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On certain subclasses of analytic functions associated with generalized struve functions

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Abstract. The goal of the present paper is to investigate some characterization for generalized Struve functions of first kind to be in the new subclasses of β uniformly starlike and β uniformly convex functions of order α. Further we point out some consequences of our main results.

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1 Introduction

Denote by $\mathcal{A}$ the class of analytic functions in the unit disc $\mathbb{U} = \{z : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{U}. \tag{1}$$

Also denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are normalized by $f(0) = 0 = f'(0) - 1$ and also univalent in the unit disc $\mathbb{U} = \{z : |z| < 1\}$. A function $f \in \mathcal{A}$ is said to be starlike of order α $(0 \leq \alpha < 1)$ if and only if $\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U})$. This function class is denoted by $\mathcal{S}^*(\alpha)$. We also write $\mathcal{S}^*(0) = \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ such that $f(\mathbb{U})$ is starlike with respect to the origin. A function $f \in \mathcal{A}$ is said to be convex of order α $(0 \leq \alpha < 1)$ if and only if $\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U})$. This class is denoted by $\mathcal{K}(\alpha)$. Further, $\mathcal{K} = \mathcal{K}(0)$, the well-known standard class of convex functions. We remark that, according to the Alexander duality theorem [1] the function $f : \mathbb{U} \to \mathbb{C}$ is convex of order α, where $0 = \alpha < 1$, if and only if $z \to zf'(z)$ is starlike of order α. We note that every starlike (and hence convex) function of the form (1) is in fact close-to-convex, and every close-to-convex function is univalent. However, if a function is starlike then it is not necessary that it will be close-to-convex with respect to a particular convex function.

The class $\beta - \mathcal{UCV}$ was introduced by Kanas et al. [14], where its geometric definition and connections with the conic domains were considered. The class $\beta - \mathcal{UCV}$ was defined purely by geometrically as a subclass of univalent functions that map each circular arc contained in the unit disk $\mathbb{U}$ with a center $\xi, |\xi| \leq \beta$ $(0 \leq$

$\beta < 1$), onto a convex arc. The notion of $\beta-$ uniformly convex function is a natural extension of the classical convexity. Observe that, if $\beta = 0$ then the center ξ is the origin and the class $\beta - \mathcal{UCV}$ reduces to the class of convex univalent functions $\mathcal{K}$. Moreover for $\beta = 1$, the class $\beta - \mathcal{UCV}$ corresponds to the class $\mathcal{UCV}$ introduced by Goodman [12,13] and studied extensively by Rønning [21,22]. The class $\beta - \mathcal{S}_P$ is related to the class $\beta - \mathcal{UCV}$ by means of the well-known Alexander equivalence between the usual classes of convex $\mathcal{K}$ and starlike $\mathcal{S}^*$ functions. Further the analytic criteria for functions in these classes are given below.

For $-1 < \alpha \leq 1$ and $\beta \geq 0$, a function $f \in \mathcal{A}$ is said to be in the class
(i) $\beta-$ uniformly starlike functions of order α, denoted by $\mathcal{S}_P(\alpha, \beta)$, if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)} - \alpha\right) > \beta\left|\frac{zf'(z)}{f(z)} - 1\right|, \ z \in \mathbb{U} \tag{2}$$

and
(ii) $\beta-$ uniformly convex functions of order α, denoted by $\mathcal{UCV}(\alpha, \beta)$, if it satisfies the condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right) > \beta\left|\frac{zf''(z)}{f'(z)}\right|, \ z \in \mathbb{U}. \tag{3}$$

Indeed it follows from (2) and (3) that

$$f \in \mathcal{UCV}(\alpha, \beta) \Leftrightarrow zf' \in \mathcal{S}_P(\alpha, \beta). \tag{4}$$

Remark 1. It is of interest to state that $\mathcal{UCV}(\alpha, 0) = \mathcal{K}(\alpha)$ and $\mathcal{S}_P(\alpha, 0) = \mathcal{S}^*(\alpha)$.

Motivated by the above definitions we define the following subclasses of $\mathcal{A}$ due to Murugusundaramoorthy and Magesh [18].

For $0 \leq \lambda < 1$, $0 \leq \alpha < 1$ and $\beta \geq 0$, we let $\mathcal{S}_P(\lambda, \alpha, \beta)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1) and satisfying the analytic criterion

$$\Re\left(\frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} - \alpha\right) > \beta\left|\frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} - 1\right|, \quad z \in \mathbb{U}, \tag{5}$$

and also, let $\mathcal{UCV}(\lambda, \alpha, \beta)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1) and satisfying the analytic criterion

$$\Re\left(\frac{f'(z) + zf''(z)}{f'(z) + \lambda zf''(z)} - \alpha\right) > \beta\left|\frac{f'(z) + zf''(z)}{f'(z) + \lambda zf''(z)} - 1\right|, \quad z \in \mathbb{U}. \tag{6}$$

We further let $\mathcal{TS}_P(\lambda, \alpha, \beta) = \mathcal{S}_P(\lambda, \alpha, \beta) \cap \mathcal{T}$ and $\mathcal{UCT}(\lambda, \alpha, \beta) = \mathcal{UCV}(\lambda, \alpha, \beta) \cap \mathcal{T}$ where $\mathcal{T}$ denotes the subclass of $\mathcal{A}$ consisting of functions whose nonzero coefficients from second on, is given by

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n. \tag{7}$$

$\mathcal{S}_P(0, \alpha, 0) \equiv \mathcal{T}^*(\alpha)$ and $\mathcal{UCT}(0, \alpha, 0) \equiv \mathcal{C}(\alpha)$ are the class of starlike and convex functions of order α ($0 \leq \alpha < 1$), introduced and studied by Silverman [23]. Suitably

specializing the parameters one can define various subclasses defined in [2, 7, 23, 27, 28]. Now we recall the following necessary and sufficient conditions for functions f to be in the function classes $\mathcal{S}_P(\lambda,\alpha,\beta)$, $\mathcal{TS}_P(\lambda,\alpha,\beta)$, $\mathcal{UCV}(\lambda,\alpha,\beta)$ and $\mathcal{UCT}(\lambda,\alpha,\beta)$ due to Murugusundaramoorthy and Magesh [18].

Theorem 1 (see [18]). *A function $f(z)$ of the form (1) is in $\mathcal{S}_P(\lambda,\alpha,\beta)$ if*

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ |a_n| \leq 1-\alpha. \tag{8}$$

Theorem 2 (see [18]). *A function $f(z)$ of the form (1) is in $\mathcal{UCV}(\lambda,\alpha,\beta)$ if*

$$\sum_{n=2}^{\infty}n[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ |a_n| \leq 1-\alpha. \tag{9}$$

Theorem 3 (see [18]). *A function $f(z)$ of the form (7) is in $\mathcal{TS}_P(\lambda,\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ |a_n| \leq 1-\alpha. \tag{10}$$

Theorem 4 (see [18]). *A function $f(z)$ of the form (7) is in $\mathcal{UCT}(\lambda,\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty}n[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ |a_n| \leq 1-\alpha. \tag{11}$$

It is well known that the special functions (series) play an important role in geometric function theory, especially in the solution by de Branges [10] of the famous Bieberbach conjecture.The surprising use of special functions (hypergeometric functions) has prompted renewed interest in function theory in the last few decades. There is an extensive literature dealing with geometric properties of different types of special functions, especially for the generalized, Gaussian hypergeometric functions [9, 15, 17, 24, 29] and the Bessel functions [3–6, 16].

We recall here the Struve function of order p (see [19, 30]), denoted by $\mathcal{H}_p$, is given by

$$\mathcal{H}_p(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+\frac{3}{2})\ \Gamma(p+n+\frac{3}{2})}\left(\frac{z}{2}\right)^{2n+p+1}, \forall z\in\mathbb{C} \tag{12}$$

which is the particular solution of the second order non-homogeneous differential equation

$$z^2\omega''(z)+z\omega'(z)+(z^2-p^2)\omega(z)=\frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+\frac{1}{2})} \tag{13}$$

where p is unrestricted real (or complex) number. The solution of the non-homogeneous differential equation

$$z^2\omega''(z)+z\omega'(z)-(z^2+p^2)\omega(z)=\frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+\frac{1}{2})} \tag{14}$$

is called the modified Struve function of order p and is defined by the formula

$$\mathcal{L}_p(z) = -ie^{-ip\pi/2}\mathcal{H}_p(iz) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+\frac{3}{2})\ \Gamma(p+n+\frac{3}{2})} \left(\frac{z}{2}\right)^{2n+p+1}, \forall z \in \mathbb{C}.$$

Let the second order non-homogeneous linear differential equation [30] (also see [19] and references cited therein),

$$z^2\omega''(z) + bz\omega'(z) + [cz^2 - p^2 + (1-b)p]\omega(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+\frac{b}{2})} \tag{15}$$

where $b, p, c \in \mathbb{C}$, which is natural generalization of Struve equation. It is of interest to note that when $b = c = 1$, then we get the Struve function (12) and for $c = -1,\ b = 1$ the modified Struve function (14). This permits us to study the Struve and modified Struve functions. Now, denote by $w_{p,b,c}(z)$ the generalized Struve function of order p given by

$$w_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (c)^n}{\Gamma(n+\frac{3}{2})\ \Gamma(p+n+\frac{b+2}{2})} \left(\frac{z}{2}\right)^{2n+p+1}, \forall z \in \mathbb{C},$$

which is the particular solution of the differential equation (15). Although the series defined above is convergent everywhere, the function $\omega_{p,b,c}$ is generally not univalent in $\mathbb{U}$. Now, consider the function $u_{p,b,c}$ defined by the transformation

$$u_{p,b,c}(z) = 2^p\sqrt{\pi}\Gamma\left(p + \frac{b+2}{2}\right)\ z^{\frac{-p-1}{2}}\ \omega_{p,b,c}\left(\sqrt{z}\right), \quad \sqrt{1} = 1.$$

By using well known Pochhammer symbol (or the shifted factorial) defined, in terms of the familiar Gamma function, by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & (n=0), \\ a(a+1)(a+2)\cdots(a+n-1) & (n \in \mathbb{N} = \{1,2,3,\ldots\}) \end{cases}$$

we can express $u_{p,b,c}(z)$ as

$$\begin{aligned} u_{p,b,c}(z) &= \sum_{n=0}^{\infty} \frac{(-c/4)^n}{(m)_n\ (3/2)_n}\ z^n \\ &= b_0 + b_1 z + b_2 z^2 + ... + b_n z^n + ..., \end{aligned}$$

where $m = \left(p + \frac{b+2}{2}\right) \neq 0, -1, -2, \ldots$. This function is analytic on $\mathbb{C}$ and satisfies the second-order inhomogeneous linear differential equation

$$4z^2u''(z) + 2(2p+b+3)zu'(z) + (cz + 2p + b)u(z) = 2p + b.$$

For convenience throughout in the sequel, we use the following notations

$$w_{p,b,c}(z) = w_p(z) \qquad u_{p,b,c}(z) = u_p(z), \qquad m = p + \frac{b+2}{2}$$

and for if $c < 0, m > 0$ $(m \neq 0, -1, -2, \dots)$ let

$$zu_p(z) = z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} z^n = z + \sum_{n=2}^{\infty} b_{n-1} z^n \tag{16}$$

and

$$\Psi(z) = z(2 - u_p(z)) = z - \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} z^n \tag{17}$$

Mapping properties of various subclasses of analytic and univalent functions are potentially useful in a number of widespread areas of the mathematical, physical and engineering sciences. In particular, in order to solve such applied problems that are expressible in terms of functions of a complex variable, but that exhibit inconvenient geometrical shapes, we can appropriately choose one or the other of such mappings and thereby transform the inconvenient geometrical shape into a much more convenient and easy-to-handle geometrical shape. Several mapping properties of the function classes $\beta-\mathcal{UST}$ and $\beta-\mathcal{UCV}$ involving hypergeometric functions were studied by Srivastava et al [26] (also see [9,15,17,24,29]) and references cited therein. Recently Yagmur and Orhan [30] (see [19]) have determined various sufficient conditions for the parameters p, b and c such that the functions $u_{p,b,c}(z)$ or $z \to zu_{p,b,c}(z)$ to be univalent, starlike, convex and close to convex in the open unit disk. Motivated essentially by the aforementioned works and by work of Baricz [3–6], in our present investigation, we determined sufficient conditions for the family of Struve functions $(zu_p(z))$ in order to belong to the classes $\mathcal{TS}_P(\lambda, \alpha, \beta)$ and $\mathcal{UCT}(\lambda, \alpha, \beta)$ in the open unit disk $\mathbb{U}$. We also proved that those sufficient conditions are necessary for functions of the form (17). Further we deduce several interesting corollaries and consequences by suitably applying our main results.

2 Main results and their consequences

Lemma 1 (see [19])**.** *If $b, p, c \in \mathbb{C}$ and $m \neq 0, -1, -2, ...,$ then the function u_p satisfies the recursive relation*

$$2zu_p'(z) + u_p(z) + \frac{cz}{2m} u_{p+1}(z) = 1$$

for all $z \in \mathbb{C}$.

Theorem 5. *If $c < 0, m > 0$ $(m \neq 0, -1, -2, ...)$, then the sufficient condition for $zu_p(z) \in \mathcal{TS}_P(\lambda, \alpha, \beta)$ is*

$$[1 + \beta - \lambda(\alpha + \beta)]u_p'(1) + (1 - \alpha)u_p(1) \leq 2(1 - \alpha). \tag{18}$$

Moreover (18) is necessary and sufficient for $\Psi(z)$, given by (17) to be in $\mathcal{TS}_P(\lambda, \alpha, \beta)$.

Proof. According to Theorem 3, we must show that

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} \leq (1-\alpha). \tag{19}$$

Now,

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\\
&=\sum_{n=2}^{\infty}[(n-1)\{1+\beta-\lambda(\alpha+\beta)\}+(1-\alpha)]\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\\
&=[1+\beta-\lambda(\alpha+\beta)]\sum_{n=2}^{\infty}\frac{(n-1)((-c/4))^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}+(1-\alpha)\sum_{n=2}^{\infty}\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\\
&=[1+\beta-\lambda(\alpha+\beta)]u_p'(1)+(1-\alpha)[u_p(1)-1].
\end{aligned}$$

But the last expression is bounded from above by $1-\alpha$ if and only if (18) holds. Since

$$z(2-u_p(z))=z-\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}z^n \tag{20}$$

the necessity of (18) for $z(2-u_p(z))$ to be in $\mathcal{TS}_P(\lambda,\alpha,\beta)$ follows from Theorem 3. □

Theorem 6. *If $c<0, m>0$ ($m\neq 0,-1,-2,...,$ then the sufficient condition for $zu_p(z)\in\mathcal{UCT}(\lambda,\alpha,\beta)$ is*

$$[1+\beta-\lambda(\alpha+\beta)]u_p''(1)+[3+2\beta-\alpha-2\lambda(\alpha+\beta)]u_p'(1)+(1-\alpha)u_p(1)\leq 2(1-\alpha). \tag{21}$$

Moreover (21) is necessary and sufficient for $\Psi(z)$, given by (17) to be in $\mathcal{UCT}(\lambda,\alpha,\beta)$.

Proof. In view of Theorem 4, we need to show that

$$\sum_{n=2}^{\infty}n[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\leq(1-\alpha).$$

If we let $g(z)=zu_p(z)$, then we have $g'(1)=u_p'(1)+u_p(1)$ and $g''(1)=u_p''(1)+2u_p'(1)$. Further we notice that

$$\begin{aligned}
&\sum_{n=2}^{\infty}n[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\\
&=[1+\beta-\lambda(\alpha+\beta)]\sum_{n=2}^{\infty}n^2\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}
\end{aligned}$$

$$
\begin{aligned}
&-(\alpha+\beta)(1-\lambda)\sum_{n=2}^{\infty} n\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} \\
&= [1+\beta-\lambda(\alpha+\beta)]\left\{\sum_{n=2}^{\infty} n(n-1)\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\right\} \\
&\quad +[1+\beta-\lambda(\alpha+\beta)-(\alpha+\beta)(1-\lambda)]\left\{\sum_{n=2}^{\infty} n\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\right\} \\
&= [1+\beta-\lambda(\alpha+\beta)]g''(z)+(1-\alpha)[g'(z)-1],
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=2}^{\infty} n[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} \\
&= [1+\beta-\lambda(\alpha+\beta)](u_p''(1)+u_p'(1))+(1-\alpha)(u_p'(1)+u_p(1)-1) \\
&= [1+\beta-\lambda(\alpha+\beta)]u_p''(1)+[3+2\beta-2\lambda(\alpha+\beta)-\alpha]u_p'(1)+(1-\alpha)[u_p(1)-1].
\end{aligned}
$$

The last expression is bounded from above by $(1-\alpha)$ if and only if (21) holds. By Theorem 4, the condition (21) is also necessary for $z(2-u_p(z)) = \Psi(z) \in \mathcal{UCT}(\lambda,\alpha,\beta)$. □

Remark 2. In particular when $\lambda = 0$ and $\beta = 0$ the conditions given in (18) and (21) yield the results obtained in [30].

By taking $\lambda = 0$ and $\alpha = 0$, we state the following results for the function classes $\mathcal{TS}_P(0,0,\beta) \equiv \mathcal{TS}_P(\beta)$ and $\mathcal{UCT}(0,0,\beta) \equiv \mathcal{UCT}(\beta)$ defined in [27].

Corollary 1. *If $c<0, m>0$ $(m \neq 0,-1,-2,...,$ then*
(i) the sufficient condition for $zu_p(z) \in \mathcal{TS}_P(\beta)$ is

$$(1+\beta)u_p'(1)+u_p(1) \leq 2,$$

moreover it is necessary and sufficient for functions $\Psi(z) = z(2-u_p(z))$ to be in $\mathcal{TS}_P(\beta)$
(ii) the sufficient condition for $zu_p(z) \in \mathcal{UCT}(\beta)$ is

$$(1+\beta)u_p''(1)+(3+2\beta)u_p'(1)+u_p(1) \leq 2,$$

moreover it is necessary and sufficient for functions $\Psi(z) = z(2-u_p(z))$ to be in $\mathcal{UCT}(\beta)$.

By taking $\lambda = 0$, we deduce results for the function class defined in [7].

Corollary 2. *If $c<0, m>0$ $(m \neq 0,-1,-2,...,$ then*
(i) the sufficient condition for $zu_p(z) \in \mathcal{TS}_P(\alpha,\beta)$ is

$$(1+\beta)u_p'(1)+(1-\alpha)u_p(1) \leq 2(1-\alpha),$$

(ii) the sufficient condition for $zu_p(z) \in \mathcal{UCT}(\alpha, \beta)$ *is*

$$(1+\beta)u_p''(1) + (3+2\beta-\alpha)u_p'(1) + (1-\alpha)u_p(1) \leq 2(1-\alpha).$$

Further the above conditions are necessary and sufficient for functions of the form (17).

3 Inclusion Properties

For functions $f \in \mathcal{A}$ given by (1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathcal{U}. \tag{22}$$

Now, we considered the linear operator

$$\mathcal{I}(c, m) : \mathcal{A} \to \mathcal{A}$$

defined by

$$\mathcal{I}(c, m)f(z) = zu_{p,b,c}(z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} a_n\ z^n \tag{23}$$

where $m = p + \frac{(b+2)}{2} \neq 0$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^{\tau}(A, B)$ $(\tau \in \mathbb{C}\backslash\{0\},\ -1 \leq B < A \leq 1)$ if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(A-B)\tau - B[f'(z) - 1]} \right| < 1 \quad (z \in \mathbb{U}).$$

The class $\mathcal{R}^{\tau}(A, B)$ was introduced earlier by Dixit and Pal [11]. If we put

$$\tau = 1,\ A = \beta \text{ and } B = -\beta\ (0 < \beta \leq 1),$$

we obtain the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\left| \frac{f'(z) - 1}{f'(z) + 1} \right| < \beta \quad (z \in \mathbb{U}; 0 < \beta \leq 1),$$

which was studied by (among others) Padmanabhan [20] and Caplinger and Causey [8]. Making use of the following lemma, we will study the action of the Struve function on the class $\mathcal{UCT}(\lambda, \alpha, \beta)$.

Lemma 2 (see [11])**.** *If* $f \in \mathcal{R}^{\tau}(A, B)$ *is of form* (1)*, then*

$$|a_n| \leq (A-B)\frac{|\tau|}{n}, \quad n \in \mathbb{N} \setminus \{1\}. \tag{24}$$

The bound given in (24) is sharp.

Theorem 7. *Let $c < 0, m > 0$ $(m \neq 0, -1, -2, ...)$. If $f \in \mathcal{R}^{\tau}(A, B)$ and if the inequality*

$$(A-B)|\tau| \left\{[1+\beta-\lambda(\alpha+\beta)]u_p'(1) + (1-\alpha)[u_p(1)-1]\right\} \leq 1-\alpha \tag{25}$$

is satisfied, then $\mathcal{I}(c,m)(f) \in \mathcal{UCT}(\lambda, \alpha, \beta)$.

Proof. Let f of the form (1) belong to the class $\mathcal{R}^{\tau}(A, B)$. By virtue of Theorem 4, it suffices to show that

$$L(\alpha,\beta,\lambda) = \sum_{n=2}^{\infty} n[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)] \ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}|a_n| \leq 1-\alpha.$$

Since $f \in \mathcal{R}^{\tau}(A, B)$ then by Lemma 2 we have,

$$|a_n| \leq (A-B)\frac{|\tau|}{n}.$$

Hence

$$\begin{aligned} L(\alpha,\beta,\lambda) &= \sum_{n=2}^{\infty} n[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)] \ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}|a_n| \\ &\leq (A-B)|\tau| \left[\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)] \ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\right]. \end{aligned} \tag{26}$$

Further, proceeding as in Theorem 5, we get

$$L(\alpha,\beta,\lambda) \quad \leq (A-B)|\tau| \left\{[1+\beta-\lambda(\alpha+\beta)]u_p'(1) + (1-\alpha)[u_p(1)-1]\right\}.$$

But this last expression is bounded above by $1-\alpha$ if and only if (25) holds. □

Theorem 8. *Let $c < 0, m > 0$ $(m \neq 0, -1, -2, ...)$ then*

$$\mathcal{L}(m,c,z) = \int_0^z (2-u_p(t))dt$$

is in $\mathcal{UCT}(\lambda, \alpha, \beta)$ if and only if

$$[1+\beta-\lambda(\alpha+\beta)]u_p'(1) + (1-\alpha)u_p(1) \leq 2(1-\alpha). \tag{27}$$

Proof. Since

$$\mathcal{L}(m,c,z) = z - \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} \frac{z^n}{n}$$

then by Theorem 4 we need only to show that

$$\sum_{n=2}^{\infty} n[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ \frac{(-c/4)^{n-1}}{n(m)_{n-1}\ (3/2)_{n-1}} \leq 1-\alpha.$$

That is, let

$$\mathcal{P}(m,c,z)=\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}$$

Now by proceeding as in Theorem 5, we get

$$\mathcal{P}(m,c,z)=[1+\beta-\lambda(\alpha+\beta)]u_p'(1)+(1-\alpha)[u_p(1)-1].$$

which is bounded from above by $1-\alpha$ if and only if (27) holds. □

Remarks. If we put $c=-1$ and $b=1$ in above theorems we obtain results analogous to ones discussed in this paper. Further by taking $\beta=0$ and specializing the parameter λ we can state various interesting results (as proved in above theorems) for the subclasses studied in the literature [2, 7, 23, 27, 28].

References

[1] Alexander J. W. *Functions which map the interior of the unit circle upon simple regions.* Annals of Mathematics, 1915, **17**, No. 1, 12—22.

[2] Altintas O., Owa S. *On subclasses of univalent functions with negative coefficients.* Pusan Kyŏngnam Math.J., 1988, **4**, 41–56.

[3] Baricz A. *Geometric properties of generalized Bessel functions.* Publ. Math. Debrecen, 2008, **73** (1-2), 155–178.

[4] Baricz A. *Geometric properties of generalized Bessel functions of complex order.* Mathematica, 2006, **48** (71) (1), 13–18.

[5] Baricz A. *Generalized Bessel functions of the first kind.* PhD thesis, Babes-Bolyai University, Cluj-Napoca, 2008.

[6] Baricz A. *Generalized Bessel functions of the first kind.* Lecture Notes in Math., vol. 1994, Springer-Verlag, 2010.

[7] Bharati R., Parvatham R., Swaminathan A. *On subclasses of uniformly convex functions and corresponding class of starlike functions.* Tamkang J.Math., 1997, **26** (1), 17–32.

[8] Caplinger T. R., Causey W. M. *A class of univalent functions.* Proc. Amer. Math. Soc., 1973, **39**, 357–361.

[9] Cho N. E., Woo S. Y., Owa S. *Uniform convexity properties for hypergeometric functions.* Fract. Cal. Appl. Anal., 2002, **5** (3), 303–313.

[10] de Branges L. *A proof of the Bierberbach conjucture.*Acta. Math., 1985, **154**, 137–152.

[11] Dixit K. K., Pal S. K. *On a class of univalent functions related to complex order.* Indian J. Pure Appl. Math., 1995, **26** (9), 889–896.

[12] Goodman A. W. *On uniformly convex functions.* Ann. Polon. Math., 1991, **56**, 87–92.

[13] Goodman A. W. *On uniformly starlike functions.* J.Math.Anal.Appl., 1991, **155**, 364–370.

[14] Kanas S., Wisniowska A. *Conic regions and k-uniform convexity.* J. Comput. Appl. Math., 1999, **105**, 327–336.

[15] Merkes E., Scott B. T. *Starlike hypergeometric functions.* Proc. Amer. Math. Soc., 1961, **12**, 885–888.

[16] Mondal S. R., Swaminathan A. *Geometric properties of Generalized Bessel functions.* Bull. Malays. Math. Sci. Soc., 2012, **35** (1), 179–194.

[17] Mostafa A. O. *A study on starlike and convex properties for hypergeometric functions.* Journal of Inequalities in Pure and Applied Mathematics, 2009, **10** (3), Art. 87, 1–8.

[18] Murugusundaramoorthy G., Magesh N. *On certain subclasses of analytic functions associated with hypergeometric functions.* Appl. Math. Letters, 2011, **24**, 494–500.

[19] Orhan H., Yagmur N. *Geometric properties of generalized Struve functions.* The International Congress in Honour of Professor Hari M. Srivastava, Bursa, Turkey, August, 2012.

[20] Padmanabhan K. S. *On a certain class of functions whose derivatives have a positive real part in the unit disc.* Ann. Polon. Math., 1970, **23**, 73–81.

[21] Rønning F. *Uniformly convex functions and a corresponding class of starlike functions.* Proc. Amer. Math. Soc., 1993, **118**, 189–196.

[22] Rønning F. *Integral representations for bounded starlike functions.* Annal. Polon. Math., 1995, **60**, 289–297.

[23] Silverman H. *Univalent functions with negative coefficients.* Proc. Amer. Math. Soc., 1975, **51**, 109–116.

[24] Silverman H. *Starlike and convexity properties for hypergeometric functions.* J. Math. Anal. Appl., 1993, **172**, 574–581.

[25] Srivastava H. M., Murugusundaramoorthy G., Janani T. *Uniformly starlike functions and uniformly convex functions associated with the Struve function* (to appear).

[26] Srivastava H. M., Murugusundaramoorthy G., Sivasubramanian S. *Hypergeometric functions in the parabolic starlike and uniformly convex domains*, Integral Transforms Spec. Funct. 2007, **18**, 511–520.

[27] Subramanian K. G., Murugusundaramoorthy G., Balasubrahmanyam P., Silverman H. *Subclasses of uniformly convex and uniformly starlike functions.* Math. Japonica, 1995, **42** (3), 517–522.

[28] Subramanian K. G., Sudharsan T. V., Balasubrahmanyam P., Silverman H. *Classes of uniformly starlike functions.* Publ. Math. Debrecen., 1998, **53** (3–4), 309–315.

[29] Swaminathan A. *Certain suffienct conditions on Gaussian hypergeometric functions.* Journal of Inequalities in Pure and Applied Mathematics., 2004, **5(4)**, Art. 83, 1–10.

[30] Yagmur N., Orhan H. *Starlikeness and convexity of generalized Struve functions.* Abstract and Appl. Anal. 2013, Article ID 954513, 6 pages.

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Rational bases of $GL(2,\mathbb{R})$-comitants and of $GL(2,\mathbb{R})$-invariants for the planar system of differential equations with nonlinearities of the fourth degree

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Abstract. This paper is devoted to the construction of minimal rational bases of $GL(2,\mathbb{R})$-comitants and minimal rational bases of $GL(2,\mathbb{R})$-invariants for the bidimensional system of differential equations with nonlinearities of the fourth degree. For this system, three minimal rational bases of $GL(2,\mathbb{R})$-comitants and two minimal rational bases of $GL(2,\mathbb{R})$-invariants were constructed. It was established that any minimal rational basis of $GL(2,\mathbb{R})$-comitants contains 13 comitants and each minimal rational basis of $GL(2,\mathbb{R})$-invariants contains 11 invariants.

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1 Definitions and notations

Let us consider the system of differential equations with nonlinearities of the fourth degree

$$\frac{dx}{dt} = P_1(x,y) + P_4(x,y), \quad \frac{dy}{dt} = Q_1(x,y) + Q_4(x,y), \tag{1}$$

where $P_i(x,y)$, $Q_i(x,y)$ are homogeneous polynomials of degree i in x and y with real coefficients.

The goal of this paper is to construct minimal rational bases of GL-comitants as well as GL-invariants for the above system. It is known (see for instance [1, 2]) that invariant polynomials with respect to the group $GL(2,\mathbb{R})$ could be used to characterize some geometric proprieties of system (1). And clearly the knowledge of the elements of minimal rational bases essentially limits the number of invariant polynomials which could be used in the study of this system.

System (1) can be written in the following coefficient form:

$$\begin{aligned} \frac{dx}{dt} &= cx + dy + gx^4 + 4hx^3y + 6kx^2y^2 + 4lxy^3 + my^4, \\ \frac{dy}{dt} &= ex + fy + nx^4 + 4px^3y + 6qx^2y^2 + 4rxy^3 + sy^4. \end{aligned} \tag{2}$$

We denote by A the 14-dimensional coefficient space of system (1), by $\mathbf{a} \in A$ the vector of coefficients $\mathbf{a} = (c,d,e,f,g,h,k,l,m,n,p,q,r,s)$, by $\boldsymbol{q} \in \mathcal{Q} \subseteq Aff(2,\mathbb{R})$

a nondegenerate linear transformation of the phase plane of system (1), by $\mathbf{q}$ the transformation matrix and by $r_{\boldsymbol{q}}(\mathbf{a})$ a linear representation of coefficients of the transformed system in the space A.

Definition 1 (see [1, 2]). *A polynomial $\mathcal{K}(\mathbf{a},\mathbf{x})$ in coefficients of system* (1) *and coordinates of the vector $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ is called a comitant of system* (1) *with respect to the group $\mathcal{Q}$ if there exists a function $\lambda : \mathcal{Q} \to \mathbb{R}$ such that*

$$\mathcal{K}(r_{\boldsymbol{q}}(\mathbf{a}), \mathbf{q}\mathbf{x}) \equiv \lambda(\boldsymbol{q})\mathcal{K}(\mathbf{a},\mathbf{x})$$

for every $\boldsymbol{q} \in \mathcal{Q}$, $\mathbf{a} \in A$ and $\mathbf{x} \in \mathbb{R}^2$.

If $\mathcal{Q}$ is the group $GL(2,\mathbb{R})$ of nondegenerate linear transformations

$$\mathbf{u} = \mathbf{q}\mathbf{x}, \quad \Delta_{\mathbf{q}} = \det \mathbf{q} \neq 0 \tag{3}$$

of the phase plane of system (1), where $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ is a vector of new phase variables and $\mathbf{q} = \begin{pmatrix} q_1^1 & q_2^1 \\ q_1^2 & q_2^2 \end{pmatrix}$ is the transformation matrix, then the comitant is called $GL(2,\mathbb{R})$-comitant or center-affine comitant. In what follows only $GL(2,\mathbb{R})$-comitants are considered. If a comitant does not depend on coordinates of the vector $\mathbf{x}$, then it is called invariant.

The function $\lambda(\boldsymbol{q})$ is called a multiplicator. It is known [1] that the function $\lambda(\boldsymbol{q})$ has the form $\lambda(\boldsymbol{q}) = \Delta_{\mathbf{q}}^{-\chi}$, where χ is an integer, which is called the weight of the comitant $\mathcal{K}(\mathbf{a},\mathbf{x})$. If $\chi = 0$, then the comitant is called absolute, otherwise it is called relative.

According to [1] if a GL-comitant $\mathcal{K}(\mathbf{a},\mathbf{x})$ is a non-homogeneous polynomial with respect to $\mathbf{x}$ and $\mathbf{a}$, then each its homogeneity is also a GL-comitant. So in what follows we shall consider only homogeneous invariant polynomials.

We say that a comitant $\mathcal{K}(\mathbf{a},\mathbf{x})$ has the character $(\rho;\ \chi;\ \delta)$ if it has the weight χ, the degree δ with respect to coefficients of system (1) and the degree ρ with respect to coordinates of the vector $\mathbf{x}$.

Every comitant $\mathcal{K}(\mathbf{a},\mathbf{x})$ of system (1) of the character $(\rho;\ \chi;\ \delta)$ can be represented in the form

$$\mathcal{K}(\mathbf{a},\mathbf{x}) = T_0(\mathbf{a})x^\rho + T_1(\mathbf{a})x^{\rho-1}y + \ \dots\ + T_{\rho-1}(\mathbf{a})xy^{\rho-1} + T_\rho(\mathbf{a})y^\rho,$$

where $T_i(\mathbf{a})$ are polynomials in coefficients of the system. The polynomial $T_0(\mathbf{a})$ is called the semi-invariant of the comitant $\mathcal{K}(\mathbf{a},\mathbf{x})$ and is denoted by $S\mathcal{K}(\mathbf{a})$. Thus,

$$S\mathcal{K}(\mathbf{a}) = \frac{1}{\rho!} \cdot \frac{\partial^\rho \mathcal{K}(\mathbf{a},\mathbf{x})}{\partial x^\rho}.$$

Definition 2. *A set $\mathcal{S}$ of comitants (invariants) is called a rational basis on $\mathcal{M} \subseteq A$ of comitants (invariants) for system (1) with respect to the group $\mathcal{Q}$ if any comitant (invariant) of system (1) with respect to the group $\mathcal{Q}$ can be expressed as a rational function of elements of the set $\mathcal{S}$.*

Definition 3. *A rational basis on $\mathcal{M} \subseteq A$ of comitants (invariants) for system (1) with respect to the group $\mathcal{Q}$ is called minimal if by the removal from it of any comitant (invariant) it ceases to be a rational basis.*

We say that $GL(2,\mathbb{R})$-comitants (invariants) of a set $\mathcal{S}$ are polynomial independent if there is no identity between them of the form $\mathcal{P}(\mathcal{K}_i) \equiv 0$, where $\mathcal{P}(\mathcal{K}_i)$ is a polynomial in elements of the set $\mathcal{S}$.

Definition 4 (see [3])**.** *Let φ and ψ be homogeneous polynomials in coordinates of the vector $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ of the degrees ρ_1 and ρ_2, respectively. The polynomial*

$$(\varphi,\psi)^{(j)} = \frac{(\rho_1 - j)!(\rho_2 - j)!}{\rho_1!\rho_2!} \sum_{i=0}^{j} (-1)^i \binom{j}{i} \frac{\partial^j \varphi}{\partial x^{j-i}\partial y^i} \frac{\partial^j \psi}{\partial x^i \partial y^{j-i}}$$

is called the transvectant of index j of polynomials φ and ψ.

Property 1 (see [4])**.** *If polynomials φ and ψ are $GL(2,\mathbb{R})$-comitants of system* (1) *with the characters $(\rho_\varphi;\ \chi_\varphi;\ \delta_\varphi)$ and $(\rho_\psi;\ \chi_\psi;\ \delta_\psi)$, respectively, then the transvectant of index $j \le \min\{\rho_\varphi, \rho_\psi\}$ is a $GL(2,\mathbb{R})$-comitant of system* (1) *with the character $(\rho_\varphi + \rho_\psi - 2j;\ \chi_\varphi + \chi_\psi + j;\ \delta_\varphi + d_\psi)$. If $j > \min\{\rho_\varphi, \rho_\psi\}$, then $(\varphi,\psi)^{(j)} = 0$.*

$GL(2,\mathbb{R})$-comitants of the first degree with respect to coefficients of system (1) have the form

$$R_i = P_i(x,y)y - Q_i(x,y)x,\ S_i = \frac{1}{i}\left(\frac{\partial P_i(x,y)}{\partial x} + \frac{\partial Q_i(x,y)}{\partial y}\right),\ i = 1,4.$$

By using the comitants R_i and S_i $(i = 1,4)$, and the notion of transvectant the following $GL(2,\mathbb{R})$-comitants and invariants of system (1) were constructed (in the list below, the bracket "$[\![$" is used in order to avoid placing the otherwise necessary parenthesis "(" (up to six)):

$$K_1 = R_4, \quad K_2 = S_4, \quad K_3 = (R_4,R_4)^{(4)}, \quad K_4 = (R_4,R_4)^{(2)},$$
$$K_5 = (R_4,S_4)^{(3)}, \quad K_6 = (R_4,S_4)^{(2)}, \quad K_7 = (R_4,S_4)^{(1)},$$
$$K_8 = (S_4,S_4)^{(2)}, \quad K_{10} = [\![R_4,R_4)^{(4)},R_4)^{(1)},$$
$$K_{13} = [\![R_4,R_4)^{(2)},R_4)^{(1)}, \quad K_{17} = [\![R_4,S_4)^{(3)},S_4)^{(2)},$$
$$K_{18} = [\![R_4,S_4)^{(3)},S_4)^{(1)}, \quad K_{21} = [\![S_4,S_4)^{(2)},S_4)^{(1)}, \quad Q_1 = R_1,$$
$$Q_2 = S_1, \quad Q_3 = (R_4,R_1)^{(2)}, \quad Q_4 = (R_4,R_1)^{(1)}, \quad Q_5 = (S_4,R_1)^{(2)},$$
$$Q_6 = (S_4,R_1)^{(1)}, \quad Q_7 = (R_1,R_1)^{(2)}, \quad Q_{19} = [\![R_4,R_1)^{(2)},R_1)^{(2)},$$
$$Q_{20} = [\![R_4,R_1)^{(2)},R_1)^{(1)}, \quad Q_{21} = [\![S_4,R_1)^{(2)},R_1)^{(1)},$$
$$Q_{43} = [\![R_4,R_1)^{(2)},R_1)^{(2)},R_1)^{(1)},$$
$$I_1 = S_1, \quad I_2 = (R_1,R_1)^{(2)}, \quad I_3 = [\![R_1,Q_5)^{(1)},Q_5)^{(1)},$$

$$
\begin{gathered}
I_4 = [\![Q_{19}, R_1)^{(1)}, Q_{19})^{(1)}, \\
J_1 = [\![R_4, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, \\
J_2 = [\![S_4, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, \\
J_3 = [\![R_4, R_1)^{(2)}, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, \\
J_4 = [\![R_4, R_1)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, \\
J_6 = [\![S_4, R_1)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, \\
J_{19} = [\![R_4, R_1)^{(2)}, R_1)^{(2)}, Q_5)^{(1)}, \\
J_{20} = [\![R_4, R_1)^{(2)}, R_1)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, \\
J_{43} = [\![R_4, R_1)^{(2)}, R_1)^{(2)}, R_1)^{(1)}, Q_5)^{(1)}, \\
\widetilde{J}_1 = [\![R_4, Q_{19})^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)}, \\
\widetilde{J}_2 = [\![S_4, Q_{19})^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)}, \\
\widetilde{J}_3 = [\![R_4, R_1)^{(2)}, Q_{19})^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)}, \\
\widetilde{J}_4 = [\![R_4, R_1)^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)}, \\
\widetilde{J}_5 = [\![S_4, R_1)^{(2)}, Q_{19})^{(1)}, \\
\widetilde{J}_6 = [\![S_4, R_1)^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)}, \\
\widetilde{J}_{20} = [\![R_4, R_1)^{(2)}, R_1)^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)}, \\
\widetilde{J}_{21} = [\![S_4, R_1)^{(2)}, R_1)^{(1)}, Q_{19})^{(1)}.
\end{gathered}
$$

2 Rational bases of $GL(2,\mathbb{R})$-comitants

2.1 The case $K_1 \not\equiv 0 \quad (R_4 \not\equiv 0)$

Theorem 1. *The set of $GL(2,\mathbb{R})$-comitants*

$$\{\mathbf{K_1},\ K_2,\ K_3,\ K_4,\ K_5,\ K_6,\ K_7,\ K_{10},\ K_{13},\ Q_1,\ Q_2,\ Q_3,\ Q_4\} \tag{4}$$

is a minimal rational basis of $GL(2,\mathbb{R})$-comitants for system (1) *of differential equations with nonlinearities of the fourth degree on $\mathcal{M} = \{\, a \in A \mid\ K_1 \not\equiv 0\}$.*

Proof. Firstly we will show that the set of comitants $\{K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_{10}, K_{13}, Q_1, Q_2, Q_3, Q_4\}$ is a rational basis of $GL(2,\mathbb{R})$-comitants when $K_1 \not\equiv 0$. Let the $GL(2,\mathbb{R})$-comitant $K_1 \not\equiv 0$. By using the transformation:

$$
\begin{aligned}
u &= \frac{1}{5K_1(\mathbf{a},\mathbf{w})} \cdot \frac{\partial K_1(\mathbf{a},\mathbf{w})}{\partial w_1} \cdot x + \frac{1}{5K_1(\mathbf{a},\mathbf{w})} \cdot \frac{\partial K_1(\mathbf{a},\mathbf{w})}{\partial w_2} \cdot y, \\
v &= -w_2 x + w_1 y,
\end{aligned} \tag{5}
$$

where $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbb{R}^2$, system (1) can be brought to the system:

$$\begin{aligned}
\frac{du}{dt} &= \frac{K_1Q_2 + 2Q_4}{2K_1}u + \frac{-K_4Q_1 + 2K_1Q_3}{2K_1^2}v + \frac{4}{5}K_2u^4 + \\
&+ \frac{12K_7 + 10K_4}{5K_1}u^3v + \frac{-6K_2K_4 + 12K_1K_6 - 30K_{13}}{5K_1^2}u^2v^2 + \\
&+ \frac{10K_1^2K_3 - 15K_4^2 + 4K_1^2K_5 - 6K_4K_7 - 4K_2K_{13}}{5K_1^3}uv^3 + \\
&+ \frac{-K_1^2K_{10} + K_4K_{13}^2}{K_1^4}v^4, \\
\frac{dv}{dt} &= -Q_1u + \frac{K_1Q_2 - 2Q_4}{2K_1}v - K_1u^4 + \frac{4}{5}K_2u^3v + \\
&+ \frac{12K_7 - 15K_4}{5K_1}u^2v^2 + \frac{-6K_2K_4 + 12K_1K_6 + 20K_{13}}{5K_1^2}uv^3 + \\
&+ \frac{-10K_1^2K_3 + 15K_4^2 + 16K_1^2K_5 - 24K_4K_7 - 16K_2K_{13}}{20K_1^3}v^4.
\end{aligned} \tag{6}$$

According to [5, Lemma 4] any $GL(2,\mathbb{R})$-comitant $\mathcal{K}(\mathbf{a},\mathbf{x})$ of system (1) coincides with the semi-invariant $S\mathcal{K}$ of any comitant $\mathcal{K}$ calculated for system (6) in which coordinates of the vector $\mathbf{w}$ are replaced, respectively, with coordinates of the vector $\mathbf{x}$. In other words

$$\mathcal{K}(\mathbf{a},\mathbf{x}) = \frac{1}{\rho!} \cdot \left. \frac{\partial^\rho \mathcal{K}\left(\mathbf{b}\left(w_1, w_2\right), \mathbf{u}\right)}{\partial u^\rho} \right|_{\substack{w_1 = x \\ w_2 = y}}, \tag{7}$$

where $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{q} \cdot \mathbf{x}$, $\mathbf{q}$ is the matrix of transformation (5) and $\mathbf{b}$ is the vector of coefficients of system (6). So any $GL(2,\mathbb{R})$-comitant $\mathcal{K}(\mathbf{a},\mathbf{x})$ can be represented as a rational function of comitants (4) where denominator is a nonnegative integer power of the comitant K_1. Thus, the set of comitatns (4) is a rational basis of $GL(2,\mathbb{R})$-comitants for system (1) on $\mathcal{M} = \{\mathbf{a} \in A \mid K_1 \not\equiv 0\}$.

