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The q.Zariski topology on the quasi-primary spectrum of a ring

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Abstract. Let R be a commutative ring with identity. We topologize q.Spec(R), the quasi-primary spectrum of R, in a way similar to that of defining the Zariski topology on the prime spectrum of R, and investigate the properties of this topological space. Rings whose q.Zariski topology is respectively T_0, T_1, irreducible or Noetherian are studied, and several characterizations of such rings are given.

Mathematics subject classification: 13A99, 54B99.

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1 Introduction

Let R denote a commutative ring with identity. The Zariski topology on the prime spectrum Spec(R), the set of prime ideals of R, play an important role in the fields of commutative algebra, algebraic geometry and lattice theory. For each ideal I of R, the set $V(I) = \{p \in \mathrm{Spec}(R) \mid p \supseteq I\}$ satisfies the axioms for the closed sets of the Zariski topology on Spec(R) (see for example, Atiyah and Macdonald [1]). In the literature, there are many different topologies of commutative or noncommutative rings ([2, 5, 6]).

About a quarter of a century later, in [3] the notion of quasi-primary ideals as a generalization of the notion of primary ideals was introduced. A proper ideal q of R is called quasi-primary if $rs \in q$, for $r, s \in R$, implies that either $r \in \sqrt{q}$ or $s \in \sqrt{q}$. Equivalently, q is quasi-primary if and only if $\sqrt{q}$ is prime [3, Definition 2, p. 176]. In this case, q is said to be p-quasi-primary where $p = \sqrt{q}$. In the sequel, we introduce and study a topology on quasi-primary spectrum q.Spec(R), the set of all quasi-primary ideals of R, such that the Zariski topology is a subspace of this topology. We investigate the interplay between the properties of this space and the algebraic properties of the ring under consideration. In particular, assuming suitable conditions for each result, we investigated when this space is T_0 (Theorem 4(4)), T_1 (Theorem 4(5)), Noetherian (Theorem 5) or irreducible (Theorem 6 and Corollary 1).

2 Main Results

Throughout, R is a commutative ring with $1_R \neq 0_R$. We denote the set of all quasi-primary ideals of R by q.Spec(R) and define the variety of an ideal I of R as follows:

$$V^{\mathbf{q}}(I) = \{q \in \text{q.Spec}(R) \mid \sqrt{q} \supseteq I\}.$$

The following lemma shows that the set $\mathcal{T}(R) = \{V^{\mathbf{q}}(I) \mid I \text{ is an ideal of } R\}$ satisfies the axioms for closed sets in a topological space on q.Spec(R), called q.Zariski topology.

The proof of the next result is easy and so it is omitted.

Lemma 1. *For any ideals I, J and I_λ $(\lambda \in \Lambda)$ of a ring R, the following hold.*

(1) $V^{\mathbf{q}}(R) = \emptyset$ *and* $V^{\mathbf{q}}(0) = \text{q.Spec}(R)$.

(2) $\bigcap\limits_{\lambda\in\Lambda} V^{\mathbf{q}}(I_\lambda) = V^{\mathbf{q}}(\sum\limits_{\lambda\in\Lambda} I_\lambda)$.

(3) $V^{\mathbf{q}}(I) \cup V^{\mathbf{q}}(J) = V^{\mathbf{q}}(I \cap J)$.

Let Y be a subset of q.Spec(R) for a ring R. We will denote the intersection of all elements in Y by $\xi(Y)$ and the closure of Y in q.Spec(R) with respect to the q.Zariski topology by $cl(Y)$. Also the set of all p-quasi-primary ideals of a ring R is denoted by q.Spec$_p(R)$.

Next we offer some descriptions for the two proper ideals I and J of R that will be useful in the sequel.

Lemma 2. *Let I and J be proper ideals of a ring R. Then the following hold.*

(1) $V^{\mathbf{q}}(I) = V^{\mathbf{q}}(\sqrt{I})$.

(2) $V^{\mathbf{q}}(I) \subseteq V^{\mathbf{q}}(J)$ *if and only if* $\sqrt{J} \subseteq \sqrt{I}$, *and if* $J \subseteq I$, *then* $V^{\mathbf{q}}(I) \subseteq V^{\mathbf{q}}(J)$.

(3) $V^{\mathbf{q}}(I) = \bigcup\limits_{I\subseteq p\in Spec(R)} \text{q.Spec}_p(R)$.

(4) Let Y be a subset of q.Spec(R). *Then* $Y \subseteq V^{\mathbf{q}}(I)$ *if and only if* $I \subseteq \sqrt{\xi(Y)}$.

Consider the surjective map $\phi : \text{q.Spec}(R) \to \text{Spec}(R)$ given by $\phi(q) = \sqrt{q}$ for every $q \in \text{q.Spec}(R)$. In the following result we ghather some properties of this map.

Proposition 1. *Let R be a ring.*

(1) The map ϕ is continuous with respect to the q.Zariski topology; more precisely, $\phi^{-1}(V(I)) = V^{\mathbf{q}}(I)$ *for every ideal I of R.*

(2) $\phi(V^{\mathbf{q}}(I)) = V(I)$ *and* $\phi(\text{q.Spec}(R) - V^{\mathbf{q}}(I)) = \text{Spec}(R) - V(I)$ *i.e. ϕ is both closed and open.*

(3) ϕ is injective if and only if it is a homeomorphism.

Proof. (1). Let I be an ideal of R. Then

$$\begin{aligned} q \in \phi^{-1}(V(I)) &\Leftrightarrow \phi(q) \in V(I) \\ &\Leftrightarrow \sqrt{q} \supseteq I \\ &\Leftrightarrow q \in V^{\mathbf{q}}(I). \end{aligned}$$

(2). As we have seen in (1), $\phi(V^{\mathbf{q}}(I)) = \phi(\phi^{-1}(V(I))) = \phi \circ \phi^{-1}(V(I)) = V(I)$ as ϕ is surjective. Similarly,

$$\begin{aligned} \phi(\text{q.Spec}(R) - V^{\mathbf{q}}(I)) &= \phi(\phi^{-1}(\text{Spec}(R)) - \phi^{-1}(V(I))) \\ &= \phi(\phi^{-1}(\text{Spec}(R) - V(I))) \\ &= \phi \circ \phi^{-1}(\text{Spec}(R) - V(I)) \\ &= \text{Spec}(R) - V(I). \end{aligned}$$

(3). This follows from (2). $\square$

Theorem 1. *For any ring R, the following are equivalent:*

(1) q.Spec(R) *is connected;*

(2) Spec(R) *is connected;*

(3) The ring R contains no idempotent other than 0 *and* 1.

Proof. (1) $\Rightarrow$ (2). Suppose q.Spec(R) is a connected space. By Proposition 1, the map ϕ is surjective and continuous and so Spec(R) is also a connected space.

(2) $\Rightarrow$ (1). Suppose, on the contrary, that q.Spec(R) is disconnected. There exists a non-empty proper subset W of q.Spec(R) that is both closed and open. By Proposition 1, $\phi(W)$ is a non-empty subset of Spec(R) that is also clopen. To complete the proof, it suffices to show that $\phi(W)$ is a proper subset of Spec(R), and so Spec(R) is disconnected, a contradiction. Since W is an open set, we have $W = \text{q.Spec}(R) - V^{\mathbf{q}}(I)$ for some ideal I of R and hence Proposition 1 shows that $\phi(W) = \text{Spec}(R) - V(I)$. It follows that $\phi(W)$ is a proper subset of Spec(R). Otherwise, if $\phi(W) = \text{Spec}(R)$, then $V(I) = \emptyset$, and so $I = R$. We conclude from this fact that $W = \text{q.Spec}(R)$ which is impossible.
(2) $\Leftrightarrow$ (3) is a well-known fact, for example [1, Exercise 22, p.14]. $\square$

For any ideal I of R, we define $\Lambda_R(I) = \text{q.Spec}(R) - V^{\mathbf{q}}(I)$ as an open set of q.Spec(R). Also for any $a \in R$, we mean $\Lambda_R(a)$ by $\Lambda_R(Ra)$. Clearly, $\Lambda_R(0) = \emptyset$ and $\Lambda_R(1) = \text{q.Spec}(R)$. Following result shows that the set $B = \{\Lambda_R(a) \mid a \in R\}$ is a base for the q.Zariski topology on q.Spec(R).

Theorem 2. *Let R be a ring and $B = \{\Lambda_R(a) \mid a \in R\}$. Then the set B forms a base for the q.Zariski topology on* q.Spec(R).

Proof. We may assume that $\text{q.Spec}(R) \neq \emptyset$. Let O be an open subset in $\text{q.Spec}(R)$. Thus $O = \text{q.Spec}(R) - V^{\mathbf{q}}(I)$ for some ideal I of R. Therefore

$$\begin{aligned} O &= \text{q.Spec}(R) - V^{\mathbf{q}}(I) = \text{q.Spec}(R) - V^{\mathbf{q}}(\sum_{a\in I} Ra) \\ &= \text{q.Spec}(R) - \bigcap_{a\in I} V^{\mathbf{q}}(Ra) \\ &= \bigcup_{a\in I} \Lambda_R(a). \end{aligned}$$

It follows that the set B forms a base for the q.Zariski topology on $\text{q.Spec}(R)$. $\square$

Theorem 3. *Let R be a ring and $a, b \in R$.*

(1) $\Lambda_R(a) = \emptyset$ if and only if a is a nilpotent element of R.

(2) $\Lambda_R(a) = \text{q.Spec}(R)$ if and only if a is a unit element of R.

(3) For each pair of ideals I and J of R, $\Lambda_R(I) = \Lambda_R(J)$ if and only if $\sqrt{I} = \sqrt{J}$ if and only if $V^{\mathbf{q}}(I) = V^{\mathbf{q}}(J)$.

(4) $\Lambda_R(ab) = \Lambda_R(a) \cap \Lambda_R(b)$.

(5) $\text{q.Spec}(R)$ is quasi-compact.

(6) For any $c \in R$, $\Lambda_R(c)$ is qusi-compact, that is, every open covering of $\Lambda_R(c)$ has a finite subcovering.

(7) An open subset of $\text{q.Spec}(R)$ is quasi-compact if and only if it is a finite union of sets $\Lambda_R(a)$.

Proof. (1). Let $a \in R$. Then

$$\begin{aligned} \emptyset &= \Lambda_R(a) = \text{q.Spec}(R) - V^{\mathbf{q}}(Ra) \\ &\Leftrightarrow V^{\mathbf{q}}(Ra) = \text{q.Spec}(R) \\ &\Leftrightarrow \sqrt{q} \supseteq Ra \ for\ every\ q \in \text{q.Spec}(R) \\ &\Leftrightarrow a\ is\ in\ every\ prime\ ideal\ of\ R \\ &\Leftrightarrow a\ is\ a\ nilpotent\ element\ of\ R. \end{aligned}$$

(2).

$$\begin{aligned} \Lambda_R(a) &= \text{q.Spec}(R) \\ &\Leftrightarrow a \notin \sqrt{q}\ for\ all\ q \in \text{q.Spec}(R) \\ &\Rightarrow a \notin q\ for\ all\ q \in Max(R) \\ &\Rightarrow a\ is\ unit. \end{aligned}$$

Conversely, it is clear that a unit element a of R is not contained in any quasi-primary ideal of R. That is, $\Lambda_R(a) = \text{q.Spec}(R)$.

(3) is clear by Lemma 2(2).
(4). Let $q \in V^{\mathbf{q}}(Rab)$. Then

$$\begin{aligned} \sqrt{q} \supseteq \sqrt{Rab} &= \sqrt{Ra} \cap \sqrt{Rb} \\ &\Leftrightarrow \sqrt{q} \supseteq \sqrt{Ra} \text{ or } \sqrt{q} \supseteq \sqrt{Rb} \\ &\Leftrightarrow q \in V^{\mathbf{q}}(Ra) \text{ or } q \in V^{\mathbf{q}}(Rb) \\ &\Leftrightarrow q \in V^{\mathbf{q}}(Ra) \cup V^{\mathbf{q}}(Rb). \end{aligned}$$

It follows that $V^{\mathbf{q}}(Rab) = V^{\mathbf{q}}(Ra) \cup V^{\mathbf{q}}(Rb)$, as required.
(5). Let $\text{q.Spec}(R) = \underset{i\in I}{\cup} \Lambda_R(J_i)$, where $\{J_i\}_{i\in I}$ is a family of ideals of R. We clearly have $\Lambda_R(R) = \text{q.Spec}(R) = \Lambda_R(\sum_{i\in I} J_i)$. Thus, by the part (3), $R = \sqrt{\sum_{i\in I} J_i}$ and hence, $1 \in \sum_{i\in I} J_i$. So there exist $i_1, i_2, \cdots, i_n \in I$ such that $1 \in \sum_{k=1}^{n} J_{i_k}$, that is $R = \sum_{k=1}^{n} J_{i_k}$. Consequently, $\text{q.Spec}(R) = \Lambda_R(R) = \Lambda_R(\sum_{k=1}^{n} J_{i_k}) = \overset{n}{\underset{k=1}{\cup}} \Lambda_R(J_{i_k})$.
(6). Let $c \in R$. For any open covering of $\Lambda_R(c)$, there is a family $\{a_i \mid a_i \in R, i \in I\}$ of elements of R such that $\Lambda_R(c) \subseteq \underset{i\in I}{\cup} \Lambda_R(a_i)$, since $B = \{\Lambda_R(a_i) \mid a_i \in R, i \in I\}$ forms a base for the q.Zariski topology on $\text{q.Spec}(R)$, by Theorem 2.
It is clear that the map $\phi : \text{q.Spec}(R) \to \text{Spec}(R)$ given by $\phi(q) = \sqrt{q}$ is surjective, and so there exists a finite subset I' of I such that $\Lambda_R(c) \subseteq \underset{i\in I'}{\cup} \Lambda_R(a_i)$, because $\phi(\Lambda_R(a)) = \text{Spec}(R) - V(a)$ is quasi-compact by [1, Exercise 1.17 p. 12]
(7). The sufficiency follows by exactly the same argument as (6). For the necessity, if an open subspace Y of $\text{q.Spec}(R)$ is a union of a finite number of sets $\Lambda_R(Ra)$, then any open cover $\{\Lambda_R(Ra_i)\}_{i\in I}$ of Y induces an open cover for each of the $\Lambda_R(Ra)$. By (6), each of those will have a finite subcover and these subcovers yield a finite subcover of $\text{q.Spec}(R)$.

□

A topological space $(X;\tau)$ is said to be a T_0-space if for each pair of distinct points a, b in X, either there exists an open set containing a and not b, or there exists an open set containing b and not a. It has been shown that a topological space is T_0 if and only if the closures of distinct points are distinct. Also, a topological space $(X;\tau)$ is called a T_1-space if every singleton set $\{x\}$ is closed in $(X;\tau)$. Clearly every T_1-space is a T_0-space.

Theorem 4. *Let R be a ring, $Y \subseteq \text{q.Spec}(R)$ and let $q \in \text{q.Spec}_p(R)$. Then*

(1) $V^{\mathbf{q}}(\xi(Y)) = cl(Y)$. In particular, $cl(\{q\}) = V^{\mathbf{q}}(q)$.

(2) If $(0) \in Y$, then Y is dense in $\text{q.Spec}(R)$.

(3) The set $\{q\}$ is closed in $\text{q.Spec}(R)$ if and only if

(i) p is a maximal element in $\{\sqrt{q'} \mid q' \in \text{q.Spec}(R)\}$, and

(ii) $\text{q.Spec}_p(R) = \{q\}$.

(4) The following statements are equivalent:

(i) $\text{q.Spec}(R)$ *is a* T_0*-space;*

(ii) the map $\phi : \text{q.Spec}(R) \to \text{Spec}(R)$*, given by* $\phi(q) = \sqrt{q}$*, is injective;*

(iii) $\text{q.Spec}(R) = \text{Spec}(R)$.

(5) $\text{q.Spec}(R)$ *is a* T_1*-space if and only if* $\text{q.Spec}(R)$ *is a* T_0*-space and* $\text{q.Spec}(R) = \text{Spec}(R) = Max(R)$ *(where* $Max(R)$ *is the set of all maximal ideals of* R*).*

(6) Let $(0) \in \text{q.Spec}(R)$*. Then* $\text{q.Spec}(R)$ *is a* T_1*-space if and only if* (0) *is the only quasi-primary ideal of* R*.*

(7) Let R *be a domain. If* $\text{q.Spec}(R)$ *is a* T_1*-space, then* R *is a field.*

Proof. (1). Let $q \in Y$. Then $\xi(Y) \subseteq q \subseteq \sqrt{q}$. Therefore $q \in V^{\mathfrak{q}}(\xi(Y))$ and so $Y \subseteq V^{\mathfrak{q}}(\xi(Y))$. Next, let $V^{\mathfrak{q}}(I)$ be any closed subset of $\text{q.Spec}(R)$ containing Y. Then $\sqrt{q} \supseteq I$ for every $q \in Y$ and hence $\sqrt{\xi(Y)} \supseteq I$.
It follows that $\sqrt{q'} \supseteq \sqrt{\xi(Y)} \supseteq I$ for every $q' \in V^{\mathfrak{q}}(\xi(Y))$ and so $V^{\mathfrak{q}}(\xi(Y)) \subseteq V^{\mathfrak{q}}(I)$. Thus $V^{\mathfrak{q}}(\xi(Y))$ is the smallest closed subset of $\text{q.Spec}(R)$ containing Y , hence $V^{\mathfrak{q}}(\xi(Y)) = cl(Y)$.
(2) is trivial by (1).
(3). Suppose that $\{q\}$ is closed. Then, by (1), $\{q\} = V^{\mathfrak{q}}(q)$. Assue that $q' \in \text{q.Spec}(R)$ such that $\sqrt{q'} \supseteq p$. Hence, $q' \in V^{\mathfrak{q}}(q) = \{q\}$, and so $\text{q.Spec}_p(R) = \{q\}$. Conversely, assume that (i) and (ii) hold. Let $q' \in cl(\{q\})$. Then $\sqrt{q'} \supseteq q$ by (1). It follows from (i) that $\sqrt{q'} = \sqrt{q} = p$ and hence $q' = q$ by (ii). This yields $cl(\{q\}) = \{q\}$.
(4). (i)⇒(ii) Suppose $q, q' \in \text{q.Spec}(R)$ such that $\sqrt{q} = \sqrt{q'}$ and $q \neq q'$. Since $\text{q.Spec}(R)$ is a T_0-space, there is an element $a \in R$ such that $q \in \Lambda_R(a)$ and $q' \notin \Lambda_R(a)$. Thus $\sqrt{q} \not\supseteq Ra$ and $\sqrt{q'} \not\supseteq Ra$, a contradiction. Thus the map ϕ is injective.
(ii)⇒(iii) is clearly true and (iii)⇒(i) will be obtained by [1, Exercise 18(iv) p. 13].
(5) is easy to check from the definition and the parts $(3), (4)$.
(6). Let $\text{q.Spec}(R)$ be a T_1-space. By the part (5), the ideal (0) is maximal and hence (0) is the only quasi-primary ideal of R. The converse follows from the definition and the part (3).
(7) follows from the part (6). □

A topological space X is said to be Noetherian if the open subsets of X satisfy the ascending chain condition. Since closed subsets are complements of open subsets, it comes to the same thing to say that the closed subsets of X satisfy the descending chain condition. Also a nonempty subset C of a topological space X is said to be irreducible if C can not be written as the union of two distinct closed sets.

Theorem 5. *Let R be a ring.*

(1) *If R is a Noetherian ring, then* q.Spec(R) *is a Noetherian topological space.*

(2) *$V^{\mathfrak{q}}(q)$ is an irreducible closed subset of* q.Spec(R) *for every quasi-primary ideal q of R.*

(3) *If I is an ideal of R such that $V^{\mathfrak{q}}(I)$ is an irreducible closed set, then there exists an irreducible ideal J of R such that $V^{\mathfrak{q}}(I) = V^{\mathfrak{q}}(J)$.*

(4) *If I is an ideal of R and* q.Spec(R) *is a Noetherian topological space, then $V^{\mathfrak{q}}(I) = \bigcup\limits_{t=1}^{k} V^{\mathfrak{q}}(I_t)$ where $V^{\mathfrak{q}}(I_t)$ are irreducible closed sets and I_k are irreducible ideals of R.*

(5) *If I is an ideal of a Noetherian ring R, then $V^{\mathfrak{q}}(I)$ can be written as a finite union of irreducible closed sets $V^{\mathfrak{q}}(I_t)$, $1 \le t \le k$ such that for each t, I_t is an irreducible ideal of R.*

Proof. (1). Let $V^{\mathfrak{q}}(I_1) \supseteq V^{\mathfrak{q}}(I_2) \supseteq V^{\mathfrak{q}}(I_3) \supseteq \cdots$ be a chain of closed sets of q.Spec(R), where $\{I_t\}_{t=1}^{\infty}$ is a family of ideals of R. We conclude from Lemma 2(2) that $\sqrt{I_1} \subseteq \sqrt{I_2} \subseteq \sqrt{I_3} \subseteq \cdots$, and since R is a Noetherian ring, there exists a positive integer n such that for each positive integer $m \ge n$, $\sqrt{I_n} = \sqrt{I_m}$. Consequently, again by using Lemma 2(1), we have $V^{\mathfrak{q}}(I_n) = V^{\mathfrak{q}}(\sqrt{I_n}) = V^{\mathfrak{q}}(\sqrt{I_m}) = V^{\mathfrak{q}}(I_m)$, which completes the proof.
(2). It is clear that a singleton subset and its closure in q.Spec(R) are both irreducible. Now, the proof will be obtained by Theorem 4.
(3). Let $A = \{L \mid L$ is an ideal of R such that $V^{\mathfrak{q}}(I) = V^{\mathfrak{q}}(L)\}$. By Zorn's lemma, the set A has a maximal element, say J. We claim that J is irreducible. Assume, on the contrary, that $J = J_1 \cap J_2$ for some ideals J_1 and J_2 of R. Then $V^{\mathfrak{q}}(I) = V^{\mathfrak{q}}(J) = V^{\mathfrak{q}}(J_1 \cap J_2) = V^{\mathfrak{q}}(J_1) \cup V^{\mathfrak{q}}(J_2)$ and so $V^{\mathfrak{q}}(I)$ is equal to $V^{\mathfrak{q}}(J_1)$ or $V^{\mathfrak{q}}(J_2)$, since $V^{\mathfrak{q}}(I)$ is irreducible. It is a contradiction, since J is a maximal element of A and $J \subseteq J_1$ and $J \subseteq J_2$.
(4). According to [4, Exercise 4.11], every closed subset can be written as a union of finitely many irreducible closed sets in a Noetherian topological space. Now the part (3) completes the proof.
(5). By the part (1), q.Spec(R) is a Noetherian topological space and hence the assertion follows from the part (4). □

Theorem 6. *Let R be a ring and $Y \subseteq$ q.Spec(R). Then $\xi(Y)$ is a quasi-primary ideal of R if and only if Y is an irreducible space.*

Proof. Suppose $\xi(Y)$ is a quasi-primary ideal of R. Let $Y \subseteq Y_1 \cup Y_2$ where Y_1 and Y_2 are two closed subsets of q.Spec(R). Then there exist two ideals I and J of R such that $Y_1 = V^{\mathfrak{q}}(I)$ and $Y_2 = V^{\mathfrak{q}}(J)$. Thus, $Y \subseteq V^{\mathfrak{q}}(I) \cup V^{\mathfrak{q}}(J) = V^{\mathfrak{q}}(I \cap J)$. It implies, by Lemma 2(4), that $I \cap J \subseteq \sqrt{\xi(Y)}$. It follows that either $I \subseteq \sqrt{\xi(Y)}$ or $J \subseteq \sqrt{\xi(Y)}$, since $\sqrt{\xi(Y)}$ is prime. Again by using Lemma 2(4), we conclude that either $Y \subseteq V^{\mathfrak{q}}(I) = Y_1$ or $Y \subseteq V^{\mathfrak{q}}(J) = Y_2$. Thus Y is irreducible. Conversely, assume that Y is an irreducible space. Let $ab \in \xi(Y)$ for some $a, b \in R$. Suppose, on the contrary, that $Ra \nsubseteq \sqrt{\xi(Y)}$ and $Rb \nsubseteq \sqrt{\xi(Y)}$. By Lemma 2(4), $Y \nsubseteq V^{\mathfrak{q}}(Ra)$ and $Y \nsubseteq V^{\mathfrak{q}}(Rb)$. Let $q \in Y$. Then $\sqrt{q} \supseteq \sqrt{\xi(Y)} \supseteq Rab$. This means that either $Ra \subseteq \sqrt{q}$ or $Rb \subseteq \sqrt{q}$. So, by Lemma 2(1),(2), we have either $V^{\mathfrak{q}}(q) \subseteq V^{\mathfrak{q}}(Ra)$ or $V^{\mathfrak{q}}(q) \subseteq V^{\mathfrak{q}}(Rb)$. Therefore, $Y \subseteq V^{\mathfrak{q}}(Ra) \cup V^{\mathfrak{q}}(Rb)$ and hence $Y \subseteq V^{\mathfrak{q}}(Ra)$ or $Y \subseteq V^{\mathfrak{q}}(Rb)$ as Y is irreducible. It is a contradiction. □

Corollary 1. *Let R be a ring.*

(1) Let I be an ideal of R. Then $V(I)$ is irreducible in q.Spec(R) *if and only if $I \in$* q.Spec(R).

(2) If R is a domain, then q.Spec(R) *is irreducible.*

Proof. (1). Since $\sqrt{I} = \xi(V(I))$, Theorem 6 shows that $\sqrt{I}$ is quasi-primary if and only if $V(I)$ is irreducible. On the other hand, it is easy to see that $I \in \text{q.Spec}(R)$ if and only if $\sqrt{I} \in \text{q.Spec}(R)$. It completes the proof.
(2). Since (0) is a prime ideal of R, we have $\xi(\text{q.Spec}(R)) \subseteq (\xi(\text{Spec}(R)) = (0)$. Thus $\xi(\text{q.Spec}(R))$ is a quasi-primary ideal of R and hence the result follows from Theorem 6. □

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Unified Approach to Starlike and Convex Functions Involving Poisson Distribution Series

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Abstract. The motivation behind present paper is to establish connection between analytic univalent functions $\mathcal{T}S_p(\zeta,\gamma,\delta)$ and $UC\mathcal{T}(\zeta,\gamma,\delta)$ by applying Hadamard product involving Poisson distribution series. We likewise consider an integral operator connection with this series.

Mathematics subject classification: 30C45.
Keywords and phrases: Starlike functions, Convex functions, Poisson distribution series, Convolution operator, Conic domains..

1 Introduction

We letting $\mathfrak{A}$ denote the class of functions $\mathfrak{f}$ of the form:

$$\mathfrak{f}(z) = z + a_2 z^2 + a_3 z^3 + ... \tag{1}$$

which are analytic in $\mathbb{U}$ and $\mathfrak{S}$ the subclass of $\mathfrak{A}$ which includes univalent functions normalized by conditions $\mathfrak{f}(0) = 0 = \mathfrak{f}'(0) - 1$. Let $\mathcal{T}$ be the subclass of $\mathfrak{A}$ consisting of functions whose non zero coefficient of the form second on, given by (see [19])

$$\mathfrak{f}(z) = z - \sum_{n=2}^{\infty} a_n z^n. \tag{2}$$

Kanas and Wisniowska [11] introduced the class $\delta - UCV$ which includes geometric aspect in connection with conic domains. The family $\delta - UCV$ is of extraordinary enthusiasm for it contains some notable, just as new, classes of univalent functions. The class $\delta - UCV$ map each circular arc contained in the unit disk $\mathbb{U}$ with a center ξ, $|\xi| \leq \delta\,(0 \leq \delta < 1)$, onto a convex arc. The notion of δ-uniformly convex function is straightforward expansion of classical convexity. In 2011, Murugusundaramoorthy and Magesh [13] unified the classes $S_p(\gamma,\delta)$ and $UCV(\gamma,\delta)$ into the classes $S_p(\zeta,\gamma,\delta)$ and $UCV(\zeta,\gamma,\delta)$ which is defined as, a function $f \in \mathcal{A}$ is said to in the class δ-uniformly starlike functions of order γ, denoted by $S_p(\zeta,\gamma,\delta)$ if it satisfies analytic criterion

$$Re\left\{\frac{zf'(z)}{(1-\zeta)f(z)+\zeta zf'(z)} - \gamma\right\} > \delta\left|\frac{zf'(z)}{(1-\zeta)f(z)+\zeta zf'(z)} - 1\right|,\ z \in \mathbb{U} \tag{3}$$

and the $f \in \mathcal{A}$ is said to in the class δ-uniformly convex functions of order γ, denoted by $UCV(\zeta, \gamma, \delta)$ if it satisfies analytic criterion

$$Re\left\{\frac{f'(z)+zf''(z)}{f'(z)+\zeta zf''(z)}-\gamma\right\} > \delta\left|\frac{f'(z)+zf''(z)}{f'(z)+\zeta zf''(z)}-1\right|, \ z \in \mathbb{U}. \tag{4}$$

We note that $\mathcal{T}S_p(\zeta, \gamma, \delta) = S_p(\zeta, \gamma, \delta) \cap \mathcal{T}$ and $UC\mathcal{T} = UCV \cap \mathcal{T}$.

Remark 1. From among the many choices of ζ, γ, δ which would provide the following known subclasses:
1) $\mathcal{T}S_p(0, \gamma, \delta) = \mathcal{T}S_p(\gamma, \delta)$ (see [4]),
2) $\mathcal{T}S_p(0, 0, \delta) = \mathcal{T}S_p(\delta)$ (see [20]),
3) $\mathcal{T}S_p(0, \gamma, 1) = \mathcal{T}S_p(\gamma)$ (see [4]),
4) $\mathcal{T}S_p(\zeta, \gamma, 0) = \mathcal{T}(\zeta, \gamma)$ (see [2],[16]),
5) $\mathcal{T}S_p(0, \gamma, 0) = \mathcal{T}^*(\gamma)$ (see [19]),
6) $UC\mathcal{T}(0, \gamma, \delta) = UC\mathcal{T}(\gamma, \delta)$ (see [4]),
7) $UC\mathcal{T}(0, 0, \delta) = UC\mathcal{T}(\delta)$ (see [21]),
8) $UC\mathcal{T}(0, \gamma, 1) = UC\mathcal{T}(\gamma)$ (see [4]),
9) $UC\mathcal{T}(\zeta, \gamma, 0) = \mathcal{C}(\zeta, \gamma)$ (see [2]),
10) $UC\mathcal{T}(0, \gamma, 0) = \mathcal{C}(\gamma)$ (see [19]).

2 Preliminary Results

A remarkably large number of special functions (series) have been presented in geometric function theory. Among those special functions, due mainly to greater abstruseness of their properties, Bieberbach conjecture have found special attention in various problems of geometric function theory. Recently, a large number of special functions involving hypergeometric functions and their various extension (or generalizations) have been investigated, see also ([3],[5],[6],[8],[9],[15],[18],[22],[23]).

