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Generalized hypergeometric systems and the fifth and sixth Painlevé equations

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Abstract. This paper concerns (generalized) hypergeometric systems associated with the fifth and sixth Painlevé equations, which are the second order nonlinear ordinary differential equations. The Painlevé equations govern monodromy preserving deformations of certain second order linear scalar equations. We reduce these scalar equations to generalized hypergeometric systems.

Mathematics subject classification: 34M55.
Keywords and phrases: Reduction problems, Painlevé equations, Bäcklund transformations.

1 Introduction

In some problems of the general theory of ordinary differential equations (ODEs) it is very efficient to study systems of ODEs rather than single scalar equations. The benefit is that the problem can be studied by using the matrix calculus and most likely can easily be generalized. Thus, the methods of reduction of a linear differential equation with a finite number of regular and irregular singularities to a system of linear differential equations of some canonical form are needed. In general, the reduction problems are difficult and only partial results are available in this direction, see for instance [1, 9].

The current paper concerns the study of the second order linear differential equation

$$\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = 0, \tag{1}$$

where $p_1(x)$ and $p_2(x)$ are certain rational functions (exact formulas are given in the sections below). The isomonodromy deformations of equation (1) with such choice of coefficients lead to the famous fifth and sixth Painlevé equations [12]. The solutions of these equations, the Painlevé transcendents, are nonlinear special functions which appear in many areas of modern mathematics and mathematical physics (random matrix theory, algebraic geometry, integrable systems, topological field theories and many others). The Painlevé equations are second order nonlinear differential equations of the form

$$\frac{d^2\lambda}{dt^2} = R\left(t, \lambda, \frac{d\lambda}{dt}\right),$$

where R is a rational function, having the Painlevé property, which is that their general solutions possess no movable critical points (see, for instance, [7] for definitions). Moreover, the Painlevé transcendents are not expressible in terms of classical linear special functions. Nowadays, the interest in the Painlevé equations is growing due to numerous applications.

There are many systems one can associate with the scalar differential equation (1). In this paper we are interested in the systems of the form

$$(x-B)\frac{dY}{dx} = AY, \tag{2}$$

where the matrix A does not depend on x and the diagonal elements of the matrix B include all singularities of equation (1). We call such systems generalized hypergeometric systems. If the matrix B is diagonal, then we call the system a hypergeometric system following [9]. We remark that the systems we consider can also be viewed as generalized Okubo systems, but we want to distinguish apparent singularities (there is a holomorphic basis of solutions at such points) and include them as elements of the matrix B. Apparent singularities, as will be discussed later on, play a special role in monodromy preserving deformations of equation (1), and hence we are interested in studying the problem of reduction (1) to generalized hypergeometric systems. Systems of the type (2) recently appeared in the study of the Heun equation [3].

In this paper, we first consider equation (1) with 4 regular singularities $x = 0, 1, \infty, t$ and one apparent singularity λ. The scalar equation (1) is Fuchsian in this case, and the algorithm of reduction is known [9]. We explicitly compute the hypergeometric system (2), where the 4×4 matrix B is diagonal $(0, 1, t, \lambda)$ and the constant matrix A is the sum of a lower triangular matrix and a nilpotent matrix having elements i, $i+1$ equal to 1 and all others equal to zero. If the parameter t moves in the complex plane, the isomonodromy deformations of (1) (deformations which preserve the monodromy group of the equation) lead to the sixth Painlevé equation (P_{VI}) for the function $\lambda(t)$. From the works of Okamoto, Noumi and others it is known that the parameter space of the sixth Painlevé equation admits the action of the extended affine Weyl group. The corresponding action of the group on solutions of (P_{VI}) is known as the action of the group of Bäcklund transformations. One of such Bäcklund transformations was recently rederived in [4] from the integral transformation of 2×2 system. Thus, we are interested to understand the action of this transformation on the hypergeometric system. This gives a new insight into the nature of the Painlevé equations and their Bäcklund transformations. In particular, the action of the Bäcklund transformation gives a new hypergeometric system with a new apparent singularity and different eigenvalues and diagonal elements.

It is also possible [5] to consider other 4×4 systems, called Okubo systems, equivalent to equation (1), but the apparent singularity is not singled out there in the diagonal matrix B as in the hypergeometric system we consider. Other types of systems of differential equations associated with the sixth Painlevé equation and the action of the Bäcklund transformations on them are considered in [10, 11]. Other

Painlevé equations are also studied from this perspective, see for instance the paper [2] concerning the fourth Painlevé equation. We also remark that equation (1) gives the Heun equation as the result of the confluence process when the apparent singularity tends to one of 4 other regular singularities of (1) and the 3×3 hypergeometric system associated with the Heun equation was useful in finding the integral transformations between its solutions [3].

As the result of the confluence process when one of regular singularities of (1) associated with (P_{VI}) coalesces with another regular singularity, we get a linear equation the isomonodromy deformations of which give the fifth Painlevé equation (P_V). In this case, equation (1) has two regular singularities $x = 0, \infty$, one irregular singularity $x = 1$ and one apparent singularity λ (which becomes the solution of (P_V) viewed as a function of the deformation parameter t). We introduce a generalized hypergeometric system and compute it explicitly for the linear system associated with the fifth Painlevé equation. The system we present encodes the information of the generalized Riemann scheme (information about the singularities of the scalar equation) in elements of the matrix B, which is not diagonal in this case, and diagonal elements and the eigenvalues of the matrix A. We remark that the generalized Okubo type systems have been recently studied in [8], but as remarked above, the apparent singularity does not appear in the diagonal elements of the matrix B.

The paper is organized as follows. In the following two sections we consider the problems outlined above in detail. The main results and open problems are summarized in the last section.

2 A hypergeometric system associated with the sixth Painlevé equation

Equation (1) with

$$p_1(x) = \frac{1-\theta_0}{x} + \frac{1-\theta_1}{x-1} + \frac{1-\theta_2}{x-t} - \frac{1}{x-\lambda}, \tag{3}$$

$$p_2(x) = \frac{k_1(k_2+1)}{x(x-1)} + \frac{\lambda(\lambda-1)\mu}{x(x-1)(x-\lambda)} - \frac{t(t-1)H_{VI}}{x(x-1)(x-t)}, \tag{4}$$

$$\begin{aligned} t(t-1)H_{VI} = k_1(k_2+1)(\lambda-t) + \lambda(\lambda-1)(\lambda-t)\mu^2 - \\ -(\theta_0(\lambda-1)(\lambda-t) + \theta_1\lambda(\lambda-t) + (\theta_2-1)\lambda(\lambda-1))\mu \end{aligned} \tag{5}$$

and

$$k_1 + k_2 + \theta_0 + \theta_1 + \theta_2 = 0 \tag{6}$$

is a Fuchsian equation with 4 regular singularities $x = 0, 1, \infty, t$ and one apparent singularity λ.

The sixth Painlevé equation is the following nonlinear ordinary differential equation of second order for the unknown function $\lambda(t)$:

$$\lambda'' = \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)(\lambda')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\lambda' +$$

$$\frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left(\alpha+\beta\frac{t}{\lambda^2}+\gamma\frac{t-1}{(\lambda-1)^2}+\delta\frac{t(t-1)}{(\lambda-t)^2}\right),$$

where $'$ stands for the derivation with respect to the independent variable t and α, β, γ, δ are complex parameters.

One of standard ways to derive the sixth Painlevé equation is to study monodromy preserving deformations of a second order Fuchsian differential equation on $\mathbb{P}^1$ with four regular singular points and one apparent singularity [12], i.e., to consider deformations of equation (1) with (3)–(6). This leads to a system of partial differential equations, and the compatibility condition gives a Hamiltonian system

$$\frac{d\lambda}{dt}=\frac{\partial H_{VI}}{\partial\mu},\quad \frac{d\mu}{dt}=-\frac{\partial H_{VI}}{\partial\lambda} \tag{7}$$

and, hence, (P_{VI}) for the function $\lambda(t)$ with

$$\alpha=\frac{(2k_1+\theta_0+\theta_1+\theta_2-1)^2}{2},\ \ \beta=-\frac{\theta_0^2}{2},\ \ \gamma=\frac{\theta_1^2}{2},\ \ \delta=\frac{1-\theta_2^2}{2}.$$

The reader is referred to [7, 12] for further details.

Each element of the hypergeometric system (2) is written as

$$(x-\lambda_j)y_j'=\sum_{k=1}^{4}\alpha_{j,k}y_k,\ \ j\in\{1,\dots,4\},$$

where $\lambda_1=0,\ \lambda_2=1,\ \lambda_3=t,\ \lambda_4=\lambda$. The matrix B in (2) is diagonal with finite singularities of (1) on the diagonal. The matrix A in (2) is independent of x and we impose condition that it is the sum of a lower triangular matrix and a nilpotent matrix having elements i, $i+1$ equal to 1 and all others equal to zero. Hence, $\alpha_{i,j}=0,\ j>i+1$, and $\alpha_{i,i+1}=1$. Because of the special form of the system, we can find successively

$$\begin{aligned}
&y_2=xy_1'+f_0(x)y_1,\\
&y_3=x(x-1)y_1''+g_1(x)y_1'+g_0(x)y_1,\\
&y_4=x(x-1)(x-t)y_1'''+h_2(x)y_1''+h_1(x)y_1'+h_0(x)y_1
\end{aligned}$$

with some functions f, g, h of x depending on the coefficients of the matrix A and, thus, we can easily find the fourth order differential equation for the first component y_1 of the vector Y. Next, we can find conditions on the coefficients when it is reduced to equation (1) with (3), (4). The elements below the diagonal are extremely cumbersome and we do not write them here[1]. However, by direct computations and using the algorithm of [1] it can be verified that the following statement holds true.

Proposition 1. *The diagonal elements of the hypergeometric system associated with equation (1) with (3)–(6) are* $\theta_0,\ \theta_1-1,\ \theta_2-2,\ -1$ *and the eigenvalues are given by* $-2,\ -1,\ -k_1,\ k_1+\theta_0+\theta_1+\theta_2-1$.

[1] The pdf file with the matrix of the hypergeometric system is available at www.mimuw.edu.pl/∼filipuk/files/ForPaper.pdf

This proposition shows that each diagonal element of the matrix A is equal to the characteristic exponent at the respective regular singular point modulo integers. Also we have that two of the eigenvalues of the matrix A are equal to the characteristic exponents at infinity of equation (1). This, in turn, implies that the local and global behaviour of solutions of the scalar equation and the system does not change.

It is well known that the parameter space of (P_{VI}) admits the action of the extended affine Weyl group of type $D_4^{(1)}$ (see [11] and references therein). It is generated by several basic transformations. By a Bäcklund transformation we mean a transformation of dependent variables and parameters that leaves system (7) invariant. The following transformation is one of generators of the group of Bäcklund transformations. Let us define new variables $\tilde{\lambda}$, $\tilde{\mu}$ as follows:

$$\tilde{\lambda} = \lambda + \frac{k_1}{\mu},\ \tilde{\mu} = \mu,\ \tilde{k}_1 = -k_1,\ \tilde{\theta}_0 = k_1 + \theta_0,\ \tilde{\theta}_1 = k_1 + \theta_1,\ \tilde{\theta}_2 = k_1 + \theta_2. \tag{8}$$

Then one can verify directly that, if the pair (λ, μ) satisfies the Hamiltonian system (7), then the pair $(\tilde{\lambda}, \tilde{\mu})$ again satisfies the same system with new parameters $\tilde{\theta}_0,\ \tilde{\theta}_1,\ \tilde{\theta}_2, \tilde{k}_1$.

As shown in [4], this transformation appears in the result of the integral transformation of the 2×2 linear Fuchsian system. Other generators of the group of Bäcklund transformations appear in the result of simple gauge transformations [6].

Next we study the action of transformation (8) on the hypergeometric system.

Theorem 1. *The Bäcklund transformation (8) induces a new hypergeometric system associated with (P_{VI}) with $B = \operatorname{diag}(0, 1, t, \lambda + k_1/\mu)$ and a new matrix A which has elements $\theta_0 + k_1,\ \theta_1 + k_1 - 1,\ \theta_2 + k_1 - 2,\ -1$ on the diagonal and eigenvalues equal to $-2,\ -1,\ k_1,\ 2k_1 + \theta_0 + \theta_1 + \theta_2 - 1$.*

3 A generalized hypergeometric system associated with the fifth Painlevé equation

We consider equation (1) with

$$p_1(x) = \frac{1 - k_0}{x} + \frac{\eta_1 t}{(x-1)^2} + \frac{1 - \theta_1}{x - 1} - \frac{1}{x - \lambda}, \tag{9}$$

$$p_2(x) = \frac{k}{x(x-1)} - \frac{tH_V}{x(x-1)^2} + \frac{\lambda(\lambda - 1)\mu}{x(x-1)(x-\lambda)}, \tag{10}$$

$$tH_V = \lambda(\lambda-1)^2\mu^2 - (k_0(\lambda-1)^2 + \theta_1\lambda(\lambda-1) - \eta_1 t\lambda)\mu + k(\lambda - 1) \tag{11}$$

and

$$4k = (k_0 + \theta_1)^2 - k_\infty^2. \tag{12}$$

The generalized Riemann scheme giving local exponents at regular and irregular singularities of equation (1) with (9)–(12) is given in [12]. The monodromy preserving deformations lead to the Hamiltonian system (7) for the Hamiltonian H_V and, hence, to the fifth Painlevé equation given by

$$\lambda'' = \left(\frac{1}{2\lambda} + \frac{1}{\lambda - 1}\right)(\lambda')^2 - \frac{1}{t}\lambda' + \frac{(\lambda-1)^2}{t^2}\left(\alpha\lambda + \beta\frac{1}{\lambda}\right) + \frac{\gamma}{t}\lambda + \delta\frac{\lambda(\lambda+1)}{\lambda-1}$$

with

$$\alpha = \frac{k_\infty^2}{2},\ \beta = -\frac{k_0^2}{2},\ \gamma = \eta_1(1+\theta_1),\ \delta = -\frac{\eta_1^2}{2}.$$

for the function $\lambda(t)$.

Since equation (1) is not Fuchsian as in (P_{VI}) case above, the algorithm of [1] is not applicable and we need to find a new type of system to reduce the equation. We introduce the following generalized hypergeometric system.

Theorem 2. *The generalized hypergeometric system of equation (1) with (9)–(12) is given by*

$$\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x-1 & 0 & 0 \\ 0 & \eta_1 t & x-1 & 0 \\ 0 & 0 & 0 & x-\lambda \end{pmatrix}\begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{pmatrix} = \begin{pmatrix} k_0 & 1 & 0 & 0 \\ \alpha_{2,1} & \theta_1 & 1 & 0 \\ \alpha_{3,1} & \alpha_{3,2} & -2 & 1 \\ \alpha_{4,1} & \alpha_{4,2} & \alpha_{4,3} & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

with

$$\alpha_{2,1} = \lambda\mu - \mu + k_0(\theta_1 + \eta_1 t + 1/\lambda - 1) - k - tH_V,$$

$$\lambda(\lambda-1)\alpha_{4,3} = k_0(\lambda-1)^2 - \lambda(1 - \theta_1 + \eta_1 t - \lambda - \theta_1\lambda + (\lambda-1)^2\mu) =: q_1,$$

$$\lambda^2(\lambda-1)\alpha_{4,2} = q_1(k_0(\lambda-1) + \lambda(1+\theta_1+\mu-\lambda\mu)),$$

$$\alpha_{4,1} = \frac{q_1}{\lambda^2(\lambda-1)}\Big\{k_0^2(\lambda-1) + k_0\lambda\Big[\theta_1 + \eta_1 t + (\lambda-2)(\lambda-1)\mu\Big] -$$

$$-\lambda^2\Big[k + \mu\big(\theta_1 + \eta_1 t - \theta_1\lambda + (\lambda-1)^2\mu\big)\Big]\Big\},$$

$$(\lambda-1)\alpha_{3,2} = 1 + \theta_1 + (1+k_0)\eta_1 t + k(\lambda-1)^2 - \lambda - q_2\lambda -$$

$$-q_3(\lambda-1)\mu + \lambda(\lambda-1)^3\mu^2,$$

$$q_2 = \theta_1 + k_0\eta_1 t,\quad q_3 = k_0 - (2k_0 + \theta_1 + \eta_1 t)\lambda + (k_0+\theta_1)\lambda^2,$$

$$\lambda^2\alpha_{3,1} = \lambda^2(2k(\lambda-1) - q_4\mu + (\lambda-1)^2(2\lambda-1)\mu^2) -$$

$$-k_0^2(1+q_5\lambda) + k_0\lambda(q_6 + \eta_1 t(1 + \eta_1(1 + \lambda(\lambda\mu - 2)) + q_7(\lambda-1))),$$

$$q_4 = \theta_1 + \eta_1 t + \lambda - 3\theta_1\lambda - 2\eta_1 t\lambda + 2\theta_1\lambda^2 - 1,$$

$$q_5 = \eta_1 t\lambda + \lambda(\lambda-1)^2\mu - 1,\quad q_6 = \theta_1(1-\lambda)(1+\lambda^2\mu),$$

$$q_7 = 1 - 2\mu + \lambda(k + \mu(3 - 2\lambda + \lambda(\lambda-1)\mu)).$$

Substituting $y_1 = y$ into the system we require that y solves equation (1) with (9)–(12). A routine calculation shows that the matrix A in the system has eigenvalues $-2,\ -1,\ (k_0+\theta_1-k_\infty)/2,\ (k_0+\theta_1+k_\infty)/2$ which encode the information of the generalized Riemann scheme in [12]. We note that the action of the Bäcklund transformations of (P_V) on the sytem can also be studied similarly to (P_{VI}) case.

4 Conclusions

We have computed the hypergeometric system associated with the sixth Painlevé equation via (1) and studied the action of a particular Bäcklund transformation on it. We introduced a new type of systems, the generalized hypergeometric system, and reduced equation (1) associated with the fifth Painlevé equation to it. The generalized hypergeometric systems give a new type of reduction problems and are worth of further study. The generalized hypergeometric systems for other Painlevé equations and the confluence process are currently under investigation and will be published elsewhere. We expect that the hypergeometric systems could be applied to other problems concerning the Painlevé equations. It is an open (and computationally difficult) problem to study the (generalized) hypergeometric systems for the (degenerate) Garnier systems and examine their symmetries. There is some evidence [10] that new symmetries of the Garnier systems may not exist and, so, the hypergeometric systems could shed some more light on this problem.

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Vague BF-algebras

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Abstract. In this paper, by using the concept of vague sets and BF-algebra we introduce the notions of vague BF-algebra. After that we state and prove some theorems in vague BF-algebras, α-cut and vague-cut. The relationship between these notions and crisp subalgebras are studied.

Mathematics subject classification: 06F35, 03G25, 03E72.
Keywords and phrases: (anti) fuzzy BF-algebras, vague sets, vague BF-algebra, vague-cut, (artinian, noetherian) BF-algebras.

1 Introduction

It is known that mathematical logic is a discipline used in sciences and humanities with different point of view. Non-classical logic takes the advantage of the classical logic (two-valued logic) to handle information with various facts of uncertainty. The non-classical logic has become a formal and useful tool for computer science to deal with fuzzy information and uncertain information.

Y. Imai and K. Iseki [7] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Recently, Andrzej Walendziak defined a BF-algebra [12].

The notion of vague set theory was introduced by W. L. Gau and D. J. Buehrer [3], as a generalizations of Zadeh's fuzzy set theory [13]. In [1], R. Biswas applied the notion to group theory and introduced vague groups.

Now, in this note we use the notion of vague set to establish the notions of vague BF-algebras; then we obtain some related results which have been mentioned in the abstract.

2 Preliminaries

In this section, we present now some preliminaries on the theory of vague sets (VS). In his pioneer work [13], Zadeh proposed the theory of fuzzy sets. Since then it has been applied in wide varieties of fields like Computer Science, Management Science, Medical Sciences, Engineering problems, etc. to list a few only.

Let $U = \{u_1, u_2, ..., u_n\}$ be the universe of discourse. The membership function for fuzzy sets can take any value from the closed interval $[0; 1]$. An fuzzy set A is defined as the set of ordered pairs $A = \{(u; \mu_A(u)) \mid u \in U\}$ where $\mu_A(u)$ is the grade of membership of element u in set A. The greater $\mu_A(u)$, the greater is the truth of the statement that 'the element u belongs to the set A'. But Gau and Buehrer [3] pointed out that this single value combines the 'evidence for u' and the 'evidence against u'. It does not indicate the 'evidence for u' and the 'evidence against u', and it does not also indicate how much there is of each. Consequently, there is a genuine necessity of a different kind of fuzzy sets which could be treated as a generalization of Zadeh's fuzzy sets [13].

Definition 1. A vague set A in the universe of discourse U is characterized by two membership functions given by:

1. A truth membership function $t_A : U \to [0, 1]$,
2. A false membership function $f_A : U \to [0, 1]$,

where $t_A(u)$ is a lower bound of the grade of membership of u derived from the 'evidence for u', and $f_A(u)$ is a lower bound of the negation of u derived from the 'evidence against u' and $t_A(u) + f_A(u) \leq 1$. Thus the grade of membership of u in the vague set A is bounded by a subinterval $[t_A(u), 1 - f_A(u)]$ of $[0, 1]$. This indicates that if the actual grade of membership is $\mu(u)$, then

$$t_A(u) \leq \mu(u) \leq 1 - f_A(u).$$

The vague set A is written as

$$A = \{(u, [t_A(u), f_A(u)]) \mid u \in U\},$$

where the interval $[t_A(u), 1 - f_A(u)]$ is called the 'vague value' of u in A and is denoted by $V_A(u)$.

It is worth to mention here that interval-valued fuzzy sets (i-v fuzzy sets) [14] are not vague sets. In i-v fuzzy sets, an interval-valued membership value is assigned to each element of the universe considering the 'evidence for u' only, without considering 'evidence against u'. In vague sets both are independently proposed by the decision maker. This makes a major difference in the judgment about the grade of membership.

Definition 2. (see [1]). A vague set A of a set U is called

1) the zero vague set of U if $t_A(u) = 0$ and $f_A(u) = 1$ for all $u \in U$,

2) the unit vague set of U if $t_A(u) = 1$ and $f_A(u) = 0$ for all $u \in U$,

3) the α-vague set of U if $t_A(u) = \alpha$ and $f_A(u) = 1 - \alpha$ for all $u \in U$, where $\alpha \in (0, 1)$.

Let $D[0, 1]$ denote the family of all closed subintervals of $[0, 1]$. Now we define refined minimum (briefly, $rmin$) and order " $\leq$ " on elements $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$ of $D[0, 1]$ as:

$$rmin(D_1, D_2) = [min\{a_1, a_2\}, min\{b_1, b_2\}],$$

$$D_1 \leq D_2 \Longleftrightarrow a_1 \leq a_2 \quad \wedge \quad b_1 \leq b_2.$$

Similarly we can define $\geq$, $=$ and $rmax$. Then the concept of $rmin$ and $rmax$ could be extended to define $rinf$ and $rsup$ of infinite number of elements of $D[0,1]$.

It is that $L = \{D[0,1], rinf, rsup, \leq\}$ is a lattice with universal bounds $[0,0]$ and $[1,1]$.

For $\alpha, \beta \in [0,1]$ we now define (α,β)-cut and α-cut of a vague set.

Definition 3. (see [1]). Let A be a vague set of a universe X with the true-membership function t_A and false-membership function f_A. The (α,β)-cut of the vague set A is a crisp subset $A_{(\alpha,\beta)}$ of the set X given by

$$A_{(\alpha,\beta)} = \{x \in X \mid V_A(x) \geq [\alpha,\beta]\},$$

where $\alpha \leq \beta$.

Clearly $A(0,0) = X$. The (α,β)-cuts are also called vague-cuts of the vague set A.

Definition 4. (see [1]). The α-cut of the vague set A is a crisp subset A_α of the set X given by $A_\alpha = A_{(\alpha,\alpha)}$.

Note that $A_0 = X$ and if $\alpha \geq \beta$ then $A_\beta \subseteq A_\alpha$ and $A_{(\beta,\alpha)} = A_\alpha$. Equivalently, we can define the α-cut as

$$A_\alpha = \{x \in X \mid t_A(x) \geq \alpha\}.$$

Definition 5. Let f be a mapping from the set X to the set Y and let B be a vague set of Y. The inverse image of B, denoted by $f^{-1}(B)$, is a vague set of X which is defined by $V_{f^{-1}(B)}(x) = V_B(f(x))$ for all $x \in X$.

Conversely, let A be a vague set of X. Then the image of A, denoted by $f(A)$, is a vague set of Y such that:

$$V_{f(A)}(y) = \begin{cases} rsup_{z \in f^{-1}(y)} V_A(z) & \text{if} \quad f^{-1}(y) = \{x : f(x) = y\} \neq \emptyset, \\ [0,0] & \text{otherwise.} \end{cases}$$

Definition 6. A vague set A of BF-algebra X is said to have the sup property if for any subset $T \subseteq X$ there exists $x_0 \in T$ such that

$$V_A(x_0) = rsup_{t \in T} V_A(t).$$

Definition 7. (see [12]). A BF-algebra is a non-empty set X with a consonant 0 and a binary operation $*$ satisfying the following axioms:

(I) $x * x = 0$,

(II) $x * 0 = x$,

(III) $0 * (x * y) = (y * x)$,

for all $x, y \in X$.

Example 1. (see [12]). (a) Let $\mathbf{R}$ be the set of real numbers and let $A = (\mathbf{R}; *, 0)$ be the algebra with the operation $*$ defined by

$$x * y = \begin{cases} x & \text{if } \quad y = 0, \\ y & \text{if } \quad x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then A is a BF-algebra.

(b) Let $A = [0; \infty)$. Define the binary operation $*$ on A as follows: $x*y = |x-y|$, for all $x, y \in A$. Then $(A; *, 0)$ is a BF-algebra.

Proposition 1. *(see [12]). Let X be a BF-algebra. Then for any x and y in X, the following hold:*

(a) $0 * (0 * x) = x$ *for all* $x \in A$;
(b) if $0 * x = 0 * y$, *then* $x = y$ *for any* $x, y \in A$;
(c) if $x * y = 0$, *then* $y * x = 0$ *for any* $x, y \in A$.

Definition 8. (see [13]). A non-empty subset S of a BF-algebra X is called a subalgebra of X if $x * y \in S$ for any $x, y \in S$.

A mapping $f : X \longrightarrow Y$ of BF-algebras is called a BF-homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

Definition 9. (see [2]). Let μ be a fuzzy set in a BF-algebra X. Then μ is called a fuzzy BF-subalgebra of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

3 Vague BF-algebras

From now on $(X, *, 0)$ is a BF-algebra, unless otherwise is stated.

Definition 10. A vague set A of X is called a vague BF-algebra of X if it satisfies the following condition:

$$V_A(x * y) \geq rmin\{V_A(x), V_A(y)\}$$

for all $x, y \in X$, that is

$$t_A(x * y) \geq \min\{t_A(x), t_A(y)\},$$

$$1 - f_A(x * y) \geq \min\{1 - f_A(x), 1 - f_A(y)\}.$$

Example 2. Let $X = \{0, 1, 2\}$ be a set with the following table:

$*$	0	1	2
0	0	1	2
1	1	0	0
2	2	0	0

Then $(X, *, 0)$ is a BF-algebra, but is not a $BCH/BCI/BCK$-algebra.

Define

$$t_A(x) = \begin{cases} 0.7 & \text{if} \quad x = 0, \\ 0.3 & \text{if} \quad x \neq 0 \end{cases}$$

and

$$f_A(x) = \begin{cases} 0.2 & \text{if} \quad x = 0, \\ 0.4 & \text{if} \quad x \neq 0. \end{cases}$$

It is routine to verify that $A = \{(x, [t_A(x), f_A(x)]) \mid x \in X\}$ is a vague BF-algebra of X.

Lemma 1. *If A is a vague BF-algebra of X, then $V_A(0) \geq V_A(x)$, for all $x \in X$.*

Proof. For all $x \in X$, we have $x * x = 0$, hence

$$V_A(0) = V_A(x * x) \geq rmin\{V_A(x), V_A(x)\} = V_A(x). \qquad \square$$

Proposition 2. *Let A be a vague BF-algebra of X and let $n \in \mathcal{N}$. Then:*

(i) $V_A(\prod^n x * x) \geq V_A(x)$, *for any odd number* n,

(ii) $V_A(\prod^n x * x) = V_A(x)$, *for any even number* n,

where $\prod^n x * x = \overbrace{x * x * \ldots * x}^{n-times}$.

Proof. Let $x \in X$ and assume that n is odd. Then $n = 2k - 1$ for some positive integer k. We prove by induction, definition and above lemma imply that $V_A(x*x) = V_A(0) \geq V_A(x)$. Now suppose that $V_A(\prod^{2k-1} x * x) \geq V_A(x)$. Then by assumption

$$\begin{aligned} V_A(\prod^{2(k+1)-1} x * x) &= V_A(\prod^{2k+1} x * x) = \\ &= V_A(\prod^{2k-1} x * (x * (x * x))) = \\ &= V_A(\prod^{2k-1} x * x) \geq V_A(x). \end{aligned}$$

Which proves (i). Similarly we can prove (ii). $\square$

Theorem 1. *Let A be a vague BF-algebra of X. If there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n \to \infty} V_A(x_n) = [1, 1],$$

then $V_A(0) = [1, 1]$.

Proof. By Lemma 1, we have $V_A(0) \geq V_A(x)$, for all $x \in X$, thus $V_A(0) \geq V_A(x_n)$, for every positive integer n. Since $t_A(0) \leq 1$ and $1 - f_A(0) \leq 1$, then we have $V_A(0) = [t_A(0), 1 - f_A(0)] \leq [1, 1]$. Consider

$$V_A(0) \geq \lim_{n\to\infty} V_A(x_n) = [1, 1].$$

Hence $V_A(0) = [1, 1]$. □

μ is called an antifuzzy BF-subalgebra of X if $\mu(x * y) \leq \max\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

In the next proposition we state the relationship between vague BF-algebra and fuzzy BF-algebras.

Proposition 3. *A vague set $A = \{(u, [t_A(u), f_A(u)]) \mid u \in X\}$ of X is a vague BF-algebra of X if and only if t_A be a fuzzy BF-subalgebra of X and f_A be an antifuzzy BF-subalgebra of X.*

Proof. The proof is straightforward. □

Theorem 2. *The family of vague BF-algebras forms a complete distributive lattice under the ordering of vague set.*

Proof. Let $\{V_i \mid i \in I\}$ be a family of vague BF-algebra of X. Since $[0, 1]$ is a completely distributive lattice with respect to the usual ordering in $[0, 1]$, it is sufficient to show that $\bigcap V_i = [\bigwedge t_i, \bigvee f_i]$ is a vague BF-algebra. Let $x, y \in X$. Then

$$\begin{aligned}
(\bigwedge t_i)(x * y) &= \inf\{t_i(x * y) \mid i \in I\} \geq \\
&\geq \inf\{\min\{t_i(x), t_i(y)\} \mid i \in I\} = \\
&= \min(\inf\{t_i(x) \mid i \in I\}, \inf\{t_i(y) \mid i \in I\}) = \\
&= \min(\bigwedge t_i(x), \bigwedge t_i(y)),
\end{aligned}$$

also we have

$$\begin{aligned}
(\bigvee f_i)(x * y) &= \sup\{f_i(x * y) \mid i \in I\} \leq \\
&\leq \sup\{\max\{f_i(x), f_i(y)\} \mid i \in I\} = \\
&= \max(\sup\{f_i(x) \mid i \in I\}, \sup\{f_i(y) \mid i \in I\}) = \\
&= \max(\bigvee f_i(x), \bigvee f_i(y)).
\end{aligned}$$

Hence $\bigcap V_i = [\bigwedge t_i, \bigvee f_i]$ is a vague BF-algebra. which proves the theorem. □

Proposition 4. *Zero vague set, unit vague set and α-vague set of X are trivial vague BF-algebras of X.*

Proof. Let A be a α-vague set of X. For $x, y \in X$ we have

$$t_A(x * y) = \alpha = \min\{\alpha, \alpha\} = \min\{t_A(x), t_A(y)\},$$

$$1 - f_A(x * y) = \alpha = \min\{\alpha, \alpha\} = \min\{1 - f_A(x), 1 - f_A(y)\}.$$

By above proposition it is clear that A is a vague BF-algebra of X. The proof of other cases is similar. □

Theorem 3. *Let A be a vague BF-algebra of X. Then for $\alpha \in [0, 1]$, the α-cut A_α is a crisp subalgebra of X.*

Proof. Let $x, y \in A_\alpha$. Then $t_A(x), t_A(y) \geq \alpha$, and so $t_A(x*y) \geq \min\{t_A(x), t_A(y)\} \geq \alpha$. Thus $x * y \in A_\alpha$. □

Theorem 4. *Let A be a vague BF-algebra of X. Then for all $\alpha, \beta \in [0, 1]$, the vague-cut $A_{(\alpha,\beta)}$ is a (crisp) subalgebra of X.*

Proof. Let $x, y \in A_{(\alpha,\beta)}$. Then $V_A(x), V_A(y) \geq [\alpha, \beta]$, and so $t_A(x), t_A(y) \geq \alpha$ and $1 - f_A(x), 1 - f_A(y) \geq \beta$. Then $t_A(x * y) \geq \min\{t_A(x), t_A(y)\} \geq \alpha$, and $1 - f_A(x * y) \geq \min\{1 - f_A(x), 1 - f_A(y)\} \geq \beta$. Thus $x * y \in A_{(\alpha,\beta)}$. □

The subalgebra $A_{(\alpha,\beta)}$ is called vague-cut subalgebra of X.

Proposition 5. *Let A be a vague BF-algebra of X. Two vague-cut subalgebras $A_{(\alpha,\beta)}$ and $A_{(\delta,\epsilon)}$ with $[\alpha, \beta] < [\delta, \epsilon]$ are equal if and only if there is no $x \in X$ such that $[\alpha, \beta] \leq V_A(x) \leq [\delta, \epsilon]$.*

Proof. In contrary, let $A_{(\alpha,\beta)} = A_{(\delta,\epsilon)}$ where $[\alpha, \beta] < [\delta, \epsilon]$ and there exists $x \in X$ such that $[\alpha, \beta] \leq V_A(x) \leq [\delta, \epsilon]$. Then $A_{(\delta,\epsilon)}$ is a proper subset of $A_{(\alpha,\beta)}$, which is a contradiction.

Conversely, suppose that there is no $x \in X$ such that $[\alpha, \beta] \leq V_A(x) \leq [\delta, \epsilon]$. Since $[\alpha, \beta] < [\delta, \epsilon]$, then $A_{(\delta,\epsilon)} \subseteq A_{(\alpha,\beta)}$. If $x \in A_{(\alpha,\beta)}$, then $V_A(x) \geq [\alpha, \beta]$ by hypothesis we get that $V_A(x) \geq [\delta, \epsilon]$. Therefore $x \in A_{(\delta,\epsilon)}$, then $A_{(\alpha,\beta)} \subseteq A_{(\delta,\epsilon)}$. Hence $A_{(\delta,\epsilon)} = A_{(\alpha,\beta)}$. □

Theorem 5. *Let $\mid X \mid < \infty$ and A be a vague BF-algebra of X. Consider the set $V(A)$ given by*

$$V(A) := \{V_A(x) \mid x \in X\}.$$

Then $A_{(\alpha,\beta)}$ are the only vague-cut subalgebras of X, where $(\alpha, \beta) \in V(A)$.

Proof. Let $[a_1, a_2] \notin V(A)$, where $[a_1, a_2] \in D[0, 1]$. If $[\alpha, \beta] < [a_1, a_2] < [\delta, \epsilon]$, where $[\alpha, \beta], [\delta, \epsilon] \in V(A)$, then $A_{(\alpha,\beta)} = A_{(a_1,a_2)} = A_{(\delta,\epsilon)}$. If $[a_1, a_2] < [a_1, b]$ where

$$[a_1, b] = rmin\{(x, y) \mid (x, y) \in V(A)\},$$

then $A_{(a_1,a_2)} = X = A_{(a_1,b)}$. Hence for any $[a_1, a_2] \in D[0, 1]$, the vague-cut subalgebra $A_{(a_1,b)}$ is one of the $A_{(\alpha,\beta)}$ for $(\alpha, \beta) \in V(A)$. □

Theorem 6. *Any subalgebra S of X is a vague-cut subalgebra of some vague BF-algebra of X.*

Proof. Define

$$t_A(x) = \begin{cases} \alpha & \text{if } x \in S, \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_A(x) = \begin{cases} 1-\alpha & \text{if } x \in S, \\ 1 & \text{otherwise.} \end{cases}$$

It is clear that

$$V_A(x) = \begin{cases} [\alpha, \alpha] & \text{if } x \in S, \\ [0,0] & \text{otherwise,} \end{cases}$$

where $\alpha \in (0,1)$. It is clear that $S = A_{(\alpha,\alpha)}$. Let $x, y \in X$. We consider the following cases:

1) If $x, y \in S$, then $x * y \in S$ therefore

$$V_A(x * y) = [\alpha, \alpha] = rmin\{V_A(x), V_A(y)\}.$$

2) If $x, y \notin S$, then $V_A(x) = [0,0] = V_A(y)$ and so

$$V_A(x * y) \geq [0,0] = rmin\{V_A(x), V_A(y)\}.$$

3) If $x \in S$ and $y \notin S$, then $V_A(x) = [\alpha, \alpha]$ and $V_A(y) = [0,0]$. Thus

$$V_A(x * y) \geq [0,0] = rmin\{[\alpha,\alpha],[0,0]\} = rmin\{V_A(x), V_A(y)\}.$$

Therefore A is a vague BF-algebra of X. □

Theorem 7. *Let S be a subset of X and A be a vague set of X which is given in the proof of above theorem. If A is a vague BF-algebra of X, then S is a (crisp) subalgebra of X.*

Proof. Let A be a vague BF-algebra of X and $x, y \in S$. Then $V_A(x) = [\alpha, \alpha] = V_A(y)$, thus

$$V_A(x * y) \geq rmin\{V_A(x), V_A(y)\} = rmin\{[\alpha,\alpha],[\alpha,\alpha]\} = [\alpha,\alpha].$$

which implies that $x * y \in S$. □

Theorem 8. *Let A be a vague BF-algebra of X. Then the set*

$$X_{V_A} := \{x \in X \mid V_A(x) = V_A(0)\}$$

is a (crisp) subalgebra of X.

Proof. Let $a, b \in X_{V_A}$. Then $V_A(a) = V_A(b) = V_A(0)$, and so

$$V(a * b) \geq rmin\{V_A(a), V_A(b)\} = V_A(0).$$

Then X_{V_A} is a subalgebra of X. □

Theorem 9. *Let N be the vague set of X which is defined by:*

$$V_N(x) = \begin{cases} [\alpha, \alpha] & \text{if } x \in N, \\ [\beta, \beta] & \text{otherwise,} \end{cases}$$

for $\alpha, \beta \in [0,1]$ with $\alpha \geq \beta$. Then N is a vague BF-algebra of X if and only if N is a (crisp) subalgebra of X. Moreover, in this case $X_{V_N} = N$.

Proof. Let N be a vague BF-algebra of X. Let $x, y \in X$ be such that $x, y \in N$. Then

$$V_N(x * y) \geq rmin\{V_N(x), V_N(y)\} = rmin\{[\alpha, \alpha], [\alpha, \alpha]\} = [\alpha, \alpha]$$

and so $x * y \in N$.

Conversely, suppose that N is a (crisp) subalgebra of X, let $x, y \in X$.

(i) If $x, y \in N$ then $x * y \in N$, thus

$$V_N(x * y) = [\alpha, \alpha] = rmin\{V_N(x), V_N(y)\}.$$

(ii) If $x \notin N$ or $y \notin N$, then

$$V_N(x * y) \geq [\beta, \beta] = rmin\{V_N(x), V_N(y)\}.$$

This shows that N is a vague BF-algebra of X.

Moreover, we have

$$X_{V_N} := \{x \in X \mid V_N(x) = V_N(0)\} = \{x \in X \mid V_N(x) = [\alpha, \alpha]\} = N. \qquad \square$$

Proposition 6. *Let X and Y be BF-algebras and f be a BF-homomorphism from X into Y and G be a vague BF-algebra of Y. Then the inverse image $f^{-1}(G)$ of G is a vague BF-algebra of X.*

Proof. Let $x, y \in X$. Then

$$\begin{aligned} V_{f^{-1}(G)}(x * y) &= V_G(f(x * y)) = \\ &= V_G(f(x) * f(y)) \geq \\ &\geq rmin\{V_G(f(x)), V_G(f(y))\} = \\ &= rmin\{V_{f^{-1}(G)}(x), V_{f^{-1}(G)}(y)\}. \qquad \square \end{aligned}$$

Proposition 7. *Let X and Y be BF-algebras and f be a BF-homomorphism from X onto Y and D be a vague BF-algebra of X with the* sup *property. Then the image $f(D)$ of D is a vague BF-algebra of Y.*

Proof. Let $a, b \in Y$, let $x_0 \in f^{-1}(a), y_0 \in f^{-1}(b)$ such that

$$V_D(x_0) = rsup_{t \in f^{-1}(a)} V_D(t), \qquad V_D(y_0) = rsup_{t \in f^{-1}(b)} V_D(t).$$

Then by the definition of $V_{f(D)}$, we have

$$\begin{aligned} V_{f(D)}(x * y) &= rsup_{t \in f^{-1}(a*b)} V_D(t) \geq \\ &\geq V_D(x_0 * y_0) \geq \\ &\geq rmin\{V_D(x_0), V_D(y_0) = \\ &= rmin\{rsup_{t \in f^{-1}(a)} V_D(t), rsup_{t \in f^{-1}(b)} V_D(t)\} = \\ &= rmin\{V_{f(D)}(a), V_{f(D)}(b)\}. \qquad \square \end{aligned}$$

4 Artinian and Noetherian *BF*-algebras

Definition 11. A BF-algebra X is said to be *Artinian* if it satisfies the descending chain condition on subalgebras of X (simply written as DCC), that is, for every chain $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ of subalgebras of X, there is a natural number i such that $I_i = I_{i+1} = \cdots$.

