1^{n-1}, y)) =$$
$$= \varphi(f(c_1^{n-1}, f(a_n, a_2^{n-1}, x)), f(c_1^{n-1}, f(a_n, a_2^{n-1}, y))) =$$
$$= \varphi(f(f(c_1^{n-1}, a_n), a_2^{n-1}, x)), f(f(c_1^{n-1}, a_n), a_2^{n-1}, y))) = \varphi(x, y)$$

for all $c_1, ..., c_{n-1} \in G$.

Thus every $\varphi \in \Phi$ is a left invariant deviation of an n–group (G, f) . Hence also (φ, k, c_2^n) is a left invariant deviation for every $k = 1, 2, ..., n$ and all $c_1, ..., c_{n-1} \in G$. Obviously (φ, k, c_2^n) is also left invariant on $(G,*)$ and $(\varphi,k,c_2^n)\in\Phi$. Therefore $(\varphi,1,a_2^n)$, $(\varphi,2,a_n,a_2^{n-1})\in\Phi$, which proves (b).

Conversely, if a topology $\mathcal{T}$ is induced by the family Φ of deviations satisfying (a) and (b), then, by Theorem 4.1, $(G,*)=ret_{a_2^{n-1}}(G,f)$ is a binary topological group. Similarly as in the proof of Theorem 4.2 from $(\varphi,2,a_n,a_2^{n-1})\in\Phi$ follows that the translation $\beta(x)=f(a_n,x,a_2^{n-1})$ is continuous. Proposition 3.3 completes the proof. $\square$

5. Embedding of topological n–semigroups

The necessary and sufficient conditions for the embedding of topological semigroup in topological group are described by N. J. Rothman (cf. [14]) and F. Christoph (cf. [3]). In this section we give some generalizations of these results.

As it is well known (cf. for example [13] or [6]) an n–semigroup (G,f) is called *semiabelian* or $(1,n)$*–commutative* if

$$f(x,a_2^{n-1},y)=f(y,a_2^{n-1},x)$$

holds for all $x,y,a_2,...,a_{n-1}\in G$, and *cancellative* if

$$f(a_1^{i-1},x,a_{i+1}^n)=f(a_1^{i-1},y,a_{i+1}^n)\implies x=y$$

for all $i=1,2,...,n$ and $x,y,a_1,...,a_n\in G$. Every n–group is obviously cancellative.

Now we use the construction of the quotient n–group presented during the Gomel's algebraic conference (1995) by A. M. Gal'mak and V. V. Mukhin.

Let (G,f) be a cancellative semiabelian n–semigroup. Then the relation

$$\langle x,y\rangle\sim\langle z,t\rangle\iff f_{(2)}(\overset{(n-1)}{y},\overset{(n)}{z})=f_{(2)}(\overset{(n-1)}{t},\overset{(n)}{x})$$

defined on $G\times G$ is an equivalence relation. Indeed, the reflexivity and symmetry are obvious. We prove the transitivity.

Let $\langle x,y\rangle\sim\langle z,t\rangle$ and $\langle z,t\rangle\sim\langle u,v\rangle$. Then

$$f_{(2)}(\overset{(n-1)}{y},\overset{(n)}{z})=f_{(2)}(\overset{(n-1)}{t},\overset{(n)}{x})\quad\text{and}\quad f_{(2)}(\overset{(n-1)}{t},\overset{(n)}{u})=f_{(2)}(\overset{(n-1)}{v},\overset{(n)}{z}).$$

Hence

$$f_{(3)}(\overset{(n-1)}{t}, \overset{(n)}{x}, \overset{(n-1)}{v}) = f_{(3)}(\overset{(n-1)}{y}, \overset{(n)}{z}, \overset{(n-1)}{v}) = f_{(3)}(\overset{(n-1)}{y}, \overset{(n-1)}{v}, \overset{(n)}{z}) =$$
$$= f_{(3)}(\overset{(n-1)}{y}, \overset{(n-1)}{t}, \overset{(n)}{u}) = f_{(3)}(\overset{(n-1)}{t}, \overset{(n-1)}{y}, \overset{(n)}{u}),$$

which by the cancellativity gives $f_{(2)}(\overset{(n-1)}{x}, \overset{(n)}{v}) = f_{(2)}(\overset{(n-1)}{y}, \overset{(n)}{u})$.

Since (G, f) is semiabelian, then

$$f_{(2)}(\overset{(n-1)}{x}, \overset{(n)}{v}) = f_{(2)}(\overset{(n-1)}{v}, \overset{(n)}{x}),$$

and in the consequence

$$f_{(2)}(\overset{(n-1)}{v}, \overset{(n)}{x}) = f_{(2)}(\overset{(n-1)}{y}, \overset{(n)}{u}),$$

which proves the transitivity.

In the set $G^* = G \times G/\sim$ of all equivalence classes $\langle x_i, y_i \rangle$ we define the new n–ary operation

$$f^*(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = \langle f(x_1^n), f(y_1^n) \rangle .$$

If $\langle x_i, y_i\rangle \sim \langle s_i, t_i\rangle$ for all $i = 1, 2, ..., n$, then also

$$f_{(2)}(\overset{(n-1)}{y_i}, \overset{(n)}{s_i}) = f_{(2)}(\overset{(n-1)}{t_i}, \overset{(n)}{x_i})$$

and

$$f(f_{(2)}(\overset{(n-1)}{y_1}, \overset{(n)}{s_1}), \ldots, f_{(2)}(\overset{(n-1)}{y_n}, \overset{(n)}{s_n})) = f(f_{(2)}(\overset{(n-1)}{t_1}, \overset{(n)}{x_1}), \ldots, f_{(2)}(\overset{(n-1)}{t_n}, \overset{(n)}{x_n})) .$$

But every semiabelian n–semigroup is also medial (see [10]), i.e. it satisfies

$$f(f(x_{11}^{1n}), f(x_{21}^{2n}), \ldots, f(x_{n1}^{nn})) = f(f(x_{11}^{n1}), f(x_{12}^{n2}), \ldots, f(x_{1n}^{nn})) .$$

Then the last identity may be written in the form

$$f_{(2)}\Big(\overset{(n-1)}{f(y_1^n)}, \overset{(n)}{f(s_1^n)} \Big) = f_{(2)}\Big(\overset{(n-1)}{f(t_1^n)}, \overset{(n)}{f(x_1^n)} \Big),$$

which proves that

$$\langle f(x_1^n), f(y_1^n) \rangle \sim \langle f(s_1^n), f(t_1^n) \rangle .$$

Hence the operation f^* is well defined. It is clear that this operation is also associative and $(1,n)$–commutative.

