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Some extensions of the Bühlmann-Straub credibility formulae

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Abstract. The paper presents some extensions of the Bühlmann-Straub credibility model. In the sequel we describe covariance structures leading to credibility formulae of the updating type, where the new credibility adjusted premium can be computed as a weighted average of the premium quoted in the previous period and the claims in this period. The credibility formula of the updating type is introduced for a wider class of models from the credibility theory, where the risk parameter does not remain the same ever time, and its properties are studied. Also, the expected values (the means) and credibility formulae of the updating type are emphasized. Finally we establish an application which shows that these formulae are attractive from practical point of view, because easy recursive formulae for the computation of the credibility weights (factors) from the Bühlmann-Straub model, can be derived.

Mathematics subject classification: 62P05.
Keywords and phrases: Credibility weights, means and covariances.

1 Introduction

In most models considered in the credibility theory we assume that the risk parameter remains the same over time. If this is not the case, one considers recursive procedures, the formulae of the *updating type*. These are closely related to the theory of Kalman filtering, where it is assumed that the parameters in a linear model themselves arise from a linear process. Because in some models, the covariances between claim sizes are such that credibility formulae arise of the *updating type*, expressing the premium as a mixture of the claims and the credibility premium of the previous observation period, the article presents these formulae and gives an application which characterizes expected values and covariances leading to credibility formulae of the *updating type*. The examples considered show special cases of credibility formulae of the updating type. Finally, is presented an application which shows that there are easy recursive formulae for the computation of the credibility weights from the Bühlmann-Straub model.

2 Theory

One of the Bühlmann-Straub assumptions is that (for this model, each contract $j = 1, \ldots, k$ of the portfolio is the average of a group of contracts, where the weight (size) $w_{j1}, \ldots, w_{jt}$ of the group j is now changing in time; we assume that all contracts have common expectation of the claim size as a function of the risk parameter θ; in addition, apart from the weighting factor w, the variance is also the same function of the risk parameter; these assumptions express the common characteristics of the risk under consideration; so the Bühlmann-Straub assumptions can be formulated as follows:

(**BS**$_1$): $E[X_{jq}|\theta_j] = \mu(\theta_j)$, $j = 1, \ldots, k$, $q = 1, \ldots, t$; $Var[X_{jr}|\theta_j] = \sigma^2(\theta_j)/w_{jr}$, $r = 1, \ldots, t$, $j = 1, \ldots, k$, where all $w_{jr} > 0$; $Cov[X_{jr}, X_{jq}|\theta_j] = 0$, $j = 1, \ldots, k$, $r, q = 1, \ldots, t$, $r \neq q$;

(**BS**$_2$): the contracts $j = 1, \ldots, k$ (i.e. the couples $(\theta_j, \underline{X}_j)$) are independent; the variables $\theta_1, \ldots, \theta_k$ are identically distributed; the observations X_{jr} have finite variance), conditionally given the risk parameter θ_j, claim sizes in different time periods are uncorrelated, that is: $Cov(X_{jr}, X_{jr'}|\theta_j) = 0$, $\forall r, r' = \overline{1,t}$, $r < r'$. The obvious advantage of this assumption is that only two parameters:

$$a \overset{def}{=} Var[\mu(\theta_j)] \overset{def}{=} Var[E(X_{jr}|\theta_j)] \quad \text{and} \quad s^2 = E[\sigma^2(\theta_j)] \overset{def}{=} E[Var(X_{jr}|\theta_j)],$$

$(r = \overline{1,t})$ have to be estimated to determine the whole covariance matrix $Cov[\underline{X}_j]$, because:

$$Cov[\underline{X}_j] \overset{def}{=} [Cov(X_{jr}, X_{jr'})]_{\substack{r,r'=\overline{1,t} \\ r<r'}} =$$

$$= \begin{pmatrix} Cov(X_{j1}, X_{j1}) & Cov(X_{j1}, X_{j2}) & \ldots & Cov(X_{j1}, X_{jt}) \\ Cov(X_{j1}, X_{j2}) & Cov(X_{j2}, X_{j2}) & \ldots & Cov(X_{j2}, X_{jt}) \\ \vdots & \vdots & \vdots & \vdots \\ Cov(X_{j1}, X_{jt}) & Cov(X_{j2}, X_{jt}) & \ldots & Cov(X_{jt}, X_{jt}) \end{pmatrix}$$

But

$$Cov(X_{jr}, X_{jr}) = E[Cov(X_{jr}, X_{jr}|\theta_j)] + Cov[E(X_{jr}|\theta_j), E(X_{jr}|\theta_j)] =$$
$$= E[Var(X_{jr}|\theta_j)] + Cov[\mu(\theta_j), \mu(\theta_j)] = s^2 + Var[\mu(\theta_j)] = s^2 + a, \quad \forall r = \overline{1,t}$$

and

$$Cov(X_{jr}, X_{jr'}) = E[Cov(X_{jr}, X_{jr'}|\theta_j)] + Cov[E(X_{jr}|\theta_j), E(X_{jr'}|\theta_j)] =$$
$$= E(0) + Cov[\mu(\theta_j), \mu(\theta_j)] = 0 + Var[\mu(\theta_j)] = 0 + a = a, \quad \forall r, r' = \overline{1,t},\ r < r',$$

such that we get

$$Cov[\underline{X}_j] = \begin{pmatrix} s^2 + a & a & \ldots & a \\ a & s^2 + a & \ldots & a \\ \vdots & \vdots & \vdots & \vdots \\ a & a & \ldots & s^2 + a \end{pmatrix},$$

where $\underline{X}'_j = (X_{j1}, X_{j2}, \ldots, X_{jt})$ with $j = \overline{1,k}$. But in practice it is quite conceivable that these claim sizes are correlated, such that estimates for their covariances have to be given. For the situation of one contract j to be embedded in a collective of contracts, the classical credibility results have the intuitively appealing form:

$$M^a_{t+1} = (1 - z_j)M_0 + z_j M_j,$$

expressing the premium for contract j and period $(t+1)$ as a mixture of collective and individual experience. In some models the co-variances between the claims sizes are such that credibility formulae arise of the updating type, expressing the premium as a mixture of the claims and the credibility premium of the previous period. These formulae are also attractive because easy recursive formulae for the credibility factors can be derived.

Definition 1 (*Credibility formulae of the updating type)***.** A linear credibility formula is said to be of the *updating type* if there is a sequence $z_1, z_2, \ldots$ of real numbers such that:

$$M^a_{t+1} = (1 - z_t)M^a_t + z_t X_t, \quad t = 1, 2, \ldots \tag{1}$$

with M^a_t the linearized credibility premium for X_t given $X_1, X_2, \ldots, X_{t-1}$.

Remark. A condition equivalent to (1) is that:

$$M^a_{t+1} - M^a_t = z_t(X_t - M^a_t),$$

which shows that the premium adjustment from year t to year $(t+1)$ is proportional to the excess, positive or negative, of claims over premiums in year t.

3 Results and discussion

The following *application* characterizes *expected values* and *covariances* leading to *credibility formulae of the updating type.*

Application 1 (*Means and covariances leading to credibility formulae of the updating type*). Let the numbers c_{tq}, $q = \overline{1,t}$ denote the weights of the claim experience in year q for the (linearized) credibility premium M^a_t in year t, $t = 1, 2, \ldots$, and c_{t0} the constant term, such that:

$$M^a_{t+1} = c_{t0} + \sum_{q=1}^{t} c_{tq} X_q. \tag{2}$$

Then the credibility formulae M^a_t are of the updating type, if and only if there exists a number m and sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ with $b_q > 0$ such that for all $q, r = 1, 2, \ldots,$

$$E(X_r) = m, \tag{3}$$

$$Cov(X_r, X_q) = \begin{cases} a_r,\ r < q & (r = \overline{1, q-1}), \\ b_r,\ r = q, & \\ a_q,\ r > q & (r = \overline{q+1, t}). \end{cases} \tag{4}$$

Proof. Using the system of equations:

$$E[\mu(\theta)] - c_0 - \sum_{q=1}^{t} c_q E[X_q] = 0 \tag{5}$$

and

$$Cov[\mu(\theta), X_r] = \sum_{q=1}^{t} c_q Cov[X_r, X_q], \quad r = \overline{1,t} \tag{6}$$

from *the original credibility model of Bühlmann* (see *the observation*, which we end Application 1) determining *the optimal credibility estimator* in the proof of *Bühlmann's optimal credibility estimator*, applied to X_{t+1} rather than $\mu(\theta)$, we see that the weights c_{tq} and the means / covariances must obey the following relations:

$$E[X_{t+1}] = c_{t0} + \sum_{q=1}^{t} c_{tq} E[X_q] \tag{7}$$

and

$$Cov[X_{t+1}, X_r] = \sum_{q=1}^{t} c_{tq} Cov[X_r, X_q], \quad r = 1, t. \tag{8}$$

We write the condition (7) as

$$E[X_{t+1}] = E\left[c_{t0} + \sum_{q=1}^{t} c_{tq} X_q\right],$$

that is (see (2)):

$$E[X_{t+1}] = E[M_{t+1}^a], \quad t = 1, 2, \ldots$$

We have

$$E[M_{t+1}^a] = E[X_{t+1}], \quad t = 1, 2, \ldots \tag{9}$$

Condition (9) expresses that M_{t+1}^a is unbiased. For the 'only if' – part of the application, suppose that the credibility formulae M_t^a are of the updating type. Taking expectation in (1) gives:

$$E[M_{t+1}^a] = (1 - z_t)E[M_t^a] + z_t E[X_t] = (1 - z_t)E[X_t] + z_t E[X_t] = E[X_t],\ t = 1, 2, \ldots$$

So

$$E[M_{t+1}^a] = E[X_t], \quad t = 1, 2, \ldots \tag{10}$$

From (9) and (10) it follows that $E[X_{t+1}] = E[X_t]$ for all t which proves (3). Replacing the $M_t^{a'}s$ in (1) with their definition (2), that is:

$$\begin{aligned} M_{t+1}^a &\overset{(1)}{=} (1 - z_t)\left[c_{t-1,0} + \sum_{q=1}^{t-1} c_{t-1,q} X_q\right] + z_t X_t = \\ &= (1 - z_t)c_{t-1,0} + \sum_{q=1}^{t-1} (1 - z_t) \cdot c_{t-1,q} X_q + z_t X_t \end{aligned} \tag{11}$$

and comparing the coefficients of the $X_q's$ (see (11) and (2)) one gets:

$$c_{t0}+\sum_{q=1}^{t-1}c_{tq}X_q+c_{tt}X_t=(1-z_t)c_{t-1,0}+\sum_{q=1}^{t-1}(1-z_t)c_{t-1,q}X_q+z_tX_t.$$

We have

$$\begin{cases} c_{t0}=(1-z_t)c_{t-1,0}, \\ c_{tq}=(1-z_t)c_{t-1,q}, \quad q=\overline{1,t-1}, \\ c_{tt}=z_t. \end{cases}$$

So

$$c_{tq}=(1-z_t)c_{t-1,q}, \quad q=\overline{0,t-1} \tag{12}$$

and

$$c_{tt}=z_t. \tag{13}$$

Inserting (12) in (8) and again applying (8) for $(t-1)$ one obtains:

$$Cov[X_r,X_{t+1}]\overset{(8)}{=}\sum_{q=1}^{t}c_{tq}Cov[X_q,X_r]=\sum_{q=1}^{t-1}c_{tq}Cov[X_q,X_r]+c_{tt}Cov[X_t,X_r]=$$

$$=\sum_{q=1}^{t-1}(1-z_t)c_{t-1,q}Cov[X_q,X_r]+z_tCov[X_t,X_r]=z_tCov[X_t,X_r]+(1-z_t)\times$$

$$\times\sum_{q=1}^{t-1}c_{t-1,q}Cov[X_q,X_r]=z_t\sum_{q=1}^{t-1}c_{t-1,q}Cov[X_q,X_r]+\sum_{q=1}^{t-1}c_{t-1,q}Cov[X_q,X_r]-$$

$$-z_t\sum_{q=1}^{t-1}c_{t-1,q}\cdot Cov[X_q,X_r]=Cov[X_t,X_r]=Cov[X_r,X_t],\ \text{for } r=\overline{1,t-1}.$$

Therefore we may write

$$Cov[X_r,X_{t+1}]=Cov[X_r,X_t]=a_r, \quad r=\overline{1,r-1},$$
$$Cov[X_r,X_t]=b_r.$$

For the 'if'- part of the application, assume that (3) and (4) hold. Then to prove (1), we have to show that (12) and (13) hold again. From (2) we get using (8) for each $r=\overline{1,t-1}$:

$$\sum_{q=1}^{t-1}c_{tq}Cov[X_q,X_r]\overset{(8)}{=}Cov[X_r,X_{t+1}]-c_{tt}Cov[X_r,X_t]=a_r-c_{tt}Cov[X_r,X_t]=$$

$$=Cov[X_r,X_t]-c_{tt}Cov[X_r,X_t]=(1-c_{tt})Cov[X_r,X_t]=(1-c_{tt})\sum_{q=1}^{t-1}c_{t-1,q}\times$$

$$\times Cov[X_q,X_r]=\sum_{q=1}^{t-1}(1-c_{tt})\cdot c_{t-1,q}\cdot Cov[X_q,X_r].$$

So

$$c_{tq} = (1 - z_t)C_{t-1,q}; \quad q = \overline{1, t-1}, \tag{14}$$

where $z_t \overset{(not)}{=} c_{tt}$.

This formula also holds because of (7):

$$(7) \Leftrightarrow m = c_{t0} + \sum_{q=1}^{t-1}(1 - z_t)c_{t-1,q} \cdot m + z_t \cdot m \Leftrightarrow c_{t0} =$$

$$= (1 - z_t) \cdot m - (1 - z_t) \cdot \left[\sum_{q=1}^{t-1} c_{t-1,q}\right] \cdot m \Leftrightarrow c_{t0} = (1 - z_t) \cdot c_{t-1,0},$$

because from (7) applied to $(t-1)$ we conclude that: $m = c_{t-1,0} + \sum_{q=1}^{t-1} c_{t-1,q}m$, that is: $c_{t-1,0} = m - \left[\sum_{q=1}^{t-1} c_{t-1,q}\right]$, so we conclude that indeed (14) also holds for $t = 0$. Therefore we may write:

$$M_{t+1}^a = c_{t0} + \sum_{q=1}^{t-1} c_{tq}X_q + c_{tt}X_t = (1 - z_t)c_{t-1,0} + \sum_{q=1}^{t-1}(1 - z_t)c_{t-1,q}X_q + c_{tt}X_t =$$

$$= (1 - z_t)\left[c_{t-1,0} + \sum_{q=1}^{t-1} c_{t-1,q}X_q\right] + c_{tt}X_t = (1 - z_t)M_t^a + z_tX_t.$$

□

A covariance matrix such as in (4) can be depicted as follows:

$$Cov[\underline{X}] = (Cov(X_r, X_q))_{r,q=\overline{1,t}} = \begin{pmatrix} b_1 & a_1 & a_1 & \dots & a_1 & \dots & a_1 \\ a_1 & b_2 & a_2 & \dots & a_2 & \dots & a_2 \\ a_1 & a_2 & b_3 & \dots & a_3 & \dots & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_1 & a_2 & a_3 & \dots & b_q & \dots & a_q \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & a_q & \dots & b_r \end{pmatrix}.$$

As a *special case*, when $Cov[X_i, X_j] = a + \delta_{ij}s^2$ like in Bühlmann's models we obtain $z_t = at/(at + s^2)$ leading to *uniform* credibility weights:

$$c_{t1} = c_{t2} = \ldots = c_{tt} = at/(at + s^2).$$

Another *special case* of credibility formulae of the updating type arises when $z_t = z$, $t = 1, 2, \ldots$; then one obtains *geometric* credibility weights:

$$\begin{aligned} c_{tq} &= (1 - z_t)c_{t-1,q} = (1 - z)c_{t-1,q} = (1 - z)(1 - z_{t-1})c_{t-2,q} = \\ &= (1 - z)^2 c_{t-2,q} = \ldots = (1 - z)^{t-q} c_{qq} = (1 - z)^{t-q} z_q = (1 - z)^{t-q} z \end{aligned}$$

If the means and covariances are as in Application 1 there are easy recursive formulae for the computation of the credibility weights z_t.

Observation 1 (*The original credibility model of Bühlmann*). In the original credibility model of Bühlmann, we consider one contract with unknown and fixed risk parameter θ, during a period of t years. The yearly claim amounts are noted by $X_1, \ldots, X_t$. The risk parameter θ is supposed to be taken from some structure distribution $U(\cdot)$. It is assumed that, for given $\theta = \theta$, the claims are conditionally independent and identically distributed with known common distribution function $F_{X|\theta}(x, \theta)$. For this model we want to estimate the net premium $\mu(\theta) = E[X_r|\theta = \theta]$, $r = \overline{1, t}$ as well as X_{t+1} for a contract with risk parameter θ.

We present the following result:

Bühlmann's optimal credibility estimator. Suppose $X_1, \ldots, X_t$ are random variables with finite variation, which are, for given $\theta = \theta$, conditionally independent and identically distributed with already known common distribution function $F_{X|\theta}(x, \theta)$. The structure distribution function is $U(\theta) = P[\theta \leq \theta]$. Let D represent the set of non-homogeneous linear combinations $g(\cdot)$ of the observable random variables $X_1, \ldots, X_t$: $g(\underline{X}') = c_0 + c_1 X_1 + \ldots + c_t X_t$. Then the solution of the problem: $\operatorname*{Min}_{g \in D} E\{[\mu(\theta) - g(X_1, \ldots, X_t)]^2\}$ is: $g(X_1, \ldots, X_t) = M^a = z\overline{Z} + (1 - z)m$, where $\underline{X}' = (X_1, \ldots, X_t)$ is the vector of observations, $z = at/(s^2 + at)$, is the resulting credibility factor, $\overline{X} = \dfrac{1}{t}\displaystyle\sum_{i=1}^{t} X_i$ is the individual estimator, and a, s^2 and m are the structural parameters as defined by the following formulae: $m = E[X_r] = E[\mu(\theta)]$, $r = \overline{1, t}$, $a = Var\{E[X_r|\theta]\} = Var[\mu(\theta)]$, $r = \overline{1, t}$, $\sigma^2(\theta) = Var[X_r|\theta]$, $r = \overline{1, t}$, $s^2 = E\{Var[X_r|\theta]\} = E[\sigma^2(\theta)]$, $r = \overline{1, t}$. If $\mu(\theta)$ is replaced by X_{t+1} in the above minimization problem, exactly the same solution M^a is obtained, since the co-variations with $\underline{X}$ are the same.

Proof. We have to solve the following minimization problem:

$$\operatorname*{Min}_{c_0, \ldots, c_t} E\left\{\left[\mu(\theta) - c_0 - \sum_{r=1}^{t} c_r X_r\right]^2\right\}.$$

Since the above problem is the minimum of a positive definite quadratic form, it suffices to find a solution with all partial derivatives equal to zero. Taking the partial derivative with respect to c_0 we get the equation: $E\left[\mu(\theta) - c_0 - \displaystyle\sum_{r=1}^{t} c_r X_r\right] = 0$

(see (5)). Using $m = E[X_r] = E[\mu(\theta)]$, we may solve this equation for c_0 and insert the result in the minimization problem. We get:

$$\underset{c_1,\dots,c_t}{\text{Min}}\; E\left\{\left[\mu(\theta) - m - \sum_{r=1}^{t} c_r(X_r - m)\right]^2\right\}.$$

Taking the derivative with respect to c_q, $q = 1,\dots,t$ leads to the equation: $E\left\{-2\left[\mu(\theta) - m - \sum_{r=1}^{t} c_r(X_r - m)\right]\cdot(X_q - m)\right\} = 0$, $q = 1,\dots,t$. This is equivalent to: $Cov[\mu(\theta), X_q] = \sum_{r=1}^{t} c_r Cov(X_q, X_r)$, $q = 1,\dots,t$ (see (6)) . Since $Cov(X_q, X_r) = a + \delta_{rq}s^2$ and $Cov[\mu(\theta), X_q] = a$ and since the system of equations is symmetrical in $c_1,\dots,c_t$ one finds from: $Cov[\mu(\theta), X_q] = \sum_{r=1}^{t} c_r Cov(X_q, X_r)$, $q = 1,\dots,t$ that: $c_1 = c_2 = \dots = c_t = a/(s^2 + at)$. Now introducing $z = at/(s^2 + at)$, from $E\left[\mu(\theta) - c_0 - \sum_{r=1}^{t} c_r X_r\right] = 0$ we see that $c_0 = (1 - z)\cdot m$, so M^a is optimal.

Application 2 (*Expressions for credibility weights*). Under the conditions of the previous application, and writing $s_t = b_t - a_t$ the credibility weights z_t can be calculated by means of:

$$\begin{cases} z_1 = a_1/(a_1 + s_1), \\ z_t = (a_t - a_{t-1} + z_{t-1}s_{t-1})/(a_t - a_{t-1} + z_{t-1}s_{t-1}),\ t = 2, 3, \dots \end{cases}$$

Proof. Equation (12) for $t = q = 1$, together with (8) and (4), gives the expression for z_1:

$$z_1 = c_{11} = \frac{Cov(X_2, X_1)}{Cov(X_1, X_1)} = \frac{a_1}{b_1} = \frac{a_1}{s_1 + a_1}.$$

Equation (8) for $r = t$, together with (12) gives:

$$\begin{aligned} Cov[X_t, X_{t+1}] &= c_{tt}Cov[X_t, X_t] + \sum_{q=1}^{t-1} c_{tq}Cov[X_q, X_t] = \\ &= z_t Var[X_t] + (1 - z_t)\cdot\sum_{q=1}^{t-1} c_{t-1,q}\cdot Cov[X_q, X_t]. \end{aligned} \tag{15}$$

The summation in (15) can be rewritten as:

$$
\begin{aligned}
&c_{t-1,t-1}Cov[X_{t-1},X_t]+\sum_{q=1}^{t-2}c_{t-1,q}Cov[X_q,X_t]=z_{t-1}Cov[X_{t-1},X_t]+\\
&+\sum_{q=1}^{t-2}c_{t-1,q}a_q=z_{t-1}Cov[X_{t-1},X_t]+\sum_{q=1}^{t-2}c_{t-1,q}Cov[X_q,X_{t-1}]=\\
&=z_{t-1}Cov[X_{t-1},X_t]-c_{t-1,t-1}\cdot Cov[X_{t-1},X_{t-1}]+\sum_{q=1}^{t-1}c_{t-1,q}Cov[X_q,X_{t-1}]=\\
&\overset{(8)}{=}z_{t-1}\{Cov[X_{t-1},X_t]-Var[X_{t-1}]\}+Cov[X_t,X_{t-1}],\ t\geq 2
\end{aligned}
\tag{16}
$$

Inserting (16) in (15) and again because of (4) one gets for (15):

$$
\begin{aligned}
a_t=Cov[X_t,X_{t+1}]&=z_tb_t+(1-z_t)\{z_{t-1}(a_{t-1}-b_{t-1})+a_{t-1}\}=\\
&=z_t(s_t+a_t)+(1-z_t)(a_{t-1}-z_{t-1}s_{t-1}),\ t\geq 2.
\end{aligned}
$$

We have

$$a_t=z_t(s_t+a_t)+(1-z_t)(a_{t-1}-z_{t-1}s_{t-1}),$$

that is

$$z_t=(a_t-a_{t-1}+z_{t-1}s_{t-1})/(a_t-a_{t-1}+z_{t-1}s_{t-1}+s_t),$$

□

4 Conclusions

The paper describes covariance structures leading to credibility formulae of the updating type, where the new credibility adjusted premium can be computed as a weighted average of the premium quoted in the previous period and the claims in this period. So, the credibility formulae of the updating type for the credibility factors from the Bühlmann-Straub model can be derived. In other models from the credibility theory, the covariances between the claims sizes are such that credibility formulae arise of the *updating type*, expressing the premium as a mixture of the claims and the credibility premium of the previous observation period.

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Postoptimal analysis of multicriteria combinatorial center location problem

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Abstract. A multicriteria variant of a well known combinatorial MINMAX location problem with Pareto and lexicographic optimality principles is considered. Necessary and sufficient conditions of an optimal solution stability of such problems to the initial data perturbations are formulated in terms of binary relations. Numerical examples are given.

Mathematics subject classification: 90C27, 90C29, 90C31, 90C47.
Keywords and phrases: Center location problem, Pareto optimal trajectory, lexicographically optimal trajectory, perturbing matrix, trajectory stability, binary relations, stability criteria.

1 Introduction

Many problems of design, planning and management in technical and organizational systems have a pronounced multicriteria character. Multiobjective models appeared in these cases are reduced to the choice of "best" (in a certain sense) values of variable parameters from some discrete aggregate of the given quantities. Therefore recent interest of mathematicians in multicriteria discrete optimization problems keeps very high, as confirmed by the intensive publishing activity (see, e.g., bibliography [1], which contains 234 references).

While solving practical optimization problems, it is necessary to take into account various kinds of uncertainty such as lack of input data, inadequacy of mathematical models to real processes, rounding off, calculation errors, etc. Therefore widespread use of discrete optimization models in the last decades stimulated many experts to investigate various aspects of incorrect problems theory and, in particular, to the questions of stability. The most important results in this topic are concerned with postoptimal and parametric behavior analysis of the solutions of the optimization problems with respect to variation of their input data. Generally the technique of such analysis is based on using the properties of multi-valued functions. Such research methods are elaborated in detail and covered in literature about optimization problems with a continuous set of feasible solutions. Numerous articles are devoted to the analysis of conditions when problem possesses some property of invariance under the problem parameters perturbations (see, e. g., [2–5]).

The main difficulty while studying stability of discrete optimization problems is the essential complexity of discrete models. They behave unpredictable even for small changes of initial data. There are a lot of papers (see, e. g., [6–15]) devoted to the analysis of scalar and vector (multicriteria) discrete optimization problems sensitivity to parameters perturbations. The present work continues our investigations of different stability types of such problems with various partial criteria and optimality principles (see, e. g., [16–23]). The here multicriteria variant of the well-known center location problem (p-center problem) is considered. Some necessary and sufficient conditions of lexicographic and Pareto optima stability under perturbations of initial data are obtained. Numerical examples are given.

2 Basic definitions and notations

Problems of finding the "best" location of equipment and facilities abound in practical situations. Often such problems are formulated as extreme problems in graphs and networks. In particular, if a graph represents a road network with its vertices representing communities, one may have the problem of locating optimally a hospital, fire station or any other emergency service facility. The criterion of optimality may justifiably be taken to be the minimization of the distance (traveling time or other costs) from the facility to the most remote vertex of the graph, i. e. the optimization of the worst-case. In a more general problem, a large number of such facilities may be required to be located. For instance, in the problems which involve the location of emergency facilities it is required to minimize the largest travel distance to any consumer from its nearest facility (center). If there are several costs criteria which have to be minimized, the vector variant of the center location problem arises. Let us consider this problem in the following formulation.

Let $N_m = \{1, 2, \ldots, m\}$ be the set of possible points (centers) of suppliers (equipment, storehouses, facility, etc.) location, N_n be consumers (clients) location, $A = (a_{ijk}) \in \mathbb{R}^{m\times n\times s}$ be the cost matrix a_{ijk}. The cost is connected with delivery of required quantity of products from point $i \in N_m$ to point $j \in N_n$ with criterion $k \in N_s$.

On the set T of nonempty subsets (trajectories) $T \subset 2^{N_m}$, $|T| \geq 2$, let the vector function

$$f(t, A) = (f_1(t, A), f_2(t, A), \ldots, f_s(t, A))$$

be defined with "bottle neck" (MINMAX) criteria:

$$f_k(t, A) = \max_{j\in N_n} \min_{i\in t} a_{ijk} \to \min_{t\in T},\ k \in N_s.$$

We give the traditional definition of the set of Pareto optimal trajectories:

$$P^s(A) = \{t \in T : \forall t' \in T\backslash\{t\}\ \ (t \underset{A,P}{\not\succ} t')\},$$

where

$$t \underset{A,P}{\succ} t' \ \Leftrightarrow\ f(t, A) \geq f(t', A) \& f(t, A) \neq f(t', A),$$

and the sign $\underset{A,P}{\overline{\succ}}$ is a negation of the relation $\underset{A,P}{\succ}$. The set $P^s(A)$ is nonempty for any matrix $A \in \mathbb{R}^{m\times n\times s}$ as $1 < |T| < \infty$.

The set of lexicographically optimal trajectories is denoted by the formula:

$$L^s(A) = \{t \in T \ : \ \forall t' \in T \ \ (t \underset{A,L}{\overline{\succ}} t')\},$$

where

$$t \underset{A,L}{\succ} t' \ \Leftrightarrow \ \exists\, l \in N_s \ (f_l(t,A) > f_l(t',A) \ \& \ l = \min\{k \in N_s : f_k(t,A) \neq f_k(t',A)\}),$$

and the sign $\underset{A,L}{\overline{\succ}}$ is a negation of the relation $\underset{A,L}{\succ}$. It is easy to see that $L^s(A) \subseteq P^s(A)$ for any matrix $A \in \mathbb{R}^{m\times n\times s}$.

Thus, two multicriteria center location problems appear: with Pareto principle of optimality, i. e. the problem of finding the set $P^s(A)$, and with lexicographic principle of optimality, i. e. the problem of finding the set $L^s(A)$.

In particular in scalar case ($s = 1$) we get the well-known p-center problem [24–27], i.e. minimax location problem:

$$\max_{j\in N_n} \min_{i\in t} a_{ij} \to \min$$

$$t \in T, \ |t| = p,$$

where p is an integer number, which satisfies the inequalities $1 \le p \le m-1$. Thereby in this problem the situation is modeled, when it is required to locate p facilities in N_m possible points to minimize the largest travel distance to any consumer from its nearest facility.

It is known (see, e. g., [28]), that the set of lexicographically optimal trajectories $L^s(A)$ can be defined as the result of solving the sequence of s scalar problems

$$L^s_k(A) = \operatorname{Arg\,min}\{f_k(t,A) \mid t \in L^s_{k-1}(A)\}, \qquad k \in N_s, \tag{1}$$

where $L^s_0(A) = T$, $\operatorname{Arg\,min}\{\cdot\}$ is the set of all optimal trajectories of the corresponding scalar optimization problem. Hence the following inclusions

$$T \supseteq L^s_1(A) \supseteq L^s_2(A) \supseteq \ldots \supseteq L^s_s(A) = L^s(A) \tag{2}$$

are true.

Perturbations of the vector criterion $f(t,A)$ parameters are modeled by adding matrix A to the matrices of the set

$$\Omega(\varepsilon) = \{A' \in \mathbb{R}^{m\times n\times s} \ \ : \ \ ||A'|| < \varepsilon\},$$

where $\varepsilon > 0$, $||A'|| = \max\{|a'_{ijk}| \ : \ (i,j,k) \in N_m \times N_n \times N_s\}$, $A' = (a'_{ijk})$. The set $\Omega(\varepsilon)$ is called the set of perturbing matrices.

Pareto optimal trajectory $t \in P^s(A)$ is called stable if

$$\exists \varepsilon > 0 \quad \forall A' \in \Omega(\varepsilon) \quad (t \in P^s(A + A')).$$

Lexicographically optimal trajectory t is called stable if

$$\exists \varepsilon > 0 \quad \forall A' \in \Omega(\varepsilon) \quad (t \in L^s(A + A')).$$

To prove stability criteria, we consider a number of evident properties and also formulate and prove 4 lemmas.

3 Properties

Directly from definitions of the binary relations $t \underset{A,P}{\succ} t'$ and $t \underset{A,L}{\succ} t'$ follows

Property 1. *If* $t \underset{A,P}{\succ} t'$*, then* $t \underset{A,L}{\succ} t'$.

For any indexes $k \in N_s,\ j \in N_n$ and trajectory t put

$$N_{jk}(t, A) = \{l \in t \ : \ f_k(t, A) = g_{jk}(t, A) = a_{ljk}\},$$

$$J_k(t, A) = \{j \in N_n \ : f_k(t, A) = g_{jk}(t, A)\},$$

where

$$g_{jk}(t, A) = \min_{i \in t} a_{ijk}.$$

Next properties directly follow from these notions.

Property 2. *If* $q \in J_k(t, A)$*, then* $f_k(t, A) = g_{qk}(t, A)$.

Property 3. *If* $q \in J_k(t, A)$ *and* $p \in N_{qk}(t, A)$*, then* $f_k(t, A) = g_{qk}(t, A) = a_{pqk}$.

Property 4. $N_{jk}(t, A) \neq \emptyset$ *if and only if* $j \in J_k(t, A)$.

Property 5. *If* $N_{jk}(t, A) = \emptyset$*, then* $g_{jk}(t, A) < f_k(t, A)$.

Property 6. *If* $g_{jk}(t, A) > g_{jk}(t', A)$*, then there exists an index* $p \in t' \backslash t$ *such that* $g_{jk}(t', A) = g_{jk}(t' \backslash t, A) = a_{pjk}$.

For any index $k \in N_s$ we define several binary relations on the set of trajectories T

$$t \underset{A,k}{\vdash} t' \ \Leftrightarrow \ t \underset{A,k}{\models\!\!\sim} t' \underset{A,k}{\models\!\!\approx} t,$$

$$t \underset{A,k}{\models\!\!\sim} t' \ \Leftrightarrow \ \forall j \in J_k(t, A) \ \ (N_{jk}(t, A) \supseteq N_{jk}(t', A)),$$

$$t' \underset{A,k}{\models\!\!\approx} t \ \Leftrightarrow \ J_k(t', A) \supseteq J_k(t, A).$$

Furthermore, we will use binary relations

$$t \underset{A}{\vdash} t' \;\Leftrightarrow\; \forall k \in N_s \;\; (t \underset{A,k}{\vdash} t'),$$

$$t \underset{A}{\sim} t' \;\Leftrightarrow\; f(t,A) = f(t',A).$$

By virtue of continuity of the function $g_{jk}(t,A)$ in parameters space $\mathbb{R}^m$ from the relations $N_{jk}(t,A) \supseteq N_{jk}(t',A) \neq \emptyset$ the formula follows

$$\exists \varepsilon > 0 \;\; \forall A' \in \Omega(\varepsilon) \;\; (g_{jk}(t, A+A') \leq g_{jk}(t', A+A')). \tag{3}$$

Therefore the following property holds

Property 7. *If for any index $k \in N_s$ the relation $t \underset{A,k}{\vdash\!\sim} t'$ holds, then*

$$\exists \varepsilon > 0 \;\; \forall A' \in \Omega(\varepsilon) \;\; \forall k \in N_s \;\; \forall j \in J_k(t, A+A') \;\; (g_{jk}(t, A+A') \leq g_{jk}(t', A+A')).$$

Property 8. *If $t \underset{A}{\vdash} t'$, then*

$$\exists \varepsilon > 0 \;\; \forall A' \in \Omega(\varepsilon) \;\; \forall k \in N_s \;\; (f_k(t, A+A') \leq f_k(t', A+A')).$$

Property 9. *If $t \underset{A}{\vdash} t'$, then there exists a number $\varepsilon > 0$ such that for any perturbing matrix $A' \in \Omega(\varepsilon)$ the following relation holds*

$$t \underset{A+A',P}{\overline{\succ}} t'.$$

Property 10. *If any of the following conclusions holds for trajectories t and t'*

$$(i) \;\; f_1(t', A) > f_1(t, A),$$

$$(ii) \;\; \exists r \in N_{s-1} \;\; (f_{r+1}(t', A) > f_{r+1}(t, A) \;\&\; \forall k \in N_r \;\; (t \underset{A,k}{\vdash} t')),$$

then the formula

$$\exists \varepsilon > 0 \;\; \forall A' \in \Omega(\varepsilon) \; (t \underset{A+A',L}{\overline{\succ}} t') \tag{4}$$

is true

Proof. If $f_1(t', A) > f_1(t, A)$, then in view of continuity of the function $f_k(t, A)$ in parameters space $\mathbb{R}^{m \times n}$ we have

$$\exists \varepsilon > 0 \;\; \forall A' \in \Omega(\varepsilon) \;\; (f_1(t', A+A') > f_1(t, A+A')).$$

Hence (4) holds. □

Now let condition (ii) hold. Then, using $t \underset{A,k}{\not\sim} t'$, $k \in N_r$, in view of (3) we get

$$\exists \varepsilon' > 0 \ \ \forall A' \in \Omega(\varepsilon') \ \ \forall k \in N_r \ \ \forall j \in J_k(t, A + A') \ \ (g_{jk}(t, A + A') \leq g_{jk}(t', A + A')).$$

Therefore

$$\exists \varepsilon' > 0 \ \ \forall A' \in \Omega(\varepsilon') \ \ \forall k \in N_r \ \ (f_k(t, A + A') \leq f_k(t', A + A')). \tag{5}$$

In addition, since $f_{r+1}(t', A) > f_{r+1}(t, A)$ it follows that

$$\exists \varepsilon'' > 0 \ \ \forall A' \in \Omega(\varepsilon'') \ \ (f_{r+1}(t', A + A') > f_{r+1}(t, A + A')). \tag{6}$$

Assuming $\varepsilon = \min\{\varepsilon', \varepsilon''\}$, we derive (4) from (5) and (6).

4 Lemmas

Set

$$\overline{P^s}(A) = T \setminus P^s(A).$$

Lemma 1. *If $t^0 \in P^s(A)$, $t^0 \underset{A}{\sim} t$ and there exists an index $r \in N_s$ such that $t \underset{A,r}{\overline{\approx}} t^0$, then the trajectory t^0 is not stable.*

Proof. From $t \underset{A,r}{\overline{\approx}} t^0$ it follows that there exists an index $q \in J_r(t^0, A) \backslash J_r(t, A)$. Therefore according to property 4 $N_{qr}(t, A) = \emptyset$. Hence using property 5 we have $g_{qr}(t, A) < f_r(t, A)$ and applying property 2 we derive $f_r(t^0, A) = g_{qr}(t^0, A)$. Thus, taking into account $t^0 \underset{A}{\sim} t$ we obtain $g_{qr}(t^0, A) > g_{qr}(t, A)$. Hence in view of property 6 there exists an index $p \in t \backslash t^0$, such that

$$g_{qr}(t, A) = g_{qr}(t \backslash t^0, A) = a_{pqr}. \tag{7}$$

For any number $\varepsilon > 0$ we build elements of the perturbing matrix $A^0 = (a^0_{ijk}) \in \Omega(\varepsilon)$ of size $m \times n \times s$ by the rule

$$a^0_{ijk} = \begin{cases} \alpha, & \text{if } i \in t^0, \ j = q, \ k = r, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < \alpha < \varepsilon$. We show that $t^0 \in \overline{P^s}(A + A^0)$. According to the matrix construction the following equalities hold

$$g_{qr}(t^0, A + A^0) = g_{qr}(t^0, A) + \alpha,$$

$$g_{jr}(t^0, A + A^0) = g_{jr}(t^0, A) \text{ for } j \neq q,$$

$$g_{jr}(t, A + A^0) = g_{jr}(t, A) \text{ for } j \neq q,$$

and by (7) it follows that

$$g_{qr}(t, A + A^0) = g_{qr}(t, A).$$

Hence we derive

$$f_r(t^0, A+A^0) = \max_{j\in N_n} g_{jr}(t^0, A+A^0) = \max\{g_{qr}(t^0, A+A^0),\ \max_{j\neq q} g_{jr}(t^0, A+A^0)\} =$$

$$= \max\{g_{qr}(t^0, A) + \alpha,\ \max_{j\neq q} \min_{i\in t^0} a_{ijq}\} = f_r(t^0, A) + \alpha,$$

$$f_r(t, A+A^0) = \max_{j\in N_n} g_{jr}(t, A+A^0) = \max\{g_{qr}(t, A),\ \max_{j\neq q} \min_{i\in t} a_{ijr}\} = f_r(t, A).$$

It follows from these equalities that

$$f_r(t^0, A+A^0) > f_r(t, A+A^0). \tag{8}$$

Furthermore, taking into account the construction of the perturbing matrix A^0 and the relation $t^0 \underset{A}{\sim} t$ the following equalities are evident

$$f_k(t^0, A+A^0) = f_k(t, A+A^0) \ \text{ for } \ k \neq r. \tag{9}$$

Therefore

$$t^0 \underset{A+A^0,P}{\succ} t.$$

Thus we have

$$\forall \varepsilon > 0 \ \ \exists A^0 \in \Omega(\varepsilon) \ \ (t^0 \in \overline{P^s}(A+A^0)), \tag{10}$$

i. e. trajectory t^0 is not stable. □

Lemma 2. *If $t^0 \in P^s(A)$, $t^0 \underset{A}{\sim} t$ and there exists an index $r \in N_s$ such that $t^0 \underset{A,r}{\overline{\approx}} t$, then trajectory t^0 is not stable.*

Proof. We assume that $t \underset{A,r}{\not\approx} t^0$. Otherwise t^0 is not stable by virtue of Lemma 1. Since $t^0 \underset{A,r}{\overline{\approx}} t$, then in view of $t \underset{A,r}{\not\approx} t^0$ there exists an index $q \in J_r(t, A) \supseteq J_r(t^0, A)$ such that $p \in N_{qr}(t, A) \backslash N_{qr}(t^0, A)$.

For any number $\varepsilon > 0$ we build elements of the perturbing matrix $A^0 = (a^0_{ijk}) \in \Omega(\varepsilon)$ of size $m \times n \times s$ by the rule

$$a^0_{ijk} = \begin{cases} -\alpha, & \text{if } i = p,\ j = q,\ k = r, \\ -\alpha, & \text{if } i \in t,\ j \in N_n \backslash \{q\},\ k = r, \\ 0 & \text{otherwise}, \end{cases}$$

where $0 < \alpha < \varepsilon$.

Let us prove that $t^0 \in \overline{P^s}(A+A^0)$. It suffices to prove that relations (8) and (9) are valid. Taking into account the construction of matrix A^0 and the relation $t^0 \underset{A}{\sim} t$ equalities (9) are evident.

Further let us prove inequalities (8). Since $p \in N_{qr}(t, A)$, then using properties 2 and 3 we obtain $f_r(t, A) = g_{qr}(t, A) = a_{pqr}$. Hence according to the construction of matrix A^0 it follows that

$$g_{qr}(t, A + A^0) = g_{qr}(t, A) - \alpha = f_r(t, A) - \alpha,$$

$$g_{jr}(t, A + A^0) = g_{jr}(t, A) - \alpha \quad \text{for} \quad j \neq q.$$

Therefore we derive

$$f_r(t, A + A^0) = \max_{j \neq N_n} g_{jr}(t, A + A^0) = \max\{g_{qr}(t, A + A^0),\ \max_{j \neq q} g_{jr}(t, A + A^0)\} =$$

$$= \max\{f_r(t, A) - \alpha,\ \max_{j \neq q}(g_{jr}(t, A) - \alpha)\} = f_r(t, A) - \alpha = f_r(t^0, A) - \alpha. \tag{11}$$

Further let us prove that $f_r(t^0, A + A^0) = f_r(t^0, A)$.

Taking into account the construction of matrix A^0 the following inequalities are evident

$$g_{jr}(t^0, A + A^0) \leq g_{jr}(t^0, A), \quad j \in N_n.$$

Furthermore, using $p \notin N_{qr}(t^0, A)$ and $q \in J_r(t^0, A)$, we have

$$g_{qr}(t^0, A + A^0) = g_{qr}(t^0, A) = f_r(t^0, A).$$

Thus in view of $f_r(t^0, A) \geq g_{jr}(t^0, A) \geq g_{jr}(t^0, A + A^0)$ for $j \in N_n$ we derive

$$\begin{aligned} f_r(t^0, A + A^0) &= \max_{j \in N_n} g_{jr}(t^0, A + A^0) = \\ &= \max\{g_{qr}(t^0, A),\ \max_{j \neq q} g_{jr}(t^0, A + A^0)\} = f_r(t^0, A). \end{aligned} \tag{12}$$

Combining (11) and (12), we obtain inequality (8). Thus we derive formula (10). Consequently the trajectory t^0 is not stable. □

Set

$$\overline{L^s}(A) = T \setminus L^s(A).$$

Lemma 3. *If $t^0 \in L^s(A)$ and there exist $r \in N_s$ and $t \in L^s_r(A)$ such that $t \underset{A,r}{\overline{\approx}} t^0$, then the trajectory t^0 is not stable.*

Proof. This lemma con be proved in analogous way as Lemma 1. It can be done by constructing a perturbing matrix A^0 the same way as in proof of lemma 1 and repeating all arguments. Thus the inequality (8) is true.

Moreover, since $t^0, t \in L^s_r(A)$, then the following inequalities hold for $r > 1$

$$f_k(t^0, A) = f_k(t, A),\ k \in N_{r-1}.$$

Therefore, taking into account the construction of matrix A^0, we obtain

$$f_k(t^0, A + A^0) = f_k(t, A + A^0),\ \in N_{r-1}. \tag{13}$$

Hence

$$t^0 \underset{A+A^0,L}{\succ} t, \tag{14}$$

Summarizing we derive the formula

$$\forall \varepsilon > 0 \ \ \exists A^0 \in \Omega(\varepsilon) \ \ (t^0 \in \overline{L^s}(A + A^0)), \tag{15}$$

i. e. the trajectory $t^0 \in L^s(A)$ is not stable. $\square$

Lemma 4. *If $t^0 \in L^s(A)$ and there exist $r \in N_s$ and $t \in L^s_r(A)$ such that $t^0 \underset{A,r}{\overline{\vdash\!\sim}} t$, then the trajectory t^0 is not stable.*

Proof. If we construct a matrix A^0 by the same rules as in lemma 2 and carry out the same reasoning, then we conclude that the inequalities (8) are true. Moreover, taking into account $t^0, t \in L^s_r(A)$ we obtain equalities (13). Hence we have (14).

Thus, formula (15) is valid, i. e. the trajectory $t^0 \in L^s(A)$ is not stable. $\square$

5 Theorems

For any trajectory t^0 set

$$Q^s(t^0, A) = \{t \in T \ : \ t^0 \underset{A}{\sim} t\}.$$

Theorem 1. *A trajectory $t^0 \in P^s(A)$ is stable if and only if the formula*

$$\forall t \in Q^s(t^0, A) \ \ (t^0 \underset{A}{\vdash} t) \tag{16}$$

is valid.

Proof. Necessity. Let a trajectory $t^0 \in P^s(A)$ be stable. Assume that formula (16) is not true. Then there exist $r \in N_s$ and $t \underset{A}{\sim} t^0$ such that $t^0 \underset{A}{\overline{\vdash}} t$, i. e. one of the following relations holds: $t^0 \underset{A,r}{\overline{\vdash\!\sim}} t$ or $t \underset{A,r}{\overline{\vdash\!\approx}} t^0$. Therefore according to lemmas 1 and 2 the trajectory t^0 is not stable. Contradiction.

