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About one explicit-difference scheme for solving the plane problem for two-component medium

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Abstract. The finite-difference scheme for plane dynamical problem of the theory of elasticity of two-component medium in displacements is obtained. The stability of this scheme by means of Niemann conditions is studied. Is found the maximal time step in dependence on the space step for which the stability is kept.

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The research of wave processes in many components continuous media represents a great interest for seismology, construction, the research of the dynamic behavior of the various mixtures of the soils etc. The works [1–6] are dedicated to the construction of mathematical models of such media. M.A. Biot in his works [1–3] proposed a rather general approach in the linear mechanics of deformation and distribution of acoustic waves in porous two-component medium.

The non-numerous works [7–13] devoted to the solution of concrete problems on the basis of M.A. Biot's equations refer exclusively to the simplest kinds of two-components media (mixture of two isotropically solid bodies, isotropically solid body and liquids, a liquid and a gas), the first stage of the solution of the problem doesn't provoke any difficulties being the determination of the speeds of the wave types appeared.

The purpose of the present article is the estimation of the time step providing the stability of one explicit finite-difference scheme for the plane dynamical problem of the theory of elasticity of two-component medium. Non-stationary processes in every layer are described by equations of the theory of elasticity: the equations of motion, the Hooke's law and the Cauchy relations.

The relations between stresses and deformations in conditions of plane deformation are the following:

$$\sigma_{xx} = -\alpha_2 + (\lambda_1 + 2\mu_1)\,\varepsilon_{xx} + \lambda_1 \varepsilon_{yy} + (\lambda_3 + 2\mu_3)\,q_{xx} + \lambda_3 q_{yy};$$

$$\sigma_{yy} = -\alpha_2 + \lambda_1 \varepsilon_{xx} + (\lambda_1 + 2\mu_1)\,\varepsilon_{yy} + \lambda_3 q_{xx} + (\lambda_3 + 2\mu_3)\,q_{yy};$$

$$\pi_{xx} = \alpha_2 + (\lambda_2 + 2\mu_2)\,q_{xx} + \lambda_2 q_{yy} + (\lambda_4 + 2\mu_3)\,\varepsilon_{xx} + \lambda_4 \varepsilon_{yy};$$

$$\pi_{yy} = \alpha_2 + \lambda_2 q_{xx} + (\lambda_2 + 2\mu_2)\,q_{yy} + \lambda_4 \varepsilon_{xx} + (\lambda_4 + 2\mu_3)\,\varepsilon_{yy};$$

$$\sigma_{xy} = 2\,(\mu_1 \varepsilon_{xy} + \mu_3 q_{xy}) - \lambda_5\,(h_{xy} - h_{yx})\,;$$

$$\begin{aligned}
\sigma_{yx} &= 2\left(\mu_1\varepsilon_{xy}+\mu_3 q_{xy}\right)+\lambda_5\left(h_{xy}-h_{yx}\right);\\
\pi_{xy} &= 2\left(\mu_2 q_{xy}+\mu_3\varepsilon_{xy}\right)-\lambda_5\left(h_{xy}-h_{yx}\right);\\
\pi_{xy} &= 2\left(\mu_2 q_{xy}+\mu_3\varepsilon_{xy}\right)+\lambda_5\left(h_{xy}-h_{yx}\right).
\end{aligned}\tag{1}$$

The behavior of the elastic system is described by the equations of motion:

$$\begin{aligned}
\frac{\partial\sigma_{x\,x}}{\partial\,x}+\frac{\partial\sigma_{x\,y}}{\partial\,y}-\frac{\partial\pi_0}{\partial\,x} &= \rho_{11}\frac{\partial^2 u_1}{\partial\,t^2}+\rho_{12}\frac{\partial^2 u_2}{\partial\,t^2}+b\left(\frac{\partial u_1}{\partial\,t}-\frac{\partial u_2}{\partial\,t}\right);\\
\frac{\partial\sigma_{y\,x}}{\partial\,x}+\frac{\partial\sigma_{y\,y}}{\partial\,y}-\frac{\partial\pi_0}{\partial\,y} &= \rho_{11}\frac{\partial^2 v_1}{\partial\,t^2}+\rho_{12}\frac{\partial^2 v_2}{\partial\,t^2}+b\left(\frac{\partial v_1}{\partial\,t}-\frac{\partial v_2}{\partial\,t}\right);\\
\frac{\partial\pi_{x\,x}}{\partial\,x}+\frac{\partial\pi_{x\,y}}{\partial\,y}+\frac{\partial\pi_0}{\partial\,x} &= \rho_{12}\frac{\partial^2 u_1}{\partial\,t^2}+\rho_{22}\frac{\partial^2 u_2}{\partial\,t^2}-b\left(\frac{\partial u_1}{\partial\,t}-\frac{\partial u_2}{\partial\,t}\right);\\
\frac{\partial\pi_{y\,x}}{\partial\,x}+\frac{\partial\pi_{y\,y}}{\partial\,y}+\frac{\partial\pi_0}{\partial\,y} &= \rho_{12}\frac{\partial^2 v_1}{\partial\,t^2}+\rho_{22}\frac{\partial^2 v_2}{\partial\,t^2}-b\left(\frac{\partial v_1}{\partial\,t}-\frac{\partial v_2}{\partial\,t}\right),
\end{aligned}\tag{2}$$

where $u_i, v_i\ \ (i=1,2)$ are the components of the displacement vector of firm phases; $\sigma_{xx}, \sigma_{xy}, \sigma_{yx}, \sigma_{yy}, \pi_{xx}, \pi_{xy}, \pi_{yx}, \pi_{yy}$ are the components of the stress tensor; ε_{xx}, ε_{xy}, h_{yx}, ε_{yy}, q_{xx}, q_{xy}, h_{yx}, q_{yy} are the components of deformation; ρ_{11}, ρ_{22} are the effective component masses at their relative motion; $\rho_{11}+\rho_{12}=\rho_1, \rho_{22}+\rho_{12}=\rho_2, \rho_{12}$ is the ,,connecting parameter" between the components of the mixture or the additional apparent mass; $\alpha_2=\lambda_3-\lambda_4$is the constant with the dimension of stress; $\lambda_j\ \ \mu_j, (j=\overline{1,5})$ are the Lame constants; $\rho_1, \rho_2$are the densities of phases; b is the diffusion coefficient

$$\pi_0=\rho_1/\rho\alpha_2(q_x+q_y)+\rho_1/\rho\alpha_2(\varepsilon_x+\varepsilon_y).$$

The relations between the deformations and displacements are the following

$$\begin{aligned}
&\varepsilon_{xx}=\frac{\partial u_1}{\partial x},\quad \varepsilon_{xy}=\frac{\partial u_1}{\partial y}+\frac{\partial v_1}{\partial x},\quad \varepsilon_{yy}=\frac{\partial v_1}{\partial y};\\
&q_{xx}=\frac{\partial u_2}{\partial x},\quad q_{xy}=\frac{\partial u_2}{\partial y}+\frac{\partial v_2}{\partial x},\quad q_{yy}=\frac{\partial v_2}{\partial y};\\
&h_{xy}=\frac{\partial v_1}{\partial x}+\frac{\partial u_2}{\partial y},\quad h_{yx}=\frac{\partial u_1}{\partial y}+\frac{\partial v_2}{\partial x}.
\end{aligned}\tag{3}$$

Let us consider the formulation of the problem in displacements. To obtain this formulation we substitute the relations (1) and (3) in the equations of motion. After some simple transformations the obtained equations can be presented in the form:

$$A_{11}\frac{\partial^2 u_1}{\partial x^2}+A_{12}\frac{\partial^2 u_1}{\partial y^2}+(A_{11}-A_{12})\frac{\partial^2 v_1}{\partial x\partial y}+B_{11}\frac{\partial^2 u_2}{\partial x^2}+B_{12}\frac{\partial^2 u_2}{\partial y^2}+(B_{11}-B_{12})\frac{\partial^2 v_2}{\partial x\partial y}=$$

$$=\rho_{11}\frac{\partial^2 u_1}{\partial t^2}+\rho_{12}\frac{\partial^2 u_2}{\partial t^2}+b\left(\frac{\partial u_1}{\partial t}+\frac{\partial u_2}{\partial t}\right);$$

$$A_{21}\frac{\partial^2 u_1}{\partial x^2}+A_{22}\frac{\partial^2 u_1}{\partial y^2}+(A_{21}-A_{22})\frac{\partial^2 v_1}{\partial x\partial y}+B_{21}\frac{\partial^2 u_2}{\partial x^2}+B_{22}\frac{\partial^2 u_2}{\partial y^2}+(B_{21}-B_{22})\frac{\partial^2 v_2}{\partial x\partial y}=$$
$$=\rho_{12}\frac{\partial^2 u_1}{\partial t^2}+\rho_{22}\frac{\partial^2 u_2}{\partial t^2}-b\left(\frac{\partial u_1}{\partial t}+\frac{\partial u_2}{\partial t}\right);$$
$$A_{11}\frac{\partial^2 v_1}{\partial x^2}+A_{12}\frac{\partial^2 v_1}{\partial y^2}+(A_{11}-A_{12})\frac{\partial^2 u_1}{\partial x\partial y}+B_{11}\frac{\partial^2 v_2}{\partial x^2}+B_{12}\frac{\partial^2 v_2}{\partial y^2}+(B_{11}-B_{12})\frac{\partial^2 u_2}{\partial x\partial y}=$$
$$=\rho_{11}\frac{\partial^2 v_1}{\partial t^2}+\rho_{12}\frac{\partial^2 v_2}{\partial t^2}+b\left(\frac{\partial v_1}{\partial t}+\frac{\partial v_2}{\partial t}\right); \tag{4}$$
$$A_{21}\frac{\partial^2 v_1}{\partial x^2}+A_{22}\frac{\partial^2 v_1}{\partial y^2}+(A_{21}-A_{22})\frac{\partial^2 u_1}{\partial x\partial y}+B_{21}\frac{\partial^2 v_2}{\partial x^2}+B_{22}\frac{\partial^2 v_2}{\partial y^2}+(B_{21}-B_{22})\frac{\partial^2 u_2}{\partial x\partial y}=$$
$$=\rho_{12}\frac{\partial^2 v_1}{\partial t^2}+\rho_{22}\frac{\partial^2 v_2}{\partial t^2}-b\left(\frac{\partial v_1}{\partial t}+\frac{\partial v_2}{\partial t}\right),$$

where $A_{11}=\lambda_1+2\mu_1-\rho_2\alpha_2/\rho$; $A_{12}=\mu_1-\lambda_5$; $A_{21}=\lambda_2+2\mu_2+\rho_1\alpha_2/\rho$; $A_{22}=\mu_2-\lambda_5$; $B_{11}=\lambda_3+2\mu_3-\rho_1\alpha_2/\rho$; $B_{12}=\mu_3+\lambda_5$; $B_{21}=\lambda_4+2\mu_3+\rho_2\alpha_2/\rho$; $B_{22}=\mu_3+\lambda_5$.

Further it will be convenient to split this system into two systems. The first system describes the processes connected with elastic properties of the medium. The second system describes the dissipative properties of the medium. So the systems differ only in right-hand parts. The first system contains the second derivatives with respect to time and the second system contains the first derivatives with respect to time.

Let us consider the following explicit finite-difference scheme for numerical solving the first system of equations.

Let us consider the rectangular grid with the steps Δx with respect to variable x, Δy with respect to time variable. We'll denote by $f_{ij}^k=f(i\Delta x,\ j\Delta y,\ k\Delta t)$ the values of function f in the nodes of the grid and approximate the derivatives with finite-difference relations

$$\frac{\partial^2 f}{\partial t^2}\sim\frac{f_{n,m}^{k+1}-2f_{n,\,m}^{k}+f_{n,\,m}^{k-1}}{\Delta t^2}=\left(f_{m,\,n}^{k}\right)_{\bar{t}t};$$
$$\frac{\partial^2 f}{\partial y^2}\sim\frac{f_{n,\,m+1}-2f_{n,\,m}+f_{n,\,m-1}}{\Delta y^2}=\left(f_{m,\,n}^{k}\right)_{\bar{y}y};$$
$$\frac{\partial^2 f}{\partial x\partial y}\sim\frac{f_{n+1,\,m+1}-f_{n-1,\,m+1}-f_{n+1,\,m-1}+f_{n-1,\,m-1}}{4\,\Delta\,x\;\Delta\,y}=\left(f_{m,\,n}^{k}\right)_{\bar{x}y},$$

as a result we obtain the following discrete system of equations:

$$A_{11}\left(u_{1m,n}^k\right)_{\bar{x}x}+A_{12}\left(u_{1m,n}^k\right)_{\bar{y}y}+(A_{11}-A_{12})\left(v_{1m,n}^k\right)_{\bar{x}y}+B_{11}\left(u_{2m,n}^k\right)_{\bar{x}x}+$$
$$+B_{12}\left(u_{2m,n}^k\right)_{\bar{y}y}+(B_{11}-B_{12})\left(v_{2m,n}^k\right)_{\bar{x}y}=\rho_{11}\left(u_{1m,n}^k\right)_{\bar{t}t}+\rho_{12}\left(u_{2m,n}^k\right)_{\bar{t}t};$$

$$A_{21}\left(u^k_{1m,n}\right)_{\bar{x}x}+A_{22}\left(u^k_{1m,n}\right)_{\bar{y}y}+(A_{21}-A_{22})\left(v^k_{1m,n}\right)_{\bar{x}y}+B_{21}\left(u^k_{2m,n}\right)_{\bar{x}x}+$$
$$+B_{22}\left(u^k_{2m,n}\right)_{\bar{y}y}+(B_{21}-B_{22})\left(v^k_{2m,n}\right)_{\bar{x}y}=\rho_{12}\left(u^k_{1m,n}\right)_{\bar{t}t}+\rho_{22}\left(u^k_{2m,n}\right)_{\bar{t}t};$$
$$A_{11}\left(v^k_{1m,n}\right)_{\bar{x}x}+A_{12}\left(v^k_{1m,n}\right)_{\bar{y}y}+(A_{11}-A_{12})\left(u^k_{1m,n}\right)_{\bar{x}y}+B_{11}\left(v^k_{2m,n}\right)_{\bar{x}x}+$$
$$+B_{12}\left(v^k_{2m,n}\right)_{\bar{y}y}+(B_{11}-B_{12})\left(u^k_{2m,n}\right)_{\bar{x}y}=\rho_{11}\left(v^k_{1m,n}\right)_{\bar{t}t}+\rho_{12}\left(v^k_{2m,n}\right)_{\bar{t}t}; \quad (5)$$
$$A_{21}\left(v^k_{1m,n}\right)_{\bar{x}x}+A_{22}\left(v^k_{1m,n}\right)_{\bar{y}y}+(A_{21}-A_{22})\left(u^k_{1m,n}\right)_{\bar{x}y}+B_{21}\left(v^k_{2m,n}\right)_{\bar{x}x}+$$
$$+B_{22}\left(v^k_{2m,n}\right)_{\bar{y}y}+(B_{21}-B_{22})\left(u^k_{2m,n}\right)_{\bar{x}y}=\rho_{12}\left(v^k_{1m,n}\right)_{\bar{t}t}+\rho_{22}\left(v^k_{2m,n}\right)_{\bar{t}t}.$$

We do not consider here the initial and boundary conditions supposing that the grid is unboundly continuous with respect to $x$and y.

Let us study the stability of finite-difference scheme (5) by means of Neumann condition [14]. We find the solution of equations (5) in the form

$$u^k_{i,\,m,\,n}\;=\;\gamma^k e^{i\,\alpha\,m}e^{i\,\beta\,n}u_{i\,0};\quad v^k_{i,\,m,\,n}\;=\;\gamma^k e^{i\,\alpha\,m}e^{i\,\beta\,n}v_{i\,0}\quad (i=1,2). \quad (6)$$

As a result we obtain the characteristic equation. After the following notations

$$\omega\;=\;-\frac{\gamma-2+\frac{1}{\gamma}}{\Delta\,t^2}\;; \quad (7)$$

$$\xi=-\frac{e^{i\alpha}-2+e^{-i\alpha}}{\Delta\,x^2}=\frac{2\,(1-\cos\;\alpha)}{\Delta\,x^2}=\frac{4\,\sin^2\,\frac{\alpha}{2}}{\Delta\,x^2}\;;\quad \zeta\;=\;-\frac{\sin\alpha\,\sin\beta}{\Delta x\Delta y}\;;$$
$$\eta=-\frac{e^{i\beta}-2+e^{-i\beta}}{\Delta\,y^2}=\frac{2\,(1-\cos\;\beta)}{\Delta\,y^2}=\frac{4\,\sin^2\,\frac{\beta}{2}}{\Delta\,y^2}\;, \quad (8)$$

the characteristic equation can be written in the form:

$$\begin{vmatrix} a_1^{\xi\eta}-\rho_{11}\omega & b_1^{\xi\eta}-\rho_{12}\,\omega & (A_{11}-A_{12})\zeta & (B_{11}-B_{12})\zeta \\ a_2^{\xi\eta}-\rho_{12}\omega & b_2^{\xi\eta}-\rho_{22}\omega & (A_{21}-A_{22})\zeta & (B_{21}-B_{22})\zeta \\ (A_{11}-A_{12})\zeta & (B_{11}-B_{12})\zeta & a_1^{\eta\xi}-\rho_{11}\omega & b_1^{\eta\xi}-\rho_{12}\,\omega \\ (A_{21}-A_{22})\zeta & (B_{21}-B_{22})\zeta & a_2^{\eta\xi}-\rho_{12}\omega & b_2^{\eta\xi}-\rho_{22}\omega \end{vmatrix}\;=\;0,$$

where $a_i^{fg}=A_{i1}f+A_{i2}g;\quad b_i^{fg}=B_{i1}f+B_{i2}g.$

The necessary Neumann condition of stability is that $|\gamma|\leq 1$ for all eight roots γ calculated from (7), where ω are the four roots of the last equation. From the last equation we obtain the equation of the fourth order with respect to ω:

$$\omega^4+a^*\omega^3+b^*\omega^2+c^*\omega+d^*=0, \quad (9)$$

where $a^*,b^*,c^*,d^*\in R, a^*\neq 0.$

This equation can be written in the form:

$$(\omega^2 + a^*\omega/2)^2 = (a^{*2}/4 - b^*)\omega^2 - c^*\omega - d^*.$$

Let us add to both parts of the equation the term$(\omega^2 + a^*\omega/2)^2 y + y^2$, then

$$(\omega^2 + a^*\omega/2 + y/2)^2 = (a^{*2}/4 - b^* + y)\omega^2 + (a^*y/2 - c^*)\omega + y^2/4 - d^*.$$

We'll find y in such a way that the right-hand part of the equation would be a perfect trinomial square. After the following notations $A_*^2 = a^{*2}/4 - b^* + y, \quad B_*^2 = y^2/4 - d^*$, $2A_*B_* = a^*y^2/2 - c^*$, this condition can be written in the form $4A_*^2B_*^2 = (2A_*B_*)^2$. So we've obtained the resolvable equation. If y_0 is the root of the last equation, then the solution of equation (9) reduces to the solution of the following two equations $\omega^2 + a^*\omega/2 + y_0/2 = A_*\omega + B_*$ and $\omega^2 + a^*\omega/2 + y_0/2 = -A_*\omega - B_*$. The examination of the roots of these equations by means of computer showed that all four roots are real and positive.

Let us consider the equation (7). It can be written in the form

$$\gamma^2 - \left(2 - \omega\,\Delta\,t^2\right)\ \gamma\ + 1 = 0. \tag{10}$$

It is easy to realize that in the case of real ω one of the following situations is possible:

– if the next condition is fulfilled

$$0 \le \omega\ \Delta\,t^2 \le 4, \tag{11}$$

then both roots are complex and their modules are equal to 1;

– if the condition (11) is not fulfilled, then both roots are real one of them is less than 1, but another is greater than1 (the product of the roots is equal to 1).

Thus, even if one of the values ω_i $(i = \overline{1,4})$ does not satisfy (11), then among the eight roots γ of the characteristic equation (5) there is necessarily one with module greater than 1. According to Neumann condition the finite-difference scheme (5) will be unstable. If the values ω_i satisfy the condition (11), then the modules of all eight roots will be equal to 1. Hence, the finite-difference scheme (5) without boundary conditions will be stable.

The examination of the roots of the equation (9) makes it possible to say that the maximum value of the greatest of them is achieved at the corner point of the rectangle $0 \le \xi \le 4/\Delta x^2\ , \quad 0 \le \eta \le 4/\Delta y^2$.

If the maximum value of the function $\omega\,(\xi,\ \eta)$ is achieved at the corner point $\xi = 4/\Delta x^2\ , \eta = 4/\Delta y^2$, then the following estimation is fulfilled:

$$\omega \le \left(\Omega + \sqrt{\Omega^2 - 4\Delta^*(AC - BD)}\right)/(2\Delta^*), \tag{12}$$

where $\Omega = A\rho_{22} + C\rho_{11} - (B + D)\rho_{12}, \quad \Delta^* = \rho_{11}\rho_{22} - \rho_{12}^2, \quad A = 4(A_{11}/\Delta x^2 + A_{12}/\Delta y^2), \quad B = 4(B_{11}/\Delta x^2 + B_{12}/\Delta y^2), \quad C = 4(B_{21}/\Delta x^2 + B_{22}/\Delta y^2), \quad D = 4(A_{21}/\Delta x^2 + A_{22}/\Delta y^2)$.

It is evident that $\Omega = \Delta^*(a^2+b^2)$, where

$$a^2 = \left(\Theta + \sqrt{\Theta^2 - 4\Delta^*(A_{11}B_{21} - B_{11}A_{21})}\right) \Big/ (2\Delta^*);$$

$$b^2 = \left(\Sigma + \sqrt{\Sigma^2 - 4\Delta^*(A_{12}B_{22} - B_{12}A_{22})}\right) \Big/ (2\Delta^*);$$

$$\Theta = A_{11}\rho_{22} + B_{21}\rho_{11} - (A_{21}+B_{11})\rho_{12}; \quad \Sigma = A_{12}\rho_{22} + B_{22}\rho_{11} - (A_{22}+B_{12})\rho_{12},$$

as a result we obtain

$$\omega \leq \frac{a^2+b^2}{2}\left(\frac{4}{\Delta x^2} + \frac{4}{\Delta y^2}\right) + \frac{a^2-b^2}{2}\left|\frac{4}{\Delta x^2} - \frac{4}{\Delta y^2}\right|. \tag{13}$$

According to the condition (11) the stability of the finite-difference scheme without boundary conditions will take place if the step with respect to time variable will satisfy the following condition:

$$\Delta t = \frac{h}{\sqrt{a^2+b^2}}, \quad if \;\; \Delta\, x = \Delta\, y = h; \tag{14}$$

$$\Delta t = \frac{\Delta x \Delta y}{\sqrt{a^2\Delta y^2 + b^2\Delta x^2}}, \quad if \quad \Delta x \leq \Delta y; \tag{15}$$

$$\Delta t = \frac{\Delta x \Delta y}{\sqrt{a^2\Delta x^2 + b^2\Delta y^2}}, \quad if \;\; \Delta x \geq \Delta y. \tag{16}$$

Now we'll take in consideration the dissipative terms in the system (4). The right-hand parts of the second system can be approximated by the relation

$$\frac{\partial f}{\partial t} \sim \frac{f^{k+1}_{n,m} - f^k_{n,m}}{\Delta t} = (f^k_{n,m})_{\bar{t}}, \tag{17}$$

where f is one of the functions $u_i,\ v_i,\ i = 1, 2$.

The values of these additional terms (in comparison with elastic model) are taking into account in the construction of transmission formulas for the next time moment $t_{k+1} = t_k + \Delta t$.

The finite-difference scheme for system (2) in the operator form can be written in the following form

$$U^{k+1} = [E + \tau(A_I + A_{II})]\ U^k, \tag{18}$$

where $A_I,\ A_{II}$ are difference operators with chosen approximation of the right-hand parts.

Let τ_I and τ_{II} be the time steps that provide the stability of these systems, i.e. the conditions $\|E + \tau_I A_I\| \leq 1$; $\|E + \tau_{II} A_{II}\| \leq 1$ are fulfilled for some norm of the difference operator. Then, if the step τ, verifies the inequality

$$\tau\left(\frac{1}{\tau_I} + \frac{1}{\tau_{II}}\right) \leq 1; \tag{19}$$

then the condition

$$\|E + \tau\,(A_I + A_{II})\| \le 1, \tag{20}$$

is fulfilled, i.e. the stability of the finite-difference scheme for system of equations (4).

In reality, from the identity $E+\tau\ \ (A_I + A_{II}) = r_I\,(E + \tau_I A_I)+r_{II}\,(E + \tau_{II} A_{II})+ (1 - r_I - r_{II})\,E$ (here $r_I \tau_I = \tau$, $r_{II}\tau_{II} = \tau$) and from the convexity of the norm follows that

$$\|E + \tau\ \ (A_I + A_{II})\| \le r_I\|E + \tau_I A_I\| + r_{II}\|E + \tau_{II} A_{II}\| +$$

$$+|\ 1 - r_I - r_{II}|\ \ \le r_I + r_{II} + |\ 1 - r_I - r_{II}|\ .$$

Hence, the inequality (20) will be fulfilled, if $1 - r_I - r_{II} \ge 0$. As $r_I = \tau/\tau_I$, $r_{II} = \tau/\tau_{II}$, then the last condition consider with (19).

As it was mentioned above the stability of the finite-difference scheme for elastic model is provided by conditions (15) and (16), i.e.

$$\frac{1}{\tau_I} = \frac{\sqrt{a^2\Delta y^2 + b^2\Delta x^2}}{\Delta x\Delta y} \quad \text{i}f\ \ \Delta x \le \Delta y;$$

$$\frac{1}{\tau_I} = \frac{\sqrt{a^2\Delta x^2 + b^2\Delta y^2}}{\Delta x\Delta y} \quad \text{i}f\ \ \Delta x \ge \Delta y. \tag{21}$$

Let us obtain the value τ_{II}, which provides the stability of the corresponding scheme.

With the help of auxiliary value $(\lambda - 1)/\Delta t = -\mu$, we obtain the characteristic equation in the following form:

$$\begin{vmatrix} a_1^{\xi\eta} - \rho_{11}\mu & b_1^{\xi\eta} - \rho_{12}\,\mu & (A_{11} - A_{12})\zeta & (B_{11} - B_{12})\zeta \\ a_2^{\xi\eta} - \rho_{12}\mu & b_2^{\xi\eta} - \rho_{22}\mu & (A_{21} - A_{22})\zeta & (B_{21} - B_{22})\zeta \\ (A_{11} - A_{12})\zeta & (B_{11} - B_{12})\zeta & a_1^{\eta\xi} - \rho_{11}\mu & b_1^{\eta\xi} - \rho_{12}\,\mu \\ (A_{21} - A_{22})\zeta & (B_{21} - B_{22})\zeta & a_2^{\eta\xi} - \rho_{12}\mu & b_2^{\eta\xi} - \rho_{22}\mu \end{vmatrix} = 0. \tag{22}$$

From this determinantal equation we obtain

$$(\mu - \xi - \eta)\ \ \left[\mu^2 - \frac{4}{3}\,(\xi + \eta)\,\mu + \frac{4}{3}\left(\xi\eta - \zeta^2\right)\right] = 0.$$

According to the above notations $\lambda = 1 - \mu\Delta\,t$ and, hence, the necessary condition of stability $|\lambda| \le 1$ is reduced to the inequality

$$1 - \mu * \Delta\,t \ge -1 \tag{23}$$

where $\mu*$ is the maximal value of the greatest root and α, β are arbitrary.

The maximal value of the greatest root for arbitrary α, β was studied by means of computer in rectangle $0 \le \xi \le 4/\Delta x^2$, $\quad 0 \le \eta \le 4/\Delta y^2$. The maximal value of the greatest root is achieved at a corner point.

If the maximal value of the function $\mu(\xi, \ \eta)$ is achieved at the corner point $\xi = 4/\Delta x^2$, $\eta = 4/\Delta y^2$, then the following estimation is fulfilled:

$$\mu^* \le \frac{AC - BD}{b(A + C + B + D)}. \tag{24}$$

Hence, from (19) we obtain:

$$\Delta t \le 2/\mu*; \quad \tau_{II} = \frac{(AC - BD)}{2b(A + B + C + D)}. \tag{25}$$

So, from the condition of stability (19) for equal grid steps $\Delta\, x = \Delta\, y = h$, we obtain

$$\tau = \frac{\tau_I + \tau_{II}}{\tau_I \tau_{II}}. \tag{26}$$

Thus, in comparison with "pure" elastic model the calculations of the dissipative problem by means of explicit finite-difference scheme must be effectuated with a smaller time step.

It is obvious that the application of the explicit difference scheme is expedient only in rather narrow range of dissipative coefficient, when the ratio $b/h$is small. We shall notice that in the case of small values of b, the attributing of the dissipative terms in finite-difference equations loses sense as the coefficients of difference viscosity of this scheme are values of the order h^2.

In the case when $b >> 1$ it is expedient to consider independently a dissipative system of equations instead of system (4) The elasticity will play a role of small correction for the solution.

The carried out research allows to hope that the stability of calculations with the time step verifying condition (26) will take place. However, as the research was carried out without taking into account boundary conditions, it requires experimental examination. Such examination was carried out. As an example the problem of impact of the rectangular domain on a rigid barrier was considered. The calculated formulas were received in the boundary nodes of the grid. This explicit finite-difference scheme was successfully approved under test problems. The acceptable coordination of the compared results and obtaining the converging solutions by reducing the grid step testify their reliability and closeness to the exact solution. The realization of numerical experiments with different grids (when $h \to 0$) has allowed to estimate the actual speed of convergence of the difference scheme and to optimize the number of nodes of integration to achieve the acceptable accuracy by the minimal expenses of computer time and operative memory resources.

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Classification of planar quadratic differential systems with center of symmetry and multiple infinite singular point

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We classify the family of planar quadratic differential systems with a center of symmetry and two invariant straight lines according to the topology of their phase portraits. The case of the existence of at most two distinct infinite singular points is considered. For each of the classes obtained we give necessary and sufficient conditions in terms of algebraic invariants and comitants. A program was implemented for computer calculations.

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Cyclic planar random evolution with four directions

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Abstract. A four-direction cyclic random motion with constant finite speed v in the plane R^2 driven by a homogeneous Poisson process of rate $\lambda > 0$ is studied. A fourth-order hyperbolic equation with constant coefficients governing the transition law of the motion is obtained. A general solution of the Fourier transform of this equation is given. A special non-linear automodel substitution is found reducing the governing partial differential equation to the generalized fourth-order ordinary Bessel differential equation, and the fundamental system of its solutions is explicitly given.

Mathematics subject classification: Primary 60G99, secondary 60J25, 60K99.
Keywords and phrases: Cyclic random evolution, finite speed, transition law, higher-order hyperbolic equations, generalized Bessel equation, fundamental system of solutions.

1 Introduction

Various models of Markovian random evolutions performed by a particle moving at chance with a constant finite speed are fairly attractive subject, which many researchers have been dealing with. Such an interest is mostly due to the fact that a great deal of practically important applied models in statistical physics, biology, transport processes and engineering (see, for instance, Tolubinsky [15], Ratanov [14], Papanicolaou [13], Brooks [1], Kolesnik [6] and the bibliography therein) can be described and studied in terms of random evolutions.

The one-dimensional motions are the most studied models in which one often managed to obtain the explicit forms of distributions (see Foong [3], Foong and Kanno [4], Orsingher [10], Ratanov [14], Kolesnik [7]) or the estimates of their normal approximations (see Brooks [1]). As far as their multidimensional counterparts are concerned, only a few particular planar random evolutions were studied so far (see Kolesnik [5], Orsingher and San Martini [12], Kolesnik and Turbin [8], Orsingher [11], Kolesnik and Orsingher [9], Di Crescenzo [2]). By this, an explicit form of the distribution was obtained only for the planar random motion with four mutually orthogonal directions without reflection (see Orsingher [11]).

The planar random evolutions performed by a particle changing the directions of its motion in a cyclic way are of a special interest because various cyclic processes are

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rather broadly used for modelling real phenomena. For example, in the well-known statistical problem of discovering a random signal in a multi-channel system the optimal strategy is just the cyclic choice of the channels. In biology the behaviour of tetramers obeys a cyclic scheme too.