Now let

$$x = f_{(\cdot)}\Big(a, \overset{(n-1)(n-2)}{d}, \overset{(n-1)(n-1)}{c}\Big)$$

and

$$y = f_{(\cdot)}\Big(b, \overset{(n-1)(n-1)}{d}, \overset{(n-1)n}{c}\Big),$$

where a,b,c,d are fixed elements from G. Then, using $(1,n)$–commutativity, we obtain

$$f_{(\cdot)}\Big(\underbrace{f(y, \overset{(n-1)}{d}), \ldots, f(y, \overset{(n-1)}{d})}_{(n-1)-times}, \overset{(n)}{a}\Big) =$$

$$= f_{(\cdot)}\Big(\underbrace{\underbrace{b, \overset{(n-1)(n-1)}{d}, \overset{(n-1)n}{c}, \overset{(n-1)}{d}}, \ldots, \underbrace{b, \overset{(n-1)(n-1)}{d}, \overset{(n-1)n}{c}, \overset{(n-1)}{d}}}_{(n-1)-times}, \overset{(n)}{a}\Big) =$$

$$= f_{(\cdot)}\Big(\overset{(n-1)}{b}, \overset{(n)}{a}, \overset{(n-1)^2 n}{d}, \overset{(n-1)^2 n}{c}\Big) = W_1$$

and

$$f_{(\cdot)}\Big(\overset{(n-1)}{b}, \underbrace{f(x, \overset{(n-1)}{c}), \ldots, f(x, \overset{(n-1)}{c})}_{n-times}\Big) =$$

$$f_{(\cdot)}\Big(\overset{(n-1)}{b}, \underbrace{\underbrace{a, \overset{(n-1)(n-1)}{d}, \overset{(n-1)(n-2)}{c}, \overset{(n-1)}{c}}, \ldots, \underbrace{a, \overset{(n-1)(n-1)}{d}, \overset{(n-1)(n-2)}{c}, \overset{(n-1)}{c}}}_{n-times}\Big)$$

$$= f_{(\cdot)}\Big(\overset{(n-1)}{b}, \overset{(n)}{a}, \overset{(n-1)^2 n}{d}, \overset{(n-1)^2 n}{c}\Big) = W_2\,.$$

Since $W_1 = W_2$, then

$$f_{(\cdot)}\Big(\underbrace{f(y, \overset{(n-1)}{d}), ..., f(y, \overset{(n-1)}{d})}_{(n-1)-times}, \overset{(n)}{a}\Big) = f_{(\cdot)}\Big(\overset{(n-1)}{b}, \underbrace{f(x, \overset{(n-1)}{c}), ..., f(x, \overset{(n-1)}{c})}_{n-times}\Big)$$

which proves that

$$\langle\, f(x, \overset{(n-1)}{c}),\, f(y, \overset{(n-1)}{d})\,\rangle = \langle a, b\rangle\,,$$

i.e.

$$f^*(\,\langle x,y\rangle,\underbrace{\langle c,d\rangle,\,\ldots,\langle c,d\rangle}_{n-1 times})=\langle a,b\rangle\,.$$

Hence for all $\langle a,b\rangle,\langle c,d\rangle\in G^*$ the last equation has the solution $\langle x,y\rangle\in G^*$.

In the similar way we prove that for all $\langle a,b\rangle,\langle c,d\rangle\in G^*$ there exists $\langle x,y\rangle\in G^*$ such that

$$f^*(\underbrace{\langle c,d\rangle,\,\ldots,\langle c,d\rangle}_{(n-1)-times},\langle x,y\rangle\,)=\langle a,b\rangle\,.$$

This proves (cf. [18]) that (G^*,f^*) is a semiabelian n–group.

The map $p(x)=\langle x,x\rangle$ is a homomorphic embedding of an n–semigroup (G,f) in an n–group (G^*,f^*). Indeed,

$$p(f(x_1^n))=\langle f(x_1^n)\,,\,f(x_1^n)\rangle=$$
$$=f^*(\langle x_1,x_1\rangle,\,\ldots,\langle x_n,x_n\rangle)=f^*(p(x_1),\,\ldots,p(x_n))$$

and $p(x)=p(y)$ implies $\langle x,x\rangle=\langle y,y\rangle$, i.e.

$$f_{(2)}(\overset{(n-1)}{x},\overset{(n)}{y})=f_{(2)}(\overset{(n-1)}{y},\overset{(n)}{x})=f_{(2)}(\overset{(n-1)}{x},\overset{(n-1)}{y},x)\,,$$

which by the cancellativity gives $x=y$. Thus the following lemma is true.

Lemma 5.1. *Every semiabelian cancellative n-semigroup may be embedded into a semiabelian n-group.* □

Lemma 5.2. *If φ is a left invariant deviation of a cancellative semiabelian n–semigroup (G,f), then*

$$\varphi_G(\,\langle x,y\rangle,\langle z,t\rangle\,)=\varphi(f_{(2)}(\overset{(n-1)}{t},\overset{(n)}{x})\,,\,f_{(2)}(\overset{(n-1)}{y},\overset{(n)}{z})\,)$$

is a left invariant deviation on G^* such that $\varphi_G(p(x),p(y))=\varphi(x,y)$.

Proof. From the definition of φ_G follows $\varphi_G(\,\langle x,x\rangle,\langle x,x\rangle\,)=0$ and $\varphi_G(\,\langle x,y\rangle,\langle z,t\rangle\,)=\varphi_G(\,\langle z,t\rangle,\langle x,y\rangle\,)$.

Moreover, if $\langle x,y\rangle\sim\langle u,v\rangle$, where $\langle x,y\rangle,\langle u,v\rangle\in G\times G$, then

$$f_{(2)}(\overset{(n-1)}{v},\overset{(n)}{x})=f_{(2)}(\overset{(n-1)}{y},\overset{(n)}{u})$$

and

$$\begin{aligned}\varphi_G(\,\langle x,y\rangle,\langle z,t\rangle\,) &= \varphi(f_{(2)}(\overset{(n-1)}{t},\overset{(n)}{x})\,,\,f_{(2)}(\overset{(n-1)}{y},\overset{(n)}{z})\,) = \\
&= \varphi(f_{(3)}(\overset{(n-1)}{v},\overset{(n-1)}{t},\overset{(n)}{x})\,,\,f_{(2)}(\overset{(n-1)}{v},\overset{(n-1)}{y},\overset{(n)}{z})\,) = \\
&= \varphi(f_{(3)}(\overset{(n-1)}{t},\overset{(n-1)}{v},\overset{(n)}{x})\,,\,f_{(3)}(\overset{(n-1)}{v},\overset{(n-1)}{y},\overset{(n)}{z})\,) = \\
&= \varphi(f_{(3)}(\overset{(n-1)}{t},\overset{(n-1)}{y},\overset{(n)}{u})\,,\,f_{(3)}(\overset{(n-1)}{v},\overset{(n-1)}{y},\overset{(n)}{z})\,) = \\
&= \varphi(f_{(3)}(\overset{(n-1)}{y},\overset{(n-1)}{t},\overset{(n)}{u})\,,\,f_{(3)}(\overset{(n-1)}{y},\overset{(n-1)}{v},\overset{(n)}{z})\,) = \\
&= \varphi(f_{(2)}(\overset{(n-1)}{t},\overset{(n)}{u})\,,\,f_{(2)}(\overset{(n-1)}{v},\overset{(n)}{z})\,) = \varphi_G(\,\langle u,v\rangle,\langle z,t\rangle\,)\end{aligned}$$

which proves that φ_G is well defined.

