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Fuzzy subquasigroups with respect to a *s*-norm

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Abstract. In this paper the notion of idempotent fuzzy subquasigroups with respect to a s-norm is introduced and some related properties are investigated. Then properties of homomorphic image and inverse image of fuzzy subquasigroups respect to a s-norm are discussed. Next some properties of direct product of fuzzy subquasigroups with respect to a s-norm are presented. Finally abnormalization of fuzzy subquasigroups with respect to a s-norm is studied.

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1 Introduction

During the last decade, there have been many applications of quasigroups in different areas, such as cryptography [12], modern physics [13], coding theory, cryptology [17], geometry [11].

The notion of fuzzy sets was first introduced by Zadeh [19]. Fuzzy set theory has been developed in many directions by many scholars and has evoked great interest among mathematicians working in different fields of mathematics, and also there have been wide-ranging applications of the theory of fuzzy sets, from the design of robots and computer simulation to engineering and water resources planning. Rosenfeld [14] introduced the fuzzy sets in the realm of group theory. Since then many mathematicians have been involved in extending the concepts and results of abstract algebra to the broader framework of the fuzzy settings. Triangular norms were introduced by Schweizer and Sklar [15, 16] to model the distances in probabilistic metric spaces. In fuzzy sets theory triangular norm (t-norm) and triangular co-norm (t-conorm or s-norm) are extensively used to model the logical connectives: conjunction (AND) and disjunction (OR) respectively. There are many applications of triangular norms in several fields of Mathematics, and Artificial Intelligence [10]. Dudek [6] introduced the notion of a fuzzy subquasigroup and studied some of its properties. Dudek and Jun [7] introduced the notion of an idempotent fuzzy subquasigroup with respect to a t-norm and discussed some of its properties. In this paper the notion of idempotent fuzzy subquasigroups with respect to a s-norm is introduced, and some related properties are investigated. Relationship between T-fuzzy subquasigroups and S-fuzzy subquasigroups of quasigroups is given. Some

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properties of direct product of fuzzy subquasigroups with respect to a s-norm are also discussed.

2 Preliminaries

In this section we first review some elementary aspects that are necessary for this paper:
A groupoid $(G, \cdot)$ is called a *quasigroup* if for any a , $b \in G$ each of the equations $a \cdot x = b$, $x \cdot a = b$ has a unique solution in G. A quasigroup may be also defined as an algebra $(G, \cdot, \backslash, /)$ with three binary operations $\cdot$, $\backslash$, $/$ satisfying the following identities:

$$(x \cdot y)/y = x, \quad x \backslash (x \cdot y) = y,$$
$$(x/y) \cdot y = x, \quad x \cdot (x \backslash y) = y.$$

The operations $\backslash$ and $/$ are called *left and right division*. In abstract algebra, a quasigroup is a algebraic structure resembling a group in the sense that "division" is always possible. Quasigroups differ from groups mainly in that they need not be associative. A nonempty subset S of a quasigroup $\mathcal{G} = (G, \cdot, \backslash, /)$ is called a *subquasigroup* if it is closed with respect to these three operations, that is, if $x * y \in S$ for all x, $y \in S$ and $* \in \{\cdot, \backslash, /\}$.

The class of all equasigroups forms a variety. This means that a homomorphic image of an equasigroup is an equasigroup. Also every subset of an equasigroup closed with respect to these three operations is an equasigroup.

For the general development of the theory of quasigroups the unipotent quasigroups, i.e., quasigroups with the identity $x \cdot x = y \cdot y$, play an important role. These quasigroups are connected with Latin squares which have one fixed element in the diagonal [5]. Such quasigroups may be defined as quasigroups G with the special element θ satisfying the identity $x \cdot x = \theta$. Obviously, θ is uniquely determined and it is an idempotent, but, in general, it is not the (left, right) neutral element.

To avoid repetitions we use the following convention: *a quasigroup* $\mathcal{G}$ always denotes an equasigroup $(G, \cdot, \backslash, /)$; G always denotes a nonempty set.

A mapping $\mu : G \to [0, 1]$ is called a *fuzzy set* in a quasigroup $\mathcal{G}$. For any fuzzy set μ in G and any $t \in [0, 1]$ we define set

$$L(\mu; t) = \{x \in G \mid \mu(x) \leq t\},$$

which is called *lower t-level cut* of μ.

Definition 1. [6] A fuzzy set μ in G is called a *fuzzy subquasigroup* of $\mathcal{G}$ if

$$\mu(x * y) \geq \min(\mu(x), \mu(y))$$

for all x, $y \in G$ and $* \in \{\cdot, \backslash, /\}$.

Proposition 1. *[6] A fuzzy set μ of a quasigroup $\mathcal{G} = (G, \cdot, \backslash, /)$ is a fuzzy subquasigroup if and only if for every $\alpha \in [0, 1]$, μ_α is empty or a subquasigroup of G.*

Proposition 2. *[6] If μ is a fuzzy subquasigroup of a unipotent quasigroup $(G, \cdot, \backslash, /, \theta)$, then $\mu(\theta) \geq \mu(x)$ for all $x \in G$.*

Definition 2. [15] A s-norm is a mapping $S : [0,1] \times [0,1] \to [0,1]$ that satisfies the following conditions:

(S1) $S(x, 0) = x$,

(S2) $S(x, y) = S(y, x)$,

(S3) $S(x, S(y, z)) = S(S(x, y), z)$,

(S4) $S(x, y) \leq S(x, z)$ whenever $y \leq z$

for all x, y, $z \in [0, 1]$.

Definition 3. Given a t-norm T and a s-norm S, T and S are *dual (with respect to the negation ′)* if and only if $(T(x, y))' = S(x', y')$.

Proposition 3. *Conjunctive(AND) operator is a t-norm T and disjunctive(OR) operator is its dual s-norm S.*

3 Fuzzy subquasigroups with respect to a s-norm

Definition 4. The set of all idempotents with respect to S, i.e., the set $E_S = \{x \in [0; 1] \mid S(x, x) = x\}$, is a subsemigroup of $([0, 1], S)$. If $Im(\mu) \subseteq E_S$, then a fuzzy set μ is called *an idempotent with respect to a s-norm S* (briefly, a S-idempotent).

Definition 5. Let S be a s-norm. A fuzzy set μ in G is called a *fuzzy subquasigroup of $\mathcal{G}$ with respect to a s-norm S* (briefly, S-fuzzy subquasigroup) if

$$\mu(x * y) \leq S(\mu(x), \mu(y))$$

for all x, $y \in G$ and $* \in \{\cdot, \backslash, /\}$. If a S-fuzzy subquasigroup μ of $\mathcal{G}$ is an idempotent, we say that μ is an idempotent S-fuzzy subquasigroup of $\mathcal{G}$.

Example 1. Let $G = \{0, a, b, c\}$ be a quasigroup with the following Cayley Table:

$\cdot$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let S_m be a s-norm defined by $S_m(x, y) = \min\{x + y, 1\}$ for all x, $y \in [0, 1]$. Define a fuzzy set μ in $\mathcal{G}$ by

$$\mu(x) = \begin{cases} 1, & \text{if x} \in \{\text{0, a, b}\} \\ 0, & otherwise. \end{cases}$$

It is easy to see that μ satisfies

$$\mu(x * y) \leq S_m(\mu(x), \mu(y))$$

for all x, $y \in G$, and $Im(\mu) \subseteq E_{S_m}$. Hence μ is an idempotent fuzzy subquasigroup of $\mathcal{G}$ with respect to S_m.

The following three propositions are obvious.

Proposition 4. *If a fuzzy set μ is an idempotent with respect to a s-norm S, then $S(x, y) = \max\{x, y\}$ for all x, $y \in Im(\mu)$.*

Proposition 5. *Let a fuzzy set μ on a quasigroup $\mathcal{G}$ be an idempotent with respect to a s-norm S. If each nonempty level set μ_α is a subquasigroup of $\mathcal{G}$, then μ is a S-idempotent fuzzy subquasigroup.*

Proposition 6. *Let μ be a S-fuzzy subquasigroup of $\mathcal{G}$ and $\alpha \in [0, 1]$.*

(a) *If $\alpha = 0$, then $L(\mu; \alpha)$ is either empty or a subquasigroup of $\mathcal{G}$.*

(b) *If $S = \max$, then $L(\mu; \alpha)$ is either empty or a subquasigroup of $\mathcal{G}$.*

Theorem 1. *Let S be a s-norm. If each nonempty level subset $L(\mu; \alpha)$ of μ is a subquasigroup of $\mathcal{G}$, then μ is a S-fuzzy subquasigroup of $\mathcal{G}$.*

Proof. Assume that every nonempty level subset $L(\mu; \alpha)$ of μ is a subquasigroup of $\mathcal{G}$. If there exist x, $y \in \mathcal{G}$ such that $\mu(x * y) >$ S $(\mu(x), \mu(y))$, then by taking $t_0 := \frac{1}{2}\{\mu(x * y) + S(\mu(x), \mu(y))\}$, we have $x \in L(\mu; t_0)$ and $y \in L(\mu; t_0)$. Since μ is a subquasigroup of $\mathcal{G}$, $x * y \in L(\mu; t_0)$, $\mu(x * y) \leq t_0$, a contradiction. Hence μ is a S-fuzzy subquasigroup of $\mathcal{G}$. □

Definition 6. Let $\mathcal{G}$ be a quasigroup and a family of fuzzy sets $\{\mu_i \mid i \in I\}$ in a quasigroup $\mathcal{G}$. Then the union $\bigvee_{i \in I} \mu_i$ of $\{\mu_i \mid i \in I\}$ is defined by

$$(\bigvee_{i \in I} \mu_i)(x) = \sup\{\mu_i(x) \mid i \in I\},$$

for each $x \in \mathcal{G}$.

Theorem 2. *If $\{\mu_i \mid i \in I\}$ is a family of fuzzy subquasigroups of a quasigroup $\mathcal{G}$ with respect to S, then $\bigvee_{i \in I} \mu(x_i)$ is a fuzzy subquasigroup of $\mathcal{G}$ with respect to S.*

Proof. Let $\{\mu_i \mid i \in I\}$ be a family of fuzzy subquasigroups of $\mathcal{G}$ with respect to S. For x, $y \in \mathcal{G}$, we have

$$\begin{aligned}
(\bigvee_{i \in I} \mu_i)(x * y) &= \sup\{\mu_i(x * y) \mid i \in I\} \\
&\leq \sup\{S(\mu_i(x), \mu_i(y)) \mid i \in I\} \\
&= S(\sup\{\mu_i(x) \mid i \in I\}, \sup\{\mu_i(y) \mid i \in I\}) \\
&= S(\bigvee_{i \in I} \mu_i(x), \bigvee_{i \in I} \mu_i(y)).
\end{aligned}$$

Hence $\bigvee_{i \in I}$ is a fuzzy subquasigroup of $\mathcal{G}$ with respect to S. □

Definition 7. Let f be a mapping on $\mathcal{G}$. If v is a fuzzy set in $f(\mathcal{G})$, then the fuzzy set $\mu = v \circ f$ (i.e., $(v \circ f)(x) = v(f(x))$) in $\mathcal{G}$ is called the *preimage* of v under f.

Theorem 3. *An onto homomorphism preimage of a S-fuzzy subquasigroup of $\mathcal{G}$ is a S-fuzzy subquasigroup.*

Proof. Let $f : \mathcal{G}_1 \to \mathcal{G}_2$ be an onto homomorphism of quasigroups. If v is a S-fuzzy subquasigroup of $\mathcal{G}_2$ and μ is the preimage of v under f, then for any x, $y \in \mathcal{G}_1$, we have

$$\begin{aligned}
\mu(x * y) &= (v \circ f)(x * y) = v(f(x * y)) \\
&\leq S(v(f(x)), v(f(y))) \\
&= S((v \circ f)(x), (v \circ f)(y)) \\
&= S(\mu(x), \mu(y)).
\end{aligned}$$

This shows that μ is a fuzzy subquasigroup of $\mathcal{G}_1$ with respect to a s-norm S. $\square$

Definition 8. Let μ be a fuzzy set in a quasigroup $\mathcal{G}$ and let f be a mapping defined on $\mathcal{G}$. Then the fuzzy set μ^f in $f(\mathcal{G})$ defined by

$$\mu^f(y) = \inf_{x \in f^{-1}(y)} \mu(x) \quad \forall y \in f(\mathcal{G})$$

is called the *image* of μ under f. A fuzzy set μ in $\mathcal{G}$ has the inf *property* if for any subset $A \subseteq \mathcal{G}$, there exists $a_0 \in A$ such that $\mu(a_0) = \inf_{a \in A} \mu(a)$.

Theorem 4. *An onto homomorphism image of a fuzzy subquasigroup with the* inf *property is a fuzzy subquasigroup.*

Proof. Let $f : \mathcal{G}_1 \to \mathcal{G}_2$ be an epimorphism of $\mathcal{G}_1$ and μ a fuzzy subquasigroup of $\mathcal{G}_1$ with the inf property. Consider $f(x)$, $f(y) \in \mathrm{f}(\mathcal{G}_1)$. Now, let x_0, $\mathrm{y}_0 \in \mathrm{f}^{-1}(f(x))$ be such that

$$\mu(x_0) = \inf_{t \in f^{-1}(f(x))} \mu(t)$$

and

$$\mu(y_0) = \inf_{t \in f^{-1}(f(y))} \mu(t)$$

respectively. Then we can deduce that

$$\begin{aligned}
\mu^f(f(x) * f(y)) &= \inf_{t \in f^{-1}(f(x) * f(y))} \mu(t) \\
&\leq \max\{\mu(x_0), \mu(y_0)\} \\
&= \max\{\inf_{t \in f^{-1}(f(x))} \mu(t), \inf_{t \in f^{-1}(f(y))} \mu(t)\} \\
&= \max\{\mu^f(f(x)), \mu^f(f(y))\}.
\end{aligned}$$

Consequently, μ^f is a fuzzy subquasigroup of $\mathcal{G}_2$. $\square$

Definition 9. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two quasigroups and let f be a function from $\mathcal{G}_1$ into $\mathcal{G}_2$. If ν is a fuzzy set in $\mathcal{G}_2$, then the *preimage* of ν under f is the fuzzy set in $\mathcal{G}_1$ defined by

$$f^{-1}(\nu)(x) = \nu(f(x)) \quad \forall \;\; x \in \mathcal{G}_1.$$

Theorem 5. *Let $f : \mathcal{G}_1 \to \mathcal{G}_2$ be an epimorphism of quasigroups. If ν is a S-fuzzy subquasigroup in $\mathcal{G}_2$, then $f^{-1}(\nu)$ is a S-fuzzy subquasigroup in $\mathcal{G}_1$.*

Proof. Let x, $y \in \mathcal{G}_1$, then

$$\begin{array}{rcl} f^{-1}(\nu)(x*y) &=& \nu(f(x*y)) \\ &\leq& S(\nu(f(x), f(y)) \\ &=& S(\nu(f(x)), \nu(f(y))) \\ &=& S(f^{-1}(\nu)(x), f^{-1}(\nu)(y)). \end{array}$$

Hence $f^{-1}(\nu)$ is a S-fuzzy quasigroup in $\mathcal{G}_1$. □

Definition 10. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be quasigroups and f a function from $\mathcal{G}_1$ into $\mathcal{G}_2$. If ν is a fuzzy set in $\mathcal{G}_2$, then the *image* of μ under f is the fuzzy set in $\mathcal{G}_1$ defined by
$f(\mu)(x) = \begin{cases} \inf_{x \in f^{-1}(y)} \mu(x), & \text{if } f^{-1}(y) \neq \emptyset \;, \\ 0, & \text{otherwise,} \end{cases}$
for each $y \in \mathcal{G}_2$.

Theorem 6. *Let $f : \mathcal{G}_1 \to \mathcal{G}_2$ be an onto homomorphism of quasigroups. If μ is a S-fuzzy subquasigroup in $\mathcal{G}_1$, then $f(\mu)$ is a S-fuzzy subquasigroup in $\mathcal{G}_2$.*

Proof. Let y_1, $y_2 \in \mathcal{G}_2$, then

$$\{x| \; x \in f^{-1}(y_1 * y_2)\} \subseteq \{x_1 * x_2)| x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\},$$

and hence

$$\begin{array}{rcl} f(\mu)(y_1 * y_2) &=& \inf\{\mu(x) \mid f^{-1}(y_1 * y_2)\} \\ &\leq& \inf\{S(\mu(x_1), \mu(x_2)) \mid x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\} \\ &=& S(\inf\{\mu(x_1) \mid x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\} \\ &,& \inf\{\mu(x_2) \mid x_2 \in f^{-1}(y_2)\}) \\ &=& S(f(\mu)(y_1), f(\mu)(y_2)). \end{array}$$

Hence $f(\mu)$ is a S-fuzzy subquasigroup in $\mathcal{G}_2$. □

Definition 11. A s-norm S on [0,1] is called a continuous s-norm if S is a continuous function which maps [0,1]×[0,1] to [0, 1] with respect to the usual topology. Obviously, the function "max" is a continuous s-norm.

Theorem 7. *Let S be a continuous s-norm and f be a homomorphism on $\mathcal{G}$. If μ is a S-fuzzy subquasigroup of $\mathcal{G}$, then μ^f is a S-fuzzy subquasigroup of $f(\mathcal{G})$.*

Proof. The proof is obtained dually by using the notion of s-norm S instead of t-norm T in [7]. $\square$

Lemma 1. *Let T be a t-norm. Then s-norm S can be defined as*

$$S(x,y) = 1 - T(1-x, 1-y).$$

Proof. Straightforward. $\square$

Theorem 8. *A fuzzy set μ of a quasigroup $\mathcal{G}$ is a T-fuzzy subquasigroup of $\mathcal{G}$ if and only if its complement μ^c is a S-fuzzy subquasigroup of $\mathcal{G}$.*

Proof. Let μ be a T-fuzzy subquasigroup of $\mathcal{G}$. For $x,\, y \in \mathcal{G}$, we have

$$\begin{aligned}
\mu^c(x*y) &= 1-\mu(x*y)\\
&\leq 1-T(\mu(x),\mu(y))\\
&= 1-T(1-\mu^c(x),1-\mu^c(y))\\
&= S(\mu^c(x),\mu^c(y)).
\end{aligned}$$

Hence μ^c is a S-fuzzy subquasigroup of $\mathcal{G}$. The converse is similar. $\square$

Theorem 9. *Let S be a s-norm. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be quasigroups and let $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$ be the direct product quasigroup of $\mathcal{G}_1$ and $\mathcal{G}_2$. Let λ be a fuzzy subquasigroup of a quasigroup $\mathcal{G}_1$ with a s-norm S and μ a fuzzy quasigroup of a quasigroup $\mathcal{G}_2$ also with the s-norm S. Then $\nu = \lambda \times \mu$ is a fuzzy quasigroup of $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$ with the s-norm S which is defined by*

$$\nu(x_1,x_2) = (\lambda\times\mu)(x_1,x_2) = S(\lambda(x_1),\mu(x_2)).$$

Moreover, if λ and μ are S-idempotent, then $\lambda \times \mu$ is also S-idempotent.