Theorem 10. *Let X be a BF-algebra. Then each vague BF-algebra of X has finite values if and only if X is Artinian.*

Proof. Suppose that each vague BF-algebra of X has finite values. If X is not Artinian, then there is a strictly descending chain

$$G = I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$$

of subalgebras of X, where $I_i \supset I_j$ expresses $I_i \supseteq I_j$ but $I_i \neq I_j$. We now construct the vague set $B = [t_A, f_A]$ of X by

$$t_A(x) := \begin{cases} \dfrac{n}{n+1} & \text{if} \quad x \in I_n \setminus I_{n+1},\ n = 1, 2, \cdots, \\ 1 & \text{if} \quad x \in \bigcap\limits_{n=1}^{\infty} I_n, \end{cases}$$
$$f_A(x) := 1 - t_A(x).$$

We first prove that B is a vague BF-algebra of X. For this purpose, we need to verify that t_A is a fuzzy subalgebra of X. We assume that $x, y \in X$. Now, we consider the following cases:

Case 1: $x, y \in I_n \setminus I_{n+1}$. In this case, $x, y \in I_n$, and $x * y \in I_n$. Thus

$$t_A(x * y) \geq \frac{n}{n+1} = \min\{t_A(x),\, t_A(y)\}.$$

Case 2: $x \in I_n \setminus I_{n+1}$ and $y \in I_m \setminus I_{m+1} (n < m)$. In this case, $x, y \in I_n$, and $x * y \in I_n$. Thus

$$t_A(x * y) \geq \frac{n}{n+1} = \min\{t_A(x),\, t_A(y)\}.$$

Case 3: $x \in I_n \setminus I_{n+1}$ and $y \in I_m \setminus I_{m+1} (n > m)$. In this case, $x, y \in I_m$, and $x * y \in I_m$. Thus

$$t_A(x * y) \geq \frac{m}{m+1} = \min\{t_A(x),\, t_A(y)\}.$$

Therefore t_A is a fuzzy subalgebra of X. This shows that B is a vague BF-algebra of X, but the values of B are infinite, which is a contradiction. Thus X is Artinian.

Conversely, suppose that X is Artinian. If there is a vague BF-algebra $B = [t_A, f_A]$ of X with $|Im(B)| = +\infty$, then $|Im(t_A)| = +\infty$ or $|Im(f_A)| = +\infty$. Without loss of generality, we may assume that $Im(t_A) = +\infty$. Select $s_i \in Im(t_A)$ $(i = 1, 2 \cdots)$ and $s_1 < s_2 < \cdots$. Then $U(t_A;\, s_i)(i = 1, 2, \cdots)$ are subalgebras of X and $U(t_A;\, s_1) \supseteq U(t_A;\, s_2) \supseteq \cdots$ with $U(t_A;\, s_i) \neq U(t_A;\, s_{i+1})$ $(i = 1, 2, \cdots)$, a contradiction. Similar for $\mathrm{Im}(f_A)$. The proof is completed. □

Definition 12. A BF-algebra X is said to be *Noetherian* if every subalgebra of X is finitely generated. X is said to satisfy *the ascending chain condition (briefly, ACC)* if for every ascending sequence $I_1 \subseteq I_2 \subseteq \cdots$ of subalgebras of X there is a natural number n such that $I_i = I_n$, for all $i \geq n$.

Theorem 11. *X is Noetherian if and only if for any vague BF-algebra A, the set $Im(B)$ is a well ordered subset, that is, $(Im(t_A), \leq)$ and $(Im(f_A), \geq)$ are well ordered subsets of* $[0, 1]$, *respectively.*

Proof. ($\Rightarrow$) Suppose that X is Noetherian. For any chain $t_1 > t_2 > \cdots$ of $Im(t_A)$, let $t_0 = \inf\{t_i | \, i = 1, 2, \cdots\}$. Then $I := \{x \in X | \, t_A(x) > t_0\}$ is a subalgebra of X, and so I is finitely generated. Let $I = (a_1, \cdots, a_k]$. Then $t_A(a_1) \wedge \cdots \wedge t_A(a_k)$ is the least element of the chain $t_1 > t_2 > \cdots$. Thus $(Im(t_A), \leq)$ is a well ordered subset of $[0, 1]$. By using the same argument as above, we can easily show that $(Im(f_A), \geq)$ is a well ordered subset of $[0, 1]$. Therefore, $Im(B)$ is a well ordered subset.

($\Leftarrow$) Let $Im(B)$ be a well ordered subset. If X is not Noetherian, then there is a strictly ascending sequence of subalgebras of X such that $I_1 \subset I_2 \subset \cdots$. We construct the bipolar fuzzy set $B = [t_A, \, f_A]$ of X by

$$\begin{aligned} t_A(x) &:= \begin{cases} \dfrac{1}{n} & \text{if} \quad x \in I_n - I_{n-1}, \; n = 1, 2, \cdots, \\ 0 & \text{if} \quad x \notin \bigcup\limits_{n=1}^{\infty} I_n, \end{cases} \\ f_A(x) &:= \; 1 - t_A(x) \end{aligned}$$

where $I_0 = \emptyset$. By using similar method as the necessity part of Theorem 18, we can prove that B is a vague BF-algebra of X. Because $Im(B)$ is not well ordered, which is a contradiction. This completes the proof. $\square$

5 Conclusions

In the present paper, we have introduced the concept of vague BF-algebras and investigated some of their useful properties.

In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as groups, semigroups, rings, nearrings, semirings (hemirings), lattices and Lie algebras. Our obtained results can be perhaps applied in engineering, soft computing or even in medical diagnosis [11].

In future work the vague ideals and quotient of BF-algebras by using these vague ideals will be presented.

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A heuristic algorithm for the two-dimensional single large bin packing problem

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Abstract. In this paper, we propose a heuristic algorithm based on concave corner (BCC) for the two-dimensional rectangular single large packing problem (2D-SLBPP), and compare it against some heuristic and metaheuristic algorithms from the literature. The experiments show that our algorithm is highly competitive and could be considered as a viable alternative, for 2D-SLBPP. Especially for large test problems, the algorithm could get satisfied results more quickly than other approaches in literature.

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1 Introduction

Packing problem involves many industrial applications. For example, wood or class industries, ship building, textile and leather industry etc. All of these applications can be formalized as packing problem [1], for more extensive and detailed descriptions of packing problems, please refer to [1–4].

In this paper, we discuss the two-dimensional single large bin packing problem (2D-SLBPP). The problem could be described as follows:

Given a rectangular board with fixed size and a set of rectangular pieces. The research of 2D-SLBPP is how to pack rectangular pieces orthogonally on the board, in the meantime, try to decrease the worst of the board with no two pieces overlap.

2 A new heuristic packing algorithm for single bin packing

2.1 Placement strategy based on Concave Corner (BCC)

Before the description of our algorithm, suppose the width and height of rectangular board are W and H. Without loss of generality, all parameters are regarded as integer. The pieces should be packed with edges parallel to the edges of the board and couldn't be rotated by 90^o.

For constructing our heuristic algorithm, we propose some definitions and rules.

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Some definitions

1. Let C_i denote the "Concave corner" (CC), see Figure 1, the CC is composed by two edges, and the size of the angle is 90^o, at the same time, the CC does not belong to any $piece_i$.

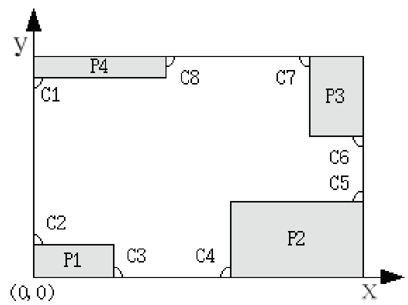

Figure 1. The example of *Concave corner*

2. Define the U, which is stated as formula (1):

$$U = \{U_1, U_2, \ldots, U_k\}; \quad U_i \cap U_j = \varnothing, \quad i \neq j, \tag{1}$$

where the U_i is a set of the CC, and k is the number of non-connected domains in the board, for example, see Figure 2, before the P_5 is packed onto the board, we have $U = U_1$, $U_1 = \{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8\}$, after packing the P_5, the U_1 should be divided into two areas, then we have: $U = \{U_1^{'},\ U_2^{'}\}$, $U_1^{'} = \{C_1, C_7, C_8, C_9, C_{12}\}$, $U_2^{'} = \{C_2, C_3, C_4, C_{10}, C_{11}\}$. Obviously, before packing any piece onto the board, there exists 4 CC and $k = 1$.

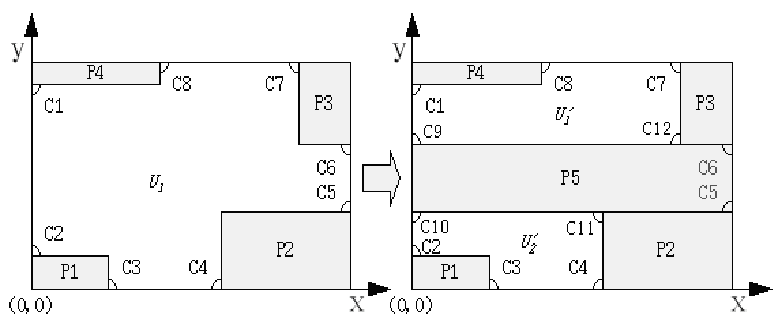

Figure 2. Dividing U_1 into $U_1^{'}$ and $U_2^{'}$

3. Define the edge of the U_i: Before any piece is packed onto the board, let l_U_1 denote the left edge of the U_1, and $r\_U_1 = W$ denote the right edge of the U_1. Similarly, we could define the t_U_1 and b_U_1 to denote the top edge and bottom edge of the U_1. So if U_i was divided into $U_e^{'}$ and $U_f^{'}$, we should update these parameters using the formulae (2-5):

$$l\_U_e^{'} = Min\{x_j^{'}\},\ 1 \leqq j \leqq s,\ \ C_j^{'} \subset U_e^{'},\ \ x_j^{'}\ \ is\ \ x - coordinate\ of\ the\ C_j^{'}; \tag{2}$$

$$r\_U_e^{'} = Max\{x_j^{'}\},\ 1 \leqq j \leqq s,\ \ C_j^{'} \subset U_e^{'},\ \ x_j^{'}\ \ is\ \ x - coordinate\ of\ the\ C_j^{'}; \tag{3}$$

$$b\_U_e^{'} = Min\{x_j^{'}\},\ 1 \leqq j \leqq s,\ \ C_j^{'} \subset U_e^{'},\ \ y_j^{'}\ \ is\ \ y - coordinate\ of\ the\ C_j^{'}; \tag{4}$$

$$t\_U_e^{'} = Max\{x_j^{'}\},\ 1 \leqq j \leqq s,\ \ C_j^{'} \subset U_e^{'},\ \ y_j^{'}\ \ is\ \ y - coordinate\ of\ the\ C_j^{'}; \tag{5}$$

where s is the number of CC in $U_e^{'}$.

Note. After a new piece is packed onto the board, if no U_i was divided, the edges of U_i should not be changed.

4. When a new piece $piece_i$ is packed onto the board, let s denote the number of edges which is touched with some packed $piece_h$ for the position of one C_k, if the $piece_i$ could be packed, then we compute the parameter $pFit\_C_k$ using formula (6):

$$pFit\_C_k = \sum_{j=1}^{s} p_j, \tag{6}$$

if the $piece_i$ is packed onto C_k with corner of the piece (query every U_m), the $piece_i$ touches one edge of the U_m ($p_j = 2$) and the $piece_i$ touches one edge of the other packed piece ($p_j = 1$), see Figure 3.

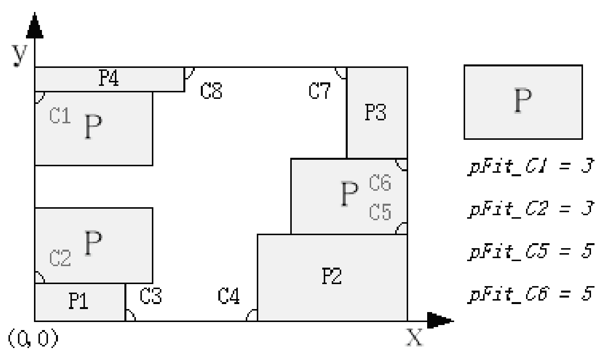

Figure 3. Computation of $pFit\_C_i$ in every U_k

5. Define the edge distance of C_k:

$$ed\_C_k = Min\{the\ distance\ between\ vertex\ of\ C_k\ with\ the\ edge\ of\ U_e\} \tag{7}$$

if $C_k \subset U_e$.

Packing rules. When $piece_i$ is packed onto the board, compute the $pFit\_C_k$ of every C_k, and select the packing position with maximal $pFit\_C_k$, if $pFit\_C_k$ are equal to each other, then select the position with ed_C_k is the shortest, then if the ed_C_k is the same, select the position randomly.

After completing the definitions and packing rules, we construct the heuristic algorithm based on the concave corner (BCC) as algorithm 1:

Algorithm 1 heuristic Packing (packing sequence)

```
s ⇐ 0;
i ⇐ 0;
while packing sequence is not null do
   get piece_i from the packing sequence;
   using the packing rules mentioned above to get good position for piece_i;
   if good position exists then
      pack the piece_i into the board at the good position;
      remove the piece_i from packing sequence;
      s ⇐ s + area of picec_i;
      continue;
   end if
   i ⇐ i + 1;
end while
return s;
```

2.2 Random search

Since the result of the heuristic packing (BCC) depends on the order of the packing sequence, so we import a random search to enhance the quality of the solution, which is described as follows:

Algorithm 2 middle Heuristic (origin data of all pieces, maxcall)

```
produce a packing sequence according to the area of all pieces from big to small;
best ⇐ 0;
area ⇐ 0;
swapLimit ⇐ pieces number × 1 / 3;
for i = 0 to maxcall do
   area = heuristicPacking(packing sequence);
   if area equals to the total area of all pieces then
      break;
   end if
   if area > best then
      best ⇐ area;
   end if
   for j = 0 to swapLimit do
      swap the pieces order of packing sequence randomly;
   end for
end for
```

3 Experiments

We implemented our algorithm by C++ programming language, and the 21 rectangular packing instances coming from [5] are used. For evaluating the algorithm more reasonably, we set the maxcall is 1000 and run the program 100 times. our experiments are run on a IBM T400 notebook PC with 2.26 GHZ CPU, GRASP is introduced in [6] and tabu search algorithm is presented in [7] (TABU), both GRASP and TABU were run on a Pentium III at 800 MHz, which is almost thrice as slow as ours. The test results are listed in Table 1.

Table 1. Comparisons of the average filling rate (FR) and the average running time (T)

Instance	Area	Number of items	GRASP		TABU		BCC	
			FR(%)	T(s)	FR(%)	T(s)	FR(%)	T(s)
1	400	16	100	0.94	100	0.42	100	0.13
2	400	17	96.5	9.28	100	4.23	96.5	1.10
3	400	16	100	0.06	100	0.95	100	0.24
4	600	25	98.33	19.44	100	0.44	96.83	2.47
5	600	25	99.5	17.36	100	4.16	99	2.27
6	600	25	100	0.71	100	0.0	99.33	1.97
7	1800	28	98.06	26.80	100	4.91	96.72	3.21
8	1800	29	97.5	37.35	100	10.11	95.56	3.33
9	1800	28	98.56	30.92	100	5.52	97.33	2.58
10	3600	49	98	102.05	99.44	45.27	96.83	9.43
11	3600	49	97.89	110.79	99	67.59	98.19	9.18
12	3600	49	98.44	94.41	99.44	51.11	98.47	8.72
13	5400	73	98.3	212.07	98.93	135.97	97.63	26.22
14	5400	73	98.39	231.56	99.28	96.80	97.39	28.38
15	5400	73	98.37	231.24	99.54	82.06	97.39	26.50
16	9600	97	98.65	480.44	99.46	240.39	98.06	53.76
17	9600	97	98.47	465.49	98.42	399.86	98.21	57.33
18	9600	97	98.44	478.02	99.64	206.78	98.02	52.89
19	38400	196	98.08	3760.14	99.03	3054.38	98.35	311.56
20	38400	197	98.8	2841.96	99.34	1990.70	98.80	360.42
21	38400	196	98.29	3700.99	98.61	5615.75	98.39	324.49

4 Conclusion

In this paper, a heuristic algorithm BCC based on the random search method for the two-dimensional single large bin packing problem is proposed, the experiments show that our algorithm is highly competitive and could be considered as a viable alternative for 2D-SLBPP. Especially for large test problems, the algorithm could get satisfied results more quickly than other approaches in the literature. Furthermore, if BCC could combine with some appropriate intelligent optimization methods, we think that it could get better optimal solutions in acceptable time.

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On isotopy, parastrophy and orthogonality of quasigroups

K. K. Shchukin

Abstract. This paper contains new results on conditions of an isotopy of two quasigroups and their orthogonality to parastrophes. The structure of parastrophe group of a quasigroup is defined. The results of this paper complement investigations of V. D. Belousov in [1, 2] and continue studies from [3].

Mathematics subject classification: 20N05.
Keywords and phrases: Quasigroup, isotopy, parastrophy, orthogonality.

To the 85 Anniversary of V. D. Belousov (1925–1988)

1 Main results

1. Every quasigroup $(Q, \cdot)$ defines three permutations on the set Q. These are left $L_a(y) = ay$ and right $R_a(y) = ya$ translations for all $a, y \in Q$. A middle one J_a and its inversion J_a^{-1} are defined by $xJ_a(x) = a$, $J_a^{-1}(x)x = a$, $x, a \in Q$ respectively. A quasigroup $(Q, *)$ is conjugate to a quasigroup $(Q, \cdot)$ if $x * y = yx$ is true for all $x, y \in Q$. It is evident that $L_a^*(y) = R_a(y)$ for all $a, y \in Q$, so $L_a^* = R_a$ and $L_a = L_a^{**} = R_a^*$.

Theorem 1 (see [3]). *Let $(Q, \cdot)$ and $(Q, \circ)$ be quasigroups and (φ, ψ, χ) be an ordered triple of permutations on the set Q.*

(i) The formula $\chi(xy) = \varphi(x) \circ \psi(y)$, for all $x, y \in Q$, defines an isotopy of $(Q, \cdot)$ and $(Q, \circ)$ if and only if

$$\psi J_a \varphi^{-1}(\varphi(x)) = J^{\circ}_{\chi(a)}(\varphi(x))$$

for all $x, y \in Q$, $xy = a$.

The equalities $\varphi = \psi = \chi$ define an isomorphism of these quasigroups:

$$\chi J_a \chi^{-1}(\chi(x)) = J^{\circ}_{\chi(a)}(\chi(x))$$

for all $x, y \in Q$, $xy = a$.

(ii) the formula $\chi(xy) = \psi(y) \circ \varphi(x)$, for all $x, y \in Q$, defines an anti-isotopy of $(Q, \cdot)$ and $(Q, \circ)$ if and only if

$$\psi J_a \varphi^{-1}(\varphi(x)) = (J^{\circ}_{\chi(a)})^{-1}(\varphi(x))$$

for all $x, y \in Q, \ \ xy = a$.

The equalities $\varphi = \psi = \chi$ *define an anti-isomorphism of* $(Q, \cdot)$ *and* $(Q, \circ)$ *if and only if*

$$\chi J_a \chi^{-1}(\chi(x)) = (J^{\circ}_{\chi(a)})^{-1}(\chi(x))$$

for all $x, y \in Q, \ \ xy = a$.

(iii) *There are equivalences of an isotopy* (φ, ψ, χ) *of the quasigroups* $(Q, \cdot)$ *and* $(Q, \circ)$ *for all* $x, y \in Q$: $\chi(xy) = \varphi(x) \circ \psi(y) \iff \chi L_x \psi^{-1}(y) = L^{\circ}_{\varphi(x)}(y) \iff \chi R_y \varphi^{-1}(x) = R^{\circ}_{\psi(y)}(x)$.

Proof. The statement (i) is established by the following chain of equivalences: $\chi(xy) = \varphi(x) \circ \psi(y) \Leftrightarrow \chi(a) = \varphi(x) \circ J^{\circ}_{\chi(a)}\varphi(x) \Leftrightarrow J^{\circ}_{\chi(a)}\varphi(x) = \psi(y) = \psi J_a \varphi^{-1}(\varphi(x)) \Leftrightarrow J^{\circ}_{\chi(a)}\varphi(x) = \psi J_a \varphi^{-1}(\varphi x)$ for all $x, y \in Q$, putting $xy = a$, where a depends on x, y. The case $\varphi = \psi = \chi$ reduces to three equivalent conditions of isomorphism of $(Q, \cdot)$ and $(Q, \circ)$.

The statement (ii) is verified like (i): $\chi(xy) = \psi(y) \circ \varphi(x) \Leftrightarrow \chi(a) = \psi(y) \circ J^{\circ}_{\chi(a)}(\psi(y)) \Leftrightarrow J^{\circ}_{\chi(a)}\psi(y) = \varphi(x) = \varphi J_a^{-1}\psi^{-1}(y) \Leftrightarrow (J^{\circ}_{\chi(a)})^{-1}\varphi(x) = \psi J_a \varphi^{-1}(\varphi(x))$ for all $x, y \in Q, xy = a$. Three equivalent conditions of anti-isomorphism of the quasigroups $(Q, \cdot)$ and $(Q, \circ)$ follow by $\varphi = \psi = \chi$. □

We consider the *signature* $(Q, \cdot)$ of a finite quasigroup $(Q, \cdot)$ of order n as an ordered triple of signs:

$$signature\ (Q, \cdot) = (sign\ Q_L, sign\ Q_R, sign\ Q_J),$$

where $Q_L = L_1 \dots L_n$, $Q_R = R_1 \dots R_n$, $Q_J = J_1 \dots J_n$ are the products of translations of $(Q, \cdot)$.

As it is known, a complete associated group of a quasigroup is generated by all left, right and middle translations of this quasigroup [1].

From Theorem 1 we easy obtain

Corollary 1. *a) Isomorphic or anti-isomorphic quasigroups have isomorphic or anti-isomorphic complete associated groups, respectively.*

b) Let $(Q, \circ)$ *be an isotope or an anti-isotope of a finite quasigroup* $(Q, \cdot)$ *of order* n. *There are the following formulas (cf.*(iii)*):*

Signature $(Q, \circ) = (sign(\chi\psi)^n sign Q_L, sign(\chi\varphi)^n sign Q_R, sign(\varphi\psi)^n sign Q_J)$ *by an isotopy* $\chi(x, y) = \varphi(x) \circ \psi(y)$

To get the formula of signature $(Q, \circ)$ *of an anti-isotope it is sufficient only to exchange the first and the second components of the formula for isotopy* (i).

There is the equality *signature* $(Q, \circ) =$ *signature* $(Q, \cdot)$ in both cases (i) and (ii) for $n = 2m$ or $\varphi = \psi = \chi$.

2. We preserve here the notation of the paper [3] (see also [4, p. 13–14]). If $\alpha = (\odot)$ is a quasigroup operation, then α, $\beta = * = \alpha^*$, $\gamma = \alpha^{-1}$, $\delta = {}^{-1}\alpha$,

$\varepsilon = {}^{-1}(\alpha^{-1}) = \gamma^*$, $\eta = ({}^{-1}\alpha)^{-1} = \delta^*$ will denote the inverse operations of the quasigroup $(Q, \odot) = Q(\alpha)$ and $\prod = \{\alpha, \beta, \gamma, \delta, \varepsilon, \eta\}$.

Let the composition $\theta'' \circ \theta'$ mean the application of θ'' to the inverse operation defined θ', then $\theta'' \circ \theta' = \theta \in \prod$ for all $\theta', \theta'' \in \prod$ (cf. [4, p. 14]).

In general a non-commutative quasigroup can have six pairwise different inverse operations. It is easy to check in general case $\alpha \circ \theta = \theta = \theta \circ \alpha$ for all $\theta \in \prod$ and $\alpha = \alpha \circ \alpha = \beta \circ \beta = \gamma \circ \gamma = \delta \circ \delta$, $\varepsilon \circ \varepsilon = \eta$, $\eta \circ \eta = \varepsilon$, $\varepsilon \circ (\varepsilon \circ \varepsilon) = \alpha = (\varepsilon \circ \varepsilon) \circ \varepsilon$, $\varepsilon^{-1} = \eta$, $\delta \circ \varepsilon = \beta = \gamma \circ \eta$, etc [4].

We can now construct the multiplication table of $(\prod, \circ)$, using the received formulas and an algorithm of [4]. This is Table 1 for a non-commutative quasigroup with six pairwise distinct parastrophes, and otherwise $(\prod, \circ)$ is isomorphic to a subgroup of the symmetric group S_3.

Each $\theta \in \prod$ defines the parastrophe $(Q, \theta) = Q(\theta)$ of a quasigroup $(Q, \odot) = Q(\alpha)$ and the parastrophy $(Q, \odot) = Q(\alpha) \xrightarrow{\theta} Q(\theta)$ as a mapping. An (ordered) sixtuple $\prod(Q(\alpha)) = (Q(\alpha), Q(\beta), Q(\gamma), Q(\delta), Q(\varepsilon), Q(\eta))$ is called a parastrophe system of the quasigroup $(Q, \odot) = Q(\alpha)$. The diagram

$$\begin{tikzcd}
(Q,\odot) = Q(\alpha) \arrow[rr, "\theta'"] \arrow[dr, "\theta"'] & & Q(\theta') \arrow[dl, "\theta''"] \\
& Q(\theta'' \circ \theta') &
\end{tikzcd}$$

of the action of parastrophies on the system $\prod(Q(\alpha))$ is commutative and $Q(\theta'' \circ \theta') = Q(\theta)$. So all parastrophs of the quasigroup $(Q, \odot) = Q(\alpha)$ form a group $(\prod, \cdot)$ relative to the action on the system $\prod(Q(\alpha))$. It is isomorphic to the group $(\prod, \circ)$.

Theorem 2. *The group $(\prod, \cdot)$ of parastrophies acting on $\prod(Q(\alpha))$ is isomorphic to the group $(\prod, \circ)$ relative to the composition of taking of inverse operations of the quasigroup $(Q, \odot) = Q(\alpha)$. Both these group are isomorphic to some subgroup of the symmetric group S_3. Table 1 serves as the multiplication table for a quasigroup with pairwise distinct parastrophes.*

$\cdot$	α	β	γ	δ	ε	η
α	α	β	γ	δ	ε	η
β	β	α	ε	η	γ	δ
γ	γ	η	α	ε	δ	β
δ	δ	ε	η	α	β	γ
ε	ε	δ	β	γ	η	α
η	η	γ	δ	β	α	ε

Table 1

Remark 1. We will denote the conjugation as $\beta\theta$ instead of θ^* using the second row $\beta\theta = \theta^*$, $\theta \in \Pi$, of the multiplication table.

In the paper [3] it is proved:

The action of an isotopy (φ, ψ, λ) on a quasigroup $(Q, \cdot) = Q(\alpha)$ induces identically an isotopy $\theta(\varphi, \psi, \lambda)$ on each $Q(\theta) \in \Pi(Q(\alpha))$.

The results of this action are presented by the following table:

$Q(\alpha)$	$Q(\beta)$	$Q(\gamma)$	$Q(\delta)$	$Q(\varepsilon)$	$Q(\eta)$
(φ, ψ, χ)	(ψ, φ, χ)	(φ, χ, ψ)	(χ, ψ, φ)	(χ, φ, ψ)	(ψ, χ, φ)

Table 2

We use the second table and also the natural commutative diagram for $\theta \in \Pi$:

$$\begin{tikzcd} Q(\alpha) \arrow[r, "\theta"] \arrow[d, "(\varphi,\psi,\chi)"'] & Q(\theta) \arrow[d, "(\lambda,\mu,\nu)"] \\ Q(\alpha(\circ)) \arrow[r, "\theta"] & Q(\theta(\circ)) \end{tikzcd}$$

(where $(Q, \cdot) = Q(\alpha)$ and λ, μ, ν depend on θ) to derive six conditions of the permutability of the isotopy and parastrophy:

$\alpha(\varphi, \psi, \chi) = (\varphi, \psi, \chi)\alpha$	$\delta(\varphi, \psi, \chi) = (\chi, \psi, \varphi)\delta$
$\beta(\varphi, \psi, \chi) = (\psi, \varphi, \chi)\beta$	$\varepsilon(\varphi, \psi, \chi) = (\chi, \varphi, \psi)\varepsilon$
$\gamma(\varphi, \psi, \chi) = (\varphi, \chi, \psi)\gamma$	$\eta(\varphi, \psi, \chi) = (\psi, \chi, \varphi)\eta$

Table 3

The full multiplication table of the parastrophies and the isotopies of a quasigroup is the following:

$\cdot$	(φ, ψ, χ)	(ψ, φ, χ)	(φ, χ, ψ)	(χ, ψ, φ)	(χ, φ, ψ)	(ψ, χ, φ)
α	$(\varphi, \psi, \chi)\alpha$	$(\psi, \varphi, \chi)\alpha$	$(\varphi, \chi, \psi)\alpha$	$(\chi, \psi, \varphi)\alpha$	$(\chi, \varphi, \psi)\alpha$	$(\psi, \chi, \varphi)\alpha$
β	$(\psi, \varphi, \chi)\beta$	$(\varphi, \psi, \chi)\alpha$	$(\chi, \varphi, \psi)\varepsilon$	$(\psi, \chi, \varphi)\eta$	$(\varphi, \chi, \psi)\gamma$	$(\chi, \psi, \varphi)\delta$
γ	$(\varphi, \chi, \psi)\gamma$	$(\psi, \chi, \varphi)\eta$	$(\varphi, \psi, \chi)\alpha$	$(\chi, \varphi, \psi)\varepsilon$	$(\chi, \psi, \varphi)\delta$	$(\psi, \varphi, \chi)\beta$
δ	$(\chi, \psi, \varphi)\delta$	$(\chi, \varphi, \psi)\varepsilon$	$(\psi, \chi, \varphi)\eta$	$(\varphi, \psi, \chi)\alpha$	$(\psi, \varphi, \chi)\beta$	$(\varphi, \chi, \psi)\gamma$
ε	$(\chi, \varphi, \psi)\varepsilon$	$(\chi, \psi, \varphi)\delta$	$(\psi, \varphi, \chi)\beta$	$(\varphi, \chi, \psi)\gamma$	$(\psi, \chi, \varphi)\eta$	$(\varphi, \psi, \chi)\alpha$
η	$(\psi, \chi, \varphi)\eta$	$(\varphi, \chi, \psi)\gamma$	$(\chi, \psi, \varphi)\delta$	$(\psi, \varphi, \chi)\beta$	$(\varphi, \psi, \chi)\alpha$	$(\chi, \varphi, \psi)\varepsilon$

Table 4

Recall that each of the products of a parastrophy with an isotopy and of an isotopy with a parastrophy is called an isostrophy (see [2, p. 28]).

Corollary 2. *of the mappings. This group G is semi-direct S_P by S_Π i.e. G is isomorphic to the holomorph $HolS_3 = S_3 \cdot AutS_3$. Each quasigroup $(Q, \odot) = Q(\alpha)$ has no more than 36 pairwise different isostrophies. The number of these isostrophies depends on order of the group $(\Pi, \cdot)$.*

It follows from Theorem 2 and Table 4.

3. According to [2] two quasigroups $(Q, \cdot)$ and $(Q, \circ)$ are mutually orthogonal if and only if the system of the equations $xy = a$, $x \circ y = b$ is identically resolved for all $a, b \in Q$. In this case it is denoted $(Q, \cdot) \perp (Q, \circ)$ or $(Q, \circ) \perp (Q, \cdot)$.

In [2] V. D. Belousov investigated he question on orthogonality of a quasigroup to its parastrophes. In order to continue this idea we use another equivalent definition of orthogonality of quasigroups.

Proposition 1. $(Q,\cdot)\perp(Q,\circ)$ *is true if and only if at least one of two equations*

$$L_x^{\circ}L_x^{-1}(a)=b, \tag{L}$$

$$R_y^{\circ}R_y^{-1}(a)=b \tag{R}$$

is identically resolved for all $a,b\in Q$.

Theorem 3. *Let* $\Pi(Q(\alpha))=(Q(\alpha),Q(\beta),Q(\gamma),Q(\delta),Q(\varepsilon),Q(\eta))$ *be the parastrophe system of a quasigroup* $(Q,\cdot)=Q(\alpha)$. *The following statements are valid:*

(i) $Q(\alpha)\perp Q(\gamma)\Leftrightarrow Q(\beta)\perp Q(\varepsilon)\Leftrightarrow$ *the equation* $L_x^2(b)=a$ *is identically resolved for all* $a,b\in Q$,

(ii) $Q(\alpha)\perp Q(\delta)\Leftrightarrow Q(\beta)\perp Q(\eta)\Leftrightarrow$ *the equation* $R_y^2(b)=a$ *is identically resolved for all* $a,b\in Q$,

(iii) $Q(\alpha)\perp Q(\beta)\Leftrightarrow$ *the equation* $L_xR_x^{-1}(b)=a$ *is identically resolved for all* $a,b\in Q$.

Proof. We use Proposition 1 and representation of parastrophes of a quasigroup $(Q,\cdot)=Q(\alpha)$ (see [1]).

(i) The equation (L) is fulfilled by $L_x^{\circ}=L_x^{\gamma}=L_x^{-1}$. It is also evident that $Q(\alpha)\perp Q(\gamma)\Leftrightarrow Q(\beta)\perp Q(\varepsilon)$ since the the equalities $Q(\beta\alpha)=Q(\beta)$ and $Q(\beta\gamma)=Q(\varepsilon)$ are true (see Table 1).

(ii) The equation (R) will be realized by $R_y^{\circ}=R_y^{\delta}=R_y^{-1}$. It is also evident that $Q(\alpha)\perp Q(\delta)\Leftrightarrow Q(\beta)\perp Q(\eta)$ since the equalities $Q(\beta\alpha)=Q(\beta)$ and $Q(\beta\delta)=Q(\eta)$ are true (see Table 1).

(iii) The equation (L) will be fulfilled by $L_x^0=L_x^{\beta}=R_x$. □

Corollary 3. *Let* $(Q,\cdot)$ *be a finite quasigroup. At least one from the conditions* $(i),(ii),(iii)$ *of Theorem 3 is broken if some permutation from* $L_x^2,R_y^2,L_xR_x^{-1}$ *contains a transposition* (a,b), $a,b\in Q$.

Example 1. The left translations $L_1=(1)$, $L_2=(12)(345)$, $L_3=(13524)$, $L_4=(14325)$, $L_5=(15423)$ define a loop $(Q,\cdot)$ of order five. $(Q,\cdot)=Q(\alpha)$ is non-orthogonal to $Q(\gamma),Q(\delta)$ and $Q(\beta)$ since $L_2=R_2=(12)(345)$.

There are some addifional conditions for a quasigroup by which it is orthogonal to some its parastrophes. Such identities are investigated in [2] where seven minimal identities are determined. We use below some of these identities to prove Theorem 3:

Conditions of Theorem 3	Supplimentary identities	Reorganized conditions of Theorem 3
$(i)\ L_x^2(b) = a$	$(x \cdot xy)x = y$	$R_x^{-1}(b) = a$
	$x(x \cdot xy) = y$	$L_x^{-1}(b) = a$
$(ii)\ R_y^2(b) = a$	$(xy \cdot y)y = x$	$R_y^{-1}(b) = a$
	$y(xy \cdot y) = x$	$L_y^{-1}(b) = a$
$(iii)\ L_x R_x^{-1}(b) = a$	$x \cdot xy = yx$	$L_x^{-1}(b) = a$

Table 5

It should be noted that there exist quasigroups which are orthogonal to some their parastrophes and non-parastrophes.

Example 2. A finite cyclic group $(Q, \cdot) = Q(\alpha)$ has only two parastrophes $Q(\gamma)$ and $Q(\delta)$. By Theorem 3 $Q(\alpha) \perp Q(\gamma)$ and $Q(\alpha) \perp Q(\delta)$ if $CardQ > 2$ is an odd number.

Moreover a quasigroup may exist a non-parastrophe $(Q, \circ)$ of which is orthogonal to the group $Q(\alpha)$. This situation is demonstrated by the following 3×3-Latin squares:

$$[\alpha] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}, \quad [\gamma] = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}, \quad [\delta] = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix},$$

$$[\circ] = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}, \quad [\alpha, \circ] = \begin{bmatrix} 12 & 23 & 31 \\ 21 & 32 & 13 \\ 33 & 11 & 22 \end{bmatrix},$$

Table 6

where $[\alpha] \perp [\gamma]$, $[\alpha] \perp [\delta]$ and $[\alpha] \perp [\circ]$.

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On stability of Pareto-optimal solution of portfolio optimization problem with Savage's minimax risk criteria

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Abstract. A multicriteria Boolean optimization problem consisting in an efficient choice of a Pareto-optimal portfolio of investor's assets that uses the Savage's minimax risk criteria is considered. Upper and lower attainable bounds of the stability radius of such portfolio with regard to independent changes of elements of a risk matrix are obtained.

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1 Introduction

In recent years, interest towards multi-objective decision-making processes under uncertainty and risk has grown dramatically. It can be explained by numerous applications of such problems in game theory, mathematical economics, optimal control, investment analysis, banking, insurance business, etc. Widespread occurrence of discrete optimization models has conditioned the interest of many experts to the study of various types of stability aspects, parametric and post-optimal analysis problems of both scalar (one-criterion) and vector (multicriteria) discrete optimization (see, for example, monographs [1–3], reviews [4–6], and annotated bibliographies [7, 8]).

One of the well-known approaches to investigation of the stability of discrete optimization problems is aimed at obtaining the so-called quantitative stability characteristics. This approach consists in finding the limit level of perturbations of initial problem data which do not change the studied original solution. As a rule, the perturbed parameters are the vector criterion coefficients. The majority of results in this research area are related to stability radius formula for Pareto-optimal (efficient) solutions of vector linear optimization problems [9, 10], in particular, Boolean problems [11], game theory problems [12, 13], and also for the stability radius of a lexicographic optimum of certain Boolean problems with linear criteria [14, 15].

This paper deals with obtaining upper and lower attainable bounds of the stability radius of a Pareto-optimal solution of portfolio optimization problem with Savage's minimax risk criteria.

2 Basic definitions and auxiliary statements

Let us consider the vector variant of the portfolio optimization problem, i.e. the problem of financial investments management, based on Markovitz's "portfolio theory" [16, 17] (see also the bibliography in [18]). To this end, we introduce the following notations:

$N_n = \{1, 2, ..., n\}$ – assets (shares, companies' bonds, real estate etc.),

N_m – economic strategies of an investor,

R – three-dimensional risk matrix (missed opportunities) of $m \times n \times s$ size with elements r_{ijk} from $\mathbf{R}$,

r_{ijk} – risk quantity of an investor choosing strategy $i \in N_m$ and asset $j \in N_n$ with criterion $k \in N_s$,

$x = (x_1, x_2, ..., x_n)^T \in X \subset \{0, 1\}^n$ – investor's portfolio of assets.

$$x_j = \begin{cases} 1, & \text{if the investor chooses an asset } j, \\ 0 & \text{otherwise.} \end{cases}$$

Presumably, each investor's portfolio x from a given portfolio set X assures expected total profit p and does not exceed the total amount of available capital c, i. e. for each portfolio $x = (x_1, x_2, ..., x_n)^T \in X$ the conditions

$$\sum_{j \in N_n} p_j \, x_j \geq p, \quad \sum_{j \in N_n} c_j x_j \leq c,$$

hold, where p_j is the expected profit of asset j, c_j is the cost of asset j.

Along with three-dimensional matrix $R = [r_{ijk}] \in \mathbf{R}^{m \times n \times s}$ we use its two-dimensional sections $R_k \in \mathbf{R}^{m \times n}$, $k \in N_s$.

Let the following vector function

$$f(x, R) = (f_1(x, R_1), f_2(x, R_2), ..., f_s(x, R_s))$$

be defined over the set X with *Savage's minimax risk* (extreme pessimism) *criteria* [19, 20], (see also [21–23])

$$f_k(x, R_k) = \max_{i \in N_m} \sum_{j \in N_n} r_{ijk} x_j \to \min_{x \in X}, \qquad k \in N_s.$$

We consider the problem of finding Pareto set $P^s(R)$, where a *Pareto-optimal (efficient) portfolios (solutions)* is regarded as *portfolio optimization problem* $Z^s(R)$:

$$P^s(R) = \{x \in X : \; P^s(x, R) = \emptyset\},$$

where $P^s(x, R) = \{x' \in X : \; x \underset{R}{\succ} x'\}$, whereas symbol $\underset{R}{\succ}$ is a binary relation defined over the set X as follows:

$$x \underset{R}{\succ} x' \quad \Leftrightarrow \quad g(x, x', R) \geq \mathbf{0} \;\; \& \;\; g(x, x', R) \neq \mathbf{0},$$

where $\mathbf{0} = (0,\ 0,\ \dots,\ 0) \in \mathbf{R}^s$, $g(x,\ x',\ R) = (g_1(x,\ x',\ R_1),\ \ g_2(x,\ x',\ R_2),\ \dots,\ g_s(x,\ x',\ R_s))$, $g_k(x,\ x',\ R_k) = f_k(x,\ R_k) - f_k(x',\ R_k) = \max\limits_{i \in N_m} R_{ik}x - \max\limits_{i \in N_m} R_{ik}x'$, $k \in N_s$, and $R_{ik} = (r_{i1k},\ r_{i2k},\ \dots,\ r_{ink})$ is row i of matrix $R_k \in \mathbf{R}^{m \times n}$.

