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On topological semi-hoops

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Abstract. We investigate topological structuers on a semi-hoop A and under conditions show that there exists a topology $\mathcal{T}$ on A such that $(A, \mathcal{T})$ is a topological semi-hoop. We prove that for each cardinal number α, there exists a topological semi-hoop of order α. Finally, the separation axioms on topological semi-hoops are study and show that for any infinite cardinal number α there exists a Hausdorff topological semi-hoop of order α with non-trivial topology.

1. Introduction

Algebra and topology, the two fundamental domains of mathematics, play complementary roles. Topology studies continuity and convergence and provides a general framework to study the concept of a limit. Algebra studies all kinds of operations and provides a basis for algorithms and calculations. Many of the most important objects of mathematics represent a blend of algebraic and of topological structures. Topological function spaces and linear topological spaces in general, topological groups and topological fields and topological lattices are objects of this kind. Very often an algebraic structure and a topology come naturally together. The rules that describe the relationship between a topology and algebraic operation are almost always transparent and natural the operation has to be continuous, jointly continuous, jointly or separately. In the 20th century many topologists and algebraists have contributed to topological algebra. In this paper, we introduce the notion of topological semi-hoop and derive here conditions that imply a semi-hoop to be a topological semi-hoop. We prove that for each cardinal number α, there exists at least a topological semi-hoop of order α. Also, we study separation axioms on topological semi-hoop and show that for any infinite cardinal number α there exists a Hausdorff topological semi-hoop of order α with non-trivial topology. We prove that a Hausdorff topological semi-hoop algebra exists and we try to study some properties of it. Also, we investigate that under what conditions a topological semi-hoop can be a Hausdorff, connected, T_0 and T_1-spaces.

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2. Preliminaries

In this section, we recollect some definitions and results which will be used in this paper and we shall not cite them every time they are used.

Definition 2.1. An algebra $(A, \odot, \rightarrow, \wedge, 1)$ of type $(2, 2, 2, 0)$ is called a *semi-hoop* if it satisfies the following conditions:

$(SH1)$ $(A, \wedge, 1)$ is a $\wedge$-semilattice with upper bound 1,
$(SH2)$ $(A, \odot, 1)$ is a commutative monoid,
$(SH3)$ $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$, for all $x, y, z \in A$.

On a semi-hoop A we define $x \leqslant y$ if and only if $x \rightarrow y = 1$. It is easy to see that " $\leqslant$ " is a partial order relation on A and for any $x \in A$, $x \leqslant 1$. A semi-hoop A is *bounded* if there exists an element $0 \in A$ such that $0 \leqslant x$, for all $x \in A$. We let $x^0 = 1$, $x^n = x^{n-1} \odot x$, for all $n \in \mathbb{N}$. In a bounded semi-hoop A, we define the negation $'$ on A by, $x' = x \rightarrow 0$, for all $x \in A$. If $(x')' = x$, for all $x \in A$, then the bounded semi-hoop A is said to have the *Double Negation Property*, or (DNP) for short. A semi-hoop A is called a *hoop* if $x \odot (x \rightarrow y) = y \odot (y \rightarrow x)$, for all $x, y \in A$. A semi-hoop A is called a $\sqcup$*-semi-hoop*, if $x \sqcup y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$ and $\sqcup$ is a join operation on A.

The following proposition provides some properties of semi-hoops.

Proposition 2.2. (cf. [7]) *Let A be a semi-hoop. Then the following hold, for all $x, y, z \in A$:*

(i) $x \odot y \leqslant z$ *if and only if* $x \leqslant y \rightarrow z$,
(ii) $x \odot y \leqslant x, y$,
(iii) $x \leqslant y \rightarrow x$,
(iv) $x \odot (x \rightarrow y) \leqslant y$,
(v) $x \leqslant y$ *implies* $z \rightarrow x \leqslant z \rightarrow y$,
(vi) $x \leqslant y$ *implies* $y \rightarrow z \leqslant x \rightarrow z$,
(vii) $(x \rightarrow y) \leqslant (y \rightarrow z) \rightarrow (x \rightarrow z)$,
$(viii)$ $(x \rightarrow y) \odot (y \rightarrow z) \leqslant (x \rightarrow z)$.

Remark 2.3. (cf. [7]) $\sqcup$*-semi-hoop* $(A, \sqcup, \wedge)$ *is a distributive lattice.*

Definition 2.4. Let A be a semi-hoop. A non-empty subset F of A is called a *filter* of A if,

$(F1)$ $x, y \in F$ implies $x \odot y \in F$,
$(F2)$ $x \leqslant y$ and $x \in F$ imply $y \in F$, for any $x, y \in A$.

We use $\mathcal{F}(A)$ to denote the set of all filters of A. Clearly, $1 \in F$, for all $F \in \mathcal{F}(A)$. $F \in \mathcal{F}(A)$ is called a *proper filter* if $F \neq A$. It can be easily seen that, if A is a bounded semi-hoop, then a filter is proper if and only if it is not containing 0.

Let $(A, \odot, \rightarrow, \wedge, 1)$ be a semi-hoop and $F \in \mathcal{F}(A)$. We define a binary relation $\sim_F$ on A by $x \sim_F y$ if and only if $x \rightarrow y$, $y \rightarrow x \in F$, for any $x, y \in A$. Then $\sim_F$

is a congruence on A. Let $A/F = \{x/F \mid x \in A\}$, where $x/F = \{y \in A \mid x \sim_F y\}$. Then the binary relation $\leqslant$ on A/F which is defined by:

$$x/F \leqslant y/F \text{ if and only if } x \to y \in F.$$

is a partial order relation on A/F. Thus $(A/F, \otimes, \rightsquigarrow, \sqcap, 1_{A/F})$ is a semi-hoop, where for any $x, y \in A$:

$$1_{A/F} = 1/F,\ x/F \otimes y/F = (x \odot y)/F,\ x/F \rightsquigarrow y/F = (x \to y)/F$$
$$\text{and} \quad x/F \sqcap y/F = (x \wedge y)/F.$$

Recall that a set X with a family $\mathcal{T}$ of it's subsets is called a *topological space*, denoted by $(X, \mathcal{T})$, if $X, \emptyset \in \mathcal{T}$ and $\mathcal{T}$ is closed under finite intersections and arbitrary unions. The members of $\mathcal{T}$ are called *open sets* of X and the complement of $U \in \mathcal{T}$, that is U^c, is said to be a *closed set*. If B is a subset of X, the smallest closed set containing B is called the *closure* of B and denoted by $\overline{B}$. A subfamily $\{U_\alpha\}$ of $\mathcal{T}$ is said to be a *base* of U if for each $x \in U \in \mathcal{T}$, there exists an α such that $x \in U_\alpha \subseteq U$, or equivalently, each $U \in \mathcal{T}$ is the union of members of $\{U_\alpha\}$. A subset P of topological space $(X, \mathcal{T})$ is said to be a *neighborhood* of $x \in X$ if there exists an open set U such that $x \in U \subseteq P$. Now, let $(A, \mathcal{T})$ be a topological space. We have the following separation axioms in $(A, \mathcal{T})$:

T_0: For each $x, y \in A$ and $x \neq y$, at least one of them has an open neighborhood not containing the other.

T_1: For each $x, y \in A$ and $x \neq y$, there exists two open sets U and V such that $x \in U$ and $y \notin U$, and $y \in V$ and $x \notin V$.

T_2: For each $x, y \in A$ and $x \neq y$, both have disjoint open neighborhoods U and V such that $x \in U$ and $y \in V$.

3. Topological semi-hoops

Definition 3.1. Let $\mathcal{T}$ be a topology on semi-hoop $(A, \odot, \to, \wedge, 1)$ and let $*$ be one of the operations $\odot, \to, \wedge$. Then

- (i) $(A, *, 1)$ is called *right topological semi-hoop* if for each $a \in A$, the map $r_a : A \to A$, defined by $x \to x * a$ is continuous, or equivalently, for any $x \in A$ and each open neighborhood U of $x * a$, there exists an open neighborhood V of x such that $V * a \subseteq U$. In this case, we also call that operation $*$ is continuous in first variable.
- (ii) $(A, *, \mathcal{T})$ is called *topological semi-hoop*, if $* : A \times A \hookrightarrow A$ is continuous, or equivalently, if for any $x, y \in A$ and any open neighborhood W of $x * y$, there exist two open sets U and V such that $x \in U$, $y \in V$ and $U * V \subseteq W$.
- (iii) $(A, \mathcal{T})$ is called *(right)topological semi-hoop*, if $(A, \odot, \to, \wedge, \mathcal{T})$ is (right) topological semi-hoop.

For $U, V \subseteq A$ we define $U \odot V$, $U \to V$ and $U \wedge V$ as follows:

$$U \odot V = \{x \odot y \mid x \in U, y \in V\}, \quad U \to V = \{x \to y \mid x \in U, y \in V\}$$

$$\text{and} \quad U \wedge V = \{x \wedge y \mid x \in U, y \in V\}$$

Example 3.1. (cf. [4]) (i) Let $A = \{0, a, b, c, 1\}$ and

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

$x \wedge y = x \odot (x \to y)$. By routine calculations, A with these operations is a bounded semi-hoop. Define the topology $\mathcal{T} = \{\emptyset, \{0\}, \{a, b\}, \{1, c\}, \{a, b, c, 1\}, A\}$. Then it is easy to see that $(A, \mathcal{T})$ is a topological semi-hoop.

(ii) Let $A = \{a, b, 1\}$ be a chain. Then define, for any $x, y \in A$, $x \wedge y = \min\{x, y\}$ and the operations $\odot$ and $\to$ on A as follows:

$\odot$	1	a	b
1	1	a	b
a	a	a	a
b	b	a	a

$\to$	1	a	b
1	1	a	b
a	1	1	1
b	1	b	1

It is easy to see that A with these operations is a semi-hoop. We define the topology $\mathcal{T} = \{\emptyset, \{a\}, A\}$. Then by routine calculations, $(A, \odot, \to, \wedge, \mathcal{T})$ is a right topological semi-hoop. But $(A, \to, \mathcal{T})$ is not one topological semi-hoop. Because $1 \to a = a \in \{a\}$ such that A and $\{a\}$ are two open sets of 1 and a, respectively, such that $A \to \{a\} = A \nsubseteq \{a\}$.

Theorem 3.2. *Let $(A, \to, \mathcal{T})$ be a topological semi-hoop. If $\{1\}$ is an open set, then $(A, \mathcal{T})$ is a topological semi-hoop.*

Proof. Let $\{1\}$ be an open set and $x \in A$. Since $(A, \to, \mathcal{T})$ is a topological semi-hoop and $x \to x = 1 \in \{1\}$, there is an open sets U such that $x \in U$, $x \to U = \{1\}$ and $U \to x = \{1\}$, which implies that $U = \{x\}$. Hence, $\mathcal{T}$ is a discrete topology on A and so $(A, \mathcal{T})$ is a topological semi-hoop. □

Theorem 3.3. *Let $(A, \odot, \to, \wedge, 1)$ be a semi-hoop and $\mathcal{F}$ be a family of filters which is closed under intersections. Then there exists a topology $\mathcal{T}$ on A such that $(A, \mathcal{T})$ is a topological semi-hoop.*

Proof. Define $\mathcal{T} = \{U \subseteq A \mid \forall x \in U, \exists F \in \mathcal{F}(\mathcal{A}) \text{ such that } x/F \subseteq U\}$. For each $x \in A$ and $F \in \mathcal{F}$, the set $x/F \in \mathcal{T}$, because if y is an arbitrary element of x/F, then $y \in y/F = x/F$. It is easy to see that $\mathcal{T}$ is a topology on A. We prove that $* \in \{\odot, \rightarrow, \wedge\}$ is continuous. For this, suppose $x * y \subseteq U \in \mathcal{T}$ such that $* \in \{\odot, \rightarrow, \wedge\}$. Then for some $F \in \mathcal{F}$, $(x * y)/F \subseteq U$, and so $x/F * y/F \subseteq U$. Since x/F and y/F are two open neighborhoods of x and y, respectively, such that $x/F * y/F \subseteq (x * y)/F \subseteq U$. Hence, $*$ is continuous. Therefore, $(A, \mathcal{T})$ is a topological semi-hoop. □

Theorem 3.4. *Let $(A, \odot, \rightarrow, \wedge, \mathcal{T})$ be a topological semi-hoop such that, for any $\emptyset \neq U \in \mathcal{T}$, $1 \in U$ and $a \notin A$. Suppose $A_a = A \cup \{a\}$. Then there exists a topology $\mathcal{T}_a$ on A_a such that $(A_a, \mathcal{T}_a)$ is a topological semi-hoop.*

Proof. Define the operation $\sqcap$, $\otimes$ and $\rightsquigarrow$ on A_a as follows,

$$x \otimes y = \begin{cases} x \odot y & \text{if } x \in A, y \in A \\ a & \text{if } x \in A_a, y = a \\ a & \text{if } x = a, y \in A_a \end{cases}, \quad x \rightsquigarrow y = \begin{cases} x \rightarrow y & \text{if } x \in A, y \in A \\ a & \text{if } x \in A, y = a \\ 1 & \text{if } x = a, y \in A_a \end{cases}$$

$$x \sqcap y = x \otimes (x \rightsquigarrow y)$$

By routine calculation, we can see that $(A_a, \otimes, \rightsquigarrow, \sqcap, 1)$ is a semi-hoop. It is easy to verify that $\mathcal{T}_a = \{U \cup \{a\} \mid U \in \mathcal{T}\} \cup \{\emptyset\}$ is a topology on A_a. Now, we prove that $(A_a, \mathcal{T}_a)$ is a topological semi-hoop. For this, we prove that $\otimes$ and $\rightsquigarrow$ are continuous.

Let $x \otimes y \in U \cup \{a\}$. In the following cases, we find two sets $V, W \in \mathcal{T}_a$ such that $x \in V$, $y \in W$ and $V \otimes W \subseteq U \cup \{a\}$.

Case 1. If $x, y \in A$, then $x \otimes y = x \odot y \in U$. Since $\odot$ is continuous, there exist $V, W \in \mathcal{T}$ such that $x \in V$, $y \in W$ and $x \odot y \in V \odot W \subseteq U$. If $z_1 \in V \cup \{a\}$ and $z_2 \in W \cup \{a\}$, then $z_1 \otimes z_2 \in \{z_1 \odot z_2, a\} \subseteq U \cup \{a\}$. Hence, $V \cup \{a\} \otimes W \cup \{a\} \subseteq U \cup \{a\}$.

Case 2. If $x = a$ and $y \in A$, then $x = a \in \{a\} \in \mathcal{T}_a$, $y \in A_a \in \mathcal{T}_a$ and $\{a\} \otimes A_a = \{a\} \subseteq U \cup \{a\}$.

Case 3. If $x = y = a$, then $x = y = a \in \{a\} \in \mathcal{T}_a$ and $\{a\} \otimes \{a\} = \{a\} \in U \cup \{a\}$.

These Cases prove that $(A_a, \otimes, \mathcal{T}_a)$ is a topological semi-hoop.

Now, we prove that $\rightsquigarrow$ is continuous. For this, let $x \rightsquigarrow y \in U \cup \{a\}$. In the following cases, we find two sets $V, W \in \mathcal{T}_a$ such that $x \in V$, $y \in W$ and $V \rightsquigarrow W \subseteq U \cup \{a\}$.

Case 1. If $x, y \in A$, then $x \rightsquigarrow y = x \rightarrow y \in U$. Since $\rightarrow$ is continuous, there exist $V, W \in \mathcal{T}$ such that $x \in V$, $y \in W$ and $x \rightarrow y \in V \rightarrow W \subseteq U$. If $z_1 \in V \cup \{a\}$ and $z_2 \in W \cup \{a\}$, since, for any $U \in \mathcal{T}$, $1 \in U$, then $z_1 \rightsquigarrow z_2 \in \{z_1 \rightarrow z_2, a, 1\} \subseteq U \cup \{a\}$. Hence, $V \cup \{a\} \rightsquigarrow W \cup \{a\} \subseteq U \cup \{a\}$.

Case 2. If $x = a$ and $y \in A$, then $x = a \in \{a\} \in \mathcal{T}_a$, $y \in A_a \in \mathcal{T}_a$ and $\{a\} \rightsquigarrow A_a = \{1\} \subseteq U \cup \{a\}$.

Case 3. If $x \in A$ and $y = a$, then $x \in A_a \in \mathcal{T}_a$, $y = a \in \{a\} \in \mathcal{T}_a$ and $A_a \rightsquigarrow \{a\} = \{a, 1\} \subseteq U \cup \{a\}$.

Case 4. If $x = y = a$, then $x = y = a \in \{a\} \in \mathcal{T}_a$ and $\{a\} \rightsquigarrow \{a\} = \{1\} \in U \cup \{a\}$.

These Cases prove that $(A_a, \rightsquigarrow, \mathcal{T}_a)$ is a topological semi-hoop. According to definition of $\sqcap$, since $\otimes$ and $\rightsquigarrow$ are continuous, it is clear that $\sqcap$ is continuous, too. Therefore, $(A_a, \mathcal{T}_a)$ is a topological semi-hoop. □

Theorem 3.5. *For any $n \geqslant 2$ there exists a topological semi-hoop of order n.*

Proof. Let A be a semi-hoop of order $n \geqslant 1$. It is clear that, $\mathcal{T} = \{A, \emptyset\}$ is a topology on A, and so $(A, \mathcal{T})$ is a topological semi-hoop. Now, suppose $x \notin A$. Define $A_x = A \cup \{x\}$. Then by Theorem 3.4, there exist the operations $\sqcap$, $\otimes$, $\rightsquigarrow$ and topology $\mathcal{T}_x$ on A_x such that $(A_x, \mathcal{T}_x)$ is a topological semi-hoop. Since $\mathcal{T}_x = \{\emptyset, \{x\}, A_x\}$, it is clear that $\mathcal{T}_x$ is a non-trivial topology on A_x. □

Theorem 3.6. *For any countable set A such that $1 \in A$, there exists a topological semi-hoop algebra on A.*

Proof. Consider $A = \{x_0 = 1, x_1, x_2, \ldots\}$ as a countable subset and define the operation $\wedge$, $\odot$ and $\rightarrow$ on A as follows,

$$x_i \wedge x_j = x_i \odot x_j = x_{\max\{i,j\}} \qquad \text{and} \qquad x_i \rightarrow x_j = \begin{cases} 1 & \text{if } i \geqslant j \\ x_j & \text{if } i < j \end{cases}$$

$$x_i \leqslant x_j \text{ if and only if } x_i \rightarrow x_j = 1.$$

By routine calculation, we can see that $(A, \odot, \rightarrow, \wedge, 1)$ is a semi-hoop. The set $F_n = \{1, x_1, \ldots, x_n\}$, for any $n \geqslant 1$ is a filter of A. Let $B = \{F_n \mid n \geqslant 1\}$. By Theorem 3.3, there is a non-trivial topology $\mathcal{T}$ on A such that $(A, \mathcal{T})$ is a topological semi-hoop. □

Theorem 3.7. *Let $(A, \odot, \rightarrow, \wedge, 1, \mathcal{T})$ be a topological semi-hoop and α be a cardinal number. If $|\, A \,| \leqslant \alpha$, then there exists a topological semi-hoop $(B, \otimes, \rightsquigarrow, \sqcap, 1, \mathcal{U})$ such that $|\, B \,| \geqslant \alpha$, $1 \in U \in \mathcal{U}$ and A is a subalgebra of B.*

Proof. Let Γ be a collection of a topological semi-hoops $(H, \circ, \dashrightarrow, \cap, 1, \mathcal{U})$ such that for any $A \subseteq H$ we have $\circ\,|_A = \odot$, $\dashrightarrow|_A = \rightarrow$ and $\cap\,|_A = \wedge$.

The following relation is a partial order on Γ:

$$(H,\circ,\dashrightarrow,\cap,1,\mathcal{U}) \leqslant (K,\star,\looparrowright,\sqcap,1,\mathcal{V}) \Leftrightarrow H \subseteq K, \star|_H = \circ, \looparrowright|_H = \dashrightarrow, \sqcap|_H = \cap,\ \mathcal{U} \subseteq \mathcal{V}.$$

Let $\sum = \{(H_i, \circ_i, \dashrightarrow_i, \cap_i, 1, \mathcal{U}_i) \mid i \in I\}$ be a chain in Γ. Put $H = \bigcup_{i\in I} H_i$ and $\mathcal{U} = \bigcup_{i\in I} \mathcal{U}_i$. If x and y are two elements of H, since $\sum$ is a chain, then for some $i \in I$, $x, y \in H_i$. Define $x \circ y = x \circ_i y$, $x \dashrightarrow y = x \dashrightarrow_i y$ and $x \cap y = x \cap_i y$. We prove that $\circ$, $\dashrightarrow$ and $\cap$ are operations on H. Suppose $x, y \in H_i \cap H_j$. Since $\sum$ is a chain, $H_i \subseteq H_j$ or $H_j \subseteq H_i$. Without the lost of generality, assume that $H_i \subseteq H_j$. Let $* \in \{\circ, \dashrightarrow, \cap\}$. Then $*_j\,|_{H_i} = *_i$. So $x *_j y = x *_i y$. This proves that $*$ is an operation on H. Now, it is easy to see that $(H, \circ, \dashrightarrow, \cap, 1)$ is a semi-hoop such that $\circ\,|_A = \odot$, $\dashrightarrow|_A = \rightarrow$ and $\cap\,|_A = \wedge$.

On the other hand, since $\sum$ is a chain, $\mathcal{U}$ is a topology on H. We prove that $(H, \circ, \dashrightarrow, \cap, 1, \mathcal{U})$ is a topological semi-hoop. Let $* \in \{\circ, \dashrightarrow, \cap\}$ and $x*y \in U \in \mathcal{U}$.

Then there exists an $i \in I$ such that $x * y = x *_i y \in U \in \mathcal{U}_i$. Since $*_i$ is continuous in $(H_i, \mathcal{U}_i)$, there are $V, W \in U_i$ such that $x \in V$, $y \in W$ and $V *_i W \subseteq U$. This proves that $*$ is continuous in $(H, \mathcal{U})$. Thus, $(H, \circ, \dashrightarrow, \cap, 1, \mathcal{U})$ is an upper bound for $\sum$. By Zorn's Lemma, Γ has a maximal element. Suppose $(B, \otimes, \rightsquigarrow, \sqcap, 1, \mathcal{U})$ is a maximal element of Γ. We prove that $\mid B \mid \geqslant \alpha$. If $\mid B \mid < \alpha$, then for some non-empty set C, $\mid B \cup C \mid = \alpha$. Take $a \in C - B$ and put $H = B \cup \{a\}$. Then by Theorem 3.4, it is easy to see that H with the following operations

$$x \bullet y = \begin{cases} x \otimes y & \text{if } x \in B, y \in B \\ a & \text{if } x \in H, y = a \\ a & \text{if } x = a, y \in H \end{cases} \qquad x \curvearrowright y = \begin{cases} x \rightsquigarrow y & \text{if } x \in B, y \in B \\ a & \text{if } x \in B, y = a \\ 1 & \text{if } x = a, y \in H \end{cases}$$

$$\text{and} \quad x \sqcap_1 y = x \bullet (x \curvearrowright y)$$

is a semi-hoop. The set $\mathcal{D} = \mathcal{U} \cup \{\{a\}\}$ is a subbase for a topology $\mathcal{V}$ on H. Similar to the proof of Theorem 3.4, we can see that, $(H, \mathcal{V})$ is a topological semi-hoop. But $(H, \bullet, \curvearrowright, \sqcap_1, \mathcal{V})$ is a member of Γ that $(B, \otimes, \rightsquigarrow, \sqcap, 1, \mathcal{U}) < (H, \bullet, \curvearrowright, \sqcap_1, \mathcal{V})$, which is a contradiction. Therefore, $\mid B \mid \geqslant \alpha$ and A is a subalgebra of B. $\square$

Theorem 3.8. *Let α be an infinite cardinal number. Then there is a topological semi-hoop of order α.*

Proof. Let X be a set of cardinality α, $0, 1 \in X$, $A = \{x_0 = 1, x_1, x_2, \ldots\}$ – a countable subset of X such that $0 \notin A$. Similar to Theorem 3.6, define the operation $\wedge$, $\odot$, $\rightarrow$ and $\leqslant$ on A as follows,

$$x_i \wedge x_j = x_i \odot x_j = x_{\max\{i,j\}} \qquad \text{and} \qquad x_i \rightarrow x_j = \begin{cases} 1 & \text{if } i \geqslant j \\ x_j & \text{if } i < j \end{cases}$$

$$x_i \leqslant x_j \text{ if and only if } x_i \rightarrow x_j = 1.$$

By routine calculation, we can see that $(A, \odot, \rightarrow, \wedge, 1)$ is a semi-hoop. The set $F_n = \{1, x_1, \ldots, x_n\}$, for any $n \geqslant 1$ is a filter of A. Let $B = \{F_n \mid n \geqslant 1\}$. By Theorem 3.3, there is a non-trivial topology $\mathcal{T}$ on A such that $(A, \mathcal{T})$ is a topological semi-hoop. Now, define the binary operation $\otimes$, $\rightsquigarrow$ and $\sqcap$ on X as follows,

$$x \otimes y = \begin{cases} x \odot y & \text{if } x \in A, y \in A \\ x & \text{if } x \notin A, y \in A \\ y & \text{if } x \in A, y \notin A \\ 0 & \text{if } x \notin A, y \notin A \end{cases} \qquad x \rightsquigarrow y = \begin{cases} x \rightarrow y & \text{if } x \in A, y \in A \\ y & \text{if } x \in A, y \notin A \\ 1 & \text{if } x \notin A, y \in A \\ 1 & \text{if } x, y \notin A, x = y \\ 1 & \text{if } x, y \notin A \cup \{0\}, x \neq y \\ 1 & \text{if } x = 0, y \notin A \cup \{0\} \\ 0 & \text{if } x \notin A \cup \{0\}, y = 0 \end{cases}$$

$$\text{and} \quad x \sqcap y = \begin{cases} 0 & \text{if } x, y \notin A \cup \{0\}, x \neq y \\ x \otimes (x \rightsquigarrow y) & \text{otherwise.} \end{cases}$$

By routine calculation, we can see that $(X, \otimes, \rightsquigarrow, \sqcap, 0, 1)$ is a bounded semi-hoop of order α and the set $C = \mathcal{T} \cup \{\{x\} \mid x \notin A\}$ is a subbase for a topology $\mathcal{U}$ on X. Since $\{1\} \notin \mathcal{U}$, $\mathcal{U}$ is a non-trivial topology on X. In the following cases we will show that $(X, \otimes, \rightsquigarrow, \sqcap, \mathcal{U})$ is a topological semi-hoop. For this, let $x \otimes y \in U \in C$.

Case 1. If $x, y \in A$, then $x \otimes y = x \odot y \in U \in \mathcal{T}$. Since $\odot$ is continuous in $(A, \mathcal{T})$, there are $V, W \in \mathcal{T}$ containing x, y, respectively, such that $V \odot W \subseteq U$. Hence, $V \otimes W \subseteq U$, which implies that $\otimes$ is continuous in $(X, \mathcal{U})$.

Case 2. If $x \notin A$ and $y \in A$, then $x \otimes y = \{x\} \subseteq U$. Now $\{x\}$ and A, both, belong to $\mathcal{U}$ and $x \in \{x\}$, $y \in A$ and $\{x\} \otimes A = \{x\} \subseteq U$.

Case 3. If $x \in A$ and $y \notin A$, then A and $\{y\}$ are two elements of $\mathcal{U}$ such that $x \in A$, $y \in \{y\}$ and $x \otimes y = \{y\}$, and so $A \otimes \{y\} = \{y\} \subseteq U$.

Case 4. If $x, y \notin A$, then $x \otimes y = \{0\} \subseteq U$. Then $\{x\}$ and $\{y\}$ are two open sets in $\mathcal{U}$ which contains x, y, respectively, and $\{x\} \otimes \{y\} = \{0\} \subseteq U$.

These Cases prove that $(X, \otimes, \mathcal{U})$ is a topological semi-hoop.

Now, we prove that $\rightsquigarrow$ is continuous. For this, let $x \rightsquigarrow y \in U$. In the following cases, we find two sets $V, W \in \mathcal{U}$ such that $x \in V$, $y \in W$ and $V \rightsquigarrow W \subseteq U$.

Case 1. If $x, y \in A$, then $x \rightsquigarrow y = x \to y \in U \in \mathcal{T}$. Since $\to$ is continuous in $(A, \mathcal{T})$, there are $V, W \in \mathcal{T}$ containing x, y, respectively, such that $V \to W \subseteq U$. Hence, $V \rightsquigarrow W \subseteq U$, which implies that $\rightsquigarrow$ is continuous in $(X, \mathcal{U})$.

Case 2. If $x \in A$ and $y \notin A$, then $x \rightsquigarrow y = \{y\} \subseteq U$. Thus, A and $\{y\}$ are two elements of $\mathcal{U}$ such that $x \in A$, $y \in \{y\}$ and $x \rightsquigarrow y = \{y\}$, and so $A \rightsquigarrow \{y\} = \{y\} \subseteq U$.

Case 3. If $x \notin A$ and $y \in A$, then $x \rightsquigarrow y = \{1\} \subseteq U$. Now $\{x\}$ and A, both, belong to $\mathcal{U}$ and $x \in \{x\}$, $y \in A$ and $\{x\} \rightsquigarrow A = \{1\} \subseteq U$.

Case 4. If $x, y \notin A$ and $x = y$, then $x \rightsquigarrow y = \{1\} \subseteq U$. Then $\{x\}$ is an open set in $\mathcal{U}$ which contains x and $\{x\} \rightsquigarrow \{x\} = \{1\} \subseteq U$.

Case 5. If $x, y \notin A \cup \{0\}$ and $x \neq y$, then $x \rightsquigarrow y = \{1\} \subseteq U$. Then $\{x\}$ and $\{y\}$ are two open sets in $\mathcal{U}$ which contains x, y, respectively, and $\{x\} \rightsquigarrow \{y\} = \{1\} \subseteq U$.

Case 6. If $x = 0$ and $y \notin A \cup \{0\}$, then $x \rightsquigarrow y = \{1\} \subseteq U$. Then $\{0\}$ and $\{y\}$ are two open sets in $\mathcal{U}$ which contains x, y, respectively, and $\{x\} \rightsquigarrow \{y\} = \{1\} \subseteq U$.

Case 7. If $x \notin A \cup \{0\}$ and $y = 0$, then $x \rightsquigarrow y = \{0\} \subseteq U$. Then $\{x\}$ and $\{0\}$ are two open sets in $\mathcal{U}$ which contains x, y, respectively, and $\{x\} \rightsquigarrow \{y\} = \{0\} \subseteq U$.

These Cases prove that $(X, \rightsquigarrow, \mathcal{U})$ is a topological semi-hoop. According to definition of $\sqcap$, since $\otimes$ and $\rightsquigarrow$ are continuous, then $\sqcap$ is continuous, too. Therefore, there is a topological semi-hoop of order α. $\square$

Theorem 3.9. *Let α be an infinite cardinal number. Then there is a right topological semi-hoop of order α, which is not a topological semi-hoop.*

Proof. Let A be a set with cardinal number α such that $0, 1 \in A$. Consider a countable subset $A_1 = \{x_0 = 1, x_1, x_2, \ldots\}$ of A and define

$$x_i \wedge x_j = x_i \odot x_j = x_{\max\{i,j\}} \quad \text{and} \quad x_i \to x_j = \begin{cases} 1 & \text{if } i \geqslant j \\ x_j & \text{if } i < j \end{cases}$$

$$x_i \leqslant x_j \text{ if and only if } x_i \to x_j = 1$$

By routine calculations, we can see that $(A_1, \odot, \to, \wedge, 1)$ is a semi-hoop. If $U_i = \{x_i, x_{i+1}, x_{i+2}, \ldots\}$, then $B = \{U_i \mid i = 1, 2, 3, \ldots\}$ is a base for a topology $\mathcal{T}_{A_1}$ on A_1. We prove that $(A_1, \odot, \to, \wedge, 1, \mathcal{T}_{A_1})$ is a right topological semi-hoop. For this, let $x_i \odot x_j \in U \in \mathcal{T}_{A_1}$. If $i \leqslant j$, then $x_i \odot x_j = x_j \in U$. Since $x_j \in U_j$, we have $x_j \in U_j \cap U$, then $x_i \odot x_j = x_i \odot (U_j \cap U) = U_j \cap U \subseteq U$. By the similar way, if $i > j$, then $x_i \odot x_j = x_i \in U$. Since $\odot$ is commutative, $x_j \odot x_i \in x_j \odot (U_i \cap U) = U_i \cap U \subseteq U$. Hence, $(A_1, \odot, \mathcal{T}_{A_1})$, and so $(A_1, \odot, \wedge, \mathcal{T}_{A_1})$ is a topological semi-hoop. Now, suppose $x_i \to x_j \in U \in \mathcal{T}_{A_1}$. If $i \geqslant j$, then $x_i \to x_j = 1 \in U$. Since A_1 is only open neighborhood of $\{1\}$, $U = A_1$. Clearly, $x_j \in A_1$ and $x_i \to A_1 \subseteq A_1 = U$. If $i < j$, then $x_i \to x_j = x_j \in U$. Since B is a base for $\mathcal{T}_{A_1}$, $x_j \in U_j \subseteq U$. Since $i < j$, $x_i \to U_j = U_j \subseteq U$. Therefore, $(A_1, \to, \mathcal{T}_{A_1})$ is a right topological semi-hoop. But this space is not a topological semi-hoop, because $x_1 \in U_1$, $x_2 \in U_2$ and $x_1 \to x_2 = x_2 \in U_2$, but $1 = x_2 \to x_2 \in U_1 \to U_2 \notin U_2$. Similar to Theorem 3.8, let A with the following operations,

$$x \otimes y = \begin{cases} x \odot y & \text{if } x \in A_1, y \in A_1 \\ x & \text{if } x \notin A_1, y \in A_1 \\ y & \text{if } x \in A_1, y \notin A_1 \\ 0 & \text{if } x \notin A_1, y \notin A_1 \end{cases} \qquad x \rightsquigarrow y = \begin{cases} x \to y & \text{if } x \in A_1, y \in A_1 \\ y & \text{if } x \in A_1, y \notin A_1 \\ 1 & \text{if } x \notin A_1, y \in A_1 \\ 1 & \text{if } x, y \notin A_1, x = y \\ 1 & \text{if } x, y \notin A_1 \cup \{0\}, x \neq y \\ 1 & \text{if } x = 0, y \notin A_1 \cup \{0\} \\ 0 & \text{if } x \notin A_1 \cup \{0\}, y = 0 \end{cases}$$

$$\text{and} \quad x \sqcap y = \begin{cases} 0 & \text{if } x, y \notin A \cup \{0\}, x \neq y \\ x \otimes (x \rightsquigarrow y) & \text{otherwise.} \end{cases}$$

Then $(A, \otimes, \rightsquigarrow, \sqcap, 0, 1)$ is a bounded semi-hoop. As the proof of Theorem 3.8, we can prove that $\mathcal{B} = \mathcal{T}_{A_1} \cup \{\{x\} \mid x \notin A_1\}$ is a subbase for a topology $\mathcal{U}$ on A such that $(A, \otimes, \rightsquigarrow, \sqcap, 0, 1, \mathcal{U})$ is a right topological bounded semi-hoop. But $\rightsquigarrow$ is not continuous in $(A, \mathcal{U})$, because $\to$ is not continuous in $(A_1, \mathcal{T}_{A_1})$. □

4. Hausdorff topological semi-hoops

Theorem 4.1. *Let $(A, \mathcal{T})$ be a topological semi-hoop and $\mathcal{F}(A)$ a basis of $\mathcal{T}$. Then, for all $x \in A$, $x^2 = x$ if and only if $(A, \mathcal{T})$ is a T_0-space.*

Proof. $(\Rightarrow)$ Let $x^2 = x$, for all $x \in A$. Then $x^n = x$, for all $n \in \mathbb{N}$. Suppose $x, y \in A$ and $x \neq y$. Since $\mathcal{F}(A)$ is a basis of $\mathcal{T}$, the filters $\langle x \rangle$ and $\langle y \rangle$ are two open neighborhoods of x and y, respectively. If $y \in \langle x \rangle$ and $x \in \langle y \rangle$, then there exist $n, m \in \mathbb{N}$ such that $x^n \leqslant y$ and $y^m \leqslant x$. Hence, $x \leqslant y$ and $y \leqslant x$, and so $x = y$, which is a contradiction.
$(\Leftarrow)$ Let $(A, \mathcal{T})$ be a T_0-space and $x \in A$. If $x^2 \neq x$, then there exists $U \in \mathcal{F}(A)$ such that $x \in U$ and $x^2 \notin U$ or there exists $V \in \mathcal{F}(A)$ such that $x \notin V$ and $x^2 \in V$. But both statements are not correct, because U and $V \in \mathcal{F}(A)$. □

Theorem 4.2. *Let $(A, \mathcal{T})$ be a topological semi-hoop and U be an open neighborhood of* 1. *Then,*

(i) *if for each $x \in A$, $U \to x$ is an open neighborhood of x, then $(A, \mathcal{T})$ is T_0-space,*

(ii) *if for each $x \in A$, $U \odot x$ is an open neighborhood of x, then $(A, \mathcal{T})$ is T_0-space.*

Proof. (i). Let $x, y \in A$ and $x \neq y$. Then $U \to x \in \mathcal{T}$ and $U \to y \in \mathcal{T}$. If $x \in U \to y$ and $y \in U \to x$, then by Proposition 2.2(v), $y \leqslant x$ and $x \leqslant y$. Hence, $x = y$, which is a contradiction. Therefore, $(A, \mathcal{T})$ is T_0-space.

(ii). The proof is similar (i). □

Theorem 4.3. *Let $(A, \to, \mathcal{T})$ be a topological semi-hoop. Then $(A, \to, \mathcal{T})$ is T_0-space if and only if for any $1 \neq x \in A$, there exist $U \in \mathcal{T}$ such that $x \in U$ and $1 \notin U$.*

Proof. Let $x, y \in A$ and $x \neq y$. Then $x \to y \neq 1$ or $y \to x \neq 1$. Without the lost of generality, suppose $x \to y \neq 1$. Then there exist a $U \in \mathcal{T}$ such that $x \to y \in U$ and $1 \notin U$. Since $\to$ is continuous, there are $P, Q \in \mathcal{T}$ such that $x \in P$, $y \in Q$ and $P \to Q \subseteq U$. If $x \in Q$, then $1 = x \to x \in P \to Q \subseteq U$, which is a contradiction. So, $x \notin Q$. Hence, $(A, \to, \mathcal{T})$ is T_0-space. The proof of converse is clear. □

Theorem 4.4. *If α is an infinite cardinal number, then there is a T_0 topological semi-hoop of order α, which it's topology is non-trivial.*

Proof. Let $(A, \odot, \to, \wedge, \mathcal{T})$ and $(X, \otimes, \rightsquigarrow, \sqcap, \mathcal{U})$ be topological semi-hoops in Theorem 3.8. It is clear that $\mathcal{U}$ is non-trivial. Let $x \in X - \{1\}$. If $x \in A$, then for some $n \geqslant 1$, $x \notin F_n$. Hence, $x \in x/F_n \in \mathcal{U}$ and $1 \notin x/F_n$. If $x \notin A$, then $x \in \{x\} \in \mathcal{U}$ and $1 \notin \{x\}$. Now, by Theorem 4.3, $(X, \otimes, \rightsquigarrow, \sqcap, \mathcal{U})$ is a topological semi-hoop of order α. □

In the next example, we have a topological semi-hoop that is T_1-space.

Example 4.2. Let A be a $\sqcup$-semi-hoop such that $x^2 = x$, for all $x \in A$. Suppose $A_a = \{x \in A \mid x \geqslant a\}$ and $B = \{A_a \mid a \in A\}$. We claim that B is a basis of a topology on A. For this, it is clear that $x \in A_x$, for all $x \in A$. Suppose $x \in A_a \cap A_b$, for $a, b \in A$. Then $a \leqslant x$ and $b \leqslant x$. Since A is a $\sqcup$-semi-hoop, we have $a \sqcup b \leqslant x$, and so $x \in A_{a \sqcup b}$. Also, if $x \in A_{a \sqcup b}$, then $a, b \leqslant a \sqcup b \leqslant x$. Hence, $x \in A_a \cap A_b$. Thus, $A_a \cap A_b = A_{a \sqcup b}$. Therefore, B is a basis of a topology $\mathcal{T}$ on A. Now, we prove that $(A, \odot, \mathcal{T})$ is a T_1 topological semi-hoop. For this, let $x, y, c \in A$ such that $x \odot y \in A_c$. Then $c \leqslant x \odot y$. By Proposition 2.2(ii), $c \leqslant x \odot y \leqslant x, y$, then $x, y \in A_c$. Thus, $x \odot y \in A_c \odot A_c$. Let $z \in A_c \odot A_c$. Then there exist $a, b \in A_c$ such that $z = a \odot b$. Since $a, b \geqslant c$, we have $a \odot b \geqslant c \odot c$. Then by assumption, $a \odot b \geqslant c$. Hence, $z = a \odot b \in A_c$, and so $A_c \odot A_c \subseteq A_c$. Then $(A, \odot, \mathcal{T})$ is a topological semi-hoop. Now, suppose $x, y \in A$ such that $x \neq y$. Then $x \in A_x$ and $y \in A_y$. If $y \in A_x$ and $x \in A_y$, then $x \leqslant y$ and $y \leqslant x$. This implies that $x = y$, which is a contradiction. Therefore, $(A, \odot, \mathcal{T})$ is a T_1 topological semi-hoop.

Theorem 4.5. *Let $(A, \to, \mathcal{T})$ be a topological semi-hoop. Then $(A, \mathcal{T})$ is a T_1-space if and only if it is a T_0-space.*

Proof. Let $(A, \mathcal{T})$ be a T_0-space and $x \neq y$. Then $x \to y \neq 1$ or $y \to x \neq 1$. Without the lost of generality, suppose $x \to y \neq 1$. Then there exist a $U \in \mathcal{T}$ such that $x \to y \in U$ and $1 \notin U$ or $x \to y \notin U$ and $1 \in U$. First assume that $x \to y \in U$ and $1 \notin U$. Since $\to$ is continuous, there are $P, Q \in \mathcal{T}$ such that $x \in P$, $y \in Q$ and $P \to Q \subseteq U$. If $x \in Q$, then $1 = x \to x \in P \to Q \subseteq U$, which is a contradiction. Similarly, $y \notin P$. Now, if $1 \in U$ and $x \to y \notin U$, then since $1 = x \to x = y \to y \in U$, there are $V, W \in \mathcal{T}$ such that $x \in V$ and $y \in W$ such that $V \to V \subseteq U$ and $W \to W \subseteq U$. If $y \in V$, then $x \to y \in V \to V \subseteq U$, which is a contradiction. Similarly, $x \notin W$. Therefore, $(A, \mathcal{T})$ is a T_1-space. The proof of converse is clear. □

Corollary 4.6. *If α is an infinite cardinal number, then there is a T_1 topological semi-hoop of order α which it's topology is non-trivial.*

Proof. By Theorems 4.4 and 4.5, the proof is clear. □

Theorem 4.7. *Let $(A, \to, \mathcal{T})$ be a topological semi-hoop. Then the following statements are equivalent:*

(i) *$(A, \to, \mathcal{T})$ is Hausdorff.*

(ii) *$\{1\}$ is closed.*

(iii) *for any $1 \neq x \in A$, there exist two open sets U and V of 1 and x, respectively, such that $U \cap V = \emptyset$.*

(iv) *$(A, \to, \mathcal{T})$ is T_1-space.*

Proof. $(i) \Rightarrow (ii)$. Since A is Hausdorff, it is clear that $\{1\}$ is closed.

$(ii) \Rightarrow (iii)$. Let $\{1\}$ be closed and $x \neq 1$. Then $1 \to x = x \in A - \{1\} \in \mathcal{T}$. Since $\to$ is continuous, there exist two open neighborhoods U and V of 1 and x, respectively, such that $U \to V \subseteq A - \{1\}$. If $z \in U \cap V$, then $1 = z \to z \in U \to V \subseteq A - \{1\}$, which is a contradiction. Therefore, $U \cap V = \emptyset$.

$(iii) \Rightarrow (iv)$. Let $x, y \in A$ and $x \neq y$. Then $x \to y \neq 1$ or $y \to x \neq 1$. Without the lost of generality, suppose $x \to y \neq 1$. By (iii), there exist two disjoint open sets U and V which contain $x \to y$ and 1, respectively. Since $\to$ is continuous, there are $P, Q \in \mathcal{T}$ such that $x \in P$, $y \in Q$ and $P \to Q \subseteq U$. If $x \in Q$, then $1 = x \to x \in P \to Q \subseteq U$, which is a contradiction. So, $x \notin Q$. Similarly, $y \notin P$. Hence, $(A, \to, \mathcal{T})$ is T_1-space.

$(iv) \Rightarrow (i)$. Let $x, y \in A$ and $x \neq y$. Then $x \to y \neq 1$ or $y \to x \neq 1$. Without the lost of generality, suppose $x \to y \neq 1$. Since $\mathcal{T}$ is a T_1-space, there exist two open neighborhoods U and V of $x \to y$ and 1, respectively, such that $1 \notin U$ and $x \to y \notin V$. Since $\to$ is continuous, there exist $P, Q \in \mathcal{T}$ such that $x \in P$, $y \in Q$ and $P \to Q \subseteq U$. If $z \in P \cap Q$, then $1 = z \to z \in U$, which is a contradiction, and so $P \cap Q = \emptyset$. By the similar way, other case is clear. Therefore, $(A, \to, \mathcal{T})$ is Hausdorff. □

Corollary 4.8. *If α is an infinite cardinal number, then there is a Hausdorff topological semi-hoop of order α, which it's topology is non-trivial.*

Proof. By Corollary 4.6 and Theorem 4.7, the proof is clear. $\square$

Suppose A is a semi-hoop algebra and F is a proper filter of A. Define $\sum = \{U \in \mathcal{F}(A) \mid \exists\ F \in \mathcal{F}(A)$ such that $F \subseteq U\}$ and $f : \sum \hookrightarrow \mathcal{F}(A/F)$ is a map such that $f(U) = U/F$, for all $U \in \sum$. Then it is easy to prove that f is a one to one corresponding among $\sum$ and $\mathcal{F}(A/F)$.

Let $\mathcal{T}$ be a topology on semi-hoop algebra A, $F \in \mathcal{F}(A)$ and $\pi : A \hookrightarrow A/F$ be canonical epimorphism. Then the set $\widetilde{\mathcal{T}} = \{U \subseteq A/F \mid \pi^{-1}(U) \in \mathcal{T}\}$ is a topology on A/F. $\widetilde{\mathcal{T}}$ is called *quotient topology.*

It is easy to see that $\pi_F : (A, \mathcal{T}) \hookrightarrow (A/F, \widetilde{\mathcal{T}})$ is continuous. Also, it is easy to prove that if $* \in \{\odot, \rightarrow, \wedge\}$ and $(A, *, \mathcal{T})$ is a topological semi-hoop algebra, then $(A/F, *, \widetilde{\mathcal{T}})$ is a topological quotient semi-hoop algebra provided $\pi_F : A \hookrightarrow A/F$ is an open map.

Proposition 4.9. *Let $(A, \rightarrow, \mathcal{T})$ be a topological semi-hoop, $F \in \mathcal{F}(A)$ and $\widetilde{\mathcal{T}}$ be quotient topology on A/F. If $\pi_F : A \hookrightarrow A/F$ is an open map, then*

(*i*) *F is open if and only if $(A/F, \widetilde{\mathcal{T}})$ is discrete.*

(*ii*) *F is closed if and only if $(A/F, \rightarrow, \widetilde{\mathcal{T}})$ is Hausdorff.*

Proof. (*i*). Let F be open. Since $\pi_F : A \hookrightarrow A/F$ is an open map, the set $\pi_F(F) = 1/F$ belongs to $\widetilde{\mathcal{T}}$. Since $(A/F, \rightarrow, \widetilde{\mathcal{T}})$ is a topological semi-hoop, by Theorem 3.2, $(A/F, \widetilde{\mathcal{T}})$ is discrete. Conversely, suppose $(A/F, \widetilde{\mathcal{T}})$ is discrete. Then $1/F$ is an open set. Since $\pi_F : A \hookrightarrow A/F$ is continuous, $F = {\pi_F}^{-1}(1/F) \in \mathcal{T}$.

(*ii*). ($\Rightarrow$) By assumption, F is closed, then F^c is open. Thus, for any $x, y \in A$, if $x \rightarrow y \in F^c$, then there are two open neighborhood U and V of x and y, respectively, such that $U \rightarrow V \subseteq F^c$, because $\rightarrow$ is continuous. Also, since π is open, so $\pi(U)$ and $\pi(V)$ are two open neighborhoods of x/F and y/F, respectively, such that $\pi(U) \rightarrow \pi(V) \subseteq \pi(U \rightarrow V) \subseteq \pi(F^c)$. If $z/F \in \pi(U) \cap \pi(V)$, then $1/F = z/F \rightarrow z/F \in \pi(U) \rightarrow \pi(V) \subseteq \pi(F^c)$, which is a contradiction. Therefore, $(A/F, \rightarrow, \widetilde{\mathcal{T}})$ is Hausdorff.

($\Leftarrow$) Since A/F is Hausdorff, the set $\{1/F\}$ is closed in A/F, and so $F = \pi^{-1}(1/F)$ is closed in A. $\square$

5. Connected topological semi-hoop

A topological space A is said to be *disconnected* if it is the union of two disjoint non-empty open sets. Otherwise, A is said to be *connected.* Also, $(A, \mathcal{T})$ is called *locally connected* at $x \in A$, if for every open subset V containing x, there exists a connected, open subset U with $x \in U \subseteq V$. *Connected component*, a maximal subset of a topological space that can not be covered by the union of two disjoint open sets. The components of any topological space X form a partition of X, they are disjoint, non-empty, and their union is the whole space. A topological space X is *totally disconnected* if the connected components in X are the one-point sets. Also, we know that the image of a connected space under a continuous map is connected and a finite cartesian product of connected spaces is connected (cf. [10]).

Proposition 5.1. *Let $(A, \mathcal{T})$ be a topological semi-hoop. If C is connected components of* 1*, then C is a closed filter of A.*

Proof. Let C be connected component of 1 and $x \in C$. Since $\odot$ is continuous, $x \odot C$ is a connected set which contains x. Since $x \in (x \odot C) \cap C$, the set $(x \odot C) \cup C$ is a connected set containing 1. Hence, $(x \odot C) \cup C \subseteq C$. This implies that $x \odot C \subseteq C$, so $C \odot C \subseteq C$. Now, suppose $x \leqslant y$ and $x \in C$, then $1 = x \to y \in C \to y$. Since $\to$ is continuous, $C \to y$ is a connected set. Hence, $C \to y \subseteq C$. So, $y = 1 \to y \in C \to y \subseteq C$. Therefore, C is a filter of A. Since C is component, clearly it is closed. $\square$

Recall a semi-hoop A is *locally finite* if only filters of A are $\{1\}$ and A.

Theorem 5.2. *Let $(A, \mathcal{T})$ be a topological locally finite semi-hoop. Then $(A, \mathcal{T})$ is connected or totally disconnected.*

Proof. Suppose $(A, \mathcal{T})$ is not connected. Let C be connected component of 1. Then by Proposition 5.1, C is a closed filter of A. Since $(A, \mathcal{T})$ is not connected, $C = \{1\}$. Let C_x be connected component of $x \in A$. Since $\to$ is continuous, $x \to C_x$ is a connected set containing $1 = x \to x$. Hence, $x \to C_x \subseteq C = \{1\}$. Thus, $x \to C_x = 1$. By the similar way, $C_x \to x = 1$. This implies that $C_x = \{x\}$, and so $(A, \mathcal{T})$ is totally disconnected. $\square$

Lemma 5.3. *Let A be a semi-hoop and $F \in \mathcal{F}(A)$. Then $x \odot a = y \odot b$, for some $a, b \in F$ if and only if $x/F = y/F$ in A/F.*

Proof. ($\Rightarrow$) Let $x \odot a = y \odot b$, for some $a, b \in F$. Since $x \odot a \leqslant y \odot b$, by Proposition 2.2(i), $a \leqslant x \to (y \odot b)$. Since $F \in \mathcal{F}(A)$ and $a \in F$, by (F2), $x \to (y \odot b) \in F$. Moreover, by Proposition 2.2(ii) and (v), $y \odot b \leqslant y$, and so $x \to (y \odot b) \leqslant x \to y$. Since $F \in \mathcal{F}(A)$ and $x \to (y \odot b) \in F$, by (F2), $x \to y \in F$. By the similar way, $y \to x \in F$. Therefore, $x/F = y/F$.
($\Leftarrow$) Let $x/F = y/F$, for $x, y \in A$. Then $x \to y \in F$ and $y \to x \in F$. Thus, there exists $a \in F$ such that $y \to x = a$. Since $a \to (y \to x) = 1$, by (SH3),

$(a \odot y) \to x = 1$, and so $a \odot y \leqslant x$. Then

$$\begin{array}{rll} & [a \odot (x \to y)] \to [x \to (a \odot y)] & \text{by (SH3)} \\ = & (x \to y) \to [a \to (x \to (a \odot y))] & \text{by (SH3)} \\ = & (x \to y) \to [(x \odot a) \to (a \odot y))] & \text{by (SH3)} \\ = & (x \to y) \to [x \to (a \to (a \odot y))] & \text{by (SH3)} \\ = & (x \odot (x \to y)) \to (a \to (a \odot y)) & \text{by Proposition 2.2(iv) and (vi)} \\ \geqslant & y \to (a \to (a \odot y)) & \text{by (SH3)} \\ = & (a \odot y) \to (a \odot y) & \\ = & 1 & \end{array}$$

Then $a \odot (x \to y) \leqslant x \to (a \odot y)$. Since $F \in \mathcal{F}(A)$, $a, x \to y \in F$, by (F1), $a \odot (x \to y) \in F$ and by (F2), $x \to (a \odot y) \in F$. Then there exists $b \in F$ such that $x \to (a \odot y) = b$. Thus, $b \to [x \to (a \odot y)] = 1$. By (SH3), $(b \odot x) \to (a \odot y) = 1$, and so $b \odot x \leqslant a \odot y$. By the similar way, $a \odot y \leqslant b \odot x$. Therefore, $a \odot y = b \odot x$. □

Proposition 5.4. *Let $(A, \mathcal{T})$ be a topological semi-hoop and C be a connected component of* 1 *in A. Then the following statements hold,*

(*i*) *if D is a closed subset of A/C such that $\pi^{-1}(D)$ is disconnected, then D is disconnected,*

(*ii*) *if $(A, \mathcal{T})$ is disconnected, then $(A/C, \widetilde{\mathcal{T}})$ is disconnected.*

Proof. (*i*). $\pi^{-1}(D) = X \cup Y$, where X, Y are two non-empty disjoint closed subsets of $\pi^{-1}(D)$ and hence, A. It is clear that $X \subseteq \pi^{-1}(\pi(X))$. Let $z \in \pi^{-1}(\pi(X))$. Then there exists $x \in X$ such that $x/F = z/F$. By Lemma 5.3, $x \odot a = z \odot b$, for some $a, b \in C$. Given C_x and C_z, two connected component of x and z, respectively. Then $x \odot a \in x \odot C \subset C_x$ and $z \odot b \in z \odot C \subset C_z$. Since $z \odot b = x \odot a$, $C_x \cap C_z \neq \emptyset$. Hence, $C_x \cup C_z$ is connected. This means that $C_x = C_z$, and so $z \in X$. Therefore, $X = \pi^{-1}(\pi(X))$. Since X is closed in A, $\pi(X)$ is closed in A/C. By the similar way, $Y = \pi^{-1}(\pi(Y))$ and $\pi(Y)$ is a closed subset of A/C. On the other hand, $\pi^{-1}(\pi(X) \cap \pi(Y)) = X \cap Y = \emptyset$ implies that $\pi(X) \cap \pi(Y) = \emptyset$. So, $D = \pi(X) \cup \pi(Y)$, where $\pi(X)$ and $\pi(Y)$ are two disjoint closed subsets of A/C. Hence, D is disconnected.

(*ii*). Let $(A, \mathcal{T})$ be disconnected. Since $\pi^{-1}(A/C) = A$, by (i), $(A/C, \widetilde{\mathcal{T}})$ is disconnected. □

Theorem 5.5. *Let $(A, \mathcal{T})$ be a topological semi-hoop, C be a connected component of* 1 *in A and $\pi : A \to A/C$ be open canonical epimorphism. Then A/C is totally disconnected.*

Proof. Let C be the connected component of 1 in A. Then by Proposition 5.1, C is a closed filter of A. Let K be a connected component of $1/C$ in A/C. If $1/C \neq x/C$, for some $x/C \in A/C$, then C is a proper subset of $\pi^{-1}(K)$. Hence, $\pi^{-1}(K)$ is not connected. Since K is closed in A/C, by Proposition 5.4,

K is disconnected, which is contradiction. Therefore, $K = \{1\}$. Suppose K_x is connected component of x/C in A/C. Since $\rightarrow$ is continuous in A/C, $K_x \rightarrow x/C$ is connected and $1/C \in K_x \rightarrow x/C$. Then $K_x \rightarrow x/C \subseteq K = \{1/C\}$. Similarly, $x/C \rightarrow K_x \subseteq K = \{1/C\}$. Hence, for each $y/C \in K_x$, $y/C \rightarrow x/C = 1/C$ and $x/C \rightarrow y/C = 1/C$. So, $x/C = y/C$. This implies that $K_x = \{x/C\}$. Therefore, A/C is totally disconnected. $\square$

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References

[1] **R.A. Borzooei, G.R. Rezaei, N. Kouhestani**, *On (semi)topologica BL-algebras*, Iranian J. Math. Sci. Informatics **6** (2011), 59 − 77.

[2] **R.A. Borzooei, G.R. Rezaei, N. Kouhestani**, *Metrizability on (semi)topological BL-algebras*, Soft Computing **16** (2012), 1681 − 1690.

[3] **R.A. Borzooei, G.R. Rezaei, N. Kouhestani**, *Separation axioms in (semi)-topological quotient BL-algebras*, Soft Computing **16** (2012), 1219 − 1227.

[4] **R.A. Borzooei, M. Aaly Kologani**, *Local and perfect semihoops*, J. Intelligent Fuzzy Syst. **29** (2015), 223 − 234.

[5] **N. Bourbaki**, *Topologie Générale*, Hermann, Paris, (1960).

[6] **A. Dvurečenskij, P. Aglianò**, *Type decomposition of linear pseudo hoops and it's applications*, J. Pure Appl. Algebra **211** (2007), 851 − 861.

[7] **F. Esteva, L. Godo, Hájek, F. Montagna**, *Hoops and fuzzy logic*, J. Logic Comput. **13** (2003), 532 − 555.

[8] **G. Georgescu, L. Leustean, V. Preoteasa**, *Pseudo-hoops*, J. Multiple-Valued Logic and Soft Comput. **11** (2005), 153 − 184.

[9] **T. Husain**, *Introduction to topological groups*, WB Sunders Company, (1966).

[10] **J. R. Munkres**, *Topology a first course*, Prentice-Hall, **23** (1975).

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On some generalized ideals in ternary semigroups

Mohammad Yahya Abbasi and *Sabahat Ali Khan*

Abstract. We characterize the relationship between minimal m-right, minimal (p, q)-lateral, minimal n-left ideal and m-right simple, (p, q)-lateral simple, n-left simple ternary semigroups. Further, some existing results of regular ternary semigroups are studied.

1. Preliminaries

The idea of invesigation of n-ary algebras i.e., the sets with one n-ary operation was given by Kasner [5]. Investigation of ideals in ternary semigroup was initiated by Sioson [8]. He also defined regular ternary semigroups.

A non-empty set S with a ternary operation $S \times S \times S \to S$, written as $(x_1, x_2, x_3) \mapsto [x_1, x_2, x_3]$, is called a *ternary semigroup* if it satisfies the following identity, for any $x_1, x_2, x_3, x_4, x_5 \in S$,

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [[x_1x_2[x_3x_4x_5]].$$

For any positive integers m and n with $m \leqslant n$ and any elements $x_1, x_2, \ldots, x_{2n+1}$ of a ternary semigroup, we can write

$$[x_1x_2 \ldots x_{2n+1}] = [x_1x_2 \ldots .[[x_mx_{m+1}x_{m+2}]x_{m+3}x_{m+4}] \ldots x_{2n+1}].$$

Example 1.1. [1] The set

$$S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a ternary semigroup under matrix multiplication.

Definition 1.2. A non-empty subset I of a ternary semigroup S is called:
- a *left ideal* of S if $SSI \subseteq I$,
- a *lateral ideal* of S if $SIS \subseteq I$,
- a *right ideal* of S if $ISS \subseteq I$,
- a *two-sided ideal* if it is left and right ideal of S,
- an *ideal* of S if I is a left, right and lateral ideal of S.

An ideal I of a ternary semigroup S is called *proper* if $I \neq S$ and *idempotent* if $III = I$.

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Proposition 1.3. *Let S be a ternary semigroup and $a \in S$. Then the principal*

(1) *left ideal generated by $'a'$ is given by $L(a) = SSa \cup \{a\}$*

(2) *right ideal generated by $'a'$ is given by $R(a) = aSS \cup \{a\}$*

(3) *lateral ideal generated $'a'$ is given by $M(a) = SaS \cup SSaSS \cup \{a\}$*

(4) *ideal generated by $'a'$ is given by $I(a) = aSS \cup SaS \cup SSaSS \cup SSa \cup \{a\}$.*

Definition 1.4. An element a in a ternary semigroup S is called *regular* if there exists an element $x \in S$ such that $axa = a$.

2. Main results

Definition 2.1. Let S be a ternary semigroup. Then a ternary subsemigroup
• R is called an *m-right ideal* of S if $RS^{2m} \subseteq R$.
• M is called a *(p, q)-lateral ideal* of S if $(S^pMS^q \cup S^{p+1}MS^{q+1}) \subseteq M$.
• L is called an *n-left ideal* of S if $S^{2n}L \subseteq L$.
where m, n, p, q are positive integers and $p + q$ is an even positive integer.

S is called an m-right (resp. (p, q)-lateral, n-left) *simple* if S is a unique m-right (resp. (p, q)-lateral, n-left) ideal of S.

Example 2.2. Let S be a set of all strictly lower triangular matrices of order 6 over $\mathbb{Z}_0^-$, the set of all non-positive integers, i.e.,

$$S = \{(a_{ij})_{6\times 6} \mid a_{ij} = 0 \ if \ i \leqslant j \ and \ a_{ij} \in \mathbb{Z}_0^- \ if \ i > j\}.$$

Then S is a ternary semigroup under the usual multiplication of matrices over $\mathbb{Z}_0^-$ while S is not a semigroup under the same operation. It is easy to see that

$$M_{gen} = \{(a_{ij}) \in S : a_{43} = a \in \mathbb{Z}_0^- \ and \ a_{ij} = 0 \ otherwise\}.$$

is a ternary subsemigroup of S and M_{gen} is a $(3, 1)$-lateral ideal of S. Now

$$SM_{gen}S = \{(a_{ij}) \,|\, a_{51}, a_{52}, a_{61}, a_{62} \in \mathbb{Z}_0^- \ and \ a_{ij} = 0 \ otherwise\} \nsubseteq M_{gen}.$$

Therefore M_{gen} is not a lateral ideal of S.

Example 2.3. Let S be a set of all strictly upper triangular matrices of order 7 over $\mathbb{Z}_0^-$, i.e.,

$$S = \{(a_{ij})_{7\times 7} \,|\, a_{ij} = 0 \ if \ i \geqslant j \ and \ a_{ij} \in \mathbb{Z}_0^- \ if \ i < j\}.$$

Then S is a ternary semigroup under the usual multiplication of matrices over $\mathbb{Z}_0^-$ while S is not a semigroup under the same operation. Then it is easy to see that

$$\mathcal{M} = \{(a_{ij}) \in S \,|\, a_{45} \in \mathbb{Z}_0^- \ and \ a_{ij} = 0 \ otherwise\}$$

is a ternary subsemigroup of S and $\mathcal{M}$ is a $(3,3)$-lateral ideal of S. Now

$$S\mathcal{M}S = \{(a_{ij}) \in S \,|\, a_{16}, a_{17}, a_{26}, a_{27}, a_{36}, a_{37} \in \mathbb{Z}_0^- \textit{ and } a_{ij} = 0 \textit{ otherwise}\} \nsubseteq M,$$

$$S^2\mathcal{M}S^2 \cup S^3\mathcal{M}S^3 = \{(a_{ij}) \in S \,|\, a_{17}, a_{27} \in \mathbb{Z}_0^{;} \textit{ and } a_{ij} = 0 \textit{ otherwise}\} \nsubseteq M,$$

$$S^3\mathcal{M}S \cup S^4\mathcal{M}S^2 = \{(a_{ij}) \in S \,|\, a_{16}, a_{17} \in \mathbb{Z}_0^- \textit{ and } a_{ij} = 0 \textit{ otherwise}\} \nsubseteq M.$$

Therefore $\mathcal{M}$ is not an $(1,1)$-lateral, $(2,2)$-lateral and $(3,1)$-lateral ideal of S.

Remark 2.4. We know that for a right ideal R, a lateral ideal M and a left ideal L of a ternary semigroup S, $RML \subseteq R \cap M \cap L$. But this result is not true for an m-right ideal R, an (p,q)-lateral ideal M and an n-left ideal L of a ternary semigroup S.

Lemma 2.5. *Let S be a ternary semigroup.*

(1) *Let $\{R_i : i \in I\}$ be a family of m-right ideals of S. Then $\bigcap_{i\in I} R_i$ is also an m-right ideal of S if $\bigcap_{i\in I} R_i \neq \emptyset$.*

(2) *Let $\{M_i : i \in I\}$ be a family of (p,q)-lateral ideals of S. Then $\bigcap_{i\in I} M_i$ is also a (p,q)-lateral ideal of S if $\bigcap_{i\in I} M_i \neq \emptyset$.*

(3) *Let $\{L_i : i \in I\}$ be a family of n-left ideals of S. Then $\bigcap_{i\in I} L_i$ is also an n-left ideal of S if $\bigcap_{i\in I} L_i \neq \emptyset$.*

Theorem 2.6. *Let S be a ternary semigroup. Then*

(1) *Every m-right ideal is an $(m+m_1)$-right ideal of S, where m_1 is a non-negative integer.*

(2) *Every (p,q)-lateral ideal is a $(p+p_1, q+q_1)$-lateral ideal of S, where p_1 and q_1 are non-negative integers and p_1+q_1 is even.*

(3) *Every n-left ideal is an $(n+n_1)$-left ideal of S, where n_1 is a non-negative integer.*

Proof. (2). We have

$$\begin{aligned} S^{p+p_1}MS^{q+q_1} \cup S^{p+p_1+1}MS^{q+q_1+1} &\subset S^{p+p_1-2}MS^{q+q_1-2} \cup S^{p+p_1-1}MS^{q+q_1-1} \\ &\subset \ldots \subset S^{p+1}MS^{q+1} \cup S^pMS^q \subset M, \end{aligned}$$

if p_1, q_1 are odd, and

$$S^{p+p_1}MS^{q+q_1} \cup S^{p+p_1+1}MS^{q+q_1+1} \subset \ldots \subset S^pMS^q \cup S^{p+1}MS^{q+1} \subset M,$$

if p_1, q_1 are even.