Now, for all $\langle x,y\rangle,\langle z,t\rangle \in G\times G$ we have

$$\begin{aligned}\varphi_G(\,\langle x,y\rangle,\langle z,t\rangle\,) &= \varphi(f_{(2)}(\overset{(n-1)}{t},\overset{(n)}{x})\,,\,f_{(2)}(\overset{(n-1)}{y},\overset{(n)}{z})\,) = \\
&= \varphi(f_{(3)}(\overset{(n-1)}{v},\overset{(n-1)}{t},\overset{(n)}{x})\,,\,f_{(3)}(\overset{(n-1)}{v},\overset{(n-1)}{y},\overset{(n)}{z})\,) \le \\
&\le \varphi(f_{(3)}(\overset{(n-1)}{v},\overset{(n-1)}{t},\overset{(n)}{x})\,,\,f_{(3)}(\overset{(n-1)}{y},\overset{(n-1)}{t},\overset{(n)}{u})\,) \\
&\quad + \varphi(f_{(3)}(\overset{(n-1)}{y},\overset{(n-1)}{t},\overset{(n)}{u})\,,\,f_{(3)}(\overset{(n-1)}{v},\overset{(n-1)}{y},\overset{(n)}{z})\,) = \\
&= \varphi(f_{(3)}(\overset{(n-1)}{t},\overset{(n-1)}{v},\overset{(n)}{x})\,,\,f_{(3)}(\overset{(n-1)}{t},\overset{(n-1)}{y},\overset{(n)}{u})\,) \\
&\quad + \varphi(f_{(3)}(\overset{(n-1)}{y},\overset{(n-1)}{t},\overset{(n)}{u})\,,\,f_{(3)}(\overset{(n-1)}{y},\overset{(n-1)}{v},\overset{(n)}{z})\,) = \\
&= \varphi(f_{(2)}(\overset{(n-1)}{v},\overset{(n)}{x})\,,\,f_{(2)}(\overset{(n-1)}{y},\overset{(n)}{u})\,) + \varphi(f_{(2)}(\overset{(n-1)}{t},\overset{(n)}{u})\,,\,f_{(2)}(\overset{(n-1)}{v},\overset{(n)}{z})\,) = \\
&= \varphi_G(\,\langle x,y\rangle,\langle u,v\rangle\,) + \varphi_G(\,\langle u,v\rangle,\langle z,t\rangle\,).\end{aligned}$$

Hence φ_G is a deviation on G^*.

To prove that φ_G is left invariant observe that for all $i=1,...,n-1$, and $a_i,b_i,a_{n-1},x,y,u,v\in G$ we have

$$\varphi_G\Big(f(\langle a_1,b_1\rangle,...,\langle a_{n-1},b_{n-1}\rangle,\langle x,y\rangle),\ f(\langle a_1,b_1\rangle,...,\langle a_{n-1},b_{n-1}\rangle,\langle u,v\rangle)\Big)$$

$$= \varphi_G\Big(\langle f(a_1^{n-1},x), f(b_1^{n-1},y)\rangle, \langle f(a_1^{n-1},u), f(b_1^{n-1},v)\rangle \Big) =$$

$$= \varphi\Big(f_{(2)}(\underbrace{f(b_1^{n-1},v),...,f(b_1^{n-1},v)}_{(n-1)-times}, \underbrace{f(a_1^{n-1},x),...,f(a_1^{n-1},x)}_{n-times}),$$

$$f_{(2)}(\underbrace{f(b_1^{n-1},y),...,f(b_1^{n-1},y)}_{(n-1)-times}, \underbrace{f(a_1^{n-1},u),...,f(a_1^{n-1},u)}_{n-times})\Big).$$

By the associativity and $(1,n)$–commutativity of f, the last formula may be written in the form

$$\varphi\Big(f_{(.)}(\dots, \overset{(n-1)}{v}, \overset{(n)}{x}), f_{(.)}(\dots, \overset{(n-1)}{y}, \overset{(n)}{u})\Big),$$

which, together with the fact that φ is left invariant, implies

$$\varphi\Big(f_{(2)}(\overset{(n-1)}{v}, \overset{(n)}{x}), f_{(2)}(\overset{(n-1)}{y}, \overset{(n)}{u})\Big) = \varphi_G(\langle x,y\rangle, \langle u,v\rangle).$$

This proves that φ_G is a left invariant deviation on G^*.

Moreover

$$\varphi_G(p(x),p(y)) = \varphi_G(\langle x,x\rangle, \langle y,y\rangle) = \varphi\Big(f_{(2)}(\overset{(n-1)}{y}, \overset{(n)}{x}), f_{(2)}(\overset{(n-1)}{x}, \overset{(n)}{y})\Big)$$

$$= \varphi\Big(f_{(2)}(\overset{(n-1)}{y}, \overset{(n-1)}{x}, x), f_{(2)}(\overset{(n-1)}{x}, \overset{(n-1)}{y}, y)\Big) =$$

$$= \varphi\Big(f_{(2)}(\overset{(n-1)}{y}, \overset{(n-1)}{x}, x), f_{(2)}(\overset{(n-1)}{y}, \overset{(n-1)}{x}, y)\Big) = \varphi(x,y),$$

which completes our proof. □

Theorem 5.3. *A cancellative semiabelian n–semigroup (G,f) with a topology $\mathcal{T}$ may be topologically embedded in a topological n–group if and only if a topology $\mathcal{T}$ is induced by a some family of left invariant deviations defined on G.*

Proof. If a cancellative semiabelian n–semigroup (G,f) with a topology $\mathcal{T}$ is topologically embedded in a topological n–group (H,f) with a topology $\mathcal{T}_H$, then $\mathcal{T}_H$ is induced by some family Φ of deviations such that

$$\varphi(f(z,a_2^{n-1},x), f(z,a_2^{n-1},y)) = \varphi(x,y),$$

where $x, y, z \in H$ and $a_2, ..., a_n$ is a right neutral sequence of an n–group H (Theorem 4.3). Since in an n–group H for all $a_2, ..., a_{n-1} \in H$ there exists $a_n \in H$ such that $a_2, ..., a_n$ is a right neutral sequence, then in the above formula all $x, y, z, a_2, ..., a_{n-1}$ are arbitrary. This proves that all $\varphi \in \Phi$ are left invariant deviations.

Conversely, if a topology $\mathcal{T}$ on a cancellative semiabelian n–semigroup (G, f) is induced by a some family Φ of left invariant deviations, then every φ_G defined in Lemma 5.2 is a left invariant deviation on G^*. By Theorem 4.3 the family $\{\varphi_G\}_{\varphi\in\Phi}$ induces on G^* the topology $\mathcal{T}_G$ such that G^* is a topological n–group and $p(x) = \langle x, x\rangle$ is a topological embedding of $(G, f, \mathcal{T})$ in $(G^*, f^*, \mathcal{T}_G)$. □

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Characteristic of ordered Menger systems of multiplace functions

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Abstract

In this article the abstract characterization of Menger systems and some Menger T-systems of multiplace functions is given. These multiplace functions can have different number of variables.

1. Introduction

In his work [1] K. Menger has formulated the problem of abstract characterization of sets of functions of several variables on which the operation of superposition is given and the relation of continuation of functions is marked. This problem for functions with the number of variables $n = 1$ was solved by B. M. Schein [2] and for fixed $n \geq 2$ was examined by V. S. Trokhimenko. But in a general case, when n takes different natural values it is open until the present moment. In this article the author solves the word problem for the so-called Menger systems and some Menger T-systems of multiplace functions. The results of the work were partially presented during the Colloquium on Semigroups in Szeged (1994).