In space $\mathbf{R}^d$ of an arbitrary dimension $d \in \mathbf{N}$ we set the l_∞-metric, i.e. as the norm of vector $z = (z_1,\ z_2,\ ...,\ z_d) \in \mathbf{R}^d$ we understand the number

$$\|z\| = \max\{|z_j| : \quad j \in N_d\},$$

and as the norm of matrix we understand the norm of a vector composed of all matrix elements. Thus the inequalities $\|R\| \geq \|R_k\| \geq \|R_{ik}\|$ holds for any $i \in N_m$ and $k \in N_s$.

As usual (see, for example,[6, 9–12]), *stability radius* of portfolio $x^0 \in P^s(R)$ is defined as follows:

$$\rho^s(x^0, R) = \begin{cases} \sup\ \Xi, & \text{if } \ \Xi \neq \emptyset, \\ 0, & \text{if } \ \Xi = \emptyset, \end{cases}$$

where

$$\Xi = \{\varepsilon > 0 : \ \forall R' \in \Omega(\varepsilon) \quad (x^0 \in P^s(R + R'))\},$$

$$\Omega(\varepsilon) = \{R' \in \mathbf{R}^{m \times n \times s} : \ \|R'\| < \varepsilon\}.$$

Here $\Omega(\varepsilon)$ is the set of *perturbing matrices*, and $Z^s(R + R')$ is the *perturbed problem*.

The following lemma is evident.

Lemma. *Let $x^0 \in P^s(R)$, $\varphi > 0$. If for any perturbing matrix $R' \in \Omega(\varphi)$ and any solution $x \in X \setminus \{x^0\}$ index $q \in N_s$ exists, such that the inequality $g_q(x,\ x^0,\ R_q + R'_q) > 0$ holds, then $x^0 \in P^s(R + R')$ for any $R' \in \Omega(\varphi)$.*

It is also quite evident that for any matrix $R_k \in \mathbf{R}^{m \times n}$ and any solutions $x^0,\ x \in X$ the following inequalities are true:

$$R_{ik}x - R_{i^0k}x^0 \geq -\|R_k\|\ \|x + x^0\|^*, \qquad i,\ \ i^0 \in N_m, \qquad k \in N_s, \tag{1}$$

where $\|z\|^* = \sum\limits_{j \in N_n} |z_j|$, $z = (z_1,\ z_2,\ ...,\ z_n)^T$.

3 Stability radius bounds

For portfolio $x^0 \in P^s(R)$ we introduce the following notations:

$$\varphi = \min_{x \in X \setminus \{x^0\}} \ \max_{k \in N_s} \ \min_{i^0 \in N_m} \ \max_{i \in N_m} \ \frac{R_{ik}x - R_{i^0k}x^0}{\|x + x^0\|^*},$$

$$\psi = \min_{x \in X \setminus \{x^0\}} \ \max_{k \in N_s} \ \min_{i^0 \in N_m} \ \max_{i \in N_m} \ \frac{R_{ik}x - R_{i^0k}x^0}{\|x - x^0\|^*}.$$

Theorem. *For stability radius $\rho^s(x^0,\ R)$, $s \geq 1$, of a Pareto-optimal portfolio x^0 of problem $Z^s(R)$ the following bounds are true:*

$$\varphi \leq \rho^s(x^0,\ R) \leq \psi.$$

Proof. Let $x^0 \in P^s(R)$. The formula

$$\forall x \in X \setminus \{x^0\} \quad (x^0 \notin P^s(x^0,\ R))$$

obviously holds. Hence with account of inequality $\|x + x^0\|^* \geq \|x - x^0\|^* > 0$, this results in $\psi \geq \varphi \geq 0$.

To prove Theorem, firstly it is necessary to prove that $\rho^s(x^0,\ R) \geq \varphi$, which is evident if $\varphi = 0$. Let $\varphi > 0$. According to the definition of φ for any portfolio $x \in X \setminus \{x^0\}$, there is such index $q \in N_s$ that

$$\min_{i^0 \in N_m} \max_{i \in N_m} (R_{iq}x - R_{i^0q}x^0) \geq \varphi\ \|x + x^0\|^*. \tag{2}$$

Further, taking into account (1), for any perturbing matrix $R' \in \Omega(\varphi)$ and any $k \in N_s$, we have:

$$g_k(x,\ x^0,\ R_k + R'_k) = \max_{i \in N_m}(R_{ik} + R'_{ik})x - \max_{i \in N_m}(R_{ik} + R'_{ik})x^0 =$$

$$= \min_{i^0 \in N_m} \max_{i \in N_m}(R_{ik}x - R_{i^0k}x^0 + R'_{ik}x - R'_{i^0k}x^0) \geq$$

$$\geq \min_{i^0 \in N_m} \max_{i \in N_m}(R_{ik}x - R_{i^0k}x^0) - \|R'_k\|\|x + x^0\|^*.$$

Hence, in view of $\varphi > \|R'\| \geq \|R'_q\|$ inequality (2) implies

$$g_q(x,\ x^0,\ R_q + R'_q) > 0.$$

Therefore, due to Lemma we have $x^0 \in P^s(R + R')$ for any perturbing matrix $R' \in \Omega(\varphi)$, i.e. the inequality $\rho^s(x^0,\ R) \geq \varphi$ is true.

Further, we prove the inequality $\rho^s(x^0,\ R) \leq \psi$. In accordance with the definition of ψ there is such portfolio $x \in X \setminus \{x^0\}$ that the following inequalities are true:

$$\psi\|x - x^0\|^* \geq \min_{i^0 \in N_m} \max_{i \in N_m}(R_{ik}x - R_{i^0k}x^0), \qquad k \in N_s. \tag{3}$$

Now, setting $\varepsilon > \psi$, we consider the perturbing matrix $R^0 = [r^0_{ijk}] \in \mathbf{R}^{m\times n\times s}$ whose elements are defined as follows:

$$r^0_{ijk} = \begin{cases} \delta, & \text{if } \ i \in N_m,\ x^0_j \geq x_j,\ k \in N_s, \\ \\ -\delta, & \text{if } \ i \in N_m,\ x^0_j < x_j,\ k \in N_s, \end{cases}$$

where $\psi < \delta < \varepsilon$. Then $\|R^0\| = \|R_k^0\| = \|R_{ik}^0\| = \delta$ where $i \in N_m, \quad k \in N_s$. In addition, all rows $R_{ik}^0, \quad i \in N_m$, of matrix R_k^0 are equal and consist of components δ and $-\delta$ for any index $k \in N_s$. Therefore, denoting this row by B (it only depends on x and x^0), we have

$$B(x - x^0) = -\delta\|x - x^0\|^*, \qquad \|B\| = \delta.$$

Hence, in view of (3), for any index $k \in N_s$, we obtain

$$g_k(x,\ x^0,\ R_k + R_k^0) = \max_{i\in N_m}(R_{ik} + B)x - \max_{i\in N_m}(R_{ik} + B)x^0 =$$

$$= \max_{i\in N_m} R_{ik}x - \max_{i\in N_m} R_{ik}x^0 + B(x - x^0) = \min_{i^0\in N_m}\max_{i\in N_m}(R_{ik}x - R_{i^0k}x^0) + B(x - x^0) \leq$$

$$\leq (\psi - \delta)\ \|x - x^0\|^* < 0.$$

Thus, the binary relation $x^0 \underset{R+R^0}{\succ} x$ holds. Therefore, for any $\varepsilon > \psi$ there is such perturbing matrix $R^0 \in \Omega(\varepsilon)$ that Pareto-optimal portfolio x^0 of problem $Z^s(R)$ looses its Pareto-optimality in the perturbed problem $Z^s(R + R^0)$, i.e. $x^0 \notin P^s(R + R^0)$. Therefore $\rho^s(x^0,\ R) \leq \psi$. $\square$

The upper bound ψ of the stability radius $\rho^s(x^0,\ R)$ indicated in Theorem is attainable, since for $m = 1$ our problem $Z^s(R)$ is transformed into a vector (s-criteria) Boolean programming problem with linear criteria:

$$R_k x \to \min_{x\in X}, \qquad k \in N_s, \tag{4}$$

whereas the upper bound turns into the form

$$\rho^s(x^0,\ R) \leq \psi = \min_{x\in X\setminus\{x^0\}}\ \max_{k\in N_s}\ \frac{R_k(x - x^0)}{\|x - x^0\|^*},$$

where R_k is k-th row of matrix $R \in \mathbf{R}^{s\times n}$. It is known [6, 10] that the right-hand side of this ratio is the expression of the stability radius of $x^0 \in P^s(R)$ of problem (4). Therefore, if $m = 1$, we have $\rho^s(x^0,\ R) = \psi$, that assures the attainability of this upper bound.

It is also quite evident that the lower bound φ is also attainable. Indeed, let the equality $\|x+x^0\|^* = \|x-x^0\|^*$ be true for any $x \in X\setminus\{x^0\}$, then $\rho^s(x^0,\ R) = \varphi = \psi$.

So we have the following corollary of Theorem, which shows that the radius of stability of Pareto-optimal portfolio $x^0 \in P^s(R)$ can be equal to the lower positive bound φ and may not coincide with the upper bound ψ.

Corollary 1. *There exists a class of problems $Z^s(R)$ such that for the solution $x^0 \in P^s(R)$ the following correlations are true:*

$$0 < \rho^s(x^0,\ R) = \varphi < \psi. \tag{5}$$

Proof. Let $\varphi > 0$. The inequality $\varphi < \psi$ is true if $\|x + x^0\|^* > \|x - x^0\|^*$ holds for any vector $x \in X \setminus \{x^0\}$. To prove the equality $\rho^s(x^0,\ R) = \varphi$ in accordance with Theorem, it is sufficient to identify the class of problems for which the inequality $\rho^s(x^0,\ R) \leq \varphi$ is true. Further exposition is devoted to this.

The definition of $\varphi > 0$ entails such vector $\widehat{x} \in X \setminus \{x^0\}$ that

$$\varphi\|\widehat{x} + x^0\|^* \geq g_k(\widehat{x},\ x^0,\ R_k), \qquad k \in N_s. \tag{6}$$

Further exposition will be for any index $k \in N_s$.

We introduce the following notations:

$$i(x^0) = \arg\max\{R_{ik}x^0 :\ i \in N_m\},$$

$$i(\widehat{x}) = \arg\max\{R_{ik}\widehat{x} :\ i \in N_m\},$$

$$\Delta = \|\widehat{x} + x^0\|^* - \|\widehat{x} - x^0\|^* > 0.$$

Further, we assume that the inequality holds:

$$(R_{i(\widehat{x})k} - R_{i(x^0)k})\widehat{x} > \varphi\Delta, \tag{7}$$

which entails the inequality $i(x^0) \neq i(\widehat{x})$, since $\varphi\Delta > 0$ holds.

For any number $\varepsilon > \varphi$ we define the elements of the section R_k^0 of the perturbing matrix R^0 by the rule

$$r_{ijk}^0 = \begin{cases} \delta, & \text{if} \quad i = i(x^0), \quad x_j^0 = 1, \\ -\delta, & \text{if} \quad i = i(x^0), \quad x_j^0 = 0, \\ -\delta, & \text{if} \quad i \in N_m \setminus \{i(x^0)\}, \quad \widehat{x}_j = 1, \\ 0 & \text{otherwise}, \end{cases}$$

where

$$\min\left\{\varepsilon,\ \frac{1}{\Delta}(R_{i(\widehat{x})k} - R_{i(x^0)k})\widehat{x}\right\} > \delta > \varphi. \tag{8}$$

Noteworthy, the last inequalities are correct because of (7).

Due to the structure of the section R_k^0 we have

$$R_{ik}^0\widehat{x} = -\delta\|\widehat{x}\|^*, \qquad i \in N_m \setminus \{i(x^0)\}, \tag{9}$$

$$R_{i(x^0)k}^0 x^0 = \delta\|x^0\|^*, \tag{10}$$

$$\|R_k^0\| = \|R^0\| = \delta, \quad R^0 \in \Omega(\varepsilon).$$

Moreover, the equality holds:

$$R_{i(x^0)k}^0\widehat{x} = \delta(\Delta - \|\widehat{x}\|^*). \tag{11}$$

Indeed, let us denote the sets:

$$Q_1 = \{j \in N_n : \quad \widehat{x}_j = x_j^0 = 1\},$$
$$Q_2 = \{j \in N_n : \quad \widehat{x}_j = 1, \quad x_j^0 = 0\}.$$

Then the following equalities are obvious:

$$|Q_1| = \Delta/2,$$
$$|Q_2| = \|\widehat{x}\|^* - \Delta/2,$$
$$R^0_{i(x^0)k}\widehat{x} = \delta|Q_1| - \delta|Q_2|,$$

from which the inequality (11) ensues.

Further, we will prove that $g_k(\widehat{x},\ x^0,\ R_k + R_k^0) < 0$. In line with (10) we have

$$f_k(x^0,\ R_k + R_k^0) = \max_{i \in N_m}(R_{ik} + R^0_{ik})x^0 = f_k(x^0,\ R_k) + \delta\|x^0\|^*. \tag{12}$$

We will prove that the equality is true:

$$f_k(\widehat{x},\ R_k + R_k^0) = f_k(\widehat{x},\ R_k) - \delta\|\widehat{x}\|^*. \tag{13}$$

Using (9), we have

$$f_k(\widehat{x},\ R_k + R_k^0) = \max\Big\{(R_{i(\widehat{x})k} + R^0_{i(\widehat{x})k})\widehat{x}, \quad \max_{i \neq i(\widehat{x})}(R_{ik} + R^0_{ik})\widehat{x}\Big\} =$$
$$= \max\Big\{(f_k(\widehat{x},\ R_k) - \delta\|\widehat{x}\|^*), \quad \max_{i \neq i(\widehat{x})}(R_{ik} + R^0_{ik})\widehat{x}\Big\}.$$

Thus, taking into account the obvious inequalities

$$f_k(\widehat{x},\ R_k) - \delta\|\widehat{x}\|^* \geq (R_{ik} + R^0_{ik})\widehat{x}, \qquad i \in N_m \setminus \{i(x^0),\ i(\widehat{x})\},$$

to prove (13) we must prove that

$$f_k(\widehat{x},\ R_k) - \delta\|\widehat{x}\|^* \geq (R_{i(x^0)k} + R^0_{i(x^0)k})\widehat{x}.$$

To this end, using (8) and (11), we have

$$f_k(\widehat{x},\ R_k) - \delta\|\widehat{x}\|^* - (R_{i(x^0)k} + R^0_{i(x^0)k})\widehat{x} = (R_{i(\widehat{x})k} - R_{i(x^0)k})\widehat{x} - \delta\|\widehat{x}\|^* -$$
$$- R^0_{i(x^0)k}\widehat{x} > \delta(\Delta - \|\widehat{x}\|^*) - R^0_{i(x^0)k}\widehat{x} = 0.$$

At last, consistently applying (12), (13), (6) and (8), we obtain

$$g_k(\widehat{x},\ x^0,\ R_k + R_k^0) = g_k(\widehat{x},\ x^0,\ R_k) - \delta\|\widehat{x} + x^0\|^* \leq (\varphi - \delta)\|\widehat{x} + x^0\|^* < 0.$$

Because of that such inequality is true for any $k \in N_s$, that $x^0 \underset{R+R^0}{\succ} \widehat{x}$.

Therefore, the formula

$$\forall \varepsilon > \varphi \quad \exists R^0 \in \Omega(\varepsilon) \qquad (x^0 \notin P^s(R + R^0))$$

holds, which because of the vector $x^0 \in P^s(R)$ results in the inequality $\rho^s(x^0,\ R) \leq \varphi$. In summary, we get proof that correlation (5) is valid. $\square$

We give a numeric example proving Corollary 1.

Example. Let $m = 2$, $n = 3$, $k = 1$; $X = \{x^0,\ x^1\}$, $x^0 = (1,\ 1,\ 0)^T$, $\widehat{x} = (0,\ 1,\ 1)^T$;

$$R = \begin{pmatrix} -5 & 2 & 2 \\ 1 & -1 & 0 \end{pmatrix}.$$

Then $f(x^0,\ R) = 0$, $f(\widehat{x},\ R) = 4$, i. e. x^0 is the optimal portfolio of the problem $Z^1(R)$; $\|\widehat{x} + x^0\|^* = 4$, $\|\widehat{x} - x^0\|^* = 2$, $i(x^0) = 2$, $i(\widehat{x}) = 1$. So $\varphi = 1$, $\psi = 2$, $(R_{i(\widehat{x})k} - R_{i(x^0)k})\widehat{x} = 5 > 2 = \varphi(\|\widehat{x} + x^0\|^* - \|\widehat{x} - x^0\|^*)$.

By Theorem $\rho^1(x^0,\ R) \geq 1$. On the other hand, if

$$R^0 = \begin{pmatrix} 0 & -\delta & -\delta \\ \delta & \delta & -\delta \end{pmatrix},$$

where $1 < \delta < 2.5$, then $\|R^0\| = \delta$ and $f(x^0,\ R + R^0) = 2\delta > 4 - 2\delta = f(\widehat{x},\ R + R^0)$.

As a result we have that $x^0 \notin P^1(R + R^0)$. Hence $\rho^1(x^0,\ R) \leq 1$. Thus, by Theorem we have $\rho^1(x^0,\ R) = \varphi = 1 < \psi = 2$.

Pareto-optimal portfolio $x^0 \in P^s(R)$ is called *stable*, if $\rho^s(x^0,\ R) > 0$. In addition, let us introduce the traditional Smale set $Sm^s(R)$ [24], i.e. the set of *strongly efficient portfolios*:

$$Sm^s(R) = \{x \in X :\ \ \forall x' \in X \setminus \{x\}\ \ \exists q \in N_s \quad (f_q(x',\ R_q) > f_q(x,\ R_q))\}.$$

Apparently, $Sm^s(R) \subseteq P^s(R)$ for any matrix $R \in \mathbf{R}^{m\times n\times s}$ and $Sm^s(R)$ can be empty.

Corollary 2. *Pareto-optimal portfolio $x^0 \in P^s(R)$ is stable iff $x^0 \in Sm^s(R)$.*

Proof. Sufficiency. Let Pareto-optimal portfolio x^0 of problem $Z^s(R)$ be strongly efficient. Then for any $x \in X \setminus \{x^0\}$ we have

$$\xi(x) = \max_{k\in N_s}\ \min_{i^0\in N_m}\ \max_{i\in N_m}\ \frac{R_{ik}x - R_{i^0k}x^0}{\|x + x^0\|^*} = \max_{k\in N_s}\ \frac{f_k(x,\ R_k) - f_k(x^0,\ R_k)}{\|x + x^0\|^*} > 0.$$

Therefore, by Theorem, we have $\rho^s(x^0,\ R) \geq \varphi = \min\limits_{x\in X\setminus\{x^0\}} \xi(x) > 0$, i.e. portfolio $x^0 \in P^s(R)$ is stable.

Necessity. Let portfolio $x^0 \in P^s(R)$ be stable. Then, according to Theorem, we obtain $\psi \geq \rho^s(x^0,\ R) > 0$. Therefore, for any portfolio $x \in X \setminus \{x^0\}$ we have

$$\max_{k\in N_s}\ \frac{f_k(x,\ R_k) - f_k(x^0,\ R_k)}{\|x - x^0\|^*} > 0.$$

It means that for any $x \in X \setminus \{x^0\}$ there is such index $q \in N_s$, that $f_q(x,\ R_q) > f_q(x^0,\ R_q)$, i.e. $x^0 \in Sm^s(R)$. □

Since from the equality $\varphi = 0$ the equality $\psi = 0$ ensues, then the following corollary results from Theorem:

Corollary 3. *If $x^0 \in P^s(R)$, then $\rho^s(x^0,\ R) = 0$ if $\varphi = 0$.*

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On quasiidenties of torsion free nilpotent loops

Alexandru Covalschi

Abstract. It is proved that any loop which contains an infinite cyclic group and does not contain infinite number of relative prime periodic elements has an infinite and independent basis of quasiidentities. In particular, any torsion free nilpotent loop has an infinite and independent basis of quasiidentities.

Mathematics subject classification: 17D05, 20N05.
Keywords and phrases: Quasigroup, loop, quasiidentities, basis of quasiidentities, independent basis, coverage.

One of the classical directions of investigation of algebraic systems in their general theory is the quasivariety theory of algebraic systems, founded by A.I. Malcev [1–3].

This paper studies the problem of existence of an independent basis of quasiidentities for certain loops. It is proved that if L is a loop which contains an infinite cyclic group and does not contain an infinite number of prime periodic elements, then the quasiidentities of L have an independent and infinite basis of quasiidentities. In particular, every torsion-free nilpotent loop has an infinite and independent basis of quasiidentities and the quasivariety generated by it has infinity of coverages.

1 Main notions and denotations

A quasigroup is an algebra with the basic set Q and with three basic binary operations $\cdot$, $/$, $\backslash$ defined on it which satisfy the identities

$$x \cdot (x\backslash y) = x\backslash(x \cdot y) = (y/x) \cdot x = (y \cdot x)/x = y.$$

If a quasigroup Q has such an element e that $e \cdot x = x \cdot e = x$ for all $x \in Q$, then Q is called a *loop* and e is called its unity (see [4] or [5]). Therefore, we consider a loop Q as an algebra with three basic operations of the quasigroup Q and one null basic operation e.

Let a be a non-unity element of a loop L. If some product of m factors, each equal to the element a, is equal to the unity element $e \in L$, then a is called *relative m-periodic*. In particular, if m is a prime number then the relative m-periodic element a is called *relative prime periodic*. If the loop L does not contain a periodic element, then they say that it is *torsion free*.

A quasiidentity (quasigroupoid quasiidentity) of variables $x_1, \ldots, x_n$ is a universal formula which has the form

$$(\forall x_1 \ldots x_n) \&_{i \in I} u_i(x_1, \ldots, x_n) = u'_i(x_1, \ldots, x_n) \Rightarrow$$

$$u(x_1, \ldots, x_n) = u'(x_1, \ldots, x_n),$$

where I is a finite set of indices, u_i, u'_i, u, u' are quasigrupoid words of variables $x_1, \ldots, x_n$. When writing quasiidenties, the symbols $\forall x_1 \ldots x_n$ are usually omitted. As the equality of two words $u = v$ in the loop class is equivalent to $u/v = e$, the quasiidentities written in the loop signature are studied as $\&_{i \in I} u_i = e \Rightarrow u = e$.

A quasigroup class formed only of quasigroups in which the quasiidentities of a given system of quasiidentities are true is called a *quasivariety*.

A system Σ of quasiidentities is called *independent* if no quasiidentity of Σ results from all the rest. A *basis* of the system Σ is such a subsystem $\Sigma' \subseteq \Sigma$ that any quasiidentity from Σ results from the overall of the quasiidentities from Σ'.

A quasivariety N is called a *coverage* of quasivariety M if $M \subset N$ and for any quasivariety K the inclusions $M \subseteq K \subset N$ imply $M = K$.

As usual, prime numbers are denoted by $p_i, i \in \Sigma = \{0, 1, 2, \ldots\}$, the infinite cyclic group - by Z, the cyclic group of order p_i - by Z_{p_i} , the quasivariety generated by a quasigroup Q - by qQ. The set of all natural numbers will be denoted by N.

2 The basic results

We shall say that the quasiidentity $\Phi(x_1, \ldots, x_n) = \&_{i=1}^m u_i(x_1, \ldots, x_n) = u'_i(x_1, \ldots, x_n) \Rightarrow u(x_1, \ldots, x_n) = u'(x_1, \ldots, x_n))$ is compatible in the quasigroup Q if the formula $\varphi(x_1, \ldots, x_n) = (\&_{i=1}^m u_i = u'_i \& u = u')$ is compatible in Q, that is, there are such values $x_1 = a_1, \ldots, x_n = a_n$ of the variables in Q that the following equalities are true:

$$u_1(a_1, \ldots, a_n) = u'_1(a_1, \ldots, a_n), \ldots, u_m(a_1, \ldots, a_n) = u'_m(a_1, \ldots, a_n),$$

$$u(a_1, \ldots, a_n) = u'(a_1, \ldots, a_n).$$

Lemma 1. *The conjunction of a finite number of quasiidentities compatible in any quasigroup is equivalent to one quasiidentity.*

Proof. It is sufficient to prove the lemma for the conjunction of two quasiidentities φ_1 and φ_2. Let the following equalities be:

$$\varphi_1 = (\&_{i=1}^m u_i(x_1, ..., x_k) = u'_i(x_1, \ldots, x_k) \Rightarrow u(x_1, \ldots, x_k) = u'(x_1, \ldots, x_k));$$

$$\varphi_2 = (\&_{i=1}^m v_i(y_1, \ldots, y_s) = v'_i(y_1, \ldots, y_s) \Rightarrow v(y_1, \ldots, y_s) = v'(y_1, \ldots, y_s)).$$

We shall show that the formula $\varphi_1 \ \& \ \varphi_2$ is equivalent to the quasiidentity $\varphi = (\&_{i=1}^m u_i(x_1, \ldots, x_k) = u'_i(x_1, \ldots, x_k) \ \& \ \&_{j=1}^m v_j(y_1, \ldots, y_s) = v'_j(y_1, \ldots, y_s) \Rightarrow u(x_1, \ldots, x_k) v(y_1, \ldots, y_s) = u'(x_1, \ldots, x_k) v'(y_1, \ldots, y_s))$.

Indeed, let the formula φ_1 & φ_2 be true in the quasigroup Q. We assume that the left side of the quasiidentity φ is true in Q for the substitutions $x_i \to a_i$ $(i = 1, \ldots, k)$, $y_j \to b_j$ $(j = 1, \ldots, s)$, where $a_1, \ldots, a_k,\ b_1, \ldots, b_s \in Q$. As the quasiidentities φ_1 and φ_2 are true in Q, we have $(u(a_1, \ldots, a_k) = u'(a_1, \ldots, a_k))$ and $(v(b_1, \ldots, b_s) = v'(b_1, \ldots, b_s))$. Therefore $u(a_1, \ldots, a_k)v(b_1, \ldots, b_s) = u'(a_1, \ldots, a_k)v'(b_1, \ldots, b_s)$. Thus, the quasiidentity φ is a consequence of the formula φ_1 & φ_2.

Conversely, let the quasiidentity φ be true in the quasigroup Q. We show that the quasiidentity φ_1 is true in the quasigroup Q. We assume that the left side of the quasiidentity φ_1 is true in Q for the substitutions $x_i \to a_i$ $(i = 1, \ldots, k)$, where $a_1, \ldots, a_k \in Q$. As the quasiidentity φ_2 is compatible in any quasigroup, and thus in the quasigroup Q, then for certain substitutions $y_j \to b_j$ $(j = 1, \ldots, s)$, where $b_1, \ldots, b_s \in Q$, we have the equalities: $v_j(b_1, \ldots, b_s) = v'_j(b_1, \ldots, b_s)$ $(j = 1, \ldots, s)$, $v(b_1, \ldots, b_s) = v'(b_1, \ldots, b_s)$.

As a result, the left side of the quasiidentity φ is true in the quasigroup Q for the substitutions $x_i \to a_i$ $(i = 1, \ldots, k)$, $y_j \to b_j$ $(j = 1, \ldots, s)$. As the quasiidentity φ is true in the quasigroup Q, then from $u(a_1, \ldots, a_k)v(b_1, \ldots, b_s) = u'(a_1, \ldots, a_k)v'(b_1, \ldots, b_s)$ it follows that $u(a_1, \ldots, a_k) = u'(a_1, \ldots, a_k)$. Similarly, we can show that φ_2 is true in the quasigroup Q. Thus, the formula φ_1 & φ_2 is a consequence of the formula φ. This completes the proof of Lemma 1. □

As any quasiidentity is compatible in any loop then from Lemma 1 follows.

Corollary 1. *In the class of loops the conjunction of a finite number of quasiidentities is equivalent to one quasiidentity.*

Lemma 2. *Let quasiidentity φ be true in a quasigroup Q and let the quasivariety qQ, generated by the quasigroup Q, contain an infinite cyclic group Z. Then the set of all prime cyclic groups Z_{p_i} in which φ is not true is finite.*

Proof. Let's assume that the statement of the lemma is not true, and thus, the set

$$I = \{i \in \Sigma | Z_{p_i} \vdash \urcorner\varphi\}$$

is infinite. Let

$$\varphi = (\&_{i=1}^{m} u_i(x_1, \ldots, x_k) = u'_i(x_1, \ldots, x_k) \Rightarrow u(x_1, \ldots, x_k) = u'(x_1, \ldots, x_k)).$$

We study the finite representative quasigroup

$$L = lp(x_1, \ldots, x_n \| u_i(x_1, \ldots, x_n) = u'_i(x_1, \ldots, x_n),\ i = 1, \ldots, m$$

from qQ generated by elements $x_1, \ldots, x_n$ with the defining relations $u_i(x_1, \ldots, x_n) = u'_i(x_1, \ldots, x_n)$, $i = 1, \ldots, m$.

As for $i \in I$ the quasiidentity φ is false in the cyclic group Z_{p_i}, then there are such elements $a_1, \ldots, a_n \in Z_{p_i}$ that $u_i(a_1, \ldots, a_k) = u_i^{'}(a_1, \ldots, a_k)$ $(i = 1, \ldots, m)$, but $u(a_1, \ldots, a_n) \neq u'(a_1, \ldots, a_n)$, that is $u(a_1, \ldots, a_n)^{-1}u'(a_1, \ldots, a_n) \neq e$ or written simpler still $u^{-1}u' \neq e$. According to Dik's Theorem [3], there is a homomorphism

$\theta_i : L \to Z_{p_i}$ for which $\theta_i(u^{-1}u') \neq e$, $(i \in I)$. By Theorem 1 [1, p.73] there is such a homomorphism $\theta_i : L \to \prod_{i\in I} Z_{p_i}$ that $\theta(a)(i) = \theta_i(a)$, for any $a \in L$ and any $i \in I$. The set I is infinite. Then $\theta_i(u^{-1}u')$ is an element of infinite order of group $\theta(L)$. As $\theta(L)$ is a finitely generated abelian group, $\theta(L)$ can be decomposed in the direct product of cyclic groups. As the element $\theta(u^{-1}(u')$ has a finite order, we conclude that there is a homomorphism $\Psi : \theta(L) \to Z$ so $e \neq \psi\theta(u^{-1}u') = u(\psi\theta(x_1), \dots, \psi\theta(x_n))^{-1}u^{'}(\psi\theta(x_1, \dots, \psi\theta(x_n))$, that is $u(\psi\varphi(a_1), \dots, \psi\varphi(a_n))^{-1} \neq u'(\psi\varphi(a_1, \dots, \psi(x_a))$.

Therefore, for values of variables $x_1 = \psi\varphi(a_1), \dots, x_n = \psi\varphi(a_n)$ we obtained that the quasiidentity φ is false in the infinite cyclic group Z.Contradiction. This completes the proof of Lemma 2. □

Let Σ be an independent system of quasiidentities. Then for any formula $\varphi \in \Sigma$ there is a quasigroup Q_φ, so $Q_\varphi| =\neg\varphi$, but $Q_\varphi| = \psi$ for any formula $\psi \in \Sigma\backslash\{\varphi\}$ by the definition of independent system of quasiidentities. We call the set $\{Q_\varphi|\varphi \in \Sigma\}$ *the system corresponding* to the independent system Σ.

Lemma 3. *Suppose there is a quasivariety N of quasigroups definted by an infinite and independent system of compatible quasiidentities $\{\varphi_i|i \in I \subseteq \Sigma\}$ with the corresponding system of quasigroups $\{Q_i|i \in I\}$. If a subquasivariety $M \subseteq N$ can be defined in the quasivariety N such that for some bijective application $\alpha : I \to \Sigma$ we have $Q_i \vdash \psi_{\alpha(j)}$ for all $j \in I\backslash\{i\}$. Then the quasivariety M has an infinite and independent basis of quasiidentities in the class of all quasigroups.*

Proof. Let α be a bijective application from I on Σ. Let's denote $\Sigma = \{\varphi_i\&\psi_{\alpha(i)}|i \in I\}$. Obviously, any quasiidentity from Σ is true in any quasigroup from M. Conversely, if in the quasigroup Q all formulas from Σ are true, then $Q \in M$. Therefore, the set Σ defines the quasivariety M in the class of all quasigroups. As all formulas from $\Sigma\backslash\{\varphi_i\&\psi_{\alpha(i)}\}$ are true in Q and the formula $\varphi_i\&\psi_{\alpha(i)}$ is false in the quasigroup Q_i, then Σ is an independent system of quasiidentities. By Lemma 1 each formula from Σ is equivalent to a quasiidentity. Hence the system Σ is equivalent to a system Σ' of quasiidentities. As Σ is independent and infinite, it results that $\Sigma\prime$ is also independent and infinite. This completes the proof of Lemma 3. □

Theorem. *If the loop L contains an infinite cyclic group and does not contain an infinity of p_i-periodic elements, then the quasiidentity qL generated by the loop L has an infinite and independent basis of quasiidentities.*

Proof. Denote by I the set of all indices $i \in \Sigma$ of prime numbers for which the loop L does not contain relative p_i-periodic elements. According to the hypothesis, the set I is infinite and for any $i \in I$ the quasiidentity $x^{p_i} = e \Rightarrow x = e$ is true in the loop L, where by u^{p_i} we understand the p^i fold product of the element u written as $(\dots(uu \cdot u)\dots u)u$. Let $\Sigma = \{\psi_i|i \in \Sigma\}$ be a set of quasiidentities (some of them may coincide) which defines the quasivariety qL and N - the quasivariety of loops defined by the independent system $\{x^{pi} = e \Rightarrow x = e|i \in I\}$ of quasiidentities. As every quasiidentity of this system is true in the loop L, then there is the inclusion $qL \subseteq N$. Let ψ_i $(i \in \Sigma)$ be an arbitrary quasiidentity from Σ:

$$\psi_i = (\&_{i=1}^m u_i(x_1, \ldots, x_k) = e \Rightarrow u(x_1, \ldots, x_k) = e);$$

we shall denote $M_i = \{Z_{p_k}| =^\neg \psi_i \in I|Z_{p_k}| =^\neg \psi_i\}$. By Lemma 2 the set M_i is finite. We construct the quasiidentity ψ'_i, corresponding to the quasiidentity ψ_i, as follows: if $M_i = \varnothing$ then we consider $\psi'_i = \psi_i$ and if $M_i \neq \varnothing$ then we consider

$$\psi'_i = (\&_{i=1}^m u_i(x_1, \ldots, x_k) = e \Rightarrow (\ldots(u(x_1, \ldots, x_k))^{p_1})\ldots)^{p_m} = e,$$

where $M_i = \{p_{i_1}, \ldots, p_{i_m}\}$.

We show that the quasiidentities ψ'_i and ψ_i are equivalent in the class N. Obviously, ψ'_i is a consequence of the quasiidentity ψ_i. In particular, this results in the quasiidentity ψ'_i be true in each of the cyclic groups Z_{p_j}, $j \in I \backslash \{i_1, \ldots, i_m\}$. Obviously, if $j \in \Sigma \backslash \{i_1, \ldots, i_m\}$ then the quasiidentity ψ'_i is true in the cyclic group Z_{p_j}. Hence for every $j \in \Sigma$ the quasiidentity ψ'_i is true in the cyclic group Z_{p_j}.

Let there be the loop $Q \in N$ and assume that the quasiidentity ψ'_i is true in the loop Q. Let the left side of the quasiidentity ψ_i be true in Q for the substitutions $x_i \rightarrow a_i, i = 1, \ldots, n$. As ψ'_i is true in Q, we have $(\ldots(u(a_1, \ldots, a_n)^{p_i})\ldots)^{p_{i_m}} = e$.

Now, applying the quasiidentities $x^{p_{ik}} = e \Rightarrow x = e, k = i_1, \ldots, i_m$, which are true in every loop from the quasivariety N, from the last equality we obtain $u(a_1, \ldots, a_n) = e$. Therefore, the quasiidentity ψ_i is true in the loop Q. Hence in the class N the quasiidentities ψ_i and ψ'_i are equivalent and ψ'_i is true in the cyclic group Z_{p_j} for any $j \in I$. The set $\{Z_{p_j}|j \in I\}$ is the system corresponding to the independent system of quasiidentities $\{x^{p_i} = e \Rightarrow x = e)|i \in I\}$.

From here by Lemma 3 it results that the quasivariety qL has an infinite and independent basis of quasiidentities. This completes the proof of Theorem. □

Corollary 2. *Every torsion-free nilpotent loop has an infinite and independent basis of quasiidentities.*

3 Applications

1. From local Malcev Theorem's the following *coverage criterion* of quasivarieties results: *If the quasivariety M has an independent and infinite basis of quasiidentities, then M has an infinity of coverages.* The detailed proof of this statement can be found, for instance, in [6].

According to Corollary 2 and the coverage criterion of quasivarieties, we obtain the following statement.

If L is a torsion free nilpotent loop of any rank, then the quasivariety qL has infinity of coverages in the latices of loop quasivarieties.

2. Let $M_{2\times 2}(K)$ be the vector space of square matrices with elements from associative ring K. We define multiplication and division in $M_{2\times 2}(K)$ by formulas:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} a+x & b+y \\ c+z & d+t+(x-a)(yc-bz) \end{pmatrix},$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} / \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} a-x & b-y \\ c-z & d-t+(a-2x)(yc-bz) \end{pmatrix}.$$

It is easy to see that the set $M_{2\times 2}(K)$ forms a commutative loop with respect to multiplication and division. The unity is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$ $= \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$. Denote that loop by L. As the ring K satisfies the identity $0 \cdot x = x$ then from formulas which defined the operations it follows that elements of the form $\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$ belong to the centre of loop L. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$, $C = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ be arbitrary elements of L. We compute its associator $(A,B,C) = (AB \cdot C)/(A \cdot BC) = \begin{pmatrix} 0 & 0 \\ 0 & (anz - ayp + 2mbz - 2myc + xbp - xnc) \end{pmatrix}$. Hence the associator (A,B,C) belongs to the centre of L. Consequently, the loop L is nilpotent of class 2. As K is a ring of characteristic zero then it is easy to see that L is a torsion free loop. Then any subloop of cartesian product of loop L is torsion free. From here it follows by Corollary 2 that any free loop of quasivariety generated by L has an infinite and independent basis of quasiidentities.

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Center problem for a class of cubic systems with a bundle of two invariant straight lines and one invariant conic

Dumitru Cozma

Abstract. For a class of cubic differential systems with a bundle of two invariant straight lines and one invariant conic it is proved that a weak focus is a center if and only if the first four Liapunov quantities $L_j,\ j=\overline{1,4}$ vanish.

Mathematics subject classification: 34C05.
Keywords and phrases: Cubic differential system, center-focus problem, invariant algebraic curve, integrability.

1 Introduction

In this paper we consider the cubic system of differential equations

$$\begin{aligned} \dot{x} &= y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \dot{y} &= -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y), \end{aligned} \tag{1}$$

in which all variables and coefficients are assumed to be real. The origin $O(0,0)$ is a singular point of a center or a focus type for (1), i.e. a weak focus. The purpose of this paper is to find verifiable conditions for $O(0,0)$ to be a center.

It is known that the origin is a center for system (1) if and only if it has in some neighborhood of $O(0,0)$ a holomorphic integrating factor of the form

$$\mu = 1 + \sum \mu_j(x,y).$$

There exists a formal power series $F(x,y) = \sum F_j(x,y)$ such that the rate of change of $F(x,y)$ along trajectories of (1) is a linear combination of polynomials $\{(x^2+y^2)^j\}_{j=2}^{\infty}$:

$$\frac{dF}{dt} = \sum_{j=2}^{\infty} L_{j-1}(x^2+y^2)^j.$$

The quantities $L_j,\ j=\overline{1,\infty}$, are polynomials in the coefficients of system (1) called Liapunov quantities. The order of the weak focus $O(0,0)$ is r if $L_1 = L_2 = \ldots = L_{r-1} = 0$ but $L_r \neq 0$.

The origin is a center for (1) if and only if $L_j = 0,\ j = \overline{1,\infty}$. By the Hilbert's basis theorem there exists a natural number N such that the infinite system $L_j = 0,\ j=\overline{1,\infty}$, is equivalent with a finite system $L_j = 0,\ j = \overline{1,N}$. The number N is known only for quadratic systems $N = 3$ [11] and for cubic systems with only homogeneous cubic nonlinearities $N = 5$ [16, 20]. If the cubic system (1) contains both quadratic and cubic nonlinearities, the problem of the center was solved only in some particular cases (see for instance [1, 2, 4, 6–10, 13, 14, 17, 18]).

In this paper we solve the problem of the center for cubic differential system (1) assuming that (1) has two invariant straight lines and one invariant conic passing through one singular point, i.e. forming a bundle. The paper is organized as follows. Results concerning the relation between integrability, invariant algebraic curves and Liapunov quantities are presented in Section 2. In Section 3 we find eight sufficient series of conditions for the existence of a bundle of two invariant straight lines and one invariant conic. In Section 4 we obtain sufficient conditions for the existence of a center and finally we give the proof of the main result: a weak focus $O(0,0)$ is a center for a class of cubic systems (1) with a bundle of two invariant straight lines and one invariant conic if and only if the first four Liapunov quantities vanish.