Next we will show that this basis is minimal. Indeed, suppose the contrary that the rational basis (4) is not a minimal one. This means that among the comitants K_i and Q_j there exists a polynomial identity $\mathcal{P}(K_i, Q_j) \equiv 0$ in $\mathbb{R}[x,y]$. On the other hand, since each K_i or Q_j is well determined by its semi-invariant SK_i or SQ_j, respectively, we conclude that the identity $\mathcal{P}(SK_i, SQ_j) \equiv 0$ must also hold. To calculate the expressions for these semi-invariants, for simplicity we apply the following substitution: $c = \frac{D+F}{2}$, $d = E$, $e = -C$,$f = \frac{F-D}{2}$, $g = \frac{4P+H}{5}$, $h = \frac{K+2Q}{10}$, $k = \frac{3L+4R}{30}$, $l = \frac{M+S}{5}$, $m = N$,$n = -G$, $p = \frac{P-H}{5}$, $r = \frac{2R-L}{10}$, $s = \frac{4S-M}{5}$, $q = \frac{4Q-3K}{30}$.

By using these substitutions system (2) is written in the form:

$$\begin{aligned}\frac{dx}{dt} &= \frac{D+F}{2}x + Ey + \frac{4P+H}{5}x^4 + \frac{4Q+2K}{5}x^3y + \\ &\quad \frac{4R+3L}{5}x^2y^2 + \frac{4S+4M}{5}xy^3 + Ny^4, \\ \frac{dy}{dt} &= -Cx + \frac{F-D}{2}y - Gx^4 + \frac{4P-4H}{5}x^3y + \\ &\quad \frac{4Q-3K}{5}x^2y^2 + \frac{4R-2L}{5}xy^3 + \frac{4S-M}{5}y^4.\end{aligned} \tag{8}$$

For system (8) the comitants R_1, S_1, R_4 and S_4 have the following form

$$\begin{aligned}&R_1 = Cx^2 + Dxy + Ey^2, \quad S_1 = F \\ &R_4 = Gx^5 + Hx^4y + Kx^3y^2 + Lx^2y^3 + Mxy^4 + Ny^5, \\ &S_4 = Px^3 + Qx^2y + Rxy^2 + Sy^3,\end{aligned} \tag{9}$$

For system (8) semi-invariants of comitants listed in the theorem have the form:

$$\begin{aligned}&SK_1 = G, \\ &SK_2 = P, \\ &SK_3 = \frac{1}{50}(3K^2 - 8HL + 20GM), \\ &SK_4 = -\frac{1}{25}(2H^2 - 5GK), \\ &SK_5 = -\frac{1}{10}(LP - KQ + 2HR - 10GS), \\ &SK_6 = \frac{1}{30}(3KP - 4HQ + 10GR), \\ &SK_7 = -\frac{1}{15}(3HP - 5GQ), \\ &SK_{10} = -\frac{1}{250}(-3HK^2 + 8H^2L + 5GKL - 50GHM + 250G^2N), \\ &SK_{13} = -\frac{1}{250}(4H^3 - 15GHK + 25G^2L), \\ &SQ_1 = C, \\ &SQ_2 = F, \\ &SQ_3 = \frac{1}{10}(10EG - 2DH + CK), \\ &SQ_4 = \frac{1}{10}(5DG - 2CH).\end{aligned} \tag{10}$$

Next in order to prove the impossibility of the polynomial identity $\mathcal{P}(SK_i, SQ_j) \equiv 0$ we use Table 1 in which the sign " $+$ " indicates that the respective parameter is contained in the expression of the semi-invariant SK_i or SQ_j, and the sign " $-$ " indicates that the respective parameter is missing from the expression of the semi-invariant SK_i or SQ_j.

Table 1

Semi-invariant	Parameters of system (8)													
	C	D	E	F	G	H	K	L	M	N	P	Q	R	S
SK_1	$-$	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SK_2	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$-$	$-$	$-$
SK_3	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$
SK_4	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SK_5	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$-$	$-$	$+$	$+$	$+$	$+$
SK_6	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$-$	$-$	$-$	$+$	$+$	$+$	$-$
SK_7	$-$	$-$	$-$	$-$	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$	$-$	$-$
SK_{10}	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$
SK_{13}	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$
SQ_1	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SQ_2	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SQ_3	$+$	$+$	$+$	$-$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SQ_4	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$

We observe that the parameter S is contained only in semi-invariant SK_5 and hence the identity $\mathcal{P}(SK_i, SQ_j) \equiv 0$ must be homogeneous in SK_5. This means that this semi-invariant could be removed from the list due to the parameter S and we denote this couple by $\langle SK_5, S\rangle$. Examining the remaining table after the removal of the line corresponding to the semi-invariant SK_5 and of the column corresponding to the parameter S, by the same reason we get the couple $\langle SK_6, R\rangle$ which allows us to remove the line corresponding to the semi-invariant SK_6 and the column corresponding to the parameter R. In the same way, we obtain the couples $\langle SK_7, Q\rangle$, $\langle SK_2, P\rangle$, $\langle SK_{10}, N\rangle$, $\langle SQ_3, E\rangle$, $\langle SQ_4, D\rangle$, $\langle SQ_2, F\rangle$, $\langle SQ_1, C\rangle$, $\langle SK_3, M\rangle$, $\langle SK_{13}, L\rangle$, $\langle SK_4, K\rangle$, $\langle SK_1, G\rangle$. It follows that the set of comitants listed in Theorem 1 are polynomial independent. So if by the removal from it of any comitant it ceases to be a rational basis. This proves that the set of comitants listed in Theorem 1 is a minimal rational basis of $GL(2,\mathbb{R})$-comitants for system (1). $\square$

2.2 The case $K_2 \not\equiv 0$ $(S_4 \not\equiv 0)$

Theorem 2. *The set of $GL(2,\mathbb{R})$-comitants*

$$\{K_1,\ \mathbf{K}_2,\ K_5,\ K_6,\ K_7,\ K_8,\ K_{17},\ K_{18},\ K_{21},\ Q_1,\ Q_2,\ Q_5,\ Q_6\} \tag{11}$$

is a minimal rational basis of $GL(2,\mathbb{R})$-comitants for system (1) *of differential equations with nonlinearities of the fourth degree on $\mathcal{M} = \{\mathbf{a} \in A \mid K_2 \not\equiv 0\}$.*

Proof. Firstly we will show that the set of comitants $\{K_1, K_2, K_5, K_6, K_7, K_8, K_{17}, K_{18}, K_{21}, Q_1, Q_2, Q_5, Q_6\}$ is a rational basis of $GL(2,\mathbb{R})$-comitants when $K_2 \not\equiv 0$.

The proof of this theorem is completely the same as the proof of previous theo-

rem. Let the $GL(2,\mathbb{R})$-comitant $K_2 \not\equiv 0$. By using the transformation:

$$\begin{aligned} u &= \frac{1}{3K_2(\mathbf{a},\mathbf{w})}\cdot\frac{\partial K_2(\mathbf{a},\mathbf{w})}{\partial w_1}\cdot x + \frac{1}{3K_2(\mathbf{a},\mathbf{w})}\cdot\frac{\partial K_2(\mathbf{a},\mathbf{w})}{\partial w_2}\cdot y, \\ v &= -w_2 x + w_1 y, \end{aligned} \tag{12}$$

system (1) can be brought to the system:

$$\begin{aligned} \frac{du}{dt} &= \frac{K_2Q_2+2Q_6}{2K_2}u + \frac{-K_8Q_1+2K_2Q_5}{2K_2^2}v + \frac{4K_2^2-5K_7}{5K_2}u^4 + \\ &+\frac{4K_2K_6-2K_1K_8}{K_2^2}u^3v + \frac{-30K_2^2K_5+6K_2^2K_8+45K_7K_8-30K_1K_{21}}{5K_2^3}u^2v^2 + \\ &+\frac{-30K_2K_6K_8+15K_1K_8^2+20K_2^2K_{18}-4K_2^2K_{21}+20K_7K_{21}}{5K_2^4}uv^3 + \\ &+\frac{8K_2^2K_5K_8-9K_7K_8^2-4K_2^3K_{17}-4K_2K_6K_{21}+8K_1K_8K_{21}}{4K_2^5}v^4, \\ \frac{dv}{dt} &= -Q_1u + \frac{K_2Q_2-2Q_6}{2K_2}v - K_1u^4 + \frac{4K_2^2+20K_7}{5K_2}u^3v + \\ &+\frac{-6K_2K_6+3K_1K_8}{K_2^2}u^2v^2 + \frac{20K_2^2K_5+6K_2^2K_8-30K_7K_8+20K_1K_{21}}{5K_2^3}uv^3 + \\ &+\frac{30K_2K_6K_8-15K_1K_8^2-20K_2^2K_{18}-16K_2^2K_{21}-20K_7K_{21}}{20K_2^4}v^4. \end{aligned} \tag{13}$$

According to [5, Lemma 4] it follows that the set of comitants (11) forms a rational basis of $GL(2,\mathbb{R})$-comitants for system (1). The minimality results from the expressions of semi-invariants of comitants (11), calculated for system (8), which are the following:

$$\begin{aligned} SK_1 &= G, \\ SK_2 &= P, \\ SK_5 &= -1/10(LP-KQ+2HR-10GS), \\ SK_6 &= 1/30(3KP-4HQ+10GR), \\ SK_7 &= -1/15(3HP-5GQ), \\ SK_8 &= -2/9(Q^2-3PR), \\ SK_{17} &= -1/30(30NP^2-10MPQ+2LQ^2+4LPR-3KQR+2HR^2- \\ &\quad -3KPS+4HQS-10GRS), \\ SK_{18} &= 1/30(6MP^2-4LPQ+KQ^2+3KPR-2HQR- \\ &\quad -6HPS+10GQS), \\ SK_{21} &= -1/27(2Q^3-9PQR+27P^2S), \\ SQ_1 &= C, \end{aligned} \tag{14}$$

$$SQ_2 = F,$$
$$SQ_5 = 1/3(3EP - DQ + CR),$$
$$SQ_6 = 1/6(3DP - 2CQ).$$

Next in order to prove the impossibility of the polynomial identity $\mathcal{P}(SK_i, SQ_j) \equiv 0$ we use Table 2 in which the sign " $+$ " indicates that the respective parameter is contained in the expression of the semi-invariant SK_i or SQ_j, and the sign " $-$ " indicates that the respective parameter is missing from the expression of the semi-invariant SK_i or SQ_j.

Table 2

Semi-invariant	Parameters of system (8)													
	C	D	E	F	G	H	K	L	M	N	P	Q	R	S
SK_1	$-$	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SK_2	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$-$	$-$	$-$
SK_5	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$-$	$-$	$+$	$+$	$+$	$+$
SK_6	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$-$	$-$	$-$	$+$	$+$	$+$	$-$
SK_7	$-$	$-$	$-$	$-$	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$	$-$	$-$
SK_8	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$-$
SK_{17}	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
SK_{18}	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$-$	$+$	$+$	$+$	$+$
SK_{21}	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$
SQ_1	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SQ_2	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SQ_5	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$-$
SQ_6	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$-$	$-$

In the same way as in the proof of Theorem 1 we obtain the couples $\langle SK_{17}, N\rangle$, $\langle SK_{18}, M\rangle$, $\langle SK_5, L\rangle$, $\langle SK_6, K\rangle$, $\langle SK_7, H\rangle$, $\langle SK_1, G\rangle$, $\langle SQ_2, F\rangle$, $\langle SQ_5, E\rangle$, $\langle SQ_6, D\rangle$, $\langle SQ_1, C\rangle$, $\langle SK_8, R\rangle$, $\langle SK_2, P\rangle$,$\langle SK_{21}, S\rangle$.

From Table 2 it follows that the comitants (11) are polynomial independent. □

2.3 The case $Q_1 \not\equiv 0$ $(R_1 \not\equiv 0)$

Theorem 3. *The set of $GL(2,\mathbb{R})$-comitants*

$$\{K_1,\ K_2,\ \mathbf{Q}_1,\ Q_2,\ Q_3,\ Q_4,\ Q_5,\ Q_6,\ Q_7,\ Q_{19},\ Q_{20},\ Q_{21},\ Q_{43}\} \tag{15}$$

is a minimal rational basis of $GL(2,\mathbb{R})$-comitants for system (1) *of differential equations with nonlinearities of the fourth degree on $\mathcal{M} = \{\mathbf{a} \in A \mid Q_1 \not\equiv 0\}$.*

Proof. Firstly we will show that the set of comitants $\{K_1, K_2, Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_{19}, Q_{20}, Q_{21}, Q_{43}\}$ is a rational basis of $GL(2,\mathbb{R})$-comitants when $Q_1 \not\equiv 0$.

The proof of this theorem is completely the same as the proof of Theorem 1. Let the $GL(2,\mathbb{R})$-comitant $Q_1 \not\equiv 0$. By using the transformation:

$$\begin{aligned} u &= \frac{1}{2Q_1(a,w)} \cdot \frac{\partial Q_1(a,w)}{\partial w_1} \cdot x + \frac{1}{2Q_1(a,w)} \cdot \frac{\partial Q_1(a,w)}{\partial w_2} \cdot y, \\ v &= -w_2 x + w_1 y, \end{aligned} \tag{16}$$

system (1) can be brought to the system:

$$\begin{aligned} \frac{du}{dt} &= \frac{Q_2}{2}u + \frac{Q_7}{2Q_1}v + \\ &+ \frac{4K_2Q_1 - 5Q_4}{5Q_1}u^4 + \frac{20Q_1Q_3 - 12Q_1Q_6 - 10K_1Q_7}{5Q_1^2}u^3v + \\ &+ \frac{12Q_1^2Q_5 - 6K_2Q_1Q_7 + 15Q_4Q_7 - 30Q_1Q_{20}}{5Q_1^3}u^2v^2 + \\ &+ \frac{-20Q_1Q_3Q_7 + 2Q_1Q_6Q_7 + 5K_1Q_7^2 + 20Q_1^2Q_{19} - 4Q_1^2Q_{21}}{5Q_1^4}uv^3 + \\ &+ \frac{4Q_1Q_7Q_{20} - Q_4Q_7^2 - 4Q_1^2Q_{43}}{4Q_1^5}v^4, \\ \frac{dv}{dt} &= -Q_1u + \frac{Q_2}{2}v - K_1u^4 + \\ &+ \frac{4K_2Q_1 + 20Q_4}{5Q_1}u^3v + \frac{15K_1Q_7 - 30Q_1Q_3 - 12Q_1Q_6}{5Q_1^2}u^2v^2 + \\ &+ \frac{12Q_1^2Q_5 - 6K_2Q_1Q_7 - 10Q_4Q_7 + 20Q_1Q_{20}}{5Q_1^3}uv^3 + \\ &+ \frac{20Q_1Q_3Q_7 + 8Q_1Q_6Q_7 - 5K_1Q_7^2 - 20Q_1^2Q_{19} - 16Q_1^2Q_{21}}{20Q_1^4}v^4. \end{aligned} \tag{17}$$

According to [5, Lemma 4] it follows that the set of comitants (15) forms a rational basis of $GL(2,\mathbb{R})$-comitants for system (1).

The minimality results from expressions of semi-invariants of comitants (15), which are the following:

$$\begin{aligned} &SK_1 = G, \\ &SK_2 = P, \\ &SQ_1 = C, \\ &SQ_2 = F, \\ &SQ_3 = 1/10(10EG - 2DH + CK), \\ &SQ_4 = 1/10(5DG - 2CH), \\ &SQ_5 = 1/3(3EP - DQ + CR), \\ &SQ_6 = 1/6(3DP - 2CQ), \end{aligned} \tag{18}$$

$$
\begin{aligned}
SQ_7 &= -1/2(D^2 - 4CE),\\
SQ_{19} &= 1/10(10E^2G - 4DEH + D^2K + 2CEK - 2CDL + 2C^2M),\\
SQ_{20} &= -1/20(-10DEG + 2D^2H + 4CEH - 3CDK + 2C^2L),\\
SQ_{21} &= -1/6(-3DEP + D^2Q + 2CEQ - 3CDR + 6C^2S),\\
SQ_{43} &= 1/20(10DE^2G - 4D^2EH - 4CE^2H + D^3K + 6CDEK - 4CD^2L -\\
&\quad - 4C^2EL + 10C^2DM - 20C^3N).
\end{aligned}
$$

Next in order to prove the impossibility of the polynomial identity $\mathcal{P}(SK_i, SQ_j) \equiv 0$ we use Table 3 in which the sign " $+$ " indicates that the respective parameter is contained in the expression of the semi-invariant SK_i or SQ_j, and the sign " $-$ " indicates that the respective parameter is missing from the expression of the semi-invariant SK_i or SQ_j.

Table 3

Semi-invariant	Parameters of system (8)													
	C	D	E	F	G	H	K	L	M	N	P	Q	R	S
SK_1	$-$	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SK_2	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$-$	$-$	$-$
SQ_1	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SQ_2	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SQ_3	$+$	$+$	$+$	$-$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SQ_4	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SQ_5	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$-$
SQ_6	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$-$	$-$
SQ_7	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SQ_{19}	$+$	$+$	$+$	$-$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$
SQ_{20}	$+$	$+$	$+$	$-$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$
SQ_{21}	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$
SQ_{43}	$+$	$+$	$+$	$-$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$

In the same way as in the proof of previous theorems we obtain the couples $\langle SQ_2, F\rangle$, $\langle SQ_{21}, S\rangle$, $\langle SQ_5, R\rangle$, $\langle SQ_6, Q\rangle$, $\langle SK_2, P\rangle$, $\langle SQ_{43}, N\rangle$, $\langle SQ_{19}, M\rangle$, $\langle SQ_{20}, L\rangle$, $\langle SQ_3, K\rangle$, $\langle SQ_4, H\rangle$, $\langle SK_1, G\rangle$, $\langle SQ_7, E\rangle$,$\langle SQ_1, C\rangle$.

From Table 3 it follows that the comitants (15) are polynomial independent. □

3 Rational bases of $GL(2,\mathbb{R})$-invariants

3.1 The case $I_3 \neq 0$

Theorem 4. *The set of $GL(2,\mathbb{R})$-invariants*

$$\{I_1,\ I_2,\ \mathbf{I}_3,\ J_1,\ J_2,\ J_3,\ J_4,\ J_6,\ J_{19},\ J_{20},\ J_{43}\} \tag{19}$$

is a minimal rational basis of $GL(2,\mathbb{R})$-invariants for system (1) *of differential equations with nonlinearities of the fourth degree on $\mathcal{M} = \{\,\mathbf{a} \in A \mid\ I_3 \neq 0\}$.*

Proof. Firstly we will show that the set of invariants $\{I_1,\ I_2,\ I_3,\ J_1,\ J_2,\ J_3,\ J_4,\ J_6,\ J_{19},\ J_{20},\ J_{43}\}$ is a rational basis of $GL(2,\mathbb{R})$-invariants when $I_3 \neq 0$.

By using the transformation:

$$\begin{aligned} u &= \frac{1}{I_3(a,w)} \cdot \frac{\partial Q_{21}(a,w)}{\partial w_1} \cdot x + \frac{1}{I_3(a,w)} \cdot \frac{\partial Q_{21}(a,w)}{\partial w_2} \cdot y, \\ v &= \frac{\partial Q_5(a,w)}{\partial w_1} \cdot x + \frac{\partial Q_5(a,w)}{\partial w_2} \cdot y, \end{aligned} \tag{20}$$

system (1) can be brought to the system:

$$\begin{aligned} \frac{du}{dt} &= \frac{I_1}{2}u - \frac{I_2}{2I_3}v + \frac{4I_3J_2 + 5J_4}{5I_3}u^4 + \\ &\quad \frac{-10I_2J_1 - 20I_3J_3 + 12I_3J_6}{5I_3^2}u^3v + \frac{-6I_2I_3J_2 - 15I_2J_4 - 30I_3J_{20}}{5I_3^3}u^2v^2 + \\ &\quad \frac{-4I_3^3 + 5I_2^2J_1 + 20I_2I_3J_3 - 2I_2I_3J_6 + 20I_3^2J_{19}}{5I_3^4}uv^3 + \\ &\quad + \frac{I_2^2J_4 + 4I_2I_3J_{20} + 4I_3^2J_{43}}{4I_3^5}v^4, \\ \frac{dv}{dt} &= I_3u + \frac{I_1}{2}v - J_1u^4 + \frac{4I_3J_2 - 20J_4}{5I_3}u^3v + \\ &\quad + \frac{15I_2J_1 + 30I_3J_3 + 12I_3J_6}{5I_3^2}u^2v^2 + \frac{-6I_2I_3J_2 + 10I_2J_4 + 20I_3J_{20}}{5I_3^3}uv^3 + \\ &\quad \frac{-16I_3^3 - 5I_2^2J_1 - 20I_2I_3J_3 - 8I_2I_3J_6 - 20I_3^2J_{19}}{20I_3^4}v^4. \end{aligned} \tag{21}$$

From system (21), it follows that any $GL(2,\mathbb{R})$-invariant of system (1) with $I_3 \neq 0$ can be represented as a rational function of invariants (19). So the set of $GL(2,\mathbb{R})$-invariants (19) forms a rational basis for system (1) with $I_3 \neq 0$.

To prove the minimality we write the expressions of invariants (19). By using the notation $U = \frac{1}{3}(EQ - DR + 3CS)$ and $V = \frac{1}{3}(3EP - DQ + CR)$ for the invariants (19) we have:

$$\begin{aligned} I_1 &= F, \\ I_2 &= \frac{1}{2}(-D^2 + 4CE), \\ I_3 &= -CU^2 + DUV - EV^2, \\ J_1 &= GU^5 - HU^4V + KU^3V^2 - LU^2V^3 + MUV^4 - NV^5, \\ J_2 &= PU^3 - QU^2V + RUV^2 - SV^3, \\ J_3 &= \frac{1}{10}[(10EG - 2DH + CK)U^3 + (-6EH + 3DK - 3CL)U^2V + \\ &\quad + (3EK - 3DL + 6CM)UV^2 + (-EL + 2DM - 10CN)V^3], \end{aligned}$$

$$
\begin{aligned}
J_4 &= \frac{1}{10}[(5DG-2CH)U^5+(-10EG-3DH+4CK)U^4V+\\
&+(8EH+DK-6CL)U^3V^2+(-6EK+DL+8CM)U^2V^3+\\
&+(4EL-3DM-10CN)UV^4+(-2EM+5DN)V^5],\\
J_6 &= \frac{1}{6}[(3DP-2CQ)U^3+(-6EP-DQ+4CR)U^2V+\\
&+(4EQ-DR-6CS)UV^2+(-2ER+3DS)V^3],\\
J_{19} &= \frac{1}{10}[(10E^2G-4DEH+D^2K+2CEK-2CDL+2C^2M)U+\\
&+(-2E^2H+2DEK-D^2L-2CEL+4CDM-10C^2N)V],\\
J_{20} &= \frac{1}{20}[(10DEG-2D^2H-4CEH+3CDK-2C^2L)U^3+\\
&+(-20E^2G+2DEH+D^2K+2CEK-5CDL+8C^2M)U^2V+\\
&+(8E^2H-5DEK+D^2L+2CEL+2CDM-20C^2N)UV^2+\\
&+(-2E^2K+3DEL-2D^2M-4CEM+10CDN)V^3],\\
J_{43} &= \frac{1}{20}[(10DE^2G-4D^2EH-4CE^2H+D^3K+6CDEK-4CD^2L-4C^2EL+\\
&+10C^2DM-20C^3N)U+(-20E^3G+10DE^2H-4D^2EK-4CE^2K+\\
&+D^3L+6CDEL-4CD^2M-4C^2EM+10C^2DN)V].
\end{aligned}
\tag{22}
$$

Next, we write the expressions of the highest power of S in the invariants (22), denoted by EI_i and EJ_i

$$
\begin{aligned}
EI_1 &= F,\\
EI_2 &= \frac{1}{2}(-D^2+4CE),\\
EI_3 &= -C^3,\\
EJ_1 &= C^5G,\\
EJ_2 &= C^3P,\\
EJ_3 &= \frac{1}{10}C^3(10EG-2DH+CK),\\
EJ_4 &= -\frac{1}{10}C^5(-5DG+2CH),\\
EJ_6 &= -\frac{1}{6}C^3(-3DP+2CQ),\\
EJ_{19} &= \frac{1}{10}C(10E^2G-4DEH+D^2K+2CEK-2CDL+2C^2M),\\
EJ_{20} &= -\frac{1}{20}C^3(-10DEG+2D^2H+4CEH-3CDK+2C^2L),\\
EJ_{43} &= -\frac{1}{20}C(-10DE^2G+4D^2EH+4CE^2H-D^3K-6CDEK+4CD^2L+\\
&\quad +4C^2EL-10C^2DM+20C^3N).
\end{aligned}
\tag{23}
$$

In the following table the sign " $+$ " indicates that the respective parameter is contained in the expression of the highest power of S in the invariants (22) and the sign " $-$ " indicates that the respective parameter is missing from the expression of the highest power of S.

Table 4

Invariant	Parameters of system (8)												
	C	D	E	F	G	H	K	L	M	N	P	Q	R
I_1	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
I_2	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
I_3	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
J_1	$+$	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
J_2	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$-$	$-$
J_3	$+$	$+$	$+$	$-$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$
J_4	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
J_6	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$-$
J_{19}	$+$	$+$	$+$	$-$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$
J_{20}	$+$	$+$	$+$	$-$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$
J_{43}	$+$	$+$	$+$	$-$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$

According to Table 4 we obtain the couples $\langle J_6, Q\rangle$, $\langle J_2, P\rangle$, $\langle J_{43}, N\rangle$, $\langle J_{19}, M\rangle$, $\langle J_{20}, L\rangle$, $\langle J_3, K\rangle$, $\langle J_4, H\rangle$, $\langle J_1, G\rangle$, $\langle I_1, F\rangle$, $\langle I_2, E\rangle$, $\langle J_2, C\rangle$.

From Table 4 it follows that the invariants (19) are polynomial independent. □

3.2 The case $I_4 \neq 0$

Theorem 5. *The set of $GL(2,\mathbb{R})$-invariants*

$$\left\{I_1,\ I_2,\ \mathbf{I}_4,\ \widetilde{J}_1,\ \widetilde{J}_2,\ \widetilde{J}_3,\ \widetilde{J}_4,\ \widetilde{J}_5,\ \widetilde{J}_6,\ \widetilde{J}_{20},\ \widetilde{J}_{21}\right\} \tag{24}$$

is a minimal rational basis of $GL(2,\mathbb{R})$-invariants for system (1) *of differential equations with nonlinearities of the fourth degree on $\mathcal{M} = \{\mathbf{a} \in A \mid I_4 \neq 0\}$.*

Proof. Firstly we will show that the set of invariants $\{I_1,\ I_2,\ I_4,\ \widetilde{J}_1,\ \widetilde{J}_2,\ \widetilde{J}_3,\ \widetilde{J}_4,\ \widetilde{J}_5,\ \widetilde{J}_6,\ \widetilde{J}_{20},\ \widetilde{J}_{21}\}$ is a rational basis of $GL(2,\mathbb{R})$-invariants when $I_4 \neq 0$.

By using the transformation:

$$\begin{aligned} u &= \frac{1}{I_4(a,w)} \cdot \frac{\partial Q_{43}(a,w)}{\partial w_1} \cdot x + \frac{1}{I_4(a,w)} \cdot \frac{\partial Q_{43}(a,w)}{\partial w_2} \cdot y, \\ v &= \frac{\partial Q_{19}(a,w)}{\partial w_1} \cdot x + \frac{\partial Q_{19}(a,w)}{\partial w_2} \cdot y, \end{aligned} \tag{25}$$

system (1) can be brought to the system:

$$\frac{du}{dt} = \frac{I_1}{2}u - \frac{I_2}{2I_4}v + \frac{4I_4\widetilde{J}_2 + 5\widetilde{J}_4}{5I_4}u^4 + \frac{-10I_2\widetilde{J}_1 - 20I_4\widetilde{J}_3 + 12I_4\widetilde{J}_6}{5I_4^2}u^3v +$$

$$+\frac{6I_2I_4\widetilde{J}_2+15I_2\widetilde{J}_4+12I_4^2\widetilde{J}_5+30I_4\widetilde{J}_{20}}{5I_4^3}u^2v^2+$$
$$+\frac{5I_2^2\widetilde{J}_1+20I_2I_4\widetilde{J}_3-2I_2I_4\widetilde{J}_6-4I_4^2\widetilde{J}_{21}}{5I_4^4}uv^3+\frac{4I_4^3+I_2^2\widetilde{J}_4+4I_2I_4\widetilde{J}_{20}}{4I_4^5}v^4, \quad (26)$$
$$\frac{dv}{dt}=I_4u+\frac{I_1}{2}v-\widetilde{J}_1u^4+\frac{4I_4\widetilde{J}_2-20\widetilde{J}_4}{5I_4}u^3v+\frac{15I_2\widetilde{J}_1+30I_4\widetilde{J}_3+12I_4\widetilde{J}_6}{5I_4^2}u^2v^2+$$
$$+\frac{-6I_2I_4\widetilde{J}_2+10I_2\widetilde{J}_4-12I_4^2\widetilde{J}_5+20I_4\widetilde{J}_{20}}{5I_4^3}uv^3+$$
$$+\frac{-5I_2^2\widetilde{J}_1+20I_2I_4\widetilde{J}_3+8I_2I_4\widetilde{J}_6+16I_4^2\widetilde{J}_{21}}{20I_4^4}v^4.$$

From system (26), it follows that any $GL(2,\mathbb{R})$-invariant of system (1) with $I_4\neq 0$ can be represented as a rational function of invariants (24). So the set of $GL(2,\mathbb{R})$-invariants (24) forms a rational basis for system (1) with $I_4\neq 0$.

To prove the minimality we write the expressions of invariants (24). By using the notation $U=\frac{1}{10}(2E^2H-2DEK+D^2L+2CEL-4CDM+10C^2N)$ and $V=\frac{1}{10}(10E^2G-4DEH+D^2K+2CEK-2CDL+2C^2M)$ for the invariants (24) we have:

$$\begin{aligned}
I_1&=F,\\
I_2&=\frac{1}{2}(-D^2+4CE),\\
I_4&=-CU^2+DUV-EV^2,\\
\widetilde{J}_1&=GU^5-HU^4V+KU^3V^2-LU^2V^3+MUV^4-NV^5,\\
\widetilde{J}_2&=PU^3-QU^2V+RUV^2-SV^3,\\
\widetilde{J}_3&=\frac{1}{10}[(10EG-2DH+CK)U^3+(-6EH+3DK-3CL)U^2V+\\
&\quad+(3EK-3DL+6CM)UV^2+(-EL+2DM-10CN)V^3],\\
\widetilde{J}_4&=\frac{1}{10}[(5DG-2CH)U^5+(-10EG-3DH+4CK)U^4V+\\
&\quad+(8EH+DK-6CL)U^3V^2+(-6EK+DL+8CM)U^2V^3+\\
&\quad+(4EL-3DM-10CN)UV^4+(-2EM+5DN)V^5], \qquad (27)\\
\widetilde{J}_5&=\frac{1}{3}[(3EP-DQ+CR)U+(-EQ+DR-3CS)V],\\
\widetilde{J}_6&=\frac{1}{6}[(3DP-2CQ)U^3+(-6EP-DQ+4CR)U^2V+\\
&\quad+(4EQ-DR-6CS)UV^2+(-2ER+3DS)V^3],\\
\widetilde{J}_{20}&=\frac{1}{20}[(10DEG-2D^2H-4CEH+3CDK-2C^2L)U^3+\\
&\quad+(-20E^2G+2DEH+D^2K+2CEK-5CDL+8C^2M)U^2V+
\end{aligned}$$

$$+ (8E^2H - 5DEK + D^2L + 2CEL + 2CDM - 20C^2N)UV^2 +$$
$$+ (-2E^2K + 3DEL - 2D^2M - 4CEM + 10CDN)V^3],$$
$$\widetilde{J}_{21} = \frac{1}{6}[(3DEP - D^2Q - 2CEQ + 3CDR - 6C^2S)U +$$
$$+ (-6E^2P + 3DEQ - D^2R - 2CER + 3CDS)V].$$

Table 5

Invariant	Parameters of system (8)												
	C	D	E	F	G	H	K	L	M	P	Q	R	S
I_1	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
I_2	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
I_4	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$\widetilde{J}_1$	$+$	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$\widetilde{J}_2$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$-$	$-$	$-$
$\widetilde{J}_3$	$+$	$+$	$+$	$-$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$
$\widetilde{J}_4$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$\widetilde{J}_5$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$-$
$\widetilde{J}_6$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$-$	$-$
$\widetilde{J}_{20}$	$+$	$+$	$+$	$-$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$
$\widetilde{J}_{21}$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$

Next, we write the expressions of the highest power of N in the invariants (24), denoted by EI_i and $E\widetilde{J}_i$

$$\begin{aligned}
EI_1 &= F,\\
EI_2 &= \frac{1}{2}(-D^2 + 4CE),\\
EI_4 &= C^5,\\
E\widetilde{J}_1 &= C^{10}G,\\
E\widetilde{J}_2 &= C^6P,\\
E\widetilde{J}_3 &= \frac{1}{10}C^6(10EG - 2DH + CK),\\
E\widetilde{J}_4 &= -\frac{1}{10}C^{10}(-5DG + 2CH),\\
E\widetilde{J}_5 &= \frac{1}{3}C^2(3EP - DQ + CR),\\
E\widetilde{J}_6 &= -\frac{1}{6}C^6(-3DP + 2CQ),\\
E\widetilde{J}_{20} &= -\frac{1}{20}C^6(-10DEG + 2D^2H + 4CEH - 3CDK + 2C^2L),\\
E\widetilde{J}_{21} &= -\frac{1}{6}C^2(-3DEP + D^2Q + 2CEQ - 3CDR + 6C^2S).
\end{aligned} \tag{28}$$

In Table 5 the sign " + " indicates that the respective parameter is contained in the expression of the highest power of N in the invariants (27) and the sign " − " indicates that the respective parameter is missing from the expression of the highest power of N.

According to Table 5 we obtain the couples $\left\langle \widetilde{J}_{21}, S\right\rangle$, $\left\langle \widetilde{J}_{5}, R\right\rangle$, $\left\langle \widetilde{J}_{6}, Q\right\rangle$, $\left\langle \widetilde{J}_{2}, P\right\rangle$, $\left\langle \widetilde{J}_{20}, L\right\rangle$, $\left\langle \widetilde{J}_{3}, K\right\rangle$, $\left\langle \widetilde{J}_{4}, H\right\rangle$, $\left\langle \widetilde{J}_{1}, G\right\rangle$, $\langle I_1, F\rangle$, $\langle I_2, E\rangle$, $\left\langle \widetilde{J}_{2}, C\right\rangle$.

From Table 5 it follows that the invariants (24) are polynomial independent. □

The aim of constructing systems of the form (6), (13), (17), (21), (26) the minimal rational basis of $GL(2,\mathbb{R})$-comitants (4), (11), (15) and the minimal rational basis of $GL(2,\mathbb{R})$-invariants (19), (24), is to use them in the qualitative study of systems (1), for example in establishing the invariant center conditions (center-focus problem) [6]. For the center-focus problem, when system (1) satisfies the conditions $I_1 = 0$, $I_2 > 0$, by a linear transformation and time scaling system (1) can be brought to the form

$$\begin{aligned}\frac{dx}{dt} &= y + gx^4 + 4hx^3y + 6kx^2y^2 + 4lxy^3 + my^4,\\ \frac{dy}{dt} &= -x + nx^4 + 4px^3y + 6qx^2y^2 + 4rxy^3 + sy^4.\end{aligned} \tag{29}$$

For this system the first two Lyapunov quantities have the form:

$$\begin{aligned}G_8 = \frac{1}{16}(&-7gh - 18hk - 3gl - 18kl - 3hm - 7lm + 7gn + 3kn + 8hp + 7np+\\ &+ 3gq - 3mq + 18pq - 8lr + 3nr + 18qr - 3ks - 7ms + 3ps + 7rs),\end{aligned} \tag{30}$$

$$\begin{aligned}G_{14} = \frac{1}{23040}(&121121g^3h + 92952gh^3 + 579516g^2hk + 234576h^3k + 866556ghk^2+\\ &+ 436752hk^3 + 55419g^3l + 199032gh^2l + 414072g^2kl + 635472h^2kl+\\ &+ 720900gk^2l + 393984k^3l + 132776ghl^2 + 572976hkl^2 + 18888gl^3+\\ &+ 165168kl^3 + 109545g^2hm + 39096h^3m + 340704ghkm + 272052hk^2m+\\ &+ 134135g^2lm + 158040h^2lm + 390264gklm + 299772k^2lm + 182280hl^2m+\\ &+ 64232l^3m + 33831ghm^2 + 58644hkm^2 + 38277glm^2 + 60048klm^2+\\ &+ 3807hm^3 + 1161lm^3 - 121121g^3n - 7470gh^2n - 351561g^2kn+\\ &+ 139428h^2kn - 321066gk^2n - 72792k^3n - 48588ghln + 254376hkln-\\ &- 32318gl^2n + 125508kl^2n - 65376g^2mn + 29970h^2mn - 123066gkmn-\\ &- 36774k^2mn + 94068hlmn + 60354l^2mn - 14175gm^2n - 5589km^2n-\\ &- 35181ghn^2 + 648hkn^2 - 23859gln^2 + 15660kln^2 + 5589hmn^2+\\ &+ 14175lmn^2 - 1161gn^3 - 3807kn^3 + 23034g^2hp - 97344h^3p+\\ &+ 313896ghkp + 347400hk^2p + 40242g^2lp - 162048h^2lp + 428040gklp+\\ &+ 487080k^2lp - 80320hl^2p - 11520l^3p + 46452ghmp + 113256hkmp+\end{aligned}$$

$$+ 149092glmp + 279240klmp + 2970hm^2p + 27666lm^2p - 324699g^2np-$$
$$- 118440h^2np - 662136gknp - 330300k^2np - 146544hlnp - 54728l^2np-$$
$$- 125730gmnp - 123552kmnp - 14175m^2np - 31050hn^2p - 27666ln^2p-$$
$$- 1161n^3p - 144360ghp^2 - 107184hkp^2 + 5928glp^2 + 149616klp^2-$$
$$- 40008hmp^2 + 54728lmp^2 - 267810gnp^2 - 292836knp^2 - 60354mnp^2-$$
$$- 73408hp^3 + 11520lp^3 - 64232np^3 - 30849g^3q + 252324gh^2q-$$
$$- 49158g^2kq + 703080h^2kq + 73008gk^2q + 128304k^3q + 310056ghlq+$$
$$+ 1324944hklq + 102468gl^2q + 684072kl^2q + 25665g^2mq + 155844h^2mq+$$
$$+ 100188gkmq + 121392k^2mq + 438120hlmq + 292836l^2mq + 15201gm^2q+$$
$$+ 35802km^2q + 3807m^3q - 200160ghnq + 75384hknq - 185496glnq+$$
$$+ 143784klnq + 66744hmnq + 123552lmnq - 64179gn^2q - 35802kn^2q+$$
$$+ 5589mn^2q - 431796g^2pq - 454320h^2pq - 923256gkpq - 530064k^2pq-$$
$$- 550560hlpq - 149616l^2pq - 122928gmpq - 143784kmpq - 15660m^2pq-$$
$$- 351144hnpq - 279240lnpq - 60048n^2pq - 538980gp^2q - 684072kp^2q-$$
$$- 125508mp^2q - 165168p^3q + 7740ghq^2 + 401760hkq^2 - 15084glq^2+$$
$$+ 530064klq^2 + 189540hmq^2 + 330300lmq^2 - 259758gnq^2 - 121392knq^2+$$
$$+ 36774mnq^2 - 713160hpq^2 - 487080lpq^2 - 299772npq^2 - 189432gq^3-$$
$$- 128304kq^3 + 72792mq^3 - 393984pq^3 + 168018g^2hr + 666504ghkr+$$
$$+ 614952hk^2r + 191146g^2lr + 104256h^2lr + 749928gklr + 713160k^2lr+$$
$$+ 173568hl^2r + 73408l^3r + 126756ghmr + 251208hkmr + 184020glmr+$$
$$+ 351144klmr + 22194hm^2r + 31050lm^2r - 219297g^2nr - 36504h^2nr-$$
$$- 443952gknr - 189540k^2nr + 20016hlnr + 40008l^2nr - 93582gmnr-$$
$$- 66744kmnr - 5589m^2nr - 22194hn^2r - 2970ln^2r - 3807n^3r-$$
$$- 49776ghpr + 99936hkpr + 193328glpr + 550560klpr - 20016hmpr+$$
$$+ 146544lmpr - 419316gnpr - 438120knpr - 94068mnpr - 173568hp^2r+$$
$$+ 80320lp^2r - 182280np^2r - 364176g^2qr - 239760h^2qr - 755784gkqr-$$
$$- 401760k^2qr - 99936hlqr + 107184l^2qr - 91080gmqr - 75384kmqr-$$
$$- 648m^2qr - 251208hnqr - 113256lnqr - 58644n^2qr - 1012968gpqr-$$
$$- 1324944kpqr - 254376mpqr - 572976p^2qr - 614952hq^2r - 347400lq^2r-$$
$$- 272052nq^2r - 436752q^3r + 93816ghr^2 + 239760hkr^2 + 197256glr^2+$$
$$+ 454320klr^2 + 36504hmr^2 + 118440lmr^2 - 173106gnr^2 - 155844knr^2-$$
$$- 29970mnr^2 - 104256hpr^2 + 162048lpr^2 - 158040npr^2 - 518724gqr^2-$$
$$- 703080kqr^2 - 139428mqr^2 - 635472pqr^2 + 97344lr^3 - 39096nr^3-$$
$$- 234576qr^3 + 11250g^3s + 200274gh^2s + 96999g^2ks + 518724h^2ks+$$

$$
\begin{aligned}
&+236754gk^2s+189432k^3s+282868ghls+1012968hkls+104322gl^2s+\\
&+538980kl^2s+114479g^2ms+173106h^2ms+308670gkms+259758k^2ms+\\
&+419316hlms+267810l^2ms+34470gm^2s+64179km^2s+1161m^3s-\\
&-98214ghns+91080hkns-107026glns+122928klns+93582hmns+\\
&+125730lmns-34470gn^2s-15201kn^2s+14175mn^2s-83769g^2ps-\\
&-197256h^2ps-98592gkps+15084k^2ps-193328hlps-5928l^2ps+\\
&+107026gmps+185496kmps+23859m^2ps-184020hnps-149092lnps-\\
&-38277n^2ps-104322gp^2s-102468kp^2s+32318mp^2s-18888p^3s+\\
&+159288ghqs+755784hkqs+98592glqs+923256klqs+443952hmqs+\\
&+662136lmqs-308670gnqs-100188knqs+123066mnqs-749928hpqs-\\
&-428040lpqs-390264npqs-236754gq^2s-73008kq^2s+321066mq^2s-\\
&-720900pq^2s-114479g^2rs-93816h^2rs-159288gkrs-7740k^2rs+\\
&+49776hlrs+144360l^2rs+98214gmrs+200160kmrs+35181m^2rs-\\
&-126756hnrs-46452lnrs-33831n^2rs-282868gprs-310056kprs+\\
&+48588mprs-132776p^2rs-666504hqrs-313896lqrs-340704nqrs-\\
&-866556q^2rs-200274gr^2s-252324kr^2s+7470mr^2s-199032pr^2s-\\
&-92952r^3s+114479ghs^2+364176hks^2+83769gls^2+431796kls^2+\\
&+219297hms^2+324699lms^2-114479gns^2-25665kns^2+65376mns^2-\\
&-191146hps^2-40242lps^2-134135nps^2-96999gqs^2+49158kqs^2+\\
&+351561mqs^2-414072pqs^2-168018hrs^2-23034lrs^2-109545nrs^2-\\
&-579516qrs^2-11250gs^3+30849ks^3+121121ms^3-55419ps^3-\\
&-121121rs^3). \qquad (31)
\end{aligned}
$$