Recently, Porwal [16] introduced a power series as

$$\chi(p, z) = z + \sum_{n=2}^{\infty} \frac{e^{-p}p^{n-1}}{(n-1)!} z^n, \quad z \in \mathbb{U}, \tag{5}$$

where $p > 0$. Further Porwal [16] defined a series

$$\varphi(p, z) = 2z - \chi(p, z) = z - \sum_{n=2}^{\infty} \frac{e^{-p}p^{n-1}}{(n-1)!} z^n, \quad z \in \mathbb{U}. \tag{6}$$

The convolution (or Hadamard product) of two series

$$(\mathfrak{f} * \mathfrak{g})(z) = (\mathfrak{g} * \mathfrak{f})(z) = \sum_{n=2}^{\infty} a_n b_n z^n.$$

Porwal and Kumar [17] introduced the linear operator $\mathfrak{I}(p)\mathfrak{f} : \mathfrak{A} \to \mathfrak{A}$ defined by using the Hadamard product as

$$\mathfrak{I}(p)\mathfrak{f} = \chi(p,z) * f(z) = z + \sum_{n=2}^{\infty} \frac{e^{-p}p^{n-1}}{(n-1)!} a_n z^n, \quad z \in \mathbb{U}. \tag{7}$$

Altinkaya and Yalcin [1] gave obligatory conditions for the Poisson distribution series belonging to the class $\mathcal{T}(\gamma,\delta)$. Murugusundaramoorthy et al.[14] investigated some characterization for Poisson distribution series. In recent times, the univalent function theorists have shown good affinity towards Possion distribution series by relating it with the area of geometric function theory (see also,[10] [12],[16],[17]). To prove our results, we will need the following results.

Theorem 1. *[13] A function $\mathfrak{f}(z)$ of the form (1) is in $\mathcal{T}S_p(\zeta,\gamma,\delta)$ if and only if*

$$\sum_{n=2}^{\infty} \Big[n(1+\delta) - (\gamma+\delta)(1+n\zeta-\zeta) \Big] |a_n| \le 1-\gamma. \tag{8}$$

Theorem 2. *[13] A function $\mathfrak{f}(z)$ of the form (1) is in $UC\mathcal{T}(\zeta,\gamma,\delta)$ if and only if*

$$\sum_{n=2}^{\infty} n \Big[n(1+\delta) - (\gamma+\delta)(1+n\zeta-\zeta) \Big] |a_n| \le 1-\gamma. \tag{9}$$

Inspired by results between various subclasses of analytic univalent functions by utilizing hypergeometric functions ([9],[15],[22]), Bessel functions ([3],[5],[6],[8]) and Struve functions ([23]), we established connections between the classes $UC\mathcal{T}(\zeta,\gamma,\delta)$ and $\mathcal{T}S_p(\zeta,\gamma,\delta)$ by applying the above mentioned results (8), (9) and convolution operator given by (7).

3 Main Results

Theorem 3. *The function $\chi(p,z)$ is in $\mathcal{T}S_p(\zeta,\gamma,\delta)$ if*

$$pe^p[(1+\delta) - \zeta(\gamma+\delta)] \le 1-\gamma \tag{10}$$

holds for $p > 0$. Moreover $\varphi(p,z)$ belongs to $\mathcal{T}S_p(\zeta,\gamma,\delta)$ if and only if (10) holds.

Proof. In view of Theorem 1, it is sufficient to show that

$$\sum_{n=2}^{\infty} \Big[n(1+\delta) - (\gamma+\delta)(1+n\zeta-\zeta) \Big] \frac{p^{n-1}}{(n-1)!} e^{-p} \le 1-\gamma.$$

Let

$$\Omega_1(p,\zeta,\gamma,\delta) = \sum_{n=2}^{\infty} \Big[n(1+\delta) - (\gamma+\delta)(1+n\zeta-\zeta) \Big] \frac{p^{n-1}}{(n-1)!} e^{-p}$$

$$
\begin{aligned}
&= e^{-p} \sum_{n=2}^{\infty} \Big[n(1+\delta) - (\gamma+\delta)(1+n\zeta-\zeta) \Big] \frac{p^{n-1}}{(n-1)!} \\
&= e^{-p} \left[\{(1+\delta) - \zeta(\gamma+\delta)\} \sum_{n=2}^{\infty} \frac{p^{n-1}}{(n-2)!} + (1-\gamma) \sum_{n=2}^{\infty} \frac{p^{n-1}}{(n-1)!} \right] \\
&= e^{-p} \Big[\{(1+\delta) - \zeta(\gamma+\delta)\} p e^{p} + (1-\gamma)(e^{p}-1) \Big] \\
&= [(1+\delta) - \zeta(\gamma+\delta)]p + (1-\gamma)(1-e^{-p}).
\end{aligned}
$$

But the last expression is bounded above by $1-\gamma$, if (10) holds. Since

$$
\varphi(p,z) = 2z - \chi(p,z) = z - \sum_{n=2}^{\infty} \frac{e^{-p} p^{n-1}}{(n-1)!} z^{n} \tag{11}
$$

the necessary of (10) for $2z - \chi(p,z)$ to be in $\mathcal{T}S_p(\zeta,\gamma,\delta)$ follows from Theorem 1. □

Remark 2. Putting $\delta = 0$ in Theorem 3, we obtain the result investigated by Porwal [16] Theorem 3.

Corollary 1. *The function $\chi(p,z)$ is in $\mathcal{T}S_p(\gamma,\delta)$ if*

$$
pe^{p}(1+\delta) \leq 1-\gamma \tag{12}
$$

holds for $p > 0$.

Corollary 2. *The function $\chi(p,z)$ is in $\mathcal{T}S_p(\gamma)$ if*

$$
pe^{p} \leq 1-\gamma \tag{13}
$$

holds for $p > 0$.

Corollary 3. *The function $\chi(p,z)$ is in $\mathcal{T}S_p(\zeta,\gamma,\delta)$ if*

$$
e^{p}\Big[\{(1+\delta) - \zeta(\gamma+\delta)\} p \Big] \leq 1-\gamma \tag{14}
$$

holds for $p > 0$.

Theorem 4. *The function $\chi(p,z)$ is in $UC\mathcal{T}(\zeta,\gamma,\delta)$ if*

$$
e^{p}\Big(\{(1+\delta) - \zeta(\gamma+\delta)\} p^{2} + \{3(1+\delta) - (1+2\zeta)(\gamma+\delta)\} p \Big) \leq 1-\gamma \tag{15}
$$

holds for $p > 0$.

Proof. In view of Theorem 2, it is sufficient to show that

$$\sum_{n=2}^{\infty} n\Big[n(1+\delta)-(\gamma+\delta)(1+n\zeta-\zeta)\Big]\frac{p^{n-1}}{(n-1)!}e^{-p} \le 1-\gamma.$$

Let

$$\begin{aligned}
&\Omega_2(p,\zeta,\gamma,\delta)\\
&=\sum_{n=2}^{\infty} n\Big[n(1+\delta)-(\gamma+\delta)(1+n\zeta-\zeta)\Big]\frac{p^{n-1}}{(n-1)!}e^{-p}\\
&=e^{-p}\sum_{n=2}^{\infty} n\Big[n(1+\delta)-(\gamma+\delta)(1+n\zeta-\zeta)\Big]\frac{p^{n-1}}{(n-1)!}\\
&=e^{-p}\Bigg[\{(1+\delta)-\zeta(\gamma+\delta)\}\Bigg(\sum_{n=2}^{\infty}\frac{p^{n-1}}{(n-3)!}+3\sum_{n=2}^{\infty}\frac{p^{n-1}}{(n-2)!}+\sum_{n=2}^{\infty}\frac{p^{n-1}}{(n-1)!}\Bigg)\\
&\qquad+\{\zeta(\gamma+\delta)-(\gamma+\delta)\}\Bigg(\sum_{n=2}^{\infty}\frac{p^{n-1}}{(n-2)!}+\sum_{n=2}^{\infty}\frac{p^{n-1}}{(n-1)!}\Bigg)\Bigg]\\
&=e^{-p}\Big(\{(1+\delta)-\zeta(\gamma+\delta)\}p^2e^p+\{3(1+\delta)-(1+2\zeta)(\gamma+\delta)\}pe^p\\
&\qquad\quad+(1-\gamma)(e^p-1)\Big)\\
&=\Big(\{(1+\delta)-\zeta(\gamma+\delta)\}p^2+\{3(1+\delta)-(1+2\zeta)(\gamma+\delta)\}p\\
&\qquad\quad+(1-\gamma)(1-e^{-p})\Big).
\end{aligned}$$

But the last expression is bounded above by $1-\gamma$, if (15) holds. □

Remark 3. Putting $\delta=0$ in Theorem 4, we obtain the result investigated by Porwal [16] Theorem 4.

Corollary 4. *The function $\chi(p,z)$ is in $UC\mathcal{T}(\gamma,\delta)$ if*

$$pe^p\Big[(1+\delta)p+2\delta-\gamma+3\Big]\le 1-\gamma \tag{16}$$

holds for $p>0$.

Corollary 5. *The function $\chi(p,z)$ is in $UC\mathcal{T}(\gamma)$ if*

$$pe^p(p-\gamma+3)\le 1-\gamma \tag{17}$$

holds for $p>0$.

Corollary 6. *The function $\chi(p,z)$ is in $UC\mathcal{T}(\zeta,\gamma,\delta)$ if*

$$e^p\Big(\{(1+\delta)-\zeta(\gamma+\delta)\}p^2+\{3(1+\delta)-(1+2\zeta)(\gamma+\delta)\}p\Big)\le 1-\gamma \tag{18}$$

holds for $p>0$.

4 Inclusion Properties

A function $f \in \mathcal{A}$ is said to in the class $\mathcal{R}_\nu^\tau(\delta)$, if it satisfies the inequality

$$\left|\frac{(1-\delta)\frac{f(z)}{z}+\nu f'(z)-1}{2\tau(1-\delta)+(1-\nu)\frac{f(z)}{z}+\nu f'(z)-1}\right|<1, \ \ (z\in\mathbb{U})$$

where $\tau \in \mathbb{C}\setminus\{0\}, \delta<1, 0<\nu\leq 1$.

The class was introduced by Swaminathan [18]. for $\nu = 1$ the class is reduces to familiar class introduced by Dixit and Pal [7]. Making use of following lemma, we will prove inclusion result on the class $UC\mathcal{T}(\zeta,\gamma,\delta)$.

Lemma. *If $f \in \mathcal{R}_\nu^\tau(\delta)$ is of the form (1) then*

$$|a_n| = \frac{2|\tau|(1-\delta)}{1+\nu(n-1)}, \ \ n\in\mathbb{N}\setminus\{1\}. \tag{19}$$

The bounds given in (4) is sharp.

Theorem 5. *Let $p>0$, $\tau\in\mathbb{C}\setminus\{0\}, \delta<1$ and $0<\nu\leq 1$. If $f\in\mathcal{R}_\nu^\tau(\delta)$, then $\mathfrak{I}(p,z)f \in UC\mathcal{T}(\zeta,\gamma,\delta)$ if and only if*

$$\Big[\{(1+\delta)-\zeta(\gamma+\delta)\}p+(1-\gamma)(1-e^{-p})\Big] \leq \frac{\nu(1-\gamma)}{2|\tau|(1-\delta)}. \tag{20}$$

Proof. In view of Lemma 4 it is sufficient to show that

$$\sum_{n=2}^{\infty} n\Big[n(1+\delta)-(\gamma+\delta)(1+n\zeta-\zeta)\Big]\frac{p^{n-1}}{(n-1)!}e^{-p}|a_n| \leq 1-\gamma.$$

Since $f\in\mathcal{R}_\nu^\tau(\delta)$, then by Lemma 4, we have

$$|a_n| = \frac{2|\tau|(1-\delta)}{1+\nu(n-1)}.$$

Let

$$\begin{aligned}
&\Omega_3(p,\zeta,\gamma,\delta)\\
&\quad=\sum_{n=2}^{\infty} n\Big[n(1+\delta)-(\gamma+\delta)(1+n\zeta-\zeta)\Big]\frac{p^{n-1}}{(n-1)!}e^{-p}|a_n| \leq 1-\gamma\\
&\quad=\sum_{n=2}^{\infty} n\Big[n(1+\delta)-(\gamma+\delta)(1+n\zeta-\zeta)\Big]\frac{p^{n-1}}{(n-1)!}e^{-p}\frac{2|\tau|(1-\delta)}{1+\nu(n-1)}
\end{aligned}$$

Since $1+\nu(n-1)\geq \nu n$

$$\Omega_3(p,\zeta,\gamma,\delta)$$

$$\leq \frac{2|\tau|(1-\delta)}{\nu}\sum_{n=2}^{\infty}\Big[n(1+\delta)-(\gamma+\delta)(1+n\zeta-\zeta)\Big]\frac{p^{n-1}}{(n-1)!}e^{-p}$$
$$\leq \frac{2|\tau|(1-\delta)}{\nu}\Big[\{(1+\delta)-\zeta(\gamma+\delta)\}p+(1-\gamma)(1-e^{-p})\Big].$$

But the last expression is bounded by $1-\gamma$, if (20) holds. □

Corollary 7. *Let $p>0$, $\tau\in\mathbb{C}\setminus\{0\},\delta<1$. If $f\in\mathcal{R}_1^{\tau}(\delta)$, then $\Im(p,z)f\in UC\mathcal{T}(\zeta,\gamma,\delta)$ if and only if*

$$\Big[\{(1+\delta)-\zeta(\gamma+\delta)\}p+(1-\gamma)(1-e^{-p})\Big]\leq\frac{(1-\gamma)}{2|\tau|(1-\delta)}. \tag{21}$$

Corollary 8. *Let $p>0$, $\tau\in\mathbb{C}\setminus\{0\},\delta<1$. If $f\in\mathcal{R}_1^{\tau}(\delta)$, then $\Im(p,z)f\in UC\mathcal{T}(\gamma,\delta)$ if and only if*

$$\Big[(1+\delta)p+(1-\gamma)(1-e^{-p})\Big]\leq\frac{(1-\gamma)}{2|\tau|(1-\delta)}. \tag{22}$$

Theorem 6. *Let $p>0$, $\tau\in\mathbb{C}\setminus\{0\},\delta<1$ and $0<\nu\leq 1$. If $f\in\mathcal{R}_\nu^{\tau}(\delta)$, then $\Im(p,z)f\in\mathcal{T}S_p(\zeta,\gamma,\delta)$ if and only if*

$$\Big[\{(1+\delta)+(1-\zeta)(\gamma+\delta)\}(1-e^{-p})-\frac{(\gamma+\delta)}{p}(1-e^{-p}-pe^{-p})\Big]\leq\frac{\nu(1-\gamma)}{2|\tau|(1-\delta)}. \tag{23}$$

Proof. The proof of Theorem 6 is similar to the proof of Theorem 5, therefore we omit the details involved. □

Corollary 9. *Let $p>0$, $\tau\in\mathbb{C}\setminus\{0\},\delta<1$. If $f\in\mathcal{R}_1^{\tau}(\delta)$, then $\Im(p,z)f\in\mathcal{T}S_p(\zeta,\gamma,\delta)$ if and only if*

$$\Big[\{(1+\delta)+(1-\zeta)(\gamma+\delta)\}(1-e^{-p})-\frac{(\gamma+\delta)}{p}(1-e^{-p}-pe^{-p})\Big]\leq\frac{(1-\gamma)}{2|\tau|(1-\delta)}. \tag{24}$$

Corollary 10. *Let $p>0$, $\tau\in\mathbb{C}\setminus\{0\},\delta<1$. If $f\in\mathcal{R}_1^{\tau}(\delta)$, then $\Im(p,z)f\in\mathcal{T}S_p(\gamma,\delta)$ if and only if*

$$\Big[\{(1+\delta)+(\gamma+\delta)\}(1-e^{-p})-\frac{(\gamma+\delta)}{p}(1-e^{-p}-pe^{-p})\Big]\leq\frac{(1-\gamma)}{2|\tau|(1-\delta)}. \tag{25}$$

5 An Integral Operator

In this section, we define a particular integral operator $\mathcal{I}(p,z)$ as follows:

$$\mathcal{I}(p,z)=\int_0^z\frac{\chi(p,s)}{s}ds. \tag{26}$$

Theorem 7. *If $p>0$, then $\mathcal{I}(p,z)$ defined by (26) is in $UC\mathcal{T}(\zeta,\gamma,\delta)$ if and only if*

$$pe^p[(1+\delta)+(1-\zeta)(\gamma+\delta)]\leq 1-\gamma. \tag{27}$$

Proof. It is easy to see that

$$\mathcal{I}(p,z) = z - \sum_{n=2}^{\infty} \frac{e^{-p}p^{n-1}}{n!} z^n, \tag{28}$$

In view of Theorem 1 it is sufficient to show that

$$\sum_{n=2}^{\infty} \Big[n(1+\delta) - (\gamma+\delta)(1+n\zeta-\zeta)\Big] \frac{p^{n-1}}{n!} e^{-p} \leq 1-\gamma.$$

Let

$$\begin{aligned}
\Omega_4(p,\zeta,\gamma,\delta) &= \sum_{n=2}^{\infty} \Big[n(1+\delta) - (\gamma+\delta)(1+n\zeta-\zeta)\Big] \frac{p^{n-1}}{(n-1)!} e^{-p} \\
&= e^{-p} \sum_{n=2}^{\infty} \Big[n(1+\delta) - (\gamma+\delta)(1+n\zeta-\zeta)\Big] \frac{p^{n-1}}{(n-1)!} \\
&= e^{-p} \left[\{(1+\delta) - \zeta(\gamma+\delta)\} \sum_{n=2}^{\infty} \frac{p^{n-1}}{(n-2)!} + (1-\gamma) \sum_{n=2}^{\infty} \frac{p^{n-1}}{(n-1)!}\right] \\
&= e^{-p}\Big[\{(1+\delta) - \zeta(\gamma+\delta)\} p e^{p} + (1-\gamma)(e^{p}-1)\Big] \\
&= [(1+\delta) - \zeta(\gamma+\delta)]p + (1-\gamma)(1-e^{-p}).
\end{aligned}$$

But the last expression is bounded by $1-\gamma$, if (27) holds. □

Theorem 8. *If $p > 0$, then $\mathcal{I}(p,z)$ defined by (26) is in $\mathcal{TS}_p(\zeta,\gamma,\delta)$ if and only if*

$$\Big[\{(1+\delta) + (1-\zeta)(\gamma+\delta)\}(1-e^{-p}) - \frac{(\gamma+\delta)}{p}(1-e^{-p}-pe^{-p})\Big] \leq (1-\gamma). \tag{29}$$

Proof. The proof of Theorem 8 is similar to the proof of Theorem 7, therefore we omit the details involved. □

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Numerical Implementation of Daftardar-Gejji and Jafari Method to the Quadratic Riccati Equation

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Abstract. The solution of quadratic Riccati differential equations can be found by classical numerical methods like Runge-Kutta method and the forward Euler method. Batiha *et al.* [7] applied variational iteration method (VIM) for the solution of General Riccati Equation. In the paper of El-Tawil *et al.* [19] they used the Adomian decomposition method (ADM) to solve the nonlinear Riccati equation. In [3] Abbasbandy applied Iterated He's homotopy perturbation method for solving quadratic Riccati differential equation. In [2] Abbasbandy used the Homotopy perturbation method to get an analytic solution of the quadratic Riccati differential equation, and a comparison with Adomian's decomposition method was presented. In [1] Abbasbandy employed VIM to find the solution of the quadratic Riccati equation by using Adomian's polynomials. Tan and Abbasbandy [30] employed the Homotopy Analysis Method (HAM) to find the solution of the quadratic Riccati equation. Batiha [5] used the multistage variational iteration method (MVIM) to solve the quadratic Riccati differential equation.

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1 Introduction

A strong tool to introduce real-life phenomena is differential equations but, in most cases, numerical or theoretical solutions are difficult to find, in recent years many numerical methods have been introduced to solve nonlinear differential equations,[4, 8, 31].

The solution of quadratic Riccati differential equations can be found by classical numerical methods like Runge-Kutta method and the forward Euler method. Batiha *et al.* [7] applied variational iteration method (VIM) for the solution of General Riccati Equation. In the paper [19] El-Tawil *et al.* used the Adomian decomposition method (ADM) to solve the nonlinear Riccati equation. In [3] Abbasbandy applied Iterated He's homotopy perturbation method for solving quadratic Riccati differential equation. In [2] Abbasbandy used the Homotopy perturbation method to get an analytic solution of the quadratic Riccati differential equation, and a comparison with Adomian's decomposition method was presented. In [1] Abbasbandy employed VIM to find the solution of the quadratic Riccati equation by using Adomian's polynomials. Tan and Abbasbandy [30] employed the Homotopy Analysis

Method (HAM) to find the solution of the quadratic Riccati equation. Batiha [5] used the multistage variational iteration method (MVIM) to solve the quadratic Riccati differential equation.

The purpose of this paper is to use the Daftardar-Gejji and Jafari method (DJM) to get the solution of quadratic Riccati differential equations and to present a comparison between VIM, ADM, HPM, and exact solution to prove the power of DJM to solve nonlinear differential equations.

2 The Daftardar–Gejji and Jafari Method

Daftardar-Gejji and Jafari method (DJM) was first introduced by Daftardar-Gejji and Jafari [16] in 2006, it has been proved that this method is a better technique for solving different kinds of nonlinear equations [6, 9–11, 13–15, 20–23, 29]. DJM has been used to create a new predictor-corrector method [17, 18]. Noor et al. [24–28] used DJM to create numerical methods to handle algebraic equations.

Here the Daftardar-Gejji and Jafari method will be discussed, which was successfully used to solve differential equations and nonlinear equations in the form:

$$y = f + L(y) + N(y), \tag{1}$$

where L, N are linear and non-linear operators, respectively, and f is a given function. The solution of Eq. (1) has the form:

$$y = \sum_{i=0}^{\infty} y_i. \tag{2}$$

Suppose we have

$$\begin{aligned} H_0 &= N(y_0), & (3)\\ H_m &= N\left(\sum_{i=0}^{m} y_i\right) - N\left(\sum_{i=0}^{m-1} y_i\right), & (4) \end{aligned}$$

then we get

$$\begin{aligned} H_0 &= N(y_0), & (5)\\ H_1 &= N(y_0 + y_1) - N(y_0), & (6)\\ H_2 &= N(y_0 + y_1 + y_2) - N(y_0 + y_1), & (7)\\ H_3 &= N(y_0 + y_1 + y_2 + y_3) - N(y_0 + y_1 + y_2) + \cdots . & (8) \end{aligned}$$

Thus $N(y)$ is decomposed as:

$$N\left(\sum_{i=0}^{\infty} y_i\right) = N(y_0) + N(y_0 + y_1) - N(y_0) + N(y_0 + y_1 + y_2) - N(y_0 + y_1)$$

$$+N(y_0 + y_1 + y_2 + y_3) - N(y_0 + y_1 + y_2) + \cdots . \tag{9}$$

So, the recurrence relation is of the following form:

$$\begin{aligned} y_0 &= f \\ y_1 &= L(y_0) + H_0 \\ y_{m+1} &= L(y_m) + H_m, \quad m = 1, 2, \cdots . \end{aligned} \tag{10}$$

Since L is linear, then:

$$\sum_{i=0}^{m} L(y_i) = L\left(\sum_{i=0}^{m} y_i\right). \tag{11}$$

So,

$$\begin{aligned} \sum_{i=0}^{m+1} y_i &= \sum_{i=0}^{m} L(y_i) + N\left(\sum_{i=0}^{m} y_i\right) \\ &= L\left(\sum_{i=0}^{m} y_i\right) + N\left(\sum_{i=0}^{m} y_i\right), \quad m = 1, 2, \cdots . \end{aligned} \tag{12}$$

Thus,

$$\sum_{i=0}^{\infty} y_i = f + L\left(\sum_{i=0}^{\infty} y_i\right) + N\left(\sum_{i=0}^{\infty} y_i\right). \tag{13}$$

The $k-$ term solution is given by the following form:

$$y = \sum_{i=0}^{k-1} y_i. \tag{14}$$

3 Convergence of the DJM

In this section, we will introduce the condition of convergence of DJM.

Lemma 1. *[9] If N is $C^{(\infty)}$ in a neighborhood of u_0 and $||N^{(n)}(u_0)|| \leq L$, for any n and for some real $L > 0$ and $||u_i|| \leq M < e^{-1}, i = 1, 2, ...,$ then the series $\sum_{n=0}^{\infty} H_n$ is absolutely convergent and*

$$||H_n|| \leq LM^n e^{n-1}(e-1), \quad n = 1, 2, ...$$

Lemma 2. *[9] If N is $C^{(\infty)}$ and $||N^{(n)}(u_0)|| \leq M \leq e^{-1}, \forall n$, then the series $\sum_{n=0}^{\infty} H_n$ is absolutely convergent.*

4 Numerical Implementation

4.1 Example 1

In this example, we shall consider the quadratic Riccati equation in the form:

$$y'(t) = 2y(t) - y^2(t) + 1, \quad y(0) = 0. \tag{15}$$

The exact solution was found to be (see Fig. 1) [19]:

$$y(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right). \tag{16}$$

If you expand Eq. (16) by Taylor expansion about $t = 0$ we get:

$$y(t) = t + t^2 + \frac{1}{3}t^3 - \frac{1}{3}t^4 - \frac{7}{15}t^5 - \frac{7}{45}t^6 + \frac{53}{315}t^7 + \frac{71}{315}t^8 + \cdots. \tag{17}$$

Bulut and Evans [12] applied the decomposition method to solve Eq. 15 and they found:

$$y(t) = t + t^2 + \frac{1}{3}t^3 - \frac{1}{3}t^4 - \frac{7}{15}t^5 - \frac{1}{5}t^6 + \frac{163}{315}t^7 - \frac{62}{315}t^8 + \cdots. \tag{18}$$

Abbasbandy [2] used Homotopy perturbation method (HPM) for quadratic Riccati differential equation and got:

$$y(t) = t + t^2 + \frac{1}{3}t^3 - \frac{1}{3}t^4 - \frac{7}{15}t^5 - \frac{7}{45}t^6 + \frac{53}{315}t^7 - \frac{221}{1260}t^8 + \cdots. \tag{19}$$

Abbasbandy [1] applied three iterates variational iteration methods (VIM) for Eq. (15) and found the result:

$$y(t) = t + t^2 + \frac{1}{3}t^3 - \frac{1}{3}t^4 - \frac{7}{15}t^5 - \frac{7}{45}t^6 + \frac{53}{315}t^7 - \frac{673}{2520}t^8 + \cdots. \tag{20}$$

To solve quadratic Riccati differential equation (15) by DJM, we integrate Eq. (15) and use initial condition $y(0) = 0$, to get:

$$y(t) = \int_0^t 2y(t) - y(t)^2 + 1dt. \tag{21}$$

By using algorithm (10) we have:

$$y_0 = 0, \quad y_1 = t, \quad y_2 = -\frac{1}{3}\,t^2\,(t-3)\,, \;\; y_3 = -\frac{t^3\left(5\,t^4 - 35\,t^3 + 21\,t^2 + 210\,t - 210\right)}{315},$$

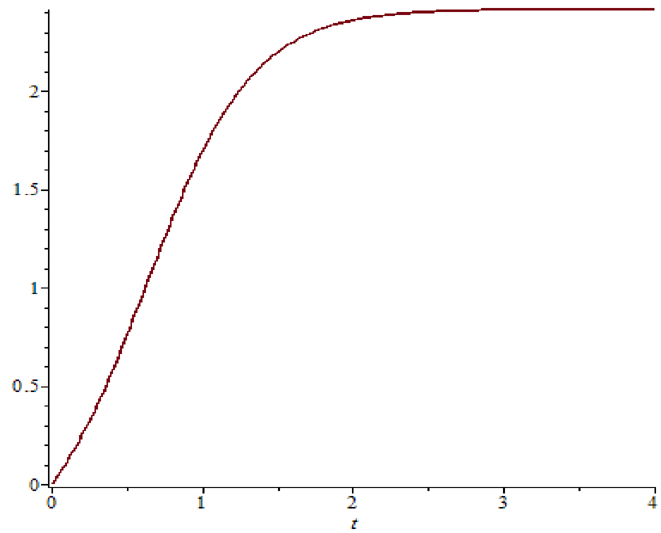

Figure 1. *The exact solution of Eq.15*

$$y_4 = -\frac{1}{170270100}t^4(2860\,t^{11} - 42900\,t^{10} + 189420\,t^9 + 90090\,t^8 - 2388204\,t^7 + \\ +2234232\,t^6 + 11171160\,t^5 - 6891885\,t^4 - 41081040\,t^3 + 3783780\,t^2 + \\ +90810720\,t - 56756700),$$

$$\vdots$$

Thus,

$$\sum_{i=0}^{4} y_i = t - \frac{1}{3}\,t^2\,(t-3) - \frac{t^3\left(5\,t^4 - 35\,t^3 + 21\,t^2 + 210\,t - 210\right)}{315} \\ -\frac{1}{170270100}t^4(2860\,t^{11} - 42900\,t^{10} + 189420\,t^9 + 90090\,t^8 \\ -2388204\,t^7 + 2234232\,t^6 + 11171160\,t^5 - 6891885\,t^4 \\ -41081040\,t^3 + 3783780\,t^2 + 90810720\,t - 56756700). \tag{22}$$

Using Taylor expansion to expand y_6 about $t = 0$ gives:

$$y(t) = t + t^2 + \frac{1}{3}t^3 - \frac{1}{3}t^4 - \frac{7}{15}t^5 - \frac{7}{45}t^6 + \frac{7}{45}t^7 - \frac{83}{315}t^8 \cdots. \tag{23}$$

4.2 Example 2

Here, we will check the following Riccati equation:

$$y'(t) = -y^2(t) + 1, \quad y(0) = 0. \tag{24}$$

The exact solution for the Riccati equation above is [19]:

$$y(t) = \frac{e^{2t} - 1}{e^{2t} + 1}. \tag{25}$$

When we expand Eq. (25) by Taylor expansion about $t = 0$ we get:

$$y(t) = t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \frac{62}{2835}t^9 - \frac{1382}{155925}t^{11} + \frac{21844}{6081075}t^{13} + \cdots. \tag{26}$$

To solve quadratic Riccati differential equation (24) by DJM, we integrate Eq. (24) and use initial condition $y(0) = 0$, to get:

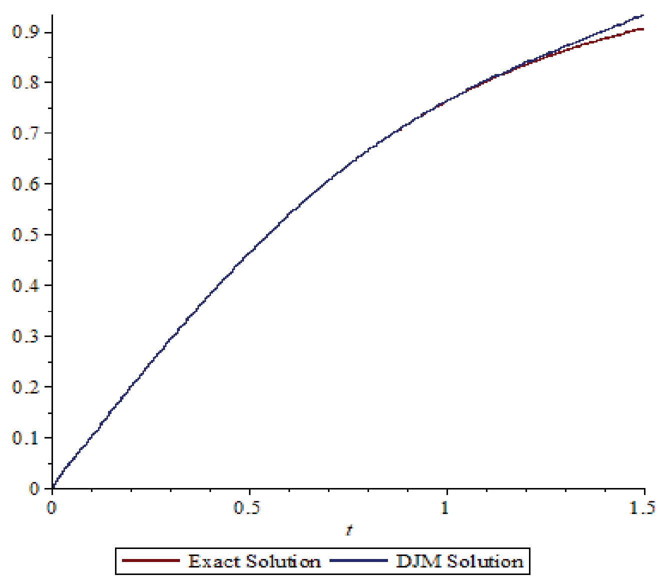

Figure 2. *The comparison between the y_4 of DJM and the exact solution*

$$y(t) = \int_0^t -y(t)^2 + 1 dt. \tag{27}$$

By using algorithm (10) we have:

$$y_0 = 0, \quad y_1 = t, \quad y_2 = -\frac{1}{3}t^3, \quad y_3 = -\frac{t^5\left(5\,t^2 - 42\right)}{315},$$

$$y_4 = -\frac{t^7\left(715\,t^8 - 13860\,t^6 + 109746\,t^4 - 570570\,t^2 + 1621620\right)}{42567525}$$

$$\vdots$$

Thus,

$$\begin{aligned}\sum_{i=0}^{4} y_i &= t - 1/3\,t^3 - \frac{t^5\left(5\,t^2 - 42\right)}{315} \\ &\quad - \frac{t^7\left(715\,t^8 - 13860\,t^6 + 109746\,t^4 - 570570\,t^2 + 1621620\right)}{42567525}. \end{aligned} \tag{28}$$

Using Taylor expansion to expand y_4 about $t = 0$ gives:

$$y(t) = t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \frac{38}{2835}t^9 + \cdots . \tag{29}$$

5 Numerical Results and Discussion

In this section, we will show the numerical solutions of quadratic Riccati differential equation.

