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FOUNDER OF THE MOLDAVIAN SCIENTIFIC
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This volume is dedicated to the 80th birthday of Petru S. Soltan, a Full Member of the Academy of Sciences of Moldova and Honorary Member of the Academy of Sciences of Romania, an outstanding personality in science, culture and politics. A prominent mathematician, Petru S. Soltan has made a very distinguished scientific career. He is also an indefatigable organizer of mathematical activities. The school he built in applied mathematics became well-known and is his great contribution to the development of mathematics in Moldova.

The main scientific interests of Professor P. Soltan lie in the fields of applied mathematics, geometry and topology. More detailed information about his scientific activity can be found in "Buletinul Academiei de Ştiinţe a Republicii Moldova. Matematica", No. 3(37), 2001, 123–134.

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On some operations in the lattice of submodules determined by preradicals

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Abstract. In the lattice $\boldsymbol{L}(_RM)$ of all submodules of a module $_RM$ four operations are defined using the standard preradicals: α-product, ω-product, α-coproduct and ω-coproduct. Some properties of these operations, as well as some connections with the lattice operations of $\boldsymbol{L}(_RM)$ are indicated. For characteristic submodules these operations were studied in the work [5].

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1 Definitions and preliminary facts

Let R be an associative ring with unity and R-Mod be the category of unitary left R-modules. For an arbitrary module $_RM \in R$-Mod we denote by $\boldsymbol{L}(_RM)$ the lattice of all submodules of $_RM$. A submodule $N \in \boldsymbol{L}(_RM)$ is called *characteristic* (fully invariant) in M if $f(N) \subseteq N$ for every R-endomorphism $f:\ _RM \to {_RM}$. The lattice of all characteristic submodules of $_RM$ will be denoted by $\boldsymbol{L}^{ch}(_RM)$.

A *preradical* r of R-Mod by definition is a subfunctor of identity functor of R-Mod (i.e. $r(M) \subseteq M$ and $f\big(r(M)\big) \subseteq r(M')$ for every module $M \in R$-Mod and every R-morphism $f: M \to M'$). Obviously, $r(M)$ is a characteristic submodule of $_RM$. Moreover, the submodule $N \in \boldsymbol{L}(_RM)$ is characteristic in $_RM$ if and only if there exists a preradical r of R-Mod such that $N = r(M)$.

If $r\big(r(M)\big) = r(M)$ for every $M \in R$-Mod, then r is called *idempotent* preradical; if $r\big(M \,/\, _RM)\big) = 0$ for every $M \in R$-Mod, then r is called a *radical*.

We denote by R-pr the family of all preradicals of the category R-Mod. Two operations „$\wedge$" and „$\vee$" are defined in R-pr by the following rules:

$$\big(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha\big)(X) = \bigcap_{\alpha\in\mathfrak{A}} r_\alpha(X), \qquad \big(\bigvee_{\alpha\in\mathfrak{A}} r_\alpha\big)(X) = \sum_{\alpha\in\mathfrak{A}} r_\alpha(X)$$

for every $X \in R$-Mod and every family of preradicals $\{r_\alpha \,|\, \alpha \in \mathfrak{A}\} \subseteq R$-pr. Then R-pr $(\wedge, \vee)$ possesses all properties of a complete lattice with the exception that it is not necessarily a set (in general case R-pr is a class), so it is called the "big lattice" of preradicals of R-Mod. In this lattice a special role is played by the following two types of preradicals. For every pair $N \subseteq M$, where $N \in \boldsymbol{L}(_RM)$, we define the functions α_N^M and ω_N^M by the rules:

$$\alpha_N^M(X) = \sum_{f\,:\,M \to X} f(N), \qquad \omega_N^M(X) = \bigcap_{f\,:\,X \to M} f^{-1}(N),$$

for every $X \in R$-Mod. The following facts are well known ([1,2]).

Proposition 1.1. 1) α_N^M *and* ω_N^M *are preradicals of R-Mod;*

2) $\alpha_N^M(M)$ *is the least characteristic submodule of* ${}_RM$ *containing* N*;*

3) $\omega_N^M(M)$ *is the largest characteristic submodule of* ${}_RM$ *contained in* N. □

Proposition 1.2. *If* $N \in \boldsymbol{L}^{ch}({}_RM)$, *then* $\alpha_N^M(M) = N$ *and* $\omega_N^M(M) = N$. *Moreover, for a preradical* $r \in R$*-pr we have:*

$$r(M) = N \quad \Leftrightarrow \quad \alpha_N^M \ \leq \ r \ \leq \ \omega_N^M. \qquad \square$$

So for a submodule $N \in \boldsymbol{L}^{ch}({}_RM)$ the preradical α_N^M is the least among preradicals $r \in R$-pr with the property $r(M) = N$. Dually, ω_N^M is the largest among preradicals $r \in R$-pr with $r(M) = N$.

Now we mention two particular cases:

a) *the idempotent preradical* r^M, defined by module ${}_RM$

$$r^M(X) = \sum_{f\,:\,M \to X} Im\, f \text{ – the } \textit{trace} \text{ of } M \text{ in } X \text{ (i.e. } r^M = \alpha_M^M);$$

b) *the radical* r_M defined by ${}_RM$

$$r_M(X) = \bigcap_{f\,:\,M \to X} Ker\, f \text{ – the } \textit{reject} \text{ of } M \text{ in } X \text{ (i.e. } r_M = \omega_0^M).$$

The following two operations in R-pr are very important in the theory of preradicals:

1) *the product* of preradicals $r,\, s \in R$-pr:

$$(r \cdot s)(X) = r\big(s(X)\big);$$

2) *the coproduct* of preradicals $r,\, s \in R$-pr:

$$(r : s)(X) \,/\, r(X) = s\big(X \,/\, r(X)\big)$$

for every $X \in R$-Mod.

2 α-product of submodules

Using preradicals of the form α_N^M the following operation is introduced in the lattice $\boldsymbol{L}({}_RM)$ of all submodules of an arbitrary module $M \in R$-Mod.

Definition 2.1. Let $M \in R$-Mod and $K, N \in \boldsymbol{L}({}_RM)$. The following submodule of M:

$$K \cdot N = \alpha_K^M(N) = \sum_{f\,:\,M \to N} f(K)$$

will be called the α-*product* in M of submodules K and N.

This operation was considered in [3] for the investigation of prime modules. The continuation of these studies can be found in [4]. For characteristic submodules $K, N \in \boldsymbol{L}^{ch}({}_RM)$ this operation coincides with the α-product defined in [5] by the rule: $K \cdot N = \alpha_K^M\, \alpha_N^M(M)$.

Some simple properties of α-product are indicated in the following statement.

Proposition 2.1. 1) *$K \cdot N \subseteq N$ and $K \cdot N$ is a characteristic submodule in N;*

2) *If $N \in \boldsymbol{L}^{ch}({}_RM)$, then $K \cdot N \in \boldsymbol{L}^{ch}({}_RM)$ for every $K \in \boldsymbol{L}({}_RM)$;*

3) *If $K \in \boldsymbol{L}^{ch}({}_RM)$, then $K \cdot N \subseteq K$, therefore $K \cdot N \subseteq K \cap N$;*

4) *If $K = 0$, then $0 \cdot N = 0$ for every $N \in \boldsymbol{L}({}_RM)$; if $N = 0$, then $K \cdot 0 = 0$ for every $K \in \boldsymbol{L}({}_RM)$;*

5) *If $N = M$, then for every $K \in \boldsymbol{L}({}_RM)$ the submodule $K \cdot M = \sum\limits_{f \,:\, M \to M} f(K)$ is the least characteristic submodule of M containing K;*

6) *If $K = M$, then for every $N \in \boldsymbol{L}({}_RM)$ we have $M \cdot N = \sum\limits_{f \,:\, M \to N} f(M) = r^M(N)$.* □

Proposition 2.2. *The operation of α-product is monotone in both variables:*

$$K_1 \subseteq K_2 \ \Rightarrow\ K_1 \cdot N \subseteq K_2 \cdot N \qquad \forall\, N \in \boldsymbol{L}({}_RM);$$
$$N_1 \subseteq N_2 \ \Rightarrow\ K \cdot N_1 \subseteq K \cdot N_2 \qquad \forall\, K \in \boldsymbol{L}({}_RM). \qquad \square$$

The following two results explore the associativity of α-product and are indicated in [3] (Lemma 2.1). For convenience we give also the sketch of proofs.

Proposition 2.3. *The following relation is true:*

$$(K \cdot N) \cdot L \ \subseteq\ K \cdot (N \cdot L)$$

for every $K,\, N,\, L \in \boldsymbol{L}({}_RM)$.

Proof. Every pair of morphisms $f : M \to N$, $g : M \to L$ determines a morphism $h = gf : M \to L$ and since by definition $N \cdot L = \sum\limits_{g \,:\, M \to L} g(N)$ we have $g\big(f(m)\big) \in N \cdot L$. So we can consider that $h \in Hom_R(M,\, N \cdot L)$. For every $a \in K$ we have $f(a) \in K \cdot N$ and $g\big(f(a)\big) \in (K \cdot N) \cdot L$. Therefore we obtain $g\big(f(a)\big) = h(a) \in K \cdot (N \cdot L) = \sum\limits_{h \,:\, M \to N \cdot L} h(K)$, proving the statement. □

Proposition 2.4. *If M is a projective module, then the operation of α-product in $\boldsymbol{L}({}_RM)$ is associative:*

$$(K \cdot N) \cdot L = K \cdot (N \cdot L)$$

for every $K,\, N,\, L \in \boldsymbol{L}({}_RM)$.

Proof. We consider the module $U = \dot{\sum\limits_{g \,:\, M \to L}} N_g$, $N_g = N$, with canonical projections $p_g : U \to U_g$. We can define the mapping:

$$h : U \to N \cdot L, \quad h(x) = \sum_{g \,:\, M \to L} g\big(p_g(x)\big) \in N \cdot L = \sum_{g \,:\, M \to L} g(N),\ x \in U.$$

Then h is an epimorphism, since every element of $N \cdot L$ by definition has the form $\sum\limits_{i=1}^{t} g_i(n_{g_i})$. By projectivity of ${}_RM$ for every $f : M \to N \cdot L$ there exists a morphism

$\overline{f} : M \to U$ such that $f = h\overline{f}$. For every $a \in K$ we have $f(a) \in K \cdot (N \cdot L)$ and $f(a) = h\overline{f}(a) = \sum_{g : M \to L} g\big(p_g \overline{f}(a)\big)$. Since $p_g \overline{f} \in Hom_R(M, N)$, we obtain $p_g \overline{f}(a) \in K \cdot N$ and then by definition $\sum_{g : M \to L} g\big(p_g \overline{f}(a)\big) \in (K \cdot N) \cdot L$, therefore $f(a) \in (K \cdot N) \cdot L$. This proves that $K \cdot (N \cdot L) \subseteq (K \cdot N) \cdot L$, and the inverse inclusion follows from Proposition 2.3. □

In continuation we will study some relations between the operation of α-product and the lattice operations of $\boldsymbol{L}({}_R M)$. For that we need the following fact on the operation ,,$\vee$" (join) in the lattice R-pr.

Lemma 2.5. *For every submodules $N, K \in \boldsymbol{L}({}_R M)$ the following relation is true:*

$$\alpha^M_{N+K} = \alpha^M_N \ \vee \ \alpha^M_K .$$

Proof. For every $X \in R$-Mod by definitions it follows:

$$(\alpha^M_N \ \vee \ \alpha^M_K)\,(X) = \alpha^M_N(X) + \alpha^M_K(X) =$$

$$= \Big(\sum_{f : M \to X} f(N) \Big) + \Big(\sum_{f : M \to X} f(K) \Big) = \sum_{f : M \to X} f(N + K) = \alpha^M_{N+K}(X).$$ □

Proposition 2.6. *For every module $M \in R$-Mod the operation of α-product is left distributive with respect to the sum of submodules:*

$$(K_1 + K_2) \cdot N = (K_1 \cdot N) + (K_2 \cdot N)$$

for every $K_1, K_2, N \in \boldsymbol{L}({}_R M)$.

Proof. Using Lemma 2.5 and definitions, we obtain:

$$(K_1 + K_2) \cdot N = \alpha^M_{K_1 + K_2}(N) = \big(\alpha^M_{K_1} \ \vee \ \alpha^M_{K_2}\big)(N) =$$

$$= \alpha^M_{K_1}(N) + \alpha^M_{K_2}(N) = (K_1 \cdot N) + (K_2 \cdot N).$$ □

Proposition 2.7. *For every submodules $K, N_1, N_2 \in \boldsymbol{L}({}_R M)$ the following relation is true: $K \cdot (N_1 + N_2) \supseteq (K \cdot N_1) + (K \cdot N_2)$. If $N_1 \cap N_2 = 0$, then we have: $K \cdot (N_1 \oplus N_2) = (K \cdot N_1) \oplus (K \cdot N_2)$.*

Proof. The first relation follows from the monotony of α-product (Proposition 2.2). In the second relation it is sufficient to verify the inclusion ($\subseteq$). Let $i_1, i_2\,(p_1, p_2)$ be the canonical injections (projections) of a direct sum $N_1 \oplus N_2$. Every morphism $f : M \to N_1 \oplus N_2$ can by uniquely represented as $i_1 g + i_2 h$, where $g = p_1 f : M \to N_1$, $h = p_2 f : M \to N_2$. For every $a \in K$ we have $f(a) \in K \cdot (N_1 \oplus N_2)$. But at the same time

$$f(a) = p_1 f(a) + p_2 f(a) = g(a) + h(a) \in (K \cdot N_1) \ \oplus \ (K \cdot N_2),$$

proving the needed inclusion. □

We conclude this section with the remark on the particular case when ${}_RM = {}_RR$, i.e. $\boldsymbol{L}({}_RR)$ is the lattice of left ideals of the ring R. For every $I,\, J \in \boldsymbol{L}({}_RR)$ we have:

$$I \cdot J = \alpha_I^R(J) = \sum_{f\,:\,R \to J} f(I) = \sum_{j \in J} I \cdot j = IJ,$$

so the α-product of left ideals coincides with the ordinary product of left ideals in R.

3 ω-product of submodules

In a similar mode as in the previous case we will now define another operation in the lattice $\boldsymbol{L}({}_RM)$ with the help of preradicals of the forme ω_N^M (Section 1).

Definition 3.1. Let $M \in R$-Mod and $K,\, N \in \boldsymbol{L}({}_RM)$. The following submodule of M:

$$K \odot N = \omega_K^M(N) = \bigcap_{f: N \to M} f^{-1}(K)$$

will be called the *ω-product* in M of submodules K and N, i.e. $K \odot N = \{n \in N \,|\, f(n) \in K \text{ for every } f : N \to M\}$.

In the case when $K,\, N \in \boldsymbol{L}^{ch}({}_RM)$ this operation coincides with the ω-product of characteristic submodules, defined in [5] by the rule: $K \odot N = \omega_K^M\, \omega_N^M(M)$.

Now we formulate some elementary properties of ω-product in $\boldsymbol{L}({}_RM)$.

Proposition 3.1. 1) *$K \odot N \subseteq N$ and $K \odot N$ is a characteristic submodule in N;*

2) *If $N \in \boldsymbol{L}^{ch}({}_RM)$, then $K \odot N \in \boldsymbol{L}^{ch}({}_RM)$ for every $K \in \boldsymbol{L}({}_RM)$;*

3) *$K \odot N \subseteq K$, therefore $K \odot N \subseteq K \cap N$;*

4) *$0 \odot N = 0$, $K \odot 0 = 0$;*

5) *$K \odot M = \omega_K^M(M) = \bigcap_{f: M \to M} f^{-1}(K)$ is the largest characteristic submodule of M contained in K; therefore if $K \in \boldsymbol{L}^{ch}({}_RM)$, then $K \odot M = K$;*

6) *$M \odot N = \omega_M^M(N) = \bigcap_{f: N \to M} f^{-1}(M) = N$.* □

Proposition 3.2. *The operation of ω-product is monotone in both variables:*

$$K_1 \subseteq K_2 \;\Rightarrow\; K_1 \odot N \subseteq K_2 \odot N \qquad \forall\, N \in \boldsymbol{L}({}_RM);$$

$$N_1 \subseteq N_2 \;\Rightarrow\; K \odot N_1 \subseteq K \odot N_2 \qquad \forall\, K \in \boldsymbol{L}({}_RM).$$ □

We remark that if $K \in \boldsymbol{L}^{ch}({}_RM)$, then $K \cdot N \subseteq K \odot N$ for every $N \in \boldsymbol{L}({}_RM)$, since $\alpha_K^M \leq \omega_K^M$ and $\alpha_K^M(N) \subseteq \omega_K^M(N)$, so we have:

$$K \cdot N \subseteq K \odot N \subseteq K \cap N.$$

Proposition 3.3. *For every submodules $K,\, N,\, L \in \boldsymbol{L}({}_RM)$ the following relation is true:*

$$(K \odot N) \odot L \supseteq K \odot (N \odot L).$$

Proof. Let $l \in K \odot (N \odot L)$. By definition this means that:

1) $l \in (N \odot L)$, i.e. $g(l) \in N$ for every $g : L \to M$;

2) $h(l) \in K$ for every $h : N \odot L \to M$.

We must verify that

$$l \in (K \odot N) \odot L \;=\; \{x \in L \,|\, f\big(g(x)\big) \in K \quad \forall f : N \to M, \;\; \forall g : L \to M\}.$$

For every pair of morphisms $g : L \to M$ and $f : N \to M$ we can define the morphism $h : N \odot L \to M$ by the rule:

$$h(m) = f\big(g(m)\big) \quad \forall\, m \in N \odot L,$$

using the fact that $g(m) \in N$ by the definition of $N \odot L$.

From $l \in (K \odot N) \odot L$ it follows $h(l) \in K$, therefore $h(l) = f\big(g(l)\big) \in K$ for every $f : N \to M$ and $g : L \to M$, but this means that $l \in (K \odot N) \odot L$. □

For the study of relation between ω-product and intersection in $\boldsymbol{L}({}_RM)$ the following remark is useful.

Lemma 3.4. *For every submodules* $N, K \in \boldsymbol{L}({}_RM)$ *the following relation is true:*

$$\omega^M_{N \cap K} \;=\; \omega^M_N \;\wedge\; \omega^M_K.$$

Proof. By definitions, for every module $X \in R$-Mod we have:

$$(\omega^M_N \;\wedge\; \omega^M_K)\,(X) \;=\; \omega^M_N(X) \;\bigcap\; \omega^M_K(X) =$$

$$= \big\{x \in X \mid f(x) \in N \;\; \forall f : X \to M\big\} \;\bigcap\; \big\{x \in X \mid f(x) \in K \;\; \forall f : X \to M\big\} =$$

$$= \big\{x \in X \mid f(x) \in N \;\bigcap\; K \;\; \forall f : X \to M\big\} = \omega^M_{N \cap K}(X). \qquad \square$$

Proposition 3.5. *For every module* $M \in R$*-Mod the operation of* ω*-product is left distributive with respect to the intersection of submodules:*

$$(K_1 \;\bigcap\; K_2) \odot N \;=\; (K_1 \odot N) \;\bigcap\; (K_2 \odot N).$$

Proof. Applying Lemma 3.4 we obtain:

$$(K_1 \;\bigcap\; K_2) \odot N \;=\; \omega^M_{K_1 \cap K_2}(N) \;=\; (\omega^M_{K_1} \;\wedge\; \omega^M_{K_2})\,(N) \;=$$

$$= \omega^M_{K_1}(N) \;\bigcap\; \omega^M_{K_2}(N) \;=\; (K_1 \odot N) \;\bigcap\; (K_2 \odot N). \qquad \square$$

In the particular case when ${}_RM = {}_RR$ we have the specification of ω-product in the lattice $\boldsymbol{L}({}_RR)$ of left ideals of the ring R. For every left ideals $J, I \in \boldsymbol{L}({}_RR)$ we obtain:

$$J \odot I = \omega^R_J(I) = \bigcap_{f : I \to R} f^{-1}(J) = \{i \in I \mid f(i) \in J \;\; \forall f : {}_RI \to {}_RR\} \;\subseteq\; J \;\bigcap\; I.$$

4 $\boldsymbol{\alpha}$-coproduct of submodules

The following two operations which will be introduced in continuation are in some sense dual to the previous operations (α-product and ω-product) and are obtained by replacing the product of preradicals with its coproduct (Section 1).

Definition 4.1. Let $M \in R$-Mod and $N, K \in \boldsymbol{L}({}_R M)$. The following submodule of M:

$$(N:K) = \pi_N^{-1}\big(\alpha_K^M(M/N)\big) = \{m \in M \mid m + N \in \sum_{f\,:\,M \to M/N} f(K)\}$$

will be called the *α-coproduct* in M of submodules N and K, where $\pi_N : M \to M/N$ is the natural morphism. In other form:

$$(N:K)/N = \alpha_K^M(M/N).$$

Some properties of α-coproduct are collected in

Proposition 4.1. *Let $M \in R$-Mod and $N, K \in \boldsymbol{L}({}_R M)$. Then:*

1) $(N:K) \supseteq N + K$;
2) *$(N:K)/N$ is a characteristic submodule in M/N; if $N \in \boldsymbol{L}^{ch}({}_R M)$, then $(N:K) \in \boldsymbol{L}^{ch}({}_R M)$;*
3) *If $N+K = M$ (in particular, if $N = M$ or $K = M$), then $(N:K) = M$;*
4) *If $N = 0$, then for every $K \in \boldsymbol{L}({}_R M)$ the submodule $(0:K)$ is the least characteristic submodule of M containing K; therefore if $K \in \boldsymbol{L}^{ch}({}_R M)$, then $(0:K) = K$;*
5) *If $K = 0$, then $(N:0) = N$ for every $N \in \boldsymbol{L}({}_R M)$.* □

Proposition 4.2. *The operation of α-coproduct is monotone in both variables:*

$$N_1 \subseteq N_2 \;\Rightarrow\; (N_1:K) \subseteq (N_2:K) \qquad \forall\, K \in \boldsymbol{L}({}_R M);$$
$$K_1 \subseteq K_2 \;\Rightarrow\; (N:K_1) \subseteq (N:K_2) \qquad \forall\, N \in \boldsymbol{L}({}_R M).$$ □

Proposition 4.3. *If the module $M \in R$-Mod is projective, then for every submodules $N, K, L \in \boldsymbol{L}({}_R M)$ the following relation is true:*

$$\big((N:K):L\big) \subseteq \big(N:(K:L)\big).$$

Proof. Let $m \in \big((N:K):L\big)$. Then by definition we have $m + (N:K) \in \alpha_L^M\big(M/(N:K)\big)$, i.e. $m + (N:K) = \sum_{g_i : M \to M/(N:K)} g_i(l_i)$, where $l_i \in L$. Since ${}_R M$ is projective, for every morphism $g_i : M \to M/(N:K)$ there exists a morphism $f_i : M \to M/N$ such that $\varphi f_i = g_i$, where $\varphi : M/N \to M/(N:K)$ is the epimorphism determined by the inclusion $N \subseteq (N:K)$ $\big($i.e. $\varphi(m+N) = m + (N:K)\big)$. Therefore:

$$m + (N : K) = \sum_{g_i : M \to M/(N:K)} g_i(l_i) =$$
$$= \sum_{f_i : M \to M/N} (\varphi f_i)(l_i) = \varphi\Big(\sum_{f_i : M \to N} f_i(l_i) \Big) \in M / (N : K).$$

Considering the inverse image in M / N we have:

$$(m + N) - \sum_{f_i : M \to M/N} f_i(l_i) \in Ker\, \varphi = (N : K) / N = \alpha_K^M (M / N),$$

and so $(m+N) - \sum_{f_i : M \to M/N} f_i(l_i) = \sum_{f_j : M \to M/N} f_j(k_j)$, where $k_j \in K$. Therefore:

$$m + N = \sum_{f_i : M \to M/N} f_i(l_i) + \sum_{f_j : M \to M/N} f_j(k_j) \in \alpha^M_{(K:L)}(M / N),$$

since $l_i \in L \subseteq (K : L)$ and $k_j \in K \subseteq (K : L)$. By definition this means that $m \in \big(N : (K : L)\big)$. □

Proposition 4.4. *For every submodules N, K_1, $K_2 \in \boldsymbol{L}({}_RM)$ the following relation is true:*

$$\big(N : (K_1 + K_2)\big) = (N : K_1) + (N : K_2),$$

i.e. the α-coproduct is right distributive with respect to the sum of submodules.

Proof. By Lemma 2.5 we have $\alpha^M_{K_1+K_2} = \alpha^M_{K_1} \vee \alpha^M_{K_2}$, therefore:

$$\big(N : (K_1 + K_2)\big) / N = \alpha^M_{K_1+K_2}(M / N) = \alpha^M_{K_1}(M / N) + \alpha^M_{K_2}(M / N) =$$

$$= \big[(N : K_1) / N\big] + \big[(N : K_2) / N\big] = \big[(N : K_1) + (N : K_2)\big] / N,$$

which implies the statement. □

Now we concretize the operation of α-coproduct for the lattice of left ideals $\boldsymbol{L}({}_RR)$ of the ring R.

Proposition 4.5. *For every left ideals $N, K \in \boldsymbol{L}({}_RR)$ the following relation is true:*

$$(N : K) = KR + N.$$

Proof. By definition $(N : K) = \pi_N^{-1}\big(\alpha^R_K(R / N)\big)$, where $\pi_N : R \to R / N$ is the natural morphism. Since $Hom_R(R, R / N) \cong R / N$, we have

$$\alpha^R_K(R / N) = \sum_{f : R \to R/N} f(K) = \sum_{r \in R} K(r + N) = K\big(\sum_{r \in R} (r + N) \big) =$$

$$= K(R / N) = (KR + N) / N,$$

therefore $(N : K) = KR + N$. □

If K is an ideal, then $(N : K) = N + K$ for every $N \in \boldsymbol{L}({}_RR)$. So in the lattice $\boldsymbol{L}^{ch}({}_RR)$ of two-sided ideals of R the α-coproduct coincides with the ordinary sum of ideals.

In particular from Proposition 4.5 it follows also that

$$(N:K)L = (KR+N)L = KRL + NL = KL + NL = \\ = (K+N)L = (N+K)L = (K:N)L$$

for every $N, K, L \in \boldsymbol{L}({}_RR)$.

5 ω-coproduct of submodules

In this section we consider an operation in $\boldsymbol{L}({}_RM)$ similar to the α-coproduct replacing α_N^M by ω_N^M.

Definition 5.1. Let $M \in R$-Mod and $N, K \in \boldsymbol{L}({}_RM)$. The following submodule of M:

$$(N \odot K) = \pi_N^{-1}\big(\omega_K^M(M/N)\big) = \{m \in M \mid m + N \in \bigcap_{f : M/N \to M} f^{-1}(K)\} =$$

$$= \{m \in M \mid f(m+N) \in K \ \ \forall f : M/N \to M\}$$

will be called the *ω-coproduct* in M of submodules N and K, where $\pi_N : M \to M/N$ is the natural morphism. Therefore:

$$(N \odot K)/N = \omega_K^M(M/N) = \bigcap_{f : M/N \to M} f^{-1}(K).$$

The ω-coproduct $(N \odot K)$ can be expressed in other form ([3]), using the fact that there exists a bijection between the morphisms $g : M \to M$ with the condition $g(N) = 0$, and all morphisms $f : M/N \to M$. Taking this into account, we can present the ω-coproduct as follows:

$$(N \odot K) = \big\{m \in M \mid g(m) \in K \ \ \forall g : M \to M, \ \ g(N) = 0\big\}.$$

If $N, K \in \boldsymbol{L}^{ch}({}_RM)$ then this operation coincides with the ω-coproduct of characteristic submodules defined in [5] by the rule:

$$(N \odot K) = (\omega_N^M : \omega_K^M)(M).$$

This operation (in other notations and other order of terms) was used in [3] for the study of coprime modules. The continuation of these studies is in [6], where coprime preradicals and coprime modules are investigated. As in the previous cases we start with some elementary properties of this operation.

Proposition 5.1. 1) $(N \odot K) \supseteq N$ *and* $(N \odot K)/N$ *is a characteristic submodule of* M/N*;*

2) *If* $N = M$, *then* $(M \odot K) = M$ *for every* $K \in \boldsymbol{L}({}_RM)$*;*

3) *If* $K = M$, *then* $(N \odot M) = M$ *for every* $N \in \boldsymbol{L}({}_RM)$*;*

4) *If* $N = 0$, *then* $(0 \odot K)$ *is the largest characteristic submodule contained in* K *for every* $K \in \boldsymbol{L}({}_RM)$*; so if* $K \in \boldsymbol{L}^{ch}({}_RM)$*, then* $(0 \odot K) = K$*;*

5) *If* $K = 0$*, then* $(N \odot 0) = \pi_N^{-1}\Big(\bigcap\limits_{f\,:\,M/N \to M} Ker f \Big)$ *for every* $N \in \boldsymbol{L}({}_RM)$*, where* $\pi_N : M \to M / N$ *is the natural morphism;*

6) *If* $N \in \boldsymbol{L}^{ch}({}_RM)$*, then* $(N \odot K) \in \boldsymbol{L}^{ch}({}_RM)$ *for every* $K \in \boldsymbol{L}({}_RM)$*;*

7) *If* $N, K \in \boldsymbol{L}^{ch}({}_RM)$*, then* $(N \odot K) \supseteq K$*, therefore* $(N \odot K) \supseteq N + K$*.* □

Proposition 5.2. *The operation of* ω*-coproduct is monotone in both variables:*

$$N_1 \subseteq N_2 \quad \Rightarrow \quad (N_1 \odot K) \subseteq (N_2 \odot K) \qquad \forall\, K \in \boldsymbol{L}({}_RM);$$

$$K_1 \subseteq K_2 \quad \Rightarrow \quad (N \odot K_1) \subseteq (N \odot K_2) \qquad \forall\, N \in \boldsymbol{L}({}_RM).$$ □

Two results on associativity of this operation are mentioned in [3] (Lemma 4.1). We remind these statements with short proofs.

Proposition 5.3. *For every* $M \in R$*-Mod the relation*

$$\big((N \odot K) \odot L\big) \subseteq \big(N \odot (K \odot L)\big)$$

is true, where $N, K, L \in \boldsymbol{L}({}_RM)$*.*

Proof. By definition we have:

$m \in \big((N \odot K) \odot L\big) \Leftrightarrow g(m) \in L \quad \forall\, g : M \to M, \quad g(N \odot K) = 0;$

$m \in \big(N \odot (K \odot L)\big) \Leftrightarrow f(m) \in (K \odot L) \quad \forall\, f : M \to M, \quad f(N) = 0 \Leftrightarrow$

$\Leftrightarrow hf(m) \in L \quad \forall\, h : M \to M, \quad h(K) = 0$ and $\forall\, f : M \to M, \quad f(N) = 0.$

If $m \in \big((N \odot K) \odot L\big)$ and we have a pair of morphisms $f, h : M \to M$ such that $f(N) = 0$ and $h(K) = 0$, then by definition $f(N \odot K) \subseteq K$ and so $hf(N \odot K) = 0$. By assumption, $hf(m) \in L$ for every such pair of morphisms, and by definition this means that $m \in \big(N \odot (K \odot L)\big)$. □

Proposition 5.4. *If* ${}_RM$ *is injective and artinian, then the operation of* ω*-coproduct in* $\boldsymbol{L}({}_RM)$ *is associative:*

$$\big((N \odot K) \odot L\big) = \big(N \odot (K \odot L)\big),$$

for every $N, K, L \in \boldsymbol{L}({}_RM)$*.*

Proof. Since ${}_RM$ is artinian there exists a finite number of endomorphisms $f_1, \dots, f_n : M \to M$ with $f_j(N) = 0$ such that $(N \odot K) = \bigcap\limits_{j=1}^{n} f_j^{-1}(N)$. We define the morphism $t : M / (N \odot K) \to \prod\limits_{1}^{n} (M / K)$ by the rule: $t\big(m + (N \odot K)\big) = \big(f_1(m) + K, \dots, f_n(m) + K\big)$ and observe that t is a monomorphism.

Let $m \in \big(N \odot (K \odot L)\big)$, i.e. $hf(m) \in L$ for every $f, h : M \to M$ with $f(N) = 0$ and $h(K) = 0$. Let $g : M \to M$ be an arbitrary morphism with $g(N \odot K) = 0$. Then g can be expressed in the form $g = g' \cdot \pi_{(N \odot K)}$, where $\pi_{(N \odot K)} : M \to M/(N \odot K)$ is natural and $g' \in Hom_R\big(M/(N \odot K), M\big)$. Since M is injective and t is mono, there exists a morphism $q : \prod_1^n (M/K) \to M$ such that $g' = qt$.

Now we consider the morphisms $u_j = i_j \pi_K : M \to \prod_1^n (M/K)$ $(j = 1, \dots, n)$, where $\pi_K : M \to M/K$ is natural, and $i_j : M/K \to \prod_1^n (M/K)$ are the canonical injections. Then:

$$g(m) = q\,t\,\pi_{(N \odot K)}(m) = q\,t\big(m + (N \odot K)\big) = q(f_1(m) + K, \dots, f_n(m) + K) =$$

$$= q\big(\pi_K f_1(m), \dots, \pi_K f_n(m)\big) = q\big(i_1 \pi_K f_1(m) + \dots + i_n \pi_K f_n(m)\big) =$$

$$= q\big(u_1 f_1(m), \dots, u_n f_n(m)\big) = q\,u_1\,f_1(m) + \dots + q\,u_n\,f_n(m),$$

where the morphism $h_j = q\,u_j : M \to M$ has the property $h_j(K) = 0$, and the morphisms f_j are given with $f_j(N) = 0$. From the assumption that $m \in \big(N \odot (K \odot L)\big)$ we obtain $q\,u_j\,f_j(m) \in L$ for every $j = 1, \dots, n$, so $g(m) \in L$ for every $g : M \to M$ with $g(N \odot K) = 0$. By definition this means that $m \in \big((N \odot K) \odot L\big)$, proving the inclusion $(\supseteq)$, the inverse inclusion is true by Proposition 5.3. □

Now we will prove the right distributivity of ω-product in $\boldsymbol{L}({}_R M)$ with respect to the intersection of submodules.

Proposition 5.5. *For every submodules N, K_1, $K_2 \in \boldsymbol{L}({}_R M)$ the following relation is true:*

$$\big(N \odot (K_1 \cap K_2)\big) = (N \odot K_1) \cap (N \odot K_2).$$

Proof. By Lemma 3.4 we have $\omega^M_{K_1 \cap K_2} = \omega^M_{K_1} \wedge \omega^M_{K_2}$, therefore:

$$\big(N \odot (K_1 \cap K_2)\big)/N = \omega^M_{K_1 \cap K_2}(M/N) = \omega^M_{K_1}(M/N) \cap \omega^M_{K_2}(M/N) =$$

$$= \big[(N \odot K_1)/N\big] \cap \big[(N \odot K_2)/N\big] = \big[(N \odot K_1) \cap (N \odot K_2)\big]/N,$$

which implies the statement. □

Remark. The distributivity relations from Propositions 2.6, 3.5, 4.4 and 5.5 can be generalized to infinite distributivity, i.e. the following relations are true:

$$\Big(\sum_{\alpha \in \mathfrak{A}} K_\alpha\Big) \cdot N = \sum_{\alpha \in \mathfrak{A}} (K_\alpha \cdot N), \quad \Big(\bigcap_{\alpha \in \mathfrak{A}} K_\alpha\Big) \odot N = \bigcap_{\alpha \in \mathfrak{A}} (K_\alpha \odot N),$$

$$\Big(N : \Big(\sum_{\alpha \in \mathfrak{A}} K_\alpha\Big)\Big) = \sum_{\alpha \in \mathfrak{A}} (N : K_\alpha), \quad \Big(N \odot \Big(\bigcap_{\alpha \in \mathfrak{A}} K_\alpha\Big)\Big) = \bigcap_{\alpha \in \mathfrak{A}} (N \odot K_\alpha).$$

Finally, we will specify the form of ω-coproduct in the lattice $\boldsymbol{L}({}_RR)$ of left ideals of R. Let $N,\, K \in \boldsymbol{L}({}_RR)$. By definition we have:

$$(N \odot K) = \big\{a \in R \mid g(a) \in K \quad \forall\, g : {}_RR \to {}_RR \text{ with } g(N) = 0\big\}.$$

If for $g : {}_RR \to {}_RR$ we denote $a_g = g(1_R)$, then $g(a) = a \cdot a_g$ for every $a \in R$ and $Ker\, g = \{a \in R \mid a \cdot a_g = 0\} = (0 : a_g)_l$ (left annihilator of a_g). The condition $g(N) = 0$ means that $N \cdot a_g = 0$, i.e. $a_g \in (0 : N)_r$ (right annihilator of N).

If $a \in (N \odot K)$, then $g(a) \in K$, i.e. $a \cdot a_g \in K$ or $a \in (K : a_g)_l$ for every $g : {}_RR \to {}_RR$ with $a_g \in (0 : N)_r$. So we obtain that $a \in \big(K : (0 : N)_r\big)_l$. Therefore:

$$(N \odot K) = \big(K : (0 : N)_r\big)_l.$$

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Estimation of the number of one-point expansions of a topology* which is given on a finite set

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Abstract. Let X be a finite set and τ be a topology on X which has precisely m open sets. If $t(\tau)$ is the number of possible one-point expansions of the topology τ on $Y = X \bigcup \{y\}$, then $\frac{m\cdot(m+3)}{2} - 1 \geq t(\tau) \geq 2\cdot m + log_2 m - 1$ and $\frac{m\cdot(m+3)}{2} - 1 = t(\tau)$ if and only if τ is a chain (i.e. it is a linearly ordered set) and $t(\tau) = 2\cdot m + log_2 m - 1$ if and only if τ is an atomistic lattice.

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1 Introduction

The present article is a continuation of the article [1].

The basic result of this article is Theorem 2, in which for any topology given on a finite set, estimations of the number of one-point expansions are obtained.

To the proof of this theorem we applied the following algorithm, which is proved in the article [1] and which allows to obtain any topology τ_1 that is a one-point expansion of the topology τ_0 given on a finite set.

Let τ_0 be some topology given on a finite set X_0 and $Y = X_0 \bigcup \{y\}$.

1. We choose arbitrarily $V_0 \in \tau_0$.
2. We choose arbitrarily $U_0 \in \tau_0$ such that $U_0 \subseteq \bigcap\limits_{V\in\tau_0, V\nsubseteq V_0} V$ (consider that $\bigcap\limits_{V\in\emptyset} V = X_0$).
3. We determine the topology
$$\tilde{\tau_1}(V_0, U_0) = \{V \in \tau \mid V \subseteq V_0\} \cup \{U \cup \{y\} \mid U \in \tau, U \supseteq U_0\}.$$

2 Main results

Assume that (X, τ) is a topological space.

*If $Y = X \bigcup \{y\}$ then a topology $\widetilde{\tau}$ on the set Y is called a one-point expansion of the topology $\tau = \widetilde{\tau}|_X$.

Definition 1. A subset $\mathbf{B} \subseteq \tau$ is called *a base of the topological space* (X, τ) if any open set is a union of some sets from $\mathbf{B}$.

Definition 2. *A weight of the topological space* (X, τ) is the minimal cardinal number m for which there exists a base of the topological space (X, τ) of cardinality m.

Definition 3. *The minimal base* of the topological space (X, τ) is any base which has cardinality equal to the weight of the space (X, τ).

Theorem 1. *If X is a finite set and τ is a topology on X, then the topological space (X, τ) has the unique minimal base.*

Proof. For each element $x \in X$ we consider the set $V(x) = \bigcap\limits_{U \in \tau, x \in U} U$ and let $\mathcal{B} = \{V(x) \mid x \in X\}$.

Let's show that $\mathcal{B}$ is a base in the topological space (X, τ).

From the finiteness of the set τ it follows that $V(x) \in \tau$ for any $x \in X$, and hence, $\mathcal{B} \subseteq \tau$.

If now $U \in \tau$, from the definition of the set $V(x)$ it follows that $V(x) \subseteq U$ for any $x \in U$. Then $U = \bigcup\limits_{x \in U} \{x\} \subseteq \bigcup\limits_{x \in U} V(x) \subseteq U$, and hence, $\bigcup\limits_{x \in U} V(x) = U$.

From the randomness of U it follows that $\mathcal{B}$ is a base in the topological space (X, τ).

Let's show that $\mathcal{B}$ is a minimal base in the topological space (X, τ), i.e. that its cardinality is equal to the weight of the topological space (X, τ).

Let $\mathcal{B}'$ be some minimal base of the topological space (X, τ) and $x \in X$. As $V(x) \in \tau$ and $x \in V(x)$ then there exists $V' \in \mathcal{B}'$ such that $x \in V' \subseteq V(x)$. From the definition of the set $V(x)$ it follows that $V(x) \subseteq V'$, and hence, $V(x) = V' \in \mathcal{B}'$. From the randomness of the element $x \in X$ it follows that $\mathcal{B} = \{V(x) \mid x \in X\} \subseteq \mathcal{B}'$.

Then $\mathcal{B} \subset \mathcal{B}'$ or $\mathcal{B} = \mathcal{B}'$.

If $\mathcal{B} \subset \mathcal{B}'$, then from the finiteness of the set $\mathcal{B}'$ it follows that cardinality of the set $\mathcal{B}$ will be less than cardinality of the set $\mathcal{B}'$. We have received a contradiction with the choice of the base $\mathcal{B}'$.

Hence $\mathcal{B} = \mathcal{B}'$.