Hence in all the two cases, M is a $(p+p_1, q+q_1)$-lateral ideal of S.

Proofs of (1) and (3) are similar. □

Corollary 2.7. *Let S be a ternary semigroup and A be its ternary subsemigroup. If A is a (p,q)-lateral ideal of S. Then, for any positive integer n:*

(1) *A will be an (np, nq)-lateral ideal of S.*

(2) *A will be a (p^n, q^n)-lateral ideal of S.*

Lemma 2.8. *For any non-empty subset A of a ternary semigroup S*

(1) AS^{2m} *is an m-right ideal of S,*

(2) $S^pAS^q \cup S^{p+1}AS^{q+1}$ *is a (p,q)-lateral ideal of S,*

(3) $S^{2n}A$ *is an n-left ideal of S.*

Lemma 2.9. *For any non-empty subset A of a ternary semigroup S*

(1) $(A\cup A^3\cup A^5\cup\ldots\cup A^{2m-1})\cup AS^{2m}$ *is the smallest m-right ideal of S containing A,*

(2) $(A\cup A^3\cup A^5\cup\ldots\cup A^{p+q-1})\cup(S^{p+1}AS^{q+1}\cup S^pAS^q)$ *is the smallest (p,q)-lateral ideal of S containing A,*

(3) $(A\cup A^3\cup A^5\cup\ldots\cup A^{2n-1})\cup S^{2n}A$ *is the smallest n-left ideal of S containing A,*

where m,n,p,q are positive integers and $p+q$ is an even positive integer.

Proof. (1). Let $R=(\bigcup_{i=1}^m A^{2i-1})\cup AS^{2m}$ and $x,y,z\in R$. Clearly $A\subseteq R$.

If $x,y,z\in\bigcup_{i=1}^m A^{2i-1}$, then $xyz\in A^r$. So, $xyz\in\bigcup_{i=1}^m A^{2i-1}$ for $r\leqslant 2m-1$, and we have $xyz\in AS^{2m}$ for $r>2m-1$.

If $x,y,z\in AS^{2m}$, then obviously $xyz\in AS^{2m}$. Therefore R is a ternary subsemigroup of S.

To show R is an m-right ideal of S. We have

$$\begin{aligned} RS^{2m} &= ((\textstyle\bigcup_{i=1}^m A^{2i-1})\cup AS^{2m})S^{2m} = (\bigcup_{i=1}^m A^{2i-1})S^{2m}\cup(AS^{2m})S^{2m} \\ &\subseteq AS^{2m}\subseteq R. \end{aligned}$$

Finally it remains to prove that R is the smallest m-right ideal of S containing A. For this suppose that R_1 is an m-right ideal of S containing A. Then

$$R=(\textstyle\bigcup_{i=1}^m A^{2i-1})\cup AS^{2m}\subseteq(\bigcup_{i=1}^m R_1^{2i-1})\cup R_1S^{2m}\subseteq R_1\cup R_1=R_1.$$

Hence R is the smallest m-right ideal of S containing A.

(2). Let $M=(\bigcup_{i=1}^m A^{2i-1}\cup(S^pAS^q)\cup S^{p+1}AS^{q+1})$, where $p+q=2m$, and $x,y,z\in M$. Clearly $A\subseteq M$. Now we have following two cases:

Case 1: $x,y,z\in\bigcup_{i=1}^m A^{2i-1}$, then $xyz\in A^n$. If $n\leqslant p+q-1$, then we have $xyz\in(\bigcup_{i=1}^m A^{2i-1})$. If $n>p+q-1$, then $xyz\in(S^pAS^q\cup S^{p+1}AS^{q+1})$.

Case 2: $x,y,z\in S^pAS^q\cup S^{p+1}AS^{q+1}$. Then, as it is easy to show, $xyz\in S^pAS^q\cup S^{p+1}AS^{q+1}$.

Therefore M is a ternary subsemigroup of S. It is easy verify that M is a (p,q)-lateral ideal of S.

Finally it remains to prove that M is the smallest (p,q)-lateral ideal of S containing A. For this suppose that M_1 is a (p,q)-lateral ideal of S containing A. Then

$$\begin{aligned} M &= (\textstyle\bigcup_{i=1}^{p+q-1} A^i\cup(S^pAS^q\cup S^{p+1}AS^{q+1})) \\ &\subseteq(\textstyle\bigcup_{i=1}^{p+q-1} M_1^i\cup(S^pM_1S^q\cup S^{p+1}M_1S^{q+1}))\subseteq M_1. \end{aligned}$$

Hence M is the smallest (p,q)-lateral ideal of S containing A.

The proof of (3) is analogous. $\square$

Furthermore, for any $a \in S$ we have:

$R(a) = aS^{2m} \cup \{a, a^3, a^5, \ldots, a^{2m-1}\}$ is an m-right ideal generated by a;

$M(a) = (S^{p+1}aS^{q+1} \cup S^paS^q) \cup \{a, a^3, a^5, \ldots, a^{p+q-1}\}$ is a (p,q)-lateral ideal generated by a;

$L(a) = S^{2n}a \cup \{a, a^3, a^5, \ldots, a^{2n-1}\}$ is an n-left ideal generated by a.

Theorem 2.10. *Let A and B be ternary subsemigroups of S such that $A \subseteq B$ and $B^3 = B$. If A is a (p,q)-lateral ideal of S, then it is a lateral ideal of B.*

Proof. Suppose A and B are two ternary subsemigroups of S such that $A \subseteq B$ and $B^3 = B$. If A is an (p,q)-lateral ideal of S, then $S^{p+1}AS^{q+1} \cup S^pAS^q \subseteq A$. Now we have $BAB \cup B^2AB^2 = BAB^3 \cup B^3BABB^3$ or $B^3AB \cup B^3BABB^3$. Proceed in this way, we get $BAB \cup B^2AB^2 = B^pAB^q \cup B^{p+1}AB^{q+1}$. Now

$$BAB \cup B^2AB^2 = B^pAB^q \cup B^{p+1}AB^{q+1} \subseteq S^pAS^q \cup S^{p+1}AS^{q+1} \subseteq A.$$

This shows that A is a lateral ideal of B. $\square$

Corollary 2.11. *If S is a ternary semigroup such that $S^3 = S$, then every its (p,q)-lateral ideal is its lateral ideal.*

Corollary 2.12. *An idempotent (p,q)-lateral ideal of a ternary semigroup S is its lateral ideal.*

Theorem 2.13. *Let S be a ternary semigroup. Then:*

(1) *An m-right ideal R is minimal if and only if $aS^{2m} = R$ for all $a \in R$.*

(2) *A (p,q)-lateral ideal M is minimal if and only if $(S^paS^q \cup S^{p+1}aS^{q+1}) = M$ for all $a \in M$.*

(3) *An n-left ideal L is minimal if and only if $S^{2n}a = L$ for all $a \in L$.*

Proof. (2) Suppose that a (p,q)-lateral ideal M is minimal. Let $a \in M$. Then $S^paS^q \cup S^{p+1}aS^{q+1} \subseteq S^pMS^q \cup S^{p+1}MS^{q+1} \subseteq M$. By Lemma 2.8(2), we have $S^paS^q \cup S^{p+1}aS^{q+1}$ is a (p,q)-lateral ideal of S. As M is minimal (p,q)-lateral ideal of S therefore $S^paS^q \cup S^{p+1}aS^{q+1} = M$.

Conversely, suppose that $S^paS^q \cup S^{p+1}aS^{q+1} = M$ for all $a \in M$. Let M' be a (p,q)-lateral ideal of S contained in M. Let $m \in M'$. Then $m \in M$. By assumption, we have $S^pmS^q \cup S^{p+1}mS^{q+1} = M$ for all $m \in M$. Now $M = S^pmS^q \cup S^{p+1}mS^{q+1} \subseteq S^pM'S^q \cup S^{p+1}M'S^{q+1} \subseteq M'$. This implies $M \subseteq M'$. Thus, $M = M'$. Hence, M is a minimal (p,q)-lateral ideal of S.

Proofs of (1) and (3) are similar. $\square$

Theorem 2.14. *Let S be a ternary semigroup. Then:*

(1) *S is an m-right simple if and only if $aS^{2m} = S$ for all $a \in S$.*

(2) *S is a (p,q)-lateral simple if and only if $S^p aS^q \cup S^{p+1}aS^{q+1} = S$ for all $a \in S$.*

(3) *S is an n-left simple if and only if $S^{2n}a = S$ for all $a \in S$.*

Proof. (2) Assume that S is a (p,q)-lateral simple, we have that S is a minimal (p,q)-lateral ideal of S. By the Theorem 2.13(2), $S^p aS^q \cup S^{p+1}aS^{q+1} = S$ for all $a \in S$.

Conversely, suppose that $S^p aS^q \cup S^{p+1}aS^{q+1} = S$ for all $a \in S$. By the Theorem 2.13(2), S is a minimal (p,q)-lateral ideal of S, and therefore S is a (p,q)-lateral simple.

Proofs of (1) and (3) are analogous. □

Lemma 2.15. *If R is an m-right ideal of S and T is a ternary subsemigroup of S and if T is an m-right simple such that $T \cap R \neq \emptyset$, then $T \subseteq R$.*

Proof. Assume that T is an m-right simple such that $T \cap R \neq \emptyset$. Let $a \in T \cap R$. By Lemma 2.8, we have $aT^{2m} \cap T$ is an m-right ideal of T. This implies that $aT^{2m} \cap T = T$. Hence $T \subseteq aT^{2m} \subseteq RS^{2m} \subseteq R$, so $T \subseteq R$. □

Lemma 2.16. *If M is a (p,q)-lateral ideal of S and T is a ternary subsemigroup of S and if T is a (p,q)-lateral simple such that $T \cap M \neq \emptyset$, then $T \subseteq M$.*

Proof. Proof is similar to the Lemma 2.15. □

Lemma 2.17. *If L is an n-left ideal of S and T is a ternary subsemigroup of S and if T is an n-left simple such that $T \cap L \neq \emptyset$, then $T \subseteq L$.*

Proof. Proof is similar to the Lemma 2.15. □

Theorem 2.18. *Let S be a ternary semigroup. Then:*

(1) *If an m-right ideal R of S is an m-right simple ternary semigroup, then R is a minimal m-right ideal of S.*

(2) *If a (p,q)-lateral ideal M of a ternary semigroup S is a (p,q)-lateral simple ternary semigroup, then M is a minimal (p,q)-lateral ideal of S.*

(3) *If an n-left ideal L of a ternary semigroup S is an n-left simple ternary semigroup, then L is a minimal n-left ideal of S.*

Proof. (2) Assume that M is a (p,q)-lateral simple. Let A be a (p,q)-lateral ideal of S such that $A \subseteq M$. Then $A \cap M \neq \emptyset$, it follows from Lemma 2.16, that $M \subseteq A$. Hence $A = M$, so M is a minimal (p,q)-lateral ideal of S.

(1) and (3) can be proved analogously. □

Theorem 2.19. *Let S be a regular ternary semigroup. Then:*

(1) *Every m-right ideal is a right ideal.*

(2) *Every (p,q)-lateral ideal is a lateral ideal.*

(3) *Every n-left ideal is a left ideal.*

Proof. (2) Let M be a (p,q)-lateral ideal of S and $a \in SMS \cup SSMSS$. Then there exists $x_1, x_2, x_3, x_4, x_5, x_6 \in S$ and $m_1, m_2 \in M$ such that $a = x_1 m_1 x_2$ or $a = x_3 x_4 m_2 x_5 x_6$. Since S is regular, for any $m_1, m_2 \in M$ there exists $x_7, x_8 \in S$ such that $m_1 = m_1 x_7 m_1$ or $m_2 = m_2 x_8 m_2$. Hence $a = x_1 m_1 x_7 m_1 x_2$ or $a = x_3 x_4 m_2 x_8 m_2 x_5 x_6$. Therefore $a \in SMSMS \subseteq S^3 MS$ or $a \in S^2 MSMS^2 \subseteq S^4 MS^2$. Thus, by the property of regularity, we see that $a \in S^p MS^q$ or $a \in S^{p+1} MS^{q+1}$ implies $a \in S^p MS^q \cup S^{p+1} MS^{q+1}$. As M is a (p,q)-lateral ideal, it implies $a \in S^p MS^q \cup S^{p+1} MS^{q+1} \subseteq M$. Therefore $SMS \cup S^2 MS^2 \subseteq M$ and hence M is a lateral ideal of S.

Proofs of (1) and (3) are similar. □

Theorem 2.20. *If a ternary semigroup S is an m-right and an n-left simple. Then it is regular.*

Proof. Suppose that S is an m-right and an n-left simple. Let $a \in S$. Then by the Theorem 2.14(1) and (3), $aS^{2m} = S$ and $S^{2n} a = S$. Now

$$a \in S = aS^{2m} = aS^{2(m-1)} S^2 = aS^{2(m-1)} SS^{2n} a = aS^{2(m-1)} S^3 S^{2(n-1)} a \subseteq aSa.$$

This shows that $a \in aSa$. Hence for any $a \in S$ there exists $x \in S$ such that $a = axa$. Therefore a is regular. Hence S is regular. □

References

[1] **V.N. Dixit and S. Diwan**, *A note on quasi and bi-ideals in ternary semigroups*, Int. J. Math. Sci. **18** (1995), 501 – 508.

[2] **M.K. Dubey and R. Anuradha**, *On generalized quasi-ideals and bi-ideals in ternary semigroups*, J. Math. Appl. **37** (2014), 27 – 37.

[3] **W.A. Dudek**, *Idempotents in n-ary semigroups*, Southeast Asian Bull. Math., **25** (2001), 97 – 104.

[4] **W.A. Dudek, I. Groździńska**, *On ideals in regular n-semigroups*, Mat. Bilten (Skopje), **29** (1979-1980), 35 – 44.

[5] **E. Kasner**, *An extension of the group concept*, Bull. Amer. Math. Soc. **10** (1904), 290 – 291.

[6] **O.V. Kolesnikov**, *Inverse n-semigroups*, Comment. Math. **21** (1980), $101-108$.

[7] **F.M. Sioson**, *On regular algebraic systems*, Proc. Japan Acad. **35** (1963), $283-286$.

[8] **F.M. Sioson**, *Ideal theory in ternary semigroups*, Math. Japan. **10** (1965), $63-84$.

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Unit and unitary Cayley graphs for the ring of Gaussian integers modulo n

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Abstract. Let $\mathbb{Z}_n[i]$ be the ring of Gaussian integers modulo n and $G(\mathbb{Z}_n[i])$ and $G_{\mathbb{Z}_n[i]}$ be the unit graph and the unitary Cayley graph of $\mathbb{Z}_n[i]$, respectively. In this paper, we study $G(\mathbb{Z}_n[i])$ and $G_{\mathbb{Z}_n[i]}$. Among many results, it is shown that for each positive integer n, the graphs $G(\mathbb{Z}_n[i])$ and $G_{\mathbb{Z}_n[i]}$ are *Hamiltonian*. We also find a necessary and sufficient condition for the unit and unitary Cayley graphs of $\mathbb{Z}_n[i]$ to be Eulerian.

1. Introduction

Finding the relationship between the algebraic structure of rings using properties of graphs associated to them has become an interesting topic in the last years. There are many papers on assigning a graph to a ring, see [1], [3], [4], [5], [7], [6], [8], [10], [11], [12], [17], [19], and [20].

Let R be a commutative ring with non-zero identity. We denote by U(R), J(R) and Z(R) the group of units of R, the Jacobson radical of R and the set of zero divisors of R, respectively. The *unitary Cayley graph* of R, denoted by G_R, is the graph whose vertex set is R, and in which $\{a, b\}$ is an edge if and only if $a - b \in U(R)$. The unit graph $G(R)$ of R is the *simple graph* whose vertices are elements of R and two distinct vertices a and b are adjacent if and only if a + b in U(R) . There are many papers where these two concepts have been discussed. See for examples [4], [8], [19], [20], [22] and [23].

The following facts are well known, see for examples Silverman (2006), [2] and [16]. The set of all complex numbers $a + ib$, where a and b are integers, form an Euclidean domain with the usual complex number operations and Euclidian norm $N(a+ib) = a^2+b^2$. This domain will be denoted by $\mathbb{Z}[i]$ and will be called the ring of Gaussian integers. Let n be a natural number and let (n) be the principal ideal generated by n in $\mathbb{Z}[i]$, and let $\mathbb{Z}_n = \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{n-1}\}$ be the ring of integers modulo n. Then the factor ring $\frac{\mathbb{Z}[i]}{(n)}$ is isomorphic to $\mathbb{Z}_n[i]$, which implies that $\mathbb{Z}_n[i]$ is a principal ideal ring. The ring $\mathbb{Z}_n[i]$ is called the ring of *Gaussian integers modulo n*. Let p be a prime integer. Then $p \equiv 1 (\text{mod } 4)$ if and only if there are integers a , b such that $p = a^2 + b^2$ if and only if there exists an integer c such that

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$c^2 \equiv -1 (\mathrm{mod}\ p)$. Moreover, if n is a natural number, then there exist integers a and b, relatively prime to p such that $p^n = a^2 + b^2$, and there exists an integer z such that $z^2 \equiv -1 (\mathrm{mod}\ p^n)$. It was shown that $\overline{a} + i\overline{b}$ is a unit in $\mathbb{Z}_n[i]$ if and only if $\overline{a}^2 + \overline{b}^2$ is a unit in $\mathbb{Z}_n$. If $n = \prod_{j=1}^{s} a_j^{k_j}$ is the prime power decomposition of the positive integer n, then $\mathbb{Z}_n[i]$ is the direct product of the rings $\mathbb{Z}_{a_j^{k_j}}[i]$. Also if $m = t^k$ for some prime t and positive integer k, then $\mathbb{Z}_m[i]$ is local ring if and only if $t = 2$ or $t \equiv 3 (\mathrm{mod}\ 4)$.

In this article, some properties of the graphs $G(\mathbb{Z}_n[i])$ and $G_{\mathbb{Z}_n[i]}$ are studied. The diameter, the girth, chromatic number, clique number and independence number, in terms of n, are found. Moreover, we prove that for each $n > 1$, the graphs $G(\mathbb{Z}_n[i])$ and $G_{\mathbb{Z}_n[i]}$ are *Hamiltonian*. We also find a necessary and sufficient condition for the unit and unitary Cayley graphs of $\mathbb{Z}_n[i]$ to be Eulerian.

A *local ring* is a ring with exactly one maximal ideal. A local ring with finitely many maximal ideals is called $semi-local$ ring. For classical theorem and notations in commutative algebra, the interested reader is referred to [9].

Throughout this paper all graphs are simple (with no loop and multiple edges). For a graph G, $V(G)$ and $E(G)$ denote the vertex set and edge set of G respectively. The set of vertices adjacent to a vertex v in the graph G is denoted by $N(v)$. The degree $deg(v)$ of a vertex v in the graph G is the number of edges of G incident with v. The graph G is called $k-regular$ if all vertices of G have degree k, where k is a fixed positive integer. A *walk* (of length k) in a graph G between two vertices a, b is an alternating sequence $a = v_0, e_1, v_1, e_2, \ldots, e_k, v_k = b$ of vertices and edges in G, denoted by

$$a = v_0 \longrightarrow v_1 \longrightarrow \ldots \longrightarrow v_k = b,$$

such that $e_i = \{v_{i-1}, v_i\}$ for all $1 \leqslant i \leqslant k$. If the vertices in a walk are all distinct, it defines a *path* in G. A *trail* between two vertices a, b is a walk between a and b without repeated vertices. A *cycle* of a graph is a path such that the start and end vertices are the same. A *Hamiltonian* path (cycle) in G is a path (cycle) in G that visits every vertex exactly once. A graph is called *Hamitonian* if it contains a Hamiltonian cycle. Also a graph G is called *connected* if for any vertices a and b of a graph G there is a path between a and b. A connected graph G is called *Eulerian* if there exists a closed trail containing every edge of G. For distinct vertices a and b of a graph G, let $d(a,b)$ be the length of a shortest path from a to b; if no such paths exists, we set $d(a,b) = \infty$. The *diameter* of G is defined as

$$diam(G) = sup\{d(a,b); a, b \in V(G)\}.$$

The *girth* of G, denoted by $gr(G)$ is the length of a shortest cycle in G, ($gr(G) = \infty$ if G contain no cycle). For a positive integer r, a graph is called $r - partite$ if the vertex set admits a partition into r classes such that vertices in the same partition class are not adjacent. A $r-$partite graph is called *complete* if every two vertices in different parts are adjacent. The complete $2 - partite$ graph (also called the *complete bipartite* graph) with exactly two partitions of size n and m, is denoted by $K_{n,m}$. A graph G is called a *complete graph* if every two distinct vertices in G are adjacent. A complete graph with n vertices is denoted by K_n. A *clique* of a graph is a complete subgraph. A maximum clique is a clique of the largest possible size in a given graph. the *clique number*, $\omega(G)$ of a graph G is the number of vertices in a maximum clique in G. An independent set in a graph is a set of pairwise non-adjacent vertices. The *independence number*, $\alpha(G)$ of a graph G is the size of a largest independet set of G. A subset M of the edge set of G, is called a *matching* in G if no two of the edges in M are adjacent. In other words, if for any two edges e and f in M, both the end vertices of e are different from the end vertices of f. A *perfect matching* of a graph G is a matching of G containing $\frac{n}{2}$ edges, the largest possible, meaning perfect matchings are only possible on graphs with an even number of vertices. A perfect matching sometimes called a *complete matching* or $1-factor$. A *coloring* of a graph is a labeling of the vertices with colors such that no two adjacent vertices have the same color. The smallest number of colors need to color the vertices of a graph G is called its *chromatic number*, and denoted by $\chi(G)$. Let G_1 and G_2 be two vertex-disjoint graphs. The *tensor product* or *Kronecker product* of G_1 and G_2 is denoted by $G_1 \otimes G_2$. That is, $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$; two distinct vertices (a,b) and (c,d) are adjacent if and only if a is adjacent to c in G_1 and b adjacent to d in G_2. We refer the reader to [13] and [15] for general references on graph theory.

2. The unit and unitary Cayley graphs for $\mathbb{Z}_{t^n}[i]$

In this section we find the diameter and girth of the unit and unitary Cayley graphs of $\mathbb{Z}_{t^n}[i]$ where t is a prime integer. Three cases are considered: When $t = 2$, $t \equiv 3 (\text{mod } 4)$ or $t \equiv 1 (\text{mod } 4)$. In this article p and q will denote prime integers such that $p \equiv 1 (\text{mod } 4)$ and $q \equiv 3 (\text{mod } 4)$.

2.1. The unit and unitary Cayley graphs for $\mathbb{Z}_{2^n}[i]$

Proposition 2.1. [4, Proposition 2.2]

(*a*) *Let R be a ring. Then G_R is a regular graph of degree $|U(R)|$.*

(*b*) *Let S be a local ring with mamximal ideal m. Then G_S is a complete mutipartite graph whose partite sets are the cosets of m in S. In paticular, G_S is a compelete graph if and only if S is a field.*

Lemma 2.2. *For each positive integer n, $G_{\mathbb{Z}_{2^n}[i]}$ is a complete bipartite graph $K_{2^{2n-1},2^{2n-1}}$.*

Proof. For each positive integer n, $\mathbb{Z}_{2^n}[i]$ is a local ring with its only maximal ideal $m = (\bar{1} + \bar{1}i)$ and the number of units in $\mathbb{Z}_{2^n[i]}$ is 2^{2n-1}, see [2] and [14]. Since $|\frac{\mathbb{Z}_{2^n}[i]}{(\bar{1}+\bar{1}i)}| = 2$, by Proposition 2.1, we conclude that $G_{\mathbb{Z}_{2^n}[i]}$ is a complete bipartite graph $K_{2^{2n-1},2^{2n-1}}$. $\square$

Lemma 2.3. [20, Lemma 4.1] *Let R be a finite ring. For $j \in R$, the following statements are equivalent:*

(a) $j \in J_R$

(b) $j + u \in U(R)$ *for any* $u \in U(R)$.

Theorem 2.4. [19, Theorem 2.6] *Let R be a finite ring. Then the following statements hold.*

(a) *If (R, m) is a local ring of even order, then $G(R) \cong G_R$.*

(b) *If R is a ring of odd order, then $G(R) \ncong G_R$*

Proposition 2.5. [19, Corollary 2.3] *Let R be a finite ring. Then $2 \in U(R)$ if and only if $|R|$ is odd.*

Theorem 2.6. *Let R be a finite ring and $R \cong R_1 \times R_2 \times \cdots \times R_n$. Then the following statements are equivalent:*

(i) $2 \in J(R)$.

(ii) $G_R = G(R)$.

(iii) *For every i with $1 \leqslant i \leqslant n$, $|R_i|$ is even.*

Proof. Let $2 \in J(R)$ and a, b be two distict elements of R. Since

$$(a - b) + (a + b) = 2a.$$

By Lemma 2.3,

$$a - b \in U(R) \quad \text{if and only if} \quad a + b \in U(R).$$

This means that $G_R = G(R)$.

Now supoose that $G_R = G(R)$ and $R \cong R_1 \times R_2 \times \cdots \times R_n$. If $n = 1$ then by Proposition 2.5, we deduce that $|R|$ is even. Now assume that $n > 1$. Since $G_R = G(R)$, we have for every i with $1 \leqslant i \leqslant n$, $G_{R_i} = G(R_i)$. Hence by the first case, for every i with $1 \leqslant i \leqslant n$, $|R_i|$ is even.

Finally, if for every i with $1 \leqslant i \leqslant n$, $|R_i|$ be even. Then by Proposition 2.5, we have $2 \notin U(R_i)$; $1 \leqslant i \leqslant n$. This implies that $2 \in J(R_i)$, and therefore $2 \in J(R)$. This completes the proof. □

Corollary 2.7. *For each positive integer n, $G_{\mathbb{Z}_{2^n}[i]} = G(\mathbb{Z}_{2^n}[i])$.*

Proof. Since $|\mathbb{Z}_{2^n}[i]|$ is even, by Proposition 2.5, we have $2 \notin U(\mathbb{Z}_{2^n}[i])$. Therefore $2 \in J(\mathbb{Z}_{2^n}[i])$. By using Theorem 2.6 we conclude that $G_{\mathbb{Z}_{2^n}[i]} = G(\mathbb{Z}_{2^n}[i])$. □

Corollary 2.8. *Let n be a positive integer. Then the following statements hold:*

(i) $diam(G_{\mathbb{Z}_{2^n}[i]}) = diam(G(\mathbb{Z}_{2^n}[i])) = 2$

(ii) $gr(G_{\mathbb{Z}_{2^n}[i]}) = gr(G(\mathbb{Z}_{2^n}[i])) = 4.$

Proof. For each positive integer n, $G_{\mathbb{Z}_{2^n}} = G(\mathbb{Z}_{2^n}[i])$ is a complete bipartite graph with $|\mathbb{Z}_{2^n}[i]| \geq 4$, Hence

$$diam(G_{\mathbb{Z}_{2^n}[i]}) = diam(G(\mathbb{Z}_{2^n}[i])) = 2$$

and

$$gr(G_{\mathbb{Z}_{2^n}[i]}) = gr(G(\mathbb{Z}_{2^n}[i])) = 4. \qquad \square$$

2.2. The unit and unitary Cayley graphs for $\mathbb{Z}_{q^n}[i], q \equiv 3(\mathrm{mod}4)$

Theorem 2.9. *Let n be a positive integer. Then the following statements hold:*

(i) $G_{\mathbb{Z}_{q^n}[i]}$ *is a complete* q^2- *partite graph.*

(ii) $G_{\mathbb{Z}_{q^n}[i]} \ncong G(\mathbb{Z}_{q^n}[i])$

Proof. If $q \equiv 3(\mathrm{mod}\ 4)$, then $\mathbb{Z}_q[i]$ is a field with q^2 elements see [2]. By Proposition 2.1, $G_{\mathbb{Z}_q[i]}$ is a complete graph with q^2 vertices. If $n > 1$, then $\mathbb{Z}_{q^n[i]} \cong \frac{\mathbb{Z}[i]}{(q^n)}$ is a local ring with maximal ideal $m = (\overline{q})$ see [2]. Also, the number of units in $\mathbb{Z}_{q^n}[i]$ is $q^{2n} - q^{2n-2}$, see [14]. Clearly, $|\frac{\mathbb{Z}_{q^n}[i]}{m}| = q^2$. Hence by proposition 2.1, $G_{\mathbb{Z}_{q^n}[i]}$ is a complete q^2- partite graph.

Since $|\mathbb{Z}_{q^n}[i]|$ is odd, by Theorem 2.4, $G_{\mathbb{Z}_{q^n}[i]} \ncong G(\mathbb{Z}_{q^n}[i])$ $\square$

Corollary 2.10. *For each positive integer n, the following statements hold:*

(i) $diam(G_{\mathbb{Z}_{q^n[i]}} = \begin{cases} 1 & for \quad n = 1 \\ 2 & for \quad n > 1 \end{cases}.$

(ii) $diam(G(\mathbb{Z}_{q^n[i]})) = 2.$

(iii) $gr(G_{\mathbb{Z}_{q^n[i]}}) = gr(G(\mathbb{Z}_{q^n}[i])) = 3.$

Proof. Let $n = 1$, then $G(G_{\mathbb{Z}_q[i]})$ is a complete graph with q^2 vertices. This implies that $diam(G_{\mathbb{Z}_q[i]}) = 1$ and $gr(G_{\mathbb{Z}_q[i]}) = 3$. Also in this case $G(\mathbb{Z}_q[i])$ is a complete $\frac{q^2+1}{2}-$ partite graph. Thus

$$diam(G(\mathbb{Z}_q[i])) = 2 \text{ and } gr(G(\mathbb{Z}_q[i])) = 3.$$

Now suppose that $n > 1$. In this case, $G_{\mathbb{Z}_{q^n}[i]}$ is a complete q^2- partite graph. Therefore,

$$diam(G_{\mathbb{Z}_{q^n}[i]}) = 2 \text{ and } gr(G_{\mathbb{Z}_{q^n}[i]}) = 3.$$

Since, $G(\dfrac{\mathbb{Z}_{q^n}[i]}{(q)})$ is a complete $\frac{q^2+1}{2}-$ partite graph, we obtain that

$$diam(G(\mathbb{Z}_{q^n[i]})) = 2 \text{ and } gr(G(\mathbb{Z}_{q^n[i]})) = 3. \qquad \square$$

2.3. The unit and unitary Cayley graphs for $\mathbb{Z}_{p^n}[i], p \equiv 1(\mathrm{mod} 4)$

Theorem 2.11. *Let n be a positive integer. Then the following statements hold:*

(*i*) $diam(G_{\mathbb{Z}_{p^n}[i]}) = diam(G(\mathbb{Z}_{p^n}[i])) = 2.$

(*ii*) $gr(G_{\mathbb{Z}_{p^n}[i]}) = gr(G(\mathbb{Z}_{p^n}[i])) = 3.$

Proof. Let p be a prime integer that is congruent to 1 modulo 4. Then there exist integer numbers a, b such that

$$p = a^2 + b^2 = (a + ib)(a - ib)$$

and

$$\mathbb{Z}_p[i] \cong \frac{\mathbb{Z}[i]}{(p)} \cong (\frac{\mathbb{Z}[i]}{(a+ib)}) \times (\frac{\mathbb{Z}[i]}{(a-ib)}).$$

Also the ideals $(a + ib)$ and $(a - ib)$ are the only maximal ideals in $\mathbb{Z}_p[i]$ see [2]. The number of units in $\mathbb{Z}_p[i]$ is $(p - 1)^2$, see [14]. By [19, Theorem 3.5], we have

$$diam(G_{\mathbb{Z}_{p^n}[i]}) = diam(G(\mathbb{Z}_{p^n}[i])) = 2.$$

On the other hand, in view of the proof of [8, Proposition 5.10] and [4, Theorem3.2], we obtain

$$gr(G_{\mathbb{Z}_{p^n}[i]}) = gr(G(\mathbb{Z}_{p^n}[i])) = 3.$$

To investigate the more general case, let $p \equiv 1(\mathrm{mod}\ 4)$, $n > 1$. By an argoment similar to that above, we conclude that

$$\mathbb{Z}_{p^n}[i] \cong \frac{\mathbb{Z}[i]}{(p^n)} \cong (\frac{\mathbb{Z}[i]}{((a+ib)^n)}) \times (\frac{\mathbb{Z}[i]}{((a-ib)^n)}) \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n}.$$

The number of units in $\mathbb{Z}_{p^n}[i]$ is $(p^n - p^{n-1})^2$, see [14]. Note that, $\mathbb{Z}_{p^n}$ is a local ring with only maximal ideal, $m = (p)$, and hence $|\frac{\mathbb{Z}_{p^n}}{m}| = p$. Hence by [19, Theorem 3.5], we have that

$$diam(G_{\mathbb{Z}_{p^n}[i]}) = diam(G(\mathbb{Z}_{p^n}[i])) = 2.$$

On the other hand, in view of the proof of [8, Proposition 5.10] and [4, Theorem3.2], we obtain $gr(G_{\mathbb{Z}_{p^n}[i]}) = gr(G(\mathbb{Z}_{p^n}[i])) = 3.$ □

3. The unit and unitary Cayley graphs for $\mathbb{Z}_n[i]$

In this section, the integers q_j and p_s are used implicitly to denote primes congruent to 3 modulo 4 and primes congruent to 1 modulo 4 respectively. The general case is now investigated.

3.1. Diameter and girth for the graphs $G_{\mathbb{Z}_n[i]}$ and $G(\mathbb{Z}_n[i])$

Now we find the diameter and girth of the unit and unitary Cayley graphs of $G(\mathbb{Z}_n[i])$ where $n > 1$ is an integer.

Remark 3.1. If R is a finite commutative ring, then $R \cong R_1 \times R_2 \times \cdots \times R_t$, where each R_i is a finite commutative local ring with maximal ideal m_i by [9, Theorem 8.7]. This decomposition is unique up to permutation of factors. Since $(u_1, \ldots, u_t)$ is a unit of R if and only if each u_i is a unit in R_i, we see immediately that

$$G_R \cong G_{R_1} \otimes G_{R_2} \cdots \otimes G_{R_t} \quad \text{and} \quad G(R) \cong G(R_1) \otimes G(R_2) \cdots \otimes G(R_t)$$

We denote by K_i the (finite) residue field $\dfrac{R_i}{m_i}$ and $f_i = |K_i|$. We also assume (after appropriate permutation of factors) that $f_1 \leqslant f_2 \leqslant \ldots \leqslant f_t$.

Remark 3.2. If $n = 2^k \times \prod_{j=1}^{m} q_j^{\alpha_j} \times \prod_{s=1}^{l} p_s^{\beta_s}$ is the prime power decomposition of the positive integer n, then $\mathbb{Z}_n[i]$ is the direct product of the rings

$$\mathbb{Z}_n[i] \cong \mathbb{Z}_{2^k}[i] \times \prod_{j=1}^{m} \mathbb{Z}_{q_j^{\alpha_j}}[i] \times \prod_{s=1}^{l} \mathbb{Z}_{p_s^{\beta_s}}[i].$$

Also the number of units in $\mathbb{Z}_n[i]$ is

$$2^{2k-1} \times \prod_{j=1}^{m} (q_j^{2\alpha_j} - q_j^{2\alpha_j - 2}) \times \prod_{s=1}^{l} (p_s^{\beta_s} - p_s^{\beta_s - 1})^2 \text{ see [2] and [14].}$$

Theorem 3.3. *Let $n > 1$ be an integer with at least two distinct prime factors. Then* $diam(G_{\mathbb{Z}_n[i]}) = diam(G(\mathbb{Z}_n[i])) = \begin{cases} 2 & \textit{for} \quad 2 \nmid n, \\ 3 & \textit{for} \quad 2 \mid n. \end{cases}$

Proof. Let $n = 2^k \times \prod_{j=1}^{m} q_j^{\alpha_j} \times \prod_{s=1}^{l} p_s^{\beta_s}$. By Remark 3.2,

$$\mathbb{Z}_n[i] \cong \mathbb{Z}_{2^k}[i] \times \prod_{j=1}^{m} \mathbb{Z}_{q_j^{\alpha_j}}[i] \times \prod_{s=1}^{l} \mathbb{Z}_{p_s^{\beta_s}}[i].$$

This shows that, $\mathbb{Z}_n[i]$ is isomorphic to a direct product of finite local rings, R_i such that for every i, $|\dfrac{R_i}{m_i}| = 2$ or q_j^2 or p_s. Since $n > 1$ is an integer with at least two distinct prime factors, we have $J(\mathbb{Z}_n[i]) \neq \{0\}$.

By [4, Theorem 3.1], we conclude that

$$diam(G_{\mathbb{Z}_n[i]}) = \begin{cases} 2 & \text{for} \quad 2 \nmid n, \\ 3 & \text{for} \quad 2 \mid n. \end{cases}$$

On the other hand, by [8, Theorem 5.7] we have

$$diam(G(\mathbb{Z}_n[i])) = \begin{cases} 2 & \text{for} \quad 2 \nmid n, \\ 3 & \text{for} \quad 2 \mid n. \end{cases} \qquad \square$$

Theorem 3.4. *Let $n > 1$ be an integer with at least two distinct prime factors. Then*

$$gr(G_{\mathbb{Z}_n[i]}) = \begin{cases} 4 & for \quad 2 \nmid n \\ 3 & for \quad 2 \mid n \end{cases}$$

and

$$gr(G(\mathbb{Z}_n[i])) \in \{3, 4\}.$$

Proof. By an argoment similar to that above, we conclude that

$$\mathbb{Z}_n[i] \cong \mathbb{Z}_{2^k}[i] \times \prod_{j=1}^{m} \mathbb{Z}_{q_j^{\alpha_j}}[i] \times \prod_{s=1}^{l} \mathbb{Z}_{p_s^{\beta_s}}[i].$$

Thus, by [4, Theorem 3.2], we obtain

$$gr(G_{\mathbb{Z}_n[i]}) = \begin{cases} 4 & \text{for} \quad 2 \nmid n, \\ 3 & \text{for} \quad 2 \mid n. \end{cases}$$

On the other hand, $J(\mathbb{Z}_n[i]) \neq \{0\}$. Thus, in view of the proof of [8, Theorem 5.10], we have $gr(G(\mathbb{Z}_n[i])) \in \{3, 4\}$. □

3.2. When are $G_{\mathbb{Z}_n[i]}$ and $G(\mathbb{Z}_n[i])$ Hamiltonian or Eulerian?

In the following, we prove that for each integer $n > 1$, the graphs $G(\mathbb{Z}_n[i])$ and $G_{\mathbb{Z}_n[i]}$ are *Hamiltonian*

Theorem 3.5. *For each integer $n > 1$, the graphs $G(\mathbb{Z}_n[i])$ and $G_{\mathbb{Z}_n[i]}$ are Hamitonian.*

Proof. Let $n > 1$ be an integer. By Corollary 2.10, Corollary 2.8, Theorem 2.11 and Theorem 3.3, the graphs ,$G(\mathbb{Z}_n[i])$ and $G_{\mathbb{Z}_n[i]}$ are connected. Thus by [23, Theorem 2.1], $G(\mathbb{Z}_n[i])$ is Hamiltonian graph. On the other hand, by [21, Lemma 4], we conclude that $G_{\mathbb{Z}_n[i]}$ is Hamiltonian graph. □

Now, we are going to find a necessary and sufficient condition for $G(\mathbb{Z}_n[i])$ and $G_{\mathbb{Z}_n[i]}$ to be Eulerian. we recall the following well-known propossition.

Proposition 3.6. *A connected graph G is Eulerian if and only if the degree of each vertex of G is even.*

Theorem 3.7. *Let $n > 1$ be an integer. Then the following statements hold:*

- (i) *The graph $G(\mathbb{Z}_n[i])$ is Eulerian if and only if $2 \mid n$.*
- (ii) *The graph $G_{\mathbb{Z}_n[i]}$ is Eulerian if and only if $2 \mid n$.*

Proof. First Suppose that $G(\mathbb{Z}_n[i])$ and $G_{\mathbb{Z}_n[i]}$ are Eulerian. Since these graphs are connected, by Propossition 3.6 we deduce that the degree of each vertex of $G(\mathbb{Z}_n[i])$ and $G_{\mathbb{Z}_n[i]}$ are even. On the other hand

$$\mathbb{Z}_n[i] \cong \mathbb{Z}_{2^k}[i] \times \prod_{j=1}^{m} \mathbb{Z}_{q_j^{\alpha_j}}[i] \times \prod_{s=1}^{l} \mathbb{Z}_{p_s^{\beta_s}}[i]$$

and so

$$|U(\mathbb{Z}_n[i])| = 2^{2k-1} \times \prod_{j=1}^{m}(q_j^{2\alpha_j} - q_j^{2\alpha_j-2}) \times \prod_{s=1}^{l}(p_s^{\beta_s} - p_s^{\beta_s-1})^2.$$

Since $G_{\mathbb{Z}_n[i]}$ and $G(\mathbb{Z}_n[i])$ are $|U(\mathbb{Z}_n[i])|-$regular graph by Proposition 2.1, and [8, Proposition 2.4], we deduce that $\mathbb{Z}_n[i]$ has a direct factor of the form $\mathbb{Z}_{2^k}[i]$, and so $2 \mid n$. Conversely, suppose that $2 \mid n$. Thus $|\mathbb{Z}_n[i)|$ is even. Hence by Proposition 2.5, $2 \notin U(\mathbb{Z}_n[i])$. On the other hand, $G_{\mathbb{Z}_n[i]}$ and $G(\mathbb{Z}_n[i])$ are connected and $|U(\mathbb{Z}_n[i])|-$regular graphs by Proposition 2.1 and [8, Proposition 2.4]. This means that

$$|U(\mathbb{Z}_n[i])| = 2^{2k-1} \times \prod_{j=1}^{m}(q_j^{2\alpha_j} - q_j^{2\alpha_j-2}) \times \prod_{s=1}^{l}(p_s^{\beta_s} - p_s^{\beta_s-1})^2$$

is even and so the degree of each vertex of $G_{\mathbb{Z}_n[i]}$ and $G(\mathbb{Z}_n[i])$ are even, and therefore these graphs are Eulirian. □

3.3. Some graph invariants of $G_{\mathbb{Z}_n[i]}$ and $G(\mathbb{Z}_n[i])$

In the following, we study chromatic, clique and independence numbers of the Graphs $G_{\mathbb{Z}_n[i]}$ and $G(\mathbb{Z}_n[i])$.

Theorem 3.8. *Let $n > 1$ be an integer and $n = 2^k \times \prod_{j=1}^{m} q_j^{\alpha_j} \times \prod_{s=1}^{l} p_s^{\beta_s}$.*

(*i*) *If $2 \mid n$, then $\chi(G_{\mathbb{Z}_n[i]}) = \omega(G_{\mathbb{Z}_n[i]}) = 2$ and $\alpha(G_{\mathbb{Z}_n[i]}) = \dfrac{n^2}{2}$.*

(*ii*) *If $2 \nmid n$, then*

$$\chi(G_{\mathbb{Z}_n[i]}) = \omega(G_{\mathbb{Z}_n[i]}) = \min\{p_s, q_j^2 \mid 1 \leqslant s \leqslant l,\ 1 \leqslant j \leqslant m,\ p_s \mid n, q_j \mid n\}$$

and

$$\alpha(G_{\mathbb{Z}_n[i]}) = \frac{n^2}{\min\{p_s, q_j^2 \mid 1 \leqslant s \leqslant l,\ 1 \leqslant j \leqslant m,\ p_s \mid n,\ q_j \mid n\}}.$$

Proof. Let $2 \mid n$, and k, be the biggest positive integer such that $2^k \mid n$. Since

$$\mathbb{Z}_n[i] \cong \mathbb{Z}_{2^k}[i] \times \prod_{j=1}^{m} \mathbb{Z}_{q_j^{\alpha_j}}[i] \times \prod_{s=1}^{l} \mathbb{Z}_{p_s^{\beta_s}}[i],$$

thus $\mathbb{Z}_n[i]$ has a direct factor of the form $\mathbb{Z}_{2^k}[i]$. Since $|\dfrac{\mathbb{Z}_{2^k}[i]}{m}| = 2$, by [4, Proposition 6.1], we conclude that $\chi(G_{\mathbb{Z}_n[i]}) = \omega(G_{\mathbb{Z}_n[i]}) = 2$ and $\alpha(G_{\mathbb{Z}_n[i]}) = \dfrac{n^2}{2}$.
Now suppose that $2 \nmid n$. This yields that $\mathbb{Z}_n[i]$ is isomorphic to a direct product of finite local rings, R_i such that for every i, $|\dfrac{R_i}{m_i}| = q_j^2$ *or* p_s. Thus by [4, Proposition 6.1], we have

$$\chi(G_{\mathbb{Z}_n[i]}) = \omega(G_{\mathbb{Z}_n[i]}) = \min\{p_s, q_j^2 \mid 1 \leqslant s \leqslant l, 1 \leqslant j \leqslant m, p_s \mid n, q_j \mid n\}$$

and

$$\alpha(G_{\mathbb{Z}_n[i]}) = \frac{n^2}{\min\{p_s, q_j^2 \mid 1 \leqslant s \leqslant l, 1 \leqslant j \leqslant m, p_s \mid n, q_j \mid n\}}. \qquad \square$$

Proposition 3.9. [13, Corollary 16.6] *Every nonempty regular bipartite graph has a perfect matching*

Lemma 3.10. [18, Lemma 2.3] *If G is a bipartite graph with a perfect matching and H is a Hamiltonian graph, then $\alpha(G \otimes H) = \dfrac{|V(G)| \times |V(H)|}{2}$.*

Theorem 3.11. *Let $n > 1$ be an integer and $n = 2^k \times \prod_{j=1}^m q_j^{\alpha_j} \times \prod_{s=1}^l p_s^{\beta_s}$.*

(*i*) *If $2 \mid n$, then $\chi(G(\mathbb{Z}_n[i])) = \omega(G(\mathbb{Z}_n[i])) = 2$ and $\alpha(G(\mathbb{Z}_n[i])) = \dfrac{n^2}{2}$*

(*ii*) *If $2 \nmid n$, then*

$$\chi(G(\mathbb{Z}_n[i])) = \omega(G(\mathbb{Z}_n[i])) = \frac{1}{2^{m+l}} \times \prod_{j=1}^{m} (q_j^{2\alpha_j} - q_j^{2\alpha_j - 2}) \times \prod_{s=1}^{l} (p_s^{\beta_s} - p_s^{\beta_s - 1})^2 + m + 2l$$

and $\quad \alpha(G(\mathbb{Z}_n[i])) \leqslant \dfrac{n^2}{2}$.

Proof. Let $n = 2^k \times \prod_{j=1}^m q_j^{\alpha_j} \times \prod_{s=1}^l p_s^{\beta_s}$. Then

$$\mathbb{Z}_n[i] \cong \mathbb{Z}_{2^k}[i] \times \textstyle\prod_{j=1}^m \mathbb{Z}_{q_j^{\alpha_j}}[i] \times \prod_{s=1}^l \mathbb{Z}_{p_s^{\beta_s}}[i].$$

Assume that $2 \mid n$. Then by Proposition 2.5, $2 \notin U(\mathbb{Z}_n[i])$. Hence, in view of the proof of [22, Theorem 2.2], we have

$$\chi(G(\mathbb{Z}_n[i])) = \omega(G(\mathbb{Z}_n[i])) = 2.$$

Since $2 \mid n$, $\mathbb{Z}_n[i]$ has a direct factor of the form $\mathbb{Z}_{2^k}[i]$. Moreover, $G(\mathbb{Z}_{2^k}[i])$ is a nonempty regular graph. Thus, by Proposition 3.9 $G(\mathbb{Z}_{2^k}[i])$ has a perfect matching. On the otherhand, by Theorem 3.5, $G(\prod_{j=1}^m \mathbb{Z}_{q_j^{\alpha_j}}[i] \times \prod_{s=1}^l \mathbb{Z}_{p_s^{\beta_s}}[i])$ is Hamiltonian graph. Therefore, by Lemma 3.10,

$$\alpha(G(\mathbb{Z}_n[i])) = \frac{n^2}{2}.$$

Now suppose that $2 \nmid n$. Thus $2 \in U(\mathbb{Z}_n[i])$. By an argoment similar to that above, we conclude that

$$\chi(G(\mathbb{Z}_n[i])) = \omega(G(\mathbb{Z}_n[i])) = \frac{1}{2^{m+l}} \times \prod_{j=1}^{m} (q_j^{2\alpha_j} - q_j^{2\alpha_j - 2}) \times \prod_{s=1}^{l} (p_s^{\beta_s} - p_s^{\beta_s - 1})^2 + m + 2l.$$

Let $n = \prod_{j=1}^m \mathbb{Z}_{q_j^{\alpha_j}}[i] \times \prod_{s=1}^l \mathbb{Z}_{p_s^{\beta_s}}[i]$. Then

$$\mathbb{Z}_n[i] \cong \textstyle\prod_{j=1}^m \mathbb{Z}_{q_j^{\alpha_j}}[i] \times \prod_{s=1}^l \mathbb{Z}_{p_s^{\beta_s}}[i]$$

and so we have

$$\mathbb{Z}_2[i] \times \mathbb{Z}_n[i] \cong \mathbb{Z}_2[i] \times \prod_{j=1}^{m} \mathbb{Z}_{qj^{\alpha_j}}[i] \times \prod_{s=1}^{l} \mathbb{Z}_{p_s^{\beta_s}}[i].$$

Thus,

$$\alpha(G(\mathbb{Z}_2[i] \times \mathbb{Z}_n[i])) \cong \alpha(G(\mathbb{Z}_2[i] \times \prod_{j=1}^{m} \mathbb{Z}_{q_j^{\alpha_j}}[i] \times \prod_{s=1}^{l} \mathbb{Z}_{p_s^{\beta_s}}[i])).$$

Now by part (i), we coclude that

$$\alpha(G(\mathbb{Z}_2[i] \times \mathbb{Z}_n[i])) = 2n^2.$$

On the other hand,

$$\alpha(G(\mathbb{Z}_2[i] \times \mathbb{Z}_n[i])) \geq \alpha(G(\mathbb{Z}_2[i])) \times |\mathbb{Z}_n[i]| = 4 \times \alpha(G(\mathbb{Z}_n[i])).$$

This implies that $\alpha(G(\mathbb{Z}_n[i])) \leqslant \dfrac{n^2}{2}$. $\square$

References

[1] **G. Aalipour, S. Akbari**, *On the Cayley graph of a commutative ring with respect to its zero-divisors*, Comm. Algebra **44** (2016), 1443 – 1459.

[2] **E.A. Abu Osba, S. Al-Addasi, N. Abu Jaradeh**, *Zero divisor graph for the ring of Gaussian integers modulo n*, Comm. Algebra **36** (2008), 3865 – 3877.

[3] **M. Afkhami, K. Khashyarmanesh, R. Zohreh**, *Some results on the annihilator graph of a commutative ring*, Czechoslovak Math. J. **67(142)** (2017), 151 – 169.

[4] **R. Akhtar, T. Jackson-Henderson, R. Karpman, M. Boggess, I. Jiménez, A. Kinzel, D. Pritikin**, *On the unitary Cayley graph of a finite ring*, Electron. J. Combin. **16** (2009), no. 1, Research Paper 117, 13 pp.

[5] **D.F. Anderson, A. Badawi**, *The total graph of a commutative ring*, J. Algebra **320** (2008), 2706 – 2719.

[6] **D.F. Anderson, A. Badawi**, *The generalized total graph of a commutative ring*, J. Algebra Appl. **12** (2013), no.5, paper 1250212.

[7] **D.F. Anderson, P.S. Livingston**, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999), 434 – 447.

[8] **N. Ashrafi, H.R. Maimani, M.R. Pournaki, S. Yassemi**, *Unit graphs associated with rings*, Comm. Algebra **38** (2010), 2851 – 2871.

[9] **M.F. Atiyah, I.G. Macdonald**, *Introduction to commutative algebra*, Addison-Wesley Publishing Co. (1969).

[10] **A. Badawi**, *On the annihilator graph of a commutative ring*, Comm. Algebra **42** (2014), 108 – 121.

[11] **A. Badawi**, *On the dot product graph of a commutative ring*, Comm. Algebra **43** (2015), 43 – 50.

[12] **Z. Barati, K. Khashyarmanesh, F. Mohammadi, Kh. Nafar**, *On the associated graphs to a commutative ring*, J. Algebra Appl. **11** (2012), no.2, paper 1250037.

[13] **J.A. Bondy, U.S.R. Murty** *Graph theory with applications*, American Elsevier Publishing Co. (1976).

[14] **J.T. Cross**, *The Euler φ-function in the Gaussian integers*, Amer. Math. Monthly **90** (1983), 518 – 528.

[15] **R. Diestel**, *Graph theory*, Springer-Verlag (2001).

[16] **G. Dresden, W.M. Dymáček**, *Finding factors of factor rings over the Gaussian integers*, Amer. Math. Monthly **112** (2005), 602 – 611.

[17] **R.P. Grimaldi**, *Graphs from rings*, Congr. Numer. **71** (1990), 95 – 103.

[18] **P. Jha, K. Pranava, S. Klavžar**, *Independence in direct-product graphs*, Ars Combin. **50** (1998), 53 – 63.

[19] **K. Khashyarmanesh, M.R. Khorsandi**, *A generalization of the unit and unitary Cayley graphs of a commutative ring*, Acta Math. Hungar. **137** (2012), no. 4, 242 – 253.

[20] **D. Kiani, M.M.A. Aghaei**, *On the unitary Cayley graph of a ring*, Electron. J. Combin. **19** (2012), no. 2, Paper 10, 10 pp.

[21] **C. Lanski, A. Maróti**, *Ring elements as sums of units*, Cent. Eur. J. Math. **7** (2009), 395 – 399.

[22] **H.R. Maimani, M.R. Pournaki, S. Yassemi**, *Weakly perfect graphs arising from rings* Glasg. Math. J. **52** (2010), 417 – 425.

[23] **H.R. Maimani, M.R. Pournaki, S. Yassemi**, *Necessary and sufficient conditions for unit graphs to be Hamiltonian*, Pacific J. Math. **249** (2011), 419 – 429.

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On ordered semigroups containing covered one-sided ideals

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Abstract. In this paper, the notion of covered left ideals of ordered semigroups will be introduced, and it is proved that the set of all covered left ideals of a given ordered semigroup is a sublattice of the lattice of all left ideals if the ordered semigroup. And then the structure of ordered semigroups containing covered left ideals will be studied. For the results of covered right ideals of ordered semigroups can be considered similarly.

1. Preliminaries

A proper left ideal M of a semigroup (without order) S is said to be a *covered left ideal* of S if

$$M \subseteq S(S \backslash M).$$

This notion was first introduced and studied by I. Fabrici [2]. Indeed, the author studied the structure of semigroups containing covered one-sided ideals. The purpose of this paper is to extend Fabrici's results to ordered semigroups. In fact, we introduce the concept of covered one-sided ideals of ordered semigroups, and study the structure of ordered semigroups containing covered one-sided ideals. For the concept of covered two-sided ideals of ordered semigroups can be found in [1].

A *partially ordered semigroup* (or simply an *ordered semigroup*) is a semigroup $(S, \cdot)$ together with a partial order $\leqslant$ that is *compatible* with the semigroup operation, meaning that, for any $x, y, z \in S$,

$$x \leqslant y \text{ implies } zx \leqslant zy \text{ and } xz \leqslant yz.$$

For A, B nonempty subsets of an ordered semigroup $(S, \cdot, \leqslant)$, the set product AB is defined to be the set of all elements xy in S where $x \in A$ and $y \in B$. We write $(A]$ for the set of all elements x in S such that $x \leqslant a$ for some $a \in A$, i.e.,

$$(A] = \{x \in S \mid x \leqslant a \text{ for some } a \in A\}.$$

In particular, we write Ax for $A\{x\}$, and similarly for $\{x\}A$. It is observed that the following hold:

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(1) $A \subseteq (A]$;

(2) $A \subseteq B \Rightarrow (A] \subseteq (B]$;

(3) $(A](B] \subseteq (AB]$;

(4) $(A \bigcup B] = (A] \bigcup (B]$;

(5) $((A]] = (A]$.

Let $(S, \cdot, \leqslant)$ be an ordered semigroup. Analogousely to the concept in lattice ordered rings (see [4], p. 142), a non-empty subset A of S is called a *left* (resp. *right*) *ideal* of S if it satisfies the following conditions:

(i) $SA \subseteq A$ (resp. $AS \subseteq A$);

(ii) for any $x \in A$ and $y \in S$, $y \leqslant x$ implies $y \in A$.

If A is both a left and a right ideal of S, then A is called a *two-sided ideal*, or simply an *ideal* of S. A left ideal A of S is called a *proper left* ideal of S if $A \subset S$. The symbol $\subset$ stands for proper subset of sets. For proper right ideals and proper ideals of S are defined similarly. A proper left (resp. right, two-sided) ideal A of S is said to be *maximal* if for any left (resp. right, two-sided) ideal B of S, $A \subset B \subseteq S$ implies $B = S$. Finally, if S does not contain proper left (resp. right, two-sided) ideals, we call it *left* (resp. *right, two-sided*) *simple*. It is easy to see that the union or intersection of two two-sided ideals of S is a two-sided ideal of S. For any element a of S the principal left ideal generated by a of S is of the form

$$L(a) = (a \bigcup Sa].$$

Analogousely to the concept of Green's relations in semigroups, the equivalence relation $\mathcal{L}$ is defined on S by, for any a, b in S,

$$a\mathcal{L}b \Longleftrightarrow L(a) = L(b).$$

The $\mathcal{L}$-class containing a in S will be denoted by L_a. The set of all $\mathcal{L}$-classes of S forms a quasi-ordered:

$$L_a \preceq L_b \Longleftrightarrow L(a) \subseteq L(b).$$

The symbol $L_a \prec L_b$ means $L_a \preceq L_b$, but $L_a \neq L_b$.

Let a be any element of an ordered semigroup $(S, \cdot, \leqslant)$. If L_a is not maximal, then $L_a \prec L_b$ for some b in S. Then $L(a) \subset L(b)$. We now have the following lemma.

Lemma 1.1. *Let a be any element of an ordered semigroup $(S, \cdot, \leqslant)$. If $L(a)$ is not proper subset of any principal left ideal of S, then L_a is maximal with respect to the quasi-order $\preceq$.*

Lemma 1.2. *Let L be a subset of an ordered semigroup $(S, \cdot, \leqslant)$. Then L is a maximal $\mathcal{L}$-class (with respect to $\preceq$) of S if and only if $S \backslash L$ is a maximal left ideal of S.*

Proof. Assume first that L is a maximal $\mathcal{L}$-class of S. Let $L = L_a$ for some $a \in S$. Let $y \in S$ and $x \in S \backslash L_a$. If $yx \in L_a$, then, by $x \notin L_a$, we have $L(a) = L(yx) \subset L(x)$. That is $L_a \prec L_x$, which contradicts to the assumption. Hence $S(S \backslash L_a) \subseteq S \backslash L_a$. Let $x \in S \backslash L_a$ and $y \in S$ be such that $y \leqslant x$. Then $L_y \preceq L_x$. If $y \in L_a$, then L_y is a maximal $\mathcal{L}$-class of S; hence $L_a = L_x$. This is a contradiction. Thus $y \in S \backslash L_a$. This shows that $S \backslash L_a$ is a left ideal of S. To show that $S \backslash L_a$ is a maximal left ideal of S, we suppose that there is a left ideal A of S such that $(S \backslash L_a) \subset A$. Let $z \in A \backslash (S \backslash L_a)$, and thus $L(a) = L(z)$. If $b \in L_a$, then

$$L(b) = L(a) = L(z) \subseteq A.$$

Thus $L_a \subseteq A$, and $S = A$.

Conversely, assume that $S \backslash L$ is a maximal left ideal of S. Choose a in $S \backslash (S \backslash L)$. We have $L = L_a$. To see this, let $x \in L_a$. If $x \in S \backslash L$, then

$$a \in L(a) = L(x) \subseteq S \backslash L.$$

This is a contradiction. Thus $x \in L$, and $L_a \subseteq L$. Let $x \in L$. Since $S \backslash L \subset (S \backslash L) \bigcup L(x)$, we have $(S \backslash L) \bigcup L(x) = S$. Similarly, $(S \backslash L) \bigcup L(a) = S$. Since $x \in (S \backslash L) \bigcup L(a)$, we have $x \in L(a)$, and so $L(x) \subseteq L(a)$. Similarly, $L(a) \subseteq L(x)$. Thus $L(x) = L(a)$, and $x \in L_a$. Therefore, $L \subseteq L_a$. Finally, by Lemma 1.1, it suffices to show that $L(a)$ is not proper subset of $L(x)$ for all x in S. Let $x \in S$. If $x \in L$, then $L(a) = L(x)$. If $x \in S \backslash L$, then $L(x) \subseteq S \backslash L$; hence $L(a) \nsubseteq L(x)$. This completes the proof. $\square$

2. Covered left ideals in ordered semigroups

We begin this section with the definition of covered left ideals of an ordered semigroup. For the concept of covered right ideals of an ordered semigroups can be defined dually, and the results for ordered semigroups containing covered right ideals are left-right dual.

Definition 2.1. *A proper left ideal L of an ordered semigroup $(S, \cdot, \leqslant)$ is called a covered left ideal of S if*

$$L \subseteq (S(S \backslash L)].$$

In this example we consider an ordered semigroup from [7] and [8].

Example 2.2. Let $(S, \cdot, \leqslant)$ be an ordered semigroup such that $S = \{a, b, c, d, e\}$ and

$\cdot$	a	b	c	d	e
a	a	b	a	a	a
b	a	b	a	a	a
c	a	b	a	a	a
d	a	b	a	a	a
e	a	b	a	a	e

$$\leqslant = \{(a,a),(a,b),(a,e),(b,b),(c,b),(c,c),(c,e),(d,d),(d,b),(d,e),(e,e)\}.$$

The covering relation is given by:

$$< = \{(a,b),(a,e),(c,b),(c,e),(d,b),(d,e)\}.$$

The left ideals of S are $\{a\}$, $\{a,c\}$, $\{a,d\}$, $\{a,c,d\}$, $\{a,b,c,d\}$, $\{a,c,d,e\}$ and S. It can be observed that the covered left ideals of S are $\{a\}$, $\{a,c\}$, $\{a,d\}$, $\{a,c,d\}$.

In this example we consider an ordered semigroup from [9].

Example 2.3. Let $(S,\cdot,\leqslant)$ be an ordered semigroup such that $S = \{a,b,c,d,f\}$ and

$\cdot$	a	b	c	d	e
a	b	d	a	b	e
b	d	b	b	d	e
c	d	b	c	d	e
d	b	d	d	b	e
e	e	e	e	e	e

$$\leqslant = \{(a,a),(b,b),(b,c),(b,e),(c,c),(d,a),(d,d),(d,e),(e,e)\}.$$

The covering relation is given by:

$$< = \{(b,c),(b,e),(d,a),(d,e)\}.$$

The left ideals of S are $\{b,d,e\}$, $\{a,b,d,e\}$ and S, and the covered left ideal of S is $\{b,d,e\}$.

Now, we will prove that the set of all covered left ideals of an ordered semigroup is a sublattice of the lattice of all left ideals.

Proposition 2.4. *If L_1 and L_2 are different proper left ideals of an ordered semigroup $(S,\cdot,\leqslant)$ such that $L_1 \bigcup L_2 = S$, then both L_1, L_2 are not covered left ideals of S.*

Proof. Assume that L_1 and L_2 are different proper left ideals of an ordered semigroup $(S,\cdot,\leqslant)$ such that $L_1 \bigcup L_2 = S$. Sine $L_1 \bigcup L_2 = S$, we have $S \backslash L_1 \subseteq L_2$ and $S \backslash L_2 \subseteq L_1$. If L_1 is a covered left ideal of S, then

$$L_1 \subseteq (S(S\backslash L_1)] \subseteq (SL_2] \subseteq L_2.$$

Since $L_1 \bigcup L_2 = S$, it follows that $S = L_2$. This is a contradiction. Similarly, L_2 is a covered left ideal of S implies $S = L_1$. This is a contradiction. Thus the assertion is proved. □

The following corollary is a consequence of Proposition 2.4.

Corollary 2.5. *If an ordered semigroup $(S, \cdot, \leqslant)$ contains more than one maximal left ideals, then non of them is a covered left ideal of S.*

Proposition 2.6. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. If L_1 and L_2 are covered left ideals of S, then $L_1 \bigcup L_2$ is a covered left ideal of S.*

Proof. Assume that L_1 and L_2 are covered left ideals of S; thus $L_1 \subseteq (S(S \backslash L_1)]$ and $L_2 \subseteq (S(S \backslash L_2)]$. Let $x \in L_1 \bigcup L_2$. If $x \in L_1$, then, by $L_1 \subseteq (S(S \backslash L_1)]$, we have $x \in (Sa]$ for some $a \in S \backslash L_1$. If $a \in S \backslash (L_1 \bigcup L_2)$, then $x \in (S(S \backslash (L_1 \bigcup L_2))]$. If $a \in L_1 \bigcup L_2$, then $a \in L_2$; hence $a \in (Sb]$ for some b in $S \backslash L_2$. We have

$$x \in (Sa] \subseteq ((S](Sb]] = (SSb] \subseteq (Sb].$$

If $b \in L_1$, then $a \in L_1$. This is a contradiction. Thus $b \in S \backslash (L_1 \bigcup L_2)$, and so $x \in (S(S \backslash (L_1 \bigcup L_2))]$. Similarly, $x \in L_2$ implies $x \in (S(S \backslash (L_1 \bigcup L_2))]$. This proves that $L_1 \bigcup L_2$ is a covered left ideal of S. □

Proposition 2.7. *Let L be a left ideal of an ordered semigroup $(S, \cdot, \leqslant)$. If L_1 is a covered left ideal of S, then $L_1 \cap L$ is a covered left ideal of S.*

Proof. If L_1 is a covered left ideal of S, then $L_1 \subseteq (S(S \backslash L_1)]$; hence

$$L_1 \cap L \subseteq L_1 \subseteq (S(S \backslash L_1)] \subseteq (S(S \backslash (L_1 \cap L))].$$

This shows that $L_1 \cap L$ is a covered left ideal of S. □

Corollary 2.8. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. If L_1 and L_2 are covered left ideals of S, then $L_1 \cap L_2$ is a covered left ideal of S.*

We now state the main theorem of this section followed by Proposition 2.6 and Corollary 2.8.

Theorem 2.9. *The set of all covered left ideals of an ordered semigroup $(S, \cdot, \leqslant)$ is a sublattice of the lattice of all left ideals of $(S, \cdot, \leqslant)$.*

3. Ordered semigroups with covered left ideals

The purpose of this section is to study the structure of ordered semigroups containing covered left ideals.

Theorem 3.1. *An ordered semigroup $(S, \cdot, \leqslant)$ with the cardinal $|S| > 1$ contains no covered left ideals if and only if S is a union of disjoint minimal left ideals.*

Proof. Assume first that S contains no covered left ideals. Let $a \in S$. If $a \notin (Sa]$, then $(Sa] \subseteq (S(S\backslash(Sa])]$. Thus $(Sa]$ is a covered left ideal of S. This is a contradiction. Hence $a \in (Sa]$. Let L be a proper left ideal of S. If $L \subset (Sa]$, then L is a covered left ideal of S. This is a contradiction. Hence $(Sa]$ is a minimal left ideal of S. Let $a, b \in S$ such that $a \neq b$ and $(Sa] \neq (Sb]$. If $L = (Sa] \cap (Sb]$, then L is a proper subset of $(Sa]$ or $(Sb]$. Thus L is a covered left ideal of S. This is a contradiction. Hence $(Sa] \cap (Sb] = \emptyset$. Therefore, $S = \bigcup_{i \in I}(Sa_i]$.