Table 1. Numerical comparisons between exact solution and y_5 of DJM

t	Exact solution	y_5 of DJM	absolute error
0.1	0.1102951967	0.1102951631	3.360E-8
0.2	0.2419767992	0.2419752509	1.548E-6
0.3	0.3951048481	0.3950932308	1.162E-5
0.4	0.5678121656	0.5677733164	3.885E-5
0.5	0.7560143925	0.7559368137	7.758E-5
0.6	0.9535662155	0.9534634383	1.028E-4
0.7	1.1529489660	1.1528561200	9.285E-5
0.8	1.3463636550	1.3463068680	5.679E-5
0.9	1.5269113120	1.5268938270	1.748E-5
1.0	1.6894983900	1.6895510560	5.266E-5

Table 1 shows the comparison between the y_5 of DJM and the exact solution for example 1. Figure 2 shows the comparison between the y_4 of DJM and the exact solution for example 2. We can see the good accuracy of DJM compared to the exact solution, but we can note that it's accurate only for small t.

6 Conclusions

In this paper, we show a new application of the Daftardar-Gejji and Jafari method (DJM) to get the solution of the quadratic Riccati differential equation. In this paper, we use the Maple Package to calculate the series obtained from the DJM. It may be concluded that DJM is a powerful tool for finding analytical and numerical solutions for the Riccati differential equation.

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On differentially prime subsemimodules

Ivanna Melnyk

Abstract. The paper is devoted to the investigation of the notion of a differentially prime subsemimodule of a differential semimodule over a commutative semiring, which generalizes the notion of differentially prime ideal of a ring. The characterization of differentially prime subsemimodules is given. The interrelation between differentially prime subsemimodules and different types of differential subsemimodules and ideals is studied.

Mathematics subject classification: 16Y60, 13N99.
Keywords and phrases: differential semimodule, differential subsemimodule, differentially prime subsemimodule, quasi-prime subsemimodule.

1 Introduction

The notion of a derivation for semirings is defined in [3] as an additive map satisfying the Leibnitz rule. Recently in [2, 13] and [11] the authors investigated different properties of semiring derivations, differential semirings, i.e. semirings considered together with a derivation, and differential ideals of such rings. Prime subsemimodules of semimodules over semirings were introduced and studied in [1]. Differentially prime ideals were introduced in [8] for differential, not necessarily commutative, rings. Differentially prime submodules of modules over associative rings were studied in [10].

The rapid development of semiring and semimodule theory in recent years motivates a further study into properties of differential semirings, differential semimodules, semiring ideals and subsemimodules defined by similar conditions. The objective of this paper is to investigate differentially prime subsemimodules of semimodules equipped with derivations over commutative semimodules, and their interrelation with other types of subsemimodules.

For the sake of completeness some definitions and properties used in the paper will be given here. For more information see [3–5, 9].

Let R be a nonempty set and let $+$ and $\cdot$ be binary operations on R. An algebraic system $(R, +, \cdot)$ is called a *semiring* if $(R, +, 0)$ is a commutative monoid, $(R, \cdot)$ is a semigroup and multiplication distributes over addition from either side. A semiring $(R, +, \cdot)$ is said to be *commutative* if $\cdot$ is commutative on R.

Zero $0_R \in R$ is called (*multiplicatively*) *absorbing* if $a \cdot 0_R = 0_R \cdot a = 0$ for all $a \in R$. An element $1_R \in R$ is called *identity* if $a \cdot 1_R = 1_R \cdot a = a$ for all $a \in R$. Suppose $1_R \neq 0_R$, otherwise $R = \{0\}$ if zero is absorbing.

Throughout the paper, we assume that all semirings are commutative with identity, $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$ denotes the set of non-negative integers.

An *ideal* of a semiring R is a nonempty set $I \neq R$ which is closed under addition $+$ and satisfies the condition $ra \in I$ for all $a \in I$, $r \in R$. An ideal I of a semiring R is called *subtractive* (or *k-ideal*) if $a \in I$ and $a + b \in I$ imply $b \in I$.

Let R be a semiring with $1_R \neq 0_R$. A *semimodule over a semiring* R (or R-*semimodule*) is a nonempty set M together with two operations $+\colon M \times M \to M$ and $\cdot\colon R \times M \to M$ such that $(M, +)$ is a commutative monoid with 0_M, $(M, \cdot)$ is a semigroup, $(r+s)m = rm+sm$ for all $r, s \in R$, $m \in M$, $r(m_1+m_2) = rm_1+rm_2$ for all $r \in R$, $m_1, m_2 \in M$, $0_R \cdot m = r \cdot 0_M = 0_M$ for all $r \in R$ and $m \in M$, $1_R \cdot m = m$ for all $m \in M$.

A subset N of an R-semimodule M is called a *subsemimodule* of M if $m+n \in N$ and $rm \in N$ for any $m, n \in N$, and $r \in R$. A subsemimodule N of an R-semimodule M is called *subtractive* or *k-subsemimodule* if $m_1 \in N$ and $m_1 + m_2 \in N$ imply $m_2 \in N$. So $\{0_M\}$ is a subtractive subsemimodule of M.

Let R be a semiring. A map $\delta\colon R \to R$ is called a *derivation on* R [3] if $\delta(a+b) = \delta(a) + \delta(b)$ and $\delta(ab) = \delta(a)b + a\delta(b)$ for any $a, b \in R$. A semiring R equipped with a derivation δ is called a *differential semiring* with respect to the derivation δ (or δ-*semiring*), and is denoted by (R, δ) [2].

For an element $r \in R$ denote by $r^{(0)} = r$, $r' = \delta(r)$, $r'' = \delta(r')$, $r^{(n)} = \delta\left(r^{(n-1)}\right)$, for any $n \in \mathbb{N}_0$. An ideal I of the semiring R is called *differential* if the set I is differentially closed under δ, i.e. $\delta(r) \in I$ for any $r \in I$. The set of all derivations of an element $r \in R$ $r^{(\infty)} = \{r^{(n)} | n = 0, 1, 2, 3 \ldots\}$ is differentially closed. The ideal $[r] = \left(r^{(\infty)}\right) = (r, r', r'', \ldots)$ of R, generated by the set $r^{(\infty)}$, is differentially generated by $r \in R$; it is the smallest differential ideal containing the element $r \in R$ [11].

Let M be a semimodule over the differential semiring (R, δ). A map $d\colon M \to M$ is called a *derivation* of the semimodule M, associated with the semiring derivation $\delta\colon R \to R$ (or a δ-derivation), if $d(m+n) = d(m) + d(n)$ and $d(rm) = \delta(r)m + rd(m)$ for any $m, n \in M$, $r \in R$. A R-semimodule M together with a derivation $d\colon M \to M$ is called a *differential semimodule* (or d-δ-*semimodule*) and is denoted by (M, d).

A subsemimodule N of the R-semimodule M is called *differential* if $d(N) \subseteq N$. Any differential semimodule has two trivial differential subsemimodules: $\{0_M\}$ and itself.

For an element $m \in M$ denote by $m^{(0)} = m$, $m' = d(m)$, $m'' = d(m')$, $m^{(n)} = d\left(m^{(n-1)}\right)$, for any $n \in \mathbb{N}_0$. Moreover, let $m^{(\infty)} = \{m^{(n)} | n \in \mathbb{N}_0\}$. It is easy to see that the set $m^{(\infty)}$ is differentially closed. The subsemimodule $[m] = \left(m^{(\infty)}\right) = (m, m', m'', \ldots)$ is the smallest differential subsemimodule of M containing $m \in M$.

A subsemimodule P of a subsemimodule M is called prime if for any ideal I of R and any submodule N of M the inclusion $IN \subseteq P$ implies $N \subseteq P$ or $I \subseteq (P : M)$. Prime subsemimodules are extensively investigated in [1].

2 Differentially prime subsemimodules

Definition. Let S be a multiplicatively closed subset of R. A non-empty subset X of the semimodule M is called an S-*closed* subset of M if $sx \in X$ for every $s \in S$ and $x \in X$.

Quasi-prime ideals of differential rings were introduced and studied in [6, 7], its generalizations to differential modules, semirings and semimodules were studied by different authors, e.g.[11, 12, 14, 15].

Definition. A differential subsemimodule N of the left differential semimodule M is called *quasi-prime* if it is maximal differential subsemimodule of M disjoint from some S-closed subset of M.

For instance, every prime differential subsemimodule is quasi-prime, because the complement of the prime subsemimodule is an S-closed subset of M, where the role of S is played by the set $\{1\}$.

In the case of a regular semimodule, we obtain the notion of quasi-prime ideal of a semiring. For differential semiring ideals it is known that every maximal among differential ideals not meeting some multiplicatively closed subset of the semiring is quasi-prime. The analogue of this fact holds for differential semimodules: every maximal among differential subsemimodules of an arbitrary differential semimodule is quasi-prime.

Definition. A differential k-subsemimodule P of M is called *differentially prime* if for any $r \in R$, $m \in M$, $k \in \mathbb{N}_0$, $rm^{(k)} \in P$ implies $r \in (P : M)$ or $m \in P$.

Theorem 1. *Every quasi-prime k-subsemimodule N of M is differentially prime.*

Proof. Let N be a quasi-prime subsemimodule of M. Suppose that there exist $r \in R$, $m \in M$ such that $r \in R \setminus (N : M)$, $m \in M \setminus N$ and $[r] \cdot [m] \subseteq N$. Since N is maximal among the differential submodule not meeting some S-closed subset X of M, for differential ideal $(N : M) + [r]$ and differential subsemimodule $N + [m]$ the maximality of N implies $((N : P) + [r]) \cap S \neq \varnothing$ and $(N + [m]) \cap X \neq \varnothing$. As a result, there exist $s \in S$, $x \in X$ such that $s \in (N : M) + [r]$ and $x \in N + [m]$. Since X is an S-closed subset of M and $s \in S$, $x \in X$ implies that there exists $n \in \mathbb{N}_0$ such that $sx^{(n)} \in X$. Then $sx^{(n)} \in ((N : M) + [r]) \cdot (N + [m]) \subseteq N$. It follows that $sx^{(n)} \in X \bigcap N \neq \varnothing$, which contradicts the original assumption. Therefore, N is differentially prime. $\square$

Definition. Let $S \neq \varnothing$ be a subset of R. A subset S is called *d-multiplicatively closed* if for any $a, b \in S$ there exists $n \in \mathbb{N}_0$ such that $ab^{(n)} \in S$.

Definition. Let S be a d-multiplicatively closed subset of R. A subset $X \subseteq M$ is called *Sd-multiplicatively closed* if for any $s \in S$, $x \in X$ there exists $n \in \mathbb{N}_0$ such that $sx^{(n)} \in X$.

Proposition 1. *A k-subsemimodule $N \subseteq M$ is differentially prime if and only if $M \setminus N$ is Sd-multiplicatively closed.*

Proof. Suppose $X = M \setminus N$, $S = R \setminus (N : M)$, $N \subseteq M$ is differentially prime and there exist $s \in S$ and $x \in M \setminus N$ such that for all $n \in \mathbb{N}_0$, $sx^{(n)} \notin M \setminus N$. Then $s \in (N : M)$ or $x \in N$, which contradicts $s \in S$.

Conversely, suppose $X = M \setminus N$ is Sd-multiplicatively closed, and for all $n \in \mathbb{N}_0$, $sx^{(n)} \notin X$ for some $s \in S$ and $x \in X$. Then $sx^{(n)} \in N$, and so $s \in (N : M)$, which is a contradiction. □

Theorem 2. *For a differential k-subsemimodule P of M, $P \neq M$ the following conditions are equivalent:*

1. *P is differentially prime;*
2. *For any $r \in R$, $m \in M$, $k, l \in \mathbb{N}_0$, $r^{(l)}m^{(k)} \in P$ implies $r \in (P : M)$ or $m \in P$;*
3. *For any $r \in R$, $m \in M$, $[r] \cdot [m] \subseteq P$ implies $r \in (P : M)$ or $m \in P$;*
4. *For any differential k-ideal I of R and any differential k-subsemimodule N of M, $IN \subseteq P$ implies $N \subseteq P$ or $I \subseteq (P : M)$.*

Proof. ($1 \Longrightarrow 2$) Suppose $r^{(l)}m^{(k)} \in P$ for any $k, l \in \mathbb{N}_0$. Denote $t = l + k$. For $t = 0$ we have $r^{(0)}m^{(0)} = rm \in P$. Therefore, $d(rm) = (rm)' \in P$. For a subtractive subsemimodule P, we have $(rm)' = r'm + rm' \in P$, $rm' \in P$. Hence, $r'm \in P$.

Consider $(rm^{(k)})' = r'm^{(k)} + rm^{(k+1)}$ for all $k \in \mathbb{N}_0$. As before, $(rm^{(k)})' \in P$, $rm^{(k+1)} \in P$ imply $r'm^{(k)} \in P$, by subtractiveness of P.

In a similar way, from $(r'm^{(k-1)})' = r''m^{(k-1)} + r'm^{(k)} \in P$, $r'm^{(k)} \in P$ and subtractiveness of P we obtain $r''m^{(k)} \in P$, etc.

($2 \Longrightarrow 1$) Obvious when $l = 0$.

($2 \Longrightarrow 3$) Note that $[r] = \sum_{l \in \mathbb{N}_0} Rr^{(l)}$, $[m] = \sum_{k \in \mathbb{N}_0} Rm^{(k)}$, and so $[r] \cdot [m] = \sum_{k,l \in \mathbb{N}_0} Rr^{(l)}m^{(k)}$.

If $[r] \cdot [m] \subseteq P$ then $\sum_{k,l \in \mathbb{N}_0} Rr^{(l)}m^{(k)} \subseteq P$, in particular $r^{(l)}m^{(k)} \in P$. Hence, $r \in (P : M)$ or $m \in P$.

($3 \Longrightarrow 2$) Suppose for any $r \in R$, $m \in M$, $[r] \cdot [m] \subseteq P$ implies $r \in (P : M)$ or $m \in P$. Prove that for any $r \in R$, $m \in M$, $k, l \in \mathbb{N}_0$, $r^{(l)}m^{(k)} \in P$ implies $r \in (P : M)$ or $m \in P$.

If $r^{(l)}m^{(k)} \in P$, then $\sum_{k,l \in \mathbb{N}_0} Rr^{(l)}m^{(k)} \subseteq P$. Therefore, $[r] \cdot [m] \subseteq P$, which follows $r \in (P : M)$ or $m \in P$.

($3 \Longrightarrow 4$). Suppose for any $r \in R$, $m \in M$, $[r] \cdot [m] \subseteq P$ implies $r \in (P : M)$ or $m \in P$, and let $IN \subseteq P$, where I is an arbitrary differential ideal of R and N is an arbitrary differential subsemimodule of M.

Suppose $N \nsubseteq P$ or $I \nsubseteq (P : M)$. There exists $x \in N$, $x \notin P$, and $r \in I$, $r \notin (P : M)$. Clearly, $[r] \cdot [x] \subseteq IN \subseteq P$. Therefore, $r \in (P : M)$ or $m \in P$, which is a contradiction.

($4 \Longrightarrow 3$) is obvious. □

Theorem 3. *Let S be d-multiplicatively closed subset of R, X be Sd-multiplicatively closed subset of M, and let N be a differential subsemimodule of M, maximal in $M \setminus N$.*

If the ideal $(N : M)$ is differentially maximal in $R \setminus S$, then N is a differentially prime subsemimodule of M.

Proof. Suppose that there exist $r \in R$, $m \in M$ and $k \in \mathbb{N}_0$ such that $rm^{(k)}) \in N$, $r \notin (N : M)$, and $m \notin N$. It is clear that $N \subset N + [m]$ and $(N : M) \subset (N : M) + [r]$.

Since N is maximal among the differential subsemimodules not meeting some Sd-closed subset X, $(N + [m]) \cap X \neq \varnothing$. Since $(N : M)$ is maximal among the differential ideals of R not meeting some d-multiplicatively closed subset S, $((N : M) + [r]) \cap S \neq \varnothing$. Therefore there exist $a \in S$, $x \in X$ such that $a \in (N : M) + [r]$ and $x \in N + [m]$. On the other hand, since X is a Sd-multiplicatively closed subset, then $a \in S$, $x \in X$ implies the existence of $n \in \mathbb{N}_0$ such that $ax^{(n)} \in X$. Therefore $x^{(n)} \in (N + [m]) \cap X$. Then $ax^{(n)} \in ((N : M) + [r]) \cdot (N + [m]) = (N : M)N + (N : M) \cdot [m] + [r] \cdot N + [r] \cdot [m] \subseteq N$. Therefore, $ax^{(n)} \in N \bigcap X \neq \varnothing$, but it contradicts the assumption that $X \bigcap N = \varnothing$. Hence N is a differentially prime subsemimodule. $\square$

Corollary 1. *Let P be differentially prime ideal of R, $S = R \setminus P$, X be Sd-multiplicatively closed subset of R, let a differential subsemimodule N be maximal in $M \setminus X$. If N is prime, then $(N : M) = P$.*

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Strong stability for multiobjective investment problem with perturbed minimax risks of different types and parameterized optimality

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Abstract. A multicriteria investment Boolean problem of minimizing lost profits with parameterized efficiency and different types of risks is formulated. The lower and upper bounds on the radius of the strong stability of efficient portfolios are obtained. Several earlier known results regarding strong stability of Pareto efficient and extreme portfolios are confirmed.

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1 Introduction

Many problems of making multi-purpose decisions (individual or group) in management, planning and design can be formulated as multicriteria discrete optimization problems. A characteristic feature of such problems is the inaccuracy of the initial parameters. This inaccuracy is due to the influence of various factors of uncertainty and randomness: the inadequacy of the mathematical models used real processes, measurement or rounding errors and other factors. To manage financial investments, G. Markovitz [1] developed an optimization model that demonstrates how an investor, choosing a portfolio of assets, can minimize the degree of risk for a given expected income level. This formulation involves the use of statistical and expert assessments of risks (financial, environmental, etc.) as input data. It is well known that complex calculations of such quantities are accompanied by large number of errors, which leads to a high degree of uncertainty of the initial information. Under these conditions, the question naturally arises about the plausibility of results obtained in solving such problems, which makes necessary to conduct a post-optimal analysis of the stability of solutions to perturbations of parameters.

Modern research on the stability of multicriteria discrete optimization problems is carried out in two directions: qualitative and quantitative. Within the framework of the first direction, the authors concentrate their attention on the definition and study of various types of stability (see monograph [2], and surveys [3,4]), establishing a connection between different types of stability as well as on the search and description of the region of stability of the problem [5, 6]. The second direction is focused on obtaining estimates of permissible changes in the initial data of the problem, at which a certain predetermined property of optimal solutions is preserved [7–12], and on the development of algorithms for calculating these estimates [13–15].

Our current work continues research towards a similar direction, with focus on a different optimality principle, namely, the so-called parameterized efficient solutions and their strong stability properties are investigated. The paper is organized as follows. In Section 2, we introduce basic concepts and formulate the problem. Section 3 contains auxiliary technical statements required for the proof of the main result. As a result of the parametric analysis, in Section 4 the lower and upper bounds on strong stability radius are obtained in the case with arbitrary Hölder's norms specified in the three spaces of the problem's initial data. Some previously known facts are confirmed in Section 5.

2 Problem formulation and basic definitions

Consider a multicriteria discrete variant of the investment optimization problem with the following parameters specified below: let

$N_n=\{1, 2, \dots, n\}$ be a variety of alternatives (investment assets);

N_m be a set of possible financial market states (market situations, scenarios);

N_s be a set of possible risks;

r_{ijk} be a numerical measure of economic risk of type $k \in N_s$ if investor chooses project $j \in N_n$ given the market is in state $i \in N_m$;

$R= [r_{ijk}] \in \mathbf{R}^{m\times n\times s}$ be a matrix specifying risks;

$x= (x_1, x_2, \dots, x_n)^T \in \mathbf{E}^n$ be an investment portfolio, where $\mathbf{E}=\{0, 1\}$, and

$$x_j=\begin{cases} 1 & \text{if investor chooses project } j, \\ 0 & \text{otherwise;} \end{cases}$$

$X \subset \mathbf{E}^n$ be a set of all admissible investment portfolios, i.e. those whose realization provides the investor with the expected income and does not exceed his/her initial capital;

$\mathbf{R}^m$ be a financial market state space; $\mathbf{R}^n$ be a portfolio space; $\mathbf{R}^s$ be a risk space.

In our model, we assume that the risk measure is addictive, i.e. the total risk of one portfolio is a sum of risks of the projects included in the portfolio. The risk of each project can be measured, for instance, by means of the associated implementation cost.

Efficiency of a chosen portfolio (Boolean vector) $x \in X$, $|X| \geq 2$, is evaluated by a vector objective function

$$f(x, \mathrm{R}) = (f(x, R_1), f(x, R_2), \dots, f(x, R_s))^T,$$

with each partial objective representing minimax Savage's risk criterion [17]:

$$f(x, R_k) = \max_{i\in N_m} r_{ik}x = \max_{i\in N_m} \sum_{j\in N_n} r_{ijk}x_j \to \min_{x\in X}, \quad k \in N_s,$$

$$r_{ik}=(r_{i1k},\ r_{i2k},\dots,r_{ink})\in\mathbf{R}^n,\quad i\in N_m,\quad k\in N_s.$$

In the formula above, $R_k\in\mathbf{R}^{m\times n}$ represents the k-th cut of the risk matrix $R=[r_{ijk}]\in\mathbf{R}^{m\times n\times s}$ with rows r_{ik}.

Certainly, the problem has practical interest due to its multicriteria nature and the criteria that could be interpreted as maximum risk minimizing attitude of an investor to market instability and uncertainty.

For arbitrary $v\in\mathrm{N}$ (dimension of a space), we define the Pareto dominance [16] between two vectors as the following binary relation in the real vector valued space $\mathbf{R}^v$: $y\succ y'\iff y\geq y'\ \&\ y\neq y'$,where $y=(y_1,y_2,\dots,y_v)^T\in\mathrm{R}^v$, and $y'=(y'_1,y'_2,\dots,y'_v)^T\in\mathrm{R}^v$.

Let $\emptyset\neq I\subseteq N_s$. Denote R_I a submatrix of the risk matrix $R=[r_{ijk}]\in\mathbf{R}^{m\times n\times s}$ consisting of $h=|I|$ cuts with numbers of the set I, i.e.

$$R_I=(R_{k_1},R_{k_2},\dots,R_{k_h})^T\in\mathbf{R}^{m\times n\times h},$$
$$I=\{k_1,k_2,\dots,k_h\},\ 1\leq k_1<k_2<\dots<k_h\leq s.$$

Thus for a fixed non-empty I and chosen $x\in X$, we have a vector function

$$f(x,R_I)=(f(x,R_{k_1}),f(x,R_{k_2}),\dots,f(x,R_{k_h}))^T,$$

with components being type of Savage's minimax risk criterion [17]:

$$f\left(x,R_k\right)=\max_{i\in N_m}r_{ik}x\ \rightarrow\min_{x\in X},\quad k\in I.$$

An investor in the conditions of economic instability and uncertainty of the market state is extremely cautious, optimizing the total risk of the portfolio in the most unfavorable situation, namely when the risk is maximum. Such caution is appropriate because any investment is the exchange of a certain current value for a possibly uncertain future income. Obviously, this approach is dictated by the safest and most protective rule prescribing to assume the worst.

Let $u\in N_s$ and $N_s=\bigcup_{v\in N_u}I_v$ be a partition of the set N_s in u non-empty subsets (types of risks), i.e. $I_v\neq\emptyset$, $v\in N_u$, and $i\neq j\implies I_i\bigcap I_j=\emptyset$.

Such partition may naturally arise in the situation when risks can be classified to the different groups, e.g. financial, industrial, ecological etc. Another situation with different types of risks may appear if risk measurement scales are different, e.g. some risks are measured on a monetary scale whereas the others are measured on various subjective preference scales.

As following definition shows, inside a group of a certain type, Pareto dominance binary relation is used while comparing portfolios. For the given partition, we introduce a set of $(I_1,I_2,\dots,I_u)$-efficient portfolios according to the following formula:

$$G^s_m(R,I_1,I_2,\dots,I_u)=\{x\in X\colon\exists v\in N_u\ \ (X(x,R_{I_v})=\emptyset)\},\tag{1}$$

where

$$X\left(x,R_{I_v}\right)=\left\{x'\in X\colon f\left(x,R_{I_v}\right)\succ f\left(x',R_{I_v}\right)\right\}.\tag{2}$$

For brevity, we sometimes refer to the set of $(I_1, I_2, \ldots, I_u)$ – efficient portfolios as $G_m^{su}(R)$ and name them *efficient*. It is easy to see that the set of efficient portfolios is non-empty.

In one particular case, if $u = 1$, i.e. $I{=}N_s$, any N_s – efficient portfolio $x \in G_m^s(R, N_s)$ is also Pareto efficient (optimal). Therefore, the set $G_m^s(R, N_s)$ is identical to the Pareto set [18] defined as follows:

$$P_m^s(R) = \{x \in X\colon X(x, R) = \emptyset\},$$

where

$$X(x, R) = \left\{x' \in X\colon f(x, R) \geq f\left(x', R\right) \ \ \& \ \ f(x, R) \neq f\left(\mathrm{x}', R\right)\right\}.$$

In another particular case, if $u{=}$ s, i.e. $I_v = \{v\}$ for $v \in N_u = N_s$, the set $G_m^s(R, \{1\}, \{2\}, \ldots, \{s\})$ is a set of all the so-called extreme portfolios (see e.g. [19]). The set of extreme portfolios is defined as

$$E_m^s(R) = \{x \in X\colon \exists k \in N_s \ (X(x, R_k) = \emptyset\},$$

where

$$X(x, R_k) = \left\{x' \in X\colon f(x, R_k) > f\left(x', R_k\right)\right\}.$$

The choice of extreme portfolios can be interpreted as finding best solutions for each of s criteria, and then combining them into one set. The vector composed of optimal objective values constitutes the ideal vector that is of great importance in theory and methodology of multiobjective optimization [19].

The problem of finding the set of efficient portfolios

$$G_m^s(R, I_1, I_2, \ldots, I_u) = G_m^{su}(R)$$

is referred to as multicriteria investment Boolean problem with Savage's risk criteria of different types and denoted by $Z_m^s(R, I_1, I_2, \ldots, I_u)$, or shortly, $Z_m^{su}(R)$.

For the fixed non-empty $I \subseteq N_s$, we introduce the following sets:

$$P(R_I) = \{x \in X\colon X(x, R_I) = \emptyset\},$$

$$E(R_I) = \{x \in X\colon \exists k \in I \ (X(x, R_k) = \emptyset\},$$

where

$$X(x, R_I) = \left\{x' \in X\colon f(x, R_I) \succ f\left(x', R_I\right)\right\}.$$

In particular, for fixed $k \in N_s$ and $I = \{k\}$, $|I| = 1$, the two sets $P(R_k)$ and $E(R_k)$ are identical. Both sets represent a set of optimal portfolios for the scalar problem with respect to the k-th risk:

$$f(x, R_k) = \max_{i \in N_m} r_{ik} x \ \rightarrow \min_{x \in X}.$$

Due to (1), we have the following equality:

$$G_m^s(R, I_1, I_2, \ldots, I_u) = \{x \in X\colon \exists v \in N_u \ (x \in P(R_{I_v}))\}. \tag{3}$$

Therefore, we have

$$G_m^s(R, I_1, I_2, \ldots, I_u) = \bigcup_{v\in N_u} P(R_{I_v}), \quad \bigcup_{v\in N_u} I_v = N_s.$$

Obviously, all the sets specified above are non-empty for any risk matrix $R \in \mathbf{R}^{m\times n\times s}$.

We will perturb the elements of the three–dimensional risk matrix $R \in \mathbf{R}^{m\times n\times s}$ by adding elements of the risk perturbing matrix $R' \in \mathbf{R}^{m\times n\times s}$. Thus the problem $Z_m^{su}(R + R')$ with perturbed risks has the following form:

$$f(x, R + R') \to \min_{x\in X}.$$

The set of $(I_1, I_2, \ldots, I_u)$ – efficient portfolios in the perturbed problem is denoted by $G_m^s(R + R', I_1, I_2, \ldots, I_u)$, or shortly $G_m^{su}(R + R')$.

Recall that Hölder's norm l_p (also known as p-norm) in vector space $\mathbf{R}^n$ is the number

$$\|a\|_p = \begin{cases} \left(\sum_{j\in N_n} |a_j|^p\right)^{\frac{1}{p}} & \text{if } \ 1 \le p < \infty, \\ \max\{|a_j| \ : j \in N_n\} & \text{if } \ p = \infty, \end{cases}$$

where $a = (a_1, a_2, ..., a_n)^T \in \mathbf{R}^n$.