From the randomness of the base $\mathcal{B}'$ it follows that the minimal base of the topological space (X, τ) is unique, and moreover, this minimal base can be received by the method which is specified in the beginning of the proof of the theorem. □

Proposition 1. *Let X be a finite set and $\tau = \{\emptyset = W_1, \ldots, W_n = X\}$ be a topology on the set X. If the topology τ is a chain (i.e. it is a linearly ordered set), then τ has precisely $\dfrac{n \cdot (n+3)}{2} - 1$ one-point expansions.*

Proof. As τ is a chain we can consider that $W_i \subset W_{i+1}$ for all $1 \leq i < n$.

If $1 \leq i \leq n-1$ and $V_0 = W_i$ (designations for V_0 and U_0 are given in Algorithm 1), then $\bigcap\limits_{V \in \tau_0, V \not\subseteq W_i} V = \bigcap\limits_{j=i+1}^{n} W_j = W_{i+1}$. Then U_0 can take $i+1$ values, namely, it can be any W_j for $1 \leq j \leq i+1$.

If $V_0 = X$ then $\bigcap\limits_{V\in\tau_0, V\nsubseteq X} V = \bigcap\limits_{V\in\emptyset} V = X$. Then U_0 can take n values. As

$$\sum_{i=1}^{n-1}(i+1) + n = \frac{(n+2)\cdot(n-1)}{2} + n =$$

$$\frac{n^2+n-2}{2} + n = \frac{n^2+n+2n-2}{2} = \frac{n\cdot(3+n)}{2} - 1$$

then we have $\dfrac{n\cdot(n+3)}{2} - 1$ various pairs (V_0, U_0) and hence the topology τ has precisely $\dfrac{n\cdot(n+3)}{2} - 1$ various one-point expansions.

The proposition is completely proved. □

Definition 4. As it is usual (see, for example, [3]), a lattice $(L, \leq)$ is called *a distributive lattice* if $\inf\{a, \sup\{b, c\}\} = \sup\{\inf\{a, b\}, \inf\{a, c\}\}$ for any $a, b, c \in L$.

Definition 5. As it is usual, a nonzero element a of a lattice $(L, \leq)$ with zero is called *an atom* if between 0 and a there are no other elements of the lattice $(L, \leq)$.

Definition 6. As it is usual, a lattice $(L, \leq)$ with zero is called (see, for example, [2]) *an atomistic lattice* if for any nonzero element $a \in L$ there exists a finite set $S \subseteq L$ of atoms of the lattice L such that $a = sup\, S$.

Remark 1. From ([3, VIII, §2, Lemma 2] it follows that in any distributive, atomistic lattice $(L, \leq)$ for any element $a \in L$ there exists the unique set $S \subseteq L$ of atoms of the lattice L for which $a = inf S$.

Remark 2. It is known that if (X, τ) is a topological space then $(\tau, \leq)$ is a distributive lattice with zero $0 = \emptyset$, in which $\sup\{U, V\} = U \bigcup V$ and $\inf\{U, V\} = U \bigcap V$.

Proposition 2. *Let X be a finite set and τ be a topology on the set X. If τ is an atomistic lattice and $\{W_1, \ldots, W_n\}$ is the set of all atoms of the lattice τ, then the topology τ has precisely $2^{(n+1)} + n - 1$ one-point expansions.*

Proof. Let $Y = \{y_1, \ldots, y_n\}$ and $\tau' = \{M \mid M \subseteq Y\}$ be the discrete topology on the set Y.

If we map each subset $M = \{y_{i_1}, \ldots, y_{i_k}\} \in \tau'$ of the set Y on the subset $\bigcup\limits_{j=1}^{k} W_{i_j} \in \tau$ of the set X, then we define a mapping $\psi : \tau' \to \tau$.

As the lattices τ' and τ are distributive and atomistic lattices, then (see Remark 1) in each of them we shall present any element uniquely as the supremum of some set of atoms. And as the sets $\{W_1, \ldots, W_n\}$ and $\{\{y_1\}, \ldots, \{y_n\}\}$ are sets of all atoms in the lattices τ and τ', accordingly, then the mapping $\psi : \tau' \to \tau$ is a lattice isomorphism. Then (see [1], Theorem 2.6) the topologies τ and τ' have the same number of one-point expansions and hence (see [1], Theorem 2.7) the topology τ has precisely $2^{(n+1)} + n - 1$ one-point expansions.

The proposition is completely proved. □

Theorem 2. *Let X be a finite set and τ be a topology on X which has precisely m open sets. If $t(\tau)$ is the number of possible one-point expansions of the topology τ on the set $Y = X \bigcup \{y\}$, then the following statements are true:*

$\mathcal{A}$) $\dfrac{m \cdot (m+3)}{2} - 1 \geq t(\tau) \geq 2 \cdot m + log_2 m - 1$;

$\mathcal{B}$) $\dfrac{m \cdot (m+3)}{2} - 1 = t(\tau)$ *if and only if τ is a chain (i.e. it is a linearly ordered set);*

$\mathcal{C}$) $t(\tau) = 2 \cdot m + log_2 m - 1$ *if and only if τ is an atomistic lattice.*

Proof. $\mathcal{A}$) Let (see Theorem 1) $\{V_1, \ldots, V_k)\}$ be the minimal base in the topological space (X, τ).

As any $U \in \tau$ can be presented as the union of some sets from $\{V_1, \ldots, V_k\}$, then the number of all open sets in the topological space (X, τ) does not exceed the number 2^k of all subsets of the set $\{V_1, \ldots, V_k\}$, and hence, $m \leq 2^k$.

For every $1 \leq i \leq k$ we consider the set $U_i = \bigcup\limits_{U \in \tau, V_i \not\subseteq U} U$.

From the construction of minimal base (see the proof of Theorem 1) it follows that there exists a subset $\{x_1, \ldots, x_k\}$ of the set X such that $V_i = V(x_i) = \bigcap\limits_{U \in \tau, x_i \in U} U$ for $1 \leq i \leq k$. Then for any $1 \leq i \leq k$ it follows that $x_i \in U$ if and only if $V_i \subseteq U$ for any $U \in \tau$, and hence, $U_i = \bigcup\limits_{U \in \tau, x_i \notin U} U$.

Prove first that $U_i \neq U_j$ for $i \neq j$.

We assume the contrary, i.e. that $U_s = U_l$ for some $s \neq l$.

Then from the minimality of the base $\{V_1, \ldots, V_k)\}$ in the topological space (X, τ) it follows that $V_s \neq V_l$. Let (for definiteness) $V_s \not\subseteq V_l$. Then $x_s \notin V_l$ (otherwise $V_s = \bigcap\limits_{U \in \tau, x_s \in U} U \subseteq V_l$), and hence, $V_s \subseteq \bigcup\limits_{U \in \tau, x_s \notin U} U = U_s = U_l$. Then $x_l \in V_l \subseteq U_l$.

We obtain a contradiction with the construction of the sets U_l, and hence, $U_i \neq U_j$ for $i \neq j$.

Now let's apply Algorithm 1 for calculation of the number $t(\tau)$ of one-point expansions of the topology τ.

The following 3 cases are possible:

1. $V_0 = X$;

2. $V_0 = U_i$ for some $1 \leq i \leq k$;

3. $V_0 \notin \{X\} \bigcup \{U_i \mid 1 \leq i \leq k\}$.

Consider each of these cases separately.

1. Let $V_0 = X$. Then $\bigcap\limits_{V \not\subseteq V_0} V = \bigcap\limits_{V \in \emptyset} V = X$, and hence, U_0 can take n values. Then the number of all pairs (X, U_0), where $U_0 \subseteq X$, is equal to m.

2. Now let $V_0 = U_i$ for some $1 \leq i \leq k$. Then

$$\bigcup_{U \in \tau, U \not\subseteq V_0} U = \bigcup_{U \in \tau, U \not\subseteq U_i} U = \bigcup_{U \in \tau, x_i \notin U_i} U = V_i \neq \emptyset,$$

and hence, in this case the set U_0 can take not less than two values.

Then the number of all pairs (U_i, U_0) for $1 \leq i \leq k$ and $U_0 \subseteq \bigcap\limits_{V\in\tau, V\not\subseteq U_i} V$ is not less than $2 \cdot k$.

3. Let $V_0 \notin \{X\}\bigcup\{U_1, \ldots, U_k\}$. Then $\emptyset \subseteq \bigcap\limits_{V\in\tau, V\not\subseteq V_0} V$, and hence, U_0 can take not less than one value. Then the number of all pairs (V_0, U_0) is not less than $1 \cdot (m-k-1) = m-k-1$.

Then the number of all pairs (V_0, U_0) will be not less than

$$m + 2 \cdot k + m - k - 1 = 2m + k - 1 \geq 2m + \log_2 m - 1.$$

Then (see [1], Theorem 2.7) the topology τ has not less than $2m + \log_2 m - 1$ one-point expansions, i.e. $t(\tau) \geq 2m + \log_2 m - 1$.

Now let's show that $t(\tau) \leq \dfrac{m \cdot (m+3)}{2} - 1$.

Let $\tau = \{W_1, \ldots, W_m\}$ be such a numbering of the set τ that $W_i \not\subseteq W_j$ for $j < i$ (such a numbering of the set τ is possible as the set τ is finite). Then the set $\{W_j \in \tau \mid W_j \subseteq W_i\}$ has no more than i subsets for every $1 \leq i \leq m$.

If $V_0 = W_i$ for $1 \leq i \leq m-1$, then $U_0 \subseteq \bigcap\limits_{V\in\tau, V\not\subseteq W_i} V \subseteq W_{i+1}$, and hence it has no more than $i+1$ subsets. And as for $V_0 = X$ the set $U_0 \subseteq X$ has m subsets, then the number of all pairs (V_0, U_0) is no more than

$$(\sum_{i=1}^{m-1}(i+1)) + m = \frac{(m+2)\cdot(m-1)}{2} + m =$$

$$\frac{m^2 + 2m - m - 2 + 2m}{2} = \frac{m \cdot (m+3)}{2} - 1.$$

Then the topology τ has no more than $\dfrac{m \cdot (3+m)}{2} - 1$ one-point expansions.

So, we have proved that $2 \cdot m + log_2 m - 1 \leq t(\tau) \leq \dfrac{m \cdot (m+1)}{2} + m - 1$.

The statement $\mathcal{A}$) is proved.

$\mathcal{B}$) If τ is a chain then (see Proposition 1) $\dfrac{m \cdot (m+3)}{2} - 1 = t(\tau)$.

If τ is not a chain, then $W_k \not\subseteq W_{k+1}$ (the definition of W_i at the end of the proof of the statement $\mathcal{A}$)) for some $1 < k < m$. Then the number of possible values for the set U_0 if $V_0 = W_k$ is strictly less than $k+1$, and hence,

$$t(\tau) < (\sum_{i=1}^{m-1}(i+1)) + m = \frac{(m+2)\cdot(m-1)}{2} + m =$$

$$\frac{m^2 + 2m - m - 2 + 2m}{2} = \frac{m \cdot (m+3)}{2} - 1.$$

The statement $\mathcal{B}$) is proved.

$\mathcal{C}$) If τ is an atomistic lattice and n is the number of atoms, then (see Proposition 2) $t(\tau) = 2^{(n+1)} + n - 1$. So in this case $m = 2^n$, then $t(\tau) = 2 \cdot m + log_2 m - 1$.

If τ is not an atomistic lattice, then the set of all atoms is not a base of a topological space (X, τ), and hence, there exists $1 \leq i \leq k$ such that V_i (definition of sets V_j see in the beginning of the proof of statement $\mathcal{A}$)) is not an atom. Then there exists $\emptyset \neq V' \in \tau$ such that $V' \subset V_i$. As $\bigcup\limits_{U \in \tau, x_i \notin U_i} U = V_i$ (see the beginning of the proof of the case 2 of the statement $\mathcal{A}$)), then the set $\{U \in \tau \mid U \not\subseteq \bigcup\limits_{U \in \tau, x_i \notin U_i} U\}$ contains not less than three subsets from τ, instead of two as we considered in the proof of the statement $\mathcal{A}$) (see the case 2). Hence, in this case we have that $t(\tau) > 2m + \log_2 m - 1$.

The statement $\mathcal{C}$) is proved, and hence, the theorem is proved completely. □

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Convex Solids with Hyperplanar Midsurfaces for Restricted Families of Chords

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Abstract. We provide new characteristic properties of convex quadrics in $\mathbb{R}^n$ in terms of hyperplanarity of midsurfaces of convex solids for restricted families of chords. These properties are based on various auxiliary characterizations of convex quadrics that involve hyperplane supports and plane quadric sections.

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1 Introduction

A classical result of convex geometry states that a convex body $K \subset \mathbb{R}^n$, $n \geq 2$, is a solid ellipsoid (solid ellipse if $n = 2$) provided the middle points of every family of parallel chords of K lie in a hyperplane (see Brunn [4, pp. 59–61] for $n = 2$, Blaschke [3, p. 159] for $n = 3$, and Busemann [5, p. 92] for all $n \geq 3$). Gruber [7] refined this result by proving, in particular, that a convex body $K \subset \mathbb{R}^n$ is a solid ellipsoid if there is an open nonempty subset T of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ such that for every unit vector $e \in T$, the middle points of all chords of K parallel to e belong to a hyperplane. Another refinement was suggested in 2009 by Erwin Lutwak, who posed the following problem: Is it true that a convex body $K \subset \mathbb{R}^n$ is a solid ellipsoid provided there is a point $p \in \operatorname{int} K$ and a scalar $\delta > 0$ such that, for every chord $[u, v]$ of K through p, the middle points of all chords of K which are parallel to $[u, v]$ and lie at a distance δ or less from $[u, v]$ belong to a hyperplane?

In this paper, we establish similar characterizations of convex quadric hypersurfaces (briefly, convex quadrics) among all convex hypersurfaces in $\mathbb{R}^n$. By a *convex solid* in $\mathbb{R}^n$ we mean an n-dimensional closed convex set $K \subset \mathbb{R}^n$ distinct from the whole space. A *convex hypersurface* in $\mathbb{R}^n$ is the boundary of a convex solid. This definition includes a hyperplane and a pair of parallel hyperplanes.

In a standard way, a *quadric* (or a *second degree hypersurface*) in $\mathbb{R}^n$ is the locus of points $x = (\xi_1, \ldots, \xi_n)$ that satisfy a quadratic equation

$$F(x) \equiv \sum_{i,k=1}^{n} a_{ik}\xi_i\xi_k + 2\sum_{i=1}^{n} b_i\xi_i + c = 0, \tag{1}$$

where at least one a_{ik} is distinct from zero and $a_{ik} = a_{ki}$ for all $i, k = 1, \ldots, n$. We say that a convex hypersurface $S \subset \mathbb{R}^n$ is a *convex quadric* provided there is a real quadric $Q \subset \mathbb{R}^n$ and a connected component U of $\mathbb{R}^n \setminus Q$ such that U is a convex set and S is the boundary of U. As proved in [17], a convex hypersurface $S \subset \mathbb{R}^n$ is a convex quadric if and only if there is a Cartesian coordinate system $\xi_1, \ldots, \xi_n$ for $\mathbb{R}^n$ such that S can be expressed as the locus of points $x = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ which satisfy one of the equations

$$
\begin{array}{ll}
a_1\xi_1^2 + \cdots + a_k\xi_k^2 = 1, & 1 \le k \le n, \\
a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_k\xi_k^2 = 1, \ \xi_1 \ge 0, & 2 \le k \le n, \\
a_1\xi_1^2 = 0, & \\
a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_k\xi_k^2 = 0, \ \xi_1 \ge 0, & 2 \le k \le n, \\
a_1\xi_1^2 + \cdots + a_{k-1}\xi_{k-1}^2 = \xi_k, & 2 \le k \le n,
\end{array}
$$

where all scalars a_i involved are positive. In particular, convex quadrics in $\mathbb{R}^n$ that contain no lines are ellipsoids, elliptic paraboloids, sheets of elliptic hyperboloids on two sheets, and sheets of elliptic cones. Various characteristic properties of convex quadrics are given in [13, 15–17]. In particular, the following assertions will be of use below.

(A) ([15]) The boundary of a convex solid $K \subset \mathbb{R}^n$, $n \ge 3$, is a convex quadric if and only if there is a point $p \in \operatorname{int} K$ such that every section of $\operatorname{bd} K$ by a 2-dimensional plane through p is a convex quadric curve.

(B) ([16]) Given a line-free convex solid $K \subset \mathbb{R}^n$ and a point $p \in \mathbb{R}^n$, $n \ge 3$, all proper bounded sections of $\operatorname{bd} K$ by 2-dimensional planes through p are ellipses if and only if the set $\operatorname{bd} K \setminus \big((p + \operatorname{rec} K) \cup (p - \operatorname{rec} K)\big)$ lies in a convex quadric, where $\operatorname{rec} K$ denotes the recession cone of K (see definitions below).

2 Main Results

We need some definitions to formulate the main results. A *chord* of the convex solid K is a line segment $[u, v]$, $u \ne v$, such that $[u, v] = K \cap \langle u, v\rangle$, where $\langle u, v\rangle$ denotes the line through u and v. We will say that both $[u, v]$ and $\langle u, v\rangle$ are *parallel* to a unit vector $e \in \mathbb{R}^n$ if $u - v$ is a nonzero multiple of e. A convex solid K has chords if and only if it is distinct from a closed halfspace. By a *plane* of dimension m we mean a translate of an m-dimensional subspace of $\mathbb{R}^n$. A plane L *properly* intersects the solid K if L intersects both the boundary $\operatorname{bd} K$ and the interior $\operatorname{int} K$ of K.

The *recession cone* of a convex solid $K \subset \mathbb{R}^n$ is defined by

$$\operatorname{rec} K = \{y \in \mathbb{R}^n : x + \alpha y \in K \text{ whenever } x \in K \text{ and } \alpha \ge 0\}.$$

It is well-known that $\operatorname{rec} K$ is a closed convex cone with apex o, the origin of $\mathbb{R}^n$; furthermore, $\operatorname{rec} K$ is distinct from $\{o\}$ if and only if K is unbounded. The subset $\mathbb{S}^{n-1} \setminus (\operatorname{rec} K \cup -\operatorname{rec} K)$ of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ consists of *non-recessional* unit vectors for K. Equivalently, a unit vector $e \in \mathbb{R}^n$ is non-recessional for K if and only if the intersection of K with any line parallel to e is either bounded or empty. Obviously, K has non-recessional unit vectors if and only if K is distinct from a closed halfspace of $\mathbb{R}^n$.

For any plane $L \subset \mathbb{R}^n$ which is complementary to the *linearity space* of K, defined by

$$\operatorname{lin} K = \operatorname{rec} K \cap (-\operatorname{rec} K),$$

the convex solid K can be expressed as the direct sum

$$K = \operatorname{lin} K \oplus (K \cap L),$$

and $K \cap L$ is a closed convex set containing no lines (see, e. g., [19] for general references on convex sets).

Theorem 1. *Given a convex solid $K \subset \mathbb{R}^n$, $n \geq 2$, distinct from a closed halfspace of $\mathbb{R}^n$ and an open nonempty subset T of $\mathbb{S}^{n-1} \setminus (\operatorname{rec} K \cup -\operatorname{rec} K)$, the following conditions are equivalent:*

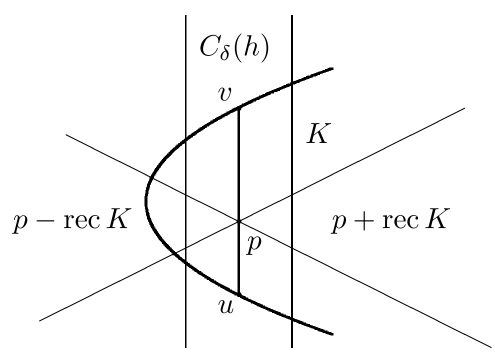
1) *for every unit vector $e \in T$, the middle points of all chords of K which are parallel to e belong to a hyperplane,*

2) $\operatorname{bd} K$ *is a convex quadric.*

Problem 1. Is it true that Theorem 1 still holds if condition 1) is replaced by the following weaker condition:

1′) for every unit vector $e \in T$, there is a scalar $\lambda = \lambda(e) \in (0,1)$ such that the points dividing in the same ratio λ all chords of K which are parallel to e belong to a hyperplane.

The answer to Problem 1 is affirmative in the following two cases: K is a convex body in $\mathbb{R}^n$ (see [7]), K is a convex solid in $\mathbb{R}^n$ and $T = \mathbb{S}^{n-1} \setminus (\operatorname{rec} K \cup -\operatorname{rec} K)$ (see [15]). The papers [7, 15] also contain results which involve a weaker version of 1′), with $\lambda \in [0,1]$ instead of $\lambda \in (0,1)$.



In what follows, we consider double cones $(p + \operatorname{rec} K) \cup (p - \operatorname{rec} K)$ with apices $p \in \mathbb{R}^n$, as depicted above.

Definition 1. Let δ be a positive scalar, $K \subset \mathbb{R}^n$ a convex solid, p a point in $\operatorname{int} K$, and $h = [u, v]$ a chord of K through p. Denote by $C_\delta(h)$ the closed circular cylinder of radius δ centered about the line $\langle u, v\rangle$, and by $\mathcal{F}_\delta(h)$ the family of all chords of K which are parallel to h and lie in $C_\delta(h)$. Furthermore, let

$$\Omega_\delta(p) = \cup\,(C_\delta(h) \cap \operatorname{bd} K),$$

where the union is taken over all chords h of K that contain p.

Clearly, $\Omega_\delta(p)$ is a closed neighborhood of $\operatorname{bd} K \setminus \big((p + \operatorname{rec} K) \cup (p - \operatorname{rec} K)\big)$ in $\operatorname{bd} K$.

Theorem 2. *Given a convex solid $K \subset \mathbb{R}^n$, $n \ge 2$, distinct from a closed halfspace of $\mathbb{R}^n$, a point $p \in \operatorname{int} K$, and a scalar $\delta > 0$, the following conditions are equivalent:*

1) *for every chord h of K that contains p, the middle points of all chords from $\mathcal{F}_\delta(h)$ belong to a hyperplane,*

2) *the set $\Omega_\delta(p)$ lies in a convex quadric.*

If K is a convex body in $\mathbb{R}^n$, then $\operatorname{rec} K = \{o\}$, implying the equality $\Omega_\delta(p) = \operatorname{bd} K$ for any given point $p \in \operatorname{int} K$. Therefore Theorem 2 implies the following corollary, which gives an affirmative solution to Lutwak's problem.

Corollary 1. *A convex body $K \subset \mathbb{R}^n$, $n \ge 2$, is a solid ellipsoid if and only if there is a point $p \in \operatorname{int} K$ and a scalar $\delta > 0$ such that for every chord h of K which contains p, the middle points of all chords from $\mathcal{F}_\delta(h)$ belong to a hyperplane.*

Remark 1. We observe that the scalar δ in Theorem 2 and Corollary 1 cannot be chosen as a function of h. Indeed, if K is a 3-dimensional octahedron, given by

$$K = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : |\xi_1| + |\xi_2| + |\xi_3| \le 1\},$$

then for any chord h of K that contains the origin o, there is a scalar $\delta = \delta(h) > 0$ such that the middle points of all chords from $\mathcal{F}_\delta(h)$ belong to a plane through o.

Problem 2. Is it true that Theorem 2 still holds if condition 1) is replaced by the following weaker condition:

1″) for any chord h of K that contains p, there is a scalar $\lambda = \lambda(e) \in (0, 1)$ such that the points dividing in the same ratio λ all chords from $\mathcal{F}_\delta(h)$ belong to a hyperplane.

The proofs of Theorem 1 and 2 are based on some auxiliary statements. The first one complements Theorem 1 from [17] by giving new characteristic properties of quadrics $Q \subset \mathbb{R}^n$ with at least one convex connected component of $\mathbb{R}^n \setminus Q$ in terms of local convexity and local supports. In what follows, a quadric $Q \subset \mathbb{R}^n$ is called *proper* provided its complement $\mathbb{R}^n \setminus Q$ has two or more connected components, which happens when Q, given by (1), is a hyperplane or both sets

$$\{x \in \mathbb{R}^n : F(x) > 0\} \quad \text{and} \quad \{x \in \mathbb{R}^n : F(x) < 0\}$$

are nonempty.

We will say that a proper quadric $Q \subset \mathbb{R}^n$ is *locally convex* at a point $u \in Q$ if there is an open ball $U_\rho(u) \subset \mathbb{R}^n$ with center u and radius $\rho > 0$ such that $Q \cap U_\rho(u)$ is a piece of a convex hypersurface. Similarly, a proper quadric $Q \subset \mathbb{R}^n$ is *locally supported* at $u \in Q$ if there is an open ball $U_\rho(u) \subset \mathbb{R}^n$ and a hyperplane $H \subset \mathbb{R}^n$ through u such that $Q \cap U_\rho(u)$ lies in a closed halfspace of $\mathbb{R}^n$ bounded by H.

Theorem 3. *For a proper quadric $Q \subset \mathbb{R}^n$, $n \geq 2$, the following conditions are equivalent:*

1) *Q is locally convex at a certain point $u \in Q$,*
2) *Q is locally supported at a certain point $u \in Q$,*
3) *at least one of the connected components of $\mathbb{R}^n \setminus Q$ is a convex set,*
4) *Q is the union of at most four convex quadrics,*
5) *there is a Cartesian coordinate system $\xi_1, \dots, \xi_n$ for $\mathbb{R}^n$ such that Q can be expressed as the locus of points $x = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ which satisfy one of the equations*

$$F_1(x) \equiv a_1\xi_1^2 + \cdots + a_k\xi_k^2 = 1, \qquad 1 \leq k \leq n, \tag{2}$$

$$F_2(x) \equiv a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_k\xi_k^2 = 1, \qquad 2 \leq k \leq n, \tag{3}$$

$$F_3(x) \equiv a_1\xi_1^2 = 0, \tag{4}$$

$$F_4(x) \equiv a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_k\xi_k^2 = 0, \qquad 2 \leq k \leq n, \tag{5}$$

$$F_5(x) \equiv a_1\xi_1^2 + \cdots + a_{k-1}\xi_{k-1}^2 = \xi_k, \qquad 2 \leq k \leq n, \tag{6}$$

where all scalars a_i involved are positive.

There is a certain analogy between Theorem 3 and respective properties of convex hypersurfaces. Indeed, if S is the boundary of an open nonempty connected set $X \subset \mathbb{R}^n$, then S is a convex hypersurface provided X is locally supported at every point $u \in S$ (see [6]). Similarly, S is a convex hypersurface if X is locally convex at every point $u \in S$ (see [10, 18]). On the other hand, Theorem 3 deals with local convexity and local support of Q at a *single* point.

The next two results characterize convex quadrics in terms of their 2-dimensional planar sections.

Theorem 4. *Let $K \subset \mathbb{R}^n$, $n \geq 3$, be a convex solid, p a point in* $\operatorname{int} K$, *and T an open nonempty subset of $\mathbb{S}^{n-1} \setminus (\operatorname{rec} K \cup -\operatorname{rec} K)$. The following conditions are equivalent:*

1) $\operatorname{bd} K$ *is a convex quadric,*

2) *for every* 2*-dimensional plane* L *through* p *which properly intersects* K *such that the subspace* $L - p$ *meets* T*, the section* $L \cap \operatorname{bd} K$ *is a convex quadric curve.*

Remark 2. Theorem 4 refines, with essential modifications of proofs, the respective statements from [15], given there for the case $T = \mathbb{S}^{n-1} \setminus (\operatorname{rec} K \cup -\operatorname{rec} K)$. It is unknown whether Theorem 4 remains true for any choice of the point p in $\mathbb{R}^n$ (compare with Problem 1 from [17]).

Obvious changes in the proof of Theorem 4 allow us to generalize the following assertion of Petty [12]: the boundary of a convex body $K \subset \mathbb{R}^n$ is an ellipsoid provided there is a line $l \subset \mathbb{R}^n$ such that all proper sections of $\operatorname{bd} K$ by 2-dimensional planes parallel to l are ellipses. Given a line $l \subset \mathbb{R}^n$ and a scalar $\delta > 0$, denote by $\mathcal{P}_\delta(l)$ the family of all 2-dimensional planes in $\mathbb{R}^n$ which are parallel to l and whose distance from l is less than δ.

Theorem 5. *Let* $K \subset \mathbb{R}^n$*,* $n \geq 3$*, be a convex solid,* l *a line that meets* $\operatorname{int} K$ *and is parallel to a unit vector from* $\mathbb{S}^{n-1} \setminus (\operatorname{rec} K \cup -\operatorname{rec} K)$*, and* δ *a positive scalar. The following conditions are equivalent:*

1) $\operatorname{bd} K$ *is a convex quadric,*

2) *for any* 2*-dimensional plane* $L \in \mathcal{P}_\delta(l)$ *properly intersecting* K*, the section* $L \cap \operatorname{bd} K$ *is a convex quadric curve.* □

Remark 3. The condition that l is parallel to a unit vector from $\mathbb{S}^{n-1} \setminus (\operatorname{rec} K \cup -\operatorname{rec} K)$ is essential in Theorem 5. Indeed, let C be the unit cube in the coordinate hyperplane $\xi_1 = 0$ of $\mathbb{R}^n$ and l be the ξ_1-axis of $\mathbb{R}^n$. Denote by K the Cartesian product of C and l. Clearly, K is a convex solid with $\operatorname{rec} K = l$ and any proper section of $\operatorname{bd} K$ by a 2-dimensional plane parallel to l is a pair of parallel lines, which is a degenerate convex quadric curve.

Alonso and Martín [1] proved that if $L_1, L_2, L_3 \subset \mathbb{R}^n$, $n \geq 3$, are three pairwise distinct $(n-1)$-dimensional subspaces and $K \subset \mathbb{R}^n$ a centrally symmetric convex body such that every proper section of $\operatorname{bd} K$ by a hyperplane parallel to one of these subspaces is an $(n-1)$-dimensional ellipsoid, then $\operatorname{bd} K$ is an ellipsoid itself. They also observed that the assumption on central symmetry of K here cannot be omitted. Indeed, if $K_\alpha \subset \mathbb{R}^3$, $0 < |\alpha| \leq 2$, is a convex body, given by

$$K_\alpha = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 + \alpha xyz \leq 1, \max\{|x|, |y|, |z|\} \leq 1\},$$

then any proper section of $\operatorname{bd} K_\alpha$ by a plane parallel to one of the coordinate subspaces $x = 0$, $y = 0$, and $z = 0$ is an ellipse (see [1] for other examples). Our next theorem extends the result of Alonso and Martín to the case of any convex body in $\mathbb{R}^n$.

Theorem 6. *If $L_1, L_2, L_3, L_4 \subset \mathbb{R}^n$, $n \ge 3$, are four pairwise distinct $(n-1)$-dimensional subspaces and $K \subset \mathbb{R}^n$ a convex body such that every proper section of* $\operatorname{bd} K$ *by a hyperplane parallel to one of these subspaces is an $(n-1)$-dimensional ellipsoid, then* $\operatorname{bd} K$ *is an ellipsoid itself.*

It would be interesting to generalize Theorem 6 to the case of convex quadrics. In what follows, $\operatorname{rbd} M$ and $\operatorname{rint} M$ denote, respectively, the relative boundary and the relative interior of a closed convex set $M \subset \mathbb{R}^n$.

3 Auxiliary Lemmas

If a proper quadric $Q \subset \mathbb{R}^n$ is given by (1), then a point $u \in Q$ is called *regular* provided the gradient vector

$$\nabla F(u) = \left(\tfrac{\partial F(u)}{\partial \xi_1}, \dots, \tfrac{\partial F(u)}{\partial \xi_n}\right),$$

the *normal* to Q at u, is distinct from the zero vector o; otherwise u is *singular*. The standard classification of quadrics in $\mathbb{R}^n$ (see, e. g., [2]) immediately implies that a description of a *proper* quadric $Q \subset \mathbb{R}^n$, given by (1), can be reduced to one of the canonical equations

$$a_1\xi_1^2 + \cdots + a_k\xi_k^2 = 1, \qquad 1 \le k \le n, \tag{7}$$
$$a_1\xi_1^2 + \cdots + a_r\xi_r^2 - a_{r+1}\xi_{r+1}^2 - \cdots - a_k\xi_k^2 = 1, \qquad 1 \le r < k \le n, \tag{8}$$
$$a_1\xi_1^2 = 0, \tag{9}$$
$$a_1\xi_1^2 + \cdots + a_r\xi_r^2 - a_{r+1}\xi_{r+1}^2 - \cdots - a_k\xi_k^2 = 0, \qquad 1 \le r < k \le n, \tag{10}$$
$$a_1\xi_1^2 + \cdots + a_{k-1}\xi_{k-1}^2 = \xi_k, \qquad 1 < k \le n, \tag{11}$$
$$a_1\xi_1^2 + \cdots + a_r\xi_r^2 - a_{r+1}\xi_{r+1}^2 - \cdots - a_{k-1}\xi_{k-1}^2 = \xi_k, \qquad 1 \le r < k-1 < n, \tag{12}$$

where all scalars a_i involved are positive. The following lemma routinely follows from (7)–(12).

Lemma 1. *A proper quadric $Q \subset \mathbb{R}^n$ has singular points if and only if its canonical equation is expressed by* (9) *or* (10)*. The set of singular points of Q is given by $\xi_1 = 0$ if Q is described by* (9) *and by $\xi_1 = \cdots = \xi_k = 0$ if Q is described by* (10)*.* □

If $u = (\mu_1, \dots, \mu_n)$ is a regular point of a proper quadric $Q \subset \mathbb{R}^n$, then the linear equation in $x = (\xi_1, \dots, \xi_n)$,

$$\nabla F(u)\cdot(x-u) \equiv \tfrac{\partial F(u)}{\partial \xi_1}(\xi_1 - \mu_1) + \cdots + \tfrac{\partial F(u)}{\partial \xi_n}(\xi_n - \mu_n) = 0, \tag{13}$$

defines the hyperplane through u which is orthogonal to $\nabla F(u)$; it is called *tangent* to Q at u. Since a proper quadric is differentiable at any regular point, we immediately obtain the following lemma.

Lemma 2. *If a proper quadric $Q \subset \mathbb{R}^n$ is locally supported by a hyperplane G at a regular point $u \in Q$, then G is tangent to Q at u.* □

Lemma 3. *The middle points of all chords of a quadric $Q \subset \mathbb{R}^n$ which are parallel to a given chord $[a, c]$ of Q belong to a hyperplane.*

Proof. Assume that Q is given by (1). The line $l = \langle a, c\rangle$ can be expressed as

$$l = \{z + tv \in \mathbb{R}^n : t \in \mathbb{R}\}, \quad v \neq o,$$

where z is the middle point of $[a, c]$ and $v = a - c$. Equivalently, $x = (\xi_1, \dots, \xi_n) \in l$ if and only if

$$\xi_i = \phi_i + t\nu_i, \quad t \in \mathbb{R}, \quad i = 1, \dots, n, \tag{14}$$

where $z = (\phi_1, \dots, \phi_n)$ and $v = (\nu_1, \dots, \nu_n)$. To determine the values of t for which $x \in l \cap Q$, we substitute $\xi_1, \dots, \xi_n$ from (14) into (1) and arrange the powers of t. The result is a quadratic equation in t,

$$A(v)\, t^2 + 2B(v, z)\, t + C(z) = 0, \tag{15}$$

where

$$A(v) = \sum_{i,k=1}^{n} a_{ik}\nu_i\nu_k, \quad B(v, z) = \tfrac{1}{2}\sum_{i=1}^{n} \tfrac{\partial F(z)}{\partial \xi_i}\, \nu_i, \quad C(z) = F(z). \tag{16}$$

Then a and c correspond to opposite non-zero solutions t_0 and $-t_0$ of (15), which is possible if and only if $A(v)\, C(z) < 0$ and $B(v, z) = 0$. Hence

$$\sum_{i=1}^{n} \Big(\sum_{k=1}^{n} a_{ik}\phi_k + b_i\Big)\nu_i = \tfrac{1}{2}\sum_{i=1}^{n} \tfrac{\partial F(z)}{\partial \xi_i}\nu_i = B(v, z) = 0.$$

Equivalently,

$$\sum_{k=1}^{n} \Big(\sum_{i=1}^{n} a_{ik}\nu_i\Big)\phi_k + \sum_{i=1}^{n} b_i\nu_i = 0. \tag{17}$$

Interpreted as an equation in $\phi_1, \dots, \phi_n$, (17) describes a hyperplane, H, because at least one of the scalars

$$c_k = \sum_{i=1}^{n} a_{ik}\nu_i, \quad k = 1, \dots, n,$$

is distinct from zero. Indeed, assuming $c_1 = \cdots = c_n = 0$, we would obtain

$$A(v) = c_1\nu_1 + \cdots + c_n\nu_n = 0,$$

which is impossible because of $A(v)\, C(z) < 0$. If $[a', c']$ is a chord of Q that is parallel to $[a, c]$, then v is a nonzero multiple of $a' - c'$, which implies that

$$\langle a', c'\rangle = \{z' + tv \in \mathbb{R}^n : t \in \mathbb{R}\},$$

where $z' = (\phi'_1, \dots, \phi'_n)$ is the middle point of $[a', c']$. Repeating the argument above, we obtain that $\phi'_1, \dots, \phi'_n$ satisfy (17), which gives $z' \in H$. □

4 Proof of Theorem 3

1) $\Rightarrow$ 2) Assume that Q is locally convex at a point $u \in Q$; that is, $Q \cap U_\rho(u)$ is a piece of a convex hypesurface $S \subset \mathbb{R}^n$ for a suitable scalar $\rho > 0$. By a convexity argument, there is a hyperplane H supporting S at u. Therefore, $Q \cap U_\rho(u)$ lies in a closed halfspace of $\mathbb{R}^n$ bounded by H, which implies that Q is locally supported at u.

2) $\Rightarrow$ 3) Choosing a suitable orthonormal basis $e_1, \dots, e_n$ for $\mathbb{R}^n$, we may suppose that Q is described by one of the equations (7)–(12). Put $u = (\mu_1, \dots, \mu_n)$ and denote by H a hyperplane that supports $Q \cap U_\rho(u)$ for a suitable choice of $\rho > 0$.

(a) If Q is expressed by (7), then Q itself is a convex quadric and the connected component $\{x \in \mathbb{R}^n : F_1(x) < 1\}$ of $\mathbb{R}^n \setminus Q$ is an open convex set.

(b) Suppose that Q is given by (8). From Lemma 1 it follows that u is a regular point of Q. Choosing suitable orthogonal bases $e'_1, \dots, e'_r$ and $e'_{r+1}, \dots, e'_n$ for the subspaces span $(e_1, \dots, e_r)$ and span $(e_{r+1}, \dots, e_n)$, respectively, we may assume that Q is still expressed by (8) and

$$u = (\mu_1, 0, \dots, 0, \mu_{r+1}, 0, \dots, 0), \quad \mu_1 > 0, \ \mu_{r+1} \geq 0,$$

with $a_1\mu_1^2 - a_{r+1}\mu_{r+1}^2 = 1$. The section of Q by the 2-dimensional subspace $L_1 =$ span (e_1, e_{r+1}) is a hyperbola, whose arm E_1 containing u is given by

$$a_1\xi_1^2 - a_{r+1}\xi_{r+1}^2 = 1, \ \xi_1 > 0, \quad \xi_2 = \dots = \xi_r = \xi_{r+2} = \dots = \xi_n = 0.$$

By Lemma 2, H is tangent to Q at u. Due to (13), H is expressed as

$$a_1\mu_1(\xi_1 - \mu_1) - a_{r+1}\mu_{r+1}(\xi_{r+1} - \mu_{r+1}) = 0.$$

Equivalently,

$$a_1\mu_1\xi_1 - a_{r+1}\mu_{r+1}\xi_{r+1} = 1.$$

We are going to show that $r = 1$. Indeed, assume for a moment that $r \geq 2$. Then the section of Q by the r-dimensional plane

$$L_2 = \{(\xi_1, \dots, \xi_n) : \xi_{r+1} = \mu_{r+1}, \xi_{r+2} = \dots = \xi_n = 0\}$$

is the r-dimensional ellipsoid, E_2, described by

$$a_1\xi_1^2 + \dots + a_r\xi_r^2 = 1 + a_{r+1}\mu_{r+1}^2, \quad \xi_{r+1} = \mu_{r+1}, \ \xi_{r+2} = \dots = \xi_n = 0.$$

From $a_1\xi_1^2 + \dots + a_r\xi_r^2 = a_1\mu_1^2$ it follows that $|\xi_1| \leq \mu_1$.

We state that E_1 and E_2 lie in the opposite closed halfspaces of $\mathbb{R}^n$ determined by H. Indeed, since the set $B_1 \subset L_1$ given by

$$a_1\xi_1^2 - a_{r+1}\xi_{r+1}^2 \geq 1, \ \xi_1 > 0, \quad \xi_2 = \dots = \xi_r = \xi_{r+2} = \dots = \xi_n = 0,$$

is strictly convex, the point

$$\left(\tfrac{\xi_1+\mu_1}{2}, 0, \dots, 0, \tfrac{\xi_{r+1}+\mu_{r+1}}{2}, 0, \dots, 0\right)$$

belongs to $\operatorname{rint} B_1$ provided the point $x = (\xi_1, 0, \dots, 0, \xi_{r+1}, 0, \dots, 0) \in E_1$ is distinct from u. Hence

$$a_1\big(\tfrac{\xi_1+\mu_1}{2}\big)^2 - a_{r+1}\big(\tfrac{\xi_{r+1}+\mu_{r+1}}{2}\big)^2 \ge 1,$$

which results in

$$a_1\mu_1\xi_1 - a_{r+1}\mu_{r+1}\xi_{r+1} \ge 1,$$

with equality if and only if $\xi_1 = \mu_1$ and $\xi_{r+1} = \mu_{r+1}$.