Proof. The proof is obtained dually by using the notion of s-norm S instead of t-norm T in [7]. $\square$

Theorem 10. *Let λ and μ be fuzzy sets in a unipotent quasigroup $\mathcal{G}$ such that $\lambda\times\mu$ is a S-fuzzy subquasigroup of $\mathcal{G}\times\mathcal{G}$, then*

(i) *either $\lambda(\theta) \leq \lambda(x)$ or $\mu(\theta) \leq \mu(x)$ $\forall x \ \in \mathcal{G}$.*

(ii) *If $\lambda(\theta) \leq \lambda(x)$ $\forall x \ \in \mathcal{G}$, then either $\mu(\theta) \leq \lambda(x)$ or $\mu(\theta) \leq \mu(x)$.*

(iii) *If $\mu(\theta) \leq \mu(x)$, $\forall x \in \mathcal{G}$, then either $\lambda(\theta) \leq \lambda(x)$ or $\lambda(\theta) \leq \mu(x)$.*

Proof. (i) We prove it using reductio ad absurdum.

Assume $\lambda(x) < \lambda(\theta)$ and $\mu(y) < \mu(\theta)$ for some x, y $\in \mathcal{G}$. Then $(\lambda\times\mu)(x,y) = \mathrm{S}(\lambda(x),\mu(y)) < \mathrm{S}(\lambda(\theta),\mu(\theta)) = (\lambda\times\mu)(\theta,\theta)$.

This implies $(\lambda\times\mu)(x,y) < (\lambda\times\mu)(\theta,\theta)$ $\forall$ $x,\ y\ \in \mathcal{G}$.

which is a contradiction. Hence (i) is proved.

(ii) Again, we use reduction to absurdity.

Assume $\mu(\theta) > \lambda(x)$ and $\mu(\theta) > \mu(y)\ \forall\ x,\ y \in \mathcal{G}$. Then,
$(\lambda \times \mu)(\theta, \theta)$= S $(\lambda(\theta), \mu(\theta)) = \mu(\theta)$

and $(\lambda \times \mu)(x,\ y) =$ S $(\lambda(x),\ \mu(y)) < \mu(\theta) = (\lambda \times \mu)(\theta, \theta)$

$\Rightarrow (\lambda \times \mu)(x,\ y) < (\lambda \times \mu)(\theta, \theta)$,

which is a contradiction. Hence (ii) is proved.

(iii) The proof is similar to (ii).

□

Theorem 11. *Let μ and ν be fuzzy sets in a unipotent quasigroup $\mathcal{G}$ such that $\mu \times \nu$ is a S-fuzzy subquasigroup of $\mathcal{G} \times \mathcal{G}$. Then*

(a) *If $\nu(x) \geq \mu(\theta)$ for all $x \in \mathcal{G}$, then ν is a S-fuzzy subquasigroup of $\mathcal{G}$.*

(b) *If $\mu(x) \geq \mu(\theta)$ for all $x \in \mathcal{G}$ and $\nu(y) < \mu(\theta)$ for some $y \in \mathcal{G}$, then μ is a S-fuzzy subquasigroup of $\mathcal{G}$.*

Proof.

(a) If $\nu(x) \geq \mu(\theta)$ for any $x \in \mathcal{G}$, then

$$\begin{aligned} \nu(x * z) &= S(\mu(\theta), \nu(x * y)) \\ &= (\mu \times \nu)(\theta, x * y) \\ &\leq S((\mu \times \nu)(\theta, x), (\mu \times \nu)(\theta, y)) \\ &= S(S(\mu(\theta), \nu(x)), S(\mu(\theta), \nu(y))) \\ &= S(\nu(x), \nu(y)). \end{aligned}$$

Hence ν is a S-fuzzy subquasigroup of $\mathcal{G}$.

(b) Assume that $\mu(x) \geq \mu(\theta)$ for all $x \in \mathcal{G}$ and $\nu(y) < \mu(\theta)$ for $y \in \mathcal{G}$. Then $\nu(\theta) \leq \nu(y) < \mu(\theta)$. since $\mu(\theta) \leq \mu(x)$ for all $x \in \mathcal{G}$, it follows that $\nu(\theta) < \mu(x)$ for any $x \in \mathcal{G}$. Thus

$$(\mu \times \nu)(x, \theta) = S(\mu(x), \nu(\theta)) = \mu(x)\ for\ all\ x \in \mathcal{G}.$$

Thus

$$\begin{aligned} \mu(x * y) &= (\mu \times \nu)(x * y, \theta) \\ &\leq S((\mu \times \nu)(x), \theta), (\mu \times \nu)(y, \theta)) \\ &= S(S(\mu(x), \nu(\theta)), S(\mu(y), \nu(\theta))) \\ &= S(\mu(x), \mu(y)). \end{aligned}$$

Hence μ is a S-fuzzy subquasigroup of $\mathcal{G}$. □

Definition 12. Let S be a s-norm. If ν is a fuzzy set in a set A , the *weakest S-fuzzy relation* on A that is S-fuzzy relation on ν is μ_ν given by $\mu_\nu(x,y)= S(\nu(x),\nu(y))$ for all x, $y \in A$.

Theorem 12. *Let ν be a fuzzy set in a quasigroup $\mathcal{G}$ and let μ_ν be the weakest S-fuzzy relation on $\mathcal{G}$. Then ν is a S-fuzzy subquasigroup of $\mathcal{G}$ if and only if μ_ν is a S-fuzzy subquasigroup of $\mathcal{G}\times\mathcal{G}$.*

Proof. Suppose that ν is a fuzzy subquasigroup of $\mathcal{G}$. Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ $\in \mathcal{G}\times\mathcal{G}$. Then

$$\begin{aligned}
\mu_\nu(x*y) &= \mu_\nu((x_1,x_2)*(y_1,y_2)) = \mu_\nu((x_1*y_1,x_2*y_2))\\
&= S(\nu(x_1*y_1),\nu(x_2*y_2))\\
&\leq S(S(\nu(x_1),\nu(y_1)),S(\nu(x_2),\nu(y_2)))\\
&= S(S(\nu(x_1),\nu(x_2),S(\nu(y_1),\nu(y_2)))\\
&= S(\mu_n(x_1),x_2),\mu_\nu(y_1,y_2))\\
&= S(\mu_\nu((x_1,x_2)),\mu_\nu((y_1,y_2)))\\
&= S(\mu_\nu(x),\mu_\nu(y)).
\end{aligned}$$

Thus μ_ν is a fuzzy subquasigroup of $\mathcal{G}\times\mathcal{G}$.
The converse is proved similarly. □

Definition 13. A S-fuzzy subquasigroup of a unipotent quasigroup $\mathcal{G}$ is said to be *abnormal* if there exist $x \in G$ such that $\mu(x) = 0$. Note that if S-fuzzy subquasigroup μ of $\mathcal{G}$ is abnormal, then $\mu(\theta) = 0$, and hence μ is an abnormal if and only if $\mu(\theta) = 0$.

Theorem 13. *Let μ be a S-fuzzy subquasigroup of a unipotent quasigroup $\mathcal{G}$ and μ^+ be a fuzzy set in G defined by $\mu^+(x) = \mu(x) - \mu(\theta)$ for all $x \in G$. Then μ^+ is an abnormal S-fuzzysubquasigroup of $\mathcal{G}$ containing μ.*

Proof. We have $\mu^+(x) = \mu(\theta) - \mu(\theta) = 0 \leq \mu^+(x)$ for all $x \in G$. For any x, $y \in G$, we have

$$\begin{aligned}
\mu^+(x*y) &= \mu(x*y) - \mu(\theta)\\
&\leq S(\mu(x),\mu(y)) - \mu(\theta)\\
&= S(\mu(x)-\mu(\theta),\mu(y)-\mu(\theta))\\
&= S(\mu^+(x),\mu^+(y)).
\end{aligned}$$

This shows that μ^+ is a S-fuzzy subquasigroup of a unipotent quasigroup. Clearly, $\mu \subset \mu^+$. This ends the proof. □

Corollary 1. *If μ is a S-fuzzy subquasigroup of a unipotent quasigroup satisfying $\mu^+(x) = 1$ for some $x \in G$, then $\mu(x) = 1$.*

Theorem 14. *Let μ and ν be S-fuzzy subquasigroups of a unipotent quasigroup $\mathcal{G}$. If $\nu \subset \mu$ and $\mu(\theta) = \nu(\theta)$, then $\mathcal{G}_\mu \subset \mathcal{G}_\nu$.*

Proof. Assume that $\nu \subset \mu$ and $\mu(\theta) = \nu(\theta)$. If $x \in \mathcal{G}_\mu$, then $\nu(x) \leq \mu(x) = \mu(\theta) = \nu(\theta)$. Noticing that $\nu(\theta) \leq \nu(x)$ for all $x \in G$, we have $\nu(x) = \nu(\theta)$, i.e., $x \in \mathcal{G}_\nu$. The proof is complete. □

Corollary 2. *If μ and ν are abnormal S-fuzzy subquasigroups of a unipotent quasigroup $\mathcal{G}$ satisfying $\nu \subset \mu$, then $\mathcal{G}_\mu \subset \mathcal{G}_\nu$.*

Theorem 15. *A S-fuzzy subquasigroup of a unipotent quasigroup $\mathcal{G}$ is abnormal if and only if $\mu^+ = \mu$.*

Proof. The sufficiency is obvious. Assume that μ is an abnormal S-fuzzy subquasigroup of a quasigroup $\mathcal{G}$ and $x \in G$. Then $\mu^+(x) = \mu(x) - \mu(\theta) = \mu(x)$, hence $\mu^+ = \mu$. □

Theorem 16. *If μ is a S-fuzzy subquasigroup of a unipotent quasigroup, then $(\mu^+)^+ = \mu^+$.*

Proof. For any $x \in G$, we have $(\mu^+)^+(x) = \mu^+(x) - \mu^+(\theta)$, completing the proof. □

Corollary 3. *If μ is an abnormal S-fuzzy subquasigroup of a unipotent quasigroup, then $(\mu^+)^+ = \mu$.*

Theorem 17. *Let μ be a non-constant abnormal S-fuzzy subquasigroup of a unipotent quasigroup, which is minimal in the poset of abnormal S-fuzzy subquasigroup under set inclusion. Then clearly μ takes only two values 0 and 1.*

Proof. Note that $\mu(\theta) = 0$. Let $x \in G$ be such that $\mu(x) \neq 0$. It is sufficient to show that $\mu(x) = 1$. Assume that then there exists $a \in G$ such that $0 < \mu(a) < 1$. Define on G a fuzzy set ν by putting $\nu(x) = \frac{1}{2}(\mu(x) + \mu(a))$ for each $x \in G$. Then clearly ν is well-defined and for all x, $y \in G$, we have

$$\begin{aligned}
\nu(x * y) &= \frac{1}{2}\mu(x * y) + \frac{1}{2}\mu(a) \\
&\leq \frac{1}{2}(S(\mu(x), \mu(y) + \mu(a))) \\
&= S(\frac{1}{2}(\mu(x) + \mu(a)), \frac{1}{2}(\mu(y) + \mu(a))) \\
&= S(\nu(x), \nu(y)).
\end{aligned}$$

Hence ν^+ is an abnormal S-fuzzy subquasigroup of a unipotent quasigroup. Noticing that $\nu^+(\theta) = 0 < \nu^+(a) = \frac{1}{2}\mu(a) < \mu(a)$, we know that ν^+ is non-constant. From $\nu^+(a) < \mu(a)$, it follows that μ is not minimal. This proves the theorem. □

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Ideal Theory in Commutative Semirings

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Abstract. In this paper, we analyze some results on ideal theory of commutative semirings with non-zero identity analogues to commutative rings with non-zero identity. Here we will make an intensive examination of the notions of Noetherian semirings, Artinian semirings, local semirings and strongly irreducible ideals in commutative semirings. It is shown that this notion inherits most of essential properties of strongly irreducible ideals of a commutative rings with non-zero identity. Also, the relationship among the families of primary ideals, irreducible ideals and strongly irreducible ideals of a semiring R is considered.

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1 Introduction

The concept of semirings was introduced by H. S. Vandiver in 1935 and has since then been studied by many authors (see, for example, [1–3, 7, 12]). Semirings constitute a fairly natural generalization of rings, with broad applications in the mathematical foundations of computer science. The main part of this paper is devoted to stating and proving analogues to several well-known results in the theory of rings (see Sections 2 and 3).

For the sake of completeness, we state some definitions and notations used throughout. By a commutative semiring, we mean a commutative semigroup $(R, .)$ and a commutative monoid $(R, +, 0)$ in which 0 is the additive identity and $r.0 = 0.r = 0$ for all $r \in R$, both are connected by ring-like distributivity. In this paper, all semirings considered will be assumed to be commutative semirings with non-zero identity.

A subset I of a semiring R will be called an ideal if $a, b \in I$ and $r \in R$ implies $a + b \in I$ and $ra \in I$. A subtractive ideal (= k-ideal) I is an ideal such that if $x, x + y \in I$ then $y \in I$ (so $\{0\}$ is a k-ideal of R). The k-closure $\mathrm{cl}(I)$ of I is defined by $\mathrm{cl}(I) = \{a \in R : a + c = d \text{ for some } c, d \in I\}$ is an ideal of R satisfying $I \subseteq \mathrm{cl}(I)$ and $\mathrm{cl}(\mathrm{cl}(I)) = \mathrm{cl}(I)$. So an ideal I of R is a k-ideal if and only if $I = \mathrm{cl}(I)$. A prime ideal of R is a proper ideal P of R in which $x \in P$ or $y \in P$ whenever $xy \in P$. So P is prime if and only if A and B are ideals in R such that $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$ where $AB = \{ab : a \in A \text{ and } b \in B\} \subseteq A \cap B$ (see [3, Theorem 5]). A primary ideal P of R is a proper ideal of R such that, if $xy \in P$ and $x \notin P$, then $y^n \in P$ for some positive integer n. If I is an ideal of R, then the radical of I,denoted by $\mathrm{rad}(I)$, is the set of all $x \in R$ for which $x^n \in I$ for some positive integer n. This is an ideal of R, containing I, and if $1 \in R$ is the intersection of all the prime ideals of R that contain I [1]. A proper ideal I of R is said to be maximal (resp. k-maximal) if J is an ideal (resp. k-ideal) in R such that $I \subsetneqq J$, then $J = R$. An ideal I of a semiring R is strongly irreducible if for ideals J and K of R, the inclusion $J \cap K \subseteq I$ implies that either $J \subseteq I$ or $K \subseteq I$. An ideal of R is said to be irreducible if I is not the intersection of two ideals of R that properly contain it. We say that R is a Noetherian (resp. Artinian) if any non-empty set of k-ideals of R has a maximal (resp. minimal) member with respect to set inclusion. This definition is equivalent to the ascending chain condition (resp. descending chain condition) on k-ideals of R.

2 Local semirings

Let R be a semiring with non-zero identity. A non-zero element a of R is said to be semi-unit in R if there exist $r, s \in R$ such that $1 + ra = sa$. R is said to be a local semiring if and only if R has a unique maximal k-ideal.

Lemma 1. *Let I be a k-ideal of a semiring R. Then the following hold:*
(i) If a is a semi-unit element of R with $a \in I$, then $I = R$.
(ii) If $x \in R$, then $\mathrm{cl}(Rx)$ is a k-ideal of R.

Proof. (i) Is clear. To see (ii), let $y, x + y \in \mathrm{cl}(Rx)$; we show that $x \in \mathrm{cl}(Rx)$. There are elements a, b, c and d of Rx such that $x + y + a = b$ and $y + c = d$. It follows that $x + a + d = b + c$, as needed. □

Lemma 2. *Let R be a semiring with $1 \neq 0$. Then R has at least one k-maximal ideal.*

Proof. Since $\{0\}$ is a proper k-ideal of R, the set Δ of all proper k-ideals of R is not empty. Of course, the relation of inclusion, $\subseteq$, is a partial order on Δ, and by using Zorn's Lemma to this partially ordered set, a maximal k-ideal of R is just a maximal member of the partially ordered set $(\Delta, \subseteq)$. □

An ideal I of a semiring R is called a partitioning ideal (= Q-ideal) if there exists a subset Q of R such that $R = \cup\{q + I : q \in Q\}$ and if $q_1, q_2 \in Q$ then $(q_1 + I) \cap (q_2 + I) \neq \emptyset$ if and only if $q_1 = q_2$. Let I be a Q-ideal of a semiring R and let $R/I = \{q + I : q \in Q\}$. Then R/I forms a semiring under the binary operations $\oplus$ and $\odot$ defined as follows: $(q_1 + I) \oplus (q_2 + I) = q_3 + I$, where $q_3 \in Q$ is the unique element such that $q_1 + q_2 + I \subseteq q_3 + I$. $(q_1 + I) \odot (q_2 + I) = q_4 + I$, where $q_4 \in Q$ is the unique element such that $q_1q_2 + I \subseteq q_4 + I$. This semiring R/I is called the quotient semiring of R by I. By definition of Q-ideal, there exists a unique $q_0 \in Q$ such that $0 + I \subseteq q_0 + I$. Then $q_0 + I$ is a zero element of R/I. Clearly, if R is commutative , then so is R/I (see [7, 8]). The next result should be compared with [12, Corollary 2.2].

