THE 70th ANNIVERSARY OF V.D.BELOUSOV

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On the 20th of February 1995 we celebrated the 70th birthday of the great Moldavian mathematician Valentin Danilovich Belousov, the famous specialist in the field of quasigroup theory, algebraic nets and functional equations. He was not only the creator of the quasigroup school in Moldova but also the excellent teacher.

V.D.Belousov was born in Bel'tsy, Moldavia. After studies in mathematics at Kishinev pedagogical Institute (1944-1947) he spent a short time as a school teacher and then taught at Bel'tsy pedagogical Institute. In 1954-1956 V.D.Belousov was a graduate student of Moscow State University, where the famous Russian algebraist professor A.G.Kurosh was his chief. Here he defended his candidate thesis (1958). In 1960-1961 V.D.Belousov was at Wisconsin University (USA), where he worked with the famous American scientist in the area of quasigroup theory professor R.H.Bruck. Beginning with 1962 up to 1988 he was Chair of the algebra and mathematical logic department in Institute of Mathematics of Moldavian Academy of Sciences and head of the algebra and geometry chair of Kishinev

State University (1967-1977). In 1966 he defended the doctoral thesis "Systems of quasigroups with generalised identities" and in 1968 was elected Corresponding Member of the Academy of Pedagogical Sciences of USSR.

V.D.Belousov was the pioneer in the area of quasigroup and loop theory in USSR and carried the great contribution at the development of this theory. He studied many questions of general quasigroup theory, such as the groups associated with quasigroups (multiplication groups of quasigroups) derivative operations of loops, regular mapping groups and nuclei of quasigroups, autotopies and antiautotopies of quasigroups, groups of inner substitutions with respect to an element, normal subquasigroups, isotopy and crossed isotopy of quasigroups and others. V.D.Belousov also investigated different classes of quasigroups and loops, particularly IP-quasigroups, F-quasigroups, TS-quasigroups, CI-quasigroups, Stein's quasigroups, I-quasigroups and I-loops, Bol loops. Especially deep results he received relative to the distributive quasigroups. His theorem that each distributive quasigroup is isotopic to a commutative Moufang loop is well known. V.D.Belousov also studied left-distributive quasigroups and loops isotopic to them. Many results are described in his monograph "Foundations of quasigroup and loop theory" (Russian, Moscow, "Nauka", 1967).

In the important paper "*Balanced identities in quasigroups*" (Russian, Matem. sbornik, v. **70(112)**, №1, 1966, p. 55-97) V.D.Belousov found the elegant characterisation of the quasigroups with balanced identities, having established their connections with linear (over groups) quasigroups. By his great paper "*Systems of quasigroups with generalised identities*" (Russian, Uspehi mat. nauk. v. **XX**, №1 (121), 1965, p.75-146) V.D.Belousov created a new branch of quasigroup theory, having shown that the considered systems are connected with the solutions of corresponding functional equations on quasigroups. In this work he studied the systems of quasigroups with generalized identities of associativity (mediality, distributivity, transitivity) and with the generalised Stein's identity (S-systems).

A large cycle of Belousov's work is devoted to the solving of different functional equations on quasigroups, such as the functional equations of general associativity, distributivity, mediality, the functional Moufang equation. In solving of the functional equation of general associativity Belousov's theorem about four quasigroups connected by the associative law plays the fundamental role. These theorem states that all of these four quasigroups are isotopic to the same group. The through study of the functional equations of general associativity for n-ary case led V.D.Belousov to the new concept of a positional algebra of quasigroups. The elements of such an algebra are quasigroups given on the same set, their operations are superpositions, each identity is a functional equation.

Belousov published a number of works devoted to the study of n-ary quasigroups. These works laid the foundation of n-ary quasigroup theory. Many of his results in these directions are reflected in his monograph "*n-Ary quasigroups*" (Russian, Kishinev, "Stiintsa", 1972). This book contains an information about general concepts for n-ary case, different classes of n-ary quasigroups (n-groups, medial quasigroups, TS-quasigroups, Menger's quasigroups, IP-quasigroups, (i,j)-associative quasigroups), positional algebras of quasigroups, reducibility of quasigroups and different cases of the functional equation of general associativity.

The monographs "*Algebraic nets and quasigroups*" (Russian, Kishinev, "Stiintsa", 1971) and "*Configurations in algebraic nets*"

(Russian, Kishinev, "Stiintsa", 1979) by V.D.Belousov reflect his great contribution in the theory of algebraic nets (or webs as they are sometimes called) and open up a new area linking algebra, geometry and combinatorics.

Further on, by mean of original algebraic methods V.D.Belousov studied some combinatorial questions of quasigroup (or Latin square) theory, such as admissibility, prolongation, ortogonality, parastrophy-invariant ortogonality of quasigroups (Latin squares). In his work "*Systems of orthogonal operations*" (Russian and English, Mat. sbornic. v. **77(119)**, №1, 1968, p. 33-52) he established the connection between orthogonal systems of operations and orthogonal systems of quasigroups (OSQ) and studied the parastrophy transformation of an OSQ. His elegant work "*Parastrophy-orthogonal quasigroups*" (Russian, Pre-print, IM AN SSRM, Kishinev, 1983) is devoted to research of the minimal identities in quasigroups, connected with ortogonality of definite quasigroup parastrophes.

Belousov's mathematical works got worldwide recognition and have been representing a source of inspiration for further research. He published six scientific monographs and more than 130 works. His numerous pupils work in many countries. He served on the Editorial Board of "Aequationes Mathematicae" from 1967 to 1975.

V.D.Belousov was not only a scientist. His teaching activity was also important for Moldova mathematics. He was a great personality of Moldavian culture, a person full of generosity, warmth and refinement. Valentin Danilovich Belousov devoted the life to science, a life that will always be an example and impulse for his followers.

The list of Belousov's principle works one can find in the sbornic "*Quasigroups and their systems*" (Russian, Matem. Issled., vyp. **113**, Kishinev, 1990).

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About some algebraic systems related with projective planes

Evgenii A. Kuznetsov

Abstract

The present article is a survey of author's results on the investigations of algebraic structures related with projective planes; some new theorems are proved too.

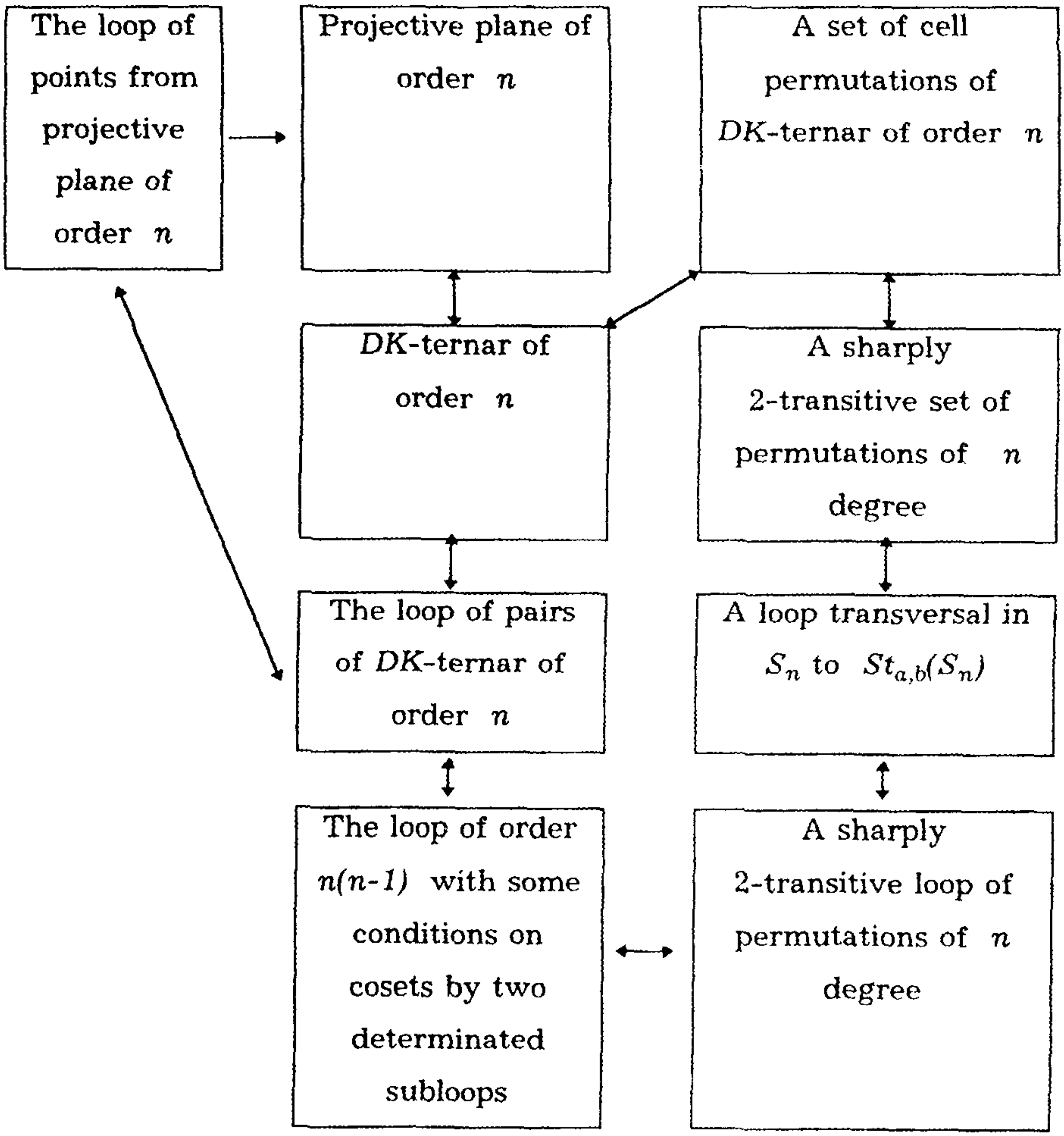
The *projective plane* is the incidence structure $<X, L, I>$ which satisfies the following axioms:

1) Given any two distinct points from X there exists just one line from L incident with both of them.

2) Given any two distinct lines from L there exists just one point from X incident with both of them.

3) There exist four points such that a line incident with any two of them is not incident with either of the remaining two.

This article is a survey of some author's results (see [2,7]) about algebraic structures related with projective planes (finite as a rule, if the contrary is not stipulated); some new theorems are proved too. The main aim of article is to demonstrate the correlations in the following scheme:



0. Necessary definitions and notations

Definition 1. [1] A system $< E,\cdot >$ is called a *quasigroup*, if for arbitrary $a,b \in E$ equations $x \cdot a = b$ and $a \cdot y = b$ have an unique solution in the set E. If in quasigroup $<E,\cdot>$ there exists element $e \in E$ such that

$$x \cdot e = e \cdot x = x$$

for any $x \in E$, then system $<E,\cdot>$ is called a *loop*.

Definition 2. [2] A system $<E,(x,t,y),0,1>$ is called a *DK-ternar* (e.g. a set E with ternary operation (x,t,y) and distinguished elements $0,1 \in E$), if the following conditions hold:

1). $(x,0,y) = x$;

2). $(x,1,y) = y$;

3). $(x,t,x) = x$;

4). $(0,t,1) = 0$;

5). If a,b,c and d are arbitrary elements from E and $a \neq b$, then the system

$$\begin{cases} (x,a,y) = c; \\ (x,b,y) = d; \end{cases}$$

has an unique solution in $E \times E$.

6). Either set E is finite, or

a) if a,b,c are arbitrary elements from E and $c \neq 0$, $(c,a,0) \neq b$, then the system

$$\begin{cases} (x,a,y) = b; \\ (x,t,y) \neq (c,t,0) \quad \forall t \in E; \end{cases}$$

has an unique solution in $E \times E$.

b) if a,b are arbitrary elements from E and $b \neq 0$, then inequality

$$(a,t,b) \neq (x,t,0) \quad \forall t \in E$$

has an unique solution in E.

If the set E is finite, then conditions 6a) and 6b) are corollaries of the conditions 1)-5) of **Definition 2**. Proof of this statement will be given later.

Definition 3. A set M of permutations on a set X is called *sharply (strongly) 2-transitive*, if for any two pairs (a,b) and (c,d) of different elements from X there exists an unique permutation $\alpha \in M$ satisfying the following conditions

$$\alpha(a) = c, \qquad \alpha(b) = d.$$

Definition 4. [3] Let G be a group and H be a subgroup in G. A complete system T of representatives of the left (right) cosets in G to H $(e = t_1 \in H)$ is called a *left (right) transversal in* G *to* H.

Let T be a transversal (left or right) in G to H. We can introduce correctly the following operations on Λ (Λ is an index set; left (right) cosets in G to H are numbered by indexes from Λ):

$$i * j = v \Leftrightarrow t_i t_j = t_v h, h \in H,$$

if T is a left transversal, and

$$i * j = w \Leftrightarrow t_i t_j = h t_w, h \in H,$$

if T is a right transversal.

Definition 5. Let T be a left (right) transversal in G to H. If the system $<\Lambda, *, 1>$ $(<\Lambda, \bullet, 1>)$ is a loop, then T is called a *left (right) loop transversal in* G *to* H.

1. Projective plane and DK-ternar

Lemma 1. *Let π be a projective plane. It is possible to introduce coordinates $(a,b),(m),(\infty)$ for points and $[a,b],[m],[\infty]$ for lines from π (where $a,b,m \in E$, E is some set with distinguished elements 0 and 1), such that for operation (x,t,y), where*

$$(x,t,y)=z \overset{def}{\Leftrightarrow} (x,y)\in[t,z],$$

the system $<E,(x,t,y),0,1>$ is a DK-ternar.

Proof. Let π be an arbitrary projective plane. Let X,Y,O,I be arbitrary four points in the general position on π.

Suppose, by definition,

$$[XY]=[\infty]; \qquad [OI]=[0];$$
$$O=(0,0); \qquad I=(1,1).$$

Then

$$[\infty]\cap[0] \overset{def}{=} (\infty).$$

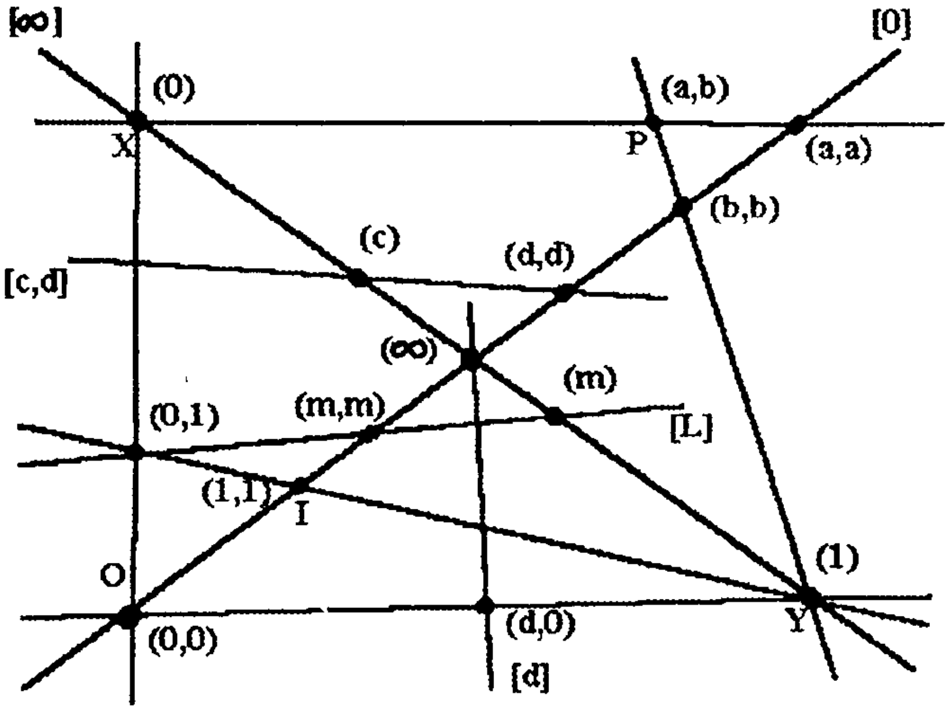
All other points of the line $[0]$ are attributed by definition by the symbols (a,a) (where $a \neq 0,1$), and different points are attributed by different symbols.

Let P be an arbitrary point from π and $P \notin [\infty]$. Let us have

$$\begin{cases} [XR]\cap[0]=(a,a); \\ [YP]\cap[0]=(b,b). \end{cases} \tag{1}$$

Then suppose, by definition,

$$P \overset{def}{=} (a,b).$$



It is evident that points of the line [0] will have their own coordinates.

Let [L] be an arbitrary line from π and $(\infty) \notin$ [L]. Let us have

$$\begin{cases} [L] = (0,1) \cup (m, m); \\ [L] \cap [\infty] = Z; \end{cases} \tag{2}$$

Then suppose by definition:

$$Z \overset{def}{=} (m).$$

In particular,

$$X = (0), \qquad Y = (1).$$

Suppose by definition:

$$(\infty) \cup (d,0) \overset{def}{=} [d]. \tag{3}$$

Finally, let [S] be an arbitrary line from π and $(\infty) \notin$ [S]. Let us have

$$\begin{cases} [S] \cap [\infty] = (c); \\ [S] \cap [0] = (d, d); \end{cases} \tag{4}$$

Then suppose by definition:

$$[S] \overset{def}{=} [c, d].$$

Let us define the ternary operation (x, t, y) by the condition of **Lemma**, e.g.

$$(x, t, y) = u \quad \overset{def}{\Leftrightarrow} \quad (x, y) \in [t, u],$$

and verify that conditions 1)-6) of **Definition 2** hold.

a). $(x,0,y) = x$.

$$(x,0,y) = u \quad \Leftrightarrow \quad (x,y) \in [0,u] \quad \Leftrightarrow$$

$$\Leftrightarrow \quad (\text{the points } (x,y), (0) \text{ and } (u,u) \text{ lie in a common line (see (4))} \quad \Rightarrow$$

$$\Rightarrow \quad (u,u) = [0] \cap [(x,y) \cup (0)] = (x,x) \quad \Rightarrow \quad u = x.$$

b). $(x,1,y) = y$.

The proof is analogous to that of a).

c). $(x,t,x) = x$.

$$(x,t,x) = u \quad \Leftrightarrow \quad (x,x) \in [t,u] \quad \Leftrightarrow$$

$$\Leftrightarrow \quad (\text{the points } (x,x), (t) \text{ and } (u,u) \text{ lie in a common line (see (4))} \quad \Rightarrow$$

$$\Rightarrow \quad u = x.$$

d). $(0,t,1) = t$.

$$(0,t,1) = u \quad \Leftrightarrow \quad (0,1) \in [t,u] \quad \Leftrightarrow$$

$$\Leftrightarrow \quad (\text{the points } (0,1), (t) \text{ and } (u,u) \text{ lie in a common line (see (4))} \quad \Rightarrow$$

$$\Rightarrow \quad (u = t \text{ (see(2))}.$$

e). Let a,b,c,d be arbitrary elements from E and $a \neq b$. Then we have

$$\begin{cases} (x,a,y) = c; \\ (x,b,y) = d; \end{cases} \Leftrightarrow \begin{cases} (x,y) \in [a,c] \\ (x,y) \in [b,d] \end{cases} \Leftrightarrow \quad (x,y) = [a,c] \cap [b,d].$$

There exists an unique such point (x,y) in the projective plane π.

f). If E is a finite set, then the proof is completed. Let E be an infinite set, a,b,c arbitrary elements from E and $c \neq 0, (c,a,0) \neq b$. Then we have

$$\begin{cases} (x,a,y) = b; \\ (x,t,y) \neq (c,t,0) \quad \forall t \in E; \end{cases} \Leftrightarrow \begin{cases} (x,y) \in [a,b] \\ (c,0) \notin [a,b] \\ (x,y) \cup (c,0) \neq [t,u] \quad \forall t,u \in E \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (x,y) \in [a,b] \\ (x,y) \cup (c,0) = [c] \end{cases} \Leftrightarrow (x,y) = [a,b] \cap [c].$$

There exists an unique such point (x,y) in the projective plane π.

The proof of the condition 6b) of **Definition 2** is analogous to that of 6a). Thus the system $< E,(x,t,y),0,1 >$ is a DK-ternar. □

Lemma 2. *Let* $< E,(x,t,y),0,1 >$ *be a DK-ternar and* a *be an arbitrary fixed element from E,* $a \neq 0,1$. *Then the system* $< E,(x,a,y) >$ *is a quasigroup.*

Proof. Let the conditions of **Lemma** hold. Then we have for arbitrary $b,c \in E$

$$(x,a,c) = c \quad \Leftrightarrow \quad \begin{cases} (x,a,y) = c; \\ y = b; \end{cases} \quad \Leftrightarrow \quad \begin{cases} (x,a,y) = c; \\ (x,1,y) = b; \end{cases}$$

There exists the unique solution (x_0,b) of the last system in $E \times E$. Then the equation $(x,a,b) = c$ has a unique solution x_0 in E. The reasoning for the equation $(b,a,y) = c$ is analogous. □

Lemma 3. *Let the conditions 1)-5) of* ***Definition 2*** *hold true and the set E be finite. Then conditions 6a) and 6b) of* ***Definition 2*** *hold.*

Proof. Let conditions of **Lemma** hold. Let a,b,c be arbitrary elements from E, $c \neq 0$, $(c,a,0) \neq b$. We will demonstrate that the system

$$\begin{cases} (x,a,y) = b; \\ (x,t,y) \neq (c,t,0) \quad \forall t \in E; \end{cases}$$

has an unique solution in $E \times E$ (e.g. the condition 6a) of **Definition 2** holds).

Let us study the system

$$\begin{cases} (x,a,y) = b; \\ (x,t,y) = (c,t,0); \end{cases}$$

for every fixed $t \in E - \{a\}$. This system has a unique solution (x_t, y_t) in $E \times E$. Let us assume that

$$\begin{cases} t_1 \neq t_2; \\ (x_{t_1}, y_{t_1}) \equiv (x_{t_2}, y_{t_2}); \end{cases}$$

Then the system

$$\begin{cases} (x,t_1,y) = (c,t_1,0); \\ (x,t_2,y) = (c,t_2,0); \end{cases}$$

has two distinct solutions $(c,0)$ and $(x_{t_1}, y_{t_1}) \equiv (x_{t_2}, y_{t_2})$ in $E \times E$ ($(x_{t_1}, y_{t_1}) \neq (c,0)$, since $(x_{t_1}, a, y_{t_1}) = b \neq (c,a,0)$). It contradicts the condition 5) of **Definition 2**. So

$$(x_{t_1}, y_{t_1}) \neq (x_{t_2}, y_{t_2}) \quad \Leftrightarrow \quad t_1 \neq t_2.$$

Then

$$\operatorname{card}\{(x_t, y_t) \mid t \in E - \{a\}\} = \operatorname{card}\{E - \{a\}\} = n - 1,$$

where $n = \operatorname{card} E$. From the other side,

$$\operatorname{card}\{(x,y) \mid (x,a,y) = b\} = n,$$

since this number is equal to the number of cells with the element b in table of the operation (x,a,y) (see **Lemma 2** too). So there exists an unique pair $(x_0,y_0)\in E\times E$ that satisfies the following system:

$$\begin{cases} (x_0,a,y_0)=b; \\ (x_0,y_0)\neq(x_t,y_t) \quad \forall t\in E-\{a\}; \end{cases}$$

This pair (x_0,y_0) is the unique solution of the initial system from the condition 6a) of **Definition 2**, e.g. this condition holds.

Proof of the condition 6b) is analogous to that of 6a). $\square$

Let us introduce the following binary operation (x,∞,y) on E:

$$(x,\infty,0)\overset{def}{=}0,$$

$$\begin{cases} (x,\infty,y)=u; \\ (x,y)\neq(u,0); \end{cases} \overset{def}{\Leftrightarrow} (x,t,y)\neq(u,t,0) \quad \forall t\in E.$$

As we can see from the condition 6b) of **Definition 2**, the operation (x,∞,y) is defined correctly.

Lemma 4. *Operation (x,∞,y) satisfies the following conditions:*

1) $$\begin{cases} (x,\infty,y)=(u,\infty,v); \\ (x,y)\neq(u,v); \end{cases} \Leftrightarrow (x,t,y)\neq(u,t,v) \quad \forall t\in E. \tag{5}$$

2) $(x,\infty,x)=0.$

3) *There exists an unique solution in $E\times E$ of the system*

$$\begin{cases} (x,a,y)=b; \\ (x,\infty,y)=c; \end{cases}$$

for an arbitrary fixed $a,b,c\in E$.

4) *System $<E,(x,\infty,y)>$ is a quasigroup.*

Proof. 1). Let

$$\begin{cases} (x,\infty,y)=(u,\infty,v)=d; \\ (x,y)\neq(u,v); \end{cases}$$

Then we have by the definition of the operation (x,∞,y):

$$(x,t,y)\neq(d,t,0) \quad \forall t\in E, \tag{6}$$

$$(u,t,v)\neq(d,t,0) \quad \forall t\in E. \tag{7}$$

Assume that there exists $t_0\in E$ such that

$$(x,t_0,y)=(u,t_0,v)=w_0. \tag{8}$$

Then the system

$$\begin{cases} (x,t_0,y)=w_0; \\ (x,t,y)\neq(d,t,0) \quad \forall t\in E; \end{cases}$$

has two distinct solutions: (x,y) and (u,v) (see (6)-(8)). It contradicts condition 6a) of **Definition 2,** since

$$(x,t,y)\neq(u,t,v) \quad \forall t\in E.$$

Conversely, let

$$(x_0,t,y_0)\neq(u_0,t,v_0) \quad \forall t\in E. \tag{9}$$

Then we have (when $t=0,1$)

$$x_0\neq u_0, \qquad y_0\neq v_0,$$

i.e. $(x_0,y_0)\neq(u_0,v_0)$.

Let

$$(x_0,\infty,y_0)=d.$$

Then we have by the definition of the operation (x,∞,y):

$$(x_0,t,y_0)\neq(d,t,0) \quad \forall t\in E. \tag{10}$$

Let us assume that there exists $t_0\in E$ such that

$$(u_0,t_0,v_0)=(d,t_0,0)=z_0. \tag{11}$$

Then the system

$$\begin{cases} (x,t_0,y)=z_0; \\ (x,t,y)\neq(x_0,t,y_0) \quad \forall t\in E; \end{cases}$$

has two distinct solutions: (u_0, v_0) and $(d,0)$ (see (9)-(11)). It contradicts the condition 6a) of **Definition 2,** since

$$(u_0, t, v_0) \neq (d, t, 0) \quad \forall t \in E.$$

Then we have by the definition of the operation (x, ∞, y):

$$(u_0, \infty, v_0) = d = (x_0, \infty, y_0).$$

2). By the definition of the operation (x, ∞, y) we have

$$(0, \infty, 0) = 0.$$

If $x \neq 0$, then

$$(0, t, 0) = 0 \neq x = (x, t, x) \quad \forall t \in E,$$

(see condition 3) of **Definition 2**) and thus

$$(x, \infty, x) = (0, \infty, 0) = 0$$

(see p. 1) of this **Lemma**).

3). Let a,b,c be arbitrary fixed elements from E.

<u>Case A:</u> $c = 0$.

Then the system from the condition 3) of **Lemma** has the following form

$$\begin{cases} (x, a, y) = b; \\ (x, \infty, y) = 0; \end{cases} \tag{12}$$

It is easy to see that the pair $(x, y) = (b, b)$ is a solution of system (12). Let us assume that there exists other solution $(x', y') \neq (b, b)$ of the system (12). Then we have

$$\begin{cases} (x', a, y') = b; \\ (x', \infty, y') = 0 = (b, \infty, b); \end{cases} \Leftrightarrow \begin{cases} (x', a, y') = b; \\ (x', t, y') \neq (b, t, b) = b \quad \forall t \in E; \end{cases}$$

It is impossible, since there exists an unique solution $(b\ b)$ of the system (12).

<u>Case B:</u> $(c, a, 0) = b$.

Then the system from the condition 3) of Lemma has the following form

$$\begin{cases} (x,a,y)=b=(c,a,0); \\ (x,\infty,y)=c=(c,\infty,0); \end{cases} \tag{13}$$

It is easy to see that the pair $(x,y)=(c,0)$ is a solution of the system (13). Let us assume that there exists other solution $(x',y')\neq(c,0)$ of the system (13). Then we have

$$\begin{cases} (x',a,y')=(c,a,0); \\ (x',\infty,y')=(c,\infty,0); \end{cases} \Leftrightarrow \begin{cases} (x',a,y')=(c,a,0); \\ (x',t,y')\neq(c,t,0) \quad \forall t\in E; \end{cases}$$

It is impossible, since there exists an unique solution of the system (13).

<u>Case C:</u> $c\neq 0$ and $(c,a,0)\neq b$.

Then the system from the condition 3) of **Lemma** has the following form

$$\begin{cases} (x,a,y)=b; \\ (x,t,y)\neq(c,t,0) \quad \forall t\in E; \end{cases} \tag{14}$$

System (14) has an unique solution in $E\times E$ (see the condition 6a) of **Definition 2**).

4). Proof is analogous to that of **Lemma 3**. □

Let us introduce points $(a,b),(m),(\infty)$ and lines $[a,b],[m],[\infty]$ (where $a,b,m\in E$) and define an incident relation I between points and lines by the following way (see [2]):

$$\begin{gathered} (a,b)I[c,d] \Leftrightarrow (a,c,b)=d, \\ (a,b)I[d] \Leftrightarrow (a,\infty,b)=d, \\ (a)I[c,d] \Leftrightarrow a=c, \\ (a)I[\infty],\quad (\infty)I[d],\quad (\infty)I[\infty], \\ (a,b)I[\infty]\Leftrightarrow(a)I[d]\Leftrightarrow(\infty)I[c,d]\Leftrightarrow \text{False}. \end{gathered} \tag{15}$$

Lemma 5. *The incidence system* $<P, L, I>$, *where*

$$P=\{(a,b),(m),(\infty) \mid a,b,m\in E\},$$

$$L=\{[a,b],[m],[\infty] \mid a,b,m\in E\},$$

I *is the incidence relation from (15)*

is a projective plane.

Proof. Let us verify the axioms of projective plane.

1). <u>An arbitrary two distinct lines are intersected in unique point.</u>

a). <u>The lines</u> $[a,b]$ <u>and</u> $[c,d]$:

If $a=c$, then we have from (15):

$$[a,b]\cap[c,d]\equiv[a,b]\cap[a,d]=(a).$$

If we assume that there exists a point (x,y) which lies both on lines $[a,b]$ and $[a,d]$, then

$$\begin{cases}(x,y)I[a,b];\\(x,y)I[a,d];\end{cases}\Leftrightarrow\begin{cases}(x,a,y)=b;\\(x,a,y)=d;\end{cases}\Rightarrow\quad b=d,$$

i.e. $[a,b]\equiv[a,d]$. It is impossible since $[a,b]$ and $[a,d]$ are distinct lines.

If $a\neq c$, then we have

$$\begin{cases}(x,y)I[a,b];\\(x,y)I[c,d];\end{cases}\Leftrightarrow\begin{cases}(x,a,y)=b;\\(x,c,y)=d;\end{cases}$$

By the condition 5) from **Definition 2** there exists an unique such point (x,y).

b). <u>The lines</u> $[a,b]$ <u>and</u> $[d]$:

We have

$$\begin{cases}(x,y)I[a,b];\\(x,y)I[d];\end{cases}\Leftrightarrow\begin{cases}(x,a,y)=b;\\(x,\infty,y)=d;\end{cases}$$

As we can see from the statement 3) of **Lemma 4** there exists an unique such point (x, y).

c). The lines $[a,b]$ and $[\infty]$, $[m]$ and $[d]$, $[m]$ and $[\infty]$.

We have

$$[a,b] \cap [\infty] = (a),$$
$$[m] \cap [d] = (\infty),$$
$$[m] \cap [\infty] = (\infty).$$

2). There exists an unique common line for arbitrary two distinct points.

a). The points (a,b) and (c,d):

If there exists an element $t_0 \in E$ such that

$$(a, t_0, b) = (c, t_0, d) = f, \tag{16}$$

then we have

$$(a,b) \cup (c,d) = [t_0, f].$$

As we can see from the condition 5) of **Definition 2**, only one element $t_0 \in E$ with the condition (16) may exist.

If

$$(a, t, b) \neq (c, t, d) \quad \forall t \in E,$$

then by the statement 1) of **Lemma 4** we have

$$(a, \infty, b) = (c, \infty, d) = h,$$

and

$$(a,b) \cup (c,d) = [h].$$

b). The points (a,b) and (m), (a,b) and (∞), (m) and (n), (m) and (∞).

We have

$$(a,b) \cup (m) = [m, (a, m, b)],$$
$$(a,b) \cup (\infty) = [(a, \infty, b)],$$
$$(m) \cup (n) = [\infty],$$
$$(m) \cup (\infty) = [\infty].$$

3). <u>There exist four points in a common position.</u>

These points are $(0,0),(1,0),(0)$ and (∞). Really, we have

$$(0,0)\cup(1,0)=[1,0],\qquad (1,0)\cup(0)=[0,1],$$
$$(0,0)\cup(0)=[0,0],\qquad (1,0)\cup(\infty)=[1],$$
$$(0,0)\cup(\infty)=[0],\qquad (0)\cup(\infty)=[\infty].$$

2. Cell permutations and pair loop of *DK*-ternar

Lemma 6. *Let the system* $<E,(x,t,y),0,1>$ *be a DK-ternar. Let* a,b *be arbitrary elements from E and* $a\neq b$. *Then any unary operation*

$$\alpha_{a,b}(t)=(a,t,b) \tag{17}$$

is a permutation on the set E.