If $x \in E_2$, then from $|\xi_1| \le \mu_1$ and $\xi_{r+1} = \mu_{r+1}$ we obtain

$$a_1\mu_1\xi_1 - a_{r+1}\mu_{r+1}\xi_{r+1} \le a_1\mu_1^2 - a_{r+1}\mu_{r+1}^2 = 1,$$

with equality if and only if $\xi_1 = \mu_1$. Summing up, $E_1 \cap U_\rho(u)$ and $E_2 \cap U_\rho(u)$ lie in the opposite closed halfspaces of $\mathbb{R}^n$ bounded by H such that $E_1 \cap H = E_2 \cap H = \{u\}$, in contradiction with the choice of $U_\rho(u)$. Hence $r = 1$, and, by proved in [17], the connected component $\{x \in \mathbb{R}^n : F_2(x) > 1\}$ of $\mathbb{R}^n \setminus Q$ is an open convex set.

(c) If Q is given by (9), then Q is the hyperplane described by $\xi_1 = 0$ and both open halfspaces $\xi_1 > 0$ and $\xi_1 < 0$ are the connected components of $\mathbb{R}^n \setminus Q$.

(d) Assume that Q is expressed by (10). Since any point

$$x = (0, \dots, 0, \xi_{k+1}, \dots, \xi_n) \in Q$$

is the apex of a "double cone" $Q \cap \operatorname{span}(e_{k+1}, \dots, e_n)$, which cannot be locally supported at x, at least one of the coordinates $\mu_1, \dots, \mu_k$ of u must be distinct from 0. From Lemma 1 it follows that u is a regular point of Q. By Lemma 2, H is tangent to Q at u. Choosing suitable orthogonal bases $e_1', \dots, e_r'$ and $e_{r+1}', \dots, e_n'$ for the subspaces $\operatorname{span}(e_1, \dots, e_r)$ and $\operatorname{span}(e_{r+1}, \dots, e_n)$, respectively, we may assume that Q is still expressed by (10) and

$$u = (\mu_1, 0, \dots, 0, \mu_{r+1}, 0, \dots, 0), \quad \mu_1 > 0,\ \mu_{r+1} > 0,$$

with $a_1\mu_1^2 - a_{r+1}\mu_{r+1}^2 = 0$. Clearly, the section of Q by the 2-dimensional subspace $L_1 = \operatorname{span}(e_1, e_{r+1})$ is a double cone. Denote by E_1 the arm of this cone given by

$$a_1\xi_1^2 - a_{r+1}\xi_{r+1}^2 = 0,\ \xi_1 > 0,\ \xi_{r+1} > 0,\ \ \xi_2 = \dots = \xi_r = \xi_{r+2} = \dots = \xi_n = 0.$$

Then $u \in E_1$. Hence H is given by

$$a_1\mu_1(\xi_1 - \mu_1) - a_{r+1}\mu_{r+1}(\xi_{r+1} - \mu_{r+1}) = 0,$$

or

$$a_1\mu_1\xi_1 - a_{r+1}\mu_{r+1}\xi_{r+1} = 0.$$

We are going to show that $r = 1$. Indeed, assume for a moment that $r \ge 2$. Then the section of Q by the r-dimensional plane

$$L_2 = \{(\xi_1, \dots, \xi_n) : \xi_{r+1} = \mu_{r+1}, \xi_{r+2} = \dots = \xi_n = 0\}$$

is the r-dimensional ellipsoid, E_2, described by

$$a_1\xi_1^2 + \cdots + a_r\xi_r^2 = a_{r+1}\mu_{r+1}^2, \quad \xi_{r+1} = \mu_{r+1},\ \xi_{r+2} = \cdots = \xi_n = 0.$$

Similarly to case (b) above, one can show that $E_1 \cap U_\rho(u)$ and $E_2 \cap U_\rho(u)$ lie in distinct closed halfspaces of $\mathbb{R}^n$ determined by H such that $E_1 \cap H = E_2 \cap H = \{u\}$, in contradiction with the choice of $U_\rho(u)$. Hence $r = 1$. As shown in [17], the connected component $\{x \in \mathbb{R}^n : F_4(x) > 0\}$ of $\mathbb{R}^n \setminus Q$ is a convex set.

(e) If Q is expressed by (11), then Q itself is a convex quadric and the connected component $\{x \in \mathbb{R}^n : F_5(x) < \xi_k\}$ of $\mathbb{R}^n \setminus Q$ is a convex set.

(f) Finally, assume that Q is expressed by (12). From Lemma 1 it follows that u is a regular point of Q. So, H is tangent to Q at u. Choosing suitable orthogonal bases $e_1', \ldots, e_r'$ and $e_{r+1}', \ldots, e_n'$ for the subspaces $\operatorname{span}(e_1, \ldots, e_r)$ and $\operatorname{span}(e_{r+1}, \ldots, e_n)$, respectively, we may assume that Q is still expressed by (12) and

$$u = (\mu_1, 0, \ldots, 0, \mu_{r+1}, 0, \ldots, 0, \mu_k, 0, \ldots, 0),$$

where $a_1\mu_1^2 - a_{r+1}\mu_{r+1}^2 = \mu_k$. Due to (13), H is expressed as

$$2a_1\mu_1(\xi_1 - \mu_1) - 2a_{r+1}\mu_{r+1}(\xi_{r+1} - \mu_{r+1}) - (\xi_k - \mu_k) = 0.$$

Equivalently,

$$\xi_k = 2a_1\mu_1\xi_1 - 2a_{r+1}\mu_{r+1}\xi_{r+1}.$$

The section of Q by the 2-dimensional plane $L_1 = u + \operatorname{span}(e_1, e_k)$ is a parabola, E_1, given by

$$\xi_k = a_1\xi_1^2 - a_{r+1}\mu_{r+1}^2,\ \xi_{r+1} = \mu_{r+1},\ \xi_i = 0 \text{ for all } i \in \{1, \ldots, n\} \setminus \{1, r+1, k\}.$$

Similarly, the section of Q by another 2-dimensional plane, $L_2 = u + \operatorname{span}(e_{r+1}, e_k)$ also is a parabola, E_2, given by

$$\xi_k = a_1\mu_1^2 - a_{r+1}\xi_{r+1}^2,\ \xi_1 = \mu_1,\ \xi_i = 0 \text{ for all } i \in \{1, \ldots, n\} \setminus \{1, r+1, k\}.$$

Clearly, $E_1 \cap U_\rho(u)$ and $E_2 \cap U_\rho(u)$ lie in distinct closed halfspaces of $\mathbb{R}^n$ determined by H such that $E_1 \cap H = E_2 \cap H = \{u\}$, in contradiction with the choice of $U_\rho(u)$. Hence Q cannot be given by (12).

Equivalence of conditions 1), 3)–5) follows from the proof of Theorem 1 from [17].

5 Proof of Theorem 4

The statement 1) $\Rightarrow$ 2) immediately follows from the fact that a proper section of a convex quadric by a 2-dimensional plane is a convex quadric curve. Conversely, assume that condition 2) of the theorem holds. Translating K on the vector $-p$, we may suppose that $o = p \in \operatorname{int} K$. We observe that K is distinct from a halfspace, since otherwise $\operatorname{rec} K \cup -\operatorname{rec} K = \mathbb{R}^n$ in contradiction with the choice of T. Also, we eliminate the trivial case when K is a slab between two parallel hyperplanes (implying that $\mathrm{bd}\, K$ is a degenerate convex quadric). Therefore we may assume that $\dim(\mathrm{lin}\, K) \le n-2$.

We observe that the proof of 2) $\Rightarrow$ 1) can be reduced to the case $\dim(\mathrm{lin}\, K) = 0$; that is, to the case when K contains no lines. Indeed, assuming the inequality $\dim(\mathrm{lin}\, K) \ge 1$, choose a subspace $M \subset \mathbb{R}^n$ complementary to $\mathrm{lin}\, K$ and intersecting T. Put $K' = M \cap K$ and $T' = T \cap M$. Clearly, $\mathrm{lin}\, K' = M \cap \mathrm{lin}\, K = \{o\}$ and T' is an open nonempty subset of $(M \cap \mathbb{S}^{n-1}) \setminus (\mathrm{rec}\, K' \cup -\mathrm{rec}\, K')$. Choose a 2-dimensional subspace $L \subset M$ that meets T' and properly intersects K'. From the equality $K = \mathrm{lin}\, K \oplus K'$, we obtain $L \cap \mathrm{rbd}\, K' = L \cap \mathrm{bd}\, K$. Hence condition 2) implies that $L \cap \mathrm{rbd}\, K'$ is a convex quadric curve. Therefore, K' satisfies condition 2) of the theorem (with M and T' instead of $\mathbb{R}^n$ and T, respectively). Finally, the equality $\mathrm{bd}\, K = \mathrm{lin}\, K \oplus \mathrm{rbd}\, K'$ shows that $\mathrm{bd}\, K$ is a degenerate convex quadric provided $\mathrm{rbd}\, K'$ is a convex quadric.

Our further consideration of the case $\dim(\mathrm{lin}\, K) = 0$ is organized by induction on $n\,(\ge 3)$. Let $n = 3$. Since K is line-free, there is a 2-dimensional subspace L' through l properly intersecting K such that $L' \cap K$ is bounded. Choose a pair of distinct planes L_1 and L_2 both containing l and placed so close to L' that the sets $L_1 \cap K$ and $L_2 \cap K$ are bounded. By condition 2), both sections $E_1 = L_1 \cap \mathrm{bd}\, K$ and $E_2 = L_2 \cap \mathrm{bd}\, K$ are convex quadric curves, whence they are ellipses. Let c be the midpoint of the line segment $l \cap K$ and c_1 and c_2 the centers of E_1 and E_2, respectively. Applying a suitable affine transformation, we may assume that both E_1 and E_2 are circles and the planes L_1 and L_2 are orthogonal. Clearly, the image of K under this transformation, also denoted by K, satisfies condition 2) of the theorem. Let 2δ be the length of $l \cap K$.

Choose a coordinate system (ξ_1, ξ_2, ξ_3) such that l is the ξ_3-axis, the points c, c_1, c_2 lie in the coordinate plane $\xi_3 = \sigma_3$, where σ_3 is a suitable scalar, and

$$c_1 = (\sigma_1, 0, \sigma_3), \quad c_2 = (0, \sigma_2, \sigma_3), \quad c = (0, 0, \sigma_3), \quad \sigma_1, \sigma_2, \sigma_3 \ge 0.$$

Then E_1 and E_2 are described as

$$E_1 = \{(\xi_1, 0, \xi_3) : (\xi_1 - \sigma_1)^2 + (\xi_3 - \sigma_3)^2 = \sigma_1^2 + \delta^2\},$$
$$E_2 = \{(0, \xi_2, \xi_3) : (\xi_2 - \sigma_2)^2 + (\xi_3 - \sigma_3)^2 = \sigma_2^2 + \delta^2\}.$$

Clearly, L_1 and L_2 are given by the equations $\xi_2 = 0$ and $\xi_1 = 0$, respectively.

Choose a point $v \in \mathrm{bd}\, K \setminus (L_1 \cup L_2)$ so close to l that $v/\|v\| \in T$ and a certain 2-dimensional plane L through $\langle o, v\rangle$ meets K along a bounded set and intersects each of the ellipses E_1, E_2 at precisely two points. As above, $L \cap \mathrm{bd}\, K$ is an ellipse.

We state the existence of a quadric surface $Q \subset \mathbb{R}^3$ that contains $\{v\} \cup E_1 \cup E_2$. For this, consider the family of quadrics $Q(\mu) \subset \mathbb{R}^3$, given by

$$\xi_1^2 + \xi_2^2 + \xi_3^2 + \mu\xi_1\xi_2 - 2\sigma_1\xi_1 - 2\sigma_2\xi_2 - 2\sigma_3\xi_3 + \sigma_3^2 - \delta^2 = 0,$$

where μ is a scalar parameter. Obviously, $E_i = L_i \cap Q(\mu)$, $i = 1, 2$, for all $\mu \in \mathbb{R}$. If $v = (v_1, v_2, v_3)$, then $v \notin L_1 \cup L_2$ if and only if $v_1 v_2 \ne 0$. Hence $v \in Q = Q(\mu_0)$, where

$$\mu_0 = \frac{\delta^2 - \sigma_3^2 + 2\sigma_1 v_1 + 2\sigma_2 v_2 + 2\sigma_3 v_3 - v_1^2 - v_2^2 - v_3^2}{v_1 v_2}.$$

Next, we observe that $L \cap \operatorname{bd} K \subset Q$. Indeed, a planar quadric curve is uniquely determined by any five points which do not belong to a line (see, e. g., [11, pp. 395–397]). Hence the ellipse $L \cap \operatorname{bd} K$ is uniquely determined by the five-point set $\{v\} \cup (E_1 \cap L) \cup (E_2 \cap L)$. Since $L \cap Q$ is a quadric curve containing $\{v\} \cup (E_1 \cap L) \cup (E_2 \cap L)$, one has $L \cap \operatorname{bd} K = L \cap Q \subset Q$.

Slightly rotating L about the line $\langle o, v\rangle$, we obtain a family of ellipses $L \cap \operatorname{bd} K$ that cover an open subset V of $\operatorname{bd} K$, which consists of two open "lenses" with a common endpoint v. As above, $V \subset Q$. Repeating this argument for the points $w \in V \cap Q$ with $\langle o, w\rangle$ sufficiently close to l, we obtain that both endpoints q_1 and q_2 of the line segment $K \cap l$ are interior to an open set $W \subset \operatorname{bd} K$ such that $W \cap Q = W$.

Finally, to show the inclusion $\operatorname{bd} K \subset Q$, choose any point $x \in \operatorname{bd} K$ and denote by N the 2-dimensional subspace through $\{x\} \cup l$. Since the quadric curve $N \cap Q$ and the convex quadric curve $N \cap \operatorname{bd} K$ coincide along the non-collinear set $N \cap W$ and are uniquely determined by this set, one has $N \cap \operatorname{bd} K \subset N \cap Q$. Varying N about l, we conclude that $\operatorname{bd} K \subset Q$. Since Q is locally convex at any point $x \in \operatorname{bd} K$, Theorem 3 implies that $\operatorname{bd} K$ is a convex quadric.

Let $n \geq 4$. As above, we assume that $o \in \operatorname{int} K$. To prove that $\operatorname{bd} K$ is a convex quadric in $\mathbb{R}^n$, it suffices to show that the intersection of $\operatorname{bd} K$ with any 2-dimensional subspace $L \subset \mathbb{R}^n$ is a convex quadric curve (see statement (A) from the introduction). Choose a vector $e \in T \setminus L$ and put $M = \operatorname{span}(e \cup L)$. Then M is a 3-dimensional subspace of $\mathbb{R}^n$. Since the set $T \cap M$ is open in $\mathbb{S}^{n-1} \cap M$, there is a scalar $\varepsilon > 0$ such that any 2-dimensional subspace N of M that forms with $\langle o, e\rangle$ an angle of size less than ε intersects $T \cap M$. By condition 2), $N \cap \operatorname{bd} K$ is a convex quadric curve. From the case $n = 3$ it follows that $M \cap \operatorname{bd} K$ is a 3-dimensional convex quadric. Hence $L \cap \operatorname{bd} K$ $(= L \cap M \cap \operatorname{bd} K)$ is a convex quadric curve. Therefore $\operatorname{bd} K$ is a convex quadric.

6 Proof of Theorem 1

Since Lemma 3 shows that 2) $\Rightarrow$ 1), it remains to prove the converse assertion. In what follows, given a vector $e \in T$, denote by $H(e)$ a hyperplane that contains the middle points of all chords of K which are parallel to e.

First, we consider the case $n = 2$. Choose a vector $e_0 \in T$ and a chord $[p_0, q_0]$ of K in direction e_0. Then $[p_0, q_0]$ cuts K into two planar convex solids, say K_0 and K_0'. If both K_0 and K_0' are unbounded, then, as easily seen, K is a closed slab between a pair of parallel lines, which implies that $\operatorname{bd} K$ is a degenerate convex quadric. Assume that at least one of the sets K_0 and K_0' is bounded. Denote by P a closed halfplane of $\mathbb{R}^2$ determined by the line $\langle p_0, q_0\rangle$ for which $K \cap P$ is bounded. Let e_m, $m \geq 1$, be the unit vector forming with e_0 an angle of size π/m such that the chord $[p_0, q_1(m)]$ of K in direction e_m lies in P. Clearly, there is a positive integer m_0 with the property that $e_m \in T$ for all $m \geq m_0$. Denote by $p_1(m)$ and $q_2(m)$ the points in $P \cap \mathrm{bd}\, K$ so that $[p_1(m), q_1(m)]$ and $[p_1(m), q_2(m)]$ have directions e_0 and e_m, respectively. By the assumption, $H(e_0)$ contains the middle points of the chords $[p_0, q_0]$ and $[p_1(m), q_1(m)]$, while $H(e_m)$ contains the middle points of the chords $[p_0, q_1(m)]$ and $[p_1(m), q_2(m)]$.

Since the set

$$X_5(m) = \{p_0, q_0, p_1(m), q_1(m), q_2(m)\}$$

does not belong to a line, there is a unique quadric curve $Q(m)$ containing $X_5(m)$ (see, e. g., [11, pp. 395–397]). If a point $q_k(m), k \geq 2$, is chosen and the line through $q_k(m)$ in direction e_0 intersects $H(e_0) \cap K$, then denote by $p_k(m)$ the point in $\mathrm{bd}\, K$ for which the line segment $[p_k(m), q_k(m)]$ has direction e_0. Similarly, if a point $p_k(m), k \geq 2$, is chosen and the line through $p_k(m)$ in direction e_m intersects $H(e_m) \cap K$, then denote by $q_{k+1}(m)$ the point in $\mathrm{bd}\, K$ for which $[p_k(m), q_{k+1}(m)]$ has direction e_m. By Lemma 3 and condition 1) of the theorem, the set

$$X_{2k+1}(m) = \{p_0, q_0, p_1(m), q_1(m), \ldots, p_k(m), q_k(m), q_{k+1}(m)\}$$

belongs to $Q(m) \cap \mathrm{bd}\, K$. Clearly, there is an increasing sequence of positive integers $k(m)$, $m \geq m_0$, such that $X_{2k(m)+1}(m)$ exists and the sequence of sets

$$X_{2k(m_0)+1}(m_0), X_{2k(m_0+1)+1}(m_0 + 1), \ldots,$$

tends to a dense subset of $P \cap \mathrm{bd}\, K$. Hence the arcs $P \cap Q(m_0), P \cap Q(m_0 + 1), \ldots$ converge to $P \cap \mathrm{bd}\, K$, which shows that the arc $P \cap \mathrm{bd}\, K$ is a piece of a quadric curve. Continuously translating $[p_0, q_0]$ away from P, we express $\mathrm{bd}\, K$ as the union of an increasing sequence of convex quadrics, implying that $\mathrm{bd}\, K$ is itself a convex quadric.

Let $n \geq 3$. Choose a point $p \in \mathrm{int}\, K$, and let L be a 2-dimensional plane through p which properly intersects K such that the subspace $L - p$ meets T. Then $L \cap T$ is an open subset of $L \cap (\mathbb{S}^{n-1} \setminus (\mathrm{rec}\, K \cup -\mathrm{rec}\, K))$. If $e \in L \cap (\mathbb{S}^{n-1} \setminus (\mathrm{rec}\, K \cup -\mathrm{rec}\, K))$, then, by condition 2) of the theorem, the middle points of all chords of K in direction e belong to a hyperplane $H(e)$. Clearly, $L \cap H(e)$ is a line in L such that the middle points of all chords of $K \cap L$ in direction e belong to $L \cap H(e)$. By the proved above, $L \cap \mathrm{bd}\, K$ is a convex quadric curve. Theorem 4 implies that $\mathrm{bd}\, K$ is a convex quadric.

7 Proof of Theorem 2

2) $\Rightarrow$ 1) Translating K on $-p$, we may assume that $p = o$. Denote by h is a chord of K which contains o. Then h is parallel to a unit vector $e \in \mathbb{S}^{n-1} \setminus (\mathrm{rec}\, K \cup -\mathrm{rec}\, K)$. If $\Omega_\delta(p)$ is the neighborhood of $\mathrm{bd}\, K \setminus \big((p + \mathrm{rec}\, K) \cup (p - \mathrm{rec}\, K)\big)$ in $\mathrm{bd}\, K$ that lies in a convex quadric, Q, then the cylinder $C_\delta(h)$ of radius δ centered about the line $\langle o, e\rangle$ intersects $\mathrm{bd}\, K$ within Q. By Lemma 3, the middle points of all chords from $\mathcal{F}_\delta(h)$ belong to a hyperplane.

1) $\Rightarrow$ 2) As above, we can reduce our consideration to the case when $p = o$. Furthermore, we may suppose that K is a line-free. Indeed, assume that $\dim(\mathrm{lin}\, K) \geq 1$.

Choose a chord h of K that contains o. Let $M \subset \mathbb{R}^n$ be a subspace which is complementary to $\operatorname{lin} K$ and contains h. Put $K' = M \cap K$. Clearly, $\operatorname{lin} K' = M \cap \operatorname{lin} K = \{o\}$. If H is a hyperplane that contains the middle points of all chords from $\mathcal{F}_\delta(h)$, then $H \cap M$ contains the middle points of these chords that lie in M. So, if we prove the existence of the neighborhood $\Omega'_\delta(o)$ of the set $\operatorname{rbd} K' \setminus (\operatorname{rec} K' \cup -\operatorname{rec} K')$ in $\operatorname{rbd} K'$ which lies in a convex quadric $Q' \subset M$, then from the equality $\operatorname{bd} K = \operatorname{lin} K \oplus \operatorname{rbd} K'$ we will conclude that the neighborhood $\Omega_\delta(o)$ of $\operatorname{bd} K \setminus (\operatorname{rec} K \cup -\operatorname{rec} K)$ in $\operatorname{bd} K$ lies in the convex quadric $\operatorname{lin} K \oplus Q'$.

First, we consider the case $n = 2$. Choose a chord $h = [p_0, q_0]$ of K that contains o and denote by e_0 the unit vector which is a positive scalar of $q_0 - p_0$. As above, $C_\delta(h)$ stands for the closed slab of $\mathbb{R}^2$ of width 2δ centered about the line $\langle p_0, q_0\rangle$. Denote by e_m, $m \ge 1$, the unit vector forming with e_0 an angle of size π/m. Clearly, there is a positive integer m_0 with the property that both chords $[p_0, q_1(m)]$ and $[p_{-1}(m), q_0]$ of K in direction e_m lie within $C_\delta(h)$ for all $m \ge m_0$.

Denote by $p_1(m)$, $m \ge m_0$, the point in $C_\delta(h) \cap \operatorname{bd} K$ so that $[p_1(m), q_1(m)]$ has direction e_0. By condition 1), there is a line $H(e_0)$ which contains the middle points of the chords $[p_0, q_0]$ and $[p_1(m), q_1(m)]$. Similarly, there is a line $H(e_m)$ containing the middle points of the chords $[p_{-1}, q_0(m)]$ and $[p_0, q_1(m)]$.

Since the set

$$Y_5(m) = \{p_0, q_0, p_{-1}(m), p_1(m), q_1(m)\}$$

does not belong to a line, there is a unique quadric curve $Q(m)$ containing $Y_5(m)$ (see, e. g., [11, pp. 395–397]). If a point $q_k(m), k \ge 2$, is chosen in $C_\delta(h) \cap \operatorname{bd} K$ and the line through $q_k(m)$ in direction e_0 intersects $H(e_0) \cap K$, then let $p_k(m)$ be the point in $C_\delta(h) \cap \operatorname{bd} K$ for which the line segment $[p_k(m), q_k(m)]$ has direction e_0. If a point $p_k(m), k \ge 2$, is chosen in $C_\delta(h) \cap \operatorname{bd} K$ and the line through $p_k(m)$ in direction e_m intersects both $H(e_m) \cap K$ and $C_\delta(h) \cap \operatorname{bd} K$, then denote by $q_{k+1}(m)$ the point in $C_\delta(h) \cap \operatorname{bd} K$ for which $[p_k(m), q_{k+1}(m)]$ has direction e_m.

Similarly, if a point $p_{-k}(m), k \ge 1$, is chosen in $C_\delta(h) \cap \operatorname{bd} K$ and the line through $p_{-k}(m)$ in direction e_0 intersects $H(e_0) \cap K$, then denote by $q_{-k}(m)$ the point in $C_\delta(h) \cap \operatorname{bd} K$ for which the line segment $[p_{-k}(m), q_{-k}(m)]$ has direction e_0. If a point $q_{-k}(m), k \ge 1$, is chosen in $C_\delta(h) \cap \operatorname{bd} K$ and the line through $q_{-k}(m)$ in direction e_m intersects both $H(e_m) \cap K$ and $C_\delta(h) \cap \operatorname{bd} K$, then denote by $p_{-k-1}(m)$ the point in $C_\delta(h) \cap \operatorname{bd} K$ for which $[p_{-k-1}(m), q_{-k}(m)]$ has direction e_m.

By Lemma 3 and condition 1) of the theorem, the set

$$\begin{aligned} Y_{2k+2}(m) = \{&p_0, q_0, p_1(m), q_1(m), \dots, p_k(m), q_k(m),\\ &p_{-1}(m), q_{-1}(m), \dots, p_{-k}(m), q_{-k}(m)\} \end{aligned}$$

belongs to $Q(m) \cap C_\delta(h) \cap \operatorname{bd} K$. Clearly, there is an increasing sequence of positive integers $k(m)$, $m \ge m_0$, such that $Y_{2k(m)+2}(m)$ exists for all $m \ge m_0$, and the sets

$$Y_{2k(m_0)+2}(m_0), Y_{2k(m_0+1)+2}(m_0+1), \dots,$$

tend to a dense subset of $C_\delta(h) \cap \operatorname{bd} K$. Hence the curves

$$C_\delta(h) \cap Q(m_0),\ C_\delta(h) \cap Q(m_0+1), \dots$$

converge to $C_\delta(h) \cap \mathrm{bd}\, K$, which shows that $C_\delta(h) \cap \mathrm{bd}\, K$ is a piece of a quadric curve (consisting of one or two arcs). Continuously rotating h about o, we cover $\mathrm{bd}\, K \setminus (\mathrm{rec}\, K \cup -\mathrm{rec}\, K)$ by the family of overlapping pieces $C_\delta(h) \cap \mathrm{bd}\, K$ of the same quadric curve. Hence the neighborhood $\Omega_\delta(o)$ of $\mathrm{bd}\, K \setminus (\mathrm{rec}\, K \cup -\mathrm{rec}\, K)$ in $\mathrm{bd}\, K$ lies in a convex quadric.

Let $n \geq 3$. Choose any 2-dimensional subspace L such that $L \cap K$ is bounded (this is possible since K is assumed to be line-free). Then $\mathrm{rec}\,(L \cap K) = \{o\}$. If h is a chord of $L \cap K$ and $H \subset \mathbb{R}^n$ is a hyperplane containing the middle points of all chords of K which are parallel to h and lie within the cylinder $C_\delta(h)$, then $C_\delta(h) \cap L$ is a slab of width 2δ centered about the line containing h and $L \cap H$ is a line that contains the middle points of all chords of $L \cap K$ that belong to $\mathcal{F}_\delta(h)$. Hence $L \cap K$ satisfies condition 1) of the theorem (with L instead of $\mathbb{R}^n$) By the proved above (see the case $n = 2$), $\mathrm{rbd}\,(L \cap K)$ is a convex quadric; so, it is an ellipse.

Because the argument above holds for any choice of a 2-dimensional subspace L, the set $\mathrm{bd}\, K \setminus (\mathrm{rec}\, K \cup -\mathrm{rec}\, K)$ lies in a convex quadric Q (see assertion (B) from the introduction). If K is bounded, then $\mathrm{rec}\, K = \{o\}$ and the whole hypersurface $\mathrm{bd}\, K$ is a convex quadric. Assume that K is unbounded and choose a halfline t with endpoint o that lies in $\mathrm{int}\, K$. Then (see the case $n = 2$) for any 2-dimensional subspace L that contains t, the neighborhood $\Omega_\delta(o)$ of $(L \cap \mathrm{bd}\, K) \setminus (\mathrm{rec}\, K \cup -\mathrm{rec}\, K)$ in $\mathrm{rbd}\,(L \cap K)$ lies in $L \cap Q$. Therefore the neighborhood $\Omega_\delta(o)$ of $\mathrm{bd}\, K \setminus (\mathrm{rec}\, K \cup -\mathrm{rec}\, K)$ in $\mathrm{bd}\, K$ lies in Q.

8 Proof of Theorem 6

The proof is organized by induction on $n\,(\geq 3)$. Let $n = 3$. Consider the 1-dimensional subspace $l = L_1 \cap L_2$ and choose a longest chord $[x, z]$ of K in direction l. Translating K on a suitable vector, we may suppose that the origin o of $\mathbb{R}^3$ is the middle point of $[x, z]$. By the assumption, both sections $E_1 = L_1 \cap \mathrm{bd}\, K$ and $E_2 = L_2 \cap \mathrm{bd}\, K$ are ellipses. Due to the choice of $[x, z]$, there are parallel planes M_x and M_z both supporting K such that $K \cap M_x = \{x\}$ and $K \cap M_z = \{z\}$ (see, e. g., [14]). Applying a suitable linear transformation, we may suppose that (i) L_1 and L_2 are orthogonal to each other, (ii) both ellipses E_1 and E_2 are circumferences with diameter $[x, z]$, (iii) both planes M_x and M_z are orthogonal to $[x, z]$. Clearly, the image of K under this transformation still satisfies the hypothesis of the theorem.

Choosing suitable Cartesian coordinates ξ_1, ξ_2, ξ_3 for $\mathbb{R}^3$, we may consider that x shows a positive direction of the ξ_3-axis and

$$E_1 = \{(\xi_1, 0, \xi_3) : \xi_1^2 + \xi_3^2 = \rho^2\}, \quad E_2 = \{(0, \xi_2, \xi_3) : \xi_2^2 + \xi_3^2 = \rho^2\},$$

where $\rho = \|x\|$. Clearly, L_1 and L_2 are given by the equations $\xi_2 = 0$ and $\xi_1 = 0$, respectively. Furthermore, M_x and M_z are described by $\xi_3 = \rho$ and $\xi_3 = -\rho$.

Choose a point $v \in (L_3 \cap \mathrm{bd}\, K) \setminus (L_1 \cup L_2)$ and consider the family of quadrics $Q(\mu)$ defined by

$$\xi_1^2 + \xi_2^2 + \xi_3^2 + \mu \xi_1 \xi_2 - \rho^2 = 0,$$

where μ is a scalar parameter. Clearly, $E_i = L_i \cap Q(\mu)$, $i = 1, 2$, for all $\mu \in \mathbb{R}$. If $v = (v_1, v_2, v_3)$, then $v \notin L_1 \cup L_2$ if and only if $v_1 v_2 \neq 0$. Hence $v \in Q = Q(\mu_0)$, where

$$\mu_0 = \frac{\rho^2 - v_1^2 - v_2^2 - v_3^2}{v_1 v_2}.$$

Since v lies within the slab $-\rho \leq \xi_3 \leq \rho$ and does not belong to the interior of $\operatorname{conv}(E_1 \cup E_2)$, the quadric Q is either a cylinder or an ellipsoid.

We state that the ellipse $E_3 = L_3 \cap \operatorname{bd} K$ is symmetric about o and lies in Q. Indeed, if L_3 contains $[x, z]$, then $[x, z]$ is the longest diameter of E_3, which shows that E_3 is uniquely determined by $[x, z]$ and v. In particular, E_3 is symmetric about o. Since $L_3 \cap Q$ is an ellipse containing $\{v, x, z\}$ and supported by both planes M_x and M_z, we conclude that $E_3 = L_3 \cap Q$. If L_3 does not contain $[x, z]$, then L_3 meets each of E_1, E_2 at a pair of points symmetric about o. Because E_3 is uniquely determined by v and the four points of intersection with $E_1 \cup E_2$, the ellipse E_3 is symmetric about o and lies in Q.

Considering separately the cases $l \subset L_3$ and $l \not\subset L_3$, we observe that a certain plane $u_0 + L_4$, $u_0 \in \operatorname{bd} K$, intersects the union $E_1 \cup E_2 \cup E_3$ at precisely six points, which do not belong to a line. Since the ellipse $E_4(u_0) = (u_0 + L_4) \cap \operatorname{bd} K$ is uniquely determined by these six points and since $(u_0 + L_4) \cap Q$ is also an ellipse determined by these points, one has $E_4(u_0) \subset Q$. By continuity, there is a small neighborhood U of u_0 such that the argument above holds for all $u \in U$. Clearly, the ellipses $E_4(u)$, $u \in U$, cover an open "belt" Ω of $\operatorname{bd} K$ which lies in Q. Repeating this consideration for the subspace L_1 and all points $u \in \Omega$, we obtain a wider "belt" of $\operatorname{bd} K$ which also lies in Q. Since the whole $\operatorname{bd} K$ can be expressed as the union of an increasing sequence of such "belts" obtained by the alternating consideration of translates of L_1 and L_2, we conclude that $\operatorname{bd} K \subset Q$. Therefor Q is a bounded convex quadric; that is, $\operatorname{bd} K = Q$ is an ellipsoid.

Let $n \geq 4$. Assume that the theorem is true for all $\mathbb{R}^m$, $m \leq n-1$, and let $K \subset \mathbb{R}^n$ be a convex body which satisfies its hypothesis. Translating K on a suitable vector, we may suppose that $o \in \operatorname{int} K$. Choose an $(n-1)$-dimensional subspace $P \subset \mathbb{R}^n$ such that the $(n-2)$-dimensional subspaces $P \cap L_i$, $i = 1, 2, 3, 4$, are pairwise distinct. From the hypothesis it follows that all proper sections of $P \cap \operatorname{bd} K$ by translates of the subspaces $P \cap L_i$, $i = 1, 2, 3, 4$, within P are $(n-2)$-dimensional ellipsoids. The inductive assumption gives that $P \cap \operatorname{bd} K$ is an $(n-1)$-dimensional ellipsoid. Because the family of $(n-1)$-dimensional subspaces $P \subset \mathbb{R}^n$ with the property

$$P \cap L_i \neq P \cap L_j, \quad i \neq j, \quad i, j \in \{1, 2, 3, 4\},$$

is dense in the family of all $(n-1)$-dimensional subspaces of $\mathbb{R}^n$, we obtain, by continuity, that every section of $\operatorname{bd} K$ by an $(n-1)$-dimensional subspace is an $(n-1)$-dimensional ellipsoid. This implies that $\operatorname{bd} K$ is an ellipsoid itself (see [5]).

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On the Abstract Čech Cohomology

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Abstract. The present Note is a survey of the authors' papers [11,12,14,16–18], concerning the introduction and study of the notion of the abstract Čech cohomology, as well as its applications. Here we have investigated: projective systems, injective systems, covering of a directed partially ordered set, abstract Čech cohomology, abstract Čech homology, Čech cohomology space, simplicial projective systems, de Rham cohomology space of projective systems, J-resolution of a projective system and acyclic resolution.

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1 Introduction

The Čech cohomology is a cohomology theory based on the intersection properties of open covers of a topological space. It is named for the mathematician Eduard Čech who in 1932 introduced it [2]. Let X be a topological space, and let $\mathcal{U}$ be an open cover of X. Define a simplicial complex $N(\mathcal{U})$ called the nerve of the covering, as follows: the vertices of $N(\mathcal{U})$ are all elements of $\mathcal{U}$, each pair $U_1, U_2 \in \mathcal{U}$ such that $U_1 \cap U_2 \neq \emptyset$ determines one edge, in general, there is one q-simplex for each $(q+1)$-element subset $\{U_0, ..., U_q\}$ for which $U_0 \cap ... \cap U_q \neq \emptyset$. Geometrically, the nerve $N(\mathcal{U})$ is essentially a "dual complex" (in the sense of a dual graph, or Poincaré duality) for the covering $\mathcal{U}$. The idea of Čech cohomology is that, if we choose a cover $\mathcal{U}$ consisting of sufficiently small, connected open sets, the resulting simplicial complex should be a good combinatorial model for the space X. For such a cover, the Čech cohomology of X is defined to be the simplicial cohomology of the nerve. This idea can be formalized by the notion of a *good cover*, for which every open set and every finite intersection of open sets is contractible. However, a more general approach is to take the direct limit of the cohomology groups of the nerve over the system of all possible open covers of X, ordered by refinement. For a more precise description see [20], Chap. 6, Sec. 7. Let X be a topological space, and let $\mathcal{F}$ be a presheaf of abelian groups on X. Let $\mathcal{U}$ be an open cover of X. A q-simplex σ of $\mathcal{U}$ is an ordered collection of $q+1$ sets chosen from $\mathcal{U}$ such that the intersection of all these sets is non-empty. This intersection is called the *support* of σ and is denoted $|\sigma|$. Now let $\sigma = (U_0, ..., U_q)$ be such a q-simplex. The j-th partial boundary of σ is defined to be the $(q-1)$-simplex obtained by removing the j-th set from σ, that is: $\partial_j\sigma := (U_i)_{i\in\{0,\ldots,q\},i\neq j}$. The *boundary* of σ is defined as the alternating sum of the partial boundaries: $\partial\sigma := \sum\limits_{j=0}^{q}(-1)^{j+1}\partial_j\sigma$. A q-cochain of $\mathcal{U}$ with coefficients in $\mathcal{F}$ is a map which associates to each q-simplex σ an element of $\mathcal{F}(|\sigma|)$ and we denote the set of all q-cochains of $\mathcal{U}$ with coefficients in $\mathcal{F}$ by $C^q(\mathcal{U},\mathcal{F})$. $C^q(\mathcal{U},\mathcal{F})$ is an abelian group by pointwise addition. The cochain groups can be made into a cochain complex $(C^{\cdot}(\mathcal{U},\mathcal{F}),\delta)$ by defining a coboundary operator (also called codifferential)

$$\delta_q : C^q(\mathcal{U},\mathcal{F}) \to C^{q+1}(U,F) : \omega \to \delta_q\omega, (\delta_q\omega)(\sigma) := \sum_{j=0}^{q+1}(-1)^j res_{|\sigma|}^{|\partial_j\sigma|}\omega(\partial_j\sigma),$$

(where $res_{|\sigma|}^{|\partial_j\sigma|}$ is the restriction morphism from $\mathcal{F}(|\partial_j\sigma|)$ to $\mathcal{F}(|\sigma|)$), and showing that $\delta^2 = 0$ (i.e., $\delta^{q+1}\circ\delta^q = 0$). A q-cochain is called a *q-cocycle* if it is in the kernel of δ_q and $Z^q(\mathcal{U},\mathcal{F}) := ker(\delta_q)$ is the set of all q-cocycles. Thus a q-cochain ω is a cocycle if for any $(q+1)$-simplex σ the cocycle condition $\sum\limits_{j=0}^{q+1}(-1)^j res_{|\sigma|}^{|\partial_j\sigma|}\omega(\partial_j\sigma) = 0$ holds. For example, ω is a 1-cocycle if $\forall A,B,C\in\mathcal{U}$, $\omega(B\cap C)|_{A\cap B\cap C} - \omega(A\cap C)|_{A\cap B\cap C} + \omega(A\cap B)|_{A\cap B\cap C} = 0$, where, for $U'\subset U''$, $\omega(U'')|_{U'}$ denotes $res_{U'}^{U''}$.

A q-cochain is called a *q-coboundary* if it is in the image of δ_{q-1} and $B^q(\mathcal{U},\mathcal{F})$ is the set of all q-coboundaries. For example, a 1-cochain ω is a 1-coboundary if there exists a 0-cochain ϖ such that $\forall A,B\in\mathcal{U}$, $\omega(A\cap B) = \delta_0(\varpi)(A\cap B) = \varpi(A)|_{A\cap B} - \varpi(B)|_{A\cap B}$.

The Čech cohomology of $\mathcal{U}$ with values in $\mathcal{F}$ is defined to be the cohomology of the cochain complex $(C^{\cdot}(\mathcal{U},\mathcal{F}),\delta)$. Thus the q-th Čech cohomology is given by

$$\check{H}^q(\mathcal{U},\mathcal{F}) := H^q((C^{\cdot}(\mathcal{U},\mathcal{F}),\delta)) = Z^q(\mathcal{U},\mathcal{F})/B^q(\mathcal{U},\mathcal{F}).$$

The Čech cohomology of X is defined by considering refinements of open covers. If $\mathcal{V}$ is a refinement of $\mathcal{U}$ then there is a map in cohomology $\check{H}^*(\mathcal{U},\mathcal{F}) \to \check{H}^*(\mathcal{V},\mathcal{F})$. The open covers of X form a directed set under refinement, so the above map leads to a direct system of abelian groups. The Čech cohomology of X with values in $\mathcal{F}$ is defined as the direct limit $\check{H}^*(X,\mathcal{F}) = \varinjlim\limits_{\mathcal{U}} \check{H}^*(\mathcal{U},\mathcal{F})$ of this system. Actually, the original Čech cohomology of X with coefficients in a fixed abelian group A, denoted $\check{H}^*(X;A)$, is defined as $\check{H}^*(X,\mathcal{F}_A)$ where $\mathcal{F}_A$ is the constant sheaf on X determined by A.