By using system (21) with $I_3 \neq 0$, $I_2 \neq 0$ and $I_1 = 0$, the first two Lyapunov quantities have the form:

$$
G_8 = \frac{3I_2^2I_3J_2J_{19}+2I_2I_3^2J_{20}+2I_2^3J_2J_3+7I_3^3J_{43}+4I_2^2J_6J_{20}+2I_2I_3J_6J_{43}}{I_2^4I_3^3}, \qquad (32)
$$

$$
\begin{aligned}
G_{14} = \frac{1}{900I_2^9I_3^6}(&12096I_2^5I_3^3J_1J_2+456288I_2^3I_3^5J_2J_{19}+1872I_2^5I_3^2J_1J_2J_{19}+\\
&+225360I_2^3I_3^4J_2J_{19}^2-12960I_2^5I_3J_1J_2J_{19}^2+43875I_2^3I_3^3J_2J_{19}^3+\\
&+11424I_2^6I_3J_2^3J_{19}+379648I_2^2I_3^6J_{20}-39312I_2^4I_3^3J_1J_{20}+\\
&+335880I_2^2I_3^5J_{19}J_{20}-84240I_2^4I_3^2J_1J_{19}J_{20}+113400I_2^2I_3^4J_{19}^2J_{20}+\\
&+12800I_2^5I_3^2J_2^2J_{20}-46224I_2^5I_3J_2^2J_{19}J_{20}-68256I_2^4I_3^2J_2J_{20}^2+\\
&+9720I_2^4I_3J_2J_{19}J_{20}^2-25920I_2^3I_3^2J_{20}^3+319168I_2^4I_3^4J_2J_3-
\end{aligned}
$$

$$
\begin{aligned}
&- 11232I_2^6I_3J_1J_2J_3 + 155880I_2^4I_3^3J_2J_3J_{19} - 12960I_2^6J_1J_2J_3J_{19}+\\
&+ 60750I_2^4I_3^2J_2J_3J_{19}^2 + 7616I_2^7J_2^3J_3 - 22032I_2^3I_3^4J_3J_{20}-\\
&- 38880I_2^5I_3J_1J_3J_{20} - 84240I_2^3I_3^3J_3J_{19}J_{20} - 34560I_2^6J_2^2J_3J_{20}+\\
&+ 32400I_2^5J_2J_3J_{20}^2 - 10800I_2^5I_3^2J_2J_3^2 + 40500I_2^5I_3J_2J_3^2J_{19}-\\
&- 51840I_2^4I_3^2J_3^2J_{20} + 16200I_2^6J_2J_3^3 + 13824I2^3I_3^5J_4+\\
&+ 79560I_2^3I_3^4J_4J_{19} + 89100I_2^3I_3^3J_4J_{19}^2 + 3456I_2^6I_3J_2^2J_4+\\
&+ 3744I_2^6J_2^2J_4J_{19} - 22464I_2^5I_3J_2J_4J_{20} - 25920I_2^5J_{19}J_2J_4J_{20}-\\
&- 38880I_2^4I_3J_4J_{20}^2 + 39312I_2^4I_3^3J_3J_4 + 84240I_2^4I_3^2J_3J_4J_{19}+\\
&+ 19440I_2^5I_3J_3^2J_4 + 1126592I_2I_3^7J_{43} - 43992I_2^3I_3^4J_1J_{43}+\\
&+ 459540I_2I_3^6J_{19}J_{43} - 132840I_2^3I_3^3J_1J_{19}J_{43} - 182250I_2I_3^5J_{19}^2J_{43}+\\
&+ 25792I_2^4I_3^3J_2^2J_{43} + 3744I_2^6J_1J_2^2J_{43} - 20304I_2^4I_3^2J_2^2J_{19}J_{43}-\\
&- 54000I_2^3I_3^3J_2J_{20}J_{43} - 25920I_2^5J_1J_2J_{20}J_{43} + 87480I_2^3I_3^2J_2J_{19}J_{20}J_{43}-\\
&- 174960I_2^2I_3^3J_{20}^2J_{43} - 180072I_2^2I_3^5J_3J_{43} - 71280I_2^4I_3^2J_1J_3J_{43}-\\
&- 473040I_2^2I_3^4J_3J_{19}J_{43} - 7920I_2^5I_3J_2^2J_3J_{43} + 6480I_2^4I_3J_2J_3J_{20}J_{43}-\\
&- 165240I_2^3I_3^3J_3^2J_{43} + 41184I_2^4I_3^2J_2J_4J_{43} + 51840I_2^4I_3J_2J_4J_{19}J_{43}-\\
&- 142560I_2^3I_3^2J_{20}J_4J_{43} + 25920I_2^5J_2J_3J_4J_{43} - 64224I_2^2I_3^4J_2J_{43}^2-\\
&- 25920I_2^4I_3J_1J_2J_{43}^2 - 39690I_2^2I_3^3J_2J_{19}J_{43}^2 - 220320I_2I_3^4J_{20}J_{43}^2-\\
&- 50220I_2^3I_3^2J_2J_3J_{43}^2 - 74520I_2^2I_3^3J_4J_{43}^2 + 23220I_3^5J_{43}^3 - 6912I_2^6I_3J_1J_2J_6+\\
&+ 14208I_2^4I_3^3J_2J_6J_{19} - 7488I_2^6J_1J_2J_6J_{19} + 68400I_2^4I_3^2J_2J_6J_{19}^2+\\
&+ 629376I_2^3I_3^4J_6J_{20} + 22464I_2^5I_3J_1J_6J_{20} + 364176I_2^3I_3^3J_6J_{19}J_{20}+\\
&+ 25920I_2^5J_1J_6J_{19}J_{20} + 66420I_2^3I_3^2J_6J_{19}^2J_{20} + 15232I_2^6J_2^2J_6J_{20}-\\
&- 69120I_2^5J_2J_6J_{20}^2 + 64800I_2^4J_6J_{20}^3 - 2048I_2^5I_3^2J_2J_3J_6+\\
&+ 84960I_2^5I_3J_{19}J_2J_3J_6 + 91584I_2^4I_3^2J_3J_6J_{20} + 123120I_2^4I_3J_3J_6J_{19}J_{20}+\\
&+ 34560I_2^6J_2J_3^2J_6 + 32400I_2^5J_3^2J_6J_{20} + 24192I_2^4I_3^3J_4J_6+\\
&+ 3744I_2^4I_3^2J_4J_6J_{19} - 25920I_2^4I_3J_4J_6J_{19}^2 - 22464I_2^5I_3J_3J_4J_6-\\
&- 25920I_2^5J_{19}J_3J_4J_6 + 360384I_2^2I_3^5J_6J_{43} - 41184I_2^4I_3^2J_1J_6J_{43}+\\
&+ 284256I_2^2I_3^4J_6J_{19}J_{43} - 51840I_2^4I_3J_1J_6J_{19}J_{43} - 2430I_2^2I_3^3J_6J_{19}^2J_{43}+\\
&+ 7616I_2^5I_3J_2^2J_6J_{43} - 46656I_2^4I_3J_2J_6J_{20}J_{43} + 19440I_2^3I_3J_6J_{20}^2J_{43}+\\
&+ 1584I_2^3I_3^3J_3J_6J_{43} - 25920I_2^5J_1J_3J_6J_{43} - 119880I_2^3I_3^2J_{19}J_3J_6J_{43}-\\
&- 48600I_2^4I_3J_3^2J_6J_{43} + 14976I_2^5J_2J_4J_6J_{43} - 51840I_2^4J_{20}J_4J_6J_{43}-\\
&- 13536I_2^3I_3^2J_2J_6J_{43}^2 - 42120I_2^2I_3^2J_6J_{20}J_{43}^2 - 51840I_2^3I_3J_4J_6J_{43}^2-\\
&- 26460I_2I_3^3J_6J_{43}^3 + 22848I_2^5I_3J_2J_6^2J_{19} + 35328I_2^4I_3^2J_6^2J_{20}+\\
&+ 92448I_2^4I_3J_6^2J_{19}J_{20} + 15232I_2^6J_2J_3J_6^2 + 69120I_2^5J_3J_6^2J_{20}-\\
&- 6912I_2^5I_3J_4J_6^2 - 7488I_2^5J_4J_6^2J_{19} + 63360I_2^3I_3^3J_6^2J_{43}-
\end{aligned}
$$

$$- 7488 I_2^5 J_1 J_6^2 J_{43} + 40608 I_2^3 I_3^2 J_6^2 J_{19} J_{43} + 15840 I_2^4 I_3 J_3 J_6^2 J_{43} + \\ + 30464 I_2^5 J_6^3 J_{20} + 15232 I_2^4 I_3 J_6^3 J_{43}). \tag{33}$$

We conclude that the number of terms in expressions (32) (6 terms) and (33) (126 terms) is less than the number of terms in expressions (30) (20 terms) and (31) (346 terms), respectively. Moreover, the expressions (32) and (33) given via invariants, can be used for any system of the form (1) with $I_3 \neq 0$, $I_2 \neq 0$ and $I_1 = 0$, while the expressions of the form (30) and (31) can be used only for system of the form (29).

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References

[1] Sibirsky K. S. *Introduction to the Algebraic Theory of Invariants of Differential Equations.* Manchester University Press, 1988.

[2] Vulpe N. I. *Polynomial bases of comitants of differential systems and their applications in qualitative theory*, Kishinev, Ştiinţa, 1986 (in Russian).

[3] Gurevich G. B. *Foundations of the Theory of Algebraic Invariants.* Noordhoff, Groningen, 1964.

[4] Boularas D., Calin Iu., Timochouk L., Vulpe N. *T-comitants of quadratic systems: A study via the translation invariants*, Report 96-90, Delft University of Technology, Faculty of Technical Mathematics and Informatics, 1996, 1–36.

[5] Calin Iu. *On rational bases of $GL(2,\mathbb{R})$-comitants of planar polynomial systems of differential equations.* Bul. Acad. Ştiinţe Repub. Moldova. Mat., 2003, No. 2(42), 69–86.

[6] Amelkin V. V., Lucashevich N. A, Sadovski A. P. *Nonlinear variation in systems of the second order.* Minsk, 1982 (in Russian).

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Certain classes of p-valent analytic functions with negative coefficients and (λ, p)-starlike with respect to certain points

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Abstract. In this article, we consider classes $S^*_{s,\lambda}(p,\alpha,\beta)$, $S^*_{c,\lambda}(p,\alpha,\beta)$, and $S^*_{sc,\lambda}(p,\alpha,\beta)$ of p-valent analytic functions with negative coefficients in the unit disk. They are, respectively, (λ,p)-starlike with respect to symmetric points, (λ,p)-starlike with respect to conjugate points, and (λ,p)-starlike with respect to symmetric conjugate points. Necessary and sufficient coefficient conditions for functions f belonging to these classes are obtained. Several properties such as the coefficient estimates, growth and distortion theorems, extreme points, radii of starlikeness, convexity, and integral operator are studied.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the punctured unit disk $\mathbb{U} = \{z : |z| < 1\}$. For f which belong to A, Salagean [1] introduced the following operator:

$$D^0 f(z) = f(z), \quad D^1 f(z) = Df(z) = zf'(z),$$

and

$$D^n f(z) = D\left(D^{n-1} f(z)\right) \quad (n \in \mathbb{N} = \{1, 2, 3, ...\}). \tag{2}$$

Note that

$$D^n f(z) = D\left(D^{n-1} f(z)\right) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad (n \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}). \tag{3}$$

Let $\mathcal{T}_p$ (p a fixed integer greater than 0) denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \tag{4}$$

that are holomorphic and p-valent in the unit disk $|z| < 1$.

Also let $\mathcal{T}_p$ denote the subclass of $\mathcal{S}_p$ consisting of functions that can be expressed in the form

$$f(z) = z^p - \sum_{k=1}^{\infty} |a_{k+p}| z^{k+p} \tag{5}$$

where $a_{k+p} \geq 0,\ p \in \mathbb{N} = \{1, 2, 3, ...\}\ ,\ n \in \mathbb{N}$, which are analytic in the unit disc $\mathbb{U}$.

We can write the following equalities for the functions $f(z)$ which belong to the class $\mathcal{T}_p$ (see [2]):

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= Df(z) = zf'(z) \\ &= z\left(pz^p - \sum_{k=1}^{\infty} (k+p) a_{k+p} z^{k+p-1}\right) \\ &= pz^p - \sum_{k=1}^{\infty} (k+p) a_{k+p} z^{k+p}, \\ D^2 f(z) &= D(Df(z)) = p^2 z^p - \sum_{k=1}^{\infty} (k+p)^2 a_{k+p} z^{k+p}, \\ &\cdots \\ D^\lambda f(z) &= D\left(D^{\lambda-1} f(z)\right) = p^\lambda z^p - \sum_{k=1}^{\infty} (k+p)^\lambda a_{k+p} z^{k+p}. \end{aligned} \tag{6}$$

Let S be the subclass of A consisting of functions that are regular and univalent in $\mathbb{U}$. Let S^* be the subclass of S consisting of functions starlike in $\mathbb{U}$. It is known that

$$f \in S^* \text{ if and only if } \Re\left\{\tfrac{zf'(z)}{f(z)}\right\} > 0, \quad (z \in \mathbb{U}).$$

In [3], Sakaguchi defined the class of starlike functions with respect to symmetric points as follows:

Let $f \in S$. Then f is said to be starlike with respect to symmetric points in $\mathbb{U}$ if and only if

$$\Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0, \quad (z \in \mathbb{U}), \tag{7}$$

and we denote this class by S_s^*. Obviously, it forms a subclass of close-to-convex functions and hence univalent. Moreover, this class includes the class of convex functions and odd starlike functions with respect to the origin, see Sakaguchi [3], Robertson [5], Stankiewicz [6], Wu [7] and Owa et al. [8]. El-Ashwah and Thomas in [9] introduced two other classes, namely the class S_c^* consisting of functions starlike with respect to conjugate points and S_{sc}^* consisting of functions starlike with

respect to symmetric conjugate points. The class S^*_{sc} is also studied by Chen et al. [10] (see also [11]).

In [4], Sudharsan et al. introduced the class $S^*_s(\alpha,\beta)$ of functions $f(z)\in S$ and satisfying the following condition (see also [12]):

$$\left|\frac{zf'(z)}{f(z)-f(-z)}-1\right|<\beta\left|\alpha\ \frac{zf'(z)}{f(z)-f(-z)}+1\right| \tag{8}$$

for some $0\le\alpha\le 1,\ 0<\beta\le 1,\ z\in\mathbb{U}$.

Recently, Aouf el at.[13] introduced the class $S^*_{s,n}\mathcal{T}(\alpha,\beta)$ of functions $f(z)$ being defined by (5). Then $f(z)$ is said to be n-starlike with respect to symmetric points if it satisfies the following condition:

$$\left|\frac{D^{n+1}f(z)}{D^nf(z)-D^nf(-z)}-1\right|<\beta\left|\alpha\ \frac{D^{n+1}f(z)}{D^nf(z)-D^nf(-z)}+1\right|, \tag{9}$$

where $n\in\mathbb{N}_0=\{0\}\cup\mathbb{N}, 0\le\alpha\le 1,\ 0<\beta\le 1,\ 0\le\frac{2(1-\beta)}{1+\alpha\beta}$, and $z\in\mathbb{U}$.

However, in this paper we consider the subclass $\mathcal{T}$ defined by (5).

Definition 1. Let a function $f(z)$ be defined by (5). Then $f(z)$ is said to be (λ,p)-starlike with respect to symmetric points if it satisfies the following condition:

$$\left|\frac{D^{\lambda+1}f(z)}{D^\lambda f(z)-D^\lambda f(-z)}-p\right|<\beta\left|\alpha\ \frac{D^{\lambda+1}f(z)}{D^\lambda f(z)-D^\lambda f(-z)}+p\right|, \tag{10}$$

where $\lambda\in\mathbb{N}_0=\{0\}\cup\mathbb{N},\ p\in\mathbb{N},\ 0\le\alpha\le 1,\ 0<\beta\le 1,\ 0\le\frac{2p(1-\beta)}{1+\alpha\beta}$, and $z\in\mathbb{U}$. We denote the class of functions (λ,p)-starlike with respect to symmetric points by $S^*_{s,\lambda}(p,\alpha,\beta)$.

Definition 2. Let a function $f(z)$ be defined by (5). Then $f(z)$ is said to be (λ,p)-starlike with respect to conjugate points if it satisfies the following condition:

$$\left|\frac{D^{\lambda+1}f(z)}{D^\lambda f(z)+\overline{D^\lambda f(\overline{z})}}-p\right|<\beta\left|\alpha\ \frac{D^{\lambda+1}f(z)}{D^\lambda f(z)+\overline{D^\lambda f(\overline{z})}}+p\right|, \tag{11}$$

where $\lambda\in\mathbb{N}_0=\{0\}\cup\mathbb{N},\ p\in\mathbb{N},\ 0\le\alpha\le 1,\ 0<\beta\le 1,\ 0\le\frac{2p(1-\beta)}{1+\alpha\beta}$, and $z\in\mathbb{U}$. We denote the class of functions (λ,p)-starlike with respect to conjugate points by $S^*_{c,\lambda}(p,\alpha,\beta)$.

Definition 3. Let a function $f(z)$ be defined by (5). Then $f(z)$ is said to be (λ,p)-starlike with respect to symmetric conjugate points if it satisfies the following condition:

$$\left|\frac{D^{\lambda+1}f(z)}{D^\lambda f(z)-\overline{D^\lambda f(\overline{-z})}}-p\right|<\beta\left|\alpha\ \frac{D^{\lambda+1}f(z)}{D^\lambda f(z)-\overline{D^\lambda f(\overline{-z})}}+p\right|, \tag{12}$$

where $\lambda \in \mathbb{N}_0 = \{0\} \cup \mathbb{N},\ p \in \mathbb{N},\ 0 \leq \alpha \leq 1,\ 0 < \beta \leq 1,\ 0 \leq \frac{2p(1-\beta)}{1+\alpha\beta}$, and $z \in \mathbb{U}$. We denote the class of functions (λ, p)-starlike with respect to symmetric conjugate points by $S^*_{sc,\lambda}(p, \alpha, \beta)$.

Notice that the above conditions imposed on $\alpha,\ \beta$ and p in the introduction are necessary to ensure that these classes form a subclass of S. For more classes (see in details Halim et al. [14, 15].

2 Coefficient estimates

To prove the following theorems, we will adopt the technique used by Dziok [16], and assume that $\lambda \in \mathbb{N}_0 = \{0\} \cup \mathbb{N},\ p \in \mathbb{N},\ 0 \leq \alpha \leq 1,\ 0 < \beta \leq 1,\ 0 \leq \frac{2p(1-\beta)}{1+\alpha\beta}$, and $z \in \mathbb{U}$.

Theorem 1. *Let the function $f(z)$ be defined by (5) and $D^\lambda f(z) - D^\lambda f(-z) \neq 0$ for $z \neq 0$. Then $f(z) \in S^*_{s,\lambda}(p, \alpha, \beta)$ if and only if*

$$\sum_{k=1}^{\infty} \left[k(1+\alpha\beta) + p\left((-1)^{p+k} + \beta(\alpha + 1 - (-1)^{p+k})\right)\right] (k+p)^{\lambda+1} |a_{k+p}|$$

$$\leq p^{\lambda+1} \left[(-1)^p + \beta(\alpha + 1 - (-1)^p)\right]. \tag{13}$$

Proof. Use (5), (6) and (10), that is

$$\left|\frac{D^{\lambda+1} f(z)}{D^\lambda f(z) - D^\lambda f(-z)} - p\right| < \beta \left|\alpha\ \frac{D^{\lambda+1} f(z)}{D^\lambda f(z) - D^\lambda f(-z)} + p\right|$$

$$= \left|(-1)^p p^{\lambda+1} z^p - \sum_{k=1}^{\infty} \left[k + p(-1)^{p+k}\right] (k+p)^\lambda |a_{k+p}| z^{k+p}\right|$$

$$< \beta \left|(\alpha + 1 - (-1)^p) p^{\lambda+1} z^p - \sum_{k=1}^{\infty} [\alpha k + p(\alpha + 1 - (-1)^{p+k})] |a_{k+p}| z^{k+p}\right|$$

and

$$-p^{\lambda+1} \left[(-1)^p + \beta(\alpha + 1 - (-1)^p)\right] |z|^p$$

$$+ \sum_{k=1}^{\infty} \left[k + p(-1)^{p+k} + \beta\{\alpha k + p(\alpha + 1 - (-1)^{p+k})\}\right] (k+p)^\lambda |a_{k+p}| |z|^{k+p} \leq 0.$$

Letting $|z| = 1$, we have

$$\sum_{k=1}^{\infty} \left[k + p(-1)^{p+k} + \beta\{\alpha k + p(\alpha + 1 - (-1)^{p+k})\}\right] (k+p)^\lambda |a_{k+p}| \leq 0.$$

$$-p^{\lambda+1} \left[(-1)^p + \beta(\alpha + 1 - (-1)^p)\right] \leq 0.$$

Therefore, by the maximum modulus theorem, we have $f(z) \in S^*_{s,\lambda}(p,\alpha,\beta)$.

For the converse, let us suppose that

$$\frac{\left|\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)-D^{\lambda}f(-z)}-p\right|}{\left|\alpha\ \frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)-D^{\lambda}f(-z)}+p\right|}<\beta.$$

This implies that

$$\left|\frac{-\left[-(-1)^p p^{\lambda+1}z^p+\sum_{k=1}^{\infty}\left[k+p(-1)^{p+k}\right](k+p)^{\lambda}|a_{k+p}|z^{k+p}\right]}{(\alpha+1-(-1)^p)p^{\lambda+1}z^p-\sum_{k=1}^{\infty}[\alpha k+p(\alpha+1-(-1)^{p+k})]|a_{k+p}|z^{k+p}}\right|<\beta.$$

Since $|\Re(z)|\leq|z|$ for all z, we have

$$\Re\left\{\frac{-(-1)^p p^{\lambda+1}z^p+\sum_{k=1}^{\infty}\left[k+p(-1)^{p+k}\right](k+p)^{\lambda}|a_{k+p}||z|^{k+p}}{(\alpha+1-(-1)^p)p^{\lambda+1}z^p-\sum_{k=1}^{\infty}[\alpha k+p(\alpha+1-(-1)^{p+k})]|a_{k+p}||z|^{k+p}}\right\}<\beta. \tag{14}$$

If we choose values of z on the real axis, then $\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)-D^{\lambda}f(-z)}$ is real and $D^{\lambda}f(z)-D^{\lambda}f(-z)\neq 0$ for $z\neq 0$. Upon clearing the denominator in (14) and letting $z\to 1^-$ through real values, we obtain

$$\sum_{k=1}^{\infty}\left[k+p(-1)^{p+k}\right](k+p)^{\lambda}|a_{k+p}|+\sum_{k=1}^{\infty}\beta[\alpha k+p(\alpha+1-(-1)^{p+k})](k+p)^{\lambda}|a_{k+p}|$$
$$\leq\beta(\alpha+1-(-1)^p)p^{\lambda+1}+(-1)^p p^{\lambda+1}.$$

This gives the required condition.

Corollary 1. *Let the function $f(z)$ defined by (5) be in the class $S^*_{s,\lambda}(p,\alpha,\beta)$. Then we have*

$$|a_{k+p}|\leq\frac{p^{\lambda+1}\left[(-1)^p+\beta(\alpha+1-(-1)^p)\right]}{\left[k(1+\alpha\beta)+p\left((-1)^{p+k}+\beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}}\quad(k\geq 1), \tag{15}$$

where $p\in\mathbb{N},\ \lambda\in\mathbb{N}_0$ and $z\in\mathbb{U}$.

The equality in (15) is attained for the function $f(z)$ given by

$$f_k(z)=z^p-\frac{p^{\lambda+1}\left[(-1)^p+\beta(\alpha+1-(-1)^p)\right]}{\left[k(1+\alpha\beta)+p\left((-1)^{p+k}+\beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}}z^{k+p}\quad(k\geq 1), \tag{16}$$

where $p\in\mathbb{N},\ \lambda\in\mathbb{N}_0$ and $z\in\mathbb{U}$.

Theorem 2. *Let the function $f(z)$ be defined by (5) and $D^{\lambda}f(z)-D^{\lambda}f(-z)\neq 0$ for $z\neq 0$. Then $f(z)\in S^*_{c,\lambda}(p,\alpha,\beta)$ if and only if*

$$\sum_{k=1}^{\infty}\left[k(1+\alpha\beta)+p\left(\beta(\alpha+2)-1\right)\right](k+p)^{\lambda+1}|a_{k+p}|$$
$$\leq p^{\lambda+1}\left[(\beta(\alpha+2)-1)\right]. \tag{17}$$

Corollary 2. *Let the function $f(z)$ defined by (5) be in the class $S^*_{c,\lambda}(p,\alpha,\beta)$. Then we have*

$$|a_{k+p}| \leq \frac{p^{\lambda+1}\left[(\beta(\alpha+2)-1)\right]}{\left[k(1+\alpha\beta)+(\beta(\alpha+2)-1)\right](k+p)^{\lambda+1}} \quad (k \geq 1), \tag{18}$$

where $p \in \mathbb{N},\ \lambda \in \mathbb{N}_0$ and $z \in \mathbb{U}$.

The equality in (18) is attained for the function $f(z)$ given by

$$f_k(z) = z^p - \frac{p^{\lambda+1}\left[(\beta(\alpha+2)-1)\right]}{\left[k(1+\alpha\beta)+(\beta(\alpha+2)-1)\right](k+p)^{\lambda+1}} z^{k+p} \quad (k \geq 1), \tag{19}$$

where $p \in \mathbb{N},\ \lambda \in \mathbb{N}_0$ and $z \in \mathbb{U}$.

Theorem 3. *Let the function $f(z)$ be defined by (5) and $D^\lambda f(z) - D^\lambda f(-z) \neq 0$ for $z \neq 0$. Then $f(z) \in S^*_{sc,\lambda}(p,\alpha,\beta)$ if and only if*

$$\sum_{k=1}^{\infty}\left[k(1+\alpha\beta)+p\left(\beta(1+\alpha)+(1-\beta)(-1)^{p+k}\right)\right](k+p)^{\lambda+1}|a_{k+p}|$$
$$\leq p^{\lambda+1}\left[(-1)^p+\beta(\alpha+1-(-1)^p)\right]. \tag{20}$$

Corollary 3. *Let the function $f(z)$ defined by (5) be in the class $S^*_{sc,\lambda}(p,\alpha,\beta)$. Then we have*

$$|a_{k+p}| \leq \frac{p^{\lambda+1}\left[(-1)^p+\beta(\alpha+1-(-1)^p)\right]}{\left[k(1+\alpha\beta)+p\left(\beta(1+\alpha)+(1-\beta)(-1)^{p+k}\right)\right](k+p)^{\lambda+1}} \quad (k \geq 1), \tag{21}$$

where $p \in \mathbb{N},\ \lambda \in \mathbb{N}_0$ and $z \in \mathbb{U}$.

The equality in (21) is attained for the function $f(z)$ given by

$$f_k(z) = z^p - \frac{p^{\lambda+1}\left[(-1)^p+\beta(\alpha+1-(-1)^p)\right]}{\left[k(1+\alpha\beta)+p\left(\beta(1+\alpha)+(1-\beta)(-1)^{p+k}\right)\right](k+p)^{\lambda+1}} z^{k+p} \quad (k \geq 1), \tag{22}$$

where $p \in \mathbb{N},\ \lambda \in \mathbb{N}_0$ and $z \in \mathbb{U}$.

3 Growth and Distortion theorems

Theorem 4. *Let the function $f(z)$ defined by (5) be in the class $f(z) \in S^*_{s,\lambda}(p,\alpha,\beta)$. Then we have*

$$p^i|z|^p - \frac{p^{\lambda+1}\left[(-1)^p+\beta(\alpha+1-(-1)^p)\right]}{\left[(1+\alpha\beta)+p\left((-1)^{p+1}+\beta(\alpha+1-(-1)^{p+1})\right)\right](p+1)^{\lambda-i}}|z|^{p+1}$$
$$\leq \left|D^i f(z)\right|$$
$$\leq p^i|z|^p + \frac{p^{\lambda+1}\left[(-1)^p+\beta(\alpha+1-(-1)^p)\right]}{\left[(1+\alpha\beta)+p\left((-1)^{p+1}+\beta(\alpha+1-(-1)^{p+1})\right)\right](p+1)^{\lambda-i}}|z|^{p+1} \tag{23}$$

for $z \in \mathbb{U}$, where $0 \leq i \leq n$. The result is sharp.

Proof. Note that $f(z) \in S^*_{s,\lambda}(p,\alpha,\beta)$ if and only if $D^i f(z) \in S^*_{s,\lambda-i}(p,\alpha,\beta)$, and that

$$D^i f(z) = p^i z^p - \sum_{k=1}^{\infty} (k+p)^i |a_{k+p}| z^{k+p}. \tag{24}$$

Using Theorem 1, we know that

$$\sum_{k=1}^{\infty} (k+p)^i |a_{k+p}| \leq \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right](p+1)^i}{\left[(1+\alpha\beta) + p\left((-1)^{p+1} + \beta(\alpha+1-(-1)^{p+1})\right)\right](p+1)^{\lambda}}. \tag{25}$$

That is

$$\sum_{k=1}^{\infty} (k+p)^i |a_{k+p}| \leq \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[(1+\alpha\beta) + p\left((-1)^{p+1} + \beta(\alpha+1-(-1)^{p+1})\right)\right](p+1)^{\lambda-i}}. \tag{26}$$

It follows from (24) and (26) that

$$\begin{aligned} \left|D^i f(z)\right| &\geq p^i |z|^p - |z|^{p+1} \sum_{k=1}^{\infty} (k+p)^i |a_{k+p}| \\ &\geq p^i |z|^p - \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[(1+\alpha\beta) + p\left((-1)^{p+1} + \beta(\alpha+1-(-1)^{p+2})\right)\right](p+1)^{\lambda-i}} |z|^{p+1}. \end{aligned} \tag{27}$$

Also

$$\begin{aligned} \left|D^i f(z)\right| &\leq p^i |z|^p + |z|^{p+1} \sum_{k=1}^{\infty} (k+p)^i |a_{k+p}| z^{k+p} \\ &\leq p^i |z|^p + \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[(1+\alpha\beta) + p\left((-1)^{p+1} + \beta(\alpha+1-(-1)^{p+1})\right)\right](p+1)^{\lambda-i}} |z|^{p+1}. \end{aligned} \tag{28}$$

Finally, we note that the equality in (23) is attained by the function

$$D^i f(z) = p^i z^p - \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[(1+\alpha\beta) + p\left((-1)^{p+1} + \beta(\alpha+1-(-1)^{p+1})\right)\right](p+1)^{\lambda-i}} z^{p+1} \tag{29}$$

or by

$$f(z) = z^p - \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[(1+\alpha\beta) + p\left((-1)^{p+1} + \beta(\alpha+1-(-1)^{p+1})\right)\right](p+1)^{\lambda}} z^{p+1} \tag{30}$$

and this completes the proof of Theorem 4.

Corollary 4. *Let the function $f(z)$ defined by (5) be in the class $f(z) \in S^*_{s,\lambda}(p,\alpha,\beta)$. Then we have*

$$|z|^p - \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[(1+\alpha\beta) + p\left((-1)^{p+1} + \beta(\alpha+1-(-1)^{p+1})\right)\right](p+1)^{\lambda}}|z|^{p+1}$$
$$\leq |f(z)|$$
$$\leq |z|^p + \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[(1+\alpha\beta) + p\left((-1)^{p+1} + \beta(\alpha+1-(-1)^{p+1})\right)\right](p+1)^{\lambda}}|z|^{p+1} \tag{31}$$

for $z \in \mathbb{U}$. The result is sharp for the function $f(z)$ given by (30).

Proof. For $i = 0$ in Theorem 4, we can easily show (30).

Similarly, we can prove the following result.

Theorem 5. *Let the function $f(z)$ defined by (5) be in the class $f(z) \in S^*_{c,\lambda}(p,\alpha,\beta)$. Then we have*

$$p^i|z|^p - \frac{p^{\lambda+1}\left[(\beta(\alpha+2)-1)\right]}{\left[(1+\alpha\beta) + p\left(\beta(\alpha+2)-1\right)\right](p+1)^{\lambda-i}}|z|^{p+1}$$
$$\leq \left|D^i f(z)\right|$$
$$\leq p^i|z|^p + \frac{p^{\lambda+1}\left[(\beta(\alpha+2)-1)\right]}{\left[(1+\alpha\beta) + p\left(\beta(\alpha+1)-1\right)\right](p+1)^{\lambda-i}}|z|^{p+1} \tag{32}$$

for $z \in \mathbb{U}$, where $0 \leq i \leq n$. The result is sharp, for the function $f(z)$ given by

$$D^i f(z) = p^i z^p - \frac{p^{\lambda+1}\left[(\beta(\alpha+2)-1)\right]}{\left[(1+\alpha\beta) + p\left(\beta(\alpha+2)-1\right)\right](p+1)^{\lambda-i}}z^{p+1} \tag{33}$$

or by

$$f(z) = z^p - \frac{p^{\lambda+1}\left[(\beta(\alpha+2)-1)\right]}{\left[(1+\alpha\beta) + p\left(\beta(\alpha+2)-1\right)\right](p+1)^{\lambda}}z^{p+1}. \tag{34}$$

Corollary 5. *Let the function $f(z)$ defined by (5) be in the class $f(z) \in S^*_{c,\lambda}(p,\alpha,\beta)$. Then we have*

$$|z|^p - \frac{p^{\lambda+1}\left[(\beta(\alpha+2)-1)\right]}{\left[(1+\alpha\beta) + p\left(\beta(\alpha+2)-1\right)\right](p+1)^{\lambda}}|z|^{p+1}$$
$$\leq |f(z)|$$
$$\leq |z|^p + \frac{p^{\lambda+1}\left[(\beta(\alpha+2)-1)\right]}{\left[(1+\alpha\beta) + p\left(\beta(\alpha+2)-1\right)\right](p+1)^{\lambda}}|z|^{p+1} \tag{35}$$

for $z \in \mathbb{U}$. The result is sharp, for the function $f(z)$ given by (34).

Theorem 6. *Let the function $f(z)$ defined by (5) be in the class $f(z) \in S^*_{sc,\lambda}(p,\alpha,\beta)$. Then we have*

$$p^i|z|^p - \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[(1+\alpha\beta) + p\left(\beta(1+\alpha) + (1-\beta)(-1)^{p+1}\right)\right](p+1)^{\lambda-i}}|z|^{p+1}$$
$$\leq \left|D^i f(z)\right|$$
$$\leq p^i|z|^p + \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[(1+\alpha\beta) + p\left(\beta(1+\alpha) + (1-\beta)(-1)^{p+1}\right)\right](p+1)^{\lambda-i}}|z|^{p+1} \tag{36}$$

for $z \in \mathbb{U}$, where $0 \leq i \leq n$. The result is sharp.

4 Extreme points

Theorem 7. *Let $f_p(z) = z^p$ and*

$$f_{k+p}(z) = z^p - \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[k(1+\alpha\beta) + p\left((-1)^{p+k} + \beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}} z^{k+p}, \tag{37}$$

where $k \geq 1$. Then $f(z) \in S^*_{s,\lambda}(p,\alpha,\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=0}^{\infty} \sigma_{k+p} f_{k+p}(z), \tag{38}$$

where $\sigma_{k+p} \geq 0$ $(k \geq 1)$ and $\sum_{k=0}^{\infty} \sigma_{k+p} = 1$.

Proof. Suppose

$$f(z) = \sum_{k=0}^{\infty} \sigma_{k+p} f_{k+p}(z)$$
$$= z^p - \sum_{k=1}^{\infty} \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[k(1+\alpha\beta) + p\left((-1)^{p+k} + \beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}} \sigma_{k+p} z^{k+p}. \tag{39}$$

Then we get

$$\sum_{k=1}^{\infty} \frac{\left[k(1+\alpha\beta) + p\left((-1)^{p+k} + \beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}}{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}$$
$$\bullet \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[k(1+\alpha\beta) + p\left((-1)^{p+k} + \beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}} \sigma_{k+p}$$
$$= \sum_{k=1}^{\infty} \sigma_{k+p} = 1 - \sigma_p \leq 1. \tag{40}$$

It follows from Theorem 1 that the function $f(z) \in S^*_{s,\lambda}(p,\alpha,\beta)$.

Conversely, suppose that $f(z) \in S^*_{s,\lambda}(p,\alpha,\beta)$. Again, by using Theorem 1, we can show that

$$|a_{p+k}| \leq \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[k(1+\alpha\beta) + p\left((-1)^{p+k} + \beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}} \ k \geq 1, \quad (41)$$

where $k \geq 1,\ \lambda \in \mathbb{N}_0 = \{0\} \cup \mathbb{N},\ p \in \mathbb{N},\ 0 \leq \alpha \leq 1,\ 0 < \beta \leq 1$ and $z \in \mathbb{U}$.

Setting

$$\sigma_{p+k} \leq \frac{\left[k(1+\alpha\beta) + p\left((-1)^{p+k} + \beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}}{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]} \ (k \geq 1) \quad (42)$$

and

$$\sigma_p = 1 - \sum_{k=1}^{\infty} \sigma_{k+p}, \quad (43)$$

we can see that $f(z)$ can be expressed in the form (38). This completes the proof of Theorem 7.

Corollary 6. *The extreme points of the class $S^*_{s,\lambda}(p,\alpha,\beta)$ are functions $f_{k+p}(z)$ $(k \geq 1,\ p \in \mathbb{N})$ given by Theorem 7.*

Similar to Theorem 7, we can easily prove the following theorems for $S^*_{c,\lambda}(p,\alpha,\beta)$ and $S^*_{sc,\lambda}(p,\alpha,\beta)$ classes.

Theorem 8. *Let $f_p(z) = z^p$ and*

$$f_{k+p}(z) = z^p - \frac{p^{\lambda+1}\left[(\beta(\alpha+2)-1)\right]}{\left[k(1+\alpha\beta) + (\beta(\alpha+2)-1)\right](k+p)^{\lambda+1}} z^{k+p} \ (k \geq 1), \quad (44)$$

where $p \in \mathbb{N},\ \lambda \in \mathbb{N}_0$ and $z \in \mathbb{U}$. Then $f(z) \in S^*_{c,\lambda}(p,\alpha,\beta)$ if and only if it can be expressed in the form $f(z) = \sum_{k=1}^{\infty} \sigma_{k+p} f_{k+p}(z)$, where $\sigma_{k+p} \geq 0$ $(k \geq 1)$ and $\sum_{k=1}^{\infty} \sigma_{k+p} = 1$.

Corollary 7. *The extreme points of the class $S^*_{c,\lambda}(p,\alpha,\beta)$ are functions $f_{k+p}(z)$ $(k \geq 1,\ p \in \mathbb{N})$ given by Theorem 8.*

Theorem 9. *Let $f_p(z) = z^p$ and*

$$f_{k+p}(z) = z^p - \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[k(1+\alpha\beta) + p\left(\beta(1+\alpha) + (1-\beta)(-1)^{p+k}\right)\right](k+p)^{\lambda+1}} z^{k+p} \ (k \geq 1), \quad (45)$$

where $p \in \mathbb{N},\ \lambda \in \mathbb{N}_0$ and $z \in \mathbb{U}$. Then $f(z) \in S^*_{sc,\lambda}(p,\alpha,\beta)$ if and only if it can be expressed in the form $f(z) = \sum_{k=1}^{\infty} \sigma_{k+p} f_{k+p}(z)$, where $\sigma_{k+p} \geq 0$ $(k \geq 1)$ and $\sum_{k=1}^{\infty} \sigma_{k+p} = 1$.

Corollary 8. *The extreme points of the class $S^*_{sc,\lambda}(p,\alpha,\beta)$ are functions $f_{k+p}(z)$ $(k \geq 1,\ p \in \mathbb{N})$ given by Theorem 9.*

Theorem 10. *The class $S^*_{s,\lambda}(p,\alpha,\beta)$ is closed under convex linear combination.*

Proof. Let us suppose that the functions $f_1(z)$ and $f_2(z)$ defined by

$$f_j(z) = z^p - \sum_{k=1}^{\infty} |a_{p+k,j}| z^{p+k} \ (a_{p+k,j} \geq 0,\ j = 1,2,\ z \in \mathbb{U}) \tag{46}$$

are in the class $S^*_{s,\lambda}(p,\alpha,\beta)$. Set

$$h(z) = \mu f_1(z) + (1 - \mu f_2(z)), \quad (0 \leq \mu \leq 1). \tag{47}$$

Then from (46), we can write

$$f(z) = z^p - \sum_{k=2}^{\infty} \left[\mu |a_{p+k,1}| + (1-\mu)|a_{p+k,1}|\right] z^{p+k} \ (a_{p+k,j} \geq 0,\ j = 1,2,\ z \in \mathbb{U}). \tag{48}$$

Thus, in view of Theorem 1, we can have that

$$\sum_{k=2}^{\infty} \left[k(1+\alpha\beta) + p\left((-1)^{p+k} + \beta(\alpha + 1 - (-1)^{p+k})\right)\right] (k+p)^{\lambda+1} |a_{k+p}| \left[\mu |a_{p+k,1}|\right]$$

$$+(1-\mu)|a_{p+k,1}| \leq p^{\lambda+1} \left[(-1)^p + \beta(\alpha + 1 - (-1)^p)\right],$$

which implies that $h(z) \in S^*_{s,\lambda}(p,\alpha,\beta)$ and this completes the proof of Theorem 10.

5 Radii of starlikeness and Convexity

Theorem 11. *Let the function $f(z)$ of the form (5) be in the class $S^*_{s,\lambda}(p,\alpha,\beta)$, then $f(z)$ is starlike in the disk $|z| = r_1 < 1$, where*

$$r_1 = \inf_{k \geq 1} \left(\frac{\left[k(1+\alpha\beta) + p\left((-1)^{p+k} + \beta(\alpha + 1 - (-1)^{p+k})\right)\right] (k+p)^{\lambda+1}}{p^{\lambda+1}\left[(-1)^p + \beta(\alpha + 1 - (-1)^p)\right]} \left(\frac{p}{p+k}\right)\right)^{\frac{1}{k}}. \tag{49}$$

Proof. It is ample to show that

$$\left| \tfrac{zf'(z)}{f(z)} - p \right| \leq p, \text{ for } |z| < 1$$

or equivalently,

$$\left| \frac{\sum_{k=1}^{\infty} k\ |a_{k+p}| z^k}{1 - \sum_{k=1}^{\infty} |a_{k+p}| z^k} \right| \leq p$$

which is equivalent to show that

$$\frac{\sum_{k=1}^{\infty}(k+1)|a_{k+p}||z|^k}{p} \leq 1. \tag{50}$$

As $f(z) \in S^*_{s,\lambda}(p,\alpha,\beta)$ we have from Theorem 1

$$\sum_{k=1}^{\infty} \frac{\left[k(1+\alpha\beta)+p\left((-1)^{p+k}+\beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}}{p^{\lambda+1}\left[(-1)^p+\beta(\alpha+1-(-1)^p)\right]}|a_{k+p}| \leq 1.$$

Hence, (50) is proven true if

$$\left(\frac{(p+k)|z|}{p}\right) \leq \frac{\left[k(1+\alpha\beta)+p\left((-1)^{p+k}+\beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}}{p^{\lambda+1}\left[(-1)^p+\beta(\alpha+1-(-1)^p)\right]}.$$

That is,

$$r_1 = \inf_{k\geq 1}\left(\frac{\left[k(1+\alpha\beta)+p\left((-1)^{p+k}+\beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}}{p^{\lambda+1}\left[(-1)^p+\beta(\alpha+1-(-1)^p)\right]}\left(\frac{p}{p+k}\right)\right)^{\frac{1}{k}}$$

and this ends the proof of Theorem 11.

On similar lines of Theorem 11, we can easily prove the following Theorems for $S^*_{c,\lambda}(p,\alpha,\beta)$ and $S^*_{sc,\lambda}(p,\alpha,\beta)$ classes.

Theorem 12. *Let the function $f(z)$ of the form (5) be in the class $S^*_{c,\lambda}(p,\alpha,\beta)$, then $f(z)$ is starlike in the disk $|z| = r_1 < 1$, where*

$$r_1 = \inf_{k\geq 1}\left(\frac{\left[k(1+\alpha\beta)+p\left(\beta(\alpha+2)-1\right)\right](k+p)^{\lambda+1}}{p^{\lambda+1}\left[(\beta(\alpha+2)-1)\right]}\left(\frac{p}{p+k}\right)\right)^{\frac{1}{k}}. \tag{51}$$

Theorem 13. *Let the function $f(z)$ of the form (5) be in the class $S^*_{sc,\lambda}(p,\alpha,\beta)$, then $f(z)$ is starlike in the disk $|z| = r_1 < 1$, where*

$$r_1 = \inf_{k\geq 1}\left(\frac{\left[k(1+\alpha\beta)+p\left(\beta(1+\alpha)+(1-\beta)(-1)^{p+k}\right)\right](k+p)^{\lambda+1}}{p^{\lambda+1}\left[(-1)^p+\beta(\alpha+1-(-1)^p)\right]}\left(\frac{p}{p+k}\right)\right)^{\frac{1}{k}}. \tag{52}$$

Similarly we can proved the following Results.

