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Isostrophy Bryant-Schneider Group-Invariant of Bol Loops

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Abstract. In the recent past, Grecu and Syrbu (in no order of preference) have jointly and individually reported some results on isostrophy invariants of Bol loops. Also, the Bryant-Schneider group of a loop has been found important in the study of the isotopy-isomorphy of some varieties of loops (e.g. Bol loops, Moufang loops, Osborn loops). In this current work, the Bryant-Schneider group of a middle Bol loop was linked with some of the isostrophy-group invariance results of Grecu and Syrbu. In particular, it was shown that some subgroups of the Bryant-Schneider group of a middle Bol loop are equal (or isomorphic) to the automorphism and pseudo-aumorphism groups of its corresponding right (left) Bol loop. Some elements of the Bryant-Schneider group of a middle Bol loop were shown to induce automorphisms and middle pseudo-automorphisms. It was discovered that if a middle Bol loop is of exponent 2, then, its corresponding right (left) Bol loop is a left (right) G-loop.

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1 Introduction

Let Q be a non-empty set. Define a binary operation "$\cdot$" on Q. If $x \cdot y \in Q$ for all $x, y \in Q$, then the pair $(Q, \cdot)$ is called a groupoid or magma. If the equations: $a \cdot x = b$ and $y \cdot a = b$ have unique solutions $x, y \in Q$ for all $a, b \in Q$, then $(Q, \cdot)$ is called a quasigroup. Let $(Q, \cdot)$ be a quasigroup and let there exist a unique element $e \in Q$ called the identity element such that for all $x \in Q$, $x \cdot e = e \cdot x = x$, then $(Q, \cdot)$ is called a loop. We write xy instead of $x \cdot y$ and stipulate that $\cdot$ has lower priority than juxtaposition among factors to be multiplied.

Let $(Q, \cdot)$ be a groupoid and let "a" be a fixed element in Q, then the left and right translations L_a, R_a of $a \in Q$ are respectively defined by $xL_a = a \cdot x$ and $xR_a = x \cdot a$ for all $x \in Q$. It can now be seen that a groupoid $(Q, \cdot)$ is a quasigroup if its left and right translation mappings are permutations. Thence, the inverse mappings L_x^{-1} and R_x^{-1} exist. Thus, for any quasigroup $(Q, \cdot)$, we have two new binary operations: right division $(/)$ and left division $(\backslash)$ and middle translation P_a for any fixed $a \in Q$.

$$x\backslash y = yL_x^{-1} = xP_y \qquad \text{and} \qquad x/y = xR_y^{-1} = yP_x^{-1}$$

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and note that

$$x\backslash y = z \Longleftrightarrow x \cdot z = y \qquad \text{and} \qquad x/y = z \Longleftrightarrow z \cdot y = x.$$

Consequently, $(Q, \backslash)$ and $(Q, /)$ are also quasigroups. The symmetric group $SYM(Q)$ of Q is defined as $SYM(Q) = \{U : Q \to Q \mid U \text{ is a permutation}\}$. For a loop $(Q, \cdot)$, the group generated by its left (right) translations is called the left (right) multiplication group $Mult_{\lambda(\rho)}(Q, \cdot) \leq SYM(Q)$.

$$(x/y)(z\backslash x) = x(zy\backslash x) \tag{1}$$

Middle Bol loops (MBLs) were first studied in the work of Belousov [9], where he gave identity (1) characterizing loops that satisfy the universal anti-automorphic inverse property. After this beautiful characterization by Belousov and the laying of foundations for a classical study of this structure, Gvaramiya [19] proved that a loop $(Q, \circ)$ is middle Bol loop if there exists a right Bol loop $(Q, \cdot)$ such that $x \circ y = (y \cdot xy^{-1})y$ for all $x, y \in Q$. If $(Q, \circ)$ is a middle Bol loop and $(Q, \cdot)$ is the corresponding right Bol loop, then

$$x \circ y = y^{-1}\backslash x \qquad \text{and} \qquad x \cdot y = y//x^{-1} \tag{2}$$

where for every $x, y \in Q$, '//' is the left division in $(Q, \circ)$.

Also, if $(Q, \circ)$ is a middle Bol loop and $(Q, \cdot)$ is the corresponding left Bol loop, then

$$x \circ y = x/y^{-1} \qquad \text{and} \qquad x \cdot y = x//y^{-1} \tag{3}$$

where '//' is the left division in $(Q, \circ)$. The relations in (2) and (3) and their translational forms shall be of tremendous use in the proofs of results in this current work.

Grecu [16] showed that the right multiplication group of a middle Bol loop coincides with the left multiplication group of the corresponding right Bol loop. After that, middle Bol loops resurfaced in literature in 1994 and 1996 when Syrbu [40, 41] considered them in relation to the universality of the elasticity law. In 2003, Kuznetsov [39], while studying gyrogroups (a special class of Bol loops) established some algebraic properties of middle Bol loop and designed a method of constructing a middle Bol loop from a gyrogroup.

In 2010, Syrbu [42] studied the connections between structure and properties of middle Bol loops and of the corresponding left Bol loops. It was noted that two middle Bol loops are isomorphic if and only if the corresponding left (right) Bol loops are isomorphic, and a general form of the autotopisms of middle Bol loops was deduced. Relations between different sets of elements, such as nucleus, left (right, middle) nuclei, the set of Moufang elements, the center of a middle Bol loop and left Bol loop were established. In 2012, Grecu and Syrbu [17] proved that two middle Bol loops are isotopic if and only if the corresponding right (left) Bol loops are isotopic. In 2012, Drapal and Shcherbacov [13] rediscovered the middle Bol identities in a new way. In 2013, Syrbu and Grecu [44] established a necessary and sufficient condition for the quotient loop of a middle Bol loop and of its corresponding right Bol loop to be isomorphic. In 2014, Grecu and Syrbu [18] established that the commutant (centrum) of a middle Bol loop is an AIP-subloop and gave a necessary and sufficient condition when the commutant is an invariant under the existing isostrophy between middle Bol loop and the corresponding right Bol loop and the same authors presented a study of loops with invariant flexibility law under the isostrophy of loop [43]. Osoba and Oyebo [31] further investigated the multiplication group of middle Bol loop in relation to left Bol loop while Jaiyéọlá [26, 27] studied second Smarandache Bol loops. Second Smarandache nuclei of second Smarandache Bol loops was further studied by Osoba [30] while more results on the algebraic properties of middle Bol loops using its parastrophes was presented by Oyebo and Osoba [34].

For any non-empty set Q, the set of all permutations on Q forms a group $SYM(Q)$ called the symmetric group of Q. Let $(Q,\cdot)$ be a loop and let $A, B, C \in SYM(Q)$. If

$$xA \cdot yB = (x \cdot y)C, \ \forall x, y \in Q$$

then the triple (A, B, C) is called an autotopism and such triples form a group $AUT(Q,\cdot)$ called the autotopism groups of $(Q,\cdot)$. If $A = B = C$, then A is called an automorphism of $(Q,\cdot)$ which forms a group $AUM(Q,\cdot)$ called the automorphism group of $(Q,\cdot)$.

Grecu [16] showed that right multiplication group of a middle Bol loop coincides with the left multiplication group of the corresponding right Bol loop.

Definition 1. Let $(Q,\cdot)$ be a loop.

1. A mapping $\theta \in SYM(Q,\cdot)$ is called a right special map for Q if there exists $f \in Q$ so that $(\theta, \theta L_f^{-1}, \theta) \in AUT(Q,\cdot)$.
2. A mapping $\theta \in SYM(Q,\cdot)$ is called a left special map for Q if there exists $g \in Q$ so that $(\theta R_g^{-1}, \theta, \theta) \in AUT(Q,\cdot)$.
3. A mapping $\theta \in SYM(Q)$ is called a special map for Q if there exist $f, g \in Q$ so that $(\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(Q,\cdot)$.

From Definition 1, it is clearly seen that

$$(\theta R_g^{-1}, \theta L_f^{-1}, \theta) = (\theta, \theta, \theta)(R_g^{-1}, L_f^{-1}, I),$$

which implies that θ is an isomorphism of $(Q,\cdot)$ onto some f,g-isotope of it.

Theorem 1. *[36] Let the set* $BS(Q,\cdot) = \{\theta \in SYM(Q) : \ \exists\ f, g \in Q \ni (\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(Q,\cdot)\}$, *then* $BS(Q,\cdot) \leq SYM(Q)$.

Theorem 1 is associated with Theorem 2.

Theorem 2. *(Pflugfelder [35])*

Let $(G,\cdot)$ *and* $(H,\circ)$ *be two isotopic loops. For some* $f, g \in G$*, there exists an* f,g*-principal isotope* $(G,*)$ *of* $(G,\cdot)$ *such that* $(H,\circ) \cong (G,*)$.

In a loop $(Q,\cdot)$, the set of right special maps shall be represented by $BS_\rho(Q,\cdot)$ and will be called the right Bryant-Schneider set of the loop $(Q,\cdot)$. Similarly, the set of left special maps shall be represented by $BS_\lambda(Q,\cdot)$ and called the left Bryant-Schneider set of the loop $(Q,\cdot)$. Also, the set of special maps shall be represented by $BS(Q,\cdot)$ and called the Bryant-Schneider set of the loop $(Q,\cdot)$. Going by Theorem 1, $BS(Q,\cdot)$ forms a group called the Bryant-Schneider group of the loop $(Q,\cdot)$.

Adeniran [1–3] studied the Bryant-Schneider group of conjugacy closed loops. Jaiyéọlá [20] and Jaiyéọlá et al. [21, 22] used the Bryant-Schneider group to study Smarandache loop, Osborn loop and its universality. For more on quasigroups and loops, see Jaiyéọlá [28], Shcherbacov [38] and Pflugfelder [35].

In 2015, Adeniran et al. [6] carried out a study of some isotopic characterisation of generalised Bol loops. In 2017, Jaiyéọlá et al. [23] studied the holomorphic structure of middle Bol loops and showed that the holomorph of a commutative loop is a commutative middle Bol loop if and only if the loop is a middle Bol loop and its automorphism group is abelian. Adeniran et al. [7, 8], Jaiyéọlá and Popoola [29] studied generalised Bol loops.

In 2018, Jaiyéọlá et al. [24], in furtherance to their exploit obtained new algebraic identities of middle Bol loop, where necessary and sufficient conditions for a bivariate mapping of a middle Bol loop to have RIP, LIP, RAP, LAP and flexible property were presented. In 2020, Syrbu and Grecu [43] considered loops with invariant flexibility under the isostrophy. Additional algebraic properties of middle Bol loops were announced by Jaiyéọlá et al. [25] in 2021.

In furtherance to earlier studies, the first two authors in their work [33] unveiled some algebraic characterizations of right and middle Bol loops relative to their cores. Drapal and Syrbu [14] studied middle Bruck loops and total multiplication group.

Definition 2. A groupoid (quasigroup) $(Q,\cdot)$ is said to have

1. left inverse property (LIP) if there exists a mapping $I_\lambda : x \mapsto x^\lambda$ such that $x^\lambda \cdot xy = y$ for all $x, y \in Q$.
2. right inverse property (RIP) if there exists a mapping $I_\rho : x \mapsto x^\rho$ such that $yx \cdot x^\rho = y$ for all $x, y \in Q$.
3. a right alternative property (RAP) if $y \cdot xx = yx \cdot x$ for all $x, y \in Q$.
4. a left alternative property (LAP) if $y \cdot xx = yx \cdot x$ for all $x, y \in Q$.
5. flexibility or elasticity if $xy \cdot x = x \cdot yx$ holds for all $x, y \in Q$.

Note that $I : x \mapsto x^{-1}$ when $I = I_\rho = I_\lambda$.

Definition 3. A loop $(Q,\cdot)$ is said to be

1. an automorphic inverse property loop (AIPL) if $(xy)^{-1} = x^{-1}y^{-1}$ for all $x, y \in Q$.

2. an anti-automorphic inverse property loop (AAIPL) if $(xy)^{-1} = y^{-1}x^{-1}$ for all $x, y \in Q$.

Definition 4. A loop $(Q, \cdot)$ is called a

1. right Bol loop if $(xy \cdot z)y = x(yz \cdot y)$ for all $x, y, z \in Q$.
2. left Bol loop if $(x \cdot yx)z = x(y \cdot xz)$ for all $x, y, z \in Q$.
3. middle Bol loop if $(x/y)(z\backslash x) = (x/(zy))x$ or $(x/y)(z\backslash x) = x((zy)\backslash x)$ for all $x, y, z \in Q$.

Definition 5. Let $(Q, \cdot)$ be a loop.

1. $\phi \in SYM(Q)$ is called a left pseudo-automorphism with companion $a \in Q$ if $(\phi L_a, \phi, \phi L_a) \in AUT(Q, \cdot)$. The set of left pseudo-automorphisms $PS_\lambda(Q, \cdot)$ forms a group called the left pseudo-automorphism group of $(Q, \cdot)$. See [35].
2. $\phi \in SYM(Q)$ is called a right pseudo-automorphism with companion $a \in Q$ if $(\phi, \phi R_a, \phi R_a) \in AUT(Q, \cdot)$. The set of right pseudo-automorphisms $PS_\rho(Q, \cdot)$ forms a group called the left pseudo-automorphism group of $(Q, \cdot)$. See [35].
3. $\phi \in SYM(Q)$ is called a middle pseudo-automorphism with companion $a \in Q$ if $(\phi R_a^{-1}, \phi L_{a^\lambda}^{-1}, \phi) \in AUT(Q, \cdot)$. The set of middle pseudo-automorphisms $PS_\mu(Q, \cdot)$ forms a group called the middle pseudo-automorphism group of $(Q, \cdot)$. See [44].

Definition 6. Let $(Q, \cdot)$ be a loop.

1. The left nucleus of Q is $N_\lambda = \{a \in Q : ax \cdot y = a \cdot xy \ \forall\ x, y \in Q\}$.
2. The right nucleus of Q is $N_\rho = \{a \in Q : y \cdot xa = yx \cdot a \ \forall\ x, y \in Q\}$.
3. The middle nucleus of Q is $N_\mu = \{a \in Q : ya \cdot x = y \cdot ax \ \forall\ x, y \in Q\}$.
4. The nucleus of Q is $N(Q, \ \cdot) = N_\lambda \cap N_\rho \cap N_\mu$.
5. The centrum or commutant of Q is $C(Q, \cdot) = \{a \in Q : ax = xa \ \forall x \in Q\}$.
6. The centre of Q is $Z(Q, \cdot) = N(Q, \cdot) \cap C(Q, \cdot)$.

Theorem 3. *[35] Let $(Q, \cdot)$ be an inverse property loop or MBL. Then, for any $a \in Q$:*

1. $I_\lambda R_a I_\rho = L_{a^\lambda}$.
2. $I_\rho R_a I_\rho = L_{a^\rho}$.
3. $I_\rho L_a I_\rho = R_{a^\rho}$.
4. $I_\lambda L_a I_\rho = R_{a^\lambda}$.

Lemma 1. *[35]*

1. *Let θ be a right (left) pseudo-automorphism of a loop, then $e\theta = e$.*
2. *Let θ be a right (left) pseudo-automorphism of a LIP (RIP) loop. Then, $I\theta = \theta I$.*

Here are some existing results on some isostrophy invariants of Bol loops.

Theorem 4. *(Grecu and Syrbu [17])*
Let $(Q, \circ)$ be a middle Bol loop and let $(Q, \cdot)$ and $(Q, *)$ be the corresponding right and left Bol loops, respectively.

1. $AUM(Q, \circ) = AUM(Q, \cdot) = AUM(Q, *)$.
2. $AUT(Q, \circ) \cong AUT(Q, \cdot) \cong AUT(Q, *)$.
3. $PS_\lambda(Q, \circ) \cong PS_\rho(Q, \cdot) \cong PS_\lambda(Q, *)$.

Theorem 5. *(Syrbu and Grecu [44])*
Let $(Q, \circ)$ be a middle Bol loop and let $(Q, \cdot)$ and $(Q, *)$ be the corresponding right and left Bol loops, respectively.

1. $PS_\rho(Q, \circ) = PS_\mu(Q, \cdot)$.
2. $PS_\mu(Q, \circ) = PS_\lambda(Q, \cdot)$.
3. $PS_\rho(Q, \circ) = PS_\rho(Q, \cdot)$.
4. $\alpha \in PS_\lambda(Q, \circ) \Leftrightarrow I\alpha I \in PS_\rho(Q, \circ)$.

In the current work, we shall be linking the Bryant-Schneider group of a middle Bol loop with some of the isostrophy-group invariance results in Theorem 4 and Theorem 5. In particular, it will be shown that some subgroups of the Bryant-Schneider group of a middle Bol loop are equal (or isomorphic) to the automorphism and pseudo-aumorphism groups of its corresponding right (left) Bol loop.

2 Main Results

Lemma 2. *Let (α, β, γ) be an autotopism of a middle Bol loop $(Q, \circ)$. Then $(I\beta I, I\alpha I, I\gamma I)$ is also an autotopism of $(Q, \circ)$.*

Proof. Let $(Q, \circ)$ be a middle Bol loop and (α, β, γ) be the autotopism of $(Q, \circ)$, then for all $x, y \in Q$, we have

$$x\alpha \circ y\beta = (x \circ y)\gamma \Longrightarrow [x\alpha \circ y\beta]I = (x \circ y)\gamma I \Longrightarrow [(y\beta)I \circ (x\alpha)I] = (x \circ y)\gamma I.$$

Doing $y \mapsto yI$ and $x \mapsto xI$ in the last equation, we get

$$yI\beta I \circ xI\alpha I = [(xI \circ yI)\gamma]I \implies yI\beta I \circ xI\alpha I = [(y \circ x)I\gamma]I.$$

Thus, $(I\beta I, I\alpha I, I\gamma I) \in AUT(Q, \circ)$. □

Theorem 6. *Let $(Q,\circ)$ be a middle Bol loop and let $\theta\in BS(Q,\circ)$ be such that $\theta: e\mapsto e$. For some $f,g\in Q$:*

1. $L_f^{-1}=P_g^{-1}R_gR_{g^2}^{-1}P_g$ *and* $R_g^{-1}=P_f^{-1}R_fL_{f^2}^{-1}P_f^{-1}$.
2. $\theta=\theta(f,g)\equiv\theta(f,f^{-1})$ *and* $\theta=\theta(f,g)\equiv\theta(g^{-1},g)$.

Proof. Suppose that $(Q,\circ)$ is a middle Bol loop, then $B=(IP_x^{-1},IP_x,IP_x^{-1}R_x)$ is an autotopism of $(Q,\circ)$ for all $x\in Q$. Since $A=(\theta R_g^{-1},\theta L_f^{-1},\theta)\in AUT(Q,\circ)$ for some $f,g\in Q$, then

$$A=(I\theta L_f^{-1}I,I\theta R_g^{-1}I,I\theta I)\in AUT(Q,\circ)\text{ for some }f,g\in Q. \quad (4)$$

Thus,

$$AB=(I\theta L_f^{-1}IIP_x^{-1},I\theta R_g^{-1}IIP_x,I\theta IIP_x^{-1}R_x)$$

$$=(I\theta L_f^{-1}P_x^{-1},I\theta R_g^{-1}P_x,I\theta P_x^{-1}R_x)\in AUT(Q,\circ). \quad (5)$$

Writing this in identical relation, for all $z,y\in Q$, we have
$yI\theta L_f^{-1}P_x^{-1}\circ zI\theta R_g^{-1}P_x=(y\circ z)I\theta P_x^{-1}R_x$
$\Longrightarrow y^{-1}\theta L_f^{-1}P_x^{-1}\circ z^{-1}\theta R_g^{-1}P_x=(y\circ z)^{-1}\theta P_x^{-1}R_x$

$$\Longrightarrow x/(f\backslash(y^{-1})\theta)\circ((z^{-1}\theta)/g)\backslash x=(x/(z^{-1}\circ y^{-1})\theta)\circ x. \quad (6)$$

Here, setting $y=e$ and $x=f$ in (6), we have

$$\begin{aligned}
[f/(f\backslash e)]\circ(z^{-1}\theta)/g)\backslash f &= (f/(z^{-1}\theta)f\\
\Longrightarrow f/f^{\rho}\circ zR_g^{-1}P_f &= zP_f^{-1}R_f\\
\Longrightarrow R_g^{-1}P_fL_{f/f^{\rho}} &= P_f^{-1}R_f\\
\Longrightarrow R_g^{-1} &= P_f^{-1}R_fL_{f/f^{\rho}}^{-1}P_f^{-1}\\
\Longrightarrow R_g &= P_fR_f^{-1}L_{f^2}P_f
\end{aligned}$$

So, $x\circ g=\{[f^2(x\backslash f)]/f\}\backslash f$. With $x=g$, we get $g=f^{-1}$.
Thus, $\theta=\theta(f,g)\equiv\theta(f,f^{-1})$.

Analogously, if we repeat the same procedure by setting $z=e$ and $x=g$ in (6), we have

$$\begin{aligned}
g/(f\backslash(y^{-1})\theta)\circ(e/g)\backslash g &= (g/(y^{-1})\theta)g\\
\Longrightarrow yL_f^{-1}P_g^{-1}R_{g^{\lambda}}\backslash g &= yP_g^{-1}R_g\\
\Longrightarrow L_f^{-1}P_g^{-1}R_g^2 &= P_g^{-1}R_g\\
\Longrightarrow L_f^{-1} &= P_g^{-1}R_gR_{g^2}^{-1}P_g
\end{aligned}$$

So, $f\backslash x=\{[(g/x)g]/g^2\}\backslash g$. With $x=f$, we get $f=g^{-1}$.
Thus, $\theta\equiv\theta(f,g)=\theta(g^{-1},g)$. □

Corollary 1. *Let $(Q,\circ)$ be a middle Bol loop. Any $\theta\in BS(Q,\circ)$ such that $\theta: e\mapsto e$ induces $\Phi=I\theta P_g^{-1}R_g\in SYM(Q)$ for some $g\in Q$ and the following hold:*

1. $\Phi\in BS(Q,\circ)$.
2. Φ *is a middle pseudo-automorphism with a square companion.*

Proof. Replacing $R_g^{-1}=P_f^{-1}R_fL_{f^2}^{-1}P_f^{-1}$ and $L_f^{-1}=P_g^{-1}R_gR_{g^2}^{-1}P_g$ in (5) gives $(I\theta P_g^{-1}R_gR_{g^2}^{-1}P_gP_x^{-1},I\theta P_f^{-1}R_fL_{f^2}^{-1}P_f^{-1}P_x,I\theta P_x^{-1}R_x)$ which is an autotopism of $(Q,\circ)$.

Put $x=g$ to get $(I\theta P_g^{-1}R_gR_{g^2}^{-1},I\theta P_f^{-1}R_fL_{f^2}^{-1}P_f^{-1}P_g,I\theta P_g^{-1}R_g)\in AUT(Q,\circ)$. Setting $f=g$ gives $(I\theta P_g^{-1}R_gR_{g^2}^{-1},I\theta P_g^{-1}R_gL_{g^2}^{-1},I\theta P_g^{-1}R_g)\in AUT(Q,\circ)$. Letting $\Phi=I\theta P_g^{-1}R_g$, gives $(\Phi R_{g^2}^{-1},\Phi L_{g^2}^{-1},\Phi)$ is also autotopism of $(Q,\circ)$. □

Corollary 2. *Let $(Q,\circ)$ be a middle Bol loop and $\theta\equiv\theta(f,g)\in BS(Q,\circ)$ for some $f,g\in Q$ (in which either is of order 2 i.e. $|f|=2$ or $|g|=2$) such that $\theta:e\mapsto e$. Then, θ induces an automorphism $\Phi=I\theta P_g^{-1}R_g\in SYM(Q)$ for some $g\in Q$.*

Proof. This follows from Corollary 1. □

Theorem 7. *Let $(Q,\circ)$ be a middle Bol loop. Then,*

$$BS'(Q,\circ)=\Big\{\theta\in BS(Q,\circ)\mid\theta:e\mapsto e\ \textit{and}\ (x\theta)^{-1}=\big(x^{-1}\big)\theta\Big\}$$
$$=\Big\{\theta\in SYM(Q)|\exists\ f\in Q\ \ni\ \big(\theta R_{f^{-1}}^{-1},\theta L_f^{-1},\theta\big)\in AUT(Q),\ e\theta=e\ \textit{and}$$
$$(x\theta)^{-1}=(x^{-1})\theta\ \forall\ x\in Q\Big\}=\Big\{\theta\in SYM(Q)|\exists\ g\in Q\ \ni\ \big(\theta R_g^{-1},\theta L_{g^{-1}}^{-1},\theta\big)$$
$$\in AUT(Q),\ e\theta=e\ \textit{and}\ (x\theta)^{-1}=(x^{-1})\theta\ \forall\ x\in Q\Big\}\le BS(Q,\circ).$$

Proof. Let
$BS'(Q,\circ)=\Big\{\theta\in BS(Q,\circ)\mid\theta:e\mapsto e\text{ and }(x\theta)^{-1}=\big(x^{-1}\big)\theta\Big\}\subseteq BS(Q,\circ)$.
Going by Theorem 6,

$$BS'(Q,\circ)=\Big\{\theta\in BS(Q,\circ)\mid\theta:e\mapsto e\text{ and }(x\theta)^{-1}=\big(x^{-1}\big)\theta\Big\}$$
$$=\Big\{\theta\in SYM(Q)|\exists\ f\in Q\ \ni\ \big(\theta R_{f^{-1}}^{-1},\theta L_f^{-1},\theta\big)\in AUT(Q),\ e\theta=e\text{ and}$$
$$(x\theta)^{-1}=(x^{-1})\theta\ \forall\ x\in Q\Big\}=\Big\{\theta\in SYM(Q)|\exists\ g\in Q\ \ni\ \big(\theta R_g^{-1},\theta L_{g^{-1}}^{-1},\theta\big)$$
$$\in AUT(Q),\ e\theta=e\text{ and }(x\theta)^{-1}=(x^{-1})\theta\ \forall\ x\in Q\Big\}.$$

Suppose that $\mathbb{I}$ is the identity mapping on Q, then, $e\mathbb{I}=e$ and $(g\mathbb{I})^{-1}=(g^{-1})\mathbb{I}$ $\forall g\in Q$ and $(\mathbb{I}R_e^{-1},\mathbb{I}L_e^{-1},\mathbb{I})=(\mathbb{I},\mathbb{I},\mathbb{I})\in AUT(Q,\circ)$. So, $\mathbb{I}\in BS'(Q,\circ)$. Thus, $BS'(Q,\circ)\neq\emptyset$.

Let $\alpha,\beta\in BS'(Q,\circ)$. Then, $\alpha,\beta\in BS(Q,\circ)$ and $e\alpha=e$ and $(x\alpha)^{-1}=(x^{-1})\alpha$, $e\beta=e$ and $(x\beta)^{-1}=(x^{-1})\beta,\ \forall x\in Q$.

Furthermore, there exist $f_1, g_1, f_2, g_2 \in Q$ with $g_1 = f_1^{-1}, g_2 = f_2^{-1}$ such that

$$A = (\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha), B = (\beta R_{g_2}^{-1}, \beta L_{f_2}^{-1}, \beta), B^{-1} =$$
$$(R_{g_2}\beta^{-1}, L_{f_2}\beta^{-1}, \beta^{-1}) \in AUT(Q, \circ).$$
$$AB^{-1} = (\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha)(R_{g_2}\beta^{-1}, L_{f_2}\beta^{-1}, \beta^{-1}) =$$
$$(\alpha R_{g_1}^{-1} R_{g_2}\beta^{-1}, \alpha L_{f_1}^{-1} L_{f_2}\beta^{-1}, \alpha\beta^{-1}) \in AUT(Q, \circ).$$

Let $\rho = \beta R_{g_1}^{-1} R_{g_2}\beta^{-1}$ and $\sigma = \beta L_{f_1}^{-1} L_{f_2}\beta^{-1}$ so that $(\alpha\beta^{-1}\rho, \alpha\beta^{-1}\sigma, \alpha\beta^{-1}) \in AUT(Q, \circ)$ if and only if for all $x, y \in Q$

$$x\alpha\beta^{-1}\rho \circ y\alpha\beta^{-1}\sigma = (x \circ y)\alpha\beta^{-1}. \tag{7}$$

Setting $x = e$ in Q and replacing y by $y\beta\alpha^{-1}$ in (7), we have

$$(e\alpha\beta^{-1}\rho) \circ (y\sigma) = y \Longrightarrow y\sigma L_{(e\alpha\beta^{-1}\rho)} = y \Longrightarrow \sigma = L_{(e\alpha\beta^{-1}\rho)}^{-1}.$$

Similarly, setting $y = e$ in Q and replacing x by $x\beta\alpha^{-1}$ in (7), we have

$$(x\rho) \circ (e\alpha\beta^{-1}\sigma) = x \Longrightarrow x\rho R_{(e\alpha\beta^{-1}\sigma)} = x \Longrightarrow \rho = R_{(e\alpha\beta^{-1}\sigma)}^{-1}.$$

Thus, $g = e\alpha\beta^{-1}\sigma = e\sigma = e\beta L_{f_1}^{-1} L_{f_2}\beta^{-1} = [f_2 \circ (f_1\backslash e)]\beta^{-1} = [f_2 \circ f_1^{-1}]\beta^{-1}$ and $f = e\alpha\beta^{-1}\rho = e\rho = e\beta R_{f_1^{-1}}^{-1} R_{f_2^{-1}}\beta^{-1} = eR_{f_1^{-1}}^{-1} R_{f_2^{-1}}\beta^{-1} = [(e/f_1^{-1}) \circ f_2^{-1}]\beta^{-1} = (f_1 \circ f_2^{-1})\beta^{-1}$. Then, $f^{-1} = [(f_1 \circ f_2^{-1})\beta^{-1}]^{-1} = (f_1 \circ f_2^{-1})^{-1}\beta^{-1} = (f_2 \circ f_1^{-1})\beta^{-1} = g$. Hence,

$$AB^{-1} = (\alpha\beta^{-1}\rho, \alpha\beta^{-1}\sigma, \alpha\beta^{-1}) = (\alpha\beta^{-1} R_{f^{-1}}^{-1}, \alpha\beta^{-1} L_f^{-1}, \alpha\beta^{-1}) \in AUT(Q, \circ),$$
$$e\alpha\beta^{-1} = e \text{ and } (x^{-1})\alpha\beta^{-1} = (x\alpha\beta^{-1})^{-1} \ \forall\ x \in Q. \text{ So, } \alpha\beta^{-1} \in BS'(Q, \circ).$$

Therefore, $BS'(Q, \circ) \leq BS(Q, \circ)$. □

Corollary 3. *Let $(Q, \circ)$ be a middle Bol loop. Then,*

$$AUM(Q, \circ) \leq BS'(Q, \circ) \leq BS(Q, \circ).$$

Proof. This follows from Theorem 7. □

Theorem 8. *Let $(Q, \circ)$ be a middle Bol loop and $(Q, \cdot)$ be the corresponding right Bol loop. Then, $BS'(Q, \circ) = PS_\lambda(Q, \cdot)$.*

Proof. We shall show that $\theta \in BS'(Q, \circ)$ if and only if $\theta \in PS_\lambda(Q, \cdot)$. Let $\theta \in BS'(Q, \circ)$, then $\theta \in BS(Q, \circ)$ such that $e\theta = e$. Thus, for some $f, g \in Q$, we have $(\theta\mathbb{R}_g^{-1}, \theta\mathbb{L}_f^{-1}, \theta) \in AUT(Q)$. For all $x, y \in Q$, we have

$$x\theta\mathbb{R}_g^{-1} \circ y\theta\mathbb{L}_f^{-1} = (x \circ y)\theta$$
$$\Leftrightarrow x\theta L_{g^{-1}} \circ y\theta(IP_f)^{-1} = (x \circ y)\theta$$

$$\Leftrightarrow \big(y\theta(IP_f)^{-1}\big)I\backslash x\theta L_{g^{-1}} = (y^{-1}\backslash x)\theta.$$

Set $z = y^{-1}\backslash x \Leftrightarrow x = y^{-1}\cdot z$. Then we have

$$\big(y\theta(IP_f)^{-1}\big)I\cdot z\theta = (y^{-1}\cdot z)\theta L_{g^{-1}} \Leftrightarrow \big(yI\theta(IP_f)^{-1}\big)I\cdot z\theta = (y\cdot z)\theta L_{g^{-1}}.$$

Putting $z = e$, we have $\big(yI\theta(IP_f)^{-1}\big)I\cdot e\theta = y\theta L_{g^{-1}} \Leftrightarrow \big(yI\theta(IP_f)^{-1}\big)I = y\theta L_{g^{-1}} \Leftrightarrow yI\theta(IP_f)^{-1}I = y\theta L_{g^{-1}}$. Thus, $(\theta L_{g^{-1}},\theta,\theta L_{g^{-1}}) \in AUT(Q,\cdot)$ which means that θ is a left pseudo-automorphism with companion g^{-1}.

Conversely, suppose that $\theta \in SYM(Q)$ is a left pseudo-automorphism of $(Q,\cdot)$ with companion g, then $(\theta L_g,\theta,\theta L_g) \in AUT(Q,\cdot)$. Note that $e\theta = e$ by Lemma 1. For all $x,y\in Q$, we have

$$\begin{aligned} x\theta L_g\cdot y\theta &= (x\cdot y)\theta L_g\\ \Leftrightarrow x\theta\mathbb{R}_{g^{-1}}^{-1}\cdot y\theta &= (xy)\theta\mathbb{R}_{g^{-1}}^{-1}\\ \Leftrightarrow y\theta//\big(x\theta\mathbb{R}_{g^{-1}}^{-1}\big)I &= \big(y//x^{-1}\big)\theta\mathbb{R}_{g^{-1}}^{-1}.\end{aligned}$$

Set $y//x^{-1} = z \Leftrightarrow y = z\circ x^{-1}$ for $z\in Q$. This leads us to

$$(z\circ xI)\theta = z\theta\mathbb{R}_{g^{-1}}^{-1}\circ x\theta\mathbb{R}_{g^{-1}}^{-1}I \Leftrightarrow (z\circ xI)\theta = z\theta\mathbb{R}_{g^{-1}}^{-1}\circ x\theta I\mathbb{L}_g^{-1}. \tag{8}$$

Substituting $z = e$, $xI\theta = e\mathbb{R}_{g^{-1}}^{-1}\circ x\theta I\mathbb{L}_g^{-1} \Leftrightarrow xI\theta = g\circ x\theta I\mathbb{L}_g^{-1} \Leftrightarrow xI\theta = x\theta I\mathbb{L}_g^{-1}\mathbb{L}_g \Leftrightarrow xI\theta = x\theta I$. So, (8) becomes $(z\circ xI)\theta = z\theta\mathbb{R}_{g^{-1}}^{-1}\circ xI\theta\mathbb{L}_g^{-1} \Leftrightarrow (\theta\mathbb{R}_{g^{-1}}^{-1},\theta\mathbb{L}_g^{-1},\theta)\in AUT(Q,\circ) \Rightarrow \theta\in BS(Q,\circ)$. Thus, $\theta\in BS'(Q,\circ)$. □

Lemma 3. *Let $(Q,\cdot)$ be a loop.*

1. $BS_\rho(Q,\cdot)\le BS(Q,\cdot)$ *and* $BS_\lambda(Q,\cdot)\le BS(Q,\cdot)$.
2. $BS'_\rho(Q,\cdot) = \Big\{\theta\in BS_\rho(Q,\cdot)\mid \theta: e\mapsto e\Big\}\le BS_\rho(Q,\cdot)\le BS(Q,\cdot)$.
3. $BS'_\lambda(Q,\cdot) = \Big\{\theta\in BS_\lambda(Q,\cdot)\mid \theta: e\mapsto e\Big\}\le BS_\lambda(Q,\cdot)\le BS(Q,\cdot)$.
4. $BS''_\rho(Q,\cdot) = \Big\{\theta\in BS_\rho(Q,\cdot)\mid \theta: e\mapsto e$ *and* $(x\theta)^{-1} = \big(x^{-1}\big)\theta\ \forall\ x\in Q\Big\}\le BS'_\rho(Q,\cdot)$.
5. $BS''_\lambda(Q,\cdot) = \Big\{\theta\in BS_\lambda(Q,\cdot)\mid \theta: e\mapsto e$ *and* $(x\theta)^{-1} = \big(x^{-1}\big)\theta\ \forall\ x\in Q\Big\}\le BS'_\lambda(Q,\cdot)$.

Proof.

1. The proof is similar to that of Theorem 1.
2. This follows from 1.

3. This follows from 1.
4. This follows from 2.
5. This follows from 3.

□

Theorem 9. *Let $(Q,\circ)$ be a middle Bol loop and let $(Q,\cdot)$ and $(Q,*)$ be its corresponding right and left Bol loops respectively. Then,*

1. $BS'_\rho(Q,\circ) = AUM(Q,\circ) = AUM(Q,\cdot) = AUM(Q,*)$.
2. $BS''_\lambda(Q,\circ) = AUM(Q,\circ) = AUM(Q,\cdot) = AUM(Q,*)$.

Proof. 1. Let $\theta \in BS'_\rho(Q,\circ)$, then $\theta \in BS(Q,\circ)$ i.e. for some $f \in Q$, $(\theta, \theta\mathbb{L}_f^{-1}, \theta) \in AUT(Q,\circ)$ and $\theta : e \mapsto e$. So, for all $x, y \in Q$, we have

$$\begin{gathered} x\theta \circ y\theta\mathbb{L}_f^{-1} = (x \circ y)\theta \\ \Leftrightarrow x\theta \circ y\theta(IP_f)^{-1} = (x \circ y)\theta \\ \Leftrightarrow \big(y\theta(IP_f)^{-1}\big)I \backslash x\theta = (y^{-1}\backslash x)\theta. \end{gathered}$$

Set $z = y^{-1}\backslash x \Leftrightarrow x = y^{-1} \cdot z$ for $z \in Q$ in order to get

$$yI\theta(IP_f)^{-1}I \cdot z\theta = (yz)\theta. \tag{9}$$

Substitute $z = e$ into (9), then we have $yI\theta(IP_f)^{-1}I = y\theta \Leftrightarrow \theta = I\theta(IP_f)^{-1}I$. Put this into (9) to have $(\theta, \theta, \theta) \in AUT(Q,\cdot)$. Thus, θ is an automorphism of right Bol loop $(Q,\cdot)$. Thus, $BS'_\rho(Q,\circ) \le AUM(Q,\cdot)$. By Theorem 4, $AUM(Q,\cdot) = AUM(Q,\circ)$, so, $BS'_\rho(Q,\circ) \le AUM(Q,\circ)$. But, $AUM(Q,\circ) \le BS'_\rho(Q,\circ)$ by Corollary 3. Thus, $BS'_\rho(Q,\circ) = AUM(Q,\circ) = AUM(Q,\cdot) = AUM(Q,*)$.

2. Let $\theta \in BS''_\lambda(Q,\circ)$, then $\theta \in BS(Q,\circ)$ i.e. for some $f \in Q$, $(\theta\mathbb{R}_g^{-1}, \theta, \theta) \in AUT(Q,\circ)$, $\theta : e \mapsto e$ and $I\theta = \theta I$. So, for all $x, y \in Q$, we have

$$\begin{gathered} x\theta\mathbb{R}_g^{-1} \circ y\theta = (x \circ y)\theta \\ \Leftrightarrow x\theta L_{f^{-1}} \circ y\theta = (x \circ y)\theta \\ \Leftrightarrow y\theta I \backslash x\theta L_{f^{-1}} = (y^{-1}\backslash x)\theta. \end{gathered}$$

Set $z = y^{-1}\backslash x \Leftrightarrow x = y^{-1} \cdot z$ for $z \in Q$ in order to get

$$\begin{gathered} y\theta I \cdot z\theta = (y^{-1} \cdot z)\theta L_{f^{-1}} \\ \Leftrightarrow yI\theta I \cdot z\theta = (y \cdot z)\theta L_{f^{-1}} \\ \Leftrightarrow y\theta \cdot z\theta = (y \cdot z)\theta L_{f^{-1}}. \end{gathered} \tag{10}$$

Substitute $z = e$ into (10), then we have $\theta L_{f^{-1}} = \theta$. Put this into (10) to have $(\theta, \theta, \theta) \in AUT(Q,\cdot)$. Thus θ is an automorphism of right Bol loop $(Q,\cdot)$.

Thus, $BS''_\lambda(Q,\circ) \leq AUM(Q,\cdot)$. By Theorem 4, $AUM(Q,\cdot) = AUM(Q,\circ)$, so, $BS''_\lambda(Q,\circ) \leq AUM(Q,\circ)$. But, $AUM(Q,\circ) \leq BS''_\lambda(Q,\circ)$. Thus, $BS''_\lambda(Q,\circ) = AUM(Q,\circ) = AUM(Q,\cdot) = AUM(Q,*)$.

□

Theorem 10. *Let $(Q,\circ)$ be a middle Bol loop and $(Q,\cdot)$ be the corresponding right Bol loop. Then, $PS_\rho(Q,\cdot) = PS_\lambda(Q,\circ) = PS_\rho(Q,\circ)$.*

Proof. If θ is right pseudo-automorphism of $(Q,\cdot)$ with companion g, then $(\theta,\theta R_g,\theta R_g) \in AUT(Q,\cdot)$. For all $x,y \in Q$, we have

$$\begin{aligned} x\theta \cdot y\theta R_g = (x\cdot y)\theta R_g &\Rightarrow x\theta \cdot y\theta I\mathbb{P}_g^{-1} = (x\cdot y)\theta I\mathbb{P}_g^{-1} \\ &\Rightarrow y\theta I\mathbb{P}_g^{-1}//x\theta I = (y//xI)\theta I\mathbb{P}_g^{-1}. \qquad (11) \end{aligned}$$

Set $z = y//xI \Longrightarrow y = z\circ xI$. So, (11) becomes

$$\begin{aligned} (z\circ xI)\theta I\mathbb{P}_g^{-1} = z\theta I\mathbb{P}_g^{-1}\circ x\theta I &\Rightarrow (z\circ x)\theta I\mathbb{P}_g^{-1} = z\theta I\mathbb{P}_g^{-1}\circ xI\theta I \\ &\Rightarrow (z\circ x)\theta I\mathbb{P}_g^{-1} = z\theta I\mathbb{P}_g^{-1}\circ x. \qquad (12) \end{aligned}$$

Set $z = e$ in (12) to get $\theta I\mathbb{P}_g^{-1} = \theta\mathbb{L}_{e\theta I\mathbb{P}_g^{-1}} = \theta\mathbb{L}_{g'}$. Thus, (12) becomes $(z\circ x)\theta\mathbb{L}_{g'} = z\theta\mathbb{L}_{g'}\circ x \Rightarrow (\theta\mathbb{L}_{g'},\theta,\theta\mathbb{L}_{g'}) \in AUT(Q,\circ)$. Thence, θ is left pseudo-automorphism of $(Q,\circ)$ with companion g'.

