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Properties of one-sided ideals of pseudonormed rings when taking the quotient rings

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Abstract. Let $\varphi : (R, \xi) \to \left(\widehat{R}, \widehat{\xi}\right)$ be an isomorphism of pseudonormed rings. The inequalities $\frac{\xi(a \cdot b)}{\xi(b)} \leq \widehat{\xi}(\varphi(a)) \leq \xi(a)$ are fulfilled for any $a, b \in R \setminus \{0\}$ iff there exists a pseudonormed ring $\left(\widetilde{R}, \widetilde{\xi}\right)$ such that (R, ξ) is a left ideal in $\left(\widetilde{R}, \widetilde{\xi}\right)$ and the isomorphism φ can be extended up to an isometric homomorphism $\widetilde{\varphi} : \left(\widetilde{R}, \widetilde{\xi}\right) \to \left(\widehat{R}, \widehat{\xi}\right)$.

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We will think that a pseudonormed ring is a ring R which may be non-associative and has a pseudonorm, i.e. a real function $\xi(r)$ such that the following conditions are satisfied: $\xi(-r) = \xi(r) \geq 0$; $\xi(r) = 0$ iff $r = 0$; $\xi(r_1 + r_2) \leq \xi(r_1) + \xi(r_2)$ and $\xi(r_1 \cdot r_2) \leq \xi(r_1) \cdot \xi(r_2)$ for any $r_1, r_2 \in R$.

The following isomorphism theorem is often applied in algebra and, in particular, in the ring theory:

If A is a subring of a ring R and I is an ideal of the ring R then the quotient rings $A/(A \cap I)$ and $(A + I)/I$ are isomorphic rings. In particular, if $A \cap I = 0$ then the ring A is isomorphic to the ring $(A + I)/I$, i.e. the rings A and $(A + I)/I$ possess identical algebraic properties.

Since when studying the pseudonormed rings it is necessary to take into account properties of pseudonorms besides algebraic properties there is a need to consider isomorphisms which keep pseudonorms instead of ring isomorphism. Such isomorphisms are called isometric isomorphisms.

Taking into consideration this fact the above specified isomorphism theorem not always takes place for pseudonormed rings. As it is shown in Theorem 2.1 from [1] it is impossible to tell anything more than performance of an inequality in case $A \cap I = 0$.

Therefore it is necessary to impose additional conditions on the ring A. For example, the cases when A is an ideal or one-sided ideal of the pseudonormed ring (R, ξ) are considered.

The case when A is an ideal of pseudonormed ring (R, ξ) was investigated in [1].

The present article is a continuation of the article [1]. The case when A is an one-side ideal of the pseudonormed ring (R, ξ) is investigated in the present article.

Definition 1. A homomorphism $\varphi : (R, \xi) \to \left(\widehat{R}, \widehat{\xi}\right)$ of pseudonormed rings is called an isometric homomorphism if $\widehat{\xi}\,(\varphi\,(r)) = \inf\{\,\xi\,(r + a) \mid a \in ker\varphi\}$ for all $r \in R$.

Remark 1. It is clear that if an isometric homomorphism is an isomorphism then it is an isometric isomorphism in usual sense.

Remark 2. If I is a closed ideal of a pseudonormed ring (R, ξ) then the canonical homomorphism[1] $\varepsilon : (R, \xi) \to (R, \xi)/I$ is an isometric homomorphism, and if $\varphi : (R, \xi) \to \left(\widehat{R}, \widehat{\xi}\right)$ is an isometric homomorphism of pseudonormed rings and $I = \ker \varphi$ then the pseudonormed rings $\left(\widehat{R}, \widehat{\xi}\right)$ and $(R, \xi)/I$ are isometrically isomorphic.

Definition 2. Let (R, ξ) and $\left(\widehat{R}, \widehat{\xi}\right)$ be pseudonormed rings. By analogy with the definition in [1], an isomorphism $\varphi : (R, \xi) \to \left(\widehat{R}, \widehat{\xi}\right)$ is said to be a semi-isometric isomorphism on the left (on the right) if there exists a pseudonormed ring $\left(\widetilde{R}, \widetilde{\xi}\right)$ such that the pseudonormed ring (R, ξ) is a left (right) ideal of the pseudonormed ring $\left(\widetilde{R}, \widetilde{\xi}\right)$ and the isomorphism φ can be extended up to an isometric homomorphism $\widetilde{\varphi} : \left(\widetilde{R}, \widetilde{\xi}\right) \to \left(\widehat{R}, \widehat{\xi}\right)$.

Theorem 1. *Let (R, ξ) and $\left(\widehat{R}, \widehat{\xi}\right)$ be pseudonormed rings and $\varphi : (R, \xi) \to \left(\widehat{R}, \widehat{\xi}\right)$ be an isomorphism. Then the following statements are equivalent:*

1. *The isomorphism φ is a semi-isometric isomorphism on the left;*
2. *The inequalities $\dfrac{\xi\,(b \cdot a)}{\xi\,(a)} \le \widehat{\xi}\,(\varphi\,(b)) \le \xi\,(b)$ are fulfilled for any $a, b \in R \setminus \{0\}$;*
3. *There exists a pseudonormed ring $\left(\widetilde{R}, \widetilde{\xi}\right)$ such that the pseudonormed ring (R, ξ) is a left ideal of the pseudonormed ring $\left(\widetilde{R}, \widetilde{\xi}\right)$ and the isomorphism φ can be extended up to an isometric homomorphism $\widetilde{\varphi} : \left(\widetilde{R}, \widetilde{\xi}\right) \to \left(\widehat{R}, \widehat{\xi}\right)$, and $(\ker \widetilde{\varphi})^2 = \{0\}$.*

Proof. $1 \Rightarrow 2$**.** Let $\varphi : (R, \xi) \to \left(\widehat{R}, \widehat{\xi}\right)$ be a semi-isometric isomorphism on the left. Then there exists a pseudonormed ring $\left(\widetilde{R}, \widetilde{\xi}\right)$ such that the pseudonormed

[1] i.e. homomorphism $\varepsilon : R \to R/I$ such that $\varepsilon\,(r) = r + I$.

ring (R,ξ) is a left ideal of the pseudonormed ring $\left(\widetilde{R},\widetilde{\xi}\right)$ and the isomorphism φ can be extended up to an isometric homomorphism $\widetilde{\varphi}:\left(\widetilde{R},\widetilde{\xi}\right)\to\left(\widehat{R},\widehat{\xi}\right)$.

Let $a,b\in R$ and $\varepsilon>0$. Since $\widetilde{\varphi}$ is an extension of the isomorphism φ then $R\cap\ker\widetilde{\varphi}=\ker\varphi=\{0\}$, and as R is a left ideal of $\widetilde{R}$ and $\ker\widetilde{\varphi}$ is an ideal of $\widetilde{R}$ then $d\cdot a\in R\cap\ker\widetilde{\varphi}=\{0\}$ for any $d\in\ker\widetilde{\varphi}$, i.e. $(\ker\widetilde{\varphi})\cdot a=0$.

Since $\widetilde{\varphi}:\left(\widetilde{R},\widetilde{\xi}\right)\to\left(\widehat{R},\widehat{\xi}\right)$ is an isometric homomorphism then $\widehat{\xi}\left(\widetilde{\varphi}\left(b\right)\right)\le\widetilde{\xi}\left(b\right)=\xi\left(b\right)$ and there exists an element $c\in\ker\widetilde{\varphi}$ such that $\widetilde{\xi}\left(b+c\right)<\widehat{\xi}\left(\widetilde{\varphi}\left(b\right)\right)+\varepsilon=\widehat{\xi}\left(\varphi\left(b\right)\right)+\varepsilon$. So as $c\cdot a\in\ker\widetilde{\varphi}\cdot a=0$ then

$$\xi\left(b\cdot a\right)=\widetilde{\xi}\left(b\cdot a\right)=\widetilde{\xi}\left(b\cdot a+c\cdot a\right)=\widetilde{\xi}\left((b+c)\cdot a\right)\le\widetilde{\xi}\left(b+c\right)\cdot\widetilde{\xi}\left(a\right)=$$

$$\widetilde{\xi}\left(b+c\right)\cdot\xi\left(a\right)<\left(\widehat{\xi}\left(\varphi\left(b\right)\right)+\varepsilon\right)\cdot\xi\left(a\right).$$

Since $\varepsilon>0$ is any number then $\xi\left(b\cdot a\right)\le\widehat{\xi}\left(\varphi\left(b\right)\right)\cdot\xi\left(a\right)$. It means that

$$\frac{\xi\left(b\cdot a\right)}{\xi\left(a\right)}\le\widehat{\xi}\left(\varphi\left(b\right)\right)\le\xi\left(b\right).$$

Hence $1\Rightarrow2$ is proved. □

Proof. $2\Rightarrow3$. Let (R,ξ) and $\left(\widehat{R},\widehat{\xi}\right)$ be pseudonormed rings and $\varphi:(R,\xi)\to\left(\widehat{R},\widehat{\xi}\right)$ be an isomorphism such that the inequalities $\dfrac{\xi\left(b\cdot a\right)}{\xi\left(a\right)}\le\widehat{\xi}\left(\varphi\left(b\right)\right)\le\xi\left(b\right)$ are fulfilled for any $a,b\in R$.

We shall lead the proof to some stages.

I. Construction of the ring $\widetilde{R}$ and checking some of its properties.

I.1. Let's consider a discrete ring $\widetilde{R}$ such that its additive group is the direct sum of the additive groups of the rings R and $\widehat{R}$, and the multiplication is certain as follows: $(r_1,\widehat{r}_1)\cdot(r_2,\widehat{r}_2)=(r_1\cdot r_2,\varphi\left(r_1\right)\cdot\widehat{r}_2)$.

I.2. It is easy to notice that $\widetilde{R}$ is a ring with respect to these operations of addition and multiplication, and the set $R'=\{\,(r,0)|\,r\in R\}$ is a left ideal of $\widetilde{R}$.

I.3. Let's define the mapping $\alpha:R\to\widetilde{R}$ as follows $\alpha\left(r\right)=(r,0)$ for any $r\in R$. It is easy to notice that $\alpha:R\to R'=\{\,(r,0)|\,r\in R\}$ is a ring isomorphism. Hence, if we identify an element $r\in R$ with the element $(r,0)\in R'$ then we can suppose that R is a left ideal in the ring $\widetilde{R}$.

I.4. Let's define the mapping $\widetilde{\varphi}:\widetilde{R}\to\widehat{R}$ as follows $\widetilde{\varphi}\left(r,\widehat{r}\right)=\varphi\left(r\right)$. It's easy to notice that $\widetilde{\varphi}:\widetilde{R}\to\widehat{R}$ is a ring homomorphism, and (considering I.3.) $\widetilde{\varphi}\left(r\right)=\widetilde{\varphi}\left(r,0\right)=\varphi\left(r\right)$ for any $r\in R$, i.e. $\widetilde{\varphi}|_R=\varphi$. Since $\ker\widetilde{\varphi}=\left\{(0,\widehat{r})|\,\widehat{r}\in\widehat{R}\right\}$ then $(\ker\widetilde{\varphi})^2=0$.

II. Definition of a pseudonorm $\widetilde{\xi}$ and checking some of its properties.

II.1. Let's define the real function $\widetilde{\xi}$ on the ring $\widetilde{R}$ as follows: $\widetilde{\xi}(r,\widehat{r}) = \xi\left(r - \varphi^{-1}(\widehat{r})\right) + \widehat{\xi}(\widehat{r})$.

II.2. Let's verify that $\widetilde{\xi}$ is a pseudonorm.

It is easy to notice that $\widetilde{\xi}(-\widetilde{r}) = \widetilde{\xi}(\widetilde{r}) \geq 0$ for any $\widetilde{r} \in \widetilde{R}$ and $\widetilde{\xi}(\widetilde{r}) = 0$ if and only if $\widetilde{r} = 0$, i.e. the first and second conditions of the definition of the pseudonorm are valid. Let $\widetilde{r}_1 = (r_1, \widehat{r}_1), \widetilde{r}_2 = (r_2, \widehat{r}_2) \in \widetilde{R}$. Then

$$\widetilde{\xi}(\widetilde{r}_1 + \widetilde{r}_2) = \widetilde{\xi}((r_1, \widehat{r}_1) + (r_2, \widehat{r}_2)) = \widetilde{\xi}((r_1 + r_2, \widehat{r}_1 + \widehat{r}_2)) =$$
$$= \xi\left(r_1 + r_2 - \varphi^{-1}(\widehat{r}_1 + \widehat{r}_2)\right) + \widehat{\xi}(\widehat{r}_1 + \widehat{r}_2) =$$
$$= \xi\left(r_1 + r_2 - \varphi^{-1}(\widehat{r}_1) - \varphi^{-1}(\widehat{r}_2)\right) + \widehat{\xi}(\widehat{r}_1 + \widehat{r}_2) \leq$$
$$\leq \xi\left(r_1 - \varphi^{-1}(\widehat{r}_1)\right) + \xi\left(r_2 - \varphi^{-1}(\widehat{r}_2)\right) + \widehat{\xi}(\widehat{r}_1) + \widehat{\xi}(\widehat{r}_2) = \widetilde{\xi}(\widetilde{r}_1) + \widetilde{\xi}(\widetilde{r}_2).$$

Besides that, because the inequalities $\xi(b \cdot a) \leq \widehat{\xi}(\varphi(b)) \cdot \xi(a)$ and $\widehat{\xi}(\varphi(a)) \leq \xi(a)$ for any $a, b \in R$ are true (see the statement 2 of formulation of the theorem) we have:

$$\widetilde{\xi}(\widetilde{r}_1 \cdot \widetilde{r}_2) = \widetilde{\xi}((r_1, \widehat{r}_1) \cdot (r_2, \widehat{r}_2)) = \widetilde{\xi}((r_1 \cdot r_2, \varphi(r_1) \cdot \widehat{r}_2)) =$$
$$= \xi\left(r_1 \cdot r_2 - \varphi^{-1}(\varphi(r_1) \cdot \widehat{r}_2)\right) + \widehat{\xi}(\varphi(r_1) \cdot \widehat{r}_2) =$$
$$= \xi\left(r_1 \cdot r_2 - r_1 \cdot \varphi^{-1}(\widehat{r}_2)\right) + \widehat{\xi}(\varphi(r_1) \cdot \widehat{r}_2) =$$
$$\xi\left(r_1 \cdot \left(r_2 - \varphi^{-1}(\widehat{r}_2)\right)\right) + \widehat{\xi}(\varphi(r_1) \cdot \widehat{r}_2) \leq$$
$$\leq \widehat{\xi}(\varphi(r_1)) \cdot \xi\left(r_2 - \varphi^{-1}(\widehat{r}_2)\right) + \widehat{\xi}(\varphi(r_1) \cdot \widehat{r}_2) =$$
$$= \widehat{\xi}(\varphi(r_1)) \cdot \xi\left(r_2 - \varphi^{-1}(\widehat{r}_2)\right) + \widehat{\xi}((\varphi(r_1) - \widehat{r}_1 + \widehat{r}_1) \cdot \widehat{r}_2) \leq$$
$$\leq \widehat{\xi}(\varphi(r_1)) \cdot \xi\left(r_2 - \varphi^{-1}(\widehat{r}_2)\right) + \widehat{\xi}((\varphi(r_1) - \widehat{r}_1) \cdot \widehat{r}_2) + \widehat{\xi}(\widehat{r}_1 \cdot \widehat{r}_2) \leq$$
$$\leq \widehat{\xi}(\varphi(r_1)) \cdot \xi\left(r_2 - \varphi^{-1}(\widehat{r}_2)\right) + \widehat{\xi}(\varphi(r_1) - \widehat{r}_1) \cdot \widehat{\xi}(\widehat{r}_2) + \widehat{\xi}(\widehat{r}_1) \cdot \widehat{\xi}(\widehat{r}_2) \leq$$
$$\leq \widehat{\xi}(\varphi(r_1)) \cdot \xi\left(r_2 - \varphi^{-1}(\widehat{r}_2)\right) + \xi\left(r_1 - \varphi^{-1}(\widehat{r}_1)\right) \cdot \widehat{\xi}(\widehat{r}_2) + \widehat{\xi}(\widehat{r}_1) \cdot \widehat{\xi}(\widehat{r}_2) =$$
$$= \widehat{\xi}(\varphi(r_1) - \widehat{r}_1 + \widehat{r}_1) \cdot \xi\left(r_2 - \varphi^{-1}(\widehat{r}_2)\right) +$$
$$\xi\left(r_1 - \varphi^{-1}(\widehat{r}_1)\right) \cdot \widehat{\xi}(\widehat{r}_2) + \widehat{\xi}(\widehat{r}_1) \cdot \widehat{\xi}(\widehat{r}_2) \leq$$
$$\leq \widehat{\xi}(\varphi(r_1) - \widehat{r}_1) \cdot \xi\left(r_2 - \varphi^{-1}(\widehat{r}_2)\right) + \widehat{\xi}(\widehat{r}_1) \cdot \xi\left(r_2 - \varphi^{-1}(\widehat{r}_2)\right) +$$
$$\xi\left(r_1 - \varphi^{-1}(\widehat{r}_1)\right) \cdot \widehat{\xi}(\widehat{r}_2) + \widehat{\xi}(\widehat{r}_1) \cdot \widehat{\xi}(\widehat{r}_2) \leq$$
$$\xi\left(r_1 - \varphi^{-1}(\widehat{r}_1)\right) \cdot \xi\left(r_2 - \varphi^{-1}(\widehat{r}_2)\right) + \widehat{\xi}(\widehat{r}_1) \cdot \xi\left(r_2 - \varphi^{-1}(\widehat{r}_2)\right) +$$
$$\xi\left(r_1 - \varphi^{-1}(\widehat{r}_1)\right) \cdot \widehat{\xi}(\widehat{r}_2) + \widehat{\xi}(\widehat{r}_1) \cdot \widehat{\xi}(\widehat{r}_2) =$$
$$= \left(\xi\left(r_1 - \varphi^{-1}(\widehat{r}_1)\right) + \widehat{\xi}(\widehat{r}_1)\right) \cdot \left(\xi\left(r_2 - \varphi^{-1}(\widehat{r}_2)\right) + \widehat{\xi}(\widehat{r}_2)\right) = \widetilde{\xi}(\widetilde{r}_1) \cdot \widetilde{\xi}(\widetilde{r}_2).$$

Hence the function $\widetilde{\xi}$ satisfies also the last condition of definition of pseudonorm. It means that $\widetilde{\xi}$ is a pseudonorm on the ring $\widetilde{R}$.

II.3. Since $\widetilde{\xi}(r) = \widetilde{\xi}(r,0) = \xi(r) + \widehat{\xi}(0) = \xi(r)$ for any $r \in R$ then $\widetilde{\xi}\Big|_R = \xi$.

II.4. Let's verify that $\widetilde{\varphi} : \left(\widetilde{R}, \widetilde{\xi}\right) \to \left(\widehat{R}, \widehat{\xi}\right)$ is an isometric homomorphism, i.e. $\widehat{\xi}(\widetilde{\varphi}(\widetilde{r})) = \inf\left\{\widetilde{\xi}(\widetilde{r}+\widetilde{a}) \,\Big|\, \widetilde{a} \in \ker\widetilde{\varphi}\right\}$ for any $\widetilde{r} \in \widetilde{R}$.

Let $\widetilde{r} = (r, \widehat{r}) \in \widetilde{R}$. Then $\widetilde{r}_1 = (0, \varphi(r) - \widehat{r}) \in \ker\widetilde{\varphi}$, and

$$\inf\left\{\widetilde{\xi}(\widetilde{r}+\widetilde{a}) \,\Big|\, \widetilde{a} \in \ker\widetilde{\varphi}\right\} \leq \widetilde{\xi}(\widetilde{r}+\widetilde{r}_1) = \widetilde{\xi}((r,\widehat{r}) + (0, \varphi(r) - \widehat{r})) =$$

$$= \widetilde{\xi}(r, \widehat{r} + \varphi(r) - \widehat{r}) = \widetilde{\xi}(r, \varphi(r)) =$$
$$\xi\left(r - \varphi^{-1}(\varphi(r))\right) + \widehat{\xi}(\varphi(r)) = \widehat{\xi}(\varphi(r)) = \widehat{\xi}(\widetilde{\varphi}(\widetilde{r})).$$

On the other hand, since $\widehat{\xi}(\varphi(b)) \leq \xi(b)$ for any element $b \in R$, then for any element $\widetilde{a} = (0, \widehat{a}) \in \ker\widetilde{\varphi}$ we have

$$\widetilde{\xi}(\widetilde{r}+\widetilde{a}) = \widetilde{\xi}((r+0, \widehat{r}+\widehat{a})) = \widetilde{\xi}(r, \widehat{r}+\widehat{a}) = \xi\left(r - \varphi^{-1}(\widehat{r}+\widehat{a})\right) + \widehat{\xi}(\widehat{r}+\widehat{a}) \geq$$

$$\geq \widehat{\xi}\left(\varphi\left(r - \varphi^{-1}(\widehat{r}+\widehat{a})\right)\right) + \widehat{\xi}(\widehat{r}+\widehat{a}) \geq \widehat{\xi}\left(\varphi\left(r - \varphi^{-1}(\widehat{r}+\widehat{a})\right) + \widehat{r} + \widehat{a}\right) =$$
$$= \widehat{\xi}(\varphi(r) - \widehat{r} - \widehat{a} + \widehat{r} + \widehat{a}) = \widehat{\xi}(\varphi(r)) = \widehat{\xi}(\widetilde{\varphi}(\widetilde{r})).$$

It means that $\inf\left\{\widetilde{\xi}(\widetilde{r}+\widetilde{a}) \,\Big|\, \widetilde{a} \in \ker\widetilde{\varphi}\right\} \geq \widehat{\xi}(\widetilde{\varphi}(\widetilde{r}))$.

Thus, $\inf\left\{\widetilde{\xi}(\widetilde{r}+\widetilde{a}) \,\Big|\, \widetilde{a} \in \ker\widetilde{\varphi}\right\} = \widehat{\xi}(\widetilde{\varphi}(\widetilde{r}))$, and $\widetilde{\varphi} : \left(\widetilde{R}, \widetilde{\xi}\right) \to \left(\widehat{R}, \widehat{\xi}\right)$ is an isometric homomorphism.

Hence $2 \Rightarrow 3$ is proved. □

For completion of the proof of the theorem it is necessary to verify that $3 \Rightarrow 1$. But this is obvious because the pseudonormed ring $\left(\widetilde{R}, \widetilde{\xi}\right)$ which is specified in the statement 3 satisfies all conditions from Definition 2.

Passing to antiisomorphic rings [2] from Theorem 1 easily follows:

Theorem 2. *If (R, ξ) and $\left(\widehat{R}, \widehat{\xi}\right)$ are pseudonormed rings and $\varphi : (R, \xi) \to \left(\widehat{R}, \widehat{\xi}\right)$ is an isomorphism then the following statements are equivalent:*

1. *The isomorphism φ is a semi-isometric isomorphism on the right;*

2. *The inequalities $\dfrac{\xi(b \cdot a)}{\xi(b)} \leq \widehat{\xi}(\varphi(a)) \leq \xi(a)$ are fulfilled for any $a, b \in R \setminus \{0\}$;*

[2] If R and R' are rings then a mapping $\varsigma : R \to R'$ is called an antiisomorphism when it is an isomorphism of the additive groups of these rings and $\varsigma(a \cdot b) = \varsigma(b) \cdot \varsigma(a)$ for any $a, b \in R$.

3. There exists a pseudonormed ring $\left(\widetilde{R}, \widetilde{\xi}\right)$ *such that the pseudonormed ring* (R, ξ) *is a right ideal of the pseudonormed ring* $\left(\widetilde{R}, \widetilde{\xi}\right)$ *and the isomorphism* φ *can be extended up to an isometric homomorphism* $\widetilde{\varphi} : \left(\widetilde{R}, \widetilde{\xi}\right) \to \left(\widehat{R}, \widehat{\xi}\right)$*, and* $(\ker \widetilde{\varphi})^2 = \{0\}$.

From Theorems 1 and 2 of the present article follows

Corollary 1. *If* (R, ξ) *and* $\left(\widehat{R}, \widehat{\xi}\right)$ *are pseudonormed rings and an isomorphism* $\varphi : (R, \xi) \to \left(\widehat{R}, \widehat{\xi}\right)$ *is a semi-isometric isomorphism on the left and a semi-isometric isomorphism on the right then it is semi-isometric.*

Remark 3. The ring $\widetilde{R}$ which is constructed by the proof $2 \Rightarrow 3$ (see the proof of Theorem 1) is associative when the rings R and $\widehat{R}$ are associative. Therefore Theorems 1 and 2 also are true for associative rings.

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Exact solutions for a rotational flow of generalized second grade fluids through a circular cylinder

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Abstract. In this note the velocity field and the associated tangential stress corresponding to the rotational flow of a generalized second grade fluid within an infinite circular cylinder are determined by means of the Laplace and Hankel transforms. At time $t = 0$ the fluid is at rest and the motion is produced by the rotation of the cylinder, around its axis, with the angular velocity Ωt. The velocity field and the adequate shear stress are presented under integral and series forms in terms of the generalized G-functions. Furthermore, they are presented as a sum between the Newtonian solutions and the adequate non-Newtonian contributions. The corresponding solutions for the ordinary second grade fluid and Newtonian fluid are obtained as particular cases of our solutions for $\beta = 1$, respectively $\alpha = 0$ and $\beta = 1$.

Mathematics subject classification: 76A05, 76U05.

Keywords and phrases: Generalized second grade fluid, velocity field, tangential stress, cylindrical domains.

1 Introduction

The motion of a fluid in a rotating or sliding cylinder is of interest to both theoretical and practical points of view. It is very important to study the mechanism of viscoelastic fluids flow in many industry fields, such as oil exploitation, chemical and food industry and bio-engineering [1]. Fetecau *et al.* [2] have considered the general case of helical flow of an Oldroyd-B fluid and have determined the velocity fields and the associated tangential stresses in forms of series in terms of Bessel functions. Recently fractional calculus has encountered much success in the description of complex dynamics, such as relaxation, oscillation, wave and viscoelastic behaviour. Bagley [3], He [4], Tan [5] used fractional calculus to handle various problems regarding to flow of the second grade fluid.

In this note we will study the rotational flow of a generalized second grade fluid within an infinite circular cylinder of radius R. The motion is due to the cylinder that at time $t = 0^+$, begins to rotate around its axis with the angular velocity Ωt. Exact analytic solutions of this problem are obtained by using Hankel and Laplace transforms and generalized G-functions. Some classical results can be obtained as special cases of our solutions.

2 Governing equations

The constitutive equation of an incompressible generalized second grade fluid is given by [4–6]

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A_1} + \alpha_1\mathbf{A_2} + \alpha_2\mathbf{A_1^2}, \tag{1}$$

where $\mathbf{T}$ is the Cauchy stress tensor, $-p\mathbf{I}$ denotes the indeterminate spherical stress, μ is the coefficient of viscosity, α_1 and α_2 are the normal stress moduli and $\mathbf{A_1}$ and $\mathbf{A_2}$ are the kinematic tensors defined through

$$\mathbf{A_1} = grad\,\mathbf{v} + (grad\,\mathbf{v})^T, \tag{2}$$

$$\mathbf{A_2} = D_t^\beta\mathbf{A_1} + \mathbf{A_1}(grad\,\mathbf{v}) + (grad\,\mathbf{v})^T\mathbf{A_1}. \tag{3}$$

In the above relations $\mathbf{v}$ is the velocity, the superscript T denotes the transpose operator, and D_t^β is the Riemann-Liouville fractional derivative operator defined by [7]

$$D_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t \frac{f(\tau)}{(t-\tau)^\beta}d\tau; \quad 0 < \beta \le 1, \tag{4}$$

where $\Gamma(\cdot)$ is the Gamma function. For $\beta = 1$ the generalized model reduces to classical model of second grade fluid because $D_t^1 f = df/dt$.
Since the fluid is incompressible, it can undergo only isochoric motions and hence

$$div\,\mathbf{v} = tr\,\mathbf{A_1} = 0. \tag{5}$$

If this model is required to be compatible with thermodynamics, then the material moduli must meet the following restrictions [8]

$$\mu \ge 0\ , \quad \alpha_1 \ge 0 \quad \text{and} \quad \alpha_1 + \alpha_2 = 0. \tag{6}$$

In cylindrical coordinates (r, θ, z), the rotational flow velocity is given by [2, 6]

$$\mathbf{v} = \mathbf{v}(r,t) = \omega(r,t)\mathbf{e}_\theta, \tag{7}$$

where $\mathbf{e}_\theta$ is the unit vector in the θ direction. For such flows the constraint of incompressibility is automatically satisfied.
Introducing (7) into constitutive equation, we find that

$$\tau(r,t) = (\mu + \alpha_1 D_t^\beta)(\frac{\partial}{\partial r} - \frac{1}{r})\omega(r,t), \tag{8}$$

where $\tau(r,t) = S_{r\theta}(r,t)$ is the shear stress which is different of zero. The last equation together with the equations of motion lead to the governing equation

$$\frac{\partial\omega(r,t)}{\partial t} = (\nu + \alpha D_t^\beta)(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2})\omega(r,t)\,, \quad r \in (0,R)\,,\ \ t > 0, \tag{9}$$

where $\nu = \mu/\rho$ is the kinematic viscosity, ρ is the constant density of the fluid and $\alpha = \alpha_1/\rho$.

3 On the rotational flow through an infinite circular cylinder

Let us consider an incompressible generalized second grade fluid at rest in an infinite circular cylinder of radius R. At time zero, the cylinder suddenly begins to rotate about its axis with the angular velocity Ωt. Owing to the shear, the fluid is gradually moved, its velocity being of the form (7) and governing equation is (9). The appropriate initial and boundary conditions are

$$\omega(r,0)=0\,;\quad r\in[0,R)\,,\qquad \omega(R,t)=R\Omega t\,;\quad t\geq 0. \tag{10}$$

To solve this problem we shall use as in [6, 9] the Laplace and Hankel transforms.

3.1 Calculation of the velocity field

Applying the Laplace transform to Eqs. (9) and (10) and using the Laplace transform formula for sequential fractional derivatives [7], we obtain

$$(\nu+\alpha q^{\beta})\Big(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\Big)\overline{\omega}(r,q)-q\overline{\omega}(r,q)=0, \tag{11}$$

where the image function $\overline{\omega}(r,q)=\int_0^\infty \omega(r,t)e^{-qt}dt$ of $\omega(r,t)$ has to satisfy the condition

$$\overline{\omega}(R,q)=\frac{R\Omega}{q^2}, \tag{12}$$

q being the transform parameter. In the following we denote by

$$\overline{\omega}_H(r_{1n},q)=\int_0^R r\overline{\omega}(r,q)J_1(rr_{1n})dr, \tag{13}$$

the Hankel transform of $\overline{\omega}(r,q)$, where $J_1(\cdot)$ is the Bessel function of first kind of order one and $r_{1n}\,,n=1,2,3,...$ are the positive roots of the transcendental equations $J_1(Rr)=0$.