An excellent presentation of Čech cohomology was made by Kostake Teleman in [21] (Chp. II, Sect. 18). Probably it was one of the reasons why his book was translated in German and Russian, shortly after appearing in Romanian (see [Zbl 018953902], [Zbl 018953904]). In particular, K. Teleman proved that if X is homotopy equivalent to a CW-complex, then the Čech cohomology $\check{H}^*(X;A)$ is naturally isomorphic to the singular cohomology $H^*(X;A)$. (For an arbitrary space X this fact is false: if X is the closed topologist's sine curve, then $\check{H}^0(X;\mathbb{Z}) = \mathbb{Z}$, whereas $H^0(X;\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}$). Also, K. Teleman proved that if $\mathcal{S}$ is a locally finite simplicial polyhedron, then the singular cohomology $H^*(\mathcal{S};\mathbb{Z})$ and Čech cohomology $\check{H}^*(\mathcal{S};\mathbb{Z})$ are isomorphic with the cohomology $\check{H}^*(\Sigma,\mathbb{Z})$, associated to the cover Σ of $\mathcal{S}$ with stellar neighborhoods.

If X is a differentiable manifold and the cover $\mathcal{U}$ of X is a good "cover" (i.e., all the sets $U \in \mathcal{U}$ are contractible to a point, and all finite intersections of sets in $\mathcal{U}$ are either empty or contractible to a point), then $\check{H}^*(X;\mathbb{R})$ is isomorphic to the de Rham cohomology.

If X is compact Hausdorff, then Čech cohomology (with coefficients in a discrete group) is isomorphic to Alexander-Spanier cohomology.

In the articles [5] and [6], René Deheuvels develops a theory of homology of ordered sets which is a generalization of the Čech homology and cohomology. This author starts with an Abelian category C, with products and enough injectives, and with an ordered set $\mathcal{E}$. He considers $\mathcal{E}$, as a category in the usual way, having the objects all elements of $\mathcal{E}$ and a morphism $a_1 \to a_2$ being a relation $a_1 < a_2$. He denotes by $C(\mathcal{E})$ the category of covariant functors from the category $\mathcal{E}$ to the category C. Let $C^\cdot$ be the category of cochain complexes in the category C and let the functor $C^\cdot(\mathcal{E},-) : C(\mathcal{E}) \to C^\cdot$ be defined by: $C^n(\mathcal{E},A) = \prod_{a_0>\ldots>a_n, a_i\in\mathcal{E}} A(a_0)$, $d : C^n(\mathcal{E},A) \to C^{n+1}(\mathcal{E},A)$, where if $f = (f_{a_0>\ldots>a_n}) \in C^n(\mathcal{E},A)$, $d(f)_{a_0>\ldots>a_{n+1}} = \eta^{a_1}_{a_0}(f_{a_1>\ldots>a_{n+1}}) + \sum_{i=1}^{n+1} f_{a_0>\ldots>\hat{a}_i>\ldots>a_{n+1}}$, $\eta^{a_1}_{a_0}$ being the morphism $A(a_1) \to A(a_0)$ corresponding to the relation $a_1 < a_0$. The author proves that this functor is a resolving functor, i.e., for every n there is a canonical isomorphism $R^n\Gamma_{\mathcal{E}} \simeq H^n(C^\cdot(\mathcal{E},-))$, where $\Gamma_{\mathcal{E}}$ is the inverse limit functor $\Gamma_{\mathcal{E}} := \varprojlim_{\mathcal{E}}$. The construction and this result have obvious duals. Let M be a "schéma simplicial" and let E be the set of simplices of M ordered by inclusion. If A is a constant functor then the author proves that $R^n\Gamma_{\mathcal{E}}(A)$ is isomorphic to the usual simplicial cohomology of M with coefficients in A. To define a generalized Čech cohomology and homology theory, Deheuvels introduced the notion of "order" of the ordered set $\mathcal{E}$ in the ordered set $\mathcal{E}'$. This is a function ρ in $\mathcal{E}$ with values in the set of subsets of $\mathcal{E}'$ such that if $a_2 \le a_1$ then $\rho(a_2) \supseteq \rho(a_1)$ and if $a_2' < a_1', a_2' \in \rho(a)$, then $a_1' \in \rho(a)$. Given an "order" ρ of $\mathcal{E}$ in $\mathcal{E}'$, the author defines a functor $\rho^{-1} : C(\mathcal{E}') \to C(\mathcal{E}^*)$, where $\mathcal{E}^*$ is the dual of $\mathcal{E}$, by $\rho^{-1}(A')(a) = \Gamma_{\rho(a)}(A')$. Modulo technicalities the generalized Čech cohomology in the sense of Deheuvels is now the hyperderived of the composed functor $L_{\mathcal{E}^*}\rho^{-1}$ with $L_{\mathcal{E}} = \varinjlim_{\mathcal{E}}$. The dual construction yields a generalized Čech homology theory. Let X be a topological space, let $\mathcal{E}$ be the dual of the set of non-empty open sets of X ordered by inclusion, and let $\mathcal{D}$ be the ordered set of all open coverings R of X. It is supposed that if $O \subseteq O'$ and $O' \in R$ then $O \in R$. The category $C(\mathcal{E})$ is then the category of presheaves on X. The "order" ρ of $\mathcal{D}$ in $\mathcal{E}$ is defined by $\rho(R) = R$. The corresponding generalized Čech cohomology in the Deheuvels sense is then shown to coincide with the usual Čech cohomology, at least when C has exact inductive limits. If X is a compact metric space and C is the category of abelian groups the author shows that the generalized Čech homology coincides with Steenrod homology theory.

We can see from this summary of the work [6] of René Deheuvels that this theory is indeed a very consistent generalization of the homology and cohomology Čech theories. But at the same time, it is clear that Deheuvels's theory is very sophisticated and difficult to apply and to find other examples. In addition, even the construction of this theory is very little similar to the construction of the Čech theory. This is the reason for which in 1974 the first and the third author proposed a theory of *abstract Čech cohomology* in [11] and [12], and not a generalization of the Čech theory as constructed Deheuvels, but simply following the Čech's construction. It is more easily applicable in other important situations. In the third chapter of the book [12], entitled "Simplicial complexes. Abstract Čech cohomology", this theory is developed in detail as follows: §3 Cohomology groups associated with a projective system or Čech abstract cohomology groups, §4 The exact cohomology sequence associated with a pair of projective systems, §5 Canonical projective systems, §6 Resolutions of a projective system, §7. Homology groups associated with an injective system. And in Chap. VII, the Čech homology and cohomology for a topological space are obtained as a "concretization" of the abstract Čech homology and cohomology. Then, in the thesis of the second author [15] and in his papers [14, 16, 17], as well as in the paper [18] of the third author, a number of examples and applications are given.

The present article is a synthesis paper including the results of the three authors about the abstract Čech homology and cohomology.

Finally, the authors wish to emphasize that they are impressed by recent research concerning multy-ary relations homology, studied by Academician Petru Soltan in [19] and [1]. They believe that this subject can be expanded by using abstract Čech (co)homology, as well as the theory of abstract Čech (co)homology can find one new, interesting and important application in the above mentioned field investigated by Academician Petru Soltan.

2 The authors' construction of the abstract Čech cohomology

Let $\mathcal{P} = (H_i, \alpha^i_j)_{i,j\in I}$ be a projective system of abelian groups. For our purpose we suppose that the partially ordered set $(I, \leq)$ of the indices over which the projective system $\mathcal{P}$ (or an inductive system $\mathcal{I}$) is given fulfills the following conditions:

(1) For every pair $i, j \in I$ there exists infimum $\inf(i, j)$, which is denoted by $i \wedge j$;

(2) For every subset J of I there exists $\sup J$, the supremum with respect to the relation $\leq$;

(3) There exists a minimal element $\theta \in I$, i.e., $\theta \leq i$ for every $i \in I$, (but this condition is not essential).

Definition 1. A subset J of I is called a *covering* of $(I, \leq)$ if for every $i \in I$ there exists $J_i \subset J$ such that $i \leq \sup J_i$.

In the set of coverings of $(I, \leq)$ a partial ordering can be introduced, namely, if J, J' are coverings of $(I, \leq)$, then $J' \prec J$ if for every $i' \in J'$ there exists an $i \in J$

such that $i' \leq i$.

In order to define the cohomology groups of a projective system $\mathcal{P} = (H_i, \alpha_j^i)_{i,j\in I}$, also we assume the condition

(4) The set of coverings of $(I, \leq)$ is directed with respect to the relation $\prec$, i.e., if J, J' are two arbitrary coverings of $(I, \leq)$, then there exists a covering J'' of $(I, \leq)$ such that $J'' \prec J$ and $J'' \prec J'$.

In these conditions ((1)-(4)) on the ordered set $(I, \leq)$, for a covering J of $(I, \leq)$ we can consider the cochain complex

$$C^*(J, \mathcal{P}) : ... \to C^q(\mathcal{J}, P) \xrightarrow{d^q} C^{q+1}(J, \mathcal{P}) \to ...,$$

where $C^q(J, \mathcal{P}) := \prod\limits_{i_0,...,i_q \in J; i_0\wedge...\wedge i_q \neq \theta} H_{i_0\wedge....\wedge i_q}$, and the boundary homomorphism d^q is defined by

$$(d^q t)_{i_0...i_{q+1}} = \sum_{p=0}^{q+1} (-1)^p \alpha_{i_0\wedge....\wedge i_{q+1}}^{i_0\wedge...\wedge\widehat{i_p}\wedge...\wedge i_{q+1}} t_{i_0\wedge...\wedge\widehat{i_p}\wedge...\wedge i_{q+1}}$$

for $t \in C^q(J, \mathcal{P})$.

The cohomology groups of this cochain complex are denoted by $\{(H^q(J, P)\}_q$. If J' is another covering of $(I, \leq)$ such that $J' \prec J$, one obtains, for every $q \geq 0$, a well defined homomorphism $\alpha^q_{J'\prec J} : H^q(J, \mathcal{P}) \to H^q(J', \mathcal{P})$ such that $\{H^q(J, P), \alpha^q_{J',J}\}_{J,J'\in\mathcal{A}_{(I,\leq)}}$ is an inductive system over the set $(\mathcal{A}_I, \prec)$ of all coverings of $(I, \leq)$.

In the imposed conditions there exists $\varinjlim\limits_{J\in\mathcal{A}_{(I,\leq)}} H^q(J, \mathcal{P})$, and this group, denoted by $\check{H}^q(\mathcal{P})$ or $\check{H}^q((I, \leq), \mathcal{P})$, is called the *q-th cohomology group of the projective system* $\mathcal{P}$.

If $(I, \leq)$ is an ordered set satisfying the conditions (1)-(4), denote by $\overleftarrow{(I, \leq)}\ (Ab)$ the category of projective systems of abelian groups indexed over $(I, \leq)$ and of morphisms of projective systems.

Proposition 1. *([12], Cor. 3.3, p. 100). For every integer $q \geq 0$ we can define a covariant functor*

$$\check{H}^q((I, \leq), -) : \overleftarrow{(I, \leq)}\ (Ab) \longrightarrow Ab,$$

which assigns to every projective system $\mathcal{P}$ indexed over $(I, \leq)$ the q-th abstract Čech cohomology group $\check{H}^q(\mathcal{P})$.

Proposition 2. *([12], Prop. 4.1, p. 101). For any exact sequence $0 \to P' \xrightarrow{\varphi} P \xrightarrow{\psi} P'' \to 0$ in the category $\overleftarrow{(I, \leq)}\ (Ab)$ we get the following exact sequence of abstract Čech cohomology*

$$0 \to \check{H}^0((I, \leq), \mathcal{P}') \xrightarrow{\varphi^{*0}} \check{H}^0((I, \leq), \mathcal{P}) \xrightarrow{\psi^{*0}} H^0((I, \leq), \mathcal{P}'') \to \check{H}^1((I, \leq), \mathcal{P}')....$$

Let $\mathcal{P} = (H_i, \alpha_{ij})$ be an object in the category $\underset{\longleftarrow}{(I,\leq)}(\mathrm{Ab})$. An element $t_i \in H_i$ is called a *section* of $\mathcal{P}$ over the index i. A system of sections $\{t_i\}_{i\in J\subset I}$ is called *coherent* if $\alpha_{i\wedge j,i}t_i = \alpha_{i\wedge j,j}t_j$, for any $i, j \in I$.

Definition 2. A projective system $\mathcal{P} = (H_i, \alpha_{ij})$ is called *complete* if for any coherent system of sections $\{t_i\}_{i\in J}$, there exists a section $t_k \in H_k$, $k = \sup J$, such that $\alpha_{ik}t_k = t_i$, for all $i \in J$.

The projective system $\mathcal{P} = (H_i, \alpha_{ij})$ is called *essential* if the following property is satisfied: for $k \in I$ and $t_k \in H_k$, there exists $J \subset I$ such that $k = \sup J$ and $\alpha_{ik}t_k = 0$ for every $i \in J$ implies t_k=0.

The projective system $\mathcal{P} = (H_i, \alpha_{ij})$ is called *canonical* if it is complete and essential.

Proposition 3. *([12], Prop. 5.2, p. 103). If $\mathcal{P} = (H_i, \alpha_{ij})$ is a canonical projective system over $(I, \leq)$ for which there exists $k = \sup I$, then $\check{H}^0(P) \simeq H_k$.*

Let $\mathcal{P} = (H_i, \alpha_{ij})$ be a projective system over $(I, \leq)$ satisfying the conditions (1)-(4).

Definition 3. A *cohomological resolution* of $\mathcal{P} = (H_i, \alpha_{ij})$ is an exact sequence in the category $\underset{\longleftarrow}{(I,\leq)}(\mathrm{Ab})$,

(RP):...$\rightarrow \mathcal{P}^{n-1} \overset{D^{n-1}}{\longrightarrow} \mathcal{P}^n \overset{D^n}{\longrightarrow} \mathcal{P}^{n+1} \rightarrow$ ...
such that:
1. $\mathcal{P}^{-1} = \mathcal{P}$, 2. $\mathcal{P}^n = 0$ for $n < -1$, 3. $\check{H}^q((I, \leq), \mathcal{P}^n) = 0$, for all $q \geq 1$ and $n \geq 0$.

A resolution (RP) of a projective system $\mathcal{P} = (H_i, \alpha_{ij})$ over $(I, \leq)$ induces a superior semiexact sequence (i.e., a cochain complex)

$$(*)\ldots \rightarrow \check{H}^0(\mathcal{P}^0) \overset{(D^0)^*}{\longrightarrow} \check{H}^0(\mathcal{P}^1) \overset{(D^1)^*}{\longrightarrow} \check{H}^0(\mathcal{P}^2) \overset{(D^2)^*}{\longrightarrow} \ldots$$

Theorem 1. *([12], Th. 6.1, p. 104) The abstract Čech cohomology groups of a projective system $\mathcal{P} = (H_i, \alpha_{ij})$ are isomorphic to the cohomology groups of the cochain complex (*) associated to a cohomological resolution (RP) of $\mathcal{P}$.*

Remark 1. If a projective system $\mathcal{P} = (H_i, \alpha_{ij})$ admits a cohomological resolution whose terms are canonical projective systems, then, by Proposition 3 and Theorem 1 the abstract Čech cohomology groups of $\mathcal{P}$ can be immediately determined.

The abstract Čech homology is defined by categorical duality. Let $\mathcal{I} = \{G_i, \alpha_{ij}\}_{(I,\leq)}$ be an inductive system of abelian groups over a directed partially ordered set $(I, \leq)$ satisfying conditions (1)-(4). For a covering $J \subset I$ a chain complex $C_*(J, \mathcal{I})$ is defined by taking

$$C_q(J, \mathcal{I}) := \bigoplus_{(i_0,\ldots,i_q)\in\Sigma_q} G_{i_0\wedge\ldots\wedge i_q}$$

and $d_q : C_q(J,\mathcal{I}) \to C_{q-1}(J,\mathcal{I})$ given by

$$d_q t_{i_0 \wedge \ldots \wedge i_q} = \sum_{p=0}^{q} (-1)^p \alpha_{i_0 \wedge \ldots \wedge \widehat{i_p} \wedge \ldots \wedge i_q, i_0 \wedge \ldots \wedge i_q} t_{i_0 \wedge \ldots \wedge i_q}.$$

The groups $H_q(C_*(J,\mathcal{I}))$ are denoted by $H_q(J,\mathcal{I})$. If we consider two coverings J, J' with $J' \prec J$, then for every $q \geq 0$ there exist natural homomorphisms $\alpha_{J' \prec J} : H_q(J',\mathcal{I}) \to H_q(J,\mathcal{I})$ such that a projective system $(H_q(J,\mathcal{I})\alpha_{J' \prec J}$ is obtained. The projective limit $\varprojlim_{\mathcal{J} \in \mathcal{A}_{\mathcal{I}}} H_q(J,\mathcal{I})$ is called the *q-th abstract Čech homology of the inductive system* $\mathcal{I}$ and it is denoted by $\check{H}_q(\mathcal{I})$ or by $\check{H}_q(|I|,\mathcal{I})$. The properties of abstract Čech homology are dual to those of the abstract Čech cohomology.

3 Examples

Example 1. Let (X, A) be a pair of topological spaces, with A a closed subspace of X, and let Γ be a presheaf over X. Consider the set $I := \{U | U$ open subset of X and $U \supset A\}$. This ordered set (by the inclusion relation) satisfies the conditions (1), (2) and (4). Then we consider the restriction Γ/I, and denote the q-th cohomology group of this projective system by $\check{H}^q([X, A]; \Gamma)$. One can prove that, for every $q \geq 0$, there exists a commutative diagram

$$\begin{array}{ccc} & & \check{H}^q((X, A); \Gamma) \\ & \nearrow & \downarrow \\ \check{H}^q([X, A]; \Gamma) & & \\ & \searrow & \downarrow \\ & & \check{H}^q(X; \Gamma_A) \end{array}$$

where $\check{H}^q((X, A); \Gamma)$ and $\check{H}^q(X; \Gamma_A)$ are the Čech cohomology groups with

$$\Gamma_A(U) = \begin{cases} \Gamma(U) & \text{if } U \cap A \neq \emptyset, \\ 0 & \text{if } U \subset X - A \end{cases}$$

for every open subset u of X.

Moreover, for these cohomology groups an excision theorem can be proved too:

$$\check{H}^q([X - V, A - V]; \Gamma) \simeq \check{H}^q([X, A]; \Gamma).$$

Example 2. Let

$$G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} G_3 .... \leftarrow G_n \xleftarrow{\varphi_n} G_{n+1} \leftarrow ....$$

be a sequence in the category of abelian groups. We obtain a projective system (sequence) $\mathcal{P} := \{G_n, \alpha_{n,m}\}_{n,m \in \mathbb{N}}$ by taking $\alpha_{n,m} = \varphi_n \circ \varphi_{n+1} \circ ... \circ \varphi_{m-1}$.

Let us denote $I = \{J_m = \{m, m+1, ...\}\}$, with $\mathbb{N} \supset J_1 \supset J_2 \supset ....$

For every $J_n \in I$ we define $Q_{J_n} := \varprojlim_{k \in J_n} G_k$, and if $J_n \subseteq J_m$, then we obtain a homomorphism $\alpha_{J_n J_m} : Q_{J_m} \longrightarrow Q_{J_n}$. For the projective system $\mathcal{Q} := \{Q_{J_n}, \alpha_{J_n J_m}\}$, we have:

$$\check{H}^0(\mathcal{Q}) = \varprojlim_{\mathbb{N}} \mathcal{P} = \varprojlim_{\mathbb{N}} G_n$$

and

$$\check{H}^q(\mathcal{Q}) = 0$$

for $q \geq 1$.

Example 3. If the sequence of abelian groups considered above is semiexact, we consider the projective system $\mathcal{P} = \{G_k, \alpha_{kh}\}_{h,k \in \mathbb{N}}$ with $\alpha_{kh} = 0$ for $h > k+1$.

A covering of $\mathbb{N}$ (in the sense of our definition) has the form $J = \{n_1 \leq ... \leq n_k \leq ...\}$, and we can prove that any two such coverings are cohomologically equivalent. In this case we obtain $\check{H}^0(\mathcal{P}) = 0$, and $\check{H}^1(\mathcal{P})$ is the factor group of the group of infinite dimensional matrices of the form

$$A = \begin{pmatrix} 0 & y_{12} & y_{13} & y_{13} & \cdots & \cdots \\ -y_{12} & y'_{12} & y_{23} & y_{24} & y_{24} & \cdots \\ -y_{13} & -y_{23} + y'_{13} & y'_{23} & y_{34} & y_{35} & y_{35} \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

by the group of infinite dimensional matrices of the form

$$B = \begin{pmatrix} 0 & y_{12} & y_{12} + \varphi_1(y_{23}) & y_{12} + \varphi_1(y_{23}) & \cdots & \cdots \\ -y_{12} & 0 & y_{23} & y_{23} + \varphi_2(y_{34}) & \cdots & \cdots \\ -y_{12} - \varphi_1(y_{23}) - y_{23} & 0 & y_{24} & y_{34} + \varphi_3(y_{45}) & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

where $y_{ih} \in G_i$ and $y'_{ih} \in Ker \varphi_i$.

Example 4. Let X be a Hausdorff topological space, and $\mathcal{C}\tau$ the set of the closed subsets of X. This set satisfies the conditions (1)-(4), with $A \wedge B = A \cap B$ and $\sup A_\alpha := \overline{\bigcup_\alpha A_\alpha}$, $\theta = \emptyset$, for $A, B, A_\alpha \in \mathcal{C}\tau$. If $\mathcal{I}$ is an inductive system over the set $\mathcal{C}\tau$, then $\check{H}_q(\mathcal{I}) = \varprojlim_{\mathcal{M} \in \mathcal{D}} \check{H}_q(M, I)$, where $\mathcal{D}$ is the set of all dense parts M in X. In particular,

$$\check{H}_0(I) = \varprojlim_{M \in \mathcal{D}} \big(\bigoplus_{x \in M} \mathcal{I}(x) \big).$$

In general, we cannot consider the cohomology since the coverings in the above sense do not form a directed set. However, the problem is possible in the case of some topological spaces which are not Hausdorff. For example, let X be a set with the "excluded point topology", i.e., the topology which is obtained by declaring open, in addition to X itself, all sets which do not include a given point $p \in X$. In this case the Čech cohomology and homology are less interesting than the cohomology and homology with coefficients in projective and inductive systems over the closed subsets of X.

If $\mathcal{P}$ is a projective system and $\mathcal{I}$ is an inductive system over the set of closed parts of X, then:

$\check{H}^0(\mathcal{P}) = \prod_{x\in X} \mathcal{P}(\{x,p\})$, $\check{H}^q(\mathcal{P}) = 0$ for $q \geq 1$, and

$\check{H}_0(\mathcal{I}) = \bigoplus_{x\in X} \mathcal{I}(\{x,p\})$, $\check{H}_q(\mathcal{I}) = 0$ for $q \geq 1$.

Finally, we mention that if the topological space (X,τ) has the property that for every open covering $\mathcal{U}$ of X there exists an open covering $\mathcal{U}'$ of X such that $\mathcal{U}'$ is a refinement of $\mathcal{U}$ and $\mathcal{CU}$ is a closed covering of X, then for every presheaf Γ on X there exists an isomorphism of the group $\check{H}^q(X,\Gamma)$ with the group $\check{H}_q(\mathcal{I})$, where $\mathcal{I}$ is the inductive system defined by $\mathcal{I}(A) := \Gamma(\mathcal{C}A)$ for every $A \in \mathcal{C}\tau$.

4 Application [18]

In this section, as an application of our abstract Čech cohomology, we consider some projective systems associated with a standard simplex Δ^n by taking as the indices the faces of Δ^n. As examples of such projective systems are the *simple cellular sheaves* considered in [3,4,7] and [13], and whose cohomology groups appear in the calculation of the K_G-groups of some particular G-spaces, by using the Atiyah-Hirzebruch spectral sequences. The cohomology groups of a linear simplex X with coefficients in a simple cellular sheaf are computed in [3] and [4] by means of a finite closed covering of X and using a Corollary of Leary's theorem ([9], p. 209).

We replace these coverings (which are rather complicated and which require the verification of some difficult acyclicity conditions) by a covering in the sense considered in [11] and [12], and which in fact consists only of the vertices of the standard simplex Δ^n. In this way we can calculate the cohomology groups with coefficients in a simple cellular sheaf for every linear simplex.

Let Δ^n be the n-dimensional standard simplex and let Σ be the set of the (closed) faces of Δ^n to which we add the empty set $\emptyset$. If $A_0, A_1, ..., A_n$ are the vertices of Δ^n, denote by $\Delta^n_{i_0 i_1 \ldots i_p}$ the face of Δ^n spanned by the vertices $A_{i_0}, A_{i_1}, ..., A_{i_p}$.

If $\sigma = \Delta^n_{i_0\ldots i_p}, \sigma' = \Delta^n_{j_0\ldots j_q} \in \Sigma$, we define the relation

$\sigma \leq \sigma'$ if and only if $j_0, ..., j_q \subseteq i_0, ..., i_p$, i.e., σ' is a face of σ.

This is a partial ordering on the set Σ.

Definition 4. A projective system $\mathcal{P} = (H_\sigma, \alpha^{\sigma'}_{\sigma})_{\sigma,\sigma'\in\Sigma}$ over the above partially ordered set $(\Sigma, \leq)$ is called a simplicial projective system.

Lemma 1. *The partially ordered set* $(\Sigma, \leq)$ *satisfies the conditions (1)-(4).*

Proof. (1) If $\sigma' = \Delta^n_{i_0\ldots i_p}$, $\sigma'' = \Delta^n_{j_0\ldots j_q}$, then for $\sigma = \Delta^n_{k_0,\ldots,k_r}$, with $\{k_0, ..., k_r\} = \{i_0, ..., i_p\} \cup \{j_0, ..., j_q\}$, we have $\sigma = \sigma' \wedge \sigma''$ (σ is sometimes the joint $\sigma' * \sigma''$).

(2) If $\Sigma' \subset \Sigma$, we have $\sup \Sigma' = \bigcap_{\sigma'\in\Sigma'} \sigma'$. Here it is necessary to suppose that Σ contains the empty set $\emptyset$.

Now we say that a subset Σ' of the set Σ is a *covering* if for every face $\sigma \in \Sigma$ there exists a subset $\Sigma'_\sigma \subset \Sigma'$ such that $\sup \Sigma'_\sigma \neq \emptyset$ and $\sigma \leq \sup \Sigma'_\sigma$.

If Σ', Σ'' are two coverings, then we have $\Sigma' \prec \Sigma''$ if and only if for every face $\sigma' \in \Sigma'$ there exists a face $\sigma'' \in \Sigma''$ such that $\sigma' \leq \sigma''$, i.e., σ'' is a face of σ'. Now the condition (4) is verified because if Σ', Σ'' are two coverings, then $\Sigma' \bigcap \Sigma''$ satisfies the relations $\Sigma' \bigcap \Sigma'' \prec \Sigma'$ and $\Sigma' \bigcap \Sigma'' \prec \Sigma''$. □

By Lemma 1 we can consider the abstract Čech cohomology groups $\check{H}^q(\mathcal{P}_n)$ of a simplicial projective system $\mathcal{P}_n = (H_\sigma, \alpha_\sigma^{\sigma'})_{\sigma,\sigma' \in \Sigma}$.

Now we recall from [7] and [4] that a *cellular sheaf* over Δ^n is a sheaf $\mathcal{F}$ on the topological space Δ^n with the property that for every open face $\overset{\circ}{\sigma}$ of Δ^n the restriction $\mathcal{F}/\overset{\circ}{\sigma}$ is a simple sheaf. For such a sheaf, if $\overset{\circ}{\sigma}$ and $\overset{\circ}{\sigma'}$ are two open faces of Δ^n with $\overset{\circ}{\sigma} \cap \sigma' \neq \emptyset$ then there exists a homomorphism $\varphi_{\sigma\sigma'} : \mathcal{F}/\overset{\circ}{\sigma} \to \mathcal{F}/\overset{\circ}{\sigma'}$ satisfying the condition that if $\overset{\circ}{\sigma} \cap \sigma' \cap \sigma'' \neq \emptyset$ and $\overset{\circ}{\sigma'} \cap \sigma'' \neq \emptyset$, then $\varphi_{\sigma'\sigma''} \circ \varphi_{\sigma\sigma'} = \varphi_{\sigma\sigma''}$ ([7], Prop.1). Also, from the definition of the above homomorphisms $\varphi_{\sigma\sigma'}$ one deduces that $\varphi_{\sigma\sigma}$ is the identity. Together with the Prop. 3 of [7] and remarking that if σ, σ' are two faces of Δ^n then $\overset{\circ}{\sigma} \cap \sigma' \neq \emptyset$ if and only if σ is a face of σ', i.e., if and only if $\sigma' \leq \sigma$, we obtain the following theorem.

Theorem 2. *Every simple cellular sheaf F over the standard simplex Δ^n defines a simplicial projective system $\mathcal{P}(\mathcal{F}) = \mathcal{P}_n$, and conversely, any simplicial projective system $\mathcal{P}_n$ induces a cellular simple sheaf over Δ^n.*

Theorem 3. *The abstract Čech cohomology groups of a simplicial projective system*

$$\mathcal{P}_n := \{H_{i_0\wedge\ldots\wedge i_q}, \alpha_{i_0\wedge\ldots\wedge i_q}^{i_0\wedge\ldots\wedge\widehat{i_p}\wedge\ldots\wedge i_q}\}$$

are given by :

$$\check{H}^q(\mathcal{P}_n) \cong \frac{\bigcap\limits_{0\leq i_0<i_1<\ldots<i_{q+1}\leq n} Z_{i_0 i_1\ldots i_{q+1}}}{\bigcap\limits_{0\leq i_0<i_1<\ldots<i_{q+1}\leq n} B_{i_0 i_1\ldots i_{q+1}}}$$

for $q = 0, 1, ..., n$ (and 0 otherwise), with

$$Z_{i_0\ldots i_{q+1}} = H_{i_1\wedge\ldots\wedge i_{q+1}} \prod\nolimits_{H_{i_0\wedge\ldots\wedge i_{q+1}}} \bigoplus_{p=1}^{q+1} H_{i_0\wedge\ldots\wedge\widehat{i_p}\wedge\ldots\wedge i_{q+1}}$$

,

$$B_{i_0\ldots i_{q+1}} = Im\alpha_{i_0\wedge\ldots\wedge i_{q+1}}^{i_0\wedge\ldots\wedge i_q} \prod\nolimits_{H_{i_0\wedge\ldots\wedge i_{q+1}}} \bigoplus_{p=1}^{q} Im\alpha_{i_0\wedge\ldots\wedge i_q}^{i_0\wedge\ldots\wedge\widehat{i_p}\wedge\ldots\wedge i_q}$$

where $A\prod_S B$ denotes a fibered product of A and B over S.

Remark 2. If the homomorphisms $\alpha_{i_0\wedge\ldots\wedge i_q}^{i_0\wedge\ldots\wedge\widehat{i_p}\wedge\ldots\wedge i_q}$ are all injective, then the formulas from Theorem 3 become more simple. For example, in this case we have

$$\check{H}^0(\mathcal{P}_n) \cong \bigcap_{i=0}^{n} H_i, \check{H}^n(\mathcal{P}_n) \cong \frac{H_{012\ldots n}}{H_0' + H_1' + \ldots + H_n'}$$

where $H_p' = H_{0\ldots\widehat{p}\ldots n}$.

Remark 3. For $n = 1, 2, 3$ we find some results of [3],[4]. Thus, if $n = 1$ we obtain

$$\check{H}^0(\mathcal{P}_1) \cong H_0 \prod\nolimits_{H_{01}} H_1$$

and

$$\check{H}^1(\mathcal{P}_1) \cong \frac{H_{01}}{Im\alpha_{01}^0 + Im\alpha_{01}^1},$$

which coincide respectively with the cohomology groups $H^q(\Delta^1, \mathcal{F})$, $q = 0, 1$.

We can establish a general result. Let $\mathcal{F}$ be a simple cellular sheaf over Δ^n. The calculation of the groups $H^q(\Delta^n, \mathcal{F})$ in [4] uses a closed covering $\mathcal{U} = \{U_0, \ldots, U_n\}$ with the acyclicity property $H^q(U_{i_0} \cap \ldots \cap U_{i_q}; F) = 0, q \geq 1$. Then, by Cor. 1 of [9], p. 209, the natural homomorphism $H^q(\mathcal{U}, \mathcal{F}) \to H^q(\Delta^n, \mathcal{F})$ is an isomorphism. Then the cochain complex $C^*(\mathcal{U}, \mathcal{F})$ is given by

$$C^q(\mathcal{U}, \mathcal{F}) = \prod \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_q})$$

and

$$(d^q t)_{U_{i_0}\cap\ldots\cap U_{i_q}} = \sum_{p=0}^{q+1} (-1)^p \mathcal{F}_{U_{i_0}\cap\ldots\cap U_{i_{q+1}}}^{U_{i_0}\cap\ldots\cap\widehat{U}_{i_p}\cap\ldots\cap U_{i_{q+1}}} t_{U_{i_0}\cap\ldots\cap\widehat{U}_{i_p}\cap\ldots\cap U_{i_{q+1}}}.$$

By the choice of the covering $\mathcal{U}$ and because $\mathcal{F}$ is a simple cellular sheaf one verifies easily that $C^*(\mathcal{U}, \mathcal{F})$ is equivalent to the cochain complex which appears in the proof of Theorem 3. Thus we have the following result.

Theorem 4. *If $\mathcal{F}$ is a simple cellular sheaf over the standard simplex Δ^n and if $\mathcal{P}(\mathcal{F})$ is its associated projective system by Theorem 2, then there exists a natural isomorphism*

$$H^q(\Delta^n, \mathcal{F}) \cong \check{H}^q(\mathcal{P}(\mathcal{F}))$$

for every integer q.

Remark 4. If we replace the standard simplex Δ^n by an arbitrary CW-complex, then a simple cellular sheaf also defines a projective system having as the set of indices the set of cells, but unfortunately this set does not satisfy the condition (1). But in [8] a method for the calculation of the cohomology groups of an arbitrary polyhedron with coefficients in a simple cellular sheaf by using the simplicial cohomology with local coefficients was given. This leads us to believe that it is possible to extend our method, that of abstract Čech cohomology, from the case of standard simplex Δ^n to the general case of an arbitrary polyhedron.

5 De Rham type theorems

5.1 J–resolutions of a projective system [14]

In this section we assume that $(I, \leq)$ satisfies (1)–(4) (see Section 2) and for $J \in I$ we put $\Sigma_s(J) = \times_s J$. Let $P = (H_i, \alpha_{ij})_{i,j \in I}$ be a projective system (of abelian groups) over I and consider the cochain complex of projective systems over I

$$0 \to P \xrightarrow{j} P^0 \xrightarrow{D^0} P^1 \to \ldots \to P^q \xrightarrow{D^q} P^{q+1} \to \ldots \tag{1}$$

where $j = (j_i)_{I \in I}$ and $D^q = (D^q_i)_{i \in I}$.

Definition 5. A *J–resolution* of the projective system P over I is a cochain complex (1) satisfying the following conditions:

a. there exists $J \in A_I$ such that for any $s \geq 0$ and $(i_0, i_1, \ldots, i_s) \in \Sigma_s(\mathrm{J})$,

$$\begin{aligned} 0 \to H_{i_0 \wedge i_1 \wedge \ldots \wedge i_s} &\overset{j_{i_0 \wedge i_1 \wedge \ldots \wedge i_s}}{\longrightarrow} H^0_{i_0 \wedge i_1 \wedge \ldots \wedge i_s} \to \ldots \\ \to H^q_{i_0 \wedge i_1 \wedge \ldots \wedge i_s} &\overset{D^q_{i_0 \wedge i_1 \wedge \ldots \wedge i_s}}{\longrightarrow} H^{q+1}_{i_0 \wedge i_1 \wedge \ldots \wedge i_s} \to \ldots \end{aligned}$$

is an exact sequence;

b. the sequence (1) is exact with respect to P and P^0.

Remark 5. 1. Let $f : P \to Q$ be a projective systems isomorphism. If $J \in \mathcal{A}_I$, then f induces an isomorphism from $H^q(J, P)$ to $H^q(J, Q)$, hence

$$\check{H}^q(|I|, P) \simeq \check{H}^q(|I|, Q)$$

for any $q \geq 0$.

2. If the sequence (1) is a J–resolution of the projective system P then its projective limit

$$0 \longrightarrow \varprojlim_{i \in I} H_i \xrightarrow{\varprojlim_{i\in I} j_i} \varprojlim_{i \in I} H^0_i \longrightarrow \ldots \longrightarrow \varprojlim_{i \in I} H^q_i \xrightarrow{\varprojlim_{i\in I} D^q_i} \varprojlim_{i \in I} H^{q+1}_i \longrightarrow \ldots \tag{2}$$

is a cochain complex and it is exact with respect to the terms $\varprojlim_{i \in I} H_i$ and $\varprojlim_{i \in I} H^0_i$.

Let (1) be a J–resolution of the projective system P and denote by $C^s(J, P^q)$ and $\mathcal{C}(J, P)$ the s–dimensional cochain groups associate to the systems P^q and P, respectively. By Remark 5 we can assume that $\mathcal{C}^s(J, P) \leq C^{s,0}(J, P)$ by the inclusion morphism j^s. Then the sequence

$$0 \to \mathcal{C}^s(J, P) \xrightarrow{j^s} C^{s,0}(J, P) \to \ldots \to C^{s,q}(J, P) \xrightarrow{d^{s,q}} C^{s,q+1}(J, P) \to \ldots \tag{3}$$

is exact for each $s \geq 0$, $q \geq 1$, and we have the following

Proposition 4. *If the projective system P has a J–resolution (1) then (3) is a resolution for its p–dimensional cochain group.*

Definition 6. A J–resolution of the projective system P is *acyclic* if $H^s(J, P^q) = 0$ for any $q \leq 0$ and $s \geq 1$.

We remark that if for any $J \in A_I$, (1) is an acyclic J–resolution of the projective system P then it is a resolution of P. Conversely, if there exists $J \in A_I$ such that $J \prec J'$ for each $J' \in A_I$ then any resolution of P is a J–resolution.

Now, we can state

Theorem 5. *If the projective system P admits an acyclic J–resolution of canonical projective systems then*

$$H^0(J,P) \simeq \ker \varprojlim_{i\in I} D_i^0,$$

$$H^q(J,P) \simeq \ker \varprojlim_{i\in I} D_i^q / \mathrm{Im} \varprojlim_{i\in I} D_i^{q-1} \quad for \quad q \geq 1.$$

We notice that in [15], the above isomorphism is effectively exhibited.

Denote by $P^* = \oplus_{q\geq 0} P^q$ the differential projective system associate to (1), with the codifferential $d = \oplus_{q\geq 0} D^q$. We have the following generalization of Theorem 5.

Theorem 6. *[17] Let P be a projective system over the $\wedge$–semilattice $(I, \leq)$, $J \in \mathcal{A}_I$ and (1) be a J–resolution of P by canonical projective systems. If $H^p(H^q(J, P^*)) = 0$ for all $p \geq 0$ and $q \geq 1$ then*

$$H^p(J,P) \simeq H^p(\varprojlim_{i\in I} H_i^0 P^*).$$

Now, we present some applications of this result. The first one is the following de Rham type theorem:

Theorem 7. *If for any $J \in \mathcal{A}_I$, the sequence (1) is an acyclic J–resolution of canonical projective systems for P, then the following isomorphisms occur:*

$$\check{H}^0(|I|,P) \simeq \ker \varprojlim_{i\in I} H_i^0 D_i^0,$$

$$\check{H}^q(|I|,P) \simeq \ker \varprojlim_{i\in I} D_i^q / \mathrm{Im} \varprojlim_{i\in I} D_i^{q-1} \quad for \quad q \geq 1.$$

From Remark 5 we deduce that the conclusion of Theorem 7 is still valid if there exists $J \in \mathcal{A}_I$ such that $J \prec J'$ for any $J' \in A_I$.

On the other hand, if the set I has a supremum k then from Theorem 5 we obtain

$$H^0(J,P) \simeq H_k, \qquad H^q(J,P) \simeq \ker d_k^q / \mathrm{Im} d_k^{q-1} \quad \text{for} \quad q \geq 1 \tag{4}$$

and, moreover, if $k \in J$ then $H^q(J, P) = 0$ for any $q \geq 1$.

Proposition 5. *Under the hypotheses from Theorem 7, if there exists* $\sup I = k$ *then*

$$\check{H}^0(|I|,P) \simeq H_k, \qquad \check{H}^q(|I|,P) = 0 \quad for \quad q \geq 1.$$

5.2 Existence of J–resolutions

We remark that, in order to obtain de Rham type theorems for projective systems, the existence of a J–resolution is essential. This problem is solved in [16], at least partially. First, we assume $(I, \leq)$ to be a $\wedge$–semilattice and for $i \in I$ and $J \subseteq I$ we put $i \wedge J = \{i \wedge j\}_{j \in J}$. Then the group of q–dimensional cochains of P relative to $i \wedge J$, is

$$K^q(i \wedge J) = \prod_{\Sigma_q} H_{i \wedge i_o \wedge \ldots \wedge i_q}.$$

If $i' \in I$, $i' \leq i$, then for $(i_0, \ldots, i_q) \in \Sigma_q$ and $t \in K^q(i \wedge J)$ we put

$$(\alpha^{*q}_{i'i} t)_{i' \wedge i_o \wedge \ldots \wedge i_q} = \alpha_{i' \wedge i_o \wedge \ldots \wedge i_q, i \wedge i_o \wedge \ldots \wedge i_q} t_{i \wedge i_o \wedge \ldots \wedge i_q}.$$

Thus we obtain the projective system $K^q(J,P) = \{K^q(i \wedge J), \alpha^{*q}_{i'i}\}_{i,i' \in I}$ and denote by $\partial^q : K^q(J,P) \to K^{q+1}(J,P)$ the coboundary operator. Moreover it induces a morphism of projective systems. Another such morphism is $f = (f_i)_{i \in I} : P \to K^0(i \wedge J, P)$, where $[f_i(h_i)]_{i \wedge j} = \alpha_{i \wedge j, i}(h_i)$ for $h_i \in H_i$ and $j \in J$. Then we have

Proposition 6. *Assume that there exists* $\sup I$ *and let* P *be a canonical projective system on* I*. Then the sequence*

$$0 \longrightarrow P \xrightarrow{f} K^0(J,P) \xrightarrow{(\partial^0)} \ldots \longrightarrow K^q(J,P) \xrightarrow{(\partial^q)} K^{q+1}(J.P) \longrightarrow \ldots$$

is an acyclic J*–resolution of* P *for any cofinal subset* J *of* I*.*

5.3 Canonical system associate to a projective system [15]

The determination of these groups is a hard and subtle problem and we know them explicitly in very few cases.