Conversely, if θ is a left pseudo-automorphism of $(Q,\circ)$ with companion g, then $(\theta,\theta\mathbb{L}_g,\theta\mathbb{L}_g) \in AUT(Q,\circ)$. For all $x,y \in Q$, we have

$$\begin{aligned} x\theta\circ y\theta\mathbb{L}_g = (x\circ)\theta\mathbb{L}_g &\Rightarrow x\theta IP_g\circ y\theta = (x\circ y)\theta IP_g \\ &\Rightarrow y\theta I\backslash x\theta IP_g = (yI\backslash x)\theta IP_g. \qquad (13) \end{aligned}$$

Set $z = yI\backslash x \Longrightarrow x = yI\cdot z$. So, (13) becomes

$$\begin{aligned} (yI\cdot z)\theta IP_g = y\theta I\cdot z\theta IP_g &\Rightarrow (y\cdot z)\theta IP_g = zI\theta I\cdot z\theta IP_g \\ &\Rightarrow (y\cdot z)\theta IP_g = z\theta\cdot z\theta IP_g. \qquad (14) \end{aligned}$$

Set $z = e$ in (14) to get $\theta IP_g = \theta R_{e\theta IP_g} = \theta R_{g'}$. Thus, (14) becomes $(y\cdot z)\theta R_{g'} = z\theta\cdot z\theta R_{g'} \Rightarrow (\theta,\theta R_g,\theta R_{g'}) \in AUT(Q,\cdot)$. Thence, θ is right pseudo-automorphism of $(Q,\cdot)$ with companion g'. So, $PS_\rho(Q,\cdot) = PS_\lambda(Q,\circ) = PS_\rho(Q,\circ)$ by Theorem 5. □

Theorem 11. *Let $(Q,\circ)$ be a middle Bol loop and let $(Q,\cdot)$ and $(Q,*)$ be its corresponding right and left Bol loops respectively. Then, $BS'(Q,\cdot) = PS_\rho(Q,\circ) = PS_\rho(Q,\cdot) = PS_\mu(Q,\cdot) = PS_\lambda(Q,\cdot) = PS_\mu(Q,\circ) = PS_\lambda(Q,\circ) \cong PS_\lambda(Q,*)$.*

Proof. We shall show that $\theta \in BS'(Q,\cdot)$ if and only if $\theta \in PS_\rho(Q,\circ)$. Let $\theta \in BS'(Q,\cdot)$, then $\theta \in BS(Q,\cdot)$ such that $e\theta = e$. Thus, for some $f,g \in Q$, $(\theta R_g^{-1},\theta L_f^{-1},\theta) \in AUT(Q)$. For all $x,y,\in Q$, we have

$$x\theta R_g^{-1}\cdot y\theta L_f^{-1} = (xy)\theta \Leftrightarrow x\theta\mathbb{P}_g I\cdot y\theta\mathbb{R}_{f^{-1}} = (x\cdot y)\theta$$

$$\Leftrightarrow (y\theta\mathbb{R}_{f^{-1}})//(x\theta\mathbb{P}_g I)I = \left(y//x^{-1}\right)\theta$$
$$\Leftrightarrow y\theta\mathbb{R}_{f^{-1}} = \left(y//x^{-1}\right)\theta\circ x\theta\mathbb{P}_g. \tag{15}$$

Set $z = y//x^{-1} \Leftrightarrow y = z\circ x^{-1}$. So, (15) becomes

$$\left(z\circ x^{-1}\right)\theta\mathbb{R}_{f^{-1}} = z\theta\circ x\theta\mathbb{P}_g$$
$$\Rightarrow (z\circ x)\,\theta\mathbb{R}_{f^{-1}} = z\theta\circ xI\theta\mathbb{P}_g. \tag{16}$$

Put $z = e$ in (16) to get $\theta\mathbb{R}_{f^{-1}} = I\theta\mathbb{P}_g$. Hence, (16) becomes $(z\circ x)\theta\mathbb{R}_{f^{-1}} = z\theta\circ x\theta\mathbb{R}_{f^{-1}} \Leftrightarrow (\theta,\theta\mathbb{R}_{f^{-1}},\theta\mathbb{R}_{f^{-1}})\in PS_\rho(Q,\circ)$.

Conversely, suppose that $\theta\in SYM(Q)$ is a right pseudo-automorphism of $(Q,\circ)$ with companion f^{-1}, then $(\theta,\theta\mathbb{R}_{f^{-1}},\theta\mathbb{R}_{f^{-1}})\in PS_\rho(Q,\circ)$. Note that $e\theta = e$. For all $x,y\in Q$, we have

$$x\theta\circ y\theta\mathbb{R}_{f^{-1}} = (x\circ y)\theta\mathbb{R}_{f^{-1}} \Leftrightarrow x\theta\circ y\theta L_f^{-1} = (x\circ y)\theta L_f^{-1}$$
$$\Leftrightarrow (y\theta L_f^{-1})I\backslash x\theta = (yI\backslash x)\theta L_f^{-1} \Leftrightarrow x\theta = (y\theta L_f^{-1})I\cdot(yI\backslash x)\theta L_f^{-1}.$$

Put $z = yI\backslash x \Leftrightarrow x = yI\cdot z$. Thus, the last equality is true

$$\Leftrightarrow (y\theta L_f^{-1})I\cdot z\theta L_f^{-1} = (yI\cdot z)\theta \Leftrightarrow yI\theta L_f^{-1}I\cdot z\theta L_f^{-1} = (y\cdot z)\theta.$$

Putting $z = e$ in the last equation, we get $I\theta L_f^{-1}I = \theta R_{f^{-1}}f^{-1}$ and consequently, $y\theta R_{f^{-1}}^{-1}\cdot z\theta L_f^{-1} = (y\cdot z)\theta \Leftrightarrow (\theta R_{f^{-1}}^{-1},\theta L_f^{-1},\theta)\in AUT(Q,\cdot) \Rightarrow \theta\in BS(Q,\cdot)$. Thus, $\theta\in BS'(Q.\cdot)$. So, by Theorem 4, Theorem 5, Theorem 8 and Theorem 10, $BS'(Q,\cdot) = PS_\rho(Q,\circ) = PS_\rho(Q,\cdot) = PS_\mu(Q,\cdot) = PS_\lambda(Q,\cdot) = PS_\mu(Q,\circ) = PS_\lambda(Q,\circ)\cong PS_\lambda(Q,*)$. □

Theorem 12. *Let $(Q,\circ)$ be a middle Bol loop of exponent 2 and let $(Q,\cdot)$ and $(Q,*)$ be its corresponding right and left Bol loops respectively. $(Q,\cdot)$ and $(Q,*)$ are left G-loop and right G-loop respectively.*

Proof. $(Q,\circ)$ is a middle Bol loop if and only if

$$(I\mathbb{P}_x^{-1}, I\mathbb{P}_x, I\mathbb{P}_x\mathbb{L}_x)\in AUT(Q,\circ). \tag{17}$$

Let $I\mathbb{P}_x\mathbb{L}_x = \theta$, then this implies that $I\mathbb{P}_x = \theta\mathbb{L}_x^{-1}$ and $yI\mathbb{P}_xI = y\theta\mathbb{L}_x^{-1}I \Rightarrow (y^{-1}\backslash\backslash x)^{-1} = (x\backslash\backslash y\theta)^{-1} \Rightarrow x^{-1}//y = (y\theta)I//x^{-1} \Rightarrow \mathbb{P}_x^{-1} = \theta I\mathbb{R}_x^{-1} \Rightarrow I\mathbb{P}_x^{-1} = I\theta I\mathbb{R}_x^{-1}$. Thus, $I\mathbb{P}_x^{-1} = I\theta I\mathbb{R}_x^{-1}$. Thence, (17) becomes $(I\theta I\mathbb{R}_x^{-1},\theta\mathbb{L}_x^{-1},\theta)\in AUT(Q,\circ)$. For all $a,b\in Q$, we have

$$aI\theta I\mathbb{R}_x^{-1}\circ b\theta\mathbb{L}_x^{-1} = (a\circ b)\theta$$
$$\Longrightarrow aI\theta IL_{x^{-1}}\circ b\theta(IP_x)^{-1} = (a\circ b)\theta$$
$$\Longrightarrow b\theta((IP_x)^{-1})I\backslash aI\theta IL_{x^{-1}} = (b^{-1}\backslash a)\theta$$
$$\Longrightarrow b\theta((IP_x)^{-1})I\cdot(b^{-1}\backslash x)\theta = aI\theta IL_{x^{-1}}.$$

Put $c = b^{-1}\backslash a \Longrightarrow a = b^{-1} \cdot c$ for $c \in Q$. So, from the last equation,

$$b\theta((IP_x)^{-1})I \cdot c\theta = (b^{-1} \cdot c)I\theta IL_{x^{-1}} \Longrightarrow bI\theta((IP_x)^{-1})I \cdot c\theta = (b \cdot c)I\theta IL_{x^{-1}}.$$

Note that $e\theta = e \Leftrightarrow (Q, \circ)$ is of exponent 2. Thus, setting $c = e$, then $bI\theta((IP_x)^{-1})I = bI\theta IL_{x^{-1}} \Longrightarrow ((IP_x)^{-1})I = IL_{x^{-1}}$. Thence, $bI\theta IL_{x^{-1}} \cdot c\theta = (b \cdot c)I\theta IL_{x^{-1}}$. Now, set $b = e$ to get $eL_{x^{-1}} \cdot c\theta = cI\theta IL_{x^{-1}}$, which implies that $\theta = I\theta I$. Hence, $b\theta L_x \cdot c\theta = (b \cdot c)\theta L_x \Longrightarrow (\theta L_x, \theta, \theta L_x) \in AUT(Q, \cdot)$ for all $x \in Q$. Thus, $\theta \in PS_\lambda(Q, \cdot)$ with companion $x \in Q$. Therefore, $(Q, \cdot)$ is a left G-loop.

The proof for $(Q, *)$ is similar. □

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On the solubility of a class of two-dimensional integral equations on a quarter plane with monotone nonlinearity

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Abstract. In the paper we study a class of two-dimensional integral equations on a quarter-plane with monotone nonlinearity and substochastic kernel. With specific representations of the kernel and nonlinearity, an equation of this kind arises in various fields of natural science. In particular, such equations occur in the dynamical theory of p-adic open-closed strings for the scalar field of tachyons, in the mathematical theory of the geographical spread of a pandemic, in the kinetic theory of gases, and in the theory of radiative transfer in inhomogeneous media.
We prove constructive theorems on the existence of a nontrivial nonnegative and bounded solution. For one important particular case, the existence of a one-parameter family of nonnegative and bounded solutions is also established. Moreover, the asymptotic behavior at infinity of each solution from the given family os studied. At the end of the paper, specific particular examples (of an applied nature) of the kernel and nonlinearity that satisfy all the conditions of the proven statements are given.

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1 Introduction

Consider the following class of two-dimensional integral equations on the first quarter of the plane with monotone nonlinearity:

$$\mathscr{F}(x_1,x_2)=\int_0^\infty\int_0^\infty \mathcal{P}(x_1,y_1,x_2,y_2)\,G(x_1,x_2,\mathscr{F}(\rho_1(x_1,y_1),\rho_2(x_2,y_2)))\,dy_1\,dy_2, \tag{1}$$
$$(x_1,x_2)\in\mathbb{R}_2^+:=\mathbb{R}^+\times\mathbb{R}^+,\quad \mathbb{R}^+:=[0,+\infty)$$

with respect to an unknown measurable and bounded function $\mathscr{F}(x_1,x_2)$ on $\mathbb{R}_2^+$.

In the equation (1), the kernel $\mathcal{P}(x_1,y_1,x_2,y_2)$ is a measurable real-valued function on $\mathbb{R}_4^+:=\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+$ satisfying the following conditions:

$a)$ (*minorant condition*)

there exist continuous on $\mathbb{R}_2^+$ functions $K(y_1,\,y_2)$ and $\lambda(x_1,x_2)$ with properties

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$a_1)$ $K(y_1, y_2) > 0,\ (y_1, y_2) \in \mathbb{R}_2^+, \quad K \in L_1(\mathbb{R}_2^+) \cap M(\mathbb{R}_2^+),$

$$\int_0^\infty \int_0^\infty K(y_1, y_2)\, dy_1\, dy_2 = 1, \tag{2}$$

$a_2)$ $0 \le \lambda(x_1, x_2) \le 1,\ (x_1, x_2) \in \mathbb{R}_2^+, \quad \lambda \uparrow$ by x_j on $\mathbb{R}^+, \quad j = 1, 2,$

$$(1 - \lambda(x_1, x_2))\, x_1^m x_2^\ell \in L_1\left(\mathbb{R}_2^+\right),\ m, \ell = 0, 1, \tag{3}$$

such that

$$\mathcal{P}(x_1, y_1, x_2, y_2) \ge \lambda(x_1, x_2) K(y_1, y_2), \tag{4}$$

$b)$ (*substochasticity condition*)

$$\mu(x_1, x_2) := \int_0^\infty \int_0^\infty \mathcal{P}(x_1, y_1, x_2, y_2)\, dy_1\, dy_2 \le 1, \quad \mu(x_1, x_2) \not\equiv 1,\ (x_1, x_2) \in \mathbb{R}_2^+$$

and $\sup\limits_{(x_1, x_2) \in \mathbb{R}_2^+} \mu(x_1, x_2) = 1.$

Nonlinearity $G(x_1, x_2, u)$ is a measurable real-valued function on $\mathbb{R}_2^+ \times \mathbb{R}$ ($\mathbb{R} := (-\infty, +\infty)$) satisfying Carathéodory condition with respect to the argument u (i.e., for every $u \in \mathbb{R}$ the function G is measurable in $(x_1, x_2) \in \mathbb{R}_2^+$ and for almost every $(x_1, x_2) \in \mathbb{R}_2^+$ this function is continuous in u on set $\mathbb{R}$) and some other conditions (see the statement of the main result).

The functions $\{\rho_j(u, v)\}_{j=1,2}$ in the right side of (1) satisfy the following conditions:

1) $\rho_j(u, v) \ge 0,\ (u, v) \in \mathbb{R}_2^+, \qquad \rho_j \in C(\mathbb{R}_2^+),\ j = 1, 2,$

2) $\rho_j(u, v) \uparrow$ in u on $\mathbb{R}^+$ and $\rho_j(u, v) \uparrow$ in v on $\mathbb{R}^+,\ j = 1, 2,$

3) $\rho_j(u, 0) \ge u, \quad \rho_j(u, 1) \ge u + 1,\ u \in \mathbb{R}^+,\ j = 1, 2.$

The equation (1), apart from its purely mathematical interest, has numerous important applications. First of all, we should single out the problems of mathematical physics and mathematical biology. So, very important in practical terms is a special case of the equation when $\rho_j(u, v) = u + v,\ j = 1, 2,\ (u, v) \in \mathbb{R}_2^+$ with specific representations of the kernel $\mathcal{P}$ and the nonlinearity G. Such equations arise in the dynamical theory of p-adic open-closed strings for the scalar field of tachyons, in the mathematical theory of space-time (geographical) propagation of pandemics, in the kinetic theory of gases, in the theory of radiative transfer in inhomogeneous media [1–8].

In the particular case $\rho_j(u, v) = u + v,\ j = 1, 2,\ (u, v) \in \mathbb{R}_2^+$, when the functions G and P do not depend on the variables (x_1, x_2), the equation (1) was studied in [8–10] under various restrictions on nonlinearity. It should be noted that in the one-dimensional case the corresponding nonlinear integral equation with the difference kernel $\mathcal{P}(x-y)$ on the semiaxis, for various representations of the nonlinearity was studied in detail in the papers [11–13]. We also note there are scientific papers devoted to the study of one-dimensional nonlinear integral equations on a semiaxis with a sum-difference kernel $\mathcal{P}(x,y)=\mathcal{P}_0(x-y)-\mathcal{P}_0(x+y)$, $(x,y)\in\mathbb{R}_2^+$ and with convex nonlinearity (see for instance [2, 14–16] and references therein).

In the present paper, under sufficiently general restrictions on the nonlinearity G, we prove a constructive theorem on the existence of a nonnegative nontrivial (nonzero) bounded solution on the set $\mathbb{R}_2^+$. In one important particular case, we also construct a one-parameter family of bounded solutions and establish the integral asymptotics of the constructed solutions. The proofs of the formulated theorems are based on the construction of invariant cone segments for the corresponding nonlinear monotone integral operator in the space of essentially bounded functions on the set $\mathbb{R}_2^+$, as well as on the methods developed during the systematic study of corresponding homogeneous and non-homogeneous linear integral equations on $\mathbb{R}_2^+$ with operators of almost Volterra type (when $\rho_j(u,v)=u+v,\ j=1,2,\ (u,v)\in\mathbb{R}_2^+$ these operators turn into two-dimensional Volterra operators with variable lower limits). At the end of the paper, we provide concrete particular examples of the functions $\mathcal{P}$, K, λ and G, which are of both applied and purely theoretical interest.

2 Auxiliary facts and notations

Before we prove the main result, we first study auxiliary equations and establish important and useful results for them, which will be used later.

2.1 Summable solution of a linear inhomogeneous auxiliary integral equation on a quarter-plane

Consider the following linear inhomogeneous two-dimensional integral equation:

$$f(x_1,x_2)=g(x_1,x_2)+\int_0^\infty\int_0^\infty K(y_1,y_2)\,f\left(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\right)dy_1dy_2,\quad (x_1,x_2)\in\mathbb{R}_2^+, \tag{5}$$

with respect to a nonnegative and measurable on $\mathbb{R}_2^+$ function $f(x_1,x_2)$. Here $g(x_1,x_2)$ is a measurable function on $\mathbb{R}_2^+$ and

$$g(x_1,x_2)\geq 0,\ (x_1,x_2)\in\mathbb{R}_2^+,\quad g(x_1,x_2)\downarrow \text{ in } x_j \text{ on } \mathbb{R}^+,\quad j=1,2,$$
$$\int_0^\infty\int_0^\infty g(x_1,x_2)\,x_1^m x_2^\ell\,dx_1\,dx_2<+\infty,\quad m,\ell=0,1. \tag{6}$$

For the equation (5) we consider the following simple iterations:

$$f_{n+1}(x_1,x_2)=g(x_1,x_2)+\int\limits_0^\infty\int\limits_0^\infty K(y_1,y_2)\,f_n\left(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\right)dy_1\,dy_2, \tag{7}$$
$$f_0(x_1,x_2)=g(x_1,x_2),\quad n=0,1,2,\ldots,\quad (x_1,x_2)\in\mathbb{R}_2^+.$$

Applying the method of mathematical induction it is easy to check that

$$f_n(x_1,x_2)\uparrow \quad\text{in}\quad n. \tag{8}$$

Now we prove that

$$f_n(x_1,x_2)\downarrow\ \text{in}\ x_j\ \text{on}\ \mathbb{R}^+,\quad j=1,2,\quad n=0,1,2,\ldots. \tag{9}$$

Indeed, the monotonicity of the zero approximation immediately follows from (6). Assume that (9) holds for some positive integer n. Then taking into account the conditions (6), a_1) and 2), from (7) for arbitrary $x_1,\widetilde{x}_1\in\mathbb{R}^+,\ x_1>\widetilde{x}_1$ we will have

$$f_{n+1}(x_1,x_2)\le g(\widetilde{x}_1,x_2)+\int\limits_0^\infty\int\limits_0^\infty K(y_1,y_2)\,f_n\left(\rho_1(\widetilde{x}_1,y_1),\rho_2(x_2,y_2)\right)\,dy_1\,dy_2=$$
$$=f_{n+1}(\widetilde{x}_1,x_2),\quad x_2\in\mathbb{R}^+.$$

By analogy, for arbitrary $x_2,\widetilde{x}_2\in\mathbb{R}^+,\ x_2>\widetilde{x}_2$ we get $f_{n+1}(x_1,x_2)\le f_{n+1}(x_1,\widetilde{x}_2)$, $x_1\in\mathbb{R}^+$. Therefore, (9) is valid.

Applying again induction on n we prove that

$$f_n\in L_1(\mathbb{R}_2^+),\quad n=0,1,2,\ldots. \tag{10}$$

In the case when $n=0$ the validity of (10) follows obviously from definition of zero approximation and its property (6). Assume that $f_n\in L_1(\mathbb{R}_2^+)$ for some $n\in\mathbb{N}$, then $g+f_n\in L_1(\mathbb{R}_2^+)$. On the other hand, taking into account (9), 2) and a_1), from (7) we derive the following estimation:

$$g(x_1,x_2)\le f_{n+1}(x_1,x_2)\le g(x_1,x_2)+$$
$$+\int\limits_0^\infty\int\limits_0^\infty K(y_1,y_2)f_n\left(\rho_1(x_1,0),\rho_2(x_2,0)\right)dy_1dy_2\le$$
$$\le g(x_1,x_2)+f_n(x_1,x_2)\int\limits_0^\infty\int\limits_0^\infty K(y_1,y_2)dy_1\,dy_2=g(x_1,x_2)+f_n(x_1,x_2),\ (x_1,x_2)\in\mathbb{R}_2^+,$$

whence it follows that $f_{n+1}\in L_1(\mathbb{R}_2^+)$.

Next we prove the existence of a such constant $C>0$ that

$$\int\limits_0^\infty\int\limits_0^\infty f_n(x_1,x_2)\,dx_1\,dx_2\le C,\quad n=1,2,\ldots. \tag{11}$$

Let $r_1 \geq 0$, $r_2 \geq 0$ be arbitrary numbers. Then taking into account the conditions $a_1), a_2), 1) - 3)$ and (6), from (7) we get

$$\int_{r_1}^{\infty}\int_{r_2}^{\infty} f_{n+1}(x_1,x_2)dx_1\,dx_2 \leq \int_{r_2}^{\infty}\int_{r_1}^{\infty} g(x_1,x_2)dx_1\,dx_2 +$$

$$+\int_{r_2}^{\infty}\int_{r_1}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} K(y_1,y_2)\, f_{n+1}\left(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\right) dy_1\,dy_2 dx_1\,dx_2 =$$

$$= \int_{r_2}^{\infty}\int_{r_1}^{\infty} g(x_1,x_2)dx_1\,dx_2 + \int_{0}^{\infty}\int_{0}^{\infty} K(y_1,y_2)\times$$

$$\times \int_{r_2}^{\infty}\int_{r_1}^{\infty} f_{n+1}\left(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\right) dx_1\,dx_2 dy_1\,dy_2 \leq \int_{r_2}^{\infty}\int_{r_1}^{\infty} g(x_1,x_2)dx_1\,dx_2 +$$

$$+\int_{0}^{1}\int_{0}^{\infty} K(y_1,y_2) \int_{r_2}^{\infty}\int_{r_1}^{\infty} f_{n+1}\left(\rho_1(x_1,y_1),\rho_2(x_2,0)\right) dx_1\,dx_2\,dy_1\,dy_2 +$$

$$+\int_{1}^{\infty}\int_{0}^{\infty} K(y_1,y_2) \int_{r_2}^{\infty}\int_{r_1}^{\infty} f_{n+1}\left(\rho_1(x_1,y_1),\rho_2(x_2,1)\right) dx_1\,dx_2\,dy_1\,dy_2 \leq$$

$$\leq \int_{r_2}^{\infty}\int_{r_1}^{\infty} g(x_1,x_2)\,dx_1\,dx_2 + \int_{0}^{1}\int_{0}^{1} K(y_1,y_2)\times$$

$$\times \int_{r_2}^{\infty}\int_{r_1}^{\infty} f_{n+1}\left(\rho_1(x_1,0),\rho_2(x_2,0)\right) dx_1\,dx_2\,dy_1\,dy_2 +$$

$$+\int_{0}^{1}\int_{1}^{\infty} K(y_1,y_2) \int_{r_2}^{\infty}\int_{r_1}^{\infty} f_{n+1}\left(\rho_1(x_1,1),\rho_2(x_2,0)\right)\, dx_1\,dx_2\,dy_1\,dy_2 +$$

$$+\int_{1}^{\infty}\int_{0}^{1} K(y_1,y_2) \int_{r_2}^{\infty}\int_{r_1}^{\infty} f_{n+1}\left(\rho_1(x_1,0),\rho_2(x_2,1)\right)\, dx_1\,dx_2\,dy_1\,dy_2 +$$

$$+\int_{1}^{\infty}\int_{1}^{\infty} K(y_1,y_2) \int_{r_2}^{\infty}\int_{r_1}^{\infty} f_{n+1}\left(\rho_1(x_1,1),\rho_2(x_2,1)\right)\, dx_1\,dx_2\,dy_1\,dy_2 \leq$$

$$\leq \int_{r_2}^{\infty}\int_{r_1}^{\infty} g(x_1,x_2)\,dx_1\,dx_2 + \int_0^1\int_0^1 K(y_1,y_2)\,dy_1\,dy_2 \int_{r_2}^{\infty}\int_{r_1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2 +$$

$$+\int_0^1\int_1^{\infty} K(y_1,y_2)\,dy_1\,dy_2 \int_{r_2}^{\infty}\int_{r_1+1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2 +$$

$$+\int_1^{\infty}\int_0^1 K(y_1,y_2)\,dy_1\,dy_2 \int_{r_2+1}^{\infty}\int_{r_1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2 +$$

$$+\int_1^{\infty}\int_1^{\infty} K(y_1,y_2)\,dy_1\,dy_2 \int_{r_2+1}^{\infty}\int_{r_1+1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2 \,.$$

Hence, combining similar integrals and taking into account (2), we obtain

$$\int_{r_2}^{\infty}\int_{r_1}^{\infty} f_{n+1}(x_1,x_2)dx_1\,dx_2\Bigg(\int_0^1\int_0^1 K(y_1,y_2)\,dy_1\,dy_2 + \int_0^1\int_1^{\infty} K(y_1,y_2)\,dy_1\,dy_2 +$$

$$+\int_1^{\infty}\int_0^1 K(y_1,y_2)dy_1\,dy_2 + \int_1^{\infty}\int_1^{\infty} K(y_1,y_2)\,dy_1\,dy_2 - \int_0^1\int_0^1 K(y_1,y_2)\,dy_1\,dy_2\Bigg) \leq$$

$$\leq \int_{r_2}^{\infty}\int_{r_1}^{\infty} g(x_1,x_2)\,dx_1\,dx_2 + \int_0^1\int_1^{\infty} K(y_1,y_2)\,dy_1\,dy_2 \int_{r_2}^{\infty}\int_{r_1+1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2 +$$

$$+\int_1^{\infty}\int_0^1 K(y_1,y_2)\,dy_1\,dy_2 \int_{r_2+1}^{\infty}\int_{r_1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2 +$$

$$+\int_1^{\infty}\int_1^{\infty} K(y_1,y_2)\,dy_1\,dy_2 \int_{r_2+1}^{\infty}\int_{r_1+1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2.$$

We introduce the following notations

$$\alpha_0 := \int_0^1\int_0^1 K(y_1,\, y_2)\,dy_1\,dy_2, \qquad \alpha_1 := \int_0^1\int_1^{\infty} K(y_1,\, y_2)\,dy_1\,dy_2,$$

$$\alpha_2 := \int_1^{\infty}\int_0^1 K(y_1,\, y_2)\,dy_1\,dy_2, \qquad \alpha_3 := \int_1^{\infty}\int_1^{\infty} K(y_1,\, y_2)\,dy_1\,dy_2.$$

Then the last inequality in the above notations can be written as follows:

$$(\alpha_1+\alpha_2+\alpha_3)\int\limits_{r_2}^{\infty}\int\limits_{r_1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2 \le \int\limits_{r_2}^{\infty}\int\limits_{r_1}^{\infty} g(x_1,x_2)\,dx_1\,dx_2+$$

$$+\alpha_1\int\limits_{r_2}^{\infty}\int\limits_{r_1+1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2+\alpha_2\int\limits_{r_2+1}^{\infty}\int\limits_{r_1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2+$$

$$+\alpha_3\int\limits_{r_2+1}^{\infty}\int\limits_{r_1+1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2.$$

After some transformations we get

$$\begin{gathered}\alpha_1\int\limits_{r_2}^{\infty}\int\limits_{r_1}^{r_1+1} f_{n+1}(x_1,x_2)\,dx_1\,dx_2+\alpha_2\int\limits_{r_2}^{r_2+1}\int\limits_{r_1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2+\\ +\alpha_3\int\limits_{r_2}^{r_2+1}\int\limits_{r_1}^{r_1+1} f_{n+1}(x_1,x_2)\,dx_1\,dx_2+\alpha_3\int\limits_{r_2}^{r_2+1}\int\limits_{r_1+1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2+\\ +\alpha_3\int\limits_{r_2+1}^{\infty}\int\limits_{r_1}^{r_1+1} f_{n+1}(x_1,x_2)\,dx_1\,dx_2\le\int\limits_{r_2}^{\infty}\int\limits_{r_1}^{\infty} g(x_1,x_2)\,dx_1\,dx_2.\end{gathered}\tag{12}$$

By virtue of (9), from (12) it follows, in particular, that

$$\begin{gathered}\alpha_1\int\limits_{r_2}^{\infty} f_{n+1}(r_1+1,x_2)dx_2+\alpha_2\int\limits_{r_1}^{\infty} f_{n+1}(x_1,r_2+1)dx_1+\\ +\alpha_3 f_{n+1}(r_1+1,r_2+1)+\alpha_3\int\limits_{r_1+1}^{\infty} f_{n+1}(x_1,r_2+1)dx_1+\\ +\alpha_3\int\limits_{r_2+1}^{\infty} f_{n+1}(r_1+1,x_2)dx_2\le\int\limits_{r_2}^{\infty}\int\limits_{r_1}^{\infty} g(x_1,x_2)\,dx_1\,dx_2.\end{gathered}\tag{13}$$

Taking into account the condition (6), by Fubini's theorem [17] we can state that

$$\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}\int\limits_{r_2}^{\infty}\int\limits_{r_1}^{\infty} g(x_1,x_2)\,dx_1\,dx_2\,dr_1\,dr_2=\int\limits_{0}^{\infty}\int\limits_{0}^{\infty} g(x_1,x_2)\int\limits_{0}^{x_1} dr_1\int\limits_{0}^{x_2} dr_2\,dx_1\,dx_2=$$

$$=\int\limits_{0}^{\infty}\int\limits_{0}^{\infty} x_1\,x_2\,g(x_1,x_2)\,dx_1\,dx_2:=M_{11}<+\infty,$$

$$\int_0^\infty \int_{r_2}^\infty \int_{r_1}^\infty g(x_1,x_2)\,dx_1\,dx_2\,dr_1 = \int_{r_2}^\infty \int_0^\infty \int_{r_1}^\infty g(x_1,x_2)\,dx_1\,dr_1\,dx_2 \leq$$

$$\leq \int_0^\infty \int_0^\infty g(x_1,x_2) \int_0^{x_1} dr_1\,dx_1\,dx_2 = \int_0^\infty \int_0^\infty x_1\,g(x_1,x_2)\,dx_1\,dx_2 := M_{10} < +\infty,$$

$$\int_0^\infty \int_{r_2}^\infty \int_{r_1}^\infty g(x_1,x_2)\,dx_1\,dx_2\,dr_2 \leq \int_0^\infty \int_0^\infty x_2\,g(x_1,x_2)\,dx_1\,dx_2 := M_{01} < +\infty.$$

Therefore, from (13) we get

$$\int_1^\infty \int_1^\infty f_{n+1}(x_1,x_2)\,dx_1\,dx_2 \leq \frac{M_{11}}{\alpha_3}, \tag{14}$$

$$\int_0^\infty \int_1^\infty f_{n+1}(x_1,x_2)\,dx_1\,dx_2 \leq \frac{M_{10}}{\alpha_1}, \tag{15}$$

$$\int_1^\infty \int_0^\infty f_{n+1}(x_1,x_2)\,dx_1\,dx_2 \leq \frac{M_{01}}{\alpha_2}. \tag{16}$$

Integrating both parts of (7) over the set $[0,\,1]\times[0,\,1]$ and then using the estimates (14)–(16), we have

$$\int_0^1 \int_0^1 f_{n+1}(x_1,x_2)\,dx_1\,dx_2 \leq \int_0^1 \int_0^1 g(x_1,x_2)\,dx_1\,dx_2 +$$

$$+ \int_0^\infty \int_0^\infty K(y_1,y_2) \int_0^1 \int_0^1 f_{n+1}\left(\rho_1(x_1,y_1), \rho_2(x_2,y_2)\right)\,dx_1\,dx_2\,dy_1\,dy_2 \leq$$

$$\leq \int_0^1 \int_0^1 g(x_1,x_2)dx_1dx_2 + \int_0^1 \int_0^\infty K(y_1,y_2) \int_0^1 \int_0^1 f_{n+1}\left(x_1, \rho_2(x_2,y_2)\right) dx_1dx_2dy_1dy_2 +$$

$$+ \int_1^\infty \int_0^\infty K(y_1,y_2) \int_0^1 \int_0^1 f_{n+1}\left(x_1+1, \rho_2(x_2,y_2)\right)\,dx_1\,dx_2\,dy_1\,dy_2 \leq$$

$$\leq \int_0^1 \int_0^1 g(x_1,x_2)\,dx_1\,dx_2 + \int_0^1 \int_0^1 K(y_1,y_2) \int_0^1 \int_0^1 f_{n+1}(x_1,x_2)\,dx_1\,dx_2\,dy_1\,dy_2 +$$

$$+ \int_0^1 \int_1^\infty K(y_1,y_2) \int_0^1 \int_0^1 f_{n+1}(x_1,x_2+1)\,dx_1\,dx_2\,dy_1\,dy_2 +$$

$$+\int_1^\infty\int_0^1 K(y_1,y_2)\int_0^1\int_0^1 f_{n+1}(x_1+1,x_2)\,dx_1\,dx_2\,dy_1\,dy_2+$$

$$+\int_1^\infty\int_1^\infty K(y_1,y_2)\int_0^1\int_0^1 f_{n+1}(x_1+1,x_2+1)\,dx_1\,dx_2\,dy_1\,dy_2\leq$$

$$\leq\int_0^1\int_0^1 g(x_1,x_2)\,dx_1\,dx_2+\alpha_0\int_0^1\int_0^1 f_{n+1}(x_1,x_2)\,dx_1\,dx_2+$$

$$+\alpha_1\int_1^\infty\int_0^1 f_{n+1}(x_1,x_2)\,dx_1\,dx_2+\alpha_2\int_0^1\int_1^\infty f_{n+1}(x_1,x_2)\,dx_1\,dx_2+$$

$$+\alpha_3\int_1^\infty\int_1^\infty f_{n+1}(x_1,x_2)\,dx_1\,dx_2\leq\int_0^1\int_0^1 g(x_1,x_2)\,dx_1\,dx_2+$$

$$+\alpha_0\int_0^1\int_0^1 f_{n+1}(x_1,x_2)\,dx_1\,dx_2+\frac{\alpha_1}{\alpha_2}M_{01}+\frac{\alpha_2}{\alpha_1}M_{10}+M_{11},$$

from which we get

$$\int_0^1\int_0^1 f_{n+1}(x_1,x_2)\,dx_1\,dx_2\leq(1-\alpha_0)^{-1}\left\{\int_0^1\int_0^1 g(x_1,x_2)\,dx_1\,dx_2+\right.$$
$$\left.+\frac{\alpha_1}{\alpha_2}M_{01}+\frac{\alpha_2}{\alpha_1}M_{10}+M_{11}\right\}:=C^*<+\infty,\quad n=0,1,2,\ldots\,. \tag{17}$$

Finally, summing the inequalities (14)–(17) we obtain

$$\int_0^\infty\int_0^\infty f_{n+1}(x_1,x_2)\,dx_1\,dx_2\leq C^*+\frac{M_{10}}{\alpha_1}+\frac{M_{01}}{\alpha_2}+\frac{M_{11}}{\alpha_3}<+\infty,\ \ n=0,1,2,\ldots, \tag{18}$$

i.e. the proving inequality (11), where $C=C^*+\dfrac{M_{10}}{\alpha_1}+\dfrac{M_{01}}{\alpha_2}+\dfrac{M_{11}}{\alpha_3}$.

Consequently, the sequence of summable and monotone functions $\{f_n(x_1,x_2)\}_{n=0}^\infty$ as $n\to\infty$ almost everywhere on $\mathbb{R}_2^+$ converges to the summable function $f(x_1,x_2)$. This fact follows from (8)–(10) and (18) by B. Levi's theorem [17]. Using again B. Levi's theorem it can be stated that limit function $f(x_1,x_2)$ satisfies the equation (5) almost everywhere on $\mathbb{R}_2^+$.

From (8), (9) and (18) we also get

$$f(x_1,x_2)\geq g(x_1,x_2),\quad (x_1,x_2)\in\mathbb{R}_2^+, \tag{19}$$

$$f(x_1,x_2)\downarrow \text{ in } x_j \text{ on } \mathbb{R}^+, \quad j=1,2, \tag{20}$$

$$\int_0^\infty\int_0^\infty f(x_1,x_2)\,dx_1\,dx_2 \le C^* + \frac{M_{10}}{\alpha_1} + \frac{M_{01}}{\alpha_2} + \frac{M_{11}}{\alpha_3}. \tag{21}$$

The foregoing implies

Theorem 1. *Let the function g satisfy the conditions* (6)*, and let the kernel K have the properties a_1). Then under conditions* 1) $-$ 3) *the equation* (5) *has a nonnegative and monotonically non-increasing in each argument and summable solution. Moreover, the estimates* (19) *and* (21) *hold for the solution.*

2.2 A nontrivial solution of a linear homogeneous auxiliary integral equation on a quarter-plane

Let us introduce into consideration the inhomogeneous auxiliary integral equation

$$f^*(x_1,x_2) = 1-\lambda(x_1,x_2)+ \\ +\lambda(x_1,x_2)\int_0^\infty\int_0^\infty K(y_1,y_2)\,f^*\left(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\right)\,dy_1\,dy_2, \quad (x_1,x_2)\in\mathbb{R}_2^+ \tag{22}$$

with respect to the unknown measurable function $f^*(x_1,x_2)$, where the functions λ and K possess the properties a_2) and a_1) respectively.

Due to a_2) the function $1-\lambda(x_1,x_2)$ satisfies the conditions (6). Therefore, according to Theorem 1, the equation (5) with the free term $g(x_1,x_2) = 1-\lambda(x_1,x_2)$ has a nonnegative and monotone (with respect to each argument) and summable solution on $\mathbb{R}_2^+$. We denote this solution by $f_\lambda(x_1,x_2)$.

For the equation (22) consider the following iterations:

$$f_{n+1}^*(x_1,x_2) = 1-\lambda(x_1,x_2)+ \\ +\lambda(x_1,x_2)\int_0^\infty\int_0^\infty K(y_1,y_2)\,f_n^*\left(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\right)\,dy_1\,dy_2, \tag{23}$$

$$f_0^*(x_1,x_2) = 1-\lambda(x_1,x_2), \quad n=0,1,2,\ldots, \quad (x_1,x_2)\in\mathbb{R}_2^+.$$

By induction it is easy to show that

$$f_n^*(x_1,x_2)\uparrow \text{ in } n, \quad (x_1,x_2)\in\mathbb{R}_2^+, \tag{24}$$

$$f_n^*(x_1,x_2)\le \min\{1, f_\lambda(x_1,x_2)\}, \quad n=0,1,2,\ldots, \quad (x_1,x_2)\in\mathbb{R}_2^+. \tag{25}$$

Therefore, the sequence of functions $\{f_n^*(x_1,x_2)\}_{n=0}^\infty$ has a pointwise limit as $n\to\infty$: $\lim\limits_{n\to\infty} f_n^*(x_1,x_2) = f^*(x_1,x_2)$. In accordance with B. Levi's theorem, the limit function $f^*(x_1,x_2)$ satisfies the equation (22). It follows from (24) and (25) that

$$1-\lambda(x_1,x_2)\le f^*(x_1,x_2)\le \min\{1, f_\lambda(x_1,x_2)\}, \quad (x_1,x_2)\in\mathbb{R}_2^+, \tag{26}$$

whence, in particular, we obtain

$$f^* \in L_1(\mathbb{R}_2^+) \cap M(\mathbb{R}_2^+). \tag{27}$$

Further, we consider the corresponding homogeneous integral equation

$$S(x_1, x_2) = \lambda(x_1, x_2) \int_0^\infty \int_0^\infty K(y_1, y_2) S\left(\rho_1(x_1, y_1), \rho_2(x_2, y_2)\right) dy_1 dy_2, \tag{28}$$

$(x_1, x_2) \in \mathbb{R}_2^+$, with respect to the measurable and bounded function $S(x_1, x_2)$. Using $a_1)$, we can check directly that $f^*_{\text{triv}}(x_1, x_2) \equiv 1$ is a solution of the equation (22). On the other hand, we have proved that the equation (22), in addition to such a trivial solution, also has an integrable and bounded solution $f^*(x_1, x_2)$ (with the property (26)). It is obvious that

$$S(x_1, x_2) = f^*_{\text{triv}}(x_1, x_2) - f^*(x_1, x_2) = 1 - f^*(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}_2^+$$

is a solution of the homogeneous equation (28). From (26), in particular, we get

$$1 \geq S(x_1, x_2) \geq 0, \quad S(x_1, x_2) \not\equiv 0, \quad S(x_1, x_2) \not\equiv 1, \quad (x_1, x_2) \in \mathbb{R}_2^+, \tag{29}$$

and from (27)

$$1 - S \in L_1(\mathbb{R}_2^+) \cap M(\mathbb{R}_2^+). \tag{30}$$

Thus, for the auxiliary linear homogeneous equation (28), the following theorem holds:

Theorem 2. *Under the conditions* $a_1), a_2)$ *and* $1) - 3)$ *the linear homogeneous integral equation* (28) *has a nonnegative nontrivial measurable and bounded solution* $S(x_1, x_2)$ *on* $\mathbb{R}_2^+$. *In addition,* $S(x_1, x_2)$ *possesses the* (29) *and* (30) *properties.*

Remark 1. It is interesting to note that the proved Theorem 2 generalizes and supplements the corresponding result from [18], devoted to the study of one-dimensional integral equations with $\rho(u, v) = u + v, \quad (u, v) \in \mathbb{R}_2^+$.