Proof. Let the conditions of **Lemma** hold. We can prove the following: if $t_1\neq t_2$, then $\alpha_{a,b}(t_1)\neq\alpha_{a,b}(t_2)$. Let us assume that there exist $t_1,t_2\in E$ such that

$$\begin{cases} t_1\neq t_2; \\ (a,t_1,b)=(a,t_2,b)=k; \end{cases}$$

Then the system

$$\begin{cases} (x,t_1,y)=k; \\ (x,t_2,y)=k; \end{cases}$$

has two distinct solutions in $E\times E$: (a,b) and (k,k). It contradicts condition 5) of **Definition 2.**

Let us prove that for any $c\in E$ there exists $t_0\in E$ such that $c=\alpha_{a,b}(t_0)$. We have (see **Lemmas 4** and **5**):

$$c = \alpha_{a,b}(t_0) \quad \Leftrightarrow$$
$$\Leftrightarrow \quad c = (a, t_0, b) \quad \Leftrightarrow$$
$$\Leftrightarrow \quad (a,b) \in [t_0, c] \quad \Leftrightarrow$$
$\Leftrightarrow$ (points $(a,b), (t_0)$ and (c,c) lie
in a common line in the projective plane π). $\Leftrightarrow$
$$\Leftrightarrow \quad (t_0) = [\infty] \cap [(a,b) \cup (c,c)].$$

There exists an unique such element $t_0 \in E$. □

The permutations from **Lemma 6** are called *cell permutations.*

Lemma 7. *Cell permutations satisfy of the following conditions:*

1). *All cell permutations are distinct;*

2). *($\alpha_{a,b}$ is a fixed-point-free cell permutation)* $\Leftrightarrow$ *($(a,\infty,b) = (0,\infty,1)$).*

3). *There exists fixed-point-free permutation v on E such that we can describe all fixed-point-free cell permutations (with the identity cell permutation $\alpha_{0,1}(t)$) by the following form:*

$$\alpha(t) = (a, t, v(a)), \qquad (v(0) = 1).$$

4). *The set M of all cell permutations of DK-ternar is sharply 2-transitive on the set E.*

Proof. 1). Let us have

$$\alpha_{a,b}(t) = \alpha_{c,d}(t) \quad \forall t \in E.$$

Then

$$a = (a,0,b) = \alpha_{a,b}(0) = \alpha_{c,d}(0) = (c,0,d) = c,$$
$$b = (a,1,b) = \alpha_{a,b}(1) = \alpha_{c,d}(1) = (c,1,d) = d,$$

i.e. $(a,b) \equiv (c,d)$. Thus if $(a,b) \neq (c,d)$, then $\alpha_{a,b} \neq \alpha_{c,d}$, e.g. all cell permutations are distinct.

2). ($\alpha_{a,b}$ is a fixed-point-free cell permutation) $\Leftrightarrow$

$$\Leftrightarrow \quad (a,t,b) \neq t = (0,t,1) \quad \forall t \in E \quad \Leftrightarrow$$
$$\Leftrightarrow \quad (a,\infty,b) = (0,\infty,1)$$

(see 1) from **Lemma 4**).

3). It is a trivial corollary of 2) and the statement 4) of **Lemma 4**.

4). Let a,b,c,d be arbitrary elements of E and $a \neq b, c \neq d$. Then we have

$$\begin{cases} \alpha_{x,y}(a) = c; \\ \alpha_{x,y}(b) = d; \end{cases} \Leftrightarrow \begin{cases} (x,a,y) = c; \\ (x,b,y) = d; \end{cases}$$

By the condition 5) of **Definition 2** there exists an unique solution (x,y) of the last system; moreover, $x \neq y$, since $c \neq d$. So the set M of all cell permutations is sharply 2-transitive on E. □

Lemma 8. *Let* $M = \left\{\alpha_{a,b}\right\}_{a,b \in E}$ *be a set of permutations on the set* E *(*E *is a finite set with distinguished elements 0 and 1), and the following conditions hold:*

1) $\alpha_{0,1} \equiv \mathrm{id}$;

2) $\alpha_{a,b}(0) = a, \qquad \alpha_{a,b}(1) = b$;

3) *Set* M *is a sharply 2-transitive set of permutations on* E.

Let us suppose by definition:

$$(x,t,x) \overset{def}{=} x,$$
$$(x,t,y) \overset{def}{=} \alpha_{x,y}(t), \quad \text{if } x \neq y.$$

Then system $< E,(x,t,y),0,1 >$ *is a DK-ternar.*

Proof is a trivial verification of the conditions 1)-5) of **Definition 2**.

Let the system $<E,(x,t,y),0,1>$ be a finite DK-ternar. Let us define on set

$E\times E-\{\Delta\}=\{<a,b>|a,b\in E, a\neq b\}$ the following binary operation:

$$<x,y>\cdot<z,u>\overset{def}{=}<(x,z,y),(x,u,y)>. \tag{18}$$

Lemma 9. *The system* $<E\times E-\{\Delta\},\cdot,<0,1>>$ *is a loop.*

Proof is given in [2].

This loop is called a *pair loop of the DK-ternar* $<E,(x,t,y),0,1>$.

Lemma 10. *Let us have a finite set* E *with distinguished elements* 0 *and* 1. *Let on the set* $E\times E-\{\Delta\}$ *a binary operation* "$\cdot$" *is defined such that system* $<E\times E-\{\Delta\},\cdot,<0,1>>$ *is a loop. Then the next conditions are equivalent:*

1) *The system* $<E\times E-\{\Delta\},\cdot,<0,1>>$ *is a pair loop of some DK-ternar;*

2) *The following quasiidentities hold on* $<E\times E-\{\Delta\},\cdot,<0,1>>$*:*

a) $(<x,y>\cdot<z,u>=<v,w>)\;\Rightarrow\;(<x,y>\cdot<u,z>=<w,v>)$;

b) $(<x,y>\cdot<z,u>=<v,w>, u\neq 0)\;\Rightarrow\;(<x,y>\cdot<0,u>=<x,w>)$;

c) $(<x,y>\cdot<z,u>=<v,w>, u\neq 1)\;\Rightarrow\;(<x,y>\cdot<1,u>=<y,w>)$;

Proof is given in [2].

3. Pair loop of DK-ternar as a loop with conditions on cosets by two subloops

Lemma 11. *Let the system* $<A,\cdot,e>$ *be a finite loop of order* $n(n-1)$. *Then the following conditions are equivalent:*

1). *The loop* $<A,\cdot,e>$ *is isomorphic to the pair loop of some finite DK-ternar.*

2). *Loop* $<A,\cdot,e>$ *satisfies the following conditions:*

a). *There exist two subloops* A_0 *and* B_1 *in the loop* A, *such that*

$$\operatorname{card} A_0 = \operatorname{card} B_1 = n-1, \qquad A_0 \cap B_1 = \{e\}.$$

b). *The loop* A *may be represented in a form of disjunctive unifications of left cosets* A_i *and* B_j *by the subloops* A_0 *and* B_1 *respectively:*

$$A = \bigcup_{i\in E} A_i = \bigcup_{j\in E} B_j,$$

where E *is an index set,* $\operatorname{card} E = n$.

c). *It is true for any* $i,j \in E$:

1) *if* $i \neq j$, *then*

$$A_i \cap B_j = \{x_{ij}\},$$

and $x_{ij} \neq x_{km}$, *when* $(i,j) \neq (k,m)$; *moreover, any* $x_0 \in A$ *may be represented in that form;*

2) *if* $i = j$, *then* $A_i \cap B_j = \varnothing$.

d). *The element* $a_0 = A_1 \cap B_0$ *satisfies the following conditions:*

1) $a_0 \in N_r(A)$, *where* $N_r(A)$ *is a right kernel of the loop* A;

2) $A_i \cdot a_0 = B_i, \qquad B_j \cdot a_0 = A_j,$

e). *It is true for any* $c_0 \in A$:

$$c_0 \cdot A_i = A_j, \qquad c_0 \cdot B_i = B_j, \qquad \forall i \in E.$$

Proof.

1) $\Rightarrow$ 2). Let loop $<A,\cdot,e>$ is isomorphic to the pair loop $<E \times E - \{\Delta\},\cdot,<0,1>>$ of some finite DK-ternar. Let us verify that the conditions 1)-5) of **Lemma** hold.

1). Let us study the following subsets of the pair loop:

$$A_0 = \{<0,x>|x \in E - \{0\}\},$$
$$B_1 = \{<x,1>|x \in E - \{1\}\}.$$

If $\text{card}E = n$, then $\text{card}A_0 = \text{card}B_1 = n-1$. Since

$$<0,x> \cdot <0,y> = <0,(0,y,x)>;$$
$$<x,1> \cdot <y,1> = <(x,y,1),1>;$$
$$<0,1> \in A_0 \cap B_1;$$

then A_0 and B_1 are subloops of the pair loop. Finally, it is evident that

$$A_0 \cap B_1 = \{<0,1>\}.$$

2). Consider the following subsets of the pair loop:

$$\begin{aligned} A_i &= \{<i,y>|y \in E - \{i\},\ i \text{ is a fixed element from } E\}; \\ B_j &= \{<x,j>|x \in E - \{j\},\ j \text{ is a fixed element from } E\}; \end{aligned} \tag{19}$$

It is evident that

$$\bigcup_{i \in E} A_i = \bigcup_{\substack{i,y \in E \\ i \neq y}} <i,y> = E \times E - \{\Delta\} \equiv A;$$

$$\bigcup_{j \in E} B_j = \bigcup_{\substack{j,x \in E \\ j \neq x}} <x,j> = E \times E - \{\Delta\} \equiv A;$$

By the help of **Lemma 10** we obtain

$$<i,y_0> \cdot <0,u> = <i,w> \quad \Rightarrow \quad <i,y_0> \cdot A_0 = A_i,$$
$$<x_0,j> \cdot <u,1> = <w,j> \quad \Rightarrow \quad <x_0,j> \cdot B_1 = B_j,$$

i.e. the sets A_i and B_j are left cosets by the subloops A_0 and B_1 respectively.

3). It is evident since

$$\{<i,j>\} = A_i \cap B_j;$$
$$<i,i> \notin E \times E - \{\Delta\}.$$

4). We have

$$A_1 \cap B_0 = \{<1,0>\}$$

and by the help of **Lemma 10** we obtain

$$(<x,y>\cdot<u,z>)\cdot<1,0>=<v,w>\cdot<1,0>=<w,v>=$$
$$=<x,y>\cdot<z,u>=<x,y>\cdot(<u,z>\cdot<1,0>),$$

i.e. $<1,0> \in N_r(A)$. We have too

$$<i,y>\cdot<1,0>=<y,i> \quad \Rightarrow \quad A_i\cdot<1,0>=B_i;$$
$$<x,j>\cdot<1,0>=<j,x> \quad \Rightarrow \quad B_j\cdot<1,0>=A_j;$$

5). Let $<a_0,b_0>$ be an arbitrary element from $E \times E - \{\Delta\}$. Then we have for any $i_0 \in E$:

$$<a_0,b_0>\cdot<i_0,y>=<(a_0,i_0,b_0),(a_0,y,b_0)>=<j_0,w>,$$

i.e.

$$<a_0,b_0>\cdot A_{i_0} = A_{j_0} \quad \text{for some} \quad j_0 \in E.$$

Analogously we obtain

$$<a_0,b_0>\cdot B_j = B_k \quad \text{for some} \quad k \in E.$$

2) $\Rightarrow$ 1).

Let the conditions 1)-5) of the present **lemma** hold for the loop $<A,\cdot,e>$. Let us define the following reflection

$$\varphi: A \to E \times E - \{\Delta\};$$
$$\varphi(A_i \cap B_j) \overset{def}{=} <i,j>.$$

The reflection φ is a bijection (see the condition 3) of **lemma**). Let us define the following operation "$\cdot$" on the set $E \times E - \{\Delta\}$:

$$<i,j>\cdot<k,m> \overset{def}{=} \varphi(x_{ij}\cdot x_{km}),$$

where $x_{uv}=A_u\cap B_v$. Operation "$\cdot$" is defined correctly, since φ is a bijection. Moreover, since

$$\varphi(x_{ij}\cdot x_{km})=<i,j>\cdot<k,m>=\varphi(x_{ij})\cdot\varphi(x_{km}),$$

then φ is an isomorphism of the loop $<A,\cdot,e>$ on some pair loop $<E\times E-\{\Delta\},\cdot,<0,1>>$ (and $\varphi(e)=\varphi(A_0\cap B_1)=<0,1>$).

Let us prove that this pair loop is a pair loop of some finite DK-ternar. It is necessary to verify that the conditions 1)-3) of **Lemma 10** hold.

a). Let us have

$$<x,y>\cdot<z,u>=<v,w>.$$

Then

$$\begin{aligned} x_{vw}&=\varphi^{-1}(<v,w>)=\varphi^{-1}(<x,y>\cdot<z,u>)=\\ &=\varphi^{-1}(<x,y>)\cdot\varphi^{-1}(<z,u>)=x_{xy}\cdot x_{zu}. \end{aligned} \tag{20}$$

By the help of the condition 4) we obtain

$$x_{vw}\cdot a_0=(A_v\cap B_w)\cdot a_0=(A_v\cdot a_0)\cap(B_w\cdot a_0)=B_v\cap A_w=x_{wv}, \tag{21}$$

and

$$(x_{xy}\cdot x_{zu})\cdot a_0=x_{xy}\cdot(x_{zu}\cdot a_0). \tag{22}$$

From (20)-(22) we obtain

$$x_{wv}=x_{vw}\cdot a_0=(x_{xy}\cdot x_{zu})\cdot a_0=x_{xy}\cdot(x_{zu}\cdot a_0)=x_{xy}\cdot x_{uz},$$

i.e.

$$<w,v>=\varphi(x_{wv})=\varphi(x_{xy}\cdot x_{uz})=<x,y>\cdot<u,z>.$$

The quasiidentity 1) from **Lemma 10** holds.

b). Let us have

$$<x,y>\cdot<z,u>=<v,w>,\quad u\neq 0.$$

Then

$$x_{vw}=x_{xy}\cdot x_{zu}.$$

By means of the condition 5) we obtain

$$\begin{aligned} A_v \cap B_w = x_{vw} = x_{xy} \cdot x_{zu} = x_{xy} \cdot (A_z \cap B_u) = \\ = (x_{xy} \cdot A_z) \cap (x_{xy} \cdot B_u) = A_m \cap B_t. \end{aligned} \tag{23}$$

By virtue of the condition 3) we obtain for any $i_0 \in E$:

$$A_{i_0} = \left\{z \in A_{i_0} \cap B_j \mid j \in E - \{i_0\}\right\} \equiv \left\{x_{i_0 j} \mid j \in E - \{i_0\}\right\}.$$

But the set A_{i_0} is a left coset by the subloop A_0 and so there exists $x_{pq} \in A$ such that

$$x_{pq} \cdot A_0 = A_{i_0}. \tag{24}$$

Since $e \in A_0$ then $x_{pq} \in A_{i_0}$; e.g. $x_{pq} \equiv x_{i_0 j_0}$ for some $j_0 \in E$. Then we obtain from (24)

$$x_{i_0 j_0} \cdot A_0 = A_{i_0},$$

and since i_0 was an arbitrary element from E, then

$$x_{xy} \cdot A_0 = A_x \tag{25}$$

for any $x \in E$. From (23) and (25) it follows that

$$x_{xy} \cdot x_{0u} = x_{xy} \cdot (A_0 \cap B_u) = (x_{xy} \cdot A_0) \cap (x_{xy} \cdot B_u) = A_x \cap B_t = x_{xt}. \tag{26}$$

By the help of the conditions 2) and 3) of this **lemma** and the identities (23)-(26) we obtain

$$A_v = A_m, \qquad B_w = B_t,$$

i.e. $v = m, w = t$. In accord with (15)

$$x_{xy} \cdot x_{0u} = x_{xw},$$

i.e.

$$< x, y > \cdot < 0, u > = \varphi(x_{xy}) \cdot \varphi(x_{0u}) = \varphi(x_{xy} \cdot x_{0u}) = \varphi(x_{xw}) = < x, w >.$$

The quasiidentity 2) of **Lemma 10** holds.

c). Proof of quasiidentity 3) of **Lemma 10** is analogously to that of b). □

§4. Sharply 2-transitive sets of permutations degree n and loop transversals in S_n to $St_{a,b}(S_n)$

Let us return to the set of cell permutations of some finite DK-ternar. The following statement is true.

Lemma 12. *Let* E *be a finite set and* $|E|=n$. *The following conditions are equivalent:*

1). *A set* T *is a loop transversal in* S_n *to* $St_{a,b}(S_n)$, *where* $a,b \in E$ *are arbitrary fixed distinct elements;*

2). *A set* T *is a sharply 2-transitive set of permutations on* E;

3) *A set* T *is a sharply 2-transitive permutation loop on* E;

The permutation loop is defined in [6].

Proof is given in [7]. □

§5. Loop of points of a projective plane

In this paragraph it will be proved the definition of such binary operation on the set of points of a projective plane, which is identical to the operation of pair loop of DK-ternar corresponding to that plane. This operation will be a loop (see §2) and since the loop of points mentioned above will be called a *loop of points of a projective plane.*

Let us have a projective plane π and a DK-ternar corresponding to it (see §1). Let us demonstrate the method of a purely geometrical construction (with the help of an incidence relation only) of the point (v,w) by the points (x,y) and (z,u) (where $x \neq y, z \neq u$), where $<v,w> = <x,y> \cdot <z,u>$ in the pair loop of the DK-ternar mentioned above. The sequence of the construction will be described step by step below.

1). $X = (0), \quad O = (0,0),$
$Y = (1), \quad I = (1,1)$

are four points in a common position on the plane π.

2). $(1) \cup (1,1) = [1,1]; \quad (0,0) \cup (1,1) = [0];$
$(0) \cup (0,0) = [0,0]; \quad (0) \cup (1) = [\infty].$

3). $[0,0] \cap [1,1] = (0,1).$

4). $(0) \cup (z,u) = [0,z]; \quad (1) \cup (z,u) = [1,u].$

5). $[0,z] \cap [0] = (z,z); \quad [1,u] \cap [0] = (u,u).$

6). $(0,1) \cup (u,u) = [u,u]; \quad (0,1) \cup (z,z) = [z,z].$

7). $[u,u] \cap [\infty] = (u); \quad [z,z] \cap [\infty] = (z).$

8). $(x,y) \cup (u) = [u,(x,u,y)] \equiv [u,w];$
$(x,y) \cup (z) = [z,(x,z,y)] \equiv [z,v].$

9). $[u,w] \cap [0] = (w,w); \quad [z,v] \cap [0] = (v,v).$

10). $(0) \cup (v,v) = [0,v]; \quad (1) \cup (w,w) = [1,w].$

11). $[0,v]\cap[1,w]=(v,w)$.

The point (v,w) is constructed.

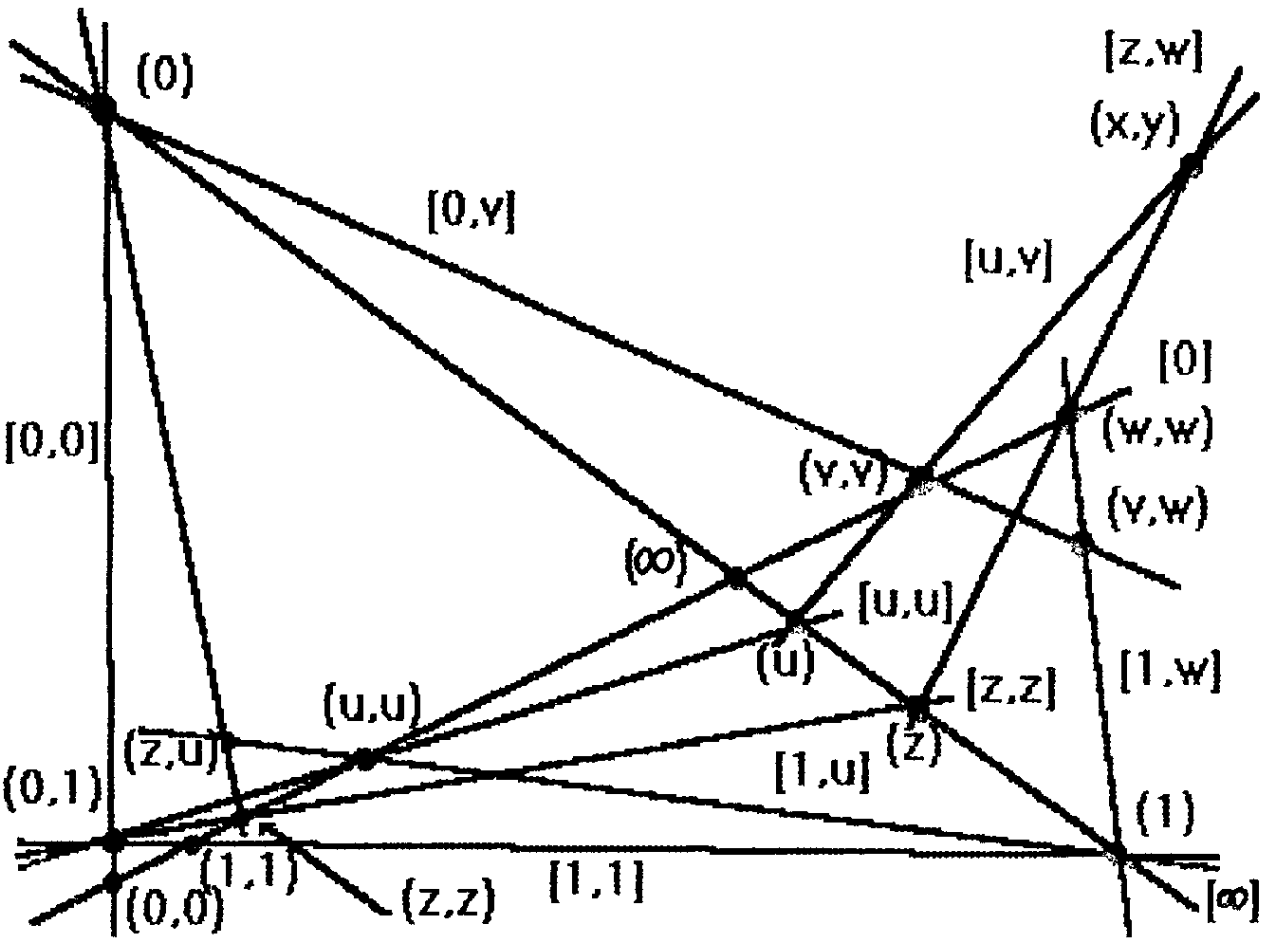


Fig. 1.

It is easy to see that we used the incidence relation only in the construction described above. Then this construction is independent from a coordinatization on the plane π and could be done without some coordinates on π.

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Isomorphisms of quasigroups isotopic to groups

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Abstract

In this note quasigroups isotopic to groups are considered. The necessary and sufficient conditions for two quasigroups isotopic to the same group to be isomorphic are found. The form of isomorphism of two quasigroups isotopic to the same group and the form of automorphism of a group isotope are given. For a T-quasigroup with an idempotent the group of automorphisms is described.

1. Introduction

We consider quasigroup operations defined on the same set Q. It will be convenient to recall some of the terminology of quasigroup theory.

The *quasigroup* $(Q,\cdot)$ is a groupoid $(Q,\cdot)$ with the unique division. For each $a \in Q$ we have two transformations of the underlying set Q. They are *called the left and the right translation by* a and they are defined by

$$L_a x = a \cdot x \quad \text{and} \quad R_a x = x \cdot a$$

for every $x \in Q$.

Since $(Q,\cdot)$ is a quasigroup, both these transformations are permutations and hence they belong to the permutations group $S(Q)$ of Q.

The *left (right) loop* is a quasigroup $(Q,\cdot)$ with the left (right) unite e (f) such that $e\cdot x=x$ $(x\cdot f=x)$ for every $x\in Q$.

The element 1 of a quasigroup $(Q,\cdot)$ is said to be *a unit* of $(Q,\cdot)$ if for every x of Q

$$1\cdot x=x\cdot 1=x.$$

A *loop* is a quasigroup with the unit.

By $Aut(Q,\cdot)$ we denote the group of all automorphisms of $(Q,\cdot)$.

For each $a\in Q$ put

$$S_a(Q)=\{\alpha\in S(Q)\,|\,\alpha a=a\}.$$

Let "$\circ$" and "$*$" be two operations defined on Q. The operation "$\circ$" is said to be isotopic to "$*$", if there exist three permutations $\alpha,\beta,\gamma\in S(Q)$ such that

$$x*y=\gamma^{-1}(\alpha x\circ\beta y) \tag{1}$$

for all $x,y\in Q$.

We also say that $(Q,*)$ and $(Q,\circ)$ are *isotopic*, or that $(Q,*)$ *is an isotope* of $(Q,\circ)$ of the form $x*y=\gamma^{-1}(\alpha x\circ\beta y)$. Shortly we write this as

$$(Q,*):\quad x*y=\gamma^{-1}(\alpha x\circ\beta y),\quad \alpha,\beta,\gamma\in S(Q),\ x,y\in Q.$$

The triple (α,β,γ) of permutations such that the relations (1) hold is called the *isotopy* of $(Q,\circ)$.

If in (1) γ is the identical permutation ε, then $(Q,*)$ is said to be the *principal isotope* of $(Q,\circ)$.

If in (1) $\alpha=\beta=\gamma$, then

$$x*y=\gamma^{-1}(\gamma x\circ\gamma y), \tag{2}$$

which means that γ is an automorphism between $(Q,*)$ and $(Q,\circ)$.

The equality (1) is equivalent to the following equality

$$x * y = \gamma^{-1}(\alpha\gamma^{-1}\gamma x \circ \beta\gamma^{-1}\gamma y), \tag{3}$$

whence we have proved the following

Theorem 1 (**[1] Theorem 1.2**). *An isotope*

$$(Q,*): \quad x * y = \gamma^{-1}(\alpha x \circ \beta y), \quad \alpha,\beta,\gamma \in S(Q), \; x,y \in Q$$

is isomorphic to the principal isotope

$$(Q,\otimes): \quad x \otimes y = \alpha\gamma^{-1}x \circ \beta\gamma^{-1}y, \quad \alpha,\beta,\gamma \in S(Q), \; x,y \in Q,$$

and γ *is the isomorphism between them.*

Theorem 2 (**[1] Theorem 1.3**). *For arbitrary fixed elements* $a,b \in Q$ *the isotope*

$$(Q,*): \; x * y = R_a^{-1}x \cdot L_b^{-1}y, \quad x,y \in Q,$$

is a loop with unit $e = b \cdot a$, *where* R_a, L_b *are translations of a quasigroup* $(Q,\cdot)$ *by* a *and* b *respectively.*

Theorem 3 (**[1] Theorem 1.4**). *If a loop* $(Q,\circ)$ *is isotopic to a group* $(Q,\cdot)$, *then it is a group isomorphic to* $(Q,\cdot)$.

A permutation $\gamma \in S(Q)$ is called *a quasiautomorphism* of a group $(Q,\cdot)$ if there exist two permutations $\alpha,\beta \in S(Q)$ such that

$$x \cdot y = \gamma^{-1}(\alpha x \cdot \beta y)$$

holds for all $x,y \in Q$.

Quasiautomorphisms of a group $(Q,\cdot)$ are described by the next

Lemma 1 (**[1] Lemma 2.5**). *A permutation* $\gamma \in S(Q)$ *is a quasiautomorphism of a group* $(Q,\cdot)$ *if and only if there exist* *element* $s \in Q$, *automorphisms* $\varphi_0, \varphi_0^{'} \in Aut(Q,\cdot)$ *such that* $\gamma = R_s\varphi_0$ *or* $\gamma = L_s\varphi_0^{'}$.

The following equalities hold in a group $(Q,\cdot)$:

$$R_a^{-1} = R_{a^{-1}};\quad L_a^{-1} = L_{a^{-1}};\quad R_aR_b = R_{ba};\quad L_aL_b = L_{ab};$$

$$\varphi R_a = R_{\varphi a}\varphi;\quad \varphi L_a = L_{\varphi a}\varphi;\quad \varphi(a^{-1}) = (\varphi a)^{-1},$$

where $\varphi \in Aut(Q,\cdot)$, a^{-1} is the inverse of a in $(Q,\cdot)$.

In abelian group $(Q,\cdot)$ we have $R_a = L_a$ for every $a \in Q$.

2. Isomorphism of group isotopes

An isotope of a group is called *a group isotope*.

Let

$$\begin{array}{l} (Q,*):\ x*y = \alpha x\cdot\beta y,\ \ \alpha,\beta \in S(Q),\ x,y \in Q, \\ (Q,\circ):\ x\circ y = \alpha_1 x\cdot\beta_1 y,\ \ \alpha_1,\beta_1 \in S(Q),\ x,y \in Q, \end{array} \tag{4}$$

be two isotopes of a group $(Q,\cdot)$.

Suppose that $(Q,*)$ and $(Q,\circ)$ are isomorphic. Then there exists a permutation $\theta \in S(Q)$ such that

$$\theta(x*y) = \theta x \circ \theta y$$

or

$$\theta(\alpha x\cdot\beta y) = \alpha_1\theta x\cdot\beta_1\theta y \tag{5}$$

for all $x,y \in Q$.

Let $1 \in Q$ be the unit of the group $(Q,\cdot)$ and $e \in Q$ be such that $\beta e = 1$. Putting in (5) $y = e$ and $\theta^{-1}x$ instead of x in (5) we obtain $\theta\alpha\theta^{-1}x = \alpha_1 x\cdot\beta_1\theta e$. Therefore

$$\alpha_1 = R_a^{-1}\theta\alpha\theta^{-1}, \tag{6}$$

where $a = \beta_1\theta e$.

From (5) using (6) we find

$$\theta(\alpha x \cdot \beta y) = R_a^{-1}\theta\alpha\theta^{-1}\theta x \cdot \beta_1\theta y = R_a^{-1}\theta\alpha x \cdot \beta_1\theta y.$$

Replacing here x for $\alpha^{-1}x$ we get

$$\theta(x \cdot \beta y) = R_a^{-1}\theta x \cdot \beta_1\theta y.$$

Putting here $x = 1$ we obtain

$$\theta\beta y = R_a^{-1}\theta 1 \cdot \beta_1\theta y.$$

Hence

$$\beta = \theta^{-1}L_b\beta_1\theta, \tag{7}$$

with $b = R_a^{-1}\theta 1$.

Analogously we will obtain a relation for α, α_1 and θ.

Let $f \in Q$ be such that $\alpha f = 1$. Putting in (5) $x = f$ and $\theta^{-1}y$ instead of y we obtain

$$\theta\beta\theta^{-1}y = \alpha_1\theta f \cdot \beta_1 y.$$

Therefore

$$\beta_1 = L_c^{-1}\theta\beta\theta^{-1} \tag{8}$$

where $c = \alpha_1\theta f$.

From (5) and (8) we find

$$\theta(\alpha x \cdot y) = \alpha_1\theta x \cdot L_c^{-1}\theta y$$

and if $y = 1$ we have

$$\theta\alpha x = \alpha_1\theta x \cdot L_c^{-1}\theta 1.$$

Hence

$$\alpha = \theta^{-1}R_d\alpha_1\theta, \tag{9}$$

where $d = L_c^{-1}\theta 1$.

Now the equality (5) can be rewritten in the following way

$$\theta(\theta^{-1}R_d\alpha_1\theta x\cdot\theta^{-1}L_b\beta_1\theta y)=\alpha_1\theta x\cdot\beta_1\theta y.$$

From this equality, replacing x for $\theta^{-1}\alpha_1^{-1}x$ and y for $\theta^{-1}\beta_1^{-1}y$ we get

$$x\cdot y=\theta(\theta^{-1}R_dx\cdot\theta^{-1}L_by). \tag{10}$$

Therefore θ is a quasiautomorphism of the group $(Q,\cdot)$.

By **Lemma 1** there exist $s\in Q$ and $\theta_0\in Aut(Q,\cdot)$ such that $\theta=R_s\theta_0$. Then (9) and (7) can be rewritten in the following way respectively:

$$\alpha=\theta_0^{-1}R_s^{-1}R_d\alpha_1R_s\theta_0, \tag{11}$$

where $d=L_c^{-1}R_s\theta_0 1=L_c^{-1}s$, $c=\alpha_1R_s\theta_0 f=\alpha_1(\theta_0 f\cdot s)$, and

$$\beta=\theta_0^{-1}R_s^{-1}L_b\beta_1R_s\theta_0, \tag{12}$$

where $b=R_a^{-1}R_s\theta_0 1=R_a^{-1}s$, $a=\beta_1R_s\theta_0 e=\beta_1(\theta_0 e\cdot s)$.

Using (11),(12) and the equality $\theta=R_s\theta_0$ we find from (10):

$$\begin{aligned}x\cdot y&=R_s\theta_0(\theta_0^{-1}R_s^{-1}R_dx\cdot\theta_0^{-1}R_s^{-1}L_by)=R_s(R_s^{-1}R_dx\cdot R_s^{-1}L_by)=\\&=(xd\cdot s^{-1})(by\cdot s^{-1})s=((x\cdot L_c^{-1}s)s^{-1})(R_a^{-1}s\cdot y)=((xc^{-1}s)s^{-1})(sa^{-1}\cdot y)=\\&=(xc^{-1})(sa^{-1}\cdot y)=(x(\alpha_1R_s\theta_0 f)^{-1}\cdot((s\cdot\beta_1R_s\theta_0 e)^{-1}))y=\\&=x(\alpha_1R_s\theta_0 f)^{-1}\cdot s\cdot(\beta_1R_s\theta_0 e)^{-1}\cdot y.\end{aligned}$$

Therefore

$$(\alpha_1R_s\theta_0 f)^{-1}\cdot s\cdot(\beta_1R_s\theta_0 e)^{-1}=1. \tag{13}$$

So, if θ is an isomorphism of quasigroups $(Q,*)$ and $(Q,\circ)$, then there exist $s\in Q$ and $\theta_0\in Aut(Q,\cdot)$ such that equalities (11)-(13) hold and $\theta=R_s\theta_0$.

Conversely, let $s\in Q$ and $\theta_0\in Aut(Q,\cdot)$ satisfy equalities (11)-(13) and $\theta=R_s\theta_0$. Then

$$R_s\theta_0(x*y) = R_s\theta_0(\theta_0^{-1}R_s^{-1}R_d\alpha_1R_s\theta_0x\cdot\theta_0^{-1}R_s^{-1}L_b\beta_1R_s\theta_0y) =$$
$$= R_s(R_s^{-1}R_d\alpha_1R_s\theta_0x\cdot R_s^{-1}L_b\beta_1R_s\theta_0y) = R_s^{-1}R_d\alpha_1R_s\theta_0x\cdot L_b\beta_1R_s\theta_0y =$$
$$=(\alpha_1(\theta_0x\cdot s)ds^{-1})(b\cdot\beta_1(\theta_0y\cdot s)) = (\alpha_1(\theta_0x\cdot s)(L_c^{-1}s\cdot s^{-1})(R_a^{-1}s\cdot\beta_1(\theta_0y\cdot s)) =$$
$$=(\alpha_1(\theta_0x\cdot s)\cdot\alpha_1(\theta_0f\cdot s)^{-1}s\cdot s^{-1})(s\cdot\beta_1(\theta_0e\cdot s)^{-1}\beta_1(\theta_0y\cdot s) =$$
$$=\alpha_1(\theta_0x\cdot s)\cdot\beta_1(\theta_0y\cdot s) = R_s\theta_0x\circ R_s\theta_0y,$$

i.e. $R_s\theta_0$ is an isomorphism of the quasigroups $(Q,*)$ and $(Q,\cdot)$.