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2. Preliminaries

Let A be a set, n – natural number. Then, every partial mapping $f : A^n \rightarrow A$ is called a *n–place function*, where A^n is the n-th Cartesian power of A. The set of all n-place functions, which are considered on A, is denoted by $F_n(A)$. Now let $(F_n(A))_{n\in I}$ be some family of sets denoted above, where $I \subseteq N$ (N is the set of natural numbers). For any $n, m_1, ..., m_n \in I$ and $f \in F_n(A)$, $g_i \in F_{m_i}(A)$ $(i = 1, ..., n)$, by $f[g_1...g_n]$ a m-place function will be denoted, where $m = \max(m_1, ..., m_n)$ and for all $a_1, ..., a_m \in A$ the following identity holds:

$$f[g_1...g_n](a_1, ..., a_m) = f(g_1(a_1, ..., a_{m_1}), ..., g_n(a_1, ..., a_{m_n}))\,. \tag{1}$$

Doing so, we consider that the left and right members of (1) are determined or not determined simultaneously. The operation

$$(f, g_1, ..., g_n)\ \xi\ f[g_1...g_n]$$

will be denoted by $\mathcal{O}_n$. It is evident that the family of operations $(\mathcal{O}_n)_{n\in I}$ satisfies the so-called condition of *superassociativity*

$$f[g_1...g_n][h_1...h_m] = f[g_1[h_1 \ldots h_{m_1}]...g_n[h_1...h_{m_n}]]\,, \tag{2}$$

where

$$f \in F_n(A), \quad g_i \in F_{m_i}(A), \quad i = 1, ..., n, \quad m = \max(m_1, ..., m_n)\,.$$

Let $(\Phi_n)_{n\in I}$ be a family of subsets, which is stable regarding operations $(\mathcal{O}_n)_{n\in I}$, where $\Phi_n \subset F_n(A)$, $n \in I$, $(\zeta_{\Phi_n}, \chi_{\Phi_n})_{n\in I}$ is a family of pairs of binary relations such that

$$(f, g) \in \zeta_{\Phi_n} \Longleftrightarrow f \subset g \tag{3}$$

$$(f, g) \in \chi_{\Phi_n} \Longleftrightarrow \mathrm{Dom}\, f \subset \mathrm{Dom}\, g \tag{4}$$

for all $f, g \in \Phi_n$, where $\mathrm{Dom} f$ is the domain of definition for f. Then the systems $(\Phi_n, \mathcal{O}_n)_{n\in I}$ will denote the *Menger T-systems*. The forms $(\Phi_n, \mathcal{O}_n, \zeta_{\Phi_n}, \chi_{\Phi_n})_{n\in I}$, $(\Phi_n, \mathcal{O}_n, \zeta_{\Phi_n})_{n\in I}$ and $(\Phi_n, \mathcal{O}_n, \chi_{\Phi_n})_{n\in I}$ will denote (respectively) *fundamentally ordered projection (f.o.p.) Menger systems, fundamentally ordered (f.o.) Menger systems and projection quasi-ordered (p.q-o.) Menger systems* of multiplace functions. If in

(1) $m = m_1 = ... = m_n$, then we come to the notion of Menger system of multiplace functions in the sense of work [6]. We similarly introduce the definition of f.o.p. and p.q-o. Menger systems of multiplace functions.

An abstract *Menger T-system of rank I* will be called a family $(G_n, \mathcal{O}_n)_{n\in I}$, where $I \subset N$, $(G_n)_{n\in I}$ are non-empty sets, such that $G_n \cap G_m = \emptyset$ for any $n, m \in I$; for every $n \in I$ $\mathcal{O}_n$ is a mapping, which brings to conformity for every $(n+1)$-th of the elements (x, y_1, y_n) from $G_n \times G_{m_1} \times ... \times G_{m_n}$, $m_1, ..., m_n \in I$; the element $x[y_1, ..., y_n]$ is from G_m, where $m = \max(m_1, ..., m_n)$, and it satisfies the identity of superassociativity of the form (2). If $m = m_1 = ... = m_n$, then we obtain the definition of an abstract Menger system [3], [6].

The Menger system of rank I is called *weakly unitary* if for every $n \in I$ the set G_n contains such elements $e_1^n, ..., e_n^n$ that for every element g from G_n the identity $g[e_1^n, ..., e_n^n] = g$ is true.

By $(T_n)_{n\in I}$ we shall denote the family of sets of polynomials for the (weakly unitary) Menger $(T-)$ system $(G_n, \mathcal{O}_n)_{n\in I}$ that is defined in the following way: let $\{x_n : n \in I\}$ be a set of different subject variables, then we can consider:

a) $x_n \in T_n$ for every $n \in I$,

b) if $t \in T_m$, $g_1, ..., g_{i-1}, g_{i+1}, ..., g_n \in G_m$, $g \in G_n$, then

$$g[g_1 ... g_{i-1} t\, g_{i+1} ... g_n] \in T_m\,,$$

for every $i = 1, ..., n$ and $m \in I$.

If $t \in T_n$ and $g \in G_n$, where $n \in I$, then by $t(g)$ we shall denote the element from G_n which is obtained as a result of realization of all operations after the substitution of g for variable x_n in the polynomial t.

Let $(G_n, \mathcal{O}_n)_{n\in I}$ be the (weakly unitary) Menger $(T-)$ system of rank I, then the family of binary relations $(\rho_n)_{n\in I}$ such that $\rho_n \subset G_n \times G_n$, is called:

– *stable*, if for all $n, m \in I$, $x, y \in G_n$, $x_i, y_i \in G_m$, $i = 1, ..., n$

$$(x, y) \in \rho_n \wedge \{(x_1, y_1), ..., (x_n, y_n)\} \subset \rho_m \Rightarrow (x[x_1 ... x_n]\,,\, y[y_1 ... y_n]) \in \rho_m,$$

– *v-regular*, if for any $n, m \in I$, $u \in G_n$, $x_i, y_i \in G_m$, $i = 1, ..., n$

$$(x_1, y_1), ..., (x_n, y_n) \subset \rho_m \Rightarrow (u[x_1 ... x_n]\,,\, u[y_1 ... y_n]) \in \rho_m,$$

– *l-regular*, if for all $n, m \in I$, $x, y \in G_n$, $z_1, ..., z_n \in G_m$
$$(x, y) \in \rho_n \Rightarrow (x[z_1...z_n]\,,\, y[z_1...z_n]) \in \rho_m.$$

A family of subsets $(W_n)_{n\in I}$ (where $W_n \subset G_n$) is called a *l-ideal*, if for all $n, m \in I$, $g \in G_n$, $\bar{y} \in G_m^n$, $x \in W_m$ and $i = 1, ..., n$ it is true that
$$g[\bar{y}\,|_i\, x] \in W_m\,,$$
where $[\bar{y}\,|_i\, x]$ denotes $(y_1, ..., y_{i-1}, x, y_{i+1}, ..., y_n)$.