The cyclic planar random evolutions have been examined by some authors. In particular, in Orsingher and San Martini [12] such a motion with three cyclically changing directions has been studied, and the explicit solutions of some initial-value problems for the governing equations have been found. The similar three-direction model have recently been investigated by Di Crescenzo [2] where, by different methods, the functional relations for the distribution of this motion have been given in terms of multidimensional convolutions.

In this paper we present a further generalization of the models mentioned above to the case of four mutually orthogonal directions changing in the cyclic way. We obtain a fourth-order hyperbolic equation governing the transition law of the motion and give its *general* solution in terms of Fourier transforms. It is important to note that the roots of corresponding characteristic equation are found *explicitly*. As an alternative approach, we were able to find a non-linear *automodel* substitution reducing governing partial differential equation to the generalized fourth-order ordinary Bessel differential equation, whose linearly independent solutions (i.e. fundamental system of solutions) are also given. It is worth to especially emphasize that we were able to find the fundamental system of solutions in an *explicit* form. This interesting fact gives us some hints for further generalizations of such types of models.

2 Description of the Motion and the Governing Equation

A particle moves with some constant finite speed v in the plane R^2. At every time instant t it can have one of the four possible directions of motion $D(t) = E_k$, where the direction E_k is orientated like the unit vector $(\cos(\pi k/2),\ \sin(\pi k/2)), k = 0, 1, 2, 3$. In other words, the particle can move parallelly to the coordinate axes OX and OY only. The motion is controlled by a homogeneous Poisson process of rate $\lambda > 0$ changing the directions according to the cyclic scheme

$$\cdots \to E_0 \to E_1 \to E_2 \to E_3 \to E_0 \to \dots$$

This means that at each Poisson-paced time moment the particle instantly changes its direction in accordance with this rule and continues its motion in the chosen direction with the same speed v until the next Poisson event occurs, then it cyclically takes on a new direction, and so on.

Denote by $Z(t) = (X(t), Y(t))$ the particle's position in the plane R^2 at some time instant $t > 0$. We are interested in studying the behaviour of the transition law of the process $Z(t)$. Introduce the joint partial densities $f_k = f_k(x, y, t),\ (x, y) \in R^2,\ t > 0$, of the particle's position and its direction as follows

$$f_k(x, y, t)\ dxdy = P\{x \le X(t) < x + dx,\ y \le Y(t) < y + dy,\ D(t) = E_k\}, \qquad (1)$$

$$k = 0, 1, 2, 3.$$

Since the random events $\{D(t) = E_k,\ k = 0, 1, 2, 3, \}$ do not intersect and form the full group of events, then the function $p = p(x, y, t),\ (x, y) \in R^2,\ t > 0$, defined as $p = f_0 + f_1 + f_2 + f_3$, represents the transition density of the motion $Z(t)$.

Our first result concerns the equation governing function p. It is given by the following theorem.

Theorem 1. *The transition density $p = p(x, y, t),\ (x, y) \in R^2,\ t > 0$, of the cyclic planar random evolution with four directions satisfies the following fourth-order hyperbolic equation with constant coefficients*

$$\left\{ \left[\left(\frac{\partial}{\partial t} + \lambda \right)^2 - v^2 \frac{\partial^2}{\partial x^2} \right] \left[\left(\frac{\partial}{\partial t} + \lambda \right)^2 - v^2 \frac{\partial^2}{\partial y^2} \right] - \lambda^4 \right\} p = 0. \tag{2}$$

Proof. The Kolmogorov equation written down for the densities (1) leads to the following hyperbolic system of four first-order PDEs

$$\frac{\partial f_k}{\partial t} = -v \cos \frac{\pi k}{2} \cdot \frac{\partial f_k}{\partial x} - v \sin \frac{\pi k}{2} \cdot \frac{\partial f_k}{\partial y} - \lambda f_k + \lambda f_{k-1},$$

$$k = 0, 1, 2, 3, \quad f_{-1} \stackrel{\text{def}}{=} f_3.$$

Computing the determinant of this system and according to Kolesnik [7], Theorem 2, we come to the conclusion that each function f_k as well as their sum satisfy hyperbolic PDE (2). $\square$

It is easy to check that the exponential substitution

$$p(x, y, t) = e^{-\lambda t} w(x, y, t) \tag{3}$$

reduces equation (2) to the equation

$$\left\{ \left(\frac{\partial^2}{\partial t^2} - v^2 \frac{\partial^2}{\partial x^2} \right) \left(\frac{\partial^2}{\partial t^2} - v^2 \frac{\partial^2}{\partial y^2} \right) - \lambda^4 \right\} w(x, y, t) = 0. \tag{4}$$

This equation will become the main subject of our further analysis.

The Fourier transform of the function $w = w(x, y, t)$

$$\mathcal{W}(\alpha, \beta, t) = \iint_{R^2} e^{i\alpha x + i\beta y} w(x, y, t)\, dx dy$$

satisfies the ordinary differential equation

$$\frac{d^4 \mathcal{W}}{dt^4} + v^2 (\alpha^2 + \beta^2) \frac{d^2 \mathcal{W}}{dt^2} + (v^4 \alpha^2 \beta^2 - \lambda^4) \mathcal{W} = 0. \tag{5}$$

Our next result concerns the general solution of equation (5). It is given by the following theorem.

Theorem 2. *The general solution of equation (5) has the form*

$$\mathcal{W}(\alpha,\beta,t) = C_0 e^{r_0 t} + C_1 e^{r_1 t} + C_2 e^{r_3 t} + C_3 e^{r_3 t}, \tag{6}$$

where C_0, C_1, C_2, C_3 *are arbitrary constants, and*

$$\begin{aligned} r_0 &= \sqrt{\frac{-v^2(\alpha^2+\beta^2) + \sqrt{v^4(\alpha^2-\beta^2)^2 + 4\lambda^4}}{2}}, \\ r_1 &= \sqrt{\frac{-v^2(\alpha^2+\beta^2) - \sqrt{v^4(\alpha^2-\beta^2)^2 + 4\lambda^4}}{2}}, \\ r_2 &= -\sqrt{\frac{-v^2(\alpha^2+\beta^2) + \sqrt{v^4(\alpha^2-\beta^2)^2 + 4\lambda^4}}{2}}, \\ r_3 &= -\sqrt{\frac{-v^2(\alpha^2+\beta^2) - \sqrt{v^4(\alpha^2-\beta^2)^2 + 4\lambda^4}}{2}}. \end{aligned} \tag{7}$$

Proof. The characteristic equation of the ordinary differential equation (5) is the bi-square equation

$$r^4 + v^2(\alpha^2+\beta^2)r^2 + (v^4\alpha^2\beta^2 - \lambda^4) = 0,$$

whose roots, as is easy to see, are given by (7). $\square$

Remark. The constants C_0, C_1, C_2, C_3 (depending on α and β) can be found from the initial conditions in each particular case.

Corollary. *The general solution* $\mathcal{P}(\alpha,\beta,t)$ *of the Fourier transform of equation (2) has the form*

$$\mathcal{P}(\alpha,\beta,t) = C_0 e^{(-\lambda+r_0)t} + C_1 e^{(-\lambda+r_1)t} + C_2 e^{(-\lambda+r_3)t} + C_3 e^{(-\lambda+r_3)t},$$

where r_0, r_1, r_2, r_3 *are given by (7). This immediately follows from (3).*

3 Fundamental System of Solutions

In this section we give an alternative approach leading to the fundamental system of solutions of equation (2). One should especially emphasize that we obtain such a system in an explicit form, unlike the solutions in terms of Fourier transforms given above.

The principal result of this section is given by the following theorem.

Theorem 3. *The fundamental system of solutions of equation (2) has the form*

$$g_i(x,y,t) = e^{-\lambda t} J^{(i)}(x,y,t), \qquad i = 0,1,2,3, \tag{8}$$

where $J^{(i)}$ *are the generalized Bessel functions*

$$J^{(0)}(x,y,t) = \sum_{k=0}^{\infty} \frac{1}{(k!)^4} \left(\frac{2\lambda}{v}\right)^{4k} \left(\frac{z}{4}\right)^{4k},$$

$$J^{(1)}(x,y,t) = \ln z + \sum_{k=1}^{\infty} \frac{1}{(k!)^4} \left(\frac{2\lambda}{v}\right)^{4k} \left(\ln z - 1 - \frac{1}{2} - \cdots - \frac{1}{k}\right) \left(\frac{z}{4}\right)^{4k},$$

$$\begin{aligned} J^{(2)}(x,y,t) = (\ln z)^2 + \sum_{k=1}^{\infty} \frac{1}{(k!)^4} \left(\frac{2\lambda}{v}\right)^{4k} \Bigg[& \left(\ln z - 1 - \frac{1}{2} - \cdots - \frac{1}{k}\right)^2 \\ & + \frac{1}{4}\left(1 + \frac{1}{2^2} + \cdots + \frac{1}{k^2}\right)\Bigg] \left(\frac{z}{4}\right)^{4k}, \end{aligned} \tag{9}$$

$$\begin{aligned} J^{(3)}(x,y,t) = (\ln z)^3 + \sum_{k=1}^{\infty} \frac{1}{(k!)^4} \left(\frac{2\lambda}{v}\right)^{4k} \Bigg[& \left(\ln z - 1 - \frac{1}{2} - \cdots - \frac{1}{k}\right)^3 \\ & + \frac{3}{4^2}\left(\ln z - 1 - \frac{1}{2} - \cdots - \frac{1}{k}\right)\left(1 + \frac{1}{2^2} + \cdots + \frac{1}{k^2}\right) \\ & - \frac{2}{4^2}\left(1 + \frac{1}{2^3} + \cdots + \frac{1}{k^3}\right)\Bigg] \left(\frac{z}{4}\right)^{4k}, \end{aligned}$$

and z is given by the equality

$$z = \left[(v^2t^2 - x^2)(v^2t^2 - y^2)\right]^{1/4}. \tag{10}$$

Proof. By means of simple but fairly unwieldy computations one can show that the automodel substitution (10) reduces partial differential equation (4) to the generalized fourth-order ordinary Bessel differential equation

$$\left\{ \mathbf{B}_z^4 - \left(\frac{2\lambda}{v}\right)^4 z^4 \right\} \psi(z) = 0, \tag{11}$$

where $\mathbf{B}_z^4$ is the generalized fourth-order Bessel differential operator

$$\mathbf{B}_z^4 = \left(z\frac{d}{dz}\right)^4.$$

According to Turbin and Plotkin [16], p.118, the solutions of equation (11) are given by the generalized Bessel functions (9). In order to check the linear independence of these functions one needs to show that their Wronskian is not zero at some arbitrary point. It is convenient to check that, for instance, at the point $z = 1$ or $z = 4$. Then taking into account (3) we obtain the statement of the theorem. □

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The commutative Moufang loops with minimum conditions for subloops II

N.I. Sandu

Abstract. It is proved that the following conditions are equivalent for an infinite nonassociative commutative Moufang loop Q: 1) Q satisfies the minimum condition for subloops; 2) if the loop Q contains a centrally solvable subloop of class s, then it satisfies the minimum condition for centrally solvable subloops of class s; 3) if the loop Q contains a centrally nilpotent subloop of class n, then it satisfies the minimum condition for centrally nilpotent subloops of class n; 4) Q satisfies the minimum condition for noninvariant associative subloops. The structure of the commutative Moufang loops, whose infinite nonassociative subloops are normal is examined.

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Keywords and phrases: Commutative Moufang loops, minimum condition for nilpotent subloops, minimum condition for solvable subloops, minimum condition for noninvariant associative subloops.

This paper is the continuation of the article [1], where the construction of the commutative Moufang loops (abbreviated CML) with the minimum condition for subloops is examined. A normal weakening for this condition is the minimum condition for the centrally solvable (centrally nilpotent) subloops of a given class. A broader question regarding these conditions is examined in Section 2, and namely, the existence in a CML of infinite centrally solvable (centrally nilpotent) subloops, possessing a property, which, by analogy with the group theory [2], will be called steady central solvability (steady central nilpotence). We will say that an infinite centrally solvable (centrally nilpotent) of the class of the loop Q is *steadily centrally solvable* (*steadily centrally nilpotent*) if any infinite centrally solvable (centrally nilpotent) subloop of the class n of loop Q contains a proper subloop of central solvability (central nilpotence) of class n. It turned out that the existence of steadily centrally solvable (centrally nilpotent) subloop of a certain given class n in CML is equivalent to the existence of an infinite decreasing series of subloops in CML. In particular it follows from here that for a CML, possessing a centrally solvable (centrally nilpotent) subloop of a certain class n, the minimum condition for subloops is equivalent to the minimum condition for subloops which have the same class of central solvability (central nilpotence) n.

It is shown in Section 3 that the minimum condition for subloops and for noninvariant associative subloops are equivalent in an infinite nonassociative CML. The infinite nonassociative CML which do not have proper infinite nonassociative subloops are described in Section 2. A weakening of the last condition is the condition for infinite nonassociative CML, when all infinite subloops are normal in them. The construction of such CML is given in Section 4.

1 Preliminaries

A *multiplicative group* $\mathfrak{M}(Q)$ of a CML Q is a group generated by all *translations* $L(x)$, where $L(x)y = xy$. The subgroup $I(Q)$ of the group $\mathfrak{M}(Q)$, generated by all the *inner mappings* $L(x,y) = L(xy)^{-1}L(x)L(y)$, is called an *inner mapping group* of the CML Q. The subloop H of the CML Q is called *normal* (*invariant*) in Q if $I(Q)H = H$.

Lemma 1.1 [3]. *The inner mappings are automorphisms in the commutative Moufang loops.*

Further we will denote by $< M >$ the subloop of the loop Q, generated by the set $M \subseteq Q$.

Lemma 1.2 [3]. *Let H and K be such loop's subloops that K is normal in $< H, K >$. Then $HK = KH =< H, K >$.*

The *associator* (a, b, c) of the elements a, b, c of the CML Q is defined by the equality $ab \cdot c = (a \cdot bc)(a, b, c)$. The identities:

$$L(x,y)z = z(z,y,x), \tag{1.1}$$

$$(x,y,z) = (y^{-1},x,z) = (y,x,z)^{-1} = (y,z,x), \tag{1.2}$$

$$(x^p, y^r, z^s) = (x,y,z)^{prs}, \tag{1.3}$$

$$(x,y,z)^3 = 1, \tag{1.4}$$

$$(xy,u,v) = (x,u,v)((x,u,v),x,y)(y,u,v)((y,u,v).y,x) \tag{1.5}$$

hold in a CML [3].

Lemma 1.3 [3]. *The periodic commutative Moufang loop is locally finite.*

Lemma 1.4 [4]. *The periodic commutative Moufang loop Q decomposes into a direct product of its maximal p-subloops Q_p, and besides, Q_p belongs to the centre $Z(Q) = \{x \in Q | (x,y,z) = 1 \forall y, z \in Q\}$ of CML Q for $p \neq 3$.*

We denote by Q_i (respect. $Q^{(i)}$) the subloop of the CML Q, generated by all associators of the form $(x_1, x_2, \ldots, x_{2i+1})$ (respect. $(x_1, \ldots, x_{3^i})^{(i)}$) where $(x_1, \ldots, x_{2i-1}, x_{2i}, x_{2i+1}) = ((x_1, \ldots, x_{2i-1}), x_{2i}, x_{2i+1})$ (respect. $(x_1, \ldots, x_{3^i})^{(i)} = ((x_1, \ldots, x_{3^{i-1}})^{(i-1)}, (x_{3^{i-1}+1}, \ldots, x_{2\cdot 3^{i-1}})^{(i-1)}, (x_{2\cdot 3^{i-1}+1}, \ldots, x_{3^i})^{(i-1)})$). The series of normal subloops $1 = Q_0 \subseteq Q_1 \subseteq \ldots \subseteq Q_i \subseteq \ldots$ (respect. $1 = Q^{(o)} \subseteq Q^{(1)} \subseteq \ldots \subseteq Q^{(i)} \subseteq \ldots$) is called the *lower central series* (respect. *derived series*) of the CML Q. We will also use for associator loop the designation $Q^{(1)} = Q'$.

The CML Q is *centrally nilpotent* (respect. *centrally solvable*) *of class* n if and only if its lower central series (respect. derived series) has the form $1 \subset Q_1 \subset \ldots \subset Q_n = Q$ (respect. $1 \subset Q^{(1)} \subset \ldots \subset Q^{(n)} = Q$).

Lemma 1.5 (Bruck-Slaby Theorem) [3]. *Let n be a positive integer, $n \geq 3$. Then every commutative Moufang loop Q which can be generated by n elements is centrally nilpotent of class at most $n-1$.*

Let M be a subset, H be a subloop of the CML Q. The subloop

$$Z_H(M) = \{x \in H|(x,u,v) = 1 \forall u,v \in M\}$$

is called the *centralizer* of the set M in the subloop H.

Lemma 1.6 [1]. *If M is a normal subloop of the subloop H of the commutative Moufang loop Q then for $a,b \in H$ $aZ_H(M) = bZ_H(M)$ if and only if $L(a,b)(a,u,v) = (b,u,v)$ for any $u,v \in M$.*

The *upper central series* of the CML Q is the series

$$1 = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \ldots \subseteq Z_\alpha \subseteq \ldots$$

of the normal subloops of the CML Q, satisfying the conditions: 1) $Z_\alpha = \sum_{\beta<\alpha} Z_\beta$ for the limit ordinal and 2) $Z_{\alpha+1}/Z\alpha = Z(Q/Z_\alpha)$ for any α.

Lemma 1.7 [3]. *The statements: 1) $x^3 \in Q$ for any $x \in Q$; 2) the quotient loop $Q/Z(Q)$ has the index 3 hold for a commutative Moufang loop Q.*

A CML Q is called *divisible* it the equation $x^n = a$ has at least one solution in Q, for any $n > 0$ and any element $a \in Q$.

Lemma 1.8 [1]. *The following conditions are equivalent for a commutative Moufang loop D: 1) D is a divisible loop; 2) D is a direct factor for any commutative Moufang loop that contains it.*

Lemma 1.9 [1]. *The following conditions are equivalent for a commutative Moufang loop Q: 1) Q satisfies the minimum condition for subloops; 2) Q is a direct product of a finite number of quasicyclic groups, lying in the centre $Z(Q)$, and a finite loop.*

2 Steadily centrally solvable (centrally nilpotent) commutative Moufang loops

Lemma 2.1. *A infinite centrally solvable (centrally nilpotent) commutative Moufang loop Q of class n contains a proper centrally solvable (centrally nilpotent) subloop of class n.*

Proof. Let us suppose the contrary, i.e., all proper subloops of the centrally solvable CML Q of class n have a class of central solvability less than n. Let us prove that in such a case the CML is finite.

As the CML Q is centrally solvable of the class n, there are such elements $a_1, \ldots, a_{3^{n-1}}$ in Q that $(a_1, \ldots, a_{3^{n-1}})^{(n-1)} \neq 1$. Due to the fact that all proper subloops of the CML Q are centrally solvable of class less than n, the elements

$a_1, \ldots, a_{3^{n-1}}$ generate the CML Q. Without violating the generality, we will suppose that all the elements $a_1, \ldots, a_{3^{n-1}}$ are different. For example, let an element a_1 have an infinite order. Then the subloop $< a_1^4, \ldots, a_{3^{n-1}} >$ is proper in Q. Now, by the identities (1.3), (1.4) we calculate

$$(a_1^4, \ldots, a_{3(n-1)})^{(n-1)} = ((a_1, \ldots, a_{3(n-1)})^{(n-1)})^4 =$$

$$= (a_1, \ldots, a_{3(n-1)})^{(n-1)} \neq 1.$$

We have obtained that the proper subloop H is centrally solvable of the class n. Contradiction. Consequently, the generators of the CML Q have a finite orders. Basing on Lemma 1.3, we conclude that the CML Q is finite. Contradiction. The second case is proved by analogy.

Corollary 2.2. *The centrally solvable (centrally nilpotent) commutative Moufang loop of class n whose proper subloops have a class of central solvability (central nilpotence) less that n is a finite loop.*

Lemma 2.3. *If a non-periodic commutative Moufang loop Q contains a finite centrally solvable (centrally nilpotent) subloop H of class n, then it contains a steadily centrally solvable (centrally nilpotent) subloop of class n.*

Proof. If a is an element of an infinite order, then by Lemma 1.6 $a^{3^k} \in Z(Q)$, where $k = 1, 2, \ldots, Z(Q)$ is the centre of the CML Q. Then the subloop $< a^{3^k}, H >$ is steadily centrally solvable (centrally nilpotent) of the class n.

Lemma 2.4. *Let a commutative Moufang loop Q, which does not satisfy the minimum condition for subloops be centrally solvable (centrally nilpotent) of the class n. Then Q possesses a proper infinite centrally solvable (centrally nilpotent) subloop of the class n.*

Proof. Let the infinite CML Q be centrally solvable of the class n and all its proper centrally solvable subloops of the class n, be finite. By Lemma 2.1 there exists a finite proper centrally solvable subloop K of the class n of the order m in the CML Q.

If L is an arbitrary normal subloop of a finite index of the CML Q, then by Lemma 1.2 LK is an infinite centrally solvable subloop of the class n and therefore $LK = Q$. By the relation

$$Q/L = LK/L \cong K(K \cap L)$$

the index of the normal subloop L is not greater than m in the CML Q. Then in the CML Q there exists a normal subloop H of a finite index. The subloop H does not possess proper normal subloops of finite index, it means that H/H' is infinite. Therefore $H'K$ is a finite subloop, and then the associator loop H' is also finite. Let us show that the subloop H is associative. Indeed, by Lemma 1.5 $aZ(H) \neq bZ(H)$ $(a, b \in H)$ if and only if there exist such elements $u, v \in H$ that $(a, u, v) \neq (b, u, v)$. Therefore the centre $Z(Q)$ has a finite index in H. The subloop H is normal in the CML Q, i.e. it is invariant regarding the inner mapping group which consists of automorphisms (Lemma 1.1). Then it is obvious that the subloop $Z(H)$ is normal in Q. We have obtained that the CML H contains a normal in Q subloop of finite index. But it contradicts the choice of subloop H. Consequently, $Z(H) = H$. Further, the set S of the elements of the group H, having simple orders, is finite. It follows from the fact that the subloop $< S > K$ (the subloop $< S >$ is normal in Q) is finite as a proper centrally solvable subloop of the class n of the CML Q. It follows from here that H is an abelian group with the minimum condition for subgroups. The second case is proved by analogy.

Corollary 2.5. *For an infinite centrally solvable (centrally nilpotent) commutative Moufang loop to be steadily centrally solvable (steadily centrally nilpotent), it is enough that it does not contain divisible subloops different from the unitary element.*

Corollary 2.6. *For an infinite periodic centrally solvable (centrally nilpotent) commutative Moufang loop Q of the class n to be steadily centrally solvable (steadily centrally nilpotent), it is necessary and sufficient that Q does not contain divisible subloops different from the unitary element.*

Proof. If Q does not contain non-trivial divisible subloops, then the necessity follows from Corollary 2.5. Conversely, for example, let the CML Q be steadily centrally solvable and let H be the maximal divisible subloop of the CML Q. By Lemma 1.7 $H \subseteq Z(Q)$. If L is a finite centrally solvable subloop of the class n, K is a quasicyclic group from H, then the subloop $< L, K >$ is centrally solvable of the class n and satisfies the minimum condition for subloops. By the mentioned above and by Lemma 2.1 it is easy to show that there exists an infinite centrally solvable subloop of the class n in the $< L, K >$ whose all subloop's proper centrally solvable subloops of the class n are finite. But it contradicts the fact that Q is steadily centrally solvable. The second case is proved by analogy. This completes the proof of Corollary 2.6.

Let us remark that the request of the periodicity of the CML Q in Corollary 2.6 is essential (example: the additive group of rational numbers).

We will call a *minimal CML* of central solvability (central nilpotence) class n any centrally solvable (centrally nilpotent) CML whose all proper subloops have a class of central solvability (central nilpotence) less than n. It follows from Lemmas 2.1 and 1.4 that these are commutative Moufang 3-loops.

Corollary 2.7. *For a commutative Moufang loop Q to be infinite centrally solvable (centrally nilpotent) of the class n, and all its proper centrally solvable (centrally nilpotent) subloops of the class n to be finite, it is necessary and sufficient that the loop Q is a direct product of quasicyclic groups and the minimal CML of the central solvability (central nilpotence) class n.*

Proof. We will examine only the case of central solvability. If an infinite CML Q is centrally solvable of class n and all its proper centrally solvable class n are finite, then by Lemma 2.4 Q satisfies the minimum condition for subloops. By Lemma 1.9 Q decomposes into a direct product of finite number of quasicyclic groups and a finite CML. Obviously, if K is a quasicyclic group and L is a minimal subloop of central solvability class n, then $Q = K \times L$. The inverse is obvious.

Lemma 2.8. *Let a commutative Moufang loop Q which does not satisfy the minimum condition for subloops be centrally solvable (centrally nilpotent) of the class n. Then Q possesses a steadily centrally solvable (steadily centrally nilpotent) subloop of class n.*

Proof. Let $Q^{(t)}$ be the last member of derived series (lower central series)

$$Q = Q^{(0)} \supset Q^{(1)} \supset \ldots \supset Q^{(t)} \supset \ldots \supset Q^{(n)} = 1$$

of the CML Q that does not satisfy the minimum condition for subloops. If there are no steadily centrally solvable (steadily centrally nilpotent) subloops of class n in the CML Q, then by Lemma 2.1 there exists a finite centrally solvable (centrally nilpotent) subloop of the class n in it. We denote it by H.

If Q is a non-periodic CML, then the statement follows from Lemma 2.3.

Let now the subloop $Q^{(t)}$ have no elements of infinite order. By (1.4) the subloop $Q^{(t+1)}$ has the degree three and by the supposition it satisfies the minimum condition for subloops. Then by Lemma 1.9 $Q^{(t+1)}$ is finite. We denote by $L/Q^{(t+1)}$ the subgroup of the abelian group $Q^{(t)}/Q^{(t+1)}$, generated by all elements of prime orders. It cannot be finite, as the group $Q^{(t)}/Q^{(t+1)}$, and then the CML $Q^{(t)}$ would also satisfy the minimum condition for subloops. We denote by Z the centralizer of the normal subloop $Q^{(t+1)}$ in the CML L. By Lemma 1.5, if $a, b \in L$, then $aZ \neq bZ$ if and only if there exist such elements u, v from $Q^{(t+1)}$ that $L(a, b)(a, u, v) \neq (b, u, v)$. The subloop $Q^{(t+1)}$ is normal in Q, then $(a, u, v) \in Q^{(t+1)}$. As $Q^{(t+1)}$ is finite, L/Z is finite. So, the subloop Z does not satisfy the minimum condition for subloops. Now it follows from the relations

$$Z/(Z \cap Q^{(t+1)}) \cong Q^{(t+1)}Z/Q^{(t+1)} \subseteq L/Q^{(t+1)}$$

that $Z/(Z \cap Q^{(t+1)})$ is an infinite abelian group. The subloop $Z \cap Q^{(t+1)}$ is contained in the centre of the CML Z, then Z is a centrally nilpotent CML of the class 2. It follows from here that the associator loop Z' is an abelian group of the exponent three. If the associator loop Z' is infinite, then $Z'H$ is an unknown subloop (the product $Z'H$ is a subloop by Lemma 1.2, as the normality of Z' in Q follows from the normality of Z in Q). But if the associator loop Z' is finite, then the subgroup K/Z' of the group Z/Z', generated by all elements of prime orders, should be infinite, as Z does not satisfy the minimum condition for subgroups. The subloop K is normal in Q as Z is normal in Q and, obviously, K contains no divisible subloops different from the unitary element. Consequently, by Corollary 2.6 HK is a steadily centrally solvable (steadily centrally nilpotent) subloop of the class n.

Lemma 2.9. *An arbitrary centrally solvable (centrally nilpotent) commutative Moufang loop Q of class n that does not satisfy the minimum condition for subloops possesses a steadily centrally solvable (steadily centrally nilpotent) subloops of central solvability (central nilpotence) class t for any $t = 1, 2, \ldots, n$.*

Proof. Let Q be a centrally nilpotent CML of class n and let $a_1, a_2, \ldots, a_{2n+1}$ be such elements from Q that $((a_1, \ldots, a_{2i+1}), a_{2i+2}, \ldots, a_{2n-1}, a_{2n}, a_{2n+1}) = 1$, but $((a_1,\dots,a_{2i+1}),a_{2i+2},\dots,a_{2n-1}) \neq 1$. Then the subloop $< (a_1,\dots,a_{2i+1}), a_{2i+2},\dots,a_{2n+1} >= H$ is centrally nilpotent of class $n-1=t$. In the case of central solvability we will examine the $(n-i)$-th member of the derived series $Q^{(n-i)}$ instead of H.

If the subloop H is not steadily centrally solvable (steadily centrally nilpotent) of class t, then by Lemma 2.1 the subloop H is finite. Let the CML Q not be periodic. Then by Lemma 2.3 Q contains a steadily centrally solvable (steadily centrally nilpotent) subloop of class t.

Let us suppose that Q is a periodic CML. Let $Q^{(i)}$ be the last member of the derived series (of the lower central series)

$$Q = Q^{(0)} \supset Q^{(1)} \supset \dots \supset Q^{(i)} \supset \dots \supset Q^{(n)} = 1$$

of the CML Q that does not satisfy the minimum condition for subloops. The subloop $Q^{(i+1)}$ satisfies the minimum condition for subloops and by (1.4) it has the index three. Then by Lemma 1.9 it is finite. We denote by $K/Q^{(i-1)}$ the subgroup of the abelian group $Q^{(i)}/Q^{(i+1)}$ generated by all elements of prime orders. The group $K/Q^{(i+1)}$ is infinite, as the CML $Q^{(i)}$ does not satisfy the minimum condition for subloops. Let us suppose that $L = KH, L_0 = Q^{(i+1)}H$. We remind that in the case of central solvability $Q^{(t)} = H$. But if $Q^{(i+1)}$ is a member of the lower central series, then the subloop L_0 is normal in L. Indeed, for that it is enough to show that if $x \in L_0, y, z \in L$, then $(x,y,z) \in L_0$. Any element from L_0 has the form ah, where $a \in Q^{(i+1)}, h \in H$, and any element from L has the form uh, where $u \in Q^{(i)}, h \in H$. If $a \in Q^{(i+1)}, u, v \in Q^{(i)}, h_1, h_2, h_3 \in H$, then by the identity (1.5) the associator (ah_1, uh_2, vh_3) may be presented as a product of the factors of the form $(a,x,y), (h_1,h_2,h_3), (u,x,y)$, where $x, y \in L$. As the subloop $Q^{(i+1)}$ is normal in Q, $(a,x,y) \in Q^{(i+1)}$. Further, it is obvious that $(h_1,h_2,h_3) \in H$. If $a \in Q^{(i)}$, then it follows from the relation $Q^{(i)}/Q^{(i+1)} \subseteq Z(Q/Q^{(i+1)})$ that $(u,x,y) \in Q^{(i+1)}$. Consequently, the subloop L_0 is normal in L.

We have already constructed such a series of elements of the CML L

$$g_1, g_2, \dots, g_r \tag{2.1}$$

that the normal subloops $L_i =< L_0, g_1, \dots, g_i >$ form s strictly ascending series $L_0 \subset L_1 \subset \dots \subset L_r$ and for any $i = 1,2,\dots,r$ the element g_i is bound by an associative law with all elements of the CML L_{i+1}. Let us show that the series (2.1) can be unlimitedly continued. We denote by Z the centralizer of the subloop L_r in L. By Lemma 1.9 if $a, b \in L$, then $aZ \neq bZ$ if and only if there exist such elements u, v from L_r that $L(a,b)(a,u,v) \neq (b,u,v)$. The CML L_r is finite and normal in L, therefore it is easy to see that L/Z is a finite CML. Then $Z/(Z \cap L_r)$ is an infinite CML. Let $g_{r+1} \in Z \backslash (Z \cap L_r)$. Then $L_r \neq< L_r, g_{r+1} >= L_{r+1}$ and the element g_{r+1} is bound by an associative law with all elements of the subloop L_r. So, the series (2.1) can be unlimitedly continued. The subloop $< H, g_1, g_2, \dots >$ is centrally solvable (centrally nilpotent) of class n and does not satisfy the minimum condition for subloops. Indeed, according to the choice of the element g_i, the quotient loop

$L_0 < g_1, \ldots, g_i, \ldots > /L_0$ is infinite, and therefore it does not satisfy the minimum condition for subloops. Consequently, the quotient loop

$$< g_1, \ldots, g_i, \ldots > /(< g_1, \ldots, g_i, \ldots > \cap L_0)$$

does not satisfy the minimum condition for subloops as well, and as L_0 is a finite CML, the subloop $< H, g_1, \ldots, g_i, \ldots >$ does not satisfy the minimum condition for subloops. It follows from Lemma 2.8 that on $< H, g_1, \ldots, g_i, \ldots >$ there exists an unknown steadily centrally solvable (steadily centrally nilpotent) subloop of class n.