3 Solubility of the main nonlinear equation. Examples

In this section, we begin to study the initial nonlinear integral equation (1), first highlighting one special case (important in applications).

3.1 One-parameter family of bounded solutions of the equation (1) in one particular case

Let the nonlinearity $G(x_1, x_2, u)$ admit a representation of the form

$$G(x_1, x_2, u) = u + \omega(x_1, x_2, u), \quad (x_1, x_2, u) \in \mathbb{R}_2^+ \times \mathbb{R}, \tag{31}$$

where $\omega(x_1, x_2, u)$ satisfies the following conditions:

I) $\omega(x_1, x_2, u) \uparrow$ in u on $\mathbb{R}^+$,

II) $\omega(x_1, x_2, u)$ satisfies the Carathéodory condition with respect to the argument u on $\mathbb{R}_2^+ \times \mathbb{R}$ (see the introduction about the Carathéodory condition),

III) $\omega(x_1, x_2, u) \geq 0, \quad (x_1, x_2, u) \in \mathbb{R}_3^+$,

IV) the supremum of ω with respect to u on $\mathbb{R}_2^+$:

$$\beta(x_1, x_2) := \sup_{u \in \mathbb{R}^+} \omega(x_1, x_2, u), \quad (x_1, x_2) \in \mathbb{R}_2^+, \tag{32}$$

possesses following properties: $\beta(x_1, x_2) \downarrow$ in x_j on $\mathbb{R}^+$, $j = 1, 2$

$$x_1^m x_2^\ell \beta(x_1, x_2) \in L_1\left(\mathbb{R}_2^+\right), \; m, \ell = 0, 1.$$

Suppose also that the kernel $\mathcal{P}(x_1, y_1, x_2, y_2)$ is linked with the functions λ and K by the relation

$$\mathcal{P}(x_1, y_1, x_2, y_2) = \lambda(x_1, x_2) K(y_1, y_2), \quad (x_1, y_1, x_2, y_2) \in \mathbb{R}_4^+. \tag{33}$$

Then the equation (1) will take the following form:

$$\begin{aligned} \mathscr{F}(x_1, x_2) = \lambda(x_1, x_2) \int_0^\infty \int_0^\infty K(y_1, y_2) \big\{ \mathscr{F}(\rho_1(x_1, y_1), \rho_2(x_2, y_2)) + \\ + \omega(x_1, x_2, \mathscr{F}(\rho_1(x_1, y_1), \rho_2(x_2, y_2))) \big\} \, dy_1 \, dy_2, \; (x_1, x_2) \in \mathbb{R}_2^+. \end{aligned} \tag{34}$$

We construct special successive approximations

$$\begin{aligned} \mathscr{F}_{n+1}^\gamma(x_1, x_2) = \lambda(x_1, x_2) \int_0^\infty \int_0^\infty K(y_1, y_2) \big\{ \mathscr{F}_n^\gamma(\rho_1(x_1, y_1), \rho_2(x_2, y_2)) + \\ + \omega\big(x_1, x_2, \mathscr{F}_n^\gamma(\rho_1(x_1, y_1), \rho_2(x_2, y_2))\big) \big\} \, dy_1 \, dy_2, \\ \mathscr{F}_0^\gamma(x_1, x_2) = \gamma S(x_1, x_2), \quad n = 0, 1, 2, \ldots, \quad (x_1, x_2) \in \mathbb{R}_2^+, \end{aligned} \tag{35}$$

where $\gamma > 0$ is an arbitrary numeric parameter.

Along with iterations (35), consider a linear inhomogeneous integral equation (5) with a free term of the form

$$g(x_1, x_2) = \beta(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}_2^+. \tag{36}$$

Due to conditions III) and IV), according to Theorem 1 the equation (5) with a free term of the form (36) has a nonnegative monotonically non-increasing and summable on $\mathbb{R}_2^+$ solution $f_\beta(x_1, x_2)$.

Below we establish several important properties that characterize the sequence $\{\mathscr{F}_n^\gamma(x_1, x_2)\}_{n=0}^\infty$ both for each value of the parameter $\gamma > 0$.

By induction on n we prove

$$\mathscr{F}_n^\gamma(x_1, x_2) \uparrow \text{ in } n, \; \gamma > 0, \; (x_1, x_2) \in \mathbb{R}_2^+, \tag{37}$$

$$\mathscr{F}_n^\gamma(x_1,x_2)\leq \gamma S(x_1,x_2)+f_\beta(x_1,x_2),\ \gamma>0,\ n=0,1,2,\dots,\ (x_1,x_2)\in\mathbb{R}_2^+. \quad (38)$$

We first prove that $\mathscr{F}_1^\gamma(x_1,x_2)\geq \mathscr{F}_0^\gamma(x_1,x_2)$ and $\mathscr{F}_1^\gamma(x_1,x_2)\leq \gamma S(x_1,x_2)+f_\beta(x_1,x_2)$, $(x_1,x_2)\in\mathbb{R}_2^+$, $\gamma>0$. Indeed, taking into account (28), (32), as well as the conditions $a_1), a_2), III)$, from (35) we have

$$\mathscr{F}_1^\gamma(x_1,x_2)\geq \lambda(x_1,x_2)\int\limits_0^\infty\int\limits_0^\infty K(y_1,\,y_2)\mathscr{F}_0^\gamma(\rho_1(x_1,y_1),\rho_2(x_2,y_2))\,dy_1\,dy_2=$$

$$=\gamma\lambda(x_1,x_2)\int\limits_0^\infty\int\limits_0^\infty K(y_1,\,y_2)S(\rho_1(x_1,y_1),\rho_2(x_2,y_2))\,dy_1\,dy_2=$$

$$=\gamma S(x_1,x_2)=\mathscr{F}_0^\gamma(x_1,x_2),$$

$$\mathscr{F}_1^\gamma(x_1,x_2)=\lambda(x_1,x_2)\times$$

$$\times\int\limits_0^\infty\int\limits_0^\infty K(y_1,\,y_2)\Big\{\gamma S\big(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\big)+f_\beta\big(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\big)+$$

$$+\omega\Big(x_1,x_2,\gamma S\big(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\big)+f_\beta\big(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\big)\Big)\Big\}\,dy_1\,dy_2\leq$$

$$\leq\gamma\lambda(x_1,x_2)\int\limits_0^\infty\int\limits_0^\infty K(y_1,\,y_2)S\big(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\big)\,dy_1\,dy_2+$$

$$+\lambda(x_1,x_2)\int\limits_0^\infty\int\limits_0^\infty K(y_1,\,y_2)f_\beta\big(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\big)\,dy_1\,dy_2+$$

$$+\beta(x_1,x_2)\lambda(x_1,x_2)\int\limits_0^\infty\int\limits_0^\infty K(y_1,\,y_2)\,dy_1\,dy_2\leq\gamma S(x_1,x_2)+\beta(x_1,x_2)+$$

$$+\int\limits_0^\infty\int\limits_0^\infty K(y_1,\,y_2)f_\beta\big(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\big)\,dy_1\,dy_2=\gamma S(x_1,x_2)+f_\beta(x_1,x_2).$$

Assume that the statements (37) and (38) are true for some $n\in\mathbb{N}$. We use again (28), (32), $a_1), a_2)$ and III). Then from (35) by virtue of I) we obtain

$$\mathscr{F}_{n+1}^\gamma(x_1,x_2)\geq\lambda(x_1,x_2)\int\limits_0^\infty\int\limits_0^\infty K(y_1,\,y_2)\Big\{\mathscr{F}_{n-1}^\gamma(\rho_1(x_1,y_1),\rho_2(x_2,y_2))+$$

$$+\,\omega(x_1,x_2,\mathscr{F}_{n-1}^\gamma(\rho_1(x_1,y_1),\rho_2(x_2,y_2)))\Big\}\,dy_1\,dy_2=\mathscr{F}_n^\gamma(x_1,x_2),$$

$$\mathscr{F}_{n+1}^{\gamma}(x_1,x_2) \le \gamma S(x_1,x_2) + f_\beta(x_1,x_2),\ n = 0,1,2,\dots,\quad (x_1,x_2)\in\mathbb{R}_2^+,\ \gamma>0,$$

whence the required assertions (37) and (38) follow.

Based on the Carathéodory condition for the function ω (see II)) it is easy to prove that for every $\gamma > 0$ each element of the sequence $\{\mathscr{F}_n^{\gamma}(x_1,x_2)\}_{n=0}^{\infty}$ is a measurable function on $\mathbb{R}_2^+$.

Thus, in view of (37) and (38) we can assert that the sequence of measurable functions on $\mathbb{R}_2^+$ $\{\mathscr{F}_n^{\gamma}(x_1,x_2)\}_{n=0}^{\infty}$ has a pointwise limit as $n\to\infty$: $\lim\limits_{n\to\infty}\mathscr{F}_n^{\gamma}(x_1,x_2) = \mathscr{F}^{\gamma}(x_1,x_2)$. By Levy's theorem, the limit function $\mathscr{F}^{\gamma}(x_1,x_2)$ satisfies the equation (34) for every $\gamma>0$. Moreover, from (37) and (38) we get that $\mathscr{F}^{\gamma}(x_1,x_2)$ satisfies the following double inequality:

$$\gamma S(x_1,x_2) \le \mathscr{F}^{\gamma}(x_1,x_2) \le \gamma S(x_1,x_2) + f_\beta(x_1,x_2),\quad (x_1,x_2)\in\mathbb{R}_2^+,\ \gamma>0. \tag{39}$$

Now we note one more important and useful property of the sequence of functions $\{\mathscr{F}_n^{\gamma}(x_1,x_2)\}_{n=0}^{\infty}$ on $\mathbb{R}_2^+$ for different values of the parameter $\gamma>0$. We prove by induction that if $\gamma_1,\ \gamma_2\in(0,+\infty)$, $\gamma_1>\gamma_2$ are arbitrary parameters, then

$$\mathscr{F}_n^{\gamma_1}(x_1,x_2) - \mathscr{F}_n^{\gamma_2}(x_1,x_2) \ge (\gamma_1-\gamma_2)S(x_1,x_2),\ n=0,1,2,\dots,\ (x_1,x_2)\in\mathbb{R}_2^+. \tag{40}$$

Indeed, when $n=0$ the required inequality is obvious. Suppose (40) is satisfied for some $n\in\mathbb{N}$. Then, using the conditions $I), a_1), a_2)$ and taking into account (28), from (35) we have

$$\mathscr{F}_{n+1}^{\gamma_1}(x_1,x_2) - \mathscr{F}_{n+1}^{\gamma_2}(x_1,x_2) = \lambda(x_1,x_2)\int_0^\infty\int_0^\infty K(y_1,\,y_2)\Big\{\mathscr{F}_n^{\gamma_1}(\rho_1(x_1,y_1),\rho_2(x_2,y_2)) -$$

$$-\mathscr{F}_n^{\gamma_2}(\rho_1(x_1,y_1),\rho_2(x_2,y_2)) + \omega(x_1,x_2,\mathscr{F}_n^{\gamma_1}(\rho_1(x_1,y_1),\rho_2(x_2,y_2))) -$$

$$-\omega(x_1,x_2,\mathscr{F}_n^{\gamma_2}(\rho_1(x_1,y_1),\rho_2(x_2,y_2)))\Big\}\,dy_1\,dy_2 \ge \lambda(x_1,x_2)\times$$

$$\times\int_0^\infty\int_0^\infty K(y_1,\,y_2)\Big\{\mathscr{F}_n^{\gamma_1}(\rho_1(x_1,y_1),\rho_2(x_2,y_2)) - \mathscr{F}_n^{\gamma_2}(\rho_1(x_1,y_1),\rho_2(x_2,y_2))\Big\}dy_1dy_2 \ge$$

$$\ge (\gamma_1-\gamma_2)\lambda(x_1,x_2)\int_0^\infty\int_0^\infty K(y_1,\,y_2)S\big(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\big)dy_1dy_2 = (\gamma_1-\gamma_2)S(x_1,x_2).$$

Letting the number $n\to\infty$ into (40), we get

$$\mathscr{F}^{\gamma_1}(x_1,x_2) - \mathscr{F}^{\gamma_2}(x_1,x_2) \ge (\gamma_1-\gamma_2)S(x_1,x_2),\quad (x_1,x_2)\in\mathbb{R}_2^+. \tag{41}$$

Since $1-S\in L_1(\mathbb{R}_2^+)\cap M(\mathbb{R}_2^+)$ and $f_\beta\in L_1(\mathbb{R}_2^+)$, then in view of (39) from the estimate below

$$|\gamma - \mathscr{F}^{\gamma}(x_1,x_2)| = |\gamma - \gamma S(x_1,x_2) + \gamma S(x_1,x_2) - \mathscr{F}^{\gamma}(x_1,x_2)| \le$$

$$\le \gamma(1-S(x_1,x_2)) + f_\beta(x_1,x_2),\ \gamma>0,\ (x_1,x_2)\in\mathbb{R}_2^+$$

we obtain the following important fact: for each $\gamma>0$ the function $\gamma-\mathscr{F}^{\gamma}\in L_1(\mathbb{R}_2^+)$.

Thus the following theorem is true.

Theorem 3. *Under conditions* $a_1), a_2), I) - IV)$ *and* $1) - 3)$*, the nonlinear integral equation* (34) *has a one-parameter family of nonnegative nontrivial measurable solutions* $\{\mathscr{F}^\gamma(x_1, x_2)\}_{\gamma\in(0,+\infty)}$ *and*

- *for all* $\gamma \in (0, +\infty)$ *the inequalities* (39) *hold,*
- *for all* $\gamma_1, \gamma_2 \in (0, +\infty)$, $\gamma_1 > \gamma_2$, (41) *takes place,*
- *for all* $\gamma \in (0, +\infty)$ *functions* $\gamma - \mathscr{F}^\gamma(x_1, x_2)$ *are summable on* $\mathbb{R}_2^+$.

Remark 2. Under the assumptions of Theorem 3, if moreover the following conditions are fulfilled

$p_1)$ $\rho_j(0, v) \geq v,\ v \in \mathbb{R}^+,\ j = 1, 2,$

$p_2)$ $\beta \in M(\mathbb{R}_2^+),$

then for any $\gamma > 0$ the solution $\mathscr{F}^\gamma(x_1, x_2)$ is bounded on the set $\mathbb{R}_2^+$.

Proof. First, we verify that $f_\beta \in M(\mathbb{R}_2^+)$. Indeed, given the monotonicity of $f_\beta(x_1, x_2)$ in x_j on $\mathbb{R}^+$, $j = 1, 2$, and also conditions $2), a_1), p_1), p_2)$, from the equation (5) with free term $g(x_1, x_2) = \beta(x_1, x_2)$ we get

$$f_\beta(x_1, x_2) \leq \sup_{(x_1,x_2)\in\mathbb{R}_2^+} \beta(x_1, x_2) + \sup_{(y_1,y_2)\in\mathbb{R}_2^+} K(y_1, y_2) \int_0^\infty\int_0^\infty f_\beta\big(\rho_1(x_1, y_1), \rho_2(x_2, y_2)\big)\, dy_1\, dy_2 \leq$$

$$\leq \sup_{(x_1,x_2)\in\mathbb{R}_2^+} \beta(x_1, x_2) + \sup_{(y_1,y_2)\in\mathbb{R}_2^+} K(y_1, y_2) \int_0^\infty\int_0^\infty f_\beta\big(\rho_1(0, y_1), \rho_2(0, y_2)\big)\, dy_1\, dy_2 \leq$$

$$\leq \sup_{(x_1,x_2)\in\mathbb{R}_2^+} \beta(x_1, x_2) + \sup_{(y_1,y_2)\in\mathbb{R}_2^+} K(y_1, y_2) \int_0^\infty\int_0^\infty f_\beta(y_1, y_2)\, dy_1\, dy_2 < +\infty,$$

whence it follows that $f_\beta \in M(\mathbb{R}_2^+)$. Consequently, from (29) and (39) we have

$$0 \leq \mathscr{F}^\gamma(x_1, x_2) \leq \gamma + \sup_{(x_1,x_2)\in\mathbb{R}_2^+} f_\beta(x_1, x_2) < +\infty,\ \gamma > 0,\ (x_1, x_2) \in \mathbb{R}_2^+.$$

□

3.2 Main result

Let us turn to the study of the original equation (1) with a common kernel $\mathcal{P}$ and a common nonlinearity $G(x_1, x_2, u)$.

First, to represent the main conditions imposed on the function G, we introduce a new function. Let $G_0(u)$ be a continuous on the set $\mathbb{R}^+$ function and

$c_1)$ $G_0(u) \uparrow$ u on $\mathbb{R}^+$, $G_0(0) = 0$,

$c_2)$ $G_0(u)$ is upward convex on $\mathbb{R}^+$, $G_0 \in C(\mathbb{R}^+)$,

c_3) there exists a number $\eta > \sup\limits_{(x_1,x_2)\in\mathbb{R}_2^+} f_\beta(x_1,x_2) := B_0$ such that

$G_0(u) \geq u,\ u \in [0,\ \eta]$.

The properties of $c_1) - c_3)$ imply the existence of a single number $\xi > \eta$ such that

$$G_0(\xi) = \xi - B_0. \tag{42}$$

The approximate graph of the function G_0 is shown in the figure.

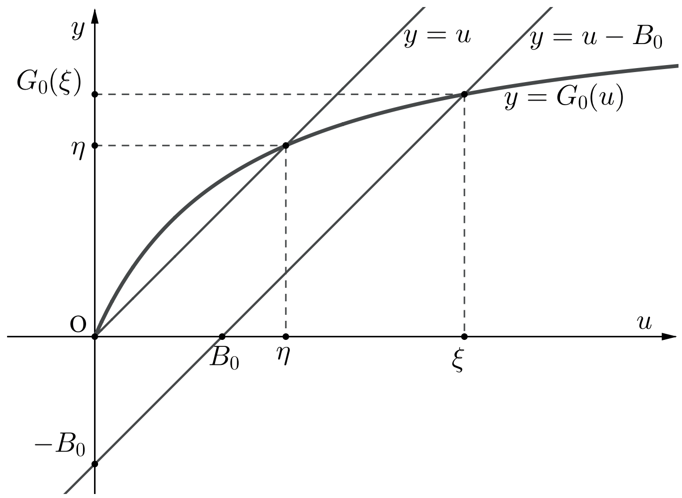

Figure. The approximate graph of the function G_0 on $[0,\ \xi]$.

Regarding the nonlinearity of $G(x_1,x_2,u)$, we assume that the following conditions are satisfied:

n_1) $G(x_1,x_2,u) \uparrow$ in u on $\mathbb{R}^+$ and $G(x_1,x_2,u)$ satisfies the Carathéodory condition on $\mathbb{R}_2^+ \times \mathbb{R}$ by argument u,

n_2) $G(x_1,x_2,u) \geq u + \omega(x_1,x_2,u),\ (x_1,x_2,u) \in \mathbb{R}_3^+$,
where ω has properties $I) - IV)$ and p_2),

n_3) $G(x_1,x_2,u) \leq G_0(u) + \beta(x_1,x_2),\ (x_1,x_2,u) \in \mathbb{R}_2^+ \times [0,\ \xi]$.

The next theorem is valid.

Theorem 4. *Let conditions* $a), b), 1) - 3), p_1), c_1) - c_3)$ *and* $n_1) - n_3)$ *be satisfied. Then the nonlinear integral equation* (1) *has a nonnegative nontrivial solution bounded on* $\mathbb{R}_2^+$.

Proof. Let $\gamma^* := \eta - B_0 > 0$. By Theorem 3 and Remark 2, to the number γ^* the bounded solution $\mathscr{F}^{\gamma^*}(x_1,x_2)$ of the equation (34) corresponds, where $\gamma^* - \mathscr{F}^{\gamma^*} \in L_1(\mathbb{R}_2^+)$ and the double inequality takes place

$$\gamma^* S(x_1,x_2) \leq \mathscr{F}^{\gamma^*}(x_1,x_2) \leq \gamma^* S(x_1,x_2) + f_\beta(x_1,x_2), \quad (x_1,x_2) \in \mathbb{R}_2^+. \tag{43}$$

By the definition of the number γ^* and the inequality $S(x_1,x_2)\leq 1,\ (x_1,x_2)\in\mathbb{R}_2^+$ from (43) it follows that

$$\mathscr{F}^{\gamma^*}(x_1,x_2)\leq\gamma^*+B_0=\eta,\ (x_1,x_2)\in\mathbb{R}_2^+. \tag{44}$$

Let us proceed to the construction of a solution to the equation (1) by successive approximations

$$\begin{aligned}\mathscr{F}_{(n+1)}(x_1,x_2)=\int\limits_0^\infty\int\limits_0^\infty\mathcal{P}(x_1,y_1,x_2,y_2)\times\\ \times G(x_1,x_2,\mathscr{F}_{(n)}(\rho_1(x_1,y_1),\rho_2(x_2,y_2))dy_1dy_2,\\ \mathscr{F}_{(0)}(x_1,x_2)=\mathscr{F}^{\gamma^*}(x_1,x_2),\ n=0,1,2,\ldots,\ (x_1,x_2)\in\mathbb{R}_2^+.\end{aligned} \tag{45}$$

We prove by induction that

$$\mathscr{F}_{(n)}(x_1,x_2)\uparrow\ \text{in } n,\ (x_1,x_2)\in\mathbb{R}_2^+. \tag{46}$$

First, note that, based on (4), (34) and condition n_2), the following chain of inequalities holds:

$$\mathscr{F}_{(1)}(x_1,x_2)\geq\int\limits_0^\infty\int\limits_0^\infty\mathcal{P}(x_1,y_1,x_2,y_2)\Big(\mathscr{F}^{\gamma^*}(\rho_1(x_1,y_1),\rho_2(x_2,y_2))+$$

$$+\omega(x_1,x_2,\mathscr{F}^{\gamma^*}(\rho_1(x_1,y_1),\rho_2(x_2,y_2)))\Big)\,dy_1\,dy_2\geq\lambda(x_1,x_2)\int\limits_0^\infty\int\limits_0^\infty K(y_1,\,y_2)\times$$

$$\times\Big(\mathscr{F}^{\gamma^*}(\rho_1(x_1,y_1),\rho_2(x_2,y_2))+\omega(x_1,x_2,\mathscr{F}^{\gamma^*}(\rho_1(x_1,y_1),\rho_2(x_2,y_2)))\Big)\,dy_1\,dy_2=$$

$$=\mathscr{F}^{\gamma^*}(x_1,x_2)=\mathscr{F}_{(0)}(x_1,x_2),\ (x_1,x_2)\in\mathbb{R}_2^+.$$

Assuming $\mathscr{F}_{(n)}(x_1,x_2)\geq\mathscr{F}_{(n-1)}(x_1,x_2),\ (x_1,x_2)\in\mathbb{R}_2^+$ for some positive integer n, due to the non-negativity of the kernel $\mathcal{P}$ and the condition n_1) from (45) we obtain that $\mathscr{F}_{(n+1)}(x_1,x_2)\geq\mathscr{F}_{(n)}(x_1,x_2),\ (x_1,x_2)\in\mathbb{R}_2^+$.

Let now prove that

$$\mathscr{F}_{(n)}(x_1,x_2)\leq\xi,\ n=0,1,2,\ldots,\ (x_1,x_2)\in\mathbb{R}_2^+. \tag{47}$$

When $n=0$ the inequality (47) is an obvious consequence of the inequalities (44) and $\eta<\xi$. Suppose (47) holds for some $n\in\mathbb{N}$. Then, in view of the conditions $b), n_1), n_3)$ and the definition of the number ξ (see (42)), from (45) we will have

$$\mathscr{F}_{(n+1)}(x_1,x_2)\leq\int\limits_0^\infty\int\limits_0^\infty\mathcal{P}(x_1,y_1,x_2,y_2)\,G(x_1,x_2,\xi)\,dy_1\,dy_2\leq$$

$$\leq (G_0(\xi) + \beta(x_1, x_2)) \int_0^\infty \int_0^\infty \mathcal{P}(x_1, y_1, x_2, y_2)\, dy_1\, dy_2 \leq$$

$$\leq (G_0(\xi) + B_0)\, \mu(x_1, x_2) \leq G_0(\xi) + B_0 = \xi, \quad (x_1, x_2) \in \mathbb{R}_2^+.$$

Thus, given that (46) and (47) hold, one can assert that the sequence of measurable on $\mathbb{R}_2^+$ functions $\{\mathscr{F}_{(n)}(x_1, x_2)\}_{n=0}^\infty$ has a pointwise limit as $n \to \infty$: $\lim\limits_{n\to\infty} \mathscr{F}_{(n)}(x_1, x_2) = \mathscr{F}(x_1, x_2)$, and the limit function $\mathscr{F}(x_1, x_2)$ satisfies the equation (1) (due to B. Levi's theorem) and the double inequality

$$\mathscr{F}^{\gamma^*}(x_1, x_2) \leq \mathscr{F}(x_1, x_2) \leq \xi, \ (x_1, x_2) \in \mathbb{R}_2^+.$$

This completes the proof. □

3.3 Examples

In the end of the work, we provide concrete illustrative examples of the functions $\{\rho_j\}_{j=1,2}$, ω, λ, K, G_0, G and $\mathcal{P}$ satisfying all assumptions of the formulated theorems.

Examples of functions $\{\rho_j\}_{j=1,2}$:

$A_1)$ $\rho_j(u, v) = u + v, \ (u, v) \in \mathbb{R}_2^+, \ j = 1, 2,$

$A_2)$ $\rho_j(u, v) = u\,(1 + \alpha_j v) + \beta_j v, \ (u, v) \in \mathbb{R}_2^+, \ j = 1, 2,$
where $\alpha_j \geq 0, \ \beta_j \geq 1$ are numerical parameters, $j = 1, 2,$

$A_3)$ $\rho_j(u, v) = (u + \varepsilon_j)e^v + 2(1 - e^{-v}), \ (u, v) \in \mathbb{R}_2^+, \ j = 1, 2,$
where $\varepsilon_j \geq 1$ is a numerical parameter, $j = 1, 2.$

Examples of functions ω:

$B_1)$ $\omega(x_1, x_2, u) = \beta(x_1, x_2)(1 - e^{-u}), \quad (x_1, x_2, u) \in \mathbb{R}_3^+,$

$B_2)$ $\omega(x_1, x_2, u) = \beta(x_1, x_2)\dfrac{u}{u+1}, \quad (x_1, x_2, u) \in \mathbb{R}_3^+.$

Examples of functions λ:

$D_1)$ $\lambda(x_1, x_2) = 1 - e^{-(x_1+x_2)}, \quad (x_1, x_2) \in \mathbb{R}_2^+,$

$D_2)$ $\lambda(x_1, x_2) = 1 - \varepsilon e^{-(x_1^2+x_2^2)}, \quad (x_1, x_2) \in \mathbb{R}_2^+, \quad 0 < \varepsilon \leq 1$ is a parameter.

Examples of kernel K:

$E_1)$ $K(y_1,\, y_2) = \dfrac{4}{\pi} e^{-(y_1^2+y_2^2)}, \quad (y_1,\, y_2) \in \mathbb{R}_2^+,$

$E_2)$ $K(y_1,\, y_2) = \int\limits_a^b e^{-(y_1+y_2)s}\, Q(s)\, ds, \quad (y_1,\, y_2) \in \mathbb{R}_2^+,$
where $Q(s) > 0$ is a continuous function on $[a,\, b), 0 < a < b \leq +\infty$, and $\int\limits_a^b \frac{G(s)}{s^2} ds = 1.$

Examples of nonlinearity G_0:

H_1) $G_0(u) = \sqrt[p]{u},\ u \in \mathbb{R}^+,\ p > 2$ is an arbitrary odd number,

H_2) $G_0(u) = d(1 - e^{-u}),\ u \in \mathbb{R}^+,\ d > 1$ is a numeric parameter.

Examples of kernel $\mathcal{P}$:

L_1) $\mathcal{P}(x_1, y_1, x_2, y_2) = \lambda(x_1, x_2)K(y_1,\ y_2),\quad (x_1, y_1, x_2, y_2) \in \mathbb{R}_4^+,$

L_2) $\mathcal{P}(x_1, y_1, x_2, y_2) = \lambda(x_1, x_2)K(y_1,\ y_2)+K_0(x_1, y_1, x_2, y_2),\ (x_1, y_1, x_2, y_2) \in \mathbb{R}_4^+,$ where $K_0(x_1, y_1, x_2, y_2) \geq 0,\ (x_1, y_1, x_2, y_2) \in \mathbb{R}_4^+$ and $\int_0^\infty\int_0^\infty K_0(x_1, y_1, x_2, y_2)\,dy_1\,dy_2 = \varepsilon(1 - \lambda(x_1, x_2)),\ 0 < \varepsilon < 1$ is a parameter.

Examples of nonlinearity G :

U_1) $G(x_1, x_2, u) = G_0(u) + \omega(x_1, x_2, u),\quad (x_1, x_2, u) \in \mathbb{R}_2^+ \times \mathbb{R},$

U_2) $G(x_1, x_2, u) = \sqrt{(u + \omega(x_1, x_2, u))(G_0(u) + \omega(x_1, x_2, u))},\ (x_1, x_2, u) \in \mathbb{R}_2^+ \times \mathbb{R},$

U_3) $G(x_1, x_2, u) = \frac{1}{2}(G_0(u) + u) + \omega(x_1, x_2, u),\quad (x_1, x_2, u) \in \mathbb{R}_2^+ \times \mathbb{R}.$

In conclusion, we note that among the above examples, the most important and most frequently encountered in applications of mathematical physics and mathematical biology are A_1), B_1), D_1), E_1), E_2), L_1), H_1), H_2) and U_1).

Remark 3. Unfortunately, the question of the uniqueness of the solution of the general nonlinear integral equation (1) in certain cone segments (functions bounded on $\mathbb{R}_2^+$) is still open problem.

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Nuclear Identification of Some New Loop Identities of Length Five

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Abstract. In this work, we discovered a dozen of new loop identities we called identities of 'second Bol-Moufang type'. This was achieved by using a generalized and modified nuclear identification model originally introduced by Drápal and Jedlička. Among these twelve identities, eight of them were found to be distinct (from well known loop identities), among which two pairs axiomatize the weak inverse property power associative conjugacy closed (WIP PACC) loop. The four other new loop identities individually characterize the Moufang identities in loops. Thus, now we have eight loop identities that characterize Moufang loops. We also discovered two (equivalent) identities that describe two varieties of Buchsteiner loops. In all, only the extra identities which the Drápal and Jedlička nuclear identification model tracked down could not be tracked down by our own nuclear identification model. The dozen laws $\{Q_i\}_{i=1}^{12}$ induced by our nuclear identification form four cycles in the following sequential format: $\left(Q_{4i-j}\right)_{i=1}^{3}$, $j = 0, 1, 2, 3$, and also form six pairs of dual identities. With the help of twisted nuclear identification, we discovered six identities of lengths five that describe the abelian group variety and commutative Moufang loop variety (in each case). The second dozen identities $\{Q_i^*\}_{i=1}^{12}$ induced by our twisted nuclear identification were also found to form six pairs of dual identities. Some examples of loops of smallest order that obey non-Moufang laws (which do not necessarily imply the other) among the dozen laws $\{Q_i\}_{i=1}^{12}$ were found.

Mathematics subject classification: 20N02, 20N05.
Keywords and phrases: Bol-Moufang type of loop, nuclear identification, Moufang loop, extra loop, Bol loop, left (right) conjugacy closed loop, Buchsteiner loop. .

1 Introduction

The first classification of the varieties of loops of Bol-Moufang was done by Fenyves in [11, 12] and concluded by Phillips and Vojtěchovský in [34, 35]. Jaíyéọlá et al. [25–27] and Ilojide et al. [14] used the identities therein to classify varieties of quasi neutrosophic triplet loops (called Fenyves BCI-Algebras) and also to study their isotopy and holomorphy. We shall refer to the identities described by the Bol-Moufang type of loops as 'first Bol-Moufang type' while we shall introduce what we call 'second Bol-Moufang type' of loops.

An identity of length four is said to be of Bol-Moufang type (first Bol-Moufang type) if:

1. It has 3 distinct variables with one of them appearing twice on both sides.

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2. The variables appear in the same order on both sides.

Coté et al. [7] and Akhtar et al. [2] classified loops of generalized Bol-Moufang type.

An identity of length four is said to be of generalized Bol-Moufang type if:

1. It has 3 distinct variables with one of them appearing twice on both sides.
2. The variables do not necessarily appear in the same order on both sides.

One of such loops of generalized Bol-Moufang type, namely Frute loops were studied by Jaíyéọlá et al. [24, 28].

An identity of length five will be said to be of second Bol-Moufang type if:

1. It has 3 distinct variables with one of them appearing 3 times.
2. The variables appear in the same order on both sides.

Two of such loops of second Bol-Moufang type are described by the identities

$$(xy\cdot x)\ \cdot xz = x((yx\cdot x)z), \qquad \text{(LWPC)}$$
$$zx\ \cdot (x\cdot yx) = (z(x\cdot xy))x \qquad \text{(RWPC)}$$

which Phillips [32] showed axiomatize the variety of loops that are weak inverse property power associative conjugacy closed (WIP PACC) loops. George et al. [13] studied loops that obey LWPC (RWPC) identity and were able to link them up with some loop identities that are not of Bol-Moufang type.

Theorem 1.1. *(George et al. [13])*

Let Q be a loop.

1. *Q is an LWPC-loop if and only if Q is an LCC-loop and $\underbrace{(xy\cdot x)x = x(yx\cdot x)}_{P_\lambda(x,y)}$.*
2. *Q is a RWPC-loop if and only if Q is an RCC-loop and $\underbrace{x(x\cdot yx) = (x\cdot xy)x}_{P_\rho(x,y)}$.*
3. *A CC-loop Q is a power associative WIP-loop if and only if Q fulfills the laws $P_\lambda(x,y)$ and $P_\rho(x,y)$.*

Drápal and Jedlička [9] investigated interactions between loop nuclei and loop identities. With the aid of nuclear identification, they considered some varieties of loops of first Bol-Moufang type and non-Bol-Moufang type in which not all the nuclei necessarily coincide. Drapal and Kinyon [10] recently used nuclear identification to obtain the identities of Osborn loops.

2 Preliminaries

A quasigroup $(Q,\cdot)$ consists of a non-empty set Q with a binary operation $\cdot$ on Q such that given $a,b\in Q$, the equations $ax=b$ and $ya=b$ have unique solutions $x,y\in Q$ respectively. We shall sometimes refer to $(Q,\cdot)$ as simply Q.

For any $x\in Q$, define the right translation map $R(x)$ and left translation map $L(x)$ of x in $(Q,\cdot)$ by $yR(x)=y\cdot x=yx$ and $yL(x)=x\cdot y=xy$, respectively. It is clear that $(Q,\cdot)$ is a quasigroup if and only if the left and right translation maps are bijections. Since the translation maps are bijections, then the inverse maps $R^{-1}(x)$ and $L^{-1}(x)$ exist and are thus defined by $yR^{-1}(x)=y/x$ and $yL^{-1}(x)=x\backslash y$.

A loop $(Q,\cdot)$ is a quasigroup with an identity element, 1, such that $1x=x1=x$, for all $x\in Q$. The right and left inverse maps $\rho: x\mapsto x^{\rho}$ and $\lambda: x\mapsto x^{\lambda}$ are unary operations that take an element x in a loop to its right and left inverses x^{ρ} and x^{λ} respectively, such that $x\cdot x^{\rho}=1=x^{\lambda}\cdot x$. A loop in which $x^{\rho}=x^{\lambda}$ for all elements x is said to have 2-sided inverse. See [5,6,17,31] for a general overview on quasigroups and loops.

A loop is a weak inverse property loop if it satisfies any one of the following identities:

$$x(yx)^{\rho}=y^{\rho}\quad\text{or}\quad (xy)^{\lambda}x=y^{\lambda}. \tag{1}$$

A loop $(Q,\cdot)$ is called a left Bol (right Bol) loop if for all $x,y,z\in Q$ it satisfies

$$(x\cdot yx)z=x(y\cdot xz), \tag{LB}$$

$$(yx\cdot z)x=y(xz\cdot x). \tag{RB}$$

A loop $(Q,\cdot)$ is called a Moufang loop if for all $x,y,z\in Q$ any of the following identities is satisfied

$$(xz\cdot x)y=x(z\cdot xy), \tag{LM}$$

$$(yx\cdot z)x=y(x\cdot zx), \tag{RM}$$

$$xy\cdot zx=(x\cdot yz)x, \tag{MM2}$$

$$xy\cdot zx=x(yz\cdot x). \tag{MM1}$$

A loop is said to be conjugacy closed (CC-loop) if it satisfies the two identities:

$$(xy)/x\ \cdot xz=x(yz), \tag{LCC}$$

$$zx\ \cdot x\backslash(yx)=(zy)x. \tag{RCC}$$

A loop $(Q,\cdot)$ is called a left central loop (LC-loop) if it satisfies the following identity for all $x,y,z\in Q$:

$$(x\cdot xy)z=x(x\cdot yz). \tag{2}$$

A loop $(Q,\cdot)$ is called a right central loop (RC-loop) if for all $x,y,z\in Q$ it satisfies the identity

$$y(zx\cdot x)=(yz\cdot x)x. \tag{3}$$

$(Q, \cdot)$ is called a central loop (C-loop) if it satisfies the identity

$$(yx \cdot x)z = y(x \cdot xz). \tag{4}$$

LC-loops, RC-loops and C-loops are among the varieties of loops of first Bol-Moufang type. Phillips and Vojtěchovský [35, 36], Kinyon et al. [37], Ramamurthi and Solarin [38], Jaiyéọlá [15,16], Adéníran and Jaiyéọlá [1], Jaiyéọlá and Adéníran [19–22] and Solarin [40], Beg [3, 4] have studied them. Fenyves [12] gave three equivalent identities that define each of LC-loops and RC-loops, and only one identity that defines C-loops. But, Phillips and Vojtěchovský [35] gave four equivalent identities that define each of LC-loops and RC-loops. Three of the four identities given by Phillips and Vojtěchovský are the same as the three already given by Fenyves. Jaíyéọlá [18] introduced and studied the generalized forms of LC-loop, RC-loop and C-loops. Jaiyéọlá and Adéníran [23] characterized Osborn-Buchsteiner loops with a new identity that is obeyed by LC-loop.

A loop $(Q, \cdot)$ is called a Buchsteiner loop, if for all $x, y, z \in Q$

$$(BUCH) \qquad x\backslash(xy \cdot z) = (y \cdot zx)/x. \tag{5}$$

A loop is power associative if subloops generated by every single element are groups.

The left alternative property (LAP) of a loop is defined as $xx \cdot y = x \cdot xy$, the right alternative property (RAP) is given by $y \cdot xx = yx \cdot x$. A loop is an alternative loop if it is left and right alternative. Flexible loops satisfy $x \cdot yx = xy \cdot x$. A loop Q is said to have the 3-power associative (3-PA) property if $xx \cdot x = x \cdot xx$.

A loop Q satisfies the left inverse property (LIP) if $x^\lambda \cdot xy = y$ and the right inverse property (RIP) if $xy \cdot y^\rho = x$. An inverse property loop is a loop that satisfies both the (LIP) and the (RIP).

The left nucleus N_λ, the middle nucleus N_μ and the right nucleus N_ρ of a loop Q are defined by

$$N_\lambda(Q) = \{a \in Q : \ a \cdot xy = ax \cdot y \ \ \forall \ x, \ y \in Q\},$$
$$N_\mu(Q) = \{a \in Q : \ xa \cdot y = x \cdot ay \ \ \forall \ x, \ y \in Q\},$$
$$N_\rho(Q) = \{a \in Q : \ xy \cdot a = x \cdot ya \ \ \forall \ x, \ y \in Q\}.$$

The intersection

$$N(Q) = N_\rho(Q) \cap N_\lambda(Q) \cap N_\mu(Q)$$

is called the nucleus of Q.

A triple of bijections $(U, \ V, \ W)$ is called an autotopism of a loop Q provided that

$$xU \cdot yV = (xy)W \tag{6}$$

for all $x, \ y \in Q$. The set of such triples forms a group $Atp(Q)$ called the autotopism group of Q.

It is easy to see that

$$a \in N_\lambda(Q) \Leftrightarrow (L(a),\ I,\ L(a)) \in Atp(Q), \tag{7}$$

$$a \in N_\mu(Q) \Leftrightarrow (R^{-1}(a),\ L(a),\ I) \in Atp(Q), \tag{8}$$

$$a \in N_\rho(Q) \Leftrightarrow (I,\ R(a),\ R(a)) \in Atp(Q). \tag{9}$$

Denote the autotopisms of (7), (8) and (9) by $\alpha_\lambda(x), \alpha_\mu(x)$ and $\alpha_\rho(x)$ respectively.

By generalizing and modifying the nuclear identification model in [9], we say a loop identity is nuclear identifiable if it can be expressed using autotopisms $\alpha_a^i(x)\alpha_b^j(x)\alpha_c^k(x)\alpha_d^l(x)$, where $i,\ j,\ k,\ l \in \{-1,\ 1\}$ and $a,\ b,\ c,\ d \in \{\lambda,\ \rho,\ \mu\}$.