Let us consider two Menger T-systems of rank $I : \mathcal{G} = (G_n, \mathcal{O}_n)_{n\in I}$ and $\mathcal{G}' = (G'_n\,,\, \mathcal{O}'_{n\in I})$. By a homomorphism $\mathcal{G}$ on $\mathcal{G}'$ we'll denote the family $(P^n)_{n\in I}$, where P^n is a mapping G_n on G'_n for every $n \in I$ such that for all $n, m_1, ..., m_n \in I$, $g \in G_n$, $g_i \in G_m$, $i = 1, ..., n$ the following identity holds:

$$P^m(g[g_1...g_n]) = P^n(g)[p^{m_1}(g_1)...P^{m_n}(g_n)]\,, \tag{5}$$

where $m = \max(m_1, ..., m_n)$. If every mapping P^n is one-to-one then such a homomorphism is called a *proper* one (or *isomorphism*). A homomorphism of the Menger T-system $\mathcal{G}$ on some Menger system of multiplace functions is called a *representation of* $\mathcal{G}$ *by functions*. The notions of homomorphism, isomorphism and representation are defined similarly for the Menger systems.

Consider the family of pairs $(\mathcal{E}_n, W_n)_{n\in I}$ on Menger ($T-$)system $(G_n, \mathcal{O}_n)_{n\in I}$, where for every $n \in I$ $\mathcal{E}_n$ is a relation of equivalence on G_n, and W_n is either an empty set or a $\mathcal{E}_n$-class; the family $(\mathcal{E}_n)_{n\in I}$ is v-regular and $(W_n)_{n\in I}$ is an l-ideal. Let $(H_a)_{a\in A_n}$ denote the family of $\mathcal{E}_n$-classes that is different from W_n, and let it be one-to-one indexed by elements of a set A_n. We find that $A_n \cap A_m = \emptyset$ for any $n, m \in I$ if $n \neq m$. Let $\{C_n : n \in I\}$ be the set of different elements that do not fall into $A = \bigcup\limits_{n\in I} A_n$. For every $n \in I$ denote by I_n the set $\{m : m \in I \wedge m < n\}$, and by I'_n – the set $I \backslash (I_n \cup \{n\})$. Let
$$B_n = \prod_{m\in I_n} A_m \times \{C_n\} \times \prod_{m\in I'_n} A_m\,, \quad \overline{A} = \prod_{n\in I} A_n\,, \quad \Im_n = \overline{A}^{\,n} \cup B_n^n$$
(for the Menger T-systems we find that $B_n = \emptyset$ and $\Im_n = \overline{A}^{\,n}$). For every element $g \in G_n$, $n \in I$, we'll determine n-place function such that:

$$(\overline{a}_1,...,\overline{a}_n,\bar{b}\,) \in P^n(g) \Longleftrightarrow$$

$$(\overline{a}_1,...,\overline{a}_n) \in \Im_n \;\wedge\; (\forall i \in I)(g[H_{\overline{a_1}<i>}...H_{\overline{a_n}<i>}] \subset H_{\bar{b}<i>}) \qquad (6)$$

where $\overline{a_k} < i >$ denotes the component of the vector $\overline{a}_k$ that belongs to the set A_i for $i \neq n$; and to the set $A_n \cup \{C_n\}$ for $i = n$. It can be shown that the family of mappings $(P^n)_{n\in I}$ (where $P^n : g \mapsto P^n(g)$) is a representation of Menger ($T-$)system $\mathcal{G}$ by multiplace functions which in future will be called the *simplest*.

3. Results

Let $(\Phi_n, \mathcal{O}_n)_{n\in I}$ be some Menger ($T-$) system of multiplace functions, $(\zeta_{\Phi_n})_{n\in I}$ and $(\chi_{\Phi_n})_{n\in I}$ be the family of binary relations that are defined by means of (3) and (4). In the future, instead of $(f,g) \in \zeta_{\Phi_n}$ and $(f,g) \in \chi_{\Phi_n}$ we'll write $f \subset_n g$ and $f \leftharpoondown_n g$ respectively.

Proposition 1. *On the Menger ($T-$) system $(\Phi_n, \mathcal{O}_n)_{n\in I}$ of multiplace functions set $(\zeta_{\Phi_n})_{n\in I}$ is the stable family of relations of order and set $(\chi_{\Phi_n})_{n\in I}$ is the l-regular family of quasi-order being for every $n \in I$; the inclusion $\zeta_{\Phi_n} \subset \chi_{\Phi_n}$ is true too.*

Proof. It is evident that ζ_{Φ_n} is an order and χ_{Φ_n} – a quasi-order on Φ_n, therefore it is necessary to verify only the conditions of stability and l-regularity.

Let $f \subset_n g$, $f_i \subset_m g_i$, $i = 1,...,n$, and $(a_1,...,a_m,c) \in f[f_1...f_n]$. Then there will exist such elements $b_1,...,b_n$ that $(a_1,...,a_n,b_i) \in f_i$, $i = 1,...,n$, and $(b_1,...,b_n,c) \in f$. Therefore, $(a_1,...,a_m,b_i) \in g_i$, $i = 1,...,n$, and $(b_1,...,b_n,c) \in g$, whence $(a_1,...,a_m,c) \in g[g_1...g_n]$. And so

$$f[f_1...f_n] \subset_m g[g_1...g_n]\,.$$

Stability of $(\zeta_{\Phi_n})_{n\in I}$ is proved.

Now assume that $f \leftharpoondown_n g$, i.e. $\mathrm{Dom}\, f \subset \mathrm{Dom}\, g$ and $f, g \in \Phi_n$. Let $h_1,...,h_n \in \Phi_m$ and $(a_1,...,a_m) \in \mathrm{Dom}\, f[h_1...h_n]$. The latter means that there exists an element c, such that $(a_1,...,a_m,c) \in f[h_1...h_n]$. Then, for some $b_1,...,b_n$: $(a_1,...,a_m,b_i) \in h_i$, $i = 1,...,n$, and $(b_1,...,b_n,c) \in f$; consequently $(b_1,...,b_n) \in \mathrm{Dom}\, f$. Therefore,

$(b_1, ..., b_n) \in \operatorname{Dom} g$. Thus, $(a_1, ..., a_m, b_i) \in h_i$, $i = 1, ..., n$, and $(b_1, ..., b_n, d) \in g$ for some element d.

The latter means that $(a_1, ..., a_m, d) \in g[h_1...h_n]$ for some d, i.e. $(a_1, ..., a_m,) \in \operatorname{Dom} g[h_1...h_n]$. Then

$$f[h_1...h_n] \leftharpoondown_m g[h_1...h_n]\,,$$

therefore, the l-regularity of family $(\chi_{\Phi_n})_{n\in I}$ is proved. The inclusion $\zeta_{\Phi_n} \subset \chi_{\Phi_n}$ is evident (for every $n \in I$). □

Proposition 2. *Let the families of relations $(\zeta_{\Phi_n})_{n\in I}$, $(\chi_{\Phi_n})_{n\in I}$ be defined on the Menger $(T-)$ system $(\Phi_n, \mathcal{O}_n)_{n\in I}$ of multiplace functions. Then they satisfy the following conditions:*

$$f_1 \subset_n f_2 \ \wedge\ g \subset_n t_1(f_1) \ \wedge\ g \subset_n t_2(g_2) \Rightarrow g \subset_n t_2(f_1)\,, \tag{7}$$

$$g_1 \subset_n f \ \wedge\ g_2 \subset_n f \ \wedge\ g_1 \leftharpoondown_n g_2 \Rightarrow g_1 \subset_n g_2\,, \tag{8}$$

$$g_1 \subset_n g_2 \ \wedge\ f \leftharpoondown_n g_1 \ \wedge\ f \leftharpoondown_n u[\overline{\omega}\,|_j\, g_2] \Rightarrow f \leftharpoondown_n u[\overline{\omega}\,|_j\, g_1]\,, \tag{9}$$

$$f[h_1...h_n] \leftharpoondown_m h_1\,, \tag{10}$$

for any $n, m \in I$, $i = 1, ..., n$, $f, f_1, f_2, g, g_1, g_2 \in \Phi_n$, $\overline{\omega} \in G_n^m$, $u, h_1, ..., h_n \in \Phi_m$, $j = 1, ..., m$, $t_1, t_2 \in T_n$.