Corollary 2.10. *For all centrally solvable (centrally nilpotent) of class n ($n \geq 2$) subloops of the commutative Moufang loop Q, that has such a subloop to be steadily centrally solvable (steadily centrally nilpotent) it is enough that all its infinite centrally solvable (centrally nilpotent) of class $n-1$ are steadily centrally solvable (steadily centrally nilpotent).*

Proof. Let L be an arbitrary infinite centrally solvable (centrally nilpotent) of class n subloop of the CML Q. If L is not steadily centrally solvable (steadily centrally nilpotent), then in the CML L there exists an infinite centrally solvable (centrally nilpotent) subloop H of class n whose all proper subloops of central solvability (central nilpotence) class n are finite. By Lemma 2.9 the CML H satisfies the minimum condition for subloops, and by Lemma 1.9 $H = D \times K$, where D is a divisible CML, lying in the centre $Z(H)$ and K is a finite CML. The CML K is centrally solvable (centrally nilpotent) of class n. Then it has a proper subloop T of central solvability (central nilpotence) class $n-1$. The subloop $T \times D$ is an infinite centrally solvable (centrally nilpotent) subloop of class $n-1$, satisfying the minimum condition for subloops. It follows from Lemma 2.9 $T \times D$ is not steadily centrally solvable (steadily centrally nilpotent). Contradiction.

Corollary 2.11. *For all infinite centrally solvable (centrally nilpotent) subloops of the commutative Moufang loop Q to be steadily centrally solvable (steadily centrally nilpotent) is necessary and sufficient that Q has no quasicyclic groups.*

The statement follows from the fact that an infinite abelian group is steadily centrally solvable if and only if it has no quasicyclic groups, as well as from Corollary 2.10.

Theorem 2.12. *If the commutative Moufang loop Q possesses a centrally solvable (centrally nilpotent) subloop S of class n (maybe finite), then the loop Q either contains a steadily centrally solvable (steadily centrally nilpotent) subloop of class n, or satisfies the minimum condition for subloops.*

Proof. Let us first suppose that CML Q is a countable p-loop and it is not centrally solvable (centrally nilpotent). In such a case, Q is the union of the countable series of finite subloops (by Lemma 1.3 the commutative Moufang p-loop is locally finite)

$$H_1 \subset H_2 \subset \ldots \subset H_k \subset \ldots,$$

where H_i is a centrally solvable (centrally nilpotent) subloop of class n. We denote by L_k the lower layer of the centre of the CML H_k. (The *lower layer* of the p-group G is the set $\{x \in Q | x^p = 1\}$). Let us now examine the CML $R =< H_1, L_2, \dots, L_k, \dots >$. The CML R is centrally solvable (centrally nilpotent) of class n. If the CML R is infinite, then is obvious that R does not satisfy the minimum condition for subloops, and by Lemma 2.9 the CML R contains a steadily centrally solvable (steadily centrally nilpotent) subloop of class n. But if the CML R is finite, then the CML $< L_1, L_2, \dots, L_k, \dots >$ is also finite. Therefore the centre $Z(Q)$ of the CML Q is different from the unitary element. The upper central series $Z_1 \subseteq Z_2, \subseteq \dots \subseteq Z_\beta \subseteq \dots$ of the CML Q stabilities on a certain ordinal number γ. If Z_γ is a centrally solvable (centrally nilpotent) CML, then the CML Q contains a steadily centrally solvable (steadily centrally nilpotent) subloop of class n. Indeed, in this case the quotient loop Q/Z_γ is a countable p-loop, and is not centrally solvable (centrally nilpotent). Then by the above-mentioned reasoning, and as the Q/Z_γ is a CML without a centre, we obtain that the CML Q/Z_γ contains a steadily centrally solvable (steadily centrally nilpotent) subloop of class n. Let it be the subloop K/Z_γ. By the definition of the derived series (of the lower central series) the subloop K is centrally solvable (centrally nilpotent) and it does not satisfy the minimum condition for subloops. Then by Lemma 2.9 the CML K contains a steadily centrally solvable (steadily centrally nilpotent) subloop of class n.

Let us now that Z_γ is not a centrally solvable (centrally nilpotent) subloop and let SZ_α be the first member of the series $SZ_1 \subset SZ_2 \subset \dots \subset SZ_\beta \dots$ which is not a centrally solvable (centrally nilpotent) subloop. If the CML SZ_β does not satisfy the minimum condition for at least one ordinal number β ($\beta < \alpha$), then by Lemma 2.9 the CML SZ_β contains an unknown steadily centrally solvable (steadily centrally nilpotent) subloop. Now suppose that for all β ($\beta < \alpha$) the subloops SZ_β satisfy the minimum condition for subloops, and denote by D the maximal divisible subloop of the CML SZ_α. By Lemma 1.9 $SZ_\alpha = D \times \overline{Z}_\alpha$, where $\overline{Z}_\alpha$ is a reduced CML. The subloops SZ_β ($\beta < \alpha$) satisfy the minimum condition, then by Lemmas 1.8, 1.7 $SZ_\beta = D_\beta \times \overline{Z}_\beta$, where D_β are divisible CML, $\overline{Z}_\beta$ are finite normal reduced subloops. Consequently, $\overline{Z}_\alpha$ is the union of an ascending series of finite normal subloops Z_β ($\beta < \alpha$). The maximal subloop $\overline{M}$ of the CML $\overline{Z}_\alpha$ that has the central solvability (central nilpotence) class n cannot be finite. Indeed, it follows from the finiteness of the subloop $\overline{M}$ that $\overline{M} \subset \overline{Z}_\beta$ for a certain $\beta < \alpha$. We denote by Z the centralizer of the subloop $\overline{Z}_\beta$ in the CML $\overline{Z}_\alpha$. If $a, b, \in \overline{Z}_\alpha$, then by Lemma 1.9 $aZ \neq bZ$ if and only if the exist such elements $u, v \in \overline{Z}_\beta$ that $L(a,b)(a,u,v) \neq (b,u,z)$. The subloop $\overline{Z}_\beta$ is normal in Q and it is finite, therefore the centralizer Z is infinite. So, there exists a non-unitary element $w \in Z \backslash \overline{M}$. The subloop $< \overline{M}, w >$ has the central solvability (central nilpotence) class n and is different from the subloop $\overline{M}$, that contradicts the choice of $\overline{M}$. Thus, $\overline{M}$ is an infinite CML. By the maximality of the divisible CML D, the CML $\overline{M}$ is a steadily centrally solvable (steadily centrally nilpotent) subloop of class n by Corollary 2.6.

Let now Q be an arbitrary CML satisfying the theorem's conditions. If a is an element of infinite order, then by Lemma 2.9 in the CML $< S, a >$ there exists a steadily centrally solvable (steadily centrally nilpotent) subloop of class n.

Let Q be a periodic CML, not centrally solvable (centrally nilpotent). By Lemma

1.4 Q decomposes into a direct product of its maximal p-subloops Q_p, besides, Q_p lies in the centre of the CML Q for $p \neq 3$. Then the subloop Q_3 is not centrally solvable (centrally nilpotent) and such a countable subloop can be found within it. By the above-mentioned, the latter contains an unknown steadily centrally solvable (steadily centrally nilpotent) subloop.

Corollary 2.13. *The following conditions are equivalent for a nonassociative commutative Moufang loop:*

1) the loop Q satisfies the minimum condition for subloops;

2) if the loop Q contains a centrally solvable subloop of class s, it satisfies the minimum condition for the centrally solvable subloops of class s;

3) if the loop Q contains a centrally nilpotent subloop of class n, it satisfies the minimum condition for the centrally nilpotent subloops of class n;

4) the loop Q satisfies the minimum condition for the associative subloops;

5) the loop Q satisfies the minimum condition for nonassociative subloops.

Corollary 2.14. *An infinite commutative Moufang loop Q, possessing a centrally solvable (centrally nilpotent) subloop H of class n, has also an infinite subloop of such type.*

Proof. Let $a \in Q$ be an element of infinite order. By Lemma 1.6 $a^{3^k} \in Z(Q)$, $k = 1, 2, \ldots$, therefore $< H, a^{3^k} >$ is an unknown subloop. If the periodic CML Q does not satisfy the minimum condition for centrally solvable (centrally nilpotent) subloops of class s, then it contains an infinite subloop of this type, as the CML Q is locally finite (Lemma 1.3). In the opposite case, by Corollary 2.13 and Lemma 1.9 $Q = D \times K$, where $D \subseteq Z(Q), K$ is a finite CML. In this case $D, H >$ is an unknown subloop.

Corollary 2.15. *Any infinite commutative Moufang loop possesses an infinite associative subloop.*

The statement follows from Corollary 2.14 and from the fact the CML is monoassociative.

Corollary 2.16. *A commutative Moufang loop with finite centrally solvable (centrally nilpotent) subloops of class n, $n = 1, 2, \ldots$, is finite itself.*

The statement is equivalent to Corollary 2.14.

In particular, the equivalence of the conditions 1), 5) of Corollary 2.13 means that each infinite nonassociative CML has an infinite nonassociative subloop different from itself with the exception of the case when it satisfies the minimum condition for subloops. It is clear that not any infinite CML with the minimum condition is an exception here. It holds true indeed.

Proposition 2.17. *The infinite nonassociative commutative Moufang loop Q does not contain its proper infinite nonassociative subloops if and only if it decomposes into a direct product of quasicyclic groups, contained in the centre $Z(Q)$ of the loop Q, and a finite nonassociative loop, generated by three elements.*

Proof. By Corollary 2.13 the CML Q satisfies the minimum condition for subloops, then by Lemma 1.9 $Q = D \times H$, where D is a direct product of a finite number of quasicyclic groups, $D \subseteq Z(Q)$, H is a finite CML. By the supposition about the CML Q, the group D contains only one quasicyclic group.

Obviously H is an nonassociative CML. If H_1 is an arbitrary proper subloop of the CML H, then by Lemma 1.2 the product DH_1 is a proper infinite subloop of the CML Q. But then DH_1 and H_1 are associative subloops. Consequently, all proper subloops of the CML Q are associative, and it follows from Lemma 1.5 [3] that H is generated by tree elements. Let now the CML Q have a decomposition $Q = D \times H$, possessing these qualities, and L be an arbitrary proper subloop of the CML Q. Obviously $D \subseteq L$. Then it follows from the decomposition $Q = D \times H$ that $L = D(L \cap H)$. As $L \neq Q$, then $L \cap H \neq H$. Then the subloop $L \cap H$, as a proper subloop of the CML H, is associative. Therefore it follows from the decomposition $L = D(L \cap H)$ that the subloop H is associative.

3 Infinite nonassociative commutative Moufang loops with minimum condition for noninvariant associative subloops

Lemma 3.1. *If an element a of an infinite order or of order three of a commutative Moufang loop Q generates a normal subloop, then it belongs to the centre $Z(Q)$ of loop Q.*

Proof. If the element $1 \neq a \in Q$ generates a normal subloop, then $L(u, v)a = a^k$ for a certain natural number k and for arbitrary fixed elements $u, v \in Q$. By (1.1) $a(a, v, u) = a^k, (a, v, u) = a^{k-1}$. If $k = 1$, then $(a, v, u) = 1$. Therefore $a \in Z(Q)$. Let us now suppose that $k > 1$. Let $a^3 = 1$. Then $k = 2$ and by (1.5) and Lemma 1.5 $a = (a, v, u), a = ((a, v, u), v, u) = 1$. We have obtained a contradiction, as $a \neq 1$. But if a has an infinite order, then by (1.4) $(a^{k-1})^3 = (a, v, u)^3 = 1$. We have obtained a contradiction again. Therefore the case of $k > 1$ is impossible. This completes the proof of Lemma 3.1.

Lemma 3.2. *The commutative Moufang loop Q, containing an element of an infinite order, is associative if and only if the subloop, generated by any element of an infinite order, is normal in Q.*

Proof. By Lemma 3.1 any element a of an infinite order of the CML Q belongs to the centre $Z(Q)$. Let b be an element of a finite order of the CML Q. Obviously the product ab has an infinite order. Again by Lemma 3.1 $ab \in Z(Q)$. Further, by (1.5) and (1.4) we have $1 = (ab, u, v) = L(a, b)(a, u, v) \cdot L(b, a)(b, u, v) = (b, L(b, a)u, L(b, a)v)$, for $u, v \in Q$. Consequently, $b \in Z(Q)$, but then the CML Q is associative.

Theorem 3.3. *If in an infinite commutative Moufang loop Q the infinite associative subloops are normal in Q, then Q is associative.*

Proof. It follows from Lemma 3.2 that it is sufficient to examine the case when the CML Q is periodic, and by Lemma 1.4 it is sufficient to examine the case when Q is a 3-loop.

Let us now first examine the case when the CML Q does not satisfy the minimum condition for subloops. By Corollary 4.5 from [1] none of its maximal elementary associative subloops H can be finite. Let

$$H = H_1 \times H_2 \times \ldots \times H_n \times \ldots$$

be the decomposition of the group H into a direct product of cyclic groups of order three. We denote by $Z_Q(H)$ the centralizer of the subloop H in Q. It is obvious that for any element a from $Z_Q(H)$ there is such an infinite subgroup $H(a) \subseteq H$ that $< a > \cap H(a) = 1$. Let $H(a) = H_1(a) \times H_2(a)$ be a decomposition of the group $H(a)$ into a direct product of infinite factors. As the cyclic group $< a >$ is the intersection of the infinite associative subloops $< a > H_1(a)$ and $< a > H_2(a)$, then $< a >$ is normal in Q. As the element a from $Z_Q(H)$ is arbitrary, we obtain that any subloop from $Z_Q(Q)$ is normal in Q, i.e. $Z_Q(H)$ is a hamiltonian CML. Then by [4] it is an associative subloop. Obviously, $H \subseteq Z_Q(H)$ and, as H_i are cyclic groups of order three, then by Lemma 3.1 $H_i \subseteq Z(Q)$, where $Z(Q)$ is the centre of the CML Q. Consequently, $Z(H) = Q$ is an associative CML.

If a CML Q satisfies the minimum condition for subloops, then by Lemma 1.9 its centre $Z(Q)$ is infinite. If a is an arbitrary element from Q, then the subloop $< a > Z(Q)$ is infinite and associative. From here and from the theorem's supposition we obtain that the subloop $< a >$ is normal in Q. Then by [4] the CML Q is associative.

Lemma 3.4. *A non-periodic commutative Moufang loop, satisfying the minimum condition for the noninvariant cycle groups, is associative.*

Proof. By Lemma 3.2 we suppose that the element a of an infinite order of the CML Q generates a noninvariant subloop. It follows from the condition of lemma that the series

$$< a > \supset < a_t > \supset < a^{t^2} > \supset \ldots \supset < a^{t^n} > \supset \ldots$$

should contain a normal subloop $< a^{t^n} >$ for any natural number t. Let t and p be two different prime numbers, $< a^{t^n} >$ and $< a^{p^k} >$ be two normal subloops corresponding to them, of such a type that u, v are such integer numbers that $ut^n + vp^k = 1$. Then

$$a = a^{ut^n + vp^k} = a^{ut^n} \cdot a^{vp^k}.$$

If x and y are arbitrary elements from Q, then by Lemma 1.1 the inner mapping $L(x, y)$ is an automorphism. Then, by the normality of the subloops $< a^{t^n} >, < a^{p^k} >$, we obtain $L(x, y)a = L(x, y)a^{ut^n} \cdot L(x, y)a^{vp^k} = (L(x, y)a^{t^n})^u \cdot (L(x, y)a^{p^k})^v \in < a >$. Consequently, the subloop $< a >$ is normal in Q. Contradiction. Then the CML Q is associative.

Theorem 3.5. *In a nonassociative commutative Moufang loop the minimum condition for subloops and the minimum condition for noninvariant associative subloops are equivalent.*

Proof. Let us suppose that the CML Q, satisfying the minimum condition for noninvariant associative subloops, does not satisfy the minimum condition for subloops. Then by Corollary 2.13 the CML Q does not satisfy the minimum condition for associative subloops. Let us show that in this case the CML Q is associative, i.e. we will obtain a contradiction. By Lemma 3.4 it is sufficient to examine the case when the CML Q is periodic, and by Lemma 1.4 when Q is a 3-loop.

As the CML Q does not satisfy the minimum condition for associative subloops, then by Corollary 4.5 from [1] Q contains an infinite direct product

$$H = H_1 \times H_2 \times \ldots \times H_n \times \ldots$$

of cyclic groups of order three. If a is an arbitrary element from the centralizer $Z_Q(H)$ of the subloop H in the CML Q, then there exists such a number $n = n(a)$ that

$$< a > \cap (H_{n+1} \times H_{n+2} \times \ldots) = 1.$$

As the CML Q satisfies the minimum condition for noninvariant associative subloops, then the infinitely descending series of associative subloops

$$S^k(a) \supset S^{k+1}(a) \subset \ldots$$

contains a normal subloop $S^l(a)(l = l(a))$, beginning with any natural $k \geq n$, where $S^k(a) =< a > (H_{k+1} \times H_{k+2} \times \ldots)$. As the intersection of all such normal subloops coincides with the subloop $< a >$, then the latter is normal in Q. But a is an arbitrary element from the centralizer $Z_Q(H)$, and it means that any normal subloop from $Z_Q(H)$ is normal. Then by [4] the CML $Z(H)$ is associative. Further, the subgroups H_i have the order three. Then it follows from Lemma 3.1 that they belong to the centre $Z(Q)$ of the CML Q. Then it follows from the definition of the centralizer $Z_Q(H)$ that $Z(Q) = Q$. Consequently, the CML Q is associative.

4 Infinite nonassociative commutative Moufang loops in which all infinite nonassociative subloops are normal

Lemma 4.1. *Let all infinite nonassociative subloops be normal in an infinite nonassociative commutative Moufang loop Q. If H is an infinite nonassociative subloop, then the quotient loop Q/H is a group.*

Proof. It is obvious that any subloop of the CML Q containing H, is normal in Q. Then the quotient loop Q/H is hamiltonian, consequently by [4] it is a group.

Proposition 4.2. *The commutative Moufang loop, in which all its infinite nonassociative subloops are normal has a finite associator loop Q'.*

Proof. Let us suppose the contrary, i.e., that the associator loop Q' is infinite. First we examine the case when Q' is nonassociative. Let H be a proper infinite nonassociative subloop of the CML Q'. Then by Lemma 4.1 Q/H is a group, i.e. $Q' \subseteq H$. Contradiction. Consequently, the associator loop Q' does not have its proper infinite nonassociative subloops. In this case, by Corollary 2.13 the CML Q' satisfies the minimum condition for subloops. But by (1.4) the associator loop Q' has degree three, therefore it is finite.

Let us now examine the case when the infinite associator loop Q' of the periodic CML is associative. Let H be a finite nonassociative subloop of the CML Q. We will examine the subloop $Q'H = \cup x_i Q', x_i \in H, i = 1, \ldots, m$. If the infinite nonassociative subloop $Q'H$ does not contain its proper infinite nonassociative subloops, then by Corollary 2.13 it satisfies the minimum condition for subloops. Taking into account (1.4), it is easy to see that the CML $Q'H$ has a finite index. Then it is finite, therefore the CML Q' is also finite. It contradicts the fact the CML $Q'H$ does not contain its proper infinite nonassociative subloops. Let $(Q'H)_1$ be the proper infinite nonassociative subloops of the CML $Q'H$. By Lemma 4.1 $Q' \subseteq (Q'H)_1$. Then $(Q'H)_1 = \cup x_i Q', i = 1, \ldots, n, n < m$. If the infinite nonassociative subloop $(Q'H)_1$ does not contain its proper infinite nonassociative subloops, then $(Q'H)_1$ is finite, as it is shown above. Contradiction. Therefore let $(Q'H)_2$ be the proper infinite nonassociative subloop of the CML $(Q'H)_1$. By Lemma 4.1 $Q' \subseteq (Q'H)_2$, therefore $(Q'H)_2 \subseteq \cup x_i Q', x_i \in H, i = 1, \ldots, r, r < n$. Applying the previous reasoning to the CML $(Q'H)_2$, after a finite number of steps we will come to infinite nonassociative subloops $(Q'H)_i$ without proper infinite nonassociative subloops. But it contradicts the statement from the previous section. Consequently, the associator loop Q' of the CML Q cannot be infinite.

Finally, let us examine the case when the CML Q is non-periodic. Obviously, the subloop H of the CML Q is nonassociative if and only if the subloop $HZ(Q)$ is nonassociative, where $Z(Q)$ is the centre of the CML Q. If the infinite nonassociative subloops of the CML Q are normal, then the infinite nonassociative subloops of the CML $Q/Z(Q)$ are normal as well. By Lemma 1.9 the CML $Q/Z(Q)$ has index three, then, according to the previous case, its associator loop $(Q/Z(Q))'$ is finite. If $a \in Z(Q)$, then $(au, v, w) = (u, v, w)$, for any $u, v, w \in Q$. It is easy to see from here that the associator loop Q' is finite.

Corollary 4.3. *If in a non-periodic commutative Moufang loop Q all the infinite nonassociative subloops are normal in Q, then its associator loop is a finite associative subloop.*

Proof. Let us suppose that the finite associator loop Q' is nonassociative. Let H be one of its minimal nonassociative subloops, and a be an element of infinite order from Q. By Lemma 1.9 a^3 belongs to the centre of the CML Q. Then by Lemma 1.2, $H < a^3 >$ is an infinite nonassociative subloop. By Lemma 4.1 $Q' \subseteq H < a^3 >$, and it is impossible if $H \neq Q'$. According to the minimality of the nonassociative CML H, it can be presented in the form of the product of the normal associative subloop L and the cyclic group $< b >$. Indeed, by the Moufang theorem [3] the CML H is generated by three elements u, v, b. By Lemma 1.5 $Q' \neq H$. Then $L =< Q', u, v >$ is a normal associative subloop and $H = L\cdot < b >$. Now let us take the CML $B\cdot < a^3 b >$. It is an infinite nonassociative subloop and, obviously, it does not contain Q'. However, by Lemma 4.1 $Q' \subseteq B$. Contradiction. Consequently, the associator loop Q' of the CML Q is associative.

Theorem 4.4. *If all infinite nonassociative subloops of a commutative Moufang loop Q are normal in it, then all nonassociative subloops are also normal in it.*

Proof. Let Q be a non-periodic CML and a be an element of an infinite order from Q. By Lemma 1.9 a^3 belongs to the centre of the CML Q. If H is a finite nonassociative subloop, then by Lemma 1.2 $< a^3 > H$ is an infinite nonassociative subloop from Q and, consequently, it is normal in Q. Therefore, H is normal in Q.

Let now Q be a periodic CML and let us suppose that the finite nonassociative subloop L is not normal in Q. The associator loop Q' is a normal subloop in Q. Therefore, by Lemma 1.9 the centralizer $Z_Q(H)$ of the subloop H in Q will be normal subloop in Q. Let us examine the set

$$C(H) = \{x \in Z_Q(H) | (x, u, v) = 1 \forall u \in Z_Q(H), \forall v \in H\}.$$

Using the identity (1.5), it is easy to show that $C(H)$ is a subloop. Moreover, it follows from the normality of the subloops $H, Z_Q(H)$, and by Lemma 1.1, that $C(H)$ is normal in Q. Indeed, if $xC(H) = yC(H)$, then $xy^{-1} \in C(H), (xy^{-1}, u, v) = 1$ for all $u \in Z_Q(H), v \in H$. Now we will use the identities (1.5), (1.1) and (1.3). We have $1 = (xy^{-1}, u, v) = L(x, y^{-1})(x, u, v) \cdot L(y^{-1}, x)(y^{-1}, u, v) = (x, L(x, y^{-1})u, L(x, y^{-1})v)(y^{-1}, L(y^{-1}, x)u, L(y^{-1}, x)v) \equiv (x, \overline{u}, \overline{v})(y^{-1}, \overline{u}, \overline{v}) = (x, \overline{u}, \overline{v})(y, \overline{u}, \overline{v})^{-1}$, $(x, \overline{u}, \overline{v}) = (y, \overline{u}, \overline{v})$ for all $u \in Z_Q(H), v \in H$. It can be proved by analogy that it follows from the equality $(x, \overline{u}, \overline{v}) = (y, \overline{u}, \overline{v})$ from all $u \in Z_Q(H), v \in H$ that $xC(H) = yC(H)$. By Proposition 4.2 the associator loop Q' is finite. Then the normal subloop $C(H)$ has a finite index in Q.

Let us show that the CML Q satisfies the minimum condition for subloops. Let us suppose the contrary. Then the subloop $C(H)$, possessing a finite index in Q, does not satisfy this condition as well. Therefore, the CML $C(H)$ has an infinite associative subloop K which decomposes into a direct product of cyclic groups of prime orders. Otherwise, by Corollary 2.13 and regardless the supposition, the CML Q would satisfy the minimum condition for subloops. It is obvious that an infinite subgroup R can be found, that intersects with L on the unitary element. Let $R = R_1 \times R_2$ be the decomposition of R into a direct product of two infinite subgroups R_1, R_2.

If S is an arbitrary associative subloop of the CML $C(H)$, then the product SL is a subloop. Indeed, by Lemma 1.2, the subloop S is normal in the CML $< S, L >$. The CML $< S, L >$ consists of all "words", composed of the elements of the set $S \cup L$. A word of the length 1 is an element of the set $S \cup L$. If u, v are words of length m, n respectively, then $u^{\epsilon_1} v^{\epsilon_2}$, where $\epsilon_1, \epsilon_2 = \pm 1$, is a word of length $\leq m + n$. It follows from the definition of the subloop $C(H)$ that if 1) $a \in S, u \in L$; 2) $a, u \in S, v \in L$, then $(a, u, v) = 1$. If $a \in S, u, v \in < S, L >$ then, using (1.2), (1.5) and the associativity of the subloop S, it can be proved by the induction on the sum of the length of the words u, v that $(a, u, v) = 1$. Then by (1.1) $L(v, u)a = a$, i.e. the subloop S is normal in $< S, L >$. Therefore $< S, L >= SL$.

By the above proved fact, the products R_1L, R_2L are subloops. As they are infinite and nonassociative, they are normal in the CML Q. Then their intersection L is also a normal subloop in Q. We have obtained a contradiction despite the supposition of the noninvariance of the subloop L. In this case, by Lemma 1.8 the CML Q decomposes into a direct product of the divisible group D, lying in the centre $Z(Q)$ of the CML Q, and the finite CML M. If $L \neq M$, then the product DL is an infinite nonassociative subloop of the CML Q, therefore the subloop L is also normal in Q. We have obtained a contradiction to the fact that L is not normal in Q. This completes the proof of Theorem 4.4.

By Corollary 4.3 a non-periodic CML whose infinite nonassociative subloops are normal in it has a finite associative associator loop. The following statement holds true for the general case.

Corollary 4.5. *If all (infinite) nonassociative subloops of an (infinite) nonassociative commutative Moufang loop Q are normal in it, then its associator loop Q' is centrally nilpotent, and the loop Q itself is centrally solvable of a class not greater than three.*

Proof. By Proposition 4.2, the associator loop Q' is finite. Then by Lemma 1.5 Q' is centrally nilpotent.

Let us suppose that the second associator loop $Q^{(2)}$ of the CML Q is nonassociative. Then any subloop that contains $Q^{(2)}$ is non-assiciative, and by Theorem 4.4, it is normal in Q. Obviously, the CML $Q/Q^{(2)}$ is hamiltonian, when it is an abelian group, by [4]. Therefore, $Q' \subseteq Q^{(2)}$, i.e. $Q' = Q^{(2)}$. But the associator loop Q' is centrally nilpotent, therefore $Q' \neq Q^{(2)}$. Contradiction. Consequently, $Q^{(2)}$ is an associative subloop, and the CML Q is centrally solvable of step not greater than three.

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On orders of elements in quasigroups

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Abstract. We study the connection between the existence in a quasigroup of (m, n)-elements for some natural numbers m, n and properties of this quasigroup. The special attention is given for case of (m, n)-linear quasigroups and (m, n)-T-quasigroups.

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1 Introduction

We shall use basic terms and concepts from books [1, 2, 11]. We recall that a binary groupoid (Q, A) with n-ary operation A such that in the equality $A(x_1, x_2) = x_3$ knowledge of any two elements of x_1, x_2, x_3 the uniquely specifies the remaining one is called a *binary quasigroup* [3]. It is possible to define a binary quasigroup also as follows.

Definition 1. *A binary groupoid $(Q, \circ)$ is called a quasigroup if for any element (a, b) of the set Q^2 there exist unique solutions $x, y \in Q$ to the equations $x \circ a = b$ and $a \circ y = b$ [1].*

An element $f(b)$ of a quasigroup $(Q, \cdot)$ is called a *left local identity element* of an element $b \in Q$, if $f(b) \cdot b = b$.

An element $e(b)$ of a quasigroup $(Q, \cdot)$ is called a *right local identity element* of an element $b \in Q$, if $b \cdot e(b) = b$.

The fact that an element e is a *left (right) identity element* of a quasigroup $(Q, \cdot)$ means that $e = f(x)$ for all $x \in Q$ (respectively, $e = e(x)$ for all $x \in Q$).

The fact that an element e is an *identity element* of a quasigroup $(Q, \cdot)$ means that $e(x) = f(x) = e$ for all $x \in Q$, i.e. all left and right local identity elements in the quasigroup $(Q, \cdot)$ coincide [1].

A quasigroup $(Q, \cdot)$ with an identity element is called a *loop.* In a loop $(Q, \cdot)$ there exists a unique identity element. Indeed, if we suppose, that 1 and e are identity elements of a loop $(Q, \cdot)$, then we have $1 \cdot e = 1 = e$.

Quasigroups are non-associative algebraic objects that, in general, do not have an identity element. Therefore there exist many ways to define the order of an element in a quasigroup.

In works [5, 6] the definition of an (n, m)-identity element of a quasigroup $(Q, \cdot)$ and some results on topological medial quasigroups with an (n, m)-identity element were given. These articles were our starting-point by the study of (m, n)-order of elements in quasigroups.

As usual $L_a : L_a x = a \cdot x$ is the left translation of quasigroup $(Q, \cdot)$, $R_a : R_a x = x \cdot a$ is the right translation of quasigroup $(Q, \cdot)$, $Mlt(Q, \cdot)$ denotes the group generated by the set of translations $\{L_x, R_y \,|\, \text{for all } x, y \in Q\}$.

An element d of a quasigroup $(Q, \cdot)$ with the property $d \cdot d = d$ is called an *idempotent element*. By ε we mean the *identity permutation*.

Definition 2. *A quasigroup $(Q, \cdot)$ defined over an abelian group $(Q, +)$ by $x \cdot y = \varphi x + \psi y + c$, where c is a fixed element of Q, φ and ψ are both automorphisms of the group $(Q, +)$, is called a T-quasigroup [9, 10].*

A quasigroup $(Q, \cdot)$ satisfying the identity $xy \cdot uv = xu \cdot yv$ is called a *medial quasigroup*. By Toyoda theorem (T-theorem) every medial quasigroup $(Q, \cdot)$ is a T-quasigroup with additional condition $\varphi\psi = \psi\varphi$ [1, 2].

A loop $(Q, \cdot)$ with the identity $x(y \cdot xz) = (xy \cdot x)z$ is called a *Moufang loop*; a loop with the identity $x(y \cdot xz) = (x \cdot yx)z$ is called a *left Bol loop*.