It is well-known that under some additional restrictions on the set $(I, \leq)$, to any projective system we can associate a canonical projective system. More precisely, we have the following

Proposition 7. *([10], Prop. 5.1, p. 102) If* $(I, \leq)$ *is a filter at left then to any projective system* P *over* $(I, \leq)$*, a canonical projective system* P^* *can be associated.*

Under some stronger conditions on $(I, \leq)$ we can get a simple relation between Čech cohomologies of P and P^*.

Theorem 8. *Let* $(I, \leq)$ *be a* $\wedge$*–semilattice such that any* $J \in \mathcal{A}_I$ *admits a refinement* $\overline{J} \in \mathcal{A}_I$ *with the property that for every* $(i_0, \ldots, i_q) \in \Sigma_q$ *there exist* $\overline{i}_0, \ldots, \overline{i}_q \in \overline{J}$ *such that* $\overline{i}_0, \ldots, \overline{i}_q \in J_{i_0 \wedge \ldots \wedge i_q}$*.*
Then the Čech cohomology groups of P *and* P^* *are isomorphic in each dimension, that is*

$$\check{H}^q(|I|, P) \simeq \check{H}^q(|I|, P^*)$$

for all $q \geq 0$*.*

5.4 Examples

Now, we use the above results to recover some classical theorems (see Introduction).

Example 9. Let M be a smooth n–dimensional manifold and denote by $\widetilde{\Lambda}^q$ its q–forms sheaf and by $\widetilde{\mathbb{R}}$ the sheaf associated to the constant presheaf on M. If $\mathcal{U} = \{U_i\}_{i\in I}$ is an open cover of M then the sheaves sequence

$$0 \longrightarrow \widetilde{\mathbb{R}} \stackrel{j}{\longrightarrow} \widetilde{\Lambda}^0 \stackrel{d}{\longrightarrow} \widetilde{\Lambda}^1 \longrightarrow \ldots \longrightarrow \widetilde{\Lambda}^q \stackrel{d}{\longrightarrow} \widetilde{\Lambda}^{q+1} \longrightarrow \ldots \tag{5}$$

is a cochain complex with respect to the exterior derivative d. (5) is exact for the terms $\widetilde{\mathbb{R}}$, $\widetilde{\Lambda}^0$ and $H^s(\mathcal{U}, \widetilde{\Lambda}^q) = 0$ for $s \geq 1$ and $q \geq 0$. If, moreover, the cover $\mathcal{U}$ is contractible then (5) is an acyclic $\mathcal{U}$–resolution of the sheaf $\widetilde{\mathbb{R}}$ and we can apply the isomorphisms (4). On the other hand, any open cover of M can be refined by a contractible one, so the set $\mathcal{A}^c$ of contractible covers of M is cofinal in the set of all open covers and then

$$\check{H}^q(M, \widetilde{\mathbb{R}}) \simeq \varprojlim_{\mathcal{U}\in\mathcal{A}^c} H^q(\mathcal{U}, \widetilde{\mathbb{R}})$$

and therefore from the isomorphism $\check{H}^q(M, \widetilde{\mathbb{R}}) \simeq \check{H}^q(M, \mathbb{R})$ we deduce the classical de Rham theorem.

Example 10. Let (X, τ) be a topological space and

$$\Delta(U): \quad \ldots \longrightarrow \Delta_q(U) \stackrel{d_{q,U}}{\longrightarrow} \Delta_{q-1}(U) \longrightarrow \ldots \longrightarrow \Delta_0(U) \longrightarrow 0$$

the singular chain sequence associate to $U \in \tau$. Denote by $\Delta^q(U)$ the group of q–dimensional singular cochains with coefficients in an abelian group G and for $l \in \Delta^q(U)$ we put $d^q_U(l) = l d_{q+1,U}$. For $V \in \tau$, $V \supset U$, we define the morphism $\alpha^q_{UV} : \Delta^q(V)$, which, to each cochain on V, associates its restriction to U. Then for each $q \geq 0$, $\Delta^q = \{\Delta^q(U), \alpha^q_{UV}\}_{U,V\in\tau}$ is a complete projective system and we obtain the cochain complex

$$\Delta^*: \quad 0 \longrightarrow \Delta^0 \longrightarrow \ldots \longrightarrow \Delta^q \stackrel{d^q}{\longrightarrow} \Delta^{q+1} \longrightarrow 0$$

where $d^q = (d^q_U)_{U\in\tau} : \Delta^q \to \Delta^{q+1}$ and $d^q_U(l) = l d_{q+1,U}$.

Now, if $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ is an open cover for X and $U \cap \mathcal{U}$ is its trace on U then we consider the group $\Delta_q(U \cap \mathcal{U})$ spanned by all singular simplexes whose images belong to $U_\alpha \in \mathcal{U}$ for some $\alpha \in I$. Denoting by $\Delta^q(U \cap \mathcal{U})$ the group of all morphisms from $\Delta_q(U \cap \mathcal{U})$ to G, as above, we can construct a complete projective system $\overline{\Delta}^q = \{\Delta^q(U \cap \mathcal{U}), \overline{\alpha}^q_{UV}\}_{U,V\in\tau}$ and the associate cochain complex $\overline{\Delta}^*$, obtained from Δ^*.

Let L^q_U be the subgroup of all elements of $\Delta^q(U \cap \mathcal{U})$ vanishing on $\Delta_q(U' \cap \mathcal{U})$ for all U' in some cover of U and consider the quotient group $\widetilde{\Delta}^q(U \cap \mathcal{U}) = \Delta^q(U \cap \mathcal{U})/L^q_U$. Then $\overline{\alpha}^q_{UV}$ and the differential $\overline{d}^q_U$ naturally induce the quotient morphisms $\widetilde{\alpha}^q_{UV}$ and

$\widetilde{d}_U^q$, respectively. Thus, from $\overline{\Delta}^*$ we obtain the following cochain complex of canonical projective systems

$$\widetilde{\Delta}^* : \quad 0 \longrightarrow \widetilde{\Delta}^0 \longrightarrow \ldots \longrightarrow \widetilde{\Delta}^q \xrightarrow{\widetilde{d}^q} \widetilde{\Delta}^{q+1} \longrightarrow 0$$

Moreover, for any cover of X, the sequences $\overline{\Delta}^*$ and $\widetilde{\Delta}^*$ are homotopy equivalent. Also, the sequence

$$0 \longrightarrow \ker \widetilde{d}^0 \xrightarrow{\hookrightarrow} \widetilde{\Delta}^0 \longrightarrow \ldots \longrightarrow \widetilde{\Delta}^q \xrightarrow{\widetilde{d}^q} \widetilde{\Delta}^{q+1} \longrightarrow 0$$

is an acyclic $\mathcal{U}$–resolution of the projective system $\ker \widetilde{d}^0$, for any contractible cover $\mathcal{U}$ of X and from the isomorphisms (4) and since the projective systems $\widetilde{\Delta}^q$ are canonical, we deduce

$$\check{H}^q(\mathcal{U}, \ker \widetilde{d}^0) \simeq \check{H}^q(\mathcal{U}, G) \simeq H^q(X, G)$$

and we find a Leray's theorem,[10], asserting that the Čech cohomology groups relative to a contractible cover of the topological space X and with coefficients in the abelian group G, are isomorphic, in each dimension, with the singular cohomology groups of X, with coefficients in G.

6 Spectral sequence [17]

Let $P^* = \oplus_{q\geq 0} P^q$ be a differential projective system over I, with the codifferential $d = \oplus_{q\geq 0} D^q$. For $J \subset I$ we consider the coboundary homomorphism associated to the projective system P^q, denoted by $d_{10}^{pq} = (d^p)^q : C^p(J, P^q) \to C^{p+1}(J, P^q)$ (see Section 2) and the homomorphism $d_{01}^{pq} = (-1)^p (\widetilde{D^q})^p : C^p(J, P^q) \to C^p(J, P^{q+1})$, where $(\widetilde{D^q})^p$ is induced by the codifferential D^q. Then the following equalities hold

$$d_{10}^{p+1,q} d_{10}^{pq} = 0, \quad d_{01}^{p,q+1} d_{01}^{pq} = 0, \quad d_{01}^{p,q+1} d_{10}^{pq} + d_{10}^{p+1,q} d_{01}^{pq} = 0 \tag{6}$$

These equalities show that for any $J \subset I$, the codifferentials d_{10}^{pq} and d_{01}^{pq} define on the group

$$C(J, P^*) = \oplus_{p\geq 0, q\geq 0} C^p(J, P^q)$$

a double complex structure and then we consider its first and second filtration given by

$$'C_p(J, P^*) = \oplus_{\overline{p}\geq p}^{q\geq 0} C^{\overline{p}}(J, P^q), \qquad ''C_q(J, P^*) = \oplus_{\overline{q}\geq q}^{p\geq 0} C^p(J, P^{\overline{q}}).$$

We remark that the first filtration $'C_p(J, P^*)$ is regular if $P^q = 0$ for $q < q_0$ for some q_0, while the second filtration $''C^p(J, P^*)$ is always regular.

Now, assume that $J \in \mathcal{A}_I$ and denote by $'E_r^{pq}(J, P^*)$ and $''E_r^{pq}(J, P^*)$ the terms of the spectral sequences corresponding to the first filtration and the second filtration of the complex $C(J, P^*)$, respectively. Then we have

Proposition 8. *If* $(I,\leq)$ *is a* $\wedge$*–semilattice then*

$$'E_1^{pq}(J,P^*)\simeq C^p(J,\mathcal{H}^q(P^*)),\qquad 'E_2^{pq}(J,P^*)\simeq H^p(J,\mathcal{H}^q(P^*))$$
$$''E_1^{pq}(J,P^*)\simeq H^p(J,P^p),\qquad ''E_2^{pq}(J,P^*)\simeq H^p(H^q(J,P^*))$$

where $\mathcal{H}^q(P^*)=\ker D^q/D^{q-1}(P^{q-1})$ *and* $H^q(J,P^*)=\oplus_{p\geq 0}H^q(J,P^p)$ *is endowed with the differential induced by* d_{01}.

One more interesting result can be obtained for the term $''E_1^{p0}(J,P^*)$, namely:

Theorem 11. *If the projective systems* P^p *are canonical for all* $p\geq 0$ *and* $(I,\leq)$ *is a* $\wedge$*–semilattice then for* $J\in\mathcal{A}_I$ *we have*

$$''E_1^{p0}(J,P^*)\simeq H^p(\varprojlim_{i\in I}(P^*)).$$

Assuming that $(I,\leq)$ is a $\wedge$–semilattice, for the term $''E_2^{p0}(J,P^*)$ we can prove the existence of a homomorphism $''E_2^{p0}(J,P^*)\longrightarrow H^p(C(J,P^*)$ if $(I,\leq)$. But if, moreover, all projective systems are canonical then we also get the homomorphisms

$$\mu^p: H^p(\varprojlim_{i\in I}(P^*))\longrightarrow H^p(C(J,P^*)$$

and we can state the following

Proposition 9. *Let* P *and* Q *be two projective systems over the* $\wedge$*–semilattice* $(I,\leq)$ *and* $J\in\mathcal{A}_I$. *If* P *and* Q *have* J*–resolutions of the form (1) by canonical projective systems then for any differential projective systems morphism* $f:(P^*)\longrightarrow(Q^*)$ *the following diagrams are commutative*

$$\begin{array}{ccc}
H^p(\varprojlim_{i\in I}(P^*)) & \xrightarrow{\mu^p} & H^p(J,P)\\
{\scriptstyle H^p(\varprojlim_{i\in I} f)}\downarrow & & \downarrow{\scriptstyle g^p}\\
H^p(\varprojlim_{i\in I}(Q^*)) & \xrightarrow[\mu^p]{} & H^p(J,Q)
\end{array}$$

where g^p *are induced by the morphism* j *corresponding to the* J*–resolution of* Q *and* f^0 *is the component* $P^0\longrightarrow Q^0$ *of* f.

The homomorphisms μ^p are also used to prove the following

Theorem 12. *If the projective systems* P^p *are canonical for all* $p\geq 0$ *and the sequences* $H^q(J,P^*)$ *are acyclic for all* $q\geq 1$ *then there exists a spectral sequence with the term* E_2 *given by*

$$E_2^{pq}\simeq H^p(J,\mathcal{H}^q(P^*))$$

and whose term E_∞ *is the graded group associated to some filtration of the sequence* $\varprojlim_{i\in I}(P^*)$.

Finally, we notice some related results concerning projective systems admitting a J–resolution. The first one is concerning the term ${}'E_2^{pq}(J,P^*)$.

Proposition 10. *Let P be a projective system over the $\wedge$–semilattice $(I,\leq)$ and $J\in\mathcal{A}_I$. If P has a J–resolution (1) then*

$$ {}'E_2^{pq}(J,P^*)=0 $$

for $q\geq 1$.

Now, using Proposition 10, we obtain the following characterization of the groups $H^p(C(J,P^*))$.

Proposition 11. *If the projective system P over the $\wedge$–semilattice $(I,\leq)$ has a J–resolution (1) then*

$$ H^p(C(J,P^*)\simeq H^p(J,P) $$

for $p\geq 0$ and $J\in\mathcal{A}_I$, where $P^*=\oplus_{q\geq 0}P^q$ is the differential projective system associated to (1).

Also, we have a result similar to the Theorem 12.

Proposition 12. *Let P be a projective system over the $\wedge$–semilattice $(I,\leq)$ and $J\in\mathcal{A}_I$. For any J-resolution (1) of P with canonical projective systems there exists a spectral sequence with the term E_2 given by*

$$ E_2^{pq}\simeq H^p(H^q(J,P^*)) $$

and whose term E_∞ is the graded group associated to the abstract Čech cohomology of P.

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Some addition theorems for rectifiable spaces

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Abstract. We establish that if a compact Hausdorff space B with the cardinality less than 2^{ω_1} is represented as the union of two non-locally compact rectifiable subspaces X and Y, then X, Y and B are separable and metrizable.

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1 Introduction

It is well-known that if the cardinality of a compact topological group X does not exceed 2^{ω} and the continuum hypothesis is satisfied, then X is separable and metrizable (see [8]). Extending this result, we show that if the cardinality of a compact Hausdorff space X is less than 2^{ω_1}, then X cannot be represented as the union of two non-locally compact rectifiable spaces. Recall that every topological group is a rectifiable space (see the definition below). Some other results in this direction are also obtained.

We use the terminology and notations from [12]. A *remainder* of a Tychonoof space X is the subspace $bX \setminus X$ of a Hausdorff compactification bX of X.

A space X is of *countable type* (respectively, of *pointwise countable type*) if every compact subspace P (respectively, any point p) of X is contained in a compact subspace $F \subset X$ with a countable base of open neighbourhoods in X. All metrizable spaces and all locally compact Hausdorff spaces, as well as all Čech-complete spaces, are of countable type [1, 2, 12].

A famous classical result on duality between properties of spaces and properties of their remainders is the following theorem of M. Henriksen and J. Isbell [14]:

Theorem 1. *A Tychonoff space X is of countable type if and only if the remainder in any (in some) Hausdorff compactification of X is Lindelöf.*

It follows from this theorem that every remainder of a metrizable space is Lindelöf.

2 Addition theorems for rectifiable spaces

Recall that a space X is *rectifiable* if there exists $e \in X$ and a homeomorphism $g : X \times X \to X \times X$ such that $g((x, e)) = (x, x)$, for every $x \in X$, and the restriction of g to the subspace $X_x = \{(x, y) : y \in X\}$ is a homeomorphism of X_x onto itself, for every $x \in X$. Every topological group is rectifiable, and every rectifiable space is homogeneous (see [9, 10]).

Theorem 2. *Suppose that B is a compact Hausdorff space such that $|B| < 2^{\omega_1}$. Suppose further that $B = X \cup Y$, where X and Y are non-locally compact rectifiable spaces. Then the spaces B, X, and Y are separable and metrizable.*

Proof. Clearly, Y and X are non-empty, since they are not locally compact. Hence, B is non-empty. By Čech-Pospišil Theorem ([3, 15], [12], Problem 3.12.11), there exists a point $a \in B$ such that B is first-countable at a. Without loss of generality, we may assume that $a \in X$. Put $Z = B \setminus X$ and $H = B \setminus Y$. The spaces X and Y are nowhere locally compact, since they are homogeneous and non-locally compact. It also follows that Z and H are nowhere locally compact. Hence, X, Y, Z, and H are dense in B.

Since X is homogeneous and X is first-countable at a, it follows that the space X is first-countable. Therefore, X is metrizable, since X is rectifiable [13]. Hence, X is a space of countable type [1], which implies that the remainder Z of X in B is Lindelöf [14].

By the Dichotomy Theorem for remainders of rectifiable spaces (see [6]), the remainder H of Y in the Hausdorff compactification B of Y is either pseudocompact, or Lindelöf. Notice that H is metrizable in any case, since, obviously $H \subset X$.

Case 1: H is pseudocompact.

Then H is compact, since H is metrizable. Therefore, H is closed in B. Hence, Y is open in B, which implies that Y is locally compact, a contradiction. Thus, Case 1 is impossible.

Case 2: H is Lindelöf.

Then H is separable, since H is metrizable. Hence, B is separable, which implies that the Souslin number of Y is countable, since Y is dense in B. It also follows that X is separable, since H is dense in X.

Lindelöfness of H also implies that Y is a space of countable type, by the theorem of Henriksen and Isbell [14]. Therefore, we can fix a non-empty compact subspace F of Y with a countable base of open neighbourhoods in Y. We have: $|F| \leq |B| < 2^{\omega_1}$. Applying one more time the Čech-Pospišil Theorem [3,12,15] we conclude that there exists a point $b \in F$ such that F is first-countable at b. Since F has a countable base of open neighbourhoods in Y, we can now conclude that the space Y is first-countable at the point b. Therefore, the space Y is first-countable, since it is homogeneous. Hence, Y is metrizable, since it is rectifiable. Finally, it follows that Y is separable, since the Souslin number of Y is countable. □

A family η of non-empty open subsets of a space X is said to be a *π-base* of X at a point $a \in X$ if every open neighbourhood of a in X contains some $V \in \eta$ (see [7]).

Here is another restriction on a compactum B under which we can obtain even a stronger conclusion:

Theorem 3. *Suppose that B is a compact Hausdorff space of countable tightness and that $B = \cup\{Y_n : n \in \omega = \{0, 1, 2, ...\}\}$, where each Y_n is dense in B and is rectifiable. Then B and each Y_n are separable and metrizable.*

Proof. Take any $y \in Y_n$. Then there exists a countable π-base ξ of B at y, since the tightness of the compactum B is countable (see [18] and [3]). Then $\eta = \{V \cap Y_n : V \in \xi\}$ is a countable π-base of the subspace Y_n at y, since Y_n is dense in B. Since Y_n is rectifiable and Y_n has a countable π-base at y, it follows from a result of A. Gul'ko [13] that the space Y_n is metrizable. Therefore, each Y_n has a σ-disjoint open base. Since Y_n is dense in B, it follows that B is first-countable and that σ-disjoint open bases in the subspaces Y_n can be extended, in a standard way, to a point-countable base in B. It remains to use a well-known deep theorem of A.S. Mischenko that every compact Hausdorff space with a point-countable base is separable and metrizable (see [12], Problem 3.12.22(f)). $\square$

The next result considerably generalizes Theorem 2.

Theorem 4. *Suppose that B is a compact Hausdorff space which doesn't admit a continuous mapping onto the Tychonoff cube I^{ω_1}. Suppose further that $B = X \cup Y$, where X and Y are non-locally compact rectifiable spaces. Then B, X, and Y are separable and metrizable.*

Proof. Clearly, Y and X are non-empty. Hence, B is non-empty. Since B cannot be continuously mapped onto the Tychonoff cube I^{ω_1}, it follows from a Theorem of B. E. Shapirovskij (see [18], [3], Theorems 2.2.20 and 3.1.9) that there exists a point $a \in B$ such that B has a countable π-base at a. Without loss of generality, we may assume that $a \in X$. Put $Z = B \setminus X$ and $H = B \setminus Y$. The spaces X and Y are nowhere locally compact, since they are homogeneous and non-locally compact. Clearly, the subspaces Z and H are nowhere locally compact as well. Thus, X, Y, Z, and H are dense in B.

Since X has a countable π-base at a and X is rectifiable, it follows that the space X is metrizable [13]. Hence, X is a space of countable type [1], which implies that the remainder Z of X in B is Lindelöf [14].

By the Dichotomy Theorem for remainders of rectifiable spaces (see [6]), the remainder H of Y in the compactification B of Y is either pseudocompact, or Lindelöf. Notice that H is metrizable, since $H \subset X$.

If H is pseudocompact, then H is compact, since H is metrizable. Therefore, H is closed in B and Y is open in B, which implies that Y is locally compact, a contradiction.

Hence, H is Lindelöf. Then H has a countable base, since H is metrizable. Hence, B has a countable π-base, since H is dense in B, which implies that Y also has a countable π-base, since Y is dense in B. Since the space Y is rectifiable, using again a theorem of A. Gul'ko in [13], we conclude that Y is metrizable. Clearly, Y is separable. It also follows that X is separable, since H is dense in X. Therefore, B is separable and metrizable, as the union of two separable metrizable subspaces (see [12], Corollary 3.1.20). □

The proof of Theorem 2 obviously contains a proof of the next statement:

Theorem 5. *Suppose that B is a compact Hausdorff space and that $B = X \cup Y$, where X and Y are non-locally compact spaces. Suppose further that X is metrizable and Y is rectifiable. Then B, X and Y are separable and metrizable.*

3 On k-gentle paratopological groups

A group G with a topology $\mathcal{T}$ is called a *paratopological* group if the multiplication $(x, y) \to x \cdot y$ is a continuous mapping of $G \times G$ onto G.

Let us call a mapping f of a space X into a space Y *k-gentle* if for every compact subset F of X the image $f(F)$ is also compact.

A group G with a topology will be called *k-gentle* if the inverse mapping $x \to x^{-1}$ is k-gentle.

Proposition 1. *Suppose that B is a compact Hausdorff space in which any non-empty G_δ-subspace has a point of countable character in this subspace. Suppose further that $B = X \cup Y$, where each $Z \in \{X, Y\}$ is a space with the following properties:*

– the space Z is not locally compact;

– if the space Z contains some point of countable character in Z, then the space Z is metrizable;

– if bZ is a Hausdorff compactification of Z, then the remainder $bZ \setminus Z$ is either pseudocompact or Lindelöf.

Then B, X, and Y are separable and metrizable.

Proof. Clearly, Y and X are non-empty, since they are not locally compact. Hence, B is non-empty. Moreover, the sets X and Y are dense in B. Thus, B is a compactification of the subspaces X and Y. There exists a point $a \in B$ such that B is first-countable at a. Without loss of generality, we may assume that $a \in X$. Then the space X is metrizable.

If $b \in X \cap Y$, for some b, then the space Y is metrizable, as a space with the countable character at b. In this case the proof is complete.

Assume that $X \cap Y = \emptyset$. Clearly, X is a space of countable type [1], since X is metrizable. It follows that the remainder Y of X in B is Lindelöf [14].

Clearly, the space X is a remainder of the space Y in B. Hence, X is either pseudocompact or Lindelöf.

Case 1: X is pseudocompact.

Then X is compact, since X is metrizable. Therefore, X is closed in B, a contradiction. Thus, Case 1 is impossible.

Case 2: X is Lindelöf.

Then X is separable, since X is metrizable. Hence, B is separable, which implies that the Souslin number of Y is countable, since Y is dense in B. Lindelöfness of X also implies that Y is a space of countable type, by the theorem of Henriksen and Isbell [14]. Therefore, we can fix a non-empty compact subspace F of Y with a countable base of open neighbourhoods in Y. By the assumptions, there exists a point $c \in F$ such that B is first-countable at c. It follows that the space Y is first-countable at c. Thus Y is a metrizable space with a countable Souslin number. Hence, X and Y are separable and metrizable. It follows that B is separable and metrizable, since B is compact and Hausdorff ([12], Corollary 3.1.20). □

Corollary 6. *Suppose that B is a compact Hausdorff space such that $|B| < 2^{\omega_1}$. Suppose further that $B = X \cup Y$, where X and Y are non-locally compact k-gentle paratopological groups. Then B, X, Y are separable, metrizable spaces, and X, Y are topological groups.*

In view of Proposition 1, this statement follows from the next proposition:

Proposition 2. *Let G be a Hausdorff k-gentle paratopological group such that G is first-countable at some point. Then:*

1) the space G is metrizable;

2) G is a topological group;

3) any remainder of G in a Hausdorff compactification bG of G is either pseudocompact or Lindelöf.

Proof. Since G is a homogeneous space, the space G is first-countable. Every first-countable Hausdorff space is a k-space ([12], Theorem 3.3.20). Hence G is a k-space. It is obvious that if a k-gentle paratopological group is a k-space, then this paratopological group is a topological group. Hence, 2) holds. Every first-countable topological group is metrizable (see [8]). Therefore, 1) holds. By the Dichotomy Theorem for remainders of topological groups (see [4, 5]), since G is a topological group, any remainder of G in a Hausdorff compactification of G is either pseudocompact or Lindelöf. Therefore, 3) holds. □

Example 7. Let X_1 be the space of all rational numbers of the interval $I = [0, 1]$. Clearly, X_1 is homeomorphic to a topological group. The space $Y_1 = I \setminus X_1$ is also homeomorphic to a topological group. Take also the topological group D^{ω_1}. Put $B = I \times D^{\omega_1}$, $X = X_1 \times D^{\omega_1}$ and $Y = Y_1 \times D^{\omega_1}$. Then X and Y are dense non-metrizable nowhere locally compact topological groups, B is a homogeneous compact Hausdorff space with the cardinality 2^{ω_1}, and $B = X \cup Y$. The space B admits a continuous mapping onto I^{ω_1} and the tightness $t(B) = \aleph_1$. Thus the respective cardinal assumptions in Theorems 2, 3, 4 and Corollary 6 are essential.

We could also modify the definitions of B, X, and Y above so that each of the spaces B, X, Y would admit a structure of a topological group.

4 On Mal'cev spaces. Some questions

A Mal'cev operation on a space X is a continuous mapping $\mu : X^3 \to X$ such that $\mu(x,x,z) = z$ and $\mu(x,y,y) = x$, for all $x,y,z \in X$. A space is called a *Mal'cev space* if it admits a Mal'cev operation (see [9–11, 16, 19]).

A homogeneous algebra on a space G is a pair of binary continuous operations $p, q : G \times G \to G$ such that $p(x,x) = p(y,y)$, and $p(x, q(x,y)) = y$, $q(x, p(x,y)) = y$ for all $x, y \in G$. If the above conditions are satisfied, then the ternary operation $\mu(x,x,z) = q(x, p(y,z))$ is a Mal'cev operation (see [9, 10]).

A biternary algebra on a space G is a pair of ternary continuous operations $\alpha, \beta : G \times G \times G \to G$ such that $\alpha(x,x,y) = y$, $\alpha(\beta(x,y,z), y, z)) = x$, and $\beta(\alpha(x,y,z), y, z)) = x$, for all $x, y, z \in G$ (see [16]).

In [9, 10] (see also [19]) it was proved that for an arbitrary space G the following conditions are equivalent:

1) G is a rectifiable space;
2) G is homeomorphic to a homogeneous algebra;
3) There exists a structure of a biternary algebra on G.

A structure of *a topological quasigroup* on a space G is a triplet of binary continuous operations $p, l, r : G \times G \times G \to G$ such that $p(x, l(x,y)) = p(r(y,x), x) = l(x, p(x,y)) = l(r(x,y), x) = r(p(y,x), x) = r(x, l(y,x)) = y$, for all $x, y \in G$. If there exists an element $e \in G$ such that $p(e,x) = p(x,e) = x$ for any $x \in G$, then we say that G is *a topological loop* and e is the identity of G. Any topological quasigroup admits the structure of a topological loop (see [16]). If $e \in G$ and $p(e,x) = x$ for any $x \in G$, then $x + y = p(y,x)$ and $x \cdot y = r(y,x)$ is a structure of a homogeneous algebra.

If $(G, \cdot)$ is a topological group with the neutral element e, then the mapping $\varphi(x,y) = (x, x^{-1} \cdot y)$ is a rectification on the space G with the neutral element e, and the mappings $p(x,y) = x^{-1} \cdot y$ and $q(x,y) = x \cdot y$ form a structure of homogeneous algebra on G. Therefore, every topological quasigroup is a rectifiable space.

A space X is called *κ-perfect* if the closure of any open subset of X is a G_δ-set in X.

Proposition 3. *Let X and Y be any pseudocompact κ-perfect subspaces of a Tychonoff space Z such that $Z = X \cup Y$, and X, Y are dense in Z. Suppose further that X and Y are Mal'cev spaces. Then the space Z is also κ-perfect.*

Proof. Let U be an open subset of the space Z. We put $V = X \cap U$ and $W = Y \cap U$. There exist two sequences $\{V_n : n \in \omega\}$ and $\{W_n : n \in \omega\}$ of open subsets of the space Z such that:

– $cl_X V = \cap\{V_n \cap X : n \in \omega\}$ and $cl_Y W = \cap\{W_n \cap Y : n \in \omega\}$;
– $V_{n+1} \subseteq V_n$ and $W_{n+1} \subseteq W_n$ for any $n \in \omega$.

Obviously, $cl_Z U = cl_Z V = cl_Z W$. We put $F + cl_z U$.

We affirm that $\cap\{V_n : n \in \omega\} \subseteq F$. Assume that $H = (\cap\{V_n : n \in \omega\}) \setminus F \neq \emptyset$. By construction, H is a G_δ-subset of Z and $H \subseteq Y$. Fix a point $b \in H$. There exists a continuous function $f : Z \longrightarrow [0,1]$ such that $b \in f^{-1}(0) \subseteq H$. Then the function $g(x) = 1/f(x)$ is a continuous unbounded function on the space X, a contradiction. Thus $\cap\{V_n : n \in \omega\} \subseteq F$ $\cap\{W_n : n \in \omega\} \subseteq F$. If $U_n = V_n \cup W_n$, then $\cap\{W_n : n \in \omega\} = F$. □

Proposition 4. *Let X and Y be pseudocompact subspaces of a compact Hausdorff space B such that X and Y are dense in B and $Z = X \cup Y$. Suppose further that X and Y are Mal'cev spaces. Then:*

1) The space B is a κ-perfect Mal'cev space.

2) There exist Mal'cev operations $\mu, \eta : B^3 \longrightarrow B$ on B such that $\mu(X^3) = X$ and $\eta(Y^3) = Y$.

3) If X is rectifiable, then B is also rectifiable, and there exists a structure of homogeneous algebra $\{+, \cdot\}$ on B such that X is a subagebra of B.

4) If X is a topological quasigroup, then there exists a structure of a topological loop on B such that X is a subloop of B.

5) If the space X is a topological group, then on there exists a structure of a topological group on B such that X is a subgroup of B.

6) $B = \beta X = \beta Y$.

Proof. Since X is a pseudocompact Mal'cev space, the Stone-Čech compactification βX of X is a compact Mal'cev space [17]. Any compact Mal'cev space is κ-perfect [10]. Thus, X and Y are κ-perfect spaces, since they are dense subspaces of κ-perfect spaces. By Proposition 3, the space B is κ-perfect. Now we need the following known fact:

Fact 1: If Z is a pseudocompact subspace of a κ-perfect compact Hausdorff space B such that Z is dense in B, then $B = \beta Z$.

Really, let F_1 and F_2 be two closed subsets of Z and $f : Z \longrightarrow \mathbb{R}$ be a continuous function such that $F_1 \subseteq f^{-1}(-2)$ and $F_\subseteq f^{-1}(2)$. There exist two open subsets U and V of B such that $U \cap Z \subseteq f^{-1}(-3,-1)$ and $V \cap Z \subseteq f^{-1}(1,3)$. Then $H = cl_B U \cap cl_B V$ is a G_δ-subset of B and, by construction, $H \subseteq B \setminus Z$. Since Z is pseudocompact, we have $H = \emptyset$. Since $F_1 \subseteq U$ and $F_2 \subseteq V$, we have $cl_B F_1 \cap cl_B F_2 = \emptyset$. Therefore $B = \beta Z$.

Thus, $B = \beta X = \beta Y$. Statements 1 and 6 are proved. The space X^n is pseudocompact for any $n \in \omega$. Hence, by virtue of Glicksberg's Theorem ([12], Problem 3.12.20(d)), any continuous binary operation on X admits continuous extension on B. Statements 2, 3 and 4 are established. □

Proposition 5. *Let X be a subalgebra of a homogeneous algebra G. If the space G is regular and Lindelöf, and the space X is of pointwise countable type and is dense in G, then there exist a separable metrizable homogeneous algebra G' and a homomorphism $g : G \longrightarrow G'$ such that $X = g^{-1}(g(X))$ and the mapping g is open and perfect. In particular, it follows that X is a Lindelöf p-space.*

Proof. By the assumptions, there is a pair of binary continuous operations $p, q : G \times G \to G$ on the space G such that:

– $p(x,x) = p(y,y)$, and $p(x, q(x,y)) = y$, $q(x, p(x,y)) = y$ for all $x, y \in G$;

– $p(x,y) \in X$ and $q(x,y) \in X$ for all $x, y \in X$.

We put $e = p(x,x)$. If $a \in G$, then $p_a(x) = p(a,x)$ and $q_a(x) = q(a,x)$ for any $x \in G$. We have $q_a^{-1} = p_a$ and $q_a(e) = a$. Thus, p_a and q_a are homeomorphisms. Moreover, $p_a(X) = q_a(X) = X$ for each $a \in X$.

Let F be a non-empty compact subspace of X with a countable base of open neighbourhoods in X. We can assume that $e \in F$. Since X is dense in G, the set F also has a countable base of open neighbourhoods in the space G. Therefore, X and G are p-spaces (see [6], Proposition 2.1).

Since F is a compact G_δ-subset of the Lindelöf algebra G, there exist a separable metrizable homogeneous algebra G' and a homomorphism $g : G \longrightarrow G'$ such that $F = g^{-1}(g(F))$ and g is a perfect mapping [10]. The quotient homomorphism of a Mal'cev algebra is an open mapping [10]. Thus, the mapping g is open. We can assume that $e' = g(e)$ and $p(z,z) = e'$ for any $z \in G'$.

Let $b \in g(X) \subseteq G'$. Fix $a \in X \cap g^{-1}(b)$. If $H = g^{-1}(e'$, then $H \subseteq F \subseteq X$ and $q_a(H) = g^{-1}(q_b(e') \subseteq X$. Thus, $g^{-1}(b) = g^{-1}(q_b(e') \subseteq X$. Therefore, $g^{-1}(g(X)) = X$. The proof is complete. □

Since a pseudocompact rectifiable space X can be considered as a subalgebra of the compact homogeneous algebra $G = \beta X$, Proposition 5 yields

Corollary 8. *If G is a pseudocompact rectifiable space of pointwise countable type, then G is compact.*

Theorem 9. *Suppose that B is a compact Hausdorff space, $B = X \cup Y$ and $X \cap Y = \emptyset$, where X and Y are non-locally compact rectifiable spaces. Then X and Y either are both pseudocompact, or are both Lindelöf p-spaces.*

Proof. Clearly, X is the remainder of Y in the Hausdorff compactification B of Y; similarly, Y is the remainder of X in the Hausdorff compactification B of X.

By the Dichotomy Theorem for remainders of rectifiable spaces (see [6]), each of the spaces X and Y is either pseudocompact, or Lindelöf. If both of them are pseudocompact, then we are done.

Assume now that at least one of the subspaces X and Y, say X, is Lindelöf.

By the Dichotomy Theorem for remainders in [6], the remainder Y of X in B is either pseudocompact or Lindelöf.

Case 1: Y is pseudocompact.

Lindelöfness of X implies that Y is a space of countable type, by the theorem of Henriksen and Isbell [14]. Then Corollary 8 implies that Y is compact. Therefore, Y is closed in B. Hence, X is open in B, which implies that X is locally compact, a contradiction. Thus, Case 1 is impossible.

Case 2: Y is Lindelöf.

Lindelöfness of X and Y implies that X and Y are spaces of countable type, by the theorem of Henriksen and Isbell [14]. Therefore, by Proposition 2.1 from [6], X and Y are Lindelöf p-spaces. The proof is complete. □

Observe that Theorem 9 doesn't generalize to homogeneous Mal'cev spaces. Indeed, a non-metrizable compactum can be represented as the union of two disjoint dense copies of Sorgenfrey line (take the "double arrow" space). It was shown in [17] that Sorgenfrey line is a Mal'cev space. It is well-known that Sorgenfrey line is not a p-space (see [5]). It is also clear that Sorgenfrey line is not pseudocompact.

Question 1. Is every rectifiable space of countable type paracompact? Normal?

Question 2. Does every rectifiable space of countable type admit a perfect mapping onto a metrizable (rectifiable) space?

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Saddle points with respect to a set

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Abstract. An extension of the concept of saddle point, a continuous property of two functions related to saddle points with respect to a set and a theorem of existence of saddle points with respect to a set are given. The paper ends with an example which shows that the proved theorems are consistent.

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1 Introduction

Let A and B be nonempty sets and $f : A \times B \to \mathbf{R}$ be a function. We remember that a point $(a, b) \in A \times B$ is said to be a saddle point of f on $A \times B$ if

$$f(x, b) \leq f(a, b) \leq f(a, y), \quad \text{for all}\,(x, y) \in A \times B. \tag{1}$$

The condition (1) is equivalent to

$$\max_{x\in A} \min_{y\in B} f(x, y) \; = \; \min_{y\in B} \max_{x\in A} f(x, y). \tag{2}$$

Let us consider a two-person zero-sum game G_f generated by the function f. This means that the first player selects a point x from A and the second player selects a point y from B. As a result of this choise, the second player pays the first one the amount $f(x, y)$. Then a point $(a, b) \in A \times B$ is a solution of the game G_f if and only if it is a saddle point of f on $A \times B$.

The first saddle point theorem was proved by von Neumann [11]. Von Neumann's theorem can be stated as follows: if A and B are finite dimensional simplices and f is a bilinear function on $A \times B$, then f has a saddle point; i.e (2) holds. M. Shiffman [14] seems to have been the first to have considered concave-convex functions in a saddle point theorem. H. Kneser [10], K. Fan [6], and C. Berge [1] (using induction and the method of separating two disjoint convex sets in Euclidian space by a hyperplane) proved saddle point theorems for concave-convex functions that are appropriately semicontinuous in one of the two variables. H. Nikaido [12], on the other hand, using Brouwer's fixed point theorem, proved the existence of a saddle point for a function satisfying the weaker algebraic condition of being quasi-concave-convex, but the stronger topological condition of being continuous in each variable. M. Sion [16] proved a very general saddle point theorem for a function which is quasi-concave and upper semicontinuous in its first variabile and quasi-convex and lower semicontinuous in its second variable in a topological vector space.

Most of the effort has been spent on relaxing the assumptions on the concave-convexity of f and also on the compactness condition for one of the sets A and B. As examples we can give the papers of H. Tuy [17–19], J. Hartung [8], U. Passy and E. Z. Prisman [13], G. H. Greco and C. D. Horvath [7], S. Simons [15], J. Yu and X. Z. Yuan [20], etc.

A little less study was dedicated to the case when the function f is defined on a proper subset M of $A \times B$ (for example P. S. Kenderov and R. E. Lucchetti [9].

This problem arises, for instance, in connection with the following two-player game. The first player wants to choose a strategy $a \in A$ such that his payoff $f(x, y)$ is maximum. This choice depends on the choice $y \in B$ of the second player. Once the leading player chooses some strategies $x \in A$, the "move" of the second player is to choose some y in the set of all the feasible strategies $y \in M_2^x \subseteq B$. Then the value

$$\underline{v} = \max_{x \in A} \min_{y \in M_2^x} f(x, y)$$

is the maximum payoff that can be guaranteed for the first player.

Analogously, for the second player, the value

$$\overline{v} = \min_{y \in B} \max_{x \in M_1^y} f(x, y)$$

is the minimum loss that can be guaranteed for the second player.

Do $\underline{v}$ and $\overline{v}$ exist ? If so, is $\underline{v} = \overline{v}$ equivalent to (1)? Therefore, in this paper we study this problem by means of the notion of *saddle point with respect to a set.*

In Section 2 we give the definition of saddle point with respect to a set and we show that this notion is effectively a generalization of the notion of saddle point. In Section 3 some properties of the function $\underline{f}$ and $\overline{f}$ defined by (4) and (5) are given. The existence of saddle point of bi-$(1_{\mathbb{R}^m}, 1_{\mathbb{R}^n})$ strictly concave-convex functions is studied in Section 4. The paper ends with an example which shows that the proved theorems are consistent.