Conversely, assume that $S = \bigcup_{i \in I} L_i$ where, for each $i \in I$, L_i is a minimal left ideal of S. We set

$$A = \bigcup_{i \in J} L_i, \quad B = \bigcup_{i \in I-J} L_i.$$

Then $S = A \bigcup B$. By Proposition 2.4, neither A nor B is a covered left ideal of S. This completes the proof. □

Theorem 3.2. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. If S is not left simple, then S contains a covered left ideal.*

Proof. Assume that S is not left simple. Then S contains a proper left ideal L. Since $(S(S\backslash L)]$ is a left ideal of S, we have $L \cap (S(S\backslash L)]$ is a proper left ideal of S. By

$$L \cap (S(S\backslash L)] \subseteq (S(S\backslash L)] \subseteq (S(S\backslash(L \cap (S(S\backslash L)]))],$$

it follows that $L \cap (S(S\backslash L)]$ is a covered left ideal of S. □

The concept of right bases of an ordered semigroup was defined in [3] as follows:

Definition 3.3. *A subset A of an ordered semigroup $(S, \cdot, \leqslant)$ is called a right base of S if it satisfies the following conditions:*

(i) $S = (A \bigcup SA]$;

(ii) *if B is a subset of A such that $S = (B \bigcup SB]$, then $B = A$.*

Here, we provide some more examples: In this example we consider an ordered semigroup from [6].

Example 3.4. Let $(S, \cdot, \leqslant)$ be an ordered semigroup such that the multiplication and the order relation are defined by:

$\cdot$	a	b	c	d	e
a	a	e	c	d	e
b	a	b	c	d	e
c	a	e	c	d	e
d	a	e	c	d	e
e	a	e	c	d	e

$$\leqslant = \{(a,a), (a,d), (b,b), (c,c), (c,e), (d,d), (e,e)\}.$$

The covering relation is given by:

$$\prec = \{(a,d),(c,e)\}.$$

We have $\{b,d\}$ is the right base of S.

In this example we consider an ordered semigroup from [5].

Example 3.5. Let $(S,\cdot,\leqslant)$ be an ordered semigroup such that the multiplication and the order relation are defined by:

$\cdot$	a	b	c	d	e
a	a	a	c	a	c
b	a	a	c	a	c
c	a	a	c	a	c
d	d	d	e	d	e
e	d	d	e	d	e

$$\leqslant = \{(a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(b,c),(b,d),(b,e),(c,c),(c,e),\\ (d,d),(d,e),(e,e)\}.$$

The covering relation is given by:

$$\prec = \{(a,b),(b,c),(b,d),(c,e),(d,e)\}.$$

The right bases of S are $\{e\}$ and $\{c\}$.

A covered left ideal L of an ordered semigroup $(S,\cdot,\leqslant)$ is called the *greatest covered left ideal* of S if every covered left ideal of S contained in L. If an ordered semigroup contains the greatest covered left ideal, we shall denote it by L^g.

To give a necessary condition so that an ordered semigroup contains one-sided bases we need the following lemma.

Lemma 3.6. *Let $(S,\cdot,\leqslant)$ be an ordered semigroup containing the greatest covered left ideal L^g. If $L^g \subset (S^2]$, then the following conditions hold:*

(1) *for every $\mathcal{L}$-class in $(S^2]\backslash L^g$ is maximal;*

(2) *$L(a) = (Sa]$ for all a in $(S^2]\backslash L^g$.*

Proof. Assume that $L^g \subset (S^2]$. Then $(S^2]\backslash L^g$ is non-empty. Frits we prove the second assertion. Let a be an element of $(S^2]\backslash L^g$. Since L^g is an left ideal of S, it follows that $L_a \subseteq (S^2]\backslash L^g$. Then $a \in (Sb]$ for some b in S. Sine $(Sb] \subseteq L(b)$, we have $L(a) \subseteq L(b)$. Suppose that $b \notin L_a$; thus $L_a \neq L_b$. If $b \in L(a)$, then $L(a) = L(b)$; hence $L_a = L_b$. This is a contradiction. Then $b \in S\backslash L(a)$. This implies $L(a) \subseteq (S(S\backslash L(a))]$. Thus $L(a)$ is a covered left ideal of S. By Proposition 2.6, $L^g \bigcup L(a)$ is a covered left ideal of S. Since $a \notin L^g$, $L^g \subset L^g \bigcup L(a)$. This is a contradiction. This shows that $b \in L_a$. Hence $L(a) \subseteq (Sb] \subseteq L(b) = L(a)$. Then $L(a) = (Sb] = L(b)$. Clearly, $(Sa] \subseteq L(a)$. If $b \leqslant a$, then $L(a) = (Sb] \subseteq (Sa]$;

hence $L(a) \subseteq (Sa]$. If $b \in (Sa]$, then $Sb \subseteq S(Sa] \subseteq (S(Sa]] \subseteq (SSa] \subseteq (Sa]$. Therefore, $L(a) = L(b) = (Sb] \subseteq (Sa]$.

We now prove the rest of the assertion. Let L_a be a $\mathcal{L}$-class in $(S^2]\backslash L^g$. Suppose that $L(a) \subseteq L(c)$ for some c in S. Then $a \in (c \bigcup Sc]$; thus $a \in (c]$ or $a \in (Sc]$. Each of the cases implies $(Sa] \subseteq (Sc]$, so $L(a) \subseteq (Sc]$. Sine $c \in S\backslash L(a)$, it follows that $L(a)$ is a covered left ideal of S; hence $L^g \subset L^g \bigcup L(a)$. This is a contradiction. This proves that any $\mathcal{L}$-class in $(S^2]\backslash L^g$ is maximal. □

Theorem 3.7. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup containing the greatest covered left ideal L^g. Then S contains a right base if satisfies the following two condition:*

(1) *$L^g \subset (S^2]$;*

(2) *any two element a, b in $S\backslash(S^2]$, neither $L_a \preceq L_b$ nor $L_b \preceq L_a$*

Proof. Assume that $L^g \subset (S^2]$ and for any two elements a, b in $S\backslash(S^2]$ are incompairable. By

$$L^g \subseteq (S(S\backslash L^g)] \subseteq (S^2] \subseteq S$$

there are two families of $\mathcal{L}$-class to consider: $C_1 = \{L_a | a \in S\backslash(S^2]\}$,
$C_2 = \{L_a | a \in (S^2]\backslash L^g\}$. We take one element from each $\mathcal{L}$-class in C_1 and C_2, and let A be the set of all elements we take, we claim that A is a right base of S. For convenience we let $L(A) = (A \bigcup SA]$. To show that $S = L(A)$, it suffices to show that L^g, $(S^2]\backslash L^g$ and $S\backslash(S^2]$ are subset of $L(A)$.

a) Let $x \in L^g$. Then $x \in (S(S\backslash L^g)]$, or equivalent $x \in (Sb]$ for some b in $S\backslash L^g$ We have $b \in L_a$ for some $a \in S\backslash(S^2]$ or $a \in (S^2]\backslash L^g$. Then by the constructing A we have $b \in L(A)$. Thus $x \in L(A)$.

b) If $x \in (S^2]\backslash L^g$, then there exists $a_1 \in A$ such that $x \in L(a_1)$; hence $x \in L(A)$.

c) If $x \in S\backslash(S^2]$, then there exists $a_2 \in A$ such that $x \in L(a_2) \subseteq L(A)$.

Finally, we show that A is the minimal subset of S such that $S = L(A)$. By Lemma 1.2, it follow that for every L_a in C_2 is maximal. Moreover, every L_a in C_1 is also maximal since for any elements a, b in $S\backslash(S^2]$, neither $L_a \preceq L_b$ nor $L_b \preceq L_a$. Now, let B be a proper subset of A such that $S = (B \bigcup SB]$. Let $x \in A\backslash B$. Then $x \leqslant y$ for some $y \in B \bigcup SB$. Since $y \in L(b)$ for some b in B, it follows that $L(x) \subset L(b)$. This contradicts to the constructing of A. The proof is completed. □

Let $(S, \cdot, \leqslant)$ be an ordered semigroup. An left ideal L of S is called the *greatest left ideal* of S if every proper left ideal of S contained in L. If an ordered semigroup $(S, \cdot, \leqslant)$ contains the greatest left ideal, we denote The left ideal by L^*.

Theorem 3.8. *Assume an ordered semigroup $(S, \cdot, \leqslant)$ contains only one maximal ideal L. If L is a covered left ideal of S, then L is the greatest left ideal of S.*

Proof. This is easy to see because if T is a proper left ideal of S, then $T \subseteq L$. Hence $L = L^*$. □

Theorem 3.9. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup with the property every proper ideal of S is a covered left ideal of S. Then either*

(1) *S contains L^*, or*

(2) *$S = (S^2]$ and for any proper left ideal L and for every left ideal $L(a) \subseteq L$, there is b in $S \backslash L$ such that $L(a) \subset L(b) \subset S$.*

Proof. If L_x and L_y are maximal $\mathcal{L}$-classes of S such that $L_x \neq L_y$, then by Lemma 1.2, we have $L_x^c = S \backslash L_x$ and $L_y^c = S \backslash L_y$ are maximal proper left ideals of S such that non of them is a covered left ideal of S. This is a contradiction. Then S contain no different maximal $\mathcal{L}$-classes; hence S contains one maximal $\mathcal{L}$-class or S does not contain maximal $\mathcal{L}$-class.

If S contains one maximal $\mathcal{L}$-class L_x. Then $L_x^c = S \backslash L_x$ is a maximal proper left ideal of S. By assumption, L_x^c is a covered left ideal of S. By Theorem 3.8, $L_x^c = L^*$.

Assume that S does not contain maximal $\mathcal{L}$-classes. We will show that $S = (S^2]$.Suppose $(S^2] \subset S$. Then there exists y in $S \backslash (S^2]$. If $L(y) = S$, then S contains a maximal $\mathcal{L}$-class. This is impossible. Then $L(y) \subset S$, and thus $L(y) \subseteq (S(S \backslash L(y))]$. Then $y \in (S^2]$. This is a contradiction.

Let L be a proper left ideal of S and let $L(a) \subseteq L$. Since $L \subseteq (S(S \backslash L)]$, there exists b in $S \backslash L$ such that $a \in (Sb]$, and hence $L(a) \subseteq L(b) \subseteq S$. Since $b \in S \backslash M$, so $L(a) \subset L(b)$. By assumption, $L(b) \subset S$. □

Theorem 3.10. *Assume an ordered semigroup $(S, \cdot, \leqslant)$ satisfies one the following two condition:*

(1) *S contains L^* which is a covered left ideal of S.*

(2) *$S = (S^2]$ and for any proper left ideal L and for every left ideal $L(a) \subseteq L$, there is b in $S \backslash L$ such that $L(a) \subseteq L(b)$.*

Then every proper left ideal of S is a covered left ideal of S.

Proof. Let L be a proper left ideal of S. First we assume that S satisfies (1). Then $L \subseteq L^*$. Since $S \backslash L^* \subseteq S \backslash L$, it follows that $L \subseteq L^* \subseteq (S(S \backslash L^*)] \subseteq (S(S \backslash L)]$. This shows that L is a covered left ideal of S.

Secondary, we assume that S satisfies (2). Let $x \in L$; thus $L(x) \subseteq L$. Then there is b in $S \backslash L$ such that $L(x) \subseteq L(b)$. We have $S = (S^2]$, so $b \in (Sd]$ for some d in S. Since $b \in S \backslash L$, $d \in S \backslash L$. Hence $x \in (Sd] \subseteq (S(S \backslash L)]$. This shows that $L \subseteq (S(S \backslash L)]$. □

Definition 3.11. *We say that the principal left ideals of an ordered semigroup $(S, \cdot, \leqslant)$ are updirected if for every $a, b \in S$ there is $c \in S$ such that $\{a, b\} \subseteq L(c)$.*

Theorem 3.12. *If all proper left ideals of an ordered semigroup $(S, \cdot, \leqslant)$ are covered, then the principal left ideals of S are updirected.*

Proof. Suppose that there exist two elements $a, b \in S$ such that there is no $c \in S$ with $\{a, b\} \subseteq L(c)$. It is sufficient to show that there exists a left ideal of S which is not covered. Let

$$L = \{x \in S \mid a \notin L(x)\}.$$

We have $L \neq \emptyset$ because $a \in L$. If $x \in S$ and $y \in L$, then $xy \in L$. Otherwise, we have $a \in L(xy) \subseteq L(y)$. This is a contradiction. If $x \in S$ and $y \in L$ such that $x \leqslant y$, then $L(x) \subseteq L(y)$. Since $a \notin L(y)$, $a \notin L(x)$. Thus $x \in L$. Hence L is a left ideal of S. Moreover, $b \in L$. Indeed, if $b \notin L$, then $a \in L(b)$; hence $\{a, b\} \subseteq L(b)$. This is a contradiction. Finally, we have to show that $b \notin (S(S \backslash L)]$. Suppose that $b \in (S(S \backslash L)]$ such that $b \leqslant xy$ for some $x \in S$ and $y \in S \backslash L$. This implies that $\{a, b\} \subseteq L(y)$ which is a contradiction. $\square$

References

[1] **T. Changpgas, P. Summaprab**, *On ordered semigroups containing covered ideals*, Commun. Algebra **44** (2016), 4104-4113.

[2] **I. Fabrici**, *Semigroups containing covered one-sided ideals*, Math. Slovaca **31** (1981), 225-231.

[3] **I. Fabrici**, *One-sided bases of semigroups*, Math. Čas. **22** (1972), 286-290.

[4] **L. Fuchs**, *Partially Ordered Algebraic Systems*, Pergamon Press, 1963.

[5] **N. Kehayopulu**, *On completely regular ordered semigroups*, Scinetiae Math. **1** (1998), 27-32.

[6] **N. Kehayopulu, M. Tsingelis**, *On subdirectly irreducible ordered semigroups*, Semigroup Forum **50** (1995), 161-177.

[7] **N. Kehayopulu**, On regular, intra-regular ordered semigroups, Pure Math. Appl. (PU.M.A.) **4** (1993), 447-461.

[8] **N. Kehayopulu, M. Tsingelis**, *On kernel of ordered semigroups*, Commun. Algebra **28** (2013), 225-229.

[9] **N. Kehayopulu, M. Tsingelis**, *A note on ordered groupoids-semigroups*, Scientiae Math. **3** (2000), 251-255.

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On the lattice of congruences on completely regular semirings

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Abstract. A semiring S is called completely regular if it is the disjunctive union of its subrings. If S is a completely regular semiring, then the Green's relation $\mathcal{H}^+$ is a congruence on S and $S/\mathcal{H}^+$ is an idempotent semiring. Let $\mathcal{V}$ be a variety of idempotent semirings. Here we characterize the lattice $C(S)$ of all congruences on S when S is completely regular and $S/\mathcal{H}^+ \in \mathcal{V}$. The lattice $C(S)$ can be embedded into the product of the lattice $\mathcal{V}(S)$ of all $\mathcal{V}$-congruences on S and the lattice $M(S)$ of all additive idempotent-separating congruences on S if and only if S is τ-modular completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$.

1. Introduction

A semigroup is called completely regular if it is the (disjunctive) union of its subgroups. Completely regular semigroups were introduced in [3] by A. H. Clifford, though he used the terminology 'semigroups admitting relative inverses' to refer to such semigroups. Such semigroups have been studied extensively. For an account of the theory of completely regular semigroups, we refer to the book [13].

A *semiring* is a $(2, 2)$ algebra $(S, +, \cdot)$ such that both the additive reduct $(S, +)$ and the multiplicative reduct $(S, \cdot)$ are semigroups and connected by the distributive laws

$$x(y + z) = xy + xz,\ (x + y)z = xz + yz.$$

An element $e \in S$ is called an *additive idempotent* if $e + e = e$. The set of all additive idempotents of S is denoted by $E^+(S)$. A semiring S is called *additive regular* if the additive reduct $(S, +)$ is a regular semigroup. By an *idempotent semiring* we mean a semiring S such that both the additive reduct $(S, +)$ and $(S, \cdot)$ are bands. If moreover, the reduct $(S, +)$ is commutative then S is called a *b-lattice*. Also we refer to [5] for the undefined terms and notions in semirings and [6] for background on semigroups.

Let us denote the Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{D}$ and $\mathcal{H}$ on the additive reduct $(S, +)$ by $\mathcal{L}^+$, $\mathcal{R}^+$, $\mathcal{D}^+$ and $\mathcal{H}^+$, respectively. Also we denote the $\mathcal{L}^+, \mathcal{R}^+, \mathcal{D}^+$ and $\mathcal{H}^+$ classes of $x \in S$ by L^+_x, R^+_x, D^+_x and H^+_x, respectively. A semiring S is called an *idempotent semiring* (*b-lattice, distributive lattice*) *of rings* if there is a congruence

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ρ on S such that the quotient semiring S/ρ is an idempotent semiring (b-lattice, distributive lattice) and each ρ-class is a ring.

Rings and distributive lattices both are semirings with commutative regular addition. So, it is interesting to consider the semirings which are subdirect products of rings and distributive lattices. Such semirings were studied by Bandelt and Petrich [1]. The study was continued by Ghosh [4] and he characterized the Clifford semirings, equivalently, the semirings which are strong distributive lattices of rings. Pastijn and Guo [12] proved that the semirings which are disjoint unions of rings form a variety and they established various structure theorems for such semirings. They proved that if S is a disjoint union of its subrings then $\mathcal{H}^+$ is an idempotent semiring congruence on S. The term 'completely regular semiring' was first used by Sen, Maity and Shum [16] to mean the semirings which are disjoint union of skew-rings (rings without commutativity of addition).

In [2], we establish several equivalent characterizations for the semirings which are the disjunctive unions of rings. Let $(S, +, \cdot)$ be the disjunctive union of its subrings. Then the additive reduct $(S, +)$ is the disjunctive union of its subgroups. For every $x \in S$, denote the zero in the subgroup $(H_x^+, +)$ of $(S, +)$ by x^o and the unique inverse of x in H_x^+ by x'. Then $x^o = x + x' = x' + x$. Hence S can be treated as an algebra $(S, +, \cdot, ')$ of type (2, 2, 1), where the reduct $(S, +, \cdot)$ is a semiring and the reduct $(S, +, ')$ is a completely regular semigroup. The following result is useful.

Lemma 1.1. [2] *Let S be a semiring. Then the following conditions are equivalent:*

(i) *S is the (disjunctive) union of its subrings;*

(ii) *for every $x, y \in S$ there exists unique $x' \in S$ such that*

$$x = x + x' + x,\ x + x' = x' + x,\ (x')' = x,\ x + y^o + x^o + y = x^o + y + x + y^o$$

and $xx^o = x^o$, *where* $x^o = x + x'$*;*

(iii) *$\mathcal{H}^+$ is an idempotent semiring congruence on S and each $\mathcal{H}^+$-class is a ring;*

(iv) *S is an idempotent semiring of rings.*

Definition 1.2. A semiring S is called *completely regular* if it satisfies either of the four equivalent conditions in Lemma 1.1.

Throughout the rest of this article, unless otherwise stated, S stands for a completely regular semiring.

It follows from a result of Kapp and Schneider [8] that the lattice $C(S)$ of all congruences on a semigroup S can be embedded in the product of certain sublattices if the semigroup S is completely simple. The problem of embedding the lattice $C(S)$ in a product of sublattices, when S is an arbitrary band of groups, was characterized by C. Spitznagel [17]. The principal tool used in these two texts is the τ-relation introduced by Reilly and Scheiblich [14]. In this last article, this relation is marked with θ ([14], Theorem 3.4). Also there are many other articles devoted to these directions [6], [7], [11].

The set of all congruences on a semiring S is a complete lattice, which we denote by $C(S)$. A sublattice L of $C(S)$ is called a modular sublattice if the lattice L is modular. It is well known that if the congruences in L commute then L is modular. The trace of a congruence ρ on a completely regular semiring S is defined by:

$$\operatorname{tr} \rho = \rho \cap (E^+(S) \times E^+(S)).$$

Define a relation τ on the lattice $C(S)$ by: for $\rho, \sigma \in C(S)$,

$$\rho \tau \sigma \text{ if } \operatorname{tr} \rho = \operatorname{tr} \sigma.$$

In Section 2, we characterize completely regular semirings S in terms of the maximum additive idempotent-separating congruence on S. In Section 3, we show that each τ-class in the lattice $C(S)$ of all congruences on an additive regular semiring S contains at most one $\mathcal{V}$-congruence on S, where $\mathcal{V}$ is a variety of idempotent semirings. We also have a necessary and sufficient condition for the greatest element of each τ-class to be a $\mathcal{V}$-congruence. In Section 4, we prove that the lattice $C(S)$ of all congruences on a τ-modular completely regular semiring S can be embedded in a certain product lattice.

Now let us fix the following notations:

$C(S)$:	the lattice of all congruences on S;
$M(S)$:	the lattice of all additive idempotent-separating congruences on S;
$D^+(S)$:	the lattice of all congruences on S that are contained in $\mathcal{D}^+$;
$\mathcal{V}(S)$:	the lattice of all $\mathcal{V}$-congruences on S;
$\rho_{\mathcal{V}}$:	the minimum $\mathcal{V}$-congruence on S;
β:	the minimum idempotent semiring congruence on S;
δ:	the minimum b-lattice congruence on S;
η:	the minimum distributive lattice congruence on S;
μ:	the maximum additive idempotent-separating congruence on S.

2. Additive idempotent separating congruences

A congruence ρ on S is called *additive idempotent-separating* if each ρ-class contains at most one additive idempotent, i.e., for every $e, f \in E^+(S)$, $e\rho f$ implies $e = f$. In this section, we characterize a completely regular semiring S by the maximum additive idempotent-separating congruence μ on itself.

In [10], Lallement proved that on a regular semigroup S, a congruence ρ is idempotent separating if and only if $\rho \subseteq \mathcal{H}$ on S. Since S is a completely regular semiring, the additive reduct $(S, +)$ is a regular semigroup, and so it follows that $\mu \subseteq \mathcal{H}^+$.

Now we have the following result.

Theorem 2.1. *Let S be an additive regular semiring and $\mathcal{V}$ be a variety of idempotent semirings. Then the following statements are equivalent:*

(*i*) *S is a completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$;*

(*ii*) *$\mu = \mathcal{H}^+ = \rho_{\mathcal{V}}$ and $x + y^o + x^o + y = x^o + y + x + y^o$ for every $x, y \in S$;*

(*iii*) *$\mu = \rho_{\mathcal{V}}$ and $x + y^o + x^o + y = x^o + y + x + y^o$ for every $x, y \in S$.*

Proof. Equivalence of (ii) and (iii) is trivial and so we omit the proof.

$(i) \Rightarrow (ii)$: Suppose that $a\mathcal{H}^+b$ in S. Then $a\rho_{\mathcal{V}}\mathcal{H}^+b\rho_{\mathcal{V}}$ in $S/\mathcal{H}^+$. Since each $\mathcal{H}^+$-class contains at most one additive idempotent, $a\rho_{\mathcal{V}} = b\rho_{\mathcal{V}}$. Hence $\mathcal{H}^+ \subseteq \rho_{\mathcal{V}}$ and it follows that $\mu \subseteq \mathcal{H}^+ \subseteq \rho_{\mathcal{V}}$. Now $S/\mathcal{H}^+ \in \mathcal{V}$ implies that $\rho_{\mathcal{V}} \subseteq \mathcal{H}^+$. Since $\mathcal{H}^+$ is an additive idempotent separating congruence, $\mathcal{H}^+ \subseteq \mu$. Thus $\mu = \mathcal{H}^+ = \rho_{\mathcal{V}}$.

$(ii) \Rightarrow (i)$: Suppose that the condition (ii) holds. Then $\mathcal{H}^+ = \rho_{\mathcal{V}}$ implies that $S/\mathcal{H}^+ \in \mathcal{V}$. Let H be an $\mathcal{H}^+$-class in S. Since $\mathcal{H}^+$ is an idempotent semiring congruence on S, H is an additive regular subsemiring of S and, by Lallement's Lemma, contains an additive idempotent. Hence $(H, +)$ is a group. Now for every $x, y \in S$, $x + y^o + x^o + y = x^o + y + x + y^o$ implies that $(H, +)$ is an abelian group. Thus H is a subring of S and so S is a completely regular semiring. $\square$

Now we have the following immediate consequence. Though it is a particular case of the above lemma, but useful.

Corollary 2.2. *Let S be any additive regular semiring. Then the following statements are equivalent.*

(*i*) *S is a completely regular semiring (b-lattice of rings, distributive lattice of rings);*

(*ii*) *$\mu = \mathcal{H}^+ = \beta\ (\ \delta,\ \eta\ )$ and $x + y^o + x^o + y = x^o + y + x + y^o$ for every $x, y \in S$;*

(*iii*) *$\mu = \beta\ (\ \delta,\ \eta\ )$ and $x + y^o + x^o + y = x^o + y + x + y^o$ for every $x, y \in S$.*

3. The relation τ on C(S)

The relation τ on $C(S)$ has many interesting properties when S is a completely regular semiring. Before coming into the main features let us first prove some lemmas.

The proof of the following result is similar to Lemma 2.1 [15], still for the sake of completeness we would like to include a proof.

Lemma 3.1. *Let S be an additive regular semiring and α be an additive idempotent separating congruence on S. Then for every $\gamma \in C(S)$, $(\alpha \vee \gamma, \gamma) \in \tau$.*

Proof. Let $\gamma \in C(S)$. Consider the relation $h = \{(a,b) \in S \times S : (a\gamma, b\gamma) \in \mathcal{H}^+\}$ on S. Then h is an equivalence relation on S and $\mathcal{H}^+ \subseteq h$ and $\gamma \subseteq h$. Let $(e,f) \in h \cap (E^+(S) \times E^+(S))$. Then $(e\gamma)\ \mathcal{H}^+\ (f\gamma)$. Since $\mathcal{H}^+$ is an additive idempotent separating congruence, $e\gamma = f\gamma$ and hence $(e,f) \in \gamma \cap (E^+(S) \times E^+(S))$. Therefore $h \cap (E^+(S) \times E^+(S)) \subseteq \gamma \cap (E^+(S) \times E^+(S))$. Since α separates additive idempotents, $\gamma \subseteq \alpha \vee \gamma \subseteq \mathcal{H}^+ \vee \gamma \subseteq h$ and consequently $\gamma \cap (E^+(S) \times E^+(S)) = h \cap (E^+(S) \times E^+(S)) = (\alpha \vee \gamma) \cap (E^+(S) \times E^+(S))$. Therefore $(\alpha \vee \gamma, \gamma) \in \tau$. □

Since $\mathcal{H}^+$ is an additive idempotent separating congruence on a completely regular semiring, we have, in particular:

Corollary 3.2. *Let S be a completely regular semiring. Then for every $\alpha \in C(S)$, $(\alpha \vee \mathcal{H}^+, \alpha) \in \tau$.*

We omit the proof of the following result, since it is similar to the proof of Theorem 2.2 [15].

Lemma 3.3. *If S is an additive regular semiring, then the relation τ is a complete lattice congruence on $C(S)$.*

Let $\mathcal{V}$ be a variety of idempotent semirings. Then a congruence σ on an additive regular semiring S is a $\mathcal{V}$-congruence if and only if σ contains $\rho_{\mathcal{V}}$, the minimum $\mathcal{V}$-congruence on S. Therefore we have:

Theorem 3.4. *Let S be an additive regular semiring and $\mathcal{V}$ be a variety of idempotent semirings. Then each τ-class in $C(S)$ contains at most one $\mathcal{V}$-congruence on S. In addition, if S is a completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$, then each τ-class contains exactly one $\mathcal{V}$-congruence.*

Proof. Let α, γ be two $\mathcal{V}$-congruences on S such that $(\alpha,\gamma) \in \tau$. Then $\rho_{\mathcal{V}} \subseteq \alpha$ and $\rho_{\mathcal{V}} \subseteq \gamma$. Let $x\alpha y$. Since $S/\rho_{\mathcal{V}}$ is an idempotent semiring it follows, by Lallement's Lemma, that there exist $e,f \in E^+(S)$ such that $e\rho_{\mathcal{V}}x$ and $f\rho_{\mathcal{V}}y$. Then $e\rho_{\mathcal{V}}x\alpha y\rho_{\mathcal{V}}f$ which implies that $(e,f) \in \alpha$. Since $(\alpha,\gamma) \in \tau$ it follows that $(e,f) \in \gamma$, and so $\rho_{\mathcal{V}} \subseteq \gamma$ implies that $x\gamma y$. Therefore $\alpha \subseteq \gamma$. Similarly we have $\gamma \subseteq \alpha$, and finally $\alpha = \gamma$.

Now suppose that S is a completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$. Then for every $\alpha \in C(S)$, $\rho_{\mathcal{V}} \subseteq \mathcal{H}^+ \subseteq \alpha \vee \mathcal{H}^+$ implies that $\alpha \vee \mathcal{H}^+$ is a $\mathcal{V}$-congruence. Also it follows from Lemma 3.1 that $\alpha \vee \mathcal{H}^+$ is in the τ-class of α. □

In particular, we have:

Corollary 3.5. *Let S be an additive regular semiring. Then each τ-class in $C(S)$ contains at most one idempotent semiring (b-lattice, distributive lattice) congruence. If moreover, S is a completely regular semiring (b-lattice of rings, distributive lattice of rings), then each τ-class contains exactly one idempotent semiring (b-lattice, distributive lattice) congruence.*

Following result shows that for every congruence α on S, the join $\alpha \vee \mathcal{H}^+$ in $C(S)$ gives important information about α.

Theorem 3.6. *Let S be a completely regular semiring and $\alpha, \gamma \in C(S)$. Then $(\alpha, \gamma) \in \tau$ if and only if $\alpha \vee \mathcal{H}^+ = \gamma \vee \mathcal{H}^+$.*

Proof. First suppose that $(\alpha, \gamma) \in \tau$. Then, by Corollary 3.2, $(\alpha \vee \mathcal{H}^+, \alpha) \in \tau$ and $(\gamma \vee \mathcal{H}^+, \gamma) \in \tau$ and it follows that $(\alpha \vee \mathcal{H}^+, \gamma \vee \mathcal{H}^+) \in \tau$. Also both $\alpha \vee \mathcal{H}^+$ and $\gamma \vee \mathcal{H}^+$ are idempotent semiring congruences and so, by Theorem 3.4, it follows that $\alpha \vee \mathcal{H}^+ = \gamma \vee \mathcal{H}^+$.

Conversely suppose that $\alpha \vee \mathcal{H}^+ = \gamma \vee \mathcal{H}^+$. Then $(\alpha \vee \mathcal{H}^+, \alpha) \in \tau$ and $(\gamma \vee \mathcal{H}^+, \gamma) \in \tau$ implies that $(\alpha, \gamma) \in \tau$. □

Following result can be proved similarly to Theorem 3.4 (ii) [14]. So we omit the proof.

Lemma 3.7. *Let S be an additive regular semiring. Then each τ-class in $C(S)$ is a complete modular sublattice of $C(S)$ with the greatest and least elements.*

The following theorem gives a necessary and sufficient condition for the greatest element of each τ-class in $C(S)$ to be a $\mathcal{V}$-congruence on S, where $\mathcal{V}$ is a variety of idempotent semirings.

Theorem 3.8. *Let S be a completely regular semiring and $\mathcal{V}$ be a variety of idempotent semirings. Then the greatest element of each τ-class in $C(S)$ is a $\mathcal{V}$-congruence if and only if $S/\mathcal{H}^+ \in \mathcal{V}$.*

Proof. First suppose that S is a completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$. Then $\mathcal{H}^+ = \rho_{\mathcal{V}}$, by Theorem 2.1. Let $\alpha \in C(S)$ and γ be the greatest element of the τ-class of α. Now, by Corollary 3.2, $(\alpha \vee \mathcal{H}^+, \alpha) \in \tau$ and so $\mathcal{H}^+ \subseteq \alpha \vee \mathcal{H}^+ \subseteq \gamma$. Thus γ is a $\mathcal{V}$-congruence.

Conversely, suppose that the greatest element of each τ-class is a $\mathcal{V}$-congruence. Since μ is the greatest element of the τ-class of Δ_S, μ is a $\mathcal{V}$-congruence. Also, on every additive regular semiring, $\mu \subseteq \mathcal{H}^+ \subseteq \rho_{\mathcal{V}}$. Therefore $\mu = \mathcal{H}^+ = \rho_{\mathcal{V}}$ and it follows, by Theorem 2.1, that $S/\mathcal{H}^+ \in \mathcal{V}$. □

Now we have the following important corollary.

Corollary 3.9. *Let S be a completely regular semiring. Then the greatest element of each τ-class is an idempotent semiring congruence.*

Moreover, the greatest element of each τ-class is a b-lattice (distributive lattice) congruence if and only if S is a b-lattice (distributive lattice) of rings.

4. Embedding of C(S) in a product lattice

Let $\mathcal{V}$ be a variety of idempotent semirings and S be a completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$. Then for every $\alpha \in C(S)$, $\mathcal{H}^+ \subseteq \alpha \vee \mathcal{H}^+$ implies that

$\alpha \vee \mathcal{H}^+ \in \mathcal{V}(S)$ and $\alpha \wedge \mathcal{H}^+ \subseteq \mathcal{H}^+$ implies that $\alpha \wedge \mathcal{H}^+$ is an additive idempotent separating congruence. Thus we have a mapping $\phi : C(S) \to \mathcal{V}(S) \times M(S)$ defined by: for every $\alpha \in C(S)$,

$$\phi(\alpha) = (\alpha \vee \mathcal{H}^+, \alpha \wedge \mathcal{H}^+).$$

Lemma 4.1. *Let $\mathcal{V}$ be a variety of idempotent semirings and S be a completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$. Then ϕ is one-to-one.*

Proof. Suppose that $\alpha, \gamma \in C(S)$ are such that $\phi(\alpha) = \phi(\gamma)$. Then $\alpha \vee \mathcal{H}^+ = \gamma \vee \mathcal{H}^+$ and $\alpha \wedge \mathcal{H}^+ = \gamma \wedge \mathcal{H}^+$. It follows, by Theorem 3.6, $(\alpha, \gamma) \in \tau$. Let $(x, y) \in \alpha$ and $e \in E^+(S) \cap H_x^+$, $f \in E^+(S) \cap H_y^+$. Then $e = x^o\ \alpha\ y^o = f$ implies that $(e, f) \in \alpha$, and so $(e, f) \in \gamma$. Now $x = (x + e)\gamma(x + f)$ and $y = (f + y)\gamma(e + y)$ together with $e\alpha f$ imply that $(e + y)\alpha(f + y) = y\alpha x = (x + e)\alpha(x + f)$, and so $(e + y)\alpha(x + f)$. Also $e\ \mathcal{H}^+\ x$ and $y\ \mathcal{H}^+\ f$ imply that $(e+y)\mathcal{H}^+(x+f)$. Thus $(e+y)\mathcal{H}^+ \wedge \alpha(x+f)$ and so $(e+y)\gamma \wedge \mathcal{H}^+(x+f)$. Hence $x\gamma(x+f)\gamma(e+y)\gamma y$ and it follows that $\alpha \subseteq \gamma$. Similarly $\gamma \subseteq \alpha$. Thus $\alpha = \gamma$. □

Theorem 4.2. *Let S be a completely regular semiring and $\alpha \in C(S)$. Then $\alpha = \bar{\alpha} \vee (\alpha \wedge \mathcal{H}^+)$, where $\bar{\alpha}$ is the smallest element of τ-class of α.*

Proof. Let $\alpha \in C(S)$. Then $\alpha \wedge \mathcal{H}^+$ is an additive idempotent separating congruence, which implies that $\alpha \wedge \mathcal{H}^+ \tau \mathcal{H}^+$ and so, by Corollary 3.2, $\bar{\alpha} \vee (\alpha \wedge \mathcal{H}^+)\ \tau\ \bar{\alpha} \vee \mathcal{H}^+\ \tau\ \bar{\alpha}\ \tau\ \alpha$. Therefore, by Theorem 3.6, $\alpha \vee \mathcal{H}^+ = [\bar{\alpha} \vee (\alpha \wedge \mathcal{H}^+)] \vee \mathcal{H}^+$. Now $\bar{\alpha}, \alpha \wedge \mathcal{H}^+ \subseteq \alpha$ implies that $[\bar{\alpha} \vee (\alpha \wedge \mathcal{H}^+)] \wedge \mathcal{H}^+ \subseteq \alpha \wedge \mathcal{H}^+$. Also, $\alpha \wedge \mathcal{H}^+ \subseteq \bar{\alpha} \vee (\alpha \wedge \mathcal{H}^+), \mathcal{H}^+$ and hence $\alpha \wedge \mathcal{H}^+ \subseteq [\bar{\alpha} \vee (\alpha \wedge \mathcal{H}^+)] \wedge \mathcal{H}^+$. Thus we have $\alpha \wedge \mathcal{H}^+ = [\bar{\alpha} \vee (\alpha \wedge \mathcal{H}^+)] \wedge \mathcal{H}^+$. Therefore $\phi(\alpha) = \phi(\bar{\alpha} \vee (\alpha \wedge \mathcal{H}^+))$ and so $\alpha = \bar{\alpha} \vee (\alpha \wedge \mathcal{H}^+)$. □

Theorem 4.3. *Let S be a completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$. Then ϕ is $\wedge$-preserving.*

Proof. We have $(\alpha \vee \mathcal{H}^+) \wedge (\gamma \vee \mathcal{H}^+)\ \tau\ (\alpha \wedge \gamma)$ and $(\alpha \wedge \gamma) \vee \mathcal{H}^+\ \tau\ (\alpha \wedge \gamma)$. Then it follows that $(\alpha \wedge \gamma) \vee \mathcal{H}^+\ \tau\ (\alpha \vee \mathcal{H}^+) \wedge (\gamma \vee \mathcal{H}^+)$. Since both $(\alpha \wedge \gamma) \vee \mathcal{H}^+$ and $(\alpha \vee \mathcal{H}^+) \wedge (\gamma \vee \mathcal{H}^+)$ are $\mathcal{V}$-congruences it follows, by Theorem 3.4, that $(\alpha \wedge \gamma) \vee \mathcal{H}^+ = (\alpha \vee \mathcal{H}^+) \wedge (\gamma \vee \mathcal{H}^+)$. Also we have $(\alpha \wedge \gamma) \wedge \mathcal{H}^+ = (\alpha \wedge \mathcal{H}^+) \wedge (\gamma \wedge \mathcal{H}^+)$. Therefore ϕ is $\wedge$-preserving. □

Corollary 4.4. *Let S be a completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$. Then $\mathcal{V}(S)$ is lattice isomorphic with $C(S)/\tau$.*

Let $\mathcal{V}$ be a variety of idempotent semirings and S be a completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$. Denote the restriction of ϕ to $D^+(S)$ by $\widetilde{\phi}$. Thus the mapping $\widetilde{\phi} : D^+(S) \to \mathcal{V}(S) \times M(S)$ is given by: for every $\alpha \in D^+(S)$,

$$\widetilde{\phi}(\alpha) = (\alpha \vee \mathcal{H}^+, \alpha \wedge \mathcal{H}^+).$$

Theorem 4.5. *Let $\mathcal{V}$ be a variety of idempotent semirings and S be a completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$. Then $\widetilde{\phi}$ is $\vee$-preserving.*

Proof. Let $\alpha, \gamma \in D^+(S)$. Then $\alpha \wedge \mathcal{H}^+$, $\gamma \wedge \mathcal{H}^+ \subseteq (\alpha \vee \gamma) \wedge \mathcal{H}^+$ implies that $(\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+) \subseteq (\alpha \vee \gamma) \wedge \mathcal{H}^+$. Suppose that $(x, y) \in (\alpha \vee \gamma) \wedge \mathcal{H}^+$. Then $(x, y) \in \alpha \vee \gamma$ implies that there exists a positive integer n and $x_i, y_i \in S$, $i = 1, 2, \ldots, n$ such that

$$x\alpha x_1 \gamma y_1 \alpha x_2 \gamma y_2 \alpha \cdots \alpha x_n \gamma y_n = y.$$

Since $\alpha, \gamma \subseteq \mathcal{D}^+$, it follows that x_i, $y_i \in D_x^+ = D_y^+$. Also $x\mathcal{H}^+ y$ implies that $H_x^+ = H_y^+$. Suppose that e is the identity element in H_x^+. Then

$$x = (e+x+e)\alpha(e+x_1+e)\tau(e+y_1+e)\alpha \cdots \alpha(e+x_n+e)\gamma(e+y_n+e) = (e+y+e) = y.$$

Since $D_x^+ = D_y^+$ is a completely simple semiring, $e + x_i + e$, $e + y_i + e \in e + D_x^+ + e = H_x^+$ for each i. Therefore we have $(x, y) \in (\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+)$ and so $(\alpha \vee \gamma) \wedge \mathcal{H}^+ \subseteq (\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+)$. Hence $(\alpha \vee \gamma) \wedge \mathcal{H}^+ = (\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+)$. Also we have $(\alpha \vee \gamma) \vee \mathcal{H}^+ = (\alpha \vee \mathcal{H}^+) \vee (\gamma \vee \mathcal{H}^+)$. This completes the proof. $\square$

Let L be a lattice, and ζ a lattice congruence on L. We say that L is ζ- *modular* if for every $a, b, c \in L$, the conditions $a \geq b$, $(a, b) \in \zeta$, $a \wedge c = b \wedge c$ and $a \vee c = b \vee c$, imply that $a = b$. A semiring S is said to be τ-modular if the lattice $C(S)$ of all congruences on S is τ-modular. Thus:

Definition 4.6. *A semiring S is called τ-modular if for every $\rho, \sigma, \xi \in C(S)$, the conditions $\sigma \subseteq \rho$, $\sigma\tau\rho$, $\rho \wedge \xi = \sigma \wedge \xi$ and $\rho \vee \xi = \sigma \vee \xi$ imply that $\rho = \sigma$.*

Lemma 4.7. *Let S be a τ-modular completely regular semiring. Then for every $\alpha, \gamma \in C(S)$, $\alpha \vee [(\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+)] = \alpha \vee [(\alpha \vee \gamma) \wedge \mathcal{H}^+]$.*

Proof. Let $\alpha, \gamma \in C(S)$. Then $(\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+) \subseteq (\alpha \vee \gamma) \wedge \mathcal{H}^+$ implies that $\alpha \vee [(\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+)] \subseteq \alpha \vee [(\alpha \vee \gamma) \wedge \mathcal{H}^+]$. Also $\alpha \vee [(\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+)] = [\alpha \vee (\alpha \wedge \mathcal{H}^+)] \vee (\gamma \wedge \mathcal{H}^+) = \alpha \vee (\gamma \wedge \mathcal{H}^+)$. Now $(\gamma \wedge \mathcal{H}^+)\tau(\alpha \vee \gamma) \wedge \mathcal{H}^+$ implies that $[\alpha \vee (\gamma \wedge \mathcal{H}^+)]\ \tau\ \alpha \vee [(\alpha \vee \gamma) \wedge \mathcal{H}^+]$. Thus by τ-modularity, it suffices to show $\gamma \vee [\alpha \vee (\gamma \wedge \mathcal{H}^+)] = \gamma \vee [\alpha \vee [(\alpha \vee \gamma) \wedge \mathcal{H}^+]]$ and $\gamma \wedge [\alpha \vee (\gamma \wedge \mathcal{H}^+)] = \gamma \wedge [\alpha \vee [(\alpha \vee \gamma) \wedge \mathcal{H}^+]]$. Now $\gamma \vee \alpha \subseteq \gamma \vee [\alpha \vee (\gamma \wedge \mathcal{H}^+)] \subseteq \gamma \vee [\alpha \vee [(\alpha \vee \gamma) \wedge \mathcal{H}^+]] \subseteq \gamma \vee [\alpha \vee (\alpha \vee \gamma)] = \gamma \vee (\alpha \vee \gamma) = \gamma \vee \alpha$, implies the first equality. For the other equality, we have $\gamma \wedge [\alpha \vee (\gamma \wedge \mathcal{H}^+)] \subseteq \gamma \wedge [\alpha \vee [(\alpha \vee \gamma) \wedge \mathcal{H}^+)]] \subseteq \gamma \wedge [\alpha \vee \mathcal{H}^+]$. Hence it suffices to show that $\gamma \wedge [\alpha \vee \mathcal{H}^+] \subseteq \gamma \wedge [\alpha \vee (\gamma \wedge \mathcal{H}^+)]$. For this, it is sufficient to show that $\gamma \wedge (\alpha \vee \mathcal{H}^+) \subseteq \alpha \vee (\gamma \wedge \mathcal{H}^+)$. Suppose that $(x, y) \in \gamma \wedge (\alpha \vee \mathcal{H}^+)$. Then $(x^o, y^o) \in \alpha \vee \mathcal{H}^+$. Since $(\alpha, \alpha \vee \mathcal{H}^+) \in \tau$, we have $(x^o, y^o) \in \alpha$. Thus $x^o = (x^o + x^o)\ \alpha\ (x^o + y^o)\ \mathcal{H}^+\ (x+y)\ \mathcal{H}^+\ (x+y)^o$, so that $(x^o, (x+y)^o) \in \alpha \vee \mathcal{H}^+$. Then $(\alpha, \alpha \vee \mathcal{H}^+) \in \tau$ implies that $(x^o, (x+y)^o) \in \alpha$, and so $(y^o, (x+y)^o) \in \alpha$. Since the $\mathcal{D}^+$-class $D_{(x+y)}$ is completely simple, we have $((x+y)^o + x + (x+y)^o)\ \mathcal{H}^+\ ((x+y)^o + y + (x+y)^o)$. Thus $x = (x^o + x + x^o)\ \alpha\ ((x+y)^o + x + (x+y)^o)\ \gamma \wedge \mathcal{H}^+ ((x+y)^o + y + (x+y)^o)\ \alpha\ (y^o + y + y^o) = y$, and so $(x, y) \in \alpha \vee (\gamma \wedge \mathcal{H}^+)$. This completes the proof. $\square$

Theorem 4.8. *Let $\mathcal{V}$ be a variety of idempotent semirings and S be a τ-modular completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$. Then ϕ is $\vee$-preserving.*

Proof. Let $\alpha, \gamma \in C(S)$. Then $(\alpha\wedge\mathcal{H}^+)\vee(\gamma\wedge\mathcal{H}^+) \subseteq (\alpha\vee\gamma)\wedge\mathcal{H}^+$. Since both $(\alpha\wedge\mathcal{H}^+)\vee(\gamma\wedge\mathcal{H}^+)$, $(\alpha\vee\gamma)\wedge\mathcal{H}^+$ are contained in $\mathcal{H}^+$, $(\alpha\wedge\mathcal{H}^+)\vee(\gamma\wedge\mathcal{H}^+)\ \tau\ (\alpha\vee\gamma)\wedge\mathcal{H}^+$. Therefore, by τ-modularity, it suffices to show that $\alpha \vee [(\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+)] = \alpha \vee [(\alpha \vee \gamma) \wedge \mathcal{H}^+]$ and $\alpha \wedge [(\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+)] = \alpha \wedge [(\alpha \vee \gamma) \wedge \mathcal{H}^+]$. First equality holds by Lemma 4.7. Since $\alpha \wedge \mathcal{H}^+ \subseteq (\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+)$, we have $\alpha \wedge \mathcal{H}^+ = \alpha \wedge (\alpha \wedge \mathcal{H}^+) \subseteq \alpha \wedge [(\alpha \wedge \mathcal{H}^+) \vee (\gamma \wedge \mathcal{H}^+)) \subseteq \alpha \wedge [(\alpha \vee \gamma) \wedge \mathcal{H}^+] = [\alpha \wedge (\alpha \vee \gamma)] \wedge \mathcal{H}^+] = \alpha \wedge \mathcal{H}^+$. Therefore $(\rho\vee\gamma)\wedge\mathcal{H}^+ = (\rho\wedge\mathcal{H}^+)\vee(\gamma\wedge\mathcal{H}^+)$. Also we have $(\rho \vee \gamma) \vee \mathcal{H}^+ = (\rho \vee \mathcal{H}^+) \vee (\gamma \vee \mathcal{H}^+)$. This completes the proof. □

Theorem 4.9. *Let $\mathcal{V}$ be a variety of idempotent semirings and S be a completely regular semiring. If $\phi : C(S) \longrightarrow \mathcal{V}(S) \times M(S)$ is an embedding, then S is τ-modular.*

Proof. Let σ, ρ be two congruences on S such that $\sigma \subseteq \rho$, $\sigma\tau\rho$ and ξ is a congruence such that $\sigma \vee \xi = \rho \vee \xi$ and $\sigma \wedge \xi = \rho \wedge \xi$. Clearly $\sigma \wedge \mathcal{H}^+ \subseteq \rho \wedge \mathcal{H}^+$, and since ϕ is an embedding, we have $(\sigma \wedge \mathcal{H}^+) \vee (\xi \wedge \mathcal{H}^+) = (\sigma \vee \xi) \wedge \mathcal{H}^+ = (\rho \vee \xi) \wedge \mathcal{H}^+ = (\rho \wedge \mathcal{H}^+) \vee (\xi \wedge \mathcal{H}^+)$. Also, $(\sigma \wedge \mathcal{H}^+) \wedge (\xi \wedge \mathcal{H}^+) = (\sigma \wedge \xi) \wedge \mathcal{H}^+ = (\rho \wedge \xi) \wedge \mathcal{H}^+ = (\rho\wedge\mathcal{H}^+)\wedge(\xi\wedge\mathcal{H}^+)$. Since, by Lemma 3.7, the τ-class of $\mathcal{H}^+$ is a modular sublattice of $C(S)$, we have $\sigma \wedge \mathcal{H}^+ = \rho \wedge \mathcal{H}^+$. Also $\sigma\ \tau\ \rho$ implies that $\sigma \vee \mathcal{H}^+ = \rho \vee \mathcal{H}^+$, by Theorem 3.6. Since ϕ is one-to-one, $\sigma = \rho$. Thus S is τ-modular. □

Now combining Theorem 4.1, 4.3, 4.8 and 4.9, we get the following result.

Theorem 4.10. *Let $\mathcal{V}$ be a variety of idempotent semirings and S be a completely regular semiring such that $S/\mathcal{H}^+ \in \mathcal{V}$. Then the function $\phi : C(S) \to \mathcal{V}(S) \times M(S)$ defined by $\phi(\alpha) = (\alpha \vee \mathcal{H}^+, \alpha \wedge \mathcal{H}^+)$ is an embedding if and only if S is τ-modular.*

References

[1] **H.J. Bandelt, M. Petrich**, *Subdirect products of rings and distributive lattices*, Proc. Edin. Math. Soc. **25** (1982), 155 – 171.

[2] **A.K. Bhuniya, R. Debnath**, *On band orthorings*, arXiv:1706.02670v1.

[3] **A.H. Clifford**, *Semigroups admitting relative inverses*, Annals Math. **42** (1941), 1037 – 1049.

[4] **S. Ghosh**, *A characterisation of semirings which are subdirect products of a distributive lattice and a ring*, Semigroup Forum **59** (1999), 106 – 120.

[5] **J.S. Golan**, *The Theory of Semirings with Applications in Mathematics and Theoretical Computer Science*, Pitman Monographs and Surveys in Pure and Applied Mathematics **54**, Longman Scientific (1992).

[6] **J.M. Howie**, *The maximum idempotent separating congruence on an inverse semigroup*, Proc. Edinburgh Math. Soc. **14** (1964), 71 – 79.

[7] **J M. Howie, G. Lallement**, *Certain fundamental congruences on a regular semigroup*, Proc. Glasgow Math. Assoc. **7** (1966), 145 – 159.

[8] **K.M. Kapp, H. Schneider**, *Completely 0-simple Semigroups*, New York, 1969.

[9] **T. Kepka**, *Structure of triabelian quasigroups*, Comment. Math. Univ. Carolin. **17** (1976), no. 2, 229 – 240.

[10] **G. Lallement**, *Congruences et équivalences de Green sur un demi-groupe régulier*, C. R. Acad. Sci. Paris **262** (1966), 613 – 616.

[11] **W.D. Munn**, *A certain sublattice of the lattice of congruences on a regular semigroup*, Proc. Cambridge Philosophic Society **60** (1964), 385 – 391.

[12] **F. Pastijn, Y.Q. Guo**, *Semirings which are unions of rings*, Science in China Ser. A: Math. **45** (2002), 172 – 195.

[13] **M. Petrich, N. R. Reilly**, *Completely Regular Semigroup*, Wiley, New York, (1999).

[14] **N.R. Reilly, H.E. Scheiblich**, *Congruences on regular semigroups*, Pacific J. Math. **23** (1967), 349 – 360.

[15] **H.E. Scheiblich**, *Certain congruence and quotient lattices related to completely 0-simple and primitive regular semigroups*, Glasow Math. J. **10** (1969), 21 – 24.

[16] **M.K. Sen, S.K. Maity, K.P. Shum**, *On Completely Regular Semirings*, Bull. Cal. Math. Soc. **98** (2006), no. 4, 319 – 328.

[17] **C. Spitznagel**, *The lattice of congruences on a band of groups*, Glasow Math. J. **14** (1973), 187 – 197.

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On prime and primary avoidance theorem for subsemimodules

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Abstract. We study some important results of prime and primary subsemimodules. We also prove the primary avoidance theorem for subsemimodules.

1. Introduction

Prime and primary submodules play crucial role in ring and module theory. These concepts were widely studied in [1], [2], [3], [6], [8], [9]. C. P. Lu in [8], proved the prime avoidance theorem for submodules. El-Atrash and Ashour in [7], proved primary avoidance theorem for submodules. Several authors have studied and explored these concepts in semimodule theory. In this paper, we study the concepts of prime and primary subsemimodules and prove several results analogous to module theory.

By a *semiring*, we mean an algebraic structure $(S, +, 0_S)$ such that $(S, \cdot)$ is a semigroup and $(S, +, 0_S)$ is a commutative monoid in which the multiplication is distributive with respect to the addition both from the left and from the right and 0_S is the additive identity of S and also $0_S x = x 0_S = 0_S$ for all $x \in S$. A nonempty subset I of a semiring S is called an *ideal* of S if $a, b \in I$ and $s \in S$, then $a + b \in I$ and $sa, as \in I$. An ideal I of a semiring S is called *subtractive* if $a, a + b \in I, b \in S$, then $b \in I$. An ideal I of a semiring S is called *prime* if $ab \in I$, then either $a \in I$ or $b \in I$. If I is an ideal of S, then the *radical* of I is defined as $Rad(I) = \sqrt{I} = \{a \in S : a^2 \in I\}$. An ideal I of a semiring S is called a *primary ideal* of S if $ab \in I$, then either $a \in I$ or $b \in \sqrt{I}$. Let S be a semiring. A *left S-semimodule* M is a commutative monoid $(M, +)$ which has a zero element 0_M, together with an operation $S \times M \to M$; denoted by $(a, x) \to ax$ such that for all $a, b \in S$ and $x, y \in M$,

1. $a(x + y) = ax + ay$,
2. $(a + b)x = ax + bx$,
3. $(ab)x = a(bx)$,
4. $0_S x = 0_M = a 0_M$.

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A proper subsemimodule N of an S-semimodule M is called *subtractive* if $a, a+b \in N$, $b \in M$ then $b \in N$. The *associated ideal* of a subsemimodule N of M is defined as $(N : M) = \{a \in S : aM \subseteq N\}$. A proper subsemimodule N of an S-semimodule M is said to be *strong subsemimodule* if for each $x \in N$ there exists $y \in N$ such that $x + y = 0$.

We shortly summarize the content of the paper: In the first section, by applying the prime avoidance theorem for subsemimodules [10], we prove the extended version of prime avoidance theorem for subsemimodules. In the second section, we prove some results on primary subsemimodules and by using the technique of efficient covering of subsemimodules, we prove the primary avoidance theorem for subsemimodules.

Throughout this paper, S will always denote a commutative semiring with identity $1 \neq 0$ and S-semimodules means semimodules.

2. Prime subsemimodules

A proper subsemimodule N of an S-semimodule M is called *prime* if whenever $rm \in N$ then $rM \subseteq N$ or $m \in N$.

We start with the following obvious results

Theorem 2.1. *If N is a maximal subsemimodule of an S-semimodule M, then N is a prime subsemimodule of M.* □

Corollary 2.2. *Let M be an S-semimodule and N be a proper subsemimodule of M. If N is a subtractive subsemimodule of M and $m \in M \setminus N$. Then the following statements holds:*

1. *$(N : M)$ is a subtractive ideal of S.*
2. *$(0 : M)$ and $(N : m)$ are subtractive ideals of S.* □

Corollary 2.3. *Let N be a prime subsemimodule of an S-semimodule M. Then for each $m \in M \setminus N$, $(N : M)$ and $(N : m)$ are prime ideals of S.* □

Theorem 2.4. *Let $N_1, N_2, \ldots, N_n$ be subsemimodules of an S-semimodule M and let N be a prime subsemimodule of M. If $\bigcap_{i=1}^{n} N_i \subseteq N$, then there exists an $1 \leqslant i \leqslant n$ such that $N_i \subseteq N$ or $(N_i : M) \subseteq (N : m)$ where $m \in M \setminus N$.*

Proof. Suppose $N_i \nsubseteq N$ and $(N_i : M) \nsubseteq (N : m)$ where $m \in M \setminus N$ and for all $1 \leqslant i \leqslant n$. For particular, $i = k$, we have $N_k \nsubseteq N$, then there exists an $m_k \in M$ such that $m_k \in N_k$ but $m_k \notin N$. Also, there exist $a_i \in (N_i : M)$ such that $a_i \notin (N : m_k)$ for all $i \neq k$. This gives $a_i m_k \in N_i$ and $a_i m_k \notin N$. Therefore, $a_i m_k \in N_i \cap N_k$ for all $i \neq k$. So $a_1 a_2 \ldots a_{k-1} a_{k+1} \ldots a_n m_k \in N_1 \cap \ldots \cap N_n \subseteq N$. This implies, $a_1 a_2 \ldots a_{k-1} a_{k+1} \ldots a_n \in (N : m_k)$. By Corollary 2.3, $(N : m_k)$ is a prime ideal. Therefore, we have $a_i \in (N : m_k)$ for $i \neq k$, a contradiction. Hence there exists an i such that $N_i \subseteq N$ or $(N_i : M) \subseteq (N : m)$, where $m \in M \setminus N$. □

Theorem 2.5. *Let M be an S-semimodule, N be an arbitrary subsemimodule of M and $N_1, N_2, \ldots, N_n$ be subtractive prime subsemimodules of M. Suppose $(N_j : M) \nsubseteq (N_i : m)$ for all $m \in M \setminus N_i$ with $i \neq j$. If $N \nsubseteq N_i$ for all i, then there exists an element $x \in N$ such that $x \notin \cup N_i$; hence, $N \nsubseteq \cup N_i$.*

Proof. Since $N \nsubseteq N_i$, then there exists $m_i \in N$ such that $m_i \notin N_i$ for all i. By Corollary 2.3, $(N_i : m_i)$ is a prime ideal of S. By the given hypothesis, there exists $r_j \in (N_j : M)$ and $r_j \notin (N_i : m_i)$ for $i \neq j$. Let $s_i = r_1 r_2 \ldots r_{i-1} r_{i+1} \ldots r_n = \prod_{j \neq i} r_j$. Let $x_i = m_i s_i$ for all i. Then $x_i = m_i s_i \in N_j$ for all $j \neq i$. But $x_i \notin N_i$ because, if $x_i \in N_i$ then $m_i s_i \in N_i$, so $s_i \in (N_i : m_i)$, a contradiction. Let $x = x_1 + x_2 + \ldots + x_n$. Then $x = x_i + \sum_{j \neq i} x_j$. Since $\sum_{j \neq i} x_j \in N_i$, therefore $x \notin N_i$ otherwise we would have $x_i \in N_i$ which is a contradiction, so $x \notin \cup N_i$. Also, $m_i \in N$ for all i, therefore $x \in N$ and hence $N \nsubseteq \cup N_i$. □

Let $N_1, N_2, \ldots, N_n$ be subsemimodules of M. Define a covering $N \subseteq N_1 \cup N_2 \cup \ldots \cup N_n$ is efficient if no N_i is superfluous for $1 \leqslant i \leqslant n$. In otherwords, we say $N = N_1 \cup N_2 \cup \ldots \cup N_n$ is an efficient union if none of the N_i's may be excluded. Any cover or union consisting of subsemimodules of M be reduced to an efficient one, called an efficient reduction, by deleting any unnecessary terms.

Theorem 2.6. (cf. [5]) *Let $N = N_1 \cup N_2 \cup \ldots \cup N_n$ be an efficient union of subtractive subsemimodules of an S-semimodule M. Then $\bigcap\limits_{i=1}^{n} N_i = \bigcap\limits_{\substack{i=1 \\ i \neq j}}^{n} N_i$ for any $j \in \{1, 2, \ldots, n\}$.* □

Proposition 2.7. (cf. [10]) *Let $N \subseteq N_1 \cup N_2 \ldots \cup N_n$ be an efficient covering consisting of subtractive subsemimodules of an S-semimodule M, where $n \geqslant 2$. If $(N_j : M) \nsubseteq (N_k : M)$ for every $j \neq k$, then no N_k for $k \in \{1, 2, \ldots, n\}$ is a prime subsemimodule of M.* □

Theorem 2.8. (The prime avoidance theorem, cf. [10])
Let M be an S-semimodule, $N_1, N_2, ..., N_n$ a finite number of subtractive subsemimodules of M and N be a subsemimodule of M such that $N \subseteq N_1 \cup N_2 \ldots \cup N_n$, $(n \geqslant 2)$. Assume that at most two of the N_i's are not prime and that $(N_j : M) \nsubseteq (N_k : M)$ for every $j \neq k$. Then, $N \subseteq N_k$ for some k. □

Now, we come to our main theorem which is a more general form of the above theorem.

Theorem 2.9. (Extended prime avoidance theorem for subsemimodules)
Let M be an S-semimodules and $N_1, N_2, ..., N_r$ be subtractive prime subsemimodues of M such that $(N_i : M) \nsubseteq (N_j : M)$ for $i \neq j$, $r \geqslant 1$. Let $m \in M$ be such that $mS + N \nsubseteq \bigcup\limits_{i=1}^{r} N_i$. Then there exists $n \in N$ such that $m + n \notin \bigcup\limits_{i=1}^{r} N_i$.

Proof. Suppose that m lies in each of $N_1, \ldots, N_k$ but none of $N_{k+1}, N_{k+2}, \ldots, N_r$. If $k = 0$, then $m = m + 0 \notin \bigcup_{i=1}^{r} N_i$ and so there is nothing to prove. Assume that it is true for $k \geqslant 1$. Now, $N \nsubseteq \bigcup_{i=1}^{k} N_i$, for otherwise by prime avoidance theorem for semimodules, we would have a contradiction. Therefore, there exists $p \in N \setminus (N_1 \cup N_2 \cup \ldots \cup N_k)$. Also, we have $N_{k+1} \cap \ldots \cap N_r \nsubseteq N_1 \cup \ldots \cup N_k$. Otherwise, since N_j is a prime subsemimodule, by prime avoidance theorem, we have $N_{k+1} \cap \ldots \cap N_r \subseteq N_j$ for some $1 \leqslant j \leqslant k$. This implies $(N_{k+1} \cap \ldots \cap N_r : M) \subseteq (N_j : M)$ for some $1 \leqslant j \leqslant k$, that is, $(N_{k+1} : M) \cap \ldots \cap (N_r : M) \subseteq (N_j : M)$ for some $1 \leqslant j \leqslant k$. Therefore, $(N_i : M) \subseteq (N_j : M)$ where $k + 1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant k$, which contradicts to the hypothesis that $(N_i : M) \nsubseteq (N_j : M)$ for $i \neq j$. Thus, there exists $b \in (N_{k+1} : M) \cap \ldots \cap (N_r : M) \setminus (N_1 : M) \cup \ldots \cup (N_k : M)$. Let $n = bp \in N$. Also, $n \in \bigcap_{j=k+1}^{r} N_j$ and $n = bp \notin N_1 \cup \ldots \cup N_k$ (if $n = bp \in N_1 \cup \ldots \cup N_k$, then we have $n \in N_i$ for some $i \in \{1, 2, \ldots k\}$, since N_i is prime, either $b \in (N_i : M)$ or $p \in N_i$ for $1 \leqslant i \leqslant k$), a contradiction. Thus, $n \in (N_{k+1} \cap \ldots \cap N_r) \setminus (N_1 \cup \ldots \cup N_k)$. Consequently, $m + n \notin \bigcup_{i=1}^{r} N_i$. □

Next, we prove that if N is a finitely generated subsemimodule of an S-semimodule M satisfying the assumption of prime avoidance theorem for subsemimodules, then there is a linear combination of the generators of N also avoids $\bigcup_{i=1}^{n} N_i$.

Theorem 2.10. *Let M be an S-semimodule and $N = \langle m_1, m_2, ..., m_r \rangle$ be a finitely generated subsemimodule of M. Let $N_1, N_2, \ldots, N_n$ be subtractive prime subsemimodules of M such that $N \nsubseteq N_i$ for each i, $1 \leqslant i \leqslant n$ and $(N_i : M) \nsubseteq (N_n : M)$ for each $i \neq j$. Then there exist $b_2, \ldots, b_r \in S$ such that $x = m_1 + b_2 m_2 + ... + b_r m_r \notin \bigcup_{i=1}^{n} N_i$.*

Proof. We prove assertion by induction on n. Without loss of generality, we suppose that $N_i \nsubseteq N_j$ for all $i \neq j$. If $n = 1$, then clearly $x = m_1 + b_2 m_2 + ... + b_r m_r \notin N_1$. So, we have done. Assume that the result is true for $(n-1)$ subtractive prime subsemimodules of M. Then there exist $c_2, c_3, \ldots, c_r \in S$ such that $y = m_1 + c_2 m_2 + ... + c_r m_r \notin \bigcup_{i=1}^{n-1} N_i$. If $y \notin N_n$, then there is nothing to prove. So assume that $y \in N_n$. If $m_2, ..., m_r \in N_n$, then from the expression for y, we have $m_1 \in N_n$ (as N_n is a subtractive), which is a contradiction to the fact that $N \nsubseteq N_n$. So for some i, $m_i \notin N_n$. Without loss of generality, suppose $i = 2$. By given hypothesis $(N_i : M) \nsubseteq (N_n : M)$ for $i \neq n$. Therefore, there exists $r_i \in (N_i : M)$ such that $r_i \notin (N_n : M)$ where $i \neq n$. Let $r = r_1 r_2 r_3 \ldots r_{n_1}$.

Then $c = m_1 + (c_2 + r)m_2 + \ldots + c_r m_r \notin \bigcup_{i=1}^{n} N_i$, which is a contradiction to our assumption. $\square$

3. The primary avoidance theorem

In this section, we study some properties of primary subsemimodules and prove primary avoidance theorem for subsemimodules.

Definition 3.1. A proper subsemimodule N of an S-semimodule M is called *primary* if whenever $am \in N$ for some $a \in S$ and $m \in M$, then $m \in N$ or $a \in \sqrt{(N : M)}$, where $\sqrt{(N : M)} = \{a \in S : a^t M \subseteq N, \text{ for some } t \in Z^+\}$.