Thus, we have proved

Theorem 4. *A permutation $\theta\in S(Q)$ is an isomorphism of isotopes $(Q,*)$ and $(Q,\circ)$ defined by (4) if and only if there exist $s\in Q$ and $\theta_0\in Aut(Q,\cdot)$ such that the relations (11)-(13) hold ,where 1 is the unit of the group $(Q,\cdot)$.*

Let

$$\begin{aligned}(Q,\underset{1}{\circ})&:\ x\underset{1}{\circ}y=\gamma^{-1}(\alpha x\cdot\beta y),\ \ \alpha,\beta,\gamma\in S(Q),\ x,y\in Q,\\ (Q,\underset{2}{\circ})&:\ x\underset{2}{\circ}y=\gamma_1^{-1}(\alpha_1x\cdot\beta_1y),\ \ \alpha_1,\beta_1,\gamma_1\in S(Q),\ x,y\in Q,\end{aligned}\tag{14}$$

be two isotopes of a group $(Q,\cdot)$ and let

$$\begin{aligned}(Q,\underset{1}{*})&:\ x\underset{1}{*}y=\alpha\gamma^{-1}x\cdot\beta\gamma^{-1}y,\ \ \alpha,\beta,\gamma\in S(Q),\ x,y\in Q,\\ (Q,\underset{2}{*})&:\ x\underset{2}{\circ}y=\alpha_1\gamma_1^{-1}x\cdot\beta_1\gamma_1^{-1}y,\ \ \alpha_1,\beta_1,\gamma_1\in S(Q),\ x,y\in Q,\end{aligned}\tag{15}$$

be isotopes of $(Q,\underset{1}{\circ})$ and $(Q,\underset{2}{\circ})$ respectively. By **Theorem 1** it follows that $\gamma(x\underset{1}{\circ}y)=\gamma x\underset{1}{*}\gamma y$ and $\gamma_1(x\underset{2}{\circ}y)=\gamma_1x\underset{2}{*}\gamma_1y$ hold for all $x,y\in Q$, e.g. γ is an isomorphism between $(Q,\underset{1}{\circ})$ and $(Q,\underset{1}{*})$, and γ_1 is an isomorphism between $(Q,\underset{2}{\circ})$ and $(Q,\underset{2}{*})$. If λ is an isomorphism between $(Q,\underset{1}{*})$ and $(Q,\underset{2}{*})$, i.e. $\lambda(x\underset{1}{*}y)=\lambda x\underset{2}{*}\lambda y$ holds for all $x,y\in Q$, then

$$\gamma_1^{-1}\lambda\gamma(x \underset{1}{\circ} y) = \gamma_1^{-1}\lambda(\gamma x \underset{1}{*} \gamma y) = \gamma_1^{-1}(\lambda\gamma x \underset{2}{*} \lambda\gamma y) = \gamma_1^{-1}\lambda\gamma x \underset{2}{\circ} \gamma_1^{-1}\lambda\gamma y,$$

whence $\gamma_1^{-1}\lambda\gamma$ is an isomorphism between $(Q,\underset{1}{\circ})$ and $(Q,\underset{2}{\circ})$.

Conversely, let θ be an isomorphism between $(Q,\underset{1}{\circ})$ and $(Q,\underset{2}{\circ})$, e.g. $\theta(x \underset{1}{\circ} y) = \theta x \underset{2}{\circ} \theta y$ holds for all $x, y \in Q$. Then for $\mu = \gamma_1\theta\gamma^{-1}$ we obtain

$$\mu(x \underset{1}{*} y) = \gamma_1\theta(\gamma^{-1}x \underset{1}{\circ} \gamma^{-1}y) = \gamma_1(\theta\gamma^{-1}x \underset{2}{\circ} \theta\gamma^{-1}y) = \gamma_1\theta\gamma^{-1}x \underset{2}{*} \gamma_1\theta\gamma^{-1}y.$$

Since x and y are arbitrary, from the last equality we conclude that μ is an isomorphism between $(Q,\underset{1}{*})$ and $(Q,\underset{2}{*})$.

So, we have proved the next

Lemma 2. *A permutation $\theta \in S(Q)$ is an isomorphism between isotopes $(Q,\underset{1}{\circ})$ and $(Q,\underset{2}{\circ})$ defined by (14), if and only if there exists an isomorphism μ between quasigroups $(Q,\underset{1}{*})$ and $(Q,\underset{2}{*})$, defined by (15), such that $\theta = \gamma_1^{-1}\mu\gamma$.*

From **Lemma 2** and **Theorem 4** we obtain

Theorem 5. *A permutation $\theta \in S(Q)$ is an isomorphism between isotopes $(Q,*)$ and $(Q,\circ)$ defined by (14), if and only if there exist $s \in Q$ and $\theta_0 \in Aut(Q,\cdot)$ such that $\theta = \gamma_1^{-1}R_s\theta_0\gamma$ and relations*

$$\begin{aligned} \alpha\gamma^{-1} &= \theta_0^{-1}R_s^{-1}R_d\alpha_1\gamma_1^{-1}R_s\theta_0, \\ \beta\gamma^{-1} &= \theta_0^{-1}R_s^{-1}L_b\beta_1\gamma_1^{-1}R_s\theta_0, \\ (\alpha_1\gamma_1^{-1}R_s\theta_0 f)^{-1} &\cdot s \cdot (\beta_1\gamma_1^{-1}R_s\theta_0 e)^{-1} = 1, \end{aligned} \tag{16}$$

hold, with $d = (\alpha_1\gamma_1^{-1}R_s\theta_0 f)^{-1} \cdot s$, $b = s \cdot (\beta_1\gamma_1^{-1}R_s\theta_0 e)^{-1}$, $f = \gamma\alpha^{-1}1$, $e = \gamma\beta^{-1}1$, where 1 is the unit of the group $(Q,\cdot)$.

If in **Theorem 5** the quasigroup $(Q,*)$ is replaced by $(Q,\circ)$, then we obtain the following

Corollary 1. *A permutation* $\theta \in S(Q)$ *is an automorphism of the isotope* $(Q,\circ): x \circ y = \gamma(\alpha x \cdot \beta y)$, $\alpha,\beta,\gamma \in S(Q)$, $x,y \in Q$ *of a group* $(Q,\cdot)$ *if and only if there exist* $s \in Q$ *and* $\theta_0 \in Aut(Q,\cdot)$ *such that* $\theta = \gamma^{-1} R_s \theta_0 \gamma$ *and relations (16) hold for* $\alpha_1 = \alpha$, $\beta_1 = \beta$, $\gamma_1 = \gamma$.

We will adapt some of the above results for right loops principal isotopic to the same group.

Let $(Q,\cdot)$ be a group with the unit 1 and a principal isotope

$$(Q,*): x*y = \alpha x \cdot \beta y, \quad \alpha,\beta, \in S(Q), \quad x,y \in Q$$

be a right loop with the unit e. Then we have

$$x = x*e = \alpha x \cdot \beta e$$

for all $x \in Q$, from which it follows that

$$\alpha = R_{(\beta e)^{-1}}.$$

Therefore

$$x*y = R_{(\beta e)^{-1}} x \cdot \beta y = x(\beta e)^{-1} \beta y = x \cdot L_{(\beta e)^{-1}} \beta y.$$

Let us consider the isotope

$$(Q,\circ): x \circ y = x \cdot L_{(\beta e)^{-1}} \beta L_e y, \quad x,y \in Q.$$

We note that

$$L_{(\beta e)^{-1}} \beta L_e 1 = 1,$$

i.e. $(Q,\circ)$ is a right loop with the unit 1. The right loop $(Q,\circ)$ is isomorphic to the right loop $(Q,*)$, since

$$L_e(x \circ y) = L_e(x \cdot L_{(\beta e)^{-1}} \beta L_e y) = L_e x \cdot L_{(\beta e)^{-1}} \beta L_e y = L_e x * L_e y.$$

From this equality and **Theorem 1** we have

Proposition 1. *Every right loop which is isotopic to a group $(Q,\cdot)$ with the unit 1 is isomorphic to a right loop*

$$(Q,\circ): x \circ y = x \cdot \alpha y, \quad \alpha, \in S_1(Q), \quad x, y \in Q$$

with the same unit 1.

The following statement is a direct corollary from **Theorem 5.**

Proposition 2. *A permutation $\theta \in S(Q)$ is an isomorphism between right loops*

$$(Q,*): \quad x * y = x \cdot \alpha y, \ \alpha \in S_1(Q), \ x, y \in Q$$

and

$$(Q,\circ): \quad x \circ y = x \cdot \beta y, \ \beta \in S_1(Q), \ x, y \in Q$$

if and only if $\theta \in Aut(Q,\cdot)$ *and* $\alpha = \theta^{-1}\beta\theta$.

Corollary 2. *For a right loop*

$$(Q,*): \quad x * y = x \cdot \alpha y, \ \alpha \in S_1(Q), \ x, y \in Q,$$

which is isotopic to a group $(Q,\cdot)$ with the unit 1 the equalities

$$Aut(Q,*) = Z_{Aut(Q,\cdot)}\{\alpha\} = \{\varphi \in Aut(Q,\cdot) \mid \varphi\alpha = \alpha\varphi\}$$

hold.

Proof. The statement follows from **Proposition 2** replacing $(Q,\circ)$ by $(Q,*)$. □

Corollary 3. *For a right loop*

$$(Q,*): \quad x * y = x \cdot \alpha y, \ \alpha \in S_1(Q), \ x, y \in Q$$

which is isotopic to a group $(Q,\cdot)$ with the unit 1 we have

$$Aut(Q,*) = \{\varepsilon\},$$

provided that

$$\alpha \notin Z_{S_1(Q)}\{Aut(Q,\cdot)\} = \{\psi \in S_1(Q) \mid \psi\varphi = \varphi\psi,\ \varphi \in Aut(Q,\cdot)\}.$$

Proof. It follows directly from **Corollary 2.** □

Corollary 4. *For a right loop*

$$(Q,*):\quad x*y = x\cdot\alpha y,\ \alpha \in S_1(Q),\ x,y \in Q$$

which is isotopic to a group $(Q,\cdot)$ *with the unit* 1 *we have*

$$Aut(Q,*) = Aut(Q,\cdot),$$

provided that

$$\alpha \in Z_{S_1(Q)}\{Aut(Q,\cdot)\} = \{\psi \in S_1(Q) \mid \psi\varphi = \varphi\psi,\ \varphi \in Aut(Q,\cdot)\}.$$

Proof. It follows from **Corollary 2.** □

3. Automorphisms of a T-quasigroup

We will use **Corollary 1** to describe automorphisms of a T-quasigroup.

We recall that a T-quasigroup $(Q,*)$ is an isotope

$$x*y = \varphi x + \psi y + g = \varphi x + R_g\psi y \tag{17}$$

of an abelian group $(Q,+)$, where $\varphi,\psi \in Aut(Q,+)$, $g \in Q$, $R_g x = x + g$ [2].

Let 0 be the zero in $(Q,+)$. Then, by **Corollary 1** a permutation $\theta \in S(Q)$ is an automorphism of the quasigroup (17) if and only if there exist $s \in Q$ and $\theta_0 \in Aut(Q,+)$ such that $\theta = R_s\theta_0$ and relations

$$\begin{aligned}
\varphi &= \theta_0^{-1} R_s^{-1} R_d \varphi R_s \theta_0, \\
R_g \psi &= \theta_0^{-1} R_s^{-1} L_b R_g \psi R_s \theta_0, \\
-\varphi s + s - R_g \psi R_s \theta_0 \psi^{-1}(-g) &= 0, \\
b &= s - R_g \psi R_s \theta_0 \psi^{-1}(-g), \\
d &= -\varphi s + s
\end{aligned} \tag{18}$$

hold.

Here $\psi^{-1}(-g)$ is a solution of the equation $R_g \psi x = 0$. Relations (18) can be simplified.

The first equality of (18) can be written as follows:

$$\varphi = \theta^{-1} R_s^{-1} R_d \varphi R_s \theta_0 = \theta^{-1} R_{-s} R_d R_{\varphi s} \varphi \theta_0 =$$
$$= \theta^{-1} R_{-s-\varphi s+s+\varphi s} \varphi \theta_0 = \theta^{-1} \varphi \theta_0.$$

Thus $\varphi\theta_0 = \theta_0\varphi$.

Further we have

$$b = s - R_g \psi R_s \theta_0 \psi^{-1}(-g) = s - R_g(\psi\theta_0\psi^{-1}(-g) + \psi s) =$$
$$= s + \psi\theta_0\psi^{-1}(-g) = \psi s - g,$$

$$R_g \psi = \theta_0^{-1} R_s^{-1} L_b R_g \psi R_s \theta_0 = \theta_0^{-1} R_{-s} R_b R_g R_{\psi s} \psi \theta_0 =$$
$$= \theta_0^{-1} R_{-s+b+g+\psi s} \psi \theta_0 = \theta_0^{-1} R_{\psi\theta_0\psi^{-1}(g)} \psi \theta_0 = \theta_0^{-1} \psi \theta_0 R_{\psi^{-1}(g)}.$$

Therefore

$$\theta_0^{-1} \psi \theta_0 = R_g \psi R^{-1}_{\psi^{-1}(g)} = R_g \psi R_{\psi^{-1}(-g)} = R_g R_{-g} \psi = \psi,$$

i.e. $\psi\theta_0 = \theta_0\psi$. Consequently θ_0 is an element of the centralizer

$$C = Z_{Aut(Q,+)}\{\varphi, \psi\}$$

of automorphisms φ and ψ in the group $Aut(Q,+)$. Granting this we obtain

$$0 = -\varphi s + s - R_g \psi R_s \theta_0 \psi^{-1}(-g) = -\varphi s + s - R_g R_{\psi s} \psi \theta_0 \psi^{-1}(-g) =$$
$$= -\varphi s + s + \theta_0 g - \psi s - g,$$

from which the equality

$$\varphi s+\psi s-s=\theta_0 g-g$$

holds.

Conversely, for $\theta_0 \in C$ and $s \in Q$ such that

$$\varphi s+\psi s-s=\theta_0 g-g$$

we find

$$\begin{aligned} R_s\theta_0(x*y) &= R_s\theta_0(\varphi x+\psi y+g)=R_s(\theta_0\varphi x+\theta_0\psi y+\theta_0 g)= \\ &=\varphi\theta_0 x+\psi\theta_0 y+\theta_0 g+s=\varphi\theta_0 x+\psi\theta_0 y+\varphi s+\psi s+g= \\ &=\varphi(\theta_0 x+s)+\psi(\theta_0 y++s)+g=\varphi R_s\theta_0 x+\psi R_s\theta_0 y+g= \\ &=R_s\theta_0 x* R_s\theta_0 y. \end{aligned}$$

Thus we have proved the following

Proposition 3. *A permutation $\theta \in S(Q)$ is an automorphism of a T-quasigroup (17) if and only if there exist $s \in Q$ and*

$$\theta_0 \in C=Z_{Aut(Q,+)}\{\varphi,\psi\}=\{\alpha \in Aut(Q,+) \mid \alpha\varphi=\varphi\alpha;\ \alpha\psi=\psi\alpha\},$$

such that $\theta=R_s\theta_0$ and $\varphi s+\psi s-s=\theta_0 g-g$.

In some cases the automorphism group of the T-quasigroup (17) can be described.

Let us denote

$$N=\{s \in Q \mid \varphi s+\psi s-s=0\};$$

$$R_N=\{R_n \mid n \in N\};$$

$$A_0=\{\theta_0 \in C=Z_{Aut(Q,+)}\{\varphi,\psi\} \mid \theta_0 g=g\}.$$

It is easy to see that $(N,+)$ is a subgroup of $(Q,+)$. From **Proposition 3** it follows that A_0 and R_N are subgroups of $Aut(Q,*)$ and A_0 is a subgroup of $Aut(Q,+)$. Since

$$0=\theta_0 0=\theta_0(\varphi s+\psi s-s)=\varphi\theta_0 s+\psi\theta_0 s-\theta_0 s$$

and

$$0=\theta_0^{-1}0=\theta_0^{-1}(\varphi s+\psi s-s)=\varphi\theta_0^{-1}s+\psi\theta_0^{-1}s-\theta_0^{-1}s$$

for $\theta_0 \in C$ and $s \in N$, then we get $\theta_0 N = N$, where

$$\theta_0 N = \{\theta_0 n \mid n \in N\}.$$

Let

$$R_s\theta_0 \in Aut(Q,*), \quad s \in Q, \quad \theta_0 \in Aut(Q,+), \quad n \in N.$$

Then

$$R_s\theta_0 R_n = R_s R_{\theta_0 n}\theta_0 = R_{\theta_0 n} R_s \theta_0.$$

So, $\alpha R_N \alpha^{-1} \subseteq R_N$ for every $\alpha \in Aut(Q,*)$, hence R_N is a normal subgroup of $Aut(Q,*)$.

It is evident that a permutation R_c is an automorphism of the group $(Q,+)$ if and only if $c = 0$, e.g. R_c is the identical permutation ε. Therefore $R_N \cap A_0 = \{\varepsilon\}$ and a semidirect product $R_N \times A_0$ is a subgroup of $Aut(Q,*)$. Also we have $Aut(Q,*) = R_N \times C$ if $g = 0$.

So we have proved the following

Proposition 4. *Let* $(Q,+)$ *be an abelian group with zero* 0, *and*

$$(Q,\otimes): \quad x \otimes y = \varphi x + \psi y, \quad \varphi, \psi \in Aut(Q,+), \quad x, y \in Q$$

be a T*-quasigroup. Then*

$$Aut(Q,\otimes) = R_N \times C,$$

where

$$C = Z_{Aut(Q,+)}\{\varphi, \psi\},$$
$$R_N = \{R_s \mid s \in Q, \ \varphi s + \psi s = 0\}.$$

We note that medial quasigroups that contain at least one idempotent [3] and transitive distributive quasigroups [4] satisfy **Proposition 4.**

4. Examples

In the following example the efficiency of **Theorem 5** and **Corollary 1** is visually demonstrated.

Let $(Q,\cdot)=\langle h\rangle$ be the infinite cyclic group which is generated by an element g, $\alpha=(h^{-1}h^{-2})$ be the transposition of elements h^{-1} and h^{-2} (i.e., $\alpha(h^{-1})=h^{-2}$, $\alpha(h^{-2})=h^{-1}$, $\alpha(x)=x$ forever $x\in Q$, $h^{-1}\neq x\neq h^{-2}$, $\alpha_1=(hh^2)$ be the transposition of elements h and h^2, I be the permutation of $(Q,\cdot)$ defined by $Ix=x^{-1}$ Consider isotopes

$$(Q,\circ):\quad x\circ y=\alpha_1 x\cdot Iy,\quad x,y\in Q,$$
$$(Q,*):\quad x*y=\alpha x\cdot Iy,\quad x,y\in Q.$$

We will prove that $(Q,*)$ and $(Q,\circ)$ are isomorphic.

Remark that in this case elements f and e from the **Theorem 5** are equal to the unit 1 of the group $(Q,\cdot)$, whence the third relation of (16) is equivalent to the equality

$$(\alpha_1 s)^{-1}s(Is)^{-1}=1.$$

Element 1 and h and only they satisfy the above equality. It is also known that $Aut(Q,\cdot)=\{\varepsilon,I\}$. By **Theorem 5** permutations $R_1\varepsilon=\varepsilon$, $R_1 I=I$, $R_h\varepsilon=R_h$, $R_h I$ and only they can be isomorphisms between $(Q,*)$ and $(Q,\circ)$. Let us verify the first two conditions of (16) for $s=1$ and $\theta_0=I$. We have $d=1$, $b=1$ and $I\alpha_1 I=\alpha$, $I=III$. The validity of the equality $I\alpha_1 I=\alpha$ is verified directly and the equality $I=III$ is trivial. Hence I is an isomorphism between $(Q,*)$ and $(Q,\circ)$.

Now we will find $Aut(Q,\circ)$. As above we can show that permutations $R_1\varepsilon=\varepsilon$, $R_1 I = I$, $R_h\varepsilon = R_h$, $R_h I$ and only they can be automorphisms of $(Q,\circ)$. The identical permutation is automorphism in every algebraic system. It is easy to see nevertheless that $R_1\varepsilon$ satisfies **Corollary 1.** As it was shown above $I\alpha_1 I = \alpha \neq \alpha_1$ then $I \notin Aut(Q,\circ)$.

For $R_h\varepsilon$ we obtain

$$b = h(Ih)^{-1} = hh = h^2, \quad d = (\alpha h)^{-1}h = h^{-2}h = h^{-1}, \quad \alpha_1 \neq R_h^{-1}R_h^{-1}\alpha_1 R_h$$

since

$$R_{h^{-1}}R_{h^{-1}}\alpha_1 R_h h = R_{h^{-1}}R_{h^{-1}}\alpha_1 h^2 = R_{h^{-1}}R_{h^{-1}}h = $$
$$= R_{h^{-1}}1 = h^{-1} \neq \alpha_1 h = h^2$$

So, $R_h \notin Aut(Q,\circ)$. For $R_h I$ we have

$$d = (\alpha_1 R_h I 1)^{-1}\cdot h = (\alpha_1 h)^{-1}\cdot h = h^{-1}h = 1$$

and

$$IR_h^{-1}R_{h^{-1}}\alpha_1 R_h I = IR_{h^{-2}}\alpha_1 R_h I \neq \alpha_1,$$

since

$$IR_{h^{-2}}\alpha_1 R_h I h^2 = IR_{h^{-1}}\alpha_1 R_h h^{-2} = IR_{h^{-2}}\alpha_1 h^{-1} =$$
$$= IR_{h^{-2}}h^{-1} = Ih^{-3} = h^3 \neq h = \alpha_1 h^2,$$

i.e. $R_h I \notin Aut(Q,\circ)$. Therefore

$$Aut(Q,\circ) = \{\varepsilon\}.$$

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Linear isotopes of small order groups

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Abstract

The first part of the results of computer investigation of linear group isotopes is given. The first section of the work is auxiliary. It contains beloning criterions to the known classes of quasigroups. The second section contains: an algorithm for description of linear group isotopes; a full list of pairwise non-isomorphic linear group isotopes up to 15 order; a full list of subquasigroups of every isotope. For each quasigroup the beloning to the known classes of quasigroups is singled out.

A groupoid $(G;\cdot)$ is called an *isotope of a groupoid* $(Q;+)$, iff there exists a triple (α,β,γ) of bijections, called an *isotopy*, such that the relation

$$\gamma(x \cdot y) = \alpha x + \beta y$$

holds. An isotope of a group is called a *group isotope*. An isotope of a group is called *linear* if every component of a corresponding isotopy is a linear transformation of the group (recall, a transformation α is said to be *linear* in the group $(Q;+)$, iff there exist an automorphism of the group and an element c such that $\alpha x = \theta x + c$ for all $x \in Q$). It is easy to verify that any groupoid isomorphic to a linear group isotope is a linear group isotope as well. So, the class of all linear group isotopes forms a variety and medial and T-quasigroups are its subvarieties. An additional information on group isotopes and linear group isotopes one can find in [1], [2] and [3].

Here, using the results of the work [3] we continue that study. Namely, in the first part of the article we give: a criterion for a linear group isotope to belong to each of 22 the most singnificant classes of quasigroups; a full list of pairwise nonisomorphic linear group isotopes up to 15 order; a number of all these isotopes of every order (≤ 15).

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1. Some necessary properties

We shall write "isotope (α,β,γ) of the groupoid $(Q;+)$" instead of isotope $(Q;*)$, defined by the equality $x*y=\gamma^{-1}(\alpha x+\beta y)$", where α, β, γ are substitutions of the set Q.

Up to isomorphism every isotope $(Q;\cdot)$ of a group $(Q;+)$ can be defined by the equality

$$xy=\varphi x+\psi y+c, \tag{1}$$

where φ, ψ are unitary substitutions of the group $(Q;+)$ (i.e. $\varphi 0=\psi 0=0$). If $(Q;\cdot)$ is linear, then φ, ψ are automorphisms of $(Q;+)$ (see [3] and **Theorem 1.6** here).

As usual, the signs f_a and e_a denote a left and right local units of a of the quasigroup operation $(\cdot)$:

$$f_a\cdot a=a\cdot e_a=a;$$

ε denotes the identical transformation of arbitrary fixed set. I_c denotes inner automorphism of the group operation $(+)$:

$$I_c x = -c + x + c.$$

A *commutation of the arbitrary operation* $(+)$ is denoted by $(\oplus)$ and defined by $x \oplus y = y + x$. A groupoid $(Q;\oplus)$ is said *to be a commutation* of a groupoid $(Q;+)$.

A class of groupoids will be called *commutation of a class* K *of groupoids* and will be denoted by K^*, if K^* consists of all commutations of groupoids from K.

A formula $\Phi(\varphi, \psi, c, +, I, \bullet, J, A)$, being an equality with no propositional constant and having the propositional variables $\varphi, \psi, c, x_1, \ldots, x_n$, binary functional variables $+, \bullet, A$ and unary ones I, J only, will be called *a beloning criterion of a linear isotope to a class* K, where K is some class of groupoids, if from the facts that c is an element of a group $(Q;+)$; φ, ψ are elements of the group $(H,\bullet) = Aut(Q,+)$; I, J are inverse operations in these two groups respectively; and $A\colon H \times Q \to Q$ is such a function, that $A(\alpha, x) = \alpha x$ is true, it follows the equivalence of predicate expressed by the formula $\forall x_1 \ldots x_n\, \Phi(\varphi, \psi, c, +, I, \bullet, J, A)$ to the beloning of the linear isotope (φ, ψ, c) of the group $(Q;+)$ to the class K.

Theorem 1.1. *If* $\Phi(\varphi, \psi, c, +, I, \bullet, J, A)$ *is a beloning criterion of a linear isotope to a class* K *of groupoids, then a beloning criterion of a linear isotope to the class* K^* *one can get by replacing every subterm of the type* $u + v$, *where* u, v *are arbitrary terms, with* $v + u$ *in the formula* $\Phi(I_c\psi, I_c\varphi, c, +, I, \bullet, J, A)$.

Proof. Let c be an element of a group $(Q;+)$, φ,ψ be elements of the group $(H,\bullet)=Aut(Q,+)$, I,J be the inverse operations in these groups and

$$A: H\times Q\to Q$$

be a function such that $A(\alpha,x)=\alpha x$. Then the predicate expressed by the formula $\forall x_1\ldots x_n\ \Phi(\varphi,\psi,c,+,I,\bullet,J,A)$ is equivalent to the beloning of linear isotope (φ,ψ,c) of the group $(Q;+)$ to a class K. Note, that c is an element of the group $(Q;\oplus)$ as well, the inverse in the groups $(Q;+)$ and $(Q;\oplus)$ for any element from the set Q is the same, and an automorphism groups of these groups coincide.

Hence, a predicate, expressed by a formula obtained in a way described in the theorem, coincide with the predicate, expressed by the formula $\forall x_1\ldots x_n\ \Phi(I_c\psi,I_c\varphi,c,\oplus,I,\bullet,J,A)$, and is equivalent to the beloning of the linear isotope $(I_c\psi,I_c\varphi,c)$ of the group $(Q;\oplus)$ to the class K. But if $(Q;\cdot)$ is such an isotope, then

$$xy=I_c\psi x\oplus I_c\varphi y\oplus c=c+I_c\varphi y+I_c\psi x,$$

whence

$$x\otimes y=c+I_c\varphi x+I_c\psi y=\varphi x+\psi y+c,$$

i.e. this is equivalent to the beloning of the linear isotope (φ,ψ,c) of the group $(Q;+)$ to the class K^*. Thus, the **theorem** is true. □

Lemma 1.2.

$$(K_1\cap\ldots\cap K_n)^*=K_1^*\cap\ldots\cap K_n^*,\quad (K_1\cup\ldots\cup K_n)^*=K_1^*\cup\ldots\cup K_n^*.$$

Lemma 1.3. *If for automorphisms* φ,ψ *of a group* $(Q;+)$ *the equality*

$$\varphi + \psi = \varepsilon$$

holds, then the group is abelian.

Proof. Really, if for some elements $u, v \in Q$ $u + v \neq v + u$, then

$$\psi^{-1}u + \varphi^{-1}v = (\varphi + \psi)\psi^{-1}u + (\varphi + \psi)\varphi^{-1}v =$$
$$= \varphi\psi^{-1}u + u + v + \psi\varphi^{-1}v \neq \varphi\psi^{-1}u + v + u + \psi\varphi^{-1}v =$$
$$= (\varphi + \psi)(\psi^{-1}u + \varphi^{-1}v) = \psi^{-1}u + \varphi^{-1}v.$$

A contradiction. $\square$

Note, that only in abelian group the mapping I ($Ix = -x$) is an automorphism.

Recall the following

Definition 1. By *right F-, right symmetrical, RIP-, right* ***Bol****, right distributive, right semimedial, right alternative quasigroup* and by *right loop* it is called a groupoid $(Q,\cdot)$, such that its commutation $(Q,\otimes)$ has the left respective condition. By F-, TS-, IP-, ***Bol****, distributive, semimedial, alternative quasigroup* is called a quasigroup fulfilling the left and right respective condition simultaneously. An idempotent TS-quasigroup (**Moufang**) is called *Steiner* (respectively *CH*-) quasigroup. A quasigroup having at least one idempotent element is said *to be a peak*. If a quasigroup has no subquasigroup exept itself then it is called *monoquasigroup*.

Theorem 1.4. *A belonging criterion of a linear isotope* (φ, ψ, c) *of a group* $(Q;+)$ *to a class of quasigroups* K *is defined by the following table*

N	*name*	*definition*	*criterion*
1	*commutative*	$xy = yx$	$\varphi = \psi$, *the group is abelian*
2	*medial*	$xy \cdot uv = xu \cdot yv$	$\varphi\psi = \psi\varphi$, *the group is abelian*
3	*idempotent*	$xx = x$	$\varphi + \psi = \varepsilon$, $c = 0$, *the group is abelian*
4	*left F*	$x \cdot yz = xy \cdot e_x z$	φ *commutes with* ψ *and with all inner automorphisms of the group*
5	*right F*	*see* ***Definition 1***	$I_c\psi$ *commutes with* φ *and with all inner automorphisms of the group*
6	*left symmetrical*	$x \cdot xy = y$	$\psi = -\varepsilon$, *the group is abelian*
7	*right symmetrical*	*see* ***Definition 1***	$\varphi = -\varepsilon$, *the group is abelian*
8	*LIP*	*exists a substitution* λ, *for which* $\lambda x \cdot xy = y$	$(I_c\psi)^2 = \varepsilon$
9	*RIP*	*see* ***Definition 1***	$\varphi^2 = \varepsilon$
10	***Mufang***	$(xy \cdot z)y = x \cdot y(e_y z \cdot y)$, $y(x \cdot yz) = (y \cdot xf_y)y \cdot z$	$\varphi^2 = (I_c\psi)^2 = \varepsilon$
11	*left* ***Bol***	$z(x \cdot zy) = R_{e_z}^{-1}(z \cdot xz) \cdot y$	$(I_c\psi)^2 = \varepsilon$
12	*right* ***Bol***	*see* ***Definition 1***	$\varphi^2 = \varepsilon$
13	*left distributive*	$x \cdot yz = xy \cdot xz$	$\varphi + \psi = \varepsilon$, $c = 0$, *the group is abelian*
14	*right distributive*	*see* ***Definition 1***	$\varphi + \psi = \varepsilon$, $c = 0$, *the group is abelian*
15	*left loop*	$f_x = f_y$	$I_c\psi = \varepsilon$
16	*right loop*	*see* ***Definition 1***	$\varphi = \varepsilon$
17	*left semimedial*	$xx \cdot yz = xy \cdot xz$	$\varphi\psi = \psi\varphi$, *the group is abelian*
18	*right semimedial*	*see* ***Definition 1***	$\varphi\psi = \psi\varphi$, *the group is abelian*
19	*primary*	*commutative, semimedial,* $yy \cdot yx = xx \cdot xx$	$\varphi = \psi$, $(\forall x)(3x = 0)$, *the group is abelian*
20	*left alternative*	$x \cdot xz = xx \cdot z$	$\varphi = I_c\psi = \varepsilon$
21	*right alternative*	*see* ***Definition 1***	$\varphi = I_c\psi = \varepsilon$
22	*elastic*	$xy \cdot x = x \cdot yx$	$\varphi\psi = \psi\varphi$, $\varphi c = \psi c$, $(\forall x, u)(I_u\varphi(-x + \varphi x) = \psi(\psi x - I_c^{-1} x))$

Proof. The points 1)-3) are evident. The points 6), 8), 11), 15), 18) follow from the points 7), 9), 12), 16), 17) correspondingly when applying **Theorem 1.1** and **Lemma 1.2**.

4) The identity $x \cdot yz = xy \cdot e_x z$ is equivalent to

$$\varphi x + \psi(\varphi y + \psi z + c) + c = \varphi(\varphi x + \psi y + c) +$$
$$+ \psi(\varphi\psi^{-1}(-\varphi x + x - c) + \psi z + c) + c,$$

i.e.

$$\varphi x + \psi\varphi y = \varphi^2 x + \varphi\psi y + \varphi c - \psi\varphi\psi^{-1}\varphi x + \psi\varphi\psi^{-1} x - \psi\varphi\psi^{-1} c,$$

this with $x = 0$ implies

$$\psi\varphi y = \varphi\psi y + \varphi c - \psi\varphi\psi^{-1} c,$$

whence $\varphi c = \psi\varphi\psi^{-1} c$ and $\psi\varphi = \varphi\psi$. Hence,

$$\varphi x + \varphi\psi y = \varphi^2 x + \varphi\psi y + \varphi c - \varphi^2 x + \varphi x - \varphi c,$$

i.e.

$$x + \psi y + c = \varphi x + \psi y + c - \varphi x + x,$$

or

$$x + v - x = \varphi x + v - \varphi x,$$

whence $I_x v = \varphi I_x(\varphi^{-1} v)$. Hence, $I_x \varphi = \varphi I_x$.