In the spaces $\mathbf{R}^n, \mathbf{R}^m$ and $\mathbf{R}^s$ we define three Hölder's norms l_p, l_q and l_t, where $p, q, t \in [1,\infty]$. So, the norm of matrix $R \in \mathbf{R}^{m\times n\times s}$ is the following number:

$$\|R\|_{pqt} = \|(\|R_1\|_{pq}, \|R_2\|_{pq}, ..., \|R_s\|_{pq})\|_t,$$

with cuts

$$\|R_k\|_{pq} = \|(\|r_{1k}\|_p, \|r_{2k}\|_p, ..., \|r_{mk}\|_p)\|_q, \quad k \in N_s.$$

For any numbers $p, q, t \in [1,\infty]$ the following inequalities are valid:

$$\|r_{ik}\|_p \le \|R_k\|_{pq} \le \|R\|_{pqt}, \quad i \in N_m, \ \ k \in N_s. \tag{4}$$

While solving investment problems, it is necessary to take into account the inaccuracy of the input information (statistical and expert risks evaluation errors) that are very common in real life. Under these conditions, it is highly recommended to get numerical bounds of possible changes to the input data that for any small perturbation the efficiency of at least one originally extreme portfolio is preserved.

Following [3], the strong stability (in terminology of [4], T_1-stability) radius of $Z_m^s(R, I_1, I_2, \ldots, I_u)$, $s, m \in \mathbf{N}$, with Hölder's norms l_p, l_q and l_t in spaces $\mathbf{R}^n, \mathbf{R}^m$ and $\mathbf{R}^s$, respectively, is defined as:

$$\rho = \rho_m^{su}(p, q, t) = \begin{cases} \sup \Xi_{pqt} & \text{if } \ \Xi_{pqt} \ne \emptyset, \\ 0 & \text{if } \ \Xi_{pqt} = \emptyset. \end{cases}$$

where

$$\Xi_{pqt} = \{\varepsilon > 0 \ : \ \forall R' \in \Omega_{pqt}(\varepsilon) \quad (G_m^{su}(R + R') \cap G_m^{su}(R) \ne \emptyset)\};$$

$\Omega_{pqt}(\varepsilon) = \{R' \in \mathbf{R}^{m\times n\times s} : \|R'\|_{pqt} < \varepsilon\}$ is the set of perturbing matrices R' with cuts $R'_k \in \mathbf{R}^{m\times n}$, $k \in N_s$;

$G^{su}_m(R + R')$ is the set of $(I_1, I_2, \ldots, I_u)$-solutions of the perturbed problem $Z^{su}_m(R + R')$;

$\|R'\|_{pqt}$ is the norm of matrix $R' = [r'_{ijk}]$.

Thus the strong stability radius of the problem $Z^{su}_m(R)$ is an extreme level of independent perturbations of elements of matrix $R \in \mathbf{R}^{m\times n\times s}$ such that the sets $G^{su}_m(R)$ and $G^{su}_m(R + R')$ are never disjoint.

Obviously, if $G^{su}_m(R) = X$, then the strong stability radius is not bounded. For this reason, the problem with $X\backslash E^m_s(R) \neq \emptyset$ is called *non-trivial*.

3 Auxiliary statements and lemmas

Let v be any of the above-numbers p, q, t. For the number v, let v^* be the number conjugate to v and defined as:

$$1/v + 1/v^* = 1, \quad 1 < v < \infty.$$

We also set $v^* = 1$ if $v = \infty$, and $v^* = \infty$ otherwise. We assume that v and v^* be taken from $[1,\infty]$, and conjugate. In addition to the above, we assume that $1/v = 0$ if $v = \infty$.

Further we will use the well-known Hölder's inequality

$$|a^T b| \leq \|a\|_v \|b\|_{v^*} \tag{5}$$

that is true for any two vectors a and b of the same dimension.

It is also well-known that Hölder's inequality becomes an equality for $1 < v < \infty$ if and only if

a) one of a or b is the zero vector;

b) the two vectors obtained from non-zero vectors a and b by raising their components' absolute values to the powers of v and v^*, respectively, are linearly dependent (proportional), and $\operatorname{sign}(a_i b_i)$ is independent of i.

When $v = 1$, (3) transforms into the following inequality:

$$|\sum_{i\in N_n} a_i b_i| \leq \max_{i\in N_n} |b_i| \sum_{i\in N_n} |a_i|.$$

The last holds as equality if, for example, b is the zero vector or if $a_j \neq 0$ for some j such that $|b_j| = \|b\|_\infty \neq 0$, and $a_i = 0$ for all $i \in N_n\backslash\{j\}$.

When $v = \infty$, (3) transforms into the following inequality:

$$|\sum_{i\in N_n} a_i b_i| \leq \max_{i\in N_n} |a_i| \sum_{i\in N_n} |b_i|.$$

The last holds as equality if, for example, b is the zero vector or if $a_i = \sigma sign(b_i)$ for all $i \in N_n$ and $\sigma \geq 0$.

It is easy to see that for any $a= (a_1, a_2, ..., a_n)^T \in \mathbf{R}^n$ with

$$|a_j| =\alpha, \quad j \in N_n,$$

the following equality holds

$$\|a\|_v=\alpha n^{1/v} \tag{6}$$

for any $v \in [1,\infty]$.

The following two lemmas can easily be proven.

Lemma 1. *Given two portfolios $x, x^0 \in X$, two market states $i, i' \in N_m$ and a fixed risk $k \in N_s$, the following statement is true for any $p, q \in [1, \infty]$:*

$$r_{ik}x - r_{i'k}x^0 \geq -\|R_k\|_{pq}\|(\|x\|_{p^*}, \|x^0\|_{p^*})\|_\nu,$$

where $R_k \in \mathbf{R}^{m\times n}$ is the k-th cut of matrix $R \in \mathbf{R}^{m\times n\times s}$ with rows $r_{1k}, r_{2k}, ..., r_{mk}$, $\nu = \min\{p^*, q^*\}$.

Proof. Let $i \neq i'$. Then, using Hölder's inequality (5), we get

$$r_{ik}x - r_{i'k}x^0 \geq -(\|r_{ik}\|_p\|x\|_{p^*} + \|r_{i'k}\|_p\|x\|_{p^*}) \geq$$

$$\geq \|(\|r_{ik}\|_p, \|r_{i'k}\|_p)\|_q \ \|(\|x\|_{p^*}, \|x^0\|_{p^*})\|_{q^*} \geq$$

$$\geq -\|R_k\|_{pq} \ \|(\|x\|_{p^*}, \|x^0\|_{p^*})\|_{q^*} \geq -\|R_k\|_{pq} \ \|(\|x\|_{p^*}, \|x^0\|_{p^*})\|_\nu.$$

For $i = i'$, using inequalities (4), and Hölder's inequality (5) we deduce

$$r_{ik}x - r_{i'k}x^0 \geq -\|r_{ik}\|_p \ \|x - x^0\|_{p^*} \geq -\|R_k\|_{pq} \ \|x - x^0\|_{p^*} \geq$$

$$\geq -\|R_k\|_{pq} \ \|(\|x\|_{p^*}, \|x^0\|_{p^*})\|_{q^*} \geq -\|R_k\|_{pq} \ \|(\|x\|_{p^*}, \|x^0\|_{p^*})\|_\nu.$$

□

From the definition of $G^s_m(R, I_1, I_2, \ldots, I_u)$, the following claim holds straightforward.

Lemma 2. *A portfolio $x \notin G^s_m(R, I_1, I_2, \ldots, I_u)$ if and only if $x \notin P(R_{I_v})$ for any index $v \in N_u$.*

4 Main result

For non-trivial problem $Z^{su}_m(R) = Z^{su}_m(R, I_1, I_2, \ldots, I_u)$, we introduce the following notation

$$\varphi = \varphi^{su}_m(p, q) = \min_{x\notin G^{su}_m(R)} \ \min_{v\in N_u} \ \max_{x'\in P(x,R_{I_v})} \ \min_{k\in I_v} \frac{g(x.x', R_k)}{\|(\|x\|_{p^*}, \|x'\|_{p^*})\|_\gamma},$$

$$\psi = \psi^{su}_m(p, q, t) = \max_{x'\in G^{su}_m(R)} \ \max_{v\in N_u} \ \min_{x\notin G^{su}_m(R)} \frac{\|[g(x, x', R_{I_v})]^+\|_t}{\|(\|x\|_{p^*}, \|x'\|_{p^*})\|_\gamma},$$

$$\chi = \chi_m^{su}(p,q,t) = n^{1/p} m^{1/q} s^{1/t} \min_{x \notin G_m^{su}(R)} \max_{v \in N_u} \max_{x' \in G_m^{su}(R)} \max_{k \in I_v} \frac{g(x,x',R_k)}{\|x - x'\|_1},$$

where

$$g(x,x',R_k) = f(x,R_k) - f(x',R_k), \quad k \in I_v,$$

$$g(x,x',R_{I_v}) = f(x,R_{I_v}) - f(x',R_{I_v}),$$

$$P(x,R_{I_v}) = P(R_{I_v}) \cap X(x,R_{I_v}),$$

$$\gamma = \min\{p^*, q^*\}.$$

Here $[y]^+ = (y_1^+, y_2^+, \dots, y_h^+)$ is a positive projection of vector $y = (y_1, y_2, \dots, y_h) \in \mathbf{R}^h$, i.e. $y_k^+ = \max\{0, y_k\}$, $k \in N_h$. It is easy to see that $\varphi,\ \psi,\ \chi \geq 0$.

Theorem 1. *Given $s, m \in \mathbf{N}$, $u \in N_s$ and $p, q, t \in [1, \infty]$, for the strong stability radius $\rho = \rho_m^{su}(p,q,t)$ of s-criteria non-trivial problem $Z_m^{su}(R)$, the following bounds are valid:*

$$0 < \max\{\varphi_m^{su}(p,q),\ \psi_m^{su}(p,q,t)\} \leq \rho_s^m(p,q,t) \leq \min\{\chi_m^{su}(p,q,t),\ \|R\|_{pqt}\}.$$

Proof. Since

$$\forall x' \in G_m^{su}(R) \quad \forall x \notin G_m^{su}(R) \quad \exists v \in N_u \quad (f(x,R_{I_v}) \succ f(x',R_{I_v})),$$

the inequalities $\psi, \chi > 0$ are evident.

Now we show that

$$\rho = \rho_m^{su}(p,q,t) \geq \varphi_m^{su}(p,q) = \varphi.$$

If $\varphi = 0$, the inequality above is evident, so we assume $\varphi > 0$.

Let the perturbing matrix $R' = [r'_{ijk}] \in \mathbf{R}^{m \times n \times s}$ with cuts R'_k, $k \in N_s$, be taken from the set $\Omega_{pqt}(\varphi)$. According to the definition of the number φ, and due to inequality (4), we obtain

$$\forall v \in N_u \quad \forall x \notin G_m^{su}(R) \quad \exists x^0 \in P(x,R_{I_v}) \quad \forall k \in I_v$$

$$\left(\frac{g(x,x^0,R_k)}{\|(\|x\|_{p^*}, \|x^0\|_{p^*})\|_\gamma} \geq \varphi > \|R'\|_{pqt} \geq \|R'_k\|_{pq} \right).$$

Thus, due to Lemma 1, for any criterion $v \in N_u$ there exists a portfolio $x^0 \neq x$ such that

$$g(x,x^0,R_k + R'_k) = f(x,R_k + R'_k) - f(x^0,R_k + R'_k) =$$

$$= \max_{i \in N_m} (r_{ik} + r'_{ik})x - \max_{i \in N_m} (r_{ik} + r'_{ik})x^0 =$$

$$= \min_{i \in N_m} \max_{i' \in N_m} (r_{ik}x + r'_{ik}x - r_{i'k}x^0 - r'_{i'k}x^0) \geq$$

$$\geq f(x,R_k) - f(x^0,R_k) - \|R'_k\|_{pq}\ \|(\|x\|_{p^*}, \|x^0\|_{p^*})\|_\gamma =$$

$$= g(x, x^0, R_k) - \|R'_k\|_{pq} \ \|(\|x\|_{p^*}, \|x^0\|_{p^*})\|_\gamma > 0, \ k \in I_v,$$

where r'_{ik} is the i-th row of the k-th cut R'_k of the matrix R'. This implies

$$x \notin P(R_{I_v} + R'_{I_v}), \ v \in N_u.$$

Therefore according to Lemma 2, we obtain that

$$x \notin G^{su}_m(R + R').$$

Summarizing and taking into account that $x \notin G^{su}_m(R)$, we conclude that for any perturbing matrix $R' \in \Omega_{pqt}(\varphi)$, any portfolio $x \in G^{su}_m(R+R')$ is also an element of $G^{su}_m(R)$, i.e. inequality $\rho \geq \varphi$ is true.

Further, we prove the lower bound

$$\rho = \rho^{su}_m(p, q, t) \geq \psi^{su}_m(p, q, t) = \psi.$$

We already know that $\psi > 0$. Therefore in order to prove $\rho \geq \psi$, it suffices to show that there exists a portfolio x^* belonging to $G^{su}_m(R) \cap G^{su}_m(R+R')$ for any perturbing matrix $R' = [r'_{ijk}] \in \Omega_{pqt}(\psi)$.

Since the problem $Z^{su}_m(R)$ is non-trivial, according to the definition of ψ, we have

$$\exists x^0 \in G^{su}_m(R) \quad \exists w \in N_u \quad \forall x \notin G^{su}_m(R)$$

$$\left(\|[g(x, x^0, R_{I_w})]^+\|_t \geq \psi \|(\|x\|_{p*}, \|x^0\|_{p*})\|_\gamma > 0\right). \tag{7}$$

Further we show that the formula

$$\forall x \notin G^{su}_m(R) \quad \forall R' \in \Omega_{pqt}(\psi) \quad (x \notin X(x^0, R_{I_w} + R'_{I_w})) \tag{8}$$

holds.

We prove this by contradiction. Assume the opposite, i.e. that formula

$$\exists \tilde{x} \notin G^{su}_m(R) \quad \exists \tilde{R} \in \Omega_{pqt}(\psi) \quad (\tilde{x} \in X(x^0, R_{I_w} + \tilde{R}_{I_w}))$$

holds. Then we get

$$f(\tilde{x}, R_{I_w} + \tilde{R}_{I_w}) \prec f(x^0, R_{I_w} + \tilde{R}_{I_w}).$$

Using Lemma 1 for any index $k \in I_w$, we obtain

$$0 \geq g(\tilde{x}, x^0, R_k + \tilde{R}_k) = f(\tilde{x}, R_k + \tilde{R}_k) - f(x^0, R_k + \tilde{R}_k) =$$

$$= \max_{i \in N_m} (r_{ik} + \tilde{r}_{ik})\tilde{x} - \max_{i \in N_m} (r_{ik} + \tilde{r}_{ik})x^0 =$$

$$= \min_{i \in N_m} \ \max_{i' \in N_m} \ (r_{ik}\tilde{x} - r_{i'k}x^0 + \tilde{r}_{ik}\tilde{x} - \tilde{r}_{i'k}x^0) \geq$$

$$\geq g(\tilde{x}, x^0, R_k) - \|\tilde{R}_k\|_{pq} \ \|(\|\tilde{x}\|_{p^*}, \|x^0\|_{p^*})\|_\gamma.$$

Therefore, we get

$$g(\tilde{x}, x^0, R_k) \le \|\tilde{R}_k\|_{pq}\ \|(\|\tilde{x}\|_{p^*}, \|x^0\|_{p^*})\|_\gamma,\ k \in I_w.$$

Then we continue

$$[g(\tilde{x}, x^0, R_k)]^+ \le \|\tilde{R}_k\|_{pq}\ \|(\|\tilde{x}\|_{p^*}, \|x^0\|_{p^*})\|_\gamma,\ k \in I_w.$$

As a result we get a formula contradicting (7)

$$\|[g(\tilde{x}, x^0, R_{I_w})]^+\|_t \le \|\tilde{R}_{I_w}\|_{pqt}\ \|(\|\tilde{x}\|_{p^*}, \|x^0\|_{p^*})\|_\gamma \le$$

$$\le \|\tilde{R}\|_{pqt}\ \|(\|\tilde{x}\|_{p^*}, \|x^0\|_{p^*})\|_\gamma < \psi\|(\|\tilde{x}\|_{p^*}, \|x^0\|_{p^*})\|_\gamma.$$

This confirms the validity of (8).

Further we show a way of selecting a portfolio $x^* \in G_m^{su}(R) \cap G_m^{su}(R+R')$ where $R' \in \Omega_{pqt}(\psi)$. If $x^0 \in G_m^{su}(R+R')$, then we get $x^* = x^0$. If $x^0 \notin G_m^{su}(R+R')$, then due to Lemma 2 we obtain $x^0 \notin P(R_{I_v} + R'_{I_v})$ for any $v \in N_u$, and in particular for a fixed $w \in N_u$ we have $x^0 \notin P(R_{J_w} + R'_{I_w})$. Then due to external stability (see [16]) of the Pareto set $P(R_{J_w} + R'_{I_w})$, one can chose a portfolio $x^* \in P(R_{J_w} + R'_{I_w})$ (and hence $x^* \in G_m^{su}(R+R')$) such that $x^* \in X(x^0, R_{I_w} + R'_{I_w})$. Taking into account (8), it is easy to see that $x^* \in G_m^{su}(R)$. Thus, we just have $\rho \ge \psi$ proven.

Further, we prove the upper bound

$$\rho = \rho_m^{su}(p,q,t) \le \chi_m^{su}(p,q,t) = \chi.$$

According to the definition of χ and due to assumption about problem's non-triviality, we have

$$\exists x^0 = (x_1^0, x_2^0, ..., x_n^0)^T \notin G_m^{su}(R) \quad \forall v \in N_u \quad \forall x \in G_m^{su}(R) \quad \forall k \in I_v$$

$$\left(\chi\|x^0 - x\|_1 \ge n^{1/p} m^{1/q} s^{1/t} g(x^0, x, R_k)\right). \tag{9}$$

Let $\varepsilon > \chi$, and let the elements of perturbing matrix $R^0 = [r_{ijk}^0] \in \mathbf{R}^{m\times n\times s}$ be defined as:

$$r_{ijk}^0 = \begin{cases} -\delta & \text{if } i \in N_m,\ \ x_j^0 = 1,\ \ k \in N_s, \\ \delta & \text{if } i \in N_m,\ \ x_j^0 = 0,\ \ k \in N_s, \end{cases}$$

where δ satisfies

$$\chi < \delta n^{1/p} m^{1/q} s^{1/t} < \varepsilon. \tag{10}$$

From the above according to (6), we get

$$\|r_{ik}^0\|_p = \delta n^{1/p}, \quad i \in N_m, \quad k \in N_s,$$

$$\|R_k^0\|_{pq} = \delta n^{1/p} m^{1/q}, \quad k \in N_s,$$

$$\|R^0\|_{pqt} = \delta n^{1/p} m^{1/q} s^{1/t},$$

$$R^0 \in \Omega_{pqt}(\varepsilon).$$

In addition, all the rows $r^0_{ik},\ i \in N_m$, of any k-th cut $R^0_k,\ k \in N_s$, are constructed identically and composed of δ and $-\delta$. So, setting $c = r^0_{ik},\ i \in N_m,\ k \in N_s$, we deduce

$$c(x^0 - x) = -\delta\|x^0 - x\|_1 < 0$$

that is true for any portfolio $x \neq x^0$. Using (9) and (10), we conclude that for any portfolio $x \in G^{su}_m(R)$ and any $v \in N_u$, the following statements are true:

$$g(x^0, x, R_k + R^0_k) = f(x^0, R_k + R^0_k) - f(x, R_k + R^0_k) =$$

$$= \max_{i \in N_m}(r_{ik} + c)x^0 - \max_{i \in N_m}(r_{ik} + c)x = \max_{i \in N_m} r_{ik}x^0 - \max_{i \in N_m} r_{ik}x + c(x^0 - x) =$$

$$= g(x^0, x, R_k) + c(x^0 - x) \leq \left(\chi(n^{1/p}m^{1/q}s^{1/t})^{-1} - \delta\right)\|x^0 - x\|_1 < 0,\ k \in I_v.$$

This implies $x \notin P(R_{I_v} + R'_{I_v})$ for any $v \in N_u$. Then due to Lemma 2 we have $x \notin G^{su}_m(R + R^0)$. Thus, for any $\varepsilon > \chi$ there exists a perturbing matrix $R^0 \in \Omega_{pqt}(\varepsilon)$ such that $G^{su}_m(R) \cap G^{su}_m(R + R^0) = \emptyset$, i.e. $\rho < \varepsilon$ for any $\varepsilon > \chi$. Hence, $\rho \leq \chi$.

Finally, we show

$$\rho = \rho^{su}_m(p, q, t) \leq \|R\|_{pqt}.$$

Let $x^0 = (x^0_1, x^0_2, ..., x^0_n)^T \notin G^{su}_m(R)$ and $\varepsilon > \|R\|_{pqt}$, and let us fix δ satisfying condition

$$0 < \delta n^{1/p}m^{1/q}s^{1/t} < \varepsilon - \|R\|_{pqt}. \tag{11}$$

We introduce an auxiliary matrix $V = [v_{ijk}] \in \mathbf{R}^{m \times n \times s}$ with cuts $V_k,\ k \in N_s$, defined as follows:

$$v_{ijk} = \begin{cases} -\delta & \text{if} \quad i \in N_m, \quad x^0_j = 1, \quad k \in N_s, \\ \delta & \text{if} \quad i \in N_m, \quad x^0_j = 0, \quad k \in N_s. \end{cases}$$

Using (6), we obtain

$$\|V_k\|_{pq} = \delta n^{1/p}m^{1/q},\ k \in N_s,$$

$$\|V\|_{pqt} = \delta n^{1/p}m^{1/q}s^{1/t}. \tag{12}$$

It is easy to see that all rows of $V_k,\ k \in N_s$, are identical and composed of δ and $-\delta$. So, we get that for any $v \in N_u$ the following formula

$$f(x^0, V_k) - f(x, V_k) = -\delta\|x^0 - x\|_1 < 0,\ k \in I_v, \tag{13}$$

is true for any $x \neq x^0$, and in particular for $x \in G^{su}_m(R)$.

Further, let $R^0 \in \mathbf{R}^{m \times n \times s}$ be a perturbing matrix with cuts $R^0_k,\ k \in N_s$, defined as:

$$R^0_k = V_k - R_k,\ k \in N_s, \tag{14}$$

i.e. $R^0 = V - R$. Using (11) and (12), we deduce

$$\|R^0\|_{pqt} \leq \|V\|_{pqt} + \|R\|_{pqt} = \delta n^{1/p}m^{1/q}s^{1/t} + \|R\|_{pqt} < \varepsilon,$$

i.e. $R^0 \in \Omega_{pqt}(\varepsilon)$.

Additionally, using (13) and (14) for any index $v \in N_u$, we have

$$g(x^0, x, R_k + R_k^0) = f(x^0, R_k + R_k^0) - f(x, R_k + R_k^0) =$$

$$= f(x^0, V_k) - f(x, V_k) = -\delta\|x^0 - x\|_1 < 0,\ k \in I_v,$$

i.e. $x \notin P(R_{I_v} + R_{I_v}^0)$ for any $v \in N_u$. Therefore, due to Lemma 2 $x \notin G_m^{su}(R + R^0)$. Summarizing, we get

$$\forall \varepsilon > \|R\|_{pqt} \quad \exists R^0 \in \Omega_{pqt}(\varepsilon) \quad \left(G_m^{su}(R) \cap G_m^{su}(R + R^0) = \emptyset\right).$$

The last implies $\rho \le \|R\|_{pqt}$. □

5 Corollaries

From theorem 1 we obtain a series of known results. For the completeness of description we list most interesting of them below. The first corollary describes strong stability bounds for an extreme case $u = 1$ where the set of efficient portfolios $G_m^s(R, N_s)$ transforms into the set of Pareto efficient portfolios $P_m^s(R)$.

Corollary 1. *[8] For $s, m \in \mathbf{N}$ and $p, q, t \in [1, \infty]$, the strong stability radius $\rho_m^{s1}(p, q, t)$ of s-criteria non-trivial problem $Z_m^s(R, N_s)$ of finding the set of Pareto efficient portfolios $P_m^s(R)$ has the following valid lower and upper bounds:*

$$0 < \max\{\varphi_m^{s1}(p, q), \psi_m^{s1}(p, q, t)\} \le \rho_m^{s1}(p, q, t) \le \min\{\chi_m^{s1}(p, q, t), \|R\|_{pqt}\},$$

where

$$\varphi_m^{s1}(p, q) = \min_{x \notin P_m^s(R)} \max_{x' \in P(x, R)} \min_{k \in N_s} \frac{g(x, x', R_k)}{\|(\|x\|_{p^*}, \|x'\|_{p^*})\|_\gamma},$$

$$\psi_m^{s1}(p, q, t) = \max_{x' \in P_m^s(R)} \min_{x \notin P_m^s(R)} \frac{\|[g(x, x', R_k)]^+\|_t}{\|(\|x\|_{p^*}, \|x'\|_{p^*})\|_\gamma},$$

$$\chi_m^{s1}(p, q, t) = n^{1/p} m^{1/q} s^{1/t} \min_{x \notin P_m^s(R)} \max_{x' \in P_m^s(R)} \max_{k \in N_s} \frac{g(x, x', R_k)}{\|x - x'\|_1}.$$

Therefore, in particular case where $p = q = t = \infty$, we have

$$0 < \max_{x' \in P_m^s(R)} \min_{x \notin P_m^s(R)} \max_{k \in N_s} \frac{g(x, x', R_k)}{\|x + x'\|_1} \le \rho_m^{s1}(\infty, \infty, \infty) \le$$

$$\le \min_{x \notin P_m^s(R)} \max_{x' \in P_m^s(R)} \max_{k \in N_s} \frac{g(x, x', R_k)}{\|x - x'\|_1}.$$

The second corollary describes strong stability bounds for another extreme case $u = s$ where the set of efficient portfolios $G_m^s(R, \{1\}, \{2\}, \dots, \{s\})$ transforms into the set of extreme portfolios $E_m^s(R)$.

Corollary 2. *[20] For $s, m \in \mathbf{N}$ and $p, q, t \in [1, \infty]$, the strong stability radius $\rho_m^{ss}(p, q, t)$ of s-criteria non-trivial problem $Z_m^s(R, \{1\}, \{2\}, \ldots, \{s\})$ of finding the set of extreme portfolios $E_m^s(R)$ has the following valid lower and upper bounds:*

$$0 < \max\{\varphi_m^{ss}(p, q), \psi_m^{ss}(p, q)\} \leq \rho_m^{ss}(p, q, t) \leq \min\{\chi_m^{ss}(p, q, t), \|R\|_{pqt}\},$$

where

$$\varphi_m^{ss}(p, q) = \min_{x \notin E_m^s(R)} \min_{k \in N_s} \max_{x' \in E(R_k)} \frac{g(x, x', R_k)}{\|(\|x\|_{p^*}, \|x'\|_{p^*})\|_\gamma},$$

$$\psi_m^{ss}(p, q) = \max_{x' \in E_m^s(R)} \max_{k \in N_s} \min_{x \notin E_m^s(R)} \frac{g(x, x', R_k)}{\|(\|x\|_{p^*}, \|x'\|_{p^*})\|_\gamma},$$

$$\chi_m^{ss}(p, q, t) = n^{1/p} m^{1/q} s^{1/t} \min_{x \notin E_m^s(R)} \max_{k \in N_s} \max_{x' \in E_m^s(R)} \frac{g(x, x', R_k)}{\|x - x'\|_1}.$$

Therefore, in particular case where $p = q = t = \infty$, we have

$$0 < \min_{x \notin E_m^s(R)} \min_{k \in N_s} \max_{x' \in E(R_k)} \frac{g(x, x', R_k)}{\|x + x'\|_1} \leq \rho_m^{ss}(\infty, \infty, \infty) \leq$$

$$\leq \min_{x \notin E_m^s(R)} \max_{k \in N_s} \max_{x' \in E_m^s(R)} \frac{g(x, x', R_k)}{\|x - x'\|_1}.$$

6 Conclusion

As a summary, it is worth mentioning that the bounds proven in Theorem 1 and corollaries, are mostly theoretical due to their analytical and enumerative structures. Even for a single objective, the difficulty of stability radius exact value calculation is a long-standing challenge pointed out in [13, 14]. In practical applications, one can try to get reasonable approximation of the bounds using some meta-heuristics, e.g. evolutionary algorithms or Monte-Carlo simulation. Another possibility to continue research in this direction is to specify some particular classes of problems where computational burden can be drastically reduced due to a unique structure of the set of efficient portfolios.

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Second order state-dependent sweeping process with unbounded perturbation

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Abstract. We establish, in the setting of an infinite dimensional Hilbert space, results concerning the existence of solutions of second order "nonconvex sweeping process" for a class of uniformly prox-regular sets depending on time and state. The perturbation considered here is general and takes the form of a sum of a single-valued Carathéodory mapping and a set-valued unbounded mapping. We deal also with a delayed perturbation, that is the external forces applied on the system in presence of a finite delay. We extend a discretization approach known for the time-dependent case to the time and state-dependent sweeping process.

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1 Introduction

The second order perturbed state-dependent nonconvex sweeping process has been a particular attraction for many authors during the last years, it takes the following form: let H be a Hilbert space, T_0 and T be two non-negative real numbers with $0 \leq T_0 < T$, and $D(t,x)$ be a nonempty closed subset of H for each $t \in [T_0,T]$ and $x \in H$. Given $b \in H$ and $a \in D(T_0,b)$, we have to find two absolutely continuous mappings $u, v : [T_0,T]$ satisfying

$$(P_F)\begin{cases} -\dot{u}(t) \in N_{D(t,v(t))}(u(t)) + F(t,v(t),u(t)), \quad \text{a.e. } t \in [T_0,T] \\ v(t) = b + \displaystyle\int_{T_0}^{t} u(s)ds, \quad u(t) = a + \displaystyle\int_{T_0}^{t} \dot{u}(s)ds, \ \forall t \in [T_0,T], \\ \qquad\qquad u(t) \in D(t,v(t)), \ \forall t \in [T_0,T], \end{cases}$$

where $N_{D(t,v(t))}(u(t))$ denotes the normal cone to $D(t,v(t))$ at the point $u(t)$, $F : [T_0,T] \times H \times H \rightrightarrows H$ is a set-valued mapping. Such problem is an extension of the so-called Moreau's sweeping process for Lagrangian system to frictionless unilateral constraints. The differential inclusion (P_F) was studied for the first time when the sets $D(t,v(t))$ are convex and compact and $F \equiv 0$ by [9], then by [17] and [21]. The nonconvex case has been considered by [16], the authors proved the existence of solutions to (P_F) for uniformly prox-regular sets $D(t,v(t))$ with absolutely continuous variation in space and Lipschitz variation in time and with a single-valued perturbation. By means of a generalized version of the Shauder's theorem, [12] provided another approach to prove the existence for uniformly prox regular and ball-compact sets $D(t, v(t))$ with absolutely continuous variation in time, without perturbation and for the perturbed problem (even in presence of a delay). The existence of solution for such problem is established by proving the convergence of the Moreau's catching-up algorithm. For other approaches, we refer to [1–6, 11, 24, 25].