2 Saddle points with respect to a set

Let A and B be nonempty sets and M be a nonempty subset of $A \times B$. We put

$$M_1 = pr_A M = \{x \in A \mid \exists y \in B \text{ such that } (x, y) \in M\},$$

$$M_2 = pr_B M = \{y \in B \mid \exists x \in A \text{ such that } (x, y) \in M\}.$$

For each $x \in M_1$ we denote by

$$M_2^x = \{y \in M_2 \,|\, (x, y) \in M\} \subseteq B$$

and for each $y \in M_2$ we denote by

$$M_1^y = \{x \in M_1 \,|\, (x, y) \in M\} \subseteq A.$$

Throughout the paper, M_1, M_2, M_2^x M_1^y, where $x \in M_1$ and $y \in M_2$, will always have this meaning.

Definition. *Let A and B be nonempty sets, M be a nonempty subset of $A \times B$ and $f : M \to \mathbf{R}$ be a function. A point $(a, b) \in M$ is called a saddle point of f with respect to M if*

$$f(x, b) \leq f(a, b) \leq f(a, y), \tag{3}$$

for all $x \in M_1^b$ and all $y \in M_2^a$.

Example 1. Let

$$M = \{(x, y) \in [0, 1] \times [0, 1] : \ x \leq y^2\} \subseteq \mathbb{R} \times \mathbb{R},$$

and $f : M \to \mathbb{R}$ be defined by $f(x, y) = -x^2 + y^4$, for all $(x, y) \in M$. For $(a, b) = (0, 0) \in M$, we have $f(x, 0) = -x^2 \leq 0 = f(0, 0) \leq y^4 = f(0, y)$, for all $x \in M_1^0 = \{0\}$, and $y \in M_2^0 = [0,\ 1]$. It follows that $(a, b) = (0, 0)$ is a saddle point of f with respect to M.$\diamond$

Example 2. Let $A = [1, 3]$, $B = [1, 2]$, $M = \{(x, 2) | x \in [1, 3]\} \bigcup \{(2, y) | y \in [1, 2]\}$ and $f : M \to \mathbf{R}$, $f(x, y) = x \cdot y$, for all $(x, y) \in M$. Because for the point $(a, b) = (2, 1) \in M$ we have $f(x, 1) = x \leq 2 = f(2, 1) \leq 2y = f(2, y)$, for all $x \in M_1^b = \{2\}$, and $y \in M_2^a = [1, 2]$, this point is a saddle point of f with respect to M.$\diamond$

If $M = A \times B$, then $M_1 = A$, $M_2 = B$ and, for each $x \in M_1$ and each $y \in M_2$, we have $M_x = A$ and $M_y = B$. It follows that if $f : A \times B \to \mathbf{R}$ is a function, then condition (3) is equivalent to condition (1). Hence the notion of saddle point with respect to a set is a generalization of the notion of saddle point.

Remark 1. If $f : A \times B \to \mathbf{R}$ is a function and $(a, b) \in A \times B$ is a saddle point of f, then (a, b) is also a saddle point of f with respect to M, for each subset M of $A \times B$ which has the property that $(a, b) \in M$.

Usually, the converse is not true, as it can be seen below:

Example 3. Let $A = [1, 3]$, $B = [1, 2]$, $M = \{(x, 2) \,|\, x \in [1, 3]\} \bigcup \{(2, y) \,|\, y \in [1, 2]\}$ and $f : A \times B \to \mathbf{R}$, $f(x, y) = x \cdot y$, for all $(x, y) \in A \times B$. Then, the point $(a, b) = (2, 1) \in M$ is a saddle point of f with respect to M (see Example 1), but $(a, b) = (2, 1) \in A \times B$ is not a saddle point of f (in the classical sense) because $f(3, 2) = 6 \not\leq 2 = f(2, 1)$.$\diamond$

3 Some properties of the functions $\underline{f}$ and $\overline{f}$

If $f : M \to \mathbf{R}$ is a continuous function and M is a compact nonempty subset of $\mathbf{R}^m \times \mathbf{R}^n$, we consider the functions $\underline{f} : M_1 \to \mathbf{R}$ and $\overline{f} : M_2 \to \mathbf{R}$ defined by

$$\underline{f}(x) \ = \ \min\{f(x, y) | y \in M_2^x\}, \text{ for all } x \in M_1, \tag{4}$$

$$\overline{f}(y) \ = \ \max\{f(x, y) | x \in M_1^y\}, \text{ for all } y \in M_2. \tag{5}$$

The following assertion holds.

Theorem 1. If $M \subseteq \mathbf{R}^m \times \mathbf{R}^n$ is a nonempty set, $f : M \to \mathbf{R}$ is a function and $(a, b) \in M$ is a saddle point of f with respect to M, then

$$\underline{f}(a) = \overline{f}(b) = f(a, b). \tag{6}$$

Proof. From $f(x, b) \leq f(a, b)$, for all $x \in M_1^b$, we conclude that $f(a, b) = \max\{f(x, b) \,|\, x \in M_1^b\} = \overline{f}(b)$, and from $f(a, b) \leq f(a, y)$, for all $y \in M_2^a$, we deduce that $f(a, b) = \min\{f(a, y) \,|\, y \in M_2^a\} = \underline{f}(a)$. Hence (6) is true. □

In the case when $A \subseteq \mathbf{R}^m$ and $B \subseteq \mathbf{R}^n$ are nonempty compact sets,

$$M = A \times B \subseteq \mathbf{R}^m \times \mathbf{R}^n$$

and $f : M \to \mathbf{R}$ is a continuous function, then the functions $\underline{f}$ and $\overline{f}$ are also continuous on $M_1 = A$, respectively, on $M_2 = B$. If M is not a cartesian product, this property is not true, as seen in the following example.

Example 4. Let $M = (\{0\} \times [0,\, 1]) \bigcup ([0,\, 3] \times \{1\})$ and $f : M \to \mathbf{R}$ be the function given by $f(x, y) = \ln(11 - x + y^2)$, for all $(x, y) \in M$. We have $M_1 = [0, 3]$, $M_2 = [0, 1]$,

$$M_2^x = \begin{cases} [0,\, 1], & \text{if } x = 0 \\ \{1\}, & \text{if } x \in]0, 3] \end{cases}, \quad M_1^y = \begin{cases} \{0\}, & \text{if } y \in [0,\, 1[\\ [1,\, 3], & \text{if } y = 1 \end{cases}.$$

The function $\underline{f} : M_1 \to \mathbf{R}$, given by

$$\underline{f}(x) = \begin{cases} \ln 11, & \text{if } x = 0 \\ \ln(12 - x), & \text{if } x \in]0, 3] \end{cases}$$

is not continuous. Moreover, $\max\{\underline{f}(x) | x \in M_1\}$ does not exist. But the function f has a saddle point with respect to M; this point is $(0, 0)$.$\diamond$

Remark 2. Let $A \subseteq \mathbf{R}^m$ and $B \subseteq \mathbf{R}^n$ be nonempty sets, M a nonempty subset of $A \times B$, $f : M \to \mathbf{R}$ be a function and $(a, b) \in M$. The following statements are true:

i) If $a \in M_1$ is a maximum point of $\underline{f}$ and $M_1^b = \{a\}$, then (a, b) is a saddle point of f with respect to M.

ii) If $b \in M_2$ is a minimum point of $\overline{f}$ and $M_2^a = \{b\}$, then (a, b) is a saddle point of f with respect to M.

Theorem 2. *Let $M \subseteq \mathbf{R}^m \times \mathbf{R}^n$ be a nonempty compact set, $a \in M_1$, $b \in M_2$ and $f : M \to \mathbf{R}$ be a continuous function. If*

i) $M_2^a = M_2$,

ii) b *is a minimum point of the function* $f(a, \cdot) : M_2 \to \mathbf{R}$,

iii) *there exists a real number* $\delta > 0$ *such that*

$$M_1 \bigcap B(a, \delta) \subseteq M_1^b,$$

then the function $\underline{f} : M_1 \to \mathbf{R}$ *defined by*

$$\underline{f}(x) = \min\{f(x,y) | y \in M_2^x\}, \text{ for all } x \in M_1,$$

is continuous at a.

Proof. We begin the proof by noticing that, for each $x \in M_1$, $y \in M_2$, the sets M_2^x and M_1^y are compact. In order to show that $\underline{f}$ is continuous at a, let $\varepsilon > 0$. Then, by the continuity of f on the compact M, there exists a positive real number δ_ε such that

$$|f(x,y) - f(u,v)| < \frac{\varepsilon}{2},$$

for each (x,y), $(u,v) \in M$ with

$$||(x,y) - (u,v)|| < \delta_\varepsilon.$$

Let $\tilde{\delta} = \min\{\delta_\varepsilon, \delta\}$ and $x \in M_1 \bigcap B(a, \tilde{\delta})$. Then, for each $y \in M_2^x$, we have $y \in M_2^a$, because $M_2^x \subseteq M_2 = M_2^a$. Also, for each $y \in M_2^x$, we have

$$|f(x,y) - f(a,y)| < \frac{\varepsilon}{2}, \tag{7}$$

because

$$||(x,y) - (a,y)|| < \tilde{\delta}.$$

It follows that

$$f(x,y) > f(a,y) - \frac{\varepsilon}{2} \geq \min\{f(a,v) | v \in M_2^a\} - \frac{\varepsilon}{2} = \underline{f}(a) - \frac{\varepsilon}{2},$$

for each $y \in M_2^x$. Since M_2^x is compact and f is continuous, we have that

$$\underline{f}(x) = \min\{f(x,y) | y \in M_2^x\} \geq \underline{f}(a) - \frac{\varepsilon}{2} > \underline{f}(a) - \varepsilon,$$

i.e.

$$\underline{f}(x) - \underline{f}(a) > -\varepsilon, \text{ for each } x \in M_1 \bigcap B(a, \tilde{\delta}). \tag{8}$$

On the other hand, from

$$M_1 \bigcap B(a, \tilde{\delta}) \subseteq M_1^b,$$

we deduce that $x \in M_1 \bigcap B(a, \tilde{\delta})$ implies $x \in M_1^b$, i.e. $b \in M_2^x$. Then, by (7), it follows that

$$|f(x,b) - f(a,b)| < \frac{\varepsilon}{2},$$

for each $x \in M_1 \bigcap B(a, \tilde{\delta})$. Hence

$$f(x,b) < f(a,b) + \frac{\varepsilon}{2},$$

for each $x \in M_1 \bigcap B(a, \tilde{\delta})$.

It follows that, for each $x \in M_1 \bigcap B(a, \tilde{\delta})$, we have

$$\underline{f}(x) = \min\{f(x,y) | y \in M_2^x\} \leq f(x,b) < f(a,b) + \frac{\varepsilon}{2} =$$

$$= \min\{f(a,y) | y \in M_2^a\} + \frac{\varepsilon}{2} = \underline{f}(a) + \frac{\varepsilon}{2},$$

by hypothesis ii). Consequently,

$$\underline{f}(x) - \underline{f}(a) < \varepsilon, \text{ for all } x \in M_1 \bigcap B(a, \tilde{\delta}). \tag{9}$$

By (8) and (9), the theorem follows. □

By analogy we have:

Theorem 3. *Let $M \subseteq \mathbf{R}^m \times \mathbf{R}^n$ be a nonempty compact set, $b \in M_2$, $a \in M_1$ and $f : M \to \mathbf{R}$ be a continuous function. If*

i) $M_1^b = M_1$,

ii) $a \in M_1$ is a maximum point of the function $f(\cdot, b) : M_1 \to \mathbf{R}$,

iii) there exists a real number $\delta > 0$ such that

$$M_2 \bigcap B(b, \delta) \subseteq M_2^a,$$

then the function $\overline{f} : M_2 \to \mathbf{R}$ defined by

$$\overline{f}(y) = \max\{f(x,y) | x \in M_1^y\}, \text{ for all } y \in M_2,$$

is continuous at b.

Remark 3. Let $A \subseteq \mathbf{R}^m$ and $B \subseteq \mathbf{R}^n$ be nonempty sets, M be a nonempty subset of $A \times B$, $f : M \to \mathbf{R}$ be a function and $(a, b) \in M$. If $M = A \times B$, then the conditions i) and iii) are self satisfied; therefore, in this case we obtain the classical theorem with respect to the continuity of the functions $\underline{f}$ and $\overline{f}$.

4 Saddle points of bi-$(1_{\mathbf{R}^m}, 1_{\mathbf{R}^n})$ strictly concave-convex functions

First we recall the notions of bi-(φ, ψ) convex set (see [3]) and bi-$(1_{\mathbf{R}^m}, 1_{\mathbf{R}^n})$ concave-convex function (see [4] and [5]).

Definition. *Let $\varphi : \mathbf{R}^m \to \mathbf{R}^m$ and $\psi : \mathbf{R}^n \to \mathbf{R}^n$ be two maps. A subset M of $\mathbf{R}^m \times \mathbf{R}^n$ is said to be bi-(φ, ψ) convex either if $M = \emptyset$ or, if for every (x, y), (x, v), (u, y) of M and every $t \in [0, 1]$ we have*

$$(\varphi(x), (1-t)\psi(y) + t\psi(v)) \in M \tag{10}$$

and

$$((1-t)\varphi(x) + t\varphi(u), \psi(y)) \in M. \tag{11}$$

We remark that if $M \subseteq \mathbf{R}^m \times \mathbf{R}^n$ is convex (in the classical sense), then M is bi-$(1_{\mathbf{R}^m}, 1_{\mathbf{R}^n})$ convex. The converse is not necessarily true (see [4]).

Example 5. The set $M = \{(x, y) \in [0, 1] \times [0, 1] \,:\, x \leq y^2\} \subseteq \mathbb{R} \times \mathbb{R}$, is bi-$(1_{\mathbb{R}}, 1_{\mathbb{R}})$ convex, but not convex. Indeed, let (x, y), (x, v), $(u, v) \in M$, and $t \in [0, 1]$. Then $0 \leq x \leq y^2 \leq 1$, $0 \leq x \leq v^2 \leq 1$, $0 \leq u \leq v^2 \leq 1$. It follows that $(x, (1-t)y + tv) \in [0, 1] \times [0, 1]$, $(1-t)x + tu, y) \in [0, 1] \times [0, 1]$, and

$$((1-t)y + tv)^2 = (1-t)^2 y^2 + 2t(1-t)yv + t^2 v^2 \geq (1-t)^2 x + 2t(1-t)x + t^2 x = x,$$

$(1-t)x + tu \leq (1-t)v^2 + tv^2 = v^2$. Hence $(1_{\mathbb{R}}(x), (1-t)1_{\mathbb{R}}(y) + t1_{\mathbb{R}}(v)) = (x, (1-t)y + tv) \in M$ and $((1-t)\,1_{\mathbb{R}}\,(x) + t1_{\mathbb{R}}\,(u)\,, 1_{\mathbb{R}}\,(y)) = ((1-t)\,x + tu, y) \in M$. On the other hand, $(1 - 1/2)\,(0, 0) + (1/2)\,(1, 1) = (1/2, 1/2) \notin M$. Consequently, the set M is bi-$(1_{\mathbb{R}}, 1_{\mathbb{R}})$ convex but not convex.$\diamond$

Definition. *Let $M \subseteq \mathbf{R}^m \times \mathbf{R}^n$ be a bi-$(1_{\mathbf{R}^m}, 1_{\mathbf{R}^n})$ convex set. A function $f : M \to \mathbf{R}$ is said to be bi-$(1_{\mathbf{R}^m}, 1_{\mathbf{R}^n})$ concave-convex (strictly concave-convex) if for every $x \in M_1$ the function $f(x, \cdot) : M_x \to \mathbf{R}$ is convex (strictly convex) and for every $y \in M_2$ the function $f(\cdot, y) : M_y \to \mathbf{R}$ is concave (strictly concave).*

Example 6. Let $M = \{(x, y) \in [0, 1] \times [0, 1] :\ x \leq y^2\} \subseteq \mathbb{R} \times \mathbb{R}$, and $f : M \to \mathbb{R}$ be defined by $f\,(x, y) = -x^2 + y^4$, for all $(x, y) \in M$. The set M is bi-$(1_{\mathbb{R}}, 1_{\mathbb{R}})$ convex (see Example 5). One can easily show that f is bi-$(1_{\mathbb{R}}, 1_{\mathbb{R}})$ strictly concave-convex (hence bi-$(1_{\mathbb{R}}, 1_{\mathbb{R}})$ concave-convex).$\diamond$

More properties of them can be found in Refs. [3–5].

Theorem 4. *Let M be a compact nonempty subset of $\mathbf{R}^m \times \mathbf{R}^n$, $(a, b) \in M$ and $f : M \to \mathbf{R}$ be a continuous function. If*

i) the set M is bi-$(1_{\mathbf{R}^m}, 1_{\mathbf{R}^n})$ convex,
ii) the function f is bi-$(1_{\mathbf{R}^m}, 1_{\mathbf{R}^n})$ strictly concave-convex,
iii) a is a maximum point of the function $\underline{f} : M_1 \to \mathbf{R}$ defined by

$$\underline{f}(x) \;=\; \min\{f(x, y) | y \in M_2^x\},\ \textit{for all } x \in M_1,$$

iv) b is a minimum point of the function $f(a, \cdot) : M_a \to \mathbf{R}$,
v) $M_2^a = M_2$,
vi) there is a real number $\delta > 0$ such that

$$B(b, \delta) \bigcap M_2 \subseteq M_2^x,\ \textit{for all}\ \ x \in M_1^b,$$

then (a, b) is a saddle-point of f with respect to M.

Proof. From hypothesis *iv*), we have

$$f(a, b) \leq f(a, y), \quad \text{for all } y \in M_2^a.$$

Let us show that

$$f(a, b) \geq f(x, b), \quad \text{for all } x \in M_1^b. \tag{12}$$

Assume, by contradiction, that there exists a point $\tilde{x} \in M_1^b$ such that

$$f(a,b) \; < \; f(\tilde{x},b).$$

Then, by the continuity of the function $f(\tilde{x},\cdot) : M_2^{\tilde{x}} \to \mathbf{R}$, there exists an open neighborhood V of the point b such that

$$f(a,b) < f(\tilde{x},y), \text{ for all } y \in V \bigcap M_2^{\tilde{x}}.$$

Without loss of generality, we can suppose that

$$V \subseteq B(b;\delta). \tag{13}$$

Since $\tilde{x} \in M_1^b$, $vi)$ and (13) imply

$$V \bigcap M_2 \subseteq B(b;\delta) \bigcap M_2 \subseteq M_2^{\tilde{x}}. \tag{14}$$

From $ii)$ and $iv)$, we deduce that

$$f(a,b) < f(a,y), \quad \text{for all } y \in M_2^a \setminus \{b\},$$

and hence

$$f(a,b) < f(a,y), \quad \text{for all } y \in M_2^a \setminus V. \tag{15}$$

On the other hand, from $iii)$ and $iv)$ we have

$$f(a,b) = \min\{f(a,y)|y \in M_2^a\} = \underline{f}(a) \geq \underline{f}(\tilde{x}) = \min\{f(\tilde{x},y)|y \in M_2^{\tilde{x}}\},$$

and hence there is a point $\tilde{y} \in M_2^{\tilde{x}}$ such that

$$f(a,b) \geq f(\tilde{x},\tilde{y}).$$

Consequently, $\tilde{y} \notin V$ and hence $\tilde{y} \in M_2^{\tilde{x}} \setminus V$.

Then, from $v)$, we obtain that

$$\tilde{y} \in M_2^{\tilde{x}} \setminus V \subseteq M_2^a \setminus V.$$

Since $M_2^a \setminus V$ is nonempty and compact and the function $f(a,\cdot) : M_2^a \to \mathbf{R}$ is continuous, then there exists $\min\{f(a,y)|y \in M_2^a \setminus V\}$ and, by (15),

$$\min\{f(a,y)|y \in M_2^a \setminus V\} \; > \; f(a,b).$$

Let

$$\varepsilon = \min\{f(a,y)|y \in M_2^a \setminus V\} \; - \; f(a,b).$$

Then, by the continuity of f and the compactness of M, there exists a real number $\mu > 0$ such that

$$|f(x,y) \; - \; f(u,v)| \; < \; \varepsilon, \tag{16}$$

for all $(x,y),(u,v)\in M$ with

$$||(x,y)-(u,v)||<\mu.$$

Let

$$t=\min\left\{\frac{3}{4},\frac{\delta}{2||\tilde{x}-a||},\frac{\mu}{2||\tilde{x}-a||}\right\},$$

and

$$x^*=(1-t)a+t\tilde{x}. \tag{17}$$

Obviously $0<t<1$. We will show that

$$f(x^*,y)>f(a,b),\quad \text{for all } y\in M_2^{x^*}. \tag{18}$$

Let $y\in M_2^{x^*}$. We distinguish two cases.

Case 1) If $y\in V$, by (14), we have

$$y\in V\bigcap M_2^{\tilde{x}},$$

i.e. $(\tilde{x},y)\in M$. Then

$$f(\tilde{x},y)>f(a,b).$$

Also, from $v)$ we have $(a,y)\in M$. In view of $i)$ and (17), we get $(x^*,y)\in M$. Since f is bi-$(1_{\mathbf{R}^m},1_{\mathbf{R}^n})$ strictly concave-convex, we have

$$f(x^*,y)=f((1-t)a+t\tilde{x},y)>(1-t)f(a,y)+tf(\tilde{x},y)\geq$$

$$\geq(1-t)f(a,b)+tf(a,b)=f(a,b),$$

because

$$f(a,y)\geq\min\{f(a,v)|v\in M_2^a\}=\underline{f}(a)=f(a,b).$$

Case 2) If $y\in M_2^{x^*}\setminus V$, from

$$||(x^*,y)-(a,y)||\;=\;||x^*-a||\;<\;\mu$$

and (16), we have

$$|f(x^*,y)-f(a,y)|<\varepsilon.$$

Then

$$f(x^*,y)=f(x^*,y)-f(a,y)+f(a,y)>-\varepsilon+f(a,y)=$$

$$=-\min\{f(a,v)|v\in M_2^a\setminus V\}\;+\;f(a,b).+f\,(a,y)\geq f(a,b).$$

Consequently, (18) is true.

By (18), it follows that

$$\underline{f}(x^*)=\min\{f(x^*,y)|y\in M_2^{x^*}\}>f(a,b)=\underline{f}(a)$$

which contradicts hypothesis $iii)$. Then (12) is true and hence (a,b) is a saddle-point of f with respect to M. □

Example 7. For

$$M = \{(x, y) \in [0, 1] \times [0, 1] : \; x \leq y^2\} \subseteq \mathbb{R} \times \mathbb{R},$$

$(a, b) = (0, 0) \in M$, and $f : M \to \mathbb{R}$ defined by

$$f(x, y) = -x^2 + y^4, \text{ for all } (x, y) \in M,$$

the hypotheses of Theorem 4 are satisfied.

Indeed, the set M is compact nonempty and f is continuous. Moreover,

i) The set M is bi-$(1_{\mathbf{R}}, 1_{\mathbf{R}})$ convex (see Example 5), hence i) holds.

ii) The function f is bi-$(1_{\mathbf{R}}, 1_{\mathbf{R}})$ strictly concave-convex (see Example 6), hence ii) is true.

iii) For each $x \in M_1 = [0, 1]$, we have

$$\underline{f}(x) = \min\{-x^2 + y^4 | y \in M_2^x\} = -x^2 + x^2 = 0,$$

hence $a = 0$ is a maximum point of $\underline{f}$ on M_1.

iv) For each $y \in M_2^0 = [0, 1]$, we have

$$f(0, y) = y^2 \geq 0 = f(0, 0),$$

hence iv) is true.

v) Since $M_2^0 = M_2 = [0, 1]$, the hypothesis v) is satisfied.

vi) For each $x \in M_1^0 = \{0\}$, and $\delta > 0$ we have

$$B(0, \delta) \cap M_2 \subseteq M_2^x = M_2^0 = [0, 1],$$

because $M_2 = [0, 1]$, hence vi) holds.

Then, in view of Theorem 4, the point $(a, b) = (0, 0)$ is a saddle point of f with respect to M (see Example 1).$\diamond$

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A heuristic algorithm for the non-oriented 2D rectangular strip packing problem

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Abstract. In this paper, we construct best fit based on concave corner strategy (BF_{BCC}) for the two-dimensional rectangular strip packing problem (2D-RSPP), and compare it with some heuristic and metaheuristic algorithms from the literature. The experimental results show that BF_{BCC} could produce satisfied packing layouts, especially for the large problem of 50 pieces or more, BF_{BCC} could get better results in shorter time.

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1 Introduction

Two-dimensional packing problem (2D-PP) has been proved to be NP-hard according to the combinatorial explosion with the problem size increasing [1]. Because the applications of the problem in business and industry are very extensive, during recent years, many researchers have provided various methods to process it. These methods could be broadly categorized into three kinds: exact algorithm, heuristic algorithm and metaheuristic algorithm [3].

Some exact algorithms could be found in [5–8]. A major drawback of these methods in that they can not provide good results for large instances of the problem [2].

During recent years, many heuristic packing algorithms have been suggested in the literature. Surveys on these methodologies for various types of the 2D rectangular packing problem could be found in [4], and these heuristic algorithms could produce good packing layout in an acceptable time, especially for large problems. The most documented heuristic approaches are the bottom-left (BL) [9] and bottom-left-fill (BLF) methods [10]. Based on these two methods, many improved algorithms have been presented in literature, see [2, 11, 12].

Now, metaheuristic algorithms have been important methods in producing packing layout for 2D-PP. These are usually hybridized algorithms involving the generation of input sequences interpreting with placement heuristics such as BL, BLF and other placement strategies, see [2, 4, 13, 14].

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In this paper, we construct *best fit based on concave corner* strategy (BF_{BCC}) for the *two-dimensional rectangular strip packing problem* (2D-RSPP), and compare it with some heuristic and metaheuristic algorithms from the literature. According to the category in [15], the problem belongs to the type of RF: the items could be rotated by 90° and no guillotine constraint is proposed. The experimental results show that the BF_{BCC} is an efficient heuristic algorithm for 2D-RSPP.

2 The problem

Two-dimensional rectangular strip packing problem (2D-RSPP) could be formulated as follows: Let W denote the width of the strip with infinite height, and $P = \{p_i(w_i, h_i), i = 1, 2, \cdots, n\}$ be a set of n rectangular pieces. Each piece p_i has width w_i and height h_i ($w_i, h_i \in Z^+$) with at least one edge no bigger than the W. The objective of the algorithm is to pack all pieces onto the strip orthogonally and pieces could be rotated by 90°, at the same time, try to minimize the used height h of the strip with no two pieces overlap, see Figure 1.

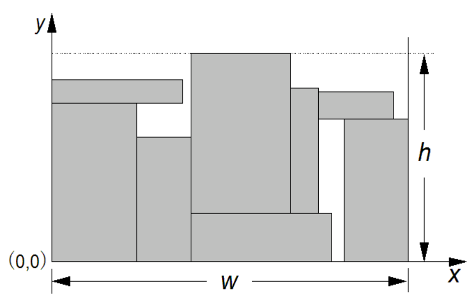

Figure 1. Two-dimensional rectangular strip packing problems

3 A new heuristic algorithm for the 2D-RSPP(BF_{BCC})

3.1 Definitions

For describing the algorithm expediently, we put the strip in the Cartesian system of coordinates, let the left-bottom vertex be superposed over the origin of the coordinate, and the right-bottom vertex in the x-axis, see Figure 1.

1. Let C_i denote the "*Concave Corner*"(CC). The CC is constructed by two edges, and the size of the angle is 90°, at the same time, the CC does not belong to any packed pieces and the corner direction is left-top or right-top.

2. The CC includes two kinds, one is "*Real Concave Corner*(RCC)" with all edges belonging to some packed pieces or the strip, it is denoted as C^+. The other kind is "*Sham Concave Corner*(SCC)" with at least one edge of the corner being the elongation line of the edges of some packed pieces, the SCC is denoted as C^-, see Figure 2.

3. Define $U = \{C_1(xC_1, yC_1), C_2(xC_2, yC_2), \cdots, C_n(xC_n, yC_n)\}$ to denote a set of *Concave Corner* before packing a piece, here, the (xC_i, yC_i) is the coordinate of the vertex of the C_i. Obviously, every C_i is a candidate position for the new piece.

4. *Temp packed height* (TH): After packing a piece onto the strip, the used height of the strip should be computed, which is denoted as TH for current state of the strip.

5. For a piece $p_i(w_i, h_i)$, define $W\_lU = \{C_i | yC_i + h_i \leq TH, w_i \geq h_i\}$, $W\_hU = \{C_i | yC_i + h_i > TH, w_i \geq h_i\}$, $H\_lU = \{C_i | yC_i + h_i \leq TH, w_i < h_i\}$ and $H\_hU = \{C_i | yC_i + h_i > TH, w_i < h_i\}$.

6. *Fitness value* of C_j for one piece: A new piece p_i with $w_i \geq h_i$ is packed onto the board at the position C_j, diagnosing whether the p_i intersects with some packed pieces or the edges of the strip, if the p_i could be deposited. Let s denote the number of edges which is touched with some packed pieces and t denotes the number of concave corner which has been occupied by the piece p_i. Then we compute the parameter $W\_pFit\_C_j$ using formula $W\_pFit\_C_j = 2s + \sum_{k=1}^{t} q_k$.

If piece p_i could be packed onto the strip with corner of the piece at C_j, then s should be computed by querying all packed pieces in the strip. After that check every CC in U: if p_i occupies a *real concave corner* then q_k is equal to 2; if p_i occupies a *sham concave corner* then q_k is equal to 1. Similarly, we could define the $H\_pFit\_C_j$ when $h_i > w_i$.

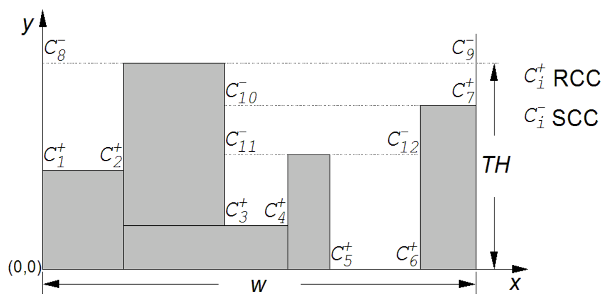

Figure 2. RCC and SCC

3.2 *Best fit based on concave corner* (BF_{BCC}) placement strategy

BF_{BCC} heuristic initially adjusts every p_i in P such that $w_i \geq h_i$ and puts all p_j with w_j bigger than the width of the strip into the front of the packing sequence P, the *startPosition* denotes the number of such pieces. Then BFF_{BCC} sorts the P from the position with subscript *startPosition* to the end by non-increasing height (resolving equal height by non-increasing width).

Before any piece is packed onto the strip, the $U = \{C_1(0, 0), C_2(W, 0)\}$ should be initialized , here C_1 is the left-bottom **concave corner** of the strip and C_2 is the right-bottom **concave corner**.

When a piece p_i being packed onto the board, the W_lU is computed if $|W\_lU| > 0$, namely, there exist positions for p_i such that the p_i does not exceed the TH and the

W_hU should not be computed, otherwise, computes the W_hU. Then exchanges the w_i with the h_i and computes H_lU, if $|H\_lU|$ is zero, then the H_hU should be computed, otherwise, H_hU need not, the following rules decide the position for the piece p_i.

Selecting Rules:

1) if $|W\_lU| > 0$ and $|H\_lU| > 0$, let W_l_{best} denote the position C_j with highest fitness value $W\_pFit\_C_j$ and lowest $yC_j + h_i$, if the positions satisfying this condition are more than one, then select the first. Then change the w_i with h_i such that $w_i < h_i$ and then the $p_i(w_i, h_i)$ would be denoted as $p_i^{'}(w_i^{'}, h_i^{'})$, let H_l_{best} denote the position C_k with highest fitness value $H\_pFit\_C_k$ and lowest $yC_k + h_i^{'}$. If $W\_pFit\_C_j > H\_pFit\_C_k$, then select the position W_l_{best} as the best position for $p_i(w_i, h_i)$, if $W\_pFit\_C_j < H\_pFit\_C_k$, then select the H_l_{best} as the best position for $p_i^{'}(w_i^{'}, h_i^{'})$, if $W\_pFit\_C_j$ equals to $H\_pFit\_C_k$, then select the position with minimal value between $yC_j + h_i$ and $yC_k + h_i^{'}$, if $yC_j + h_i = yC_k + h_i^{'}$ then select the position with minimal value between yC_j and yC_k.

2) if $|W\_lU| > 0$ and $|H\_lU| = 0$, select the position C_j with highest fitness value $W\_pFit\_C_j$ and lowest $yC_j + h_i$ as the best position for the $p_i(w_i, h_i)$.

3) if $|W\_lU| = 0$ and $|H\_lU| > 0$, exchange the w_i with h_i such that $w_i < h_i$, then select the position C_k with highest fitness value $H\_pFit\_C_k$ and lowest $yC_k + h_i^{'}$ as the best position for the piece $p_i^{'}(w_i^{'}, h_i^{'})$.

4) if $|W\_lU| = 0$ and $|H\_lU| = 0$, select the position C_t from W_hU and H_hU with lowest $yC_t + h_i$ and yC_t.

After selecting the best position for the $p_i(w_i, h_i)$ and packing it, updates the TH and U according to the current packing layout of the strip.

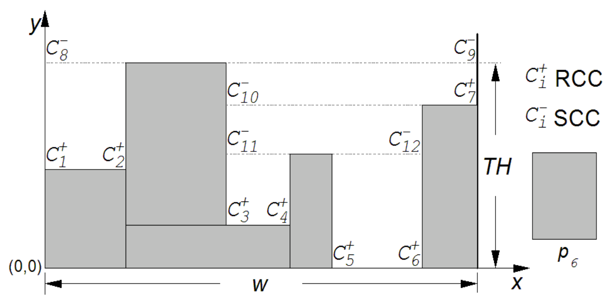

Figure 3. An example of BF_{BCC} placement strategy

An example is given in Figure 3: before the piece p_6 arrival, the $U = \{C_1^+, C_2^+, \cdots, C_7^+, C_8^-, C_9^-, \cdots, C_{12}^-\}$, which means that there exist 12 candidate positions for the piece p_6. By checking every C_i in U, we could get $W\_lU = \{C_1^+, C_2^+, C_5^+, C_6^+, C_{11}^-, C_{12}^-\}$, then exchange w_i with h_i, we have $H\_lU = \{C_1^+, C_2^+, C_3^+, C_4^+, C_5^+, C_6^+, C_{11}^-, C_{12}^-\}$, according to the **rules 1** with $|W\_lU| > 0$ and $|H\_lU| > 0$, the W_hU and the H_hU need not be computed. Then compute *Fitness value* for every "*Concave Corner*" in W_lU using formula(1), we have: $W\_pFit\_C_1^+ = W\_pFit\_C_2^+ = 10$, $W\_pFit\_C_5^+ = W\_pFit\_C_6^+ = 6$, $W\_pFit\_C_{11}^- = 5$, $W\_pFit\_C_{12}^- =$

3, so $W\_l_{best} = W\_pFit\_C_1^+$. Then exchange w_i with h_i such that $h_i > w_i$ and piece p_6 we denoted $p_6'(w_6', h_6')$. After that we have $H\_pFit\_C_3^+ = H\_pFit\_C_4^+ = 10$, $H\_pFit\_C_5^+ = H\_pFit\_C_6^+ = 6$, $H\_pFit\_C_{11}^- = 3$, $H\_pFit\_C_{12}^- = 3$, so $H\_l_{best} = C_3^+$. Here we found the fitness of W_l_{best} and H_l_{best} that are same, but $yC_1^+ + h_6 > yC_3^+ + h_6'$. So, we select C_3^+ as the best position for the piece $p_6'(w_6', h_6')$. After packing the piece p_6, U should be updated, but TH needn't be updated because $yC_3^+ + h_6'$ is no more than TH.

The whole placement heuristic algorithm could be described as Algorithm 1.

Algorithm 1 heuristic packing (packing sequence P, strip width $stripWidth$)

```
adjusts every p_i(w_i, h_i) in P such that w_i ≥ h_i;
startPosition ⇐ 0;
for i = 1 to |P| do
  if w_i ≥ stripWidth then
    exchanges p_i with p_startPosition;
    startPosition ⇐ startPosition + 1;
  end if
end for
sorts the P by non-increasing height (resolving equal height by non-increasing
width) from the position with subscript startPosition to the end;
TH ⇐ 0;
j ⇐ 0;
U ⇐ {C_1(0,0), C_2(stripWidth, 0)};
while packing sequence P is not null do
  W_lU ⇐ ∅, W_hU ⇐ ∅;
  H_lU ⇐ ∅, H_hU ⇐ ∅;
  gets p_j from the packing sequence P;
  if w_j is bigger than stripWidth then
    exchanges w_j with h_j;
  end if
  computes the W_lU, W_hU, H_lU and H_hU based on the definitions men-
  tioned above;
  selects the best position C_s for p_j from W_lU, W_hU, H_lU and H_hU according
  to the selecting rules;
  packs the p_j onto the board at the position C_s;
  removes the p_j from packing sequence P;
  if the used height of current strip is exceeded than TH then
    updates TH;
  end if
  updates U such that U includes all CC at the current state;
  j ⇐ j + 1;
end while
```

Obviously, BF_{BCC} algorithm includes the Bottom-left(BL), Best-fit-fill(BLF) algorithms etc, and it could process the "hole" easily.

4 Experiments

The test program has been coded in c++ language and run on a IBM T400 notebook PC with 2.26 GHZ CPU and 2048 MB RAM, test data coming from [2, 4] are used to compare the BF_{BCC} with some heuristic and metaheuristic algorithms. All test results except BF_{BCC} are obtained from [2], which is performed on a pc with 850 MHz CPU and 128 MB RAM. For all instances, the best solutions are shown in bold type.

Table 1 shows that the BF_{BCC} outperforms Bottom-Left, Bottom-Left-Fill and Best-Fit [2] in almost all test data, even when preordering is allowed (DW means "decreasing width" and DH means "decreasing height). The computational results of metaheuristic approaches (GA+BLF, SA+BLF) and Best-Fit [2] could be found in Table 2. We can see that BF_{BCC} could gives better results than $GA + BLF$, $SA + BLF$ and $Best - Fit$ quickly.

Table 1. Comparison of the BF_{BCC} heuristic with some heuristic algorithm (% over optimal)

Category:	C1			C2			C3			C4			C5			C6			C7		
Problem:	P1	P2	P3	P1	P2	P3	P1	P2	P3	P1	P2	P3	P1	P2	P3	P1	P2	P3	P1	P2	P3
Number:	16	17	16	25	25	25	28	29	28	49	49	49	72	73	72	97	97	97	196	197	196
BL	45	40	35	53	80	67	40	43	40	32	37	30	27	32	30	33	39	34	22	41	31
BL-DW	30	20	20	13	27	27	10	20	17	17	22	22	16	18	13	22	25	18	16	19	17
BL-DH	15	10	**5**	13	73	13	10	10	13	12	13	6.7	4.4	10	7.8	8.3	8.3	9.2	5	10	7.1
BLF	30	35	25	47	73	47	37	50	33	25	25	27	20	23	21	20	18	21	15	20	17
BLF-DW	10	15	15	13	20	20	10	13	13	10	5	10	5.6	6.7	5.6	5	4.2	4.2	4.6	3.4	2.9
BLF-DH	10	10	**5**	13	73	13	10	**6.7**	13	10	5	5	4.4	5.6	4.4	5	2.5	6.7	3.8	2.9	3.8
BF	**5**	10	20	**6.7**	**6.7**	**6.7**	**6.7**	13	10	**5**	**3.3**	**3.3**	3.3	**2.2**	3.3	2.5	**1.7**	3.3	2.9	1.7	2.1
BF_{BCC}	**5**	**5**	10	**6.7**	**6.7**	**6.7**	**6.7**	**6.7**	**6.7**	**5**	**3.3**	**3.3**	**2.2**	3.3	**1.1**	**1.7**	**1.7**	**1.7**	**1.7**	**1.3**	**1.7**

5 Conclusion

In this paper, a new heuristic algorithm (BF_{BCC}) for no-oriented 2D-SP problem has been proposed. The approach is tested on a set of instances taken from the literature and compared with some heuristic algorithms (Bottom-Left, Bottom-Left-Fill and Best Fit) and some metaheuristic algorithms (GA + BLF and SA + BLF), the experimental results show that BF_{BCC} could produce better-quality packing layouts than these algorithms, especially for the large problem of 50 pieces or more, BF_{BCC} could get better results in shorter time.