Theorem 14. *Let the function $f(z)$ of the form (5) be in the class $S^*_{s,\lambda}(p,\alpha,\beta)$, then $f(z)$ is convex in the disk $|z| = r_2 < 1$, where*

$$r_1 = \inf_{k\geq 1}\left(\frac{\left[k(1+\alpha\beta)+p\left((-1)^{p+k}+\beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}}{p^{\lambda+1}\left[(-1)^p+\beta(\alpha+1-(-1)^p)\right]}\left(\frac{p}{p+k}\right)^2\right)^{\frac{1}{k}}. \tag{53}$$

Theorem 15. *Let the function $f(z)$ of the form (5) be in the class $S^*_{c,\lambda}(p,\alpha,\beta)$, then $f(z)$ is convex in the disk $|z| = r_2 < 1$, where*

$$r_1 = \inf_{k\geq 1}\left(\frac{[k(1+\alpha\beta)+p\,(\beta(\alpha+2)-1)]\,(k+p)^{\lambda+1}}{p^{\lambda+1}\,[(\beta(\alpha+2)-1)]}\left(\frac{p}{p+k}\right)^2\right)^{\frac{1}{k}}. \tag{54}$$

Theorem 16. *Let the function $f(z)$ of the form (5) be in the class $S^*_{sc,\lambda}(p,\alpha,\beta)$, then $f(z)$ is convex in the disk $|z| = r_2 < 1$, where*

$$r_1 = \inf_{k\geq 1}\left(\frac{\left[k(1+\alpha\beta)+p\left(\beta(1+\alpha)+(1-\beta)(-1)^{p+k}\right)\right](k+p)^{\lambda+1}}{p^{\lambda+1}\,[(-1)^p+\beta(\alpha+1-(-1)^p)]}\left(\frac{p}{p+k}\right)^2\right)^{\frac{1}{k}}. \tag{55}$$

In order to establish the required results in Theorems 14, 15 and 16, it is sufficies to show that

$$\left|1+\tfrac{zf''(z)}{f'(z)}-p\right| \leq p, \text{ for } |z|<1.$$

6 Integral Operator

Definition 4. Let $f \in \mathcal{T}_p$, an integral operator $R_c(f)$ with $c > -p$ is defined by

$$R_c(f) = \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt, \quad (z \in \mathbb{U}). \tag{56}$$

Theorem 17. *Let the function $f(z)$ of the form (5) be in the class $S^*_{s,\lambda}(p,\alpha,\beta)$, then $R_c(f)$ defined by (56) be also in the class $S^*_{s,\lambda}(p,\alpha,\beta)$.*

Proof. From (56), we get

$$R_c(f) = z^p - \sum_{k=1}^{\infty}\frac{c+p}{c+p+k}|a_{k+p}|z^{k+p}. \tag{57}$$

Therefore by hypothesis

$$\sum_{k=1}^{\infty}\left[k(1+\alpha\beta)+p\left((-1)^{p+k}+\beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}\left(\frac{c+p}{c+p+k}\right)|a_{k+p}|$$

$$\leq \sum_{k=1}^{\infty}\left[k(1+\alpha\beta)+p\left((-1)^{p+k}+\beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}|a_{k+p}|$$

$$\leq p^{\lambda+1}\,[(-1)^p+\beta(\alpha+1-(-1)^p)]$$

since $f(z) \in S^*_{s,\lambda}(p,\alpha,\beta)$. Hence, by Theorem 1, $R_c(f) \in S^*_{s,\lambda}(p,\alpha,\beta)$.

Similar to Theorem 17, we can easily prove the following theorems for $S^*_{c,\lambda}(p,\alpha,\beta)$ and $S^*_{sc,\lambda}(p,\alpha,\beta)$.

Theorem 18. *Let the function $f(z)$ of the form (5) be in the class $S^*_{c,\lambda}(p,\alpha,\beta)$, then $R_c(f)$ defined by (56) be also in the class $S^*_{c,\lambda}(p,\alpha,\beta)$.*

Theorem 19. *Let the function $f(z)$ of the form (5) be in the class $S^*_{sc,\lambda}(p,\alpha,\beta)$, then $R_c(f)$ defined by (56) be also in the class $S^*_{sc,\lambda}(p,\alpha,\beta)$.*

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References

[1] Salagean G. S. *Subclasses of univalent functions.* Lecture Notes Math., 1013, Springer Verlag, Berlin, 1983, 362–372.

[2] Orhan H., Kiziltunc H. *A generalization on subfamily of p-valent functions with negative coefficients.* Appl. Math. Comp., 2004, **155**, 521–530.

[3] Sakaguchi K. *On certain univalent mapping.* J. Math. Soc. Japan, 1959, **11**, 72–75.

[4] Sudharsan T. V., Balasubrahmananyam P., Subramanian K. G. *On functions starlike with respect to symmetric and conjugate points*, Taiwanese J. Math., 1998, **2**, 57–68.

[5] Robertson M. S. *Applications of the subordination principle to univalent functions.* Pacific J. Math., 1961, **11**, 315–324.

[6] Stankiewicz J. *Some remarks on functions starlike with respect to symmetric points.* Ann. Univ. Marie Curie Sklodowska, 1965, **19**, 53–59.

[7] Wu Z. *On classes of Sakaguchi functions and Hadamard products.* Sci. Sinica Ser., 1987, **A 30**, 128–135.

[8] Owa S., Wu Z., Ren F. *A note on certain subclass of Sakaguchi functions.* Bull. Soc. Roy. Liege, 1988, **57**, 143–150.

[9] El-Ashwah R. M., Thomas D. K. *Some subclasses of close-to-convex functions.* J. Ramanujan Math. Soc., 1987, **2**, 86–100.

[10] Chen M.-P., Wu Z.-R., Zou Z.-Z. *On functions α-starlike with respect to symmetric conjugate points.* Jour. Math. Anal. App. 201, 1996, **Art. No. 0238**, 25–34.

[11] Sokół J. *Function starlike with respect to conjugate point.* Zeszyty Nauk. Politech. Rzeszowskiej Mat. Fiz., 1991, **12**, 53–64.

[12] Sokół J. *Some remarks on the class of functions starlike with respect to symmetric points.* Folia Scient. Univ. Tech. Resoviensis, 1990, **73**, 79–89.

[13] Aouf M. K., El-Ashwah R. M., El-Deeb S. M. *Certain classes of univalent functions with negative coefficients and n-starlike with respect to certain points.* Matematiqki Vesnik, 2010, **62**, No. 3, 215–226.

[14] Halim S. A., Janteng A., Darus M. *Coefficient properties for classes with negative coefficient and starlike with respect to other points.* Proceeding of The 13th Math. Sci. Nat. Symposium, UUM., 2005, **2**, 658–663.

[15] Halim S. A., Janteng A., Darus M. Classes with negative coefficient and starlike with respect to other points. Int. Math., Forum 2, 2007, **46**, 2261–2268.

[16] Dziok P. *On the convex combination of the Dziok-Srivastava operator*. Appl. Math. Comp., 2006, **188**, 1214–1220.

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Third Hankel determinant for the inverse of reciprocal of bounded turning functions

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Abstract. In this paper we obtain the best possible upper bound to the third Hankel determinants for the functions belonging to the class of reciprocal of bounded turning functions using Toeplitz determinants.

Mathematics subject classification: 30C45, 30C50.
Keywords and phrases: Univalent function, function whose reciprocal derivative has a positive real part, third Hankel determinant, positive real function, Toeplitz determinants.

1 Introduction

Let A denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions. For a univalent function in the class A, it is well known that the nth coefficient is bounded by n. The bounds for the coefficients of univalent functions give information about their geometric properties. In particular, the growth and distortion properties of a normalized univalent function are determined by the bound of its second coefficient. The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke [12] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

This determinant has been considered by many authors in the literature. For example, Noor [10] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in S with bounded boundary. Ehrenborg [4] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [7]. One can easily observe that the Fekete-Szego functional is $H_2(1)$. Fekete-Szego then further generalized the estimate

$|a_3 - \mu a_2^2|$ with μ real and $f \in S$. R. M. Ali [1] found sharp bounds for the first four coefficients and sharp estimate for the Fekete-Szego functional $|\gamma_3 - t\gamma_2^2|$, where t is real, for the inverse function of f defined as

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n,$$

when it belongs to the class of strongly starlike functions of order α $(0 < \alpha \leq 1)$ denoted by $\widetilde{ST}(\alpha)$. In the recent years several authors have investigated bounds for the Hankel determinant of functions belonging to various subclasses of univalent and multivalent analytic functions. In particular for $q = 2$, $n = 1$, $a_1 = 1$ and $q = 2$, $n = 2$, $a_1 = 1$, the Hankel determinant simplifies respectively to

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2 \quad \text{and} \quad H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

For our discussion in this paper, we consider the Hankel determinant in the case of $q = 3$ and $n = 1$, denoted by $H_3(1)$, given by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}. \tag{2}$$

For $f \in A$, $a_1 = 1$, so we have

$$H_3(1) = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2)$$

and by applying triangle inequality, we obtain

$$|H_3(1)| \leq |a_3||a_2 a_4 - a_3^2| + |a_4||a_2 a_3 - a_4| + |a_5||a_3 - a_2^2|. \tag{3}$$

The sharp upper bound to the second Hankel functional $H_2(2)$ for the subclass RT of S, consisting of functions whose derivative has a positive real part, studied by Mac Gregor [9] was obtained by Janteng [6]. It was known that if $f \in RT$ then $|a_k| \leq \frac{2}{k}$, for $k \in \{2, 3, ...\}$. Also the sharp upper bound for the functional $|a_3 - a_2^2|$ was $\frac{2}{3}$, stated in [2], for the class RT. Further, the best possible sharp upper bound for the functional $|a_2 a_3 - a_4|$ was obtained by Babalola [2] and hence the sharp inequality for $|H_3(1)|$, for the class RT. The sharp upper bound to $|H_3(1)|$ for the class of inverse of a function whose derivative has a real part was obtained by D. Vamshee Krishna et al. [14].

Motivated by the above mentioned results obtained by different authors in this direction and the results by Babalola [2], in the present paper, we seek an upper bound to the second Hankel determinant, $|t_2 t_3 - t_4|$ and an upper bound to the third Hankel determinant, for certain subclass of analytic functions defined as follows.

Definition 1. A function $f(z) \in A$ is said to be function whose reciprocal derivative has a positive real part (also called reciprocal of bounded turning function), denoted by $f \in \widetilde{RT}$, if and only if

$$Re\Big(\frac{1}{f'(z)}\Big) > 0, \ \forall z \in E. \tag{4}$$

Some preliminary Lemmas required for proving our results are as follows.

2 Preliminary Results

Let $\mathcal{P}$ denote the class of functions consisting of p, such that

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{5}$$

which are regular in the open unit disc E and satisfy $Re\Big(p(z)\Big) > 0$ for any $z \in E$. Here $p(z)$ is called the Caratheòdory function [3].

Lemma 1 (see [11, 13])**.** *If $p \in \mathcal{P}$, then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $\frac{1+z}{1-z}$.*

Lemma 2 (see [5])**.** *The power series for $p(z)$ given in (5) converges in the open unit disc E to a function in $\mathcal{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \quad \textit{for} \ \ n = 1, 2, 3....$$

and $c_{-k} = \overline{c}_k$, are all non-negative. They are strictly positive except for

$$p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k} z),$$

$\rho_k > 0$, t_k real and $t_k \neq t_j$, for $k \neq j$, where $p_0(z) = \frac{1+z}{1-z}$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$.

This necessary and sufficient condition found in [5] is due to Caratheòdory and Toeplitz. We may assume without restriction that $c_1 > 0$. On using Lemma 2, for $n = 2$, we have

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c}_1 & 2 & c_1 \\ \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} = [8 + 2Re\{c_1^2 c_2\} - 2 \mid c_2 \mid^2 - 4|c_1|^2] \geq 0,$$

which is equivalent to

$$2c_2 \;=\; c_1^2 + x(4 - c_1^2), \text{ for some } x, |x| \leq 1. \tag{6}$$

For $n = 3$,

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c}_1 & 2 & c_1 & c_2 \\ \overline{c}_2 & \overline{c}_1 & 2 & c_1 \\ \overline{c}_3 & \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} \geq 0$$

and is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \tag{7}$$

Simplifying the relations (6) and (7), we get

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \quad \text{for some } z, \text{ with } |z| \leq 1. \tag{8}$$

To obtain our results, we refer to the classical method initiated by Libera and Zlotkiewicz [8] and used by several authors in the literature.

3 Main Result

Theorem 1. *If* $f(z) \in \widetilde{RT}$ *and* $f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n$ *near* $w=0$ *is the inverse function of* f *then* $|\, t_2t_4 - t_3^2| \;\leq\; \frac{137}{288}$.

Proof. For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \widetilde{RT}$, there exists an analytic function $p \in \mathcal{P}$ in the open unit disc E with $p(0) = 1$ and $Re\Big(p(z)\Big) > 0$ such that

$$\frac{1}{f'(z)} = p(z) \Leftrightarrow 1 = f'(z)p(z). \tag{9}$$

Replacing $f'(z)$ and $p(z)$ with their equivalent series expressions in (9), we have

$$1 = \Big(1 + \sum_{n=2}^{\infty} na_n z^{n-1}\Big)\Big(1 + \sum_{n=1}^{\infty} c_n z^n\Big).$$

Upon simplification, we obtain

$$\begin{aligned} 1 = 1 &+ (c_1 + 2a_2)z + (c_2 + 2a_2c_1 + 3a_3)z^2 \\ &+ (c_3 + 2a_2c_2 + 3a_3c_1 + 4a_4)z^3 \\ &+ (c_4 + 2a_2c_3 + 3a_3c_2 + 4a_4c_1 + 5a_5)z^4 \cdots . \end{aligned} \tag{10}$$

Equating the coefficients of like powers of z, z^2, z^3 and z^4 respectively on both sides of (10), after simplifying, we get

$$a_2 = -\frac{c_1}{2}; \quad a_3 = \frac{1}{3}(c_1^2 - c_2); \quad a_4 = -\frac{1}{4}(c_3 - 2c_1c_2 + c_1^3);$$
$$a_5 = -\frac{1}{5}(c_4 - 2c_1c_3 + 3c_1^2c_2 - c_2^2 - c_1^4). \tag{11}$$

Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \widetilde{RT}$, from the definition of inverse function of f, we have

$$w = f(f^{-1}(w)) = f^{-1}(w) + \sum_{n=2}^{\infty} a_n \big(f^{-1}(w)\big)^n \Leftrightarrow w$$
$$= w + \sum_{n=2}^{\infty} t_n w^n + \sum_{n=2}^{\infty} a_n \left(w + \sum_{n=2}^{\infty} t_n w^n\right)^n.$$

After simplifying, we get

$$(t_2 + a_2)w^2 + (t_3 + 2a_2t_2 + a_3)w^3 + (t_4 + 2a_2t_3 + a_2t_2^2 + 3a_3t_2 + a_4)w^4$$
$$+(t_5 + 2a_2t_4 + 2a_2t_2t_3 + 3a_3t_3 + 3a_3t_2^2 + 4a_4t_2 + a_5)w^5 + \cdots = 0. \tag{12}$$

Equating the coefficients of like powers of w^2, w^3, w^4 and w^5 on both sides of (12), respectively, further simplification gives

$$t_2 = -a_2; \quad t_3 = -a_3 + 2a_2^2; \quad t_4 = -a_4 + 5a_2a_3 - 5a_2^3;$$
$$t_5 = -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4. \tag{13}$$

Using the values of a_2, a_3, a_4 and a_5 in (11) along with (13), upon simplification, we obtain

$$t_2 = \frac{c_1}{2}; \; t_3 = \frac{1}{6}[2c_2 + c_1^2]; \; t_4 = \frac{1}{24}[6c_3 + 8c_1c_2 + c_1^3];$$
$$t_5 = \frac{1}{120}[24c_4 + 42c_1c_3 + 22c_1^2c_2 + 16c_2^2 + c_1^4]. \tag{14}$$

Substituting the values of t_2, t_3 and t_4 from (14) in the functional $|\, t_2t_4 - t_3^2|$ for the function $f \in \widetilde{RT}$ upon simplification, we obtain

$$|\, t_2t_4 - t_3^2| = \frac{1}{144}\Big|18c_1c_3 + 8c_1^2c_2 - 16c_2^2 - c_1^4\Big|$$

which is equivalent to

$$|\, t_2t_4 - t_3^2| = \frac{1}{144}\Big|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4\Big| \tag{15}$$

$$\text{where} \quad d_1 = 18; \; d_2 = 8; \; d_3 = -16; \; d_4 = -1. \tag{16}$$

Substituting the values of c_2 and c_3 given in (6) and (8) respectively from Lemma 2 on the right-hand side of (15) and using the fact $|z| < 1$, we have

$$4|\ d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leq \Big|(d_1 + 2d_2 + d_3 + 4d_4)c_1^4 + \{2d_1c_1 + 2(d_1 + d_2 + d_3)c_1^2|x| - [(d_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3]|x|^2\}(4 - c_1^2)\Big|. \quad (17)$$

From (16) and (17), we can now write

$$(d_1 + 2d_2 + d_3 + 4d_4) = 14; \quad 2d_1 = 36; \quad 2(d_1 + d_2 + d_3) = 20;$$
$$(d_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3 = 2(c_1^2 + 18c_1 + 32). \quad (18)$$

Since $c_1 \in [0, 2]$, using the result $(c_1 + a)(c_1 + b) \geq (c_1 - a)(c_1 - b)$, where $a, b \geq 0$ in (18), we can have

$$-\{(d_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3\} \leq -2(c_1^2 - 18c_1 + 32). \quad (19)$$

Substituting the calculated values from (18) and (19) on the right-hand side of (17), we have

$$4|\ d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leq \Big|14c_1^4 + \{36c_1 + 20c_1^2|x| - 2(c_1^2 - 18c_1 + 32)|x|^2\}(4 - c_1^2)\Big|.$$

Choosing $c_1 = c \in [0, 2]$, applying triangle inequality and replacing $|x|$ by μ on the right-hand side of the above inequality

$$2|\ d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leq \Big|7c^4 + \{18c + 10c^2\mu + (c^2 - 18c + 32)\mu^2\}(4 - c^2)\Big| = F(c, \mu)\ ,\ 0 \leq \mu = |x| \leq 1 \ \text{ and } \ 0 \leq c \leq 2. \quad (20)$$

We next maximize the function $F(c, \mu)$ on the closed region $[0, 2] \times [0, 1]$. Differentiating $F(c, \mu)$ given in (20) partially with respect to μ, we obtain

$$\frac{\partial F}{\partial \mu} = [20c + 2(c - 2)(c - 16)\mu](4 - c^2) > 0. \quad (21)$$

For $0 < \mu < 1$ and for fixed c with $0 < c < 2$, from (21), we observe that $\frac{\partial F}{\partial \mu} > 0$. Therefore, $F(c, \mu)$ becomes an increasing function of μ and hence it cannot have a maximum value at any point in the interior of the closed region $[0, 2] \times [0, 1]$. Moreover, for a fixed $c \in [0, 2]$, we have

$$\max_{0 \leq \mu \leq 1} F(c, \mu) = F(c, 1) = G(c).$$

Therefore, replacing μ by 1 in $F(c, \mu)$, upon simplification, we obtain

$$G(c) = -4c^4 + 12c^2 + 128 \quad (22)$$

$$G'(c) = -16c^3 + 24c \tag{23}$$
$$G''(c) = -48c^2 + 24. \tag{24}$$

For optimum value of $G(c)$, consider $G'(c) = 0$. From (23), we get

$$c^2 = \frac{3}{2}.$$

Using the obtained value of c^2 in (24), which simplifies to give

$$G''(c) = -48 < 0.$$

Therefore, by the second derivative test, $G(c)$ has maximum value at $c = \sqrt{\frac{3}{2}} \in [0, 2]$. Substituting the value of c in the expression (22), upon simplification, we obtain the maximum value of $G(c)$ at c as

$$G_{max} = 137. \tag{25}$$

Simplifying the expressions (20) and (25)

$$| d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leq \frac{137}{2}. \tag{26}$$

From the relations (15) and (26), we obtain

$$| t_2t_4 - t_3^2| \leq \frac{137}{288}. \tag{27}$$

This completes the proof of our Theorem. □

Remark 1. It is observed that the upper bound to the second Hankel determinant of inverse of a function whose derivative has a positive real part [14] and the inverse of a function whose reciprocal derivative has a positive real part is the same.

Theorem 2. *If* $f(z) \in \widetilde{RT}$ *and* $f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n$ *near* $w = 0$ *is the inverse function of* f *then* $| t_2t_3 - t_4| = \left(\frac{13}{6}\right)^{\frac{3}{2}}$.

Proof. Substituting the values of t_2, t_3 and t_4 from (14) in $| t_2t_3 - t_4|$ for the function $f \in \widetilde{RT}$, after simplifying, we get

$$| t_2t_3 - t_4| = \frac{1}{24}| - 6c_3 - 4c_1c_2 + c_1^3|. \tag{28}$$

Substituting the values of c_2 and c_3 from (6) and (8) respectively, from Lemma 2 on the right-hand side of (28) and using the fact $| z |< 1$, after simplifying, we get

$$2| - 6c_3 - 4c_1c_2 + c_1^3| \leq \Big| - 5c_1^3 - 6(4 - c_1^2) - 10c_1(4 - c_1^2)|x|$$

$$+ 3(c_1+2)(4-c_1^2)|x|^2\Big|. \tag{29}$$

Since $c_1 = c \in [0,2]$, using the result $(c_1+a) \geq (c_1-a)$, where $a \geq 0$, applying triangle inequality and replacing $|x|$ by μ on the right-hand side of the above inequality, we have

$$\begin{aligned} 2|-6c_3-4c_1c_2+c_1^3| \leq & \mid 5c^3+6(4-c^2)+ 10c(4-c^2)\mu+3(c-2)(4-c^2)\mu^2| \\ = & F(c,\mu) \ , \ 0 \leq \mu = \mid x \mid \leq 1 \text{ and } 0 \leq c \leq 2. \end{aligned} \tag{30}$$

Next, we maximize the function $F(c,\mu)$ on the closed square $[0,2]\times[0,1]$. Differentiating $F(c,\mu)$ partially with respect to μ, we get

$$\frac{\partial F}{\partial \mu} = (4-c^2)[10c+6(c-2)\mu] > 0.$$

As described in Theorem 3, further, we obtain

$$G(c) = -8c^3+52c \tag{31}$$

$$G'(c) = -24c^2+52 \tag{32}$$

$$G''(c) = -48c. \tag{33}$$

For optimum value of $G(c)$, consider $G'(c)=0$. From (32), we get

$$c^2 = \frac{13}{6}.$$

Using the obtained value of $c=\sqrt{\frac{13}{6}} \in [0,2]$ in (33), then

$$G''(c) = -8\sqrt{78} < 0.$$

Therefore, by the second derivative test, $G(c)$ has maximum value at $c=\sqrt{\frac{13}{6}}$. Substituting the value of c in the expression (31), upon simplification, we obtain the maximum value of $G(c)$ at c as

$$G_{max} = \frac{104}{3}\sqrt{\frac{13}{6}}. \tag{34}$$

From the expressions (30) and (34), after simplifying, we get

$$|-6c_3-4c_1c_2+c_1^3| \leq \frac{52}{3}\sqrt{\frac{13}{6}}. \tag{35}$$

Simplifying the relations (28) and (35), we obtain

$$\mid t_2t_3-t_4| \leq \frac{1}{3}\left(\frac{13}{6}\right)^{\frac{3}{2}}.$$

This completes the proof of our Theorem. □

Remark 2. It is observed that the upper bound to the $|t_2t_3 - t_4|$ of a function whose derivative has a positive real part [14] and a function whose reciprocal derivative has a positive real part is the same.

The following theorem is a straight forward verification on applying the same procedure as described in Theorems 3 and 4 and the result is sharp for the values $c_1 = 0,\ c_2 = 2$ and $x = 1$.

Theorem 3. *If* $f(z) \in \widetilde{RT}$ *and* $f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n$ *near* $w = 0$ *is the inverse function of* f *then* $|\, t_3 - t_2^2| \leq \frac{2}{3}$.

Using the fact that $|c_n| \leq 2,\ \ n \in N = \{1, 2, 3, \cdots\}$, with the help of c_2 and c_3 values given in (6) and (8) respectively together with the values in (14), we at once obtain all the below inequalities.

Theorem 4. *If* $f(z) \in \widetilde{RT}$ *and* $f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n$ *near* $w = 0$ *is the inverse function of* f *then we have the following inequalities:*
$(i)\ \ |t_2| \leq 1\ \ (ii)\ \ |t_3| \leq \frac{4}{3}\ \ (iii)\ \ |t_4| \leq \frac{13}{6}\ \ (iv)\ \ |t_5| \leq \frac{59}{15}$.

Using the results of Theorems 3, 4, 5 and 6 in (3), we obtain the following corollary.

Corollary 1. *If* $f(z) \in \widetilde{RT}$ *and* $f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n$ *near* $w = 0$ *is the inverse function of* f *then* $|H_3(1)| \leq \frac{1}{3}\left[\frac{3157}{360} + \left(\frac{13}{6}\right)^{\frac{5}{2}}\right]$.

Remark 3. It is observed that the upper bound to the third Hankel determinant of a function whose derivative has a positive real part [14] and a function whose reciprocal derivative has a positive real part is the same.

References

[1] Ali R. M. *Coefficients of the inverse of strongly starlike functions.* Bull. Malays. Math. Sci. Soc. (second series), 2003, **26 (1)**, 63–71.

[2] Babalola K. O. *On $H_3(1)$ Hankel determinant for some classes of Uni-valent Functions.* Inequality Theory and Applications, 2010, **6**, 1–7.

[3] Duren P. L. *Univalent functions.* Vol. 259, of Grundlehren der Mathematischen Wissenschaften, Springer, New York, USA, 1983.

[4] Ehrenborg R. *The Hankel determinant of exponential polynomials.* Amer. Math. Monthly, 2000, **107 (6)**, 557–560.

[5] Grenander U., Szegö G. *Toeplitz forms and their applications, Second edition.* Chelsea Publishing Co., New York, 1984.

[6] Janteng A., Halim S. A., Darus M. *Coefficient inequality for a function whose derivative has a positive real part.* J. Inequal. Pure Appl. Math., 2006, **7 (2)**, 1–5.

[7] Layman J. W. *The Hankel transform and some of its properties.* J. Integer Seq., 2001, **4 (1)**, 1–11.

[8] Libera R. J., Zlotkiewicz E. J. *Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$.* Proc. Amer. Math. Soc., 1983, **87 (2)**, 251–257.

[9] Mac Gregor T. H. *Functions whose derivative have a positive real part.* Trans. Amer. Math. Soc., 1962, **104 (3)**, 532–537.

[10] Noor K. I. *Hankel determinant problem for the class of functions with bounded boundary rotation.* Rev. Roum. Math. Pures Appl., 1983, **28 (8)**, 731–739.

[11] Pommerenke Ch. *Univalent functions.* Vandenhoeck and Ruprecht, Gottingen, 1975.

[12] Pommerenke Ch. *On the coefficients and Hankel determinants of univalent functions.* J. Lond. Math. Soc., 1966, **41**, 111–122.

[13] Simon B. *Orthogonal polynomials on the unit circle, part 1.* Classical theory, AMS Colloquium Publ. 54, Part 1, American Mathematical Society, Providence, RI, 2005.

[14] Vamshee Krishna D., Venkateswarlu B., Ramreddy T. *Third Hankel determinant for the inverse of a function whose derivative has a positive real part.* Math. Stud., 2014, **42 (1)**, 54–60.

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Generating Cubic Equations as a Method for Public Encryption

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Abstract. The paper introduces a new method for public encryption in which the enciphering process is performed as generating coefficients of some cubic equation over finite ring and the deciphering process is solving the equation. Security of the method is based on difficulty of factoring problem, namely, difficulty of factoring a composite number n that serves as public key. The private key is the pair of primes p and q such that $n = pq$. The deciphering process is performed as solving cubic congruence modulo n. Finding roots of cubic equations in the fields $GF(p)$ and $GF(q)$ is the first step of the decryption. We have described a method for solving cubic equations defined over ground finite fields. The proposed public encryption algorithm has been applied to design bi-deniable encryption protocol.

Mathematics subject classification: 11T71, 11S05, 94A60.
Keywords and phrases: Cryptography, ciphering, public encryption, deniable encryption, public key, cubic equation, Galois field, factoring problem.

Dedicated to the light memory of our colleague and outstanding mathematician Galina Borisovna Belyavskaya

1 Introduction

The public-key encryption algorithm proposed by Rabin [1] uses the public key represented as the pair of integers n and $b < n$, where n is a composite number difficult for factoring; b is an arbitrary integer. To generate an appropriate number n one has to select a pair of strong [2] primes p and q and then compute the value $n = pq$.

Some secret message $M < n$ can be send to the owner of the public key (n, b) in form of the ciphertext C computed as $C = M \cdot (M + b) \pmod{n}$. Decryption of the ciphertext consists in finding roots of the quadratic congruence $x^2 + bx - C \equiv 0 \pmod{n}$. The last can be easily performed using the private key (p, q).

The Rabin cryptosystem is a provably secure public-key cryptosystem, i. e. one can formally prove that decryption of the ciphertext C without knowing the devisors of n is as difficult as factoring the value n. Paper [3] extends the class of provably secure public key cryptosystems based on the difficulty of factoring problem introducing the encryption formula $C = M^k \pmod{n}$, where k ($k \geq 2$) divides at least one of numbers $p - 1$ and $q - 1$.

Provable security is an important merit of the mentioned public-key cryptosystems. However for all of those cryptosystems the output of the decryption procedure is ambiguous, namely, deciphering process outputs several decrypted texts and only one of them is equal to the encrypted text. The minimum number of the decrypted texts is equal to three and relates to the case $k = 3$ [3].

Recently solving the quadratic congruences like $x^2 - Ax + B \equiv 0 \pmod{n}$ was used in [4] to design the public-key algorithm for encrypting simultaneously two messages into the ciphertext (A, B). That algorithm was put into the base of the sender-deniable encryption protocol. In [4] the authors mentioned potential possibility to construct algorithms for simultaneous encryption of three and four messages into the cryptogram representing the set of coefficients of the cubic and fourth-power congruences, respectively. Naturally, decryption in the last two cases consists in solving congruences like $x^3 - Ax^2 + Bx - D \equiv 0 \pmod{n}$ and $x^4 - Ax^3 + Bx^2 - Dx + E \equiv 0 \pmod{n}$.

The case of using cubic equations represents special interest since it provides potential possibility to design public encryption algorithms that are free from ambiguity of the decryption process, whereas the quadratic and fourth-power equations cannot be used for such purpose.

In this paper we consider the design of the public-encryption algorithms based on using the cubic equation. We consider details of solving cubic equations in the ground field $GF(p)$ in the case when the equations have solutions (this is defined by the design of the encryption algorithm). The described method for solving cubic equations in $GF(p)$ actually determines the decryption algorithm. It is shown that for a particular design the encryption algorithm processes one input message and the decryption procedure outputs one decrypted text, i.e. only the input message.

2 A new method for public encryption

Using the public key n one can encrypt simultaneously three different messages $M < n$, $T < n$, and $U < n$ as generating three coefficients A, B, and D of the cubic equation such that the messages M, T, and U represent three roots of the equation. Since the last values are to be roots, then the encryption is defined by the condition $(x - M)(x - T)(x - U) = x^3 - (M + T + U)x^2 + (MT + MU + TU)x - MTU = 0 \pmod{n}$.

Thus, such idea of constructing the public encryption scheme leads to the enciphering procedure that consists in computing the following three coefficients that compose the ciphertext $C = (A, B, D)$:

$$A = (M + T + U) \bmod n,$$
$$B = (MT + MU + TU) \bmod n,$$
$$D = MTU \bmod n.$$

Respectively, deciphering of the cryptogram C is to be performed as solving the cubic equation

$$x^3 - Ax^2 + Bx - D = 0 \pmod{n}. \tag{1}$$

Solving equation (1) can be performed by the owner of public key n using his private key (p, q) that represents two divisors of the modulus n. For this purpose he is to solve the cubic equation

$$x^3 - Ax^2 + Bx - D = 0 \pmod p \tag{2}$$

and the cubic equation

$$x^3 - Ax^2 + Bx - D = 0 \pmod q. \tag{3}$$

Let x_{1p}, x_{2p}, and x_{3p} be roots of equation (2) and x_{1q}, x_{2q}, and x_{3q} be roots of equation (3). Then nine roots of the equation (1) can be computed solving nine systems of the congruences of the following form

$$\begin{cases} X_{ij} \equiv x_{ip} \pmod p \\ X_{ij} \equiv x_{jq} \pmod q, \end{cases}$$

where $i, j \in \{1, 2, 3\}$. Three of the computed roots are equal to the sensible messages M, T and U that have been encrypted. Other six roots represent some random values and are to be ignored. Thus, solving cubic equations in the ground finite fields is the central part of the considered public-key encryption scheme.

3 Solving cubic equations in the ground finite field

To find roots of the cubic equation (2) over the ground field $GF(p)$ we propose to solve the equation (relative to the unknown $X \in GF(p^2)$)

$$(1, 0)X^3 - (A, 0)X^2 + (B, 0)X - (D, 0) = (0, 0) \tag{4}$$

over the extension field $GF(p^2)$ that is defined evidently in the vector form [5] with the unity element $(1, 0)$ and zero element $(0, 0)$.

Addition and multiplication of two elements $(a, b), (c, d) \in GF(p^2)$ are defined with the formulas

$$(a, b) + (c, d) = ((a + c) \bmod p, (b + d) \bmod p) \tag{5}$$

and

$$(a, b)(c, d) = ((ac + kbd) \bmod p, (bc + ad) \bmod p), \tag{6}$$

where $k \in GF(p)$ is some specified constant that is equal to a quadratic non-residue, respectively.

Substitution of the unknown x in (2) by the variable $z = x - 3^{-1}A \bmod p$ gives the following equation (like in [6]) that is identical to (2):

$$z^3 + Pz + Q = 0 \bmod p, \tag{7}$$

where $P = B - \frac{A^2}{3} \bmod p$ and $Q = \frac{AB}{3} - \frac{2A^3}{27} - D \bmod p$.

Respectively, with analogous variable substitution $\mathbf{X} = \mathbf{Z} + (3^{-1} \bmod p, 0)(A, 0)$ one can reduce equation (4) to

$$\mathbf{Z}^3 + \mathbf{P}\mathbf{Z} + \mathbf{Q} = (0, 0), \tag{8}$$

where

$$\mathbf{P} = (P, 0) = (B, 0) - \tfrac{(A,0)^2}{3} \text{ and}$$

$$\mathbf{Q} = (Q, 0) = \tfrac{(A,0)(B,0)}{3} - \tfrac{2(A,0)^3}{27} - (D, 0).$$

Since for the given coefficients $(A, 0)$, $(B, 0)$, and $(D, 0)$ the equation (4) has at least one solution, for example, $X = (M \bmod p, 0)$, then the equation (8) also has solution and using the method for solving cubic equations which is described in [6] one can derive the following formula for roots of equation (8)

$$\mathbf{Z} = (z, 0) = \alpha + \beta \tag{9}$$

and the following formula for roots of equation (4)

$$\mathbf{X} = (x, 0) = \frac{(A, 0)}{3} + \alpha + \beta, \tag{10}$$

where

$$\alpha = \sqrt[3]{-\frac{\mathbf{Q}}{2} + \sqrt{\frac{\mathbf{Q^2}}{4} + \frac{\mathbf{P^3}}{27}}}; \quad \beta = \sqrt[3]{-\frac{\mathbf{Q}}{2} - \sqrt{\frac{\mathbf{Q^2}}{4} + \frac{\mathbf{P^3}}{27}}}. \tag{11}$$

For the case under consideration ($p > 3$) there exist three different cubic roots α and three different cubic roots β. In formulas (9) and (10) one should select only pairs of the values α and β which satisfy the condition

$$\alpha\beta = -\frac{\mathbf{P}}{3}. \tag{12}$$

4 About number of roots of the cubic equation in $GF(p)$

To consider type and number of roots of the equations (7) and (8) it is useful to formulate the following preliminary statements.

Lemma 1. *Suppose a prime $p > 3$ and $\mathbf{A}$ is a cubic residue in $GF(p^2)$. Then there exist exactly three different cubic roots from $\mathbf{A}$.*

Proof. An arbitrary prime $p > 3$ can be represented as $p = 6t \pm 1$. Respectively $p^2 - 1 = 36t^2 \pm 12t \Rightarrow 3|p^2 - 1$, where $p^2 - 1$ is the order of the multiplicative group of $GF(p^2)$. The last group is a finite cyclic one, therefore it contains exactly two elements ε and ε^2 having order 3 that are non-trivial cubic roots from $(1, 0) \in GF(p^2)$.

If $\mathbf{B}$ is a cubic root from $\mathbf{A}$, then $\varepsilon\mathbf{B}$ and $\varepsilon^2\mathbf{B}$ are also cubic roots from $\mathbf{A}$. Assumption about existence of the fourth cubic root $\mathbf{B}' = \sqrt[3]{\mathbf{A}}$ leads to contradiction about existence of the third element $\varepsilon' = \mathbf{B}/\mathbf{B}' \neq (1, 0)$ having order 3, such that $\varepsilon' \neq \varepsilon$ and $\varepsilon' \neq \varepsilon^2$. $\square$

Lemma 2. *If the value* $-\frac{\mathbf{Q}}{2} \pm \sqrt{\frac{\mathbf{Q^2}}{4} + \frac{\mathbf{P^3}}{27}}$ *is a cubic residue in* $GF(p^2)$*, then the value* $-\frac{\mathbf{Q}}{2} \mp \sqrt{\frac{\mathbf{Q^2}}{4} + \frac{\mathbf{P^3}}{27}}$ *is also a cubic residue.*

Proof. We have

$$\left(-\frac{\mathbf{Q}}{2} \pm \sqrt{\frac{\mathbf{Q^2}}{4} + \frac{\mathbf{P^3}}{27}}\right)^{\frac{p^2-1}{3}} \left(-\frac{\mathbf{Q}}{2} \mp \sqrt{\frac{\mathbf{Q^2}}{4} + \frac{\mathbf{P^3}}{27}}\right)^{\frac{p^2-1}{3}} =$$

$$\left(-\frac{\mathbf{P^3}}{27}\right)^{\frac{p^2-1}{3}} = \left(-\frac{\mathbf{P}}{3}\right)^{p^2-1} = (1,0).$$

For cubic residue $\left(-\frac{\mathbf{Q}}{2} \pm \sqrt{\frac{\mathbf{Q^2}}{4} + \frac{\mathbf{P^3}}{27}}\right)$ we have $\left(-\frac{\mathbf{Q}}{2} \pm \sqrt{\frac{\mathbf{Q^2}}{4} + \frac{\mathbf{P^3}}{27}}\right)^{\frac{p^2-1}{3}} = (1,0)$. Therefore $\left(-\frac{\mathbf{Q}}{2} \mp \sqrt{\frac{\mathbf{Q^2}}{4} + \frac{\mathbf{P^3}}{27}}\right)^{\frac{p^2-1}{3}} = (1,0)$, i. e. the value $\left(-\frac{\mathbf{Q}}{2} \mp \sqrt{\frac{\mathbf{Q^2}}{4} + \frac{\mathbf{P^3}}{27}}\right)$ is a cubic residue in $GF(p^2)$. □

For some vector $\mathbf{V} = (v,u) \in GF(p^2)$ one can define $\overline{\mathbf{V}} = (v,-u) \in GF(p^2)$.

Lemma 3. *Suppose* $\mathbf{V} = (v,u) \in GF(p^2)$ *is a cubic residue and* $\mathbf{R}$ *is one of the cubic roots from* $\mathbf{V}$*. Then* $\overline{\mathbf{R}}$ *is one of cubic roots from* $\overline{\mathbf{V}}$*.*

Proof. Using formula (6) for some element $(a,b) \in GF(p^2)$ it is easy to get $\left(\overline{(a,b)}\right)^3 = \overline{(a,b)^3}$. For $\mathbf{R}$ we have $\overline{\mathbf{R}}^{\mathbf{3}} = \overline{\mathbf{R^3}} = \overline{\mathbf{V}}$. □

For other two cubic roots from $\mathbf{V}$, i.e. for $\varepsilon\mathbf{R}$ and $\varepsilon^2\mathbf{R}$, we have $\left(\overline{\varepsilon\mathbf{R}}\right)^3 = \overline{\mathbf{V}}$ and $\left(\overline{\varepsilon^2\mathbf{R}}\right)^3 = \overline{\mathbf{V}}$, therefore one can write $\sqrt[3]{\overline{\mathbf{V}}} = \overline{\sqrt[3]{\mathbf{V}}}$.

Lemma 4. *Suppose number* 3 *does not divide* $p-1$ *and* ε, $\varepsilon^2 \in GF(p^2)$ *are two non-trivial cubic roots from the unity element* $(1,0)$*. Then* $\overline{\varepsilon} = \varepsilon^2$ *and* $\overline{\varepsilon}^2 = \varepsilon$*.*

Proof. Taking into account Lemma 3 we have $\overline{\varepsilon}^3 = \overline{\varepsilon^3} = \overline{(1,0)} = (1,0)$ hence $\overline{\varepsilon}$ is one of two non-trivial roots from $(1,0)$ that differs from ε. Therefore $\overline{\varepsilon} = \varepsilon^2$ and $\overline{\varepsilon}^2 = \overline{\varepsilon^2} = \varepsilon$. □

Lemma 5. *Suppose* $a \in GF(p)$ *is a quadratic non-residue. Then for* $(a,0) \in GF(p^2)$ *we have* $\sqrt{(a,0)} = \left(0, \pm\sqrt{k^{-1}a}\right)$*, where* k *is the quadratic non-residue used to define the multiplication operation in* $GF(p^2)$ *with formula (6).*

Proof. Using formula (6) we get $\left(0, \pm\sqrt{k^{-1}a}\right)^2 = (a,0)$. □

In general case computation in (11) should be performed in the field $GF(p^2)$, since the value $\frac{\mathbf{Q^2}}{4} + \frac{\mathbf{P^3}}{27}$ can be equal to a quadratic non-residue in the field $GF(p)$.

In the case under consideration $p > 3$, therefore number 3 divides the value $p^2 - 1$ and there exist three different cubic roots in $GF(p^2)$ from each of the values $\left(-\frac{\mathbf{Q}}{2} + \sqrt{\frac{\mathbf{Q^2}}{4} + \frac{\mathbf{P^3}}{27}}\right)$ and $\left(-\frac{\mathbf{Q}}{2} - \sqrt{\frac{\mathbf{Q^2}}{4} + \frac{\mathbf{P^3}}{27}}\right)$. Three values α and three values β define all roots of (8). Types of the lasts depend on the value $\Delta = \frac{\mathbf{Q^2}}{4} + \frac{\mathbf{P^3}}{27} \bmod p$.

4.1 Δ is a quadratic non-residue in $GF(p)$

If Δ is equal to a quadratic non-residue in $GF(p)$, then in formulas (9) and (10) elements α and β are two-dimension vectors the second coordinate of which is not equal to zero. Suppose $\alpha = \mathbf{K}$ and $\beta = \overline{\mathbf{K}}$ are cubic roots from $\alpha^3 = -\mathbf{Q}/2 + \left(0, \sqrt{k^{-1}\Delta}\right)$ and $\beta^3 = -\mathbf{Q}/2 - \left(0, \sqrt{k^{-1}\Delta}\right)$, respectively. Then $\alpha' = \varepsilon\mathbf{K}$ and $\alpha'' = \varepsilon^2\mathbf{K}$ ($\beta' = \overline{\alpha'} = \overline{\varepsilon\mathbf{K}} = \varepsilon^2\overline{\mathbf{K}}$ and $\beta'' = \overline{\alpha''} = \overline{\varepsilon^2\mathbf{K}} = \varepsilon\overline{\mathbf{K}}$) are also cubic roots from α^3 (β^3).

There are possible the following two cases.

Case 1. $3 \nmid (p-1)$. In this case $\mathbf{K}\overline{\mathbf{K}} = -\mathbf{P}/3$. Indeed, $\mathbf{K}\overline{\mathbf{K}} \in GF(p)$, $\varepsilon \in GF(p^2)$, and $\mathbf{P} = (P, 0) \in GF(p)$, therefore $\mathbf{K}\overline{\mathbf{K}} \neq -\varepsilon\mathbf{P}/3$. Each of the following three pairs of the values:

1. $\alpha = \mathbf{K}$ and $\beta = \overline{\mathbf{K}}$;
2. $\alpha' = \varepsilon\mathbf{K}$ and $\beta' = \varepsilon^2\overline{\mathbf{K}}$;
3. $\alpha'' = \varepsilon^2\mathbf{K}$ and $\beta'' = \varepsilon\overline{\mathbf{K}}$;

defines one root of each of the equations (7) and (8), since $\alpha\beta = \alpha'\beta' = \alpha''\beta'' = -\mathbf{P}/3$. These three roots of (7), i.e. the values $\alpha+\beta$, $\alpha'+\beta'$ and $\alpha''+\beta''$, are contained in $GF(p)$. Indeed, for example, $\alpha' + \beta' = \varepsilon\mathbf{K} + \varepsilon^2\overline{\mathbf{K}} = \varepsilon\mathbf{K} + \overline{\varepsilon\mathbf{K}}$. Correspondingly, three roots of (8) are also contained in $GF(p)$.

Case 2. $3|(p-1)$. In this case $\varepsilon \in GF(p)$.