A Moufang loop is *diassociative*, i.e. every pair of its elements generates a subgroup; a left Bol loop is a *power-associative loop*, i.e. every its element generates a subgroup [1, 4, 11].

A left Bol loop $(Q, \cdot)$ with the identity $(xy)^2 = x \cdot (y^2 \cdot x)$ is called a *Bruck loop*. Any Bruck loop has the property $I(x \cdot y) = Ix \cdot Iy$, where $x \cdot Ix = 1$ for all $x \in Q$ [11].

Definition of the order of an element of a power-associative loop $(Q, \cdot)$ can be given as definition of the order of an element in case of groups [7].

Definition 3. *The order of an element b of the power-associative loop $(Q, \cdot)$ is the order of the cyclic group $< b >$ which it generates.*

2 $(\mathbf{m}, \mathbf{n})$-orders of elements

Definition 4. *An element a of a quasigroup $(Q, \cdot)$ has the order (m, n) (or element a is an (m, n)-element) if there exist natural numbers m, n such that $L_a^m = R_a^n = \varepsilon$ and the element a is not the (m_1, n_1)-element for any integers m_1, n_1 such that $1 \leq m_1 < m$, $1 \leq n_1 < n$.*

Remark 1. It is obvious that m is the order of the element L_a in the group $Mlt(Q, \cdot)$, n is the order of the element R_a in this group. Therefore it is possible to name the (m, n)-order of an element a as well as the (L, R)-order or the left-right-order of an element a.

Remark 2. In the theory of non-associative rings ([8]) often one uses so-called left and right order of brackets by multiplying of elements of a ring $(R, +, \cdot)$, namely $(\dots(((a_1 \cdot a_2) \cdot a_3) \cdot a_4) \dots)$ is called the left order of brackets and $(\dots(a_4 \cdot (a_3 \cdot (a_2 \cdot a_1)))\dots)$ is called the right order of brackets.

So the (m, n)-order of an element a of a quasigroup $(Q, \cdot)$ is similar to the order of an element a of a non-associative ring $(R, +, \cdot)$ with the right and the left orders of brackets respectively.

Proposition 1. *In a diassociative loop $(Q,\cdot)$ there exist only (n,n)-elements.*

Proof. If we suppose that there exists an element $a \in Q$ of diassociative loop of order (m,n), then in this case we have $L_a^m x = a\cdot(a\cdot\ldots(a\cdot x)\ldots) = a^m x = L_{a^m}x$.

Therefore $L_a^m = \varepsilon$ if and only if $a^m = 1$, where 1 is the identity element of the loop $(Q,\cdot)$. Similarly $R_a^n = \varepsilon$ if and only if $a^n = 1$.

From the last two equivalences and Definitions 3, 4 (from the minimality of numbers m,n) it follows that in a diassociative loop $m = n$, i.e. in diassociative loop there exist only (n,n)-elements. $\square$

Remark 3. It is clear that Proposition 1 is true for Moufang loops and groups since these algebraic objects are diassociative.

From Definition 4 it follows that $(1,1)$-element is the identity element of a quasigroup $(Q,\cdot)$, i.e. in this case the quasigroup $(Q,\cdot)$ is a loop.

Proposition 2. *Any $(1,n)$-element is a left identity element of a quasigroup $(Q,\cdot)$. In any quasigroup such element is unique and in this case the quasigroup $(Q,\cdot)$ is so-called a left loop i.e. $(Q,\cdot)$ is a quasigroup with a left identity element.*

Any $(m,1)$-element is a right identity element of a quasigroup $(Q,\cdot)$, the quasigroup $(Q,\cdot)$ is a right loop.

Proof. If in a quasigroup $(Q,\cdot)$ an element a has the order $(1,n)$, then $a\cdot x = L_a^{\cdot}x = x$ for all $x \in Q$. If we suppose that in a quasigroup $(Q,\cdot)$ there exist left identity elements e and f, then we obtain that equality $x\cdot a = a$, where a is some fixed element of the set Q, will have two solutions, namely, e and f are such solutions. We obtain a contradiction. Therefore in a quasigroup there exists a unique left identity element. $\square$

Using the language of quasigroup translations it is possible to re-write the definition of an (n,m)-identity element from [5,6] in the form:

Definition 5. *An idempotent element e of a quasigroup $(Q,\cdot)$ is called an (m,n)-identity element if and only if there exist natural numbers m,n such that $(L_e^{\cdot})^m = (R_e^{\cdot})^n = \varepsilon$.*

Hence any (m,n)-identity element of a quasigroup $(Q,\cdot)$ can be called as well as *idempotent element of order* (m,n) or an *idempotent* (m,n)*-element*.

Theorem 1. *A quasigroup $(Q,\cdot)$ has an (m,n)-identity element 0 if and only if there exist a loop $(Q,+)$ with the identity element 0 and permutations φ,ψ of the set Q such that $\varphi 0 = \psi 0 = 0$, $\varphi^n = \psi^m = \varepsilon$, $x\cdot y = \varphi x + \psi y$ for all $x,y \in Q$.*

Proof. Let a quasigroup $(Q,\cdot)$ have an idempotent element 0 of order (m,n). Then the isotope $(R_0^{-1}, L_0^{-1}, \varepsilon)$ of the quasigroup $(Q,\cdot)$ is a loop $(Q,+)$ with the identity element 0, i.e. $x + y = R_0^{-1}x \cdot L_0^{-1}y$ for all $x,y \in Q$ ([1]). From the last equality we have $x\cdot y = R_0 x + L_0 y$, $R_0 0 = L_0 0 = 0$. Then $\varphi = R_0$, $\psi = L_0$, $L_0^m = \psi^m = R_0^n = \varphi^n = \varepsilon$.

Conversely, let $x \cdot y = \varphi x + \psi y$, where $(Q, +)$ is a loop with the identity element 0, $\varphi 0 = \psi 0 = 0$, $\varphi^m = \psi^n = \varepsilon$. Then the element 0 is an idempotent element of quasigroup $(Q, \cdot)$ of order (m, n) since $L^{\cdot}_0 y = \psi y$, $R^{\cdot}_0 x = \varphi x$ and $(L^{\cdot}_0)^m = \psi^m = \varepsilon$, $(R^{\cdot}_0)^n = \varphi^n = \varepsilon$. □

3 (m, n)-linear quasigroups

Definition 6. *A quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \psi y$, where φ, ψ are automorphisms of a loop $(Q, +)$ such that $\varphi^n = \psi^m = \varepsilon$, will be called an (m, n)-linear quasigroup.*

Taking into consideration Theorem 1 we see that any (m, n)-linear quasigroup $(Q, \cdot)$ is a linear quasigroup over a loop $(Q, +)$ with at least one (m, n)-idempotent element.

Lemma 1. *In an (m, n)-linear quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \psi y$, where $(Q, +)$ is a group, we have*
$L^{\cdot}_a = L^+_{\varphi a}\psi$, $(L^{\cdot}_a)^k = L^+_c \psi^k$, $c = \varphi a + \psi\varphi a + \cdots + \psi^{k-1}\varphi a$,
$R^{\cdot}_a = R^+_{\psi a}\varphi$, $(R^{\cdot}_a)^r = R^+_d \varphi^r$, $d = \psi a + \varphi\psi a + \cdots + \varphi^{r-1}\psi a$.

Proof. It is well known that if $\varphi \in Aut(Q, \cdot)$, i.e. if $\varphi(x \cdot y) = \varphi x \cdot \varphi y$ for all $x, y \in Q$, then $\varphi L_x y = L_{\varphi x}\varphi y$, $\varphi R_y x = R_{\varphi y}\varphi x$. Indeed, we have $\varphi L_a x = \varphi(a \cdot x) = \varphi a \cdot \varphi x = L_{\varphi a}\varphi x$, $\varphi R_b x = \varphi(x \cdot b) = \varphi x \cdot \varphi b = R_{\varphi b}\varphi x$.

Using these last equalities we have

$$(L^{\cdot}_x)^2 = L^+_{\varphi x}\psi L^+_{\varphi x}\psi = L^+_{\varphi x + \psi\varphi x}\psi^2, \quad (L^{\cdot}_x)^3 = L^+_{(\varphi x + \psi\varphi x) + \psi^2\varphi x}\psi^3,$$

and so on. □

Proposition 3. *An element a of an (m, n)-linear quasigroup $(Q, \cdot)$ over a group $(Q, +)$ has the order (k, r), where $k, r \in N$, if and only if $\varphi a + \psi\varphi a + \cdots + \psi^{k-1}\varphi a = 0$, $\psi a + \varphi\psi a + \cdots + \varphi^{r-1}\psi a = 0$, $k = m \cdot i$, $r = n \cdot j$, where i, j are some natural numbers.*

Proof. It is possible to use Lemma 1. If an element $a \in Q$ has an order $(k, \_)$, then the permutation $L^k_a = L^+_c \psi^k$, where $c = \varphi a + \psi\varphi a + \cdots + \psi^{k-1}\varphi a$ is the identity permutation. This is possible only in two cases: (i) $L^+_c = \psi^{-k} \neq \varepsilon$; (ii) $L^+_c = \varepsilon$ and $\psi^k = \varepsilon$.

Case (i) is impossible. Indeed, if we suppose that $L^+_c = \psi^{-k}$, then we have $L^+_c 0 = \psi^{-k} 0$, where 0 is the identity element of the group $(Q, +)$. Further we have $\psi^{-k} 0 = 0$, $L^+_c 0 = 0$, $c = 0$, $L^+_c = \varepsilon$, $\psi^k = \varepsilon$. Therefore, if the element a has the order $(k, \_)$, then $L^+_c = \varepsilon$ and $\psi^k = \varepsilon$. Further, since $\psi^m = \varepsilon$, we have that $k = m \cdot i$ for some natural number $i \in N$.

Converse. If $\varphi a + \psi\varphi a + \cdots + \psi^{k-1}\varphi a = 0$, $L^+_c = \varepsilon$ and $\psi^k = \varepsilon$ for some element a, then this element has the order $(k, \_)$.

Therefore an element a of an (m, n)-linear quasigroup $(Q, \cdot)$ over a group $(Q, +)$ will have the order $(k, \_)$ if and only if $L^+_c = \varepsilon$, i.e. $c = 0$, where $c = \varphi a + \psi\varphi a + \cdots + \psi^{k-1}\varphi a$ and $\psi^k = \varepsilon$, i.e. $k = m \cdot i$ for some natural number $i \in N$.

Similarly any element a of an (m,n)-linear quasigroup $(Q,\cdot)$ over a group $(Q,+)$ will have the order $(\_,r)$ if and only if $R_d^+=\varepsilon$, i.e. $d=0$, where $d=\psi a+\varphi\psi a+\dots+\varphi^{r-1}\psi a$ and $\varphi^r=\varepsilon$. Further, since $\varphi^r=\varepsilon$, we have that $r=n\cdot j$ for some natural number $j\in N$. □

Proposition 4. *The number M of elements of order (mi,nj) in an (m,n)-linear quasigroup $(Q,\cdot)$ over a group $(Q,+)$ is equal to $|K(\varphi)\cap K(\psi)|$ where $K(\varphi)=\{x\in Q\mid \psi x+\varphi\psi x+\dots+\varphi^{nj-1}\psi x=0\}$, $K(\psi)=\{x\in Q\mid \varphi x+\psi\varphi x+\dots+\psi^{mi-1}\varphi x=0\}$.*

Proof. From Proposition 3 it follows that an element a of an (m,n)-linear quasigroup $(Q,\cdot)$ over a group $(Q,+)$ has the order (mi,nj) if and only if $\varphi a+\psi\varphi a+\dots+\psi^{mi-1}\varphi a=0$ and $\psi a+\varphi\psi a+\dots+\varphi^{nj-1}\psi a=0$.

In other words an element a of (m,n)-linear quasigroup $(Q,\cdot)$ over a group $(Q,+)$ has the order (mi,nj) if and only if $a\in K(\varphi)\cap K(\psi)$.

Therefore $M=|K(\varphi)\cap K(\psi)|$. □

Theorem 2. *Any $(2,2)$-linear quasigroup $(Q,\cdot)$ over a loop $(Q,+)$ such that all elements of $(Q,\cdot)$ have the order $(2,2)$ can be represented in the form $x\cdot y=Ix+Iy$, where $x+Ix=0$ for all $x\in Q$.*

Proof. In this case we have $(L_x^{\cdot})^2=L^+_{\varphi x}L^+_{\psi\varphi x}\psi^2=L^+_{\varphi x}L^+_{\psi\varphi x}=\varepsilon$ for any $x\in Q$. Then $\varphi x+(\psi\varphi x+0)=\varepsilon 0=0$ for all $x\in Q$. Therefore $x+\psi x=0$, $\psi x=-x=Ix$.

By analogy we have that $\varphi x=-x=Ix$ for all $x\in Q$. Indeed, $(R_x^{\cdot})^2=R^+_{\psi x}R^+_{\varphi\psi x}\varphi^2=R^+_{\psi x}R^+_{\varphi\psi x}=\varepsilon$, $\psi x+\varphi\psi x=0$, $x+\varphi x=0$, $\varphi x=Ix$. □

Remark 4. From Theorem 2 it follows that any $(2,2)$-linear quasigroup $(Q,\cdot)$ such that all elements of $(Q,\cdot)$ have the order $(2,2)$ exists only over a loop with the property $I(x+y)=Ix+Iy$ for all $x,y\in Q$, where $x+Ix=0$ for all $x\in Q$. A loop with this property is called an *automorphic-inverse property loop (AIP-loop)*.

We notice, the Bruck loops, the commutative Moufang loops, the abelian groups are AIP-loops.

4 $(\mathbf{m},\mathbf{n})$-linear T-quasigroups

Theorem 3. *If in an (m,n)-linear T-quasigroup $(Q,\cdot)$ of the form $x\cdot y=\varphi x+\psi y$ over an abelian group $(Q,+)$ the maps $\varepsilon-\varphi,\varepsilon-\psi$ are permutations of the set Q, then all elements of the quasigroup $(Q,\cdot)$ have order (m,n).*

Proof. It is easy to see that if the maps $\varepsilon-\varphi,\varepsilon-\psi$ are permutations of the set Q, then $m>1,n>1$. From Proposition 4 it follows that the number M of elements of the order (m,n) is equal to the number $|K(\varphi)\cap K(\psi)|$, where

$$K(\varphi)=\{x\in Q|(\varepsilon+\varphi+\ \dots\ +\varphi^{n-1})\psi x=0\},$$
$$K(\psi)=\{x\in Q|(\varepsilon+\psi+\ \dots\ +\psi^{m-1})\varphi x=0\}.$$

Since the map $\varepsilon-\varphi$ is a permutation of the set Q, we have: $\varepsilon+\varphi+\ \dots\ +\varphi^{n-1}=(\varepsilon+\varphi+\ \dots\ +\varphi^{n-1})(\varepsilon-\varphi)(\varepsilon-\varphi)^{-1}=(\varepsilon-\varphi+\varphi-\varphi^2+\varphi^2-\ \dots\ -\varphi^n)(\varepsilon-\varphi)^{-1}=(\varepsilon-\varphi^n)(\varepsilon-\varphi)^{-1}$. Since $\varphi^n=\varepsilon$ we obtain that $K(\varphi)=Q$.

By analogy it is proved that $K(\psi) = Q$. Therefore $K(\varphi) \cap K(\psi) = Q$. $\square$

A quasigroup $(Q, \cdot)$ with the identities $x\cdot(y\cdot z) = (x\cdot y)\cdot(x\cdot z)$, $(x\cdot y)\cdot z = (x\cdot z)\cdot(y\cdot z)$ is called a *distributive quasigroup* [1].

Corollary 1. *In any medial distributive* (m, n)*-linear quasigroup all its elements have order* (m, n).

Proof. It is known that any medial distributive quasigroup $(Q, \cdot)$ can be presented in the form $x \cdot y = \varphi x + \psi y$, where $(Q, +)$ is an abelian group and $\varphi + \psi = \varepsilon$ [1, 12]. Therefore conditions of Theorem 3 are fulfilled in any medial distributive (m, n)-linear quasigroup. $\square$

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Flow induced by a constantly accelerating plate in a Maxwell fluid

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In this note, the analytical solutions corresponding to an unsteady flow induced by a flat plate, in a Maxwell fluid, are determined. These solutions satisfy both the associate partial differential equations and all imposed initial and boundary conditions. The similar solutions corresponding to a Newtonian fluid appear as a limiting case of our solutions. Finally, some numerical results and interesting conclusions are presented.

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Kojalovich Method and Studying Abel's Equation with the one known solution

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Abstract. The problem of constructing a general solution for the Abel's equation of the special kind with a known partial solution is considered.

Mathematics subject classification: 34A30.
Keywords and phrases: Abel's differential equation of the second kind, solution.

1 Introduction

It is known that there are some classes of differential equations that are not integrable in quadratures but become integrable if some its partial solution has been found. As an example, let us consider the following Abel's equation of the second kind [1, 2]

$$yy' - y = r(x), \tag{1}$$

where $y = y(x)$ is an unknown function and $r(x)$ is some known function which will be determined below. Equation (1) is connected closely with many problems of physics, mechanics, chemistry, biology, ecology and other [1]. Some differential equations which are reduced to Abel equation are considered in [1–3].

In order to solve this equation we shall use the following special method which was developed in the textbook of Kojalovich [4].

2 Kojalovich's method

Let a function f be an integrating function of equation (1). Then, according to [4], it satisfies the following equality

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\ \frac{y+r}{y} + \frac{\partial f}{\partial \alpha_i} + \frac{\alpha_i + r}{\alpha_i} = \psi(x,y)\ \omega(x,\alpha_i), \tag{2}$$

where α_i $(i = \overline{1,n})$ are partial solutions of equation (1).

Instead of f let us consider new integrating function $F(x, y, \alpha_i)$ of the form

$$F(x,y,\alpha_i) = f(x,y,\alpha_i) + \lambda_1(x)\ \frac{\partial f}{\partial \alpha_i} + \lambda_2(x)\ \frac{\partial^2 f}{\partial \alpha_i^2}, \tag{3}$$

where $\lambda_1(x),\ \lambda_2(x)$ are unknown functions. The function F is obviously a superposition of functions of x. So its derivative may be written as

$$\frac{dF}{dx}=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\ \frac{d\ y}{d\ x}+\frac{\partial F}{\partial \alpha_i}\ \frac{d\ \alpha_i}{d\ x}=$$

$$=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\ \frac{y+r(x)}{y}+\frac{\partial F}{\partial \alpha_i}\ \frac{\alpha_i+r(x)}{\alpha_i}=$$

$$=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\ \frac{y+r(x)}{y}+\frac{\partial f}{\partial \alpha_i}\ \frac{\alpha_i+r(x)}{\alpha_i}\ +\lambda_1'(x)\ \frac{\partial f}{\partial \alpha_i}+$$

$$+\lambda_1'(x)\left(\frac{\partial^2 f}{\partial x\ \partial \alpha_i}+\frac{\partial^2 f}{\partial y\ \partial \alpha_i}\ \frac{y+r(x)}{y}+\frac{\partial^2 f}{\partial {\alpha_i}^2}\ \frac{\alpha_i+r(x)}{\alpha_i}\right)+$$

$$+\lambda_2'(x)\ \frac{\partial^2 f}{\partial {\alpha_i}^2}+\lambda_2(x)\left(\frac{\partial^3 f}{\partial x\ \partial {\alpha_i}^2}+\frac{\partial^3 f}{\partial y\ \partial {\alpha_i}^2}\ \frac{y+r(x)}{y}+\frac{\partial^3 f}{\partial {\alpha_i}^3}\ \frac{\alpha_i+r(x)}{\alpha_i}\right), \qquad (4)$$

because of (1)

$$y'=\frac{y+r(x)}{y},\quad \alpha_i'=\frac{\alpha_i+r(x)}{\alpha_i}.$$

Differentiating equation (2) by α_i we write

$$\frac{\partial^2 f}{\partial x\ \partial \alpha_i}+\frac{\partial^2 f}{\partial y\ \partial \alpha_i}\ \frac{y+r(x)}{y}+\frac{\partial^2 f}{\partial {\alpha_i}^2}\ \frac{\alpha_i+r(x)}{\alpha_i}=\psi(x,y)\ \frac{\partial \omega(x,\alpha_i)}{\partial \alpha_i}+\frac{r(x)}{{\alpha_i}^2}\ \frac{\partial f}{\partial \alpha_i}, \quad (5)$$

$$\frac{\partial^3 f}{\partial x\ \partial {\alpha_i}^2}+\frac{\partial^3 f}{\partial y\ \partial {\alpha_i}^2}\ \frac{y+r(x)}{y}+\frac{\partial^3 f}{\partial {\alpha_i}^3}\ \frac{\alpha_i+r(x)}{\alpha_i}=$$

$$=\psi(x,y)\ \frac{\partial^2 \omega(x,\alpha_i)}{\partial {\alpha_i}^2}+2\frac{r(x)}{{\alpha_i}^2}\ \frac{\partial^2 f}{\partial {\alpha_i}^2}-2\frac{r(x)}{{\alpha_i}^3}\ \frac{\partial f}{\partial \alpha_i}. \qquad (6)$$

Using relations (5) and (6) we rewrite (4) in the form

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\ \frac{y+r(x)}{y}+\frac{\partial F}{\partial \alpha_i}\ \frac{\alpha_i+r(x)}{\alpha_i}=\psi(x,y)\ \omega(x,\alpha_i)+\lambda_1'(x)\frac{\partial f}{\partial \alpha_i}+$$

$$+\lambda_1(x)\left(\psi(x,y)\frac{\partial \omega(x,\alpha_i)}{\partial \alpha_i}+\frac{r(x)}{{\alpha_i}^2}\ \frac{\partial f}{\partial \alpha_i}\right)+$$

$$+\lambda_2'(x)\frac{\partial^2 f}{\partial {\alpha_i}^2}+\lambda_2(x)\left(\psi(x,y)\ \frac{\partial^2 \omega(x,\alpha_i)}{\partial {\alpha_i}^2}+2\frac{r(x)}{{\alpha_i}^2}\ \frac{\partial^2 f}{\partial {\alpha_i}^2}-2\frac{r(x)}{{\alpha_i}^3}\ \frac{\partial f}{\partial \alpha_i}\right)=$$

$$=\psi(x,y)\ \left(\omega(x,\alpha_i)+\lambda_1(x)\ \frac{\partial \omega(x,\alpha_i)}{\partial \alpha_i}+\lambda_2(x)\ \frac{\partial^2 \omega(x,\alpha_i)}{\partial {\alpha_i}^2}\right)+$$

$$+\frac{\partial f}{\partial \alpha_i}\ \left(\lambda_1'(x)+\lambda_1(x)\ \frac{r(x)}{{\alpha_i}^2}-2\lambda_2(x)\ \frac{r(x)}{{\alpha_i}^3}\right)+\frac{\partial^2 f}{\partial {\alpha_i}^2}\ \left(\lambda_2'(x)+2\lambda_2(x)\ \frac{r(x)}{{\alpha_i}^2}\right). \quad (7)$$

As the function F is an integrating function (it is easy to verify that F satisfies the criterium (2)) then the coefficients of $\dfrac{\partial f}{\partial \alpha_i}$ and $\dfrac{\partial^2 f}{\partial {\alpha_i}^2}$ must be equal to 0. Thus the functions $\lambda_1(x)$, $\lambda_2(x)$ satisfy the following differential equations

$$\lambda_1'(x) + \lambda_1(x)\ \frac{r(x)}{{\alpha_i}^2} - 2\lambda_2(x)\ \frac{r(x)}{{\alpha_i}^3} = 0, \quad \lambda_2'(x) + 2\lambda_2(x)\ \frac{r(x)}{{\alpha_i}^2} = 0. \tag{8}$$

General solution of the second equation of the system (8) may be written in the form

$$\lambda_2(x) = C_1\ \exp(2\ I_1), \tag{9}$$

where $I_1 \equiv\ -\displaystyle\int \frac{r(x)}{{\alpha_i}^2}\ dx$ and C_1 is an arbitrary constant. Using (9) we rewrite the first equation of the system (8) in the form

$$\lambda_1'(x) + \lambda_1(x)\ \frac{r(x)}{{\alpha_i}^2} = 2C_1\ \frac{r(x)}{{\alpha_i}^3}\ \exp(2\ I_1). \tag{10}$$

Its general solution has the form

$$\lambda_1(x) = \left(C_2 + 2C_1\ \int\ \exp(I_1)\ \frac{r(x)}{{\alpha_i}^3}\ dx\right)\ \exp(I_1). \tag{11}$$

Thus, the function F satisfies the equation

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\ \frac{y + r(x)}{y} + \frac{\partial F}{\partial \alpha_i}\ \frac{\alpha_i + r(x)}{\alpha_i} = \Psi(x,y)\ \Omega(x,\alpha_i),$$

where

$$\Psi(x,y) = \psi(x,y),$$

$$\Omega(x,\alpha_i) = \omega(x,\alpha_i) + \lambda_1(x)\ \frac{\partial\omega(x,\alpha_i)}{\partial\alpha_i} + \lambda_2(x)\ \frac{\partial^2\omega(x,\alpha_i)}{\partial{\alpha_i}^2}.$$

Hence F is the integrating function for equation (1).

3 Application of the method

Let us know one partial solution α of the equation (1). Kojalovich proved [4] that there are three types of autonomous integrating functions which may be written in the form

$$f = \int^{y-\alpha_i}\ \frac{\exp(h\ u)}{u}\ du, \tag{12}$$

$$f = \frac{1}{hy}\ \exp(h(y-\alpha_i)) - \int^{y-\alpha_i}\ \frac{\exp(h\ u)}{u}\ du,$$

$$f = -\frac{1}{h\alpha_i}\ \exp(h(y-\alpha_i)) - \int^{y-\alpha_i}\ \frac{\exp(h\ u)}{u}\ du,$$

where h is a constant. We consider here only the integrating function (12).

Substituting (12) into the equation (3) (the functions $\lambda_2(x),\ \lambda_1(x)$ are determined from equations (9), (11)) we obtain

$$\frac{\exp(h(y-\alpha)+2I(x))\ C_1\ (h(y-\alpha)-1)}{(y-\alpha)^2}+\int^x \frac{\exp(hy)}{u}\ du-$$

$$-\frac{\exp(h(y-\alpha)+I(x))\ \left(C_2+2C_1\ \int_0^x \frac{\exp(I(v_1))\ r(v_1)}{\alpha^3(v_1)}\ dv_1\right)}{y-\alpha}=0, \tag{13}$$

where $C_1,\ C_2$ are arbitrary constants and

$$I(x)\equiv\ -\int_0^x \frac{r(v)}{\alpha^2(v)}\ dv.$$

Differentiating (13) yields

$$\frac{r(x)\ \exp(h(y-\alpha))}{\alpha^3\ y}\ (\alpha^2-C_2\alpha(1+h\ \alpha)\ e^{I(x)}+C_1(2+2h\ \alpha+h^2\alpha^2)\ e^{2I(x)}-$$

$$-2C_1\ \alpha(1+h\alpha)\ e^{I(x)}\ \int_0^x \frac{r(t)}{\alpha^3(t)}\ e^{I(t)}\ dt)=0.$$

Thus, the partial solution α must satisfy the following equation

$$\alpha^2=\alpha(1+h\alpha)\ e^{I(x)}\left(C_2+2C_1\ \int_0^x \frac{r(t)}{\alpha^3(t)}\ e^{I(t)}\ dt\right)-C_1(2+2h\ \alpha+h^2\alpha^2)\ e^{2I(x)}. \tag{14}$$

Solving equation (14) for $C_1=3/h,\ C_2=0$ we find

$$\alpha=-\ \frac{3\ r(x)}{1+h\ r(x)}. \tag{15}$$

Substituting (15) into equation (1) we obtain

$$x(r)=C_3+\frac{3}{h(hr+1)}+\frac{\ln(hr-2)-\ln(hr+1)}{h},$$

where C_3 is an arbitrary constant. Then equation (1) has the form

$$y(r)\ y'(r)=\frac{9(r+y(r))}{(hr-2)(hr+1)^2}. \tag{16}$$

Substituting (15) into equation (3) (functions $\lambda_1(x),\ \lambda_2(x)$ have the form (9), (11)) we obtain the general integral of equation (16)

$$\int_0^x \frac{\exp(h\ \Phi)}{\Phi}\ dt-\frac{\exp(h\ \Phi))\ (1+hr)(6r(hr-1)+3r+(hr-2)(hr+1)y(r))}{h(hr-2)(y(r)+r(3+hy(r)))^2}=C, \tag{17}$$

where

$$\Phi\equiv\ \frac{3r}{1+hr}+y(r)$$

and C is an arbitrary constant.

4 Conclusion

We have shown that using the integrating function (12) it is possible to build a new integrating function of the form (3), where functions $\lambda_1(x)$, $\lambda_2(x)$ are defined in (9), (11). Besides, the corresponding Abel's equation is obtained in the form (16) and its general integral has the form (17).

It is possible to consider integrating function of the form

$$F(x,y,\alpha_i) = f(x,y,\alpha_i) + \sum_{k=1}^{n} \lambda_k(x)\, \frac{\partial^k f}{\partial \alpha_i^k},$$

where $\lambda_k(x)$ $(k=\overline{1,n})$ are some functions of x. This functions also may be found with the help of the upper considered procedure.

This procedure may be used for solving Abel's equation of the form

$$y(r)\ y'(r) = \frac{4n^3(r+y(r))}{(hr-n)(hr+n)^2}, \tag{18}$$

where n is a natural number, h is a constant and $r = r(x)$. The corresponding partial solution of equation (18) is

$$y = -\,\frac{2nr}{n+hr}.$$

Note that equation (18) for $n=1$ was integrated in [5].

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Discrete optimal control problems on networks and dynamic games with p players

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Abstract. We consider a special class of discrete optimal control problems on networks. The dynamics of the system is described by a directed graph of passages. An additional integral-time cost criterion is given and the starting and final states of the system are fixed. The game-theoretical models for such a class of problems are formulated, and some theoretical results connected with the existence of the optimal solution in the sense of Nash are given. A polynomial-time algorithm for determining Nash equilibria is proposed. The results are applied to decision making systems and determining the optimal strategies in positional games on networks.

Mathematics subject classification: 90C47.
Keywords and phrases: Multiobjective discrete control, Optimal strategies in the sence Nash, dynamic Netwerks, dynamic c-game.

1 Introduction and problems formulations

In this paper we study control processes for time-discrete systems with finite set of states. The main results are concerned with game-theoretical approach to the following control problem [1, 2].

Let us consider a discrete dynamical system L with the set of states $X \subseteq \mathbb{R}^n$. At every time-step $t = 0, 1, 2, \dots$ the state of the system L is $x(t) \in X$, $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$. The dynamics of the system L is described as follows

$$x(t+1) = g_t(x(t), u(t)), \quad t = 0, 1, 2, \dots , \tag{1}$$

where $x(0) = x_0$ is the starting point of the dynamical system and $u(t) = (u_1(t), u_2(t), \dots, u_m(t)) \in \mathbb{R}^m$ represents the vector of control parameters (see). For any time-step t the feasible set $U_t(x(t))$ for the vector of control parameter $u(t)$ is given, i.e.,

$$u(t) \in U_t(x(t)), \quad t = 0, 1, 2, \dots .$$

Assume that in (1) the vector-functions

$$g_t(x(t), u(t)) = (g_t^1(x(t), u(t)),\ g_t^2(x(t), u(t)), \dots, g_t^n(x(t), u(t)))$$

are determined uniquely by $x(t)$ and $u(t)$ at every time-step $t = 0, 1, 2, \dots$. So, $x(t+1)$ is determined uniquely by $x(t)$ and $u(t)$.