Sufficiency. Let formula (16) hold. Let us show that trajectory $t^0 \in P^s(A)$ is stable. We consider two possible cases for an arbitrary trajectory $t \in T$.

Case 1. $t \in Q^s(t^0, A)$. Then according to the theorem condition $t^0 \underset{A}{\vdash} t$. Hence from property 9 it follows that the formula

$$\exists \varepsilon(t) > 0 \ \ \forall A' \in \Omega(\varepsilon(t)) \ (t^0 \underset{A+A',P}{\overline{\succ}} t) \tag{17}$$

is true.

Case 2. $t \in T \setminus Q^s(t^0, A)$. Therefore the relation $t^0 \underset{A}{\sim} t$ does not hold. Then there exists an index $r \in N_s$ such that $f_r(t^0, A) < f_r(t, A)$. Hence by virtue of continuity of the function $f_r(t, A)$ in $\mathbb{R}^{m\times n}$ there exists a number $\varepsilon(t)$ such that formula (17) is valid.

Summarizing both cases, we obtain

$$\exists \varepsilon^* > 0 \ \ \forall t \in T \ \ \forall A' \in \Omega(\varepsilon^*) \ \ (t^0 \underset{A+A',L}{\overline{\succ}} t),$$

where $\varepsilon^* = \min\{\varepsilon(t) : t \in T\}$, i. e. trajectory $t^0 \in L^s(A)$ is stable. □

Theorem 2. *A trajectory $t^0 \in L^s(A)$ is stable if and only if the formula*

$$\forall k \in N_s \ \ \forall t \in L^s_k(A) \ \ (t^0 \underset{A,k}{\vdash} t) \tag{18}$$

is valid.

Proof. Necessity. Let a trajectory $t^0 \in L^s(A)$ be stable. Assume that formula (18) does not hold. Then there exist $r \in N_s$ and $t \in L^s_r(A)$ such that $t^0 \underset{A,r}{\overline{\vdash}} t$. Therefore one of the following relations holds: $t^0 \underset{A,r}{\overline{|\!\sim}} t$ or $t \underset{A,r}{\overline{|\!\approx}} t^0$. Further using Lemmas 3 and 4 we conclude that trajectory $t^0 \in L^s(A)$ is not stable. Contradiction.

Sufficiency. Let formula (18) hold. We show that a trajectory $t^0 \in L^s(A)$ is stable. We consider two possible cases for an arbitrary trajectory $t \in T$.

Case 1. $t \in L^s_1(A)$. First, let $t \in L^s(A)$. Then according to the theorem condition for any index $k \in N_s$ the relation $t^0 \underset{A,r}{\vdash} t$ is valid. Therefore from properties 1 and 9 it follows that the following formula holds

$$\exists \varepsilon(t) > 0 \ \ \forall A' \in \Omega(\varepsilon(t)) \ (t^0 \underset{A+A',L}{\overline{\succ}} t). \tag{19}$$

Now, let $t \in L^s_1(A) \setminus L^s(A)$. Then there exists an index $r = r(t) \in N_s \setminus \{1\}$ such that $t \notin L^s_r(A)$ and $t \in L^s_r(A)$ for $k \in N_{r-1}$. Hence we have

$$f_{r+1}(t, A) > f_{r+1}(t^0, A) \ \& \ \forall k \in N_{r-1} \ \ (t^0 \underset{A,k}{\vdash} t).$$

Taking into account these facts and property $10(ii)$, we conclude that the following formula holds

$$\exists \varepsilon(t) > 0 \ \ \forall A' \in \Omega(\varepsilon(t)) \ \ (t \underset{A+A',L}{\succ} t^0).$$

Thus we obtain (19).

Case 2. $t \in T \setminus L^s_1(A)$. Therefore the relation

$$f_1(t, A) > f_1(t^0, A)$$

is valid. Hence formula (19) follows from property $10(i)$.

Summarizing both cases, we obtain

$$\exists \varepsilon^* > 0 \;\; \forall t \in T \;\; \forall A' \in \Omega(\varepsilon^*) \;\; (t^0 \underset{A+A',L}{\not\succ} t),$$

where $\varepsilon^* = \min\{\varepsilon(t) : t \in T\}$, i. e. trajectory $t^0 \in L^s(A)$ is stable. □

6 Corollaries

Next corollaries follow from Theorems 1 and 2.

Corollary 1. *The equality $Q^s(t^0, A) = \{t^0\}$ is the sufficient condition for a trajectory $t^0 \in P^s(A)$ to be stable.*

Corollary 2. *The formula*

$$\forall t \in Q^s(t^0, A) \;\; \forall k \in N_s \;\; (t \underset{A,k}{\approx} t^0)$$

is the necessary condition for trajectory $t^0 \in P^s(A)$ to be stable.

Corollary 3. *A sufficient condition for a trajectory $t^0 \in P^s(A)$ to be stable is that for any trajectory $t \in Q^s(t^0, A)$ and any index $k \in N_s$ the following equalities hold*

$$J_k(t^0, A) = J_k(t, A),$$

$$N_{jk}(t^0, A) = N_{jk}(t, A), \quad j \in J_k(t^0, A).$$

It is evident that the problem under consideration turns to the vector combinatorial problem with partial criteria of the form MINMIN for $n = 1$ $(A \in \mathbb{R}^{m \times s})$. Hence the following well-known result follows from Theorem 1.

Corollary 4. *[29] A trajectory $t^0 \in P^s(A)$ of the problem with partial criteria of the form MINMIN $(n = 1)$ is stable if and only if the following formula holds*

$$\forall t \in Q^s(t^0, A) \;\; \forall k \in N_s \;\; (N_k(t^0, A) \supseteq N_k(t, A)),$$

where $N_k(t, A) =$Argmin$\{a_{ik} \; : \; i \in t\}$, $A = (a_{ik}) \in \mathbb{R}^{m \times s}$.

Corollary 5. *If $|t| = 1$ for any trajectory $t \in T$ $(p = 1)$, then the equality $Q^s(t^0, A) = \{t^0\}$ is the necessary and sufficient condition for trajectory of a vector 1-center problem $t^0 \in P^s(A)$ to be stable.*

Corollary 6. *The equality $L_1^s(A) = \{t^0\}$ is the sufficient condition for a trajectory t^0 to be stable.*

Corollary 7. *If $p = 1$ (a vector 1-center problem), then the equality $L_1^s(A) = \{t^0\}$ is the necessary and sufficient condition for trajectory $t^0 \in L^s(A)$ to be stable.*

Corollary 8. *The formula*

$$\forall k \in N_s \quad \forall t \in L^s_k(A) \quad (t \underset{A,k}{\not\approx} t^0)$$

is the necessary condition for trajectory $t^0 \in L^s(A)$ to be stable.

Corollary 9. *For a trajectory $t^0 \in L^s(A)$ to be stable it is sufficient for any index $k \in N_s$ and any trajectory $t \in L^s_k(A)$ to have*

$$J_k(t^0, A) = J_k(t, A),$$

$$N_{jk}(t^0, A) = N_{jk}(t, A), \quad j \in J_k(t^0, A).$$

Corollary 10. *[30] A trajectory $t^0 \in L^s(A)$ of the problem with partial criteria of the form MINMIN ($n = 1$) is stable if and only if the following formula holds*

$$\forall k \in N_s \quad \forall t \in L^s_k(A) \quad (N_k(t^0, A) \supseteq N_k(t, A)),$$

$N_k(t, A) = Arg\min\{a_{ik} \;:\; i \in t\}, \; A = (a_{ik}) \in \mathbb{R}^{m \times s}$.

Corollary 11. *A trajectory $t^0 \in L^s(A)$ is not stable if*

$$\exists k \in N_s \quad \exists t \in L^s_k(A) \quad (J_k(t^0, A) \cap J_k(t, A) = \emptyset).$$

Corollary 12. *A trajectory $t^0 \in L^s(A)$ is not stable if*

$$\exists k \in N_s \quad \exists t \in L^s_k(A) \quad \exists j \in J_k(t^0, A) \quad (N_{jk}(t^0, A) \not\supseteq N_{jk}(t, A)).$$

7 Examples

Let us give several examples which illustrate results stated above. First, consider the example of the problem, in which each Pareto optimal trajectory is stable.

Example 1. Let $m = 2, \quad n = 2, \quad s = 2, \quad T = \{t^1, t^2, t^3\}, \quad t^1 = \{1\}, \quad t^2 = \{1, 2\}$, $t^3 = \{2\}$ and

$$A_1 = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}.$$

Then $f(t^1, A) = (0, 1), \; f(t^2, A) = (0, 1), \; f(t^3, A) = (2, 2)$. Hence $P^2(A) = \{t^1, t^2\}$, $t^1 \underset{A}{\sim} t^2$. Further, we found the sets

$$J_1(t^1, A) = J_1(t^2, A) = \{2\},$$

$$J_2(t^1, A) = J_2(t^2, A) = \{2\},$$

$$N_{21}(t^1, A) = N_{21}(t^2, A) = \{2\},$$

$$N_{22}(t^1, A) = N_{22}(t^2, A) = \{2\}.$$

Therefore we have

$$\forall k \in N_2 \quad (t^2 \underset{A,k}{\mid\!\sim} t^1 \underset{A,k}{\mid\!\approx} t^2),$$

$$\forall k \in N_2 \quad (t^1 \underset{A,k}{\mid\!\sim} t^2 \underset{A,k}{\mid\!\approx} t^1),$$

i. e. $t^2 \underset{A}{\vdash} t^1 \underset{A}{\vdash} t^2$. Hence formula (16) is valid for trajectories t^1 and t^2. Thus, by virtue of Theorem 1 trajectories t^1 and t^2 are stable.

The following example illustrates the situation when both stable and nonstable trajectories exist among Pareto optimal trajectories.

Example 2. Let $m = 3, \quad n = 2, \quad s = 2, \quad T = \{t^1, t^2, t^3, t^4\}, \quad t^1 = \{1, 2\}, \quad t^2 = \{1, 3\}, \quad t^3 = \{2, 3\},\ t^4 = \{1\}$ and

$$A_1 = \begin{pmatrix} -1 & 0 \\ 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

Then $f(t^1, A) = (0, 1),\ f(t^2, A) = (0, 1),\ f(t^3, A) = (1, 0),\ f(t^4, A) = (0, 2)$. Therefore $P^2(A) = \{t^1, t^2, t^3\},\ t^1 \underset{A}{\sim} t^2,\ Q^2(t^3) = \{t^3\}$. Taking into account the last equality and Corollary 1 we derive that trajectory t^3 is stable. Further, we found the sets

$$J_1(t^1, A) = J_1(t^2, A) = \{2\},$$

$$J_2(t^1, A) = \{1\},\ J_2(t^2, A) = \{2\}.$$

Hence we conclude that there exists index $k = 2$ such that $J_2(t^1, A) \not\subseteq J_2(t^2, A)$ and $J_2(t^2, A) \not\subseteq J_2(t^1, A)$. Hence $t^1 \underset{A,2}{\overline{\mid\!\approx}} t^2 \underset{A,2}{\overline{\mid\!\approx}} t^1$, i. e. $t^2 \underset{A}{\overline{\vdash}} t^1 \underset{A}{\overline{\vdash}} t^2$. Thus, by virtue of Theorem 1 trajectories $t^1,\ t^2$ are not stable.

Further we consider the example of the problem in which each Pareto optimal trajectory is nonstable.

Example 3. Let $m = 3, \quad n = 2, \quad s = 2, \quad T = \{t^1, t^2, t^3\}, \quad t^1 = \{1\}, \quad t^2 = \{2, 3\}, \quad t^3 = \{2\}$ and

$$A_1 = \begin{pmatrix} -1 & 0 \\ 2 & 0 \\ -1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ -2 & 1 \\ 0 & 2 \end{pmatrix}.$$

Then $f(t^1, A) = (0, 1),\ f(t^2, A) = (0, 1),\ f(t^3, A) = (2, 1)$. Therefore $P^2(A) = \{t^1, t^2\},\ t^1 \underset{A}{\sim} t^2$. Further, we found the sets

$$J_1(t^1, A) = J_1(t^2, A) = \{2\},$$

$$J_2(t^1, A) = J_2(t^2, A) = \{2\},$$

$$N_{21}(t^1, A) = \{1\},\ N_{21}(t^2, A) = \{2\}.$$

Hence $N_{21}(t^1, A) \not\subseteq N_{21}(t^2, A)$, $N_{21}(t^2, A) \not\subseteq N_{21}(t^1, A)$, i. e. there exist $k = 1$ and $j = 2$ such that $t^1 \underset{A,1}{\overline{|\!\sim}} t^2 \underset{A,1}{\overline{|\!\sim}} t^1$. Therefore $t^1 \underset{A}{\overline{\vdash}} t^2 \underset{A}{\overline{\vdash}} t^1$. Hence formula (16) is not valid for trajectories t^1 and t^2. Thus, by virtue of Theorem 1 trajectories t^1 and t^2 are not stable.

Now consider the example of the problem in which each lexicographically optimal trajectory is stable.

Example 4. Let $m = 3$, $n = 3$, $s = 2$, $T = \{t^1, t^2, t^3\}$, $t^1 = \{1, 2\}$, $t^2 = \{2, 3\}$, $t^3 = \{1, 2, 3\}$ and

$$A_1 = \begin{pmatrix} -2 & -1 & 0 \\ 2 & -1 & -1 \\ -2 & 1 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 2 & -2 \\ 1 & -2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then $f(t^1, A) = (-1, -1)$, $f(t^2, A) = (-1, 1)$, $f(t^3, A) = (-1, -1)$. Therefore $L_1^2(A) = \{t^1, t^2, t^3\} = T$, $L^2(A) = L_2^2(A) = \{t^1, t^2\}$. Further, we found the sets

$$J_1(t^1, A) = J_1(t^2, A) = J_1(t^3, A) = \{2, 3\},$$

$$J_2(t^1, A) = J_2(t^3, A) = \{1\},\ J_2(t^2, A) = \{3\},$$

$$\{1, 2\} = N_{21}(t^1, A) = N_{21}(t^3, A) \subset\ N_{21}(t^2, A) = \{2\},$$

$$N_{31}(t^1, A) = N_{31}(t^2, A) = N_{31}(t^3, A) = \{2\},$$

$$N_{12}(t^1, A) = N_{12}(t^3, A) = \{1\}.$$

Hence the following relations hold

$$t^1 \underset{A,1}{|\!\sim} t^2 \underset{A,1}{|\!\approx} t^1,\ t^1 \underset{A,1}{|\!\sim} t^3 \underset{A,1}{|\!\approx} t^1,\ t^3 \underset{A,1}{|\!\sim} t^1 \underset{A,1}{|\!\approx} t^3,\ t^3 \underset{A,1}{|\!\sim} t^2 \underset{A,1}{|\!\approx} t^3,$$

$$t^1 \underset{A,2}{|\!\sim} t^3 \underset{A,2}{|\!\approx} t^1,\ t^3 \underset{A,2}{|\!\sim} t^1 \underset{A,2}{|\!\approx} t^3.$$

Thus,

$$\forall k \in N_2 \ \ \forall t \in L_k^2(A) \ \ (t^1 \underset{A}{\vdash} t),$$

$$\forall k \in N_2 \ \ \forall t \in L_k^2(A) \ \ (t^3 \underset{A}{\vdash} t),$$

i. e. formula (18) is true. Therefore, by virtue of Theorem 2 trajectories t^1, t^3 are stable.

Further, we consider the problem in which each lexicographically optimal trajectory is not stable.

Example 5. Let $m = 2,\ n = 3,\ s = 2,\ T = \{t^1, t^2\},\ t^1 = \{1\},\ t^2 = \{2\}$ and

$$A_1 = \begin{pmatrix} 0 & -1 & 0 \\ -2 & -2 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & -2 & -1 \\ -2 & -2 & -1 \end{pmatrix}.$$

Then $f(t^1, A) = (0, -1),\ f(t^2, A) = (0, -1)$. Therefore $L_1^2(A) = \{t^1, t^2\} = T$, $L^2(A) = L_2^2(A) = \{t^1, t^2\}$. Further, we found the sets

$$J_1(t^1, A) = J_1(t^2, A) = \{3\},$$
$$\{1, 3\} = J_2(t^1, A) \not\subseteq\ J_2(t^2, A) = \{3\},$$
$$N_{31}(t^1, A) = \{1\},\ N_{31}(t^2, A) = \{2\}.$$

Hence we have

$$N_{31}(t^1, A) \not\subseteq N_{31}(t^2, A),\ N_{31}(t^2, A) \not\subseteq N_{31}(t^1, A),$$

i. e. $t^1 \underset{A,1}{\overline{\vdash\!\sim}} t^2 \underset{A,1}{\overline{\vdash\!\sim}} t^1,\ t^2 \underset{A,1}{\overline{\vdash\!\approx}} t^1$. Therefore $t^1 \underset{A}{\overline{\vdash}} t^2 \underset{A}{\overline{\vdash}} t^1$. Hence formula (18) is not valid for both lexicographically optimal trajectories t^1 and t^2. Thus, in view of Theorem 2 they are not stable.

The following example illustrates situation when both stable and nonstable trajectories exist among lexicographically optimal trajectories.

Example 6. Let $m = 2, \quad n = 3, \quad s = 2, \quad T = \{t^1, t^2, t^3\}, \quad t^1 = \{1\}, \quad t^2 = \{1, 2\}, \quad t^3 = \{2\}$ and

$$A_1 = \begin{pmatrix} -1 & -1 & -2 \\ 0 & -1 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \end{pmatrix}.$$

Then $f(t^1, A) = (-1, 1),\ f(t^2, A) = (-1, 1),\ f(t^3) = (0, 2)$. Therefore $L_1^2(A) = \{t^1, t^2\}$, $L^2(A) = L_2^2(A) = \{t^1, t^2\}$. Further, we found the sets

$$J_1(t^1, A) = J_1(t^2, A) = \{1, 2\},$$
$$J_2(t^1, A) = J_2(t^2, A) = \{1, 3\},$$
$$N_{11}(t^1, A) = N_{11}(t^2, A) = \{1\},$$
$$\{1\} = N_{21}(t^1, A) \subset\ N_{21}(t^2, A) = \{1, 2\},$$
$$N_{12}(t^1, A) = N_{12}(t^2, A) = \{1\},$$
$$\{1\} = N_{32}(t^1, A) \subset\ N_{32}(t^2, A) = \{1, 2\}.$$

Hence we derive

$$\forall k \in N_2 \quad (t^2 \underset{A}{\vdash} t^1).$$

Therefore formula (18) is valid and by virtue of Theorem 2 trajectory t^2 is stable. But

$$N_{21}(t^1, A) \not\supseteq N_{21}(t^2, A),$$

i. e. there exist index $k = 1$ and trajectory $t^2 \in L_1^2(A)$ such that $t^1 \underset{A,1}{\overline{\vdash\!\sim}} t^2$. Hence $t^1 \underset{A}{\overline{\vdash}} t^2$. Thus, formula (18) does not hold and by virtue of Theorem 2 trajectory t^1 is nonstable.

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Flow of an Unsteady Dusty Visco-Elastic Fluid Between Two Moving Plates in Frenet-Frame Field System

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Abstract. The present investigation deals with the study of an unsteady motion of a dusty viscoelastic conducting fluid under arbitrary pressure gradient between two infinite moving parallel plates. The influence of time dependent pressure gradients, i.e. impulsive, transition and motion for a finite time is considered along with the effect of the movement of the plates and the presence of uniform magnetic field. Expressions for the velocities of the fluid and particles are obtained by using the Laplace transform technique. Results are presented in graphical form. Finally the skin friction at the boundaries is calculated.

Mathematics subject classification: 76T10, 76T15.
Keywords and phrases: Frenet frame field system; parallel plates, dusty fluid; velocity of dust phase and fluid phase, conducting dusty fluid, magnetic field.

1 Introduction

The presence of dust particles in fluids has certain influence on the motion of the fluids, and such situations arise, for instance in the movement of dust-laden air, in fluidization, in the use of dust in gas cooling systems, and in sedimentation in tidal waves, powder technology, acoustics, performance of solid fuel rocket nozzles, rainerosion, guided missiles, paint spraying, etc.

The stability of the laminar flow of a dusty gas in which the dust particles are uniformly distributed has been discussed by P. G. Saffman [18] and the basic equations for the flow of dusty fluid were formulated. T. M. Nabil [16] studied the effect of couple stresses on pulsatile hydromagnetic Poiseuille flow. N. Datta [5] obtained the solutions for Pulsatile flow of heat transfer of a dusty fluid through an infinitely long annular pipe. Girish Kumar, R. K. S. Chaudhary and K. K. Singh [9] have discussed the unsteady flow of conducting dusty visco-elastic liquid through a channel, and N. C. Ghosh, B. C. Ghosh and L. Debnath [10] obtained the results for the hydromagnetic flow of a dusty visco-elastic fluid between two infinite parallel plates.

Some researchers like Kanwal [12], Truesdell [19], Indrasena [11], Purushotham [17], Bagewadi and Gireesha [1, 2] have applied differential geometry techniques to investigate the kinematical properties of fluid flows in the field of fluid mechanics.

Further, recently the authors [6–8] have studied dusty fluid flow in Frenet frame field system under varying time dependent pressure gradients.

The present investigation deals with the study of an electrically conducting dusty viscoelastic fluid flow between two infinitely extended non-conducting parallel plates in Frenet frame field system. Initially, the fluid and dust particles are assumed to be at rest. The motion of fluid is due to the influence of time dependent pressure gradient along with movement of the plates and applied uniform magnetic field. The analytical expressions are obtained for velocities of fluid and dust particles in three cases. For each case the skin friction at boundaries is obtained. The changes in the velocity profiles for different Hartmann numbers are shown graphically. section-Frenet Frame Field System

Let $\overrightarrow{s}$, $\overrightarrow{n}$, $\overrightarrow{b}$ be triply orthogonal unit vectors tangent, principal normal, binormal respectively to the spatial curves of congruences formed by fluid phase velocity and dusty phase velocity lines respectively as shown in Figure 1.

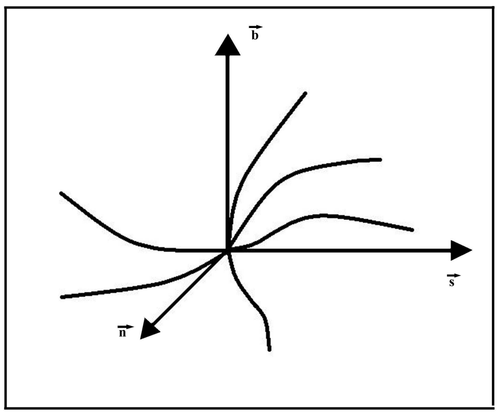

Figure 1. Frenet Frame Field System

Geometrical relations are given by Frenet formulae [3]

$$
\begin{aligned}
&i) && \frac{\partial \overrightarrow{s}}{\partial s} = k_s \overrightarrow{n}, \ \frac{\partial \overrightarrow{n}}{\partial s} = \tau_s \overrightarrow{b} - k_s \overrightarrow{s}, \ \frac{\partial \overrightarrow{b}}{\partial s} = -\tau_s \overrightarrow{n};\\
&ii) && \frac{\partial \overrightarrow{n}}{\partial n} = k'_n \overrightarrow{s}, \ \frac{\partial \overrightarrow{b}}{\partial n} = -\sigma'_n \overrightarrow{s}, \ \frac{\partial \overrightarrow{s}}{\partial n} = \sigma'_n \overrightarrow{b} - k'_n \overrightarrow{n};\\
&iii) && \frac{\partial \overrightarrow{b}}{\partial b} = k''_b \overrightarrow{s}, \ \frac{\partial \overrightarrow{n}}{\partial b} = -\sigma''_b \overrightarrow{s}, \ \frac{\partial \overrightarrow{s}}{\partial b} = \sigma''_b \overrightarrow{n} - k''_b \overrightarrow{b};\\
&iv) && \nabla.\overrightarrow{s} = \theta_{ns} + \theta_{bs}; \ \nabla.\overrightarrow{n} = \theta_{bn} - k_s; \ \nabla.\overrightarrow{b} = \theta_{nb},
\end{aligned}
\tag{1}
$$

where $\partial/\partial s$, $\partial/\partial n$ and $\partial/\partial b$ are the intrinsic differential operators along fluid phase velocity (or dust phase velocity) lines, principal normal and binormal. The functions (k_s, k'_n, k''_b) and $(\tau_s, \sigma'_n, \sigma''_b)$ are the curvatures and torsion of the above curves and

θ_{ns} and θ_{bs} are normal deformations of these spatial curves along their principal normal and binormal respectively.

2 Formulation and Solution of the Problem

The present discussion considers a dusty visco-elastic fluid bounded by two infinite flat moving plates separated by a distance h in the absence of body force. Both the fluid and the dust particle clouds are supposed to be static at the beginning. The dust particles are assumed to be spherical in shape and uniform in size. The number density of the dust particles is taken as a constant throughout the flow. It is assumed that the dust particles are electrically nonconducting and neutral. The flow is due to the influence of time dependent pressure gradient along with motion of plates and due to magnetic field of uniform strength B_0. Under these assumptions the flow will be a parallel flow in which the streamlines are along the tangential direction as shown in Figure 2.

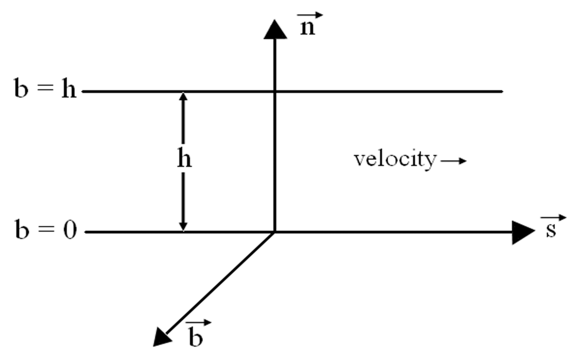

Figure 2. Geometry of the flow

For the above described flow the velocities of fluid and dust are of the form

$$\vec{u} = u_s \vec{s}, \qquad \vec{v} = v_s, \vec{s} \tag{2}$$

i.e., $u_n = u_b = 0$ and $v_n = v_b = 0$, where (u_s, u_n, u_b) and (v_s, v_n, v_b) denote the velocity components of fluid and dust respectively.

Since the flow is in between two moving plates, we can assume the velocity of both fluid and dust particles do not vary along tangential direction. Suppose the fluid extends to infinity in the principal normal direction, then the velocities of both may be neglected in this direction.

The modified Saffman's [18] equations for the dusty visco-elastic fluid with the help of equation (1) are given by:

$$\frac{\partial u_s}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial s} + \left(\alpha + \beta\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 u_s}{\partial b^2} - C_r u_s\right) + \frac{kN}{\rho}(v_s - u_s) - \frac{\sigma B_0^2}{\rho}u_s; \tag{3}$$

$$\frac{\partial v_s}{\partial t} = \frac{k}{m}(u_s - v_s). \tag{4}$$

We have the following nomenclature:

$\rho-$density of the gas, $p-$pressure of the fluid, $N-$number of density of dust particles, $k = 6\pi a\mu-$ Stoke's resistance (drag coefficient), $a-$spherical radius of dust particle, $m-$mass of the dust particle, $B_0-$the intensity of the imposed transverse magnetic field, $\sigma-$electrical conductivity of the fluid, $m/k-$relaxation time of the dust particles, α & β are the kinematic coefficients of visco-elasticity of the fluid, $t-$time, and $C_r = (\sigma_b'^2 + k_n'^2 + k_b'^2 + \sigma_b''^2)$ is called the curvature term [2].

Introducing the nondimensional quantities

$$x' = x/h, \quad y' = y/h, \quad t' = \alpha t/h^2, \quad p' = ph^2/\alpha^2\rho, \quad u_s' = u_s h/\alpha, \quad v_s' = v_s h/\alpha$$

in equations (3) and (4) and dropping the primes one can get

$$\frac{\partial u_s}{\partial t} = -\frac{\partial p}{\partial s} + \left(1 + E\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 u_s}{\partial b^2} - C_r u_s\right) + \frac{l}{w}(v_s - u_s) - M^2 u_s; \tag{5}$$

$$\frac{\partial v_s}{\partial t} = \frac{1}{w}(u_s - v_s) \tag{6}$$

where $E = \beta/h^2$ is the elastic parameter, $l = mN/\rho$, $w = m\alpha/kh^2$, $M = B_0 h\sqrt{\sigma/\mu}$ (Hartmann number).

Equations (5) and (6) are to be solved subject to the initial and boundary conditions in nondimensional form as:

$$\begin{aligned} \text{Initial condition; at } t = 0;\, u_s &= 0, v_s = 0 \\ \text{Boundary condition; for } t > 0; u_s &= f(t), \text{ at } b = 0 \\ \text{and } u_s &= g(t) \text{ at } b = 1 \end{aligned} \tag{7}$$

Let $P(t)$ be the time dependent pressure gradient to be impressed on the system for $t > 0$. So we can write

$$-\frac{\partial p}{\partial s} = P(t).$$

We define Laplace transformations of u_s and v_s as

$$U = \int_0^\infty e^{-xt} u_s dt \quad \text{and} \quad V = \int_0^\infty e^{-xt} v_s dt. \tag{8}$$

Applying the Laplace transform to equations (5) and (6) and to boundary conditions, then by using initial conditions one obtains

$$xU = P(x) + (1 + xE)\left(\frac{\partial^2 U}{\partial b^2} - C_r U\right) + \frac{l}{w}(V - U) - M^2 U; \tag{9}$$

$$xV = \frac{1}{w}(U - V); \tag{10}$$

$$U = F(x), \text{ at } b=0 \text{ and } U = G(x) \text{ at } b=1, \tag{11}$$

where $F(x)$, $G(x)$ and $P(x)$ are Laplace transforms of $f(t)$, $g(t)$ and $P(t)$ respectively.

Eliminating V from (9) and (10) we obtain the following equation

$$\frac{d^2U}{db^2} - Q^2U = -\frac{P(x)}{1+xE}, \tag{12}$$

where $Q^2 = \left(C_r + \frac{x}{1+xE} + \frac{M^2}{1+xE} + \frac{xl}{(1+xE)(1+xw)}\right)$.

CASE 1. Impulsive Motion: Consider the case of impulsive motion, in which

$$\begin{aligned} f(t) &= u_0\delta(t) \text{ at } b=0, \\ g(t) &= u_1\delta(t) \text{ at } b=1, \\ P(t) &= p_0\delta(t), \end{aligned}$$

where $\delta(t)$ is the Dirac delta function and u_0, u_1 & p_0 are constants.

The velocities of fluid and dust particle are obtained by solving the equation (12) subjected to the boundary conditions (11) as follows:

$$\begin{aligned} U &= \left[\frac{u_1\sinh(Qb) - u_0\sinh(Q(b-1))}{sinh(Q)}\right] + \\ &+ \frac{p_0}{Q^2(1+xE)}\left[\frac{\sinh(Q(b-1)) - \sinh(Qb)}{sinh(Q)} + 1\right]. \end{aligned}$$

Using U in (10) we obtain V as

$$\begin{aligned} V &= \frac{1}{(1+xw)}\left[\frac{u_1\sinh(Qb) - u_0\sinh(Q(b-1))}{sinh(Q)}\right] + \\ &+ \frac{p_0}{Q^2(1+xE)(1+xw)}\left[\frac{\sinh(Q(b-1)) - \sinh(Qb)}{sinh(Q)} + 1\right]. \end{aligned}$$

By taking the inverse Laplace transform to U and V, one can obtain

$$\begin{aligned} u_s &= 2\pi\sum_{r=0}^{\infty} r[u_0 - u_1(-1)^r]\sin(r\pi b) \times \\ &\times \left[\frac{e^{\alpha_1 t}(1+\alpha_1 E)^2(1+\alpha_1 w)^2}{\delta_1} + \frac{e^{\alpha_2 t}(1+\alpha_2 E)^2(1+\alpha_2 w)^2}{\delta_2}\right] + \\ &+ \frac{2p_0}{\pi}\sum_{r=0}^{\infty}\frac{[(-1)^r - 1]}{r}\sin(r\pi b) \times \\ &\times \left[\frac{e^{\alpha_1 t}(1+E\alpha_1)(1+w\alpha_1)^2}{\delta_1} + \frac{e^{\alpha_2 t}(1+E\alpha_2)(1+w\alpha_2)^2}{\delta_2}\right] + \end{aligned}$$

$$
\begin{aligned}
&+ \quad \left[\frac{u_1 \sinh(Xb) - u_0 \sinh(X(b-1))}{\sinh(X)}\right] + \\
&+ \quad \frac{p_0}{X^2}\left[\frac{\sinh(X(b-1)) - \sinh(Xb)}{\sinh(X)} + 1\right];
\end{aligned}
$$

$$
\begin{aligned}
v_s \quad &= \quad 2\pi \sum_{r=0}^{\infty} r[u_0 - u_1(-1)^r] \sin(r\pi b) \times \\
&\times \quad \left[\frac{e^{\alpha_1 t}(1+\alpha_1 E)^2(1+\alpha_1 w)}{\delta_1} + \frac{e^{\alpha_2 t}(1+\alpha_2 E)^2(1+\alpha_2 w)}{\delta_2}\right] + \\
&+ \quad \frac{2p_0}{\pi} \sum_{r=0}^{\infty} \frac{[(-1)^r - 1]}{r} \sin(r\pi b) \times \\
&\times \quad \left[\frac{e^{\alpha_1 t}(1+E\alpha_1)(1+w\alpha_1)}{\delta_1} + \frac{e^{\alpha_2 t}(1+E\alpha_2)(1+w\alpha_2)}{\delta_2}\right] + \\
&+ \quad \left[\frac{u_1 \sinh(Xb) - u_0 \sinh(X(b-1))}{\sinh(X)}\right] + \\
&+ \quad \frac{p_0}{X^2}\left[\frac{\sinh(X(b-1)) - \sinh(Xb)}{\sinh(X)} + 1\right].
\end{aligned}
$$

Shear stress (Skin friction): The expression for shear stress at the plates $b = 0$ and $b = 1$ are respectively given by:

$$
\begin{aligned}
D_0 \quad &= \quad 2\pi^2 \mu \sum_{r=0}^{\infty} r^2[u_0 - u_1(-1)^r] \times \\
&\times \quad \left[\frac{e^{\alpha_1 t}(1+\alpha_1 E)^2(1+\alpha_1 w)^2}{\delta_1} + \frac{e^{\alpha_2 t}(1+\alpha_2 E)^2(1+\alpha_2 w)^2}{\delta_2}\right] + \\
&+ \quad 2p_0 \mu \sum_{r=0}^{\infty} [(-1)^r - 1] \left[\frac{e^{\alpha_1 t}(1+E\alpha_1)(1+w\alpha_1)^2}{\delta_1} + \frac{e^{\alpha_2 t}(1+E\alpha_2)(1+w\alpha_2)^2}{\delta_2}\right] \times \\
&\times \quad \mu X \left[\frac{u_1 - u_0 \cosh(X)}{\sinh(X)}\right] + \frac{\mu p_0}{X}\left[\frac{cosh(X) - 1}{\sinh(X)}\right];
\end{aligned}
$$

$$
\begin{aligned}
D_1 \quad &= \quad 2\pi^2 \mu \sum_{r=0}^{\infty} r^2[u_0(-1)^r - u_1] \times \\
&\times \quad \left[\frac{e^{\alpha_1 t}(1+\alpha_1 E)^2(1+\alpha_1 w)^2}{\delta_1} + \frac{e^{\alpha_2 t}(1+\alpha_2 E)^2(1+\alpha_2 w)^2}{\delta_2}\right] + \\
&+ \quad 2p_0 \mu \sum_{r=0}^{\infty} [1 - (-1)^r] \left[\frac{e^{\alpha_1 t}(1+E\alpha_1)(1+w\alpha_1)^2}{\delta_1} + \frac{e^{\alpha_2 t}(1+E\alpha_2)(1+w\alpha_2)^2}{\delta_2}\right] \times \\
&\times \quad \mu X \left[\frac{u_1 \cosh(X) - u_0}{\sinh(X)}\right] + \frac{\mu p_0}{X}\left[\frac{1 - \cosh(X)}{\sinh(X)}\right].
\end{aligned}
$$

CASE 2. Transition Motion: We consider the case of transition motion in which

$$\begin{aligned} f(t) &= u_0 H(t) e^{-\lambda t} \ \text{ at } \ b=0, \\ g(t) &= u_1 H(t) e^{-\lambda t} \ \text{ at } \ b=1, \\ P(t) &= p_0 H(t) e^{-\lambda t} \ \ \lambda > 0, \end{aligned}$$

where $H(t)$ is the Heaviside unit step function.

Now we obtain the expressions for velocities of both fluid and dust phase as

$$\begin{aligned} u_s &= 2\pi \sum_{r=0}^{\infty} r[u_0 - u_1(-1)^r] \sin(r\pi b) \times \\ &\times \left[\frac{e^{\alpha_1 t}(1+\alpha_1 E)^2(1+\alpha_1 w)^2}{\delta_1(\alpha_1+\lambda)} + \frac{e^{\alpha_2 t}(1+\alpha_2 E)^2(1+\alpha_2 w)^2}{\delta_2(\alpha_2+\lambda)} \right] + \\ &+ e^{-\lambda t} \left[\frac{u_1 \sinh(Yb) - u_0 \sinh(Y(b-1))}{\sinh(Y)} \right] + \\ &+ \frac{2p_0}{\pi} \sum_{r=0}^{\infty} \frac{[(-1)^r - 1]}{r} \sin(r\pi b) \times \\ &\times \left[\frac{e^{\alpha_1 t}(1+E\alpha_1)(1+w\alpha_1)^2}{\delta_1(\alpha_1+\lambda)} + \frac{e^{\alpha_2 t}(1+E\alpha_2)(1+w\alpha_2)^2}{\delta_2(\alpha_2+\lambda)} \right] + \\ &+ \frac{p_0 e^{-\lambda t}}{(1-\lambda E)Y^2} \left[\frac{\sinh(Y(b-1)) - \sinh(Yb) + \sinh(Y)}{\sinh(Y)} \right]; \end{aligned}$$

$$\begin{aligned} v_s &= 2\pi \sum_{r=0}^{\infty} r[u_0 - u_1(-1)^r] \sin(r\pi b) \times \\ &\times \left[\frac{e^{\alpha_1 t}(1+\alpha_1 E)^2(1+\alpha_1 w)}{\delta_1(\alpha_1+\lambda)} + \frac{e^{\alpha_2 t}(1+\alpha_2 E)^2(1+\alpha_2 w)}{\delta_2(\alpha_2+\lambda)} \right] + \\ &+ e^{-\lambda t} \left[\frac{u_1 \sinh(Yb) - u_0 \sinh(Y(b-1))}{\sinh(Y)(1-\lambda w)} \right] + \\ &+ \frac{2p_0}{\pi} \sum_{r=0}^{\infty} \frac{[(-1)^r - 1]}{r} \sin(r\pi b) \times \\ &\times \left[\frac{e^{\alpha_1 t}(1+E\alpha_1)(1+w\alpha_1)}{\delta_1(\alpha_1+\lambda)} + \frac{e^{\alpha_2 t}(1+E\alpha_2)(1+w\alpha_2)}{\delta_2(\alpha_2+\lambda)} \right] + \\ &+ \frac{p_0 e^{-\lambda t}}{(1-\lambda E)Y^2} \left[\frac{\sinh(Y(b-1)) - \sinh(Yb) + \sinh(Y)}{\sinh(y)(1-\lambda w)} \right]. \end{aligned}$$

Shear stress (Skin friction): The shear stress at the plates $b=0$ and $b=1$ for transition motion are, respectively, given by:

$$D_0 = 2\pi^2 \mu \sum_{r=0}^{\infty} r^2 [u_0 - u_1(-1)^r] \times$$

$$
\begin{aligned}
&\times \left[\frac{e^{\alpha_1 t}(1+\alpha_1 E)^2(1+\alpha_1 w)^2}{\delta_1(\alpha_1+\lambda)} + \frac{e^{\alpha_2 t}(1+\alpha_2 E)^2(1+\alpha_2 w)^2}{\delta_2(\alpha_2+\lambda)}\right] + \\
&+ \mu Y e^{-\lambda t}\left[\frac{u_1 - u_0\cosh(Y)}{\sinh(Y)}\right] + \frac{p_0 \mu e^{-\lambda t}}{(1-\lambda E)Y}\left[\frac{\cosh(Y)-1}{\sinh(Y)}\right] + \\
&+ 2\mu p_0 \sum_{r=0}^{\infty}[(-1)^r - 1]\left[\frac{e^{\alpha_1 t}(1+E\alpha_1)(1+w\alpha_1)^2}{\delta_1(\alpha_1+\lambda)} + \frac{e^{\alpha_2 t}(1+E\alpha_2)(1+w\alpha_2)^2}{\delta_2(\alpha_2+\lambda)}\right];
\end{aligned}
$$

$$
\begin{aligned}
D_1 &= 2\pi^2\mu \sum_{r=0}^{\infty} r^2[u_0(-1)^r - u_1] \times \\
&\times \left[\frac{e^{\alpha_1 t}(1+\alpha_1 E)^2(1+\alpha_1 w)^2}{\delta_1(\alpha_1+\lambda)} + \frac{e^{\alpha_2 t}(1+\alpha_2 E)^2(1+\alpha_2 w)^2}{\delta_2(\alpha_2+\lambda)}\right] + \\
&+ Y\mu e^{-\lambda t}\left[\frac{u_1\cosh(Y) - u_0}{\sinh(Y)}\right] + \frac{p_0 \mu e^{-\lambda t}}{(1-\lambda E)Y}\left[\frac{1-\cosh(Y)}{\sinh(Y)}\right] + \\
&+ 2\mu p_0 \sum_{r=0}^{\infty}[1-(-1)^r]\left[\frac{e^{\alpha_1 t}(1+E\alpha_1)(1+w\alpha_1)^2}{\delta_1(\alpha_1+\lambda)} + \frac{e^{\alpha_2 t}(1+E\alpha_2)(1+w\alpha_2)^2}{\delta_2(\alpha_2+\lambda)}\right].
\end{aligned}
$$

CASE 3. Motion for a finite time. This case considers the motion of the plates and the pressure gradient get ceased after a finite time, Hence it can be taken as

$$
\begin{aligned}
f(t) &= u_0[H(t) - H(t-T)] \quad \text{at } b=0, \\
g(t) &= u_1[H(t) - H(t-T)] \quad \text{at } b=1, \\
P(t) &= p_0[H(t) - H(t-T)] \quad \lambda > 0,
\end{aligned}
$$

where $H(t)$ is the Heaviside unit step function. For this case the expressions for velocities of both fluid and dust phase are obtained as

$$
\begin{aligned}
u_s &= 2\pi \sum_{r=0}^{\infty} r[u_0 - u_1(-1)^r]\sin(r\pi b) \times \\
&\times \left[\frac{e^{\alpha_1 t}(1+\alpha_1 E)^2(1+\alpha_1 w)^2(1-e^{-\alpha_1 T})}{\delta_1\alpha_1} + \right. \\
&+ \left. \frac{e^{\alpha_2 t}(1+\alpha_2 E)^2(1+\alpha_2 w)^2(1-e^{-\alpha_2 T})}{\delta_2\alpha_2}\right] + \\
&+ \frac{2p_0}{\pi}\sum_{r=0}^{\infty}\frac{[(-1)^r - 1]}{r}\sin(r\pi b)\left[\frac{e^{\alpha_1 t}(1+E\alpha_1)(1+w\alpha_1)^2(1-e^{-\alpha_1 T})}{\delta_1\alpha_1} + \right. \\
&+ \left. \frac{e^{\alpha_2 t}(1+E\alpha_2)(1+w\alpha_2)^2(1-e^{-\alpha_2 T})}{\delta_2\alpha_2}\right]; \\
v_s &= 2\pi \sum_{r=0}^{\infty} r([u_0 - u_1(-1)^r]\sin(r\pi b) \times
\end{aligned}
$$

$$\begin{aligned}
&\times \left[\frac{e^{\alpha_1 t}(1+\alpha_1 E)^2(1+\alpha_1 w)(1-e^{-\alpha_1 T})}{\delta_1\alpha_1} + \right.\\
&+ \left.\frac{e^{\alpha_2 t}(1+\alpha_2 E)^2(1+\alpha_2 w)(1-e^{-\alpha_2 T})}{\delta_2\alpha_2}\right] +\\
&+ \frac{2p_0}{\pi}\sum_{r=0}^{\infty}\frac{[(-1)^r-1]}{r}\sin(r\pi b)\left[\frac{e^{\alpha_1 t}(1+E\alpha_1)(1+w\alpha_1)(1-e^{-\alpha_1 T})}{\delta_1\alpha_1} + \right.\\
&+ \left.\frac{e^{\alpha_2 t}(1+E\alpha_2)(1+w\alpha_2)(1-e^{-\alpha_2 T})}{\delta_2\alpha_2}\right].
\end{aligned}$$

Shear stress (Skin friction): The shear stress at the plates $b = 0$ and $b = 1$ for this flow are, respectively, given by:

$$\begin{aligned}
D_0 &= 2\mu\pi^2\sum_{r=0}^{\infty} r^2[u_0 - u_1(-1)^r] \times\\
&\times \left[\frac{e^{\alpha_1 t}(1+\alpha_1 E)^2(1+\alpha_1 w)^2(1-e^{-\alpha_1 T})}{\delta_1\alpha_1} + \right.\\
&+ \left.\frac{e^{\alpha_2 t}(1+\alpha_2 E)^2(1+\alpha_2 w)^2(1-e^{-\alpha_2 T})}{\delta_2\alpha_2}\right] +\\
&+ 2\mu p_0\sum_{r=0}^{\infty}[(-1)^r-1]\left[\frac{e^{\alpha_1 t}(1+E\alpha_1)(1+w\alpha_1)^2(1-e^{-\alpha_1 T})}{\delta_1\alpha_1} + \right.\\
&+ \left.\frac{e^{\alpha_2 t}(1+E\alpha_2)(1+w\alpha_2)^2(1-e^{-\alpha_2 T})}{\delta_2\alpha_2}\right];\\
D_1 &= 2\mu\pi^2\sum_{r=0}^{\infty} r^2[u_0(-1)^r - u_1] \times\\
&\times \left[\frac{e^{\alpha_1 t}(1+\alpha_1 E)^2(1+\alpha_1 w)^2(1-e^{-\alpha_1 T})}{\delta_1\alpha_1} + \right.\\
&+ \left.\frac{e^{\alpha_2 t}(1+\alpha_2 E)^2(1+\alpha_2 w)^2(1-e^{-\alpha_2 T})}{\delta_2\alpha_2}\right] +\\
&+ 2\mu p_0\sum_{r=0}^{\infty}[1-(-1)^r]\left[\frac{e^{\alpha_1 t}(1+E\alpha_1)(1+w\alpha_1)^2(1-e^{-\alpha_1 T})}{\delta_1\alpha_1} + \right.\\
&+ \left.\frac{e^{\alpha_2 t}(1+E\alpha_2)(1+w\alpha_2)^2(1-e^{-\alpha_2 T})}{\delta_2\alpha_2}\right],
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= \left[(C_r + r^2\pi^2)E + 1\right]w, \quad b_1 = C_r(w+E) + 1 + M^2 w + l + r^2\pi^2(w+E);\\
c_1 &= C_r + M^2 + r^2\pi^2, \quad \alpha_1 = \frac{-b1 + \sqrt{b_1^2 - 4a_1c_1}}{2a_1}, \quad \alpha_2 = \frac{-b1 - \sqrt{b_1^2 - 4a_1c_1}}{2a_1};\\
Y &= \sqrt{\frac{C_r(1-E\lambda)(1-\lambda w) + (M^2-\lambda)(1-\lambda w) - l\lambda}{(1-E\lambda)(1-\lambda w)}}, \quad X = C_r + M^2;
\end{aligned}$$

$$\begin{aligned} \delta_1 &= (1 - M^2E)(1+\alpha_1 w)^2 + l(1-\alpha_1^2 Ew); \\ \delta_2 &= (1 - M^2E)(1+\alpha_2 w)^2 + l(1-\alpha_2^2 Ew). \end{aligned}$$

3 Conclusions

Figures 3 to 5 show the parabolic nature of velocity profiles for the fluid and dust particles for all three cases.