We now state some existing results which we shall be using in this work.

Lemma 2.1. *[33]*

Let Q be a loop. The following are equivalent for any $x \in Q$:

1. Q is a WIPL. *2. $R^{-1}(x) = \rho L(x)\lambda$.* *3. $L^{-1}(x) = \lambda R(x)\rho$.*

Theorem 2.2. *[33]*

Let Q be a WIPL. If $(U,\ V,\ W) \in Atp(Q)$:

1. $(V,\ \lambda W\rho,\ \lambda U\rho) \in Atp(Q)$. *2. $(\rho W\lambda,\ U,\ \rho V\lambda) \in Atp(Q)$.*

In this work, we shall consider some loops of second Bol-Moufang type. We shall investigate loops of length five with two coinciding nuclei relative to weak inverse property.

3 Main Results

3.1 Nuclear Identification

Definition 3.1. *Let Q be a loop which obeys an identity $Id = Id_\alpha$ where Id is equivalently expressible by the autotopism α. Let $\alpha_\lambda(x) = (L(x),\ I,\ L(x))$, $\alpha_\rho(x) = (I,\ R(x),\ R(x))$ and $\alpha_\mu(x) = (R^{-1}(x),\ L(x),\ I)$. Then the identity $Id = Id_\alpha$ is said to be nuclear identifiable in Q if α can be expressed as $\alpha_\eta^\epsilon(x)\alpha_\xi^\omega(x)\alpha_\chi^\kappa(x)\alpha_\zeta^\psi(x)$, where $\epsilon,\ \omega,\ \kappa,\ \psi \in \{-1,\ 1\}$ and $\eta,\ \xi,\ \chi,\ \zeta \in \{\lambda,\ \rho,\ \mu\}$.*

We shall code such identity $Id = Id_\alpha$ as $(\eta,\ \xi,\ \chi,\ \zeta\ ;\epsilon,\ \omega,\ \kappa,\ \psi)$ and replace 1 and -1 by $+$ and $-$ in concrete instances. Using Definition 3.1, a dozen identities of second Bol-Moufang type which are directly or indirectly affiliated with the loop identities induced by nuclear identification in [9] (except the extra identities) are presented (cf. Table 4).

Lemma 3.2. *Let Q be a loop satisfying an identity $(\eta,\ \xi,\ \chi,\ \zeta;\ \epsilon,\ \omega,\ \kappa,\ \psi)$ such that $\zeta = \eta = \xi \neq \chi$. Then $N_\eta = N_\chi$.*

Proof. Let Q be a loop satisfying an identity $(\eta,\ \xi,\ \chi,\ \zeta;\ \epsilon,\ \omega,\ \kappa,\ \psi)$ which is equivalently expressible by the autotopism α. Then $\alpha = \alpha_\eta^\epsilon(x)\alpha_\xi^\omega(x)\alpha_\chi^\kappa(x)\alpha_\zeta^\psi(x)$, where $\epsilon,\ \omega,\ \kappa,\ \psi \in \{-1,\ 1\}$ and $\eta,\ \xi,\ \chi,\ \zeta \in \{\lambda,\ \rho,\ \mu\}$. With the hypothesis $\zeta = \eta = \xi \neq \chi$, $N_\eta = N_\chi$. □

Definition 3.3. *A triple $\alpha \in SYM(Q)^3$ of a loop Q is said to be an I-shift of an $(U,\ V,\ W) \in Atp(Q)$ if $\alpha = (\rho W\lambda,\ U,\ \rho V\lambda)$.*

Theorem 3.4. *The dozen loop identities $\{Q_i\}_{i=1}^{12}$ of Table 4 induced by the nuclear identification of Definition 3.1 form the following cycles in a WIPL:*
(Q_1, Q_5, Q_9), (Q_2, Q_6, Q_{10}), (Q_3, Q_7, Q_{11}), (Q_4, Q_8, Q_{12}) in which each of the identities is the I-shift of the preceding.

Proof. Based on Definition 3.3, for any loop Q, the I-shift of any $(U,\ V,\ W) \in Atp(Q)$ will give a triple $(\rho W\lambda,\ U,\ \rho V\lambda) \in SYM(Q)^3$ which is not necessarily in $Atp(Q)$. For $(\rho W\lambda,\ U,\ \rho V\lambda) \in Atp(Q)$, Q must be a WIPL going by Lemma 2.1 and Theorem 2.2. Note that $\rho = \lambda$ in all the loops identified by nuclear identification in Definition 3.1.

The autotopic equivalence of Q_1 is $(R^{-2}(x)L^{-1}(x)R(x), L(x), L^{-1}(x))$ and its I-shift is

$$\big(\rho L^{-1}(x)\lambda,\ R^{-2}(x)L^{-1}(x)R(x), \rho L(x)\lambda\big) = \big(R(x), R^{-2}(x)L^{-1}(x)R(x), R^{-1}(x)\big)$$

which is the autotopic equivalence of Q_5. Furthermore, the I-shift of the autotopism of Q_5 is $\big(L(x), R(x), L^2(x)R(x)L^{-1}(x)\big)$ and this characterizes Q_9. The I-shift of the autotopism of Q_9 gives the autotopism of Q_1.

Similarly, the autotopism of Q_2 is $\big(R(x), L^{-2}(x)R^{-1}(x)L(x), R^{-1}(x)\big)$ and the I-shift of this autotopism is

$$\big(\rho R^{-1}(x)\lambda, R(x), \rho L^{-2}(x)R^{-1}(x)L(x)\lambda\big) = \big(L(x), R(x), R^2(x)L(x)R^{-1}(x)\big)$$

which is the autotopism that characterizes Q_6. Computing the I-shift of the autotopism of Q_6 gives $\big(L^{-2}(x)R^{-1}(x)L(x), L(x), L^{-1}(x)\big)$ which is the autotopism for Q_{10}. The I-shift of the autotopism of Q_{10} gives the autotopism for Q_2.

The arguments for the other two cycles are similar. □

Lemma 3.5. *The dozen loop identities $\{Q_i\}_{i=1}^{12}$ of Table 4 induced by the nuclear identification of Definition 3.1 are made up of six pairs of dual identities:*
$\{Q_1, Q_2\}, \{Q_3, Q_4\}, \{Q_5, Q_{10}\}, \{Q_6, Q_9\}, \{Q_7, Q_{12}\}, \{Q_8, Q_{11}\}$,

Proof. This follows by checking the identities of $\{Q_i\}_{i=1}^{12}$ in Table 4 for duality. □

Corollary 3.6. *Let Q be a loop. The following are equivalent to each other:*
(a) Q obeys Q_1 and WIP, (b) Q obeys Q_2 and WIP, (c) Q is a Moufang loop.

Proof. This follows from Theorem 3.4. □

3.2 Loops of Second Bol-Moufang Type

Lemma 3.7.

1. *In a loop, each of the following identities Q_1, Q_3, Q_7, Q_8 implies $P_\lambda(x,y)$.*
2. *In a loop, each of the following identities Q_2, Q_4, Q_{11}, Q_{12} implies $P_\rho(x,y)$.*
3. *A loop in which any of the identittes Q_5, Q_6, Q_9, Q_{10} is obeyed is a flexible loop.*
4. *A flexible loop obeys $P_\lambda(x,y)$ and $P_\rho(x,y)$.*
5. *Any loop that obeys $P_\lambda(x,y)$ or $P_\rho(x,y)$ has 2-sided inverse.*

Proof. This is easily achieved by using the identities in a loop. □

Theorem 3.8. *Let Q be a loop.*

1. *Q is a Q_1-loop if and only if Q is a left Bol loop and $P_\lambda(x,y)$ is satisfied.*
2. *Q is a Q_2-loop if and only if Q is a right Bol loop and $P_\rho(x,y)$ is satisfied.*
3. *Q is a Q_3-loop if and only if Q is an LCC-loop and $P_\lambda(x,y)$ is satisfied.*
4. *Q is a Q_4-loop if and only if Q is an RCC-loop and $P_\rho(x,y)$ is satisfied.*
5. *Q is a Q_5-loop if and only if Q is a right Moufang loop.*
6. *Q is a Q_6-loop if and only if Q is an MM1-loop if and only if Q is a MM2-loop.*
7. *Q is a Q_7-loop if and only if Q is an RCC and $P_\lambda(x,y)$ is satisfied.*
8. *Q is a Q_8-loop if and only if Q is a Buchsteiner loop and $P_\lambda(x,y)$ is satisfied.*
9. *Q is a Q_9-loop if and only if Q is an MM1-loop if and only if Q is an MM2-loop.*
10. *Q is a Q_{10}-loop if and only if Q is an LM-loop.*
11. *Q is a Q_{11}-loop if and only if Q is a Buchsteiner loop and $P_\rho(x,y)$ is satisfied.*
12. *Q is a Q_{12}-loop if and only if Q is an LCC-loop and $P_\rho(x,y)$ is satisfied.*
13. *Q is a WIP PACC-loop if and only if Q is a Q_3-loop and a Q_4-loop if and only if Q is a Q_7-loop and a Q_{12}-loop.*
14. *Q is a Q_8-loop if and only if Q is a Q_{11}-loop.*

Proof. 1. Let Q be a Q_1-loop, then by Lemma 3.7(1), $P_\lambda(x,y)$ is satisfied. Note that

$$x(yx \cdot xz) = (x(yx \cdot x))z \Rightarrow$$
$$x(y \cdot xz) = (x \cdot yx)z.$$

Conversely, suppose Q is a left Bol loop and $P_\lambda(x,y)$ is satisfied. Then,

$$\begin{aligned} x(y\cdot xz) &= (x\cdot yx)z \Rightarrow \\ x(yx\cdot xz) &= (x(yx\cdot x))z \Rightarrow \\ x(yx\cdot xz) &= ((xy\cdot x)x)z. \end{aligned}$$

2. This follows from the mirror argument of 1.

3. Let Q be a Q_3-loop, then by Lemma 3.7(1), $P_\lambda(x,y)$. So, identity Q_3 becomes

$$\begin{aligned} (xy\cdot x)\cdot xz &= x((x\backslash((xy\cdot x)x))z) \Rightarrow \\ y\cdot xz &= x((x\backslash(yx))z) \Rightarrow \\ (xy)/x\ \cdot xz &= x(yz). \end{aligned}$$

For the converse, suppose Q is an LCC-loop, then

$$\begin{aligned} y\cdot xz &= x((x\backslash(yx))z) \Rightarrow \\ (xy\cdot x)\cdot xz &= x((x\backslash((xy\cdot x)x))z). \end{aligned}$$

This last identity becomes Q_3 since Q also satisfies $x\backslash((xy\cdot x)x) = yx\cdot x$.

4. This can be proved by mirroring the argument in 3 above.

5. Assume Q is a Q_5-loop, then by Lemma 3.7(3), Q is flexible. Thus,

$$\begin{aligned} (yx\cdot zx)x &= y((xz\cdot x)x) \Rightarrow \\ (yx\cdot zx)x &= y((x\cdot zx)x) \Rightarrow \\ (yx\cdot z)x &= y(xz\cdot x) \\ &= y(x\cdot zx). \end{aligned}$$

Conversely, suppose Q is a right Moufang loop, then

$$\begin{aligned} (yx\cdot z)x &= y(x\cdot zx) \Rightarrow \\ (yx\cdot z)x &= y(xz\cdot x) \Rightarrow \\ (yx\cdot zx)x &= y((x\cdot zx)x) \Rightarrow \\ (yx\cdot zx)x &= y((xz\cdot x)x). \end{aligned}$$

6. Let Q be a Q_6-loop, then by Lemma 3.7 (3), Q is flexible, and by Lemma 3.7, Q_6 satisfies $P_\lambda(x,y)$. Therefore,

$$\begin{aligned} (xy\cdot zx)x &= x((yz\cdot x)x) \Rightarrow \\ (xy\cdot zx)x &= ((x\cdot yz)x)x \Rightarrow \\ (xy\cdot zx) &= (x\cdot yz)x. \end{aligned}$$

Conversely, suppose Q is an MM2-loop, then

$$\begin{aligned}(xy \cdot zx) &= (x \cdot yz)x \Rightarrow \\ (xy \cdot zx)x &= ((x \cdot yz)x)x \Rightarrow \\ (xy \cdot zx)x &= x(yz \cdot x)x = x((yz \cdot x)x).\end{aligned}$$

Again, if Q is a Q_6-loop, then

$$\begin{aligned}(xy \cdot zx)x &= x((yz \cdot x)x) \Rightarrow \\ (xy \cdot zx)x &= (x(yz \cdot x))x \Rightarrow \\ xy \cdot zx &= x(yz \cdot x).\end{aligned}$$

Thus, Q is an MM1-loop. Conversely, suppose Q is a MM1-loop, then

$$\begin{aligned}(xy \cdot zx) &= x(yz \cdot x) \Rightarrow \\ (xy \cdot zx)x &= (x(yz \cdot x))x \Rightarrow \\ (xy \cdot zx)x &= x((yz \cdot x)x).\end{aligned}$$

7. Let Q be a Q_7-loop, then by Lemma 3.7(2), Q satisfies $P_\lambda(x, y)$ or $x\backslash(xy \cdot x)x) = (yx \cdot x)$. Thus,

$$\begin{aligned}(y(xz \cdot x))x &= yx \cdot (zx \cdot x) \Rightarrow \\ (y(xz \cdot x))x &= yx \cdot x\backslash((xz \cdot x)x) \Rightarrow \\ yz \cdot x &= yx \cdot x\backslash zx.\end{aligned}$$

Conversely, let Q be an RCC-loop, then

$$\begin{aligned}yz \cdot x &= yx \cdot x\backslash zx \Rightarrow \\ (y(xz \cdot x))x &= yx \cdot x\backslash((xz \cdot x)x)\end{aligned}$$

and the result follows since Q also satisfies $x\backslash(xy \cdot x)x) = (yx \cdot x)$.

8. Suppose Q is a Q_8-loop, then Q satisfies

$$\begin{aligned}x((y \cdot zx)x) &= ((xy \cdot z)x)x \Rightarrow \\ x(yz \cdot x) &= ((xy \cdot z/x)x)x \Rightarrow \\ ((x(yz \cdot x))/x)/x &= xy \cdot z/x. \qquad (10)\end{aligned}$$

By Lemma 3.7(4), Q satisfies $P_\lambda(x, y)$ or equivalently,

$$((x \cdot yx)/x)/x = x(y/x). \qquad (11)$$

Use (11) in (10) to get

$$x((yz)/x)) = xy \cdot z/x \Rightarrow$$

$$\begin{aligned} x((y\cdot zx)/x)) &= xy\cdot z \Rightarrow \\ (y\cdot zx)/x &= x\backslash(xy\cdot z). \end{aligned}$$

Conversely, suppose Q is a Buchsteiner loop and $P_\lambda(x,y)$ or equivalently $(x\cdot yx)/x)/x = x(y/x)$, then

$$\begin{aligned} (y\cdot zx)/x &= x\backslash(xy\cdot z) \Rightarrow \\ x((y\cdot zx)/x)) &= xy\cdot z \Rightarrow \\ x((yz)/x)) &= xy\cdot z/x \Rightarrow \\ ((x(yz\cdot x))/x)/x &= xy\cdot z/x \Rightarrow \\ x((y\cdot zx)x) &= ((xy\cdot z)x)x. \end{aligned}$$

9. Suppose Q is a Q_9-loop, then

$$\begin{aligned} x(xy\cdot zx) &= (x(x\cdot yz))x \Rightarrow \\ x(xy\cdot zx) &= x((x\cdot yz)x) \Rightarrow \\ xy\cdot zx &= (x\cdot yz)x. \end{aligned}$$

Thus, Q is a MM2-loop. Conversely, suppose Q is MM2-loop, then

$$\begin{aligned} xy\cdot zx &= (x\cdot yz)x \Rightarrow \\ x(xy\cdot zx) &= x((x\cdot yz)x) \Rightarrow \\ x(xy\cdot zx) &= (x(x\cdot yz))x. \end{aligned}$$

Therefore, Q is an Q_9-loop. Now, suppose Q is Q_9, then by Lemma 3.7(1),

$$\begin{aligned} x(xy\cdot zx) &= (x(x\cdot yz))x \Rightarrow \\ x(xy\cdot zx) &= x(x(yz\cdot x)) \Rightarrow \\ xy\cdot zx &= x(yz\cdot x). \end{aligned}$$

Thus, Q is an MM1-loop. Conversely, suppose Q is MMI-loop, then just reverse the process to get Q_9.

10. The proof is similar to the one in 5.

11. The Q_{11} identity is mirror to Q_8 identity, so a mirror argument will suffice.

12. Suppose Q is Q_{12}-loop, then using Lemma 3.7(2) in the Q_{12} identity, we have

$$\begin{aligned} x((x\cdot yx)z) &= (x((x\cdot yx))/x\cdot xz \Rightarrow \\ x\cdot yz &= (xy)/x\cdot xz. \end{aligned}$$

The converse is easy if we reverse the process and use the fact that Q also satisfies $P_\rho(x,y)$.

13. This follows from 7 and 12 of above and Theorem 1.1.

14. Let Q be a Q_8-loop. Then, Q is a Buchsteiner loop in which $P_\lambda(x,y)$ holds by 8. By Lemma 3.7(5), Q is a Buchsteiner loop with 2-sided inverse, hence, a WIP Buchsteiner loop. Applying Theorem 3.4, Q is a Q_{12}-loop and so $P_\rho(x,y)$ holds. Thus, Q is a Q_{11}-loop by 11. The converse is similar. Therefore, Q is a Q_8-loop if and only if Q is a Q_{11}-loop.

□

Lemma 3.9.

1. *In an LC-loop the identity $P_\rho(x,y)$ is satisfied.*

2. *In an RC-loop the identity $P_\lambda(x,y)$ is satisfied.*

3. *In a C-loop the identities $P_\rho(x,y)$ and $P_\lambda(x,y)$ are satisfied.*

4. *In an extra loop, the identities $P_\rho(x,y)$ and $P_\lambda(x,y)$ are satisfied.*

Proof. 1. Put $z = x$ in (2).

2. Put $y = x$ in (3).

3. A loop is a C-loop if and only if it is an LC-loop and RC-loop.

4. An extra loop is a C-loop.

□

Code	Identity	Label	Equivalent Form(s) ($\Leftrightarrow$)
$(\mu,\mu,\lambda,\mu;+,+,-,-)$	$x(yx\cdot xz) = ((xy\cdot x)x)z$	Q_1	LB + $P_\lambda(x,y)$
$(\mu,\mu,\rho,\mu;-,-,-,+)$	$(yx\cdot xz)x = y(x(x\cdot zx))$	Q_2	RB + $P_\rho(x,y)$
$(\mu,\mu,\lambda,\mu;+,+,+,-)$	$(xy\cdot x)\cdot xz = x((yx\cdot x)z)$	Q_3	LWPC=LCC + $P_\lambda(x,y)$
$(\mu,\mu,\rho,\mu;-,-,+,+)$	$yx\cdot(x\cdot zx) = (y(x\cdot xz))x$	Q_4	RWPC=RCC + $P_\rho(x,y)$
$(\rho,\rho,\mu,\rho;-,-,-,+)$	$(yx\cdot zx)x = y((xz\cdot x)x)$	Q_5	RM
$(\rho,\rho,\lambda,\rho;+,+,+,-)$	$(xy\cdot zx)x = x((yz\cdot x)x)$	Q_6	MM1 or MM2
$(\rho,\rho,\mu,\rho;-,-,+,+)$	$(y(xz\cdot x))x = yx\cdot(zx\cdot x)$	Q_7	RCC + $P_\lambda(x,y)$
$(\rho,\rho,\lambda,\rho;+,+,-,-)$	$x((y\cdot zx)x) = ((xy\cdot z)x)x$	Q_8	BUCH + $P_\lambda(x,y)$
$(\lambda,\lambda,\rho,\lambda;+,+,+,-)$	$x(xy\cdot zx) = (x(x\cdot yz))x$	Q_9	MM1 or MM2
$(\lambda,\lambda,\mu,\lambda;-,-,+,+)$	$x(xy\cdot xz) = (x(x\cdot yx))z$	Q_{10}	LM
$(\lambda,\lambda,\rho,\lambda;+,+,-,-)$	$(x(xy\cdot z))x = x(x(y\cdot zx))$	Q_{11}	BUCH + $P_\rho(x,y)$
$(\lambda,\lambda,\mu,\lambda;-,-,-,+)$	$x((x\cdot yx)z) = (x\cdot xy)\cdot xz$	Q_{12}	LCC + $P_\rho(x,y)$

Table 1. Summary of new loop identities induced by nuclear identifications and their equivalent forms

Theorem 3.10. *The variety* $(\eta,\ \xi,\ \chi,\ \zeta\ ;\epsilon,\ \omega,\ \kappa,\ \psi)^*$ *consists of all commutative loops in the variety* $(\eta,\ \xi,\ \chi,\ \zeta\ ;\epsilon,\ \omega,\ \kappa,\ \psi)$*, whenever* $\epsilon,\ \omega,\ \kappa,\ \psi \in \{-1, 1\}$ *and* $\eta,\ \xi,\ \chi,\ \zeta \in \{\rho, \lambda, \mu\}$*, such that* $\zeta = \eta = \xi \neq \chi$*.*

Proof. Let Q be a commutative loop.

If $(A, B, C) \in Atp(Q)$, then $(B, A, C) \in Atp(Q)$. Let $(A(x), B(x), C(x))$ be the autotopisms of the loop varieties described by identities $(\eta,\ \xi,\ \chi,\ \zeta\ ;\epsilon,\ \omega,\ \kappa,\ \psi)$ as given in Table 4 and let $(\eta,\ \xi,\ \chi,\ \zeta\ ;\epsilon,\ \omega,\ \kappa,\ \psi)^*$ be the loop varieties determined by $(B(x), A(x), C(x))$ for all $x \in Q$. Table 5 highlights the identities obtained for these varieties (where ABG stands for the variety of abelian groups and CML represents commutative Moufang loop). It can be easily verified that each of these laws describes a variety of commutative loops (abelian grousp and commutative Moufang loops). □

Corollary 3.11. *Let* Q *be a loop.*

1. *The following are equivalent:*
   (a) Q *is a commutative Moufang loop.*
   (b) Q *obeys* Q_1^* *or* Q_2^* *or* Q_5^* *or* Q_6^* *or* Q_9^* *or* Q_{10}^**.*
2. *The following are equivalent:*
   (a) Q *is an abelian group.*
   (b) Q *obeys* Q_3^* *or* Q_4^* *or* Q_7^* *or* Q_8^* *or* Q_{11}^* *or* Q_{12}^**.*

Proof. This follows from Theorem 3.10 and Table 5. □

Lemma 3.12. *The dozen loop identities* $\{Q_i^*\}_{i=1}^{12}$ *of Table 5 induced by twisted nuclear identification are made up of six pairs of dual identities:*
$\{Q_1^*, Q_2^*\}, \{Q_3^*, Q_4^*\}, \{Q_5^*, Q_{10}^*\}, \{Q_6^*, Q_9^*\}, \{Q_7^*, Q_{12}^*\}, \{Q_8^*, Q_{11}^*\}$.

Proof. This follows by checking the identities of $\{Q_i^*\}_{i=1}^{12}$ in Table 5 for duality. □

3.3 Examples and Constructions

We shall be using the GAP Package [30] and Library of GAP-LOOPS Package [29] to get some examples of non-Moufang, non-extra loops and non-CC-loops that are of second Bol-Moufang type. In GAP-LOOPS, 'LeftBolLoop(n, m)' returns the mth left Bol loop (LBL) of order $n < 17$ while 'RightBolLoop(n, m)' returns mth right Bol loop (RBL) of order $n < 17$ in the library. Similarly, 'RCCLoop(n, m)' returns the mth right conjugacy closed loop (RCCL) of order $n \leq 27$ while 'LCCLoop(n, m)' returns the mth left conjugacy closed loop (LCCL) of order $n \leq 27$ in the library.

1. Any Moufang loop obeys $Q_1, Q_2, Q_5, Q_6, Q_9, Q_{10}$.
2. Any extra loop obeys any of the identities in the set $\{Q_i\}_{i=1}^{12}$.

3. 'LeftBolLoop(8, i)', $i = 1, 2, \cdots, 6$, is an LBL, which is a non-Moufang loop (i.e. does not obey Q_5, Q_6, Q_9, Q_{10}) and obeys $P_\lambda(x, y)$ (hence a Q_1-loop). It also obeys $P_\rho(x, y)$ but is not an RBL. So, it is not a Q_2-loop. See Proposition 3.2 of [13].

4. LeftBolLoop(8, i), $i = 1, 2, \cdots, 6$, is an LCCL, which is non-Moufang and non-CC loop that obeys $P_\lambda(x, y)$ (hence a Q_3-loop). It also obeys $P_\rho(x, y)$ but is not an RCCL. So, it is not a Q_4-loop. Since it obeys $P_\rho(x, y)$ and it is an LCCL, then, it is a Q_{12}-loop. Though, it obeys $P_\lambda(x, y)$ but it is not an RCCL, hence, not a Q_7-loop. See Proposition 3.2 of [13].

5. According to ([9], Lemma 3.6), a Buchsteiner loop is an LCCL iff it is an RCCL. Assume by contradiction that LeftBolLoop(8, i), $i = 1, 2, \cdots, 6$, is a Buchsteiner loop. Since it is an LCCL, then it should be an RCCL which will be a contradiction. So, LeftBolLoop(8, i), $i = 1, 2, \cdots, 6$, is not a Buchsteiner loop. Hence, LeftBolLoop(8, i), $i = 1, 2, \cdots, 6$, is neither a Q_8-loop nor a Q_{11}-loop.

6. Consider the opposite loop of LeftBolLoop(8, i), $i = 1, 2, \cdots, 6$, i.e. LeftBolLoop(8, i)*= RightBolLoop(8, i), $i = 1, 2, \cdots, 6$. It is an RBL, which is non-Moufang loop (i.e. does not obey Q_5, Q_6, Q_9, Q_{10}) and obeys $P_\rho(x, y)$ (hence a Q_2-loop). It also obeys $P_\lambda(x, y)$ but is not an LBL. So, it is not a Q_1-loop.

7. RightBolLoop(8, i), $i = 1, 2, \cdots, 6$, is an RCCL, which is non-Moufang and non-CC loop that obeys $P_\rho(x, y)$ (hence a Q_4-loop). It also obeys $P_\lambda(x, y)$ but is not an LCCL. So, it is not a Q_3-loop. Since it obeys $P_\lambda(x, y)$ and it is an RCCL, then it is a Q_7-loop. Though, it obeys $P_\rho(x, y)$ but it is not an LCCL, hence, not a Q_{12}-loop. One of such loops was constructed in Example 2.1 of [39].

8. According to ([9], Lemma 3.6), a Buchsteiner loop is an RCCL iff it is an LCCL. Assume by contradiction that RightBolLoop(8, i), $i = 1, 2, \cdots, 6$, is a Buchsteiner loop. Since it is an RCCL, then it should be an LCCL which will be a contradiction. So, RightBolLoop(8, i), $i = 1, 2, \cdots, 6$, is not a Buchsteiner loop. Hence, RightBolLoop(8, i), $i = 1, 2, \cdots, 6$, is neither a Q_8-loop nor a Q_{11}-loop.

9. For $n = 6$:
   (a) When $m = 1$, both $P_\lambda(x, y)$ and $P_\rho(x, y)$ are satisfied by RCCLoop(n, m). Hence, it is a Q_4-loop and Q_7-loop. But it is not a $Q_1, Q_2, Q_3, Q_8, Q_{11}, Q_{12}$-loop.
   (b) When $m = 2, 3$, none of $P_\lambda(x, y)$ and $P_\rho(x, y)$ is satisfied by RCCLoop(n, m). Thus, it does not satisfy any of $\{Q_i\}_{i=1}^{12}$.

10. For $n = 8$:

(a) When $m = 1, 2, 3, 7, 8, 9, 13, 15, 16, 17, 18, 19$, none of $P_\lambda(x, y)$ and $P_\rho(x, y)$ is satisfied by RCCLoop(n, m). Thus, it does not satisfy any of $\{Q_i\}_{i=1}^{12}$

(b) When $m = 4, 5, 6, 10, 11, 12$, both $P_\lambda(x, y)$ and $P_\rho(x, y)$ are satisfied by RCCLoop(n, m). It is also an RBL. Hence, it is a Q_2, Q_4-loop and Q_7-loop. But it is not a $Q_1, Q_3, Q_8, Q_{11}, Q_{12}$-loop.

(c) When $m = 14$, RCCLoop(n, m) satisfies $P_\lambda(x, y)$ but does not satisfy $P_\rho(x, y)$. Hence, it is a Q_7-loop but it is not a $Q_1, Q_2, Q_3, Q_4, Q_8, Q_{11}, Q_{12}$-loop.

11. In Theorem 3.1 and Theorem 3.4 of [13], methods of construction of Q_3 loops were described.

Example 3.13. *Let $G = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$ and define $*$ on G as shown in Table 2. $(G, *)$ is a Q_3-loop, Q_4-loop, Q_7-loop, Q_{12}-loop, Q_8-loop, Q_{11}-loop that is power associative, not diassociative, not (Moufang, left Bol, right Bol, LC, RC, C, extra), not (left or right power alternative), right A-loop and left A-loop, not middle A-loop. $(G, *)$ is not a $Q_1, Q_2, Q_5, Q_6, Q_9, Q_{10}$-loop.*

$*$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2	3	4	1	6	7	8	5	10	11	12	9	14	15	16	13
3	3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
4	4	1	2	3	8	5	6	7	12	9	10	11	16	13	14	15
5	5	6	7	8	1	2	3	4	13	14	15	16	11	12	9	10
6	6	7	8	5	2	3	4	1	14	15	16	13	12	9	10	11
7	7	8	5	6	3	4	1	2	15	16	13	14	9	10	11	12
8	8	5	6	7	4	1	2	3	16	13	14	15	10	11	12	9
9	9	10	11	12	16	13	14	15	1	2	3	4	8	5	6	7
10	10	11	12	9	13	14	15	16	2	3	4	1	5	6	7	8
11	11	12	9	10	14	15	16	13	3	4	1	2	6	7	8	5
12	12	9	10	11	15	16	13	14	4	1	2	3	7	8	5	6
13	13	14	15	16	12	9	10	11	7	8	5	6	4	1	2	3
14	14	15	16	13	9	10	11	12	8	5	6	7	1	2	3	4
15	15	16	13	14	10	11	12	9	5	6	7	8	2	3	4	1
16	16	13	14	15	11	12	9	10	6	7	8	5	3	4	1	2

Table 2. A Q_3-loop, Q_4-loop, Q_7-loop, Q_{12}-loop, Q_8-loop, Q_{11}-loop $(G, *)$

Example 3.14. *Let $G = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$ and define $\star$ on G as shown in Table 3. $(G, \star)$ is a Q_3-loop, Q_4-loop, Q_7-loop, Q_{12}-loop, Q_8-loop, Q_{11}-loop that is power associative, not diassociative, not (Moufang, left Bol, right Bol, LC, RC, C, extra), not (left or right power alternative), right A-loop and left A-loop, not middle A-loop. $(G, \star)$ is not a $Q_1, Q_2, Q_5, Q_6, Q_9, Q_{10}$-loop. $(G, *)$ and $(G, \star)$ are neither isomorphic nor isotopic.*

3.4 Discussion, Conclusion and Future Study

Note that $P_\rho(x,y)$ and $P_\lambda(x,y)$ are satisfied by any dissociative loop (e.g. Moufang or extra loop). In fact, each of the identities in $\{Q_i\}_{i=1}^{12}$ generalizes the extra law in loops, but this is not true of the Moufang law in loops. Among all the loop identities tracked down by the nuclear identification code in (Table 1, [9]), only the Moufang identities got tracked down distinctively as new loop identities by the nuclear identification code introduced in this work (see Q_5, Q_6, Q_9, Q_{10} in Table 1). The importance of $P_\rho(x,y)$ and $P_\lambda(x,y)$ in this current work is the fact that they are associated to equivalent forms of the new loop identities which were not tracked down distinctively by our nuclear identification code (see $Q_1, Q_2, Q_3, Q_4, Q_7, Q_8, Q_{11}, Q_{12}$ in Table 1). Among all the loop identities tracked down by the nuclear identification code in (Table 1, [9]), the left (right) Bol, LCC (RCC) and Buchsteiner identities got tracked down non-distinctively as new loop identities by our nuclear identification code (see $Q_1, Q_2, Q_3, Q_4, Q_7, Q_8, Q_{11}, Q_{12}$ in Table 1). Among the 12 identities tracked down by the nuclear identification code in (Table 1, [9]), only the extra identities are missing in our own work. But our work has been able to discover:

1. eight new loop identities (i.e. $Q_1, Q_2, Q_3, Q_4, Q_7, Q_8, Q_{11}, Q_{12}$) among which the two pairs (Q_3, Q_4) and (Q_7, Q_{12}) axiomatize the weak inverse property power associative conjugacy closed (WIP PACC) loop, while Q_8 and Q_{11} were found to be equivalent.

2. four new loop identities which individually characterize the Moufang identities

$\star$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
4	4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13
5	5	6	8	7	1	2	4	3	13	14	16	15	10	9	11	12
6	6	5	7	8	2	1	3	4	14	13	15	16	9	10	12	11
7	7	8	6	5	3	4	2	1	15	16	14	13	12	11	9	10
8	8	7	5	6	4	3	1	2	16	15	13	14	11	12	10	9
9	9	10	12	11	15	16	14	13	1	2	4	3	7	8	6	5
10	10	9	11	12	16	15	13	14	2	1	3	4	8	7	5	6
11	11	12	10	9	13	14	16	15	3	4	2	1	5	6	8	7
12	12	11	9	10	14	13	15	16	4	3	1	2	6	5	7	8
13	13	14	15	16	12	11	10	9	6	5	8	7	4	3	2	1
14	14	13	16	15	11	12	9	10	5	6	7	8	3	4	1	2
15	15	16	13	14	10	9	12	11	8	7	6	5	2	1	4	3
16	16	15	14	13	9	10	11	12	7	8	5	6	1	2	3	4

Table 3. A Q_3-loop, Q_4-loop, Q_7-loop, Q_{12}-loop, Q_8-loop, Q_{11}-loop $(G, \star)$

in loops (i.e. Q_5, Q_6, Q_9, Q_{10}). Thus, we now have eight loop identities that characterize Moufang loop.

Therefore, we have been able to identify a dozen second Bol-Moufang type identities via nuclear identification, among which are first Bol-Moufang type identities (e.g. Moufang) or relations of first Bol-Moufang type identities (RB, LB) or non-Bol-Moufang type identities (LCC, RCC, Buchsteiner).

In [9], loop identities that split into at least two other loop identities (with at least three variables) were nuclear identified (e.g. extra and Moufang). Also, loop identities that do not split into at least two other loop identities (with at least three variables) were also nuclear identified (e.g. left Bol, right Bol, LCC, RCC and Buchsteiner). But, with our own nuclear identification model, loop identities that split into at least two other loop identities (with at least three variables) were nuclear identified (only Moufang) without the company of $P_\rho(x,y)$ or $P_\lambda(x,y)$. While, loop identities that do not split into at least two other loop identities (with at least three variables) were also nuclear identified (e.g. left Bol, right Bol, LCC, RCC and Buchsteiner) with the company of $P_\rho(x,y)$ or $P_\lambda(x,y)$. Thus, $P_\rho(x,y)$ and $P_\lambda(x,y)$ are distinguishing features between our own nuclear identification model and that of [9].

Note that a Q_8-loop and Q_{11}-loop are both Buchsteiner loops with 2-sided inverse. Hence, they are linked to **Buch2SI** in the following chain of varieties of Buchsteiner loops (Csorgo [8]):

$$\mathbf{BuchCS} \subset \mathbf{Buch2SI} \subset \mathbf{BuchWIP} \subset \mathbf{BuchCC}$$

where **BuchCS**, **Buch2SI**, **BuchWIP** and **BuchCC** represent the varieties of Buchsteiner with central square, Buchsteiner with 2-sided inverse, Buchsteiner with WIP and Buchsteiner that is a CC-loop respectively. The identities that describe Q_8-loop and Q_{11}-loop form two varieties of Buchsteiner loops. But we are not sure if the varieties **BuchCS**, **Buch2SI**, **BuchWIP** and **BuchCC** have single identities that describe them.

Just like the dozen laws of (Proposition 1.3, [9]) form four cycles, our dozen laws also form four cycles as well (but in a sequential manner) and also form six pairs of dual identities. Using twisted nuclear identification, the authors in [9] were able to identify six identities of lengths four that describe the abelian group variety and commutative Moufang loop variety (in each case). We also achieved a similar result in this work with the discovery of six identities of length five that describe the abelian group variety and commutative Moufang loop variety (in each case). This second dozen of identities were also found to form six pairs of dual identities.

Code	Autotopism	Identity	Label
$(\mu,\mu,\lambda,\mu;+,+,-,-)$	$(R^{-2}(x)L^{-1}(x)R(x), L(x), L^{-1}(x))$	$x(yx\cdot xz) = ((xy\cdot x)x)z$	Q_1
$(\mu,\mu,\rho,\mu;-,-,-,+)$	$(R(x), L^{-2}(x)R^{-1}(x)L(x), R^{-1}(x))$	$(yx\cdot xz)x = y(x(x\cdot zx))$	Q_2
$(\mu,\mu,\lambda,\mu;+,+,+,-)$	$(R^{-2}(x)L(x)R(x), L(x), L(x))$	$(xy\cdot x)\cdot xz = x((yx\cdot x)z)$	Q_3
$(\mu,\mu,\rho,\mu;-,-,+,+)$	$(R(x), L^{-2}(x)R(x)L(x), R(x))$	$yx\cdot(x\cdot zx) = (y(x\cdot xz))x$	Q_4
$(\rho,\rho,\mu,\rho;-,-,-,+)$	$(R(x), R^{-2}(x)L^{-1}(x)R(x), R^{-1}(x))$	$(yx\cdot zx)x = y((xz\cdot x)x)$	Q_5
$(\rho,\rho,\lambda,\rho;+,+,+,-)$	$(L(x), R(x), R^{2}(x)L(x)R^{-1}(x))$	$(xy\cdot zx)x = x((yz\cdot x)x)$	Q_6
$(\rho,\rho,\mu,\rho;-,-,+,+)$	$(R^{-1}(x), R^{-2}(x)L(x)R(x), R^{-1}(x))$	$(y(xz\cdot x))x = yx\cdot(zx\cdot x)$	Q_7
$(\rho,\rho,\lambda,\rho,+,+,-,-)$	$(L^{-1}(x), R(x), R^{2}(x)L^{-1}(x)R^{-1}(x))$	$x((y\cdot zx)x) = ((xy\cdot z)x)x$	Q_8
$(\lambda,\lambda,\rho,\lambda;+,+,+,-)$	$(L(x), R(x), L^{2}(x)R(x)L^{-1}(x))$	$x(xy\cdot zx) = (x(x\cdot yz))x$	Q_9
$(\lambda,\lambda,\mu,\lambda;-,-,+,+)$	$(L^{-2}(x)R^{-1}(x)L(x), L(x), L^{-1}(x))$	$x(xy\cdot xz) = (x(x\cdot yx))z$	Q_{10}
$(\lambda,\lambda,\rho,\lambda;+,+,-,-)$	$(L(x), R^{-1}(x), L^{2}(x)R^{-1}(x)L^{-1}(x))$	$(x(xy\cdot z))x = x(x(y\cdot zx))$	Q_{11}
$(\lambda,\lambda,\mu,\lambda;-,-,-,+)$	$(L^{-2}(x)R(x)L(x), L^{-1}(x), L^{-1}(x))$	$x((x\cdot yx)z) = (x\cdot xy)\cdot xz$	Q_{12}

Table 4. Summary of new loop identities induced by nuclear identifications

Code	Autotopism	Identity	Variety	Label
$(\mu,\mu,\lambda,\mu;+,+,-,-)^*$	$(L(x), R^{-2}(x)L^{-1}(x)R(x), L^{-1}(x))$	$x(xy\cdot zx) = y((xz\cdot x)x)$	CML	Q_1^*
$(\mu,\mu,\rho,\mu;-,-,-,+)^*$	$(L^{-2}(x)R^{-1}(x)L(x), R(x), R^{-1}(x))$	$(xy\cdot zx)x = (x(x\cdot yx))z$	CML	Q_2^*
$(\mu,\mu,\lambda,\mu;+,+,+,-)^*$	$(L(x), R^{-2}(x)L(x)R(x), L(x))$	$xy\cdot(xz\cdot x) = x(y(zx\cdot x))$	ABG	Q_3^*
$(\mu,\mu,\rho,\mu;-,-,+,+)^*$	$(L^{-2}(x)R(x)L(x), R(x), R(x))$	$(x\cdot yx)\cdot zx = ((x\cdot xy)z)x$	ABG	Q_4^*
$(\rho,\rho,\mu,\rho;-,-,-,+)^*$	$(R^{-2}(x)L^{-1}(x)R(x), R(x), R^{-1}(x))$	$(yx\cdot zx)x = ((xy\cdot x)x)z$	CML	Q_5^*
$(\rho,\rho,\lambda,\rho;+,+,+,-)^*$	$(R(x), L(x), R^{2}(x)L(x)R^{-1}(x))$	$(yx\cdot xz)x = x((yz\cdot x)x)$	CML	Q_6^*
$(\rho,\rho,\mu,\rho;-,-,+,+)^*$	$(R^{-2}(x)L(x)R(x), R^{-1}(x), R^{-1}(x))$	$((xy\cdot x)z)x = (yx\cdot x)\cdot zx$	ABG	Q_7^*
$(\rho,\rho,\lambda,\rho,+,+,-,-)^*$	$(R(x), L^{-1}(x), R^{2}(x)L^{-1}(x)R^{-1}(x))$	$x((yx\cdot z)x) = ((y\cdot xz)x)x$	ABG	Q_8^*
$(\lambda,\lambda,\rho,\lambda;+,+,+,-)^*$	$(R(x), L(x), L^{2}(x)R(x)L^{-1}(x))$	$x(yx\cdot xz) = (x(x\cdot yz))x$	CML	Q_9^*
$(\lambda,\lambda,\mu,\lambda;-,-,+,+)^*$	$(L(x), L^{-2}(x)R^{-1}(x)L(x), L^{-1}(x))$	$x(xy\cdot xz) = y(x(x\cdot zx))$	CML	Q_{10}^*
$(\lambda,\lambda,\rho,\lambda;+,+,-,-)^*$	$(R^{-1}(x), L(x), L^{2}(x)R^{-1}(x)L^{-1}(x))$	$(x(y\cdot xz))x = x(x(yx\cdot z))$	ABG	Q_{11}^*
$(\lambda,\lambda,\mu,\lambda;-,-,-,+)^*$	$(L^{-1}(x), L^{-2}(x)R(x)L(x), L^{-1}(x))$	$x(y(x\cdot zx)) = xy\cdot(x\cdot xz)$	ABG	Q_{12}^*

Table 5. Loop Identities obtained by twisted nuclear identifications

In the conclusion of [9], the authors pointed out the prospect of possibly using another nuclear identification model to track down the LC, RC and C-loop identities which their own nuclear identification model could not track down (except if a restriction in their code is expunged). It is worth mentioning that even though our own nuclear identification model could not track down the LC, RC and C-loop identities, but Lemma 3.9 informs us that LC, RC loop identities imply $P_\rho(x, y), P_\lambda(x, y)$ respectively. Thus, some other nuclear identification models for identities of length five that could track down the LC, RC and C-loop identities might exist.