Theorem 3.2. *If N is a primary subsemimodule of M and $m \in M \setminus N$, then $\sqrt{(N : m)} = \{r \in S : r^n m \in N, \text{ for some } n \in \mathbb{Z}^+\}$ is a prime ideal of S.*

Proof. Let $rs \in \sqrt{(N : m)}$ for some $r, s \in S$. Then $(rs)^n \in (N : m)$ for some positive integer n. Therefore, $r^n(s^n m) \in N$. Since N is primary, we have either $r^n \in (N : M)$ or $s^n m \in N$. Thus, $r \in \sqrt{(N : M)}$ or $s \in \sqrt{(N : m)}$. Since $\sqrt{(N : M)} \subseteq \sqrt{(N : m)}$, we get $r \in \sqrt{(N : m)}$ or $s \in \sqrt{(N : m)}$. Hence $\sqrt{(N : m)}$ is a prime ideal of S. $\square$

Theorem 3.3. *Let N be a primary subsemimodule of an S-semimodule M. Then $(N : M)$ is a primary ideal of S, and hence $\sqrt{(N : M)}$ is a prime ideal of S.*

Proof. The proof is easy and hence omitted. $\square$

Definition 3.4. Let N be a primary subsemimodule of an S-semimodule M. Then N is called a *P-primary subsemimodule* of M, when $P = \sqrt{(N : M)}$ is a prime ideal of S.

Proposition 3.5. *Let M be an S-semimodule and N be a strong subsemimodule of M and suppose $a \in S$. If P is a prime ideal of S, $a \notin P$ such that $Q = (N : a)$ is a P-primary in M, then $N = Q \cap (N + aM)$. Furthermore, N is a P-primary in $N + aM$, where $(N : a) = \{m \in M : am \in N\}$.*

Proof. Clearly, $N \subseteq Q \cap (N + aM)$. Let $x \in (N + aM) \cap Q$. Then $x = n + am$ where $n \in N$ and $m \in M$. Since N is strong, there exists $n_1 \in N$ such that $n + n_1 = 0$. Now, $x = n + am$ implies $x + n_1 = (n + n_1) + am = 0 + am$. Thus, we have $x + n_1 = am \in Q$, as x and n_1 both are in Q. Since Q is a P-primary and $a \notin P$, we have $m \in Q$, which implies $am \in N$. Therefore, $x = n + am \in N$. Hence, $(N + aM) \cap Q \subseteq N$.

Next, we show that N is a P-primary in $(N + aM)$. Let $rx \in N$ for some $r \in S$ and $x \in (N + aM) \setminus N$. Then $x = n + am$ for some $n \in N$ and $m \in M$. Since N is a strong subsemimodule of M, therefore there exist $n_1 \in N$ such that $n + n_1 = 0$. Now, adding n_1 on both sides, we have $x + n_1 = n + n_1 + am$. This implies, $rx + rn_1 = ram$ where $r \in S$. Since $ram \in N$ gives $rm \in (N : a) = Q$ and Q is P-primary. If $m \in Q$, then $x = n + am \in N$, which is a contradiction. Hence, $m \notin Q$. Therefore, $r \in P$. Therefore, N is a P- primary in $(N + aM)$. $\square$

The following theorem can be proved easily.

Theorem 3.6. *Let M and M' be S-semimodules, $f : M \longrightarrow M'$ be an epimorphism and N is a proper subsemimodue of M'. Then N is a primary subsemimodule of M' if and only if $f^{-1}(N)$ is a primary subsemimodule of M.* $\square$

Theorem 3.7. *Let M and M' be S-semimodules, $f : M \longrightarrow M'$ be an epimorphism such that $f(0) = 0$ and N be a subtractive strong subsemimodule of M. If N is a primary subsemimodule of M with $kerf \subseteq N$, then $f(N)$ is a primary subsemimodule of M'*

Proof. Let N be a primary subsemimodule of M and $ax \in f(N)$ for some $a \in S$ and $x \in M'$. Since $ax \in f(N)$, there exists an element $x' \in N$ such that $ax = f(x')$. Since f is an epimorphism and $x \in M'$, then there exists $y \in M$ such that $f(y) = x$. As $x' \in N$ and N is a strong subsemimodule of M, therefore there exists $x'' \in N$ such that $x' + x'' = 0$, which gives $f(x' + x'') = 0$. Therefore, $ax + f(x'') = 0$ or $f(ay) + f(x'') = 0$ implies $ay + x'' \in kerf \subseteq N$. Thus, we have $ay \in N$, since N is a subtractive subsemimodule of M. Since N is a primary, we conclude that $a \in \sqrt{(N : M)}$ or $y \in N$. Thus, $a \in f(\sqrt{(N : M)}) \subseteq \sqrt{f(N : M)}$ or $f(y) \in f(N)$ and hence $a \in \sqrt{(f(N) : M')}$ or $x \in f(N)$. Thus, $f(N)$ is a primary subsemimodule of M'. $\square$

Theorem 3.8. *Let $N_1, N_2, \ldots, N_n$ be subsemimodule of an S-semimodule M and let N be a primary subsemimodule of M. If $\bigcap_{i=1}^{n} N_i \subseteq N$, then there exists an $1 \leqslant i \leqslant n$ such that $N_i \subseteq N$ or $(N_i : M) \subseteq \sqrt{(N : m)}$ where $m \in M \setminus N$.*

Proof. Suppose $N_i \nsubseteq N$ and $(N_i : M) \nsubseteq \sqrt{(N : m)}$ where $m \in M \setminus N$ and for all $1 \leqslant i \leqslant n$. For, $i = k$, we have $N_k \nsubseteq N$, then there exists an $m_k \in M$ such that $m_k \in N_k$ but $m_k \notin N$. Also, there exist $a_i \in (N_i : M)$ such that $a_i \notin \sqrt{(N : m_k)}$ for all $i \neq k$. This gives $a_i m_k \in N_i$ and for every positive integer p_i, $a_i^{p_i} m_k \notin N$. Therefore, $a_i^{p_i} m_k \in N_i \cap N_k$ for all $i \neq k$. So $(a_1^{p_1} a_2^{p_2} \ldots a_{k-1}^{p_{k-1}} a_{k+1}^{p_{k+1}} \ldots a_n^{p_n}) m_k \in N_1 \cap \ldots N_n \subseteq N$. Let $l = max\{p_1, p_2, \ldots, p_{k-1}, p_{k+1}, \ldots, p_n\}$. Therefore, $(a_1 a_2 \ldots a_{k-1} a_{k+1} \ldots a_n)^l m_k \in N$. This implies, $(a_1 a_2 \ldots a_{k-1} a_{k+1} \ldots a_n)^l \in (N : m_k)$ and hence $a_1 a_2 \ldots a_{k-1} a_{k+1} \ldots a_n \in \sqrt{(N : m_k)}$. By Theorem 3.2, $\sqrt{(N : m_k)}$ is a prime ideal. Therefore, we have $a_i \in \sqrt{(N : m_k)}$ for $i \neq k$, a contradiction. Hence there exists an i such that $N_i \subseteq N$ or $(N_i : M) \subseteq \sqrt{(N : m)}$ where $m \in M \setminus N$. $\square$

Theorem 3.9. *Let N be a P-primary subsemimodule of M. Then $(N : r)$ is a P-primary subsemimodule of M containing N for all $r \in \sqrt{(N : M)} \setminus (N : M)$.*

Proof. Let $r \in \sqrt{(N:M)} \setminus (N:M)$. Clearly, $N \subseteq (N:r)$. Let $s \in S$ and $m \in M$ be such that $sm \in (N:r)$. Therefore, $srm \in N$. Since N is primary, we have either $s \in \sqrt{(N:M)}$ or $rm \in N$, that is $s^n M \subseteq N$ or $m \in (N:r)$ for some positive integer n. Hence $s^n \in ((N:r):M)$ or $m \in (N:r)$ for some positive integer n. Thus, $(N:r)$ is a primary ideal of M. Next, we show that $\sqrt{(N:M)} = \sqrt{(N:r):M}$. Since, $N \subseteq (N:r)$, we have $(N:M) \subseteq ((N:r):M)$ and therefore, $\sqrt{(N:M)} \subseteq \sqrt{((N:r):M)}$. Let $s \in \sqrt{((N:r):M)}$. Therefore, $s^n \in ((N:r):M)$, for some positive integer n. This gives, $rs^n \subseteq (N:M)$. Since N is a primary subsemimodule of M, $(N:M)$ is a primary ideal of S. Therefore, $rs^n \subseteq (N:M)$ implies $s \in \sqrt{(N:M)}$, since $r \notin (N:M)$. Thus, $\sqrt{(N:r):M} \subseteq \sqrt{(N:M)}$. Hence, $\sqrt{(N:M)} = \sqrt{(N:r):M}$. □

Theorem 3.10. *Let N be a subsemimodule of an S-semimodule M such that $N \subseteq N_1 \cup N_2$ for some subtractive subsemimodules N_1, N_2 of M. Then either $N \subseteq N_1$ or $N \subseteq N_2$.*

Proof. The proof is straightforward. □

Now, by using Theorem 2.6, we prove the following proposition.

Proposition 3.11. *Let $N \subseteq N_1 \cup N_2 \cup \ldots \cup N_n$ be an efficient union of subtractive subsemimodules of an S-semimodule M, where $n > 1$. If $\sqrt{(N_j:M)} \nsubseteq \sqrt{(N_k:M)}$ for every $j \neq k$, then no N_k for $k \in \{1, 2, \ldots, n\}$ is a primary subsemimodule of M.*

Proof. Suppose that N_k is a primary subsemimodule of M for some $1 \leqslant k \leqslant n$. Since $N \subseteq N_1 \cup N_2 \cup \ldots \cup N_n$ is an efficient covering, $N = (N \cap N_1) \cup (N \cap N_2) \cup \ldots \cup (N \cap N_n)$ is an efficient union, otherwise for some $i \neq j$, $N \cap N_i \subseteq N \cap N_j$ and this imply $N = (N \cap N_1) \cup \ldots \cup (N \cap N_{i-1}) \cup (N \cap N_{i+1}) \cup \ldots (N \cap N_n)$ and thus we get $N \subseteq N_1 \cup \ldots \cup N_{i-1} \cup N_{i+1} \cup \ldots \cup N_n$, a contradiction. Hence for every $k \in \{1, 2, ..., n\}$ there exists an element $\ell_k \in N \setminus N_k$. Also, by Theorem 2.6, we have $\bigcap_{j \neq k} (N \cap N_j) \subseteq N \cap N_k$. Since N_k is a primary subsemimodule of M, by Theorem 3.2, we have $\sqrt{(N_k:M)}$ is a prime ideal of S. By hypothesis, if $j \neq k$, $\sqrt{(N_j:M)} \nsubseteq \sqrt{(N_k:M)}$ so there exists an $s_j \in \sqrt{(N_j:M)} \setminus \sqrt{(N_k:M)}$. Now, $s = \prod_{j \neq k} s_j \in \sqrt{(N_j:M)}$ but $s = \prod_{j \neq k} s_j \notin \sqrt{(N_k:M)}$. Since $s = \prod_{j \neq k} s_j \in \sqrt{(N_1:M)}\sqrt{(N_2:M)}...\sqrt{(N_{k-1}:M)}\sqrt{(N_{k+1}:M)}...\sqrt{(N_n:M)}$ but $s = \prod_{j \neq k} s_j \notin \sqrt{(N_k:M)}$, where $s_j \in \sqrt{(N_j:M)}$, where $1 \leqslant j \leqslant n$. Therefore, for some positive integers $m_1, m_2, ...m_n$, we have $s_1^{m_1} \in (N_1:M), s_2^{m_2} \in (N_2:M), ..., s_n^{m_n} \in (N_n:M)$. Let $l = max\{m_1, m_2, ..., m_n\}$. Then for $j \neq k$, $s^l \in (N_j:M)$ but $s^l \notin (N_k:M)$. Therefore, $s^l l_k \in N \cap N_j$ for every $j \neq k$ but $s^l l_k \notin (N \cap N_k)$ because if $s^l l_k \in (N \cap N_k)$, then $sl_k \in N_k$. This gives, $l_k \in N_k$ or $s \in \sqrt{(N_k:M)}$, since N_k is primary. Therefore, $s^l l_k \notin (N \cap N_k)$, which is a contradiction to the fact that $\bigcap_{j \neq k} (N \cap N_j) \subseteq N \cap N_k$. Therefore, no N_k is primary subsemimodule of M. $\square$

Now, we come to our main theorem of this paper.

Theorem 3.12. (The Primary Avoidance Theorem)
Let $N_1, N_2, ..., N_n$ be subtractive subsemimodules of an S-semimodule M and let N be a subsemimodule of M such that $N \subseteq N_1 \cup N_2 \cup \ldots \cup N_n$. Suppose that at most two of N_k's are not primary subsemimodule of M and $\sqrt{(N_j : M)} \nsubseteq \sqrt{(N_k : M)}$ for every $j \neq k$. Then $N \subseteq N_k$ for some k.

Proof. Assume that the covering is efficient. Then $n \neq 2$. Also by Proposition 3.12, $n < 2$ (as $\sqrt{(N_j : M)} \nsubseteq \sqrt{(N_k : M)}$ for every $j \neq k$). Therefore, $n = 1$. Hence $N \subseteq N_k$ for some k. $\square$

Theorem 3.13. (Extended Version of Primary Avoidance Theorem)
Let M be an S-semimodules and $N_1, N_2, ..., N_r$ subtractive primary subsemimodues of M such that $\sqrt{(N_i : M)} \nsubseteq \sqrt{(N_j : M)}$ for $i \neq j$, $r \geqslant 1$. Let $m \in M$ be such that $mS + N \nsubseteq \bigcup_{i=1}^{r} N_i$. Then there exists $n \in N$ such that $m + n \notin \bigcup_{i=1}^{r} N_i$.

Proof. Suppose that m lies in each of $N_1, N_2, \ldots, N_k$ but in none of $N_{k+1}, N_{k+2}, \ldots, N_r$. If $k = 0$, we have $m = m+0 \notin \bigcup_{i=1}^{r} N_i$ and so there is nothing to prove. Assume that it is true for $k \geqslant 1$. Now, $N \nsubseteq \bigcup_{i=1}^{k} N_i$, for otherwise by primary avoidance theorem for semimodules, we would have a contradiction. Therefore, there exists $p \in N \setminus (N_1 \cup N_2 \cup \ldots \cup N_k)$. Thus, we have $N_{k+1} \cap \ldots \cap N_r \nsubseteq N_1 \cup \ldots \cup N_k$. Otherwise, since $N_j's$ are primary subsemimodules, by primary avoidance theorem, we have $N_{k+1} \cap \ldots \cap N_r \subseteq N_j$ for some $1 \leqslant j \leqslant k$. This implies $(N_{k+1} \cap \ldots \cap N_r : M) \subseteq (N_j : M)$ for some $1 \leqslant j \leqslant k$, gives $\sqrt{(N_{k+1} : M) \cap \ldots \cap (N_r : M)} \subseteq \sqrt{(N_j : M)}$ for some $1 \leqslant j \leqslant k$. This gives, $\sqrt{(N_{k+1} : M)} \cap \ldots \cap \sqrt{(N_r : M)} \subseteq \sqrt{(N_j : M)}$ for some $1 \leqslant j \leqslant k$. Therefore, $\sqrt{(N_i : M)} \subseteq \sqrt{(N_j : M)}$, (since $\sqrt{(N_i : M)}$'s are subtractive prime ideals for all i) where $k + 1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant k$, which contradicts to the hypothesis that $\sqrt{(N_i : M)} \nsubseteq \sqrt{(N_j : M)}$ for $i \neq j$. Thus, there exists $b \in (N_{k+1} : M) \cap \ldots \cap (N_r : M) \setminus (N_1 : M) \cup \ldots \cup (N_k : M)$. Let $n = bp$, then $n \in N$. Also, $n \in \bigcap_{j=k+1}^{r} N_j$ and $n = bp \notin N_1 \cup \ldots \cup N_k$ (since if $n = bp \in N_1 \cup \ldots \cup N_k$, then $n = bp \in N_i$ for some $1 \leqslant i \leqslant k$ and since N_i is primary, either $b \in \sqrt{(N_i : M)}$ or $p \in N_i$ for $1 \leqslant i \leqslant k$). Thus, $n \in (N_{k+1} \cap \ldots \cap N_r) \setminus (N_1 \cup \ldots \cup N_k)$. Also, $m \in N_1, N_2, \ldots N_k$, it follows that $m + n \notin \bigcup_{i=1}^{r} N_i$. $\square$

References

[1] **R.E. Atani and S.E. Atani**, *A note on uniformly primary submodules*, Novi Sad J. Math. **38(2)** (2008), 83 − 89.

[2] **R.E. Atani and S.E. Atani**, *Spectra of semimodules*, Bull. Acad. Sti. Rep. Mold. Matematica **3** (2011), 15 − 28.

[3] **A. Azizi and H. Sharif**, *On prime submodule*, Honam Mathematical J. **21(1)** (1999), 1 − 12.

[4] **M.K. Dubey**, *Prime and weakly prime ideals in semirings*, Quasigroups and Related Systems **20** (2012), 151 − 156.

[5] **M.K. Dubey and P. Sarohe**, *On 2-absorbing semimodules*, Quasigroups and Related Systems **21** (2013), 175 − 184.

[6] **M.K. Dubey and P. Sarohe**, *Generalizations of prime subsemimodules*, Southeast Asian Bull. Math. **39** (2015), 469 − 476.

[7] **M. S. El-Atrash A. E. Ashour,** *On primary compactly packed modules*, J. Islamic University of Gaza **13(2)** (2005), 117 − 128.

[8] **C.P. Lu**, *Union of prime submodules*, Houston J.Math., **23(2)** (1997), 203 − 213.

[9] **P.F. Smith**, *Primary modules over commutative rings*, Glasgow Math. J. **43** (2001), 103 − 111.

[10] **G. Yesilot and K.H. Oral**, *On prime subsemimodules of semimodules*, Internat. J. Algebra **4** (2010), 53 − 60.

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Normal submultigroups and comultisets of a multigroup

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Abstract. We study properties of normal submultigroups. It is shown that if A is a multigroup of a group X and B is a submultigroup of A, the union and intersection of comultisets of B in A are identical and equal to B.

1. Introduction

The notion of multigroup was first mentioned in [3] and defined as algebraic system that satisfied all the axioms of group except that the binary operation is multi-valued. This perspective is neither in conformity with the idea of multisets nor in alignment with other non-classical group studied in [8]. Also, the generalizations of group theory as multigroup in [5, 7, 9] are not within the framework of multiset.

The perspective of multigroups in [10, 11] seem to be better off because the notion of multiset was captured but however, do not define multigroup via count function of multiset. In [6], the concept of multigroups was introduced via count function of multiset and some properties were discussed. Further studies on the concept of multigroups via multisets can be found in [1, 2, 4].

In this paper, we study some properties of normal submultigroups, propose conjugate and normalizer in multigroups, and obtain some results. The homomorphic properties of normal submultigroups are explicated. Finally, we explore the idea of comultisets of a multigroup mentioned in [6] and deduce some results. We show that the union and intersection of comultisets of a submultigroup of a multigroup are identical and equal to the submultigroup.

2. Preliminaries

In this section, we present some existing definitions and results that are useful in the subsequent sections.

Definition 2.1. Let $X = \{x_1, x_2, \ldots, x_n, \ldots\}$ be a set. A *multiset* A over X is a cardinal-valued function, that is, $C_A : X \to N$ such that for $x \in Dom(A)$ implies

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$A(x)$ is a cardinal and $A(x) = C_A(x) > 0$, where $C_A(x)$ denoted the number of times an object x occur in A. Whenever $C_A(x) = 0$, implies $x \notin Dom(A)$. We denote the set of all multisets over X by $MS(X)$.

A multiset $A = [a, a, b, b, c, c, c]$ can be represented as $A = [a, b, c]_{2,2,3}$. Different forms of representing multiset exist other than this.

Definition 2.2. Let A and B be multisets over X. Then A is called a *submultiset* of B written as $A \subseteq B$ if $C_A(x) \leqslant C_B(x)$ for all $x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper submultiset of B and denoted as $A \subset B$. Thus $A = B$ means that $C_A(x) = C_B(x)$ for all $x \in X$. A multiset A with the property $C_A(x) = C_B(y)$ for all $x, y \in X$, is called *regular*. Otherwise it is *irregular*.

Definition 2.3. Let A and B be multisets over X. Then the *intersection* and *union* of A and B, denoted by $A \cap B$ and $A \cup B$ respectively, are defined by the rules that for any object $x \in X$,

(i) $C_{A\cap B}(x) = C_A(x) \wedge C_B(x)$,

(ii) $C_{A\cup B}(x) = C_A(x) \vee C_B(x)$,

where $\wedge$ and $\vee$ denote minimum and maximum respectively.

Definition 2.4. Let X be a group. A multiset G is called a *multigroup* of X if it satisfies the following conditions:

(i) $C_G(xy) \geqslant C_G(x) \wedge C_G(y) \quad \forall x, y \in X$,

(ii) $C_G(x^{-1}) = C_G(x) \quad \forall x \in X$,

where C_G denotes count function of G from X into a natural number $\mathbb{N}$.

For any multigroup A its inverse A^{-1} is defined by

$$C_{A^{-1}}(x) = C_A(x^{-1}) \quad \forall x \in X.$$

The set of all multigroups of X is denoted by $MG(X)$. It is worthy of note that every multigroup is a multiset but the converse is not true.

Definition 2.5. Let $A \in MG(X)$. A submultiset B of A is called a *submultigroup* of A denoted by $B \sqsubseteq A$ if B form a multigroup. A submultigroup B of A is a proper submultigroup denoted by $B \sqsubset A$, if $B \sqsubseteq A$ and $A \neq B$.

Definition 2.6. Let $\{A_i\}_{i\in I}, I = 1, \ldots, n$ be an arbitrary family of multigroups of X. Then

$$C_{\bigcap_{i\in I} A_i}(x) = \bigwedge_{i\in I} C_{A_i}(x) \quad \forall x \in X$$

and

$$C_{\bigcup_{i\in I} A_i}(x) = \bigvee_{i\in I} C_{A_i}(x) \quad \forall x \in X.$$

The family of multigroups $\{A_i\}_{i\in I}$ of X is said to have *inf/sup assuming chain* if either $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n$ or $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n$ respectively.

Definition 2.7. Let $A, B \in MG(X)$. Then the *product* of A and B denoted as $A \circ B$, is defined by

$$C_{A\circ B}(x) = \bigvee\{C_A(y) \wedge C_B(z) \mid x = yz, y, z \in X\}.$$

Proposition 2.8. *Let $A \in MG(X)$. Then*

(*i*) $A_* = \{x \in X \mid C_A(x) > 0\}$,

(*ii*) $A^* = \{x \in X \mid C_A(x) = C_A(e)\}$,

where e is the identity element of X, are subgroups of X. □

Definition 2.9. Let A and B be multisets over groups X and Y and $f : X \longrightarrow Y$ be a homomorphism. Then

(*i*) the *image* of A under f, denoted by $f(A)$, is a multiset of Y defined by

$$C_{f(A)}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} C_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

for each $y \in Y$.

(*ii*) the *inverse image* of B under f, denoted by $f^{-1}(B)$, is a multiset of X defined by $C_{f^{-1}(B)}(x) = C_B(f(x)) \forall x \in X$.

Definition 2.10. Let X and Y be groups and let $A \in MG(X)$ and $B \in MG(Y)$, respectively.

(*i*) A homomorphism f from X to Y is called a *weak homomorphism* from A to B if $f(A) \subseteq B$. If f is a weak homomorphism of A into B, then we say that, A is weakly homomorphic to B denoted by $A \sim B$.

(*ii*) An isomorphism f from X to Y is called a *weak isomorphism* from A to B if $f(A) \subseteq B$. If f is a weak isomorphism of A into B, then we say that, A is weakly isomorphic to B denoted by $A \simeq B$.

(*iii*) A homomorphism f from X to Y is called a *homomorphism* from A to B if $f(A) = B$. If f is a homomorphism of A onto B, then A is homomorphic to B denoted by $A \approx B$.

(*iv*) An isomorphism f from X to Y is called an *isomorphism* from A to B if $f(A) = B$. If f is an isomorphism of A onto B, then A is isomorphic to B denoted by $A \cong B$.

Theorem 2.11. *Let X and Y be groups and $f : X \to Y$ be an isomorphism. If $A \in MG(X)$ and $B \in MG(Y)$, then $f(A) \in MG(Y)$ and $f^{-1}(B) \in MG(X)$.* □

3. Properties of normal submultigroups

Let $A \in MG(X)$ is said to be *abelian* if $C_A(xy) = C_A(yx)$ for all $x, y \in X$. If $A, B \in MG(X)$ and $A \subseteq B$, then A is called a *normal submultigroup* of B if

$$C_A(xyx^{-1}) \geqslant C_A(y) \quad \forall x, y \in X.$$

Example 3.1. Let $X = \{e, a, b, c\}$ be a Klein 4-group such that

$$ab = c,\ ac = b,\ bc = a,\ a^2 = b^2 = c^2 = e.$$

Suppose $A = [e, a, b, c]_{3,2,3,2}$ and $B = [e, a, b, c]_{5,2,4,2}$ are multigroups of X satisfying the axioms in Definition 2.4. Clearly, $A \subseteq B$. Then A is a normal submultigroup of B since

$$C_A(aba^{-1}) = C_A(b) = 3 \geqslant C_A(b), \quad C_A(bab^{-1}) = C_A(a) = 2 \geqslant C_A(a),$$
$$C_A(cbc^{-1}) = C_A(b) = 3 \geqslant C_A(b), \quad C_A(bcb^{-1}) = C_A(c) = 2 \geqslant C_A(c).$$

Definition 3.2. Let $A \in MG(X)$ and $x, y \in X$. Then x and y are called *conjugate elements* in A if

$$C_A(x) = C_A(yxy^{-1}) \quad \forall x, y \in X.$$

Two multigroups A and B of X are *conjugate* to each other if for all $x, y \in X$,

$$C_A(y) = C_B(xyx^{-1}) \text{ and } C_B(x) = C_A(yxy^{-1}), \text{ i.e.,}$$
$$C_A(y) = C_{B^x}(y) \text{ and } C_B(x) = C_{A^y}(x).$$

Remark 3.3. If $A, B \in MG(X)$ and A is a normal submultigroup of B. Then A_* is a normal subgroup of B_* and A^* is a normal subgroup of B^*. Moreover, A is normal if and only if A^{-1} is normal.

Proposition 3.4. *Let $A, B \in MG(X)$. Then the following statements are equivalent.*

(*i*) *A is a normal submultigroup of B,*

(*ii*) $C_A(xyx^{-1}) = C_A(y)\ \ \forall x, y \in X$,

(*iii*) $C_A(xy) = C_A(yx)\ \ \forall x, y \in X$.

Proof. Straightforward. □

Proposition 3.5. *Let $A, B \in MG(X)$ such that $A \subseteq B$ and $C_A(x) = C_A(y)$ for all $x, y \in X$. Then the following assertions are equivalent.*

(*i*) *A is a normal submultigroup of B.*

(*ii*) $C_A(yx) \geqslant C_A(xy) \wedge C_B(y)\ \ \forall x, y \in X$.

Proof. $(i) \Rightarrow (ii)$. Since A is a normal submultigroup of B and $C_A(x) = C_A(y)$, by Proposition 3.4 we have $C_A(yx) = C_A(y(xy)y^{-1}) \geqslant C_A(xy) \wedge C_B(y)$ for all $x, y \in X$.

$(ii) \Rightarrow (i)$. Since $C_A(yx) \geqslant C_A(xy) \wedge C_B(y)$, $C_A(xy) \geqslant C_A(yx) \wedge C_B(y)$, it implies $C_A(xy) = C_A(yx)$. Proposition 3.4 completes the proof. $\square$

Proposition 3.6. *Let X be a group, A a submultigroup of $G \in MG(X)$ and B a submultiset of G. If A and B are conjugate, then B is a submultigroup of G.* $\square$

Proposition 3.7. *Let $A, B, C \in MG(X)$ such that A and B are normal submultigroups of C. If $A \subseteq B \subseteq C$, then $A \cap B$ and $A \cup B$ are normal submultigroups of C.* $\square$

Proposition 3.8. *Let A be a submultigroup of $B \in MG(X)$. Then A is a normal submultigroup of B if and only if $x \in X$ is constant on the conjugacy classes of A.* $\square$

Proof. Suppose that A is a normal submultigroup of B. Then

$$C_A(y^{-1}xy) = C_A(xyy^{-1}) = C_A(x) \quad \forall x, y \in X.$$

This implies that, $x \in X$ is constant on the conjugacy classes of A.

Conversely, let $x \in X$ be constant (that is, fixed) on each conjugacy classes of A. Then $C_A(xy) = C_A(xyxx^{-1}) = C_A(x(yx)x^{-1}) = C_A(yx)$ $\forall x, y \in X$. Hence, A is normal. $\square$

We now give an alternative formulation of the notion of normal submultigroup in terms of commutator of a group. First, we recall that if X is a group and $x, y \in X$, then the element $x^{-1}y^{-1}xy$ is usually depicted by $[x, y]$ and is called the *commutator* of x and y.

Theorem 3.9. *Let $A, B \in MG(X)$ such that $A \subseteq B$. Then A is a normal submultigroup of B if and only if*

(i) $C_A([x, y]) \geqslant C_A(x)$ $\forall x, y \in X$.

(ii) $C_A([x, y]) = C_A(e)$ $\forall x, y \in X$, *where e is the identity of X.*

Proof. (i). Suppose A is a normal submultigroup of B. Let $x, y \in X$, then

$$C_A(x^{-1}y^{-1}xy) \geqslant C_A(x^{-1}) \wedge C_A(y^{-1}xy) = C_A(x) \wedge C_A(x) = C_A(x).$$

Conversely, assume that A satisfies the inequality. Then for all $x, y \in X$,

$$C_A(x^{-1}yx) = C_A(yy^{-1}x^{-1}yx) \geqslant C_A(y) \wedge C_A([y, x]) = C_A(y).$$

Thus, $C_A(x^{-1}yx) \geqslant C_A(y)$ for all $x, y \in X$. Hence A is normal.

(ii). Let $x, y \in X$. Suppose A is a normal submultigroup of B. We know that A is a normal submultigroup of $B \Leftrightarrow C_A(xy) = C_A(yx) \Leftrightarrow C_A(x^{-1}y^{-1}x) =$

$C_A(y^{-1}) \Leftrightarrow C_A(x^{-1}y^{-1}xyy^{-1}) = C_A(y^{-1}) \Leftrightarrow C_A([x,y]y^{-1}) = C_A(y^{-1})$ for all $x, y \in X$. Consequently, $C_A([x,y]) = C_A(y^{-1}y) = C_A(e)$ for all $x, y \in X$.

Conversely, assume $C_A([x,y]) = C_A(e)$ for all $x, y \in X$. Then $C_A(x^{-1}y^{-1}xy) = C_A(e)$, so, $C_A((yx)^{-1}xy) = C_A(e)$. That is, $C_A(xy) = C_A(yx)$ for all $x, y \in X$. Thus, A is a normal submultigroup of B. □

Theorem 3.10. *Let A be a normal submultigroup of $G \in MG(X)$. Then $\bigcap_{x\in X} A^x$ is normal and is the largest normal submultigroup of G that is contained in A.*

Proof. Suppose $A^x \in MG(X) \forall x \in X$. Then for all $y \in X$, we observe that $\{A^x \mid x \in X\} = \{A^{xy} \mid x \in X\}$. Thus,

$$\begin{aligned}\bigwedge_{x\in X} C_{A^x}(yzy^{-1}) &= \bigwedge_{x\in X} C_A(xyzy^{-1}x^{-1}) = \bigwedge_{x\in X} C_A((xy)z(xy)^{-1}) \\ &= \bigwedge_{x\in X} C_{A^{xy}}(z) = \bigwedge_{x\in X} C_{A^x}(z) \ \ \forall y, z \in X.\end{aligned}$$

Hence, $\bigcap_{x\in X} A^x$ is a normal submultigroup of G.

Now let B be a normal submultigroup of G such that $B \subseteq A$. Then $B = B^x \subseteq A^x \forall x \in X$. Thus, $B \subseteq \bigcap_{x\in X} A^x$. Therefore, $\bigcap_{x\in X} A^x$ is the largest normal submultigroup of G that is contained in A. □

Definition 3.11. Let A be a submultigroup of $B \in MG(X)$. Then the it normalizer of A in B is the set given by

$$N(A) = \{g \in X \mid C_A(gy) = C_A(yg) \ \forall y \in X\}.$$

We now note that

$$N(A) = \{g \in X \mid C_{A^g}(y) = C_A(y) \ \forall y \in X\}.$$

It suffices to note that, $C_A(gy) = C_A(yg)$ for all $y \in X$ implies $C_A(g^{-1}yg) = C_A(y)$ for all $y \in X$. Then $C_A(g^{-1}yg) = C_A(y)$ gives $C_A(g^{-1}(gy)g) = C_A(gy)$, i.e., $C_A(yg) = C(gy)$ for all $y \in X$.

Example 3.12. Let $X = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8\}$ such that

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, g_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, g_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, g_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$g_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, g_6 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, g_7 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, g_8 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

be a group under matrix multiplication, and $A \subseteq B \in MG(X)$ such that

$$A = [g_1^{10}, g_2^5, g_3^7, g_4^5, g_5^5, g_6^5, g_7^7, g_8^8]$$

satisfying the axioms in Definition 2.4. Using Definition 3.11, $N(A) = \{g_1, g_3, g_7, g_8\}$.

Theorem 3.13. *Let A be a submultigroup of $B \in MG(X)$. Then the following assertions hold.*

(*i*) *$N(A)$ is a subgroup of X.*

(*ii*) *A is a normal submultigroup of B if and only if $N(A) = X$.*

Proof. (*i*). Let $g, h \in N(A)$. Then $C_{A^{gh}}(x) = C_{(A^h)^g}(x) = C_{A^h}(x) = C_A(x)$ for all $x \in X$ since $C_{A^g}(x) = C_A(g^{-1}xg) = C_A(x)$. Hence $gh \in N(A)$. Again, let $g \in N(A)$. We show that $g^{-1} \in N(A)$. For all $y \in X$, $C_A(gy) = C_A(yg)$ and so $C_A((gy)^{-1}) = C_A((yg)^{-1})$. Thus for all $y \in X$, $C_A(y^{-1}g^{-1}) = C_A(g^{-1}y^{-1})$ and so $C_A(yg^{-1}) = C_A(g^{-1}y)$ since $C_A(y) = C_A(y^{-1})$. Thus, $g^{-1} \in N(A)$. Hence, $N(A)$ is a subgroup of X.

(*ii*). Let A be a normal submultigroup of B and $g \in X$. Then for all $x \in X$, we have

$$C_{A^g}(x) \quad = \quad C_A(g^{-1}xg) = C_A((g^{-1}x)g) = C_A(g(g^{-1}x)) = C_A(x).$$

Thus, $C_{A^g}(x) = C_A(x)$ and so $g \in N(A)$. Therefore, $N(A) = X$.

Conversely, suppose $N(A) = X$. Let $x, y \in X$. To prove that A is normal, it is sufficient we show that $C_A(xy) = C_A(yx)$. Now

$$C_A(xy) = C_A(xyxx^{-1}) = C_A(x(yx)x^{-1}) = C_{A^{x^{-1}}}(yx) = C_A(yx),$$

where the last equality follows since $N(A) = X$ and so $x^{-1} \in N(A)$. Consequently, $C_{A^{x^{-1}}}(y) = C_A(y)$. Thus, A is a normal submultigroup of B. □

Remark 3.14. Let A be a submultigroup of $B \in MG(X)$. Then $S = N(A) = T$, if

$$S = \{x \in X \mid C_A(xy(yx)^{-1}) = C_A(e) \ \forall y \in X\}$$

and

$$T = \{x \in X \mid C_A(xyx^{-1}) = C_A(y) \ \forall y \in X\}.$$

Theorem 3.15. *Let A, B and C be multigroups of an abelian group X such that $A \subseteq B \subseteq C$. Then*

$$N(A) \cap N(B) \subseteq N(A \cap B).$$

Proof. Let $y \in N(A) \cap N(B)$. Then for any $x, y \in X$, we get $C_{A\cap B}(xy) = C_{A\cap B}(yx)$. thus, $C_{A\cap B}(xyx^{-1}) = C_{A\cap B}(y)$. Now

$$\begin{aligned} C_{A\cap B}(xyx^{-1}) &= C_A(xyx^{-1}) \wedge C_B(xyx^{-1}) = C_A(yxx^{-1}) \wedge C_B(yxx^{-1}) \\ &= C_A(y) \wedge C_B(y) = C_{A\cap B}(y). \end{aligned}$$

Thus, $y \in N(A \cap B)$. Hence, $N(A) \cap N(B) \subseteq N(A \cap B)$. □

Corollary 3.16. *Let $A, B, C \in MG(X)$ such that $A \subseteq B \subseteq C$ and $C_A(e) = C_B(e)$. Then*

$$N(A) \cap N(B) = N(A \cap B).$$

Proof. Recall that

$$\begin{aligned} N(A) &= \{x \in X \mid C_A(xy) = C_A(yx) \ \ \forall y \in X\} \\ &= \{x \in X \mid C_A(xyx^{-1}y^{-1}) = C_A(e) \ \ \forall y \in X\}. \end{aligned}$$

Let $y \in N(A \cap B)$. Then from the definition of $N(A)$, for all $x \in X$ we get

$$C_{A\cap B}(xyx^{-1}y^{-1}) = C_A(xyx^{-1}y^{-1}) \wedge C_B(xyx^{-1}y^{-1}) = C_A(e) \wedge C_B(e),$$

implies $y \in N(A) \cap N(B)$. Since $C_A(xyx^{-1}y^{-1}) = C_A(e)$ we obtain $C_A(xy) = C_A(yx)$. Similarly in the case of B because $C_A(e) = C_B(e)$. Hence $N(A) \cap N(B) = N(A \cap B)$. □

Corollary 3.17. *Let $A, B, C \in MG(X)$ such that $A \subseteq B \subseteq C$. Then*

$$N(A) \cap N(B) \subseteq N(A \circ B).$$

Proof. Let $y \in N(A) \cap N(B)$, that is $y \in N(A)$ and $y \in N(B)$. Then for all $x \in X$,

$$\begin{aligned} C_{A\circ B}(y) &= \bigvee_{y=ab} \{C_A(a) \wedge C_B(b) \mid \forall a, b \in X\} \\ &= \bigvee_{y=ab} \{C_A(x^{-1}ax) \wedge C_B(x^{-1}bx) \mid \forall a, b \in X\} \\ &\leqslant \bigvee_{x^{-1}yx=cd} \{C_A(c) \wedge C_B(d) \mid \forall c, d \in X\} \\ &= C_{A\circ B}(x^{-1}yx), \end{aligned}$$

which gives $C_{A\circ B}(y) \leqslant C_{A\circ B}(x^{-1}yx)$. The inequality holds since $y = ab \Rightarrow x^{-1}abx = cd \Rightarrow ab = xcdx^{-1} = (xcx^{-1})(xdx^{-1})$ and since $a = xcx^{-1}$ and $b = xdx^{-1}$ imply $x^{-1}ax = c$ and $x^{-1}bx = d$. Again,

$$C_{A\circ B}(x^{-1}yx) \leqslant C_{A\circ B}(x(x^{-1}yx)x^{-1}) = C_{A\circ B}(y).$$

So, $C_{A\circ B}(y) \geqslant C_{A\circ B}(x^{-1}yx)$. Thus, $C_{A\circ B}(y) = C_{A\circ B}(x^{-1}yx)$, which proves, $y \in N(A \circ B)$. Therefore, $N(A) \cap N(B) \subseteq N(A \circ B)$. □

Remark 3.18. If $A, B, C \in MG(X)$ such that $A \subseteq B \subseteq C$. Then $N(A) \subseteq N(B)$.

4. Homomorphism of normal submultigroups

In this section, we present some results on the homomorphic properties of normal submultigroups.

Theorem 4.1. *Let f be a homomorphism of an abelian group X onto an abelian group Y. Let A and B be multigroups of X such that $A \subseteq B$. Then*

$$f(N(A)) \subseteq N(f(A)).$$

Proof. Let $x \in f(N(A))$. Then $f(u) = x$ for some $u \in N(A)$. So, for all $y, z \in Y$,

$$\begin{aligned} C_{f(A)}(xyx^{-1}) &= C_A(f^{-1}(xyx^{-1})) = C_A(f^{-1}(x)f^{-1}(y)f^{-1}(x^{-1})) \\ &= C_A(f^{-1}(x)f^{-1}(y)f^{-1}(x)^{-1}) = C_A(f^{-1}(x)f^{-1}(y)(f^{-1}(x))^{-1}) \\ &= C_A(f^{-1}(f(u))f^{-1}(f(v))(f^{-1}(f(u)))^{-1}) = C_A(uvu^{-1}) \\ &= C_A(vuu^{-1}) = C_A(v) = C_A(f^{-1}(y)) = C_{f(A)}(y), \end{aligned}$$

where $v \in X$ such that $f(v) = y$. Thus, $x \in N(f(A))$, and consequently $f(N(A)) \subseteq N(f(A))$. □

Theorem 4.2. *Let $f : X \to Y$ be homomorphism of abelian groups X and Y. Let A and B be multigroups of Y such that $B \subseteq A$. Then*

$$f^{-1}(N(B)) = N(f^{-1}(B)).$$

Proof. Let $x \in f^{-1}(N(B))$. Then for all $y \in X$,

$$\begin{aligned} C_{f^{-1}(B)}(xyx^{-1}) &= C_B(f(xyx^{-1})) = C_B(f(x)f(y)f(x^{-1})) = C_B(f(x)f(y)(f(x))^{-1}) \\ &= C_B(f(y)f(x)(f(x))^{-1}) = C_B(f(y)) = C_{f^{-1}(B)}(y). \end{aligned}$$

Thus $x \in N(f^{-1}(B))$. So, $f^{-1}(N(B)) \subseteq N(f^{-1}(B))$.

Again, let $x \in N(f^{-1}(B))$ and $f(x) = u$. Then for all $v \in Y$,

$$\begin{aligned} C_B(uvu^{-1}) &= C_B(f(x)f(y)(f(x))^{-1}) = C_B(f(y)f(x)(f(x))^{-1}) \\ &= C_B(f(y)) = C_B(v), \end{aligned}$$

where $y \in X$ such that $f(y) = v$. Clearly, $u \in N(B)$, that is, $x \in f^{-1}(N(B))$. Thus, $N(f^{-1}(B)) \subseteq f^{-1}(N(B))$. Hence, $f^{-1}(N(B)) = N(f^{-1}(B))$. □

Theorem 4.3. *Let $f : X \to Y$ be an isomorphism of groups and let A be a normal submultigroup of $B \in MG(X)$. Then $f(A)$ is a normal submultigroup of $f(B) \in MG(Y)$.*

Proof. By Theorem 2.11, $f(A), f(B) \in MG(Y)$ and so, $f(A) \subseteq f(B)$. We show that $f(A)$ is a normal submultigroup of $f(B)$. Let $x, y \in Y$. Since f is an isomorphism, then for some $a \in X$ we have $f(a) = x$. Thus,

$$\begin{aligned} C_{f(A)}(xyx^{-1}) &= \bigvee_{b \in X} \{C_A(b) \mid f(b) = xyx^{-1}\} = \bigvee_{b \in X} \{C_A(a^{-1}ba) \mid f(a^{-1}ba) = y\} \\ &\geqslant \bigvee_{a^{-1}ba \in X} \{C_A(b) \mid f(b) = y\} = \bigvee_{b \in X} \{C_A(f^{-1}(y)) \mid f(b) = y\} = C_{f(A)}(y). \end{aligned}$$

Hence, $f(A)$ is a normal submultigroup of $f(B)$. □

Theorem 4.4. *Let Y be a group and $A \in MG(Y)$. If f is an isomorphism of X onto Y and B is a normal submultigroup of A, then $f^{-1}(B)$ is a normal submultigroup of $f^{-1}(A)$.*

Proof. By Theorem 2.11, $f^{-1}(A), f^{-1}(B) \in MG(X)$. Since B is a submultigroup of A, so $f^{-1}(B) \subseteq f^{-1}(A)$. Let $a, b \in X$, then we have

$$\begin{aligned} C_{f^{-1}(B)}(aba^{-1}) &= C_B(f(aba^{-1})) = C_B(f(a)f(b)(f(a))^{-1}) \\ &= C_B(f(a)(f(a))^{-1}f(b)) \\ &\geqslant C_B(e) \wedge C_B(f(b)) = C_{f^{-1}(B)}(b), \end{aligned}$$

which completes the proof. □

5. Comultisets of a multigroup

In this section, we assume that if G is a multigroup of a group X, then $G_* = X$. That is, every element of X is in G with its multiplicity or count.

Definition 5.1. Let X be a group. For any submultigroup A of a multigroup G of X, the submultiset yA of G for $y \in X$ defined by

$$C_{yA}(x) = C_A(y^{-1}x) \forall x \in A_*$$

is called the *left comultiset* of A. Similarly, the submultiset Ay of G for $y \in X$ defined by

$$C_{Ay}(x) = C_A(xy^{-1}) \forall x \in A_*$$

is called the *right comultiset* of A.

Example 5.2. Let $X = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$ be a permutation group of $\{1, 2, 3\}$ such that $\rho_0 = (1), \rho_1 = (123), \rho_2 = (132), \rho_3 = (23), \rho_4 = (13), \rho_5 = (12)$ and $G = [\rho_0^7, \rho_1^5, \rho_2^5, \rho_3^3, \rho_4^3, \rho_5^3]$ be a multigroup of X. Then $H = [\rho_0^6, \rho_1^3, \rho_2^3, \rho_3^2, \rho_4^2, \rho_5^2]$ is a submultigroup of G.

Now, we find the left comultisets of H by pre-multiplying each element of G by H.

$$\begin{aligned} \rho_0 H &= [\rho_0^6, \rho_1^3, \rho_2^3, \rho_3^2, \rho_4^2, \rho_5^2] & \rho_1 H &= [\rho_2^3, \rho_0^6, \rho_1^3, \rho_5^2, \rho_3^2, \rho_4^2] \\ \rho_2 H &= [\rho_1^3, \rho_2^3, \rho_0^6, \rho_4^2, \rho_5^2, \rho_3^2] & \rho_3 H &= [\rho_3^2, \rho_5^2, \rho_4^2, \rho_0^6, \rho_2^3, \rho_1^3] \\ \rho_4 H &= [\rho_4^2, \rho_3^2, \rho_5^2, \rho_1^3, \rho_0^6, \rho_2^3] & \rho_5 H &= [\rho_5^2, \rho_4^2, \rho_3^2, \rho_2^3, \rho_1^3, \rho_0^6] \end{aligned}$$

Similarly, the right comultisets of H are

$$\begin{aligned} H\rho_0 &= [\rho_0^6, \rho_1^3, \rho_2^3, \rho_3^2, \rho_4^2, \rho_5^2] & H\rho_1 &= [\rho_2^3, \rho_0^6, \rho_1^3, \rho_4^2, \rho_5^2, \rho_3^2] \\ H\rho_2 &= [\rho_1^3, \rho_2^3, \rho_0^6, \rho_5^2, \rho_3^2, \rho_4^2] & H\rho_3 &= [\rho_3^2, \rho_4^2, \rho_5^2, \rho_0^6, \rho_1^3, \rho_2^3] \\ H\rho_4 &= [\rho_4^2, \rho_5^2, \rho_3^2, \rho_2^3, \rho_0^6, \rho_1^3] & H\rho_5 &= [\rho_5^2, \rho_3^2, \rho_4^2, \rho_1^3, \rho_2^3, \rho_0^6] \end{aligned}$$

From Example 5.2, we notice that $H = yH$ for all $y \in X$ because a multigroup is an unordered collection. Consequently, $xH = yH$ for all $x, y \in X$.

Proposition 5.3. *Let X be a group. If A is a submultigroup of a multigroup G of X, then $yA = Ay$ for all $y \in X$.*

Proof. Assume A is a submultigroup of G. Then $\forall x \in A_*$ we have

$$C_{yA}(x) = C_A(y^{-1}x) \geqslant C_A(y) \wedge C_A(x) = C_A(x) \wedge C_A(y) = C_A(x) \wedge C_A(y^{-1}).$$

Suppose by hypothesis, $C_A(x) \wedge C_A(y) = C_A(xy)$. Then $C_{yA}(x) \geqslant C_{Ay}(x)$. Again,

$$C_{Ay}(x) = C_A(xy^{-1}) \geqslant C_A(x) \wedge C_A(y) = C_A(y) \wedge C_A(x) = C_A(y^{-1}) \wedge C_A(x).$$

By the same hypothesis, we get $C_{Ay}(x) \geqslant C_{yA}(x)$. Hence, $C_{yA}(x) = C_{Ay}(x)$, that is, $yA = Ay$. □

Remark 5.4. If A is a submultigroup of a multigroup G of a group X, then each yA (and Ay) are submultigroups of G.

Proposition 5.5. *If H is a submultigroup of $A \in MG(X)$, then the number of comultisets of H equals the cardinality of H_*.*

Proof. Recall that $H_* = \{x \in X \mid C_H(x) > 0\}$, that is, H_* is a set. Since comultisets of H is formed by pre-multiplying each element of X (since $A_* = X$) by H and $C_{yH}(x) = C_H(y^{-1}x) \forall y \in X$ must exist, hence the result follows. □

Proposition 5.6. *Let H be a submultigroup of $A \in MG(X)$. The union and intersection of the comultisets of H are comparable to H.*

Proof. $H = yH$ for all $y \in X$. Hence, the union and intersection of yH for all $y \in X$ are equal to H. □

Proposition 5.7. *Let X be a group. Any submultigroup A of a multigroup G and for any $z \in X$, the submultiset zAz^{-1}, where $C_{zAz^{-1}}(x) = C_A(z^{-1}xz)$ for each $x \in X$ is a submultigroup of G.*

Proof. Let $x, y \in X$, then we have $C_{zAz^{-1}}(e) = C_A(e)$ and

$$\begin{aligned} C_{zAz^{-1}}(xy^{-1}) &= C_A(z^{-1}xy^{-1}z) = C_A(z^{-1}xzz^{-1}y^{-1}z) \\ &\geqslant C_A(z^{-1}xz) \wedge C_A(z^{-1}y^{-1}z) = C_{zAz^{-1}}(x) \wedge C_{zAz^{-1}}(y^{-1}) \\ &= C_{zAz^{-1}}(x) \wedge C_{zAz^{-1}}(y) \end{aligned}$$

for all $z \in X$. Hence zAz^{-1} is a submultigroup of G. □

Corollary 5.8. *Let $\{A_i\}_{i \in I} \in MG(X)$, then*

(*i*) $\bigcap_{i \in I} zA_iz^{-1} \in MG(X)$ *for all* $z \in X$,

(*ii*) $\bigcup_{i \in I} zA_iz^{-1} \in MG(X)$ *for all* $z \in X$ *provided* $\{A_i\}_{i \in I}$ *have sup/inf assuming chain.* □

Proposition 5.9. *Let $A \in MG(X)$ and for all $g, h \in X$, then the following statements hold:*

(i) $Ag \circ Ag = Ag$,

(ii) $Ag \circ Ah = Ah \circ Ag$,

(iii) $(Ag \circ Ah)^{-1} = (Ah)^{-1} \circ (Ag)^{-1}$,

(iv) $(Ag \circ Ah)^{-1} = Ag \circ Ah$.

Proof. Let $g, h \in X$.

(i). From Definition 2.7, we have

$$\begin{aligned} C_{Ag\circ Ag}(x) &= \bigvee\{C_{Ag}(y) \wedge C_{Ag}(z) \mid x = yz, \forall y, z \in X\} \\ &= \bigvee_{y\in X}\{C_{Ag}(xy^{-1}) \wedge C_{Ag}(y) \mid x \in X\} = C_{Ag}(x). \end{aligned}$$

Hence, $Ag \circ Ag = Ag$.

$$\begin{aligned} (ii).\ \ C_{Ag\circ Ah}(x) &= \bigvee\{C_{Ag}(y) \wedge C_{Ah}(z) \mid x = yz, \forall y, z \in X\} \\ &= \bigvee\{C_{Ah}(z) \wedge C_{Ag}(y) \mid x = yz, y, z \in X\} = C_{Ah\circ Ag}(x). \end{aligned}$$

Hence, $Ag \circ Ah = Ah \circ Ag$.

(iii). We show that, the left and right hand sides are equal. By Definition 2.4

$$C_{(Ag\circ Ah)^{-1}}(x) = C_{Ag\circ Ah}(x^{-1}) = C_{Ag\circ Ah}(x).$$

Again, from the right hand side we get

$$\begin{aligned} C_{(Ah)^{-1}\circ(Ag)^{-1}}(x) &= \bigvee_{y\in X}\{C_{(Ah)^{-1}}(y^{-1}) \wedge C_{(Ag)^{-1}}(yx) \mid x \in X\} \\ &= \bigvee_{y\in X}\{C_{Ah}(y^{-1}) \wedge C_{Ag}(yx) \mid x \in X\} \\ &= C_{Ah\circ Ag}(x) = C_{Ag\circ Ah}(x). \end{aligned}$$

Hence, $(Ag \circ Ah)^{-1} = (Ah)^{-1} \circ (Ag)^{-1}$.

(iv). Straightforward from (iii). □

Proposition 5.10. *Let A be a commutative multigroup of a group X. Then*

(i) $Ay \circ Az = Ayz$ *for all* $y, z \in X$,

(ii) $yA \circ zA = yzA$ *for all* $y, z \in X$.

Proof. (i). Let $A \in MG(X)$ and $x \in X$, then we have

$$\begin{aligned} C_{Ay\circ Az}(x) &= \bigvee_{x=zy} \{C_{Ay}(z) \wedge C_{Az}(y) \mid \forall y, z \in X\} \\ &= \bigvee_{x=zy} \{C_A(zy^{-1}) \wedge C_A(yz^{-1}) \mid \forall y, z \in X\} \\ &= \{C_{A\cap A}((zy^{-1})(yz^{-1})) \mid \forall y, z \in X\} \\ &= \{C_A(xz^{-1}y^{-1}) \mid x = yz, \forall y, z \in X\} = \{C_{Ayz}(x) \mid x = yz, \forall y, z \in X\}. \end{aligned}$$

Hence, $Ay \circ Az = Ayz$.

(ii). Similar to (i). □

Corollary 5.11. *Let A be a multigroup of a group X and $y, z \in X$. The following statements are equivalent.*

(i) $(Ay \circ Az)^{-1} = Ay \circ Az$,

(ii) $Ay \circ Az = Ayz$.

Proof. Combining Proposition 5.9 and Proposition 5.10, the result follows. □

Theorem 5.12. *Let A be a commutative multigroup of a group X and $g, h \in X$, then $Ag \circ Ah = Agh$ if and only if $gA \circ hA = ghA$. Consequently, $Agh = ghA$.*

Proof. Let $A \in MG(X)$ and $g, h \in X$. Suppose $Ag \circ Ah = Agh$. Then

$$\begin{aligned} C_{Agh}(x) = C_{Ag\circ Ah}(x) &= \bigvee_{y\in X}(C_{Ag}(y) \wedge C_{Ah}(y^{-1}x)) = \bigvee_{y\in X}(C_A(yg^{-1}) \wedge C_A(y^{-1}xh^{-1})) \\ &= \bigvee_{y\in X}(C_A(g^{-1}y) \wedge C_A(h^{-1}y^{-1}x)) = \bigvee_{y\in X}(C_{gA}(y) \wedge C_{hA}(y^{-1}x)) \\ &= C_{gA\circ hA}(x) = C_{ghA}(x). \end{aligned}$$

So, $gA \circ hA = ghA$.

Conversely, let $gA \circ hA = ghA$. Then

$$\begin{aligned} C_{ghA}(x) = C_{gA\circ hA}(x) &= \bigvee_{y\in X}(C_{gA}(y) \wedge C_{hA}(y^{-1}x)) = \bigvee_{y\in X}(C_A(g^{-1}y) \wedge C_A(h^{-1}y^{-1}x)) \\ &= \bigvee_{y\in X}(C_A(yg^{-1}) \wedge C_A(y^{-1}xh^{-1})) = \bigvee_{y\in X}(C_{Ag}(y) \wedge C_{Ah}(y^{-1}x)) \\ &= C_{Ag\circ Ah}(x) = C_{Agh}(x). \end{aligned}$$

Thus, $Ag \circ Ah = Agh$. Hence, $Ag \circ Ah = Agh \Leftrightarrow gA \circ hA = ghA$. It follows that $Agh = ghA$. □

Proposition 5.13. *Let A be a normal submultigroup of $B \in MG(X)$. Then $C_{xA}(xz) = C_{xA}(zx) = C_A(z)$ for all $x, z \in X$.*

Proof. Let $x, z \in X$. Suppose A is a normal submultigroup of B, then by Proposition 3.4 and the fact that $C_A(xz) = C_A(zx)$, we get $C_{xA}(xz) = C_{xA}(zx) = C_A(x^{-1}zx) = C_A(z)$. Hence, $C_{xA}(xz) = C_{xA}(zx) = C_A(z)$ for all $z \in X$. □

Theorem 5.14. *Let $A, B \in MG(X)$ such that $A \subseteq B$. Then A is a normal submultigroup of B if and only if for all $x \in X$, $Ax = xA$.*

Proof. Suppose A is a normal submultigroup of B. Then for all $x \in X$, we have $C_{Ax}(y) = C_A(yx^{-1}) = C_A(x^{-1}y) = C_{xA}(y)$ for all $y \in X$. Thus, $Ax = xA$.

Conversely, let $Ax = xA$ for all $x \in X$. Then, $C_A(xy) = C_{x^{-1}A}(y) = C_{Ax^{-1}}(y) = C_A(yx)$ for all $y \in X$. Hence A is a normal submultigroup of B by Proposition 3.4. □

References

[1] **J.A. Awolola and P.A. Ejegwa**, *On some algebraic properties of order of an element of a multigroup*, Quasigroups and Related Systems **25** (2017), 21 – 26.

[2] **J.A. Awolola and A.M. Ibrahim**, *Some results on multigroups*, Quasigroups and Related Systems **24** (2016), 169 – 177.

[3] **M. Dresher and O. Oré**, *Theory of multigroups*, American J. Math. **60** (1938), 705 – 733.

[4] **A.M. Ibrahim and P.A. Ejegwa**, *A survey on the concept of multigroup theory*, J. Nigerian Asso. Mathl. Physics **38** (2016), 75 – 89.

[5] **L. Mao**, *Topological multigroups and multifields*, Int. J. Math. Combinatorics **1** (2009), 8 – 17.

[6] **Sk. Nazmul, P. Majumdar and S.K. Samanta**, *On multisets and multigroups*, Annals Fuzzy Math. Inform. **6** (2013), 643 – 656.

[7] **W. Prenowitz**, *Projective geometries as multigroups*, American J. Math. **65** (1943), 235 – 256.

[8] **A. Rosenfeld**, *Fuzzy subgroups*, J. Math. Anal. Appl. **35** (1971), 512 – 517.

[9] **B.M. Schein**, *Multigroups*, J. Algebra **111** (1987), 114 – 132.

[10] **Y. Tella and S. Daniel**, *A study of group theory in the context of multiset theory*, Int. J. Sci. Tech. **2** (2013), 609 – 615.

[11] **Y. Tella and S. Daniel**, *Symmetric groups under multiset perspective*, IOSR J. Math. **7** (2013), no. 5, 47 – 52.

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Note on the cyclic subgroup intersection graph of a finite group

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Abstract. The cyclic subgroup intersection graph of a finite group G, $\Gamma_{CSI}(G)$, is a simple graph with non-trivial cyclic subgroups as vertex set. Two cyclic subgroups are adjacent if and only if they have a non-trivial intersection. It is easy to see that $\Gamma_{CSI}(G)$ is a subgraph of the intersection graph was introduced by Csákány and Pollák many years ago. In this paper the main properties of this new graph is studied. The graph structure of the cyclic groups, dihedral groups, generalized quaternion groups and the group $Z_{p^\alpha} \times Z_{p^\beta}$ are completely determined.

1. Introduction

Throughout this paper all groups are assumed to be finite and graphs will be finite and simple. For notations not defined here, we refer the reader to [4, 7, 8]. The greatest common divisor and least common multiple of integers a and b are denoted by (a, b) and $[a, b]$, respectively. The number of positive divisors of an integer n is denoted by $d(n)$. Our calculations are done with the aid of GAP [2].

The intersection graph of a finite group G was introduced many years ago by Csákány any and Pollák [1]. The vertex set of this graph is all proper non-trivial subgroups of G and two vertices H and K are adjacent if and only if $H \cap K \neq 1$, where 1 denotes the trivial subgroup of G. In the mentioned paper, the authors proved that if G is abelian and there are two subgroups H and K in G such that there is no chain of subgroups which unites them, then G is the direct product of two simple cyclic groups. As a consequence, they proved that the diameter of this graph is at most 2, when G is an abelian group. The diameter of non-abelian, non-simple groups is at most 4. Some interesting open questions are also included in [1]. Zelinka [10], continued the study of this graph and conjectured that two finite Abelian groups with isomorphic intersection graphs are isomorphic.

Tamizh Chelvam and Sattanathan [9] continued the seminal paper of Csákány any and Pollák to introduce the subgroup intersection graph of a finite group G denoted by $\Gamma_{SI}(G)$. The vertex set of this graph is $G \backslash \{e\}$, and there is an edge

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between two distinct vertices x and y if and only if $\langle x\rangle \cap \langle y\rangle \neq 1$. As a consequence of a result in this paper, the subgroup intersection graph of a finite group G is complete if and only if G is a cyclic $p-$group or a generalized quaternion $2-$group. Moreover, the subgroup intersection graph of a finite abelian $p-$group is a union of complete graphs.

The cyclic subgroup intersection graph of G, $\Gamma_{CSI}(G)$, is another simple graph with proper non-trivial cyclic subgroups as vertex set. Two cyclic subgroups are adjacent if and only if they have a non-trivial intersection. It is easy to see that $\Gamma_{CSI}(G)$ is a subgraph of $\Gamma_{SI}(G)$.

Suppose Δ is a simple graph. Following Sabidussi [6], the $\Delta-$join of a family $\mathcal{F} = \{T_x \mid x \in V(\Delta)\}$ of simple graphs is another simple graph Γ with the following vertex and edge sets:

$$V(\Gamma) = \{(x, y) \mid x \in V(\Delta) \ \& \ y \in V(T_x)\},$$
$$E(\Gamma) = \{(x, y)(a, b) \mid xa \in E(\Delta) \ or \ x = a \ \& \ yb \in E(T_x)\}.$$

If $V(\Delta) = \{x_1, \ldots, x_n\}$ and $\mathcal{F} = \{T_1, \ldots, T_n\}$ then the $\Delta-$join of the family $\mathcal{F}$ is denoted by $\Delta[T_1, \ldots, T_n]$.

An independent set of a simple graph Γ is a subset of its vertices, no two of which are adjacent. The cardinality of an independent set in Γ of largest possible size is called the independence number of Γ. This number is denoted by $\alpha(\Gamma)$. We refer to the famous book of Harary [4] for our graph theory notations.

The aim of this paper is to investigate the main properties of the cyclic subgroup intersection graph. It is proved, among other things, that if $G = Z_{p^\alpha} \times Z_{p^\beta}$, where p is prime and α, β are two positive integers such that $\alpha \leqslant \beta$ then $\Gamma_{CSI}(G)$ is a union of the complete graphs $K_{(\beta-\alpha)p^\alpha + \frac{p^\alpha - 1}{p-1}}$ together with p copies of $K_{\frac{p^\alpha - 1}{p-1}}$, and $\Gamma_{SI}(G)$ is a union of the complete graphs $K_{p^{\alpha+\beta} - \frac{p^{2\alpha+1}+1}{p+1}}$ together with p copies of $K_{\frac{p^{2\alpha}-1}{p+1}}$.

2. Main results

Suppose G is a non-cyclic group and $A = \Gamma_{CSI}(G)$. For each $\langle a\rangle \in V(A)$, we define $T_{\langle a\rangle} = K_{\phi(|a|)}$, where ϕ denotes the Euler totient function. Then one can easily see that $\Gamma_{SI}(G)$ is an $A-$join of $\{T_{\langle x\rangle} \mid \langle x\rangle \in V(A)\}$.

Lemma 2.1. *Let G be a group of order n. Then $|V(\Gamma_{CSI}(G))| \geqslant d(n) - 2$ with equality if and only if G is cyclic.*

Proof. By [5], the number of cyclic subgroups of a group G of order n is at least $d(n)$ with equality if and only if $G \cong Z_n$, as desired. $\square$

By Lemma 2.1, the cyclic subgroup intersection graph of a cyclic group of order p^{m+1} has exactly m vertices. This proves that for each positive integer m, there exists at least a group with an $m-$vertex cyclic subgroup intersection graph.

Example 2.2. Suppose $SmallGroup(n, i)$ denotes the $i-th$ group of order n in the small group library of GAP [2]. Define $G = SmallGroup(168, 46) = (Z_7 \times A_4) : Z_2$, $H = SmallGroup(168, 38) = (Z_{42} \times Z_2) : Z_2$ and $K = SmallGroup(168, 42)$ $= PSL(3, 2)$. Then $\Gamma_{CSI}(G) \cong \Gamma_{CSI}(H) \cong \Gamma_{CSI}(K)$, but G, H and K are mutually non-isomorphic.

The previous example shows that if $\Gamma_{CSI}(G)$ and $\Gamma_{CSI}(H)$ are isomorphic then we cannot deduce that G and H are isomorphic, even in the case that one of these groups is simple.

Example 2.3. In this example the cyclic subgroup intersection graph of a dihedral group of order $2n$ will be computed. The dihedral group of order $2n$ can be presented as $D_{2n} = \langle x^n = y^2 = e, y^{-1}xy = x^{-1}\rangle$. Suppose $k_1, \ldots, k_{d(n)}$ are all divisors of n. Then

$$V(\Gamma_{CSI}(D_{2n})) = \{\langle a^{k_1}\rangle, \ldots, \langle a^{k_{d(n)-1}}\rangle, \langle b\rangle, \langle ab\rangle, \ldots, \langle a^{n-1}b\rangle\}.$$

It is easy to see that $\langle b\rangle, \langle ab\rangle, \ldots, \langle a^{n-1}b\rangle$ are pendant vertices of $\Gamma_{CSI}(D_{2n})$. Moreover, $\langle a^{k_i}\rangle$ and $\langle a^{k_j}\rangle$ are adjacent if and only if $[k_i, k_j] < n$.