Multiplying now both sides of Eq. (11) by $rJ_1(rr_{1n})$, integrating with respect to r from 0 to R and taking into account the condition (12) and the equality

$$\int_0^R r\Big[\frac{\partial^2\overline{\omega}(r,q)}{\partial r^2}+\frac{1}{r}\frac{\partial\overline{\omega}(r,q)}{\partial r}-\frac{\overline{\omega}(r,q)}{r^2}\Big]J_1(rr_{1n})dr=$$
$$=Rr_{1n}J_2(Rr_{1n})\overline{\omega}(R,q)-r_{1n}^2\overline{\omega}_H(r_{1n},q), \tag{14}$$

we find that

$$\overline{\omega}_H(r_{1n},q)=\Omega R^2 r_{1n}J_2(Rr_{1n})\frac{\nu+\alpha q^{\beta}}{q^2\Big[q+\alpha r_{1n}^2 q^{\beta}+\nu r_{1n}^2\Big]}. \tag{15}$$

Now, for a more suitable presentation of the final results, we rewrite Eq. (15) in the following equivalent form

$$\overline{\omega}_H(r_{1n},q)=\overline{\omega}_{1H}(r_{1n},q)+\overline{\omega}_{2H}(r_{1n},q)+\overline{\omega}_{3H}(r_{1n},q), \tag{16}$$

where

$$\overline{\omega}_{1H}(r_{1n},q)=\frac{\Omega R^2}{q^2 r_{1n}}J_2(Rr_{1n}),\tag{17}$$

$$\overline{\omega}_{2H}(r_{1n},q)=-\frac{\Omega R^2 J_2(Rr_{1n})}{\nu r_{1n}^3}\left(\frac{1}{q}-\frac{1}{q+\nu r_{1n}^2}\right)\tag{18}$$

and

$$\overline{\omega}_{3H}(r_{1n},q)=\alpha\Omega R^2 r_{1n}J_2(Rr_{1n})\frac{1}{q+\nu r_{1n}^2}\frac{q^{\beta-1}}{\left[q+\alpha r_{1n}^2 q^{\beta}+\nu r_{1n}^2\right]}.\tag{19}$$

Using the formula

$$\int_0^R r^2 J_1(rr_{1n})dr=\frac{R^2}{r_{1n}}J_2(Rr_{1n}),\tag{20}$$

we get that inverse Hankel transform of the function $\overline{\omega}_{1H}(r_{1n},q)$ is

$$\overline{\omega}_1(r,q)=\frac{\Omega r}{q^2}.\tag{21}$$

The inverse Hankel transforms of the functions $\overline{\omega}_{kH}(r_{1n},q)$, $\quad k=2,3$, are the functions

$$\overline{\omega}_{kH}(r,q)=\frac{2}{R^2}\sum_{n=1}^{\infty}\frac{J_1(rr_{1n})}{J_2^2(Rr_{1n})}\overline{\omega}_{kH}(r_{1n},q).\tag{22}$$

Introducing Eqs. (21) and (22) into Eq. (16) we find that the Laplace transform $\overline{\omega}(r,q)$ has the form

$$\overline{\omega}(r,q)=\frac{\Omega r}{q^2}-\frac{2\Omega}{\nu}\sum_{n=1}^{\infty}\frac{J_1(rr_{1n})}{r_{1n}^3 J_2(Rr_{1n})}\left(\frac{1}{q}-\frac{1}{q+\nu r_{1n}^2}\right)+$$
$$+2\alpha\Omega\sum_{n=1}^{\infty}\frac{r_{1n}J_1(rr_{1n})}{J_2(Rr_{1n})}\frac{1}{q+\nu r_{1n}^2}\frac{q^{\beta-1}}{\left[q+\alpha r_{1n}^2 q^{\beta}+\nu r_{1n}^2\right]}.\tag{23}$$

To obtain the velocity field $\omega(r,t)=L^{-1}\{\overline{\omega}(r,q)\}$ we will apply the discrete inverse Laplace transform method [6, 7, 9]. For this we use the expansion

$$F(q)=\frac{q^{\beta-1}}{q+\alpha r_{1n}^2 q^{\beta}+\nu r_{1n}^2}=\frac{q^{-1}}{(q^{1-\beta}+\alpha r_{1n}^2)+\nu r_{1n}^2 q^{-\beta}}=$$
$$=\sum_{k=0}^{\infty}(-\nu r_{1n}^2)^k\frac{q^{-\beta k-1}}{\left(q^{1-\beta}+\alpha r_{1n}^2\right)^{k+1}}.\tag{24}$$

Introducing (24) into (23), applying the discrete inverse Laplace transform and using the following properties

$$L^{-1}\{F_1(q)F_2(q)\} = (f_1 * f_2)(t) = \int_0^t f_1(t-s)f_2(s)ds, \tag{25}$$

where

$$f_k(t) = L^{-1}\{F_k(q)\}, \qquad k = 1, 2,$$

$$L^{-1}\Big\{\frac{q^b}{(q^a-d)^c}\Big\} = G_{a,b,c}(d,t), \quad Re\,(ac-b) > 0, \tag{26}$$

and [10]

$$G_{a,b,c}(d,t) = \sum_{j=0}^{\infty} \frac{d^j\Gamma(c+j)}{\Gamma(c)\Gamma(j+1)} \frac{t^{(c+j)a-b-1}}{\Gamma[(c+j)a-b]}, \tag{27}$$

are the generalized G-functions, we find for $\omega(r,t)$ the expression

$$\omega(r,t) = \omega_N(r,t) + 2\alpha\,\Omega \sum_{n=1}^{\infty} \frac{r_{1n}J_1(rr_{1n})}{J_2(Rr_{1n})} \sum_{k=0}^{\infty} \Big(-\nu\, r_{1n}^2\Big)^k \times$$

$$\times \int_0^t exp[-\nu r_{1n}^2(t-s)]G_{1-\beta,\,-\beta k-1,\,k+1}\Big(-\alpha r_{1n}^2, s\Big)ds, \tag{28}$$

where [2, Eq. (4.5)]

$$\omega_N(r,t) = r\Omega t - \frac{2\Omega}{\nu} \sum_{n=1}^{\infty} \frac{J_1(rr_{1n})}{r_{1n}^3 J_2(Rr_{1n})}\big[1 - exp(-\nu r_{1n}^2 t)\big], \tag{29}$$

is the similar solution for Newtonian fluids, performing the same motion.

3.2 Calculation of the shear stress

Applying the Laplace transform to Eq. (8) we find that

$$\overline{\tau}(r,q) = (\mu + \alpha_1 q^\beta)(\frac{\partial}{\partial r} - \frac{1}{r})\overline{\omega}(r,q). \tag{30}$$

The image function $\overline{\omega}(r,q)$ can be obtained using Eqs. (27)-(29) and the formula

$$L\Big\{\frac{t^a}{\Gamma(a+1)}\Big\} = \frac{1}{q^{a+1}}, \qquad a > -1. \tag{31}$$

Consequently, applying the Laplace transform to Eq. (28), differentiating the result with respect to r and using the identity

$$rJ_1'(rr_{1n}) - J_1(rr_{1n}) = -rr_{1n}J_2(rr_{1n}), \tag{32}$$

we find that

$$\frac{\partial\overline{\omega}}{\partial r} - \frac{\overline{\omega}}{r} = \frac{2\Omega}{\nu}\sum_{n=1}^{\infty}\frac{J_2(rr_{1n})}{r_{1n}^2 J_2(Rr_{1n})}\left(\frac{1}{q} - \frac{1}{q+\nu r_{1n}^2}\right) -$$

$$-2\alpha\Omega\sum_{n=1}^{\infty}\frac{r_{1n}^2 J_2(rr_{1n})}{J_2(Rr_{1n})}\sum_{k,j=0}^{\infty}\frac{\left(-\nu\, r_{1n}^2\right)^k\left(-\alpha\, r_{1n}^2\right)^j\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}\times$$

$$\times\frac{1}{q+\nu r_{1n}^2}\frac{1}{q^{k+(1-\beta)(j+1)+1}}. \tag{33}$$

Introducing (33) into (30) we get

$$\overline{\tau}(r,q) = 2\rho\Omega\sum_{n=1}^{\infty}\frac{J_2(rr_{1n})}{r_{1n}^2 J_2(Rr_{1n})}\left(\frac{1}{q} - \frac{1}{q+\nu r_{1n}^2}\right) + 2\alpha_1\Omega\sum_{n=1}^{\infty}\frac{J_2(rr_{1n})}{J_2(Rr_{1n})}\frac{q^{\beta-1}}{q+\nu r_{1n}^2} -$$

$$-2\alpha\Omega\sum_{n=1}^{\infty}\frac{r_{1n}^2 J_2(rr_{1n})}{J_2(Rr_{1n})}\sum_{k,j=0}^{\infty}\frac{\left(-\nu\, r_{1n}^2\right)^k\left(-\alpha r_{1n}^2\right)^j\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}\times$$

$$\times\left\{\frac{1}{q+\nu r_{1n}^2}\left[\frac{\mu}{q^{k+(1-\beta)(j+1)+1}} - \frac{\nu\alpha_1 r_{1n}^2}{q^{k+3+(1-\beta)j-2\beta}}\right] + \frac{\alpha_1}{q^{k+3+(1-\beta)j-2\beta}}\right\}. \tag{34}$$

Applying the inverse Laplace transform to Eq. (34), we find that the shear stress $\tau(r,t)$ has the form

$$\tau(r,t) = \tau_N(r,t) + 2\alpha_1\Omega\sum_{n=1}^{\infty}\frac{J_2(rr_{1n})}{J_2(Rr_{1n})}G_{1,\beta-1,1}(-\nu r_{1n}^2, t) -$$

$$-2\alpha\Omega\sum_{n=1}^{\infty}\frac{r_{1n}^2 J_2(rr_{1n})}{J_2(Rr_{1n})}\sum_{k,j=0}^{\infty}\frac{\left(-\nu\, r_{1n}^2\right)^k\left(-\alpha\, r_{1n}^2\right)^j\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}\times$$

$$\times\int_0^t exp[-\nu r_{1n}^2(t-s)]\left\{\frac{\mu s^{k+(1-\beta)(j+1)}}{\Gamma[k+(1-\beta)(j+1)+1]} -\right.$$

$$-\frac{\nu\alpha_1 r_{1n}^2 s^{k+2+(1-\beta)j-2\beta}}{\Gamma[k+3+(1-\beta)j-2\beta]}\Bigg\}ds\ -2\alpha\Omega\sum_{n=1}^{\infty}\frac{r_{1n}^2 J_2(rr_{1n})}{J_2(Rr_{1n})}\times$$

$$\times\sum_{k,j=0}^{\infty}\frac{\Big(-\nu\, r_{1n}^2\Big)^k\Big(-\alpha\, r_{1n}^2\Big)^j\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}\frac{\alpha_1 t^{k+2+(1-\beta)j-2\beta}}{\Gamma[k+3+(1-\beta)j-2\beta]}, \tag{35}$$

where [2, Eq. (5.3) for $\alpha = 0$]

$$\tau_N(r,t) = 2\rho\Omega\sum_{n=1}^{\infty}\frac{J_2(rr_{1n})}{r_{1n}^2 J_2(Rr_{1n})}\big[1-exp(-\nu r_{1n}^2 t)\big], \tag{36}$$

is the shear stress corresponding to a Newtonian fluid performing the same motion.

4 Special cases

Making $\beta = 1$ into Eq. (28), we obtain the velocity field

$$\omega(r,t) = \omega_N(r,t) + 2\alpha\Omega\sum_{n=1}^{\infty}\frac{r_{1n}J_1(rr_{1n})}{J_2(Rr_{1n})}\sum_{k=0}^{\infty}\Big(-\nu\, r_{1n}^2\Big)^k\times$$

$$\times\int_0^t exp[-\nu r_{1n}^2(t-s)]G_{0,-k-1,k+1}\Big(-\alpha r_{1n}^2, s\Big)ds, \tag{37}$$

corresponding to an ordinary second grade fluid, performing the same motion. Similarly, from (35), we obtain the shear stress

$$\tau(r,t) = \tau_N(r,t) + 2\alpha_1\Omega\sum_{n=1}^{\infty}\frac{J_2(rr_{1n})}{J_2(Rr_{1n})}G_{1,0,1}(-\nu r_{1n}^2, t) -$$

$$-2\alpha\Omega\sum_{n=1}^{\infty}\frac{r_{1n}^2 J_2(rr_{1n})}{J_2(Rr_{1n})}\sum_{k,j=0}^{\infty}\frac{(-\nu\, r_{1n}^2)^k(-\alpha r_{1n}^2)^j\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}\times$$

$$\times\int_0^t exp[-\nu r_{1n}^2(t-s)](\mu+\nu\alpha_1 r_{1n}^2)\frac{s^k}{\Gamma(k+1)}ds-$$

$$-2\alpha\alpha_1\Omega\sum_{n=1}^{\infty}\frac{r_{1n}^2 J_2(rr_{1n})}{J_2(Rr_{1n})}\sum_{k,j=0}^{\infty}\frac{(-\nu\, r_{1n}^2)^k(-\alpha r_{1n}^2)^j\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}\frac{t^k}{\Gamma(k+1)}, \tag{38}$$

corresponding to an ordinary second grade fluid, performing the same motion. The above relations can be simplified if we use the following relations:

$$G_{0,-k-1,k+1}(-\alpha r_{1n}^2, s) = \frac{s^k}{\Gamma(k+1)} \sum_{j=0}^{\infty} \frac{(-\alpha r_{1n}^2)^j \Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)} =$$
$$= \frac{s^k}{\Gamma(k+1)} (1+\alpha r_{1n}^2)^{-(k+1)}, \tag{39}$$

$$\sum_{k=0}^{\infty} (-\nu r_{1n}^2)^k G_{0,-k-1,k+1}(-\alpha r_{1n}^2, s) = \frac{1}{1+\alpha r_{1n}^2} \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{\nu r_{1n}^2 s}{1+\alpha r_{1n}^2} \right)^k =$$
$$= \frac{1}{1+\alpha r_{1n}^2} exp\left(-\frac{\nu r_{1n}^2 s}{1+\alpha r_{1n}^2} \right), \tag{40}$$

and

$$G_{1,0,1}(-\nu r_{1n}^2, t) = exp\Big(-\nu r_{1n}^2 t \Big). \tag{41}$$

As a result, we find the velocity field and the adequate shear stress under simplified forms

$$\omega(r,t) = r\Omega t - \frac{2\Omega}{\nu} \sum_{n=1}^{\infty} \frac{J_1(r r_{1n})}{r_{1n}^3 J_2(R r_{1n})} \left[1 - exp\left(-\frac{\nu r_{1n}^2}{1+\alpha r_{1n}^2} t \right) \right] \tag{42}$$

and

$$\tau(r,t) = 2\rho\Omega \sum_{n=1}^{\infty} \frac{J_2(r r_{1n})}{r_{1n}^2 J_2(R r_{1n})} \left[1 - \frac{1}{1+\alpha r_{1n}^2} exp\left(-\frac{\nu r_{1n}^2}{1+\alpha r_{1n}^2} t \right) \right], \tag{43}$$

which are identical to Eqs. (5.1) and (5.3) from [2].

If in Eqs. (42) and (43), we make $\alpha = 0$, then the corresponding solutions of the Newtonian fluids are recovered.

5 Conclusions

In this note, the velocity field and the adequate shear stress corresponding to the rotational flow induced by an infinite circular cylinder in an incompressible generalized second grade fluid, have been determined using Hankel and Laplace transforms. The motion is produced by the circular cylinder that at the initial moment begins to rotate around its axis with angular velocity Ωt. The solutions that have been obtained, written under integral and series forms in terms of generalized G-function, satisfy all imposed initial and boundary conditions. Furthermore, they are presented as a sum between the Newtonian solutions and the adequate non-Newtonian contributions. In the special case when $\beta = 1$, or $\beta = 1$ and $\alpha = 0$, the corresponding solutions for ordinary second grade fluid and Newtonian fluid, respectively, performing the same motion, are obtained.

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Mathematical models in regression credibility theory

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Abstract. In this paper we give the matrix theory of some regression credibility models and we try to demonstrate what kind of data is needed to apply linear algebra in the regression credibility models. Just like in the case of classical credibility model we will obtain a credibility solution in the form of a linear combination of the individual estimate (based on the data of a particular state) and the collective estimate (based on aggregate USA data). To illustrate the solution with the properties mentioned above, we shall need the well-known representation formula of the inverse for a special class of matrices. To be able to use the better linear credibility results obtained in this study, we will provide useful estimators for the structure parameters, using the matrix theory, the scalar product of two vectors, the norm and the concept of perpendicularity with respect to a positive definite matrix given in advance, an extension of Pythagoras' theorem, properties of the trace for a square matrix, complicated mathematical properties of conditional expectations and of conditional covariances.

Mathematics subject classification: 15A03, 15A12, 15A48, 15A52, 15A60, 62P05, 62J12, 62J05.
Keywords and phrases: Linearized regression credibility premium, the structural parameters, unbiased estimators.

Introduction

In this paper we give the matrix theory of some regression credibility models.

The article contains a description of the Hachemeister regression model allowing for effects like inflation.

In Section 1 we give Hachemeister's original model, which involves only one isolated contract. In this section we will give the assumptions of the Hachemeister regression model and the optimal linearized regression credibility premium is derived. Just like in the case of classical credibility model, we will obtain a credibility solution in the form of a linear combination of the individual estimate (based on the data of a particular state) and the collective estimate (based on aggregate USA data). To illustrate the solution with the properties mentioned above, we shall need the well-known representation formula of the inverse for a special class of matrices. It turns out that this procedure does not provide us with a statistic computable from the observations, since the result involves unknown parameters of the structure function. To obtain estimates for these structure parameters, for Hachemeister's classical model we embed the contract in a collective of contracts, all providing independent information on the structure distribution.

Section 2 describes the classical Hachemeister model. In the classical Hachemeister model, a portfolio of contracts is studied. Just as in Section 1, we will derive the best linearized regression credibility premium for this model and we will provide some useful estimators for the structure parameters, using a well-known representation theorem for a special class of matrices, properties of the trace for a square matrix, the scalar product of two vectors, the norm $||\cdot||^2_P$, the concept of perpendicularity $\perp$ and an extension of Pythagoras' theorem, where P is a positive definite matrix given in advance. So, to be able to use the result from Section 1, one still has to estimate the portfolio characteristics. Some unbiased estimators are given in Section 2. From the practical point of view the attractive property of unbiasedness for these estimators is stated.

1 The original regression credibility model of Hachemeister

In the original regression credibility model of Hachemeister, we consider one contract with unknown and fixed risk parameter θ, during a period of t (≥ 2) years. The yearly claim amounts are denoted by $X_1, \dots, X_t$. Suppose $X_1, \dots, X_t$ are random variables with finite variance. The contract is a random vector consisting of a random structure parameter θ and observations $X_1, \dots, X_t$. Therefore, the contract is equal to $(\theta, \underline{X}')$, where $\underline{X}' = (X_1, \dots, X_t)$. For this model we want to estimate the net premium: $\mu(\theta) = E(X_j|\theta)$, $j = \overline{1,t}$ for a contract with risk parameter θ.

Remark 1.1. In the credibility models, the pure net risk premium of the contract with risk parameter θ is defined as:

$$\mu(\theta) = E(X_j|\theta),\ \forall j = \overline{1,t}. \tag{1.1}$$

Instead of assuming time independence in the pure net risk premium (1.1) one could assume that the conditional expectation of the claims on a contract changes in time, as follows:

$$\mu_j(\theta) = E(X_j|\theta) = \underset{\sim j}{Y}'\underset{\sim}{b}(\theta),\ \forall j = \overline{1,t}, \tag{1.2}$$

where the design vector $\underset{\sim j}{Y}$ is known ($\underset{\sim j}{Y}$ is a column vector of length q, the non-random ($q \times 1$) vector $\underset{\sim j}{Y}$ is known) and where the $\underset{\sim}{b}(\theta)$ are the unknown regression constants ($\underset{\sim}{b}(\theta)$ is a column vector of length q).

Remark 1.2. Because of inflation we are not willing to assume that $E(X_j|\theta)$ is independent of j. Instead we make the regression assumption $E(X_j|\theta) = \underset{\sim j}{Y}'\underset{\sim}{b}(\theta)$.

When estimating the vector $\underset{\sim}{\beta}$ from the initial regression hypothesis $E(X_j) = \underset{\sim j}{Y}'\underset{\sim}{\beta}$ formulated by actuary, Hachemeister found great differences. He then assumed that to each of the states there was related an unknown random risk parameter θ containing the risk characteristics of that state, and that θ's from different states were independent and identically distributed. Again considering one particular state, we assume that $E(X_j|\theta) = \underset{\sim j}{Y}'\underset{\sim}{b}(\theta)$, with $E[\underset{\sim}{b}(\theta)] = \underset{\sim}{\beta}$.

Consequence of the hypothesis (1.2):

$$\underset{\sim}{\mu}^{(t,1)}(\theta) = E(\underset{\sim}{X}|\theta) = \underset{\sim}{Y}\underset{\sim}{b}(\theta), \tag{1.3}$$

where $\underset{\sim}{Y}$ is a $(t \times q)$ matrix given in advance, the so-called design matrix of full rank q $(q \leq t)$ [the $(t \times q)$ design matrix $\underset{\sim}{Y}$ is known and having full rank $q \leq t$] and where $\underset{\sim}{b}(\theta)$ is an unknown regression vector [$\underset{\sim}{b}(\theta)$ is a column vector of length q].

Observations. By a suitable choice of the $\underset{\sim}{Y}$ (assumed to be known), time effects on the risk premium can be introduced.

Examples. **1**) If the design matrix is for example chosen as follows: $\underset{\sim}{Y} = \underset{\sim}{Y}^{(t,3)} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2^2 \\ \vdots & \vdots & \vdots \\ 1 & t & t^2 \end{bmatrix}$ we obtain a quadratic inflationary trend: $\mu_j(\theta) = b_1(\theta) + jb_2(\theta) + j^2b_3(\theta)$, $j = \overline{1,t}$, where $\underset{\sim}{b}(\theta) = (b_1(\theta), b_2(\theta), b_3(\theta))'$. Indeed, by standard computations we obtain: $\underset{\sim}{\mu}^{(t,1)}(\theta) = \underset{\sim}{Y}\underset{\sim}{b}(\theta) = (1b_1(\theta) + 1b_2(\theta) + 1^2b_3(\theta),\ 1b_1(\theta) + 2b_2(\theta) + 2^2b_2(\theta), \ldots, 1b_1(\theta) + tb_2(\theta) + t^2b_3(\theta))'$ and as $\underset{\sim}{\mu}^{(t,1)}(\theta) = (\mu_1(\theta), \mu_2(\theta), \ldots, \mu_t(\theta))'$ results that is established our first assertion.

2) If the design matrix is for example chosen as follows: $\underset{\sim}{Y} = \underset{\sim}{Y}^{(t,2)} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & t \end{bmatrix}$ (the last column of 1 is omitted) a linear inflation results: $\mu_j(\theta) = b_1(\theta) + jb_2(\theta)$, $j = \overline{1,t}$, where $\underset{\sim}{b}(\theta) = (b_1(\theta), b_2(\theta))'$. The proof is similar.

After these motivating introductory remarks, we state the model assumptions in more detail.

Let $\underset{\sim}{X} = (X_1, \ldots, X_t)'$ be an observed random $(t \times 1)$ vector and θ an unknown random risk parameter. We assume that:

$$E(\underset{\sim}{X}|\theta) = \underset{\sim}{Y}\underset{\sim}{b}(\theta). \tag{H_1}$$

It is assumed that the matrices:

$$\underset{\sim}{\Lambda} = \mathrm{Cov}[\underset{\sim}{b}(\theta)](\underset{\sim}{\Lambda} = \underset{\sim}{\Lambda}^{(q\times q)}) \tag{H_2}$$

$$\underset{\sim}{\Phi} = E[\mathrm{Cov}(\underset{\sim}{X}|\theta) \; (\underset{\sim}{\Phi} = \underset{\sim}{\Phi}^{(t\times t)}) \tag{H_3}$$

are positive definite. We finally introduce: $E[\underset{\sim}{b}(\theta)] = \underset{\sim}{\beta}$.

Let $\tilde{\mu}_j$ be the credibility estimator of $\mu_j(\theta)$ based on $\underset{\sim}{X}$.

For the development of an expression for $\tilde{\mu}_j$, we shall need the following lemma.

Lemma 1.1 (Representation formula of the inverse for a special class of matrices). *Let $\underset{\sim}{A}$ be an $(r \times s)$ matrix and $\underset{\sim}{B}$ an $(s \times r)$ matrix. Then*

$$(\underset{\sim}{I} + \underset{\sim}{A}\underset{\sim}{B})^{-1} = \underset{\sim}{I} - \underset{\sim}{A}(\underset{\sim}{I} + \underset{\sim}{B}\underset{\sim}{A})^{-1}\underset{\sim}{B}, \tag{1.4}$$

if the displayed inverses exist.

Proof. We have

$$\begin{aligned}
\underset{\sim}{I} &= \underset{\sim}{I} + \underset{\sim}{A}\underset{\sim}{B} - \underset{\sim}{A}\underset{\sim}{B} = \underset{\sim}{I} + \underset{\sim}{A}\underset{\sim}{B} - \underset{\sim}{A}(\underset{\sim}{I} + \underset{\sim}{B}\underset{\sim}{A})(\underset{\sim}{I} + \underset{\sim}{B}\underset{\sim}{A})^{-1}\underset{\sim}{B} = \\
&= (\underset{\sim}{I} + \underset{\sim}{A}\underset{\sim}{B}) - (\underset{\sim}{I}\underset{\sim}{A} + \underset{\sim}{A}\underset{\sim}{B}\underset{\sim}{A})(\underset{\sim}{I} + \underset{\sim}{B}\underset{\sim}{A})^{-1}\underset{\sim}{B} = \\
&= (\underset{\sim}{I} + \underset{\sim}{A}\underset{\sim}{B}) - (\underset{\sim}{I} + \underset{\sim}{A}\underset{\sim}{B})\underset{\sim}{A}(\underset{\sim}{I} + \underset{\sim}{B}\underset{\sim}{A})^{-1}\underset{\sim}{B}
\end{aligned}$$

giving $\underset{\sim}{I} = (\underset{\sim}{I} + \underset{\sim}{A}\underset{\sim}{B})[\underset{\sim}{I} - \underset{\sim}{A}(\underset{\sim}{I} + \underset{\sim}{B}\underset{\sim}{A})^{-1}\underset{\sim}{B}]$ and multiplying this equation from the left by $(\underset{\sim}{I} + \underset{\sim}{A}\underset{\sim}{B})^{-1}$ gives (1.4).

Observation. $\underset{\sim}{I}$ denotes the $(r \times r)$ identity matrix.

The optimal choice of $\tilde{\mu}_j$ is determined in the following theorem:

Theorem 1.1. *The credibility estimator $\tilde{\mu}_j$ is given by:*

$$\tilde{\mu}_j = \underset{\sim}{Y}'_j[\underset{\sim}{Z}\hat{\underset{\sim}{b}} + (\underset{\sim}{I} - \underset{\sim}{Z})\underset{\sim}{\beta}], \tag{1.5}$$

with:

$$\hat{\underset{\sim}{b}} = (\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1}\underset{\sim}{Y})^{-1}\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1}\underset{\sim}{X}, \tag{1.6}$$

$$\underset{\sim}{Z} = \underset{\sim}{\Lambda}\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1}\underset{\sim}{Y}(\underset{\sim}{I} + \underset{\sim}{\Lambda}\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1}\underset{\sim}{Y})^{-1}, \tag{1.7}$$

where $\underset{\sim}{I}$ denotes the $q \times q$ identity matrix ($\hat{\underset{\sim}{b}} = \hat{\underset{\sim}{b}}^{(q\times 1)}$; $\underset{\sim}{Z} = \underset{\sim}{Z}^{(q\times q)}$), for some fixed j.

Proof. The credibility estimator $\tilde{\mu}_j$ of $\mu_j(\theta)$ based on $\underset{\sim}{X}$ is a linear estimator of the form

$$\tilde{\mu}_j = \gamma_0 + \underset{\sim}{\gamma}'\underset{\sim}{X}, \tag{1.8}$$

which satisfies the normal equations $\begin{cases} E(\tilde{\mu}_j) = E[\mu_j(\theta)] \\ \mathrm{Cov}(\tilde{\mu}_j, X_j) = \mathrm{Cov}[\mu_j(\theta), X_j] \end{cases}$ where γ_0 is a scalar constant, and $\underset{\sim}{\gamma}$ is a constant $(t \times 1)$ vector.

The coefficients γ_0 and $\underset{\sim}{\gamma}$ are chosen such that the normal equations are satisfied.