Future Studies Definitely, the dozen identities discovered in this work are not the only identities of second Bol-Moufang type. There is the need to know if there are some more others that can be nuclear identified like the twelve of this work. Perhaps, the extra law which we could not nuclear-identify could be nuclear-identifiable among the future loop identities of second Bol-Moufang type.

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B-spline approximation of discontinuous functions defined on a closed contour in the complex plane

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Abstract. In this paper we propose an efficient algorithm for approximating piecewise continuous functions, defined on a closed contour Γ in the complex plane. The function, defined numerically on a finite set of points of Γ, is approximated by a linear combination of B-spline functions and Heaviside step functions, defined on Γ. Theoretical and practical aspects of the convergence of the algorithm are presented, including the vicinity of the discontinuity points.

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1 Introduction and problem formulation

Let Γ be a simple closed contour in the complex plane that includes inside it the origin of coordinates and $f : \Gamma \to C$ is a function defined at the points of this contour. Let the function $f \in PC(\Gamma)$, where $PC(\Gamma)$ is the set of all continuous or piecewise continuous functions on Γ. If the function $f \in PC(\Gamma)$ is discontinuous on Γ, we consider that it has finite jump discontinuities, being left continuous at the discontinuity points.

In multiple practical situations the function f is not defined analytically, but by its values on a finite set of points. In this paper we aim to develop an efficient algorithm for approximating the function $f \in PC(\Gamma)$, defined numerically on the set $\{t_j\}$ of points belonging to the contour Γ.

The proposed approximation algorithm is based on the concept of B-spline functions, defined on the contour Γ. The spline functions, defined on the Jordan curve Γ in the complex plane, have been introduced in the paper [1] and the B-spline functions – in [2]. For B-spline functions, some properties analogous to those that occur for B-splines defined on a segment of the real axis have been proved.

For two points $t_1, t_2 \in \Gamma$ we use the notation $t_1 \prec t_2$ if when traversing the contour Γ in counterclockwise direction we meet first the point t_1, and then t_2 (see Figure 1). Let $t_1 \prec t_2 \prec \ldots \prec t_n$ $(\prec t_1)$ be a set of distinct points of the contour Γ. We denote by $\Gamma_j := arc\,[t_j, t_{j+1}]$ the set of points of the contour Γ, located between the points t_j and t_{j+1}.

Let the positive integers $m, n \geq 2$. The spline function $s(t)$ of order m, defined on the contour Γ, satisfies the following properties:

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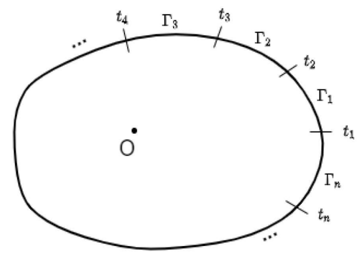

Figure 1: The type of contour and notations used

a) $s \in C^{m-2}(\Gamma)$;

b) the restriction of s on Γ_j for $j = 1, ..., n$ is a polynomial of degree $m-1$.

The set of all spline functions of the order m forms the linear space $S_{m,n}$. In [1] it is shown that any continuous function on Γ can be approximated uniformly on Γ with a linear ($m = 2$) or cubic ($m = 4$) spline function.

In [2] the B-spline functions of order $m \geq 2$ on the contour Γ are defined, based on the recursive formula

$$B_{m,j}(t) := \frac{m}{m-1}\left(\frac{t-t_j}{t_{j+m}-t_j}B_{m-1,j}(t) + \frac{t_{j+m}-t}{t_{j+m}-t_j}B_{m-1,j+1}(t)\right), \ j = 1, ..., n, \tag{1}$$

where $B_{1,j}(t) = \begin{cases} \frac{1}{t_{j+1}-t_j} & \text{if } t \in arc\,[t_j, t_{j+1}) \\ 0 & \text{otherwise} \end{cases}$. Also, it is shown that the set of B-splines $\{B_{m,1}, ..., B_{m,n}\}$ forms a basis of the space $S_{m,n}$ of spline functions on Γ. It follows that any continuous function on Γ can be uniformly approximated on Γ by a linear combination of B-spline functions. Now we intend to study what happens when the approximated function f has discontinuities on Γ.

The case when the piecewise continuous function is defined on a closed interval $[a, b]$ of the real axis is examined in [3]. In this paper it is shown that at the approximation of the discontinuous function $f \in PC([a, b])$ with spline functions of order $m \geq 2$, we do not have uniform convergence, because in the vicinity of the discontinuity points we have strong oscillations of the spline values around the values of the function f. When amplifying the number of nodes on which the spline is built, the amplitude of the oscillations does not tend to zero. When we move away from the points of discontinuity, the approximation becomes uniform and the error can be evaluated based on the relationships established at the approximation of continuous functions. Also, in [3] it is shown that the oscillating effect in the vicinity of discontinuity points can be annihilated if the approximation is constructed as a linear combination of m-order B-spline functions. Moreover, in order to construct a piecewise continuous approximation, which converges uniformly to the function f, a linear combination of B-spline functions and Heaviside step functions is considered.

Next, we apply the approach proposed in [3] and study the convergence of the linear combination of B-splines on Γ to the function $f \in PC(\Gamma)$, and as a result we present an algorithm for approximating the function f. The algorithm is efficient in the sense that it achieves a uniform approximation of the function f on the whole contour Γ, but also due to the fact that it consumes a limited amount of computational resources.

2 Approximation of function by a linear combination of B-splines

Let a closed and piecewise smooth contour Γ be the boundary of the simply connected domain $\Omega^+ \subset C$. Let the point $z = 0 \in \Omega^+$. Consider the Riemann function $z = \psi(w)$, that performs the conformal map of the domain D^- from the outside of the circle $\Gamma_0 := \{w \in C : |w| = 1\}$ onto the domain Ω^- from the outside of the contour Γ, such that $\psi(\infty) = \infty$, $\psi'(\infty) > 0$. The function $\psi(w)$ transforms the circle Γ_0 onto the contour Γ. Next, we consider that the points of the contour Γ are defined by means of the function $\psi(w)$.

Let $\{t_j\}_{j=1}^{n_B}$ be the set of distinct points of the contour Γ where the values of the function $f \in PC(\Gamma)$ are defined. We consider that the points t_j are generated based on the relation

$$t_j = \psi(w_j), \; w_j = e^{i\theta_j}, \; \theta_j = 2\pi(j-1)/n_B, \; j = 1, ..., n_B.$$

Thus, the variation of the parameter θ ensures a uniform coverage of the interval $[0, 2\pi]$ and the points t_j are distributed over the entire contour Γ.

As a set of nodes on which the B-spline functions of order m $(m \leq n_B)$ are constructed (see formula (1)), we consider the set $\left\{t_j^B\right\}_{j=1}^{n_B+m}$, where $t_j^B = t_j$, $j = 1, ..., n_B$, and $t_{n_B+1}^B = t_1^B, t_{n_B+2}^B = t_2^B, ..., t_{n_B+m}^B = t_m^B$.

We construct the approximation of the function $f(t)$ in the form $\varphi_{n_B}(t) := \sum_{k=1}^{n_B} \alpha_k B_{m,k}(t)$, where the coefficients $\alpha_k \in C$, $k = 1, ..., n_B$, are determined imposing the interpolation conditions

$$f\left(t_j^C\right) = \varphi_{n_B}\left(t_j^C\right), \; j = 1, ..., n_B. \tag{2}$$

The set of nodes of the B-spline, arranged in a certain order, is considered as interpolation points t_j^C.

The system of equations (2) can be written as $B\bar{x} = \bar{f}$, where

$$B = \{m_{j,k}\}_{j,k=1}^{n_B}, \; m_{j,k} = B_{m,k}\left(t_j^C\right), \; \bar{x} = \{\alpha_k\}_{k=1}^{n_B}, \; \bar{f} = \left\{f\left(t_j^C\right)\right\}_{j=1}^{n_B}.$$

To approximate the function $f(t)$ we use the B-spline functions of order $m \in \{2, 3, 4\}$. Based on formula (1) one can deduce the following explicit representations for the B-splines $B_{m,k}(t)$ $(k = 1, ..., n_B)$:

For $m = 2$:

$$B_{2,k}(t) = \begin{cases} \dfrac{2\left(t - t_k^B\right)}{\left(t_{k+2}^B - t_k^B\right)\left(t_{k+1}^B - t_k^B\right)} & \text{if } t \in arc\left[t_k^B, t_{k+1}^B\right) \\ \dfrac{2\left(t_{k+2}^B - t\right)}{\left(t_{k+2}^B - t_k^B\right)\left(t_{k+2}^B - t_{k+1}^B\right)} & \text{if } t \in arc\left[t_{k+1}^B, t_{k+2}^B\right) \\ 0 & \text{otherwise} \end{cases}$$

For $m = 3$:

$$B_{3,k}(t) = \begin{cases} \dfrac{3\left(t - t_k^B\right)^2}{\left(t_{k+3}^B - t_k^B\right)\left(t_{k+2}^B - t_k^B\right)\left(t_{k+1}^B - t_k^B\right)} & \text{if } t \in arc\left[t_k^B, t_{k+1}^B\right) \\ 3\left(I_1 + I_2\right) & \text{if } t \in arc\left[t_{k+1}^B, t_{k+2}^B\right) \\ \dfrac{3\left(t_{k+3}^B - t\right)^2}{\left(t_{k+3}^B - t_k^B\right)\left(t_{k+3}^B - t_{k+1}^B\right)\left(t_{k+3}^B - t_{k+2}^B\right)} & \text{if } t \in arc\left[t_{k+2}^B, t_{k+3}^B\right) \\ 0 & \text{otherwise} \end{cases}$$

where

$$I_1 := \frac{\left(t - t_k^B\right)\left(t_{k+2}^B - t\right)}{\left(t_{k+3}^B - t_k^B\right)\left(t_{k+2}^B - t_k^B\right)\left(t_{k+2}^B - t_{k+1}^B\right)},$$

$$I_2 := \frac{\left(t - t_{k+1}^B\right)\left(t_{k+3}^B - t\right)}{\left(t_{k+3}^B - t_k^B\right)\left(t_{k+3}^B - t_{k+1}^B\right)\left(t_{k+2}^B - t_{k+1}^B\right)}.$$

For $m = 4$:

$$B_{4,k}(t) = \begin{cases} \dfrac{4\left(t - t_k^B\right)^3}{\left(t_{k+4}^B - t_k^B\right)\left(t_{k+3}^B - t_k^B\right)\left(t_{k+2}^B - t_k^B\right)\left(t_{k+1}^B - t_k^B\right)} & \text{if } t \in arc\left[t_k^B, t_{k+1}^B\right) \\ 4\left(I_3 + I_4\right) & \text{if } t \in arc\left[t_{k+1}^B, t_{k+2}^B\right) \\ 4\left(I_5 + I_6\right) & \text{if } t \in arc\left[t_{k+2}^B, t_{k+3}^B\right) \\ \dfrac{4\left(t_{k+4}^B - t\right)^3}{\left(t_{k+4}^B - t_k^B\right)\left(t_{k+4}^B - t_{k+1}^B\right)\left(t_{k+4}^B - t_{k+2}^B\right)\left(t_{k+4}^B - t_{k+3}^B\right)} & \text{if } t \in arc\left[t_{k+3}^B, t_{k+4}^B\right) \\ 0 & \text{otherwise} \end{cases}$$

where

$$I_3 := \frac{t - t_k^B}{t_{k+4}^B - t_k^B}\left(I_{3,1} + \frac{\left(t - t_{k+1}^B\right)\left(t_{k+3}^B - t\right)}{\left(t_{k+3}^B - t_k^B\right)\left(t_{k+3}^B - t_{k+1}^B\right)\left(t_{k+2}^B - t_{k+1}^B\right)}\right),$$

$$I_{3,1} := \frac{\left(t - t_k^B\right)\left(t_{k+2}^B - t\right)}{\left(t_{k+3}^B - t_k^B\right)\left(t_{k+2}^B - t_k^B\right)\left(t_{k+2}^B - t_{k+1}^B\right)},$$

$$I_4 := \frac{\left(t_{k+4}^B - t\right)\left(t - t_{k+1}^B\right)^2}{\left(t_{k+4}^B - t_k^B\right)\left(t_{k+4}^B - t_{k+1}^B\right)\left(t_{k+3}^B - t_{k+1}^B\right)\left(t_{k+2}^B - t_{k+1}^B\right)},$$

$$I_5 := \frac{\left(t_{k+3}^B - t\right)^2 \left(t - t_k^B\right)}{\left(t_{k+4}^B - t_k^B\right)\left(t_{k+3}^B - t_k^B\right)\left(t_{k+3}^B - t_{k+1}^B\right)\left(t_{k+3}^B - t_{k+2}^B\right)},$$

$$I_6 := \frac{t_{k+4}^B - t}{t_{k+4}^B - t_k^B}\left(I_{6,1} + \frac{\left(t - t_{k+2}^B\right)\left(t_{k+4}^B - t\right)}{\left(t_{k+4}^B - t_{k+1}^B\right)\left(t_{k+4}^B - t_{k+2}^B\right)\left(t_{k+3}^B - t_{k+2}^B\right)}\right),$$

$$I_{6,1} := \frac{\left(t - t_{k+1}^B\right)\left(t_{k+3}^B - t\right)}{\left(t_{k+4}^B - t_{k+1}^B\right)\left(t_{k+3}^B - t_{k+1}^B\right)\left(t_{k+3}^B - t_{k+2}^B\right)}.$$

It can be seen that the B-spline functions $B_{m,k}(t)$ have the support on the curve arc $\left[t_k^B, t_{k+m}^B\right)$. This leads to a sparse matrix $B = \left\{B_{m,k}\left(t_j^C\right)\right\}_{j,k=1}^{n_B}$ in the system of equations (2). On the one hand, it can be considered as an advantage because small computational resources can be involved when calculating the solution to the system (2). On the other hand, it is possible that the determinant of the matrix B to be equal to zero.

The location of the interpolation points t_j^C on the contour Γ has a direct influence on the conditioning of the matrix $B = \{m_{j,k}\}_{j,k=1}^{n_B}$ in the system (2). In order to ensure the good conditioning of the matrix B, it is proposed the interpolation points t_j^C to be selected as follows.

For $m = 2$ we consider $t_j^C = t_{j+1}^B,\ j = 1, ..., n_B$, and in this case the matrix B has a diagonal structure with non-zero elements on the main diagonal, that means most often in practice that it is a well-conditioned matrix.

For $m = 3$ and $m = 4$ we consider $t_j^C = t_{j+2}^B,\ j = 1, ..., n_B$, and in this case, for $m = 3$, the matrix B has a bidiagonal structure, and for $m = 4$ it has a tridiagonal structure. Matrix B has non-zero diagonal and codiagonal elements and, as a rule, it is well conditioned.

After determining the solution $\alpha_k \in \mathrm{C},\ k = 1, ..., n_B$ to the system (2), we construct the approximation $\varphi_{n_B}(t) := \sum_{k=1}^{n_B} \alpha_k B_{m,k}(t)$ of the function $f(t)$ and calculate its values at points $t \in \Gamma$. In the presented approximation algorithm there are two problems:

1. The graph of the function $\varphi_{n_B}(t)$ passes through the origin of coordinates, even if $f(t) \neq 0,\ \forall t \in \Gamma$. To overcome this problem, we proceed as follows. If $f(t_0) \neq 0$, where $t_0 = \psi\left(e^{i\theta_0}\right),\ \theta_0 = 0$, then from the table with generated values of the approximation $\varphi_{n_B}(t)$ (calculated for the parameter $\theta \in [0, 2\pi)$, starting with $\theta_0 = 0$), we eliminate the first values $\varphi_{n_B}\left(\tilde{t}\right)$ for which $\left|\varphi_{n_B}\left(\tilde{t}\right) - f(t_0)\right| \geq \varepsilon_1$, where ε_1 is a small value, for example, $\varepsilon_1 = 0.01$.

2. The approximation curve $\varphi_{n_B}(t)$ is continuous, being generated as a linear combination of continuous B-spline functions. Therefore, at the points of discontinuity of the function $f(t)$, we have no "breaks" of the graph of the function $\varphi_{n_B}(t)$, but continuous connections of its values. Thus, often the graph of the function $\varphi_{n_B}(t)$ has a distorted aspect compared to the graph of the approximated function $f(t)$. Next, we present an algorithm that allows to overcome the mentioned difficulty.

3 Approximation of function through a linear combination of B-spline and Heaviside functions

We admit that the values of the function f are known at the discontinuity points t_r^d, $r = 1, ..., npd$, on the contour Γ. For the function f, defined numerically, in [4] and [5] several algorithms have been proposed for establishing the locations of the discontinuity points on Γ.

We construct the approximation φ_{n_B} in the form

$$\varphi_{n_B}^H(t) := \sum_{k=1}^{n_B} \alpha_k B_{m,k}(t) + \sum_{r=1}^{npd} \beta_r H\left(t - t_r^d\right), \qquad (3)$$

where H is the Heaviside function on the contour Γ, defined as follows

$$H\left(t - t_r^d\right) := \begin{cases} 0 & \text{if } t \in \Gamma_1 \cup ... \cup \Gamma_{s-1} \cup arc\left[t_s^B, t_r^d\right) \\ 1 & \text{if } t \in arc\left[t_r^d, t_{s+1}^B\right) \cup \Gamma_{s+1} \cup ... \cup \Gamma_{n_B} \end{cases},$$

$$\Gamma_s = arc\left[t_s^B, t_{s+1}^B\right], \ t_r^d \in \Gamma_s.$$

We determine the coefficients $\alpha_k,\ k = 1, ..., n_B$, and $\beta_r,\ r = 1, ..., npd$, from the interpolation conditions

$$f\left(t_j^C\right) = \varphi_{n_B}^H\left(t_j^C\right), \ \ j = 1, ..., n,$$

where $n := n_B + npd$, and the interpolation points $t_j^C,\ j = 1, ..., n$, are chosen as follows:

- the first n_B points $t_j^C,\ j = 1, ..., n_B$, are identical to those used to determine the solution to the system (2);
- the remaining npd points are considered as discontinuity points of the function f.

If among the points $t_j^C,\ j = 1, ..., n_B$, there are points of discontinuity $t_j^d = \psi\left(e^{i\theta_j^d}\right)$ of the function f on Γ, then instead of them we consider the points $\tilde{t}_j^d = \psi\left(e^{i\left(\theta_j^d - \varepsilon_2\right)}\right)$, where $\varepsilon_2 > 0$ is a small value, for example, $\varepsilon_2 = 0.01$. Since the function is left continuous, for a sufficiently small ε_2, it can be considered that the value of the function f at point $\tilde{t}_j^d$ coincides with its value at point t_j^d.

The term $\sum_{k=1}^{n_B} \alpha_k B_{m,k}(t)$ in relation (3) defines a continuous function on Γ, that approximates the aspect of the pieces of the graph of the function f corresponding to the arcs of the contour Γ between the points of discontinuity. The coefficients $\beta_r,\ r = 1, ..., npd$, define the "jumps" of the pieces of the graph at the discontinuity points t_r^d, so that each term $\beta_r H\left(t - t_r^d\right)$ determines the displacement of the piece of the graph $\sum_{k=1}^{n_B} \alpha_k B_{m,k}(t)$, corresponding to the points of Γ, which are located after the discontinuity point t_r^d when traversing the contour Γ in a positive direction.

4 Numerical example

Consider the Riemann function $z = \psi(w)$ that performs the conformal transformation of the set $\{w \in \mathrm{C}: \ |w| > 1\}$ on the domain Ω^- from the outside of the contour Γ as $\psi(w) = w + 1/\left(3w^3\right)$. Thus, $\psi(w)$ transforms the unit circle Γ_0 onto the astroid Γ (see Figure 2).

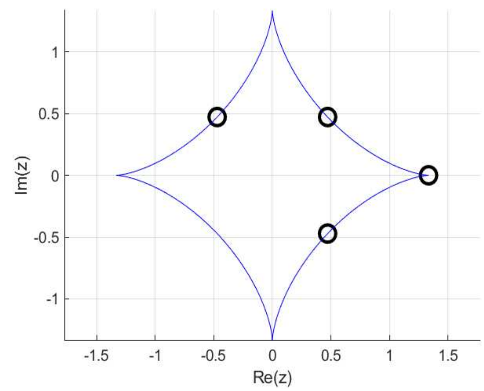

Figure 2: The contour and discontinuity points

For testing purposes, we consider the function of a complex variable f given analytically on Γ:

$$f(t) = \begin{cases} t^3 \ \text{ if } \ \theta \in (0, \zeta_1] \\ -\cos(t) \ \text{ if } \ \theta \in (\zeta_1, \zeta_2] \\ t^2 e^t \ \text{ if } \ \theta \in (\zeta_2, \zeta_3] \\ t^2 Re(2t) \ \text{ if } \ \theta \in (\zeta_3, \zeta_4] \\ t^2 Re(2t) \ \text{ if } \ \theta = 0 \end{cases},$$

where $\zeta_1 = \pi/4, \ \zeta_2 = 3\pi/4, \ \zeta_3 = 7\pi/4, \ \zeta_4 = 2\pi$. The function f has $npd = 4$ jump discontinuity points on the contour Γ corresponding to the points $t_j^d = \psi\left(e^{i\zeta_j}\right)$, $j = 1, ..., 4$ (see Figure 2 and Figure 3).

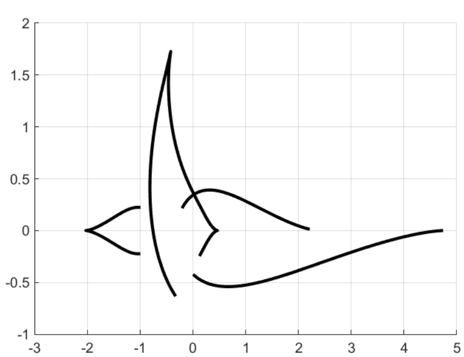

Figure 3: Graph of the function

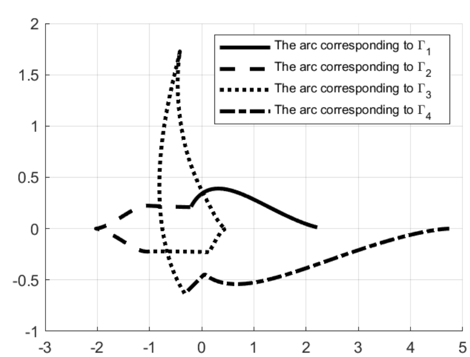

Figure 4: Combination of B-splines

The approximation algorithm takes as initial data the values f_j of the function f at the points

$$t_j = \psi\left(e^{i\theta_j}\right) \in \Gamma,\ \theta_j = 2\pi\,(j-1)/n_B,\ n_B \in \mathrm{N},\ k = 1, ..., n_B.$$

Let the number of points where the values of the function f on Γ are given be $n_B = 320$. Considering the approximation by linear combination of the form (3), where B-spline functions of order $m = 4$ are involved, we determine the solution to the system of equations $B\bar{x} = \bar{g}$, where $\bar{x} = (\alpha_1, ..., \alpha_{n_B}, \beta_1, ..., \beta_{npd})^T$, $\bar{g} = (f(t_1^c), ..., f(t_n^c))^T$, $n = n_B + npd$, and

$$B = \begin{pmatrix} B_{m,1}(t_1^c) & \cdots & B_{m,n_B}(t_1^c) & H\left(t_1^c - t_1^d\right) & \cdots & H\left(t_1^c - t_{npd}^d\right) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ B_{m,1}(t_n^c) & \cdots & B_{m,n_B}(t_n^c) & H\left(t_n^c - t_1^d\right) & \cdots & H\left(t_n^c - t_{npd}^d\right) \end{pmatrix}.$$

The coefficients $\alpha_1, ..., \alpha_{n_B}$ specify the linear combination of B-splines (see the graph in Figure 4), and the coefficients

$$\beta_1 = -0.7744 + 0.0439i,\ \beta_2 = 1.1119 - 0.0126i,$$

$$\beta_3 = 0.2962 + 0.2465i,\ \beta_4 = -2.2529 - 0.0033i,$$

establish approximations of displacements of the pieces of the graph $\sum\limits_{k=1}^{n_B} \alpha_k B_{m,k}(t)$, corresponding to the arcs between the discontinuity points (compare the data in Figure 2 and Figure 3).

For values $n_B = 160$ and $n_B = 320$ in Figure 5 and Figure 6 the error obtained at the approximation of the function f by $\varphi_{n_B}^H$ is presented. It can be seen that the maximum error decreases significantly for $n_B = 320$.

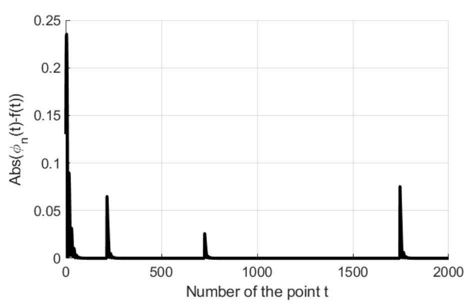

Figure 5: The approximation error for nB=160

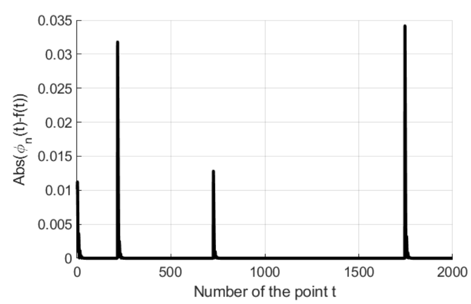

Figure 6: The approximation error for nB=320

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On recursively differentiable k-quasigroups

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Abstract. Recursive differentiability of linear k-quasigroups ($k \geq 2$) is studied in the present work. A k-quasigroup is recursively r-differentiable (r is a natural number) if its recursive derivatives of order up to r are quasigroup operations. We give necessary and sufficient conditions of recursive 1-differentiability (respectively, r-differentiability) of the k-group (Q, B), where $B(x_1, ..., x_k) = x_1 \cdot x_2 \cdot ... \cdot x_k, \forall x_1, x_2, ..., x_k \in Q$, and $(Q, \cdot)$ is a finite binary group (respectively, a finite abelian binary group). The second result is a generalization of a known criterion of recursive r-differentiability of finite binary abelian groups [4]. Also we consider a method of construction of recursively r-differentiable finite binary quasigroups of high order r. The maximum known values of the parameter r for binary quasigroups of order up to 200 are presented.

Mathematics subject classification: 20N05, 20N15, 11T71.
Keywords and phrases: k-ary quasigroup, recursive derivative, recursively differentiable quasigroup.

The notions "recursive derivative" and "recursively differentiable quasigroup" were introduced in [1], where the authors considered recursive MDS-codes (Maximum Distance Separable codes). The recursive derivative of order $t \geq 0$ of a k-ary groupoid (Q, A) is denoted by $A^{(t)}$ and is defined as follows:

$$A^{(0)} = A,$$
$$A^{(t)}(x_1^k) = A(x_{t+1}, ..., x_k, A^{(0)}(x_1^k), ..., A^{(t-1)}(x_1^k)) \text{ if } 1 \leq t < k;$$
$$A^{(t)}(x_1^k) = A(A^{(t-k)}(x_1^k), ..., A^{(t-1)}(x_1^k)) \text{ if } t \geq k, \forall x_1, ..., x_k \in Q,$$

where we denoted the sequence $x_1, x_2, ..., x_k$ by x_1^k. A k-ary quasigroup (Q, A) is called *recursively r-differentiable* if the recursive derivatives $A^{(0)}, A^{(1)}, ..., A^{(r)}$ are quasigroup operations ($r \geq 0$).

The length n of the codewords in a k-recursive code

$$C(n, A) = \{(x_1, ..., x_k, A^{(0)}(x_1^k), ..., A^{(n-k-1)}(x_1^k)) | x_1, ..., x_k \in Q\}$$

given on an alphabet Q of q elements, where $A : Q^k \to Q$ is the defining k-ary quasigroup operation, satisfies the condition $n \leq r + k + 1$, where r is the maximum order of recursive differentiability of (Q, A). On the other hand, $C(n, A)$ is an MDS-code if and only if $d = n - k + 1$, where d is the minimum Hamming distance of this code. At present it is an open problem to determine all triplets (n, d, q) of natural numbers such that there exists an MDS-code C of lenght n, on an alphabet of q elements, with $|C| = q^k$ and with the minimum Hamming distance d, for each

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$k \geq 2$. This general problem implies, in particular, the problem of determining the maximum order of recursive differentiability of finite k-ary quasigroups ($k \geq 2$).

Let $(Q, *)$ be a binary quasigroup. Denoting by $\overset{t}{*}$ the recursive derivative of order t of the operation $*$, we have:

$$x \overset{0}{*} y = x * y,$$
$$x \overset{1}{*} y = y * (x * y),$$
$$x \overset{t}{*} y = (x \overset{t-2}{*} y) * (x \overset{t-1}{*} y),\ \forall t \geq 2 \text{ and } \forall x, y \in Q.$$

It is known that there exist recursively 1-differentiable binary finite quasigroups of any order, except 1, 2, 6, and possibly 14, 18, 26 and 42 [1]. Some estimations of the maximum (known) order r of recursive differentiability of finite n-quasigroups ($n \geq 2$) are given in [1–4]. General properties of recursively differentiable binary quasigroups are studied in [4, 6, 7].

The recursive differentiability of k-ary quasigroups is closely connected to the orthogonality of the recursive derivatives [1,4,6]. It is shown in [1] that a k-quasigroup defines an MDS-code of length n if and only if its first $n-k-1$ recursive derivatives are strongly orthogonal. Hence the defining k-quasigroup operation of a recursive MDS-code of length n is recursively $(n-k-1)$-differentiable. On the other hand, it is known that a system of binary quasigroups is strongly orthogonal if and only if it is (simply) orthogonal [5]. Another "special property" of binary quasigroups is given in [1]: the recursive derivatives of order up to r of a finite binary quasigroup $(Q, *)$ are quasigroup operations if and only if $(Q, *)$ defines a recursive MDS-code of length $r+3$. So, a finite binary quasigroup $(Q, *)$ is recursively r-differentiable if and only if its recursive derivatives of order up to r are mutually orthogonal. The last statement implies the fact that there do not exist recursively 1-differentiable quasigroups of orders 2 and 6 and that $r \leq q-2$, where $q = |Q|$ and r is the order of the recursive differentiability of the quasigroup Q. Recall that there do not exist orthogonal latin squares of order 2 or 6, and the number of mutually orthogonal latin squares on a set of q elements does not exceed $q-1$ [5]. The mentioned above results imply the following lemma.

Lemma 1. *The maximum order r of recursive differentiability of a finite binary quasigroup of order q satisfies the inequality $r \leq q-2$.*

It is shown in [1] that there exist recursively $(q-2)$-differentiable finite binary quasigroups of every primary order $q \geq 3$. However, it is an open problem to find the maximum order r of recursive differentiability of finite k-ary quasigroups of order q, for $k \geq 2$ and non-primary q.

Recursive differentiability of linear n-ary quasigroups ($n \geq 2$) is studied in the present work. In particular, we give necessary and sufficient conditions of recursive 1-differentiability (respectively, r-differentiability) of an n-group (Q, B), where $B(x_1^n) = x_1 \cdot x_2 \cdot ... \cdot x_n, \forall x_1, x_2, ..., x_n \in Q$, and $(Q, \cdot)$ is a finite binary group (respectively, a finite abelian binary group). The second result is a generalization of a known criterion for finite binary abelian groups, given in [4]. Also we consider a method of construction of recursively differentiable finite binary quasigroups of high (in particular, maximum) order r. The maximum known values of the order r of recursive differentiability of finite binary quasigroups of order up to 200, are qiven in Table 1.

Lemma 2. *Let $n \geq 2$ and let (Q_i, A_i) be a recursively r_i-differentiable n-quasigroup, $i = 1, \ldots, m$. Then the direct product $(Q_1 \times ... \times Q_m, B)$,*

$$B((x_{11}^{1m}), ..., (x_{n1}^{nm})) = (A_1(x_{11}^{n1}), ..., A_m(x_{1m}^{nm})), \tag{1}$$

$\forall (x_{11}^{1m}), ..., (x_{n1}^{nm}) \in Q_1 \times ... \times Q_m$, is a recursively r-differentiable n-quasigroup, where $r = min\{r_1, ..., r_m\}$.

Proof. Remind that an n-ary groupoid (Q, B) is an n-ary quasigroup if each element u_i in the equality $B(u_1, ..., u_n) = u_{n+1}$ is uniquely determined by the remaining n elements. Hence, we get from (1) that $(Q_1 \times ... \times Q_m, B)$ is an n-quasigroup. To find the recursive derivatives of B we'll consider the following two cases:

(i) $1 \leq t < n$

$$\begin{aligned}
&B^{(t)}((x_{11}^{1m}), ..., (x_{n1}^{nm})) = \\
&= B((x_{t+1,1}^{t+1,m}), ..., (x_{n1}^{nm}), B^{(0)}((x_{11}^{1m}), ..., (x_{n1}^{nm})), ..., B^{(t-1)}((x_{11}^{1m}), ..., (x_{n1}^{nm}))) = \\
&B((x_{t+1,1}^{t+1,m}), ..., (x_{n1}^{nm}), (A_1^{(0)}(x_{11}^{n1}), ..., A_m^{(0)}(x_{1m}^{nm})), ..., (A_1^{(t-1)}(x_{11}^{n1}), ..., A_m^{(t-1)}(x_{1m}^{nm}))) = \\
&= (A_1^{(t)}(x_{11}^{n1}), ..., A_m^{(t)}(x_{1m}^{nm}));
\end{aligned}$$

(ii) $t \geq n$

$$\begin{aligned}
&B^{(t)}((x_{11}^{1m}), ..., (x_{n1}^{nm})) = B(B^{(t-n)}((x_{11}^{1m}), ..., (x_{n1}^{nm})), ..., B^{(t-1)}((x_{11}^{1m}), ..., (x_{n1}^{nm}))) = \\
&= B((A_1^{(t-n)}(x_{11}^{n1}), ..., A_m^{(t-n)}(x_{1m}^{nm})), ..., (A^{(t-1)}(x_{11}^{n1}), ..., A_m^{(t-1)}(x_{1m}^{nm}))) = \\
&= (A_1(A_1^{(t-n)}(x_{11}^{n1}), ..., A_1^{(t-1)}(x_{11}^{n1})), ..., A_m(A_m^{(t-n)}(x_{1m}^{nm}), ..., A_m^{(t-1)}(x_{1m}^{nm}))) = \\
&= (A_1^{(t)}(x_{11}^{n1}), ..., A_m^{(t)}(x_{1m}^{nm})).
\end{aligned}$$

Hence, $B^{(t)}((x_{11}^{1m}), ..., (x_{n1}^{nm})) = (A_1^{(t)}(x_{11}^{n1}), ..., A_m^{(t)}(x_{1m}^{nm}))$, for every $t \geq 1$ and every $(x_{11}^{1m}), ..., (x_{n1}^{nm}) \in Q_1 \times ... \times Q_m$. As each of the operations $A_1, ..., A_m$ is recursively r-differentiable, where $r = min\{r_1, ..., r_m\}$, we get that B is recursively r-differentiable. □

Proposition 1. *Let $(Q, \cdot)$ be a finite binary group and $n \geq 2$. The n-ary group (Q, B), where $B(x_1^n) = x_1 \cdot x_2 \cdot ... \cdot x_n, \forall x_1, x_2, ..., x_n \in Q$, is recursively 1-differentiable if and only if the mapping $x \to x^2$ is a bijection in $(Q, \cdot)$.*

Proof. The n-group (Q, B) is recursively 1-differentiable if and only if the recursive derivative $B^{(1)}$ is a quasigroup operation, i.e. if and only if in the equality

$$B^{(1)}(x_1, ..., x_n) = b, \tag{2}$$

every n elements uniquely determine the remaining $(n+1)$-th one. Taking $x_j = a_j \in Q$ in (2), for every $j = 2, \ldots, n$, we get the equation $x_1 \cdot a_2 \cdot ... \cdot a_n = b$, which has a unique solution in Q. For $i \in \{2, ..., n\}$, taking $x_j = a_j \in Q, \forall j \neq i,\ j \in \{1, ..., n\}$, we have:

$$B^{(1)}(a_1, ..., a_{i-1}, x_i, a_{i+1}, ..., a_n) = b \Leftrightarrow$$
$$\Leftrightarrow B(a_2, ..., a_{i-1}, x_i, a_{i+1}, ..., a_n, B(a_1, ..., a_{i-1}, x_i, a_{i+1}, ..., a_n)) = b \Leftrightarrow$$
$$\Leftrightarrow a_2 \cdot ... \cdot a_{i-1} \cdot x_i \cdot a_{i+1} \cdot ... \cdot a_n \cdot a_1 \cdot ... \cdot a_{i-1} \cdot x_i \cdot a_{i+1} \cdot ... \cdot a_n = b \Leftrightarrow$$
$$\Leftrightarrow (a_1 \cdot ... \cdot a_{i-1} \cdot x_i \cdot a_{i+1} \cdot ... \cdot a_n)^2 = a_1 \cdot b.$$

Hence, denoting $a_1 \cdot ... \cdot a_{i-1} \cdot x_i \cdot a_{i+1} \cdot ... \cdot a_n$ by y, we get that the n-group (Q, B) is recursively 1-differentiable if and only if, for each $b \in Q$, the equation $y^2 = b$ has a unique solution. □

Corollary 1. *There exist finite recursively 1-differentiable n-quasigroups of any odd order $q \geq 3$, for every $n \geq 2$.*

Proof. This statement follows from the fact that the mapping $x \to x^2$ is a bijection in every finite binary group of odd order $q \geq 3$. □

Theorem 1. *Let $(Q, \cdot)$ be a finite binary abelian group and let $n \geq 2, r \geq 1$ be two natural numbers. The n-ary group (Q, B), where $B(x_1^n) = x_1 \cdot x_2 \cdot ... \cdot x_n$, for every $x_1, x_2, ..., x_n \in Q$, is recursively r-differentiable if and only if the mappings $x \to x^{s_i^k}$ are bijections in the group $(Q, \cdot)$, $\forall i = 1, \ldots, n$ and $\forall k = 1, \ldots, r$, where the sequences $(s_i^k)_{k \geq 0}$ are defined as follows:*

1. $k = 0$

$$s_1^0 = ... = s_n^0 = 1;$$

2. $1 \leq k < n$

$$s_t^k = s_t^0 + ... + s_t^{k-1}, \quad \forall t = 1, \ldots, k;$$
$$s_t^k = 1 + s_t^0 + ... + s_t^{k+1}, \quad \forall t = k+1, \ldots, n;$$

3. $k \geq n$

$$s_t^k = s_t^{k-n} + ... + s_t^{k-1}, \ \forall t = 1, \ldots, n.$$

Proof. As $(Q, \cdot)$ is an abelian group and $B^{(0)}(x_1^n) = x_1 \cdot ... \cdot x_n$, the recursive derivatives $B^{(1)}$ and $B^{(2)}$ as follows:

$$B^{(1)}(x_1^n) = B(x_2, ..., x_n, B^{(0)}(x_1^n)) = x_2 \cdot ... \cdot x_n \cdot x_1 \cdot ... \cdot x_n = x_1 \cdot x_2^2 \cdot ... \cdot x_n^2;$$
$$B^{(2)}(x_1^n) = B(x_3, ..., x_n, B^{(0)}(x_1^n), B^{(1)}(x_1^n)) = x_3 \cdot ... \cdot x_n \cdot x_1 \cdot ... \cdot x_n \cdot x_1 \cdot x_2^2 \cdot ... \cdot x_n^2 =$$
$$= x_1^2 \cdot x_2^3 \cdot x_3^4 \cdot ... \cdot x_n^4.$$

Let denote $B^{(k)}(x_1^n) = x_1^{s_1^k} \cdot x_2^{s_2^k} \cdot ... \cdot x_n^{s_n^k}$, for every $k \geq 0$. To find the sequences $(s_i^k)_{k \geq 0}$, where $i = 1, \ldots, n$, we will consider the following two cases:

1. $0 \leq k < n$

$B^{(k)}(x_1^n) = B(x_{k+1}, ..., x_n, B^{(0)}(x_1^n), ..., B^{(k-1)}(x_1^n)) =$

$= x_{k+1} \cdot ... \cdot x_n \cdot x_1^{s_1^0} \cdot ... \cdot x_n^{s_n^0} \cdot ... \cdot x_1^{s_1^{k-1}} \cdot ... \cdot x_n^{s_n^{k-1}} =$

$= x_1^{s_1^0+...+s_1^{k-1}} \cdot ... \cdot x_k^{s_k^0+...+s_k^{k-1}} \cdot x_{k+1}^{1+s_{k+1}^0+...+s_{k+1}^{k-1}} \cdot ... \cdot x_n^{1+s_n^0+...+s_n^{k-1}};$

2. $k \geq n$

$B^{(k)}(x_1^n) = B(B^{(k-n)}(x_1^n), ..., B^{(k-1)}(x_1^n)) = B^{(k-n)}(x_1^n) \cdot ... \cdot B^{(k-1)}(x_1^n) =$

$= x_1^{s_1^{k-n}+...+s_1^{k-1}} \cdot ... \cdot x_n^{s_n^{k-n}+...+s_n^{k-1}}.$

The recursive derivatives $B^{(k)}$, where $k = 1, 2, \ldots, r$, are quasigroup operations if and only if the mappings $x \to x^{s_i^k}$ are bijections in the group $(Q, \cdot)$, $\forall i = 1, \ldots, n$ and $\forall k = 1, \ldots, r$.