Theorem 2.4. *The cyclic subgroup intersection graph of a finite group G is complete if and only if G is cyclic or a generalized quaternion $2-$group.*

Proof. It is well-known that a $p-$group G has a unique subgroup of order p if and only if G is cyclic or a generalized quaternion $2-$group. By this theorem, if G is cyclic or a generalized quaternion $2-$group then the intersection of non-trivial subgroups H and K contains the unique subgroup of G and so $H \cap K \neq 1$. This proves that $\Gamma_{CSI}(G)$ is complete. Conversely, if $\Gamma_{CSI}(G)$ is complete and p, q are two prime divisors of $|G|$ then there are elements a and b of orders p and q in G, respectively. Since $\langle a\rangle \cap \langle b\rangle = 1$,we lead to a contradiction. So, G is a $p-$group. Since $\Gamma_{CSI}(G)$ is complete, there is a unique subgroup of order p and by mentioned well-known result G is cyclic or a generalized quaternion $2-$group. □

Lemma 2.5. *Let G be a finite group. Then $\alpha(\Gamma_{CSI}(G))$ is the number of cyclic subgroups of a prime order.*

Proof. Suppose $\alpha = \alpha(\Gamma_{CSI}(G))$, $\{\langle a_1\rangle, \ldots, \langle a_k\rangle\}$ is the set of all cyclic subgroups of G of a prime order and $B = \{\langle b_1\rangle, \ldots, \langle b_\alpha\rangle\}$ is a given independent set of largest possible size for G. Since A is an independent set for $\Gamma_{CSI}(G)$, $k \leqslant \alpha$. Choose i, $1 \leqslant i \leqslant \alpha$, and element c_i of a prime order such that $\langle c_i\rangle \subseteq \langle b_i\rangle$. Since B is an independent set, $c_i \neq c_j$, when $i \neq j$. This shows that $\alpha \leqslant k$, proving the lemma. □

Suppose m and n are positive integers. Define:

$$I_{m,n} = \{(a, b, t) \in N^2 \times N_0 \mid a|m,\ b|n,\ 0 \leqslant t \leqslant (a, \frac{n}{b}) - 1\},$$

$$H_{a,b,t} = \{(ia + \frac{jta}{(a, \frac{n}{b})}, jb) \mid 0 \leqslant i \leqslant \frac{m}{a} - 1,\ 0 \leqslant j \leqslant \frac{n}{b} - 1\},\quad (a, b, t) \in I_{m,n}.$$

For the sake of completeness, we mention here a result in [3] which is crucial in our next result. If $ns = 0$ then we define $(mb, na, ns) = (mb, na)$.

Theorem 2.6. [3, Theorem 2] *Suppose* $s = \frac{ta}{(a,\frac{n}{b})}$. *Then,*

1. $H \leqslant Z_m \times Z_n$ *if and only if there exists* $(a, b, t) \in I_{m,n}$ *such that* $H = H_{a,b,t}$.
2. $H_{a,b,t}$ *is cyclic if and only if* $ab = (mb, na, ns)$.
3. *The number of cyclic subgroups in* $Z_m \times Z_n$ *is* $\displaystyle\sum_{a|m,b|n,(\frac{m}{a},\frac{n}{b})=1} (a, b)$.

Theorem 2.7. *Suppose* $G = Z_{p^\alpha} \times Z_{p^\beta}$, *where* p *is prime and* α, β *are two positive integers such that* $\alpha \leqslant \beta$. *Then* $\Gamma_{CSI}(G)$ *is a union of the complete graphs* $K_{(\beta-\alpha)p^\alpha+\frac{p^\alpha-1}{p-1}}$ *and* p *copies of* $K_{\frac{p^\alpha-1}{p-1}}$.

Proof. By definition of $H_{a,b,t}$ and Theorem 2.6, it can easily see that for each d, $1 \leqslant d \leqslant p-1$, the subgroups

$$\begin{aligned}
&H_{p^\alpha,p^{\beta-1},d};\\
&H_{p^\alpha,p^{\beta-k},t}, 2 \leqslant k \leqslant \alpha, t = d, p+d, \ldots, (p-1)p+d;\\
&H_{p^\alpha,p^{\beta-k'},t}, 3 \leqslant k' \leqslant \alpha, t = p^2+d, p^2+p+d, \ldots, p^2+(p^{k'-1}+p-1)p+d,
\end{aligned}$$

are cyclic subgroups containing $(dp^{\alpha-1}, p^{\beta-1})$ which gives $p-1$ cliques isomorphic to $K_{\frac{p^\alpha-1}{p-1}}$. These complete subgraphs are denoted by $\Gamma_1, \ldots, \Gamma_{p-1}$. On the other hand, the cyclic subgroups $H_{p^k,p^\beta,0}$, $0 \leqslant k \leqslant \alpha-1$ and $H_{p^{\alpha-l},p^{\beta-k'},t}$, $1 \leqslant k' \leqslant \alpha-1$, $1 \leqslant l \leq \alpha-k'$, $1 \leqslant t \leqslant p^{k'}-1$, $t \not\equiv 0 (mod\ p)$ have a common element $(p^{\alpha-1}, 0)$ and so we will have another clique of size $\frac{p^\alpha-1}{p-1}$. Note that for each k', there are $(\alpha-k')(p^{k'}-p^{k'-1})$ cyclic subgroups $H_{p^{\alpha-l},p^{\beta-k'},t}$ that gives a clique of size $\frac{p^\alpha-1}{p-1}$. The complete subgraph induced by this clique is denoted by Γ_p. We now consider the cyclic subgroups $H_{p^\alpha,p^k,t}$, $0 \leqslant k \leqslant \beta-\alpha-1$, $0 \leqslant t \leqslant p^\alpha-1$ and the cyclic subgroups $H_{p^\alpha,p^{\beta-k},t}$, $1 \leqslant k \leqslant \alpha$, $0 \leqslant t \leqslant p^k-1$ and $t \equiv 0\ (mod\ p)$. These are $(\beta-\alpha)p^\alpha + \frac{p^\alpha-1}{p-1}$ cyclic subgroups containing element $(0, p^{\beta-1})$ which gives us a clique of order $(\beta-\alpha)p^\alpha + \frac{p^\alpha-1}{p-1}$. Define Γ_{p+1} to be the complete subgraph induced by the las clique. By Theorem 2.6(3), these are all cyclic subgroups of $Z_{p^\alpha} \times Z_{p^\beta}$ and we have to show that $\Gamma_{CSI}(Z_{p^\alpha} \times Z_{p^\beta})$ is the union of $\Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_{p+1}$.

To complete the proof, we will shows that there is no edge in $\Gamma_{CSI}(Z_{p^\alpha} \times Z_{p^\beta})$ connecting a vertex in Γ_i to a vertex in Γ_j, $i \neq j$. Suppose vertices $v_1 = \langle a_1 \rangle \in V(\Gamma_i)$ and $v_2 = \langle a_2 \rangle \in V(\Gamma_j)$ that are not adjacent in $\Gamma_{CSI}(G)$. We prove that there is no vertex $u_1 = \langle b_1 \rangle$ in Γ_i to be adjacent with a vertex $u_2 = \langle b_2 \rangle$ in Γ_j. If u_1 and u_2 are adjacent in $\Gamma_{CSI}(G)$ then b_1 and b_2 will be adjacent in $\Gamma_{SI}(G)$ and since $\Gamma_{SI}(G)$ is a union of complete graphs, a_1 and $a)2$ will be adjacent in $\Gamma_{SI}(G)$

and so v_1 and v_2 are adjacent in $\Gamma_{CSI}(G)$ which is impossible. To complete our argument, we consider the following cyclic subgroups:

$$\begin{aligned} H_{p^\alpha,p^{\beta-1},d} &= \{(jdp^{\alpha-1}, jp^{\beta-1}) \mid 0 \leqslant j \leqslant p-1\};\ (1 \leqslant d \leqslant p-1), \\ H_{p^k,p^\beta,0} &= \{(ip^k, 0) \mid 0 \leqslant i \leqslant p^{\alpha-k} - i\}, \\ H_{p^\alpha,p^{\beta-1},0} &= \{(0, jp^{\beta-1}) \mid 0 \leqslant j \leqslant p^\beta - 1\}. \end{aligned}$$

We now prove that these vertices are not adjacent. Suppose $(jdp^{\alpha-1}, jp^{\beta-1}) \in H_{p^k,p^\beta,0} \cap H_{p^\alpha,p^{\beta-1},d}$. Then $jp^{\beta-1} = 0$, $0 \leqslant j \leqslant p-1$, and so $j = 0$. This shows that $(jdp^{\alpha-1}, jp^{\beta-1}) = (0,0)$. It is also clear that $H_{p^k,p^\beta,0} \cap H_{p^\alpha,p^{\beta-1},0} = \{(0,0)\}$. If $(jdp^{\alpha-1}, jp^{\beta-1}) \in H_{p^\alpha,p^{\beta-1},0}$, $1 \leqslant j,d \leqslant p-1$, then $j = 0$ and so $H_{p^\alpha,p^{\beta-1},0} \cap H_{p^\alpha,p^{\beta-1},d} = \{(0,0)\}$.

We now assume that $d' \neq d$. Choose a common element in two cyclic subgroups of the first type, say $(jdp^{\alpha-1}, jp^{\beta-1}) = (j'd'p^{\alpha-1}, j'p^{\beta-1})$. Then $j\beta^{p-1} \equiv j'\beta^{p-1}$, where $0 \leqslant j,j' \leqslant p-1$. Thus $j = j'$ and since $jdp^{\alpha-1} \equiv jd'p^{\alpha-1}$ $(mod\ p^\alpha)$. Therefore, $d = d'$ which completes our proof. $\square$

Theorem 2.8. *Suppose $G = Z_{p^\alpha} \times Z_{p^\beta}$, where p is prime and α, β are two positive integers such that $\alpha \leqslant \beta$. Then $\Gamma_{SI}(G)$ is a union of the complete graphs $K_{p^{\alpha+\beta} - \frac{p^{2\alpha+1}+1}{p+1}}$ and p copies of $K_{\frac{p^{2\alpha}-1}{p+1}}$.*

Proof. By Theorem 2.7, the graph $\Gamma_{CSI}(Z_{p^\alpha} \times Z_{p^\beta})$ is a union of $p+1$ complete graph and by definition of Γ_{SI} and Γ_{CSI}, a given component of Γ_{SI} is constructed from a component of Γ_{CSI} by adding some vertices corresponding to generators of vertices in Γ_{CSI}. So the components of Γ_{SI} will also be a complete graph. Suppose $1 \leqslant d \leqslant p-1$. By the proof of Theorem 2.7, the vertices of $p-1$ components of Γ_{CSI} are as follows:

$$\begin{aligned} H_{p^\alpha,p^{\beta-1},d} &= \{(jtp^{\alpha-1}, jp^{\beta-1});\ 0 \leqslant j \leqslant p-1\}, \\ H_{p^\alpha,p^{\beta-k},t} &= \{(jtp^{\alpha-k}, jp^{\beta-t}),\ 0 \leqslant j \leqslant p^k - 1\}, \\ H_{p^\alpha,p^{\beta-k'},t'} &= \{(jt'p^{\alpha-k'}, jp^{\beta-k'}),\ 0 \leqslant j \leqslant p^{k'} - 1\}, \end{aligned}$$

where $t \in A = \{d, p+d, \ldots, (p-1)p+d\}$, $2 \leqslant k \leqslant \alpha$ and
$t' \in B = \{p^2+d, \ldots, p^2+(p^{k'-1}-p-1)p+d\}$, $3 \leqslant k' \leqslant \alpha$.

On the other hand, $|H_{p^\alpha,p^{\beta-1},d}| = p$, $|H_{p^\alpha,p^{\beta-k},t}| = p^k$, $|A| = p$, $|H_{p^\alpha,p^{\beta-k'},t'}| = p^{k'}$, $|B| = p^{k'} - p$ and by considering the number of generators, we will have $p-1$ complete graph $K_{\frac{p^{2\alpha}-1}{p+1}}$.

By the proof of Theorem 2.7, the cyclic subgroups

$$\begin{aligned} H_{p^k,p^\beta,0} &= \{(ip^k, 0),\ 0 \leqslant i \leqslant p^{\alpha-k} - 1\}, 0 \leqslant k \leqslant \alpha - 1, \\ H_{p^{\alpha-l},p^{\beta-k},t} &= \{(ip^{\alpha-l} + jtp^{\alpha-l-k}, jp^{\beta-k}),\ 0 \leqslant i \leqslant p^l - 1,\ 0 \leqslant j \leqslant p^k - 1\} \end{aligned}$$

constitutes a $\frac{p^\alpha-1}{p-1}$−vertex component of $\Gamma_{CSI}(Z_{p^\alpha}\times Z_{p^\beta})$. By an easy calculation, one can see that the number of generators of vertices are equal to $\frac{p^{2\alpha}-1}{p+1}$. We now consider the component $K_{(\beta-\alpha)p^\alpha+\frac{p^\alpha-1}{p-1}}$ of $\Gamma_{CSI}(G)$ with the following vertices:

$$H_{p^\alpha,p^k,t} = \{(jt, jp^{\beta-(\alpha+1)}) \mid 0 \leqslant j \leqslant p^{\beta-k}-1\}$$
$$H_{p^\alpha,p^{\beta-k'},t'} = \{(jt'p^{\alpha-k'}, jp^{\beta-k'}),\ 0 \leqslant j \leqslant p^{k'}-1\},$$

where $0 \leqslant k \leqslant \beta-\alpha+1,\ 0 \leqslant t \leqslant p^\alpha-1,\ 1 \leqslant k' \leqslant \alpha,\ 0 \leqslant t' \leqslant p^{k'}-1$ and $t' \equiv 0 \pmod p$. By counting the number of generators, a component isomorphic to $K_{p^{\alpha+\beta}-\frac{p^{2\alpha+1}+1}{p+1}}$ is obtained, as desired. □

References

[1] **B. Csákány and G. Pollák**, *The graph of subgroups of a finite group*, Czechoslovak Math. J. **19** (1969), 241 − 247.

[2] *GAP – Groups, Algorithms and Programming*, Version 4, The GAP team.

[3] **M. Hampejs, N. Holighaus, L. Tóth and C. Wiesmeyr**, *Representing and counting the subgroups of the group $\mathbb{Z}_m \times \mathbb{Z}_n$*, J. Numbers **2014** (2014), Article ID 491428.

[4] **F. Harary**, *Graph Theory*, Addison-Wesley Publishing Co., London, 1969.

[5] **I.M. Richards**, *A remark on the number of cyclic subgroups of a finite group*, Amer. Math. Monthly **91** (1984), 571 − 572.

[6] **G. Sabidussi**, *Graph derivatives*, Math. Z. **76** (1961), 385 − 401.

[7] **M. Suzuki**, *Group Theory I*, Springer-Verlag, Berlin, 1982.

[8] **M. Suzuki**, *Group Theory II*, Springer-Verlag, Berlin, 1986.

[9] **T. Tamizh Chelvam and M. Sattanathan**, *Subgroup intersection graph of a group*, J. Adv. Research Pure Math. **3** (2011), 44 − 49.

[10] **B. Zelinka**, *Intersection graphs of finite abelian groups*, Czechoslovak Math. J. **25** (1975), 171 − 174.

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Representation of monoids in the category of monoid acts

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To Bernhard Banaschewski on his 90th Birthday

Abstract. The study of monoids in the category of monoid acts leads to the notion of power action. In this paper, for a monoid T, we investigate the relationship between the category T-**Act** of all T-acts and the category T-**Pwr** of all T-power acts. For a T-power act M on a commutative monoid T, we introduce the covariant functor M^{M^-} from T-**Act** to T-**Pwr** and show that the family of assignments $(\eta_A : A \to M^{M^A})_{A \in T\text{-}\mathbf{Act}}$ constitutes a natural transformation. Moreover, the Hom-functor $(M^-)^-$ and the tensor functor $M^{-\otimes-}$ from T-**Act** $\times$ T-**Act** to T-**Pwr** are naturally equivalent.

1. Introduction and preliminaries

Representation of mathematical structures is a way for better seeing of them to study. Analyzing the internalized concepts in a topos captured the interest of some mathematicians. The general notion of a mathematical object in a topos (or a category with some properties) introduces a lot of conceptions and structures obtained from its classical versions in **Set**, the category of sets ([4]). For instance, "Algebras in a Category" are some of these structures such as groups and group actions in a topos (see [2, 8]).

For a monoid T, let T-**Act** denote the category of all T-acts and act homomorphisms between them. Considering the monoid T as a category T with one object, T-**Act** is isomorphic to the functor category $\mathbf{Set}^T$ (or $[T,\mathbf{Set}]$ in another notation), hence it is a (presheaf) topos (see [3]). Here we study the structure of monoids in the category T-**Act**, so-called *T-power acts*, or *actions over monoids* in the sense of [5] which were used to construct the hypergroups. First we verify some basic properties of the power acts. In particular, the free objects in the category T-**Pwr** of all T-power acts are constructed. For a T-power act M and a T-act A over a commutative monoid T, it is shown that the set M^A of all T-act homomorphisms from A to M is a T-power act which gives the two functors M^- (contravariant) and M^{M^-} (covariant) from T-**Act** to T-**Pwr**. Also the family of assignments $(\eta_A : A \to M^{M^A})_{A \in T\text{-}\mathbf{Act}}$ constitutes a natural transformation from

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the identity functor to UM^{M^-}, where U is the forgetful functor. Finally, we prove that $(M^A)^B$ and $M^{A\otimes B}$ are naturally isomorphic in T-**Pwr** for every T-acts A and B.

Now let us briefly recall some needed notions in the sequel.

Let T be a monoid and A be a (non-empty) set. A *right T-act* on A is a map $A \times T \to A$, $(a,t) \rightsquigarrow at$, such that for every $a \in A$ and $t, s \in T$, $(at)s = a(ts)$ and $a1 = a$. The notion of *left T-act* is defined similarly. Here by a T-act we mean a right T-act unless otherwise stated. An element θ in a T-act A is said to be a *fixed element* if $\theta t = \theta$ for each $t \in T$. Let A, B be two T-acts. A map $f : A \to B$ is called a *T-act homomorphism* or simply *act homomorphism* if $f(at) = f(a)t$, for every $a \in A$ and $t \in T$. The class of all T-acts together with the T-act homomorphisms between them forms a category which is denoted by T-**Act**. For a monoid M, $H(M)$ denotes the monoid of all endomorphisms of M with the composition of mappings as its operation. To denote the image of $x \in M$ under $\sigma \in H(M)$ we will use the postfix notation. An equivalence relation θ on a T-act A is called a *T-act congruence* if $x\theta y$ implies that $xt\theta yt$, for every $x, y \in A$ and $t \in T$. The free T-act on a non-empty set X is the set $X \times T$ with the action $(x,t)s = (x,ts)$, for every $x \in X$ and $t, s \in T$. Let A be a right T-act and B be a left T-act. The *tensor product* of A and B is the set $A \otimes B := (A \times B)/\theta$, where θ is the equivalence relation on the set $A \times B$ generated by the pairs $((at,b),(a,tb))$ for $a \in A, b \in B, t \in T$. We denote $(a,b)/\theta \in A \otimes B$ by $a \otimes b$. In the case that T is a commutative monoid, every T-act can be considered as a T-biact so that there is naturally a T-act structure on the tensor product $A \otimes B$ for any two T-acts A and B (see [6, Proposition II.5.12]). For more information on the theory of acts over monoids, see [6]. Also for some required categorical ingredients we refer to [7]. Throughout the paper T stands for a monoid unless otherwise stated.

2. Monoids in the category of acts: Power action

Algebra in a category is a subject for mathematicians to study algebraic structures categorically. In this theory, a base category $\mathcal{C}$ is replaced to the category **Set** and all algebraic operations are the morphisms of $\mathcal{C}$, and homomorphisms are those morphisms in $\mathcal{C}$ such that preserve the operations in the sense of commutative diagrams in $\mathcal{C}$. Note that equations in algebras are explained as commutative diagrams. For more information we refer to [2, 4, 8].

Here we study the notion of monoid in the base category T-**Act**, where T is a monoid. Let us first recall the notion of a *monoid in an arbitrary category.* Let $\mathcal{C}$ be a category with finite products. A monoid $\langle M, \cdot, 1_M\rangle$ in $\mathcal{C}$ is an object of $\mathcal{C}$ together with two morphisms $\cdot : M \times M \to M$ called *multiplication* and $1_M : \top \to M$ called *identity,* in which $\top$ is the terminal object of $\mathcal{C}$ such that the following diagrams commute:

- Association law $((x \cdot y) \cdot z = x \cdot (y \cdot z))$:

$$\begin{tikzcd}
M \times M \times M \arrow[r, "\cdot \times id_M"] \arrow[d, "id_M \times \cdot"'] & M \times M \arrow[d, "\cdot"] \\
M \times M \arrow[r, "\cdot"'] & M
\end{tikzcd}$$

• Identity law ($x \cdot 1_M = x = 1_M \cdot x$):

$$\begin{tikzcd}
\top \times M \arrow[r, "1_M \times id_M"] \arrow[dr, "\pi_2"'] & M \times M \arrow[d, "\cdot"] & M \times \top \arrow[l, "id_M \times 1_M"'] \arrow[dl, "\pi_1"] \\
& M &
\end{tikzcd}$$

Now let M, N be two monoids in a category $\mathcal{C}$. A *homomorphism* from M to N is a morphism $f : M \to N$ in $\mathcal{C}$ such that the following diagrams commute:
• Preserving the multiplication:

$$\begin{tikzcd}
M \times M \arrow[r, "\cdot"] \arrow[d, "f \times f"'] & M \arrow[d, "f"] \\
N \times N \arrow[r, "\cdot"'] & N
\end{tikzcd}$$

• Preserving the identity:

$$\begin{tikzcd}
\top \arrow[r, "1_M"] \arrow[dr, "1_N"'] & M \arrow[d, "f"] \\
& N
\end{tikzcd}$$

All monoids in a category $\mathcal{C}$ with homomorphisms between them make a category denoted by $\mathbf{Mon}(\mathcal{C})$.

Here we are going to explain objects of the category $\mathbf{Mon}(T\text{-}\mathbf{Act})$ for a monoid T with identity 1. Let M be an object in this category. Then there is a T-action $M \times T \to M, (m,t) \rightsquigarrow mt$, with a T-act homomorphism $\cdot : M \times M \to M$. So for every $t, s \in T$ and $m, n \in M$ we have $(mt)s = m(ts)$, $m1 = m$ and $(m{\cdot}n)t = mt{\cdot}nt$. Since $1_M : \top \to M$ is a T-act homomorphism where $\top$ is considered as the one-element T-act, $1_M t = 1_M$. Finally, by the diagrams of associativity and identity, M is a monoid. Because of the kind of these equations, we use the notation m^t for mt and give the following definition. If no confusion arises, the identities of M and T are denoted by the same symbol 1.

Definition 1. Let T be a monoid. By a (*right*) T*-power act*, we mean a monoid M equipped with a map $M \times T \to M$, $(m,t) \rightsquigarrow m^t$, in such a way that the following conditions hold for all $t, s \in T$ and $m, n \in M$:

$$(mn)^t = m^t n^t, \quad (m^t)^s = m^{ts}, \quad m^1 = m, \quad 1^t = 1.$$

If T contains a zero, then m^0 is clearly a fixed element of M where M is considered as a T-act.

Note that the notion of power act is also appeared in [5] under the name of "action over monoids".

Now we describe the morphisms of the category $\mathbf{Mon}(T\text{-}\mathbf{Act})$. Let M and N be two objects of $\mathbf{Mon}(T\text{-}\mathbf{Act})$. It is easy to see that a map $f : M \to N$ is a morphism in $\mathbf{Mon}(T\text{-}\mathbf{Act})$, so-called a *$T$-power act homomorphism* or simply *power act homomorphism* if and only if $f(mn) = f(m)f(n)$, $f(1) = 1$ and $f(m^t) = f(m)^t$, for all $m, n \in M$ and $t \in T$. The category of all T-power acts with T-power act homomorphisms between them is denoted by T-$\mathbf{Pwr}$ which is isomorphic to the category $\mathbf{Mon}(T\text{-}\mathbf{Act})$.

In the following, we give some examples of power acts.

Example 1.
1. Consider the monoid $(\mathbb{N}, \cdot)$. Then every commutative monoid M with m^k to be $mm \cdots m$, k-times, for every $m \in M$ and $k \in \mathbb{N}$, is an $\mathbb{N}$-power act.

2. Given a monoid M, let T be a submonoid of $H(M)$. Then we define m^σ to be $m\sigma$, for all $m \in M$ and $\sigma \in T$. Then M is a T-power act which is called the *natural power action*.

3. Given two monoids M and T with $0 \in T$, let $\phi : T \to H(M)$ be a monoid homomorphism and $u \in M$. For every $m \in M$ and $t \neq 0$ in T, define $m^t = m\phi(t)$, and $m^0 = u$. Then M is a T-power act if and only if $u\phi(t) = u$ for all $t \in T$ and $u^2 = u$. This is called the *(ϕ, u)-power action*. In particular, the $(\mathbf{id}, 1)$-power action is said to be an *identity power action* where $\mathbf{id} : T \to H(M)$ is the constant homomorphism mapping every $t \in T$ to id_M.

Proposition 1. *Let M and T be two monoids and $0 \in T$. Then each T-power act M is of the form (ϕ, u)-power act (in the sense of Example 1(3)) for a unique monoid homomorphism $\phi : T \to H(M)$ and some $u \in M$.*

Proof. Let M be a T-power act and $t \in T$. Define $\sigma_t : M \to M$ by $m\sigma_t = m^t$ for every $m \in M$. We show that the map σ_t is a monoid homomorphism. Indeed, we have $(mn)\sigma_t = (mn)^t = m^t n^t = m\sigma_t n\sigma_t$, and $1\sigma_t = 1^t = 1$ for every $m, n \in M$. Now, define $\phi : T \to H(M)$ by $\phi(t) = \sigma_t, t \in T$. The map ϕ is a monoid homomorphism. To see this, for any $t, s \in T$ and $m \in M$, $m\sigma_{ts} = m^{ts} = (m^t)^s = m\sigma_t\sigma_s$. Thus $\phi(ts) = \sigma_{ts} = \sigma_t\sigma_s = \phi(t)\phi(s)$. Also $\phi(1) = \sigma_1 = id$. Now take $u := \phi(0)$. It is clear that $u^2 = u$ and $u\phi(t) = u$ for all $t \in T$. Then M is a (ϕ, u)-power act (see Example 1(3)). For the uniqueness of ϕ, suppose that $\psi : T \to H(M)$ is a monoid homomorphism with $m^t = m\psi(t)$, for all $m \in M$ and $t \in T$. This implies that $m\psi(t) = m\phi(t)$ for all $m \in M$ and $t \in T$ which means $\psi = \phi$. □

Here we define the notion of a bipower act.

Definition 2. Let T and S be monoids. By a (T,S)-*bipower act* M we mean a monoid M which is both (right) T and S-power acts simultaneously, in such a way that $(m^t)^s = (m^s)^t$, for every $m \in M$, $t \in T$ and $s \in S$.

Remark 1. Every (T,S)-bipower act M for two monoids T and S can be considered as a $T \times S$-power act. To this end, we define the power action $m^{(t,s)}$ to be $(m^t)^s$ for every $m \in M$, $t \in T$ and $s \in S$. Then we have:

1. $m^{(1,1)} = (m^1)^1 = m$,
2. $1^{(t,s)} = (1^t)^s = 1$,
3. $m^{(t,s)}n^{(t,s)} = (m^t)^s(n^t)^s = (m^tn^t)^s = ((mn)^t)^s = (mn)^{(t,s)}$,
4. $(m^{(t_1,s_1)})^{(t_2,s_2)} = (((m^{t_1})^{s_1})^{t_2})^{s_2} = (((m^{t_1})^{t_2})^{s_1})^{s_2} = (m^{t_1t_2})^{s_1s_2} = m^{(t_1,s_1)(t_2,s_2)}$.

By a *power act congruence* on a T-power act M we mean a monoid congruence as well as a T-act congruence on M.

Suppose that M is a T and S-power act for monoids T and S. We construct a quotient of M which is a (T,S)-bipower act. To do this, let θ be the power act congruence on M generated by the set $\theta = \{((m^t)^s, (m^s)^t) : m \in M, t \in T, s \in S\}$. Define $(m/\theta)(m'/\theta) = (mm')/\theta$, $(m/\theta)^t = m^t/\theta$ and $(m/\theta)^s = m^s/\theta$ for $m, m' \in M, t \in T, s \in S$. It is easily seen that M/θ is a (T,S)-bipower act. Hence, it follows from Remark 1 that M/θ is a $T \times S$-power act.

Lastly, we show that the power act is a universal algebraic structure and verify the existence of the free power acts. The reader is referred to [1] for some required details on universal algebra.

Let M be a T-power act. Then M can be considered as an algebra of the type $\langle \cdot, (\lambda_t)_{t\in T}, 1\rangle$, where $\cdot$ is the binary operation, λ_t is the unary operation given by $\lambda_t(m) = m^t$, for every $t \in T, m \in M$, and 1 is the nullary operation on M such that the following equations hold for every $t, s \in T$ and $x, y \in M$:

$$\lambda_t(x \cdot y) = \lambda_t(x) \cdot \lambda_t(y), \quad \lambda_s(\lambda_t(x)) = \lambda_{ts}(x), \quad \lambda_1(x) = x, \quad \lambda_t(1) = 1.$$

Therefore, the category T-**Pwr** is an equational class and then the free objects over T-acts exist in this category. We explain the construction of free T-power acts in the following.

Let A be a T-act. Consider the free monoid $Fm(A) = \{x_1x_2\cdots x_n : x_i \in A, n \in \mathbb{N}\} \cup \{1\}$ on the set A. Now we define a T-action on $Fm(A)$ by $(x_1\cdots x_n)^t = x_1^t \cdots x_n^t$, $1^t = 1$ for all $t \in T$ and $x_i \in A$, then one can easily see that $Fm(A)$ is a T-power act, and the inclusion map $i : A \to Fm(A)$ is a T-act homomorphism. If M is a T-power act and $f : A \to M$ is a T-act homomorphism, we define $\overline{f} : Fm(A) \to M$ to be $\overline{f}(x_1\cdots x_n) = f(x_1)\cdots f(x_n)$. Clearly, $\overline{f}$ is a T-power act homomorphism with $\overline{f}i = f$. Also if $g : Fm(A) \to M$ is a T-power act homomorphism with $gi = f$, then we have $g(x_1\cdots x_n) = g(x_1)\cdots g(x_n) = f(x_1)\cdots f(x_n) = \overline{f}(x_1\cdots x_n)$, for every $x_1, x_2, \ldots, x_n \in A$, that is, $\overline{f}$ is unique. Hence, $Fm(A)$ is a free monoid in the category T-**Act** on a T-act A. Then the assignment $A \rightsquigarrow Fm(A)$ defines the free functor $\mathbf{Fm} : T\text{-}\mathbf{Act} \to T\text{-}\mathbf{Pwr}$. It is worth noting that the composition of **Fm** to the free functor $\mathbf{F} : \mathbf{Set} \to T\text{-}\mathbf{Act}$, given by $X \rightsquigarrow X \times T$, gives the free functor $\mathbf{Fpwr} : \mathbf{Set} \to T\text{-}\mathbf{Pwr}$, $X \rightsquigarrow Fm(X \times T)$. Consequently, $Fm(X \times T)$ is the free T-power act on a set X.

3. Power acts over commutative monoids

This section is devoted to study T-power acts for which T is a commutative monoid. This kind of power acts displays a close relationship between Hom-functors and tensor functors.

For a T-power act M and a T-act A, let us denote $M^A := Hom_{T\text{-}\mathbf{Act}}(A, M)$, the set of all T-act homomorphisms from A to M where M is considered as a T-act. It is easily seen that the set M^A is a monoid under the operation $(f \cdot g)(a) := f(a)g(a)$, for every $f, g \in M^A, a \in A$. Note that the identity element of M^A is $1 : A \to M$ mapping every $a \in A$ to $1 \in M$. Now we get the following:

Lemma 1. *Let M be a T-power act and A be a T-act, where T is a commutative monoid. Then the monoid M^A is a T-power act together with the action $f^t(a) := (f(a))^t$, for every $f \in M^A, t \in T, a \in A$.*

Proof. Take any $f \in M^A$ and $t \in T$. First note that $f^t \in M^A$. Indeed, for every $t, s \in T, a \in A$, the commutativity of T implies that

$$f^t(as) = (f(as))^t = ((f(a))^s)^t = (f(a))^{st} = (f(a))^{ts} = ((f(a))^t)^s = (f^t(a))^s.$$

Moreover, for every $f, g \in M^A, t, s \in T$ and $a \in A$, we have:

1. $(f \cdot g)^t(a) = ((f \cdot g)(a))^t = (f(a)g(a))^t = (f(a))^t(g(a))^t = f^t(a)g^t(a) = (f^t \cdot g^t)(a)$.
2. $(f^t)^s(a) = (f^t(a))^s = ((f(a))^t)^s = f(a)^{ts} = f^{ts}(a)$.
3. $f^1(a) = (f(a))^1 = f(a)$.
4. $1^t(a) = (1(a))^t = 1^t = 1$.

This means that M^A is a T-power act. $\square$

We carry on this section with studying of the connections between the categories T-**Act** and T-**Pwr** for which T is a commutative monoid.

Proposition 2. *Let M be a T-power act on a commutative monoid T. The following assertions hold:*

(i) *There is a contravariant Hom-functor $M^- = Hom_{T\text{-}\mathbf{Act}}(-, M) : T\text{-}\mathbf{Act} \to T\text{-}\mathbf{Pwr}$ assigning each T-act A to M^A, and each T-act homomorphism $h : A \to B$*

to $M^h : M^B \to M^A$ mapping each $f \in M^B$ to $f \circ h$. Moreover, this yields a covariant Hom-functor $M^{M^-} = Hom_{T\text{-}\mathbf{Pwr}}(M^-, M) : T\text{-}\mathbf{Act} \to T\text{-}\mathbf{Pwr}$ in a natural way.

(ii) *The family of assignments $(\eta_A : A \to M^{M^A})_{A \in T\text{-}\mathbf{Act}}$ each of them assigning $a \mapsto \hat{a} : M^A \to M$, $\hat{a}(f) = f(a)$ for every $a \in A, f \in M^A$, constitutes a natural transformation from the identity functor $Id_{T\text{-}\mathbf{Act}}$ to the functor UM^{M^-} where $U : T\text{-}\mathbf{Act} \to T\text{-}\mathbf{Pwr}$ is the forgetful functor.*

Proof. (i) For every T-act A, $M^A \in T\text{-}\mathbf{Pwr}$ by Lemma 1. Considering a T-act homomorphism $h : A \to B$, we claim that M^h is a T-power act homomorphism. Clearly, M^h is a monoid homomorphism. Let $t \in T$ and $f \in M^B$. Then $M^h(f^t)(a) = (f^t \circ h)(a) = f^t(h(a)) = f(h(a)t) = f(h(at)) = (M^h(f))(at) = (M^h(f))^t(a)$, for every $a \in A$. So $M^h(f^t) = (M^h(f))^t$, as desired. Assume that $h : A \to B$ and $k : B \to C$ are homomorphisms in T-**Act** and $f \in M^C$. It follows that $M^{k \circ h}(f) = f \circ (k \circ h) = (f \circ k) \circ h = M^h(M^k(f)) = (M^h \circ M^k)(f)$. That is, $M^{k \circ h} = M^h \circ M^k$. Also clearly $M^{id_A} = id_{M^A}$. Therefore, M^- is a contravariant functor. For the second part, it suffices to note that $M^{M^-} = M^- \circ U \circ M^-$ where $U : T\text{-}\mathbf{Pwr} \to T\text{-}\mathbf{Act}$ is the forgetful functor.

(ii) First we show that the map $\hat{a} : M^A \to M$ is a morphism in T-**Pwr**, for each a in a T-act A. Let $f, g \in M^A$ and $t \in T$. Then $\hat{a}(f \cdot g) = (f \cdot g)(a) = f(a)g(a) = \hat{a}(f)\hat{a}(g)$, and $\hat{a}(f^t) = f^t(a) = (f(a))^t = (\hat{a}(f))^t$. Moreover, each η_A is a morphism in T-**Act** because $\widehat{at}(f) = f(at) = f^t(a) = \widehat{a}(f^t) = (\widehat{a})^t(f)$ for all $a \in A, t \in T, f \in M^A$. Hence, $\eta_A(at) = (\eta_A(a))^t$. It remains to prove the commutativity of the following diagram:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & M^{M^A} \arrow[d, "M^{M^f}"] \\ B \arrow[r, "\eta_B"'] & M^{M^B} \end{tikzcd}$$

Let $a \in A$, $\beta \in M^B$. We have $M^{M^f} \circ \eta_A(a)(\beta) = (\hat{a} \circ M^f)(\beta) = \hat{a}(\beta \circ f) = (\beta \circ f)(a) = \beta(f(a)) = \widehat{f(a)}(\beta) = \eta_B \circ f(a)(\beta)$, as required. $\square$

Remark 2. (i) Let Γ be a subclass of morphisms in T-**Act** and M be a T-power act for a commutative monoid T. Then one can easily check that M is a Γ-injective object in T-**Act**, i.e. injective with respect to all Γ-morphisms, if and only if the contravariant functor M^- maps every Γ-morphism to an onto morphism in T-**Pwr**.

(ii) Let $\mathcal{C}$ be the category of all contravariant functors from T-**Act** to T-**Pwr** for a commutative monoid T, and natural transformations between them. Then the assignment $M \rightsquigarrow M^-$ gives a covariant functor $T\text{-}\mathbf{Pwr} \to \mathcal{C}$. More explicitly, for every morphism $\alpha : M \to N$ in T-**Pwr**, one can define a natural transformation $\widehat{\alpha} = (\widehat{\alpha}_A)_{A \in T\text{-}\mathbf{Act}} : M^- \to N^-$ to be $\widehat{\alpha}_A(f) = \alpha \circ f$, for all $f \in M^A$. That is, for every T-act homomorphism $h : A \to B$, the following diagram commutes:

$$\begin{tikzcd} M^A \arrow[r, "\widehat{\alpha}_A"] & N^A \\ M^B \arrow[u, "M^h"] \arrow[r, "\widehat{\alpha}_B"'] & N^B \arrow[u, "N^h"'] \end{tikzcd}$$

Indeed, $N^h \circ \widehat{\alpha}_B(f) = N^h(\alpha \circ f) = (\alpha \circ f) \circ h = \alpha \circ (f \circ h) = \widehat{\alpha}_A(f \circ h) = \widehat{\alpha}_A \circ M^h(f)$, for every $f \in M^B$.

At the end, we give the following theorem which shows the relationship between Hom-functors and tensor functors.

Theorem 1. *For a T-power act M on a commutative monoid T, the Hom-functor $(M^-)^- : T$-**Act** $\times$ T-**Act** $\to T$-**Pwr** is naturally equivalent to the tensor functor $M^{-\otimes -} : T$-**Act** $\times$ T-**Act** $\to T$-**Pwr**.*

Proof. For every T-acts A and B, we define $\phi = \phi_{A,B} : M^{A \otimes B} \to (M^A)^B$ mapping each T-power act homomorphism $f : A \otimes B \to M$ to $\phi(f) : B \to M^A$, where $\phi(f)(b) : A \to M$, for every $b \in B$, maps every $a \in A$ to $f(a \otimes b)$. It follows from [6, Corollary II.5.20] that ϕ is a T-act isomorphism. Moreover, it is clear that ϕ is a monoid homomorphism. Hence, ϕ is an isomorphism in T-**Pwr**. It remains to prove the naturality of $(\phi_{A,B})_{A,B} : M^{-\otimes -} \to (M^-)^-$. Consider any T-act homomorphisms $f : A \to A'$ and $g : B \to B'$. We show that the following diagram commutes:

$$\begin{tikzcd} M^{A\otimes B} \arrow[r, "\phi_{A,B}"] & (M^A)^B \\ M^{A'\otimes B'} \arrow[u, "M^{f\otimes g}"] \arrow[r, "\phi_{A',B'}"] & (M^{A'})^{B'} \arrow[u, "(M^f)^g"'] \end{tikzcd}$$

Indeed, for every $a \in A$ and $b \in B$, we have

$$\begin{aligned} ((\phi_{A,B} \circ M^{f\otimes g})(\alpha))(b)(a) &= \phi_{A,B}(M^{f\otimes g}(\alpha))(b)(a) = M^{f\otimes g}(\alpha)(a \otimes b) \\ &= (\alpha \circ (f \otimes g))(a \otimes b) \\ &= \alpha(f(a) \otimes g(b)). \end{aligned}$$

On the other hand,

$$\begin{aligned} (((M^f)^g \circ \phi_{A',B'})(\alpha))(b)(a) &= (M^f)^g(\phi_{A',B'}(\alpha))(b)(a) \\ &= (M^f \circ \phi_{A',B'}(\alpha) \circ g)(b)(a) \\ &= M^f(\phi_{A',B'}(\alpha)(g(b)))(a) \\ &= (\phi_{A',B'}(\alpha)(g(b)) \circ f)(a) \\ &= \phi_{A',B'}(\alpha)(g(b))(f(a)) \\ &= \alpha(f(a) \otimes g(b)). \end{aligned}$$

Hence, $\phi_{A,B} \circ M^{f\otimes g} = (M^f)^g \circ \phi_{A',B'}$. $\square$

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References

[1] **S. Burris, H.P. Sankappanavar**, *A Course in Universal Algebra*, Springer-Verlag, 1981.

[2] **M.M. Ebrahimi**, *Algebra in a Grothendieck topos: injectivity in quasi-equational classes*, J. Pure Appl. Algebra **26** (1982), no. 3, 269 – 280.

[3] **M.M. Ebrahimi, M. Mahmoudi**, *The category of M-sets*, Italian J. Pure Appl. Math. **9** (2001), 123 – 132.

[4] **P.T. Johnstone**, *Sketches of an Elephant: A Topos Theory Compendium*, Vol. 1,2, Oxford: Clarendon Press, 2002.

[5] **A. Karimi Feizabadi, H. Rasouli**, *Actions over monoids and hypergroups*, Quasigroups Related Systems **23** (2015), 257 – 262.

[6] **M. Kilp, U. Knauer, A.V. Mikhalev**, *Monoids, Acts and Categories*, Walter de Gruyter, Berlin, New York, 2000.

[7] **S. MacLane**, *Categories for the working mathematicians*, Graduate Texts in Mathematics 5, Springer-Verlag, 1971.

[8] **S. Mac Lane, I. Moerdijk**, *Sheaves in Geometry and Logic*, Springer-Verlag, 1992.

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On left strongly simple ordered hypersemigroups

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Abstract. We present a structure theorem referring to the decomposition of ordered hypersemigroups into left strongly simple components, that is, into subhypersemigroups which are both simple and left quasi-regular. We prove that an ordered hypersemigroup is a semilattice of left strongly simple hypersemigroups if and only if it is a complete semilattice of left strongly simple hypersemigroups and we characterize this type of hypersemigroups in terms of intra-regular and semisimple hypersemigroups. We also characterize the chains of left strongly simple ordered hypersemigroups.

1. Introduction and prerequisites

The concept of the hypergroup introduced by the French Mathematician F. Marty at the 8th Congress of Scandinavian Mathematicians in 1933 is as follows: An hypergroup is a nonempty set H endowed with a multiplication xy such that (i) $xy \subseteq H$; (ii) $x(yz) = (xy)z$; (iii) $xH = Hx = H$ for every x, y, z in H (cf. [9]). Hundreds of papers appeared on hyperstructures since Marty introduced this concept, and in the recent years, many groups in the world investigate the hypersemigroups in research programs using the definition given by Marty. Being impossible to give a complete information regarding the bibliography, we will refer only some recent books and articles such as the [1–7, 9–11].

The present paper deals with the decomposition of ordered hypersemigroups into their hypersemigroups which are left strongly simple, that is, both simple and left quasi-regular. In this respect, we characterize the ordered hypersemigroups which are semilattices of left strongly simple hypersemigroups. We prove that for ordered hypersemigroups, the concepts of semilattices of left strongly simple hypersemigroups and complete semilattices of left strongly simple hypersemigroups are the same. Moreover, we prove that an ordered hypersemigroup S is a semilattice of left strongly simple hypersemigroups if and only if it is a union of left strongly simple hypersubsemigroups. We show that an ordered hypersemigroup S is a semilattice of left strongly simple hypersemigroups if and only if every left hyperideal of S is an intra-regular hypersubsemigroup or a semisimple hypersubsemigroup of S. This type of ordered hypersemigroups are the ordered hypersemigroups in which $a \in (S \circ a^2 \circ S \circ a]$ for every $a \in S$. Finally, we prove that the chains and the complete chains of left strongly simple ordered hypersemigroups coincide and they

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are characterized as the ordered hypersemigroups in which, for every $a, b \in S$, we have $a \in (S \circ a \circ b \circ S \circ a]$ or $b \in (S \circ a \circ b \circ S \circ b]$. The corresponding results for hypersemigroups (without order) can be also obtained as application of the results of this paper, and this is because every hypersemigroup endowed with the equality relation is an ordered hypersemigroup. Left strongly simple semigroups (without order) have been considered in [8].

Let $(S, \circ, \leq)$ be an ordered hypersemigroup. For a hypersubsemigroup T of S and a subset H of T, we denote by $(H]_T$ the subset of T defined by

$$(H]_T := \{t \in T \mid t \leq h \text{ for some } h \in H\}.$$

In particular, for $T = S$, we write $(H]$ instead of $(H]_S$. So, for $H \subseteq S$, we have

$$(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

A nonempty subset A of S is called a *left* (resp. *right*) hyperideal of S if (1) $S \circ A \subseteq A$ (resp. $A \circ S \subseteq A$) and (2) if $a \in A$ and $b \in S$, $b \leq a$, then $b \in A$. A is called a *hyperideal* of S if it is both a left and a right hyperideal of S. We denote by $L(a)$ (resp. $R(a)$) the left (resp. right) hyperideal of S generated by a, and by $I(a)$ the hyperideal of S generated by a ($a \in S$). We have $L(a) = (a \cup S \circ a]$, $R(a) = (a \cup a \circ S]$ and $I(a) = (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S]$ for every $a \in S$. A left (resp. right) hyperideal A of S is clearly a hypersubsemigroup of S i.e. $A \circ A \subseteq A$. S is called *simple* if for every hyperideal T of S, we have $T = S$. A hypersubsemigroup L of S is called *intra-regular* if for each $a \in L$ there exist $x, y \in L$ such that $a \leq x \circ a^2 \circ y$, equivalently if $a \in (L \circ a^2 \circ L]_L$ for every $a \in L$ or $A \subseteq (L \circ A^2 \circ L]_L$ for every nonempty subset A of L. An equivalence relation σ on S is called *congruence* if $(a, b) \in \sigma$ implies $(a \circ c, b \circ c) \in \sigma$ and $(c \circ a, c \circ b) \in \sigma$ for every $c \in S$, in the sense that for every $x \in a \circ c$ and every $y \in b \circ c$ we have $(x, y) \in \sigma$ and for every $x \in c \circ a$ and every $y \in c \circ b$, we have $(x, y) \in \sigma$. A congruence σ on S is called *semilattice congruence* if, for every $a, b \in S$, we have $(a^2, a) \in \sigma$ meaning that $x \in a \circ a$ implies $(x, a) \in \sigma$ and $(a \circ b, b \circ a) \in \sigma$ in the sense that if $x \in a \circ b$ and $y \in b \circ a$, then $(x, y) \in \sigma$. If σ is a semilattice congruence on S, then the σ-class $(x)_\sigma$ of S containing x is a hypersubsemigroup of S for every $x \in S$. A semilattice congruence σ on S is called *complete* if $a \leq b$ implies $(a, a \circ b) \in \sigma$, that is, if $x \in a \circ b$, then $(a, x) \in \sigma$. Recall that if σ is a complete semilattice congruence on S then, the relation $a \leq a$ implies $(a^2, a) \in \sigma$, so the complete semilattice congruences on S can be also defined as the congruences on S such that $(a \circ b, b \circ a) \in \sigma$ and $a \leq b$ implies $(a, a \circ b) \in \sigma$ for every $a, b \in S$. A hypersubsemigroup F of S is called a *hyperfilter* of S if (1) for any $a, b \in S$, $(a \circ b) \cap A \neq \emptyset$ implies $a, b \in F$ and (2) $a \in F$ and $S \ni b \geq a$ implies $b \in A$. We denote by $\mathcal{N}$ the relation on S defined by $\mathcal{N} := \{(x, y) \mid N(x) = N(y)\}$ where $N(a)$ denotes the hyperfilter of S generated by a ($a \in S$). The relation $\mathcal{N}$ is the least complete semilattice congruence on S. We say that S is a *semilattice of left strongly simple hypersemigroups* (resp. *complete semilattice of left strongly simple hypersemigroups*) if there exists a semilattice congruence (resp. complete semilattice congruence) σ on S such that the σ-class $(x)_\sigma$ of S containing x is a left strongly simple hypersubsemigroup of S for every $x \in S$. An equivalent definition is the following: The ordered hypersemigroup S is a semilattice of left strongly simple hypersemigroups if there exists a semilattice Y and a nonempty family $\{S_\alpha \mid \alpha \in Y\}$ of left strongly simple hypersubsemigroups of S such that

(1) $S_\alpha \cap S_\beta = \emptyset$ for every $\alpha, \beta \in Y$, $\alpha \neq \beta$

(2) $S = \bigcup\limits_{\alpha \in Y} S_\alpha$

(3) $S_\alpha \circ S_\beta \subseteq S_{\alpha\beta}$ for every $\alpha, \beta \in Y$.

In ordered hypersemigroups, the semilattice congruences are defined exactly as in hypersemigroups (without order) so the two definitions are equivalent. An ordered hypersemigroup S is a complete semilattice of left simple hypersemigroups if and only if in addition to (1), (2) and (3) above, we have the following:

(4) $S_\beta \cap (S_\alpha] \neq \emptyset$ implies $\beta = \alpha\beta$.

We say that S is a *chain* (resp. *complete chain*) *of left strongly simple hypersemigroups* if there exists a semilattice congruence (resp. complete semilattice congruence) σ on S such that the σ-class $(x)_\sigma$ of S containing x is a left strongly simple hypersubsemigroup of S for every $x \in S$, and the set S/σ of (all) $(x)_\sigma$-classes of S endowed with the relation $(x)_\sigma \preceq (y)_\sigma \iff (x)_\sigma = (x \circ y)_\sigma$ is a chain.

2. Main results

Definition 2.1. A hypersubsemigroup L of an ordered hypersemigroup S is called *left* (resp. *right*) *quasi-regular* if $a \in (L \circ a \circ L \circ a]_L$ (resp. $a \in (a \circ L \circ a \circ L]_L$) for every $a \in L$.

Definition 2.2. An ordered hypersemigroup S is called *left* (resp. *right*) *strongly simple* if it is simple and left (resp. right) quasi-regular.

Definition 2.3. A hypersubsemigroup L of an ordered hypersemigroup S is called *semisimple* if $a \in (L \circ a \circ L \circ a \circ L]_L$ for every $a \in L$.

It might be noted that an ordered hypersemigroup S is semisimple if and only if the hyperideals of S are idempotent, that is, for every hyperideal A of S, we have $(A \circ A] = A$.

Lemma 2.4. *An ordered hypersemigroup S is simple if and only if $(S \circ a \circ S] = S$ for every $a \in S$.*

Proof. $(\Rightarrow)$. For an element a of S, the set $(S \circ a \circ S]$ is an hyperideal of S. Indeed, the set $(S \circ a \circ S]$ is a nonempty subset of S, $S \circ (S \circ a \circ S] = (S] \circ (S \circ a \circ S] \subseteq (S^2 \circ a \circ S] \subseteq (S \circ a \circ S]$, $(S \circ a \circ S]S \subseteq (S \circ a \circ S]$, and $((S \circ a \circ S]] = (S \circ a \circ S]$. Since S is simple, we have $(S \circ a \circ S] = S$.

($\Leftarrow$). Let T be an hyperideal of S. We get an arbitrary element b of T (such an element exists since T is nonempty). Then $S \circ b \circ S \subseteq S \circ T \circ S \subseteq T$, so $(S \circ b \circ S] \subseteq (T] = T$. On the other hand, by hypothesis, we have $(S \circ b \circ S] = S$. Thus we have $S \subseteq T$, and $T = S$. □

Lemma 2.5. *An ordered hypersemigroup S is left strongly simple if and only if $a \in (S \circ b \circ S \circ a]$ for every $a, b \in S$.*

Proof. ($\Rightarrow$). Let $a, b \in S$. Since S is simple, by Lemma 2.4, we have $(S \circ b \circ S] = S$, then $a \in (S \circ b \circ S]$. On the other hand, since S is left quasi-regular, we have $a \in (S \circ a \circ S \circ a]$. Thus we get

$$\begin{aligned} a \in (S \circ a \circ S \circ a] &\subseteq (S \circ (S \circ b \circ S] \circ S \circ a] \\ &= (S \circ (S \circ b \circ S) \circ S \circ a] \\ &\subseteq (S \circ b \circ S \circ a]. \end{aligned}$$

($\Leftarrow$). If $a \in S$, by hypothesis, we have $a \in (S \circ a \circ S \circ a]$, so S is left quasi-regular. If $a, b \in S$, by hypothesis, we have

$$\begin{aligned} a \in (S \circ b \circ S \circ a] &\subseteq (S \circ b \circ S \circ (S \circ b \circ S \circ a]] \\ &= (S \circ b \circ S \circ (S \circ b \circ S \circ a)] \\ &\subseteq (S \circ b \circ S], \end{aligned}$$

thus we have $S \subseteq (S \circ b \circ S]$ and $(S \circ b \circ S] = S$, and so S is simple. □

Lemma 2.6. *If S is an intra-regular ordered hypersemigroup, then for the complete semilattice congruence $\mathcal{N}$ on S, the class $(x)_{\mathcal{N}}$ is a simple hypersubsemigroup of S for every $x \in S$.*

Which means that the intra-regular ordered hypersemigroups are complete semilattices of simple hypersemigroups.

Theorem 2.7. *Let $(S, \circ, \leq)$ be an ordered hypersemigroup and σ a complete semilattice congruence on S. Then S is left quasi-regular if and only if $(a)_\sigma$ is a left quasi-regular hypersubsemigroup of S for every $a \in S$.*

Proof. ($\Rightarrow$). Let $b \in (a)_\sigma$. Then there exist elements $u, v \in (a)_\sigma$ such that $b \leq u \circ b \circ v \circ b$. In fact: Since $b \in S$ and S is left quasi-regular, $b \leq s \circ b \circ t \circ b$ for some $s, t \in S$. Then we have

$$\begin{aligned} b &\leq s \circ b \circ t \circ (s \circ b \circ t \circ b) \leq s \circ b \circ t \circ s \circ b \circ t \circ (s \circ b \circ t \circ b) \\ &= (s \circ b \circ t \circ s) \circ b \circ (t \circ s \circ b \circ t) \circ b. \end{aligned}$$

Moreover we have $s \circ b \circ t \circ s, t \circ s \circ b \circ t \in (a)_\sigma$. In fact, since $b \leq s \circ b \circ t \circ b$ and σ is a complete semilattice congruence on S, we have $(b, b \circ s \circ b \circ t \circ b) \in \sigma$, then $(b, s \circ b \circ t \circ b) \in \sigma$. Since $(a, b) \in \sigma$, we have $(a, s \circ b \circ t \circ b) \in \sigma$. Since $(t \circ b, b \circ t) \in \sigma$, we have $(s \circ b \circ t \circ b, s \circ b^2 \circ t) \in \sigma$, then $(a, s \circ b \circ t) \in \sigma$, $(a, s \circ b \circ t \circ s) \in \sigma$, and $s \circ b \circ t \circ s \in (a)_\sigma$. Moreover, since $(a, s \circ b \circ t) \in \sigma$, we have $(a, s \circ b \circ t^2) \in \sigma$, $(a, t \circ s \circ b \circ t) \in \sigma$, and $t \circ s \circ b \circ t \in (a)_\sigma$.

($\Leftarrow$). Let $a \in S$. Since $(a)_\sigma$ is left quasi-regular, we have

$$a \in ((a)_\sigma \circ a \circ (a)_\sigma \circ a]_{(a)_\sigma} \subseteq (S \circ a \circ S \circ a],$$

so S is left quasi-regular. □

Theorem 2.8. *Let $(S, \circ, \leq)$ be an ordered hypersemigroup. The following are equivalent:*

(1) *S is a complete semilattice of left strongly simple hypersemigroups.*

(2) *S is a semilattice of left strongly simple hypersemigroups.*

(3) *S is a union of left strongly simple hypersubsemigroups of S.*

(4) *$a \in (S \circ a^2 \circ S \circ a]$ for every $a \in S$.*

(5) *Every left hyperideal of S is an intra-regular hypersubsemigroup of S.*

(6) *Every left hyperideal of S is a semisimple hypersubsemigroup of S.*

Proof. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). Let S be the union of the left strongly simple hypersubsemigroups S_α, $\alpha \in Y$, and let $a \in S$. Suppose $a \in S_\alpha$ for some $\alpha \in Y$. Since S_α is a left strongly simple hypersemigroup and $a, a^2 \in S_\alpha$, by Lemma 2.5, we have

$$a \in (S_\alpha \circ a^2 \circ S_\alpha \circ a]_{S_\alpha} \subseteq (S \circ a^2 \circ S \circ a].$$

(4) $\Rightarrow$ (5). Let L be a left hyperideal of S and $a \in L$. Since $a, a^2 \in S$, by (4), we have

$$\begin{aligned} a \in (S \circ a^2 \circ S \circ a] &\subseteq (S \circ (S \circ a^4 \circ S \circ a^2] \circ S \circ a] \\ &= (S \circ (S \circ a^4 \circ S \circ a^2) \circ S \circ a] \\ &\subseteq ((S \circ a^2) \circ a^2 \circ (S \circ a^2 \circ S \circ a)]. \end{aligned}$$

Since $a^2 \in L$, we have $S \circ a^2 \subseteq SL \subseteq L$ and $S \circ a^2 \circ S \circ a \subseteq S \circ L \subseteq L$. Thus we have $a \in (L \circ a^2 \circ L] = (L \circ a^2 \circ L]_L$, and L is intra-regular.

(5) $\Rightarrow$ (6). Let L be a left hyperideal of S and $a \in L$. Since L is intra-regular, we have

$$\begin{aligned} a \in (L \circ a^2 \circ L]_L &\subseteq (L \circ a \circ (L \circ a^2 \circ L]_L \circ L]_L \\ &= (L \circ a \circ (L \circ a^2 \circ L) \circ L]_L \\ &\subseteq (L \circ a \circ L \circ a \circ (S \circ L)]_L \\ &\subseteq (L \circ a \circ L \circ a \circ L]_L, \end{aligned}$$

and L is semisimple.

(6) $\Rightarrow$ (1). Let $a \in S$. By (6), $L(a)$ is a semisimple hypersubsemigroup of S i.e. $x \in (L(a) \circ x \circ L(a) \circ x \circ L(a)]_{L(a)} = (L(a) \circ x \circ L(a) \circ x \circ L(a)]$ for every $x \in L(a)$. Thus we have

$$\begin{aligned} a &\in (L(a) \circ a \circ L(a) \circ a \circ L(a)] \\ &= ((a \cup S \circ a] \circ a \circ (a \cup S \circ a] \circ a \circ (a \cup S \circ a]] \\ &= ((a \cup S \circ a) \circ a \circ (a \cup S \circ a) \circ a \circ (a \cup S \circ a)] \\ &= (a^2 \circ S \circ a^3 \cup S \circ a^2 \circ S \circ a]. \end{aligned}$$

Then

$$a^2 \in (a^2 \circ S \circ a^3 \cup S \circ a^2 \circ S \circ a] \circ (a] \subseteq (a^2 \circ S \circ a^4 \cup S \circ a^2 \circ S \circ a^2],$$

and

$$\begin{aligned} a^2 \circ S \circ a^3 &\subseteq (a^2 \circ S \circ a^4 \cup S \circ a^2 \circ S \circ a^2] \circ (S \circ a^3] \\ &\subseteq ((a^2 \circ S \circ a^4 \cup S \circ a^2 \circ S \circ a^2) \circ (S \circ a^3)] \\ &= (a^2 \circ S \circ a^4 \circ S \circ a^3 \cup S \circ a^2 \circ S \circ a^2 \circ S \circ a^3] \\ &\subseteq (S \circ a^2 \circ S \circ a]. \end{aligned}$$

Thus we have $a \in ((S \circ a^2 \circ S \circ a] \cup S \circ a^2 \circ S \circ a] = ((S \circ a^2 \circ S \circ a]] = (S \circ a^2 \circ S \circ a]$. Since $a \in (S \circ a^2 \circ S \circ a] \subseteq (S \circ a^2 \circ S], (S \circ a \circ S \circ a]$ for every $a \in S$, S is both intra-regular and left quasi-regular. Since S is intra-regular, by Lemma 2.6, $(x)_{\mathcal{N}}$ is a simple hypersubsemigroup of S for every $x \in S$. Since S is left quasi-regular and $\mathcal{N}$ a complete semilattice congruence of S, by Theorem 2.7, $(x)_{\mathcal{N}}$ is a left quasi-regular hypersubsemigroup of S for every $x \in S$. Since $\mathcal{N}$ is a complete semilattice congruence on S and $(x)_{\mathcal{N}}$ a left strongly simple hypersubsemigroup of S for every $x \in S$, S is a complete semilattice of left strongly simple hypersemigroups. $\square$

Theorem 2.9. *An ordered hypersemigroup S is a chain of left strongly simple hypersemigroups if and only if, for every $a, b \in S$, we have*

$$a \in (S \circ a \circ b \circ S \circ a] \text{ or } b \in (S \circ a \circ b \circ S \circ b].$$

Proof. ($\Rightarrow$). Suppose σ is a semilattice congruence on S such that $(x)_\sigma$ is a left strongly simple hypersubsemigroup of S for every $x \in S$ and the set S/σ endowed with the relation

$$(x)_\sigma \preceq (y)_\sigma \Longleftrightarrow (x)_\sigma = (x \circ y)_\sigma$$

is a chain. Let now $a, b \in S$. Since $(S/\sigma, \preceq)$ is a chain, we have $(a)_\sigma \preceq (b)_\sigma$ or $(b)_\sigma \preceq (a)_\sigma$. Let $(a)_\sigma \preceq (b)_\sigma$. Then $(a)_\sigma = (a \circ b)_\sigma$ and $\{a\}, a \circ b \subseteq (a)_\sigma$. Since $(a)_\sigma$ is a left strongly simple hypersemigroup, by Lemma 2.5, we have $a \in ((a)_\sigma \circ a \circ b \circ (a)_\sigma \circ a]_{(a)_\sigma} \subseteq (S \circ a \circ b \circ S \circ a]$. If $(b)_\sigma \preceq (a)_\sigma$, similarly we obtain $b \in (S \circ a \circ b \circ S \circ b]$.

($\Leftarrow$). Let $a \in S$. By hypothesis, we have $a \in (S \circ a^2 \circ S \circ a]$. For the semilattice congruence $\mathcal{N}$, the $\mathcal{N}$-class $(x)_{\mathcal{N}}$ is a left strongly simple hypersubsemigroup of S for every $x \in S$ (cf. the proof of (6) $\Rightarrow$ (1) in Theorem 2.8). Let now $(x)_{\mathcal{N}}, (y)_{\mathcal{N}} \in S/\mathcal{N}$. By hypothesis, we have $x \in (S \circ x \circ y \circ S \circ x]$ or $y \in (S \circ x \circ y \circ S \circ y]$. Let $x \in (S \circ x \circ y \circ S \circ x]$. Since $x \in N(x)$ and $x \le t \circ x \circ y \circ h \circ x$ for some $t, h \in S$, we have $x \circ y \subseteq N(x)$, then $N(x \circ y) \subseteq N(x)$. Let $y \in (S \circ x \circ y \circ S \circ y]$. Since $y \in N(y)$ and $y \le z \circ x \circ y \circ k \circ y$ for some $z, k \in S$, we have $x \circ y \subseteq N(y)$, so $N(x \circ y) \subseteq N(y)$. On the other hand, $x \circ y \subseteq N(x \circ y)$ implies $x, y \in N(x \circ y)$, then $N(x) \subseteq N(x \circ y)$ and $N(y) \subseteq N(x \circ y)$. Hence we have $N(x \circ y) = N(x)$ or $N(x \circ y) = N(y)$. Thus $(x)_{\mathcal{N}} = (x \circ y)_{\mathcal{N}}$ or $(y)_{\mathcal{N}} = (x \circ y)_{\mathcal{N}} = (y \circ x)_{\mathcal{N}}$, that is, $(x)_{\mathcal{N}} \preceq (y)_{\mathcal{N}}$ or $(y)_{\mathcal{N}} \preceq (x)_{\mathcal{N}}$. $\square$

Remark. An ordered hypersemigroup is a chain of left strongly simple hypersemigroups if and only if it is a complete chain of left strongly simple hypersemigroups.

Let us finish with the following examples which correspond to the definitions 2.1–2.3.

Example 2.10. We consider the ordered hypersemigroup $S = \{a, b, c, d, f\}$ defined by the hyperoperation given in the table and the order below.

$\circ$	a	b	c	d	f
a	$\{a, f\}$	$\{b, f\}$	$\{c, d, f\}$	$\{d\}$	$\{f\}$
b	$\{b, f\}$	$\{a, f\}$	$\{c, d, f\}$	$\{d\}$	$\{f\}$
c	$\{c, d, f\}$	$\{c, d, f\}$	$\{c, d, f\}$	$\{c, d, f\}$	$\{c, d, f\}$
d	$\{c, d, f\}$	$\{c, d, f\}$	$\{c, d, f\}$	$\{c, d, f\}$	$\{c, d, f\}$
f	$\{f\}$	$\{f\}$	$\{c, d, f\}$	$\{d\}$	$\{f\}$

$$\le := \{(a, a), (b, b), (c, c), (d, c), (d, d), (f, a), (f, b), (f, c), (f, f)\}.$$

The covering relation of S is the following:

$$\prec = \{(d, c), (f, a), (f, b), (f, c)\}.$$

This is a left quasi-regular ordered hypersemigroup. As the left quasi-regular ordered hypersemigroups are also semisimple, this is an example of an ordered semisimple hypersemigroups as well. It is not simple as $(S \circ c \circ S] \neq S$.

Example 2.11. The ordered hypersemigroup defined by the hyperoperation and the covering relation below is left quasi-regular (also right quasi-regular) and simple.

$\circ$	a	b	c	d	e
a	$\{a\}$	$\{a\}$	$\{a, b, c\}$	$\{a\}$	$\{a, b, c\}$
b	$\{a\}$	$\{a\}$	$\{a, b, c\}$	$\{a\}$	$\{a, b, c\}$
c	$\{a\}$	$\{a\}$	$\{a, b, c\}$	$\{a\}$	$\{a, b, c\}$
d	$\{a, b, d\}$	$\{a, b, d\}$	S	$\{a, b, d\}$	S
e	$\{a, b, d\}$	$\{a, b, d\}$	S	$\{a, b, d\}$	S

$$\prec = \{(a,b),(b,c),(b,d),(c,e),(d,e)\}.$$

We wrote this paper in the usual way, and we will come back to this paper in a forthcoming paper.

References

[1] **T. Changphas, B. Davvaz**, *Properties of hyperideals in ordered semihypergroups*, Italian J. Pure Appl. Math. **33** (2014), 425 – 432.

[2] **P. Corsini**, *Prolegomena of Hypergroup Theory*, Supplement to Riv. Mat. Pura Appl. Aviani Editore, Tricesimo, 1993.

[3] **P. Corsini, V. Leoreanu**, *Applications of hyperstructure theory*, Advances in Mathematics (Dordrecht), 5. Kluwer Academic Publishers, 2003.

[4] **B. Davvaz**, *Semihypergroup Theory*, Elsevier, 2016.

[5] **B. Davvaz, A. Dehghan-Nezhad**, *Chemical examples in hypergroups*, Ratio Mat. **14** (2003), 71 – 74.

[6] **K. Hila, B. Davvaz, K. Naka**, *On quasi-hyperideals in semihypergroups*, Comm. Algebra **39** (2011), 4183 – 4194.