Proof. Let the premise of condition (7) be valid, then, from $f_1 \subset_n f_2$, we obtain that $f_1 = f_2 \circ \underline{+}_{Dom\, f_1}$ (the restriction of function f_2 on the domain of the function f_1 is denoted by $f_2 \circ \underline{+}_{Dom\, f_1}$). From $g \subset_n t_1(f_1)$ follows $\operatorname{Dom} g \subset \operatorname{Dom} f_1$, therefore from $g \subset_n t_2(f_2)$ we obtain that

$$g = g \circ \underline{+}_{Dom\, f_1} \subset t_2(f_2) \circ_{Dom\, f_1} = t_2(f_2 \circ \underline{+}_{Dom\, f_1}) = t_2(f_1)\,.$$

Thus, $g \subset_n t_2(f_1)$. The condition (7) is proved.

Let now the premise of condition (8) be valid, then from $g_1 \subset_n f$ and $g_2 \subset_n f$ we have $g_1 = f \circ \underline{+}_{Dom\, g_1}$ and $g_2 = f \circ \underline{+}_{Dom\, g_2}$, respectively. Since $g_1 \leftharpoondown_n g_2$, then $\operatorname{Dom} g_1 \subset \operatorname{Dom} g_2$, therefore $f \circ \underline{+}_{Dom\, g_1} \subset f \circ \underline{+}_{Dom\, g_2}$, i.e. $g_1 \subset g_2$. The condition (8) is proved. The following condition (9) is proved similarly (7), therefore we must prove validity of (10).

Let $(a_1, ...a_m) \in \operatorname{Dom} f[h_1...h_n]$, therefore $(a_1, ..., a_m, c) \in f[h_1...h_n]$ for some element c, and there exists vector $(b_1, ..., b_n)$ for which

$(b_1, ..., b_n, c) \in f$ and $(a_1, ..., a_m, b_i) \in h_i$, $i = 1, ..., n$. Therefore, we obtain $(a_1, ..., a_m) \in \operatorname{Dom} h_i$. Thus, $\operatorname{Dom} f[h_1...h_n] \subset \operatorname{Dom} h_i$ which was needed to prove. □

Theorem. *A (weakly unitary) Menger $(T-)$ system of the form $(G_n, \mathcal{O}_n, \zeta_n, \chi_n)_{n\in I}$, where ζ_n, χ_n are fixed binary relations on G_n, is isomorphic to some f.o.p. Menger $(T-)$ system of multiplace functions if and only if it is a stable family of relations of order, $(\chi_n)_{n\in I}$ is a l-regular family of relations of quasi-order such that $\zeta_n \subset \chi_n$ for every $n \in I$, and the following conditions hold:*

$$g_1 \leq_n g \ \wedge \ g_2 \leq_n g \ \wedge \ g_1 \leftharpoondown_n g_2 \ \Rightarrow \ g_1 \leq_n g_2 \,, \tag{11}$$

$$g[h_1...h_n] \leftharpoondown_m h_i \,, \tag{12}$$

$$g_1 \leq_n g_2 \ \wedge \ g \leftharpoondown_n g_1 \ \wedge \ g \leftharpoondown_n u[\overline{\omega}\,|_j\, g_2] \ \Rightarrow \ g \leftharpoondown_n u[\overline{\omega}\,|_j\, g_1] \,, \tag{13}$$

for all $n, m \in I$, $i = 1, ..., n$, $j = 1, ..., m$, $g, g_1, g_2 \in G_n$, $u, h_1, ..., h_n \in G_m$, $\overline{\omega} = (\omega_1, ..., \omega_m) \in G_n^m$, where $g_1 \leq_n g_2$, $g_1 \leftharpoondown_n g_2$ mean that $(g_1, g_2) \in \zeta_n$ and $(g_1, g_2) \in \chi_n$, respectively.

Proof. The necessity of conditions of the theorem follows from Propositions 1 and 2, therefore weŠll dwell on the proof of their sufficiency. So, let the Menger (weakly unitary $T-$) system of the form $(G_n, \mathcal{O}_n, \zeta_n, \chi_n)_{n\in I}$ satisfies all conditions of the theorem. Easily can be proved that for all $n \in I$, $g, g_1, g_2 \in G_n$, $t_1, t_2 \in T_n$ the conditions:

$$g_1 \leq_n g_2 \ \wedge \ g \leftharpoondown_n t_1(g_1) \ \wedge \ g \leftharpoondown_n t_2(g_2) \ \Rightarrow \ g \leftharpoondown_n t_2(g_1) \,, \tag{14}$$

$$g_1 \leq_n g_2 \ \wedge \ g \leftharpoondown_n t_1(g_1) \ \wedge \ g \leftharpoondown_n t_2(g_2) \ \Rightarrow \ g \leftharpoondown_n t_2(g_1) \,, \tag{15}$$

are valid.

Let $\overline{G}$ denote the Cartesian power of the sets of the family $(G_n)_{n\in I}$. For every $\overline{g}$ from $\overline{G}$ we shall assign a family of pairs of the form $(\mathcal{E}_{\overline{g}<n>}, W_{\overline{g}<n>})_{n\in I}$, where $\mathcal{E}_{\overline{g}<n>} = \mathcal{E}^1_{\overline{g}<n>} \cap \mathcal{E}^2_{\overline{g}<n>}$, and $\mathcal{E}^1_{\overline{g}<n>}$, $\mathcal{E}^2_{\overline{g}<n>}$, $W_{\overline{g}<n>}$ are defined as follows:

$$(g_1, g_2) \in \mathcal{E}^1_{\overline{g}<n>} \Leftrightarrow (\forall t \in T_n)\, [\overline{g}\langle n\rangle \leq_n t(g_1) \Leftrightarrow \overline{g}\langle n\rangle \leq_n t(g_2)] \,, \tag{16}$$

$$(g_1, g_2) \in \mathcal{E}^2_{\overline{g}\langle n\rangle} \Leftrightarrow (\forall t \in T_n)\, [\overline{g}\langle n\rangle \leftharpoondown_n t(g_1) \Leftrightarrow \overline{g}\langle n\rangle \leftharpoondown_n t(g_2)] \tag{17}$$

$$W_{\overline{g}<n>} = G_n \backslash \chi_n \langle \overline{g}\langle n\rangle\rangle \,, \tag{18}$$

for all $g, g_1, g_2 \in G_n$, where

$$\chi_n\langle \overline{g}\langle n\rangle\rangle = \{x \,:\, \overline{g}\langle n\rangle \leftharpoondown_n x\}\,.$$