2 Invariant algebraic curves, Liapunov quantities, center

An algebraic curve $\Phi(x,y) = 0$ (real or complex) is said to be an invariant curve of system (1) if there exists a polynomial $K(x,y)$ such that

$$P\frac{\partial \Phi}{\partial x} + Q\frac{\partial \Phi}{\partial y} = \Phi K.$$

The polynomial K is called the cofactor of the invariant algebraic curve $\Phi = 0$. We shall consider only algebraic curves $\Phi = 0$ with Φ irreducible.

If the cubic system (1) has sufficiently many invariant algebraic curves $\Phi_j(x,y) = 0,\ j = 1,\dots,q$, then in most cases an integrating factor can be constructed in the Darboux form

$$\mu = \Phi_1^{\alpha_1}\Phi_2^{\alpha_2}\cdots\Phi_q^{\alpha_q}. \tag{2}$$

A function (2), with $\alpha_j \in \mathbb{C}$ not all zero, is an integrating factor for (1) if and only if

$$\sum_{j=1}^{q}\alpha_j K_j \equiv -\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}.$$

System (1) is called Darboux integrable if the system has a first integral or an integrating factor of the form (2).

The method of Darboux turns out to be very useful and elegant one to prove integrability for some classes of systems depending on parameters. These last years, interesting results which relate algebraic solutions, Liapunov quantities and Darboux integrability have been published (see, for example, [3, 5, 6, 9, 10, 15, 19]). The cubic systems (1) which are Darboux integrable have a center at $O(0,0)$.

Definition 1. We shall say that (Φ_j, $j = \overline{1,M}$; $L = N$) is *ILC* (I – invariant algebraic curves, L – Liapunov quantities, C – center) for (1), if the existence of M algebraic curves $\Phi_j(x,y) = 0$ and the vanishing of the focal values L_ν, $\nu = \overline{1,N}$, implies the origin $O(0,0)$ to be a center for (1).

The works [6–9,17,18] are dedicated to investigation of the problem of the center for cubic differential systems with invariant straight lines. In these papers, the problem of the center was completely solved for cubic systems with at least three invariant straight lines. The principal results of these works are gathered in the following two theorems:

Theorem 1. *$(\Phi_j(x,y),\ \Phi_j(0,0)\neq 0,\ j=\overline{1,4};\ \ L=1)$ is ILC for system (1).*

Theorem 2. *$(a_jx+b_jy+c_j,\ j=\overline{1,4};\ \ L=2)$ and $(a_jx+b_jy+c_j,\ j=\overline{1,3};\ \ L=7)$ are ILC for cubic system (1).*

The problem of the center was solved for cubic systems (1) with two homogeneous invariant straight lines and one invariant conic; for cubic systems (1) with two parallel invariant straight lines and one invariant conic [10]:

Theorem 3. *$(x\pm iy,\ \Phi;\ \ L=2)$ and $(l_j=1+a_jx+b_jy,\ j=1,2,\ l_1||l_2,\ \Phi;\ \ L=3)$, where $\Phi=0$ is an irreducible invariant conic, are ILC for system (1).*

3 Conditions for the existence of a bundle of two invariant straight lines and one invariant conic

Let the cubic system (1) have two invariant straight lines l_1, l_2 intersecting at a point (x_0,y_0). The intersection point (x_0,y_0) is a singular point for (1) and has real coordinates. By rotating the system of coordinates ($x\to x\cos\varphi-y\sin\varphi$, $y\to x\sin\varphi+y\cos\varphi$) and rescaling the axes of coordinates ($x\to\alpha x$, $y\to\alpha y$), we obtain $l_1\cap l_2=(0,1)$. In this case the invariant straight lines can be written as

$$l_j=1+a_jx-y,\ a_j\in\mathbb{C},\ j=1,2;\ \Delta_{12}=a_2-a_1\neq 0. \tag{3}$$

The straight lines (3) are invariant for (1) if and only if the following coefficient conditions are satisfied:

$$\begin{array}{l}
k=(a-1)(a_1+a_2)+g,\ \ l=-b,\ \ s=(1-a)a_1a_2,\\
m=-a_1^2-a_1a_2-a_2^2+c(a_1+a_2)-a+d+2,\ \ r=-f-1,\\
n=a_1a_2(-f-2)-(d+1),\ \ p=(f+2)(a_1+a_2)+b-c,\\
q=(a_1+a_2-c)a_1a_2-g,\ \ (a-1)^2+(f+2)^2\neq 0.
\end{array} \tag{4}$$

If the conditions (4) are satisfied then the cubic system (1) looks:

$$\begin{array}{l}
\dot{x}=y+ax^2+cxy+[d+2-a-a_1^2-(a_1+a_2)(a_2-c)]x^2y-(f+1)y^3+\\
\quad fy^2+[(a-1)(a_1+a_2)+g]x^3+[(f+2)(a_1+a_2)+b-c]xy^2\equiv P(x,y),\\
\dot{y}=-x-gx^2-dxy-by^2+(a-1)a_1a_2x^3+[g+a_1a_2(c-a_1-a_2)]x^2y+\\
\quad [(f+2)a_1a_2+d+1]xy^2+by^3\equiv Q(x,y).
\end{array} \tag{5}$$

Next for cubic system (5) we find conditions for the existence of one invariant conic passing through the same singular point $(0,1)$, i.e. forming a bundle. Let the conic curve be given by the equation

$$\Phi(x,y)\equiv a_{20}x^2+a_{11}xy+a_{02}y^2+a_{10}x+a_{01}y+1=0 \tag{6}$$

with $(a_{20}, a_{11}, a_{02}) \neq 0$ and $a_{20}, a_{11}, a_{02}, a_{10}, a_{01} \in \mathbb{R}$.

For every conic curve (6) the following quantities [12]:

$$I_1 = a_{02} + a_{20}, \quad I_2 = (4a_{02}a_{20} - a_{11}^2)/4,$$
$$I_3 = (4a_{02}a_{20} - a_{01}^2 a_{20} + a_{01}a_{10}a_{11} - a_{02}a_{10}^2 - a_{11}^2)/4$$

are invariants with respect to the translation and rotation of axes. These invariants will be taken into account classifying conics. A conic (6) is reducible into two straight lines if and only if $I_3 = 0$. If $I_2 > 0$, then (6) is an ellipse, if $I_2 < 0$ – a hyperbola and if $I_2 = 0$ – a parabola.

In order the conic (6) pass through a singular point $(0, 1)$ and form a bundle with the invariant straight lines (3), we shall assume $a_{01} = -a_{02} - 1$. In this case

$$\Phi(x, y) \equiv a_{20}x^2 + a_{11}xy + a_{10}x + (a_{02}y - 1)(y - 1) = 0. \tag{7}$$

The conic (7) is an invariant conic for (5) if and only if there exist numbers c_{20}, c_{11}, c_{02}, c_{10}, $c_{01} \in \mathbb{R}$, where $c_{10} = -a_{01}$, $c_{01} = a_{10}$, such that

$$P(x, y)\frac{\partial \Phi}{\partial x} + Q(x, y)\frac{\partial \Phi}{\partial y} \equiv \Phi(x, y)(c_{20}x^2 + c_{11}xy + c_{02}y^2 + (a_{02} + 1)x + a_{10}y). \tag{8}$$

Identifying the coefficients of $x^i y^j$ in (8), we reduce this identity to three systems of equations $\{F_{ij} = 0\}$ for the unknowns a_{20}, a_{11}, a_{02}, a_{10}, c_{20}, c_{11}, c_{02} :

$$\begin{aligned}
F_{40} &\equiv (a-1)(a_1a_2a_{11} + 2a_1a_{20} + 2a_2a_{20}) + a_{20}(2g - c_{20}) = 0,\\
F_{31} &\equiv (a-1)(2a_1a_2a_{02} + a_1a_{11} + a_2a_{11}) - (a_2a_{11} + 2a_{20})a_1^2 -\\
&\quad - (a_1a_{11} + 2a_{20})a_2^2 + (ca_{11} - 2a_{20})a_1a_2 + (2ca_1 + 2ca_2 - 2a -\\
&\quad - c_{11} + 2d + 4)a_{20} + (2g - c_{20})a_{11} = 0,\\
F_{22} &\equiv 2(c - a_1 - a_2)a_1a_2a_{02} + (2g - c_{20})a_{02} + [c(a_1 + a_2) - a_1^2 -\\
&\quad - a_2^2 + (f+1)a_1a_2 - a - c_{11} + 2d + 3]a_{11} +\\
&\quad + [2(f+2)(a_1 + a_2) + 2b - 2c - c_{02}]a_{20} = 0,\\
F_{13} &\equiv (f+2)[2a_1a_2a_{02} + (a_1 + a_2)a_{11}] + (2 + 2d - c_{11})a_{02} +\\
&\quad + (2b - c - c_{02})a_{11} - 2(f+1)a_{20} = 0,\\
F_{04} &\equiv (2b - c_{02})a_{02} - (f+1)a_{11} = 0,
\end{aligned} \tag{9}$$

$$\begin{aligned}
F_{30} &\equiv (a-1)[(a_1 + a_2)a_{10} - a_1a_2(a_{02} + 1)] - ga_{11} +\\
&\quad + (2a - 1 - a_{02})a_{20} + (g - c_{20})a_{10} = 0,\\
F_{21} &\equiv [g - c_{20} + ca_1a_2 - (a_1 + a_2)a_1a_2](-a_{02} - 1) +\\
&\quad + [c(a_1 + a_2) - a_1^2 - a_1a_2 - a_2^2 - a + d + 2 - c_{11}]a_{10} +\\
&\quad + (2c - a_{10})a_{20} + (a - d + 1 + a_{02})a_{11} - 2ga_{02} = 0,\\
F_{12} &\equiv (f+2)[(a_1 + a_2)a_{10} - a_1a_2(a_{02} + 1)] - (d + 1 - c_{11})(a_{02} + 1) -\\
&\quad - (a_{02} + 2d + 1)a_{02} + (b - c - c_{02})a_{10} + (c - b - a_{10})a_{11} + 2fa_{20} = 0,\\
F_{03} &\equiv (b - c_{02})(a_{02} + 1) + (a_{10} + 2b)a_{02} + (f+1)a_{10} - fa_{11} = 0,
\end{aligned} \tag{10}$$

$$\begin{aligned}
F_{20} &\equiv (a - a_{02} - 1)a_{10} + g(a_{02} + 1) - a_{11} - c_{20} = 0,\\
F_{11} &\equiv (a_{02} + d + 1)a_{01} + (a_{10} - c)a_{10} + 2a_{02} - 2a_{20} + c_{11} = 0,\\
F_{02} &\equiv c_{02} - (a_{10} + b)(a_{02} + 1) - fa_{10} - a_{11} = 0.
\end{aligned} \tag{11}$$

Let us denote

$$j_1 = (a_1 + a_2 - c)a_{02} + (f + 1)a_{11}, \quad j_2 = a_{02}a_1^2 + a_{11}a_1 + a_{20},$$
$$j_3 = a_{02}a_2^2 + a_{11}a_2 + a_{20}, \quad j_4 = 4a_{02}a_{20} - a_{11}^2.$$

We shall study the compatibility of the system of equations $\{(9), (10), (11)\}$ when $f + 2 \neq 0,\ I_3 \neq 0$ and split the investigation into five subcases: $\{j_1 = 0\}$, $\{j_1 \neq 0, j_2 = 0\}$, $\{j_1 j_2 \neq 0,\ j_3 = 0\}$, $\{j_1 j_2 j_3 \neq 0, j_4 = 0\}$, $\{j_1 j_2 j_3 j_4 \neq 0\}$.

Remark 1. If $a_{02} = 1$, then the system $\{(9), (10), (11)\}$ is not compatible.

Indeed, we express c_{02} from $F_{04} = 0$ of (9) and substituting in (11) we obtain

$$F_{02} \equiv (a_{10} + a_{11})(f + 2) = 0.$$

If $a_{11} + a_{10} = 0$, then $I_3 = 0$. Next we shall assume that $a_{02} - 1 \neq 0$.

3.1 Case $\mathbf{j_1 = 0}$

3.1.1. $a_{02} = a_{11} = 0$. In this case $F_{04} = 0$ and the equation $F_{13} = 0$ yields $f = -1$. We express c_{02}, c_{11} and c_{20} from (9), a_{20} from $F_{11} = 0$, g from $F_{20} = 0$ and replace in (10). Reduce the equations of (10) by b from $F_{02} = 0$, then we get

$$F_{12} \equiv (a_1 - a_{10})(a_2 - a_{10}) = 0.$$

If $a_{10} = a_1$ or $a_{10} = a_2$, then we obtain the following series of conditions

1)
$$a = 1/2,\ f = -1,\ g = (4c - 3b)/6,\ a_1 = (2c)/3,\ a_2 = (2c - 3b)/6$$

for the existence of an invariant parabola for system (5):

$$(9b^2 - 6bc - 4c^2 - 18d - 36)x^2 - 24cx + 36(y - 1) = 0.$$

3.1.2. $a_{02} = 0,\ a_{11} \neq 0$. In this case the equation $j_1 = 0$ yields $f = -1$ and $F_{04} \equiv 0$. We express c_{02}, c_{11}, c_{20} from (9) and obtain $F_{40} \equiv f_1 f_2 f_3 = 0$, where

$$f_1 = a_1 a_{11} + a_{20},\ f_2 = a_2 a_{11} + a_{20},\ f_3 = (a_1 + a_2 - c)a_{20} + (a - 1)a_{11}.$$

Let $f_1 = 0$ and reduce the equations of (10) and (11) by b from $F_{02} = 0$, d from $F_{11} = 0$ and g from $F_{20} = 0$, then we get $F_{12} \equiv (a_{11} + a_{10} - a_2)I_3 = 0$.

If $a_{11} = a_2 - a_{10}$, then we obtain the following series of conditions

2)
$$a = 0,\ d = (g^2 - 2cg - 2bg - 8)/4,\ f = -1,\ a_1 = g/2,\ a_2 = b + g$$

for the existence of an invariant conic for (5):

$$g(4b - 2c + 3g)x^2 + 2(2c - 4b - 3g)xy + 2(2b - 2c + g)x + 4y - 4 = 0.$$

The case $f_2 = 0$ can be reduced to $f_1 = 0$ if we replace a_1 with a_2.

Assume now $f_1 f_2 \neq 0$ and $f_3 = 0$. We express $a_{11} = a_1 + a_2 + b - c$ from $F_{02} \equiv F_{03} = 0$ and reduce the equations of (10) by d from $F_{11} = 0$ and g from $F_{20} = 0$, then we get

$$F_{12} \equiv (a_{10} + a_1 + b - c)(a_{10} + a_2 + b - c) = 0.$$

If $a_{10} = c - b - a_1$ or $a_{10} = c - b - a_2$, then we obtain

3)
$$c = (2b^2 - 6a - 3bp + p^2 + 3)/(b - p),\ g = (ab^2 - 2abp + ap^2 - 4a + 2)/(b - p),$$
$$d = (6ab^2 - 8a^2 - 10abp + 4ap^2 + 8a - 5b^2 + 8bp - 3p^2 - 2)/(b - p)^2,$$
$$f = -1,\ a_1 = (c - 2b + p)/3,\ a_2 = (2c - b + 2p)/3.$$

The invariant conic is

$$(a-1)px^2+(2-4a+(b-p)^2)x+(b-p)(1-y+pxy)=0.$$

3.1.3. $a_{02}\neq 0$. In this case we express c from $j_1=0$, c_{02} from $F_{04}=0$, c_{11} from $F_{13}=0$, c_{20} from $F_{22}=0$ and b, d, g from (11). Then we get $F_{12}\equiv e_1e_2(f+2)$, where $e_1=a_1a_{02}-a_1+a_{10}+a_{11},\ e_2=a_2a_{02}-a_2+a_{10}+a_{11}$.

3.1.3.1. If $e_1=0$, then $a_1=(a_{10}+a_{11})/(1-a_{02})$ and (9) becomes:
$F_{40}\equiv h_1[a_{20}(2a_{02}^2a_2-2a_{02}a_{10}-a_{02}a_{11}-2a_{02}a_2-a_{11})-a_{02}a_{11}a_2(a_{10}+a_{11})]=0,$
$F_{31}\equiv h_1[2a_{20}a_{02}(a_{02}-1)+2a_{02}^2a_{10}a_2+a_{02}^2a_{11}a_2+a_{02}a_{10}a_{11}+a_{02}a_{11}a_2+a_{11}^2]=0,$
where $h_1=(a-1)a_{02}-(f+1)a_{20}$.

Let $h_1=0$ and reduce the equations of (10) by a from $h_1=0$. Express a_{20} from $F_{30}=0$, a_{11} from $F_{21}=0$, a_{02} from $h_1=0$ and obtain the following series of conditions

4)
$$\begin{aligned}
&b=[(v+1+h)(2a_2-a_{10})(v+1)]/(hv),\\
&c=[(hv-h-v-1)a_{10}+(2-2hv^2+hv+2h+2v)a_2)/(hv),\\
&d=[(h+2v^2+3v+1)a_{10}a_2-2(hv+h+2v^2+3v+1)a_2^2-\\
&\quad -h(hv+h+3v+1)]/(hv),\quad h=2a+a_{10}a_2-2fa_2^2-4a_2^2-2,\\
&g=[a_{10}^2a_2-2(v+2)a_{10}a_2^2+ha_{10}+4(v+1)a_2^3+2ha_2]/(2h),\\
&a_1=[2hva_2+(h+v+1)(2a_2-a_{10})]/(hv),\quad v=f+1
\end{aligned}$$

for the existence of an invariant conic $(h+1)[2vy^2+(2a_2^2+2va_2^2-a_2a_{10}+h)x^2+2(a_{10}-2va_2-2a_2)xy]+2v[a_{10}x-(h+2)y+1]=0$.

Assume now $h_1\neq 0$, then from $F_{31}=0$ we find a_{20}, and the equation $F_{40}=0$ becomes $F_{40}\equiv(2a_{02}a_2+a_{11})I_3=0$.

If $a_{11}=-2a_{02}a_2$, then $F_{31}\equiv(a_{02}-1)^2a_{02}^2\neq 0$.

3.1.3.2. The case $e_2=0$ can be reduced to $e_1=0$, if we replace a_1 with a_2.

3.2 Case $\mathbf{j_1\neq 0,\ j_2=0}$

In this case $a_{20}=-a_1(a_{11}+a_1a_{02}),\ I_2<0$ and the conic is a hyperbola. If $a_{02}=0$, then $F_{04}\equiv j_1\neq 0$. Next assuming $a_{02}\neq 0$ we express c_{02}, c_{11}, c_{20} from the equations $\{F_{04}=0,\ F_{13}=0,\ F_{22}=0\}$ of (9) and b, d, g from the equations of (11). Then we get $F_{12}\equiv e_1e_2(f+2)$, where

$$e_1=a_1a_{02}-a_1+a_{10}+a_{11},\ e_2=a_2a_{02}-a_2+a_{10}+a_{11}.$$

3.2.1. Let $e_1=0$, then $I_3=0$ and the conic is reducible.

3.2.2. Assume $e_1\neq 0$ and $e_2=0$. In this case we express a_2 from $e_2=0$ and a_{10} from $F_{21}=0$. If $a_{11}=-2a_{02}a_1$, then $F_{30}=I_3\neq 0$.

Let $2a_{02}a_1+a_{11}\neq 0$, then express c from $F_{30}=0$ and

$$F_{40}\equiv F_{31}=2aa_{02}a_1+aa_{11}-a_{02}^2a_1-a_{02}a_1-a_{02}a_{11}=0.$$

If $a_{02}=a$, then $F_{40}=0$ yields $a_1=0$ and we obtain

5)
$$\begin{aligned}
&d=-a-f-3,\quad b=[(f+2)(f+a+1)a_{11}]/[a(1-a)],\\
&g=0,\quad c=[(af-2f-2)a_{11}^2+a^2(a-1)^2]/[(a^2-a)a_{11}],\\
&a_1=0,\quad a_2=[(a+f+1)a_{11}]/(a-a^2).
\end{aligned}$$

The invariant hyperbola is $(1+f+ay)a_{11}x + a(ay-1)(y-1) = 0$.

If $a_{02} \neq a$, then express a_{11} from $F_{40} = 0$ and obtain the following series of conditions for the existence of a hyperbola

6)
$$\begin{array}{l} b = -[(f+1+a_{02})(f+2)a_1]/h, \;\; a_2 = [a_1(h-a-f-1)]/h, \\ c = [(af-2f+2a_{02}-2a-2)a_1^2+h^2]/(ha_1), \;\; g = (f+3)a_1, \\ d = [(2a+f^2+5f+4)a_1^2-(a_{02}+f+3)h]/h, \;\; h = a_{02}-a. \end{array}$$

The invariant hyperbola is $a_{02}a_1^2(a-1)x^2 + a_{02}a_1(h-a+1)xy + a_1(2a - fa_{02} - 3a_{02} + f + 1)x - h(a_{02}y-1)(y-1) = 0$.

3.3 Case $\mathbf{j_1 \cdot j_2 \neq 0}$, $\mathbf{j_3 = 0}$

In this case we also obtain the series of conditions 5) and 6).

3.4 Case $\mathbf{j_1 \cdot j_2 \cdot j_3 \neq 0}$, $\mathbf{j_4 = 0}$

If $a_{02} = 0$, then $j_4 = 0$ yields $a_{11} = 0$ and $j_1 = 0$. Next assume $a_{02} \neq 0$ and from $j_4 = 0$ we find $a_{20} = a_{11}^2/(4a_{02})$. In this case $I_2 = 0$ and the conic is a parabola. We express c_{02}, c_{11}, c_{20} from the equations $\{F_{04} = 0,\ F_{13} = 0,\ F_{22} = 0\}$ of (9) and b, d, g from the equations of (11), then we obtain $F_{12} \equiv e_1 e_2 (f+2) = 0$, where
$$e_1 = a_{02}a_1 - a_1 + a_{10} + a_{11}, \;\; e_2 = a_{02}a_2 - a_2 + a_{10} + a_{11}.$$

3.4.1. Assume $e_1 = 0$, i.e. $a_{10} = a_1 - a_{02}a_1 - a_{11}$. Reduce the equations $\{F_{31} = 0, F_{30} = 0\}$ by f from $F_{21} = 0$, the equation $F_{30} = 0$ by a from $F_{31} = 0$ and express c from $F_{30} = 0$, then we obtain
$$\begin{array}{l} F_{21} \equiv a_1a_{02}(a_{02}-1) + a_{11}(a_{02}+f+1) + 2(f+2)a_2a_{02} = 0, \\ F_{31} \equiv a_{11}(2a-a_{02}-1) + 4(a-1)a_2a_{02} = 0. \end{array} \tag{12}$$

If $f = -2a$, then $a \neq 1$. Solving (12) for a_1 and a_2 we get

7)
$$\begin{array}{l} b = [a_{11}(a_{02}-2h-1)]/a_{02}, \;\; g = [a_{11}(1-2a_{02}+2h)]/(2a_{02}), \\ d = [(4ha_{02}+a_{02}-4h^2-4h-1)a_{11}^2 - 4ha_{02}^2(a_{02}-2h+1)]/(4ha_{02}^2), \\ f = -2a, \;\; c = [a_{11}^2(a_{02}-4h^2-4h-1)+8a_{02}^2h^2]/(4ha_{11}a_{02}), \\ a_1 = 0, \;\; a_2 = [a_{11}(a_{02}-2h-1)]/(4ha_{02}), \;\; h = a-1. \end{array}$$

The invariant parabola is $a_{11}^2x^2 + 4a_{02}(y-1)(a_{11}x + a_{02}y - 1) = 0$.

If $f + 2a \neq 0$, then express a_{11} from $F_{31} + F_{21} = 0$ and a_2 from $F_{21} = 0$. We obtain

8)
$$\begin{array}{l} b = [a_1(f+2)(a_{02}+f+1)]/v, \;\; a_2 = [a_1(2a-a_{02}-1)]/(2v), \\ g = [a_1(2aa_{02}-2a^2+av+3a-a_{02}v-2a_{02}-1)]/v, \;\; v = 2a+f, \\ c = [a_1^2(4a^2-2av-4a-a_{02}+4v+1)-2v^2]/(2va_1), \\ d = [a_1^2(8(a-1)^2(a_{02}+v-a)+(a-1)(2a_{02}-8va_{02}-2v^2+6v-2)+ \\ \quad +(2v-1)(a_{02}-1)v)+2v^2(2a-v-a_{02}-3)]/(2v^2). \end{array}$$

The invariant parabola is $(a-1)a_1a_{02}[(a-1)a_1x - 2vy]x + v(2aa_{02} - va_{02} - 2a_{02} + v)a_1x + v^2(a_{02}y-1)(y-1) = 0$.

3.4.2. The case $e_2 = 0$ can be reduced to $e_1 = 0$, if we replace a_1 with a_2.

3.5 Case $\mathbf{j_1 \cdot j_2 \cdot j_3 \cdot j_4 \neq 0}$

We express c_{02} from $F_{04} = 0$, c_{11} from $F_{13} = 0$, c_{20} from $F_{22} = 0$ and substitute into the equations $\{F_{40} = 0,\ F_{31} = 0\}$ of (9). Calculating the resultant of the equations $\{F_{40} = 0,\ F_{31} = 0\}$ by a we obtain

$$Res(F_{40}, F_{31}, a) = j_1 j_2 j_3 j_4.$$

In this case $Res(F_{40}, F_{31}, a) \neq 0$ and therefore the system of algebraic equations $\{(9), (10), (11)\}$ is not compatible.

Remark 2. For cubic differential system (1) we obtained 8 series of conditions for the existence of two invariant straight lines and one invariant conic passing through the same singular point (0, 1).

4 Sufficient conditions for the existence of a center

Lemma 1. *The following ten series of conditions are sufficient conditions for the origin to be a center for system (5):*

i)

$$a = 1/2,\ \ f = -1,\ \ d = (-5)/2,\ \ g = (4c - 3b)/6,\ \ a_1 = (2c)/3,$$
$$3b^2c - 2bc^2 + 9b + 12c = 0,\ \ a_2 = (2c - 3b)/6;$$

ii)

$$a = d = 0,\ \ b = -g - 2/g,\ \ c = (3g^2 - 4)/(2g),\ \ f = -1,\ \ a_1 = g/2,\ \ a_2 = -2/g;$$

iii)

$$a = (f^2 + f + 1)/(1 - f),\ \ b = (f + 2)a_1,\ \ g = -fa_1,\ \ c = (1 - 2f)a_1,$$
$$d = (2f^2 + 3f + 4)/(f - 1),\ \ f(f - 1)a_1^2 + f^2 + 3f + 2 = 0,\ \ a_2 = 0;$$

iv)

$$b = [(f^3 + (a + 5)f^2 + (7a + 5)f + 4a^2 + 2a + 2)(f + 2)u]/[(f + 1)v^2 a_2],$$
$$c = [((3 - 2a)f^3 - 2(a^2 + a - 5)f^2 - (a^2 + a - 14)f - a^2 + 4a + 5)u]/$$
$$[(f + 1)v^2 a_2],\ \ d = -2[f^3 + (a + 5)f^2 + 6(a + 1)f + 3a^2 + a + 4]/v,$$
$$g = [(f^3 + (a + 2)f^2 - 5a + 1)(a - 1)u]/[(f + 1)v^2 a_2],$$
$$a_1 = [(2a + f)u]/[(f + 1)va_2],\ \ (f + 1)v^2 a_2^2 - (a - 1)u^2 = 0,$$
$$u = f^2 + (a + 1)f + 1 - a,\ \ v = f^2 + (f + 1)(a + 3);$$

v)

$$a = (-h)/(f + 3),\ \ d = -a - f - 3,\ \ b = [(f + 2)(f + a + 1)a_{11}]/[a(1 - a)],$$
$$g = 0,\ \ c = [2(3f^2 + 2f + 3)(f + 2)h]/[(f + 3)^2(f + 1)a_{11}],\ \ h = 2f^2 + 3f - 3,$$
$$(f + 1)(f + 3)^3 a_{11}^2 + 4f(f + 2)h^2 = 0,\ \ a_1 = 0,\ \ a_2 = [(a + f + 1)a_{11}]/(a - a^2);$$

vi)

$$a = -(f^2 + 6f + 3)/3,\ \ c = -(f^4 + 14f^3 + 60f^2 + 87f + 48)/[3(f + 5)a_1],$$
$$b = -[(2f^3 + 15f^2 + 27f + 6)(f + 2)a_1]/[(f^2 + 6f + 6)(f + 1)],$$
$$d = (3f^3 + 23f^2 + 42f + 6)/[3(f + 5)],\ \ 3(f + 5)a_1^2 + (f + 1)(f^2 + 6f + 6) = 0,$$
$$g = (f + 3)a_1,\ \ a_2 = -[(f^2 + 3f - 6)a_1]/[(f + 1)(f^2 + 6f + 6)];$$

vii)

$$\begin{aligned}
&b=-[(a+\alpha t^2+f+1)(f+2)\beta]/(\alpha t),\quad a_2=[(\alpha t^2-a-f-1)\beta]/(\alpha t),\\
&c=[\alpha^2t^2+2\alpha\beta^2t^2+\beta^2(af-2f-2)]/(\alpha\beta t),\quad g=(f+3)\beta t,\\
&d=[(2a+f^2+5f+4)\beta^2-\alpha^2t^2-(a+f+3)\alpha]/\alpha,\\
&[(f+4)(f+1)+2a](f+3)(f+2)\beta^4+(af^2+10af+15a+3f^3+\\
&\quad +18f^2+28f+9)\alpha\beta^2+(af+3a+2f^2+3f-3)\alpha^2=0,\\
&t^2=[(1-a-2f-f^2)\alpha\beta^2-(3af+5a+f^3+7f^2+13f+7)\beta^4]/\\
&\quad[\alpha^3+(f+2)\alpha^2\beta^2-(f+3)\alpha\beta^4];
\end{aligned}$$

viii)

$$\begin{aligned}
&a=(18t-t^2-24)/[2t(t+2)],\quad b=[a_{11}(3t^2-14t+24)]/[t(3t-10)],\\
&c=(9t^6+144t^5-2416t^4+12912t^3-33872t^2+44352t-23040)/\\
&\quad[8a_{11}t(5t^3+14t^2-4t-24)],\quad g=[3a_{11}(4t-t^2-4)]/[t(3t-10)],\\
&d=(9t^3-74t^2+168t-192)/[4t(t+2)],\quad f=(t^2-18t+24)/[t(t+2)],\\
&4(5t-6)(t+2)^2a_{11}^2+(3t^2-14t+24)(3t-10)^2(t-2)=0,\\
&a_1=0,\quad a_2=[a_{11}(t+2)]/[2(10-3t)];
\end{aligned}$$

ix)

$$\begin{aligned}
&a=(t^3-3t-1)/[(t^2-1)t],\quad b=[(t^2+t+3)(2t+1)a_1]/[(t^2+3t-1)(1-t^2)],\\
&c=[-(t^3+6t^2+4t-4)(2t+1)(t+2)]/[(t^2+3t-1)(t+1)a_1t],\\
&f=(2t^2+2t-1)/(1-t^2),\quad g=[(t^3-t^2+5t+4)a_1]/[(t^2+3t-1)(t^2-1)],\\
&d=(5t^4+23t^3+21t^2+16t+7)/[(t^2+3t-1)(t^2-1)t],\\
&ta_1^2+(2t+1)(t+2)=0,\quad a_2=[a_1(t-2)]/(t^2+3t-1);
\end{aligned}$$

x)

$$\begin{aligned}
&b=[a_1(f+2)(a_{02}+f+1)]/v,\quad a_2=[a_1(2a-a_{02}-1)]/(2v),\\
&g=[a_1(2aa_{02}-2a^2+av+3a-a_{02}v-2a_{02}-1)]/v,\\
&c=[a_1^2(4a^2-2av-4a-a_{02}+4v+1)-2v^2]/(2va_1),\\
&d=[a_1^2(8(a-1)^2(a_{02}+v-a)+(a-1)(2a_{02}-8va_{02}-2v^2+6v-2)\\
&\quad+(2v-1)(a_{02}-1)v)+2v^2(2a-v-a_{02}-3)]/(2v^2),\\
&(2a-f^2-2f-2)(a-1)(f+2)(f+1)a_1^6-v(2a^2f^2-6a^2-2af^3-\\
&\quad-28af^2-66af-40a-f^4-12f^3-30f^2-22f-2)a_1^4+\\
&\quad+v^3(3af+3a-5f-7)a_1^2-v^5=0,\quad v=2a+f,\\
&a_{02}=[(8a^3f+16a^3+2a^2f^3+22a^2f^2+42a^2f+10a^2+2af^4+16af^3+\\
&\quad+30af^2+10af-3f^3-10f^2-8f-2)a_1^4+(2a^2f+2a^2+2a-2f^3-\\
&\quad-6f^2-5f-2)v^2a_1^2-2v^4(a+f+1)]/[a_1^2((2a^2f+2a^2+2af^3+\\
&\quad+14af^2+26af+16a+2f^4+13f^3+28f^2+24f+6)a_1^2+\\
&\quad+(2af+2a+2f^2+3f)v^2)].
\end{aligned}$$

Proof. In each of the cases **i)–x)** the system (5) has two invariant straight lines of the form (3) and one invariant conic $\Phi=0$. The system (5) has a Darboux integrating factor of the form

$$\mu=l_1^{\alpha_1}l_2^{\alpha_2}\Phi^{\alpha_3}.$$

In the case i): $\Phi=(27b^3-24bc^2-27b-16c^3-36c)x^2-12(2cx-3y+3)(3b+4c)$ and $\alpha_1=(3b+4c)^2/[2(8c^2-9b^2+3bc)],\ \alpha_2=-4,\ \alpha_3=(18b^2+21bc+8c^2)/[2(9b^2-3bc-8c^2)]$.

In the case ii): $\Phi=(g^2+1)(gx-2y)x-g(y-1-2gx)$ and $\alpha_1=3,\ \alpha_2=(2-g^2)/[2(g^2+1)],\ \alpha_3=(-5g^2-8)/[2(g^2+1)]$.

In the case iii): $\Phi=(2f+1)[f(f+2)x^2+2f(f-1)a_1xy-(f^2-1)y^2]-2f(f-1)^2a_1x+2(f^3-1)y+(f-1)^2$ and $\alpha_1=[3(f+1)]/(2f+1),\ \alpha_2=-2,\ \alpha_3=-(8f+7)/[2(2f+1)]$.

In the case iv): $\Phi=(a-1)[vwa_2x^2-2uwxy-2(v+2a-2)ux]+(f+1)[vwa_2y^2-2(2a^2+3af+a+f^2+f+1)va_2y+v^2a_2]$ and $\alpha_1=[(2a-2+v)(2a+2f+1)]/w,\ \alpha_2=-2,\ \alpha_3=-(4a^2f+16a^2+8af^2+31af+11a+4f^3+15f^2+9f+1)/(2w)$, where $w=4a^2+5af+a+f^2-f-1$.

In the case v): $\Phi=(f+3)ha_{11}xy-(f+3)^2(f+1)a_{11}x-h^2y^2+2(f^2+f-3)hy+h(f+3)$ and $\alpha_1=3,\ \alpha_2=-(f+3)^2/h,\ \alpha_3=(18-3f-5f^2)/h$.

In the case vi): $\Phi=(f^2+6f+6)(f+1)hx^2+12ha_1xy-18(f+5)(f+4)(f+1)a_1x+3(f+1)hy^2-6(f^3+9f^2+21f+3)(f+1)y-9(f+5)(f+1)$ and $\alpha_1=3,\ \alpha_2=-(f+3)^3/h,\ \alpha_3=-(5f^3+45f^2+108f+36)/h$, where $h=2f^3+18f^2+45f+21$.

In the case vii): $\Phi=(ay+\alpha t^2y-1)(1-y)\alpha t-x[(a-\alpha t^2-1)y-(a-1)\beta tx](a+\alpha t^2)\beta+\beta(1-af-a-\alpha ft^2-3\alpha t^2+f)x$ and $\alpha_1=3,\ \alpha_2=[(f+3)\alpha^3-(a-1)(f+3)\alpha^2\beta^2-((f+4)(f+1)+2a)(f+3)(f+1)\beta^6-((f+3)a+2(f+2)(f+1))(f+3)\alpha\beta^4]/[a\alpha^3+(af+a-f^2-2f+1)\alpha^2\beta^2-(4af+8a+f^3+7f^2+13f+7)\alpha\beta^4]$, $\alpha_3=[(8af+18a+2f^3+15f^2+f\alpha_2+36f+3\alpha_2+33)\alpha\beta^4-(3a+f+\alpha_2+6)\alpha^3+(f^2-3af-3a-f\alpha_2-f-2\alpha_2-9)\alpha^2\beta^2-(2af+6a+f^3+8f^2+19f+12)\beta^6]/[(a+1)\alpha^3+(af+a-f^2-f+3)\alpha^2\beta^2-(4af+8a+f^3+7f^2+14f+10)\alpha\beta^4]$.

In the case viii): $\Phi=(3t^2-14t+24)(3t-10)x^2+(5t-6)[8(t+2)a_{11}xy-8(t+2)a_{11}x-4(3t-10)(t-2)y^2+4(3t^2-18t+16)y+8(t+2)]$ and $\alpha_1=[6(t-2)^2]/[t(3t-10)],\ \alpha_2=-4,\ \alpha_3=(38t-9t^2-48)/[2t(3t-10)]$.

In the case ix): $\Phi=(t^2-t-3)(t+1)[(2t+1)x^2+2ta_1xy-t(t+2)y^2]-2t^2(t^2+2)a_1x+2t(t^4+2t^3-4t^2-5t-3)y-t^2(t^2+3t-1)(t-1)$ and $\alpha_1=(t+2)^2/(3+t-t^2),\ \alpha_2=-4,\ \alpha_3=(3t^2+7t+5)/[2(t^2-t-3)]$.

In the case x): $\Phi=(a-1)^2a_{02}a_1^2x^2-2v(a-1)a_{02}a_1xy+(v-(f+2)a_{02})va_1x+v^2(a_{02}y-1)(y-1)$ and $\alpha_1=[a_1^2(2a_{02}f^2-4a^2-2af^2-10af-4a+7fa_{02}+6a_{02}+3f+2)-2(v+a_{02}\alpha_3+a_{02}+\alpha_3-1)v^2]/(2v^2),\ \alpha_2=-4,\ \alpha_3=[a_1^4(2f^2a_{02}-4a^2-2af^2-10af-4a+7fa_{02}+6a_{02}+3f+2)-a_1^2(8a^2+4aa_{02}+10af+6fa_{02}+5a_{02}+6f^2+6f+3)v-2v^3]/[4v(a+f+1)a_{02}a_1^2]$. □

Lemma 2. *The following nine series of conditions are sufficient conditions for the origin to be a center for system (5):*

i)
$$a=1/2,\ \ a_1=c=0,\ \ d=(-3)/2,\ \ f=-1,\ \ a_2=g=(-b)/2;$$

ii)
$$a=0,\ \ b=-g-2g^{-1},\ \ a_1=c=g/2,\ \ d=(g^2-2)/2,\ \ f=-1,\ \ a_2=-2g^{-1};$$

iii)

$$a=(2b^2-3bp+p^2+2)/4,\quad d=[((b-p)^2-2)(2b-p)]/[2(b-p)],$$
$$c=(p-2b)/2,\quad g=(2b^3-5b^2p+4bp^2-6b-p^3+2p)/4,$$
$$f=-1,\quad a_1=-b+p/2,\quad a_2=p-b;$$

iv)

$$a=1/2,\quad c=3b,\quad d=(-3)/2,\quad f=-1,\quad g=b,\quad a_1=0,\quad a_2=b;$$

v)

$$b=(a-1)a_1,\quad c=(2a^2+a-1)/a_1,\quad d=4a-2a^2-3,\quad f=-1,$$
$$g=(5a-2a^2-2)/a_1,\quad a_1^2-2a+1=0,\quad a_2=a(2a^2+a-1);$$

vi)

$$b=[(3-4a)a_2]/[4(a-1)],\quad c=[(2a-3)a_2]/[2(a-1)],\quad d=2a-3,$$
$$f=(-3)/2,\quad g=(3a_2)/2,\quad a_1=a_2/(2-2a);$$

vii)

$$a=-[(4f^2+10f+h^2+6)a_2+(2f+1)h]/(2h),\quad h=a_{10}-2(f+2)a_2,$$
$$b=-[(4fa_2+6a_2+h)(f+2)]/(2f+3),\quad c=a_2+2h(f+1)/(2f+3),$$
$$d=[(4f^2+14f-h^2+12)a_2+2h(f+1)]/h,\quad a_1=-h/(2f+3),$$
$$g=[(4f^2+18f-h^2+18)a_2+h(2f+3)]/[2(2f+3)];$$

viii)

$$a=((f+2)a_2^2-f)/2,\quad b=[(f+2)(1-a_2^2)(fa_2^2-a_2^2+f+1)]/(2za_2),$$
$$c=[(2-3f-3f^2)a_2^4+2(1-3f-f^2)a_2^2+f^2+5f+4]/(2za_2),$$
$$d=[(f^2-3)a_2^4+2(1-2f-f^2)a_2^2-3(f^2+4f+5)]/(2z),$$
$$g=[(2f^2+5f+1)a_2^4+2(f^2+5f+7)a_2^2+f+1]/(2za_2),$$
$$a_1=(3fa_2^4+5a_2^4+4fa_2^2+10a_2^2+f+1)/(2za_2),\quad z=(f+1)a_2^2+f+3;$$

ix)

$$a=((f+2)a_1^2-f)/2,\quad b=[((f+2)a_1^2+f+1)(1-a_1^2)(f+2)]/(wa_1),$$
$$c=[(-3f^2-9f-4)a_1^4-(2f^2+3f+1)a_1^2+(f+1)^2]/(wa_1),$$
$$d=[(2f^2+6f+3)a_1^4-2(2f^2+13f+19)a_1^2-3(2f^2+8f+7)]/(2w),$$
$$g=[(4f^2+19f+23)a_1^4+2(2f^2+7f+5)a_1^2-f-1]/(2a_1w),$$
$$a_2=[(2a_1^2-1)(f+1)+(3f+7)a_1^4]/(2a_1w),\quad w=(2f+5)a_1^2+2f+3.$$

Proof. In each of the cases **i)–ix)** the first Liapunov quantity vanishes $L_1=0$. The system (5) along with invariant straight lines (3) has also one more invariant straight line $l_3=0$ and one invariant conic $\Phi=0$.