Suppose $\mathbf{K}\overline{\mathbf{K}} = -\varepsilon\mathbf{P}/3$. Then each of the following three pairs of the values:

1. $\alpha = \mathbf{K}$ and $\beta = \varepsilon^2\overline{\mathbf{K}}$;
2. $\alpha' = \varepsilon^2\mathbf{K}$ and $\beta' = \overline{\mathbf{K}}$;
3. $\alpha'' = \varepsilon\mathbf{K}$ and $\beta'' = \varepsilon\overline{\mathbf{K}}$;

defines one root of the equations (7) and (8), since $\alpha\beta = \alpha'\beta' = \alpha''\beta'' = -\mathbf{P}/3$. The first and second roots, i.e. the values $\alpha + \beta = \mathbf{K} + \varepsilon^2\overline{\mathbf{K}}$ and $\alpha' + \beta' = \varepsilon^2\mathbf{K} + \overline{\mathbf{K}}$, are contained in $GF(p^2)$. The third root, i.e. the value $\alpha'' + \beta'' = \varepsilon\mathbf{K} + \varepsilon\overline{\mathbf{K}} = \varepsilon(\mathbf{K} + \overline{\mathbf{K}})$, is contained in $GF(p)$.

Suppose $\mathbf{K}\overline{\mathbf{K}} = -\mathbf{P}/3$. Then each of the following three pairs of the values:

1. $\alpha = \mathbf{K}$ and $\beta = \overline{\mathbf{K}}$.

2. $\alpha' = \varepsilon\mathbf{K}$ and $\beta' = \varepsilon^2\overline{\mathbf{K}}$;

3. $\alpha'' = \varepsilon^2\mathbf{K}$ and $\beta'' = \varepsilon\overline{\mathbf{K}}$;

defines one root of the equations (7) and (8), since $\alpha\beta = \alpha'\beta' = \alpha''\beta'' = -\mathbf{P}/3$. The first root is equal to $\alpha + \beta = \mathbf{K} + \overline{\mathbf{K}}$, i.e. it is contained in $GF(p)$. The second and third roots, i.e. the values $\alpha' + \beta' = \varepsilon\mathbf{K} + \varepsilon^2\overline{\mathbf{K}} = \varepsilon\left(\mathbf{K} + \varepsilon\overline{\mathbf{K}}\right)$ and $\alpha'' + \beta'' = \varepsilon^2\mathbf{K} + \varepsilon\overline{\mathbf{K}} = \varepsilon\left(\varepsilon\mathbf{K} + \overline{\mathbf{K}}\right)$, correspondingly, are contained in $GF(p^2)$.

Thus, in Case 2 we have one root in $GF(p)$ and two roots in $GF(p^2)$.

It should be noted that in this paper there are considered cubic equations over $GF(p)$ which have solutions, therefore we do not consider the case when the value $-\mathbf{Q}/2 + \left(0, \sqrt{k^{-1}\Delta}\right)$ is a cubic non-residue in $GF(p^2)$.

4.2 Δ is a quadratic residue in $GF(p)$

If Δ is equal to a quadratic residue in $GF(p)$, then in formula (11) the values $\left(-\frac{\mathbf{Q}}{2} + \sqrt{\frac{\mathbf{Q^2}}{4} + \frac{\mathbf{P^3}}{27}}\right)$ and $\left(-\frac{\mathbf{Q}}{2} - \sqrt{\frac{\mathbf{Q^2}}{4} + \frac{\mathbf{P^3}}{27}}\right)$ are elements of $GF(p)$. We consider the following two subcases.

Case 1. $3|(p-1)$. If the number $\left(-\frac{\mathbf{Q}}{2} + \sqrt{\Delta}\right)$ is a cubic residue in $GF(p)$, then we have three cubic roots from this number and three cubic roots from $\left(-\frac{\mathbf{Q}}{2} - \sqrt{\Delta}\right)$ that are elements of $GF(p)$, hence all three roots of equation (7) and all three roots of equation (8) are elements of the field $GF(p)$. If $\left(-\frac{\mathbf{Q}}{2} - \sqrt{\Delta}\right)$ is a cubic non-residue in $GF(p)$, then the vector $(\left(-\frac{\mathbf{Q}}{2} + \sqrt{\Delta}\right), 0)$ is a cubic non-residue in $GF(p^2)$, since

$$\left(\left(-\frac{\mathbf{Q}}{2} + \sqrt{\Delta}\right), 0\right)^{\frac{p^2-1}{3}} = \left(\left(-\frac{\mathbf{Q}}{2} + \sqrt{\Delta}\right)^{\frac{p-1}{3}(p+1)}, 0\right) =$$
$$(\varepsilon^{p+1}, 0) \neq (1, 0),$$

where ε is one of two non-trivial cubic roots from 1 in $GF(p)$, and equations (7) and (8) have no solutions. However the last situation is out of the consideration of the cubic equations having a solution.

Case 2. $3 \nmid (p-1)$. In $GF(p)$ there exists one cubic root from $\left(-\frac{\mathbf{Q}}{2} + \sqrt{\Delta}\right)$ and one cubic root from $\left(-\frac{\mathbf{Q}}{2} - \sqrt{\Delta}\right)$. Let $K = \sqrt[3]{-\frac{Q}{2} + \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}}$ and $\tilde{K} = \sqrt[3]{-\frac{Q}{2} - \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}}$. In $GF(p^2)$ there exists two additional cubic roots from each of the values $\left(-\frac{\mathbf{Q}}{2} + \sqrt{\frac{\mathbf{Q^2}}{4} + \frac{\mathbf{P^3}}{27}}\right)$ (the roots $K\varepsilon$ and $K\varepsilon^2$, where $\varepsilon, \varepsilon^2 \in GF(p^2)$ are non-trivial cubic roots from the unity element $(1, 0)$) and $\left(-\frac{\mathbf{Q}}{2} - \sqrt{\frac{\mathbf{Q^2}}{4} + \frac{\mathbf{P^3}}{27}}\right)$ (the roots $\tilde{K}\varepsilon$ and $\tilde{K}\varepsilon^2$). We have $K\tilde{K} = -P/3 \bmod p = \mathbf{P}/3$ and one root of (7) is equal to $K + \tilde{K} \bmod p \in GF(p)$.

We have also $K\varepsilon\tilde{K}\varepsilon^2 = -\mathbf{P}/3$ and $K\varepsilon^2\tilde{K}\varepsilon = -\mathbf{P}/3$ that gives two roots of (7) $K\varepsilon + \tilde{K}\varepsilon^2$ and $K\varepsilon^2 + \tilde{K}\varepsilon$ that are elements of $GF(p^2)$ with the second coordinate different from zero.

4.3 $\Delta = 0$

We consider the following two cases.

Case 1. $3|(p-1)$. If $-Q/2$ is a cubic residue in $GF(p)$, then in $GF(p)$ there exists three cubic roots from $-Q/2$. Suppose these three roots are the values K, $K' = eK$, and $K'' = e^2K$, where $e, e^2 \in GF(p)$ and are non-trivial cubic roots from 1. Then, taking into account that $K^2 = -P/3$, we have the following three roots of (7): $2K$, $K' + K''$, and $K'' + K'$ that are elements of $GF(p)$, the last two roots being equal.

If $-Q/2$ is a cubic non-residue in $GF(p)$, then the vector $(-Q/2, 0)$ is a cubic non-residue in $GF(p^2)$, since

$$(-Q/2,0)^{\frac{p^2-1}{3}} = \left((-Q/2)^{\frac{p-1}{3}(p+1)}, 0\right) = \left(e^{p+1}, 0\right) \neq (1,0),$$

where $e \in GF(p)$ is a cubic root from 1, and there are no solutions for (7) and (8), therefore the last situation is out of the consideration of the cubic equations over $GF(p)$ which have solutions.

Case 2. $3 \nmid (p-1)$. In $GF(p)$ there exists one cubic root from $-Q/2$. Let $K = \sqrt[3]{-Q/2} \bmod p$. In $GF(p^2)$ there exists two additional cubic roots from $-Q/2$, namely, the roots $K' = K\varepsilon$ and $K'' = K\varepsilon^2$, where $\varepsilon, \varepsilon^2 \in GF(p^2)$ are non-trivial cubic roots from the unity element $(1,0)$. Taking into account that $K^2 = -P/3$, we have the following three roots of equation (7): $2K \in GF(p)$, $(K' + K''), (K'' + K') \in GF(p^2)$, the last two being equal.

Table 1. Number and type of roots of cubic equation (7) with condition that this equation has a solution.

Δ is a quadratic non-residue in $GF(p)$		Δ is a quadratic residue in GF(p)		$\Delta = 0$ $Q^2/4 = -P^3/27 \bmod p$	
$3 \nmid (p-1)$	$3|(p-1)$	$3|(p-1)$	$3 \nmid (p-1)$	$3|(p-1)$	$3 \nmid (p-1)$
Three different roots contained in $GF(p)$	One root in $GF(p)$ and two different roots in $GF(p^2)$	Three different roots contained in $GF(p)$	One root in $GF(p)$ and two different roots in $GF(p^2)$	Three roots contained in $GF(p)$ two of them being equal	One root in $GF(p)$ and two equal roots in $GF(p^2)$

5 Public encryption cryptoscheme free from the decryption ambiguity

To avoid the decryption ambiguity one can put the cubic equation that relates to the Case 2 from Subsections 4.1 and 4.2 into the base of public encryption algorithm. To encrypt the message M one is to generate random numbers T and U such that the value $T^2/4 - U$ is quadratic non-residue modulo p and modulo q and then compute the cryptogram in form of the coefficients of the following cubic equation

$$(x - M)(x^2 + Tx + U) = x^3 - (M + T)x^2 + (U - TM)x - MU = 0 \bmod n.$$

Thus, the enciphering procedure that consists in computing the following three coefficients that compose the ciphertext $C = (A, B, D)$:

$$\begin{aligned} A &= M + T \bmod n, \\ B &= U - TM \bmod n, \\ D &= MU \bmod n. \end{aligned} \tag{13}$$

The first step of the public encryption, i.e. finding a value that is equal to a non-residue mod$n$, cannot be surely performed without knowing prime divisors of n. Therefore a non-residue N is to be generated by owner of the public key, i.e. he generates his public key as the pair of numbers n and N. Using such public key the encryption of the message M is to be performed as follows:

1. Generate a random number T and compute the value $U = T^2/4 - N \bmod n$.
2. Compute the cryptogram $C = (A, B, D)$ using formulas (13).

Decryption of the cryptogram C consists in finding the roots of the equation (1) which are contained in Z_n. Each of the equations (2) and (3) has a unique solution in $GF(p)$ and $GF(q)$, respectively. Therefore there exists only one root of equation (1) that can be computed solving the following system of two congruences

$$\begin{cases} M \equiv M_p \bmod p \\ M \equiv M_q \bmod q, \end{cases} \tag{14}$$

where $M_p \in GF(p)$ and $M_q \in GF(q)$ are roots of equations (2) and (3), respectively. In correspondence with the Chinese remainder theorem the solution of the system (14) is

$$M = \left[M_p q \left(q^{-1} \bmod p\right) + M_q p \left(p^{-1} \bmod q\right)\right] \bmod pq.$$

One of steps of the decryption procedure is finding cubic roots in the field of the two-dimension vectors defined over the ground finite field. Next section considers this case.

6 Finding cubic roots in $GF(p^2)$

Since $3|p^2 - 1$, there exist three cubic roots from a cubic residue $\mathbf{Y}$ in $GF(p^2)$. In the case $p^2 = 7 \bmod 9$ it is rather simple to compute one cubic root $\mathbf{J} = \mathbf{Y}^{1/3}$

using the following formula

$$\mathbf{J} = \mathbf{Y}^{\frac{p^2+2}{3}}.$$

Proof that this formula works is as follows

$$\mathbf{J}^3 = (\mathbf{Y})^{\frac{p^2+2}{3}} = \mathbf{Y}\,(\mathbf{Y})^{\frac{p^2-1}{3}} = \mathbf{Y}.$$

Thus, to find a cubic root (if it exists) in the case $3|(p^2-1)$ it is sufficient to perform one exponentiation operation. Two other roots can be computed multiplying the last by the non-trivial roots from the unity element $(1,0)$. For some arbitrary prime p finding cubic roots in $GF(p^2)$ can be performed with method like that described in [7] for finding cubic roots in $GF(p)$, where $3|(p-1)$.

7 Bi-deniable hybrid-encryption protocol secure against active coercer

The public encryption algorithm proposed in Section 5 can be used for designing bi-deniable encryption protocol as follows. The idea is to include in the protocol the entity authentication stage that provides protection against active attackers, including the case of active coercer, and possibility to implement the hidden exchange of single-use public keys [9, 10]. The single-use public keys are used to agree the single-use shared key with which the secret message is derived from the ciphertext directed from sender to receiver. While using the private keys of the sender and receiver and all values sent via communication channel, after the secret communication terminates the coercive attacker is able only to disclose a fake message from the ciphertext.

Suppose $y_A = g^{x_A} \bmod p'$ and $y_B = g^{x_B} \bmod p'$, where p' is a sufficiently large prime and g is a primitive element modulo p', are public keys of the sender and receiver, correspondingly, that are to be used in frame of the ElGamal's signature scheme [11]. The values x_A and x_B are their private keys. Additionally the receiver has other public key (n, N) that is to be used in frame of the public encryption scheme described in Section 5.

The following protocol, where Alice is the sender of secret message $S < n$ and Bob is receiver, presents the bi-deniable hybrid encryption scheme.

1. Alice generates a uniformly random value $k_A < p'-1$ and computes the value $R_A = g^{k_A} \bmod p'$ and her signature $Sign_A(R_A)$ to R_A. Then she sends the values $Sign_A(R_A)$ and R_A to Bob.

2. Bob verifies the signature $Sign_A(R_A)$. If the signature is invalid he terminates the communication session. Otherwise he generates a uniformly random value $k_B < p'-1$ and computes the value $R_B = g^{k_B} \bmod p'$, his signatures $Sign_B(R_B)$ to R_B and his signature $Sign_B(R_A)$ to R_A. Then he sends the values R_B, $Sign_B(R_B)$, and $Sign_B(R_A)$ to Alice.

3. Alice verifies the signatures $Sign_B(R_B)$ and $Sign_B(R_A)$. If at least one of the signatures is invalid she terminates the communication session. Otherwise she

generates a fake message $M < n$ and encrypts simultaneously two messages S and M as follows:

3.1. Compute the common key related to the public keys y_A and y_B : $Z_{AB} = y_B^{x_A} \bmod p'$.

3.2. Compute the common single-use key related to the single-use public keys R_A and $R_B : W_{AB} = R_B^{k_A} \bmod p'$.

3.3. Compute the values $T = W_{AB}S \bmod n$ and $U = T^2/4 - N \bmod n$. Then, using the public-encryption algorithm described in Section 5, compute the cryptogram $C = (A, B, D)$ and send C to Bob.

Bob discloses the secret message using the following algorithm.

Decryption algorithm.

1. Using his private key (p, q) Bob finds the root M of equation (1) with coefficients A, B, and D taken from the cryptogram C.

2. Then Bob computes the secret message S as follows:

2.1. Compute the common single-use key related to the single-use public keys R_A and $R_B : W_{AB} = R_A^{k_B} \bmod p'$.

2.2. Compute the value $T = (A - M) \bmod n$.

2.3. Compute the secret message $S = T{W_{AB}}^{-1} \bmod n$.

Dishonest decryption algorithm:

Using Bob's private key (p, q) the coercer computes the root M from the equation (1) with coefficients taken from the cryptogram.

The coercer is able to compute the value $T = A - M \bmod n$, however he is not able to distinguish the values R_A, R_B, and T from uniformly random values and to disclose the secret message S (until he solves the discrete logarithm problem modulo p'), even if he is provided with private keys x_A and x_B.

8 Conclusion

We considered a method for computing the roots of cubic equation over the ground finite field $GF(p)$ in the case when the equation definitely has solutions. This case takes place in the public encryption scheme characterized in simultaneous encryption of three messages [8]. This scheme includes the decryption procedure that is ambiguous. Using the obtained results related to analysis of number and type of the roots of the cubic equations we have proposed a new method for public encryption based on solving the cubic equations, which is free from ambiguity of the decryption procedure. The proposed method has been used to design a new bi-deniable encryption protocol that is sufficiently practical.

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References

[1] Rabin M. O. *Digitalized signatures and public key functions as intractable as factorization.* Technical report MIT/LCS/TR-212, MIT Laboratory for Computer Science, 1979.

[2] Gordon J. *Strong primes are easy to find.* Advances in cryptology – EUROCRYPT'84, Springer-Verlag LNCS, 1985, **209**, p. 216–223.

[3] Moldovyan N. A., Moldovyan A. A. *Class of Provably Secure Information Authentication Systems.* Springer Verlag CCIS, 2007, **1**, p. 147-152.

[4] Moldovyan N. A., Moldovyan A. A., Shcherbacov V. A. *Provably Sender-Deniable Encryption Scheme.* Proceedings of The Third Conference of Mathematical Society of the Republic of Moldova (IMCS-50). Chisinau, 19-23 August, Institute of Mathematics and Computer Science, Academy of Sciences of Moldova, 2014, p. 134-141.

[5] Moldovyan N. A., Moldovyanu P. A. *Vector form of the finite fields $GF(p^m)$.* Bul. Acad. Ştiinţe Repub. Mold., Mat., 2009, No. 3, p. 57-63.

[6] Alexeev V. B. *Abel theorem in problems and solutions.* MTSNMO, Moscow, 2001 (in Russian).

[7] Nishihara N., Harasawa R., Sueyoshi Y., Kudo A. *Root computation in finite fields.* IEICE Trans. Fundamentals, 2013, **E96-A**, No. 6, p. 1081-1087.

[8] Moldovyan N. A., Moldovyan A. A., Shcherbacov V. A. *Provably sender-deniable encryption scheme.* Computer Science Journal of Moldova, 2015, **23,** No. 1(67), p. 62-71.

[9] Moldovyan A. A., Moldovyan N. A., Shcherbacov V. A. *Bi-Deniable Public-Key Encryption Protocol Secure Against Active Coercive Adversary.* Bul. Acad. Ştiinţe Repub. Mold., Mat., 2014, No. 3(76), p. 23-29.

[10] Moldovyan A. A., Moldovyan N. A. *Practical Method for Bi-Deniable Public-Key Encryption.* Quasigroups and related systems, 2014, **22**, p. 277-282.

[11] ElGamal T. *A public key cryptosystem and a signature scheme based on discrete logarithms.* IEEE Transactions on Information Theory, 1985, **IT-31**, No. 4. p. 469-472.

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Determining the Distribution of the Duration of Stationary Games for Zero-Order Markov Processes with Final Sequence of States

Alexandru Lazari

Abstract. A zero-order Markov process with final sequence of states represents a stochastic system with independent transitions that stops its evolution as soon as given final sequence of states is reached. The transition time of the system is unitary and the transition probability depends only on the destination state. We consider the following game. Initially, each player defines his distribution of the states. The initial distribution of the states is established according to the distribution given by the first player. After that, the stochastic system passes consecutively to the next state according to the distribution given by the next player. After the last player, the first player acts on the system evolution and the game continues in this way until the given final sequence of states is achieved. Our goal is to study the duration of this game, knowing the distribution established by each player and the final sequence of states of the stochastic system. It is proved that the distribution of the duration of the game is a homogeneous linear recurrent sequence and it is developed a polynomial algorithm to determine the initial state and the generating vector of this recurrence. Using the generating function, the main probabilistic characteristics are determined.

Mathematics subject classification: 65C40, 60J22, 90C39, 90C40.
Keywords and phrases: Zero-Order Markov Process, Final Sequence of States, Duration, Game, Homogeneous Linear Recurrence, Generating Function.

1 Introduction and Problem Formulation

Let L be a discrete stochastic system with finite set of states V, $|V| = \omega$. At every discrete moment of time $t \in \mathbb{N}$ the state of the system is $v(t) \in V$. The system L starts its evolution from the state v with the probability $p^*(v)$, for all $v \in V$, where $\sum\limits_{v \in V} p^*(v) = 1$.

Also, the transition from one state u to another state v is performed according to the same probability $p^*(v)$ that depends only on the destination state v, for every $u \in V$ and $v \in V$. Additionally we assume that a sequence of states $X = (x_1, x_2, \ldots, x_m) \in V^m$ is given and the stochastic system stops transitions as soon as the states $x_1, x_2, \ldots, x_m$ are reached consecutively in given order. The time T when the system stops is called evolution time of the stochastic system L with given final sequence of states X.

The stochastic system L described above represents a zero-order Markov process with final sequence of states X. Several interpretations of these Markov processes were analyzed in 1981 by Leo J. Guibas and Andrew M. Odlyzko in [8] and

G. Zbaganu in 1992 in [7]. Various problems related to such systems have been studied in [1]–[5]. Also, in these papers, polynomial algorithms for determining the main probabilistic characteristics (expectation, variance, mean square deviation, n-order moments) of evolution time of the given stochastic system L were proposed.

Next, in this paper, a generalization of this problem is studied. The following game is considered. Initially, each player $\mathcal{P}_\ell$ defines his distribution of the states $(p^{*(\ell)}(v))_{v\in V}$, $\ell = \overline{0, r-1}$. The initial distribution of the states is established according to the distribution $(p^{*(0)}(v))_{v\in V}$ given by the first player $\mathcal{P}_0$. After that, the stochastic system passes consecutively to the next state according to the distribution given by the next player. After the last player $\mathcal{P}_{r-1}$, the first player $\mathcal{P}_0$ acts on the system evolution and the game continues in this way until the given final sequence of states X is achieved. The player $\mathcal{P}_{T mod\ r}$, who acts the last on the system, is considered the winner of the game.

Our goal is to study the duration T of this game, knowing the distribution $p^{*(\ell)} = (p^{*(\ell)}(v))_{v\in V}$ established by each player $\mathcal{P}_\ell$, $\ell = \overline{0, r-1}$, and the final sequence of states X of the stochastic system L. We will prove that the distribution of the game duration T is a homogeneous linear recurrent sequence ([1],[6]) and a polynomial algorithm to determine the initial state and the generating vector of this recurrence will be developed. Having the generating vector and the initial state of the recurrence, we can use the related algorithm from [1], which was mentioned above, for determining the main probabilistic characteristics of the game duration.

2 The Main Results

2.1 Determining The Distribution of the Game Duration

In this section we will determine the distribution law of the game duration T. Initially, we consider the notations

$$X_k = \{x_k\},\ \overline{X}_k = V\backslash\{x_k\},\ \pi_k^{(\ell)} = p^{*(\ell)}(x_k), w_k^{(\ell)} = \prod_{j=2}^{k} \pi_j^{(\ell\ \oplus(-1)\oplus j)},$$

for each $k = \overline{1, m}$ and $\ell = \overline{0, r-1}$, where $c \oplus d = (c + d) \mod r$, $\forall c, d \in \mathbb{Z}$.

Let $a = (a_n)_{n=0}^{\infty}$ be the distribution of the game duration T, i.e. $a_n = \mathbb{P}(T = n)$, $n = \overline{0, \infty}$. Since $T \geq m - 1$, we have $a_n = 0$, $n = \overline{0, m-2}$. If $T = m - 1$, then $v(j) = x_{j+1}$, $j = \overline{0, m-1}$, that implies

$$a_{m-1} = \mathbb{P}(T = m-1) = \pi_1^{(0)}\pi_2^{(1)} \dots \pi_m^{(m\oplus(-1))} = \pi_1^{(0)} w_m^{(0)}.$$

We consider $\forall n \in \mathbb{Z}$. Let be $S(V) = \{A \mid A \subseteq V\}$. Denote by $P_\Phi^{(\ell)}(n)$ the probability that $T = n$, $v(j) \in \Phi_j$, $j = \overline{0, t-1}$ and the player $\mathcal{P}_\ell$ acts first, for all $\Phi = (\Phi_j)_{j=0}^{t-1} \in (S(V))^t$, $t \in \mathbb{N}$ and $\ell = \overline{0, r-1}$. We introduce the following functions

on $\mathbb{Z}$, $k=\overline{0,m}$, $\ell=\overline{0,r-1}$:

$$\begin{aligned}\alpha_k^{(\ell)}(n)&=P^{(\ell)}_{(X_1,X_2,\dots,X_{k-1},\overline{X}_k)}(n),\\ \beta_k^{(\ell)}(n)&=P^{(\ell)}_{(X_1,X_2,\dots,X_k)}(n),\\ \gamma_k^{(\ell)}(n)&=P^{(\ell)}_{(X_2,X_3,\dots,X_k)}(n).\end{aligned}\tag{1}$$

We have

$$\beta_k^{(\ell)}(n)=P^{(\ell)}_{(X_1,X_2,\dots,X_k)}(n)=a_n^{(\ell)}-\sum_{j=1}^{k}\alpha_j^{(\ell)}(n),\ k=\overline{0,m},\ell=\overline{0,r-1},\tag{2}$$

where $a_n^{(\ell)}=P^{(\ell)}_{(\ )}(n)$, $\ell=\overline{0,r-1}$.

We consider the sets

$$T_s=\{s+1\}\cup\{t\in\{2,3,\dots,s\}\mid x_{t-1+j}=x_j,\ j=\overline{1,s+1-t}\},\ s=\overline{1,m}.$$

The minimal elements from these sets are

$$t_s=\min_{k\in T_s}k,\ s=\overline{1,m}.\tag{3}$$

The value t_s represents the auto superposition level of the sequence $(x_1,x_2,\dots,x_s)$, i.e. t_s is the position in the sequence $(x_1,x_2,\dots,x_s)$ starting with which, if we overlap the same sequence, the superposed elements are equal.

Using the formula (2) for $s=\overline{1,m}$ and $\ell=\overline{0,r-1}$, we obtain

$$\begin{aligned}\gamma_s^{(\ell)}(n)&=P^{(\ell)}_{(X_2,X_3,\dots,X_s)}(n)=\\ &=\pi_2^{(\ell)}\pi_3^{(\ell\oplus 1)}\dots\pi_{t_s-1}^{(\ell\oplus(t_s-3))}P^{(\ell\oplus(t_s-2))}_{(X_{t_s},X_{t_s+1},\dots,X_s)}(n-t_s+2)=\\ &=w_{t_s-1}^{(\ell\oplus(-1))}\beta_{s+1-t_s}^{(\ell\oplus(t_s-2))}(n-t_s+2)=\\ &=w_{t_s-1}^{(\ell\oplus(-1))}\left(a_{n-t_s+2}^{(\ell\oplus(t_s-2))}-\sum_{j=1}^{s+1-t_s}\alpha_j^{(\ell\oplus(t_s-2))}(n-t_s+2)\right).\end{aligned}\tag{4}$$

Particularly, we obtain the relation

$$\gamma_1^{(\ell)}(n)=a_n^{(\ell)},\ \ell=\overline{0,r-1},\ n=\overline{0,\infty}.\tag{5}$$

The values $\alpha_k^{(\ell)}(n)$, $k=\overline{1,m}$, $\ell=\overline{0,r-1}$ are determined in the following way:

$$\alpha_1^{\ell}(n)=P^{(\ell)}_{(\overline{X}_1)}(n)=(1-\pi_1^{(\ell)})a_{n-1}^{(\ell\oplus 1)},\tag{6}$$

$$\alpha_k^{(\ell)}(n)=P^{(\ell)}_{(X_1,X_2,\dots,X_{k-1},\overline{X}_k)}(n)=\pi_1^{(\ell)}P^{(\ell\oplus 1)}_{(X_2,X_3,\dots,X_{k-1},\overline{X}_k)}(n-1)=$$

$$= \pi_1^{(\ell)} \left(P^{(\ell \oplus 1)}_{(X_2, X_3, \ldots, X_{k-1})}(n-1) - P^{(\ell \oplus 1)}_{(X_2, X_3, \ldots, X_k)}(n-1) \right) =$$
$$= \pi_1^{(\ell)} \left(\gamma_{k-1}^{(\ell \oplus 1)}(n-1) - \gamma_k^{(\ell \oplus 1)}(n-1) \right), \ k = \overline{2, m}. \tag{7}$$

Next, we obtain the recurrent formula:

$$a_n^{(\ell)} = \sum_{j=1}^{m} \alpha_j^{(\ell)}(n) = (1 - \pi_1^{(\ell)}) a_{n-1}^{(\ell \oplus 1)} + \sum_{j=2}^{m} \pi_1^{(\ell)} \left(\gamma_{j-1}^{(\ell \oplus 1)}(n-1) - \gamma_j^{(\ell \oplus 1)}(n-1) \right) =$$
$$= (1 - \pi_1^{(\ell)}) a_{n-1}^{(\ell \oplus 1)} + \pi_1^{(\ell)} \left(a_{n-1}^{(\ell \oplus 1)} - \gamma_m^{(\ell \oplus 1)}(n-1) \right) =$$
$$= a_{n-1}^{(\ell \oplus 1)} - \pi_1^{(\ell)} \gamma_m^{(\ell \oplus 1)}(n-1), \ \forall n \geq m, \ \ell = \overline{0, r-1}. \tag{8}$$

According to the relations (4) – (7), using the mathematical induction, we can prove that there exist the real coefficients $u_{jk\ell}^{(i)}$ and $v_{jk\ell}^{(i)}$, $j = \overline{1, m}$, $k = \overline{0, j-1}$, $\ell = \overline{0, r-1}$, $i = \overline{0, r-1}$ such that

$$\alpha_j^{(\ell)}(n) = \sum_{i=0}^{r-1} \sum_{k=0}^{j-1} u_{jk\ell}^{(i)} \, a_{n-1-k}^{(i)}, \ \gamma_j^{(\ell)}(n-1) = \sum_{i=0}^{r-1} \sum_{k=0}^{j-1} v_{jk\ell}^{(i)} \, a_{n-1-k}^{(i)}, \ \forall n \in \mathbb{Z}. \tag{9}$$

From the relations (5) and (6), for $i = \overline{0, r-1}$ and $\ell = \overline{0, r-1}$, we obtain

$$u_{1,0,\ell}^{(i)} = \begin{cases} 1 - \pi_1^{(\ell)}, & \text{if } \ i = \ell \oplus 1 \\ 0, & \text{if } \ i \neq \ell \oplus 1 \end{cases} \tag{10}$$

and

$$v_{1,0,\ell}^{(i)} = \begin{cases} 1, & \text{if } \ i = \ell \\ 0, & \text{if } \ i \neq \ell \end{cases} . \tag{11}$$

Using the representation (9), the formula (4) obtains the form

$$\gamma_s^{(\ell)}(n-1) = w_{t_s-1}^{(\ell \oplus (-1))} \left(a_{(n-1)-t_s+2}^{(\ell \oplus (t_s-2))} - \sum_{j=1}^{s+1-t_s} \sum_{i=0}^{r-1} \sum_{k=0}^{j-1} u_{j,k,\ell \oplus (t_s-2)}^{(i)} a_{n-t_s-k}^{(i)} \right) =$$
$$= w_{t_s-1}^{(\ell \oplus (-1))} \left(a_{(n-1)-(t_s-2)}^{(\ell \oplus (t_s-2))} - \sum_{i=0}^{r-1} \sum_{k=t_s-1}^{s-1} a_{n-1-k}^{(i)} \sum_{j=k-t_s+2}^{s+1-t_s} u_{j, \ k-t_s+1, \ \ell \oplus (t_s-2)}^{(i)} \right) =$$
$$= \sum_{i=0}^{r-1} \sum_{k=0}^{s-1} v_{sk\ell}^{(i)} \, a_{n-1-k}^{(i)}, \ s = \overline{1, m}, \ \ell = \overline{0, r-1},$$

where

$$v_{s,k,\ell}^{(i)} = \begin{cases} 0, & \text{if } \ k \leq t_s - 3 \\ 0, & \text{if } \begin{cases} k = t_s - 2, \\ i \neq \ell \oplus (t_s - 2) \end{cases} \\ w_{t_s-1}^{(\ell \oplus (-1))}, & \text{if } \begin{cases} k = t_s - 2, \\ i = \ell \oplus (t_s - 2) \end{cases} \\ -w_{t_s-1}^{(\ell \oplus (-1))} \sum\limits_{j=k-t_s+2}^{s+1-t_s} u_{j, \ k-t_s+1, \ \ell \oplus (t_s-2)}^{(i)}, & \text{if } \ k \geq t_s - 1 \end{cases} \tag{12}$$

$s=\overline{1,m}$, $k=\overline{0,s-1}$, $\ell=\overline{0,r-1}$, $i=\overline{0,r-1}$, and the formula (7) becomes

$$\alpha_s^{(\ell)}(n)=\pi_1^{(\ell)}\left(\gamma_{s-1}^{(\ell\oplus 1)}(n-1)-\gamma_s^{(\ell\oplus 1)}(n-1)\right)=$$

$$=\pi_1^{(\ell)}\left(\sum_{i=0}^{r-1}\sum_{k=0}^{s-2}v_{s-1,\ k,\ \ell\oplus 1}^{(i)}a_{n-1-k}^{(i)}-\sum_{i=0}^{r-1}\sum_{k=0}^{s-1}v_{s,\ k,\ \ell\oplus 1}^{(i)}a_{n-1-k}^{(i)}\right)=$$

$$=\sum_{i=0}^{r-1}\sum_{k=0}^{s-1}u_{sk\ell}^{(i)}\,a_{n-1-k}^{(i)},\ s=\overline{2,m},\ \ell=\overline{0,r-1},$$

where

$$u_{s,k,\ell}^{(i)}=\begin{cases}\pi_1^{(\ell)}\left(v_{s-1,\ k,\ \ell\oplus 1}^{(i)}-v_{s,\ k,\ \ell\oplus 1}^{(i)}\right), & \text{if } k\le s-2\\ -\pi_1^{(\ell)}v_{s,\ k,\ \ell\oplus 1}^{(i)}, & \text{if } k=s-1\end{cases}, \tag{13}$$

$s=\overline{2,m}$, $k=\overline{0,s-1}$, $\ell=\overline{0,r-1}$, $i=\overline{0,r-1}$. The formula (8) obtains the form

$$a_n^{(\ell)}=a_{n-1}^{(\ell\oplus 1)}-\pi_1^{(\ell)}\gamma_m^{(\ell\oplus 1)}(n-1)=a_{n-1}^{(\ell\oplus 1)}-\pi_1^{(\ell)}\sum_{i=0}^{r-1}\sum_{k=0}^{m-1}v_{m,\ k,\ \ell\oplus 1}^{(i)}a_{n-1-k}^{(i)}=$$

$$=\sum_{i=0}^{r-1}\sum_{k=0}^{m-1}q_{ik}^{(\ell)}a_{n-1-k}^{(i)},\ \forall n\ge m,\ \ell=\overline{0,r-1}, \tag{14}$$

where

$$q_{ik}^{(\ell)}=\begin{cases}1-\pi_1^{(\ell)}v_{m,\ 0,\ \ell\oplus 1}^{(\ell\oplus 1)}, & \text{if } i=\ell\oplus 1 \text{ and } k=0\\ -\pi_1^{(\ell)}v_{m,\ k,\ \ell\oplus 1}^{(i)}, & \text{otherwise}\end{cases}. \tag{15}$$

Next, we consider the column vectors $A_n=((a_n^{(\ell)})_{\ell=0}^{r-1})^T$, $n=\overline{0,\infty}$. Also, we define the matrices $Q^{(k)}=(q_{ik}^{(\ell)})_{\ell,\ i=\overline{0,\ r-1}}$, $k=\overline{0,m-1}$ and we consider the sequence $Q=(Q^{(k)})_{k=0}^{m-1}$. From the relation (14), we have the homogeneous linear recurrence $A_n=\sum\limits_{k=0}^{m-1}Q^{(k)}A_{n-1-k}$, $\forall n\ge m$, i.e. $A=(A_n)_{n=0}^{\infty}\in Rol^*[\mathcal{M}_r(\mathbb{R})][m]$ with generating vector $Q\in G^*[\mathcal{M}_r(\mathbb{R})][m](A)$. So, the vectorial sequence A is homogeneous linear recurrent on the matrix field $\mathcal{M}_r(\mathbb{R})$ with generating vector Q. Applying the results obtained in [1], we have $A\in Rol^*[\mathbb{R}][mr]$ with characteristic polynomial $H(z)=|I_r-zG_m^{[Q]}(z)|\in H^*[\mathbb{R}][mr](A)$, which implies that $a^{(\ell)}=(a_n^{(\ell)})_{n=0}^{\infty}\in Rol^*[\mathbb{R}][mr]$ and $H(z)=|I_r-zG_m^{[Q]}(z)|\in H^*[\mathbb{R}][mr](a^{(\ell)})$, $\ell=\overline{0,r-1}$. Because the game is started by player $\mathcal{P}^{(0)}$, then the distribution a of the game duration T coincides with $a^{(0)}$, i.e. $a=(a_n)_{n=0}^{\infty}\in Rol^*[\mathbb{R}][mr]$ with characteristic polynomial $H(z)=|I_r-zG_m^{[Q]}(z)|\in H^*[\mathbb{R}][mr](a)$.

Next, we will use only the relation $a\in Rol^*[\mathbb{C}][mr]$, the minimal generating vector being determined by use of the minimization method based on the matrix rank, described in [1], that is available also for degenerated homogeneous linear

recurrences. So, according to this method, we have that the minimal generating vector $q = (q_0, q_1, \ldots, q_{R-1}) \in G^*[\mathbb{C}][R](a)$ is obtained from the unique solution $x = (q_{R-1}, q_{R-2}, \ldots, q_0)$ of the system

$$A_R^{[a]} x^T = (f_R^{[a]})^T, \tag{16}$$

where

$$f_R^{[a]} = (a_R, a_{R+1}, \ldots, a_{2R-1}),\ A_n^{[a]} = (a_{i+j})_{i,j=\overline{0,n-1}},\ \forall n \in \mathbb{N}^* \tag{17}$$

and R is the rank of the matrix $A_{mr}^{[a]}$.

For this, we need to have only the values a_k, $k = \overline{0, 2mr-1}$. These values are determined the formula

$$a_k = a_k^{(0)},\ k = \overline{0, 2mr-1}, \tag{18}$$

using the relations (4) – (8) and the initial conditions

$$a_n = a_n^{(\ell)} = P^{(\ell)}(n) = 0,\ \ell = 0, r-1,\ n = \overline{0, m-2},$$

$$\alpha_k^{(\ell)}(n) = 0,\ k = \overline{1, m},\ n = \overline{0, m-1},\ \ell = \overline{0, r-1},$$

$$a_{m-1}^{(\ell)} = \pi_1^{(\ell)} w_m^{(\ell)},\ \ell = \overline{0, r-1}. \tag{19}$$

2.2 Describing the developed algorithm

In the previous subsection we theoretically grounded the following algorithm for determining the main probabilistic characteristics (the distribution $(\mathbb{P}(T = n))_{n=0}^{\infty}$, the expectation $\mathbb{E}(T)$, the variance $\mathbb{V}(T)$, the mean square deviation $\sigma(T)$ and the k-order moments $\nu_k(T)$, $k = 1, 2, \ldots$) of the game duration T.

Algorithm 1.

Input: $X = (x_1, x_2, \ldots, x_m) \in V^m$, $\pi_k^{(\ell)}$, $k = \overline{1, m}$, $\ell = \overline{0, r-1}$;
Output: $\mathbb{E}(T)$, $\mathbb{V}(T)$, $\sigma(T)$, $\nu_k(T)$, $k = \overline{1, t}$, $t \geq 2$.

1. Determine the values a_k, $k = \overline{0, 2mr-1}$, using the formula (18), the relations (4) – (8) and the initial conditions (19);

2. Find the minimal generating vector $q = (q_0, q_1, \ldots, q_{R-1}) \in G^*[\mathbb{R}][R](a)$ by solving the system (16), taking into account the relation (17);

3. Consider the distribution $a = (a_n)_{n=0}^{\infty} = (\mathbb{P}(T = n))_{n=0}^{\infty}$ of the game duration T as a homogeneous linear recurrence with the initial state $I_R^{[a]} = (a_n)_{n=0}^{R-1}$ and the minimal generating vector $q = (q_k)_{k=0}^{R-1}$, determined at the steps 1 and 2;

4. Determine the expectation $\mathbb{E}(T)$, the variance $\mathbb{V}(T)$, the mean square deviation $\sigma(T)$ and the k-order moments $\nu_k(T)$, $k = \overline{1, t}$, of the game duration T by using the corresponding algorithm from [1].

3 Conclusions

In this paper the stationary games defined on zero-order Markov processes with final sequence of states were studied and the duration of these games was analyzed. It was proved that the game duration is a discrete random variable with homogeneous linear recurrent distribution. Based on this fact, the generating function is applied for determining the main probabilistic characteristics of the game duration. The developed algorithm has polynomial time complexity. Also, the algorithm is applicable for the case when the set of the states is infinite.

References

[1] Lazari A., Lozovanu D., Capcelea M. *Dynamical deterministic and stochastic systems: Evolution, optimization and discrete optimal control* Chişinău, CEP USM, 2015, 310 p. (in Romanian).

[2] Lazari A. *Determining the Optimal Evolution Time for Markov Processes with Final Sequence of States.* Bul. Acad. Ştiinţe Repub. Moldova, Mat., 2015, No. 1(77), 115–126.

[3] Lazari A. *Evolution Time of Composed Stochastic Systems with Final Sequence of States and Independent Transitions.* Annals of the Alexandru Ioan Cuza University - Mathematics. ISSN (Online) 1221-8421, DOI: 10.2478/aicu-2014-0036, December 2014, 18 p.

[4] Lazari A. *Optimization of Zero-Order Markov Processes with Final Sequence of States.* Universal Journal of Applied Mathematics, Horizon Research Publishing, 2013, **1**, No. 3, 198–206.

[5] Lazari A. *Compositions of stochastic systems with final sequence states and interdependent transitions.* Annals of the University of Bucharest, 2013, **4(LXII)**, No. 1, 289–303.

[6] Lazari A. *Algorithms for Determining the Transient and Differential Matrices in Finite Markov Processes.* Bul. Acad. Ştiinţe Repub. Moldova, Mat., 2010, No. 2(63), 84–99.

[7] Zbaganu G. *Waiting for an ape to type a poem.* Bul. Acad. Ştiinţe Repub. Moldova, Mat., 1992, No. 2(8), 66–74.

[8] Guibas Leo J., Odlyzko Andrew M. *Periods in Strings.* Journal of Combinatorial Theory, Series A, 1981, **30**, 19–42.

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Cubic differential systems with two affine real non-parallel invariant straight lines of maximal multiplicity

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Abstract. In this article we classify all differential real cubic systems possessing two affine real non-parallel invariant straight lines of maximal multiplicity. We show that the maximal multiplicity of each of these lines is at most three. The maximal sequences of multiplicities: $m(3,3;1)$, $m(3,2;2)$, $m(3,1;3)$, $m(2,2;3)$, $m_\infty(2,1;3)$, $m_\infty(1,1;3)$ are determined. The normal forms and the corresponding perturbations of the cubic systems which realize these cases are given.

Mathematics subject classification: 34C05.
Keywords and phrases: Cubic differential system, invariant straight line, algebraic multiplicity, geometric multiplicity.

1 Introduction and the statement of main results

We consider the real polynomial system of differential equations

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \quad \gcd(P,Q) = 1 \tag{1}$$

and the vector field

$$\mathbb{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$$

associated to system (1).

Denote $n = \max\{\deg(P), \deg(Q)\}$. If $n = 3$ then system (1) is called cubic.

A curve $f(x,y) = 0$, $f \in \mathbb{C}[x,y]$ is said to be an *invariant algebraic curve* of (1) if there exists a polynomial $K_f \in \mathbb{C}[x,y]$, $deg(K_f) \leq n-1$ such that the identity $\mathbb{X}(f) \equiv f(x,y)K_f(x,y)$ holds. We will be interested in invariant algebraic curves of degree one, that is invariant straight lines $\alpha x + \beta y + \gamma = 0$, $(\alpha,\beta) \neq (0,0)$.

Definition 1 (see [5])**.** An invariant algebraic curve f of degree d for the vector field $\mathbb{X}$ has *algebraic multiplicity* m when m is the greatest positive integer such that the m-th power of f divides $E_d(\mathbb{X})$, where

$$E_d(\mathbb{X}) = det\begin{pmatrix} \upsilon_1 & \upsilon_2 & ... & \upsilon_l \\ \mathbb{X}(\upsilon_1) & \mathbb{X}(\upsilon_2) & ... & \mathbb{X}(\upsilon_l) \\ ... & ... & ... & ... \\ \mathbb{X}^{l-1}(\upsilon_1) & \mathbb{X}^{l-1}(\upsilon_2) & ... & \mathbb{X}^{l-1}(\upsilon_l) \end{pmatrix}, \tag{2}$$

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and $\upsilon_1, \upsilon_2, ..., \upsilon_l$ is a basis of $\mathbb{C}_d[x, y]$.

We note that this definition of multiplicity can be applied to the infinite line $Z = 0$ in the case when this line is not full of singular points.