We consider the following discrete optimal control problem:

Problem 1. Find T and $u(0), u(1), \ldots, u(T-1)$ which satisfy the conditions

$$\begin{cases} x(t+1) = g_t(x(t), u(t)), \quad t = 0, 1, 2, \ldots, T-1; \\ u(t) \in U_t(x(t)), \quad t = 0, 1, \ldots, T-1, \\ x(0) = x_0, \; x(T) = x_f \end{cases}$$

and minimize the objective function

$$F_{x_0 x_f}(u(t)) = \sum_{t=0}^{T-1} c_t(x(t), g_t(x(t), u(t))),$$

where $c_t(x(t), g_t(x(t), u(t))) = c_t(x(t), x(t+1))$ represents the cost of the system's passage from the state $x(t)$ to the state $x(t+1)$ at the stage $[t, t+1]$. The vectors $v(0), v(1), \ldots, v(T-1)$ generate the trajectory $x_0 = x(0), x(1), x(2), \ldots, x(T) = x_f$ which transfers the system L from the starting state x_0 to the final state x_f with minimal integral-time costs (cf. [1, 2]).

This problem can be applied in decision making systems where the dynamics of the systems are controlled by one person. Here, we formulate the game theoretic approach to this problem, i.e., we consider that the dynamics of the system is controlled by p actors (players) and it is described as follows

$$x(t+1) = g_t(x(t), u^1(t), u^2(t), \ldots, u^p(t)), \quad t = 0, 1, 2, \ldots\ , \tag{2}$$

where $x(0) = x_0$ is the starting state of system L and $u^i(t) \in \mathbb{R}^{m_i}$ represents the vector of control parameters of player i, $i = \overline{1,p}$, i.e. $i \in \{1, \ldots, p\}$. The state $x(t+1)$ of the system L at time-step $t+1$ is obtained uniquely if the state $x(t)$ at time-step t is known and the players $1, 2, \ldots, p$ fix their vectors of control parameters $u^1(t), u^2(t), \ldots, u^p(t)$, respectively. For each $i = \overline{1,p}$, the admissible sets $U_t^i(x(t))$ for the vectors of control parameters $u^i(t)$ are given, i.e.

$$u^i(t) \in U_t^i(x(t)), \quad i = \overline{1,p}, \quad t = 0, 1, 2, \ldots\ ,$$

We shall assume that the sets $U_t^i(x(t)) \quad i = \overline{1,p}$, $t = 0, 1, 2, \ldots$, are non-empty and $U_t^i(x(t)) \cap U_t^j(x(t)) = \emptyset$ for $i \neq j$, $t = 0, 1, 2, \ldots$.

Let us consider that the players $1, 2, \ldots, p$ fix their vectors of control parameters

$$u^1(t), u^2(t), \ldots, u^p(t); \; t = 0, 1, 2, \ldots,$$

respectively, and the starting state $x_0 = x(0)$ and final state x_f of the system L are known. Then for a given set of vectors $u^1(t), u^2(t), \ldots, u^p(t)$ either a unique trajectory

$$x_0 = x(0), x(1), x(2), \ldots, x(T(x_f)) = x_f$$

from x_0 to x_f exists and $T(x_f)$ represents the time step when the state x_f is reached by L, or such a trajectory does not exist.

The game version of our control problem is the following.

Problem 2. We denote by

$$F^i_{x_0x_f}(u^1(t),u^2(t),\dots,u^p(t))=\sum_{t=0}^{T(x_f)-1} c^i_t(x(t),g_t(x(t),u^1(t),u^2(t),\dots,u^p(t)))$$

the integral-time cost of system's passage from x_0 to x_f for the players i, $i=1,2,\dots,p$, if the set of vectors $u^1(t),u^2(t),\dots,u^p(t)$ generates a trajectory

$$x_0=x(0),x(1),x(2),\dots,x(T(x_f))=x_f$$

from x_0 to x_f and

$$u^i(t)=U^i(x(t)),\ t=0,1,2,\dots,T(x_f).$$

Otherwise we put

$$F^i_{x_0x_f}(u^1(t),u^2(t),\dots,u^p(t))=\infty.$$

Here

$$c^i_t(x(t),g_t(x(t),u^1(t),u^2(t),\dots,u^p(t)))=c^i_t(x(t),x(t+1))$$

represents the cost of system's passage from the state $x(t)$ to state $x(t+1)$ at the stage $[t,t+1]$. We consider the problem of finding vectors of control parameters

$$u^{1^*}(t),u^{2^*}(t),\dots,u^{i-1^*}(t),\ u^{i^*}(t),u^{i+1^*}(t),\dots,u^{p^*}(t)$$

which satisfy the following condition

$$F^i_{x_0x_f}(u^{1^*}(t),u^{2^*}(t),\dots,u^{i-1^*}(t),\ u^{i^*}(t),u^{i+1^*}(t),\dots,u^{p^*}(t))\le$$

$$\le F^i_{x_0x_f}(u^{1^*}(t),u^{2^*}(t),\dots,u^{i-1^*}(t),\ u^{i}(t),u^{i+1^*}(t),\dots,u^{p^*}(t))$$

$$\forall\, u^i(t)\in\mathbb{R}^{m_i},\ i=\overline{1,p}.$$

So, we consider the problem of finding the solution in the sense of Nash [3].

In order to determine the existence of Nash equilibria for multiobjective control in problem 2 we assume that players i and j never actively control the system at the same state in time, although for different moments of time different players may control the system at same state. This condition in the game model corresponds to the case when for any moment of time t and for an arbitrary state $x(t)\in X$ the application in (2) depends only on one of the vectors of control parameters $u^i(t)$, $i\in\{1,2.\dots.p\}$. The multiobjective control problem with a such condition allows us to regard it as a dynamic noncooperative game on a network which consists of p interacting subnetworks controlled by different players. On this network the problem is considered to control the given initial vertex (state) toward some prescribed final state. It is assumed that the costs on edges of the network depend on time, i.e. depend on order an edge is visited in the directed path from starting state to final one. We are seeking for a Nash equilibrium in this dynamic game. Polynomial-time algorithms for determining the optimal strategies of players are proposed.

This problem generalizes the classical control problems with integral-time cost criterion by a trajectory and arose as auxiliary one when solve dynamic games in positional form [4 – 10]. The main tool we shall use for studying and solving our problem is based on dynamic programming and concept of noncooperative games in positional form. A such approach for determining the optimal strategies in dynamic games for the case with constant cost functions on edges of the network has been used in [8]. Here we extend this approach for the general case of the problem.

2 Discrete optimal control problems and dynamic games with p players on networks

In this section we consider the discrete optimal control problem on networks. We formulate the game variant of the problem when the dynamics of the system is described by a directed graph of passages [4 – 6]. The graph vertices in this problem correspond to the states of the system, where the edges identify the possibility of the system to pass from one state to another. Moreover, the cost functions are associated to the edges of the graph which depend on time and express the cost of the system's passages. The graph of passages, on which edges time-depending cost functions are defined, and in which two vertices corresponding to the starting and the final states of the system are chosen, is called a *dynamic network* [6]. First, we formulate the discrete optimal control problem on dynamic networks, and then we shall extend the model according to a game-theoretical approach.

2.1 The discrete optimal control problem on networks

Let L be a dynamical system with a finite set of states X, $|X| = N$, and at every discrete moment of time $t = 0, 1, 2, \dots$ the state of the system L is $x(t) \in X$. Note, that we associate $x(t)$ with an abstract element (in Section 1, $x(t)$ represents a vector from $\mathbb{R}^n$). Two states x_0 and x_f are chosen in X, where x_0 is a the starting point of the system L, $x_0 = x(0)$, and x_f is the final state of the system, i.e., x_f is the state in which the system must be brought. The dynamics of the system is described by a directed graph of passages $G = (X, E), |E| = m$, an edge $e = (x, y)$ which signifies the possibility of passages of the system L from the state $x = x(t)$ to the state $y = x(t + 1)$ at any moment of time $t = 0, 1, 2, \dots$. This means that the edges $e = (x, y) \in E$ can be regarded as the possible values of the control parameter $u(t)$ when the state of the system is $x = x(t)$, $t = 0, 1, 2, \dots$. The next state $y = x(t + 1)$ of the system L is determined uniquely by $x = x(t)$ at the time-step t and an edge $e = (x, y) \in E(x)$, where $E(x) = \{X \mid (x, y) \in E\}$. So $E(x) = E\big(x(t)\big)$ corresponds to the admissible set $U_t\big(x(t)\big)$ for the control parameter $u(t)$ at every time-step t. To each edge $e = (x, y)$ a function $c_e(t)$ is assigned, which reflects the costs of system's passage from the state $x(t) = x \in X$ to the state $x(t + 1) = y \in X$ at any time-step $t = 0, 1, 2, \dots$ We consider the discrete optimal control problem on networks [1, 2, 7] for which the sequence of system's passages $(x(0),\, x(1)),\, (x(1),\, x(2)),\, \dots,\, (x(T(x_f) - 1),\, x(T(x_f))) \in E$, which transfers the system L from $x_0 = x(0)$ to $x_f = x(T(x_f))$ with minimal integral-time cost of the passages by a trajectory $x_0 = x(0), x(1), x(2), \dots, x(T(x_f)) = x_f$. Here, we distinguish the following variants of the problem:

1) the number of the stages (time $T(x_f)$) is fixed, i.e. $T(x_f) = T$;
2) for $T(x_f)$ is given the restriction $T(x_f) \in [T_1, T_2]$, where T_1 and T_2 are known;
3) the parameter $T(x_f)$ is unknown and must be found.

2.2 A dynamic programming approach and computational complexity

Let us assume that $T(x_f)$ is fixed, i.e. $T(x_f) = T$ (case 1). Denote by

$$F_{x_0 x_f}(T) = \min_{x_0 = x(0), x(1), \ldots, x(T) = x_f} \sum_{t=0}^{T-1} c_{(x(t), x(t+1))}(t)$$

the minimal integral-time cost of system's passages from x_0 to x_f. If the state x_f couldn't be reached by using T stages, then we put $F_{x_0 x_f}(T) = \infty$. For $F_{x_0 x(t)}(t)$ the following recursive formula can be gained

$$F_{x_0 x(t)}(t) = \min_{x(t-1) \in X_G^-(x(t))} \Big\{ F_{x_0 x(t-1)}(t-1) + c_{(x(t-1), x(t))}(t-1) \Big\},$$

where $X_G^-(y) = \{x \in X \mid e = (x, y) \in E\}$. It is easy to observe that using dynamical programming methods we could tabulate the values $F_{x_0 x(t)}(t)$, $t = 1, 2, \ldots, T$ $(F_{x_0 x(0)}(0) = 0)$. So, if T is fixed, then the problem can be solved in time $O(N^2 T)$ (Here we do not take in consideration the number of operations for calculations of the value of functions $c_e(t)$ for given t.)

In the case when $T(x_f) \in [T_1, T_2]$ the problem can be reduced to $T_2 - T_1 + 1$ problems with $T(x_f) = T_1$, $T(x_f) = T_1 + 1$, $T(x_f) = T_1 + 2, \ldots, T(x_f) = T_2$, respectively; compare the minimal integral-costs of these problem we find the best one.

Case 3) of the problem can be reduced to case 2) if we find T_1 and T_2 such that $T(x_f) \in [T_1, T_2]$. It is obvious that for positive and non-decreasing cost functions $c_e(t)$, $e \in E$, we have $T(x_f) \in [1, N-1]$, i.e., $T_1 = 1, T_2 = N - 1$. Therefore, the problem for positive and non-decreasing functions on edges can be solved in time $O(N^3)$.

2.3 A game theoretic approach for the discrete optimal control problem on networks

Now, we consider the game-theoretical versions of the problem from Section 2.1. First we formulate the stationary case of the problem and then we extend it to nonstationary one.

2.3.1 The problem of determining the optimal stationary strategies of players in dynamic c-game

Let $G = (X, E)$ be a directed graph of system's passages, where G has the property that for any vertex $x \in X \setminus \{x_f\}$ there exists a leaving edge $e = (x, y) \in$

E. Assume that the vertex set X is divided into p disjoint subsets $X_1, X_2, \dots, X_p$ $(X = \bigcup_{i=1}^{p} X_i,\ X_i \cap X_j = \emptyset, i \neq j)$ and consider vertices $x \in X_i$ as the positions of player i, $i = \overline{1,p}$. Moreover, we consider that to each edge $e = (x,y) \in E$ of the graph of passages p functions $c_e^1(t), c_e^2(t), \dots, c_e^p(t)$ are assigned, where $c_e^i(t)$ expresses the cost of system's passage from the state $x = x(t)$ to the state $y = x(t+1)$ at the stage $[t, t+1]$ for player i. Define the stationary strategies of players $1, 2, \dots, p$ as maps $s_1, s_2, \dots, s_p$ on $X_1, X_2, \dots, X_p$, respectively:

$$\begin{array}{ll} s_1 : x \mapsto y \in X_G(x) & \text{for } x \in X_1 \setminus \{x_f\}; \\ s_2 : x \mapsto y \in X_G(x) & \text{for } x \in X_2 \setminus \{x_f\}; \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ s_p : x \mapsto y \in X_G(x) & \text{for } x \in X_p \setminus \{x_f\}, \end{array}$$

where $X_G(x)$ is the set of extremals of edges $e = (x,y)$, starting in x, i.e., $X_G(x) = \{y \in X \mid e = (x,y) \in E\}$. For a given set of strategies $s = (s_1, s_2, \dots, s_p)$ denote by $G_s = (X, E_s)$ the subgraph generated by the edges $e = (x, s_i(x))$ for $x \in X \setminus \{x_f\}$ and $i = \overline{1,p}$. Then, in G_s for every vertex $x \in X \setminus \{x_f\}$ there exists a unique directed edge $e = (x,y) \in E_s$, originating in x. Obviously, for fixed $s_1, s_2, \dots, s_p$ either a unique directed path $P_s(x_0, x_f)$ from x_0 to x_f exists in G_s or such a path does not exist in G_s. In the second case, if we pass through the edges from x_0 we get a unique directed cycle C_s.

For fixed strategies $s_1, s_2, \dots, s_p$ and fixed states x_0 and x_f define the quantities
$$F^1_{x_0 x_f}(s_1, s_2, \dots, s_p),\ F^2_{x_0 x_f}(s_1, s_2, \dots, s_p), \dots,\ F^p_{x_0 x_f}(s_1, s_2, \dots, s_p)$$
in the following way. We assume that the path $P_s(x_0, x_f)$ does exist in G_s. Then it is unique and we can assign to its edges numbers $0, 1, 2, 3, \dots, k_s$, starting with the edge that begins in x_0. These numbers characterize the time steps $t_e(s_1, s_2, \dots, s_p)$ when the system passes from one state to another, if the strategies $s_1, s_2, \dots, s_p$ are applied. In this case, we put

$$F^i_{x_0 x_f}(s_1, s_2, \dots, s_p) = \sum_{e \in E(P_s(x_0, x_f))} c_e^i(t_e(s_1, s_2, \dots, s_p)), \quad i = \overline{1,p},$$

where $E(P_s(x_0, x_f))$ is the set of edges of the path $P_s(x_0, x_f)$. The set of vertices $x_0 = x(0), x(1), \dots, x(k) = x_f$ in the path $P_s(x_0, x_f)$ represents the trajectory generated by the strategies $s_1, s_2, \dots, s_p$ of the players. If there are no directed paths $P_s(x_0, x_f)$ from x_0 to x_f in H_s, then we put

$$F^i_{x_0 x_f}(s_1, s_2, \dots, s_p) = +\infty, \quad i = \overline{1,p}.$$

Problem formulation – the dynamic c-game. We consider the problem of finding the maps $s_1^*, s_2^*, \dots, s_p^*$ for which the following condition is satisfied

$$\begin{array}{l} F^i_{x_0 x_f}(s_1^*, s_2^*, \dots, s_{i-1}^*, s_i^*, s_{i+1}^*, s_p^*) \le F^*_{x_0 x_f}(s_1^*, s_2^*, \dots, s_{i-1}^*, s_i, s_{i+1}^*, s_p^*), \\ \forall s_i,\ i = \overline{1,p}. \end{array} \qquad (3)$$

So, we study the problem of finding the optimal solution in the sense of Nash [3] on $S_1 \times S_2 \times \cdots \times S_p$, where $S_i = \{s_i\colon x \mapsto y \in X_G(x) \text{ for } x \in X_i\}$, $i = \overline{1,p}$.

The functions

$$F^1_{x_0x_f}(s_1, s_2, \ldots, s_p),\ F^2_{x_0x_f}(s_1, s_2, \ldots, s_p), \ldots,\ F^p_{x_0x_f}(s_1, s_2, \ldots, s_p)$$

on $S_1 \times S_2 \times \cdots \times S_p$ define a game in the normal form with p players [3, 7].

In positional form, this game is defined by the graph G, partitions $X_1, X_2, \ldots, X_p$, vector-functions $c^i(t) = (c^i_{e_1}(t), c^i_{e_2}(t), \ldots, c^i_{e_m}(t))$, $i = \overline{1,p}$, starting and final positions x_0, x_f. We call this game the *dynamic c-game* with p players on networks $(G, X_1, X_2, \ldots, X_p, c^1(t), c^2(t), \ldots, c^p(t), x_0, x_f)$.

2.3.2 The nonstationary dynamic c-game

We define the nonstationary strategies of players on network as maps

$$\begin{array}{l}
u_1\colon (x,t) \to (y,t{+}1) \in X_G(x) \times \{t{+}1\} \quad \text{for } x \in X_1 \setminus \{x_f\},\ t = 0,1,2,\ldots,T{-}1;\\
u_2\colon (x,t) \to (y,t{+}1) \in X_G(x) \times \{t{+}1\} \quad \text{for } x \in X_2 \setminus \{x_f\},\ t = 0,1,2,\ldots,T{-}1;\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
u_p\colon (x,t) \to (y,t{+}1) \in X_G(x) \times \{t{+}1\} \quad \text{for } x \in X_p \setminus \{x_f\},\ t = 0,1,2,\ldots,T{-}1,
\end{array}$$

were T is given. Here (x,t) has the same sense as the notation $x(t)$, i.e. $(x,t) = x(t)$.

For any set of strategies $u_1, u_2, \ldots, u_p$ we define the quantities

$$\overline{F}^1_{x_0x_f}(u_1, u_2, \ldots, u_p), \overline{F}^2_{x_0x_f}(u_1, u_2, \ldots, u_p), \ldots, \overline{F}^p_{x_0x_f}(u_1, u_2, \ldots, u_p)$$

in the following way.

Let us consider that $u_1, u_2, \ldots, u_p$ generate in G a trajectory $x_0 = x(0), x(1), x(2), \ldots, x(T(x_f)) = x_f$ from x_0 to x_f where $T(x_f)$ represents the time-moment when x_f is reached. Then we set

$$\overline{F}^i_{x_0x_f}(u_1, u_2, \ldots, u_p) = \sum_{t=0}^{T(x_f)-1} c_{(x(t),x(t+1))}(t),\ i = \overline{1,p}, \quad \text{if } \ T(x_f) \le T;$$

otherwise we put $\overline{F}^i_{x_0x_f}(u_1, u_2, \ldots, u_p) = \infty$, $i = \overline{1,p}$.

We regard the problem of finding the nonstationary strategies $u_1^*, u_2^*, \ldots, u_p^*$ for which the following condition is satisfied

$$\overline{F}^i_{x_0x_f}(u_1^*, u_2^*, \ldots, u_{i-1}^*, u_i^*, u_{i+1}^*, \ldots, u_p^*) \le$$

$$\le \overline{F}^i_{x_0x_f}(u_1^*, u_2^*, \ldots, u_{i-1}^*, u_i, u_{i+1}^*, \ldots, u_p^*), \quad \forall\ u_i,\ i = \overline{1,p}.$$

So we consider the nonstationary case of determining Nash equilibria in dynamic c-game.

In the following we show that the nonstationary case of the problem can be reduced to stationary one.

3 Preliminaries and some results on determining the optimal strategies in dynamic c-games

The dynamic c-game was introduced in [9] as auxiliary problem for studying and solving a special class of positional games on networks – cyclic games [4–7]. The main results from [9–11] are related to the existence of Nash equilibria for zero-sum games on networks with constant cost functions on edges. The dynamic c-games with p players for constant cost functions on edges have been studied in [8], and the following result is given.

Theorem 1. *Let* $(G,\, X_1,\, X_2,\, \dots,\, X_p,\;\; c^1(t), c^2(t),\, \dots,\, c^p(t),\, x_0,\, x_f)$ *be a network where* G *has the property that the vertex* x_f *is attainable from* x_0*. If the components* $c^i_{e_j}(t)$ *of the vectors* $c^i(t) = (c^i_{e_1}(t), c^i_{e_2}(t), \dots, c^i_{e_m}(t)), i = \overline{1,p}$*, are positive constant functions then there exists the optimal solution in the sense of Nash* $s^*_1, s^*_2, \dots, s^*_p$ *for stationary dynamic c-game on network* $(G,\, X_1,\, X_2,\, \dots,\, X_p,\;\; c^1(t), c^2(t),\, \dots,\, c^p(t),\, x_0,\, x_f)$*.*

On the basis of this theorem we can prove the following result.

Theorem 2. *Let* $(G,\, X_1,\, X_2,\, \dots,\, X_p,\;\; c^1(t), c^2(t),\, \dots,\, c^p(t),\, x_0,\, x_f)$ *be a network where* G *contains at least a directed path with not more than* T *edges and the cost functions on edges are positive. Then for dynamic c-game on network* $(G,\, X_1,\, X_2,\, \dots,\, X_p,\;\; c^1(t), c^2(t),\, \dots,\, c^p(t),\, x_0,\, x_f)$ *there exist the optimal nonstationary strategies in the sense of Nash* $u^*_1(t), u^*_2(t), \dots, u^*_p(t)$*.*

Proof. It is sufficient to show that the nonstationary dynamic c-game on network $(G, X_1, X_2, \dots, X_p, c^1(t), c^2(t), \dots, c^p(t), x_0, x_f)$ can be reduced to stationary dynamic c-game on an auxiliary network $(\overline{G}, Y_1, Y_2, \dots, Y_p, \overline{c}^1(t), \overline{c}^2(t), \dots, \overline{c}^p(t), \overline{y}_0, \overline{y}_f)$ with constant cost functions on edges. In this auxiliary network the graph $\overline{G} = (Y, \overline{E})$ is obtained from G as follows. The vertex set Y is obtained from X when it is doubled $T+1$ times, i.e. $Y = X \times \{0, 1, 2, \dots, T\} = X \times \{0\} \cup X \times \{1\} \cup \dots \cup X \times \{T\}$. Each two subsets $X \times \{t\}$ and $X \times \{t+1\}$, $t = \overline{0, T-1}$, are connected with directed edges $\overline{e} = ((x,t), (y, t+1)) \in \overline{E}$ if $e = (x, y) \in E$. In addition each subset $X \times \{t\}$, $t = \overline{0, T-1}$ is connected with the set $X \times \{T\}$ by directed edges $\overline{e} = ((x,t), (y, T)) \in \overline{E}$ if $e = (x,y) \in E$.

The partition $Y = Y_1 \cup Y_2 \cup \dots \cup Y_p$ which determine the position sets of players and the cost functions $\overline{c}_{\overline{e}}(t)$ on edges $\overline{e} = ((x,t), (y, t+1)) \in \overline{E}$, $\overline{e} = ((x,t), (y, T)) \in \overline{E}$ are defined as follows:

$$Y_i = \{(x,t) \in E \,|\, x \in X_i,\, t = \overline{0,T}\}, \quad i = \overline{1,p};$$

$$\overline{c}_{\overline{e}}(t) = c_e(t), \quad \text{if} \quad e = (x,t) \in \overline{E},\; t = \overline{0, T-1}.$$

On this network we consider the dynamic c-game with starting position $\overline{y}_0 = (x_0, 0) \in X \times \{0\}$ and final position $(y_f, T) \in X \times \{T\}$.

It is easy to observe that for each subset $X \times \{t\}$ the time t is known. Therefore the cost functions $\overline{c}_{\overline{e}}(t)$ on auxiliary acyclic network may be considered constant functions. According to Theorem 1 for dynamic c-game on network $(G, X_1, X_2, \ldots .X_p, c^1(t), c^2(t), \ldots, c^p(t), x_0, x_f)$ there exists the optimal by Nash solution. This solution represents the optimal by Nash strategies for nonstationary case of the problem. □

On the basis of constructive proof of Theorem 2 we may propose the following algorithm for finding the optimal nonstationary strategies of players on network $(G,\, X_1,\, X_2,\, \ldots,\, X_p,\;\, c^1(t), c^2(t),\, \ldots,\, c^p(t),\, x_0,\, x_f)$. We construct the auxiliary network $(\overline{G}, Y_1, Y_2, \ldots .Y_p, \overline{c}^1(t), \overline{c}^2(t), \ldots, \overline{c}^p(t), \overline{y}_0, \overline{y}_f)$ and solve the problem of finding the optimal stationary strategies of players with constant cost functions on edges (see algorithms from [8]). The obtained solution on this network corresponds to optimal nonstationary strategies of players for dynamic c-game on $(G,\, X_1,\, X_2,\, \ldots,\, X_p,\, c^1(t),\, c^2(t),\, \ldots,\, c^p(t),\, x_0,\, x_f)$.

It is easy to observe also that if Nash equilibria for stationary case of dynamic c-game exist then the mentioned above construction with $T = N$ can be used for determining the optimal stationary strategies of players on network $(G,\, X_1,\, X_2,\, \ldots,\, X_p,\, c^1(t), c^2(t),\, \ldots,\, c^p(t),\, x_0,\, x_f)$ when the cost functions on edges are nondecreasing. A such approch for determining the optimal stationary strategies of players is developed in Sections 5 and 6.

For some practical problems may be useful also the following variant of the dynamic c-game with backward time-step account.

Let $P_s(x_0, x_f)$ be the directed path from x_0 to x_f in H_s generated by strategies $s_1, s_2, \ldots, s_p$ of players $1, 2, \ldots, p$. In 2.3.1 for an edge $e \in E(P_s(x_0, x_f))$ is defined the time-step $t_e(s_1, s_2, \ldots, s_p)$ as an order of the edge in the path $P_s(x_0, x_f)$ starting with o from x_0. To each edge $e \in E(P_s(x_0, x_f))$ we may associate also the backward time-step account $t'_e(s_1, s_2, \ldots, s_p)$ if we start number the edges with 0 from end position x_f in inverse order, i.e. $t'_e(s_1, s_2, \ldots, s_p) = k_s - t_e(s_1, s_2, \ldots, s_p)$. For fixed strategies $s_1, s_2, \ldots, s_p$ we define the quantities

$$\overline{F}^1_{x_0 x_f}(s_1, s_2, \ldots, s_p), \overline{F}^2_{x_0 x_f}(s_1, s_2, \ldots, s_p), \ldots, \overline{F}^p_{x_0 x_f}(s_1, s_2, \ldots, s_p)$$

in the following way. We put

$$\overline{F}^i_{x_0 x_f}(s^*_1, s^*_2, \ldots, s_p) = \sum_{e \in E(P_s(x_0, x_f))} t'_e(s_1, s_2, \ldots, s_p),\;\; i = \overline{1,p};$$

if in H_s there exists a path $P_s(x_0, x_f)$ from x_0 to x_f; otherwise we put

$$\overline{F}^i_{x_0 x_f}(s_1, s_2, \ldots, s_p) = \infty,\;\; i = \overline{1,p}.$$

So, we obtain a new game on network. In the case when the costs $c^i_e(t)$ are constant this problem coincides with the problem from [8]. This game can be regarded as dual problem for the dynamic c-game from 2.3.1.

4 A discrete optimization principle for dynamic networks and an algorithm for solving the problem in the case $\mathbf{p = 1}$

In this section, we consider the formulated problem on networks in the special case $p = 1$. We have introduced the problem in this case for positive and non-decreasing cost functions $c_e(t)$ on edges $e \in E$ which coincides with the discrete optimal control problem on G with starting states x_0 and final state x_f. Therefore, the optimal trajectory $x_0 = x(0), x(1), x(2), \ldots, x(T(x_f)) = x_f$ corresponds in G to the directed path $P^*(x_0, x_f)$ from x_0 to x_f. We call this path the optimal path for the dynamic network. For the path $P^*(x_0, x_f)$ contains no more then $N - 1$ edges, the problem can be solved in finite time by using dynamical programming techniques. We show that a more effective algorithm for solving this problem can be elaborated if the dynamical network satisfies the following conditions:

Problem formulation. Let us assume that the cost functions $c_e(t),\ e \in E$, in the dynamic network have the following property. If $P^*(x_0, x)$ is an arbitrary optimal path from x_0 to x which can be represented as $P^*(x_0, x) = P_1^*(x_0, y) \cup P_2^*(y, x)$, where $P_1^*(x_0, y)$ and $P_2^*(y, x)$ have no common edges, then a leading part $P_1^*(x_0, y)$ of the path $P^*(x_0, x)$ is also an optimal path of the problem in G with given starting state x_0 and final state y. If such a property holds, then we say that for the dynamic network the optimization principle is satisfied. In the case, when on network the cost functions $c_e(t), e \in E$, are positive functions and the optimization principle is satisfied, the following algorithm determines all optimal paths $P^*(x_0, x)$ from x_0 to each $x \in X$, which correspond to the optimal strategies in the problem for $p = 1$.

Algorithm 1

Preliminary step (Step 0). Set $Y = \{x_0\},\ E^* = \emptyset$. Assign to every vertex $x \in X$ two labels $t(x)$ and $F(x)$ as follows:

$$t(x_0) = 0, \quad t(x) = \infty, \quad \forall\, x \in X \setminus \{x_0\};$$
$$F(x_0) = 0, \quad F(x) = \infty, \quad \forall\, x \in X \setminus \{x_0\}.$$

General step (Step k). Find the set

$$E' = \{(x', y') \in E(Y) \,|\, F(x') + c_{(x',y')}(t(x')) = \min_{x \in Y} \min_{y \in \overline{X}(x)} \{F(x) + c_{(x,y)}(t(x))\},$$

where

$$E(Y) = \{(x, y) \in E \,|\, x \in Y,\ y \in X \setminus Y\}, \quad \overline{X}(x) = \{y \in X \setminus Y \,|\, (x, y) \in E(Y)\}.$$

Find the set of vertices $X' = \{y' \in X \setminus Y |\, (x', y') \in E'\}$. For every $y' \in X'$ select one edge $(x', y') \in E'$ and build the union $\overline{E}'$ of such edges. After that change the labels $t(y')$ and $F(y')$ for every vertex $y' \in X'$ as follows

$$t(y') = t(x') + 1, \quad F(y') = F(x') + c_{(x',y')}(t(x')), \quad \forall \ (x', y') \in \overline{E}'.$$

Replace the set Y by $Y \cup X'$ and E^* by $E^* \cup \overline{E}'$. Note $X^k = Y,\ E^k = E^*$. If $X^k \neq X$ then fix the tree $H^k = (X^k, E^k)$ and go to the next step $k+1$, otherwise fix the tree $H = (X, E^*)$ and STOP.

Note that the tree $H = (X, E^*)$ contains optimal paths from x_0 to each $x \in X$. After k steps of the algorithm the tree $H^k = (X^k, E^k)$ represents a part of H. If it is necessary to find the optimal path from x_0 to x_f, then the algorithm can be interrupted after k steps as soon as the condition $x_f \in X^k$ is satisfied, i.e., in this case the condition $X^k \neq X$ in the algorithm must be replaced by $x_f \in X^k$. The labels $F(x)$, $x \in X$, indicate the costs of optimal paths from x_0 to $x \in X$ and $t(x)$ represents the number of edges in these paths.

The correctness of the algorithm is based on the following theorem:

Theorem 3. *Let $(G, c(t), x_0, x_f)$ be a dynamic network, where the vector-function $c(t) = (c_{e_1}(t), c_{e_2}(t), \ldots, c_{e_m}(t))$ has positive and bounded components for $t \in [0, N-1]$. Moreover, let us assume that the optimization principle on the dynamic network is satisfied. Then the tree $H^k = (X^k, E^k)$ obtained after k steps of the algorithm gives the optimal paths from x_0 to every $x \in X^k$ which correspond to optimal strategies in the problem for $p = 1$.*

Proof. We prove the theorem by using the induction principle on the number of steps k of the algorithm. In the case when $k = 0$ the assertion is evident.