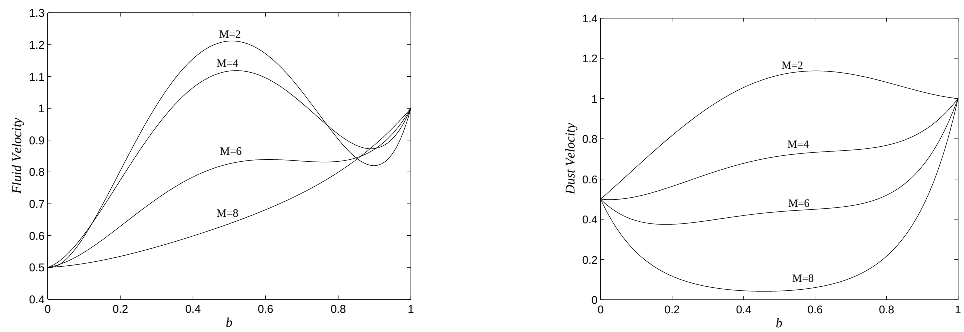

Figure 3. Variation of fluid and dust phase velocity with b (for Case 1)

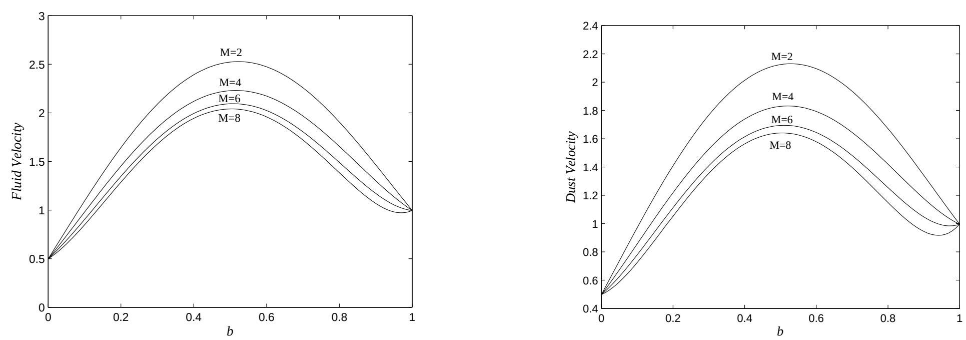

Figure 4. Variation of fluid and dust phase velocity with b (for Case 2)

According to Frenet approximation of a curve in the osculating plane the path of the curve near origin is parabolic. Hence the results obtained here are analogous to [3]. It is concluded that the velocity of fluid particles is parallel to velocity of dust particles. Also it is evident from the graphs that, as we increase the strength of the magnetic field, it has an appreciable effect on the velocities of fluid and dust particles. Further one can observe that if the magnetic field is zero then the results are in agreement with the plane Couette flow. The velocities for fluid and dust particles decreases for large values of t. We observe that if the dust is very fine then the velocities of both fluid and dust particles will be the same.

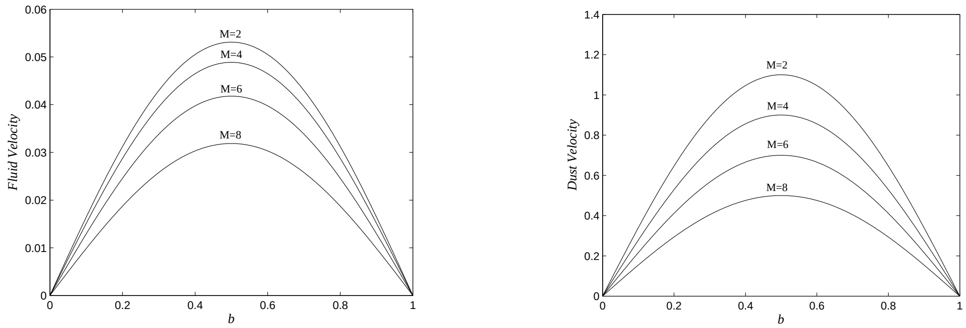

Figure 5. Variation of fluid and dust phase velocity with b (for Case 3)

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On preradicals associated to principal functors of module categories. II

A. I. Kashu

Abstract. Continuing part I (see [1]) the classes of modules and preradicals determined by the functor $U \otimes_S$- : S-$Mod \to \mathcal{A}b$ are studied, the relations between them are established and the conditions of coincidence of some preradicals are shown.

Mathematics subject classification: 16D90, 16S90, 16D40.
Keywords and phrases: Tensor product, preradical, torsion, torsion class, flat module.

Introduction

In the first part of this work [1] the classes of modules and preradicals associated to the functor $H = Hom_R(U, \text{-}) : R\text{-}Mod \to \mathcal{A}b \quad ({}_RU \in R\text{-}Mod)$ are studied. Now we will use the same methods for the investigation of similar questions for the functor of tensor product:

$$T = U \otimes_S \text{-} : S\text{-}Mod \to \mathcal{A}b,$$

where U_S is a fixed *right S-module*. The preradicals determined in S-Mod by U_S and T are elucidated, their properties and relations between them are shown. Moreover, some conditions for the coincidence of "near" preradicals are indicated. We remark that there exists a partial duality between these results and those of part I for the functor $H = Hom_R(U, \text{-})$. The main general facts on preradicals and torsions in modules can be found in the books [3–6].

1 Preradicals defined by the functor T

Let S be a ring with unity and S-Mod is the category of unitary left S-modules. We fix a *right S-module* U_S and consider the functor of tensor product, defined by U_S:

$$T = T^U = U \otimes_S \text{-} : S\text{-}Mod \to \mathcal{A}b,$$

where $\mathcal{A}b$ is the category of abelian groups.

In S-Mod we consider the following class of modules:

$$\mathcal{F}(U_S) = \{M \in S\text{-}Mod \,|\, U \otimes_S m = 0 \text{ in } U \otimes_S M \;\text{ implies }\; m = 0\},$$

where $U \otimes_S m = \{u \otimes_S m \in U \otimes_S M \,|\, u \in U\}$ for $m \in M$. A direct verification proves

Proposition 1.1. *$\mathcal{F}(U_S)$ is a pretorsionfree class (i.e. is closed under submodules and direct products), therefore it defines a <u>radical</u> t_U in S-Mod such that $\mathcal{P}(t_U) \overset{def}{=\!=} \mathcal{F}(U_S)$. For every module ${}_SM$ we have*:

$$t_U(M) = \{m \in M \,|\, U \otimes_S m = 0 \text{ in } U \otimes_S M\}. \qquad \square$$

Having the module U_S and respective functor $T = T^U$, we denote:

$$Ker T = \{M \in S\text{-}Mod \,|\, T(M) = 0\}.$$

Proposition 1.2. *$Ker\ T$ is a torsion class (i.e. is closed under homomorphic images, direct sums and extensions), therefore it defines an <u>idempotent radical</u> $\overline{t}_U$ in S-Mod such that $\mathcal{R}(\overline{t}_U) \overset{def}{=\!=} Ker T$. For every module $M \in S$-Mod we have*:

$$\overline{t}_U(M) = \textstyle\sum\{N_\alpha \subseteq M \,|\, N_\alpha \in Ker\, T\}.$$

The corresponding torsionfree class is $\mathcal{P}(\overline{t}_U) = (Ker\, T)^{\downarrow}$.

Proof. From properties of the functor T (which is right exact and preserves direct sums) follows that $\mathrm{Ker}\, T$ is a torsion class. For example, any short exact sequence in S-Mod

$$0 \to M' \xrightarrow{\varphi} M \xrightarrow{\pi} M'' \to 0$$

with $M', M'' \in Ker\, T$ implies in $\mathcal{A}b$ the exact sequence

$$T(M') \xrightarrow{T(\varphi)} T(M) \xrightarrow{T(\pi)} T(M'') \to 0$$

with $T(M') = T(M'') = 0$, therefore $T(M) = 0$. Thus the class $\mathrm{Ker}\, T$ is closed under extensions. The rest of statements are also obvious. $\square$

Next we clarify the relation between the preradicals t_U and $\overline{t}_U$. For that we study the connections between the associated classes of modules.

Proposition 1.3. $\mathcal{F}(U_S) \subseteq (Ker T)^{\downarrow}$.

Proof. Let $N \in \mathcal{F}(U_S)$. If $M \in Ker\, T$ and $f \in Hom_S(M, N)$, then for the morphism $T(f)\colon U \otimes_S M \to U \otimes_S N$ and for every $m \in M$ we have $U \otimes_S m = 0$ in $U \otimes_S M = 0$. Therefore $U \otimes_S f(m) = 0$ in $U \otimes_S N$, and from the assumption $N \in \mathcal{F}(U_S)$ now it follows $f(m) = 0$. Thus $f = 0$ and $Hom_S(M, N) = 0$ for every $M \in KerT$, i.e. $M \in (Ker\, T)^{\downarrow}$. $\square$

Proposition 1.4. $\big(\mathcal{F}(U_S)\big)^{\uparrow} = KerT$.

Proof. $(\subseteq)$ Let $M \in \big(\mathcal{F}(U_S)\big)^{\uparrow}$, i.e. $Hom_S(M, N) = 0$ for every $N \in \mathcal{F}(U_S)$. Since t_U is a radical, for every $M \in S$-Mod we have:

$$M/t_U(M) \in \mathcal{P}(t_U) = \mathcal{F}(U_S).$$

From the assumption it follows $Hom_R\big(M, M/t_U(M)\big) = 0$, therefore $M/t_U(M) = 0$, i.e. $M = t_U(M)$. This means that $U \otimes_S m = 0$ in $U \otimes_S M$ for every $m \in M$, thus $U \otimes_S M = 0$.

($\supseteq$) By Proposition 1.3 $\mathcal{F}(U_S) \subseteq (KerT)^{\downarrow}$, therefore

$$\big(\mathcal{F}(U_S)\big)^{\uparrow} \supseteq (KerT)^{\downarrow\uparrow} = KerT,$$

the last relation beeng true since $KerT$ is a torsion class (Proposition 1.2). $\square$

Proposition 1.5. *For every module U_S we have the relation $t_U \geq \overline{t}_U$ and $\overline{t}_U$ is the greatest idempotent radical contained in the radical t_U.*

Proof. By Proposition 1.3 $\mathcal{F}(U_S) \subseteq (KerT)^{\downarrow}$, i.e. $\mathcal{P}(t_U) \subseteq \mathcal{P}(\overline{t}_U)$, therefore $t_U \geq \overline{t}_U$. Moreover, from Proposition 1.4 it follows $\big(\mathcal{F}(U_S)\big)^{\uparrow\downarrow} = (Ker\,T)^{\downarrow}$ and, since $\mathcal{F}(U_S) = \mathcal{P}(t_U)$ and $(Ker\,T)^{\downarrow} = \mathcal{P}(\overline{t}_U)$, we obtain $\big(\mathcal{P}(t_U)\big)^{\uparrow\downarrow} = \mathcal{P}(\overline{t}_U)$. Thus $\mathcal{P}(\overline{t}_U)$ is the least torsionfree class, containing $\mathcal{P}(t_U)$, which is equivalent with the assertion of proposition. $\square$

Further we will show the necessary and sufficient conditions for coincidence of these two "neighbour" preradicals t_U and $\overline{t}_U$. We will need the following notion.

Definition 1. A module U_S will be called *weakly flat* if the functor $T = U \otimes_S$-preserves the short exact sequences of the form

$$0 \to t_U(M) \underset{\subseteq}{\xrightarrow{i}} M \underset{\text{nat}}{\xrightarrow{\pi}} M\,/\,t_U(M) \to 0$$

for every module $M \in S$-Mod (i.e. $T(i)$ is a monomorphism for every ${}_SM$).

Proposition 1.6. *For module U_S the following conditions are equivalent*:

1) $t_U = \overline{t}_U$;
2) *radical t_U is idempotent*;
3) $\mathcal{F}(U_S) = (KerT)^{\downarrow}$;
4) *U_S is weakly flat.*

Proof. 1) $\Leftrightarrow$ 2) $\Leftrightarrow$ 3) follow from Proposition 1.5.

2) $\Rightarrow$ 4). If t_U is idempotent, then $t_U(M) = t_U\big(t_U(M)\big)$ for every module ${}_SM$, therefore

$$t_U(M) \in \mathcal{R}(t_U) = \mathcal{R}(\overline{r}_U) = KerT,$$

thus $T\big(t_U(M)\big) = 0$. So $T(i) = 0$ and $T(i)$ is mono, where i is the inclusion $t_U(M) \subseteq M$.

4) $\Rightarrow$ 2). Let U_S be a weakly flat module. Let $m \in t_U(M)$, i.e. $U \otimes_S m = 0$ in $U \otimes_S M$. Since the subset $U \otimes_S m \subseteq U \otimes_S t_U(M)$ pass by $T(i)$ on $U \otimes_S i(m) = U \otimes_S m = 0$ in $U \otimes_S M$, and by assumption $T(i)$ is a monomorphism, we have $U \otimes_S m = 0$ in $U \otimes_S t_U(M)$. Therefore $m \in t_U\big(t_U(M)\big)$ and $t_U(M) \subseteq t_U\big(t_U(M)\big)$, i.e. t_U is idempotent. $\square$

Now we will consider the stronger condition to radical t_U: the requirement to be a *torsion* (i.e. hereditary radical).

Definition 2. The module U_S will be called *t-hereditary* if from $U \otimes_S M = 0$ it follows $U \otimes_S N = 0$ for every submodule $N \subseteq M$.

From the previous results and definitions follows

Proposition 1.7. *For module U_S the following conditions are equivalent*:

1) *radical t_U is a torsion*;
2) $t_U = \overline{t}_U$ *and class $Ker\, T$ is hereditary*;
3) $t_U = \overline{t}_U$ *and class $(Ker\, T)^{\downarrow}$ is stable*;
4) U_S *is weakly flat and t-hereditary.* □

Corollary 1.8. *If module U_S is flat then the radical t_U is a torsion.*

Proof. If U_S is flat then by definition it is weakly flat. Let $U \otimes_S M = 0$ and $N \overset{i}{\subseteq} M$. Then $T(i)$ is monomorphism, so $U \otimes_S N = 0$, i.e. U_S is t-hereditary. □

2 Relations between $(t_U, \overline{t}_U)$ and preradicals defined by ideal $J = (0 : U_S)$

As before we fix a module U_S which defines the radical t_U (Section 1). Acting by t_U to ${}_SS$ we obtain the ideal:

$$J \overset{\text{def}}{=\!=} t_U({}_SS) = \{s \in S \,|\, U \otimes_S s = 0 \ \text{ in } \ U \otimes_S S\}.$$

The isomorphism $U \otimes_S S \cong U$ show that the relation $U \otimes_S s = 0$ in $U \otimes_S S$ means that $Us = 0$, therefore the ideal

$$J = (0 : U_S) = \{s \in S \,|\, Us = 0\}$$

is the *annihilator* of module U_S. As every ideal of a ring, J determines in S-Mod the following classes of modules [1, 2, 7]:

$_J\mathcal{T} = \{M \in S\text{-}Mod \,|\, JM = M\}$;
$_J\mathcal{F} = \{M \in S\text{-}Mod \,|\, m \in M, Jm = 0 \Rightarrow m = 0\}$;
$\mathcal{A}(J) = \{M \in S\text{-}Mod \,|\, JM = 0\}$.

We remind briefly form some facts on these classes of modules.

Proposition 2.1. 1) $_J\mathcal{T}$ *is a torsion class, therefore it determines an* <u>*idempotent radical*</u> r^J *such that* $\mathcal{R}(r^J) \overset{def}{=\!=} {}_J\mathcal{T}$ *and so* $\mathcal{P}(r^J) = {}_J\mathcal{T}^{\downarrow}$;

2) $_J\mathcal{F}$ *is a torsionfree and stable class, therefore it determines a* <u>*torsion*</u> r_J *such that* $\mathcal{P}(r_J) \overset{def}{=\!=} {}_J\mathcal{F}$ *and so* $\mathcal{R}(r_J) = {}_J\mathcal{F}^{\uparrow}$;

3) $\mathcal{A}(J)$ *is a pretorsion and hereditary class, therefore it determines a* <u>*pretorsion*</u> $r_{(J)}$ *such that* $\mathcal{R}(r_{(J)}) = \mathcal{A}(J)$;

4) *$\mathcal{A}(J)$ is a pretorsionfree and cohereditary class, therefore it determines a cohereditary radical $r^{(J)}$ such that $\mathcal{P}\left(r^{(J)}\right) \overset{def}{=\!=} \mathcal{A}(J)$.* $\square$

Proposition 2.2. 1) *$r^{J} \leq r^{(J)}$ and r^{J} is the greatest idempotent radical contained in $r^{(J)}$.*

2) *$r_{J} \geq r_{(J)}$ and r_{J} is the least idempotent radical (torsion) containing $r_{(J)}$.* $\square$

Proposition 2.3. *The following conditions are equivalent*:

1) $r^{J} = r^{(J)}$;
2) *$r^{(J)}$ is idempotent*;
3) $\mathcal{A}(J) = {}_{J}\mathcal{T}^{\downarrow}$;
4) $r_{J} = r_{(J)}$;
5) *$r_{(J)}$ is a radical*
6) $\mathcal{A}(J) = {}_{J}\mathcal{F}^{\uparrow}$;
7) $J = J^2$. $\square$

Next we will study the relations between the preradicals defined by ideal $J \lhd S$ and preradicals $t_U, \overline{t}_U$ from Section 1. For that purpose it is sufficient to clarify the connections between the respective classes of modules.

Proposition 2.4. *$\mathcal{F}(U_S) \subseteq \mathcal{A}(J)$ (i.e. $\mathcal{P}(t_U) \subseteq \mathcal{P}\left(r^{(J)}\right)$, so $t_U \geq r^{(J)}$.*

Proof. Let $M \in \mathcal{F}(U_S)$. For every $j \in J$ and $m \in M$ we have:

$$U \otimes_S (j\,m) = (Uj) \otimes_S m = 0 \otimes_S m \ \text{ in } \ U \otimes_S M,$$

thus by assumption it follows $j\,m = 0$. Therefore $JM = 0$, i.e. $M \in \mathcal{A}(J)$. $\square$

Proposition 2.5. *${}_{J}\mathcal{T} \subseteq KerT$ (i.e. $\mathcal{R}(r^{J}) \subseteq \mathcal{R}(\overline{t}_U)$*, so *$r^{J} \leq \overline{t}_U$).*

Proof. If $M \in {}_{J}\mathcal{T}$, then $JM = M$ and we have

$$U \otimes_S M = U \otimes_S (JM) = UJ \otimes_S M = 0 \otimes_S M = 0,$$

thus $M \in KerT$. $\square$

From the last statement it follows that

$${}_{J}\mathcal{T}^{\downarrow} \supseteq (KerT)^{\downarrow} = \left(\mathcal{F}(U_S)\right)^{\uparrow\downarrow} \supseteq \mathcal{F}(U_S),$$

i.e. $\mathcal{P}(r^{J}) \supseteq \mathcal{P}(\overline{t}_U) \supseteq \mathcal{P}(t_U)$, which means that

$$r^{J} \leq \overline{t}_U \leq t_U.$$

In this way, we obtain the following scheme, which illustrates the relations between preradicals studied above:

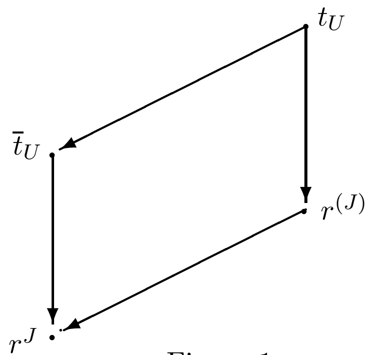


Figure 1.

The question of coincidence of all these preradicals is more complicated than in the case of functor H [1]. We remark, in particular, that the relations $t_U = r^{(J)}$ or $\overline{t}_U = r^J$ are not sufficient for the coincidence of all preradicals of Figure 1.

The relation $r^J = \overline{t}_U$ is equivalent to the inclusion $Ker\, T \subseteq {}_J\mathcal{T}$; the relation $r^{(J)} = t_U$ is equivalent to the inclusion $\mathcal{A}(J) \subseteq \mathcal{F}(U_S)$. Finally, the stronger relation $r_J = t_U$ is equivalent to the inclusion ${}_J\mathcal{T}^{\downarrow} \subseteq \mathcal{F}(U_S)$.

The general situation on classes of modules in this case is shown in Figure 2 (see next page).

3 Supplement to the case of functor H

In the part I of this work [1] we noted the fact that for the functor H is not obtained the symmetric statements for the preradicals $\big(r_I, r_{(I)}\big)$. Now we supplement the results of [1], using the above constructions for the functor T.

We remind that in part I [1] is studied the functor

$$H = H^U = Hom_R(U,\text{-}) : R\text{-}Mod \to \mathcal{A}b$$

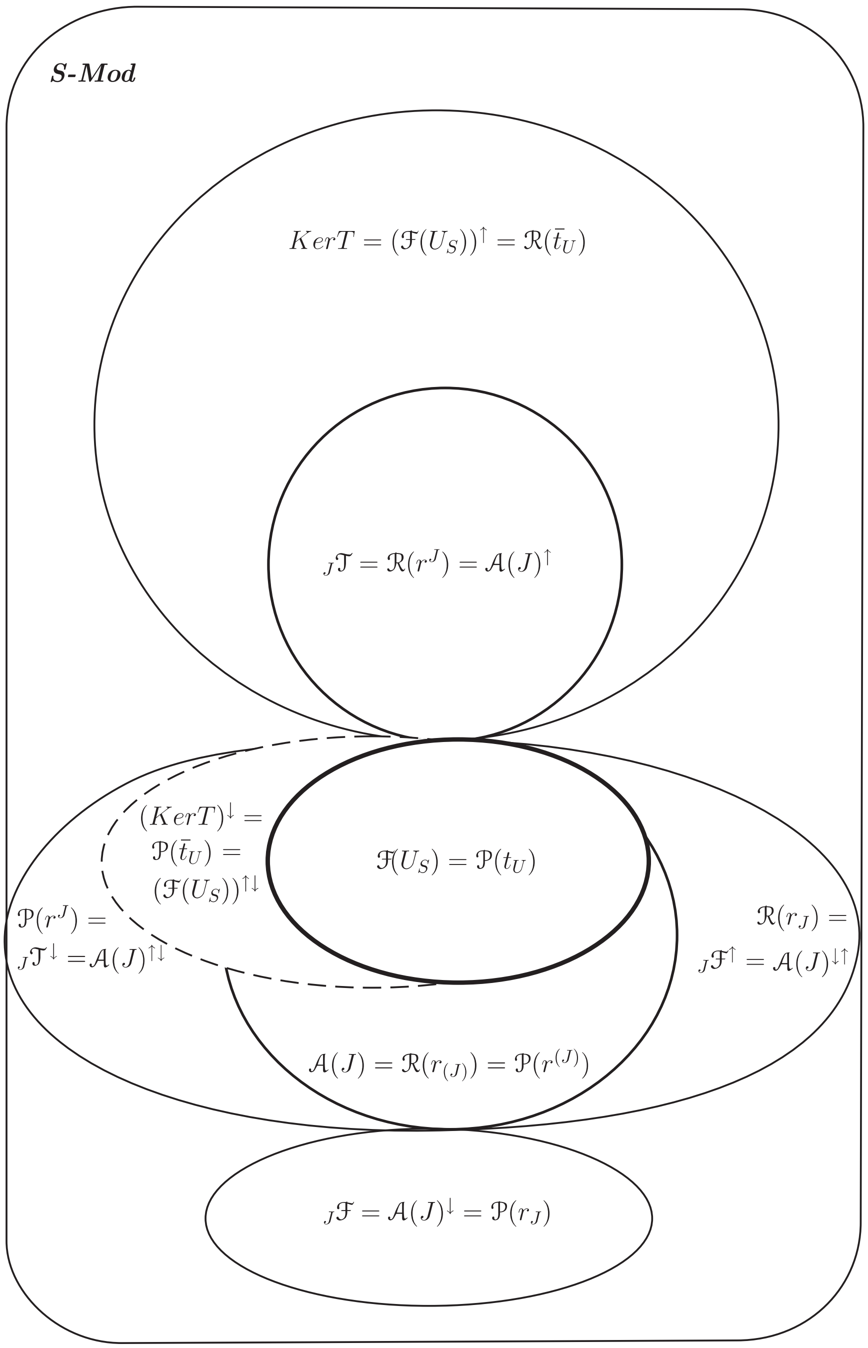
for a fixed module ${}_RU \in R$-Mod. We have the idempotent preradical r^U in R-Mod with $\mathcal{R}(r^U) = Gen\,({}_RU)$ and the idempotent radical $\overline{r}^{\,U}$ with $\mathcal{P}(\overline{r}^{\,U}) = Ker\, H$. Moreover, the trace of ${}_RU$ in R, i.e. the ideal $I = r^U({}_RR)$, determines two pairs of preradicals of different types: $\big(r^I, r^{(I)}\big)$ and $\big(r_I, r_{(I)}\big)$. We obtained the situation

$$r^I \leq r^U \leq \overline{r}^{\,U}, \quad r^I \leq r^{(I)} \leq \overline{r}^{\,U},$$

studying the conditions of coincidence of these preradicals.

Now we will construct two preradicals t_V and $\overline{t}_V$, which are related similarly with the pair $(r_I, r_{(I)})$. With this purpose for our fixed module ${}_RU \in R$-Mod we denote:

$$V_R = Hom_R(U, R),$$



S-Mod
$Ker T = (\mathcal{F}(U_S))^{\uparrow} = \mathcal{R}(\bar{t}_U)$
${}_J\mathcal{T} = \mathcal{R}(r^J) = \mathcal{A}(J)^{\uparrow}$
$(Ker T)^{\downarrow} =$
$\mathcal{P}(\bar{t}_U) =$
$(\mathcal{F}(U_S))^{\uparrow\downarrow}$
$\mathcal{F}(U_S) = \mathcal{P}(t_U)$
$\mathcal{P}(r^J) =$
${}_J\mathcal{T}^{\downarrow} = \mathcal{A}(J)^{\uparrow\downarrow}$
$\mathcal{R}(r_J) =$
${}_J\mathcal{F}^{\uparrow} = \mathcal{A}(J)^{\downarrow\uparrow}$
$\mathcal{A}(J) = \mathcal{R}(r_{(J)}) = \mathcal{P}(r^{(J)})$
${}_J\mathcal{F} = \mathcal{A}(J)^{\downarrow} = \mathcal{P}(r_J)$

Figure 2.

the dual module of ${}_RU$, which is a *right R-module*. For this module we consider the functor

$$T = T^V = V \otimes_R \text{-} : R\text{-}Mod \to \mathcal{A}b,$$

which determines the associated preradicals t_V and $\bar{t}_V$ of R-Mod, where:

1) t_V is a *radical* of R-Mod with $\mathcal{P}(t_V) \overset{\text{def}}{=\!=} \mathcal{F}(V_R) =$
$= \{M \in R\text{-}Mod \,|\, V \otimes_R m = 0 \text{ in } V \otimes_R M \Rightarrow m = 0\};$

2) $\bar{t}_V$ is an *idempotent radical* of R-Mod such that $\mathcal{R}(\bar{t}_V) \overset{\text{def}}{=\!=} Ker\,T^V$, therefore $\mathcal{P}(\bar{t}_V) = (Ker T^V)^{\downarrow}$.

From Section 1 it follows that $\mathcal{F}(V_R) \subseteq Ker T^V)^{\downarrow}$ (Proposition 1.3), thus $t_V \geq \bar{t}_V$. Moreover, $\big(\mathcal{F}(V_R)\big)^{\uparrow} = Ker T^V$, therefore $\bar{t}_V$ is the greatest idempotent radical contained in t_V (Proposition 1.5).

Now we will combine this situation with the corresponding situation defined in R-Mod by module ${}_RU$ and ideal I [1]. The purpose is to clarify the relations between preradicals studied in part I [1] and preradicals $(t_V, \bar{t}_V)$. As usual, we study the connections between the corresponding classes of modules.

Proposition 3.1. *$Ker T^V \subseteq \mathcal{A}(I)$.*

Proof. Every element $u \in U$ determines the morphism $\varphi_u : V \otimes_R M \to M$ by the rule $\varphi_u(f \otimes m) \overset{\text{def}}{=\!=} [(u)f] \cdot m$, where $f \in Hom_R(U, R)$ and $m \in M$. We have $Im\,\varphi_u = [(u)V] \cdot M$ and

$$\sum_{u \in U} Im\,\varphi_u = \sum_{u \in U} [(u)V] \cdot M = \Big(\sum_{f:U \to R} Im\, f \Big) \cdot M = I\,M.$$

If $M \in Ker T^V$, then $V \otimes_R M = 0$ and $\varphi_u = 0$ for every $u \in U$, therefore $\sum\limits_{u \in U} Im\,\varphi_u = I\,M = 0$. $\square$

Proposition 3.2. *${}_I\mathcal{F} \subseteq \mathcal{F}(V_R)$.*

Proof. Let $M \in {}_I\mathcal{F}$, i.e. from $I \cdot m = 0 \;\; (m \in M)$ it follows $m = 0$. Suppose that $V \otimes_R m = 0$ in $V \otimes_R M$. Then as in the preceding proof, for every $u \in U$ we have the morphism $\varphi_u : V \otimes_R M \to M$ such that

$$\varphi_u(V \otimes_R m) = [(u)V] \cdot m = 0.$$

Therefore

$$\sum_{u \in U} \varphi_u(V \otimes_R m) = \sum_{u \in U} [(u)V] \cdot m = I \cdot m = 0$$

and from the assumption $M \in {}_I\mathcal{F}$ it follows $m = 0$. So $M \in \mathcal{F}(V_R)$. $\square$

We remark that from Proposition 1.3 we have also the inclusion:

$$\mathcal{F}(V_R) \subseteq (Ker T^V)^{\downarrow}.$$

Corollary 3.3. *$\bar{t}_V \leq r_{(I)}$ and $r_I \geq t_V$.*

Proof. Since $Ker\,T^V = \mathcal{R}(\overline{t}_V)$ and $\mathcal{A}(I) = \mathcal{R}\big(r_{(I)}\big)$, from Proposition 3.1 we have $\mathcal{R}(\overline{t}_V) \subseteq \mathcal{R}\big(r_{(I)}\big)$, thus $\overline{t}_V \leq r_{(I)}$.

Similarly, since ${}_I\mathcal{F} = \mathcal{P}(r_I)$ and $\mathcal{F}(V_R) = \mathcal{P}(t_V)$, from Proposition 3.2 it follows $\mathcal{P}(r_I) \subseteq \mathcal{P}(t_V)$, therefore $r_I \geq t_V$. □

In this way, for the functor H we have the following relations between the associated preradicals:

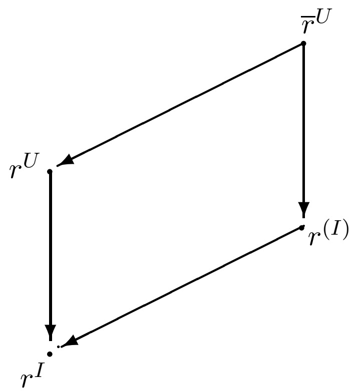


Figure 3.

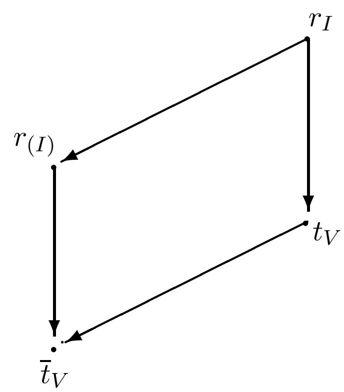


Figure 4.

The conditions of coincidence of preradicals from Figure 3 are shown in part I ([1], Proposition 4.4). A similar result is true for preradicals from Figure 4.

Proposition 3.4. *The following conditions are equivalent*:

1) $t_V = r_I$;
2) $\overline{t}_V = r_I$;
3) $\overline{t}_V = r_{(I)}$;
4) $t_V = r_{(I)}$;
5) $VI = V$.

Proof. The equivalence of conditions 1)–4) can be verified similarly to the proof of Proposition 4.4 of part I [1].

1) $\Rightarrow$ 5). Let $t_V = r_I$. Then $\mathcal{P}(t_V) = \mathcal{P}(r_I)$, i.e. $\mathcal{F}(V_R) = {}_I\mathcal{F}$. Therefore $\big(\mathcal{F}(V_R)\big)^{\uparrow} = {}_I\mathcal{F}^{\uparrow}$ where $\big(\mathcal{F}(V_R)\big)^{\uparrow} = KerT^V$, thus $KerT^V = {}_I\mathcal{F}^{\uparrow}$. From the relations

$$KerT^V \subseteq \mathcal{A}(I) \subseteq {}_I\mathcal{F}^{\uparrow}$$

we obtain $\mathcal{A}(I) = KerT^V$. Since $R/I \in \mathcal{A}(I)$, we have $R/I \in KerT^V$, i.e. $V \otimes_R (R/I) \cong V/VI = 0$, thus $V = VI$.

5) $\Rightarrow$ 1). Let $V\,I = V$. It is sufficient to show that $\mathcal{F}(V_R) = {}_I\mathcal{F}$, i.e. the inclusion $\mathcal{F}(V_R) \subseteq {}_I\mathcal{F}$. If $M \in \mathcal{F}(V_R)$ and $I \cdot m = 0$ for some $m \in M$, then:

$$V \otimes_R m = VI \otimes_R m = V \otimes_R (Im) = 0 \;\text{ in }\; V \otimes_R M.$$

From the assumption $M \in \mathcal{F}(V_R)$ now it follows $m = 0$. So $M \in {}_I\mathcal{F}$. □

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On commutative Moufang loops with some restrictions for subloops and subgroups of its multiplication groups

Natalia Lupashco

Abstract. It is proved that if an infinite commutative Moufang loop L has such an infinite subloop H that in L every associative subloop which has with H an infinite intersection is a normal subloop then the loop L is associative. It is also proved that if the multiplication group $\mathfrak{M}$ of infinite commutative Moufang loop L has such an infinite subgroup $\mathfrak{N}$ that in $\mathfrak{M}$ every abelian subgroup which has with $\mathfrak{N}$ an infinite intersection is a normal subgroup then the loop L is associative.

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When considering different classes of algebras (rings, groups, loops) it is very important to know whether they have subalgebras (systems of subalgebras) with prescribed features. For example, in [1] it is proved that *every infinite CML L contains an infinite associative subloop* and, *if all infinite associative subloops of L are normal in L, then L is associative* [2]. Similarly from the equivalence of statements 1), 2), 8) of Theorem 3.5 from [1] it follows that *every multiplication group $\mathfrak{M}$ of infinite CML L contains an infinite abelian subgroup* and if all infinite abelian subgroups of the multiplication group $\mathfrak{M}$ are normal in $\mathfrak{M}$ then CML L is associative [3].

In this work the restriction on infinite associative subloops and infinite abelian subgroups is reduced. We prove that if an infinite CML L (respect. multiplication group $\mathfrak{M}$ of infinite CML L) has such an infinite subloop H (respect. infinite subgroup $\mathfrak{N}$) that in L (respect. $\mathfrak{M}$) every associative subloop (respect. abelian subgroup) which has with H (respect. $\mathfrak{N}$) an infinite intersection is a normal subloop (respect. subgroup) then the CML L is associative.

We remind that the *commutative Moufang loop* (abbreviated *CML*) is characterized by the identity $x^2 \cdot yz = xy \cdot xz$.

The *multiplication group* $\mathfrak{M}(L)$ of a CML L is the group generated by all the *translations* $R(x)$, where $R(x)y = yx$.

The subgroup $\mathfrak{I}(L)$ of the group $\mathfrak{M}(L)$, generated by all the *inner mappings* $R(x,y) = R^{-1}(xy)R(y)R(x)$ is called the *inner mapping group* of the CML L.

A subloop H of the CML L is called *normal* in L if $x \cdot yH = xy \cdot H$ for all $x, y \in L$. Equivalently, H is normal in L if $\mathfrak{I}(Q)H = H$.

The *center* $Z(L)$ of the CML L is the normal subloop $Z(L) = \{x \in L | xy \cdot z = x \cdot yz \quad \forall y, z \in L\}$ [4].

Further we will denote by $< M >$ the subloop of the loop L, generated by the set $M \subseteq L$.

Theorem. *For an infinite CML L with multiplication group $\mathfrak{M}$ the following statements are equivalent:*

1) the CML L is associative;

2) the CML L has such an infinite subloop H that every associative subloop which hase an infinite intersection with H is a normal subloop in L;

3) the group $\mathfrak{M}$ is abelian;

4) the group $\mathfrak{M}$ has such an infinite subloop $\mathfrak{N}$ that every associative subloop which hase an infinite intersection with $\mathfrak{N}$ is a normal subgroup in $\mathfrak{M}$.

Proof. The implications 1) $\Rightarrow$ 2), 3) $\Rightarrow$ 4), 1) $\Leftrightarrow$ 3) are obvious.

2) $\Rightarrow$ 1). We suppose that H is a non-periodic subloop and let $a \in H$ be an element of infinite order. By [4] the element a^3 belongs to the center $Z(L)$ of CML L. Let b be an arbitrary element in L such that $< b > \cap < a >= 1$. The subloop $< a^3 >$ is normal in L. Then by [4] the product $< b >< a^3 >$ is a subgroup. As $< a^3 > \subseteq < b >< a^3 > \cap H$ then by statement 2) $< b >< a^3 >$ is a normal subloop in L. Let φ be an inner mapping of CML L. In CML the inner mappings are its automorphisms [4]. Then $< b >< a^3 >= \varphi(< b >< a^3 >) = \varphi(< b >)\varphi(< a^3 >) = \varphi(< b >) < a^3 >$, $\varphi(< b >) < a^3 >=< b >< a^3 >$. We have $< b > \cap < a^3 >= 1$. Then and $\varphi(< b >) \cap < a^3 >= 1$.

We denote $< a^3 >= A$, $< b >= B$. Let θ, η be the restrictions of natural homomorphism $\lambda : AB \to AB/A$ onto B and φB respectively. Obviously, $\ker\theta = B \cap A$, $\ker\eta = \varphi B \cap A$. Then from equalities $B \cap A = 1$, $\varphi B \cap A = 1$ it follows that θ, η are monomorphisms.

Let $b \in B$. Then $b = ca$ for some $c \in \varphi B$, $a \in A$. Further, $\lambda b = \lambda(ca)$, $\lambda b = \lambda c \cdot \lambda a$, $\lambda b = \lambda c \cdot \lambda 1$, $\lambda b = \lambda c$. The homomorphism λ acts onto φB as η. Hence $\lambda c = \eta c$. η is a restriction of λ onto φB and is a monomorphism of φB. Then from $\lambda b = \eta c$ it follows that $b \in \varphi B$, $B \subseteq \varphi B$. Analogously, $\varphi B \subseteq B$. Hence $\varphi B = B$. Consequently, the subloop $< b >$ is normal in L.

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Next, using the normality of $< b >$ in L by analogy it is proved that the subloop $< a >$ is normal in L. We get that any element of L generates a normal subloop in L. This means that CML L is hamiltonian. But any hamiltonian CML is associative [5]. Hence the implication 2) $\Rightarrow$ 1 holds.

Now we suppose that the abelian group H is periodic. Then H decomposes into a direct product of its maximal p-subgroups H_p. Let $H = D \times H_3$. By [6] $D \subseteq Z(L)$. The subgroup D is normal in L. If D is infinite then, as in the previous case, it is proved that CML L is hamiltonian and, consequently, is associative. If D is finite then the subgroup H_3 is infinite.

We suppose that the infinite abelian group H_3 satisfies the minimum condition for its subgroups. Then $H_3 = T \times K$, where K is a finite group and T is an infinite divisible group. By [1] $T \subset Z(L)$ and, as in the previous case the CML L is associative.

To prove the implication 2 $\Rightarrow$ 1) we have only to consider the case when the abelian group H_3 does not satisfy the minimum condition for its subgroups. In this case H_3 has an infinite abelian subgroup B which decomposes into a direct product of cyclic groups of order 3. Let $b \in B$ and let $R \subseteq B$ be such a subgroup that $< b > \cap R = 1$. Let $R = R_1 \times R_2$ be a certain decomposition of group R into a direct product of two infinite subgroups. From statement 2) it follows that the subloops R_1, $< b > \times R_1$, R_2, $< b > \times R_2$ are normal in CML L. In [2] it is proved that if in a CML an element of order 3 generates a normal subloop, then this element belongs to the center of this CML. Then $b \in Z(L)$ and, consequently, $B \subseteq Z(L)$. The subgroup B is infinite, then as in the previous cases, it may be proved that the CML L is associative. Consequently, the implication 2) $\Rightarrow$ 1) holds.

4) $\Rightarrow$ 3). We suppose that $\mathfrak{N}$ is a non-periodic subgroup and let $\alpha \in \mathfrak{N}$ be an element of infinite order. Let $C(\mathfrak{M})$ denote the center of group $\mathfrak{M}$. In [4] it is proved that the quotient group $\mathfrak{M}/C(\mathfrak{M})$ is a locally finite 3-group. Then the element α^k belongs to the center $C(\mathfrak{M})$ for some integer k. Let ε be the unity of the group $\mathfrak{M}$ and let β be an arbitrary element in $\mathfrak{M}$ such that $< \beta > \cap < \alpha >= \varepsilon$. The subgroup $< \alpha^k >$ is normal in $\mathfrak{M}$. Then the product $< \beta >< \alpha^k >$ is a subgroup. As $< \alpha^k > \subseteq < \beta >< \alpha^k > \cap \mathfrak{N}$ then by statement 4) $< \beta >< \alpha^k >$ is a normal subgroup in $\mathfrak{M}$. Let φ be an inner automorphism of group $\mathfrak{M}$. Then $< \beta >< \alpha^k >= \varphi(< \beta >< \alpha^k >) = \varphi(< \beta >)\varphi(< \alpha^k >) = \varphi(< \beta >) < \alpha^k >$, $\varphi(< \beta >) < \alpha^k >=< \beta >< \alpha^k >$. We have $< \beta > \cap < \alpha^k >= \varepsilon$. Then and $\varphi(< \beta >) \cap < a^3 >= \varepsilon$.

We denote $< \alpha^k >= \mathfrak{A}$, $< \beta >= \mathfrak{B}$. Let θ, η be the restrictions of natural homomorphism $\lambda : \mathfrak{A}\mathfrak{B} \to \mathfrak{A}\mathfrak{B}/\mathfrak{A}$ onto $\mathfrak{B}$ and $\varphi\mathfrak{B}$ respectively. Obviously, $\ker\theta = \mathfrak{B} \cap \mathfrak{A}$, $\ker\eta = \varphi\mathfrak{B} \cap \mathfrak{A}$. Then from equalities $\mathfrak{B} \cap \mathfrak{A} = \varepsilon$, $\varphi\mathfrak{B} \cap \mathfrak{A} = \varepsilon$ it follows that θ, η are monomorphisms.

Let $\beta \in \mathfrak{B}$. Then $\beta = \gamma\alpha$ for some $\gamma \in \varphi\mathfrak{B}$, $\alpha \in \mathfrak{A}$. Further, $\lambda\beta = \lambda(\gamma\alpha)$, $\lambda\beta = \lambda\gamma \cdot \lambda\alpha$, $\lambda\beta = \lambda\gamma \cdot \lambda\varepsilon$, $\lambda\beta = \lambda\gamma$. The homomorphism λ acts onto $\varphi\mathfrak{B}$ as η.

Hence $\lambda\gamma = \eta\gamma$. η is a restriction of λ onto $\varphi\mathfrak{B}$ and is a monomorphism of $\varphi\mathfrak{B}$. Then from $\lambda\beta = \eta\gamma$ it follows that $\beta \in \varphi\mathfrak{B}$, $\mathfrak{B} \subseteq \varphi\mathfrak{B}$. Analogously, $\varphi\mathfrak{B} \subseteq \mathfrak{B}$. Hence $\varphi\mathfrak{B} = \mathfrak{B}$. Consequently, the subgroup $< \beta >$ is normal in $\mathfrak{M}$.

Further, using the normality of $< \beta >$ in $\mathfrak{M}$ it is proved by analogy that the subgroup $< \alpha >$ is normal in $\mathfrak{M}$. We get that any element in $\mathfrak{M}$ generates a normal subgroup in $\mathfrak{M}$. This means that the group $\mathfrak{M}$ is hamiltonian. But any hamiltonian multiplication group of CML is abelian [3]. Hence the implication 4)$\Rightarrow$ 3) holds.