Our main purpose in this paper is to study, in an infinite dimensional Hilbert space, the second order sweeping process with two perturbations

$$(\mathcal{P})\begin{cases} -\dot{u}(t) \in N_{D(t,v(t))}(u(t)) + F(t, v(t), u(t)) + f(t, v(t), u(t)), \text{ a. e. } t \in [T_0, T]; \\ v(t) = b + \displaystyle\int_{T_0}^{t} u(s)ds; \;\; u(t) = a + \int_{T_0}^{t} \dot{u}(s)ds, \; \forall t \in [T_0, T]; \\ u(t) \in D(t, v(t)), \; \forall t \in [T_0, T], \end{cases}$$

where $F : [T_0, T] \times H \times H \rightharpoonup H$ is an upper semicontinuous set-valued map with nonempty closed convex values unnecessarily bounded and without any compactness condition and $f : [T_0, T] \times H \times H \to H$ is a Carathéodory mapping satisfying the linear growth condition. This work is motivated by the recent results obtained for the same problem by [20] and [22], where reduction approaches have been used. In [20], only a single-valued "Lipschitz" perturbation is considered, the authors reduced the problem for second order time and state-dependent sweeping process to a first order time-dependent one. They make use of the Shauder's fixed point argument in the line of the approach of [16]. Whereas the reduction approach of [22] is valid only in finite dimensional setting. Our aim in this paper is to generalize all the results obtained in the two cases, using a different approach, we weaken the hypotheses on the perturbation by taking a Carathéodory mapping satisfying a linear growth condition and an unbounded set-valued perturbation for which only the element of minimum norm satisfies a linear growth condition.

On the other hand, we extend another reduction approach, known for the time-dependent sweeping process in presence of delay; it consists to reduce a second order sweeping process with delayed perturbation to a problem without delay. We show that this approach is still valid in the case of time and state-dependent sweeping process. The paper is organized as follows. In Section 2, we recall some basic notations, definitions and useful results which are used throughout the paper. In Section 3, we provide the existence results for the problem $(\mathcal{P})$. The delayed problem is studied in the last section.

2 Notation and Preliminaries

We begin with some notations used in the paper. Let H be a real separable Hilbert space whose inner product is denoted by $\langle \cdot, \cdot \rangle$, and the associated norm by $\| \cdot \|$. We denote by $\overline{\mathbf{B}}_H$ the unit closed ball of H, $\mathcal{L}([T_0, T])$ the σ-algebra of Lebesgue measurable subsets of $[T_0, T]$ and by $\mathcal{B}(H)$ the Borel tribe on H. We denote also by $L^1_H([T_0, T])$ the space of all Lebesgue-Bochner integrable H-valued mappings defined on $[T_0, T]$, by $\mathcal{C}_H([T_0, T])$ the Banach space of all continuous mappings $u : [T_0, T] \to H$ endowed with the norm of uniform convergence.

For any nonempty closed subset S, S' of H, we denote by:
• $d(\cdot, S)$ the usual distance function associated with S;
• $\delta^*(x', S) = \sup\limits_{y\in S}\langle x', y\rangle$ the support function of S at $x' \in H$. If S is closed convex subset $d(x,S) = \sup\limits_{x'\in\overline{\mathbf{B}}_H} \big(\langle x', x\rangle - \delta^*(x', S)\big)$;
• $\mathrm{Proj}_S(u)$ the projection of u onto S defined by

$$Proj_S(u) = \{y \in S : \quad d(u,S) = \|u - y\|\},$$

is unique whenever S is closed convex;
• $\mathcal{H}$ the Hausdorff distance between S and S', defined by

$$\mathcal{H}(S,S') = \max\{\sup_{u\in S} d(u,S'), \sup_{v\in S'} d(v,S)\};$$

• $co(S)$ the convex hull of S and $\overline{co}(S)$ its closed convex hull, characterized by

$$\overline{co}(S) = \{x \in H : \ \forall x' \in H, \ \langle x', x\rangle \le \delta^*(x', S)\}.$$

Recall that $f : [T_0, T] \times H \to H$ is called a Carathéodory mapping if $f(\cdot, u)$ is measurable on $[T_0, T]$ for all $u \in H$ and $f(t,\cdot)$ is continuous on H for every $t \in [T_0, T]$. A set-valued mapping $G : H \to H$ is called :
• upper semicontinuous if, for any open subset $\mathcal{V} \subset H$, the set $\{x \in H : \ G(x) \subset \mathcal{V}\}$ is open in H;
• scalarly upper semicontinuous on H if for every $h \in H$, $\delta^*(h, G(\cdot))$ is upper semicontinuous on H.

We need in the sequel to recall some definitions and results that will be used throughout the paper. Let A be an open subset of H and $\varphi : A \to (-\infty, +\infty]$ be a lower semicontinuous function, the proximal subdifferential $\partial^P\varphi(x)$, of φ at x (see [19]) is the set of all proximal subgradients of φ at x, any $\xi \in H$ is a proximal subgradient of φ at x if there exist positive numbers η and ς such that

$$\varphi(y) - \varphi(x) + \eta\|y - x\|^2 \ge \langle \xi, y - x\rangle, \ \forall y \in x + \varsigma\overline{\mathbf{B}}_H.$$

Let x be a point of $S \subset H$, we recall (see [19]) that the proximal normal cone to S at x is defined by $N_S^P(x) = \partial^P\Psi_S(x)$, where Ψ_S denotes the indicator function of S, i.e. $\Psi_S(x) = 0$ if $x \in S$ and $+\infty$ otherwise. Note that the proximal normal cone is also given by

$$N_S^P(x) = \{\xi \in H : \ \exists \varrho > 0 \ \text{ s.t. } \ x \in \mathrm{Proj}_S(x + \varrho\xi)\}.$$

When S is a closed set one has $\partial^P d(x,S) = N_S^P(x) \cap \overline{\mathbf{B}}_H$.
If φ is a real-valued locally-Lipschitz function defined on H, the Clarke subdifferential $\partial^C\varphi(x)$ of φ at x is the nonempty convex compact subset of H given by

$$\partial^C\varphi(x) = \{\xi \in H : \ \varphi^\circ(x; v) \ge \langle \xi, v\rangle, \forall v \in H\},$$

where

$$\varphi^\circ(x;v) = \limsup_{y\to x,\ t\downarrow 0} \frac{\varphi(y+tv)-\varphi(y)}{t}$$

is the generalized directional derivative of φ at x in the direction v (see [19]). The Clarke normal cone $N_S^C(x)$ to S at $x \in S$ is defined by polarity with T_S^C, that is,

$$N_S^C(x) = \{\xi \in H : \langle \xi, v\rangle \leq 0,\ \forall v \in T_S^C\},$$

where T_S^C denotes the clarke tangent cone, and is given by

$$T_S^C = \{v \in H : d^\circ(x, S; v) = 0\}.$$

Recall now, that for a given $r \in]0,+\infty]$ the subset S is uniformly r-prox-regular (see [19]) or equivalently r-proximally smooth ([23]) if and only if for all $\overline{x} \in S$ and all $0 \neq \xi \in N_S^P(\overline{x})$ one has

$$\langle \frac{\xi}{\|\xi\|}, x - \overline{x}\rangle \leq \frac{1}{2r}\|x-\overline{x}\|^2,$$

for all $x \in S$. We make the convention $\frac{1}{r} = 0$ for $r = +\infty$. Recall that for $r = +\infty$ the uniform r-prox-regularity of S is equivalent to the convexity of S. It's well known that the class of uniformly r-prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of a Hilbert space and many other nonconvex sets (see [15, 20]). Furthermore, the following properties hold for a closed uniformly r-prox-regular set S:

- for any $N_S^P(x) = N_S^C(x) = N_S(x)$;
- the proximal subdifferential of $d(.,S)$ coincides with its Clarke subdifferential at all points $x \in H$ satisfying $d(x,S) < r$;
- for all $x \in H$ with $d(x,S) < r$, $\mathrm{Proj}_S(x)$ is a singleton of H.

The next proposition provides an upper semicontinuity property of the support function of the proximal subdifferential of the distance function to uniformly r-prox-regular sets.

Proposition 1. *Let $D : [T_0,T]\times H \rightharpoondown H$ be a uniformly r-prox regular closed valued mapping satisfying*

$$|d(u, D(t,x)) - d(v, D(s,y))| \leq ||u-v|| + v(t) - v(s) + L||x-y||$$

for all u,x,v,y in H and for all $s \leq t$ in $[T_0,T]$, where $v : [T_0,T] \to \mathbf{R}^+$ is a nondecreasing absolutely continuous function and L is a positive constant. Then the convex weakly compact valued mapping $(t,x,y) \to \partial^p d(y, D(t,x))$ satisfies the upper semicontinuity property: let (t_n,x_n) be a sequence in $[T_0,T]\times H$ converging to some $(t,x) \in [T_0,T]\times H$, and (y_n) be a sequence in H with $y_n \in D(t_n,x_n)$ for all n, converging to $y \in D(t,x)$, then, for any $z \in H$,

$$\limsup_{n\to\infty} \delta^*(z, \partial^p d(y_n, D(t_n,x_n))) \leq \delta^*(z, \partial^p d(y, D(t,x))).$$

3 Main results

The following assumption will be useful.

Assumption 1: Let $D : [T_0, T] \times H \to H$ be a set-valued mapping with nonempty closed and uniformly r-prox regular values such that:

$(\mathcal{A}_1)$ There is a positive constant L and a nondecreasing absolutely continuous function $\zeta : [T_0, T] \to \mathbf{R}_+$ such that, for all $s \le t$ in $[T_0, T]$ and $x_i, y_i \in H (i = 1, 2)$,

$$|d\big(x_1, D(t, y_1)\big) - d\big(x_2, D(s, y_2)\big)| \le \|x_1 - x_2\| + \zeta(t) - \zeta(s) + L\|y_1 - y_2\|;$$

$(\mathcal{A}_2)$ for all $(t, x) \in [T_0, T] \times H,\ D(t, x)$ is contained in a compact set Γ.

Let us start with an existence result for second order state-dependent sweeping process without perturbations, it will be used in the next theorem. The proof is a careful adaptation of Theorem 3.2 and 3.4 in [12]. Remark that, here the sets $D(t, u)$ are with absolutely continuous variation in time while in Theorem 3.2 of [12] the variation in time is Lipschitz.

Theorem 1. *Assume that Assumption 1 holds. Then, for every $b \in H$ and for every $a \in D(T_0, b)$, there exist two absolutely continuous mappings $u : [T_0, T] \to H$ and $v : [T_0, T] \to H$ satisfying*

$$\begin{cases} -\dot{u}(t) \in N_{D(t,v(t))}(u(t)), \quad a.e. \ t \in [T_0, T]; \\ v(t) = b + \displaystyle\int_{T_0}^{t} u(s)ds, \quad u(t) = a + \int_{T_0}^{t} \dot{u}(s)ds, \ \forall t \in [T_0, T]; \\ u(t) \in D(t, v(t)), \ \ \forall t \in [T_0, T], \end{cases}$$

with

$$\|\dot{u}(t)\| \le \dot{\zeta}(t)(1 + L\alpha) \quad a.e. \quad t \in [T_0, T].$$

Proof. By assumption $(\mathcal{A}_2)$, for some $\alpha > 0$ we have $D(t, x) \subset \Gamma \subset \alpha\overline{\mathbf{B}}_H$. Consider a partition of $[T_0, T]$ by the points $t_k^n = T_0 + ke_n,\ e_n = \frac{T - T_0}{n}, k \in \{0, 1, 2, ..., n\}$ and set

$$\sigma_k^n = \zeta(t_{k+1}^n) - \zeta(t_k^n)$$

and

$$\sigma^n = \max_{0 \le k \le n-1} \sigma_k^n.$$

As the sequences (σ^n) and (e_n) converge to 0, one can fix a positive integer n_0 such that for any $n \ge n_0$

$$(\sigma^n + e_n)(1 + L\alpha) < r.$$

Construction of approximate solutions: For each $t \in [t_0^n, t_1^n]$, we define

$$v_n(t) = b + (t - t_0^n)a$$

$$u_n(t) = x_0^n + \frac{\zeta(t) - \zeta(t_0^n)}{\sigma_0^n + e_n}(x_1^n - x_0^n),$$

where $x_0^n = a \in D(T_0, b)$ and $x_1^n = \text{Proj}_{D(t_1^n, v_n(t_1^n))}(x_0^n)$. Despite the absence of the convexity of the images of D, the last equality is well defined. Indeed, we have

$$\begin{aligned}
d(x_0^n, D(t_1^n, v_n(t_1^n))) &= |d(x_0^n, D(t_0^n, v_n(t_0^n))) - d(x_0^n, D(t_1^n, v_n(t_1^n)))| \\
&\leq \zeta(t_1^n) - \zeta(t_0^n) + L\|v_n(t_1^n) - v_n(t_0^n)\| \\
&\leq \sigma_0^n + Le_n\|x_0^n\| \leq (\sigma^n + e_n)(1 + L\alpha) \leq r.
\end{aligned}$$

Hence $v_n(t_0^n) = b,\ u_n(t_0^n) = a$ and for $t \in]t_0^n, t_1^n[$, we have $\dot{v}_n(t) = a$ and

$$\dot{u}_n(t) = \dot{\zeta}(t)\frac{x_1^n - x_0^n}{\sigma_0^n + e_n} \in -N_{D(t_1^n, v_n(t_1^n))}(x_1^n),$$

with

$$\|\dot{u}_n(t)\| \leq \dot{\zeta}(t)(1 + L\alpha).$$

By induction, suppose that $(v_n), (u_n)$ are well defined on $]t_0^n, t_k^n]$ with $u_n(t_k^n) = x_k^n$ and $\|\dot{u}_n(t)\| \leq \dot{\zeta}(t)(1 + L\alpha)$. For each $t \in]t_k^n, t_{k+1}^n]$, we define

$$v_n(t) = v_n(t_k^n) + (t - t_k^n)u_n(t_k^n)$$

and

$$u_n(t) = x_k^n + \frac{\zeta(t) - \zeta(t_k^n)}{\sigma_k^n + e_n}(x_{k+1}^n - x_k^n),$$

where $x_{k+1}^n = \text{Proj}_{D(t_{k+1}^n, v_n(t_{k+1}^n))}(x_k^n)$ and $d(x_k^n, D(t_{k+1}^n, v_n(t_{k+1}^n))) \leq r$.
Then for $t \in]t_k^n, t_{k+1}^n]$, we have $\dot{v}_n(t) = u_n(t_k^n)$ and

$$\dot{u}_n(t) = \dot{\zeta}(t)\frac{x_{k+1}^n - x_k^n}{\sigma_n^k + e_n} \in -N_{D(t_{k+1}^n, v_n(t_{k+1}^n))}(x_{k+1}^n),$$

with

$$\|\dot{u}_n(t)\| \leq \dot{\zeta}(t)(1 + L\alpha) \quad \text{and} \quad \|\dot{v}_n(t)\| \leq \alpha.$$

Defining for each $t \in [T_0, T]$ and each $n \geq n_0$,

$$p_n(t) = \begin{cases} t_k^n & \text{if} \quad t \in [t_k^n, t_{k+1}^n[\\ T & \text{if} \quad t = T; \end{cases}$$

$$q_n(t) = \begin{cases} T_0 & \text{if} \quad t = T_0 \\ t_{k+1}^n & \text{if} \quad t \in]t_k^n, t_{k+1}^n], \end{cases}$$

we get

$$\dot{u}_n(t) \in -N_{D(q_n(t), v_n(q_n(t)))}(u_n(q_n(t))) \ \ a.e.\ \ [T_0, T];$$

$$u_n(q_n(t)) \in D(q_n(t), v_n(q_n(t)),\ \forall [T_0, T];$$

$$v_n(t) = b + \int_{T_0}^t u_n(p_n(s))ds,\ \forall [T_0, T];$$

$$\lim_{n\to\infty} p_n(t) = \lim_{n\to\infty} q_n(t) = t,\ \ \forall [T_0, T];$$

$$\|\dot{v}_n(t)\| = \|u_n(p_n)(t))\| = ||x_k^n|| \leq \alpha, \ \ \forall\, k \leq n, \ \forall t \in [T_0, T]$$

and

$$\|\dot{u}_n(t)\| \leq \dot{\zeta}(t)(1 + L\alpha) = \rho(t). \tag{1}$$

Thus

$$\lim_{n\to\infty} ||u_n(p_n(t)) - u_n(t)|| = 0. \tag{2}$$

Convergence of approximate sequences:
We have $u_n(p_n(t)) \in D(p_n(t), v_n(p_n(t))) \subset \Gamma$, so that; $u_n(p_n(t))$ is relatively compact for every $t \in [T_0, T]$ in H, so is $(u_n(t))$ thanks to (2). By (1), $(u_n(\cdot))$ is equicontinuous. Thus (u_n) is relatively compact in $\mathcal{C}_H([T_0, T])$, consequently (u_n) converges in $\mathcal{C}_H([T_0, T])$ to the absolutely continuous mapping u. By (1) again, $(\dot{u}_n)$ weakly converges in $L^1_H[T_0, T]$ to a function z with $\|z(t) \leq \rho(t)$ a.e. in $[T_0, T]$ (see Proposition 6.2.3 in [10]) and (u_n) converges pointwise on $[T_0, T]$ with respect to the weak topology to an absolutely continuous function u and

$$u(t) = a + \int_{T_0}^{t} \dot{u}(s)ds, \ \forall [T_0, T]$$

with $\dot{u} = z$. From the convergence of (u_n) we deduce that of (v_n) to an absolutely continuous function v with

$$v(t) = b + \int_{0}^{t} u(s)ds, \ \forall [T_0, T].$$

For the rest of the demonstration we can consult the proof of Theorem 2 below. $\square$

Now, we give the main result in this section.

Theorem 2. *Assume that Assumption 1 holds. Let $F : [T_0, T] \times H \times H \rightharpoonup H$ be a set-valued map with nonempty closed convex values such that:*

$(\mathcal{A}_{F_1})$ *F is $\mathcal{L}([T_0, T]) \otimes \mathcal{B}(H) \otimes \mathcal{B}(H)$-measurable and for all $t \in [T_0, T]$, $F(t, \cdot, \cdot)$ is scalarly upper semicontinuous on $H \times H$;*

$(\mathcal{A}_{F_2})$ *there exists a real $\beta > 0$, such that, for all $(t, u, v) \in [T_0, T] \times H \times H$,*

$$d(0, F(t, u, v)) \leq \beta(1 + \|u\| + \|v\|).$$

And let $f : [T_0, T] \times H \times H \to H$ be a Carathéodory mapping satisfies

$(\mathcal{A}_f)$ *there exists a non-negative function $\gamma \in \mathbf{L}^1_{\mathbf{R}^+}([T_0, T])$ such that, for all $t \in [T_0, T]$ and for all $(u, v) \in H \times H$,*

$$\|f(t, u, v)\| \leq \gamma(t)(1 + \|u\| + \|v\|).$$

Then, for any $a, b \in H$ with $a \in D(T_0, b)$, there exist two absolutely continuous mappings $u, v : [T_0, T] \to H$ satisfying $(\mathcal{P})$.

Proof. **Step 1.** We begin by a single-valued integrable mapping $m \in L^1_H([T_0,T])$. Put for all $t \in [T_0,T]$,

$$m_1(t) = \int_{T_0}^t m(s)ds \ \ and \ \ m_2(t) = \int_{T_0}^t m_1(s)ds$$

and consider the set-valued map $C : [T_0,T] \times H \rightharpoonup H$ defined by

$$C(t,z) = D(t, z - m_2(t)) + m_1(t) \quad \forall \quad (t,z) \in [T_0,T] \times H.$$

Obviously, C satisfies $(\mathcal{A}_2)$, let verify $(\mathcal{A}_1)$. For any $w_1,\ w_2,\ z_1,\ z_2$ in H and any $s \le t$ in $[T_0,T]$, we have

$$\begin{aligned} &|d(w_1, C(t,z_1)) - d(w_2, C(s,z_2))| \\ &= |d(w_1 - m_1(t), D(t, z_1 - m_2(t))) - d(w_2 - m_1(s), D(s, z_2 - m_2(s)))| \\ &\le \|w_1 - w_2\| + \|m_1(t) - m_1(s)\| + L\|m_2(t) - m_2(s)\| + \zeta(t) - \zeta(s) + L\|z_1 - z_2\| \\ &\le \|w_1 - w_2\| + \zeta_1(t) - \zeta_1(s) + L\|z_1 - z_2\| \end{aligned}$$

where

$$\zeta_1(t) = \int_{T_0}^t \Big(\dot{\zeta}(\omega) + \|m(\omega)\| + L\int_{T_0}^{\omega} \|m(\tau)\|d\tau\Big)d\omega$$

is an absolutely continuous nondecreasing mapping. Hence, C satisfies $(\mathcal{A}_1)$, as $a \in C(T_0,b) = D(T_0,b)$, from Theorem 1, there exist two absolutely continuous mappings $x : [T_0,T] \to H$ and $y : [T_0,T] \to H$ such that

$$\begin{cases} -\dot{y}(t) \in N_{C(t,x(t))}(y(t)), \ \text{ a.e. } \ t \in [T_0,T]; \\ x(t) = b + \displaystyle\int_{T_0}^t y(s)ds, \ \ y(t) = a + \int_{T_0}^t \dot{y}(s)ds, \ \forall t \in [T_0,T]; \\ y(t) \in C(t, x(t)), \ \forall t \in [T_0,T]. \end{cases}$$

Let $u(t) = y(t) - m_1(t)$ and $v(t) = x(t) - m_2(t)$, the mappings $u(\cdot)$ and $v(\cdot)$ satisfy

$$\begin{cases} -\dot{u}(t) \in N_{D(t,v(t))}(u(t)) + m(t), \ \text{ a.e. } \ t \in [T_0,T]; \\ v(t) = b + \displaystyle\int_{T_0}^t u(s)ds, \ \ u(t) = a + \int_{T_0}^t \dot{u}(s)ds, \ \forall t \in [T_0,T]; \\ u(t) \in D(t, v(t)), \ \forall t \in [T_0,T]. \end{cases}$$

with

$$\|\dot{u}(t)\| \le \Bigg(1 + L\alpha\Bigg)\Bigg(\dot{\zeta}(t) + 2\|m(t)\| + L\int_{T_0}^s \|m(\tau)\|d\tau\Bigg)ds.$$

Step 2. For each $(t,u,v) \in [T_0,T] \times H \times H$, let $P(t,x,y)$ be the element of minimal norm of the closed convex set $F(t,x,y)$ of H, that is

$$P(t,x,y) = \mathrm{Proj}_{F(t,x,y)}(0), \ \ \forall\ (t,u,v) \in [T_0,T] \times H \times H.$$

Since F is $\mathcal{L}([T_0,T]) \otimes \mathcal{B}(H) \otimes \mathcal{B}(H)$-measurable, so $P(\cdot,\cdot,\cdot) = d(0, F(\cdot,\cdot,\cdot))$, is measurable. In view of $(\mathcal{A}_{F_2})$

$$\|P(t,x,y)\| \leq \beta(1 + \|x\| + \|y\|). \tag{3}$$

We put

$$g(t,x,y) = f(t,x,y) + P(t,x,y)$$

and

$$\Lambda(t) = \gamma(t) + \beta,$$

by (3) and $(\mathcal{A}_f)$, we get for all $(t,u,v) \in [T_0,T] \times H \times H$,

$$\|g(t,x,y)\| \leq \Lambda(t)(1 + \|x\| + \|y\|). \tag{4}$$

Construction of sequences: Consider, for every $n \in \mathbf{N}$, a partition of $[T_0,T]$ defined by $t_i^n = T_0 + i\frac{T-T_0}{n}$ $(0 \leq i \leq n)$. We are going to construct a sequence of maps $(u_n(\cdot))$ and $(v_n(\cdot))$ via Step 1, by considering a perturbation g with fixed second and third variables in each subinterval $[t_i^n, t_{i+1}^n]$. So, for $a \in D(T_0,b)$, let us consider the following problem on the interval $[T_0, t_1^n]$:

$$(P_0) \begin{cases} -\dot{u}(t) \in N_{D(t,v(t))}(u(t)) + g(t,b,a) \text{ a.e. } t \in [T_0, t_1^n] \\ v(T_0) = a, \ u(T_0) = a \in D(T_0,b) \end{cases}$$

where $g(\cdot,b,a)$ is a mapping depending only on t and is $L^1_H([T_0,t_1^n])$. By Step 1, there are two absolutely continuous mappings that we denote by $u_0^n(.), v_0^n(.) : [T_0,t_1^n] \to H$ solutions of (P_0). Now, since $u_0^n(t_1^n) \in D(t_1^n, v_0^n(t_1^n))$ is well defined in the interval $[t_1^n, t_2^n]$ the problem

$$(P_1) \begin{cases} -\dot{u}_1^n(t) \in N_{D(t,v_1^n(t))}(u_1^n(t)) + g(t, v_0^n(t_1^n), u_0^n(t_1^n)) \text{ a.e. } t \in [t_1^n, t_2^n]; \\ u_0^n(t_1^n) \in D(t_1^n, v_0^n(t_1^n)). \end{cases}$$

admits an absolutely continuous solution $(u_1^n(\cdot), v_1^n(\cdot))$ with $u_1^n(t_1^n) = u_0^n(t_1^n)$ and $v_1^n(t_1^n) = v_0^n(t_1^n)$. By induction, for each n, there exist two finite sequence of absolutely continuous mappings $u_i^n(\cdot), v_i^n(\cdot) : [t_i^n, t_{i+1}^n] \to H$ with $u_i^n(t_i^n) = u_{i-1}^n(t_i^n)$ and $v_i^n(t_i^n) = v_{i-1}^n(t_i^n)$ such that, for each $i \in \{0, ..., n-1\}$,

$$(P_i) \begin{cases} -\dot{u}_i^n(t) \in N_{D(t,v_i^n(t))}(u_i^n(t)) + g(t, v_{i-1}^n(t_i^n), u_{i-1}^n(t_i^n)) \text{ a.e. } t \in [t_i^n, t_{i+1}^n]; \\ u_{i-1}^n(t_i^n) \in D(t_i^n, v_{i-1}^n(t_i^n)), \end{cases}$$

where $u_{-1}^n(T_0) = a, v_{-1}^n(T_0) = b$ and

$$\|\dot{u}(t)\| \leq \Big(1 + L\alpha\Big)\Big(\dot{\zeta}(t) + 2\|g(t, v_{i-1}^n(t_i^n), u_{i-1}^n(t_i^n))\|$$

$$+L\int_{t_i^n}^t \|g(\tau, v_{i-1}^n(t_i^n), u_{i-1}^n(t_i^n))\| d\tau\Big),$$

a.e. $t \in [t_i^n, t_{i+1}^n]$. We define the absolutely continuous mappings $u_n, v_n : [T_0, T] \to H$ by $u_n(t) = u_i^n(t)$ and $v_n(t) = v_i^n(t)$ for all $t \in [t_i^n, t_{i+1}^n],\ i \in \{0, \cdots, n\}$. One can write

$$\begin{cases} \dot{u}_n(t) \in -N_{D(t,v_n(t))}(u_n(t)) + g(t, v_n(p_n(t)), u_n(p_n(t))) \text{ a.e. } t \in [T_0, T]; \\ v_n(t) = b + \int_{T_0}^t u_n(s)ds,\ u_n(t) = a + \int_{T_0}^t \dot{u}_n(s)ds,\ \forall t \in [T_0, T]; \\ u_n(t) \in D(t, v_n(t)),\ \forall t \in [T_0, T],\ u_n(T_0) = a,\ v_n(T_0) = b, \end{cases}$$

with a.e. $t \in [T_0, T]$

$$\|\dot{u}_n(t)\| \leq \Big(1 + L\alpha\Big)\Big(\dot{\zeta}(t) + 2\|g(t, v_n(p_n(t)), u_n(p_n(t)))\|$$
$$+L\int_{p_n(t)}^t \|g(\tau, v_n(p_n(\tau)), u_n(p_n(\tau)))\|d\tau\Big).$$

Since for all $t \in [T_0, T],\ u_n(p_n(t)) \in D(p_n(t), v_n(p_n(t)))$, then

$$\|u_n(p_n(t))\| \leq \alpha \quad \text{and} \quad \|v_n(p_n(t))\| \leq \|b\| + (T - T_0)\alpha.$$

By (4), we get for almost every $t \in [T_0, T]$

$$\|g(t, v_n(p_n(t)), u_n(p_n(t)))\| = \Big(1 + \|b\| + (T + 1)\alpha\Big)\Lambda(t) = c_1(t). \tag{5}$$

Then

$$\|\dot{u}_n(t)\| \leq \Big(1 + L\alpha\Big)\Big(\dot{\zeta}(t) + \big(2 + L\int_{T_0}^T \Lambda(\tau)d\tau)(1 + \|b\| + (T + 1)\alpha\big)\Big) = c_2(t). \tag{6}$$

Convergence of sequences: Since for each $t,\ u_n(t) \in D(t, v_n(t)) \subset \Gamma$, for all $n \in \mathbf{N}$ such that $(u_n(t))$ is relatively compact in H for every $t \in [T_0, T]$. Using Ascoli-Arzelà theorem, (u_n) is relatively compact in $\mathcal{C}_H([T_0, T])$. Then there exists a subsequence again denoted by (u_n) which converges to a mapping u. According to (6), we may suppose that $(\dot{u}_n)$ weakly converges in $L_H^1([T_0, T])$ to a mapping z with $\|z(t)\| \leq c_2(t)$ a.e. in $[T_0, T]$. Thus

$$\lim_{n\to\infty} u_n(t) = a + \lim_{n\to\infty}\int_{T_0}^t \dot{u}_n(s)ds = a + \int_{T_0}^t z(s)ds,$$

then, $u(t) = a + \int_{T_0}^t z(s)ds$. Consequently, $u(t)$ is absolutely continuous with $\dot{u} = z$. Furthermore,

$$|p_n(t) - t| \leq |t_{k+1}^n - t_k^n| = \frac{T - T_0}{n},$$

so $\lim\limits_{n\to\infty} |p_n(t) - t| = 0$ and

$$||u_n(p_n(t)) - u_n(t)|| \leq \int_{p_n(t)}^t \|\dot{u}_n(s)\|ds \leq \int_{p_n(t)}^t c_2(s)ds,$$

since $c_2 \in L^1_{\mathbf{R}_+}([T_0, T])$, we get $\lim\limits_{n\to\infty} ||u_n(p_n(t)) - u_n(t)|| = 0$, so that

$$\lim_{n\to\infty} ||u_n(p_n(t)) - u(t)|| \leq \lim_{n\to\infty} \Big(||u_n(p_n(t)) - u_n(t)|| + ||u_n(t) - u(t)|| \Big) = 0.$$