We write the normal equations as

$$E(\tilde{\mu}_j) = \underset{\sim j}{Y}'\underset{\sim}{\beta}, \tag{1.9}$$

$$\mathrm{Cov}(\tilde{\mu}_j, \underset{\sim}{X}') = \mathrm{Cov}[\mu_j(\theta), \underset{\sim}{X}']. \tag{1.10}$$

After inserting (1.8) in (1.10), one obtains

$$\underset{\sim}{\gamma}'\mathrm{Cov}(\underset{\sim}{X}) = \mathrm{Cov}[\mu_j(\theta), \underset{\sim}{X}'], \tag{1.11}$$

where

$$\begin{aligned}
\mathrm{Cov}(\underset{\sim}{X}) &= E[\mathrm{Cov}(\underset{\sim}{X}(\theta)] + \mathrm{Cov}[E(\underset{\sim}{X}(\theta)] = \\
&= \underset{\sim}{\Phi} + \mathrm{Cov}[\underset{\sim}{Y}\underset{\sim}{b}(\theta)] = \underset{\sim}{\Phi} + \mathrm{Cov}[\underset{\sim}{Y}\underset{\sim}{b}(\theta), (\underset{\sim}{Y}\underset{\sim}{b}(\theta))'] = \\
&= \underset{\sim}{\Phi} + \underset{\sim}{Y}\mathrm{Cov}[\underset{\sim}{b}(\theta), (\underset{\sim}{b}(\theta))'\underset{\sim}{Y}'] = \underset{\sim}{\Phi} + \underset{\sim}{Y}\mathrm{Cov}[\underset{\sim}{b}(\theta), (\underset{\sim}{b}(\theta))']\underset{\sim}{Y}' = \\
&= \underset{\sim}{\Phi} + \underset{\sim}{Y}\mathrm{Cov}[\underset{\sim}{b}(\theta)]\underset{\sim}{Y}' = \underset{\sim}{\Phi} + \underset{\sim}{Y}\underset{\sim}{\Lambda}\underset{\sim}{Y}'
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Cov}[\mu_j(\theta), \underset{\sim}{X}'] &= \mathrm{Cov}[\mu_j(\theta), E(\underset{\sim}{X}'(\theta)] = \mathrm{Cov}[\underset{\sim j}{Y}'\underset{\sim}{b}(\theta), (\underset{\sim}{Y}\underset{\sim}{b}(\theta))'] = \\
&= \underset{\sim j}{Y}'\mathrm{Cov}[\underset{\sim}{b}(\theta), (\underset{\sim}{b}(\theta))'\underset{\sim}{Y}'] = \underset{\sim j}{Y}'\mathrm{Cov}[\underset{\sim}{b}(\theta), (\underset{\sim}{b}(\theta))']\underset{\sim}{Y}' = \\
&= \underset{\sim j}{Y}'\mathrm{Cov}[\underset{\sim}{b}(\theta)]\underset{\sim}{Y}' = \underset{\sim j}{Y}'\underset{\sim}{\Lambda}\underset{\sim}{Y}'
\end{aligned}$$

and thus (1.11) becomes $\underset{\sim}{\gamma}'(\underset{\sim}{\Phi} + \underset{\sim}{Y}\underset{\sim}{\Lambda}\underset{\sim}{Y}') = \underset{\sim j}{Y}'\underset{\sim}{\Lambda}\underset{\sim}{Y}'$, from which

$$\begin{aligned}
\underset{\sim}{\gamma}' &= \underset{\sim j}{Y}'\underset{\sim}{\Lambda}\underset{\sim}{Y}'(\underset{\sim}{\Phi} + \underset{\sim}{Y}\underset{\sim}{\Lambda}\underset{\sim}{Y}')^{-1} = \underset{\sim j}{Y}'\underset{\sim}{\Lambda}\underset{\sim}{Y}'[(\underset{\sim}{I} + \underset{\sim}{Y}\underset{\sim}{\Lambda}\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1})\underset{\sim}{\Phi}]^{-1} = \\
&= \underset{\sim j}{Y}'\underset{\sim}{\Lambda}\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1}(\underset{\sim}{I} + \underset{\sim}{Y}\underset{\sim}{\Lambda}\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1})^{-1}.
\end{aligned}$$

Lemma 1.1. now gives

$$\begin{aligned}
\underset{\sim}{\gamma}' &= \underset{\sim j}{Y}'\underset{\sim}{\Lambda}\underset{\sim}{Y}\underset{\sim}{\Lambda}^{-1}[\underset{\sim}{I} - \underset{\sim}{Y}(\underset{\sim}{I} + \underset{\sim}{\Lambda}\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1}\underset{\sim}{Y})^{-1} \cdot \underset{\sim}{\Lambda}\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1}] = \\
&= \underset{\sim j}{Y}'[\underset{\sim}{\Lambda}\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1} - \underset{\sim}{\Lambda}\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1}\underset{\sim}{Y}(\underset{\sim}{I} + \underset{\sim}{\Lambda}\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1}\underset{\sim}{Y})^{-1}\underset{\sim}{\Lambda}\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1}] = \\
&= \underset{\sim j}{Y}'[\underset{\sim}{I} - \underset{\sim}{\Lambda}\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1}\underset{\sim}{Y}(\underset{\sim}{I} + \underset{\sim}{I} \cdot \underset{\sim}{\Lambda}\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1}\underset{\sim}{Y})^{-1} \cdot \underset{\sim}{I}]\underset{\sim}{\Lambda}\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1} = \\
&= \underset{\sim j}{Y}'(\underset{\sim}{I} + \underset{\sim}{\Lambda}\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1}\underset{\sim}{Y})^{-1}\underset{\sim}{\Lambda}\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1}
\end{aligned}$$

and, once more using Lemma 1.1

$$\begin{aligned}\underset{\sim}{\gamma}'\underset{\sim}{X} &= \underset{\sim j}{Y}'(\underset{\sim}{I}+\underset{\sim}{\Lambda}\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1}\underset{\sim}{Y})^{-1}\underset{\sim}{\Lambda}\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1}\underset{\sim}{X} = \\ &= \underset{\sim j}{Y}'\underset{\sim}{I}(\underset{\sim}{I}+\underset{\sim}{\Lambda}\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1}\underset{\sim}{Y}\underset{\sim}{I})^{-1}\underset{\sim}{\Lambda}\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1}\underset{\sim}{Y}\hat{\underset{\sim}{b}} = \underset{\sim j}{Y}'[\underset{\sim}{I}-(\underset{\sim}{I}+\underset{\sim}{\Lambda}\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1}\underset{\sim}{Y}\underset{\sim}{I})^{-1}]\hat{\underset{\sim}{b}}\end{aligned}$$

with $\hat{\underset{\sim}{b}}$ given by (1.6). According to Lemma 1.1 we obtain

$$\underset{\sim}{\gamma}'\underset{\sim}{X} = \underset{\sim j}{Y}'\{\underset{\sim}{I}-[\underset{\sim}{I}-\underset{\sim}{\Lambda}\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1}\underset{\sim}{Y}(\underset{\sim}{I}+\underset{\sim}{\Lambda}\underset{\sim}{Y}'\underset{\sim}{\Phi}^{-1}\underset{\sim}{Y})^{-1}]\}\cdot\hat{\underset{\sim}{b}} = \underset{\sim j}{Y}'\cdot\underset{\sim}{Z}\cdot\hat{\underset{\sim}{b}},$$

with $\underset{\sim}{Z}$ given by (1.7). Insertion in (1.9) gives

$$\gamma_0 + \underset{\sim j}{Y}'\underset{\sim}{Z}E(\hat{\underset{\sim}{b}}) = \underset{\sim j}{Y}'\underset{\sim}{\beta} \tag{1.12}$$

where

$$\begin{aligned}E(\hat{\underset{\sim}{b}}) &= (\underset{\sim}{Y}\underset{\sim}{\Phi}^{-1}\underset{\sim}{Y})^{-1}\underset{\sim}{Y}\underset{\sim}{\Phi}^{-1}E(\underset{\sim}{X}) = (\underset{\sim}{Y}\underset{\sim}{\Phi}^{-1}\underset{\sim}{Y})^{-1}\underset{\sim}{Y}\underset{\sim}{\Phi}^{-1}E[E(\underset{\sim}{X}|\theta)] = \\ &= (\underset{\sim}{Y}\underset{\sim}{\Phi}^{-1}\underset{\sim}{Y})^{-1}\underset{\sim}{Y}\underset{\sim}{\Phi}^{-1}\underset{\sim}{Y}E[\underset{\sim}{b}(\theta)] = (\underset{\sim}{Y}\underset{\sim}{\Phi}^{-1}\underset{\sim}{Y})^{-1}(\underset{\sim}{Y}\underset{\sim}{\Phi}^{-1}\underset{\sim}{Y})\underset{\sim}{\beta} = \underset{\sim}{\beta}\end{aligned}$$

and thus (1.12) becomes $\gamma_0 + \underset{\sim j}{Y}'\underset{\sim}{Z}\underset{\sim}{\beta} = \underset{\sim j}{Y}'\underset{\sim}{\beta}$ from which $\gamma_0 = \underset{\sim j}{Y}'(\underset{\sim}{I}-\underset{\sim}{Z})\underset{\sim}{\beta}$.

This completes the proof of Theorem 1.1.

2 The classical credibility regression model of Hachemeister

In this section we will introduce the classical regression credibility model of Hachemeister, which consists of a portfolio of k contracts, satisfying the constraints of the original Hachemeister model.

The contract indexed j is a random vector consisting of a random structure θ_j and observations $X_{j1},\dots,X_{jt}$. Therefore the contract indexed j is equal to $(\theta_j,\underline{X}'_j)$, where $\underline{X}'_j = (X_{j1},\dots,X_{jt})$ and $j=\overline{1,k}$ (the variables describing the j^{th} contract are $(\theta_j,\underline{X}'_j)$, $j=\overline{1,k}$). Just as in Section 1, we will derive the best linearized regression credibility estimators for this model.

Instead of assuming time independence in the net risk premium:

$$\mu(\theta_j) = E(X_{jq}|\theta_j), \quad j=\overline{1,k},\ q=\overline{1,t} \tag{2.1}$$

one could assume that the conditional expectation of the claims on a contract changes in time, as follows:

$$\mu_q(\theta_j) = E(X_{jq}|\theta_j) = y_{jq}\beta(\theta_j), \quad j=\overline{1,k},\ q=\overline{1,t}, \tag{2.2}$$

with y_{jq} assumed to be known and $\beta(\cdot)$ assumed to be unknown.

Observations: By a suitable choice of the y_{jq}, time effects on the risk premium can be introduced.

Examples. **1**) If for instance the claim figures are subject to a known inflation i, (2.2) becomes:

$$\mu_q(\theta_j) = E(X_{jq}|\theta_j) = (1+i)^q \cdot \beta(\theta_j), \quad j=\overline{1,k},\ q=\overline{1,t}.$$

2) If in addition the volume w_j changes from contract to contract, one could introduce the model:

$$\mu_q(\theta_j) = E(X_{jq}|\theta_j) = w_j(1+i)^q \cdot \beta(\theta_j), \quad j=\overline{1,k},\ q=\overline{1,t}$$

where w_j and i are given.

Consequence of the hypothesis (2.2):

$$\underline{\mu}^{(t,1)}(\theta_j) = E(\underline{X}_j|\theta_j) = x^{(t,n)}\underline{\beta}^{(n,1)}(\theta_j), \quad =\overline{1,k}, \tag{2.3}$$

where $x^{(t,n)}$ is a matrix given in advance, the so-called design matrix, and where the $\underline{\beta}(\theta_j)$ are the unknown regression constants. Again one assumes that for each contract the risk parameters $\underline{\beta}(\theta_j)$ are the same functions of different realizations of the structure parameter.

Observations: By a suitable choice of the x, time effects on the risk premium can be introduced.

Examples. **1**) If the design matrix is for examples chosen as follows:
$x^{(t,3)} = \begin{bmatrix} 1 & 1 & 1^2 \\ 1 & 2 & 2^2 \\ \vdots & \vdots & \vdots \\ 1 & t & t^2 \end{bmatrix}$, we obtain a quadratic inflationary trend:

$$\mu_q(\theta_j) = \beta_1(\theta_j) + q\beta_2(\theta_j) + q^2\beta_3(\theta_j), \quad j=\overline{1,k},\ q=\overline{1,t}, \tag{2.4}$$

where $\underline{\beta}^{(3,1)}(\theta_j) = (\beta_1(\theta_j), \beta_2(\theta_j), \beta_3(\theta_j))'$, with $j=\overline{1,k}$.

2) If the design matrix is for example chosen as follows:
$x^{(t,2)} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & t \end{bmatrix}$ (the last column of 1) is omitted) a linear inflation results:

$$\mu_q(\theta_j) = \beta_1(\theta_j) + q\beta_2(\theta_j), \quad j=\overline{1,k},\ q=\overline{1,t}, \tag{2.5}$$

where $\underline{\beta}^{(2,1)}(\theta_j) = (\beta_1(\theta_j), \beta_2(\theta_j))'$, with $j=\overline{1,k}$.

For some fixed design matrix $x^{(t,n)}$ of full rank n $(n < t)$, and a fixed weight matrix $v_j^{(t,t)}$, the hypotheses of the Hachemeister model are:

(H_1) The contracts $(\theta_j, \underline{X}'_j)$ are independent, the variables $\theta_1, \ldots, \theta_k$ are independent and identically distributed.

(H_2) $E(\underline{X}_j^{(t,1)}|\theta_j) = x^{(t,n)}\underline{\beta}^{(n,1)}(\theta_j)$, $j = \overline{1,k}$, where $\underline{\beta}$ is an unknown regression vector;

$\mathrm{Cov}(\underline{X}_j^{(t,1)}|\theta_j) = \sigma^2(\theta_j) \cdot v_j^{(t,t)}$, where $\sigma^2(\theta_j) = \mathrm{Var}(X_{jr}|\theta_j)$, $\forall r = \overline{1,t}$ and $v_j = v_j^{(t,t)}$ is a known non-random weight $(t \times t)$ matrix, with $\mathrm{rg} v_j = t$, $j = \overline{1,k}$.

We introduce the structural parameters, which are natural extensions of those in the Bühlmann-Straub model. We have:

$$s^2 = E[\sigma^2(\theta_j)] \tag{2.6}$$

$$a = a^{(n,n)} = \mathrm{Cov}[\underline{\beta}(\theta_j)] \tag{2.7}$$

$$\underline{b} = \underline{b}^{(n,1)} = E[\underline{\beta}(\theta_j)], \tag{2.8}$$

where $j = \overline{1,k}$.

After the credibility result based on these structural parameters is obtained, one has to construct estimates for these parameters. Write: $c_j = c_j^{(t,t)} = \mathrm{Cov}(\underline{X}_j)$, $u_j = u_j^{(n,n)} = (x'v_j^{-1}x)^{-1}$, $z_j = z_j^{(n,n)} = a(a + s^2u_j)^{-1} =$ [the resulting credibility factor for contract j], $j = \overline{1,k}$.

Before proving the linearized regression credibility premium, we first give the classical result for the regression vector, namely the GLS-estimator for $\underline{\beta}(\theta_j)$.

Theorem 2.1 (Classical regression result). *The vector $\underline{B}_j$ minimizing the weighted distance to the observations $\underline{X}_j$,*

$$d(\underline{B}_j) = (\underline{X}_j - x\underline{B}_j)'v_j^{-1}(\underline{X}_j - x\underline{B}_j),$$

reads

$$\underline{B}_j = (x'v_j^{-1}x)^{-1}x'v_j^{-1}\underline{X}_j = u_jx'v_j^{-1}\underline{X}_j \ ,$$

or

$$\underline{B}_j = (x'c_j^{-1}x)^{-1}x'c_j^{-1}\underline{X}_j \quad \textit{in case} \quad c_j = s^2v_j + xax'.$$

Proof. The first equality results immediately from the minimization procedure for the quadratic form involved, the second one from Lemma 2.1.

Lemma 2.1. (Representation theorem for a special class of matrices). *If C and V are $t \times t$ matrices, A an $n \times n$ matrix and Y a $t \times n$ matrix, and*

$$C = s^2V + YAY',$$

then

$$(Y'C^{-1}Y)^{-1} = s^2(Y'V^{-1}Y)^{-1} + A$$

and

$$(Y'C^{-1}Y)^{-1}Y'C^{-1} = (Y'V^{-1}Y)^{-1}Y'V^{-1}.$$

We can now derive the regression credibility results for the estimates of the parameters in the linear model. Multiplying this vector of estimates by the design matrix provides us with the credibility estimate for $\underline{\mu}(\theta_j)$, see (2.3).

Theorem 2.2 (Linearized regression credibility premium). *The best linearized estimate of* $E[\underline{\beta}^{(n,1)}(\theta_j)|\underline{X}_j]$ *is given by:*

$$\underline{M}_j = z_j^{(n,n)}\underline{B}_j^{(n,1)} + (I^{(n,n)} - z_j^{(n,n)})\underline{b}^{(n,1)} \tag{2.9}$$

and the best linearized estimate of $E[x^{(t,n)}\underline{\beta}^{(n,1)}(\theta_j)|\underline{X}_j]$ *is given by*:

$$x^{(t,n)}\underline{M}_j = x^{(t,n)}[z_j^{(n,n)}\underline{B}_j^{(n,1)} + (I^{(n,n)} - z_j^{(n,n)})\underline{b}^{(n,1)}]. \tag{2.10}$$

Proof. The best linearized estimate $\underline{M}_j$ of $E[\underline{\beta}(\theta_j)|\underline{X}_j]$ is determined by solving the following problem

$$\underset{\varepsilon}{Min}\ d(\varepsilon), \tag{2.11}$$

with

$$\begin{aligned} d(\varepsilon) &= ||\underline{\beta}(\theta_j) - (\underline{M}_j + \varepsilon\underline{V})||_p^2 = \\ &= E[(\underline{\beta}(\theta_j) - \underline{M}_j - \varepsilon\underline{V})'P(\underline{\beta}(\theta_j) - \underline{M}_j - \varepsilon\underline{V})], \end{aligned} \tag{2.12}$$

where $\underline{V} = \underline{V}^{(n,1)}$ is a linear combination of 1 and the components of $\underline{X}_j$, $P = P^{(n,n)}$ is a positive definite matrix given in advance and $||\cdot||_p^2$ is a norm defined by: $||\underline{X}||_p^2 = E(\underline{X}'PX)$, with $\underline{X} = \underline{X}^{(n,1)}$ an arbitrary vector.

The theorem holds in case $d'(0) = 0$ for every $\underline{V}$. Standard computations lead to

$$\begin{aligned} d(\varepsilon) &= E[(\underline{\beta}(\theta_j))'P\underline{\beta}(\theta_j)] - E[(\underline{\beta}(\theta_j))'P\underline{M}_j] - \\ &- \varepsilon E[(\underline{\beta}(\theta_j))'P\underline{V}] - E[\underline{M}_j'P\underline{\beta}(\theta_j)] + E[\underline{M}_j'P\underline{M}_j] + \\ &+ \varepsilon E[\underline{M}_j'P\underline{V}] - \varepsilon E[\underline{V}'P\underline{\beta}(\theta_j)] + \varepsilon E[\underline{V}'P\underline{M}_j] + \varepsilon^2 E[\underline{V}'P\underline{V}] \end{aligned} \tag{2.13}$$

The derivative $d'(\varepsilon)$ is given by

$$d'(\varepsilon) = -2E[\underline{V}'P(\underline{\beta}(\theta_j) - \underline{M}_j - \varepsilon\underline{V})] \tag{2.14}$$

Define reduced variables by

$$\underline{\beta}^0(\theta_j) = \underline{\beta}(\theta_j) - E[\underline{\beta}(\theta_j)] = \underline{\beta}(\theta_j) - \underline{b} \tag{2.15}$$

$$\underline{B}_j^0 = \underline{B}_j - E(\underline{B}_j) = \underline{B}_j - \underline{b}, \tag{2.16}$$

$$\underline{X}_j^0 = \underline{X}_j - E(\underline{X}_j) = \underline{X}_j - x\underline{b}. \tag{2.17}$$

Inserting $\underline{M}_j$ from (2.9) in (2.14) for $\varepsilon = 0$, we have to prove that

$$E[\underline{V}'P(\underline{\beta}(\theta_j) - Z_j\underline{B}_j - \underline{b}z_j\underline{b})] = 0, \tag{2.18}$$

for every $\underline{V}$.

Using (2.15) and (2.16), the relation (2.18) can be written as

$$E[\underline{V}'P(\underline{\beta}^0(\theta_j) - z_j\underline{B}_j^0)] = 0, \tag{2.19}$$

for every $\underline{V}$.

But since $\underline{V}$ is an arbitrary vector, with as components linear combinations of 1 and the components of $\underline{X}_j$, it may be written as

$$\underline{V} = \underline{\alpha}_0 + \underline{\alpha}_1^{(n,t)}\underline{X}_j^0. \tag{2.20}$$

Therefore one has to prove that

$$E[(\underline{\alpha}_0' + \underline{X}_j^{0'}\alpha_1')P(\underline{\beta}^0(\theta_j) - z_j\underline{B}_j^0)] = 0, \tag{2.21}$$

for every $\underline{V}$.

Standard computations lead to the following expression for the left hand side

$$\begin{aligned}
&\underline{\alpha}_0'PE[\underline{\beta}^0(\theta_j)] + E[\underline{X}_j^{0'}\alpha_1'P\underline{\beta}^0(\theta_j)] - \underline{\alpha}_0'Pz_jE(\underline{B}_j^0) - E[\underline{X}_j^{0'}\alpha_1'PZ_j\underline{B}_j^0] = \\
&= E[\underline{X}_j^{0'}\alpha_1'P(\underline{\beta}^0(\theta_j) - z_j\underline{B}_j^0)] = E\{Tr[\underline{X}_j^{0'}\alpha_1'P(\underline{\beta}^0(\theta_j) - z_j\underline{B}_j^0)]\} = \\
&= E\{Tr[\alpha_1'P(\underline{\beta}^0(\theta_j) - z_j\underline{B}_j^0)\underline{X}_j^{0'}]\} = Tr\{\alpha_1'PE[(\underline{\beta}^0(\theta_j) - z_j\underline{B}_j^0)\underline{X}_j^{0'}]\},
\end{aligned} \tag{2.22}$$

where we used the fact that $E[\underline{B}^0(\theta_j)] = 0$, $E(\underline{\beta}_j^0) = 0$ and that a scalar random variable trivially equals its trace, and also that $Tr(AB) = Tr(BA)$.

Expression (2.22) is equal to zero, as can be seen by

$$\begin{aligned}
&E[(\underline{\beta}^0(\theta_j) - z_j\underline{B}_j^0)\underline{X}_j^{0'}] = E[\underline{\beta}^0(\theta_j)\underline{X}_j^{0'}] - z_jE(\underline{B}_j^0\underline{X}_j^{0'}) = \\
&= \mathrm{Cov}\,[\underline{\beta}^0(\theta_j), \underline{X}_j^{0'}] - z_j\mathrm{Cov}(\underline{B}_j^0, \underline{X}_j^{0'}) = \\
&= \mathrm{Cov}[\underline{\beta}(\theta_j), \underline{X}_j] - z_j\mathrm{Cov}(\underline{B}_j, \underline{X}_j) = \\
&= ax' - z_j(a + s^2u_j)x' = ax' - a(a + s^2u_j)^{-1}(a + s^2u_j)x' = \\
&= ax' - ax' = 0.
\end{aligned} \tag{2.23}$$

This proves (2.9), (2.10) follows by replacing P in (2.12) by $x'Px$. So repeating the same reasoning as above we arrive at (2.10).

Remark 2.1. Here and in the following we present the main results leaving the detailed computations to the reader.

Remark 2.2. From (2.9) we see that the credibility estimates for the parameters of the linear model are given as the matrix version of a convex mixture of the classical regression result $\underline{B}_j$ and the collective result $\underline{b}$.

Theorem 2.2 concerns a special contract j. By the assumptions, the structural parameters $a, \underline{b}$ and s^2 do not depend on j. So if there are more contracts, these parameters can be estimated.

Every vector $\underline{B}_j$ gives an unbiased estimator of $\underline{b}$. Consequently, so does every linear combination of the type $\Sigma\alpha_j\underline{B}_j$, where the vector of matrices $(\alpha_j^{(n,n)})_{j=\overline{1,k}}$, is such that:

$$\sum_{j=1}^{k}\alpha_j^{(n,n)} = I^{(n,n)}. \tag{2.24}$$

The optimal choice of $\alpha_j^{(n,n)}$ is determined in the following theorem:

Theorem 2.3 (Estimation of the parameters $\underline{b}$ in the regression credibility model). *The optimal solution to the problem*

$$\underset{\underline{\alpha}}{Min}\, d(\alpha), \tag{2.25}$$

where:

$$d(\underline{\alpha}) = \left\|\underline{b} - \sum_j \alpha_j\underline{B}_j\right\|_p^2 \underline{\underline{def}}\; E\left[\left(\underline{b} - \sum_j \alpha_j\underline{B}_j\right)' P\left(\underline{b} - \sum_j \alpha_j\underline{B}_j\right)\right]$$

(the distance from $\left(\sum_j \alpha_j\underline{B}_j\right)$ *to the parameters* $\underline{b}$*),* $P = P^{(n,n)}$ *a given positive definite matrix (P is a non-negative definite matrix), with the vector of matrices* $\underline{\alpha} = (\alpha_j)_{j=\overline{1,k}}$ *satisfying (2.24), is:*

$$\underline{\hat{b}}^{(n,1)} = Z^{-1}\sum_{j=1}^{k} z_j\underline{B}_j, \tag{2.26}$$

where $Z = \sum_{j=1}^{k} z_j$ *and* z_j *is defined as:* $z_j = a(a+s^2u_j)^{-1}$, $j=\overline{1,k}$.

Proof. Using the norm $||X||_p^2 = E(X'PX)$ and the perpendicularity concept $\perp$ of two vectors $\underline{X}^{(n,1)}$ and $\underline{Y}^{(n,1)}$ defined by $\underline{X}\perp\underline{Y}$ iff $E(\underline{X}'P\underline{Y}) = 0$, we see that it is sufficient to prove that for all feasible $\underline{\alpha}$

$$(\underline{\hat{b}} - \underline{b}) \perp \left(\sum_j \alpha_j\underline{B}_j - \underline{\hat{b}}\right), \tag{2.27}$$

since then according to an extension of Pythagoras$'$ theorem

$$\underline{X}\perp\underline{Y} \Leftrightarrow ||X+Y||_p^2 = ||X||_p^2 + ||Y||_p^2,$$

we have

$$\begin{aligned} ||\underline{b} - \sum_j \alpha_j\underline{B}_j||_p^2 &= ||\underline{b} - \underline{\hat{b}} + \underline{\hat{b}} - \sum_j \alpha_j\underline{B}_j||_p^2 = \\ &= ||\underline{b} - \underline{\hat{b}}||_p^2 + ||\underline{\hat{b}} - \sum_j \alpha_j\underline{\beta}_j||_p^2 \end{aligned} \tag{2.28}$$

so for every choice of $\underline{\alpha}$ one gets

$$||\underline{b}-\hat{\underline{b}}||_p^2 \leq ||\underline{b}-\sum_j \alpha_j \underline{B}_j||_p^2. \tag{2.29}$$

So let us show now that (2.27) holds. It is clear that

$$\hat{\underline{b}}-\sum_j \alpha_j \underline{B}_j = Z^{-1}\sum_j z_j \underline{B}_j - \sum \alpha_j \underline{B}_j = \sum_j (Z^{-1}Z_j-\alpha_j)\underline{B}_j = \sum_j \gamma_j \underline{B}_j \tag{2.30}$$

with

$$\sum_j \gamma_j = \sum_j (Z^{-1}z_j-\alpha_j) = Z^{-1}\sum_j z_j - \sum_j \alpha_j = Z^{-1}Z - I = I - I = 0, \tag{2.31}$$

where $\gamma_j = Z^{-1}z_j-\alpha_j$, $j=\overline{1,k}$. To prove (2.27), we have to show that

$$\left(\sum_j \gamma_j \underline{B}_j\right) \perp (\hat{\underline{b}}-\underline{b}) \tag{2.32}$$

so that

$$E\left[\left(\sum_j \gamma_j \underline{B}_j\right)' P(\hat{\underline{b}}-\underline{b})\right] = 0. \tag{2.33}$$

The left hand side of (2.33) can successively be rewritten as follows

$$\begin{aligned}
&E\left[\left(\sum_j \underline{B}_j'\gamma_j'\right) P(\hat{\underline{b}}-\underline{b})\right] = \sum_j E(\underline{B}_j'\gamma_j' P\hat{\underline{b}}^0) = \\
&= \sum_j [E(\underline{B}_j'\gamma_j' P\hat{\underline{b}}^0) - \underline{b}'\gamma_j' P E(\hat{\underline{b}}^0)] = \\
&= \sum_j [E(\underline{B}_j'\gamma_j' P\hat{\underline{b}}^0)) - E(\underline{b}'\gamma_j' P\hat{\underline{b}}^0)] = \sum_j E[(\underline{B}_j'-\underline{b}')\gamma_j' P\hat{\underline{b}}^0] = \\
&= \sum_j E[(\underline{B}_j'-E(\underline{B}_j'))\gamma_j' P\hat{\underline{b}}^0] = \sum_j E(\underline{B}_j'^0\gamma_j' P\hat{\underline{b}}^0) = \\
&= \sum_j E(\underline{B}_j'^0\gamma_j' PZ^{-1}\cdot\sum_i z_i \underline{B}_i^0) = \sum_{j,i} E(\underline{B}_j'^0\gamma_j' PZ^{-1} z_i \underline{B}_i^0) = \\
&= \sum_{j,i} E[Tr(\underline{B}_j'^0\gamma_j' PZ^{-1} z_i \underline{B}_i^0)] = \sum_{j,i} E[Tr(\gamma_j' PZ^{-1} z_i \underline{B}_i^0 \underline{B}_j'^0)] = \\
&= \sum_{j,i} Tr[\gamma_j' PZ^{-1} z_i E(\underline{B}_i^0 \underline{B}_j'^0)] = \sum_{j,i} Tr[\gamma_j' PZ^{-1} z_i \mathrm{Cov}\,(\underline{B}_i^0, \underline{B}_j^0)] = \\
&= \sum_{j,i} Tr[\gamma_j' PZ^{-1} z_i \mathrm{Cov}\,(\underline{B}_i \underline{B}_j)] = \sum_{j,i} Tr[\gamma_j' PZ^{-1} z_i \delta_{ij}(a+s^2 u_j)] =
\end{aligned}$$

$$
\begin{aligned}
&= \sum_j Tr[\gamma_j' P Z^{-1} z_j (a + s^2 u_j)] = \\
&= \sum_j Tr[\gamma_j' P Z^{-1} z_j a (a + s^2 u_j)^{-1} (a + s^2 u_j)] = \\
&= \sum_j Tr(\gamma_j' P Z^{-1} a) = Tr\left[(\sum_j \gamma_j') P Z^{-1} a \right] = \\
&= Tr(0 P Z^{-1} a) = Tr(0) = 0,
\end{aligned} \tag{2.34}
$$

where $\hat{\underline{b}^0} = \hat{\underline{b}} - E(\hat{\underline{b}}) = \hat{\underline{b}} - \underline{b}$, $\hat{\underline{B}_j'^0} = \underline{B}_j' - E(\underline{B}_j') = \underline{B}_j' - \underline{b}'$ are the reduced variables. In (2.34) we used the fact that $E(\hat{\underline{b}^0}) = 0$ and that a scalar random variable trivially equals its trace, and also that $Tr(AB) = Tr(BA)$. The proof is complete.

Theorem 2.4 (Unbiased estimator for s^2 for each contract group). *In case the number of observations t_j in the j^{th} contract is larger than the number of regression constants n, the following is an unbiased estimator of s^2:*

$$\hat{s}_j^2 = \frac{1}{t_j - n} (\underline{X}_j - x_j \underline{B}_j)' (\underline{X}_j - x_j \underline{B}_j). \tag{2.35}$$

Corollary (Unbiased estimator for s^2 in the regression model). *Let K denote the number of contracts j, with $t_j > n$. The $E(\hat{s}^2) = s^2$, if:*

$$\hat{s}^2 = \frac{1}{K} \sum_{j; t_j > n} \hat{s}_j^2. \tag{2.36}$$

For a, we give an unbiased pseudo-estimator, defined in terms of itself, so it can only be computed iteratively:

Theorem 2.5 (Pseudo-estimator for a). *The following random variable has expected value a:*

$$\hat{a} = \frac{1}{k-1} \sum_j z_j (\underline{B}_j - \hat{\underline{b}})(\underline{B}_j - \hat{\underline{b}})'. \tag{2.37}$$

Proof. By standard computations we obtain

$$E(\hat{a}) = \frac{1}{k-1} \sum_j z_j [E(\underline{B}_j \underline{B}_j') - E(\underline{B}_j \hat{\underline{b}}') - E(\hat{\underline{b}}\, \underline{B}_j') + E(\hat{\underline{b}}\, \hat{\underline{b}}')]. \tag{2.38}$$

Since

$$
\begin{aligned}
&\mathrm{Cov}(\underline{B}_j) = \mathrm{Cov}(\underline{B}_j, \underline{B}_j') = \mathrm{Cov}[u_j x' v_j^{-1} \underline{X}_j, (u_j x' v_j^{-1} \underline{X}_j)'] = \\
&= u_j x' v_j^{-1} \mathrm{Cov}(\underline{X}_j) v_j^{-1} x u_j' = u_j x' v_j^{-1} (s^2 v_j + x a x') v_j^{-1} x u_j' = a + s^2 u_j,
\end{aligned}
$$

results that

$$E(\underline{B}_j\, \underline{B}_j') = \mathrm{Cov}(\underline{B}_j) + E(\underline{B}_j) E(\underline{B}_j') = a + s^2 u_j + \underline{b}\, \underline{b}', \tag{2.39}$$

where $E(\underline{B}_j) = E[E(\underline{B}_j|\theta_j)] = E(\underline{\beta}(\theta_j)] = \underline{b}$. Since

$$\mathrm{Cov}(\underline{B}_j, \underline{\hat{b}}') = \mathrm{Cov}(\underline{B}_j, Z^{-1}\sum_i z_i \underline{B}_i) = \left(\mathrm{Cov}\left(\sum_i Z^{-1} z_i \underline{B}_i, \underline{B}_j\right)\right)' =$$
$$= \sum_i \mathrm{Cov}(\underline{B}_j, \underline{B}_i) z_i' \cdot (Z')^{-1} = \sum_i \delta_{ij}(a + s^2 u_j) z_i'(Z')^{-1} = (a + s^2 u_j) z_j'(Z')^{-1},$$

results that

$$E(\underline{B}_j \underline{\hat{b}}') = \mathrm{Cov}(\underline{B}_j, \underline{\hat{b}}') + E(\underline{B}_j)E(\underline{\hat{b}}') = (a + s^2 u_j) z_j'(Z')^{-1} + \underline{b}\,\underline{b}', \tag{2.40}$$

where

$$E(\underline{\hat{b}}) = E\left(Z^{-1}\sum_j z_j \underline{B}_j\right) = Z^{-1}\left(\sum_j z_j\right) E(\underline{B}_j) = Z^{-1} Z \underline{b} = \underline{b}.$$

Since

$$\mathrm{Cov}(\underline{\hat{b}}, \underline{B}_j) = (\mathrm{Cov}(\underline{B}_j, \underline{\hat{b}}))' = [(a + s^2 u_j) z_j'(Z')^{-1}]' =$$
$$= Z^{-1} z_j (a + s^2 u_j) = Z^{-1} a (a + s^2 u_j)^{-1}(a + s^2 u_j) = Z^{-1} a,$$

results that

$$E(\underline{\hat{b}} \underline{B}_j') = \mathrm{Cov}(\underline{\hat{b}}, \underline{B}_j) + E(\underline{\hat{b}})E(\underline{B}_j') = Z^{-1} a + \underline{b}\,\underline{b}' \tag{2.41}$$

Since

$$\mathrm{Cov}(\underline{\hat{b}}) = \mathrm{Cov}(\underline{\hat{b}}, \underline{\hat{b}}) = \mathrm{Cov}(Z^{-1}\sum_i z_i \underline{B}_i, Z^{-1}\sum_j z_j \underline{B}_j) =$$
$$= Z^{-1}\sum_i z_i \left(\sum_j \mathrm{Cov}\,(\underline{B}_j, \underline{B}_i) z_j'\right) \cdot (Z')^{-1} =$$
$$= Z^{-1}\sum_i z_i \left(\sum_j \delta_{ij}(a + s^2 u_j) z_j'\right)(Z')^{-1} =$$
$$= Z^{-1}\left(\sum_i z_i (a + s^2 u_i) z_i'\right)(Z')^{-1} =$$
$$= Z^{-1}\left(\sum_i a(a + s^2 u_i)^{-1}(a + s^2 u_i) z_i'\right)(Z')^{-1} = Z^{-1} a Z'(Z')^{-1} = Z^{-1} a$$

results that

$$E(\underline{\hat{b}}\,\underline{\hat{b}}') = \mathrm{Cov}(\underline{\hat{b}}) + E(\underline{\hat{b}})E(\underline{\hat{b}}') = Z^{-1} a + \underline{b}\,\underline{b}'\ . \tag{2.42}$$

Now (2.37) follows from (2.38), (2.39), (2.40), (2.41) and (2.42).