Now we suppose that the abelian group $\mathfrak{N}$ is periodic. Then $\mathfrak{N}$ decomposes into a direct product of its maximal p-subgroups $\mathfrak{N}_p$. Let $\mathfrak{N} = \mathfrak{D} \times \mathfrak{N}_3$. By [2] $\mathfrak{D} \subseteq C(\mathfrak{M})$. The subgroup $\mathfrak{D}$ is normal in $\mathfrak{M}$. If $\mathfrak{D}$ is infinite then as in the previous case, we show that the group $\mathfrak{M}$ is hamiltonian and, consequently, is abelian. If $\mathfrak{D}$ is finite then the subgroup $\mathfrak{N}_3$ is infinite.

Let the infinite abelian group $\mathfrak{N}_3$ satisfy the minimum condition for its subgroups. Then $\mathfrak{N}_3 = \mathfrak{T} \times \mathfrak{K}$, where $\mathfrak{K}$ is a finite group and $\mathfrak{T}$ is an infinite divisible group. By [1] $\mathfrak{T} \subset Z(\mathfrak{M})$ and as in the previous case the CML $\mathfrak{M}$ is abelian.

To prove the implication $4 \Rightarrow 3)$ we have to consider only the case when the abelian group $\mathfrak{N}_3$ does not satisfy the minimum condition for its subgroups. In this case $\mathfrak{N}_3$ has an infinite abelian subgroup $\mathfrak{B}$, which decomposes into a direct product of cyclic groups of order 3. Let $\beta \in \mathfrak{B}$ and let $\mathfrak{R} \subseteq \mathfrak{B}$ be such a subgroup that $< \beta > \cap \mathfrak{R} = \varepsilon$. Let $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2$ be a certain decomposition of group $\mathfrak{R}$ into a direct product of two infinite subgroups. From statement 2) it follows that the subloops $\mathfrak{R}_1$, $< \beta > \times\mathfrak{R}_1$, $\mathfrak{R}_2$, $< \beta > \times\mathfrak{R}_2$ are normal in group $\mathfrak{M}$. In [3] it is proved that if in a multiplication group of CML an element of order 3 generates a normal subgroup, then this element belongs to the center of this multiplication group. Then $\beta \in C(\mathfrak{M})$ and, consequently, $\mathfrak{B} \subseteq C(\mathfrak{M})$. The subgroup $\mathfrak{B}$ is infinite then as in the previous cases it may be proved that the group $\mathfrak{M}$ is abelian. Consequently, the implication 4) $\Rightarrow$ 3) holds. This completes the proof of Theorem. $\square$

We note that the construction of arbitrary groups that satisfy the equivalence of statements 3), 4) of Theorem is described in [7]. It is easy to see that the equivalence of statements 3), 4) and the equivalence of statements 1), 2) of Theorem are proved by the same schema. But if we use the results of paper [7] to prove the equivalence of statements 3), 4), then the proof doesn't get easier but, on the contrary, it gets more complicated.

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Vector Form of the Finite Fields $GF(p^m)$ *

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Abstract. Specially defined multiplication operation in the m-dimensional vector space (VS) over a ground finite field (FF) imparts properties of the extension FF to the VS. Conditions of the vector FF (VFF) formation are derived theoretically for cases $m = 2$ and $m = 3$. It has been experimentally demonstrated that under the same conditions VFF are formed for cases $m = 4$, $m = 5$, and $m = 7$. Generalization of these results leads to the following hypotheses: for each dimension value m the VS defined over a ground field $GF(p)$, where p is a prime and $m|p-1$, can be transformed into a VFF introducing special type of the vector multiplication operations that are defined using the basis-vector multiplication tables containing structural coefficients. The VFF are formed in the case when the structural coefficients that could not be represented as the mth power of some elements of the ground field are used. The VFF can be also formed in VS defined over extension FF represented by polynomials. The VFF present interest for cryptographic application.

Mathematics subject classification: 11G20, 11T71.
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1 Introduction

The finite fields (FF) play a prominent role in the public key cryptography. They are well studied as primitives of the digital signature (DS) algorithms [1–3]. Finding discrete logarithm (DL) in a subgroup of the multiplicative group of some FF is used as the hard computational problem put into the base of DS algorithms. The security of the DS is determined by the difficulty of the DL problem.

Since all FF of the same order are isomorphic, in many cases it is sufficient to consider only the polynomial FF $GF(p^n)$ and extend the results to any possible type of the field $GF(p^n)$. However in the case of computational problems it is reasonable to take into account concrete forms of the FF representation. For example, finding DL has essentially different difficulty in various particular variants of the field $GF(p^n)$ for the same values p and n. To reduce the DL problem defined in one representation form to the DL problem defined in some other particular form of the field $GF(p^n)$ one should compute the isomorphism between these FF variants. A prominent example of the analogous situation is presented by elliptic curves (EC) over finite fields [4]. Finite groups of the EC points are isomorphic to some subgroups of the multiplicative group of some ring Z_p, where the DL problem can be solved with methods having subexponential complexity, however the best known

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methods for solving the DL problem on specially selected EC have exponential complexity [5]. At present the DS algorithms based on the difficulty of the DL problem on EC are the most computationally efficient among the DSA providing the same security level. However performing the group operation over the EC points includes the inversion operation in the field over which the EC are defined [5]. The inversion operation significantly restricts the rate of the EC-based DS algorithms.

Search of new representation forms of the FF and their use in the DS algorithms have significant importance for information security practice. In the present paper we introduce a new form of the FF $GF(p^m)$ defined over the m-dimensional finite vector spaces (VS), the vector coordinates being the elements of some ground FF $GF(p)$. In such FF, called vector FF (VFF), the multiplication operation is free of the inversion in the underlying field $GF(p)$. Therefore the use of the VFF can provide significant improvement of the DS algorithm performance [6]. In Section 2 we derive the VFF formation conditions for cases $m = 2$ and $m = 3$. In Section 3 we experimentally show that the derived conditions work for the cases $m = 4$, $m = 5$, and $m = 7$. In Section 4 we generalize the VFF formation conditions to arbitrary dimension values. In the concluding Section 5 it is underlined that the VFF can be defined over some extended FF.

In the paper the following specific term is used:

The kth-power element in some FF $GF(p^d)$, where $d \geq 1$, is an element $a \in GF(p^d)$ for which the equation $x^k = a$ has solutions in $GF(p^d)$.

2 Two- and three-dimensional vector finite fields

Let us have some m-dimensional vector space over a field $GF(p)$. Suppose $\mathbf{e}$, $\mathbf{i}$, $\dots$, $\mathbf{j}$ be some m basis vectors and $a, b, c \in GF(p)$, where $p \geq 3$, are coordinates. So this space is the set of vectors $a\mathbf{e} + b\mathbf{i} + \cdots + c\mathbf{j}$. A vector can be also represented as a set of its coordinates $(a, b, \dots, c)$. The terms $\epsilon\mathbf{v}$, where $\epsilon \in GF(p)$ and $\mathbf{v} \in \{\mathbf{e}, \mathbf{i}, \dots, \mathbf{j}\}$, are called components of the vector.

The addition of two vectors $(a, b, \dots, c)$ and $(x, y, \dots z)$ is defined as follows:

$$(a, b, \dots, c) + (x, y, \dots, z) = (a + x, b + y, \dots, c + z),$$

where "+" denotes addition operation in the field $GF(p)$. The first representation of the vectors can be interpreted as the sum of the vector components.

Let the multiplication of the vectors $(a, b, \dots, c)$ and $(x, y, \dots z)$ be defined by the formula

$$(a\mathbf{e} + b\mathbf{i} + \cdots + c\mathbf{j}) \cdot (x\mathbf{e} + y\mathbf{i} + \cdots + z\mathbf{j}) = a\mathbf{e} \cdot x\mathbf{e} + b\mathbf{i} \cdot x\mathbf{e} + \dots$$

$$\cdots + c\mathbf{j} \cdot x\mathbf{e} + a\mathbf{e} \cdot y\mathbf{i} + b\mathbf{i} \cdot y\mathbf{i} + \cdots + c\mathbf{j} \cdot y\mathbf{i} + \dots a\mathbf{e} \cdot z\mathbf{j} + b\mathbf{i} \cdot z\mathbf{j} + \cdots + c\mathbf{j} \cdot z\mathbf{j} =$$

$$= ax\mathbf{e}\cdot\mathbf{e} + bx\mathbf{i}\cdot\mathbf{e} + \cdots + cx\mathbf{j}\cdot\mathbf{e} + ay\mathbf{e}\cdot\mathbf{i} + by\mathbf{i}\cdot\mathbf{i} + \cdots + cy\mathbf{j}\cdot\mathbf{i} + \dots az\mathbf{e}\cdot\mathbf{j} + bz\mathbf{i}\cdot\mathbf{j} + \cdots + cz\mathbf{j}\cdot\mathbf{j}),$$

where each product of two basis vectors is replaced by a vector component $\epsilon\mathbf{v}$ ($\epsilon \in GF(p)$) in accordance with some given tables called basis-vector multiplication tables (BVMT). To define formation of the VFF the BVMT should be properly designed.

Let us consider two- and three-dimensional VS defined over some ground field $GF(p)$. In the case $m = 2$ the general representation of the BVMT possessing commutativity, associativity, and unit $(1, 0)$ can be described as follows:

$$\mathbf{e} \cdot \mathbf{i} = \mathbf{i} \cdot \mathbf{e} = \mathbf{i}, \quad \mathbf{e} \cdot \mathbf{e} = \mathbf{e}, \quad \mathbf{i} \cdot \mathbf{i} = \epsilon \mathbf{e},$$

where different values $\epsilon \in GF(p)$ define different variants of the multiplication operation. Each of these variants defines a finite ring of the two-dimensional vectors. Let us consider a nonzero vector $Z = a\mathbf{e}+b\mathbf{i}$. The element $Z^{-1} = x\mathbf{e}+y\mathbf{i}$ is called an inverse of Z if $Z^{-1}Z = \mathbf{e} = (1, 0)$, where 1 and 0 are the identity and zero elements in $GF(p)$. We have

$$Z^{-1}Z = (ax + \epsilon by)\mathbf{e} + (bx + ay)\mathbf{i} = 1\mathbf{e} + 0\mathbf{i}.$$

For given (a, b) there exists a unique pair $(x, y) \in GF(p) \times GF(p)$ satisfying the last equation if the system of equations

$$\begin{cases} ax + \epsilon by &= 1, \\ bx + ay &= 0 \,. \end{cases}$$

has a unique solution in $GF(p) \times GF(p)$, i.e. if $a^2 - \epsilon b^2 \neq 0$ in $GF(p)$. The last condition holds for all vectors (a, b), except (0,0), if ϵ is not the second-power element in the field $GF(p)$. In this case the vector space is a field $GF(p^2)$ the multiplicative group of which has the order

$$\Omega = p^2 - 1 = (p - 1)(p + 1).$$

Thus, in the case $m = 2$ the characteristic equation

$$a^2 - \epsilon b^2 = 0 \tag{1}$$

defines formation of the VFF $GF(p^2)$. If this equation has no solution for each pair (a, b), except $(0, 0)$, then for each nonzero vector of the two-dimensional VS defined over the field $GF(p)$ there exists its unique inverse, i.e. we have the VFF $GF(p^2)$.

In the case $m = 3$ Table 1, where $\mu \in GF(p)$ and $\epsilon \in GF(p)$, represents the BVMT possessing commutativity, associativity, and unit $(1, 0, 0)$ for arbitrary values μ and ϵ, called structural coefficients. Let us consider a nonzero vector $Z = a\mathbf{e} + b\mathbf{i} + c\mathbf{k}$. There exists its unique inverse $X = x\mathbf{e} + y\mathbf{i} + z\mathbf{k}$ if the vector equation

$$ZX = (ax + \epsilon\mu cy + \epsilon\mu bz)\mathbf{e} + (bx + ay + \mu cz)\mathbf{i} + (cx + \epsilon by + az)\mathbf{j} = 1\mathbf{e} + 0\mathbf{i} + 0\mathbf{j}$$

has a unique solution relative to the unknown X. From the last equation the following system of equations can be derived

$$\begin{cases} ax + \epsilon\mu cy + \epsilon\mu bz &= 1, \\ bx + ay + \mu cz &= 0, \\ cx + \epsilon by + az &= 0. \end{cases}$$

Table 1. The BVMT in the general case for $m = 3$

$\cdot$	$\overrightarrow{e}$	$\overrightarrow{\imath}$	$\overrightarrow{\jmath}$
$\overrightarrow{e}$	$\mathbf{e}$	$\mathbf{i}$	$\mathbf{j}$
$\overrightarrow{\imath}$	$\mathbf{i}$	$\epsilon\mathbf{j}$	$\mu\epsilon\mathbf{e}$
$\overrightarrow{\jmath}$	$\mathbf{j}$	$\mu\epsilon\mathbf{e}$	$\mu\mathbf{i}$

From this system the following characteristic equation can be easily derived

$$a^3 - 3\epsilon\mu bc \cdot a + \epsilon^2\mu b^3 + \epsilon\mu^2 c^3 = 0. \tag{2}$$

If this equation has no solutions relative to the unknown a for each pair (b, c), except $(0, 0)$, and only one solution $a = 0$ for $(b, c) = (0, 0)$, then the three-dimensional VS is an extension FF $GF(p^3)$. Denoting $B = (\epsilon^2\mu b^3 + \epsilon\mu^2 c^3)/2$ and using the well known formulas [7] for cubic equation roots we get the expression for the roots a of equation (2) in the following form

$$a = A' + A'',$$

where

$$A' = \sqrt[3]{B + \sqrt{B^2 - (\epsilon\mu bc)^3}} = \sqrt[3]{-\epsilon\mu^2 c^3},$$

$$A'' = \sqrt[3]{B - \sqrt{B^2 - (\epsilon\mu bc)^3}} = \sqrt[3]{-\epsilon^2\mu b^3}.$$

Thus, if both of the values $\epsilon\mu^2$ and $\epsilon^2\mu$ are not the third-power elements in the field $GF(p)$, then the characteristic equation (2) has no solutions relative to the unknown a for all possible pairs $(a, b) \neq (0, 0)$ and only one solution $a = 0$ for $(a, b) = (0, 0)$. It is well known that this situation is possible if $3|p - 1$.

Thus, if $3|p - 1$ and each of the products $\epsilon^2\mu$ and $\epsilon\mu^2$ is not the third-power element in the field $GF(p)$, then for each nonzero vector Z there exists its unique inverse and the VS is the VFF $GF(p^3)$. The multiplicative group of the field $GF(p^3)$ has the order

$$\Omega = p^3 - 1 = (p - 1)(p^2 + p + 1).$$

Example 1. Suppose $p = 1723$ (i.e. $3|p - 1$). Then for $\mu = 1$ and $\epsilon = 1666$ (ϵ is not the cubic element in $GF(1723)$) a vector field $GF(p^3)$ is formed in which the vector (2,3,3) is a generator of the multiplicative group of the order $\Omega = p^3 - 1 = 5115120066$.

It is easy to see that characteristic equation (1) has no solutions over FF $GF(p^d)$ for some integer $d \geq 1$, if $b \neq 0$ and ϵ is not the second-power element in $GF(p^d)$. Analogously, characteristic equation (2) has no solutions over FF $GF(p^d)$ for some integer $d \geq 1$, if $(b, c) \neq (0, 0)$ and both values $\epsilon\mu^2$ and $\epsilon^2\mu$ are not the third-power elements in the field $GF(p^d)$. Thus, the two-dimensional VFF $GF(p^{2d})$ and three-dimensional VFF $GF(p^{3d})$ can be defined over the extension FF $GF(p^d)$.

Table 2. Basis-vector multiplication table for the case $m = 4$

$\cdot$	$\vec{e}$	$\vec{\imath}$	$\vec{\jmath}$	$\vec{k}$
$\vec{e}$	$\mathbf{e}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\vec{\imath}$	$\mathbf{i}$	$\epsilon\mathbf{j}$	$\epsilon\mathbf{k}$	$\mu\epsilon\mathbf{e}$
$\vec{\jmath}$	$\mathbf{j}$	$\epsilon\mathbf{k}$	$\mu\epsilon\mathbf{e}$	$\mu\mathbf{i}$
$\vec{k}$	$\mathbf{k}$	$\mu\epsilon\mathbf{e}$	$\mu\mathbf{i}$	$\mu\mathbf{j}$

3 Formation of the vector finite fields in the case $m \geq 4$

Analysis of the cases $m = 2$ and $m = 3$ shows that vector fields are formed in the case $m|p^d - 1$, provided some of the structural coefficients are not the mth-power elements in the field $GF(p^p)$ over which the VS is defined. Validity of this VFF formation condition has been experimentally demonstrated for cases $m = 4$, $m = 5$, and $m = 7$, while using the BVMT presented in Tables 2, 3, and 4, correspondingly. Tables 2 and 3 are designed in line with the BVMT type presented by Table 1 that relates to the case $m = 3$ (note that in the case of Table 3 the vector $(\tau^{-1}, 0, 0, 0, 0)$ is unit). The analogous design is possible for the case $m = 7$, however we used a particular variant for structure of Table 4 to show that in general different types of the BVMT can be applied to define VFF.

Example 2. For prime $p = 2609$, the dimension $m = 4$ $(m|p - 1)$, and coefficients $\mu = 1$ and $\epsilon = 2222$ (ϵ is not the 4th-power element in $GF(2731)$) the vector $G_\Omega = 1\mathbf{e}+3\mathbf{i}+3\mathbf{j}+5\mathbf{k}$ is a generator of the multiplicative group of the VFF $GF(p^4)$. The vector $G_q = 392\mathbf{e} + 2173\mathbf{i} + 2545\mathbf{j} + 443\mathbf{k}$ is a generator of the cyclic subgroup having prime order $q = 3403441$.

Example 3. For prime $p = 151$, the dimension $m = 5$ $(5|p - 1)$, and coefficients $\tau = \mu = 1$ and $\epsilon = 111$ (ϵ is not the 5th-power element in $GF(151)$) the vector $G_\Omega = 1\mathbf{e} + 3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k} + 11\mathbf{u}$ is a generator of the multiplicative group of the VFF $GF(p^5)$. The vector $G_q = 141\mathbf{e} + 111\mathbf{i} + 50\mathbf{j} + 28\mathbf{k} + 142\mathbf{u}$ is a generator of the subgroup having prime order $q = 104670301$.

Example 4. For prime $p = 29$, the dimension $m = 7$ $(7|p - 1)$, and coefficient $\epsilon = 3$ (ϵ is not the 7th-power element in $GF(29)$) the vector $G_\Omega = (1, 3, 7, 5, 3, 1, 4)$ is a generator of the multiplicative group of the VFF $GF(p^7)$. The vector $G_q = (7, 10, 0, 3, 15, 14, 22)$ is a generator of the subgroup having prime order $q = 88009573$.

Theoretic results presented in Section 2 and experiments for the cases $m = 4$ and $m = 5$ give us grounds to put forward the following hypothesis.

In some finite m-dimensional VS defined over a FF $GF(p^d)$ such that $m|p^d - 1$ and $d \geq 1$ it is possible to define vector multiplication with BVMT which imparts to the VS properties of the FF $GF(p^{dm})$.

Table 3. Basis-vector multiplication table for the case $m = 5$

$\cdot$	$\vec{e}$	$\vec{\imath}$	$\vec{\jmath}$	$\vec{k}$	$\vec{u}$
$\vec{e}$	$\tau\mathbf{e}$	$\tau\mathbf{i}$	$\tau\mathbf{j}$	$\tau\mathbf{k}$	$\tau\mathbf{u}$
$\vec{\imath}$	$\tau\mathbf{i}$	$\epsilon\mathbf{j}$	$\epsilon\mathbf{k}$	$\epsilon\mathbf{u}$	$\epsilon\mu\tau^{-1}\mathbf{e}$
$\vec{\jmath}$	$\tau\mathbf{j}$	$\epsilon\mathbf{k}$	$\epsilon\mathbf{u}$	$\epsilon\mu\tau^{-1}\mathbf{e}$	$\mu\mathbf{i}$
$\vec{k}$	$\tau\mathbf{k}$	$\epsilon\mathbf{u}$	$\epsilon\mu\tau^{-1}\mathbf{e}$	$\mu\mathbf{i}$	$\mu\mathbf{j}$
$\vec{u}$	$\tau\mathbf{u}$	$\epsilon\mu\tau^{-1}\mathbf{e}$	$\mu\mathbf{i}$	$\mu\mathbf{j}$	$\mu\mathbf{k}$

Table 4. Basis-vector multiplication table for the case $m = 7$

$\cdot$	$\vec{e}$	$\vec{\imath}$	$\vec{\jmath}$	$\vec{k}$	$\vec{u}$	$\vec{v}$	$\vec{w}$
$\vec{e}$	$\mathbf{e}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	$\mathbf{u}$	$\mathbf{v}$	$\mathbf{w}$
$\vec{\imath}$	$\mathbf{i}$	$\epsilon\mathbf{k}$	$\epsilon\mathbf{v}$	$\epsilon\mathbf{j}$	$\epsilon\mathbf{e}$	$\epsilon\mathbf{w}$	$\epsilon\mathbf{u}$
$\vec{\jmath}$	$\mathbf{j}$	$\epsilon\mathbf{v}$	$\epsilon\mathbf{u}$	$\epsilon\mathbf{w}$	$\mathbf{k}$	$\epsilon\mathbf{e}$	$\mathbf{i}$
$\vec{k}$	$\mathbf{k}$	$\epsilon\mathbf{j}$	$\epsilon\mathbf{w}$	$\epsilon\mathbf{v}$	$\mathbf{i}$	$\epsilon\mathbf{u}$	$\epsilon\mathbf{e}$
$\vec{u}$	$\mathbf{u}$	$\epsilon\mathbf{e}$	$\mathbf{k}$	$\mathbf{i}$	$\mathbf{w}$	$\mathbf{j}$	$\mathbf{v}$
$\vec{v}$	$\mathbf{v}$	$\epsilon\mathbf{w}$	$\epsilon\mathbf{e}$	$\epsilon\mathbf{u}$	$\mathbf{j}$	$\mathbf{i}$	$\mathbf{k}$
$\vec{w}$	$\mathbf{w}$	$\epsilon\mathbf{u}$	$\mathbf{i}$	$\epsilon\mathbf{e}$	$\mathbf{v}$	$\mathbf{k}$	$\mathbf{j}$

The required BVMT can be constructed analogously to Tables 2 and 3 and using the unit element of $GF(p)$ as the coefficient μ and a value ϵ that is not the mth-power element in $GF(p)$. Since $m|p^d - 1$ such values ϵ exist and can be easily found.

4 Conclusion

Defining the vector multiplication operation with BVMT that contain the structural coefficients having large size and using sufficiently large values m one can define more difficult DL problem in the VFF. Therefore in such cases VFF with smaller order size can be used to design the DS algorithms. Besides, the vector multiplication operation can be implemented as parallel performing the multiplications in the FF over which the VFF is defined. These two facts provide possibility to get sufficiently high performance of the DS algorithms based on VFF.

The following problems are important for further consideration of the VFF as cryptographic primitive.

1. Proof of the hypothesis presented in the end of Section 3.

2. Proof of the generalization of the experimental results (if there exist m-dimensional VFF over $GF(p^d)$ for some values d and p such that $m|p-1$, then there exist VFF for the same value d and arbitrary values p such that $m|p-1$).

3. Development of the BVMT providing minimization of the vector multiplication complexity.

4. Detailed investigation of the DL problem difficulty in VFF and its connection with the dimension value and the size of the structural coefficients in BVMT.

Using special type of BVMT it is possible to define non-commutative rings over finite VS, which also present interest as cryptographic primitive and is a subject of independent research.

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A lower bound for a quotient of roots of factorials

Cristinel Mortici

Abstract. With the aid of asymptotic properties of polygamma functions a new lower bound is established for the quotient $\phi(r+1)/\phi(r)$ where $\phi(r) = (r!)^{1/r}$.

Mathematics subject classification: 33B15; 57Q55; 15A15.
Keywords and phrases: Gamma function; polygamma function; factorial function; complete monotonicity; approximations; permanent; $(0,1)$-matrix.

1 Introduction

In 1965, H. Minc and L. Sathre [12] have given one of the first estimations of the expression

$$\phi(r) = (r!)^{1/r}.$$

Inequalities involving the function $\phi(r)$ are of interest in themselves, but they also have important applications in the theory of $(0,1)$-matrices.

The permanent of an n-by-n matrix $A = (a_{ij})$ is defined as

$$\mathrm{Per}(A) = \sum a_{1\sigma(1)} a_{2\sigma(2)} \cdot \ldots \cdot a_{n\sigma(n)},$$

where the sum goes over every permutation σ of the set $\{1, 2, \ldots, n\}$. Although it looks similar to the determinant of matrices, the permanent is much harder to be computed. The literature on bounds for permanents is quite extensive. It was first conjectured by H. Minc [10], then proved by L.M. Brégman [4] that for a $(0,1)$-matrix with row sums $r_1, r_2, \ldots, r_n$, the following upper bound holds:

$$\mathrm{Per}(A) \le \prod_{i=1}^{n} \phi(r_i).$$

This kind of bounds and some others, see [5, 9, 11, 14], motivated many authors [12, 15, 16, 17] to introduce new inequalities involving $(r!)^{1/r}$, or the ratio $\phi(r+1)/\phi(r)$.

H. Minc and L. Sathre [13, Cor. 2] proved that for every positive integer r:

$$1 < \frac{\phi(r+1)}{\phi(r)} < 1 + \frac{1}{r}. \tag{1.1}$$

One of the main results of this paper is the following new inequality, for every $x \geq 1$,

$$\frac{\Gamma\left(x+2\right)^{1/(x+1)}}{\Gamma\left(x+1\right)^{1/x}} \geq \frac{\left(4x+4\right)^{1/(x+1)}}{\left(4x\right)^{1/x}}\left(1+\frac{1}{x}\right) > 1.$$

Since $\Gamma(r+1) = r!$ for the positive integer r, this improves the estimation from the left-hand side of (1.1).

2 The Results

In the early 18th century, famous Swiss mathematician Leonhard Euler (1707 – 1783), introduced the function

$$\Gamma\left(x\right) = \int_0^\infty t^{x-1}e^{-t}dt\ , \quad x > 0,$$

now known as the Euler's gamma function. It is the natural extension of the factorial function to every positive real number (or more exactly to $\mathbb{C}\setminus\mathbb{Z}_-$), since $\Gamma\left(n+1\right) = n!$, for every counting number n. The famous Bohr-Mollerup theorem [2, 3] states that the gamma function extends uniquely the factorial function, as $f = \Gamma$ is the only solution of the functional equation

$$f\left(x+1\right) = xf\left(x\right), \quad f\left(1\right) = 1$$

in the class of log-convex functions $f : (0,1) \to (0,1)$. (Another result of this kind says, that $f = \Gamma$ also in the case where there is such a $g : (0,1) \to \mathbb{R}$ that the function $g \circ f$ is convex in an interval $(\gamma, 1)$, $\gamma > 0$, and $g(x) = a\ln x + b$, $x \to \infty$, with some $a > 0$ and $b \in \mathbb{R}$, cf. [6]). The psi or digamma function is defined as

$$\psi\left(x\right) = \frac{d}{dx}\ln\Gamma\left(x\right) = \frac{\Gamma'\left(x\right)}{\Gamma\left(x\right)},$$

while the derivatives ψ', ψ'', ψ''', ... are called the tri-, tetra-, pentagamma functions, or simply the polygamma functions. In what follows, we use the following integral representations [1, 13, 18]

$$\psi^{(n)}\left(x\right) = \left(-1\right)^{n+1}\int_0^\infty \frac{t^n}{1-e^{-t}}e^{-tx}dt \tag{2.1}$$

and for every $\omega > 0$,

$$\frac{1}{x^\omega} = \frac{1}{\Gamma\left(\omega\right)}\int_0^\infty t^{\omega-1}e^{-tx}dt. \tag{2.2}$$

Recall that a function z is said to be completely monotonic on $(0,\infty)$ if it has derivatives of all orders and for every positive integer k and $x \geq 0$, we have

$$\left(-1\right)^k z^{(k)}\left(x\right) \geq 0.$$

This notion was introduced in 1921 by F. Hausdorff [8], under the name 'total monoton'. J. Dubourdieu [7] proved that every non-constant, completely monotonic function satisfies $(-1)^k z^{(k)}(x) > 0$. According with the well-known Hausdorff–Bernstein–Widder theorem in [18, Theorem 12a, p. 160], a function z on $(0,\infty)$ is completely monotonic if and only if there exists a non-negative measure $\mu(t)$ such that for every $x \geq 0$,

$$z(x) = \int_0^\infty e^{-xt} d\mu(t), \tag{2.3}$$

such that the integral converges for all $x > 0$. Completely monotonic functions involving the gamma function are very useful, since they produce sharp bounds for the polygamma functions. They also play a basic role in probability theory, or asymptotic and numerical analysis and in physics.

Motivated by the right-hand inequality of (1.1), we introduce the function $h : (0,\infty) \to \mathbb{R}$, by the formula

$$h(x) = x(x+1)\ln \frac{x\Gamma(x+1)^{1/(x+1)}}{(x+1)\Gamma(x)^{1/x}}.$$

Theorem 2.1. *The function h' is completely monotonic.*

Proof. We have

$$h(x) = x\ln\Gamma(x+1) - (x+1)\ln\Gamma(x) - \left(x^2+x\right)\ln\left(1+\frac{1}{x}\right).$$

Then

$$h'(x) = 2 + \ln x - (2x+1)\ln\left(1+\frac{1}{x}\right) - \psi(x) \tag{2.4}$$

and

$$h''(x) = \frac{2}{x} + \frac{1}{x+1} - 2\ln\left(1+\frac{1}{x}\right) - \psi'(x) \tag{2.5}$$

and

$$h'''(x) = \frac{2}{x} - \frac{2}{x+1} - \frac{2}{x^2} - \frac{1}{(x+1)^2} - \psi''(x). \tag{2.6}$$

Using (2.1)–(2.2), we have

$$h'''(x) = \int_0^\infty 2e^{-tx}dt - \int_0^\infty 2e^{-t(x+1)}dt -$$

$$-\int_0^\infty 2te^{-tx}dt - \int_0^\infty te^{-t(x+1)}dt + \int_0^\infty \frac{t^2}{1-e^{-t}}e^{-tx}dt,$$

or

$$h'''(x) = \int_0^\infty \varphi(t)\frac{e^{-t(x+1)}}{e^t-1}dt,$$

where

$$\varphi(t) = t^2 e^{2t} - \left(e^t - 1\right)\left(2 + t - 2e^t + 2te^t\right) =$$
$$= \sum_{n=3}^{\infty} \frac{2^{n-2}\left(n^2 - 5n + 8\right) + n - 4}{n!} t^n > 0.$$

Next we use the fact that

$$\lim_{x\to\infty} (\psi(x) - \ln x) = \lim_{x\to\infty} \psi'(x) = \lim_{x\to\infty} \psi''(x) = 0,$$

as it results from the asymptotic expansions of the polygamma functions, *e.g.,* [1, p. 259 – Rel. 6.3.18; p. 260 – Rel. 6.4.12 and 6.4.13]. Thus, from (2.4)–(2.6), we have

$$\lim_{x\to\infty} h'(x) = \lim_{x\to\infty} h''(x) = \lim_{x\to\infty} h'''(x) = 0.$$

Now, from $h''' > 0$, it results that h'' is strictly increasing. As $\lim_{x\to\infty} h''(x) = 0$, we have $h'' < 0$. Further, h' is strictly decreasing, with $\lim_{x\to\infty} h'(x) = 0$, so $h' > 0$. Finally, from (2.3) it results that h' is completely monotonic. □

Corollary 2.1. *For every $x \geq 1$, we have:*

$$\frac{\Gamma(x+1)^{1/(x+1)}}{\Gamma(x)^{1/x}} \geq 4^{\frac{-1}{x(x+1)}} \left(1 + \frac{1}{x}\right) > 1, \tag{2.7}$$

where the constant 4 is best possible.

Proof. The function h' is positive, so h is strictly increasing. In consequence, for every $x \geq 1$, we have $h(1) \leq h(x)$. As $h(1) = -\ln 4$, we obtain

$$-\ln 4 \leq x(x+1) \ln \frac{x\Gamma(x+1)^{1/(x+1)}}{(x+1)\Gamma(x)^{1/x}}.$$

By exponentiating, we get

$$4^{\frac{-1}{x(x+1)}} \leq \frac{x}{x+1} \cdot \frac{\Gamma(x+1)^{1/(x+1)}}{\Gamma(x)^{1/x}},$$

which is the conclusion. □

By using the recurrence $\Gamma(y+1) = y\Gamma(y)$ in (2.7), we can state the following

Corollary 2.2. *For every $x \geq 1$, we have:*

$$\frac{\Gamma(x+2)^{1/(x+1)}}{\Gamma(x+1)^{1/x}} \geq \frac{(4x+4)^{1/(x+1)}}{(4x)^{1/x}} \left(1 + \frac{1}{x}\right) > 1,$$

where the constant 4 is best possible.

As a consequence, this inequality can be used as a good approximation

$$\frac{\Gamma(x+2)^{1/(x+1)}}{\Gamma(x+1)^{1/x}} \approx \frac{(4x+4)^{1/(x+1)}}{(4x)^{1/x}}\left(1+\frac{1}{x}\right),$$

as we can see from numerical computations:

x	$\frac{\Gamma(x+2)^{1/(x+1)}}{\Gamma(x+1)^{1/x}}$	$\frac{(4x+4)^{1/(x+1)}}{(4x)^{1/x}}\left(1+\frac{1}{x}\right)$
10	1.084 021 393	1.072 979 624
50	1.019 047 171	1.018 278 181
125	1.007 818 486	1.007 666 066
350	1.002 829 804	1.002 806 159
500	1.001 985 892	1.001 973 593
2500	1.000 399 307	1.000 398 686

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Some more results on b-θ-open sets

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Abstract. In this paper, we consider the class of b-θ-open sets in topological spaces and investigate some of their properties. We also present and study some weak separation axioms by involving the notion of b-θ-open sets. We define the concepts of b-θ-kernal of sets and slightly b-θ-R_0 spaces. We apply them to investigate some properties of the graph functions.

Mathematics subject classification: 54C10 .
Keywords and phrases: Topological spaces, b-open sets, b-closure, b-θ-open sets.

1 Introduction

In 1996, Andrijevic [1] initiated the study of so called b-open sets. This notion has been studied extensively in recent years by many topologists (see [2–5]). In this paper, we will continue the study of related spaces by using b-θ-open [7] sets.

Throughout this paper, X and Y refer always to topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ denote the closure of A and the interior of A in X, respectively. A subset A of X is said to be b-open [1] if $A \subseteq \mathrm{cl}(\mathrm{int}(A)) \cup \mathrm{int}(\mathrm{cl}(A))$. The complement of b-open set is called b-closed. The intersection of all b-closed sets of X containing A is called the b-closure [1] of A and is denoted by $b\,\mathrm{cl}(A)$. A set A is b-closed if and only if $b\,\mathrm{cl}(A) = A$. The union of all b-open sets of X contained in A is called the b-interior of A and is denoted by $b\,\mathrm{int}(A)$. A set A is said to be b-regular [7] if it is b-open and b-closed. The family of all b-open (resp. b-closed, b-regular) sets of X is denoted by $BO(X)$ (resp. $BC(X)$, $BR(X)$). We set $BO(X, x) = \{V \in BO(X) | x \in V\}$ for $x \in X$.

2 Preliminaries

A point x of X is called a b-θ-cluster [7] point of $S \subseteq X$ if $b\,\mathrm{cl}(U) \cap S \neq \varnothing$ for every $U \in BO(X, x)$. The set of all b-θ-cluster points of S is called the b-θ-closure of S and is denoted by $b\,\mathrm{cl}_\theta(S)$. A subset S is said to be b-θ-closed if and only if $S = b\,\mathrm{cl}_\theta(S)$. The complement of a b-θ-closed set is said to be b-θ-open. The family of all b-θ-open subsets of X is denoted by $B_\theta O(X)$.

Theorem 1 (see [7]). *Let A be a subset of a topological space X. Then,*

(i) $A \in BO(X)$ *if and only if* $b\,\mathrm{cl}(A) \in BR(X)$.

(ii) $A \in BO(X)$ *if and only if* $b\,\mathrm{int}(A) \in BR(X)$.

Theorem 2 (see [7]). *For a subset A of a topological space X, the following properties hold:*

(i) If $A \in BO(X)$, *then* $b\,\mathrm{cl}(A) = b\,\mathrm{cl}_\theta(A)$,

(ii) $A \in BR(X)$ *if and only if A is b-θ-open and b-θ-closed.*

Definition 1. A topological space X is said to be b-regular [7] if for each $F \in BC(X)$ and each $x \notin F$, there exist disjoint b-open sets U and V such that $x \in U$ and $F \subseteq V$.

Theorem 3 (see [7]). *For a topological space X, the following properties are equivalent:*

(i) X is b-regular;

(ii) For each $U \in BO(X)$ *and each* $x \in U$, *there exists* $V \in BO(X)$ *such that* $x \in V \subseteq b\,\mathrm{cl}(V) \subseteq U$;

(iii) For each $U \in BO(X)$ *and each* $x \in U$, *there exists* $V \in BR(X)$ *such that* $x \in V \subseteq U$.

Definition 2. A function $f : X \to Y$ is said to be b-irresolute [6] if $f^{-1}(V) \in BO(X)$ for every $V \in BO(Y)$.

3 b-θ-open sets

Remark 1. It is easy to prove that

(i) the intersection of an arbitrary collection of b-θ-closed sets is b-θ-closed.

(ii) X and $\varnothing$ are b-θ-closed sets.

Remark 2. The following example shows that an union of any two b-θ-closed sets of X is not necessarily b-θ-closed in X.

Example 1. Let $X = \{a, b, c\}$ with topology $\tau = \{\varnothing, \{a, b\}, X\}$. Clearly, $\{a\}, \{b\}$ are b-θ-closed sets in X, but their union $\{a, b\}$ is not b-θ-closed in X.

Lemma 1 (see [7]). *Let A be a subset of a topological space (X, τ). The following hold:*

(i) If $A \in BO(X)$, *then* $b\,\mathrm{cl}(A)$ *is b-regular and* $b\,\mathrm{cl}(A) = b\,\mathrm{cl}_\theta(A)$;

(ii) A is b-regular if and only if A is b-θ-closed and b-θ-open;

(iii) A is b-regular if and only if A is $b\,\mathrm{int}(b\,\mathrm{cl}(A))$.

Lemma 2. *For any subset A of a topological space (X,τ), $b\,\mathrm{cl}_\theta(A)$ is b-θ-closed.*

Definition 3. A subset S of a topological space (X,τ) is said to be θ-completment b-open (briefly θ-c-b-open) provided there exists a subset A of X for which $X-S$ $= b\,\mathrm{cl}_\theta(A)$. We call a set θ-complement b-closed if its complement is θ-c-b-open.

Remark 3. It should be mentioned that by Lemma 2, $X-S = b\,\mathrm{cl}_\theta(A)$ is b-θ-closed and S is b-θ-open. Therefore, the equivalence of θ-c-b-open and b-θ-open is obvious from the definition.

Theorem 4. *If $A \subseteq X$ is b-open, then $b\,\mathrm{int}(b\,\mathrm{cl}_\theta(A))$ is b-θ-open.*

Proof. Since $X - b\,\mathrm{int}(b\,\mathrm{cl}(A)) = b\,\mathrm{cl}(X - b\,\mathrm{cl}(A))$, then by complements $b\,\mathrm{int}(b\,\mathrm{cl}(A))$ $= (X - b\,\mathrm{cl}(X - b\,\mathrm{cl}(A)))$. Since $X - b\,\mathrm{cl}(A)$ $(=B$, say) is b-open, $b\,\mathrm{cl}(B) = b\,\mathrm{cl}_\theta(B)$ from Lemma 1. Therefore, there exists a subset $B = X - b\,\mathrm{cl}(A)$ for which $X - b\,\mathrm{int}(b\,\mathrm{cl}(A)) = b\,\mathrm{cl}_\theta(B)$. Hence $b\,\mathrm{int}(b\,\mathrm{cl}(A))$ is b-θ-open. □

Corollary 1. *If $A \subseteq X$ is b-regular, then A is b-θ-open.*

Proof. Obvious by Lemma 1, since A is b-regular if and only if $A = b\,\mathrm{int}(b\,\mathrm{cl}(A))$. □

Theorem 5. *b-θ-open is equivalent to b-regular if and only if $b\,\mathrm{cl}_\theta(A)$ is b-regular for every set $A \subseteq X$.*

Proof. Let X be a topological space. Assume b-θ-open is equivalent to b-regular and let $A \subseteq X$. Then by Lemma 2, $X - b\,\mathrm{cl}_\theta(A)$ is b-θ-open which implies that $b\,\mathrm{cl}_\theta(A)$ is b-regular. Conversely, assume $b\,\mathrm{cl}_\theta(A)$ is b-regular for every set A. Suppose U is b-θ-open and let $A \subseteq X$ such that $X - U = b\,\mathrm{cl}_\theta(A)$. That is, $U = X - b\,\mathrm{cl}_\theta(A)$. Then, $b\,\mathrm{cl}_\theta(A)$ is b-regular and U is b-regular. Therefore, b-θ-open is equivalent to b-regular. □

Theorem 6. *If $B \subseteq X$ is b-θ-open, then B is an union of b-regular sets.*

Proof. Let B be b-θ-open and $x \in B$. Since B is b-θ-open, then there exists a set $A \subseteq X$ such that $B = X - b\,\mathrm{cl}_\theta(A)$. Because $x \notin b\,\mathrm{cl}_\theta(A)$, there exists a b-open set W for which $x \in W$ and $b\,\mathrm{cl}(W) \cap A = \varnothing$. Hence $x \in b\,\mathrm{int}(b\,\mathrm{cl}(W)) \subseteq X - b\,\mathrm{cl}_\theta(A)$, where $b\,\mathrm{int}(b\,\mathrm{cl}(W))$ $(=V$ (say)$) \in BR(X)$, that is, $B = \bigcup\{V : V \subseteq B, V \in BR(X)\}$. □

Corollary 2. *If B is b-θ-closed, then B is the intersection of b-regular sets.*

4 On b-θ-D_i (resp. b-θ-T_i) topological spaces

Now, we study some classes of topological spaces in terms of the concept of b-θ-open sets. The relations with other notions, directly or indirectly connected with these classes are investiaged.

Definition 4. A subset A of a topological space (X, τ) is called a b-θ-D-set if there are two sets $U, V \in B_\theta O(X)$ such that $U \neq X$ and $A = U - V$.

It is true that every b-θ-open set U different from X is a b-θ-D set if $A = U$ and $V = \varnothing$.

Definition 5. A topological space (X, τ) is called b-θ-D_0 if for any distinct pair of points x and y of X, there exists a b-θ-D-set of X containing one of the points but not the other.

Definition 6. A topological space (X, τ) is called b-θ-D_1 if for any distinct pair of points x and y of X, there exists a b-θ-D-set F of X containing x but not y and a b-θ-D set G of X containing y but not x.

Definition 7. A topological space (X, τ) is called b-θ-D_2 if for any distinct pair of points x and y of X, there exists disjoint b-θ-D-sets G and E of X containing x and y respectively.

Definition 8. A topological space (X, τ) is called b-θ-T_0 if for any distinct pair of points in X, there exists a b-θ-open set containing one of the points but not the other.

Definition 9. A topological space (X, τ) is called b-θ-T_1 if for any distinct pair of points x and y in X, there exists a b-θ-open set U in X containing x but not y and a b-θ-open set V in X containing y but not x.

Definition 10. A topological space (X, τ) is called b-θ-T_2 if for any distinct pair of points x and y in X, there exist b-θ-open sets U and V in X containing x and y, respectively, such that $U \cap V = \varnothing$.

Remark 4. From Definitions 4 to 10, we obtain the following diagram:

$$\begin{array}{ccccc}
b\text{–}\theta\text{–}T_2 & \Rightarrow & b\text{–}\theta\text{–}T_1 & \Rightarrow & b\text{–}\theta\text{–}T_0 \\
\Downarrow & & \Downarrow & & \Downarrow \\
b\text{–}\theta\text{–}D_2 & \Rightarrow & b\text{–}\theta\text{–}D_1 & \Rightarrow & b\text{–}\theta\text{–}D_0
\end{array}$$

Theorem 7. *If a topological space (X, τ) is b-θ-T_0, then it is b-θ-T_2.*

Proof. For any points $x \neq y$, let V be a b-θ-open set such that $x \in V$ and $y \notin V$. Then, there exists $U \in BO(X)$ such that $x \in U \subseteq b\,\mathrm{cl}(U) \subseteq V$. By Lemma 1, $b\,\mathrm{cl}(U) \in BR(X)$. Then $b\,\mathrm{cl}(U)$ is b-θ-open and also $X - b\,\mathrm{cl}(U)$ is a b-θ-open set containing y. Therefore, X is b-θ-T_2. $\square$

Theorem 8. *For a topological space (X, τ), the six properties in the diagram are equivalent.*

Proof. By Theorem 7, we have that b-θ-T_0 implies b-θ-T_2. Now we prove that b-θ-D_0 implies b-θ-T_0. Let (X, τ) be b-θ-D_0 so that for any distinct pair of points x and y of X, one of them belongs to a b-θ-D set A. Therefore, we choose $x \in A$ and $y \notin A$. Suppose $A = U - V$ for which $U \neq X$ and $U, V \in B_\theta O(X)$. This implies that $x \in U$. For the case that $y \notin A$ we have (i) $y \notin U$, (ii) $y \in U$ and $y \in V$. For (i), the space X is b-θ-T_0 since $x \in U$ but $y \notin U$. For (ii), the space X is also b-θ-T_0 since $y \in V$ but $x \notin V$. □

Let x be a point of X and V a subset of X. The set V is called a b-θ-neighbourhood of x in X if there exists a b-θ-open set A of X such that $x \in A \subseteq V$.

Definition 11. A point $x \in X$ which has only X as the b-θ-neighbourhood is called a point common to all b-θ-closed sets (briefly b-θ-*cc*).

Theorem 9. *If a topological space (X, τ) is b-θ-D_1, then (X, τ) has no b-θ-cc-point.*

Proof. Since (X, τ) is b-θ-D_1, so each point x of X is contained in a b-θ-D set $A = U - V$ and thus in U. By definition $U \neq X$ and this implies that x is not a b-θ-*cc*-point. □

Definition 12. A subset A of a topological space (X, τ) is called a quasi b-θ-closed (briefly *qbt*-closed) set if $b\,\mathrm{cl}_\theta(A) \subseteq U$ whenever $A \subseteq U$ and U is b-θ-open in (X, τ).