Theorem 1. *Let I be a proper Q-ideal of a semiring R. Then there exists a maximal k-ideal M of R with $I \subseteq M$.*

Proof. Since R/I is non-trivial, and so, by Lemma 2, has a k-maximal ideal L, which, by [5, Theorem 2.3], will have to have the form M/I for some k-ideal M of R with $I \subseteq M$. It now follows from [5, Theorem 2.14] that M is a k-maximal ideal of R. □

Lemma 3. *Let R be a semiring and let $a \in R$. Then a is a semi-unit of R if and only if a lies outside each k-maximal ideal of R.*

Proof. By Lemma 1 (i), a is a semi-unit of R if and only if $R = \mathrm{cl}Ra$. First, Suppose that a is a semi-unite of R and let $a \in M$ for some maximal k-ideal ideal M of R. Then we should have $Ra \subseteq M \subsetneqq R$, so that a could not be a semi-unit of R. Conversely, if a were not a semi-unit of R, then $1 + ra = sa$ holds for no $r, s \in R$. Hence, $1 \notin \mathrm{cl}(Ra)$ yields that $\mathrm{cl}(Ra)$ is a proper k-ideal of R by Lemma 1 (ii). By [12, Corollary 2.2], $\mathrm{cl}(Ra) \subseteq J$ for some maximal k-ideal J of R; but this would contradict the fact that a lies outside each maximal k-ideal of R. □

Theorem 2. *Let R be a semiring. Then R is a local semiring if and only if the set of non-semi-unit elements of R is a k-ideal.*

Proof. Assume that R is a local semiring with unique maximal k-ideal P. By Lemma 3, P is precisely the set of non-semi-unit elements of R. Conversely, assume that the set of non-semi-units of R is a k-ideal I of R (so $I \neq R$ since 1 is a semi-unit of R). Since R is not trivial, it has at least one maximal k-ideal: let J be one such. By Lemma 3, J consists of non-semi-units of R, and so $J \subseteq I \subsetneqq R$. Thus $I = J$ since J is k-maximal. We have thus shown that R has at least one maximal k-ideal, and for any maximal k-ideal of R must be equal to I. □

Let R be a given semiring, and let S be the set of all multiplicatively cancellable elements of R (so $1 \in S$). Define a relation $\sim$ on $R \times S$ as follows: for $(a, s), (b, t) \in R \times S$, we write $(a, s) \sim (b, t)$ if and only if $ad = bc$. Then $\sim$ is an equivalence relation on $R \times S$. For $(a, s) \in R \times S$, denote the equivalence class of $\sim$ which contains (a, s) by a/s, and denote the set of all equivalence classes of $\sim$ by R_S. Then R_S can be given the structure of a commutative semiring under oprerations for which $a/s + b/t = (ta + sb)/st$, $(a/s)(b/t) = (ab)/st$ for all $a, b \in R$ and $s, t \in S$. This new semiring R_S is called the semiring of fractions of R with respect to S; its zero element is $0/1$, its multiplicative identity element is $1/1$ and each element of S has a multiplicative inverse in R_S (see [10, 11]).

Thoughout this paper we shall assume unless otherwise stated, that **S** is the set of all multiplicatively cancellable elements of a semiring R. Now suppose that I be an ideal of a semiring R. The ideal generated by I in R_S, that is, the set of all finite sums $s_1a_1 + ..., s_na_n$ where $r_i \in R_S$ and $s_i \in I$, is called the extention of I to R_S, and it is denoted by IR_S. Again, if J is an ideal of R_S then by the contraction of J in R we mean $J \cap R = \{r \in R : r/1 \in J\}$, which is clearly an ideal of R.

We need the following lemma proved in [6, Lemma 2.3].

Lemma 4. *Assume that I, J and K are ideals of a semiring R and let L be an ideal of semiring R_S. Then the following hold:*

(i) $x \in IR_S$ if and only if it can be written in the form $x = a/c$ for some $a \in I$ and $c \in S$.

(ii) $(L \cap R)R_S = L$.

(iii) $(I \cap J)R_S = (IR_S) \cap (JR_S)$.

Lemma 5. *Let I be a k-ideal of a semiring R. Then IR_S is a k-ideal of R_S.*

Proof. Suppose that $a/s, a/s + b/t \in IR_S$; we show that $b/t \in IR_S$. By Lemma 4, there are elements $c, d \in I$ and $u, v \in S$ such that $(at+bs)/st = c/u$ and $a/t = d/w$, so $atuw + bsuw = t^2ud + sbuw = cstw \in I$; hence $bsuw \in I$ since I is a k-ideal. It follows that $b/t = (bsuw)/(tsuw) \in IR_S$, as required. □

Theorem 3. *Let R be a local semiring with unique maximal k-ideal P such that $S \cap P = \emptyset$. Then R_S is a local semiring with unique maximal k-ideal of PR_S.*

Proof. By Lemma 5 and Theorem 2, it is enough to show that PR_S is exactly the set of non-semi-units of R_S. Let $z \in R_S - PR_S$, and take any representation $z = a/s$ with $a \in R$, $s \in S$. We must have $a \notin P$, so $1 + ra = sa$ for some $r, s \in R$ by Lemma 3. It then follows from $1/1 + ((rs)/1)(a/s) = (s^2/1)(a/s)$ that a/s is a semi-unit of R_S. On the other hand, if y is a semi-unit of R_S, and $y = b/t$ for some $b \in R$, $t \in S$, then there exist $c, d \in R$ and $u, w \in S$ such that $1/1 + (bc)/(tu) = (bd)/(tw)$. It follows that $t^2uw + bctw = tubc$; hence $b \notin P$ since P is a k-ideal, and since this reasoning applies to every representation $y = b/t$ with $b \in R$, $t \in S$, of y as a formal fraction, it follows that $y \notin PR_S$, and so the proof is complete. □

For the remainder of this section we turn our attention to study some essential properties of Noetherian and Artinian semirings.

Lemma 6. *If R is a Noetherian semiring, then every proper k-ideal is a finite intersection of irreducible k-ideals.*

Proof. The proof is trivial. □

Proposition 1. *Let R be a Noetherian semiring and let I be an irreducible k-ideal of R. Then I is primary.*

Proof. The proof is straightforward (see [13, Proposition 4.34]). □

Theorem 4. *If R is a Noetherian semiring, then every proper k-ideal is a finite intersection of primary k-ideals.*

Proof. This follows from Lemma 6 and Proposition 1. $\square$

Let R be a semiring. R is called cancellative if whenever $ac = ab$ for some elements a, b and c of R with $a \neq 0$, then $b = c$. Also, we define the Jacobson radical of R, denoted by $\mathrm{Jac}(R)$, to be the intersection of all the maximal k-ideals of R. Then by [12, Corollary 2.2], the Jacobson radical of R always exists and by [5, Lemma 2.12], it is a k-ideal of R.

Theorem 5. *Let R be an Artinian cancellative semiring. Then the following hold:*

(i) Every prime k-ideal of R is k-maximal.

(ii) $\mathrm{Jac}(R) = \mathrm{rad}(0)$ *(the nilradical of R).*

Proof. (i) Assume that I is a prime k-ideal of R and let $I \subsetneqq J$ for some k-ideal J of R; we show that $J = R$. There is an element $x \in J$ with $x \notin I$. Then by Lemma 1, $\mathrm{cl}(Rx) \supseteq \mathrm{cl}(Rx^2) \supseteq ...$ is a descending chain of k-ideals of R, so $\mathrm{cl}(Rx^n) = \mathrm{cl}(Rx^{n+1})$ for some n; hence $x^n + rx^{n+1} = sx^{n+1}$ for some $r, s \in R$. Since R is cancellative and $x \neq 0$, it follows that we may cancel x^n, hence $1 + rx = sx$. Hence x is a semi-unit in J, and therefore $J = R$ by Lemma 1. (ii) follows from (i). $\square$

Lemma 7. *Let R be a semiring. Then the following hold:*

(i) Let $P_1, ..., P_n$ be prime k-ideals and let I be an ideal of R contained in $\bigcup_{i=1}^{n} P_i$. Then $I \subseteq P_i$ for some i.

(ii) Let $I_1, ..., I_n$ be ideals and let P be a prime ideal containing $\bigcap_{i=1}^{n} I_i$. Then $I_i \subseteq P$ for some i. If $P = \bigcap_{i=1}^{n}$, then $P = I_i$ for some i.

Proof. (i) The proof is straightforward by induction on n (see [13, Theorem 3.61]).

(ii) Suppose $I_i \nsubseteq P$ for all i. Then there exist $x_i \in I_i$, $x_i \notin P$ $(1 \leq i \leq n)$, and therefore $x_1x_2...x_n \in I_1I_2...I_n \subseteq \bigcap_{i=1}^{n} I_i$; but $x_1x_2...x_n \notin P$; hence $\bigcap_{i=1}^{n} I_i \nsubseteq P$ which is a contradiction. Finally, if $P = \bigcap_{i=1}^{n} I_i$, then $I_i \subseteq P$ and hence $P = I_i$ for some i. $\square$

Theorem 6. *An Artinian cancellative semiring has only a finite number of maximal k-ideals.*

Proof. Consider the set of all finite intersections $P_1 \cap ... \cap P_n$, where the P_i are maximal k-ideals (note that an intersection of a family of k-ideals of R is a k-ideal by [5, Lemma 2.12]). This set has a minimal element, say $Q_1 \cap ... \cap Q_s$; hence for any maximal k-ideal Q we have $Q \cap Q_1 \cap ... \cap Q_s = Q_1 \cap ... \cap Q_s$, and therefore $Q_1 \cap ... \cap Q_s \subseteq Q$. By Lemma 7, $Q_i \subseteq Q$ for some i, hence $Q = Q_i$ since Q_i is k-maximal, as required. $\square$

3 Strongly irreducible ideals

In this section we list some basic properties concerning strongly irreducible ideals. The results of this section should be compared with [9].

Theorem 7. *Let I be an ideal of a semiring R. Then the following hold:*

(i) If I is strongly irreducible, then I is irreducible.

(ii) If R is Noetherian and I is a strongly irreducible k-ideal of R, then I is primary.

Proof. (i) Assume that I is strongly irreducible and let J and K be ideals of R such that $J \cap K = I$. Then $J \cap K \subseteq I$, so either $J \subseteq I$ or $K \subseteq I$, and it then follows that either $I = J$ or $I = K$, so I is irreducible.

(ii) This follows from (i) and Proposition 1. □

Proposition 2. *Let I be an ideal of a semiring R. Then the following hold:*

(i) To show that I is strongly irreducible, it suffices to show that if Ra and Rb are cyclic ideals of R such that $Ra \cap Rb \subseteq I$, then either $a \in I$ or $b \in I$.

(ii) If I is a prime ideal of R, then I is strongly irreducible.

Proof. (i) Let J and K be ideals of R such that $J \cap K \subseteq I$; we show that either $J \subseteq I$ or $k \subseteq I$. Suppose $J \nsubseteq I$. Then there exists $a \in J$ such that $a \notin I$. Then for all $b \in K$ it holds $Ra \cap Rb \subseteq J \cap K \subseteq I$, so $b \in I$, as required.

(ii) Assume that I is prime and let J and K be ideals of R such that $J \cap K \subseteq I$. Since $IJ \subseteq I \cap J \subseteq I$, I prime gives either $J \subseteq I$ or $K \subseteq I$, as needed. □

Lemma 8. *Let I be a Q-ideal of a semiring R. If J, K and L are k-ideals of R containing I, then $(J/I) \cap (K/I) = L/I$ if and only if $J \cap K = L$.*

Proof. Suppose that $(J/I) \cap (K/I) = L/I$; we show that $J \cap K = L$. Let $x \in J \cap K$. Then there exist $q_1 \in Q$ and $a \in I$ such that $x = q_1 + a$, so $q_1 \in Q \cap J$ and $q_1 \in Q \cap K$ since J, K are k-ideals; hence $q_1 + I \in (J/I) \cap (K/I) = L/I$ by [5, Proposition 2.2]. Therefore, $q_1 \in L$; thus $x \in L$ since L is a k-ideal. So, $J \cap K \subseteq L$. Now suppose that $z \in L$. Then $z = q_2 + b$ for some $q_2 \in Q$ and $b \in I$. It follows that $q_2 + I \in L/I = (J/I) \cap (K/I)$, so $q_2 \in K \cap J$; hence $z \in K \cap J$. Thus $L = J \cap K$. The other implication is similar. □

Theorem 8. *Let R be a semiring, I a Q-ideal of R and J a strongly irreducible k-ideal of R with $I \subseteq J$. Then J/I is a strongly irreducible ideal of R/I.*

Proof. Let N and M be ideals of R/I such that $N \cap M \subseteq J/I$. Then there are k-ideals K, H of R such that $N = K/I$ and $M = H/I$ by [5, Theorem 2.3]; hence Lemma 8 gives $K \cap H \subseteq J$. Since J is strongly irreducible it follows that either $K \subseteq J$ or $H \subseteq J$; hence either $N = K/I \subseteq J/I$ or $M = H/I \subseteq J/I$ by [5, Lemma 2.13 (ii)]. So J/I is strongly irreducible. □

Lemma 9. *Let I be a primary ideal of a semiring R with* $\mathrm{rad}(I) = P$. *Then the following hold:*

(i) IR_S is a primary ideal of R_S.

(ii) If $P \cap S = \emptyset$, then $IR_S \cap R = I$.

(iii) If $P \cap S = \emptyset$, then $(IR_S :_{R_S} PR_S) = (I :_R P)R_S$.

(iv) If $P \cap S = \emptyset$ and J is an ideal of R such that $JR_S \subseteq IR_S$, then $J \subseteq I$.

Proof. (i) Let $a/s, b/t \in R_S$, where $a, b \in R$, $s, t \in S$, be such that $(ab)/(st) \in IR_S$ but $a/s \notin IR_S$. Then there exist $e \in I$ and $z \in S$ such that $abz = ste \in I$ but $az \notin I$ (otherwise, $a/s = (az)/(zs) \in IR_S$); hence I primary gives $b^n \in I$ for some positive integer n. It follows that $b^n/t^n = (b/t)^n \in IR_s$, as needed.

(ii) Clearly, $I \subseteq IR_S \cap R$. Let $a \in IR_S \cap R$. Then there are elements $b \in I$ and $s \in S$ such that $a/1 = b/s$, so $as = b \in I$; hence I primary gives $a \in I$, as required.

(iii) Clearly, $(I :_R P)R_S \subseteq (IR_S :_{R_S} PR_S)$. For the other direction, let $a/s \in (IR_S :_{R_S} PR_S)$, where $a \in R$, $s \in S$. It suffices to show that $ab \in I$ for every $b \in P$. There are elements $c \in I$ and $t \in S$ such that $(a/s)(b/1) = (ab)/s = c/t$, so $abt = sc \in I$ but $t \notin P$; hence $ab \in I$, and so the proof is complete.

(iv) Let $a \in J$. As $a/1 \in JR_S$, there are elements $b \in I$ and $t \in S$ such that $a/1 = b/t$, so $at = b \in I$; hence I primary gives $a \in I$. Thus $J \subseteq I$. □

Proposition 3. *Let I be an ideal of a semiring R. If IR_S is a strongly irreducible ideal of R_S, then $IR_S \cap R$ is an irreducible ideal of R.*

Proof. Assume that IR_S is strongly irreducible and let J and K be ideals of R such that $J \cap K \subseteq IR_S \cap R$. Then $JR_S \cap KR_S \subseteq IR_S$ by Lemma 4. It follows that either $JR_S \subseteq IR_S$ or $KR_S \subseteq IR_S$, so either $J \subseteq IR_S \cap R$ or $K \subseteq IR_S \cap R$, as required. □

Theorem 9. *Let I be a strongly irreducible primary ideal of a semiring R such that* $\mathrm{Rad}(I) \cap S = \emptyset$. *Then IR_S is strongly irreducible.*

Proof. Assume that I is a strongly irreducible primary ideal of R and let H and G be ideals of R_S such that $H \cap G \subseteq IR_S$. Then $(H \cap R) \cap (G \cap R) \subseteq IR_S \cap R = I$ by Lemma 9. So either $H \cap R \subseteq I$ or $G \cap R \subseteq I$ since I is strongly irreducible. Therefore it follows that either $G = (G \cap R)R_S \subseteq IR_S$ or $H = (H \cap R)R_S \subseteq IR_S$ by Lemma 4, and hence IR_S is strongly irreducible. □

Theorem 10. *Assume that I is a primary ideal of a semiring R with* $\mathrm{Rad}(I) \cap S = \emptyset$ *and let IR_S be strongly irreducible ideal of R_S. Then I is strongly irreducible.*

Proof. By Lemma 9 and Proposition 3, $IR_S \cap R = I$ is a strongly irreducible ideal of R. □

Lemma 10. *Assume that R is a semiring and let $a, b \in R$. Then the following hold:*

(i) $Ra \cap Rb = (Ra : Rb)b = (Rb : Ra)a$. Moreover, if I is an ideal of R such that $I \subseteq Ra$, then $I = (I : Ra)a$.

(ii) $\mathrm{cl}(Ra) \cap Rb = (\mathrm{cl}(Ra) : Rb)b$.

(iii) If I is a k-ideal of R such that $I \subseteq Ra$, then $I = (I : \mathrm{cl}(Ra))a$.

Proof. (i) Clearly, $(Ra : Rb)b \subseteq Ra \cap Rb$. For the other direction, if $z \in Ra \cap Rb$, then $z = ra = sb$ for some $r, s \in R$. It is clear that $s \in (Ra : Rb)$; hence $z \in (Ra : Rb)b$. By symmetry it follows that $Ra \cap Rb = (Rb : Ra)a$. For the last statement, assume that I is an ideal of R such that $I \subseteq Ra$. Then it is clear that $(I : Ra)a \subseteq I$, and if $x \in I \subseteq Ra$, then $x = ta$ for some $t \in R$, so $t \in (I : Ra)$; thus $x = ta \in (I : Ra)a$, as required.

(ii) Since the inclusion $(\mathrm{cl}(Ra) : Rb)b \subseteq (\mathrm{cl}(Ra) \cap Rb$ is clear, we will prove the reverse inclusion. Let $y \in \mathrm{cl}(Ra) \cap Rb$. Then there are elements r, s and t of R such that $y + ra = sa$ and $y = tb$, so $tb \in \mathrm{cl}(Ra)$ gives $t \in (\mathrm{cl}(Ra) : Rb)$. Therefore, we must have $y \in (\mathrm{cl}(Ra) : Rb)b$.

(iii) It is clear that $(I : \mathrm{cl}(Ra))a \subseteq I$. For the other containment, assume that $z \in I \subseteq Ra$, so $z = ra$ for some $r \in R$. Let $c \in \mathrm{cl}(Ra)$. Then $c + ta = sa$, so $rc + rta = rsa$; hence $rc \in I$ since I is a k-ideal. It follows that $r \in (I : \mathrm{cl}(Ra))$; hence $z \in (I : \mathrm{cl}(Ra))a$, as needed. $\square$

Theorem 11. *Let R be a local semiring with unique maximal k-ideal P and let I be a strongly irreducible P-primary k-ideal in R. Assume that $I \subsetneqq (I : P)$. Then the following hold:*

(i) $(I : P) = \mathrm{cl}(Rx)$ for some $x \in R$.

(ii) For each k-ideal J in R either $J \subseteq I$ or $(I : P) \subseteq J$.