In the case i): $l_3=bx-2,\quad \Phi=(b^2-1)x^2+4y-4$.

In the case ii): $l_3=(g^2+2)(2x+gy)+2g,\quad \Phi=(g^2+4)(gx^2-2xy)+2(g^2+2)x-2g(y-1)$.

In the case iii): $l_3=(b-p)(bx-1)+by,\quad \Phi=p(2b^2-3bp+p^2-2)x^2+4(b-p)[(px-1)y-bx+1]$.

In the case iv): $l_3=1+bx,\quad \Phi=x^2-4bxy+8bx-4y+4$.

In the case v): $l_3 = (1-a)a_1x + (2a-1)y + 1$, $\Phi = a(a-1)a_1x^2 - (2a-1)(ay+1)x + a_1(y-1)$.

In the case vi): $l_3 = a_2x + 2(a-1)y + 1$, $\Phi = (2a-1)[2(a-1)x^2 - y^2] - a_2x + 2ay - 1$.

In the case vii): $l_3 = [(2f+3)y + hx]a_{10} + h$, $\Phi = [(4f^2a_2 + 10fa_2 + h^2a_2 + 6a_2 + 2fh + 3h)x^2 - 2h^2xy - 2h(f+1)y^2](a_{10} - a_2) + h^2a_{10}x + h(2fa_{10} + a_{10} + 2a_2)y + h^2$.

In the case viii): $l_3 = ((f+3)a_2^2 + f + 1)(a_2^2x - 2a_2y - x) - 2za_2$, $\Phi = (f+2)(a_2^4 - 1)a_2x^2 - ((3f+5)a_2^2 - f - 1)(a_2^2+1)xy - 2x((f+3)a_2^2 + f + 1) + 2[(z-2)y^2 - (z-4)y - 2]a_2$.

In the case ix): $l_3 = ((f+3)a_1^2 + f + 1)[(a_1^2 - 1)x - 2a_1y] + 2a_1w$, $\Phi = (a-1)a_1a_{02}[(a-1)a_1x - 2vy]x + v(2aa_{02} - va_{02} - 2a_{02} + v)a_1x + v^2(a_{02}y - 1)(y-1)$, where $a_{02} = -[(f+2)^2a_1^4 + (2f^2 + 6f + 3)a_1^2 + (f+1)^2]/w$, $v = 2a + f$.

By Theorem 1 in each of these cases the origin is a center. □

Lemma 3. *The following two series of conditions are sufficient conditions for the origin to be a center for system (5):*

i)

$$a = (-2f)/3,\ \ b = (f+2)a_2,\ \ c = [3(-22f-41)]/[a_2(13f+24)],\ \ g = 0,$$
$$d = (-f-9)/3,\ \ (9f^2 + 12f - 9)a_2^2 + (f+3)^2 = 0,\ \ f^2 - 3f - 9 = 0,\ \ a_1 = 0;$$

ii)

$$a = [-f(f^2 + 7f + 9)]/v,\ \ b = [-f(f+3)(f+2)a_1]/u,$$
$$c = [(2f^3 + 13f^2 + 48f + 54)u]/[(f+3)^2a_1v],\ \ g = (f+3)a_1,$$
$$d = [-(2f^2 + 17f + 24)f^2]/[v(f+3)],\ \ v(f+3)^2a_1^2 + u^2 = 0,$$
$$a_2 = [(f^2 - 9f - 18)a_1]/u,\ \ u = f^2 - 3f - 9,\ \ v = f^2 + 12f + 18.$$

Proof. In each of the cases **i)** and **ii)** the first Liapunov quantity vanishes $L_1 = 0$. The system (5) along with invariant straight lines (3) has also two more invariant straight lines $l_3 = 0$, $l_4 = 0$ and one invariant conic $\Phi = 0$.

In the case i): $l_{3,4} = a_2b_j(55f + 102)x + (b_jy + 1)(87 + 47f - (8f + 15)b_j)$, where b_j, $j = 3, 4$ are the solutions of the equation $3(48f + 89)b_j^2 - 3(185f + 343)b_j + 521f + 966 = 0$ and $\Phi = a_2(10fy - 18f + 18y - 33)x + (6fy + 2f + 12y + 3)(y - 1)$.

In the case ii): $l_3 = 3(f+3)^2(f+2)a_1x - u(3fy + f + 3y + 3)$, $l_4 = (f+3)^2(2f+3)a_1x - u(2fy + f + 3y + 3)$ and $\Phi = 2(2f+3)^2u^2x^2 + a_1v[2f(2f+3)^3y - (f+3)(7f+12)v]x - uv(y-1)(8f^2y + 24fy + 18y + v)$.

By Theorem 1 in each of these cases the origin is a center. □

Theorem 4. ($l_j = 1 + a_jx - y$, $j = 1, 2$, Φ; $L = 4$), *where* $f + 2 \neq 0$ *and* $\Phi = 0$ *is an invariant conic of the form (7), is ILC for system (1), i.e. the order of a weak focus is at most four.*

Proof. To prove the theorem, we compute the first four Liapunov quantities L_j, $j = \overline{1,4}$ in each series of conditions **1)–8)** using the algorithm described in [19]. In the expressions for L_j we will neglect denominators and non-zero factors.

In the case **1)** the first Liapunov quantity is $L_1 = 6(3b + 4c)d - (6b^2c - 4bc^2 - 27b - 36c)$. From $L_1 = 0$ we find d and replacing into the expression for L_2, we obtain $L_2 = f_1 f_2$, where $f_1 = c,\ f_2 = 3b^2c - 2bc^2 + 9b + 12c$. If $f_1 = 0$, then we are in the conditions of Lemma 2, i), if $f_2 = 0$, then Lemma 1, i).

In the case **2)** the vanishing of the first Liapunov quantity gives $b = -g - 2/g$. Then $L_2 = f_1 f_2$, where $f_1 = 2c - g,\ f_2 = 2cg - 3g^2 + 4$. If $f_1 = 0$, then we are in the conditions of Lemma 2, ii), if $f_2 = 0$, then Lemma 1, ii).

In the case **3)** the first Liapunov quantity is $L_1 = g_1 g_2$, where $g_1 = 4a - 2b^2 + 3bp - p^2 - 2,\ g_2 = a^2(b^2 - 2bp + p^2 - 4) + a(b^2 - bp + 4) - b^2 + bp - 1$.

If $g_1 = 0$, then Lemma 2, iii). Assume $g_1 \neq 0$ and calculate L_2. The resultant of the polynomials g_2 and L_2 by b is

$$Res(g_2, L_2, b) = 144a^6(2a-1)^{15}(2a^3 - a^2 - p^2).$$

If $a = 0$, then $g_2 = 0$ yields $p = (b^2+1)/b$ and $I_3 = 0$. If $a = 1/2$, then $g_2 = 0$ yields $p = -b$ and we are in the conditions of Lemma 2, iv). If $p^2 = a^2(2a-1)$ and $g_2 = 0$, then Lemma 2, v).

In the case **4)** the first Liapunov quantity is $L_1 = g_1 g_2$, where $g_1 = 4(f+2)^2 a_2^3 - 4(f+2)a_{10}a_2^2 - (4af + 8a - 2 - a_{10}^2)a_2 + (2a + 2f + 1)a_{10},\ g_2 = a_2^2(-af^2 - 2af + a - f^3 - 4f^2 - 6f - 5) + a_2 a_{10}(af + f^2 + 2f + 2) + (af - a + f^2 + f + 1)(a-1)$.

Assume $g_1 = 0$. If $a_{10} = 2(f+2)a_2$ and $f = (-3)/2$, then Lemma 2, vi). If $g_1 = 0$ and $a_{10} \neq 2(f+2)a_2$, then Lemma 2, vii).

Let $g_1 \neq 0$ and $g_2 = 0$. If $a_2 = 0$, then $g_2 = 0$ yields $a = (f^2 + f + 1)/(1-f)$ and $L_2 = (f-1)a_{10}^2 + 4f(f+1)(f+2)$. If $L_2 = 0$, then Lemma 1, iii).

If $a_2 \neq 0$ and $a = (-f^2 - 2f - 2)/f$, then $g_2 = 0$ yields $f = (-2)/(a_2^2 + 1)$. In this case $L_2 = f_1 f_2$, where $f_1 = (a_2^2 + 1)a_{10} - 6a_2^3 + 2a_2,\ f_2 = 2a_{10}a_2 - 3a_2^2 + 1$. If $f_1 = 0$, then Lemma 1, iv) and if $f_2 = 0$, then Lemma 2, viii) $(f = (-2)/(a_2^2+1))$.

Assume $a_2 \neq 0$ and $a \neq (-f^2 - 2f - 2)/f$. From $g_2 = 0$ we find a_{10} and replace into the expression for L_2. We obtain $L_2 = h_1 h_2$, where $h_1 = 2a + f - (f+2)a_2^2$, $h_2 = (f+1)[f^2 + (f+1)(a+3)]^2 a_2^2 - (a-1)[f^2 + (a+1)f + 1 - a]^2$.

If $h_1 = 0$, then Lemma 2, viii) and if $h_2 = 0$, then Lemma 1, iv).

In the case **5)** the vanishing of the first Liapunov quantity gives $a_{11} = [a^2(a-1)^2]/(1 - 2f - f^2 - a)$. In this case $L_2 = f_1 f_2$, where $f_1 = (f+3)a + 2f^2 + 3f - 3,\ f_2 = 3a + 2f$. If $f_1 = 0$, then Lemma 1, v). Let $f_1 \neq 0$ and $f_2 = 0$, then $a = (-2f)/3$. We calculate $L_3 = h_1 h_2$, where $h_1 = 5f + 6,\ h_2 = f^2 - 3f - 9$. If $h_1 = 0$, then $L_4 \neq 0$ and if $h_2 = 0$, then Lemma 3, i).

In the case **6)** we denote $a_1 = \beta t$, $a_{02} = \alpha t^2 + a$ and calculate L_1.

Let $\alpha = \beta^2$, then $L_1 = 0$ yields $a = (-f^2 - 6f - 3)/3$. The second Liapunov quantity is $L_2 = f_1 f_2$, where $f_1 = 3(f+5)\beta^2 t^2 + (f^2 + 6f + 6)(f+1),\ f_2 = 3(f+1)\beta^2 t^2 - (3f^3 + 19f^2 + 33f + 15)$.

If $f_1 = 0$, then Lemma 1, vi). Assume $f_1 \neq 0$ and let $f_2 = 0$, then we find t^2 and replacing into the expression for L_3, we obtain $L_3 = h_1 h_2$, where $h_1 = f^3 + 9f^2 + 18f + 9,\ h_2 = 5f^3 + 30f^2 + 54f + 24$. If $h_1 = 0$, then Lemma 3, ii), if $h_2 = 0$, then $L_4 \neq 0$ and therefore the origin is a focus.

Let now $\alpha \neq \beta^2$ and $\alpha = -(f+3)\beta^2$, then $L_1 = 0$ yields $a = (-f^2 - 4f - 5)/(f+1)$ and $L_2 = 3(f+3)(f+1)\beta^2 t^2 + 3f^2 + 20f + 27$. We find t^2 from $L_2 = 0$ and replacing into the expression for L_3, we obtain $L_3 = u_1u_2$, where $u_1 = 8f + 15$, $u_2 = 7f + 12$. If $u_1 = 0$, then Lemma 3, ii); if $u_2 = 0$, then $L_4 \neq 0$ and therefore $O(0,0)$ is a focus.

Assume now $(\alpha + (f + 3)\beta^2)(\alpha - \beta^2) \neq 0$. Then from $L_1 = 0$ we find t^2 and replacing into the expression for L_2, we obtain $L_2 = f_1f_2$, where $f_1 = [(f + 4)(f + 1)+2a](f+3)(f+2)\beta^4+(af^2+10af+15a+3f^3+18f^2+28f+9)\alpha\beta^2+(af+3a+2f^2+3f-3)\alpha^2$, $f_2 = (f^2+6f+3+6a)(f+2)\beta^4-\alpha\beta^2(3af+6a+f^2+4f+3)-(3a+2f)\alpha^2$.

If $f_1 = 0$, then Lemma 1, vii). Assume $f_1 \neq 0$, $f_2 = 0$ and calculate L_3. The resultant of f_2 and L_3 by β is

$$Res(f_2, L_2, \beta) = v_1v_2v_3v_4v_5v_6,$$

where $v_1 = f + 1$, $v_2 = 3a + f^2 + 6f + 3$, $v_3 = 6a + f^2 + 6f + 3$, $v_4 = (f + 1)a + f^2 + 4f + 5$, $v_5 = af^2 + 12af + 18a + f^3 + 7f^2 + 9f$, $v_6 = 75(f + 2)a^2 + 150af^2 + 390af + 180a + 75f^3 + 240f^2 + 209f + 54$.

Let $v_1 = 0$, then $L_2 = 0$ yields $\alpha = [2(1-3a)\beta^2]/(3a-2)$ and $L_3 = w_1w_2$, where $w_1 = 7a - 3$, $w_2 = 15a^2 - 12a + 2$. If $w_1 = 0$, then $L_1 \equiv 28\beta^2t^2 + 25 \neq 0$ and if $w_2 = 0$, then $L_4 \neq 0$. Therefore the origin is a focus.

Assume $v_1 \neq 0$, $v_2 = 0$, i.e. $a = (-f^2 - 6f - 3)/3$. Then $L_2 = 0$ yields $\alpha = -[(f^2 + 6f + 3)(f + 2)\beta^2]/[(f + 3)(f + 1)]$ and $L_3 = w_1w_2$, where $w_1 = f+6$, $w_2 = 5f^2+10f+3$. If $w_1 = 0$, then Lemma 3, ii) and if $w_2 = 0$, then $L_4 \neq 0$.

Let $v_1v_2 \neq 0$, $v_3 = 0$, then $a = (-f^2 - 6f - 3)/6$. The vanishing of the second Liapunov quantity gives $\alpha = -[f(f^2 + 6f + 7)\beta^2]/(f^2 + 2f + 3)$ and $L_3 = w_1w_2$, where $w_1 = f^2 + 6f + 6$, $w_2 = 5f^3 - 9f + 6$. If $w_1 = 0$, then Lemma 3, ii) and if $w_2 = 0$, then $L_4 \neq 0$.

Assume $v_1v_2v_3 \neq 0$, $v_4 = 0$, then $a = (-f^2 - 4f - 5)/(f + 1)$. In this case from $L_2 = 0$, we find $\alpha = -[(f^2+13f+18)\beta^2]/(f^2+10f+15)$ and $L_3 = 31f^2+122f+121$ has not real roots.

Let $v_1v_2v_3v_4 \neq 0$, $v_5 = 0$, then $a = [f(-f^2-7f-9)]/(f^2+12f+18)$. We calculate $L_2 = z_1z_2$, where $z_1 = (f^2-3f-9)\alpha+\beta^2(f^3+8f^2+21f+18)$, $z_2 = (f^2+6f+6)\beta^2+f\alpha$. If $z_1 = 0$, then Lemma 3, ii); if $z_2 = 0$, then $L_3 = 107f^3 + 426f^2 + 540f + 216$. Let $L_3 = 0$, then $L_4 \neq 0$.

Assume $v_1v_2v_3v_4v_5 \neq 0$, $v_6 = 0$ and calculate L_3 and L_4. Solve the system of equations $\{L_3 = 0, L_4 = 0\}$ by α and a, then $v_6 = 0$ has not real solutions.

In the case **7)** we calculate the first two Liapunov quantities and the resultant of them by a_{11}, then we get

$$Res(L_1, L_2, a_{11}) = f_1f_2f_3f_4,$$

where $f_1 = 2a - 1$, $f_2 = 2aa_{02} + 2a - 3a_{02} - 1$, $f_3 = 4a^2 - 4aa_{02} - 4a + 3a_{02} + 1$, $f_4 = 4a^2a_{02} + 20a^2 + 2aa_{02}^2 - 26aa_{02} - 24a + a_{02}^2 + 16a_{02} + 7$.

If $f_1 = 0$, then $a = 1/2$ and $a_{02} = a_{11}^2$. In this case $L_1 = L_2 = 0$ and $L_3 \neq 0$.

If $f_2 = 0$, then $a_{02} = (1 - 2a)/(2a - 3)$ and $L_1 = a(2a - 3)a_{11}^2 + 2a^2 - 3a + 1$. Let $L_1 = 0$, then $a_{11}^2 = (2a^2 - 3a + 1)/[a(3 - 2a)]$. In this case $L_2 \neq 0$ and therefore the origin is a focus.

If $f_3 = 0$, then $a_{02} = (2a - 1)^2/(4a - 3)$. The first Liapunov quantity is $L_1 = 32a^4 - 80a^3 + 32a^2a_{11}^2 + 72a^2 - 36aa_{11}^2 - 28a + 9a_{11}^2 + 4$. Let $L_1 = 0$ and express a_{11}^2, then $L_2 \neq 0$.

Assume $f_4 = 0$. This equation admits the parametrization $a = (18t - 24 - t^2)/[2t(t+2)]$, $a_{02} = (16t - 20 - 3t^2)/[2(t+2)]$. In this case L_1 looks $L_1 = g_1 g_2$, where $g_1 = 4(5t-6)(t+2)^2 a_{11}^2 + (3t^2 - 14t + 24)(3t-10)^2(t-2)$, $g_2 = 2(t+2)^2(t-4)(t-6)a_{11}^2 - (3t^2 - 14t + 24)(3t-10)^2(t-2)$. If $g_1 = 0$, then Lemma 1, viii). Let $g_1 \neq 0$ and $g_2 = 0$, then express a_{11}^2 from $g_2 = 0$ and calculate L_2. We obtain that $L_2 \neq 0$.

In the case **8)** we calculate the first Liapunov quantity and denote $w \equiv (2a^2 f + 2a^2 + 2af^3 + 14af^2 + 26af + 16a + 2f^4 + 13f^3 + 28f^2 + 24f + 6)a_1^2 + (2af + 2a + 2f^2 + 3f)(2a+f)^2$. If $w = 0$, then $L_1 = f_1 f_2$, where $f_1 = f + 1$, $f_2 = (f+1)a^2 - 4af - 6a - f^3 - 6f^2 - 9f - 3$. If $f_1 = 0$, then $L_2 = 0$ yields $a_{02} = (2a^2 - 3a + 1)/(1 - 3a)$ and Lemma 2, ix).

Assume $f_1 \neq 0$ and $f_2 = 0$. The equation $f_2 = 0$ admits the parametrization $a = (t^3 - 3t - 1)/[t(t^2 - 1)]$, $f = (1 - 2t - 2t^2)/(t^2 - 1)$. We calculate the second Liapunov quantity $L_2 = [(t^2 + 3t - 1)(t-1)t]a_{02} - (t^2 - t - 3)(t+2)(t+1)$ and if $L_2 = 0$, then Lemma 1, ix).

Let $w \neq 0$, then from $L_1 = 0$ we find a_{02} and substituting in L_2 we get $L_2 = g_1 g_2 g_3$, where $g_1 = 2a + f - (f+2)a_1^2$, $g_2 = 2a + 2f + 1$, $g_3 = (2a - f^2 - 2f - 2)(a-1)(f+2)(f+1)a_1^6 - v(2a^2 f^2 - 6a^2 - 2af^3 - 28af^2 - 66af - 40a - f^4 - 12f^3 - 30f^2 - 22f - 2)a_1^4 + v^3(3af + 3a - 5f - 7)a_1^2 - v^5$.

If $g_1 = 0$, then Lemma 2, ix). If $g_1 \neq 0$, $g_2 = 0$, then $a = (-2f - 1)/2$ and $L_3 \neq 0$. Assume $g_1 g_2 \neq 0$ and $g_3 = 0$, then Lemma 1, x). $\square$

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Method for constructing one-point expansions of a topology* on a finite set and its applications

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Abstract. The article consists of two parts. In the first part we present an algorithm which allows to receive, for any topology τ which is given on a set X from n elements, all topologies on the set $X \bigcup \{y\}$ each of which induces the topology τ on the set X. In the second part (as an example) this algorithm is applied for calculation of the number of topologies on the set Y each of which induces the discrete topology on the set X.

Mathematics subject classification: 54A10.

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Introduction

The history of researches of the problem about the number of topologies on finite sets and some results received by different authors are given in [1].

The works [1] and [2] contain an extended list of articles, which are devoted to this problem.

At present the number of all topologies on sets having no more than 18 elements is known. These numbers are given in the following table, which can be find in [1] and [2].

*If $Y = X \bigcup \{y\}$ then a topology $\widetilde{\tau}$ on the set Y is called one-point expansion of the topology $\tau = \widetilde{\tau}|_X$.

The number of elements of the set X	The number of topologies on the set X
0	1
1	1
2	4
3	29
4	355
5	6942
6	209527
7	9535241
8	642779354
9	63260289423
10	8977053873043
11	1816846038736192
12	519355571065774021
13	207881393656668953041
14	115617051977054267807460
15	88736269118586244492485121
16	93411113411710039565210494095
17	134137950093337880672321868725846
18	261492535743634374805066126901117203

This article adjoins the works in which this problem is studied. However, this question is investigated from other point of view.

Namely, we consider a topology on a set from $n+1$ elements as one-point expansion of a topology given on a set from n elements.

1 Justification of the algorithm

1.1. Theorem. *Let τ be a topology on a finite set X and let $\tilde{\tau}$ be such a topology on $Y = X \bigcup \{y\}$ that $\tilde{\tau}|_X = \tau$. Then there exist such $V_0 \in \tau$ and $U_0 \in \tau$ that the following statements are valid:*

1. $U_0 \subseteq \bigcap\limits_{V \not\subseteq V_0, V \in \tau} V$,

2. $\tilde{\tau} = \{V \in \tau | \, V \subseteq V_0\} \cup \{U \cup \{y\} | \, U \in \tau, U \supseteq U_0\}$.

Proof. We take $V_0 = \bigcup\limits_{\tilde{V} \in \tilde{\tau}, y \notin \tilde{V}} \tilde{V}$ and $U_0 = \bigcap\limits_{\tilde{V} \in \tilde{\tau}, y \in \tilde{V}} (\tilde{V} \backslash \{y\})$.

As $y \notin V_0 = \bigcup\limits_{\tilde{V} \in \tilde{\tau}, y \notin \tilde{V}} \tilde{V} \in \tilde{\tau}$, then $V_0 = V_0 \cap X \in \tilde{\tau}|_X = \tau$.

Besides

$$U_0 = \bigcap_{\tilde{V} \in \tilde{\tau}, y \in \tilde{V}} (\tilde{V} \backslash \{y\}) = (\bigcap_{\tilde{V} \in \tilde{\tau}, y \in \tilde{V}} \tilde{V}) \backslash \{y\} = (\bigcap_{\tilde{V} \in \tilde{\tau}, y \in \tilde{V}} \tilde{V}) \cap X \in \tilde{\tau}|_X = \tau.$$

Prove the first statement.

Let $V' \in \tau$ and $V' \nsubseteq V_0$. Then there exists $\tilde{U}' \in \tilde{\tau}$ such that $V' = X \bigcap \tilde{U}'$. As $V' \nsubseteq V_0 = \bigcup\limits_{\tilde{V} \in \tilde{\tau}, y \notin \tilde{V}} \tilde{V}$, then $y \in \tilde{U}'$, and hence,

$$U_0 = \bigcap_{\tilde{V} \in \tilde{\tau}, y \in \tilde{V}} (\tilde{V} \backslash \{y\}) \subseteq \tilde{U}' \setminus \{y\} = \tilde{U}' \bigcap X = V'.$$

From arbitrariness of the set V' it follows that $U_0 \subseteq \bigcap\limits_{V \nsubseteq V_0, V \in \tau} V$.

The first the statement is proved.

Now prove the second statement, i.e.

$$\tilde{\tau} = \{V \in \tau | \, V \subseteq V_0\} \cup \{U \cup \{y\} | \, U \in \tau, U \supseteq U_0\}.$$

Let $\tilde{W} \in \tilde{\tau}$. If $y \notin \tilde{W}$, then from the definition of V_0 it follows that $\tilde{W} \subseteq V_0 = \bigcup\limits_{\tilde{V} \in \tilde{\tau}, y \notin \tilde{V}} \tilde{V}$, and as $\tilde{W} = \tilde{W} \cap X \in \tilde{\tau}|_X = \tau$, then

$$\tilde{W} \in \{V \in \tau | \, V \subseteq V_0\} \subseteq \{V \in \tau | \, V \subseteq V_0\} \cup \{U \cup \{y\} | \, U \in \tau, U \supseteq U_0\}.$$

If $y \in \tilde{W}$, then $\tilde{W} \backslash \{y\} = \tilde{W} \cap X \in \tilde{\tau}|_X = \tau$. Besides, $\tilde{W} \backslash \{y\} \supseteq \bigcap\limits_{\tilde{V} \in \tilde{\tau}, y \in \tilde{V}} (\tilde{V} \backslash \{y\}) = U_0$. Then

$$\tilde{W} = (\tilde{W} \backslash \{y\}) \cup \{y\} \in \{U \cup \{y\} | \, U \in \tau, U \supseteq U_0\} \subseteq$$

$$\{V \in \tau | \, V \subseteq V_0\} \cup \{U \cup \{y\} | \, U \in \tau, U \supseteq U_0\}.$$

So, we have shown that $\tilde{\tau} \subseteq \{V \in \tau | \, V \subseteq V_0\} \cup \{U \cup \{y\} | \, U \in \tau, U \supseteq U_0\}$.

Now let

$$W \in \{V \in \tau | \, V \subseteq V_0\} \cup \{U \cup \{y\} | \, U \in \tau, U \supseteq U_0\}.$$

If $W \in \{V \in \tau | \, V \subseteq V_0\}$, then there exists $\tilde{W} \in \tilde{\tau}$ such that $W = \tilde{W} \cap X$.

As $y \notin V_0$, then $V_0 \subseteq X$, and hence $W = \tilde{W} \cap X \supseteq \tilde{W} \cap V_0$. As $W \subseteq \tilde{W}$ and $W \subseteq V_0$, then $W \subseteq \tilde{W} \cap V_0$, and hence $W = \tilde{W} \cap V_0$. Besides, $V_0 = \bigcup\limits_{\tilde{V} \in \tilde{\tau}, y \notin \tilde{V}} \tilde{V} \in \tilde{\tau}$, and hence, $W = \tilde{W} \cap V_0 \in \tilde{\tau}$.

If $W \in \{U \cup \{y\} | \, U \in \tau, U \supseteq U_0\}$ then there exists such $U' \in \tau$ that $U' \supseteq U_0$ and $U' \cup \{y\} = W$. As $U' \in \tau = \tilde{\tau}|_X$ then there exists $\tilde{W}' \in \tilde{\tau}$ such that $U' = W' \cap X$.

As $U_0 = \bigcap\limits_{\tilde{V} \in \tilde{\tau} y \in \tilde{V}} (\tilde{V} \backslash \{y\})$, then from finiteness of the set $\tilde{\tau}$ it follows that $U_0 \cup \{y\} = \bigcap\limits_{\tilde{V} \in \tilde{\tau}, y \in \tilde{V}} \tilde{V} \in \tilde{\tau}$. Then $y \in \tilde{W}' \cup U_0 \cup \{y\}$, and

$$W = U' \cup \{y\} = U' \cup U_0 \cup \{y\} =$$

$$((\tilde{W}' \cap X) \cup (U_0 \cup \{y\}) \cap X) \cup ((\tilde{W}' \cup (U_0 \cup \{y\})) \cap \{y\}) =$$

$$(\tilde{W}' \cup (U_0 \cup \{y\})) \cap (X \cup \{y\}) = \tilde{W}' \cup (U_0\{y\}) \in \tilde{\tau}.$$

Therefore, $\tilde{\tau} \supseteq \{V \in \tau | V \subseteq V_0\} \cup \{U \cup \{y\} | U \in \tau, U \supseteq U_0\}$, and hence, $\tilde{\tau} = \{V \in \tau | V \subseteq V_0\} \cup \{U \cup \{y\} | U \in \tau, U \supseteq U_0\}$.

The theorem is completely proved.

1.2. Theorem. *Let τ be a topology on a set X and $V_0 \in \tau$. Consider a set $U_0 \in \tau$ such that $U_0 \subseteq \bigcap\limits_{V \in \tau, V \not\subseteq V_0} V$ (we assume that $\bigcap\limits_{V \in \emptyset} V = X$). Then*

$$\tilde{\tau}(V_0, U_0) = \{V \in \tau | V \subseteq V_0\} \cup \{U \cup \{y\} | U \in \tau, U \supseteq U_0\}$$

is a topology on the set $Y = X \bigcup \{y\}$, and $\tilde{\tau}(V_0, U_0)|_X = \tau$.

Proof. Prove first that $\tilde{\tau}(V_0, U_0)$ is a topology on the set Y.

As $\emptyset \subseteq V_0$, then $\emptyset \in \{V \in \tau | V \subseteq V_0\} \subseteq \tilde{\tau}$. Besides, as $X \in \tau$ and $U_0 \subseteq X$, then $X \in \{U | U \in \tau, U \supseteq U_0\}$, and hence, $Y = X \cup \{y\} \in \tilde{\tau}$.

Now let $A, B \in \tilde{\tau}$, then:

– If $A, B \in \{V \in \tau | V \subseteq V_0\}$, then $A \cap B \in \tau$ and $A \cap B \subseteq V_0$, and hence, $A \cap B \in \tilde{\tau}$.

– If $A \in \{V \in \tau | V \subseteq V_0\}$ and $B \in \{U \cup \{y\} | U \in \tau, U \supseteq U_0\}$, then $A \in \tau$ and $B \backslash \{y\} \in \tau$, and as $A \subseteq V_0$ and $B \backslash \{y\} \supseteq U_0$, then $A \cap (B \backslash \{y\}) \in \tau$. As $y \notin A$, then $A \cap B = A \cap (B \backslash \{y\}) \subseteq V_0$, and hence, $A \cap B \in \tilde{\tau}$.

It is similarly proved that $A \cap B \in \tilde{\tau}$ if $B \in \{V \in \tau | V \subseteq V_0\}$ and

$$A \in \{U \cup \{y\} | U \in \tau, U \supseteq U_0\}.$$

– If $A, B \in \{U \cup \{y\} | U \in \tau, U \supseteq U_0\}$, then $A \backslash \{y\} \in \tau$, $B \backslash \{y\} \in \tau$ and $A \backslash \{y\} \supseteq U_0$, $B \backslash \{y\} \supseteq U_0$. As τ is a topology on the set X, then $(A \backslash \{y\}) \cap (B \backslash \{y\}) \in \tau$. Besides, as $(A \backslash \{y\}) \cap (B \backslash \{y\}) \supseteq U_0$, then $A \cap B \in \{U \cup \{y\} | U \in \tau, U \supseteq U_0\} \subseteq \tilde{\tau}$, and hence $A \cap B \in \tilde{\tau}$.

So, we have checked that $A \cap B \in \tilde{\tau}$, for any $A, B \in \tilde{\tau}$.

Now let $\{A_\gamma | \gamma \in \Gamma\} \subseteq \tilde{\tau}$. If $A_\gamma \in \{V \in \tau | V \subseteq V_0\}$ for any $\gamma \in \Gamma$, then $\bigcup\limits_{\gamma \in \Gamma} A_\gamma \in \tau$ and $\bigcup\limits_{\gamma \in \Gamma} A_\gamma \subseteq V_0$, and hence $\bigcup\limits_{\gamma \in \Gamma} A_\gamma \in \tilde{\tau}$.

If there exists $\gamma_0 \in \Gamma$ such that $A_{\gamma_0} \notin \{V \in \tau | V \subseteq V_0\}$, then $A_{\gamma_0} \in \{U \cup \{y\} | U \in \tau, U \supseteq U_0\}$, and hence, $A_{\gamma_0} = U_{\gamma_0} \cup \{y\}$, where $U_{\gamma_0} \in \tau$. Then $\bigcup\limits_{\gamma \in \Gamma} A_\gamma \supseteq A_{\gamma_0} \supseteq U_0$ and $(\bigcup\limits_{\gamma \in \Gamma} A_\gamma) \backslash \{y\} = \bigcup\limits_{\gamma \in \Gamma} (A_\gamma \backslash \{y\})$.

Let $\gamma \in \Gamma$. If $A_\gamma \in \{V \in \tau | V \subseteq V_0\} \subseteq \tau$, then $A_\gamma \backslash \{y\} = A_\gamma \in \tau$ and if $A_\gamma \notin \{V \in \tau | V \subseteq V_0\}$, then there exists $V_\gamma \in \tau$ such that $V_\gamma \supseteq U_0$ and $A_\gamma = V_\gamma \cup \{y\}$. But then $A_\gamma \backslash \{y\} = V_\gamma \in \tau$.

So, we have proved that $A_\gamma \backslash \{y\} \in \tau$ for any $\gamma \in \Gamma$. Having put $V_\gamma = A_\gamma \backslash \{y\}$ for those $\gamma \in \Gamma$, receive

$$\bigcup_{\gamma \in \Gamma} A_\gamma = A_{\gamma_0} \cup (\bigcup_{\gamma \notin \Gamma, \gamma \neq \gamma_0} A_\gamma) = (V_{\gamma_0} \cup \{y\}) \cup (\bigcup_{\gamma \notin \Gamma \gamma \neq \gamma_0} A_\gamma \backslash \{y\}) =$$

$$V_{\gamma_0} \cup (\bigcup_{\gamma\in\Gamma \gamma\neq\gamma_0} V_\gamma)) \cup \{y\} = (\bigcup_{\gamma\in\Gamma} V_\gamma) \cup \{y\}.$$

As $\bigcup\limits_{\gamma\in\Gamma} V_\gamma \in \tau$ and $\bigcup\limits_{\gamma\in\Gamma} V_\gamma \supseteq U_0$, then, $\bigcup\limits_{\gamma\in\Gamma} A_\gamma \in \tilde{\tau}$.

So, we have proved that $\tilde{\tau}(V_0, U_0)$ is a topology on the set Y.

Now prove that $\tilde{\tau}(V_0, U_0)|_X = \tau$.

Let $U \in \tilde{\tau}(V_0, U_0)|_X$. Then there exists $\tilde{U} \in \tilde{\tau}(V_0, U_0)$ such that $U = \tilde{U} \cap X$. If $\tilde{U} \in \{V \in \tau | V \subseteq V_0\}$, then $\tilde{U} \in \tau$ and $y \notin \tilde{U}$. Then $U = \tilde{U} \cap X = \tilde{U} \in \tau$.

Now let $\tilde{U} \in \{U \cup \{y\} | U \in \tau, U \supseteq U_0\}$. Then $\tilde{U} \backslash \{y\} \in \tau$, and hence, $U = \tilde{U} \cap X = \tilde{U} \backslash \{y\} \in \tau$. From arbitrariness of U it follows that we have the inclusion $\tilde{\tau}(V_0, U_0)|_X \subseteq \tau$. Now show the inverse inclusion.

Let $V' \in \tau$. Two cases are possible:

1) $V' \subseteq V_0$;

2) $V' \nsubseteq V_0$.

If $V' \subseteq V_0$, then $V' \in \{V \in \tau | V \subseteq V_0\} \subseteq \tilde{\tau}(V_0, U_0)$, and $y \notin V'$. Then $V' = V' \cap X \in \tilde{\tau}(V_0, U_0)|_X$.

If $V' \nsubseteq V_0$, then $V' \notin \{V \in \tau | V \subseteq V_0\}$, and according to the condition of the theorem we have that $V' \supseteq \bigcap\limits_{V\in\tau, V\nsubseteq V_0} V \supseteq U_0$. Then, from the definition of the topology $\tilde{\tau}(V_0, U_0)$ it follows that $V' \cup \{y\} \in \tilde{\tau}(V_0, U_0)$.

Besides, as $V' \subseteq X$, then $V' = (V' \cup \{y\}) \cap X \in \tilde{\tau}(V_0, U_0)|_X$. From arbitrariness of V' it follows that $\tilde{\tau}(V_0, U_0)|_X \supseteq \tau$, and hence, $\tilde{\tau}(V_0, U_0)|_X = \tau$.

The theorem is completely proved.

1.3. Theorem. *Let X be a finite set, $\tilde{\tau}$ and $\tilde{\tau}'$ be such topologies on the set $Y = X \bigcup \{y\}$ that $\tilde{\tau}|_X = \tilde{\tau}'|_X = \tau$. If $V_0, U_0, V_0', U_0' \in \tau$, $\tilde{\tau} = \{V \in \tau | V \subseteq V_0\} \cup \{U \cup \{y\} | U \in \tau, U \supseteq U_0\}$ and $\tilde{\tau}' = \{V \in \tau | V \subseteq V_0'\} \cup \{U \cup \{y\} | U \in \tau, U \supseteq U_0'\}$, then $\tilde{\tau} \neq \tilde{\tau}'$ if and only if $(V_0, U_0) \neq (V_0', U_0')$.*

Proof. Necessity. We assume the contrary, i.e. $\tilde{\tau} \neq \tilde{\tau}'$, but $(V_0, U_0) = (V_0', U_0')$.

Then $V_0 = V_0'$ and $U_0 = U_0'$, hence,

$$\tilde{\tau} = \{V \in \tau | V \subseteq V_0\} \cup \{U \cup \{y\} | U \in \tau, U \supseteq U_0\} =$$

$$= \{V \in \tau | V \subseteq V_0'\} \cup \{U \cup \{y\} | U \in \tau, U \supseteq U_0'\} = \tilde{\tau}'.$$

Receive a contradiction with the assumption that $\tilde{\tau} \neq \tilde{\tau}'$.

Hence $(V_0, U_0) \neq (V_0', U_0')$.

Sufficiency. We assume the contrary, i.e. $\tilde{\tau} = \tilde{\tau}'$ and $(V_0, U_0) \neq (V_0', U_0')$.

If $V_0 \neq V_0'$, then $V_0 \nsubseteq V_0'$, or $V_0' \nsubseteq V_0$.

We assume, for definiteness, that $V_0 \nsubseteq V_0'$. Then $V_0 \in \{V \in \tau | V \subseteq V_0\} \subseteq \tilde{\tau}$ and $V_0 \notin \{V \in \tau | V \subseteq V_0'\}$.

As any set from $\{U \cup \{y\} | U \supseteq U_0'\}$ contains y and $y \notin V_0$, then

$$V_0 \notin \{V \in \tau | V \subseteq V_0'\} \cup \{U \cup \{y\} | U \in \tau, U \supseteq U_0\} = \tilde{\tau}',$$

and hence, in this case $\tilde{\tau} \neq \tilde{\tau}'$.

If $U_0 \neq U_0'$, then $U_0 \nsubseteq U_0'$, or $U_0' \subseteq U_0$.

We assume, for definiteness, that $U_0 \nsubseteq U_0'$. Then $U_0' \cup \{y\} \in \{U \cup \{y\} | U \in \tau, U \supseteq U_0'\} \subseteq \tilde{\tau}'$, and $U_0' \cup \{y\} \notin \{U \cup \{y\} | U \in \tau, U \supseteq U_0\}$. As any set from $\{V \in \tau | V \subseteq V_0\}$ does not contain y and $y \in U_0' \cup \{y\}$, then

$$U_0' \cup \{y\} \notin \{V \in \tau | V \subseteq V_0\} \cup \{U \cup \{y\} | U \in \tau, U \supseteq U_0\} = \tilde{\tau},$$

and hence, $\tilde{\tau} \neq \tilde{\tau}'$ in this case, too.

Therefore $\tilde{\tau} \neq \tilde{\tau}'$.

The theorem is completely proved.

1.4. Remark. We notice that if $\tilde{\tau}$ and $\tilde{\tau}'$ are such topologies on the set $Y = X \bigcup \{y\}$ that $\tilde{\tau}|_X \neq \tilde{\tau}'|_X$, then $\tilde{\tau} \neq \tilde{\tau}'$. Therefore any extensions on the set Y of various topologies set on the set X will be various.

So, from Theorems 1.2 and 1.3 the following algorithm for the construction of all topologies on the set $Y = X \bigcup \{y\}$ follows, knowing all topologies on the finite set X.

1.5. Algorithm.

1. We choose any topology τ_0 set on the set X;

2. We choose arbitrarily a subset $V_0 \in \tau_0$;

3. We choose arbitrarily such subset $U_0 \in \tau_0$ that $U_0 \subseteq \bigcap\limits_{V \in \tau_0, V \nsubseteq V_0} V$ (consider that $\bigcap\limits_{V \in \emptyset} V = X$);

4. We determine the topology
$$\tilde{\tau}(V_0, U_0) = \{V \in \tau | V \subseteq V_0\} \cup \{U \cup \{y\} | U \in \tau, U \supseteq U_0\}.$$

2 Application of the algorithm for calculation of the number of some topologies

2.1. Definition. As it is usual, a partially ordered set $(X, \leq)$ is called a *lattice* if for any elements $a, b \in X$ there exists $inf\{a, b\}$ and $sup\{a, b\}$.