5) By **Theorem 1.1**, **Lemma 1.2** and the just proved point 4) the identity $xy \cdot z = xf_z \cdot yz$ is equivalent to a conjuction of the commutation of the automorphisms $I_c\varphi, I_c\psi$ and the equality

$$-x + v + x = -I_c\psi x + v + I_c\psi x.$$

From the second condition it follows that

$$-x + I_c\psi u + x = I_c\psi(-x + u + x),$$

i.e. $I_x I_c \psi = I_x \psi I_c$, this with $x = -c$ implies $\psi I_c = I_c \psi$. Applying the first condition we have $I_c\varphi I_c\psi = I_c^2\psi\varphi$, i.e. $\varphi I_c\psi = I_c\psi\varphi$.

9) The parameter identity $xy \cdot \rho y = x$ is equivalent to a conjuction of the equalities $\varphi^2 = \varepsilon$ and $\rho = \psi^{-1} I I_c R_{\varphi c} \varphi\psi$. Really, if the identity holds and $y = 0$, then, accounting (1), we have $\varphi^2 x + a = x$

for some $a \in Q$. This means that $\varphi^2 = \varepsilon$ and $a = 0$. So, the identity can be rewritten as

$$x + \varphi\psi y + \varphi c + \psi\rho y + c = x.$$

Consequently, $\rho = \psi^{-1} I L_c R_{\varphi c} \varphi\psi$. Conversly, let the above equalities hold, then

$$xy \cdot \rho y = \varphi(\varphi x + \psi y + c) + \psi\psi^{-1}(-\varphi c - \varphi\psi y - c) + c =$$
$$= x + \varphi\psi y + \varphi c - \varphi c - \varphi\psi y - c + c = x.$$

7) To the proof of the point 9) we have to add that $\rho = \varepsilon$ i.e. $\varphi = -\varepsilon = I$ (since, as it is easy to see, $\varphi c = -c$). It means, in particular, that the group is abelian.

10) From the first equality with $y = z = 0$ and accounting (1) we have that

$$\varphi^3 x + a = \varphi x + b$$

for all $x \in Q$ and for some elements $a, b \in Q$. It is easy to see that the last equality is equivalent to $a = b$ and $\varphi^3 = \varphi$. So, $\varphi^2 = \varepsilon$ By **Theorem 1.1** from the second equality and from the just proved assertion we have $(I_c\psi)^2 = \varepsilon$. It is easy to verify that the converse statement is true as well.

12) The right **Bol** identity is the following:

$$(yz \cdot x)z = y \cdot L_{f_z}^{-1}(zx \cdot z).$$

When $x = z = 0$, we have a relation $\varphi^3 y + a = \varphi y + b$ for all $y \in Q$ and for some elements $a, b \in Q$. So, $\varphi^2 = \varepsilon$. It is easy to verify that the converse statement is true as well.

13), 14) The quasigroup is idempotent, so, $\varphi + \psi = \varepsilon$, $c = 0$ and the group is abelian. The converse statement is evident.

16) The relation (1) implies $e_x = \psi^{-1}(-\varphi x + x + c)$. The existence of a right identity means that $e_x = e_y$ for all $x, y \in Q$. It is easy to see, that the equality $e_x = e_0$ is equivalent to $\varphi = \varepsilon$.

17) In the linear group isotope the left semimedial identity is equivalent to

$$\varphi\psi x + \varphi c + \psi\varphi y = \varphi\psi y + \varphi c + \psi\varphi x, \tag{2}$$

this with $y = 0$ implies $\psi\varphi = \varphi I_c \psi$. Then the equality (2) will be rewritten as $x + y = y + x$, i.e. $(Q;+)$ is abelian. Hence, $I_c = \varepsilon$, and then $\varphi\psi = \psi\varphi$.

19) By 1), 17), 18) $\varphi = \psi$ and $(Q;+)$ is abelian. Then the identity

$$yy \cdot yx = xx \cdot xx$$

is equivalent to $3\varphi^2 y = 3\varphi^2 x$, or $3y = 3x$. Replacing with $y = 0$, we have

$$(\forall x)(3x = 0).$$

That, in particular, means the truth of the equality $3y = 3x$.

20) The identity $x \cdot xz = xx \cdot z$ is equivalent to

$$\varphi x + \psi\varphi x + \psi u + c = \varphi^2 x + \varphi\psi x + \varphi c + u.$$

If $x = u = 0$, then $\varphi c = c$ and then the additional equality $x = 0$ gives the relation $\psi u + c = c + u$, i.e. $I_c \psi = \varepsilon$. Since $\psi = I_c^{-1}$, then

$$\varphi x + I_c^{-1}\varphi x + I_c^{-1} u + c = \varphi^2 x + \varphi I_c^{-1} x + \varphi c + u,$$

i.e. $\varphi x + c = \varphi^2 x + \varphi c$. Accounting $\varphi c = c$ we have $\varphi = \varepsilon$, whence, in particular, it follows that $\varphi c = c$.

21) By 15), 16), 20) the left alternativity (and then right alternativity) of linear group isotopes may hold exactly for loops.

22) The identity $xy \cdot x = x \cdot yx$ is equivalent to

$$\varphi^2 x + \varphi\psi y + \varphi c + \psi x = \varphi x + \psi\varphi y + \psi^2 x + \psi c$$

If $x=y=0$, then $\varphi c=\psi c$, and then, with $x=0$, we shall obtain $\varphi\psi=\psi\varphi$. Then

$$\varphi^2 x+u+\psi c+\psi x=\varphi x+u+\psi^2 x+\psi c,$$

or

$$-u-\varphi x+\varphi^2 x+u=\psi^2 x+\psi c-\psi x-\psi c,$$

i.e.

$$I_u\varphi(-x+\varphi x)=\psi(\psi x-I_c^{-1}x). \qquad \square$$

The theorem implies the following result immedately.

Corollary 1.5. *The following classes of linear isotopes coinside:*

a) left semimedial = right semimedial = semimedial = medial;

b) left distributive = right distributive = distributive = idempotent;

c) CH = TS;

*d) left **Bol** = LIP;*

*e) right **Bol** = RIP;*

*f) **Bol** = **Mufang** = IP;*

g) left alternative = right alternative = alternative = loops = groups;

h) distributive Steiner quasigroups = Steiner quasigroups.

The pairs $<\varphi,\psi>$, $<\tilde{\varphi},\tilde{\psi}>$ of the unitary substitutions are said to be *midle-isoequal* (*left-isoequal, right-isoequal*) in a group $(Q;+)$, if there exist elements $a,b\in Q$, such that the isotopes $(Q;\cdot)$ and $(Q;*)$, defined by the equalities $xy=\varphi x+a+\psi y$ and $x*y=\tilde{\varphi}x+a+\tilde{\psi}y$ (respectively $xy=a+\varphi x+\psi y$ and $x*y=b+\tilde{\varphi}x+\tilde{\psi}y$, $xy=\varphi x+\psi y+a$ and $x*y=\tilde{\varphi}x+\tilde{\psi}y+b$), are isomorphic.

We recall some results, obtained by F. Sokhatsky in [3], in the following statement.

Theorem 1.6. *The following assertions are true.*

a) An isomorphism of group isotopes implies an isomorphism of the corresponding groups.

b) There exists a bijection between the sets of all isotopes of isomorphic groups such that the corresponding isotopes are isomorphic.

c) For every element 0 of a group isotope $(Q;f)$ *there exists exactly one quadruple* $(+,\alpha_1,\alpha_2,c)$, *such that* $(Q;+)$ *is a group with a neutral element 0, and* α_1,α_2 *are unitary substitutions of the group* $(Q;+)$ *and*

$$f(x,y)=a+\alpha_1 x+\alpha_2 y, \qquad g(x,y)=b+\beta_1 x+\beta_2 y \tag{3}$$

hold (the right side of the equality is called "left canonical decomposition").

d) If (3) are canonical decompositions of $(Q;f)$ *and* $(Q;g)$ *respectively, then the group isotopes are isomorphic if and only if there exist* $c\in Q$, $\theta\in Aut(Q,+)$, *such that:*

$$\theta b=a+\alpha_1 c+\alpha_2 c-c,$$
$$\theta\beta_1 x=c-\alpha_2 c-\alpha_1 c+\alpha_1(\theta x+c)+\alpha_2 c-c,$$
$$\theta\beta_2 x=c-\alpha_2 c+\alpha_2(\theta x+c)-c.$$

e) If pairs $<\varphi,\psi>$ *and* $<\tilde\varphi,\tilde\psi>$ *of unitary substitutions are left-isoequal in a group* $(Q;+)$, *then there exists a bijection between the set of all isotopes of the type* (φ,ψ,L_a^{-1}) *and the set of all isotopes of the type* $(\tilde\varphi,\tilde\psi,L_b^{-1})$ *of the group* $(Q;+)$ *such that the corresponding isotopes are isomorphic.*

f) Every subquasigroup of an isotope of a group is a right coset of the group by some of its subgroup.

Theorem 1.7. *Linear isotopes* (φ, ψ, a) *and* $(\widetilde{\varphi}, \widetilde{\psi}, b)$ *of a group* $(Q;+)$ *are isomorphic if and only if there exist such* $c \in Q$, $\theta \in Aut(Q,+)$, *that*

$$\theta b = \varphi c + \psi c + a - c, \quad \theta\widetilde{\varphi} = \varphi\theta, \quad I_{\varphi c}\theta\widetilde{\psi} = \psi\theta.$$

Proof. Let us denote the operations in the given isotopes by f and g respectively. Then

$$f(x,y) = a + I_a\varphi x + I_a\psi y, \qquad g(x,y) = b + I_b\widetilde{\varphi} x + I_b\widetilde{\psi} y.$$

By **Theorem 1.6** d) $(Q;f) \cong (Q;g)$ if and only if there exist $c \in Q, \theta \in Aut(Q;+)$ such that

$$\theta b = a + I_a\varphi c + I_a\psi c - c,$$
$$\theta I_b\widetilde{\varphi} = I_{I_a\varphi c + I_a\psi c - c} I_a\varphi\theta,$$
$$\theta I_b\widetilde{\psi} = I_{I_a\psi c - c} I_a\psi\theta.$$

Accounting that $\theta I_b = I_{\theta b}\theta$, we'll substitute the first equality into the second and the third ones:

$$\theta\widetilde{\varphi} = \varphi\theta, \qquad I_{\varphi c}\theta\widetilde{\psi} = \psi\theta.$$

It remains to simplify the first equality. □

Theorem 1.8. *If pairs* $<\varphi, \psi>$ *and* $<\widetilde{\varphi}, \widetilde{\psi}>$ *of unitary substitutions are right-isoequal in a group* $(Q;+)$, *then there exists a bijection between the set of all isotopes of the type* $(\varphi, \psi, R_a^{-1})$ *and the set of all isotopes of the type* $(\widetilde{\varphi}, \widetilde{\psi}, R_b^{-1})$ *of the group, such that the corresponding isotopes are isomorphic.*

Proof. Really, then there exist $a, b \in Q$, for which the isotopes $(Q;\cdot)$ and $(Q;\times)$, defined by the equalities

$$xy = \varphi x + \psi y + a, \qquad x \times y = \widetilde{\varphi} x + \widetilde{\psi} y + b,$$

are isomorphic. Then $(Q;\bullet) \cong (Q;\otimes)$, where

$$x \bullet y = yx = \varphi y + \psi x + a = a \oplus \psi x \oplus \varphi y,$$
$$x \otimes y = b \oplus \widetilde{\psi} x \oplus \widetilde{\varphi} y,$$

i.e. $< \psi, \varphi >$ and $< \widetilde{\psi}, \widetilde{\varphi} >$ are left-isoequal in the group $(Q;\oplus)$. By **Theorem 1.6** e) there exists a bijection between the set of isotopes of the type $(Q;f)$ and $(Q;g)$, where

$$f(x,y) = c \oplus \psi x \oplus \varphi y,$$
$$g(x,y) = d \oplus \widetilde{\psi} x \oplus \widetilde{\varphi} y,$$

for which the respective isotopes are isomorphic.

Then the commutations of the corresponding isotopes are isomorphic, but these are the isotopes $(\varphi, \psi, R_c^{-1})$ and $(\widetilde{\varphi}, \widetilde{\psi}, R_d^{-1})$ of the group $(Q;+)$. $\square$

To describe the algorithm given below we have to cite the following evident assertion.

Proposition 1.9. *If* $(H;+)$ *is a subgroup of the group* $(Q;+)$, *then* $H + a = H + b$ *if and only if* $(a - b) \in H$.

2. A description of isotopes

Let us describe all linear group isotopes up to the 15-th order up to isomorphism. The obtained isotopes will be classified acccoding to the known classes of quasigroups; the full list of subquasigroups of every isotope will be given.

By **Theorem 1.6** a),b) the problem can be solved for every of 28 groups of the indicated order separately (up to isomorphism). Let a group $(Q;+)$ be given, i.e. an order, a neutral element, generators and a **Cayley table** are known. Let us apply the following algorithm to the group.

Algorithm. First, using generators, we construct a sequence of the formation of all other elements (besides the neutral element). We verify also whether or not the group is abelian; we construct a table of all inverse elements of the group. We find all subgroups (except the group itself) of the given group, looking through all proper subsets, whose number of elements is a factor of the order of the group. It is enough to verify the closure of the subset under the main operation only. Furthermore, using **Proposition 1.9** we find all right cosets of the group by all subgroups, other than the given group. We find all automorphisms of the group. For this purpose consider all injectional mappings from the generator set into the group and their extentions to endomorphisms, i.e. using the properties $\varphi 0 = 0$, $\varphi(x+y) = \varphi x + \varphi y$. If its kernel is trivial, then it is an automorphism.

Remember the actions of all automorphisms on all elements of the group, and also find the identity automorphism (it moves no one of the generators in contrast to the others).

If the group is abelian, we find the automorphism $I = -\varepsilon$ as well (it maps every generator to its inverse and only I does it). Furthermore, we construct **Cayley table** for automorphisms. If φ and ψ are automorphisms of the group, then the only automorphism $\varphi\psi$ acts on every of the generators t, as φ acts on ψt. Construct a table for inverse automorphisms by verifying, if a composition coinsides with ε.

Let us find the correspondence $c \to I_c$ *(it is enough to verify* the actions of the automorphisms on generators). We create a table A of the size $k \times k$, where k is the number of all automorphisms and fill with "+". Consider, in turn, all pairs of automorphisms. Let $<\varphi,\psi>$ be a next in turn pair. If the respective box in the table A contains a sign "-", we consider the next pair. Otherwise, we create a table B, filled with "+", of the size being equal to the order of the group. For all pairs $<\theta,c> \in Aut(Q;+) \times Q$ we do the following: if

$$(\theta^{-1}\varphi\theta \neq \varphi) \vee ((I_{\varphi c}\theta)^{-1}\psi\theta \neq \psi)$$

then we put "-" in the box of the table A, corresponding to the pair $<\theta^{-1}\varphi\theta,(I_{\varphi c}\theta)^{-1}\psi\theta>$; otherwise, for every element m of the group (when the corresponding box contains "+" in the table B), if $n=\theta^{-1}(\varphi c+\psi c+m-c)$ has a number, which is greater than the number of the element m, then in the table B we put "-" in the box corresponding to the element n. As the result, we get all triples $<\varphi,\psi,a>$, where a runs the set of all elements of the group having the sign "+" in the table B.

Having run all the table A we obtain, according to **Theorems 1.7, 1.8,** a list of all linear isotopes of the group $(Q;+)$. Using **Theorems 1.4, 1.6** f), we select in this list the isotopes from the known classes of quasigroups and find all subquasigroups of every isotope.

This algorithm was applied to all 28 groups up to the 15-th order using a personal computer. Linear isotopes of the first and the second orders are isomorphic to groups of the same order.

Thus, we consider linear isotopes of order greater than two. To account the results on such groups, we need some designations.

The elements of the group H be denoted as follows:

- "0",...,"$m-1$", if $H = Z_m$;
- "xy", where $<x, y>$ is a corresponding vector, if $H = Z_m \times Z_2$ or $H = Z_3 \times Z_3$;
- $4a+2b+c$, where $<a,b,c>$ is a corresponding vector, if $H = Z_2 \times Z_2 \times Z_2$;
- "xy", if $H = D_m$ (diedr group) and a corresponding element is obtained after the application of y symmetries with respect to a fixed axis of the m-angle and then of x elementary turns in the fixed direction;
- "1","i","j","k","-1","$-i$","$-j$","$-k$" as usual, if $H = Q_8$;
- "xy", where φ is a corresponding element and $\varphi 1 = x$, $\varphi 2 = y$, if $H = A_4$ is an alternating group;
- "mn", where $a^m b^n$ is a corresponding element, if

$$H = G_{12} = \{a^m b^n \mid m = 0,1,2,3; n = 0,1,2; a^4 = b^3 = 1; ba = ab^2\}.$$

As sequences of the generators select the following: "1" in cyclic groups; "4","2","1" in $Z_2 \times Z_2 \times Z_2$; "$i$","$j$" in Q_8; "13","21" in A_4; "10","01" in the others. The automorphisms will be denoted by a sequence of images of the generators.

All right cosets by all subgroups (except the group itself) will be numbered. The writing "N: $a_1,...,a_k + b_1,...,b_l$" in the next paragraph means that the number N is denoted a subgroup H created by the generators $a_1,...,a_k$, and the numbers $N+1,...,N+l$ are the cosets $H+b_1,...,H+b_l$ respectively.

Group Z_3. 1: 0+1,2.

Group Z_4. 1: 0+1,2,3; 5: 2+1.

Group Z_5. 1: 0+1,...,4.

Group Z_6. 1: 0+1,...,5; 7: 3+1,2; 10: 2+1.

Group Z_7. 1: 0+1,...,6.

Group Z_8. 1: 0+1,...,7; 9: 4+1,2,3; 13: 2+1.

Group Z_9. 1: 0+1,...,8; 10: 3+1,2.

Group Z_{10}. 1: 0+1,...,9; 11: 5+1,...,4; 16: 2+1.

Group Z_{11}. 1: 0+1,...,10.

Group Z_{12}. 1: 0+1,...,11; 13: 6+1,...,5; 19: 4+1,2,3; 23: 3+1,2; 26: 2+1.

Group Z_{13}. 1: 0+1,...,12.

Group Z_{14}. 1: 0+1,...,13; 15: 7+1,...,6; 22: 2+1.

Group Z_{15}. 1: 0+1,...,14; 16: 5+1,...,4; 21: 3+1,2.

Group $Z_2 \times Z_2$. 1: 00+01,10,11; 5: 01+10; 7: 10+01; 9: 11+01.

Group $Z_4 \times Z_2$. 1: 00+01,10,...,31; 9: 01+10,20,30; 13: 20+01,10,11; 17: 21+01,10,11; 21: 01,20+10; 23: 10+01; 25: 11+01.

Group $Z_6 \times Z_2$. 1: 00+01,10,...,51; 13: 01+10,20,...,50; 19: 30+01,10,...,21; 25: 31+01,10,...,21; 31: 20+01,10,11; 35: 01,30+10,20; 38: 01,20+10; 40: 10+01; 42: 11+01.

Group $Z_3 \times Z_3$. 1: 00+01,02,...,22; 10: 01+10,20; 13: 10+01,02; 16: 11+01,02; 19: 12+01,02.

Group $Z_2 \times Z_2 \times Z_2$. 1: 0+1,...,7; 9: 1+2,4,6; 13: 2+1,4,5; 17: 3+1,4,5; 21: 4+1,2,3; 25: 5+1,2,3; 29: 6+1,2,3; 33: 7+1,2,3; 37: 1,2+4; 39: 1,4+2; 41: 1,6+2; 43: 2,4+1; 45: 2,5+1; 47: 3,4+1; 49: 3,5+1.

Group D_3. 1: 00+01,10,...,21; 7: 01+10,11; 10: 11+01,20; 13: 21+01,10; 16: 10+01;

Group D_4. 1: 00+01,10,...,31; 9: 01+10,11,20; 13: 11+01,20,21; 17: 20+01,10,11; 21: 21+01,10,30; 25: 31+01,10,11; 29: 01,20+10; 31: 10+01; 33: 11,20+01.

Group D_5. 1: 00+01,10,...,41; 11: 01+10,11,20,21; 16: 11+01,20,21,30; 21: 21+01,10,30,31; 26 31+01,10,11,40; 31: 41+01,10,11,20; 36: 10+01.

Group D_6. 1: 00+01,10,...,51; 13: 01+10,11,...,30; 19: 11+01,20,21,30,31; 25: 21+01,10,30,31,40; 31· 30: +01,10,...,21; 37: 31+01,10,11,40,41; 43: 41+01,10,11,20,50; 49: 51+01,10,...,21; 55: 20+01,10,11; 59: 01,30+10,11; 62: 11,30+01,20; 65: 21,30+01,10; 68: 01,20+10; 70: 10+01; 72: 11,20+01.

Group D_7. 1: 00+01,10,...,61; 15 01+10,11,...,31; 22: 11+01,20,21,...,40; 29: 21+01,10,30,31,40,41; 36: 31+01,10,11,40,41,50; 43: 41+01,10,11,20,50,51; 50: 51+01,10,...,21,60; 57: 61+01,10,...,30; 64: 10+01.

Group Q_8. 1: **1+i,j,...,-k**; 9: **-1+i,j,k**; 13: **i+j**; 15: **j+i**; 17: **k+i**.

Group A_4. 1: 12+13,14,...,43; 13: 21+13,14,31,32,34; 19: 34+13,14,...,24; 25: 43+13,14,...,24; 31: 13+21,23,24; 35: 23+13,14,41; 39: 24+13,14,31; 43: 32+13,14,21; 47: 21,34+13,14.

Group G_{12}. 1: 00+01,02,...,32; 13: 20+01,02,...,12; 19: 01+10,20,30; 23: 10+01,02; 26: 11+01,02; 29: 12+01,02; 32: 01,20+10.

Now we number the sequences of the numbers of cosets (regardless of the groups). The notations "N: $a_1+b_1\cdot c_1,\dots,a_k+b_k\cdot c_k$" in the next paragraph means that with the number N (this will be an integer or an integer with a letter) a sequence

$$a_1, a_1+c_1, \dots, a_1+b_1c_1, a_2, a_2+c_2, \dots, a_2+b_2c_2, \dots, a_k+b_kc_k$$

is denoted. Here, instead of "$a+b\cdot 1$","$a+1\cdot c$", "$a+0\cdot c$","$a+b\cdot c, d+e\cdot f$" (where $d=a+bc$) we write "$a+b$","$a\oplus c$, a","$a+b\cdot c+e\cdot f$" respectively. The empty sequence will be denoted with the number 1.

2a: 1; 2b: 11; 2c: 17; 2d: 18; 2e: 19; 2f: 20; 2g: 21; 2h: 22; 2i: 23; 2k: 33; 2l: 34; 2m: 35; 2n: 36; 2o: 37; 2p: 39; 2q: 45; 2r: 46; 2s: 47; 2t: 48; 2u: 49; 2v: 50; 2w: 64; 2x: 65; 3a(4c): 1,5; 3b: 1,9; 3c: 1⊕9; 3d: 1,13; 3e: 1,16; 3f: 1,19; 3g: 1,25; 3h: 1,36; 3i: 1,39; 3j: 1,45; 3k: 1,64; 3l: 3⊕8; 3m: 3,16; 3n: 3,36; 3o: 3,64; 3p: 4,17; 3q: 4,37; 3r: 4,65; 3s: 5,16; 3t: 5,36; 3u: 5,49; 3v: 5,64; 3w: 6⊕4; 3x: 6,37; 3y: 6,65; 3z: 7,36; 3A: 7,64; 3B: 8,37; 3C: 8,65; 3D: 9,36; 3E: 9,64; 3F: 10,37; 3G: 10,65; 3H: 11,48; 3I: 11,64; 3J: 12,47; 3K: 12,65; 3L: 13,64; 3M: 14,65; 3N: 20⊕7; 3O: 20,33; 3P: 22⊕5; 3Q: 22,33; 3R: 32,43; 3S: 33,43; 3T: 58,71; 4a: 1,2; 4b: 1,4; 4c(3a): 1,5; 4d: 1,6; 4e: 1,7; 4f: 1,8; 4g: 31+1; 4h: 33+1; 5a: 1+2; 5b: 1,4,7; 5c: 1,5,9; 5d: 1,6,8; 5e: 10+2; 5f: 16+2; 5g: 21+2; 5h: 35+2; 6a: 1,7⊕3; 6b: 1,7⊕9; 6c: 1,11⊕5; 6d: 1,11,36; 6e: 1,13,39; 6f: 1,15⊕7; 6g: 1,15,64; 6h: 1,16⊕3; 6i: 1,16⊕5; 6j: 1,31⊕4; 6k: 2⊕5,17; 6l: 2⊕9,37; 6m: 2,15,65; 6n: 2,31,48; 6o: 3,31,49; 7a: 1,9⊕4; 7b: 1,9⊕6; 7c: 1+2·8; 7d: 1,13⊕8; 7e: 1,13,45; 7f: 1,13,47; 7g: 1,17,31; 7h: 1,25,39; 7i: 1,25,45; 7j: 1+2·24; 7k: 1,33,45; 7l: 2+2·8; 7m: 2,18,32; 7n: 2,26,48; 7o: 3⊕8⊕7; 7p: 3,19,31; 7q: 4⊕8⊕5; 7r: 4⊕9,47; 7s: 4,20,32; 7t: 5,23,49; 7u: 6⊕4⊕8; 7v: 7⊕4⊕7; 7w: 7+2·12; 7x: 8+2·4; 7y: 8⊕4⊕5; 7z: 8+2·12; 7A: 8,12,49; 7B: 10,23,49; 7C: 12⊕6,47; 8a: 1,2,39; 8b: 1,6,45; 8c: 1,7,49; 8d: 1,8,45; 8e: 5,8,49; 8f: 6⊕5,48; 9a: 1+3; 9b: 1,2,7,8; 9c: 1,4,6,7; 9d: 31+3, 10a: 1,5,7,9; 10b: 33⊕6+1⊕3; 10c: 58,69⊕2+1; 11a: 1,5,9⊕7; 11b: 1,6,8,19;

12a: 1,2,9+1; 12b: 1,2,13+1; 12c: 1,4⊕9+1; 12d: 1,4,33⊕3; 12e: 1,5⊕8⊕2; 12f: 1,6,9+1; 12g: 3,4⊕7+1; 12h: 3,8⊕3+1; 12i: 7,8⊕3+1; 13a: 1,13⊕6,32; 13b: 1,31,55,70; 13c: 2,14⊕5,32; 13d: 2,32,56,71; 13e: 3,15⊕4,32; 13f: 3,33,57,70; 13g: 4,34,58,71; 13h: 5,17⊕3,33; 13i: 9,33,55,70; 13j: 11⊕6⊕5,33; 13k: 11,35,57,70; 14a: 1,12,31,47; 14b: 2,7,31,48; 15a: 1+4; 15b: 16+4; 16a: 1,13⊕6⊕4⊕3; 16b: 1,13⊕6⊕4⊕9; 16c: 1,19,31⊕4⊕5; 16d: 1,25⊕6⊕4⊕7; 16e: 1,31,55,65⊕5; 16f: 4,16⊕4⊕3,33; 16g: 5,35,55⊕6⊕9; 16h: 5,35,55,66⊕4; 16i: 6,36,56⊕4,71; 16j: 8,32,58+1,71; 16k: 10+2·6+1,33; 16l: 12,36,58⊕2,71; 17a: 1,9+3·4; 17b: 1,13+2·6,47; 17c: 1+2·12⊕8,45; 17d: 4+2·9⊕6,47; 17e: 9⊕9+1⊕9,47; 18a: 1,9⊕4+2·2; 18b: 1,13⊕8+2·2; 18c: 1,17,29+2·2; 18d: 1,25,39⊕6⊕4; 18e: 2⊕8+2·3⊕2; 18f: 3,19,30+1⊕3; 18g: 6⊕4+2·3⊕2; 18h: 6,18,29⊕3⊕2; 18i: 7⊕4⊕3+1⊕3; 18j: 7,19,30+1⊕3; 18k: 8,20,30⊕2+1; 19a: 1,2,13+1,39; 19b: 1,2,31+1⊕3; 19c: 1,4,33⊕3,49; 19d: 1,7,31+2·2; 19e: 1,8,31⊕3+1; 20a: 1,7+3·3; 20b: 1⊕9+3·3; 21: 1,2,5,6,39; 22: 2,4,6,17; 23: 1,13,31,43⊕4; 24a: 4⊕9,47+2; 24b: 9⊕9,47+2; 25a: 1+3·3+2; 25b: 1,5,9⊕7+2; 26a: 1+2,16+2; 26b: 1,4,7⊕9+2; 27a: 1,31,55,68+2·2; 27b: 3,33,57,69+1⊕3; 27c: 6,36,56,68⊕3⊕2; 27d: 9,33,55,68+2·2; 27e: 11,35,57,69+1⊕3; 28: 3,5,8⊕2,31,49; 29a: 1+3·2+3; 29b: 1,5,9,31⊕4+2; 29c: 1,6,8⊕8+3; 29d: 1+4·5+2; 30: 1+6; 31a: 1,9+3·4+2·2; 31b: 1+2·8⊕4⊕8+2·2; 31c: 1+2·12⊕8+2·6⊕4; 31d: 2,9⊕9⊕4⊕7⊕3⊕2; 32a: 1+5·2,36; 32b: 2+4·2+1,37; 33: 1,11+5·5; 34a: 1,13⊕6⊕4+3·3; 34b: 1,31,55⊕4+2·3⊕5; 34c: 9,33,55⊕5⊕3⊕4⊕3; 35a: 1,5⊕5⊕3,47+2; 35b: 6,7,9⊕9,47+2; 36: 1+7; 37a: 1,4,5,8⊕5+3; 37b: 1,6+6; 38a: 1,2,5,6⊕7+3,39; 38b: 1,2,7,8,31+4; 39a: 1,13+3·6⊕4⊕3+2·2; 39b: 1,13,31⊕6,55⊕4⊕9+2·2;

39c: 2,13,32⊕6,56⊕3⊕9⊕3⊕2; 40: 1+8; 41: 1,9+5·4+2·2; 42a: 2+2·4,32+2·2,56⊕5,71; 42b: 2+2·4,32+2·2,56,66⊕5; 42c: 4+2,16+2⊕2⊕3,33; 42d: 4+2·4,32+2·2,58⊕7⊕6; 42e: 10+2⊕4+2⊕4+1,33; 43a: 1+7·2,64; 43b: 2+6·2+1,65; 44: 1,15+7·7; 45: 1,13+3·6+4·4; 46: 2,3,8⊕3+1⊕6,47+2; 47a: 1+5·2+5; 47b: 1+5·3+5; 48: 1+10; 49a: 1+3·4+3·2⊕4+3; 49b: 1,5,9,25⊕3+1⊕2⊕4+2⊕5; 50: 1,2,5,6,9+1,31+1⊕3+2; 51: 1+2,8⊕3⊕2,31,43⊕4+2; 52a: 1+11; 52b: 1+8⊕7+2; 53: 1+12; 54: 2+2·4⊕3,32+3·2,56⊕3⊕9⊕3⊕2; 55: 1+7·2+7; 56: 1,9+7·4+6·2; 57: 1,13+7·6⊕4+3·3+2·2; 58: 1+11,31+6; 59: 1+3·4+6·2⊕3+1⊕2⊕4+3+2·2; 60: 1+20; 61: 1+22.

Now we give a full list of the obtained isotopes. Every linear isotope (φ,ψ,c) will be imagined as follows: a sign for φ, comma (","), a sign for ψ, comma, a sign for c, a sequence of the signs "i" and central dot ("·") ("i" on the k-th place means the truth of the k-th property from the given below in the paragraph (*) list, and "·" the property is false), a number of the sequence of proper subquasigroups of the isotope, fullstop (" "). If c is a neutral element of the group, then the sign for neutral element and the comma before it are omitted; if φ is the same as in the privious isotope, then the sign for φ and the comma after it are omitted; if both, then the signs for φ and for ψ coincide; the sign "/c" will be written instead of "φ,ψ,c". If all properties are false, only one central dot will be written. If the proper subquasigroups in the isotope are absent, a number of the sequence of subquasigroups (i.e. 1) will be omitted.

Note, that the numeration of the subquasigroup sequences are selected in such a way that the upper semilattices of subquasigroups (the order is inclusion) are isomorphic for different isotopes iff in the corresponding integers are the same.

Commutativity of an isotope one can verify using the equality $\varphi=\psi$. An isotope is: idempotent (a peak) iff there are (is) all (at least one) one-element subsets in the full list of subquasigroups; a monoquasigroup if there is no subquasigroups; left (right) symmetrical, iff the group is abelian and the equality $\psi=-\varepsilon$ (respectively $\varphi=-\varepsilon$) holds (it is enough to verify it on the generators only); primary, iff the equality $\varphi=\psi$ only in groups Z_3 and $Z_3\times Z_3$; a right loop, if the equality $\varphi=\varepsilon$ is true. There is no point in giving mediality for cyclic (Z_3-Z_{15}) and nonabelian (D_3-D_7,Q_8,A_4,G_{12}) groups. The subsets of left and right F-quasigroups in the set of isotopes on abelian groups coincide with the subset of medial ones. Left loops in that set are defined by the equality $\psi=\varepsilon$.

(*) Hence, with the signs "i" and "·" we denote

- for isotopes of cyclic groups 1) *LIP*-quasigroups, 2) *RIP*-quasigroups and 3) elastic quasigroups;
- for isotopes of non-cyclic abelian groups 1) medial quasigroups, 2) *LIP*-quasigroups, 3) *RIP*-quasigroups and 4) elastic quasigroups;
- for isotopes of nonabelian groups 1) - 2) left and right F-quasigroups, 3) *LIP*-quasigroups, 4) *RIP*-quasigroups, 5) left loops and 6) elastic quasigroups.

Group Z_3. 1,1iii2a. 2ii·2a. 2,1ii·2a. 2iii5a. /1iii.

Group Z_4. 1,1iii3a. 3ii·3a. 3,1ii·3a. 3iii3a.

Group Z_5. 1,1iii2a. 2·i·2a. 3·i·2a. 4ii·2a. 2,1i··2a. 2··i2a. 3·2a. 4i·i15a. /1i··i. 3,1i··2a. 2·2a. 3··i15a. /1··i. 4i··2a. 4,1ii·2a. 2·ii15a. /1·i·. 3·i·2a. 4iii2a.

Group Z_6. 1,1iii6a. 5ii·6a. 5,1ii·6a. 5iii29a. /1iii2b.