Lemma 3. *[7] Let A be any subset of a topological space X. Then $x \in b\,\mathrm{cl}_\theta(A)$ if and only if $V \cap A = \varnothing$ for every $V \in BR(X, x)$.*

Theorem 10. *For a topological space (X, τ), the following properties hold:*

(i) For each pair of points x and y in X, $x \in b\,\mathrm{cl}_\theta(\{y\})$ implies $y \in b\,\mathrm{cl}_\theta(\{x\})$;

(ii) For each $x \in X$, the singleton $\{x\}$ is qbt-closed in (X, τ).

Proof. (i): Let $y \notin b\,\mathrm{cl}_\theta(\{x\})$. This implies that there exists $V \in BO(X, y)$ such that $b\,\mathrm{cl}(V) \cap \{x\} = \varnothing$ and $X - b\,\mathrm{cl}(V) \in BR(X, x)$ which means that $x \notin b\,\mathrm{cl}_\theta(\{y\})$.
(ii): Suppose that $U \in B_\theta O(X)$. This implies that there exists $V \in BO(X)$ such that $x \in V \subseteq b\,\mathrm{cl}(V) \subseteq U$. Now we have $b\,\mathrm{cl}_\theta(\{x\}) \subseteq b\,\mathrm{cl}_\theta(V) = b\,\mathrm{cl}(V) \subseteq U$. □

Definition 13. A topological space (X, τ) is said to be b-θ-$T_{1/2}$ if every *qbt*-closed set is b-θ-closed.

Theorem 11. *For a topological space (X, τ), the following are equivalent:*

(i) (X, τ) is b-θ-$T_{1/2}$;

(ii) (X, τ) is b-θ-T_1.

Proof. (i)⇒(ii): For distinct points x, y of $X, \{x\}$ is *qbt*-closed by Theorem 10. By hypothesis, $X - \{x\}$ is b-θ-open and $y \in X - \{x\}$. By the same token, $x \in X - \{y\}$ and $X - \{y\}$ is b-θ-open. Therefore, (X, τ) is b-θ-T_1.

(ii)⇒(i): Suppose that A is a *qbt*-closed set which is not b-θ-closed. There exists $x \in b\,\mathrm{cl}_\theta(A) - A$. For each $a \in A$, there exists a b-θ-open set V_a such that $a \in V_a$ and $x \notin V_a$. Since $A \subseteq \bigcup_{a\in V_a} V_a$ and $\bigcup_{a\in V_a} V_a$ is b-θ-open, we have $b\,\mathrm{cl}_\theta(A) \subseteq \bigcup_{a\in V_a} V_a$. Since $x \in b\,\mathrm{cl}_\theta(A)$, there exists $a_0 \in A$ such that $x \in V_{a_0}$. But this is a condradiction. □

Recall that a topological space (X, τ) is called b-T_2 [2] if for any distinct pair of points x and y in X, there exist b-open sets U and V in X containing x and y respectively such that $U \cap V = \varnothing$.

Theorem 12. *For a topological space (X, τ), the following are equivalent:*

(i) (X, τ) *is* b-θ-T_2*;*

(ii) (X, τ) *is* b-T_2.

Proof. (i)⇒(ii): This is obvious since every b-θ-open set is b-open.

(ii)⇒(i): Let x and y be distinct points of X. There exist b-open sets U and V such that $x \in U$, $y \in V$ and $b\,\mathrm{cl}(U) \cap b\,\mathrm{cl}(V) = \varnothing$. Since $b\,\mathrm{cl}(U)$ and $b\,\mathrm{cl}(V)$ are b-regular, then they are b-θ-open and hence (X, τ) is b-θ-T_2. □

Definition 14. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be weak b-irresolute [8] if for each $x \in X$ and each $V \in BO(Y, f(x))$, there exists a $U \in BO(X, x)$ such that $f(U) \subseteq b\,\mathrm{cl}(V)$.

Remark 5. A function $f : (X, \tau) \to (Y, \sigma)$ is weak b-irresolute if and only if $f^{-1}(V)$ is b-θ-closed (resp. b-θ-open) in (X, τ) for every b-θ-closed (resp. b-θ-open) set V in (Y, σ).

Theorem 13. *If $f : (X, \tau) \to (Y, \sigma)$ is a weak b-irresolute surjective function and E is a b-θ-D set in Y, then the inverse image of E is a b-θ-D set in X.*

Proof. Let E be a b-θ-D-set in Y. Then there are b-θ-open sets U and V in Y such that $E = U - V$ and $U \neq Y$. By weak b-irresoluteness of f, $f^{-1}(U)$ and $f^{-1}(V)$ are b-θ-open in X. Since $U \neq Y$, we have $f^{-1}(U) \neq X$. Hence $f^{-1}(E) = f^{-1}(U) - f^{-1}(V)$ is a b-θ-D-set in X. □

Theorem 14. *If (Y, σ) is a b-θ-D_1 space and $f : (X, \tau) \to (Y, \sigma)$ is a weak b-irresolute bijection, then (X, τ) is b-θ-D_1.*

Proof. Suppose that Y is a b-θ-D_1 space. Let x and y be any pair of distinct poins in X. Since f is injective and Y is b-θ-D_1, there exists b-θ-D sets U and V of Y containing $f(x)$ and $f(y)$, respectively such that $f(y) \notin U$ and $f(x) \notin V$. By Theorem 13, $f^{-1}(U)$ and $f^{-1}(V)$ are b-θ-D sets in X containing x and y, respectively such that $y \notin f^{-1}(U)$ and $x \notin f^{-1}(V)$. This implies that X is a b-θ-D_1 space. □

Theorem 15. *For a topological space (X, τ), the following statements are equivalent:*

(i) (X, τ) *is b-θ-D_1;*

(ii) For each pair of distinct points $x, y \in X$, there exists a weak b-irresolute surjective function $f : (X, \tau) \to (Y, \sigma)$, where Y is a b-θ-D_1 space such that $f(x)$ and $f(y)$ are distinct.

Proof. (i)⇒(ii): For every pair of distinct points of X, it suffices to take the identity function $f : (X, \tau) \to (X, \tau)$.

(ii)⇒(i): Let x and y be any pair of distinct points in X. By hypothesis, there exists a surjective weak b-irresolute function f of the space X into a b-θ-D_1 space Y such that $f(x) \neq f(y)$. Therefore, there exist disjoint b-θ-D sets U and V of Y containing $f(x)$ and $f(y)$, respectively such that $f(y) \notin U$ and $f(x) \notin V$. Since f is weak b-irresolute and surjective, by Theorem 13, $f^{-1}(U)$ and $f^{-1}(V)$ are b-θ-D sets in X containing x and y, respectively such that $y \notin f^{-1}(U)$ and $x \notin f^{-1}(V)$. Hence X is a b-θ-D_1 space. □

5 Further properties

Definition 15. Let A be a subset of a topological space (X, τ). The b-θ-kernel of $A \subseteq X$ denoted by $bker_\theta(A)$, is defined to be the set
$\bigcap \{O : O \in B_\theta O(X, \tau) \text{ and } A \subseteq O\} = \{x : b\,\mathrm{cl}_\theta(\{x\}) \cap A \neq \varnothing\}$.

Definition 16. A topological space (X, τ) is said to be slightly b-θ-R_0 if $\bigcap\{b\,\mathrm{cl}_\theta(\{x\}) : x \in X\} = \varnothing$.

Theorem 16. *A topological space (X, τ) is slightly b-θ-R_0 if and only if $bker_\theta(\{x\}) \neq X$ for any $x \in X$.*

Proof. Necessity. Let the space (X, τ) be slightly b-θ-R_0. Assume that there is a point y in X such that $bker_\theta(\{y\}) = X$. Then $y \notin O$ which is some proper b-θ-open subset of X. This implies that $y \in \bigcap\{b\,\mathrm{cl}_\theta(\{x\})\colon x \in X\}$. But this is a contradiction.

Sufficiency. Now assume that $bker_\theta(\{x\}) \neq X$ for any $x \in X$. If there exists a point y in X such that $y \in \bigcap\{b\,\mathrm{cl}_\theta(\{x\})\colon x \in X\}$, then every b-θ-open set containing y must contain every point of X. This implies that the space X is the unique b-θ-open set containing y. Hence $b\,\mathrm{cl}_\theta(\{x\}) = X$ which is a contracdiction . Therefore, (X, τ) is slightly b-θ-R_0. □

Theorem 17. *If the topological space X is slightly b-θ-R_0 and Y is any topological space, then the product $X \times Y$ is slightly b-θ-R_0.*

Proof. By showing that $\bigcap\{b\,\mathrm{cl}_\theta(\{x, y\})\colon (x, y) \in X \times Y\} = \varnothing$ we are done. We have $\bigcap\{b\,\mathrm{cl}_\theta(\{x, y\})\colon (x, y) \in X \times Y\} \subseteq \bigcap\{b\,\mathrm{cl}_\theta(\{x\}) \times b\,\mathrm{cl}_\theta(\{y\})\colon (x, y) \in X \times Y\} = \bigcap\{b\,\mathrm{cl}_\theta(\{x\})\colon x \in X\} \times \bigcap\{b\,\mathrm{cl}_\theta(\{y\})\colon y \in Y\} \subseteq \varnothing \times Y = \varnothing$. □

Definition 17. A function $f : (X,\tau) \to (Y,\sigma)$ is R-continuous (resp. θ-R-b-continuous, R-b-continuous) if for each $x \in X$ and each b-open subset V of Y containing $f(x)$, there exists an open subset U of X containing x such that $\mathrm{cl}(f(U)) \subseteq V$ (resp. $b\,\mathrm{cl}_\theta(f(U)) \subseteq V$, $b\,\mathrm{cl}(f(U)) \subseteq V$).

Definition 18. A function $f : X \to Y$ is said to be b-open [6] if $f(U)$ is b-open in Y for every open set U of X.

Remark 6. (i): Since $A \subseteq b\,\mathrm{cl}(A) \subseteq b\,\mathrm{cl}_\theta(A)$ for any set A, θ-R-b-continuity implies R-b-continuity.

(ii): Since the b-closure and b-θ-closure operate agree on b-open sets (Lemma 1) it follows that if $f : (X,\tau) \to (Y,\sigma)$ is R-b-continuous and b-open, then f is θ-R-b-continuous.

Definition 19. The graph $G(f)$ of a function $f : (X,\tau) \to (Y,\sigma)$ is said to be strongly b-θ-closed if for each point $(x,y) \in (X \times Y) - G(f)$, there exists subsets $U \in BO(X,x)$ and $V \in B_\theta O(Y,y)$ such that $(b\,\mathrm{cl}(U) \times V) \cap G(f) = \varnothing$.

Lemma 4. *The graph $G(f)$ of $f : (X,\tau) \to (Y,\sigma)$ is strongly b-θ-closed in $X \times Y$ if and only if for each point $(x,y) \in (X \times Y) - G(f)$, there exist $U \in BO(X,x)$ and $V \in B_\theta O(Y,y)$ such that $f(b\,\mathrm{cl}(U)) \cap V = \varnothing$.*

Proof. It follows immediately from Definition 19. □

Recall that a topological space (X,τ) is called b-T_1 [2] if for any distinct pair of points x and y in X, there is a b-open set U in X containing x but not y and a b-open set V in X containing y but not x.

Theorem 18. *If $f : (X,\tau) \to (Y,\sigma)$ is θ-R-b-continuous, weak b-irresolute and Y is b-T_1, then $G(f)$ is strongly b-θ-closed.*

Proof. Assume $(x,y) \in (X \times Y) - G(f)$. Since $y \neq f(x)$ and Y is b-T_1, there exists a b-open set V of Y such that $f(x) \in V$ and $y \notin V$. The θ-R-b-continuity of f implies the existence of an open subset U of X containing x such that $b\,\mathrm{cl}_\theta(f(U)) \subseteq V$. Therefore, $(x,y) \in b\,\mathrm{cl}(U) \times (Y - b\,\mathrm{cl}_\theta\, f(U))$ which is disjoint from $G(f)$ because if $a \in b\,\mathrm{cl}(U)$, then since f is a weak b-irresolute function, $f(a) \in f(b\,\mathrm{cl}(U)) \subseteq b\,\mathrm{cl}_\theta\, f(U)$. Note that $Y - b\,\mathrm{cl}_\theta(f(U))$ is b-θ-open. □

Theorem 19. *Let $f : (X,\tau) \to (Y,\sigma)$ be a weak b-irresolute function. Then f is θ-R-b-continuous if and only if for each $x \in X$ and each b-closed subset F of Y with $f(x) \notin F$, there exists an open subset U of X containing x and a b-θ-open subset V of Y with $F \subseteq V$ such that $f(b\,\mathrm{cl}(U)) \cap V = \varnothing$.*

Proof. Necessity. Let $x \in X$ and F be a b-closed subset of Y with $f(x) \in Y - F$. Since F is θ-R-b-continuous, there exists an open subset U of X containing x such that $b\,\mathrm{cl}_\theta(f(U)) \subseteq Y - F$. Let $V = Y - b\,\mathrm{cl}_\theta(f(U))$. Then V is b-θ-open and $F \subseteq V$.

Since f is weak b-irresolute, $f(b\,\mathrm{cl}(U)) \subseteq b\,\mathrm{cl}_\theta(f(U))$. Therefore $f(b\,\mathrm{cl}(U)) \cap V = \varnothing$.

Sufficiency. Let $x \in X$ and let V be a b-open subset of Y with $f(x) \in V$. Let $F = Y - V$. Since $f(x) \notin F$ there exists an open subset U of X containing x and a b-θ-open subset W of Y with $F \subseteq W$ such that $f(b\,\mathrm{cl}(U)) \cap W = \varnothing$. Then $f(b\,\mathrm{cl}(U)) \subseteq Y - W$, thus $b\,\mathrm{cl}_\theta(f(U)) \subseteq b\,\mathrm{cl}_\theta(Y - W) = Y - W \subseteq Y - F = V$. Therefore, f is θ-R-b-continuous. □

Corollary 3. *Let X and Y be topological spaces and $f : (X, \tau) \to (Y, \sigma)$ be a weak b-irresolute function. Then f is θ-R-b-continuous if and only if for each $x \in X$ and each b-open subset V of Y containing $f(x)$, there exists an open subset U of X containing x such that $b\,\mathrm{cl}_\theta(f(b\,\mathrm{cl}(U))) \subseteq V$.*

Proof. Assume f is θ-R-b-continuous. Let $x \in X$ and let V be a b-open subset of Y with $f(x) \in V$. Then there exists an open subset U of X containing x such that $b\,\mathrm{cl}_\theta(f(U)) \subseteq V$. By hypothesis of f, we have $b\,\mathrm{cl}_\theta(f(b\,\mathrm{cl}(U))) \subseteq b\,\mathrm{cl}_\theta(b\,\mathrm{cl}_\theta(f(U))) = b\,\mathrm{cl}_\theta(f(U)) \subseteq V$. Thus, $b\,\mathrm{cl}_\theta(f(b\,\mathrm{cl}(U))) \subseteq V$. The converse implication is immediate. □

Definition 20. A topological space (X, τ) is said to be b-R_1 if for $x, y \in X$ with $b\,\mathrm{cl}(\{x\}) \neq b\,\mathrm{cl}(\{y\})$, there exist disjoint b-open sets U and V such that $b\,\mathrm{cl}(\{x\}) \subseteq U$ and $b\,\mathrm{cl}(\{y\}) \subseteq V$.

Lemma 5. *A topological space X is b-R_1 if and only if $b\,\mathrm{cl}_\theta(\{x\}) = bcl(\{x\})$ for all $x \in X$.*

Proof. Necessity. Generally we have $b\,\mathrm{cl}(\{x\}) \subseteq b\,\mathrm{cl}_\theta(\{x\})$ for all $x \in X$. Suppose that $y \notin b\,\mathrm{cl}(\{x\})$ for any $x \in X$. Then there exists $A \in BO(X, y)$ such that $A \cap \{x\} = \varnothing$. Since X is b-R_1 and $b\,\mathrm{cl}(\{x\}) \neq b\,\mathrm{cl}(\{y\})$, there exist b-open sets U and V such that $b\,\mathrm{cl}(\{x\}) \subseteq U$, $b\,\mathrm{cl}(\{y\}) \subseteq V$ and $U \cap V = \varnothing$. Since $U \in BO(X, x)$ and $V \in BO(X, y)$, then $b\,\mathrm{cl}(U) \cap b\,\mathrm{cl}(V) = \varnothing$. This implies $b\,\mathrm{cl}(\{x\}) \cap b\,\mathrm{cl}(V) = \varnothing$ and hence $\{x\} \cap b\,\mathrm{cl}(V) = \varnothing$. Therefore $y \notin bcl_\theta(\{x\})$ and thus $b\,\mathrm{cl}_\theta(\{x\}) \subseteq b\,\mathrm{cl}(\{x\})$.

Sufficiency. Let $x, y \in X$ with $b\,\mathrm{cl}(\{x\}) \neq b\,\mathrm{cl}(\{y\})$. Then there exists a $k \in b\,\mathrm{cl}(\{x\})$ such that $k \notin b\,\mathrm{cl}(\{y\})$. Since $k \in b\,\mathrm{cl}(\{x\}) = b\,\mathrm{cl}_\theta(\{x\})$, then $U \cap \{x\} \neq \varnothing$ for every $U \in BR(X, k)$ and hence $b\,\mathrm{cl}(\{x\}) \subseteq U$. Since $k \notin b\,\mathrm{cl}(\{y\}) = b\,\mathrm{cl}_\theta(\{y\})$, there exists $U \in BR(X, k)$ such that $U \cap \{y\} = \varnothing$. Since $U \in BO(X, k)$, $U \cap b\,\mathrm{cl}(\{y\}) = \varnothing$ and hence $bcl(\{y\}) \subseteq X \backslash U$. Therefore, there exists disjoint b-open sets U and $X \backslash U$ such that $b\,\mathrm{cl}(\{x\}) \subseteq U$ and $b\,\mathrm{cl}(\{y\}) \subseteq X \backslash U$. □

Proposition 1. *A space X is b-R_1 if and only if for each b-open set A and each $x \in A$, $b\,\mathrm{cl}_\theta(\{x\}) \subseteq A$.*

Proof. Necessity. Assume X is b-R_1. Suppose that A is a b-open subset of X and $x \in A$. Let y be an arbitrary element of $X - A$. Since X is b-R_1, then $b\,\mathrm{cl}_\theta(\{y\}) = b\,\mathrm{cl}(\{y\}) \subseteq X - A$. Hence we have that $x \notin b\,\mathrm{cl}_\theta(\{y\})$ and so $y \notin b\,\mathrm{cl}_\theta(\{x\})$. It follows that $b\,\mathrm{cl}_\theta(\{x\}) \subseteq A$.

Sufficiency. Assume now that $y \in b\,\mathrm{cl}_\theta(\{x\}) - b\,\mathrm{cl}(\{x\})$ for some $x \in X$. Then there exists a b-open set A containing y such that $b\,\mathrm{cl}(A) \cap \{x\} \neq \varnothing$ but $A \cap \{x\} = \varnothing$. Then $b\,\mathrm{cl}_\theta(\{y\}) \subseteq A$ and $b\,\mathrm{cl}_\theta(\{y\}) \cap \{x\} = \varnothing$. Hence $x \notin b\,\mathrm{cl}_\theta(\{y\})$. Thus $y \notin b\,\mathrm{cl}_\theta(\{x\})$. By this contradiction, we obtain $b\,\mathrm{cl}_\theta(\{x\}) = b\,\mathrm{cl}(\{x\})$ for each $x \in X$. Thus, X is b-R_1. □

Theorem 20. *If $f : (X,\tau) \to (Y,\sigma)$ is a θ-R-b-continuous surjection, then (Y,σ) is a b-R_1 space.*

Proof. Let V be a b-open subset of Y and $y \in V$. Let $x \in X$ such that $y = f(x)$. Since f is θ-R-b-continuous, there exists an open subset U of X containing x such that $b\,\mathrm{cl}_\theta(f(U)) \subseteq V$. Then $b\,\mathrm{cl}_\theta(\{y\}) \subseteq b\,\mathrm{cl}_\theta(f(U)) \subseteq V$. Therefore, by Proposition 1, Y is b-R_1. □

We give some basic properties of θ-R-b-continuous functions concerning composition and restriction.

Theorem 21. *If $f : (X,\tau) \to (Y,\sigma)$ is continuous and $g : (Y,\sigma) \to (Z,\gamma)$ is θ-R-b-continuous, then $g \circ f\colon (X,\tau) \to (Z,\gamma)$ is θ-R-b-continuous.*

Proof. Let $x \in X$ and W be a b-open subset of Z containing $g(f(x))$. Since g is θ-R-b-continuous, there exists an open subset V of Y containing $f(x)$ such that $b\,\mathrm{cl}_\theta(g(V)) \subseteq W$. Since f is continuous, there exists an open subset U of X containing x, $f(U) \subseteq V$; hence $b\,\mathrm{cl}_\theta(g(f(U))) \subseteq W$. Therefore $g \circ f$ is θ-R-continuous. □

Theorem 22. *Let $f : (X,\tau) \to (Y,\sigma)$ and $g : (Y,\sigma) \to (Z,\gamma)$ be functions. If $g \circ f\colon (X,\tau) \to (Z,\gamma)$ is θ-R-b-continuous and f is fn open surjection, then g is θ-R-b-continuous.*

Proof. Let $y \in Y$ and W be a b-open subset of Z containing $g(y)$. Since f is surjective, there exists $x \in X$ such that $y = f(x)$. Since $g \circ f$ is θ-R-b-continuous, there exists an open subset U of X containing x such that $b\,\mathrm{cl}_\theta(g(f(U))) \subseteq W$. Note that $f(U)$ is an open set containing y. Therefore g is θ-R-b-continuous. □

Theorem 23. *If $f : (X,\tau) \to (Y,\sigma)$ is θ-R-b-continuous and $A \subseteq X$, then $f|_A\colon A \to Y$ is θ-R-b-continuous.*

Proof. Let $x \in A$ and let V be any b-open subset of Y containing $f(x)$ $(= f|_A(x))$. Since f is θ-R-b-continuous, there exists an open subset U of X containing x such that $b\,\mathrm{cl}_\theta(f(U)) \subseteq V$. Put $O = U \cap A$, then O is an open subset of A containing x such that $b\,\mathrm{cl}_\theta(f|_A(O)) = b\,\mathrm{cl}_\theta(f(O)) \subseteq b\,\mathrm{cl}_\theta(f(U)) \subseteq V$. Therefore $f|_A : A \to Y$ is θ-R-b-continuous. □

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Singular limits of solutions to the Cauchy problem for second order linear differential equations in Hilbert spaces

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Abstract. We study the behavior of solutions to the problem

$$\begin{cases}\varepsilon\Big(u_\varepsilon''(t)+A_1u_\varepsilon(t)\Big)+u_\varepsilon'(t)+A_0u_\varepsilon(t)=f(t), \quad t>0,\\ u_\varepsilon(0)=u_0, \quad u_\varepsilon'(0)=u_1,\end{cases}$$

in the Hilbert space H as $\varepsilon\to 0$, where A_1 and A_0 are two linear selfadjoint operators.

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1 Introduction

Let H be a real Hilbert space endowed with the inner product $(\cdot,\cdot)$ and the norm $|\cdot|$. Let $A_i : D(A_i)\to H$, $i=0,1$, be two linear self-adjoint, positive defined operators. Consider the following Cauchy problem:

$$\begin{cases}\varepsilon\left(u_\varepsilon''(t)+A_1u_\varepsilon(t)\right)+u_\varepsilon'(t)+A_0u_\varepsilon(t)=f_\varepsilon(t), \quad t\in(0,T),\\ u_\varepsilon(0)=u_{0\varepsilon}, \quad u_\varepsilon'(0)=u_{1\varepsilon},\end{cases} \qquad (P_\varepsilon)$$

where $\varepsilon>0$ is a small parameter($\varepsilon\ll 1$), $u_\varepsilon, f_\varepsilon : [0,T)\to H$.

We will investigate the behavior of solutions $u_\varepsilon(t)$ to the perturbed system (P_ε) when $\varepsilon\to 0$, $u_{0\varepsilon}\to u_0$ and $f_\varepsilon\to f$. We will establish a relationship between solutions to the problem (P_ε) and the corresponding solutions to the following unperturbed system:

$$\begin{cases}v'(t)+A_0v(t)=f(t), \quad t\in(0,T),\\ v(0)=u_0.\end{cases} \qquad (P_0)$$

In our study we will use the following conditions:

(H1) *The operator $A_0 : D(A_0)\subseteq H\to H$ is self-adjoint and positive defined, i.e. there exists $\omega_0>0$ such that*

$$(A_0u,u)\geq\omega_0\,|u|^2, \quad \forall u\in D(A_0);$$

(H2) *The operator $A_1 : D(A_1)\subseteq H\to H$ is self-adjoint, positive defined and there exists $\alpha>1$ such that*:

(i) $D(A_0^\alpha) \subseteq D(A_1)$;

(ii) $A_1\Big[D(A_0^{2\alpha-1})\Big] \subseteq D(A_0^{\alpha-1})$;

(iii) $A_1 A_0^{\alpha-1}u = A_0^{\alpha-1}A_1 u, \quad \forall u \in D(A_0^{2\alpha-1})$;

(iv) *there exists* $\omega_2 > 0$ *and* $\omega_3 > 0$ *such that*

$$\omega_2\,|u|^2 \leq (A_1 u, u) \leq \omega_3\,(A_0^\alpha u, u)\,, \quad \forall u \in D(A_0^{2\alpha-1}).$$

The definition and properties of operator A^α can be found in [2].

If, in some topology, $u_\varepsilon(t)$ tends to the corresponding solutions $v(t)$ of the unperturbed system (P_0) as $\varepsilon \to 0$, then the system (P_0) is called *regularly perturbed.* In the opposite case, the system (P_0) is called *singularly perturbed.* In the last case, a subset of $[0,\infty)$, in which the solution $u_\varepsilon(t)$ has a singular behavior relative to ε, arises. This subset is called *the boundary layer.* The function which defines the singular behavior of the solution $u_\varepsilon(t)$ within the boundary layer is called *the boundary layer function.*

Our approach is based on two key points. The first one is the relationship between the solutions to the problems (P_ε) and (P_0). The second key point consists in obtaining a priori estimates for the solutions to the problems (P_ε), estimates which are uniform with respect to the small parameter ε.

In what follows we will need some notations. Let $k \in \mathbb{N}^\star$, $1 \leq p \leq +\infty$, $(a,b) \subset (-\infty,+\infty)$ and let X be a Banach space. We denote by $W^{k,p}(a,b;X)$ the Banach space of all vectorial distributions $u \in D'(a,b;X)$, $u^{(j)} \in L^p(a,b;X)$, $j = 0,1,\ldots,k$, endowed with the norm

$$\|u\|_{W^{k,p}(a,b;X)} = \left(\sum_{j=0}^{k} \|u^{(j)}\|^p_{L^p(a,b;X)}\right)^{1/p}$$

for $p \in [1,\infty)$ and

$$\|u\|_{W^{k,\infty}(a,b;X)} = \max_{0\leq j\leq k} \|u^{(j)}\|_{L^\infty(a,b;X)}$$

for $p = \infty$.

In the particular case $p = 2$, we denote $W^{k,2}(a,b;X) = H^k(a,b;X)$. If X is a Hilbert space, then $H^k(a,b;X)$ is also a Hilbert space with the inner product

$$(u,v)_{H^k(a,b;X)} = \sum_{j=0}^{k} \int_a^b \Big(u^{(j)}(t), v^{(j)}(t)\Big)_X\, dt.$$

For each fixed $s \in \mathbb{R}$, $k \in \mathbb{N}$ and $p \in [1,\infty]$, we define the Banach space

$$W_s^{k,p}(a,b;H) = \{f : (a,b) \to H; f^{(l)}(\cdot)e^{-st} \in L^p(a,b;X),\ l = 0,\ldots,k\},$$

with the norm

$$\|f\|_{W_s^{k,p}(a,b;X)} = \|fe^{-st}\|_{W^{k,p}(a,b;X)}.$$

2 Existence of strong solutions to both (P_ε) and (P_0)

Theorem 1. [1] *Let $T>0$ and let us assume that A_0 satisfies the condition* (**H1**). *If $u_0 \in D(A_0)$ and $f \in W^{1,1}(0,T;H)$, then there exists a unique strong solution $v \in W^{1,\infty}(0,T;H)$ to the problem (P_0). Moreover, v satisfies*

$$|v(t)| + \left(\int_0^t \left|A_0^{1/2}u(s)\right| ds\right)^{1/2} \leq |u_0| + \int_0^t |f(s)|\, ds, \quad \forall t \in [0,T],$$

$$\left|v'(t)\right| \leq |A_0 u_0 - f(0)| + \int_0^t \left|f'(s)\right| ds, \quad \forall t \in [0,T].$$

Theorem 2. [1] *Let $T>0$. Let us assume that $A : D(A) \subset H \to H$ is linear self-adjoint and positive defined. If $u_0 \in D(A)$, $u_1 \in H$ and $f \in W^{1,1}(0,T;H)$, then there exists a unique function $u : [0,T] \to H$ such that: $u \in W^{2,\infty}(0,T;H)$, $A^{1/2}u \in W^{1,\infty}(0,T;H)$, $Au \in L^\infty(0,T;H)$, $A^{1/2}u$ and u' are differentiable on the right in H for every $t \in [0,T)$ and*

$$\frac{d^+}{dt}\frac{du}{dt}(t) + \frac{du}{dt}(t) + Au(t) = f(t), \quad t \in [0,T), \tag{1}$$

$$u(0) = u_0, \quad u'(0) = u_1. \tag{2}$$

In what follows this function will be called *the strong solution* to the problem (1), (2).

3 A priori estimates for solutions to the problem (P_ε)

Consider the following problem:

$$\begin{cases} \varepsilon\left(u_\varepsilon''(t) + A_1 u_\varepsilon(t)\right) + u_\varepsilon'(t) + A_0 u_\varepsilon(t) = f(t), & t \in (0,T), \\ u_\varepsilon(0) = u_0, \quad u_\varepsilon'(0) = u_1. \end{cases} \tag{3}$$

Lemma 1. [4] *Let $T>0$. Suppose that, for each $\varepsilon \in (0,1)$, the operator $A(\varepsilon) = (\varepsilon A_1 + A_0) : D\left(A(\varepsilon)\right) \subseteq H \to H$ is self-adjoint and satisfies*

$$(A(\varepsilon)u, u) \geq \omega\, |u|^2, \quad \forall u \in D\left(A(\varepsilon)\right), \quad \omega > 0, \quad \varepsilon \in (0,1]. \tag{4}$$

If $f \in W^{1,1}(0,T;H), u_0 \in D\left(A(\varepsilon)\right)$, $u_1 \in H$, then the unique strong solution, u_ε, of the problem (3) *satisfies*

$$\|A^{1/2}(\varepsilon)u_\varepsilon\|_{C([0,t];\,H)} + \|u_\varepsilon'\|_{L^2(0,\,t;\,H)} \leq C(\omega)\, M(t), \tag{5}$$

for each $t \in [0,T]$ and each $\varepsilon \in (0,1/2]$. If, in addition, $u_1 \in D\left(A^{1/2}(\varepsilon)\right)$, then

$$\|u_\varepsilon'\|_{C([0,t];\,H)} + \|A^{1/2}(\varepsilon)\, u_\varepsilon'\|_{L^2(0,\,t;\,H)} \leq C(\omega)\, M_1(t), \tag{6}$$

for each $t \in [0,T]$, *and each* $\varepsilon \in (0,1]$, *and*

$$\|A(\varepsilon)u_\varepsilon\|_{L^\infty(0,\,t;\,H)} \le C(\omega)M_1(t), \quad \forall t \in [0,T], \quad \forall \varepsilon \in (0,1], \tag{7}$$

where $C(\omega)$ *is a constant depending on* ω,

$$M(t) = M(t, u_0, u_1, f) = \left|A^{1/2}(\varepsilon)u_0\right| + |u_1| + \|f\|_{W^{1,1}(0,t;\,H)} + |f(0)|$$

and

$$M_1(t) = M_1(t, u_0, u_1, f) = \left|A^{1/2}(\varepsilon)u_1\right| + |A(\varepsilon)u_0| + \|f\|_{W^{1,1}(0,t;\,H)} + |f(0)|.$$

Let u_ε be a strong solution of the problem (3) and let us denote by

$$z_\varepsilon(t) = u'_\varepsilon(t) + \alpha e^{-t/\varepsilon}, \quad \alpha = f(0) - u_1 - A(\varepsilon)u_0. \tag{8}$$

Lemma 2. [4] *Let* $T > 0$ *and let us assume that, for each* $\varepsilon \in (0,1)$, *the operator* $A(\varepsilon) = \varepsilon A_1 + A_0$ *is self-adjoint and satisfies* (4). *If* $u_1,\ f(0) - A(\varepsilon)u_0 \in D\left(A(\varepsilon)\right)$ *and* $f \in W^{2,1}(0,T;H)$, *then there exists* $C(\omega) > 0$, *such that the function* z_ε, *defined by* (8), *satisfies*

$$\|A^{1/2}(\varepsilon)z_\varepsilon\|_{C([0,\,t];\,H)} + \|z'_\varepsilon\|_{C([0,\,t];\,H)} + \left\|A^{1/2}(\varepsilon)z'_\varepsilon\right\|_{L^2(0,\,t;\,H)}$$
$$\le C(\omega)\, M_2(t), \quad \forall t \in [0,T], \quad \forall \varepsilon \in (0,1], \tag{9}$$

where

$$M_2(t) = |A(\varepsilon)f(0) - A^2(\varepsilon)u_0| + \|f\|_{W^{2,1}(0,t;H)} + |A(\varepsilon)u_1| + |f'(0)|.$$

4 The relationship between the solution to (P_ε) and (P_0)

Now we are going to establish the relationship between the solution to the problem (P_ε) and the corresponding solution to the problem (P_0). To this end, we begin by defining the transformation kernel which realizes this relationship.

Namely, for $\varepsilon > 0$, let us denote

$$K(t,\tau,\varepsilon) = \frac{1}{2\,\varepsilon\,\sqrt{\pi}}\left(K_1(t,\tau,\varepsilon) + 3K_2(t,\tau,\varepsilon) - 2K_3(t,\tau,\varepsilon)\right),$$

where

$$K_1(t,\tau,\varepsilon) = \exp\left\{\frac{3t - 2\tau}{4\varepsilon}\right\}\lambda\left(\frac{2t-\tau}{2\sqrt{\varepsilon t}}\right),$$
$$K_2(t,\tau,\varepsilon) = \exp\left\{\frac{3t + 6\tau}{4\varepsilon}\right\}\lambda\left(\frac{2t+\tau}{2\sqrt{\varepsilon t}}\right),$$
$$K_3(t,\tau,\varepsilon) = \exp\left\{\frac{\tau}{\varepsilon}\right\}\lambda\left(\frac{t+\tau}{2\sqrt{\varepsilon t}}\right), \quad \lambda(s) = \int_s^\infty e^{-\eta^2}\,d\eta.$$

Lemma 3. [3]. *The function* $K \in C([0,\infty) \times [0,\infty)) \cap C^2((0,\infty) \times (0,\infty))$ *has the following properties:*

(*i*) $K(t,\tau,\varepsilon) > 0, \quad \forall t \geq 0, \quad \forall \tau \geq 0\,;$

(*ii*) *For every continuous* $\varphi : [0,\infty) \to H$, *with* $|\varphi(t)| \leq M \exp\{\gamma\, t\}$, *we have*:

$$\lim_{t\to 0} \left\| \int_0^\infty K(t,\tau,\varepsilon)\varphi(\tau)d\tau - \int_0^\infty e^{-\tau}\varphi(2\varepsilon\tau)d\tau \right\|_H = 0,$$

for every $\varepsilon \in \left(0, (2\,\gamma)^{-1}\right)$;

(*iii*)

$$\int_0^\infty K(t,\tau,\varepsilon)d\tau = 1, \quad \forall t \geq 0.$$

(*iv*) *For every* $q \in [0,1]$, *there exists* $C > 0$ *and* $\varepsilon_0 > 0$, *depending on* q, *such that:*

$$\int_0^\infty K(t,\tau,\varepsilon)\,|t-\tau|^q\,d\tau \leq C\,\varepsilon^{q/2}\,\left(1+\sqrt{t}\right)^q,\ \forall t \geq 0, \forall \varepsilon \in (0,1]\,;$$

(*v*) *Let* $p \in (1,\infty]$ *and* $f : [0,\infty) \to H$, $f \in W^{1,p}(0,\infty;H)$. *There exist* $C > 0$, *and* ε_0 *depending on* p, *such that*

$$\left\| f(t) - \int_0^\infty K(t,\tau,\varepsilon)f(\tau)d\tau \right\|_H$$

$$\leq C\,\|f\,'\|_{L^p(0,\infty;H)}\,\left(1+\sqrt{t}\right)^{\frac{p-1}{p}}\,\varepsilon^{(p-1)/2p},\ \forall t \geq 0,\ \forall \varepsilon \in (0,1].$$

(*vi*) *For every* $q > 0$ *and* $\alpha \geq 0$, *there exists* $C(q,\alpha) > 0$ *such that*

$$\int_0^t \int_0^\infty K(\tau,\theta,\varepsilon)\,e^{-q\,\theta/\varepsilon}\,|\tau-\theta|^\alpha\,d\theta\,d\tau \leq C(q,\alpha)\,\varepsilon^{1+\alpha},$$

for each $t \geq 0$, *and each* $\varepsilon > 0$.

Theorem 3. [4] *Suppose that* $A(\varepsilon)$ *satisfies* **(H1)**, *let* $f \in L^\infty_c(0,\infty;H)$ *and let* $u_\varepsilon \in W^{2,\infty}_c(0,\infty;H)$ *be the strong solution to the problem* (3), *with* $Au_\varepsilon \in L^\infty_c(0,\infty;H)$, *for some* $c \geq 0$. *Then the function* w_ε, *defined by*

$$w_\varepsilon(t) = \int_0^\infty K(t,\tau,\varepsilon)\,u_\varepsilon(\tau)\,d\tau,$$

is the strong solution to the problem

$$\begin{cases} w'_\varepsilon(t) + A(\varepsilon)w_\varepsilon(t) = F_0(t,\varepsilon), & t > 0, \\ w_\varepsilon(0) = \varphi_\varepsilon, \end{cases} \tag{10}$$

where

$$\varphi_\varepsilon = \int_0^\infty e^{-\tau}u_\varepsilon(2\varepsilon\tau)d\tau, \quad F_0(t,\varepsilon) = f_0(t,\varepsilon)u_1 + \int_0^\infty K(t,\tau,\varepsilon)\,f(\tau)\,d\tau,$$

$$f_0(t,\varepsilon) = \frac{1}{\sqrt{\pi}}\left[2\exp\left\{\frac{3t}{4\varepsilon}\right\}\lambda\left(\sqrt{\frac{t}{\varepsilon}}\right) - \lambda\left(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\right)\right].$$

5 The limit of the solutions to the problem (P_ε) as $\varepsilon \to 0$

In this section we will study the behavior of the solutions to the problem (P_ε) as $\varepsilon \to 0$.

Theorem 4. *Let $T > 0$ and $p \in (1, \infty]$. Suppose that the operators A_0 and A_1 satisfy conditions* **(H1)** *and* **(H2)**. *If*

$$u_0 \in D(A_0), u_{0\,\varepsilon} \in D(A_0^{2\alpha-1}), \quad u_{1\,\varepsilon} \in D(A_0^{\alpha-1}), f, A_0^{\alpha-1} f_\varepsilon \in W^{1,p}(0,T;H),$$

then there exist constants $\varepsilon_0 = \varepsilon_0(\omega_0) \in (0,1)$ and $C = C(T, p, \omega_0, \omega_2, \omega_3, \alpha) > 0$ such that

$$||u_\varepsilon - v||_{C([0,T];H)} \le C\left(\mathcal{M}_{3\varepsilon}\, \varepsilon^\beta + \big|u_{0\,\varepsilon} - u_0\big| + \big|\big|f_\varepsilon - f\big|\big|_{L^p(0,T;H)}\right), \tag{11}$$

for all $\varepsilon \in (0, \varepsilon_0]$, where u_ε and v are the strong solutions to problems (P_ε) and (P_0) respectively,
$\beta = \min\{1/4, (p-1)/2p\}$ *and*

$$\mathcal{M}_{3\varepsilon} = \big|A_0^{(3\alpha-2)/2} u_{0\,\varepsilon}\big| + |A_0^{\alpha-1} u_{1\,\varepsilon}| + ||A_0^{\alpha-1} f_\varepsilon||_{W^{1,p}(0,T;H)},$$

If in addition, $u_{1\,\varepsilon} \in D\big(A_0^{\alpha/2}\big)$, then

$$||u_\varepsilon - v||_{C([0,T];H)} \le C\left(\mathcal{M}_{4\,\varepsilon}\, \varepsilon^{(p-1)/2p} + \big|u_{0\,\varepsilon} - u_0\big| + \big|\big|f_\varepsilon - f\big|\big|_{L^p(0,T;H)}\right), \tag{12}$$

$\varepsilon \in (0, \varepsilon_0]$ *and*

$$\begin{aligned} &||A_0^{1/2} u_\varepsilon - A_0^{1/2} v||_{L^2(0,\,T;\,H)} \\ &\le C\left(\mathcal{M}_{4\,\varepsilon}\, \varepsilon^\beta + \big|u_{0\,\varepsilon} - u_0\big| + \big|\big|f_\varepsilon - f\big|\big|_{L^p(0,T;H)}\right), \end{aligned} \tag{13}$$

$\varepsilon \in (0, \varepsilon_0]$ *where*
$\beta = \min\{1/4, (p-1)/2p\}$ *and*

$$\begin{aligned} \mathcal{M}_{4\,\varepsilon} = &\big|A_0^{(3\alpha-2)/2} u_{0\,\varepsilon}\big| + |A_0 u_{0\,\varepsilon}| + |A_1 u_{0\,\varepsilon}| + \big|A_0^{\alpha/2} u_{1\,\varepsilon}\big| \\ &+ \big|A_0^{\alpha-1} u_{1\,\varepsilon}\big| + ||A_0^{\alpha-1} f_\varepsilon||_{W^{1,p}(0,\,T;\,H)}. \end{aligned}$$

Proof. During the proof of this theorem, we will agree to denote all constants $\varepsilon_0(\omega_0)$ and $C = C(T, p, \omega_0, \omega_2, \omega_3, \alpha) > 0$ be ε_0 and C respectively.