Let us assume that the theorem holds for any $k \leq r$ and let us show that it is true for $k = r+1$. If $H^r = (H^r, E^r)$ is the tree obtained after r steps and $H^{r+1} = (X^{r+1}, E^{r+1})$ is the tree obtained after $r+1$ steps of the algorithm, then $X^\circ = X^{r+1} \setminus X^r$ and $E^\circ = E^{r+1} \setminus E^r$ represents the vertex set and edge set obtained by the algorithm at the step $r+1$. Let us show that if y' is an arbitrary vertex of X°, then in H^{r+1} the unique directed path $P^*(x_0, y')$ from x_0 to y' is optimal. Indeed, if this is not the case, then there exists an optimal path $Q(x_0, y')$ from x_0 to x', which does not contain the edge $e = (z', y') \in E^\circ$. The path $Q(x_0, y')$ can be represented as $Q(x_0, y') = Q^1(x_0, x') \cup \{(x', y)\} \cup Q^2(y, y')$, where x' is the last vertex of the path $Q(x_0, y')$ belonging to X^r when we pass from x_0 to y'. It is easy to observe that if the conditions of the theorem hold then

$$\text{cost}\,(Q(x_0, y')) \geq \text{cost}\,(P^*(x_0, y')),$$

where

$$\text{cost}\,(Q(x_0, y')) = \sum_{t=0}^{m_Q} c_{e_t}(t),$$

$e_0, e_1, \ldots, e_{m_Q}$ are the corresponding edges of the directed path $Q(x_0, y')$ when we pass from x_0 to y' and

$$\text{cost}\,(P^*(x_0, y')) = \sum_{t=0}^{m_p} c_{e'_t}(t),$$

were $e'_0, e'_1, \ldots, e'_{m_p}$ are the corresponding edges of the directed path $P^*(x_0, y')$ when we pass from x_0 to y'. This means that the tree $H^{r+1} = (X^{r+1}, E^{r+1})$ contains an optimal path from x_0 to every $y' \in X^{r+1}$. $\square$

Remark 1. *In analoguos way can be proposed the algorithm for solving the problem with backward time-step account in the case when the optimization principle on network is satisfied. Here the optimization principle should be defined as follows: every part* $P_2^*(y, x_f)$ *of an arbitrary optimal path* $P^*(x, x_f) = P_1^*(x, y) \cup P_2^*(y, x_f)$ $(E(P_1^*(x, y)) \cap E(P_2^*(x, y)) = \emptyset)$ *is optimal one.*

Algorithm 1 is an extension of Dijkstra's Algorithm. Furthermore, such an algorithm we develop for the dynamic c-game with p players in the case when the optimization principle is satisfied with respect to each player. In the next section, we define the optimization principle on dynamic networks $(G, X_1, X_2, \ldots, X_n, c^1(t), c^2(t), \ldots, c^p(t), x_0, x_f)$ with respect to player i.

5 The optimization principle for dynamic networks with p players and determining Nash equilibria for stationary case of the problem

In this section we extend the optimization principle for stationary case of the problem on dynamic networks with p players. We define the optimization principle with respect to player i, $i \in \{1, 2, \ldots, p\}$, on dynamic networks $(G, X_1, X_2, \ldots, X_p, c^1(t), c^2(t), \ldots, c^p(t), x_0, x_f)$.

We denote by E^i the subset of edges from E starting in verteces $x \in X_i$, i.e., $E^i = \{(x, y) \in E \,|\, x \in X_i\}$, $i = \overline{1, p}$. Hereby, the set E^i represents the admissible set of system's passages from the states $x \in X_i$ to the state $y \in X$ for the player i. Furthermore, the set E^i indicates the set of edges of player i. By E_{s_i} we denote the subset of E generated by a fixed strategy s_i of player $i, i \in \{1, 2, \ldots, p\}$, i.e., $E_{s_i} = \{(x, y) \in E^i \,|\, x \in X_i, y = s_i(x)\}$.

Let $s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_p$ be the set of strategies of the players $1, 2, \ldots, i-1, i+1, \ldots, p$ and let $G_{S \setminus s_i} = (X, E_{S \setminus s_i})$ be the subgraph of G, where

$$E_{S \setminus s_i} = E_{s_1} \cup E_{s_2} \cup \cdots \cup E_{s_i - 1} \cup E^i \cup E_{s_i + 1} \cup \cdots \cup E_{s_p}.$$

The graph $G_{S \setminus s_i}$ represents the subgraph of G generated by the set of edges of player i and edges of E when the players $1, 2, \ldots, i-1, i+1, \ldots, p$ fix their strategies $s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_p$, respectively. On $G_{S \setminus s_i}$ we consider the single objective control problem with respect to cost functions $c_e^i(t)$ of player i, starting vertex x_0 and final vertex $x \in X$.

Definition 1. *Let us assume that for any given set of strategies*

$$s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_p$$

the cost functions $c_e^i(t)$, $e \in E_{S \setminus s_i}$ *in* $G_{S \setminus s_i}$ *have the property that if an arbitrary optimal path* $P^*(x_0, x)$ *can be represented as* $P^*(x_0, z) = P_1^*(x_0, z) \cup P_2^*(z, x)$ $(P_1^*(x_0, z)$ *and* $P_2^*(z, x)$ *have no common edges), then the leading part* $P_1^*(x_0, z)$ *of* $P^*(x_0, x)$ *is an optimal one. We call this property the optimization principle for dynamic networks with respect to player* i.

Note that if $c_e^i(t)$, $i = \overline{1,p}$, $e \in E$ are constant positive functions then the optimization principle for dynamic c-game is valid. It is easy to observe that in the case when

$$c_e^i(t) = f^i(t), \ \ i = \overline{1,p}, \, e \in E, \tag{4}$$

where $f^1(t), f^2(t), \dots, f^p(t)$ are arbitrary positive and non-decreasing functions, the optimization principle for dynamic c-game is also satisfied with respect to each player. If the dynamic network has the structure of a graph without directed cycles then $f^1(t), f^2(t), \dots, f^p(t)$ in (4) may be arbitrary non-decreasing functions. In the case when G has the structure of a k-partite directed graph without directed cycles, the optimization principle is satisfied for arbitrary positive cost functions.

Theorem 4. *Let $(G, X_1, X_2, \dots, X_p, c^1(t), c^2(t), \dots, c^p(t), x_0, x_f)$ be the dynamic network with p players for which the vertex x_f in G is attainable from x_0 and for any vertex $x \in X$ there exists an edge $e = (x, y) \in E$. Assume that the vector-functions $c^i(t) = (c^i_{e_1}(t), c^i_{e_2}(t), \dots, c^i_{e_N}(t)), i = \overline{1,p}$, have positive and nondecreasing components. Moreover, let us assume that the optimization principle on the dynamic network is satisfied with respect to each player. Then, in the dynamic c-game on networks $(G, X_1, X_2, \dots, X_p, \, c^1(t), c^2(t), \dots, c^p(t), x_0, x_f)$ for the players $1, 2, \dots, p$ there exists an optimal solution in the sense of Nash $s_1^*, s_2^*, \dots, s_p^*$.*

Proof. Let us regard the auxiliary network $(\overline{G}, Y_1, Y_2, \dots, Y_p, \overline{c}^1(t), \overline{c}^2(t), \dots, \overline{c}^p(t), \overline{y}_0, \overline{y}_f)$ from Section 3 when $T = N$ (see the proof of Theorem 2). As we have already noted for the dynamic c-game on this network there exist the optimal by Nash stationary strategies $\overline{s}_1^*, \overline{s}_2^*, \dots, \overline{s}_p^*$ which generate in $\overline{G} = (Y, \overline{E})$ a trajectory $\overline{y}_0 = (x_0, 0), (x_1, 1), (x_2, 2), \dots, (x_{T(x_f)}, T(x_f)) = \overline{y}_f$ from $\overline{y}_0$ to $\overline{y}_f$. The construction given below shows that $x_0, x_1, \dots, x_{T(x_f)} = x_f$ correspond to a trajectory generated by an optimal stationary strategies $s_1^*, s_2^*, \dots, s_p^*$ for dynamic c-game on network $(G, X_1, X_2, \dots, X_p, c^1(t), c^2(t), \dots, c^p(t), x_0, x_f)$. The stationary strategies $s_1^*, s_2^*, \dots, s_p^*$ can be obtained from $\overline{s}_1^*, \overline{s}_2^*, \dots, \overline{s}_p^*$ as follows.

Algorithm 2

Preliminary step (Step 0). Set $W^0 = \{(x_0, 0), (x_1, 1), \dots, (x_{T(x_f)}, T(x_f))\}$, and $X^0 = \{x_0, x_1, \dots, x_{T(x_f)}\}$. For every $x_t \in X$, $t = \overline{0, T(x_f)}$, we put $s_i^*(x_t) = x_{t+1}$ if $x_t \in X_i$, $i \in \{1, 2, \dots, p\}$.

General step (Step k). If $X^{k-1} = X$ then STOP; otherwise we find the set

$$W_{\overline{s}^*}(X^{k-1}) = \{(x,t) \in (X \setminus X^{k-1}) \times \{1, 2, \dots, N\} \,\Big|\, \overline{s}^*(x,t) \in W^{k-1}$$
$$\text{for } (x,t) \in Y_i, \, i = \overline{1,p}\}.$$

If $W_{\overline{s}^*}(X^{k-1}) = \emptyset$ then for every $x \in X \setminus X^{k-1}$ we put $s_i^*(x) = z$ where $\overline{s}_i^*(x,t) = (z, t+1)$ with $(x,t) \in Y_i$ and minimal t, $i \in \{1, 2, \dots, p\}$. In the case $W_{\overline{s}^*}(X^{k-1}) \neq \emptyset$ we find a vertex $(x', t') \in W_{\overline{s}^*}(X^{k-1})$ with a minimal t' for given x'. Then we form the sets $W^k = W^{k-1} \cup \{x', t'\}$, $X^k = X^{k-1} \cup \{x'\}$ and go to next step.

It is easy to observe that if the condition of the theorem holds then the verteces $x_0, x_1, \dots, x_{T(x_f)}$ are different and the stationary strategies $s_1^*, s_2^*, \dots, s_p^*$ represent the optimal solution in the sense of Nash for dynamic c-game on $(G, X_1, X_2, \dots, X_p, c^1(t), c^2(t), \dots, c^p(t), x_0, x_f)$. □

So the optimal by Nash solution for stationary case of the problem in the a case when the conditions of Theorem 4 hold can be find in the following way. We construct the auxiliary network $(\overline{G}, Y_1, Y_2, \dots, Y_p, \overline{c}^1(t), \overline{c}^2(t), \dots, \overline{c}^p(t), \overline{y}_0, \overline{y}_f)$ we find the optimal stationary strategies on this network by using the algorithm from [8]. Then we apply algorithm 2 and find optimal stationary strategies $s_1^*, s_2^*, \dots, s_p^*$.

6 Tree of optimal paths in dynamic c-game

In [11] is shown that if the cost functions c_e^i, $i = \overline{1,p}$, on edges $e \in E$ are constant and the final position x_f in G is attainable from each $x \in X$ then there exist the optimal strategies $s_1^*, s_2^*, \dots, s_p^*$ such that the graph $H_{S^*} = (X, E_{S^*})$ generated by these strategies has the structure of a directed tree with sink vertex x_f. Moreover $s_1^*, s_2^*, \dots, s_p^*$ represent the solution of the dynamic c-game on network $(G, X_1, X_2, \dots, X_p, c^1(t), c^2(t), \dots, c^p(t), x, x_f)$ with an arbitrary starting position $x \in X$ and final position x_f. This means that optimal strategies of players for considered case does not depend on starting position $x_0 \in X$. In general case for arbitrary cost function on edges the optimal strategies of players depend on starting position x_0.

Let us consider the dynamic c-game on network $(G, X_1, X_2, \dots, X_p, c^1(t), c^2(t), \dots, c^p(t), x_0, x_f)$ for which the optimization principle is satisfied with respect to each player and the cost function on edges are non-decreasing functions. We show that if every vertex $x \in X$ in G is attainable from x_0 then there exists a tree $H^* = (X, E^*)$ with root vertex x_0 such that H^* gives all optimal paths $P_{H^*}(x_0, x)$ from x_0 to $x \in X$. A unique directed path $P_{H^*}(x_0, x)$ from x_0 to an arbitraty $x \in X$ in H^* corresponds to a solution $s_1^*, s_2^*, \dots, s_p^*$ of the game on network $(G, X_1, X_2, \dots, X_p, c^1(t), c^2(t), \dots, c^p(t), x_0, x_f)$ with starting position x_0 and final position x. But for different verteces x and y the directed paths $P'_{H^*}(x)(x_0, x)$ and $P''_{H^*}(x_0, y)$ in H^* correspond to different optimal strategies of players $\overline{s}_1^*, \overline{s}_2^*, \dots, \overline{s}_p^*$ and $\overline{\overline{s}}_1^*, \overline{\overline{s}}_2^*, \dots, \overline{\overline{s}}_p^*$ in different dynamic c-games with starting vertex x_0 and final positions x, y, respectively.

Theorem 5. *Let $(G, X_1, X_2, \dots, X_p, c^1, c^2, \dots c^p, x_0, x_f)$ be the dynamic network with p players for which in G any vertex $x \in X$ is attainable from x_0 and vector-functions $c^i(t) = (c^i_{e_1}(t), c^i_{e_2}(t), \dots, c^i_{e_m}(t))$, $i = \overline{1,p}$, have non-negative and non-decreasing components. Moreover, let us consider that the optimization principle for the dynamic network is satisfied with respect to each player. Then, in G there exists a tree $H^* = (X, E^*)$ for which any vertex $x \in X$ is attainable from x_0, and a unique directed path $P_{H^*}(x_0, x_f)$ from x_0 to x in H^* corresponds to an optimal strategies $s_1^*, s_2^*, \dots, s_p^*$ of players in dynamic c-games on network*

$(G, X_1, X_2, \dots, X_p, c^1, c^2, \dots, c^p, x_0, x)$ *with starting position* x_0 *and final position* x. *For different verteces* x *and* y *the optimal paths* $P'_{H^*}(x_0, x)$ *and* $P''_{H^*}(x_0, y)$ *correspond to different strategies of players* $\overline{s}_1^*, \overline{s}_2^*, \dots, \overline{s}_p^*$ *and* $\overline{\overline{s}}_1^*, \overline{\overline{s}}_2^*, \dots, \overline{\overline{s}}_p^*$ *in different games with starting vertex* x_0 *and final positions* x, y, *respectively.*

Proof. According to Theorem 4 in G for any vertex $x \in X$ there exists the optimal path $P_{s^*}(x_0, x)$ from x_0 to x which corresponds to the optimal strategies of the players in the dynamic c-game with starting position x_0 and final position x. Let us select all vertices $x \in X$ for which optimal paths in G contain not more than one edge. Obviously, the graph $H^1 = (X^1, E^1)$ generated by these paths has the structure of a directed tree with root vertex x_0. If $X^1 = X$ the assertion is proved. If $X^1 \neq X$, then we select the vertices $x \in X \setminus X^1$ for which there exist optimal paths $P_{H^*}(x_0, x)$ from x_0 to x which contain two edges. According to Lemma 1 each of the paths $P^1_{H^*}(x_0, y)$, representing the part of the optimal paths $P_{H^*}(x_0, x)$ in G is an optimal one. Therefore, each of the optimal paths $P_{H^1}(x_0, x)$ can be regarded as the path which contains one part of the paths $P^1_{H^*}(x_0, y)$. If we add to H^1 the last edges of the optimal paths $P_{H^*}(x_0, x)$ with vertices x we obtain the tree $H^2 = (X^2, E^2)$ with root vertex x_0. In H^2 any directed path $P_{H^2}(x_0, x)$ from x_0 to x is an optimal path. If $X^2 = X$, the theorem is proved. If $X^2 \neq X$, then select the vertices $x \in X$ for which there exist optimal paths $P_{H^*}(x_0, x)$ from x_0 to x containing three edges. In an analogous way, we find the tree $H^3 = (X^3, E^3)$ and so on. In a finite number of steps, we find the tree $H^q = (X^q, E^q)$ for which $X^q = X$ and for any vertex $x \in X$ the unique directed path $P_{H^q}(x_0, x)$ from x_0 to x in H^q is an optimal one. □

Note that if for the problem with backward time-step account we define optimization principle in analogous way then the optimal strategies of players in this game satisfy the same property as the optimal stationary strategies of players in the problem with constant cost functions on edges. This means that there exist the optimal stationary strategies $s_1^*, s_2^*, \dots, s_p^*$ such that the graph $H_{s^*} = (X, E_{s^*})$ generated by these strategies has the structure of a directed graph with sink vertex x_f. So, $s_1^*, s_2^*, \dots, s_p^*$ represent the solution of dynamic c-game on network $(G, X_1, X_2, \dots, X_p, c^1, c^2, \dots c^p, x, x_f)$ with an arbitrary starting position x and final position x_f.

7 The algorithm for determining the tree of optimal paths in a dynamic c-game on acyclic networks

Assumption. We regard a dynamic c-game with p players and assume that in G any vertex $x \in X$ is attainable from x_0. Moreover, let us assume that the optimization principle is satisfied with respect to each player and the functions $c^i_e(t)$, $e \in E$, $i = \overline{1,p}$, have positive and non-decreasing components.

We propose an algorithm for determining the tree of optimal paths $H^* = (X, E^*)$ when G has no directed cycles, i.e. G is an acyclic graph. We assume that the positions of the network are numbered with $0, 1, 2, \dots, N-1$ according to partial order determined by the structure of acyclic graph G. This means that if $y > x$ then there is directed path $P(y,x)$ from x to y. The algorithm consists of N steps and construct a sequence of trees $H^k = (X^k, E^k)$, $k = \overline{0, N-1}$, such that at the final step $k = N-1$ we obtain $H^{N-1} = H^*$.

Algorithm 3

Preliminary step (step 0). Set $H^\circ = (X^\circ, E^\circ)$, where $X^\circ = \{x_0\}$, $E^\circ = \emptyset$. Assign to every vertex $x \in X$ a set of labeles $F^1(x), F^2(x), \dots, F^p(x), t(x)$ as follows:

$$
\begin{array}{ll}
F^i(x_0) = 0, & i = \overline{1,p}, \\
F^i(x) = \infty, & \forall x \in X \setminus \{x_0\},\ i = \overline{1,p}, \\
t(x_0) = 0, & \\
t(x) = \infty, & \forall x \in X \setminus \{x_0\}.
\end{array}
$$

General step (step k). Find in $X \setminus X^{k-1}$ the least vertex x^k and the set of incoming edges $E^-(x^k) = \{(x^r, x^k) \in E \,|\, x^r \in X^{k-1}\}$ for x^k. If $|E^-(x^k)| = 1$ then go to a); otherwise go to b):

a) Find a unique vertex y such that $e' = (y, x^k) \in E^-(x^k)$ and calculate

$$
\begin{array}{l}
F^i(x^k) = F^i(y) + c^i_{(y,x^k)}(t(y)), \quad i = \overline{1,p}; \\
t(x^k) = t(y) + 1.
\end{array}
\tag{5}
$$

After that form the sets $X^k = X^k \cup \{x^k\}$, $E^k = E^{k-1} \cup \{x^k\}$ and put $H^k = (X^k, E^k)$. If $k < N-1$ then go to the next step $k+1$; otherwise fix $E^* = E^{N-1}$, $H^* = (X, E^*)$ and stop.

b) Select the greatest vertex $z \in X^{k-1}$ such that in graph $H^k = \Big(X^{k-1} \cup \{x^k\},\ E^{k-1} \cup E^-(x^k)\Big)$ there exist at least two parallel directed paths $P'(z, x^k)$, $P''(z, x^k)$ from z to x^k without common edges, i.e. $E(P'(z,x^k)) \cap E(P''(z,x^k)) = \emptyset$. Let $e' = (x^r, x^k)$ and $e'' = (x^s, x^k)$ be the respective edges of these paths with common end vertex in x^k. So, $e', e'' \in E^-(x^k)$. For the vertex z we determine i_z such that $z \in X_{i_z}$.

If

$$
F^{i_z}(x^r) + c^{i_z}_{(x^r,x^k)}(t(x^r)) \le F^{i_z}(x^s) + c^{i_z}_{(x^s,x^k)}(t(x^s))
$$

then we delete the edge $e'' = (x^s, x^k)$ from $E^-(x^k)$ and from G; otherwise we delete the edge $e' = (x^r, x^k)$ from $E^-(x^k)$ and from G. After that check again the condition $|E^-(x^k)| = 1$? If $|E^-(x^k)| = 1$ then go to a) otherwise go to b). □

Remark 2. *The values $F^i(x)$, $i = \overline{1,p}$, for $x \in X$ in the algorithm 3 express the respective costs of the players in dynamic c-game with starting position x_0 and final position x.*

Note that the version of the problem with backward time-step account can be solved using algorithm 3. This algorithm finds the optimal strategies of players and constructs the tree of optimal paths $H_{s^*} = (X, E_{s^*})$ with sink vertex x_f if the optimization principle is satisfied with respect to each players. We explain the algorithm for the case of acyclic networks.

Algorithm 4

Preliminary step (step 0). Set $\overline{H}^{\circ} = (\overline{X}^{\circ}, \overline{E}^{\circ})$, where $\overline{X}^{\circ} = \{x_f\}$, $\overline{E}^{\circ} = \emptyset$. Assign to every vertex a set of labels $\overline{F}^1(x), \overline{F}^2(x), \dots, \overline{F}^p(x), t'(x)$ as follows:

$$\begin{array}{ll} \overline{F}^i(x_f) = 0, & i = \overline{1,p}, \\ \overline{F}^i(x) = \infty, & \forall x \in X \setminus \{x_f\},\ i = \overline{1,p}, \\ t'(x_f) = 0, & \\ t'(x) = \infty, & \forall x \in X \setminus \{x_f\}. \end{array}$$

General step (step k). Find a vertex $y^k \in X \setminus X^k$ which satisfies the condition

$$X^+(y^k) \subseteq X^{k-1},$$

where $X^+(y^k) = \{x \in X \,|\, (x^k, y^k) \in E\}$. Denote $E^+(y^k) = \{(y^k, x) \in E \,|\, x \in X^+(y^k)\}$ and select an edge (y^k, x^k) which satisfies the condition

$$\overline{F}^{i_k}(y^k) + c^{i_k}_{(y^k,x^k)}(t'(x^k)) = \min_{x^k \in X^+(y^k)} \{\overline{F}^{i_k}(y^k) + c^{i_k}_{(y^k,x)}(t'(x))\} \text{ if } y^k \in X_{i_k}.$$

After that we calculate

$$\begin{array}{l} \overline{F}^i(y^k) = \overline{F}^i(y^k) + c^i_{(y^k,x^k)}(t'(x^k)),\ \ i = \overline{1,p}, \\ t'(y^k) = t'(x^k) + 1. \end{array}$$

Form the sets $\overline{X}^k = \overline{X}^{k-1} \cup \{y^k\}$, $\overline{E}^k = \overline{E}^{k-1} \cup \{(y^k, x^k)\}$ and put $H^* = (X^k, E^k)$. If $k < N - 1$ then go to the next step $k + 1$; otherwise fix $E^* = \overline{E}^{n-1}$, $H_{s^*} = (X, E_{s^*})$ and stop.

Remark 3. *The values $\overline{F}^i(x)$, $i = \overline{1,p}$, for $x \in X$ in the algorithm 4 express the respective costs of players in the dynamic c-game with starting position x and final position x_f.*

Example

Let us consider the stationary dynamic c-game of two players. The game is determined by the network given in Fig. 2. This network consists of directed graph $G = (X, E)$ with partition $X = X_1 \cup X_2$, $X_1 = \{0, 2, 5, 6\}$, $X_2 = \{1, 3, 4\}$, starting position $x_0 = 0$, final position $x_f = 6$ and the cost functions of players 1 and 2 given in paranthesis in Fig. 1.

It is easy to check that the optimization principle for this dynamic network is satisfied with respect to each player. Therefore if we use algorithm 2 we obtain:

Step 0: $H^{\circ} = (\{0\}, \emptyset)$; $X^{\circ} = \{0\}$; $E^{\circ} = \emptyset$; $F^1(0) = 0$; $F^2(0) = 0$; $t(0) = 0$; $F^i(x) = \infty$ for $x \neq 0$, $i = 1, 2$; $t(x) = \infty$ for $x \neq 0$.

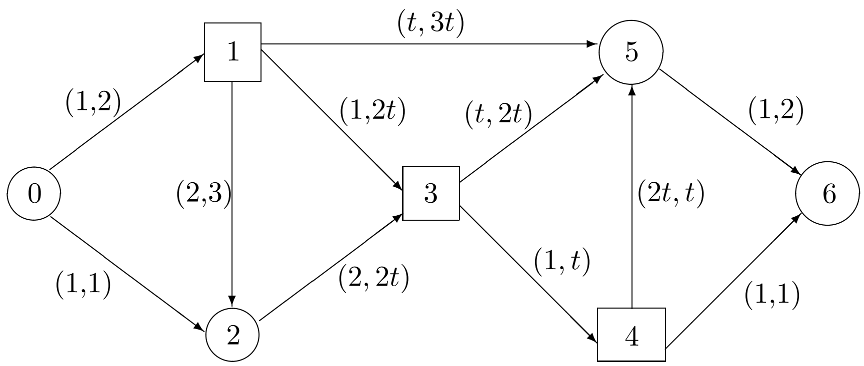

Fig. 1

Step 1: $x^1 = 1$; $E^-(1) = \{(0,1)\}$; $F^1(0) = 0$; $F^2(0) = 0$; $F^1(1) = 1$; $F^2(1) = 2$; $t(0) = 0$; $t(1) = 1$; $F^i(x) = 0$ for $x \neq 0, 1$ and $i = 1, 2$; $t(x) = \infty$ for $x \neq 0, 1$.

After step 1 we have $H^1 = (\{0,1\}, \{(0,1)\})$, i.e. $X^1 = \{0,1\}$, $E^1 = \{0,1\}$.

Step 2: $x^2 = 2$; $E^-(2) = \{(0,2), (1,2)\}$. Since $E^-(2) \neq 1$ we have the case b): $z = 0$; $P'(0,2) = \{(0,2)\}$, $P''(0,2) = \{(0,1), (0,2)\}$; $e^1 = (0,2)$; $e'' = (1,2)$ and $i_z = 1$ because $0 \in X_1$. For e' and e'' the following condition holds:

$$F^1(0) + c^1_{(0,2)}(0) \leq F^1(1) + c^1_{(1,2)}(1), \quad \text{i.e.} \quad 1 \leq 2.$$

Therefore we delete (1,2) from $E^-(2)$ and obtain $E^-(2) = \{(0,2)\}$ (case a)). We calculate $F^1(2) = F^1(0) + c^1_{(0,2)}(0) = 1$; $F^2(2) = F^2(0) + c^2_{(0,2)}(0) = 1$;

$t(2) = t(0) + 1 = 1$.

After step 2 we obtain $H^2 = (\{0,1,2\}, \{(0,1), (0,2)\})$; $F^1(0) = 0$; $F^2(0) = 0$; $F^1(1) = 1$; $F^2(1) = 2$; $F^1(2) = 1$; $F^2(2) = 1$; $F^i(x) = \infty$ for $x \neq 0, 1, 2$, $i = 1, 2$;

$t(0) = 0$; $t(1) = 1$; $t(2) = 1$; $t(x) = \infty$ for $x \neq 0, 1, 2$.

Step 3: $x^3 = 3$; $E^-(3) = \{(1,3), (2,3)\}$. So, we have $E^-(3) \neq 1$ (case b): $z = 0$; $P'(0,3) = \{(0,1), (1,3)\}$; $P''(0,3) = \{(0,2), (2,3)\}$; $e^1 = (1,3)$; $e'' = (2,3)$; $i_z = 0$. For e' and e'' we have

$$F^1(1) + c^1_{(1,3)}(1) \leq F^1(2) + c^1_{(2,3)}(1)$$

Therefore we delete (2,3) from $E^-(3)$. So, $E^-(3) = \{(1,2)\}$ and $F^1(3) = F^1(1) + c^1_{(1,3)}(1) = 1 + 1 = 2$; $F^2(3) = F^2(1) + c^2_{(1,3)}(1) = 2 + 2 = 4$; $t(3) = t(1) + 1 = 2$.

We delete the $H^3 = (\{0,1,2,3\}, \{(0,1), (0,2), (1,3)\})$; $F^1(0) = 0$; $F^2(0) = 0$; $F^1(1) = 1$; $F^2(1) = 2$; $F^1(2) = 1$; $F^2(2) = 1$; $F^1(3) = 2$; $F^2(3) = 4$; $F^i(x) = \infty$ for $x \in \{4,5,6\}$, $i = 1, 2$;

$t(0) = 1$; $t(1) = 1$; $t(2) = 1$; $t(3) = 2$; $t(4) = t(5) = t(6) = \infty$.

Step 4: $x^4 = 4$; $E^-(3) = \{(3,4)\}$. Therefore we obtain $H^4 = (\{0,1,2,3,4\}, \{(0,1), (0,2), (1,3), (3,4)\})$; $F^1(0) = 0$; $F^2(0) = 0$; $F^1(1) = 1$; $F^2(1) = 2$;

$F^1(2) = 1$; $F^2(2) = 1$; $F^1(3) = 2$; $F^2(3) = 4$; $F^1(4) = 3$; $F^2(4) = 8$; $F^1(5) = \infty$; $F^2(5) = \infty$; $F^1(6) = \infty$; $F^2(6) = \infty$;

$t(0) = 0$; $t(1) = 1$; $t(2) = 1$; $t(3) = 2$; $t(4) = 3$; $t(5) = t(6) = \infty$.

Step 5: $x^5 = 5$; $E^-(3) = \{(1,5),\,(3,5),\,(4,5)\}$. Since $E^-(3) = 3$ we have case b): $z = 3$; $P'(3,5) = \{(3,5)\}$; $P''(3,5) = \{(3,4),\,(4,5)\}$; $e' = (3,5)$; $e'' = (4,5)$; $i_z = 2$. For e' and e'' we have

$$F^2(3) + c^2_{(3,5)}(2) \le F^2(4) + c^2_{(4,5)}(3).$$

We delete the edge (4,5) from $E^-(4)$ and obtain $E^-(4) = \{(1,5),(3,5)\}$. For the edges $e' = (1,5)$ and $e'' = (3,5)$ we find $z = 1$, $i_z = 2$ and the paths $P'(1,5)$, $P''((1,3),\,(3,5))$. Since the following condition holds:

$$F^2(1) + c^2_{(1,5)}(1) \le F^2(3) + c^2_{(3,5)}(2).$$

We delete the edge (3,5) from $E^-(4)$ and obtain $E^-(4) = \{(1,5)\}$. So

$$F^1(5) = F^1(1) + c^1_{(1,5)}(1) = 2;\ F^2(5) = F^2(1) + c^2_{(1,5)}(1) = 5;\ t(5) = t(1) + 1 = 2.$$

After step 5 we obtain $H^5 = (\{0,1,2,3,4,5\},\ \{(0,1),\,(0,2),\,(1,3),\,(3,4),\,(1,5)\})$; $F^1(0) = 0$; $F^2(0) = 0$; $F^1(1) = 1$; $F^2(1) = 2$; $F^1(2) = 1$; $F^2(2) = 1$; $F^1(3) = 2$; $F^2(3) = 4$; $F^1(4) = 3$; $F^2(4) = 8$; $F^1(5) = 2$; $F^2(5) = 5$; $F^1(6) = \infty$; $F^2(6) = \infty$;

$t(0) = 0$; $t(1) = 1$; $t(2) = 1$; $t(3) = 2$; $t(4) = 3$; $t(5) = 2$.