Definition 2 (see [5])**.** An invariant algebraic curve $f = 0$ of degree d of the vector field $\mathbb{X}$ has *geometric multiplicity* m if m is the largest integer for which there exists a sequence of vector fields $(\mathbb{X}_i)_{i>0}$ of bounded degree, converging to $h\mathbb{X}$, for some polynomial h, not divisible by f, such that each $\mathbb{X}_r$ has m distinct invariant algebraic curves, $f_{r,1} = 0, ..., f_{r,m} = 0$, of degree at most d, which converge to $f = 0$ as r goes to infinity. If we set $h = 1$ in the definition above, then we say that the curve has *strong geometric multiplicity* m.

In [5] it is proved that the notions of algebraic and geometric multiplicity are equivalent.

By present a great number of works have been dedicated to the investigation of polynomial differential systems with invariant straight lines.

The problem of estimating the number of invariant straight lines which a polynomial differential system can have was considered in [1]; the problem of coexistence of the invariant straight lines and limit cycles was examined in $\{[16] : n = 2\}$, $\{[9], n = 3\}$, [8]; the problem of coexistence of the invariant straight lines in cubic systems and singular points of center type was studied in [6], [7], [17].

The classification of all cubic systems with the maximum number of invariant straight lines, including the line at infinity, and taking into account their multiplicities, is given in [10].

In [1] it was proved that the cubic system (1) can have at most eight affine invariant straight lines. The cubic systems with exactly eight and exactly seven distinct affine invariant straight lines have been studied in [10], [11], with invariant straight lines of total geometric (parallel) multiplicity eight (seven) - in [2], [3], [4] ([18]), and with six real invariant straight lines along two (three) directions - in [13], [14]. The cubic systems with degenerate infinity and invariant straight lines of total parallel multiplicity six were investigated in [15]. In [19] it was shown that in the class of cubic differential systems the maximal multiplicity of an affine real straight line (of the line at infinity) is seven.

In this paper the cubic systems with two affine real non-parallel invariant straight lines of maximal multiplicity are classified.

Denote by $\mathbb{CSL}_k^*$ ($\mathbb{CSL}_{2(r)}^{\times}$) the class of cubic systems with exactly k distinct (with exactly 2 real non-parallel) affine invariant straight lines.

Definition 3. We say that $(\mu_1, \mu_2, ..., \mu_k; \mu_\infty)$, where $\mu_j \in \mathbb{N}^*$, $j = 1, \ldots, k, \infty$, $\mu_j \geq \mu_{j+1}, j = 1, \ldots, k-1$, is a sequence of multiplicities of invariant straight lines in the class $\mathbb{CSL}_k^*$ if in $\mathbb{CSL}_k^*$ there exists a system with invariant affine straight lines $l_1, ..., l_k$ which have respectively the multiplicities $\mu_1, \mu_2, ..., \mu_k$ and the line at infinity has the multiplicity μ_∞.

Definition 4. The sequence of multiplicities $(\mu_1, \mu_2, ..., \mu_k; \mu_\infty)$ is called maximal with respect to the component $j, j \in \{1, 2, ..., k, \infty\}$ if $(\mu_1, \mu_2, ..., \mu_j + 1, ..., \mu_k; \mu_\infty)$

is not a sequence of multiplicities of invariant straight lines in the class $\mathbb{CSL}_k^*$. We will denote this sequence by $m_j(\mu_1, \mu_2, ..., \mu_k; \mu_\infty)$. The sequence of the type $m_j(\mu_1, \mu_2, ..., \mu_k; \mu_\infty)$ is called partially maximal. If the sequence $(\mu_1, \mu_2, ..., \mu_k; \mu_\infty)$ is maximal with respect to all components, then it is called maximal (or totally maximal) and is denoted by $m(\mu_1, \mu_2, ..., \mu_k; \mu_\infty)$.

Our main results are the following:

Main Theorem *Any cubic system having two affine non-parallel real invariant straight lines of the (partially) maximal multiplicity* $m(\mu_1, \mu_2; \mu_\infty)$ $(m_\infty(\mu_1, \mu_2; \mu_\infty))$ *via an affine transformation and time rescaling can be written as one of the following forms:*

$m(3,3;1)$	1)	$\dot{x} = x^3, \quad \dot{y} = y(x^2 + ay + by^2),\ b \neq 0;$
$m(3,2;2)$	2.1)	$\dot{x} = ax^3, \quad \dot{y} = y^2, a \neq 0;$
	2.2)	$\dot{x} = x(ax^2 + y), \quad \dot{y} = y^2, \ a \neq 0;$
$m(3,1;3)$	3.1)	$\dot{x} = x^2(ax + by), \quad \dot{y} = y, \ a \neq 0;$
	3.2)	$\dot{x} = x(ay + b), \quad \dot{y} = y(x^2 + ay + b), \ b \neq 0;$
$m(2,2;3)$	4)	$\dot{x} = x, \quad \dot{y} = y(1 + bxy), \ b \neq 0;$
$m_\infty(2,1;3)$	5.1)	$\dot{x} = x^2(a + bx + cy), \quad \dot{y} = y, \quad c(a^2 + b^2) \neq 0;$
	5.2)	$\dot{x} = x, \quad \dot{y} = y(1 + ax + bx^2 + cxy), \quad a(b^2 + c^2) \neq 0;$
	5.3)	$\dot{x} = x(1 + ax + bx^2 + cxy), \quad \dot{y} = y, \quad c(a^2 + b^2) \neq 0;$
	5.4)	$\dot{x} = x(1 + ay), \quad \dot{y} = y(1 + bx + ay + cx^2), \quad abc \neq 0;$
$m_\infty(1,1;3)$	6.1)	$\dot{x} = x, \quad \dot{y} = y(a + bx + cy + dx^2 + exy + fy^2),$ $(a^2 + c^2 + f^2)(d^2 + e^2 + f^2)(a^2 + b^2 + d^2)((a-1)^2 + c^2 + f^2)\cdot$ $((a-1)^2 + b^2 + d^2)((a-1)^2 + (c^2d - bce + b^2f)^2) \neq 0;$
	6.2)	$\dot{x} = x(a + by), \quad \dot{y} = y(c + dx + ey + x^2), \quad a(c^2 + e^2)((a - c)^2 + (b - e)^2) \neq 0;$
	6.3)	$\dot{x} = x(a + by + cxy + y^2),\ \dot{y} = -y(d + ex + c^2x^2 + cxy),$ $ad(c^2 + e^2 + (a + d)^2)((a + d)^2 + (bc - e)^2) \neq 0;$
	6.4)	$\dot{x} = x(a + by + cxy + dy^2),\ \dot{y} = \alpha y(1 + bx + cx^2 + dxy),$ $\alpha a(c^2 + d^2)(\alpha - a) \neq 0.$

2 The proof of the Main Theorem

2.1 The maximal algebraic multiplicity of the affine invariant straight lines

The goal of this section is to determine the maximal algebraic multiplicity of the invariant straight lines for the cubic systems with two affine real non-parallel invariant straight lines.

We consider the cubic differential system

$$\begin{cases} \dot{x} = P_0 + P_1(x,y) + P_2(x,y) + P_3(x,y) \equiv P(x,y), \\ \dot{y} = Q_0 + Q_1(x,y) + Q_2(x,y) + Q_3(x,y) \equiv Q(x,y), \end{cases} \tag{3}$$

where $P_k = \sum\limits_{i+j=k} a_{ij}x^iy^j$ and $Q_k = \sum\limits_{i+j=k} b_{ij}x^iy^j$, $a_{ij}, b_{ij} \in \mathbb{R}$, $k = \overline{0,3}$.

Suppose that

$$yP_3(x,y) - xQ_3(x,y) \not\equiv 0, \qquad \gcd(P,Q) = 1, \tag{4}$$

i.e. at infinity the system (3) has at most four distinct singular points and the right-hand sides of (3) do not have common divisors of degree greater than 0.

Let the system (3) have two real non-parallel invariant straight lines l_1, l_2. By an affine transformation we can make them to be described by equations $x = 0$ and $y = 0$, respectively. Then, the system (3) looks as

$$\begin{cases} \dot{x} = x(a_{10} + a_{20}x + a_{11}y + a_{30}x^2 + a_{21}xy + a_{12}y^2), \\ \dot{y} = y(b_{01} + b_{11}x + b_{02}y + b_{21}x^2 + b_{12}xy + b_{03}y^2). \end{cases} \tag{5}$$

We denote by μ_1 the multiplicity of the line $x = 0$, by μ_2 the multiplicity of the line $y = 0$ and by μ_∞ the multiplicity of the line at infinity.

Applying Definition 1, first we determine the maximal algebraic multiplicity of the line $x = 0$, secondly the maximal algebraic multiplicity of the line $y = 0$ and the third step consists in the determination of the maximal algebraic multiplicity of the line at infinity $Z = 0$.

2.1.1 The maximal algebraic multiplicity of the line $x = 0$

In this subsection, we compute the maximal algebraic multiplicity of the invariant straight line $x = 0$ of the system (5). For this purpose, we calculate the determinant $E_1(\mathbb{X})$ from Definition 1. For (5) the determinant $E_1(\mathbb{X})$ is a polynomial in x and y of degree 8. To determine the maximal algebraic multiplicity of the line $x = 0$, we write it in the form:

$$\begin{aligned} E_1(\mathbb{X}) = x(&A_1(y) + A_2(y)x + A_3(y)x^2 + A_4(y)x^3 + A_5(y)x^4 \\ &+A_6(y)x^5 + A_7(y)x^6 + A_8(y)x^7). \end{aligned} \tag{6}$$

Thus for system (5) we have $A_1(y) = -yA_{11}(y)A_{12}(y)$, where $A_{11}(y) = b_{01} + b_{02}y + b_{03}y^2$ and $A_{12}(y) = a_{10}^2 - a_{10}b_{01} + 2a_{10}a_{11}y - 2a_{10}b_{02}y + a_{11}^2y^2 + 2a_{10}a_{12}y^2 + a_{12}b_{01}y^2 - a_{11}b_{02}y^2 - 3a_{10}b_{03}y^2 + 2a_{11}a_{12}y^3 - 2a_{11}b_{03}y^3 + a_{12}^2y^4 - a_{12}b_{03}y^4$.

The algebraic multiplicity μ_1 of the invariant straight line $x = 0$ is at least two if the identity $A_1(y) \equiv 0$ holds. From conditions (4) the polynomial $A_{11}(y)$ is not identically zero, i.e. $|b_{01}| + |b_{02}| + |b_{03}| \neq 0$, therefore it is necessary that $A_{12}(y)$ be identically zero. The identity $A_{12}(y) \equiv 0$ holds if one of the following six sets of conditions is satisfied:

$$a_{10} = a_{11} = a_{12} = 0; \tag{7}$$

$$a_{11} = a_{12} = b_{02} = b_{03} = 0, b_{01} = a_{10},\ a_{10} \neq 0; \tag{8}$$

$$a_{10} = a_{12} = b_{03} = 0, b_{02} = a_{11},\ a_{11} \neq 0; \tag{9}$$

$$a_{12} = b_{03} = 0, b_{01} = a_{10}, b_{02} = a_{11},\ a_{10}a_{11} \neq 0; \tag{10}$$

$$a_{10} = 0, b_{01} = a_{11}(b_{02} - a_{11})/a_{12}, b_{03} = a_{12},\ a_{12} \neq 0; \tag{11}$$

$$b_{01} = a_{10}, b_{02} = a_{11}, b_{03} = a_{12},\ a_{10}a_{12} \neq 0. \tag{12}$$

Lemma 1. *For cubic differential system* $\{(5), (4)\}$ *the algebraic multiplicity* μ_1 *of the invariant straight line* $x = 0$ *is at least two if and only if one of the following six sets of conditions* (7), (8), (9), (10), (11), (12) *is satisfied.*

We will examine each of the cases (7), (8), (9), (10), (11) and (12) separately.

1) *Conditions* (7).

The algebraic multiplicity is at least three ($\mu_1 \geq 3$) if the identity $A_2(y) \equiv 0$ holds. Here we have $A_2(y) = yA_{11}(a_{20}b_{01} + 2a_{20}b_{02}y + a_{21}b_{02}y^2 + 3a_{20}b_{03}y^2 + 2a_{21}b_{03}y^3)$. The identity $A_2(y) \equiv 0$ holds if one of the following two series of conditions is satisfied:

$$a_{20} = a_{21} = 0; \tag{13}$$

$$a_{20} = b_{02} = b_{03} = 0, a_{21} \neq 0. \tag{14}$$

Under the conditions $\{(4), (7), (13)\}$, the cubic system (5) looks as

$$\begin{aligned} &\dot{x} = a_{30}x^3, \quad \dot{y} = y(b_{01} + b_{11}x + b_{02}y + b_{21}x^2 + b_{12}xy + b_{03}y^2), \\ &a_{30}(|b_{01}| + |b_{02}| + |b_{03}|) \neq 0. \end{aligned} \tag{15}$$

For this system $A_3(y) = a_{30}yA_{11}(y)(b_{01} + 2b_{02}y + 3b_{03}y^2) \not\equiv 0$, so in this case the multiplicity of the invariant straight line $x = 0$ is three.

If the conditions $\{(4), (7), (14)\}$ occur, then the system (5) looks as:

$$\dot{x} = x^2(a_{30}x + a_{21}y), \quad \dot{y} = y(b_{01} + b_{11}x + b_{21}x^2 + b_{12}xy), \ a_{21}a_{30}b_{01} \neq 0. \tag{16}$$

The algebraic multiplicity of the line $x = 0$ can not be greater than three, because $A_3(y) = b_{01}y(a_{30}b_{01} - a_{21}(2a_{21} - b_{12})y^2) \not\equiv 0$.

2) *Conditions* (8):

$A_2(y) = -a_{10}^2y(2(a_{20} - b_{11}) + 3(a_{21} - b_{12})y) \equiv 0 \Rightarrow$

$$b_{11} = a_{20}, b_{12} = a_{21} \tag{17}$$

$\Rightarrow A_3(y) = -3a_{10}^2(a_{30} - b_{21})y \not\equiv 0$, therefore μ_1 can not be greater than three.

In the conditions $\{(4), (8), (17)\}$ the system (5) takes the form

$$\begin{aligned} &\dot{x} = x(a_{10} + a_{20}x + a_{30}x^2 + a_{21}xy), \\ &\dot{y} = y(a_{10} + a_{20}x + b_{21}x^2 + a_{21}xy), \ a_{10}(b_{21} - a_{30}) \neq 0. \end{aligned} \tag{18}$$

3) *Conditions* (9).

The identity $A_2(y) = y(a_{20}b_{01}^2 - a_{11}(a_{11}a_{20} + 2a_{21}b_{01} - a_{11}b_{11} - b_{01}b_{12})y^2 - 2a_{11}^2(a_{21} - b_{12})y^3) \equiv 0$ holds if one of the following two series of conditions is satisfied:

$$a_{20} = 0, \ b_{11} = a_{21}b_{01}/a_{11}, \ b_{12} = a_{21}; \tag{19}$$

$$b_{01} = 0, \ b_{11} = a_{20}, \ b_{12} = a_{21}, \ a_{20} \neq 0. \tag{20}$$

Under the conditions {(9), (19)} we write the system (5) as

$$\begin{aligned}&\dot{x} = x(a_{11}y + a_{30}x^2 + a_{21}xy),\\ &\dot{y} = y(a_{11}b_{01} + a_{21}b_{01}x + a_{11}^2y + a_{11}b_{21}x^2 + a_{11}a_{21}xy)/a_{11}, \quad b_{21} - a_{30} \neq 0.\end{aligned} \tag{21}$$

Here $A_3(y) = y(a_{30}b_{01}^2 - a_{11}a_{30}b_{01}y - 2a_{11}^2(a_{30} - b_{21})y^2) \not\equiv 0$, therefore the multiplicity μ_1 can not be greater than three.

If the conditions {(9), (20)} are satisfied then the cubic system (5) obtains the following form

$$\begin{aligned}&\dot{x} = x(a_{20}x + a_{11}y + a_{30}x^2 + a_{21}xy),\\ &\dot{y} = y(a_{20}x + a_{11}y + b_{21}x^2 + a_{11}y + a_{21}xy), \quad a_{11}a_{20}(b_{21} - a_{30}) \neq 0.\end{aligned} \tag{22}$$

The algebraic multiplicity of the line $x = 0$, for the system (22), can not be greater than three, because $A_3(y) = 2a_{11}^2(b_{21} - a_{30})y^3 \not\equiv 0$.

4) *Conditions* (10):

$A_2(y) = -y(a_{10} + a_{11}y)(2a_{10}(a_{20} - b_{11}) + (a_{11}a_{20} + 3a_{10}a_{21} - a_{11}b_{11} - 3a_{10}b_{12})y + 2a_{11}(a_{21} - b_{12})y^2) \equiv 0 \Rightarrow \{b_{11} = a_{20}, b_{12} = a_{21}\} \Rightarrow A_3(y) = y(b_{21} - a_{30})(a_{10} + a_{11}y)(3a_{10} + 2a_{11}y) \not\equiv 0$, so $\mu_1 = 3$. The cubic system (5) looks as

$$\begin{aligned}&\dot{x} = x(a_{10} + a_{20}x + a_{11}y + a_{30}x^2 + a_{21}xy),\\ &\dot{y} = y(a_{10} + a_{20}x + a_{11}y + b_{21}x^2 + a_{21}xy), \ a_{10}a_{11}(b_{21} - a_{30}) \neq 0.\end{aligned} \tag{23}$$

5) *Conditions* (11).

The identity $A_2(y) = y(a_{11} + a_{12}y)(a_{11}a_{20}(a_{11} - b_{02})^2 + 2a_{12}a_{20}(a_{11} - b_{02})^2y + a_{12}(3a_{11}^2a_{21} - 3a_{11}a_{12}a_{20} + 2a_{12}a_{20}b_{02} - 4a_{11}a_{21}b_{02} + a_{21}b_{02}^2 + 2a_{11}a_{12}b_{11} - a_{12}b_{02}b_{11} - a_{11}^2b_{12} + a_{11}b_{02}b_{12})y^2 - 2a_{11}a_{12}^2(a_{21} - b_{12})y^3 - a_{12}^3(a_{21} - b_{12})y^4)/a_{12}^2 \equiv 0$ holds if one of the following four series of conditions is satisfied:

$$a_{20} = 0, b_{02} = 2a_{11}, b_{12} = a_{21}; \tag{24}$$

$$a_{20} = 0, b_{11} = a_{21}(b_{02} - a_{11})/a_{12}, b_{12} = a_{21}, b_{02} \neq 2a_{11}; \tag{25}$$

$$a_{11} = 0, a_{20} \neq 0, b_{02} = 0, b_{12} = a_{21}; \tag{26}$$

$$a_{11} \neq 0, a_{20} \neq 0, b_{02} = a_{11}, b_{11} = a_{20}, b_{12} = a_{21}. \tag{27}$$

a) The conditions {(11), (24)} lead us to the system

$$\begin{aligned}&\dot{x} = x(a_{11}y + a_{30}x^2 + a_{21}xy + a_{12}y^2), \quad \dot{y} = y(a_{11}^2 + 2a_{11}a_{12}y\\ &+a_{12}b_{21}x^2 + a_{12}a_{21}xy + a_{12}^2y^2)/a_{12}, \ b_{21} - a_{30} \neq 0,\end{aligned} \tag{28}$$

for which $A_3(y) = y(a_{11}^4a_{30} + 2a_{11}^3a_{12}a_{30}y + a_{12}(a_{11}^2a_{12}b_{21} - a_{11}^2a_{21}^2 + 2a_{11}a_{12}a_{21}b_{11} - a_{12}^2b_{11}^2)y^2 - 2a_{11}a_{12}^3(a_{30} - b_{21})y^3 - a_{12}^4(a_{30} - b_{21})y^4)/a_{12}^2 \not\equiv 0$, so $\mu_1 = 3$.

b) Under the conditions (25) we have $A_3(y) = y(a_{11} + a_{12}y)(a_{11}a_{30}(a_{11} - b_{02})^2 + a_{12}a_{30}(3a_{11} - 2b_{02})(a_{11} - b_{02})y - a_{12}^2(3a_{11} - b_{02})(a_{30} - b_{21})y^2 - a_{12}^3(a_{30} - b_{21})y^3)/a_{12}^2 \not\equiv 0$, therefore in this case $\mu_1 = 3$. The cubic system (5) has the form

$$\begin{aligned}&\dot{x} = x(a_{11}y + a_{30}x^2 + a_{21}xy + a_{12}y^2), \quad \dot{y} = y(a_{11}(b_{02} - a_{11}) +\\ &a_{21}(b_{02} - a_{11})x + a_{12}b_{02}y + a_{12}b_{21}x^2 + a_{12}a_{21}xy + a_{12}^2y^2)/a_{12},\\ &(b_{21} - a_{30})(b_{02} - 2a_{11}) \neq 0.\end{aligned} \tag{29}$$

c) When conditions $\{(11), (26)\}$ hold we have $A_3(y) = -a_{12}y^3(2a_{20}^2 - 3a_{20}b_{11} + b_{11}^2 + a_{12}a_{30}y^2 - a_{12}b_{21}y^2) \not\equiv 0$ and one obtains the following system

$$\begin{aligned} \dot{x} &= x(a_{20}x + a_{30}x^2 + a_{21}xy + a_{12}y^2), \\ \dot{y} &= y(b_{11}x + b_{21}x^2 + a_{21}xy + a_{12}y^2), \quad a_{20}a_{12}(b_{21} - a_{30}) \neq 0. \end{aligned} \tag{30}$$

The multiplicity μ_1 is equal to three.

d) The conditions $\{(11), (27)\}$ lead us to the system

$$\begin{aligned} \dot{x} &= x(a_{20}x + a_{11}y + a_{30}x^2 + a_{21}xy + a_{12}y^2), \\ \dot{y} &= y(a_{20}x + a_{11}y + b_{21}x^2 + a_{21}xy + a_{12}y^2), \quad a_{11}a_{12}a_{20}(b_{21} - a_{30}) \neq 0. \end{aligned} \tag{31}$$

For system (31) we have $A_3(y) = (b_{21} - a_{30})(a_{11} + a_{12}y)(2a_{11} + a_{12}y)y^3 \not\equiv 0$ and therefore $\mu_1 = 3$.

6) *Conditions* (12):

$A_2(y) = -y(a_{10}+a_{11}y+a_{12}y^2)(2a_{10}a_{20}-2a_{10}b_{11}+a_{11}a_{20}y+3a_{10}a_{21}y-a_{11}b_{11}y-3a_{10}b_{12}y+2a_{11}a_{21}y^2-2a_{11}b_{12}y^2+a_{12}a_{21}y^3-a_{12}b_{12}y^3) \equiv 0 \Rightarrow \{b_{11} = a_{20}, b_{12} = a_{21}\} \Rightarrow A_3(y) = y(b_{21} - a_{30})(a_{10} + a_{11}y + a_{12}y^2)(3a_{10} + 2a_{11}y + a_{12}y^2) \not\equiv 0$. Therefore $\mu_1 = 3$. In this case the cubic system (5) looks as

$$\begin{aligned} \dot{x} &= x(a_{10} + a_{20}x + a_{11}y + a_{30}x^2 + a_{21}xy + a_{12}y^2), \\ \dot{y} &= y(a_{10} + a_{20}x + a_{11}y + b_{21}x^2 + a_{21}xy + a_{12}y^2), \ a_{10}a_{12}(b_{21} - a_{30}) \neq 0. \end{aligned} \tag{32}$$

In this way we have proved the following two lemmas.

Lemma 2. *Let the cubic system* $\{(3), (4)\}$ *have two affine real non-parallel invariant straight lines. Then the maximal algebraic multiplicity of one of these lines is at most three.*

Lemma 3. *For cubic differential system* $\{(5), (4)\}$ *the algebraic multiplicity of the invariant straight line* $x = 0$ *is three if and only if it has one of the following forms:* (15), (16), (18), (21), (22), (23), (28), (29), (30), (31), (32).

2.1.2 The maximal algebraic multiplicity of the line $y = 0$

In this subsection for the systems, enumerated in Lemma 3, we determine the maximal algebraic multiplicity of the line $y = 0$. For this purpose, we write the determinant $E_1(\mathbb{X})$ from Definition 1 in the form:

$$\begin{aligned} E_1(\mathbb{X}) = y(&B_1(x) + B_2(x)y + B_3(x)y^2 + B_4(x)y^3 + B_5(x)y^4 \\ &+B_6(x)y^5 + B_7(x)y^6 + B_8(x)y^7). \end{aligned} \tag{33}$$

The algebraic multiplicity μ_2 of the invariant straight line $y = 0$ is at least two if the identity $B_1(x) \equiv 0$ holds.

Taking into account the condition (4), for each of the systems (16), (18), (22), (23), (31), (32), the polynomial $B_1(x)$ is not identically zero, therefore $\mu_2 = 1$.

In the case of system (15) the identity $B_1(x) \equiv 0$, where $B_1(x) = a_{30}x^3(b_{01}^2 + 2b_{01}b_{11}x - 3a_{30}b_{01}x^2 + b_{11}^2x^2 + 2b_{01}b_{21}x^2 - 2a_{30}b_{11}x^3 + 2b_{11}b_{21}x^3 - a_{30}b_{21}x^4 + b_{21}^2x^4)$ holds if one of the following two series of conditions is satisfied

$$b_{01} = b_{11} = b_{21} = 0; \tag{34}$$

$$b_{01} = b_{11} = 0, b_{21} = a_{30}. \tag{35}$$

The conditions (34) imply $B_2(x) = -a_{30}^2x^5(3b_{02} + 2b_{12}x) \equiv 0 \Rightarrow$

$$b_{02} = b_{12} = 0. \tag{36}$$

Under the conditions (34) and (36) the multiplicity is $\mu_2 = 3$. The cubic system (15) has the form $\dot{x} = a_{30}x^3$, $\dot{y} = b_{03}y^3$, $a_{30}b_{03} \neq 0$. This system is an element of the class $\mathbb{CSL}_4^*$ and for it we have $m(3, 3, 1, 1; 1)$ (see [10]).

The conditions (35) $\Rightarrow B_2(x) = a_{30}^2b_{12}x^6 \equiv 0 \Rightarrow b_{12} = 0 \Rightarrow B_3(x) = a_{30}x^3(2b_{02}^2 + a_{30}b_{03}x^2) \not\equiv 0$, therefore the multiplicity is $\mu_2 = 3$ and the system (15) takes the form

$$\dot{x} = a_{30}x^3, \quad \dot{y} = y(b_{02}y + a_{30}x^2 + b_{03}y^2), \quad a_{30}b_{03} \neq 0. \tag{37}$$

For the system (21) we have $B_1(x) = a_{30}x^3(a_{11}^2b_{01}^2 + 2a_{11}a_{21}b_{01}^2x - b_{01}(3a_{11}^2a_{30} - a_{21}^2b_{01} - 2a_{11}^2b_{21})x^2 - 2a_{11}a_{21}b_{01}(a_{30} - b_{21})x^3 - a_{11}^2b_{21}(a_{30} - b_{21}))/a_{11}^2$ and $\{B_1(x) \equiv 0, (4)\} \Rightarrow$

$$b_{01} = b_{21} = 0,\ a_{11}a_{21}a_{30} \neq 0 \tag{38}$$

$\Rightarrow B_2(x) = -a_{30}^2x^5(3a_{11} + 2a_{21}x) \not\equiv 0$, $\mu_2 = 2$.

In the case of system (28) we have $B_1(x) = a_{30}x^3(a_{11}^4 + 2a_{11}^2a_{12}b_{11}x - a_{12}(3a_{11}^2a_{30} - a_{12}b_{11}^2 - 2a_{11}^2b_{21})x^2 - 2a_{12}^2b_{11}(a_{30} - b_{21})x^3 - a_{12}^2b_{21}(a_{30} - b_{21})x^4)/a_{12}^2 \equiv 0 \Rightarrow$

$$a_{11} = b_{11} = b_{21} = 0, \tag{39}$$

$\Rightarrow B_2(x) = -2a_{21}a_{30}^2x^6 \equiv 0 \Rightarrow a_{21} = 0 \Rightarrow B_3(x) = -3a_{12}a_{30}^2x^5 \not\equiv 0, \quad \mu_2 = 3$. The system (28) looks as:

$$\dot{x} = x(a_{30}x^2 + a_{12}y^2), \quad \dot{y} = a_{12}y^3, \quad a_{30}a_{12} \neq 0. \tag{40}$$

For the system (29) we get $B_1(x) = a_{30}x^3(a_{11}^2(a_{11} - b_{02})^2 + 2a_{11}a_{21}(a_{11} - b_{02})^2x + (a_{11} - b_{02})(a_{11}a_{21}^2 + 3a_{11}a_{12}a_{30} - a_{21}^2b_{02} - 2a_{11}a_{12}b_{21})x^2 + 2a_{12}a_{21}(a_{11} - b_{02})(a_{30} - b_{21})x^3 - a_{12}^2b_{21}(a_{30} - b_{21})x^4)/a_{12}^2$. The identity $B_1(x) \equiv 0$ holds if at least one of the following two sets of conditions is satisfied:

$$a_{11} = a_{21} = b_{21} = 0, \tag{41}$$

$$b_{02} = a_{11}, b_{21} = 0, a_{11} \neq 0. \tag{42}$$

When conditions (41) ((42)) hold the polynomial $B_2(x) = -3a_{30}^2b_{02}x^5$ ($B_2(x) = -a_{30}^2x^5(3a_{11} + 2a_{21}x)$) is not identically zero, therefore $\mu_2 = 2$.

Consider now the system (30). We have: $B_1(x) = -x^4(a_{20}+a_{30}x)(a_{20}b_{11}-b_{11}^2+2a_{30}b_{11}x-2b_{11}b_{21}x+a_{30}b_{21}x^2-b_{21}^2x^2) \equiv 0 \Rightarrow$

$$b_{11} = 0,\ b_{21} = 0 \tag{43}$$

$\Rightarrow B_2(x) = -a_{21}x^4(a_{20}+a_{30}x)(a_{20}+2a_{30}x) \equiv 0 \Rightarrow a_{21} = 0 \Rightarrow B_3(x) = -a_{12}x^3(a_{20}+a_{30}x)(2a_{20}+3a_{30}x) \not\equiv 0,\ \mu_2 = 3$. The cubic system (30) looks as:

$$\dot{x} = x(a_{20}x + a_{30}x^2 + a_{12}y^2), \quad \dot{y} = a_{12}y^3, \quad a_{12}a_{20}a_{30} \neq 0. \tag{44}$$

The transformation $X = y, Y = x$ reduces (40) and (44) to a system of the form (37).

Lemma 4. *For cubic differential system* $\{(5),\ (4)\}$ *the algebraic multiplicity of the invariant straight lines* $x = 0$ *and* $y = 0$ *are respectively* $\mu_1 = 3$ *and* $\mu_2 \geq 2$ *if and only if it has one of the forms: 1)* $\{(15),\ (34)\}$, *2)* $\{(15),\ (35)\}$, *3)* $\{(21),\ (38)\}$, *4)* $\{(28),\ (39)\}$, *5)* $\{(29),\ (41)\}$, *6)* $\{(29),\ (42)\}$, *7)* $\{(30),\ (43)\}$.

Lemma 5. *In the class of cubic systems* $\{(5),\ (4)\} \in \mathbb{CSL}^{\times}_{2(r)}$ *the algebraic multiplicity of the invariant straight lines* $x = 0$ *and* $y = 0$ *is three if and only if it has the form* (37).

2.2 Classification of cubic differential systems with two affine real non-parallel invariant straight lines and the line at infinity of maximal algebraic multiplicity

In this section for cubic system $\{(5),\ (4)\} \in \mathbb{CSL}^{\times}_{2(r)}$ we establish the partially maximal sequences of multiplicities of the type $m_\infty(\mu_1, \mu_2; \mu_\infty)$.

We fix $\mu_1 \in \{1, 2, 3\}$ and $\mu_2 \in \{1, 2, 3\}$, $\mu_1 \geq \mu_2$ and we will determine the maximal multiplicity of the line at infinity such that the sequence $(\mu_1, \mu_2; \mu_\infty)$ should be maximal in the third component. We will investigate the cases:

1. $m(3, 3; \mu_\infty)$, **2.** $m_\infty(3, 2; \mu_\infty)$, **3.** $m_\infty(3, 1; \mu_\infty)$, **4.** $m_\infty(2, 2; \mu_\infty)$, **5.** $m_\infty(2, 1; \mu_\infty)$, **6.** $m_\infty(1, 1; \mu_\infty)$.

We consider the cubic system $\{(5),\ (4)\} \in \mathbb{CSL}^{\times}_{2(r)}$ and its associated homogeneous system

$$\begin{cases} \dot{x} = x(a_{10}Z^2 + a_{20}xZ + a_{11}yZ + a_{30}x^2 + a_{21}xy + a_{12}y^2), \\ \dot{y} = y(b_{01}Z^2 + b_{11}xZ + b_{02}yZ + b_{21}x^2 + b_{12}xy + b_{03}y^2). \end{cases} \tag{45}$$

For (45) we write $E_1(\mathbb{X})$ in the form

$$\begin{aligned} E_1(\mathbb{X}) = C_0(x,y) + C_1(x,y)Z + C_2(x,y)Z^2 + C_3(x,y)Z^3 + C_4(x,y)Z^4 \\ + C_5(x,y)Z^5 + C_6(x,y)Z^6 + C_7(x,y)Z^7 + C_8(x,y)Z^8, \end{aligned} \tag{46}$$

where $C_j(x, y), j = \overline{0, 8}$ are polynomials in x and y.

The algebraic multiplicity of the line at infinity is $\mu_\infty \in \mathbb{N}^*$ if μ_∞ is the maximal number such that $Z^{(\mu_\infty - 1)}$ divides $E_1(\mathbb{X})$.

2.2.1 Case $m(3,3;\mu_\infty)$

To investigate the maximal algebraic multiplicity of the line at infinity for the system (37) (see Lemma 5), we consider the homogenized system

$$\dot{x} = a_{30}x^3, \quad \dot{y} = y(b_{02}yZ + a_{30}x^2 + b_{03}y^2), \quad a_{30}b_{03} \neq 0. \tag{47}$$

For (47) we have $C_0(x,y) = a_{30}b_{03}x^3y^3(a_{30}x^2 + 3b_{03}y^2) \not\equiv 0$, therefore the algebraic multiplicity of the line at infinity is one and in the class $\mathbb{CSL}^{\times}_{2(r)}$ we have the maximal sequence $m(3,3;1)$.

Lemma 6. *Via an affine transformation and time rescaling any cubic system having two non-parallel real invariant straight lines of the maximal multiplicity $m(3,3;1)$, can be brought to the form*

$$\dot{x} = x^3, \quad \dot{y} = y(ay + x^2 + by^2),\ b \neq 0. \tag{48}$$

2.2.2 Case $m_\infty(3,2;\mu_\infty)$

According to Lemma 4, the cubic system {(5), (4)} admits the invariant straight lines $x = 0$ and $y = 0$ of the multiplicities three and two respectively if the cubic system has one of the following seven forms:

1) {(15), (34)}, 2) {(15), (35)}, 3) {(21), (38)}, 4) {(28), (39)},
5) {(29), (41)}, 6) {(29), (42)}, 7) {(30), (43)}.

Case 1) {(15), (34)}. Under the condition (34) the cubic system (15) looks as

$$\dot{x} = a_{30}x^3, \quad \dot{y} = y^2(b_{02} + b_{12}x + b_{03}y),\ a_{30}(|b_{02}| + |b_{03}|) \neq 0. \tag{49}$$

For homogeneous system associated to the system (49) we have $C_0(x,y) = -a_{30}x^3y^2(2b_{12}x + 3b_{03}y)(a_{30}x^2 - b_{12}xy - b_{03}y^2) \equiv 0 \Rightarrow b_{03} = b_{12} = 0 \Rightarrow C_1(x,y) = -3a_{30}^2b_{02}x^5y^2 \not\equiv 0$, therefore the multiplicity of the line at infinity is two. The system (49) takes the form $\dot{x} = a_{30}x^3$, $\dot{y} = b_{02}y^2$, $b_{02}a_{30} \neq 0$, and after the time rescaling we can write it as

$$\dot{x} = ax^3, \quad \dot{y} = y^2,\ a \neq 0 \tag{50}$$

(see system 2.1) of the Main Theorem).

From the above it follows for system (50) that $m_\infty(3,2;2) = m(3,2;2)$.

In the Cases 2), 4), 5), 6), 7) we have respectively

$$\dot{x} = a_{30}x^3, \quad \dot{y} = y(a_{30}x^2 + b_{02}y + b_{12}xy + b_{03}y^2),\ a_{30}(b_{02}^2 + b_{03}^2 + b_{12}^2) \neq 0,$$
$$C_0(x,y) = a_{30}x^3y^2(b_{12}x + b_{03}y)(a_{30}x^2 + 2b_{12}xy + 3b_{03}y^2) \not\equiv 0,\ \mu_\infty = 1;$$

$$\dot{x} = x(a_{30}x^2 + a_{21}xy + a_{12}y^2), \quad \dot{y} = y^2(a_{21}x + a_{12}y),\ a_{12}a_{30} \neq 0,$$
$$C_0(x,y) = -a_{30}x^3y^2(2a_{21}a_{30}x^3 + a_{21}^2x^2y + 3a_{12}a_{30}x^2y + 2a_{12}a_{21}xy^2$$
$$+a_{12}^2y^3) \not\equiv 0,\ \mu_\infty = 1;$$

$$\dot{x} = x(a_{30}x^2 + a_{12}y^2), \quad \dot{y} = y^2(a_{12}y + b_{02}),\ a_{12}a_{30} \neq 0,$$
$$C_0(x,y) = -a_{12}a_{30}x^3y^3(3a_{30}x^2 + a_{12}y^2) \not\equiv 0,\ \mu_\infty = 1;$$

$\dot{x} = x(a_{30}x^2 + a_{11}y + a_{21}xy + a_{12}y^2), \quad \dot{y} = y^2(a_{11} + a_{21}x + a_{12}y),$
$a_{12}a_{30} \neq 0, \;\; C_0(x,y) = -a_{30}x^3y^2(2a_{21}a_{30}x^3 + a_{21}^2x^2y + 3a_{12}a_{30}x^2y$
$+2a_{12}a_{21}xy^2 + a_{12}^2y^3) \not\equiv 0, \;\; \mu_\infty = 1;$

$\dot{x} = x(a_{20}x + a_{30}x^2 + a_{21}xy + a_{12}y^2), \quad \dot{y} = y^2(a_{21}x + a_{12}y),$
$a_{12}a_{20}a_{30} \neq 0, \quad C_0(x,y) = -a_{30}x^3y^2(2a_{21}a_{30}x^3 + a_{21}^2x^2y +$
$3a_{12}a_{30}x^2y + 2a_{12}a_{21}xy^2 + a_{12}^2y^3) \not\equiv 0, \;\; \mu_\infty = 1.$

Case 3) {(21), (38)}. In this case $C_0(x,y) = -a_{21}a_{30}x^5y^2(2a_{30}x + a_{21}y) \equiv 0 \Rightarrow a_{21} = 0 \Rightarrow C_1(x,y) = -3a_{11}a_{30}^2x^5y^2 \not\equiv 0, \;\; \mu_\infty = 2$. The system {(21), (38)} obtains the form $\dot{x} = x(a_{11}y + a_{30}x^2), \;\; \dot{y} = a_{11}y^2, \;\; a_{11}a_{30} \neq 0$, and after time rescaling we can write it as

$$\dot{x} = x(y + ax^2), \quad \dot{y} = y^2, \;\; a \neq 0 \tag{51}$$

(see system 2.2) of the Main Theorem).

For system (51) we have $m_\infty(3,2;2) = m(3,2;2)$.

Lemma 7. *Any cubic system of the class $\mathbb{CSL}^{\times}_{2(r)}$ with invariant straight lines of the maximal multiplicity $m(3,2;2)$ via an affine transformation and time rescaling can be written in form* (50) *or* (51).

2.2.3 Case $m_\infty(3,1;\mu_\infty)$

The following cubic systems: (15), (16), (18), (21), (22), (23), (28), (29), (30), (31), (32) possess the invariant straight lines $x = 0$ and $y = 0$ of the multiplicity $\mu_1 = 3$ and $\mu_2 = 1$, respectively (see Lemma 3). Proceeding as in the previous case and taking into account the condition (4), we will examine each system separately.

System (15). For this system we have $C_0(x,y) = -a_{30}x^3yC_{01}(x,y)C_{02}(x,y)$, where $C_{01}(x,y) = a_{30}x^2 - b_{21}x^2 - b_{12}xy - b_{03}y^2$, $C_{02} = (b_{21}x^2 + 2b_{12}xy + 3b_{03}y^2)$. If $C_{01}(x,y) \equiv 0$, then the infinity is degenerate for (15). Let $C_{01}(x,y) \not\equiv 0$, i.e. $|a_{30} - b_{21}| + |b_{12}| + |b_{03}| \neq 0$, and $C_{02}(x,y) \equiv 0$. Then, $b_{03} = b_{12} = b_{21} = 0 \Rightarrow C_1(x,y) = -a_{30}^2x^5y(2b_{11}x + 3b_{02}y) \equiv 0 \Rightarrow b_{02} = b_{11} = 0 \Rightarrow C_2(x,y) = -3a_{30}^2b_{01}x^5y \not\equiv 0, \;\; \mu_\infty = 3$. Under the above conditions the system (15) takes the form

$$\dot{x} = a_{30}x^3, \qquad \dot{y} = b_{01}y, \quad a_{30}b_{01} \neq 0. \tag{52}$$

System (16). In this case: $\{(4),\ C_0(x,y) = -x^4y((a_{30} - b_{21})x + (a_{21} - b_{12})y)(a_{30}b_{21}x^2 + 2a_{30}b_{12}xy + a_{21}b_{12}y^2) \equiv 0\} \Rightarrow \{|a_{30} - b_{21}| + |a_{21} - b_{12}| \neq 0,\ b_{21} = b_{12} = 0\} \Rightarrow C_1(x,y) = -b_{11}x^4y(a_{30}x + a_{21}y)(2a_{30}x + a_{21}y) \equiv 0 \Rightarrow b_{11} = 0 \Rightarrow C_2(x,y) = -b_{01}x^3y(a_{30}x + a_{21}y)(3a_{30}x + 2a_{21}y) \not\equiv 0, \;\; \mu_\infty = 3$. The system (16) has the form

$$\dot{x} = x^2(a_{30}x + a_{21}y), \;\; \dot{y} = b_{01}y, \;\; a_{30}b_{01} \neq 0. \tag{53}$$

Note that the system (52) is a particular case of the system (53), and after time rescaling the last system can be written in the form

$$\dot{x} = x^2(ax + by), \quad \dot{y} = y, \;\; a \neq 0 \tag{54}$$

(see system 3.1) of Main Theorem). The system (54) has not the third affine invariant straight line because $E_1(\mathbb{X}) = -x^3y(-a + 3a^2x^2 + 5abxy + 2b^2y^2)$. The conic $f \equiv -a + 3a^2x^2 + 5abxy + 2b^2y^2 = 0$ is reducible in $\mathbb{C}[x, y]$ only if $b = 0$, i.e. $f = a(-1 + 3ax^2)$, but $f = 0$ is not invariant for $\{(54),\ b = 0\}$. For system (54) we get $m_\infty(3, 1; 3) = m(3, 1; 3)$.

Remark 1. For the homogeneous systems associated to (18) (respectively, (21), (22), (23)) the polynomial $C_0(x, y)$ has the form $C_0(x, y) = (b_{21} - a_{30})x^5y(a_{30}b_{21}x^2 + 2a_{21}a_{30}xy + a_{21}^2y^2)$ and for these systems identity $C_0(x, y) \equiv 0$ holds if one of the following two series of conditions is satisfied:

$$A)\ a_{21} = b_{21} = 0 \qquad \text{and} \qquad B)\ a_{21} = a_{30} = 0.$$

System (18). In conditions A) (B)) we have $C_1(x, y) = -2a_{20}a_{30}^2x^6y \equiv 0$ ($C_1(x, y) = a_{20}b_{21}^2x^6y \equiv 0$) $\Rightarrow a_{20} = 0 \Rightarrow C_2(x, y) = -3a_{10}a_{30}^2x^5y \not\equiv 0$ ($C_2(x, y) = a_{10}b_{21}^2x^5y \not\equiv 0$), $\mu_\infty = 3$. We obtain the following two systems:

$$\dot{x} = x(a_{30}x^2 + a_{10}), \quad \dot{y} = a_{10}y, \ \ a_{10}a_{30} \neq 0; \tag{55}$$

$$\dot{x} = a_{10}x, \quad \dot{y} = y(b_{21}x^2 + a_{10}), \ \ a_{10}b_{21} \neq 0. \tag{56}$$

The system (55) has four affine invariant straight lines: $l_1 = x$, $l_2 = y$, $l_{3,4} = x \pm \sqrt{-a_{10}/a_{30}}$ which, together with the line at infinity, form a sequence of multiplicities $(3, 1, 1, 1; 3)$.