□

Corollary 2. [4] *A finite binary abelian group $(Q, \cdot)$ is recursively r-differentiable $(r \geq 1)$ if and only if the mappings $x \to x^{s_i^k}$ are bijections, $\forall i = 1, 2$ and $\forall k = 1, ..., r$, where the sequences $(s_1^k)_{k\geq 0}$ and $(s_2^k)_{k\geq 0}$ are defined as follows:*

$s_1^0 = s_2^0 = 1;\ s_1^1 = 1, s_2^1 = 2;\ s_i^k = s_i^{k-2} + s_i^{k-1}, \forall k \geq 2, \forall i = 1, 2.$

Note that $(s_1^k)_{k\geq 0}$ and $(s_2^k)_{k\geq 0}$ are Fibonacci sequences.

We will give bellow an algorithm of construction of binary linear (over $\mathbb{Z}_n$) quasigroups, which are recursively differentiable of high order.

Lemma 3. [7] *If $(Q, *)$ is a binary quasigroup then, for every $x, y \in Q$ and $\forall s \geq 1$,*

$$x \overset{s}{*} y = y \overset{s-1}{*} (x * y). \tag{3}$$

Lemma 4. *Let $a \in \mathbb{Z}\backslash\{0\}$ and $x * y = ax + y, \forall x, y \in \mathbb{Z}$. The following statements hold:*

1. *There exist $u_s,\ v_s \in \mathbb{Z}$ such that $x \overset{s}{*} y = u_s x + v_s y,\ \forall x, y \in \mathbb{Z}, \forall s \geq 1$;*

2. *If $n \geq 2$ is a natural number, $k \in \{1, ..., n-1\}$ and $a = n-k$, then there exists $b_{s+2} \in \mathbb{Z}$ such that $v_{s+2} = nb_{s-2} + (-kc_s + c_{s+1})$, for $\forall s \geq 1$, where c_s and c_{s+1} are the rests from dividing v_s and v_{s+1} by n, respectively.*

Proof. 1. In this case $x \overset{1}{*} y = y * (x * y) = ax + (a+1)y, \forall x, y \in \mathbb{Z}$. Denoting $x \overset{s-1}{*} y = u_{s-1}x + v_{s-1}y$ and using the mathematical induction and (3), we get $x \overset{s}{*} y = u_{s-1}y + v_{s-1}(ax + y) = av_{s-1}x + (u_{s-1} + v_{s-1})y$.

2. As $x \overset{s+2}{*} y = (x \overset{s}{*} y) * (x \overset{s+1}{*} y) = (au_s + u_{s+1})x + (av_s + v_{s+1})y$, the following equalities hold:

$$v_{s+2} = av_s + v_{s+1} = (n-k)(nb_s + c_s) + (nb_{s+1} + c_{s+1}) =$$
$$= n(nb_s + c_s - kb_s + b_{s+1}) + (-kc_s + c_{s+1}),$$

where c_s and c_{s+1} are the rests from dividing v_s and v_{s+1} by n, respectively.

□

Now, let consider the operation $x * y = \overline{a}x + y$ on the ring $\mathbb{Z}_n$ of integers modulo n, where $(a, n) = 1$. Then $(\mathbb{Z}_n, *)$ is a quasigroup and, according to the previous lemma, there exist $\overline{u_s}, \overline{v_s} \in \mathbb{Z}_n$ such that $x \overset{s}{*} y = \overline{u_s}x + \overline{v_s}y, \forall s \geq 0$.

Theorem 2. *Let $n \geq 2, a = n-k, k \in \{1, ..., n-1\}, (a, n) = 1$ and $x * y = \overline{a}x + y, \forall x, y \in \mathbb{Z}_n$. If, for some $s \geq 1$, the recursive derivatives $(\mathbb{Z}_n, \overset{s}{*})$ and $(\mathbb{Z}_n, \overset{s+1}{*})$, where $x \overset{i}{*} y = \overline{u_i}x + \overline{v_i}y,\ i = s, s+1$, are quasigroups, then $(\mathbb{Z}_n, \overset{s+2}{*})$ is a quasigroup if and only if $(-kc_s + c_{s+1}, n) = 1$, where c_s and c_{s+1} are the rests from dividing v_s and v_{s+1} by n, respectively.*

Proof. We have: $x \overset{s+2}{*} y = \overline{u_{s+2}}x + \overline{v_{s+2}}y = \overline{av_{s+1}}x + (\overline{u_{s+1}} + \overline{v_{s+1}})y$, so $\overline{v_{s+2}} = \overline{u_{s+1}} + \overline{v_{s+1}} = \overline{-kc_s + c_{s+1}}$, where c_s and c_{s+1} are the rests from dividing v_s and v_{s+1} by n, respectively. If $(\mathbb{Z}_n, \overset{s}{*})$ and $(\mathbb{Z}_n, \overset{s+1}{*})$ are quasigroups, then $(av_{s+1}, n) = 1$, hence $(\mathbb{Z}_n, \overset{s+2}{*})$ is a quasigroup if and only if $(-kc_s + c_{s+1}, n) = 1$. □

Using Theorem 2, we get, for example, that the quasigroups $(\mathbb{Z}_7, *), x*y = 4x+y$, and $(\mathbb{Z}_{11}, *), x*y = 3x+y$, are recursively 5- and 9-differentiable, respectively. Recall that the order r of recursive differentiability of a binary quasigroup, defined on a set of q elements, satisfies the inequality $r \leq q-2$. The following corollary gives all values of the element a such that the quasigroup $(\mathbb{Z}_p, *)$, where $x * y = \overline{a}x + y, \forall x, y \in \mathbb{Z}_p$, is recursively differentiable of maximum order, for each odd prime p, up to 19.

Corollary 3. *Let $(\mathbb{Z}_n, *)$, where $x * y = \overline{a}x + y, \forall x, y \in \mathbb{Z}_n$, be a quasigroup. The following statements hold:*

1. *$(\mathbb{Z}_3, *)$ is recursively 1-differentiable if and only if $a = 1$;*
2. *$(\mathbb{Z}_5, *)$ is recursively 3-differentiable if and only if $a = 3$;*
3. *$(\mathbb{Z}_7, *)$ is recursively 5-differentiable if and only if $a = 1$ or 4;*
4. *$(\mathbb{Z}_{11}, *)$ is recursively 9-differentiable if and only if $a = 3$ or 4;*
5. *$(\mathbb{Z}_{13}, *)$ is recursively 11-differentiable if and only if $a = 5, 8$ or 11;*
6. *$(\mathbb{Z}_{17}, *)$ is recursively 15-differentiable if and only if $a = 7$ or 10;*
7. *$(\mathbb{Z}_{19}, *)$ is recursively 17-differentiable if and only if $a = 1, 5$ or 7.*

The known estimations $r_0 \leq r$ of the order r of recursive differentiability of binary finite quasigroups of order $q \leq 200$ are given in the following Table 1. In the cell with coordinates (m, k) we give the known value of the parameter r for quasigroups of order $m + k$. Remark that the cell $(0, 0)$ contains the known value of r for the quasigroups of order 200. An analogous table containing the maximum known length of recursive MDS-codes, defined by quasigroups of order up to 100, is given in [2] and we use it in the first ten lines of Table 1.

	0	1	2	3	4	5	6	7	8	9
0(200)	$r \geq 2$	0	0	1	2	3	0	5	6	7
10	1	9	1	11	?	1	14	15	?	17
20	2	2	1	21	2	23	?	25	2	27
30	1	29	30	1	1	3	1	35	1	2
40	1	39	?	41	2	1	1	45	1	47
50	4	1	2	51	3	3	5	4	4	57
60	2	59	3	5	62	4	3	65	3	3
70	4	69	6	71	3	3	3	5	4	77
80	5	79	3	81	4	4	4	3	6	87
90	3	5	4	3	4	4	4	95	4	7
100	2	99	1	101	6	1	1	105	1	107
110	1	1	5	111	1	3	2	1	1	5
120	1	119	1	1	2	123	1	125	126	1
130	1	129	1	5	1	1	6	135	1	137
140	2	1	1	9	1	3	1	1	2	147
150	1	149	6	1	1	3	1	155	1	1
160	3	5	1	161	2	1	1	165	1	167
170	1	1	2	171	1	3	5	1	1	177
180	2	179	1	1	6	3	1	9	2	1
190	1	189	1	191	1	1	2	195	1	197

Table 1. Estimations of the parameter r (order of recursive differentiability) in the case of binary quasigroups

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Limits of solutions to the semilinear plate equation with small parameter

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Abstract. We study the existence of the limits of solutions to the semilinear plate equation with boundary Dirichlet condition with a small parameter coefficient of the second order derivative in time. We establish the convergence of solutions to the perturbed problem and their derivatives in spacial variables to the corresponding solutions to the unperturbed problem as the small parameter tends to zero.

Mathematics subject classification: 35B25, 35K15, 35L15, 34G10.
Keywords and phrases: a priory estimate, boundary layer, semilinear plate equation, singular perturbation.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with the smooth boundary $\partial\Omega$. Consider the following initial boundary value problem for the plate equation:

$$\begin{cases} \varepsilon\, u_{tt}(x,t) + u_t(x,t) + \Delta^2 u(x,t) + B(u(t)) = f(x,t), \;\; (x,t) \in \Omega \times (0,T), \\ u\big|_{t=0} = u_0(x), \quad u_t\big|_{t=0} = u_1(x), \quad x \in \Omega, \\ u\big|_{x\in\partial\Omega} = \dfrac{\partial u}{\partial \bar{\nu}}\Big|_{x\in\partial\Omega} = 0, \quad t \geq 0, \end{cases} \tag{P_ε}$$

where $\bar{\nu}$ is the outer normal vector to $\partial\Omega$ and ε is a small positive parameter.

We study the behaviour of the solutions to the problem (P_ε) as $\varepsilon \to 0$. It is natural to expect that the solutions to the problem (P_ε) tend to the corresponding solutions to the following unperturbed problem:

$$\begin{cases} v_t(x,t) + \Delta^2 v(x,t) + B(v(t)) = f(x,t), \;\; (x,t) \in \Omega \times (0,T), \\ v\big|_{t=0} = u_0(x), \quad x \in \Omega, \\ v\big|_{x\in\partial\Omega} = \dfrac{\partial v}{\partial \bar{\nu}}\Big|_{x\in\partial\Omega} = 0, \quad t \geq 0, \end{cases} \tag{P_0}$$

as $\varepsilon \to 0$.

We investigate two cases: the first case when the operator B is Lipschitzian and the second case when the operator B is monotone.

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The main results are contained in Theorems 8 and 9. Under some conditions on u_0, u_1 and f we prove that

$$u \to v \quad \text{in} \quad C([0,T]; L^2(\Omega)) \cap L^\infty(0,T; H^2(\Omega)), \quad \text{as} \quad \varepsilon \to 0. \tag{1}$$

This means that the perturbation (P_ε) of the system (P_0) is regular in the indicated norms. At the same time, we prove that

$$u' - v' - \alpha\, e^{-t/\varepsilon} \to 0 \text{ in } C([0,T]; L^2(\Omega)) \cap L^\infty(0,T; H^2(\Omega)),\ \alpha \neq 0,\ \text{as } \varepsilon \to 0. \tag{2}$$

It means that the derivatives of the solutions to the problem (P_ε) do not converge to the derivatives of the corresponding solutions to the problem (P_0), as $\varepsilon \to 0$. The relation (2) shows that the derivative u' has a singular behaviour, as $\varepsilon \to 0$, in the neighborhood of $t = 0$. This singular behaviour is determined by the function $\alpha\, e^{-\tau/\varepsilon}$, which is *the boundary layer function* and the neighborhood of $t = 0$ is *the boundary layer* for u'.

The proofs of the relations (1) and (2) are based on two key points. The first one is the relationship between the solutions to the problem (P_0) and (P_ε) in the linear case (see Lemma 3 and Theorem 7). The second key point is the *a priori* estimates of the solutions to the problem (P_ε), which are uniform relative to the small parameter ε (see Lemmas 1 and 2).

The singularly perturbed nonlinear problems of hyperbolic-parabolic type were studied by many authors. Without pretending to a complete list of the papers in this area, we mention the works [4–11] containing a wide list of references.

In what follows, we use some notations. For $m \in [1, \infty)$ denote by

$$L^m(\Omega) = \{f : \text{a.e.}\, \Omega \to \mathbb{C};\ \int_\Omega |f(x)|^m\, dx < \infty\},$$

the Banach space, endowed with the norm

$$||f||_{L^m(\Omega)} = \Big(\int_\Omega |f(x)|^m\, dx\Big)^{1/m}$$

and for $m = \infty$ denote by

$$L^\infty(\Omega) = \{f : \text{a.e.}\, \Omega \to \mathbb{C};\ \text{ess sup}_\Omega\, |f(x)| < \infty\}$$

the Banach space, endowed with the norm

$$||f||_{L^\infty(\Omega)} = \text{ess sup}_\Omega\, |f(x)|.$$

By $L^m_{loc}(\Omega)$ denote the space of integrable functions on each compact $K \subset\subset \Omega$. Denote by $W^{l,m}(\Omega)$ the Banach space of all elements of $L^m(\Omega)$ whose derivatives $\partial^\alpha u$ in the sense of distributions up to the order l belong to $L^m(\Omega)$. The norm in $W^{l,m}(\Omega)$ is defined as

$$||u||_{W^{l,m}(\Omega)} = \Big(\int_\Omega \sum_{|\alpha| \leq l} |\partial^\alpha u|^m\, dx\Big)^{1/m}.$$

By $W^{l,m}_{loc}(\Omega)$ denote the local Sobolev space, i.e. a function $u \in W^{l,m}_{loc}(\Omega)$ if $u \in W^{l,m}(K)$ for every compact $K \subset\subset \Omega$.

For $k \in \mathbb{N}$ we denote by $H^k(\Omega)$ ($H^0(\Omega) := L^2(\Omega)$) the usual real Hilbert spaces equipped with the following scalar product and norm:

$$\Big(u, v\Big)_{H^k(\Omega)} = \int\limits_\Omega \sum_{|\alpha|\leq k} \partial^\alpha u(x) \partial^\alpha v(x) dx, \quad ||u||_{H^k(\Omega)} = \Big(\int\limits_\Omega \sum_{|\alpha|\leq k} |\partial^\alpha u(x)|^2\, dx\Big)^{1/2}.$$

Denote by $H^k_0(\Omega)$ the closure of $C^\infty_0(\Omega)$ in the norm of the space $H^k(\Omega)$. By $H^{-k}(\Omega)$ denote the dual space of $H^k_0(\Omega)$, i.e. $H^{-k}(\Omega) = (H^k_0(\Omega))'$.

Denote by V the space $V = \{u \in H^2(\Omega); u|_{\partial\Omega} = \dfrac{\partial u}{\partial \bar\nu}|_{\partial\Omega} = 0\}$, endowed with the norm of the space $H^2(\Omega)$, and by V' the dual space of the space V. We will write $\langle \cdot, \cdot\rangle$ to denote the pairing between V' and V. Also denote by

$$(u, v) = \int\limits_\Omega u(x)v(x)dx, \quad |u| = ||u||_{L^2(\Omega)}, \quad ||u|| = ||u||_{H^2(\Omega)}.$$

Let X be a Banach space. For $k \in \mathbb{N}$, $p \in [1, \infty)$ and $(a, b) \subset (-\infty, +\infty)$ we denote by $W^{k,p}(a, b; X)$ the usual Sobolev space of the vectorial distributions $W^{k,p}(a, b; X) = \{f \in D'(a, b, X); f^{(l)} \in L^p(a, b; X), l = 0, 1, \ldots, k\}$ equipped with the norm

$$||f||_{W^{k,p}(a,b;X)} = \Big(\sum_{l=0}^k ||f^{(l)}||^p_{L^p(a,b;X)}\Big)^{1/p}.$$

For each $k \in \mathbb{N}, W^{k,\infty}(a, b; X)$ is the Banach space equipped with the norm

$$||f||_{W^{k,\infty}(a,b;X)} = \max_{0\leq l\leq k} ||f^{(l)}||_{L^\infty(a,b;X)}.$$

For $s \in \mathbb{R}$, $k \in \mathbb{N}$ and $p \in [1, \infty]$ we also denote by

$$W^{k,p}_s(a, b; H) = \{f : (a, b) \mapsto H; f^{(l)}(\cdot)\, e^{s\,t} \in L^p(a, b; X), l = 0, \ldots, k\}$$

the Banach space, endowed with norms $||f||_{W^{k,p}_s}(a, b; X) = ||f e^{s\,t}||_{W^{k,p}(a,b;X)}$.

2 Solvability of the problems (P_ε) and (P_0)

The framework of our investigations will be determined by the following conditions:

(B1) *The operator* $B : D(B) \subseteq L^2(\Omega) \mapsto L^2(\Omega)$ *verifies the condition:* $V \subset D(D)$ *and there exists a constant* $L > 0$ *such that*

$$|B(u_1) - B(u_2)| \leq L\, ||u_1 - u_2||_{H^2(\Omega)}, \quad \forall u_1,\, u_2 \in V;$$

(B2) *The operator* B *possesses the Fréchet derivative* B' *in* V*, so that there exist some constants* $L_0 \geq 0$ *and* $L_1 \geq 0$ *such that*

$$\big|\big(B'(u_1) - B'(u_2)\big)\, v\big| \leq L_1\, ||u_1 - u_2||_{H^2(\Omega)}\, ||v||_{H^2(\Omega)}, \quad \forall u_1,\, u_2,\, v \in V,$$

$$\big|B'(u)\,v\big| \le L_0\,|v|, \quad \forall u \in V, \quad \forall v \in L^2(\Omega);$$

(B3) *The operator $B : D(B) \subseteq L^2(\Omega) \to L^2(\Omega)$ is $H^2(\Omega)$ local lipschitzian, i.e. $V \subset D(B)$ and for every $R > 0$ there exists $L(R) \ge 0$ such that*

$$|B(u_1) - B(u_2)| \le L(R)\,||u_1 - u_2||_{H^2(\Omega)}, \quad \forall u_i \in V,\ ||u_i||_{H^2(\Omega)} \le R,\ i = 1, 2,$$

and B is Fréchet derivative of some convex and positive functional $\mathcal{B}$ with $V \subset D(\mathcal{B})$.

The hypothesis that operator B is Fréchet derivative of some convex and positive functional implies, in particular, that the operator B is monotone and verifies the condition

$$\frac{d}{dt}\mathcal{B}(u(t)) = \big(B(u(t)), u'(t)\big), \quad t \in [a,b] \subset \mathbb{R},$$

for $u \in C([a,b], V) \cap C^1([a,b], L^2(\Omega))$ (see [13]).

(B4) *The operator B possesses the Fréchet derivative B' in V and for every $R > 0$ there exists a constant $L_1(R) \ge 0$ such that*

$$\big|\big(B'(u_1) - B'(u_2)\big)v\big| \le L_1(R)\,||u_1 - u_2||_{H^2(\Omega)}\,||v||_{H^2(\Omega)}, \quad \forall u_1,\, u_2,\, v \in V,$$
$$||u_i||_{H^2(\Omega)} \le R, \quad i = 1, 2.$$

Firstly we remind the definitions of solutions to the problems (P_ε) and (P_0) and the existence theorems for solutions to the considered problems.

Definition 1. *Let $T > 0$, $f \in L^2(0,T;V')$ and $B : D(B) \subseteq L^2(\Omega) \to V'$. A function $u \in L^2(0,T;V \bigcap D(B))$ with $u' \in L^2(0,T;L^2(\Omega))$ and $u'' \in L^2(0,T : V')$ is called solution to the problem (P_ε) if u satisfies the equality*

$$\begin{cases} \varepsilon\langle u''(t), \eta\rangle + (u'(t), \eta) + \big(\Delta u(t), \Delta\eta\big) + \big(B\big(u(t)\big), \eta\big) = (f(t), \eta), \\ \forall \eta \in V, \text{ a.e. } t \in [0,T], \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{3}$$

Definition 2. *Let $T > 0$, $f \in L^2(0,T;V')$ and $B : D(B) \subseteq L^2(\Omega) \to V'$. A function $v \in L^2(0,T;V \cap D(B))$ with $v' \in L^2(0,T;V')$ is called solution to the problem (P_0) if v satisfies the equality*

$$\begin{cases} \langle v'(t), \eta\rangle + \big(\Delta v(t), \Delta\eta\big) + \big(B\big(v(t)\big), \eta\big) = (f(t), \eta), \forall \eta \in V, \quad \text{a.e. } t \in [0,T], \\ v(0) = u_0. \end{cases} \tag{4}$$

Remark 1. *For $u \in L^2(0,T;V), u' \in L^2(0,T;L^2(\Omega))$, and $u'' \in L^2(0,T;V')$ it follows that $u \in C([0,T];L^2(\Omega))$ and $u' \in C([0,T];V')$. Consequently, the initial conditions from (3) are understood in the following sense:*

$$|u(t) - u_0| \to 0, \quad ||u'(t) - u_1||_{V'} \to 0, \text{ as } t \to 0.$$

Similarly, for $v \in L^2(0,T;V)$ *with* $v' \in L^2(0,T:V')$, *it follows that* $v \in C([0,T];V)$, *consequently, the initial conditions from* (4) *are understood in the following sense* $|v(t)-u_0| \to 0$ *as* $t \to 0$.

Using the methods developed in [2] and [3], in [8] the following theorems are proved.

Theorem 1. *Let* $T>0$. *Suppose that condition* **(B1)** *is fulfilled. If* $u_0 \in H^4(\Omega)\cap V$, $u_1 \in V$, *and* $f \in W^{1,1}(0,T;L^2(\Omega))$ *then there exits a unique solution to the problem* (P_ε) *such that* $u \in W^{2,\infty}(0,T;L^2(\Omega))$, $\Delta\, u \in W^{1,\infty}(0,T;L^2(\Omega))$, $\Delta^2 u \in L^\infty(0,T;L^2(\Omega))$.

The function $t \in [0,T) \mapsto u'(t) \in L^2(\Omega)$ *is derivable to the right and the equalitiy*

$$\frac{d^+u'}{dt}(t) = f(t_0) - \Delta^2 u(t) - B(u(t)) - u'(t), \quad t \in [0,T),$$

is true. The function $t \in [0,T] \mapsto \Delta^2 u(t)$ *is weakly continuous in* $L^2(\Omega)$ *and the equality*

$$\frac{d}{dt}\big(\Delta^2 u(t), u(t)\big) = 2\left(\Delta^2 u(t), u'(t)\right), \quad t \in [0,T),$$

is true.

If, in addition, $u_1 \in H^4(\Omega)\cap V$, $f(0)-B(u_0)-\Delta^2 u_0 - u_1 \in V$, $f \in W^{2,1}(0,T;L^2(\Omega))$ *and condition* **(B2)** *is fulfilled, then* $u \in W^{3,\infty}(0,T;L^2(\Omega))$ *and* $\Delta u \in W^{2,\infty}(0,T;L^2(\Omega))$.

Theorem 2. *Let* $T>0$. *Suppose that condition* **(B3)** *is fulfilled. If* $u_0 \in H^4(\Omega)\cap V, u_1 \in V$ *and* $f \in W^{1,1}(0,T;L^2(\Omega)$, *then there exists a unique solution to the problem* (P_ε) *such that* $u \in C^2([0,T];L^2(\Omega))$, $u' \in C^1([0,T];V)$, $\Delta^2 u \in C([0,T];L^2(\Omega))$.

If, in addition, $u_1 \in H^4(\Omega)\cap V$, $f(0)-B(u_0)-\Delta^2 u_0 - u_1 \in V$, $f \in W^{2,1}(0,T;L^2(\Omega))$ *and condition* **(B4)** *is fulfilled, then* $u \in W^{3,\infty}(0,T;L^2(\Omega))$, $\Delta u \in W^{2,\infty}(0,T;L^2(\Omega))$.

Theorem 3. *Let* $T>0$. *Suppose that condition* **(B1)** *is fulfilled. If* $u_0 \in H^4(\Omega)\cap V$ *and* $f \in W^{1,1}(0,T;L^2(\Omega))$, *then there exits a unique solution to the problem* (P_0). *The function* $t \in [0,T) \mapsto v(t) \in L^2(\Omega)$ *is derivable to the right, verifies the equality*

$$\frac{d^+v}{dt}(t) + \Delta^2 v(t) + B(v(t)) = f(t), \quad t \in [0,T),$$

and the estimates

$$||v(t)||_{C([0,t];L^2(\Omega))} + ||v||_{L^2(0,t;V)} + ||v'||_{L^\infty(0,t;L^2(\Omega))} + ||v'||_{L^2(0,t;V)} \leq$$

$$\leq C\,\widetilde{M}_0(t)\,e^{\gamma\,t}, \ \forall t \in [0,T],$$

are true with C *and* γ *depending on* L, n, Ω, *and*

$$\widetilde{M}_0(t) = |u_0| + |B(u_0)| + |\Delta^2 u_0| + ||f||_{W^{1,2}(0,t;L^2(\Omega))}.$$

Remark 2. *In the conditions of Theorem 3,* $v \in C([0,T];L^2(\Omega))$, $v' \in L^\infty(0,T;L^2(\Omega))$, *the term* $\langle v'(t),\eta\rangle$ *in* (4) *can be expressed in the form* $(v'(t),\eta)$.

Theorem 4. *Let* $T > 0$. *Suppose that condition* **(B3)** *is fulfilled. If* $u_0 \in H^4(\Omega)\cap V$ *and* $f \in W^{1,1}(0,T;L^2(\Omega))$, *then there exists a unique solution to the problem* (P_0) *such that* $v \in C^1([0,T];L^2(\Omega))\cap C([0,T];V)$ *and the following estimates*

$$\big|\big|v\big|\big|_{C^1([0,t];L^2(\Omega))} + ||v||_{C([0,t];V)} + ||v'||_{L^2(0,t;V)} \le C\,\widetilde{M}_1(t), \quad \forall t\in[0,T],$$

hold, where $\widetilde{M}_1(t) = \big|u_0\big| + \big|\Delta^2 u_0\big| + ||f||_{W^{1,1}(0,t;H)} + |B(0)|\,t$.

3 A priori estimates for the solutions to the problem (P_ε)

In this section we prove some *a priori* estimates for the solutions to the problem (P_ε), which are uniform relative to the small values of the parameter ε.

Firstly we remind the following theorems.

Theorem 5. [14] *Let* $\Omega\subset\mathbb{R}^n$ *be an open and bounded set with the compact boundary of class* C^2. *If* $u,\Delta u \in L^2(\Omega)$, *then* $u\in H^2(\Omega)$ *and there exists a constant* $C_0(n,\Omega)$ *such that*

$$||u||_{H^2(\Omega)} \le C_0\big(||\Delta u||_{L^2(\Omega)} + ||u||_{L^2(\Omega)}\big). \tag{5}$$

Theorem 6. [1] *Let* $\Omega\subset\mathbb{R}^n$ *be an open and bounded set. For* $n > m\,l$ *if* $q \le \dfrac{m\,n}{n-m\,l}$ *and for* $n = m\,l$, $\forall q$, *the following inequality*

$$||u||_{L^q(\Omega)} \le C(q,m,n,\Omega)\,||u||_{W^{l,m}(\Omega)}, \quad \forall u\in W^{l,m}(\Omega)$$

is true.

For $n < m\,l$ *we have*

$$\max_{x\in\overline{\Omega}} |u(x)| \le C(q,m,n,\Omega)\,||u||_{W^{l,m}(\Omega)}, \quad \forall u\in W^{l,m}(\Omega).$$

In what follows, denote by $u(t)=u(t,\cdot)$, $u'(t)=u_t(t,\cdot)$.

Lemma 1. *Let* $u_0\in H^4(\Omega)\cap V$, $u_1\in V$, $f\in W^{1,2}(0,\infty;L^2(\Omega))$ *and condition* **(B1)** *is fulfilled. Then there exist some positive constants* $C = C(n,\Omega,L)$ *and* $\gamma(n,\Omega,L)$ *such that for every solution* u *to the problem* (P_ε) *the following estimates*

$$||u||_{C^1([0,t];L^2(\Omega))} + ||\Delta u||_{W^{1,\infty}(0,t;L^2(\Omega))} + ||u||_{W^{2,2}(0,t;L^2(\Omega))} \le$$

$$\le C M(t)\, e^{\gamma\, t}, \quad t\ge 0, \quad \varepsilon \in \Big(0,\frac{1}{2}\Big], \tag{6}$$

hold, where

$$M(t) = |\Delta^2 u_0| + |u_1| + |B(u_0)| + ||f||_{W^{1,2}(0,t;L^2(\Omega))}. \tag{7}$$

If, in addition, condition **(B2)** *is fulfilled and* $u_0, u_1, \alpha \in H^4 \cap V$, $f \in W^{2,2}(0,\infty; L^2(\Omega))$, *then there exist some positive constants* $\gamma = \gamma(n, \Omega, L, L_0, L_1)$, $C = C(n, \Omega, L, L_0, L_1)$ *such that for the function* z, *defined by*

$$z(t) = u'(t) + \alpha e^{-t/\varepsilon}, \quad \alpha = f(0) - u_1 - \Delta^2 u_0 - B(u_0), \tag{8}$$

the following estimates

$$||z||_{W^{1,\infty}(0,t;L^2(\Omega))} + ||z||_{W^{1,\infty}(0,t;V)} + ||z||_{W^{2,2}(0,t;L^2(\Omega))} \leq$$

$$\leq C\, M_0(t), \quad t \geq 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big], \tag{9}$$

are true with

$$M_0(t) = \big|\Delta^2 u_0\big| + \big|\Delta^2 u_1\big| + \big|\Delta^2 \alpha\big| + \big|\big|f\big|\big|_{W^{2,2}(0,\,t;L^2(\Omega))} + M^2(t)\, e^{2\,\gamma\, t}. \tag{10}$$

If $B = 0$, *then* $\gamma = 0$ *in* (6) *and in* (9).

Proof. *Proof of the estimate* (6). In what follows let us agree to denote all constants depending on n, Ω, L, L_0 and L_1 by the same constant C. Due to Theorem 1 we have that $u \in W^{2,\infty}(0,t;L^2(\Omega))$, $\Delta u \in W^{1,\infty}(0,t;L^2(\Omega))$, $\Delta^2 u \in L^\infty(0,t;L^2(\Omega))$ for every $t > 0$.

Let us denote by

$$E(u;t) = \varepsilon\, |u'(t)|^2 + |u(t)|^2 + 2\,\varepsilon\big(u(t), u'(t)\big) + |\Delta u(t)|^2 +$$

$$+2\,(1-\varepsilon)\int_0^t |u'(s)|^2\, ds + 2\int_0^t |\Delta u(s)|^2\, ds, \quad t \geq 0. \tag{11}$$

The direct computations show that for every solution to the problem (P_ε) the following equality

$$\frac{d}{dt} E(u;t) = 2\,\Big(f(t) - B(u), u(t) + u'(t)\Big), \quad a.e. \quad t \in [0, \infty), \tag{12}$$

is fulfilled. According to the condition **(B1)** and (5), we have

$$|B(u)| \leq |B(0)| + L\,||u(t)|| \leq |B(0)| + L\, C_0\,\big(|u(t)| + |\Delta u(t)|\big)$$

and

$$|u(t)|^2 + |\Delta u(t)|^2 \leq$$

$$\leq 2\left[\varepsilon\, |u'(t)|^2 + |u(t)|^2 + 2\,\varepsilon\big(u(t), u'(t)\big)\right] + |\Delta u(t)|^2 \leq 2\, E(u;t), \quad \varepsilon \in \Big(0, \frac{1}{2}\Big].$$

Then, we get

$$\Big|\Big(f(t) - B(u), u(t) + u'(t)\Big)\Big| \leq \big(|f(t)| + |B(0)| + L\,||u(t)||_{H^2(\Omega)}\big)\,\big(|u(t)| + |u'(t)|\big) \leq$$

$$\leq \Big[|f(t)| + |B(0)| + 2\sqrt{2}\, L\, C_0\, E^{1/2}(u;t)\Big] \left(\sqrt{2}\, E^{1/2}(u;t) + |u'(t)|\right) \leq$$

$$\leq \frac{1-\varepsilon}{2}\, |u'(t)|^2 + \frac{4}{1-\varepsilon} \left[2\, E(u;t) + \Big(|f(t)| + |B(0)| + 2\sqrt{2}\, L\, C_0\, E^{1/2}(u;t) \Big)^2 \right] \leq$$

$$\leq \frac{1-\varepsilon}{2}\, |u'(t)|^2 + \frac{4}{1-\varepsilon} \left[1 + 8\, L^2\, C_0^2 \right] E(u;t) + C\, (|f(t)| + |B(0)|)^2 \leq$$

$$\leq \gamma\, E(u;t) + C\, (|f(t)| + |B(0)|)^2 +$$

$$+ \frac{1-\varepsilon}{2}\, \frac{d}{dt} \int\limits_0^t |u'(s)|^2\, ds, \quad t \geq 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big], \tag{13}$$

where $\gamma = 8\,(1 + 8\, L^2\, C_0^2)$.

Therefore, from (12) it follows that

$$\frac{d}{dt} \left[E(u;t) - (1-\varepsilon) \int\limits_0^t |u'(s)|^2\, ds \right] \leq$$

$$\leq \gamma\, E(u;t) + C\, (|f(t)| + |B(0)|)^2, \quad t \geq 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big]. \tag{14}$$

As

$$E(u;t) \leq 2\, E_0(u;t), \quad \text{where} \quad E_0(u;t) = E(u;t) - (1-\varepsilon) \int\limits_0^t |u'(s)|^2\, ds, \tag{15}$$

then from (14) we obtain

$$\frac{d}{dt} \left[e^{-2\gamma\, t}\, E_0(u;t) \right] \leq C\, (|f(t)| + |B(0)|)^2\, e^{-2\,\gamma\, t}, \quad t \geq 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big].$$

Integrating this inequality, we get

$$E_0(u;t) \leq E_0(u;0)\, e^{2\gamma\, t} + C \int\limits_0^t (|f(s)| + |B(0)|)^2\, e^{2\,\gamma\,(t-s)}\, ds, \quad t \geq 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big].$$

From the last inequality it follows that

$$|u(t)| + |\Delta u(t)| + ||u'||_{L^2(0,t;L^2(\Omega))} + ||\Delta u||_{L^2(0,t;L^2(\Omega))} \leq$$

$$\leq C\, M(t)\, e^{\gamma\, t}, \quad t \geq 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big]. \tag{16}$$

To prove the estimate (6) let us denote by $u_h(t) = h^{-1}(u(t+h) - u(t))$, $h > 0$. For every solution to the problem (P_ε) the equality

$$\frac{d}{dt} E(u_h;t) = 2\left(F_h(t), u_h'(t) + u_h(t) \right), \quad a.e. \quad t \in [0, \infty), \tag{17}$$

is true, where

$$F_h(t) = f_h(t) - h^{-1}\Big((Bu)(t+h) + (Bu)(t)\Big). \tag{18}$$

Due to the condition (5), proceeding as in the proof of the estimate (13), we get

$$\Big|\Big(F_h(t), u_h'(t) + u_h(t)\Big)\Big| \le \big(|u_h(t)| + |u_h'(t)|\big)\,\big(|f_h(t)| + L\,||u_h(t)||_{H^2(\Omega)}\big) \le$$

$$\le \big(|u_h(t)| + |u_h'(t)|\big)\,\Big(|f_h(t)| + L\,C_0\,\big(|u_h(t)| + |\Delta u_h(t)|\big)\Big) \le$$

$$\le \gamma\,E(u_h;t) + C\,|f_h(t)|^2 + \frac{1-\varepsilon}{2}\,\frac{d}{dt}\int\limits_0^t |u_h'(s)|^2\,ds, \quad t \ge 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big]. \tag{19}$$

Consequently,

$$\frac{d}{dt}\Big[e^{-2\gamma\,t}\,E_0(u_h;t)\Big] \le C\,|f_h(t)|^2\,e^{-2\,\gamma\,t}, \quad t \ge 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big].$$

Integrating the last equality on $(0,t)$, we get

$$E_0(u_h;t) \le E_0(u_h;0)\,e^{2\gamma\,t} + C\int\limits_0^t |f_h(s)|^2\,e^{2\,\gamma\,(t-s)}\,ds, \quad t \ge 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big]. \tag{20}$$

Since for $1 \le p < \infty, k \in \mathbb{N}$ and $u \in W^{1,p}(0,T;H^k(\Omega))$ the inequality

$$\int_0^t ||u_h(\tau)||^p_{H^k(\Omega)}d\tau \le \int_0^t ||u'(\tau)||^p_{H^k(\Omega)}d\tau, \quad t \in [0,\infty), \tag{21}$$

is true (see [2]), then

$$\int_0^t |f_h(s)|^2\,ds \le \int_0^t |f'(s)|^2\,ds, \quad t \in [0,\infty). \tag{22}$$

As $u'(0) = u_1$, $\varepsilon u''(0) = f(0) - u_1 - \Delta^2 u_0 - B(u_0)$, then

$$E_0(u',0) \le C\,M(t). \tag{23}$$

Using the estimates (22), (23) and passing to the limit in the inequality (20) as $h \to 0$ we obtain the estimate

$$|u'(t)| + |\Delta u'(t)| + ||u''||_{L^2(0,t;L^2(\Omega))} + ||\Delta u'||_{L^2(0,t;L^2(\Omega))} \le$$

$$\le C\,M(t)\,e^{\gamma\,t}, \quad t \ge 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big]. \tag{24}$$

Finally, from (16) and (24) the inequality (6) follows.

It is easy to see from the proof, that in the case of $B = 0$, $\gamma = 0$.

Proof of the estimate (9). Under the conditions of the Lemma, if u is a solution to the problem (P_ε), then $\big(B(u)\big)' \in W^{1,1}(0,t;L^2(\Omega))$ for every $t>0$ and $\varepsilon \in \Big(0,\dfrac{1}{2}\Big]$. Indeed, due to the conditions **(B2)** and (5), we have

$$\big|\big(B(u(t))\big)'\big| = \big|B'((u(t))u'(t)\big| \le L_0\big|u'(t)\big|, \quad t \ge 0, \tag{25}$$

and for $u_h(t) = h^{-1}(u(t+h)-u(t))$, $h>0$ and $t>0$, the estimate

$$\Big|h^{-1}\Big(\big(B'(u(t))\big)\,u'(t)\Big)_h\Big| \le$$

$$\le \Big|h^{-1}\Big(B'\big(u(t+h)\big) - B'\big(u(t)\big)\Big)\,u'(t+h)\Big| + \Big|B'(u(t))\,u_h'(t)\Big| \le$$

$$\le L_1\,C_0^2\,\Big(\big|\Delta u_h(t)\big| + \big|u_h(t)\big|\Big)\,\Big(|\Delta u'(t+h)| + |u'(t+h)|\Big) + L_0\,|u_h'(t)|, \quad t \ge 0, \tag{26}$$

is valid.

Using the estimate (6) and inequality (21), from (25) and (26) we deduce that $\big(B(u)\big)' \in W^{1,2}(0,t;L^2(\Omega))$ and

$$\Big|\Big|\Big(\big(B'(u(t))\big)\Big)'\Big|\Big|_{L^2(0,T;L^2(\Omega))} \le$$

$$\le C\;M(t)\,e^{\gamma t}\,\Big(||\Delta u'||_{L^2(0,t;L^2(\Omega))} + ||u'||_{L^2(0,t;L^2(\Omega))}\Big) + L_0\,||u''||_{L^2(0,t;L^2(\Omega))},$$

$$\le C\,M^2(t)\,e^{2\gamma t}, \quad t \ge 0, \quad \varepsilon \in \Big(0,\frac{1}{2}\Big].$$

Therefore, $\big(B(u)\big)' \in W^{1,1}(0,t;L^2(\Omega))$ for $\varepsilon \in \Big(0,\dfrac{1}{2}\Big]$ and every $t>0$. If $u_1+\alpha \in H^4(\Omega)\cap V$ and $f \in W^{2,1}(0,t;L^2(\Omega))$, then, in virtue of Theorem 1, the function z, defined by (8), is the solution in $L^2(\Omega)$ to the problem

$$\begin{cases} \varepsilon\,z''(t) + z'(t) + \Delta^2 z(t) = \mathcal{F}(t,\varepsilon), & \text{a. e.} \quad t \ge 0, \\ z(0) = u_1 + \alpha, \quad z'(0) = 0, \end{cases} \tag{27}$$

with

$$\mathcal{F}(t,\varepsilon) = f'(t) - \Big(B\big(u(t)\big)\Big)' + e^{-t/\varepsilon}\,\Delta^2\alpha \tag{28}$$

and z possesses the properties:

$$z \in W^{2,\infty}(0,T;L^2(\Omega)), \quad \Delta z \in W^{1,\infty}(0,T;L^2(\Omega)), \quad \Delta^2 z \in L^\infty(0,T;L^2(\Omega)).$$

Furthermore

$$||\mathcal{F}(t,\varepsilon)||_{L^2(0,t;L^2(\Omega))} \le C\big(||f||_{W^{2,2}(0,t;L^2(\Omega))} + M^2(t)\,e^{2\gamma t}\big), \quad t \ge 0, \quad \varepsilon \in \Big(0,\frac{1}{2}\Big].$$

In the same way, as the estimate (16) was obtained in the case $B = 0$, we get the estimate

$$|z(t)| + |\Delta z(t)| + ||z'||_{L^2(0,t;L^2(\Omega))} + ||\Delta z||_{L^2(0,t;L^2(\Omega))} \leq$$
$$\leq C\, M_0(t), \quad t \geq 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big]. \tag{29}$$

Also, similarly as the estimate (24) was proved in the case $B = 0$, we prove the estimate

$$|z'(t)| + |\Delta z'(t)| + ||z''||_{L^2(0,t;L^2(\Omega))} + ||\Delta z'||_{L^2(0,t;L^2(\Omega))} \leq$$
$$\leq C\, M_0(t), \quad t \geq 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big]. \tag{30}$$

Finally, from (29) and (30) the inequality (9) follows. Lemma 1 is proved.