Group Z_7. 1,1iii2a. 2·i·2a. 3·i·2a. 4·i·2a. 5·i·2a. 6ii·2a. 2,1i··2a. 2··i2a. 3·2a. 4·2a. 5·2a. 6i·i30a. /1i··. 3,1i··2a. 2·2a. 3··i2a. 4·2a. 5··i30a. /1··. 6i··2a. 4,1i··2a. 2·2a. 3·2a. 4··i30a. /1··i. 5·2a. 6i··2a. 5,1i··2a. 2·2a. 3··i30a. /1··. 4·2a. 5··i2a. 6i··2a. 6,1ii·2a. 2·ii30a. /1·i·. 3·i·2a. 4·i·2a. 5·i·2a. 6iii2a.

Group Z_8. 1,1iii7a. 3ii·7a. 5ii·7a. 7ii·7a. 3,1ii·7a. 3iii7a. 5ii·7a. 7ii·7a. 5,1ii·7a. 3ii·7a. 5iii7a. 7ii·7a. 7,1ii·7a. 3ii·7a. 5ii·7a. 7iii7a

Group Z_9. 1,1iii3c. 2·i·3c. 4·i·3c. 5·i·3c. 7·i·3c. 8i··3c. 2,1i··3c. 2··i25a. /1··i. 4·3c. 5··i25a. /1··. 7·3c. 8i·i52a. /1i··. /3i·i5e. 4,1i··3c. 2·3c. 4··i3c. 5·3c. 7·3c. 8i··3c. 5,1i··3c. 2··i25a. /1··. 4·3c. 5··i52a. /1··i. /3··i5e. 7·3c. 8i·i25a. /1i··. 7,1i··3c. 2·3c. 4·3c. 5·3c. 7··i3c. 8i··3c. 8,1ii·3c. 2·ii52a. /1·i·. /3·ii5e. 4·i·3c. 5·ii25a. /1·i·. 7·i·3c. 8iii25a. /1iii.

Group Z_{10}. 1,1iii6c. 3·i·6c. 7·i·6c. 9ii·6c. 3,1i··6c. 3··i47a. /1··i2c. 7·6c. 9i··6c. 7,1i··6c. 3·6c. 7··i6c. 9i·i47a. /1i··2c. 9,1ii·6c. 3·i·6c. 7·ii47a. /1·i·2c. 9iii6c.

Group Z_{11}. 1,1iii2a. 2·i·2a. 3·i·2a. 4·i·2a. 5·i·2a. 6·i·2a. 7·i·2a. 8·i·2a. 9·i·2a. 10ii·2a. 2,1i··2a. 2··i2a. 3·2a. 4·2a. 5·2a. 6·2a. 7·2a. 8·2a. 9·2a. 10i·i48a. /1i··. 3,1i··2a. 2·2a. 3··i2a. 4·2a. 5·2a. 6·2a. 7·2a. 8·2a. 9··i48a. /1··. 10i··2a. 4,1i··2a. 2·2a. 3·2a. 4··i2a. 5·2a. 6·2a. 7·2a. 8··i48a. /1··. 9·2a. 10i··2a. 5,1i··2a. 2·2a. 3·2a. 4·2a. 5··i2a. 6·2a. 7··i48a. /1··. 8·2a. 9·2a. 10i··2a. 6,1i··2a. 2·2a. 3·2a. 4·2a. 5·2a. 6··i48a. /1··i. 7·2a. 8·2a. 9·2a. 10i··2a. 7,1i··2a. 2·2a. 3·2a. 4·2a. 5··i48a. /1··. 6·2a. 7··i2a. 8·2a. 9·2a. 10i··2a. 8,1i··2a. 2·2a. 3·2a. 4··i48a. /1··. 5·2a. 6·2a. 7·2a. 8··i2a. 9·2a. 10i··2a. 9,1i··2a. 2·2a. 3··i48a. /1··. 4·2a. 5·2a.

6·2a. 7·2a. 8·2a. 9··i2a. 10i··2a. 10,1ii·2a. 2·ii48a. /1·i·. 3·i·2a. 4·i·2a. 5·i·2a. 6·i·2a. 7·i·2a. 8·i·2a. 9·i·2a. 10iii2a.

Group Z_{12}. 1,1iii16a. 5ii·16a. 7ii·16a. 11ii·16a. 5,1ii·16a. 5iii49a. /1iii3P. 7ii·16a. 11ii·49a. /1ii·3N. 7,1ii·16a. 5ii·16a. 7iii16a. 11ii·16a. 11,1ii·16a. 5ii·49a. /1ii·3N. 7ii·16a. 11iii49a. /1iii3P.

Group Z_{13}. 1,1iii2a. 2·i·2a. 3·i·2a. 4·i·2a. 5·i·2a. 6·i·2a. 7·i·2a. 8·i·2a. 9·i·2a. 10·i·2a. 11·i·2a. 12ii·2a. 2,1i··2a. 2··i2a. 3·2a. 4·2a. 5·2a. 6·2a. 7·2a. 8·2a. 9·2a. 10·2a. 11·2a. 12i·i53a. /1i··. 3,1i··2a. 2·2a. 3··i2a. 4·2a. 5·2a. 6·2a. 7·2a. 8·2a. 9·2a. 10·2a. 11··i53a. /1··. 12i··2a. 4,1i··2a. 2·2a. 3·2a. 4··i2a. 5·2a. 6·2a. 7·2a. 8·2a. 9·2a. 10··i53a. /1··. 11·2a. 12i··2a. 5,1i··2a. 2·2a. 3·2a. 4·2a. 5··i2a. 6·2a. 7·2a. 8·2a. 9··i53a. /1··. 10·2a. 11·2a. 12i··2a. 6,1i··2a. 2·2a. 3·2a. 4·2a. 5·2a. 6··i2a. 7·2a. 8··i53a. /1··. 9·2a. 10·2a. 11·2a. 12i··2a. 7,1i··2a. 2·2a. 3·2a. 4·2a. 5·2a. 6·2a. 7··i53a. /1··i. 8·2a. 9·2a. 10·2a. 11·2a. 12i··2a. 8,1i··2a. 2·2a. 3·2a. 4·2a. 5·2a. 6··i53a. /1··. 7·2a. 8··i2a. 9·2a. 10·2a. 11·2a. 12i··2a. 9,1i··2a. 2·2a. 3·2a. 4·2a. 5··i53a. /1··. 6·2a. 7·2a. 8·2a. 9··i2a. 10·2a. 11·2a. 12i··2a. 10,1i··2a. 2·2a. 3·2a. 4··i53a. /1··. 5·2a. 6·2a. 7·2a. 8·2a. 9·2a. 10··i2a. 11·2a. 12i··2a. 11,1i··2a. 2·2a. 3··i53a. /1··. 4·2a. 5·2a. 6·2a. 7·2a. 8·2a. 9·2a. 10·2a. 11··i2a. 12i··2a. 12,1ii·2a. 2·ii53a. /1·i·. 3·i·2a. 4·i·2a. 5·i·2a. 6·i·2a. 7·i·2a. 8·i·2a. 9·i·2a. 10·i·2a. 11·i·2a. 12iii2a.

Group Z_{14}. 1,1iii6f. 3·i·6f. 5·i·6f. 9·i·6f. 11·i·6f. 13ii·6f. 3,1i··6f. 3··i6f. 5··i55a. /1·2i. 9·6f. 11·6f. 13i··6f. 5,1i··6f. 3··i55a. /1·2i. 5··i6f. 9·6f. 11·6f. 13i··6f. 9,1i··6f. 3·6f. 5·6f. 9··i6f. 11·6f. 13i·i55a. /1i··2i. 11,1i··6f. 3·6f. 5·6f. 9·6f. 11··i55a. /1··i2i. 13i··6f. 13,1ii·6f. 3·i·6f. 5·i·6f. 9·ii55a. /1·i·2i. 11·i·6f. 13iii6f.

Group Z_{15}. 1,1iii6i. 2·i·6i. 4ii·6i. 7·i·6i. 8·i·6i. 11ii·6i. 13·i·6i. 14ii·6i. 2,1i··6i. 2··i29d. /1··i2e. 4i··47b. /1i··2h. 7·6i. 8·29d. /1·2c. 11i··29d. /1i··2d.

13·6i. 14i·i61a. /1i··. /3i··5g. /5i·i15b. 4,1ii·6i. 2·i·47b. /1·i·2h. 4iii6i. 7·ii47b. /1·i·2i. 8·i·6i. 11ii·6i. 13·i·6i. 14ii·6i. 7,1i··6i. 2·6i. 4i·i47b. /1i··2i. 7··i6i. 8·6i. 11i··6i. 13·6i. 14i··47b. /1i··2h. 8,1i··6i. 2·29d. /1·2c. 4i··6i. 7·6i. 8··i61a. /1··i. /3··i5g. /5··i15b. 11i··29d. /1i··2e. 13·47b. /1·2h. 14i··29d. /1i··2f. 11,1ii·6i. 2·i·29d. /1·i·2d. 4ii·6i. 7·i·6i. 8·i·29d. /1·i·2e. 11iii29d. /1iii2f. 13·i·6i. 14ii·29d. /1ii·2c. 13,1i··6i. 2·6i. 4i··6i. 7·6I. 8·47b. /1·2h. 11i··6i. 13··i47b. /1··i2i. 14i··6i. 14,1ii·6i. 2·ii61a. /1·i·. /3·i·5g. /5·ii15b. 4ii·6i. 7·i·47b. /1·i·2h. 8·i·29d. /1·i·2f. 11ii·29d. /1ii·2c. 13·i·6i. 14iii29d. /1iii2d.

Group $Z_2 \times Z_2$. 0110,0110iiii3b. 0111··i·4a. /01··i·.1001iii·3b. 1011·ii·4a. /10·ii·. 0111,0110·i··4a. /01·i··. 0111i··i2a. 1001ii··2a. 1110i··i9a. /01i···. 1001,0110iii·3b. 0111i·i·2a. 1001iiii10a.

Group $Z_4 \times Z_2$. 1001,1001iiii31a. 1021·ii·18b. 1101iii·17a. 1121i·i·7d. 3001iii·31a. 1021,1001iii·18b. 1021iiii18b. 1101·ii·7d. 1121··i·7d. 3001iii·18b. 3021iii·18b. 1101,1001iii·17a. 1021·ii·7d. 1101iiii17a. 1121··i·7d. 3001iii·17a. 3101iii·17a. 1121,1001ii··7d. 1021·i··7d. 1101·i··7d. 1121i··i7d. 3001ii··7d. 3121i···7d. 3001,1001iii·31a. 1021iii·18b. 1101iii·17a. 1121i·i·7d. 3001iiii31a.

Group $Z_6 \times Z_2$. 1001,1001iiii39a. 1031iii·16c. 1130i·i·6j. 2130iii·16d. 2131i·i·6j. 5001iii·39a. 1031,1001iii·16c. 1031iiii16c. 1101·ii·19e. /01·ii·2m. 1130··i·19b. /01··i·2m. 2130·ii·19b. /10·ii·2n. 2131··i·19e. /10··i·2n. 5001iii·16c. 5031iii·16c. 1130,1001ii··6j. 1031·i··19b. /01·i··2m. 1130i··i6j. 2130·i··19d. /10·i··2n. 2131i···38b. /01i···2m. 4131i··i38b. /01i···2m. 5001ii··6j. 5130i···6j. 2130,1001iii·16d. 1031·ii·19b. /10·ii·2n. 1130··i·19d. /10··i·2n. 2130iiii49b. /10iiii3S. 2131··i·50a. /01··i·5h. /10··i·4h. /11··i·. 4130iii·16d. 5001iii·49b. /10iii·3R. 5031·ii·50a. /10·ii·. /20·ii·4g. /30·ii·5h. 2131,1001ii··6j. 1031·i··19e. /10·i··2n. 1130i···38b. /01i···2m. 2130·i··50a.

/01·i··5h. /10·i··4h. /11·i··. 2131i··i29b. /10i··i2k. 4131i···6j. 5001ii··29b. /10ii··2l. 5130i··i58a. /01i···5h. /10i···. /20i··i9d. 5001,1001iii·39a. 1031iii·16c. 1130i·i·6j. 2130iii·49b. /10iii·3R. 2131i·i·29b. /10i·i·2l. 5001iiii59a. /10iiii10b.

Group $Z_3 \times Z_3$. 0110,0110iiii29c. /01iiii2g. 0111··i·2a. 0112··i·2a. 0120··i·2a. 0121··i·26a. /01··i·. 0122··i·3f. 0211··i·11b. /01··i·2f. 0212··i·3e. 0220iii·6h. 0221··i·5a. /01··I·. 0222··i·2a. 1001iii·6h. 1002·ii·2a. 1011··i·2a. 1012·ii·3e. 1021··i·5a. /10··i·. 1022·ii·11b. /10·ii·2g. 1102·ii·3f. 1112··i·2a. 1121··i·5a. /10··i·. 1202·ii·26b. /01·ii·. 1222··i·5b. /10··i·. 2002iii·29c. /01iii·2f. 2012··i·2a. 2022··i·2a. 2122··i·2a. 0111,0110·i··2a. 0111i··i2a. 0112·5d. /01·. 0120·5a. /01·. 0121·2a. 0122·2a. 0220·i··5a. /01·I··. 0221·2a. 0222i···2a. 1001ii··2a. 1012·i··2a. 1021·5d. /10·. 1112i···2a. 1120·5a. /10·. 1220i··i40a. /01i···. 2002ii··2a. 2110i···2a. 2221i···2a. 0112,0110·i··2a. 0111·5d. /01·. 0112i··i2a. 0120·2a. 0121·2a. 0122·5a. /01·. 0220·i··2a. 0221i···2a. 0222·5a. /01·. 1001ii··2a. 1011·2a. 1022·i··5a. /10·i··. 1110i···2a. 1120·5c. /10·. 1222i··i40a. /01i···. 2002ii··2a. 2111i···2a. 2220i···2a. 0120,0110·i··2a. 0111·5a. /01·. 0112·2a. 0120i··i2a. 0121·2a. 0122·5c. /01·. 0210i···2a. 0211·5a. /01·. 0212·2a. 1001ii··2a. 1012·i··5d. /10·I··. 1022·i··2a. 1112·5c. /10·. 1121i···2a. 1211i··i40a. /01i···. 2002ii··2a. 2122i···2a. 2212i···2a. 0121,0110·i··26a. /01·I··. 0111·2a. 0112·2a. 0120·2a. 0121i··i25b. /01i··i. 0122·2a. 0210·5a. /01·. 0211·2a. 0212i···3e. 0220·i··11a. /01·i··2c. 1001ii··3e. 1002·i··2a. 1011·5d. /10·. 1022·i··2a. 1110·5a. /10·. 1120·2a. 1210i··i52b. /01I···. /11i··i5f. 1211·5b. /10·. 2002ii·i25b. /01ii··. 2120i···3e. 0122,0110·i··3f. 0111·2a. 0112·5a. /01·. 0120·5c. /01·. 0121·2a. 0122i··i3f. 0210·2a. 0211i···3f. 0212·5a. /01·. 0220·i··3f. 1001ii··3f. 1002·i··2a. 1012·i··5a. /10·i··. 1021·2a. 1110·5c. /10·. 1120i···3f. 1121·2a. 1210·5b. /10·. 2002ii··3f. 2210i···3f. 1001,0110iii·6h.

0111i·i·2a. 0112i·i·2a. 0120i·i·2a. 0121i·i·3e. 0122i·i·3f. 1001iiii20b. 2002iii·20b. 2002,0110iii·29c. /01iii·2f. 0111i·i·2a. 0112i·i·2a. 0120i·i·2a. 0121i·ii25b. /01i·i·. 0122i·i·3f. 1001iii·20b. 2002iiii60a. /01iiii5e.

Group $Z_2 \times Z_2 \times Z_2$. 124,124iiii31c. 125··i·19a. /1··i·2p. 126··i·7e. 134··i·7h. 136··i·4b. /1··I·. 137··i·2a. 146··i·4e. /1··i·. 147··i·8c. /1··i·2v. 156··i·3j. 157··i·2a. 165··i·8a. /1··i·2p. 174iii·18d. 236··i·12d. /1··I·. 237··i·3g. 241·ii·19c. /1·ii·2v. 243··i·2a. 245··i·4a. /1··I·. 247·ii·7j. 256··i·2a. 263··i·12d. /2··I·. 265··i·4a. /2··I·. 273··i·4e. /2··I·. 276··i·4e. /1··I·. 326iii·17c. 351·ii·7k. 376i·i·7i. 421iii·31c. 125,124·i··19a. /1·i··2p. 125i··i6e. 126·12c. /1·. 127·3d. 134·8a. /1·2p. 135·3I. 136·2a. 137·4b. /1·. 142·4e. /1·. 143·4e. /1·. 146·4e. /1·. 147·4e. /1·. 152·2a. 153·2a. 156·2a. 157·2a. 162·4b. /1·. 163·2a. 164·8a. /1·2p. 172·2a. 173·4b. /1·. 174·i··8a. /1·i··2p. 214·4a. /2·. 217·2a. 234·4a. /1·. 237·4b. /1·. 241·i··2a. 243·4d. /1·. 247·i··4f. /1·i··. 251·4b. /1·. 253·4d. /1·. 254·4a. /1·. 261·2a. 263·4d. /2·. 267·2a. 271·4e. /1·. 273·9c. /1·. /2·. /3·. 274·9b. /1·. /2·. /3·. 314·4a. /2·. 316·2a. 324·12b. /1·. 326·i··12c. /1·i··. 346·4f. /1·. 354·4a. /1·. 364·9b. /1·. /2·. /3·. 376·2a. 413·4e. /2·. 421ii··6e. 423·i··3d. 431·i··3i. 453·2a. 463·4f. /2·. 516·3a. /4·. 524i··i38a. /1i···2p. 526·37a. /1·. /4·. /5·. 534·21a. /1·2p. /4·2p. /5·2p. 546·3a. /1·. 576·3a. /1·. 126,124·i··7e. 125·12c. /1·. 126i··i7e. 134·4b. /1·. 135·2a. 136·2a. 137·4a. /1·. 142·4e. /2·. 143·9c. /1·. /2·. /3·. 147·9b. /1·. /2·. /3·. 152·4b. /2·. 153·8b. /2·2q. 156·8d. /2·2q. 157·4a. /2·. 163·4d. /1·. 164·2a. 165·4f. /1·. 172·2a. 174·i··8d. /1·i··2r. 175·2a. 234·4b. /1·. 235·2a. 241·i··2a. 243·4b. /1·. 245·2a. 247·i··4a. /1·i··. 253·2a. 254·2a. 261·4b. /2·. 263·2a. 265·2a. 267·4a. /2·. 271·4e. /2·. 274·4e. /1·. 324i···7e. 325·12c. /1·. 341·4d. /1·. 345·4f. /1·. 354·8d. /1·2r. 361·9c. /1·. /2·. /3·. 367·9b. /1·. /2·. /3·. 421ii··7e. 423·i··7e. 425·i··12c. /4·i··. 523·12e. /1·. 623ii··7e. 136,124·i··4b. /1·i··. 125·2a. 126·2a.

127·4a. /1·. 136i··i2a. 137·4a. /1·. 142·4b. /2·. 143·4d. /2·. 147·4a. /2· 152·4e. /2·. 153·9c. /1·. /2·. /3·. 157·9b. /1·. /2·. /3·. 162·2a. 164·4f. /1·. 165·2a. 173·4d. /1·. 174·i··2a. 175·4f. /1·. 243·2a. 247·i··4a. /1·i··. 254·4f. /1·. 261·4e. /2·. 267·9b. /1·. /2·. /3·. 273i···2a. 345i···2a. 357·4a. /1·. 421ii··2a. 517i··i36a. /1i···. 652i···2a. 764i···2a. 137,124·i··2a. 125·4b. /1·. 126·4a. /1·. 127·2a. 136·4a. /1·. 137i··i2a. 142·4d. /2·. 143·4b. /2·. 146·4a. /2·. 152·9c. /1·. /2·. /3·. 153·4e. /2·. 156·9b. /1·. /2·. /3·. 163·2a 164·2a. 165·4f. /1·. 172·4d. /1·. 174·i··4f. /1·I··. 175·2a. 241·i··4d. /1·i··. 245·2a. 253i···2a. 256·4a. /1·. 265·4e. /1·. 276·4a. /2·. 364i···2a. 376·9b. /1·. /2·. /3·. 421ii··2a. 516i··i36a. /1I···. 672i···2a. 745i···2a. 421,124iii·31c. 125i·i·6e. 126i·i·7e. 136i·i·2a. 137i·i·2a. 421iiii56a.

Group D_3. 1001,1001iiiiii20a. /01i·ii··6k. 10i··i··3s. 1011,1001·ii·i·3e. /01··i···3p. /10·3s. /20·3m. 2001,1001·iiii·6b. /01··ii··22a. /10···i··3m. /11··ii··2c.

Group D_4. 1001,1001iiiiii41a. /01iiii··31d. /10iiii··18j. 1011i··i··7g. /01i·ii··7z. 1011,1001·ii·i·7g. /01··i···7s. /10·ii···7w. 1011·7g. /01··i···7m. /10·7w. 1021,1001iiiii·18c. /01iiii··18h. /10iiii·i18j. 1011i··i··7g. /01i·ii··7s. 3001,1001iiiii·31b. /01iiii·i31d. /10iiii··18f. /11iiii··18k. 1011···i··7g. /01··ii··7s. 3011,1001·iiii·7g. /01··ii··7z. /10·iii··7p. 1011···i··7g. /01··ii··7m. /11··ii··7z.

Group D_5. 1001,1001iiiiii33a. /01i·ii··6l. /10i··i··3D. 2001i··i··6d. /01i··i··6l. 1011,1001·ii·i·3h. /01··i···3q. /10·3D. /20·3z. /30·3t. /40·3n. 2001·3h. /01·3B. 2001,1001·ii·i·6d. /01··i···32b. /10·3t. /11··i···2o. 2001·6d. /01·6l. /10·3z. /11·3q. 3001,1001·ii·i·6d. /01··i···6l. /10·3z. /11··i···3F. 2001·6d. /01·32b. /10·3n. /11·2o. 4001,1001·iiii·6d. /01··ii··6l. /10···i··3n. /11··ii··3x. 2001···i··32a. /01···i··6l. /10···i··2n. /11···i··3F.

Group D_6. 1001,1001iiiii57a. /01i·ii··39c. /10i··i··27e. 1011i··i··13b. /01i·ii··16l. /20iiii··34c. 1011,1001·ii·i·13b. /01··i···13g. /10·13k. /20·13i. 1011·13b. /01··i···13d. /10·13k. /20·ii···13i. 1021,1001·ii·i·27a. /01··i···27c. /10·27e. /20·27d. 1011·13b. /01··i···13g. /10·13k. /20·ii···13i. 1031,1001iiiii·34b. /01i·ii··16j. /10i··i··13k. 1011i··i··13b. /01i·ii··16i. /20iiii·i34c. 5001,1001·iiii·39b. /01··ii··54a. /10···i··27b. /11··ii··10c. 1011···i··13b. /01··ii··3T. /11··ii··42a. /20·iii··16g. 5011,1001·iiii·16e. /01··ii··3T. /10···i··13f. /21··ii··42d. 1011···i··13b. /01··ii··42b. /11··ii··3T. /20·iii··16h.

Group D_7. 1001,1001iiiii44a. /01i·ii··6m. /10i··i··3L. 2001i··i··6g. /01i··i··6m. 3001i··i··6g. /01i··i··6m. 1011,1001·ii·i·3k. /01··i···3r. /10·3L. /20·3I. /30·3E. /40·3A. /50·3v. /60·3o. 2001·3k. /01·3G. 3001·3k. /01·3K. 2001,1001·ii·i·6g. /01··i···43b. /10·3A. /11··i···2x. 2001·6g. /01·6m. /10·3v. /11·3r. 3001·6g. /01·6m. /10·3I. /11·3G. 3001,1001·ii·i·6g. /01··i···6m. /10·3v. /11··i···3M. 2001·6g. /01·43b. /10·3I. /11·2x. 3001·6g. /01·6m. /10·3E. /11·3r. 4001,1001·ii·i·6g. /01··i···6m. /10·3I. /11··i···3C. 2001·6g. /01·6m. /10·3E. /11·3M. 3001·6g. /01·43b. /10·3o. /11·2x. 5001,1001·ii·i·6g. /01··i···6m. /10·3E. /11··i···3y. 2001·6g. /01·6m. /10·3o. /11·3C. 3001·43a. /01·6m. /10·2w. /11·3M. 6001,1001·iiii·6g. /01··ii··6m. /10···i··3o. /11··ii··3K. 2001···i··43a. /01···i··6m. /10···i··2w. /11···i··3y. 3001···i··6g. /01···i··6m. /10···i··3L. /11···i··3C.

Group Q_8. ij,ij··ii··7c. /i···i··7u. /k··ii··7y. ik··i··12a. /i···i··. ji·iiii·7c. /i··ii··7o. /k·iii··7q. jk···i··12a. /i··ii··12h. /j···i··. /k··ii··. ik,ij··i···12f. /i·. /j·12i. /k··i···. ik·3b. /i·3w. ji·ii·i·3b. /i··i···3l. kj·37b. /I·. i··j,ij··i···7c. /i·7l. /j·7v. ik·12f. /i·. ji·ii·i·7c. /i··i···7o. /k·ii···7y. jk·12f. /i··i···12g. /j·. /k··i···.

ik·12f. /i·. ji·ii·i·7c. /i··i···7o. /k·ii···7y. jk·12f. /i··i···12g. /j·. /k··i···. ji,iji·ii··7c. /ii··i··7v. iki··i··3b. jiiiiiii18a. /iiiii··18g. j-i,ij··ii··7c. /i···i··7v. ik···i··3b. jiiiiii·18a. /iiiii··18e. /jiiii·i18i. jki··i··7b. /ii·ii··7x.

Group A_4. 1321,1321iiiiii45a. /13i··i··6o. /21i·ii··17d. 1421i·ii··23a. /23i··i··7B. 1334,1321·ii·i·6k. /13·28a. /14·6n. /21··i···3J. /23·3H. /24·2u. 1421··i···14a. /21··i···2s. /23·2u. /24·8f. 1421,1321·iiii·23a. /13···i··14b. /21··ii··7r. /23···i··2u. /34··ii··7C. 1421··ii··51a. /13··ii··2j. /21··ii··24a. /23···I··. /34···i··35b. 2321,1321·ii·i·7f. /13·2t. /14·8e. /21··i···7r. /34··i···7C. 1421··i···35a. /13··i···. /23·. /34·24b. /43·46a. 2421,1321·iiii·17b. /13···i··3u. /21··ii··17d. /34··ii··17e. 1421··ii··7f. /14··ii··7t. /23···i··7A. /24···i··7n.

Group G_{12}. 1001,1001iiiiii34a. /01i··i··13e. /10i·ii··16k. 3001iiii··34a. /01i··i··13e. /10i·ii··16f. 1002,1001·iiii·16b. /01···i··13c. /10··ii··42e. /11··ii··3Q. 3001·iii··16b. /01···i··13c. /10··ii··42c. /11··ii··3O. 1101,1001·ii·i·13a. /01·13e. /02·13c. /10··i···13j. 3001·ii···13a. /01·13e. /02·13c. /10··i···13h. 3001,1001iiiii·34a. /01i··i··13e. /10i·ii··16f. 3001iiii·i34a. /01i··i··13e. /10i·ii··16k. 3002,1001·iiii·16b. /01···i··13c. /10··ii··42c. /11··ii··3O. 3001·iii··16b. /01···i··13c. /10··ii··42e. /11··ii··3Q. 3101,1001·ii·i·13a. /01·13e. /02·13c. /10··i···13h. 001·ii···13a. /01·13e. /02·13c. /10··i···13j.

Hence, we can sum up the number characteristics (up to isomorphism) of group isotopes up to the 15-th order in the following table. In the table it is denoted with V_{4m} the direct product of the group Z_{2m} by the group Z_2, with Z_2^3 the direct product of the group V_4 by the group Z_2, and with Z_3^2 the direct product of the group Z_3 by itself.

order	2	3	4		5	6		7	8				
Group	Z_2	Z_3	Z_4	V_4	Z_5	Z_6	D_3	Z_7	Z_8	V_8	Z_2^3	D_4	Q_8
a number of linear isotopes	1	5	4	15	19	5	11	41	16	28	341	28	47

9		10		11	12					13	14		15
Z_9	Z_3^2	Z_{10}	D_5	Z_{11}	Z_{12}	V_{12}	D_6	A_4	G_{12}	Z_{13}	Z_{14}	D_7	Z_{15}
48	183	19	37	109	20	75	44	43	44	155	41	79	95

Thus, there exist exactly 1554 linear isotopes of the group up to the 15-th order inclusively. One can conclude that there exist exactly 61 upper semilattices of subquasigroups of the linear isotopes up to isomorphism.

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Sharply 2-transitive permutation groups. 1

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Abstract

In this article a sharply 2-transitive permutation groups on some set E (finite or infinite) are studied.

Sharply 2-transitive permutation groups were described by Zassenhaus in [1,2]. He proved (see [3] too), for example, that sharply 2-transitive permutation group G on a finite set of symbols E is a group G^* of linear transformations of some near-field $<E,+,\bullet>$:

$$G^* = \{\alpha_{a,b} \mid \alpha_{a,b}(t) = a \cdot t + b,\ a \neq 0,\ a,b,t \in E\}.$$

In the case when the set E is infinite, the problem of description of sharply 2-transitive permutation groups on E is opened. Some investigations in this direction were pursued in [4,5,6,7]. The same problem was formulated by Mazurov in [8, № 11.52].

In this work we try to describe some new approach to problem mentioned above by means of transversals in groups. Necessary definitions and propositions may be found in [9] and in the author's article [13] in this issue too.

§1. Preliminary lemmas and a partition on cases

Let G be a sharply 2-transitive permutation group on an arbitrary set E.

Lemma 1. *All elements of order 2 from G are in one and the same class of conjugate elements.*

Proof was given in [3]. □

Since G is a sharply 2-transitive permutation group, then only the identity permutation **id** fixes more than one symbol from E. So we obtain the following two cases:

<u>**Case 1.**</u> *Every element of order 2 from G is a fixed-point-free permutation on E.*

<u>**Case 2.**</u> *Every element of order 2 from G has exactly one fixed point from E.*

Lemma 2. *Let α and β be distinct elements of order 2 from G. Then the permutation $\gamma = \alpha\beta$ is a fixed-point-free permutation on E.*

Proof was given in [3]. □

Let 0 and 1 be some distinguished distinct elements from E. Denote

$$H_0 = St_0(G).$$

§2. A loop transversal in group G and its properties

Lemma 3. *In both of cases 1 and 2 there exists a left transversal T in G to H_0, which consists from* **id** *and elements of order 2.*

Proof. By the definition (see [9,10] a complete system T of representatives of the left (right) cosets in G to H_0 is called a *left (right) transversal in G to H_0*.

If case 1 takes place, then we define the following set of permutations from G

$$T=\{t_j\}_{j\in E};$$

$$t_j=\begin{pmatrix}0 & j & \dots\\ j & 0 & \dots\end{pmatrix}, \text{ if } j\neq 0; \tag{1}$$

$$t_0=\mathrm{id}.$$

Then T is a left transversal in G to H_0 and for any $j\neq 1$

$$t_j^2=\begin{pmatrix}0 & j & \dots\\ 0 & j & \dots\end{pmatrix}=\mathrm{id},$$

since only the identity permutation **id** fixes more than one symbol from E. So all nonidentity elements from T have order 2.

Let the case 2 takes place. Note (see proof in [3]), that for any given $i_0\in E$ there exists an unique element $\alpha\in G$ of order 2 such that $\alpha(i_0)=i_0$. So there exists an unique element $\alpha_0\in G$ of order 2 such that $\alpha(0)=0$; moreover, $\alpha_0\in H_0$. Then define the following set

$$T=\{t_j\}_{j\in E};$$

$$t_j=\begin{pmatrix}0 & j & \dots\\ j & 0 & \dots\end{pmatrix}, \text{ if } j\neq 0; \tag{2}$$

$$t_0=\alpha_0.$$

Then T is a left transversal in G to H_0 and further proof is analogous to the same in case 1. □

Lemma 4. *Transversal T is a normal (invariant) subset in the group G.*

Proof. Let case 1 takes place. We have for any $j \in E$ and $\pi \in G$

$$\pi t_j \pi^{-1} = t_k h, \tag{3}$$

where $k \in E, h \in H_0$ (since T is a left transversal in G to H_0). If $j = 0$, then

$$\pi t_0 \pi^{-1} = \pi \cdot \mathrm{id} \cdot \pi^{-1} = \mathrm{id} = t_0.$$

If $j \neq 0$, then we have from (3)

$$h = t_k^{-1} \cdot (\pi t_j \pi^{-1}). \tag{4}$$

By means of **Lemma 2** we obtain: product in the right part of (4) has to be equal to **id**. Then we obtain

$$h = t_k^{-1} \cdot (\pi t_j \pi^{-1}) = \mathrm{id},$$

since $h \in H_0$. Then for any $j \in E$ and $\pi \in G$ we have from (3)

$$\pi t_j \pi^{-1} = t_k,$$

i.e.

$$\pi T \pi^{-1} \subseteq T.$$

From the last equality we have

$$T \subseteq \pi^{-1} T \pi = \pi' T \pi'^{-1},$$

where $\pi' = \pi^{-1} \in G$. So, for any $\pi \in G$ we have

$$T \subseteq \pi T \pi^{-1},$$

i.e.

$$T = \pi T \pi^{-1}.$$

Then T is a normal subset in G.

Proof in case 2 is analogous to that in case 1. □

Lemma 5. *Set T is:*

a loop transversal in G to H_0 in case 1;

a stable transversal [10] in G to H_0 in case 2.

Proof. As we can see from **Lemma 4**,

$$T = \pi T \pi^{-1}$$

for any $\pi \in G$. Then for any $\pi \in G$ the set $T^{\pi} = \pi T \pi^{-1} = T$ is a left transversal in G to H_0. So by means of [10, **theorem 2.1**] we obtain that T is a loop (correspondingly, stable) transversal in G to H_0. □

We can correctly introduce (see [9,10]) the following operation on the set E:

$$i \cdot j = k \overset{def}{\Leftrightarrow} t_i t_j = t_k h, \quad h \in H_0.$$

Then we obtain from **Lemma 5** (see [9,10] too) that the system $< E,\cdot,0 >$ is:

a loop with the identity element 0 in case 1;

a quasigroup with the right identity element 0 in case 2.