Step 6: $x^6 = 6$; $E^-(6) = \{(4,6),\,(5,6)\}$; $E^-(6) \neq 1$ we have case b): $z = 1$; $P'(1,6) = \{(1,5),\,(5,6)\}$; $P''(1,6) = \{(1,3),\,(3,4),\,(4,6)\}$; $e' = (5,6)$; $e'' = (4,6)$; $i_7 = 2$. For e' and e'' the following condition holds:

$$F^2(5) + c^2_{(5,6)}(2) \le F^2(4) + c^2_{(4,6)}(3).$$

We delete (4,6) from $E^-(4)$. Therefore we obtain

$$F^1(6) = F^1(5) + c^1_{(5,6)}(2) = 3;\ F^2(6) = F^2(5) + c^2_{(5,6)}(2) = 7;\ t(6) = t(5) + 1 = 3.$$

Finally we obtain $H^6 = H^* = (\{0,1,2,3,4,5,6\},\ \{(0,1),\,(0,2),\,(1,3),\,(3,4),\,(1,5),\,(5,6)\})$; $F^1(0) = 0$; $F^2(0) = 0$; $F^1(1) = 1$; $F^2(1) = 2$; $F^1(2) = 1$; $F^2(2) = 1$; $F^1(3) = 2$; $F^2(3) = 4$; $F^1(4) = 3$; $F^2(4) = 8$; $F^1(5) = 2$; $F^2(5) = 5$; $F^1(6) = 3$; $F^2(6) = 7$;

$t(0) = 0$; $t(1) = 1$; $t(2) = 1$; $t(3) = 2$; $t(4) = 3$; $t(5) = 2$; $t(6) = 3$.

So, for the dynamic c-game on our network we obtain the tree of optimal paths given in Fig. 2. For the case of dynamic c-game with backward time-step account we obtain the tree of optimal paths given in Fig. 3.

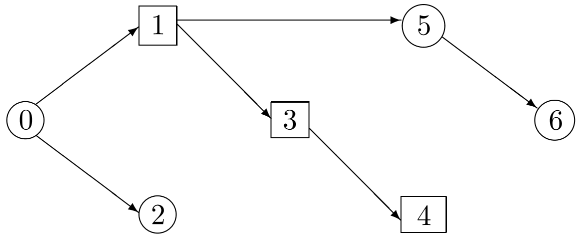

Fig. 2

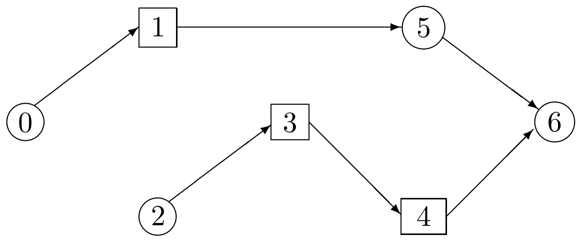

Fig. 3

7.1 Computational complexity and correctness of the algorithm

Note that if for a given $e \in E$, $t \in \{0, 1, 2, \dots, N-1\}$ and $i \in \{1, 2, \dots, p\}$ the value $c_e^i(t)$ can be calculated in time k then the algorithm determines the tree of optimal paths $H^* = (X, E^*)$ in time $O(pN^3k)$. Indeed, each general step of the algorithm requires $O(pN^2k)$ operations and the maximal number of the steps is $N-1$. So the computational complexity of the algorithm is $O(pN^3k)$.

The correctness of algorithm 2 can be proved in the same way as the correctness of algorithm 1 if we use the induction principle on number of steps k. In the case $k = 1$ the correctness of the algorithm is evident. Assume that algorithm 2 finds the tree optimal paths for $k = r$ and let us show that it finds the tree of optimal paths for the case $k = r + 1$. Denote by $H^r = (X^r, E^r)$ the tree obtained after r steps and by $H^{r+1} = (X^{r+1}, E^{r+1})$ denote the tree obtained after $r + 1$ steps. So, $X' = X^{r+1} \setminus X^r$, $E' = E^{r+1} \setminus E^r$.

Let x^{r+1} be the vertex from X' and consider the stationary dynamic c-game on network $(G, X_1, X_2, \dots, X_p, c^1(t), c^2(t), \dots, c^p(t), x_0, x^{r+1})$ with starting position x_0 and final position x^{r+1}.

According to induction principle each path from x_0 to $y \in X^r$ in H^r is optimal one. This means that to reach a position $y \in X$ with the best costs for the players in the game each player should pass through the edges of the unique directed path from x_0 to y in H^r. But to reach the vertex $x^{r+1} \in X'$ from x_0 with the best costs for the players in the game each player should pass through the edge $e' = (y, x^{r+1}) \in E'$ as soon vertex y is reached. A such best solution is well-provid by conditions of the algorithm in the case when the optimization principle on network is satisfied with respect to each player.

So, a directed path $P_{H^{r+1}}(x_0, x^{r+1})$ in the $H^{r+1} = (X^{r+1}, E^{r+1})$ corresponds to a optimal solutions $s_1^*, s_2^*, \dots, s_p^*$ in dynamic c-game.

8 Generalizations

For our problem formulated in Section 3 we have assumed that the time of passages $\tau_{x,y}$ of the system L from one state $x \in X$ to another state $y \in X$ for every $(x, y) \in E$ is equal to 1. It is easy to observe that all the results of the paper can be extended to the problem in general case where different edges (x, y) and (x', y') of the graph of passages G may have different time of passages $\tau_{xy}, \tau_{x'y'}$ and each of them may be different from 1. Theorems 1–5 for the problem in general case hold, too, and can be proved analogously. Therefore, if we replace in the algorithm the relation in (5) by the following relation

$$t(y') = t(x') + \tau_{x'y'},$$

then we obtain an algorithm for solving the problem in general form. The computational complexity of the algorithm in this general case is also $O(pN^3k)$ operations, where k is the running-time for the calculation of the value $c_e^i(t)$ for given i, e and t.

A more general mathematical model of a dynamic c-game may be obtained when the positions of the players are changing in time, i.e. for any moment of time $t = 0, 1, 2, \dots$, the partitions $X = X_1(t) \cup X_2(t) \cup \dots \cup X_p(t)$ $(X_i(t) \cap X_j(t) = \emptyset,\ i \neq j)$ are given. Using the dynamical decomposition of the network from [6, 12] this problem can be reduced to the problem formulated in Section 2.

All formulated problems in the paper may be studied also in the cases when the optimality criterion is considered in the sense of Pareto [12]. The results can be seen in a wider sense as a continuation of [13] regarding game-theoretical approaches on networks [14, 15] including global structures for such problems [16].

Note that some generalizations of routing and flow problems by using game-theoretical approach have been used in [17, 18]. But the generalizations from [17, 18] is not related to dynamic games in positional form.

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On I-radicals

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Abstract. In this paper I-radicals are studied. Rings are characterized with the help of I-radicals. For example, each I-radical over a left perfect ring splits if and only if this ring is a direct sum of finitely many left perfect rings, the Jacobson radicals of which are maximal ideals of them.

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As usual, all rings are associative with $1 \neq 0$, all modules are unitary, $J(R)$ denotes the Jacobson radical of a ring R. The category of all left R-modules (right R-modules) will be denoted by $R-\mathrm{Mod}$ ($\mathrm{Mod}-R$).

A subset I of a ring R is left (right) T-nilpotent whenever for every sequence $a_1, a_2, \dots$ in I there is an n such that $a_n \dots a_2 a_1 = 0$ ($a_1 a_2 \dots a_n = 0$).

A ring R is said to be left (right) perfect if $J(R)$ is right (left) T-nilpotent and $R/J(R)$ is semisimple.

A preradical r is said to be a hereditary preradical in case r is a left exact preradical.

A preradical r is said to be a hereditary torsion in case r is a left exact radical.

A hereditary torsion r of $R-\mathrm{Mod}$ is an S-torsion if there exists a left ideal H of R satisfying the following condition $\{I \text{ is a left ideal of } R \mid I + H = R\} = \{I \text{ is a left ideal of } R \mid r(R/I) = R/I\}$ (see [8]).

It is well known that for each left (right) ideal D of R r_D is an idempotent radical of $R-\mathrm{Mod}$ ($\mathrm{Mod}-R$), where

$$r_D(M) = \sum \{N \mid N \text{ is a submodule of } M, DN = N\}$$

$$(r_D(M) = \sum \{N \mid N \text{ is a submodule of } M, ND = N\})$$

for every left (right) R-module M [6].

A preradical r is said to be an I-radical if $r = r_D$ for some left (right) ideal D of R.

If R is a ring, then the lattice of all I-radicals of $R-\mathrm{Mod}$ is denoted by $\mathrm{Ir}(l, R)$ [6].

We shall say that a preradical r of $R-\mathrm{Mod}$ splits if for each left R-module M $r(M)$ is a direct summand of M.

Let R be a ring and let M be a right R-module. For each $m \in M$ we define the following subset of R

$$\mathrm{Ann}_r(m) = \{x \in R \mid mx = 0\}.$$

Lemma 1. *Let I be a two-sided ideal of a ring R. Then the set of right ideals $E_I = \{T \mid T+I = R\}$ is a radical filter if and only if the set $S_I = \{a \mid a \in R,\ aR+I = R\}$ satisfies the following conditions:*

1) S_I is multiplicatively closed;

2) if $s \in S_I$ and $a \in R$ then there exist $s' \in S_I$ and $a' \in R$ such that $sa' = as'$.

Proof. E_I has a basis consisting of principal right ideals (for example, $\{aR \mid a \in S_I\}$ is a basis). Now we consider the conditions S1 – S4 [3, Proposition 15.1]. S2 – S3 are clear. To verify S1 we take into account that $1 \in S_I$. The property S4 is immediate from the fact that $st \in S_I$ implies that $s \in S_I$ [5]. □

Theorem 1. *Let I be a two-sided ideal of R and $S_I = \{a \mid a \in R, aR + I = R\}$. Then r_I is a hereditary torsion if and only if the following conditions are fulfilled:*

1) S_I is multiplicatively closed;

2) if $s \in S_I$ and $a \in R$ then there exist $s' \in S_I$ and $a' \in R$ such that $sa' = as'$;

3) for every sequence $\{a_i\}_{i=1}^{\infty}$ (where $a_i \in I$ for each $i = 1, 2, \ldots$)

$$\bigcup_{i=1}^{\infty} Ann_r(a_i a_{i-1} \ldots a_1) + I = R.$$

Proof. ($\Rightarrow$) Let I be a two-sided ideal and r_I be a hereditary torsion. Then the radical filter for r_I is the set $E_I = \{T \mid T \text{ is a right ideal of } R, T + I = R\}$. In accordance with Lemma 1 conditions 1 – 2 are fulfilled. Suppose that condition 3 does not hold true. Then there exists a sequence $\{a_i\}_{i=1}^{\infty}$ (where $a_i \in I$ for each $i = 1, 2, \ldots$) such that $\bigcup\limits_{i=1}^{\infty} \mathrm{Ann}(a_i a_{i-1} \ldots a_1) + I \neq R$. Let F be a free module with free basis $\{x_i\}_{i=1}^{\infty}$ and P be a submodule of F spanned by $\{x_i - x_{i+1}a_i\}_{i=1}^{\infty}$. Then $r_I(F/P) = F/P$ but the submodule $\overline{x}_1 R$ of F/P does not belong to $T(r_I)$. This contradicts the assumption that r_I is a hereditary torsion.

($\Leftarrow$) Let I be a two-sided ideal of R satisfying conditions 1–3 of the Theorem. Then in accordance with Lemma 1 $E_I = \{T \mid T \text{ is a right ideal of } R, T+I = R\}$ is a radical filter. Let α is a hereditary torsion corresponding to the radical filter E_I. If $\alpha \neq r_I$ then there exists a right module N such that $r_I(N) = N$ and $\alpha(N) \neq N$. Put $M = N/\alpha(N)$. Then $M \in T(r_I)$ and $M \in F(\alpha)$. The last relation means that for every $m \in M \setminus \{0\}$ $\mathrm{Ann}_r(m) + I \neq R$. On the other hand since $M \in T(r_I)$, for every element $x \in M \setminus \{0\}$ there exist $x_i^{(1)} \in M$ and $a_i^{(1)} \in I$ ($i = 1, \ldots, n_1$) such that $x = \sum\limits_{i=1}^{n_1} x_i^{(1)} a_i^{(1)}$. At least one of the elements $x_i^{(1)} a_i^{(1)}$ ($i = 1, \ldots, n_1$) is non-zero. Suppose that $x_1^{(1)} a_1^{(1)} \neq 0$. Reasoning similarly we have that $x_1^{(1)} = \sum\limits_{i=1}^{n_2} x_i^{(2)} a_i^{(2)} \neq 0$. Hence $x_1^{(1)} a_1^{(1)} = \sum\limits_{i=1}^{n_2} x_i^{(2)} a_i^{(2)} a_1^{(1)} \neq 0$. Therefore there exists i, for example $i = 1$, such that $x_1^{(2)} a_1^{(2)} a_1^{(1)} \neq 0$. Going on we obtain the sequence $\{x_1^{(i)} a_1^{(i)} a_1^{(i-1)} \ldots a_1^{(1)}\}_{i=1}^{\infty}$ of non-zero elements belonging to M, where $a_1^{(i)} \in I$ for each $i = 1, 2, \ldots$. Property 3 shows that for the sequence $\{a_1^{(i)}\}_{i=1}^{\infty}$ there exists k such that $\mathrm{Ann}_r(a_1^{(k)}a_1^{(k-1)}\dots a_1^{(1)})+I=R$. Since $\mathrm{Ann}_r(x_1^{(k)}a_1^{(k)}a_1^{(k-1)}\dots a_1^{(1)}) \supseteq \mathrm{Ann}_r(a_1^{(k)}a_1^{(k-1)}\dots a_1^{(1)})$, $\mathrm{Ann}_r(y)+I=R$, where $y=x_1^{(k)}a_1^{(k)}a_1^{(k-1)}\dots a_1^{(1)} \neq 0$. Thus, $0 \neq y \in \alpha(M)$. It means that $M \notin F(\alpha)$. But $M \in F(\alpha)$. We have a contradiction. $\square$

Theorem 2. *Let R be a commutative ring. Then each I-radical is a hereditary torsion if and only if $R/J(R)$ is a von Neumann regular ring and $J(R)$ is left T-nilpotent.*

Proof. ($\Leftarrow$) Let $J(R)$ be left T-nilpotent and $R/J(R)$ be a von Neumann regular ring. Since conditions 1–2 of Theorem 1 are satisfied for every commutative ring, we have to verify condition 3 of Theorem 1 for an arbitrary two-sided ideal $I \neq R$.

Let $\{a_i\}_{i=1}^{\infty}$ be any sequence such that $a_i \in I$ for each $i=1,2,\dots$. Suppose that there exist infinitely many elements a_i belonging to $J(R)$. Then taking into consideration that R is commutative and $J(R)$ is left T-nilpotent, it is obvious that $\bigcup\limits_{i=1}^{\infty} \mathrm{Ann}_r(a_n a_{n-1}\dots a_1) = R$. Hence $\bigcup\limits_{n=1}^{\infty} \mathrm{Ann}_r(a_n a_{n-1}\dots a_1) + I = R$. Therefore assume that $a_i \notin J(R)$ for any $i \geq k$, where $k \in \mathbb{N}$. Since $R/J(R)$ is a von Neumann regular ring, there exist elements $x_i \in R$, $g_i \in J(R)$ for each $i \geq k$ such that $a_i(x_i a_i - 1) = g_i$. There exists $m \in N$ such that $g_m \dots g_k = 0$ $(m \geq k)$ because $J(R)$ is left T-nilpotent. Hence $(g_m \dots g_k)(x_m a_m - 1)\dots(x_k a_k - 1) = 0$. It is clear that $(x_m a_m - 1)\dots(x_k a_k - 1) = a \pm 1$ for some $a \in I$. Thus $\mathrm{Ann}_r(a_m \dots a_1) + I = R$.

($\Rightarrow$) Suppose that every I-radical is a hereditary torsion. Since every idempotent radical over a commutative ring corresponding torsion theory to which is cogenerated by a simple module is an I-radical ([4, Proposition 2]), every such an idempotent radical is a hereditary torsion. Therefore the idempotent radical r corresponding torsion theory to which is cogenerated by the class of all simple modules is also a hereditary torsion because it is an intersection of hereditary torsions [1, p.51]. For each maximal ideal M of R $R/M \in F(r)$. Therefore $\{R\}$ is a radical filter for r. This means that $T(r)=\{0\}$. Hence for each non-zero R-module N there exists a simple module P such that $\mathrm{Hom}_R(N,P) \neq 0$. Therefore N contains a maximal submodule. Thus every non-zero module N contains a maximal submodule. Now apply Theorem 1.8 [7]. Therefore $J(R)$ is left T-nilpotent and $R/J(R)$ is a von Neumann regular ring. $\square$

Theorem 3. *Let R be a ring. Then the following statements are equivalent:*

(1) Every preradical of $\mathrm{Mod}-R$ *is an I-radical;*

(2) Every hereditary preradical of $\mathrm{Mod}-R$ *is an I-radical;*

(3) soc *of* $\mathrm{Mod}-R$ *is an I-radical;*

(4) R is semisimple.

Proof. (3) $\Rightarrow$ (4) Let soc of $\mathrm{Mod}-R$ be an I-radical. Then $\mathrm{soc} = r_S$ for some two-sided ideal S of R. Then $r_S(R/M) = \mathrm{soc}(R/M) = R/M$ for any maximal right ideal M of R. It follows from this that $(R/M)S = R/M$ for any maximal right ideal M of R. Hence $(S+M)/M = R/M$, i.e. $S+M=R$ for any maximal right ideal M of R. Thus $S=R$. Then $RS=RR=R$. Therefore $\mathrm{soc}(R)=R$.

(4) $\Rightarrow$ (1) Let R be semisimple. Then every right R-module M is projective. Now apply Proposition 1.4.4 [1] and we have that $r(M) = Mr(R)$ for every right R-module M, where r is an arbitrary preradical of Mod $-R$. It follows from this that every preradical of Mod $-R$ is an I-radical.

(1) $\Rightarrow$ (2). This is clear.

(2) $\Rightarrow$ (3). This is clear. □

Theorem 4. *Let R be a ring. If every hereditary torsion of* Mod $-R$ *is an I-radical then R is left perfect.*

Proof. If a hereditary torsion is an I-radical then it is an S-torsion [8]. Now apply Corollary 3 [8]. □

Theorem 5. *Let R be a ring satisfying the following conditions:*

$$R/J(R) \cong T_1 \times \ldots \times T_n \textit{ for some simple rings}$$

$$T_1, \ldots, T_n \textit{ and } J(R) \textit{ is right } T\textit{-nilpotent.}$$

Then the following statements are equivalent:

(A) Each I-radical splits;

(B) Each atom of the lattice $\mathrm{Ir}(l, R)$ *splits;*

(C) $R = R_1 \dot{\rightarrow} + \ldots \dot{\rightarrow} + R_n$, where $R_i/J(R_i)$ is simple for every $i \in \{1, \ldots, n\}$.

Proof. $(A) \Rightarrow (B)$ This is clear.

$(B) \Rightarrow (C)$ Assume that each atom of $\mathrm{Ir}(l, R)$ splits. By Theorems 4–5 [6], the lattice $\mathrm{Ir}(l, R)$ has n atoms $r_1, \ldots, r_n$. Then $r_i = r_{I_i}$ for every $i \in \{1, \ldots, n\}$, where I_i is an idempotent ideal (see Theorem 9 [6]). Let $i \in \{1, \ldots, n\}$. Then

$$R = r_i(R) \oplus H_i, \tag{1}$$

where H_i is a left ideal of R. By Proposition 2 [6], $r_i(R) = I_i R = I_i$. Taking into consideration (1), we have that $I_i \oplus H_i = R$. This implies

$$I_i = Re_i, \tag{2}$$

where e_i is an idempotent of R.

Therefore $\{e_1, \ldots, e_n\}$ is a set of idempotents of the ring. Let's show that all these idempotents are pairwise orthogonal. To prove this we shall show that $I_i I_j = 0$ for $i \neq j$, $i, j \in \{1, \ldots, n\}$. Really, in view of splitingness we have

$$I_j = r_i(I_j) \oplus L_{ij}, \tag{3}$$

where L_{ij} is a left ideal of R. By Proposition 2 [6]

$$r_i(I_j) = I_i I_j. \tag{4}$$

By (3)–(4),

$$I_j = I_i I_j \oplus L_{ij}. \tag{5}$$

It follows from (1), (2), (5) that

$$R = I_i I_j \oplus L_{ij} \oplus H_j. \tag{6}$$

By (6),

$$I_i I_j = Re_{ij}, \tag{7}$$

where e_{ij} is an idempotent of R.

Since r_i and r_j are atoms, $n_R = r_i \wedge r_j$, where n_R is 0 in $\mathrm{Ir}(l, R)$ [6].

Taking into account the proof of Theorem 1 [6],

$$r_i \wedge r_j = r_{I_i} \wedge r_{I_j} = r_{I_i I_j}.$$

Therefore $n_R = r_{I_i I_j}$. By Proposition 1 [6] $I_i I_j$ is right T-nilpotent. By (7), $e_{ij} \in I_i I_j$. Since $I_i I_j$ is right T-nilpotent, $e_{ij}^s = 0$ for some $s \in \mathbb{N}$. Since e_{ij} is an idempotent, $e_{ij} = e_{ij}^s$. Hence $e_{ij} = 0$. It follows from (7) that $I_i I_j = 0$. Since $e_i e_j \in I_i I_j$, $e_i e_j = 0$. We shall show that $R = I_1 + \ldots + I_n$. Since $\{r_1, \ldots, r_n\}$ is the set of atoms of $\mathrm{Ir}(l, R)$ (see [6]),

$$r_R = u_R = r_1 \vee \ldots \vee r_n = r_{I_1} \vee \ldots \vee r_{I_n} = r_{I_1 + \ldots + I_n}$$

(see proof of Theorem 1 [6]).

By Proposition 1 [6], $R = I_1 + \ldots + I_n$, i.e. $R = Re_1 + \ldots + Re_n$. Thus, since idempotents $e_1, \ldots, e_n$ are pairwise orthogonal, the set $\{e_1, \ldots, e_n\}$ is complete. Therefore we have the ring decomposition $R = I_1 \oplus \ldots \oplus I_n$.

Then

$$R/J(R) \cong I_1/J(I_1) \times \ldots \times I_n/J(I_n). \tag{8}$$

Since $R/J(R) \cong T_1 \times \ldots \times T_n$ for some simple rings $T_1, \ldots, T_n$, $R/J(R) \cong T_1 \times \ldots \times T_n$ is an indecomposable ring decomposition. It follows from (8) that $I_i/J(I_i)$ is a simple ring for each $i \in \{1, \ldots, n\}$ (see Proposition 7.8 [2]). It means that we have proved $(B) \Rightarrow (C)$.

$(C) \Rightarrow (A)$ Assume (C). Let $r \in \mathrm{Ir}(l, R)$. Then $r = r_I$ for some ideal I of R (see Remark 1 [5]). Let $\{e_1, \ldots, e_n\}$ be the set of idempotents for the decomposition $R = R_1 \oplus \ldots \oplus R_n$. Since $R_i/J(R_i)$ is simple, either $Ie_i + J(R_i) = J(R_i)$ or $Ie_i + J(R_i) = R_i$.

Set $A = \bigl\{i \in \{1, \ldots, n\} \mid Ie_i + J(R_i) = R_i\bigr\}$, $B = \{1, \ldots, n\} \setminus A$.

By Proposition 1 [6],

$$r_{Ie_i + J(R_i)} = n_R, \ \text{ if } \ i \in B; \qquad r_{Ie_i + J(R_i)} = r_{R_i}, \ \text{ if } \ i \in A.$$

Then

$$r_I = r_{Ie_1 \oplus \ldots \oplus Ie_n} = r_{Ie_1} \vee \ldots \vee r_{Ie_n} = r_{Ie_1 + J(R_1)} \vee \ldots \vee r_{Ie_n + J(R_n)} =$$

$$= \bigvee_{i \in A} r_{Ie_i + J(R_i)} \vee \bigvee_{i \in B} r_{Ie_i + J(R_i)} = \bigvee_{i \in A} r_{R_i} \vee n_R = r_{\bigoplus\limits_{i \in A} R_i}.$$

Since $\bigoplus\limits_{i\in A} R_i$ is an idempotent ideal of R, it follows from Proposition 2 [6] that for each left R-module M

$$r_I(M) = r_{\bigoplus\limits_{i\in A} R_i}(M) = \left(\oplus_{i\in A} R_i\right) M.$$

Hence $M = r_I(M) \oplus \left(\bigoplus\limits_{i\in B} R_i\right) M$. □

Corollary 1. *Let R be a left perfect ring. Then each atom of the lattice* $\mathrm{Ir}(l,R)$ *splits if and only if the ring R is a direct sum of finitely many left perfect rings, the Jacobson radicals of which are maximal ideals of them.*

Corollary 2. *Let R be a left perfect ring. Then each I-radical of* $R-\mathrm{Mod}$ *splits if and only if the ring R is a direct sum of finitely many left perfect rings, the Jacobson radicals of which are maximal ideals of them.*

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On natural classes of R-modules in the language of ring R

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Abstract. Every natural class of left R-modules is closed, i.e. is completely described by special set of left ideals of R (natural set). Some characterizations of such sets are shown. The complementation operator of sets is defined and its properties permit to transfer some results on natural classes to the lattice of left ideals of R.

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Introduction

Various types of classes of modules play an important role in the theory of radicals. For example, every idempotent radical r of R-mod can be described by each of the classes $\mathcal{R}(r)$ and $\mathcal{P}(r)$ of r-torsion or r-torsion free modules. The classes of such types are characterized by closure properties (under submodules, homomorphic images, extensions, etc.) [1–3]. In the literature numerous types of classes of modules with special properties are studied. In particular, very intensively are investigated the *natural classes* ($\equiv$ saturated classes) of R-modules, which are closed under submodules, direct sums and injective envelopes [4–8].

In the present note an attempt is made to transfer some results on natural classes of R-modules in to lattice $\mathbb{L}({}_RR)$ of left ideals of ring R, using some facts from [9–11] on the relation between classes of left R-modules and sets of left ideals of ring R. This is possible since every natural class of modules is closed, i.e. can be characterized by special set of left ideals of R. Some descriptions of such sets are indicated. The operator of complementation of natural classes is transfered in the lattice $\mathbb{L}({}_RR)$, some properties and applications are shown. In particular, the lattice R-Nat of natural sets is boolean.

1 Natural classes and natural sets

Let R be an arbitrary ring with unity and R-mod be the category of unitary left R-modules. We consider the *abstract* classes of R-modules (i.e. $M \in \mathcal{K}$, $M \cong N$ implies $N \in \mathcal{K}$).

Definition 1. *The abstract class $\mathcal{K} \subseteq R$-Mod is called* **natural** *if it is closed with respect to submodules, direct sums and injective envelopes* (*or essential extensions*).

The natural ($\equiv$ saturated) classes were studied in a series of works, in particular in [4–8]. We are interested in some aspects related to the description of such classes in the language of ring R. Firstly we remind the following known fact.

Lemma 1.1. *If the class $\mathcal{K} \subseteq R$-Mod is closed under submodules, finite direct sums and injective envelopes, than $\mathcal{K}$ is closed also under extensions (i.e. if in the exact sequence $0 \to A \to B \to C \to 0$ modules $A, C \in \mathcal{K}$, then $B \in \mathcal{K}$).*

Proof. Consider injective envelopes $X = E(A)$, $Y = E(C)$ and the diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "k"'] & B \arrow[r, "g"] \arrow[dl, dotted, "h"'] \arrow[d, dotted, "t"] & C \arrow[r] \arrow[d, "l"] & 0 \\
0 \arrow[r] & X \arrow[r] & X \oplus Y \arrow[r] & Y \arrow[r] & 0,
\end{tikzcd}$$

where k and l are injections. Since X is injective, there exists h: $B \to X$ with $hf = k$. Define t: $B \to X \oplus Y$ by $t(b) = \Big(h(b), lg(b)\Big)$, obtaining the commutative diagram, where k and l are mono, so t is mono. Now from $A, C \in \mathcal{K}$ we conclude that $X, Y \in \mathcal{K}$, $X \oplus Y \in \mathcal{K}$ and $B \in \mathcal{K}$. $\square$

Corollary 1.2. *Every natural class of R-modules is closed under extensions.*

In particular, the classes of the form $\mathcal{P}(r) = \{{}_R M \mid r(M) = 0\}$ for a torsion r can be described as classes closed under submodules, direct product and injective envelopes, but these conditions that it is closed under extensions [2, 3, 10].

Now we are going to expose some general facts on the relation between classes of modules $\mathcal{K} \subseteq R$-Mod and some sets $\mathcal{E} \subseteq \mathbb{L}({}_R R)$ of left ideals of R [9–11]. As above we denote by $\mathbb{L}({}_R R)$ the lattice of left ideals of R. We define the following two operators:

1) if $\mathcal{K} \subseteq R$-Mod we denote

$$\Gamma(\mathcal{K}) = \{(0:m) \mid m \in M,\ M \in \mathcal{K}\}, \text{ where } (0:m) = \{a \in R \mid am = 0\};$$

2) if $\mathcal{E} \subseteq \mathbb{L}({}_R R)$ then by definition

$$\Delta(\mathcal{E}) = \{M \in R\text{-Mod} \mid (0:m) \in \mathcal{E} \ \ \forall\, m \in M\}.$$

Definition 2 [9, 10]. *The class $\mathcal{K} \subseteq R$-Mod is called* **closed** *if $\mathcal{K} = \Delta\Gamma(\mathcal{K})$. The set $\mathcal{E} \subseteq \mathbb{L}({}_R R)$ is called* **closed** *if $\mathcal{E} = \Gamma\Delta(\mathcal{E})$.*

Lemma 1.3 [9, 10]. *The class $\mathcal{K} \subseteq R$-Mod is closed if and only if is satisfies the condition:*

(A_1) $M \in \mathcal{K} \Leftrightarrow Rm \in \mathcal{K} \ \ \forall\, m \in M$;

(or: (A_1') $M \in \mathcal{K} \Rightarrow m \in M$ and (A_1'') $Rm \in \mathcal{K}\ \forall\, m \in M \Rightarrow M \in \mathcal{K}$).

The set $\mathcal{E} \subseteq \mathbb{L}({}_R R)$ is closed if and only if is satisfied the condition:

(a_1) $I \in \mathcal{E} \Rightarrow (I:a) \in \mathcal{E} \ \ \forall a \in R$ (where $(I:a) = \{b \in R \mid ba \in I\}$).

Lemma 1.4 [9, 10]. *The operators Γ and Δ define a bijection (preserving inclusions) between closed classes of R-Mod and closed sets of left ideals of $\mathbb{L}({}_RR)$. If $\mathcal{K}$ and $\mathcal{E}$ correspond each other then: $I \in \mathcal{E} \Leftrightarrow R/I \in \mathcal{K}$.*

For the pair $(\mathcal{K}, \mathcal{E})$ with $\mathcal{K} = \Delta(\mathcal{E})$ and $\mathcal{E} = \Gamma(\mathcal{K})$ a series of closure properties of $\mathcal{K}$ can be translated in the language of R as properties of the set $\mathcal{E}$. The most important examples of such properties of $\mathcal{K}$ are the closeness under:

(A_2) homomorphic images;
(A_3) direct sums;
(A_4) direct products;
(A_5) extensions;
(A_6) essential extensions ($\equiv$ stability of $\mathcal{K}$).

In parallels the following conditions of the set $\mathcal{E} \subseteq \mathbb{L}({}_RR)$ are considered:

(a_2) $I \in \mathcal{E},\ J \in \mathbb{L}({}_RR),\ J \supseteq I \Rightarrow J \in \mathcal{E}$;
(a_3) $I, J \in \mathcal{E} \Rightarrow I \cap J \in \mathcal{E}$;
(a_4) $I_\alpha \in \mathcal{E} (\alpha \in \mathfrak{A}) \Rightarrow \bigcap\limits_{\alpha \in \mathfrak{A}} I_\alpha \in \mathcal{E}$;
(a_5) $I \in \mathcal{E},\ J \in \mathbb{L}({}_RR),\ J \subseteq I, (J : i) \in \mathcal{E}\ \forall i \in I \Rightarrow J \in \mathcal{E}$;
(a_6) $J \in I,\ (J : i) \in \mathcal{E}\ \ \forall i \in I,\ I/J \subseteq^* R/J \Rightarrow J \in \mathcal{E}$
(where $\subseteq^*$ is the essential inclusion).

Theorem 1.5 [9, 10]. *Let $(\mathcal{K}, \mathcal{E})$ be a pair with $\mathcal{K} = \Delta(\mathcal{E})$ and $\mathcal{E} = \Gamma(\mathcal{K})$. The class $\mathcal{K}$ satisfies the condition (A_n) if and only if the set $\mathcal{E}$ satisfies the condition (a_n) for $n = 2, 3, 4, 5, 6$.*

Proof. These statements are proved in [9] and [10]. For convenience we verify the case $n = 6$, which presents here a special interest.