Proof. (i) By hypothesis, $I \subsetneqq (I : P)$, so there exists $x \in (I : P) - I$; we show that $(I : P) = \mathrm{cl}(Rx)$. Suppose not. Let $y \in (I : P) - \mathrm{cl}(Rx)$. Since $(\mathrm{cl}(Rx) : Ry) \neq R$, [12, Corollary 2.2], [4, Lemma 2.1], Lemma 1 and Lemma 10 gives $\mathrm{cl}(Rx) \cap Ry = (\mathrm{cl}(Rx) : Ry)y \subseteq yP \subseteq I$. However, I strongly irreducible implies that either $\mathrm{cl}(Rx) \subseteq I$ or $Ry \subseteq I$, hence $y \in I$. Therefore, $(I : P) \subseteq I \cup \mathrm{cl}(Rx)$. For the other direction, it suffices to show that $\mathrm{cl}(Rx) \subseteq (I : P)$. Let $d \in \mathrm{cl}(Rx)$. Then $d + tx = ux$ for some $u, t \in R$, so $d \in (I : P)$ since $x \in (I : P)$ and $(I : P)$ is a k-ideals by [4, Lemma 2.1]. Thus, $(I : P) = I \cup \mathrm{cl}(Rx)$. Since by [4, Lemma 2.2], if an ideal is the union of two k-ideals, then it is equal to one of them, we must have $(I : P) \subseteq \mathrm{cl}(Rx)$ or $(I : P) \subseteq I$ which is a contradiction. Therefore, $(I : P) = \mathrm{cl}(Rx)$, so (i) holds.

(ii) It may clearly be assumed that $J \nsubseteq I$, so it remains to show that $\mathrm{cl}(Rx) = (I : P) \subseteq J$; that is, that $x \in J$ since J is a k-ideal. For this, if $x \notin J$, then let $b \in J$, so $x \notin Rb$ and $(Rb : Rx) \neq R$. Therefore [12, Corollary 2.2] and Lemma 10 gives $Rx \cap Rb = (Rb : Rx)x \subseteq xP \subseteq I$, hence $Rb \subseteq I$ since I is a strongly irreducible ideal and $Rx \nsubseteq I$. Since this holds for each $b \in J$, it follows that $J \subseteq I$, and this is a contradiction. Therefore $x \in J$, hence (ii) holds. $\square$

Corollary 1. *Let I be a strongly irreducible k-ideal in a local Noetherian semiring R with the unique maximal k-ideal of $\mathrm{rad}(I) = P$, and assume that $I \neq P$ and $P \cap S = \emptyset$. Then the following hold:*

(i) $(IR_S :_R PR_S)R_S = \mathrm{cl}(R_S X)$ for some $X \in R_S$.

(ii) For each k-ideal L in R_S either $L \subseteq IR_S$ or $(IR_S :_{R_S} PR_S) \subseteq L$.

Proof. By Theorem 3, R_S is a local semiring with unique maximal k-ideal PR_S and by Lemma 5, Lemma 9 and Theorem 9, IR_S is a PR_S-primariy strongly irreducible k-ideal of R_S. So (i) and (ii) follows immediately from Theorem 11. $\square$

Proposition 4. *Let I be a strongly irreducible k-ideal in a local Noetherian semiring R with the unique maximal k-ideal of $\mathrm{rad}(I) = P$, and assume that $I \neq P$ and $P \cap S = \emptyset$. Then the following hold:*

(i) $(I :_R P)R_S = \mathrm{cl}(R_S X)$ for some $X \in R_S$.

(ii) For each k-ideal J in R either $J \subseteq I$ or $(I : P)R_S \subseteq JR_S$.

Proof. This follows from Lemma 9 and Corollary 1. $\square$

Proposition 5. *Let R be a local semiring with unique maximal k-ideal P and let I be a strongly irreducible P-primary k-ideal of R with $P \neq I$. Then I and $(I : P)$ are comparable (under containment) to all ideals in R; in fact, $I = \bigcup\{J : J \text{ is a } k\text{-ideal in } R \text{ and } J \subsetneqq (I : P)\}$ and $(I : P) = \bigcap\{J : J \text{ is a } k\text{-ideal in } R \text{ and } I \subsetneqq J\}$.*

Proof. As $(I : P)$ is a k-ideal of R, we must have $(I : P) = \bigcap\{J : J \text{ is a k-ideal in R and } I \subsetneqq J\}$ by Theorem 11 (ii). Also, if J is a k-ideal in R such that $J \subsetneqq (I : P)$, then $(I : P) \nsubseteq J$, so $J \subseteq I$ by Proposition 4 (ii); hence $I = \bigcup\{J : J \text{ is a k-ideal in R and } J \subsetneqq (I : P)\}$ since I is a k-ideal. $\square$

Theorem 12. *Let I be an ideal of a local Noetherian semiring R with $\mathrm{rad}(I) \cap S = \emptyset$. Then I is a non-prime strongly irreducible k-ideal if and only if there exist k-ideals J and P of R such that $I \subsetneqq J \subseteq P$ and: (1) P is prime; (2) I is P-primary; and, (3) for all k-ideals L in R either $L \subseteq I$ or $JR_S \subseteq LR_S$. Also if this holds, then $JR_S = (IR_S :_{R_S} PR_S)$. In particular, a local Noetherian semiring R contains a non-prime strongly irreducible k-ideal if and only if there exists a k-ideal of R satisfying these conditions.*

Proof. Proposition 4 and Proposition 5 gives a non-prime strongly irreducible k-ideal in a local Notherian semiring satisfies the stated conditions. For the converse, assume that I is a P-primary k-ideal of R. By Theorem 10, it suffices to show that IR_S is strongly irreducible. Let L and T be ideals in R_S such that $L \cap T \subseteq IR_S$. If $L \nsubseteq IR_S$ and $T \nsubseteq IR_S$, then it follows from Lemma 4 and Lemma 9 that $(L \cap R) \nsubseteq I$ and $(T \cap R) \nsubseteq I$. By assumption, $IR_S \subseteq JR_S \subseteq T \cap L$, and this is a contradiction. Therefore, either $L \subseteq IR_S$ or $T \subseteq IR_S$, hence IR_S is strongly irreducible.

Finally, the ideal JR_S is clearly uniquely determined by the properties (1) $IR_S \subsetneqq JR_S \subseteq PR_S$ and (2) for all k-ideals L in R_S either $L \subseteq IR_S$ or $JR_S \subseteq L$. Since $(IR_S : PR_S)$ also has these properties by Corollary 1 (ii), $JR_S = (IR_S : PR_S)$. $\square$

Theorem 13. *Let I be an irreducible P-primary k-ideal over a local Noetherian semiring R with the unique maximal k-ideal of P, and assume that $I \neq P$ and $P \cap S = \emptyset$. Then I is strongly irreducible if and only if I comparable to all k-ideals of R.*

Proof. By Corollary 1 and Proposition 4, it is enough to show that an irreducible ideal that is comparable to all k-ideals in R is strongly irreducible. To see that, by Theorem 12, it suffices to show that $(I : P)$ is comparable to all k-ideals in R. For this, if J is a k-ideal of R that is not contained in I, then $I \subsetneqq J$, by hypothesis. Since I is irreducible, it follows that $(I : P) \subseteq J$. $\square$

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A closed form asymptotic solution for the FitzHugh-Nagumo model

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Abstract. By means of a change of unknown function and independent variable, the Cauchy problem of singular perturbation from electrophysiology, known as the FitzHugh-Nagumo model, is reduced to a regular perturbation problem (Section 1). Then, by applying the regular perturbation technique to the last problem and using an existence, uniqueness and asymptotic behavior theorem of the second and third author, the models of asymptotic approximation of an arbitrary order are deduced (Section 2). The closed-form expressions for the solution of the model of first order asymptotic approximation and for the time along the phase trajectories are derived in Section 3. In Section 4, by applying several times the method of variation of coefficients and prime integrals, the closed-form solution of the model of second order asymptotic approximation is found. The results from this paper served to the author to study (elsewhere) the relaxation oscillations versus the oscillations in two and three times corresponding to concave limit cycles (canards).

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1 Mathematical model

The FitzHugh-Nagumo (F-N) model is the Cauchy problem $x(0) = x^0$, $y(0) = y^0$ for the system of ordinary differential equations

$$\frac{dx}{dt_1} = c\left(x + y - x^3/3\right), \quad \frac{dy}{dt_1} = -(x + by - a)/c,$$

where $x, y : \mathbb{R} \to \mathbb{R}$, $x = x(t1)$, $y = y(t1)$ are the state functions, $t1$, the time, stands for the independent variable, and $a, b, c \in \mathbb{R}$ are three real parameters. For asymptotically fixed a, b and c, the dynamics generated by this model and its changes with respect to the parameters, i.e. the static, dynamic and perturbed bifurcation, were investigated analytically in several papers among which we quote [1, 2] and numerically, by the methods from [3, 4].

The present study continues these investigations with the asymptotic behavior of the phase portrait of the N-S model as $\mu = c^{-2} \to 0$, when a and b remain asymptotically fixed. For $\mu \neq 0$ the F-N model is a singular perturbation problem, which by the change $(x,y,t_1,c) \to (z,y,t,\mu)$, $z=x$, $y=y$, $t=t1/c$, $\mu=c^{-2}$, reads

$$\mu\frac{dz}{dt}=z+y-z^3/3,\ \frac{dy}{dt}=-z-by+a,\ z(0,\mu)=z^0,\ y(0,\mu)=y^0. \tag{1}$$

The problem (1) is a particular case of the singular perturbation problem

$$\mu\frac{dz}{dt}=F(z,y,\mu),\quad \frac{dy}{dt}=f(z,y,\mu),\quad z(0)=z^0,\quad y(0)=y^0, \tag{2}$$

where z^0 and y^0 are asymptotically fixed. Problems of type (2) were intensively studied by methods of classical qualitative theory of ordinary differential equations by the school of A.N. Tikhonov and A.B. Vasil'eva. They used the boundary layer functions method, which, in applications leads to cumbersome computations. This is why we preferred another way, namely to reduce (1) to a regular perturbation problem, which is developed in the following.

By means of the transform $(z,y)\leftrightarrow(\eta,\varsigma)$, $\varsigma=z+y-z^3/3$, $\eta=z$, the inverse of which reads $y=\varsigma-\eta+\frac{1}{3}\eta^3$, $z=\eta$ and using the chain rule (of differentiation of composite functions)

$$\frac{dz}{dt}=\frac{\partial z}{\partial\varsigma}\cdot\frac{d\varsigma}{dt}+\frac{\partial z}{\partial\eta}\cdot\frac{d\eta}{dt}=\frac{d\eta}{dt},\quad \frac{dy}{dt}=\frac{\partial y}{\partial\varsigma}\cdot\frac{d\varsigma}{dt}+\frac{\partial y}{\partial\eta}\cdot\frac{d\eta}{dt}=\frac{d\varsigma}{dt}+\left(-1+\eta^2\right)\frac{d\eta}{dt},$$

problem (1) becomes

$$\begin{cases}\mu\dfrac{d\eta}{dt}=\varsigma,\\[2ex] \dfrac{d\varsigma}{dt}+\left(-1+\eta^2\right)\dfrac{d\eta}{dt}=-\eta+a-b\left(\varsigma-\eta+\dfrac{1}{3}\eta^3\right),\end{cases}$$

$$\begin{cases}\varsigma(0,\mu)=z^0+y^0-\dfrac{1}{3}\left(z^0\right)^3\equiv\varsigma^0,\\ \eta(0,\mu)=z^0\equiv\eta^0,\end{cases}$$

or, equivalently,

$$\begin{cases}\mu\cdot\dfrac{d\eta}{dt}=\varsigma,\\[2ex] \mu\cdot\dfrac{d\varsigma}{dt}=a\mu+(-1+b)\mu\eta+(1-\mu b)\varsigma-\dfrac{1}{3}b\mu\eta^3-\varsigma\eta^2,\end{cases}\quad \begin{cases}\varsigma(0,\mu)=\varsigma^0,\\[2ex] \eta(0,\mu)=\eta^0.\end{cases} \tag{3}$$

By means of the change of variable $t=\mu\tau$ and taking into account that $\frac{d}{dt}=\frac{1}{\mu}\cdot\frac{d}{d\tau}$, the singular perturbation problem (3) becomes the problem of regular perturbations

$$\begin{cases}\dfrac{d\eta}{d\tau}=\varsigma,\\[2ex] \dfrac{d\varsigma}{d\tau}=a\mu-(1-b)\mu\eta+(1-\mu b)\varsigma-\dfrac{1}{3}b\mu\eta^3-\varsigma\eta^2,\end{cases}\quad \begin{cases}\varsigma(0,\mu)=\xi^0,\\[2ex] \eta(0,\mu)=\eta^0.\end{cases} \tag{4}$$

2 Models of asymptotic approximation

The use of the time recalling $\tau = t/\mu$ may be embarrassing: it is appropriate to inner asymptotic approximations for singular perturbation two-point problems or to singular perturbation Cauchy problems which possess one asymptotic boundary layer or asymptotic initial layer respectively. The new time τ is the inner independent variable. In these cases t is assumed to be small, namely of order of μ as $\mu \to 0$. For larger t and τ, the inner component of the asymptotic solution looses its importance. In problems of the type (2) there is an infinity of interval layers as t is increased beyond 0.

Therefore $\tau = t/\mu$ can be large as t is other very small but $t >> \mu$, or $t >> 1$. This means that our study is appropriate to t larger than the order used to stretch the boundary or to initial layers. On the other hand, we expect that the "inner" component (i.e. corresponding to τ) of the asymptotic solution be important as τ is increased, because if crosses other and other interval layers. In other words, we expect that the problem involving τ has an "outer" role too, i.e. it takes the role of the problem (1) in t. As far as small τ is concerned, the problem in τ plays the role of a genuine inner problem, corresponding to $t << \mu$ or $t \sim \mu$. These are the reasons for suspecting that (4) is a good approximation of (2) for every t, irrespective of its order. The numerical results based on (4) confirmed this assumption.

Further we use a convenient variant of one (unpublished) result of the second and third authors.

Theorem 1. *Assume that in the Cauchy problem*

$$\frac{dx}{dt} = f(t, x, \mu), x(t_0) = x^0, \tag{5}$$

the function $f(t,x,\mu)$is continuous and satisfies in x (uniformly in t and μ) the condition Lipschitz on the domain

$$0 \equiv \left\{(t, x, \mu) \mid |t - t_0| \leq a, \left|\left|x - x^0\right|\right| \leq b, |\mu - \mu_0| \leq 0\right\}.$$

Then:

I. 1) there exists a unique continuous solution $x(t,\mu)$ of problem (5) on the compact $[t_0, t_0 + H] \times [\mu_0 - c, \mu_0 + c]$, where $H = \min\left\{a, \dfrac{b}{\mu}\right\}$, $M = \max\limits_{0} |f(t, x, \mu)|$;

2) the solution $x(t, \mu_0)$ is defined on $[t_0, t_0 + H_0]$, where $H_0 = \min\left\{a, \dfrac{b}{M_0}\right\}$, $M_0 = \max\limits_{0} |f(t, x, \mu_0)|$. There exists $c_0 \in (0, c)$ such that $x(t, \mu)$ is defined on $[t_0, t_0 + H_0] \times [\mu_0 - c_{01}\mu_0 - c_0]$ and $\lim\limits_{\mu \to \mu_0} x(t, \mu) = x(t, \mu_0)$, uniformly in $t \in [t_0, t_0 + H_0]$;

II. 1) if, in addition, exist and are continuous f_x, f_μ on D, then there exists the derivative $\dfrac{\partial x}{\partial \mu}(t, \mu)$,denoted by X, which is differentiable with respect to t, and it is the solution of the Cauchy problem

$$\frac{dX}{dt} = f_x(t, x(t, \mu), \mu), \ X + f_\mu(t, x(t, \mu), \mu), \ X(t_0) = 0. \tag{6}$$

If, in addition, f possesses bounded partial derivatives up to the order $n + 1$ in x and μ, then

$$x(t, \mu) = x(t, \mu_0) + (\mu - \mu_0) \frac{\partial x}{\partial \mu}(t, \mu_0) + \cdots + \frac{(\mu - \mu_0)^n}{n!} \frac{\partial^n x}{\partial \mu^n}(t, \mu_0) + \varepsilon_{n+1}(t, \mu) \tag{7}$$

where $\varepsilon_{n+1}(t,\mu) = O\left(|\mu-\mu_0|^{n+1}\right)$.

Let us use this theorem by denoting $\dfrac{\partial^k x}{\partial \mu^k}(t,\mu_0)$ by $x_k(t)$ and replacing x in (5) by (7) we obtain

$$\begin{aligned} &\frac{d}{dt}\left[x_0(t) + (\mu-\mu_0)x_1(t) + \cdots + \frac{(\mu-\mu_0)^n}{n!}x_n(t) + \varepsilon_{n+1}(t,\mu)\right] = \\ &= f\left(t, x_0(t) + (\mu-\mu_0)x_1(t) + \cdots + \frac{(\mu-\mu_0)^n}{n!}x_n(t), \mu\right) + \varepsilon_{n+1}(t,\mu) \end{aligned} \tag{8}$$

and

$$x_0(t_1) + (\mu-\mu_0)x_1(t_0) + ... + \frac{(\mu-\mu_0)}{n!}x_n(t_0) + \varepsilon_{n+1}(t_0,\mu) = x^0. \tag{9}$$

From (8) and (9), by matching, we deduce the problems satisfied by $x_k(t)$, $k = \overline{0,n}$, (they are the models of regular asymptotic approximation of order k), namely: from (8) for $\mu = \mu_0$, we obtain

$$\frac{dx_0}{dt} = f(t, x_0, \mu_0), \quad x_0(t_0) = x^0;$$

differentiating (8) with respect to μ and taking $\mu = \mu_0$ it follows

$$\frac{dx_1}{dt} = \frac{\partial t}{\partial x}(t, x_0, \mu_0)x_1 + \frac{\partial f}{\partial \mu}(t, x_0, \mu_0), \quad x_1(t_0) = 0;$$

differentiating (8) two times with respect to μ and taking $\mu = \mu_0$, we have

$$\begin{aligned} \frac{dx_2}{dt} &= \frac{\partial f}{\partial x}(t, x_0, \mu_0)x_2 + \frac{\partial^2 f}{\partial x^2}(t, x_0, \mu_0)(x_1, x_1) + \\ &+ \frac{\partial^2 f}{\partial x \partial \mu}(t, x_0, \mu_0)x_1 + \frac{\partial^2 f}{\partial \mu^2}(t, x_0, \mu_0), \quad x_2(t_0) = 0 \end{aligned}$$

and so on.