[7] **S. Lekkoksung**, *On weakly semiprime hyperideals in semihypergroups*, Int. J. Algebra **6** (2012), 613 – 616.

[8] **M. S. Mitrović**, *Semilattices of Archimedean Semigroups*, University of Nis, 2003.

[9] **J. Mittas**, *Hypergroupes canoniques valués et hypervalués*, Math. Balkanica **1** (1971), 181 – 185.

[10] **B. Pibaljommee, K. Wannatong, B. Davvaz**, *An investigation on fuzzy hyperideals of ordered semihypergroups*, Quasigroups Related Systems **23** (2015), 297 – 308.

[11] **T. Vougiouklis**, *Hyperstructures and their Representations*, Hadronic Press, 1994.

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A note on semisymmetry

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Abstract. J.D.H. Smith showed how to replace homotopies between quasigroups by homomorphism between semisymmetric quasigroups. This is a semisymmetrization and it replaces a quasigroup by a semisymmetric structure defined on its Cartesian cube. The reason for a semisymmetrization is that homomorphisms behave more regularly than homotopies.

A thorough survey of properties of Smith's semisymmetrization is given in this paper. Also, new semisymmetrizations, which replace a quasigroup by semisymmetric structures defined on its Cartesian square are suggested.

1. Introduction

For a plausible category of quasigroups, it seems that homotopies between quasigroups, taken as morphisms, are better choice than homomorphisms (see [3] and [9]). However, homomorphisms are sometimes easier to work with. For example, isotopies (bijective homotopies) do not preserve units – every quasigroup is isotopic to a loop (quasigroup with a unit) but is not necessarily a loop itself. This note is about turning homotopies into homomorphisms.

Smith, [6], proved that there is an adjunction from the category of semisymmetric quasigroups with homomorphisms to the category of quasigroups with homotopies. Also, he proved in [6] that the latter category is isomorphic to a subcategory of the former category, and in [7], that every T algebra, for T being the monad defined by the above adjunction, is isomorphic to the image of a semisymmetric quasigroup under the comparison functor.

These results, especially the embedding of the category of quasigroups with homotopies into the category of semisymmetric quasigroups with homomorphisms, could be of interest to a working universal algebraist. Our intention is to make them more accessible to such a reader and to indicate a possible misusing. Also, we give a proof that the comparison functor is full, which completes the proof of monadicity of the adjunction.

At the end of the paper, we show that there is a more economical way to embed the category of quasigroups with homotopies into the category of semisymmetric quasigroups with homomorphisms. One could get an impression, due to [6], that for such an embedding it is necessary to have a semisymmetrization functor that

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is a right adjoint in an adjunction. If one is interested just in this embedding and not in reflectivity (see the end of Section 4), then this new semisymmetrization suits as any other.

We assume that the reader is familiar with the notions of category, functor and natural transformation. If not, we suggest to consult [5] for these notions. All other relevant notions from Category theory are introduced at the appropriate places in the text.

2. Quasigroups

We start by recapitulating a few basic facts about quasigroups.

One way to define a *quasigroup* is that it is a grupoid $(Q;\cdot)$ satisfying:

$$\forall ab\exists_1 x\ \ (x\cdot a=b) \qquad \text{and} \qquad \forall ab\exists_1 x\ \ (a\cdot x=b)$$

Uniqueness of the solution of the equation $x\cdot a=b$ ($a\cdot x=b$) enables one to define right (left) division operation $x=b/a$ ($x=a\backslash b$) which is also a quasigroup (short for: $(Q;/)$ is a quasigroup). We can define three more operations:

$$x*y=y\cdot x \qquad\qquad x/\!\!/y=y/x \qquad\qquad x\backslash\!\!\backslash y=y\backslash x$$

dual to $\cdot, /, \backslash$ respectively. They are also quasigroups. The six operations $\cdot, /, \backslash, *, /\!\!/$ and $\backslash\!\!\backslash$ are *parastrophes* of $\cdot$ (and of each other).

A function $f:Q\to R$ between the base sets of quasigroups $(Q;\cdot)$ and $(R,\cdot)$ is a *homomorphism* iff:

$$f(x)\cdot f(y)=f(x\cdot y)$$

and *isomorphism* if f is a bijection as well.

A triple $\bar f=(f_1,f_2,f_3)$ of functions ($f_i:Q\to R$) is a *homotopy* iff:

$$f_1(x)\cdot f_2(y)=f_3(x\cdot y)$$

which implies (and is implied by any of):

$$\begin{array}{ll} f_3(x)/f_2(y)=f_1(x/y) & \qquad f_2(x)/\!\!/f_3(y)=f_1(x/\!\!/y) \\ f_1(x)\backslash f_3(y)=f_2(x\backslash y) & \qquad f_3(x)\backslash\!\!\backslash f_1(y)=f_2(x\backslash\!\!\backslash y) \end{array}$$

If all three components of $\bar f$ are bijections, then $\bar f$ is an *isotopy*.

***

We can also define a quasigroup as an algebra $(Q;\cdot,/,\backslash)$ with three binary operations: multiplication $(\cdot)$, right and left division. The axioms that a quasigroup satisfies are (xy is short for $(x\cdot y)$):

$$\begin{array}{ll} xy/y=x & \qquad x\backslash xy=y \\ (x/y)y=x & \qquad x(x\backslash y)=y \end{array} \tag{Q}$$

For obvious reasons, such quasigroups are called *equational, primitive* or *equasigroups.*

Thus, we have the variety of all quasigroups. Another important variety is the variety of *semisymmetric* quasigroups, defined by one of the following five equivalent axioms (in addition to (Q)):

$$x \cdot yx = y \tag{2.1}$$

$$xy \cdot x = y \tag{2.2}$$

$$x/y = yx$$

$$x\backslash y = yx$$

$$x\backslash y = x/y$$

Smith, [6], defined a *semisymmetrization* of a quasigroup $\mathbb{Q} = (Q; \cdot, /, \backslash)$ as a one–operation quasigroup $\mathbb{Q}^{\Delta} = (Q^3; \circ)$ where the binary operation $\circ$ is defined by:

$$(x_1, x_2, x_3) \circ (y_1, y_2, y_3) = (y_3/x_2, y_1\backslash x_3, x_1 y_2) \tag{2.3}$$

and proved that, for any quasigroup $\mathbb{Q}$, the semisymmetrization $\mathbb{Q}^{\Delta}$ of $\mathbb{Q}$ is a semisymmetric quasigroup.

3. Twisted quasigroups

For our purpose, there is a better way to define a quasigroup. In this definition the *twisted quasigroup* is an algebra $(Q; /\!\!/, \backslash\!\!\backslash, \cdot)$ satisfying appropriate paraphrasing of the above quasigroup axioms (Q):

$$y /\!\!/ xy = x \qquad\qquad xy \backslash\!\!\backslash x = y$$

$$(y /\!\!/ x)y = x \qquad\qquad x(y \backslash\!\!\backslash x) = y$$

We have the following symmetry result, lacking for quasigroups defined as $(Q; \cdot, /, \backslash)$.

Proposition 3.1. *An algebra* $(Q; /\!\!/, \backslash\!\!\backslash, \cdot)$ *is a* twisted quasigroup *iff* $(Q; \backslash\!\!\backslash, \cdot, /\!\!/)$ *is a* twisted quasigroup *iff* $(Q; \cdot, /\!\!/, \backslash\!\!\backslash)$ *is a* twisted quasigroup.

Analogously, we have the paraphrasing of axioms for *semisymmetric twisted semisymmetric quasigroups*: (2.1),(2.2) and

$$x /\!\!/ y = xy$$

$$x \backslash\!\!\backslash y = xy$$

$$x \backslash\!\!\backslash y = x /\!\!/ y$$

The last three identities we shorten to symbolic identities: $/\!\!/ = \cdot, \backslash\!\!\backslash = \cdot, \backslash\!\!\backslash = /\!\!/$.

There is also a result corresponding to Proposition 3.1:

Proposition 3.2. *An algebra* $(Q; /\!\!/, \backslash\!\!\backslash, \cdot)$ *is a* semisymmetric twisted quasigroup *iff* $(Q; \backslash\!\!\backslash, \cdot, /\!\!/)$ *is a* semisymmetric twisted quasigroup *iff* $(Q; \cdot, /\!\!/, \backslash\!\!\backslash)$ *is a* semisymmetric twisted quasigroup.

***

Using twisted quasigroups we can see how a (twisted) semisymmetrization (defined below), which we call ∇, 'works'.

Let us start with three single–operation quasigroups $(Q; \cdot), (Q; /\!\!/)$ and $(Q; \backslash\!\!\backslash)$, where $/\!\!/$ and $\backslash\!\!\backslash$ are duals of appropriate division operations of $\cdot$. We can define direct (Cartesian) product $(Q; /\!\!/) \times (Q; \backslash\!\!\backslash) \times (Q; \cdot)$ and an operation $\otimes$ on Q^3 such that

$$(x_1, x_2, x_3) \otimes (y_1, y_2, y_3) = (x_1 /\!\!/ y_1, x_2 \backslash\!\!\backslash y_2, x_3 y_3) \tag{3.4}$$

defines multiplication in the direct product. Therefore $(Q^3; \otimes)$ is a quasigroup.

Define also a permutation $' : Q^3 \to Q^3$ by $(x_1, x_2, x_3)' = (x_2, x_3, x_1)$. It follows that $(x_1, x_2, x_3)'' = (x_3, x_1, x_2)$ and $(x_1, x_2, x_3)''' = (x_1, x_2, x_3)$. Define another operation $\nabla_3 : Q^3 \times Q^3 \to Q^3$ by $\bar{x} \nabla_3 \bar{y} = \bar{x}' \otimes \bar{y}''$, where $\bar{u} = (u_1, u_2, u_3)$. The groupoid $(Q^3; \nabla_3)$ is also a quasigroup, so there are appropriate division operations of ∇_3 and their duals ∇_1 and ∇_2:

$$\bar{x} \nabla_3 \bar{y} = \bar{z} \quad \text{iff} \quad \bar{y} \nabla_1 \bar{z} = \bar{x} \quad \text{iff} \quad \bar{z} \nabla_2 \bar{x} = \bar{y}.$$

Therefore $(Q^3; \nabla_1, \nabla_2, \nabla_3)$ is a twisted quasigroup.

Let us calculate ∇_1.

$$\begin{aligned} \bar{z} = (z_1, z_2, z_3) &= \bar{x} \nabla_3 \bar{y} = (x_1, x_2, x_3)' \otimes (y_1, y_2, y_3)'' \\ &= (x_2, x_3, x_1) \otimes (y_3, y_1, y_2) = (x_2 /\!\!/ y_3, x_3 \backslash\!\!\backslash y_1, x_1 y_2). \end{aligned}$$

Therefore

$$\bar{x} = (y_2 /\!\!/ z_3, y_3 \backslash\!\!\backslash z_1, y_1 z_2) = (y_2, y_3, y_1) \otimes (z_3, z_1, z_2) = \bar{y}' \otimes \bar{z}'' = \bar{y} \nabla_3 \bar{z}$$

i.e. $\nabla_1 = \nabla_3$ (and consequently $\nabla_2 = \nabla_3$) hence $(Q^3; \nabla_1, \nabla_2, \nabla_3)$ is semisymmetric twisted quasigroup. So we recognize ∇_3 as a twisted analogue of Smith's $\circ$ (see identity (2.3)). Let us call $\mathbb{Q}^\nabla = (Q^3; \nabla_1, \nabla_2, \nabla_3)$ a *twisted semisymmetrization* of $\mathbb{Q}$.

For (f_1, f_2, f_3) being a homotopy from $\mathbb{Q}$ to $\mathbb{R}$, we also have:

$$\begin{aligned} (f_1 \times f_2 \times f_3)\,(\bar{x} \nabla_3 \bar{y}) &= (f_1 \times f_2 \times f_3)\,(\bar{x}' \otimes \bar{y}'') \\ &= (f_1(x_2 /\!\!/ y_3), f_2(x_3 \backslash\!\!\backslash y_1), f_3(x_1 \cdot y_2)) \\ &= (f_2 x_2 /\!\!/ f_3 y_3, f_3 x_3 \backslash\!\!\backslash f_1 y_1, f_1 x_1 \cdot f_2 y_2) \\ &= (f_2 x_2, f_3 x_3, f_1 x_1) \otimes (f_3 y_3, f_1 y_1, f_2 y_2) \\ &= (f_1 x_1, f_2 x_2, f_3 x_3)' \otimes (f_1 x_1, f_2 x_2, f_3 x_3)'' \\ &= (f_1 \times f_2 \times f_3)\,(\bar{x}) \nabla_3 (f_1 \times f_2 \times f_3)\,(\bar{y}), \end{aligned}$$

so $f_1 \times f_2 \times f_3$ is a homomorphism.

4. The categories Qtp and P

This section follows the lines of [6] with some adjustments. The main novelty is a proof of [6, Corollary 5.3]. We try to keep to the notation introduced in [6]. However, we write functions and functors to the left of their arguments.

For a fixed, large enough universe U, a quasigroup $\mathbb{Q} = (Q; \cdot, /, \backslash)$ is *small* when Q belongs to U (see [5, I.2]). Let **Qtp** be the category with objects all small quasigroups $\mathbb{Q} = (Q; \cdot, /, \backslash)$ and arrows all homotopies. The identity homotopy on $\mathbb{Q}$ is the triple $(\mathbf{1}_Q, \mathbf{1}_Q, \mathbf{1}_Q)$, where $\mathbf{1}_Q$ is the identity function on Q, and the composition of homotopies $(f_1, f_2, f_3)\colon \mathbb{P} \to \mathbb{Q}$ and $(g_1, g_2, g_3)\colon \mathbb{Q} \to \mathbb{R}$ is the homotopy $(g_1 \circ f_1, g_2 \circ f_2, g_3 \circ f_3)\colon \mathbb{P} \to \mathbb{R}$.

Let **P** be the category with objects all small semisymmetric quasigroups and arrows all quasigroup homomorphisms. For every arrow $f\colon \mathbb{Q} \to \mathbb{R}$ of **P**, the triple (f, f, f) is a homotopy between $\mathbb{Q}$ and $\mathbb{R}$.

Let Σ be a functor from **P** to **Qtp**, which is identity on objects. Moreover, let Σf, for a homomorphism f, be the homotopy (f, f, f).

The category **P** is a full subcategory of the category **Q** with objects all small quasigroups and arrows all quasigroup homomorphisms. The functor Σ is just a restriction of a functor from **Q** to **Qtp**, which is defined in the same manner.

An *adjunction* is given by two functors, $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$, and two natural transformations, the *unit* $\eta\colon \mathbf{1_C} \xrightarrow{\cdot} GF$ and the *counit* $\varepsilon\colon FG \xrightarrow{\cdot} \mathbf{1_D}$, such that for every object C of **C** and every object D of **D**

$$G\varepsilon_D \circ \eta_{GD} = \mathbf{1}_{GD}, \quad \text{and} \quad \varepsilon_{FC} \circ F\eta_C = \mathbf{1}_{FC}.$$

These two equalities are called *triangular identities.* The functor F is a *left adjoint* for the functor G, while G is a *right adjoint* for the functor F.

That $\Sigma : \mathbf{P} \to \mathbf{Qtp}$ has a right adjoint is shown as follows. Let $/\!\!/$ and $\backslash\!\!\backslash$ be defined as at the beginning of Section 3. For $\mathbb{Q}$ a quasigroup, let $\nabla_3 \colon Q^3 \times Q^3 \to Q^3$ be defined as in Section 3, i.e., for every $\bar{x} = (x_1, x_2, x_3)$ and $\bar{y} = (y_1, y_2, y_3)$

$$\bar{x} \nabla_3 \bar{y} = (x_2 /\!\!/ y_3, x_3 \backslash\!\!\backslash y_1, x_1 \cdot y_2).$$

That $(Q^3; \nabla_3)$ is a semisymmetric quasigroup follows from the fact that the structure $(Q^3; \nabla_1, \nabla_2, \nabla_3)$ is a semisymmetric twisted quasigroup, which is shown in Section 3. The semisymmetric quasigroup $(Q^3; \nabla_3)$ is the semisymmetrization $\mathbb{Q}^\Delta$ of $\mathbb{Q}$ defined at the end of Section 2 (see (2.3)).

Let $\Delta\colon \mathbf{Qtp} \to \mathbf{P}$ be a functor, which maps a quasigroup $\mathbb{Q}$ to the semisymmetric quasigroup $(Q^3; \nabla_3)$. A homotopy (f_1, f_2, f_3) is mapped by Δ to the product $f_1 \times f_2 \times f_3$, which is a homomorphism as it is shown at the end of Section 3. By the functoriality of product, we have that Δ preserves identities and composition, and it is indeed a functor. A proof of the following proposition is given in [6, Theorem 5.2].

Proposition 4.1. *The functor Δ is a right adjoint for Σ.*

Moreover, every component of the counit of this adjunction is epi (i.e. right cancellable) and the semisymmetrization is one-one. This is sufficent for **Qtp** to be isomorphic to a subcategory of **P**. This is one way how to establish this fact using the previous proposition. However, if the goal was just to establish that **Qtp** is isomorphic to a subcategory of **P**, this adjunction is not necessary at all, which is shown below.

A functor $F\colon \mathbf{C} \to \mathbf{D}$ is *faithful* when for every pair $f, g\colon A \to B$ of arrows of $\mathbf{C}$, $Ff = Fg$ implies $f = g$. An arrow $f\colon A \to B$ of $\mathbf{C}$ is *epi* when for every pair $g, h\colon B \to C$ of arrows of $\mathbf{C}$, the equality $g \circ f = h \circ f$ implies $g = h$. The following lemmas will help us to prove that **Qtp** is isomorphic to a subcategory of **P**.

Lemma 4.2. *The functor Δ is faithful.*

Proof. For homotopies (f_1, f_2, f_3) and (g_1, g_2, g_3) from $\mathbb{Q}$ to $\mathbb{R}$, if $f_1 \times f_2 \times f_3$ and $g_1 \times g_2 \times g_3$ are equal as homomorphisms from $\Delta\mathbb{Q}$ to $\Delta\mathbb{R}$ in **P**, then for every $i \in \{1, 2, 3\}$, $f_i = g_i$. Hence, these homotopies are equal in **Qtp**. $\square$

Alternatively, by [5, IV.3, Theorem 1, Part (i)] (see also [2, Section 4, Proposition 4.1] for an elegant proof of a related result) one may establish that Δ is faithful by relying on Proposition 4.1. It suffices to prove that for every object $\mathbb{Q}$ of **Qtp**, the component $\varepsilon_{\mathbb{Q}}$ of the counit of the adjunction established in Proposition 4.1 is epi. The arrow $\varepsilon_{\mathbb{Q}}$ is defined as the triple (π_1, π_2, π_3), where $\pi_i\colon Q^3 \to Q$ is the ith projection. Let $g, h\colon \mathbb{Q} \to \mathbb{R}$ be a pair of arrows of **Qtp** such that $g \circ \varepsilon_{\mathbb{Q}} = h \circ \varepsilon_{\mathbb{Q}}$. This means that for every $i \in \{1, 2, 3\}$ we have that $g_i \circ \pi_i = h_i \circ \pi_i$. Hence, the function g_i is equal to the function h_i, since the function π_i is right cancellable. (However, the homotopy $\varepsilon_{\mathbb{Q}}$ need not have a right inverse in **Qtp**.)

Lemma 4.3. *If $(Q; \cdot, /, \backslash)$ and $(Q; \cdot', /', \backslash')$ are two different quasigroups, then there are $x, y \in Q$ such that*

$$x \cdot y \neq x \cdot' y.$$

Proof. Suppose that for every $x, y \in Q$, $x \cdot y = x \cdot' y$ holds. Then for every $z, t \in Q$ we have

$$z/t = ((z/t) \cdot' t)/'t = ((z/t) \cdot t)/'t) = z/'t.$$

Analogously, we prove that for every $u, v \in Q$, $u\backslash v = u\backslash' v$. Hence, $(Q; \cdot, /, \backslash)$ and $(Q; \cdot', /', \backslash')$ are the same, which contradicts the assumption. $\square$

Lemma 4.4. *The functor Δ is one-one on objects.*

Proof. Suppose that $(Q; \cdot, /, \backslash)$ and $(Q'; \cdot', /', \backslash')$ are two different quasigroups. If Q and Q' are different sets, then $\Delta\mathbb{Q}$ and $\Delta\mathbb{Q}'$ are different. If $Q = Q'$, then, by Lemma 4.3, there are x and y in this set such that $x \cdot y \neq x \cdot' y$. Hence, the operations ∇_3 for $\Delta\mathbb{Q}$ and $\Delta\mathbb{Q}'$ differ when applied to (x, x, x) and (y, y, y). $\square$

As a corollary of these two lemmas we have the following result.

Proposition 4.5. *The category* **Qtp** *is isomorphic to a subcategory of* **P***; namely, to its image under the functor* Δ.

As we have shown by the proof of Lemma 4.2, Proposition 4.5 is independent of Proposition 4.1. The adjunction, together with this embedding of **Qtp** in **P**, says that the category **P** reflects in **Qtp** in the following sense. A subcategory **A** of **B** is *reflective* in **B**, when the inclusion functor from **A** to **B** has a left adjoint called a *reflector* (see [5, IV.3]). The adjunction is called a *reflection* of **B** in **A**.

Propositions 4.1 and 4.5 say that **Qtp** may be considered as a reflective subcategory of **P**. The functor Σ is a reflector and the adjunction between Σ and Δ is a reflection of **P** in **Qtp**. However, this does not mean that the Δ-image of **Qtp** is an *iso-full* subcategory of **P**, i.e. that two quasigroups are isotopic in **Qtp** if and only if their semisymmetrizations are isomorphic in **P**. Im, Ko and Smith, [4, first paragraph in the introduction], refer to [6] for this iso-fullness. However, this is not considered at all in [6] and the question of fullness or iso-fullness of the image of **Qtp** in **P** remains open. The reader should be aware of this potential missusing of these results.

5. Monadicity of Δ

For $F: \mathbf{C} \to \mathbf{D}$ a left adjoint for $G: \mathbf{D} \to \mathbf{C}$, and η and ε, the unit and counit of this adjunction, a GF-algebra is a pair (C, h), where C is an object of $\mathbf{C}$ and $h: GFC \to C$ is an arrow of $\mathbf{C}$ such that the following equalities hold.

$$h \circ GFh = h \circ G\varepsilon_{FC}, \qquad h \circ \eta_C = \mathbf{1}_C.$$

A morphism of GF-algebras (C, h) and (C', h') is given by an arrow $f: C \to C'$ of $\mathbf{C}$ such that $f \circ h = h' \circ GFf$.

The category $\mathbf{C}^{GF}$ has GF-algebras as objects and morphisms of GF-algebras as arrows. The *comparison functor* $K: \mathbf{D} \to \mathbf{C}^{GF}$ is given by

$$KD = (GD, G\varepsilon_D), \qquad Kf = Gf.$$

In many cases the comparison functor is an isomorphism or an equivalence (i.e. there is a functor from $\mathbf{C}^{GF}$ to $\mathbf{D}$ such that both compositions with K are naturally isomorphic to the identity functors). The right adjoint of an adjunction or an adjunction are called *monadic* when the comparison functor is an isomorphism (see [5, VI.3], also [8, Section 4.2]). Some other authors (see [1, Section 3.3]) call an adjunction monadic (tripleable) when K is just an equivalence.

In the case of adjoint situation involving Σ and Δ, the comparison functor $K: \mathbf{Qtp} \to \mathbf{P}^{\Delta\Sigma}$ is just an equivalence. To prove this, by [5, IV.4, Theorem 1] it suffices to prove that K is full and faithful, and that every GF-algebra is isomorphic to $K\mathbb{Q}$ for some quasigroup $\mathbb{Q}$. The faithfulness of K follows from 4.2 since the arrow function K coincides with the arrow function Δ. That every GF-algebra is isomorphic to $K\mathbb{Q}$ for some quasigroup $\mathbb{Q}$ is proven in [7, Section 10, Theorem 33].

A functor $F\colon \mathbf{C} \to \mathbf{D}$ is *full* when for every pair of objects C_1 and C_2 of $\mathbf{C}$ and every arrow $g\colon FC_1 \to FC_2$ of $\mathbf{D}$ there is an arrow $f : C_1 \to C_2$ of $\mathbf{C}$ such that $g = Ff$. It remains to prove that K is full. For this we use the following lemma.

Lemma 5.1. *Every arrow of $\boldsymbol{P}^{\Delta\Sigma}$ from $K\mathbb{Q}$ to $K\mathbb{R}$ is of the form $f_1 \times f_2 \times f_3$, for (f_1, f_2, f_3) a homotopy from $\mathbb{Q}$ to $\mathbb{R}$.*

Proof. For quasigroups $\mathbb{Q}$ and $\mathbb{R}$ we have that $K\mathbb{Q} = (\Delta\mathbb{Q}, \pi_1 \times \pi_2 \times \pi_3)$ and $K\mathbb{R} = (\Delta\mathbb{R}, \pi_1 \times \pi_2 \times \pi_3)$. So, let

$$f\colon (\Delta\mathbb{Q}, \pi_1 \times \pi_2 \times \pi_3) \to (\Delta\mathbb{R}, \pi_1 \times \pi_2 \times \pi_3)$$

be an arrow of $\mathbf{P}^{\Delta\Sigma}$. Since f is a morphism of $\Delta\Sigma$-algebras, we have that

$$f \circ (\pi_1 \times \pi_2 \times \pi_3) = (\pi_1 \times \pi_2 \times \pi_3) \circ (f \times f \times f)$$

as functions from $(Q^3)^3$ to R^3.

For $i \in \{1, 2, 3\}$ and $u \in Q$, let $f_i(u) = \pi_i(f(u, u, u))$. Moreover, let (x, y, z) be an arbitrary element of Q^3. Apply the both sides of the above equality to $((x, x, x), (y, y, y), (z, z, z)) \in (Q^3)^3$ in order to obtain

$$f(x, y, z) = (\pi_1(f(x, x, x)), \pi_2(f(y, y, y)), \pi_3(f(z, z, z))) = (f_1(x), f_2(y), f_3(z)).$$

Hence, $f = f_1 \times f_2 \times f_3$ and since it is a homomorphism from $\Delta\mathbb{Q}$ to $\Delta\mathbb{R}$, we have for every $\bar{x}, \bar{y} \in Q^3$

$$(f_1 \times f_2 \times f_3)(\bar{x})\ \nabla_3\ (f_1 \times f_2 \times f_3)(\bar{y}) = (f_1 \times f_2 \times f_3)(\bar{x}\ \nabla_3\ \bar{y}).$$

By restricting this equality to the third component, we obtain $f_1(x_1) \cdot f_2(y_2) = f_3(x_1 \cdot y_2)$, and hence (f_1, f_2, f_3) is a homotopy from $\mathbb{Q}$ to $\mathbb{R}$. □

6. A new semisymmetrization

Definition 6.1. An algebra $(Q; /\!\!/, \backslash\!\!\backslash)$ is a *biquasigroup* iff $/\!\!/(\backslash\!\!\backslash)$ is the dual of the right (left) division operation of a quasigroup operation $\cdot$ on Q.

A biquasigroup is *semisymmetric* iff $\backslash\!\!\backslash = /\!\!/$.

Proposition 6.2. *An algebra $(Q; /\!\!/, \backslash\!\!\backslash)$ is a biquasigroup iff $(Q; \backslash\!\!\backslash, \cdot)$ is a biquasigroup iff $(Q; \cdot, \backslash\!\!\backslash)$ is a biquasigroup.*

Proposition 6.3. *An algebra $(Q; /\!\!/, \backslash\!\!\backslash)$ is a semisymmetric biquasigroup iff $(Q; \backslash\!\!\backslash, \cdot)$ is a semisymmetric biquasigroup iff $(Q; \cdot, \backslash\!\!\backslash)$ is a semisymmetric biquasigroup.*

Let us start with three single–operation quasigroups $(Q; \cdot), (Q; /\!\!/)$ and $(Q; \backslash\!\!\backslash)$, where $/\!\!/$ and $\backslash\!\!\backslash$ are duals of appropriate division operations of $\cdot$. We can define direct (Cartesian) product $(Q; /\!\!/) \times (Q; \backslash\!\!\backslash)$ and an operation $\otimes$ on Q^2 such that

$$(x_1, x_2) \otimes (y_1, y_2) = (x_1 /\!\!/ y_1, x_2 \backslash\!\!\backslash y_2)$$

defines multiplication in the direct product. Therefore $(Q^2;\otimes)$ is a quasigroup.

Define also a permutation $' : Q^2 \to Q^2$ by $(x_1,x_2)' = (x_2,x_1)$. Define another operation $\nabla : Q^2 \times Q^2 \to Q^2$ by $\hat{x}\nabla\hat{y} = R_{\hat{y}}(\hat{x}') \otimes L_{\hat{x}}(\hat{y}')$, where $\hat{u}$ is (u_1,u_2), $L_{\hat{x}}(\hat{y}) = (x_1 \cdot y_1, y_2)$ and $R_{\hat{y}}(\hat{x}) = (x_1, x_2 \cdot y_2)$. The groupoid $(Q^2;\nabla)$ is also a quasigroup, moreover a semisymmetric one. Therefore $(Q^2;\nabla,\nabla)$ is a semisymmetric biquasigroup.

Let us define:

$$x \,①\, y = x /\!\!/ y \qquad x \,②\, y = x \backslash\!\!\backslash y \qquad x \,③\, y = x \cdot y$$

Then the definition of ∇_{12}, which we abbreviate just by ∇, is:

$$(x_1,x_2)\nabla_{12}(y_1,y_2) = (x_2 \,①\, (x_1 \,③\, y_2), (x_1 \,③\, y_2) \,②\, y_1).$$

There are two more alternative semisymmetrizations with corresponding definitions in $(Q^2;\backslash\!\!\backslash,\cdot)$ (respectively $(Q^2;\cdot,/\!\!/)$):

$$(x_1,x_2)\nabla_{23}(y_1,y_2) = (x_2 \,②\, (x_1 \,①\, y_2), (x_1 \,①\, y_2) \,③\, y_1)$$

$$(x_1,x_2)\nabla_{31}(y_1,y_2) = (x_2 \,③\, (x_1 \,②\, y_2), (x_1 \,②\, y_2) \,①\, y_2).$$

The indexing of operations is used to emphasize the symmetry.

In this section we introduce a new semisymmetrization functor from **Qtp** to **P**. This leads to another subcategory of **P** isomorphic to **Qtp**. We start with an auxiliary result.

Lemma 6.4. *The third component f_3 of a homotopy is determined by the first two components f_1 and f_2.*

Proof. Let $\mathbb{Q}$ be a quasigroup. For every element $x \in Q$ there are $y, z \in Q$ such that $x = y \cdot z$ (e.g. $x = y \cdot (y\backslash x)$). Hence, $f_3(x) = f_1(y) \cdot f_2(z)$. □

Let $\Gamma\colon \mathbf{Qtp} \to \mathbf{P}$ be a functor defined on objects so that $\Gamma\mathbb{Q}$ is a semisymmetric quasigroup $(Q^2;\nabla)$ whose elements are pairs (x_1,x_2), abbreviated by $\hat{x}$, and ∇ is defined so that

$$(x_1,x_2)\nabla(y_1,y_2) = (x_2 /\!\!/ (x_1 \cdot y_2), (x_1 \cdot y_2) \backslash\!\!\backslash y_1).$$

(It is straightforward to check that $(\hat{y}\nabla\hat{x})\nabla\hat{y} = \hat{y}\nabla(\hat{x}\nabla\hat{y}) = \hat{x}$, hence $\Gamma\mathbb{Q}$ is a semisymmetric quasigroup.)

A homotopy (f_1,f_2,f_3) is mapped by Γ to the product $f_1 \times f_2$, which is a homomorphism:

$$\begin{aligned}(f_1 \times f_2)(\hat{x}) \,\nabla\, (f_1 \times f_2)(\hat{y}) &= (f_2(x_2) /\!\!/ (f_1(x_1) \cdot f_2(y_2)), (f_1(x_1) \cdot f_2(y_2)) \backslash\!\!\backslash f_1(y_1)) \\ &= (f_1(x_2 /\!\!/ (x_1 \cdot y_2)), f_2((x_1 \cdot y_2) \backslash\!\!\backslash y_1)) \\ &= (f_1 \times f_2)(\hat{x}\nabla\hat{y}).\end{aligned}$$

By the functoriality of product, we have that Γ preserves identities and composition, and it is indeed a functor.

The functor Γ is not a right adjoint for Σ since a right adjoint is unique up to isomorphism and $\Gamma\mathbb{Q}$ is not isomorphic to $\Delta\mathbb{Q}$ for every object $\mathbb{Q}$ of **Qtp**. However, this adjunction is not necessary for the faithfulness of Γ.

Lemma 6.5. *The functor* Γ *is faithful.*

Proof. We proceed as in the second proof of Lemma 4.2. If (f_1, f_2, f_3) and (g_1, g_2, g_3) are two homotopies from $\mathbb{Q}$ to $\mathbb{R}$, then $\Gamma(f_1, f_2, f_3) = \Gamma(g_1, g_2, g_3)$ means that $f_1 \times f_2 = g_1 \times g_2$. Hence, $f_1 = g_1$ and $f_2 = g_2$, and by Lemma 6.4, $f_3 = g_3$. $\Box$

The functor Γ, as defined, is not one-one on objects. For example,

$$(\{0,1\}, +, +, +) \quad \text{and} \quad (\{0,1\}, \oplus, \oplus, \oplus),$$

where $+$ is addition mod 2 and $x \oplus y = x + y + 1$, are mapped by Γ to the same object of **P**. To remedy this matter, one may redefine Γ so that

$$\Gamma\mathbb{Q} = (Q^2 \times \{\mathbb{Q}\}, \nabla),$$

where $\mathbb{Q}$, as the third component of every element, guarantees that Γ is one-one on objects. The operation ∇ is defined as above, just neglecting the third component. Hence, **Qtp** may be considered as another subcategory of **P**.

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References

[1] **M. Barr and C. Wells**, *Toposes, Triples and Teories*, Springer, Berlin, 1985.

[2] **K. Došen**, *On sets of premises*, preprint (2014), arXiv:1412.6783.

[3] **A.A. Gvaramiya**. *Category of quasigroups with homotopies*, Comptes rendus de l'Académie Bulgare des Sciences **38** (1985), 663 – 666.

[4] **B. Im, H-J. Ko and J.D.H. Smith**, *Semisymmetrizations of abelian group isotopies*, Taiwanese J. Math. **11** (2007), 1529 – 1534.

[5] **S. Mac Lane**, *Categories for the Working Mathematician*, Springer, Berlin, 1998.

[6] **J.D.H. Smith**, *Homotopy and semisymmetry of quasigroups*, Algebra Universalis **38** (1997), 175 – 184.

[7] **J.D.H. Smith**, *Quasigroup homotopies, semisymmetrization, and reversible automata*, Intern. J. Algebra and Comput. **7** (2008), 1203 – 1221.

[8] **J.D.H. Smith and A.B. Romanowska**, *Post-Modern Algebra*, Wiley, New York, 1999.

[9] **G. Voutsadakis**, *Categorical models and Quasigroup Homotopies*, Theory and Applications of Categories **11** (2003), 1 – 14.

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The congruence $\mathscr{Y}^*$ on completely regular semirings

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Abstract. We investigate the congruence generated by $\mathscr{Y}$ on completely regular semirings and get that $\mathscr{Y}^* \in [\epsilon, \nu]$ on completely regular semirings.

1. Introduction

The study of the structure of semigroups and semirings are essentially influenced by the study of the congruences defined on them. We know that the set of all congruences defined on a semiring or a semigroup is a partially ordered set with respect to inclusion and relative to this partial order it forms a lattice, the lattice of congruences $\mathscr{C}(S)$ on S. In 1999, Petrich and Reilly [8] defined a relation $\mathscr{Y}$ on a completely regular semigroup S by: for $a, b \in S$;

$$a \mathscr{Y} b \quad \text{if and only if} \quad V(a) = V(b).$$

Under certain special conditions of semigroups, $\mathscr{Y}$ was proved to be the least Clifford congruence on S. It was proposed by them as an open problem that, what can be said about $\mathscr{Y}^*$, the congruence generated by $\mathscr{Y}$ on a completely regular semigroup. Recently, in 2011, C. Guo, G. Liu and Y. Guo solved this open problem in their paper [1]. They proved that $\mathscr{Y}^* \in [\epsilon, \nu]$ on completely regular semigroups. Furthermore, they gave a description of $\mathscr{Y}^*$ on completely simple semigroups and normal cryptogroups, respectively. The main aim of this paper is to further extend these ideas on completely regular semirings.

The preliminaries and prerequisites we need for this paper are discussed in Section 2. In Section 3 we study some properties of orthodox completely regular semirings and finally in Section 4 we characterize the relation $\mathscr{Y}^*$ on completely regular semirings.

2. Preliminaries

A *semiring* $(S, +, \cdot)$ is a type (2, 2)-algebra such that the semigroup reducts $(S, +)$ and $(S, \cdot)$ are connected by distributive laws, i.e., $a(b+c) = ab+ac$ and $(b+c)a =$

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$ba + ca$ for all $a, b, c \in S$. Here the additive reduct $(S, +)$ of the semiring $(S, +, \cdot)$ is not necessarily commutative. An element a in a semiring $(S, +, \cdot)$ is said to be *additively regular* if there exists an element $x \in S$ such that $a + x + a = a$.

Following [5], we say that an element a of a semiring $(S, +, \cdot)$ is *completely regular* if there exists $x \in S$ such that $a = a+x+a$, $a+x = x+a$ and $a(a+x) = a+x$. A semiring S is said to be *completely regular* if every element of S is completely regular.

Let τ be a relation on a semiring S. Define the relation τ^e on S by: for $a, b \in S$;

$$a\tau^e b \text{ if and only if } a = x + c + y,\ b = x + d + y \text{ for some } x, y \in S^0 \text{ and } c\tau d.$$

Also, we define $\tau^\natural$ by $\tau^\natural = \left((\tau \cup \tau^{-1} \cup \epsilon)^e\right)^t$, where ϵ is the equality congruence and η^t denotes the transitive closure of η.

Following [5], a semiring $(S, +, \cdot)$ is called a *skew-ring* if its additive reduct $(S, +)$ is a group, not necessarily an abelian group. A semiring $(S, +, \cdot)$ is said to be a *b-lattice* [5] if $(S, \cdot)$ is a band and $(S, +)$ is a semilattice. If $(S, +, \cdot)$ is a semiring, we denote Green's relations on the semigroup $(S, +)$ by $\mathscr{L}^+$, $\mathscr{R}^+$, $\mathscr{J}^+$, $\mathscr{D}^+$ and $\mathscr{H}^+$. In fact, the relations $\mathscr{L}^+$, $\mathscr{R}^+$, $\mathscr{J}^+$, $\mathscr{D}^+$ and $\mathscr{H}^+$ are all congruences on the multiplicative reduct $(S, \cdot)$. Thus, if any one of these happens to be a congruence on the additive reduct $(S, +)$, it will be a congruence on the semiring $(S, +, \cdot)$. A completely regular semiring S is said to be *completely simple* [5] if $\mathscr{J}^+ = S \times S$. A congruence ξ on a semiring S is called a *b-lattice congruence (idempotent semiring congruence)* if S/ξ is a b-lattice (respectively, an idempotent semiring). A semiring S is said to be a *b-lattice* (*idempotent semiring*) Y *of semirings* $S_\alpha (\alpha \in Y)$ if S admits a b-lattice congruence (respectively, an idempotent semiring congruence) ξ on S such that $Y = S/\xi$ and each S_α is a ξ-class. We write $S = (Y; S_\alpha)$.

First we prove the following result.

Theorem 2.1. *The following conditions on a semiring are equivalent:*

(*i*) *S is completely regular;*
(*ii*) *every $\mathscr{H}^+$-class is a skew-ring;*
(*iii*) *S is union* (*disjoint*) *of skew-rings;*
(*iv*) *S is a b-lattice of completely simple semirings;*
(*v*) *S is an idempotent semiring of skew-rings.*

Proof. From [5, Theorem 3.6], it follows that first four conditions are equivalent.

(*i*) $\Rightarrow$ (*v*): Let S be a completely regular semiring. Then by [5, Theorem 3.6], it follows that each $\mathscr{H}^+$-class is a skew-ring. Let x^0 be the zero of the skew-ring H_x, where H_x is the $\mathscr{H}^+$-class containing the element $x \in S$. To complete the prove it suffices to show that $\mathscr{H}^+$ is an idempotent semiring congruence on S. For this let $a\,\mathscr{H}^+\,b$ and $c \in S$. Then $a^0 = b^0$. Now $(a+c)^0 = (a+c)(a+c)^0 = a(a+c)^0 + c(a+c)^0 = a^0(a+c)^0 + c(a+c)^0 = (a^0+c)(a+c)^0 = (a^0+c)^0(a+c)^0 = (a^0+c)^0(a+c) = (a^0+c)^0 a + (a^0+c)^0 c = (a^0+c)^0 a^0 + (a^0+c)^0 c = (a^0+c)^0(a^0+c) = (a^0+c)^0$. Similarly, we can show that $(b+c)^0 = (b^0+c)^0$. Thus, $(a+c)^0 = (a^0+c)^0 = (b^0+c)^0 = (b+c)^0$. This implies $a+c\,\mathscr{H}^+\,b+c$. Dually,

$c+a\,\mathscr{H}^+\,c+b$. Hence $\mathscr{H}^+$ is a congruence on $(S,+)$. Since $\mathscr{H}^+$ is a congruence on $(S,\cdot)$, it follows that $\mathscr{H}^+$ is a congruence on the semiring S. Clearly, $2a\,\mathscr{H}^+\,a$ and $a^2\,\mathscr{H}^+\,a$. Hence $S/\mathscr{H}^+$ is an idempotent semiring. Consequently, S is an idempotent semiring of skew-rings.

$(v) \Rightarrow (i)$: This is obvious. □

Throughout this paper, we always let $E^+(S)$ be the set of all additive idempotents of the semiring S. Observe that the distributive laws imply that whenever the set $E^+(S)$ is non-empty, it forms an ideal of the multiplicative reduct $(S,\cdot)$ of S. If $a \in S$ is additively regular, we denote the set of all inverse elements of a in the semigroup $(S,+)$ by $V^+(a)$. Also we denote the least skew-ring congruence by σ and the least b-lattice of skew-ring congruence by ν on a semiring S. We always let $S=(Y;S_\alpha)$ be a completely regular semiring, where Y is a b-lattice and S_α $(\alpha \in Y)$ is a completely simple semiring. For other notation and terminology not given in this paper, the reader is referred to the texts of Howie [3], Golan [4], and Petrich and Reilly [8].

Next we introduce some results which can be proved in a similar way as completely regular semigroup (see for example Theorem II.4.5 in [8]).

Theorem 2.2. *Let $S=(Y;S_\alpha)$ be completely regular semiring. Then $\mathscr{J}^+=\mathscr{D}^+$.*

Lemma 2.3. *For any completely regular semiring S,*

$$\nu = \{(f,g) \mid f,g \in E^+(S) \text{ and } f\,\mathscr{D}^+\,g\}^\natural.$$

Proof. Let $\eta = \{(f,g) \mid f,g \in E^+(S),\ f\,\mathscr{D}^+\,g\}^\natural$.

Clearly, $\eta \subseteq \mathscr{D}^+$ and each $\mathscr{D}^+$- class of S/η contains a unique additive idempotent. Hence S/η is a b-lattice of skew-rings and $\nu \subseteq \eta$. On the other hand, S/ν is a b-lattice of skew-rings so that $\{(f,g) \mid f,g \in E^+(S), f\,\mathscr{D}^+\,g\} \subseteq \nu$, which implies $\{(f,g) \mid f,g \in E^+(S), f\,\mathscr{D}^+\,g\}^e \subseteq \nu$. Thus, $\nu = \{(f,g) \mid f,g \in E^+(S),\ f\,\mathscr{D}^+\,g\}^\natural$. □

Lemma 2.4. *Let $S=(Y;S_\alpha)$ be a completely regular semiring and $a \in S_\alpha$, $b \in S_\beta$, where $\beta \leqslant \alpha$. Then,*

(i) $a\,\mathscr{L}^+\,(b+a)$, $a\,\mathscr{R}^+\,(a+b)$,

(ii) $a = a+(b+a)^0 = (a+b)^0+a$.

Proof. Follows similarly from [8, Corollary II.4.3.]. □

3. The relation $\mathscr{Y}$

We call a semiring $(S,+,\cdot)$ an *orthodox* semiring if the additive reduct $(S,+)$ is orthodox, i.e., $E^+(S)$ forms an ideal of S. We show that the relation $\mathscr{Y}$ and ν are equivalent on an orthodox completely regular semiring.

Let S be a completely regular semiring. Define a relation $\mathscr{Y}$ on S by: for $a,b \in S$;

$$a\,\mathscr{Y}\,b \text{ if and only if } V^+(a)=V^+(b).$$

We need the following result.

Lemma 3.1. *Let $S = (Y; S_\alpha)$ be an orthodox completely regular semiring, where Y is a b-lattice and $S_\alpha (\alpha \in Y)$ is a completely simple semiring. Then $\Big(E^+(S_\alpha), +\Big)$ is a rectangular band for all $\alpha \in Y$ and for any two elements $a, b \in S_\alpha$, $e \in E^+(S_\beta)$, $a + b = a + e + b$, where $\alpha, \beta \in Y$ such that $\beta \leqslant \alpha$.*

Proof. Follows similarly from [8, Lemma II.5.2]. □

Theorem 3.2. *Let $S = (Y; S_\alpha)$ be an orthodox completely regular semiring and $a, b \in S$. Then the following conditions are equivalent:*
- (i) $a \mathscr{Y} b$.
- (ii) *There exists* $e, f, g, h \in E^+(S)$ *with* $a = e + b + f$ *and* $b = g + a + h$.
- (iii) $a = a^0 + b + a^0$ *and* $b = b^0 + a + b^0$.

Proof. $(i) \Rightarrow (ii)$: At first we suppose that $a \mathscr{Y} b$ for $a, b \in S$. Then $V^+(a) = V^+(b)$. Let $x \in V^+(a)$. Then $x \in V^+(b)$, i.e., $a = a + x + a$, $x + a + x = x$ and $b = b + x + b$, $x + b + x = x$.

Thus, $a = (a+x)+b+(x+a) = e+b+f$, where $e = a+x$, $f = x+a \in E^+(S)$. Similarly, $b = g + a + h$, for some $g, h \in E^+(S)$.

$(ii) \Rightarrow (iii)$: We have $a^0+e+b+f+a^0 = a^0+a+a^0 = a$ for some $e, f \in E^+(S)$. Then $a \mathscr{D}^+ b$. Let $a, b \in S_\alpha$, $e \in S_\beta$ and $f \in S_\gamma$. Then $\beta, \gamma \leqslant \alpha$. Now, by Lemma 3.1, $a^0 + e + b = a^0 + b$. Similarly, $b + f + a^0 = b + a^0$. Hence, we have, $a^0 + b + a^0 = a$. Similarly, $b^0 + a + b^0 = b$.

$(iii) \Rightarrow (i)$: Let, $x \in V^+(a)$. Then, using Lemma 3.1, we have

$$\begin{aligned} b &= b^0 + a + b^0 = b^0 + a + x + a + b^0 \\ &= (b^0 + a + b^0) + x + (b^0 + a + b^0) = b + x + b. \end{aligned}$$

Similarly, $x + b + x = x$. Hence, $x \in V^+(b)$ and thus, $V^+(a) \subseteq V^+(b)$. By symmetry, it follows that $V^+(b) \subseteq V^+(a)$. Thus, $a \mathscr{Y} b$. □

Theorem 3.3. *Let $S = (Y; S_\alpha)$ be a completely regular semiring. Then $\mathscr{Y}$ is the least b-lattice of skew-rings congruence on S if and only if S is orthodox.*

Proof. By [1, Theorem 1.6], we have $\mathscr{Y}$ is the least semilattice of groups congruence on the semigroup reduct $(S, +)$ if and only if $(S, +)$ is orthodox. To complete the proof it remains to show that $\mathscr{Y}$ is a congruence on $(S, \cdot)$. For this let $a \mathscr{Y} b$ and $c \in S$. Then $a = a^0 + b + a^0$ and $b = b^0 + a + b^0$. This implies $ca = ca^0 + cb + ca^0 = (ca)^0 + cb + (ca)^0$ and $cb = cb^0 + ca + cb^0 = (cb)^0 + ca + (cb)^0$ and hence $ca \mathscr{Y} cb$. Similarly, we can show that $ac \mathscr{Y} bc$. Consequently, $\mathscr{Y}$ is a congruence on S. Since S is completely regular, it follows that $S/\mathscr{Y}$ is also completely regular. Moreover, since $(S/\mathscr{Y}, +)$ is semilattice of groups, one can easily prove that $S/\mathscr{Y}$ is a b-lattice of skew-rings, i.e., $\mathscr{Y}$ is the least b-lattice of skew-ring congruence on S. □

Theorem 3.4. *Let $S = (Y; S_\alpha)$ be an orthodox completely regular semiring. Then $\mathscr{D}^+ = \mathscr{H}^+ \mathscr{Y}$.*

Proof. Let $a\,\mathscr{D}^+\,b$ for $a, b \in S$. Now, we have, by Lemma 3.1, $b = b^0 + b + b^0 = b^0 + (a^0 + b + a^0) + b^0$. Again, $(a^0 + b + a^0)^0 + b + (a^0 + b + a^0)^0 = a^0 + b + a^0$. Hence, $(a^0 + b + a^0)\,\mathscr{Y}\,b$. Again, since $a^0 = (a^0 + b + a^0)^0$ we have $a\,\mathscr{H}^+\,(a^0 + b + a^0)$. Thus we have, $a\,(\mathscr{H}^+\,\mathscr{Y})\,b$ and hence $\mathscr{D}^+ \subseteq \mathscr{H}^+\,\mathscr{Y}$. The reverse inclusion is obvious. This completes the proof. □

We highlight a very interesting result based on the congruences that we have discussed so far.

Theorem 3.5. *Let $S = (Y; S_\alpha)$ be a completely regular semiring, where Y is a b-lattice and $S_\alpha\,(\alpha \in Y)$ is a completely simple semiring. Then the following conditions are equivalent:*

(*i*) *S is orthodox,*

(*ii*) *S is a spined product of an idempotent semiring and a b-lattice of skew-rings,*

(*iii*) *S satisfies the identity $a^0 + b^0 = (a + b)^0$.*

Proof. (*i*) $\Rightarrow$ (*ii*): Let π_1 (respectively, π_2) be the natural projection of $S/\mathscr{H}^+$ (respectively, $S/\mathscr{Y}$) onto Y. Let A be the spined product of $S/\mathscr{H}^+$ and $S/\mathscr{Y}$. Then, for any $a \in S_\alpha$, $\pi_1(a\mathscr{H}^+) = \pi_2(a\mathscr{Y}) = \alpha$.

We define a mapping, $\phi : S \to A$ by $\phi(a) = (a\mathscr{H}^+, a\mathscr{Y})$ for all $a \in S$. Clearly, ϕ is a semiring homomorphism.

Let $a, b \in S$ such that $\phi(a) = \phi(b)$. This implies $(a\mathscr{H}^+, a\mathscr{Y}) = (b\mathscr{H}^+, b\mathscr{Y})$, i.e., $a\,\mathscr{H}^+\,b$ and $a\,\mathscr{Y}\,b$, i.e., $a^0 = b^0$ and $a = a^0 + b + a^0, b = b^0 + a + b^0$. Therefore, $a = a^0 + b + a^0 = b^0 + b + b^0 = b$ and hence ϕ is injective.

To show ϕ is surjective, let $b, c \in S$ such that $(b\mathscr{H}^+, c\mathscr{Y}) \in A$. Then, $\pi_1(b\mathscr{H}^+) = \pi_2(c\mathscr{Y}) = \alpha$, say, so that $b, c \in S_\alpha$. Hence, $b\,\mathscr{D}^+\,c$. Now, by Theorem 3.4, $b\,(\mathscr{H}^+\,\mathscr{Y})\,c$. This implies $b\,\mathscr{H}^+\,a\,\mathscr{Y}\,c$ for some $a \in S$, i.e., $b\mathscr{H}^+ = a\mathscr{H}^+$ and $a\mathscr{Y} = c\mathscr{Y}$.

Hence, $\phi(a) = (a\mathscr{H}^+, a\mathscr{Y}) = (b\mathscr{H}^+, c\mathscr{Y})$, which implies that ϕ is surjective. Consequently, ϕ is an isomorphism.

(*ii*) $\Rightarrow$ (*iii*): Let S be a spined product of an idempotent semiring I and a b-lattice of skew-rings T. Since every idempotent semiring and every b-lattice of skew-rings satisfies the identity $x^0 + y^0 = (x + y)^0$ and therefore so does S.

(*iii*) $\Rightarrow$ (*i*): If S satisfies the identity $a^0 + b^0 = (a + b)^0$, then for any two elements $e, f \in E^+(S)$, we have $e^0 + f^0 = (e + f)^0$, i.e., $e + f = (e + f)^0 \in E^+(S)$. Hence S is orthodox. □

Corollary 3.6. *Let S be an orthodox completely regular semiring. Then $\mathscr{H}^+ \cap \mathscr{Y} = \epsilon$, where ϵ is the equality relation on S.*

4. The interval which $\mathscr{Y}^*$ belongs to

So far we have discussed the nature and properties of the relation $\mathscr{Y}$ on a special kind of completely regular semirings. In the following section, we try to describe

$\mathscr{Y}^*$ on completely regular semirings without any other special conditions.

Following [6, Theorem 3.1] we describe the structure of completely simple semiring.

Let R be a skew-ring, $(I, \cdot)$ and $(\Lambda, \cdot)$ are bands such that $I \cap \Lambda = \{o\}$ and $P = (p_{\lambda,i})$ be a matrix over R, $i \in I, \lambda \in \Lambda$ under the assumptions

(i) $p_{\lambda,o} = p_{o,i} = 0$,

(ii) $p_{\lambda\mu,kj} = p_{\lambda\mu,ij} - p_{\nu\mu,ij} + p_{\nu\mu,kj}$,

(iii) $p_{\mu\lambda,jk} = p_{\mu\lambda,ji} - p_{\mu\nu,ji} + p_{\mu\nu,jk}$,

(iv) $ap_{\lambda,i} = p_{\lambda,i}a = 0$,

(v) $ab + p_{o\mu,io} = p_{o\mu,io} + ab$,

(vi) $ab + p_{\lambda o,oj} = p_{\lambda o,oj} + ab$, for all $i, j, k \in I$, $\lambda, \mu, \nu \in \Lambda$ and $a, b \in R$.

On $S = I \times R \times \Lambda$, we define '+' and '$\cdot$' by

$$(i, a, \lambda) + (j, b, \mu) = (i, a + p_{\lambda,j} + b, \mu)$$

and

$$(i, a, \lambda) \cdot (j, b, \mu) = (ij, -p_{\lambda\mu,ij} + ab, \lambda\mu).$$

Then $(S, +, \cdot)$ is a semiring which is called a *Rees matrix semiring* and is denoted by $\mathscr{M}(I, R, \Lambda; P)$. The authors in [6] proved (Theorem 3.1) that a semiring S is a completely simple semiring if and only if S is isomorphic to a Rees matrix semiring.

Next, we give a description of least skew-ring congruence to determine the interval of $\mathscr{Y}^*$ on completely regular semirings.

Lemma 4.1. *Let $S = (Y; S_\alpha)$ be a completely regular semiring, where Y is a b-lattice and S_α $(\alpha \in Y)$ is a completely simple semiring and $a, b \in S_\alpha$. Then the following statements are equivalent.*

(i) $a \mathscr{Y} b$,

(ii) $a = e+b+f$ and $b = g+a+h$ for any $e, f, g, h \in E^+(S)$ with $e \mathscr{R}^+ a \mathscr{L}^+ f$ and $g \mathscr{R}^+ b \mathscr{L}^+ h$,

(iii) $a = (a+x)^0 + b + (x+a)^0$ and $b = (b+x)^0 + a + (x+b)^0$ for any $x \in S_\alpha$.

Proof. $(i) \Rightarrow (ii)$: Let $a = (i, s, \lambda)$, $b = (j, t, \mu) \in S_\alpha$ and $a \mathscr{Y} b$. For any $e = (i, -p_{\delta,i}, \delta)$, $f = (k, -p_{\lambda,k}, \lambda) \in E^+(S_\alpha)$, we have $e \mathscr{R}^+ a \mathscr{L}^+ f$.

Let $c = (k, -p_{\lambda,k} - s - p_{\delta,i}, \delta) \in S_\alpha$. Then

$$\begin{aligned} a + c + a &= (i, s, \lambda) + (k, -p_{\lambda,k} - s - p_{\delta,i}, \delta) + (i, s, \lambda) \\ &= (i, s + p_{\lambda,k} - p_{\lambda,k} - s - p_{\delta,i} + p_{\delta,i} + s, \lambda) \\ &= (i, s, \lambda) = a. \end{aligned}$$

Since, S_α is a completely simple semiring, we have $c + a + c = c$. This implies, $c \in V^+(a)$.

Since $a \mathscr{Y} b$, we have $c \in V^+(b)$. Hence $b + c + b = b$, i.e., $(j, t, \mu) + (k, -p_{\lambda,k} - s - p_{\delta,i}, \delta) + (j, t, \mu) = (j, t + p_{\mu,k} - p_{\lambda,k} - s - p_{\delta,i} + p_{\delta,j} + t, \mu) = (j, t, \mu)$.

So we get, $t = -p_{\delta,j} + p_{\delta,i} + s + p_{\lambda,k} - p_{\mu,k}$. Then

$$\begin{aligned} e+b+f &= (i,-p_{\delta,i},\delta)+(j,-p_{\delta,j}+p_{\delta,i}+s+p_{\lambda,k}-p_{\mu,k},\mu)+(k,-p_{\lambda,k},\lambda)\\ &= (i,-p_{\delta,i}+p_{\delta,j}-p_{\delta,j}+p_{\delta,i}+s+p_{\lambda,k}-p_{\mu,k}+p_{\mu,k}-p_{\lambda,k},\lambda)\\ &= (i,s,\lambda)=a. \end{aligned}$$

Similarly, we can prove for any $g,h\in E^+(S_\alpha)$ with $g\,\mathscr{R}^+\,b\,\mathscr{L}^+\,h$, $b=g+a+h$.

$(ii)\Rightarrow(iii)$: For $x,a,b\in S_\alpha$, by Lemma 2.4(i), we have $(a+x)\,\mathscr{R}^+\,a\,\mathscr{L}^+\,(x+a)$ and $(b+x)\,\mathscr{R}^+\,b\,\mathscr{L}^+\,(x+b)$. This implies $(a+x)^0\,\mathscr{R}^+\,a\,\mathscr{L}^+\,(x+a)^0$ and $(b+x)^0\,\mathscr{R}^+\,b\,\mathscr{L}^+\,(x+b)^0$. Hence by (ii), $a=(a+x)^0+b+(x+a)^0$ and $b=(b+x)^0+a+(x+b)^0$.

$(iii)\Rightarrow(i)$: Let $c\in V^+(a)$ for $a,b\in S_\alpha$. Then, $c\in S_\alpha$, $(a+c),(c+a)\in E^+(S_\alpha)$. By (iii),

$$\begin{aligned} b &= (b+c)^0+a+(c+b)^0\\ &= (b+c)^0+a+c+a+(c+b)^0\\ &= (b+c)^0+(a+c)^0+b+(c+a)^0+c+(a+c)^0+b+(c+a)^0+(c+b)^0\\ &= (b+c)^0+(a+c)^0+b+(c+a)+c+(a+c)+b+(c+a)^0+(c+b)^0\\ &= (b+c)^0+(a+c)^0+b+c+b+(c+a)^0+(c+b)^0\\ &= (b+c)^0+b+c+b+(c+b)^0 \quad \text{[by Lemma 2.4 and Lemma 3.1]}\\ &= b+c+b. \end{aligned}$$

This implies $c\in V^+(b)$ and hence $V^+(a)\subseteq V^+(b)$. By symmetry, we get $V^+(a)=V^+(b)$. This completes the proof. □

Following [6, Definition 5.1] a normal subgroup N of $(R,+)$ (where R is a skew-ring) is said to be a *skew-ideal* of R if $a\in N$ implies $ca,ac\in N$ for all $c\in R$.

Notation 4.2. Let $S=\mathscr{M}(I,R,\Lambda;P)$ be a Rees matrix semiring over a skew-ring R. Let $\langle P\rangle$ denote the smallest skew-ideal of R generated by the elements of P.

Lemma 4.3. *Let $S=\mathscr{M}(I,R,\Lambda;P)$ be a completely simple semiring. Define a relation σ on S as: for all $a,b\in S$;*

$$a\,\sigma\,b \quad \textit{if and only if} \quad (g-h)\in\langle P\rangle,$$

where $a=(i,g,\lambda)$, $b=(j,h,\mu)\in S$. Then σ is the least skew-ring congruence on S.

Proof. The relation σ is obviously reflexive and symmetric.

Let $a\,\sigma\,b$ and $b\,\sigma\,c$ where $a,b,c\in S$. Also, let $a=(i,g,\lambda)$, $b=(j,h,\mu)$ and $c=(k,t,\delta)\in S$. Then $(g-h)\in\langle P\rangle$ and $(h-t)\in\langle P\rangle$. This implies $(g-t)\in\langle P\rangle$. Hence $a\,\sigma\,c$. Thus, σ is transitive and hence σ is an equivalence relation on S.

Next we prove that σ is compatible with respect to the operations in S. Let $a,b\in S$ such that $a\,\sigma\,b$. Then we have, $(g-h)\in\langle P\rangle$, where $a=(i,g,\lambda)$, $b=(j,h,\mu)\in S$. Let $c=(k,t,\delta)\in S$ be arbitrary. Therefore, $a+c=(i,g,\lambda)+(k,t,\delta)=(i,g+p_{\lambda,k}+t,\delta)$. Similarly, $b+c=(j,h,\mu)+(k,t,\delta)=(j,h+p_{\mu,k}+t,\delta)$.

Now, $(g+p_{\lambda,k}+t)-(h+p_{\mu,k}+t)=g+p_{\lambda,k}-p_{\mu,k}-h$. Again, $(g-h)\in\langle P\rangle$ implies $-h+g\in\langle P\rangle$, i.e., $p_{\lambda,k}-p_{\mu,k}-h+g+p_{\mu,k}-p_{\lambda,k}\in\langle P\rangle$, i.e., $g+p_{\lambda,k}-p_{\mu,k}-h+g+p_{\mu,k}-p_{\lambda,k}-g\in\langle P\rangle$. Also, $g+p_{\mu,k}-p_{\lambda,k}-g\in\langle P\rangle$.

Thus, $g+p_{\lambda,k}-p_{\mu,k}-h\in\langle P\rangle$. Hence, $(a+c)\,\sigma\,(b+c)$. Similarly, it can be shown that $(c+a)\,\sigma\,(c+b)$.

Again, $ac=(i,g,\lambda)(k,t,\delta)=(ik,-p_{\lambda\delta,ik}+gt,\lambda\delta)$ and $bc=(jk,-p_{\mu\delta,jk}+ht,\mu\delta)$. Now, $(g-h)\in\langle P\rangle$ implies $(gt-ht)\in\langle P\rangle$, i.e., $-p_{\lambda\delta,ik}+gt-ht+p_{\lambda\delta,ik}\in\langle P\rangle$, i.e., $-p_{\lambda\delta,ik}+gt-ht+p_{\mu\delta,jk}-p_{\mu\delta,jk}+p_{\lambda\delta,ik}\in\langle P\rangle$. Since, $-p_{\mu\delta,jk}+p_{\lambda\delta,ik}\in\langle P\rangle$ it follows that $-p_{\lambda\delta,ik}+gt-ht+p_{\mu\delta,jk}\in\langle P\rangle$. Therefore, $(ac)\,\sigma\,(bc)$. Similarly, $(ca)\,\sigma\,(cb)$. Consequently, σ is a congruence on $(S,+,\cdot)$.

Next we show that σ is a skew-ring congruence on S. If we can show that there is a unique additive idempotent in S/σ, then we are done. For this it is enough to prove that all additive idempotents of S are σ related.

Let $e,f\in E^+(S)$. Then $e=(i,-p_{\lambda,i},\lambda)$ and $f=(j,-p_{\mu,j},\mu)$. Now, $-p_{\lambda,i}+p_{\mu,j}\in\langle P\rangle$ implies that $e\,\sigma\,f$. This proves that σ is a skew-ring congruence on S.

At last, we prove that σ is the least skew-ring congruence on S. For this let ξ be any skew-ring congruence on S. Then both σ and ξ are group congruences on $(S,+)$. Moreover, by [1, Lemma 2.3], it follows that σ is the least group congruence on $(S,+)$. Thus, we must have $\sigma\subseteq\xi$. Consequently, σ is the least skew-ring congruence on S. This completes the proof. □

Lemma 4.4. *Let $S=(Y;S_\alpha)$ be a completely regular semiring where Y is a b-lattice and $S_\alpha\,(\alpha\in Y)$ is a completely simple semiring. If ν is the least b-lattice of skew-rings congruence on S, then $\mathscr{Y}^*\subseteq\nu$.*

Proof. Let $a,b\in S$ and $a\mathscr{Y}b$. Then there exists some $\alpha\in Y$ such that $a,b\in S_\alpha$. Let $a=(i,g,\lambda)$, $b=(j,h,\mu)$. By Lemma 4.1, we get

$$(i,g,\lambda)=(i,-p_{\delta,i},\delta)+(j,h,\mu)+(k,-p_{\lambda.k},\lambda),$$

since $(i,-p_{\delta,i},\delta)\,\mathcal{R}^+\,(i,g,\lambda)\,\mathcal{L}^+\,(k,-p_{\lambda,k},\lambda)$.

It follows that $g=-p_{\delta,i}+p_{\delta,j}+h+p_{\mu,k}-p_{\lambda,k}$ whence $g+p_{\lambda,k}-p_{\mu,k}-h-p_{\delta,j}+p_{\delta,i}=0$, where 0 is the zero of R. Taking $k=\delta=o$, we have $(g-h)=0\in\langle P\rangle$. Then by Lemma 4.3, it follows that $a\,\sigma_\alpha\,b$, where σ_α is the least skew-ring congruence on S_α. Hence $\mathscr{Y}|_{S_\alpha}\subseteq\sigma_\alpha$ for all $\alpha\in Y$.

Let $\nu|_{S_\alpha}=\nu_\alpha$. Then $\nu=\bigcup_{\alpha\in Y}\nu_\alpha$. Since S_α/ν_α is a skew-ring, it follows that $\sigma_\alpha\subseteq\nu_\alpha$ for all $\alpha\in Y$. Therefore, $\mathscr{Y}|_{S_\alpha}\subseteq\sigma_\alpha\subseteq\nu_\alpha$ for all $\alpha\in Y$ and hence $\mathscr{Y}^*\subseteq\nu$. □

Definition 4.5. A congruence ξ on a semiring S is said to be an *additive idempotent pure congruence* if $a\,\xi\,e$ with $a\in S$ and $e\in E^+(S)$ implies that $a\in E^+(S)$.

Theorem 4.6. *Let $S=\mathscr{M}(I,R,\Lambda;P)$ be a completely simple semiring. Then $\mathscr{Y}$ is the greatest additive idempotent pure congruence on S.*

Proof. Clearly, $\mathscr{Y}$ is an equivalence relation. Let $a,b\in S$ and $a\,\mathscr{Y}\,b$. By Lemma 4.1, for any $x,c\in S$, $a=(a+x+c)^0+b+(x+c+a)^0$ and $b=(b+x+c)^0+a+(x+c+b)^0$.