The convergence of the sequence $(u_n(p_n(\cdot))$ to $(u(\cdot))$ is obtained.
From the convergence of $(u_n(\cdot))$ we deduce that of $(v_n(\cdot))$ to an absolutely continuous function $v(\cdot)$ with

$$v(t) = b + \int_{T_0}^{t} u(s)ds, \ \forall t \in [T_0, T]$$

and

$$\lim_{n\to\infty} ||v_n(p_n(t)) - v_n(t)|| = 0.$$

Let us set for all $t \in [T_0, T]$,

$$f(t, v_n(p_n(t)), u_n(p_n(t))) = l_n(t)$$

and

$$P(t, v_n(p_n(t)), u_n(p_n(t))) = \eta_n(\cdot).$$

By the continuity of the mapping $f(t, \cdot, \cdot)$ we get $l_n(t)$ converges to $l(t) = f(t, u(t), v(t))$ and

$$\|l(t)\| \leq \Big(1 + \|b\| + (T+1)\alpha\Big)\gamma(t).$$

On the other hand, for all $n \geq n_0$ and for all $t \in [T_0, T]$, we have

$$\|\eta_n(t)\| \leq \|\Big(1 + \|b\| + (T+1)\alpha\Big)\beta,$$

so $(\eta_n(\cdot))$ is bounded, taking a subsequence if necessary, we may conclude that $(\eta_n(\cdot))$ weakly converges to some mapping $\eta \in L^1_H([T_0, T])$ with

$$\|\eta(t)\| \leq \Big(1 + \|b\| + (T+1)\alpha\Big)\beta.$$

Now, we proceed to prove that

$$\dot{u}(t) \in -N_{D(t,v(t))}(u(t)) + F(t, v(t), u(t)) + f(t, v(t), u(t)) \ a.e.\ t \in [T_0, T].$$

First, we check that $u(t) \in D(t, v(t))$. For every $t \in [T_0, T]$ and for every n, we have

$$\begin{aligned} d(u_n(t), D(t, v(t))) &\leq ||u_n(t) - u_n(p_n(t))|| + d(u_n(p_n(t)), D(t, v(t))) \\ &\leq ||u_n(t) - u_n(p_n(t))|| + \mathcal{H}(D(p_n(t), v_n(p_n(t))), D(t, v(t))) \\ &\leq ||u_n(t) - u_n(p_n(t))|| + |\zeta(t) - \zeta(p_n(t))| + L||v_n(p_n(t)) - v_n(t)||, \end{aligned}$$

Passing to the limit when $n \to \infty$, in the preceding inequality, we get $u(t) \in D(t, v(t))$. According to (5) and (6), we obtain

$$\| -\dot{u}_n(t) + l_n(t) + \eta_n(t)\| \le c_1(t) + c_2(t) := \lambda(t),$$

so

$$-\dot{u}_n(t) + l_n(t) + \eta_n(t) \in \lambda(t)\overline{\mathbf{B}}_H$$

since

$$-\dot{u}_n(t) + l_n(t) + \eta_n(t) \in N_{D(t,v_n(t))}(u_n(t)),$$

we get

$$-\dot{u}_n(t) + l_n(t) + \eta_n(t) \in \lambda(t)\partial d(u_n(t), D(t, v_n(t))).$$

Remark that $(-\dot{u}_n + l_n + \eta_n, \eta_n)$ weakly converges in $L^1_{H\times H}([T_0, T])$ to $(-\dot{u} + l + \eta, \eta)$. An application of the Mazur's Theorem to $(-\dot{u}_n + l_n + \eta_n, \eta_n)$ provides a sequence (w_n, ζ_n) with

$$w_n \in co\{-\dot{u}_m + l_m + \eta_m : \ m \ge n\} \quad \text{and} \quad \zeta_n \in co\{\eta_m : \ m \ge n\}$$

such that (w_n, ζ_n) converges strongly in $L^1_{H\times H}([0, T])$ to $(-\dot{u} + l + \eta, \eta)$. We can extract from (w_n, ζ_n) a subsequence which converges a.e. to $(-\dot{u} + l + \eta, \eta)$. Then, there is a Lebesgue negligible set $S \subset [0, T]$ such that for every $t \in [0, T] \setminus S$

$$\begin{aligned} -\dot{u}(t) + l(t) + \eta(t) &\in \bigcap_{n\ge 0} \overline{\{w_m(t) : \ m \ge n\}} \\ &\subset \bigcap_{n\ge 0} \overline{co}\{-\dot{u}_m(t) + l_m(t) + \eta_m(t) : \ m \ge n\}, \end{aligned} \tag{7}$$

$$\eta(t) \in \bigcap_{n\ge 0} \overline{\{\zeta_m(t) : \ m \ge n\}} \subset \bigcap_{n\ge 0} \overline{co}\{\eta_m(t) : \ m \ge n\}. \tag{8}$$

Fix any $t \in [0, T] \setminus S, n \ge n_0$ and $\mu \in H$, then the relation (7) gives

$$\begin{aligned} \langle \mu, -\dot{u}(t) + l(t) + \eta(t)\rangle &\le \limsup_{n\to\infty} \delta^*(\mu, \lambda(t)\partial d(u_n(t), D(t, v_n(t)))) \\ &\le \delta^*(\mu, \lambda(t)\partial d(u(t), D(t, v(t)))), \end{aligned}$$

where the first inequality follows from the characterization of convex hull and the second one follows from Proposition 1. Taking the supremum over $\mu \in H$, we deduce that

$$\delta(-\dot{u}(t) + l(t) + \eta(t), \lambda(t)\partial d(u(t), D(t, v(t)))) =$$

$$\delta^{**}(-\dot{u}(t) + l(t) + \eta(t), \lambda(t)\partial d(u(t), D(t, v(t)))) \le 0$$

which entails

$$-\dot{u}(t) + l(t) + \eta(t) \in \lambda(t)\partial d(u(t), D(t, v(t))) \subset N_{D(t,v(t))}(u(t)).$$

Further, the relation (8) gives

$$\langle \mu, \eta(t)\rangle \leq \limsup_{n\to\infty} \delta^*(\mu, F(t, v_n(p_n(t)), u_n(p_n(t)))),$$

since $\delta^*(\mu, F(t,\cdot,\cdot))$ is upper semicontinuous on $H \times H$ then

$$\langle \mu, \eta(t)\rangle \leq \delta^*(\mu, F(t, v(t), u(t))),$$

so, we get $d(\eta(t), F(t, v(t), u(t))) \leq 0$, because F has closed convex values. Consequently $\eta(t) \in F(t, v(t), u(t))$ a.e $t \in [T_0, T]$. Then

$$\dot{u}(t) \in -N_{D(t,v(t))}(u(t)) + F(t, v(t), u(t)) + f(t, v(t), u(t)).$$

This completes the proof of the theorem. □

Remark 1. As in [22], the result remains valid if we replace the uniformly r-prox regular sets by a family of equi-uniformly subsmooth sets.

In the next theorem we prove the existence of solution on the whole interval $\mathbf{R}_+ = [0 + \infty[$.

Theorem 3. *Let $D : \mathbf{R}_+ \times H \to H$ be a set-valued mapping with nonempty closed and uniformly r-prox regular values such that:*

(*i*) *There is a positive constant L and a nondecreasing absolutely continuous function $\zeta : \mathbf{R}_+ \to \mathbf{R}_+$ such that, for all $s \leq t$ in $\mathbf{R}_+$ and $x_i, y_i \in H (i = 1, 2)$,*

$$|d\big(x_1, D(t, y_1)\big) - d\big(x_2, D(s, y_2)\big)| \leq \|x_1 - x_2\| + \zeta(t) - \zeta(s) + L\|y_1 - y_2\|;$$

(*ii*) *for all $(t, x) \in \mathbf{R}_+ \times H$, $D(t, x)$ is contained in a compact set Γ.*

Let $F : \mathbf{R}_+ \times H \times H \rightharpoonup H$ be a set-valued map with nonempty closed convex values such that:

(*iii*) *F is $\mathcal{L}(\mathbf{R}_+) \otimes \mathcal{B}(H) \otimes \mathcal{B}(H)$-measurable and for all $t \in \mathbf{R}_+$, $F(t,\cdot,\cdot)$ is scalarly upper semicontinuous on $H \times H$;*

(*vi*) *there exists a non-negative function $\beta(\cdot) \in L^\infty_{loc}(\mathbf{R}_+)$, such that, for all $(t, u, v) \in \mathbf{R}_+ \times H \times H$,*

$$d(0, F(t, u, v)) \leq \beta(t)(1 + \|u\| + \|v\|).$$

Then, for any $a, b \in H$ with $a \in D(T_0, b)$, there exist two absolutely continuous mappings $u, v : \mathbf{R}_+ \to H$ satisfying

$$(\mathcal{P}_{\mathbf{R}_+}) \begin{cases} -\dot{u}(t) \in N_{D(t,v(t))}(u(t)) + F(t, v(t), u(t)), \quad a.e.\ t \in \mathbf{R}_+; \\ v(t) = b + \displaystyle\int_{T_0}^t u(s)ds, \quad u(t) = a + \int_{T_0}^t \dot{u}(s)ds, \ \forall t \in \mathbf{R}_+; \\ u(t) \in D(t, v(t)), \ \forall t \in \mathbf{R}_+. \end{cases}$$

Proof. Since $\mathbf{R}_+ = \bigcup_{k\in\mathbf{N}} [k, k+1]$, for all $k \in \mathbf{N}$ applying Theorem 2 on each interval $[k, k+1]$, there exist two absolutely continuous mappings $u^k, v^k : [k, k+1] \to H$ satisfying

$$\begin{cases} -\dot{u}^k(t) \in N_{D(t,v^k(t))}(u^k(t)) + F(t, v^k(t), u^k(t)), \ \text{ a.e. } t \in [k, k+1]; \\ u^k(t) \in D(t, v^k(t)), \ \forall t \in [k, k+1], ; \ u^k(k) = u^{k-1}(k) \text{ and } v^k(k) = v^{k-1}(k). \end{cases}$$

Let $u : \mathbf{R}_+ \to H$ and $v : \mathbf{R}_+ \to H$ be defined by $u(t) = u^k(t)$ and $v(t) = v^k(t)$ for $t \in [k, k+1], \ k \in \mathbf{N}$, then it is easy to conclude that u, v are absolutely continuous solutions of the problem $(\mathcal{P}_{\mathbf{R}_+})$. This completes the proof of the theorem. $\square$

4 Delayed sweeping process

Now, we proceed, in the infinite dimensional setting, to an existence result for second order functional differential inclusion governed by the time and state-dependent nonconvex sweeping process, that is when the perturbation contains a finite delay. This problem was addressed by [22] using the discretization approach based on the Moreau's catching-up algorithm. Here, we provide another technique initiated in [10] for the first order time-dependent case, which consists to subdivide the interval $[0, T]$ in a sequence of subintervals and to reformulate the problem with delay to a sequence of problems without delay and apply the results known in this case. For second order functional problems regarding the time-dependent sweeping process, we refer to [7,8]. We will extend this approach for the case of time and state-dependent sweeping process with unbounded delayed perturbation. For a question of clarity and shortness, we will restrict ourselves to Theorem 2 for uniformly prox-regular sets and one set-valued perturbation, but it is clear that this remains valid for equi-uniformly subsmooth sets as well as for the sum of two perturbations.

Let $\tau > 0$ be a positive number and $\mathcal{C}_0 = \mathcal{C}_H([-\tau, 0])$ (resp. $\mathcal{C}_T = \mathcal{C}_H([-\tau, T])$ the Banach space of H-valued continuous functions defined on $[-\tau, 0]$ (resp. $[-\tau, T]$) equipped with the norm of uniform convergence. Let $u : [-\tau, T] \to H$, then for every $t \in [0, T]$ we define the function $u_t = \mathcal{T}(t)u$ on $[-\tau, 0]$ by $(\mathcal{T}(t)u)(s) = u(t+s), \ \forall s \in [-\tau, 0]$. Clearly, if $u \in \mathcal{C}_T$, then $u_t \in \mathcal{C}_0$ and the mapping $u \to u_t$ is continuous.

Consider the following problem

$$(\mathcal{P}_\tau) \quad \begin{cases} -\dot{u}(t) \in N_{D\big(t,v(t)\big)}(u(t)) + G\Big(t, \mathcal{T}(t)v, \mathcal{T}(t)u\Big) \ \text{ a.e. } t \in [0, T]; \\ u(t) = \psi(0) + \displaystyle\int_0^t \dot{v}(s)ds, \ \ v(t) = \varphi(0) + \displaystyle\int_0^t u(s)ds, \ \ \forall t \in [0, T]; \\ v(t) \in D\big(t, u(t)\big), \ \ \forall t \in [0, T]; \\ u \equiv \psi \ \text{ and } \ v \equiv \varphi \ \text{ on } [-\tau, 0]. \end{cases}$$

Theorem 4. *Assume that $D : [0, T] \times H \rightharpoonup H$ satisfies Assumption 1 and let $G : [0, T] \times \mathcal{C}_0 \times \mathcal{C}_0 \rightharpoonup H$ be a set-valued mapping with nonempty closed convex values such that:*

($\mathcal{A}_{G_1}$) *G is $\mathcal{L}([0,T]) \otimes \mathcal{B}(\mathcal{C}_0) \otimes \mathcal{B}(\mathcal{C}_0)$-measurable and for all $t \in \mathbf{R}_+$, $G(t,\cdot,\cdot)$ is scalarly upper semicontinuous on $\mathcal{C}_0 \times \mathcal{C}_0$;*

($\mathcal{A}_{G_2}$) *there exists a real $\beta > 0$, such that, for all $(t,\varphi,\psi) \in [T_0,T] \times \mathcal{C}_0 \times \mathcal{C}_0$,*

$$d(0, G(t,\varphi,\psi)) \leq \beta(1 + \|\varphi(0)\| + \|\psi(0)\|).$$

Then for every $(\varphi,\psi) \in \mathcal{C}_0 \times \mathcal{C}_0$ verifying $\psi(0) \in D(0,\varphi(0))$, there exist two absolutely continuous mappings $u : [0,T] \to H$ and $v : [0,T] \to H$ satisfying $(\mathcal{P}_\tau)$.

Proof. Let $a = \psi(0)$ and $b = \varphi(0)$, then $a \in D(0,b)$. We consider the same partition of $[0,T]$ by the points $t_k^n = ke_n$, $e_n = \frac{T}{n}, (k = 0,1,...,n)$. For each $(t,u,v) \in [-\tau, t_1^n] \times H \times H$, we define $f_0^n : [-\tau, t_1^n] \times H \to H$, $g_0^n : [-\tau, t_1^n] \times H \to H$ by

$$f_0^n(t,v) = \begin{cases} \varphi(t) & \forall t \in [-\tau,0], \\ \varphi(0) + \frac{n}{T} t (v - \varphi(0)) & \forall t \in]0, t_1^n], \end{cases}$$

$$g_0^n(t,u) = \begin{cases} \psi(t) & \forall t \in [-\tau,0], \\ \psi(0) + \frac{n}{T} t (u - \psi(0)) & \forall t \in]0, t_1^n]. \end{cases}$$

We have $f_0^n(t_1^n, v) = v$ and $g_0^n(t_1^n, v) = u$ for all $(u,v) \in H \times H$. Observe that the mapping $(u,v) \to \Big(\mathcal{T}(t_1^n) f_0^n(\cdot,v), \mathcal{T}(t_1^n) g_0^n(\cdot,u)\Big)$ from $H \times H$ to $\mathcal{C}_0 \times \mathcal{C}_0$ is nonexpansive since for all $(v_1,v_2) \in H \times H$

$$\|\mathcal{T}(t_1^n) f_0^n(\cdot,v_1) - \mathcal{T}(t_1^n) f_0^n(\cdot,v_2)\|_{\mathcal{C}_0} =$$

$$\sup_{s \in [-\tau,0]} \|f_0^n(s + t_1^n, v_1) - f_0^n(s + t_1^n, v_2)\| =$$

$$\sup_{s \in \left[-\tau + \frac{T}{n}, \frac{T}{n}\right]} \|f_0^n(s,v_1) - f_0^n(s,v_2)\| =$$

$$\sup_{0 \leq s \leq \frac{T}{n}} \left\| \frac{n}{T} s (v_1 - \varphi(0)) - \frac{n}{T} s (v_2 - \varphi(0)) \right\| =$$

$$\sup_{0 \leq s \leq \frac{T}{n}} \left\| \frac{n}{T} s (v_1 - v_2) \right\| = \|v_1 - v_2\|.$$

Similarly, for all $(u_1,u_2) \in H \times H$ we get

$$\|\mathcal{T}(t_1^n) g_0^n(\cdot,u_1) - \mathcal{T}(t_1^n) g_0^n(\cdot,u_2)\|_{\mathcal{C}_0} = \|u_1 - u_2\|.$$

Hence the mapping $(u,v) \to \Big(\mathcal{T}(t_1^n) f_0^n(\cdot,v), \mathcal{T}(t_1^n) g_0^n(\cdot,v)\Big)$ from $H \times H$ to $\mathcal{C}_0 \times \mathcal{C}_0$ is nonexpansive, so the set-valued mapping with nonempty closed convex values $G_0^n : [0,t_1^n] \times H \times H \rightharpoonup H$ defined by

$$G_0^n(t,u,v) = G(t, \mathcal{T}(t_1^n) f_0^n(\cdot,v), \mathcal{T}(t_1^n) g_0^n(\cdot,u))$$

is globally measurable and scalarly upper semicontinuous on $H \times H$, thanks to by $(\mathcal{A}_{G_1})$ and

$$d(0, G_0^n(t,v,u) = d(0, G(t, \mathcal{T}(t_1^n) f_0^n(\cdot, v), \mathcal{T}(t_1^n) g_0^n(\cdot, u))$$

$$\leq \beta(1 + \|v\| + \|u\|),$$

for all $(t,v,u) \in [0,t_1^n] \times H \times H$ since, $\mathcal{T}(t_1^n) f_0^n(0,v) = u, \mathcal{T}(t_1^n) g_0^n(0,u) = v$. Hence G_0^n verifies conditions of Theorem 2, then there exist two absolutely continuous mappings $u_0^n : [0,t_1^n] \to H$ and $v_0^n : [0,t_1^n] \to H$ such that

$$\begin{cases} -\dot{u}_0^n(t) \in N_{D(t,v_0^n(t))}(u_0^n(t)) + G_0^n(t, v_0^n, u_0^n) \quad \text{a.e on} \quad [0,t_1^n]; \\ v_0^n(t) = b + \displaystyle\int_0^t u_0^n(s)\,ds, \;\; u_0^n(t) = a + \int_0^t \dot{u}_0^n(s)\,ds \;\; \forall t \in [0,t_1^n]; \\ u_0^n(t) \in D(t, v_0^n(t)) \;\; \forall t \in [0,t_1^n]; \\ v_0^n(0) = b = \varphi(0), \;\; u_0^n(0) = a = \psi(0), \end{cases}$$

with

$$\|v_0^n(t)\| \leq \|b\| + T\alpha, \;\; \|u_0^n(t)\| \leq \alpha, \;\; \|\dot{u}_0^n(t)\| \leq c_2.$$

Set

$$v_n(t) = \begin{cases} \varphi(t) & \forall t \in [-\tau, 0], \\ v_0^n(t) & \forall t \in]0, t_1^n], \end{cases}$$

$$u_n(t) = \begin{cases} \psi(t) & \forall t \in [-\tau, 0], \\ u_0^n(t) & \forall t \in]0, t_1^n]. \end{cases}$$

Then, u_n and v_n are well defined on $[-\tau, t_1^n]$, with $v_n = \varphi$, $u_n = \psi$ on $[-\tau, 0]$, and

$$\begin{cases} -\dot{u}_n(t) \in N_{D(t,v_n(t))}(u_n(t)) + G_0(t, v_n(t), u_n(t)) \quad \text{a.e on} \quad [0,t_1^n]; \\ v_n(t) = b + \displaystyle\int_0^t u_n(s)\,ds, \\ u_n(t) = a + \displaystyle\int_0^t \dot{u}_n(s)\,ds, \;\; \forall t \in [0,t_1^n]; \\ u_n(t) \in D(t, v_n(t)), \;\; \forall t \in [0,t_1^n]; \\ v_n(0) = b = \varphi(0), \;\; u_n(0) = a = \psi(0), \end{cases}$$

By induction, suppose that u_n and v_n are defined on $[-\tau, t_k^n]$ $(k \geq 1)$ with $v_n = \varphi, u_n = \psi$ on $[-\tau, 0]$ and satisfy

$$v_n(t) = \begin{cases} v_0^n(t) = b + \displaystyle\int_0^t u_n(s)ds \;\; \forall t \in [0, t_1^n], \\ v_1^n(t) = v_n(t_1^n) + \displaystyle\int_{t_1^n}^t u_n(s)\,ds \;\; \forall t \in]t_1^n, t_2^n], \\ \cdots \\ v_{k-1}^n(t) = v_n(t_{k-1}^n) + \displaystyle\int_{t_{k-1}^n}^t u_n(s)\,ds \;\; \forall t \in \left]t_{k-1}^n, t_k^n\right], \end{cases}$$

$$u_n(t)=\begin{cases} u_0^n(t)=b+\int_0^t \dot{u}_n(s)\,ds \ \ \forall t\in[0,t_1^n]; \\ u_1^n(t)=u_n(t_1^n)+\int_{t_1^n}^t \dot{u}_n(s)\,ds \ \ \forall t\in\left]t_1^n,t_2^n\right]; \\ \cdots \\ u_{k-1}^n(t)=u_n\left(t_{k-1}^n\right)+\int_{t_{k-1}^n}^t \dot{u}_n(s)\,ds \ \ \forall t\in\left]t_{k-1}^n,t_k^n\right], \end{cases}$$

u_n and v_n are solutions of

$$\begin{cases} -\dot{u}_n(t)\in N_{D(t,v_n(t))}(u_n(t))+G\Big(t,\mathcal{T}(t_k^n)f_{k-1}^n(\cdot,v_n(t)),\mathcal{T}(t_k^n)g_{k-1}^n(\cdot,u_n(t))\Big); \\ v_n(t)=v_{k-1}^n(t)=v_n\left(t_{k-1}^n\right)+\int_{t_{k-1}^n}^t u_n(s)\,ds; \\ u_n(t)=u_{k-1}^n(t)=u_n\left(t_{k-1}^n\right)+\int_{t_{k-1}^n}^t \dot{u}_n(s)\,ds; \\ u_n(t)\in D(t,v_n(t)) \end{cases}$$

on $\left]t_{k-1}^n,t_k^n\right]$, where f_{k-1}^n and g_{k-1}^n are defined for any $(v,u)\in H\times H$ as follows

$$f_{k-1}^n(t,v)=\begin{cases} v_n(t) & \forall t\in\left[-\tau,t_{k-1}^n\right], \\ v_n\left(t_{k-1}^n\right)+\frac{n}{T}\left(t-t_{k-1}^n\right)\left(v-v_n\left(t_{k-1}^n\right)\right) & \forall t\in\left]t_{k-1}^n,t_k^n\right], \end{cases} \tag{9}$$

$$g_{k-1}^n(t,u)=\begin{cases} u_n(t) & \forall t\in\left[-\tau,t_{k-1}^n\right], \\ u_n\left(t_{k-1}^n\right)+\frac{n}{T}\left(t-t_{k-1}^n\right)\left(u-u_n\left(t_{k-1}^n\right)\right) & \forall t\in\left]t_{k-1}^n,t_k^n\right]. \end{cases} \tag{10}$$

Similarly we can define $f_k^n,g_k^n:\left[-\tau,t_{k+1}^n\right]\times H\to H$ as

$$f_k^n(t,v)=\begin{cases} v_n(t) & \forall t\in[-\tau,t_k^n], \\ v_n(t_k^n)+\frac{n}{T}(t-t_k^n)(v-v_n(t_k^n)), & \forall t\in\left]t_k^n,t_{k+1}^n\right], \end{cases}$$

$$g_k^n(t,u)=\begin{cases} u_n(t) & \forall t\in[-\tau,t_k^n], \\ u_n(t_k^n)+\frac{n}{T}(t-t_k^n)(u-u_n(t_k^n)) & \forall t\in\left]t_k^n,t_{k+1}^n\right], \end{cases}$$

for any $(u,v)\in H\times H$. Note that for all $(u,v)\in H\times H$,

$$\mathcal{T}(t_{k+1}^n)f_k^n(0,v)=f_k^n\left(t_{k+1}^n,v\right)=v,$$

$$\mathcal{T}(t_{k+1}^n)g_k^n(0,u)=g_k^n\left(t_{k+1}^n,u\right)=u.$$

Note also that, for all $(u_1,v_1),(u_2,v_2)\in H\times H$, we have

$$\left\|\mathcal{T}(t_{k+1}^n)f_k^n(\cdot,v_1)-\mathcal{T}(t_{k+1}^n)f_k^n(\cdot,v_2)\right\|_{\mathcal{C}_0}=$$

$$\sup_{s\in[-\tau,0]}\left\|f_k^n\left(s+t_{k+1}^n,v_1\right)-f_k^n\left(s+t_{k+1}^n,v_2\right)\right\|=$$

$$\sup_{s\in\left[-\tau+\frac{(k+1)T}{n},\frac{(k+1)T}{n}\right]} \left\|f_k^n\left(s,u_1\right)-f_k^n\left(s,u_2\right)\right\|,$$

and

$$\left\|\mathcal{T}(t_{k+1}^n)g_k^n\left(\cdot,u_1\right)-\mathcal{T}(t_{k+1}^n)g_k^n\left(\cdot,u_2\right)\right\|_{\mathcal{C}_0}=$$
$$\sup_{s\in[-\tau,0]}\left\|g_k^n\left(s+t_{k+1}^n,u_1\right)-g_k^n\left(s+t_{k+1}^n,u_2\right)\right\|=$$
$$\sup_{s\in\left[-\tau+\frac{(k+1)T}{n},\frac{(k+1)T}{n}\right]} \left\|g_k^n\left(s,u_1\right)-g_k^n\left(s,u_2\right)\right\|.$$

We distinguish two cases:

(1) if $-\tau+\dfrac{(k+1)\,T}{n}<\dfrac{kT}{n}$, we have

$$\sup_{s\in\left[-\tau+\frac{(k+1)T}{n},\frac{(k+1)T}{n}\right]} \left\|f_k^n\left(s,v_1\right)-f_k^n\left(s,v_2\right)\right\|=$$
$$\sup_{s\in\left[\frac{kT}{n},\frac{(k+1)T}{n}\right]} \left\|f_k^n\left(s,v_1\right)-f_k^n\left(s,v_2\right)\right\|=$$
$$\sup_{\frac{kT}{n}\le s\le\frac{(k+1)T}{n}} \left\|\frac{n}{T}\left(s-t_k^n\right)\left(v_1-v_2\right)\right\|=\left\|v_1-v_2\right\|$$

and

$$\sup_{s\in\left[-\tau+\frac{(k+1)T}{n},\frac{(k+1)T}{n}\right]} \left\|g_k^n\left(s,u_1\right)-g_k^n\left(s,u_2\right)\right\|=$$
$$\sup_{s\in\left[\frac{kT}{n},\frac{(k+1)T}{n}\right]} \left\|g_k^n\left(s,u_1\right)-g_k^n\left(s,u_2\right)\right\|=$$
$$\sup_{\frac{kT}{n}\le s\le\frac{(k+1)T}{n}} \left\|\frac{n}{T}\left(s-t_k^n\right)\left(u_1-u_2\right)\right\|=\left\|u_1-u_2\right\|;$$

(2) if $\frac{kT}{n}\le-\tau+\frac{(k+1)T}{n}\le\frac{(k+1)T}{n}$, we have

$$\sup_{s\in\left[-\tau+\frac{(k+1)T}{n},\frac{(k+1)T}{n}\right]} \left\|f_k^n\left(s,v_1\right)-f_k^n\left(s,v_2\right)\right\|=$$
$$\sup_{s\in\left[\frac{kT}{n},\frac{(k+1)T}{n}\right]} \left\|f_k^n\left(s,v_1\right)-f_k^n\left(s,v_2\right)\right\|=$$
$$\sup_{\frac{kT}{n}\le s\le\frac{(k+1)T}{n}} \left\|\frac{n}{T}(s-t_k^n)\left(v_1-v_2\right)\right\|=\left\|v_1-v_2\right\|$$

and

$$\sup_{s\in\left[-\tau+\frac{(k+1)T}{n},\frac{(k+1)T}{n}\right]} \left\|g_k^n\left(s,u_1\right)-g_k^n\left(s,u_2\right)\right\|=$$

$$\sup_{s\in\left[\frac{kT}{n},\frac{(k+1)T}{n}\right]}\left\|g_k^n\left(s,u_1\right)-g_k^n\left(s,u_2\right)\right\|=$$

$$\sup_{\frac{kT}{n}\leq s\leq\frac{(k+1)T}{n}}\left\|\frac{n}{T}(s-t_k^n)\left(u_1-u_2\right)\right\|=\left\|u_1-u_2\right\|.$$

So the mapping $(v,u)\to\Big(\mathcal{T}(t_{k+1}^n)f_k^n\left(\cdot,v\right),\mathcal{T}(t_{k+1})g_k^n\left(\cdot,u\right)\Big)$ from $H\times H$ to $\mathcal{C}_0\times\mathcal{C}_0$ is nonexpansive. Hence the set-valued mapping $G_k^n:\left[t_k^n,t_{k+1}^n\right]\times H\times H\rightharpoonup H$ defined by

$$G_k^n\left(t,u,v\right)=G\Big(t,\mathcal{T}(t_{k+1}^n)f_k^n\left(.,u\right),\mathcal{T}(t_{k+1}^n)g_k^n\left(.,v\right)\Big)$$

globally measurable and scalarly upper semicontinuous on $H\times H$, with nonempty closed convex values. As above we can easily check that

$$d(0,G_k^n\left(t,v,u\right)\leq\left(1+\|u\|+\|v\|\right),\ \forall\left(t,u,v\right)\in\left[t_k^n,t_{k+1}^n\right]\times H\times H.$$

Applying Theorem 2, there exist two absolutely continuous mappings $u_k^n:\left[t_k^n,t_{k+1}^n\right]\to H$ and $v_k^n:\left[t_k^n,t_{k+1}^n\right]\to H$ such that

$$\begin{cases}-\dot{u}_k^n\left(t\right)\in N_{D(t,v_k^n(t))}\left(u_k^n(t)\right)+G_k^n(t,v_k^n\left(t\right),u_k^n\left(t\right))\ \text{ a.e. on }\left[t_k^n,t_{k+1}^n\right];\\ v_k^n\left(t\right)=v_n\left(t_k^n\right)+\displaystyle\int_{t_k^n}^t u_k^n\left(s\right)ds,\ \ \forall t\in\left[t_k^n,t_{k+1}^n\right];\\ u_k^n\left(t\right)=u_n\left(t_k^n\right)+\displaystyle\int_{t_k^n}^t \dot{u}_k^n\left(s\right)ds,\ \ \forall t\in\left[t_k^n,t_{k+1}^n\right];\\ u_k^n(t)\in D(t,u_k^n\left(t\right))\ \ \forall t\in\left[t_k^n,t_{k+1}^n\right],\end{cases}$$

with

$$\left\|u_k^n\left(t\right)\right\|\leq\alpha,\ \ \left\|v_k^n\left(t\right)\right\|\leq\|b\|+T\alpha,\ \ \left\|\dot{u}_k^n\left(t\right)\right\|\leq c_2(t).$$