Lemma 2. *Suppose the condition* **(B3)** *is fulfilled. If* $u_0 \in H^4(\Omega) \cap V$, $u_1 \in V$ *and* $f \in W^{1,2}(0,\infty; L^2(\Omega))$, *then for every solution* u *to the problem* (P_ε) *the following estimates*

$$||u||_{C^1([0,\,t];L^2(\Omega))} + ||\Delta u||_{C^1([0,\,t];L^2(\Omega))} + \big|\big|\Delta u'\big|\big|_{L^2(0,t;L^2(\Omega))} + |\mathcal{B}(u)|^{1/2} \leq$$
$$\leq C(\mathbf{m})\, M_1(t)\, e^{\gamma(\mathbf{m})\,t}, \quad t \geq 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big]. \tag{31}$$

are true, where

$$M_1(t) = \big|\Delta^2 u_0\big| + \big|\Delta u_1\big| + \big|\big|f\big|\big|_{W^{1,2}(0,\,t;L^2(\Omega))} + |\mathcal{B}(u_0)|^{1/2} \tag{32}$$

and

$$\mathbf{m} = |\Delta u_0| + |u_1| + |\mathcal{B}(u_0)|^{1/2} + ||f||_{L^2(0,\infty;L^2(\Omega))}.$$

If, in addition, condition **(B4)** *is fulfilled and* $u_0, u_1, \alpha \in H^4(\Omega) \cap V$ *and* $f \in W^{2,2}(0,\infty; L^2(\Omega))$, *then for the function* z, *defined by* (8), *the estimates*

$$||\Delta z||_{C([0,\,t];L^2(\Omega))} + ||z'||_{C([0,\,t];L^2(\Omega))} + \big|\big|\Delta z'\big|\big|_{L^2(0,\,t;\,L^2(\Omega))} \leq$$
$$\leq C\, M_2(t)\, e^{\gamma(\mathbf{m})\,t}, \quad t \geq 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big]. \tag{33}$$

are true, where $C = C(\mathbf{m}, ||B'(0||)$ *and*

$$M_2(t) = M_1^2(t)\, e^{2\,\gamma(\mathbf{m})\,t} + ||f||_{W^{2,2}(0,t;L^2(\Omega))} + |\Delta^2 \alpha|. \tag{34}$$

Proof. *Proof of the estimate* (31). Due to Theorem 2 we have that $u \in C^2([0,T]; L^2(\Omega))$, $u' \in C^1([0,t]; V)$, $\Delta^2 u \in C([0,t]; L^2(\Omega))$ for every $t > 0$.

Denote by

$$E_1(u;t) = \varepsilon\, |u'(t)|^2 + |\Delta u(t)|^2 + 2 \int_0^t |u'(s)|^2 ds + 2\, \mathcal{B}(u(t)).$$

Then for every solution u to the problem (P_ε), we have

$$\frac{d}{dt} E_1(u;t) = 2\Big(f(t), u'(t)\Big), \quad t \geq 0.$$

Integrating this inequality, we obtain

$$E_1(u;t) \leq E_1(u;0) + 2\int_0^t |f(s)|\,|u'(s)|\,ds \leq \int_0^t |f(s)|^2\,ds + \int_0^t |u'(s)|^2\,ds, \quad t \geq 0.$$

Therefore, we get the estimate

$$\big|\big|\Delta u\big|\big|_{C([0,t;L^2(\Omega))} + ||u'||_{L^2(0,t;\,L^2(\Omega))} + \Big(\mathcal{B}\big(u(t)\big)\Big)^{1/2} \leq$$

$$\leq C\left(E_1^{1/2}(u,0) + ||f||_{L^2(0,t;L^2(\Omega))} + |\mathcal{B}(u_0)|^{1/2}\right), \quad t \geq 0, \quad \varepsilon \in (0,1].$$

As $||u||_{L^2(\Omega)} \leq C(n,\Omega)\,||\Delta u||_{L^2(\Omega)}$ for $u \in V$, then from the last inequality the estimate

$$\big|\big|u\big|\big|_{C([0,t];L^2(\Omega))} + \big|\big|\Delta u\big|\big|_{C([0,t];L^2(\Omega))} + ||u'||_{L^2(0,t;L^2(\Omega))} + \Big(\mathcal{B}\big(u(t)\big)\Big)^{1/2} \leq$$

$$\leq C\,\mathbf{m}, \quad t \geq 0, \quad \varepsilon \in (0,1), \tag{35}$$

follows.

Let $u_h(t) = h^{-1}\big(u(t+h) - u(t)\big)$, $h > 0$, $t \geq 0$ and the functional $E(u,t)$ is defined by (11). For every solution u to the problem (P_ε) the equality (17) is true with $F_h(t)$ defined by (18).

Due to (5), conditions **(B3)** and the estimate (35), proceeding as in the proof of the estimate (19), we obtain

$$\left|\Big(F_h(t), u_h'(t) + u_h(t)\Big)\right| \leq \big(|u_h(t)| + |u_h'(t)|\big)\big(|f_h(t)| + L(\mathbf{m})\,||u_h(t)||_{H^2(\Omega)}\big) \leq$$

$$\leq \big(|u_h(t)| + |u_h'(t)|\big)\Big(|f_h(t)| + L(\mathbf{m})\big(|u_h(t)| + |\Delta u_h(t)|\big)\Big) \leq$$

$$\leq \gamma(\mathbf{m})\,E(u_h;t) + C(\mathbf{m})\,|f_h(t)|^2 + \frac{1-\varepsilon}{2}\,\frac{d}{dt}\int_0^t |u_h'(s)|^2\,ds, \quad t \geq 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big].$$

Consequently, for $E_0(u;t)$, defined by (15), we have

$$\frac{d}{dt}\left[e^{-2\gamma(\mathbf{m})\,t}\,E_0(u_h;t)\right] \leq C(\mathbf{m})\,|f_h(t)|^2\,e^{-2\,\gamma\,t}, \quad t \geq 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big].$$

Integrating the last equality on $(0,t)$, we get

$$E_0(u_h;t) \le E_0(u_h;0)\, e^{2\,\gamma(\mathbf{m})\,t} + C(\mathbf{m}) \int\limits_0^t |f_h(s)|^2\, e^{2\,\gamma(\mathbf{m})\,(t-s)}\, ds, \quad t\ge 0, \quad \varepsilon \in \Big(0,\frac{1}{2}\Big].$$

In what follows, proceeding as in the proof of the estimate (24), we get the estimate

$$||u'||_{C([0,t];L^2(\Omega))} + ||\Delta u'||_{C([0,t];L^2(\Omega))} + ||\Delta u'||_{L^2(0,t;L^2(\Omega))} \le$$
$$\le C(\mathbf{m})\, M_1(t)\, e^{\gamma(\mathbf{m})\,t}, \quad t\ge 0, \quad \varepsilon\in\Big(0,\frac{1}{2}\Big], \tag{36}$$

with $M_1(t)$ from (32). Finally, from (35) and (36) the inequality (31) follows.

Proof of the estimate (33). Under the conditions of Lemma we have $\big(B(u)\big)' \in W^{1,1}(0,t;L^2(\Omega))$ for every $t>0$. Indeed, due to Thorem 2, $u\in W^{3,\infty}(0,t;L^2(\Omega))$ and $\Delta u \in W^{2,\infty}(0,t;L^2(\Omega))$ for every $t>0$. Therefore, using the condition **(B4)** and the estimate (31), we deduce

$$\big|\big(B(u(t))\big)'\big| = \big|B'(u(t))\,u'(t)\big| \le C\big(L_1(\mathbf{m}) + ||B'(0)||\big)\, ||u'(t)||_{H^2(\Omega)}, \quad t>0.$$

For $h>0$, $t>0$ and $u_h(t) = h^{-1}\big(u(t+h)-u(t)\big)$ we have

$$\Big|h^{-1}\Big(\big(B(u(t))\big)'\Big)_h\Big| \le$$
$$\le \Big|h^{-1}\Big(B'\big(u(t+h)\big) - B'\big(u(t)\big)\Big)\,u'(t+h)\Big| + \Big|B'\big(u(t)\big)\,u_h'(t)\Big| \le$$
$$\le L_1(\mathbf{m})\,M_1(t)\,e^{\gamma(\mathbf{m})\,t}\,||u_h(t)||_{H^2(\Omega)} + C\,\big(L_1(\mathbf{m}) + ||B'(0)||\big)\,||u_h'||_{H^2(\Omega)} \le$$
$$\le C\,\Big(L_1(\mathbf{m})\,M_1(t)\,e^{\gamma(\mathbf{m})\,t} + ||B'(0)||\Big)\,\big(||u_h(t)||_{H^2(\Omega)} + ||u_h'||_{H^2(\Omega)}\big), \quad t>0. \tag{37}$$

In virtue of (22), (31) and (37), we conclude that $\Big(\big(B'(u)\big)\Big)' \in W^{1,2}(0,t;L^2(\Omega))$ for every $t>0$ and

$$\Big|\Big|\Big(\big(B'(u(t))\big)\Big)'\Big|\Big|_{L^2(0,t;L^2(\Omega))} \le$$
$$\le C(\mathbf{m}, ||B'(0||)\, M_1^2(t)\, e^{\gamma(\mathbf{m})\,t}, \quad t>0, \quad \varepsilon\in\Big(0,\frac{1}{2}\Big]. \tag{38}$$

From (38) it follows that the function $\mathcal{F}$, which is defined by (28), belongs to $W^{1,1}(0,t;L^2(\Omega))$, for every $t>0$, and

$$||\mathcal{F}(t,\varepsilon)||_{L^2(0,t;L^2(\Omega))} \le C(\mathbf{m}, ||B'(0||)\, M_2(t)\, e^{\gamma(\mathbf{m})\,t}, \quad t\ge 0, \quad \varepsilon\in\Big(0,\frac{1}{2}\Big]. \tag{39}$$

According to Theorem 2, for every $t>0$, the function z possesses the following properties: $z\in W^{2,\infty}(0,t;L^2(\Omega))$, $\Delta z \in W^{1,\infty}(0,t;L^2(\Omega))$, $\Delta^2 z \in L^\infty(0,t;L^2(\Omega))$. The estimate (33) is obtained in the same way as the estimate (9) was obtained, using (31) and (39). Lemma 2 is proved.

4 Relationship between solutions to the problems (P_ε) and (P_0) in the linear case

In this section we establish the relationship between solutions to the problems (P_ε) and (P_0) in the linear case, i.e. in the case when the term $B(u)$ in the problems (P_ε) and (P_0) is missing. This relationship was inspired by the work [12]. Firstly we give some properties of the kernel $K(t,\tau,\varepsilon)$ of the transformation which realizes this connection.

For $\varepsilon > 0$ denote by

$$K(t,\tau,\varepsilon) = \frac{1}{2\sqrt{\pi}\,\varepsilon}\Big(K_1(t,\tau,\varepsilon) + 3\,K_2(t,\tau,\varepsilon) - 2\,K_3(t,\tau,\varepsilon)\Big),$$

where

$$K_1(t,\tau,\varepsilon) = \exp\Big\{\frac{3\,t-2\,\tau}{4\,\varepsilon}\Big\}\lambda\Big(\frac{2\,t-\tau}{2\sqrt{\varepsilon\,t}}\Big), \quad K_2(t,\tau,\varepsilon) = \exp\Big\{\frac{3\,t+6\,\tau}{4\,\varepsilon}\Big\}\lambda\Big(\frac{2\,t+\tau}{2\sqrt{\varepsilon\,t}}\Big),$$

$$K_3(t,\tau,\varepsilon) = \exp\Big\{\frac{\tau}{\varepsilon}\Big\}\lambda\Big(\frac{t+\tau}{2\sqrt{\varepsilon\,t}}\Big), \quad \lambda(s) = \int\limits_s^\infty e^{-\eta^2}\,d\eta.$$

The properties of the kernel $K(t,\tau,\varepsilon)$ are collected in the following lemma.

Lemma 3 [9] *The function $K(t,\tau,\varepsilon)$ is the solution to the problem*

$$\begin{cases} K_t(t,\tau,\varepsilon) = \varepsilon K_{\tau\tau}(t,\tau,\varepsilon) - K_\tau(t,\tau,\varepsilon), \quad \forall t>0, \quad \forall \tau > 0, \\ \varepsilon\,K_\tau(t,0,\varepsilon) - K(t,0,\varepsilon) = 0, \quad \forall t \geq 0 \\ K(0,\tau,\varepsilon = \dfrac{1}{2\,\varepsilon}\exp\Big\{-\dfrac{\tau}{2\,\varepsilon}\Big\}, \quad \forall \tau \geq 0, \end{cases}$$

from $C([0,\infty)\times[0,\infty))\cap C^2((0,\infty)\times(0,\infty))$ and possesses the following properties:

(i) *$K(t,\tau,\varepsilon) > 0, \quad \forall t \geq 0, \quad \forall \tau \geq 0,$ and $\displaystyle\int_0^\infty K(t,\tau,\varepsilon)\,d\tau = 1, \quad \forall t \geq 0;$*

(ii) *Let $q \in [0,1]$. Then $\displaystyle\int_0^\infty K(t,\tau,\varepsilon)\,|t-\tau|^q\,d\tau \leq C\,\big(\varepsilon + \sqrt{\varepsilon\,t}\big)^q$, $\forall \varepsilon > 0$, $\forall t \geq 0$;*

(iii) *Let $\gamma > 0$ and $q \in [0,1]$. There exist C_1, C_2 and ε_0, all of them positive and depending on γ and q, such that the following estimates are fulfilled:*

$$\int_0^\infty K(t,\tau,\varepsilon)\,e^{\gamma\tau}|t-\tau|^q\,d\tau \leq C_1\,e^{C_2 t}\,\varepsilon^{q/2}, \quad \forall \varepsilon \in (0,\varepsilon_0], \ \forall t > 0;$$

(iv) *Let $p \in (1,\infty]$ and $f:\,[0,\,\infty) \to H$, $f(t) \in W^{1,p}(0,\infty;H)$. Then*

$$\Big|f(t) - \int_0^\infty K(t,\tau,\varepsilon)f(\tau)d\tau\Big| \leq C(p)\,\|f'\|_{L^p(0,\infty;H)}\,\big(\varepsilon + \sqrt{\varepsilon\,t}\big)^{\frac{p-1}{p}}, \ \forall \varepsilon > 0, \ \forall t \geq 0.$$

Theorem 7.[9] *Suppose that* $f \in L^\infty_\gamma(0,\infty;L^2(\Omega))$, $u \in W^{2,\infty}_\gamma(0,\infty;L^2(\Omega)) \cap L^\infty_\gamma(0,\infty;V)$ *and* $\Delta^2 u \in L^{2,\infty}_\gamma(0,\infty : V')$ *is the solution to the problem*

$$\begin{cases} \varepsilon(u''(t),\eta) + (u'(t),\eta) + \big(\Delta u(t),\Delta\eta\big) = (f(t),\eta), \ \forall \eta \in V, \quad a.\ e.\ t \in [0,\infty), \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases}$$

then for $0 < \varepsilon < (4\,\gamma)^{-1}$ *the function*

$$w_0(t) = \int_0^\infty K(t,\tau,\varepsilon) u(\tau) d\tau$$

is solution to the problem

$$\begin{cases} (w_0'(t),\eta) + \big(\Delta w_0(t),\Delta\eta\big) = (F_0(t,\varepsilon)\,u_1,\eta), \ \forall \eta \in V, \qquad a.e.\ t \in [0,\infty), \\ w_0 = \varphi_\varepsilon, \end{cases}$$

where

$$F_0(t,\varepsilon) = f_0(t,\varepsilon)\,u_1 + \int_0^\infty K(t,\tau,\varepsilon)\,f(\tau) d\tau,$$

$$f_0(t,\varepsilon) = \frac{1}{\sqrt{\pi}}\Big[2\,\exp\Big\{\frac{3\,t}{4\,\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big) - \lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big], \quad \varphi_\varepsilon = \int_0^\infty e^{-\tau} u(2\,\varepsilon\,\tau) d\tau.$$

Moreover, $w_0 \in W^{2,\infty}_{\mathrm{loc}}(0,\infty;L^2(\Omega)) \cap L^\infty_{\mathrm{loc}}(0,\infty;V)$.

5 Behaviour of solutions to the problem (P_ε)

In this section we prove the main results concerning the behavior of the solutions to the problem (P_ε) as $\varepsilon \to 0$ relative to solution to the corresponding unperturbed problem (P_0).

Theorem 8. *Let* $T > 0$ *and* $p \in [2,\infty]$. *Assume that* **(B1)** *is fulfilled. If* $u_0 \in H^4(\Omega) \cap V$, $u_1 \in V$ *and* $f \in W^{1,p}(0,T;L^2(\Omega))$, *then there exist constants* $C = C(L,T,p,\Omega,n) > 0$ *and* $\varepsilon_0 = \varepsilon_0(L,p,\Omega,n)$ *such that*

$$||u - v||_{C([0,T];L^2(\Omega))} \leq C\,M(T)\,\varepsilon^\beta, \quad t \in [0,T], \quad \varepsilon \in (0,\varepsilon_0], \tag{40}$$

$$||u - v||_{L^\infty(0,T;V)} \leq C\,M(T)\,\varepsilon^\beta, \quad t \in [0,T], \quad \varepsilon \in (0,\varepsilon_0], \tag{41}$$

where u *and* v *are solutions to the problems* (P_ε) *and* (P_0), *respectively,*

$$M(T) = |\Delta^2 u_0| + ||u_1|| + |B(u_0)| + ||f||_{W^{1,p}(0,T;L^2(\Omega))}, \tag{42}$$

$$\beta = \begin{cases} 1/2 \text{ if } f = 0, \\ (p-1)/(2\,p) \text{ if } f \neq 0. \end{cases}$$

If, in addition, condition **(B2)** *is fulfilled and* $u_0, u_1, \alpha \in H^4(\Omega) \cap V$ *and* $f \in W^{2,p}(0,T; L^2(\Omega))$*, then there exist constants* $\varepsilon_0 = \varepsilon_0(L, L_0)$*,* $\varepsilon_0 \in (0,1)$*,* $\gamma = \gamma(L, L_0, L_1)$*,* $C = C(p, L, L_0, L_1)$ *such that*

$$||u' - v' + \alpha\, e^{-t/\varepsilon}||_{C([0,T]\,;L^2(\Omega))} + ||u' - v' + \alpha\, e^{-t/\varepsilon}||_{L^2(0,T\,;H^2(\Omega))} \leq$$

$$\leq C\, M_0(T)\, e^{\gamma\, t}\, \varepsilon^{\beta}, \quad t \geq 0, \quad \varepsilon \in (0, \varepsilon_0], \tag{43}$$

with $M_0(T)$ *defined by* (10).

Proof. In this section, we agree to denote by C all constants depending on T, p, Ω, n, L, L_0 and L_1. For every $f \in W^{k,p}(0,T; L^2(\Omega))$ then there exists the extension $\widetilde{f} : [0,\infty) \mapsto L^2(\Omega)$ such that

$$||\widetilde{f}||_{W^{k,p}(0,\infty;L^2(\Omega))} \leq C(T,p)\, ||f||_{W^{k,p}(0,T;L^2(\Omega))}. \tag{44}$$

If we denote by $\widetilde{U}$ the unique solution to the problem (P_ε), defined on $(0,\infty)$ instead of $(0,T)$ and $\widetilde{f}$ instead of f, then, from Theorem 1 and Lemma 1, it follows that $\widetilde{U} \in W^{2,\infty}(0,\infty; L^2(\Omega))$, $\widetilde{U}' \in L^2(0,\infty; L^2(\Omega))$, $\Delta^2 \widetilde{U} \in L^\infty(0,\infty; L^2(\Omega))$. Due to the estimates (24), for $\widetilde{U}$ we obtain the following estimates

$$\big|\big|\tilde{U}'\big|\big|_{C([0,t];L^2(\Omega))} + \big|\big|\Delta \tilde{U}'\big|\big|_{L^\infty(0,t;L^2(\Omega))} \leq C\, M(T)\, e^{\gamma t}, \quad t \geq 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big], \tag{45}$$

with $M(T)$ from (42) and γ from (13).

By Theorem 7, the function W defined by $W(t) = \displaystyle\int_0^\infty K(t,\tau,\mu)\, \widetilde{U}(\tau)\, d\tau$, is a solution to the problem

$$\begin{cases} W'(t) + \Delta^2 W(t) = F(t,\varepsilon), \quad \text{a.e.} \quad t > 0, \quad \text{in} \quad L^2(\Omega), \\ W(0) = \varphi_\varepsilon, \end{cases} \tag{46}$$

where

$$F(t,\varepsilon) = f_0(t,\varepsilon)\, u_1 + \int_0^\infty K(t,\tau,\varepsilon)\, \widetilde{f}(\tau)\, d\tau - \int_0^\infty K(t,\tau,\varepsilon)\, B(\widetilde{U}(\tau))\, d\tau,$$

$$\varphi_\varepsilon = \int_0^\infty e^{-\tau}\, \widetilde{U}(2\,\varepsilon\tau)\, d\tau.$$

Denote by $R(t,\varepsilon) = \widetilde{V}(t) - W(t)$, where $\widetilde{V}$ is the solution to the problem (P_0) with $\tilde{f}$ instead of f, $T = \infty$ and W is the solution to the problem (46). Then, due to

Theorem 2, $R(\cdot,\varepsilon)\in W^{2,\infty}_{\text{loc}}(0,\infty;\,L^2(\Omega))\cap L^2(0,\infty;V)$ and R is a solution in $L^2(\Omega)$ to the problem

$$\begin{cases} R'(t,\varepsilon)+\Delta^2 R(t,\varepsilon)+B\big(\widetilde{V}(t)\big)-B\big(W(t)\big)=\mathcal{F}(t,\varepsilon), \quad \text{a.e.} \quad t>0,\\ R(0,\varepsilon)=u_0-W(0), \end{cases} \tag{47}$$

where

$$\mathcal{F}(t,\varepsilon)=\tilde{f}(t)-\int_0^\infty K(t,\tau,\varepsilon)\tilde{f}(\tau)\,d\tau-f_0(t,\varepsilon)\,u_1+$$

$$+B\big(\widetilde{U}(t)\big)-B\big(W(t)\big)+\int_0^\infty K(t,\tau,\varepsilon)\Big[B\big(\widetilde{U}(\tau)\big)-B\big(\widetilde{U}(t)\big)\Big]\,d\tau. \tag{48}$$

In what follows, we need the following two Lemmas, which will be proved after the proof of the estimates (40) and (41).

Lemma 4. *Assume the conditions of Theorem 8 are fulfilled. Then there exist constants $C=C(L,\Omega,n)$, $C_0=C_0(L,\Omega,n)$ and $\varepsilon_0=\varepsilon_0(L,\Omega,n)$ such that following estimates*

$$|\widetilde{U}(t)-W(t)|\le C\,M(T)\,\varepsilon^{1/2}\,e^{C_0\,t}, \quad t\ge 0, \quad \varepsilon\in(0,\varepsilon_0], \tag{49}$$

$$||\widetilde{U}(t)-W(t)||_{L^\infty(0,t;V)}\le C\,M(T)\,\varepsilon^{1/2}\,e^{C_0\,t}, \quad t\ge 0, \quad \varepsilon\in(0,\varepsilon_0], \tag{50}$$

are true with $M(T)$ from (42).

Lemma 5. *Assume the conditions of Theorem 8 are fulfilled. Then there exist constants $C=C(L,\Omega,n)$, $c_0=c_0(L,\Omega,n)$ and $\varepsilon_0=\varepsilon_0(L,\Omega,n)$ such that for the solution to the problem* (47) *the following estimates*

$$||R||_{C([0,\,t];\,L^2(\Omega))}+||\Delta R||_{L^2(0,t;L^2(\Omega))}\le$$

$$\le C\,M(T)\,e^{c_0\,t}\,\varepsilon^{(p-1)/(2\,p)}, \quad t\ge 0, \quad \varepsilon\in(0,\varepsilon_0], \tag{51}$$

$$||R||_{L^\infty(0,\,t;H^2(\Omega))}\le C\,M(T)\,e^{c_0\,t}\,\varepsilon^{(p-1)/(2\,p)}, \quad t\ge 0, \quad \varepsilon\in(0,\varepsilon_0], \tag{52}$$

are true with $M(T)$ from (42).

From the last two lemmas we deduce that

$$||\tilde{U}-\tilde{V}||_{C([0,t];L^2(\Omega))}\le||\tilde{U}-W||_{C([0,t];L^2(\Omega))}+||R||_{C([0,t];L^2(\Omega))}\le$$

$$\le C\,M(T)\,e^{C_0\,t}\,\varepsilon^{\beta}, \quad t\ge 0, \quad \varepsilon\in(0,\varepsilon_0].$$

Since $u(t)=\tilde{U}(t)$, $v(t)=\tilde{V}(t)$, for all $t\in[0,T]$, then we have

$$|u(t)-v(t)|=\big|\tilde{U}(t)-\tilde{V}(t)\big|\le C\,M(T)\,\varepsilon^{\beta}, \quad t\in[0,T], \quad \varepsilon\in(0,\varepsilon_0]. \tag{53}$$

Concequently, from (53) the estimate (40) follows. Similarly, using (50) and (52), we obtain the estimate (41).

Proof of Lemma 4. Using the properties (**i**), (**ii**) and (**iii**) from Lemma 3, the estimate (45) and the Hölder's inequality, we get

$$|\widetilde{U}(t) - W(t)| \leq \int\limits_0^\infty K(t,\tau,\varepsilon)\left|\widetilde{U}(t) - \widetilde{U}(\tau)\right| d\tau \leq$$

$$\leq \int\limits_0^\infty K(t,\tau,\varepsilon)\Big|\int\limits_\tau^s \left|\widetilde{U}'(\xi)\right| d\xi\Big|\, d\tau \leq C\,M(T)\int\limits_0^\infty K(t,\tau,\varepsilon)\Big|\int\limits_\tau^t e^{\gamma\,\xi}\, d\xi\Big|\, d\tau \leq$$

$$\leq C\,M(T)\int\limits_0^\infty K(t,\tau,\varepsilon)\,|\tau - t|\left[e^{\gamma\,t} + e^{\gamma\,\tau}\right] d\tau \leq$$

$$\leq C\,M(T)\left[e^{\gamma\,t}\int\limits_0^\infty K(t,\tau,\varepsilon)\,|\tau - t|\, d\tau + \int\limits_0^\infty K(t,\tau,\varepsilon)\,|\tau - t|\, e^{\gamma\,\tau}\, d\tau\right] \leq$$

$$\leq C\,M(T)\,e^{C_2\,t}\,\varepsilon^{1/2}, \quad t \geq 0, \quad \varepsilon \in (0,\varepsilon_0]. \tag{54}$$

Thus, the estimate (49) is proved.

In the same way, using properties (**i**), (**i**) and (**iii**) from Lemma 3, the estimate (45) and the Hölder's inequality, we get

$$\left|\Delta\widetilde{U}(t) - \Delta W(t)\right| \leq \int\limits_0^\infty K(t,\tau,\varepsilon)\left|\Delta\widetilde{U}(t) - \Delta\widetilde{U}(\tau)\right| d\tau \leq$$

$$\leq \int\limits_0^\infty K(t,\tau,\varepsilon)\Big|\int\limits_\tau^s \left|\Delta\widetilde{U}'(\xi)\right| d\xi\Big|\, d\tau \leq C\,M(T)\,e^{C_2\,t}\,\varepsilon^{1/2}, \quad t \geq 0, \quad \varepsilon \in (0,\varepsilon_0]. \tag{55}$$

Due to Theorem 5, we have that

$$||\widetilde{U} - W||_{L^\infty(0,t;V)} = ||\widetilde{U} - W||_{L^\infty(0,t;H^2(\Omega))} \leq$$

$$\leq C\left[||\widetilde{U} - W||_{L^\infty(0,t;L^2(\Omega))} + ||\Delta\widetilde{U} - \Delta W||_{L^\infty(0,t;L^2(\Omega))}\right].$$

From the last inequality, using (49) and (55), we get (50). Lemma 4 is proved.

Proof of Lemma 5. *Proof of the estimate* (51). Multiplying scalarly in $L^2(\Omega)$ the equation (47) by R and using the condition **(B1)** and Theorem 5 we obtain the inequality

$$\frac{d}{dt}|R(t,\varepsilon)|^2 + 2\,|\Delta R(t,\varepsilon)|^2 \leq 2\,|\mathcal{F}(t,\varepsilon)|\,|R(t,\varepsilon)| + 2\,L\,||R(t,\varepsilon)||_{H^2(\Omega)}\,|R(t,\varepsilon)| \leq$$

$$\leq 2\,|\mathcal{F}(t,\varepsilon)|\,|R(t,\varepsilon)| + C_0\,L\,\big(|R(t,\varepsilon)| + |\Delta R(t,\varepsilon)|\big)\,|R(t,\varepsilon)|, \quad t \geq 0,$$

from which it follows that

$$\frac{d}{dt}|R(t,\varepsilon)|^2 + |\Delta R(t,\varepsilon)|^2 \leq 2\,|\mathcal{F}(t,\varepsilon)|^2 + 2\,\gamma_1\,|R(t,\varepsilon)|^2, \quad t \geq 0,$$

or

$$\frac{d}{dt}\Big[|R(t,\varepsilon)|^2\,e^{-2\,\gamma_1\,t}\Big] + |\Delta R(t,\varepsilon)|^2\,e^{-2\,\gamma_1\,t} \leq 2\,|\mathcal{F}(t,\varepsilon)|^2\,e^{-2\,\gamma_1\,t}, \quad t \geq 0,$$

with some γ_1 depending on L and constant C_0 from Theorem 5. Integrating on $(0,t)$ the last equality, we deduce

$$|R(t,\varepsilon)| + ||\Delta R(\cdot,\varepsilon)||_{L^2(0,t;L^2(\Omega))} \leq$$

$$\leq C\,\Big[|R(0,\varepsilon)| + ||\mathcal{F}(\cdot,\varepsilon)|_{L^2(0,t;L^2(\Omega))}\Big]\,e^{\gamma_1\,t}, \quad \forall t \geq 0, \tag{56}$$

where $\mathcal{F}(t,\varepsilon)$ is defined by (48). In what follows, we will estimate the right side of (56). Using (45), we get

$$\big|R(0,\varepsilon)\big| \leq \int\limits_0^\infty e^{-\tau}\Big|\tilde{U}(2\varepsilon\,\tau) - u_0\Big|\,d\tau \leq \int\limits_0^\infty e^{-\tau}\int\limits_0^{2\,\varepsilon\,\tau}\big|\tilde{U}'(\xi)\big|\,d\xi\,d\tau \leq$$

$$\leq C\,M(T)\,\varepsilon\int\limits_0^\infty \tau\,e^{-\tau}\,d\tau = C\,M(T)\,\varepsilon, \quad \varepsilon \in \Big(0,\frac{1}{2}\Big]. \tag{57}$$

Using the property (**iv**) from Lemma 3 and (44), we deduce

$$\Big|\tilde{f}(t) - \int\limits_0^\infty K(t,\tau,\varepsilon)\tilde{f}(\tau)\,d\tau\Big| \leq C\,\|\tilde{f}'\|_{L^p(0,\infty\,;\,L^2(\Omega))}\,(\varepsilon + \sqrt{\varepsilon\,t})^{(p-1)/p} \leq$$

$$\leq C\,\|f'\|_{L^p(0,T\,;\,L^2(\Omega))}\,\big(\varepsilon + \sqrt{\varepsilon\,t}\big)\big)^{(p-1)/p}, \quad t \geq 0, \quad \varepsilon > 0. \tag{58}$$

Since $e^{\xi}\lambda(\sqrt{\xi}) \leq C, \quad \forall \xi \geq 0$, then the following estimates

$$\int\limits_0^t \exp\Big\{\frac{3\xi}{4\,\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{\xi}{\varepsilon}}\Big)d\xi \leq C\,\varepsilon\int\limits_0^\infty e^{-\xi/4}\,d\xi \leq C\,\varepsilon, \quad t \geq 0, \quad \varepsilon > 0,$$

$$\int\limits_0^s \lambda\Big(\frac{1}{2}\sqrt{\frac{\xi}{\varepsilon}}\Big)\,d\xi \leq \varepsilon\int_0^\infty \lambda\Big(\frac{1}{2}\sqrt{\xi}\Big)\,d\xi \leq C\,\varepsilon, \quad t \geq 0, \quad \varepsilon > 0$$

hold. Consequently

$$\Big|\int\limits_0^t f_0(\xi,\varepsilon)\,u_1 d\xi\Big| \leq C\,\varepsilon|u_1|, \quad t \geq 0, \quad \varepsilon > 0. \tag{59}$$

Using **(B1)**, (5) and the estimates (49) and (50), we get the following estimates

$$\big|B(\widetilde{U}(t)) - B(W(t))\big| \le$$
$$\le L\,\|\widetilde{U}(t) - W(t)\|_{H^2(\Omega)} \le C\,M(T)\,\varepsilon^{1/2}\,e^{c_0\,t}, \quad t \ge 0, \varepsilon \in (0, \varepsilon_0]. \tag{60}$$

Similarly as the estimate (54) was obtained, we get

$$\int\limits_0^{\infty} K(t,\tau,\varepsilon)\,\big|B(\widetilde{U}(\tau)) - B(\widetilde{U}(t))\big|\,d\tau \le C\,M(T)\,e^{C_2\,t}\,\varepsilon^{1/2}, \quad t \ge 0, \quad \varepsilon \in (0,\varepsilon_0]. \tag{61}$$

Using (58), (59), (60) and (61), from (48) we get

$$\big|\mathcal{F}(\tau,\varepsilon)\big| \le C\,M(T)\,e^{C_2\,t}\,\varepsilon^{(p-1)/(2\,p)}, \quad t \ge 0, \quad \varepsilon \in (0,\varepsilon_0].$$

Consequently,

$$\Big(\int\limits_0^t \big|\mathcal{F}(\tau,\varepsilon)\big|^2\,d\tau\Big)^{1/2} \le C\,M(T)\,e^{C_2\,t}\,\varepsilon^{(p-1)/(2\,p)}, \quad t \ge 0, \quad \varepsilon \in (0,\varepsilon_0]. \tag{62}$$

From (56), using (57) and (62) we get the estimate (51).

Proof of the estimate (52). From Theorem 3 it follows that $R \in W^{1,2}_{\text{loc}}(0,t;V) \cap W^{1,\infty}_{\text{loc}}(0,t;L^2(\Omega))$ and $\Delta^2 R \in L^2_{\text{loc}}(0,\infty;L^2(\Omega))$. Moreover the function $t \mapsto (\Delta^2 R(t,\varepsilon), R(t,\varepsilon))$ is an absolutely continuous function on $[0,T]$ for every $T > 0$ and

$$\frac{d}{dt}(\Delta^2 R(t,\varepsilon), R(t,\varepsilon)) = 2(\Delta^2 R(t,\varepsilon), R'(t,\varepsilon)), \quad \text{a. e.} \quad t > 0.$$

Multiply the equation (47) by $\Delta^2 R(t,\varepsilon)$ and then integrate on $(0,t)$ to get

$$\big|\Delta R(t,\varepsilon\big|^2 + 2\int\limits_0^t \big|\Delta^2 R(s,\varepsilon)\big|^2\,ds =$$
$$= \big|\Delta R(0,\varepsilon)\big|^2 + 2\int\limits_0^t \big(\mathcal{F}(s,\varepsilon) - B\big(\widetilde{V}(s)\big) + B\big(W(s)\big), \Delta^2 R(s,\varepsilon)\big)\,ds, \quad t \ge 0.$$

Therefore,

$$|\Delta R(t,\varepsilon)|^2 + \int\limits_0^t \big|\Delta^2 R(s,\varepsilon)\big|^2\,ds \le$$
$$\le \big|\Delta R(0,\varepsilon)\big|^2 + \int\limits_0^t \Big[\big|\mathcal{F}(s,\varepsilon)\big|^2 + \big|B\big(\widetilde{V}(s)\big) - B\big(W(s)\big)\big|^2\Big]\,ds, \quad t \ge 0.$$

From the last inequality, using (62) and (51), we obtain

$$\big|\Delta R(t,\varepsilon\big| + ||\Delta^2 R||_{L^2(0,t;L^2(\Omega))} \leq$$

$$\leq C\Big[|\Delta R(0,\varepsilon)| + ||\mathcal{F}||_{L^2(0,t;L^2(\Omega))} + L\,||R||_{L^2(0,t;L^2(\Omega))}\Big] \leq$$

$$\leq C\Big[|\Delta R(0,\varepsilon)| + M(T)\, e^{C_2\, t}\, e^{(p-1)/(2\,p)}\Big], \quad t>0, \quad \varepsilon\in(0,\varepsilon_0]. \tag{63}$$

Using (45), we get

$$|\Delta R(0,\varepsilon)| \leq \int\limits_0^\infty e^{-s}\,|\Delta(\tilde{U}(2\,\varepsilon\, s) - u_0))|\,ds \leq$$

$$\leq \int\limits_0^\infty e^{-s} \int\limits_0^{2\,\varepsilon\, s} |\Delta \tilde{U}'(\tau)|\,d\tau\, ds \leq C\, M(T)\,\varepsilon, \quad \varepsilon \leq \frac{\gamma}{4}. \tag{64}$$

From (63) and (64) it follows that

$$\big|\Delta R(t,\varepsilon\big| \leq C\, M(T)\, e^{C_2\, t}\, e^{(p-1)/(2\,p)}, \quad t>0, \quad \varepsilon\in(0,\varepsilon_0]. \tag{65}$$

As, due to Theorem 5, we have that

$$||R||_{L^\infty(0,t;V)} = ||R||_{L^\infty(0,t;H^2(\Omega))} \leq C_0\,\big[||R||_{L^\infty(0,t;L^2(\Omega))} + ||\Delta R||_{L^\infty(0,t;L^2(\Omega))}\big],$$

then using (51) and (65) we get (52). Lemma 5 is proved.

Proof of the estimate (43). According to Lemma 1, the function $\tilde{z}$, defined as

$$\tilde{z}(t) = \tilde{U}'(t) + \alpha e^{-t/\varepsilon}, \quad \alpha = \tilde{f}(0) - u_1 - \Delta^2 u_0 - B(u_0),$$

is solution to the problem (27) with

$$\mathcal{F}(t,\varepsilon) = \tilde{f}'(t) - \Big(B\big(\tilde{U}(t)\big)\Big)' + e^{-t/\varepsilon}\,\Delta^2\alpha$$

and $\tilde{z}$ satisfies the following estimate

$$||\tilde{z}||_{W^{1,\infty}(0,t;L^2(\Omega))} + ||\tilde{z}||_{W^{1,\infty}(0,t;V)} + ||\tilde{z}||_{W^{2,2}(0,t;L^2(\Omega))} \leq$$

$$\leq C\, M_0(t), \quad t\geq 0, \quad \varepsilon\in\Big(0,\frac{1}{2}\Big], \tag{66}$$

wherein, due to inequality (44), with the same $M_0(t)$ from (10).

As $\tilde{z}'(0)=0$, then according to Theorem 7, the function

$$w_1(t) = \int\limits_0^\infty K(t,\tau,\varepsilon)\,\tilde{z}(\tau)\,d\tau,$$

is solution to the following problem:

$$\begin{cases} w_1'(t)+\Delta^2 w_1(t)=F_1(t,\varepsilon), \quad \text{a. e.} \quad t>0, \quad \text{in} \quad L^2(\Omega),\\ w_1(0)=\varphi_{1\varepsilon}, \end{cases}$$

for $0<\varepsilon\le\varepsilon_0$, where

$$F_1(t,\varepsilon)=\int_0^\infty K(t,\tau,\varepsilon)\tilde f\,'(\tau)\,d\tau-\int_0^\infty K(t,\tau,\varepsilon)\big(B(\tilde U)\big)'(\tau)\,d\tau+$$

$$+\int_0^\infty K(t,\tau,\varepsilon)\,e^{-\tau/\varepsilon}\,d\tau\,\Delta^2\alpha,\quad \varphi_{1\varepsilon}=\int_0^\infty e^{-\tau}\tilde z(2\varepsilon\tau)\,d\tau.$$

Using the properties (**i**), (**ii**) and (**iii**) from Lemma 3 and the estimate (66) and proceeding as in the proof of estimate (54), we get

$$\big|\tilde z(t)-w_1(t)\big|\le\int_0^\infty K(t,\tau,\varepsilon)\big|\tilde z(t)-\tilde z(\tau)\big|\,d\tau\le\int_0^\infty K(t,\tau,\varepsilon)\Big|\int_\tau^t\big|\tilde z\,'(s)\big|\,ds\Big|\,d\tau\le$$

$$\le C\,M_0(T)\,e^{C_2\,t}\,\varepsilon^{1/2},\quad t\ge0,\quad \varepsilon\in(0,\varepsilon_0]. \tag{67}$$

In the same way, using (66), we obtain the estimate

$$\big|\big|\tilde z-w_1\big|\big|_{L^\infty(0,\,t;H^2(\Omega))}\le C\,M_0(T)\,e^{C_2\,t}\,\varepsilon^{1/2},\quad t\ge0,\quad \varepsilon\in(0,\varepsilon_0]. \tag{68}$$

Let $v_1(t)=v'(t)$, where v is solution to the problem (P_0) with $\tilde f$ instead of f, $T=\infty$.