Hence, $c+a = c+(a+x+c)^0+b+(x+c+a)^0 = (c+a+x)^0+c+b+(x+c+a)^0$, by Lemma 2.4 (ii). Similarly, $c+b = (c+b+x)^0+(c+a)+(x+c+b))^0$. This implies $(c+a)\,\mathscr{Y}\,(c+b)$. Dually, it follows that $(a+c)\,\mathscr{Y}\,(b+c)$.

We now show that $(ac)\,\mathscr{Y}\,(bc)$. Let $a,b,c \in S$ and $a\,\mathscr{Y}\,b$. Then there exists some $\alpha \in Y$ such that $a,b \in S_\alpha$. Let $a=(i,x,\lambda)$, $b=(j,y,\mu)$ and $c=(k,z,\nu)$.

By Lemma 4.1, $a = e_1+b+f_1$ for all $e_1, f_1 \in E^+(S)$ with $e_1\,\mathscr{R}^+\,a\,\mathscr{L}^+\,f_1$, i.e., $(i,x,\lambda) = (i,-p_{t,i},t)+(j,y,\mu)+(s,-p_{\lambda,s},\lambda)$, for all $t \in \Lambda$ and for all $s \in I$, i.e., $(i,x,\lambda) = (i,-p_{t,i}+p_{t,j}+y+p_{\mu,s}-p_{\lambda,s},\lambda)$, for all $t \in \Lambda$ and for all $s \in I$, i.e., $x = -p_{t,i}+p_{t,j}+y+p_{\mu,s}-p_{\lambda,s}$ for all $t \in \Lambda$ and for all $s \in I$. ...(1)

We also note that $xz = yz$ for any $z \in S$. ...(2)

Again, $(i,x,\lambda)(k,z,\nu) = (i,-p_{t,i},t)(k,z,\nu)+(j,y,\mu)(k,z,\nu)+(s,-p_{\lambda,s},\lambda)(k,z,\nu)$,
i.e., $(ik,-p_{\lambda\nu,ik}+xz,\lambda\nu) = (ik,-p_{t\nu,ik},t\nu)+(jk,-p_{\mu\nu,jk}+yz,\mu\nu)+(sk,-p_{\lambda\nu,sk},\lambda\nu)$,
i.e., $(ik,-p_{\lambda\nu,ik}+xz,\lambda\nu) = (ik,-p_{t\nu,ik}+p_{t\nu,jk}-p_{\mu\nu,jk}+yz+p_{\mu\nu,sk}-p_{\lambda\nu,sk},\lambda\nu)$,
i.e., $-p_{\lambda\nu,ik}+xz = -p_{t\nu,ik}+p_{t\nu,jk}-p_{\mu\nu,jk}+yz+p_{\mu\nu,sk}-p_{\lambda\nu,sk}$,
i.e., $-p_{\lambda\nu,ik}+xz = -p_{t\nu,ik}+p_{t\nu,jk}-p_{\mu\nu,jk}+p_{\mu\nu,sk}-p_{\lambda\nu,sk}+yz$. ...(3)

Now, let $e=(ik,-p_{\delta,ik},\delta)$, $f=(l,-p_{\lambda\nu,l},\lambda\nu) \in E^+(S)$. Then $e\,\mathscr{R}^+\,(ac)\,\mathscr{L}^+\,f$.
Now,

$$\begin{aligned} e+bc+f &= (ik,-p_{\delta,ik},\delta)+(jk,-p_{\mu\nu,jk}+yz,\mu\nu)+(l,-p_{\lambda\nu,l},\lambda\nu) \\ &= ik,-p_{\delta,ik}+p_{\delta,jk}-p_{\mu\nu,jk}+yz+p_{\mu\nu,l}-p_{\lambda\nu,l},\lambda\nu) \\ &= (ik,-p_{\delta\nu,ik}+p_{\delta\nu,i}-p_{\delta,i}+p_{\delta,j}-p_{\delta\nu,j}+p_{\delta\nu,jk}-p_{\mu\nu,jk}+ \\ &\qquad yz+p_{\mu\nu,lk}-p_{\mu,lk}+p_{\mu,l}-p_{\lambda,l}+p_{\lambda,lk}-p_{\lambda\nu,lk},\lambda\nu) \end{aligned}$$

i.e., $e+bc+f = (ik,-p_{\delta\nu,ik}+p_{\delta\nu,jk}-p_{\mu\nu,jk}+yz+p_{\mu\nu,lk}-p_{\lambda\nu,lk},\lambda\nu)$, ...(4)

[By putting once $t=\delta$ and $t=\delta\nu$ and equating in (1) and again by putting $s=l$ and $s=lk$ and equating in (1) we obatin (4)]

Now, by substituting $t=\delta$ and $s=l$ in (3) we can obtain

$$\begin{aligned} -p_{\lambda\nu,ik}+xz &= -p_{\delta\nu,ik}+p_{\delta\nu,jk}-p_{\mu\nu,jk}+p_{\mu\nu,lk}-p_{\lambda\nu,lk}+yz \\ &= -p_{\delta\nu,ik}+p_{\delta\nu,jk}-p_{\mu\nu,jk}+yz+p_{\mu\nu,lk}-p_{\lambda\nu,lk}. \end{aligned}$$

Therefore,

$$\begin{aligned} e+bc+f &= (ik,-p_{\delta\nu,ik}+p_{\delta\nu,jk}-p_{\mu\nu,jk}+yz+p_{\mu\nu,lk}-p_{\lambda\nu,lk},\lambda\nu) \\ &= (ik,-p_{\lambda\nu,ik}+xz,\lambda\nu) \\ &= ac. \end{aligned}$$

Thus, we see that $ac = e+bc+f$ for any $e,f \in E^+(S)$ with $e\,\mathscr{R}^+\,(ac)\,\mathscr{L}^+\,f$. Similarly, we can show that $bc = g+ac+h$ for any $g,h \in E^+(S)$ with $g\,\mathscr{R}^+\,(bc)\,\mathscr{L}^+\,h$. Consequently, $\mathscr{Y}$ is a congruence on the semiring S.

Next we show that $\mathscr{Y}$ is an additive idempotent pure congruence on S. Let $a \in S$ with $a=(i,g,\lambda) \in S$, $e=(k,-p_{\lambda,k},\lambda) \in E^+(S)$ and $a\,\mathscr{Y}\,e$. Then $V^+(a) = V^+(e)$. By Lemma 4.1, for $f=(i,-p_{\lambda,i},\lambda)$, $h=(k,-p_{\lambda,k},\lambda) \in E^+(S)$ with $f\,\mathcal{R}^+\,a\,\mathcal{L}^+\,h$, we have $a = f+e+h = (i,-p_{\lambda,i},\lambda)+(k,-p_{\lambda,k},\lambda)+(k,-p_{\lambda,k},\lambda) = (i,-p_{\lambda,i},\lambda) \in E^+(S)$. Thus $\mathscr{Y}$ is an additive idempotent pure congruence on S.

Let η be any additive idempotent pure congruence on S. Let $a,b \in S$ such that $a\,\eta\,b$. Then by [1, Theorem 2.5], it follows that $a\,\mathscr{Y}\,b$. Hence, $\eta \subseteq \mathscr{Y}$, which proves that $\mathscr{Y}$ is the greatest additive idempotent pure congruence on S. $\square$

Theorem 4.7. *Let $S = (Y; S_\alpha)$ be a completely regular semiring, where Y is a b-lattice and $S_\alpha\,(\alpha \in Y)$ is a completely simple semiring. Then $\mathscr{Y}^* = \epsilon$ on S if and only if for each $\alpha \in Y$, ϵ_α is the unique additive idempotent pure congruence on S_α, where ϵ is the trivial congruence.*

Proof. First suppose that for each $\alpha \in Y$, ϵ_α is the unique additive idempotent pure congruence on S_α. Since $\mathscr{Y}$ is the greatest additive idempotent pure congruence on S, it follows that $\mathscr{Y}|_{S_\alpha} = \epsilon_\alpha$ on S_α. Hence $\mathscr{Y}^* = \epsilon$.

Conversely, let $\mathscr{Y}^* = \epsilon$. Now since $\mathscr{Y} \subseteq \mathscr{Y}^* = \epsilon$ and $\mathscr{Y}$ is reflexive on S, it follows that $\mathscr{Y} = \epsilon$ on S. This implies $\mathscr{Y}|_{S_\alpha} = \epsilon_\alpha$ and hence by Theorem 4.6, it follows that ϵ_α is the unique additive idempotent pure congruence on S_α for each $\alpha \in Y$. □

Combining Theorem 3.4, Lemma 4.4 and Theorem 4.7 we get the following result.

Theorem 4.8. *Let S be a completely regular semiring. Then $\mathscr{Y}^* \in [\epsilon, \nu]$, where ϵ is the equality congruence and ν is the least b-lattice of skew-ring congruence on S.*

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References

[1] **C. Guo, G. Liu and Y. Guo**, *The congruence $\mathscr{Y}^*$ on completely regular semigroups*, Commun. Algebra **39** (2011), 2082 – 2096.

[2] **U. Hebisch and H.J. Weinert**, *Semirings, Algebra Theory and Applications in Computer Science*, Series in Algebra **5**, World Scientific Singapore, 1998.

[3] **J.M. Howie** *Introduction to the theory of semigroups*, Academic Press (1976).

[4] **J.S. Golan**, *The Theory of Semirings with Applications in Mathematics and Theoretical Computer Science*, Pitman Monographs and Surveys in Pure and Applied Mathematics **54**, Longman Scientific (1992).

[5] **M.K. Sen, S.K. Maity and K.P. Shum**, *On completely regular semirings*, Bull. Cal. Math. Soc. **98** (2006), no. 4, 319 – 328.

[6] **M.K. Sen, S.K. Maity and H.J. Weinert**, *Completely simple semirings*, Bull. Cal. Math. Soc. **97** (2005), no. 2, 163 – 172.

[7] **M. Petrich**, *Regular semigroups which are subdirect products of a band and a semilattice of groups*, Glasgow Math. J. **14** (1973), 27 – 49.

[8] **M. Petrich and N.R. Reilly**, *Completely Regular Semigroups*, Wiley, New York, 1999.

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Behrends-Humble simple maps are regular

Lon Mitchell

Abstract. We consider simple binary operations in the sense of Behrends and Humble. We prove that a groupoid (magma) with such a map is regular. As a consequence, a division groupoid with simple binary operation is a quasigroup.

Let G be a groupoid (magma) with binary operation φ. The map φ induces maps $\varphi_n : G^{n+1} \to G^n$ by

$$\varphi_n(s_0, s_1, \ldots, s_n) = \big(\varphi(s_0, s_1), \varphi(s_1, s_2), \ldots, \varphi(s_{n-1}, s_n)\big).$$

Let Φ_n be the composition $\varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_n$. For an integer $k \geqslant 2$, we say that φ is *k-simple* if $\Phi_k(s_0, \ldots, s_k) = \varphi(s_0, s_k)$ for all $s_0, \ldots, s_k \in G$ and that φ is *simple* if it is k-simple for some k.

Simple maps were first studied by Ehrhard Behrends and Steve Humble [1]. Michael Jones, Brittany Shelton, and the author recently proved that any groupoid with simple binary operation is medial [4]. In this note, we establish that any groupoid with simple binary operation is regular and show that any division groupoid with simple binary operation is a quasigroup. We also offer remarks on cancellation groupoids and 2-simple maps.

Theorem 1. *If G is a groupoid with simple binary operation φ, then G is regular.*

Proof. Suppose that φ is n-simple for some integer n, $a, b \in G$ and $\varphi(a, x) = \varphi(b, x)$ for some $x \in G$. Given any other $y \in G$, we find that

$$\begin{aligned}\varphi(a, y) &= \Phi_n(a, x, \ldots, x, y) = \Phi_{n-1}\big(\varphi(a, x), \varphi(x, x), \ldots, \varphi(x, x), \varphi(x, y)\big)\\ &= \Phi_{n-1}\big(\varphi(b, x), \varphi(x, x), \ldots, \varphi(x, x), \varphi(x, y)\big) = \Phi_n(b, x, \ldots, x, y) = \varphi(b, y).\end{aligned}$$

Similarly, $\varphi(x, a) = \varphi(x, b)$ implies $\varphi(y, a) = \varphi(y, b)$ for all $y \in G$. □

Theorem 2. *A division groupoid with simple binary operation is a quasigroup.*

Proof. If $(G, \cdot)$ is a division groupoid with simple binary operation, it is medial [4] and regular. Thus there exists a binary operation $+$ on G such that $(G, +)$ is an Abelian group and there exist commuting surjective endomorphisms f and g of $(G, +)$ and an element $c \in G$ such that $xy = f(x) + g(y) + c$ for all $x, y \in G$ [2].

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Let 0 be the identity element of $(G,+)$. For $q_0,\dots,q_n \in Q$, by simplicity,

$$f(q_0)+g(q_n)+c = \\ f^n(q_0)+\binom{n}{1}f^{n-1}g(q_1)+\cdots+\binom{n}{n-1}fg^{n-1}(q_{n-1})+g^n(q_n) \\ +\big((f+g)+(f+g)^2+\cdots+(f+g)^{n-1}\big)(c)+c.$$

If we let the q_i all be 0, we find $\big((f+g)+(f+g)^2+\cdots+(f+g)^{n-1}\big)(c)=0$. Next, letting all of the q_i except q_0 or q_n be 0, we find $g=g^n$ and $f=f^n$. Since f and g are also surjective, they must be automorphsims. By the Bruck-Murdoch-Toyoda Theorem [3], (G,φ) is a quasigroup. □

Theorem 3. *If (G,φ) is a groupoid and φ is 2-simple, then (G,φ) is a semigroup.*

Proof. If φ is 2-simple, (G,φ) is medial [4]. Then, for any $a,b,c\in G$,

$$\varphi\big(a,\varphi(b,c)\big)=\Phi_2\big(a,\varphi(b,b),\varphi(b,c)\big)=\varphi\big(\varphi(a,\varphi(b,b)),\varphi(b,c)\big) \\ =\varphi\big(\varphi(a,b),\varphi(\varphi(b,b),c)\big)=\Phi_2\big(\varphi(a,b),\varphi(b,b),c\big)=\varphi\big(\varphi(a,b),c\big),$$

so that φ is associative. □

If G is a cancellation groupoid with simple binary operation, then G is medial [4]. As a result, there exists a medial quasigroup $(Q,\cdot)$ such that G is a dense subgroupoid of Q; moreover, G and Q satisfy the same identities [5, 2]. In particular, Q has simple binary operation. Let n be a positive integer such that the operation of $(Q,\cdot)$ is n-simple. Define Φ_n as above using $\varphi(x,y)=xy$. If $x,y\in G$, $q\in Q$, and $xq\in G$, then $yq\in G$, since $yq=\Phi_n(y,x,\dots,x,q)=\Phi_{n-1}(yx,xx,\dots,xx,xq)$ and $yx,xx,xq\in G$. Can $G\neq Q$?

References

[1] **E. Behrends and S. Humble**, *Triangle mysteries*, Math. Intelligencer **35** (2013), no.2, 10 – 15.

[2] **J. Ježek and T. Kepka**, *Medial groupoids*, Rozpravy Československé Akad. Věd Řada Mat. Přírod. Věd **93** (1983), no. 2.

[3] **E.N. Kuz'min and I.P. Shestakov**, *Non-associative structures*, Algebra VI, Encyclopedia Math. Sci. **57** (1995), 197 – 280.

[4] **L. Mitchell, M.A. Jones and B. Shelton**, *Abelian and non-Abelian triangle mysteries*, Amer. Math. Monthly **123** (2016), 808 – 813.

[5] **M. Sholander**, *On the existence of the inverse operation in alternation groupoids*, Bull. Amer. Math. Soc. **55** (1949), 746 – 757.

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Regularity of ternary semihypergroups

Krisanthi Naka and Kostaq Hila

Abstract. We study some properties of regular ternary semihypergroups, completely regular ternary semihypergroups, intra-regular ternary semihypergroups and characterize them by using various hyperideals of ternary semihypergroups.

1. Introduction and preliminaries

In 1965, Sioson [14] studied ideal theory in ternary semigroups. In [4, 5] Dudek et. al. studied the ideals in n-ary semigroups. In 1995, Dixit and Dewan [3] introduced and studied some properties of ideals and quasi-(bi-)ideals in ternary semigroups. Other important results on ternary semigroups are obtained in [12, 13, 16, 15].

Hyperstructure theory was introduced in 1934, when F. Marty [11] defined hypergroups based on the notion of hyperoperation, began to analyze their properties and applied them to groups. In the following decades and nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics by many mathematicians. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Davvaz et al. in [2] considered a class of algebraic hypersystems which represent a generalization of semigroups, hypersemigroups and n-ary semigroups.

In this paper we extend the notion of regularity in ternary semihypergroups and we study some properties of regular ternary semihypergroups, completely regular ternary semihypergroups, intra-regular ternary semihypergroups and characterize them by using various hyperideals of ternary semihypergroups extending those for ternary semigroups.

Recall first the basic terms and definitions from the ternary semihypergroups theory.

Definition 1.1. A map $f : H \times H \times H \to \mathcal{P}^*(H)$ is called *ternary hyperoperation* on the set H, where H is a nonempty set and $\mathcal{P}^*(H)$ denotes the collection of all nonempty subsets of H.

A *ternary hypergroupoid* is called the pair (H, f) where f is a ternary hyperoperation on the set H.

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If A, B, C are nonempty subsets of H, then we define

$$f(A,B,C) = \bigcup_{a\in A, b\in B, c\in C} f(a,b,c).$$

A ternary hypergroupoid (H, f) is called a *ternary semihypergroup* if for all $a_1, a_2, \ldots, a_5 \in H$, we have

$$f(f(a_1,a_2,a_3),a_4,a_5) = f(a_1, f(a_2,a_3,a_4),a_5) = f(a_1,a_2,f(a_3,a_4,a_5)).$$

A nonempty subset T of H is called a *ternary subsemihypergroup* of H if and only if $f(T,T,T) \subseteq T$.

Definition 1.2. Let (H, f) be a ternary semihypergroup. Then H is called a *ternary hypergroup* if for all $a, b, c \in H$, there exist $x, y, z \in H$ such that:

$$c \in f(x,a,b) \cap f(a,y,b) \cap f(a,b,z).$$

Definition 1.3. Let (H, f) be a ternary hypergroupoid. Then

1. (H, f) is $(1,3)$-*commutative* if for all $a_1, a_2, a_3 \in H, f(a_1,a_2,a_3) = f(a_3,a_2,a_1)$;
2. (H, f) is $(2,3)$-*commutative* if for all $a_1, a_2, a_3 \in H, f(a_1,a_2,a_3) = f(a_1,a_3,a_2)$;
3. (H, f) is $(1,2)$-*commutative* if for all $a_1, a_2, a_3 \in H, f(a_1,a_2,a_3) = f(a_2,a_1,a_3)$;
4. (H, f) is *commutative* if for all $a_1, a_2, a_3 \in H$ and for all $\sigma \in \mathbb{S}_3, f(a_1,a_2,a_3) = f(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$.

Definition 1.4. A ternary semihypergroup (H, f) is said to have a *zero element* if there exists an element $0 \in H$ such that for all $a, b \in H, f(0,a,b) = f(a,0,b) = f(a,b,0) = \{0\}$. An element $e \in H$ is called *left* (*right*) *identity* element of H if for all $a \in H$, $f(a,e,e) = \{a\}(f(e,e,a) = \{a\})$. An element $e \in H$ is called an *identity* element of H if for all $a \in H$, $f(a,e,e) = f(e,e,a) = f(e,a,e) = \{a\}$.

Definition 1.5. Let (H, f) be a ternary semihypergroup. A nonempty subset I of a ternary semihypergroup H is called a *left* (*right, lateral*) *hyperideal* of H if

$$f(H,H,I) \subseteq I(f(I,H,H) \subseteq I, f(H,I,H) \subseteq I).$$

A nonempty subset I of H is called a *hyperideal* of H if it is a left, right and lateral hyperideal of H. A nonemtpy subset I of H is called *two-sided hyperideal* of H if it is a left and right hyperideal of H. A lateral hyperideal I of H is called a *proper lateral hyperideal* of H if $I \neq H$. A left hyperideal I of H is called *idempotent* if $f(I,I,I) = I$.

Example 1.6. Let $H = \{a, b, c, d, e, g\}$ and $f(x, y, z) = (x*y)*z$ for all $x, y, z \in H$, where $*$ is defined by the table:

$*$	a	b	c	d	e	g
a	a	$\{a,b\}$	c	$\{c,d\}$	e	$\{e,g\}$
b	b	b	d	d	g	g
c	c	$\{c,d\}$	c	$\{c,d\}$	c	$\{c,d\}$
d	d	d	d	d	d	d
e	e	$\{e,g\}$	c	$\{c,d\}$	e	$\{e,g\}$
g	g	g	d	d	g	g

Then (H, f) is a ternary semihypergroup. Clearly, $I_1 = \{c, d\}$, $I_2 = \{c, d, e, g\}$ and H are lateral hyperideals of H.

Let (H, f) be a ternary semihypergroup. It is clear that the intersection of all lateral hyperideals of a ternary subsemihypergroup T of H containing a nonempty subset A of T is the *lateral hyperideal of H generated by A*.

For every element $a \in H$, the left, right, lateral, two-sided and hyperideal generated by a are respectively given by

$$\begin{aligned}
\langle a\rangle_l &= \{a\} \cup f(H, H, a),\\
\langle a\rangle_r &= \{a\} \cup f(a, H, H),\\
\langle a\rangle_m &= \{a\} \cup f(H, a, H) \cup f(H, H, a, H, H),\\
\langle a\rangle_t &= \{a\} \cup f(H, H, a) \cup f(a, H, H) \cup f(H, H, a, H, H),\\
\langle a\rangle &= \{a\} \cup f(H, H, a) \cup f(a, H, H) \cup f(H, a, H) \cup f(H, H, a, H, H).
\end{aligned}$$

Definition 1.7. Let (H, f) be a ternary semihypergroup. A proper hyperideal P of H is called *prime hyperideal* of H if $f(A, B, C) \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$ for any three hyperideals A, B, C of H.

A proper hyperideal P of H is said to be *strongly irreducible*, if for hyperideals T and K of H, $T \cap K \subseteq P$ implies that $T \subseteq P$ or $K \subseteq P$

A proper hyperideal A of a ternary semihypergroup H is called a *semiprime hyperideal* of H if $f(I, I, I) \subseteq A$ implies $I \subseteq A$ for any hyperideal I of H.

A proper hyperideal A of a ternary semihypergroup H is called *completely semiprime hyperideal* of H if $f(x, x, x) \subseteq A$ implies that $x \in A$ for any element $x \in A$.

Definition 1.8. A ternary subsemihypergroup B of a ternary semihypergroup H is called a *bi-hyperideal* of H if $f(B, H, B, H, B) \subseteq B$.

2. Regular ternary semihypergroups

Definition 2.1. A ternary semihypergroup H is said to be *regular* if for each $a \in H$, there exists an element $x \in H$ such that $a \in f(a, x, a)$.

A ternary semihypergroup H is called regular if all of its elements are regular.

It is clear that every ternary hypergroup is a regular ternary semihypergroup.

The ternary semihypergroup of the Example 1.6 is regular ternary semihypergroup.

We note that every left and right hyperideal of a regular ternary semihypergroup may not be a regular ternary semihypergroup; however, for a lateral hyperideal of a regular ternary semihypergroup, we have the following lemma:

Lemma 2.2. *Every lateral hyperideal of a regular ternary semihypergroup H is a regular ternary semihypergroup.*

Proof. Let L be a lateral hyperideal of a regular ternary semihypergroup H. Then for every $a \in L$, there exists $x \in H$ such that $a \in f(a, x, a)$. Now $a \in f(a, x, a) \subseteq f(a, x, f(a, x, a)) \subseteq f(a, f(x, a, x), a) \subseteq f(a, L, a)$. So there exists $b \in L$ such that $a \in f(a, b, a)$. This implies that L is a regular ternary semihypergroup. $\square$

Obviously, every hyperideal of a regular ternary semihypergroup H is a regular ternary semihypergroup.

Theorem 2.3. *Let (H, f) be a ternary semihypergroup. Then the following statements are equivalent:*

(1) *H is regular.*

(2) *For any right hyperideal R, lateral hyperideal M and left hyperideal L of H, $f(R, M, L) = R \cap M \cap L$.*

(3) *For $a, b, c \in H$, $f(\langle a\rangle_r, \langle b\rangle_m, \langle c\rangle_l) = \langle a\rangle_r \cap \langle b\rangle_m \cap \langle c\rangle_l$.*

(4) *For $a \in H$, $f(\langle a\rangle_r, \langle a\rangle_m, \langle a\rangle_l) = \langle a\rangle_r \cap \langle a\rangle_m \cap \langle a\rangle_l$.*

Proof. (1) $\Rightarrow$ (2). Let H be a regular ternary semihypergroup. Let R, M and L be a right hyperideal, a lateral hyperideal and a left hyperideal of H respectively. Then clearly, $f(R, M, L) \subseteq R \cap M \cap L$. Now for $a \in R \cap M \cap L$, we have $a \in f(a, x, a)$ for some $x \in H$. This implies that $a \in f(a, x, a) \subseteq f(f(a, x, a), x, f(a, x, a)) \subseteq f(R, M, L)$. Thus we have $R \cap M \cap L \subseteq f(R, M, L)$. So we find that $f(R, M, L) = R \cap M \cap L$.

Clearly, (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4).

It remains to show that (4) $\Rightarrow$ (1).

Let $a \in H$. Clearly, $a \in \langle a\rangle_r \cap \langle a\rangle_m \cap \langle a\rangle_l = f(\langle a\rangle_r, \langle a\rangle_m, \langle a\rangle_l)$. Then we have, $a \in f(f(a, H, H) \cup \{a\}, f(H, a, H) \cup f(H, H, a, H, H) \cup \{a\}, f(H, H, a) \cup \{a\}) \subseteq f(a, H, a)$. So we find that $a \in f(a, H, a)$ and hence there exists an element $x \in H$ such that $a \in f(a, x, a)$. This implies that a is regular and hence H is regular. $\square$

Corollary 2.4. *Let (H, f) be a ternary semihypergroup. Then the following statements are equivalent:*

(1) *H is regular.*

(2) *For any right hyperideal R and left hyperideal L of H, $f(R,H,L) = R \cap L$.*

(3) *For $a, b \in H$, $f(\langle a \rangle_r, H, \langle b \rangle_l) = \langle a \rangle_r \cap \langle b \rangle_l$.*

(4) *For $a \in H$, $f(\langle a \rangle_r, H, \langle a \rangle_l) = \langle a \rangle_r \cap \langle a \rangle_l$.*

Theorem 2.5. *A ternary semihypergroup H is regular if and only if every hyperideal of H is idempotent.*

Proof. Let H be a regular ternary semihypergroup and I be any hyperideal of H. Then $f(I,I,I) \subseteq f(H,H,I) \subseteq I$. Let $a \in I$. Then there exists $x \in H$ such that $a \in f(a,x,a) \subseteq f(a,x,f(a,x,a))$. Since I is a hyperideal and $a \in I$, $f(x,a,x) \subseteq I$. Thus $a \in f(a,x,a) \subseteq f(a,x,f(a,x,a)) \subseteq f(I,I,I)$. Consequently, $I \subseteq f(I,I,I)$ and hence $f(I,I,I) = I$, that is I is idempotent.

Conversely, suppose that every hyperideal of H is idempotent. Let A, B and C be three hyperideals of H. Then $f(A,B,C) \subseteq f(A,H,H) \subseteq A, f(A,B,C) \subseteq f(H,B,H) \subseteq B$ and $f(A,B,C) \subseteq f(H,H,C) \subseteq C$. This implies that $f(A,B,C) \subseteq A \cap B \cap C$. Also, $f(A \cap B \cap C, A \cap B \cap C, A \cap B \cap C) \subseteq f(A,B,C)$. Again, since $A \cap B \cap C$ is a hyperideal of H, $f(A \cap B \cap C, A \cap B \cap C, A \cap B \cap C) = A \cap B \cap C$. Thus $A \cap B \cap C \subseteq f(A,B,C)$ and hence $A \cap B \cap C = f(A,B,C)$. Therefore, by Theorem 2.3, H is a regular ternary semihypergroup. □

Theorem 2.6. *A commutative ternary semihypergroup H is regular if and only if every hyperideal of H is semiprime.*

Proof. Let H be a commutative regular ternary semihypergroup and I be any hyperideal of H such that $f(A,A,A) \subseteq I$ for any hyperideal A of H. From Theorem 2.3, it follows that $f(A,A,A) = A$. Consequently, $A \subseteq I$ and hence I is a semiprime hyperideal of H.

Conversely, suppose that every hyperideal of a commutative ternary semihypergroup H is semiprime. Let $a \in H$. Then $f(a,H,a)$ is a hyperideal of H. Now by hypothesis, $f(a,H,a)$ is a semiprime hyperideal of H. If $f(a,H,a) = H$, then we are done. Now suppose that $f(a,H,a) \neq H$. Then

$$\begin{aligned} f(\langle a \rangle, \langle a \rangle, \langle a \rangle) &= f(f(H,H,a) \cup f(a,H,H) \cup f(H,a,H) \cup \\ &\quad \cup f(H,H,a,H,H) \cup \{a\}, f(H,H,a) \cup f(a,H,H) \cup \\ &\quad \cup f(H,a,H) \cup f(H,H,a,H,H) \cup \{a\}, f(H,H,a) \cup \\ &\quad \cup f(a,H,H) \cup f(H,a,H) \cup f(H,H,a,H,H) \cup \{a\}) \\ &\subseteq f(a,H,a) \end{aligned}$$

that is, $f(\langle a \rangle, \langle a \rangle, \langle a \rangle) \subseteq f(a,H,a)$. This implies that $\langle a \rangle \subseteq f(a,H,a)$, since $f(a,H,a)$ is a semiprime hyperideal of H. Consequently, $a \in f(a,x,a)$ for some $x \in H$ and hence H is a regular ternary semihypergroup. □

Let N be the *nuclear hyperideal* of a ternary semihypergroup (H,f), that is the intersection of all hyperideals in H, N_r the intersection of all right hyperideals in H, N_m the intersection of all lateral hyperideals of H, and N_l the intersection of all left hyperideals of H.

Theorem 2.7. *Let (H, f) be a ternary semihypergroup and let $N = N_r = N_m = N_l \neq \emptyset$. Then H is regular if and only if N is regular ternary semihypergroup.*

Proof. If H is regular, then clearly N is also regular as a hyperideal.

Conversely, suppose that N is a regular hyperideal of H, so that for any right hyperideal R, lateral hyperideal M, and left hyperideal L of H,

$$N \cup f(R, M, L) = R \cap M \cap L.$$

Since $f(N, N, N)$ is both a right and left hyperideal, then

$$f(R, M, L) \subseteq f(N, N, N) \subseteq N.$$

Whence $f(R, M, L) = R \cap M \cap L$. □

Corollary 2.8. *Let (H, f) be a ternary semihypergroup and let $N = N_r = N_m = N_l \neq \emptyset$. Then H is regular if and only if every hyperideal of H is regular.*

Proof. If H is regular, then N is a regular hyperideal. Hence any hyperideal I which necessary contains N is also a regular hyperideal.

Conversely, if every hyperideal of H is regular, then N is regular. Thus by the previous Theorem 2.7, H is regular ternary semihypergroup. □

Theorem 2.9. *Let (H, f) be a ternary semihypergroup and I a hyperideal of H. The following statements are equivalent:*

(1) *I is a regular hyperideal of H;*

(2) *For every $a \in H$, $I \cup f(\langle a \rangle_r, \langle a \rangle_m, \langle a \rangle_l) = I \cup (\langle a \rangle_r \cap \langle a \rangle_m \cap \langle a \rangle_l)$;*

(3) *For every $a \in H \backslash I$, either $a \in f(a, a_1, a, a_2, a)$ or $a \in f(a, b_1, b_2, a, b_3, b_4, a)$, for some $a_1, a_2, b_1, b_2, b_3, b_4 \in H$.*

Proof. (1) ⇒ (2). Suppose that I is a regular hyperideal. Then for each $a \in H$,

$$I \cup (\langle a \rangle_r \cap \langle a \rangle_m \cap \langle a \rangle_l) \subseteq \langle I \cup \langle a \rangle_r \rangle_r, \langle I \cup \langle a \rangle_m \rangle_m, \langle I \cup \langle a \rangle_l \rangle_l.$$

Moreover, since each of the three sets on the right side contains I, then we have

$$\begin{aligned} & I \cup (\langle a \rangle_r \cap \langle a \rangle_m \cap \langle a \rangle_l) \subseteq \langle I \cup \langle a \rangle_r \rangle_r \cap \langle I \cup \langle a \rangle_m \rangle_m \cap \langle I \cup \langle a \rangle_l \rangle_l \\ & = I \cup f(I \cup \langle a \rangle_r, I \cup \langle a \rangle_m, I \cup \langle a \rangle_l) \\ & = I \cup f(I, I \cup \langle a \rangle_m, I \cup \langle a \rangle_l) \cup f(\langle a \rangle_r, I, I \cup \langle a \rangle_l) \cup f(\langle a \rangle_r, \langle a \rangle_m, I) \cup f(\langle a \rangle_r, \langle a \rangle_m, \langle a \rangle_l) \\ & = I \cup f(\langle a \rangle_r, \langle a \rangle_m, \langle a \rangle_l) \subseteq I \cup (\langle a \rangle_r \cap \langle a \rangle_m \cap \langle a \rangle_l). \end{aligned}$$

(2) $\Rightarrow$ (3). We note that

$$\begin{aligned}\langle I\cup\langle a\rangle_r\rangle_r &= \langle I\cup\langle a\rangle_r\rangle_r\cap H\cap H = I\cup f(\langle I\cup\langle a\rangle_r\rangle_r, H, H)\\ &= I\cup f(I,H,H)\cup f(\langle a\rangle_r, H,H)\cup f(I,H,H,H,H)\cup\\ &\quad \cup f(\langle a\rangle_r, H,H,H,H)\\ &= I\cup f(I,H,H)\cup f(a,H,H,H)\cup f(a,H,H,H,H)\cup\\ &\quad \cup f(I,H,H,H,H)\cup\\ &\quad \cup f(a,H,H,H,H)\cup f(a,H,H,H,H,H,H)\\ &= I\cup f(I,H,H)\cup f(a,H,H)\cup f(a,H,H,H,H)\\ &= \langle I\cup f(a,H,H)\rangle_r = I\cup f(a,H,H).\end{aligned}$$

In the same manner, we obtain

$$\begin{aligned}\langle I\cup\langle a\rangle_m\rangle_m &= \langle I\cup f(H,a,H)\rangle_m = I\cup f(H,a,H)\cup f(H,H,a,H,H),\\ \langle I\cup\langle a\rangle_l\rangle_l &= \langle I\cup f(H,H,a)\rangle_l = I\cup f(H,H,a).\end{aligned}$$

Then

$$\begin{aligned}&\langle I\cup f(a,H,H)\rangle_r\cap\langle I\cup f(H,a,H)\rangle_m\cap\langle I\cup f(H,H,a)\rangle_l\\ &= I\cup f(\langle I\cup f(a,H,H)\rangle_m, \langle I\cup f(H,a,H)\rangle_m, \langle I\cup f(H,H,a)\rangle_l)\\ &= I\cup f(a,H,H,H,a,H,H,H,a)\cup f(a,H,H,H,H,a,H,H,H,H,a)\\ &= I\cup f(a,H,a,H,a)\cup f(a,H,H,a,H,H,a).\end{aligned}$$

The result now follows.

(3) $\Rightarrow$ (1). Let R be an arbitrary right hyperideal, M an arbitrary lateral hyperideal, L an arbitrary left hyperideal of H all containing I. Let us assume that I satisfies the condition (3). It is clear that,

$$I\cup f(R,M,L)\subseteq R\cap M\cap L.$$

Let $a\in R\cap M\cap L$. By (3), then $a\in I$ or $a\in f(f(a,a_1,a,a_2,a)$ or $a\in f(a,b_1,b_2,a,b_3,b_4,a)$ for some $a_1,a_2,b_1,b_2,b_3,b_4\in H$. We note also that in the second and third cases we have:

$$\begin{aligned}&a\in f(a,a_1,a,a_1,a,a_2,a,a_2,a) = f(f(a,a_1,a_2), f(a_1,a,a_2), f(a,a_2,a)),\\ &a\in f(a,b_1,b_2,a,b_1,b_2,a,b_3,b_4,a,b_3,b_4,a) =\\ &\quad = f(f(a,b_1,b_2), f(a,b_1,b_2), a, f(b_3,b_4,a), f(b_3,b_4,a)).\end{aligned}$$

Hence in the last two cases we have

$$a\in f(f(a,x_2,x_3), f(y_1,a,y_3), f(z_1,z_2,a)),$$

for some $x_2,x_3,y_1,y_2,z_1,z_2\in H$. Whence, in any case we have:

$$a\in I\cup f(R,M,L)$$

and therefore $I \cup f(R, M, L) = R \cap M \cap L$. □

Theorem 2.10. *Let (H, f) be a ternary semihypergroup and I a regular hyperideal of a H. Then, for any right hyperideal R, lateral hyperideal M, and left hyperideal L of H, if $f(R, M, L) \subseteq I$, then $R \cap M \cap L \subseteq I$.*

Proof. Suppose $f(R, M, L) \subseteq I$ and I is a regular hyperideal. Then

$$
\begin{aligned}
&R \cap M \cap L \subseteq \langle I \cup R\rangle_r \cap \langle I \cup M\rangle_m \cap \langle I \cup L\rangle_l \\
&I \cup f(\langle I \cup R\rangle_r, \langle I \cup M\rangle_m, \langle I \cup L\rangle_l) = I \cup f(I, \langle I \cup M\rangle_m, \langle I \cup L\rangle_l) \\
&= f(R, I, \langle I \cup L\rangle_l) \cup f(R, M, I) \cup f(R, M, L) \subseteq I.
\end{aligned}
$$
□

Corollary 2.11. *A regular and strongly irreducible hyperideal is always prime.*

Corollary 2.12. *Every regular hyperideal is prime.*

Definition 2.13. Let (H, f) be a ternary semihypergroup and Q a nonempty subset of H. Then Q is called a *quasi-hyperideal* of H if and only if

$$
\begin{gathered}
f(Q, H, H) \cap f(H, Q, H) \cap f(H, H, Q) \subseteq Q \text{ and} \\
f(Q, H, H) \cap f(H, H, Q, H, H) \cap f(H, H, Q) \subseteq Q.
\end{gathered}
$$

Theorem 2.14. *Let (H, f) be a regular ternary semihypergroup and Q be a nonempty subset of H. Then Q is a quasi-hyperideal if and only if*

$$f(Q, H, Q, H, Q) \cap f(Q, H, H, Q, H, H, Q) \subseteq Q.$$

Proof. Let H be a regular ternary semihypergroup and Q be a quasi-hyperideal of H. Then

$$
\begin{gathered}
f(Q, H, Q, H, Q) \cap f(Q, H, H, Q, H, H, Q) \subseteq f(H, H, Q),\ f(Q, H, H), \text{ and} \\
f(H, Q, H) \cup f(H, H, Q, H, H)
\end{gathered}
$$

and hence

$$
\begin{aligned}
&f(Q, H, Q, H, Q) \cap f(Q, H, H, Q, H, H, Q) \subseteq \\
&\subseteq f(H, H, Q) \cap (f(H, Q, H) \cup f(H, H, Q, H, H)) \cap f(Q, H, H) \subseteq Q.
\end{aligned}
$$

Conversely, suppose that H is regular and

$$f(Q, H, Q, H, Q) \cap f(Q, H, H, Q, H, H, Q) \subseteq Q.$$

Then

$$
\begin{aligned}
&f(Q, H, H) \cap (f(H, Q, H) \cup f(H, H, Q, H, H)) \cap f(H, H, Q) \\
&= f(f(Q, H, H), f(H, Q, H) \cup f(H, H, Q, H, H), f(H, H, Q)) \\
&= f(f(Q, H, H), f(H, Q, H), f(H, H, Q)) \cup f(f(Q, H, H), f(H, H, Q, H, H), \\
&f(H, H, Q)) \subseteq f(Q, H, Q, H, Q) \cup f(Q, H, H, Q, H, H, Q) \subseteq Q.
\end{aligned}
$$
□

Theorem 2.15. *Let (H,f) be a regular ternary semihypergroup and Q_1,Q_2,Q_3 be three quasi-hyperideals of H. Then $f(Q_1,Q_2,Q_3)$ is a quasi-hyperideal.*

Proof.

$$\begin{aligned}&f(f(Q_1,Q_2,Q_3),H,f(Q_1,Q_2,Q_3),H,f(Q_1,Q_2,Q_3)) \cup f(f(Q_1,Q_2,Q_3),H,H,\\&f(Q_1,Q_2,Q_3),H,H,f(Q_1,Q_2,Q_3))\\&= f(f(Q_1,f(Q_2,Q_3,H),Q_1,f(Q_2,Q_3,H),Q_1),Q_2,Q_3) \cup\\&\cup f(f(Q_1,f(Q_2,Q_3,H),H,Q_1,f(Q_2,Q_3,H),H,Q_1),Q_2,Q_3) \subseteq\\&\subseteq f(Q_1,Q_2,Q_3).\end{aligned}$$

□

Corollary 2.16. *The family of all quasi-hyperideals of a regular ternary semihypergroup is a ternary semihypergroup.*

Theorem 2.17. *Let (H,f) be a ternary semihypergroup. If for every quasi-hyperideal Q of H, $f(Q,Q,Q)=Q$, then H is a regular ternary semihypergroup.*

Proof. Let R be a right hyperideal of H, L be a left hyperideal of H and M be a lateral hyperideal of H. By Theorem 2.2 [9], $R\cap M\cap L$ is a quasi-hyperideal of H. Then by hypothesis, we have

$$R\cap M\cap L = f(R\cap M\cap L, R\cap M\cap L, R\cap M\cap L) \subseteq f(R,M,L).$$

On the other hand, $f(R,M,L)\subseteq R\cap M\cap L$. Therefore we have $f(R,M,L) = R\cap M\cap L$. By Theorem 2.3(2), H is a regular ternary semihypergroup. □

Theorem 2.18. *Let (H,f) be a ternary semihypergroup. The following statements are equivalent:*

(1) *H is regular;*

(2) *For every bi-hyperideal B of H, $f(B,H,B,H,B)=B$;*

(3) *For every quasi-hyperideal Q of H, $f(Q,H,Q,H,Q)=Q$.*

Proof. (1) $\Rightarrow$ (2). Let us assume that H is regular and B be a bi-hyperideal of H. Let $b\in B$. From regularity of H, there exists $x\in H$, such that $b\in f(b,x,b)$. Thus, $B\subseteq f(B,H,B)$. We have $b\in f(b,x,b)\subseteq f(b,x,f(b,x,b))\subseteq f(B,H,f(B,H,B)) = f(B,H,B,H,B)$. Therefore, $B\subseteq f(B,H,B,H,B)$. On the other hand, since B is a bi-hyperideal of H, we have $f(B,H,B,H,B)\subseteq B$. Thus, $f(B,H,B,H,B)=B$.

(2) $\Rightarrow$ (3). It is clear by Lemma 4.2 [9] since every quasi-hyperideal is a bi-hyperideal.

(3) $\Rightarrow$ (1). Let R be a right hyperideal of H, L be a left hyperideal of H and M be a lateral hyperideal of H. By Theorem 2.2 [9], $Q=R\cap M\cap L$ is a quasi-hyperideal of H. By (3) we have $f(Q,H,Q,H,Q)=Q$. Thus $R\cap M\cap L = Q = f(Q,H,Q,H,Q) =\subseteq f(R,H,M,H,L)\subseteq f(R,M,L)$. But $f(R,M,L)\subseteq R\cap M\cap L$. Therefore, since $f(R,M,L)=R\cap M\cap L$, by Theorem 2.3(2), H is a regular ternary semihypergroup. □

Corollary 2.19. *Let (H, f) be a ternary semihypergroup. The following statements are equivalent:*

(1) *H is regular;*

(2) *For every bi-hyperideal B of H, $f(B, H, B) = B$;*

(3) *For every quasi-hyperideal Q of H, $f(Q, H, Q) = Q$.*

Theorem 2.20. *Let (H, f) be a ternary semihypergroup. If for every bi-hyperideal B of H, $f(B, B, B) = B$, then H is a regular ternary semihypergroup.*

Proof. The proof is a corollary of Theorem 2.17. □

Theorem 2.21. *Let (H, f) be a regular ternary semihypergroup. Then a ternary subsemihypergroup B of H is bi-hyperideal if and only if B is a quasi-hyperideal of H.*

Proof. Let H be a regular ternary semihypergroup and B a bi-hyperideal of H. By Theorem 2.3, we have $f(R \cap M \cap L) = f(R, M, L)$ for every right hyperideal R, lateral hyperideal M and left hyperideal L. Thus

$$\begin{aligned}
& f(B, H, H) \cap (f(H, B, H) \cup f(H, H, B, H, H)) \cap f(H, H, B) \\
= {} & f(f(B, H, H), f((f(H, B, H) \cup f(H, H, B, H, H)), f(H, H, B, H, H)) \\
= {} & f(B, f(H, H, H), B, f(H, H, H), B) \cup f(B, f(H, H, H), H, B, f(H, H, H), H, B) \\
\subseteq {} & f(B, H, B, H, B) \cup f(B, H, H, B, H, H, B) \\
\subseteq {} & B \cup f(B, H, B) = B \cup B = B.
\end{aligned}$$

Therefore, B is a quasi-hyperideal of H.

Conversely, let B be a quasi-hyperideal of H. Then, by Lemma 4.2 [9], B is a bi-hyperideal of H. □

Corollary 2.22. *Let (H, f) be a regular ternary semihypergroup. A ternary subsemihypergroup B of H is bi-hyperideal of H if and only if B is the intersection of a right hyperideal, a lateral hyperideal and a left hyperideal of H.*

Theorem 2.23. *Let (H, f) be a ternary semihypergroup. The following statements are equivalent:*

(1) *H is regular;*

(2) *$M \cap B = f(B, M, B)$ for every lateral hyperideal M and for every bi-hyperideal B of H;*

(3) *$M \cap Q = f(Q, M, Q)$ for every lateral hyperideal M and for every bi-hyperideal Q of H.*

Proof. (1) $\Rightarrow$ (2). Let M be a lateral hyperideal of H and B a bi-hyperideal of H. We have $f(B,M,B) \subseteq f(H,M,H) \subseteq M$. By Corollary 2.19, we have $f(B,M,B) \subseteq f(B,H,B) = B$. Therefore, $f(B,M,B) \subseteq M \cap B$. Let $a \in M \cap B$. Since H is regular, there exists $h \in H$ such that $a \in f(a,h,a)$. We have $a \in f(a,h,a) \subseteq f(f(a,h,a),h,a) = f(a,f(h,a,h),a) \subseteq f(B,M,B)$. It follows that $M \cap B \subseteq f(B,M,B)$. Therefore $f(B,M,B) = M \cap B$.

(2) $\Rightarrow$ (3). It is clear since every quasi-hyperideal is a bi-hyperideal.

(3) $\Rightarrow$ (1). Let Q be a quasi-hyperideal of H. By (3) it follows that $Q = H \cap Q = f(Q,H,Q)$. By Corollary 2.19, it follows that H is a regular ternary semihypergroup. □

In the sequel, the following results hold. The proof of them is straightforward, so we omit it.

Theorem 2.24. *Let (H,f) be a ternary semihypergroup. The following statements are equivalent:*

(1) *H is regular;*

(2) *$B \cap L \subseteq f(B,H,L)$ for every bi-hyperideal B of H and for every left hyperideal L;*

(3) *$Q \cap L \subseteq f(Q,H,L)$ for every quasi-hyperideal Q of H and for every left hyperideal L;*

(4) *$B \cap R \subseteq f(R,H,B)$ for every bi-hyperideal B of H and for every right hyperideal R;*

(5) *$Q \cap R \subseteq f(R,H,Q)$ for every quasi-hyperideal Q of H and for every right hyperideal R.*

Theorem 2.25. *Let (H,f) be a ternary semihypergroup. The following statements are equivalent:*

(1) *H is regular;*

(2) *$B_1 \cap B_2 \subseteq f(B_1,H,B_2) \cap f(B_2,H,B_1)$ for every bi-hyperideals B_1, B_2 of H;*

(3) *$B \cap Q \subseteq f(B,H,Q) \cap f(Q,H,B)$ for every bi-hyperideal B and for every quasi-hyperideal Q of H;*

(4) *$B \cap L \subseteq f(B,H,L) \cap f(L,H,B)$ for every bi-hyperideal B of H and for every left hyperideal L;*

(5) *$Q \cap L \subseteq f(Q,H,L) \cap f(L,H,Q)$ for every quasi-hyperideal Q of H and for every left hyperideal L;*

(6) *$R \cap L \subseteq f(R,H,L) \cap f(L,H,R)$ for every right hyperideal R of H and for every left hyperideal L;*

(7) $B \cap R \subseteq f(R, H, B) \cap f(B, H, R)$ *for every bi-hyperideal* B *of* H *and for every right hyperideal* R*;*

(8) $Q \cap R \subseteq f(R, H, Q) \cap f(Q, H, R)$ *for every quasi-hyperideal* Q *of* H *and for every right hyperideal* R*.*

3. Completely regular and intra-regular ternary semihypergroups

Definition 3.1. Let (H, f) be a ternary semihypergroup. An element $a \in H$ is said to be *left* (resp. *right*) *regular* if there exists an element $x \in H$ such that $a \in f(x, a, a)$ (resp. $a \in f(a, a, x)$). An element $a \in H$ is said to be *completely regular* if it is left regular, right regular and regular.

If all the elements of a ternary semihypergroup H are left (resp. right, completely) regular, then H is called left (resp. right, completely) regular.

The ternary semihypergroup of the Example 1.6 is a completely regular ternary semihypergroup.

Theorem 3.2. *A ternary semihypergroup* (H, f) *is left (resp. right) regular if and only if every left (resp. right) hyperideal of* H *is completely semiprime.*

Proof. Let H be a left regular ternary semihypergroup and L be any left hyperideal of H. Suppose that $f(a, a, a) \subseteq L$ for $a \in H$. Since H is left regular, there exists an element $x \in H$ such that $a \in f(x, a, a) \subseteq f(x, f(x, a, a), a) \subseteq f(x, x, f(a, a, a)) \subseteq f(H, H, L) \subseteq L$. Thus L is completely semiprime.

Conversely, suppose that every left hyperideal of H is completely semiprime. Now for any $a \in H$, $f(H, a, a)$ is a left hyperideal of H. Then by hypothesis, $f(H, a, a)$ is a completely semiprime hyperideal of H. Now $f(a, a, a) \subseteq f(H, a, a)$. Since $f(H, a, a)$ is completely semiprime, it follows that $a \in f(H, a, a)$. So there exists an element $x \in H$ such that $a \in f(x, a, a)$. Consequently, a is left regular. Since a is arbitrary, it follows that H is left regular.

Similarly, it can be proved the theorem for the right regularity. □

Proposition 3.3. *A ternary semihypergroup* (H, f) *is completely regular if and only if* $a \in f(a, a, H, a, a)$ *for all* $a \in H$*.*

Proof. Suppose that H is a completely regular ternary semihypergroup. Let $a \in H$. Then, by the definition, we have that $a \in f(a, a, H)$ and $a \in f(H, a, a)$, that is $a \in f(a, a, H) \cap f(H, a, a)$. Since H is completely regular, there exists an element $x \in H$ such that $a \in f(a, x, a)$. So we have

$$\begin{aligned} a \in f(a, x, a) &\subseteq f(f(a, a, H), x, f(H, a, a)) \subseteq \\ &\subseteq f(a, a, f(H, x, H), a, a) \subseteq f(a, a, H, a, a). \end{aligned}$$

Conversely, suppose that for any $a \in H$, $a \in f(a, a, H, a, a)$. Then

1. $a \in f(a, a, H, a, a) \subseteq f(a, f(a, H, a), a) \subseteq f(a, H, a)$, that is H is regular.
2. $a \in f(a, a, H, a, a) \subseteq f(f(a, a, H), a, a) \subseteq f(H, a, a)$, that is H is left regular.
3. $a \in f(a, a, H, a, a) \subseteq f(a, a, f(H, a, a)) \subseteq f(a, a, H)$, that is H is right regular. Therefore H is completely regular. □

Theorem 3.4. *A ternary semihypergroup (H, f) is completely regular if and only if every bi-hyperideal of H is completely semiprime.*

Proof. Suppose that H is completely regular ternary semihypergroup. Let B be any bi-hyperideal of H. Let $f(b, b, b) \subseteq B$ for $b \in B$. Since H is completely regular, from Proposition 3.3, it follows that $b \in f(b, b, H, b, b)$. This implies that there exists $x \in H$ such that

$$\begin{aligned} b &\in f(b, b, x, b, b) \subseteq f(b, f(b, b, x, b, b), x, f(b, b, x, b, b), b) = \\ &= f(b, b, b, f(x, b, b, x), b, f(b, b, x, b, b), x, b, b, b) \\ &= f(b, b, b, f(x, b, b, x), b, b, b, f(x, b, b, x), b, b, b) \subseteq f(B, H, B, H, B) \subseteq B. \end{aligned}$$

This shows that B is completely semiprime.

Conversely, suppose that every bi-hyperideal of H is completely semiprime. Since every left and right hyperideal of a ternary semihypergroup H is a bi-hyperideal of H, it follows that every left and right hyperideal of H is completely semiprime. Consequently, we have from Theorem 3.2 that H is both left and right regular.

Let $a \in H$. We consider $f(a, H, a)$. Let $x, y, z \in f(a, H, a)$ and $h_1, h_2 \in H$. Then for some $h_0, h_0^{'}, h_0^{''} \in H$ we have:

$$\begin{aligned} f(x, h_1, y, h_2, z) &\subseteq f(f(a, h_0, a), h_1, f(a, h_0^{'}, a), h_2, f(a, h_0^{''}, a)) \\ &\subseteq f(a, f(h_0, a, h_1, a, h_0^{'}, a, h_2, a, h_0^{''}), a) \\ &\subseteq f(a, H, a). \end{aligned}$$

This implies that $f(f(a, H, a), H, f(a, H, a), H, f(a, H, a)) \subseteq f(a, H, a)$. That is, $f(a, H, a)$ is a bi-hyperideal of H. Since $f(a, a, a) \subseteq f(a, H, a)$ and $f(a, H, a)$ is completely semiprime, it follows that $a \in f(a, H, a)$, for all $a \in H$. That is H is regular. This completes the proof. □

Theorem 3.5. *If (H, f) is a completely regular ternary semihypergroup, then every bi-hyperideal of H is idempotent.*

Proof. Let H be a completely regular ternary semihypergroup and B be a bi-hyperideal of H. Since H is a completely regular ternary semihypergroup, it is also a regular ternary semihypergroup. Let $b \in B$. Then there exists $x \in H$ such that $b \in f(b, x, b)$. This implies that $b \in f(B, H, B)$ and hence $B \subseteq f(B, H, B)$. Also $f(B, H, B) \subseteq f(B, H, B, H, B) \subseteq B$. Thus we find that $B = f(B, H, B)$. Again, we have from Proposition 3.3 that $b \in f(b, b, H, b, b) \subseteq f(B, B, H, B, B)$.

This implies that $B \subseteq f(B, B, H, B, B) = f(B, f(B, H, B), B) = f(B, B, B) \subseteq B$. Consequently, $f(B, B, B) = B$. $\square$

Definition 3.6. A ternary semihypergroup (H, f) is called *intra-regular* if for each element $a \in H$, there exist elements $x, y \in H$ such that $a \in f(x, a, a, a, y)$.

Theorem 3.7. [9, Theorem 6.4] *Let (H, f) be a ternary semihypergroup. Then the following statements are equivalent:*

(1) *H is intra-regular;*

(2) *For every left hyperideal L, lateral hyperideal M and right hyperideal R of H, $L \cap M \cap R \subseteq f(L, M, R)$.*

Proposition 3.8. *Let (H, f) be an intra-regular ternary semihypergroup. Then a non-empty subset I of H is a hyperideal of H if and only if I is a lateral hyperideal of H.*

Proof. Clearly, if I is a hyperideal of H, then I is a lateral hyperideal of H.

Conversely, let I be a lateral hyperideal of an intra-regular ternary semihypergroup. Let $a \in I$ and $s, t \in H$. Then $a \in H$ and hence there exist elements $x, y \in H$ such that $a \in f(x, a, a, a, y)$. Now $f(s, t, a) \subseteq f(s, t, f(x, a, a, a, y)) \subseteq f(H, I, H) \subseteq I$ and $f(a, s, t) \subseteq f(f(x, a, a, a, y), s, t) \subseteq f(H, I, H) \subseteq I$. This implies that I is both a left hyperideal and a right hyperideal of H. Consequently, I is an hyperideal of H. $\square$

Lemma 3.9. *Every lateral hyperideal of an intra-regular ternary semihypergroup (H, f) is an intra-regular ternary semihypergroup.*

Proof. Let L be a lateral hyperideal of an intra-regular ternary semihypergroup H. Then for each $a \in L$, there exist $x, y \in H$ such that $a \in f(x, a, a, a, y)$. Now $a \in f(x, a, a, a, y) \subseteq f(x, f(x, a, a, a, y), f(x, a, a, a, y), f(x, a, a, a, y), y)$ $\subseteq f(f(x, x, a, a, a, y, y), f(a, a, a), f(y, x, a, a, a, y, y)) \subseteq f(L, f(a, a, a), L)$. This implies that there exist $u, v \in L$ such that $a \in f(u, f(a, a, a), v)$. Consequently, L is an intra-regular ternary semihypergroup. $\square$

From the Proposition 3.8 we have the following corollary:

Corollary 3.10. *Every hyperideal of an intra-regular ternary semihypergroup H is an intra-regular ternary semihypergroup.*

Theorem 3.11. *Let I be a hyperideal of an intra-regular ternary semihypergroup H and J be a hyperideal of I. Then J is a hyperideal of the entire ternary semihypergroup H.*

Proof. It is sufficient to show that J is a lateral hyperideal of H. Let $a \in J \subseteq I$ and $s, t \in H$. Then $f(s, a, t) \subseteq I$. We have to show that $f(s, a, t) \subseteq J$. From Corollary 3.10, it follows that I is an intra-regular ternary semihypergroup. Thus there exist $u, v \in I$ such that $f(s,a,t) \subseteq f(u, f(s,a,t), f(s,a,t), f(s,a,t), v) \subseteq f(f(u,s,a,t,s), a, f(t,s,a,t,v)) \subseteq f(I,J,I) \subseteq J$. Consequently, J is a lateral hyperideal of H. $\square$

Theorem 3.12. *A ternary semihypergroup (H,f) is intra-regular if and only if every hyperideal of H is completely semiprime.*

Proof. Let H be an intra-regular ternary semihypergroup and I be a hyperideal of H. Let $f(a,a,a) \subseteq I$ for $a \in H$. Since H is intra-regular, there exist $x, y \in H$ such that $a \in f(x, f(a,a,a), y) \subseteq I$. Consequently, I is completely semiprime.

Conversely, suppose that every hyperideal of H is completely semiprime. Let $a \in H$. Then $f(a,a,a) \subseteq \langle f(a,a,a)\rangle$. This implies that $a \in \langle f(a,a,a)\rangle$, since $\langle f(a,a,a)\rangle$ is completely semiprime.

Now $\langle f(a,a,a)\rangle = f(H,H,f(a,a,a)) \cup f(f(a,a,a),H,H) \cup f(H,f(a,a,a),H) \cup f(H,H,f(a,a,a),H,H) \cup f(a,a,a)$. So we have the following cases:

If $a \in f(H,H,f(a,a,a))$, then $f(a,a,a) \subseteq f(H,H,f(a,a,a),a,a)$. Hence $a \in f(H,H,H,H,f(a,a,a),a,a) \subseteq f(H,H,H,a,a,a,H) \subseteq f(H,f(a,a,a),H)$.

If $a \in f(f(a,a,a),H,H)$, then $f(a,a,a) \subseteq f(a,a,f(a,a,a),H,H)$. Hence $a \in f(a,a,f(a,a,a),H,H,H,H) \subseteq f(H,a,a,a,H,H,H) \subseteq f(H,f(a,a,a),H)$.

If $a \in f(H,f(a,a,a),H)$, then we are done.

If $a \in f(H,H,f(a,a,a),H,H)$, then $f(a,a,a) \subseteq f(a,H,H,f(a,a,a),H,H,a)$.

Hence

$$\begin{aligned} a &\in f(H,H,a,H,H,f(a,a,a),H,H,a,H,H) \\ &\subseteq f(H,H,H,f(a,a,a),H,H,H) \subseteq f(H,f(a,a,a),H). \end{aligned}$$

If $a \in f(a,a,a)$, then

$$a \in f(a,a,a) \subseteq f(f(a,a,a), f(a,a,a), f(a,a,a)) \subseteq f(H,f(a,a,a),H).$$

So we find that in any case, H is intra-regular. $\square$

References

[1] **B. Davvaz, W.A. Dudek and S. Mirvakili**, *Neutral elements, fundamental relations and n-ary hypersemigroups*, Int. J. Algebra Comput. **19** (2009), 567 – 583.

[2] **B. Davvaz, W.A. Dudek and T. Vougiouklis**, *A Generalization of n-ary algebraic systems*, Commun. Algebra **37** (2009), 1248 – 1263.

[3] **V.N. Dixit and S. Dewan**, *A note on quasi and bi-ideals in ternary semigroups*, Int. J. Math. Math. Sci. **18** (1995), 501 – 508.

[4] **W.A. Dudek**, *On divisibility in n-semigroups*, Demonstr. Math. **13** (1980), 355 – 367.

[5] **W.A. Dudek and I. Groździńska**, *On ideals in regular n-semigroups*, Mat. Bilten **3(4)** (1980), 29 – 30.

[6] **K. Hila, B. Davvaz and K. Naka**, *On hyperideal structure of ternary semihypergroups*, Iran. J. Math. Sci. Inform. **9** (2014), 78 – 95.

[7] **M. Kapranov, I. M. Gelfand and A. Zelevinskii**, *Discriminants, resultants and multidimensional determinants.* Birkhauser, Berlin (1994).

[8] **K. Naka and K. Hila**, *Some properties of hyperideals in ternary semihypergroups*, Math. Slovaca **63** (2013), 449 – 468.

[9] **K. Naka and K. Hila**, *On the structure of quasi-hyperideals and bi-hyperideals in ternary semihypergroups*, Afr. Mat. **26** (2015), 1573 – 1591.

[10] **K. Naka and K. Hila**, *On some special classes of hyperideals in ternary semihypergroups*, Util. Math. **98** (2015), 97 – 112.

[11] **F. Marty**, *Sur une generalization de la notion de group*, 8th Congres Math. Scandinaves, Stockholm, (1934), 45 – 49.

[12] **M.L. Santiago**, *Regular ternary semigroups*, Bull. Calcutta Math. Soc. **82** (1990), 67 – 71.

[13] **M.L. Santiago and S. Sri Bala**, *Ternary semigroups*, Semigroup Forum **81** (2010), 380 – 388.

[14] **F.M. Sioson**, *Ideal theory in ternary semigroups*, Math. Japonica **10** (1965), 63–64.

[15] **M. Shabir and M. Bano**, *Prime bi-ideals in ternary semigroups*, Quasigroups Relat. Syst. **16** (2008), 239 – 256.

[16] **M. Shabir and Sh. Bashir**, *Prime ideals in ternary semigroups*, Asian Eur. J. Math. **2** (2009), 141 – 154.

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On state equality algebras

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Abstract. We show that every state-morphism operator on an equality algebra is an internal state operator on it and prove that the converse is correct for the linearly ordered equality algebras under a special condition. Then we show that there is a one-to-one correspondening between congruence relations on a state-morphism (linearly ordered state) equality algebra and state-morphism (state) deductive systems on it. Moreover, we define the notion of homomorphism on equality algebras and we investigate the relation between state operators and state-morphism operators with equality-homomorphism. Finally, we characterize the simple and semisimple classes of state-morphism equality algebras.

1. Introduction

Equality algebras were introduced in [8] by Jenei, that the motivation cames from EQ-algebra [13]. State MV-algebras were introduced by Flaminio and Montagna as MV-algebras with internal states [6]. Di Nola and Dvurečenskij introduced the notion of state-morphism MV-algebra which is a stronger variation of a state MV-algebra [4]. State BCK-algebras and state-morphism BCK-algebras have been defined and studied by Borzooei, Dvurečenskij and Zahiri [2]. Recently, the state equality algebras and state-morphism equality algebras have been introduced in [3]. Now we prove that every state-morphism operator on an equality algebra is an internal state operator on it, and we prove the converse is true for a linearly ordered equality algebra under a special condition. Also, we remove the condition of [3, Th. 6.8] and [3, prop. 5.7(3)] and state them in general case. We introduce a deductive system on state (state-morphism) equality algebra and we investigate some related results. Then we show that for any linearly ordered sate (state-morphism) equality algebra (A, σ), there is a one-to-one correspondence between $Con(A, \sigma)$ and $IDS(A_\sigma)$ $(SDS(A_\sigma))$. We show that every internal state operator on an equality algebra is a state-morphism if it is equality-homomorphism. Finally, we study some classes of state-morphism equality algebras such as simple and semisimple state-morphism equality algebras.

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2. Preliminaries

In this section, we recall basic definitions and results relevant to equality algebra which will be used in the next sections.

Definition 2.1. (cf. [8]) An *equality algebra* is an algebra $(A, \wedge, \sim, 1)$ of type $(2, 2, 0)$ such that the following axioms are fulfilled for all $a, b, c \in A$:
(E_1) $(A; \wedge, 1)$ is a meet-semilattice with top element 1,
(E_2) $a \sim b = b \sim a$,
(E_3) $a \sim a = 1$,
(E_4) $a \sim 1 = a$,
(E_5) $a \leqslant b \leqslant c$ implies $a \sim c \leqslant b \sim c$ and $a \sim c \leqslant a \sim b$,
(E_6) $a \sim b \leqslant (a \wedge c) \sim (b \wedge c)$,
(E_7) $a \sim b \leqslant (a \sim c) \sim (b \sim c)$,
where $a \leqslant b$ iff $a \wedge b = b$.

Let $(A, \wedge, \sim, 1)$ be an equality algebra. A subset $D \subseteq A$ is called a *deductive system* of A if for all $a, b \in A$, (DS_1): $1 \in D$, (DS_2): $a \in D$ and $a \leqslant b$ implies $b \in D$, (DS_3): $a, a \sim b \in D$ implies $b \in D$.

A deductive system D of an equality algebra A is *proper* if $D \neq A$. The set of all deductive systems of A is denoted by DS(A). An equality algebra A is called *simple* if $DS(A) = \{\{1\}, A\}$. A non-empty subset S of an equality algebra $(A, \wedge, \sim, 1)$ which is closed under $\sim$ is called a *subalgebra* of A and the set of all subalgebras of A is denoted by $Sub(A)$. We know that $\sim$ is higher priority than the operation $\wedge$ (it means that first we calculate the operation $\wedge$ then apply the operation $\sim$). For simplify, some times we write $a \sim (a \wedge b) = a \sim a \wedge b$. The operations $\rightarrow$ (called *implication*) and $\leftrightarrow$ (called *equivalence*) on equality algebra A are defined as follows:

$$a \rightarrow b = a \sim (a \wedge b) \quad , \quad a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a).$$

If there exists *zero element* $0 \in A$ such that $0 \leqslant a$ (i.e, $0 \rightarrow a = 1$), for all $a \in A$, then A is called a *bounded equality algebra* and it is denoted by $(A, \wedge, \sim, 0, 1)$.