System (21). Assume the conditions B) hold, then the system (21) is degenerate, i.e. $\deg(\gcd(P, Q)) > 0$ (see (4)). Let $a_{30} \neq 0$. Then, A) $\Rightarrow C_1(x, y) = -3a_{11}a_{30}^2x^5y^2 \not\equiv 0$, $\mu_\infty = 2$.

System (22). When the set of conditions A) (B)) is satisfied, then $C_1(x, y) = a_{20}b_{21}^2x^6y \not\equiv 0$ ($C_1(x, y) = -a_{30}^2x^5y(2a_{20}x + 3a_{11}y) \not\equiv 0$), $\mu_\infty = 2$.

System (23). Under the conditions A) we have $C_1(x, y) = -a_{30}^2x^5y(2a_{20}x + 3a_{11}y) \not\equiv 0$, so $\mu_\infty = 2$. In the case of conditions B): $C_1(x, y) = a_{20}b_{21}^2x^6y \equiv 0 \Rightarrow a_{20} = 0$, $C_2(x, y) = b_{21}x^3y(a_{10}b_{21}x^2 + 2a_{11}^2y^2) \not\equiv 0$, $\mu_\infty = 3$. The system (23) takes the form

$$\dot{x} = x(a_{11}y + a_{10}), \quad \dot{y} = y(b_{21}x^2 + a_{11}y + a_{10}), \quad a_{10}a_{11}b_{21} \neq 0. \tag{57}$$

It is easy to show that for the systems (28), (29), (30), (31), (32) the algebraic multiplicity of the line at infinity is one.

Note that systems (56) and (57) may be combined in one system which after an affine transformation and time rescaling can be writing in the form

$$\dot{x} = x(ay + b), \quad \dot{y} = y(x^2 + ay + b), \ \ b \neq 0 \tag{58}$$

(see system 3.2) of the Main Theorem). For system (58) only the lines $x = 0$ and $y = 0$ are affine invariant straight lines as $E_1(\mathbb{X}) = x^3y(3b^2 + 5aby + bx^2 + 2a^2y^2)$ and the algebraic curve $3b^2 + 5aby + bx^2 + 2a^2y^2 = 0$ is not invariant for (58). For system (58) we have $m_\infty(3, 1; 3) = m(3, 1; 3)$.

Lemma 8. *Any cubic system of the class $\mathbb{CSL}^{\times}_{2(r)}$ with invariant straight lines of the maximal multiplicity $m(3, 1; 3)$ via an affine transformation and time rescaling can be written in the form* (54) *or* (58).

2.2.4 Case $m_\infty(2,2;\mu_\infty)$

In Section 2.2.2 we have obtained the canonical forms of the systems (see Lemma 7) which have the maximal sequence $m(3,2;2)$. For each of these systems the affine invariant straight line $x=0$ ($y=0$) has the algebraic multiplicity three (two) and the line at infinity l_∞ has multiplicity two. The Poincaré transformation $z=1/x,\ u=y/x$ sends: the line $x=0$ into the line at infinity of the phase plane Ozu, the line at infinity of the phase plane Oxy into the line $z=0$, the line $y=0$ into the line $u=0$, and preserves the multiplicities. This transformation reduces the systems (50) and (51) to the cubic systems, respectively

$$\dot z=-az,\quad \dot u=-u(a-zu); \tag{59}$$

$$\dot z=-z(a+zu),\quad \dot u=-au. \tag{60}$$

Putting in (59) ((60)) $z=x,u=y,t=-\tau/a,a=-1/b$ ($z=y,u=x,t=-\tau/a,a=1/b$) we obtain the system

$$\dot x=x,\quad \dot y=y(1+bxy),\quad b\neq 0. \tag{61}$$

Lemma 9. *Any cubic system of the class $\mathbb{CSL}^{\times}_{2(r)}$ with straight lines of the maximal multiplicity $m(2,2;3)$ via an affine transformation and time rescaling can be written in the form* (61).

2.2.5 Case $m_\infty(2,1;\mu_\infty)$

We will examine the sets of conditions (7)–(12) under which the cubic system (5) admits the invariant straight lines $x=0$ and $y=0$ of multiplicities $\mu_1=2$ and $\mu_2=1$, respectively.

1) Conditions (7).

When for cubic system (5) the conditions (7) hold we have $C_0(x,y)=-x^2yC_{01}(x,y)\cdot C_{02}(x,y)$, where $C_{01}(x,y)=((a_{30}-b_{21})x^2+(a_{21}-b_{12})xy-b_{03}y^2)$, $C_{02}(x,y)=(a_{30}b_{21}x^3+2a_{30}b_{12}x^2y+(3a_{30}b_{03}+a_{21}b_{12})xy^2+2a_{21}b_{03}y^3)$.

Taking into account conditions (4) the polynomial $C_{01}(x,y)$ can not be identically zero, so we will require for $C_{02}(x,y)$ to be identically zero. In this case the multiplicity is $\mu_\infty\geq 2$ if one of the following three series of conditions is satisfied

$$a_{30}=a_{21}=0; \tag{62}$$

$$a_{30}=b_{12}=b_{03}=0,\ a_{21}\neq 0; \tag{63}$$

$$b_{21}=b_{12}=b_{03}=0,\ a_{30}\neq 0. \tag{64}$$

The conditions $\{(62),(4)\}$ give us $C_1(x,y)=a_{20}x^2y(b_{21}x^2+b_{12}xy+b_{03}y^2)(b_{21}x^2+2b_{12}xy+3b_{03}y^2)\not\equiv 0,\ \mu_\infty=2$.

For conditions $\{(63),\ (4)\}$ we get $C_1(x,y)=x^3y(a_{20}b_{21}^2x^3+(a_{21}b_{02}b_{21}-a_{21}^2b_{11})xy^2-2a_{21}^2b_{02}y^3)\equiv 0\Rightarrow b_{02}=b_{11}=b_{21}=0\Rightarrow C_2(x,y)=-2a_{21}^2b_{01}x^3y^3\not\equiv 0$, $\mu_\infty=3$. The system $\{(5),\ (4)\}$ has the form

$$\dot x=x^2(a_{20}+a_{21}y),\quad \dot y=b_{01}y,\quad a_{20}a_{21}b_{01}\neq 0. \tag{65}$$

In the case of conditions $\{(64),(4)\}$ we have: $C_1(x,y) = -x^3y(a_{30}x + a_{21}y) \cdot (2a_{30}b_{11}x^2 + 3a_{30}b_{02}xy + a_{21}b_{11}xy + 2a_{21}b_{02}y^2) \equiv 0 \Rightarrow b_{11} = b_{02} = 0 \Rightarrow C_2(x,y) = -b_{01}x^3y(a_{30}x + a_{21}y)(3a_{30}x + 2a_{21}y) \neq 0,\ \mu_\infty = 3$. We obtain the following cubic system

$$\dot{x} = x^2(a_{30}x + a_{21}y + a_{20}), \quad \dot{y} = b_{01}y, \quad a_{30}a_{21}b_{01} \neq 0. \tag{66}$$

After time rescaling $t = \tau/b_{01}$ the systems (65) and (66) can be combined into the system

$$\dot{x} = x^2(a + bx + cy), \quad \dot{y} = y, \ \ c(a^2 + b^2) \neq 0. \tag{67}$$

(see the system 5.1) of Main Theorem).

2) Conditions (8).

Taking into account (4) the polynomial $C_0(x,y) = -x^4y((a_{30} - b_{21})x + (a_{21} - b_{12})y)(a_{30}b_{21}x^2 + 2a_{30}b_{12}xy + a_{21}b_{12}y^2)$ is identically zero if one of the following three series of conditions is fulfilled: $a_{30} = a_{21} = 0$, i.e. (62), and

$$a_{30} = b_{12} = 0, a_{21} \neq 0; \tag{68}$$

$$b_{21} = b_{12} = 0, a_{30} \neq 0. \tag{69}$$

Under the conditions (62) we have: $\{(4);\ C_1(x,y) = a_{20}x^4y(b_{21}x + b_{12}y)(b_{21}x + 2b_{12}y) \equiv 0\} \Rightarrow \{(4);\ a_{20} = 0\} \Rightarrow C_2(x,y) = a_{10}x^3y(b_{21}x + b_{12}y)(b_{21}x + 2b_{12}y) \not\equiv 0$, $\mu_\infty = 3$. The cubic system looks as

$$\dot{x} = a_{10}x, \quad \dot{y} = y(a_{10} + b_{11}x + b_{21}x^2 + b_{12}xy), \quad a_{10}(b_{21}^2 + b_{12}^2) \neq 0. \tag{70}$$

The conditions (68) give us $C_1(x,y) = x^4y(a_{20}b_{21}^2x^2 - a_{21}^2b_{11}y^2)$. The multiplicity is $\mu_1 = 2$, $\mu_2 = 1$ and $\mu_\infty \geq 3$, if $b_{11} = a_{20} = 0, b_{21} \neq 0$ or $b_{11} = b_{21} = 0, a_{20} \neq 0$. Thus, we have the following two systems, respectively

$$\dot{x} = x(a_{10} + a_{21}xy), \quad \dot{y} = y(a_{10} + b_{21}x^2), \quad a_{10}b_{21}a_{21} \neq 0; \tag{71}$$

$$\dot{x} = x(a_{10} + a_{20}x + a_{21}xy), \quad \dot{y} = a_{10}y, \quad a_{10}a_{20}a_{21} \neq 0. \tag{72}$$

For $\{(71),(4)\}$ ($\{(72),(4)\}$) the polynomial $C_2(x,y) \equiv a_{10}x^3y(b_{21}x - a_{21}y)(b_{21}x + 2a_{21}y)$ ($C_2(x,y) \equiv -2a_{10}a_{21}^2x^3y^3$) is not identically zero, therefore $\mu_\infty = 3$.

For conditions (69): $C_1(x,y) = -b_{11}x^4y(a_{30}x + a_{21}y)(2a_{30}x + a_{21}y) \equiv 0 \Rightarrow b_{11} = 0$; $\{b_{11} = 0,(4)\} \Rightarrow C_2(x,y) \equiv -a_{10}x^3y(a_{30}x + a_{21}y)(3a_{30}x + 2a_{21}y) \neq 0$, $\mu_\infty = 3 \Rightarrow$

$$\dot{x} = x(a_{10} + a_{20}x + a_{30}x^2 + a_{21}xy), \quad \dot{y} = a_{10}y, \quad a_{10}a_{21}a_{30} \neq 0. \tag{73}$$

The system $\{(70),\ b_{11} = 0,\ b_{21}b_{12} \neq 0\}$ (respectively, (71) and $\{(73),\ a_{20} = 0,\ a_{30}a_{21} \neq 0\}$) has the affine straight lines $l_1 = x$, $l_2 = y$, $l_3 = b_{21}x + b_{12}y$ (respectively, $l_3 = b_{21}x - a_{21}y$ and $l_3 = a_{30}x + a_{21}y$) and it realizes the sequence of multiplicities $(2,1,1;3)$. If for differential system (70): $b_{11} = b_{21} = 0$ ($b_{11} = b_{12} = 0$), then $\mu_1 = 3 > 2$ ($\mu_2 = 2 > 1$). Let $a_{10}b_{11}(b_{21}^2 + b_{12}^2) \neq 0$, then, after the time rescaling and change of notation of the coefficients, we can write (70) in the form

$$\dot{x} = x, \quad \dot{y} = y(1 + ax + bx^2 + cxy), \quad a(b^2 + c^2) \neq 0 \tag{74}$$

(see the system 5.2) of Main Theorem).

After time rescaling and change of notation of the coefficients, the systems (72) and (73) can be combined into the system

$$\dot{x} = x(1 + ax + bx^2 + cxy), \quad \dot{y} = y, \quad c(a^2 + b^2) \neq 0. \tag{75}$$

(see the system 5.3) of Main Theorem).

3) Conditions (9).

Taking into account (4) the polynomial $C_0(x, y) = -x^4y((a_{30} - b_{21})x + (a_{21} - b_{12})y)(a_{30}b_{21}x^2 + 2a_{30}b_{12}xy + a_{21}b_{12}y^2)$ is identically zero if one of the conditions (62), (68), (69) is satisfied.

When the conditions $\{(62),\ (4)\}$ ($\{(68),\ (4)\}$ and $\{(69),\ (4)\}$) hold we obtain $C_1(x, y) = x^3y(b_{21}x + b_{12}y)(a_{20}b_{21}x^2 + 2a_{20}b_{12}xy + a_{11}b_{12}y^2) \not\equiv 0$ (respectively, $C_1(x, y) = x^3y(a_{20}b_{21}^2x^3 - a_{21}^2b_{11}xy^2 + 2a_{11}a_{21}b_{21}xy^2 - 2a_{11}a_{21}^2y^3) \not\equiv 0$ and $C_1(x, y) = -x^3y(a_{30}x + a_{21}y)(2a_{30}b_{11}x^2 + 3a_{11}a_{30}xy + a_{21}b_{11}xy + 2a_{11}a_{21}y^2) \not\equiv 0$), $\mu_\infty = 2$.

4) Conditions (10).

In this case we get $C_0(x, y) = -x^4y((a_{30} - b_{21})x + (a_{21} - b_{12})y)(a_{30}b_{21}x^2 + 2a_{30}b_{12}xy + a_{21}b_{12}y^2)$ and $C_0(x, y)$ is identically zero if at least one of the conditions (62), (68), (69) is satisfied.

For conditions (62) we find $\{(4),\ C_1(x, y) = x^3y(b_{21}x + b_{12}y)(a_{20}b_{21}x^2 + 2a_{20}b_{12}xy + a_{11}b_{12}y^2) \equiv 0\} \Rightarrow \{(4),\ a_{20} = b_{12} = 0\} \Rightarrow C_2(x, y) = b_{21}x^3y(a_{10}b_{21}x^2 + 2a_{11}^2y^2) \not\equiv 0,\ \mu_\infty = 3$. The cubic system looks as

$$\dot{x} = x(a_{11}y + a_{10}), \quad \dot{y} = y(b_{21}x^2 + b_{11}x + a_{11}y + a_{10}), \quad a_{10}a_{11}b_{21} \neq 0. \tag{76}$$

If $b_{11} = 0$, then the invariant straight line $x = 0$ of (76) has multiplicity $\mu_1 = 3$. Let $b_{11} \neq 0$. Via rescaling the time and change of notation of coefficients, the system (76) can be reduced to the system

$$\dot{x} = x(1 + ay), \quad \dot{y} = y(1 + bx + ay + cx^2), \quad abc \neq 0 \tag{77}$$

(see the system 5.4) of Main Theorem).

In the cases (68) and (69) we have respectively $C_1(x, y) \equiv x^3y(a_{20}b_{21}^2x^3 - a_{21}^2b_{11}xy^2 + 2a_{11}a_{21}b_{21}xy^2 - 2a_{11}a_{21}^2y^3) \neq 0$ and $C_1(x, y) = -x^3y(a_{30}x + a_{21}y) \cdot (2a_{30}b_{11}x^2 + (3a_{11}a_{30} + a_{21}b_{11})xy + 2a_{11}a_{21}y^2) \not\equiv 0$, thus μ_∞ can not be greater than two.

5) Conditions (11) *and Conditions* (12). Taking into account (4), in each of this conditions, we have $C_0(x, y) = -x^2y((a_{30} - b_{21})x + (a_{21} - b_{12})y)(a_{30}b_{21}x^4 + 2a_{30}b_{12}x^3y + (3a_{12}a_{30} + a_{21}b_{12} - a_{12}b_{21})x^2y^2 + 2a_{12}a_{21}xy^3 + a_{12}^2y^4) \not\equiv 0,\ \mu_\infty = 1$.

Lemma 10. *Any cubic system of the class $\mathbb{CSL}^{\times}_{2(r)}$ with straight lines of the partially maximal multiplicity $m_\infty(2, 1; 3)$ via an affine transformation and time rescaling can be written in one of the following four forms* (67), (70), (75) *and* (77).

2.2.6 Case $m_\infty(1,1;\mu_\infty)$

We consider the homogenized system associated to the system (5)

$$\begin{cases} \dot{x} = x(a_{10}Z^2 + a_{20}xZ + a_{11}yZ + a_{30}x^2 + a_{21}xy + a_{12}y^2), \\ \dot{y} = y(b_{01}Z^2 + b_{11}xZ + b_{02}yZ + b_{21}x^2 + b_{12}xy + b_{03}y^2). \end{cases} \tag{78}$$

For (78) we have $C_0(x,y) = -xyC_{01}(x,y)C_{02}(x,y)$, where $C_{01}(x,y) = (a_{30}-b_{21})x^2 + (a_{21}-b_{12})xy + (a_{12}-b_{03})y^2$ and $C_{02}(x,y) = (a_{30}b_{21}x^4 + 2a_{30}b_{12}x^3y + (3a_{30}b_{03} + a_{21}b_{12} - a_{12}b_{21})x^2y^2 + 2a_{21}b_{03}xy^3 + a_{12}b_{03}y^4)$. If $C_{01} \equiv 0$, then the system (78) has degenerate infinity. Let $C_{01} \not\equiv 0$. The identity $C_{02}(x,y) \equiv 0$ holds if at least one of the following four series of conditions is fulfilled

$$a_{30} = a_{21} = a_{12} = 0; \tag{79}$$

$$a_{30} = a_{21} = b_{21} = b_{03} = 0,\ a_{12} \neq 0; \tag{80}$$

$$a_{30} = b_{03} = 0, b_{12} = a_{12}b_{21}/a_{21}; \tag{81}$$

$$b_{21} = b_{12} = b_{03} = 0, a_{30} \neq 0. \tag{82}$$

1) Conditions {(79), (4)}: $C_1(x,y) = -xyC_{01}(x,y)(a_{20}b_{21}x^3 + 2a_{20}b_{12}x^2y + 3a_{20}b_{03}xy^2 + a_{11}b_{12}xy^2 + 2a_{11}b_{03}y^3) \equiv 0 \Rightarrow$

$$a_{20} = a_{11} = 0 \tag{83}$$

or

$$a_{20} = b_{12} = b_{03} = 0, a_{11} \neq 0. \tag{84}$$

For conditions {(83), (4)} we have the system

$$\begin{gathered} \dot{x} = a_{10}x, \quad \dot{y} = y(b_{01} + b_{11}x + b_{02}y + b_{21}x^2 + b_{12}xy + b_{03}y^2), \\ a_{10}(b_{21}^2 + b_{12}^2 + b_{03}^2)(b_{01}^2 + b_{02}^2 + b_{03}^2) \neq 0 \end{gathered} \tag{85}$$

for which $C_2(x,y) = -a_{10}xyC_{01}(x,y)(b_{21}x^2 + 2b_{12}xy + 3b_{03}y^2) \not\equiv 0$, $\mu_\infty = 3$, and for conditions {(84), (4)} the cubic system looks as

$$\dot{x} = x(a_{10} + a_{11}y), \quad \dot{y} = y(b_{01} + b_{11}x + b_{02}y + b_{21}x^2), \ \ a_{10}a_{11}b_{21}(b_{01}^2 + b_{02}^2) \neq 0. \tag{86}$$

For (86) we find $C_2(x,y) = b_{21}x^3y(a_{10}b_{21}x^2 + a_{11}^2y^2 + a_{11}b_{02}y^2) \not\equiv 0$, $\mu_\infty = 3$. Via rescaling the time an change of notation of coefficients, (85) can be reduced to the system

$$\dot{x} = x,\ \dot{y} = y(a + bx + cy + dx^2 + exy + fy^2),\ \ (a^2 + c^2 + f^2)(d^2 + e^2 + f^2) \neq 0 \tag{87}$$

(see the system 6.1) of Main Theorem).

In 6.1) the condition $(a^2 + b^2 + d^2)((a-1)^2 + (c^2d - bce + b^2f)^2) \neq 0$ means that the system (87) has only the following two affine invariant straight lines $x = 0$, $y = 0$ and the condition $((a-1)^2 + c^2 + f^2)((a-1)^2 + b^2 + d^2) \neq 0$ means that each of these affine straight lines has the algebraic multiplicity one.

2) Conditions {(80), (4)}. The polynomial $C_1(x,y) = xy^3(2a_{20}b_{12}^2x^3 - b_{12}(a_{12}a_{20} + a_{12}b_{11} - a_{11}b_{12})x^2y - a_{12}^2b_{02}y^3)$ is identically zero if one of the following two series of conditions is satisfied

$$b_{02} = b_{12} = 0; \tag{88}$$

$$a_{20} = b_{02} = 0, b_{11} = a_{11}b_{12}/a_{12},\ b_{12} \neq 0. \tag{89}$$

The conditions {(88), (4)} and {(89), (4)} lead us, respectively, to the following two systems

$$\dot{x} = x(a_{12}y^2 + a_{20}x + a_{11}y + a_{10}), \quad \dot{y} = y(b_{11}x + b_{01}),\ a_{12}b_{01}(a_{10}^2 + a_{20}^2) \neq 0, \tag{90}$$

$C_2(x,y) = -a_{12}xy^3(b_{11}(a_{20} + b_{11})x^2 + a_{12}b_{01}y^2) \not\equiv 0,\ \mu_\infty = 3;$

$$\dot{x} = x(a_{12}y^2 + a_{11}y + a_{10}),\ \dot{y} = y(a_{12}b_{12}xy + a_{11}b_{12}x + a_{12}b_{01})/a_{12}, a_{10}b_{12}b_{01} \neq 0, \tag{91}$$

$C_2(x,y) = -xy^3(-2a_{10}b_{12}^2x^2 + a_{12}b_{01}b_{12}xy + a_{12}^2b_{01}y^2) \not\equiv 0,\ \mu_\infty = 3.$

Via an affine transformation of coordinates and time rescaling (90) can be reduced to the system

$$\dot{x} = x(a + by),\ \ \dot{y} = y(c + dx + ey + x^2),\ \ a(c^2 + e^2) \neq 0 \tag{92}$$

(see system 6.2) of the Main Theorem). In 6.2) the inequality $(a-c)^2 + (b-e)^2 \neq 0$ means that $\mu_1 = 1$.

Note that (86) modulo time rescaling is a particular case of the system (92).

3) Conditions {(81), (4)}. In this case the polynomial $C_1(x,y) = -xy(a_{21}x + a_{12}y)(-a_{20}a_{21}b_{21}^2x^4 - 2a_{12}a_{20}b_{21}^2x^3y + (a_{21}^3b_{11} + a_{12}a_{20}a_{21}b_{21} - a_{11}a_{21}^2b_{21} - a_{21}^2b_{02}b_{21} + a_{12}a_{21}b_{11}b_{21} - a_{11}a_{12}b_{21}^2)x^2y^2 + 2a_{21}^3b_{02}xy^3 + a_{12}a_{21}^2b_{02}y^4)/a_{21}^2$ is identically zero if one of the following three series of conditions is satisfied:

$$b_{11} = b_{02} = b_{21} = 0, a_{20} \neq 0; \tag{93}$$

$$a_{20} = b_{02} = 0, a_{12} = -a_{21}^2/b_{21}; \tag{94}$$

$$a_{20} = b_{02} = 0, b_{11} = a_{11}b_{21}/a_{21}. \tag{95}$$

The conditions (93), (94), (95) give us, respectively, the systems:

$$\dot{x} = x(a_{10} + a_{20}x + a_{11}y + a_{21}xy + a_{12}y^2),\ \dot{y} = b_{01}y,\ b_{01}(a_{10}^2 + a_{20}^2)(a_{21}^2 + a_{12}^2) \neq 0 \tag{96}$$

with $C_2(x,y) \equiv -b_{01}xy^3(a_{21}x + a_{12}y)(2a_{21}x + a_{12}y) \neq 0;$

$$\begin{aligned} \dot{x} &= x(a_{10}b_{21} + a_{11}b_{21}y + a_{21}b_{21}xy - a_{21}^2y^2)/b_{21}, \\ \dot{y} &= y(b_{01} + b_{11}x + b_{21}x^2 - a_{21}xy),\ a_{10}b_{01} \neq 0 \end{aligned} \tag{97}$$

with $C_2(x,y) \equiv xy(a_{10}b_{21}^4x^4 - 2a_{10}a_{21}b_{21}^3x^3y + a_{21}^2b_{11}^2b_{21}x^2y^2 + a_{10}a_{21}^2b_{21}^2x^2y^2 - a_{21}^2b_{01}b_{21}^2x^2y^2 - 2a_{11}a_{21}b_{11}b_{21}^2x^2y^2 + a_{11}^2b_{21}^3x^2y^2 + 2a_{21}^3b_{01}b_{21}xy^3 - a_{21}^4b_{01}y^4)/b_{21}^2 \neq 0;$

$$\begin{aligned} \dot{x} &= x(a_{10} + a_{11}y + a_{21}xy + a_{12}y^2), \\ \dot{y} &= y(a_{21}b_{01} + a_{11}b_{21}x + a_{21}b_{21}x^2 + a_{12}b_{21}xy)/a_{21} \end{aligned} \tag{98}$$

with $C_2(x,y) \equiv -xy(a_{21}x + a_{12}y)(-a_{10}a_{21}b_{21}^2x^3 - a_{10}a_{21}^2b_{21}x^2y - 2a_{10}a_{12}b_{21}^2x^2y + 2a_{21}^3b_{01}xy^2 + a_{12}a_{21}b_{01}b_{21}xy^2 + a_{12}a_{21}^2b_{01}y^3)/a_{21}^2 \neq 0$. Thus, in the case of conditions $\{(81), (4)\}$ the multiplicity μ_∞ is three.

Via an affine transformation of coordinates and time rescaling (96) can be reduced to the system (85). If $a_{21} = 0$, then the system (97) is modulo time rescaling a particular case of the system (92). Let $a_{21} \neq 0$. Then, after the time rescaling $t \to -b_{21}t/a_{21}^2$, the system (97) has the form

$$\dot{x} = x(a + by + cxy + y^2),\ \dot{y} = -y(d + ex + c^2x^2 + cxy),\ \ ad \neq 0, \tag{99}$$

where $a = -a_{10}b_{21}/a_{21}^2, b = -a_{11}b_{21}/a_{21}^2, c = -b_{21}/a_{21}, d = b_{01}b_{21}/a_{21}^2); e = b_{11}b_{21}/a_{21}^2)$ (see the system 6.3) of the Main Theorem). In 6.3) the condition $c^2 + e^2 + (a+d)^2 \neq 0$ ($(a+d)^2 + (bc-e)^2 \neq 0$) means that $\mu_2 = 1$ (only $x = 0$ and $y = 0$ are affine invariant straight lines for 6.3)).

If $b_{21} = 0$, then the system (98) modulo affine transformation and time rescaling is a particular case of the system (87). Let $b_{21} \neq 0$. The time rescaling $t \to b_{21}t/(a_{21}b_{01})$ reduces (98) to the following system

$$\dot{x} = x(a + by + cxy + dy^2),\ \dot{y} = \alpha y(1 + bx + cx^2 + dxy),\ \ \alpha a(c^2 + d^2) \neq 0, \tag{100}$$

where $a = a_{10}b_{21}/(a_{21}b_{01})$, $b = a_{11}b_{21}/(a_{21}b_{01})$, $c = b_{21}/b_{01}$, $d = a_{12}b_{21}/(a_{21}b_{01})$, $\alpha = b_{21}/a_{21}$ (see the system 6.4) of the Main Theorem). In 6.4) the inequality $\alpha - a \neq 0$ means that the differential system has only the affine invariant straight lines $x = 0$ and $y = 0$.

4) Conditions $\{(82), (4)\}$:

$C_1(x,y) = -xy(a_{30}x^2 + a_{21}xy + a_{12}y^2)(2a_{30}b_{11}x^3 + (3a_{30}b_{02} + a_{21}b_{11})x^2y + 2a_{21}b_{02}xy^2 + a_{12}b_{02}y^3) \equiv 0 \Rightarrow b_{11} = b_{02} = 0 \Rightarrow C_2(x,y) = -b_{01}xy(a_{30}x^2 + a_{21}xy + a_{12}y^2)(3a_{30}x^2 + 2a_{21}xy + a_{12}y^2) \not\equiv 0 \Rightarrow \mu_\infty = 3$. The cubic system looks as:

$$\begin{aligned} \dot{x} = x(a_{10} + a_{20}x + a_{11}y + a_{30}x^2 + a_{21}xy + a_{12}y^2),\ \dot{y} = b_{01}y, \\ a_{30}b_{01}(a_{10}^2 + a_{20}^2 + a_{30}^2) \neq 0. \end{aligned} \tag{101}$$

Modulo affine transformation the system (101) is a particular case of the system (85).

Lemma 11. *Any cubic system of the class* $\mathbb{CSL}_{2(r)}^{\times}$ *with straight lines of the partially maximal multiplicity* $m_\infty(1,1;3)$ *via an affine transformation and time rescaling can be written in one of the following four forms* (87), (92), (99) *and* (100).

The proof of the Main Theorem follows from Lemmas 8–11.

2.3 Geometric multiplicity

In this section for the normal forms given in Main Theorem we construct the corresponding perturbed cubic systems which show that for invariant straight lines ($x = 0$, $y = 0$ and $Z = 0$) the algebraic and geometric multiplicities coincide.

1) $m(3,3;1)$: $\dot{x} = x^3$, $\dot{y} = y(x^2 + ay + by^2)$, $b \neq 0$.

The perturbed cubic system is

$\dot{x} = x(x - a\epsilon + 2bx\epsilon^2)(x + a\epsilon + 2bx\epsilon^2)$, $\dot{y} = y(x^2 + ay + by^2 + a^2\epsilon^2 + 3bx^2\epsilon^2 + 4aby\epsilon^2 + +4b^2y^2\epsilon^2 + a^2b\epsilon^4 + 4ab^2y\epsilon^4 + 4b^3y^2\epsilon^4 - 4b^3x^2\epsilon^6)$, $b \neq 0$.

The invariant straight lines are $l_1 = x$, $l_2 = y$, $l_3 = x - a\epsilon + 2bx\epsilon^2$, $l_4 = x + a\epsilon + 2bx\epsilon^2$, $l_5 = y - x\epsilon + a\epsilon^2 + 2by\epsilon^2 - 2bx\epsilon^3$, $l_6 = y + x\epsilon + a\epsilon^2 + 2by\epsilon^2 + 2bx\epsilon^3$.

If $\epsilon \to 0$, then the invariant straight lines $l_1, l_3, l_4 \to l_1$ and $l_2, l_5, l_6 \to l_2$.

2.1) $m(3,2;2)$: $\dot{x} = ax^3$, $\dot{y} = y^2, a \neq 0$.

The perturbed cubic system is $\dot{x} = ax(x-\epsilon)(x+\epsilon)$, $\dot{y} = y(y-\epsilon)(\epsilon y+1), a \neq 0$.

The invariant straight lines are $l_1 = x$, $l_2 = y$, $l_3 = x - \epsilon$, $l_4 = x + \epsilon$, $l_5 = y - \epsilon$, $l_6 = \epsilon y + 1$.

If $\epsilon \to 0$, then the invariant straight lines $l_1, l_3, l_4 \to l_1$; $l_2, l_5 \to l_2$ and $l_6 \to l_\infty$.

2.2) $m(3,2;2)$: $\dot{x} = x(ax^2 + y)$, $\dot{y} = y^2$, $a \neq 0$.

The perturbed cubic system is $\dot{x} = x(ax^2 + y + \epsilon - a\epsilon^4)$, $\dot{y} = y(y+\epsilon)(1 + ay\epsilon^2 - a\epsilon^3), a \neq 0$.

The invariant straight lines are $l_1 = x$, $l_2 = y$, $l_3 = x - y\epsilon$, $l_4 = x + y\epsilon$, $l_5 = y - \epsilon$, $l_6 = ay\epsilon^2 - a\epsilon^3 + 1$.

If $\epsilon \to 0$, then the invariant straight lines $l_1, l_3, l_4 \to l_1$; $l_2, l_5 \to l_2$ and $l_6 \to l_\infty$.

3.1) $m(3,1;3)$: $\dot{x} = x^2(ax + by)$, $\dot{y} = y$, $a \neq 0$.

The perturbed cubic system is

$\dot{x} = x(ax^2 + bxy - a\epsilon^2 + 4a^2x^2\epsilon^2 + 4abxy\epsilon^2 + 2b^2y^2\epsilon^2 - 4a^2\epsilon^4 + 4a^3x^2\epsilon^4 + 4a^2bxy\epsilon^4 + ab^2y^2\epsilon^4 - 4a^3\epsilon^6)$, $\dot{y} = y(-1 + by\epsilon - 2a\epsilon^2)(1 + by\epsilon + 2a\epsilon^2)$, $a \neq 0$.

The invariant straight lines are $l_1 = x$, $l_2 = y$, $l_3 = x - \epsilon + 2ax\epsilon^2 + by\epsilon^2 - 2a\epsilon^3$, $l_4 = x + \epsilon + 2ax\epsilon^2 + by\epsilon^2 + 2a\epsilon^3$, $l_5 = by\epsilon - 2a\epsilon^2 - 1$, $l_6 = by\epsilon + 2a\epsilon^2 + 1$.

If $\epsilon \to 0$, then invariant straight lines $l_1, l_3, l_4 \to l_1$ and $l_5, l_6 \to l_\infty$.

3.2) $m(3,1;3)$: $\dot{x} = x(ay + b)$, $\dot{y} = y(x^2 + ay + b)$, $b \neq 0$.

The perturbed cubic system is

$\dot{x} = -x(-b - ay - 4b^2\epsilon^2 + bx^2\epsilon^2 - 4aby\epsilon^2 - 2a^2y^2\epsilon^2 - 4b^3\epsilon^4 + 4b^2x^2\epsilon^4 - 4ab^2y\epsilon^4 - a^2by^2\epsilon^4 + 4b^3x^2\epsilon^6)$, $\dot{y} = y(b + x^2 + ay + 4b^2\epsilon^2 + 3bx^2\epsilon^2 + 4aby\epsilon^2 + a^2y^2\epsilon^2 + 4b^3\epsilon^4 + 4ab^2y\epsilon^4 + a^2by^2\epsilon^4 - 4b^3x^2\epsilon^6)$, $b \neq 0$.

The invariant straight lines are $l_1 = x$, $l_2 = y$, $l_3 = x - ay\epsilon + 2bx\epsilon^2$, $l_4 = x + ay\epsilon + 2bx\epsilon^2$, $l_5 = x\epsilon - 2b\epsilon^2 - ay\epsilon^2 + 2bx\epsilon^3 - 1$, $l_6 = x\epsilon + 2b\epsilon^2 + ay\epsilon^2 + 2bx\epsilon^3 + 1$.

If $\epsilon \to 0$, then invariant straight lines $l_1, l_3, l_4 \to l_1$ and $l_5, l_6 \to l_\infty$.

4) $m(2,2;3)$: $\dot{x} = x$, $\dot{y} = y(1 + bxy)$, $b \neq 0$.

The perturbed cubic system is $\dot{x} = -x(x\epsilon - 1)(x\epsilon + 1)$, $\dot{y} = y(1 + bxy + by^2\epsilon - y^2\epsilon^4)$, $b \neq 0$.

The invariant straight lines are $l_1 = x$, $l_2 = y$, $l_3 = x + \epsilon y$, $l_4 = by + x\epsilon^2 - y\epsilon^3$, $l_5 = x\epsilon + 1$, $l_6 = x\epsilon - 1$.

If $\epsilon \to 0$, then $l_1, l_3 \to l_1$; $l_2, l_4 \to l_2$ and $l_5, l_6 \to l_\infty$.

5.1) $m_\infty(2,1;3)$: $\dot{x} = x^2(a + bx + cy)$, $\dot{y} = y$, $c(a^2 + b^2) \neq 0$.

The perturbed cubic system is $\dot{x} = x(a + bx + cy)(x + \epsilon)$, $\dot{y} = -y(-1 + \epsilon y)(1 + \epsilon y)$, $c(a^2 + b^2) \neq 0$.

The invariant straight lines are $l_1 = x$, $l_2 = y$, $l_3 = x + \epsilon$, $l_4 = \epsilon y - 1$, $l_5 = \epsilon y + 1$.

If $\epsilon \to 0$, then $l_1, l_3 \to l_1$ and $l_4, l_5 \to l_\infty$.

5.2) $m_\infty(2,1;3)$: $\dot{x} = x$, $\dot{y} = y(1 + ax + bx^2 + cxy)$, $a(b^2 + c^2) \neq 0$.

The perturbed cubic system is

$\dot{x} = -x(-1 + x\epsilon)(1 + x\epsilon)$, $\dot{y} = y(1 + ax + bx^2 + cxy + ay\epsilon + bxy\epsilon + cy^2\epsilon - x^2\epsilon^2)$, $a(b^2 + c^2) \neq 0$.

The invariant straight lines are $l_1 = x$, $l_2 = y$, $l_3 = x + \epsilon y$, $l_4 = \epsilon x + 1$, $l_5 = \epsilon x - 1$.

If $\epsilon \to 0$, then $l_1, l_3 \to l_1$ and $l_4, l_5 \to l_\infty$.

5.3) $m_\infty(2,1;3)$: $\dot{x} = x(1 + ax + bx^2 + cxy)$, $\dot{y} = y$, $c(a^2 + b^2) \neq 0$;

The perturbed cubic system is

$\dot{x} = x(1 + ax + bx^2 + cxy + ay\epsilon + bxy\epsilon + cy^2\epsilon - y^2\epsilon^2)$, $\dot{y} = -y(-1 + y\epsilon)(1 + y\epsilon)$, $c(a^2 + b^2) \neq 0$.

The invariant straight lines are $l_1 = x$, $l_2 = y$, $l_3 = x + \epsilon y$, $l_4 = \epsilon y + 1$, $l_5 = \epsilon y - 1$.

If $\epsilon \to 0$, then $l_1, l_3 \to l_1$ and $l_4, l_5 \to l_\infty$.

5.4) $m_\infty(2,1;3)$: $\dot{x} = x(1 + ay)$, $\dot{y} = y(1 + bx + ay + cx^2)$, $abc \neq 0$.

The perturbed cubic system is

$\dot{x} = x(1 + x\epsilon)(1 + ay - x\epsilon)$, $\dot{y} = y(c^2 + bc^2x + c^3x^2 + ac^2y + 2bc\epsilon + 2b^2cx\epsilon + 2bc^2x^2\epsilon + abcy\epsilon - 2ac^2xy\epsilon + b^2\epsilon^2 + 4c\epsilon^2 + b^3x\epsilon^2 + 4bcx\epsilon^2 + b^2cx^2\epsilon^2 + 3c^2x^2\epsilon^2 + 4acy\epsilon^2 - abcxy\epsilon^2 + 2a^2cy^2\epsilon^2 + 4b\epsilon^3 + 4b^2x\epsilon^3 + 2bcx^2\epsilon^3 + 2aby\epsilon^3 + ab^2xy\epsilon^3 - 2acxy\epsilon^3 + 4\epsilon^4 + 4bx\epsilon^4 - b^2x^2\epsilon^4 + 4ay\epsilon^4 + 4abxy\epsilon^4 - 4bx^2\epsilon^5 + 4axy\epsilon^5 - 4x^2\epsilon^6)/(c + b\epsilon + 2\epsilon^2)^2$, $c \neq 0$.

The invariant straight lines are $l_1 = x$, $l_2 = y$, $l_3 = -cx - bx\epsilon + ay\epsilon - 2x\epsilon^2$, $l_4 = x\epsilon + 1$, $l_5 = c + b\epsilon - cx\epsilon + 2\epsilon^2 - bx\epsilon^2 + 2ay\epsilon^2 - 2x\epsilon^3$.

If $\epsilon \to 0$, then $l_1, l_3 \to l_1$ and $l_4, l_5 \to l_\infty$.

6.1) $m_\infty(1,1;3)$: $\dot{x} = x$, $\dot{y} = y(a + bx + cy + dx^2 + exy + fy^2)$, $(a^2 + c^2 + f^2)(d^2 + e^2 + f^2)(a^2 + b^2 + d^2)((a-1)^2 + c^2 + f^2)((a-1)^2 + b^2 + d^2)((a-1)^2 + (c^2d - bce + b^2f)^2) \neq 0$;

The perturbed cubic system is $\dot{x} = x(\epsilon x + 1)(\epsilon x - 1)$, $\dot{y} = y(a + bx + cy + dx^2 + exy + fy^2)$.

The invariant straight lines are $l_1 = x$, $l_2 = y$, $l_3 = \epsilon x + 1$, $l_4 = \epsilon x - 1$.

If $\epsilon \to 0$, then $l_3, l_4 \to l_\infty$.

6.2) $m_\infty(1,1;3)$: $\dot{x} = x(a + by)$, $\dot{y} = y(c + dx + ey + x^2)$, $a(c^2 + e^2)((a - c)^2 + (b - e)^2) \neq 0$.

The perturbed cubic system is

$\dot{x} = -x(1 + x\epsilon)(-a - by + x\epsilon)$, $\dot{y} = y(a^5c + a^5dx + a^5x^2 + a^5ey + 2a^4cd\epsilon - a^5x\epsilon + a^6x\epsilon + 2a^4d^2x\epsilon + 2a^4dx^2\epsilon + 2a^4dey\epsilon - a^4bxy\epsilon - a^4exy\epsilon + 2a^5c\epsilon^2 + 2a^3c^2\epsilon^2 + a^3cd^2\epsilon^2 -$

$2a^4dx\epsilon^2 + 4a^5dx\epsilon^2 + 2a^3cdx\epsilon^2 + a^3d^3x\epsilon^2 + a^5x^2\epsilon^2 + 2a^3cx^2\epsilon^2 + a^3d^2x^2\epsilon^2 + 2a^5ey\epsilon^2 + 2a^3cey\epsilon^2 + a^3d^2ey\epsilon^2 - a^3bdxy\epsilon^2 + a^4bdxy\epsilon^2 - 2a^3dexy\epsilon^2 + a^3bey^2\epsilon^2 + a^4bey^2\epsilon^2 + 2a^4cd\epsilon^3 + 2a^2c^2d\epsilon^3 - 2a^5x\epsilon^3 + 2a^6x\epsilon^3 - 2a^3cx\epsilon^3 + 2a^4cx\epsilon^3 - a^3d^2x\epsilon^3 + 3a^4d^2x\epsilon^3 + 2a^2cd^2x\epsilon^3 + 2a^2cdx^2\epsilon^3 + 2a^4dey\epsilon^3 + 2a^2cdey\epsilon^3 - a^4bxy\epsilon^3 + a^5bxy\epsilon^3 + 2a^3bcxy\epsilon^3 + a^3bd^2xy\epsilon^3 - 2a^4exy\epsilon^3 - 2a^2cexy\epsilon^3 - a^2d^2exy\epsilon^3 + a^2bdey^2\epsilon^3 + a^3bdey^2\epsilon^3 + a^5c\epsilon^4 + 2a^3c^2\epsilon^4 + ac^3\epsilon^4 - 2a^4dx\epsilon^4 + 3a^5dx\epsilon^4 - 2a^2cdx\epsilon^4 + 4a^3cdx\epsilon^4 + ac^2dx\epsilon^4 - a^5x^2\epsilon^4 + ac^2x^2\epsilon^4 - a^3d^2x^2\epsilon^4 + a^5ey\epsilon^4 + 2a^3cey\epsilon^4 + ac^2ey\epsilon^4 + 2a^4bdxy\epsilon^4 + abcdxy\epsilon^4 + 3a^2bcdxy\epsilon^4 - 2a^3dexy\epsilon^4 - 2acdexy\epsilon^4 - ab^2cy^2\epsilon^4 - 2a^2b^2cy^2\epsilon^4 - a^3b^2cy^2\epsilon^4 + a^3bey^2\epsilon^4 + a^4bey^2\epsilon^4 + abcey^2\epsilon^4 + a^2bcey^2\epsilon^4 - a^5x\epsilon^5 + a^6x\epsilon^5 - 2a^3cx\epsilon^5 + 2a^4cx\epsilon^5 - ac^2x\epsilon^5 + a^2c^2x\epsilon^5 - 2a^4dx^2\epsilon^5 - 2a^2cdx^2\epsilon^5 + a^5bxy\epsilon^5 + a^2bcxy\epsilon^5 + 3a^3bcxy\epsilon^5 + bc^2xy\epsilon^5 + 2abc^2xy\epsilon^5 - a^4exy\epsilon^5 - 2a^2cexy\epsilon^5 - c^2exy\epsilon^5 - a^5x^2\epsilon^6 - 2a^3cx^2\epsilon^6 - ac^2x^2\epsilon^6)/(a(a^2 + ad\epsilon + a^2\epsilon^2 + c\epsilon^2)^2)$.