Denote by $R_1(t,\varepsilon)=v_1(t)-w_1(t)$. Then the function $R_1(t,\varepsilon)$ is solution to the problem

$$\begin{cases} R_1'(t,\varepsilon)+\Delta^2R_1(t,\varepsilon)=\mathcal{F}_1(t,\varepsilon), \quad \text{a. e.} \quad t>0, \quad \text{în} \quad L^2(\Omega)\\ R_1(0,\varepsilon)=R_{10}=:f(0)-\Delta^2u_0-B(u_0)-\varphi_{1\varepsilon}, \end{cases}$$

where

$$\mathcal{F}_1(t,\varepsilon)=\tilde f\,'(t)-\int_0^\infty K(t,\tau,\varepsilon)\,\tilde f'(\tau)\,d\tau+\int_0^\infty K(t,\tau,\varepsilon)\,e^{-\tau/\varepsilon}\,d\tau\,\Delta^2\alpha-$$

$$-\big(B(v)\big)'(t)+\int_0^\infty K(t,\tau,\varepsilon)\,\big(B(\tilde U_\varepsilon)\big)'(\tau)d\tau. \tag{69}$$

Due to the conditions of Theorem 8, similarly as the inequality (56) was obtained, and the estimates (57), (62), we get the inequality

$$||R_1||_{C([0,t];\,H)}+||\Delta R_1||_{L^2(0,t;L^2(\Omega))}\le$$

$$\leq C\Big[|R_{10}| + ||\mathcal{F}_1(\cdot,\varepsilon)|_{L^2(0,t;L^2(\Omega))}\Big]\, e^{\gamma_1 t}, \quad \forall t \geq 0. \tag{70}$$

and the estimates

$$|R_{10}| \leq \int\limits_0^\infty e^{-\tau}\, |\tilde{z}(2\,\varepsilon\tau) - \tilde{z}(0)|\, d\tau \leq$$

$$\leq \int\limits_0^\infty e^{-\tau} \int_0^{2\varepsilon\tau} |\tilde{z}'_\varepsilon(s)|\, ds\, d\tau \leq C\, M_0(T)\, \varepsilon, \quad \varepsilon \in (0,\varepsilon_0], \tag{71}$$

$$||\mathcal{F}_1(\cdot,\varepsilon)|_{L^2(0,t;L^2(\Omega))} \leq C\, M_0(T)\, e^{C_2\, t}\, \varepsilon^{(p-1)/(2\,p)}, \quad t \geq 0, \quad \varepsilon \in (0,\varepsilon_0]. \tag{72}$$

From (70), using (71) and (72), we get the estimate

$$||R_1||_{C([0,t];L^2(\Omega))} + ||\Delta R_1||_{L^2(0,t;L^2(\Omega))} \leq C\, M_0(T)\, e^{C_2\, t}\, \varepsilon^{(p-1)/(2\,p)}, \tag{73}$$

$t \geq 0, \quad \varepsilon \in (0,\varepsilon_0]$.

Finally, due to (5), from (67), (68) and (73) the estimate (43) follows. Theorem 8 is proved.

Similarly, using Theorems 3 and 4 instead of Theorems 1 and 2 and Lemma 2 instead of Lemma 1, the following theorem is proved.

Theorem 9. *Let $T > 0$ and $p \in [2,\infty]$. Assume that* **(B3)** *is fulfilled. If $u_0 \in H^4(\Omega) \cap V$, $u_1 \in V$ and $f \in W^{1,p}(0,T;L^2(\Omega))$, then there exist constants $C = C(\mathbf{m},T,p,\Omega,n) > 0$ and $\varepsilon_0 = \varepsilon_0(\mathbf{m},p,\Omega,n)$, such that*

$$||u - v||_{C([0,T];L^2(\Omega))} \leq C\, M(T)\, \varepsilon^\beta, \quad t \in [0,T], \quad \varepsilon \in (0,\varepsilon_0], \tag{74}$$

$$||u - v||_{L^\infty(0,T;V)} \leq C\, M(T)\, \varepsilon^\beta, \quad t \in [0,T], \quad \varepsilon \in (0,\varepsilon_0], \tag{75}$$

where u and v are solutions to the problems (P_ε) and (P_0), respectively,

$$M(T) = |\Delta^2 u_0| + ||u_1|| + |B(u_0)| + |\mathcal{B}(u_0)|^{1/2} + ||f||_{W^{1,p}(0,T;L^2(\Omega))},$$

$$\mathbf{m} = |\Delta u_0| + |u_1| + |\mathcal{B}(u_0)|^{1/2} + ||f||_{L^2(0,T;L^2(\Omega))}, \quad \beta = \begin{cases} 1/2 \text{ if } f = 0, \\ (p-1)/(2\,p) \text{ if } f \neq 0. \end{cases}$$

If, in addition, condition **(B4)** *is fulfilled and $u_0, u_1, \alpha \in H^4(\Omega) \cap V$ and $f \in W^{2,p}(0,T;L^2(\Omega))$, then there exist constants $\varepsilon_0 = \varepsilon_0(L,L_0)$, $\varepsilon_0 \in (0,1)$, $\gamma = \gamma(L,L_0,L_1)$, $C = C(p,L,L_0,L_1)$ such that*

$$||u' - v' + \alpha\, e^{-t/\varepsilon}||_{C([0,T];L^2(\Omega))} + ||u' - v' + \alpha\, e^{-t/\varepsilon}||_{L^2(0,T;H^2(\Omega))} \leq$$

$$\leq C\, M_2(T)\, e^{\gamma\, t}\, \varepsilon^\beta, \quad t \geq 0, \quad \varepsilon \in (0,\varepsilon_0] \tag{76}$$

with $M_2(T)$ defined by (34).

6 An Examples

In this section, we present some applications of Theorems 8 and 9, which are determined by different operators B.

The Lipschitzian case. Let the operator B be one of the following: $B(u) = |u|$, or $B = |\nabla u|$, or $B(u) = \sin u$. In these cases it is easy to check that for the operator B the conditions (**B1**) are fulfilled. Consequently, for every $T > 0$ and every $p \in [2, \infty]$, if $u_0 \in H^4(\Omega) \cap V$, $u_1 \in V$ and $f \in W^{1,2}(0, T; L^2(\Omega))$, then from Theorem 8 the estimates (40) and (41) follow.

For $B(u) = \sin u$, due to Theorem 6, condition **(B2)** is fulfilled if $1 \leq n \leq 12$. Indeed, for $n = 1, 2, 3, 4$, Theorem 6 ensures the fulfillment of the condition (**B2**). For $n > 4$, using the Hölder's inequality and Theorem 6, we have that

$$\int\limits_{\Omega} \left|\big(B'(u_1) - B'(u_2)\big)\, v\right|^2 dx \leq \int\limits_{\Omega} \left|\big(\cos(u_1) - \cos(u_2)\big)\, v\right|^2 dx \leq$$

$$\leq 4 \int\limits_{\Omega} \left|\sin\big((u_1 - u_2)/2\big)\, v\right|^2 dx \leq C \int\limits_{\Omega} \big|u_1 - u_2\big|\, |v|^2\, dx \leq$$

$$\leq C \Big(\int\limits_{\Omega} \big|u_1 - u_2\big|^{2\,n/(n-4)}\, dx\Big)^{(n-4)/(2\,n)} \times \Big(\int\limits_{\Omega} |v|^{4\,n/(n+4)}\, dx\Big)^{(n+4)/(2\,n)} \leq$$

$$\leq C\, ||u_1 - u_2||_{H^2(\Omega)}\, ||v||^2_{L^{4\,n/(n+4)}(\Omega)} \leq C\, ||u_1 - u_2||_{H^2(\Omega)}\, ||v||^2_{H^2(\Omega)}, \quad \text{if} \quad 5 \leq n \leq 12.$$

Therefore, if $u_0, u_1, \alpha \in H^4(\Omega) \cap V$ and $f \in W^{2,p}(0, T; L^2(\Omega))$, then the estimate (43) also holds. It means that

$$u \to v \quad \text{in} \quad C([0, T]; L^2(\Omega)) \cap L^\infty(0, T; H^2(\Omega)), \quad \text{as} \quad \varepsilon \to 0. \tag{77}$$

At the same time, the relation (43) shows that in this case the derivative u' of the solution to the problem (P_ε) does not converge to the derivative v' of the solution to the problem (P_0). In this case the derivative u' has a singular behavior in the neighborhood of the point $t = 0$ as $\varepsilon \to 0$. This singular behavior is described by the function $\alpha\, e^{-t/\varepsilon}$, which is *the boundary layer function* for u'. If $\alpha = 0$, then

$$u' \to v' \quad \text{in} \quad C([0, T]; L^2(\Omega)), \quad \text{as} \quad \varepsilon \to 0. \tag{78}$$

The monotone case. Let $B : D(B) = L^2(\Omega) \cap L^{2(q+1)}(\Omega) \mapsto L^2(\Omega)$, $B(u) = b\, |u|^q\, u$, $b > 0$.

Then the operator B is the Fréchet derivative of the convex and positive functional $\mathcal{B}$, defined as follows

$$D(\mathcal{B}) = L^{q+2}(\Omega) \cap L^2(\Omega), \quad \mathcal{B}u = \frac{b}{q+2} \int_\Omega |u(x)|^{q+2}\, dx$$

and the Fréchet derivative of the operator B is defined by the relations

$$D\big(B'(u)\big) = \{v \in L^2(\Omega) : u^q\, v \in L^2(\Omega)\}, \quad B'(u)v = b\,(q+1)|u|^q\, v.$$

In what follows, to check the fulfillment of the condition **(B3)** for the operator B we apply Theorem 6.

If $n>4$ and $q\in[0,4/(n-4)]$, then using the Hölder's inequality and Theorem 6, we get

$$||Bu_1-Bu_2||^2_{L^2(\Omega)}=b^2\int\limits_\Omega \Big||u_1(x)|^q u_1(x)-|u_2(x)|^q u_2(x)\Big|^2\,dx\le$$

$$\le C(q,b)\int\limits_\Omega \big|u_1(x)-u_2(x)\big|^2\Big(\big|u_1(x)\big|^{2q}+\big|u_2(x)\big|^{2\,q}\Big)\,dx\le$$

$$\le C(q,n,b)||u_1-u_2||^2_{L^{2\,n/(n-4)}(\Omega)}\Big(||u_1||^{2\,q}_{L^{q\,n/2}(\Omega)}+||u_2||^{2\,q}_{L^{q\,n/2}(\Omega)}\Big)\le$$

$$\le C(q,b,n,\Omega)||u_1-u_2||^2_{H^2(\Omega)}\Big(||u_1||^{2q}_{H^2(\Omega)}+||u_2||^{2q}_{H^2(\Omega)}\Big),\quad u_1,\,u_2\in V. \tag{79}$$

Similarly, using the Hölder's inequality and Theorem 6, it is not difficult to prove the estimate (79) in the case $q\in[0,\infty]$ for $n=1,2,3,4$.

Thus, if

$$\begin{cases} b\ge 0, \\ q\in[0,4/(n-4)], \quad \text{if} \quad n>4, \\ q\in[0,\infty], \quad \text{if} \quad n=1,2,3,4 \end{cases} \tag{80}$$

then the operator B verifies condition **(B3)**.

Finally, if $u_0\in H^4(\Omega)\cap V$, $u_1\in V$ and $f\in W^{1,p}(0,T;L^2(\Omega))$ and conditions (80) are met, then, by virtue of Theorem 9, the estimates (74) and (75) and hence the relations (77) and are also valid.

If $n>4$ and $q\in[1,4/(n-4)]$, then, according to Theorem 6, we have

$$||\big(B'(u_1)-B'(u_2)\big)v||^2_{L^2(\Omega)}=b^2(q+1)^2\int_\Omega \Big||u_1(x)|^q-|u_2(x)|^q\Big|^2\,|v(x)|^2\,dx\le$$

$$\le C(q,b)\int_\Omega |u_1(x)-u_2(x)|^2\Big(|u_1(x)|^{2(q-1)}+|u_2(x)|^{2(q-1)}\Big)\,|v(x)|^2\,dx\le$$

$$\le C(q,b)\,||v||^2_{L^{2n/(n-4)}(\Omega)}\,||u_1-u_2||^2_{L^{2n/(n-(n-4)q)}(\Omega)}\times$$

$$\times\Big(||u_1||^{2(q-1)}_{L^{2n/(n-4)}(\Omega)}+||u_2||^{2(q-1)}_{L^{2n/(n-4)}(\Omega)}\Big)\le$$

$$\le C(n,q,b,\Omega,\omega)\,||u_1-u_2||^2_{H^2(\Omega)}\,||v||^2_{H^2(\Omega)}\Big(||u_1||^{2(q-1)}_{H^2(\Omega)}+||u_2||^{2(q-1)}_{H^2(\Omega)}\Big). \tag{81}$$

Involving the Hölder's inequality and Theorem 6, we get the inequality (81) in the cases $n=1,2,3,4$ and $q\ge 1$. Therefore, if

$$\begin{cases} b\ge 0, \\ q\in[1,4/(n-4)] \quad \text{if} \quad n>4, \\ q\in[1,\infty] \quad \text{if} \quad n=1,2,3,4, \end{cases}$$

then the operator B verifies the condition (**B4**). Therefore, if $u_0, u_1, \alpha \in H^4(\Omega) \cap V$ and $f \in W^{2,p}(0,T; L^2(\Omega))$, then the estimate (76) is fulfilled. Also, as in the Lipschitzian case, this relationship shows that the derivative u' of solution to the problem (P_ε) does not converge to the derivative v' of solution to the problem (P_0). In this case the derivative u' has a singular behavior in the neighborhood of the point $t = 0$ as $\varepsilon \to 0$. This singular behavior is described by the function $\alpha\, e^{-t/\varepsilon}$, which is *the boundary layer function* for u'. If $\alpha = 0$, then as in the Lipschitzian case the relation (78) is true.

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Asymptotic Behavior of Homogeneous Linear Recurrent Processes and Their Perturbations

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Abstract. In this paper the impact of small perturbations on asymptotic evolution of homogeneous linear recurrent processes is investigated. Analytical methods for describing homogeneous linear recurrent systems, from convergence, periodicity and boundedness perspective, are presented. These methods are based on Jury Stability Criterion and the classification of the roots of minimal characteristic polynomial in relation to unit disc.

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1 Introduction

The main goal of this paper is to study the impact of small perturbations on asymptotic evolution of homogeneous linear recurrent processes.

It is started with definitions and main properties of homogeneous linear recurrent processes. The direct formula for the states and the formula for generating function are given. Also, the linear combination and the product are presented as algebraic operations over the set of homogeneous linear recurrences.

Next, the definition of minimality, over a given set, is introduced. Inequalities for the dimension of the linear combination and product are presented. We formulate the minimization method based on matrix rank definition and the minimization method by elimination of characteristic zeros.

After that, we are interested in asymptotic behavior of homogeneous linear recurrences. The convergence criteria and the efficient formula for calculating the limit are given. The Jury Stability Criterion is proposed as alternative, for the case when the characteristic roots are not known.

Next, we continue with investigation of the main probabilistic characteristics of homogeneous linear recurrent distributions. The top of interest is represented by efficient methods for finding the expectation, the variance, the standard deviation, the moments of order n, the median and the mode of these distributions.

The last section is devoted to the perturbations generated by deviations in initial state or deviations in generating vector components. Also, mixed perturbations are considered. The asymptotic stability is studied and the maximal perturbation impact is estimated.

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2 Homogeneous Linear Recurrent Processes

The homogeneous linear recurrences and their main properties were intensively studied in [3] and [4]. Next, they will be briefly recalled and new extensions will be presented. These results will represent the ground of a new analytical method for studying the small perturbations and their impact on asymptotic evolution.

2.1 Main Definitions and Properties

A non-degenerate homogeneous linear m-recurrence over a set K is defined as a sequence $a = \{a_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$ that satisfies the recurrence

$$a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k},\ \forall n \geq m,$$

for a given positive integer m, generating vector $q = (q_k)_{k=0}^{m-1} \in K^m$ and initial state $I_m^{[a]} = (a_n)_{n=0}^{m-1}$, where $q_{m-1} \neq 0$.

The function $G^{[a]}(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ is the generating function and the function $G_t^{[a]}(z) = \sum\limits_{n=0}^{t-1} a_n z^n$ is the partial generating function of order t of the sequence a.

For this sequence a with generating vector q, the unit characteristic polynomial $H_m^{[q]}(z) = 1 - zG_m^{[q]}(z)$ and the characteristic equation $H_m^{[q]}(z) = 0$ are defined. Every polynomial $H_{m,\alpha}^{[q]}(z) = \alpha H_m^{[q]}(z)$ is, also, considered a characteristic polynomial of a.

The set $G[K][m](a)$ represents the set of all generating vectors of length m and $H[K][m](a)$ represents the set of characteristic polynomials of degree m of the sequence a. The set $Rol[K][m]$ is the set of all non-degenerate homogeneous linear m-recurrences over K.

Additionally, the sets $Rol[K] = \bigcup\limits_{m=1}^{\infty} Rol[K][m]$, $G[K](a) = \bigcup\limits_{m=1}^{\infty} G[K][m](a)$ and $H[K](a) = \bigcup\limits_{m=1}^{\infty} H[K][m](a)$ are considered.

Next, it is considered that the set K is a subfield of $\mathbb{C}$. The following theorem, theoretically grounded in [4], describes the generating function as a simple formula:

Theorem 1. *Let $a \in Rol[K][m]$ and $q = (q_k)_{k=0}^{m-1} \in G[K][m](a)$. The generating function is a rational fraction for which the following formula holds:*

$$G^{[a]}(z) = \frac{G_m^{[a]}(z) - z\sum\limits_{k=0}^{m-1} q_k z^k G_{m-1-k}^{[a]}(z)}{H_m^{[q]}(z)}.$$

Also, the following result presents us the direct formula for calculating the terms of a homogeneous linear recurrence:

Theorem 2. *Let $a \in Rol[K][m]$ with generating vector $q \in G[K][m](a)$ and characteristic polynomial $H^{[q]}_{m,\alpha}(z) = \prod\limits_{k=0}^{p-1}(z-z_k)^{s_k}$, where $z_i \neq z_j$, $\forall i \neq j$. Considering for convenience $0^0 = 1$, the direct formula for calculating the terms of sequence a is*

$$a_n = I^{[a]}_m \cdot ((B^{[a]})^T)^{-1} \cdot (\beta^{[a]}_n)^T, \ \forall n \in \mathbb{N},$$

where $\beta^{[a]}_n = \left(n^j z_k^{-n}\right)_{k=\overline{0,p-1},\ j=\overline{0,s_k-1}}$, $\forall n \in \mathbb{N}$, and $B^{[a]} = (\beta^{[a]}_i)_{i=0}^{m-1}$.

Another important result from [4] is the fact that the linear combination and product are algebraic operations over $Rol[K]$. More exactly, the next theorems hold.

Theorem 3. *Let $a^{(j)} \in Rol[K]$, $P_j(z) \in H[K](a^{(j)})$ and $\alpha_j \in \mathbb{C}$, $j = \overline{1,t}$. Then $a = \sum\limits_{k=1}^{t} \alpha_k a^{(k)} \in Rol[K]$ and $P(z) = lcm(P_1(z), P_2(z), \ldots, P_t(z)) \in H[K](a)$.*

Theorem 4. *Consider that $a \in Rol[K][m]$, $b \in Rol[K][1]$, $(q_0) \in G[K][1](b)$ and $P(z) \in H[K][m](a)$. Then, $ab = (a_n b_n)_{n=0}^{\infty} \in Rol[K][m]$ and $P(q_0 z) \in H[K](ab)$.*

Theorem 5. *Consider $a \in Rol[\mathbb{C}][m_1]$, $b \in Rol[\mathbb{C}][m_2]$, $u \in G[\mathbb{C}][m_1](a)$ and $v \in G[\mathbb{C}][m_2](b)$. Let $z_0, z_1, \ldots, z_{p-1}$ be all distinct complex roots, of multiplicity $s_0, s_1, \ldots, s_{p-1}$ correspondingly, of the polynomial $H^{[u]}_{m_1}(z)$; $z^*_0, z^*_1, \ldots, z^*_{p^*-1}$ be all distinct complex roots, of multiplicity $s^*_0, s^*_1, \ldots, s^*_{p^*-1}$ correspondingly, of the polynomial $H^{[v]}_{m_2}(z)$. Then, $ab \in Rol[\mathbb{C}]$ and*

$$P(z) = lcm(\{(z - z_k z^*_r)^{s_k + s^*_r - 1} | k = \overline{0,p-1}, r = \overline{0,p^*-1}\}) \in H[\mathbb{C}](ab).$$

2.2 Minimization Methods

The non-zero sequence (with at least one non-zero element) $a \in Rol[K]$ is called m-minimal over K if $a \in Rol[K][m]$ and $a \notin Rol[K][t]$, $\forall t < m$. In this case, the number m represents the dimension of the sequence a over K and it is denoted $dim[K](a) = m$. The dimension of the zero sequence is considered 0.

It is obvious that $dim[K](a) \leq m$, $\forall a \in Rol[K][m]$. Also, if $K_1 \subseteq K_2$ and $a \in Rol[K_1]$, then $a \in Rol[K_2]$ and $dim[K_2](a) \leq dim[K_1](a)$.

According to Theorem 3, if $a^{(k)} \in Rol[K]$ and $\alpha_k \in \mathbb{C}$, $k = \overline{1,t}$, then

$$dim[K]\left(\sum_{k=1}^{t} \alpha_k a^{(k)}\right) \leq \sum_{k=1}^{t} dim[K](a^{(k)}).$$

Additionally, from Theorem 5, for $\forall a^{(k)} \in Rol[\mathbb{C}]$, $k = \overline{1,t}$, we have the inequality

$$dim[\mathbb{C}]\left(\prod_{k=1}^{t} a^{(k)}\right) \leq \prod_{k=1}^{t} dim[\mathbb{C}](a^{(k)}).$$

It is known from [4] that the minimal generating vector is unique, i.e.

$$|G[K][dim[K](a)](a)| = 1.$$

This unique minimal generating vector determines the unique minimal unit characteristic polynomial $P(z) \in H[K][dim[K](a)](a)$. We may omit the word "unit" and consider $P(z)$ as the minimal characteristic polynomial of a. This polynomial allows us to describe the set of all characteristic polynomials in the following way:

$$H[K](a) = \{Q(z) \in K[z] \mid Q(z) \vdots P(z),\ Q(0) \neq 0\}.$$

The minimization problem consists in finding the dimension of the non-zero sequence a and its minimal generating vector over K. According to [4], there are two minimization methods over $\mathbb{C}$: the minimization method based on matrix rank definition and the minimization method by elimination of characteristic zeros.

Theorem 6. *If $a \in Rol[\mathbb{C}][m]$, then $dim[\mathbb{C}](a) = R = rank(A_m^{[a]})$ and the minimal generating vector is $q = (q_0, q_1, \ldots, q_{R-1}) \in G[\mathbb{C}][R](a)$, where the reverse vector $x = (q_{R-1}, q_{R-2}, \ldots, q_0)$ is the unique solution of the system with linear equations $A_R^{[a]} x^T = (f_R^{[a]})^T$ with free terms $f_R^{[a]} = (a_R, a_{R+1}, \ldots, a_{2R-1})$ and the system matrix $A_R^{[a]} = (a_{i+j})_{i,j=\overline{0,R-1}}$.*

Theorem 7. *Let $a \in Rol[\mathbb{C}][m]$, $x = I_m^{[a]}((B^{[a]})^T)^{-1} = (A_{k,j})_{k=\overline{0,p-1},\ j=\overline{0,s_k-1}}$, t_k be the number of zeros from the end of $(A_{k,j})_{j=\overline{0,s_k-1}}$, $k = \overline{0,p-1}$ and $t = \sum_{k=0}^{p-1} t_k$. Then $dim[\mathbb{C}](a) = m - t$ and $Q(z) = \dfrac{P(z)}{\prod_{k=0}^{p-1}(z-z_k)^{t_k}} \in H[\mathbb{C}][m-t](a)$, where z_k, $k = \overline{0,p-1}$, are all distinct roots of the polynomial $P(z) \in H[\mathbb{C}][m](a)$.*

These methods also can be used for minimization over a subset K of $\mathbb{C}$. Having determined the minimal characteristic polynomial over $\mathbb{C}$, the second step is to find a multiple of minimal degree for it, through the divisors of characteristic polynomial over K, which has the free term -1 and the rest of coefficients belonging to K.

The minimization method based on matrix rank definition is more applicable than the minimization method by elimination of characteristic zeros, because it does not suppose to know the complex roots of the characteristic polynomial.

3 Asymptotic Behavior of Homogeneous Linear Recurrences

In this section, the asymptotic behavior of homogeneous linear recurrences is studied. The convergence criteria and the efficient formula for calculating the limit are given. The Jury Stability Criterion is proposed as alternative, for the case when the characteristic roots are not known.

3.1 Convergence Criterion Based on Characteristic Zeros

According to [4], the convergence criterion is given by the following theorem. Practically, the classification of the roots of minimal characteristic polynomial gives us the information about the asymptotic behavior of given homogeneous linear recurrent process.

Theorem 8. *Consider $a \in Rol[\mathbb{C}][m]$ a non-zero sequence with $dim[\mathbb{C}](a) = m$ and $P(z) \in H[\mathbb{C}][m](a)$. Let $z_0, z_1, \ldots, z_{p-1}$ be all distinct roots of the polynomial $P(z)$, of corresponding multiplicity $s_0, s_1, \ldots, s_{p-1}$. The sequence a is convergent if and only if $|z_k| > 1$ or ($z_k = 1$ and $s_k = 1$), $k = \overline{0, p-1}$.*

In other words, the minimal characteristic polynomial of the convergent sequence $a \in Rol[\mathbb{C}][m]$ has at most one simple root equal to 1. The rest of the roots lie outside of the unit disc.

Moreover, if a is convergent, the limit can be easily calculated. We have $\lim\limits_{n\to\infty} a_n = 0$ in the case when $P(1) \neq 0$, and $\lim\limits_{n\to\infty} a_n = (I_m^{[a]}((B^{[a]})^T)^{-1})_{t_0}$ in the case when $P(1) = 0$. Next, according to minimization method by elimination of characteristic zeros, we have $\lim\limits_{n\to\infty} a_n \neq 0$ when $P(1) = 0$. In this situation, to avoid the need for knowing the roots of minimal characteristic polynomial, the sequence a is transformed into a linear $(m-1)$ – recurrence with a constant inhomogeneity.

Theorem 9. *Let*

$$a \in Rol[\mathbb{C}][m],\ P(z) = H_m^{[p]}(z) \in H[\mathbb{C}][m](a),\ P(1) = 0,$$

where $m = dim[\mathbb{C}](a) \geq 2$. Then, the sequence a is a linear $(m-1)$ – recurrence over $\mathbb{C}$, generated by vector $q = (q_0, q_1, \ldots, q_{m-2})$ and inhomogeneity

$$r_{m-1} = a_{m-1} - \sum_{k=0}^{m-2} q_k a_{m-2-k},$$

where

$$q_k = \sum_{j=0}^{k} p_j - 1,\ k = \overline{0, m-2}.$$

If, additionally, a is convergent, then

$$\lim_{n\to\infty} a_n = \frac{r_{m-1}}{1 - \sum\limits_{k=0}^{m-2} q_k} \neq 0.$$

If there is at least one root, of the minimal characteristic polynomial, which lies inside of the unit disc, then a diverges to infinity. The same thing happens when there is at least one multiple root on the unit circle. Instead, if all the roots are simple roots of unity, then a is periodic. When all the roots are simple roots of unity or lie outside of the unit disc, then a is bounded.

3.2 Jury Stability Criterion

Let $a \in Rol[\mathbb{C}][m]$ with minimal characteristic polynomial

$$P(z) = H_m^{[p]}(z) \in H[\mathbb{C}][m](a).$$

The Jury Stability Criterion, described in [1] and [2], can be applied for studying the localization of the roots of reciprocal polynomial $P^*(z)$ of $P(z)$ in relation to unit circle, without finding the roots. Basically, the calculations are organized as a table, where

- the columns correspond to monomials of $P^*(z)$, ordered in descending order by exponent;
- the first row contains the coefficients of $P^*(z)$;
- each further even row $2k+2$ contains the numbers from previous row in reverse order;
- each further odd row $2k+3$ is calculated by subtracting α times the previous even row from the previous odd row, where $\alpha = \beta_{2k+2}/\beta_{2k+1}$, β_{2k+2} is the first element from previous even row $2k+2$ and β_{2k+1} is the first element from previous odd row $2k+1$;
- the table is expanded until the last row of the table contains only one non-zero element.

Since $\beta_1 = 1 > 0$, then for every negative value from the sequence $\beta_1, \beta_3, \beta_5, \ldots$ the polynomial $P^*(z)$ has one root outside of the unit disc, i.e. the polynomial $P(z)$ has one root inside the unit disc. So, for stability, it is needed all these values $\beta_1, \beta_3, \beta_5, \ldots$ to be non-negative.

A particular additional result, which is involved from [1], is the fact that we need to have at least $P(1) > 0$, $P(-1) > 0$ and $|p_{m-1}| < 1$ in order all the roots of $P(z)$ lie outside of unit disc. For instance, based on [3], this does not happen when $P(z) \in \mathbb{Z}[z]$. Instead, the homogeneous linear recurrent distributions satisfy this property.

4 Homogeneous Linear Recurrent Distributions

Let consider a nonnegative integer random variable ξ with probabilistic distribution $rep(\xi) = a = (a_n)_{n=0}^{\infty}$. This means that a_n represents the probability that random variable ξ has the value n, for each $n = 0, 1, 2, \ldots$, i.e. $a_n = \mathbb{P}(\xi = n)$, $n = \overline{0, \infty}$.

According to [4], the main probabilistic characteristics of random variable ξ are: the expectation $\mathbb{E}(\xi)$, the moments $\nu_n(\xi) = \mathbb{E}(\xi^n)$ $(n = \overline{1, \infty})$, the variance $\mathbb{V}(\xi) = \nu_2(\xi) - \nu_1^2(\xi)$ and the standard deviation $\sigma(\xi) = \sqrt{\mathbb{V}(\xi)}$. Two additional probabilistic characteristics, that are useful for solving various stochastic problems, are the mode μ, for which $a_\mu = \max_{n\in\mathbb{N}} a_n$, and the median m_0, that satisfies the double inequality $\mathbb{P}(\xi < m_0) < \frac{1}{2} \le P(\xi \le m_0)$, equivalent with $\sum_{k=0}^{m_0-1} a_k < \frac{1}{2} \le \sum_{k=0}^{m_0} a_k$.

Both, the mode μ and the median m_0, can be found by successive search algorithm, i.e. by checking consecutively the values $a_0,\ a_1,\ a_2,\ \ldots$, until the median is found or the maximum number of iterations for finding the mode is reached.

For finding the median, the maximum number of iterations of the successive search algorithm is $N(\xi) = [\mathbb{E}(\xi) + \sigma(\xi)\sqrt{2}]$. The algorithm starts with setting $\psi_0 = a_0$ and continues with calculation of the value $\psi_n = \psi_{n-1} + a_n$ at each step $n = 1, 2, \ldots$, until the inequality $\psi_n \ge \frac{1}{2}$ becomes true.

Similarly, for finding the mode, the maximum number of iterations of the successive search algorithm is $n_s(\xi) = \left[\mathbb{E}(\xi) + \frac{\sigma(\xi)}{\sqrt{a_s}}\right]$, where s is the smallest index for which $a_s > 0$. The mode μ is that index which satisfies the equality $a_\mu = \max_{s\le n\le n_s(\xi)} a_n$.

We can easily note that the successive search algorithm for finding the mode of the random variable ξ depends on the main probabilistic characteristics $\mathbb{E}(\xi)$ and $\sigma(\xi)$. In general case, these values can be obtained from generating function $G_\xi(z) = G^{[a]}(z)$ using the formulas:

$$\mathbb{E}(\xi) = G'_\xi(1),\ \mathbb{V}(\xi) = G''_\xi(1) + G'_\xi(1) - (G'_\xi(1))^2,\ \sigma(\xi) = \sqrt{\mathbb{V}(\xi)}.$$

Next, we consider the homogeneous linear recurrent distributions, i.e. the case when $a = rep(\xi) \in Rol[\mathbb{C}]$. It is known that $a \in Rol[\mathbb{R}]$ and $dim[\mathbb{R}][a] = dim[\mathbb{C}][a]$. Moreover, since distributions are convergent to 0, the minimal characteristic polynomial does not have the root $z = 1$. In this case, the moments can be found in an easier way, using the following theorem from [4]:

Theorem 10. *Let ξ be a random variable with distribution $a = rep(\xi) \in Rol[\mathbb{R}][m]$ and generating vector $q \in G[\mathbb{R}][m](a)$. Then $c^{(k)} = (n^k a_n)_{n=0}^\infty \in Rol[\mathbb{R}][M_k]$, $q^{(k)} \in G[\mathbb{R}][M_k](c^{(k)})$ and*

$$\nu_k(\xi) = G^{[c^{(k)}]}(1),\ \forall k \ge 1,$$

where $M_k = m(k+1)$ and

$$H_{M_k}^{[q^{(k)}]}(z) = (H_m^{[q]}(z))^{k+1} \in H[\mathbb{R}][M_k](c^{(k)}).$$

In consequence, $\mathbb{E}(\xi)$ and $\sigma(\xi)$ can be calculated too, using the relations

$$\mathbb{E}(\xi) = \nu_1(\xi),\ \mathbb{V}(\xi) = \nu_2(\xi) - \nu_1^2(\xi),\ \sigma(\xi) = \sqrt{\mathbb{V}(\xi)}.$$

5 Perturbations and Their Asymptotic Behavior

We consider the homogeneous linear recurrence $a \in Rol[\mathbb{R}][m]$ with initial state $I_m^{[a]} = (a_n)_{n=0}^{m-1}$, generating vector $q \in G[\mathbb{R}][m](a)$ and the corresponding characteristic polynomial $H_m^{[q]}(z) \in H[\mathbb{R}][m](a)$. Perturbations are defined as deviations in the evolution of a, caused by small deviations in the parameters, i.e. deviations of initial state elements and deviations of generating vector components.

5.1 Perturbations Generated by Deviations in Initial State

Initially, we consider only deviations in initial state $I_m^{[a]}$ of the homogeneous linear recurrence a, without any change in generating vector q. In this case, the perturbed recurrence represents a new homogeneous linear recurrence $b \in Rol[\mathbb{R}][m]$ with initial state $I_m^{[b]} = (b_n)_{n=0}^{m-1}$ and the same generating vector $q \in G[\mathbb{R}][m](b)$, where

$$b_n = a_n + \Delta_n,\ n = \overline{0, m-1}.$$

The perturbation is given by the sequence $\epsilon = (\epsilon_n)_{n=0}^{\infty}$, where $\epsilon_n = b_n - a_n$, $n = \overline{0, \infty}$. We have $\epsilon_n = \Delta_n$, $n = \overline{0, m-1}$. Also, applying Theorem 3, we obtain $\epsilon \in Rol[\mathbb{R}][m]$ and $q \in G[\mathbb{R}][m](\epsilon)$. So,

$$\epsilon \in Rol[\mathbb{R}][m],\ q \in G[\mathbb{R}][m](\epsilon),\ I_m^{[\epsilon]} = (\Delta_n)_{n=0}^{m-1}.$$

The perturbation $\epsilon = (\epsilon_n)_{n=0}^{\infty}$ is considered asymptotically stable if and only if $\lim\limits_{n\to\infty} \epsilon_n = 0$. The convergence of ϵ can be studied according to Section 3.

As a remark, the asymptotical stability of perturbation ϵ does not depend on deviation in initial state. Since the components of generating vector q are not changed, the characteristic roots are not changed too. This means that the asymptotic behavior of the perturbed recurrence is exactly the same as asymptotic behavior of the original recurrence.

The maximal impact of the perturbation ϵ is represented by the positive value $\epsilon^* = \max\limits_{n=\overline{0,\infty}} |\epsilon_n|$. Even if ϵ is asymptotically stable, it might have a big enough maximal perturbation impact.

In order to study the maximal impact of the asymptotically stable perturbation ϵ, we can consider the sequence $\epsilon^2 = (\epsilon_n^2)_{n=0}^{\infty}$. Since $\epsilon^2 = \epsilon \cdot \epsilon$ and $\epsilon \in Rol[\mathbb{R}][m]$, we have

$$\epsilon^2 \in Rol[\mathbb{R}],\ dim[\mathbb{R}][\epsilon^2] \le (dim[\mathbb{R}][\epsilon])^2 \le m^2.$$

In consequence, $\epsilon^2 \in Rol[\mathbb{R}][m^2]$ and its minimal generating vector can be obtained using the minimization method based on matrix rank definition.

Next, using Theorem 1, the value $s = \sum\limits_{n=0}^{\infty} \epsilon_n^2 = G^{[\epsilon^2]}(1)$ can be calculated. If ξ is a random variable with distribution $p = \epsilon^2/s$, then its mode μ and its probability p_μ can be found using the successive search algorithm, described in Section 4. In the end, we obtain the maximal perturbation impact $\epsilon^* = \sqrt{sp_\mu}$.

5.2 Perturbations Generated by Deviations in Generating Vector

Now, we consider only deviations in generating vector q of the homogeneous linear recurrence a, without any change in initial state $I_m^{[a]}$. In this case, the perturbed recurrence represents a new homogeneous linear recurrence $b \in Rol[\mathbb{R}][m]$ with initial state $I_m^{[b]} = I_m^{[a]}$ and the generating vector $r = (r_n)_{n=0}^{\infty} \in G[\mathbb{R}][m](b)$, where

$$r_n = q_n + \delta_n,\ n = \overline{0, m-1}.$$

The perturbation is given by the sequence $\varepsilon = (\varepsilon_n)_{n=0}^{\infty}$, where $\varepsilon_n = b_n - a_n$, $n = \overline{0, \infty}$. Applying Theorem 3, we obtain

$$\varepsilon \in Rol[\mathbb{R}],\ dim[\mathbb{R}](\varepsilon) \le dim[\mathbb{R}](a) + dim[\mathbb{R}](b) \le m + m = 2m.$$

In consequence, $\varepsilon \in Rol[\mathbb{R}][2m]$ and its minimal generating vector can be obtained using the minimization method based on matrix rank definition.

The perturbation $\varepsilon = (\varepsilon_n)_{n=0}^{\infty}$ is considered asymptotically stable if and only if $\lim_{n\to\infty} \varepsilon_n = 0$. The convergence of ε can be studied according to Section 3.

As a remark, the perturbation ε can be asymptotically stable even if the initial recurrence a is not convergent. This happens when $dim[\mathbb{R}](\varepsilon) < dim[\mathbb{R}](a)$ and all roots of the minimal characteristic polynomial of a over $\mathbb{R}$ which are not greater than 1 in absolute value disappear from the list of all roots of the minimal characteristic polynomial of ε over $\mathbb{R}$.

Similarly to Section 5.1, in order to study the maximal impact $\varepsilon^* = \max_{n=\overline{0,\infty}} |\varepsilon_n|$ of the asymptotically stable perturbation ε, we can consider the sequence ε^2, obtaining $\varepsilon^2 \in Rol[\mathbb{R}][4m^2]$. Its minimal generating vector can be obtained using the minimization method based on matrix rank definition too.

5.3 Mixed Perturbations

Mixed perturbations are generated by both types of deviations: deviations in initial state $I_m^{[a]}$ and deviations in generating vector q of the homogeneous linear recurrence a. The perturbed recurrence represents a new homogeneous linear recurrence $c \in Rol[\mathbb{R}][m]$ with initial state $I_m^{[c]}$ and the generating vector $r = (r_n)_{n=0}^{\infty} \in G[\mathbb{R}][m](c)$, where

$$c_n = a_n + \Delta_n,\ r_n = q_n + \delta_n,\ n = \overline{0, m-1}.$$

The perturbation is given by the sequence $\varepsilon = (\varepsilon_n)_{n=0}^{\infty}$, where

$$\varepsilon_n = c_n - a_n,\ n = \overline{0, \infty}.$$

We can study mixed perturbations using results from Section 5.1 and Section 5.2. The deviation in initial state and the deviation in generating vector can be performed consecutively, one by one, in the following way.

Let $b \in Rol[\mathbb{R}][m]$ be the perturbed recurrence, generated by deviation in initial state, i.e.

$$q = (q_n)_{n=0}^{m-1} \in G[\mathbb{R}][m](b),$$
$$b_n = a_n + \Delta_n,\ n = \overline{0, m-1}.$$

Its perturbation is represented by the sequence $\epsilon = (\epsilon_n)_{n=0}^{\infty}$, where

$$\epsilon_n = b_n - a_n,\ n = \overline{0, \infty}.$$

Next, the perturbed recurrence $c \in Rol[\mathbb{R}][m]$ is obtained from $b \in Rol[\mathbb{R}][m]$ by applying the given deviation in generating vector $q = (q_n)_{n=0}^{\infty} \in G[\mathbb{R}][m](b)$:

$$c_n = b_n,\ r_n = q_n + \delta_n,\ n = \overline{0, m-1}.$$

The corresponding perturbation is represented by the sequence $\zeta = (\zeta_n)_{n=0}^{\infty}$, where

$$\zeta_n = c_n - b_n,\ n = \overline{0, \infty}.$$

The mixed perturbation $\varepsilon = (\varepsilon_n)_{n=0}^{\infty}$ represents the sum of these two perturbations from decomposition:

$$\varepsilon_n = c_n - a_n = (c_n - b_n) + (b_n - a_n) = \zeta_n + \epsilon_n,\ n = \overline{0, \infty}.$$

So, based on Theorem 3, it is also a homogeneous linear recurrence. As consequence, the asymptotic behavior and the maximal perturbation impact can be studied similarly.

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