Proposition 2.2. (cf. [3, 8]) *Let $(A, \wedge, \sim, 1)$ be an equality algebra. Then the following hold for all $a, b, c \in A$:*
(E_8) $a \sim b \leqslant a \rightarrow b \leqslant a \leftrightarrow b$,
(E_9) $a \leqslant (a \sim b) \sim b$,
(E_{10}) $a \sim b = 1$ *iff* $a = b$,
(E_{11}) $a \rightarrow b = 1$ *iff* $a \leqslant b$,
(E_{12}) $a \rightarrow b = 1$ *and* $b \rightarrow a = 1$ *implies* $a = b$,
(E_{13}) $a \leqslant b \rightarrow a$,
(E_{14}) $a \leqslant (a \rightarrow b) \rightarrow b$,
(E_{15}) $a \rightarrow b \leqslant (b \rightarrow c) \rightarrow (a \rightarrow c)$,
(E_{16}) $a \leqslant b \rightarrow c$ *iff* $b \leqslant a \rightarrow c$,

(E_{17}) $a \to (b \to c) = b \to (a \to c)$,
(E_{18}) $x \leqslant y$ *implies* $y \to z \leqslant x \to z$,
(E_{19}) $x \leqslant y$ *implies* $z \to x \leqslant z \to y$,
(E_{20}) $b \leqslant a \sim a \wedge b$, $a \sim b \leqslant a \sim a \wedge b$,
(E_{21}) $a \leqslant (a \sim a \wedge b) \sim b$, $b \leqslant (a \sim a \wedge b) \sim b$,
(E_{22}) $((a \to b) \to b) \to b = a \to b$.

Proposition 2.3. (cf. [3, 5]) *Let* $(A, \wedge, \sim, 1)$ *be an equality algebra and* $D \in DS(A)$*. Then the following hold for all* $a, b \in A$*:*
(*i*) *if* $a, a \to b \in D$*, then* $b \in D$*,*
(*ii*) *if* $a, b \in D$*, then* $a \sim b \in D$ *and* $a \wedge b \in D$*,*
(*iii*) *if* A *is linearly ordered, then* $a \sim b \in D$ *iff* $a \leftrightarrow b \in D$ *iff* $b \to a, a \to b \in D$*.*

Proposition 2.4. (cf. [3]) *Every deductive system of an equality algebra* A *is a subalgebra of* A*.*

Proposition 2.5. (cf. [3, 9]) *Let* A *be an equality algebra and* $Con(A)$ *be the set of all congruence relations on* A*. Then the following hold:*
(*i*) *For any* $D \in DS(A)$*, the relation* θ_D *on* A *which is defined by* $(a, b) \in \theta_D \Leftrightarrow a \sim b \in D$*, is a congruence relation on* A*.*
(*ii*) *If* $\theta \in Con(A)$*, then* $[1]_\theta = \{a \in A : (a, 1) \in \theta\}$ *is a deductive system of* A*.*

For $D \in DS(A)$ and $\theta_D \in Con(A)$, we denote the set of all equivalence classes of θ_D by $A/D = \{a/D : a \in A\}$.

Theorem 2.6. (cf. [3, 9]) *Let* $(A, \wedge, \sim, 1)$ *be an equality algebra. Then there is a one-to-one correspondence between* $DS(A)$ *and* $Con(A)$*.*

Theorem 2.7. (cf. [3, 5]) *Let* $(A, \wedge, \sim, 1)$ *be an equality algebra and* $D \in DS(A)$*. Then* $(A/D, \wedge_D, \sim_D, 1_D)$ *is an equality algebra with the following operations:*

$$a/D \wedge_D b/D = (a \wedge b)/D \ , \qquad a/D \sim_D b/D = (a \sim b)/D.$$

In the following we recall definitions of internal state and state-morphism operators and their properties. For more details, see [3].

Definition 2.8. (cf. [3]) Let $(A, \wedge, \sim, 1)$ be an equality algebra. Then (A, σ) is called an *internal state equality algebra* if $\sigma : A \to A$ is a unary operator on A such that for all $a, b \in A$ the following conditions are satisfied:
(S_1) $\sigma(a) \leqslant \sigma(b)$, whenever $a \leqslant b$,
(S_2) $\sigma(a \sim a \wedge b) = \sigma((a \sim a \wedge b) \sim b) \sim \sigma(b)$,
(S_3) $\sigma(\sigma(a) \sim \sigma(b)) = \sigma(a) \sim \sigma(b)$,
(S_4) $\sigma(\sigma(a) \wedge \sigma(b)) = \sigma(a) \wedge \sigma(b)$.

In the following, we replace internal state equality algebra by state equality algebra.

For any state equality algebra (A, σ), the $Ker(\sigma)$ is defined as $\{a \in A | \sigma(a) = 1\}$. The state σ is called *faithful*, if $Ker(\sigma) = \{1\}$. The set of all internal states on an equality algebra A denote by $S(A)$. Clearly $S(A) \neq \emptyset$. In fact, the identity map 1_A is a faithful state on A. If A is linearly ordered, then $Id_A \in S(A)$.

Proposition 2.9. (cf. [3]) *Let $(A, \wedge, \sim, 1)$ be an state equality algebra. Then for all $a, b \in A$ the following hold:*
(1) $\sigma(1) = 1$,
(2) $\sigma(\sigma(a)) = \sigma(a)$,
(3) $\sigma(A) = \{a \in A : a = \sigma(a)\}$,
(4) $\sigma(A)$ *is a subalgebra of* A,
(5) $Ker(\sigma) \in DS(A)$,
(6) $Ker(\sigma)$ *is a subalgebra of* A,
(7) $Ker(\sigma) \cap \sigma(A) = \{1\}$.

Definition 2.10. (cf. [3])Let $(A, \wedge, \sim, 1)$ be an equality algebra. Then (A, σ) is called a *state-morphism equality algebra* if $\sigma : A \to A$ is a unary operator on A such that for all $a, b \in A$ the following conditions are satisfied:
(SM_1) $\sigma(a \sim b) = \sigma(a) \sim \sigma(b)$,
(SM_2) $\sigma(a \wedge b) = \sigma(a) \wedge \sigma(b)$,
(SM_3) $\sigma(\sigma(a)) = \sigma(a)$.

The set of all state-morphisms on an equality algebra A is denoted by $SM(A)$. Clearly $SM(A) \neq \emptyset$. Indeed, if A is an equality algebra, then the constant map $1_A(a) = 1$ and the identity map $Id_A(a) = a$ are state-morphism operators on A.

Proposition 2.11. (cf. [3]) *Let (A, σ) be a state-morphism equality algebra. Then the following hold:*
(1) $Ker(\sigma) \in DS(A)$,
(2) $Ker(\sigma) = \{\sigma(a) \sim a : a \in A\}$,
(3) *If* $Ker(\sigma) = \{1\}$, *then* $\sigma = Id_A$,
(4) *If A is a simple equality algebra, then* $SM(A) = \{1_A, Id_A\}$.

3. (State) deductive systems in equality algebras

In this section, by considering the notion of deductive system, we define the concept of state deductive system on state (state morphism) equality algebras then prove that the quotient algebra constructed with a state deductive system of a state-morphism (and linearly ordered state) equality algebra (A, σ) is a state-morphism (and state) equality algebra. Finally, we show that a deductive system on a state-morphism (and linearly ordered state) equality algebra define a congruence relation on (A, σ) and there is a one-to-one correspondence between $SDS(A_\sigma)$ $(IDS(A_\sigma))$ and $Con(A, \sigma)$.

Theorem 3.1. *Let X be a subset of an equality algebra A.*
(i) *The deductive system generated by X which is denoted by $\langle X \rangle$ is*

$$\langle X \rangle = \{a \in A \mid \exists n \in \mathbb{N} \textit{ and } x_1, \ldots, x_n \in X \textit{ st. } x_1 \to (x_2 \to \ldots(x_n \to a)\ldots) = 1\}$$

(*ii*) *If D is a deductive system of A and $S \subseteq A$, then*

$$\langle D \cup S \rangle = \{a \in A \mid \exists n \in \mathbb{N} \text{ and } s_1, \ldots, s_n \in S \text{ st. } s_1 \to (s_2 \to (...(s_n \to a)...)) \in D\}$$

Proof. It follows from [5, Prop. 4.3] and [11, Prop. 2.2.7]. □

For each x belonging to an equality algebra A, the deductive system generated by $\{x\}$ is called *principal deductive system.* Clearly,

$$\langle x \rangle = \{a \in A \mid x^n \to a = 1, \text{ for some } n \in \mathbb{N}\},$$

where $x^0 \to b = b$, $x^n \to b = x \to (x^{n-1} \to b)$.

Definition 3.2. A proper deductive system D of an equality algebra A is called

- *prime* if $a \sim a \wedge b \in D$ or $b \sim b \wedge a \in D$, for all $a, b \in A$,
- *maximal* if there is not any proper deductive system strictly containing D.

An equality algebra A is called *semisimple* if $Rad(A) = \bigcap_{D \in Max(A)} D = \{1\}$. The set of all prime (maximal) deductive systems of an equality algebra A is denoted by $Pr(A)(Max(A))$.

Proposition 3.3. *Any proper deductive system of a bounded equality algebra A is contained in a maximal deductive system of A.*

Proof. It is an immediate consequence of Zorn's Lemma. □

Example 3.4. (*i*). Let $A = \{0, a, b, 1\}$ be a poset with $0 < a, b < 1$. Then $(A, \wedge, \sim, 1)$ is an equality algebra with the operation $\sim$ on A, given as follows:

$\sim$	0	a	b	1
0	1	b	a	0
a	b	1	0	a
b	a	0	1	b
1	0	a	b	1

Then $DS(A) = \{\{1\}, \{a, 1\}, \{b, 1\}, A\}$, $Pr(A) = \{\{a, 1\}, \{b, 1\}\}$ and $Max(A) = \{\{a, 1\}, \{b, 1\}\}$. Also by Theorem 3.1, $\langle 0 \rangle = A$, $\langle a \rangle = \{a, 1\}$, $\langle b \rangle = \{b, 1\}$ and $\langle 1 \rangle = \{1\}$.

(*ii*). Let $B = \{0, b, 1\}$ be a chain such that $0 < b < 1$. Then $(B, \wedge, \sim, 1)$ is an equality algebra with the operation $\sim$ on B, given as follows:

$\sim$	0	b	1
0	1	b	0
b	b	1	b
1	0	b	1

Then $DS(B) = \{\{1\}, B\}$, $Pr(B) = \{1\}$ and $Max(B) = \{1\}$. By Theorem 3.1, $\langle 0\rangle = \langle b\rangle = B, \langle 1\rangle = \{1\}$.

(iii). Let $C = \{0, a, b, 1\}$ be a poset with $0 < a < b < 1$. Then $(C, \wedge, \sim, 1)$ is an equality algebra with the operation $\sim$ on C, given as follows:

$\sim$	0	a	b	1
0	1	a	0	0
a	a	1	a	a
b	0	a	1	b
1	0	a	b	1

Then $DS(C) = \{\{1\}, \{b, 1\}, A\}$, $Pr(C) = \{\{1\}, \{b, 1\}\}$ and $Max(C) = \{\{b, 1\}\}$. Also $\langle 0\rangle = \langle a\rangle = C$, $\langle b\rangle = \{b, 1\}$, $\langle 1\rangle = \{1\}$.

Proposition 3.5. *Let D be a proper deductive system of an equality algebra A. Then the following are equivalent:*

(i) *D is maximal.*

(ii) *For all $x \in A \setminus D$, $\langle D \cup \{x\}\rangle = A$.*

(iii) *For all $x \in A \setminus D$, $x^n \to a \in D$ for any $a \in A$.*

Proof. $(i) \Rightarrow (ii)$. If $x \in A \setminus D$, then $D \subset \langle D \cup \{x\}\rangle$. Since D is maximal, we get $\langle D \cup \{x\}\rangle = A$.

$(ii) \Rightarrow (i)$. Assume that F is a proper deductive system of A such that $D \subset F$. Hence there is $x \in F \setminus D$, and so by (ii), $\langle D \cup \{x\}\rangle = A$. Then $F = A$, that is a contradiction.

$(ii) \Leftrightarrow (iii)$. It is clearly by Theorem 3.1(ii). □

Proposition 3.6. *Let A be an equality algebra. The subalgebra S of A is a deductive system of A, if $a \in S$ and $b \in A\backslash S$ implies $a \wedge b \in A\backslash S$ and $a \sim b \in A\backslash S$.*

Proof. Let S be a subalgebra of A. Since $1 = a \sim a \in S$, thus (DS_1) satisfied. If $a \in S$ and $a \leqslant b$, then $a \wedge b = a \in S$. Assume that $b \notin S$. Since $a \in S$ and $b \in A \setminus S$ then a ∧ b ∈ A\S, which is a contradiction. Hence $b \in S$. Thus (DS_2) satisfied. Now, let $a, a \sim b \in S$, but $b \notin S$. Hence by assumption $a \sim b \in A \setminus S$, which is a contradiction. Thus $b \in S$. So (DS_3) is satisfied. □

Example 3.7. Let A be the equality algebra in Example 3.4(i). Then

$$Sub(A) = \{\{1\}, \{0, 1\}, \{a, 1\}, \{b, 1\}, A\}.$$

Clear that any member of $Sub(A)$ is a deductive system, except $\{0, 1\}$. It follows that Proposition 3.6 is not satisfied for subalgebra $\{0, 1\}$.

Proposition 3.8. *Let A be an equality algebra. Then the following hold.*

(i) *If A is linearly ordered and $a \in A$, then $A(a) = \{x \in A \mid a \leqslant x\}$ is a subalgebra of A.*

(ii) *If A is bounded, then $A_0 = \{a \in A \mid a \sim 0 = 0\}$ is a proper deductive system and subalgebra of A.*

Proof. (i). Let $a \in A$. Clearly, $A(a)$ is closed under $\wedge$. Put $x, y \in A(a)$. Since A is linearly ordered, we assume $x \leqslant y$. Now by (E_{13}) and (E_2) we get $a \leqslant x \leqslant y \rightarrow x = y \sim (y \wedge x) = y \sim x = x \sim y$. Hence $x \sim y \in A(a)$. For $y \leqslant x$, with a similar way, the result satisfies.

(ii). Since $1 \sim 0 = 0$, we get $1 \in A_0$. Let $a \in A_0$ and $a \leqslant b$. Then by (E_5), $b \sim 0 \leqslant a \sim 0 = 0$ and so $b \sim 0 = 0$. Thus $b \in A_0$. Now let $a, a \sim b \in A_0$. By (E_7), $b \sim 0 \leqslant (a \sim b) \sim (a \sim 0) = (a \sim b) \sim 0 = 0$. Hence $b \sim 0 = 0$ and so $b \in A_0$. Therefore, A_0 is a proper deductive system. Also, by Proposition 2.4, A_0 is a subalgebra of A. □

Proposition 3.9. *Let D be a proper deductive system of an equality algebra A. Then the following hold:*

(i) *D is prime iff A/D is a linearly ordered equality algebra,*

(ii) *if D is prime, then $\{F \in DS(A) \mid D \subseteq F\}$ is linearly ordered by inclusion.*

Proof. (i). For any $a, b \in A$, $a \sim a \wedge b \in D$ iff $(a \sim a \wedge b)/D = 1/D$ iff $a/D \sim_D a/D \wedge_D b/D = 1/D$ iff $a/D = a/D \wedge_D b/D$ iff $a/D \leqslant b/D$. By the similar way $b \sim b \wedge a \in D$ iff $b/D \leqslant a/D$. Hence D is prime iff A/D is a linearly ordered equality algebra.

(ii). Let $F, G \in \{F \in DS(A) \mid D \subseteq F\}$. If F and G are incomparable, then there exist $a \in F \setminus G$ and $b \in G \setminus F$. Since D is prime, by (i), A/D is linearly ordered. Then we can assume $a/D \leqslant b/D$, and so $a \sim a \wedge b \in D \subseteq F$. Since $a \in F$, by (DS_2), $a \wedge b \in F$ and since $a \wedge b \leqslant b$, by (DS_1) we get $b \in F$, which is a contradiction. Hence $F \subseteq G$ or $G \subseteq F$. □

Proposition 3.10. *Let A be an equality algebra. Then A is a linearly ordered iff each proper deductive systems of A are prime.*

Proof. Let A be a linearly ordered equality algebra. Then we have $a \leqslant b$ or $b \leqslant a$, for all $a, b \in A$. Thus for any proper $D \in DS(A)$, $a \sim a \wedge b = 1 \in D$ or $b \sim b \wedge a = 1 \in D$ and so D is prime. Conversely, by the assumption, $\{1\}$ is prime and so by Proposition 3.9, $A/\{1\} = A$ is a linearly ordered equality algebra. □

Corollary 3.11. *An equality algebra A is linearly ordered iff the set $DS(A)$ is linearly ordered by inclusion.*

Proof. It follows from Propositions 3.10 and 3.9(ii). □

Proposition 3.12. *Let A be an equality algebra. Then $D \in Max(A)$ iff A/D is simple.*

Proof. Let $D \in Max(A)$. If A/D is not simple, then there is $a \in A$ such that $\langle a/D \rangle \neq 1/D$. So $a \notin D$ and $D \subset \langle D \cup \{a\} \rangle$, which is a contradiction with the maximality of D. Hence A/D is simple. The converse is obvious. □

In the follows, we define the notion of state deductive system on state equality algebras.

Definition 3.13. Let (A,σ) be an state equality algebra. A deductive system D of A is called a *state deductive system* of A if $\sigma(D) \subseteq D$ (i.e., $a \in D$ implies $\sigma(a) \in D$). The set of all state deductive systems on state equality algebra (A,σ) is denoted by $IDS(A_\sigma)$. A proper state deductive system of (A,σ) is called a *maximal state deductive system* if there is no proper deductive system strictly containing it. The set of all maximal state deductive systems of (A,σ) is denoted by $IMax(A_\sigma)$. The intersection of all the maximal state deductive system of (A,σ) is denoted by $Rad(A,\sigma)$. Clearly, $Ker(\sigma)$ is a state deductive system of any state equality algebra.

Example 3.14. (i). Let A be the equality algebra in Example 3.4(i). Then $\sigma_1 : A \to A$ which is defined by $\sigma_1(0) = 0, \sigma_1(a) = 1, \sigma_1(b) = 0, \sigma_1(1) = 1$ is an state on A. We can check $\{b,1\} \in DS(A)$, but $\{b,1\} \notin IDS(A_{\sigma_1})$. Since $b \in \{b,1\}$ but $\sigma_1(b) = 0 \notin \{b,1\}$. Then $Rad(A) = \{1\}$ and $Rad(A,\sigma) = \{a,1\}$.

(ii). Let C be the equality algebra of Example 3.4(iii). Then $\sigma_1 : C \to C$ which is defined by $\sigma_1(0) = 0, \sigma_1(a) = a, \sigma_1(b) = a, \sigma_1(1) = 1$ is an state on C. We can check $\{b,1\} \in DS(C)$, but $\{b,1\} \notin IDS(C_{\sigma_1})$. Since $b \in \{b,1\}$ but $\sigma_1(b) = a \notin \{b,1\}$. Therefore $Rad(A) = \{b,1\}$ and $Rad(A,\sigma_1) = \{1\}$.

Example 3.15. (i). $\{1\}$ and A are state deductive systems of any state equality algebra (A,σ).

(ii). In any linearly ordered state equality algebra (A, Id_A), every $D \in DS(A)$ is a state deductive system of (A,σ). Then $Rad(A) = Rad(A,\sigma)$.

(iii). If C is the equality algebra in Example 3.4(iii). Then $\sigma : C \to C$ which is defined by $\sigma(0) = 0, \sigma(a) = a, \sigma(b) = 1, \sigma(1) = 1$ is an state on C. Then we can see that $D \in DS(C)$ iff $D \in IDS(C_\sigma)$, Since $x \in D$ follows $\sigma(x) \in D$. Then $Rad(A) = Rad(A,\sigma)$.

(iv). If A is the equality algebra of Example 3.4(i), then $\sigma : A \to A$ which is defined by $\sigma(0) = a, \sigma(a) = a, \sigma(b) = 1, \sigma(1) = 1$ is an state on A. Then we can see that $D \in DS(A)$ iff $D \in IDS(A_\sigma)$. Since $x \in D$ follows $\sigma(x) \in D$. Then $Rad(A) = Rad(A,\sigma)$.

Example 3.16. Let $(A, \wedge_A, \sim_A, 1_A)$ and $(B, \wedge_B, \sim_B, 1_B)$ be two equality algebras. Then $C = A \times B = \{(a,b) \in A \times B \mid a \in A, b \in B\}$ with operations $\wedge$, $\sim$, 1 as follows : $(a,b) \wedge (a^{'}, b^{'}) = (a \wedge_A a^{'}, b \wedge_B b^{'})$, $(a,b) \sim (a^{'}, b^{'}) = (a \sim_A a^{'}, b \sim_B b^{'})$, $1 = (1_A, 1_B)$, for all $(a,b), (a^{'}, b^{'}) \in C$, is an equality algebra.

Let $\sigma_1 : A \to A$ and $\sigma_2 : B \to B$ are states on A and B, respectively. Then $\sigma : C \to C$ which is defined by $\sigma(a,b) = (\sigma_1(a), \sigma_2(b))$ is an state on C, for all $(a,b) \in C$. Let $D_1 \in DS(A)$ and $D_2 \in DS(B)$. Then $D_1 \times D_2 \in DS(C)$ is a state deductive system of (C,σ) if for all $(a,b) \in D_1 \times D_2$ we get $\sigma(a,b) \in D_1 \times D_2$. Hence $D_1 \times D_2 \in IDS(C_\sigma)$ iff $D_1 \in IDS(A\sigma_1)$ and $D_2 \in IDS(B\sigma_2)$.

Proposition 3.17. *Let (A,σ) be an state equality algebra. Then*

(i) $\sigma(a \to b) \leqslant \sigma(a) \to \sigma(b)$, *for any* $a, b \in A$,

(ii) *if A is linearly ordered, then* $\sigma(a \sim b) \leqslant \sigma(a) \sim \sigma(b)$ *and* $\sigma(a \wedge b) = \sigma(a) \wedge \sigma(b)$.

Proof. (i). By (E_{21}) we have $a \leqslant (a \sim a \wedge b) \sim b$, so by (S_1), we get $\sigma(a) \leqslant \sigma((a \sim a \wedge b) \sim b)$. Now (E_{18}) follows $\sigma((a \sim a \wedge b) \sim b) \to \sigma(b) \leqslant \sigma(a) \to \sigma(b)$. Thus by (S_2), $\sigma(a \sim a \wedge b) = \sigma((a \sim a \wedge b) \sim b) \sim \sigma(b) \leqslant \sigma((a \sim a \wedge b) \sim b) \to \sigma(b)$. So $\sigma(a \to b) \leqslant \sigma(a) \to \sigma(b)$.

(ii). Since A is linearly ordered, assume that $a \leqslant b$. Then by $a \sim b \leqslant b \to a$ and (i), we get $\sigma(a \sim b) = \sigma(b \to a) \leqslant \sigma(b) \to \sigma(a) = \sigma(a) \sim \sigma(b)$. Moreover, if $a \leqslant b$ $(b \leqslant a)$ then by (S_1), $\sigma(a) \leqslant \sigma(b)$ $(\sigma(b) \leqslant \sigma(a))$. So $\sigma(a \wedge b) = \sigma(a) \wedge \sigma(b)$. □

Proposition 3.18. *Let (A, σ) be an state equality algebra and $S \subseteq A$. Then*

$$Fix(S) = \{a \in A \mid \sigma(a) \to s = s, \textit{for all } s \in S\}$$

is a state deductive system of (A, σ).

Proof. Obviously, $1 \in Fix(S)$. Let $a \in Fix(S)$ and $a \leqslant b$. Then $\sigma(a) \to s = s$. Hence by Definition 2.8(S_1) and (E_{18}), $\sigma(a) \leqslant \sigma(b)$ and so $\sigma(b) \to s \leqslant \sigma(a) \to s = s$, which implies that $\sigma(b) \to s = s$. Thus $b \in Fix(S)$. Let $a, a \sim b \in Fix(S)$. Then $\sigma(a) \to s = s$ and $\sigma(a \sim b) \to s = s$. Since $a \sim b \leqslant a \to b$, by Definition 2.8 and (E_{18}) we get $s \leqslant \sigma(a \to b) \to s \leqslant \sigma(a \sim b) \to s = s$. Hence $\sigma(a \to b) \to s = s$. Now by Proposition 3.17, we get $\sigma(a \to b) \leqslant \sigma(a) \to \sigma(b)$ and so $(\sigma(a) \to \sigma(b)) \to s = s$. Since $(\sigma(a) \to \sigma(b)) \to (\sigma(a) \to s) = s$ thus we have $s \leqslant \sigma(b) \to s \leqslant (\sigma(a) \to \sigma(b)) \to (\sigma(a) \to s) = s$, that follows $b \in Fix(S)$. Finally, let $a \in Fix(S)$. So $\sigma(a) \to s = s$. By $s \leqslant \sigma(\sigma(a)) \to s = \sigma(a) \to s = s$, we get $\sigma(a) \in Fix(S)$. Hence $Fix(S) \in IDS(A_\sigma)$. □

Definition 3.19. Let (A, σ) be an state equality algebra. If $S \subseteq A$, then $\langle\langle S \rangle\rangle$ is the *state deductive system generated by S*.

Proposition 3.20. *Let (A, σ) be an state equality algebra. If $D \in DS(A)$,*

$$\langle\langle D \rangle\rangle = \{a \in A \mid \exists n \in \mathbb{N}, \exists x_1, \ldots, x_n \in D \textit{ st. } \sigma(x_1) \to (...(\sigma(x_n) \to a)...) \in D\}.$$

Proof. Let
$S = \{a \in A \mid \exists n \in \mathbb{N}, \exists x_1, \ldots, x_n \in D \textit{ st. } \sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_n) \to a)...)) \in D\}$.

First, we show that $D \subseteq S$. For any $d \in D$, since $1 \in D$ and $\sigma(1) = 1 \in D$ we get $\sigma(1) \to d = 1 \to d = d \in D$ and so $d \in S$. Now we prove that S is a state deductive system of (A, σ). Since for all $x \in D, \sigma(x) \to 1 = 1 \in D$, by definition of S, $1 \in S$. Now, let $a \in S$ and $a \leqslant b$. Then there are $n \in \mathbb{N}$ and $x_1, x_2, ..., x_n \in D$ such that $\sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_n) \to a)...)) \in D$. Since $a \leqslant b$ and $D \in DS(A)$, from (E_{19}),

$$\sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_n) \to a)...)) \leqslant \sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_n) \to b)...))$$

it follows that $\sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_n) \to b)...)) \in D$. So $b \in S$. Finally, let $a, a \sim b \in S$. Then there are $m, n \in \mathbb{N}$, $x_1, x_2, ..., x_m \in D$ and $y_1, y_2, ..., y_n \in D$ such that

$$\sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_m) \to a)...)) \in D$$

and $\sigma(y_1) \to (\sigma(y_2) \to (...(\sigma(y_n) \to (a \sim b))...)) \in D$. Since $a \sim b \leqslant a \to b$, we get $\sigma(y_1) \to (\sigma(y_2) \to (...(\sigma(y_n) \to (a \to b))...)) = Z \in D$. Now from (E_{19}) and (E_{17}), we have $\sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_m) \to a)...)) \leqslant \sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_m) \to (\sigma(y_1) \to (\sigma(y_2) \to (...(\sigma(y_n) \to (Z \to b)))...))$. So

$$Z \to \sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_m) \to (\sigma(y_1) \to (\sigma(y_2) \to (...(\sigma(y_n) \to b))...)) \in D$$

and $Z \in D$. Hence by definition of S, $b \in S$. Thus S is a deductive system of A. Now, we prove that S is a state deductive system of A. For any $a \in S$, there are $x_1, x_2, ..., x_n \in D$ such that $\sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_n) \to a)...)) = Y \in D$. Hence $Y \to (\sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_n) \to a)...))) = 1 \in D$ and $\sigma(Y \to (\sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_n) \to a)...)))) = \sigma(1) = 1 \in D$. By using Propositions 3.17 and 2.9 (2), $\sigma(Y) \to (\sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_n) \to \sigma(a))...))) = 1 \in D$. From $Y \in D$, by definition of S, $\sigma(a) \in S$. Finally we show that S is the smallest state deductive system of A containing D. Let $F \in IDS(A_\sigma)$ such that $D \subseteq F$. Assume $a \in S$, if $a = 1$, then $S \subseteq F$. Otherwise there are $x_1, x_2, ..., x_n \in D \subseteq F$ such that $\sigma(x_1) \to (\sigma(x_2) \to (...(\sigma(x_n) \to a)...)) \in D \subseteq F$. Since F is a state deductive system of A, thus $\sigma(x_1), \sigma(x_2), ..., \sigma(x_n) \in F$, so $a \in F$. Hence S is the smallest state deductive system of A containing D, that is $\langle\langle D \rangle\rangle = S$. □

Proposition 3.21. *Let D be a state deductive system of an state equality algebra (A, σ) and $x \in A$. Then*

$$\langle\langle D \cup \{x\} \rangle\rangle = \{a \in A \mid \sigma^m(x) \to (x^n \to a) \in D,\ \exists m, n \in \mathbb{N}\}.$$

A state deductive system M of a bounded state equality algebra is maximal iff for any $x \notin M$, there are $m, n \in \mathbb{N}$ such that $\sigma^m(x) \to (x^n \to 0) \in M$.

Proof. Set $S = \{a \in A \mid \sigma^m(x) \to (x^n \to a) \in D,\ \exists m, n \in \mathbb{N}\}$. First, we show that $\{D \cup \{x\}\} \subseteq S$. Let $y \in \{D \cup \{x\}\}$, if $y = x$ then $y \in S$. Otherwise $y \in D$, from $y \leqslant x \to y$ follows $x \to y \in D$. So $y \in S$. Now we prove that S is a state deductive system of (A, σ). Obviously, $1 \in S$. Let $a \in S$ and $a \leqslant b$. Then there are $m, n \in \mathbb{N}$ such that $\sigma^m(x) \to (x^n \to a) \in D$. By (E_{19}), $\sigma^m(x) \to (x^n \to b) \in D$. So $b \in S$. Now, let a and $a \sim b \in S$. Then there are $m, n, s, t \in \mathbb{N}$ such that $\sigma^m(x) \to (x^n \to a) \in D$ and $\sigma^s(x) \to (x^t \to (a \sim b)) \in D$. Since $a \sim b \leqslant a \to b$, thus $\sigma^m(x) \to (x^n \to (a \to b)) = Y \in D$. By routine proof we get $\sigma^m(x) \to (x^n \to a) \leqslant \sigma^m(x) \to (x^n \to (\sigma^s(x) \to (x^t \to (Y \to b))))$. Thus $\sigma^{m+s}(x) \to (x^{n+t} \to (Y \to b))) \in D$. On the other hand we have $Y \in D$ and so $b \in S$. Hence S is a deductive system of A. Moreover, for any $a \in S$ there are $m, n \in \mathbb{N}$ such that $\sigma^m(x) \to (x^n \to a) = Y \in D$. Then $Y \to (\sigma^m(x) \to (x^n \to a)) = 1 \in D$. By Propositions 2.9(1) and 3.17, we have $1 = \sigma(1) = \sigma(Y \to (\sigma^m(x) \to (x^n \to a)) \leqslant \sigma(Y) \to (\sigma\sigma^m(x) \to (\sigma\sigma^n(x) \to \sigma(a)))$. Since $Y \in D$ and D is state, we get $\sigma(Y) \in D$ and so by definition of S, $\sigma(a) \in S$. Hence S is a state deductive system of A, that is $S = \langle\langle D \cup \{x\} \rangle\rangle$. For proof of the second part, we assume that M is maximal and $x \notin M$. Then by maximality of M, $\langle\langle M \cup \{x\} \rangle\rangle = A$. Since A is bounded, we get $0 \in \langle\langle M \cup \{x\} \rangle\rangle$. Thus there are $m, n \in \mathbb{N}$ such that $\sigma^m(x) \to (x^n \to 0) \in M$. The converse is evident. □

Remark 3.22. Obviously, Propositions 3.20 and 3.21 hold for any state-morphism equality algebra, too.

Definition 3.23. Let (A, σ) be an state equality algebra and θ be a congruence relation on A. Then θ is called a *congruence relation* on (A, σ) if $(a, b) \in \theta$ implies $(\sigma(a), \sigma(b)) \in \theta$. The set of all congruence on (A, σ) denote by $Con(A, \sigma)$.

In the following, we show that if (A, σ) is a linearly ordered state equality algebra, there is a bijection between $IDS(A_\sigma)$ and $Con(A, \sigma)$.

Proposition 3.24. *Let (A, σ) be a linearly ordered state equality algebra. Then the following hold:*

(*i*) *if $D \in IDS(A_\sigma)$, then $\theta_D = \{(a, b) \in A \times A \mid a \sim b \in D\}$ is a congruence relation on (A, σ),*

(*ii*) *if $\theta \in Con(A, \sigma)$, then $[1]_\theta = \{a \in A \mid (a, 1) \in \theta\}$ is a state deductive system of (A, σ) (that is $[1]_\theta \in IDS(A_\sigma\ ))$.*

Proof. (i). Let $D \in IDS(A_\sigma)$. By Proposition 2.5(i), θ_D is a congruence relation of A. Let $(a, b) \in \theta_D$. Then $a \sim b \in D$, by Definition 3.13, we get $\sigma(a \sim b) \in D$. Now since A is linearly ordered, so by Proposition 3.17, $\sigma(a) \sim \sigma(b) \in D$. Thus $(\sigma(a), \sigma(b)) \in \theta_D$. Hence θ_D is a congruence relation on (A, σ).

(ii) Let θ be a congruence relation on (A, σ). By Proposition 2.5(ii), $[1]_\theta$ is a deductive system of A. Let $a \in [1]_\theta$. Then $(a, 1) \in \theta$. Since $\theta \in Con(A, \sigma)$, thus $(\sigma(a), \sigma(1)) \in \theta$. From $\sigma(1) = 1$ follows $(\sigma(a), 1) \in \theta$ and so $\sigma(a) \in [1]_\theta$. Thus $[1]_\theta$ is a state deductive system of (A, σ). □

Theorem 3.25. *Let (A, σ) be a linearly ordered state equality algebra. Then there is a one-to-one correspondence between $IDS(A_\sigma)$ and $Con(A, \sigma)$.*

Proof. Define $f : Con(A, \sigma) \to IDS(A_\sigma)$ by $f(\theta) = [1]_\theta$. By Theorem 2.6 and Proposition 3.24, f is an one-to-one correspondence between $IDS(A_\sigma)$ and $Con(A, \sigma)$. Then the proof is complete. □

Theorem 3.26. *Let (A, σ) be a linearly ordered state equality algebra. If $D \in IDS(A_\sigma)$, then $\sigma' : A/D \to A/D$ is an state on A/D with $\sigma'(a/D) = \sigma(a)/D$.*

Proof. First, we show that σ' is well defined. Let $a/D = b/D$. Then $a \sim b \in D$ and so $\sigma(a \sim b) \in D$. By Proposition 3.17, $\sigma(a) \sim \sigma(b) \in D$ and so $\sigma(a)/D = \sigma(b)/D$. Hence $\sigma'(a/D) = \sigma'(b/D)$. Now we prove σ' is an state. For the proof of (S_1), let $a/D \leqslant b/D$. Then $a/D \sim (a/D \wedge b/D) = 1/D$ and so $a \sim (a \wedge b) \in D$. By Definition 3.13, we get $\sigma(a \sim (a \wedge b)) \in D$. Also, by Proposition 3.17, $\sigma(a) \sim \sigma(b) \wedge \sigma(b) \in D$. Thus $\sigma(a)/D \leqslant \sigma(b)/D$ and so $\sigma'(a/D) \leqslant \sigma'(b/D)$. For the proof of (S_2),

$$\begin{aligned}\sigma'(a/D \sim a/D \wedge b/D) &= \sigma'((a \sim a \wedge b)/D) = \sigma(a \sim a \wedge b)/D \\ &= (\sigma((a \sim a \wedge b) \sim b) \sim \sigma(b))/D \\ &= \sigma((a \sim a \wedge b) \sim b)/D \sim \sigma(b)/D \\ &= \sigma'((a/D \sim a/D \wedge b/D) \sim b/D) \sim \sigma'(b/D).\end{aligned}$$

For the proof of (S_3),

$$\begin{aligned}\sigma^{'}(\sigma^{'}(a/D) \sim \sigma^{'}(b/D)) &= \sigma^{'}(\sigma(a)/D \sim \sigma(b)/D) = \sigma^{'}((\sigma(a) \sim \sigma(b))/D)\\ &= (\sigma(\sigma(a) \sim \sigma(b)))/D = (\sigma(a) \sim \sigma(b))/D\\ &= \sigma(a)/D \sim \sigma(b)/D = \sigma^{'}(a/D) \sim \sigma^{'}(b/D).\end{aligned}$$

Also (S_4) satisfies since

$$\begin{aligned}\sigma^{'}(\sigma^{'}(a/D) \wedge \sigma^{'}(b/D)) &= \sigma^{'}(\sigma(a)/D \wedge \sigma(b)/D) = \sigma^{'}((\sigma(a) \wedge \sigma(b))/D)\\ &= \sigma(\sigma(a) \wedge \sigma(b))/D = (\sigma(a) \wedge \sigma(b))/D\\ &= \sigma(a)/D \wedge \sigma(b)/D = \sigma^{'}(a/D) \wedge \sigma^{'}(b/D).\end{aligned}$$

Finally (S_5) satisfies since

$$\sigma^{'}(\sigma^{'}(a/D)) = \sigma^{'}(\sigma(a)/D) = \sigma(\sigma(a))/D = \sigma(a)/D = \sigma^{'}(a/D).$$ □

Note that in Proposition 3.26, $\sigma^{'}$ is faithful if $Ker(\sigma^{'}) = \{x/D \mid \sigma^{'}(x/D) = 1/D\} = \{1/D\}$ i.e., $Ker(\sigma^{'}) = \{x/D \mid \sigma(x) \in D\}$.

Corollary 3.27. *Let (A, σ) be a linearly ordered state equality algebra. Then $\sigma^{'} : A/K \to A/K$ is an state on A/K such that $K = Ker(\sigma)$.*

Proof. Since $Ker(\sigma)$ is a state deductive system of (A, σ), so the result follows from Theorem 3.26. □

Definition 3.28. Let (A, σ) be a state-morphism equality algebra. A deductive system D of A is called the *state-morphism deductive system* of A if $\sigma(D) \subseteq D$, i.e., if $a \in D$ implies $\sigma(a) \in D$.

The set of all state-morphism deductive systems on a state-morphism equality algebra (A, σ) denote by $SDS(A_\sigma)$ and the set of all maximal state-morphism deductive systems of (A, σ) denote by $SMax(A_\sigma)$.

Remark 3.29. Clearly, by Theorem 4.6(i) and Definition 2.10, the above results proved for linearly ordered state equality algebra hold for state-morphism equality algebra.

Proposition 3.30. *Let (A, σ) be a state-morphism equality algebra and D be a deductive system of A. Then D is a prime state deductive system of (A, σ) iff $(A/D, \sigma^{'})$ is a linearly ordered state-morphism equality algebra.*

Proof. It follows by Proposition 3.9, Remark 3.29 and Theorem 3.26. □

Definition 3.31. Let (A, σ) be a state-morphism (an state) equality algebra. A subalgebra S of A is called *state subalgebra* if $a \in S$ implies $\sigma(a) \in S$.

Example 3.32. (i). If A is the equality algebra in Example 3.4(i), then σ_1 and $\sigma_2 : A \to A$ defined by $\sigma_1(0) = 0$, $\sigma_1(a) = 1, \sigma_1(b) = 0, \sigma_1(1) = 1$ and $\sigma_2(0) = a$, $\sigma_2(a) = a, \sigma_2(b) = 1, \sigma_2(1) = 1$ are state-morphisms on A. Also, $\{0, 1\}$ is a state subalgebra of (A, σ_1), which is not a state subalgebra of (A, σ_2), since $\sigma_2(0) = a \notin \{0, 1\}$.

(ii). Let C be an equality algebra. We know that $(C, 1_C)$ is a state-morphism equality algebra. Then every subalgebra of C is a state subalgebra of $(C, 1_C)$.

Remark 3.33. Let (A, σ) be a bounded state-morphism equality algebra. If A is linearly ordered, $a \in A$ and $a \leqslant \sigma(a)$, then by Proposition 3.8(i), $A(a)$ is a state subalgebra. Moreover, if $\sigma(0) = 0$, then by Proposition 3.8(ii), A_0 is a state deductive system. Since for any $a \in A_0$, $a \sim 0 = 0$. By Definition 2.10, $\sigma(a \sim 0) = \sigma(a) \sim \sigma(0) = \sigma(0)$, then we get $\sigma(a) \sim 0 = 0$. Thus $\sigma(a) \in A_0$.

Proposition 3.34. *Every state deductive system of an state equality algebra (A, σ) is a state subalgebra of (A, σ).*

Proof. By Proposition 2.4 and Definition 3.31, the proof is clear. □

4. Some properties of state equality algebra and state-morphism equality algebras

In the following, we state some properties of state equality algebra and state-morphism equality algebra. We proved every state-morphism operator on an equality algebra is a state operator on it and the converse is true for a linearly ordered equality algebra under a condition.

Proposition 4.1. *Let (A, σ) be a linearly ordered state equality algebra. The map $\sigma' : A/Ker(\sigma) \to A/Ker(\sigma)$ defined by $\sigma'(a/Ker(\sigma)) = \sigma(a)/Ker(\sigma)$, is a state on $A/Ker(\sigma)$, for any $a \in A$.*

Proof. First, we show that σ' is well defined. For this, let $K = Ker(\sigma)$ and $a/K = b/K$. Then $a \sim b \in K$ and so $\sigma(a \sim b) = 1$. Since A is linearly ordered, by Proposition 3.17, $\sigma(a) \sim \sigma(b) = 1$ and this conclude that $\sigma(a) = \sigma(b)$. Hence $\sigma'(a/K) = \sigma'(b/K)$. Now by Theorem 2.7, Definition 2.8 and Proposition 3.17, the proof is complete. □

Proposition 4.2. *Let (A, σ) be a state equality algebra and $Ker(\sigma)$ be prime. Then $\sigma(A)$ is linearly ordered.*

Proof. For all $a, b \in A$, $a \sim a \wedge b \in Ker(\sigma)$ or $b \sim b \wedge a \in Ker(\sigma)$. So $\sigma(a \sim a \wedge b) = 1$ or $\sigma(b \sim b \wedge a) = 1$. From $a \wedge b \leqslant a, b$ and Proposition 3.17, we get $\sigma(a) \sim \sigma(a \wedge b) = 1$ or $\sigma(b) \sim \sigma(b \wedge a) = 1$. Hence $\sigma(a) \leqslant \sigma(b)$ or $\sigma(b) \leqslant \sigma(a)$. Thus $\sigma(A)$ is linearly ordered. □

Example 4.3. Let A be the equality algebra in Example 3.4(i) and $\sigma = Id_A$. Then (A, σ) is a state equality algebra. But $Ker(\sigma) = \{1\}$ is not prime. Since by Proposition 4.2, $\sigma(A)$ is not linearly ordered ($\sigma(a) \nleq \sigma(b)$).

Proposition 4.4. *Let (A, σ) be a linearly ordered state equality algebra. Then the following statements are equivalent:*

(i) $\sigma(a \to b) = \sigma(a) \to \sigma(b)$,

(ii) $\sigma(a \sim b) = \sigma(a) \sim \sigma(b)$.

Proof. (i) $\Rightarrow$ (ii). Since A is linearly ordered, we can assume $a \leqslant b$. By (S_1), we get $\sigma(a) \leqslant \sigma(b)$ and so

$$\begin{aligned}\sigma(b \sim a) &= \sigma(b \sim b \wedge a) = \sigma(b \to a) = \sigma(b) \to \sigma(a)\\ &= \sigma(b) \sim \sigma(b) \wedge \sigma(a) = \sigma(b) \sim \sigma(a).\end{aligned}$$

For $b \leqslant a$ the proof is similarly.

(ii) $\Rightarrow$ (i). By Proposition 3.17,

$$\sigma(a \to b) = \sigma(a \sim a \wedge b) = \sigma(a) \sim \sigma(a \wedge b) = \sigma(a) \sim (\sigma(a) \wedge \sigma(b)) = \sigma(a) \to \sigma(b).$$

□

Proposition 4.5. *Let (A, σ) be a state equality algebra, σ be faithful and for any $a, b \in A$, $\sigma((a \sim a \wedge b) \sim b) = \sigma((b \sim b \wedge a) \sim a)$. Then*

(i) *$a < b$ implies $\sigma(a) < \sigma(b)$,*

(ii) *if A is linearly ordered, then $\sigma(a) = a$, for all $a \in A$.*

Proof. (i). Let $a < b$. By (S_1) we have $\sigma(a) \leqslant \sigma(b)$. Assume $\sigma(a) = \sigma(b)$. Then by (S_2) and assumption,

$$\begin{aligned}\sigma(a \sim b) &= \sigma(b \sim b \wedge a) = \sigma((b \sim b \wedge a) \sim a) \sim \sigma(a)\\ &= \sigma((a \sim a \wedge b) \sim b) \sim \sigma(a) = \sigma(b) \sim \sigma(a) = 1.\end{aligned}$$

So $a \sim b \in Ker(\sigma) = \{1\}$ and it follows $a = b$, which is a contradiction with $a < b$. Then $\sigma(a) < \sigma(b)$.

(ii). Let for all $a \in A$, $\sigma(a) \neq a$. Since A is linearly ordered, $\sigma(a) < a$ or $a < \sigma(a)$. By (i) we get $\sigma(\sigma(a)) < \sigma(a)$ or $\sigma(a) < \sigma(\sigma(a))$, which is a contradiction with (S_3). Hence $\sigma(a) = a$. □

Proposition 4.5, is not true for any state equality algebra. In Example 3.4(iii), with $\sigma_1 : C \to C$ defined by $\sigma_1(0) = 0, \sigma_1(a) = a, \sigma_1(b) = a, \sigma_1(1) = 1$, (C, σ_1) is a linearly ordered state equality algebra with $Ker(\sigma) = \{1\}$. But $\sigma(b) = a \neq b$, since $\sigma((a \sim a \wedge b) \sim b) \neq \sigma((b \sim b \wedge a) \sim a)$.

Theorem 4.6. *Let A be an equality algebra. Then*

(i) *any state-morphism on A is a state on A,*

(ii) *if (A, σ) is a linearly ordered state equality algebra in which for all $a, b \in A$ $\sigma((a \sim a \wedge b) \sim b) = \sigma((b \sim b \wedge a) \sim a)$, then σ is a state-morphism on A.*

Proof. (i) Let σ be a state-morphism operator on A. Clearly, (S_1) satisfies. Since $b \leqslant (a \to b) \to b)$, by (SM_1),

$$\begin{aligned}\sigma(a \sim a \wedge b) &= \sigma(a \to b) = \sigma(((a \to b) \to b) \to b) \\ &= \sigma(((a \sim a \wedge b) \sim b) \sim b) = \sigma((a \sim a \wedge b) \sim b) \sim \sigma(b).\end{aligned}$$

Thus (S_2) satisfies. Also (S_3) and (S_4) follow from $(SM_1) - (SM_3)$.

(ii). Let σ be a state operator on A and $a \leqslant b$. By (S_1) we have $\sigma(a) \leqslant \sigma(b)$. Then by (S_2),

$$\begin{aligned}\sigma(a \sim b) &= \sigma(b \sim b \wedge a) = \sigma((b \sim b \wedge a) \sim a) \sim \sigma(a) \\ &= \sigma((a \sim a \wedge b) \sim b) \sim \sigma(a) = \sigma(b) \sim \sigma(a) = \sigma(a) \sim \sigma(b).\end{aligned}$$

For $b \leqslant a$, with the similar proof, σ is a state-morphism operator on A. Finally, (SM_2) and (SM_3) follow from Propositions 3.17 and 2.9(2). □

Definition 4.7. (cf. [3]) Let A be an equality algebra and $a \in A$. Then

(i) A is called $(\sim_a)$-*involutive*, if for all $b \in A$, $((b \sim a) \sim a) = b$,

(ii) $x \in A$ is called a-*regular* if $(x \sim a) \sim a = x$,

(iii) A is called *involutive* if $A = Reg_a(A)$, for all $a \in A$, where $Reg_a(A)$ is the set of all a-regular elements of A.

Example 4.8. (1). Any equality algebra A is $(\sim_1)$-involutive and $A = Reg_1(A)$ (for all $b \in A, ((b \sim 1) \sim 1) = b$).
(2). Let A be the equality algebra in Example 3.4(i). Then A is $(\sim_a)$-involutive, for all a $\in$ A and $A = Reg_a(A)$.
(3). Let B be the equality algebra in Example 3.4(ii). Then B is $(\sim_0)$-involutive since $((0 \sim 0) \sim 0) = 0$, $((b \sim 0) \sim 0) = b$, $((1 \sim 0) \sim 0) = 1$. But B is not $(\sim_b)$-involutive, since $((0 \sim b) \sim b) = 1 \neq 0$.

Corollary 4.9. *Let A be a linearly ordered involutive equality algebra. Then σ is a state on A iff σ is a state-morphism on A.*

Proof. Since A is involutive, we get $(a \sim b) \sim b = a$, for all $a, b \in A$. Then by Theorem 4.6, the proof is complete. □

Example 4.10. Let C be the linearly ordered equality algebra of Example 3.4(iii). Then $\sigma_1, \sigma_2 : C \to C$ defined by $\sigma_1(0) = 1, \sigma_1(a) = 1, \sigma_1(b) = 1, \sigma_1(1) = 1$ and $\sigma_2(0) = 0, \sigma_2(a) = a, \sigma_2(b) = 1, \sigma_2(1) = 1$ are two state-morphisms on C. By Theorem 4.6, σ_1 and σ_2 are states on C. Moreover, $\sigma_3 : C \to C$ which is defined by $\sigma_3(0) = 0, \sigma_3(a) = a, \sigma_3(b) = a, \sigma_3(1) = 1$ is a state on C but it is not a state-morphism on C. Since $\sigma_3(a \sim b) \neq \sigma_3(a) \sim \sigma_3(b)$. Also, Theorem 4.6$(ii)$ is not satisfied, since $\sigma_3((b \sim b \wedge a) \sim a) = \sigma_3(1) = 1 \neq a = \sigma_3((a \sim a \wedge b) \sim b)$.

Proposition 4.11. *Let (A, σ) be a state-morphism equality algebra and $a \in A$. If $x \in Reg_a(A)$, then $\sigma(x) \in Reg_{\sigma(a)}(A)$.*

Proof. It is clearly by Definitions 4.7(ii) and 2.10. □

Proposition 4.12. *Let (A, σ) be a state-morphism equality algebra. Then $Ker(\sigma)$ is prime iff $\sigma(A)$ is linearly ordered.*

Proof. If $Ker(\sigma)$ is prime, then the proof is similar to the proof of Proposition 4.2. Conversely, assume that for all $a, b \in A$, $\sigma(a) \leqslant \sigma(b)$ or $\sigma(b) \leqslant \sigma(a)$. Let $a \sim a \wedge b \notin Ker(\sigma)$. Then $\sigma(a \sim a \wedge b) \neq 1$ and so by (SM_1) and (SM_2), $\sigma(a) \sim \sigma(a \wedge b) \neq 1$. Thus $\sigma(a) \nleq \sigma(b)$ and so by assumption, $\sigma(b) \leqslant \sigma(a)$. Hence $\sigma(b \sim b \wedge a) = 1$ and $b \sim b \wedge a \in Ker(\sigma)$. □

Proposition 4.13. *Let (A, σ) be a state-morphism equality algebra and $K = Ker(\sigma)$. Then*
(i) *$a/K \leqslant b/K$ iff $\sigma(a) \leqslant \sigma(b)$,*
(ii) *$a/K = b/K$ iff $\sigma(a) = \sigma(b)$.*

Proof. Applying Theorem 2.7 and Definition 2.10, we get
(i). $a/K \leqslant b/K$ iff $a/K = (a \wedge b)/K$ iff $(a \sim (a \wedge b))/K = 1/K$ iff $a \sim (a \wedge b) \in K$ iff $\sigma(a \sim (a \wedge b)) = 1$ iff $\sigma(a) \sim (\sigma(a) \wedge \sigma(b)) = 1$ iff $\sigma(a) \leqslant \sigma(b)$.
(ii). $a/K = b/K$ iff $(a \sim b)/K = 1/K$ iff $a \sim b \in K$ iff $\sigma(a \sim b) = 1$ iff $\sigma(a) \sim \sigma(b) = 1$ iff $\sigma(a) = \sigma(b)$. □

Proposition 4.14. *Let σ and μ be two state-morphisms on equality algebra A such that $Ker(\sigma) = Ker\mu$ and $Im\sigma = Im\mu$. Then $\sigma = \mu$.*

Proof. By Proposition 2.11, for all $a \in A$, $\sigma(a) \sim a \in Ker(\sigma) = Ker\mu$. Then $\mu(\sigma(a) \sim a) = 1$ and so we have $\mu(\sigma(a)) \sim \mu(a) = 1$. From $\sigma(a) \in Im\sigma = Im\mu$ follows $\mu(\sigma(a)) = \sigma(a)$. Hence $\sigma(a) \sim \mu(a) = 1$, that means $\sigma(a) = \mu(a)$. □

Theorem 4.15. *If (A, σ) is a state-morphism equality algebra, then*

$$A = \langle Ker(\sigma) \cup Im\sigma \rangle.$$

Proof. Obviously, $\langle Ker(\sigma) \cup Im\sigma \rangle \subseteq A$. Since $Ker(\sigma) \in Ds(A)$ and $Im\sigma \subseteq A$, thus by Theorem 3.1(ii), $\langle Ker(\sigma) \cup Im\sigma \rangle = \{a \in A \mid \sigma(a_1) \to (\sigma(a_2) \to (...(\sigma(a_n) \to a)...)) \in Ker(\sigma), \text{for some } \ a_1, ...a_n \in A\}$. Let a be an arbitrary element of A, by Proposition 2.11, $\sigma(a) \sim a \in Ker(\sigma)$. Since $\sigma(a) \sim a \leqslant \sigma(a) \to a$, then $\sigma(a) \to a \in Ker(\sigma)$ such that $\sigma(a) \in Im\sigma$ and so $a \in \langle Ker(\sigma) \cup Im\sigma \rangle$. Hence $A = \langle Ker(\sigma) \cup Im\sigma \rangle$. □

5. Equality-homomorphisms and their relation with the state-morphism operator

In this section, we define a homomorphism between two equality algebras and we state some related results. Then we prove that an state on an equality algebra, is a state-morphism if it is an equality-homomorphism.

Definition 5.1. *Let* $(A, \wedge, \sim, 1)$ *and* $(A^{'}, \wedge^{'}, \sim^{'}, 1^{'})$ *be two equality algebras. The map* $f : A \to A^{'}$ *is called an equality-homomorphism, if the following hold, for all* $a, b \in A$:

(H_1) $f(a \sim b) = f(a) \sim^{'} f(b)$,

(H_2) $f(a \wedge b) = f(a) \wedge^{'} f(b)$.

If $f : A \to A^{'}$ is a homomorphism of equality algebras, then f is called an *equality-endomorphism*. The set $Kerf = \{a \in A \mid f(a) = 1^{'}\}$ is called a *kernel* of f.

It is clear that every equality-homomorphism, is a BCK $\wedge$-semilattice homomorphism.

Proposition 5.2. *Let* $f : A \to A^{'}$ *be a bounded equality-homomorphism and* $f(0) = 0^{'}$. *Then:*

(i) $f(1) = 1^{'}$,

(ii) f *is monotone,*

(iii) $f(x \sim 0) = f(x) \sim^{'} 0^{'}$,

(iv) $Kerf$ *is a proper deductive system of* A,

(v) Imf *is a subalgebra of* $A^{'}$,

(vi) f *is injective iff* $Kerf = \{1\}$,

(vii) *if* $D^{'} \in DS(A^{'})$, *then* $f^{-1}(D^{'}) \in DS(A)$,

$(viii)$ *if* f *is surjective and* $Kerf \subseteq D \in DS(A)$, *then* $f(D) \in DS(A^{'})$.

Proof. The proofs of $(i) - (vi)$ are straightforward.

(vii). Assume that $D^{'} \in DS(A^{'})$. Since $f(1) = 1^{'} \in D^{'}$, thus $1 \in f^{-1}(D^{'})$. Let $a \in f^{-1}(D^{'})$ and $a \leqslant b$. Then $f(a) \in D^{'}$ and $f(a) \leqslant^{'} f(b)$. Thus $f(b) \in D^{'}$. Let $a, a \sim b \in f^{-1}(D^{'})$. Then $f(a \sim b) \in D^{'}$, by equality-homomorphism f, we get $f(a) \sim^{'} f(b) \in D^{'}$. So $f(a) \in D^{'}$ follows that $f(b) \in D^{'}$, thus $b \in f^{-1}(D^{'})$. Thus $f^{-1}(D^{'})$ is a deductive system of A.

$(viii)$. Since $1 \in D$, by (i), $1^{'} \in f(D)$. Let $a^{'}, b^{'} \in A^{'}$. If $a^{'} \in f(D)$ and $a^{'} \leqslant b^{'}$. Then there exists $a \in D$ such that $f(a) = a^{'}$. Since f is surjective, there exists $b \in A$ such that $b^{'} = f(b)$. So $f(a) \leqslant^{'} f(b)$ follows that $f(a) \to^{'} f(b) = 1$ and so $f(a \to b) = 1$, thus $a \to b \in Kerf \subseteq D$. Then $b \in D$, so $b^{'} = f(b) \in f(D)$. Let $a^{'}, a^{'} \sim^{'} b^{'} \in f(D)$. Then there are $a, z \in D$ such that $f(a) = a^{'}$ and $f(z) = a^{'} \sim^{'} b^{'}$. Since f is surjective, so there is $b \in A$, such that $f(b) = b^{'}$. So $f(z) = a^{'} \sim^{'} b^{'} = f(a) \sim^{'} f(b) = f(a \sim b)$. Thus $1 = f(z) \sim^{'} f(a \sim b) = f(z \sim (a \sim b))$. Then $z \sim (a \sim b) \in Kerf \subseteq D$ follows that $b \in D$ and so $b^{'} = f(b) \in f(D)$. Then $f(D)$ is a deductive system of $A^{'}$. □

Theorem 5.3. *If* $f : A \to A^{'}$ *is a surjective equality-homomorphism, then there is a bijective correspondence between* $\{D \mid D \in DS(A), Kerf \subseteq D\}$ *and* $DS(A^{'})$.

Proof. By Proposition 5.2(vii) and $(viii)$, $f : \{D \mid D \in DS(A), Kerf \subseteq D\} \to DS(A^{'})$ such that $D \longmapsto f(D)$ and $f^{-1} : Ds(A^{'}) \to \{D \mid D \in DS(A), Kerf \subseteq D\}$ such that $D^{'} \longmapsto f^{-1}(D^{'})$ are well defined functions. Now we will show $f(f^{-1}(D^{'})) = D^{'}$ and $f^{-1}(f(D)) = D$. Since f is surjective, then $f(f^{-1}(D^{'})) = D^{'}$. It is clear that $D \subseteq f^{-1}(f(D))$. Assume that $a \in f^{-1}(f(D))$ then $f(a) \in$

$f(D)$, so there is $x \in D$ such that $f(a) = f(x)$ then $f(a) \sim f(x) = 1$. By Definition 5.1, $f(a \sim x) = 1$ so $a \sim x \in Kerf \subseteq D$. since $x \in D$ we get $a \in D$, thus $f^{-1}(f(D)) \subseteq D$. So $f^{-1}(f(D)) = D$. □

Theorem 5.4. *Let A be an equality algebra. Then $f : A \to A$ is a state-morphism operator iff f is an equality-endomorphism with $f(a) \sim a \in Kerf$, for all $a \in A$.*

Proof. Let f be a state-morphism operator. Then by Definition 2.10, (H_1) and (H_2) satisfies. Also by (SM_3), we get, $1 = f(f(a)) \sim f(a) = f(f(a) \sim a)$. It follows $f(a) \sim a \in Kerf$.

Conversely, let f be an equality-endomorphism. By Definition 5.1, (SM_1) and (SM_2) satisfies. By the assumption, for all $a \in A, f(a) \sim a \in Kerf$. Thus $f(f(a) \sim a) = 1$. From (H_1), we get $1 = f(f(a) \sim a) = f(f(a)) \sim f(a)$ that it follows $f(f(a)) = f(a)$. So (SM_3) satisfies. □

Corollary 5.5. *If A is a simple equality algebra, then every equality-endomorphism $f : A \to A$ is a state-morphism operator, if $f = 1_A$ or $f = Id_A$.*

Proof. Assume f is an equality-endomorphism. Then By Theorem 5.4, f is a state-morphism operator if $f(a) \sim a \in Ker\sigma$ for any $a \in A$. Since A is simple so $Ker(\sigma) = \{1\}$ or $Ker(\sigma) = A$. Then $f = Id_A$ or $f = 1_A$. □

Example 5.6. Let A be the equality algebra as Example 3.4(i). Then $f : A \to A$ is an equality-endomorphism by define $f(0) = 0, f(a) = b, f(b) = a, f(1) = 1$. But f is not a state-morphism operator on A.

Lemma 5.7. *Let $f : A \to A$ be an endomorphism on equality algebra A and for all $a \in A, f(a) \sim a \in Kerf$. Then f is a state operator on A.*

Proof. By Theorems 5.4 and 4.6(i), f is an state on A. □

The converse of proposition 5.7 is not true. In Example 3.4(iii), $\sigma : C \to C$ which is defined by $\sigma(0) = 0, \sigma(a) = a, \sigma(b) = 1, \sigma(1) = 1$ is an state on the linearly ordered equality algebra C, but σ is not equality-endomorphism ($\sigma_2(a \sim b)$ $\neq \sigma_2(a) \sim \sigma_2(b)$).

Lemma 5.8. *Let σ be an state operator on a linearly ordered equality algebra A such that for all $a, b \in A$, $\sigma((a \sim a \wedge b) \sim b) = \sigma((b \sim b \wedge a) \sim a)$. Then σ is an equality-endomorphism with $\sigma^2 = \sigma$.*

Proof. By Theorems 4.6(ii) and 5.4, the proof is complete. □

Theorem 5.9. *Let $f : A \to A$ be an equality-endomorphism on an equality algebra A. Then the following are equivalent.*

(i) *f is a state operator on A.*

(ii) *f is a state-morphism operator on A.*

Proof. By Lemmas 5.7 and 5.8 and Theorem 5.4, the proof is clear. □

Theorem 5.10. *Let (A, σ) be a state-morphism equality algebra. Then,*

(i) *$\sigma(A)$ is a simple subalgebra of A iff $Ker(\sigma) \in SMax(A_\sigma)$,*

(ii) *(A, σ) is a simple state-morphism equality algebra iff A is a simple equality algebra,*

(iii) *if $\sigma(A)$ is a semisimple subalgebra of A, then the intersection of all maximal state-morphism deductive systems of (A, σ) is a subset of $Ker(\sigma)$.*

Proof. (i). Let (A, σ) be a state-morphism equality algebra. Then by Theorem 5.4, σ is an equality-endomorphism, which implies that $A/Ker(\sigma) \cong \sigma(A)$. Thus $Ker(\sigma) \in SMax(A_\sigma)$ iff $A/Ker(\sigma)$ is simple iff $\sigma(A)$ is simple.

(ii). Let (A, σ) be a simple state-morphism. Then $Ker(\sigma) \in SDS(A_\sigma)$ and so $Ker(\sigma) = \{1\}$ or $Ker(\sigma) = A$. Hence $\sigma = Id_A$ or $\sigma = 1_A$. In this case every deductive system of A is state. Thus $\{1\}$ and A are only deductive systems of A. Therefore, A is simple. Conversely, let A be a simple equality algebra. Then A has only two deductive systems, $\{1\}$ and A which they are state. Hence (A, σ) is a simple state-morphism equality algebra.

(iii). Let $\sigma(A)$ be a semisimple subalgebra of A. Then by Definition 3.2,

$$\bigcap_{I \in SMax(\sigma(A))} I = \{1\}.$$

Since $A/Ker(\sigma) \cong \sigma(A)$, then $A/Ker(\sigma)$ is a semisimple equality algebra. So $\cap\{D : Ker(\sigma) \subseteq D \in SMax(A)\} = 1/Ker\sigma$. Now we show that D is state. Let $D \in SMax(A_\sigma)$ and $Ker\sigma \subseteq D$. Then by Proposition 2.11, for all $a \in D$, $\sigma(a) \sim a \in Ker\sigma \subseteq D$. Therefore, $\sigma(a) \in D$. $\square$

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References

[1] **R.A. Borzooei, M. Aaly Kologani, O. Zahiri**, *State of hoops*, Math. Slovaca **67** (2017), 1 – 16.

[2] **R.A. Borzooei, A. Dvurečenskij, O. Zahiri**, *State BCK-algebras and state-morphism BCK-algebras*, Fuzzy Sets Syst. **244** (2014), 86 – 105.

[3] **L.C. Ciungu**, *Internal states on equality algebras*, Soft Comput. **19** (2015), 939 – 953.

[4] **A. Di Nola, A. Dvurečenskij**, *State-morphism MV-algebras*, Ann. Pure Appl. Logic. **161** (2009), 161 – 173.

[5] **A. Dvurečenskij, O. Zahiri**, *Pseudo equality algebras: Revision*, Soft Comput. **20** (2016), 2091 – 2101.

[6] **T. Flaminio, F. Montagna**, *MV-algebras with internal states and probabilistic fuzzy logics*, Inter. J. Approx. Reasoning **50** (2009), 138 – 152.

[7] **P. Hájek**, *Metamathematics of fuzzy Logic*, Kluwer Academic Publishers, (1998).

[8] **S. Jenei**, *Equality algebras*, Studia Logica **100** (2012), 1201 – 1209.

[9] **S. Jenei, L. Kóródi**, *Pseudo-equality algebras*, Arch. Math. Logic **52** (2013), 469 – 481.

[10] **J. Kühr**, *Pseudo-BCK semilattices*, Demonstr. Math. **40** (2007), 495 – 516.

[11] **J. Kühr**, *Pseudo-BCK-algebras and related structures*, Thesis, Palacky University Olomouc, (2007).

[12] **D. Mundici**, *Averaging the truth-value in Lukasiewicz sentential logic*, Studia Logica **55** (1995), 113 – 127.

[13] **V. Novák, B. De Baets**, *EQ-algebras*, Fuzzy Sets Syst. **160** (2009), 2956 – 2978.

[14] **Y.S. Huang**, *BCI- algebra*, Science Press, China. (2006).

[15] **F. Zebardasta, R.A. Borzooei, M. Aaly Kologani**, *Results on equality algebras*, Inform. Sci. **381** (2017), 270 – 282.

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