Table 2. Comparison of the BF_{BCC} heuristic with BF and some metaheuristic methods (GA + BLF, SA + BLF)

					GA+BLF		SA+BLF		Best Fit		BF_{BCC}		GA_{best}-BF_{BCC}		SA_{best}-BF_{BCC}		BF-BF_{BCC}	
Data set	Cat.	Problem	Number	Optimal height	Best	Time(s)	Best	Time(s)	Sol.	Time(s)	Sol.	Time(s)	Abs.	%Impv.	Abs.	%Impv.	Abs.	%Impv.
Hopper	C1	P1	16	20	**20**	3.4	**20**	1.1	21	< 0.01	21	0.01	-1	-5.0	-1	-5.0	**0**	**0**
		P2	17	20	**21**	0.5	**21**	0.8	22	< 0.01	**21**	0.01	**0**	**0**	**0**	**0**	**1**	**4.5**
		P3	16	20	**20**	7.1	**20**	0.8	24	< 0.01	22	0.01	-2	-10.0	-2	-10	**2**	**8.3**
	C2	P1	25	15	**16**	1.3	**16**	6.5	**16**	< 0.01	**16**	0.02	**0**	**0**	**0**	**0**	**0**	**0**
		P2	25	15	**16**	2.2	**16**	13.9	**16**	< 0.01	**16**	0.03	**0**	**0**	**0**	**0**	**0**	**0**
		P3	25	15	**16**	1.0	**16**	13.6	**16**	< 0.01	**16**	0.02	**0**	**0**	**0**	**0**	**0**	**0**
	C3	P1	28	30	**32**	7.4	**32**	20.3	**32**	< 0.01	**32**	0.03	**0**	**0**	**0**	**0**	**0**	**0**
		P2	29	30	**32**	12.4	**32**	22.5	34	< 0.01	**32**	0.05	**0**	**0**	**0**	**0**	**2**	**5.9**
		P3	28	30	**32**	11.6	**32**	18.3	33	< 0.01	**32**	0.04	**0**	**0**	**0**	**0**	**1**	**3**
	C4	P1	49	60	64	35	64	65	**63**	< 0.01	**63**	0.16	**1**	**1.6**	**1**	**1.6**	**0**	**0**
		P2	49	60	63	48	64	46	**62**	< 0.01	**62**	0.13	**1**	**1.6**	**2**	**3.1**	**0**	**0**
		P3	49	60	**62**	61	63	70	**62**	< 0.01	**62**	0.14	**0**	**0**	**1**	**1.6**	**0**	**0**
	C5	P1	72	90	95	236	94	501	93	0.01	**92**	0.40	**3**	**3.2**	**2**	**2.1**	**1**	**1.1**
		P2	73	90	95	440	95	285	**92**	0.01	93	0.37	**2**	**2.1**	**2**	**2.1**	-1	-1.1
		P3	72	90	95	150	95	425	93	0.01	**91**	0.29	**4**	**4.2**	**4**	**4.2**	**2**	**2.2**
	C6	P1	97	120	127	453	127	854	123	0.01	**122**	0.67	**5**	**3.9**	**5**	**3.9**	**1**	**0.8**
		P2	97	120	126	866	126	680	**122**	0.01	**122**	0.64	**4**	**3.2**	**4**	**3.2**	**0**	**0**
		P3	97	120	126	946	126	912	124	0.01	**122**	0.61	**4**	**3.2**	**4**	**3.2**	**2**	**1.6**
	C7	P1	196	240	255	4330	255	4840	247	0.01	**244**	4.11	**11**	**4.3**	**11**	**4.3**	**3**	**1.2**
		P2	197	240	251	5870	253	5100	244	0.01	**243**	4.31	**8**	**3.2**	**10**	**4.0**	**1**	**0.4**
		P3	196	240	254	5050	255	6520	245	0.01	**244**	3.39	**10**	**3.9**	**11**	**4.3**	**1**	**0.4**
Burke		N1	10	40	**40**	1.02	**40**	0.24	45	< 0.01	44	< 0.01	-4	-10	-4	-10	**1**	**2.2**
		N2	20	50	**51**	9.2	52	8.14	53	< 0.01	54	0.02	-3	-0.59	-2	-3.8	-1	-0.3
		N3	30	50	**52**	2.6	**52**	39.5	**52**	< 0.01	54	0.04	-2	-3.8	-2	-3.8	-2	-3.9
		N4	40	80	**83**	12.6	**83**	84	**83**	< 0.01	**83**	0.11	**0**	**0**	**0**	**0**	**0**	**0**
		N5	50	100	106	52.3	106	228	**105**	0.01	106	0.15	**0**	**0**	**0**	**0**	-1	-1
		N6	60	100	103	261	103	310	103	0.01	**102**	0.19	**1**	**1**	**1**	**1**	**1**	**1**
		N7	70	100	106	671	106	554	107	0.01	**103**	0.35	**3**	**2.8**	**3**	**2.8**	**4**	**3.7**
		N8	80	80	85	1142	85	810	84	0.01	**82**	0.39	**3**	**3.5**	**3**	**3.5**	**2**	**2.4**
		N9	100	150	155	4431	155	1715	**152**	0.01	155	0.61	**0**	**0**	**0**	**0**	-3	-2.0
		N10	200	150	154	2×10^4	154	6066	**152**	0.02	**152**	2.79	**2**	**1.3**	**2**	**1.3**	**0**	**0**
		N11	300	150	155	8×10^4	155	3×10^4	**152**	0.03	154	6.68	**1**	**0.5**	**1**	**0.6**	-2	-0.7
		N12	500	300	313	4×10^5	312	6×10^4	**306**	0.06	**306**	26.88	**7**	**2.2**	**6**	**1.9**	**0**	**0**
		N13	3152	960	-	-	-	-	964	1.37	**962**	3291.20					**2**	**0.2**

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The Variational Approach to Nonlinear Evolution Equations

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Abstract. In this paper, we present a few recent existence results via variational approach for the Cauchy problem

$$\frac{dy}{dt}(t) + A(t)y(t) \ni f(t), \quad y(0) = y_0, \quad t \in [0,T],$$

where $A(t) : V \to V'$ is a nonlinear maximal monotone operator of subgradient type in a dual pair (V, V') of reflexive Banach spaces. In this case, the above Cauchy problem reduces to a convex optimization problem via Brezis–Ekeland device and this fact has some relevant implications in existence theory of infinite-dimensional stochastic differential equations.

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1 Introduction

Consider the Cauchy problem

$$\begin{aligned} &\frac{dy}{dt}(t) + A(t)y(t) \ni f(t), \quad t \in (0,T), \\ &y(0) = y_0, \end{aligned} \tag{1.1}$$

where $y : [0,T] \to V$, $\frac{dy}{dt} : (0,T) \to V'$, $f : (0,T) \to V'$ and $y_0 \in H$. Here V is a real reflexive Banach space with the dual V' and H is a real Hilbert space such that $V \subset H \subset V'$ algebraically and topologically.

The scalar product on H and the duality pairing between V and V' are both denoted by $(\cdot,\cdot)$ and the latter coincides with the scalar product of H on $H \times H \subset V \times V'$. Here $A(t) : V \to V'$, $t \in (0,T)$, is a family of maximal monotone operators on $V \times V'$ of the form (see, e.g.,[4])

$$A(t) = \partial\varphi(t,\cdot) \quad \text{a.e. } t \in (0,T), \tag{1.2}$$

where $\varphi(t,\cdot) : V \to \overline{\mathbb{R}}^* =]-\infty,+\infty]$ is a family of convex and lower-semicontinuous functions and $\partial\varphi(t,\cdot) : V \to V'$ is the subdifferential of $\varphi(t,\cdot)$ (see, e.g., [4,5]).

By strong solution to (1.1) we mean a measurable function $y : (0,T) \to V$ which is H-valued continuous and V'-absolutely continuous on $[0,T]$ and satisfies a.e. equation (1.1) on $(0,T)$ along with the initial value condition $y(0) = y_0 \in H$.

We recall that

$$\partial\varphi(t,y)=\{z\in V';\ \varphi(t,y)\le\varphi(t,u)+(y-u,z),\ \forall u\in V\},\ y\in V, \tag{1.3}$$

and the conjugate function $\varphi^*(t,\cdot):V'\to\overline{\mathbb{R}}^*$ is defined by (see, e.g., [4,5])

$$\varphi^*(t,z)=\sup\{(y,z)-\varphi(t,y);\ y\in V\},\ \forall z\in V'. \tag{1.4}$$

We recall also the duality relations (see [5])

$$\varphi(t,y)+\varphi^*(t,z)\ \ge\ (y,z),\quad \forall y\in V,\ z\in V', \tag{1.5}$$

$$\varphi(t,y)+\varphi^*(t,z)\ =\ (y,z)\quad \text{iff } z\in\partial\varphi(t,y). \tag{1.6}$$

By virtue of (1.5) and (1.6), we may rewrite equation (1.1) as

$$\frac{dy}{dt}(t)+z(t)=f(t),\quad \varphi(t,y(t))+\varphi^*(t,z(t))=(y(t),z(t)),t\in(0,T).$$

Equivalently,

$$\varphi(t,y(t))+\varphi^*\left(t,f(t)-\frac{dy}{dt}(t)\right)=\left(y(t),f(t)-\frac{dy}{dt}(t)\right),\quad \text{a.e. } t\in(0,T). \tag{1.7}$$

In other words, any strong solution y to (1.1) can be viewed as solution to the minimization problem

$$\operatorname{Min}\left\{\int_0^T\!(\varphi(t,y(t))+\varphi^*\left(t,f(t)-\frac{dy}{dt}(t)\right)-\left(y(t),f(t)-\frac{dy}{dt}(t)\right)dt\right\}. \tag{1.8}$$

The exact formulation of (1.8) will be given later on, but is easily seen that if one takes the minimum in (1.8) on the space

$$\mathcal{W}_p=\left\{y\in L^p(0,T;V),\ \frac{dy}{dt}\in L^{p'}(0,T;V');\ \frac{1}{p}+\frac{1}{p'}=1,\ 1<p<\infty\right\},$$

then (1.8) reduces to the convex optimization problem

$$\operatorname{Min}\left\{\int_0^T\left(\varphi(t,y(t))+\varphi^*(t,f(t)-\frac{dy}{dt}(t)-(y(t),f(t))\right)dt +\frac{1}{2}(|y(T)|^2-|y_0)|^2);\ y\in\mathcal{W}_p\right\}. \tag{1.9}$$

Conversely, one might expect that every solution y to problem (1.9) is a strong solution to the Cauchy problem (1.1) and we shall see that this is indeed the case under suitable assumptions on $A(t)=\partial\varphi(t,\cdot)$. This is the fundament idea behind the variational approach to equation (1.1), which goes back to the influential works [7,8] of Brezis and Ekeland (see, also, Nayrolles [12]). Now this is known as Brezis &

Ekeland principle and it was a fertile idea used later on in a variety of situations (see [1–3, 9–11, 13, 14]). Though, in general, the equivalence of problems (1.1) and (1.9) is still an open problem and it is not true in general, this principle leads to a variational formulation of a large class of nonlinear Cauchy problems which, from the point of view of mathematical physics and numerical computation, represents a great advantage. As a matter of fact, by this device the Cauchy problem (1.1) reduces to a convex optimization problem for which a large set of strategies which belong to convex analysis are applicable. This approach is, in particular, useful for the time-dependent Cauchy problems of the form (1.1) for which a complete existence theory is known only in a few situations requiring either time-regularity of the operator $A(t)$ or polynomial growth conditions from V to V' (see [4], Section 4.4). In applications to stochastic differential equations of the form

$$\begin{aligned} &dX + AX(t)dt = dW(t), \quad t \in (0,T), \\ &X(0) = y_0, \end{aligned} \tag{1.10}$$

in a probability space $\{\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\}$, where $A = \partial\varphi$ and W is a Wiener process on H, we are lead to an equation of the form (1.1) by the transformation $y = X - W_t$ and get the random differential equation

$$\begin{aligned} &\frac{dy}{dt} + A(y + W(t)) = 0, \quad t \in (0,T), \\ &y(0) = y_0. \end{aligned} \tag{1.11}$$

Since the function $t \to W(t)$ is not smooth, we are lead to an equation of the form (1.1) when $A(t)y \equiv A(y + W(t))$ for which the standard existence theory is not applicable but which can be reformulated in terms of (1.9). This problem, which is discussed in details in [4, 6], represents a viable and promising approach to the existence theory of infinite-dimensional nonlinear stochastic differential equations.

Notation and definitions

If Y is a Banach space with the norm $\|\cdot\|_Y$, we denote by $L^p(0,T;Y)$, $1 \leq p \leq \infty$, the space of all Y-measurable functions $u : (0,T) \to Y$ with $\|u\|_Y \in L^p(0,T)$. By $C([0,T];Y)$ we denote the space of all continuous Y-valued functions on $[0,T]$ and by $W^{1,p}([0,T];Y)$ the Sobolev space $\big\{y \in L^p(0,T;Y);\, \frac{d}{dt} \in L^p(0,T);Y)\big\}$, where $\frac{dy}{dt}$ is taken in the sense of vectorial distributions on $(0,T)$. Equivalently, $W^{1,p}([0,T];Y)$ is the space of absolutely continuous functions $u : [0,T] \to Y$ which are a.e. differentiable and $\frac{d}{dt} \in L^p(0,T;Y)$. (See [4, 5].)

Everywhere in the following, $\mathcal{O}$ is an open and bounded subset of the Euclidean space $\mathbb{R}^d$, $d \geq 1$, with smooth boundary $\partial\mathcal{O}$ (of class C^2, for instance) and $W^{k,p}(\mathcal{O})$, $k \in \mathbb{N}$, $1 \leq p \leq \infty$, are standard Sobolev spaces on $\mathcal{O}$, i.e.,

$$W^{k,p}(\mathcal{O}) = \{u \in L^p(\mathcal{O});\ D^\alpha u \in L^p(\mathcal{O}),\ |\alpha| \leq k\}. \tag{1.12}$$

$W_0^{k,p}(\mathcal{O})$ is the subset of functions in $W^{k,p}(\mathcal{O})$ which are of trace zero on $\partial\mathcal{O}$.

We set $H_0^1(\mathcal{O}) = W_0^{1,2}(\mathcal{O})$.

A multivalued function (graph) $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ is said to be *maximal monotone* if it is monotone, that is, $(v_1 - v_2)(u_1 + u_2) \geq 0$ for $v_i \in \beta(u_i)$, $i = 1, 2$, and the range of $u \to u + \beta(u)$ is all of $\mathbb{R}$. Any maximal monotone graph $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ is of the form $\beta = \partial j$, where $j : \mathbb{R} \to]-\infty, +\infty]$ is convex and lower-semicontinuous. (This is the potential function corresponding to β.)

2 Existence for the Cauchy problem (1.1)

2.1 The main results

We study here problem (1.1) under the following hypotheses:

(i) *V is a real Banach space with the dual V' and H is a real Hilbert space such that $V \subset H \subset V'$ algebraically and topologically.*

Denote by $(\cdot, \cdot)$ the pairing between V and V' and, respectively, the scalar product on H. The norms of V, V' and H are denoted by $\|\cdot\|_V$, $\|\cdot\|_{V'}$ and $|\cdot|_H$.

(ii) *$A(t)y = \partial\varphi(t, y)$ a.e. $t \in (0, T)$, $\forall y \in V$, where $\varphi : (0, T) \times V \to \mathbb{R}$ is measurable in t on $(0, T)$ and lower-semicontinuous on V with respect to y. There are $\alpha_1, \alpha_2 > 0$, $\gamma_1, \gamma_2 \in \mathbb{R}$ and $1 < p_1 \leq p_2 < \infty$ such that*

$$\gamma_1 + \alpha_2\|u\|_V^{p_1} \leq \varphi(t, u) \leq \gamma_2 + \alpha_2\|u\|_V^{p_2},\ \forall u \in V,\ \text{a.e. } t \in (0, T). \tag{2.1}$$

Instead of (ii) we consider the following alternative weaker assumption on φ.

(iii) *$A(t) = \partial\varphi(t, \cdot)$, where $\varphi : (0, T) \times H \to \mathbb{R}$ is measurable in t, convex and lower-semicontinuous in y on H and for each $M > 0$ there is $C_M > 0$ independent of t such that*

$$\varphi(t, u) \leq C_M \quad \text{a.e. } t \in (0, T),\ \|u\|_V \leq M, \tag{2.2}$$

$$\gamma_1 + \alpha_1\|u\|_V^{p_1} \leq \varphi(t, u), \quad \forall u \in V,\ \text{a.e. } t \in (0, T). \tag{2.3}$$

It should be mentioned that both hypotheses (ii), (iii) imply that $\varphi(t, \cdot)$ is continuous on V for almost all $t \in (0, T)$ but no differentiability conditions so $A(t) = \partial\varphi(t, \cdot)$ might be multivalued as well.

Hypothesis (iv) below is a symmetry condition on $u \to \varphi(t, u)$ for large $\|u\|_V$.

(iv) *There are $C_1, C_2 \in R^+$ such that*

$$\varphi(t, -u) \leq C_1\varphi(t, u) + C_2, \quad \forall u \in V,\ \text{a.e. } t \in (0, T). \tag{2.4}$$

Theorems 2.1, 2.2 below are the main results.

Theorem 2.1. *Under hypotheses* (i), (ii), (iv)*, for each $y_0 \in V$ and $f \in L^{p_1'}(0, T; V')$ there is a unique strong solution to* (1.1) *satisfying*

$$y \in L^{p_1}(0, T; V) \cap C([0, T]; H) \cap W^{1,p_2'}([0, T]; V'), \tag{2.5}$$

where $\frac{1}{p_i}+\frac{1}{p_i'}=1,\ 1<p_i<\infty, i=1,2.$

Moreover, y is the solution to the minimization problem

$$\begin{aligned}\text{Min}\Big\{ \int_0^T \Big(\varphi(t,u(t))+\varphi^*\Big(t,f(t)-\frac{du}{dt}(t)\Big)-(u(t),f(t))\Big)dt\\ +\frac{1}{2}\,|u(T)|_H^2;\ u\in L^{p_1}(0,T;V)\cap W^{1,p_2'}([0,T];V'),\ u(0)=y_0\Big\}.\end{aligned}\tag{2.6}$$

We have also

Theorem 2.2. *Under hypotheses* (i), (iii), (iv) *for each $y_0\in V$ and $f\in L^{p_1'}(0,T;V')$ there is a unique strong solution to* (1.1) *such that*

$$y^*\in L^{p_1}(0,T;V)\cap C([0,T];H)\cap W^{1,1}([0,T];V).\tag{2.7}$$

Moreover, y^* is the solution to the minimization problem (2.6)*.*

2.2 Examples to PDEs

Now, we pause briefly to see how Theorems 2.1 and 2.2 apply to a few standard parabolic nonlinear boundary value problems.

Example 2.1. (Semilinear parabolic equations) Consider the boundary value problem

$$\begin{aligned}&\frac{\partial y}{\partial t}-\Delta y+\beta(t,y)\ni f(t,\xi), && t\in(0,T),\xi\in\mathcal{O},\\ &y(0,\xi)=y_0(\xi), && \xi\in\mathcal{O},\\ &y(t,\xi)=0, && \text{on } (0,T)\times\partial\mathcal{O}.\end{aligned}\tag{2.8}$$

Here $\mathcal{O}\subset\mathbb{R}^d$, $d\geq 1$, is a bounded open domain with smooth boundary $\partial\mathcal{O}$ and $\beta:(0,T)\times\mathbb{R}\to 2^{\mathbb{R}}$ is a maximal monotone graph in y for almost all $t\in(0,T)$ and is measurable in t.

Denote by $j(t,\cdot):\mathbb{R}\to\mathbb{R}$ the potential associated with $\beta(t,\cdot)$ that is,

$$\partial_r j(t,r)=\beta(t,r),\quad \forall r\in\mathbb{R},\ t\in(0,T),$$

and assume that

$$\begin{aligned}\gamma_1+\alpha_1|r|^{p_1}&\leq j(t,r)\leq\gamma_2+\alpha_2|r|^{p_2}, && \forall r\in\mathbb{R},\\ j(t,-r)&\leq C_1 j(t,r)+C_2, && \forall r\in\mathbb{R},\end{aligned}\tag{2.9}$$

where $1<p_1\leq p_2<\infty$ and $C_1,\alpha_1,\alpha_2>0,\ \gamma_1,\gamma_2,C_2\in\mathbb{R}$.

We apply Theorem 2.1, where $V=H_0^1(\mathcal{O})\cap L^{p_1}(\mathcal{O}),\ V'=H^{-1}(\mathcal{O})+L^{p_1'}(\mathcal{O})$, where $H^{-1}(\mathcal{O})=(H_0^1(\mathcal{O}))'$ is the dual of $H_0^1(\mathcal{O})$, and φ is the function

$$\varphi(t,u)=\int_{\mathcal{O}}\left(\frac{1}{2}\,|\nabla u(\xi)|^2+j(t,u(\xi))\right)d\xi,\quad \forall u\in V,\ t\in(0,T).\tag{2.10}$$

Then, by Theorem 2.1, we obtain that

Corollary 2.1. *Under assumptions* (2.8) *for each* $f \in L^{p_2'}(0,T;H^{-1}(\mathcal{O})+L^{p_1'}(\mathcal{O}))$ *and* $y_0 \in V$ *there is a unique solution* y *to* (2.8) *which satisfies*

$$y \in L^{p_1}(0,T;H_0^1(\mathcal{O})\cap L^{p_1}(\mathcal{O}))\cap C([0,T];L^2(\mathcal{O})), \tag{2.11}$$

$$\frac{\partial y}{\partial t} \in L^{p_2'}(0,T;H^{-1}(\mathcal{O})+L^{p_1'}(\mathcal{O})). \tag{2.12}$$

Similarly, by Theorem 2.2 we have

Corollary 2.2. *Assume that, instead of* (2.8), *the function* j *satisfies the weaker assumption* $j(t,-r) \le C_1 j(t,r) + C_2$ *for all* $r \in \mathbb{R}$ *and*

$$\left\{\begin{array}{rcll} \gamma_1+\alpha_2|r|^{p_1} & \le & j(t,r), & \forall r\in\mathbb{R},\ t\in(0,T),\\ j(t,r) & \le & C_M, & \forall |r|\le M,\ \forall M>0,\ t\in(0,T). \end{array}\right. \tag{2.13}$$

Then, for $f \in L^{p_1'}(0,T;H^{-1}(\mathcal{O})+L^{p_1'}(\mathcal{O}))$ *and* $y_0 \in V$, *there is a unique solution* y *to* (2.8) *satisfying* (2.11) *and*

$$\frac{\partial y}{\partial t} \in L^{1}(0,T;H^{-1}(\mathcal{O})+L^{p_1'}(\mathcal{O})). \tag{2.14}$$

The conjugate φ^* to the function φ is given by

$$\varphi^*(t,v)=\sup\left\{(u,v)-\int_{\mathcal{O}}\left(\frac{1}{2}|\nabla u|^2+j(t,u)\right)d\xi;\ u\in H_0^1(\mathcal{O}))\right\}$$

and, by Fenchel's duality theorem, we have after some calculation (see [5], p. 219)

$$\varphi^*(t,v)=\inf_u\left\{\frac{1}{2}\,\|v+u\|^2_{H^{-1}(\mathcal{O})}+\int_{\mathcal{O}} j^*(t,u)d\xi\right\},$$

which is just the Moreau regularization of the function $u \to \int_\Omega j^*(t,u)d\xi$ in the space $H^{-1}(\mathcal{O})$. Then, by Theorems 2.1 and 2.2 it follows that the solution y given by Corollaries 2.1 and 2.2 are given by

$$y=\arg\min\left\{\int_0^T\left(\varphi(t,u)+\varphi^*\left(t,f-\frac{du}{dt}\right)\right)dt+\frac{1}{2}\int_\Omega u^2(T,\xi)d\xi\right\},$$

where φ, φ^* are as above.

Example 2.2. (The porous media equation) Consider the equation

$$\begin{array}{ll} \dfrac{\partial y}{\partial t}-\Delta\beta(t,y)\ni f & \text{in } (0,T)\times\mathcal{O},\\ y(0,\xi)=y_0(\xi) & \text{in } \mathcal{O},\\ \beta(t,y)=0 & \text{on } (0,T)\times\partial\mathcal{O}, \end{array} \tag{2.15}$$

where $\beta : (0,T)\times\mathbb{R}\to 2^{\mathbb{R}}$ is measurable in t and maximal monotone in $y \in \mathbb{R}$.

Assume that condition (2.8) holds for

$$\begin{aligned} \frac{2d}{d+2} < p_1 \quad &\leq \quad p_2 < \infty \qquad \text{if } d > 2, \\ 1 < p_1 \quad &\leq \quad p_2 < \infty \qquad \text{if } d = 1, 2. \end{aligned} \tag{2.16}$$

We shall apply here Theorem 2.1 for

$$H = H^{-1}(\mathcal{O}), \quad V = L^{p_1}(\mathcal{O})$$

and

$$\varphi(t,y) = \left\{ \begin{array}{ll} \displaystyle\int_{\mathcal{O}} j(t,y(\xi))d\xi \text{ if } y \in H^{-1}(\mathcal{O}) \text{ and } j(t,y) \in L^1(\mathcal{O}), \\ +\infty \qquad \text{otherwise.} \end{array} \right\} \tag{2.17}$$

The space V' is, in this case, the dual of $V = L^{p_1}(\mathcal{O}) \subset H^{-1}(\mathcal{O})$ (by the Sobolev embedding theorem we have this inclusion) with the pivot space $H^{-1}(\mathcal{O})$ endowed with the scalar product $\langle u, v \rangle_{-1} = \int_{\mathcal{O}} u(-\Delta)^{-1} v \, d\xi$, where $D(\Delta) = H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$. It is easily seen that $\Delta^{-1} V' \subset L^{p_2}(\mathcal{O})$. Then, as easily follows, $\partial\varphi(t,y) = -\Delta \partial j(t,y)$ (see, e.g.,[4], p. 68), we obtain

Corollary 2.3. *Under assumptions* (2.16)*, for each* $y_0 \in L^{p_1}(\mathcal{O})$ *and* $f \in L^{p_1'}(0,T;H^{-1}(\mathcal{O})) \subset L^{p_1'}(0,T;V')$ *there is a unique solution* y *to* (2.15) *such that*

$$y \in L^{p_1}((0,T) \times \mathcal{O}), \; \frac{dy}{dt} \in L^{p_2'}(0,T;V'), \tag{2.18}$$

$$\begin{aligned} \frac{\partial y}{\partial t} = \Delta\eta \textit{ in } (0,T) \times \mathcal{O}; \quad \eta \in L^{p_2}((0,T) \times \mathcal{O}), \\ \eta \in \beta(y) \textit{ a.e. in } (0,T) \times \mathcal{O}. \end{aligned} \tag{2.19}$$

Example 2.3. (Parabolic nonlinear BVP of divergence type) Consider the equation

$$\begin{aligned} &\frac{\partial y}{\partial t} - \text{div } a(t, \nabla y) = f \qquad \text{in } (0,T) \times \mathcal{O}; \\ &y = 0 \qquad\qquad\qquad\qquad\quad \text{on } (0,T) \times \partial\mathcal{O}. \end{aligned} \tag{2.20}$$

Here $a(t,r) = \partial j(t,r) : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$, where $j(t,\cdot) : \mathbb{R}^d \to \mathbb{R}$ is convex, continuous $y \in \mathbb{R}^d$ and measurable in t. Moreover, there are $\alpha_i > 0$, $\gamma_i \in \mathbb{R}$, $i = 1, 2$, and $1 < p_1 \leq p_2 < \infty$, such that

$$\begin{aligned} r_1 + \alpha_1 \|r\|_{\mathbb{R}^d}^{p_1} \quad &\leq \quad j(t,r) \leq \gamma_2 + \alpha_2 \|r\|_{\mathbb{R}^d}^{p_2}, \quad \forall r \in \mathbb{R}^d, \\ \frac{j(t,r)}{j(t,-r)} \quad &\leq \quad C, \qquad\qquad\qquad\qquad\qquad \forall r \in \mathbb{R}. \end{aligned} \tag{2.21}$$

One applies Theorem 2.1 for $V = W_0^{1,p_1}(\mathcal{O})$, $V' = W^{-1,p_1'}(\mathcal{O})$, $H = L^2(\mathcal{O})$ if $p_1 \geq 2$ and $V = W_0^{1,p_1}(\mathcal{O}) \cap L^2(\mathcal{O})$ if $1 < p_1 < 2$.

In this case, $\varphi : (0,T) \times V \to \mathbb{R}$ is defined by

$$\varphi(t,y) = \begin{cases} \int_{\mathcal{O}} j(t, \nabla y(\xi)) d\xi & \text{if } j(t,\nabla y) \in L^1(\mathcal{O}), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.22}$$

We obtain

Corollary 2.4. *Under assumptions* (2.21) *for all* $y_0 \in W_0^{1,p_1}(\mathcal{O})$ *and* $f \in L^{p_1'}(0,T;V')$ *there is a unique solution* y *to* (2.20)

$$y \in L^{p_1}(0,T;V) \cap C([0,T]; L^2(\mathcal{O})), \tag{2.23}$$

$$\frac{\partial y}{\partial t} \in L^{p_2'}(0,T;V'). \tag{2.24}$$

Remark 2.1. Multivalued functions β arise naturally if one attempts to treat parabolic equations with discontinuous monotone nonlinearities. For instance, the equation

$$\begin{aligned} &\frac{\partial y}{\partial t} - \Delta y + \beta_0(t,y) = f(t,\xi) && \text{in } (0,T) \times \mathcal{O}, \\ &y(0,\xi) = y_0(\xi), && \xi \in \mathcal{O}, \\ &y(t,\xi) = 0, && (t,\xi) \in (0,T) \in \partial\mathcal{O}, \end{aligned}$$

where $r \to \beta_0(t,r)$ is monotonically increasing and discontinuous in $r = r_j$, can be put in the form (2.8), where

$$\beta(t,r) = \begin{cases} \beta_0(t,r), & r \neq r_j, \\ [\beta_0(t,r_j - 0),\ \beta_0(t, r_j + 0)], & r = r_j, \end{cases}$$

and for which Corollary 2.1 is applicable.

Multivalued functions $\partial\varphi(t,\cdot)$ arise also in the treatment of parabolic problems with free boundary. For instance, the free boundary problem

$$\begin{aligned} &\frac{\partial y}{\partial t} - \Delta y = f && \text{in } \{y > 0\}, \\ &y \geq 0 && \text{in } (0,T) \times \mathcal{O}, \\ &y(0,\xi) = y_0(\xi) && \text{in } \mathcal{O}, \\ &y = 0 && \text{on } (0,T) \times \partial\mathcal{O}, \end{aligned}$$

can be written in the form (1.1), where $V = H_0^1(\mathcal{O})$, $H = L^2(\mathcal{O})$ and

$$\varphi(t,y) = \begin{cases} \frac{1}{2} \int_{\mathcal{O}} |\nabla y|^2 d\xi & \text{if } y \in K, \\ +\infty & \text{otherwise,} \end{cases}$$

where $K = \{y \in H_0^1(\mathcal{O});\ y \geq 0 \text{ a.e. in } \mathcal{O}\}$. (See, e.g., [4], p. 209.)

Remark 2.2. Equation (2.15) is relevant in dynamics of fluid flows in porous media as well as in that of underground water flows. In this later case, (2.15) reduces to the Richards equation which, in presence of a transport term, is written as

$$\frac{\partial y}{\partial t} - \Delta\beta(t,y) + \text{div } K(t,y) = 0.$$

Remark 2.3. In the works [10,13,14] there are several examples of physical problems which are reduced to variational problems by the above procedure, as well as in the recent book [11] by N. Ghoussoub. In particular, in the work [13] the doubly nonlinear equation $\partial\psi\left(\frac{dy}{dt}\right) + \partial\varphi(y) \ni f,\ y(0) = y_0$ is studied via the above Brezis–Ekeland principle.

There are some recent extensions of the Brezis–Ekeland principle to nonlinear equations of the form

$$\frac{dy}{dt} + Ay \ni f, \quad t \in (0,T),\ y(0) = y_0, \tag{2.25}$$

where A is a maximal monotone operator of potential type. This representation of the Cauchy problem (2.25) as a variational problem is via Fitzpatrick function [9]. For a presentation of this approach we refer to the work of A. Visintin [14] (See also the monograph [11].)

3 Proofs

3.1 Proof of Theorem 2.1

Without loss of generality, we may assume that $y_0{=}0$. This can be achieved by shifting the initial data y_0 to origin via the transformation $y{\to}y{-}y_0$.

For simplicity, we shall write $y' = \frac{dy}{dt}\cdot$

As noticed earlier in Introduction, we have (see (1.6), (1.7))

$$\varphi(t,y(t)) + \varphi^*(t,f(t)-y'(t)) = (f(t)-y'(t),y(t)) \text{ a.e. } t \in (0,T),$$

while

$$\varphi(t,z(t)) + \varphi^*(t,f(t)-z'(t)) - (f(t)-z'(t),z(t)) \geq 0 \text{ a.e. } t \in (0,T),$$

for all $z \in L^{p_1}(0,T;V) \cap W^{1,p_2'}([0,T];V')$. Therefore, we are lead to the optimization problem

$$\begin{aligned}\text{Min}\Big\{ \int_0^T &(\varphi(t,y(t)){+}\varphi^*(t,f(t){-}y'(t)){-}(f(t){-}y'(t),y(t)))dt;\\ &y \in L^{p_1}(0,T;V) \cap W^{1,p_2'}([0,T];V'),\ y(0)=0\Big\}.\end{aligned} \tag{3.26}$$

However, since the integral $\int_0^T (y'(t),y(t))dt$ might not be well defined, taking into account that (see, e.g., [4], p. 23)

$$\frac{1}{2}\,\frac{d}{dt}\,\|y(t)\|_V^2 = (y'(t),y(t)) \qquad \text{a.e. } t \in (0,T),$$

for each $y \in L^{p_1}(0,T;V) \cap W^{1,p_2'}([0,T];V')$, we shall replace (3.26) by the following convex optimization problem

$$\begin{aligned}\text{Min}\Big\{\int_0^T(\varphi(t,y(t)))+\varphi^*(t,f(t)-y'(t))-(f(t),y(t))dt+\frac{1}{2}\,\|y(T)\|_V^2;\\ y \in L^{p_1}(0,T;V)\cap W^{1,p_2'}([0,T];V'),\ y(0)=0, y(T)\in H\Big\},\end{aligned} \tag{3.27}$$

which is well defined because, as easily follows by Hypothesis (ii), we have, by virtue of the conjugacy formulae,

$$\overline{\gamma}_1 + \overline{\alpha}_1\|\theta\|_{V'}^{p_2'} \le \varphi^*(t,\theta) \le \overline{\gamma}_2 + \overline{\alpha}_2\|\theta\|_{V'}^{p_1'},\ \forall \theta \in V' \text{ a.e. } t \in (0,T). \tag{3.28}$$

We prove now that problem (3.27) has a solution y^*, which is also a solution to (1.1). To this end, we set $d^* = \inf$ (3.27) and choose a sequence

$$\{y_n\} \subset L^{p_1}(0,T;V) \cap W^{1,p_2'}([0,T];V')$$

such that $y_n(0) = 0$ and

$$\begin{aligned} d^* \le & \int_0^T(\varphi(t,y_n(t))+\varphi^*(t,f(t)-y_n'(t))-(f(t),y_n(t))dt+\frac{1}{2}\,|y_n(T)|_H^2\\ &\le d^* + \frac{1}{n},\ \forall n \in \mathbb{N}.\end{aligned} \tag{3.29}$$

By Hypothesis (ii) and by (3.28), we see that

$$\|y_n\|_{L^{p_1}(0,T;V)} + \|y_n'\|_{L^{p_2'}(0,T;V')} \le C, \quad \forall n \in \mathbb{N},$$

and, therefore, on a subsequence, we have

$$\begin{aligned} y_n &\to y && \text{weakly in } L^{p_1}(0,T;V),\\ y_n' &\to y' && \text{weakly in } L^{p_2'}(0,T;V'),\\ y_n(T) &\to y(T) && \text{weakly in } H.\end{aligned} \tag{3.30}$$

Inasmuch as the functions $y \to \int_0^T \varphi(t,y(t))dt$, $z \to \int_0^T \varphi^*(t,f(t)-z'(t))dt$ and $y_1 \to |y_1|_H^2$ are weakly lower-semicontinuous in $L^{p_1}(0,T;V)$, $L^{p_2'}(0,T;V')$ and H, respectively, letting n tend to zero into (3.29), we see that

$$\int_0^T(\varphi(t,y(t))+\varphi^*(t,f(t)-y'(t))-(f(t),y(t)))dt + \frac{1}{2}\,|y(T)|_H^2 = d^*, \tag{3.31}$$

that is, y is solution to (3.27). Now, we are going to prove that $d^* = 0$. To this aim, we invoke the duality theorem for optimal convex control problems (see [5]). Namely, we have

$$d^* + \min(\mathrm{P}_1^*) = 0, \tag{3.32}$$

where (P_1^*) is the dual optimization problem corresponding to (3.27), that is,

$$(\mathrm{P}_1^*)\ \ \mathrm{Min}\Big\{ \int_0^T \{\varphi(t,-p(t)) + \varphi^*(t, f(t)+p'(t)) + (f(t),p(t)))dt + \frac{1}{2}\,|p(T)|_H^2;$$
$$p \in L^{p_1'}(0,T;V) \cap W^{1,p_2'}(0,T;V')\Big\}.$$

Clearly, for $p=-y$, we get $\min(\mathrm{P}_1^*) \le d^*$ and so, by (3.32), we see that

$$\min(\mathrm{P}_1^*) \le 0. \tag{3.33}$$

On the other hand, if $\widetilde{p}$ is optimal in (P_1^*), we have

$$(\widetilde{p}',\widetilde{p}) \in L^1(0,T),\ \int_0^T (\widetilde{p}',\widetilde{p})dt = \frac{1}{2}\,(|\widetilde{p}(T)|_H^2 - \frac{1}{2}\,|\widetilde{p}(0)|_H^2). \tag{3.34}$$

Indeed, we have

$$-(\widetilde{p}'(t),\widetilde{p}(t)) \le \varphi(t,-\widetilde{p}(t)) + \varphi^*(t,f(t)+p'(t)) + (f(t),\widetilde{p}(t)) \text{ a.e. } t\in[0,T]$$

and

$$(\widetilde{p}'(t)+f(t),\widetilde{p}(t)) \le \varphi(t,\widetilde{p}(t)) + \varphi^*(t,f(t)+\widetilde{p}'(t)) \text{ a.e. } t\in[0,T].$$

Since $\varphi(t,-\widetilde{p}) \in L^1(0,T)$, by Hypothesis (iv), it follows that $\varphi(t,\widetilde{p}) \in L^1(0,T)$, too, and therefore $(\widetilde{p}',\widetilde{p}) \in L^1(0,T)$, as claimed.

Now, since

$$\frac{1}{2}\,\frac{d}{dt}\,|\widetilde{p}(t)|_H^2 = (\widetilde{p}'(t),\widetilde{p}(t)) \text{ a.e. } t\in(0,T),$$

we get (3.34), as claimed. This means that

$$\begin{aligned}\min(\mathrm{P}_1^*) \;&=\; \int_0^T (\varphi(t,-\widetilde{p}(t)) + \varphi^*(t,f(t)+\widetilde{p}'(t)) + (f(t)+\widetilde{p}'(t),\widetilde{p}(t))dt \\ &\quad + \frac{1}{2}\,\|\widetilde{p}(0)\|_H^2 \ge 0\end{aligned}$$

by virtue of (1.5)–(1.6). Then, by (3.33), we get $d^*=0$, as claimed.

The same relation (3.34) follows for y^* and so,

$$\frac{1}{2}\,(|y^*(t)|_H^2 - |y^*(s)|_H^2) = \int_s^t ((y^*)'(\tau),y^*(\tau))d\tau, \quad 0\le s\le t\le T.$$

This implies that $y\in C([0,T];H)$ and

$$\frac{1}{2}\,|y^*(T)|^2 = \int_0^T ((y^*)'(\tau),y^*(\tau))d\tau.$$

Substituting the latter into (3.31), we have that y^* is solution to (3.31) and also that

$$\int_0^T ((\varphi(t,y^*(t)) + \varphi^*(t,f(t)(y^*)'(t)) - (f(t)-(y^*)'(t),y^*(t)) = 0.$$

Hence,

$$\varphi(t, y^*(t)) + \varphi^*(t, f(t)(y^*)'(t)) - (f(t) - (y^*)'(t), y^*(t)) = 0 \quad \text{a.e } t \in (0, T)$$

and, therefore, $(y^*(t))' + \partial\varphi(t, y^*(t)) \ni f(t)$ a.e. $t \in (0, T)$, as claimed.

The uniqueness of a solution y^* satisfying (1.1) is immediate by monotonicity of $u \to \partial\varphi(t, u)$ because, for two such solutions y_1^*, y_2^*, we have therefore

$$\frac{d}{dt}\, \|y_1^*(t) - y_2^*(t)\|_H^2 \le 0 \quad \text{a.e. } t \in (0, T)$$

and, since $y_1^* - y_2^*$ is H-valued continuous and $y_1^*(T) - y_2^*(T) = 0$, we infer that $y_1^* - y_2^* \equiv 0$, as claimed. This completes the proof of Theorem 2.1.

3.2 Proof of Theorem 2.2

First we note that, by hypothesis (iii), part (2.2), we have for all $N > 0$

$$\varphi^*(t, v) \ge N\|v\|_{V'} - C_N, \quad \forall v \in V'.$$

This implies that

$$\lim_{\|v\|_{V'} \to \infty} \frac{\varphi^*(t, v)}{\|v\|_{V'}} = +\infty \ \text{ uniformly in } t. \tag{3.35}$$

Now, coming back to (3.29), we see by (2.2) and (3.35) that

$$\|y_n\|_{L^{p_1}(0,T;V)} \le C, \quad \forall n, \tag{3.36}$$

and, by the Pettis weak compacity theorem in $L^1(0, T; V;)$ (see, e.g., [4]), we have that

$$\{f - y_n'\}_n \text{ is weakly compact in } L^1(0, T; V'. \tag{3.37}$$

Hence, on a subsequence, again denoted n, we have

$$\begin{array}{lcll} y_n & \to & y & \text{weakly in } L^{p_1}(0, T; V), \\ y_n' & \to & y' & \text{weakly in } L^1(0, T; V'). \end{array}$$

Then, letting $n \to \infty$ into (3.29), we see that $y \in W^{1,1}([0, T]; V') \cap L^{p_1}(0, T; V)$ is solution to (3.27), that is, (3.31) holds.