Since, by Theorem 1, the vector field associated with problem (4) is analytic with respect to μ at $\mu = \mu_0 = 0$, the solution of (4) possesses converging series of powers of μ

$$\eta(\tau,\mu) = \sum_{k\geq 0} \frac{\mu^k}{k!}\frac{\partial^k \eta}{\partial \mu^k}(\tau, 0), \qquad \varsigma(\tau,\mu) = \textstyle\sum_{k\geq 0} \frac{\mu^k}{k!}\frac{\partial^k \varsigma}{\partial \mu^k}(\tau,0). \tag{10}$$

Denoting $\eta_k(\tau) = \dfrac{\partial^k \eta}{\partial \mu^k}(\tau, 0)$, $\varsigma_k(\tau) = \dfrac{\partial^k \varsigma}{\partial \mu^k}(\tau, 0)$ and introducing (10) in (4) it follows the Cauchy problem

$$\begin{cases} \dfrac{d}{d\tau}\left[\sum_{k\geq 0}\dfrac{\mu^k}{k!}\eta_k(\tau)\right] = \sum_{k\geq 0}\dfrac{\mu^k}{k!}\varsigma_k(\tau), \\ \dfrac{d}{d\tau}\left[\sum_{k\geq 0}\dfrac{\mu^k}{k!}\varsigma_k(\tau)\right] = a\mu - (1-b)\mu\left[\sum_{k\geq 0}\dfrac{\mu^k}{k!}\eta_k(\tau)\right] + \\ +(1-\mu b)\left[\sum_{k\geq 0}\dfrac{\mu^k}{k!}\varsigma_k(\tau)\right] - \dfrac{1}{3}b\mu\left[\sum_{k\geq 0}\dfrac{\mu^k}{k!}\eta_k(\tau)\right]^3 - \\ \qquad -\left[\sum_{k\geq 0}\dfrac{\mu^k}{k!}\varsigma_k(\tau)\right]\left[\sum_{k\geq 0}\dfrac{\mu^k}{k!}\eta_k(\tau)\right]^2, \end{cases} \tag{11}$$

$$\sum_{k\geq 0}\frac{\mu^k}{k!}\eta_k(0) = \eta^0, \quad \sum_{k\geq 0}\frac{\mu^k}{k!}\varsigma_k(0) = \varsigma^0, \tag{12}$$

whence, by matching, from (12) the models of asymptotic approximation of order k are immediately deduced.

The model of the first approximation reads

$$\begin{cases} \dfrac{d\eta_0}{d\tau} = \varsigma_0, \\ \dfrac{d\varsigma_0}{d\tau} = \varsigma_0 - \varsigma_0\eta_0^2, \end{cases} \qquad \begin{cases} \eta_0(0) = \eta^0, \\ \varsigma_0(0) = \varsigma^0. \end{cases} \tag{13}$$

Like in any regular perturbation problem, (13) could be, formally, deduced by letting $\mu = 0$ in (12). The model of the second order asymptotic approximation (which, formally, follows by taking $\mu = 0$ in the derivative of (12) 1,2 with respect to μ) is

$$\begin{cases} \dfrac{d\eta_1}{d\tau} = \varsigma_1, \\ \dfrac{d\varsigma_1}{d\tau} = a - (1-b)\eta_0 - \varsigma_0 + \varsigma_1 - \dfrac{1}{3}b\eta_0^3 - \varsigma_1\eta_0^2 - \varsigma_0 2\eta_0\eta_1, \end{cases} \qquad \begin{cases} \eta_1(0) = 0, \\ \varsigma_1(0) = 0. \end{cases} \tag{14}$$

Similarly, differentiating (12) 1,2 two times with respect to μ and taking $\mu = 0$ in the obtained equation, we obtain the model of the second order asymptotic approximation

$$\begin{cases} \dfrac{d\eta_2}{d\tau} = \varsigma_2, \\ \dfrac{d\varsigma_2}{d\tau} = -2(1-b)\varsigma_1 - 2b\varsigma_1 + \varsigma_2 - \dfrac{2}{3}b\varsigma\eta_0^2\eta_1 - \varsigma_2\eta_0^2 - 2\varsigma_1\varsigma\eta_0\eta_1 - 2\varsigma_0(\eta_2\eta_1 + \eta_1\eta_1), \end{cases}$$

$$\eta_2(0) = 0, \ \varsigma_2(0) = 0. \tag{15}$$

Apparently, the equations (13) 1,2 are simpler in form than (14) 1,2 and (15) 1,2 but they have cubic nonlinearities. In addition the conditions (13) 3 are nonhomogenous. The equations in (14) and (15) look more complicated but, in fact, they are affine and the associated conditions (14) 3, (15) 3 are homogenous. Hence the model (13) is the most difficult to be solved, at least in principle.

Proposition 1. *As $\mu \to 0$, the limit cycle of the dynamical system associated with the model* (13) *contains two parallel straightlines $y = y_0$.*

Indeed, (13) implies

$$\frac{d}{d\tau}\left(\varsigma_0 - \eta_0 + \eta_0^3/3\right) = 0, \tag{16}$$

therefore $\varsigma_0 - \eta_0 + \eta_0^3/3 = \varsigma_0^0 - \eta_0^0 - \eta_0^3/3$, i.e. $y_0 = y^0$. This shows that as $\mu \to 0$, some portions of the trajectory limit cycle, are straightlines. In Section 3 we show that they are situated between the two external (stable) branches of the infinitecline $\varsigma + \eta - \eta^3/3 = 0$ (written, equivalently, as $\varsigma = 0$).

3 Model of the first asymptotic approximation

The dynamical system associated with (13) has an infinity of equilibria; their locus is the y_0 – axis. The closed-form or (algebraically) implicit form of each nonconstant solution of (13) can be found immediately by eliminating ς_0 between the equations of (13). We obtain

$$\frac{d^2\eta_0}{d\tau^2} = \left(1 - \eta_0^2\right)\frac{d\eta_0}{d\tau}, \quad \eta_0|_{\tau=0} = \eta^0, \quad \frac{d\eta_0}{d\tau}|_{\tau=0} = \varsigma^0,$$

or, equivalently, denoting (only in Section 3 and 4) $c = \varsigma^0 - \eta^0 + \dfrac{\eta^{03}}{3}$, integrating this equation and taking into account the initial conditions, it follows

$$\frac{d\eta_0}{d\tau} = \eta_0 - \frac{\eta_0^3}{3} + c, \quad \eta_0|_{\tau=0} = \eta^0. \tag{17}$$

Since, by (16) $\varsigma^0 = 0$ implies $\eta_0 = \eta^0$, i.e. $\left(\varsigma^0, \eta^0\right)$ corresponds to a point $\left(z^0, y^0\right)$ situated on the infinitecline *y=0*, we assume *ς0 ≠0*. Let us also remark that $c = y^0$.

Case $c = 0$. The solution of (17) is

$$\eta_0(\tau) = \frac{\sqrt{3}\eta^0 e^\tau}{\sqrt{|\eta^{02}e^{2\tau} + 3 - \eta^{02}|}}, \tag{18}$$

and from (13)1, it follows

$$\varsigma_0(\tau) = \frac{\sqrt{3}\eta^0\left(3 - \eta^{02}\right)e^\tau}{|\eta^{02}e^{2\tau} + 3 - \eta^{02}|^{3/2}}, \quad \text{with} \quad \varsigma^0 = \eta^0 - \frac{\eta^{03}}{3} \neq 0. \tag{19}$$

The relations (18) and (19) represent the parametic form of the solutions of (13). Whence, as expected by (16), the closed form

$$\varsigma_0 = \eta_0 - \frac{\eta_0^3}{3}, \tag{20}$$

or, coming back to the phase functions y and z, we have

$$\begin{cases} y(t) = 0, \\ z(t) = \dfrac{z^0\sqrt{3}e^{t/\mu}}{\sqrt{z^0 e^{2t/\mu} + 3 - z^{02}}}, \end{cases} \quad z^0 \neq 0, \pm\sqrt{3}.$$

The form (20) shows that, in the case *c=0,* the trajectory starting at a point of the z-axis is a portion of that axis, namely that one for which $z^0 \neq 0, \pm\sqrt{3}$ and $e^{2t/\mu} \neq \left(z^{02}-3\right)/z^0$.

Case $c \neq 0$. Equation $\eta_0^3 - 3\eta_0 - 3c = 0$, defining the equilibria of (17) has the discriminant $\Delta = -1 + \dfrac{9c^2}{4}$. Hence, for it has three real mutually distinct roots $\eta_{01,}$; η_{02} ; η_{03}, for $\Delta = 0$ it has a double root and a simple root (a triple root is not possible, because it should be null, whereas $c \neq 0$); for $\Delta > 0$, there exists a unique real root η_{00}.

Subcase $\Delta < 0$. Take by convention $\eta_{01} < \eta_{02} < \eta_{03}$. From (17) it follows the (implicit, algebraic) form of its solution

$$\frac{\left|\eta_0 - \eta_{01}\right|^A \ \left|\eta_0 - \eta_{03}\right|^D}{\left|\eta_0 - \eta_{02}\right|^{-B}} = ke^{-\tau/\varepsilon}, \tag{21}$$

or, equivalently, the closed form of τ as a function of η_0 , where

$$A = \frac{1}{\left(\eta_{01} - \eta_{02}\right)\left(\eta_{01} - \eta_{03}\right)} > 0, \quad B = \frac{1}{\left(\eta_{02} - \eta_{01}\right)\left(\eta_{02} - \eta_{03}\right)} < 0,$$

$$D = \frac{1}{\left(\eta_{03} - \eta_{01}\right)\left(\eta_{03} - \eta_{02}\right)} > 0$$

and k is a constant obtained by taking $\tau = 0$ in (21). Since, by (16) $\varsigma_0 - \eta_0 + \dfrac{\eta_0^3}{3} = c$ is a prime integral it follow that ς_0 is a function of η_0. Consequently, it is sufficient to determine only η_0 (because ς_0 follows).

Subcase $\Delta > 0$. Equation (17) is equivalent (in the class of nonconstant solutions) to anyone among the following three forms

$$\frac{d\eta_0}{d\tau} = \eta_0 - \frac{\eta_0^3}{3} + c, \qquad \frac{d\eta_0}{c + \eta_0 - \dfrac{\eta_0^3}{3}} = d\tau, \qquad \int_{\eta^0}^{\eta_0} \frac{d\eta_0}{c + \eta_0 - \dfrac{\eta_0^3}{3}} = \tau - \tau_0.$$

In this way, by expressing τ as a function of η_0, we obtain the closed-form solution. Taking into account the decomposition in simple fractions

$$\frac{1}{\eta_0 + c - \frac{1}{3}\eta_0^3} = \frac{1}{\eta_{00}^2 - 1}\left[\frac{1}{\eta_0 - \eta_{00}} + \frac{\frac{1}{3}\eta_0 + \frac{2}{3}\eta_{00}}{\frac{1}{3}\eta_0^2 + \frac{1}{3}\eta_{00}\eta_0 + \left(\frac{1}{3}\eta_{00}^2 - 1\right)}\right]$$

the solution of (17) becomes successively

$$\tau = \int_{\eta^0}^{\eta_0} \frac{d\eta_0}{\eta_0 + c - \frac{1}{3}\eta_0^3} =$$

$$= \frac{1}{\eta_{00}^2 - 1}\int_{\eta^0}^{\eta_0}\left[\frac{1}{\eta_0 - \eta_{00}} + \frac{1}{2}\frac{\dfrac{2}{3}\eta_0 + \dfrac{1}{3}\eta_{00}}{\dfrac{1}{3}\eta_0^2 + \dfrac{1}{3}\eta_{00}\eta_0 + \left(\dfrac{1}{3}\eta_{00}^2 - 1\right)} + \right.$$

$$\left. + \frac{1}{3}\left(\frac{1}{\dfrac{1}{3}\left(\eta_0 + \dfrac{1}{2}\eta_{00}\right)^2 + \dfrac{1}{4}\eta_{00}^2 - 1}\right)\right] = \frac{1}{\eta_{00}^2 - 1}\ \left[\ln\left[\frac{\eta_0 - \eta_{00}}{\eta^0 - \eta_{00}}\right] + \right.$$

$$+\frac{1}{2}\ln\left|\frac{\frac{1}{3}\eta_0^2+\frac{1}{3}\eta_{00}\eta_0+\frac{1}{3}\eta_0^2-1}{\frac{1}{3}\left(\eta^0\right)^2+\frac{1}{3}\eta_{00}\eta^0+\frac{1}{3}\eta_{00}^2-1}\right|-$$

$$-\frac{3\eta_{00}}{2\sqrt{3\left(\frac{1}{4}\eta_{00}^2-1\right)}}\left(arctg\frac{\left(\eta^0+\frac{1}{2}\eta_{00}\right)}{\sqrt{3\left(\frac{1}{4}\eta_{00}^2-1\right)}}\right)-arctg\frac{\left(\eta_0+\frac{1}{2}\eta_{00}\right)}{\sqrt{3\left(\frac{1}{4}\eta_{00}^2-1\right)}}\Bigg)\Bigg].$$

Subcase $\mathbf{\Delta=0}$. If $\Delta=0$ and the equation $\eta_0+c-\frac{\eta_0^3}{3}=0$ has a simple root η_{01} and a double root η_{02}, hence $\eta_0+c-\frac{\eta_0^3}{3}=\frac{-1}{3}\left(\eta_0-\eta_{01}\right)\left(\eta_0-\eta_{02}\right)^2$, then

$$\frac{1}{\eta_0+c-\frac{1}{3}\eta_0^3}=-\frac{3}{\left(\eta_{01}-\eta_{02}\right)^2}\cdot\frac{1}{\eta_0-\eta_{01}}\cdot\frac{-3}{\left(\eta_{01}-\eta_{02}\right)^2}\cdot\frac{\eta_0-\eta_{01}+2\eta_{02}}{\left(\eta_0-\eta_{02}\right)^2},$$

implying the closed-form solution τ as a function of η_0

$$\tau=\int_{\eta^0}^{\eta_0}\frac{d\eta_0}{\eta_0+c-\frac{1}{3}\eta_0^3}=\int_{\eta^0}^{\eta_0}\frac{-3}{\left(\eta_{01}-\eta_{02}\right)^2}\cdot\frac{1}{\eta_0-\eta_{01}}d\eta_0-$$

$$-\frac{3}{\left(\eta_{01}-\eta_{02}\right)^2}\int_{\eta^0}^{\eta_0}\left[\frac{1}{\eta_0-\eta_{02}}+\frac{-\eta_{01}+3\eta_{02}}{\left(\eta_0-\eta_{02}\right)^2}\right]d\eta_0=$$

$$=-\frac{3}{\left(\eta_{01}-\eta_{02}\right)^2}\left[\ln\frac{\eta^0-\eta_{01}}{\eta_0-\eta_{01}}+\ln\left[\frac{\eta^0-\eta_{02}}{\eta_0-\eta_{02}}\right]-\right.$$

$$\left.-\left(-\eta_{01}+3\eta_{02}\right)\cdot\left(\frac{1}{\eta_0-\eta_{02}}-\frac{1}{\eta^2-\eta_{02}}\right)\right].$$

4 Model of the second order asymptotic approximation

The system (14), can be successively written as

$$\frac{d^2\eta_1}{d\tau^2}=a-\eta_0-b\left(\varsigma_0-\eta_0+\frac{\eta_0^3}{3}\right)+\varsigma_1\left(1-\eta_0^2\right)-2\varsigma_0\eta_0\eta_1=$$

$$=a-\eta_0-bc+\left(1-\eta_0^2\right)\frac{d\eta_1}{d\tau}\ -2\eta_0\frac{d\eta_0}{d\tau}\eta_1=$$

$$=a-\eta_0-bc+\left(1-\eta_0^2\right)\frac{d\eta_1}{d\tau}-\eta_1\frac{d\eta_0^2}{d\tau}=$$

$$=a-\eta_0-bc+\left(1-\eta_0^2\right)\frac{d\eta_1}{d\tau}+\eta_1\frac{d\left(1-\eta_0^2\right)}{d\tau},$$

hence

$$\frac{d^2\eta_1}{d\tau^2} = a - \eta_0 - bc + \frac{d\left[\eta_1\left(1-\eta_0^2\right)\right]}{d\tau}. \tag{22}$$

Further we solve the Cauchy problem for this equation by applying several times the method of variation of coefficients. Thus, the linear equation corresponding to (22) is

$$\frac{d^2\eta_1}{d\tau^2} = \frac{d\left[\eta_1\left(1-\eta_0^2\right)\right]}{d\tau}, \tag{23}$$

which can be written as

$$\frac{d\eta_1}{d\tau} = \eta_1\left(1-\eta_0^2\right) + C_1, \tag{24}$$

where C_1 is an arbitrary constant. The linear equation corresponding to (24) is

$$\frac{d\eta_1}{d\tau} = \eta_1\left(1-\eta_0^2\right),$$

whence $\ln|\eta_1| = \int\left(1-\eta_0^2\right)d\tau = \int \frac{d\varsigma_0}{d\tau}\cdot\frac{1}{\varsigma_0}d\tau = \int\frac{d\varsigma_0}{\varsigma_0} = \ln|\varsigma_0| + C_2$, i.e. $\eta_1 = K_2\varsigma_0$. Then, the method of variation of coefficients applied to (24), where C_1 is a constant and K_2 is a function of τ, implies $K_2(\tau) = \int\frac{C_1 d\tau}{\varsigma_0} + C_3$, where C_3 is a constant. Therefore, $\eta_1(\tau) = \varsigma_0\left[C_1\int\frac{d\tau}{\varsigma_0} + C_3\right]$ is the general solution of (23). In order to find the solution of (22) we apply again the method of variation of coefficients to find

$$\frac{d\eta_1}{d\tau} = \frac{d\varsigma_0}{d\tau}\left[C_1\int\frac{d\tau}{\varsigma_0} + C_3\right] + \varsigma_0\left[\frac{dC_1}{d\tau}\int\frac{d\tau}{\varsigma_0} + \frac{dC_3}{d\tau} + \frac{C_1}{\varsigma_0}\right] =$$

$$= \frac{d\varsigma_0}{d\tau}\left[C_1\int\frac{d\tau}{\varsigma_0} + C_3\right] + \frac{dC_1}{d\tau}\varsigma_0\int\frac{d\tau}{\varsigma_0} + \varsigma_0\frac{dC_3}{d\tau} + C_1 =$$

$$= \varsigma_0\left(1-\eta_0^2\right)\left[C_1\int\frac{d\tau}{\varsigma_0} + C_3\right] + C_1,$$

and impose

$$\varsigma_0\frac{dC_3}{d\tau} + \varsigma_0\frac{dC_1}{d\tau}\int\frac{d\tau}{y_0} = 0, \tag{25}$$

$$\begin{aligned}
\frac{d^2\eta_1}{d\tau^2} &= \frac{d}{d\tau}\left\{\varsigma_0\left(1-\eta_0^2\right)\left[C_1\int\frac{d\tau}{\varsigma_0} + C_3\right]\right\} + \frac{dC_1}{d\tau} = \\
&= \frac{d}{d\tau}\left[\eta_1\left(1-\eta_0^2\right)\right] + \frac{dC_1}{d\tau}\left[1 + \left(1-\eta_0^2\right)y_0\int\frac{d\tau}{y_0}\right] + y_0\left(1-\eta_0^2\right)\cdot\frac{dC_3}{d\tau} = \\
&= \frac{d}{d\tau}\left(1-\eta_0^2\right) + a - \eta_0 - bc.
\end{aligned} \tag{26}$$

Hence from (25) and (26), we have successively $\frac{dC_1}{d\tau} = a - \eta_0 - bc$, i.e. $\frac{dC_1}{d\eta_0}\cdot\frac{d\eta_0}{d\tau} = a - \eta_0 - bc$, therefore $\frac{dC_1}{d\eta_0} = \frac{a-\eta_0-bc}{\varsigma_0}$, which implies $\frac{dC_1}{d\eta_0} = \frac{a-\eta_0-bc}{c+\eta_0-\frac{\eta_0^3}{3}}$, whence $C_1(\eta_0) =$

$\psi(\eta_0) + C_{10}$, where $\psi(\eta_0)$ is an integral which can be computed immediately, for instance by using some formulae from [5]. Then the relation (25) reads

$$\frac{dC_3}{d\eta_0}\frac{d\eta_0}{d\tau} = -\frac{dC_1}{d\eta_0}\frac{d\eta_0}{d\tau}\int\left(c+\eta_0-\frac{\eta_0^3}{3}\right)^{-2}d\eta_0$$

and, by integration with respect to η_0, we have

$$C_3(\eta_0) - C_3 = -\int\left\{\frac{a-\eta_0-bc}{c+\eta_0-\frac{\eta_0^3}{3}}\int\frac{d\eta_0}{\left(c+\eta_0-\frac{\eta_0^3}{3}\right)^2}\right\}d\eta_0\ ,$$

i.e. $C_3(\eta_0) = \varphi(\eta_0) + C_{30}$, where $\varphi(\eta_0)$ is an integral which can be computed by using appropriate formulae from [5]. Finally, η_1 reads

$$\eta_1(\tau) = y_0\left[(\psi(\eta_0)+C_{10})\int\frac{d\tau}{y_0}+\varphi(\eta_0)+C_{30}\right], \tag{27}$$

where C_{10} and C_{30} are constants related to y_0^0 and η_0^0. Thus η_1 and y_1 are completely determined.