Remark 2.3. Another unbiased estimator for a is the following:

$$\hat{a} = \frac{1}{(w^2. - \sum w_j^2)} \left\{ \frac{1}{2} \sum_{i,j} w_i w_j (\underline{B}_i - \underline{B}_j)(\underline{B}_i - \underline{B}_j)' - \hat{s}^2 \sum_{j=1}^{k} w_j (w. - w_j) u_j \right\}, \tag{2.43}$$

where w_j is the volume of the risk for the j^{th} contract, $j = \overline{1,k}$ and $w. = \sum_j w_j$.

Proof. Complicate and tedious computations lead to

$$(w^2. - \sum_j w_j^2) E(\hat{a}) = \frac{1}{2} \left\{ \sum_{i,j} w_i w_j E[(\underline{B}_i - \underline{B}_j) \cdot (\underline{B}_i - \underline{B}_j)'] \right\} - E(\hat{s}^2) \cdot$$

$$\cdot \sum_j w_j (w. - w_j) u_j = \frac{1}{2} \{ \sum_{i,j} w_i w_j [E(\underline{B}_i \, \underline{B}_i') - E(\underline{B}_i \, \underline{B}_j') -$$

$$-E(\underline{B}_j \, \underline{B}_i') + E(\underline{B}_j \, \underline{B}_j')]\} - s^2 \left(\sum_j w_j w. u_j - \sum_j w_j^2 u_j \right) =$$

$$= \tfrac{1}{2} \{ \sum_{i,j} w_i w_j [a + s^2 u_i + \underline{b}\,\underline{b}' - \delta_{ij}(a + s^2 u_j) - \underline{b}\,\underline{b}' - \delta_{ij}(a + s^2 u_j) -$$

$$-\underline{b}\,\underline{b}' + a + s^2 u_j + \underline{b}\,\underline{b}']\} - s^2 \sum_j w_j w. u_j + s^2 \sum_j w_j^2 u_j =$$

$$= \tfrac{1}{2} \cdot 2 w. w. a + \tfrac{1}{2} s^2 \sum_i w_i u_i w. - \frac{1}{2} 2 \sum w_j \cdot \sum_i w_i \delta_{ij} (a + s^2 u_j) +$$

$$+ \tfrac{1}{2} s^2 \sum_j w_j u_j w. - s^2 \sum w_j u_j w. + s^2 \sum_j w_j^2 u_j =$$

$$= w^2. a - \sum_j w_j^2 a = (w^2. - \sum_j w_j^2) a$$

Thus we have proved our assertion.

Observation. This estimator is a statistic; it is not a pseudo-estimator. Still, the reason to prefer (2.37) is that this estimator can easily be generalized to multi-level hierarchical models. In any case, the unbiasedness of the credibility premium disappears even if one takes (2.43) to estimate a.

3 Conclusions

The article contains a credibility solution in the form of a linear combination of the individual estimate (based on the data of a particular state) and the collective estimate (based on aggregate USA data). This idea is worked out in regression credibility theory.

In case there is an increase (for instance by inflation) of the results on a portfolio, the risk premium could be considered to be a linear function in time of the type $\beta_0(\theta) + t\beta_1(\theta)$. Then two parameters $\beta_0(\theta)$ and $\beta_1(\theta)$ must be estimated from the observed variables. This kind of problem is named regression credibility. This model arises in cases where the risk premium depends on time, e.g. by inflation. The one could assume a linear effect on the risk premium as an approximation to the real growth, as is also the case in time series analysis.

These regression models can be generalized to get credibility models for general regression models, where the risk is characterized by outcomes of other related variables.

This paper contains a description of the Hachemeister regression model allowing for effects like inflation. If there is an effect of inflation, it is contained in the claim figures, so one should use estimates based on these figures instead of external data. This can be done using Hachemeister's regression model.

In this article the regression credibility result for the estimates of the parameters in the linear model is derived. After the credibility result based on the structural parameters is obtained, one has to construct estimates for these parameters.

The matrix theory provided the means to calculate useful estimators for the structure parameters. The property of unbiasedness of these estimators is very appealing and very attractive from the practical point of view.

The fact that it is based on complicated mathematics, involving linear algebra, needs not bother the user more than it does when he applies statistical tools like discriminant analysis, scoring models, SAS and GLIM.

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Ore extensions over 2-primal Noetherian rings

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Abstract. Let R be a ring and σ an automorphism of R. We prove that if R is a 2-primal Noetherian ring, then the skew polynomial ring $R[x;\sigma]$ is 2-primal Noetherian. Let now δ be a σ-derivation of R. We say that R is a δ-ring if $a\delta(a) \in P(R)$ implies $a \in P(R)$, where $P(R)$ denotes the prime radical of R. We prove that $R[x;\sigma,\delta]$ is a 2-primal Noetherian ring if R is a Noetherian $\mathbb{Q}$-algebra, σ and δ are such that R is a δ-ring, $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$ and $\sigma(P) = P$, P being any minimal prime ideal of R. We use this to prove that if R is a Noetherian $\sigma(*)$-ring (i.e. $a\sigma(a) \in P(R)$ implies $a \in P(R)$), δ a σ-derivation of R such that R is a δ-ring and $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$, then $R[x;\sigma,\delta]$ is a 2-primal Noetherian ring.

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1 Introduction

A ring R always means an associative ring. $\mathbb{Q}$ denotes the field of rational numbers. $Spec(R)$ denotes the set of prime ideals of R. $MinSpec(R)$ denotes the set of minimal prime ideals of R. $P(R)$ and $N(R)$ denote the prime radical and the set of nilpotent elements of R, respectively. Let I and J be any two ideals of a ring R. Then $I \subset J$ means that I is strictly contained in J. Let I be an ideal of a ring R such that $\sigma^m(I) = I$ for some integer $m \geq 1$, we denote $\cap_{i=1}^{m}\sigma^i(I)$ by I^0.

This article concerns the study of Ore extensions in terms of 2-primal rings. 2-primal rings have been studied in recent years and the 2-primal property is being studied for various types of rings. In [18], G. Marks discusses the 2-primal property of $R[x;\sigma,\delta]$, where R is a local ring, σ is an automorphism of R and δ is a σ-derivation of R.

Recall that a σ-derivation of R is an additive map $\delta : R \to R$ such that $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$, for all $a, b \in R$. In case σ is the identity map, δ is called just a derivation of R. For example for any endomorphism τ of a ring R and for any $a \in R$, $\varrho : R \to R$ defined as $\varrho(r) = ra - a\tau(r)$ is a τ-derivation of R.

Let σ be an endomorphism of a ring R and $\delta : R \to R$ any map. Let $\phi : R \to M_2(R)$ be a homomorphism defined by

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$$\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}, \quad \text{for all} \quad r \in R.$$

Then δ is a σ-derivation of R.

Also let $R = K[x]$, K a field. Then the formal derivative d/dx is a derivation of R.

Minimal prime ideals of 2-primal rings have been discussed by Kim and Kwak in [15] and Shin in [20]. 2-primal near rings have been discussed by Argac and Groenewald in [2]. Recall that a ring R is called 2-primal if the set of nilpotent elements of R coincides with the prime radical of R (G. Marks [18]), or equivalently if its radical contains every nilpotent element of R, or if $P(R)$ is a completely semiprime ideal of R. An ideal I of a ring R is called completely semiprime if $a^2 \in I$ implies $a \in I$ for $a \in R$.

We also note that a reduced ring (i. e. a ring with no nonzero nilpotent elements) is 2-primal and a commutative ring is also 2-primal. For further details on 2-primal rings, we refer the reader to [5, 11, 14, 15, 20].

Recall that $R[x; \sigma, \delta]$ is the skew polynomial ring with coefficients in R in which multiplication is subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We denote $R[x; \sigma, \delta]$ by $O(R)$. In case σ is the identity map, we denote the ring of differential operators $R[x; \delta]$ by $D(R)$, if δ is the zero map, we denote the skew polynomial ring $R[x; \sigma]$ by $S(R)$.

Recall that in Krempa [16], a ring R is called σ-rigid if there exists an endomorphism σ of R with the property that $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$. In [17], Kwak defines a $\sigma(*)$-ring R to be a ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$ and establishes a relation between a 2-primal ring and a $\sigma(*)$-ring. The property is also extended to the skew-polynomial ring $S(R)$.

Remark 1. If R is a ring and σ an automorphism of R such that R is a $\sigma(*)$-ring, then R is 2-primal.

Proof. We will show that $P(R)$ is a completely semiprime ideal of R. Let $a \in R$ be such that $a^2 \in P(R)$. Then $a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(P(R)) = P(R)$. Therefore $a\sigma(a) \in P(R)$ and hence $a \in P(R)$. □

In Theorem 12 of [17], Kwak has proved that if R is a $\sigma(*)$-ring such that $\sigma(P(R)) = P(R)$, then $R[x; \sigma]$ is 2-primal if and only if $P(R)[x; \sigma] = P(R[x; \sigma])$.

Hong, Kim and Kwak have proved in Corollary 2.8 of [13] that if R is a 2-primal ring and every simple singular left R-module is p-injective, then every prime ideal of R is maximal. In particular, every prime factor ring of R is a simple domain.

It is known (Theorem 1.2 of Bhat [5]) that if R is 2-primal Noetherian $\mathbb{Q}$-algebra and δ is a derivation of R, then $D(R)$ is 2-primal. We also note that if R is a Noetherian ring, then even $R[x]$ need not be 2-primal.

Example 1. Let $R = M_2(\mathbb{Q})$, the set of 2×2 matrices over $\mathbb{Q}$. Then $R[x]$ is a prime ring with non-zero nilpotent elements and, so can not be 2-primal.

Let now R be a 2-primal ring. Is $O(R)$ also a 2-primal ring? For the time being we are not able to answer this question, but towards this we have the following.

Let R be a ring, σ be an automorphism of R and δ be a σ-derivation of R. We say that R is a δ-ring if $a\delta(a) \in P(R)$ implies $a \in P(R)$. We note that a ring with identity is not a δ-ring. We ultimately prove the following:

1. Let R be a 2-primal Noetherian ring. Then $S(R)$ is 2-primal Noetherian. This is proved in Theorem 2.

2. Let R be a Noetherian $\mathbb{Q}$-algebra. Let σ be an automorphism of R and δ a σ-derivation of R such that R is a δ-ring, $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$; $\sigma(P) = P$ for all $P \in MinSpec(R)$ and $\delta(P(R)) \subseteq P(R)$. Then $O(R)$ is 2-primal Noetherian. This is proved in Theorem 6.

3. Let R be a Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ be a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$ and R is a δ-ring. Then $R[x; \sigma, \delta]$ is 2-primal Noetherian.

Before proving (2) and (3) above, we find a relation between the minimal prime ideals of R and those of the Ore extension $O(R)$, where R is a Noetherian $\mathbb{Q}$-algebra, σ an automorphism of R and δ a σ-derivation of R. This is proved in Theorem 3.

Ore-extensions including skew-polynomial rings and differential operator rings have been of interest to many authors. See [1, 3, 4, 6–8, 12, 16, 17].

2 Skew polynomial ring $S(R)$

Recall that an ideal I of a ring R is called σ-invariant if $\sigma(I) = I$. Also I is called completely prime if $ab \in I$ implies $a \in I$ or $b \in I$ for $a, b \in R$. We also note that in a right Noetherian ring R, $MinSpec(R)$ is finite (Theorem 2.4 of Goodearl and Warfield [10]), and for any $P \in MinSpec(R)$, $\sigma^t(P) \in MinSpec(R)$ for all integers $t \geq 1$. Let $MinSpec(R) = \{P_1, P_2, \ldots, P_n\}$. Let $\sigma^{m_i}(P_i) = P_i$, for some positive integers m_i, $1 \leq i \leq n$, and $u = m_1.m_2...m_n$. Then $\sigma^u(P_i) = P_i$ for all $P_i \in MinSpec(R)$. We use same u henceforth, and as mentioned in introduction above, we denote $\cap_{i=1}^{u}\sigma^i(P)$ by P^0, P being any minimal prime ideal of R.

Proposition 1. *Let R be a right Noetherian ring. Let σ be an automorphism of R. Then $\sigma(N(R)) = N(R)$.*

Proof. Denote $N(R)$ by N. We have $\sigma(N) \subseteq N$ as R is right Noetherian, therefore, $\sigma(N)$ is a nilpotent ideal of R by Theorem 5.18 of Goodearl and Warfield [10]. Now let $n \in N$. Then σ being an automorphism of R implies that there exists $a \in R$ such that $n = \sigma(a)$. Now $I = \sigma^{-1}(N) = \{a \in R \text{ such that } \sigma(a) = n \in N\}$ is an ideal of R. Now I is nilpotent, so $I \subseteq \sigma(N)$, which implies that $N \subseteq \sigma(N)$. Hence $\sigma(N) = N$. □

Proposition 2. *Let R be a Noetherian ring and σ an automorphism of R. Then $S(N(R)) = N(S(R))$.*

Proof. It is easy to see that $S(N(R)) \subseteq N(S(R))$. We will show that $N(S(R)) \subseteq S(N(R))$. Let $f = \sum_{i=0}^{m} x^i a_i \in N(S(R))$. Then $f(S(R)) \subseteq N(S(R))$, and $f(R) \subseteq N(S(R))$. Let $(f(R))^k = 0$, $k > 0$. Then equating leading term to zero, we get $(x^m a_m R)^k = 0$. This implies on simplification that

$$x^{km}\sigma^{(k-1)m}(a_m R) \cdot \sigma^{(k-2)m}(a_m R) \cdot \sigma^{(k-3)m}(a_m R) \ldots a_m R = 0.$$

Therefore,

$$\sigma^{(k-1)m}(a_m R) \cdot \sigma^{(k-2)m}(a_m R) \cdot \sigma^{(k-3)m}(a_m R) \ldots a_m R = 0 \subseteq P,$$

for all $P \in MinSpec(R)$. Now there are two cases:

1. $u \geq m$.
2. $m \geq u$.

If $u \geq m$, then we have

$$\sigma^{(k-1)u}(a_m R) \cdot \sigma^{(k-2)u}(a_m R) \cdot \sigma^{(k-3)u}(a_m R) \ldots a_m R \subseteq P.$$

This implies that $\sigma^{(k-j)u}(a_m R) \subseteq P$, for some j, $1 \leq j \leq k$, i.e. $a_m R \subseteq \sigma^{-(k-j)u}(P) = P$. So we have $a_m R \subseteq P$, for all $P \in MinSpec(R)$. Therefore, $a_m \in P(R) = N(R)$. Now $x^m a_m \in S(N(R)) \subseteq N(S(R))$ implies that $\sum_{i=0}^{m-1} x^i a_i \in N(S(R))$, and with the same process, in a finite number of steps, it can be seen that $a_i \in P(R) = N(R)$, $0 \leq i \leq m-1$. Therefore $f \in S(N(R))$. Hence $N(S(R)) \subseteq S(N(R))$ and the result follows. The other case is similar. □

Theorem 1. (Theorem 2.4, (2) of Bhat [4]) *Let R be a Noetherian ring and σ an automorphism of R. Then $P \in MinSpec(S(R))$ if and only if there exists $L \in MinSpec(R)$ such that $S(P \cap R) = P$ and $P \cap R = L^0$.*

Proof. Let $L \in MinSpec(R)$. Then $\sigma^u(L) = L$ for some integer $u \geq 1$. Then by Lemma 10.6.12 of McConnell and Robson [19] and by Theorem 7.27 of Goodearl and Warfield [10], $S(L^0) \in MinSpec(S(R))$.

Conversely suppose that $P \in MinSpec(S(R))$. Then $P \cap R = U^0$ for some $U \in Spec(R)$ and U contains a minimal prime ideal U_1. Now $P \supseteq S(R)U_1^0$, which is a prime ideal of $S(R)$. Hence $P = S(R)U_1^0$. □

We are now in a position to prove the main result of this section in the form of the following Theorem.

Theorem 2. *Let R be a 2-primal Noetherian ring. Then $S(R)$ is 2-primal Noetherian.*

Proof. R is Noetherian implies $S(R)$ is Noetherian follows from Hilbert Basis Theorem, namely Theorem 1.12 of Goodearl and Warfield [10]. Now R is 2-primal implies $N(R) = P(R)$ and Proposition 1 implies that $\sigma(N(R)) = N(R)$. Therefore $S(N(R))$ and $S(P(R))$ are ideals of $S(R)$ and $S(N(R)) = S(P(R))$. Now by Proposition 2 $S(N(R)) = N(S(R))$.

We now show that $S(P(R)) = P(S(R))$. It is easy to see that $S(P(R)) \subseteq P(S(R))$. Now let g $= \sum_{i=0}^{t} x^i b_i \in P(S(R))$. Then $g \in P_i$, for all $P_i \in MinSpec(S(R))$. Now Theorem 1 implies that there exists $U_i \in MinSpec(R)$ such that $P_i = S((U_i)^0)$. Now it can be seen that P_i are distinct implies that U_i are distinct. Therefore $g \in S((U_i)^0)$. This implies that $b_i \in (U_i)^0 \subseteq U_i$. Thus we have $b_i \in U_i$, for all $U_i \in MinSpec(R)$. Therefore $b_i \in P(R)$, which implies that $g \in S(P(R))$. Therefore $P(S(R)) \subseteq S(P(R))$, and hence $S(P(R)) = P(S(R))$.

Thus we have $P(S(R)) = S(P(R)) = S(N(R)) = N(S(R))$. Hence $S(R)$ is 2-primal. □

Question 1. *Let R be a 2-primal ring. Is $S(R)$ 2-primal? The main difficulty is that Proposition 2 and Theorem 1 do not hold.*

3 Ore extension $O(R)$

We begin with the following definition:

Definition 1. Let R be a ring. Let σ be an automorphism of R and δ a σ-derivation of R. We say that R is a δ-ring if $\delta(a) \in P(R)$ implies $a \in P(R)$.

Recall that an ideal I of a ring R is called δ-invariant if $\delta(I) \subseteq I$. If an ideal I of R is σ-invariant and δ-invariant, then $O(I)$ is an ideal of $O(R)$ as for any $a \in I$, $\sigma^j(a) \in I$ and $\delta^j(a) \in I$ for all positive integers j.

Gabriel proved in Lemma 3.4 of [9] that if R is a Noetherian $\mathbb{Q}$-algebra and δ is a derivation of R, then $\delta(P) \subseteq P$, for all $P \in MinSpec(R)$. We generalize this for σ-derivation δ of R and give a structure of minimal prime ideals of $O(R)$ in the following Theorem.

Theorem 3. *Let R be a Noetherian $\mathbb{Q}$-algebra. Let σ be an automorphism of R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for $a \in R$. Then:*

1. *$P_1 \in MinSpec(R)$ such that $\sigma(P_1) = P_1$ implies $O(P_1) \in MinSpec(O(R))$.*
2. *$P \in MinSpec(O(R))$ such that $\sigma(P \cap R) = P \cap R$ implies $P \cap R \in MinSpec(R)$.*

Proof. **(1)** Let $P_1 \in MinSpec(R)$ with $\sigma(P_1) = P_1$. Let $T = R[[t;\sigma]]$, the skew power series ring. We note that multiplication in $R[[t;\sigma]]$ is determined by the computation $ax = x\sigma(a)$ for all $a \in R$. Now we know that

$$e^{t\delta} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\delta^n$$

and it can be seen that $e^{t\delta}$ is an automorphism of T. Now $P_1T \in Spec(T)$. Suppose if possible that $P_1T \notin MinSpec(T)$ and $P_2 \subset P_1T$ be a minimal prime ideal of T. Then $P_2 \cap R \subset P_1T \cap R = P_1$, which is not possible as $P_1 \in MinSpec(R)$. Therefore $P_1T \in MinSpec(T)$. We also know that $(e^{t\delta})^k(P_1T) \in MinSpec(T)$ for all integers $k \geq 1$. Now T is Noetherian by Exercise (1ZA(c)) of Goodearl and Warfield [10], and therefore, Theorem 2.4 of Goodearl and Warfield [10] implies that $MinSpec(T)$ is finite. So there exists an integer $n \geq 1$ such that $(e^{t\delta})^n(P_1T) = P_1T$, i. e. $(e^{nt\delta})(P_1T) = P_1T$. But R is a $\mathbb{Q}$-algebra, therefore, $e^{t\delta}(P_1T) = P_1T$. Now for any $a \in P_1$, $a \in P_1T$ also, and so $e^{t\delta}(a) \in P_1T$, i. e.

$$a + t\delta(a) + (t^2/2!)\delta^2(a) + \cdots \in P_1T,$$

which implies that $\delta(a) \in P_1$. Therefore $\delta(P_1) \subseteq P_1$.

Now on the same lines as in Theorem 2.22 of Goodearl and Warfield [10], it can be easily seen that $O(P_1) \in Spec(O(R))$. Suppose that $O(P_1) \notin MinSpec(O(R))$, and $P_2 \subset O(P_1)$ is a minimal prime ideal of $O(R)$. Then we have $P_2 = O(P_2 \cap R) \subset O(P_1) \in MinSpec(O(R))$. Therefore $P_2 \cap R \subset P_1$, which is a contradiction as $P_2 \cap R \in Spec(R)$. Hence $O(P_1) \in MinSpec(O(R))$.

(2) Let $P \in MinSpec(O(R))$ with $\sigma(P \cap R) = P \cap R$. Then on the same lines as in Theorem 2.22 of Goodearl and Warfield [10], it can be seen that $P \cap R \in Spec(R)$ and $O(P \cap R) \in Spec(O(R))$. Therefore $O(P \cap R) = P$. We now show that $P \cap R \in MinSpec(R)$. Suppose that $U \subset P \cap R$, and $U \in MinSpec(R)$. Then $O(U) \subset O(P \cap R) = P$. But $O(U) \in Spec(O(R))$ and, $O(U) \subset P$, which is not possible. Thus we have $P \cap R \in MinSpec(R)$. □

Recall that in Proposition 1.11 of Shin [20], it has been proved that a ring R is 2-primal if and only if each minimal prime ideal of R is a completely prime ideal.

Proposition 3. *Let R be a 2-primal ring. Let σ be an automorphism of R and δ a σ-derivation of R such that $\delta(P(R)) \subseteq P(R)$. If $P \in MinSpec(R)$ is such that $\sigma(P) = P$, then $\delta(P) \subseteq P$.*

Proof. Let $P \in MinSpec(R)$. Now P is a completely prime ideal, therefore, for any $a \in P$, there exists $b \notin P$ such that $ab \in P(R)$ by Corollary 1.10 of Shin [20]. Now $\delta(P(R)) \subseteq P(R)$, and therefore $\delta(ab) \in P(R)$; i. e. $\delta(a)\sigma(b) + a\delta(b) \in P(R) \subseteq P$. Now $a\delta(b) \in P$ implies that $\delta(a)\sigma(b) \in P$. Now $\sigma(P) = P$ implies that $\sigma(b) \notin P$ and since P is completely prime in R, we have $\delta(a) \in P$. Hence $\delta(P) \subseteq P$. □

Theorem 4. *Let R be a ring. Let σ be an automorphism of R and δ a σ-derivation of R such that R is a δ-ring and $\delta(P(R)) \subseteq P(R)$. Then R is 2-primal.*

Proof. Define a map $\rho : R/P(R) \to R/P(R)$ by $\rho(a + P(R)) = \delta(a) + P(R)$ for $a \in R$ and $\tau : R/P(R) \to R/P(R)$ a map by $\tau(a + P(R)) = \sigma(a) + P(R)$ for $a \in R$, then it can be seen that τ is an automorphism of R/P(R) and ρ is a τ-derivation of R/P(R). Now $a\delta(a) \in P(R)$ if and only if $(a + P(R))\rho(a + P(R)) = P(R)$ in R/P(R). Thus as in Proposition 5 of Hong, Kim and Kwak [12], R is a reduced ring and, therefore as mentioned in introduction, R is 2-primal. □

Proposition 4. *Let R be a ring. Let σ be an automorphism of R and δ a σ-derivation of R. Then:*

1. *For any completely prime ideal P of R with $\sigma(P) = P$ and $\delta(P) \subseteq P$, $O(P)$ is a completely prime ideal of $O(R)$.*

2. *For any completely prime ideal U of $O(R)$, $U \cap R$ is a completely prime ideal of R.*

Proof. **(1)** Let P be a completely prime ideal of R. Now let $f(x) = \sum_{i=0}^{n} x^i a_i \in O(R)$ and $g(x) = \sum_{j=0}^{m} x^j b_j \in O(R)$ be such that $f(x)g(x) \in O(P)$. Suppose $f(x) \notin O(P)$. We will show that $g(x) \in O(P)$. We use induction on n and m. For $n = m = 1$, the verification is easy. We check for $n = 2$ and $m = 1$. Let $f(x) = x^2a + xb + c$ and $g(x) = xu + v$. Now $f(x)g(x) \in O(P)$ with $f(x) \notin O(P)$. The possibilities are $a \notin P$ or $b \notin P$ or $c \notin P$ or any two out of these three do not belong to P or all of them do not belong to P. We verify case by case.

Let $a \notin P$. Since $x^3\sigma(a)u + x^2(\delta(a)u + \sigma(b)u + av) + x(\delta(b)u + \sigma(c)u + bv) + \delta(c)u + cv \in O(P)$, we have $\sigma(a)u \in P$, and so $u \in P$. Now $\delta(a)u + \sigma(b)u + av \in P$ implies $av \in P$, and so $v \in P$. Therefore $g(x) \in O(P)$.

Let $b \notin P$. Now $\sigma(a)u \in P$. Suppose $u \notin P$, then $\sigma(a) \in P$ and therefore a, $\delta(a) \in P$. Now $\delta(a)u+\sigma(b)u+av \in P$ implies that $\sigma(b)u \in P$ which in turn implies that $b \in P$, which is not the case. Therefore we have $u \in P$. Now $\delta(b)u+\sigma(c)u+bv \in P$ implies that $bv \in P$ and therefore $v \in P$. Thus we have $g(x) \in O(P)$.

Let $c \notin P$. Now $\sigma(a)u \in P$. Suppose $u \notin P$, then as above a, $\delta(a) \in P$. Now $\delta(a)u + \sigma(b)u + av \in P$ implies that $\sigma(b)u \in P$. Now $u \notin P$ implies that $\sigma(b) \in P$; i. e. $b, \delta(b) \in P$. Also $\delta(b)u + \sigma(c)u + bv \in P$ implies $\sigma(c)u \in P$ and therefore $\sigma(c) \in P$ which is not the case. Thus we have $u \in P$. Now $\delta(c)u + cv \in P$ implies $cv \in P$, and so $v \in P$. Therefore $g(x) \in O(P)$.

Now suppose the result is true for k, $n = k > 2$ and $m = 1$. We will prove for $n = k + 1$. Let $f(x) = x^{k+1}a_{k+1} + x^k a_k + \cdots + xa_1 + a_0$, and $g(x) = xb_1 + b_0$ be such that $f(x)g(x) \in O(P)$, but $f(x) \notin O(P)$. We will show that $g(x) \in O(P)$. If $a_{k+1} \notin P$, then equating coefficients of x^{k+2}, we get $\sigma(a_{k+1})b_1 \in P$, which implies that $b_1 \in P$. Now equating coefficients of x^{k+1}, we get $\sigma(a_k)b_1 + a_{k+1}b_0 \in P$, which implies that $a_{k+1}b_0 \in P$, and therefore $b_0 \in P$. Hence $g(x) \in O(P)$.

If $a_j \notin P$, $0 \leq j \leq k$, then using induction hypothesis, we get that $g(x) \in O(P)$. Therefore the statement is true for all n. Now using the same process, it can be easily seen that the statement is true for all m also.

(2) Let U be a completely prime ideal of $O(R)$. Suppose $a, b \in R$ are such that $ab \in U \cap R$ with $a \notin U \cap R$. This means that $a \notin U$ as $a \in R$. Thus we have $ab \in U \cap R \subseteq U$, with $a \notin U$. Therefore we have $b \in U$, and thus $b \in U \cap R$. □

Corollary 1. *Let R be a ring and σ an automorphism of R. Then:*

1. *For any completely prime ideal P of R with $\sigma(P) = P$, $S(P)$ is a completely prime ideal of $S(R)$.*

2. *For any completely prime ideal U of $S(R)$, $U \cap R$ is a completely prime ideal of R.*

Corollary 2. *Let R be a ring, σ an automorphism of R and δ a σ-derivation of R such that R is moreover a δ-ring and $\delta(P(R)) \subseteq P(R)$. Let $P \in MinSpec(R)$ be such that $\sigma(P) = P$. Then $O(P)$ is a completely prime ideal of $O(R)$.*

Proof. R is 2-primal by Theorem 4, and so by Proposition 3 $\delta(P) \subseteq P$. Further more as mentioned in Proposition 3 above, P is a completely prime ideal of R. Now use Proposition 4, and the proof is complete. □

We now prove the following Theorem, which is crucial in proving Theorem 6.