Using **(H1)** and the properties of A_0^α proved in [2], we can state that there exists a constant $C(\omega_0, \alpha)$ such that:

$$\Big(A_0^\alpha u, u\Big) \ge C(\omega_0, \alpha) \parallel u \parallel^2, \quad u \in D(A_0^\alpha). \tag{14}$$

Using **(H1)**, **(H2)** and (14), since $u_{0\varepsilon} \in D(A_0^{2\alpha-1})$, $u_{1\varepsilon} \in D(A_0^{\alpha-1})$

$$(A_0^{3\alpha-2}u_{0\varepsilon}, u_{0\varepsilon}) = (A_0^{2\alpha-2}A_0^{\alpha/2}u_{0\varepsilon}, A_0^{\alpha/2}u_{0\varepsilon}) \geq C(\omega_0, \alpha)(A_0^{\alpha}u_{0\varepsilon}, u_{0\varepsilon})$$

$$= C(\omega_0, \alpha)(A_0^{\alpha-1}A_0^{1/2}u_{0\varepsilon}, A_0^{1/2}u_{0\varepsilon}) \geq C^2(\omega_0, \alpha)(A_0u_{0\varepsilon}, u_{0\varepsilon});$$
$$(A_0^{\alpha-1}u_{1\varepsilon}, A_0^{\alpha-1}u_{1\varepsilon}) = (A_0^{\alpha-1}A_0^{(\alpha-1)/2}u_{1\varepsilon}, A_0^{(\alpha-1)/2}u_{1\varepsilon})$$
$$\geq C(\omega_0, \alpha)(A_0^{\alpha-1}u_{1\varepsilon}, u_{1\varepsilon}) \geq C^2(\omega_0, \alpha)(u_{1\varepsilon}, u_{1\varepsilon}). \tag{15}$$

Let us also observe that, for $\alpha > 1$, we have $D(A_0^{2\alpha-1}) \subset D(A_0^{\alpha})$. Thus, from (**H2**), we get

$$(\lambda I + A_0^{\alpha-1})A_1u = A_1(\lambda I + A_0^{\alpha-1})u, \quad u \in D(A_0^{2\alpha-1}), \quad \lambda \geq 0,$$

which implies

$$(\lambda I + A_0^{\alpha-1})^{-1}A_1^{-1}u = A_1^{-1}(\lambda I + A_0^{\alpha-1})^{-1}u, \quad \forall u \in D(A_0^{2\alpha-1}), \quad \forall \lambda \geq 0.$$

Since A_1^{-1} is bounded and commutes with the resolvent of $A_0^{\alpha-1}$, we can state that

$$\left[A_0^{\alpha-1}\right]^{1/2}A_1^{-1}u = A_1^{-1}\left[A_0^{\alpha-1}\right]^{1/2}u, \quad \forall u \in D(A_0^{\alpha-1}).$$

So, if $u \in D(A_0^{\alpha-1})$, then $A_1^{-1}\left[A_0^{\alpha-1}\right]^{1/2}u \in D(A_1)$. Thus

$$A_1\left[A_0^{(\alpha-1)/2}A_1^{-1}\right]u = A_0^{(\alpha-1)/2}u, \quad \forall u \in D(A_0^{\alpha-1}).$$

Taking $u \in D(A_0^{2\alpha-1})$, from (ii) of (**H2**), we get $A_1u \in D(A_0^{\alpha-1})$, which finally implies

$$A_1A_0^{(\alpha-1)/2}u = A_0^{(\alpha-1)/2}A_1u, \quad \forall u \in D(A_0^{2\alpha-1}).$$

Using (iv) of (**H2**) and the last inequality, we get

$$|(A_1u, v)| = |(A_0^{(\alpha-1)/2}A_1u, A_0^{-(\alpha-1)/2}v)| = |(A_1A_0^{(\alpha-1)/2}u, A_0^{-(\alpha-1)/2}v)|$$

$$\leq \sqrt{(A_1A_0^{(\alpha-1)/2}u, A_0^{(\alpha-1)/2}u)(A_1A_0^{-(\alpha-1)/2}v, A_0^{-(\alpha-1)/2}v)}$$
$$\leq \omega_3\sqrt{(A_0^{\alpha}A_0^{(\alpha-1)/2}u, A_0^{(\alpha-1)/2}u)(A_0^{\alpha}A_0^{-(\alpha-1)/2}v, A_0^{-(\alpha-1)/2}v)}$$
$$= \omega_3|A_0^{\alpha-1/2}u||A_0^{1/2}v|, \quad \forall u, v \in D(A_0^{2\alpha-1}). \tag{16}$$

If $f_\varepsilon \in W^{l,p}(0, T; H)$ with $p \in (1, \infty]$ and $l \in \mathbb{N}^\star$, we have that $f_\varepsilon \in C([0, T]; H)$ and there exists an extension $\tilde{f}_\varepsilon \in W^{l,p}(0, \infty; H)$ such that

$$||\tilde{f}_\varepsilon||_{C([0,\infty);H)} + ||\tilde{f}_\varepsilon||_{W^{l,p}(0,\infty;H)} \leq C(T,p,l)\,||f_\varepsilon||_{W^{l,p}(0,T;H)}. \tag{17}$$

Let us denote by $\tilde{u}_\varepsilon$ the unique strong solution to the problem (P_ε) and by $\tilde{v}$ the unique strong solution to the problem (P_0), substituting $(0,T)$ by $(0,\infty)$ and f_ε by $\tilde{f}_\varepsilon$. From Theorem 2, we have

$$\begin{cases} \tilde{u}_\varepsilon \in W^{2,\infty}(0,T;H),\ A^{1/2}(\varepsilon)\tilde{u}_\varepsilon \in W^{1,\infty}(0,T;H), \\ A(\varepsilon)\tilde{u}_\varepsilon \in L^\infty(0,T;H),\ \forall T \in (0,\infty). \end{cases}$$

From Lemma 1 and (15), it follows that

$$\begin{cases} \tilde{u}_\varepsilon \in W^{2,\infty}(0,\infty;H), \quad A_0^{1/2}\tilde{u}_\varepsilon \in W^{1,2}(0,\infty;H), \\ A(\varepsilon)\tilde{u}_\varepsilon \in L^\infty(0,\infty;H). \end{cases}$$

Moreover, due to this lemma and inequalities (15) and (17), we get

$$||A_0^{1/2}\tilde{u}_\varepsilon||_{C([0,t];\,H)} + ||\tilde{u}'_\varepsilon||_{L^2(0,t;\,H)} \leq C\,\mathcal{M}_{3\varepsilon}, \quad t \geq 0, \quad \varepsilon \in (0,\varepsilon_0]. \tag{18}$$

If, in addition, $u_{1\,\varepsilon} \in D\big(A_0^{\alpha/2}\big)$, then

$$||\tilde{u}'_\varepsilon||_{C([0,t];\,H)} + ||A_0^{1/2}\,\tilde{u}'_\varepsilon||_{L^2(0,t;\,H)} \leq C\,\mathcal{M}_{4\,\varepsilon}, \quad t \geq 0, \quad \varepsilon \in (0,\varepsilon_0]. \tag{19}$$

Proof of (11). According to Theorem 3, the function

$$w_\varepsilon(t) = \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{u}_\varepsilon(\tau)\,d\tau, \tag{20}$$

is the strong solution to the problem

$$\begin{cases} w'_\varepsilon(t) + A(\varepsilon)w_\varepsilon(t) = F(t,\varepsilon), \quad t > 0, \quad \text{în} \quad H, \\ w_\varepsilon(0) = w_0, \end{cases} \tag{21}$$

for $0 < \varepsilon \leq \varepsilon_0$, where

$$\begin{cases} F(t,\varepsilon) = f_0(t,\varepsilon)\,u_{1\,\varepsilon} + \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{f}_\varepsilon(\tau)\,d\tau, \\[2ex] f_0(t,\varepsilon) = \frac{1}{\sqrt{\pi}}\Big[2\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big) - \lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big], \\[2ex] w_0 = \int_0^\infty e^{-\tau}\tilde{u}_\varepsilon(2\varepsilon\tau)d\tau. \end{cases} \tag{22}$$

Using Holder's inequality, properties (i)-(v) of Lemma 3 and (18), we obtain

$$\big|\big|\tilde{u}_\varepsilon(t) - w_\varepsilon(t)\big|\big|_H = \Big|\Big|\tilde{u}_\varepsilon(t) - \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{u}_\varepsilon(\tau)\,d\tau\Big|\Big|_H$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon)\big|\big|\tilde{u}_\varepsilon(t) - \tilde{u}_\varepsilon(\tau)\big|\big|_H\,d\tau \leq \int_0^\infty K(t,\tau,\varepsilon)\Big|\int_t^\tau ||\tilde{u}'_\varepsilon(s)||_H\,ds\Big|\,d\tau$$

$$\leq \|\widetilde{u}_\varepsilon'\|_{L^2(0,\,\infty;\,H)} \int_0^\infty K(t,\tau,\varepsilon)\,|t-\tau|^{1/2}\,d\tau \leq C\,\mathcal{M}_{3\varepsilon}\,\varepsilon^{1/4}, t\in[0,T], \quad \varepsilon\in(0,\varepsilon_0].$$

Then it follows

$$||\tilde{u}_\varepsilon - w_\varepsilon||_{C([0,T];\,H)} \leq C\,\mathcal{M}_{3\varepsilon}\,\varepsilon^{1/4}, \quad \varepsilon\in(0,\varepsilon_0]. \tag{23}$$

Let us denote by $R(t,\varepsilon)=\tilde{v}(t)-w_\varepsilon(t)$, which clearly is the strong solution in H to the problem

$$\begin{cases} R'(t,\varepsilon)+A_0R(t,\varepsilon)=\varepsilon\,A_1w_\varepsilon(t)+\mathcal{F}(t,\varepsilon), \quad t>0,\\ R(0,\varepsilon)=R_0, \end{cases} \tag{24}$$

where $R_0=u_0-w_0$ and

$$\mathcal{F}(t,\varepsilon)=\tilde{f}(t)-\int_0^\infty K(t,\tau,\varepsilon)\tilde{f}_\varepsilon(\tau)\,d\tau - f_0(t,\varepsilon)\,u_{1\,\varepsilon}. \tag{25}$$

Taking the inner product by R in the equation (24) and then integrating, we obtain

$$|R(t,\varepsilon)|^2+2\int_0^t \left|A_0^{1/2}R(s,\varepsilon)\right|^2 ds$$

$$= \left|R_0\right|^2+2\int_0^t \left|\mathcal{F}(s,\varepsilon)\right|\left|R(s,\varepsilon)\right| ds + 2\,\varepsilon\int_0^t \big(A_1w_\varepsilon(s),R(s,\varepsilon)\big)\,ds, \quad t\geq 0.$$

Using (16), from the last equality, we get

$$|R(t,\varepsilon)|^2+\int_0^t \left|A_0^{1/2}R(s,\varepsilon)\right|^2 ds$$

$$\leq \left|R_0\right|^2+2\int_0^t \left|\mathcal{F}(s,\varepsilon)\right|\left|R(s,\varepsilon)\right| ds + \varepsilon^2\int_0^t \left|A_0^{\alpha-1/2}w_\varepsilon(s)\right|^2 ds, \quad t\geq 0. \tag{26}$$

From (26), we obtain

$$|R(t,\varepsilon)|+\Big(\int_0^t \left|A_0^{1/2}R(s,\varepsilon)\right|^2 ds\Big)^{1/2}$$

$$\leq \left|R_0\right|+\int_0^t \left|\mathcal{F}(s,\varepsilon)\right| ds + \varepsilon\Big(\int_0^t \left|A_0^{\alpha-1/2}w_\varepsilon(s)\right|^2 ds\Big)^{1/2}, \quad t\geq 0, \quad \varepsilon\in(0,\varepsilon_0]. \tag{27}$$

From (18), it follows that

$$\left|R_0\right| \leq \int_0^\infty e^{-s}\left|\tilde{u}_\varepsilon(2\varepsilon s)-u_{0\,\varepsilon}\right| ds + |u_{0\varepsilon}-u_0|$$

$$\le \int_0^\infty e^{-s} \int_0^{2\varepsilon s} \left|\tilde u'_\varepsilon(\tau)\right| d\tau\, ds + |u_{0\,\varepsilon} - u_0|$$

$$\le C\,\mathcal{M}_{3\varepsilon}\,\varepsilon^{1/2} + |u_{0\,\varepsilon} - u_0|, \quad \varepsilon \in (0,\varepsilon_0]. \tag{28}$$

Using property (v) of Lemma 3, from (17), we have

$$\Big|\tilde f(t) - \int_0^\infty K(t,\tau,\varepsilon)\, \tilde f_\varepsilon(\tau)\, d\tau\Big| \le \big|\tilde f(t) - \tilde f_\varepsilon(t)\big|$$

$$+ \int_0^\infty K(t,\tau,\varepsilon)\, \big|\tilde f_\varepsilon(t) - \tilde f_\varepsilon(\tau)\big|\, d\tau$$

$$\le \big|\tilde f(t) - \tilde f_\varepsilon(t)\big| + C(T,p)\, \|f'_\varepsilon\|_{L^p(0,T;\,H)}\, \varepsilon^{(p-1)/2\,p}, \quad t \ge 0, \quad \varepsilon \in (0,\varepsilon_0]. \tag{29}$$

As $e^\tau \lambda(\sqrt{\tau}) \le C, \quad \tau \ge 0$, we have

$$\int_0^t \exp\Big\{\frac{3\tau}{4\varepsilon}\Big\} \lambda\Big(\sqrt{\frac{\tau}{\varepsilon}}\Big) d\tau \le C\,\varepsilon \int_0^{\frac{t}{\varepsilon}} e^{-\tau/4}\, d\tau \le C\,\varepsilon \int_0^\infty e^{-\tau/4}\, d\tau \le C\varepsilon, \quad t \ge 0,$$

$$\int_0^t \lambda\Big(\frac{1}{2}\sqrt{\frac{\tau}{\varepsilon}}\Big)\, d\tau \le \varepsilon \int_0^\infty \lambda\Big(\frac{1}{2}\sqrt{\tau}\Big)\, d\tau \le C\,\varepsilon, \quad t \ge 0.$$

Hence

$$\Big|\int_0^t f_0(\tau,\varepsilon)\, d\tau\, u_{1\,\varepsilon}\Big| \le C\,\varepsilon\,|u_{1\,\varepsilon}|, \quad t \ge 0. \tag{30}$$

Using (29) and (30), we get

$$\int_0^t \big|\mathcal{F}(s,\varepsilon)\big|\, ds$$

$$\le C\left(\mathcal{M}_{3\varepsilon}\,\varepsilon^{(p-1)/2p} + \big|\big|f_\varepsilon - f\big|\big|_{L^p(0,T;H)}\right), \quad t \in [0,T], \quad \varepsilon \in (0,\varepsilon_0]. \tag{31}$$

Let us denote by $\tilde y_\varepsilon = A_0^{\alpha-1}\tilde u_\varepsilon$. Since $A_0^{\alpha-1}u_{0\,\varepsilon} \in D(A_0^\alpha), \quad A_0^{\alpha-1}u_{1\,\varepsilon} \in H, \quad A_0^{\alpha-1}f_\varepsilon \in W^{1,p}(0,T;H)$, from Lemma 1, we can state:

$$||A_0^{1/2}\tilde y_\varepsilon||_{C([0,t];\,H)} + ||\tilde y'_\varepsilon||_{L^2(0,t;\,H)} \le C\,\mathcal{M}_{3\varepsilon}, \quad t \ge 0, \quad \varepsilon \in (0,1/2]. \tag{32}$$

As the operator $A_0^{\alpha-1/2}$ is closed, then, using (32), we obtain

$$\big|A_0^{\alpha-1/2} w_\varepsilon(t)\big|$$

$$\le \int_0^\infty K(t,\tau,\varepsilon)\,\big|A_0^{1/2}\tilde y_\varepsilon(\tau)\big|\, d\tau \le C\,\mathcal{M}_{3\varepsilon}, t \ge 0, \varepsilon \in (0,\varepsilon_0]. \tag{33}$$

Thanks to (28), (31) and (33), from (27) it follows that

$$||R||_{C([0,T];H)} + \big|\big|A_0^{1/2}R\big|\big|_{L^2(0,T;H)}$$
$$\leq \Big(\mathcal{M}_{3\varepsilon}\,\varepsilon^{(p-1)/2p} + \big|u_{0\,\varepsilon} - u_0\big| + \big|\big|f_\varepsilon - f\big|\big|_{L^p(0,T;H)}\Big), \quad \varepsilon \in (0,\varepsilon_0]. \tag{34}$$

Finally, from (23) and (34), it follows that

$$||\tilde{u}_\varepsilon - \tilde{v}||_{C([0,T];H)} \leq ||\tilde{u}_\varepsilon - w_\varepsilon||_{C([0,T];H)} + ||R||_{C([0,T];H)}$$
$$\leq C\Big(\mathcal{M}_{3\varepsilon}\,\varepsilon^{\beta} + \big|u_{0\,\varepsilon} - u_0\big| + \big|\big|f_\varepsilon - f\big|\big|_{L^p(0,T;H)}\Big), \quad \varepsilon \in (0,\varepsilon_0]. \tag{35}$$

According to Theorems 1 and 2, we have that $u_\varepsilon(t) = \tilde{u}_\varepsilon(t)$ and $\tilde{v}(t) = v(t)$ for $t \in [0,T]$. Therefore, from (35), we deduce (11).

Proof of (12). If $u_{1\,\varepsilon} \in D\big(A_0^{\alpha/2}\big)$, from (19), we get

$$\big|\big|\tilde{u}_\varepsilon(t) - w_\varepsilon(t)\big|\big|_H = \Big|\Big|\tilde{u}_\varepsilon(t) - \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{u}_\varepsilon(\tau)\,d\tau\Big|\Big|_H$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon)\big|\big|\tilde{u}_\varepsilon(t) - \tilde{u}_\varepsilon(\tau)\big|\big|_H\,d\tau \leq \int_0^\infty K(t,\tau,\varepsilon)\Big|\int_t^\tau ||\tilde{u}'_\varepsilon(s)||_H\,ds\Big|\,d\tau$$

$$\leq \|\tilde{u}'_\varepsilon\|_{C([0,\infty);H)}\int_0^\infty K(t,\tau,\varepsilon)\,|t-\tau|\,d\tau \leq C\,\mathcal{M}_{4\,\varepsilon}\,\varepsilon^{1/2}, \quad t \in [0,T], \quad \varepsilon \in (0,\varepsilon_0].$$

This yields

$$||\tilde{u}_\varepsilon - w_\varepsilon||_{C([0,T];H)} \leq C\,\mathcal{M}_{4\,\varepsilon}\,\varepsilon^{1/2}, \quad \varepsilon \in (0,\varepsilon_0].$$

As, for $p \in (1;\infty]$, we have $(p-1)/2p \leq 1/2$, the proof of (12) follows in the same way as the proof of (11).

Proof of (13). Using properties (i), (iii) and (iv) of Lemma 3 and (19), we get

$$\big|A_0^{1/2}\big(\tilde{u}_\varepsilon(t) - w_\varepsilon(t)\big)\big| \leq \int_0^\infty K(t,\tau,\varepsilon)\,\big|A_0^{1/2}\big(\tilde{u}_\varepsilon(t) - \tilde{u}_\varepsilon(\tau)\big)\big|\,d\tau$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon)\,\Big|\int_\tau^t \big|\big|A_0^{1/2}\tilde{u}'_\varepsilon(s)\big|\big|_H\,ds\Big|\,d\tau$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon)\,|t-\tau|^{1/2}\,\Big|\int_\tau^t \big|\big|A_0^{1/2}\tilde{u}'_\varepsilon(s)\big|\big|_H^2\,ds\Big|^{1/2}\,ds\Big|\,d\tau$$

$$\leq C\,\mathcal{M}_{4\,\varepsilon}\,\varepsilon^{1/4}, \quad t \geq 0, \quad \varepsilon \in (0, \varepsilon_0].$$

As $u_\varepsilon(t) = \tilde{u}_\varepsilon(t)$, $t \in [0, T]$, therefore

$$\big|\big|A_0^{1/2}\big(u_\varepsilon - w_\varepsilon\big)\big|\big|_{C([0,T];\,H)} \leq C\,\mathcal{M}_{4\,\varepsilon}\,\varepsilon^{1/4}, \quad \varepsilon \in (0, \varepsilon_0]. \tag{36}$$

From (34), it follows that

$$\big|\big|A_0^{1/2} R\big|\big|_{L^2(0,T)\,;H)}$$
$$\leq \Big(\mathcal{M}_{4\varepsilon}\,\varepsilon^{(p-1)/2\,p} + \big|u_{0\,\varepsilon} - u_0\big| + \big|\big|f_\varepsilon - f\big|\big|_{L^p(0,T;H)}\Big), \quad \varepsilon \in (0, \varepsilon_0]. \tag{37}$$

Finally, (36) and (37) imply (13) and this completes the proof. □

Theorem 5. *Let $T > 0$ and $p \in (1, \infty]$. Suppose that the operators A_0 and A_1 satisfy* **(H1)** *and* **(H2)**. *If $u_0, A_0 u_0, f(0) \in D(A_0)$, $\quad u_{1\varepsilon}, A_0 u_{0\varepsilon}, A_1 u_{0\varepsilon}, f_\varepsilon(0) \in D(A_0^{2\alpha-1}), f, A_0^{\alpha-1} f_\varepsilon \in W^{2,p}(0,T;H)$, then there exist constants $\varepsilon_0 = \varepsilon_0(\omega_0) \in (0,1)$ and $C = C(T, p, \omega_0, \omega_2, \omega_3, \alpha) > 0$ such that*

$$\big|\big|u_\varepsilon' - v' + h_\varepsilon\, e^{-\frac{t}{\varepsilon}}\big|\big|_{C([0,T];\,H)} \leq C\,\Big(\mathcal{M}_{5\,\varepsilon}\,\varepsilon^{(p-1)/2p} + D_\varepsilon\Big), \tag{38}$$

$$\big|\big|A_0^{1/2}\big(u_\varepsilon' - v' + h_\varepsilon\, e^{-\frac{t}{\varepsilon}}\big)\big|\big|_{L^2(0,T;\,H)} \leq C\,\Big(\mathcal{M}_{5\,\varepsilon}\,\varepsilon^{\beta} + D_\varepsilon\Big), \tag{39}$$

where v and u_ε are the strong solutions to problems (P_0) and (P_ε) respectively, $\beta = \min\{1/4, (p-1)/2p\}$, $h_\varepsilon = f_\varepsilon(0) - u_{1\,\varepsilon} - A(\varepsilon) u_{0\,\varepsilon}$,

$$D_\varepsilon = \big|\big|f_\varepsilon - f\big|\big|_{W^{1,\,p}(0,T;\,H)} + \big|A_0(u_{0\varepsilon} - u_0)\big|,$$

$$\mathcal{M}_{5\,\varepsilon} = \big|A_0^\alpha h_\varepsilon\big| + \big|A_0^{\alpha-1} A_1 h_\varepsilon\big| + \big|A_0^\alpha u_{1\varepsilon}\big|$$
$$+ \big|A_0^{\alpha-1} A_1 u_{1\varepsilon} + |A_1 u_{0\varepsilon}| + \big|\big|A_0^{\alpha-1} f_\varepsilon\big|\big|_{W^{2,p}(0,T;\,H)}.$$

Proof. During this proof, for $\tilde{u}_\varepsilon$, $\tilde{v}$, $\tilde{f}$ and $\tilde{f}_\varepsilon$ we will use the same notations as in the proof of Theorem 4. Let us denote by

$$\tilde{z}_\varepsilon(t) = \tilde{u}_\varepsilon'(t) + \alpha_\varepsilon e^{-\frac{t}{\varepsilon}}, \quad \alpha_\varepsilon = f_\varepsilon(0) - u_{1\,\varepsilon} - A(\varepsilon) u_{0\,\varepsilon}.$$

If $u_{1\,\varepsilon} + \alpha_\varepsilon \in D\big(A_0^{2\alpha-1})\big) \subseteq D(A_0^\alpha)$ and $f \in W^{2,1}(0,T;H)$, then, due to (15) and (17), $u_{1\,\varepsilon} + \alpha_\varepsilon \in D\big(A(\varepsilon)\big)$ and $\tilde{f} \in W^{2,1}(0,\infty;H)$. According to Theorem 2, $\tilde{z}_\varepsilon$ is the strong solution in H to the problem

$$\begin{cases} \varepsilon \tilde{z}_\varepsilon''(t) + \tilde{z}_\varepsilon'(t) + A(\varepsilon)\tilde{z}_\varepsilon(t) = \tilde{\mathcal{F}}(t,\varepsilon), \quad t > 0, \\ \tilde{z}_\varepsilon(0) = f_\varepsilon(0) - A(\varepsilon) u_{0\,\varepsilon}, \quad \tilde{z}_\varepsilon'(0) = 0, \end{cases} \tag{40}$$

where

$$\tilde{\mathcal{F}}(t,\varepsilon) = \tilde{f}'_\varepsilon(t) + e^{-t/\varepsilon} A(\varepsilon)\alpha_\varepsilon$$

and

$$\tilde{z}_\varepsilon \in W^{2,\infty}(0,\infty;H), \quad A_0^{1/2}\tilde{z}_\varepsilon \in W^{1,2}(0,\infty;H), \quad A(\varepsilon)\tilde{z}_\varepsilon \in L^\infty(0,\infty;H).$$

From Lemma 2 it follows that

$$\begin{aligned} & ||A_0^{1/2}\tilde{z}_\varepsilon||_{C([0,\,\infty];\,H)} + ||\tilde{z}'_\varepsilon||_{C([0,\,\infty);\,H)} \\ & + \big|\big|A_0^{1/2}\tilde{z}'_\varepsilon\big|\big|_{L^2(0,\,\infty;\,H)} \le C\,\mathcal{M}_{5\,\varepsilon}, \quad \varepsilon \in (0,\varepsilon_0]. \end{aligned} \tag{41}$$

According to Theorem 3 the function

$$w_{1\,\varepsilon}(t) = \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{z}_\varepsilon(\tau)d\tau$$

is a strong solution to the problem

$$\begin{cases} w'_{1\,\varepsilon}(t) + A(\varepsilon)w_{1\varepsilon}(t) = \mathcal{F}_1(t,\varepsilon), \quad t>0, \quad \varepsilon \in (0,\varepsilon_0] \\ w_{1\,\varepsilon}(0) = \int_0^\infty e^{-\tau}\,\tilde{z}_\varepsilon(2\,\varepsilon\,\tau)d\tau, \end{cases}$$

where

$$\mathcal{F}_1(t,\varepsilon) = \int_0^\infty K(t,\tau,\varepsilon)\Big(\tilde{f}'_\varepsilon(\tau)\,d\tau + e^{-\frac{\tau}{\varepsilon}}\,A(\varepsilon)\alpha_\varepsilon\Big)\,d\tau.$$

Moreover,

$$\big|A_0^{1/2}w_{1\,\varepsilon}(t)\big| \le \int_0^\infty K(t,\tau,\varepsilon)\,\big|A_0^{1/2}\tilde{z}_\varepsilon(\tau)\big|\,d\tau \le C\mathcal{M}_{5\,\varepsilon}, \quad t \ge 0. \tag{42}$$

Using properties (i), (iii)-(v) of Lemma 3 and(41), we get

$$\begin{aligned} \big|\big|\tilde{z}_\varepsilon(t) - w_{1\,\varepsilon}(t)\big|\big|_H &= \Big|\Big|\tilde{z}_\varepsilon(t) - \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{z}_\varepsilon(\tau)\,d\tau\Big|\Big|_H \\ &\le \int_0^\infty K(t,\tau,\varepsilon)\big|\big|\tilde{z}_\varepsilon(t) - \tilde{z}_\varepsilon(\tau)\big|\big|_H\,d\tau \\ &\le \int_0^\infty K(t,\tau,\varepsilon)\Big|\int_t^\tau ||\tilde{z}'_\varepsilon(s)||_H\,ds\Big|\,d\tau \end{aligned}$$

$$\le \|\tilde{z}'_\varepsilon\|_{C([0,\infty);\,H)}\int_0^\infty K(t,\tau,\varepsilon)\,|t-\tau|\,d\tau \le C\,\mathcal{M}_{5\,\varepsilon}\,\varepsilon^{1/2}, \quad t \in [0,T], \quad \varepsilon \in (0,\varepsilon_0]$$

and

$$\big|\big|A_0^{1/2}\big(\tilde{z}_\varepsilon(t) - w_{1\,\varepsilon}(t)\big)\big|\big|_H = \Big|\Big|A_0^{1/2}\tilde{z}_\varepsilon(t) - \int_0^\infty K(t,\tau,\varepsilon)\,A_0^{1/2}\tilde{z}_\varepsilon(\tau)\,d\tau\Big|\Big|_H$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon)\big|\big|A_0^{1/2}\big(\tilde{z}_\varepsilon(t)-\tilde{z}_\varepsilon(\tau)\big)\big|\big|_H\,d\tau$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon)\Big|\int_t^\tau ||A_0^{1/2}\tilde{z}'_\varepsilon(s)||_H\,ds\Big|\,d\tau$$

$$\leq \|A_0^{1/2}\tilde{z}'_\varepsilon\|_{L^2(0,\,\infty;\,H)}\int_0^\infty K(t,\tau,\varepsilon)\,|t-\tau|^{1/2}\,d\tau$$

$$\leq C\,\mathcal{M}_{5\,\varepsilon}\,\varepsilon^{1/4},\quad t\in[0,T],\quad \varepsilon\in(0,\varepsilon_0],$$

from which it follows that

$$||\tilde{z}_\varepsilon-w_{1\,\varepsilon}||_{C([0,T];\,H)}\leq C\,\mathcal{M}_{5\,\varepsilon}\,\varepsilon^{1/2},\quad \varepsilon\in(0,\varepsilon_0],\tag{43}$$

$$\big|\big|A_0^{1/2}\big(\tilde{z}_\varepsilon-w_{1\,\varepsilon}\big)\big|\big|_{L^2(0,T;\,H)}\leq C\,\mathcal{M}_{5\,\varepsilon}\,\varepsilon^{1/4},\quad \varepsilon\in(0,\varepsilon_0].\tag{44}$$

Let $R_1(t,\varepsilon)=\tilde{v}'(t)-w_{1\varepsilon}(t)$. If $f(0)-A_0u_0\in D(A_0)$ and $f\in W^{2,1}(0,T;H)$, then, according to Theorem 1, $\tilde{v}\in W^{2,\infty}(0,\infty;H)$, $A_0^{1/2}\tilde{v}\in W^{1,2}(0,\infty;H)$. Therefore $R_1\in W^{1,\infty}(0,\infty;H)$ and

$$\begin{cases} R_1'(t,\varepsilon)+A_0R_1(t,\varepsilon)=\tilde{f}'(t)-\mathcal{F}_1(t,\varepsilon)+\varepsilon A_1w_{1\,\varepsilon}(t), & t>0,\\ R_1(0,\varepsilon)=f(0)-A_0u_0-w_{1\,\varepsilon}(0). \end{cases}$$

Similarly to (27), we deduce inequality

$$\big|R_1(t,\varepsilon)\big|+\Big(\int_0^t\big|A_0^{1/2}R_1(s,\varepsilon)\big|^2\,ds\Big)^{1/2}\leq\big|R_1(0,\varepsilon)\big|$$

$$+\int_0^t\big|\tilde{f}'(s)-\mathcal{F}_1(s,\varepsilon)\big|\,ds+\varepsilon\Big(\int_0^t\big|A_0^{\alpha-1/2}w_{1\,\varepsilon}(s)\big|^2\,ds\Big)^{1/2},\quad t\geq 0.\tag{45}$$

Using (41), for $R_1(0,\varepsilon)$, we get

$$\big|R_1(0,\varepsilon)\big|\leq\big|f(0)-f_\varepsilon(0)\big|+\big|A_0(u_0-u_{0\,\varepsilon})\big|$$

$$+\varepsilon\,\big|A_1u_{0\,\varepsilon}\big|+\int_0^\infty e^{-s}\big|\tilde{z}_\varepsilon(2\,\varepsilon\,s)-\tilde{z}_\varepsilon(0)\big|\,ds$$

$$\leq C\,D_\varepsilon+\varepsilon\,\big|A_1u_{0\,\varepsilon}\big|+\mathcal{M}_{5\,\varepsilon}\,\varepsilon\leq C\,D_\varepsilon+\mathcal{M}_{5\,\varepsilon}\,\varepsilon,\quad \varepsilon\in(0,\varepsilon_0].\tag{46}$$

As

$$\big|\tilde{f}'(s)-\mathcal{F}_1(s,\varepsilon)\big|\leq\big|\tilde{f}'(s)-\tilde{f}'_\varepsilon(s)\big|+\int_0^\infty K(s,\tau,\varepsilon)\,\big|\tilde{f}'_\varepsilon(\tau)-\tilde{f}'_\varepsilon(s)\big|\,d\tau$$

$$+\int_0^\infty K(s,\tau,\varepsilon)\,e^{-\frac{\tau}{\varepsilon}}\,d\tau\,\big|\,A(\varepsilon)\alpha_\varepsilon\big|,$$

then, due to property (iv) and (vi) of Lemma 3, it follows:

$$\int_0^t \big|\tilde f'(s) - \mathcal{F}_1(s,\varepsilon)\big|\,ds \le C\Big(D_\varepsilon + \mathcal{M}_{5\,\varepsilon}\,\varepsilon^{(p-1)/2p} + \big|\,A(\varepsilon)\alpha_\varepsilon\big|\,\varepsilon\Big)$$

$$\le C\Big(D_\varepsilon + \mathcal{M}_{5\,\varepsilon}\,\varepsilon^{(p-1)/2p}\Big), \quad t\in[0,T], \quad \varepsilon\in(0,\varepsilon_0]. \tag{47}$$

Let us denote by $\tilde y_{1\varepsilon} = A_0^{\alpha-1}\tilde z_\varepsilon$. Since $A_0^{\alpha-1} z_\varepsilon(0) \in D(A_0^\alpha)$, $A_0^{\alpha-1} f_\varepsilon \in W^{1,p}(0,T;H)$, from Lemma 1, we can state the estimate:

$$||A_0^{1/2}\tilde y_{1\varepsilon}||_{C([0,t];\,H)} + ||\tilde y_{1\varepsilon}'||_{L^2(0,\,t;\,H)} \le C\,\mathcal{M}_{5\varepsilon}, \quad t\ge 0, \quad \varepsilon\in(0,1/2]. \tag{48}$$

As the operator $A_0^{\alpha/2}$ is closed, then, using (48), we obtain

$$\big|A_0^{\alpha-1/2} w_{1\varepsilon}(t)\big| \le \int_0^\infty K(t,\tau,\varepsilon)\,\big|A_0^{1/2}\tilde y_\varepsilon(\tau)\big|\,d\tau$$

$$\le C\,\mathcal{M}_{5\varepsilon}, \quad t\ge 0, \quad \varepsilon\in(0,\varepsilon_0]. \tag{49}$$

Using (42), (46), (47), from (45), we get

$$\big|\big|R_1\big|\big|_{C([0,T];\,H)} + \big|\big|A_0^{1/2}R_1\big|\big|_{L^2(0,\,T;\,H)} \le C\Big(D_\varepsilon$$

$$+\mathcal{M}_{5\,\varepsilon}\,\varepsilon^{(p-1)/2p}\Big), \quad \varepsilon\in(0,1]. \tag{50}$$

Finally, as (43), (44) and (45) imply (38) and (39), the proof is complete. □

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Free Moufang loops and alternative algebras

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Abstract. It is proved that any free Moufang loop can be embedded in to a loop of invertible elements of some alternative algebra. Using this embedding it is quite simple to prove the well-known result: if three elements of Moufang loop are bound by the associative law, then they generate an associative subloop. It is also proved that the intersection of the terms of the lower central series of a free Moufang loop is the identity and that a finitely generated free Moufang loop is Hopfian.

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This work offers another way of examining Moufang loops, and namely, with the help of alternative algebras. It is well known that for an alternative algebra A with unit the set $U(A)$ of all invertible elements of A forms a Moufang loop with respect to multiplication. It is known also that if L is a Moufang loop, then its loop algebra FL is not always alternative, i.e. the Moufang laws are not always true in FL [1]. These themes are stated in survey [2] and [3] in details.

However, let L be a free Moufang loop. It is shown that if we factor the loop algebra FL by some ideal I, then FL/I will be an alternative algebra and the loop L will be embedded in to the loop of invertible elements of algebra FL/I. This is a positive answer to the question raised in [4]: is it true that any Moufang loop can be imbedded into a homomorphic image of a loop of type $U(A)$ for a suitable unital alternative algebra A? The equivalent version of this question is: whether the variety generated by the loops of type $U(A)$ is a proper subvariety of the variety of all Moufang loops?

The findings of this paper also give a partial positive answer to a more general question (see, for example, [3]): is it true that any Moufang loop can be embedded into a loop of type $U(A)$ for a suitable unital alternative algebra A? A positive answer to this question was announced in [5]. Here, in fact, the answer to this question is negative: in [4] the author constructed a Moufang loop, which is not embedded into a loop of invertible elements of any alternative algebra.

Using this embedding it is quite simple to prove the well-known Moufang Theorem: if three elements of Moufang loop are bound by the associative law, then they generate an associative subloop. The Magnus Theorem for groups, stating that the intersection of the terms of the lower central series of a free group is the identity, is well known. This paper proves an analogous result for free Moufang loops. It also proves that a finitely generated free Moufang loop is Hopfian.

1 Preliminaries

A loop $(L, \cdot) \equiv L$ is called *IP-loop* if the laws ${}^{-1}x \cdot xy = yx \cdot x^{-1} = y$ are true in it, where ${}^{-1}xx = xx^{-1} = 1$. In IP- loops ${}^{-1}x = x^{-1}$ and $(xy)^{-1} = y^{-1}x^{-1}$. A loop is *Moufang* if it satisfies the law

$$x(y \cdot zy) = (xy \cdot z)y. \tag{1}$$

Every Moufang loop is an IP-loop. A subloop H of a loop L is called *normal* in L if

$$xH = Hx, \quad x \cdot yH = xy \cdot H, \quad H \cdot xy = Hx \cdot y \tag{2}$$

for every $x, y \in L$.

For elements x, y, z of a loop, the *commutator* (x, y) and the *associator* (x, y, z) are defined by

$$xy = (yx)(x, y), \quad (xy)z = (x(yz))(x, y, z). \tag{3}$$

The set of all elements z of a loop L which commute and associate with all elements of L, so that for all a, b in L, $(a, z) = 1$, $(z, a, b) = 1$, $(a, z, b) = 1$, $(a, b, z) = 1$ is a normal subloop $Z(L)$ of L, called its *center*.

If $Z_1(L) = Z(L)$, then the normal subloops $Z_{i+1}(L) : Z_{i+1}(L)/Z_i(L) = Z(L/Z_i(L))$ are inductively determined. A loop L is called *centrally nilpotent of class n* if its *upper central series* has the form $\{1\} \subset Z_1(L) \subset \ldots \subset Z_{n-1}(L) \subset Z_n(L) = L$.

If H is a normal subloop of a loop L, there is a unique smallest normal subloop M of L such that H/M is a part of the center of L/M, and we write $M = [H, L]$. From here it follows that M is the normal subloop of L generated by the set $\{(x, z), (z, x, y), (x, z, y), (x, y, z) | \forall z \in H, \forall x, y \in L\}$. The *lower central series* of L is defined by $L_1 = L$, $L_{i+1} = [L_i, L]$ $(i \geq 1)$ [2]. Consequently, L_{n+1} *is the normal subloop of L generated by the set* $\{(g, x), (g, x, y), (x, g, y), (x, y, g) | \forall g \in L_n$, $\forall x, y \in L\}$.

Let F be a field and L be a loop. Let us examine the *loop algebra FL*. This is a free F-module with the basis $\{q | q \in L\}$ and the product of the elements of this basis is determined as their product in loop L. Let H be a normal subloop of loop L. We denote the ideal of algebra FL, generated by the elements $1 - h$ $(h \in H)$ by ωH. If $H = L$, then ωL is called the *augmentation ideal* of algebra FL [2].

2 Embedding of free Moufang loops in to alternative algebras

Let us determine the homomorphism of F-algebras φ: $FL \to F(L/H)$ by the rule $\varphi(\sum \lambda_q q) = \sum \lambda_q Hq$. Takes place

Lemma 1. *Let H, H_1, H_2 be normal subloops of loop L. Then*

1) $Ker\varphi = \omega H$;

2) $1 - h \in \omega H$ if and only if $h \in H$;

3) if the elements h_i generate the subloop H, then the elements $1 - h_i$ generate the ideal ωH; if $H_1 \neq H_2$, then $\omega H_1 \neq \omega H_2$; if $H_1 \subset H_2$, then $\omega H_1 \subset \omega H_2$; if $H = \{H_1, H_2\}$, then $\omega H = \omega H_1 + \omega H_2$;

4) $\omega L = \{\sum_{q\in L} \lambda_q q | \sum_{q\in L} \lambda_q = 0\}$*;*

5) $FL/\omega H \cong F(L/H), \quad \omega L/\omega H \cong \omega(L/H)$*;*

6) the augmentation ideal is generated as F*-module by the elements of the form* $1-q$ $(q \in L)$*.*

Proof. 1) As the mapping φ is F-linear, then by (2) for $h \in H, q \in L$ we have $\varphi((1-h)q) = Hq - H(hq) = Hq - Hq = 0$, i.e. $\omega H \subseteq \mathrm{Ker}\varphi$. Let now $K = \{k_j | j \in J\}$ be a complete system of representatives of cosets of loop L modulo the normal subloop H and let $\varphi(r) = 0$. We present r as $r = u_1k_1 + \ldots + r_tk_t$, where $u_i = \sum_{h\in H} \lambda_h^{(i)} h, k_i \in K$. Then $0 = \varphi(r) = \varphi(u_1)\varphi(k_1) + \ldots + \varphi(u_t)\varphi(k_t) = (\sum_{h\in H} \lambda_h^{(1)})\varphi(k_1) + \ldots + (\sum_{h\in H} \lambda_h^{(t)})\varphi(k_t)$. As $\varphi(k_1), \ldots, \varphi(k_t)$ are pairwise distinct, then for all i $\sum_{h\in H} \lambda_h^{(i)} = 0$. Hence $-u_i = \sum_{h\in H} \lambda_h^{(i)}(1-h) - \sum_{h\in H} \lambda_h^{(i)} = \sum_{h\in H} \lambda_h^{(i)}(1-h)$ is an element from ωH. Consequently, $\mathrm{Ker}\varphi \subseteq \omega H$, and then $\ker\varphi = \omega H$.

2) If $q \notin H$, then $Hq \neq H$. Then $\varphi(1-q) = H - Hq \neq 0$, i.e. by 1) $1 - q \notin \mathrm{Ker}\varphi = \omega H$.

3) Let elements $\{h_i\}$ generate subloop H and I be an ideal, generated by the elements $\{1 - h_i\}$. Obviously $I \subseteq \omega H$. Conversely, let $g \in H$ and $g = g_1g_2$, where g_1, g_2 are words from h_i. We suppose that $1 - g_1, 1 - g_2 \in I$. Then $1 - g = (1 - g_1)g_2 + 1 - g_2 \in I$, i.e. $\omega H \subseteq I$. Hence $I = \omega H$. Let $H_1 \neq H_2$ (respect. $H_1 \subset H_2$) and $g \in H_1, g \notin H_2$. Then by 1) $1 - g \in \omega H_1$, but $1 - g \notin \omega H_2$. Hence $\omega H_1 \neq \omega H_2$ (respect. $\omega H_1 \subset \omega H_2$). If $H = \{H_1, H_2\}$, then by the first statement of 3) $\omega H = \omega H_1 + \omega H_2$.

4) We denote $R = \{\sum_{q\in L} \lambda_q q | \sum_{q\in L} \lambda_q = 0\}$. Obviously, $\omega L \subseteq R$. Conversely, if $r \in R$ and $r = \sum_{q\in L} \lambda_q q$, then $-r = -\sum_{q\in L} \lambda_q q = (\sum_{q\in L} \lambda_q)1 - \sum_{q\in L} \lambda_q q = \sum_{q\in L} \lambda_q(1-q) \in \omega L$, i.e. $R \subseteq \omega Q$. Hence $\omega L = R$.

5) Mapping $\varphi : FL \to F(L/H)$ is a homomorphism of loop algebras and as by 1) $\mathrm{Ker}\varphi = \omega H$, then $FL/\omega H \cong F(L/H)$. Now from 4) it follows that $\omega L/\omega H \cong \omega(L/H)$.

6) As $(1-q)q' = (1-qq') - (1-q')$, then the augmentation ideal ωL is generated by elements of the form $1-q$, where $q \in L$. This completes the proof of Lemma 1. $\square$

Lemma 2. *Let* $(L, \cdot)$ *be an* IP*-loop and let* φ *be a homomorphism of the algebra* $(FL, +, \cdot)$*. Then the* A*-homomorphism image at* φ *of the loop* $(L, \cdot)$ *will be a loop.*

Proof. We denote the A-homomorphism at image φ of the loop $(L, \cdot)$ by $(\overline{L}, \star)$. It follows from the IP-loop identity $x^{-1} \cdot xy = y$ that $\varphi(x^{-1}) = (\varphi x)^{-1}$ and $(\varphi x^{-1}) \star (\varphi x \star \varphi y) = \varphi y$, $(\varphi x)^{-1} \star (\varphi x \star \varphi y) = \varphi y$, $\overline{x}^{-1} \star (\overline{x} \star \overline{y}) = \overline{y}$. Let $\overline{a}, \overline{b} \in \overline{L}$. It is obvious that the equation $\overline{a} \star x = \overline{b}$ is always solvable and as $\overline{a}^{-1} \star (\overline{a} \star x) = \overline{a}^{-1} \star \overline{b}$, $x = \overline{a}^{-1} \star \overline{b}$, then it is uniquely solvable. It can be shown by analogy that the equation $y \star \overline{a} = \overline{b}$ is also uniquely solvable. Therefore, $(\overline{L}, \star)$ is a loop, as required.

Now, before we pass to the presentation of the basic results, we give the construction of *free* IP*-loop* with the set of free generators $X = \{x_1, x_2, \ldots\}$, using ideas from [2]. To the set X we add the disjoint set $\{x_1^{-1}, x_2^{-1}, \ldots\}$. Let us examine all groupoid words $L(X)$ from the set $\{x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots\}$ relative to the multiplication $(\cdot)$ and let e denote the empty word. For the words from $L(X)$ we define the *inverse words*: 1) for $x_i \in X$ the inverse will be x_i^{-1}, and for x_i^{-1} the inverse will be x_i, i.e. $(x_i^{-1})^{-1} = x_i$; 2) if $u \cdot v \in L(X)$, then $(u \cdot v)^{-1} = v^{-1} \cdot u^{-1}$. Further, we define two words u, v in $F(X)$ to be *Moufang-equivalent*, $u \approx v$ if one can be obtained from other by a sequence of substitutions, each of which replaces a subword $(rs \cdot r)t$ by $r(s \cdot rt)$ and vice-versa, where r, s, t are any words in $F(X)$. By a *contraction* μ of a word in $F(X)$ we mean the substitution at a subword of the form $u^{-1}(vw)$ or $(wv)u^{-1}$, where $u \approx v$, by w. The action ν, opposite to contraction μ we call the *expansion*.

For words w, w' in $F(X)$ we define the (μ, ν)-equivalence $w \cong w'$ if one word can be obtained from the other one by a finite sequence of substitutions, each of which is either a contraction μ or an expansion ν or a single use of the Moufang law (1). The relation $\cong$ will be, obviously, a relation of equivalence on $L(X)$. Moreover, it will be congruence, as if a word $(u_1 u_2 \ldots u_n)_\alpha$ is given when α is some parentheses distributions, obtained from words $u_1, u_2, \ldots, u_n$, then the replacement of the word u_i, $i = 1, 2, \ldots, n$, with words or equivalence can be realized applying to the given word a finite number of transformations of the above described form.

With the multiplication $\{u\} \cdot \{v\} = \{uv\}$ and the inverse $\{u\}^{-1} = \{u^{-1}\}$ of congruence classes we obtain a loop with the unity $\{e\}$, as the quotient loop $L(X)/\cong$ satisfies the laws $x^{-1} \cdot xy = y$, $yx \cdot x^{-1} = y$. Moreover, $L(X)/\cong$ will be a free Moufang loop on $\{x_i\}$, $i = 1, 2, 3, \ldots$, the set of free generators of X. We identify $\{x_i\}$ with x_i and we denote $L(X)/\cong$ by $L_X(\mathfrak{M})$.

Similarly to $F(X)$, we introduce the Moufang-equivalence, transformations μ, ν and (μ, ν)-equivalence for words in $L_X(\mathfrak{M})$. We define a word in $L(\mathfrak{M})$ to be a *reduced word* if no reductions of type μ of it are possible. If $w \in L(\mathfrak{M})$, then the number $l(w)$ of the variables in X, contained in w, will be called the *length* of the word w. Now let us show that if $w \to w_1$ and $w \to w_2$ are any reductions of type μ of a word w, then there is a word w_3 obtained from each of w_1, w_2 by a sequence of reductions of type μ. We use induction on the length of w. If $l(w) = 1$, w is already a reduced word. If $l(w) = n$ and $w = u \cdot v$ where u, v are subwords of w, then $l(u) < n$, $l(v) < n$. If both reductions $w \to w_1$ and $w \to w_2$ take place in the same subword, say u, then induction on length applied to u yields the result. If the two reductions take place in separate subwords, then applying both gives the w_3 needed. This leaves the last case where at least one of the reductions $w \to w_1$ and $w \to w_2$ involves both subwords u, v of w. Then w has, for example, the form $w = u^{-1}(uv)$. Therefore $w = v$ and thus $l(w) < n$, then by inductive hypothesis the statement is true.

Using this statement, one may prove by induction on length that any word w has reduced words regarding the reductions μ and all such reduced words belong to unique class of Moufang-equivalence. Then, induction on the number of reductions and expansions connecting a pair of congruent words shows that congruent words have the same reduced words.