An alternative procedure is: from (14) it follows

$$\frac{d\varsigma_1}{d\eta_0} = \frac{a-\eta_0-bc}{c+\eta_0-\frac{\eta_0^3}{3}} + \frac{d}{d\eta_0}\left[\eta_1\left(1-\eta_0^2\right)\right],$$

hence

$$\frac{d}{d\eta_0}\left[\varsigma_1-\eta_1\left(1-\eta_0^2\right)\right] = \frac{a-\eta_0-bc}{c+\eta_0-\frac{\eta_0^3}{3}},$$

whence the prime integral

$$\varsigma_1 = \eta_1\left(1-\eta_0^2\right) + \int\frac{a-\eta_0-bc}{c+\eta_0-\frac{\eta_0^3}{3}}d\eta_0 + K,$$

is a computation of η_1 and y_1 in terms of τ.

In order to determine ς_1 and η_1 as functions of η_0, we use (14) to obtain

$$\frac{d\eta_1}{d\eta_0}\cdot\frac{d\eta_0}{d\tau} = \eta_1\left(1-\eta_0^2\right) + \int\frac{a-\eta_0-bc}{c+\eta_0-\frac{\eta_0^3}{3}}d\eta_0 + K,$$

$$\frac{d\eta_1}{d\eta_0} = \eta_1\frac{\left(1-\eta_0^2\right)}{c+\eta_0-\frac{\eta_0^3}{3}} + \frac{1}{c-\eta_0-\frac{\eta_0^3}{3}}\cdot\int\frac{a-\eta_0-bC}{c+\eta_0-\frac{\eta_0^3}{3}}d\eta_0 + \frac{K}{c+\eta_0-\frac{\eta_0^3}{3}},$$

The last equation (affine in η_1) can be solved by the method of the variation of coefficients. This ends the determination of the closed-form of the solution of model (14), also by the alternative procedure.

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Non-fundamental 2-isohedral tilings of the sphere *

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Abstract. The investigation of 2-isohedral tilings of the 2-dimensional sphere is continued. In previous works all the fundamental 2-isohedral tilings of the sphere have been enumerated. Here non-fundamental 2-isohedral tilings of the sphere are obtained from the fundamental ones using the method of gluing disks. For all the 7 countable series of isometry groups of the sphere the classification of normal non-fundamental tilings is given in a table of pictures. For non-normal tilings only numerical results are given. For the 7 separate isometry groups of the sphere numerical results are also shown.

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1 Introduction

There are a lot of works where tilings of the two-dimensional sphere with transitivity properties are investigated. For survey of earlier results and the classification of isohedral, isogonal and isotoxal tilings of the sphere see [1]. In [2] 2-isotoxal tilings of the sphere have been researched. General methods of finding k-isohedral ($k \geq 2$) tilings of a two-dimensional constant curvature space (i. e. the Euclidean plane, the sphere and the hyperbolic plane) that use the known isohedral tilings have been developed by the author in [3]. Applying these methods the author has obtained the complete classification of 2-isohedral tilings of the sphere in some tables of pictures. A part of these results (and namely, normal fundamental 2-isohedral tilings of the sphere) can be found in [4, 5]. Note that the same methods using Delaney–Dress symbols have been implemented [6] in algorithms and computer programmes, the numerical results of [6] concerning 2-isohedral tilings of the sphere coincide with the ours. In the present paper we show the classification of normal non-fundamental tilings of the sphere for all 7 infinite series of isometry groups of the 2-dimensional sphere.

Consider a tiling W of the 2-dimensional sphere by topological disks and a discrete isometry group G of the sphere. The tiling W is called k-isohedral with respect to the group G if G maps W onto itself and all the tiles of G form exactly k transitivity classes under the group G. Two pairs (W, G) and (W', G') are said to be of the same Delone class (or equivariant type [6]) if there exists a homeomorphism φ of the

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sphere with $\varphi(W) = W'$ and $G' = \varphi G \varphi^{-1}$. The above and some below concepts and definitions hold for all three 2-dimensional spaces of constant curvature (see [3, 7]).

In a tiling of the sphere by disks a vertex (an edge) is defined as a connected component of the intersection of two or more different disks which is (is not) a single point. A Delone class (W, G) is called $(h_1, h_2, \dots, h_k)$-transitive if the group G acts h_i times transitively on the i-th class of tiles of the tiling W, $i = 1, 2, \dots, k$. If $h_1 = h_2 = \dots = h_k = 1$, the Delone class (W, G) is called fundamental, otherwise non-fundamental.

For the description of discrete isometry groups of the 2-dimensional sphere we use here the Conway's orbifold symbol, which is equivalent to the Macbeath's group signature. Remind the explanation of the orbifold symbol as it is given in [8]. Let X be one of the three 2-dimensional spaces of constant curvature and G be an isometry group of X with a compact fundamental domain. Consider the quotient $M = X/G$, which is a compact 2-dimensional manifold, maybe with boundary. Any point $x \in X$ with the non-trivial stabilizer group $G_x = \{g \in G | g(x) = x\}$ gives rise to a cone point $(\bar{x}, v)$ of degree v if G_x is a rotation group of order v, or to a corner point of degree v if G_x is a dihedral group generated by a v-fold rotation and a reflection. Here $\bar{x}$ denotes the equivalence class of points containing x. The orbifold symbol can be obtained by specifying the following four items:

(O1) The number of handles h if M is orientable, or the number of cross-caps k otherwise.

(O2) The system of branching numbers for all cone points.

(O3) The number of boundary components q.

(O4) For each boundary component B one must list the branching numbers of all corner points lying on B in a cyclic order. If M is orientable one must list the corner points of each boundary component in the order induced by a fixed orientation of the underlying manifold.

The rules for writing down the orbifold symbol are the following: First, if M is orientable with h handles one writes h small circles: $\circ \circ \circ \cdots$. Second, the branching numbers for all cone points are listed. Next, each boundary component is indicated by a star: $*$, followed by the list of branching numbers encountered while going around the boundary component. Finally, if M is non-orientable with k cross-caps one writes k crosses: $\times \times \times \cdots$.

There are 7 countable series of isometry groups of the sphere given by the following orbifold symbols: nn, $n\times$, $n*$, $*nn$, $22n$, $2*n$, $*22n$ where $n = 1, 2, \dots$. Also there are 7 separate isometry groups of the sphere with the following orbifold symbols: 322, $3*2$, $*332$, 432, $*432$, 532, $*532$.

A tiling of the sphere is called normal if it satisfies the following conditions [1]:

SN1. Each tile is a topological disk.

SN2. The intersection of any sets of tiles is a connected (possibly empty) set.

SN3. Each edge of the tiling has two endpoints which are vertices of the tiling.

In a normal tiling every tile contains at least three edges on its boundary and the valence of each vertex is at least three. The works [4, 5] contain the complete enumeration of fundamental 2-isohedral tilings on the sphere, where the normal tilings are shown in figures, for the rest of tilings (both without digons and containing digons) the numerical results are given.

Now all the fundamental Delone classes of 2-isohedral tilings on the sphere by disks are known. The method of finding non-fundamental Delone classes is the same as the method proposed in [9] for finding non-fundamental Delone classes of isohedral tilings on the Euclidean plane.

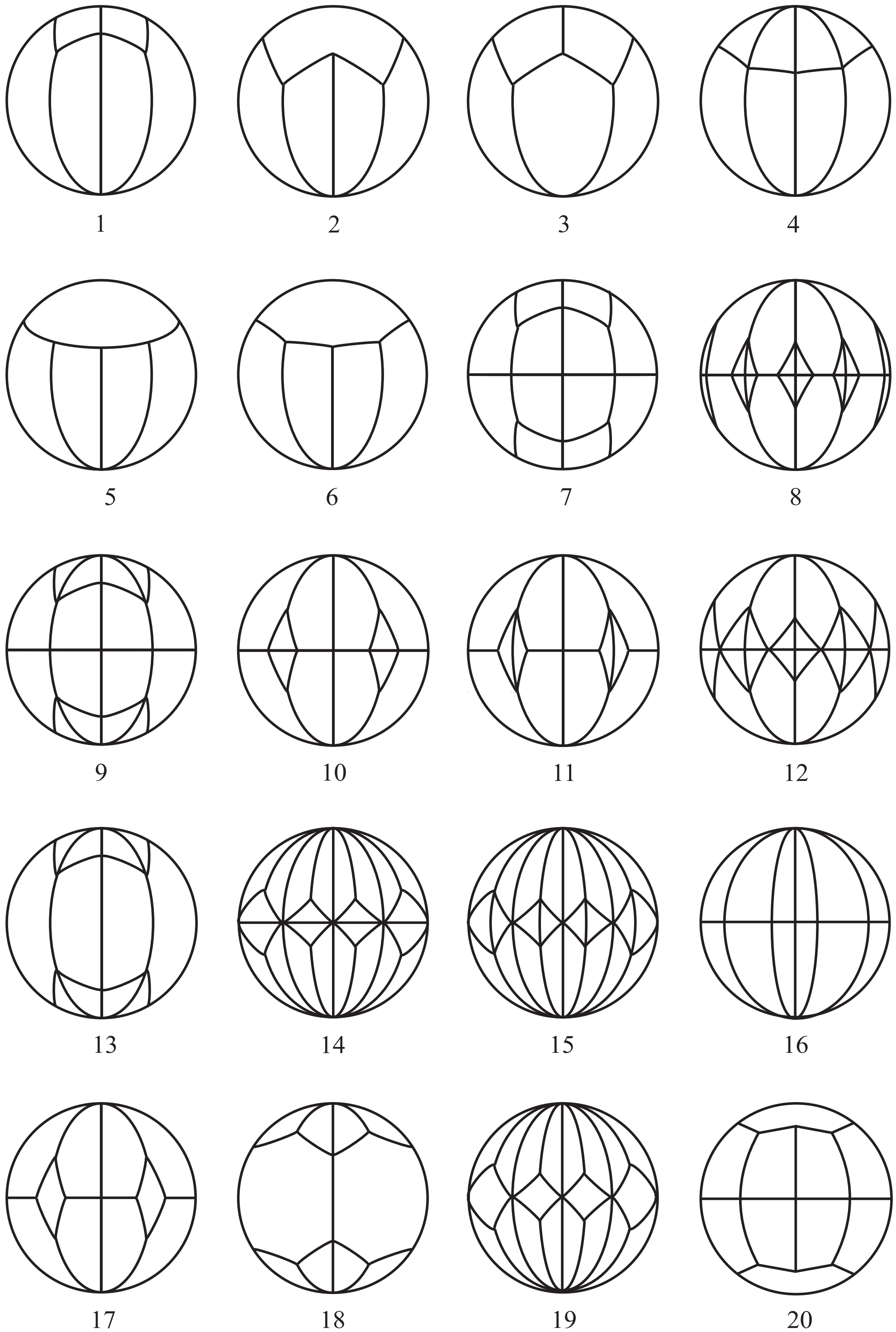
Let W be a tiling of the sphere by disks which is 2-isohedral with respect to a fundamental isometry group G. Let O be a vertex or the midpoint of an edge of the tiling W. If the order h of the stabilizer group $G_0 \subset G$ coincides with the number of tiles from W that contain O, we say the point O is good for gluing. Then glue (unite) all these tiles yielding a new disk. Do such a gluing at each point from the orbit $\{O_G\}$. As a result we obtain a new 2-isohedral tiling W' of the sphere by disks, the group G acts h times transitively on the set of new (glued) disks, so the Delone class of the pair (W', G) is non-fundamental.

Applying the gluing method to all the fundamental Delone classes of 2-isohedral tilings on the sphere by disks the author has obtained all the possible non-fundamental Delone classes of 2-isohedral tilings on the sphere by disks. Because of a large number of the resulted Delone classes, here we give pictures for normal tilings and numerical data for non-normal tilings. Besides, in the present paper we show pictures only for 7 infinite series of isometry groups, one representative tiling from each series of Delone classes is drawn (for either $n = 4$ or $n = 8$). As to the 7 separate isometry groups of the sphere, here we give numerical results and plan to publish the pictures in a further paper.

For the series of groups $*nn$ there is 1 series of (1,2)-transitive Delone classes of normal 2-isohedral tilings of the sphere (Fig. 1) and 5 series of (1,2)-transitive tilings containing digonal disks, 1 series of (1,$2n$)-transitive Delone classes of normal tilings (Fig. 2), 2 series of 2-transitive normal tilings (Fig. 3, 4) and 3 series of 2-transitive tilings containing digons, 1 series of (2,$2n$)-transitive normal tilings (Fig. 5), 1 series of $2n$-transitive tilings, each consisting of two disks; altogether there are 14 series of tilings (including 5 series of normal ones).

For the series of groups nn there is 1 series of (1,n)-transitive Delone classes of normal tilings (Fig. 6) and 1 series of n-transitive tilings, each consisting of two disks; altogether there are 2 series of tilings.