The invariant straight lines are $l_1 = x$, $l_2 = y$, $l_3 = x\epsilon + 1$, $l_4 = a^3 + a^2d\epsilon - a^2x\epsilon + a^3\epsilon^2 + ac\epsilon^2 - adx\epsilon^2 + aby\epsilon^2 + a^2by\epsilon^2 - a^2x\epsilon^3 - cx\epsilon^3$.

If $\epsilon \to 0$, then $l_3, l_4 \to l_\infty$.

6.3) $m_\infty(1,1;3)$: $\dot{x} = x(a + by + cxy + y^2)$, $\dot{y} = -y(d + ex + c^2x^2 + cxy)$, $ad(c^2 + e^2 + (a+d)^2)((a+d)^2 + (bc-e)^2) \neq 0$.

The perturbed cubic system is

$\dot{x} = x(a + by + cxy + y^2 - bcxy\epsilon + exy\epsilon - a^2\epsilon^2 - ab^2\epsilon^2 + 2abcx\epsilon^2 - 2aex\epsilon^2 - ac^2x^2\epsilon^2 - aby\epsilon^2 - b^3y\epsilon^2 - acxy\epsilon^2 - 2bexy\epsilon^2 - ay^2\epsilon^2 - b^2y^2\epsilon^2 - a^2b\epsilon^3 - 2a^2cx\epsilon^3 + 2abex\epsilon^3 + 2abc^2x^2\epsilon^3 - 2acex^2\epsilon^3 - ab^2y\epsilon^3 - 2aexy\epsilon^3 + b^2exy\epsilon^3 - aby^2\epsilon^3 - 2a^2bcx\epsilon^4 + 2a^2ex\epsilon^4 - a^2c^2x^2\epsilon^4 + ab^2c^2x^2\epsilon^4 + 2abcex^2\epsilon^4 + 2abexy\epsilon^4 - a^2bc^2x^2\epsilon^5 + 2a^2cex^2\epsilon^5 + a^2exy\epsilon^5)$, $\dot{y} = y(-d - ex - c^2x^2 - cxy - acx\epsilon + b^2cx\epsilon + 2bc^2x^2\epsilon - 2cex^2\epsilon + bcxy\epsilon - exy\epsilon + ad\epsilon^2 + b^2d\epsilon^2 + b^3cx\epsilon^2 - 2bcdx\epsilon^2 + b^2ex\epsilon^2 + 2dex\epsilon^2 - 2ac^2x^2\epsilon^2 + b^2c^2x^2\epsilon^2 + c^2dx^2\epsilon^2 + 2bcex^2\epsilon^2 - acxy\epsilon^2 + 2b^2cxy\epsilon^2 + 2cdxy\epsilon^2 + dy^2\epsilon^2 + abd\epsilon^3 - a^2cx\epsilon^3 + 2acdx\epsilon^3 + 2abex\epsilon^3 - 2bdex\epsilon^3 + abc^2x^2\epsilon^3 - 2bc^2dx^2\epsilon^3 + 2cdex^2\epsilon^3 - 2bcdxy\epsilon^3 + b^2exy\epsilon^3 + 2dexy\epsilon^3 - a^2bcx\epsilon^4 + 2abcdx\epsilon^4 + a^2ex\epsilon^4 - 2adex\epsilon^4 - a^2c^2x^2\epsilon^4 + ab^2c^2x^2\epsilon^4 + ac^2dx^2\epsilon^4 - b^2c^2dx^2\epsilon^4 + 2abcex^2\epsilon^4 - 2bcdex^2\epsilon^4 - 2b^2cdxy\epsilon^4 + 2abexy\epsilon^4 - 2bdexy\epsilon^4 - ady^2\epsilon^4 - b^2dy^2\epsilon^4 - a^2bc^2x^2\epsilon^5 + abc^2dx^2\epsilon^5 + 2a^2cex^2\epsilon^5 - 2acdex^2\epsilon^5 + a^2exy\epsilon^5 - 2adexy\epsilon^5 - abdy^2\epsilon^5)$.

The invariant straight lines are $l_1 = x$, $l_2 = y$, $l_3 = 1 + cx\epsilon + y\epsilon$, $l_4 = -1 + cx\epsilon + y\epsilon + a\epsilon^2 + b^2\epsilon^2 - 2bcx\epsilon^2 + 2ex\epsilon^2 + ab\epsilon^3 + acx\epsilon^3 - b^2cx\epsilon^3 - 2bex\epsilon^3 - ay\epsilon^3 - b^2y\epsilon^3 + abcx\epsilon^4 - 2aex\epsilon^4 - aby\epsilon^4$.

If $\epsilon \to 0$, then $l_3, l_4 \to l_\infty$.

6.4) $m_\infty(1,1;3)$: $\dot{x} = x(a + by + cxy + dy^2)$, $\dot{y} = \alpha y(1 + bx + cx^2 + dxy)$, $\alpha a(c^2 + d^2)(\alpha - a) \neq 0$.

The perturbed cubic system is

$\dot{x} = -x(-a - by - cxy - dy^2 - axy\alpha\epsilon^2 + ax^2\alpha^2\epsilon^2 - 2xy\alpha^2\epsilon^2)$, $\dot{y} = -y\alpha(-1 - bx - cx^2 - dxy - axy\epsilon^2 + y^2\epsilon^2 + ax^2\alpha\epsilon^2 - 2xy\alpha\epsilon^2 - x^2\alpha^2\epsilon^2)$.

The invariant straight lines are $l_1 = x$, $l_2 = y$, $l_3 = 1 - y\epsilon + x\alpha\epsilon$, $l_4 = -1 - y\epsilon + x\alpha\epsilon$.

If $\epsilon \to 0$, then the lines $l_3, l_4 \to \infty$.

References

[1] Artes J., Grünbaum B. and Llibre J. *On the number of invariant straight lines for polynomial differential systems.* Pacific J. of Math., 1998, **184**, No. 2, 207–230.

[2] Bujac C. *One subfamily of cubic systems with invariant lines of total multiplicity eight and with two distinct real infinite singularities.* Bul. Acad. Ştiinţe Repub. Mold., Mat., 2015, No. 1(77), 48–86.

[3] Bujac C., Vulpe N. *Cubic systems with invariant straight lines of total multiplicity eight and with three distinct infinite singularities.* Qual. Theory Dyn. Syst., 2015, **14**, No. 1, 109–137.

[4] Bujac C., Vulpe N. *Cubic systems with invariant lines of total multiplicity eight and with four distinct infinite singularities.* Journal of Mathematical Analysis and Applications, 2015, **423**, No. 2, 1025–1080.

[5] Christopher C., Llibre J., Pereira J. V. *Multiplicity of invariant algebraic curves in polynomial vector fields.* Pacific J. of Math., 2007, **229**, No. 1, 63–117.

[6] Cozma D. *Integrability of cubic systems with invariant straight lines and invariant conics.* Chişinău, Ştiinţa, 2013, 240 p.

[7] Cozma D., Şubă A. *The solution of the problem of center for cubic differential systems with four invariant straight lines.* An. Ştiinţ. Univ. "Al. I. Cuza"(Iaşi), 1998, **44**, suppl. 517–530.

[8] Suo Guangjian, Sun Jifang. *The n-degree differential system with $(n-1)(n+1)/2$ straight line solutions has no limit cycles.* Proc. of Ordinary Differential Equations and Control Theory, Wuhan. 1987, 216–220 (in Chinese).

[9] Kooij R. *Cubic systems with four line invariants, including complex conjugated lines.* Math. Proc. Camb. Phil. Soc., 1995, **118**, No. 1, 7–19.

[10] Llibre J., Vulpe N. *Planar cubic polynomial differential systems with the maximum number of invariant straight lines.* Rocky Mountain J. Math., 2006, **36**, No 4, 1301–1373.

[11] Lyubimova R. A. *About one differential equation with invariant straight lines.* Differential and integral equations, Gorky Universitet, 1984, **8**, 66–69; 1997, **1**, 19–22 (in Russian).

[12] Mironenko V. I. *Linear dependence of functions along solutions of differential equations.* Beloruss. Gos. Univ., Minsk, 1981. 104 p. (in Russian).

[13] Puţuntică V., Şubă A. *The cubic differential system with six real invariant straight lines along two directions.* Studia Universitatis, 2008, No 8(13), 5–16.

[14] Puţuntică V., Şubă A. *The cubic differential system with six real invariant straight lines along three directions.* Bul. Acad. Ştiinţe Repub. Mold., Mat., 2009, No 2(60), 111–130.

[15] Repeşco V. *Cubic systems with degenerate infinity and straight lines of total parallel multiplicity six.* ROMAI J., 2013, **9**, No. 1, 133–146.

[16] Rychkov G. S. *The limit cycles of the equation $u(x+1)du = (-x+ax^2+bxu+cu+du^2)dx$.* Differentsial'nye Uravneniya, 1972, **8**, No 12, 2257–2259 (in Russian).

[17] Şubă A., Cozma D. *Solution of the problem of the center for cubic differential system with three invariant straight lines in generic position.* Qual. Th. of Dyn. Systems, 2005, **6**, 45–58.

[18] Şubă A., Repeşco V., Puţuntică V. *Cubic systems with invariant affine straight lines of total parallel multiplicity seven.* Electron. J. Diff. Equ., 2013, Vol. 2013 (2013), No. 274, 1–22. http://ejde.math.txstate.edu/

[19] Şubă A., and Vacaraş O. *Cubic differential systems with an invariant straight line of maximal multiplicity.* Annals of the University of Craiova. Mathematics and Computer Science Series, 2015, **42**, No. 2, 427–449.

[20] Zhang Xikang. *Number of integral lines of polynomial systems of degree three and four.* J. of Nanjing University, Math. Biquartely, 1993, **10**, 209–212.

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A Parametric Scheme for Online Uniform-Machine Scheduling to Minimize the Makespan

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Abstract. In this paper, we consider the Online Uniform Machine Scheduling problem in the case when speed $s_i = 1$ for $i = n-k+1, \ldots, n$ and $S_i = s, 1 \leq s \leq 2$ for $i = 1, \ldots, k$, where k is a constant, and we propose a parametric scheme with an asymptotic worst-case behavior (when m tends to infinity).

Mathematics subject classification: 34C05.
Keywords and phrases: Online Scheduling, Uniform Parallel Machine, worst-case behavior, parametric scheme.

1 Introduction

In this paper, we study the classic problem of scheduling jobs *online* on m uniform machines $(M_1, M_2, \ldots M_m)$ with speeds $(s_1, s_2, \ldots, s_m)$ without preemption: jobs arrive one at a time, according to a linear ordering (a list) σ, with known processing times and must immediately be scheduled on one of the machines, without knowledge of what jobs will come afterwards, or how many jobs are still to come; all machines can perform the same tasks, according to distinct speeds. However, the way jobs are ordered inside the list σ has no correlation with the starting times which are assigned to them in the schedule: some future (in the list σ) job may come to start earlier than the current one, because what we do here is only distributing the jobs among the machines.

We denote by J_j the jth job in the list s, and say that job J_j arrives at *step* j according to s. We denote by p_j the processing time of job J_j. If job p_j is assigned to machine M_i, then p_j/s_i time units are required to process this job.

The quality of an online algorithm A is measured by its competitive ratio, defined as the smallest number c such that, for every list of jobs σ which describes jobs together with their arrival order, we have $F(A, \sigma) \leq c \cdot Opt(\sigma)$, where $F(A, \sigma)$ denotes the makespan of the schedule which derives from application of algorithm A to the list σ, and $Opt(\sigma)$ denotes the makespan of some optimal schedule of the jobs of σ, computed while considering σ as a set of jobs, and not as an ordering. We may also say that $Opt(\sigma)$ is the optimal value of the *offline* scheduling problem induced by the jobs contained in the list σ. The algorithm A is said to be c-competitive.

The online *Multi-machine Scheduling* problem for identical machines (they are all provided with the same speed) was first investigated by Graham, who showed

that the *List* algorithm (LS) which always puts the next job on the least loaded machine is exactly $(2 - 1/m)$-competitive [2].

In the case of uniform machines Cho and Sahni [1] proved that the LS algorithm has a worst-case bound of $(3m-1)/(m+1)$ for $m \geq 3$. When $s_i = 1, i = 1, \ldots, m-1$ and $s_m > 1$, Cho and Sahni also showed that the LS algorithm has a worst-case bound c of $1 + (m-1) \cdot (\min(2, s)/(m+s-1)) \leq 3 - 4/(m+1)$, and the bound $3 - 4/(m+1)$ is achieved when $s = 2$. Li and Shi [3] proved that the LS algorithm is the best possible one for $m \leq 3$, and proposed an algorithm that is significantly better than the LS algorithm when $s_i = 1, i = 1, \ldots, m-1$ and $s_m = 2, m \geq 4$. The algorithm has a worst-case bound of 2.8795 for a big m. For the same problem Cheng, Ng and Kotov [4] proposed a 2.45-competitive algorithm for any $m \geq 4$ and any $s_m, 1 \leq s_m = s \leq 2$. Also, some results in the case of fixed number of machines can be found in [5–7]. It should be mentioned that the worst-case behavior of all previous algorithms occurs when m tends to infinity.

In this paper we use ideas of reserved classes and a dynamic lower bound of the optimal solution from [8, 9].

2 A Parametric Scheme for the OnLine Uniform Machine Scheduling Problem

Before presenting the main results, we introduce some notations.

1. m denotes the total number of machines;
2. k denotes the number of machines with a speed $1 < s \leq 2$, k is a constant.

We are going to describe here a strategy (an algorithm) which will allow us to assign for any index j the job J_j with processing time p_j which arrives at step j ($j = 1, \ldots, Length(s)$) according to the list ordering σ to some machine $M_i, i = 1, \ldots, m$. We shall do in such a way that J_j will then be scheduled immediately after the end of the latest job which was assigned to M_i. As a matter of fact, since no precedence relation is imposed to the jobs, jobs assigned to a same machine will be consecutively run, without any idle time. So, any time we have to deal with a current job J_j of the input list σ, we denote by:

1. $L_{i,j}$ the current load of machine i before assigning job J_j;
2. $L^*_{i,j}$ the current load of machine i after assigning job J_j;
3. V_j the theoretical optimal makespan for the *offline* scheduling problem induced by the job set $J(j) = \{J_1, \ldots, J_j\}$ made of the jobs which arrived no later than step j.

It is easy to check that, if we denote by $q_1, .., , q_j$ the processing time of the jobs of $J(j)$, sorted by decreasing order, which means that we have: $q1 \geq q2 \geq \cdots \geq q_j$, then we may state:

Lemma 1. *The following inequalities hold:*

1. $V_j \geq (q_1 + q_2 + \cdots + q_j)/(m - k + s \cdot k)$;
2. $V_j \geq q_1/s$;
3. $V_j \geq \min\{(q_k + q_{k+1})/s, q_{k+1}\}$.

Proof. Left to the reader. It is important to notice that the last inequality $V_j \geq \min\{(q_k + q_{k+1})/s, q_{k+1}\}$ derives from the hypothesis $1 \leq s \leq 2$. As a matter of fact, it will be the only place, inside our reasoning process, where the hypothesis plays a role.

So, for any step value j, we set:

$$LB_j = \max\{(q_1 + q_2 + \cdots + q_j)/(m - k + s \cdot k), q_1/s, \min\{(q_k + q_{k+1})/s, q_{k+1}\}\}. \quad (1)$$

Clearly, LB_j is a lower bound for the optimal *offline* makespan related to step j and we have: $LB_{j-1} \leq LB_j$ (LB_j is monotonic). □

2.1 The Assignment Process Assign

We suppose now that some positive number α is given together with three integral numbers R, m_1 and m_2 in such a way that:

$$(1 + \alpha) \cdot s \cdot k + (1 + \alpha/2) \cdot m_1 \geq s \cdot k + m_1 + m_2, \quad (2)$$

$$k + m_1 + m_2 = m, \quad (3)$$

$$m_2 = R \cdot k, \quad (4)$$

$$R \geq \log_{1+\alpha/2}((1 + \alpha/2)/(2 + \alpha - s)). \quad (5)$$

It is easy to see that, if we fix k, s and α, and if we require R and m_1 to take the smallest possible values, then R, m, m_1 and m_2 are completely determined by k, s and α.

This assumption about the way the machine number m may be decomposed, allows us to split the machine set machines into three classes:

1. machines with speed s are called *Fast*;
2. we pick up m_1 machines among the $m - k$ machines with speed 1 and call them *Normal*;
3. the m_2 remaining machines with speed 1 are called *Reserved* and the $m_2 = R \cdot k$ *Reserved* machines are split into R groups $G_0, \ldots, G_{R-1}$, each group containing exactly k machines.

By the same way, we say that job J_j, which arrives at step j is:

1. *Small* if its processing time p_j is at most equal to $(1+\alpha/2)\cdot LB_j$;
2. *Large* else.

Finally, we say that this job J_j fits machine M_i, $i = 1\dots m$, if $L_{i,j} + p_j/s_i \leq (2+\alpha)\cdot LB_j$.

We easily see that:

Lemma 2. *If Large job J_j does not fit machine i from class Fast then $L_{i,j} > (1+\alpha)\cdot LB_j$.*

Proof. It comes in a straightforward way from the fact that $p_j/s_i = p_j/s \leq LB_j$. □

Doing this allows us to describe our online algorithm *Assign*, which will work on any instance of the *Online Uniform Machine Scheduling* Problem such that m, k, s may be written according to the relations (2)-(5). The main idea here is that at any step j, we are going to be able to assign job J_j to some machine $i(j)$ in such a way that we keep the following inequality: $\max_i L^*_{i,j} \leq (2+\alpha)\cdot LB_j$. While doing this will happen to be easy in the case when j is a *Small* job, the trick will be to show that, if j is a *Large* job, we may, by conveniently switching machines inside the *Normal* and *Reserved* classes, do in such a way that if j does not fit any of machine of classes *Fast* and *Normal*, then it fits at least some machine in current group G_0, whose machines are, at any time during the process, provided with current labels in $\{1, ..., k\}$. It is important to understand here that the status *Normal* or *Reserved* of a given machine with speed 1 is not going to be fixed, and will be evolving all throughout the process.

Algorithm Assign

Initialization: Set $n = 1$; (*n denotes the index of the current target *Reserved* machine in group G_0; machines in every group G_R are indexed from 0 to $k-1$*); Set $j = 0$; $LB_j = 0$;

Read(σ);
While σ is non empty do
$j := j + 1$;
Read the current job J_j and perform Step j as follows:
Update LB_j according to formula (1).
If job J_j fits some machine i in classes *Fast* and *Normal*
  then (I1)
    assign j to this machine i
  Else
    If $n < k$ then (I2)
      Assign job J_j on the machine (with label) n in G_0;
      Let i_0 be the machine from class *Normal* with minimal
      current load. Switch machines n and i_0 between groups *Normal*
      and G_0 in such a way that machine i_0 comes in G_0 with label n,

and machine n is put into class *Normal*. Set $n = n + 1$;
If $n = k$ then (I3)
Update the labeling of groups $G_0, \ldots, G_{R-1}$ in such a way that group $r, 1 \leq r \leq R-1$, becomes group $r-1$, and group 0 becomes group $R-1$. Set $n = 1$.

2.2 Worst Case Performance of Assign

The *Assign* algorithm works on an instance $(M_1, M_2, \ldots, M_m; s_1, s_2, \ldots, s_m)$ of the *Online Uniform Machine Scheduling* Problem which is such that:

1. $s_i = s \in [1, 2]$ for $i = 1, ..., k; s_i = 1$ for $i = k+1, ..., m$;
2. m may be decomposed as a sum $m = k + m_1 + m_2 = k + m_1 + k \cdot R$ with m_1, m_2, R as in (2)-(5).

We are now going to show that, if k, α is fixed and if m is large enough, then the competitive ratio of *Assign* is no more than $(2+\alpha)$.More specifically, we are going to prove that, if a job list s is some input for *Assign*, then the makespan $F(Assign, \sigma)$ of the schedule which is computed by *Assign* does not exceed $(2+a) \cdot LB(\sigma)$, where $LB(\sigma)$ denotes the lower bound for $Opt(\sigma)$ which may be derived from the list s according to Lemma 1.

Lemma 3. *At every step j during the execution of the Assign algorithm there exists either a machine i in class Fast such that $L_{i,j} \leq (1+\alpha) \cdot LB_j$ or a machine i from class Normal such that $L_{i,j} \leq (1+\alpha/2) \cdot LB_j$.*

Proof. Let us suppose the converse, which means that, at some step j, we have, for any *Fast* machine i: $L_{i,j} > (1+\alpha) \cdot LB_j$, and for any *Normal* machine i : $L_{i,j} > (1+\alpha/2) \cdot LB_j$. It means that $p_1 + p_2 + \cdots + p_j = s \cdot \sum_{i \in Fast} L_{i,j} + \sum_{i \in \text{Normal} \cup \text{Reserved}} L_{i,j} > k \cdot s \cdot (1+\alpha) \cdot LB_j + m_1 \cdot (1+\alpha/2) \cdot LB_j$. But Lemma 1 tells us that $p_1 + p_2 + \cdots + p_j s \leq s \cdot k + m_1 + m_2) \cdot LB_j$, while relation 2 tells us that $k \cdot s \cdot (1+\alpha) + m_1 \cdot (1+\alpha/2) \geq (s \cdot k + m_1 + m_2)$. We deduce a contradiction and conclude. □

We deduce:

Lemma 4. *If current job J_j is a Small job then there is a machine from class Fast or Normal such that job J_j fits with it.*

Proof. Let us apply above Lemma 2 and consider a machine i as in the statement of Lemma 2. If i is Fast, then $L_{i,j} \leq (1+\alpha) \cdot LB_j$. We deduce from the fact that $p_j/s_i = p_j/s \leq LB_j$ that $L_{i,j} + p_j/s_i \leq (2+\alpha) \cdot LB_j$ and the result. If i is Normal, then $L_{i,j} \leq (1+\alpha/2) \cdot LB_j$, and $L_{i,j} + p_j/s_i = L_{i,j} + p_j \leq (2+\alpha) \cdot LB_j$. We conclude. □

Given some input job list σ: let us denote by $j(1), ..., j(Q)$ the steps when process *Assign* performs instructions (I2) or (I3) while running σ. Clearly, those instructions are performed according to some kind of cyclic scheme, and every index $q = 1, ..., Q$ may be written as $q = h + t \cdot k + T \cdot k \cdot R$, where $h \in \{0, ..., k-1\}$ and $t \in \{0, ..., R-1\}$, $T \geq 0$, with the following meaning: when performing (I2) or (I3), *Assign* deals with the job group which was originally group G_t, and, inside this group, deals with machine with label h.

For every $q = 1, ..., Q$, we denote by $i(q)$ the related target machine, which is, at this time, a *Reserved* machine located in current group G_0, with index h.

We may notice that:

– instruction (I3) occurs every time t is incremented: $t \to t+1$;

– original group G_0 takes again label 0 every time T is incremented: $T \to T+1$.

We claim:

Lemma 5. *For $q = 1, ..., Q$, we have $L_{i(q),j(q)} \leq (2 + \alpha - s) \cdot LB_{j(q)}$. (*)*

Proof. Let us consider $q = h + t \cdot k + T \cdot k \cdot R$, and try to prove above inequality (*). Obviously, (*) is true in case $T = 0$, since all machines from class *Reserved* are empty. So we may suppose $T \geq 1$. After assigning a *Large* current job $J_{j(q-k\cdot R)}$ to the machine $j(q-k\cdot R) = h$ in current group G_0, we switch machine $j(q-k\cdot R)$ with some *Normal* machine i_0 according to instruction (I2). Since we could not assign job $J_{j(q-k\cdot R)}$ neither to a *Fast* nor to a *Normal* machine, Lemma 3 tells us that there is a machine i in class *Normal* such that: $L_{i,j(q-k\cdot R)} \leq (1 + \alpha/2) \cdot LB_{j(q-k\cdot R)}$. So, this inequality also holds for the machine i_0 which becomes machine h in group G_0. We deduce that the load, after instruction (I2) has been performed, of machine h in group G_0 is bounded by $(1 + \alpha/2) \cdot LB_{j(q-k\cdot R)}$. This machine is going to keep with the same load until we arrive to step $q = h + t \cdot k + T \cdot k \cdot R$ and at this time this machine corresponds to machine $i(q)$. So we may state:

$$L_{i(q),j(q)} \leq (1 + \alpha/2) \cdot LB_{j(q-k\cdot R)}. \tag{6}$$

On the one hand, we see that, for any value $q \geq k+1$, we have been provided with *Large* (at the time when they arrived) $k+1$ jobs $J_{j(q)}, ..., J_{j(q-k)}$, all with processing times respectively larger than $(1 + \alpha/2) \cdot LB_{j(q)}, ..., (1 + \alpha/2) \cdot LB_{j(q-k)}$, which means, because of the monotonicity of LBj, all with processing times larger than $(1+\alpha/2)\cdot LB_{j(q-k)}$. It comes from the relation $LB_j \geq \min\{(q_k+q_{k+1})/s, q_{k+1}\}\}$ of Lemma 1, that $(1 + \alpha/2) \cdot LB_{j(q-k)} < LB_{j(q)}$. We may propagate this relation and get:

$$(1 + \alpha/2)^R \cdot LB_{j(q-R\cdot k)} \leq LB_{j(q)}. \tag{7}$$

Combining (6) and (7) yields: $L_{i(q),j(q)} \leq (1 + \alpha/2)^{1-R} \cdot LB_{j(q-R\cdot k)}$. We deduce (*) if $(1 + \alpha/2)^{1-R} \leq 2 + \alpha - s$, that means if $R \geq \log_{1+\alpha/2}((1 + \alpha/2)/(2 + \alpha - s))$. We conclude since this last inequality is part of our hypothesis (equation (5)). □

Theorem 1. *Let us suppose that σ is given and that our Online Uniform Machine Scheduling instance is such that m, k, s may be written according to the relations*

(2)-(5). Then, for any input job list s, the Assign algorithm works in such a way that: $F(Assign, \sigma) \leq (2+\alpha) \cdot Opt(\sigma)$. *That means that its competitive ratio does not exceed* $(2+\sigma)$ *in the case of such instance.*

Proof. Lemma 4 tells us that, if, at any step j, current job J_j is *Small*, then it is possible to assign it to some machine in *Normal* $\cup$ *Fast* in such a way that the resulting makespan does not exceed $(2+\alpha) \cdot LB_j$. By the same way, if J_j is *Large* and fits with some *Fast* machine, then it is possible, according to the mere definition of fitness, to assign it to this machine in such a way that the resulting makespan does not exceed $(2+\alpha) \cdot LB_j$. Finally, Lemma 2 and 5 tell us that if if J_j is *Large* and cannot be assigned to some *Fast* machine, then *Reserved* machine i with label n in group G_0 is such that $L_{i,j} \leq (2+\alpha-s) \cdot LB_j$. Since Algorithm *Assign* assigns job J_j to machine i, we see that the resulting load $L^*_{i,j}$ does not exceed $(2+\alpha-s) \cdot LB_j + p_j$. Since Lemma 1 tells us that $p_j \leq s \cdot LB_j$, we deduce that the makespan which results from assigning job J_j to machine i does not exceed $(2+\alpha) \cdot LB_j$. In any case, we see that we are able to bound, at the end of every iteration of *Assign*, the current makespan by $(2+\alpha) \cdot LB_j$. Since LB_j is a lower bound of the optimal makespan related to the *offline Uniform Machine Scheduling* problem induced by the job set $J(j) = \{J_1, ..., J_j\}$, we conclude. □

Theorem 2. *Given the speed s value,* $1 < s \leq 2$, *and the number k of machines with speed s. Then, for any value* $\alpha > 0$, *there exists* m_0 *such that if an Online Uniform Machine Scheduling instance, defined with k machines with speed s and* $m-k$ *machines with speed* 1, *is such that* $m \geq m_0$, *then the Assign algorithm may be applied to this instance in such a way that, for any input job list* σ: $F(Assign, \sigma) \leq (2+\alpha) \cdot Opt(\sigma)$.

Proof. It comes in a straightforward way from the fact that, if m is large enough, then it is possible to compute R, m_1, m_2 in such a way that relations (2)-(5) hold. □

Remark. It should be mentioned that it is possible to reverse the way we have been using inequalities (2)-(5) in order to get a lower bound for the worst-case performance of the *Assign* algorithm. First, we may notice that we may generate input job lists σ, such that (*) inequality is going to hold as an equality, which will means that the worst case performance of *Assign* is going to converge to $(2+\alpha) \cdot LB(\sigma)$ when the size of s is going to increase. On the other side, we may, while starting from m_2, k and s, derive α, R and m_1 according to (2.5), and with minimal values. Indeed, when m_2 (and R) is fixed, the smallest value of α which ensures (*), is the value α_1 such that $(1+\alpha_1/2)^{1-R} \leq (2+\alpha_1-s)$. We may consider an example, related to $s = 2, k = 1, m_2 = 7$. In such a case, we derive from (2)-(5): $R = m_2 = 7, m_1 = 31$ and $m = 39, \alpha \approx 0.41$. Therefore for any $m \geq 39$ the proposed algorithm provides W.C.P. of at least $2.41 \cdot LB(\sigma)$.

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References

[1] Cho Y., Sahni S. *Bounds for list schedules on uniform processors.* SIAM Journal on Computing, 1980, **9**, 91–103.

[2] Graham R. L. *Bounds on multiprocessing timing anomalies.* SIAM Journal on Applied Mathematics, 1969, **17**, 263–269.

[3] Li R., Shi L. *An on-line algorithm for some uniform processor scheduling.* SIAM Journal on Computing, 1998, **27**, 414–422.

[4] Cheng T. C. E., Ng C. T., Kotov Vladimir. *A new algorithm for online uniform-machine scheduling to minimize the makespan.* Information Processing Letters, 2006, **99**, 102–105.

[5] Liu Ming, Xu Yinfeng, Chu Chengbin, Zheng Feifeng. *Online scheduling on two uniform machines to minimize the makespan.* Theoretical Computer Science, 2009, **410 (21–23)**, 2099–2109.

[6] Angelelli E., Speranza Maria Grazia, Tuza Z. *Semi-online scheduling on two uniform processors.* Theoretical Computer Science, 2008, **393**, 211–219.

[7] Wen J., Du D. *Preemptive on-line scheduling for two uniform processors.* Operations Research Letters, 1998, **23**, 113–116.

[8] Kellerer H., Kotov V. *An efficient algorithm for bin stretching.* Operations Research Letters, 2013, **41(4)**, 343–346.

[9] Kellerer H., Kotov V., Gabay M. *An efficient algorithm for semi-online multiprocessor scheduling with given total processing time.* Journal of Scheduling 2015, **18**, 623–630.

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On the absence of finite approximation relative to model completeness in propositional provability logic

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Abstract. In the present paper we consider the expressibility of formulas in the provability logic GL and related to it questions of the model completeness of system of formulas. We prove the absence of a finite approximation relative to model completeness in GL.

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Keywords and phrases: Expressibility, model completeness, provabilty logic, diagonalizable algebra.

1 Introduction

Artificial Intelligence (AI) systems simulating human behavior are often called intelligent agents. These intelligent agents exhibit somehow human-like intelligence. Intelligent agents typically represent human cognitive states using underlying beliefs and knowledge modeled in a knowledge representation language, specifically in the context of decision making [1]. In the present paper we investigate some functional properties of the underlying knowledge representation language of intelligent agents which are based on the provability logic GL [2].

The notion of model completeness of systems of formulas was proposed in [6, 7]. In the present paper we prove the propositional provability logic of Gödel-Löb (GL) is not finitely approximable relative to model completeness.

2 Definitions and notations

Provability logic. We consider the propositional provability logic GL whose formulas are based on propositional variables $p, q, r, \ldots$ and logical connectives $\&, \vee, \supset, \neg, \Delta$, its axioms are the classical ones together with the following Δ-formulas:

$$\Delta(p \supset q) \supset (\Delta p \supset \Delta q), \quad \Delta(\Delta p \supset p) \supset \Delta p, \quad \Delta p \supset \Delta\Delta p,$$

and the rules of inference are the rules of: 1) substitution; 2) the modus ponens, and 3) the necessity, which allows to get formula ΔA if we already get formula A. The normal extensions of the propositional provability logic GL are defined as usual [2].

Diagonalizable algebras. A diagonalizable algebra [4] is a universal algebra of the form $\mathfrak{A} =< M; \&, \vee, \supset, \neg, \Delta >$, where $< M; \&, \vee, \supset, \neg >$ is a boolean algebra, and the unary operation Δ satisfies the relations

$$\Delta(\Delta x \supset x) = \Delta x,\ \Delta(x \& y) = (\Delta x \& \Delta y),\ \Delta 1_{\mathfrak{A}} = 1_{\mathfrak{A}},$$

where $1_{\mathfrak{A}}$ is the unit of $\mathfrak{A}$, which is denoted also by 1 in case the confusion is avoided.

Diagonalizable algebras are known to be algebraic models for provability logic and its extensions [5]. Obviously we can interpret any formula of the calculus of GL on any diagonalizable algebra $\mathfrak{A}$. As usual a formula F is said to be valid on $\mathfrak{A}$ if for any evaluation of variables of F with elements of $\mathfrak{A}$ the value of the formula on $\mathfrak{A}$ is $1_{\mathfrak{A}}$. The set of all valid formulas on $\mathfrak{A}$, denoted by $L\mathfrak{A}$ and referred to as the logic of the algebra $\mathfrak{A}$, forms an extension $L\mathfrak{A}$ of the provability logic GL [5].

An extension L of GL is called tabular if there is a finite diagonalizable algebra $\mathfrak{A}$ such that $L = L\mathfrak{A}$.

Expressibility and model completeness. The formula $F(p_1, \ldots, p_n)$ is a model for the Boolean function $f(x_1, \ldots, x_n)$ if for any ordered set $(\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \{0, 1\}, i = 1, \ldots, n$, we have $F(\alpha_1, \ldots, \alpha_n) = f(\alpha_1, \ldots, \alpha_n)$, where logical connectors from F are interpreted in a natural way on the two-valued Boolean algebra [6, 7].

They say the formula F is expressible in the logic L via a system of formulas Σ if F can be obtained from variables and Σ applying finitely many times 2 kinds of rules: a) the rule of weak substitution, b) the rule of passing to equivalent formula in L [3].

The system of formulas Σ is called model complete in the logic L if at least a model for every Boolean function is expressible via Σ in the logic L. System Σ is model pre-complete in L if Σ is not model complete in L, but for any formula F which is not expressible in L via Σ the system $\Sigma \cup \{F\}$ is already model complete in L [8].

The logic L is finitely approximable with respect to model completeness if for any system of formulas Σ which is not model complete in L there is a tabular extension of L in which Σ is also model incomplete.

3 Preliminary results

First let mention an obvious fact:

Proposition. *If a system of formulas Σ is complete with respect to expressibility of formulas in the logic GL then it is also model complete in GL.*

Let us consider the following system of formulas:

$$\{p \& \neg q, \Delta p\}. \tag{1}$$

Lemma 1. *The system of formulas* (1) *is model complete in any tabular extension of the propositional provability logic GL.*

Proof. Note that for any finite diagonalizable algebra $\mathfrak{A}$ there exists a positive integer k such that the following equivalence is valid in the logic $L\mathfrak{A}$

$$\Delta^k(p\&\neg p) \sim (p \supset p),$$

which shows the tautology $p \supset p$ is expressible in the logic $L\mathfrak{A}$ via system of formulas (1). It remains to observe the system

$$\{(p \supset p)\} \cup \{\Delta p, p\&\neg q\}$$

is complete in the logic $L\mathfrak{A}$, so by Proposition it is also model complete in $L\mathfrak{A}$. □

Let $\mathfrak{M}^*$ the diagonalizable algebra of sequences of the form $\alpha = (\mu_1, \mu_2, \ldots)$, where $\mu_i \in \{0, 1\}$ $(i = 1, 2, \ldots)$ and the operations $\&, \vee, \supset, \neg$ made term by term as Boolean functions on the set of $\{0, 1\}$, and $\Delta\alpha$ is a sequence $(\nu_1, \nu_2, \ldots)$, where $\nu_i = (\mu_1\& \cdots \&\mu_i)$ $(i = 1, 2, \ldots)$. The logic $L\mathfrak{M}^*$ coincides with the extension of provability logic generated by the formula

$$\Delta(\Box p \supset q) \vee \Delta(\Box q \supset p),$$

where $\Box p$ means $p\&\Delta p$.

Lemma 2. *Let L be any intermediate logic between GL and $L\mathfrak{M}^*$. The system of formulas* (1) *is not model complete in the propositional provability logic L.*

Proof. Realy, the system of formulas (1) is not model complete in $L\mathfrak{M}^*$ since formulas of the system (1) conserves the relation $x \neq 1$ on the algebra $\mathfrak{M}^*$, and the formula $(p \supset p)$ does not. □

4 Main result

Now we are able to formulate the main result of the present work.

Theorem. *Let L be any intermediate logic between GL and $L\mathfrak{M}^*$. The propositional provability logic L is not finitely approximable with respect to model completeness.*

Proof. The proof results from the above Lemmas 1 and 2. □

Taking into account our previous result [9] together with these new findings we can conclude that traditional algorithm for determining model completeness of systems of formulas in GL is impossible to find out.

References

[1] Subrata Das. *Foundations of Decision-Making Agents: Logic, Probability and Modality.* World Scientific Publishing Company, 2008.

[2] Boolos G. S. *The Logic of Provability.* Cambridge University Press, 1995.

[3] Kuznetsov A. V. *On detecting non-deductibility and non-expressibility.* Logical Deduction, Nauka, Moscow, 1979, 5–33 (in Russian).

[4] Magari R. *The diagonalizable algebras.* Boll. Unione Matem. Italiana 1975 **12**, 117–125.

[5] Solovay R. M. *Provability interpretations of modal logic.* Israel J. Math., 1975, **25**, 287–304.

[6] Tolstova Yu. N. *Modelling l-valued logic in k-valued logic* ($k > 1$). Problemy Kibernet., 1967, **18** 67–82 (in Russian); English translation: *Systems Theory Res.*, 1968, **18**, 62–76.

[7] Covalgiu O. *Modeling the classical logic in the 3-valued extension of the provability-intuitionistic logic.* Bul. Acad. Ştiinţe Repub. Moldova, Mat., 1990, No. 2(2), 9–15 (in Russian).

[8] Covalgiu O., Ratsa M. *On model completeness for the classes of the functions 3-valued Δ-pseudo-Boolean algebra.* An. Şt. Univ. Ovidius, ser. Matematica, 1994, **2**, 76–79.

[9] Izbaş O., Rusu A. *On non-tabular m-pre-complete classes of formulas in the propositional provability logic.* An. Ştiinţ. Univ. "Ovidius" Constanţa, Ser. Mat., 2006, **14(1)**, 91–98.

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Alexei Caşu (Kashu)

On his 75th Anniversary

On August 20, 2015, an outstanding algebraist, one of the oldest members of the Editorial Board of our journal, Professor Alexei Caşu turns 75. The influence of Caşu's results on the development of algebra in Moldova is significant.

Alexei Caşu was born on 20 August, 1940, in Sadaclia village, Basarabeasca district, Republic of Moldova. Starting his studies in 1947, he graduated from the secondary school in his native village in 1957. During the years 1958–1963 he studied at the Faculty of Physics and Mathematics of the Moldova State University (Chişinău). After graduating from the university he served for two years in the army (1963–1965). Alexei Caşu joined the staff of the Institute of Physics and Mathematics of the Academy of Science of Moldova in 1965 and continues his association with this institution till the present.

His first appointment at the institute was as a laboratory assistant (1965–1967), then promoted to a Junior Researcher (1967–1971), Senior Researcher (1971–1991), and Leading Researcher (1991–1992). He served as Deputy Director of the Institute of Mathematics and Computer Science from 1992 to 1998. Since 1998 Alexei Caşu has been a Principal Researcher.

His main research field has been algebra, specializing in the theory of rings, modules and categories. The problems he has studied are mainly related to radicals

and torsions in module categories. He obtained in this domain a large number of results, among which the following should be mentioned.

A relationship was established between (pre) radicals and classes of R-modules, on the one hand, and special sets of left ideals of the ring R, on the other hand, thereby generalizing the classical results of well-known algebraists such as P. Gabriel, S. E. Dickson, J.-M. Maranda, etc. It is remarkable that much later the methods of Alexei Caşu were used in order to describe important new classes (natural, prenatural, etc.) of modules, introduced and investigated by J. Dauns.

Another cycle of works is devoted to the study of radicals and torsions in some special constructions, such as Morita contexts and adjoint situations. Here the main role is played by methods of category theory: the behavior of torsions and localizations on the transition from one category to another is analyzed using principal functors of a category of modules. Thus he continued and essentially improved some results, ideas and methods of K. Morita and J. Lambek that lead to the remarkable equivalences of some special subcategories of categories of modules.

A problem was solved concerning the (anti-) isomorphism of lattices of submodules on action of principal functors, showing necessary and sufficient conditions to obtain such a situation. Some quotient rings were described which were constructed by torsions using bicommutators of modules that determine these torsions. In the case of Morita contexts an elegant relationship was obtained which refers to lattices of torsions, namely it was proved that special sublattices of these lattices are isomorphic.

The latest results refer to (pre) radicals accompanying principal functors of a category of modules: properties, relationships, criteria of coincidence. The technique of (pre) radicals in R-Mod allows one to define four new operations in a lattice of characteristic submodules, that opens up new research perspectives in this area.

Based on the obtained results Alexei Caşu defended his Ph. D thesis in 1969 at the Moldova State University (Chişinău), and then Doctor of Science (habilitation thesis) in 1991 at the State University of St. Petersburg. He was awarded the title of Senior Researcher in 1978, Full Professor in 2000. In 1986 he attended LOMI (Leningrad Department of Steklov Institute of Mathematics of the USSR Academy of Sciences) for one year in St. Petersburg (former Leningrad). He had been an invited professor for cycles of lectures for 1–2 months: Tashkent, 1978; Iaşi, 1991; Lviv, 1987 and 2005.

Professor Alexei Caşu is the author of more than 130 publications, including articles, preprints, synthesis papers, surveys and four books (2 monographs and 2 textbooks for universities). Some of his published results were included in several basic monographs written by J. Golan, B. Stenstrom, L. Bican, T. Kepka, P. Nemec, L. A. Skornyakov. Numerous references to his results can be found in works published by scientists from Austria, Hungary, Ukraine, Germany, Romania, Czech Republic, Sweden, Russia, etc.

Professor Alexei Caşu is undeniably one of the leading specialists in the field of rings, modules and categories. He took part with invited lectures and communications in international conferences in various countries: Austria, Germany, Hungary,

Poland, Bulgaria, Romania, Russian Federation, Ukraine. As the organizer, moderator and person with responsibility for programs and sections he participated in all conferences organized by the Mathematical Society of the Republic of Moldova and numerous international specialized conferences in Ukraine, Romania and Moldova, among which are the international conference on radicals (Chişinău, 2003), national algebra conferences in Romania and international conferences on algebra in Ukraine. Also as organizer he took part in some national conferences dedicated to the memory of V. Andrunachievici and C. Sibirschi (Chişinău, 2007). He gave plenary lectures during some major international algebraic conferences: St. Petersburg (1997, in memory of D. K. Faddeev), Moscow (1998, in memory of A. G. Kurosh). He was an invitated speaker at the workshop held in Warsaw, Stefan Banach International Mathematical Center, 2009.

Alexei Caşu contributed much through his scientific work, the organization and development of research in mathematics, and in the preparation of highly qualified staff. Over more than 47 years Alexei Caşu taught general, special and optional courses for students in their forth and fifth years or master students at the Faculty of Mathematics and Computer Science of the Moldova State University. Since 2009 he has collaborated with the University of Academy of Sciences of Moldova. Two handbooks for students, master and PhD students were published.

Professor Caşu was a scientific advisor of two PhD students, who defended their theses (and one of them later defended a habilitation thesis).

Professor Alexei Caşu held several positions related to research work: vice-president of the Mathematical Society of Moldova (2000–2009); Chief of scientific seminar "Algebra and mathematical logic" (2000–2008); member of the NCAA (National Council for Accreditation and Attestation) experts in mathematics; member of the Executive Board of the journal "Bul. Acad. Ştiinţe Repub. Mold. Mat."; member of the Editorial Board of the journal "Algebra and Discrete Mathematics" (Ukraine); reviewer of zbMATH, Berlin (1980–2015); member of the Editorial Board (and also one of the authors) of the book "Academician Vladimir Andrunachievici" (ASM, Chişinău, 2009).

Scientific results obtained by Alexei Caşu are highly appreciated by experts from our country, as well as abroad, he was official reviewer of over 30 PhD theses and 3 habilitation theses. For valuable results in research and the training of highly qualified scientific staff Professor Alexei Caşu was awarded the Prize "Academician Constantin Sibirschi" (2006), the medal "Dimitrie Cantemir" and the Order "Labour Glory" (2010).

Professor Alexei Caşu is totally devoted to mathematics and the propagation of its ideas, beauty and importance, which have brought him the deep respect and gratitude of the mathematical scientific community.