Any word in $L_X(\mathfrak{M})$ has a reduced words. A *normal form* of a word u in $L_X(\mathfrak{M})$ is a reduced word of the least length. Clear by every word in $L_X(\mathfrak{M})$ has a normal form. Let $u(x_1, x_2, \ldots, x_k)$, $u(y_1, y_2, \ldots, y_n)$, where $x_i, y_j \in X \cup X^{-1}$, be two words of normal form of u of length $l(u)$. $L_X(\mathfrak{M})$ is a free loop. Assume, for example, $y_1 \notin \{x_1, x_2, \ldots, x_k\}$, then $u(x_1, x_2, \ldots, x_k) = u(1, y_2, \ldots, y_n)$. The length of $u(1, y_2, \ldots, y_n)$ is strict by less than $l(u)$. But this contradicts the minimum condition for $l(u)$. Consequently, all words of normal form of the same word in $L_X(\mathfrak{M})$ have the same free generators in their structure. This completes the proof of the following statement. $\square$

Lemma 3. *Any word in $L_X(\mathfrak{M})$ has a reduced word that belongs to the unique class of Moufang - equivalence, two words are (μ, ν)-equivalent if and only if they have the same reduced words and all words of normal form of the same word in $L_X(\mathfrak{M})$ have the same free generators in their structure.*

Now we consider a loop algebra FM of free Moufang loop $(M, \cdot) \equiv M$ over an arbitrary field F. Let $\overline{M} = \{\overline{u} = 1 - u | u \in M\}$ and we define the *circle composition* $\overline{u} \circ \overline{v} = \overline{u} + \overline{v} - \overline{u} \cdot \overline{v}$. Then $(M, \circ)$ is a loop, denoted sometimes as $\overline{M}$. The identity $\overline{1}$ of $\overline{M}$ is the zero of FM, $\overline{1} = 1 - 1$, and the inverse of $\overline{u}$ is $\overline{u}^{-1} = 1 - u^{-1}$ as $\overline{u} \circ \overline{1} = 1 - u + 0 - (1 - u)0 = 1 - u = \overline{u}$, $\overline{1} \circ \overline{u} = \overline{u}$, $\overline{u} \circ \overline{u}^{-1} = \overline{u} + \overline{u}^{-1} - \overline{u}\overline{u}^{-1} = 1 - u + 1 - u^{-1} - (1 - u)(1 - u^{-1}) = 0$, $\overline{u}^{-1} \circ \overline{u} = 0$. Let $\overline{u}, \overline{v} \in \overline{M}$. Then $\overline{u} \circ \overline{v} = \overline{u} + \overline{v} - \overline{uv} = 1 - u + 1 - v - (1 - u)(1 - v) = 1 - uv = 1 - \overline{uv}$. Hence $\overline{M}$ is closed under the composition $(\circ)$ and

$$\overline{u} \circ \overline{v} = 1 - uy. \tag{4}$$

Further, by (4) $\overline{u}^{-1} \circ (\overline{u} \circ \overline{v}) = 1 - u^{-1}(uv) = 1 - v = \overline{v}$ and $(\overline{v} \circ \overline{u}) \circ \overline{u}^{-1} = \overline{v}$. From here it follows that $(\overline{M}, \circ)$ is a loop. We call it the *circle loop* corresponding to the loop $(M, \cdot)$.

We define the one-to-one mapping $\overline{\varphi} : M \to \overline{M}$ by $\overline{\varphi}(a) = \overline{a}$. For $a, b \in M$ by (4) we have $\overline{\varphi}(ab) = 1 - ab = \overline{a} \circ \overline{b} = \varphi(a) \circ \varphi(b)$. Hence $\overline{\varphi}$ is an isomorphism of the loop M upon the loop $\overline{M}$. Then, by Lemma 2, it follows that $\overline{\varphi}$ induces the isomorphism φ of the loop algebra FM upon the loop algebra $F\overline{M}$ by the rule $\varphi(\Sigma_{u \in M} \alpha_u u) = \Sigma_{u \in M} \alpha_u (\overline{\varphi}(u)) = \Sigma_{u \in M} \alpha_u \overline{u}$.

Clear by if the loop M is generated by free generators $x_1, x_2, \ldots$, then the loop $\overline{M}$ is generated by free generators $\overline{x}_1, \overline{x}_2, \ldots$, the isomorphism $\varphi : FM \to F\overline{M}$ is defined by mappings $x_i \to \overline{x}_i$ and a word u in M has a normal form if and only if the corresponding word $\overline{u}$ also has a normal form. This completes the proof of the following lemma.

Lemma 4. *Let FM be a loop algebra of a free Moufang loop $(M, \cdot)$ with free generators $x_1, x_2, \ldots$ and let $\overline{M} = \{\overline{u} = 1 - u | u \in M\}$ be the corresponding loop under the circle composition $\overline{u} \circ \overline{v} = \overline{u} + \overline{v} - \overline{uv}$. Then the mappings $x_i \to \overline{x}_i$ define an isomorphism φ of the loop algebra FM upon the loop algebra $F\overline{M}$ by the rule $\varphi(\Sigma \alpha_u u) = \Sigma \alpha_u (\overline{\varphi}(u)) = \Sigma \alpha_u \overline{u}$, $\alpha_u \in F$, $u \in M$, and a word in the loop $(M, \cdot)$ has a normal form if and only if the word φu has a normal form in the loop $(\overline{M}, \circ)$.*

From now on, according to Lemma 4 for the algebra FM we will consider only monomials of normal form. Let $u \in FM$ and let φ be the isomorphism defined in Lemma 4. We denote $\varphi(u) = \overline{u}$. If $u = \Sigma\alpha_i u_i$, $\alpha_i \in F$, $u_i \in M$, is a polynomial in FM then we denote $\mathfrak{c}(u) = \Sigma\alpha_i$. Clear by $\mathfrak{c}(u) = \mathfrak{c}(\overline{u})$, where $\overline{u} = \Sigma\alpha_i\overline{u}_i$.

Let $(a, b, c) = ab \cdot c - a \cdot bc$ be the associator in algebra. If the free Moufang loop M is non-associative, then from the definition of loop algebra it follows that the equalities

$$(a, b, c) + (b, a, c) = 0, \quad (a, b, c) + (a, c, b) = 0 \quad \forall a, b, c \in L, \tag{5}$$

do not always hold in algebra FM. Let $I(M)$ denote the ideal of algebra FM, generated by all the elements of the left part of equalities (5). It follows from the definition of loop algebra and di-associativity of Moufang loops that $FM/I(M)$ will be an alternative algebra. We remind that an algebra A is called *alternative* if the identities $(x, x, y) = (y, x, x) = 0$ hold in it. Hence we proved

Lemma 5. *Let FM and $F\overline{M}$ be the loop algebras of a free Moufang loop $(M, \cdot)$ and its corresponding circle loop $(\overline{M}, \circ)$ and let $I(M, \cdot)$, $I(\overline{M}, \circ)$ be the ideals of FM and $F\overline{M}$ respectively, defined above. Then $I(M) = I(\overline{M})$ and for any $\overline{u} \in I(\overline{M})$ and $\mathfrak{c}(\overline{u}) = 0$.*

Proof. We denote $v_1 = v_1(u_{11}, u_{12}, u_{13}) = (u_{11}, u_{12}, u_{13}) + (u_{12}, u_{11}, u_{13})$, $v_2 = v_2(u_{21}, u_{22}, u_{23}) = (u_{21}, u_{22}, u_{23}) + (u_{21}, u_{23}, u_{22})$, where $u_{ij} \in M$, $i = 1, 2$, $j = 1, 2, 3$. Then, as an F-module, the ideal $I(M)$ is generated by elements of the form

$$w(d_1, \dots, d_k, v_i, d_{k+1}, \dots, d_m),$$

where $i = 1, 2$ and $d_1, \dots, d_m$ are monomials from FM.

Let $w = w(d_1, \dots, d_k, v_1, d_{k+1}, \dots, d_m)$. Then by (4)

$$w = w(d_1, \dots, d_k, (u_{11}, u_{12}, u_{13}) + (u_{12}, u_{11}, u_{13}), d_{k+1}, \dots, d_m) =$$

$$w(d_1, \dots, d_k, u_{11}u_{12} \cdot u_{13}, d_{k+1}, \dots, d_m) -$$

$$w(d_1, \dots, d_k, u_{11} \cdot u_{12}u_{13}, d_{k+1}, \dots, d_m) +$$

$$w(d_1, \dots, d_k, u_{12}u_{11} \cdot u_{13}, d_{k+1}, \dots, d_m) -$$

$$w(d_1, \dots, d_k, u_{12} \cdot u_{11}u_{13}, d_{k+1}, \dots, d_m) =$$

$$-(1 - w(d_1, \dots, d_k, u_{11}u_{12} \cdot u_{13}, d_{k+1}, \dots, d_m)) +$$

$$(1 - w(d_1, \dots, d_k, u_{11} \cdot u_{12}u_{13}, d_{k+1}, \dots, d_m)) -$$

$$(1 - w(d_1, \dots, d_k, u_{12}u_{11} \cdot u_{13}, d_{k+1}, \dots, d_m)) +$$

$$(1 - w(d_1, \dots, d_k, u_{12} \cdot u_{11}u_{13}, d_{k+1}, \dots, d_m)) =$$

$$\overline{w}(\overline{d}_1, \dots, \overline{d}_k, (\overline{u}_{11} \circ \overline{u}_{12}) \circ \overline{u}_{13}, \overline{d}_{k+1}, \dots, \overline{d}_m) -$$

$$\overline{w}(\overline{d}_1, \dots, \overline{d}_k, \overline{u}_{11} \circ (\overline{u}_{12} \circ \overline{u}_{13}), \overline{d}_{k+1}, \dots, \overline{d}_m) +$$

$$\overline{w}(\overline{d}_1, \ldots, \overline{d}_k, (\overline{u}_{12} \circ \overline{u}_{11}) \circ \overline{u}_{13}, \overline{d}_{k+1}, \ldots, \overline{d}_m) -$$
$$\overline{w}(\overline{d}_1, \ldots, \overline{d}_k, \overline{u}_{12} \circ (\overline{u}_{11} \circ \overline{u}_{13}), \overline{d}_{k+1}, \ldots, \overline{d}_m) =$$
$$\overline{w}(\overline{d}_1, \ldots, \overline{d}_k, \overline{v}_2, \overline{d}_{k+1}, \ldots, \overline{d}_m).$$

Similarly, $w(d_1, \ldots, d_k, v_2, d_{k+1}, \ldots, d_m) = \overline{w}(\overline{d}_1, \ldots, \overline{d}_k, \overline{v}_2, \overline{d}_{k+1}, \ldots, \overline{d}_m)$. Hence $I(M) \subseteq I(\overline{M})$.

Conversely, we consider a polynomial in $f\overline{M}$ of the form $\overline{w}(\overline{d}_1, \ldots, \overline{d}_k, \overline{v}_i, \overline{d}_{k+1}, \ldots, \overline{d}_m)$. It is clear that $\overline{w} \in I(\overline{M})$ and any element $\overline{z} \in I(\overline{M})$ will be represented as the sum of a finite number of polynomials of such a form. We have $\mathfrak{c}(\overline{v}_i) = 0$, then $\mathfrak{c}(\overline{w}) = 0$ and, consequently, $\mathfrak{c}(\overline{z}) = 0$. Now, let for example $\overline{v}_i = \overline{v}_1$. By (4) we get $\overline{v}_1 = (\overline{u}_{11} \circ \overline{u}_{12}) \circ \overline{u}_{13} = \overline{u}_{11} \circ (\overline{u}_{12} \circ \overline{u}_{13}) = 1 - u_{11}u_{12} \cdot u_{13} - (1 - u_{11} \cdot u_{12}u_{13}) = -u_{11}u_{12} \cdot u_{13} + u_{11} \cdot u_{12}u_{13} = -(u_{11}, u_{12}, u_{13}) = -v_1$. Further, by the relation $\overline{x} \circ \overline{y} = 1 - xy$ in an expression $\overline{w}$ we pass from the operation $(\circ)$ to the operation $(\cdot)$. Then $\overline{w}$ can be written as the sum of a finite number of monomials, each of them containing the associators v_i in its structure. Then $\overline{w} \in I(M)$, and hence $\overline{z} \in I(M)$, $I(\overline{M}) \subseteq I(M)$. Consequently, $I(\overline{M}) = I(M)$. This completes the proof of Lemma 5. □

Theorem 1. *Let $(M, \cdot)$ be a free Moufang loop, let F be an arbitrary field and let $\varphi : FM \to FM/I(M)$ be the natural homomorphism of the algebra FM upon the alternative algebra $FM/I(M)$. Then the image $\varphi(M, \cdot) = (\overline{M}, \star)$ of the loop $(M, \cdot)$ will be an isomorphism of these loops.*

Proof. Any Moufang loop is an IP-loop, so by Lemma 2 the image of the loop $(M, \cdot)$ under the A-homomorphism $\varphi : FM \to FM/I(M)$ will be a loop $(\overline{M}, \star)$. Let H be a normal subloop of loop $(M, \cdot)$ that corresponds to φ. Then $1 - H \subseteq I(M)$. We suppose that $H \neq \{1\}$ and let $1 \neq u(x_1, \ldots, x_k) \in H$ be a word in free generators $x_1, \ldots, x_k$ of the normal form. Then the length $l(u) > 0$. By (4) we write $1 - u(x_1, \ldots, x_k)$ in generators $\overline{x}_1, \ldots, \overline{x}_k$ with respect to the circle composition $(\circ)$, $1 - u(x_1, \ldots, x_k) = \overline{u}(\overline{x}_1, \ldots, \overline{x}_k)$. As $1 - u(x_1, \ldots, x_k) \in I(M)$ then by Lemma 5 $\overline{u}(\overline{x}_1, \ldots, \overline{x}_k) \in I(\overline{M})$ and $\overline{u}(\overline{x}_1, \ldots, \overline{x}_k) = \overline{u}$ has a normal form. Hence $l(\overline{u}) > 0$ and, consequently, $\mathfrak{c}(\overline{u}) = 1$. But by Lemma 1 $\mathfrak{c}(\overline{u}) = 0$. We get a contradiction with $\mathfrak{c}(\overline{u}) = 1$. Hence our supposition that $H \neq \{1\}$ is false. This completes the proof of Theorem 1. □

Remark. The proof of Lemma 3 has a constructive character for free Moufang loops. But Lemma 3 holds for algebras of Ω-words (see, for example, [6]). Any relatively free Moufang loop is an algebra of Ω-words. From here it follows that Lemma 3 is true for any relatively free Moufang loop. Then it is easy to see that the main result of this paper (Theorem 1) holds for every relatively free Moufang loop.

Further we identify the loop $(\overline{M}, \star)$ with $(M, \cdot)$. Then every element in $FM/I(M)$ has the form $\sum_{q \in M} \lambda_q q$, $\lambda_q \in F$. Further for the alternative algebra $FM/I(M)$ we use the notation FM and we call them *"loop algebra"* (in quote marks). Let H be a normal subloop of M. We denote the ideal of "loop algebra"

FM, generated by the elements $1-h$ $(h \in H)$ by ωH. If $H = M$, then ωM will be called the *"augmentation ideal"* (in quote marks) of "loop algebra" FM. Let us determine the homomorphism φ of F-algebra FM by the rule $\varphi(\sum \lambda_q q) = \sum \lambda_q Hq$. Similarly to Lemma 1 we proved

Proposition 1. *Let H be a normal subloops of a free Moufang loop M and let FM and ωM be, respectively, the "loop algebra" and the "augmentation ideal" of M. Then*

1) $\omega H \subseteq Ker\varphi$;

2) $1-h \in Ker\varphi$ if and only if $h \in H$;

3) $\omega M = \{\sum_{q\in M} \lambda_q q | \sum_{q\in M} \lambda_q = 0\}$;

4) the "augmentation ideal" ωM is generated as F-module by elements of the form $1-q$ $(q \in M)$.

Let $\overline{\omega M}$ denote the augmentation ideal (without quote marks) of $\overline{FM}$. Then from 4) of Lemma 1 and 3) of Proposition 1 it follows that

$$\omega M = \overline{\omega M}/I(M). \tag{6}$$

Any Moufang loop L has a representation $L = L/H$, where L is a free Moufang loop. As we have noted above, in [4] Moufang loops L are constructed that are not embedded into a loop of invertible elements of any alternative algebras. Then for such normal subloop H of L $\mathrm{Ker}\varphi = FL$ and by 2) of Proposition 7 the inclusion $\omega H \subset \mathrm{Ker}\varphi$ is strict.

We mention that Proposition 1 holds also for Moufang loops for which Theorem 1 is true.

3 Some corollaries

Now we consider Moufang loops with the help of alternative algebras, using the embedding of Moufang loops in alternative algebras from Theorem 6. It is obvious that from the identities $(x,y,z) = -(y,x,z)$, $(x,y,z) = -(x,z,y)$, which hold in any alternative algebra, follows

Lemma 6. *Let Q be a loop, let FQ be its "loop algebra" and let $a,b,c \in Q$. Then, if $(a,b,c) = 0$, then $(a',b',c') = 0$, where a',b',c' are obtained from a,b,c with some substitution or with the change of some loop elements a,b,c for the inverse.*

In an arbitrary alternative algebra the identities

$$(x^2,y,z) = x(x,y,z) + (x,y,z)x, \tag{7}$$

$$(x,yx,z) = x(x,y,z), \tag{8}$$

$$(x,xy,z) = (x,y,z)x \quad [7] \tag{9}$$

hold true, the linearization of the last leads to the identities

$$(x,yt,z) + (t,yx,z) = x(t,y,z) + t(x,y,z), \tag{10}$$

$$(x,ty,z) + (t,xy,z) = (x,y,z)t + (t,y,z)x. \tag{11}$$

Proposition 2. *(Moufang Theorem) If three elements a, b, c of Moufang loop Q are bounded by the associative law $ab \cdot c = a \cdot bc$, then they generate an associative subloop.*

Proof. Obviously, it is sufficient to show that if there are arbitrary monomials $u_i = u_i(x_1, x_2, x_3), i = 1, 2, 3,$ of the "loop algebra" $FL_X(\mathfrak{M})$ from the generators $x_1, x_1^{-1}, x_2, x_2^{-1}, x_3, x_3^{-1}$ of the free Moufang loop $L_X(\mathfrak{M})$, then the equality $(\overline{u}_1, \overline{u}_2, \overline{u}_3) = 0$ holds true for $a, b, c \in Q$ in the "loop algebra" FQ where $\overline{u}_i = u_i(a, b, c)$. We prove the proposition by induction on the number $n = l(u_1)+l(u_2)+l(u_3)$, where $l(u_i)$ is the length of word u_i of loop $L_X(\mathfrak{M})$. If $n = 3$, then the statement follows from Lemma 6. Let now $n > 3$ and the equality $(\overline{v}_1, \overline{v}_2, \overline{v}_3) = 0$ holds true for the words $v_1(x_1, x_2, x_3), v_2(x_1, x_2, x_3), v_3(x_1, x_2, x_3)$ of loop $L_X(\mathfrak{M})$ such that $l(v_1) + l(v_2) + l(v_3) < n$. Then by the inductive hypothesis the associator $(\overline{u}_1, \overline{u}_2, \overline{u}_3)$ does not depend on the parentheses places in the words u_i. Let us now consider the two possible cases.

1. The words u_i have, for example, the form $u_1 = x_1^k,\ u_2 = x_2^r,\ u_3 = x_3^s$. Taking into account Lemma 6, we consider that $k > 0$. If $k = 2n$, then by (7) and by the inductive hypothesis $(a^{2n}, \overline{u}_2, \overline{u}_3) = a^n(a^n, \overline{u}_2, \overline{u}_3) + (a^n, \overline{u}_2, \overline{u}_3)a^n = 0$. Let now $k = 2n + 1$. Then by (11), by the inductive hypothesis and the previous case $(a^k, \overline{u}_2, \overline{u}_3) = (a^{2n}a, \overline{u}_2, \overline{u}_3) = (a^{2n}, \overline{u}_2 a, \overline{u}_3) - (\overline{u}_2, a, \overline{u}_3)a^{2n} - (a^{2n}, a, \overline{u}_3)\overline{u}_2 = (a^{2n}a, \overline{u}_2, \overline{u}_3) = 0$.

2. Two words from u_1, u_2, u_3 have in their structure a variable of the form x_i or x_i^{-1}. Taking into account the property of IP-loop $(xy)^{-1} = y^{-1}x^{-1}$ and Lemma 6, it is sufficient to consider the case when these words have the variable x_i in their structure. We suppose, for example, that $u_1 = v_1x_1 \cdot w_1, u_2 = v_2x_1 \cdot w_2$, where v_1, w_1, v_2, w_2 can be missing. Then by the identities (8) - (11) and by the inductive hypothesis we have $(\overline{u}_1, \overline{u}_2, \overline{u}_3) = (\overline{u}_3, \overline{u}_1, \overline{u}_2) = (\overline{u}_3, \overline{v}_1 a \cdot \overline{w}_1, \overline{v}_2 a \cdot \overline{w}_2) = -(\overline{v}_1 a, \overline{u}_3\overline{w}_1, \overline{v}_2 a \cdot \overline{w}_2) + (\overline{u}_3, \overline{w}_1, \overline{v}_2 a \cdot \overline{w}_2)(\overline{v}_1 a) + (\overline{v}_1 a, \overline{w}_1, \overline{v}_2 a \cdot w_2)\overline{u}_3 = -(\overline{v}_1 a, \overline{u}_3\overline{w}_1, \overline{v}_2 a \cdot \overline{w}_2) = -(\overline{u}_3\overline{w}_1, \overline{v}_2 a \cdot \overline{w}_2, \overline{v}_1 a) = (\overline{v}_2 a, \overline{u}_3\overline{w}_1 \cdot \overline{w}_2, \overline{v}_1 a) - (\overline{u}_3\overline{w}_1, \overline{w}_2, \overline{v}_1 a)(\overline{v}_2 a) - (\overline{v}_2 a, \overline{w}_2, \overline{v}_1 a)(\overline{u}_3\overline{w}_1) = (\overline{v}_2 a, \overline{u}_3\overline{w}_1 \cdot \overline{w}_2, \overline{v}_1 a) = (\overline{v}_2 a, t, \overline{v}_1 a) = (t, \overline{v}_1 a, \overline{v}_2 a) = -(a, \overline{v}_1 t, \overline{v}_2 a) + t(a, \overline{v}_1, \overline{v}_2 a) + a(t, \overline{v}_1, \overline{v}_2 a) = -(a, \overline{v}_1 t, \overline{v}_2 a) = (a, \overline{v}_2 a, \overline{v}_1 t) = a(a, \overline{v}_2, \overline{v}_1 t) = 0$. This completes the proof of Proposition 2. $\square$

If we apply the Proposition 2 to the equality $a \cdot ab = aa \cdot b$, which follows from (1), we get

Corollary. *The Moufang loop is di-associative, i.e. any its two elements generate an associative subloop.*

Let $L_X(\mathfrak{M})$ be a free Moufang loop with the set of free generators X. By Lemma 3 every word in $L_X(\mathfrak{M})$ can be presented as a reduced word in different ways. As $L_X(\mathfrak{M})$ is a free loop, all reduced words of the same element in $L_X(\mathfrak{M})$ have the same free generators in their structure. Hence, their number is finite. The reduced words of element w in $L_X(\mathfrak{M})$ of the least length will be called *normal reduced words* of w. Hence every word in $L_X(\mathfrak{M})$ has normal reduced words. We will call the normal reduced words u, v in $L_X(\mathfrak{M})$ *l-homogeneous* if u, v have the same length, $l(u) = l(v)$ with respect to the variables $y \in X \cup X^{-1}$.

By the definition of the loop algebra $FL_X(\mathfrak{M})$ any element in $FL_X(\mathfrak{M})$ has the form $\sum \alpha_g g, g \in L_X(\mathfrak{M})$, and only a finite number of coefficients $\alpha_g \in F$ differ from zero. We have introduced earlier a notion of l-homogeneity for the monomials g_j. We extend it to the polynomials of algebra $FL_X(\mathfrak{M})$. It can be done, as $FL_X(\mathfrak{M})$ is a free F-module with free generators g_j. Then the algebra $FL_X(\mathfrak{M})$ decomposes into a direct sum of *l-homogeneous submodules*, consisting of l-homogeneous polynomials.

By 3) of Lemma 1 the augmentation ideal $\omega L_X(\mathfrak{M})$ of the loop algebra $FL_X(\mathfrak{M})$ is generated by the set $\overline{X} = \{1 - x_i | \forall x_i \in X \cup X^{-1}\}$. If $u(x_1, x_2, \dots \dots, x_k\}$, where $x_i \in X \cup X^{-1}$ is a normal reduced word in $L_X(\mathfrak{M})$, then the monomial $u(\overline{x}_1, \overline{x}_2, \dots, \overline{x}_k)$ in $\omega L_X(\mathfrak{M})$ will be called *normal reduced with respect to the generating set $\overline{X}$*. We transfer the notions of length and l-homogeneity of monomials $u(x_1, x_2, \dots, x_k\}$ to monomials $u(\overline{x}_i, \dots, \overline{x}_k)$.

Lemma 7. *Let $FL_X(\mathfrak{M})$ be a loop algebra of the free Moufang loop $L_X(\mathfrak{M})$ with a set of free generators X and let $u(x_1, \dots, x_k)$ be a normal reduced word in the variables $x_1, \dots, x_k \in X \cup X^{-1}$ of length $l(u)$. Then*

1) the polynomial $1 - u(x_1, \dots, x_k)$ of the augmentation ideal $\omega L_X(\mathfrak{M})$ is represented as a sum of normal reduced monomials of $\omega L_X(\mathfrak{M})$ in variables $\overline{x}_1, \dots, \overline{x}_k \in \overline{X}$ whose lengths do not exceed $l(u)$, and in this representation there is only one monomial of length $l(u)$ which has the form $\pm u(\overline{x}_1, \dots, \overline{x}_k)$;

2) $\omega L_X(\mathfrak{M})$ is generated as F-module by the normal reduced monomials from the set of generators $\overline{X}$ and decomposes into a direct sum of l-homogeneous submodules $\omega L_X(\mathfrak{M}) = \bigoplus_{i \in I} (\omega L_X(\mathfrak{M}))_i$.

Proof. 1) We will prove by induction on length $l(u)$. Let $x_1, x_2 \in X \cup X^{-1}$. We have $(1 - x_1)(1 - x_2) = 1 - x_1 - x_2 + x_1 x_2 = (1 - x_1) - (1 + x_2) - (1 - x_1 x_2)$, $1 - x_1 x_2 = (1 + x_1) + (1 + x_2) - (1 - x_1)(1 - x_2)$, $1 - x_1 x_2 = \overline{x}_1 + \overline{x}_2 - \overline{x}_1 \overline{x}_2$. Hence the statement of lemma for $l(u) = 2$ holds. Let us now consider the normal reduced loop word $u(x_1, \dots, x_k)$ of length $l(u) > 2$. We expand the expression $u(1 + x_1, \dots, 1 + x_k)$ and get

$$u(\overline{x}_1, \dots, \overline{x}_k) = 1 + \sum x_j + \sum v_2(x_{j_1}, x_{j_2}) + \dots$$
$$\dots + \sum v_r(x_{j_1}, \dots, x_{j_r}) + \dots + u(x_1, \dots, x_k), \tag{12}$$

where $v_r(x_{j_1}, \dots, x_{j_r})$ is a loop word, containing in its structure r $(r \le l(u) - 1)$ generators $x_{j_1}, \dots, x_{j_r} \in \{x_1, \dots, x_k\}$. We consider that the loop words $v_r(x_{j_1}, \dots \dots, x_{j_r})$ are reduced, as in the opposite case we can bring them to this form. It is easy to see by induction on k that the right part of the equality (12) contains even number of monomials. That is why $u(\overline{x}_1, \dots, \overline{x}_k)$ can be presented as a sum of terms of the form $1 - v_r(x_{j_1}, \dots, x_{j_r})$ or $1 - u(x_1, \dots, x_n)$. Then it follows from (12) that $1 - u(x_1, \dots, x_k) = \sum \epsilon (1 - v_r(x_{j_1}, \dots, x_{j_r})) + u(\overline{x}_1, \dots, \overline{x}_n)$, where $\epsilon = \pm 1, r \le l(u) - 1$. Using the inductive hypothesis for the monomials $1 - v_r(x_{j_1}, \dots, x_{j_r})$ we obtain from here the troth of 1).

2) We have proved above that the algebra $FL_X(\mathfrak{M})$ decomposes into a direct sum of l-homogeneous submodules $FL_X(\mathfrak{M}) = \bigoplus_{i\in I}(FL_X(\mathfrak{M}))_i$. From 6) of Lemma 1 and 1) of this lemma it follows that $\omega L_X(\mathfrak{M}) = \sum_{i\in I}(\omega L_X(\mathfrak{M}))_i$. Let $u(\overline{x}_1,\dots,\overline{x}_k) \in (\omega L_X(\mathfrak{M}))_i \cap (\omega L_X(\mathfrak{M}))_j$, where $i \neq j$. From the definition of a normal reduced word with respect to the set $\overline{X}$ it follows that $u(x_1,\dots,x_k) \in (\omega L_X(\mathfrak{M}))_i \cap (\omega L_X(\mathfrak{M}))_j$. From here it follows that $u(x_1,\dots,x_k) = 0$. Then $u(\overline{x}_1,\dots,\overline{x}_k) = 0$, as well. Hence $\omega L_X(\mathfrak{M}) = \bigoplus_{i\in I}(\omega L_X(\mathfrak{M}))_i$. This completes the proof of Lemma 7. □

Now, according to Theorem 1 we transfer the notions of length, l-homogeneity of polynomials and l-homogeneity submodules of augmentation ideal of loop algebra for polynomials of "augmentation ideal" of "loop algebra".

Lemma 8. *Let $\omega L_X(\mathfrak{M})$ be the "augmentation ideal" of "loop algebra" of free Moufang loop $L_X(\mathfrak{M})$. Then*

1) $\omega L_X(\mathfrak{M})$ decomposes into a direct sum of l-homogeneous submodules $\omega L_X(\mathfrak{M}) = \bigoplus_{i\in I}(\omega L_X(\mathfrak{M}))_i$;

2) the intersection of the l-homogeneous submodules of $\omega L_X(\mathfrak{M})$ is the zero.

Proof. Expanding the expression we obtain that $(1-a, 1-b, 1-c) = -(a,b,c)$. Then from the definition (5) of ideal $I(L_X(\mathfrak{M}))$ of loop algebra $FL_X(\mathfrak{M})$ it follows that this ideal is generated by elements of the form

$$v(d_1,\dots,d_k,\overline{w}_i,d_{k+1},\dots,d_m),$$

where $i = 1,2$ and $d_1,\dots,d_m$ are normal reduced words from $FL_X(\mathfrak{M})$. Now, by the relations $yz = z-(1-y)z,\ zy = z-z(1-y),\ yz = (1+y)z-z,\ zy = z(1+y)-z$, and by 1) of Lemma 4 it is easy to see that the ideal $I(L_X(\mathfrak{M}))$ is generated as F-module by l-homogeneous polynomials of the form

$$v(\overline{b}_1,\dots,\overline{b}_r,\overline{w}_i,\overline{b}_{r+1},\dots,\overline{b}_s), \tag{13}$$

where $\overline{b}_i$ are normal reduced monomials from $\omega L_X(\mathfrak{M})$ with respect respect to the set $\overline{X}$.

By 2) of Lemma 7 the augmentation ideal $\omega L_X(\mathfrak{M})$ decomposes into a direct sum of l-homogeneous submodules $\omega L_X(\mathfrak{M}) = \bigoplus_{i\in I}(\omega L_X(\mathfrak{M}))_i$. Then the ideal $I(L_X(\mathfrak{M}))$ decomposes into a direct sum of l-homogeneous submodules $I(L_X(\mathfrak{M})) = \bigoplus_{i\in I}(I(L_X(\mathfrak{M})))_i$ as well. From here it follows that the decomposition of algebra $\omega L_X(\mathfrak{M})$ into a direct sum of submodules $(\omega L_X(\mathfrak{M}))$ induces a similar decomposition also for the quotient algebra $\omega L_X(\mathfrak{M})/I(L_X(\mathfrak{M}))$: $\omega L_X(\mathfrak{M})/I(L_X(\mathfrak{M})) = \bigoplus_{i\in I}((\omega L_X(\mathfrak{M}))_i \cap I(FL_X(\mathfrak{M})))$, which by (6) is the "augmentation ideal" of the "loop algebra" $FL_X(\mathfrak{M})/FL_X(\mathfrak{M})$. This completes the proof of item 1). The item 2) follows from item 1) and item 2) of Lemma 7. □

Theorem 2. *The intersection of the terms of the lower central series of a free Moufang loop $L_X(\mathfrak{M})$ is the identity.*

Proof. We denote $L_X(\mathfrak{M}) = Q$, $\omega L_X(\mathfrak{M}) = B$. Let $Q = Q_0 \supseteq Q_1 \supseteq \ldots \supseteq Q_n \supseteq \ldots$ be the lower central series of free Moufang loop Q. We have to prove that

$$\cap_{n=0}^{\infty} Q_n = 1. \tag{14}$$

Really, let $B^0 = B, B^n = \sum_{i+j=n} B^i \cdot B^j$. By 2) of Lemma 8 it is easy to see that $\cap_{n=0}^{\infty} B^n = 0$. Further, $D_n = \{g \in Q | 1 - g \in B^n\}$ is a normal subloop of loop Q, as this is the kernel of homomorphism, induced by natural homomorphism $FL_X(\mathfrak{M}) \to FL_X(\mathfrak{M})/B^n$. From the relation $\cap_{n=0}^{\infty} B^n = 0$ it follows that $\cap_{n=0}^{\infty} D_n = 1$. Now to prove (14) it is sufficient to show that $Q_n \subseteq D_n$. We will prove this by induction on n. We have $Q_0 = Q = D_0$. Let $a, b \in Q$ and we suppose that the element $g_n \in Q_n$ belongs to D_n. Then $u_n = 1 - g_n \in B^n$, $v = 1 - a \in B^0, w = 1 - b \in B^0$. Any Moufang loop is an IP-loop. Then from (3) we get $1 - (g_n, a, b) = 1 - (g_n a \cdot b)(g_n \cdot ab)^{-1} = (g_n \cdot ab - g_n a \cdot b)(g_n \cdot ab)^{-1} = ((1 - g_n)((1 - a)(1 - b)) - (((1 - g_n)(1 - a))(1 - b))(g_n \cdot ab)^{-1} = (u_n \cdot vw - u_n v \cdot w)(g_n \cdot ab)^{-1} = (u_n \cdot vw - u_n v \cdot w) - (u_n \cdot vw - u_n v \cdot w)(1 - (g_n \cdot ab)^{-1}) \in B^{n+1}$. By analogy we prove that $1 - (a, b_n, b) \in B^{n+1}$, $1 - (a, b, g_n) \in B^{n+1}$, $1 - (g_n, a) \in B^{n+1}$. Then $(g_n, a, b), (a, g_n, b), (a, b, g_n), (g_n, a) \in D_{n+1}$. But as shown at the beginning of this paper elements of the form $(g_n, a, b), (a, g_n, b), (a, b, g_n), (g_n, a)$ generate the normal subloop Q_{n+1}. Then $Q_{n+1} \subseteq D_{n+1}$. Consequently, $\cap_{n=0}^{\infty} Q_n = 1$. This completes the proof of Theorem 2. □

We remind now that a loop Q is called a *Hopfian loop* and it has a Hopfian property if it can't be isomorphic to any of its quotient loop. Obviously, any finite loop is Hopfian, but no free loop of infinite rank F_∞ can be Hopfian. Really, if $x_1, x_2, \ldots, x_i, \ldots$ is a free generators for F_∞, then the map $x_1 \to 1, x_i \to x_{i-1}$ $(x > 1)$ defines an endomorphism on with non-trivial kernel.

Proposition 3. *A finitely generated centrally nilpotent Moufang loop L is Hopfian.*

Proof. Let us consider a normal subloop $N \neq 1$ of the loop L such that $\overline{L} = L/N$ is isomorphic to L. We must come to a contradiction. For that we will prove that no element $g \neq 1$ of the loop L can be mapped into the unit of the loop $\overline{L}$. In [8] it is proved that the loop L is residually finite. Then let K be a normal subloop of L of index n, not containing g. We denote by $K^\star$ the intersection of all normal subloops of L of index $\leq n$.Then the subloop $K^\star$ also has a finite index $n^\star$ in L and also doesn't contain g. Under a homomorphic mapping of L on $\overline{L}$, the subloop $K^\star$ is mapped on subloop $K^{\star\star}$ of loop $\overline{L}$. As the index of a finite loop is not augmented by a homomorphic mapping, $K^{\star\star}$ will contain subloop $\overline{K}^\star$ of loop $\overline{L}$, which corresponds to $K^\star \subseteq L$ under an isomorphic mapping of L on $\overline{L}$. In such a way the inverse image of $\overline{K}^\star$ in L (denoted by P) should be contained in $K^\star$. On the other hand, P contains N and, consequently, g is not mapped on 1 (under a natural homomorphism of L on $\overline{L}$). □

Lemma 9. *A loop L has a Hopfian property if and only if it has a set of fully invariant normal subloops, whose quotient loop has a Hopfian property and whose intersection is trivial.*

Proof. The necessity is trivial. To prove this, it is enough to denote by φ some endomorphism on loop L and by N we denote the fully invariant normal subloop of L, whose quotient loop is Hopfian. As $\varphi N \subseteq N$ and $\varphi L = L$ then φ induces an endomorphism on of L/N. According to the supposion, it is an automorphism of loop L/N, so that $\ker\varphi \subseteq N$. It means that the intersection at any set of such fully invariant subloops contains $\ker\varphi$. If the intersection is trivial, then $\ker\varphi$ is trivial and φ is an automorphism, as required. $\square$

Combining (14), Proposition 3 and Lemma 9 we get

Theorem 3. *Any finitely generated free Moufang loop is Hopfian.*

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On π-quasigroups isotopic to abelian groups

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Abstract. A π-quasigroup is a quasigroup satisfying one of the seven minimal identities from the V.Belousov's classification given in [1]. Some general results about π-quasigroups isotopic to groups are obtained by V. Belousov and A. Gwaramija in [1] and [2]. π-Quasigroups isotopic to abelian groups are investigated in this paper.

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Let Q be a nonempty set and let $\Sigma(Q)$ be the set of all binary quasigroup operations defined on Q. V. Belousov (see [1]) found all nontrivial identities $w_1 = w_2$ in $Q(\Sigma)$ having the length $|w_1| + |w_2| = 5$ ($|w|$ is the number of free elements in the word w), called minimal identities. He proved that, using transformation to inverse operations, every minimal identity can be transformed into the form:

$$A(x, B(x, C(x, y))) = y.$$

Using multiplication of operations, the last identity can be rewritten in abbreviated form as $ABC = E$, where $E(x, y) = y, \forall x, y \in Q$, is the right selector.

Minimal nontrivial identities imply the orthogonality of participating operations. It is known that two quasigroup operations A and B, defined on a set Q, are orthogonal if and only if there exists a quasigroup operation C on Q, such that $CBA^{-1} = E$ [1]. Hence, if $A, B, C \in Q(\Sigma)$ and $ABC = E$,we have $A \perp B^{-1}$, $B \perp C^{-1}$ and $C \perp A^{-1}$.

A quasigroup $Q(A)$ is called a π-quasigroup of type $[\alpha, \beta, \gamma]$, where $\alpha, \beta, \gamma \in S_3$, if it satisfies the identity ${}^{\alpha}A^{\beta}A^{\gamma}A = E$.

V. Belousov considered the following transformations of types on S_3^3: $f[\alpha, \beta, \gamma] = [\beta, \gamma, \alpha]$ and $h[\alpha, \beta, \gamma] = [r\gamma, r\beta, r\alpha]$, where $r = (23)$. The transformations f and h generate the group $S^0 = \{\varepsilon, f, f^2, h, fh, f^2h\} \cong S_3$. Two types $T = [\alpha, \beta, \gamma]$ and $T' = [\alpha', \beta', \gamma']$ are called parastrophically equivalent if there exist $g \in S^0$ and $\theta \in S_3$ such that $T' = gT\theta$. If the types T and T' are parastrophically equivalent then we'll denote $T \sim T'$. The binary relation " $\sim$ " is an equivalence on S_3^3 and $S_3^3/\sim$ consists of 7 classes [1]. A system of representatives of the seven equivalence classes is:

$T_1 = [\varepsilon, \varepsilon, \varepsilon]$, $T_2 = [\varepsilon, \varepsilon, l]$, $T_4 = [\varepsilon, \varepsilon, lr]$, $T_6 = [\varepsilon, l, lr]$, $T_{10} = [\varepsilon, lr, l]$,
$T_8 = [\varepsilon, rl, lr]$, $T_{11} = [\varepsilon, lr, rl]$, where $l = (13)$, $r = (23)$, $s = (12)$.

Two minimal identities

$${}^{\alpha}A(x, {}^{\beta}A(x, {}^{\gamma}A(x, y))) = y,$$

$$^{\alpha'}A(x,^{\beta'}A(x,^{\gamma'}A(x,y))) = y,$$

where $A \in \Sigma(Q)$, are called parastrophically equivalent if the types $T = [\alpha, \beta, \gamma]$ and $T' = [\alpha', \beta', \gamma']$ are parastrophically equivalent.

Denoting $A =$ " $\cdot$ " the following identities, which correspond to the seven types, respectively, were obtained by V. Belousov in [1]:

No.	Type	Identity in $Q(\cdot)$	
1	$T_1 = [\varepsilon, \varepsilon, \varepsilon]$	$x(x \cdot xy) = y$	
2	$T_2 = [\varepsilon, \varepsilon, l]$	$x(y \cdot yx) = y$	
3	$T_4 = [\varepsilon, \varepsilon, lr]$	$x \cdot xy = yx$	Stein's 1st law
4	$T_6 = [\varepsilon, l, lr]$	$xy \cdot x = y \cdot xy$	Stein's 2nd law
5	$T_{10} = [\varepsilon, lr, l]$	$xy \cdot yx = y$	Stein's 3d law
6	$T_8 = [\varepsilon, rl, lr]$	$xy \cdot y = x \cdot xy$	Schröder's 1st law
7	$T_{11} = [\varepsilon, lr, rl]$	$yx \cdot xy = y$	Schröder's 2nd law

The same classification was obtained independently by F. E. Bennett in 1989 (see, for example, [3] and [4]).

π-Quasigroups isotopic to groups have been investigated by V. Belousov in [1] and by V. Belousov and A. Gwaramija in [2]. In particular, they proved that the groups which are isotopic to π-quasigroups of type $T_4 = [\varepsilon, \varepsilon, lr]$ (i.e. to Stein quasigroups) or to π-quasigroups of type $T_6 = [\varepsilon, l, lr]$, are metabelian. Also it is proved in [1] that if a group $Q(+)$ is isotopic to a π-quasigroup of type $T_8 = [\varepsilon, rl, lr]$ then $Q(+)$ is abelian of exponent 2. More, every finite group of exponent 2 is isotopic to a π-quasigroup of type T_8. π-Quasigroups of other types, isotopic to groups, are considered in [1] as well. We'll consider below π-quasigroups isotopic to abelian groups.

Let $Q(\cdot)$ be a π-quasigroups of type $T_1 = [\varepsilon, \varepsilon, \varepsilon]$, i.e. a quasigroup satisfying the identity

$$x(x \cdot xy) = y. \tag{1}$$

Such quasigroups are also called C_3-quasigroups. Suppose that $Q(\cdot)$ is isotopic to an abelian group, and for $a, b \in Q$ consider the LP-isotopes $(\cdot)^{(R_a^{-1}, L_b^{-1}, \varepsilon)}$ and $(+) = (\cdot)^{(R_0^{-1}, L_{f_0}^{-1}, \varepsilon)}$ where $0 = b \cdot a$ and $f_0 \cdot 0 = 0$. According to Albert's theorem, these two LP-isotopes are abelian groups (as loops which are isotopic to groups), so $Q(+)$ where $x + y = R_0^{-1}(x) \cdot L_{f_0}^{-1}(y)$, for every $x, y \in Q$, is an abelian group with the neutral element $0 = f_0 \cdot 0$. Let denote now $L_{f_0}^{-1}$ by λ. Then $x + y = R_0^{-1}(x) \cdot \lambda(y)$ and $x \cdot y = R_0(x) + \lambda^{-1}(y)$, for every $x, y \in Q$, so the identity (1) takes the form $R_0(x) + \lambda^{-1}(R_0(x) + \lambda^{-1}(R_0(x) + \lambda^{-1}(y))) = y$ or, after replacing $R_0(x)$ by x:

$$x + \lambda^{-1}(x + \lambda^{-1}(x + \lambda^{-1}(y))) = y. \tag{2}$$

Taking $x = 0$, from (2) it follows $\lambda^3 = \varepsilon$. Also the equality (2) implies $x + \lambda^{-1}(x + \lambda^{-1}(y)) = \lambda(I(x) + y)$ or, replacing $\lambda^{-1}(y)$ by y:

$$x + \lambda^{-1}(x + y) = \lambda(I(x) + \lambda(y)), \tag{3}$$

where $I : Q \to Q$, $I(x) = -x$ (in the abelian group $Q(+)$). Taking $y = 0$, (3) implies

$$x + \lambda^{-1}(x) = \lambda I(x), \tag{4}$$

for every $x \in Q$, as $\lambda(0) = L_{f_0}^{-1}(0) = 0$.

Let consider now a new operation on Q denoted by " $\circ$ " and defined as follows:

$$x \circ y = \lambda(x) + x + I(y), \tag{5}$$

$\forall x, y \in Q$.

Proposition 1. *The grupoid $Q(\circ)$ is a quasigroup isotopic to $Q(+)$.*

Proof. From (4) it follows $\lambda^{-1}(x) = \lambda I(x) + I(x)$, $\forall x \in Q$ so $\lambda^{-1}I(x) = \lambda(x) + x, \forall x \in Q$, and then $x \circ y = \lambda(x) + x + I(y) = \lambda^{-1}I(x) + I(y)$, $\forall x, y \in Q$, i.e. $(\circ) = (+)^{(\lambda^{-1}I,I,\varepsilon)}$. □

Proposition 2. *Let $Q(\cdot)$ be a π-quasigroup of type T_1, isotopic to an abelian group and let $Q(+)$ and $Q(\circ)$ be its isotopes defined above. The following conditions are equivalent:*

1. *$\lambda I = I\lambda$;*
2. *$\lambda \in AutQ(+)$;*
3. *$\lambda \in AutQ(\circ)$;*
4. *$I \in AutQ(\circ)$;*
5. *$Q(+)$ satisfies the equality $\lambda^2(x) + \lambda(x) + x = 0$, $\forall x \in Q$;*
6. *$Q(\circ)$ is a medial quasigroup.*

Proof. 1.⇒ 2.: If $\lambda I = I\lambda$ then from (3) and (4) it follows $x+\lambda^{-1}(x+y) = \lambda(I(x)+\lambda(y)) = \lambda I(x+I\lambda(y)) = \lambda I(x+\lambda I(y)) = \lambda I(x+y+\lambda^{-1}(y)) = x+y+\lambda^{-1}y+\lambda^{-1}(x+y+\lambda^{-1}(y)) = x + \lambda I(y) + \lambda^{-1}(x + \lambda I(y))$, so $\lambda^{-1}(x + y) = \lambda I(y) + \lambda^{-1}(x + \lambda I(y))$, which implies

$$\lambda^{-1}(x + y) + \lambda(y) = \lambda^{-1}(x + \lambda I(y)).$$

Denoting $x + y$ by z, from the last equality it follows $\lambda^{-1}(z) + \lambda(y) = \lambda^{-1}(z + I(y) + \lambda I(y)) = \lambda^{-1}(z + \lambda^{-1}(y))$, so replacing y by $\lambda(y)$ and using the equality $\lambda^3 = \varepsilon$, we get: $\lambda^{-1}(z) + \lambda^{-1}(y) = \lambda^{-1}(z + y)$, $\forall z, y \in Q$, i.e. $\lambda \in \mathrm{Aut}Q(+)$.