It is easy to see that $(\mathcal{E}_{\overline{g}<n>})_{n\in I}$ is the v-regular family of relations of equivalency and $(W_{\overline{g}<n>})_{n\in I}$ is an l-ideal family of $\mathcal{E}_{\overline{g}<n>}$-classes, if $W_{\overline{g}<n>} \neq \emptyset$ for every $n \in I$. Therefore, as it is shown in the previous part, the family $(\mathcal{E}_{\overline{g}<n>}, W_{\overline{g}<n>})_{n\in I}$ defines the simplest representation of the system $(G_n, \mathcal{O}_n)_{n\in I}$ with the help of multiplace functions, which we denote by $(P^n_{\overline{g}})_{n\in I}$. If $\overline{u}, \overline{v} \in \overline{G}$, $\overline{u} \neq \overline{v}$, then for every $g \in G_n$, $n \in I$ we'll find that the n-place functions $P^n_{\overline{u}}(g)$ and $P^n_{\overline{v}}(g)$ are given on disjoint sets. Let now

$$P^n(g) = \bigcup_{\overline{u}\in\overline{G}} P^n_{\overline{u}}(g),$$

then the family $(P^n)_{n\in I}$ is a representation of $(G_n, \mathcal{O}_n)_{n\in I}$ with the help of multiplace functions.

Let's prove that for every $n \in I$ and any $g_1, g_2 \in G_n$

$$g_1 \leftharpoondown_n g_2 \Longleftrightarrow \mathrm{Dom}\, P^n(g_1) \subset \mathrm{Dom}\, P^n(g_2)\,.$$

Indeed, if $\mathrm{Dom}\, P^n(g_1) \subset \mathrm{Dom}\, P^n(g_2)$, then it means that

for every $\overline{u} \in \overline{G}$:

$$\mathrm{Dom}\, P^n_{\overline{u}}(g_1) \subset \mathrm{Dom}\, P^n_{\overline{u}}(g_2)$$

for every $\overline{u} \in \overline{G}$.

The latter inclusion means that for $\overline{a_1}, ..., \overline{a_n} \in \overline{A}^n$ such that $(\overline{a_1}, ..., \overline{a_n}) \in \Im^n$, the implication

$$(\overline{a_1}, ..., \overline{a_n}) \in \mathrm{Dom}\, P^n_{\overline{u}}(g_1) \;\Rightarrow\; (\overline{a_1}, ..., \overline{a_n}) \in \mathrm{Dom}\, P^n_{\overline{u}}(g_2) \tag{19}$$

is valid. This condition, as it is easily to see, is equivalent to

$$(\forall\, \overline{b})\, (\exists\, \overline{c})\;\; (\overline{a_1}, ..., \overline{a_n}, \overline{b}) \in P^n_{\overline{u}}(g_1) \;\Rightarrow\; (\overline{a_1}, ..., \overline{a_n}, \overline{c}) \in P^n_{\overline{u}}(g_1)\,,$$

which, in its turn means

$$(\forall\, \overline{b})\, (\exists\, \overline{c})\, \Big[(\forall\, i \in I)\; \Big(g_1[H_{\overline{a_1}<i>} ... H_{\overline{a_n}<i>}] \subset H_{\overline{b}<i>} \Big) \Rightarrow \\ \Rightarrow (\forall k \in I)\; (\; g_2[H_{\overline{a_1}<k>} ... H_{\overline{a_n}<k>}] \subset H_{\overline{c}<k>}\;) \,]\,. \tag{20}$$

It can be proved that (20) is equivalent to the formula

$$(\,\forall\,(\overline{x_1},...,\overline{x_n})\in D_n)\,(\forall k\in I)$$
$$(\,g_1[\overline{x_1}\langle k\rangle...\overline{x_n}\langle k\rangle]\notin W_{\overline{u}<k>}\ \Rightarrow\ g_2[\overline{x_1}\langle k\rangle...\overline{x_n}\langle k\rangle]\notin W_{\overline{u}<k>}\,)\ , \tag{21}$$

where

$$D_n=\overline{G}^{\,n}\cup E_n^n\,,\quad E_n=\prod_{m\in I_n}G_m\times\{e_n\}\times\prod_{m\in I_n'}G_m\,,$$

(for the weakly unitary Menger T-system we suppose $E_n=\emptyset$), $e_n\notin G_n$ and $g[e_n...e_n]=g$ for every $g\in G_n$ by the definition. Let the condition (21) fulfill: $\overline{x_1},...,\overline{x_n}\in E_n^n$, $k=n$, and let $\overline{u}$ be an element from $\overline{G}$ such that $\overline{u}\langle n\rangle=g_1$, then we obtain:

$$g_1[e_n...e_n]\notin W_{g_1}\Rightarrow g_2[e_n...e_n]\notin W_{g_1}\,, \tag{22}$$

i.e.

$$g_1\notin W_{g_1}\Rightarrow g_2\notin W_{g_1}\,.$$

So $g_1\notin W_{g_1}$ is true for every $g_1\in G_n$, and (22) can be written as $g_2\in W_{g_1}$, i.e. $g_1 \leftharpoondown_n g_2$.

Conversely, let

(a) $g_1 \leftharpoondown_n g_2$,

(b) $g_1[\overline{x_1}\langle k\rangle...\overline{x_n}\langle k\rangle]\notin W_{\overline{u}<k>}$

for some $\overline{u}\in\overline{G}$, $k\in I$, $\overline{x_1},...,\overline{x_n}\in D_n$.

So (χ_n) is an l-regular family, and from (a) we obtain

$$g_1[\overline{x_1}\langle k\rangle...\overline{x_n}\langle k\rangle]\leftharpoondown_k g_2[\overline{x_1}\langle k\rangle...\overline{x_n}\langle k\rangle]\,.$$

The condition (b) means that

$$\overline{u}\langle k\rangle\leftharpoondown_k g_1[\overline{x_1}\langle k\rangle...\overline{x_n}\langle k\rangle]\,,$$

therefore, due to transitivity of χ_n we have

$$\overline{u}\langle k\rangle\leftharpoondown_k g_2[\overline{x_1}\langle k\rangle...\overline{x_n}\langle k\rangle]\,,$$

i.e. $g_2[\overline{x_1}\langle k\rangle...\overline{x_n}\langle k\rangle]\notin W_{\overline{u}<k>}$. Thus, (21) is proved. Hence,

$$\mathrm{Dom}\ P^n_{\overline{u}}(g_1)\subset\mathrm{Dom}\ P^n_{\overline{u}}(g_2)$$

for every $\overline{u}\in\overline{G}$, i.e. $\mathrm{Dom}\ P^n(g_1)\subset\mathrm{Dom}\ P^n(g_2)$. Let's prove now that for every $n\in I$ and any $g_1,g_2\in G_n$ the condition $g_1\leq_n (g_2)$ is valid if and only if the inclusion $P^n(g_1)\subset P^n(g_2)$ is true.