Theorem 5. *Let R be a ring, σ an automorphism of R and δ a σ-derivation of R such that R is a δ-ring and $\delta(P(R)) \subseteq P(R)$ and $\sigma(P) = P$ for all $P \in MinSpec(R)$. Then $O(R)$ is 2-primal if and only if $O(P(R)) = P(O(R))$.*

Proof. Let $O(R)$ be 2-primal. Now by Corollary 2 $P(O(R)) \subseteq O(P(R))$. Let $f(x) = \sum_{j=0}^{n} x^j a_j \in O(P(R))$. Now R is a 2-primal subring of $O(R)$ by Theorem 4, which implies that a_j is nilpotent and thus $a_j \in N(O(R)) = P(O(R))$, and so we have $x^j a_j \in P(O(R))$ for each j, $0 \leq j \leq n$, which implies that $f(x) \in P(O(R))$. Hence $O(P(R)) = P(O(R))$.

Conversely suppose $O(P(R)) = P(O(R))$. We will show that $O(R)$ is 2-primal. Let $g(x) = \sum_{i=0}^{n} x^i b_i \in O(R)$, $b_n \neq 0$, be such that $(g(x))^2 \in P(O(R)) = O(P(R))$. We will show that $g(x) \in P(O(R))$. Now leading coefficient $\sigma^{2n-1}(a_n)a_n \in P(R) \subseteq P$, for all $P \in MinSpec(R)$. Now $\sigma(P) = P$ and since R is 2-primal by Theorem 4, therefore, P is completely prime. Therefore we have $a_n \in P$, for all $P \in MinSpec(R)$; i. e. $a_n \in P(R)$. Now since $\delta(P(R)) \subseteq P(R)$ and $\sigma(P) = P$ for all $P \in MinSpec(R)$, we get $(\sum_{i=0}^{n-1} x^i b_i)^2 \in P(O(R)) = O(P(R))$ and as above we get $a_{n-1} \in P(R)$. With the same process in a finite number of steps we get $a_i \in P(R)$ for all i, $0 \leq i \leq n$. Thus we have $(g(x)) \in O(P(R))$, i. e. $(g(x)) \in P(O(R))$. Therefore $P(O(R))$ is a completely semiprime ideal of $O(R)$. Hence $O(R)$ is 2-primal. □

Theorem 6. *Let R be a Noetherian $\mathbb{Q}$-algebra, σ an automorphism of R and δ a σ-derivation of R such that R is a δ-ring, $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$; $\sigma(P) = P$ for all $P \in MinSpec(R)$ and $\delta(P(R)) \subseteq P(R)$. Then $O(R)$ is 2-primal.*

Proof. Let $P_1 \in MinSpec(R)$. Then it is given that $\sigma(P_1) = P_1$, and therefore Theorem 3 implies that $O(P_1) \in MinSpec(O(R))$. Similarly for any $P \in MinSpec(O(R))$ such that $\sigma(P \cap R) = P \cap R$ Theorem 3 implies that $P \cap R \in MinSpec(R)$. Therefore, $O(P(R)) = P(O(R))$, and now the result is obvious by using Theorem 5. □

Corollary 3. *Let R be a Noetherian $\mathbb{Q}$-algebra, σ an automorphism of R and δ a σ-derivation of R such that R is a δ-ring, $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$ and $\sigma(P) = P$ for all $P \in MinSpec(R)$. Then $O(R)$ is 2-primal.*

Proof. Let $P_1 \in MinSpec(R)$ with $\sigma(P_1) = P_1$. Then as in the proof of Theorem 3 $\delta(P_1) \subseteq P_1$, and therefore $\delta(P(R)) \subseteq P(R)$. Now the rest is obvious using Theorem 6. ☐

Theorem 7. *Let R be a Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ be a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$ and R is a δ-ring. Then $R[x; \sigma, \delta]$ is 2-primal Noetherian.*

Proof. We show that $\sigma(U) = U$ for all $U \in MinSpec(R)$. Suppose $U = U_1$ is a minimal prime ideal of R such that $\sigma(U) \neq U$. Let $U_2, U_3, \ldots, U_n$ be the other minimal primes of R. Now $\sigma(U)$ is also a minimal prime ideal of R. Renumber so that $\sigma(U) = U_n$. Let $a \in \cap_{i=1}^{n-1} U_i$. Then $\sigma(a) \in U_n$, and so $a\sigma(a) \in \cap_{i=1}^{n} U_i = P(R)$. Therefore $a \in P(R)$, and thus $\cap_{i=1}^{n-1} U_i \subseteq U_n$, which implies that $U_i \subseteq U_n$ for some $i \neq n$, which is impossible. Hence $\sigma(U) = U$. Now the rest is obvious. ☐

We now have the following question:

Question 2. *If R is a Noetherian $\mathbb{Q}$-algebra (even commutative), σ is an automorphism of R and δ is a σ-derivation of R. Is $O(R)$ 2-primal? The main problem is to get Theorem 5 satisfied.*

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The $GL(2,\mathbb{R})$−orbits of the homogeneous polynomial differential systems

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Abstract. In this work, we study the generic homogeneous polynomial differential system $\dot{x}_1 = P_k(x_1, x_2)$, $\dot{x}_2 = Q_k(x_1, x_2)$ under the action of the center-affine group of transformations of the phase space, $GL(2,\mathbb{R})$. We show that if the dimension of the $GL(2,\mathbb{R})-$ orbits of this system is smaller than four, then $deg(GCD(P_k, Q_k)) \geq k-1$.

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1 Center-affine transformations

We consider the system

$$\dot{x_1} = P_k(x_1, x_2),\ \dot{x_2} = Q_k(x_1, x_2), \tag{1}$$

where P_k, Q_k are homogeneous polynomials of degree k:

$$P_k = \sum_{i+j=k} a_{ij} x_1^i x_2^j,\ Q_k = \sum_{i+j=k} b_{ij} x_1^i x_2^j.$$

Denote by $\mathbb{E}$ the space of coefficients

$$\mathbf{e} = (\mathbf{a};\mathbf{b}) = (a_{k,0}, a_{k-1,1}, ..., a_{0k};\ b_{k,0}, b_{k-1,1}, ..., b_{0k})$$

of system (1) and by $GL(2,\mathbb{R})$ the group of center-affine transformations of the phase space $O\mathbf{x}$, $\mathbf{x} = (x_1, x_2)$.

Applying in (1) the transformations $\mathbf{X} = q\mathbf{x}$, where $\mathbf{X} = (X_1, X_2)$, $q \in GL(2,\mathbb{R})$, i.e.

$$q = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix};\ \alpha_{ij} \in \mathbb{R},\ det(q) \neq 0,\ q^{-1} = \frac{1}{det(q)} \begin{pmatrix} \alpha_{22} & -\alpha_{12} \\ -\alpha_{21} & \alpha_{11} \end{pmatrix},$$

we obtain the system

$$\dot{X}_1 = P_k^*(X_1, X_2),\ \dot{X}_2 = Q_k^*(X_1, X_2), \tag{2}$$

where

$$P_k^* = \alpha_{11} \cdot P_k(q^{-1}\mathbf{x}) + \alpha_{12} \cdot Q_k(q^{-1}\mathbf{x}) = \sum_{i=0}^{k} a_{k-i,i}^* X_1^{k-i} X_2^i,$$

$$Q_k^* = \alpha_{21} \cdot P_k(q^{-1}\mathbf{x}) + \alpha_{22} \cdot Q_k(q^{-1}\mathbf{x}) = \sum_{i=0}^{k} b_{k-i,i}^* X_1^{k-i} X_2^i.$$

The coefficients $\mathbf{e}^*$ of the system (2) can be expressed linearly by the coefficients of the system (1): $\mathbf{e}^* = \Lambda_{(q)}(\mathbf{e})$, $det\Lambda_{(q)} \neq 0$. The set $\Lambda = \{\Lambda_{(q)} | q \in GL(2,\mathbb{R})\}$ forms a 4-parameter linear group with the operation of composition. It is called the representation of the group $GL(2,\mathbb{R})$ in the space of coefficients $\mathbb{E}$ of system (1).

The set $O(\mathbf{e}) = \{\Lambda_{(q)}(\mathbf{e}) |\, q \in GL(2,\mathbb{R}\}$ is called the $GL(2,\mathbb{R})-$orbit of the point $\mathbf{e} \in \mathbb{E}$ or of the differential system (1) corresponding to this point.

Let

$$q_1^t = \begin{pmatrix} exp(t) & 0 \\ 0 & 1 \end{pmatrix},\ q_2^t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},\ q_3^t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},\ q_4^t = \begin{pmatrix} 1 & 0 \\ 0 & exp(t) \end{pmatrix}$$

and $G_l = \{q_l^t |\, t \in \mathbb{R}\} \subset GL(2,\mathbb{R})$, $l = \overline{1,4}$. Denote $g_l^t = \Lambda_{(q_l^t)}$. Il is obvious that $\Lambda_l = \{g_l^t\}$, $l = \overline{1,4}$, are the linear representations in $\mathbb{E}$ of the subgroups G_l respectively. Each of the pairs $(\mathbb{E}, \{g_l^t\})$, $l = \overline{1,4}$, corresponds to a flow defined in $\mathbb{E}$ by the following systems of linear equations:

$$\frac{d\mathbf{e}}{dt} = \left(\frac{dg_l^t(\mathbf{e})}{dt}\right)|_{t=0} = A^{(l)} \cdot \mathbf{e},\ l = \overline{1,4}. \tag{3}$$

If we represent the matrix $A^{(l)}$ of dimension $(2k+2) \times (2k+2)$ as four quadratic blocks of dimensions $(k+1) \times (k+1)$: $A^{(l)} = \begin{pmatrix} A_l & B_l \\ C_l & D_l \end{pmatrix}$ and if denote by O the matrix null, and by I the unity matrix, both of dimensions $(k+1)\times(k+1)$, we get :

$$A_1 = -diag(k-1, k-2, \ldots, 1, 0, -1), B_1 = C_1 = O,$$
$$D_1 = -diag(k, k-1, \ldots, 1, 0);$$

$$A_2 = -\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ k & 0 & 0 & \cdots & 0 & 0 \\ 0 & k-1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & k-2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

$$B_2 = I,\ C_2 = O,\ D_2 = A_2;$$

$$A_3 = -\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & k \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$B_3 = O,\ C_3 = I,\ D_3 = A_3;$$

$$A_4 = -diag(0, 1, 2, \ldots, k),\ B_4 = C_4 = O,\ D_4 = -diag(-1, 0, 1, 2, \ldots, k-1).$$

Let $\mathbf{v}_l$, $l = \overline{1,4}$, be the vector fields defined in $\mathbb{E}$ by the systems (3) and $L_{\mathbf{v}}$ be the derivative in the direction of the vector $\mathbf{v}$. Setting $\mathbf{w} = [\mathbf{u}, \mathbf{v}]$, where $L_{\mathbf{w}} = L_{\mathbf{u}}L_{\mathbf{v}} - L_{\mathbf{v}}L_{\mathbf{u}}$, it is easy to verify that the vector fields $\mathbf{v}_l$, $l = \overline{1,4}$, generate a Lie algebra. Following [1, 2] the dimension of the orbit $O(\mathbf{e})$ is equal to the dimension of this algebra applying to the element $\mathbf{e}$, i.e. to the rank of a matrix $M_k = \big(\mathbf{v}_l(\mathbf{e})|\, l = \overline{1,4}\big)$ of the dimensions $4 \times (2k+2)$. The classification of some polynomial systems according to the dimensions of their $GL(2,\mathbb{R})-$orbits was done in [2–11].

Denote $\mathbf{v}_l(\mathbf{e}) = (A_{k0}^{(l)}, A_{k-1,1}^{(l)}, \ldots, A_{0k}^{(l)}; B_{k0}^{(l)}, B_{k-1,1}^{(l)}, \ldots, B_{0k}^{(l)})$, $l = \overline{1,4}$. Taking into account that $\mathbf{v}_l(\mathbf{e}) = A^{(l)} \cdot \mathbf{e}$, the coordinates of vectors $\mathbf{v}_l(\mathbf{e})$ can be represented by coefficients of the system (1) as follows:

$$A_{k-i,i}^{(1)} = -(k-i-1)a_{k-i,i},\ B_{k-i,i}^{(1)} = -(k-i)b_{k-i,i},\ i = \overline{0,k};$$

$$A_{k0}^{(2)} = b_{k0},\ A_{k-i,i}^{(2)} = b_{k-i,i} - (k-i+1)a_{k-i+1,i-1},$$

$$B_{k0}^{(2)} = 0,\ B_{k-i,i}^{(2)} = -(k-i+1)b_{k-i+1,i-1},\ i = \overline{1,k};$$

$$A_{k-i,i}^{(3)} = -(i+1)a_{k-i-1,i+1},\ A_{0k}^{(3)} = 0,$$

$$B_{k-i,i}^{(3)} = a_{k-i,i} - (i+1)b_{k-i-1,i+1},\ B_{0k}^{(3)} = a_{0k},\ i = \overline{0,k-1};$$

$$A_{k-i,i}^{(4)} = -ia_{k-i,i},\ B_{k-i,i}^{(4)} = -(i-1)b_{k-i,i},\ i = \overline{0,k}.$$

For $k = 0$ and $k = 1$ the matrix M_k becomes

$$M_0 = \begin{pmatrix} a_{00} & 0 \\ b_{00} & 0 \\ 0 & a_{00} \\ 0 & b_{00} \end{pmatrix},\quad M_1 = \begin{pmatrix} 0 & a_{01} & -b_{10} & 0 \\ b_{10} & b_{01} - a_{10} & 0 & -b_{10} \\ -a_{01} & 0 & a_{10} - b_{01} & a_{01} \\ 0 & -a_{01} & b_{10} & 0 \end{pmatrix}.$$

By direct calculations, we obtain the following two theorems:

Theorem 1. *Let $k = 0$ and d be the dimension of the $GL(2,\mathbb{R})-$orbit $O(\mathbf{e})$ of the system (1). Then,*

$d = 0$, iff $P_0 = Q_0 = 0$ and

$d = 2$ in other cases.

Theorem 2. *Let $k=1$ and d be the dimension of the $GL(2,\mathbb{R})-$orbit $O(\mathbf{e})$ of the system (1). Then,*
$d=0$, *iff* $a_{10}-b_{01}=a_{01}=b_{10}=0$ *and*
$d=2$ *in other cases.*

Let $GCD(P,Q)$ be the greatest common divisor of the polynomials P and Q. The main result of this paper is the following theorem.

Theorem 3. *If the dimension of the $GL(2,\mathbb{R})-$orbit of the differential system (1) is smaller than four, then $deg(GCD(P,Q)) \geq k-1$.*

Next, in this work we will suppose that

$$k \geq 2 \text{ and } |P_k(x_1,x_2)|+|Q_k(x_1,x_2)| \not\equiv 0. \tag{4}$$

2 One lemma

Let $\tau \in \{0,1,2,...,k\}$. Consider the polynomial

$$f = z_1x^k + z_2x^{k-1}+\ldots+z_{k+1},\ z_i \in \mathbb{C},\ i=\overline{1,k+1} \tag{5}$$

and the $(k+1)\times(k+1)$-matrix $\tilde{A}$ defined by :

$$\begin{array}{l} \tilde{a}_{i,i-1}=(k-i+2)\xi_1\xi_2,\ i=\overline{2,k+1};\ \tilde{a}_{i,i+1}=-i,\ i=\overline{1,k}; \\ \tilde{a}_{ii}=(k-\tau-i+1)\xi_1+(\tau-i+1)\xi_2,\ i=\overline{1,k+1}; \\ \tilde{a}_{il}=0,\ |i-l|>1, \end{array} \tag{6}$$

where ξ_1, ξ_2 are constant. It is easy to show that

$$k \leq rank(\tilde{A}) \leq k+1. \tag{7}$$

Lemma 1. *If the vector*

$$Z=(z_1,z_2,...,z_{k+1})^{tr} \tag{8}$$

is a solution of the equation

$$\tilde{A}Z=0, \tag{9}$$

then (5) has the form

$$f=c\cdot(x+\xi_1)^{k-\tau}(x+\xi_2)^{\tau}, \tag{10}$$

where c is a constant.

Proof. Without loss of generality we can assume that $\tau \in \{0,\ 1,\ 2,...,[k/2]\}$, where by $[k/2]$ we denoted the integer part of the number $k/2$.

Let $\tilde{R}=\tilde{A}\tilde{Z}=(\tilde{r}_1,\ldots,\tilde{r}_{k+1})^{tr}$, where $\tilde{Z}=(\tilde{z}_1,\ldots,\tilde{z}_{k+1})^{tr}$ and

$$\tilde{z}_i=\sum_{\mu=0}^{i-1} C_{k-\tau}^{i-\mu-1}C_{\tau}^{\mu}\xi_1^{i-\mu-1}\xi_2^{\mu}, \tag{11}$$

if $1 \le i \le \tau+1$;

$$\tilde{z}_i = \sum_{\mu=0}^{\tau} C_{k-\tau}^{i-\mu-1} C_\tau^\mu \xi_1^{i-\mu-1} \xi_2^\mu, \tag{12}$$

if $\tau+1 < i \le k-\tau+1$ and

$$\tilde{z}_i = \sum_{\mu=0}^{k-i+1} C_{k-\tau}^{i-\tau+\mu-1} C_\tau^{\tau-\mu} \xi_1^{i-\tau+\mu-1} \xi_2^{\tau-\mu}, \tag{13}$$

if $k-\tau+1 < i \le k+1$.

We will prove that the vector $\tilde{Z}$ with the coordinates (11)−(13) is a solution of the equation (9).

a) Let $1 \le i \le \tau$. Taking into consideration (6) and (11), we obtain:

$$\tilde{r}_1 = \tilde{a}_{12} \cdot \tilde{z}_2 + \tilde{a}_{11} \tilde{z}_1 = -((k-\tau)\xi_1 + \tau\xi_2) + ((k-\tau)\xi_1 + \tau\xi_2) \cdot 1 = 0;$$

$$\tilde{r}_i = \tilde{a}_{i,i+1}\tilde{z}_{i+1} + \tilde{a}_{i,i-1}\tilde{z}_{i-1} + \tilde{a}_{i,i}\tilde{z}_i = -i\tilde{z}_{i+1} + (k-i+2)\sum_{\mu=0}^{i-2} C_{k-\tau}^{i-\mu-2} C_\tau^\mu \xi_1^{i-\mu-1}\xi_2^{\mu+1} +$$

$$+(k-\tau-i+1)\sum_{\mu=0}^{i-1} C_{k-\tau}^{i-\mu-1} C_\tau^\mu \xi_1^{i-\mu}\xi_2^{\mu} + (\tau-i+1)\sum_{\mu=0}^{i-1} C_{k-\tau}^{i-\mu-1} C_\tau^\mu \xi_1^{i-\mu-1}\xi_2^{\mu+1} =$$

$$= -i\tilde{z}_{i+1} + (k-i+2)\sum_{\mu=1}^{i-1} C_{k-\tau}^{i-\mu-1} C_\tau^{\mu-1} \xi_1^{i-\mu}\xi_2^{\mu} +$$

$$+(k-\tau-i+1)\sum_{\mu=1}^{i-1} C_{k-\tau}^{i-\mu-1} C_\tau^\mu \xi_1^{i-\mu}\xi_2^{\mu} + (k-\tau-i+1) C_{k-\tau}^{i-1} C_\tau^0 \xi_1^i \xi_2^0 +$$

$$+(\tau-i+1)\sum_{\mu=0}^{i-2} C_{k-\tau}^{i-\mu-1} C_\tau^\mu \xi_1^{i-\mu-1}\xi_2^{\mu+1} + (\tau-i+1) C_{k-\tau}^0 C_\tau^{i-1} \xi_1^0 \xi_2^i =$$

$$= -i\tilde{z}_{i+1} + \sum_{\mu=1}^{i-1} [(k-i+2) C_{k-\tau}^{i-\mu-1} C_\tau^{\mu-1} + (k-\tau-i+1) C_{k-\tau}^{i-\mu-1} C_\tau^\mu +$$

$$+(\tau-i+1) C_{k-\tau}^{i-\mu} C_\tau^{\mu-1}] \xi_1^{i-\mu}\xi_2^{\mu} + i C_{k-\tau}^{k-\tau-i} C_\tau^0 \xi_1^i \xi_2^0 + i C_{k-\tau}^0 C_\tau^i \xi_1^0 \xi_2^i =$$

$$= -i\tilde{z}_{i+1} + \sum_{\mu=1}^{i-\mu} C_{k-\tau}^{i-\mu} C_\tau^\mu \left[(k-i+2) \cdot \frac{\mu(i-\mu)}{(k-\tau-i+\mu+1)(\tau-\mu+1)} + \right.$$

$$\left. +(k-\tau-i+1) \cdot \frac{i-\mu}{k-\tau-i+\mu+1} + (\tau-i+1) \cdot \frac{\mu}{\tau-\mu+1}\right] \xi_1^{i-\mu}\xi_2^{\mu} +$$

$$+iC^i_{k-\tau}C^0_\tau\xi_1^i\xi_2^0+iC^0_{k-\tau}C^i_\tau\xi_1^0\xi_2^i=-i\tilde{z}_{i+1}+i\biggl(C^i_{k-\tau}C^0_\tau\xi_1^i\xi_2^0+\sum_{\mu=1}^{i-1}C^{i-\mu}_{k-\tau}C^\mu_\tau\xi_1^{i-\mu}\xi_2^\mu+$$
$$+C^0_{k-\tau}C^i_\tau\xi_1^0\xi_2^i\biggr)=-i\tilde{z}_{i+1}+i\tilde{z}_{i+1}=0.$$

b) $\boldsymbol{i=\tau+1}$**.** From formulae (6), (11) and (12) we get:

$$\tilde{r}_{\tau+1}=\tilde{a}_{\tau+1,\tau}\tilde{z}_\tau+\tilde{a}_{\tau+1,\tau+1}\tilde{z}_{\tau+1}-\tilde{a}_{\tau+1,\tau+2}\tilde{z}_{\tau+2}=(k-\tau+1)\xi_1\xi_2\tilde{z}_\tau+$$
$$+(k-2\tau)\sum_{\mu=0}^{\tau}C^{\tau-\mu}_{k-\tau}C^\mu_\tau\xi_1^{\tau-\mu+1}\xi_2^\mu-(\tau+1)\sum_{\mu=0}^{\tau}C^{\tau-\mu+1}_{k-\tau}C^\mu_\tau\xi_1^{\tau-\mu+1}\xi_2^\mu=$$
$$=(k-\tau+1)\xi_1\xi_2\tilde{z}_\tau+(k-2\tau)\sum_{\mu=1}^{\tau}C^{\tau-\mu}_{k-\tau}C^\mu_\tau\xi_1^{\tau-\mu+1}\xi_2^\mu-$$
$$-(\tau+1)\sum_{\mu=1}^{\tau}C^{\tau-\mu+1}_{k-\tau}C^\mu_\tau\xi_1^{\tau-\mu+1}\xi_2^\mu=(k-\tau+1)\xi_1\xi_2\tilde{z}_\tau+$$
$$+\xi_1\xi_2\sum_{\mu=0}^{\tau-1}\left[(k-2\tau)\frac{\tau-\mu}{\mu+1}-(\tau+1)\frac{k-2\tau+\mu+1}{\tau-\mu}\cdot\frac{\tau-\mu}{\mu+1}\right]C^{\tau-\mu-1}_{k-\tau}C^\mu_\tau\xi_1^{\tau-\mu-1}\xi_2^\mu=$$
$$=(k-\tau+1)\xi_1\xi_2\tilde{z}_\tau-(k-\tau+1)\xi_1\xi_2\tilde{z}_\tau=0.$$

c) $\boldsymbol{\tau+2\le i\le k-\tau}$**.** In this case the formulae (6) and (12) give us:

$$\tilde{r}_i=-i\tilde{z}_{i+1}+(k-i+2)\xi_1\xi_2\tilde{z}_{i-1}+[(k-\tau-i+1)\xi_1+(\tau-i+1)\xi_2]\tilde{z}_i=$$
$$=-i\tilde{z}_{i+1}+(k-i+2)\sum_{\mu=0}^{\tau}C^{i-\mu-2}_{k-\tau}C^\mu_\tau\xi_1^{i-\mu-1}\xi_2^{\mu+1}+$$
$$+(\tau-i+1)\sum_{\mu=0}^{\tau}C^{i-\mu-1}_{k-\tau}C^\mu_\tau\xi_1^{i-\mu-1}\xi_2^{\mu+1}+(k-\tau-i+1)\sum_{\mu=0}^{\tau}C^{i-\mu-1}_{k-\tau}C^\mu_\tau\xi_1^{i-\mu}\xi_2^\mu=$$
$$=-i\tilde{z}_{i+1}+(k-i+2)\sum_{\mu=1}^{\tau}C^{i-\mu-2}_{k-\tau}C^\mu_\tau\xi_1^{i-\mu-1}\xi_2^{\mu+1}+(\tau-i+1)\sum_{\mu=0}^{\tau-1}C^{i-\mu-1}_{k-\tau}C^\mu_\tau\xi_1^{i-\mu-1}\xi_2^{\mu+1}+$$
$$+(k-\tau-i+1)\sum_{\mu=1}^{\tau}C^{i-\mu-1}_{k-\tau}C^\mu_\tau\xi_1^{i-\mu}\xi_2^\mu+(k-\tau-i+1)C^{i-1}_{k-\tau}\xi_1^i=$$
$$-i\tilde{z}_{i+1}+iC^i_{k-\tau}\xi_1^i+\sum_{\mu=1}^{\tau}\biggl[(k-i+2)\frac{(i-\mu)\mu}{(k-\tau-i+\mu+1)(\tau-\mu+1)}+$$

$$+(\tau-i+1)\frac{\mu}{\tau-\mu+1}+(k-\tau-i+1)\frac{i-\mu}{k-\tau-i+\mu+1}\Bigg]C_{k-\tau}^{i-\mu}C_\tau^\mu\xi_1^{i-\mu}\xi_2^\mu=$$
$$=-i\tilde{z}_{i+1}+i\tilde{z}_{i+1}=0.$$

d) $\boldsymbol{i=k-\tau+1}$. From (6), (12) and (13) we obtain:

$$\tilde{r}_{k-\tau+1}=(\tau+1)\sum_{\mu=0}^{\tau}C_{k-\tau}^{\mu+1}C_\tau^\mu\xi_1^{k-\tau-\mu}\xi_2^{\mu+1}-(k-2\tau)\sum_{\mu=0}^{\tau}C_{k-\tau}^{\mu}C_\tau^\mu\xi_1^{k-\tau-\mu}\xi_2^{\mu+1}-$$

$$-(k-\tau+1)\sum_{\mu=0}^{\tau-1}C_{k-\tau}^{\tau-\mu-1}C_\tau^\mu\xi_1^{k-2\tau+\mu+1}\xi_2^{\tau-\mu}=(\tau+1)\sum_{\mu=0}^{\tau-1}C_{k-\tau}^{\mu+1}C_\tau^\mu\xi_1^{k-\tau-\mu}\xi_2^{\mu+1}-$$

$$-(k-2\tau)\sum_{\mu=0}^{\tau-1}C_{k-\tau}^{\mu}C_\tau^\mu\xi_1^{k-\tau-\mu}\xi_2^{\mu+1}-(k-\tau+1)\sum_{\mu=0}^{\tau-1}C_{k-\tau}^{\mu}C_\tau^{\mu+1}\xi_1^{k-\tau-\mu}\xi_2^{\mu+1}=$$

$$=\sum_{\mu=0}^{\tau-1}C_{k-\tau}^{\mu}C_\tau^\mu\xi_1^{k-\tau-\mu}\xi_2^{\mu+1}\left[(\tau+1)\frac{k-\tau-\mu}{\mu+1}-(k-2\tau)-(k-\tau+1)\frac{\tau-\mu}{\mu+1}\right]=0.$$

e) $\boldsymbol{k-\tau+2\le i\le k+1.}$

$$\tilde{r}_i=-i\tilde{z}_{i+1}+(k-i+2)\sum_{\mu=0}^{k-i+2}C_{k-\tau}^{i-\tau+\mu-2}C_\tau^{\tau-\mu}\xi_1^{i-\tau+\mu-1}\xi_2^{\tau-\mu+1}+$$

$$+(k-\tau-i+1)\sum_{\mu=0}^{k-i+1}C_{k-\tau}^{i-\tau+\mu-1}C_\tau^{\tau-\mu}\xi_1^{i-\tau+\mu}\xi_2^{\tau-\mu}+$$

$$+(\tau-i+1)\sum_{\mu=0}^{k-i+1}C_{k-\tau}^{i-\tau+\mu-1}C_\tau^{\tau-\mu}\xi_1^{i-\tau+\mu-1}\xi_2^{\tau-\mu+1}=$$

$$-i\tilde{z}_{i+1}+(k-i+2)\sum_{\mu=1}^{k-i+1}C_{k-\tau}^{i-\tau+\mu-2}C_\tau^{\mu}\xi_1^{i-\tau+\mu-1}\xi_2^{\tau-\mu+1}+$$

$$+(k-\tau-i+1)\sum_{\mu=0}^{k-i}C_{k-\tau}^{i-\tau+\mu-1}C_\tau^{\mu}\xi_1^{i-\tau+\mu}\xi_2^{\tau-\mu}+$$

$$+(\tau-i+1)\sum_{\mu=1}^{k-i+1}C_{k-\tau}^{i-\tau+\mu-1}C_\tau^{\mu}\xi_1^{i-\tau+\mu-1}\xi_2^{\tau-\mu+1}=-i\tilde{z}_{i+1}+$$

$$+\sum_{\mu=0}^{k-i}C_{k-\tau}^{i-\tau+\mu}C_\tau^\mu\xi_1^{i-\tau+\mu}\xi_2^{\tau-\mu}\Bigg[(k-i+2)\frac{(i-\tau+\mu)(\tau-\mu)}{(k-i-\mu+1)(\mu+1)}+$$

$$+(k-\tau-i+1)\frac{i-\tau+\mu}{k-i-\mu+1}+(\tau-i+1)\frac{\tau-\mu}{\mu+1}\Bigg]=-i\tilde{z}_{i+1}+i\tilde{z}_{i+1}=0.$$

Hence, taking into account (7), the rank of the matrix $\tilde{A}$ is equal to k and therefore the general solution of the matrix equation (9) has the form $Z=\{c\tilde{Z}|c\in\mathbb{C}\}$. □

Corollary 1. *If* $\mathbf{Z}=\mathbf{a}$ $\;(\mathbf{Z}=\mathbf{b})$*, where*

$$\mathbf{a}=(a_{k0},\,a_{k-1,1},...,a_{0k})\;(\mathbf{b}=(b_{k0},\,b_{k-1,1},...,b_{0k})),\tag{14}$$

is a solution of the matrix equation (9) then the first (second) equation of (1) has the form

$$\dot{x}=c\cdot(x+\xi_1y)^{k-\tau}(x+\xi_2y)^{\tau},\;\;(\dot{y}=c\cdot(x+\xi_1y)^{k-\tau}(x+\xi_2y)^{\tau}).$$

3 Proof of Theorem 3

Applying to the system (1) the transposition of coordinates

$$x_1\rightarrow x_2,\;\;x_2\rightarrow x_1\tag{15}$$

we obtain

$$\dot{x}_1=Q_k(x_2,x_1),\;\;\dot{x}_2=P_k(x_2,x_1).\tag{16}$$

Denote by $\mathbf{v}_l^*$, $l=\overline{1,4}$, the vector fields associated to the differential system (16).

Remark 1. The equalities $\alpha\mathbf{v}_1+\beta\mathbf{v}_2+\gamma\mathbf{v}_3+\delta\mathbf{v}_4=0$ and $\delta\mathbf{v}_1^*+\gamma\mathbf{v}_2^*+\beta\mathbf{v}_3^*+\alpha\mathbf{v}_4^*=0$ are equivalent.