2.2. Definition. Lattices $(X, \leq)$ and $(Y, \leq)$ are called:

– *isomorphic* if there exists such a bijection $f : X \to Y$ that $f(inf\{a, b\}) = inf\{f(a), f(b)\}$ and $f(sup\{a, b\}) = sup\{f(a), f(b)\}$, for any elements $a, b \in X$;

– *antiisomorphic* if there exists such a bijection $f : X \to Y$ that $f(inf\{a, b\}) = sup\{f(a), f(b)\}$ and $f(sup\{a, b\}) = inf\{f(a), f(b)\}$, for any elements $a, b \in X$.

2.3. Definition. If (X, τ_1) and (Y, τ_2) are topological spaces then the topologies τ_1 and τ_2 are called:

– *lattice isomorphic* if the lattices $(\tau_1, \subseteq)$ and $(\tau_2, \subseteq)$ are isomorphic;
– *lattice antiisomorphic* if the lattices $(\tau_1, \subseteq)$ and $(\tau_2, \subseteq)$ are antiisomorphic.

2.4. Remark. If X is a finite set, (X, τ) is a topological space and $\tau' = \{X \setminus V \big| V \in \tau\}$, it is easy to notice that τ' is a topology on the set X which is lattice antiisomorphic with the topology τ.

2.5. Proposition. *Let X be a finite set and $Y = X \bigcup \{y\}$. If τ is a topology on the set X and $\tau' = \{X \setminus V \big| V \in \tau\}$, then τ' is a topology on the set X and τ and τ' have the same number of expansions on the set Y.*

Proof. Let Ω and Ω' be sets of all expansions of topologies τ and τ' on the set Y, accordingly. Define the following mapping $\psi : \Omega \to \Omega'$:

map each topology $\widehat{\tau} \in \Omega$ onto the topology $\psi(\widehat{\tau}) = \widehat{\tau}' = \{Y \setminus \widehat{V} \big| \widehat{V} \in \widehat{\tau}\}$. As

$$\psi(\widehat{\tau})\big|_X = \{(Y \setminus \widehat{V}) \bigcap X \big| \widehat{V} \in \widehat{\tau}\} = \{X \setminus (\widehat{V} \bigcap X) \big| \widehat{V} \in \widehat{\tau}\} = \{X \setminus V) \big| V \in \tau\} = \tau',$$

then $\psi(\widehat{\tau}) \in \Omega'$.

If $\widehat{\tau}' \in \Omega'$, then $\widehat{\tau} = \{Y \setminus V \big| V \in \widehat{\tau}'\} \in \Omega$ and $\psi(\widehat{\tau}) = \widehat{\tau}'$, and hence, $\psi : \Omega \to \Omega'$ is a surjective mapping.

Besides if $\widehat{\tau_1} \neq \widehat{\tau_2}$, then

$$\psi(\widehat{\tau_1}) = \{Y \setminus V \big| V \in \widehat{\tau_1}\} \neq \{Y \setminus U \big| U \in \widehat{\tau_1}\} = \psi(\widehat{\tau_2}),$$

and hence, $\psi : \Omega \to \Omega'$ is injective mapping, i.e. $\psi : \Omega \to \Omega'$ is an bijective mapping.

The proposition is completely proved.

2.6. Theorem. *Let τ' and τ'' be such topologies on finite sets X and Z, accordingly, that they are lattice isomorphic or lattice antiisomorphic. If $\widetilde{X} = X \bigcup \{y\}$ and $\widetilde{Z} = Z \bigcup \{y\}$, then the topologies τ' and τ'' have the same number of expansions on the sets $\widetilde{X}$ and $\widetilde{Z}$, accordingly.*

Proof. First we consider the case when the topologies τ' and τ'' are lattice isomorphic. Let $f : (\tau', \subseteq) \to (\tau'', \subseteq)$ be a corresponding lattice isomorphism.

If $\Omega_1 = \{(V', U') \big| V' \in \tau', U' \in \tau'$, and $U' \subseteq \bigcap\limits_{V \in \tau', V \nsubseteq V'} V\}$ and $\Omega_2 = \{(V'', U'') \big| V'' \in \tau'', U'' \in \tau''$, and $U'' \subseteq \bigcap\limits_{W \in \tau'', W \nsubseteq V''} W\}$, then we define the mapping $\Psi : \Omega_1 \to \Omega_2$ as follows: $\Psi((V', U')) = (f(V'), f(U'))$.

As $f : (\tau', \subseteq) \to (\tau'', \subseteq)$ is a lattice isomorphism, then $U \subseteq V$ if and only if $f(U) \subseteq f(V)$ for any $U, V \in \tau_1$.

If $(V', U') \in \Omega_1$, then $U' \subseteq \bigcap\limits_{V \in \tau', V \nsubseteq V'} V$, and hence,

$$f(U') \subseteq \bigcap_{V \in \tau', V \nsubseteq V'} f(V) = \bigcap_{W \in \tau'', W \nsubseteq f(V')} W,$$

i.e. $\Psi((V', U')) = (f(V'), f(U')) \in \Omega_2$.

The injectivity of the mapping $\Psi : \Omega_1 \to \Omega_2$ follows from the injectivity of the mapping $f : \tau' \to \tau''$.

If $(V'', U'') \in \Omega_2$, then $U'' \subseteq \bigcap\limits_{W\in\tau',\, W\nsubseteq V''} W$. Then

$$f^{-1}(U'') \subseteq \bigcap_{W\in\tau'',\, W\nsubseteq V''} f^{-1}(W) = \bigcap_{V\in\tau',\, V\nsubseteq f^{-1}(V')} V,$$

and hence, $(f^{-1}(V''), f^{-1}(U'')) \in \Omega_1$, and

$$\Psi((f^{-1}(V''), f^{-1}(U''))) = (f(f^{-1}(V'')), f(f^{-1}(U''))) = (V'', U'').$$

Therefore, $\Psi : \Omega_1 \to \Omega_2$ is a bijection.

So, we have proved that the sets Ω_1 and Ω_2 have the same number of elements.

From Theorems 1.1, 1.2 and 1.3 it follows that the number of expansions of the topology τ' on the set $\widetilde{X}$ is equal to the number of elements of the set Ω_1, and the number of expansions of the topology τ'' on the set $\widetilde{Z}$ is equal to the number of elements of the set Ω_2. Hence the number of expansions of the topology τ' on the set $\widetilde{X}$ is equal to the number of expansions of the topology τ'' on the set $\widetilde{Z}$.

The theorem is proved for the case when topologies τ' and τ'' are lattice isomorphic.

If the topologies τ' and τ'' are lattice antiisomorphic, then it is easy to notice that the topology $\tau_1' = \{X \setminus V \big| V \in \tau'\}$ will be lattice isomorphic to topology τ''. Then, according to proved above, the topologies τ_1' and τ'' have the same number of expansions on the sets $\widetilde{X}$ and $\widetilde{Z}$, accordingly. According to Proposition 2.5, the topologies τ_1' and τ' have the same number of expansions on the sets $\widetilde{X}$, and hence, the topologies τ_1' and τ'' have the same number of expansions on the sets $\widetilde{X}$ and $\widetilde{Z}$, accordingly.

The theorem is completely proved.

2.7. Theorem. [1] *If X is a set from n elements and $Y = X \bigcup \{y\}$, then on the set Y precisely $2^{n+1} + n - 1$ topologies are present, each of which induces the discrete topology on the set X.*

Proof. If τ is the discrete topology on the set X, then $\tau = \{V \big| V \subseteq X\}$. For any subset $V_0 \in \tau$ we consider the sets $\widetilde{U}(V_0) = \{U \in \tau \big| U \subseteq \bigcap\limits_{V\in\tau, V\nsubseteq V_0} V\}$ and $\Omega(V_0) = \{(V_0, U) \big| U \in \widetilde{U}(V_0)\}$.

The following 3 cases are possible:

1. $V_0 = X$;
2. $V_0 \in \{X \setminus \{x\} \big| x \in X\}$;

[1] The proof of this theorem given below shows the way of using the mentioned above algorithm for calculation of one-point expansions for some topologies. Though, other and probably shorter proofs of this theorem can be. The referee kindly informed authors of this work about one of such proofs.

3. $V_0 \in \tau \setminus (\{X\} \bigcup \{X \setminus \{x\} \big| x \in X\})$.

Consider each of these cases separately.

1. Let $V_0 = X$. As $\{V \in \tau \big| V \nsubseteq V_0 = X\} = \emptyset$, then the set

$$\widetilde{U}(X) = \{U \in \tau \big| U \subseteq \bigcap_{V \in \emptyset} V\} = \{U \in \tau \big| U \subseteq X\} = \tau,$$

contains precisely 2^n subsets of the set X. Then the set $\Omega(X) = \{(X, U) \big| U \in \widetilde{U}(X)\}$ contains precisely 2^n elements.

2. Let $V_0 \in \{X \setminus \{x\} \big| x \in X\}$. As $\{V \in \tau \big| V \nsubseteq X \setminus \{x\}\} = \{A \subseteq X \big| x \in A\}$, then the set

$$\widetilde{U}(X \setminus \{x\}) = \{U \in \tau \big| U \subseteq \bigcap_{V \in \{A \subseteq X \big| x \in A\}} V\} = \{U \in \tau \big| U \subseteq \{x\}\} = \{\emptyset, \{x\}\}$$

contains precisely 2 subsets of the set X. Then the set

$$\Omega(X \setminus \{x\}) = \{(X \setminus \{x\}, U) \big| U \in \widetilde{U}(X \setminus \{x\})$$

contains precisely 2 elements for any $x \in X$, and hence the set $\bigcup\limits_{x \in X} \Omega(X \setminus \{x\})$ contains precisely $2 \cdot n$ elements.

3. Now let $V_0 \in \tau \setminus (\{X\} \bigcup \{X \setminus \{x\} \big| x \in X\})$. Then $\{x_1\} \nsubseteq V_0$ and $\{x_2\} \nsubseteq V_0$ for the some elements $x_1, x_2 \in X$, and hence,

$$\widetilde{U}(V_0) = \{U \in \tau \big| U \subseteq \bigcap_{V \nsubseteq V_0} V\} \subseteq \{x_1\} \bigcap \{x_2\} = \{\emptyset\}$$

contains only $\emptyset$. Therefore the set $\Omega(V_0) = \{(V_0, \emptyset)\}$ contains precisely 1 element for any $V_0 \in \tau \setminus (\{X\} \bigcup \{X \setminus \{x\} \big| x \in X\})$. Then the set $\bigcup\limits_{V_0 \in \tau \setminus (\{X\} \bigcup \{X \setminus \{x\} \big| x \in X\})} \Omega(V_0)$ contains precisely $2^n - 1 - n$ elements.

From Theorems 1.1, 1.2 and 1.3 it follows that the number of topologies on the set $Y = X \bigcup \{y\}$ each of which induces the topology τ on the set X is equal to the number of elements of the set

$$\{(V, U) \big| V, U \in \tau, U \subseteq \bigcap_{W \in \tau, W \nsubseteq V} W\} =$$

$$\Omega(X) \bigcup \Omega(X \setminus \{x\}) \bigcup \Big(\bigcup_{V_0 \in \tau \setminus (\{X\} \bigcup \{X \setminus \{x\} \big| x \in X\})} \Omega(V_0) \Big),$$

i.e. it is equal to $2^n + 2 \cdot n + 2^n - 1 - n = 2^{n+1} + n - 1$.

The theorem is completely proved.

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Topological rings with at most two nontrivial closed ideals

Valeriu Popa

Abstract. In this paper, we describe the Hausdorff topological rings with identity in which every nontrivial closed ideal is topologically maximal, respectively, strongly topologically maximal, and the Hausdorff topological rings with identity which have no more than two nontrivial closed ideals.

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Keywords and phrases: Topological rings, closed ideals, topologically maximal ideals.

Introduction

In [4], F. Perticani determined the structure of (discrete) commutative rings with identity in which every nontrivial ideal (i. e., distinct from the zero ideal and the whole ring) is maximal. He proved that such a ring, E, has at most two distinct nontrivial ideals, and if E is not simple, then either it is isomorphic to a product of two fields or it is obtained as extension of a one-dimensional vector space over some field, considered as ring with zero multiplication, by the same field in such a way that the mentioned vector space structure coincides with the structure determined by the exact sequence defining the corresponding extension.

We consider here analogous questions in the more general context of topological rings. To be precise, we describe the (not necessarily commutative) topological rings with identity in which every nontrivial closed ideal is topologically maximal, respectively, strongly topologically maximal. We also determine the topological rings with identity which have no more than two nontrivial closed ideals.

Throughout the paper, all topological rings considered are assumed to be Hausdorff. If E is a topological ring and A is an ideal of E, we denote by $\overline{A}$ the closure of A in E, by $ann_E(A)$ the annihilator of A in E, and by $ann^l_E(A)$ and $ann^r_E(A)$ the left annihilator and the right annihilator of A in E, respectively. If B is a closed ideal of E satisfying $A \subset B$, we denote by $ann_E(B/A)$ the annihilator of the quotient E-bimodule B/A in E. Also, the symbol $\cong$ stands for topological isomorphism.

1 Topological rings in which every nontrivial closed ideal is topologically maximal

As mentioned in Introduction, F. Perticani described in his paper [4] the commutative rings with identity in which every nontrivial ideal is maximal. The purpose of the present paper is to extend the results obtained in [4] to topological rings. We begin by introducing, for topological rings, the analogue of the notion of maximal ideal.

Definition 1. Let E be a topological ring. A closed ideal M of E is said to be topologically maximal if M is proper (i. e., $M \neq E$) and for every closed ideal C of E such that $M \subset C$, either $C = M$ or $C = E$.

Definition 2. A topological ring E is said to be topologically simple in case E is nonzero and has no nontrivial closed (two-sided) ideals.

We will need the following analogue of the well known characterization of maximal ideals.

Lemma 1. *Let E be a topological ring. A closed ideal M of E is topologically maximal if and only if E/M is topologically simple.*

Proof. Let M be a closed ideal of E, and let π denote the canonical projection of E onto E/M.

If M is topologically maximal and if C' is a closed ideal of E/M, then $\pi^{-1}(C')$ is a closed ideal of E and $M \subset \pi^{-1}(C')$, so that $\pi^{-1}(C')$ coincides with either M or E. As $C' = \pi(\pi^{-1}(C'))$, it follows that C' coincides with either the zero ideal or the whole ring E/M.

For the converse, let C be a closed ideal of E such that $M \subset C$. Then $(E/M) \setminus \pi(C) = \pi(E \setminus C)$. Since π is open, it follows that $\pi(C)$ is closed in E/M, and hence $\pi(C)$ coincides with either the zero ideal or E/M. As $C = \pi^{-1}(\pi(C))$, we conclude that either $C = M$ or $C = E$. $\square$

We proceed now to study the structure of topological rings in which every nontrivial closed ideal is topologically maximal.

Lemma 2. *Let E be a topological ring in which every nontrivial closed ideal is topologically maximal. If A and B are different nontrivial closed ideals of E, then $\overline{A + B} = E$ and $A \cap B = \{0\}$.*

Proof. Since A and B are contained in $\overline{A + B}$, the relation $\overline{A + B} \neq E$ would imply $A = \overline{A + B} = B$, because A and B have to be topologically maximal. Similarly, since $A \cap B$ is contained in A and in B, the relation $A \cap B \neq \{0\}$ would imply $A = A \cap B = B$, because $A \cap B$ has to be topologically maximal. $\square$

Lemma 3. *Let E be a topological ring with identity, and let A and B be nontrivial closed ideals of E such that $\overline{A + B} = E$ and $A \cap B = \{0\}$. Then $ann_E(A) = B$ and $ann_E(B) = A$.*

Proof. Since AB and BA are contained in $A \cap B$, we have $A \subset ann_E(B)$ and $B \subset ann_E(A)$. To show the inverse inclusions, pick any $u \in ann_E(B)$ and $v \in ann_E(A)$. Since $\overline{A+B} = E$, we can write $1 = \lim_{\lambda \in L}(a_\lambda + b_\lambda)$, where $(a_\lambda)_{\lambda \in L}$ is a net in A and $(b_\lambda)_{\lambda \in L}$ is a net in B [2, Proposition 1.6.3.]. It follows that

$$u = u \lim_{\lambda \in L}(a_\lambda + b_\lambda) = \lim_{\lambda \in L} ua_\lambda \in A$$

and

$$v = v \lim_{\lambda \in L}(a_\lambda + b_\lambda) = \lim_{\lambda \in L} vb_\lambda \in B.$$

Consequently, $ann_E(A) = B$ and $ann_E(B) = A$. □

With these preparations, we have

Theorem 1. *A topological ring with identity in which every nontrivial closed ideal is topologically maximal cannot have more than two different nontrivial closed ideals.*

Proof. Let E be a topological ring with identity in which every nontrivial closed ideal is topologically maximal, and assume A, B and C are different nontrivial closed ideals of E. By Lemma 2, we have $\overline{A+B} = E$, so that $1 = \lim_{\lambda \in L}(a_\lambda + b_\lambda)$, where $(a_\lambda)_{\lambda \in L}$ is a net in A and $(b_\lambda)_{\lambda \in L}$ is a net in B. Pick any nonzero $c \in C$. The multiplication by c being continuous, it follows that

$$c = c \cdot \lim_{\lambda \in L}(a_\lambda + b_\lambda) = \lim_{\lambda \in L} c \cdot (a_\lambda + b_\lambda) \in \overline{C \cdot A + C \cdot B}.$$

But $C \cdot A \subset C \cap A$ and $C \cdot B \subset C \cap B$. Since $C \cap A = \{0\} = C \cap B$ by Lemma 2 and since E is Hausdorff, this proves that $\overline{CA + CB} = \{0\}$, so $c = 0$, a contradiction. Consequently, E cannot have more than two different nontrivial closed ideals. □

Next we consider the case of topological rings with two nontrivial closed ideals.

Theorem 2. *Let E be a topological ring with identity having two different nontrivial closed ideals. The following statements are equivalent:*

(i) *E has exactly two different nontrivial closed ideals, and these ideals are not comparable with respect to inclusion.*

(ii) *Every nontrivial closed ideal of E is topologically maximal.*

(iii) *There exist two different nontrivial closed ideals A, B of E such that the following conditions hold:*

 (1) *$\overline{A+B} = E$ and $A \cap B = \{0\}$;*

 (2) *A and B are topologically simple rings.*

Proof. Clearly, (i) implies (ii). Assume (ii), and let A and B be different nontrivial closed ideals of E. The fact that condition (1) of (iii) is satisfied follows from Lemma 2. In particular, we can write $1 = \lim_{\lambda\in L}(a_\lambda + b_\lambda)$, where $(a_\lambda)_{\lambda\in L}$ is a net in A and $(b_\lambda)_{\lambda\in L}$ is a net in B. It also follows from Lemma 3 that $ann_E(A) = B$ and $ann_E(B) = A$. To see that A is a topologically simple ring, let I be an arbitrary nonzero closed ideal of A. For any $x \in E$ and $y \in I$, we have

$$xy = x[\lim_{\lambda\in L}(a_\lambda + b_\lambda)]y = x(\lim_{\lambda\in L} a_\lambda y) = \lim_{\lambda\in L}(xa_\lambda)y \in I$$

and

$$yx = y[\lim_{\lambda\in L}(a_\lambda + b_\lambda)]x = (\lim_{\lambda\in L} ya_\lambda)x = \lim_{\lambda\in L} y(a_\lambda x) \in I,$$

so I is an ideal of E. In view of (ii), we must have $I = A$. The proof that B is a topologically simple ring is similar, so condition (2) of (iii) also holds.

Assume (iii). The ideals A and B, whose existence is claimed in (iii), cannot be comparable with respect to inclusion because $A \cap B = \{0\}$. It also follows from Lemma 3 that $ann_E(A) = B$ and $ann_E(B) = A$. To see that A and B are the unique different nontrivial closed ideals of E, pick an arbitrary closed ideal C of E. Then $A \cap C$ is a closed ideal of A and $B \cap C$ is a closed ideal of B. Since A and B are topologically simple rings, it follows that $A \cap C$ coincides with either $\{0\}$ or A and $B \cap C$ coincides with either $\{0\}$ or B. We distinguish cases. If $A \cap C = A$ and $B \cap C = B$, we have $A \subset C$ and $B \subset C$, so that $E = \overline{A + B} \subset C$, and hence in this case $C = E$. Next assume $A \cap C = \{0\}$ and $B \cap C = \{0\}$. Since $AC,\ CA \subset A \cap C$, we have $AC = \{0\} = CA$, so that $C \subset ann_E(A) = B$. In a similar way, $C \subset ann_E(B) = A$. As $A \cap B = \{0\}$, it follows that in this case $C = \{0\}$. Now assume $A \cap C = \{0\}$ and $B \cap C = B$. As we have seen, the relation $A \cap C = \{0\}$ gives $C \subset B$. Since the relation $B \cap C = B$ gives $B \subset C$, it follows that in this case $C = B$. Finally, if $A \cap C = A$ and $B \cap C = \{0\}$, we get in a similar way $C = A$. Consequently, E admits only two different nontrivial closed ideals, namely A and B. □

In view of Theorem 2, it would be interesting to know when a topological ring with exactly two nontrivial closed ideals is topologically isomorphic to the direct product of those ideals. To answer this question, we need a new

Definition 3. Let E be a topological ring and M a closed ideal of E. We say M is strongly topologically maximal if M is topologically maximal and if for any closed ideal C of E, $M + C$ is closed in E.

Lemma 4. *Let E be a topological ring. A proper closed ideal M of E is strongly topologically maximal if and only if for each closed ideal C of E such that $C \not\subset M$ one has $M + C = E$.*

Proof. Assume M is strongly topologically maximal, and let C be an arbitrary closed ideal of E such that $C \not\subset M$. Since $M + C$ is closed in E and properly contains M, we must have $M + C = E$.

Assume the converse. Given an arbitrary closed ideal C of E, we then have $M + C = M$ if $C \subset M$ and $M + C = E$ if $C \not\subset M$, so that $M + C$ is closed in E. It is also clear that M is topologically maximal. □

We have the following

Theorem 3. *Let E be a topological ring with identity having two different nontrivial closed ideals A and B. Every nontrivial closed ideal of E is strongly topologically maximal if and only if A and B are topologically simple rings, and $E \cong A \times B$.*

Proof. If every nontrivial closed ideal of E is strongly topologically maximal, it follows from Theorem 2 that $A \cap B = \{0\}$, $\overline{A + B} = E$, and A, B are topologically simple rings. Further, $\overline{A + B} = A + B$ by Lemma 4, and hence $E \cong A \times B$ by [1, Ch. III, §6, Exer. 6].

Now assume that A and B are topologically simple rings, and that there is an isomorphism of topological rings $h : E \to A \times B$. Set $A' = h^{-1}(A \times \{0\})$ and $B' = h^{-1}(\{0\} \times B)$. It follows that $A' + B' = E$ and $A' \cap B' = \{0\}$, so that, by Theorem 2, A' and B' are the only nontrivial closed ideals of E. In particular $\{A', B'\} = \{A, B\}$. If C is an arbitrary closed ideal of E such that $C \not\subset A$, then C coincides with either B or E, so that $A + C = E$, and hence A is strongly topologically maximal by Lemma 4. Clearly, the same holds also for B. □

2 Topological bimodule structures induced by ideal extensions

Let $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$ be an exact sequence of abstract rings and homomorphisms of rings, that is such that $\ker(\varphi) = \{0\}$, $\mathrm{im}(\varphi) = \ker(\psi)$, and $\mathrm{im}(\psi) = B$. As is well known (see [3] or [4]), if $A^2 = \{0\}$, then A can be given a bimodule structure over B.

We establish here a topological version of this fact.

Definition 4. Let A and B be arbitrary topological rings. A topological ring E is said to be an ideal extension of A by B if there exist continuous ring homomorphisms $\varphi : A \to E$ and $\psi : E \to B$ such that the following conditions hold:

(i) φ is injective and open onto its image;

(ii) ψ is surjective and open;

(iii) $\mathrm{im}(\varphi) = \ker(\psi)$.

If, in addition, E has an identity, then it is called a unital ideal extension of A by B.

Clearly, if $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$ is a unital ideal extension of A by B, then B has an identity too and ψ is unital.

As usual, when we want to emphasize explicitly the homomorphisms $\varphi : A \to E$ and $\psi : E \to B$ making E an ideal extension of A by B, we identify E with the sequence $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$.

Lemma 5. *Let $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$ be an ideal extension of A by B. Then $ann_A(A)$ can be turned into a topological bimodule over B.*

Proof. The multiplication of E determines a B-bimodule structure on $ann_A(A)$ in the following way. Let $a \in ann_A(A)$ and $b \in B$ be arbitrary. Since ψ is surjective, there is $c \in E$ such that $b = \psi(c)$. Then $\varphi(a)c$ and $c\varphi(a)$ belong to $\varphi(A)$ because $\varphi(A)$ is an ideal of E. Given any $x \in A$, we have $(\varphi(a)c)\varphi(x) = \varphi(a)(c\varphi(x)) = 0$ and $\varphi(x)(\varphi(a)c) = \varphi(xa)c = \varphi(0)c = 0$, so that in fact $\varphi(a)c \in ann_{\varphi(A)}\big(\varphi(A)\big)$. Similarly, $c\varphi(a) \in ann_{\varphi(A)}\big(\varphi(A)\big)$. Set $ab = \varphi^{-1}\big(\varphi(a)c\big)$ and $ba = \varphi^{-1}\big(c\varphi(a)\big)$. To see that the products ab and ba are well defined, let c' be another element in E such that $\psi(c') = b$. Then $c - c' \in \ker(\psi) = \mathrm{im}(\varphi)$, and since $a \in ann_A(A)$ and hence $\varphi(a) \in ann_{\varphi(A)}\big(\varphi(A)\big)$, we have $\varphi(a)(c - c') = 0 = (c - c')\varphi(a)$. Consequently, ab and ba are well defined. It is now easy to see that $ann_A(A)$ is a bimodule over B, with respect to its addition induced from A and scalar multiplications defined above. Moreover, the addition is, clearly, continuous.

Let us show that the left scalar multiplication is continuous. The case of the right scalar multiplication is similar. Fix any elements $a \in ann_A(A)$ and $b \in B$, and any neighbourhood V of zero in A. Also choose $c \in E$ such that $\psi(c) = b$. Since φ is open onto its image, $\varphi(V)$ is a neighbourhood of zero in $\varphi(A)$. Now, since $\varphi(A)$ is a topological left E-module, there exist a neighbourhood U of zero in E and a neighbourhood W of zero in $\varphi(A)$ such that

$$UW \subset \varphi(V), \quad U\varphi(a) \subset \varphi(V) \quad \text{and} \quad cW \subset \varphi(V).$$

As φ is continuous and ψ is open, $\varphi^{-1}(W)$ is a neighbourhood of zero in A and $\psi(U)$ is a neighbourhood of zero in B. By the definition of the left scalar multiplication, we then have

$$\psi(U)\big(\varphi^{-1}(W) \cap ann_A(A)\big) \subset V \cap ann_A(A), \qquad \psi(U)a \subset V \cap ann_A(A)$$

and

$$b\big(\varphi^{-1}(W) \cap ann_A(A)\big) \subset V \cap ann_A(A),$$

so the left scalar multiplication $(\beta, \alpha) \to \beta\alpha$ from $B \times ann_A(A)$ to $ann_A(A)$ is continuous at $(0, 0)$, and the mappings $\beta \to \beta a$ from B to $ann_A(A)$ and $\alpha \to b\alpha$ from $ann_A(A)$ to $ann_A(A)$ are continuous at 0. Since a and b were arbitrary, it follows from [5, (2.16)] that the left scalar multiplication is continuous. □

Definition 5. The topological B-bimodule structure of $ann_A(A)$ described above will be referred to as the topological B-bimodule structure determined by the sequence $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$.

Corollary 1. *Let A and B be topological rings, and let $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$ be an ideal extension of A by B.*

(i) *If A contains a closed ideal K such that $(A/K)^2 = \{0\}$, then A/K can be turned into a topological bimodule over B.*

(ii) *If B_0 is a closed ideal of B such that $\psi^{-1}(B_0)^2 = \{0\}$, then $\psi^{-1}(B_0)$ can be turned into a topological bimodule over B/B_0.*

Proof. (i) Let $\lambda : A \to A/K$ and $\varrho : E \to E/K$ be the canonical projections. As is well known, there exist continuous ring homomorphisms $\hat{\varphi} : A/K \to E/K$ and $\hat{\psi} : E/K \to B$ such that $\varrho \circ \varphi = \hat{\varphi} \circ \lambda$ and $\psi = \hat{\psi} \circ \varrho$. Moreover, $\hat{\varphi}$ and $\hat{\psi}$ are open onto their images,

$$\ker(\hat{\varphi}) = \ker(\varrho \circ \varphi)/K = \{0\}, \qquad \operatorname{im}(\hat{\psi}) = \operatorname{im}(\psi)$$

and

$$\ker(\hat{\psi}) = \ker(\psi)/K = \varphi(A)/K = \operatorname{im}(\hat{\varphi}).$$

Consequently, the homomorphisms $\hat{\varphi} : A/K \to E/K$ and $\hat{\psi} : E/K \to B$ make E/K an ideal extension of A/K by B. Since $(A/K)^2 = \{0\}$, it follows from Lemma 5 that A/K can be given a topological bimodule structure over B.

(ii) Let $\eta : \psi^{-1}(B_0) \to E$ be the canonical injection of $\psi^{-1}(B_0)$ into E and $\pi : B \to B/B_0$ the canonical projection of B onto B/B_0. Then η and $\pi \circ \psi$ are open onto their images, η is injective, $\pi \circ \psi$ is surjective, and $\operatorname{im}(\eta) = \psi^{-1}(B_0) = \ker(\pi \circ \psi)$, so that η and $\pi \circ \psi$ transform E into an ideal extension of $\psi^{-1}(B_0)$ by B/B_0. By Lemma 5, $\psi^{-1}(B_0)$ can be given a topological bimodule structure over B/B_0. □

Definition 6. The topological B-bimodule structure of A/K described above is referred to as the topological B-bimodule structure determined on A/K by the sequence $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$.

Similarly, the topological B/B_0-bimodule structure of $\psi^{-1}(B_0)$ described above is referred to as the topological B/B_0-bimodule structure determined on $\psi^{-1}(B_0)$ by the sequence $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$.

Lemma 6. *Let $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$ be an ideal extension of A by B. Then $ann^l_A(A)$ can be turned into a topological right B-module. Similarly, $ann^r_A(A)$ can be turned into a topological left B-module.*

Proof. The multiplication by scalars in $ann^l_A(A)$ (respectively, $ann^r_A(A)$) is given by $ab = \varphi^{-1}(\varphi(a)c)$ (respectively, $ba = \varphi^{-1}(c\varphi(a))$) for $a \in ann^l_A(A)$ (respectively, $a \in ann^r_A(A)$), $b \in B$, and $c \in E$ with $b = \psi(c)$. □

Definition 7. The topological B-module structure of $ann^l_A(A)$ (respectively, $ann^r_A(A)$) described above is referred to as the topological B-module structure determined on $ann^l_A(A)$ (respectively, $ann^r_A(A)$) by the sequence $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$.

3 Topological rings with only one nontrivial closed ideal

In this section, we relate the study of topological rings with only one nontrivial closed ideal to an extension problem, although the cohomology theory for topological rings is not constructed yet.

We use the following simple

Lemma 7. *Let E be a ring and A a nonzero ideal of E. If*

$$ann^l_E(A) = \{0\} = ann^r_E(A),$$

then for any nonzero $x \in A$, AxA is nonzero.

Proof. Pick any nonzero $x \in A$. Since $x \notin ann^r_E(A)$, there exists $a \in A$ such that $ax \neq 0$. Similarly, since $ax \notin ann^l_E(A)$, there exists $a' \in A$ such that $axa' \neq 0$. Hence $AxA \neq \{0\}$. □

Definition 8. Let E be a topological ring. A topological module (respectively, bimodule) A over E is said to be topologically simple in case A is nonzero and has no nontrivial closed submodules (respectively, subbimodules).

Theorem 4. *Let E be a topological ring with identity having only one nontrivial closed ideal A. Then E/A is a unital topologically simple ring, and E can be viewed as an ideal extension $A \xrightarrow{\eta} E \xrightarrow{\pi} E/A$ of A by E/A, where η is the canonical injection and π is the canonical projection, such that exactly one of the following conditions hold:*

(i) *$ann_E(A) = \{0\}$ and A is a topologically simple ring;*

(ii) *$ann_E(A) = A$ and A, with the structure given by the sequence $A \xrightarrow{\eta} E \xrightarrow{\pi} E/A$, is a unital topologically simple E/A-bimodule.*

Proof. Consider the natural exact sequence $A \xrightarrow{\eta} E \xrightarrow{\pi} E/A$, where η is the canonical injection and π is the canonical projection. As is well known, η and π are continuous and open onto their images. Now, since A is the only nontrivial closed ideal of E, it is clear that E/A is a unital topologically simple ring. Further, since E has an identity, we cannot have $ann_E(A) = E$, so that either $ann_E(A) = \{0\}$ or $ann_E(A) = A$. Assume the former, and consider the one-sided annihilators $ann^l_E(A)$ and $ann^r_E(A)$. Clearly, $ann^l_E(A)$ and $ann^r_E(A)$ are closed ideals of E. As in the case of $ann_E(A)$, we have $ann^l_E(A) \neq E$ and $ann^r_E(A) \neq E$. On the other hand, either of equalities $ann^l_E(A) = A$ or $ann^r_E(A) = A$ implies $ann_E(A) = A$, in contradiction with our assumption that $ann_E(A) = \{0\}$. Therefore we must have $ann^l_E(A) = \{0\} = ann^r_E(A)$. To see that A is a topologically simple ring, pick an arbitrary nonzero closed ideal B of A and any nonzero element $b \in B$. By Lemma 7, $\overline{AbA}$ is a nonzero closed ideal of A satisfying $\overline{AbA} \subset B$. Since A is an ideal of E, it then follows that $\overline{AbA}$ is a nonzero closed ideal of E, whence $\overline{AbA} = A$, so $B = A$. Consequently, A is a topologically simple ring, and hence in this case we are led to (i).

Now consider the latter case when $ann_E(A) = A$. By using the sequence $A \xrightarrow{\eta} E \xrightarrow{\pi} E/A$, it follows from Lemma 5 that A can be turned into a topological bimodule over E/A. Moreover, if $a \in A$, then $a \cdot \pi(1) = a \cdot 1 = a$ and $\pi(1) \cdot a = 1 \cdot a = a$, so this bimodule is unital. Pick an arbitrary nonzero closed E/A-subbimodule C of A. Taking into account the definition of scalar multiplications, we see that for any $x \in E$,

$$xC = \pi(x)C \subset C \qquad \text{and} \qquad Cx = C\pi(x) \subset C,$$

so C is a nonzero closed ideal of E contained in A, whence $C = A$. Since the E/A-subbimodule C was picked arbitrarily, it follows that A is a topologically simple E/A-bimodule, and hence in this case we have (ii). $\square$

We next show that the converse is also true.

Theorem 5. *Let A and B be topologically simple rings, and let $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$ be an ideal extension of A by B. If $ann_E(\varphi(A)) = \{0\}$, then E has only one nontrivial closed ideal, namely $\varphi(A)$.*

Proof. Clearly, $\varphi(A)$ is a nontrivial closed ideal of E. Moreover, if K is a closed ideal of E such that $\varphi(A) \subset K$, then $B \setminus \psi(K) = \psi(E \setminus K)$ is open in B, so that $\psi(K)$ is closed in B. Since B is topologically simple, it follows that either $\psi(K) = \{0\}$ or $\psi(K) = B$, and hence either $K = A$ or $K = E$. Consequently, the ideal $\varphi(A)$ is topologically maximal.

Now, let C be an arbitrary closed ideal of E. Then $\varphi(A) \cap C$ is a closed ideal of $\varphi(A)$. If $\varphi(A) \cap C \neq \{0\}$, we must have $\varphi(A) \cap C = \varphi(A)$ because $\varphi(A)$ is a topologically simple ring. It follows that $\varphi(A) \subset C$, and hence C coincides with either $\varphi(A)$ or E because $\varphi(A)$ is topologically maximal in E. Suppose $\varphi(A) \cap C = \{0\}$. Since $\varphi(A)C$ and $C\varphi(A)$ are contained in $\varphi(A) \cap C$, it follows that $\varphi(A)C = \{0\} = C\varphi(A)$, so $C \subset ann_E\big(\varphi(A)\big)$, and hence $C = \{0\}$. Thus E has only one nontrivial closed ideal. $\square$

Theorem 6. *Let A be a topological ring with $ann_A(A) = A$, let B be a topologically simple ring with identity, and let $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$ be a unital ideal extension of A by B. If A is a topologically simple B-bimodule relative to the bimodule structure determined by the sequence $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$, then E has only one nontrivial closed ideal, namely $\varphi(A)$.*

Proof. Clearly, $\varphi(A)$ is a nonzero closed ideal of E. Moreover, since $E/\varphi(A)$ is topologically isomorphic to B, $\varphi(A)$ is topologically maximal by Lemma 1. Let C be a nonzero closed ideal of E. It is easy to see that $\varphi(A) \cap C$ is then a B-subbimodule of $\varphi(A)$. We cannot have $\varphi(A) \cap C = \{0\}$. For, otherwise it would follow that $\overline{\varphi(A) + C} = E$, since $\overline{\varphi(A) + C}$ would then properly contain $\varphi(A)$. Hence there would exist a net $(a_\lambda)_{\lambda \in L}$ of elements in A and a net $(c_\lambda)_{\lambda \in L}$ of elements in C with $\lim_{\lambda \in L}\big((\varphi(a_\lambda) + c_\lambda\big) = 1$. For any $\lambda, \lambda' \in L$, we would have

$$\big(\varphi(a_\lambda) + c_\lambda\big)\big(\varphi(a_{\lambda'}) + c_{\lambda'}\big) = \varphi(a_\lambda)c_{\lambda'} + c_\lambda\varphi(a_{\lambda'}) + c_\lambda c_{\lambda'} \in C$$

since $\varphi(A)$ has zero multiplication. Taking the limit first relative to λ and then relative to λ', we would obtain that $1 \in C$, so $C = E$, in contradiction with our assumption that $\varphi(A) \cap C = \{0\}$. Thus $\varphi(A) \cap C \neq \{0\}$, and hence $\varphi(A) \cap C = \varphi(A)$ because $\varphi(A)$ is a topologically simple B-bimodule. It follows that $\varphi(A) \subset C$, so that C must coincide with either $\varphi(A)$ or E, because $\varphi(A)$ is topologically maximal in E. Consequently, E has only one nontrivial closed ideal. □

4 Topological rings with only two different nontrivial closed ideals

In this section, we turn our attention to topological rings with exactly two nontrivial closed ideals. First we consider the case when the corresponding ideals are incomparable with respect to inclusion or, equivalently, disjoint.

Theorem 7. *Let E be a topological ring with identity having only two different nontrivial closed ideals. Assume that these ideals are not comparable with respect to inclusion, and let A denote one of them. Then A is a topologically simple ring, $ann_E(A) \neq \{0\}$, E/A is a topologically simple ring with identity, and E can be viewed as a unital ideal extension $A \xrightarrow{\eta} E \xrightarrow{\pi} E/A$ of A by E/A, where η is the canonical injection and π is the canonical projection.*

Proof. The assertion follows from Theorem 2 and Lemma 3. □

Theorem 8. *Let A be a topologically simple ring, let B be a topologically simple ring with identity, and let $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$ be a unital ideal extension of A by B such that $ann_E\big(\varphi(A)\big) \neq \{0\}$. Then E has exactly two nontrivial closed ideals, namely $\varphi(A)$ and $ann_E\big(\varphi(A)\big)$.*

Proof. Clearly, $\varphi(A)$ is a nontrivial closed ideal of E. Moreover, since $E/\varphi(A) \cong B$, $\varphi(A)$ is topologically maximal in E. Further, since E is unital, we must have $ann_E\big(\varphi(A)\big) \neq E$, so $ann_E\big(\varphi(A)\big)$ is a nontrivial closed ideal of E as well.

Let C be an arbitrary nonzero closed ideal of E. Then $\varphi(A) \cap C$ is a closed ideal of $\varphi(A)$. Since $\varphi(A)$ is topologically simple, it follows that either $\varphi(A) \cap C = \{0\}$ or $\varphi(A) \cap C = \varphi(A)$. Assume the former holds. Since $\varphi(A)C$ and $C\varphi(A)$ are contained in $\varphi(A) \cap C$, we conclude that $C \subset ann_E\big(\varphi(A)\big)$. But, since C is nonzero, $\overline{\varphi(A) + C}$ properly contains $\varphi(A)$, so $\overline{\varphi(A) + C} = E$. It follows that

$$\begin{aligned} ann_E\big(\varphi(A)\big) &= E \cdot ann_E\big(\varphi(A)\big) = \overline{(\varphi(A) + C) \cdot ann_E\big(\varphi(A)\big)} \\ &= \overline{C \cdot ann_E\big(\varphi(A)\big)} \subset C, \end{aligned}$$

and hence $C = ann_E\big(\varphi(A)\big)$.