For the series of groups $*22n$ there are 10 series of (1,2)-transitive Delone classes of normal 2-isohedral tilings (Fig. 7–16), 4 series of (1,2)-transitive non-normal tilings without digonal disks and 19 series of (1,2)-transitive tilings containing digons; 3 series of (1,4)-transitive normal tilings (Fig. 17–19), 2 series of (1,4)-transitive non-normal tilings without digonal disks and 4 series of (1,4)-transitive tilings containing digons; 1 series of (1,$2n$)-transitive normal tilings (Fig. 20), 2 series of (1,$2n$)-transitive non-normal tilings without digonal disks and 3 series of (1,$2n$)-transitive tilings containing digons; 8 series of 2-transitive normal tilings (Fig. 21–28), 4 series of 2-transitive non-normal tilings without digonal disks and 16 series of 2-transitive tilings containing digons; 4 series of (2,4)-transitive normal tilings (Fig. 29–32), 2 series of (2,4)-transitive non-normal tilings without digonal disks and 9 series of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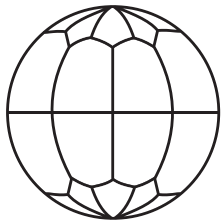
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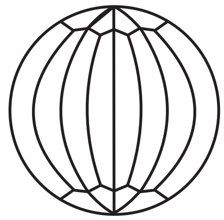
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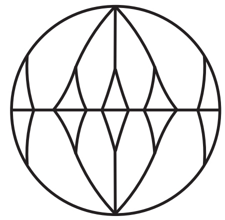
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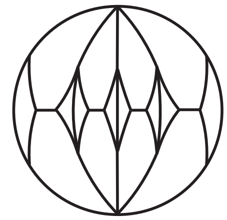
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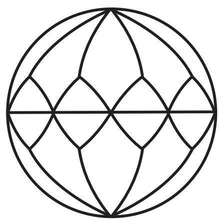
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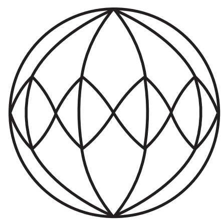
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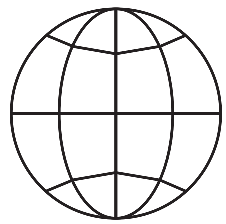
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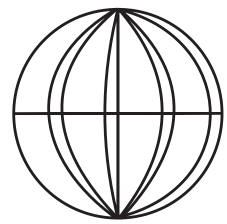
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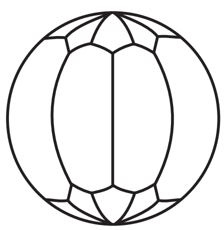
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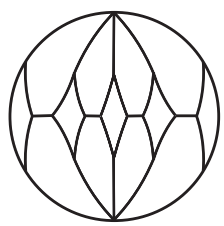
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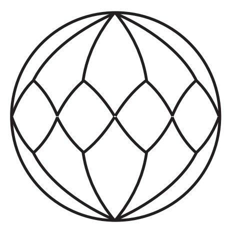
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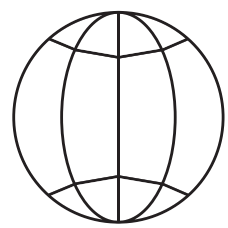
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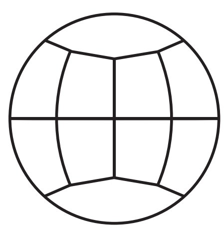
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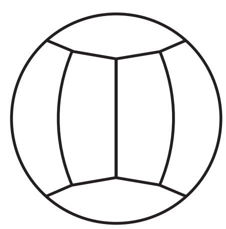
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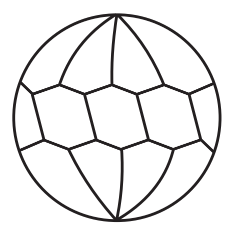
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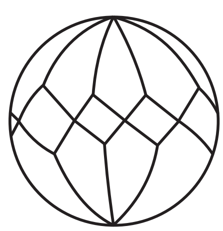
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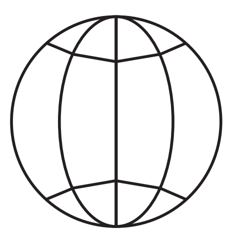
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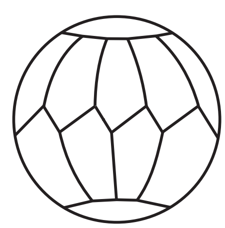
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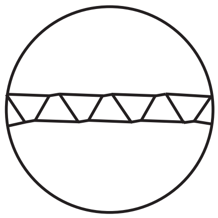
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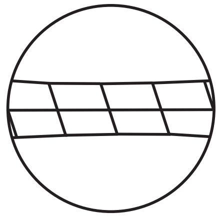
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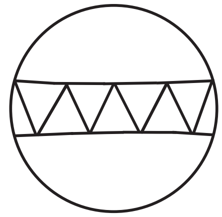
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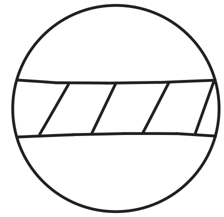
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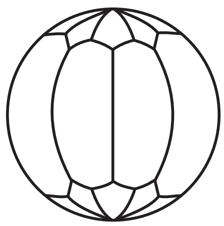
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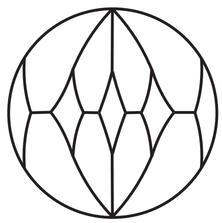
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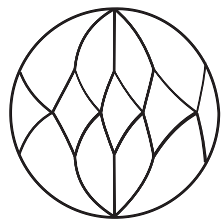
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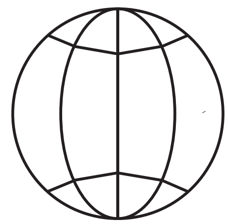
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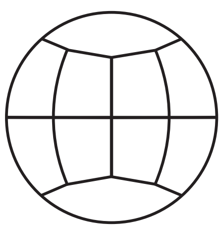
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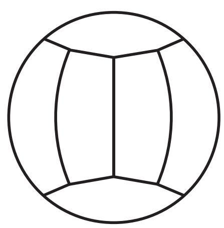
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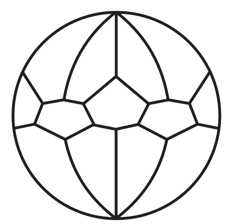
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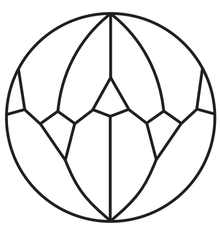
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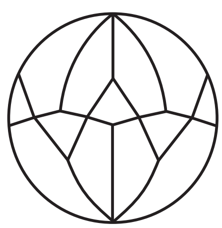
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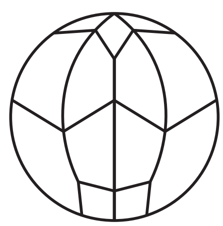
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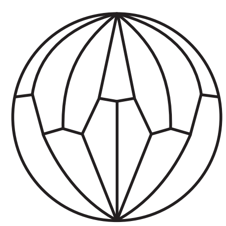
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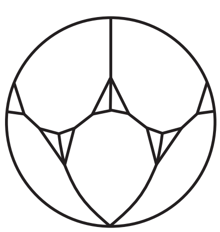
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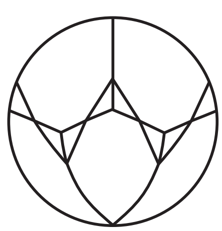
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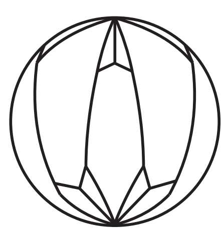
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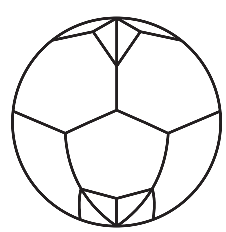
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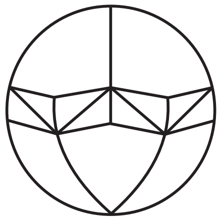
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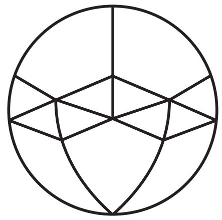
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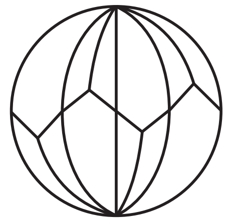
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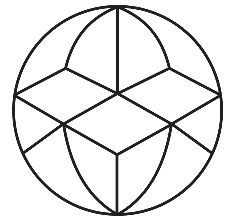
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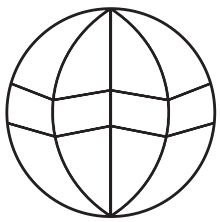
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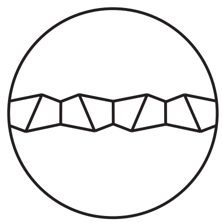
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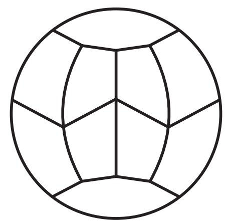
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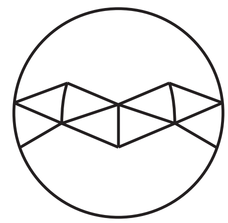
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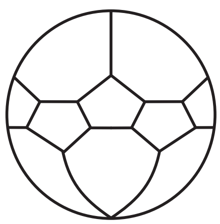
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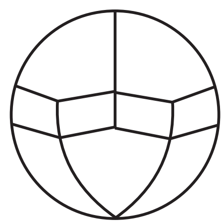
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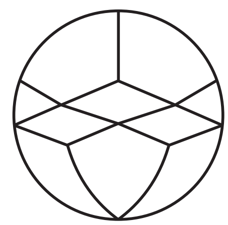
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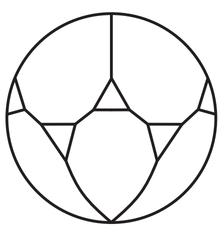
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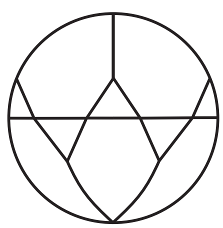
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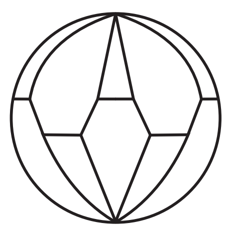
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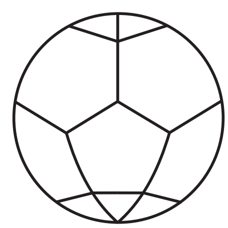
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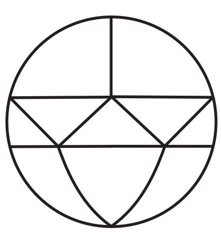
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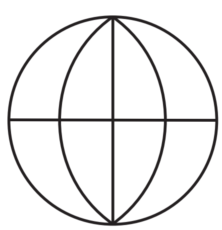
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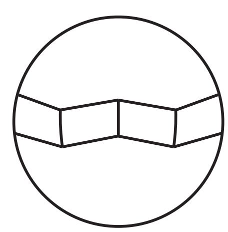
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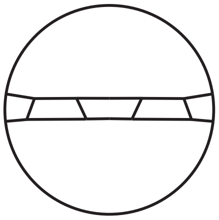

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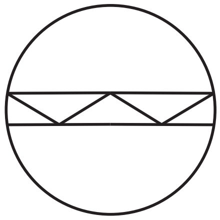

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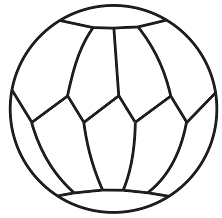

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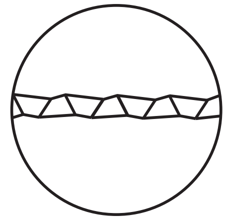

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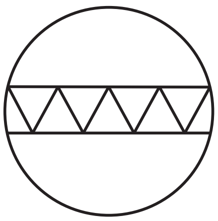

85

(2,4)-transitive tilings containing digons; 2 series of (2,2n)-transitive normal tilings (Fig. 33, 34), 2 series of (2, 2n)-transitive non-normal tilings without digonal disks and 5 series of (2,2n)-transitive tilings containing digons; 2 series of 4-transitive tilings containing digonal disks; 1 series of (4,2n)-transitive normal tilings (Fig. 35) and 2 series of (4,2n)-transitive tilings containing digons; altogether there are 105 series of tilings (including 29 series of normal ones).

For the series of groups 22n there are 4 series of (1,2)-transitive Delone classes of normal 2-isohedral tilings (Fig. 36–39), 2 series of (1,2)-transitive non-normal tilings without digonal disks and 11 series of (1,2)-transitive tilings containing digons; 4 series of (1,n)-transitive normal tilings (Fig. 40–43), 2 series of (1,n)-transitive non-normal tilings without digonal disks and 5 series of (1,n)-transitive tilings containing digons; 2 series of 2-transitive tilings containing digonal disks; 1 series of (2,n)-transitive normal tilings (Fig. 44) and 2 series of (2,n)-transitive tilings containing digons; altogether there are 33 series of tilings (including 9 series of normal ones).

For the series of groups $n*$ there are 4 series of (1,2)-transitive Delone classes of normal 2-isohedral tilings (Fig. 45–48), 2 series of (1,2)-transitive non-normal tilings without digonal disks and 7 series of (1,2)-transitive tilings containing digons; 1 series of (1,n)-transitive normal tilings (Fig. 49) and 2 series of (1,n)-transitive tilings containing digonal disks; 2 series of 2-transitive tilings containing digons; 1 series of (2,n)-transitive normal tilings (Fig. 50) and 2 series of (2,n)-transitive tilings containing digonal disks; altogether there are 21 series of tilings (including 6 series of normal ones).

For the series of groups $2{*}n$ there are 15 series of (1,2)-transitive Delone classes of normal 2-isohedral tilings (Fig. 51–65), 4 series of (1,2)-transitive non-normal tilings without digonal disks and 28 series of (1,2)-transitive tilings containing digons; 3 series of (1,2n)-transitive normal tilings (Fig. 66–68), 4 series of (1,2n)-transitive non-normal tilings without digonal disks and 6 series of (1,2n)-transitive tilings containing digons; 10 series of 2-transitive normal tilings (Fig. 69–78) and 10 series of 2-transitive tilings containing digonal disks; 4 series of (2,$2n$)-transitive normal tilings (Fig. 79–82) and 4 series of (2,$2n$)-transitive tilings containing digons; altogether there are 88 series of tilings (including 32 series of normal ones).

For the series of groups $n\times$ there are 3 series of (1,n)-transitive Delone classes of normal 2-isohedral tilings (Fig. 83–85) and 2 series of (1,n)-transitive tilings containing digonal disks; altogether there are 5 series of tilings (including 3 series of normal ones).

For the group $*332$ there are 12 (1,2)-transitive Delone classes of normal 2-isohedral tilings, 2 (1,2)-transitive non-normal tilings without digonal disks and 19 (1,2)-transitive tilings containing digons; 3 (1,4)-transitive normal tilings and 3 (1,4)-transitive tilings containing digonal disks; 3 (1,6)-transitive normal tilings, 2 (1,6)-transitive non-normal tilings without digonal disks and 4 (1,6)-transitive tilings containing digons; 12 2-transitive normal tilings, 2 2-transitive non-normal tilings without digonal disks and 14 2-transitive tilings containing digons; 4 (2,4)-transitive normal tilings and 5 (2,4)-transitive tilings containing digonal disks; 7 (2,6)-transitive normal tilings, 2 (2,6)-transitive non-normal tilings without digonal disks and 6 (2,6)-transitive tilings containing digons; 1 (4,6)-transitive normal tiling and 2 (4,6)-transitive tilings containing digonal disks; 2 6-transitive normal tilings; altogether there are 105 tilings (including 44 normal ones).

For the group 332 there are 4 (1,2)-transitive Delone classes of normal 2-isohedral tilings and 7 (1,2)-transitive tilings containing digonal disks; 9 (1,3)-transitive normal tilings, 2 (1,3)-transitive non-normal tilings without digonal disks and 6 (1,3)-transitive tilings containing digons; 1 (2.3)-transitive normal tilings and 2 (2,3)-transitive tilings containing digonal disks; 2 3-transitive normal tilings; altogether there are 33 tilings (including 16 normal ones).

For the group $*432$ there are 23 (1,2)-transitive Delone classes of normal 2-isohedral tilings, 4 (1,2)-transitive non-normal tilings without digonal disks and 36 (1,2)-transitive tilings containing digons; 5 (1,4)-transitive normal tilings and 4 (1,4)-transitive tilings containing digonal disks; 3 (1,6)-transitive normal tilings, 2 (1,6)-transitive non-normal tilings without digonal disks and 4 (1,6)-transitive tilings containing digons; 3 (1,8)-transitive normal tilings, 2 (1,8)-transitive non-normal tilings without digonal disks and 4 (1,8)-transitive tilings containing digons; 22 2-transitive normal tilings, 4 2-transitive non-normal tilings without digonal disks and 28 2-transitive tilings containing digons; 7 (2,4)-transitive normal tilings and 8 (2,4)-transitive tilings containing digonal disks; 7 (2,6)-transitive normal tilings, 2 (2,6)-transitive non-normal tilings without digonal disks and 6 (2,6)-transitive tilings containing digons; 7 (2,8)-transitive normal tilings, 2 (2,8)-transitive non-normal tilings without digonal disks and 6 (2,8)-transitive tilings containing digons; 1 (4,6)-transitive normal tiling and 2 (4,6)-transitive tilings containing digonal disks; 1 (4,8)-transitive normal tiling and 2 (4,8)-transitive tilings containing digons; 3 (6,8)-transitive normal tilings; altogether there are 198 tilings (including 82 normal ones).

For the group 432 there are 7 (1,2)-transitive Delone classes of normal 2-isohedral tilings and 10 (1,2)-transitive tilings containing digonal disks; 9 (1,3)-transitive normal tilings, 2 (1,3)-transitive non-normal tilings without digonal disks and 6 (1,3)-transitive tilings containing digons; 9 (1,4)-transitive normal tilings, 2 (1,4)-transitive non-normal tilings without digonal disks and 6 (1,4)-transitive tilings containing digons; 1 (2,3)-transitive normal tiling and 2 (2,3)-transitive tilings containing digonal disks; 1 (2,4)-transitive normal tiling and 2 (2,4)-transitive tilings containing digonal disks; 3 (3,4)-transitive normal tilings; altogether there are 60 tilings (including 30 normal ones).

For the group $3*2$ there are 19 (1,2)-transitive Delone classes of normal 2-isohedral tilings, 2 (1,2)-transitive non-normal tilings without digonal disks and 19 (1,2)-transitive tilings containing digons; 4 (1,3)-transitive normal tilings and 3 (1,3)-transitive tilings containing digonal disks; 7 (1,4)-transitive normal tilings, 2 (1,4)-transitive non-normal tilings without digonal disks and 4 (1,4)-transitive tilings containing digons; 5 2-transitive normal tilings, 2 2-transitive non-normal tilings without digonal disks and 6 2-transitive tilings containing digons; 4 (2,3)-transitive normal tilings and 3 (2,3)-transitive tilings containing digonal disks; 1 (2,4)-transitive normal tiling and 4 (2,4)-transitive tilings containing digons; 3 (3,4)-transitive normal tilings; altogether there are 88 tilings (including 43 normal ones).

For the group $*532$ there are 23 (1,2)-transitive Delone classes of normal 2-isohedral tilings, 4 (1,2)-transitive non-normal tilings without digonal disks and 36 (1,2)-transitive tilings containing digons; 5 (1,4)-transitive normal tilings and 4 (1,4)-transitive tilings containing digonal disks; 3 (1,6)-transitive normal tilings, 2 (1,6)-transitive non-normal tilings without digonal disks and 4 (1,6)-transitive tilings containing digons; 3 (1,10)-transitive normal tilings, 2 (1,10)-transitive non-normal tilings without digonal disks and 4 (1,10)-transitive tilings containing digons; 22 2-transitive normal tilings, 4 2-transitive non-normal tilings without digonal disks and 28 2-transitive tilings containing digons; 7 (2,4)-transitive normal tilings and 8 (2,4)-transitive tilings containing digonal disks; 7 (2,6)-transitive normal tilings, 2 (2,6)-transitive non-normal tilings without digonal disks and 6 (2,6)-transitive tilings containing digons; 7 (2,10)-transitive normal tilings, 2 (2,10)-transitive non-normal tilings without digonal disks and 6 (2,10)-transitive tilings containing digons; 1 (4,6)-transitive normal tiling and 2 (4,6)-transitive tilings containing digonal disks; 1 (4,10)-transitive normal tiling and 2 (4,10)-transitive tilings containing digons; 3 (6,10)-transitive normal tilings; altogether there are 198 tilings (including 82 normal ones).

For the group 532 there are 7 (1,2)-transitive Delone classes of normal 2-isohedral tilings and 10 (1,2)-transitive tilings containing digonal disks; 9 (1,3)-transitive normal tilings, 2 (1,3)-transitive non-normal tilings without digonal disks and 6 (1,3)-transitive tilings containing digons; 9 (1,5)-transitive normal tilings, 2 (1,5)-transitive non-normal tilings without digonal disks and 6 (1,5)-transitive tilings containing digons; 1 (2,3)-transitive normal tiling and 2 (2,3)-transitive tilings containing digonal disks; 1 (2,5)-transitive normal tiling and 2 (2,5)-transitive tilings containing digonal disks; 3 (3,5)-transitive normal tilings; altogether there are 60 tilings (including 30 normal ones).

Remark that in the pictures of tilings for simplicity straight-line segments are drawn instead of some arcs.

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Discrete Optimal Control Problem with Varying Time of States Transactions of Dynamical System and Algorithm for its solving

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Abstract. We consider time-discrete systems with finite set of states. The starting and the final states of dynamical system are given. The discrete optimal control problem with integral-time cost criterion by a trajectory is studied. An algorithm for solving the problem with varying time of states transactions is proposed. The running time of the proposed algorithm is estimated.

Mathematics subject classification: 90C47.
Keywords and phrases: Time-discrete systems, optimal control, dynamic programming, transit-time function.