Thus, by induction, we can construct two continuous mappings $u_n,v_n:[-\tau,T]\to H\times H$ with

$$v_n\left(t\right)=\begin{cases}\varphi\left(t\right) & \forall t\in\left[-\tau,0\right],\\ v_k^n\left(t\right) & \forall t\in\left]t_k^n,t_{k+1}^n\right],\ \forall k=0,\cdots,n-1;\end{cases}$$

$$u_n\left(t\right)=\begin{cases}\psi\left(t\right) & \forall t\in\left[-\tau,0\right],\\ u_k^n\left(t\right) & \forall t\in\left]t_k^n,t_{k+1}^n\right],\ \forall k=0,\cdots,n-1,\end{cases}$$

such that their restriction on each interval $\left[t_k^n,t_{k+1}^n\right]$ is a pair solution to

$$\begin{cases}-\dot{u}\left(t\right)\in N_{D(t,v(t))}\left(u(t)\right)+G(t,\mathcal{T}(t_{k+1}^n)f_k^n\left(.,v\left(t\right)\right),\mathcal{T}(t_{k+1}^n)g_k^n\left(.,u\left(t\right)\right));\\ v\left(t\right)=v_n\left(t_k^n\right)+\displaystyle\int_{t_k^n}^t u\left(s\right)ds,\ u\left(t\right)=u_n\left(t_k^n\right)+\displaystyle\int_{t_k^n}^t \dot{u}\left(s\right)ds\\ u\left(t\right)\in D(t,v\left(t\right)).\end{cases}$$

Let $h_k^n : \left[t_k^n, t_{k+1}^n\right] \times \mathcal{C}_0 \times \mathcal{C}_0$ be the element of minimal norm of G_k^n, then

$$\begin{cases} h_k^n\left(t, v_k^n\left(t\right), u_k^n\left(t\right)\right) \in G_k^n(t, v_k^n\left(t\right), u_k^n\left(t\right)) \ \text{ a.e. on } \ \left[t_k^n, t_{k+1}^n\right], \\ -\dot{u}_k^n\left(t\right) \in N_{D(t,v_k^n(t))}\left(u_k^n(t)\right) + h_k^n\left(t, v_k^n\left(t\right), u_k^n\left(t\right)\right) \ \text{ a.e. on } \ \left[t_k^n, t_{k+1}^n\right], \\ v_k^n\left(t_k^n\right) = v_n\left(t_k^n\right), u_k^n\left(t_k^n\right) = u_n\left(t_k^n\right) \\ u_k^n(t) \in D(t, v_k^n\left(t\right)), \ \forall t \in \left[t_k^n, t_{k+1}^n\right]. \end{cases}$$

Let set for notational convenience, $h_n(t,v,u) = h_k^n(t,v,u)$, $\theta_n\left(t\right) = t_{k+1}^n$ and $\delta_n\left(t\right) = t_k^n$, for all $t \in \left]t_k^n, t_{k+1}^n\right]$. Then we get for almost every $t \in [0,T]$

$$\begin{cases} h_n(t, v_n, u_n) \in G(t, \mathcal{T}(\theta_n\left(t\right)) f^n_{\frac{n}{T}\delta_n(t)}\left(., v_n\left(t\right)\right), \mathcal{T}(\theta_n\left(t\right)) g^n_{\frac{n}{T}\delta_n(t)}\left(., u_n\left(t\right)\right)); \\ -\dot{u}_n\left(t\right) \in N_{D(t,v_n(\theta_n(t)))}\left(u_n(\theta_n\left(t\right))\right) + h_n\left(t, v_n(t), u_n(t)\right); \\ v_n\left(0\right) = b = \varphi\left(0\right), \ u_n\left(0\right) = a = \psi\left(0\right) \in D\left(0, b\right), \\ u_n\left(t\right) \in D\left(t, v_n\left(\theta_n\left(t\right)\right)\right), \forall t \in [0,T] \end{cases}$$

with for all $t \in [0,T]$

$$\begin{aligned} & d\Big(0, G(t, \mathcal{T}(\theta_n\left(t\right)) f^n_{\frac{n}{T}\delta_n(t)}\left(., v_n\left(t\right)\right), \mathcal{T}(\theta_n\left(t\right)) g^n_{\frac{n}{T}\delta_n(t)}\left(., u_n\left(t\right)\right)\Big) \\ & \qquad \leq \beta\left(1 + \|u_n\left(t\right)\| + \|v_n\left(t\right)\|\right). \end{aligned}$$

We claim that $\mathcal{T}(\theta_n\left(t\right)) f^n_{\frac{n}{T}\delta_n(t)}\left(., v_n\left(t\right)\right)$ and $\mathcal{T}(\theta_n\left(t\right)) g^n_{\frac{n}{T}\delta_n(t)}\left(., u_n\left(t\right)\right)$ pointwise converge on $[0,T]$ to $\mathcal{T}(t)v$ and $\mathcal{T}(t)u$ respectively in $\mathcal{C}_0$. The proof is similar to the one given in Theorem 2.1 in [14].
Further, as $\|v_n\left(t\right)\| \leq \|b\| + T\alpha$, $\|\dot{u}(t)\| \leq c_2(t)$ and

$$\begin{aligned} \|h_n\left(t, v_n(t), u_n(t)\right)\| &\leq \beta\Big(1 + \|u_n(t)\| + \|v_n(t)\|\Big) \\ &\leq \beta\big(1 + \|b\| + (1+T)\alpha\big). \end{aligned}$$

We can proceed as in Theorem 2 to conclude the convergence of (u_n) and (v_n) to the solution of $(\mathcal{P}_\tau)$. $\square$

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Upper Bounds for the Number of Limit Cycles for a Class of Polynomial Differential Systems Via The Averaging Method

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Abstract. In this paper, we study the number of limit cycles of polynomial differential systems of the form

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - \varepsilon(h_1(x)\, y^{2\alpha} + g_1(x)\, y^{2\alpha+1} + f_1(x)\, y^{2\alpha+2}) \\ \qquad - \varepsilon^2(h_2(x)\, y^{2\alpha} + g_2(x)\, y^{2\alpha+1} + f_2(x)\, y^{2\alpha+2}) \end{cases}$$

where m, n, k and α are positive integers, h_i, g_i and f_i have degree n, m and k, respectively for each $i = 1, 2$, and ε is a small parameter. We use the averaging theory of first and second order to provide an accurate upper bound of the number of limit cycles that bifurcate from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$. We give an example for which this bound is reached.

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1 Introduction and statement of the main results

One of the main problems in the theory of ordinary differential equations is the study of the existence of limit cycles, their number and stability. A limit cycle of a differential equation is a periodic orbit in the set of all isolated periodic orbits of the differential equation. The second part of the 16th Hilbert's problem (see [8]) is related to the least upper bound on the number of limit cycles of polynomial vector fields having a fixed degree.

Many of the results on the limit cycles of polynomial differential systems have been obtained by considering limit cycles that bifurcate from a single degenerate singular point (i.e. from a Hopf bifurcation), which are called small amplitude limit cycles, see Lloyd [14]. There are partial results concerning the maximum number of small-amplitude limit cycles for Liénard polynomial differential systems. The number of small-amplitude limit cycles gives a lower bound for the maximum number of limit cycles that a polynomial differential system can have. There are many results concerning the existence of small-amplitude limit cycles for the following generalization of the classical Liénard polynomial differential system

$$\dot{x} = y \qquad \text{and} \quad \dot{y} = -g(x) - f(x)y \tag{1}$$

where $f(x)$ and $g(x)$ are polynomials in the variable x of degrees n and m, respectively. We denote by $H(m,n)$ and $\hat{H}(m,n)$ the maximum number of limit cycles that system (1) can have and the maximum number of small-amplitude limit cycles that system(1) can have, respectively. The first number is usually called Hilbert number for system (1). Since the work of Liénard [10] to the present time several authors have found particular values of these numbers H and $\hat{H}$, to find a survey about these values see [13]. The authors of [12] computed the maximum number of limit cycles $\hat{H}_k(m,n)$ of system(1) that bifurcate from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$, using the averaging theory of order k. More specifically it was found that $\hat{H}_1(m,n) = [(n+m-1)/2]$. In order to find the maximum number of limit cycles it is interesting to know what families of system (1) have a center. This is because we can perturb these centers and control the number of small-amplitude limit cycles or the number of limit cycles that bifurcate from the periodic orbits of these centers, (see [5, 6]). We recall that a singular point is a center if there is an open neighborhood consisting, besides the singularity, of periodic orbits. The center problem consists in determining what families of a given system have a center. For more information about the Hilbert's 16th problem and related topics see [9]. Now we are citing some results about the limit cycles on Liénard differential systems (see [12]) In 1928, Liénard proved that if $m = 1$ and $F(x) = \int_0^x f(s)ds$ is a continuous odd function, which has a unique root at $x = a$ and is monotone increasing for $x \geq a$, then equations (1.2) have a unique limit cycle. In 1977 Lins, de Melo and Pugh [11] stated the conjecture that if $f(x)$ has degree $n \geq 1$ and $g\,(x) = x$ then system (1) has at most $[n/2]$ limit cycles. They prove this conjecture for $n = 1, 2$. In 1998 Gasull and Torregrosa [4] obtained upper bounds for $\hat{H}(7,6), \hat{H}(6,7)$, $\hat{H}(7,7)$ and $\hat{H}(4,20)$. In 2010, Llibre et al, computed the maximum number of limit cycles $\hat{H}_k(m,n)$ of system (1) that bifurcate from the periodic orbits of the linear centre $\dot{x} = y, \dot{y} = -x$, using the averaging theory of order k, for $k = 1, 2, 3$. In 2014 B. Garca, J. Llibre, and J. S. Pérez del Rio 1001[3] using the averaging theory of first and second order, they studied the maximum number of medium amplitude limit cycles bifurcating from the linear center $\dot{x} = y, \dot{y} = -x$ of the more generalized polynomial Liénard differential systems of the form

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - \varepsilon(h_1\,(x) + p_1\,(x)\,y + q_1\,(x)\,y^2) \\ \qquad - \varepsilon^2(h_2\,(x) + p_2\,(x) + q_2\,(x)\,y^2) \end{cases}$$

where h_1, h_2, p_1, q_1, p_2 and q_2 have degree n.

In this work using the averaging theory, we study the maximum number of limit cycles which can bifurcate from the periodic orbits of a linear center perturbed inside the class of generalized polynomial Liénard differential equations

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - \varepsilon(h_1\,(x)\,y^{2\alpha} + g_1\,(x)\,y^{2\alpha+1} + f_1\,(x)\,y^{2\alpha+2}) \\ \qquad - \varepsilon^2(h_2\,(x)\,y^{2\alpha} + g_2\,(x)\,y^{2\alpha+1} + f_2\,(x)\,y^{2\alpha+2}) \end{cases} \tag{2}$$

where m, n, k and α are positive integers, h_i, g_i and f_i have degree n, m and k, respectively for each $i = 1, 2$, and ε is a small parameter.

Let $[\cdot]$ denote the integer part function. Our main result is the following one.

Theorem 1. *For $|\varepsilon|$ sufficiently small, the maximum number of limit cycles of the polynomial differential systems* (2) *bifurcating from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$, using the averaging theory*
(a) of first order is

$$\lambda_1 = \left[\frac{m}{2}\right],$$

(b) of second order is

$$\lambda = \max\left\{\left[\frac{m}{2}\right]; \left[\frac{m-1}{2}\right] + \left[\frac{n}{2}\right] + \alpha; \left[\frac{m-1}{2}\right] + \left[\frac{k}{2}\right] + 1 + \alpha\right\}.$$

The proof of the above theorem is given in Section 3.

2 The averaging theory of first and second order

In this section we present the basic results from the averaging theory that we shall need for proving the main results of this paper. The averaging theory up to second order for studying specifically periodic orbits was developed in [1, 2]. It is summarized as follows.

Consider the differential system

$$\dot{x}(t) = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 R(t, x, \varepsilon),$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}$ are continuous functions, $T-$periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. Assume that the following hypotheses hold.
(i) $F_1(t, \cdot) \in C^2(D), F_2(t, \cdot) \in C^1(D)$ for all $t \in \mathbb{R}, F_1, F_2, R$ are locally Lipschitz with respect to x, and R is twice differentiable with respect to ε.
We define $F_{k0} : D \to \mathbb{R}$ for $k = 1, 2$ as

$$\begin{aligned} F_{10}(x) &= \frac{1}{T}\int_0^T F_1(s, x)ds, \\ F_{20}(x) &= \frac{1}{T}\int_0^T (D_x F_1(s, x))\, y_1(s, x) + F_2(s, x)ds, \end{aligned}$$

where

$$y_1(s, x) = \int_0^s F_1(t, x)dt.$$

(ii) For an open and bounded set $V \subset D$ and for each $\varepsilon \in (-\varepsilon f, \varepsilon f)\backslash\{0\}$, there exists $a_\varepsilon \in V$ such that $F_{10}(a_\varepsilon) + \varepsilon F_{20}(a_\varepsilon) = 0$ and $d_B(F_{10} + \varepsilon F_{20}, V, a_\varepsilon) \neq 0$.

Then, for $|\varepsilon| > 0$ sufficiently small there exists a $T-$periodic solution $x(.,\varepsilon)$ of the system such that $x(0,\varepsilon) \to a_\varepsilon$ as $\varepsilon \to 0$.

The expression $d_B(F_{10} + \varepsilon F_{20}, V, a_\varepsilon) \neq 0$ means that the Brouwer degree of the function $F_{10} + \varepsilon F_{20} : V \to \mathbb{R}^n$ at the fixed point a_ε is not zero. A sufficient condition of this inequality holding is that the Jacobian of the function $F_{10} + \varepsilon F_{20}$ at a_ε is not zero.

If F_{10} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20}$ are mainly the zeros of F_{10} for ε sufficiently small. In this case the previous result provides the *averaging theory of first order.*

If F_{10} is identically zero and F_{20} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20}$ are mainly the zeros of F_{20} for ε sufficiently small. In this case the previous result provides the *averaging theory of second order.*

3 Proof of Theorem 1

For the proof we shall use the first order averaging theory as it was stated in Section 2. We write system (2) in polar coordinates (r,θ) given by $x = r\cos\theta$ and $y = r\sin\theta$. In this way, system (2) will become written in the standard form for applying the averaging theory. If we write

$$
\begin{aligned}
h_1(x) &= \sum_{i=0}^{n} a_i x^i, \ g_1(x) = \sum_{i=0}^{m} c_i x^i, \ f_1(x) = \sum_{i=0}^{k} d_i x^i, \\
h_2(x) &= \sum_{i=0}^{n} A_i x^i, \ g_2(x) = \sum_{i=0}^{m} C_i x^i, \ f_2(x) = \sum_{i=0}^{k} D_i x^i
\end{aligned}
$$

then, system (2) becomes

$$
\begin{cases}
\dot{r} = -\varepsilon E_1(r,\theta) - \varepsilon^2 H_1(r,\theta), \\
\dot{\theta} = -1 - \dfrac{\varepsilon}{r} E_2(r,\theta) - \dfrac{\varepsilon^2}{r} H_2(r,\theta),
\end{cases}
$$

where

$$
\begin{aligned}
E_1(r,\theta) = \sum_{i=0}^{n} a_i h_{i,2\alpha+1}(\theta) r^{2\alpha+i} + \sum_{i=0}^{k} d_i h_{i,2\alpha+3}(\theta) r^{2\alpha+i+2} + \\
+ \sum_{i=0}^{m} c_i h_{i,2\alpha+2}(\theta) r^{2\alpha+i+1}, \\
H_1(r,\theta) = \sum_{i=0}^{n} A_i h_{i,2\alpha+1}(\theta) r^{2\alpha+i} + \sum_{i=0}^{k} D_i h_{i,2\alpha+3}(\theta) r^{2\alpha+i+2} + \\
+ \sum_{i=0}^{m} C_i h_{i,2\alpha+2}(\theta) r^{2\alpha+i+1},
\end{aligned}
$$

$$E_2(r,\theta)=\sum_{i=0}^{n}a_ih_{i+1,2\alpha}(\theta)\,r^{2\alpha+i}+\sum_{i=0}^{k}d_ih_{i+1,2\alpha+2}(\theta)\,r^{2\alpha+i+2}+$$
$$+\sum_{i=0}^{m}c_ih_{i+1,2\alpha+1}(\theta)\,r^{2\alpha+i+1},$$
$$H_2(r,\theta)=\sum_{i=0}^{n}A_ih_{i+1,2\alpha}(\theta)\,r^{2\alpha+i}+r^2\sum_{i=0}^{k}D_ih_{i+1,2\alpha+2}(\theta)\,r^{2\alpha+i+2}+$$
$$+\sum_{i=0}^{m}C_ih_{i+1,2\alpha+1}(\theta)\,r^{2\alpha+i+1},$$

where $h_{i,\alpha}(\theta)=\cos^i\theta\sin^i\theta$ Taking θ as the new independent variable, system (2) becomes

$$\frac{dr}{d\theta}=\varepsilon F_1(r,\theta)+\varepsilon^2F_2(r,\theta)+O\left(\varepsilon^3\right), \tag{3}$$

where

$$\begin{aligned} F_1(r,\theta) &= E_1(r,\theta), \\ F_2(r,\theta) &= H_1(r,\theta)-\frac{1}{r}E_1(r,\theta)\,E_2(r,\theta). \end{aligned} \tag{4}$$

First we shall study the limit cycles of the differential equation (3) using the averaging theory of first order. Therefore, by Section 2 we must study the simple positive zeros of the function

$$F_{10}(r)=\frac{1}{2\pi}\int_0^{2\pi}F_1(r,\theta)\,d\theta.$$

For every one of these zeros we will have a limit cycle of the polynomial differential system (2). If $F_{10}(r)$ is identically zero, applying the theory of averaging of second order (see again Section 2) every simple positive zero of the function

$$F_{20}(r)=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{d}{dr}F_1(r,\theta)\left(\int_0^{\theta}F_1(r,s)ds\right)+F_2(r,\theta)\right)d\theta,$$

will provide a limit cycle of the polynomial differential system (2).

3.1 Proof of statement (a) of Theorem 1

Taking into account the expression of (4), in order to obtain F_{10} is necessary to evaluate the integrals of the form

$$\int_0^{\pi}\cos^i\theta\sin^j\theta d\theta$$

In the following lemma we compute these integrals.

Lemma 1. *Let* $h_{i,j}(\theta) = \cos^i\theta \sin^j\theta$ *and* $\delta_{i,j}(\theta) = \int\limits_0^\theta h_{i,j}(s)\,ds$ *Then*

$$\delta_{i,j}(2\pi) = \begin{cases} 0 & \text{if } i \text{ is odd or } j \text{ is odd,} \\ \frac{(j-1)(j-3)\dots 1}{(j+i)(j+i-2)\dots(i+2)} \frac{1}{2^{i-1}} \binom{i}{\frac{i}{2}} \pi & \text{if } i \text{ and } j \text{ are even,} \end{cases} \tag{5}$$

where $\binom{i}{\frac{i}{2}} = \dfrac{i!}{\left(\frac{i}{2}!\right)^2}$

Proof. Using the integrals 12 and 13 given at the appendix with $\theta = 2\pi$ and taking into account that $h_{i,j}(2\pi) = 0$ if $j \neq 0$ we have that

$$\delta_{i,2j}(2\pi) = \frac{(2j-1)(2j-3)\dots 1}{(2j+i)(2j+i-2)(i+2)} \delta_{i,0}(2\pi), \delta_{i,2j+1}(2\pi) = 0. \tag{6}$$

Again, using the integrals 10 and 11 given in the appendix, with $\theta = 2\pi$, we have that $\delta_{2i,0}(2\pi) = \frac{(2i-1)(2i-3)}{2^i i!} 2\pi$ and $\delta_{2i+1,0}(2\pi) = 0$, Substituting $\delta_{2i,0}(2\pi)$ and $\delta_{2i+1,0}(2\pi)$ given as above into (6) we obtain (5). Using this lemma we shall obtain in the next proposition the function $F_{10}(r)$: ☐

Proposition 1. *We have*

$$F_{10}(r) = \frac{r^{2\alpha+1}}{2\pi} \sum_{i=0}^{\left[\frac{m}{2}\right]} c_{2i}\, \delta_{2i,2\alpha+2}(2\pi)\, r^{2i}. \tag{7a}$$

Proof. The function $F_{10}(r)$ is given by

$$\begin{aligned} F_{10}(r) &= \frac{1}{2\pi} \int\limits_0^{2\pi} \sum_{i=0}^{n} a_i h_{i,2\alpha+1}(\theta)\, r^{2\alpha+i} d\theta + \frac{1}{2\pi} \int\limits_0^{2\pi} \sum_{i=0}^{k} d_i h_{i,2\alpha+3}(\theta)\, r^{2\alpha+i+2} d\theta \\ &\quad + \frac{1}{2\pi} \int\limits_0^{2\pi} \sum_{i=0}^{m} c_i h_{i,2\alpha+2}(\theta)\, r^{2\alpha+i+1} d\theta. \end{aligned}$$

Using lemma 1, we obtain

$$\int\limits_0^{2\pi} h_{i,2\alpha+1}(\theta)\, d\theta = \int\limits_0^{2\pi} h_{i,2\alpha+3}(\theta)\, d\theta = 0, \ \forall i \in \mathbb{N}.$$

Then

$$F_{10}(r) = \frac{1}{2\pi} \int\limits_0^{2\pi} \sum_{i=0}^{m} c_i h_{i,2\alpha+2}(\theta)\, r^{2\alpha+i+1} d\theta$$

$$
\begin{aligned}
&= \int_0^{2\pi} \sum_{\substack{i=0\\ i \text{ odd}}}^{m} c_i h_{i,2\alpha+2}(\theta)\, r^{2\alpha+i+1} + \sum_{\substack{i=0\\ i \text{ even}}}^{m} c_i h_{i,2\alpha+2}(\theta)\, r^{2\alpha+i+1} d\theta \\
&= \sum_{i=0}^{\left[\frac{m-1}{2}\right]} c_{2i+1} \int_0^{2\pi} h_{2i+1,2\alpha+2}(\theta)\, r^{2\alpha+2i+2} + \sum_{i=0}^{\left[\frac{m}{2}\right]} c_{2i} \int_0^{2\pi} h_{2i,2\alpha+2}(\theta)\, r^{2\alpha+2i+1} d\theta.
\end{aligned}
$$

Again, using lemma 1, we conclude that $\int_0^{2\pi} h_{2i+1,2\alpha+2}(\theta)\, d\theta = 0$, then

$$
F_{10}(r) = \frac{r^{2\alpha+1}}{2\pi} \sum_{i=0}^{\left[\frac{m}{2}\right]} c_{2i} \delta_{2i,2\alpha+2}(2\pi)\, r^{2i}.
$$

□

From Proposition 1, the polynomial $F_{10}(r)$ has at most $\lambda_1 = \left\{\left[\frac{m}{2}\right]\right\}$ positive roots, and we can choose c_{2i} in such a way that $F_{10}(r)$ has exactly λ_1 simple positive roots, hence the statement (a) of Theorem 1 is proved.

3.2 Proof of statement (b) of Theorem 1

Now using the results stated in Section 2 we shall apply the second order averaging theory to the previous differential equation. For this we put $F_{10}(r) \equiv 0$, which is equivalent to

$$
c_i = 0, \quad \text{for all } i \text{ even}. \tag{8}
$$

We must study the simple positive zeros of the function

$$
F_{20}(r) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{d}{dr} F_1(r,\theta) \left(\int_0^{\theta} F_1(r,s) ds \right) + F_2(r,\theta) \right) d\theta.
$$

We split the computation of the function $F_{20}(r)$ in two pieces, i.e. we define $2\pi F_{20}(r) = \Phi(r) + \Psi(r)$, where

$$
\begin{aligned}
\Phi(r) &= \int_0^{2\pi} \frac{d}{dr} F_1(r,\theta) \left(\int_0^{\theta} F_1(r,s) ds \right) d\theta, \\
\Psi(r) &= \int_0^{2\pi} F_2(r,\theta)\, d\theta = \int_0^{2\pi} \left(H_1(r,\theta) - \frac{1}{r} E_1(r,\theta)\, E_2(r,\theta) \right) d\theta.
\end{aligned}
$$

First we compute the integrals $\int_0^{2\pi} \delta_{i,j}(\theta) h_{p,q}(\theta) d\theta$, in the following lemma.

Lemma 2. *Let* $\eta_{i,j}^{p,q}(2\pi) = \int_0^{2\pi} \delta_{i,j}(\theta) h_{p,q}(\theta) d\theta$. *Then the following equalities hold:*

a) The integral $\eta_{2i+1,0}^{p,q}(2\pi)$ *is zero if* p *is odd or* q *is even, and is equal to*

$$\frac{1}{2i+1}\left(\sum_{l=0}^{i-1}\frac{2^l j(j-1)....(j-l+1)}{(2i-1)(2i-3)...(2i-2l-1)}\delta_{2i+p+2l-2;q+1}(2\pi)\right)$$
$$+\frac{1}{2i+1}\delta_{2i+p;q+1}(2\pi)$$

if p *is even and* q *is odd.*

b) The integral $\eta_{2i+1,2j+1}^{p,q}(2\pi)$ *is zero if* p *is odd or* q *is odd, and is equal to*

$$-\frac{1}{2j+2i+2}\left(\sum_{l=1}^{j-1}\frac{\left(2^l j(j-1)....(j-l+1)\right)\delta_{2i+p+2;2j-2l+q}(2\pi)}{(2j+2i)(2j+2i-2)...(2j+2i-2l+2)}\right)$$
$$-\frac{1}{2j+2i+2}\delta_{2i+p+2,2j+q}(2\pi)$$

if p *is even and* q *is even.*

c) The integral $\eta_{2i,2j+1}^{p,q}(2\pi)$ *is zero if* p *is even or* q *is odd, and is equal to*

$$-\frac{1}{2j+2i+1}\left(\sum_{l=1}^{j-1}\frac{\left(2^l j(j-1)....(j-l+1)\right)\delta_{2i+p+1;2j-2l+q}(2\pi)}{(2j+2i-1)(2j+2i-3)...(2j+2i-2l+1)}\right)$$
$$-\frac{1}{2j+2i+1}\delta_{2i+p+1,2j+q}(2\pi)$$

if p *is odd and* q *is even.*

(d) The integral $\eta_{2i+1,2j}^{p,q}(2\pi)$ *is zero if* p *is odd or* q *is even, and is equal to*

$$-\frac{1}{2j+2i+1}\left(\sum_{l=1}^{j-1}\frac{((2j-1)(2j-3)....(2j-2l+1))\delta_{2i+p+2;2j-2l+q-1}(2\pi)}{(2j+2i-1)(2j+2i-3)...(2j+2i-2l+1)}\right)$$
$$-\frac{1}{2j+2i+1}\delta_{2i+p+2;2j+q+1}(2\pi)$$
$$+\frac{(2j-1)(2j-3)....1}{(2j+2i+1)(2j+2i-1)...(2i+3)}\eta_{2i+1,0}^{p,q}(2\pi)$$

if p *is even and* q *is odd.*

Proof. Using the integral 12 of the appendix and taking into account $h_{i,j}(\theta)\,h_{p,q}(\theta) = h_{i+p,j+q}(\theta)$, we have

$$\eta^{p,q}_{2i+1,0}(2\pi) = \frac{1}{2i+1}\sum_{l=0}^{i-1}\frac{2^l j(j-1)....(j-l+1)}{(2i-1)(2i-3)...(2i-2l-1)}\int_0^{2\pi} h_{2i+p+2l-2;q+1}(\theta)d\theta + \frac{1}{2i+1}\int_0^{2\pi} h_{2i+p;q+1}(\theta)d\theta.$$

By using lemma 2, statement (a) follows. Using the integral 14 of the appendix and taking into account $h_{i,j}(\theta)\,h_{p,q}(\theta) = h_{i+p,j+q}(\theta)$, we have

$$\eta^{p,q}_{2i+1,2j+1}(2\pi) = -\frac{1}{2j+2i+2}\int_0^{2\pi} h_{2i+p+2,2j+q}(\theta)d\theta - \frac{1}{2j+2i+2}\left(\sum_{l=1}^{j-1}\frac{2^l j(j-1)....(j-l+1)}{(2j+2i)(2j+2i-2)...(2j+2i-2l+2)} * \int_0^{2\pi} h_{2i+p+2;2j-2l+q}(\theta)d\theta\right)$$

and

$$\eta^{p,q}_{2i,2j+1}(2\pi) = -\frac{1}{2j+2i+1}\int_0^{2\pi} h_{2i+p+1,2j+q}(\theta)d\theta - \frac{1}{2j+2i+1}\left(\sum_{l=1}^{j-1}\frac{2^l j(j-1)....(j-l+1)}{(2j+2i-1)(2j+2i-3)...(2j+2i-2l+1)} * \int_0^{2\pi} h_{2i+p+1;2j-2l+q}(\theta)d\theta.\right)$$

Using again lemma 2, statement (b), (c) follows. Using the integral 12 and 13 of the appendix and taking into account $h_{i,j}(\theta)\,h_{p,q}(\theta) = h_{i+p,j+q}(\theta)$ and using lemma 2, we obtain

$$\begin{aligned}\eta^{p,q}_{2i+1,2j}(2\pi) = {} & \frac{(2j-1)(2j-3)....1}{(2j+2i+1)(2j+2i-1)...(2i+3)}\eta^{p,q}_{2i+1,0}(2\pi) \\ & -\frac{1}{2j+2i+1} \\ & *\left(\sum_{l=1}^{j-1}\frac{((2j-1)(2j-3)....(2j-2l+1))\,\delta_{2i+p+2;2j-2l+q-1}(2\pi)}{(2j+2i-1)(2j+2i-3)...(2j+2i-2l+1)}\right) \\ & -\frac{1}{2j+2i+1}\left(\delta_{2i+p+2;2j+q+1}(2\pi)\right).\end{aligned}$$

Hence statement (d) of lemma 2 is proved. □

Proposition 2. *The integral $\Phi(r)$ can be expressed by*

$$\Phi(r) = r^{4\alpha+1} P_1\left(r^2\right).$$

where $P_1\left(r^2\right)$ is a polynomial in the variable r^2 of degree

$$\lambda_2 = \max\left\{\left[\frac{n}{2}\right] + \left[\frac{m-1}{2}\right]; \left[\frac{k}{2}\right] + \left[\frac{m-1}{2}\right] + 1\right\}.$$