In the latter case when $\varphi(A) \cap C = \varphi(A)$, we have $\varphi(A) \subset C$. Since $\varphi(A)$ is topologically maximal in E, it follows that C coincides with either $\varphi(A)$ or E. □

In the following, we consider the case of topological rings with exactly two nontrivial closed ideals and such that the corresponding ideals are comparable with respect to inclusion. We first determine under what conditions the topological rings of this type can be realized as ideal extensions of a topologically simple ring by a topological ring with only one nontrivial closed ideal.

Theorem 9. *Let E be a topological ring with identity having only two different nontrivial closed ideals A and B. If $A \subset B$, then E/A is a topological ring with identity containing only one nontrivial closed ideal, and E can be viewed as an ideal extension $A \xrightarrow{\eta} E \xrightarrow{\pi} E/A$ of A by E/A, where η is the canonical injection and π is the canonical projection, such that exactly one of the following conditions hold:*

(i) *$ann_E(A) = \{0\}$ and A is a topologically simple ring;*

(ii) *$ann_E(A) = A$ and A, with the structure given by the sequence $A \xrightarrow{\eta} E \xrightarrow{\pi} E/A$, is a topologically simple (E/A)-bimodule;*

(iii) *$ann_E(A) = B$, $\overline{B^2}$ coincides with either A or B, and A, with the structure given by the sequence $A \xrightarrow{\eta} E \xrightarrow{\pi} E/A$, is a topologically simple (E/A)-bimodule;*

(iv) *$B^2 = \{0\}$ and the topological E/B-bimodule B, determined by the sequence $A \xrightarrow{\eta} E \xrightarrow{\pi} E/A$, has only one nontrivial closed subbimodule.*

Proof. Since A and B are the only nontrivial closed ideals of the unital ring E, it follows that $ann_E(A)$ coincides with one of the ideals $\{0\}$, A, or B. Now, if $ann_E(A) = \{0\}$, we must have $ann^l_E(A) = \{0\} = ann^r_E(A)$. For, if one of the ideals $ann^l_E(A)$ or $ann^r_E(A)$ coincided with either A or B, it would follow that $ann_E(A) \neq \{0\}$. Pick an arbitrary nonzero closed ideal C of A, and let $c \in C$ be a nonzero element. It follows from Lemma 7 that $\overline{AcA}$ is a nonzero closed ideal of A and hence of E, so $\overline{AcA} = A$, whence $C = A$. Consequently, A is a topologically simple ring, and hence in this case we are led to (i). Next, if $ann_E(A) = A$, it follows from Lemma 5 that A can be turned into a topological bimodule over E/A. Since every closed subbimodule of A is a closed ideal of E, we deduce that A is a topologically simple E/A-bimodule. Thus in this case we have (ii). Further, assume $ann_E(A) = B$. If $B^2 \neq \{0\}$, it follows from our hypothesis that $\overline{B^2}$ coincides with either A or B. Since, as above, A can be turned into a topologically simple E/A-bimodule, in this case we must have (iii). Finally, if $B^2 = \{0\}$, it follows from Corollary 1 that B can be turned into a topological bimodule over $(E/A)/(B/A) \cong E/B$. Let C be a closed subbimodule of B. We see that for any $x \in E$,

$$xC = \big((x+A) + B/A\big)C \subset C \quad \text{and} \quad Cx = C\big((x+A) + B/A\big) \subset C.$$

It follows that C is a closed ideal of E contained in B, so C must coincides with one of the ideals $\{0\}$, A, or B. Consequently, the topological E/A-bimodule B has only one nontrivial closed subbimodule, and hence in this case we are led to (iv). □

Theorem 10. *Let A be a nonzero topological ring, let B be a topological ring with identity having only one nontrivial closed ideal B_0, and let $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$ be a unital ideal extension of A by B satisfying one of the following conditions:*

(i) $ann_E(\varphi(A)) = \{0\}$ *and* A *is a topologically simple ring;*

(ii) $ann_E(\varphi(A)) = \varphi(A)$ *and* A *is a topologically simple* B*-bimodule relative to the structure given by the sequence* $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$;

(iii) $ann_E(\varphi(A)) = \psi^{-1}(B_0)$, $\overline{\psi^{-1}(B_0)^2}$ *coincides with either* $\varphi(A)$ *or* $\psi^{-1}(B_0)$, *and* A *is a topologically simple* B*-bimodule relative to the structure given by the sequence* $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$;

(iv) $\psi^{-1}(B_0)^2 = \{0\}$ *and the topological* (B/B_0)*-bimodule* $\psi^{-1}(B_0)$, *determined by the sequence* $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$, *has only one nontrivial closed subbimodule.*

Then E *has exactly two nontrivial closed ideals, namely* $\varphi(A)$ *and* $\psi^{-1}(B_0)$.

Proof. It is clear that $\varphi(A) \neq \{0\}$ and that $\psi^{-1}(B_0)$ is the only closed ideal of E satisfying $\varphi(A) \subsetneq \psi^{-1}(B_0) \subsetneq E$. Pick an arbitrary closed ideal C of E. If $C \cap \varphi(A) = \varphi(A)$, then $\varphi(A) \subset C$, so that C coincides with one of the ideals $\varphi(A)$, $\psi^{-1}(B_0)$, or E. Assume $C \cap \varphi(A) \neq \varphi(A)$. We shall show that, in any of cases (i)-(iv), $C = \{0\}$. First observe that we must have $C \cap \varphi(A) = \{0\}$. Indeed, this is clear in case (i) holds, since then $C \cap \varphi(A)$ is a closed ideal of the topologically simple ring $\varphi(A)$. Further, in either of cases (ii) or (iii) $C \cap \varphi(A)$ is a closed B-subbimodule of the topologically simple B-bimodule $\varphi(A)$, and so $C \cap \varphi(A) = \{0\}$. Finally, in case (iv) holds, it is clear that $C \cap \varphi(A)$ is a closed (B/B_0)-subbimodule of $\psi^{-1}(B_0)$, so $C \cap \varphi(A) = \{0\}$ because $\psi^{-1}(B_0)$ has only one nontrivial closed subbimodule, namely $\varphi(A)$. This proves that in any of cases (i)-(iv), $C \cap \varphi(A) = \{0\}$. Now, since $C \cdot \varphi(A)$ and $\varphi(A) \cdot C$ are contained in $C \cap \varphi(A)$, it follows that $C \subset ann_E(\varphi(A))$. In particular, $C = \{0\}$ if (i) holds. In case (ii) holds, C becomes a closed subbimodule of the topologically simple B-bimodule $\varphi(A)$, so again $C = \{0\}$. Further, in case (iv) holds, we clearly have $ann_E(\varphi(A)) = \psi^{-1}(B_0)$, so $C = \{0\}$ by our hypothesis that $\varphi(A)$ is the only nontrivial closed (B/B_0)-subbimodule of $\psi^{-1}(B_0)$ and the fact that $C \cap \varphi(A) = \{0\}$. Assume (iii). If we had $C \neq \{0\}$, it would follow that $\overline{C + \varphi(A)} = \psi^{-1}(B_0)$, which would imply

$$\overline{C^2} = \overline{(\overline{C + \varphi(A)})(\overline{C + \varphi(A)})} = \overline{\psi^{-1}(B_0)^2}.$$

But then, in case $\overline{\psi^{-1}(B_0)^2} = \varphi(A)$, we would have $\varphi(A) = \overline{C^2} \subset C \cap \varphi(A)$. Similarly, in case $\overline{\psi^{-1}(B_0)^2} = \psi^{-1}(B_0)$, we would have $C = \psi^{-1}(B_0)$. In both cases the derived conclusion is in contradiction with the fact that $C \cap \varphi(A) = \{0\}$. $\square$

Next we complete the picture by determining under what conditions topological rings with exactly two nontrivial closed ideals can be realized as extensions of a topological ring with only one nontrivial closed ideal by a topologically simple ring.

Definition 9. Let E be a topological ring. A closed ideal M of E is said to be a topologically minimal ideal of E if $M \neq \{0\}$ and for every closed ideal C of E such that $C \subset M$, either $C = \{0\}$ or $C = M$.

Theorem 11. *Let E be a topological ring with identity having only two different nontrivial closed ideals A and B. If $A \subset B$, then E/B is a topologically simple ring with identity, and E can be viewed as an ideal extension $B \xrightarrow{\eta} E \xrightarrow{\pi} E/B$ of B by E/B, where η is the canonical injection and π is the canonical projection, such that exactly one of the following conditions hold:*

(i) *The ideals $\overline{AB}$, $\overline{BA}$ and $ann_E(B/A)$ coincide with A, and B has only one nontrivial closed ideal;*

(ii) *$\overline{AB} = A = \overline{BA}$, $ann_E(B/A) = B$, A is a topologically minimal ideal of B, and the topological E/B-bimodule B/A, determined by the sequence $B \xrightarrow{\eta} E \xrightarrow{\pi} E/B$, is topologically simple;*

(iii) *$\overline{AB} = A = ann^r_E(B)$, $B^2 = B$, and A is a topologically maximal ideal of B and a unital topologically simple left E/B-module relative to the structure given by the sequence $B \xrightarrow{\eta} E \xrightarrow{\pi} E/B$;*

(iv) *$\overline{BA} = A = ann^l_E(B)$, $B^2 = B$, and A is a topologically maximal ideal of B and a unital topologically simple right E/B-module relative to the structure given by the sequence $B \xrightarrow{\eta} E \xrightarrow{\pi} E/B$;*

(v) *$ann_E(B) = A$, $ann_E(B/A) = A$, $\overline{B^2} = B$, and A is a topologically maximal ideal of B and a unital topologically simple E/B-bimodule relative to the structure given by the sequence $B \xrightarrow{\eta} E \xrightarrow{\pi} E/B$;*

(vi) *$ann_E(B) = A$, $ann_E(B/A) = B$, and A and B/A are unital topologically simple E/B-bimodules relative to the structures given by the sequence $B \xrightarrow{\eta} E \xrightarrow{\pi} E/B$;*

(vii) *$ann_E(B) = B$, and the topological E/B-bimodule B, determined by the sequence $B \xrightarrow{\eta} E \xrightarrow{\pi} E/B$, has only one nontrivial closed subbimodule.*

Proof. Since A and B are the only nontrivial closed ideals of E and since $A \subset B$, it is clear that E/B is a topologically simple ring. It is also clear that $ann_E(B)$ coincides with one of the ideals $\{0\}$, A, or B.

We first consider the case when $ann_E(B) = \{0\}$. Then, clearly, at least one of the ideals $\overline{AB}$ and $\overline{BA}$ is nonzero. Suppose first that $\overline{AB}$ and $\overline{BA}$ are both nonzero. Since $\overline{AB}$ and $\overline{BA}$ are contained in A, it follows that $\overline{AB} = A = \overline{BA}$. In particular, since A is the smallest nonzero closed ideal of E, we conclude that $ann^l_E(B) = \{0\} = ann^r_E(B)$. Further, since $A \subset ann_E(B/A)$, we have either $ann_E(B/A) = A$ or $ann_E(B/A) = B$. Assume the former holds. Then we must have $ann^l_E(B/A) = A$ and $ann^r_E(B/A) = A$. For, if we had either $ann^l_E(B/A) = B$ or $ann^r_E(B/A) = B$, it would follow that $ann_E(B/A) = B$, a contradiction. Thus $ann^l_E(B/A) = A = ann^r_E(B/A)$. Pick an arbitrary nonzero closed ideal C of B. Given any nonzero $c \in C$, it follows from Lemma 7 that $\overline{BcB}$ is a nonzero ideal of B and hence of E, whence $\overline{BcB}$ coincides with either A or B. Consequently, if $C \subset A$, we must have $C = A$.

Suppose $C \not\subset A$, and pick any $c \in C \setminus A$. Since $ann^l_E(B/A) = A$, there exists $b \in B$ such that $cb \notin A$. Similarly, since $ann^r_E(B/A) = A$, there exists $b' \in B$ such that $b'cb \notin A$. It follows that $\overline{BcB} = B$, so $C = B$, and hence in this case we are led to (i). Now assume the latter case when $ann_E(B/A) = B$ holds. Then $(B/A)^2 = \{0\}$, so that, by Corollary 1, B/A can be turned into a topological bimodule over E/B by setting

$$(b + A)(x + B) = (b + A)(x + A) = bx + A$$

and

$$(x + B)(b + A) = (x + A)(b + A) = xb + A$$

for all $b \in B$ and $x \in E$. To see that this bimodule is topologically simple, pick an arbitrary closed E/B-subbimodule C' of B/A. Letting $\varphi : B \to B/A$ be the canonical projection, set $C = \varphi^{-1}(C')$. Since, for any $c \in C$ and $x \in E$, we have $cx + A = (c + A)(x + B) \in C'$ and $xc + A = (x + B)(c + A) \in C'$, it follows that C is a proper closed ideal of E containing A, so C coincides with either A or B, which proves that C' is trivial in B/A. Further, given any nonzero $a \in A$, we deduce by Lemma 7 and the fact that $B^2 \subset A$, that $\overline{BaB}$ is a nonzero closed ideal of B and hence of E, which is contained in A, whence $\overline{BaB} = A$. It follows that A is a topologically minimal ideal of B, so in this case we have (ii).

Now let us suppose that $\overline{AB} \neq \{0\}$ and $\overline{BA} = \{0\}$. Then, clearly, $\overline{AB} = A$ and $\overline{B^2} \neq \{0\}$. If we had $\overline{B^2} = A$, it would follow that $\overline{AB} = \overline{\overline{B^2}B} = \overline{B\overline{B^2}} = \overline{BA} = \{0\}$, a contradiction. Thus $\overline{B^2} = B$, and hence $ann^r_E(B) = A$. By using the sequence $B \xrightarrow{\eta} E \xrightarrow{\pi} E/B$, we see from Lemma 6 that A can be turned into a topological left E/B-module. If C is a closed submodule of A, then C is clearly a closed ideal of E contained in A, so either $C = \{0\}$ or $C = A$. This proves that A is a topologically simple E/B-module. Now let C be a closed ideal of B properly containing A, and pick any $c \in C \setminus A$. Since $\overline{B^2} = B$, there is $b \in B$ such that $bc \notin A$. Analogously, there is $b' \in B$ such that $bcb' \notin A$. It follows that $\overline{BcB}$ is a closed ideal of B, and hence of E, which properly contains A, so $\overline{BcB} = B$, whence $C = B$. Consequently, A is topologically maximal in B, and thus in this case we have (iii).

Similarly, in the remaining case when $\overline{AB} = \{0\}$ and $\overline{BA} \neq \{0\}$, we have (iv).

Next we consider the case when $ann_E(B) = A$. It follows from Lemma 5 that A can be turned into a topological bimodule over E/B by setting $a(x + B) = ax$ and $(x + B)a = xa$ for all $a \in A$ and $x \in E$. Letting C be a nonzero closed E/B-subbimodule of A, pick any $c \in C$ and $x \in E$. Since $cx = c(x + B) \in C$ and $xc = (x + B)c \in C$, we see that C is an ideal of E, which gives $C = A$. Hence A is a topologically simple E/B-bimodule. Further, let us consider $ann_E(B/A)$. We must have either $ann_E(B/A) = A$ or $ann_E(B/A) = B$. If the former holds, then $B^2 \not\subset A$, so that $\overline{B^2} = B$. We also deduce as above that $ann^l_E(B/A) = A = ann^r_E(B/A)$. Let C be an arbitrary closed ideal of B properly containing A, and pick any $c \in C \setminus A$. Since $ann^l_E(B/A) = A$, there exists $b \in B$ such that $cb \notin A$. Similarly, since $ann^r_E(B/A) = A$, there exists $b' \in B$ such that $b'cb \notin A$. It follows that $\overline{BcB}$ is a closed ideal of E which is not contained in A, so $\overline{BcB} = B$, whence $C = B$, proving that A is topologically maximal in B. Hence in this case we are led to (v). Now assume the latter case when $ann_E(B/A) = B$ holds. Then $(B/A)^2 = \{0\}$, so B/A can be turned into a topological bimodule over E/B. As above, we can see that if C' is arbitrary nonzero closed E/B-subbimodule of B/A and $\varphi : B \to B/A$ is the canonical projection, then $\varphi^{-1}(C')$ coincides with either A or B. Consequently, in this case we are led to (vi).

Now we consider the case when $ann_E(B) = B$. By Lemma 5, B can be turned into a topological bimodule over E/B. If C is a nontrivial closed E/B-subbimodule of B, it is easy to see that C is an ideal of E, and so we must have $C = A$. Thus in this case we are led to (vii). $\square$

Theorem 12. *Let A be a topological ring having a nontrivial closed ideal A_0, let B be a topologically simple ring with identity, and let $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$ be a unital ideal extension of A by B satisfying one of the following conditions:*

(i) $\overline{A_0A} = A_0 = \overline{AA_0}$, $ann_E\big(\varphi(A)/\varphi(A_0)\big) = \varphi(A_0)$, *and A has only one nontrivial closed ideal;*

(ii) $\overline{A_0A} = A_0 = \overline{AA_0}$, $ann_E\big(\varphi(A)/\varphi(A_0)\big) = \varphi(A)$, *$A_0$ is a topologically minimal ideal of A, and the topological B-bimodule A/A_0, determined by the sequence $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$, is topologically simple;*

(iii) $\varphi(\overline{A_0A}) = \varphi(A_0) = ann^r_E\big(\varphi(A)\big)$, $A^2 = A$, *and A_0 is a topologically maximal ideal of A and a unital topologically simple left B-module relative to the structure given by the sequence $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$;*

(iv) $\varphi(\overline{AA_0}) = \varphi(A_0) = ann^l_E\big(\varphi(A)\big)$, $A^2 = A$, *and A_0 is a topologically maximal ideal of A and a unital topologically simple right B-module relative to the structure given by the sequence $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$;*

(v) $ann_E\big(\varphi(A)\big) = \varphi(A_0)$, $ann_E\big(\varphi(A)/\varphi(A_0)\big) = \varphi(A_0)$, $\overline{A^2} = A$, *and A_0 is a topologically maximal ideal of A and a unital topologically simple B-bimodule relative to the structure given by the sequence $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$;*

(vi) $ann_E\big(\varphi(A)\big) = \varphi(A_0)$, $ann_E\big(\varphi(A)/\varphi(A_0)\big) = \varphi(A)$, *and A_0 and A/A_0 are unital topologically simple B-bimodules relative to the structures given by the sequence $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$;*

(vii) $ann_E\big(\varphi(A)\big) = \varphi(A)$, *and the topological B-bimodule A, determined by the sequence $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$, has only one nontrivial closed subbimodule.*

Then E has exactly two nontrivial closed ideals, namely $\varphi(A_0)$ and $\varphi(A)$.

Proof. Clearly, $\varphi(A_0)$ and $\varphi(A)$ are distinct nontrivial closed ideals of E satisfying $\varphi(A_0) \subset \varphi(A)$. Moreover, $\varphi(A)$ is topologically maximal in E because $E/\varphi(A) \cong B$. Let C be an arbitrary closed ideal of E. If $C \cap \varphi(A) = \varphi(A)$, then $\varphi(A) \subset C$, so that C coincides with one of the ideals $\varphi(A)$ or E since $\varphi(A)$ is topologically maximal.

Assume $C \cap \varphi(A) \neq \varphi(A)$. We first show that in any of cases (i) - (vii), $C \cap \varphi(A)$ coincides with either $\{0\}$ or $\varphi(A_0)$. Indeed, this is clear in case (i) holds because $C \cap \varphi(A)$ is a closed ideal of $\varphi(A)$.

Assume (ii) holds, and suppose $C \cap \varphi(A) \neq \{0\}$. Since $ann_E\big(\varphi(A)\big) = \{0\}$, we cannot have $\big(C \cap \varphi(A)\big)\varphi(A) = \{0\} = \varphi(A)\big(C \cap \varphi(A)\big)$. On the other hand, $\big(C \cap \varphi(A)\big)\varphi(A)$ and $\varphi(A)\big(C \cap \varphi(A)\big)$ are contained in $\varphi(A_0)$ by our hypothesis that $ann_E\big(\varphi(A)/\varphi(A_0)\big) = \varphi(A)$. Since $\varphi(A_0)$ is topologically minimal in $\varphi(A)$, it follows that either $\big(C \cap \varphi(A)\big)\varphi(A)$ or $\varphi(A)\big(C \cap \varphi(A)\big)$ coincides with $\varphi(A_0)$, whence $\varphi(A_0) \subset C \cap \varphi(A)$. As the B-bimodule $\varphi(A)/\varphi(A_0)$ is topologically simple, we deduce that $C \cap \varphi(A)$ coincides with $\varphi(A_0)$.

In the following, we consider (iii), (iv), (v) and (vi) simultaneously. By hypotheses, in every of cases (iii), (iv) and (v) we have $\overline{\varphi(A)^2} = \varphi(A)$. We first show that if (vi) holds, then $\overline{\varphi(A)^2} = \varphi(A_0)$. Indeed, since $ann_E\big(\varphi(A)/\varphi(A_0)\big) = \varphi(A)$, we have $\overline{\varphi(A)^2} \subset \varphi(A_0)$, so that $\overline{\varphi(A)^2}$ is a closed B-subbimodule of $\varphi(A_0)$. Moreover, $\overline{\varphi(A)^2} \neq \{0\}$ because $ann_E\big(\varphi(A)\big) = \varphi(A_0)$. Since $\varphi(A_0)$ is a topologically simple B-bimodule, we get $\overline{\varphi(A)^2} = \varphi(A_0)$.

Now, in every of cases (iii), (iv) and (v), if $C \cap \varphi(A) \subset \varphi(A_0)$, we must have either $C \cap \varphi(A) = \{0\}$ or $C \cap \varphi(A) = \varphi(A_0)$ because $C \cap \varphi(A)$ is a B-submodule (respectively, B-subbimodule) of $\varphi(A_0)$ and $\varphi(A_0)$ is topologically simple. We next show that $C \cap \varphi(A) \not\subset \varphi(A_0)$ leads to a contradiction. Indeed, suppose $C \cap \varphi(A) \not\subset \varphi(A_0)$, so that $\overline{(C \cap \varphi(A)) + \varphi(A_0)}$ properly contains $\varphi(A_0)$. Consequently, in every of cases (iii), (iv) and (v), we have $\overline{(C \cap \varphi(A)) + \varphi(A_0)} = \varphi(A)$ because $\varphi(A_0)$ is topologically maximal in $\varphi(A)$. Further, in case (vi) holds, it is easy to see that $\overline{(C \cap \varphi(A)) + \varphi(A_0)}/\varphi(A_0)$ is a nonzero closed B-subbimodule of $\varphi(A)/\varphi(A_0)$. Since $\varphi(A)/\varphi(A_0)$ is topologically simple, it follows that $\overline{(C \cap \varphi(A)) + \varphi(A_0)}/\varphi(A_0) = \varphi(A)/\varphi(A_0)$, so again $\overline{(C \cap \varphi(A)) + \varphi(A_0)} = \varphi(A)$. We then have

$$\overline{\varphi(A)^2} = \overline{(\overline{(C \cap \varphi(A)) + \varphi(A_0)})\varphi(A)} = \overline{(C \cap \varphi(A))\varphi(A)} \subset C \cap \varphi(A).$$

Therefore either of equalities $\overline{\varphi(A)^2} = \varphi(A)$ or $\overline{\varphi(A)^2} = \varphi(A_0)$ together with the fact that $\overline{(C \cap \varphi(A)) + \varphi(A_0)} = \varphi(A)$ gives $C \cap \varphi(A) = \varphi(A)$, a contradiction.

Finally, if (vii) holds, then clearly $C \cap \varphi(A)$ is a B-subbimodule of $\varphi(A)$, and hence $C \cap \varphi(A)$ must coincide with either $\{0\}$ or $\varphi(A_0)$.

Thus, in any of cases (i)-(vii), $C \cap \varphi(A)$ coincides with either $\{0\}$ or $\varphi(A_0)$. Now, since $C\varphi(A)$ and $\varphi(A)C$ are contained in $C \cap \varphi(A)$, we have $C \subset ann_E\big(\varphi(A)\big)$ if $C \cap \varphi(A) = \{0\}$ and $C \subset ann_E\big(\varphi(A)/\varphi(A_0)\big)$ if $C \cap \varphi(A) = \varphi(A_0)$. It follows that if $C \cap \varphi(A) = \{0\}$, then $C = \{0\}$ in any of cases (i)-(vii). Similarly, if $C \cap \varphi(A) = \varphi(A_0)$, then $C = \varphi(A_0)$ in any of cases (i)-(vii). $\square$

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Convex Quadrics

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Abstract. We introduce and describe convex quadrics in $\mathbb{R}^n$ and characterize them as convex hypersurfaces with quadric sections by a continuous family of hyperplanes.

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1 Introduction and main results

Characterizations of ellipses and ellipsoids among convex bodies in the plane or in space became an established topic of convex geometry on the turn of 20th century. Comprehensive surveys on various characteristic properties of ellipsoids in the Euclidean space $\mathbb{R}^n$ are presented in [9] and [13] (see also [10]). Similar characterizations of unbounded convex quadrics, like paraboloids, sheets of elliptic hyperboloids or elliptic cones, are given by a short list of sporadic results (see, e. g., [1, 2, 15, 16]). Furthermore, even a classification of convex quadrics in $\mathbb{R}^n$ for $n \geq 4$ is not established (although it is used in [15, 16] without proof). Our goal here is to introduce and to describe convex quadrics in $\mathbb{R}^n$ and to provide a characteristic property of these hypersurfaces in terms of hyperplane sections.

In what follows, by a *convex solid* we mean an n-dimensional closed convex set in $\mathbb{R}^n$, distinct from the entire space (convex *bodies* are compact convex solids). As usual, $\operatorname{bd} K$ and $\operatorname{int} K$ denote, respectively, the boundary and interior of a convex solid K. A *convex hypersurface* (a *surface* if $n = 3$ or a *curve* if $n = 2$) is the boundary of a convex solid. This definition includes a hyperplane or a pair of parallel hyperplanes.

In a standard way, a *quadric hypersurface* (or a *second degree hypersurface*) in $\mathbb{R}^n$, $n \geq 2$, is the locus of points $x = (\xi_1, \dots, \xi_n)$ that satisfy a quadratic equation

$$\sum_{i,k=1}^{n} a_{ik}\xi_i\xi_k + 2\sum_{i=1}^{n} b_i\xi_i + c = 0, \tag{1}$$

where not all a_{ik} are zero. We say that a convex hypersurface $S \subset \mathbb{R}^n$ is a *convex quadric* provided there is a real quadric hypersurface $Q \subset \mathbb{R}^n$ and a convex component U of $\mathbb{R}^n \setminus Q$ such that S is the boundary of U. This definition allows us to include into considerations convex hypersurfaces like sheets of elliptic cones and sheets of elliptic hyperbolids, and not only ellipsoids and elliptic paraboloids.

The following theorem plays a key role in the description of convex quadrics.

Theorem 1. *The complement of a real quadric hypersurface $Q \subset \mathbb{R}^n$, $n \geq 2$, is the disjoint union of four or fewer open sets; at least one of these components is convex if and only if the canonical form of Q is given by one of the equations*

$$\begin{aligned}
&a_1\xi_1^2 + \cdots + a_k\xi_k^2 = 1, && 1 \leq k \leq n,\\
&a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_k\xi_k^2 = 1, && 2 \leq k \leq n,\\
&a_1\xi_1^2 = 0, &&\\
&a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_k\xi_k^2 = 0, && 2 \leq k \leq n,\\
&a_1\xi_1^2 + \cdots + a_{k-1}\xi_{k-1}^2 = \xi_k, && 2 \leq k \leq n,
\end{aligned}$$

where all scalars a_i involved are positive.

Corollary 1. *A convex hypersurface $S \subset \mathbb{R}^n$, $n \geq 2$, is a convex quadric if and only if S can be described in suitable Cartesian coordinates $\xi_1, \dots, \xi_n$ by one of the conditions*

$$\begin{aligned}
&a_1\xi_1^2 + \cdots + a_k\xi_k^2 = 1, && 1 \leq k \leq n,\\
&a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_k\xi_k^2 = 1,\ \ \xi_1 \geq 0, && 2 \leq k \leq n,\\
&a_1\xi_1^2 = 0, &&\\
&a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_k\xi_k^2 = 0,\ \ \xi_1 \geq 0, && 2 \leq k \leq n,\\
&a_1\xi_1^2 + \cdots + a_{k-1}\xi_{k-1}^2 = \xi_k, && 2 \leq k \leq n,
\end{aligned}$$

where all scalars a_i involved are positive.

In what follows, a *plane* of dimension m in $\mathbb{R}^n$ is a translate of an m-dimensional subspace. We say that a plane L *properly* intersects a convex solid K provided L intersects both sets $\operatorname{bd} K$ and $\operatorname{int} K$.

A well-known result of convex geometry states that the boundary of a convex body $K \subset \mathbb{R}^n$ is an ellipsoid if and only if there is a point $p \in \operatorname{int} K$ such that all sections of $\operatorname{bd} K$ by 2-dimensional planes through p are ellipses (see [3,12] for $n = 3$ and [7, pp. 91–92] for $n \geq 3$). This result is generalized in [15] by showing that the boundary of a convex solid $K \subset \mathbb{R}^n$ is a convex quadric if and only if there is a point $p \in \operatorname{int} K$ such that all sections of $\operatorname{bd} K$ by 2-dimensional planes through p are convex quadric curves. In this regard, we pose the following problem (solved in [6,11] for the case of convex bodies).

Problem 1. *Given a convex solid $K \subset \mathbb{R}^n$, $n \geq 3$, and a point $p \in \mathbb{R}^n$, is it true that either* $\operatorname{bd} K$ *is a convex quadric or K is a convex cone with apex p provided all proper sections of* $\operatorname{bd} K$ *by 2-dimensional planes through p are convex quadric curves?*

Kubota [12] proved that, given a pair of bounded convex surfaces in $\mathbb{R}^3$, one being enclosed by the other, if all planar sections of the bigger surface by planes tangent to the second surface are ellipses, then the containing surface is an ellipsoid. Independently, Bianchi and Gruber [4] established the following far-reaching assertion: If K is a convex body in $\mathbb{R}^n$, $n \geq 3$, and $\delta(u)$ is a continuous real-valued function on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ such that for each vector $u \in S^{n-1}$ the hyperplane $H(u) = \{x \mid x \cdot u = \delta(u)\}$ intersects $\operatorname{bd} K$ along an $(n-1)$-dimensional ellipsoid, then $\operatorname{bd} K$ is an ellipsoid. Our second theorem extends this assertion to the case of convex solids.

Theorem 2. *Let K be a convex solid in $\mathbb{R}^n$, $n \geq 3$, and $\delta(u)$ be a continuous real-valued function on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ such that for each vector $u \in S^{n-1}$ the hyperplane $H(u) = \{x \mid x \cdot u = \delta(u)\}$ either lies in K or intersects* $\operatorname{bd} K$ *along an $(n-1)$-dimensional convex quadric. Then* $\operatorname{bd} K$ *is a convex quadric.*

2 Proof of Theorem 1

Let $Q \subset \mathbb{R}^n$ be a real quadric hypersurface. Choosing a suitable orthogonal basis, we may suppose that Q has one of the following canonical forms:

$$\begin{array}{lll}
A_k \ : & \xi_1^2 + \cdots + \xi_k^2 = 1, & 1 \leq k \leq n, \\
B_{k,r} : & \xi_1^2 + \cdots + \xi_k^2 - \xi_{k+1}^2 - \cdots - \xi_r^2 = 1, & 1 \leq k < r \leq n, \\
C_k \ : & \xi_1^2 + \cdots + \xi_k^2 = 0, & 1 \leq k \leq n, \\
D_{k,r} : & \xi_1^2 + \cdots + \xi_k^2 - \xi_{k+1}^2 - \cdots - \xi_r^2 = 0, & 1 \leq k < r \leq n, \\
E_{k,r} : & \xi_1^2 + \cdots + \xi_k^2 - \xi_{k+1}^2 - \cdots - \xi_{r-1}^2 = \xi_r, & 1 \leq k < r \leq n.
\end{array}$$

First, we exclude the trivial cases $Q = A_1$ (when Q is a pair of parallel hyperplanes) and $Q = C_k$ (when Q is an $(n-k)$-dimensional subspace). Furthermore, the proof can be reduced to the case when Q has one of the forms $A_n, B_{k,n}, D_{k,n}, E_{k,n}$, since otherwise Q is a cylinder generated by a lower-dimensional quadric of the same type.

We are going to express each of the hypersurfaces A_n, $B_{k,n}$, $D_{k,n}$, $E_{k,n}$ as the set of revolution of a respective lower-dimensional surface. To describe these revolutions, choose any subspaces L_1, L_2, and L_3 of $\mathbb{R}^n$ such that $L_1 \subset L_2 \subset L_3$ and

$$\dim L_1 = m - 1, \quad \dim L_2 = m, \quad \dim L_3 = m + 1, \quad 2 \leq m \leq n - 1.$$

Let M be the 2-dimensional subspace of L_3 orthogonal to L_1. Given a point $y \in L_2$, put $M_y = y + M$ and denote by z the point of intersection of L_1 and M_y (z is the orthogonal projection of y on L_1). Let C_y be the circumference in M_y with center z and radius $\|y - z\|$. We say that a set $X \subset L_3$ is the *set of revolution* of a set $Y \subset L_2$ about L_1 within L_3 provided $X = \cup_{y \in Y} C_y$. A set $Z \subset \mathbb{R}^n$ is called *symmetric* about a subspace $N \subset \mathbb{R}^n$ if for any point $x \in Z$ and its orthogonal projection u on N, the point $2u - x$ lies in Z.

In these terms, we formulate three lemmas (the first one being obvious). In what follows, $\langle e_1, \dots, e_k \rangle$ means the span of vectors $e_1, \dots, e_k$.

Lemma 1. *If Y is a subset of L_2 and X is the set of revolution of Y about L_1 within L_3, then X is symmetric about L_2 and any component of X is the set of revolution of a suitable component of Y about L_1 within L_3.* □

Lemma 2. *If a set $Y \subset L_2$ is symmetric about L_1 and X is the set of revolution of Y about L_1 within L_3, then X is a convex set if and only if Y is a convex set.*

Proof. Without loss of generality, we may put $L_3 = \mathbb{R}^n$. Choose an orthonormal basis $e_1, \dots, e_n$ for $\mathbb{R}^n$ such that

$$L_1 = \langle e_1, \dots, e_{n-2} \rangle \quad \text{and} \quad L_2 = \langle e_1, \dots, e_{n-1} \rangle.$$

Clearly, $x = (\xi_1, \dots, \xi_n)$ belongs to X if and only if there is a point

$$y = (\xi_1, \dots, \xi_{n-2}, \xi'_{n-1}, 0) \in Y \quad \text{where} \quad \xi'_{n-1} = \sqrt{\xi_{n-1}^2 + \xi_n^2}.$$

If X is convex, then Y is convex due to $Y = X \cap L_2$. Conversely, let Y be convex. Choose any points $a = (\alpha_1, \dots, \alpha_n)$ and $b = (\beta_1, \dots, \beta_n)$ in X and a scalar $\lambda \in [0,1]$. We intend to show that $c = (1-\lambda)a + \lambda b \in X$. Let

$$a' = (\alpha_1, \dots, \alpha_{n-2}, \alpha'_{n-1}, 0), \quad b' = (\beta_1, \dots, \beta_{n-2}, \beta'_{n-1}, 0),$$

and

$$c' = ((1-\lambda)\alpha_1 + \lambda\beta_1, \dots, (1-\lambda)\alpha_{n-2} + \lambda\beta_{n-2}, (1-\lambda)\alpha'_{n-1} + \lambda\beta'_{n-1}, 0)$$

be points in Y, where

$$\alpha'_{n-1} = \sqrt{\alpha_{n-1}^2 + \alpha_n^2}, \quad \text{and} \quad \beta'_{n-1} = \sqrt{\beta_{n-1}^2 + \beta_n^2}.$$

Then $a', b' \in Y$ and $c' = (1-\lambda)a' + \lambda b' \in Y$ due to convexity of Y. Because Y is symmetric about L_1, we have

$$((1-\lambda)\alpha_1 + \lambda\beta_1, \dots, (1-\lambda)\alpha_{n-2} + \lambda\beta_{n-2}, \mu, 0) \in Y$$

for any scalar μ with $|\mu| \le (1-\lambda)\alpha'_{n-1} + \lambda\beta'_{n-1}$. Let

$$y = ((1-\lambda)\alpha_1 + \lambda\beta_1, \dots, (1-\lambda)\alpha_{n-2} + \lambda\beta_{n-2}, \rho, 0),$$

where

$$\rho = \sqrt{\big((1-\lambda)\alpha_{n-1} + \lambda\beta_{n-1}\big)^2 + \big((1-\lambda)\alpha_n + \lambda\beta_n\big)^2}.$$

From $\alpha_{n-1}\beta_{n-1} + \alpha_n\beta_n \le \alpha'_{n-1}\beta'_{n-1}$, we obtain $\rho \le (1-\lambda)\alpha'_{n-1} + \lambda\beta'_{n-1}$, which gives $y \in Y$. Clearly, the point

$$z = ((1-\lambda)\alpha_1 + \lambda\beta_1, \dots, (1-\lambda)\alpha_{n-2} + \lambda\beta_{n-2}, 0, 0)$$

is the orthogonal projection of y on L_1. The equalities $\|c - z\| = \|y - z\| = \rho$ imply that $c \in C_y \subset X$. Hence X is convex. $\square$

Lemma 3. *Within $\mathbb{R}^n$, $n \ge 3$, we have*

1) *A_n is the set of revolution of $A_{n-1} \subset \langle e_1, \dots, e_{n-1} \rangle$ about $\langle e_1, \dots, e_{n-2} \rangle$,*

2) *$B_{k,n}$ is the set of revolution of $B_{k,n-1} \subset \langle e_1, \dots, e_{n-1}\rangle$ about $\langle e_1, \dots, e_{n-2}\rangle$, $1 \le k \le n-2$,*

3) *$D_{k,n}$ is the set of revolution of $D_{k,n-1} \subset \langle e_1, \dots, e_{n-1}\rangle$ about $\langle e_1, \dots, e_{n-2}\rangle$, $1 \le k \le n-2$,*

4) *$B_{k,n}$ is the set of revolution of $B_{k-1,n-1} \subset \langle e_2, \dots, e_n\rangle$ about $\langle e_3, \dots, e_n\rangle$, $2 \le k \le n-1$,*

5) *$D_{k,n}$ is the set of revolution of $D_{k-1,n-1} \subset \langle e_2, \dots, e_n\rangle$ about $\langle e_3, \dots, e_n\rangle$, $2 \le k \le n-1$.* □

Proof. 1) Given a point $x = (\xi_1, \dots, \xi_n) \in A_n$, put

$$y = (\xi_1, \dots, \xi_{n-2}, \sqrt{\xi_{n-1}^2 + \xi_n^2}, 0), \quad z = (\xi_1, \dots, \xi_{n-2}, 0, 0). \tag{2}$$

Then $y \in A_{n-1} \subset \langle e_1, \dots, e_{n-1}\rangle$ and z is the orthogonal projection of y on $\langle e_1, \dots, e_{n-2}\rangle$. From

$$\|x - z\| = \|y - z\| = \sqrt{\xi_{n-1}^2 + \xi_n^2}$$

we see that $x \in C_y$. So, A_n lies in the revolution of A_{n-1} about $\langle e_1, \dots, e_{n-2}\rangle$. Conversely, if $y = (\eta_1, \dots, \eta_{n-1}, 0)$ is a point in $A_{n-1} \subset \langle e_1, \dots, e_{n-1}\rangle$ and $z = (\eta_1, \dots, \eta_{n-2}, 0, 0)$ is the orthogonal projection of y on $\langle e_1, \dots, e_{n-2}\rangle$, then any point u from the circle $C_y \subset y + \langle e_{n-1}, e_n\rangle$ can be written as

$$u = (\eta_1, \dots, \eta_{n-2}, \gamma_{n-1}, \gamma_n), \quad \text{where} \quad \gamma_{n-1}^2 + \gamma_n^2 = \eta_{n-1}^2.$$

Clearly, $u \in A_n$, which shows that A_n contains the set of revolution of A_{n-1} about $\langle e_1, \dots, e_{n-2}\rangle$.

Cases 2)–5) are considered similarly, where the points y and z are defined, respectively, by (2) in cases 2) and 3), and by

$$y = (0, \sqrt{\xi_1^2 + \xi_2^2}, \xi_3, \dots, \xi_n), \quad z = (0, 0, \xi_3, \dots, \xi_n)$$

in cases 4) and 5). □

Proof of Theorem 1. Our further consideration is organized by induction on n. The cases $n = 2$ and $n = 3$ follow immediately from the well-known properties of quadric curves and surfaces. Suppose that $n \ge 4$. Assuming that the conclusion of Theorem 1 holds for all $m < n$, let the quadric hypersurface $Q \subset \mathbb{R}^n$ have one of the forms $A_n, B_{k,n}, D_{k,n}, E_{k,n}$. We consider these forms separately.

Case 1. Let $Q = A_n$. By Lemma 3, A_n can be obtained from

$$A_2 = \{(\xi_1, \xi_2) \mid \xi_1^2 + \xi_2^2 = 1\} \subset \langle e_1, e_2\rangle$$

by consecutive revolutions of $A_i \subset \langle e_1, \dots, e_i\rangle$ about $\langle e_1, \dots, e_{i-1}\rangle$ within the subspace $\langle e_1, \dots, e_{i+1}\rangle$, $i = 2, \dots, n-1$. Since both components of $\langle e_1, e_2\rangle \setminus A_2$ are symmetric about the line $\langle e_1 \rangle$, Lemmas 1 and 2 imply that $\mathbb{R}^n \setminus A_n$ consists of two components; one of them, given by $\xi_1^2 + \cdots + \xi_n^2 < 1$, is convex.