By Remark 1, in order to determine the orbits of dimension two and three it is sufficient to examine the following two cases:

$$\mathbf{v}_1-\delta\mathbf{v}_4=0,\quad \alpha\mathbf{v}_1+\beta\mathbf{v}_2-\mathbf{v}_3+\delta\mathbf{v}_4=0.$$

3.1 The case $\mathbf{v}_1-\delta\mathbf{v}_4=0$.

Let $\mathbf{v}_1(\mathbf{e})-\delta\mathbf{v}_4(\mathbf{e})=0$ or

$$(A^{(1)}-\delta A^{(4)})\mathbf{e}=0.\tag{17}$$

Because $\mathbf{e}\neq 0$ (see (4)) the equality (17) is realized for those δ for which $det(A^{(1)}-\delta A^{(4)})=0$, i.e.

$$-(k-1)^2(1+k\delta)(\delta+k)\prod_{\nu=2}^{k}[(\nu-1)\delta+\nu-k]^2=0.$$

By the assumption (4), $k\geq 2$. If $\delta=-1/k$ ($\delta=-k$), then $det(D_1-\delta D_4)\neq 0$ ($det(A_1-\delta A_4)\neq 0$), but the matrix $A_1-\delta A_4$ ($D_1-\delta D_4$) has on the principal diagonal unique element equal to zero and this element is placed on $(k+1,k+1)$ $((1,1))$. In these cases equality (17) leads us to the systems

$$\dot{x_1} = a_{0k}x_2^k,\ \dot{x_2} = 0; \tag{18}$$

$$\dot{x_1} = 0,\ \dot{x_2} = b_{k0}x_1^k. \tag{19}$$

Let $\delta = (k-\nu)/(\nu-1)$. For this δ both matrixes $A_1 - \delta A_4$ and $D_1 - \delta D_4$ have on the principal diagonal only one element equal to zero: first on the cells $(\nu,\nu),\ \nu = \overline{2,k}$, and second on the cells $(\nu+1,\nu+1),\ \nu = \overline{2,k}$. Taking into account (17), we obtain the systems

$$\dot{x}_1 = a_{k-\nu+1,\nu-1}x_1 \cdot F,\ \dot{x}_2 = b_{k-\nu,\nu}x_2 \cdot F,\ F = x_1^{k-\nu}x_2^{\nu-1},\ \nu = \overline{2,k}. \tag{20}$$

Remark 2. Substitutions (15) reduce system (19) to one of the form (18).

3.2 The case $\mathbf{v}_3 = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \delta\mathbf{v}_4$.

In this subsection we will determine the systems (1), $k \geq 2$, for which there exist numbers α, β and δ such that

$$\mathbf{v}_3 = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \delta v_4. \tag{21}$$

Denote

$$M = A^{(3)} - \alpha A^{(1)} - \beta A^{(2)} - \delta A^{(4)},\ \mathbf{a} = (a_{k0},\ldots,a_{0k}),\ \mathbf{b} = (b_{k0},\ldots,b_{0k}).$$

Then

$$M = \begin{pmatrix} A & I \\ -\beta I & D \end{pmatrix},\ A = A_3 - \alpha A_1 - \beta A_2 - \delta A_4,\ D = A + (\alpha-\delta)I,\ \mathbf{e} = (\mathbf{a},\mathbf{b}).$$

We have to find α, β and δ such that the matrix equation

$$M\mathbf{e} = 0 \qquad \text{or} \qquad \begin{cases} A\mathbf{a} = -\mathbf{b}, \\ [A + (\alpha-\delta)I]\mathbf{b} = \beta\mathbf{a} \end{cases} \tag{22}$$

have nontrivial solutions with respect to $\mathbf{e}$.

From (22) it follows that $\mathbf{a}$ and $\mathbf{b}$ verify the same matrix equation:

$$SZ = 0, \tag{23}$$

where $S = A^2 + (\alpha-\delta)A + \beta I$, $dimS = (k+1)\times(k+1)$, and Z is the vector (8).

The matrix S has the following elements:

$$s_{11} = (k-1)(k\alpha^2 - \alpha\delta - \beta), \quad s_{12} = -2(k-1)\alpha, \quad s_{13} = 2;$$

$$s_{21} = 2k(k-1)\alpha\beta,\ s_{22} = (k-1)\left[(k-2)\alpha^2 + \alpha\delta - 3\beta\right],$$
$$s_{23} = -4\left[(k-2)\alpha + \delta\right],\ s_{24} = 6;$$

$$s_{i,i-2} = (k-i+2)(k-i+3)\beta^2,\ s_{i,i-1} = 2(k-i+2)[(k-i+1)\alpha + (i-2)\delta]\beta,$$
$$s_{i,i} = [(k-i)\alpha + (i-1)\delta]\cdot[(k-i+1)\alpha + (i-2)\delta] - [(2i-1)k - 2(i-1)^2 - 1]\beta,$$
$$s_{i,i+1} = -2i[(k-i)\alpha + (i-1)\delta],\ s_{i,i+2} = i(i+1),\ i = \overline{2, k-1};$$
$$s_{k,k-2} = 6\beta^2,\ s_{k,k-1} = 4[\alpha + (k-2)\delta]\cdot\beta,\ s_{k,k} = (k-1)[\alpha\delta + (k-2)\delta^2 - 3\beta],$$
$$s_{k,k+1} = -2k(k-1)\delta;$$
$$s_{k+1,k-1} = 2\beta^2,\ s_{k+1,k} = 2(k-1)\delta\beta,\ s_{k+1,k+1} = (k-1)(k\delta^2 - \alpha\delta - \beta);$$
$$s_{ij} = 0,\ i,j = \overline{1, k+1}, |i-j| > 2.$$

The rank of S verifies the inequalities $k-1 \le rank(S) \le k+1$ and the determinant ($\Delta = det(S)$) is equal to

$$\begin{aligned}&\Delta = (k-1)^4(\beta+\alpha\delta)^2[(k+1)^2\beta - (\alpha - k\delta)(k\alpha-\delta)]\times\\ &\prod_{j=0}^{m-2}[(2j+1)^2\beta + ((m+j)\alpha + (m-j-1)\delta)\times\\ &((m-j-1)\alpha + (m+j)\delta)]^2\end{aligned} \tag{24}$$

if $k = 2m$ and

$$\begin{aligned}&\Delta = 4m^4(\alpha+\delta)^2(\beta+\alpha\delta)^2[(k+1)^2\beta - (\alpha - k\delta)(k\alpha-\delta)]\times\\ &\prod_{j=1}^{m-1}[4j^2\beta + ((m+j)\alpha + (m-j)\delta)\times\\ &((m-j)\alpha + (m+j)\delta)]^2\end{aligned} \tag{25}$$

if $k = 2m+1$.

Denote

$$A_{1,2} = A - \frac{\delta - \alpha \pm \sqrt{(\delta-\alpha)^2 - 4\beta}}{2}I.$$

We have that $A_1A_2 = A_2A_1$, $k \le rankA_{1,2} \le k+1$ and in (23) that $S = A_2A_1$. Hence, every solution of the matrix equation

$$A_1Z = 0 \tag{26}$$

or

$$A_2Z = 0 \tag{27}$$

is also a solution of the equation (23).

Next we will analyse each of the cases when the determinant Δ of the matrix S is equal to zero and will indicate the systems (1) of which coefficients (14) verify the matrix equation (23), i.e. each of the vectors $\mathbf{a}$ and $\mathbf{b}$ verifies at least one of the equations (26), (27).

3.2.1. $\beta = (\alpha - k\delta)(k\alpha - \delta)/(k+1)^2$. Let

$$\xi_1 = (\alpha - k\delta)/(k+1), \quad \xi_2 = (k\alpha - \delta)/(k+1).$$

Then $\beta = \xi_1\xi_2$ and

$$A_1 = A + \xi_1 I, \quad A_2 = A + \xi_2 I. \tag{28}$$

Setting in (6) $\tau = k$, we obtain that $\tilde{A} = A_1$. Therefore, $det A_1 = 0$ and $ker A_1 = \{c\mathbf{Z_1} | c = const\}$, where $\mathbf{Z_1}$ has coordinates (11).

If $A_2 \neq A_1$, i.e. $\alpha + \delta \neq 0$, then from (24), (25) and $\Delta = det S = det A_1 \cdot det A_2$ it follows that $det A_2 \neq 0$. Thus, in this case, in order that the dimension of the $GL(2,\mathbb{R})-$orbit of the system (1) be smaller than four it is necessary that its coefficients (14) (**a** and **b**) verify the equation (26). By Lemma 1 $f = c(x+\xi_2)^k$ and by Corollary 1, we have

$$\begin{cases} \dot{x_1} = c_1 \cdot F(x_1, x_2), \ \dot{x_2} = c_2 \cdot F(x_1, x_2); \ c_1, c_2 = const, \\ \\ F = [(k+1)x_1 + (k\alpha - \delta)x_2]^k . \end{cases} \tag{29}$$

3.2.2. $\beta = -\alpha\delta$. In this case, we put $\xi_1 = \alpha, \ \xi_2 = -\delta$. Then $A_1 = A + \alpha I, \ A_2 = A - \delta I$ and setting in (6) $\tau = 0$ ($\tau = 1$), we have that $A_1 = \tilde{A}$ ($A_2 = \tilde{A}$) and $f = c_1(x+\xi_1)^k$ ($f = c_2(x+\xi_1)^{k-1}(x+\xi_2)$). If $\tau = 0$ ($\tau = 1$) the vector $\mathbf{Z_1}$ ($\mathbf{Z_2}$) with the coordinates (12) ((12), (13)) is a solution of the equation (26) ((27)). The solutions $\mathbf{Z_1}$ and $\mathbf{Z_2}$ are linear independent and therefore $c_1\mathbf{Z_1} + c_2\mathbf{Z_2}$; $c_1, c_2 = const$, is the general solution of (23). This implies (1) to have the form

$$\begin{cases} \dot{x_1} = (ax_1 + bx_2)F(x_1, x_2), \ \dot{x_2} = (cx_1 + dx_2)F(x_1, x_2); \\ \\ a, b, c, d = const, \ F(x_1, x_2) = (x_1 + \alpha x_2)^{k-1}. \end{cases} \tag{30}$$

3.2.3. $k = 2m$,

$$\beta = -((m-j-1)\alpha + (m+j)\delta)((m+j)\alpha + (m-j-1)\delta)/(2j+1)^2, \ j = \overline{0, m-2}.$$

Let

$$\xi_1 = -[(m-j-1)\alpha + (m+j)\delta]/(2j+1),$$

$$\xi_2 = [(m+j)\alpha + (m-j-1)\delta]/(2j+1).$$

Then $\beta = \xi_1\xi_2$ and the equalities (28) hold. Setting in (6) $\tau = m+j$ ($\tau = m+j+1$), we obtain that $\tilde{A} = a_1$ ($\tilde{A} = a_2$), and the relations $(11)-(13)$ lead us to the polynomial

$$f = c(x+\xi_1)^{m-j}(x+\xi_2)^{m+j} \ \left(f = c(x+\xi_1)^{m-j-1}(x-\xi_2)^{m+j+1}\right).$$

Hence, for $\tau = m-j$ ($\tau = m-j-1$) the vector Z_1 (Z_2) with the coordinates $(11)-(13)$ is a solution of the equation (26) ((27)). The vectors Z_1 and Z_2 are linear independent which implies the differential system (1) to be written as:

$$\begin{cases} \dot{x_1} = (ax_1 + bx_2) \cdot F, \ \dot{x_2} = (cx_1 + dx_2) \cdot F, \\ \\ F = [(2j+1)x_1 - ((m-j-1)\alpha + (m+j)\delta)x_2]^{m-j-1} \times \\ \\ [(2j+1)x_1 + ((m+j)\alpha + (m-j-1)\delta)x_2]^{m+j}, \ j = \overline{0, m-2}. \end{cases} \tag{31}$$

3.2.4. Let $k = 2m+1$ and $\beta = \xi_1\xi_2$, where

$$\xi_1 = -\left[(m-j)\alpha + (m+j)\delta\right]/(2j),$$

$$\xi_2 = \left[(m+j)\alpha + (m-j)\delta\right]/(2j),\ \ j = \overline{1, m-1}.$$

In these conditions equalities (28) hold. If $\tau = m+j$ ($\tau = m+j+1$), then the vector Z_1 (Z_2) with the coordinates (11)$-$(13) is a solution of the equation (26) ((27)) and the polynomial (5) looks as:

$$f = c(x+\xi_1)^{m-j+1}(x+\xi_2)^{m+j}\ \ (f = c(x+\xi_1)^{m-j}(x-\xi_2)^{m+j+1}).$$

The solutions $c_1Z_2 + c_2Z_2$; $c_1c_2 = const$ of the equation (23) lead us to the following system

$$\begin{cases} \dot{x_1} = (ax_1+bx_2)\cdot F,\ \ \dot{x_2} = (cx_1+dx_2)\cdot F, \\ F = \left[2jx_1 - ((m-j)\alpha + (m+j)\delta)x_2\right]^{m-j} \times \\ \left[2jx_1 + ((m+j)\alpha + (m-j)\delta)x_2\right]^{m+j},\ \ j = \overline{1, m-1}. \end{cases} \tag{32}$$

3.2.5. $\alpha + \delta = 0$. Let

$$\delta = -\alpha,\ \ \xi_1 = \alpha - \sqrt{\alpha^2 - \beta},\ \ \xi_2 = \alpha + \sqrt{\alpha^2-\beta}.$$

Substituting in (11)$-$(13) $\tau = m$ ($\tau = m+1$), we obtain that the vector Z_1 (Z_2) with these coordinates is a solution of the equation (26) ((27)), where A_1 and A_2 are given in (28). The polynomial f looks as:

$$f = c(x+\xi_1)^{m+1}(x+\xi_2)^{m}\ \ (f = c(x+\xi_1)^{m}(x+\xi_2)^{m+1}).$$

This case leads us to the following differential system

$$\begin{cases} \dot{x_1} = (ax_1+bx_2)\cdot F,\ \ \dot{x_2} = (cx_1+dx_2)\cdot F, \\ F = (x_1^2 + 2\alpha x_1x_2 + \beta x_2^2)^m. \end{cases} \tag{33}$$

Theorem 3 is proved.

From proving Theorem 3 follows

Theorem 4. *In order that the dimension of the $GL(2,\mathbb{R})-$orbit of the system (1) be smaller than four it is necessary (up to transformation (15)) that the system (1) have one of the forms (18),(20),(29)–(33).*

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Hopf bifurcations analysis of a three-dimensional nonlinear system

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Abstract. Bifurcations analysis of a $3D$ nonlinear chaotic system, called the T system, is treated in this paper, extending the work presented in [5] and [6]. The system T belongs to a class of cvasi-metriplectic systems having the same Poisson tensor and the same Casimir.

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1 Introduction

The nonlinear differential systems are studied both from theoretical point of view and from the point of view of their potential applications in various areas. The nonlinear systems present many times the property of sensitivity with respect to the initial conditions (some authors consider this property sufficient for a system to be chaotic). Applications of such systems have been found lately in secure communications [1, 4]. Of the pioneering papers which proposed to use the chaotic systems in communications are the papers of Pecora and Carrol [8, 9]. Consequently, an appropriate chaotic system can be chosen from a catalogue of chaotic systems to optimize some desirable factors, idea suggested in [4].

These facts led us to study a new 3D polynomial differential system given by:

$$\dot{x} = a(y - x), \;\; \dot{y} = (c - a)x - axz, \;\; \dot{z} = -bz + xy, \tag{1}$$

with a, b, c real parameters and $a \neq 0$. Call it the T system. Some results regarding the T system are already presented in [5] and [6], where we pointed out two particular cases. The system T is related to the Lorenz, Chen and Lü system [3] being a small generalization of the latter one.

The paper is organized as follows. In the first Section we recall some results regarding the stability of equilibria. In Section 2 we present the pitchfork and Hopf bifurcations occurring at the equilibrium points in the general case. Also, we show that the T system belongs to a class of cvasi-metriplectic systems.

2 Equilibrium points of the system

Because the dynamics of the system is characterized by the existence and the number of the equilibrium points as well as of their type of stability, we recall in the

Figure 1. a) The orbit of the Lorenz system for $a = 10,\ b = 8/3,\ c = 28$ and the initial condition $(-0.04,\ -0.3,\ 0.52)$ (left); b) The orbit of the Chen system for $a = 35,\ b = 3,\ c = 28$ and the initial condition $(-0.1,\ 0.1,\ 0.4)$ (right)

Figure 2. a) The orbit of the Lü system for $a = 36, b = 3, c = 19$ and the initial condition $(-1, 0.1, 4)$ (left) ; b) The orbit of the T system for $(a, b, c) = (2.1, 0.6, 30)$ and the initial condition $(0.1, -0.3, 0.2)$ (right)

following the equilibrium points of the system T and their stability.

Proposition 1. *If* $\frac{b}{a}(c-a) > 0$*, then the system* T *possesses three equilibrium isolated points:*

$$O(0,0,0), E_1(x_0, x_0, c/a-1), E_2(-x_0, -x_0, c/a-1)$$

where $x_0 = \sqrt{\frac{b}{a}(c-a)}$, *and for* $b \neq 0, \frac{b}{a}(c-a) \leq 0$, *the system* T *has only one isolated equilibrium point, namely* $O(0,0,0)$,

Theorem 1. *For* $b \neq 0$ *the following statements are true:*

a) If $(a > 0, b > 0, c \leq a)$*, then* $O(0,0,0)$ *is asymptotically stable;*

b) If $(b < 0)$ *or* $(a < 0)$ *or* $(a > 0, c > a)$*, then* $O\,(0,0,0)$ *is unstable.*

For the other two equilibria, $E_{1,2}(\pm x_0, \pm x_0, c/a-1)$, for $b/a(c-a) > 0$, using the Routh-Hurwitz conditions we have:

Theorem 2. *([5]) If the conditions* $a+b > 0$, $ab(c-a) > 0$, $b(2a^2+bc-ac) > 0$ *hold, the equilibrium points* $E_{1,2}(\pm x_0, \pm x_0, c/a-1)$, *are asymptotically stable.*

3 Pitchfork and Hopf bifurcations of the system T

Consider the parameter a as bifurcation parameter.

a) *Bifurcations at O(0,0,0)*

Proposition 2. *([7]) If* $\beta = a - c = 0$ *the equilibrium O(0,0,0) of the system* T *undergoes a pitchfork bifurcation that generates the asymptotic stable equilibrium point O(0,0,0) if* $a > c$*, and for* $a < c$ *three equilibria: O(0,0,0) (unstable),* $E_{1,2}(\pm x_0, \pm x_0, c/a-1)$ *(locally stable).*

Notice that the equilibrium O(0,0,0) can not undergo a Hopf bifurcation because the roots of the characteristic polynomial of the Jacobian matrix of the system T at $O(0,0,0)$ are $\lambda_1 = -b$, $\lambda_2 = \frac{1}{2}\left(-a-\sqrt{4ac-3a^2}\right)$, $\lambda_3 = \frac{1}{2}\left(-a+\sqrt{4ac-3a^2}\right)$ and the last two roots can not be purely imaginary because $a \neq 0$.

b) Bifurcations of the equilibria E_1 *and* E_2*.*

We observe that the characteristic polynomial in this case is:

$$f(\lambda) := \lambda^3 + \lambda^2(a+b) + bc\lambda + 2ab(c-a) \tag{2}$$

Because $ab(c-a) > 0$, the system T does not undergo pitchfork bifurcations at $E_{1,2}$, so we study the Hopf bifurcations at these points. The following proposition characterizes the imaginary roots of (2).

Proposition 3. *Consider* $\frac{b}{a}(c-a)>0$. *The polynomial (2) has one real negative root and two purely imaginary roots if and only if* $(a,b,c)\in\Omega$, *where*

$$\Omega=\{(a,b,c)\in\mathbb{R}^3 \mid a>b>0, 2a^2+bc=ac\}.$$

In this case the roots are: $\lambda_1=-a-b$, $\lambda_{2,3}=\pm i\omega, \omega:=\sqrt{bc}$.

In the following we show that the system T undergoes a Hopf bifurcation at E_1 (for E_2 is similar). Remember that a is the bifurcation parameter.

From $2a^2+bc=ac$ one gets $a=a_s:=\dfrac{c\pm\sqrt{c^2-8bc}}{4}$. Denote $\lambda:=\alpha\,(a)\pm i\omega\,(a)$ the complex roots depending on the bifurcation parameter a of (2). If $a=a_s$ and $(a,b,c)\in\Omega$, from the above Proposition 3, we have $\lambda_1=-a-b$ and $\lambda_{2,3}=\pm i\omega$ with $\omega=\sqrt{bc}$.

From (2) it follows that:

$$Re\left(\frac{d\lambda\,(a)}{da}\Big|_{a=a_s,\lambda=i\sqrt{bc}}\right)=Re\left(\frac{bc-4ab}{2bc-2i\sqrt{bc}\,(a+b)}\right)=\frac{bc-4ab}{2bc+2\,(a+b)^2}\neq 0$$

for $c\neq 8b$ because $c-4a=\pm\sqrt{c^2-8bc}$, $(b,c>0)$.

In the following we compute the index number K from the Hopf bifurcation theorem [2], employing the central manifold theory.

Using the transformation $(x,y,z)\to(X_1,Y_1,Z_1)$,

$$x=X_1+x_0,\ y=Y_1+x_0,\ z=Z_1+c/a-1, \tag{3}$$

the system T leads to:

$$\dot{X}_1=a(Y_1-X_1),\ \dot{Y}_1=-ax_0Z_1-aX_1Z_1,\ \dot{Z}_1=x_0(X_1+Y_1)-bZ_1+X_1Y_1 \tag{4}$$

or, in the normal form:

$$\begin{pmatrix}\dot{X}_1\\ \dot{Y}_1\\ \dot{Z}_1\end{pmatrix}=\begin{pmatrix}-a & a & 0\\ 0 & 0 & -ax_0\\ x_0 & x_0 & -b\end{pmatrix}\begin{pmatrix}X_1\\ Y_1\\ Z_1\end{pmatrix}+\begin{pmatrix}0\\ -aX_1Z_1\\ X_1Y_1\end{pmatrix}. \tag{5}$$

The eigenvalues of the Jacobian matrix J,

$$J=\begin{pmatrix}-a & a & 0\\ 0 & 0 & -ax_0\\ x_0 & x_0 & -b\end{pmatrix}$$

are, respectively $\lambda_1=-a-b<0$, $\lambda_{2,3}=\pm i\omega, \omega=\sqrt{bc}$, $b>0$, $c>0$, with corresponding eigenvectors:

$$v_1\left(1,\frac{-c+2a}{c},-\frac{2}{c}\sqrt{c-2a-b}\right),\quad v_2\left(-\frac{a}{\omega}qx_0+iqx_0,-\frac{a}{\omega}x_0,i\right),$$

$$v_3\left(-\frac{a}{\omega}qx_0 - iqx_0, -\frac{a}{\omega}x_0, -i\right)$$

with $q = \dfrac{a^2}{\omega^2+a^2}$. Then the vectors $v_2' = \dfrac{v_2+v_3}{2} = \left(-\dfrac{a}{\omega}qx_0, -\dfrac{a}{\omega}x_0, 0\right)$, $v_3' = \dfrac{v_2-v_3}{2i} = (qx_0, 0, 1)$ and $v_1\left(1, \dfrac{-c+2a}{c}, -\dfrac{2}{c}\sqrt{c-2a-b}\right)$ lead us to the transformation:

$$\begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} = P \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

where

$$P = \begin{pmatrix} 1 & -\frac{a}{\omega}qx_0 & qx_0 \\ \frac{-c+2a}{c} & -\frac{a}{\omega}x_0 & 0 \\ -\frac{2}{c}\sqrt{c-2a-b} & 0 & 1 \end{pmatrix}$$

or, equivalently,

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = P^{-1} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix}$$

where

$$P^{-1} = \frac{1}{d}\begin{pmatrix} c & -qc & -qx_0c \\ \frac{-c+2a}{ax_0}\omega & -\frac{c+2\sqrt{c-2a-b}\,qx_0}{ax_0}\omega & q\frac{c-2a}{a}\omega \\ 2\sqrt{c-2a-b} & -2q\sqrt{c-2a-b} & c+qc-2qa \end{pmatrix}$$

with $d = c + qc - 2qa + 2\sqrt{c-2a-b}\,qx_0$.

Then the system (5) reads:

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = P^{-1}JP \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + P^{-1}\begin{pmatrix} 0 \\ -aX_1Z_1 \\ X_1Y_1 \end{pmatrix}, \tag{6}$$

or, equivalently

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \begin{pmatrix} 2a\frac{-c+a}{c} & 0 & 0 \\ 0 & 0 & \omega \\ 0 & -\omega & 0 \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}, \tag{7}$$

where

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = P^{-1}\begin{pmatrix} 0 \\ -a\left(X - \frac{a}{\omega}qx_0Y + qx_0Z\right)\left(-\frac{2}{c}\sqrt{c-2a-b}\,X + Z\right) \\ \left(X - \frac{a}{\omega}qx_0Y + qx_0Z\right)\left(\frac{-c+2a}{c}X - \frac{a}{\omega}x_0Y\right) \end{pmatrix}, \tag{8}$$

or

$$g_1 = \frac{1}{d}qca\left(X - \frac{a}{\omega}qx_0Y + qx_0Z\right)\left(-\frac{2}{c}\sqrt{c-2a-b}X + Z\right) -$$

$$-\frac{1}{d}qx_0c\left(aX - \frac{a}{\omega}qx_0Y + qx_0Z\right)\left(\frac{-c+2a}{c}X - \frac{a}{\omega}x_0Y\right),$$

$$g_2 = \frac{1}{d}\frac{c+2\sqrt{c-2a-b}qx_0}{x_0}\omega\left(X - \frac{a}{\omega}qx_0Y + qx_0Z\right)\left(-\frac{2}{c}\sqrt{c-2a-b}X + Z\right) +$$

$$\frac{1}{d}q\frac{c-2a}{a}\omega\left(aX - \frac{a}{\omega}qx_0Y + qx_0Z\right)\left(\frac{-c+2a}{c}X - \frac{a}{\omega}x_0Y\right),$$

$$g_3 = \frac{2}{d}q\sqrt{c-2a-b}a\left(X - \frac{a}{\omega}qx_0Y + qx_0Z\right)\left(-\frac{2}{c}\sqrt{c-2a-b}X + Z\right) +$$

$$+\frac{1}{d}\left(c+qc-2qa\right)\left(aX - \frac{a}{\omega}qx_0Y + qx_0Z\right)\left(\frac{-c+2a}{c}X - \frac{a}{\omega}x_0Y\right).$$

Consider the 2-dimensional central manifold at the origin given by:

$$X = h\left(Y, Z\right) = AY^2 + BYZ + CZ^2 + \ldots \tag{9}$$

Then, replacing (9) in (7) and taking into account that $\dot{X} = 2AY\dot{Y} + B\dot{Y}Z + BY\dot{Z} + 2CZ\dot{Z}$, obtained from (9), one gets:

$$\dot{X} = BZ^2\omega - BY^2\omega - 2C\omega YZ + 2A\omega ZY + \ldots \tag{10}$$

On the other hand, from (7) we have

$$\dot{X} = \alpha_1Y^2 + \alpha_2YZ + \alpha_3Z^2 + \ldots \tag{11}$$

where

$$\alpha_1 = -2aA + 2\frac{a^2}{c}A - \frac{1}{d}q^2x_0^3c\frac{a^2}{\omega^2},$$

$$\alpha_2 = \frac{1}{d}q^2x_0^3c\frac{a}{\omega} - \frac{1}{d}q^2ca^2\frac{1}{\omega}x_0 - 2aB + 2\frac{a^2}{c}B,$$

$$\alpha_3 = 2\frac{a^2}{c}C + \frac{1}{d}q^2cax_0 - 2aC.$$

Then, equalling the coefficients of the terms Y^2, Z^2, YZ in the above relations (10) and (11), one gets

$$Y^2 : -\omega B = -2aA + 2\frac{a^2}{c}A - \frac{1}{d}q^2x_0^3c\frac{a^2}{\omega^2},$$

$$YZ : 2\omega A - 2C\omega = \frac{1}{d}q^2x_0^3c\frac{a}{\omega} - \frac{1}{d}q^2ca^2\frac{1}{\omega}x_0 - 2aB + 2\frac{a^2}{c}B,$$

$$Z^2 : B\omega = 2\frac{a^2}{c}C + \frac{1}{d}q^2cax_0 - 2aC.$$

Finally we get

$$A=-\frac{1}{4}q^2c^2x_0\frac{-\omega^2ca^3+\omega^2c^2a^2-\omega^4c^2+\omega^2cx_0^2a^2-4x_0^2a^4c+2x_0^2a^3c^2+2x_0^2a^5}{\left(\omega^2c^2-2ca^3+c^2a^2+a^4\right)d\omega^2\left(-a+c\right)},$$

$$B=\frac{1}{2}q^2x_0c^2\frac{2x_0^2ca+a^3-ca^2+\omega^2c-x_0^2a^2}{\omega d\left(\omega^2c^2-2ca^3+c^2a^2+a^4\right)}a,$$

$$C=-\frac{1}{4}q^2c^2x_0\frac{2x_0^2c^2a+5ca^3-3c^2a^2-\omega^2c^2-cx_0^2a^2-2a^4}{\left(\omega^2c^2-2ca^3+c^2a^2+a^4\right)d\left(-a+c\right)}.$$

Hence, the system (8) restricted to the central manifold is given by:

$$\begin{pmatrix}\dot{Y}\\ \dot{Z}\end{pmatrix}=\begin{pmatrix}0&\omega\\-\omega&0\end{pmatrix}\begin{pmatrix}Y\\Z\end{pmatrix}+\begin{pmatrix}g^2\left(Y,Z\right)\\g^3\left(Y,Z\right)\end{pmatrix},$$

where $g^2\left(Y,Z\right)=g_2\left(h\left(Y,Z\right),Y,Z\right),\ \ g^3\left(Y,Z\right)=g_3\left(h\left(Y,Z\right),Y,Z\right).$

Now K can be computed as follows [2]:

$$K(Y,Z)=\frac{1}{16}\left[\frac{\partial^3g^2}{\partial Y^3}+\frac{\partial^3g^2}{\partial Y\partial Z^2}+\frac{\partial^3g^3}{\partial Y^2\partial Z}+\frac{\partial^3g^3}{\partial Z^3}\right]+$$
$$+\frac{1}{16\omega}\left[\frac{\partial^2g^2}{\partial Y\partial Z}\left(\frac{\partial^2g^2}{\partial Y^2}+\frac{\partial^2g^2}{\partial Z^2}\right)-\frac{\partial^2g^3}{\partial Y\partial Z}\left(\frac{\partial^2g^3}{\partial Y^2}+\frac{\partial^2g^3}{\partial Z^2}\right)-\right.$$
$$\left.-\frac{\partial^2g^2}{\partial Y^2}\frac{\partial^2g^3}{\partial Y^2}+\frac{\partial^2g^2}{\partial Z^2}\frac{\partial^2g^3}{\partial Z^2}\right].$$

Then

$$K=K(0,0)=-\frac{1}{8d}x_0caqC+\frac{1}{4d}x_0a^2qC-\frac{1}{4d}x_0cq^2C+\frac{1}{d}a\sqrt{c-2a-b}Cq+$$
$$+\frac{1}{d}\frac{x_0}{c}aCq^2b-\frac{1}{8d}cAqx_0-\frac{1}{4d\omega}a^2Bqx_0+\frac{1}{8d\omega}caBq^2x_0+\frac{1}{4d\omega}a^3Bqx_0-$$
$$-\frac{1}{2d\omega c}q^2a^2Bbx_0-\frac{1}{2d\omega c}q^2a^3Bx_0-\frac{1}{8d\omega}ca^2Bx_0-\frac{1}{8d\omega}ca^2Bqx_0-\frac{3}{8}\frac{C}{d}cqx_0+$$
$$+\frac{3}{4}\frac{C}{d}aqx_0-\frac{1}{8d^2\omega^4}c^2q^4x_0^4a^3-\frac{1}{2d^2\omega^4}q^4a^5x_0^4-\frac{1}{2d^2\omega^4}q^4\sqrt{c-2a-b}a^5x_0^3+$$
$$+\frac{1}{4d^2\omega^4}cq^4\sqrt{c-2a-b}a^4x_0^3+\frac{1}{4d^2\omega^4}cq^3\sqrt{c-2a-b}a^4x_0^3+$$
$$+\frac{1}{2d^2\omega^2}cq^4a^3x_0^2-\frac{1}{d^2\omega^2}q^4a^4x_0^2-\frac{1}{2d^2\omega^2}q^4a^3x_0^2b+\frac{1}{2d^2}x_0^2caq^4-\frac{1}{d^2}x_0^2a^2q^4-$$
$$-\frac{1}{2d^2}x_0^2aq^4b-\frac{1}{8d^2}x_0^2c^2a^2\frac{q^3}{\omega^2}+\frac{1}{4d^2}x_0^2ca^3\frac{q^3}{\omega^2}-\frac{1}{8d^2}c^2aq^2-\frac{1}{8d^2}x_0^4c^2a\frac{q^4}{\omega^2}+$$
$$+\frac{1}{2d^2}x_0^4ca^2\frac{q^4}{\omega^2}-\frac{1}{4d^2}x_0^3cq^4\sqrt{c-2a-b}-\frac{1}{8d^2}x_0^2c^2q^3-\frac{1}{2d^2}x_0^4a^3\frac{q^4}{\omega^2}+$$

$$+\frac{1}{2d^2}x_0^3aq^4\sqrt{c-2a-b}+\frac{1}{4d^2}x_0^2caq^3+\frac{1}{2d^2\omega^4}cq^4x_0^4a^4+$$
$$+\frac{1}{4d^2\omega^2}cq^3x_0^3a^2\sqrt{c-2a-b}+\frac{1}{8d^2\omega^4}c^2q^2x_0^4a^3+\frac{A}{d}\sqrt{c-2a-b}aq+\frac{1}{8dx_0}c\omega B+$$
$$+\frac{1}{d}\frac{x_0}{c}a^2q^2C-\frac{1}{8d}x_0\frac{c}{a}q^2\omega B-\frac{A}{d}\frac{x_0}{c}q^2a^2-\frac{A}{d}\frac{x_0}{c}q^2ab-\frac{3}{8}\frac{A}{d}x_0cqa+\frac{3}{4}\frac{A}{d}x_0qa^2+$$
$$+\frac{1}{4}\frac{A}{d}x_0cq^2+\frac{1}{2d}\frac{x_0}{c}aq^2\omega B+\frac{1}{2d}\frac{x_0}{c}\omega Bq^2b+\frac{1}{4d}Aaqx_0+\frac{1}{8d\omega}caBqx_0.$$

Concluding we have the theorem:

Theorem 3. *If* $a=a_s:=\dfrac{c\pm\sqrt{c^2-8bc}}{4}$, $c\neq 8b$, *the equation (2) has a negative solution* $\lambda_1=-a-b<0$ *together with two purely imaginary roots* $\lambda_{2,3}=\pm i\omega, \omega=\sqrt{bc}$ *such that* $R:=Re\left(\dfrac{d\lambda(a)}{da}\Big|_{a=a_s,\lambda=i\omega}\right)\neq 0$. *Consequently, if* $K\neq 0$, *the equilibrium* E_1 *of the system* T *undergoes a Hopf bifurcation. In addition, the periodic orbits that bifurcate from the equilibrium* E_1 *for* a *in the neighborhood of* a_s, *are stable if* $K<0$, *and unstable if* $K>0$. *The direction of bifurcation are above (bellow)* a_s *if* $RK<0$ *(*$RK>0$*).*

Remark 1. In the particular case $a=2.1$, $b=1.806$, $c=30$ we have $K=-2.815\times 10^{-3}$, $R=0.28$. So the periodic orbits that bifurcate from the equilibria $E_{1,2}$ are stable and the bifurcation is above a_s.