Lemma 6. *Let's define the following permutation representation $\hat{G}$ of a group G by the left cosets to H_0 with the help of a left transversal in G to H_0:*

$$\hat{g}(x) = y \overset{def}{\Leftrightarrow} g t_x H_0 = t_y H_0$$

Then we have

$$\hat{G} \cong G \quad and \quad \hat{g}(x) = g(x)$$

for any $x \in E$.

Proof. Let all conditions of the **Lemma** hold. Then we have

$$\hat{g}(u) = v,$$
$$gt_u H_0 = t_v H_0,$$
$$gt_u = t_v h, \quad h \in H_0,$$
$$gt_u(0) = t_v h(0) = t_v(0),$$
$$g(u) = v,$$

i.e. $\hat{g}(u) = g(u)$ for any $u \in E$. So the reflection $\varphi: \hat{g} \to g$ is an isomorphism between groups $\hat{G}$ and G. □

Lemma 7. *The following identities hold on* $<E,\cdot,0>$:

1. $x \cdot x = 0$;
2. $x \cdot (x \cdot y) = y$;
3. $x/y = y/x$;
4. $x \cdot (y \cdot (x \cdot z)) = (x \cdot (y \cdot x)) \cdot z$; *(left **Bol** identity)*
5. *System* $<E,\cdot,0>$ *is a left G-quasigroup.*

Proof. All definitions see in [11].

1. We have for any $x \in E$

$$t_0 = \mathrm{id} = t_x^2 = t_x t_x = t_{x \cdot x} h, \quad h \in H_0,$$

i.e. $x \cdot x = 0$.

2. We have for any $x, y \in E$

$$t_x t_y = t_{x \cdot y} h, \quad h \in H_0;$$

$$t_y = t_x^{-1} t_{x\cdot y} h = t_x t_{x\cdot y} h = t_{x\cdot(x\cdot y)} h', \quad h' \in H_0;$$

i.e. $x\cdot(x\cdot y) = y$.

3. We have

$$x\cdot(x\cdot y) = y.$$

Since system $<E,\cdot,0>$ is a quasigroup (in both of cases 1 and 2) then we can replace: $x = {}^z\!/_y$. Then we obtain for any $y, z \in E$

$$({}^z\!/_y)\cdot z = y;$$

$${}^z\!/_y = {}^y\!/_z;$$

4. Let us denote

$$h_{x,y} \overset{def}{=} t_{x\cdot y}^{-1} t_x t_y = t_{x\cdot y} t_x t_y.$$

Therefore

$$h_{x,y}^{-1} = (t_{x\cdot y} t_x t_y)^{-1} = t_y^{-1} t_x^{-1} t_{x\cdot y}^{-1} = t_y t_x t_{x\cdot y}.$$

Then we obtain by **Lemma 6** and [9]:

$$h_{x,y}(u) = t_{x\cdot y} t_x t_y(u) = (x\cdot y)\cdot(x\cdot(y\cdot u)); \tag{5}$$

$$h_{x,y}^{-1}(u) = t_y t_x t_{x\cdot y}(u) = y\cdot(x\cdot((x\cdot y)\cdot u)) \tag{6}$$

for any $u \in E$. From **Lemma 4** we obtain for any $x, y \in E$

$$t_x t_y t_x = t_x t_y t_x^{-1} = t_z,$$

where

$$z = t_z(0) = t_x t_y t_x(0) = x\cdot(y\cdot x)$$

Now we have

$$t_{x\cdot(y\cdot x)} = t_x t_y t_x = t_x t_{y\cdot x} h_{y,x} = t_{x\cdot(y\cdot x)} h_{x,y\cdot x} h_{y,x}.$$

So

$$h_{x,y\cdot x} = h_{y,x}^{-1}. \tag{7}$$

From (5)-(7) it follows that for any $x, y, u \in E$

$$(x\cdot(y\cdot x))\cdot(x\cdot((y\cdot x)\cdot u)) = x\cdot(y\cdot((y\cdot x)\cdot u)).$$

Since system $<E,\cdot,0>$ is a quasigroup (in both of cases 1 and 2) then we can replace: $w=(y\cdot x)\cdot u$. Then we obtain

$$(x\cdot(y\cdot x))\cdot(x\cdot w)=x\cdot(y\cdot w).$$

Finally, we can replace: $z=x\cdot w$. Then we have for any $x,y,z\in E$

$$(x\cdot(y\cdot x))\cdot z=x\cdot(y\cdot(x\cdot z),$$

since

$$x\cdot z=x\cdot(x\cdot w)=w,$$

(see 2.). We proved that the system $<E,\cdot,0>$ is:

left Bol loop in case 1;

left Bol quasigroup in case 2;

(see definitions in [11, 12]).

5. We have for any $a\in E$

$$t_x t_y=t_{x\cdot y}h,\quad h\in H_0;$$
$$t_a t_x t_y t_a^{-1}=t_a t_{x\cdot y}ht_a^{-1},\quad h\in H_0;$$
$$t_a t_x t_a^{-1}\cdot t_a t_y t_a^{-1}=t_a t_{x\cdot y}t_a^{-1}\cdot t_a h t_a^{-1},\quad h\in H_0,$$
$$t_{a\cdot(x\cdot a)}t_{a\cdot(y\cdot a)}t_a=t_{a\cdot((x\cdot y)\cdot a)}t_a h,\quad h\in H_0;$$
$$\varphi_a(x)\cdot R_a(\varphi_a(y))=R_a(\varphi_a(x\cdot y)),$$

where

$$\varphi_a(u)=a\cdot(u\cdot a) \qquad (*)$$

is a permutation on E. Then φ_a is a left pseudoautomorphism with the companion a. Moreover, any element $a\in E$ is a companion of the left pseudoautomorphism φ_a of the form (*); i.e. system $<E,\cdot,0>$ is a left G-quasigroup [11]. □

Let us return to the subgroup H_0 of the group G. Any nonidentity element $h\in H_0$ is a fixed-point-free permutation on the set $E-\{0\}$. Moreover, since G is a sharply 2-transitive permutation group on E, then H_0 is a sharply transitive permutation group on

$E-\{0\}$. So for a distinguished element $1 \in E\ (1 \neq 0)$ and for any $j \in E-\{0\}$ there exists an unique element $h_j \in H_0$ such that $h_j(1)=j$. Then we can define correctly the following operation on E:

$$\begin{aligned} i * j &\overset{def}{=} k \Leftrightarrow h_i(j)=k, \quad if\ i \neq 0; \\ 0 * j &\overset{def}{=} 0; \end{aligned} \tag{8}$$

Lemma 8. *The following statements are true:*

1. $x*0=0; \quad x*1=1*x=x;$
2. $<E-\{0\},*,1> \cong H_0;$
3. $x*(y \cdot z)=(x*y)\cdot(x*z);$
4. *The system* $<E,\cdot,0>$ *is a left special quasigroup.*

Proof. Nesessary definitions are in [11].

1. We have for any $x \in E-\{0\}$

$$x*0=u \Leftrightarrow u=h_x(0)=0 \Rightarrow x*0=0,$$
$$x*1=v \Leftrightarrow v=h_x(1)=x \Rightarrow x*1=x;$$
$$1*x=w \Leftrightarrow w=h_1(x);$$

But $h_1(1)=1$, since $h_1 \equiv \mathrm{id}$. So we obtain

$$w=h_1(x)=x,$$

i.e. $1*x=x$.

2. Let us define the following reflection

$$\varphi: E-\{0\} \to H_0,$$
$$\varphi(x) \overset{def}{=} h_x.$$

It is easy to see that φ is a bijection; moreover

$$\varphi(x)\varphi(y)=h_xh_y=h_z=\varphi(z),$$

where

$$z = h_z(1) = h_x h_y(1) = h_x(y) = x * y,$$

i.e.

$$\varphi(x)\varphi(y) = \varphi(x * y),$$

and φ is an isomorphism.

3. Since T is a normal subset in G (see **Lemma 4**) then for any $i \in E$ and $h_u \in H_0$ we obtain

$$h_u t_i h_u^{-1} = t_k,$$

where

$$k = t_k(0) = h_u t_i h_u^{-1}(0) = h_u t_i(0) = h_u(i) = u * i.$$

So, for any $i \in E$ and $u \in E - \{0\}$ we obtain

$$h_u t_i h_u^{-1} = t_{u*i}. \tag{9}$$

Then we have for any $u \in E - \{0\}$:

$$t_x t_y = t_{x \cdot y} h, \quad h \in H_0,$$

$$h_u t_x t_y h_u^{-1} = h_u t_{x \cdot y} h h_u^{-1}, \quad h \in H_0,$$

$$h_u t_x h_u^{-1} h_u t_y h_u^{-1} = h_u t_{x \cdot y} h_u^{-1} h_u h h_u^{-1}, \quad h \in H_0,$$

$$t_{u*x} t_{u*y} = t_{u*(x \cdot y)} h', \quad h' \in H_0,$$

$$(u * x) \cdot (u * y) = u * (x \cdot y). \tag{10}$$

Finally,

$$(0 * x) \cdot (0 * y) = 0 \cdot 0 = 0 = 0 * (x \cdot y).$$

4. We can write the equality (10) in the following form

$$h_u(x) \cdot h_u(y) = h_u(x \cdot y)$$

for any $u, x, y \in E$. So, any permutation $h_u \in H_0$ is an automorphism of the system $< E, \cdot, 0 >$. Then the permutation $h_{x,y}$ (see (2)) is an automorphism of the system $< E, \cdot, 0 >$ for any $x, y \in E$. Since

$$h_{x,y} \equiv L_{x \cdot y}^{-1} L_x L_y,$$

then system $< E, \cdot, 0 >$ is a left special quasigroup. $\square$

Lemma 9.

$$G = \{\alpha_{a,b} \mid \alpha_{a,b}(x) = a \cdot ((a \cdot b) * x),\ a \neq b,\ a,b \in E\}.$$

Proof. Since T is a left loop transversal (in case 1) or a stable transversal (in case 2) in G to H_0, then we can respresent any element $g \in G$ in the form

$$g = t_a h_c = t_a h_{a \cdot b},$$

where $t_a \in T$, $h_{a \cdot b} \in H_0$, $a \neq b$. So we obtain for any $x \in E$

$$g(x) = t_a h_{a \cdot b}(x) = t_a((a \cdot b) * x) = a \cdot ((a \cdot b) * x) \equiv \alpha_{a,b}(x);$$

moreover,

$$g(0) = a \cdot 0 = a, \qquad g(1) = a \cdot (a \cdot b) = b,$$

i.e.

$$g(x) = \alpha_{g(0),g(1)}(x). \tag{11}$$

□

Let us define the following two operations on E:

$$(x,a,y) \overset{def}{=} x \cdot ((x \cdot y) * a) = \alpha_{x,y}(a), \tag{12}$$

$$(x,\infty,y) \overset{def}{=} x \cdot y. \tag{13}$$

Definition. [11] Two operations $< E,\cdot >$ and $< E,\bullet >$ are called *orthogonal*, if the system

$$\begin{cases} x \cdot y = a, \\ x \bullet y = b \end{cases}$$

has an unique solution in $E \times E$ for any given $a,b \in E$.

Lemma 10. *The following statements are true:*

1). $$(x,0,y)=x;\quad (x,1,y)=y;\quad (x,t,x)=x;\quad (0,t,1)=t;\quad (x,\infty,0)=x;\quad (x,\infty,x)=0;$$

2). *The operations* (x,a,y) *and* (x,∞,y) *are orthogonal for any* $a\in E$;

3). *The operations* (x,a,y) *and* (x,b,y) *are orthogonal for any given* $a,b\in E, a\neq b$.

Proof. 1). We have

$$(x,0,y)=x\cdot 0=x;$$
$$(x,1,y)=x\cdot(x\cdot y)=y;$$
$$(x,\infty,0)=x\cdot 0=x;$$
$$(x,t,x)=x\cdot(0*t)=x\cdot 0=x;$$
$$(x,\infty,x)=x\cdot x=0;$$
$$(0,t,1)=0\cdot((0\cdot 1)*t)=\alpha_{0,1}(t).$$

But by means of (11) we obtain

$$(0,t,1)=\alpha_{0,1}(t)=\mathrm{id}(t)=t.$$

2). Let a be an arbitrary given element from E. Then we have for any given $b,c\in E$:

$$\begin{cases}(x,a,y)=b;\\(x,\infty,y)=c;\end{cases}\Leftrightarrow\begin{cases}x\cdot((x\cdot y)*a)=b;\\ x\cdot y=c;\end{cases}\Leftrightarrow$$
$$\Leftrightarrow\begin{cases}x\cdot(c*a)=b;\\ y=x\cdot c;\end{cases}\Leftrightarrow\begin{cases}x={}^{b}\!/_{(c*a)};\\ y=({}^{b}\!/_{(c*a)})\cdot c;\end{cases}$$

i.e. there exists an unique solution $(x,y)=({}^{b}\!/_{(c*a)},({}^{b}\!/_{(c*a)})\cdot c)$ of the initial system in $E\times E$.

3). Let $a,b\,(a\neq b)$ be an arbitrary given elements from E. Then we have for any given $c,d\in E$: if $c\neq d$, then

$$\begin{cases}(x,a,y)=c;\\(x,b,y)=d;\end{cases}\Leftrightarrow\begin{cases}x\cdot((x\cdot y)*a)=c;\\x\cdot((x\cdot y)*b)=d;\end{cases}\Leftrightarrow\begin{cases}\alpha_{x,y}(a)=c;\\\alpha_{x,y}(b)=d;\end{cases}$$

Since G is a sharply 2-transitive permutation group on E, then by **Lemma 9** we obtain that there exists an unique such pair $(x,y)\in E\times E$.

If $c=d$, then we have

$$\begin{cases}(x,a,y)=c;\\(x,b,y)=c;\end{cases}\Leftrightarrow\begin{cases}x\cdot((x\cdot y)*a)=c;\\x\cdot((x\cdot y)*b)=c;\end{cases}\Leftrightarrow\begin{cases}(x\cdot y)*a=x\cdot c;\\(x\cdot y)*b=x\cdot c;\end{cases}\Rightarrow$$

$$(x\cdot y)*a=(x\cdot y)*b$$

$$((x\cdot y)*a)\cdot((x\cdot y)*b)=0$$

$$(x\cdot y)*(a\cdot b)=0$$

Since $a\neq b$, then $a\cdot b\neq 0$. So we get

$$x\cdot y=0,$$

i.e. $x=y$. Then pair $(x,y)=(c,c)$ is an unique solution of the initial system. □

As we can see from **Lemma 10**, the system $<E,(x,t,y),0,1>$ is a DK-ternar [13] without the conditions 6a) and 6b) of **Definition 2** from [13], and the operation (x,∞,y) is a supplemented operation to it.

Lemma 11. *The following statements are true:*

1). *The operation* (x,a,y) *is a quasigroup for any given* $a\neq 0,1$;

2). $(x,(u,z,v),y)=((x,u,y),z,(x,v,y))$;

3). *The permutation* $\alpha_{a,b}$ *is an automorphism of the operation* (x,c,y) *for any given* $a,b,c\in E$, $a\neq b$; *i.e. any operation* (x,c,y) *admits the sharply 2-transitive automorphism group* G.

Proof. 1). It is proved in [13].

2). As we can see from **Lemma 9**,

$$G=\{\alpha_{a,b}|\alpha_{a,b}(t)=(a,t,b),, a\neq b, a,b\in E\}.$$

Then we have for any $x,y,u,v,z\in E, x\neq y, u\neq v$:

$$\alpha_{x,y}\cdot\alpha_{u,v}(z)=\alpha_{w,s}(z) \tag{14}$$

for some $w,s\in E, w\neq s$. We obtain from (14)

$$\alpha_{x,y}\cdot\alpha_{u,v}(z)=\alpha_{x,y}((u,z,v))=(x,(u,z,v),y);$$
$$w=\alpha_{w,s}(0)=\alpha_{x,y}\cdot\alpha_{u,v}(0)=\alpha_{x,y}(u)=(x,u,y);$$
$$s=\alpha_{w,s}(1)=\alpha_{x,y}\cdot\alpha_{u,v}(1)=\alpha_{x,y}(v)=(x,v,y),$$

So

$$(x,(u,z,v),y)=((x,u,y),z,(x,v,y)).$$

When $x=y$ or $u=v$ the last identity is a trivial corollary of 1), **Lemma 10**.

3). Let a,b,c be arbitrary given elements from E, $a\neq b$. Then we have from 2)

$$(a,(x,c,y),b)=((a,x,b),c,(a,y,b)),$$
$$\alpha_{a,b}((x,c,y))=(\alpha_{a,b}(x),c,\alpha_{a,b}(y)),$$

i.e. the permutation $\alpha_{a,b}$ is an automorphism of the operation (x,c,y). □

Lemma 12. *The operations* (x,a,y) *and* $(x,0,y)=y\cdot x$ *are orthogonal for any* $a\in E$.

Proof. We prove at first the following identity

$$((x\cdot y)\cdot x)*u^{-1}=((x\cdot u)\cdot z)*(u\cdot(x\cdot z))^{-1}, \tag{15}$$

for any $x,u,z\in E, u\neq 0, u\neq x\cdot z$. Really, we have from (5) for any $t\neq 0$

$$(x\cdot u)\cdot(x\cdot(u\cdot t))=h_{x,u}(t)=h_{h_{x,u}(1)}(t)=h_{x,u}(1)*t=((x\cdot u)\cdot(x\cdot(u\cdot 1)))*t,$$

$$(x\cdot u)\cdot(x\cdot(u\cdot 1))=((x\cdot u)\cdot(x\cdot(u\cdot t)))*t^{-1}. \tag{16}$$

If $t=u$, then we have from (16)

$$(x\cdot u)\cdot(x\cdot(u\cdot 1))=((x\cdot u)\cdot x)*u^{-1}. \tag{17}$$

If $t=u\cdot(x\cdot z)$, then we obtain from (16)

$$(x\cdot u)\cdot(x\cdot(u\cdot 1))=((x\cdot u)\cdot z)*(u\cdot(x\cdot z))^{-1}. \tag{18}$$

The identity (15) follows from (17)-(18).

Further on, we have for any given $a,b,c\in E$

a). If $c=0$, then

$$\begin{cases}(x,a,y)=b;\\(x,\Diamond,y)=0;\end{cases}\Leftrightarrow\begin{cases}x\cdot((x\cdot y)*a)=b;\\ y\cdot x=0;\end{cases}\Leftrightarrow x=y=b,$$

i.e. the pair $(x,y)=(b,b)$ is an unique solution of the initial system.

b). If $a=0$ then

$$\begin{cases}(x,a,y)=b;\\(x,\Diamond,y)=c;\end{cases}\Leftrightarrow\begin{cases}x=b;\\ y\cdot x=c;\end{cases}\Leftrightarrow(x,y)=(b,c/b).$$

c). Let $a\neq 0, c\neq 0$. Then

$$\begin{cases}(x,a,y)=b;\\(x,\Diamond,y)=c;\end{cases}\Leftrightarrow\begin{cases}x\cdot((x\cdot y)*a)=b;\\ y\cdot x=c;\end{cases}\Leftrightarrow\begin{cases}x\cdot((x\cdot y)*a)=b;\\ x=y\cdot c;\end{cases}\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}(y\cdot c)\cdot(((y\cdot c)\cdot y)*a)=b;\\ x=y\cdot c;\end{cases}\Leftrightarrow\begin{cases}((y\cdot c)\cdot y)*a=(y\cdot c)\cdot b;\\ x=y\cdot c;\end{cases}$$

Let us denote: $z=c^{-1}*y$, $b'=c^{-1}*b$. Then the last system is equivalent to the following system

$$\begin{cases}(z\cdot 1)\cdot z=((z\cdot 1)\cdot b')*a^{-1};\\ x=(c*z)\cdot c;\end{cases} \tag{19}$$

If $u=1$ then we obtain from (15)

$$(x\cdot 1)\cdot x=((x\cdot 1)\cdot z)*(1\cdot(x\cdot z))^{-1}, \tag{20}$$

for any $x, z \in E, x \cdot z \neq 1$. Using (20) in (19), we obtain the following system

$$\begin{cases} ((z \cdot 1) \cdot b') * (1 \cdot (z \cdot b')) = ((z \cdot 1) \cdot b') * a \;; \\ x = (c * z) \cdot c; \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \left[\begin{array}{l} 1 \cdot (z \cdot b') = a; \\ (z \cdot 1) \cdot b' = 0; \end{array}\right. \\ x = (c * z) \cdot c; \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} (c \;\; * y) \cdot (c \;\; * b) = 1 \cdot a, \\ (c^{-1} * y) \cdot 1 = c^{-1} * b; \end{array}\right. \\ x = y \cdot c; \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \left[\begin{array}{l} y \cdot b = c * (1 \cdot a); \\ y \cdot c = b; \end{array}\right. \\ x = y \cdot c; \end{cases} \Leftrightarrow \left[\begin{array}{l} \begin{cases} x = \left({}^{(c*(1 \cdot a))}\!/\!_{b} \right) \cdot c, \\ y = {}^{(c*(1 \cdot a))}\!/\!_{b}; \end{cases} \\ \begin{cases} x = b; \\ y = {}^{b}\!/\!_{c}; \end{cases} \end{array}\right.$$

Assume that $(x, y) = (b, {}^{b}\!/\!_{c})$, then we obtain from the initial system:

$$b \cdot ((b \cdot ({}^{b}\!/\!_{c})) * a) = b,$$
$$(b \cdot ({}^{b}\!/\!_{c})) * a = 0.$$

Since $a \neq 0$, then

$$b \cdot ({}^{b}\!/\!_{c}) = 0,$$
$${}^{b}\!/\!_{c} = b,$$
$$b = c \cdot b,$$
$$c = 0.$$

But $c \neq 0$ by the conditions of the case c). Then we obtain: the pair $(x, y) = \left(\left({}^{(c*(1 \cdot a))}\!/\!_{b} \right) \cdot c, {}^{(c*(1 \cdot a))}\!/\!_{b} \right)$ is an unique solution of the initial system in the case c). Proof is completed. □

Remark. Note that the collection P of operations $(x, a, y), a \in E$ and (x, ∞, y) (or $(x, \Diamond, y)$) is a complete system of orthogonal operations, i.e. there is no such an operation $x \otimes y$ which is orthogonal to all operations from P.

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On superassociative group isotopes

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Abstract

It is proved that every Menger quasigroup (grouplike Menger algebra) is an Ω-algebra. Relations between such algebraic notions as: homomorphism, subquasigroup, congruence relations and so on of a Menger quasigroup and the corresponding notions of its decomposition algebra are found. A criterion for the group isotope to be a superassociative is established. Some main algebraic notions in superassociative group isotopes are considered.

One of the well known generalization of the binary associativity is the superassociativity. It is an abstract characteristic of the class of all $(n+1)$-ary Menger algebras of n-ary transformations of a set. When $n=1$, a Menger algebra is a semigroup of transformations and the superassociativity is the binary associativity (see [1]). In this connection works appear, where algebraic structure of Menger algebras and grouplike Menger algebras (i.e. superassociative quasigroups) are present. For example, the works of V.S. Trokhimenko [2], W.A.Dudek [3], Ya.N. Yaroker [5], H.L. Skala [6] and so on.

The main purpose of the article is to begin the study of superassotiative group isotopes, but it is necessary to consider some modification of the fundamental results of Menger algebras. We do it in sections **1** and **2**. The principal results of section 3 are: a canonical decomposition of an arbitrary group isotope and its uniqueness (**Lemma 3.1**); conditions for a group isotope to be superassociative and linear (**Theorem 3.2** and **Corollary 3.3**); conditions for a transformation of the basic set to be a homomorphism or an isomorphism of superassociative group isotopes of the same group (**Theorem 3.6**) as well as an endomorphism or an automorphism of a superassociative group isotope of a group (**Corollary 3.7**); criteria for a subset to be a subquasigroup and a normal subquasigroup of a superassociative group isotope (**Theorem 3.8**).

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1. General notes

A groupoid $(Q;f)$ of an arity $n+1$ is called *superassociative* or *Menger algebra of rang n*, if the *superassociative law* holds:

$$f(f(x,y_1,\dots,y_n),z_1,\dots,z_n)=f(x,f(y_1,z_1,\dots,z_n),\dots,f(y_n,z_1,\dots,z_n)) \quad (1)$$

Let $(Q;f)$ be a Menger algebra. A binary groupoid $(Q;\cdot)$, defined by

$$x\cdot y=f(x,y,\dots,y) \quad (2)$$

is associative. It follows from the equality (1), when $y_1=y_2=\dots=y_n=y$ and $z_1=z_2=\dots=z_n=z$. $(Q;\cdot)$ is called a *diagonal semigroup* of $(Q;f)$.

To expound the text we firstly recall (see [8]), that *i-th shift defined by a* of the *(n+1)*-ary groupoid $(Q;f)$ is a transformation $\lambda_{i,a}$ defined by

$$\lambda_{i,a}(x) = f(\underbrace{a,\ldots,a}_{i\ \text{times}},x,a,\ldots,a)$$

"Shift" is understood as a translation where all elements defining it are equel.

If the *i*-th shift defined by an element *a* is a substitution of the set *Q*, then the element *a* is called *i-invertible*, and if $\lambda_{i,a}(x)$ is an identity transformation, then it is called *i-unit* element of the groupoid $(Q;f)$; 0-invertible (0-unit) and *n*-invertible (*n*-unit) elements also will be called *right invertible (right unit) and left invertible (left unit)* respectively.

Lemma 1.1. *If a is a left (right) invertible element in a binary semigroup* $(Q;\cdot)$, *then the element* $e_l = \lambda^{-1}_{0,a}(a)$ $(e_r = \lambda^{-1}_{1,a}(a))$ *is its left (right) unit and the element* $a_l^{-1} = \lambda^{-2}_{0,a}(a)$ $(a_r^{-1} = \lambda^{-2}_{1,a}(a))$ *is a left (right) inverse of a.*

Proof. We shall prove the lemma for the "right" case only, since the proof of the "left" case is dual. The assotiative law implies the following equalities

$$\lambda_{0,a}(x\cdot y) = (x\cdot y)\cdot a = x\cdot(y\cdot a) = x\cdot\lambda_{0,a}(y)$$

Replacing x with $\lambda^{-1}_{0,a}(x)$ and applying $\lambda^{-1}_{0,a}(x)$ to the both side of this equality we have

$$\lambda^{-1}_{0,a}(x\cdot y) = x\cdot\lambda^{-1}_{0,a}(y).$$

Using this equality we can infer the following equalities

$$x \cdot e = x \cdot \lambda_{0,a}^{-1}(a) = \lambda_{0,a}^{-1}(x \cdot a) = \lambda_{0,a}^{-1}\lambda_{0,a}(x) = x$$

$$a \cdot a^{-1} = a \cdot \lambda_{0,a}^{-2}(a) = \lambda_{0,a}^{-1}(a \cdot \lambda_{0,a}^{-1}(a)) = \lambda_{0,a}^{-1}(a \cdot e) = \lambda_{0,a}^{-1}(a) = e$$

The lemma is proved.

Thus, the following assertion is evident.

Corollary 1.2. *A binary semigroup has a left (right) unit iff it has a left (right) invertible element. A binary semigroup has a unit iff it has an invertible element.*

It is easy to see that the right shift of a superassociative groupoid will be a right shift of its diagonal semigroup. The same is true for the right invertible and right unit elements. This permits to establish the truth of the following statement.

Corollary 1.3. *A superassociative groupoid has a right unit if and only if it has a right invertible element.*

We define n-ary operation $[...]$ on the set Q by

$$[x_1,\ldots,x_n] \overset{def}{=} f(e,x_1,\ldots,x_n), \qquad (3)$$

where e is a right unit of the superassociative groupoid $(Q;f)$. Using this relation it is easy to prove the following statement, which is a generalization of **Theorem 3.8** from [4].

Theorem 1.4. *If an $(n+1)$-ary superassociative groupoid $(Q;f)$ has at least one right invertible element a, then its diagonal semigroup operation $(\cdot)$ defined by (2), the operation $[...]$ defined by (3) with $e=\lambda_{0,a}^{-1}(a)$, and the operation f are connected by the following relations:*

$$f(x, z_1, \ldots, z_n) = x \cdot [z_1, \ldots, z_n], \tag{4}$$

$$[y_1, \ldots, y_n] \cdot z = [y_1 \cdot z, \ldots, y_n \cdot z]. \tag{5}$$

And conversely, if an associative operation $(\cdot)$ *is right distributive under some* n*-ary operation* $[\ldots]$, *then the operation* f *defined by (4) will be superassociative.*

In this case $(Q;\cdot,[\ldots])$ will be called a *decomposition algebra* of the superassociative groupoid $(Q;f)$.

Proof. By **Lemma 1.3**, the element e is a right unit of the diagonal semigroup, so the equalities (4) and (5) follow from the equalities (1) with $y_1 = y_2 = \ldots = y_n = e$ and $x = e$, $z_1 = z_2 = \ldots = z_n = z$ respectively. The converse statement is a partial case of **Lemma 3.7** from [4]. □

For example, if $(Q;+,\cdot)$ is a ring, then the ternary groupoid $(Q;f)$ defined by

$$f(x,y,z) = x \cdot (y+z)$$

is a Menger algebra. Some other examples can be found in [4].

This theorem implies immediately the following result.

Corollary 1.5. *Let a superassociative groupoid has a right invertible element* a, *then its diagonal semigroup has a left unit if and only if the operation* $[\ldots]$ *defined by the equality (3) with* $e = \lambda_{0,a}^{-1}(a)$ *is idempotent.*

Proof. Let the diagonal semigroup of the groupoid $(Q;f)$ has left unit, then it coincides with a right unit e of the semigroup. According to (2), the equality $e \cdot x = x$ is equivalent to the equality $f(e,x,\dots x)=x$, which, in turn, is equivalent to the equality $[x,\dots,x]=x$. The corollary is proved. □

Corollary 1.6. [7] *A superassociative groupoid* $(Q;f)$ *of the arity* $n+1$ *is a quasigroup if and only if there exist a group* $(Q;\cdot)$ *and an idempotent quasigroup* $(Q;[\dots])$ *such that the relations (4), (5) hold.*

Proof. In the quasigroup $(Q;f)$ every element is right invertible and, hence, according to **Theorem 1.4** the relations (4), (5) hold. Since the operation $[\dots]$ is defined by the equality (3), then it is a quasigroup as well. It remains to use the following statement, which is a corollary of **Theorem 1** from [11]. □

Proposition 1.7. *If one of functions* f, g, h *is a repetition-free superposition of two others and two of them are quasigroups, then the third one will be a quasigroup as well.*

A superassociative quasigroup is called *grouplike Menger algebra*. The following assertion is a corollary of **Lemma 1.1** and **Theorem 1.4**.

Corollary 1.8. *The diagonal semigroup of a superassociative groupoid is a group if and only if every element of the groupoid is right invertible and the diagonal semigroup has a left invertible element.*

To prove this corollary note that a semigroup is a group if it has a unit and every element has a right inverse.

Ya.N. Yaroker in [5] has found another criterion: a diagonal semigroup is a group if and only if the Menger algebra has no proper s- and v-ideal, that is iff the diagonal semigroup has no proper left and right ideals.

Corollary 1.9. *If in a superassociative groupoid the diagonal semigroup is a group, then its decomposition algebra is an Ω-group.*

The truth of the corollary follows directly from **Corollary 1.8** and the definition of an Ω-group: an algebra $(Q,+,\Omega)$ is called an *Ω-group*, if $(Q;+)$ is a group and $h(0,\dots,0)=0$ for all operations $h\in\Omega$.

A ring is a Ω-group as well, while the operation from $\Omega=\{\cdot\}$ is distributive under the group operation of the ring, but in the superassociative groupoid the situation is quite the reverse. From **Corollary 1.9** and the conclusions from [9] one can get a number of results for such Ω-groups.

2. On some algebraic notions

Let us consider a connection between the algebraic notions of a Menger quasigroup and its decomposition algebra.

Theorem 2.1. *A subset of a grouplike Menger algebra is a subquasigroup of it if and only if it is a subquasigroup of its decomposition algebra.*

Proof. Let $(Q;f)$ be a grouplike Menger algebra of an arity $n+1$ and let $(Q,\cdot,[\ldots])$ be its decomposition algebra.

If H is a subquasigroup of $(Q;f)$, then for arbitrary elements a, b from H $a\cdot b=f(a,b,\ldots b)\in H$ and $a^{-1}\in H$, since it is a solution of the equation

$$b=x\cdot a=f(x,a,\ldots,a).$$

So, the set H is a subgroup of the diagonal group $(Q;\cdot)$. In particular, it means that the unit e of the group is in the set H, and then for arbitrary elements $a_1,\ldots,a_n,b$ from the set H the results $[a_1,\ldots,a_n]=f(e,a_1,\ldots,a_n)\in H$ and a solution of the equation

$$b=[a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_n]=f(e,a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_n)$$

belongs to the set H. Thus, the set H is a subquasigroup of the algebra $(Q,\cdot,[\ldots])$.

Conversly, if a subset H of the set Q is a subquasigroup of the quasigroup algebra $(Q,\cdot,[\ldots])$, then the equality (4) implies that the subset H is closed under the operation f. The solution of the equation

$$b=f(x,a_1,\ldots,a_n)=x\cdot[a_1,\ldots,a_n]$$

belongs to the set H since H is a subgroup of the diagonal group. The equation

$$f(a_0,a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_n)=b$$

one can rewrite as

$$[a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_n]=a_0^{-1}\cdot b,$$

then an element x exists, it is unique and belongs to H as soon as the elements $a_0,a_1,\ldots,a_n,b$ belong to the set H. The theorem is proved. □

Theorem 2.2. *A mapping* φ *from one grouplike algebra onto the other will be homomorphic (isomorphic) if and only if* φ *is a homomorphic (isomorphic) mapping between the corresponding decomposition algebras.*

Proof. Let $(Q;f)$, (G,h) be grouplike algebras and let $(Q,\cdot,[\ldots])$ and $(G,+,[\ldots])$ be their decomposition algebras, then

$$\varphi(x\cdot y)=\varphi f(x,y,\ldots y)=h(\varphi x,\varphi y,\ldots,\varphi y)=\varphi x+\varphi y;$$
$$\varphi([x_1,\ldots,x_n])=\varphi f(e,x_1,\ldots,x_n)=h(\varphi e,\varphi x_1,\ldots,\varphi x_n)=$$
$$=h(0,\varphi x_1,\ldots,\varphi x_n)=[\varphi x_1,\ldots,\varphi x_n].$$

The converse is evident.