Proof. First, we have

$$\begin{aligned}
F_1(r,\theta) &= \sum_{\substack{i=0 \\ \text{i odd}}}^{n} a_i h_{i,2\alpha+1}(\theta) r^{2\alpha+i} + \sum_{\substack{i=0 \\ i \text{ odd}}}^{k} d_i h_{i,2\alpha+3}(\theta) r^{2\alpha+i+2} \\
&\quad + \sum_{\substack{i=0 \\ i \text{ odd}}}^{m} c_i h_{i,2\alpha+2}(\theta) r^{2\alpha+i+1} + \sum_{\substack{i=0 \\ i \text{ even}}}^{n} a_i h_{i,2\alpha+1}(\theta) r^{2\alpha+i} \\
&\quad + \sum_{\substack{i=0 \\ i \text{ even}}}^{k} d_i h_{i,2\alpha+3}(\theta) r^{2\alpha+i+2} + \sum_{\substack{i=0 \\ i \text{ even}}}^{m} c_i h_{i,2\alpha+2}(\theta) r^{2\alpha+i+1} \\
&= \sum_{i=0}^{\left[\frac{n-1}{2}\right]} a_{2i+1} h_{2i+1,2\alpha+1}(\theta) r^{2\alpha+2i+1} + \sum_{i=0}^{\left[\frac{n}{2}\right]} a_{2i} h_{2i,2\alpha+1}(\theta) r^{2\alpha+2i} \\
&\quad + \sum_{i=0}^{\left[\frac{k-1}{2}\right]} d_{2i+1} h_{2i+1,2\alpha+3}(\theta) r^{2\alpha+2i+3} + \sum_{i=0}^{\left[\frac{k}{2}\right]} d_{2i} h_{2i,2\alpha+3}(\theta) r^{2\alpha+2i+2} \\
&\quad + \sum_{i=0}^{\left[\frac{m-1}{2}\right]} c_{2i+1} h_{2i+1,2\alpha+2}(\theta) r^{2\alpha+2i+2}.
\end{aligned}$$

Next we calculate the terms of this integral. First we have that

$$\begin{aligned}
\frac{d}{dr} F_1(r,\theta) &= \sum_{i=0}^{\left[\frac{n-1}{2}\right]} (2\alpha+2i+1) a_{2i+1} h_{2i+1,2\alpha+1}(\theta) r^{2\alpha+2i} \\
&\quad + \sum_{i=0}^{\left[\frac{k-1}{2}\right]} (2\alpha+2i+3) d_{2i+1} h_{2i+1,2\alpha+3}(\theta) r^{2\alpha+2i+2} \\
&\quad + \sum_{i=0}^{\left[\frac{m-1}{2}\right]} (2\alpha+2i+2) c_{2i+1} h_{2i+1,2\alpha+2}(\theta) r^{2\alpha+2i+1} \\
&\quad + \sum_{i=0}^{\left[\frac{n}{2}\right]} (2\alpha+2i) a_{2i} h_{2i,2\alpha+1}(\theta) r^{2\alpha+2i-1}
\end{aligned}$$

$$+\sum_{i=0}^{\left[\frac{k}{2}\right]}(2\alpha+2i+2)\,d_{2i}h_{2i,2\alpha+3}\left(\theta\right)r^{2\alpha+2i+1}$$

Then

$$\begin{aligned}\int_0^\theta F_1(r,s)ds &= \sum_{i=0}^{\left[\frac{n-1}{2}\right]} a_{2i+1}\delta_{2i+1,2a+1}\left(\theta\right)r^{2\alpha+2i+1}\\ &+\sum_{i=0}^{\left[\frac{k-1}{2}\right]} d_{2i+1}\delta_{2i+1,2\alpha+3}\left(\theta\right)r^{2\alpha+2i+3}\\ &+\sum_{i=0}^{\left[\frac{m-1}{2}\right]} c_{2i+1}\delta_{2i+1,2\alpha+2}\left(\theta\right)r^{2\alpha+2i+2}\\ &+\sum_{i=0}^{\left[\frac{n}{2}\right]} a_{2i}\delta_{2i,2\alpha+1}\left(\theta\right)r^{2\alpha+2i}\\ &+\sum_{i=0}^{\left[\frac{k}{2}\right]} d_{2i}\delta_{2i,2\alpha+3}\left(\theta\right)r^{2\alpha+2i+2}.\end{aligned}$$

By using lemma 2, from the 25 main products of $\Phi\left(r\right)$ only the following 4 are not zero when we integrate them between 0 and 2π. So the terms of $\Phi\left(r\right)$ which will contribute to $F_{20}(r)$ are :

$$\begin{aligned}\Phi\left(r\right) &= \sum_{i=0}^{\left[\frac{n}{2}\right]}\sum_{p=0}^{\left[\frac{m-1}{2}\right]}(2\alpha+2i)\,a_{2i}c_{2p+1}\eta_{2p+1,2\alpha+2}^{2i,2\alpha+1}\left(2\pi\right)r^{4\alpha+2i+2p+1}\\ &+\sum_{i=0}^{\left[\frac{k}{2}\right]}\sum_{p=0}^{\left[\frac{m-1}{2}\right]}(2\alpha+2i+2)\,d_{2i}c_{2p+1}\eta_{2p+1,2\alpha+2}^{2i,2\alpha+3}\left(2\pi\right)r^{4\alpha+2i+2p+3}\\ &+\sum_{i=0}^{\left[\frac{m-1}{2}\right]}\sum_{p=0}^{\left[\frac{n}{2}\right]}(2\alpha+2i+2)\,c_{2i+1}a_{2p}\eta_{2p,2\alpha+1}^{2i+1,2\alpha+2}\left(2\pi\right)r^{4\alpha+2i+2p+1}\\ &+\sum_{i=0}^{\left[\frac{m-1}{2}\right]}\sum_{p=0}^{\left[\frac{k}{2}\right]}(2\alpha+2i+2)\,c_{2i+1}d_{2p}\eta_{2p,2\alpha+3}^{2i+1,2\alpha+2}\left(2\pi\right)r^{4\alpha+2i+2p+3}\\ &= r^{4\alpha+1}P_1\left(r^2\right)\end{aligned}$$

where P_1 is polynomial in the variable r^2 of degree λ_2,

$$\lambda_2=\max\left\{\left[\frac{m-1}{2}\right]+\left[\frac{n}{2}\right];\left[\frac{m-1}{2}\right]+\left[\frac{k}{2}\right]+1\right\}.$$

Finally, we obtain $\Phi(r)$ is a polynomial in the variable r^2 of the form

$$\Phi(r) = r^{4\alpha+1} P_1\left(r^2\right).$$

This completes the proof of the Proposition 2. $\square$

In order to complete the computation of $F_{20}(r)$ we must determine the function $\Psi(r)$.

Proposition 3. *The integral $\Psi(r)$ can be expressed by*

$$\Psi(r) = r^{2\alpha+1}\left(P_2\left(r^2\right) + r^{2\alpha} P_3\left(r^2\right)\right)$$

where $P_2\left(r^2\right)$ is a polynomial in the variable r^2 of degree

$$\lambda_1 = \left[\frac{m}{2}\right],$$

$P_3\left(r^2\right)$ is a polynomial in the variable r^2 of degree

$$\lambda_3 = \max\left\{\left[\frac{m-1}{2}\right] + \left[\frac{n}{2}\right]; \left[\frac{m-1}{2}\right] + \left[\frac{k}{2}\right] + 1\right\}.$$

Proof. Firstly we calculate,

$$\begin{aligned}\int_0^{2\pi} H_1(r,\theta)\, d\theta &= \sum_{i=0}^{n} A_i r^{2\alpha+i} \int_0^{2\pi} h_{i,2\alpha+1}(\theta)\, d\theta + \sum_{i=0}^{k} D_i r^{2\alpha+i+2} \int_0^{2\pi} h_{i,2\alpha+3}(\theta)\, d\theta \\ &\quad + \sum_{i=0}^{m} C_i r^{2\alpha+i+1} \int_0^{2\pi} h_{i,2\alpha+2}(\theta)\, d\theta.\end{aligned}$$

Using lemma 2, we conclude that $\int_0^{2\pi} h_{i,2\alpha+1}(\theta)\, d\theta = \int_0^{2\pi} h_{i,2\alpha+3}(\theta)\, d\theta = 0$, and we have

$$\int_0^{2\pi} H_1(r,\theta)\, d\theta = \sum_{\substack{i=0 \\ \text{i even}}}^{m} C_i r^{2\alpha+i+1} \int_0^{2\pi} h_{i,2\alpha+2}(\theta)\, d\theta = \sum_{i=0}^{[\frac{m}{2}]} C_i r^{2\alpha+i+1} \int_0^{2\pi} h_{i,2\alpha+2}(\theta)\, d\theta.$$

Then

$$\begin{aligned}\int_0^{2\pi} H_1(r,\theta)\, d\theta &= \pi \sum_{i=0}^{[\frac{m}{2}]} C_{2i} \delta_{2i,2\alpha+2}(2\pi)\, r^{2\alpha+2i+1} \\ &= r^{2\alpha+1} P_2\left(r^2\right)\end{aligned}$$

where P_2 is a polynomial in the variable r^2 of degree λ_1.

Finally, we shall study the contribution of the second part $\int_0^{2\pi} \frac{1}{r} E_1(r,\theta) E_2(r,\theta)\, d\theta$ of $F_2(r,\theta)$ to $F_{20}(r)$. Using the expressions of $E_1(r,\theta)$ and $E_2(r,\theta)$ and taking into account that $c_i = 0$ for all i even, we have :

$$\begin{aligned} E_1(r,\theta) = & \sum_{i=0}^{\left[\frac{n-1}{2}\right]} a_{2i+1} h_{2i+1,2\alpha+1}(\theta)\, r^{2\alpha+2i+1} + \sum_{i=0}^{\left[\frac{k-1}{2}\right]} d_{2i+1} h_{2i+1,2\alpha+3}(\theta)\, r^{2\alpha+2i+3} \\ & + \sum_{i=0}^{\left[\frac{m-1}{2}\right]} c_{2i+1} h_{2i+1,2\alpha+2}(\theta)\, r^{2\alpha+2i+2} + \sum_{i=0}^{\left[\frac{n}{2}\right]} a_{2i} h_{2i,2\alpha+1}(\theta)\, r^{2\alpha+2i} \\ & + \sum_{i=0}^{\left[\frac{k}{2}\right]} d_{2i} h_{2i,2\alpha+3}(\theta)\, r^{2\alpha+2i+2} \end{aligned}$$

and

$$\begin{aligned} E_2(r,\theta) = & \sum_{p=0}^{\left[\frac{n-1}{2}\right]} a_{2p+1} h_{2p+2,2\alpha}(\theta)\, r^{2\alpha+2p+1} + \sum_{p=0}^{\left[\frac{k-1}{2}\right]} d_{2p+1} h_{2p+2,2\alpha+2}(\theta)\, r^{2\alpha+2p+3} \\ & + \sum_{p=0}^{\left[\frac{m-1}{2}\right]} c_{2p+1} h_{2p+2,2\alpha+1}(\theta)\, r^{2\alpha+2p+2} + \sum_{p=0}^{\left[\frac{n}{2}\right]} a_{2p} h_{2p+1,2\alpha}(\theta)\, r^{2\alpha+2p} \\ & + \sum_{p=0}^{\left[\frac{k}{2}\right]} d_{2p} h_{2p+1,2\alpha+2}(\theta)\, r^{2\alpha+2p+2}. \end{aligned}$$

Using Lemma 2, from the 25 main products of $\int_0^{2\pi} \frac{1}{r} E_1(r,\theta) E_2(r,\theta)\, d\theta$, only the following 4 are not zero when we integrate them between 0 and 2π, So the terms which will contribute to $F_{20}(r)$ are

$$\begin{aligned} \int_0^{2\pi} \frac{1}{r} E_1(r,\theta) E_2(r,\theta)\, d\theta = & \sum_{i=0}^{\left[\frac{n}{2}\right]} \sum_{p=0}^{\left[\frac{m-1}{2}\right]} a_{2i} c_{2p+1} \delta_{2i+2p+2,4\alpha+2}(2\pi)\, r^{4\alpha+2i+2p+1} \\ & + \sum_{i=0}^{\left[\frac{k}{2}\right]} \sum_{p=0}^{\left[\frac{m-1}{2}\right]} d_{2i} c_{2p+1} \delta_{2i+2p+2,4\alpha+2}(2\pi)\, r^{4\alpha+2i+2p+3} \\ & + \sum_{i=0}^{\left[\frac{m-1}{2}\right]} \sum_{p=0}^{\left[\frac{n}{2}\right]} c_{2i+1} a_{2p} \delta_{2i+2p+2,4\alpha+2}(2\pi)\, r^{4\alpha+2i+2p+1} \\ & + \sum_{i=0}^{\left[\frac{m-1}{2}\right]} \sum_{p=0}^{\left[\frac{k}{2}\right]} c_{2i+1} d_{2p} \delta_{2i+2p+2,4\alpha+2}(2\pi)\, r^{4\alpha+2i+2p+3} \\ & = r^{4\alpha+1} P_3\left(r^2\right) \end{aligned}$$

where P_3 is a polynomial in the variable r^2 of degree

$$\lambda_3 = \max\left\{ \left[\frac{m-1}{2}\right] + \left[\frac{n}{2}\right]; \left[\frac{m-1}{2}\right] + \left[\frac{k}{2}\right] + 1 \right\}.$$

Then, we obtain $\Psi(r)$ is a polynomial in the variable r^2

$$\Psi(r) = r^{2\alpha+1}\left(P_2\left(r^2\right) + r^{2\alpha} P_3\left(r^2\right)\right)$$

of degree

$$\lambda_{\Psi(r)} = \max\{\lambda_1, \lambda_3 + \alpha\}.$$

Finally, we obtain $F_{20}(r)$ is a polynomial in the variable r^2 of the form

$$F_{20}(r) = \frac{r^{2\alpha+1}}{2\pi}\left(r^{2\alpha}P_1\left(r^2\right) + P_2\left(r^2\right) + r^{2\alpha}P_3\left(r^2\right)\right).$$

To find the real positive roots of F_{20} we must find the zeros of a polynomial in r^2 of degree $\lambda = \max\{\lambda_1, \lambda_2 + \alpha, \lambda_3 + \alpha\}$. This yields that F_{20} has at most λ real positive roots. Hence, Theorem 1 is proved. Moreover, we can choose the coefficients a_i,c_i, d_i,A_i,C_i,D_i in such a way that F_{20} has exactly λ real positive roots. This completes the proof of Theorem 1. □

4 Example

We consider the differential system 2 with $k = n = 1, m = 3, \alpha = 1$

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - \varepsilon(\left(-\frac{118}{65} + x\right)y^2 + \left(\left(-\frac{13}{427}x + \frac{1}{61}x^3\right)\right)y^3 + (1+x)\,y^4) \\ -\varepsilon^2\left(\left(-1 - \frac{1}{4}x\right)y^2 + \left(\frac{1}{80} + \frac{967}{34\,160}x^2 + \frac{1}{8}x^3\right)y^3 - xy^4\right) \end{cases} \tag{9}$$

An easy computation shows that $F_{10}(r)$ is identically zero, so to look for the limit cycles, we must solve the equation $F_{20}(r) = 0$ which is equivalent to

$$-\frac{1}{1280}r^3\left(r^6 - 6r^4 + 11r^2 - 6\right) = 0$$

This equation has exactly three positive roots $r_1 = 1, r_2 = \sqrt{2}, r_3 = \sqrt{3}$. According with Theorem 1, that system (9) has exactly three limit cycles bifurcating from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$.

5 Appendix

In this appendix, we recall some formulas used during this article; for more details see [7]. For $i \geq 0$ and $j \geq 0$, we have

$$\begin{aligned} \int_0^\theta \cos^i s \sin^j s ds &= \frac{\cos^{i-1}\theta \sin^{j+1}\theta}{i+j} + \frac{i-1}{i+\alpha}\int_0^\theta \cos^{i-2} s \sin^j s ds \\ &= \frac{\cos^{i-1}\theta \sin^{j+1}\theta}{i+j} + \frac{\alpha-1}{i+\alpha}\int_0^\theta \cos^{i} s \sin^{j-2} s ds, \end{aligned} \tag{10}$$

$$\begin{aligned}\int_0^\theta \cos^{2i} s ds &= \frac{\sin\theta}{2i}\sum_{l=1}^{i-1}\frac{(2i-1)(2i-3)....(2i-2l+1)}{2^l(i-1)(i-2).(i-l)}\cos^{2i-2l-1}\theta \\ &+\frac{\sin\theta}{2i}\cos^{2i-1}\theta+\frac{(2i-1)(2i-3)....1}{2^i i!}\theta \\ &= \frac{1}{2^{2i-1}}\sum_{l=0}^{i-1}\binom{2i}{l}\frac{\sin 2(i-l)\theta}{2(i-l)}+\frac{1}{2^{2i}}\binom{2i}{i}\theta,\end{aligned} \tag{11}$$

$$\begin{aligned}\int_0^\theta \cos^{2i+1} s ds &= \frac{\sin\theta}{2i+1}\sum_{l=1}^{i-1}\frac{2^{l+1}i(i-1).....(i-l)}{(2i-1)(2i-3)....(2i-2l-1)}\cos^{2i-2l-2}\theta \\ &+\frac{\sin\theta}{2i+1}\cos^{2i}\theta \\ &= \frac{1}{2^{2i}}\sum_{l=0}^{i-1}\binom{2i+1}{l}\frac{\sin(2i-2l+1)\theta}{(2i-2l+1)},\end{aligned} \tag{12}$$

where $\binom{2i}{p}=\frac{2i!}{p!(2i-p)!}$

$$\begin{aligned}&\int_0^\theta \cos^i s\sin^{2j} s ds \\ &= -\frac{\cos^{i+1}\theta}{2j+1}\sum_{l=1}^{j-1}\frac{(2j-1)(2j-3)...(2j-2l+1)}{(2j+i-2)(2j+i-4)...(2j+i-2l)}\sin^{2j-2l-1}\theta \\ &+\frac{(2j-1)(2j-3)...1}{(2j+i)(2j+i-2)...(i+2)}\int_0^\theta \cos^i s ds,\end{aligned} \tag{13}$$

$$\begin{aligned}&\int_0^\theta \cos^i s\sin^{2j+1} s ds \\ &= -\frac{\cos^{i+1}\theta}{2j+i+1}\sum_{l=1}^{j-1}\frac{2^l j(j-1).....(j-l+1)}{(2j+i-1)(2j+i-3)...(2j+i-2l+1)}\sin^{2j-2l}\theta \\ &-\frac{\cos^{i+1}\theta}{2j+i+1}\sin^{2\alpha}\theta.\end{aligned} \tag{14}$$

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Methods of construction of Hausdorff extensions

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Abstract. In this paper we study the extensions of Hausdorff spaces generated by discrete families of open sets.

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1 Introduction

Any space is considered to be a Hausdorff space. We use the terminology from [3]. For any completely regular space X denote by βX the Stone-Čech compactification of the space X.

Fix a space X. A space eX is an extension of the space X if X is a dense subspace of eX. If eX is a compact space, then eX is called a compactification of the space X. The subspace $eX \setminus X$ is called a remainder of the extension eX.

Denote by $Ext(X)$ the family of all extensions of the space X. If X is a completely regular space, then by $Ext_\rho(X)$ we denote the family of all completely regular extensions of the space X. Obviously, $Ext_\rho(X) \subset Ext(X)$. Let $Y, Z \in Ext(X)$ be two extensions of the space X. We consider that $Z \leq Y$ if there exists a continuous mapping $f : Y \longrightarrow Z$ such that $f(x) = x$ for each $x \in X$. If $Z \leq Y$ and $Y \leq Z$, then we say that extensions Y and Z are equivalent and there exists a unique homeomorphism $f : Y \longrightarrow Z$ of Y onto Z such that $f(x) = x$ for each $x \in X$. We identify the equivalent extensions. In this case $Ext(X)$ and Ext_ρ are partially ordered sets.

Let τ be an infinite cardinal. Denote by $O(\tau)$ the set of all ordinal numbers of cardinality $< \tau$. We consider that τ is the first ordinal number of the cardinality τ. For any $\alpha \in O(\tau)$ we put $O(\alpha) = \{\beta \in O(\tau) : \beta < \alpha\}$. In this case $O(\tau)$ is well ordered set such that $|O(\tau)| = \tau$ and $|O(\alpha)| < \tau$ for every $\alpha \in O(\tau)$.

A point $x \in X$ is called a $P(\tau)$-point of the space X if for any non-empty family γ of open subsets of X for which $x \in \cap\gamma$ and $|\gamma| < \tau$ there exists an open subset U of X such that $x \in U \subset \cap\gamma$. If any point of X is a $P(\tau)$-point, then we say that X is a $P(\tau)$-space.

Any point is an $\aleph_0$-point. If $\tau = \aleph_1$, then the $P(\tau)$-point is called the P-point.

2 Hausdorff extensions of discrete spaces

Let τ be an infinite cardinal. Let E be a discrete space of the cardinality $\geq \tau$.

A family η of subsets of E is called τ-centered if the family η is non-empty, $\cap\eta = \emptyset$, $\emptyset \notin \eta$ and for any subfamily $\zeta \subset \eta$, with cardinality $|\zeta| < \tau$, there exists $l \in \eta$ such that $L \subset \cap\zeta$.

Two families η and ζ of subsets of the space E are almost disjoint if there exist $L \in \eta$ and $Z \in \zeta$ such that $L \cap Z = \emptyset$.

Any family of subsets is ordered by the following order: $L \preceq H$ if and only if $H \subset L$. Relative to this oder some families of sets are well-ordered.

Proposition 1. *Let $k = |E| \geq \tau$ and $\Sigma\{k^m : m < \tau\} = k$. Then on E there exists a set Ω of well-ordered almost disjoint τ-centered families such that $|\Omega| = k^\tau$ and $|\eta| = \tau$ for each $\eta \in \Omega$.*

Proof. We fix an element $0 \in E$. For every $\alpha \in O(\tau)$ we put $E_\alpha = E$ and $0_\alpha = 0$. Then $E^\tau = \Pi\{E_\alpha : \alpha \in O(\tau)\}$. For each $x = (x_\alpha : \alpha \in O(\tau)) \in E^\tau$ we put $\phi(x) = sup\{0, \alpha : x_\alpha \neq 0_\alpha\}$. Obviously, $0 \leq \phi(x) \leq \tau$. Let $D = \{x = (x_\alpha : \alpha \in O(\tau)) \in E^\tau : \phi(x) < \tau\}$. By construction, $|D| = \Sigma\{k^m : m < \tau\} = k$ and $|E^\tau| = k^\tau$. Since $|E| = |D|$, we can fix a one-to-one mapping $f : E \longrightarrow D$. Fix a point $x = (x_\alpha : \alpha \in O(\tau)) \in E^\tau$. For any $\beta \in O(\tau)$ we put $V(x, \beta) = \{y = (y_\alpha : \alpha \in O(\tau)) \in E^\tau : y_\alpha = x_\alpha$ for every $\alpha \leq \beta\}$ and $\eta_x = \{L(x, \beta) = f^{-1}(D \cap V(x, \beta) : \beta \in 0(\tau)\}$. Then $\Omega = \{\eta_x : x \in E^\tau\}$ is the desired set of τ-centered families. □

Remark 1. Let $|E| = k \geq \tau$. Since on E there exists k mutually disjoint subsets of cardinality τ, on E there exists a set Φ of well-ordered almost disjoint τ-centered families such that $|\Phi| \geq k$ and $|\eta| = \tau$ for each $\eta \in \Phi$.

Fix a set Φ of almost disjoint τ-centered families of subsets of the set E. We put $e_\Phi E = E \cup \Phi$. On $e_\Phi E$ we construct two topologies.

Topology $T^s(\Phi)$. The basis of the topology $T^s(\Phi)$ is the family $\mathcal{B}^s(\Phi) = \{U_L = L \cup \{\eta \in \Phi : H \subset L$ for some $H \in \eta\} : L \subset E\}$.

Topology $T_m(\Phi)$. For each $x \in E$ we put $B_m(x) = \{\{x\}\}$. For every $\eta \in \Phi$ we put $B_m(\eta) = \{V_{(\eta,L)} = \{\eta\} \cup L : L \in \eta\}$. The basis of the topology $T_m(\Phi)$ is the family $\mathcal{B}_m(\Phi) = \cup\{B_m(x) : x \in e_\Phi E\}$.

Theorem 1. *The spaces $(e_\Phi E, T^s(\Phi))$ and $(e_\Phi E, T_m(\Phi))$ are Hausdorff zero-dimensional extensions of the discrete space E, and $T^s(\Phi) \subset T_m(\Phi))$. In particular, $(e_\Phi E, T^s(\Phi)) \leq (e_\Phi E, T_m(\Phi))$.*

Proof. The inclusion $T^s(\Phi) \subset T_m(\Phi))$ follows from the constructions of the topologies $T^s(\Phi)$ and $T_m(\Phi))$. If $L \in \eta \in \Phi$, then $\eta \in clL$. Hence the set E is dense in the spaces $(e_\Phi E, T^s(\Phi))$ and $(e_\Phi E, T_m(\Phi))$. If the families $\eta, \zeta \in \Phi$ are distinct, then there exist $L \in \eta$ and $Z \in \zeta$ such that $L \cap Z = \emptyset$. Then $U_L \cap U_Z = \emptyset$. If $L \subset E$ and $|L| < \tau$, then L is an open-and-closed subset of the spaces $(e_\Phi E, T^s(\Phi))$ and $(e_\Phi E, T_m(\Phi))$. Hence the topologies $T^s(\Phi)$ and $T_m(\Phi)$ are discrete on E and the spaces $(e_\Phi E, T^s(\Phi))$ and $(e_\Phi E, T_m(\Phi))$ are Hausdorff extensions of the discrete space E. Since the sets U_L and $V_{(\eta,L)}$ are open-and-closed in the topologies $T^s(\Phi)$ and $T_m(\Phi))$, respectively, the spaces $(e_\Phi E, T^s(\Phi))$ and $(e_\Phi E, T_m(\Phi))$ are zero-dimensional. □

Theorem 2. *The spaces $(e_\Phi E, T^s(\Phi))$ and $(e_\Phi E, T_m(\Phi))$ are $P(\tau)$-spaces.*

Proof. Fix $\eta \in \Phi$. If $\zeta \subset \eta$ and $|\zeta| < \tau$, then there exists $L(\zeta) \in \eta$ such that $L(\zeta) \subset \cap\zeta$. From this fact immediately follows that $(e_\Phi E, T_m(\Phi))$ is a $P(\tau)$-space. Assume that $\{L_\mu : \mu \in M\}$ is a family of subsets of E, $|M| < \tau$, $\eta \in \Phi$ and $\eta \in \cap\{L_\mu : \mu \in M\}$. Then there exists $L \in \eta$ such that $L \subset \cap\{L_\mu : \mu \in M\}$. Thus $\eta \in U_L \in \cap\{U_{L_\mu} : \mu \in M\}$. From this fact immediately follows that $(e_\Phi E, T^s(\Phi))$ is a $P(\tau)$-space. □

Corollary 1. *If $T^s(\Phi) \subset \mathcal{T} \subset T_m(\Phi))$, then $(e_\Phi E, \mathcal{T})$ is a Hausdorff extension of the discrete space E, and $(e_\Phi E, T^s(\Phi)) \leq (e_\Phi E, \mathcal{T}) \leq (e_\Phi E, T_m(\Phi))$.*

Theorem 3. *The space $(e_\Omega E, T^s(\Omega))$, where Ω is the set of well-ordered almost disjoint τ-centered families from Proposition 1, is a zero-dimensional paracompact space with character $\chi(e_\Omega E, T^s(\Omega)) = \tau$ and weight $\Sigma\{|E|^m : m < \tau\}$.*

Proof. We consider that $E = D$. The family $\mathcal{B} = \{\{x\} : x \in D\} \cup \{V(x,\beta) : x \in E^\tau, \beta \in O(\tau)\}$ is a base of the topology $T^s(\Omega)$. If $U, V \in \mathcal{B}$, then either $U \subset V$, or $V \subset U$, or $U \cap V = \emptyset$. From the A. V. Arhangel'skii Theorem [1] it follows that $(e_\Omega E, T^s(\Omega))$ is a zero-dimensional paracompact space. □

3 Construction of Hausdorff extensions

Let τ be an infinite cardinal. Fix a $P(\tau)$-space X. Let $\gamma = \{H_\mu : \mu \in M\}$ be a discrete family of non-empty open subsets of the space X and $\tau \leq |M|$. For any $\mu \in M$ we fix a point $e_\mu \in U_\mu$ and a family $\xi_\mu = \{H_{(\mu,\alpha)} : \alpha \in O(\tau)\}$ of open subsets of X such that $e_\mu \in \cap\xi_\mu$ and $H_{(\mu,\beta)} \subset H_{(\mu,\alpha)} \subset H_\mu$ for all $\alpha \in O(\tau)$ and $\beta \in O(\alpha)$. Then $E = \{e_\mu : \mu \in M\}$ is a discrete closed subspace of the space X.

Consider the Hausdorff extension rE of the space E. We put $e_{rE}X = X \cup (rE \backslash E)$. In $e_{rE}X$ we construct the topology $\mathcal{T} = T(\gamma, E, \xi_\mu, \tau)$ as follows:

– we consider X as an open subspace of $e_{(E,Y)}X$;

– let T_X be the topology of X and T_{rE} be the topology of the space rE;

– if $V \in T_{rE}$, then $e_\alpha V = V \cup \{H_{(\mu,\alpha)} : e_\mu \in V\}$;

– $\mathcal{B} = T_X \cup \{e_\alpha V : V \in T_{rE}\}$ is an open base of the topology $\mathcal{T} = T(\gamma, E, \xi_\mu, \tau)$.

Theorem 4. *The space $(e_{(E,Y)}X, T(\gamma, E, \xi_\mu, \tau))$ is a Hausdorff extension of the space X.*

Proof. If $V, W \in T_{rE}$, then:

– $e_\alpha W \subset e_\alpha V$ if and only if $W \subset V$;

– $e_\alpha W \cap e_\alpha V = \emptyset$ if and only if $W \cap tV = \emptyset$;

– $e_\alpha V \cap rE = V$.

These facts and Theorem 1 complete the proof. □

Theorem 5. *If rE is a $P(\tau)$-space, then $(e_{(E,Y)}X, T(\gamma, E, \xi_\mu, \tau))$ is a $P(\tau)$-space too. Moreover, $\chi(e_{(E,Y)}X, T(\gamma, E, \xi_\mu, \tau)) = \chi(X) + \chi(rE)$ and*

$$w(e_{(E,Y)}X, T(\gamma, E, \xi_\mu, \tau)) = w(X) + w(rE).$$

Proof. Follows immediately from the construction of the sets $e_\alpha V$. □

Theorem 6. *Assume that the spaces rE and X are zero-dimensional, and the sets $H_{(\mu,\alpha)}$ are open-and-closed in X. Then:*

1. $(e_{(E,Y)}X, T(\gamma, E, \xi_\mu, \tau))$ is a zero-dimensional space.

2. The space $(e_{(E,Y)}X, T(\gamma, E, \xi_\mu, \tau))$ is paracompact if and only if the spaces rE and X are paracompact.

Proof. If the set V is open-and-closed in rE and the sets $H_{(\mu,\alpha)}$ are open-and-closed in X, then the sets $e_\alpha V$ are open-and-closed in $(e_{(E,Y)}X, T(\gamma, E, \xi_\mu, \tau))$. If $\{V_\lambda : \lambda \in L\}$ is a discrete cover of rE, and $\alpha(\lambda) \in O(\tau)$, then $\{e_{\alpha(\lambda)}V_\lambda : \lambda \in L\}$ is a discrete family of open-and-closed sets. This fact completes the proof. □

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