In the following we show that the system T belongs to a class of cvasi-metriplectic systems.

Definition 1. Consider X a vector field on $\mathbb{R}^3$. The vector field X is called cvasi-metriplectic, if there exists a Poisson tensor field P, a cvasi-metric tensor field g, a Hamilton function H and a Casimir function S associated to P, such that:

$$X=P\nabla H+g\nabla S. \tag{12}$$

Consider the vector field $X=(x_1,x_2,x_3)$ given by:

$$x_1=-a_1x+a_1y,\ x_2=a_2x+a_3y-a_4xz+a_6,\ x_3=-a_5z+xy, \tag{13}$$

with $a_i, i=\overline{1,6}$, real numbers.

Remark 2. 1. If $a_1=a, a_2=c, a_3=-1, a_4=1, a_5=b, a_6=0$, the dynamical system associated to X is the Lorenz system.

2. If $a_1=a, a_2=c-a, a_3=c, a_4=1, a_5=b, a_6=0$, the dynamical system associated to X is the Chen (or Chen 1) system.

3. If $a_1=a, a_2=0, a_3=c, a_4=1, a_5=b, a_6=0$, the dynamical system associated to X is the Lü system.

4. If $a_1 = a, a_2 = c - a, a_3 = 0, a_4 = a, a_5 = b, a_6 = 0$, the dynamical system associated to X is the T system.

5. If $a_1 = a, a_2 = c - a, a_3 = c, a_4 = 1, a_5 = b, a_6 = m$, the dynamical system associated to X is the Chen 2 system.

Proposition 4. *The vector field X (13) has the cvasi-metriplectic representation given by:*

$$X = P\nabla H + g\nabla S \tag{14}$$

where

$$P = \begin{pmatrix} 0 & a_1 & 0 \\ -a_1 & 0 & -x \\ 0 & x & 0 \end{pmatrix}, g = \begin{pmatrix} -a_1 & 0 & 0 \\ 0 & \varepsilon & -\dfrac{a_3 y + a_6}{a_1} \\ 0 & -\dfrac{a_3 y + a_6}{a_1} & \dfrac{a_5 z}{a_1} \end{pmatrix} \tag{15}$$

$$H(x, y, z) = \frac{1}{2}(y^2 + a_4 z^2) - a_2 z, \quad S(x, y, z) = \frac{1}{2}x^2 - a_1 z \quad \textit{with} \quad \varepsilon \in \mathbb{R}.$$

The proof is immediately.

Observe that the Poisson tensor field P and the Casimir S are the same for the all above five systems, consequently, the systems belong to the same class of cvasi-metriplectic systems. In addition, for the system T, the tensor g is in diagonal form.

4 Conclusions

In this paper we further investigated a nonlinear differential system with three equilibrium points, origin and another two. In the origin, the system displays a pitchfork bifurcation and in the other two equilibrium points a Hopf bifurcation. The system belongs to a class of cvasi-metriplectic systems.

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A numerical approximation of the free-surface heavy inviscid flow past a body

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Abstract. The object of this paper is to apply the Complex Variable Boundary Element Method (CVBEM) for solving the problem of the bidimensional heavy fluid flow over an immersed obstacle, of smooth boundary, situated near the free surface in order to obtain the perturbation produced by its presence and the fluid action on it. Using the complex variable, complex perturbation potential, complex perturbation velocity and the Cauchy's formula the problem is reduced to an integro-differential equation with boundary conditions. For solving the integro-differential equation a complex variable boundary elements method with linear elements is developed. We use linear boundary elements for discretize smooth curve, and free surface, in fact we approximate them with polygonal lines formed by segments, and we choose for approximating the unknown on each element a linear model that uses the nodal values of the unknown. Finite difference schemes are used for eliminating the derivatives that appear. The problem is finally reduced to a system of linear equations in terms of nodal values of the components of the velocity field. All coefficients in the mentioned system are analytically calculated. Those arising from singular integrals are evaluated using generalized Cauchy integrals. After solving the system we obtain the velocity and further the local pressure coefficient and the fluid action over the obstacle can be deduced. For evaluating the coefficients and for solving the system to which the problem is reduced, we can use a computer code.

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1 Introduction

Let us consider a uniform steady potential plane free surface flow of a heavy inviscid fluid past an arbitrary wing (obstacle) immersed in the immediate proximity of the free surface. Assuming that the boundary $\Gamma$of the wing is smooth enough to avoid the existence of some angular points (and implicitly of a Kutta type condition), we intend to set up a numerical procedure-backed by a CVBEM, for determining the perturbation induced by the presence of the obstacle (wing) and the action exerted by the fluid on this obstacle. The objective is to find the fluid velocity field and the local pressure coefficient. Using a CVBEM with linear boundary elements the problem is finally reduced to a system of linear equations. This problem is solved in [2] using Schwarz principle, without a free-surface discretization, and in paper [1] by means of linear boundary elements, but for obtaining the system's coefficients, a theorem which makes connection between the analytic function $\omega(z)$, defined by the contour integral $\omega(z) = \int\limits_{\Gamma} \frac{h(\zeta)}{\zeta - z} d\zeta$, and $\omega\prime(z)$ is used. In the herein paper other techniques are used for evaluating system's coefficients. For a better understanding, a short presentation of the problem is considered necessary, and it is made according to [2].

By splitting the velocity potential Φ into the unperturbed (uniform) stream potential and the perturbation (due to the obstacle) potential, and using dimensionless variables we have $\Phi(x,y) = x + \varphi(x,y)$, where $\varphi(x,y)$ is the perturbation potential which satisfies the Laplace equation $\Delta\varphi(x,y) = 0,\ x \in (-\infty, +\infty),\ y \in (-\infty, 0)$.

Assuming, at the beginning that the free surface can be approximated by the real axis Ox, by linearizing the Bernoulli's integral, the following boundary condition on free surface holds:

$$\frac{\partial^2\varphi}{\partial x^2} + k_0 \frac{\partial\varphi}{\partial y} = 0,\ x \in (-\infty, +\infty),\ y = 0,\ (x,y) \notin \Gamma, \tag{1}$$

where $k_0 = \frac{1}{Fr^2}$, $Fr = \frac{U}{\sqrt{gL}}$, L and U being the characteristic length and velocity.

On the surface of the immersed wing the slip condition becomes

$$\frac{\partial\varphi}{\partial n} |_{\Gamma} = -n_x \tag{2}$$

where $\overline{n}(n_x, n_y)$ is the outward unit normal drawn on Γ while, on far field, $\lim\limits_{x\to\infty} \varphi(x,y) = 0$.

By introducing the stream (perturbation) function $\psi(x,y)$, and by using the complex variable $z = x + iy$ and the complex (perturbation) velocity, $w = u - iv$, ($u = \frac{\partial\varphi}{\partial x}$, $v = \frac{\partial\varphi}{\partial y}$), the complex (perturbation) potential $f(z) = \varphi(x,y) + i\psi(x,y)$ satisfies relations:

$$\frac{df}{dz} = \frac{\partial\varphi}{\partial x} + i\frac{\partial\psi}{\partial x} = w, \quad \operatorname{Re}\frac{d^2 f}{dz^2} = \frac{\partial^2\varphi}{\partial x^2}, \quad \operatorname{Im}\frac{df}{dz} = -\frac{\partial\varphi}{\partial y}.$$

Hence the previous conditions (1) and (2) become

$$\operatorname{Im}\left(i\frac{d^2 f}{dz^2} - k_0\frac{df}{dz}\right) = 0, \quad \text{for} \quad z = x \in R\ (y = 0);$$

$$\operatorname{Re}\left(\frac{df}{dz}(n_x + in_y)\right) = -n_x, \quad \text{on} \quad \Gamma \tag{3}$$

By introducing the holomorphic (in the flow domain) function F, defined by:

$$F(z) = i\frac{d^2 f}{dz^2} - k_0\frac{df}{dz} = i\frac{dw}{dz} - k_0 w \tag{4}$$

we get:

$$\operatorname{Im} F(z) = 0,\ \text{for}\ z = x \in R. \tag{5}$$

2 The Boundary Integro-Differential Equation

As $\lim\limits_{|z|\to\infty} F(z) = 0$, the use of the Cauchy's formula for the whole domain (the lower half plane without the obstacle domain) allows us to write

$$\frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty}\frac{F(\varsigma)}{\varsigma - z}d\varsigma = F(z) - \frac{1}{2\pi i}\int\limits_{\Gamma}\frac{F(\varsigma)}{\varsigma - z}d\varsigma \tag{6}$$

Replacing the expression of F from (4) in the above relation and using (5) we can write:

$$\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\frac{\mathrm{Re}(F(\varsigma))}{\varsigma - z}d\varsigma = -\frac{dw(z)}{dz} - ik_0 w(z) + \frac{1}{2\pi i}\int\limits_{\Gamma}\left(k_0 i\frac{w(\varsigma)}{\varsigma - z} + \frac{w(\varsigma)}{(\varsigma - z)^2}\right)d\varsigma \tag{7}$$

3 The Discrete Equation

For solving the integro-differential equation (7) a complex variable boundary elements method with linear elements will be developed. As regards the term $\frac{dw(z)}{dz}$ an appropriate finite difference scheme will be used. Following the same steps as in [2], the border Γ is discretized by choosing a set of control points of affixes z_i, $i = \overline{1, N}$. Consequently the smooth curve Γ is approximated by a polygonal line made by segments L_j, $j = \overline{1, N}$, whose edges have the affixes z_j, z_{j+1}, $j = \overline{1, N}$, $z_{N+1} = z_1$. Using linear boundary elements (L_j) and a linear approximation for $w(z)$ of the type (see [3])

$$\widetilde{w}(\varsigma) = w(z_j)\frac{\varsigma - z_{j+1}}{z_j - z_{j+1}} + w(z_{j+1})\frac{z_j - \varsigma}{z_j - z_{j+1}}, \quad j = \overline{1, N} \tag{8}$$

(precisely all the elements with index $N+1$ are seen as having the index 1), by denoting $w(z_i) = w_i$ and by introducing the additional denotations

$$a_j(z) = \int\limits_{L_j}\frac{\varsigma - z_{j+1}}{(z_j - z_{j+1})(\varsigma - z)^2}d\varsigma; \qquad b_{j+1}(z) = \int\limits_{L_j}\frac{z_j - \varsigma}{(z_j - z_{j+1})(\varsigma - z)^2}d\varsigma \tag{9}$$

$$c_j(z) = \int\limits_{L_j}\frac{k_0 i(\varsigma - z_{j+1})}{(z_j - z_{j+1})(\varsigma - z)}d\varsigma; \qquad d_{j+1}(z) = \int\limits_{L_j}\frac{k_0 i(z_j - \varsigma)}{(z_j - z_{j+1})(\varsigma - z)}d\varsigma,$$

$$m_j = a_j + c_j, \quad n_{j+1} = b_{j+1} + d_{j+1}, \quad A_j = m_j + n_j, \ j = \overline{1, N} \tag{10}$$

equation (7) gets the form:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{Re}\,(F\,(\varsigma))}{\varsigma-z}d\varsigma+\frac{dw\,(z)}{dz}+ik_0w\,(z)=\frac{1}{2\pi i}\sum_{j=1}^{N}w_jA_j \tag{11}$$

The involved integrals may be analytically evaluated, and so the above unknowns coefficients. Making the effective calculations and considering that $z_0 = z_N$, we get for $j = \overline{1,N}$ the following expression for them:

$$A_j=\left[\frac{1+ik_0\,(z-z_{j+1})}{z_j-z_{j+1}}\right]\ln\left(\frac{z_{j+1}-z}{z_j-z}\right)+\left[\frac{-1+ik_0\,(z_{j-1}-z)}{z_{j-1}-z_j}\right]\ln\left(\frac{z_j-z}{z_{j-1}-z}\right) \tag{12}$$

Now if we let $z \to z_i \in \Gamma$, $i = \overline{1,N}$, backed on the results of [4], we obtain:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{Re}\,(F\,(\varsigma))}{\varsigma-z_i}d\varsigma+\frac{dw\,(z_i)}{dz}+ik_0w\,(z_i)=\frac{1}{2\pi i}\sum_{j=1}^{N}w_jA_{ij} \tag{13}$$

The two indexes point out that the limits of coefficients (11), when $z \to z_i \in \Gamma$, are considered.

Concerning the coefficients from (11), their calculation is performed by imposing effectively $z \to z_i \in \Gamma$ in the previous expressions of A_j,so in (12). Except the elements originated from the integrals calculated on segments $\Gamma_{i-1}$and Γ_i, which become singular, this implies a simple replacement of z with z_i. With regard to the coefficients coming from the singular integral, we shall use some results obtained in [5] for the evaluation of a principal value (in the Cauchy sense) of a singular integral of the type $\int_\Gamma \frac{f\,(\xi)}{(\xi-z)^2}d\xi$ (Γ being a closed segmentary smooth curve) and the equality $\lim_{z\to z_i}(z-z_i)\log(z-z_i)=0$ (see [6]). We get the following expressions:

$$A_{ij}=\left[\frac{1+ik_0\,(z_i-z_{j+1})}{z_j-z_{j+1}}\right]\ln\left(\frac{z_{j+1}-z_i}{z_j-z_i}\right)+\left[\frac{-1+ik_0\,(z_{j-1}-z_i)}{z_{j-1}-z_j}\right]\ln\left(\frac{z_j-z_i}{z_{j-1}-z_i}\right),$$
$$j\neq i-1,i,i+1,$$
$$A_{ii}=ik_0\ln\left(\frac{z_{i+1}-z_i}{z_{i-1}-z_i}\right)+\frac{1+\ln|z_{i-1}-z_i|}{z_{i-1}-z_i}+\frac{-1+\ln|z_{i+1}-z_i|}{z_i-z_{i+1}},$$

$$A_{ii-1}=\left[\frac{-1+ik_0\,(z_{i-2}-z_i)}{z_{i-2}-z_{i-1}}\right]\ln\left(\frac{z_{i-1}-z_i}{z_{i-2}-z_i}\right)+\frac{1+\ln|z_{i-1}-z_i|}{z_i-z_{i-1}}$$

$$A_{ii+1}=\left[\frac{1+ik_0\,(z_i-z_{i+2})}{z_{i+1}-z_{i+2}}\right]\ln\left(\frac{z_{i+2}-z_i}{z_{i+1}-z_i}\right)+\frac{1+\ln|z_{i+1}-z_i|}{z_{i+1}-z_i} \tag{14}$$

where $i,j = \overline{1,N}$, while by index $N+1$ we should understand 1, by $N+2$ we understand 2, by 0 we understand N, by -1 we understand $N-1$.

The complex velocity derivative at node i is approximated by a backward finite difference scheme, namely $\dfrac{dw(z_i)}{dz} = \dfrac{w(z_i) - w(z_{i-1})}{z_i - z_{i-1}}$, and is replaced in (13). We obtain the following system of equations:

$$\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\frac{\mathrm{Re}\,(F(\varsigma))}{\varsigma - z_i}d\varsigma + \frac{w_i - w_{i-1}}{z_i - z_{i-1}} + ik_0 w_i = \frac{1}{2\pi i}\sum_{j=1}^{N} w_j A_{ij},\, i = \overline{1,N} \tag{15}$$

or the equivalent form:

$$\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\frac{\mathrm{Re}\,(F(\varsigma))}{\varsigma - z_i}d\varsigma = \sum_{j=1}^{N} w_j \widetilde{A}_{ij},\ \ i = \overline{1,N} \tag{16}$$

where

$$\widetilde{A}_{ij} = -\frac{1}{2\pi k_0}A_{ij},\ \ j \neq i,\ \ j \neq i+1;\ \ \widetilde{A}_{ii} = -\frac{1}{2\pi k_0}\left(A_{ii} - \frac{2\pi i}{z_i - z_{i-1}}\right) - 1;$$
$$\widetilde{A}_{ii-1} = -\frac{1}{2\pi k_0}\left(A_{ii-1} + \frac{2\pi i}{z_i - z_{i-1}}\right) \tag{17}$$

By denoting with v_n, v_s the normal and the tangential, respectively, components of the perturbation velocity we can write that, on the border, $w = (v_n - iv_s)(n_x + in_y)$ while on Γ, $v_n = -n_x$, so that $w = (-n_x - iv_s)(n_x + in_y)$. Equation (16) becomes:

$$\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\frac{\mathrm{Re}\,(F(\varsigma))}{\varsigma - z_i}d\varsigma. = \sum_{j=1}^{N}\left(-n_x^j - iv_s^j\right)\left(n_x^j + in_y^j\right)\widetilde{A}_{ij} \tag{18}$$

As the perturbation vanishes at far field we can accept that

$$\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\frac{\mathrm{Re}\,(F(\varsigma))}{\varsigma - z_i}d\varsigma = \frac{1}{2\pi}\int\limits_{a}^{b}\frac{\mathrm{Re}\,(F(\varsigma))}{\varsigma - z_i}d\varsigma \tag{19}$$

Taking into account that on the free surface the following condition holds: $\mathrm{Re}\,(F(z)) = -\dfrac{1}{k_0}\dfrac{\partial^2 u}{\partial x^2} - k_0 u,\ z = x,$ and choosing $M+1$ equidistant nodes on it, $x_0 = a,\, x_k = a + k\dfrac{b-a}{M},\, k = \overline{1,M}$, in order to obtain a discretization of the free surface into M isoparametric linear boundary elements we get:

$$\frac{1}{2\pi}\int\limits_{a}^{b}\frac{\mathrm{Re}\,(F(\varsigma))}{\varsigma - z_i}d\varsigma = \frac{1}{2\pi}\sum_{l=0}^{M-1}\int\limits_{x_l}^{x_{l+1}}\frac{\mathrm{Re}\,(F(\varsigma))}{\varsigma - z_i}d\varsigma = \frac{1}{2\pi}\sum_{l=0}^{M-1}\int\limits_{x_l}^{x_{l+1}}\frac{-\frac{1}{k_0}\frac{\partial^2 u}{\partial x^2} - k_0 u}{x - z_i}dx \tag{20}$$

For the isoparametric linear boundary element $[x_l, x_{l+1}]$ we have: $x = x_l + t\left(x_{l+1} - x_l\right), \quad u = u_l + t\left(u_{l+1} - u_l\right) \quad t \in [0,1]$.

Following the calculations we have

$$\int_{x_l}^{x_{l+1}} \frac{-\frac{1}{k_0}\frac{\partial^2 u}{\partial x^2} - k_0 u}{x - z_i} dx = B_{li} u_l + C_{li}\left(u_{l+1} - u_l\right) = \left(B_{li} - C_{li}\right) u_l + C_{li} u_{l+1} \qquad (21)$$

where

$$B_{li} = -k_0\left(x_{l+1} - x_l\right) \int_0^1 \frac{dt}{x_l + t\left(x_{l+1} - x_l\right) - z_i}$$

and

$$C_{li} = -k_0\left(x_{l+1} - x_l\right) \int_0^1 \frac{tdt}{x_l + t\left(x_{l+1} - x_l\right) - z_i}.$$

Concerning the integrals

$$I_0 = \int_0^1 \frac{dt}{x_l + t\left(x_{l+1} - x_l\right) - z_i}$$

and

$$I_1 = \int_0^1 \frac{tdt}{x_l + t\left(x_{l+1} - x_l\right) - z_i},$$

they could be expressed analytically, precisely we have

$$I_0 = \frac{1}{x_{l+1} - x_l} \ln\left(\frac{x_{l+1} - z_i}{x_l - z_i}\right), \quad I_1 = \frac{1}{x_{l+1} - x_l} - \frac{x_l - z_i}{x_{l+1} - x_l} I_0,$$

where for the complex logarithm the main branch is considered. So, coefficients that arise in (21) have expressions:

$$B_{li} = -k_0 \ln\left(\frac{x_{l+1} - z_i}{x_l - z_i}\right), \quad C_{li} = -k_0\left[1 - I_0\left(x_l - z_i\right)\right] \qquad (22)$$

Finally, denoting by $B'_{li} = \frac{1}{2\pi}\left(B_{li} - C_{li}\right) = \frac{k_0}{2\pi}\left[1 - I_0\left(x_{l+1} - z_i\right)\right]$, $C'_{li} = \frac{1}{2\pi} C_{li} = \frac{-k_0}{2\pi}\left[1 - I_0\left(x_l - z_i\right)\right]$, (20) becomes:

$$\frac{1}{2\pi}\int_a^b \frac{\operatorname{Re}\left(F\left(\varsigma\right)\right)}{\varsigma - z_i} d\varsigma = \sum_{l=0}^{M-1}\left[B'_{li} u_l + C'_{li} u_{l+1}\right] \qquad (23)$$

For sake of simplicity we consider $v_s^i = v_i,\ i = \overline{1, N}$, and using the above relation, and (19) we obtain the equivalent form for system (18):

$$\sum_{l=0}^{M-1} \left[B'_{li} u_l + C'_{li} u_{l+1}\right] = \sum_{j=1}^{N} \left(-n_x^j - iv_j\right)\left(n_x^j + in_y^j\right) \widetilde{A}_{ij} \tag{24}$$

As the number of unknowns $N + M + 1$ is greater than the number of equations for "closing" the system we should now perform $z \to x_k,\ k = \overline{0, M}$ in relations (11) and (12). So we get

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\mathrm{Re}\left(F\left(\varsigma\right)\right)}{\varsigma - z_i} d\varsigma + \frac{dw\left(x_k\right)}{dx} + ik_0 w\left(x_k\right) = \frac{1}{2\pi i} \sum_{j=1}^{N} w_j \widehat{A}_{kj} \tag{25}$$

where $\widehat{A}_{kj}$ are the nonsingular integrals whose exact expressions are:

$$\widehat{A}_{kj} = \left[\frac{1 + ik_0\left(x_k - z_{j+1}\right)}{z_j - z_{j+1}}\right] \ln\left(\frac{z_{j+1} - x_k}{z_j - x_k}\right) +$$

$$+ \left[\frac{-1 + ik_0\left(z_{j-1} - x_k\right)}{z_{j-1} - z_j}\right] \ln\left(\frac{z_j - x_k}{z_{j-1} - x_k}\right). \tag{26}$$

Then through (24) and a forward finite difference scheme for the complex velocity derivative of first M control points on the free surface, we get:

$$\sum_{l=0}^{M-1} \left[B'_{lk} u_l + C'_{lk} u_{l+1}\right] + \frac{w\left(x_k\right) - w\left(x_{k+1}\right)}{x_k - x_{k+1}} + ik_0 w\left(x_k\right) =$$

$$= \frac{1}{2\pi i} \sum_{j=1}^{N} w_j \widehat{A}_{kj}, \quad k = \overline{0, M-1} \tag{27}$$

For $x_k = x_M$ a backward finite difference $\dfrac{dw\left(x_M\right)}{dx} = \dfrac{w\left(x_M\right) - w\left(x_{M-1}\right)}{x_M - x_{M-1}}$ is to be envisaged. Hence

$$\sum_{l=0}^{M-1} \left[B'_{lM} u_l + C'_{lM} u_{l+1}\right] + \frac{w\left(x_M\right) - w\left(x_{M-1}\right)}{x_M - x_{M-1}} + ik_0 w\left(x_M\right) = \frac{1}{2\pi i} \sum_{j=1}^{N} w_j \widehat{A}_{Mj} \tag{28}$$

where the coefficients B'_{lk} and C'_{lk} have analogous expressions with those arising in (24), the only one difference being that now, for all nonsingular integrals (i.e., when x_k is not a node of the element on which the integral is calculated), a natural logarithm of a real number is implied.

Thus,

$$B'_{li} = \frac{k_0}{2\pi}\left[1 - I_0\left(x_{l+1} - z_i\right)\right], C'_{li} = \frac{-k_0}{2\pi}\left[1 - I_0\left(x_l - z_i\right)\right], \tag{29}$$

with $I_0 = \dfrac{1}{x_{l+1}-x_l}\ln\left|\dfrac{x_{l+1}-x_k}{x_l-x_k}\right|$, for $l \neq k-1$, $l \neq k$ when $k=\overline{1,M-1}$; $l \neq 0$ when $k=0$; $l \neq M-1$ when $k=M$. For the singular integrals, by using their finite parts, we finally get:

$$B'_{k-1k} = B'_{kk} = B'_{00} = B'_{M-1M} = \frac{k_0}{2\pi}; \quad C'_{k-1k} = C'_{kk} = C'_{00} = C'_{M-1M} = \frac{-k_0}{2\pi} \tag{30}$$

By replacing in (27) and (28) the expression of the complex velocity on the boundary, as function of the perturbation velocity components, and using the denotation s for v evaluated on the free surface (for avoiding any confusion), we can write for $k=\overline{0,M-1}$

$$\sum_{l=0}^{M-1}\left[B_{lk}u_l + C'_{lk}u_{l+1}\right] + \frac{u_k - is_k - u_{k+1} + is_{k+1}}{x_k - x_{k+1}} + ik_0\left(u_k - is_k\right) =$$

$$= \frac{1}{2\pi i}\sum_{j=1}^{N}\left(-n_x^j - iv_j\right)\left(n_x^j + in_y^j\right)\widehat{A}_{kj}, \tag{31}$$

respectively, for $k=M$,

$$\sum_{l=0}^{M-1}\left[B_{lM}u_l + C'_{lM}u_{l+1}\right] + \frac{u_M - is_M - u_{M-1} + is_{M-1}}{x_M - x_{M-1}} + ik_0\left(u_M - is_M\right) =$$

$$= \frac{1}{2\pi i}\sum_{j=1}^{N}\left(-n_x^j - iv_j\right)\left(n_x^j + in_y^j\right)\widehat{A}_{Mj}. \tag{32}$$

In this way we have obtained the rest of the $M+1$ equations that ensures the mathematical coherence of our mathematical problem, i.e., the solving of the system for the components of the perturbation velocity on the free surface and on the border (boundary) of the obstacle. The final system which should be solved is made by equations (24), (31) and (32).

For the outward normal components at the control points on the boundary, we also have expressions depending on points coordinates: $n_x^j = \dfrac{\operatorname{Im} ag\left(z_j - z_{j+1}\right)}{\left|z_j - z_{j+1}\right|}$; $n_y^j = \dfrac{-\operatorname{Re} al\left(z_j - z_{j+1}\right)}{\left|z_j - z_{j+1}\right|}$, and consequently all the coefficients which are present in the final system can be expressed as functions of the discretization nodes coordinates. Their calculation, system's solution and evaluation of fluid action over the body, expressed by the local pressure coefficient, can be performed by a computer, irrespective of the obstacle shape and the discretization mesh used for the boundaries.

After solving the system the problem is reduced at, it is also possible to find the shape of the unknown free surface using the velocity field and the Bernoulli relation (1). But this will be the objective of a further work.

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The sets of the classes $\widetilde{\mathrm{M}}_{p,k}$ and their subsets

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Abstract. In this paper the sets of the classes $\widetilde{M}_{p,k}$ having the Darboux property in the generalized metric spaces (E, l) are considered. Certain properties for these sets and their subsets in the generalized metric spaces (E, l) and in the Cartesian space have been given here.

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1 Introduction

Let (E, l) be a generalized metric space. E denotes here an arbitrary non-empty set, and l is a non-negative real function defined on the Cartesian product $E_0 \times E_0$ of the family E_0 of all non-empty subsets of the set E.