From this point, the proof is identical with that of Theorem 2.1.

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On Wallman compactifications of T_0-spaces and related questions

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Abstract. We study the compactification of the Wallman-Shanin type of T_0-spaces. We have introduced the notion of compressed compactification and proved that any compressed compactification is of the Wallman-Shanin type. The problem of the validity of the equality $\omega(X \times Y) = \omega X \times \omega Y$ is examined. Two open questions have arisen.

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1 Introduction. Preliminaries

Any space is considered to be a T_0-space. We use the terminology from [6, 9]. By $|A|$ we denote the cardinality of a set A, $w(X)$ be the weight of a space X, $\mathbb{N} = \{1, 2, ...\}$. The intersection of τ open sets is called a G_τ-set. For any space X denote by $P_\tau X$ the set X with the topology generated by the G_τ-sets of the space X.

Let τ be an infinite cardinal. A space X is called *τ-subtle* if on X the closed G_τ-sets form a closed base.

Let X be a dense subspace of a space Y. The space Y is called a *compressed extension* of the space X if for some infinite cardinal τ the set X is dense in the space $P_\tau Y$ and Y is τ-subtle. The cardinal τ is called the *index of compressing* of the extension Y of X and put $ic(X \subset Y) \leq \tau$.

Any completely regular space is $\aleph_0$-subtle, i.e. is τ-subtle for any infinite cardinal τ.

Example 1.1. Let τ be an uncountable cardinal, $I = [0, 1]$ and L be a dense subset of I^τ of the cardinality $\leq \tau$. Denote by $\mathcal{T}_1$ the topology of the Tychonoff cube I^τ and $\mathcal{T}$ is the topology generated by the open base $\mathcal{T}_1 \cup \{U \setminus L : U \in \mathcal{T}_1\}$. Denote by X the set I^τ with the topology $\mathcal{T}$. The set L is closed in X. If $m < \tau$, H is a G_m-set of X and $L \subseteq H$, then the set H is dense in X. Thus X is a Hausdorff space which is not m-subtle for any $m < \tau$.

Example 1.2. A space X is called feebly compact if any locally finite family of open non-empty sets is finite. Let Y be an $\aleph_0$-subtle extension of the feebly compact space X. Then Y is a compressed extension of the space X and $ic(X \subset Y) \leq \tau$.

Example 1.3. A completely regular space X is feebly compact if and only if it is pseudocompact. Thus any completely regular extension Y of a pseudocompact space X is compressed and $ic(X \subseteq Y) = \aleph_0$.

Example 1.4. Let Y be the one-point Alexandroff compactification of an uncountable discrete space X. Then $ic(X \subseteq Y) = \aleph_0$.

Definition 1.5. *A family $\mathcal{L}$ of subsets of a space X is called a WS-ring if $\mathcal{L}$ is a family of closed subsets of X and $F \cap H, F \cup H \in \mathcal{L}$ for any $F, H \in \mathcal{L}$.*

Definition 1.6. A family $\mathcal{L}$ of subsets of a space X is called a WF-ring if $\mathcal{L}$ is a WS-ring and $X \setminus F = \cup\{H \in \mathcal{L} : H \cap F = \emptyset\}$ for any $F \in \mathcal{L}$.

The family $\mathcal{F}(X)$ of closed subsets of a space X is a WS-ring. The family $\mathcal{F}(X)$ is a WF-ring if and only if X is a T_1-space.

Definition 1.7. *A g-compactification of a space X is a pair (Y, f), where Y is a compact T_0-space, $f : X \to Y$ is a continuous mapping, the set $f(X)$ is dense in Y and for any point $y \in Y \setminus f(X)$ the set $\{y\}$ is closed in Y. If f is an embedding, then we say that Y is a compactification of X and consider that $X \subseteq Y$, where $f(x) = x$ for any $x \in X$.*

Fix a WS-ring $\mathcal{L}$ of a space X. For any $x \in X$ we put $\xi(x, \mathcal{L}) = \{F \in \mathcal{L} : x \in F\}$. Denote by $M(\mathcal{L}, X)$ the family of all ultrafilters $\xi \in \mathcal{L}$. Let $\omega_{\mathcal{L}} X = M(\mathcal{L}, X) \cup \{\xi \subseteq \mathcal{L} : \xi = \xi(x, y)$ for some $x \in X\}$. Consider the mapping $\omega_{\mathcal{L}} : X \to \omega_{\mathcal{L}} X$, where $\omega_{\mathcal{L}} X = \xi(x, \mathcal{L})$ for any $x \in X$. On $\omega_{\mathcal{L}} X$ consider the topology generated by the closed base $< \mathcal{L} >= \{< H >= \{\xi \in \omega_{\mathcal{L}} X : H \in \xi\} : H \in \mathcal{L}\}$.

Theorem 1.8 (M. Choban, L. Calmuţchi [5]). *If $\mathcal{L}$ is a WS-ring of a space X, then:*

1. $(\omega_{\mathcal{L}} X, \omega_{\mathcal{L}})$ is a g-compactification of the space X.

2. $< H >= cl_{\omega_{\mathcal{L}} X} \omega_{\mathcal{L}}(H)$, $H = \omega_{\mathcal{L}}^{-1}(< H >)$ and $< H > \cap \omega_{\mathcal{L}}(X) = \omega_{\mathcal{L}}(H)$ for any $H \in \mathcal{L}$.

3. $\mathcal{L}$ is a WF-ring if and only if $\omega_{\mathcal{L}} X$ is a T_1-space.

Definition 1.9. *A g-compactification (Y, f) of a space X is called a Wallman-Shanin g-compactification of the space X if $(X, f) = (\omega_{\mathcal{L}} X, \omega_{\mathcal{L}})$ for some WS-ring $\mathcal{L}$.*

Definition 1.10. *A g-compactification (Y, f) of a space X is called a Wallman-Frink g-compactification of the space X if $(X, f) = (\omega_{\mathcal{L}} X, \omega_{\mathcal{L}})$ for some WF-ring $\mathcal{L}$.*

The compactifications of the Wallman-Shanin type were introduced by N. A. Shanin [10] and studied by many authors (see [1, 5, 7, 11–14] and the references in these articles). Any Wallman-Frink g-compactification is a Wallman-Shanin g-compactification. The Wallman compactification $\omega X = \omega_{\mathcal{F}(X)} X$ is a Wallman-Shanin compactification of X. The compactification ωX is a Wallman-Frink compactification if and only if X is a T_1-space. There exists Hausdorff compactifications of discrete spaces which are not Wallman-Shanin compactifications [11, 13].

2 Comparison of the *WS*-rings

Following [7] and [14] on the family $\mathcal{F}(X)$ of closed subsets of a space X consider the binary relation $\sim$: $A \sim B$ if and only if the set $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is relatively compact in X, i.e. its closure in X is compact.

For any family $\mathcal{L} \subseteq \mathcal{F}(X)$ we put $m\mathcal{L} = \{F \in \mathcal{F}(X) : F \sim A \text{ for some } A \in \mathcal{L}\}$. A family $\mathcal{L}$ is called maximal if $\mathcal{L} = m\mathcal{L}$.

Lemma 2.1. *If $\mathcal{L}$ is a WS-ring of subsets of a space X, then $m\mathcal{L}$ is a WS-ring too.*

Proof. Follows from the relations $(A \cap B)\triangle(F \cap H) \subseteq (A\triangle F) \cup (B\triangle H)$ and $(A \cup B)\triangle(F \cup H) \subseteq (A\triangle F) \cup (B\triangle H)$.

Let $\mathcal{L}$ and $\mathcal{M}$ be WS-rings of closed subsets of a space X. We put $\mathcal{L} \leq \mathcal{M}$ if $\mathcal{L} \subseteq \mathcal{M}$ and for each $\xi \in \omega_{\mathcal{M}}X$ we have $\xi \cap \mathcal{L} \in \omega_{\mathcal{L}}X$.

Lemma 2.2. *Let $\mathcal{L}$ and $\mathcal{M}$ be WS-rings of closed subsets of a space X and $\mathcal{L} \leq \mathcal{M}$. Then there exists a unique continuous mapping $\varphi : \omega_{\mathcal{M}}X \to \omega_{\mathcal{L}}X$ such that $\omega_{\mathcal{L}}(x) = \varphi(\omega_{\mathcal{M}}(x))$ for any $x \in X$, i.e. $\omega_{\mathcal{L}} = \varphi \circ \omega_{\mathcal{M}}$.*

Proof. By definition for any $\xi \in \omega_{\mathcal{M}}X$ we have $\xi \cap \mathcal{L} \in \omega_{\mathcal{L}}X$. We put $\varphi(\xi) = \xi \cap \mathcal{L}$. Thus φ is a mapping of $\omega_{\mathcal{M}}X$ into $\omega_{\mathcal{L}}X$. Obviously, $\varphi(\omega_{\mathcal{M}}X) = \omega_{\mathcal{L}}X$.

If $x \in X$, then $\xi(x, \mathcal{L}) = \xi(x, \mathcal{M}) \cap \mathcal{L}$. Hence $\omega_{\mathcal{L}}(x) = \varphi(\omega_{\mathcal{M}}(x))$. For any $F \in \mathcal{L}$ we have $\varphi^{-1}(\{\xi \in \omega_{\mathcal{L}}X : F \in \xi\}) = \{\eta \in \omega_{\mathcal{L}}X : F \in \eta\}$. Thus the mapping φ is continuous.

If $f : \omega_{\mathcal{M}}X \to \omega_{\mathcal{L}}X$ is a continuous mapping and $\omega_{\mathcal{L}} = f \circ \omega_{\mathcal{M}}$, then $f^{-1}(\{\xi \in \omega_{\mathcal{L}}X : F \in \xi\}) = \{\eta \in \omega_{\mathcal{L}}X : F \in \eta\}$ for any $F \in \mathcal{L}$. Thus $f = \varphi$. The proof is complete.

Theorem 2.3. *Let $\mathcal{L}$ be a WS-ring and a closed base of a space X and $F \in \mathcal{L}$ for any closed compact subset F of X. Then $(\omega_{\mathcal{L}}X, \omega_{\mathcal{L}}) = (\omega_{m\mathcal{L}}X, \omega_{m\mathcal{L}})$. Moreover, $\omega_{\mathcal{L}}X$ is a compactification of the space X.*

Proof. For any $\xi \in \omega_{\mathcal{L}}X$ we put $\varphi(\xi) = \xi \cap \mathcal{L}$.

Claim 1. $\varphi(\xi) \in \omega_{\mathcal{L}}X$.

Let $F \in \mathcal{L}$ and $F \notin \xi$. Then there exists $H \in \xi$ such that $F \cap H = \emptyset$. Since $H \in m\mathcal{L}$, we have $H \sim \Phi$ for some $\Phi \in \mathcal{L}$. Hence, there exists a closed compact subset $\Phi_1 \in \mathcal{L}$ such that $H\triangle\Phi\triangle\Phi_1$.

Case 1. $\Phi_1 \in \xi$.

In this case $\cap\xi \neq \emptyset$ and there exists a point $x \in \Phi_1 \subseteq X$ such that $\xi = \xi(x, m\mathcal{L})$. In this case $\varphi(\xi) = \xi(x, \mathcal{L}) \in \omega_{\mathcal{L}}X$.

Case 2. $\Phi_1 \notin \xi$.

In this case there exists $H_1 \in \xi$ such that $H_1 \subseteq H$ and $H_1 \cap \Phi_1 = \emptyset$. Since $\mathcal{L}$ is a base, there exists $H_2 \in \mathcal{L}$ such that $H_1 \subseteq H_2$ and $H_2 \cap \Phi_1 = \emptyset$. Then $H_2 \in \xi$ and $H_2 \cap F = \emptyset$. Thus $H_2 \in \varphi(\xi)$ and $H_2 \cap F = \emptyset$. Hense $\varphi(\xi)$ is a maximal filter in $\mathcal{L}$, i.e. $\varphi(\xi) \in \omega_{\mathcal{L}}X$. Claim 1 is proved.

By virtue of Lemma 2.2, $\varphi : \omega_{m\mathcal{L}}X \to \omega_{\mathcal{L}}X$ is the unique continuous mapping for which $\omega_{\mathcal{L}} = \varphi \circ \omega_{m\mathcal{L}}$.

Claim 2. If $\xi \in \omega_{\mathcal{L}}X$, $F \in \xi$, $H \in \mathcal{L}$, $\cap\xi = \emptyset$ and $F \sim H$, then $H \in \xi$.

There exists a compact subset $\Phi \in \mathcal{L}$ such that $F \triangle H \subseteq \Phi$. Let $H \notin \xi$. Then there exists $L \in \xi$ such that $L \subseteq F, L \cap H = \emptyset$ and $L \cap \Phi = \emptyset$. Then $F \subseteq H \cup \Phi$ and $L \cap (H \cup \Phi) = \emptyset$, a contradiction.

Claim 3. $\varphi : \omega_{m\mathcal{L}}X \to \omega_{\mathcal{L}}X$ is a homeomorphism.

Let $\xi_1, \xi_2 \in \omega_{m\mathcal{L}}X$, $\xi_1 \neq \xi_2$ and $\eta = \varphi(\xi_1) = \varphi(\xi_2)$. In this case $\cap\eta = \emptyset$. Thus there exist $H_1 \in \xi_1 \setminus \xi_2$ and $H_2 \in \xi_2 \setminus \xi_1$ such that $H_1 \cap H_2 = \emptyset$. Since $H_1 \cap H_2 \in m\mathcal{L}$, there exist $F_1, F_2 \in \mathcal{L}$ and a compact subset $\Phi \in \mathcal{L}$ such that $F_1 \sim H_1, F_2 \sim H_2, F_1 \triangle H_1 \subseteq \Phi, F_2 \triangle H_2 \subseteq \Phi$. By virtue of Claim 2, we have $F_1 \in \xi_1$ and $F_2 \in \xi_2$. Then $F_1, F_2 \in \eta$ and $F_1 \cap F_2 \subseteq \Phi$, i.e. $\Phi \in \eta$, a contradiction. Therefore φ is a one-to-one mapping. Let $H \in m\mathcal{L}$ and $H_1 = \{\xi \in \omega_{m\mathcal{L}}X : H \in \xi\}$. Assume that $\eta \in \omega_{m\mathcal{L}}X$ and $H \notin \eta$.

Case 1. $\cap\eta \neq \emptyset$.

In this case $\eta = \xi(x, m\mathcal{L})$ for some $x \in X$ and $x \notin H$. Since $\mathcal{L}$ is a base of X, there exists $F \in \mathcal{L}$ such that $x \notin F$ and $H \subseteq F$. Thus $\varphi(\eta) \notin cl_{\omega_{\mathcal{L}}X}\varphi(H_1)$.

Case 2. $\cap\eta = \emptyset$.

In this case there exists $F \in \mathcal{L}$ such that $H \sim F$. We can assume that $H \subseteq F$. Then $F \notin \eta$ and $\varphi(\eta) \notin cl_{\omega_{\mathcal{L}}X}\varphi(H_1) \subseteq \{\xi \in \omega_{\mathcal{L}}X : F \in \xi\}$. Therefore the set $\varphi(H_1)$ is closed for any $H \in m\mathcal{M}$. Since $\{H_1 : H \in m\mathcal{M}$ is a closed base of $\omega_{\mathcal{L}}X$, the mapping φ is closed. Hence φ is a homeomorphism. The proof is complete.

Let $\mathcal{L}$ and $\mathcal{M}$ be WS-rings of closed subsets of a space X. We put $\mathcal{L} << \mathcal{M}$ if for any two sets $F_1, F_2 \in \mathcal{L}$, with the empty intersection $F_1 \cap F_2 = \emptyset$, there exist two sets $H_1, H_2 \in \mathcal{M}$ such that $H_1 \cap H_2$ is a compact subset of X, $F_1 \subseteq H_1$ and $F_2 \subseteq H_2$. If $\mathcal{L} << \mathcal{M}$ and $\mathcal{M} << \mathcal{L}$, then we put $\mathcal{L} \approx \mathcal{M}$.

Proposition 2.4. *Let $\mathcal{L}$, $\mathcal{M}$ be two WS-rings and closed bases of a space X and $F \in \mathcal{L} \cap \mathcal{M}$ for any closed compact subset F of X. The next assertions are equivalent:*

1. *$\mathcal{L} << \mathcal{M}$.*
2. *$m\mathcal{L} << m\mathcal{M}$.*
3. *$m\mathcal{L} << \mathcal{M}$.*

Proof. Let $\mathcal{L} << \mathcal{M}$. Assume that $F_1, F_2 \in m\mathcal{L}$ and $F_1 \cap F_2 = \emptyset$. By virtue of Theorem 2.3, we have $cl_{\omega_{\mathcal{L}}X}F_1 \cap cl_{\omega_{\mathcal{L}}X}F_1 = \emptyset$. Then there exist two sets $L_1, L_2 \in \mathcal{L}$ such that $L_1 \cap L_2 = \emptyset$, $F_1 \subseteq L_1$ and $F_2 \subseteq L_2$. Since $\mathcal{L} << \mathcal{M}$, there exist two sets $H_1, H_2 \in \mathcal{M}$ such that $H_1 \cap H_2$ is a compact subset of X, $F_1 \subseteq L_1 \subseteq H_1$ and $F_2 \subseteq L_2 \subseteq H_2$. Therefore $m\mathcal{L} << \mathcal{M}$ and $m\mathcal{L} << m\mathcal{M}$. The implications $1 \to 3 \to 2 \to 3$ are proved. Theorem 2.3 completes the proof.

Proposition 2.5. *Let $\omega_{\mathcal{L}}X$ and $\omega_{\mathcal{M}}X$ be Hausdorff compactifications of a space X and $F \in \mathcal{M}$ for any compact subset F of X. The next assertions are equivalent:*

1. *There exists a continuous mapping $\varphi : \omega_{\mathcal{M}}X \to \omega_{\mathcal{L}}X$ such that $\varphi(x) = x$ for any $x \in X$.*
2. *$\mathcal{L} << \mathcal{M}$.*
3. *$m\mathcal{L} << \mathcal{M}$.*
4. *$m\mathcal{L} << m\mathcal{M}$.*

Proof. Let $\varphi : \omega_{\mathcal{M}}X \to \omega_{\mathcal{L}}X$ be a continuous mapping and $\varphi(x) = x$ for any $x \in X$. Fix $F_1, F_2 \in \mathcal{L}$ such that $F_1 \cap F_2 = \emptyset$. Then $cl_{\omega_{\mathcal{L}}X}F_1 \cap cl_{\omega_{\mathcal{L}}X}F_2 = \emptyset$. Since φ is a continuous mapping, then $cl_{\omega_{\mathcal{L}}X}F_1 \cap cl_{\omega_{\mathcal{L}}X}F_2 = \emptyset$. The family $\{cl_{\omega_{\mathcal{M}}X}H : H \in \mathcal{M}\}$ is a closed base of a compact space $\omega_{\mathcal{M}}X$. Thus there exists $H_1, H_2 \in \mathcal{M}$ such that $H_1 \cap H_2 = \emptyset, F_1 \subseteq H_1, F_2 \subseteq H_2$. Implication $1 \to 2$ is proved.

Assume that $\mathcal{L} << \mathcal{M}$. There exist two continuous mappings $f : \beta X \to \omega_{\mathcal{L}}X$ and $g : \beta X \to \omega_{\mathcal{M}}X$ of the Stone-*Čech* compactification βX of X such that $f(x) = g(x) = x$ for any $x \in X$. It is sufficient to prove that $\varphi(x) = f(g^{-1}(x))$ is a singleton for any $x \in \omega_{\mathcal{M}}X$. Let $y \in \omega_{\mathcal{L}}X$ and $x_1, x_2 \in \varphi(y)$ be two distinct points of $\omega_{\mathcal{L}}X$. Obviously, $y \in \omega_{\mathcal{M}}X \setminus X$ and $\varphi(y) \subseteq \omega_{\mathcal{L}}X$. There exists $F_1, F_2 \in \mathcal{L}$ such that $x_1 \in cl_{\omega_{\mathcal{L}}}F_1, x_2 \in cl_{\omega_{\mathcal{L}}}F_2$ and $F_1 \cap F_2 = \emptyset$. Let $H_1, H_2 \in \mathcal{M}$, $H = H_1 \cap H_2$ be a compact subset of X, $F_1 \subseteq H_1$, $F_2 \subseteq H_2$. Then $H = cl_{\omega_{\mathcal{M}}X}H_1 \cap cl_{\omega_{\mathcal{M}}X}H_2$. Let $\Phi_1 = f^{-1}(x_1)$ and $\Phi_2 = f^{-1}(x_2)$. Then $y \in g(\Phi_1) \cap g(\Phi_2)$. Since $\Phi_1 \subseteq cl_{\beta X}F_1$ and $g(cl_{\beta X}F_1) = cl_{\omega_{\mathcal{M}}X}F_1$ we have $g(\Phi_1) \subseteq cl_{\omega_{\mathcal{M}}X}F_1 \subseteq cl_{\omega_{\mathcal{M}}X}H_1$ and $g(\Phi_2) \subseteq cl_{\omega_{\mathcal{M}}X}F_2 \subseteq cl_{\omega_{\mathcal{M}}X}H_2$. Hence $Y \in H \subseteq X$, a contradiction. Implication $2 \to 1$ is proved. Proposition 2.4 completes the proof.

Corollary 2.6. *Let $\omega_{\mathcal{L}}X$ and $\omega_{\mathcal{M}}X$ be Hausdorff compactifications of a space X. Then $\omega_{\mathcal{L}}X = \omega_{\mathcal{M}}X$ if and only if $\mathcal{L} \approx \mathcal{M}$.*

3 On compressed compactification

Teorem 3.1. *If (Y, f) is a compressed g-compactification of a space X, then (Y, f) is a Wallman-Shanin g-compactification of the space X.*

Proof. Let τ be a cardinal number for which:

– $f(X)$ is dense in $P_\tau Y$;

– the closed G_τ-subsets of Y form a closed base of the space Y.

We put $Z = f(X)$. Denote by $\mathcal{F}_\tau(Y)$ the family of all closed G_τ-subsets of Y. By construction, $\mathcal{L} = \{f^{-1}(H) : H \in \mathcal{F}_\tau(Y)\}$ is a WS-ring of closed subsets of the space X.

Claim 1. If $H \in \mathcal{F}_\tau(Y)$, then $H = cl_Y(H \cap Z)$.

Obviously, $cl_Y(H \cap Z) \subseteq H$. Let $y \in H$, U be an open subset of Y and $y \in U$. Then $Y = U \cap H$ is a G_τ-subset of Y. Since Z is G_τ-dense in Y, we have $V \cap Z \neq \emptyset$. Hence $U \cap (H \cap Z) \supseteq V \cap Z \neq \emptyset$ and $y \in cl_Y(H \cap Z)$. Claim is proved.

Claim 2. $(Y, f) = (\omega_{\mathcal{L}}X, \omega_{\mathcal{L}})$.

Let $\xi \in \omega_{\mathcal{L}}X$.

Case 1. $\cap\xi \neq \emptyset$.

There exists $x \in X$ such that $\xi = \xi(x, \mathcal{L})$. In this case we put $\varphi(\xi) = f(x)$

Case 2. $\cap\xi = \emptyset$.

In this case ξ is an $\mathcal{L}$-ultrafilter. Let $\xi = \{L_\alpha : \alpha \in A\}$. For each $\alpha \in A$ there exists a unique $H_\alpha \in \mathcal{F}_\tau(Y)$ such that $L_\alpha = f^{-1}(H_\alpha)$. By construction, $\eta = \{H_\alpha : \alpha \in A\}$ is an $\mathcal{F}_\tau(Y)$-ultrafilter and $\cap\eta = \emptyset$. There exists a unique point $y \in Y \setminus Z$ such that $y \in \cap\eta$. We put $\varphi(\xi) = y$.

The mapping $\varphi : \omega_{\mathcal{L}}X \to Y$ of $\omega_{\mathcal{L}}X$ onto Y is constructed. Obviously, the mapping φ is one-to-one. By construction, $\varphi(\omega_{\mathcal{L}}(x)) = f(x)$ for any $x \in X$ and $\varphi(\{\xi \in \omega_{\mathcal{L}}X : f^{-1}(H) \in \xi\} = H$ for each $H \in \mathcal{F}_\tau(Y)$. Hence the mapping φ is a homeomorphism. The proof is complete.

Corollary 3.2. *Let Y be a Hausdorff compactification of a space X and the space X is G_τ-dense in Y. Then Y is a Wallman-Frink compactification of the space X.*

Corollary 3.3 (R. A. Alo, H. L. Shapiro [1], E. Wajch [14]). *Let X be a pseudo-compact space. Then any Hausdorff compactification Y of X is a Wallman-Frink compactification.*

Corollary 3.4. *Let (Y, f) be a Hausdorff g-compactification of a feebly compact space X. Then (Y, f) is a Wallman-Frink g-compactification of the space X.*

For any discrete uncountable space the family of compressed Hausdorff compactifications is large. Moreover, this fact is valid for Hausdorff paracompact locally compact non-Lindelöf spaces.

Theorem 3.5. *Let X be a Hausdorff locally compact space which contains an uncountable discrete family of open non-empty subsets. Assume that $dimX = 0$. Then the family $\mathcal{B}$ of all compressed Hausdorff compactifications of X is uncountable and $\beta X = sup\mathcal{B}$.*

Proof. Fix $n \geq 2$. There exists a family $\{X_\mu : \mu \in M\}$ of open-and-closed subsets of X such that for any $\mu \in M$ the set X_μ is compact and there exist n distinct points $b_{1\mu}, b_{2\mu}, ..., b_{n\mu} \in X_\mu$. The sets $\{B_i = \{b_{i\mu} : \mu \in M\} : i \leq n\}$ are closed and disjoint. Fix n distinct points $b_1, b_2, ..., b_n \in \beta X \setminus X$. Since the sets $\{cl_{\beta X} B_i : i \leq n\}$ are disjoint we can assume that $b_{im} \notin \cup\{cl_{\beta X} B_j : j \leq n, j \neq i\}$ for any $i \leq n$. Fix n open-and-closed subsets $\{H_i : i \leq n\}$ of $\beta X \setminus X$ such that $b_i \in H_i, H_i \cap H_j = \emptyset$, $H_i \cap cl_{\beta X} B_j = \emptyset$ for any $i \neq j$ and $\beta X \setminus X = \cup\{H_i : i \leq n\}$. Then there exists a compactification Y of X and a continuous mapping $f : \beta X \to Y$ such that $f(x) = x$ for any $x \in X$ and $f^{-1}(f(b_i)) = H_i$ for any $i \leq n$. The compactification Y is compressed. By construction, the compressed compactifications $\mathcal{B}$ of X separate the points of βX. Thus $\beta X = sup\mathcal{B}$. The proof is complete.

4 Cartesian products of compactifications

Let A be a non-empty set, $\{X_\alpha : \alpha \in A\}$ be a family of non-empty spaces, $X = \Pi\{X_\alpha : \alpha \in A\}$, $(b_\alpha X_\alpha, \varphi_\alpha)$ be a family of g-compactifications of given spaces X_α. Then $bX = \Pi\{b_\alpha X_\alpha : \alpha \in A\}$ and the mapping $\varphi : X \to bX$, where $\varphi((x_\alpha : \alpha \in A)) = (\varphi_\alpha(x_\alpha) : \alpha \in A)$ for any $(x_\alpha : \alpha \in A) \in X$, is a g-compactification of the space X. If each $b_\alpha X_\alpha$ is a compactification of the space X_α, then bX is a compactification of the space X. Let $\mathcal{L}_\alpha$ be a WS-ring of closed sets of the space X_α. We put $\mathcal{L}' = \{\Pi\{H_\alpha : \alpha \in A\} : H_\alpha \in \mathcal{L}_\alpha, \alpha \in A\}$, $\mathcal{L} = \{H_1 \cup H_2 \cup ... \cup H_n : H_1, H_2, ..., H_n \in \mathcal{L}', n \in \mathbb{N}\}$.

Now we put $\mathcal{L} = \otimes\{\mathcal{L}_\alpha : \alpha \in A\}$.

Theorem 4.1. *The family $\mathcal{L}$ of closed subsets of the space X is a WS-ring and $\omega_{\mathcal{L}} X = \Pi\{\omega_{\mathcal{L}_\alpha} X_\alpha : \alpha \in A\}$.*

Proof. Obviously, $\mathcal{L}$ is a WS-ring of closed subsets of the space X.

Let ξ be an $\mathcal{L}'$-filter. Obviously $e(\xi) = \{H \cup F : H \in \xi, F \in \mathcal{L}\}$ is a $\mathcal{L}$-filter. Moreover, ξ is an $\mathcal{L}'$-ultrafilter if and only if $e(\xi)$ is an $\mathcal{L}$-ultrafilter.

Let $x = (x_\alpha : \alpha \in A\}$, $\xi_\alpha \subseteq \mathcal{L}_\alpha$ and $\xi = \{\Pi\{H_\alpha : \alpha A\} : H_\alpha \in \xi_\alpha, \alpha \in A\}$. Then:
– if any ξ_α is an $\mathcal{L}_\alpha$-filter, then ξ is an $\mathcal{L}'$-filter;
– ξ_α is an $\mathcal{L}_\alpha$-ultrafilter if and only if ξ is an $\mathcal{L}'$-ultrafilter;
– $\xi_\alpha = \xi(x_\alpha, \mathcal{L}_\alpha)$ for any $\alpha \in A$ if and only if $\xi = \xi(x, \mathcal{L}')$ and $e(\xi) = \xi(x, \mathcal{L})$.
These facts complete the proof.

Theorem 4.2. *If $|A| \geq 2$, $\omega X = \Pi\{\omega X_\alpha : \alpha \in A\}$ and any X_α is an infinite T_1-space, then any space X_α is countably compact.*

Proof. Fix $\beta \in A$. Assume that the space X_β is not countably compact. Then X_β contains an infinite, discrete and closed subset $F = \{b_n : n \in \mathbb{N}\}$.

Since $\omega Z_1 = cl_{\omega Z} Z_1$ for any closed subspace Z_1 of a T_0-space Z, we can assume that $X_\beta F$.

We put $Y_\beta = \Pi\{\omega X_\alpha : \alpha \in A \setminus \{\beta\}\}$. Obviously, $X = X_\beta \times Y_\beta$ and $\omega X = \omega X_\beta \times \omega Y_\beta$.

If the space Y_β is not countably compact, then Y_β contains a discrete infinite space Z and $\omega(X_\beta \times Z) = cl_{\omega X}(X_\beta \times Z) = (\omega X_\beta \times \omega Z)$, a contradiction with the Glicksberg's theorem ([6], Problem 3.12.20(d)). Thus we can assume that the space Y_β is countably compact.

In the space Y_β fix a set $L = \{c_n : n \in \mathbb{N}\}$, where $c_n \neq c_m$ for distinct $n, m \in \mathbb{N}$. The set $\Phi = \{(b_n, c_n) : n \in \mathbb{N}\}$ is closed and discrete in X. Projection $p : X_\beta \times Y_\beta \to X_\beta$ is a continuous closed mapping. Fix an ultrafilter ξ of closed subsets of the space X for which $\Phi \in \xi$ and $\cap\xi = \emptyset$. Then $p(\xi) = \{p(H) : H \in \xi\}$ is an ultrafilter of closed subsets of the space X_β. If $\cap p(\xi) \neq \emptyset$, then there exists a unique point $b \in X_\beta$ for which $\{b\} = \cap p(\xi)$. In this case $\{b\} \times Y_\beta \in \xi$ and $\cap\xi = \emptyset$. Since $\Phi \in \xi$, there exists a unique $n \in \mathbb{N}$ such that $b = b_n$ and $(b_n, c_n) \in H \cap (\{b\} \times Y_\beta)$ for each $H \in \xi$, a contradiction with $\cap\xi = \emptyset$. Thus $\cap p(\xi) = \emptyset$. Hence there exists a unique $b \in \omega X_\beta \setminus X_\beta$ for which $\{b\} = \cap\{cl_{X_\beta} H : H \in p(\xi)\}$.

Since $\omega X = \omega X_\beta \times \omega Y_\beta$, there exists a unique $c \in \omega Y_\beta \setminus Y_\beta$ such that $(b, c) \in \cap\{cl_{\omega X} H : H \in \xi\}$. In this case $X_\beta \times \{c\} \in \xi$. There exists a unque $n \in \mathbb{N}$ and some $H \in \xi$ such that $(b_n, c_n) \in \Phi \cap (X_\beta \times \{c\})$ and $H \cap (X_\beta \times \{c\}) = \emptyset$. Then $(b, c) \in cl_{\omega X} H \cap cl_{\omega X}(X_\beta \times \{c\})$ and $cl_{\omega X} H \cap cl_{\omega X}(X_\beta \times \{c\})$, a contradiction. The proof is complete.

Theorem 4.3. *Let $f : X \to Y$ be a continuous closed mapping of a space X onto a space Y. Then there exists a unique continuous mapping $\omega f : \omega X \to \omega Y$ such that $f = \omega f | X$. Moreover, the mapping ωf is closed.*

Proof. If ξ is an ultrafilter of closed subsets of X, then $\omega f(\xi) = \{f(H) : H \in \xi\}$ is an ultrafilter of closed subsets of Y. The mapping ωf is constructed.

Let τ be an infinite cardinal number. A space X is called initial τ-compact if any open cover γ of X of the cardinality $\leq \tau$ contains a finite subcover.

We say that the sequential character $s\chi(X) < \tau$ if for any non-closed subset H of X there exist a subset $Y \subseteq X$ and a point $x \in X \setminus H$ such that $x \in Y$, $x \in cl_X(H \cap Y)$ and $\chi(Y, x) < \tau$. A space X is sequential if and only if $s\chi(X) \leq \aleph_0$.

Theorem 4.4. *Let τ be an infinite cardinal number, X be an initial τ-compact space, Y be a compact space and $s\chi(Y) \leq \tau$. Then:*

1. The projection $p : X \times Y \to Y$, where $p(x, y) = y$ for each $(x, y) \in X \times Y$, is a continuous closed-and-open mapping.

2. There exists a continuous bijection $\varphi : \omega(X \times Y) \to \omega X \times Y$ such that $\varphi(x, y) = (x, y)$ for all $(x, y) \in X \times Y$.

Proof. It is well known that the projection p is continuous and open.

Let $y_0 \in Y$, W be an open subset of $X \times Y$ and $p^{-1}(y_0) = X \times \{y_0\} \subseteq W$. We put $V = \{y \in Y : p^{-1}(y) \subseteq W\}$. Obviously, $y_0 \in V$. We affirm that the set V is open in Y. Suppose that the set V is not open in Y. Then the set $Y \setminus V$ is not closed in Y. Thus there exist a point $z \in V$ and a subspace $Z \subseteq Y$ such that $z \in Z$, $z \in cl_Z(Z \cap (Y \setminus V))$ and $\chi(Z, z) \leq \tau$. We fix an open base $\{V_\alpha : \alpha \in A\}$ of the space Z at the point z such that $|A| \leq \tau$. For any $\alpha \in A$ consider the set $U_\alpha = \cup\{U : U \textit{ is open } X,\ U \times V_\alpha \subseteq W\}$. Obviously $X = \cup\{U_\alpha : \alpha \in A\}$. Since X is τ-compact and $|A| \leq \tau$, there exists a finite set $B \subseteq A$ such that $X = \cup\{U_\alpha : \alpha \in B\}$. There exists an element $\beta \in A$ for which $V_\beta \subseteq \cap\{V_\alpha : \alpha \in B\}$. Then $U_\beta \supseteq \cup\{U_\alpha : \alpha \in B\} = X$. Hence $U_\beta = X$ and $X \times V_\beta = U_\beta \times V_\beta \subseteq W$. Therefore $z \in V_\beta \subseteq V$ and $z \notin cl_Y(Y \setminus V)$, a contradiction. Assertion 1 is proved.

Consider the projection $f : X \times Y \to X$. The mappings f and p are continuous open-and-closed. Then there exist two continuous closed mappings $\omega f : \omega(X \times Y) \to \omega X$ and $\omega p : \omega(X \times Y) \to \omega Y$ such that$f = \omega f | X \times Y$ and $p = \omega p | X \times Y$. Consider the continuous mapping $\varphi : \omega(X \times Y) \to \omega X \times Y$ for which $\varphi(z) = (\omega f(z), \omega p(z))$ for each $z \in \omega(X \times Y)$. By construction, we have $\varphi(z) = (f(x, y), p(x, y)) = (x, y) = z$ for each $z = (x, y) \in X \times Y \subseteq \omega(X \times Y)$. Fix $z \in \omega(X \times Y) \setminus (X \times Y)$. Then there exists a unique ultrafilter ξ of closed subsets of $X \times Y$ for which $\{z\} = \cap\{cl_{\omega(X \times Y)} H : H \in \xi\}$. The family $p(\xi) = \{g(H) : H \in \xi\}$ is an ultrafilter of closed subsets of the space Y. There exists a unique point $y(z) = \omega g(z) \in \cap\{cl_Y g(H) : H \in \xi\}$. In this case $X(\xi) = X \times \{y(z)\} \in \xi$. Thus $\overline{\xi} = \{H \cap X(\xi) : H \in \xi\} \subseteq \xi$ is an ultrafilter of closed subsets of the subspace $X(\xi)$ of $X \times Y$.

Let ξ, η be two ultrafilters of closed subsets of the space $X \times Y$, $z \in \cap\{cl_{\omega(X \times Y)} H : H \in \xi\}$ and $z' \in \cap\{cl_{\omega(X \times Y)} H : H \in \eta\}$. Assume that $y(z) = y(z')$. Then $X(\xi) = X(\eta)$ and there there exist $H \in \overline{\xi}$ and $L \in \overline{\eta}$ such that $H \cap L = \emptyset$. Since $f | X(\xi) : X(\xi) \to X$ is a homeomorphism, $f(\xi) = f(\overline{\xi})$, $f(\eta) = f(\overline{\eta})$ and $f(H) \cap f(L) = \emptyset$. Thus $f(\xi) \neq f(\eta)$ and $\omega f(z) = \cap\{cl_{\omega X} f(M) : M \in \xi\} \neq \cap\{cl_{\omega X} f(P) : P \in \xi\eta\} = \omega f(z')$. Therefore φ is a bijection. The proof is complete.

Corollary 4.6. *Let τ be an infinite cardinal number, X be an initial τ-compact normal space, Y be a compact Hausdorff space and $s\chi(Y) \leq \tau$. Then:*

1. $\omega(X \times Y) = \omega X \times Y$.

2. $X \times Y$ is an initial τ-compact normal space.

Remark 4.7. Let X be a first countable normal countably compact not paracompact space and $Y = \beta X$. By virtue of Tamano's Theorem (see [6], Theorem 5.1.38), the space $X \times Y$ is not normal. Then $\omega X = \beta X$ and $\omega(X \times Y) \neq \beta(X \times Y) = \omega X \times Y$. Thus the restriction $s\chi(Y) \leq \tau$ in the above assertions is essential.

5 Remainders of compactifications

The main result of the section is the following theorem.

Theorem 5.1 *For any space Y the following assertions are equivalent:*

1. Y is a T_1-space.

2. There exists a T_0-space X such that the spaces Y and $\omega X \setminus X$ are homeomorphic.

3. There exists a T_1-space X such that the spaces Y and $\omega X \setminus X$ are homeomorphic.

Proof. Let X be a T_0- space and $Y = \omega X \setminus X$. Any ultrafilter of closed sets ξ represents a point $\xi \in \omega X$ for which the set $\{\xi\}$ is closed in ωX. Thus Y is a T_1-space. Implication $2 \to 1$ is proved. Implication $3 \to 2$ is obvious.

Let Y be a non-empty T_1-space. If Y is compact, then we put $Z = Y$. Let Y be a non-compact space. Consider a point $b \notin Y$. In this case Y is an open subspace of the space $Z = Y \cup \{b\}$, where the base of the space Z at the point b is the family $\{Z \setminus \Phi : \Phi$ *is a closed compact subset of* $Y\}$. By construction Z is a compact T_1-space. Fix an infinite cardinal number $\tau \geq w(Z)$. Denote by $W(\tau^+)$ the space of all ordinal numbers of the cardinality $\leq \tau$ in the topology generated by the linear order. Then $W(\tau^+)$ is a normal initial τ-compact space and $\omega W(\tau^+) \setminus (\tau^+) = \{c\}$ is a singleton.

If the space Y is compact, we consider the space $X = W(\tau^+) \times Y$ as a subspace of the compact space $\omega W(\tau^+) \times Z$. Further, if the space Y is not compact, then we consider the space $X = (W(\tau^+) \times Y) \cup \{(c, b)\}$ as a subspace of the compact space $\omega W(\tau^+) \times Z$.

Since the space X is initial τ-compact and $s\chi(Z) \leq \tau$, the mapping $g : X \longrightarrow Z$, where $g(z, y) = y$ for any $(z, y) \in X$, is continuous and open-and-closed. Hence $\omega X = \omega W(\tau^+) \times Z$. By construction, the spaces $\omega X \setminus X = \{c\} \times Y$ and Y are homeomorphic. The proof is complete.

Any Hausdorff locally compact space is a Wallman remainder of some normal space.

Question 1. *Under which conditions a completely regular space is a Wallman remainder of some normal space?*

Question 2. *Under which conditions a T_1-space is a Wallman remainder of some completely regular (regular, Hausdorff) space?*

Other problems about remainders of spaces have been examined recently in [2–4].

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