Corollary 2.3. *Any endomorphism (automorphism) of a grouplike algebra is an endomorphism (automorphism) of the corresponding decomposition algebra and vice versa*

Recall, a congruence of a quasigroup is called *normal*, if the corresponding quotient-groupoid is a quasigroup also; a subquasigroup is called *normal*, if it is a class by a normal congruence.

In a group every congruence is normal and exactly one of its classes is a normal subgroup, namely, the class containing the unit element of the group. In a quasigroup this is not so. There exist infinite quasigroups having non normal congruences and there exist quasigroups having congruence with no subquasigroup as its class. In an idempotent quasigroup every class by a normal congruence is a subquasigroup. There is the same situation in the theory of n-ary groups and polygroups as in the theory of quasigroups. The following resoning shows, that in a grouplike Menger algebra the uniqueness is the same as in the group theory. Immediatly from **Theorem 2.2** we have the truth of the following statements.

Corollary 2.4. *Any normal congruence of a grouplike algebra is a normal congruence of the corresponding decomposition algebra and vice versa.*

Corollary 2.5. *Exactly one of congruence classes of a grouplike algebra is a subquasigroup, namely, the class containing the unit element of the diagonal group.*

Corollary 2.6. *Every normal congruence of a Menger quasigroup is an equivalence relation corresponding to a partition by the normal subgroups of the diagonal group, which are normal subquasigroups of the decomposition algebra.*

A full description of all congruences (including one-side congruences) in grouplike Menger algebras was obtained by V.S.Trokhimenko in [2].

3. Superassociative group isotopes

In this section superassociative group isotopes are under consideration.

A group isotope or an *isotope of a group* $(G;\bullet)$ of the arity $n+1$ is a groupoid $(Q;f)$ defined by

$$f(x_0,x_1,\dots,x_n)=\gamma^{-1}(\gamma_0x_0\bullet\gamma_1x_1\bullet\dots\bullet\gamma_nx_n),\qquad(6)$$

where $\gamma_0,\ldots,\gamma_n,\gamma$ are bijections between the sets Q and G. If all of the bijections are linear transformations of the group $(G;\circ)$ (α is *linear*, iff $\alpha x=\theta x+a$ for some $a\in G$ and an automorphism θ of the group $(G;\circ)$), then the isotope $(G;f)$ is called *linear*. It is easy to prove that a groupoid being isomorphic to a linear group isotope is a linear group isotope as well. The following statement is true.

Lemma 3.1. *Let $(Q;f)$ be an $(n+1)$-ary isotope of a group, then for any element e of the set Q there exists exactly one sequence of operations $(\cdot,\alpha_0,\ldots,\alpha_n,a)$ such that $(Q;\cdot)$ is a group with a unit element e; $\alpha_0,\ldots,\alpha_n$ are unitary substitutions of the set Q, i.e. $\alpha_i(e)=e$, $i=0,1,\ldots,n$; $a\in Q$ and the following equality*

$$f(x_0,x_1,\ldots,x_n)=\alpha_0x_0\cdot\alpha_1x_1\cdot\ldots\cdot\alpha_nx_n\cdot a \tag{7}$$

holds. The isotope $(Q;f)$ is linear iff $\alpha_0,\ldots,\alpha_n$ are aumorphisms of $(Q;\cdot)$.

In this case, let us use the following terminology: a decomposition (7) will be called *e-canonical*, permutations $\alpha_0,\ldots,\alpha_n$ will be called *decomposition coefficients*, and the element a will be called a *free member.*

Proof. Let (6) holds. On the set Q we define a group operation $(+)$:

$$x+y=\gamma^{-1}(\gamma x\bullet\gamma y)$$

Then we rewrite the equality (6) as follows:

$$f(x_0,\ldots,x_n)=\gamma^{-1}\gamma_0x_0+\gamma^{-1}\gamma_1x_1+\ldots+\gamma^{-1}\gamma_nx_n).$$

Replacing $(+)$ with $(\cdot)$, where $x\cdot y=x-e+y$, i.e. $x+y=(x+e)\cdot y$, and $\gamma^{-1}\gamma x_n$ by α_n, $\gamma^{-1}\gamma x_iR_e$ by α_i, $i=0,1,\ldots,n-1$, we have

$$f(x_0,x_1,\ldots,x_n)=\alpha_0x_0\cdot\alpha_1x_1\cdot\ldots\cdot\alpha_nx_n. \tag{8}$$

It is clear, the operations $(\cdot)$ and $(+)$ are isomorphic and the unit element in $(Q;\cdot)$ is e. Suppose, that $\alpha_0,\alpha_1,\ldots,\alpha_{i-1}$ are unitary and $\alpha_ie\neq e$, $i=0,1,\ldots,n$, then we consider the following notation

$$\alpha'_i\,x_i=\alpha_ix_i\cdot(\alpha_ie)^{-1},\qquad \alpha'_{i+1}\,x_{i+1}=(\alpha_ie)\cdot(\alpha_{i+1}x_{i+1}).$$

As a result we have the relation

$$f(x_0,x_1,\ldots,x_n)=\alpha_0x_0\cdot\ldots\cdot\alpha_{i-1}x_{i-1}\cdot\alpha'_i\,x_i\cdot\alpha'_{i+1}\,x_{i+1}\cdot\alpha_{i+2}x_{i+2}\cdot\ldots\cdot\alpha_nx_n,$$

where the first substitutions $\alpha_0,\ldots,\alpha_{i-1},\alpha'_i$ are unitary. After the finite number of the steps we obtain the equality (8), where the substitutions $\alpha_0,\ldots,\alpha_{n-1}$ are unitary. Consider a new notation:

$$\alpha'_n\,x_n=\alpha_nx_n\cdot(\alpha_ne)^{-1},\qquad \text{and}\quad a=\alpha_ne$$

Thus, we obtain the equality (7). To prove the uniqueness we assume that the decomposition

$$f(x_0,x_1,\ldots,x_n)=\beta_0x_0\oplus\beta_1x_1\oplus\ldots\oplus\beta_nx_n\oplus b$$

is e-canonical as well, i.e. $e\oplus x=x\oplus e=x$, $\beta_0e=\beta_1e=\ldots=\beta_ne=e$. Then $a=f(e,\ldots,e)=b$ and

$$\alpha_nx_n\cdot a=f(e,\ldots,e,x_n)=\beta_nx_n\oplus b.$$

So, we obtain

$$\alpha_0x_0\cdot\alpha_nx_n\cdot a=f(x_0,e,\ldots,e,x_n)=\beta_0x_0\oplus\beta_nx_n\oplus b=\beta_0x_0\oplus(\alpha_nx_n\cdot a)$$

for all $x,y\in Q$. Replacing $\alpha_nx_n\cdot a$ with y we have

$$\alpha_0x_0\cdot y=\beta_0x_0\oplus y.$$

Since the element e is a common unit of the operations $(\cdot)$ and $(\oplus)$, then from the last equality and $y=e$ we have $\alpha_0=\beta_0$, and so the operations $(\cdot)$ and $(\oplus)$ coincide. Hence,

$$\alpha_ix\cdot a\cdot a^{-1}=f(e,\ldots,x,e,\ldots,e)\cdot a^{-1}=(\beta_ix\oplus b)\cdot a^{-1}=\beta_ix\cdot a\cdot a^{-1}=\beta_ix$$

for all $x\in Q$, then $\alpha_0=\beta_0$ for all $i=0,1,\ldots,n$. **Lemma** is proved.

Using this lemma we may establish the truth of the following

Theorem 3.2. *A group isotope* (Q,f) *is superassociative if and only if*

$$f(x_0,x_1,\ldots,x_n)=x_0\cdot\alpha_1x_1\cdot\ldots\cdot\alpha_nx_n \tag{9}$$

for some unitary substitutions $\alpha_1,\ldots,\alpha_n$ *of the group* $(Q,\cdot)$, *which satisfy the following conditions:*

$$\alpha_1y\cdot\alpha_2y\cdot\ldots\cdot\alpha_ny=y, \tag{10}$$

$$\alpha_1y\cdot\ldots\cdot\alpha_{i-1}y\cdot\alpha_i(x\cdot y)=\alpha_ix\cdot\alpha_1y\cdot\ldots\cdot\alpha_iy,\quad i=1,\ldots,n. \tag{11}$$

Proof. Let a superassociative quasigroup (Q,f) of arity $n+1$ be an isotope of some group and let (7) be its e-canonical decomposition, where e is a unit of its diagonal group. From the uniqueness of the e-canonical decomposition of the group isotope $(Q;+)$, from the equalities (4), (7) and the idempotence of the operation [...] the relations $\alpha_0=e$, $(\bullet)=(\cdot)$ and

$$[y_1,\ldots,y_n]=\alpha_1y_1\cdot\alpha_2y_2\cdot\ldots\cdot\alpha_ny_n, \tag{12}$$

follow, since

$$e\overset{Corol.1.5}{=}[e,\ldots,e]\overset{(3)}{=}f(e,e,\ldots,e)\overset{Lemmma\,3.1}{=}a.$$

The idempotence of [...] is equivalent to the identity (10). Thus, the distributivity relation (5) will be rewritten as follows:

$$\alpha_1x_1\cdot\alpha_2x_2\cdot\ldots\cdot\alpha_nx_n\cdot y=\alpha_1(x_1\cdot y)\cdot\alpha_2(x_2\cdot y)\cdot\ldots\cdot\alpha_n(x_n\cdot y).$$

Setting $x_j=e$ for all $j\neq i$ in the last equality we obtain

$$\alpha_ix\cdot y=\alpha_1(y)\cdot\ldots\cdot\alpha_{i-1}(y)\cdot\alpha_i(x\cdot y)\cdot\alpha_{i+1}(y)\cdot\ldots\cdot\alpha_n(y)$$

In the left side of it we replace y with its value from the equality (10), and then we cancel it out of $\alpha_{i+1}y\cdot\ldots\cdot\alpha_ny$. As a result we obtain the relation (11).

Converse, let the operation f on the set Q be defined by the equality (9) by such unitary substitutions $\alpha_0,\dots,\alpha_n$ of the group $(Q,\cdot)$, that the relations (10) and (11) hold. To prove the superassociativity of the operation f we defined an operation $[\dots]$ by (12). Hence, we obtain the following equalities:

$$
\begin{aligned}
&[x_1,\dots,x_n]\cdot y = \\
&\overset{(10)}{=} \alpha_1 x_1\cdot\alpha_2 x_2\cdot\ldots\alpha_n x_n\cdot y \overset{(11)}{=} \\
&= \alpha_1 x_1\cdot\alpha_2 x_2\cdot\ldots\alpha_n x_n\cdot\alpha_1 y\cdot\alpha_2 y\cdot\ldots\alpha_n y \overset{(11)}{=} \\
&= \alpha_1 x_1\cdot\alpha_2 x_2\cdot\ldots\alpha_{n-1} x_{n-1}\cdot\alpha_1 y\cdot\alpha_2 y\cdot\ldots\alpha_{n-1} y\cdot\alpha_n(x_n\cdot y) \overset{(11)}{=} \\
&= \alpha_1 x_1\cdot\alpha_2 x_2\cdot\ldots\alpha_{n-2} x_{n-2}\cdot\alpha_1 y\cdot\alpha_2 y\cdot\ldots\alpha_{n-2} y\cdot\alpha_{n-1}(x_{n-1}\cdot y)\cdot\alpha_n(x_n\cdot y) = \\
&\overset{(11)}{=}\ldots \overset{(11)}{=} \alpha_1 x_1\cdot\alpha_1 y\cdot\alpha_2(x_2\cdot y)\cdot\ldots\alpha_n(x_n\cdot y) \overset{(12)}{=} \\
&= \alpha_1(x_1\cdot y)\cdot\alpha_2(x_2\cdot y)\cdot\ldots\alpha_n(x_n\cdot y) = \\
&= [x_1\cdot y,\dots,x_n\cdot y].
\end{aligned}
$$

By **Theorem 1.4** the triple $(Q,\cdot,[\dots])$ is a decomposition of the superassociative groupoid (Q,f), which is a quasigroup, since the operation f is a repetition-free superposition of the quasigroup operations $(\cdot)$ and $[\dots]$ (see **Proposition 1.7**). **Theorem** is proved. □

Corollary 3.3. *For any superassociative group isotope (Q,f) with a canonical decomposition (9) the following statements are equivalent:*

1) the isotope (Q,f) is linear;

2) α_2 is an automorphism of the group $(Q,\cdot)$;

3) the group $(Q,\cdot)$ is commutative.

Proof. The relations (11) with $i = 2$ give the following equality:

$$\alpha_1 y \cdot \alpha_2 (x \cdot y) = \alpha_2 x \cdot \alpha_1 y \cdot \alpha_2 y,$$

which implies that 2) and 3) are equivalent. If, in addition, the group $(Q,\cdot)$ is commutative, then the relations (11) mean that every of the substitutions α_i is an automorphism of the group.

Corollary 3.4. *An $(n+1)$-ary isotope (Q,f) of an abelian group will be superassociative if and only if there exist an abelian group $(Q;+)$ and a sequence $\alpha_1,\alpha_2,\ldots,\alpha_n$ of its automorphisms such that the following equalities are true*

$$f(x_0,x_1,\ldots,x_n) = x_0 + \alpha_1 x_1 + \ldots + \alpha_n x_n, \quad \alpha_1 + \ldots + \alpha_n = \varepsilon$$

By **Theorem 2.2**, an isomorphism of grouplike algebras implies an isomorphism of its diagonal groups. It is easy to prove that the converse is true as well.

Proposition 3.5. *If groups are isomorphic, then there exists a bijection between the sets of superassociative isotopes of this groups such that the corresponding isotopes are isomorphic.*

Hence, from here on it suffices to consider superassociative isotopes of the same arbitrary fixed group $(Q,\cdot)$. An isomorphism criterion for them is in the following

Theorem 3.6. *A transformation θ of the set Q is a homomorphism (isomorphism) of a superassociative isotope (Q,f) in a superassociative isotope (Q,h) of a group $(Q,\cdot)$ with* *the canonical decompositions* $(\varepsilon,\alpha_1,\dots,\alpha_n)$ *and* $(\varepsilon,\beta_1,\dots,\beta_n)$ *respectively if and only if* θ *is an endomorphism (automorphism) of the group* $(Q,\cdot)$ *and the following relations are true*

$$\theta\beta_i=\alpha_i\theta,\quad i=1,2,\dots,n.$$

Proof. From **Theorem 2.2** and the relation (12) it follows that the group isotopes are homomorphic (isomorphic) if and only if the transformation θ is an endomorphism (automorphism) of the diagonal group $(Q,\cdot)$ and the following relation fulfils:

$$\theta(\beta_1x_1\cdot\beta_2x_2\cdot\ldots\cdot\beta_nx_n)=\alpha_1\theta x_1\cdot\alpha_2\theta x_2\cdot\ldots\cdot\alpha_n\theta x_n$$

for all $x_1,x_2,\dots,x_n\in Q$. In particular, when $x_j=e$ for all $j\neq i$, we obtain the necessary relations. The converse is evident. $\square$

Corollary 3.7. *A transformation* θ *of a set* Q *is an endomorphism (automorphism) of a superassociative isotope* (Q,f) *with a canonical decomposition (9) if and only if* θ *is an endomorphism (automorphism) of the diagonal group* $(Q,\cdot)$ *and commutes with every coefficient of the canonical decomposition.*

In the following assertion we shall describe the subquasigroups of the superassociative group isotopes.

Theorem 3.8. *A subset of a superassociative group isotope is a subquasigroup of it if and only if it is a subgroup of the diagonal group and invariant under all components of its canonical decomposition.*

Proof. Let (Q,f) be a superassociative isotope and (9) be its canonical decomposition. After **Theorem 2.1** it remains to elucidate which conditions are necessary for a subgroup H of the diagonal group to be a subquasigroup of the quasigroup $(Q;[\ldots])$, where (12) defines the operation $[\ldots]$. Hence, the subgroup H is a subquasigroup of $(Q;[\ldots])$ iff in the equality

$$\alpha_1 x_1 \cdot \alpha_2 x_2 \cdot \ldots \cdot \alpha_n x_n = h$$

any n elements from the set H uniquely determine the $(n+1)$-th element which is in the set H as well. The first part of this assertion is fulfilled, since $(Q;[\ldots])$ is a quasigroup. It remains to show the belonging of this element to the set H. From the last equality with $h_j = e$ for all $j \neq i$ we have the following statement: in the equality $\alpha_j h_j = h$ the elements h_j and h belong to the set H simultaneously, that is $\alpha_i H = H$. The reverse is obvious. The **theorem** is proved. □

Corollary 3.9. *A subset of a superassociative group isotope is its normal subquasigroup if and only if it is a normal subgroup of the diagonal group and it is invariant under all components of its canonical decomposition.*

Comparing the results of the article [12] with the assertions given here we have the following properties for superassociative isotopes of the cyclic groups:

1) *superassociative cyclic group isotopes are nonisomorphic, if their canonical decomposition groups coincide and the corresponding sequences of coefficients are different;*

2) *endomorphisms, automorphisms, subquasigroups, normal subquasigroups, congruences of a superassociative isotope of a cyclic group are the same as in the group;*

3) *there exist exactly* $\left((p-1)^n+(-1)^n(p-1)\right)/p$ *of* $(n+1)$*-ary* $(n>1)$ *superassociative group isotopes of prime order* p *up to isomorphism.*

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THE INTERPRETATION AND EQUIVALENCE OF THE VARIETY OF n-GROUPS

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Abstract

In this paper the varieties of n-ary groups are studied from the point of view of the interpretation and equivalence.

In this paper we deal with the interpretation and equivalence of the variety of n-groups. First of all, we shall recall some definitions and theorems from the theory of universal algebras (cf. [4]). We use the letter A to denote the universe of the algebra $\underline{A}$. Let F be a similarity type of an algebra $\underline{A}$. The subset of all n-ary function symbols in F is denoted by F_n for $n = 0,1,2,\ldots$

Definition 1. Let F be a similarity type. Let $\omega = \{0,1,2,\ldots,n,\ldots\}$. By a term (of type F) we mean an element of the term algebra $T_F(\omega)$. We put $z_n = (n)$, and the terms z_n $(n \in \omega)$ are called *variables*.

For every $n \geq 0$ the term algebra $T_F(n)$ is the subalgebra of $T_F(\omega)$ generated by $Z_n = \{z_0, z_1, \ldots, z_{n-1}\}$. If $n = 0$, then $Z_0 = \emptyset$. If $F_0 = \emptyset$, then the algebra $T_F(0)$ does not exist.

Assume that $\underline{A}$ is an algebra of type F. If $p \in T_F(\omega)$ then $p^{\underline{A}}$ denotes the term operation of the algebra $\underline{A}$ determined by the term p. We use the symbols $\mathrm{Clo}(\underline{A})$ and $\mathrm{Pol}(\underline{A})$ to denote the clones of term operations and polynomial operations of an algebra $\underline{A}$, respectively.

Definition 2. Two algebras $\underline{A}$ and $\underline{B}$ are called *term equivalent* if

(i) $A = B$,

(ii) $\mathrm{Clo}(\underline{A}) = \mathrm{Clo}(\underline{B})$.

Definition 3. Two algebras $\underline{A}$ and $\underline{B}$ are called *polynomially equivalent* if

(i) $A = B$,

(ii) $\mathrm{Pol}(\underline{A}) = \mathrm{Pol}(\underline{B})$..

Definition 4. Let V and W be varieties of respective similarity types F and G. By an *interpretation of V in W* a mapping $D: F \to T_G(\omega)$ is meant satisfying the following conditions:

(D1) If $f \in F_n$ and $n > 0$, then $D(f) = f_D \in T_G(n)$;

(D2) If $f \in F_0$, then $D(f) = f_D \in T_G(1)$ and the equation $f_D(z_0) \approx f_D(z_1)$ is valid in W;

(D3) For every algebra $\underline{A} \in W$, the algebra $\underline{A}^D = (A, f_D^{\underline{A}}\ (f \in F))$ belongs to the variety V.

Definition 5. By an *equivalence of varieties V and W* a pair of interpretations (D, E) is meant satisfying the following conditions:

1) D is an interpretation of V in W;

2) E is an interpretation of W in V;

3) $\forall \underline{A} \in W: \quad \underline{A}^{DE} = \underline{A};$

4) $\forall \underline{B} \in V: \quad \underline{B}^{ED} = \underline{B}.$

Now we present the fundamental definitions and theorems from the theory of n-groups (cf. [1], [5], [6]). For simplicity of notation, it will be convenient to abbreviate $x_1, \ldots, x_k$, as x_1^k. If $x_1 = x_2 = \ldots = x_k = x$, then we write x^k.

Definition 6. An algebra $\underline{A} = (A, f)$ endowed with an n-ary operation $f: A^n \to A$ $(n \geq 2)$ is called an *n-groupoid* (a *groupoid* for $n = 2$).

Definition 7. An n-groupoid $\underline{A} = (A, f)$ is said to be an *n-group* if the following conditions hold:

(1) for all $x_1, \ldots, x_{2n-1} \in A$ the associative law

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

holds for all $i, j \in \{1, \ldots, n\}$;

(2) for every $k \in \{1, \ldots, n\}$ and for all $x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \in A$ there exists an uniquely element $z_k \in A$ such that

$$f(x_1^{k-1}, z_k, x_{k+1}^n) = x_0.$$

Theorem 1 (cf. [2]). *An n-groupoid $\underline{A} = (A, f)$ is an n-group if and only if*

$$f(x_1^n) = x_1 \cdot \alpha(x_2) \cdot \alpha^2(x_3) \cdot \ldots \cdot \alpha^{n-2}(x_{n-1}) \cdot a \cdot x_n \tag{3}$$

for all $x_1, \ldots, x_n \in A$, where the following conditions hold:

(i) $(A, \cdot, ^{-1}, e)$ *is a group;*

(ii) $a \in A$ *is a fixed element;*

(iii) $\alpha \in Aut(A,\cdot,^{-1},e)$, $\alpha(a)=a$, $\alpha^{n-1}(x)=a\cdot x\cdot a^{-1}$ *for every* $x \in A$.

Definition 8. An algebra $\underline{A}^{(n)}=(A,\cdot,^{-1},e,\alpha,\alpha^{-1},a)$ fulfilling the conditions (3) and (i)-(iii) of **Theorem 1** is called an *α-algebra associated with the n-group* $\underline{A}=(A,f)$

For simplicity, this algebra will be denoted by $\underline{A}^{(n)}=(A,\cdot,\alpha,a)$.

From now on, we consider n-groups with $n \geq 3$.

Theorem 2 (cf. [1]). *If* $\underline{A}=(A,f)$ *is an n-group, then there is a unary operation* $x \to \bar{x}$ *on the set* A *such that*

$$f(\bar{x},x^{n-2},y)=y \quad \text{and} \quad f(y,x^{n-2},\bar{x})=y \tag{4}$$

for all $x,y \in A$. *Conversely, if a nonempty set A carries an n-ary operation* f *and a unary operation* $x \to \bar{x}$ *satisfying the conditions (1) and (4), then* $\underline{A}=(A,f)$ *is an n-group.*

In view of **Theorem 2**, an n-group will often be defined as an algebra $\underline{A}=(A,f,-)$ equipped with an n-ary operation f and an unary operation $x \to \bar{x}$ satisfying the conditions (1) and (4). The element $\bar{x}$ is usually called a *skew element*.

Let $\underline{A}=(A,f,-)$ be an n-group. Let $p \in A$ be an arbitrary fixed element. Using the Sokolov method (cf. [6]) we can construct an α-algebra $\underline{A}^{(n)}=(A,\cdot,\alpha,a)$ associated with the n-group $\underline{A}=(A,f,-)$ in the following way.

Define the binary operation according to the formula

$$x\cdot y=f(x,p^{n-2},y)$$

for all $x, y \in A$. Let $\alpha: A \to A$ be the mapping defined by

$$\alpha(x) = f(\overline{p}, x, p^{n-2})$$

for $x \in A$. Finally, let us take

$$a = f(\overline{p}^n).$$

The α-algebra $\underline{A}^{(n)} = (A, \cdot, \alpha, a)$ associated with the n-group $\underline{A} = (A, f, ^{-})$ and constructed by means of the above method will be denoted by $\underline{A}_p^{(n)} = (A, \cdot, \alpha, a)$.

From **Theorem 2** it follows that the class of all n-groups is a variety which will be denoted by V_n. It is easy to verify that

$$p(x, y, z) = f(x, \overline{y}, y^{n-3}, z)$$

is a Malcev term for V_n. Thus the variety V_n is congruence-permutable.

For $n \geq 3$ we define an algebra

$$\underline{A}^{(n)} = (A, \cdot, ^{-1}, e, \alpha, \alpha^{-1}, a) \tag{5}$$

fulfilling the following conditions:

(i) $(A, \cdot, ^{-1}, e)$ is a group;

(ii) $\forall x, y \in A$: $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$;

(iii) $\forall x \in A$: $\alpha(\alpha^{-1}(x)) = \alpha^{-1}(\alpha(x)) = x$;

(iv) $\alpha(a) = a$;

(v) $\forall x \in A$: $\alpha^{n-1}(x) = a \cdot x \cdot a^{-1}$.

For simplicity, this algebra will be denoted by $\underline{A}^{(n)} = (A, \cdot, \alpha, a)$.

Definition 9. The algebra $\underline{A}^{(n)}$ defined by (5) is called an *α-algebra.*

For an arbitrary fixed $n \geq 3$ the class of all α-algebras $\underline{A}^{(n)}$ is a Malcev variety and it will be denoted by W_n.

Theorem 3. *The variety* V_n *is interpretable into the variety* W_n.

Proof. Let us consider the similarity types $F = \{f, -\}$ and $G = \{\cdot, ^{-1}, e, \alpha, \alpha^{-1}, a\}$ of the varieties V_n and W_n, respectively. We define the mapping $D: F \to T_G(\omega)$ according to the formulas:

$$D(f) = z_1 \cdot \alpha(z_2) \cdot \alpha^2(z_3) \cdot \ldots \cdot \alpha^{n-2}(z_{n-1}) \cdot a \cdot z_n$$

where $z_1, z_2, \ldots, z_n \in T_G(n)$ are variables, and

$$D(-) = a^{-1} \cdot \alpha^{n-2}(z^{-1}) \cdot \ldots \cdot \alpha^2(z^{-1}) \cdot \alpha(z^{-1})$$

where $z \in T_G(1)$ is a variable. It follows immediately that conditions (D1) and (D2) of **Definition 4** are satisfied. Assume that $\underline{A} = (A, \cdot, ^{-1}, e, \alpha, \alpha^{-1}, a) \in W_n$. Let us take

$$f(x_1^n) = x_1 \cdot \alpha(x_2) \cdot \alpha^2(x_3) \cdot \ldots \cdot \alpha^{n-2}(x_{n-1}) \cdot a \cdot x_n,$$

$$\bar{x} = a^{-1} \cdot \alpha^{n-2}(x^{-1}) \cdot \ldots \cdot \alpha^2(x^{-1}) \cdot \alpha(x^{-1})$$

for all $x, x_1, \ldots, x_n \in A$. It is easy to check that $\underline{A}^D = (A, f, -) \in V_n$. Consequently, the condition (D3) holds.

Corollary 1. *For every α-algebra $\underline{A} \in W_n$ there exists an n-group $\underline{A}^D \in V_n$ such that* $\mathrm{Clo}(\underline{A}^D) \subset \mathrm{Clo}(\underline{A})$.

The variety W_n is not interpretable into the variety V_n.

Example 1. Consider the **Klein** group with the universe $A = \{e, a, b, c\}$. The mapping

$$\alpha(e) = e, \quad \alpha(a) = b, \quad \alpha(b) = a, \quad \alpha(c) = c$$

is an automorphism of the above group. We define the 3-group $\underline{A} = (A, f, {}^{-})$ as follows

$$f(x_1, x_2, x_3) = x_1 \alpha(x_2) x_3$$

for all $x_1, x_2, x_3 \in A$. The 3-group $\underline{A}$ has the two different one-element 3-subgroups with the subinverses $\{e\}$ and $\{c\}$. On the other hand, there are no α-algebras of the variety W_3 with two different one-element α-subalgebras.

Definition 10. Let us consider an algebra $\underline{A} = (A, f, {}^{-}, p)$ fulfilling the following conditions:

(i) the reduct $(A, f, {}^{-})$ is an n-group,

(ii) p is a constant 0-ary operation such that $\bar{p} = p$.

An algebra $\underline{A} = (A, f, {}^{-}, p)$ is called an *n-group with constant.*

The class V_n^0 of all n-groups with constant is a Malcev variety.

Let W_n^0 be a class of all α-algebras $\underline{A}^{(n)} = (A, \cdot, {}^{-1}, e, \alpha, \alpha^{-1}, a)$ of the variety W_n such that $a = e$. In this case we write $\underline{A}^{(n)} = (A, \cdot, {}^{-1}, e, \alpha, \alpha^{-1})$. Since any α-algebra $\underline{A}^{(n)} = (A, \cdot, {}^{-1}, e, \alpha, \alpha^{-1})$ belongs to the class of Ω-groups (cf. [3]) with $\Omega = \{\alpha, \alpha^{-1}\}$, it will be referred to as an α-group. The class W_n^0 is a Malcev variety.

Theorem 4. *The varieties V_n^0 and W_n^0 are equivalent.*

Proof. Let $F=\{f,-,p\}$ and $G=\{\cdot,{}^{-1},e,\alpha,\alpha^{-1}\}$ be similarity types of the varieties V_n^0 and W_n^0, respectively. We define the mapping $D: F \to T_G(\omega)$ as follows:

$$D(f)=z_1\cdot\alpha(z_2)\cdot\alpha^2(z_3)\cdot\ldots\cdot\alpha^{n-2}(z_{n-1})\cdot z_n,$$

where $z_1,\ldots,z_n \in T_G(n)$ are variables,

$$D(-)=\alpha^{n-2}(z^{-1})\cdot\ldots\cdot\alpha^2(z^{-1})\cdot\alpha(z^{-1}),$$

$$D(p)=z\cdot z^{-1},$$

where $z\in T_G(1)$ is a variable.

The conditions (D1) and (D2) are satisfied. Suppose that $\underline{A}^{(n)}=(A,\cdot,{}^{-1},e,\alpha,\alpha^{-1})\in W_n^0$. Let us put:

$$f(x_1^n)=x_1\cdot\alpha(x_2)\cdot\alpha^2(x_3)\cdot\ldots\cdot\alpha^{n-2}(x_{n-1})\cdot x_n,$$

$$\bar{x}=\alpha^{n-2}(x^{-1})\cdot\ldots\cdot\alpha^2(x^{-1})\cdot\alpha(x^{-1}),$$

$$p=e$$

for all $x,x_1,\ldots,x_n\in A$. It is easy to check that $(\underline{A}^{(n)})^D=(A,f,-,e)\in V_n^0$. Thus, D is an interpretation of V_n^0 in W_n^0. Next, we define the mapping $E: G\to T_F(\omega)$ as follows:

$$E(\cdot)=f(z_1,p^{n-2},z_2)$$

where $z_1,z_2\in T_F(2)$ are variables,

$$E({}^{-1})=f(p,\bar{z},z^{n-3},p),$$

$$E(e)=f(\bar{z},z^{n-2},p),$$

$$E(\alpha)=f(p,z,p^{n-2}),$$

$$E(\alpha^{-1})=f(p^{n-2},z,p)$$

where $z\in T_F(1)$ is a variable.

The mapping E satisfies the conditions (D1) and (D2).

Suppose that $\underline{A}=(A,f,-,p)\in V_n^0$. Let us put:

$$x_1\cdot x_2=f(x_1,p^{n-2},x_2),$$

$$x^{-1} = f(p, \bar{x}, x^{n-3}, p),$$
$$e = p,$$
$$\alpha(x) = f(p, x, p^{n-2}),$$
$$\alpha^{-1}(x) = f(p^{n-2}, x, p)$$

for all $x, x_1, x_2 \in A$.

Using the Sokolov method it is easy to check that $\underline{A}^E = (A, \cdot, ^{-1}, e, \alpha, \alpha^{-1}) \in W_n^0$. Thus E is an interpretation of W_n^0 in V_n^0. A straightforward computation shows that $(\underline{A}^{(n)})^{DE} = \underline{A}^{(n)}$ and $\underline{A}^{ED} = \underline{A}$. The pair of interpretations (D, E) is an equivalence of varieties V_n^0 and W_n^0.

As an immediate consequence of **Theorem 4** we obtain the following corollary.

Corollary 2. *If $\underline{A}$ is an n-group with constant p, then the algebras $\underline{A}$ and $\underline{A}_p^{(n)}$ are term equivalent.*

Proposition 1. Let $\underline{A} = (A, f, \bar{\ }, p)$ *be a 3-group with an arbitrary fixed constant* p *(not necessarily $p = \bar{p}$). The algebras $\underline{A}$ and $\underline{A}_p^{(3)} = (A, \cdot, \alpha, a)$ are term equivalent.*

Proof. Since

$$x \cdot y = f(x, p, y),$$

$$x^{-1} = f(\overline{p}, \overline{x}, \overline{p}),$$
$$e = \overline{p},$$
$$\alpha(x) = f(\overline{p}, x, p),$$
$$\alpha^{-1}(x) = f(p, x, \overline{p}),$$
$$a = f(\overline{p}, \overline{p}, \overline{p})$$

for all $x, y \in A$, we conclude that $\mathrm{Clo}(\underline{A}_p^{(3)}) \subset \mathrm{Clo}(\underline{A})$. We have

$$f(x_1, x_2, x_3) = x_1 \cdot \alpha(x_2) \cdot a \cdot x_3,$$
$$\overline{x} = a^{-1} \cdot \alpha(x^{-1}),$$
$$p = a^{-1}$$

for all $x, x_1, x_2, x_3 \in A$. Consequently, $\mathrm{Clo}(\underline{A}) \subset \mathrm{Clo}(\underline{A}_p^{(3)})$. □

On the whole, an n-group $\underline{A}$ with an arbitrary fixed constant p (not necessarily $p = \overline{p}$) is not term equivalent to an α-algebra $A_p^{(n)}$ for $n > 3$. Consider a suitable example.

Example 2. Let Z_4 be the additive group of the integers modulo 4. Consider the 4-group $\underline{A} = (Z_4, f, -, 1)$ with constant 1 defined according to the formula

$$f(x_1^4) = x_1 + x_2 + x_3 + x_4 + 2$$

for all $x_1, x_2, x_3, x_4 \in Z_4$. Define the mapping

$$\varphi(0) = 0, \quad \varphi(1) = 3, \quad \varphi(2) = 2, \quad \varphi(3) = 1.$$

The mapping φ is an automorphism of the algebra $\underline{A}_1^{(4)} = (Z_4, +, \mathrm{id}_{Z_4}, 2)$, but it is not an automorphism of the 4-group $\underline{A}$. Thus the algebras $\underline{A}$ and $\underline{A}_1^{(4)}$ are not term equivalent.