$(\Rightarrow)$ Let $\mathcal{K}$ satisfy (A_6) and consider the situation of condition (a_6). From $(J : i) \in \mathcal{E}$ for every $i \in I$ follows $I/J \in \mathcal{K}$ and now the condition $I/J \subseteq^* R/J$ implies $R/J \in \mathcal{K}$, i.e. $J \in \mathcal{E}$.

$(\Leftarrow)$ Let $\mathcal{E}$ satisfy (a_6), $M \in \mathcal{K}$ and $M \subseteq^* N$. For an element $n \in N \setminus M$ we have $0 \neq Rn \subseteq N$ and $M \cap Rn \neq 0$. Denote: $I = (M : n)$, $J = (0 : n)$. Then $In = M \cap Rn \neq 0$ and $I \supseteq J$. Moreover, for every $i \in I$ we obtain $in \in M$, $M \in \mathcal{K}$, therefore $(0 : in) = \Big((0 : n) : i\Big) = (J : i) \in \mathcal{E}$. It is easy to verify that $I/J \subseteq^* R/J$. Now we are in the situation of condition (a_6) and so $J \in \mathcal{E}$, i.e. $(0 : n) \in \mathcal{E}$ for every $n \in N$, therefore $N \in \mathcal{K}$ and $\mathcal{K}$ is stable (condition (A_6)). $\square$

In continuation we will apply these results for the investigation of natural classes of modules, taking into account that is true

Proposition 1.6. *Every natural class of modules is closed.*

Proof. Let $\mathcal{K}$ be an arbitrary natural class of R-Mod. Since $\mathcal{K}$ is hereditary, we have (A_1') and now we verify (A_1''). Let $M \in R$-Mod and $Rm \in \mathcal{K}$ for every $m \in M$. We consider the family $\mathfrak{P}$ of independent sets of submodules of M. Then $\mathfrak{P} \neq \emptyset$ and is inductive, therefore by Zorn's lemma it possesses a maximal element $\mathcal{F} = \{M_\alpha \,|\, \alpha \in \mathfrak{A}\}$. We denote $N = E\Big(\bigoplus\limits_{\alpha \in \mathfrak{A}} M_\alpha\Big) \bigcap M$. Since $M_\alpha \in \mathcal{K}$ for every

$\alpha \in \mathfrak{A}$, we obtain $N \in \mathcal{K}$. If $N \cap P = 0$ for $0 \neq P \subseteq M$, then from the choice of M follows the existence of $0 \neq M' \subseteq P$ with $M' \in \mathcal{K}$. But then $\mathcal{F} \cup \{M'\}$ is independent, in contradiction with maximality of $\mathcal{F}$. This shows that $N \subseteq^{*} M$, therefore $M \in \mathcal{K}$. □

So every natural class $\mathcal{K}$ is completely described by the corresponding set $\mathcal{E} = = \Gamma(\mathcal{K}) \subseteq \mathbb{L}({}_RR)$. We will show some characterizations of such sets of left ideals.

Definition 3. *The set $\mathcal{E} \subseteq \mathbb{L}({}_RR)$ is called* **natural** *if it satisfies the conditions $(a_1), (a_3)$ and (a_6).*

Theorem 1.7. *The operators Γ and Δ define a bijection (preserving inclusions) between the natural classes of R-Mod and natural sets of $\mathbb{L}({}_RR)$.*

Proof. Let $\mathcal{K}$ be a natural class and $\mathcal{E} = \Gamma(\mathcal{K})$. Then $\mathcal{K}$ is closed (Prop. 1.6), therefore $\mathcal{E}$ is closed (condition (a_1)). So we can apply Theorem 1.5: since $\mathcal{K}$ satisfies (A_3) and (A_6), the set $\mathcal{E}$ satisfies (a_3) and (a_6), i.e. $\mathcal{E}$ is a natural set.

Let now $\mathcal{E}$ be a natural set and $\mathcal{K} = \Delta(\mathcal{E})$. Then $\mathcal{E}$ is closed, therefore $\mathcal{K}$ is closed (condition (A_1)) and by Theorem 1.5 $\mathcal{K}$ satisfies (A_3) and (A_6), i.e. $\mathcal{K}$ is a natural class.

From Lemma 1.4 it is clear that the indicated correspondences define a bijection which preserves the inclusions. □

Now we show two important examples of natural classes, related to the theory of radicals in R-Mod.

Example 1. For every torsion ($\equiv$ hereditary radical) r the class $\mathcal{P}(r) = \{M \in R\text{-Mod} \mid r(M) = 0\}$ of r-torsion free modules, as we remark above, is characterized by properties (A_1), (A_4) and (A_6) (which imply (A_5)). Since from (A_1) and (A_4) follows (A_3), the class $\mathcal{P}(r)$ is natural. The corresponding set of left ideals $\mathcal{E} = \Gamma\Big(\mathcal{P}(r)\Big)$ is described by the conditions (a_1), (a_4) and (a_6) (which imply (a_5)). Such sets were called *cofilters* [9, 10] (dual to the filters which describe the class $\mathcal{R}(r)$), or *torsion-free sets* [7].

Example 2. For every stable torsion (i.e. $\mathcal{R}(r)$ is stable) the class $\mathcal{R}(r) = \{M \in R\text{-Mod} \mid r(M) = M\}$ of r-torsion modules is described by properties (A_1), (A_2), (A_3) and (A_6) (which implies (A_5)). Therefore $\mathcal{R}(r)$ in this case is a natural class. The corresponding set $\mathcal{E} = \Gamma\Big(\mathcal{R}(r)\Big)$ is characterized by conditions (a_1), (a_2), (a_3) and (a_6) (which imply (a_5)).

We will call such sets *stable filters* ($\equiv$ Gabriel filters with (a_6), which translates the stability of $\mathcal{R}(r)$). Obviouly, Γ and Δ determine a bijection between stable torsions and stable filters.

2 Operators of complementation

For investigations of natural classes of R-Mod the operator of complementation plays an essential role. It is defined as follows:

$$\mathcal{K}^{\perp} = \{M \in R\text{-Mod} \mid M \text{ is without nonzero submodules from } \mathcal{K}\}.$$

The set R-nat of all natural classes of R-Mod is a complete lattice with respect to the operations $\bigwedge$ and $\bigvee$, naturally defined ([1, 2], etc.). Moreover, R-nat by operator $(\ )^{\perp}$ becomes a *boolean lattice* [1].

The double complement of the class $\mathcal{K}$ is:

$$\mathcal{K}^{\perp\perp} = \{M \in R\text{-Mod} \mid \forall\, 0 \neq N \subseteq M,\ \exists\, 0 \neq P \subseteq N,\ P \in \mathcal{K}\},$$

i.e. every nonzero submodule of M contains a nonzero submodule from $\mathcal{K}$.

Let $\mathcal{K}$ be a hereditary class of R-modules. Then $\mathcal{K}^{\perp}$ is closed: it is hereditary and if $M \in R$-Mod, $Rm \in \mathcal{K}^{\perp}$ for every $m \in M$, then $M \in \mathcal{K}^{\perp}$ (if not, then M contains a nonzero submodule from $\mathcal{K}$ and by hereditarity it contains some cyclic submodule from $\mathcal{K}$, contradiction). Moreover, $\mathcal{K}^{\perp}$ is in that case a natural class ([8, Theorem 6]), the class $\mathcal{K}^{\perp\perp}$ is the least natural class containing $\mathcal{K}$, so the relation $\mathcal{K} = \mathcal{K}^{\perp\perp}$ is true if and only if $\mathcal{K}$ is natural [1, 8].

In continuation the operator $(\ )^{\perp}$ for classes of R-modules will be translated in the language of ring R (for sets of left ideals) and some properties of this operator will be shown.

For the set $\mathcal{E} \subseteq \mathbb{L}({}_R R)$ we define:

$$\mathcal{E}^{\perp} = \{I \in \mathbb{L}({}_R R) \mid (I : a) \notin \mathcal{E}\ \ \forall\, a \notin I\}$$

(i.e. this set contains the left ideals without nontrivial quotients from $\mathcal{E}$). Now we verify that the operators of complementation (for classes and for sets) are perfectly concordant with the mappings Γ and Δ.

Proposition 2.1. *Let $(\mathcal{K}, \mathcal{E})$ be a pair with $\mathcal{E} = \Gamma(\mathcal{K})$ and $\mathcal{K} = \Delta(\mathcal{E})$ (in this case $I \in \mathcal{E} \Leftrightarrow R/I \in \mathcal{K}$, Lemma 1.4). Then the following relations are true:*

$$\Gamma(\mathcal{K}^{\perp}) = \mathcal{E}^{\perp} \qquad (1)$$

$$\Delta(\mathcal{E}^{\perp}) = \mathcal{K}^{\perp} \qquad (2)$$

$$\begin{tikzcd} \mathcal{K} \arrow[r, shift left, "\Gamma"] \arrow[d] & \mathcal{E} \arrow[l, shift left, "\Delta"] \arrow[d] \\ \mathcal{K}^{\perp} \arrow[r, dotted, shift left, "\Gamma"] & \mathcal{E}^{\perp} \arrow[l, dotted, shift left, "\Delta"] \end{tikzcd}$$

Proof. (1) $(\subseteq)$ Firstly we verify the inclusion $\Gamma(\mathcal{K}^{\perp}) \subseteq \mathcal{E}^{\perp}$. Let $I \in \Gamma(\mathcal{K}^{\perp})$, i.e. there exists $M \in \mathcal{K}^{\perp}$ and $m \in M$ such that $(0 : m) = I$. We must show that $(I : a) \notin \mathcal{E}$ for every $a \notin I$.

Suppose the contrary: there exists $a \notin I = (0 : m)$ such that $(I : a) \in \mathcal{E}$. Then $am \notin I$ and $R/(I : a) \cong Ram \in \mathcal{K}$, in contradiction with $M \in \mathcal{K}^{\perp}$. This proves that $I \in \mathcal{E}^{\perp}$.

$(\supseteq)$ To prove the inverse inclusion of (1), let $I \in \mathcal{E}^{\perp}$. We must verify that there exists $M \in \mathcal{K}^{\perp}$ and $m \in M$ with $I = (0 : m)$. For that we consider $M = R/I$ and $m = 1 + I$, where $(0 : m) = I$. It remains to show that $R/I \in \mathcal{K}^{\perp}$.

If $R/I \notin \mathcal{K}^{\perp}$, then there exists $0 \neq J/I \subseteq R/I$ with $J/I \in \mathcal{K}$. Then for $\overline{0} \neq a + I \in J/I$ from $J/I \in \mathcal{K}$ follows $R(a + I) \cong R/(I : a) \in \mathcal{K}$, i.e. $(I : a) \in \mathcal{E} (a \notin I)$, in contradiction with $I \in \mathcal{E}^{\perp}$. So we have $R/I \in \mathcal{K}^{\perp}$.

(2) ($\subseteq$) Let $M \in \Delta(\mathcal{E}^{\perp})$, i.e. $(0:m) \in \mathcal{E}^{\perp}$ for every $m \in M$. Then M is without nonzero submodules from $\mathcal{K}$: in the contrary we have $0 \neq Rm \subseteq M$, $Rm \in \mathcal{K}$ and from $Rm \cong R/(0:m)$ follows $R/(0:m) \in \mathcal{K}$, i.e. $(0:m) \in \mathcal{E}$, $m = 0$, contradiction.

($\supseteq$) Let $M \in \mathcal{K}^{\perp}$. We must verify that $M \in \Delta(\mathcal{E}^{\perp})$, i.e. for every $m \in M$ we have $(0:m) \in \mathcal{E}^{\perp}$. Suppose the contrary: there exists $m \in M$ such that $(0:m) \notin \mathcal{E}^{\perp}$. Then there exists $a \notin (0:m)$ (i.e. $am \neq 0$) such that $((0:m):a) = (0:am) \in \mathcal{E}$. Therefore $R/(0:am) \cong Ram \in \mathcal{K}$, contradiction with $M \in \mathcal{K}^{\perp}$. So $(0:m) \in \mathcal{E}^{\perp}$ for every $m \in M$, i.e. $M \in \Delta(\mathcal{E}^{\perp})$. □

This proposition permits us to transfer in $\mathbb{L}({}_RR)$ some results on classes of modules avoiding the direct proofs.

It is obvious that in conditions of Prop. 2.1 $\mathcal{E}^{\perp}$ and $\mathcal{K}^{\perp}$ are closed. Moreover, is true

Corollary 2.2. *If the set $\mathcal{E} \subseteq \mathbb{L}({}_RR)$ is closed then $\mathcal{E}^{\perp}$ is a natural set.*

Proof. If $\mathcal{E}$ is closed, then the class $\mathcal{K} = \Delta(\mathcal{E})$ is closed, therefore $\mathcal{K}^{\perp}$ is natural, so the set $\Gamma(\mathcal{K}^{\perp}) = \mathcal{E}^{\perp}$ is natural (Theorem 1.7). □

Corollary 2.3. *If $\mathcal{E} \subseteq \mathbb{L}({}_RR)$ is a closed set, then $\mathcal{E}^{\perp\perp}$ is the least natural set containing $\mathcal{E}$, therefore the relation $\mathcal{E} = \mathcal{E}^{\perp\perp}$ is true if and only if $\mathcal{E}$ is a natural set.*

Proof. Follows from the known fact: $\mathcal{K}^{\perp\perp}$ is the least natural class containing $\mathcal{K} = \Delta(\mathcal{E})$. □

We denote by R-Nat the family of natural sets of left ideals of R. It can be transformed in a complete lattice with order relation $\subseteq$ (inclusion) and with lattice operations $\bigwedge$ and $\bigvee$, defined as follows:

$$\bigwedge_{\alpha \in \mathfrak{A}} \mathcal{E}_\alpha = \bigcap_{\alpha \in \mathfrak{A}} \mathcal{E}_\alpha, \quad \bigvee_{\alpha \in \mathfrak{A}} \mathcal{E}_\alpha = \bigcap \big\{ \mathcal{F} \in R\text{-Nat} \,\big|\, \mathcal{F} \supseteq \mathcal{E}_\alpha \;\; \forall \alpha \in \mathfrak{A} \big\}.$$

Since the mappings Γ and Δ define a bijection (preserving order) between R-nat and R-Nat (Theorem 1.7) we have

Corollary 2.4. *The lattices R-nat and R-Nat are isomorphic, therefore R-Nat is a boolean lattice, where $\mathcal{E}^{\perp}$ is a complement of $\mathcal{E} \in R$-Nat.*

Using the fact that for the natural class $\mathcal{K}$ the relation $\mathcal{K} = \mathcal{K}^{\perp\perp}$ is true, in the article [7] a characterization of natural sets is shown by the following condition:

(a_6') If $I \notin \mathcal{E}$ than there exists $J \in \mathbb{L}({}_RR)$, $J \supsetneqq I$ such that $(I:a) \notin \mathcal{E}$ for every $a \in J \setminus I$.

It is formulated in $\mathbb{L}({}_RR) \setminus \mathcal{E}$ and translating them for $\mathcal{E}$ we obtain:

(a_6'') If $I \in \mathbb{L}({}_RR)$ and for every $J \supsetneqq I$ there exists $a \in J \setminus I$ with $(I:a) \in \mathcal{E}$, then $I \in \mathcal{E}$.

From the definition of the operator $(\ )^{\perp}$ we have:

$$\mathcal{E}^{\perp\perp} = \{ I \in \mathbb{L}({}_RR) \,|\, \forall a \notin I, \; (I:a) \notin \mathcal{E}^{\perp} \} =$$

$$= \{ I \in \mathbb{L}({}_RR) \,|\, \forall a \notin I, \; \exists b \notin (I:a), \; \big((I:a):b\big) \in \mathcal{E} \}.$$

If $\mathcal{E}$ is a closed set, then to be natural the inclusion $\mathcal{E}^{\perp\perp} \subseteq \mathcal{E}$ which is necessary can be expressed as follows:

(a_6''') If $I \in \mathbb{L}({}_R R)$ and for every $a \notin I$ there exists $b \notin (I : a)$ such that $(I : ba) \in \mathcal{E}$, then $I \in \mathcal{E}$.

Now we can supplement Proposition 2.2 of [7] by

Proposition 2.5. *The following conditions on the set $\mathcal{E} \subseteq \mathbb{L}({}_R R)$ are equivalent:*

1) *$\mathcal{E}$ is a natural set;*
2) *$\mathcal{E}$ satisfies the conditions (a_1), (a_2) and (a_6') [7];*
3) *$\mathcal{E}$ satisfies the conditions (a_1), (a_2) and (a_6'');*
4) *$\mathcal{E}$ satisfies the conditions (a_1), (a_2) and (a_6''').*

In conclusion we remark that in the definitions of cofilters and of stable filters (see examples 1 and 2) the condition (a_6) can be replaced by each of the conditions (a_6'), (a_6'') or (a_6''').

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Financial programming model and sustainable development

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Abstract. In this article an attempt to examine sustainable development in the framework of financial programming model will be considered. In this connection the financial programming approach will be described on the whole and the place of sustainability in it will be determined.

Mathematics subject classification: 91B64.
Keywords and phrases: Mathematical modeling, financial programming, sustainable development.

The base of the financial programming model consists in accounting macroeconomic framework, which covers the main sectors of national economy: private, government, monetary and foreign. Economic identities describe budgetary constrains of each mentioned sectors and formed a basis of accounting framework. In order to obtain closed system of equations from mentioned identities, behavioural equations, determining interrelation between principal economic variables are specified. The behavioural equations determining interrelation between economic variables are specified so that to add accounting identities up to the closed system of equations. The variables identified in this framework are subdivided on exogenous, endogenous and policy. The combination of variables, economic relations and identities forms economic model, which is called to prove policy decisions. For the realisation of financial programming model (development a the financial program) it is necessary to execute the forecast of exogenous variables, to define precisely values of target variables and to solve model for policy variables which will provide desirable values for the target variables.

Will be considered generalized approach of financial programming model, which gathered, monetary and grows approach [1]. The resulting merged approach contains three fundamental purposes of financial programs: the balance of payments, inflation and growth rate of real gross domestic product inside the consistent framework. Dynamic aspects of this model will be presented in finite differences. Four investigated sectors are production, government, monetary and foreign. The production sector will be defined by Cobb-Douglas production function, relationships between population and environmental degradation like [2], capital and prices. The monetary sector will be determined by demand and supply for money. The government sector will be defined by budgetary constraint. The foreign sector will be specified by equations for export, import, net foreign assets and change in international reserves.

In a sequel the model will be presented. We start with the production sector, which is assumed to own all factors of production and earn all income.

Production sector

$$\Delta y_t^* = \Delta K_t^{1-\alpha} \Delta L_t^{\alpha} / (P_{t-1} + \Delta P_t), \tag{1}$$

$$\Delta I_t^p = s(Y_{t-1} + \Delta Y_{t-1} - T_t) - \Delta M_t^d - \Delta F_t^p + \Delta D_t^p, \tag{2}$$

$$\Delta I_t^g = (T_t - C_t^g) - \Delta F_t^g + \Delta D_t^g = 0, \tag{3}$$

$$\Delta Y_t = P_{t-1} \cdot \Delta y_t^* + y_{t-1}^* \cdot \Delta P_t, \tag{4}$$

$$\Delta P_t = (1-\theta) \cdot \Delta P_{dt}^* + \theta \cdot \Delta \hat{e}_t \bar{\cdot} P_t^z, \tag{5}$$

$$\Delta K_t = I_t - \delta \cdot \Delta K_t, \tag{6}$$

$$\Delta I_t = \Delta I_t^p + \Delta I_t^g, \tag{7}$$

$$\Delta L_t = P_{cr} \cdot (L_t - Emigr_t) - P_{dc} \cdot L_t, \tag{8}$$

$$\Delta E_t = P_{dez} \cdot Y_t + P_s(L_t + L_{ue}) - Cur_{ef} \cdot K_t - Auto_{cur} \cdot E_t. \tag{9}$$

First equation says that the change in real GDP is equal to the well-balanced change in capital and labor, deflated by price index. Here K_t is the nominal capital in year t, L_t is the number of employer in year t, L_{ue} is the number of unemployment, $0 \leq \alpha \leq 1$ is the coefficient of Cobb-Douglas production function with constant retain to scale; Δy_t^* is the change in real GDP in year t-target variable, P_t, ΔP_t are the GDP deflator and inflation in year t.

Second equation asserts that the change in private investment $\Delta I_t^p$is equal to savings (nominal $GDP - Y_{t-1} - T_t$) mines the change in demand for money ΔM_t^d and the change in foreign assets to private sector ΔF_t^p plus the change in domestic credits to private sector $\Delta \hat{D}_t^p$ in year t; $\Delta \hat{D}_t^p$ is the policy variable. Third equation declares that the change in government investment $I_t^g$is equal to collected taxes T_t mines the government consumption C_t^g, mines the change in net foreign assets to government sector ΔF_t^g plus the change in domestic credits to government sector $\Delta \hat{D}_t^g$ in year t. Fourth equation states that the change in nominal GDP – ΔY_t is equal to the change in real GDP in current year Δy_t^* (target variable), multiplied by the price index in previous year P_{t-1} plus the inflation in current year ΔP_t multiplied by the real GDP in previous year y_{t-1}^*. Fifth equation announces that ΔP_t which is equal to a linear combination between the domestic price index $(1-\theta) \cdot \Delta P_{dt}^*$ – the target variable, and the price index for import in local currency $\Delta \hat{e}_t \cdot \bar{P}_t^z \quad (0 \leq \theta \leq 1)$, $\Delta \hat{e}_t$ is the exchange rate modification – the policy variable. Sixth equation declares that the change in capital ΔK_t is equal to the investment in current year I_t discounted by the corresponding rate of depreciation ΔK_t, $\Delta \hat{e}_t$ is the rate of depreciation, θ is the share of importable in domestic prices. Seventh equation represents the investment identity, which states that the total investment ΔI_t is equal to the sum of private ΔI_t^p and government ΔI_t^g investment. Eighth equation asserts that the change in the employment population ΔL_t is equal to the population growth $P_{cr}(L_t - Emig)$

mines the population decease $P_{dc} \cdot L_t$, P_{dc}, P_{cr}, $Emig$ are the population decease, population growth, and emigration rate, respectively. Ninth equation states that the change in the environmental degradation ΔE_t is equal to the environmental degradation owing to economic development $P_{dez} \cdot Y_t$ and environmental degradation owing to social development $Pl_s \cdot L_t$ mines the environmental clean up due to the state protection $Cur_{ef} \cdot I_t$ and due to self clean up $Auto_{cur} \cdot E_t$.

Monetary sector

$$\Delta M_t^d = \nu \cdot \Delta Y_t, \tag{10}$$

the change in money demand ΔM_t^d is equal to the change in nominal GDP ΔY_t multiplied by the constant inverse to the income velocity of money ν.

$$\Delta M_t^s \equiv \Delta R_t^* + \Delta \hat{D}_t^p + \Delta \hat{D}_t^g, \tag{11}$$

the change in money supply ΔM_t^s is equal to the change in foreign reserves ΔR_t^* plus the change in domestic credits to private $\Delta \hat{D}_t^p$ and government $\Delta \hat{D}_t^g$ sectors

$$\Delta M_t^d = \Delta M_t^d = \Delta M_t \tag{12}$$

and the money flow equilibrium is mentioned continue on the money market.

Foreign sector

From the budgetary constraint of foreign sector the balance of payment is defined:

$$\Delta R_t^* \equiv X_t - Z_t - (\Delta F_t^p + \Delta F_t^g). \tag{13}$$

The net foreign assets are exogenously expressed in the foreign exchange:

$$\Delta F_t^p = \Delta \bar{F}_t^p \cdot (1 + \Delta \hat{e}_t), \tag{14}$$

$$\Delta F_t^g = \Delta \bar{F}_t^g (1 + \Delta \hat{e}_t). \tag{15}$$

Here $\Delta \bar{F}_t^p$ and $\Delta \bar{F}_t^g$ are exogenously determined net foreign assets expressed in the foreign exchange, destined to private and governmental sectors, respectively, $\Delta \hat{e}_t$ is exchange rate modification-policy variable.

$$X_t = X_{t-1} + (X_{t-1} + c) \cdot \Delta \hat{e}_t - c \cdot \Delta P_{dt}^*, \tag{16}$$

$$Z_t = Z_{t-1} + (Z_{t-1} - b) \cdot P_t^{*z} \cdot \Delta \hat{e}_t + b \cdot \Delta P_{dt}^* + a \cdot \Delta y_t^*, \tag{17}$$

X_t is the export volume, Z_t is the import volume, $\bar{P}_t^{*z}$ is the exogenously determined price index for import, ΔP_{dt}^* is the change in domestic price index -– the target variable, a is the marginal propensity to import, b is the coefficient of response of import to relative prices, c is the coefficient of response of export to relative prices, X_{t-1}, Z_{t-1} are the previous year volume of export and import, respectively.

Model extension

The ceiling on expansion of total domestic credit is accompanied by a subceiling on the expansion of credit of the governmental sector. This subceiling assists in monitoring the overall credit ceiling, and ensures that the availability of credit to the public sector subseiling is not be diminished by the overall credit ceiling. Formally, this implies a secondary target such as ΔD_t^{p*} which can be achieved, according to $\Delta \hat{D}_t^g = \Delta \hat{D}_t - \Delta D_t^{p*}$.

The targeted expansion of private credit would be derived from the relationship such as $\Delta D_t^{p*} = (D_t^p / Y_t)_{n-1} \cdot \Delta Y_t$.

Since, from the government budget constraint, $T_t - C_t^g \equiv \Delta \bar{F}_t^g - \Delta \hat{D}_t^g$, the governmental sector must adjust to this programmed deficit by increasing revenue and/or reducing expenditures.

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The Schauder basis in symmetrically normed ideals of operators

E. Spinu

Abstract. In this paper we build a basis in a separable symmetrically normed ideal.

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It is well known that every Banach space with Shauder basis is separable. Converse proposition, as P.Enflo showed in 1973 [1] is not true. In the present work the problem of the existence of a Schauder basis in separable symmetrically normed ideals is considered. It is found that all such ideals have a basis. For particular case, symmetrically normed Lorentz ideals $\Upsilon_{p,q}$, a basis was built in [2].
The terminology of the article is based on [3].

Theorem. *Let $\{\phi_j\}_{j=1}^{\infty}$ be an orthonormal basis in a Hilbert space H. A sequence of linear continuous operators $\{A_n\}_{n=1}^{\infty}$ of the form*

$$A_{m^2+j} = \begin{cases} (\cdot, \phi_{m+1})\phi_j, & 1 \leq j \leq m+1 \\ (\cdot, \phi_{2m+2-j})\phi_{m+1}, & m+1 < j \leq 2m+1 \end{cases}, m = 0, 1, \dots$$

forms a basis in every symmetrically normed ideal.

Proof. Let Υ be a separable symmetrically normed ideal. Since the ideal Υ is separable there is a symmetrically normed function $\Phi(x)$ so that $\Upsilon = \Upsilon_{\Phi}^{(0)}$. For every operator $A \in \Upsilon_{\Phi}^{(0)}$ we can write the Schmidt representation: $A = \sum_{j=1}^{\infty} s_j(A)(\cdot, x_j)y_j$. For every $\epsilon > 0$ we can choose $n_0 \in \mathbf{N}$ such that $||A - A_{n_0}|| < \epsilon/2$, where $A_{n_0} = \sum_{j=1}^{n_0} s_j(A)(\cdot, x_j)y_j$. For every $0 < \delta < 1$ and $\forall j \in \mathbf{N}$ there are $u_j, v_j \in span\{\phi_j\}_{j=1}^{\infty}$ such as $||x_j - u_j|| < \delta, ||y_j - v_j|| < \delta$. We have $||(\cdot, x_j)y_j - (\cdot, u_j)v_j||_{\Phi} \leq ||(\cdot, x_j - u_j)y_j||_{\Phi} + ||(\cdot, u_j)(v_j - y_j)||_{\Phi} \leq 3\delta.$

If we take $\delta = \frac{\epsilon}{2n_o s_1(A)}$ and $B_{n_0} = \sum_{j=1}^{n_0} s_j(A)(\cdot, u_j)v_j \in \overline{span\{A_n\}_{n=1}^{\infty}}$ we get that $||A_{n_0} - B_{n_0}|| \leq \epsilon/2.$ Thus $||A - B_{n_0}||_{\Phi} \leq ||A - A_{n_0}||_{\Phi} + ||A_{n_0} - B_{n_0}||_{\Phi} < \epsilon$. Hence, $A \in \overline{span\{A_n\}_{n=1}^{\infty}}$, in other words, the sequence $\{A_n\}_{n=1}^{\infty}$ is complete in Υ. We show that the sequence $\{A_n\}_{n=1}^{\infty}$ is minimal. To prove that it is sufficient to show that this system has a biorthogonal one.

Define $F_{m^2+j} = sp(XA^*_{m^2+j})$, where $X \in \Upsilon_{\Phi}^{(0)}$, $sp(A) = \sum_{j=1}^{\infty}(A\phi_j, \phi_j)$ and $\{\phi_j\}_{j=1}^{\infty}$ is a basis in H.

It is easy to note that F_{m^2+j} is a linear bounded operator on $\Upsilon_{\Phi}^{(0)}$ and

$$F_{m^2+j} = sp(XA^*_{m^2+j}) = \begin{cases} 1, & m = r, j = s \\ 0, & m^2 + j \neq r^2 + s \end{cases}.$$

It follows that $\{F_{m^2+j}\}$ and $\{A_{m^2+j}\}$ are a biorthogonal system.

We consider the sequence of projectors $\{\mathfrak{P}_n\}_{n=1}^{\infty}$ of the form

$$\mathfrak{P}_n(A) = \sum_{j=1}^{n} F_j(A) A_j \mathfrak{P}_{m^2}(A) = \sum_{k=1}^{m}\sum_{j=1}^{m} sp(A(\cdot,\phi_k)\phi_j)(\cdot,\phi_j)\phi_k =$$

$$= \sum_{k=1}^{m}\sum_{j=1}^{m}(A\phi_j,\phi_k)(\cdot,\phi_j)\phi_k = P_m A P_m,$$

where $P_m x = \sum_{j=1}^{m}(x,\phi)j)\phi_j$, $x = \sum_{j=1}^{\infty}(x,\phi_j)\phi_j$ and $||P_m|| = 1$. We therefore have $||\mathfrak{P}_{m^2}(A)|| = ||P_m A P_m||_\Phi \le ||A||_\Phi$ Hence, $||\mathfrak{P}_{m^2}|| \le 1$. Let $1 \le j \le m+1$. Then we have

$$\mathfrak{P}_{m^2+j}(A) = P_m A P_m + \sum_{r=1}^{j} sp(A(\cdot,\phi_r)\phi_{m+1})(\cdot,\phi_{m+1})\phi_r = P_m A P_m +$$

$$+ \sum_{r=1}^{j}(A\phi_{m+1},\phi_r)(\cdot,\phi_{m+1})\phi_r = P_m A P_m + P_j A(P_{m+1} - P_m).$$

So, $||\mathfrak{P}_{m^2+j}(A)|| \le 3||A||_\Phi$, $\forall A \in \Upsilon_\Phi^{(0)}$. Let $m+2 \le j \le 2m+1$. Then we have

$$\mathfrak{P}_{m^2+j}(A) = P_{m+1} A P_{m+1} - \sum_{r=1}^{2m+1-j} sp(A(\cdot,\phi_r)\phi_{m+1})(\cdot,\phi_{m+1})\phi_r =$$

$$= P_{m+1} A P_{m+1} - P_{2m+1-j} A(P_{m+1} - P_m).$$

So, $||\mathfrak{P}_{m^2+j}(A)|| \le 3||A||_\Phi$, $\forall A \in \Upsilon_\Phi^{(0)}$.

Thus, $||\mathfrak{P}_n|| \le 3$ $(n = 1, 2...)$. By criterion of basis in the Banach space [4], we obtain that $\{A_n\}_{n=1}^{\infty}$ is a basis of the Banach space Υ.

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