2.⇒ 1.: If $\lambda \in \mathrm{Aut}Q(+)$ then $\lambda(-x) = -x$, $\forall x \in Q$, i.e. $\lambda I = I\lambda$.

1.⇒ 3.: Using Proposition 1, we get: $\lambda I = I\lambda \Rightarrow \lambda \in \mathrm{Aut}Q(+)$, so $\lambda(x \circ y) = \lambda(\lambda^{-1}I(x) + I(y)) = I(x) + \lambda I(y) = \lambda^{-1}I\lambda(x) + I\lambda(y) = \lambda(x) \circ \lambda(y)$, $\forall x, y \in Q$, so $\lambda \in \mathrm{Aut}Q(\circ)$.

3.⇒ 1.: $\lambda \in \mathrm{Aut}Q(\circ) \Leftrightarrow \lambda(x \circ y) = \lambda(x) \circ \lambda(y)$, $\forall x, y \in Q \Leftrightarrow \lambda(\lambda^{-1}I(x) + I(y)) = \lambda^{-1}I\lambda(x) + I\lambda(y)$, $\forall x, y \in Q$. Taking $x = 0$, the last equality implies $\lambda I(y) = I\lambda(y)$, $\forall y \in Q$, i.e. $\lambda I = I\lambda$.

1.$\Leftrightarrow$ 5.: Using (4), we have: $\lambda I = I\lambda \Leftrightarrow x + \lambda^{-1}(x) = I\lambda(x)$, $\forall x \in Q$, $\Leftrightarrow$ $\lambda^2(x) + \lambda(x) + x = 0$, $\forall x \in Q$.

1.$\Rightarrow$ 4.: According to Proposition 1, $x \circ y = \lambda^{-1}I(x) + I(y)$, $\forall x, y \in Q$. If $\lambda I = I\lambda$, then $I(x \circ y) = I(\lambda^{-1}I(x) + I(y)) = \lambda^{-1}(x) + y$, $\forall x, y \in Q$, and $I(x) \circ I(y) = \lambda^{-1}I(I(x)) + I(I(y)) = \lambda^{-1}(x) + y$, $\forall x, y \in Q$, so $I(x \circ y) = I(x) \circ I(y)$, $\forall x, y \in Q$, i.e. $I \in \mathrm{Aut}Q(\circ)$.

4.$\Rightarrow$ 1.: If $I \in \mathrm{Aut}Q(\circ)$, then $I(x \circ y) = I(x) \circ I(y)$, $\forall x, y \in Q$, $\Rightarrow I(\lambda^{-1}I(x) + I(y)) = \lambda^{-1}\, II(x) + I(I(y)) = \lambda^{-1}(x) + y, \Rightarrow I\lambda^{-1}I(x) + y = \lambda^{-1}(x) + y$, $\forall x, y \in Q$, $\Rightarrow I\lambda^{-1}I = \lambda^{-1}, \Rightarrow I\lambda = \lambda I$.

6.$\Rightarrow$ 1.: Remark that from (5) it follows $x \circ x = \lambda(x), \forall x \in Q$. If $Q(\circ)$ is a medial quasigroup, i.e. if $Q(\circ)$ satisfies the identity $(x \circ y) \circ (u \circ v) = (x \circ u) \circ (y \circ v)$, then $\lambda(x \circ y) = (x \circ y) \circ (x \circ y) = \lambda(x) \circ \lambda(y), \Rightarrow \lambda \in \mathrm{Aut}Q(\circ) \Rightarrow \lambda I = I\lambda$.

1.$\Rightarrow$ 6.: If $\lambda I = I\lambda$, then $\lambda \in \mathrm{Aut}Q(+)$, so $\lambda I, I \in \mathrm{Aut}Q(+)$ where $Q(+)$ is an abelian group and $(\lambda^{-1}I)I = \lambda^{-1} = I(\lambda^{-1}I)$, i.e. $Q(\circ)$, where $x \circ y = \lambda^{-1}I(x) + I(y)$, $\forall x, y \in Q$, is a medial quasigroup. □

Proposition 3. *Let $Q(\cdot)$ be an isotope of an abelian group, $0 \in Q$, $f_0 \cdot 0 = 0$, $\lambda = L_{f_0}^{-1}$, $(+) = (\cdot)^{(R_0^{-1}, \lambda, \varepsilon)}$, and let $\lambda I = I\lambda$, where $I : Q \to Q, I(x) + x = 0$. Then $Q(\cdot)$ is a π-quasigroup of type T_1 if and only if $Q(+)$ satisfies the condition $\lambda^2(x) + \lambda(x) + x = 0$, $\forall x \in Q$.*

Proof. If $Q(\cdot)$ is a π-quasigroup of type T_1, isotopic to an abelian group and $\lambda I = I\lambda$ then, according to Proposition 2, $Q(+)$ satisfies the condition $\lambda^2(x) + \lambda(x) + x = 0$, $\forall x \in Q$.

Conversely, if $\lambda^2(x) + \lambda(x) + x = 0$, $\forall x \in Q$, then $\lambda \in \mathrm{Aut}Q(+)$ and $\lambda^2 + \lambda + \varepsilon = \omega$, where $\omega : Q \to Q, \omega(x) = 0$, $\forall x \in Q$, $\Rightarrow \lambda^3 - \varepsilon = (\lambda - \varepsilon)(\lambda^2 + \lambda + \varepsilon) = (\lambda_\varepsilon)\omega = \omega$ (in the ring of endomorphisms of $Q(+)$), as $\lambda^3 = \varepsilon$. Moreover, $\lambda^2(x) + \lambda(x) + x = 0$, $\forall x \in Q \Rightarrow \lambda^2(x) + \lambda(x) + x + y = y$, $\forall x, y \in Q$, $\Rightarrow x + \lambda^{-1}(x + \lambda^{-1}(x + \lambda^{-1}(y))) = y$, $\forall x, y \in Q$, $\Rightarrow x(x \cdot xy) = y$, $\forall x, y \in Q$ (see (2)), so $Q(\cdot)$ is a π-quasigroup of type T_1. □ □

Corollary. *Let $Q(\cdot)$ be an isotope of an abelian group $Q(+) \cong Z_2^k$, for some positive integer k, with the isotopy $(R_0^{-1}, L_{f_0}^{-1}, \varepsilon)$, where 0 is the neutral element of $Q(+)$ and $f_0 \cdot 0 = 0$. Then $Q(\cdot)$ is a π-quasigroup of type T_1 if and only if $Q(+)$ satisfies the condition $\lambda^2(x) + \lambda(x) + x = 0$, $\forall x \in Q$, where $\lambda = L_{f_0}^{-1}$.*

Proof. Indeed, in this case $I = \varepsilon$, so $\lambda I = I\lambda$. □

Proposition 4. *Let $Q(\cdot)$ be a π-quasigroup of type T_1, isotopic to an abelian group $Q(+)$, $(+) = (\cdot)^{(R_0^{-1}, \lambda, \varepsilon)}$ where $0 \in Q$, $f_0 \cdot 0 = 0$, $\lambda = L_{f_0}^{-1}$. The following conditions are equivalent:*

1. *$Q(\cdot)$ has a left unit;*
2. *$Q(\circ)$, where "$\circ$" is defined in (5), is idempotent;*
3. *$Q(+)$ satisfies the equality $x + x + x = 0$, $\forall x \in Q$.*

Proof. 1.$\Leftrightarrow$ 2.: According to the definition (5), $x \circ x = \lambda(x) + x + I(x)$, $\forall x \in Q$. So $x \circ x = x, \forall x \in Q \Leftrightarrow \lambda = \varepsilon \Leftrightarrow \lambda^{-1} = \varepsilon \Leftrightarrow L_{f_0}(x) = x, \forall x \in Q \Leftrightarrow f_0 \cdot x = x, \forall x \in Q$, i.e. $Q(\cdot)$ has the left unit f_0.

2.$\Leftrightarrow$ 3.: $x \circ x = x, \forall x \in Q \Leftrightarrow \lambda = \varepsilon \Leftrightarrow x + x = x + \lambda^{-1}(x) = \lambda I(x) = I(x)$, $\forall x \in Q$ (see (4)), i.e. $x + x + x = 0, \forall x \in Q$. $\square$

Denote Id$Q(\circ) = \{x \in Q \mid x \circ x = x\}$, i.e. the set of all idempotents of $Q(\circ)$.

Proposition 5. *If $\lambda I = I\lambda$, then IdQ$(\circ)$ is a subquasigroup of $Q(\circ)$.*

Proof. If $\lambda I = I\lambda$, then $\lambda \in$ Aut$Q(+)$, so for every $x, y \in$ Id$Q(\circ)$ we have:

$$(x \circ y) \circ (x \circ y) = \lambda(x \circ y) = \lambda(x) \circ \lambda(y) = (x \circ x) \circ (y \circ y) = x \circ y,$$

i.e. $x \circ y \in$ Id$Q(\circ)$. Moreover, if $a, b \in$ Id$Q(\circ)$ and $a \circ x = b$, then (as $Q(\circ)$ is a medial quasigroup) we have:

$$a \circ (x \circ x) = (a \circ a) \circ (x \circ x) = (a \circ x) \circ (a \circ x) = b \circ b = b,$$

hence $x \circ x = x$, i.e. the solution x of the equation $a \circ x = b$ is in Id$Q(\circ)$, for every $a, b \in$ Id$Q(\circ)$. Analogously we get that the solution of the equation $x \circ a = b$ belongs to Id$Q(\circ)$, for every $a, b \in$ Id$Q(\circ)$. $\square$

Remark. If $\lambda I = I\lambda$, then Id$Q(\circ) \subseteq \{x \in Q \mid x + x + x = 0\}$.

Proof. Indeed, if $x \in$ Id$Q(\circ)$, then $x = x \circ x = \lambda(x) + x + I(x) = \lambda(x)$, $\forall x \in Q$. On the other hand, from (4) it follows $x + x = x + \lambda^{-1}(x) = \lambda I(x) = I(x)$, $\forall x \in Q$ $\Rightarrow x + x + x = 0$, $\forall x \in Q$. $\square$

Proposition 6. *If $|Q| < \infty$, then IdQ$(\circ) = \{0\}$ if and only if $\lambda I = I\lambda$.*

Proof. If $\lambda I = I\lambda$ and $x \in$ Id$Q(\circ) \backslash \{0\}$, then x is an element of order 3 in $Q(+)$ (see the remark above). But it is known that there exist the following possibilities for the order $|Q|$ of a finite π-quasigroup of type T_1: $|Q| = 4$, $|Q| \equiv 1$ or $4(\mathrm{mod}12)$, or $|Q| \equiv 1(\mathrm{mod}3)$, i.e. $|Q|$ is not divisible by 3. Consequently, if $\lambda I = I\lambda$, then Id$Q(\circ) = \{0\}$.

Conversely, let Id$Q(\circ) = \{0\}$ and $|Q| < \infty$. As Ker$(\lambda - \varepsilon) = \{x \in Q | \lambda(x) = x\}$ $= \{x \in Q | x \circ x = x\}$, we have: $(\lambda - \varepsilon)(x) = (\lambda - \varepsilon)(y) \Rightarrow \lambda(x - y) = x - y \Rightarrow x - y \in$ Ker$(\lambda - \varepsilon) \Rightarrow x - y = 0 \Rightarrow x = y$, hence $\lambda - \varepsilon$ is injective and, as Q is finite, it follows that $\lambda - \varepsilon$ is a bijection. On the other hand, $\lambda^3 = \varepsilon \Rightarrow \omega = \lambda^3 - \varepsilon = (\lambda - \varepsilon)(\lambda^2 + \lambda + \varepsilon)$ $\Rightarrow \lambda^2 + \lambda + \varepsilon = (\lambda - \varepsilon)^{-1}\omega = \omega$, where $\omega : Q \to Q, \omega(x) = 0, \forall x \in Q$, hence according to Proposition 2, $\lambda I = I\lambda$. $\square$

Example. The quasigroup $Q(\cdot)$, where $Q = \{0, 1, 2, 3\}$ and

$\cdot$	0	1	2	3
0	3	1	0	2
1	0	2	3	1
2	1	3	2	0
3	2	0	1	3

is a π-quasigroup of type T_1 and is isotopic to the Klein group $K_4 = Q(+)$ (0 is the neutral element of $Q(+)$): $x \cdot y = R_0(x) + \lambda^{-1}(y)$, where $R_0 = (0321), \lambda = (132)$. Remark that the quasigroup $Q(\circ)$, where $x \circ y = \lambda(x) + x + I(y)$, is defined by the following table:

$\circ$	0	1	2	3
0	0	1	2	3
1	2	3	0	1
2	3	2	1	0
3	1	0	3	2

As $I = \varepsilon$ we have $\lambda I = I\lambda$, hence $\lambda \in \mathrm{Aut}Q(+)$ and $\lambda \in \mathrm{Aut}Q(\circ)$. The conditions $|Q| < \infty$ and $\lambda I = I\lambda$ give $\mathrm{Id}Q(\circ) = \{0\}$.

Proposition 7. *If a π-quasigroup $Q(\cdot)$ of type $T_2 = [\varepsilon, \varepsilon, l]$ is isotopic to an abelian group $Q(\oplus)$, then for every $b \in Q$ there exists an isomorphic copy $Q(+) \cong Q(\oplus)$ such that $x \cdot y = IL_b^3(x) + L_b(y) + b$, $\forall x, y \in Q$, where $I : Q \to Q$, $I(x) = -x$, $\forall x \in Q$.*

Proof. Let $Q(\cdot)$ be a π-quasigroup of type $T_2 = [\varepsilon, \varepsilon, l]$, isotopic to an abelian group. Then, for every $a, b \in Q$, the LP-isotope $Q(+)$, where $x + y = R_a^{-1}(x) + L_b^{-1}(y)$, $\forall x, y \in Q$, is an abelian group as well. Denote its neutral element $b \cdot a$ by 0. The quasigroup $Q(\cdot)$ satisfies the identity

$$x(y \cdot yx) = y. \tag{6}$$

Using the definition of " + ", the identity (6) takes the form $R_a(x) + L_b(R_a(y) + L_b(R_a(y) + L_b(x))) = y$ or, after replacing $y \to R_a^{-1}(y)$ and $x \to L_b^{-1}(x)$: $R_aL_b^{-1}(x) + L_b(y + L_b(y + x)) = R_a^{-1}(y)$, which implies:

$$L_b(y + L_b(y + x)) = R_a^{-1}(y) + IR_aL_b^{-1}(x). \tag{7}$$

Taking $y = 0$ in (7) we get $L_b^2(x) = b + IR_aL_b^{-1}(x) \Rightarrow L_b^3(x) = b + IR_a(x), \forall x \in Q, \Rightarrow R_a(x) = b + IL_b^3(x), \forall x \in Q$, $x \cdot y = IL_b^3(x) + L_b(y) + b$, $\forall x, y \in Q$. $\square$

Proposition 8. *A quasigroup $Q(\cdot)$, isotopic to a group $Q(\oplus)$ and having an idempotent 0, is a π-quasigroup of type $T_2 = [\varepsilon, \varepsilon, l]$ if and only if there exists an isomorphic copy $Q(+) \cong Q(\oplus)$ such that $x \cdot y = IL_0^3(x) + L_0(y)$ and $L_0(y + L_0(y + x)) = L_0^2(x) + L_0^{-3}I(y)$, for every $x, y \in Q$.*

Proof. If $Q(\cdot)$ is a π-quasigroup of type $T_2 = [\varepsilon, \varepsilon, l]$ and 0 is an idempotent of $Q(\cdot)$, then the LP-isotope $(+) = (\cdot)^{(R_0^{-1}, L_0^{-1}, \varepsilon)}$ is a group with unit 0. Using the definition of " + " the identity $x(y \cdot yx) = y$ takes the form:

$$R_0(x) + L_0(R_0(y) + L_0(R_0(y) + L_0(x))) = y,$$

or, after replacing $R_0(y)$ by y and $L_0(x)$ by x:

$$L_0(y + L_0(y + x)) = IR_0L_0^{-1}(x) + R_0^{-1}(y). \tag{8}$$

For $y = 0$ the last equality implies $L_0^2(x) = IR_0L_0^{-1}(x)$ for every $x \in Q$, so $R_0 = IL_0^3$ and then (8) implies

$$L_0(y + L_0(y + x)) = L_0^2(x) + L_0^{-3}I(y).$$

At the same time we get that $x \cdot y = IL_0^3(x) + L_0(y)$.

Conversely, let $Q(\cdot)$ be the quasigroup defined by the last equality, where $Q(+)$ is a group, 0 is an idempotent of $Q(\cdot)$ and let the equality $L_0(y + L_0(y + x)) = L_0^2(x) + L_0^{-3}I(y)$ holds. Then $x(y \cdot yx) = IL_0^3(x) + L_0(IL_0^3(y) + L_0(IL_0^3(y) + L_0(x))) = IL_0^3(x) + L_0^3(x) + L_0^{-3}I^2L_0^3(y) = y$, i.e. $Q(\cdot)$ is a π-quasigroup of type T_2. $\square$

It was proved in [2] by V.Belousov and A.Gwaramiya that every group G which is isotopic to a π-quasigroup of type $T_4 = [\varepsilon, \varepsilon, lr]$ (i.e. to a Stein quasigroup) is metabelian (i.e. $[x, y] \in Z$ for every $x, y \in G$). It was also proved by V.Belousov in [1] that if a group $Q(\cdot)$ is isotopic to a π-quasigroup of type $T_6 = [\varepsilon, l, lr]$, then $Q(\cdot)$ is metabelian.

Proposition 9. *If a π-quasigroup $Q(\cdot)$ of type $T_6 = [\varepsilon, l, lr]$ is isotopic to an abelian group $Q(\oplus)$, then there exists an element $0 \in Q$ and an isomorphic copy $Q(+) \cong Q(\oplus)$ such that $x \cdot y = R_0(x) + \varphi R_0(y), \forall x, y \in Q$, where $\varphi \in AutQ(+)$.*

Proof. Let $Q(\cdot)$ be a π-quasigroup of type $T_6 = [\varepsilon, l, lr]$, i.e. let $Q(\cdot)$ be a quasigroup with the identity

$$x \cdot yx = yx \cdot y. \tag{9}$$

Then for $y = f_x$, where $f_x x = x, \forall x \in Q$, we have $x^2 = x \cdot f_x \Rightarrow f_x = x \Rightarrow x = f_x \cdot x = x \cdot x$, i.e. $Q(\cdot)$ is idempotent. Let $0 \in Q$ and consider the LP-isotope $(+) = (\cdot)^{(R_0^{-1}, L_0^{-1}, \varepsilon)}$. It is clear that $Q(+)$ is an abelian group with the neutral element $0 = 0 \cdot 0$. Now, using the equality $x \cdot y = R_0(x) + L_0(y)$, the identity (9) takes the form

$$R_0(x) + L_0(R_0(y) + L_0(x)) = R_0(R_0(y) + L_0(x)) + L_0(y),$$

$\forall x, y \in Q$, hence replacing $R_0(y) \to y$ and $L_0(x) \to x$, we get

$$R_0L_0^{-1}(x) + L_0(y + x) = R_0(y + x) + L_0R_0^{-1}(y),$$

which implies

$$L_0(y+x) + IR_0(y+x) = IR_0L_0^{-1}(x) + L_0R_0^{-1}(y). \tag{10}$$

Taking $x = 0$ in (10) we get:

$$L_0(y) + IR_0(y) = L_0R_0^{-1}(y) \tag{11}$$

for all $y \in Q$. For $y = I(x)$, the equality (10) implies $0 = IR_0L_0^{-1}(x) + L_0R_0^{-1}I(x), \forall x \in Q$, i.e. $R_0L_0^{-1} = L_0R_0^{-1}I$. Denoting $L_0R_0^{-1}$ by φ, we get $\varphi I = \varphi^{-1}$ and $L_0 = \varphi R_0$, so $x \cdot y = R_0(x) + \varphi R_0(y), \forall x, y \in Q$. On the other hand, using (11), the equality (10) takes the form $L_0R_0^{-1}(y+x) = IR_0L_0^{-1}(x) + L_0R_0^{-1}(y)$, i.e. $\varphi(x+y) = I\varphi^{-1}(x) + \varphi(y) = I\varphi I(x) + \varphi(y)$, $\forall x, y \in Q$.

As $\varphi(0) = L_0R_0^{-1}(0)$, taking $y = 0$ in the last equality, we get $\varphi = I\varphi^{-1}$, so $\varphi(x+y) = \varphi(x) + \varphi(y), \forall x, y \in Q$, i.e. $\varphi \in \mathrm{Aut}Q(+)$. □

Proposition 10. *Let $Q(+)$ be an abelian group with the neutral element 0, $\varphi \in AutQ(+)$ and $\varphi^2 = I$, where $I(x) = -x, \forall x \in Q$. If the isotope $Q(\cdot)$, where $(+) = (\cdot)^{(R_0^{-1}, R_0^{-1}\varphi^{-1}, \varepsilon)}$, is idempotent then $Q(\cdot)$ is a π-quasigroup of type T_6.*

Proof. Using the definition of "$\cdot$", we have $x \cdot y = R_0(x) + \varphi R_0(y)$, $\forall x, y \in Q$. If $Q(\cdot)$ is idempotent then $z \cdot z = z, \forall z \in Q$, so $R_0(z) + \varphi R_0(z) = z, \Rightarrow z + IR_0(z) = \varphi R_0(z), \forall z \in Q$. Taking $z = y + \varphi(x)$ in the last equality, we get $y + \varphi(x) + IR_0(y + \varphi(x)) = \varphi R_0(y + \varphi(x)) \Rightarrow \varphi(x) + IR_0(y + \varphi(x)) = \varphi R_0(y + \varphi(x)) + I(y) \Rightarrow \varphi(x) + \varphi^2 R_0(y + \varphi(x)) = \varphi R_0(y + \varphi(x)) + \varphi^2(y) \Rightarrow x + \varphi R_0(y + \varphi(x)) = R_0(y + \varphi(x)) + \varphi(y)$, $\forall x, y \in Q$. Now, replacing $x \to R_0(x)$ and $y \to R_0(y)$, we get:

$$R_0(x) + \varphi R_0(R_0(y) + \varphi R_0(x)) = R_0(R_0(y) + \varphi R_0(x)) + \varphi R_0(y),$$

$\forall x, y \in Q$, i.e. $x \cdot yx = yx \cdot y, \forall x, y \in Q$. So $Q(\cdot)$ is a π-quasigroup of type T_6. □

Proposition 11. *If $Q(\cdot)$ is a π-quasigroup of type $T_{10} = [\varepsilon, lr, l]$, isotopic to an abelian group, $a \in Q$ and $(+) = (\cdot)^{(R_a^{-1}, L_a^{-1}, \varepsilon)}$, then there exists a complete substitution θ of $Q(+)$ such that $x \cdot y = R_a x + R_a^{-1} I\theta y$, for every $x, y \in Q$, where $Ix = -x$, $\forall x \in Q$.*

Proof. The quasigroup $Q(\cdot)$ satisfies the identity $xy \cdot yx = y$ so, using the equality $x \cdot y = R_a x + L_a y$, we get $R_a(R_a x + L_a y) + L_a(R_a y + L_a x) = y$ or, after replacing $R_a x$ by x and $L_a y$ by y:

$$R_a(x+y) + L_a(R_a L_a^{-1}(y) + L_a R_a^{-1}(x)) = L_a^{-1}(y). \tag{12}$$

Taking $x = a^2$ (the unit of the group $Q(+)$), from (12) it follows:

$$R_a(y) + L_a R_a L_a^{-1}(y) = L_a^{-1}(y), \tag{13}$$

or, replacing y by $L_a(y)$:

$$R_aL_a(y) + L_aR_a(y) = y. \tag{14}$$

Now, taking $y = a^2$ in (12), we have: $R_ax + L_a^2R_a^{-1}x = a$ and, replacing x by R_ax in the last equality, we get $R_a^2x + L_a^2x = a$. From (14) it follows $y + IR_aL_a(y) = L_aR_a(y)$, $\forall y \in Q$, where $I(x) = -x$, $\forall x \in Q$, so IR_aL_a is a complete substitution of $Q(+)$. Finally, denoting IR_aL_a by θ, we get $L_a = R_a^{-1}I\theta$ and $x \cdot y = R_ax + R_a^{-1}I\theta y$, $\forall x, y \in Q$. □

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Academician Vladimir Arnautov – 70th anniversary

Academician Vladimir Arnautov is a Moldavian mathematician and an irrefutable leader of the Moldavian school of topological algebra, which made an important contribution to the topological algebra and to the education of new generations of highly-qualified specialists. In the middle of the summer of 2009 professor Vladimir Arnautov turned 70. This was an excellent opportunity for us, his colleagues and friends, to stop on our own mathematical and personal path and to bring once more into light his life and activity up to this moment.

Vladimir Arnautov was born July 30, 1939 in Bolgrad (Romania, now Ukraine), being the second son (of six children) in a Bulgarian family. His father Ivan Stepanovich Arnautov was a technician at the local communication service. His mother Vera Simionovna Arnautova (Stadnitskaya) was a housewife and her principal occupation was the education of children.

Bolgrad (abbreviated from of Bolgarian town, which was called, until 1818, Tabacu) a little town in the south part of Bassarabia was in the 19th century the residence of the Bassarabian Bulgarians evacuated from Bulgaria who had been subjugated by Ottoman Empire until the 19th century.

In 1956 he successfully finished the local secondary (ten-year) school. The mathematical form and formulas, logical deduction charmed him, and without doubt he decided to continue the mathematical studies. In 1956 he started his university education at the Faculty of Physics and Mathematics (now Faculty of Mathematics and Computer Sciences) of Chişinău State University (now State University of Moldova).

During the student days he was influenced by the talented teachers and schoolarls as Professors C. Sibirschi, A. Zamorzaev, B. Shcherbacov, I. Parovicenco, C. Shchukin and other. Soon he became one of the best students. At the same time he actively contributed to various public organizations. In particular, 1959-1961 he was the Chairman of the Council of the Student Scientific Society of the University. In 1961 V. Arnautov successfully graduated from the Chişinău University. His Master Thesis *"About some classes of completely regular spaces"* was published in the Scientific Notes of the Chişinău University, Mathematics, vol. 1, 1962, p.13-18. His scientific and active public involvement made possible the obtaining of the University Academic Council recommendation for continuing his post-graduate advanced studies in Mathematics at the newly-created Institute of Mathematics and Physics (now Institute of Mathematics and Computer Sciences and Institute of Applied Physics). At the Institute a good fortune has brought him together with the Academician Vladimir Andrunachievici, one of the founders of the Academy of Sciences of Moldova, the first director of the Institute of Mathematics and Physics and the founder of the Moldavian algebraical school. V. Andrunachievici became scientific supervisor of the young scientist.

Academician V. Andrunachievici was one of the best specialists in the abstract theory of radicals. The scientific interest of the supervisor and his own topological knowledge determined the direction of Arnautov's further mathematical investigations: the theory of radicals of the topological rings.

The abstract theory of radicals had already been applied in the theory of topological rings in some works of I. Kaplansky, H. Leptin, D. Zelinsky and other mathematicians. However, they left out of account that these radicals, as a rule, are not closed in the topological rings.

First of all, V. Arnautov proposed the concept of the topological radical.

Let Φ be an associative and commutative topological ring and $\mathcal{K}$ be a class of topological algebras over the topological ring Φ with the following properties:

– if A is an ideal of some algebra $B \in \mathcal{K}$, then $A \in \mathcal{K}$;

– if $B \in K$ and A is a closed ideal of B, then the factor-algebra $B/A \in \mathcal{K}$.

If Φ is the discrete ring of integers $\mathbb{Z}$, then $\mathcal{K}$ is a class of topological rings. We assume that any ideal of the algebra B is a subalgebra of the algebra B.

We say that a radical ρ is defined over the class $\mathcal{K}$ if ρ is a correspondence of $\mathcal{K}$ into $\mathcal{K}$ for which:

1R. $\rho(A)$ is a closed ideal of the algebra $A \in \mathcal{K}$ (it is called the ρ-radical of A).

2R. $\rho(\rho(A)) = \rho(A)$ for any $A \in \mathcal{K}$.

3R. If $A, B \in \mathcal{K}$ and $\varphi : A \to B$ is a continuous homomorphism, then $\varphi(\rho(A)) \subseteq \rho(B)$.

4R. If $A \in \mathcal{K}$, then $\rho(A/\rho(A)) = \{0\}$.

Let ρ be a radical over the given class $\mathcal{K}$ of topological algebras. If $A \in \mathcal{K}$ and $\rho(A) = \{0\}$, then the algebra A is called ρ-semisimple. If $\rho(A) = A$, then A is called a ρ-radical algebra. Thus the factor-algebra $A/\rho(A)$ is the replica of the algebra $A \in \mathcal{K}$ in the class $\mathcal{K}_\rho$ of all ρ-semisimple algebras from $\mathcal{K}$.

If $\mathcal{K}$ is a class of discrete algebras, then each topological radical over the class $\mathcal{K}$ is an abstract radical and vice versa.

Consequently, many facts and notions of the abstract theory are their analogues for the topological case. However:

– for some abstracts radicals there exist a more than one topological variants;

– in some classes of topological algebras distinct radicals can coincide.

Therefore, the topological theory of radicals is a new fundamental area of mathematics with new concepts and techniques which have important and fruitful applications in other branches of algebra, topology and mathematics, in general. V. Arnautov has introduced a number of new notions and gave a number of original and deep results. The following problem was arisen one of first.

Problem 1. *Find the radical properties of distinct classes of topological rings.*

A property $\mathcal{P}$ is a radical property in the class $\mathcal{K}$ of topological rings if there exists a radical ρ such that $\rho(A) = A$ if and only if $A \in \mathcal{K}$ is a topological ring with the property $\mathcal{P}$.

One of the first remarkable results was: the property of a topological ring to contain a non-zero topological nilpotent ideal in any non-trivial continuous homomorphic image is a radical property. This radical was named the Baer-McCoy or the Baer radical. That radical was comprehensively studied in the thesis *"On the theory of topological rings"* for a doctor's degree which was defended in March 1965 at the Institute of Mathematics of the Siberian Branch of the Academy of Sciences of USSR.

For a long time that property was the only non-trivial radical property in the class of all topological rings. Then various radical properties were found in the classe of all topological rings, including:

– the property of a topological ring to be a locally nilpotent ring generates the Levitzky radical;

– to be a nil ring is a radical property which generates the Koethe radical;

– the property to be a quasi-regular ring is a radical property and generates the Jacobson radical;

– the property to be a Boolean ring generates the Boolean radical, etc.

The analogues of some topological radicals in the concrete classes of topological rings were well-known earlier. V. Arnautov has constructed them in the class of all topological rings.

The coincidence of some topological radicals in special classes of spaces was an unexpected fact. For example, in the class of compact rings the topological radicals of Baer-McCoy and Jacobson coincide with the topological quasi-regular radical. This fact confirms the initial assumption of interdependence of algebraical properties of topological radicals and topological properties of classes of topological rings. Moreover, as applications the structural descriptions of some classes of topological rings were obtained.

The methods of the theory of radicals are important for the study of the algebraical properties of the completion $\breve{R}$ of a topological ring R. V. Andrunachievici and V. Arnautov established that for any topological ring R the following assertions are equivalent:

– in the topological ring R any one-sided ideal is trivial and only zero is a generalized zero divisor;

– R is a ring with unity and any element of R is invertible in $\breve{R}$.

The next curious fact immediately follows from this result: a locally compact topological ring without non-trivial one-sided ideals is a topological field.

Some fundamental investigations of V. Arnautov were made jointly with his scientific tutor, Academician V. Andrunachievici, others with his gifted post-graduate student, Mihail Ursul, who then created new interesting directions of research and educated new generations of highly-qualified mathematicians. Interesting results about topological radicals where proved by Mihail Vodinchar and Trinh Dang Khoi, when they were post-graduate students of Professor V. Arnautov.

A voluminous outline of the investigations of the topological radicals and of the radical properties was presented in the review: V. I. Arnautov, *The Theory of Radicals of Topological Rings*, Mathematica Japonica 47:3(1998), 439–544.

In 1946 Professor A. A. Markov, in one of his articles, arose the next question: is it true that on each infinite group there exists a non-discrete Hausdorff topology?

Markov's question generates in the more general aspect the following problems.

Problem 2. *Under which conditions on a given universal algebra A there exist some (only one, two) topologies with the given property?*

Problem 3. *Find algebraical properties of a universal algebra A which can be characterized by the properties of the lattice $LT(A)$ of the topologies on the algebra A. In particular, under which conditions the lattice $LT(A)$ has coatoms?*

Problem 4. *Let A be a subalgebra of a universal algebra B and $\mathcal{T}$ be a Hausdorff topology on A. Under which conditions on B there exists a Hausdorff topology $\mathcal{T}'$ such that A is a topological subalgebra of B?*

The Problems 3 and 4 are more difficult if on the space of operations some topology is fixed and, in particular, for modules or for algebras over some topological ring Φ.

The history of solution of Markov's Problem for groups is long and surprising. First, it was established that the positive answer for commutative groups follows from the theory of characters (A. Kertesz and T. Szele, 1956). Then, in 1977, S. Shelah, using forcing method, constructed an uncountable group without non-discrete topologies. In 1980 A. Olšanskii observed that the infinite countable group $A(m,n)/C^m$, where $A(m,n)$ is the infinite countable group constructed by S. Adian in 1975, has not non-discrete Hausdorff topologies.

Markov's Problem for rings was solved by V. Arnautov by 1972. He obtained the following valuable results:

1. On any infinite countable ring there exist non-discrete Hausdorff topologies.

2. On each infinite commutative associative ring there exists a Hausdorff non-discrete topology.

3. There exists an infinite ring on which only the anti-discrete topology $\{0, A\}$ is non-discrete.

Obviously, the Kertesz-Szele's Theorem for commutative groups follows from the Arnautov's results.

For construction on rings the Hausdorff non-discrete topologies with distinct proprieties V. Arnautov developed interesting combinatorial methods. These facts and some structural properties of topological rings constitute the content of his doctor's science degree thesis defended in 1972 at the Institute of Mathematics of the Siberian Branch of the Academy of Sciences of USSR.

The topology constructed by the characters is totally bounded or metrizable and locally totally bounded, but the topologies proposed by V. Arnautov are metrizable only for countable rings and, as a rule, not locally totally bounded. This fact arises the next major problems.

Problem 5. *Under which conditions on an algebra there exists some compact Hausdorff topology?*

Problem 6. *Under which conditions on an algebra there exits a topology generated by a linear ordering?*

Problem 7. *Let τ be an infinite cardinal. Under which conditions on an algebra there exists a Hausdorff non-discrete P_τ-topology?*

The topology is called a P_τ-topology, where τ is an infinite cardinal, if the intersection of τ open sets is an open set.

The Problems 5 - 7 relate to the problem.

Problem 8. *Find the interdependence of algebraical properties of algebra A and properties of topologies on the algebra A.*

Under the guidance of Professor V. Arnautov remarkable results concerning the problems 5 - 8 were obtained by Mihail Ursul, Pavel Chircu, Victor Vizitiu, Elena Marin, Valeriu Popa, Kirill Filippov, Dilfuza Yunusova, Anatolie Topală.

The topological free algebras are important algebraical objects. Interesting results about the properties of topological free rings in concrete classes of topological rings were proposed by V.Arnautov and his post-graduate students Ştefan Alexei, Reli Calistru and Stelian Dumitrashcu (the later was a post-graduate student of M. Cioban too).

A lasting, active and efficient collaboration has been established between Professor V. Arnautov and the Moskow mathematicians Professors Alexander V. Mikhalev and Sergei T. Glavatsky.

In connection with Problem 4 they examined the next two problems.

Problem 9. *Under which conditions on the semigroup ring the topology of the ring and the topology of the semigroup can be simultaneously extended?*

Problem 10. *Under which condition the topology of the ring admits some extension over its ring of quotients?*

The following monographs constitute the final result of this collaboration:

1. Arnautov V. I, Vodinchar M. I., Mikhalev A. V, *Introduction to the Theory of Topological Rings and Modules,* – Ştiinţa: Chişinău, 1981, 175 p. (In Russian).

2. Arnautov V. I, Vodinchar M. I., Glavatsky S. T., Mikhalev A. V, *Constructions of the Topological Rings and Modules,* – Ştiinţa: Chişinău, 1988, 168 p. (In Russian).

3. Arnautov V. I, Glavatsky S. T., Mikhalev A. V, *Introduction to the Theory of Topological Rings and Modules,* – Marcel Dekker: New York-Basel, 1996, 502 p.

In the mentioned books the results of the authors of books and of their former students constitute a significent part and the impact of this works on the development of Topological Algebra is considerable.

The last scientific researches of Professor V. Arnautov are dedicated to the investigation of the lattice of topologies of groups and rings. One general method of construction of neighbour pairs of topologies was found. Two topologies on the algebra A form a neighbour pair if between them other topologies do not exist. In particular, any coatom and the maximal topology of the lattice $LT(A)$ form a neighbour pair. If A is an algebra over a discrete ring or a group, then the maximal topology of the lattice $LT(A)$ is discrete. Mentioned facts confirm the importance of this concept and method. Professor V. Arnautov proved that the lattice $LT(A)$, where A is a linear space over field of reals, does not contain Hausdorff coatoms. In this case the maximal element of the lattice $LT(A)$ is not discrete.

Academician V. Arnautov has published more than 160 research papers and 3 monographs. Having a good prestige in the world of mathematics, Professor Vladimir Arnautov has been invited at more then 40 prestigious international conferences in Algebra and Topology (Russia, Belarus, Ukraine, Poland, Austria, etc). He passionately and skillfully has organized in collaboration with colleagues several (about 20) national and international conferences on Algebra, Topology and Topological Algebra. For instance, in 1984 and 1986 Professor V. Arnautov in collaboration with Professors A. Arhangel'skii, M. Cioban and A. Mikhalev organized in Tiraspol the well-known workshops "Topological Algebra" which had a considerable influence on the development of Topological Algebra and General Topology. In particular, these workshops have established close contacts between many algebraical and topological schools of the former USSR (from Moscow, Saint Petersburg, Novosibirsk, Tomsk, Yekaterinburg, Ukraine, Byelorussia, Moldova, Estonia, etc).

Thus in 1961 Professor V. Arnautov steady and full of energy began the scholarly activity and didactic carrier. During 1964–1967 and 1967–1970 he was respectively scientific worker and superior scientific worker at the Institute of Mathematics and Computer Sciences of the Academy of Sciences of Moldova (IMCS ASM). Between 1970–1978 he was the head of the laboratory of IMCS ASM. In 1978 he becomes the full Professor. Between 1978–1988 and 1990–1993 he was the deputy director of IMI ASM for research problems. In 1984 he was elected the corresponding member of the Academy of Sciences of Moldova. In the period 1990–1993 he was the associate member of the Presidium of the Academy of Sciences of Moldova. Between 1993–1999 he was the principal scientific worker and since 1999 he is the head of Department of Theoretical Mathematics of IMCS of ASM. In 2007 Professor V. Arnautov was elected the full member of the Academy of Sciences of Moldova, the highest scientific forum of the Republic of Moldova and the highest recognition which a scholar may receive in the native country.

The contribution of Professor V. Arnautov to the education of new generations of highly-qualified mathematicians is enormous. He has trained 13 doctors of sciences and Ph.D's. To his colleagues and former students he has an inspiration not only as a mathematician, but as a human being.

Professor V. Arnautov is an active member of many state communities and commissions. He was a member of the Commission of Experts and of the Scientific Council of the Higher Certifying Committee of USSR for the academic degree and rank. During 1973–1977 he was a Chairman of the Council of Young Researchers of the Republic of Moldova. Now he is a Chairman of the Council and a member of the Experts Commission of CNAA (National Council for Accreditation and Attestation).

Professor Vladimir Arnautov was awarded the Prize of the Moldovian Komsomol (1972) for the young researchers, the prize "Academician Constantin Sibirschi" (2001). He is a "Honoured scientist of the Republic of Moldova" and is awarded with the "Honour Diploma of the Presidium of the Supreme Soviet of MSSR", medal "Distinction in Labour" and order "Glory of Labour".

Professor V. Arnautov is a member of the Moldavian Mathematics Society, American Mathematical Society and of the Editorial Board of the Bulletin of the Academy of Science of Moldova, Mathematics.

At the age of 70, full of vigor and optimism, the academician Vladimir Arnautov is a prominent personality and continues an active presence in the academic community of the Republic of Moldova. We wish him a good health, prosperity and new accomplishments in his prodigious scientific and didactic activities: "Happy Birthday to You, Happy returns of the Day".

Mitrofan Cioban
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President of the Mathematical Society of the Republic of Moldova

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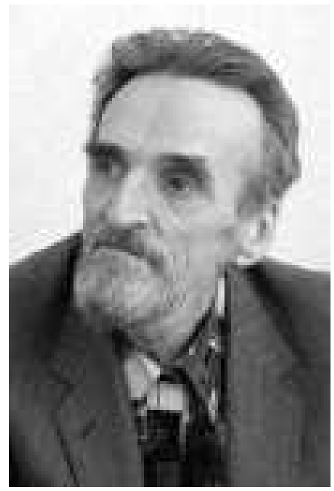
Academician Iurie Reabuhin – 70th anniversary



Iurii Mihailovici Reabuhin was born on February 8, 1939 in Moscow, Russia. Graduate of high school N 4 (Chisinau, 1956), and of faculty of physics and mathematics, Moldova State University (Chisinau, 1961), post-graduate student, Institute of Mathematics with Computer Center of Moldova Academy of Sciences (1965).

PhD (1965), Habilitat doctor (1971), University professor (1978), Corresponding Member (1989) and Academician (1993) of Moldova Academy of Sciences.

Distinctions: Laureate of Moldova State Prize in domain of science and technics (1972), Honoured Public Education Worker (1984), Honoured Science Worker (1988).

Research activity: Institute of Mathematics and Informatics, Moldova Academy of Sciences – laboratory assistant (1961–1963), research worker, senior research worker (1963–1970), head of the department Algebra and Mathematical Logic (1970–1993), principal research worker (since 1993).

Professor Iurii Reabuhin wrote his first scientific article in geometry under the supervision of the well known geometrician Alexandr Zamorzaev. In 1961 he met a great specialist in Algebra Vladimir Andrunakievici, and they formed a tandem which is now famous all over the algebraic world. They say, in the person of Iurii Reabuhin mathematics lost a good geometrician but gained a great algebraist.

The scientific interests of professor Reabuhin turned up in the theory of rings, algebras and modules. One circle of works deals with the theory of radicals of rings and algebras. He was one of the first to prove that the radical, generated by a hereditary property, is hereditary. Studying lower radicals, professor Reabuhin was the first to indicate an unbreakable Kurosh chain, and together with academician Andrunakievici showed that the break of this chain in the case of non-associative algebras is a rare event.

Professor Reabuhin showed that it is possible to come to subnilpotent and special radicals from category considerations. He constructed an example of a subnilpotent radical which is not special and later showed that there is a whole "heap", by an expression of the famous algebraist K. Zhevlakov, of such radicals.

Another circle of scientific works of professor Reabuhin is connected with algebras without nilpotent elements. He and Andrunakievici proved that every associative ring without nilpotent elements is decomposed as a subdirect product of rings without zero divisors. Prof. Reabuhin had solved completely the problem of description of non-necessary associative algebras (over an arbitrary associative commutative ring with identity) which are decomposed as a subdirect product of algebras without zero divisors, and generalized some theorems of Gerchikov and Weierstrass. He also showed that it is possible to develop the theory of radicals even in "bad" categories.

These and a lot of other results are exposed in the monograph of V. Andrunakievici and I. Reabuhin "Radicals of algebras and structure theory".

One more circle of works of Reabuhin is devoted to the search of generalizations of classical Noether-Laseeker additive ideal theory to the non-commutative case. In a large series of works professors Reabuhin and Andrunakievici defined axiomatically the notion of primarity and studied different properties of primary ideals and their intersections: "existence" (every ideal has of a primary decomposition, that is every ideal is an intersection of a finite number of primary ideals), "intersection" (intersection of a finite number of primary ideals with the same radical is a primary ideal), "uniqueness" (two primary decompositions have the same set of radicals). It turns out that under some very natural conditions the only one "good" primarity is "tertiarity", studied by Lesiur and Croisot.

If the restrictions are slightly modified one gains the "primality" of Fuchs, which, however, does not coincide with classical "primarity" even in commutative case. The developed theory is valid not only for rings, but also for groups, semigroups, modules and systems with quotients, the last being introduced by Reabuhin and Andrunakievici.

We must confess that some ideas of pioneer in traditional researches of algebraists of the Republic of Moldova belong to academician Iurii Reabuhin. These ideas not only extended traditional themes of investigations, but also generated new directions of researches of Chişinau algebraic scool. Here are just some aspects which enrich algebraic investigations with new results and methods in the theory of rings and algebras non-necessary associative: general constructions of locally nilpotent algebras, locally finite dimensional algebras, description of some varieties of algebras, analysis of marked varieties of associative algebras, cardinality of minimal varieties. Some results and applications lead to solution of some special problems of ring theory. Here we may remak, for example, that Baer and Levitzky are different, the cardinality of minimal variety is immense (continuum) and some applications can be extended to the non-associative case. The methods, described and applied by academician Iurii Reabuhin leave you thunderstruck not only by originality, but also by their contents. They contain vast possibilities of new extensions and applications, open new directions which were successfully capitalized by his disciples.

Beside academician Iurii Reabuhin you always feel a special devotion to algebraic school, to its prestige at national and international level. These nontraditional directions together with obtained results had fortified the image of our algebraic school and had amplified investigations in rings and algebras with different conditions of finiteness.

Now, when Academician Iurii Reabuhin is 70 years old, we are proud of his achievements and wish good health, prosperity and successes in his scientific and didactic activity.

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