Proposition 2. *Let* $\underline{A} = (A, f, \bar{\ })$ *be an* n*-group, and let* $\underline{A}^{(n)} = (A, \cdot, \alpha, a)$ *be an* α*-algebra associated with the* n*-group* $\underline{A}$*. Then the algebras* $\underline{A}$ *and* $\underline{A}^{(n)}$ *are polynomially equivalent.*

Proof. Since

$$f(x_1^n) = x_1 \cdot \alpha(x_2) \cdot \alpha^2(x_3) \cdot \ldots \cdot \alpha^{n-2}(x_{n-1}) \cdot a \cdot x_n,$$
$$\bar{x} = a^{-1} \cdot \alpha^{n-2}(x^{-1}) \cdot \ldots \cdot \alpha^2(x^{-1}) \cdot \alpha(x^{-1}),$$

for all $x, x_1, \ldots, x_n \in A$, we conclude that $\mathrm{Pol}(\underline{A}) \subset \mathrm{Pol}(\underline{A}^{(n)})$. Note that $\bar{e} = a^{-1}$. It is easy to prove that

$$x \cdot y = f(x, e^{n-3}, \bar{e}, y),$$
$$x^{-1} = f(e, \bar{x}, x^{n-3}, e),$$
$$\alpha(x) = f(e, x, e^{n-3}, \bar{e}),$$
$$\alpha^{-1}(x) = f(\bar{e}, e^{n-3}, x, e)$$

for all $x, y \in A$. Therefore $\mathrm{Pol}(\underline{A}^{(n)}) \subset \mathrm{Pol}(\underline{A})$. $\square$

Corollary 3. *Let* $\underline{A} = (A, f, \bar{\ })$ *be an* n*-group, and let* $\underline{A}^{(n)} = (A, \cdot, \alpha, a)$ *be an* α*-algebra associated with the* n*-group* $\underline{A}$*. Then there exists a constant* $p \in A$ *such that the algebra* $\underline{A}' = (A, f, \bar{\ }, p)$ *is term equivalent to the* α*-algebra* $\underline{A}^{(n)}$*.*

It is sufficient to take $p = e$. $\square$

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ON DISTRIBUTIVE n-ARY GROUPS

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Abstract

The classes of medial n-groups, distributive n-groups and autodistributive n-groups are described. These are the classes of n-ary groups $(n \geq 3)$ in which the unary operation $\bar{}\ : x \to \bar{x}$ ($z = \bar{x}$ is a unique solution of the equation $f(x, x, \ldots x, z) = x$ in an n-ary group) plays an important role.

1. Introduction

As it is well known [11], [8], [4], an n-ary group $(n \geq 3)$ may be defined as an n-ary semigroup (G, f) with a special unary operation $\bar{}\ : x \to \bar{x}$, i.e. as an universal algebra $(G, f, \bar{}\ )$ of type $(n,1)$. Since the equation

$$f(x, x, \ldots, x, z) = x$$

has in any n-ary group (G, f) a unique solution $z = \bar{x}$, then the operation $\bar{}\ : x \to \bar{x}$ is a uniquely defined by the operation f. The element $z = \bar{x}$ is called *skew to x*. Obviously $x = \bar{x}$ iff x is an idempotent. In general $\bar{x} \neq \bar{y}$, but in some n-ary groups (G, f) there exists an element z such that $z = \bar{x}$ for all $x \in G$. All such n-ary groups are derived (cf. [7]) from a binary group of the exponent $k | (n-2)$.

In this paper we describe some classes of n-ary groups in which the operation $\bar{}\ : x \to \bar{x}$ plays a very important role.

Because for $n=2$ such groups are trivial, we consider only the case $n \geq 3$. Used terminology and notion are standard.

2. Medial n-groups

From the proof of **Theorem 3** in [10] it follows that any medial n-ary group satisfies the identity

$$\overline{f(x_1^n)} = f(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n). \tag{1}$$

Hence an n-ary group (G, f) is medial iff it is Abelian as an algebra $(G, f, \bar{\ })$ of type $(n,1)$. On the other hand, one can prove (cf. [4]) that n-ary group (G, f) is medial iff there exists $a \in G$ such that

$$f(x, \overset{(n-2)}{a}, y) = f(y, \overset{(n-2)}{a}, x)$$

for all $x, y \in G$, i.e. iff the binary retract of (G, f) is commutative (cf. [4], [6]).

Note that the identity (1) is satisfied also in some non-medial n-ary groups. For example, (1) holds in the 8-group derived from the group S_3. It is also satisfied in all idempotent n-ary groups.

Let $x = \bar{x}^{(0)}$ and let $\bar{x}^{(s+1)}$ be the skew element to $\bar{x}^{(s)}$, where $s \geq 0$. In other words, let $\bar{x}^{(1)} = \bar{x}$, $\bar{x}^{(2)} = \bar{\bar{x}}$ etc. For example, in a 4-group (G, f) derived from the additive group Z_8, we have

$$\bar{x} = 6x (\operatorname{mod} 8), \quad \bar{\bar{x}} = 4x (\operatorname{mod} 8), \quad \bar{x}^{(s)} = 0$$

for every $s \geq 3$. But in the n-ary group (Z, f) derived from the additive group of integers we have $\bar{x}^{(s)} \neq \bar{x}^{(t)}$ for all $s \neq t$.

If $\bar{x}^{(s)} = x$, then

$$ord_n(x) = ord_n(\overline{x}^{(t)})$$

for any natural t, where $ord_n(x)$ denotes the n-ary order of x, i.e. the minimal natural number p (if it exists) such that $x^{<p>} = x$. By $x^{<s>}$ we mean x if $s = 0$, and $f(x^{<s-1>}, x, \dots, x)$ if $s > 1$ (cf. [3] or [5]).

One can prove (cf. [3]) that $\overline{x}^{(m)} = x$ iff $ord_n(x)$ divides

$$\frac{1-(2-n)^m}{n-1} = \sum_{k=0}^{m-1}(2-n)^k.$$

In particular $\overline{\overline{x}} = x$ iff $ord_n(x)$ divides $n-3$. Hence in any ternary group (G, f) we have $\overline{\overline{x}} = x$ for all $x \in G$. Note also that if the n-ary order of x is finite, then

$$ord_n(x) = ord_n(\overline{x})$$

iff $ord_n(x)$ and $n-2$ are relatively prime.

It is clear that if an n-ary group (G, f) satisfies (1), then for all $s \geq 0$ it satisfies also

$$\overline{f(x_1^n)}^{(s)} = f(\overline{x}_1^{(s)}, \overline{x}_2^{(s)}, \dots, \overline{x}_n^{(s)}).$$

Therefore if an n-ary group (G, f) satisfies (1), then the mapping ϕ_s defined by the formula

$$\phi_s(x) = \overline{x}^{(s)}$$

is an n-ary endomorphism of (G, f). Obviously $\phi_s\phi_t = \phi_{s+t}$ and ϕ_k is the identity endomorphism of (G, f) iff $\overline{x}^{(k)} = x$ for all $x \in G$. Thus the set of all ϕ_s forms the cyclic subsemigroup of the semigroup $End(G, f)$.

Moreover, the relation ρ_s defined on (G, f) by the formula $(x, y) \in \rho_s$ iff $\overline{x}^{(s)} = \overline{y}^{(s)}$, i.e. iff $\phi_s(x) = \phi_s(y)$ is a congruence on (G, f). Obviously, $\rho_0 \leq \rho_s \leq \rho_t$ for any $s \leq t$.

If the set

$$E_s = \{x \in G \mid x = \phi_s(x)\}$$

is non-empty, then it is an n-ary subgroup of an n-ary group (G, f) with (1). It is clear that $E_1 \subset E_s \subset E_{st}$ and $E_s \cap E_{s+1} = E_1$.

Similarly, it is not difficult to verify that if (1) holds in (G, f), then for any s the set

$$G^{(s)} = \{\overline{x}^{(s)} \mid x \in G\}$$

is an n-ary subgroup of (G, f). Moreover, $G^{(s+t)} = (G^{(s)})^{(t)}$ for all $s, t \in N$ and $G \supseteq G^{(1)} \supseteq G^{(2)} \supseteq \ldots$ Obviously, for any finite n-ary group there exists $t \in N$ such that $G^{(s)} = G^{(t)}$ for all $s \geq t$. On the other hand, the n-ary group (G, f) derived from the additive group of all integers is an example of an n-ary group with $G^{(s)} \neq G^{(t)}$ for $s \neq t$ ($G^{(s)}$ contains all integers which are divided by $(n-2)^s$).

3. Distributive n-groups

Let (G, f) be an n-ary group in which the n-ary operation f is distributive with respect to itself , i.e. an n-ary group in which the identity

$$f(x_1^{i-1}, f(y_1^n), x_{i+1}^n) = f(f(x_1^{i-1}, y_1, x_{i+1}^n), f(x_1^{i-1}, y_2, x_{i+1}^n), \ldots, f(x_1^{i-1}, y_n, x_{i+1}^n)),$$

holds for all $i = 1, 2, \ldots, n$. Such groups are called *autodistributive n-groups* (cf. [5]). One can prove (cf. [5], **Theorem 3**) that every autodistributive n-group (G, f) satisfies

$$\overline{f(x_1^n)} = f(x_1^{i-1}, \overline{x}_i, x_{i+1}^n), \tag{2}$$

where $i=1,2,\dots,n$. An n-ary group (G,f) satisfying (2) will be called *distributive*.

Let (G,f) be an n-ary semigroup with a unary operation ϕ such that $f(x,x,\dots x,\phi(x))=x$ for all $x\in G$. If for any $i=1,2,\dots,n$ holds also the identity

$$\phi(f(x_1^n))=f(x_1^{i-1},\phi(x_i),x_{i+1}^n),$$

then (G,f,ϕ) is a (f/ϕ)-algebra in the sense of H.J.Hoehnke [12]. If (G,f) is an n-ary group, then we have a distributive n-group, because $\phi(x)=\bar{x}$.

Proposition 1. *Let (G,f) be an n-ary semigroup with the above defined unary operation ϕ. Then (G,f) is a distributive n-group iff it is a cancellative n-semigroup.*

Proof. Suppose that an n-semigroup (G,f) is cancellative. Then

$$f(x_1^{i-1},a,x_{i+1}^n)=f(x_1^{i-1},b,x_{i+1}^n)$$

implies $a=b$ (cf. [5]). If ϕ is distributive with respect to f, then

$$f(\overset{(n-2)}{x},\phi(x),x)=\phi(f(\overset{(n)}{x}))=f(\overset{(n-1)}{x},\phi(x))=x.$$

Thus for $y\in G$ we have

$$f(y,\overset{(n-3)}{x},\phi(x),x)=f(y,\overset{(n-3)}{x},\phi(x),f(\overset{(n-2)}{x},\phi(x),x))=$$
$$=f(f(y,\overset{(n-3)}{x},\phi(x),x),\overset{(n-3)}{x},\phi(x),x),$$

which (by cancellation) gives

$$f(y,\overset{(n-3)}{x},\phi(x),x)=y.$$

Similarly

$$f(\overset{(n-2)}{x},\phi(x),y)=f(f(\overset{(n-2)}{x},\phi(x),x),\overset{(n-3)}{x},\phi(x),y))=$$
$$=f(\overset{(n-2)}{x},\phi(x),f(\overset{(n-2)}{x},\phi(x),y)),$$

implies

$$f(\overset{(n-2)}{x},\phi(x),y)=y$$

Hence for all $x,y\in G$ we have

$$f(y,\overset{(n-3)}{x},\phi(x),x)=f(\overset{(n-2)}{x},\phi(x),y)=y,$$

which proves (cf. [4], [8]) that (G,f) is a distributive n-group and $\phi(x)=\bar{x}$. The converse is obvious. □

As it is well known (cf. [4], [8]) for $n>2$ an n-ary group may be defined as an n-semigroup (G,f) with a unary operation $^{-}: x\rightarrow\bar{x}$ in which so-called **Dornte**'s identities

$$f(y,\overset{(n-j-1)}{x},\bar{x},\overset{(j-1)}{x})=f(y,\overset{(i-1)}{x},\bar{x},\overset{(n-i-1)}{x},y)=y \tag{3}$$

hold for all $i,j=1,2,\dots,n-1$.

Using these identities and (2) it is not difficult to verify that the following lemma is true.

Lemma 1. *In any distributive n-group* $x=\bar{x}^{(n-1)}$ *and* $\bar{x}=\bar{x}^{(n)}$.

Corollary 1. *Any distributive n-group satisfies (1). Moreover, for $x\in G$ we have also* $x=f(\bar{x},\bar{x},\dots,\bar{x})=\overline{f(x,x,\dots,x)}$.

Lemma 2. *In distributive n-groups*

(a) $x^{<k>}=\bar{x}^{(n-1-k)}$,

(b) $\bar{x}^{<k>}=(\bar{x}^{<k+1>})^{<1>}$,

$$(c)\quad f(x_1^{i-1}, x_i^{<k>}, x_{i+1}^n) = (f(x_1^n))^{<k>}$$

for all $k = 0,1,\ldots,n-1$ *and* $i = 1,2,\ldots,n$.

Proof. We prove only *(a)*. For $k=0$ this condition is obvious. If it holds for some $t < n-1$, then for $t+1$ we have

$$x^{<t+1>} = f(x^{<t>}, \overset{(n-1)}{x}) = f(\bar{x}^{(n-1-t)}, \overset{(n-1)}{x}) =$$
$$= f(\bar{x}^{(n-2-t)}, \overset{(n-2)}{x}, \bar{x}) = \bar{x}^{(n-1-(t+1))},$$

which completes the proof of *(a)*.

The condition *(c)* is a simple consequence of (2) and *(a)*. □

Corollary 2. *If in a distributive* n*-group* (G,f) $ord_n(x) = p$ *for some* $x \in G$, *then* $\bar{x}^{(p)} = x$ *and* $x^{<k>} = \bar{x}^{(p-k)}$ *for* $k = 0,1,\ldots,p$.

Corollary 3. *If* p *is a minimal natural number such that* $x = \bar{x}^{(p)}$ *for some element* x *of a distributive* n*-group, then* $ord_n(x) = p$.

Proof. See the proof of **Corollary 10** from [5]. □

Lemma 3. *All elements of a distributive* n*-group have the same finite* n*-ary order which divides* $n-1$.

Proof. As a simple consequence of **Lemma 2** (a) we obtain $x^{<n-1>} = x$. This shows that all elements of a distributive n-group have a finite n-ary order which is a divisor of $n-1$ (cf. [3]).

Now, if $ord_n(x) = t$, $ord_n(y) = s$, then

$$x = f(x, \bar{y}, \overset{(n-2)}{y}) = f(x, \bar{y}, \overset{(n-3)}{y}, y^{<s>}) = (f(x, \bar{y}, \overset{(n-3)}{y}, y))^{<s>} = x^{<s>},$$

by (3) and **Lemma 2** . Therefore $t|s$. Similarly we obtain $y = y^{<t>}$ and $s|t$. Hence $s = t$, which proves that all elements have the same n-ary order. □

Theorem 1. *Any distributive n-group is a set-theoretic union of disjoint cyclic and isomorphic autodistributive n-groups without proper subgroups.*

Proof. Let $ord_n(x) = t$ and let C_x be an n-ary subgroup generated by x. Then $C_x = \{x, x^{<1>}, x^{<2>}, \ldots, x^{<t-1>}\}$. Since all elements have the same n-ary order, then C_x has no proper subgroups and any two subgroups C_x and C_y are isomorphic. Such subgroups are autodistributive by **Theorem 4** *from* [5]. (This fact follows also from our **Corollary 12**). □

Corollary 4. *A distrivutive n-group is idempotent or has no any idempotents.*

Theorem 2. *Let x be an arbitrary element of a distributive n-group (G, f). Then C_x is the normal subgroup of the retract $ret_x(G, f)$ and every coset of C_x in $ret_x(G, f)$ is an n-ary subgroup of (G, f) isomorphic to (C_x, f).*

Proof. Let $(G, \bullet) = ret_x(G, f)$, i.e. let $a \bullet b = f(a, \overset{(n-2)}{x}, b)$ for all $a, b \in G$ (cf. [9]). Then $x^{<k>} = x^{k+1}$ in $(G, \bullet)$. Moreover, $C_x = \{x, x^2, \ldots, x^t\}$ and C_x is a cyclic subgroup of the order $t = ord_n(x)$ in $(G, \bullet)$. It is normal, because by (2) and (3) we get

$$a \bullet x = f(a, \overset{(n-1)}{x}) = f(\bar{x}, \overset{(n-2)}{x}, f(a, \overset{(n-1)}{x})) =$$
$$= f(\overset{(n-1)}{x}, f(a, \overset{(n-2)}{x}, \bar{x})) = f(\overset{(n-1)}{x}, a) = x \bullet a$$

for all $a \in G$.

Moreover, for every $k = 1,2,\dots,t$ we have also

$$a \bullet x^{(k)} = f(a, \overset{(n-2)}{x}, \bar{x}^{(k)}) = f(\bar{a}^{(k-1)}, \overset{(n-2)}{x}, \bar{x}) = \bar{a}^{(k-1)},$$

which gives $a \bullet C_x = C_a$ for every $a \in G$. This completes our proof.

Since by **Corollary 2** $C_x = \{x, \bar{x}, \bar{x}^{(2)}, \dots, x^{(t-1)}\}$, where $t = ord_n(x)$. Then $\bar{x}^{(s)} = \bar{y}^{(s)}$ implies $\bar{x}^{(t-1)} = \bar{y}^{(t-1)}$ and, in the consequence, $x = y$. This proves that in a distributive n-group any endomorphism $\phi\ x\ \ \bar{x}^{(\ )}$ is one-to-one and there is only t different endomorphisms ϕ_s. Obviously any such endomorphism is also "onto" because for every $x \in G$ there exists $y = \bar{x}^{(t-s)} \in G$ such that $x = \phi_s(y)$. Thus $\phi_0, \phi_1, \dots, \phi_{t-1}$ form a cyclic subgroup in the group $Aut(G,f)$ of all automorphism of (G,f). Since $\phi(\bar{x}) = \overline{\phi(x)}$ for all automorphisms of an arbitrary n-ary group , then this subgroup is invariant in the group $Aut(G,f)$. Obviously any ϕ_s is a splitting-automorphism in the sense of Plonka [13].

Thus we obtain the following result.

Proposition 2. *If (G,f) is a distributive n-group , then the operation $\bar{\ }$: $x \to \bar{x}$ induces the cyclic subgroup in the group $Aut(G,f)$ of automorphisms of (G,f). Moreover, this subgroup is invariant in the group $Aut(G,f)$ and in the group of all splitting-automorphisms of (G,f).*

From the above results it follows also that $G = E_s = G^{(s)}$ for any distributive n-group (G,f). Thus the class V_s of n-ary groups (G,f) such that $G = E_s$ (cf. [6], **Problem 4**) contains the class of distributive n-groups. The class of distributive n-groups is also contained in the class of all n-ary groups satisfying the descending chain condition for $G^{(s)}$ (cf. [6], **Problem 5**).

The class of all n-ary groups (for fixed n) is a variety (cf. [11], [8]). The class of all distributive n-groups is a subvariety of this variety. From **Theorem 1** it follows that any free n-group in this subvariety is a set-theoretic union of disjoint cyclic autodistributive n-groups with $n-1$ elements which have no proper subgroups, but in general this n-group is not autodistributive.

Theorem 3. *An n-ary group (G,f) is distributive iff it has the form*

$$f(x_1^n) = x_1 \bullet \theta x_2 \bullet \theta^2 x_3 \bullet \ldots \bullet \theta^{n-2} x_{n-1} \bullet x_n \bullet b, \tag{4}$$

where b is a fixed central element of a group $(G,\bullet)$ with the identity e, $b^{n-1} = e$, θ is an automorphism of $(G,\bullet)$, $\theta b = b$, $x \bullet \theta x \bullet \theta^2 x \bullet \ldots \bullet \theta^{n-2} x = e$ and $\theta^{n-1} x = x$ for all $x \in G$.

Proof. According to the well known **Gluskin-Hosszu theorem** any n-ary group has the form $(G,f) = der_{\theta,b}(G,\bullet)$ (cf. for example [9]), i.e. for any n-ary group (G,f) there exist a group $(G,\bullet)$, $b \in G$, and an automorphism θ of $(G,\bullet)$ such that $\theta b = b$, $\theta^{n-1} x = b \bullet x \bullet b^{-1}$ for all $x \in G$ and

$$f(x_1^n) = x_1 \bullet \theta x_2 \bullet \theta^2 x_3 \bullet \ldots \bullet \theta^{n-1} x_n \bullet b.$$

If θ and b are as in our theorem, then direct computations show that $\bar{x} = x \bullet b^{n-2}$ for all $x \in G$ and $(G,f) = der_{\theta,b}(G,\bullet)$ is a distributive n-group.

Conversely, if $(G,f) = der_{\theta,b}(G,\bullet)$ is a distributive n-group and e is the identity of $(G,\bullet)$, then (2) and (3) imply

$$x \bullet b = f(x, \overset{(n-1)}{e}) = f(\bar{e}, \overset{(n-2)}{e}, f(x, \overset{(n-1)}{e})) =$$
$$= f(\overset{(n-1)}{e}, f(x, \overset{(n-2)}{e}, \bar{e})) = f(\overset{(n-1)}{e}, x) = \theta^{n-1}x \bullet b,$$

which shows that θ^{n-1} is an identity mapping and $x \bullet b = b \bullet x$ for all $x \in G$.

Since $e = f(\bar{e}, \overset{(n-1)}{e}) = \bar{e} \bullet b$, then b^{-1} is skew to e. Thus

$$e = f(\bar{x}, \overset{(n-2)}{x}, e) = f(\overset{(n-1)}{x}, \bar{e}) = x \bullet \theta x \bullet \theta^2 x \bullet \ldots \bullet \theta^{n-2} x,$$

and in particular $e = b^{n-1}$, which completes our proof. □

Corollary 5. *If* $(G,f) = der_{\theta,b}(G,\bullet)$ *is a distributive* n*-group, then* $ord_n(x) = ord_2(b)$ *for all* $x \in G$.

Proof. Since $e = x \bullet \theta x \bullet \theta^2 x \bullet \ldots \bullet \theta^{n-2} x$ for all $x \in G$, then

$$x^{<k>} = f(x^{<k-1>}, \overset{(n-1)}{x}) = x^{<k-1>} \bullet \theta(x \bullet \theta x \bullet \theta^2 x \bullet \ldots \bullet \theta^{n-2} x) \bullet b =$$
$$= x^{<k-1>} \bullet b = \ldots = x \bullet b^k,$$

then $x^{<k>} = e$ iff $b^k = e$. Hence $ord_n(x) = ord_2(b)$. □

Corollary 6. *A distributive* n*-group* (G,f) *is idempotent iff* $(G,f) = der_{\theta,e}(G,\bullet)$.

Corollary 7. *An n-ary group* $(G,f)=der_{\xi,b}(G,\bullet)$, *where* ξ *is an identity mapping, is distributive iff the exponent of* $(G,\bullet)$ *divides* $n-1$.

Corollary 8. *A ternary group* (G,f) *is distributive iff there exist a commutative group* $(G,\bullet)$ *and an element* $b\in G$ *such that* $b=b^{-1}$ *and* $f(x,y,z)=x\bullet y^{-1}\bullet z\bullet b$.

Proof. If a ternary group $(G,f)=der_{\theta,b}(G,\bullet)$ is distributive, then $b^2=e$ and $x\bullet\theta x=e$. Hence $\theta x=x^{-1}$ and $(G,\bullet)$ is a commutative group. The converse is obvious. □

Corollary 9. *The class of all distributive 3-groups is a proper subvariety of a variety of medial 3-groups.*

Theorem 4. *For any* $n\geq 3$ *there exists a medial distributive n-group which is not derived from any group of the arity* $k<n$.

Proof. Let Z_p be the additive group of rests modulo $p=t^{n-1}-1$, where $t\geq 2$ and $(t-1)|(n-1)$. Then $\theta x\equiv tx(\bmod p)$ is an automorphism of the group Z_p such that $\theta^{n-1}x=x$ for all $x\in Z_p$ and $\theta b=b$ for $b=1+t+t^2+\ldots+t^{n-2}$. The n-ary group $(G,f)=der_{\theta,b}(Z_p,+_p)$ is medial, because the creasing group Z_p is commutative (cf. [4], [6]). Since, in this n-group $\bar{x}\equiv x-b(\bmod p)$ for all $x\in Z_p$, then it is also distributive.

Suppose now that our n-ary group (Z_p, f) is derived from some k-ary group (Z_p, g). Then $n = s(k-1)+1$,

$$f(x_1^{s(k-1)+1}) = g(\ldots g(g(x_1^k), x_{k+1}^{2k+1}), \ldots, x_{(s-1)(k-1)+2}^{s(k-1)+1}), \tag{5}$$

and for $a = 0$ there exists $d \in Z_p$ (d is skew to 0 in (Z_p, g)) such that for all $b \in Z_p$ we have (cf. (3))

$$g(b, \overset{(k-2)}{0}, d) = g(d, \overset{(k-2)}{0}, b)$$

and

$$f(b, \overset{(n-2)}{0}, d) = g(d, \overset{(k-2)}{0}, b, \overset{(n-k)}{0}).$$

For $b = 1$ the last identity gives $(1+d) \equiv (d + t^{k-1})(\mathrm{mod}\, p)$, i.e. $(t^{k-1} - 1) \equiv 0(\mathrm{mod}\, p)$, which for $t \geq 2$ and $2 \leq k < n$ is impossible. Obtained contradiction proves that our n-group is not derived from any k-group of the arity $k < n$, which finish the proof □

Observe that for $t = 2$ the n-ary group constructed in the above proof is idempotent. Thus the following statement is true.

Corollary 10. *For any $n \geq 3$ there exists a medial idempotent distributive n-group which is not derived from any group of the arity $k < n$.*

4. Autodistributive n-groups

Any commutative autodistributive n-ary group (G,f) may be considered as an algebra (G,f,f) of type (n,n). In this case it is an (n,n)-ring in the sense of G.Cupona [2] and G.Crombez [1]. It is also a special case of (f/g)-algebras described by H.J.Hoehnke [12].

Since a commutative idempotent n-ary group is autodistributive, then for any natural $n \geq 3$ there exists an (n,n)-ring in which all elements are identities of this (n,n)-ring.

Theorem 5. *An n-group (G,f) is autodistributive iff it has the form*

$$f(x_1^n) = x_1 \bullet \theta x_2 \bullet \theta^2 x_3 \bullet \ldots \bullet \theta^{n-2} x_{n-1} \bullet x_n \bullet b,$$

where b is a fixed element of a commutative group $(G,\bullet)$ with the identity e, θ is an automorphism of $(G,\bullet)$ such that $\theta b = b$, $x \bullet \theta x \bullet \theta^2 x \bullet \ldots \bullet \theta^{n-2} x = e$ and $\theta^{n-1} x = x$ for all $x \in G$.

Proof. Direct computations show that any n-group $(G,f) = der_{\theta,b}(G,\bullet)$, where $(G,\bullet)$, θ and b are as in our theorem, is autodistributive.

Conversely, if (G,f) is an autodistributive n-group, then (by **Theorem 3** from [5]) it is also distributive and has the form described in our **Theorem 3**.

Moreover, the autodistributivity of (G,f) implies

$$\theta x \bullet b \bullet \theta y \bullet b = f(f(e,x,\overset{(n-2)}{e}),y,\overset{(n-2)}{e}) =$$
$$= f(f(e,y,\overset{(n-2)}{e}),f(x,y,\overset{(n-2)}{e}),f(e,y,\overset{(n-2)}{e}),\ldots,f(e,y,\overset{(n-2)}{e})) =$$
$$= \theta y \bullet b \bullet \theta x \bullet \theta^2 y \bullet b \bullet \ldots \bullet \theta^n y \bullet b \bullet b =$$
$$= \theta y \bullet b \bullet \theta x \bullet \theta^2 (y \bullet \theta y \bullet \ldots \bullet \theta^{n-2} y) \bullet b^n =$$
$$= \theta y \bullet b \bullet \theta x \bullet b,$$

which gives the commutativity of $(G,\bullet)$. This completes our proof. □

Corollary 11. *Any autodistributive n-group is medial.*

Comparing the above result and **Theorem 3** we obtain

Corollary 12. *A distributive n-group is autodistributive iff it is $<\theta,b>$-derived from a commutative group, i.e. iff it is medial.*

This together with **Corollary 7** gives the following characterization of autodistributive n-groups which are b-derived from a some binary group

Corollary 13. *An n-ary group $(G,f) = der_{\xi,b}(G,\bullet)$, where ξ is an identity mapping, is autodistributive iff the group $(G,\bullet)$ is commutative and its exponent divides $n-1$.*

Thus for $n < 7$ all distributive n-groups b-derived from a some binary group are autodistributive. For $n \geq 7$ there are distributive b-derived n-groups which are not autodistributive. As an example of such 7-groups we may consider 7-groups b-derived from the symmetric group S_3.

Observe that **Corollaries 9** and **12** give the following connection between distributive and autodistributive 3-groups.

Corollary 14. *Any distributive 3-group is autodistributive and vice versa.*

Theorem 6. *For any $n>3$ there exists a non-reducible idempotent distributive n-group which is not autodistributive.*

Proof. Let C be the field of complex number. It is not difficult to verify that $G=C^3$ with the multiplication defined by the formula

$$(x,y,z)\bullet(a,b,c)=(x+a,y+b,z+c)$$

is a non-commutative group with the identity $e=(0,0,0)$. The map $\theta(x,y,z)=(\alpha x,\alpha^2 y,\alpha z)$, where α is a primitive $(n-1)$th root of unity, is an automorphism of $(G,\bullet)$ such that $\theta^{n-1}x=x$ and $x\bullet\theta x\bullet\theta^2 x\bullet\ldots\bullet\theta^{n-2}x=e$ for all $x\in G$. Thus an n-group $(G,f)=der_{\theta,e}(G,\bullet)$ is idempotent and distributive (**Theorem 3**). Obviously it is not autodistributive (**Corollary 12**).

Now we prove that this n-group is not derived from any group of the arity $k<n$. Indeed, if an n-group $(G,f)=der_{\theta,e}(G,\bullet)$ is derived from a some binary group with the identity c, then for all $x\in G$ we have

$$f(x,\overset{(n-1)}{c})=f(c,x,\overset{(n-2)}{c})=x,$$

which implies $x\bullet\theta c=c\bullet\theta x$. This for $x=e$ gives $\theta c=c$. Hence $c=e$ and $\theta x=x$ for every $x\in G$, which is incompatible with the definition of θ. Thus this n-group is not derived from a binary group.

If it is derived from some k-ary ($k>2$) group (G,g), then $n=s(k-1)+1$, $s\geq 2$ and (5) holds. Moreover, **Dornte**'s identities for (G,g) and (5) show that

$$f(\bar{x},\overset{(k-2)}{x},x,\overset{(k-2)}{x},\bar{x},\ldots,\overset{(k-2)}{x},\bar{x})=f(x,\bar{x},\overset{(k-3)}{x},x,\overset{(k-2)}{x},\bar{x},\ldots,\overset{(k-2)}{x},\bar{x})$$

for all $x\in G$, where $\bar{x}$ denotes the skew element in (G,g). Hence $\bar{x}\bullet\theta x=x\bullet\theta\bar{x}$, which for $x=e$ gives $\bar{e}=\theta\bar{e}$. Therefore $\bar{e}=e$ and $\theta x=x$, which is incompatible with the definition of θ. This contradiction completes the proof. □

Theorem 7. *There exist non-reducible and non-idempotent distributive n-groups which are not autodistributive.*

Proof. Let K be a fixed field of the characteristic $p\neq 0$. As in the proof of the previous theorem, it is not difficult to verify that $G=K^3$ with the multiplication

$$(x,y,z)\bullet(a,b,c)=(x+a,y+xc+b,z+c)$$

is a non-commutative group with the identity $e=(0,0,0)$. Moreover, for any natural $m\geq 2$ such that $p\nmid m$, the map $\theta(x,y,z)=(\alpha x,y,\beta z)$, where $\alpha\beta=1$ and α is a primitive mth root of unity of **K**, is an automorphism of $(G,\bullet)$ such that $\theta b=b$ for $b=(0,1,0)$. Since $\theta^m x=x$, $x\bullet\theta x\bullet\theta^2 x\bullet\ldots\bullet\theta^{n-2}x=e$ and $b\bullet x=x\bullet b$ for all $x\in G$ and $n=pm+1$, then an n-group $(G,f)=der_{\theta,b}(G,\bullet)$ is distributive but not autodistributive.

In a similar way as in the previous proof we can see that this n-group is not derived from any binary group. It is not derived from any k-ary $(k>2)$ group, too. Indeed , if it is derived from a some k-ary group (G,g), then as in the previous proof $n=s(k-1)+1$, $s\geq 2$ and $\bar{x}\bullet\theta x=x\bullet\theta\bar{x}$ for all $x\in G$. From this identity it follows that $\bar{e}=(0,y,0)$ and $\bar{a}=(1,u,1)$ for $a=(1,0,1)$. Therefore

$$f(\overset{(k-2)}{a},\bar{a},e,\overset{(k-2)}{e},\bar{e},\overset{(k-2)}{e},\ldots,\overset{(k-2)}{e},\bar{e})=e \tag{6}$$

implies $0=1+\alpha+\alpha^2+..+\alpha^{k-2}$. Thus $\alpha^{k-1}=1$ and $k-1=tm$, because α is a primitive mth root of unity. Hence $pm=n-1=stm$, and in the consequence $s=p$. Therefore $n=p(k-1)+1$. Thus from (6) for $a=e$ we obtain $e=(\bar{e})^p\bullet b$, which is impossible because $py+1=0$ has no any solutions in **K**. This contradiction proves that our n-ary group is not reducible to any k-group. This completes the proof. □

From the above proof it follows that non-reducible non-idempotent distributive n-groups which are not autodistributive exist for some $n\geq 7$. For $n=4,5,6$ this problem is open.

Corollary 15. *For any $n\geq 3$ there exist an autodistributive n-group which is not derived from any group of the arity $m<n$.*

Proof. Such n-groups are constructed in the proof of the **Theorem 4**. □

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