Case 2. Let $Q = B_{k,n}$, $1 \le k \le n-1$. If $k = 1$, then Lemma 3 implies that $B_{1,n}$ can be obtained from

$$B_{1,2} = \{(\xi_1, \xi_2) \mid \xi_1^2 - \xi_2^2 = 1\} \subset \langle e_1, e_2 \rangle$$

by consecutive revolutions of $B_{1,i} \subset \langle e_1, \dots, e_i \rangle$ about $\langle e_1, \dots, e_{i-1} \rangle$ within the subspace $\langle e_1, \dots, e_{i+1} \rangle$, $i = 2, \dots, n-1$. Since all three components of $\langle e_1, e_2 \rangle \setminus B_{1,2}$ are symmetric about the line $\langle e_1 \rangle$, Lemmas 1 and 2 imply that $\mathbb{R}^n \setminus B_{1,n}$ consists of three components; two of them, given, respectively, by

$$\xi_1 > \sqrt{\xi_2^2 + \cdots + \xi_n^2 + 1} \quad \text{and} \quad \xi_1 < -\sqrt{\xi_2^2 + \cdots + \xi_n^2 + 1},$$

are convex. If $k \ge 2$, then $B_{k,n}$ can be obtained from

$$B_{1,2} = \{(\xi_k, \xi_{k+1}) \mid \xi_k^2 - \xi_{k+1}^2 = 1\} \subset \langle e_k, e_{k+1} \rangle$$

in two steps. First, we obtain $B_{k,k+1} \subset \mathbb{R}^{k+1} = \langle e_1, \dots, e_{k+1} \rangle$ by consecutive revolutions of $B_{i,i+1} \subset \langle e_{k+1-i}, e_{k+2-i}, \dots, e_{k+1} \rangle$ about $\langle e_{k+2-i}, \dots, e_{k+1} \rangle$ within $\langle e_{k-i}, e_{k+1-i}, \dots, e_{k+1} \rangle$, $i = 1, 2, \dots, k-1$. The complement of

$$B_{2,3} = \{(\xi_{k-1}, \xi_k, \xi_{k+1}) \mid \xi_{k-1}^2 + \xi_k^2 - \xi_{k+1}^2 = 1\}$$

in $\langle e_{k-1}, e_k, e_{k+1} \rangle$, consists of two components, both symmetric about $\langle e_k, e_{k+1} \rangle$. Since none of these components is convex, Lemmas 1 and 2 imply that $\mathbb{R}^{k+1} \setminus B_{k,k+1}$ consists of two components, both symmetric about any k-dimensional coordinate subspace of $\mathbb{R}^{k+1}$, but none of them convex.

Second, we obtain $B_{k,n}$ from $B_{k,k+1}$ by consecutive revolutions of $B_{k,j} \subset \langle e_1, \dots, e_j \rangle$ about $\langle e_1, \dots, e_{j-1} \rangle$ within $\langle e_1, \dots, e_{j+1} \rangle$, $j = k+1, \dots, n-1$. As above, $\mathbb{R}^n \setminus B_{k,n}$ consists of two components, none of them convex.

Case 3. Let $Q = D_{k,n}$, $1 \le k \le n-1$. If $k = 1$, then $D_{1,n}$ can be obtained from

$$D_{1,2} = \{(\xi_1, \xi_2) \mid \xi_1^2 - \xi_2^2 = 0\} \subset \langle e_1, e_2 \rangle$$

by consecutive revolutions of $D_{1,i} \subset \langle e_1, \dots, e_i \rangle$ about $\langle e_1, \dots, e_{i-1} \rangle$ within the subspace $\langle e_1, \dots, e_{i+1} \rangle$, $i = 2, \dots, n-1$. The complement of

$$D_{1,3} = \{(\xi_1, \xi_2, \xi_3) \mid \xi_1^2 - \xi_2^2 + \xi_3^2 = 0\}$$

in $\langle e_1, e_2, e_3 \rangle$ consists of tree components, all symmetric about $\langle e_1, e_2 \rangle$. Since two of these components are convex, Lemmas 1 and 2 imply that $\mathbb{R}^n \setminus D_{1,n}$ consists of three components; two of them, given, respectively, by

$$\xi_1 > \sqrt{\xi_2^2 + \cdots + \xi_n^2} \quad \text{and} \quad \xi_1 < -\sqrt{\xi_2^2 + \cdots + \xi_n^2},$$

are convex.

Since the case $k = n-1$ is reducible to that of $k = 1$ (by reordering $e_1, e_2, \dots, e_n$ as $e_n, e_{n-1}, \dots, e_1$), we may assume that $2 \le k \le n-2$. Then $D_{k,n}$ can be obtained from

$$D_{2,3} = \{(\xi_{k-1}, \xi_k, \xi_{k+1}) \mid \xi_{k-1}^2 + \xi_k^2 - \xi_{k+1}^2 = 0\} \subset \langle e_{k-1}, e_k, e_{k+1}\rangle$$

in two steps. First, we obtain $D_{2,n-k+2} \subset \langle e_{k-1}, e_k, \dots, e_n\rangle$ by consecutive revolutions of $D_{2,i} \subset \langle e_{k-1}, e_k, \dots, e_i\rangle$ about $\langle e_{k-1}, e_k, \dots, e_{i-1}\rangle$ within $\langle e_{k-1}, e_k, \dots, e_{i+1}\rangle$, $i = k+1, \dots, n-1$. Clearly, $\langle e_{k-1}, e_k, e_{k+1}\rangle \setminus D_{2,3}$ consists of three components; two of them,

$$\xi_{k+1} > \sqrt{\xi_{k-1}^2 + \xi_k^2} \quad \text{and} \quad -\xi_{k+1} < \sqrt{\xi_{k-1}^2 + \xi_k^2},$$

are convex and symmetric to each other about $\langle e_{k-1}, e_k\rangle$. Hence $\langle e_{k-1}, e_k, e_{k+1}, e_{k+2}\rangle \setminus D_{3,4}$ consists of two components, none of them convex. Lemmas 1 and 2 imply that $\mathbb{R}^{n-k+2} \setminus D_{2,n-k+2}$ consists of two components, none of them convex.

Next, we obtain $D_{k,n}$ from $D_{2,n-k+2}$ by consecutive revolutions of the surface $D_{i,n-k+i} \subset \langle e_{k-i+1}, \dots, e_n\rangle$ about $\langle e_{k-i+2}, \dots, e_n\rangle$ within $\langle e_{k-i}, \dots, e_n\rangle$, $i = 2, \dots, k-1$. As above, $\mathbb{R}^n \setminus D_{k,n}$ consists of two components, none of them convex.

Case 4. Let $Q = E_{k,n}$, $1 \le k \le n-1$. Clearly, $E_{k,n}$ is the graph of a real-valued function φ on $\mathbb{R}^{n-1} = \langle e_1, \dots, e_{n-1}\rangle$, given by

$$\xi_n = \varphi(\xi_1, \dots, \xi_{n-1}) = \xi_1^2 + \dots + \xi_k^2 - \xi_{k+1}^2 - \dots - \xi_{n-1}^2.$$

Hence $\mathbb{R}^n \setminus E_{k,n}$ has two components. The Hessian $\left(\frac{\partial^2 \varphi}{\partial \xi_i \partial \xi_j}\right)$ is a diagonal $n \times n$-matrix, with 2's on its first k diagonal entries and -2's on the other $n-k-1$ diagonal entries. Therefore, φ is not concave, being convex if and only if $k = n-1$. So, $\mathbb{R}^n \setminus E_{k,n}$ has a convex component if and only if $k = n-1$; this component is given by $\xi_1^2 + \dots + \xi_{n-1}^2 < \xi_n$. □

3 Proof of Theorem 2

In what follows, the origin of $\mathbb{R}^n$ is denoted by o. We say that a plane L *supports* a convex solid K provided L intersects K such that $L \cap \operatorname{int} K = \varnothing$. The *recession cone* of K is defined by

$$\operatorname{rec} K = \{e \in \mathbb{R}^n \mid x + \alpha e \in K \text{ for all } x \in K \text{ and } \alpha \ge 0\}.$$

It is well-known that $\operatorname{rec} K \ne \{o\}$ if and only if K is unbounded; K is called *line-free* if it contains no line. Finally, $\operatorname{rint} M$ and $\operatorname{rbd} M$ denote the relative interior and the relative boundary of a convex set $M \subset \mathbb{R}^n$.

Under the assumptions of Theorem 2, we divide the proof into a sequence of lemmas.

Lemma 4. *If K contains a line, then* $\operatorname{bd} K$ *is a convex quadric cylinder.*

Proof. If l is a line in K, then K is the direct sum $\langle u_0 \rangle \oplus (K \cap H(u_0))$, where $\langle u_0 \rangle$ is the 1-dimensional subspace spanned by a unit vector u_0 parallel to l. By the assumption, $\operatorname{bd} K \cap H(u_0)$ is an $(n-1)$-dimensional convex quadric. Hence $\operatorname{bd} K = \langle u_0 \rangle \oplus (\operatorname{bd} K \cap H(u_0))$ is a convex quadric cylinder. □

Due to Lemma 4, we may further assume that K is line-free. Then no hyperplane lies in K; so, every hyperplane $H(u)$, $u \in S^{n-1}$, properly intersects K.

Lemma 5. *For any $(n-2)$-dimensional plane L supporting K, there is a hyperplane $H(u)$, $u \in S^{n-1}$, that contains L.*

Proof. Let P be the 2-dimensional subspace orthogonal to L and π be the orthogonal projection of $\mathbb{R}^n$ on P. Clearly, the intersection $L \cap P$ is a singleton, say $\{v\}$. The set $M = \pi(K)$ is convex, $\operatorname{rint} M = \pi(\operatorname{int} K)$, and $v \in \operatorname{rbd} M$. Choose an orientation in P and denote by l a line in P that supports M at v. Let u_0 be the unit vector in P orthogonal to l such that u_0 is an outward unit normal to M at v. Let m denote the line through v orthogonal to l, and T be the open halfplane of P bounded by l and disjoint from M.

Assume, for contradiction, that no line $l(u) = P \cap H(u)$, $u \in P \cap S^{n-1}$, contains v. In particular, the line $l(u_0)$ is distinct from l. Continuously rotating the unit vector u from the initial position u_0 in a positive direction along $P \cap S^{n-1}$, we obtain a continuous family of lines each of them missing v. This is possible only if the parallel lines $l(u_0)$ and $l(-u_0)$ intersect m at points that belong to the opposite open halflines with common apex v. Hence one of the lines $l(u_0), l(-u_0)$ entirely lies in T, thus missing M, which is impossible due to $K \cap H(u_0) \neq \varnothing$ and $K \cap H(-u_0) \neq \varnothing$. □

We recall that a convex solid $K \subset \mathbb{R}^n$ is called *strictly convex* if $\operatorname{bd} K$ contains no line segments. Furthermore, K is called *regular* provided any point $x \in \operatorname{bd} K$ belongs to a unique hyperplane supporting K.

Lemma 6. *If K is neither strictly convex nor regular, then* $\operatorname{bd} K$ *is a sheet of an elliptic cone.*

Proof. First, we are going to show that if K is not regular, then K is not strictly convex. Indeed, suppose that K is not regular and choose a singular point $x \in \operatorname{bd} K$. Then there are distinct hyperplanes G_1 and G_2 both supporting K at x. Choose a hyperplane G through $G_1 \cap G_2$ supporting K and different from each of G_1 and G_2. Let $L \subset G$ be an $(n-2)$-dimensional plane through x which is distinct from $G_1 \cap G_2$. By Lemma 5, there is a hyperplane $H(u)$ containing L. Because $H(u)$ meets $\operatorname{int} K$, the point x is singular for the $(n-1)$-dimensional convex quadric $E(u) = \operatorname{bd} K \cap H(u)$. According to Corollary 1, $E(u)$ must be a sheet of an $(n-1)$-dimensional elliptic cone. Choosing a line segment in $E(u)$, we conclude that K is not strictly convex.

Now, assume that K is not strictly convex and choose a line segment $[x, z] \subset \operatorname{bd} K$. By Lemma 5, there is a hyperplane $H(u_0)$ containing the line through x and z. Since the $(n-1)$-dimensional convex quadric $E(u_0) = \operatorname{bd} K \cap H(u_0)$ is line-free and contains a line segment, it should be a sheet of an $(n-1)$-dimensional elliptic cone. Let v be the apex of $E(u_0)$. Denote by h_1 the halfline $[v, x)$ and choose another halfline $h_2 = [v, w) \subset E(u_0)$ such that the 2-dimensional plane through $h_1 \cup h_2$ intersects $\operatorname{int} K$ (this is possible since $H(u_0)$ meets $\operatorname{int} K$). Let P_2 be a hyperplane supporting K with the property $h_2 \subset P_2$. By the above, $h_1 \not\subset P_2$.

Choose a halfline h with apex v tangent to K and so close to h_1 that $h \not\subset P_2$. Let P be a hyperplane through h which supports K. By Lemma 5, there is a hyperplane $H(u)$ that meets $\operatorname{int} K$ and contains h. Since the section $E(u) = \operatorname{bd} K \cap H(u)$ is bounded by both P and P_2, the point v is singular for $E(u)$. As above, $E(u)$ is a sheet of an $(n-1)$-dimensional elliptic cone. Hence $h \subset \operatorname{bd} K$. Varying h and h_2, we obtain by the argument above that every tangent halfline of K at v lies in $\operatorname{bd} K$. This shows that K is a convex cone with apex v. Finally, choose a hyperplane $H(u_1)$ that properly intersects K along a bounded set (this is possible since K is line-free). By the assumption, $\operatorname{bd} K \cap H(u_1)$ is an $(n-1)$-dimensional ellipsoid. So, $\operatorname{bd} K$ is a sheet of an elliptic cone with apex v generated by $\operatorname{bd} K \cap H(u_1)$. □

Lemma 7. *Let K be strictly convex and regular. There are hyperplanes $H(u_1)$ and $H(u_2)$, $u_1, u_2 \in S^{n-1}$, such that both sections $\operatorname{bd} K \cap H(u_1)$ and $\operatorname{bd} K \cap H(u_2)$ are $(n-1)$-dimensional ellipsoids whose intersection is an $(n-2)$-dimensional ellipsoid.*

Proof. Since K is line-free, there is a 2-dimensional subspace P such that the orthogonal projection, M, of K on P is a line-free closed convex set (see, e.g., [14]). Choose any orientation in P. Denote by $\mathcal{F}$ the family of lines $l(u) = P \cap H(u)$, $u \in P \cap S^{n-1}$, such that $M \cap l(u)$ is bounded. Let $l(u_0)$ be one of these lines. Put $[v, w] = M \cap l(u_0)$. The line $l(u_0)$ cuts M into 2-dimensional closed convex subsets, M' and M'', at least one of them, say M', being compact. If there is a line $l(u) \in \mathcal{F}_0 = \mathcal{F} \setminus \{l(u_0)\}$ which intersects the open line segment $]v, w[$, then the respective hyperplanes $H(u)$ and $H(u_0)$ have the desired property.

Assume that no line $l(u) \in \mathcal{F}_0$ intersects $]v, w[$. We state that no line $l(u) \in \mathcal{F}_0$ intersects $\operatorname{rint} M'$. Indeed, if a line $l(u_1) \in \mathcal{F}_0$ intersected $\operatorname{rint} M'$, then, rotating u about $P \cap S^{n-1}$ from the initial position u_1, we would find a line $l(u_2)$ supporting M at v or at w (which is impossible since $\operatorname{int} K \cap H(u_2) \neq \varnothing$). In a similar way, no line $l(u) \in \mathcal{F}_0$ intersects $\operatorname{rint} M''$ if M'' is bounded.

This argument shows that M'' should be unbounded, since otherwise no line $l(u) \in \mathcal{F}_0$ intersects $\operatorname{rint} M = \operatorname{rint} M' \cup \operatorname{rint} M'' \cup]v, w[$, which is impossible due to $\operatorname{int} K \cap H(u_2) \neq \varnothing$. Rotating u about $P \cap S^{n-1}$ in a positive direction from the initial position u_0, we observe that the lines $l(u) \in \mathcal{F}_0$ cover the whole unbounded branch of $\operatorname{rbd} M''$ with endpoint v. Rotating u about $P \cap S^{n-1}$ in a negative direction from the initial position u_0, we see that the lines $l(u) \in \mathcal{F}_0$ cover the second unbounded branch of $\operatorname{rbd} M''$, with endpoint w. This implies the existence of lines $l(u_3), l(u_4) \in \mathcal{F}_0$ such that the line segments $M \cap l(u_3)$ and $M \cap l(u_4)$ have a common interior point. The respective $(n-1)$-dimensional ellipsoids $\operatorname{bd} K \cap H(u_3)$ and $\operatorname{bd} K \cap H(u_4)$ satisfy the conclusion of the lemma. □

Lemma 8. *Let K be strictly convex and regular. If $\operatorname{bd} K$ contains an open piece of a real quadric hypersurface, then $\operatorname{bd} K$ is a convex quadric.*

Proof. Let A be an open piece of a real quadric hypersurface $Q \subset \mathbb{R}^n$ which lies in $\operatorname{bd} K$. We state that $\operatorname{bd} K \subset Q$. Assume, for contradiction, that $\operatorname{bd} K \not\subset Q$, and choose a maximal (under inclusion) open piece B of $\operatorname{bd} K \cap Q$ that contains A. Let $U_r(x) \subset \mathbb{R}^n$ be an open ball with center $x \in B$ and radius $r > 0$ such that $\operatorname{bd} K \cap U_r(x) \subset B$. Continuously moving x towards $\operatorname{bd} K \setminus B$, we find points $x_0 \in B$ and $z_0 \in \operatorname{bd} K \setminus B$ with the property $\operatorname{bd} K \cap U_r(x_0) \subset B$ and $\|x_0 - z_0\| = r$.

Let G be the hyperplane through z_0 which supports K (G is unique since K is regular). Denote by $\mathcal{G}$ the family of $(n-2)$-dimensional planes $L \subset G$ that contain z_0 and are distinct from the $(n-2)$-dimensional plane $L_0 \subset G$ tangent to $U_r(x_0) \cap G$ at z_0. Due to Lemma 5, any plane $L \in \mathcal{G}$ lies in a respective hyperplane $H_L(u)$. By continuity, there is a scalar $t > 0$ so small that the union of $(n-1)$-dimensional convex quadrics $E_L(u) = \operatorname{bd} K \cap H_L(u)$, $L \in \mathcal{G}$, is dense in the hypersurface t-neighborhood $\operatorname{bd} K \cap U_t(z_0)$ of z_0. Each $E_L(u)$ has a nontrivial strictly convex intersection with B. Since $E_L(u)$ is a unique convex quadric containing $E_L(u) \cap B$, we conclude that $E_L(u) \subset Q$. By continuity,

$$\operatorname{bd} K \cap U_t(z_0) \subset \operatorname{cl}\big(\bigcup_{L \in \mathcal{G}} E_L(u)\big) \subset Q.$$

Hence $\operatorname{bd} K \cap U_t(z_0) \subset B$, contrary to the choice of $z_0 \in \operatorname{bd} K \setminus B$. Thus $\operatorname{bd} K \subset Q$. Because $\operatorname{int} K$ is a convex component of $\mathbb{R}^n \setminus Q$, the hypersurface $\operatorname{bd} K$ is a convex quadric. ☐

Lemma 9. *Let E_1 and E_2 be $(n-1)$-dimensional ellipsoids in $\mathbb{R}^n$, $n \geq 3$, which lie, respectively, in hyperplanes H_1 and H_2 of $\mathbb{R}^n$ such that $E = E_1 \cap E_2$ is an $(n-2)$-dimensional ellipsoid. For any point $v \in \mathbb{R}^n \setminus (H_1 \cup H_2)$, there is a quadric hypersurface Q that contains $\{v\} \cup E_1 \cup E_2$.*

Proof. Choose an orthonormal basis for $\mathbb{R}^n$ such that

$$\begin{aligned} E &= \{(0, 0, \xi_3, \dots, \xi_n) \mid \xi_3^2 + \dots + \xi_n^2 = 1\}, \\ E_1 &= \{(\xi_1, 0, \xi_3, \dots, \xi_n) \mid (\xi_1 - \rho_1)^2 + \xi_3^2 + \dots + \xi_n^2 = \rho_1^2 + 1\}, \\ E_2 &= \{(0, \xi_2, \xi_3, \dots, \xi_n) \mid (\xi_2 - \rho_2)^2 + \xi_3^2 + \dots + \xi_n^2 = \rho_2^2 + 1\}, \end{aligned}$$

where $\rho_1 > 0$ and $\rho_2 > 0$. Then H_1 and H_2 are described by the equations $\xi_2 = 0$ and $\xi_1 = 0$, respectively. Consider the family of quadric hypersurfaces $Q(\mu) \subset \mathbb{R}^n$ given by

$$\xi_1^2 + \dots + \xi_n^2 + 2\mu\xi_1\xi_2 - 2\rho_1\xi_1 - 2\rho_2\xi_2 - 1 = 0,$$

where $\mu \in \mathbb{R}$. We have $E_i = H_i \cap Q(\mu)$, $i = 1, 2$. The point $v = (\nu_1, \dots, \nu_n)$ belongs to $\mathbb{R}^n \setminus (H_1 \cup H_2)$ if and only if $\nu_1\nu_2 \neq 0$. Then $v \in Q(\mu_0)$ provided

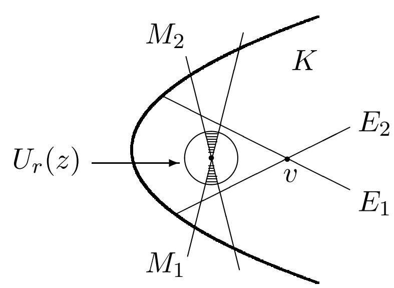
$$\mu_0 = (1 + 2\rho_1\nu_1 + 2\rho_2\nu_2 - \nu_1^2 - \dots - \nu_n^2)/(2\nu_1\nu_2). \qquad ☐$$

Lemma 10. *If K is strictly convex and regular, then* $\operatorname{bd} K$ *contains an open piece of a quadric hypersurface.*

Proof. We proceed by induction on $n\,(\geq 3)$. Let $n = 3$. By Lemma 7, there are planes $H(u_1)$ and $H(u_2)$ such that both sections $E_1 = \operatorname{bd} K \cap H(u_1)$ and $E_2 = \operatorname{bd} K \cap H(u_2)$ are ellipses, with precisely two points, say v and w, in common. The set $\operatorname{bd} K \setminus (E_1 \cup E_2)$ consists of four open pieces, at least three of them being bounded because K is line-free. We choose any of these pieces if K is bounded, and choose the piece opposite to the unbounded one if K is unbounded. Denote by Γ the chosen piece. Let L be a plane through $[v, w]$ that misses Γ and is distinct from both $H(u_1)$ and $H(u_2)$. There is a neighborhood $\Omega \subset \operatorname{bd} K$ of v such that for any point $z \in \Gamma \cap \Omega$, the plane L_z through z parallel to L intersects each of the ellipses E_1 and E_2 at two distinct points.

Choose a point $z \in \Gamma \cap \Omega$ and denote by P_z the plane through z that supports K (P_z is unique since K is regular), and by l_z the line through z parallel to $[v, w]$. Let $\mathcal{F}_\alpha$, $\alpha > 0$, be the family of planes through l_z forming with L_z an angle of size α or less. By continuity, the neighborhood Ω and the scalar α can be chosen so small that for any given plane $M \in \mathcal{F}_\alpha$, every plane $H(u)$ through the line $M \cap P_z$ intersects each of the ellipses E_1 and E_2 at two distinct points. Furthermore, we can find a scalar $r > 0$ such that for any plane $H(u)$ trough z, the convex quadric curve $\operatorname{bd} K \cap H(u)$ intersects the closed curve $\operatorname{bd} K \cap S_r(z)$ at two points, where $S_r(z) \subset \mathbb{R}^3$ is the sphere of radius r centered at z.

Due to Lemma 9, there is a quadric surface Q containing $\{z\} \cup E_1 \cup E_2$. By the above, given a plane $M \in \mathcal{F}_\alpha$, every plane $H(u)$ through the line $M \cap P_z$ intersects $\operatorname{bd} K$ along an ellipse, which has five points in Q (namely, z and two on each ellipse E_i, $i = 1, 2$). Since an ellipse is uniquely defined by five points in general position, the ellipse $E(u) = \operatorname{bd} K \cap H(u)$ lies in Q for any choice of a plane $H(u)$ through the line $M \cap P_z$, where $M \in \mathcal{F}_\alpha$. This argument shows the existence of two open "triangular" regions in $\operatorname{bd} K \cap Q \cap U_r(z)$ which have a common vertex z and are bounded by a pair of planes $M_1, M_2 \in \mathcal{F}_\alpha$ (see the shaded sectors of $\operatorname{bd} K \cap U_r(z)$ in the figure below). Hence the case $n = 3$ is proved.



Suppose that the inductive statement holds for all $m \leq n - 1$, $n \geq 4$, and let $K \subset \mathbb{R}^n$ be a line-free, strictly convex and regular solid that satisfies the hypothesis of Theorem 2. Since the case when K is compact is proved in [4], we may assume that K is unbounded. Then the recession cone $\operatorname{rec} K$ contains halflines and is line-free. Choose a halfline $h \subset \operatorname{rec} K$ with endpoint o such that the $(n-1)$-dimensional subspace $L \subset \mathbb{R}^n$ orthogonal to h satisfies the condition $L \cap \operatorname{rec} K = \{o\}$. Then any proper section of K by a hyperplane parallel to L is bounded (see, e.g., [14]).

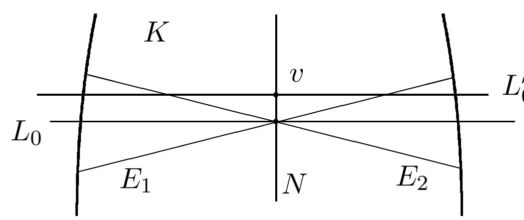
Because the set $\Delta = \{\delta(u)u \mid u \in L \cap S^{n-1}\}$ is compact, we can choose a hyperplane L_0 parallel to L and properly intersecting K so far from Δ that every hyperplane $H(u)$, $u \in L \cap S^{n-1}$, intersects $\operatorname{rint}(K \cap L_0)$. Since any section

$$\operatorname{bd} K \cap H(u) \cap L_0, \quad u \in L \cap S^{n-1},$$

is an $(n-2)$-dimensional convex quadric, $K \cap L_0$ satisfies the hypothesis of Theorem 2 (with L_0 instead of $\mathbb{R}^n$). By the inductive assumption, $\operatorname{rbd}(K \cap L_0)$ contains a relatively open piece of an $(n-1)$-dimensional quadric, and Lemma 8 implies that $\operatorname{bd} K \cap L_0$ is an $(n-1)$-dimensional ellipsoid. Let $G \subset L_0$ be an $(n-2)$-dimensional plane through the center of $K \cap L_0$. By continuity and the argument above, there is an $\varepsilon > 0$ such that the hyperplanes L_1 and L_2 through G forming with L_0 an angle of size ε also intersect $\operatorname{bd} K$ along $(n-1)$-dimensional ellipsoids E_1 and E_2, respectively. Denote by N the hyperplane through G parallel to h, and choose a point $v \in (\operatorname{bd} K \cap N) \setminus (L_1 \cup L_2)$ so close to L_0 that the hyperplane L_0' through v parallel to L_0 satisfies the following conditions (see the figure below):

a) $\operatorname{bd} K \cap L_0'$ is an $(n-1)$-dimensional ellipsoid,

b) L_0' intersects the relative interior of each of the $(n-1)$-dimensional solid ellipsoids $K \cap L_1$ and $K \cap L_2$.



By Lemma 9, there is a real quadric hypersurface Q that contains $\{v\} \cup E_1 \cup E_2$. Since the $(n-1)$-dimensional ellipsoid $E_0' = \operatorname{bd} K \cap L_0'$ is uniquely determined by the set $\{v\} \cup (E_1 \cap L_0') \cup (E_2 \cap L_0')$, we have $E_0' \subset \operatorname{bd} K \cap Q$. By continuity, there is a $\beta > 0$ such that any hyperplane L' through G that forms with L_0' an angle of size β or less satisfies conditions a) and b) above; whence $\operatorname{bd} K \cap L'$ is an $(n-1)$-dimensional ellipsoid that lies in $\operatorname{bd} K \cap Q$. The union of such ellipsoids $\operatorname{bd} K \cap L'$ covers an open piece of Q that lies in $\operatorname{bd} K$. $\square$

Summing up the statements of Lemmas 4–10, we conclude that $\operatorname{bd} K$ is a convex quadric.

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On spaces related to the Navier-Stokes equations

Valery Dryuma

Abstract. Some examples of multidimensional Riemann metrics related to the Navier-Stokes and the Euler equations are constructed. Their properties are discussed.

Mathematics subject classification: 34C14.
Keywords and phrases: Riemann metrics, Navier-Stokes equations, Euler equations.

1 6D-metrics and the NS-equations

With the Navier-Stokes equations

$$\frac{\partial}{\partial t}\vec{V} + \left(\vec{V}\cdot\vec{\nabla}\right)\vec{V} = \mu\Delta\vec{V} + \vec{\nabla}f, \quad \vec{\nabla}\cdot\vec{V} = 0, \tag{1}$$

where $\vec{V} = (U(\vec{x},t), V(\vec{x},t), W(\vec{x},t))$ is the fluid velocity, $P(\vec{x},t)$ is the pressure and μ is the viscosity, $\vec{x} = (x,y,z)$ presented as the conditions of compatibility

$$H_y(\vec{x},t) - E_x(\vec{x},t) = 0, \quad H_z(\vec{x},t) - B_x(\vec{x},t) = 0, \; E_z(\vec{x},t) - B_y(\vec{x},t) = 0, \tag{2}$$

where the functions $H(\vec{x},t),\ E(\vec{x},t),\ B(\vec{x})$ have the form

$$H(\vec{x},t) = f_x(\vec{x},t),\ E(\vec{x},t) = f_y(\vec{x},t),\ B(\vec{x},t) = f_z(\vec{x},t),$$

we can associate 6D-metrics

$$ds^2 = -2\,B(\vec{x},t)dt\,dv + 2\,E(\vec{x},t)dt\,dw + 2\,H(\vec{x},t)dv\,dw -$$

$$-2\left(\int\frac{\partial}{\partial y}H(\vec{x},t)dz\right)dw^2 + dt\,dx + dv\,dy + dw\,dz \tag{3}$$

having fifteen components.

Nine of them are equal to zero if the functions $H(\vec{x},t),\ E(\vec{x},t),\ B(\vec{x},t)$ satisfy the conditions (2). The remaining six components $R_{vv},\ R_{vw},\ R_{ww},\ R_{tt},\ R_{tv},\ R_{tw}$ are expressed in terms of the functions $H(\vec{x},t)$, $E(\vec{x},t)$, $B(\vec{x},t)$ and their derivatives.

Properties of the 6-dimensional space with the metrics (3) can be used for the study of properties of the N-S-equations.

Example. The metrics (3) is Ricci-flat $R_{ik}=0$ if the velocity components of fluid have the form

$$U(\vec{x},t)=-1/2\,x\frac{\partial}{\partial z}F(z,t),\ V(\vec{x},t)=-1/2\,\left(\frac{\partial}{\partial z}F(z,t)\right)y,\ W(\vec{x},t)=F(z,t),$$

and the function $F(z,t)$ satisfies the equation (see [1])

$$-F_{zzt}(z,t)-F(z,t)F_{zzz}(z,t)+\mu\,F_{zzzz}(z,t)=0. \tag{4}$$

In this case the functions $H(\vec{x},t), E(\vec{x},t)$ and $B(\vec{x},t)$ have the form

$$B(\vec{x},t)=F_t(z,t)+F(z,t)F_z(z,t)-\mu\,F_{zz}(z,t),$$

$$H(\vec{x},t)=-1/2\,xF_{zt}(z,t)+1/4\,xF_z(z,t)^2-1/2\,xF(z,t)F_{zz}(z,t)+1/2\,\mu\,xF_{zzz}(z,t),$$

$$E(\vec{x},t)=-1/2\,yF_{zt}(z,t)+1/4\,yF_z(z,t)^2-1/2\,yF(z,t)F_{zz}(z,t)+1/2\,\mu\,yF_{zzz}(z,t).$$

2 12D-metrics and the Euler equations

The Euler system of equations, which is the limit case $\mu=0$ of the NS-equations, is considered as a part of conditions $R_{ik}=0$ on the Ricci tensor of the Riemann metrics of the 12D space in local coordinates $(x,y,z,t,u,v,w,p,\xi,\eta,\chi,\rho)$:

$$\begin{gathered}
ds^2=dv\,d\rho+c(\vec{x},t)d\rho^2+du\,d\chi+a(\vec{x},t)d\chi^2+\\
+\Big((V(\vec{x},t))^2-f(\vec{x},t)\Big)\,dp^2+dz\,d\xi+dy\,dp+dy\,d\chi+dt\,d\eta+2\,\beta(\vec{x})d\eta\,d\rho+\\
+2\,V(\vec{x},t)dp\,d\rho+2\,W(\vec{x},t)V(\vec{x},t)dp\,d\xi+2\,U(\vec{x},t)W(\vec{x},t)dw\,d\xi+\\
+2\,U(\vec{x},t)V(\vec{x},t)dw\,dp+2\,U(\vec{x},t)dw\,d\eta+2\,V(\vec{x},t)dp\,d\chi+2\,U(\vec{x},t)dw\,d\rho+\\
+2\,\epsilon(\vec{x})d\eta\,d\chi+2\,W(\vec{x},t)d\xi\,d\chi+\Big((U(\vec{x},t))^2-f(\vec{x},t)\Big)\,dw^2+2\,b(\vec{x},t)d\chi\,d\rho+\\
+2\,U(\vec{x},t)dw\,d\chi+2\,V(\vec{x},t)dp\,d\eta+2\,W(\vec{x},t)d\xi\,d\rho+2\,W(\vec{x},t)d\xi\,d\eta+\alpha(\vec{x})d\eta^2+\\
+dx\,dw+\Big((W(\vec{x},t))^2-f(\vec{x},t)\Big)\,d\xi^2+dx\,d\rho,
\end{gathered}$$

where $f(\vec{x},t)$ is the pressure of fluid.

Such a metric has 45 components of the Ricci tensor. 21 components from 45 are equal to zero on solutions of the Euler equations.

Let us consider some examples of solutions of the Euler equations defined by the condition $T=R_{ik}\cdot R^{ik}=0$ on scalar invariant of the metric.

Proposition. On the solutions of the Euler equations

1. (Shanko [2])

$$U(x,y,z,t)=\frac{y-x+tx}{t^2},\quad V(x,y,z,t)=\frac{ty+y-2\,x}{t^2},$$

$$W(x,y,z,t) = -2\,\frac{z}{t}, \quad f(x,y,z,t) = -P_0 - 1/2\,\frac{x^2+y^2}{t^4} + 3\,\frac{z^2}{t^2};$$

2.

$$U(x,y,z,t) = -\cos(x)\cos(z), \quad V(x,y,z,t) = -2\,\cos(x)\sin(z),$$

$$W(x,y,z,t) = -\sin(x)\sin(z), \quad f(x,y,z,t) = 1/4\,\cos(2\,x) - 1/4\,\cos(2\,z) + P_0(t);$$

3. (Aristov, Polyanin [1])

$$U(x,y,z,t) = \frac{\sin(kx)}{A\,(\cos(kx)\cos(ky)-1)}, \; V(x,y,z,t) = \frac{\sin(ky)}{A\,(\cos(kx)\cos(ky)-1)},$$

$$f(x,y,z,t) = -\frac{1}{A^2\,(\cos(kx)\cos(ky)-1)}, \quad W(x,y,z,t) = 0$$

the invariant $T = 0$.

3 14D-metrics and the NS-equations

Proposition. With the full system of the NS-equations we can associate the 14D space in local coordinates

$$\mathtt{x}^a = [x,y,z,t,u,v,w,p,\xi,\eta,\chi,\rho,q,\delta]$$

equipped with the Riemann metrics of the form

$$\begin{gathered}
ds^2 = -\int V_t(\vec{x},t)dy\,d\xi^2 - \int W_t(\vec{x},t)dz\,d\eta^2 + dv\;dq + s(\vec{x},t)dq^2 + \\
+2\,\kappa(\vec{x},t)d\delta^2 + c(\vec{x},t)d\rho^2 + 2\,\Big((U(\vec{x},t))^2 - \mu\,U_x(\vec{x},t) - f(\vec{x},t)\Big)\,dp\;d\rho + \\
+2\,U(\vec{x},t)dp\;d\chi - \int U_t(\vec{x},t)dx\,dp^2 + 2\,\epsilon(\vec{x},t)dq\;d\delta + 2\,h(\vec{x},t)d\rho\;d\delta + \\
+2\,\tau(\vec{x},t)d\rho\;dq + 2\,W(\vec{x},t)d\chi\;d\delta + 2\,V(\vec{x},t)d\chi\;dq + 2\,U(\vec{x},t)d\chi\;d\rho + \\
+2\,\big(W(\vec{x},t)^2 - \mu\,W_z(\vec{x},t) - f(\vec{x},t)\big)\,d\eta\;d\delta + 2\,\left(V(\vec{x},t)W(\vec{x},t) - \mu\,V_z(\vec{x},t)\right)d\eta\;dq + \\
+2\,\left(U(\vec{x},t)W(\vec{x},t) - \mu\,U_z(\vec{x},t)\right)d\eta\;d\rho + 2\,\left(V(\vec{x},t)W(\vec{x},t) - \mu\,W_y(\vec{x},t)\right)d\xi\;d\delta + \\
+2\,\big(V(\vec{x},t)^2 - \mu\,V_y(\vec{x},t) - f(\vec{x},t)\big)\,d\xi\;dq + 2\,\left(U(\vec{x},t)V(\vec{x},t) - \mu\,U_y(\vec{x},t)\right)d\xi\;d\rho + \\
+2\,\left(U(\vec{x},t)W(\vec{x},t) - \mu\,W_x(\vec{x},t)\right)dp\;d\delta + 2\,\left(U(\vec{x},t)V(\vec{x},t) - \mu\,V_x(\vec{x},t)\right)dp\;dq + \\
+2\,W(\vec{x},t)d\eta\;d\chi + dy\;d\xi + 2\,V(\vec{x},t)d\xi\;d\chi + dt\;d\chi + dw\;d\delta + \\
+dx\;dp + dz\;d\eta + du\;d\rho. \qquad (5)
\end{gathered}$$

It has 56 components of the Ricci tensor. 28 from them are equal to zero on solutions of the NS-equations.

Proposition. Due to the condition (2) the simplest scalar invariant of the metric (5) $T = R_{ij} \cdot R^{ij} = 0$, but the invariant $S = R_{ijkl} \cdot R^{ijkl}$ of the metrics (5) is not equal to zero and has the form

$$S = 5\,U_{xt}(\vec{x},t)^2 - 4\,U_{xx}(\vec{x},t)U_{tt}(\vec{x},t) + 2\int V_{xxt}(\vec{x},t)dy \int U_{yyt}(\vec{x},t)dx -$$

$$-4\,V_{xx}(\vec{x},t)\int U_{tt}(\vec{x},t)dx - 4\int V_{xtt}(\vec{x},t)dy U_{yy}(\vec{x},t) + 8\,V_{xt}(\vec{x},t)U_{yt}(\vec{x},t) +$$

$$+2\int W_{xxt}(\vec{x},t)dz \int U_{zzt}(\vec{x},t)dx - 4\,W_{xx}(\vec{x},t)\int U_{ztt}(\vec{x},t)dx -$$

$$-4\int W_{xtt}(\vec{x},t)dz U_{zz}(\vec{x},t) + 8\,W_{x}(\vec{x},t)U_{z}(\vec{x},t) + 5\,V_{yt}(\vec{x},t)^2 - 4\,V_{yy}(\vec{x},t)V_{tt}(\vec{x},t) +$$

$$+2\int W_{yyt}(\vec{x},t)dz \int V_{zzt}(\vec{x},t)dy - 4\,W_{yy}(\vec{x},t)\int V_{ztt}(\vec{x},t)dy -$$

$$-4\int W_{ytt}(\vec{x},t)dz V_{zz}(\vec{x},t) + 8\,W_{yt}(\vec{x},t)V_{zt}(\vec{x},t) + 5\,W_{zt}(\vec{x},t)^2 - 4\,W_{zz}(\vec{x},t)W_{tt}(\vec{x},t).$$

On solution of the NS-equations

$$U(x,y,z,t) = \frac{y - x + tx - K(z,t)t\sin(t^{-1})}{t^2},$$

$$V(x,y,z,t) = \frac{y + ty - 2\,x - K(z,t)t\sin(t^{-1}) - \cos(t^{-1})K(z,t)t}{t^2},$$

$$W(x,y,z,t) = -2\,\frac{z}{t}, \quad f(x,y,z,t) = -P_0 - 1/2\,\frac{x^2+y^2}{t^4} + 3\,\frac{z^2}{t^2},$$

where the function $K(z,t)$ satisfies the equation

$$\frac{\partial}{\partial t}K(z,t) = 2\,\frac{z\frac{\partial}{\partial z}K(z,t)}{t} + \mu\,\frac{\partial^2}{\partial z^2}K(z,t) + J(t)$$

with arbitrary $J(t)$, the invariant S takes the form $S = 96\dfrac{5t^2-4}{t^6}$ and the space with the metrics (5) is not Ricci-flat, but its scalar curvature $R = 0$ and invariant $T = 0$.

References

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