Indeed, if $P^n(g_1)\subset P^n(g_2)$, then for every $\overline{u}\in\overline{G}$ we have

$P^n_{\overline{u}}(g_1) \subset P^n_{\overline{u}}(g_2)$. This inclusion means that

$$(\overline{a_1}, ..., \overline{a_n}, \overline{b}) \in P^n_{\overline{u}}(g_1) \Rightarrow (\overline{a_1}, ..., \overline{a_n}, \overline{b}) \in P^n_{\overline{u}}(g_2) \tag{23}$$

for any $\overline{a_1}, ..., \overline{a_n}, \overline{b} \in \overline{A}$, where $(\overline{a_1}, ..., \overline{a_n}) \in \Im_n$. According to the definition of the simplest representation the condition (22) can be rewritten as follows:

$$(\forall i \in I)\,(\, g_1[H_{\overline{a_1}<i>} ... H_{\overline{a_n}<i>}] \subset H_{\overline{b}<i>}\,) \Rightarrow$$

$$\Rightarrow (\forall k \in I)\,(\, g_2[H_{\overline{a_1}<k>} ... H_{\overline{a_n}<k>}] \subset H_{\overline{b}<k>}\,)\,, \tag{24}$$

for all $\overline{a_1}, ... \overline{a_n}, \overline{b} \in \overline{A}$, where $(\overline{a_1}, ..., \overline{a_n}) \in \Im_n$. One can check that (24) is equivalent to the formula:

$$(\forall (\overline{x_1}, ..., \overline{x_n} \in D_n)\,(\forall k \in I)\ (g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle] \notin W_{\overline{u}<k>} \Rightarrow$$
$$\Rightarrow g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle] \equiv g_2[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle]\ (\mathcal{E}_{\overline{u}} < k >)\,) \tag{25}$$

Assume that $\overline{x_1}, ..., \overline{x_n} \in E^n_n$, $k = n$ in the condition (25) and let $\overline{u}$ be an element from $\overline{G}$, such that $\overline{u}\langle n\rangle = g_1$, then we obtain $g_1 \equiv g_2(\mathcal{E}_{g_1})$, whence it follows: $g_1 \equiv g_2(\mathcal{E}^1_{g_1})$. The latter according to formula (16), means that

$$(\forall t \in T_n)\ (\, g_1 \leq_n t(g_1) \Longleftrightarrow g_1 \leq_n t(g_2)\,). \tag{26}$$

Let t be the variable x_n, then from (26) it follows that $g_1 \leq_n g_2$.

Conversely, suppose that $g_1 \leq_n g_2$ and

$$g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle] \notin W_{\overline{u}<k>}$$

for any $\overline{u} \in \overline{G}$, $(\overline{x_1}, ..., \overline{x_n}) \in D_n$, $k \in I$. We must prove that

$$g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle] \equiv g_2[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle]\ (\mathcal{E}_{\overline{u}<k>}) \tag{27}$$

is valid.

For this purpose we must check if the condition (27) is valid for every relation $\mathcal{E}^i_{\overline{u}<k>}$, $i = 1, 2$, which is defined with the help of the formulas (16) and (17). Let

$$\overline{u}\langle k\rangle \leq_k\ t(g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle])$$

for some $t \in T_k$. Since the family $(\zeta_n)_{n\in I}$ is stable, then from $g_1 \leq_n g_2$

we obtain

$$t(g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle]) \leq_k t(g_2[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle])\,,$$

therefore

$$\overline{u}\langle k\rangle \leq_k t(g_2[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle])\,.$$

Suppose now that

$$\text{(c)} \quad \overline{u}\langle k\rangle \leq_k t(g_2[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle])$$

is valid, where $t \in T_k$.

Since $g_1 \leq_n g_2$ and

$$g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle] \notin W_{\overline{u}<k>}$$

then it is evident that

$$g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle] \leq_k g_2[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle]$$

and

$$\overline{u}\langle k\rangle \leftharpoondown_k g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle]\,,$$

whence considering (c) and (15) we obtain

$$\overline{u}\langle k\rangle \leq_k t(g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle])\,.$$

This means that (27) is true for $\mathcal{E}^1_{\overline{u}<k>}$.

Now, let

$$\text{(d)} \quad \overline{u}\langle k\rangle \leftharpoondown_k t(g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle])$$

be valid for some $t \in T_k$.

Since $g_1 \leq_n g_2$ then as it has been stated above

$$t(g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle]) \leq_k t(g_2[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle])\,,$$

therefore,

$$t(g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle]) \leftharpoondown_k t(g_2[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle])\,,$$

hence,

$$\text{(e)} \quad \overline{u}\langle k\rangle \leftharpoondown_k t(g_2[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle])\,.$$

Conversely, let (c) be valid. From $g_1 \leq_n g_2$ we obtain

$$g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle] \leq_k g_2[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle]\,,$$

therefore, according to (14),

$$\overline{u}\langle k\rangle \leftharpoondown_k \ t(g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle])$$

it will be true. Thus, (27) is true for $\mathcal{E}^2_{\overline{u}<k>}$. Thus, (25) is valid because, as it has been stated above, it is equivalent to $P^n_{\overline{u}}(g_1) \subset P^n_{\overline{u}}(g_2)$, where $\overline{u} \in \overline{G}$. Thus $P^n(g_1) \subset P^n(g_2)$, $n \in I$, $g_1, g_2 \in G_n$, which was needed to prove.

Finally, let us suppose that $P^n(g_1) = P^n(g_2)$ for some $g_1, g_2 \in G_n$. Then $P^n(g_1) \subset P^n(g_2)$ and $P^n(g_2) \subset P^n(g_1)$. Therefore $g_1 \leq_n g_2$ and $g_2 \leq_n g_1$, whence $g_1 = g_2$, since ζ_n is an order. So we have proved that the (weakly unitary $T-$) Menger system $\mathcal{G}$ is isomorphic to f.o.p. Menger ($T-$) system of multiplace functions. The theorem is proved. □

Corollary 1. *A (weakly unitary $T-$) Menger system of the form $(G_n, \mathcal{O}_n, \zeta_n)_{n\in I}$ (where $\zeta_n \subset G_n \times G_n$) is isomorphic to some f.o. ($T-$) Menger system of multiplace functions if and only if $(\zeta_n)_{n\in I}$ is a stable family of relations of order, satisfying the condition:*

$$g_1 \leq_n g_2 \ \wedge \ g \leq_n t_1(g_1) \ \wedge \ g \leq_n t_2(g_2) \ \Rightarrow \ g \leq_n t_2(g_1) \qquad (28)$$

for all $n \in I$, $g, g_1, g_2 \in G_n$, $t_1, t_2 \in T_n$.

Proof. Supposing $\chi_n = \delta_n \circ \zeta_n$, where $\delta_n \subset G_n \times G_n$ and

$$(g_1, g_2) \in \delta_n \Longleftrightarrow (\exists\, t \in T_n)\ g_1 = t(g_2)\,,$$

we come to the conclusion that the system $(G_n, \mathcal{O}_n, \zeta_n, \chi_n)_{n\in I}$ satisfies all the conditions of the theorem. □

Corollary 2. *A (weakly unitary $T-$) Menger system of the form $(G_n, \mathcal{O}_n, \chi_n)_{n\in I}$ (where $\chi_n \subset G_n \times G_n$) is isomorphic to some p.q-o. ($T-$) Menger system of multiplace functions if and only if $(\chi_n)_{n\in I}$ is a l-regular family of relations of quasi-order satisfying the condition* (12).

Proof. Supposing $\zeta_n = \Delta_{G_n}$ in the theorem, where Δ_{G_n} is the identical relation on G_n, we obtain the present corollary. □

We must remark that analogous results for semigroups [2] and for Menger algebras [4] may obtain from the proved theorem.

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