Let k be any, but fixed positive real number, and let a, b be arbitrary non-negative real functions defined in a certain right-hand side neighbourhood of 0 such that

$$a(r) \xrightarrow[r\to 0^+]{} 0 \quad \text{and} \quad b(r) \xrightarrow[r\to 0^+]{} 0. \tag{1}$$

We say that a pair (A, B) of sets of the family E_0 is (a, b)-clustered at the point p of the space (E, l) if 0 is the cluster point of the set of all numbers $r > 0$ such that the sets $A \cap S_l(p, r)_{a(r)}$ and $B \cap S_l(p, r)_{b(r)}$ are non-empty.

The sets $S_l(p, r)_{a(r)}$, $S_l(p, r)_{b(r)}$ (see [12]) denote here, respectively, so-called $a(r)$-, $b(r)$-neighbourhoods of the sphere $S_l(p, r)$ with the centre at the point $p \in E$ and the radius $r > 0$ in the space (E, l).

The tangency relation $T_l(a, b, k, p)$ of sets of the family E_0 in the generalized metric space (E, l) is defined as follows (see [12]):

$$\begin{aligned} T_l(a, b, k, p) = \Big\{ (A, B) :\ & A, B \in E_0, \text{ the pair } (A, B) \text{ is } (a, b)\text{-clustered} \\ & \text{at the point } p \text{ of the space } (E, l) \text{ and} \\ & \frac{1}{r^k} l(A \cap S_l(p, r)_{a(r)}, B \cap S_l(p, r)_{b(r)}) \xrightarrow[r\to 0^+]{} 0 \Big\}. \end{aligned} \tag{2}$$

If $(A, B) \in T_l(a, b, k, p)$, then we say that the set $A \in E_0$ is (a, b)-tangent (or shortly: is tangent) of order k to the set $B \in E_0$ at the point p of the space (E, l).

Let ρ be an arbitrary metric of the set E. By $d_\rho A$ we shall denote the diameter of the set $A \in E_0$, and by $\rho(A,B)$ the distance of sets $A, B \in E_0$ in the metric space (E,ρ), i.e.

$$d_\rho A = \sup\{\rho(x,y) : x,\, y \in A\} \quad \text{and} \quad \rho(A,B) = \inf\{\rho(x,y) : x \in A,\ y \in B\} \tag{3}$$

for $A, B \in E_0$.

Let f be any subadditive increasing real function defined in a certain right-hand side neighbourhood of 0 such that $f(0) = 0$.

By $\mathfrak{F}_{f,\rho}$ we will denote the class of all functions l fulfilling the conditions:

1^0 $\quad l : E_0 \times E_0 \longrightarrow \langle 0, \infty)$,

2^0 $\quad f(\rho(A,B)) \le l(A,B) \le f(d_\rho(A \cup B)) \quad \text{for} \quad A, B \in E_0$.

It is easy to show that every function $l \in \mathfrak{F}_{f,\rho}$ generates in the set E the metric l_0 defined by the formula:

$$l_0(x,y) = l(\{x\},\{y\}) = f(\rho(x,y)) \quad \text{for} \quad x, y \in E. \tag{4}$$

We say (see [5]) that the set $A \in E_0$ has the Darboux property at the point p of the generalized metric space (E,l), and we shall write this as: $A \in D_p(E,l)$ if there exists a number $\tau > 0$ such that $A \cap S_l(p,r) \neq \emptyset$ for $r \in (0,\tau)$.

It is easy to notice that, if the set $A \in E_0$ has the Darboux property at the point p of the generalized metric space (E,l), then every set $E_0 \ni B \supset A$ has also this property at the point p of this space, i.e.

$$(A \in D_p(E,l) \ \wedge \ A \subset B \in E_0) \ \Rightarrow \ B \in D_p(E,l). \tag{5}$$

In this paper we shall consider some problems concerning the sets of the classes $\widetilde{M}_{p,k}$ on the Darboux property at the point p of the generalized metric spaces (E,l) for the function $l \in \mathfrak{F}_{f,\rho}$. Some theorems for the sets of the classes $\widetilde{M}_{p,k}$ will be given.

2 On the sets of the classes $\widetilde{M}_{p,k}$

Let ρ be a metric of the set E, and let A be any set of the family E_0 of all non-empty subsets of the set E. By A' we shall denote the set of all cluster points of the set A of the family E_0.

Let us put

$$\rho(x,A) = \inf\{\rho(x,y) : \ y \in A\} \quad \text{for} \quad x \in E. \tag{6}$$

The classes of sets $\widetilde{M}_{p,k}$, mentioned in the Introduction of this paper, are defined as follows (see [5]):

$$\begin{aligned} \widetilde{M}_{p,k} = \Big\{ & A \in E_0 : \ p \in A' \text{ and there exists } \mu > 0 \text{ such that} \\ & \text{for an arbitrary } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that} \\ & \text{for every pair of points } (x,y) \in [A,p;\mu,k] \\ & \text{if } \rho(p,x) < \delta \text{ and } \frac{\rho(x,A)}{\rho^k(p,x)} < \delta, \text{ then } \frac{\rho(x,y)}{\rho^k(p,x)} < \varepsilon \Big\}, \end{aligned} \tag{7}$$

where

$$[A, p; \mu, k] = \{(x, y) : \ x \in E, \ y \in A \ \text{and} \ \mu\rho(x, A) < \rho^k(p, x) = \rho^k(p, y)\}. \tag{8}$$

Let A, B be arbitrary sets of the family E_0.

Lemma 1. *If* $A \subset B$, *then* $[A, p; \mu, k] \subset [B, p; \mu, k]$.

Proof. Let us assume that $(x, y) \in [A, p; \mu, k]$ for $x \in E$ and $y \in A$. Hence and from the definition of the set $[A, p; \mu, k]$ it results

$$\mu\rho(x, A) < \rho^k(p, x) = \rho^k(p, y). \tag{9}$$

Because $A \subset B$, then $\rho(x, B) \leq \rho(x, A)$ for $x \in E$. From here and from (9) it follows that

$$\mu\rho(x, B) < \rho^k(p, x) = \rho^k(p, y) \quad \text{for} \quad x \in E \ \text{and} \ y \in B,$$

which yields $(x, y) \in [B, p; \mu, k]$. Therefore the inclusion $[A, p; \mu, k] \subset [B, p; \mu, k]$ is satisfied. □

Using this lemma we prove the following theorem:

Theorem 1. *If* $A \in E_0$ *is an arbitrary subset of the set* $B \in \widetilde{M}_{p,k}$ *and* $p \in A'$, *then* $A \in \widetilde{M}_{p,k}$.

Proof. Let us assume that $B \in \widetilde{M}_{p,k}$. From here and from the definition of the classes of sets $\widetilde{M}_{p,k}$ it follows that for an arbitrary $\varepsilon > 0$ there exists a number $\delta_1 > 0$ such that for every pair of points $(x, y) \in [B, p; \mu, k]$

$$\frac{\rho(x, y)}{\rho^k(p, x)} < \varepsilon \tag{10}$$

if only

$$\rho(p, x) < \delta_1 \quad \text{and} \quad \frac{\rho(x, B)}{\rho^k(p, x)} < \delta_1. \tag{11}$$

We shall prove that for an arbitrary $\varepsilon > 0$ there exists $\delta_2 > 0$ such that for every pair of points $(x, y) \in [A, p; \mu, k]$ the inequality (10) is fulfilled if

$$\rho(p, x) < \delta_2 \quad \text{and} \quad \frac{\rho(x, A)}{\rho^k(p, x)} < \delta_2. \tag{12}$$

If the inequalities (11) are fulfilled, then from here and from Lemma 1 of this paper it follows that the inequality (10) is satisfied for every pair of points $(x, y) \in [A, p; \mu, k]$.

Let us put $\delta_2 = \min(\frac{1}{\mu}, \delta_1)$. Hence, from the assumption that $p \in A'$ and from the condition (12) we obtain the inequality (10), what means that $A \in \widetilde{M}_{p,k}$. This ends the proof. □

In the paper [10] was proved the following (see Theorem 4.3):

Theorem 2. *If $l \in \mathfrak{F}_{f,\rho}$, the sets $A, B \in E_0$ on the Darboux property at the point p of the space (E, l) are subsets of a certain set $C \in \widetilde{M}_{p,k}$ and the functions a, b fulfil the condition*

$$\frac{a(r)}{r^k} \xrightarrow[r\to 0^+]{} 0 \quad \text{and} \quad \frac{b(r)}{r^k} \xrightarrow[r\to 0^+]{} 0, \tag{13}$$

then $(A, B) \in T_l(a, b, k, p)$.

From above theorem, Theorem 1 of this paper, and from symmetry and transitivity of the tangency relation $T_l(a, b, k, p)$ result the following corollaries:

Corollary 1. *If $l \in \mathfrak{F}_{f,\rho}$, the functions a, b fulfil the condition* (13), *then*

$$(A, B) \in T_l(a, b, k, p) \quad \wedge \quad (B, A) \in T_l(a, b, k, p) \tag{14}$$

for arbitrary sets $A, B \in E_0$ such that $A \subset B$, $A \in D_p(E, l)$ and $B \in \widetilde{M}_{p,k}$.

Corollary 2. *If $l \in \mathfrak{F}_{f,\rho}$, the functions a, b fulfil the condition* (13), *then for arbitrary sets $A, B, C \in E_0$ such that $A \subset B$, $A \in D_p(E, l)$, $B \in \widetilde{M}_{p,k}$ and $C \in \widetilde{M}_{p,k} \cap D_p(E, l)$*

$$(A, C) \in T_l(a, b, k, p) \quad \Leftrightarrow \quad (B, C) \in T_l(a, b, k, p). \tag{15}$$

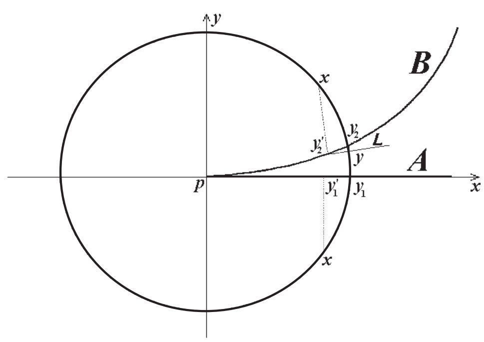

Figure 1

Below we shall give the examples of sets of the class $\widetilde{M}_{p,k}$ which are tangent in the two-dimensional Cartesian space $E = \mathbf{R}^2$.

Example 1. Let $E = \mathbf{R}^2$ be the two-dimensional Cartesian space. Let φ be an increasing differentiable function defined in a certain right-hand side neighbourhood of 0 such that $\varphi(0) = 0$.

Let $A \subset E$, $B \subset E$ be the sets of the form (see Figure 1)

$$A = \{(t, 0) : \ t \geq 0\}, \quad B = \{(t, \varphi^{k+1}(t)) : \ t \geq 0, \ k \in \mathbf{N}\}. \tag{16}$$

The sets A, B defined by the formula (16) are the sets of the class $\widetilde{M}_{p,k}$, where $p = (0, 0)$.

In the paper [11] I proved (see Example 2.1) that B is the set of the class $\widetilde{M}_{p,k}$.

Now we shall prove that the set A defined by (16) also belongs to the class $\widetilde{M}_{p,k}$. In the first place we shall prove that for an arbitrary $\varepsilon > 0$ there exists $\delta_1 > 0$ such that for every pair of points $(x, y_1) \in [A, p; \mu, k]$

$$\frac{\rho(x, y_1)}{\rho^k(p, x)} < \varepsilon, \tag{17}$$

when

$$r = \rho(p, x) < \delta_1 \quad \text{and} \quad \frac{\rho(x, A)}{\rho^k(p, x)} < \delta_1. \tag{18}$$

Let y_1' be the projection of the point $x \in E$ the set A, i.e. such point of the set A that $\rho(x, y_1') = \rho(x, A)$. Because $x = (t, \pm\sqrt{r^2 - t^2})$ for $0 \le t < r$, then

$$\rho(y_1', y_1) = r - t = \sqrt{(r-t)^2} \le \sqrt{(r+t)(r-t)} = \sqrt{r^2 - t^2} = \rho(x, y_1'),$$

that is to say,

$$\rho(y_1', y_1) \le \rho(x, A). \tag{19}$$

Let $\mu = 2$, $\delta_1 = \min\left(\frac{1}{2}, \frac{\varepsilon}{2}\right)$. Hence, from (18), (19) and from the triangle inequality we have

$$\frac{\rho(x, y_1)}{\rho^k(p, x)} \le \frac{\rho(x, y_1') + \rho(y_1', y_1)}{\rho^k(p, x)} \le \frac{2\rho(x, A)}{\rho^k(p, x)} < \varepsilon,$$

which yields the inequality (17). From here it follows that the set $A \in \widetilde{M}_{p,k}$.

Let now φ be a increasing function of the class C_1 (homogenous function together with 1^{st} derivative) defined in a certain right-hand side neighbourhood of 0 such that $\varphi(0) = 0$. Using the de L′Hospital′s rule and mathematical induction we can easily prove that

$$\frac{\varphi^{k+1}(t)}{t^k} \xrightarrow[t \to 0^+]{} 0 \quad \text{for} \quad k \in \mathbf{N}. \tag{20}$$

From this it follows immediately

$$\frac{\varphi^{2k+2}(t)}{t^{2k}} \xrightarrow[t \to 0^+]{} 0 \quad \text{for} \quad k \in \mathbf{N}. \tag{21}$$

Example 2. Similarly as in Example 1, let $E = \mathbf{R}^2$ be the two-dimensional Cartesian space, and let A, B be sets defined by (16). Let $f = \mathrm{id}$, where id is the identity function defined in a right-hand side neighbourhood of 0. Let moreover a, b be functions defined in a right-hand side neighbourhood of 0 and filgilling the condition (13).

We shall prove that $(A, B) \in T_l(a, b, k, p)$ for $k \in \mathbf{N}$ and $p = (0, 0)$. Let y_2 be an arbitrary point of the set B. Then $y_2 = (t, \varphi^{k+1}(t))$ and

$$r = \rho(p, y_2) = \sqrt{t^2 + \varphi^{2k+2}(t)}. \tag{22}$$

Hence it follows that $y_1 = (\sqrt{t^2+\varphi^{2k+2}(t)}, 0) \in A \cap S_\rho(p,r)$, where $S_\rho(p,r)$ denotes the sphere with the centre at the point p and the radius r in the metric space (E,ρ). From the assumptions on the funcion φ and from (22) it follows that $r \to 0^+$ if and only if $t \to 0^+$. Hence and from (20), (21), (22) for $r > 0$ we have

$$\frac{1}{r^{2k}}\rho^2(y_1,y_2) = \frac{(\sqrt{t^2+\varphi^{2k+2}(t)}-t)^2+\varphi^{2k+2}(t)}{(t^2+\varphi^{2k+2}(t))^k}$$

$$= 2\frac{t^2+\varphi^{2k+2}(t)-t\sqrt{t^2+\varphi^{2k+2}(t)}}{(t^2+\varphi^{2k+2}(t))^k}$$

$$= 2\frac{\varphi^{2k+2}(t)+t^2-t\sqrt{t^2+\varphi^{2k+2}(t)}}{t^{2k}}\ \frac{1}{(1+\varphi^{2k+2}(t)/t^2)^k} \xrightarrow[t\to 0^+]{}$$

$$2\left(\frac{\varphi^{2k+2}(t)}{t^{2k}}+\frac{t-\sqrt{t^2+\varphi^{2k+2}(t)}}{t^{2k-1}}\right) = 2\left(\frac{\varphi^{2k+2}(t)}{t^{2k}}-\frac{\varphi^{2k+2}(t)}{t^{2k-1}(\sqrt{t^2+\varphi^{2k+2}(t)}+t)}\right)$$

$$= 2\left(\frac{\varphi^{2k+2}(t)}{t^{2k}}-\frac{\varphi^{2k+2}(t)}{t^{2k}(\sqrt{1+\varphi^{2k+2}(t)/t^2}+1)}\right)$$

$$= 2\frac{\varphi^{2k+2}(t)}{t^{2k}}\left(1-\frac{1}{1+\sqrt{1+\varphi^{2k+2}(t)/t^2}}\right) \xrightarrow[t\to 0^+]{} \left(\frac{\varphi^{k+1}(t)}{t^k}\right)^2 \xrightarrow[t\to 0^+]{} 0,$$

that is to say,

$$\frac{1}{r^k}\rho(y_1,y_2) \xrightarrow[r\to 0^+]{} 0. \tag{23}$$

From here, from the inequality

$$d_\rho(A\cup B) \le d_\rho A + d_\rho B + \rho(A,B) \quad \text{for} \quad A,B \in E_0, \tag{24}$$

from the fact that $f = \mathrm{id}$ and from Theorem 2.1 of the paper [8] we obtain

$$\begin{aligned}\frac{1}{r^k}l(A\cap S_l(p,r)_{a(r)}, B\cap S_l(p,r)_{b(r)}) &\le \frac{1}{r^k}d_\rho((A\cap S_l(p,r)_{a(r)})\cup(B\cap S_l(p,r)_{b(r)}))\\ &\le \frac{1}{r^k}d_\rho(A\cap S_l(p,r)_{a(r)}) + \frac{1}{r^k}d_\rho(B\cap S_l(p,r)_{b(r)})\\ &\quad + \frac{1}{r^k}\rho(A\cap S_l(p,r)_{a(r)}, B\cap S_l(p,r)_{b(r)})\\ &\le \frac{1}{r^k}d_\rho(A\cap S_l(p,r)_{a(r)}) + \frac{1}{r^k}d_\rho(B\cap S_l(p,r)_{b(r)}) + \frac{1}{r^k}\rho(y_1,y_2) \xrightarrow[r\to 0^+]{} 0,\end{aligned}$$

i.e.,

$$\frac{1}{r^k} l(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0. \tag{25}$$

Because the pair of sets (A, B) is (a, b)-clustered at the point p of the metric space (E, ρ), then from here and from (25) it follows that $(A, B) \in T_l(a, b, k, p)$ for $k \in \mathbf{N}$.

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Queuing system evolution in phase merging scheme*

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Abstract. We study asymptotic average scheme for semi-Markov queuing systems using compensating operator of the corresponding extended Markov process. The peculiarity of our queuing system is that the series scheme is considered with phase merging procedure.

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Keywords and phrases: Queuing systems, semi-Markov process, phase merging, average scheme, weak convergence, compensating operator.

1 Introduction

The queuing system (QS) of $[SM|M|1|\infty]^N$ type means that the input flow is described by a semi-Markov process, the service time is exponentially distributed, there are N servers connected by a route probability matrix. So the queuing networks is considered with a semi-Markov flow. The peculiarity of our queuing system is that the series scheme is considered with phase merging procedure [1]. The average algorithm is established for the queuing process (QP) described the number of claims at every node. Analogously problem was investigated in work [1].

2 Preliminaries

The regular semi-Markov process $k^\varepsilon(t)$, $t \geq 0$ on the standard phase space (E, E) in the series scheme, with the small series parameter $\varepsilon \to 0$ ($\varepsilon > 0$), given by the semi-Markov kernel [1, 3, 4].

$$Q^\varepsilon(\kappa, B, t) = P^\varepsilon(\kappa, B)G_\kappa(t), \kappa \in E, B \in e, t \geq 0. \tag{1}$$

The stochastic kernel

$$P^\varepsilon(\kappa, B) = P(\kappa, B) + \varepsilon P_1(\kappa, B). \tag{2}$$

The stochastic kernel $P(\kappa, B)$ is coordinated with the split phase space

$$E = \bigcup_{k=1}^{N} E_k, E_k \bigcap E_k = \oslash, k \neq k', \tag{3}$$

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as follows

$$P(\kappa, E_k) = \delta_k(\kappa) := \begin{cases} 1, \kappa \in E_k \\ 0, \kappa \notin E_k \end{cases} \tag{4}$$

The perturbing kernel $P_1(\kappa, B)$ provides the transition probabilities of the embedded Markov chain k_n^ε, $n \geq 0$, between classes of states E_k, $1 \leq k \leq N$, which tend to zero as $\varepsilon \to 0$.

The renewal moments τ_n, $n \geq 0$, are defined by the distribution functions

$$G_\kappa(t) = P(\theta_{n+1} \leq t | k_n^\varepsilon = \kappa) =: P(\theta_\kappa \leq t), \tag{5}$$

here $\theta_{n+1} = \tau_{n+1} - \tau_n$, $n \geq 0$, are the sojourn times. For more details of semi-Markov process see monograph [1, Ch 1].

Introduce the mean values of sojourn time

$$g(\kappa) := E\theta_\kappa = \int_0^\infty \bar{G}_\kappa(t)dt, \bar{G}_\kappa(t) := 1 - G_\kappa(t), \tag{6}$$

and the average intensities

$$q(\kappa) = 1/g(\kappa), \kappa \in E. \tag{7}$$

In what follows the associated Markov process $k^0(t)$, $t \geq 0$, given by the generator

$$Q\varphi(\kappa) = q(\kappa) \int_E P(\kappa, dy)[\varphi(y) - \varphi(\kappa)], \tag{8}$$

is uniformly ergodic in every class E_k, $k \in \widehat{E}$, $\hat{E} = \{1, 2, \ldots, N\}$ with the stationary distributions $\pi_k(d\kappa)$, $k \in \widehat{E}$. The corresponding embedded Markov chain $k_n^0 = k^0(\tau_n)$, $n \geq 0$, is uniformly ergodic also with the stationary distributions $\rho_k(d\kappa)$, $k \in \widehat{E}$. Note that the following relations are valid:

$$\pi_k(d\kappa)q(\kappa) = q_k\rho_k(d\kappa), q_k = \int_{E_1} \pi_k(d\kappa)q(\kappa). \tag{9}$$

According to Theorem 4.1 [1, § 4.2.1, p.108] the merged process $\nu(k^\varepsilon(t/\varepsilon))$ converges weakly as $\varepsilon \to 0$, to the Markov process $\hat{k}(t)$, $t \geq 0$, on the merged phase space $\hat{E} = \{1, 2, \ldots, N\}$, given by the generative matrix $\hat{Q} = [\hat{q}_{kr}; k, r \in \hat{E}]$.

We assume that the merged Markov process $\hat{k}(t)$, $t \geq 0$, is ergodic with the stationary distribution $\hat{\pi} = (\hat{\pi}_k, k \in \hat{E})$.

3 Queuing process in the networks

The evolution of claims in the networks on $\hat{E} = \{1, 2, \ldots, N\}$ is defined by the route matrix P_0 and the intensity vector of exponential service time $\mu = (\mu_k, k \in \hat{E})$.

The queuing process in average scheme is considered in the following normalizing form:

$$U^\varepsilon(t) = \varepsilon^2 \rho^\varepsilon(t/\varepsilon^2), t \geq 0, \varepsilon > 0, \tag{10}$$

where $\rho^\varepsilon(t) = (\rho_k^\varepsilon(t),\ k \in \hat{E})$ is the vector with the components $\rho_k^\varepsilon(t)$ – number of claims at node $k \in \hat{E}$ at time t.

The queuing process $U^\varepsilon(t)$ in average scheme is considered under the following assumptions.

A1: The queuing networks is open, that means the route matrix satisfies the condition:

$$p_{k0}^0 := 1 - \sum_{r=1}^{N} p_{kr}^0, \max_{k \in \hat{E}} p_{k0}^0 > 0. \tag{11}$$

A2: There exists nonnegative solution of the evolutionary equation

$$dU^0(t)/dt = C(U^0(t)), U^0(0) = u_0, \tag{12}$$

where the velocity vector

$$C(u) = (C_k(u), k \in \hat{E}), \tag{13}$$

is defined by its components

$$C_k(u) = \gamma_k(u) + \lambda_k, \quad \gamma_k(u) = \sum_{r=1}^{N} \mu_r u_r [p_{rk} - \delta_{rk}], \lambda_k = \hat{\pi}_k q_k.$$

Theorem 1. *Under the assumptions A1-A2 the weak convergence* $U^\varepsilon(t) \Rightarrow U^0(t), \varepsilon \to 0,$ *takes place.*

Corollary 1. *Let exist an equilibrium point* $u^0 \geq 0$ *satisfying*

$$C(u^0) = 0. \tag{14}$$

Then under initial condition $U^\varepsilon(0) \Rightarrow u_0, \varepsilon \to 0,$ *the weak convergence* $U^\varepsilon(t) \Rightarrow u_0, \varepsilon \to 0,$ *takes place.*

Remark 1. The vector $\tilde{\pi} = (\tilde{\pi}_k := q\hat{\pi}_k q_k,\ k \in \hat{E}),\ q^{-1} = \sum_{k \in \hat{E}} \hat{\pi}_k q_k$ describes the stationary distribution of the Markov process $\tilde{k}(t)$, $t \geq 0$, defined by the generating matrix (see [1, Theorem 4.1])

$$\tilde{Q} = [p_{kr}, k, r \in \hat{E}], p_{kr} = \int_{E_1} \rho_k(d\kappa) P_1(\kappa, E_r). \tag{15}$$

Indeed (see [1,(4.17) and (4.19)],

$$\sum_k \hat{\pi}_k q_k p_{kr} = \sum_k \hat{\pi}_k q_k \hat{p}_k \hat{p}_{kr} = \sum_k \hat{\pi}_k \hat{q}_k \hat{p}_{kr} = \sum_k \hat{\pi}_k \hat{q}_{kr} = 0. \tag{16}$$

4 Proof of Theorem. Compensating operator

The extended Markov renewal process

$$u_n^\varepsilon = u^\varepsilon(\tau_n^\varepsilon), k_n^\varepsilon = k^\varepsilon(\tau_n^\varepsilon), \tau_n^\varepsilon = \varepsilon^2\tau_n, n \geq 0, \tag{17}$$

is characterized by the compensating operator (CO) (see [1,Ch 1, 2])

$$L^\varepsilon\varphi(u,\kappa) = \varepsilon^{-2}q(\kappa)E[\varphi(u_{n+1}^\varepsilon, k_{n+1}^\varepsilon) - \varphi(u,\kappa)]u_n^\varepsilon = u, k_n^\varepsilon = \kappa. \tag{18}$$

The key step in asymptotic analysis of the QS is to construct an asymptotic expansion of the CO (18).

Lemma 1. *The CO (18) can be represented in the following form*

$$L^\varepsilon\varphi(u,\kappa) = \varepsilon^{-2}q(\kappa)[G^\varepsilon(\kappa)P^\varepsilon D^\varepsilon(k) - I, \tag{19}$$

where

$$G^\varepsilon(\kappa) = \int_0^\infty G_\kappa(dt)\Gamma_t^\varepsilon. \tag{20}$$

The semigroup Γ_t^ε is defined by the generator

$$\Gamma^\varepsilon\varphi(u) = \sum_{k,r=1}^{N} \gamma_{kr}(u)[\varphi(u+\varepsilon^2 e_{rk}) - \varphi(u)], \tag{21}$$

$$e_{kr} := e_r - e_k, e_k := (\delta_{kl}, l \in \hat{E}).$$

The operators $D^\varepsilon(k)$, $k \in \hat{E}$, are defined by

$$D^\varepsilon(k)\varphi(u) = \phi(u+\varepsilon^2 e_{rk}), k \in \hat{E}. \tag{22}$$

The operator

$$P^\varepsilon = P + \varepsilon P_1, \tag{23}$$

where

$$P\varphi(\kappa) = \int_E P(\kappa,dy)\varphi(y), P_1\varphi(\kappa) = \int_E P_1(\kappa,dy)\varphi(y). \tag{24}$$

Proof of Lemma 1. The representation (19) is direct conclusion of the equality

$$u_{n+1}^\varepsilon - u_n^\varepsilon = \beta^\varepsilon(\theta_{n+1}) + \varepsilon^2 e_{n+1},$$

where $\beta^\varepsilon(t)$, $t \geq 0$, is the Markov process given by the generator (21).

Lemma 2. *The CO (19) admits the following asymptotic expansion on the test-function $\phi(u,\kappa) \in C^3(R^d)$ uniformly in $\kappa \in E$:*

$$L^\varepsilon(k)\varphi(u,\kappa) = [\varepsilon^{-2}Q + \varepsilon^{-1}Q_1 + Q_2(\kappa) + \theta_L^\varepsilon(\kappa)]\varphi(u,\kappa), \tag{25}$$

where

$$Q\varphi(\kappa) = q(\kappa)\int_E P(\kappa,dy)[\varphi(y) - \varphi(\kappa)], \tag{26}$$

$$Q_1\varphi(\kappa) = q(\kappa)\int_E P_1(\kappa,dy)\varphi(y), \tag{27}$$

$$\lambda(\kappa) = (\lambda_k(\kappa), k \in \hat{E}), \lambda_k(\kappa) = q(\kappa)\delta_k(\kappa), Q_2(\kappa)\varphi(u) = [\gamma(u) + \lambda(\kappa)]\varphi'(u) \tag{28}$$

and the negligible term $\theta_L^\varepsilon(\kappa)\varphi(u) \to 0$ as $\varepsilon \to 0$, $\varphi(u) \in C^3(R^N)$.

Proof of Lemma 2. The following identity is used below:

$$GD - I = G - I + D - I + (G - I)(D - I),$$

and asymptotic expansion on the test- function $\varphi(u) \in C^3(R^N)$

$$\varepsilon^{-2}q(\kappa)[G^\varepsilon(\kappa) - I]P\varphi(u) = [q(\kappa)G(\kappa)P + \theta_g^\varepsilon(\kappa)P]\varphi(u),$$

$$\varepsilon^{-2}q(\kappa)P[D^\varepsilon(k) - I]\varphi(u) = [q(\kappa)PD(k) + \theta_d^\varepsilon(\kappa)P]\varphi(u),$$

$$\varepsilon^{-2}q(\kappa)\varepsilon P_1[D^\varepsilon(k) - I]\varphi(u) = [\varepsilon q(\kappa)P_1D(k) + \varepsilon\theta_{dl}^\varepsilon(\kappa)P_1]\varphi(u),$$

$$\varepsilon^{-2}q(\kappa)[G^\varepsilon(\kappa) - I]P^\varepsilon[D^\varepsilon(k) - I]\varphi(u) = \theta_{gd}^\varepsilon(\kappa)P^\varepsilon\varphi(u)$$

is a negligible term.

The limit operator in the theorem is defined by a solution of singular perturbation problem for the truncated operator

$$L_0^\varepsilon = \varepsilon^{-2}Q + \varepsilon^{-1}Q_1 + Q_2(\kappa). \tag{29}$$

Lemma 3. *The limit operator L in the theorem is defined by formulae (see [1, Proposition 5.3., p.146]:*

$$L = \hat{\Pi}\Pi Q_2(\kappa)\Pi\hat{\Pi}, \tag{30}$$

where the projectors Π and $\hat{\Pi}$ act as follows:

$$\Pi\varphi(\kappa) = \sum_{k=1}^N \hat{\varphi}_k l_k(\kappa), \hat{\varphi}_k = \int_{E_1} \pi_k(d\kappa)\varphi(\kappa), k \in \hat{E},$$

$$\hat{\Pi}\hat{\varphi}(\kappa) = \sum_{k=1}^N \hat{\pi}_k\hat{\varphi}_k.$$

Corollary 2. *The limit operator L in Theorem is defined as follows*

$$L\varphi(u) = C(u)\varphi'(u) = \sum_{k=1}^{N} C_k(u)\varphi'_k(u), \varphi'_k(u) := \partial\varphi(u)/\partial u_k, \qquad (31)$$

where $C(u) = \gamma(u) + \lambda, C_k(u) = \gamma_k(u) + \hat{\pi}_k q_k, \lambda = (\hat{\pi}_k q_k, k \in \hat{E})$.

The last step of the proof of theorem is realized by using Theorem 6.6 from [1, Ch. 6, p.202].

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