1 Introduction and Problem Formulation

In this paper we study the discrete optimal control problem with varying time of states transaction of the dynamical system. This problem generalizes the classical optimal control problem with unit time of states transactions [1, 2].

The statement of the problem is the following.

Let L be a time discrete system with a finite set of states $X \subseteq R^n$, where at every discrete moment of time $t = 0, 1, 2, \dots$ the state of L is $x(t) \in X$. The starting state $x_0 = x(0)$ and the final state x_f are fixed. Assume that the dynamical system should reach the final state x_f at the time moment $T(x_f)$ such that

$$T_1 \leq T(x_f) \leq T_2$$

where T_1 and T_2 are given. The control of the time-discrete system L at each time-moment $t = 0, 1, 2, \dots$ for an arbitrary state $x(t)$ is made by using the vector of control parameter $u(t)$ for which a feasible set $U_t(x(t))$ is given, i.e. $u(t) \in U_t(x(t))$. In addition we assume that for arbitrary t and $x(t)$ on $U_t(x(t))$ is defined an integer function

$$\tau : U_t(x(t)) \to N$$

which gives to each control $u(t) \in U_t(x(t))$ an integer value $\tau(u(t))$. This value represents the time of system's passage from the state $x(t)$ to the state $x(t+\tau(u(t)))$ if the control $u(t) \in U_t(x(t))$ has been applied at the moment t for given state $x(t)$.

Assume that the dynamics of the system is described by the following system of difference equations

$$\begin{cases} t_{j+1} &= t_j + \tau(u(t_j)), \\ x(t_{j+1}) &= g_{t_j}(x(t_j),\ u(t_j)), \\ & u(t_j) \in U_{t_j}(x(t_j)), \\ & j = 0,\ 1,\ 2,\ \dots\ , \end{cases} \tag{1}$$

where

$$t_0 = 0,\ x(t_0) = 0 \tag{2}$$

is a starting representation of the dynamical system L.

We suppose that the functions g_t and τ in (1) are known and t_{j+1} and $x(t_{j+1})$ are determined uniquely by $x(t_j)$ and $u(t_j)$ at every step $j = 0,\ 1,\ 2,\ \dots$.

Let $u(t_j),\ j = 0,\ 1,\ 2,\ \dots,$ be a control, which generates the trajectory

$$x(0),\ x(t_1),\ x(t_2),\ \dots x(t_k),\ \dots\ .$$

Then either this trajectory passes trough the final state x_f and $T(x_f) = t_k$ represents the time-moment when the final state x_f is reached or this trajectory does not pass trough x_f.

For an arbitrary control we define the quantity

$$F_{x_0,\ x_f}(u(t)) = \sum_{j=0}^{k-1} c_{t_j}(x(t_j),\ g_{t_j}(x(t_j),\ u(t_j))) \tag{3}$$

if the trajectory

$$x(0),\ x(t_1),\ x(t_2),\ \dots x(t_k),\ \dots$$

passes through the final state x_f i.e. $T(x_f) = t_k$; otherwise we put

$$F_{x_0,\ x_f}(u(t)) = \infty.$$

Here $c_{t_j}(x(t_j),\ g_{t_j}(x(t_j),\ u(t_j))) = c_{t_j}(x(t_j),\ x(t_{j+1}))$ represents the cost of system's passage from the state $x(t_j)$ to the state $x(t_{j+1})$ at the stage $[j,\ j+1]$.

We consider the following control problem:

Problem 1. To find time-moments $t_0 = 0,\ t_1,\ t_2,\ \dots,\ t_k$ and vectors of control parameters $u(t_0),\ u(t_1),\ u(t_2),\ \dots,\ u(t_{k-1})$ which satisfy conditions (1), (2) and minimize functional (3).

In the following we develop a mathematical tool for solving this problem. We show that a simple modification of time expanded network method from [3–5] allows to elaborate efficient algorithm for solving the considered problem.

2 Algorithm for solving the problem based on Dynamic Programming and Time-Expanded Network method

Here we develop the dynamic programming algorithm for solving Problem 1 in the case when T is fixed, i.e. $T_1 = T_2 = T$. The proposed algorithm can be argued in the same way as the algorithm from Section 1.

We denote by $F^*_{x_0,x(t_k)}$ the minimal integral-time cost of system's passage from the starting state $x_0 = x(0)$ to the state $x = x(t_k) \in X$ by using exactly t_k units of time. So,

$$F^*_{x_0,x(t_k)} = \sum_{j=0}^{k-1} c_{t_j}(x^*(t_j),\ g_{t_j}(x^*(t_j),\ u^*(t_j)))$$

where

$$x(0) = x^*(0),\ x^*(t_1),\ x^*(t_2),\ \dots,\ x^*(t_{k-1}),\ x^*(t_k)$$

is the optimal trajectory from $x_0 = x^*(0)$ to $x^*(t_k)$, generated by optimal control

$$u^*(0),\ u^*(t_1),\ u^*(t_2),\ \dots,\ u^*(t_{k-1})$$

where

$$t_0 = 0;$$

$$t_{j+1} = t_j + \tau(u^*(t_j)),\ j = 0,\ 1,\ 2,\ \dots,\ k-1.$$

If for given $x \in X$ there is no trajectory from x_0 to $x = x(t_k)$ such that x may be reached by using t_k units of time then we put $F^*_{x_0,\ x(t_k)} = \infty$.

For $F^*_{x_0,\ x(t_k)}$ the following recursive formula can be gained:

$$F^*_{x_0x(t_j)} = \begin{cases} \min\limits_{x(t_{j-1}) \in X^-(x(t_j))} \left\{ F^*_{x_0x(t_{j-1})} + c_{t_{j-1}}(x(t_{j-1}),\ x(t_j)) \right\} & \text{if } X^-(x(t_j)) \neq \varnothing, \\ \infty & \text{if } X^-(x(t_j)) = \varnothing, \end{cases}$$

$$j = 1,\ 2,\ \dots,$$

where

$$t_0 = 0,$$

$$F^*_{x_0x(0)} = \begin{cases} 0 & if\ x(0) = x_0, \\ \infty & if\ x(0) \neq x_0 \end{cases}$$

and

$$X^-(x(t_j)) = \{x(t_{j-1}) \in X \mid x(t_j) = g_{t_{j-1}}(x(t_{j-1}),\ u(t_{j-1})),$$
$$t_j = t_{j-1} + \tau(u(t_{j-1})),\ u(t_{j-1}) \in U_{t_{j-1}}(x(t_j))\}.$$

If $F^*_{x_0x(t)}$, $t = 0,\ 1,\ 2,\ \dots,\ T$, are known then the optimal control

$$u^*(0),\ u^*(t_1),\ u^*(t_2),\ \dots,\ u^*(t_{k-1})$$

and the optimal trajectory

$$x(0) = x^*(0),\ x^*(t_1),\ x^*(t_2),\ \ldots,\ x^*(t_{k-1}),\ x(t_k) = x(T)$$

from x_0 to x_f can be found in the following way.

Find $t_{k-1},\ u^*(t_{k-1})$ and $x^*(t_{k-1}) \in X^-(x(t_k))$ such that

$$F^*_{x_0 x^*(t_k)} = F^*_{x_0 x^*(t_{k-1})} + c_{t_{k-1}}(x^*(t_{k-1}),\ g_{t_{k-1}}(x^*(t_{k-1}),\ u^*(t_{k-1}))),$$

where $t_k = t_{k-1} + \tau(u^*(t_{k-1}))$.

After that find $t_{k-2},\ u^*(t_{k-2})$ and $x^*(t_{k-2}) \in X^-(x_{t_{k-1}})$ such that

$$F^*_{x_0 x^*(t_{k-1})} = F^*_{x_0 x^*(t_{k-2})} + c_{t_{k-2}}(x^*(t_{k-2}),\ g_{t_{k-2}}(x^*(t_{k-2}),\ u^*(t_{k-2}))),$$

where $t_{k-1} = t_{k-2} + \tau(u^*(t_{k-2}))$.

Using $k-1$ steps we find the optimal control $u^*(0),\ u^*(t_1),\ u^*(t_2),\ \ldots,\ u^*(t_{k-1})$ and the trajectory $x(0),\ x^*(t_1),\ x^*(t_2),\ \ldots,\ x^*(t_{k-1}),\ x(t_k) = x(T)$.

In order to argue the algorithm we shall use time-expanded network with a simple modification. First we ground the algorithm when $T_2 = T_1 = T$ and then we show that the general case of the problem with $T_2 > T_1$ can be reduced to the case with fixed T.

Assume that $T_2 = T_1 = T$ and construct a time-expanded network with the structure of acyclic directed graph $\overline{G} = (Y,\ \overline{E})$ where Y consists of $T+1$ copies of the set of states X corresponding to the time moments $t = 0,\ 1,\ 2,\ \ldots,\ T$. So,

$$Y = Y^0 \cup Y^1 \cup Y^2 \cup \ldots \cup Y^T \quad (Y^t \cap Y^l = \varnothing,\ t \neq l),$$

where $Y^t = (X,\ t)$ corresponds to the set of states of dynamical system at the time moment $t = 0,\ 1,\ 2,\ \ldots,\ T$. This means that $Y^t = \{(x,\ t) \mid x \in X\},\ t = 0,\ 1,\ 2,\ \ldots,\ T$ the graph $\overline{G}$ is represented in Fig. 1, where at each moment of time $t = 0,\ 1,\ 2,\ \ldots,\ T$ we can see all copies of vertex set X.

We define the set of edges $\overline{E}$ of the graph $\overline{G}$ in the following way.

If at given moment of time $t_j \in [0,\ T]$ for given state $x = x(t_j)$ of dynamical system there exists a vector of control parameters $u(t_j) \in U_{t_j}(x(t_j))$ such that

$$z = x(t_{j+1}) = g_{t_j}(x(t_j),\ u(t_j)),$$

where

$$t_{j+1} = t_j + \tau(u(t_j)),$$

then $((x,\ t_j),\ (z,\ t_{j+1})) \in \overline{E}$, i.e. in $\overline{G}$ we connect the vertex $y_j = (x,\ t_j) \in Y^{t_j}$ with the vertex $y_{j+1} = (z,\ t_{j+1})$ (see Fig. 1). To this edge $\overline{e} = ((x,\ t_j),\ (z,\ t_{j+1}))$ we associate in $\overline{G}$ a cost $c_{\overline{e}} = c_{t_j}(x(t_j),\ x(t_{j+1}))$.

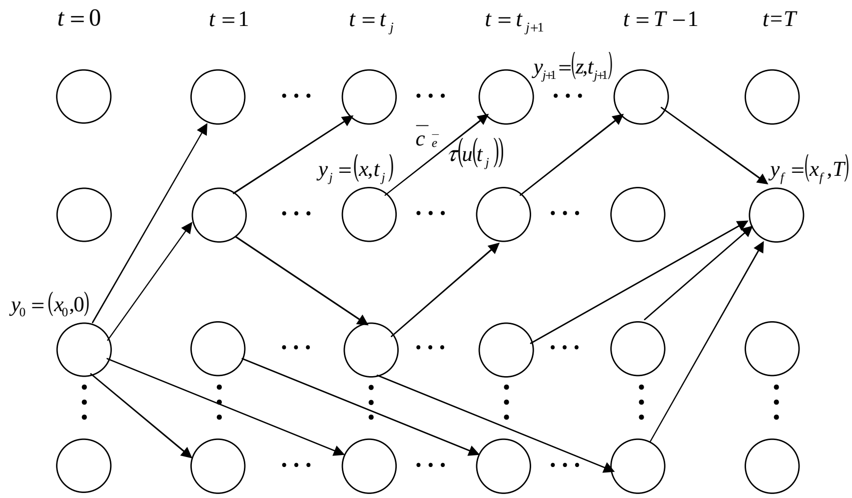

Figure 1.

The following lemma holds

Lemma 1. *Let* $u(t_0),\ u(t_1),\ u(t_2),\ \ldots,\ u(t_{k-1})$ *be a control of the dynamical system in Problem 1, which generates a trajectory*

$$x_0 = x(t_0),\ x(t_1),\ x(t_2),\ \ldots,\ x(t_k) = x_f$$

from x_0 *to* x_f*, where*

$$t_0 = 0,\ t_{j+1} = t_j + \tau(u(t_j)),\ j = 0,\ 1,\ 2,\ \ldots,\ k-1;$$
$$u(t_j) \in U_t(x(t_j)),\ j = 0,\ 1,\ 2,\ \ldots,\ k-1;$$
$$t_k = T.$$

Then in $\overline{G}$ *there exists a directed path*

$$P_{\overline{G}}(y_0,\ y_f) = \{y_0 = (x_0,\ 0),\ (x_1,\ t_1),\ (x_2,\ t_2),\ \ldots,\ (x_k,\ T) = y_f\}$$

from y_0 *to* y_f*, where*

$$x_j = x(t_j),\ j = 0,\ 1,\ 2,\ \ldots,\ k;$$

and $x(t_k) = x_f$*, i.e.* $t(x_f) = t_k = T$*. So, between the set of states of the trajectory* $x_0 = x(t_0),\ x(t_1),\ x(t_2),\ \ldots,\ x(t_k) = x_f$ *and the set of vertices of directed path* $P_{\overline{G}}(y_0,\ y_f)$ *there exists a bijective mapping*

$$(x_j,\ t_j) \Leftrightarrow x(t_j),\ j = 0,\ 1,\ 2,\ \ldots,\ k,$$

such that $x_j = x(t_j), j = 0,\ 1,\ 2,\ \ldots,\ k$*, and*

$$\sum_{j=0}^{k-1} c_{t_j}(x(t_j),\ x(t_{j+1})) = \sum_{j=0}^{k-1} \overline{c}_{(x_j,\ t_j),\ (x_{j+1},\ t_{j+1})}(t_j),$$

where $t_0 = 0,\ x_0 = x(t_0)$*, and* $x_f = x(t_k),\ t_k = T$*.*

Proof. In Problem 1 an arbitrary control $u(t_j)$ for given state $x(t_j) \in U_{t_j}(x(t_j))$ at given moment of time t_j uniquely determines the next state $x(t_{j+1})$. So, $u(t_j)$ can be identified with a unique passage $(x(t_j),\ x(t_{j+1}))$ from the state $x(t_j)$ to the state $x(t_{j+1})$. In $\overline{G} = (Y,\ \overline{E})$ this passage corresponds to a unique directed edge $((x_j,\ t_j),\ (x_{j+1},\ t_{j+1}))$ which connects vertices $(x_j,\ t_j)$ and $(x_{j+1},\ t_{j+1})$; the cost of this edge is $\overline{c}((x_j,\ t_j),\ (x_{j+1},\ t_{j+1}))(t_j) = c_{t_j}(x(t_j),\ x(t_{j+1}))$. This one-to-one correspondence between the control $u(t_j)$ at given moment of time and the directed edge $\overline{e} = ((x_j,\ t_j),\ (x_{j+1},\ t_{j+1})) \in \overline{E}$ implies the existence of bijective mapping between the set of trajectories from the starting state x_0 to the final state x_f in Problem 1 and the set of directed paths from y_0 to y_f in $\overline{G}$, which preserves the integral-time costs. □

Corollary. *If $u^*(t_j),\ j = 0,\ 1,\ 2,\ \ldots,\ k-1$ is the optimal control of the dynamical system in Problem 1, which generates a trajectory*

$$x_0 = x^*(0),\ x^*(t_1),\ x^*(t_2),\ \ldots,\ x^*(t_k) = x_f$$

from x_0 to x_f, then in $\overline{G}$ the corresponding directed path

$$P^*_{\overline{G}}(y_0,\ y_f) = \{y_0 = (x_0,\ 0),\ (x^*_1,\ t_1),\ (x^*_2,\ t_2),\ \ldots,\ (x^*_k,\ t_k) = y_T\}$$

is the minimal integral cost directed path from y_0 to y_f and vice-versa.

On the basis of the results mentioned above we can propose the following algorithm for solving Problem 1.

Algorithm. Determining the optimal solution to Problem 1 based on the time-expanded network method

1. We construct the auxiliary time-expanded network consisting of directed acyclic graph $\overline{G} = (Y,\ \overline{E})$, cost function $\overline{c}:\ \overline{E} \to R^1$ and given starting and final vertices y_0 and y_f.
2. Find in G the directed path $P^*_{\overline{G}}(y_0,\ y_f)$ from starting vertex y_0 to final vertex y_f with minimal sum of edge's costs.
3. We determine the control $u^*(t_j),\ j = 0,\ 1,\ 2,\ \ldots,\ k-1$, which corresponds to directed path $P^*_{\overline{G}}(y_0,\ y_f)$ from y_0 to y_f. Then $u^*(t_j),\ j = 0,\ 1,\ 2,\ \ldots,\ k-1$, is a solution to Problem 1.

This algorithm finds the solution to the control problem with fixed time $T(x_f) = T$ of system's passage from starting state to final one. In the case $T_1 \le T(x_f) \le T_2\ \ (T_2 > T_1)$ Problem 1 can be solved by its reducing to $T_2 - T_1 + 1$ problems with $T = T_1,\ T_1 + 1,\ \ldots,\ T_2$ and finding the best solution to these problems.

In general, if we construct the auxiliary acyclic directed graph $\overline{G} = (Y,\ \overline{E})$ with $T = T_2$ then in $\overline{G}$ the tree of optimal path from starting vertex $y_0 = (x_0,\ 0)$ to an arbitrary vertex $y = (x,\ t) \in Y$ can be found. This tree allows us to find the solution to the control problem with given starting state and an arbitrary state $x = x(t)$ with $t = 0,\ 1,\ 2,\ \dots,\ T_2$; in particular the solution to Problem 1 with $T_1 \le T(x_f) \le T_2$ can be obtained.

Denote by $GT^*_{y_0} = (Y^*,\ E^*_{y_0})$ the tree of optimal directed paths with root vertex $y_0 = (x_0,\ 0)$, which gives all optimal directed paths from y_0 to an arbitrary attainable directed vertex $y = (x,\ t) \in Y$. As we have noted this tree allows us to find in the control problem all optimal trajectory from starting state $x_0 = x(0)$ to an arbitrary reachable state $x = x(t)$ at given moment of time $t \in [0,\ T]$.

In G we can also find the tree of optimal directed paths $GT^0_{y_f} = (Y^0,\ E^0_{y_0})$ with sink vertex $y_f = (x_f,\ T)$, which gives all possible optimal directed paths from an arbitrary $y = (x,\ t) \in Y$ to sink vertex $y_f = (x_f,\ T)$. This mean that in the control problem we can find all possible optimal trajectories with starting state $x = x(t)$ at given moment of time $t \in [0,\ T]$ to the final state $x_f = x(T)$.

If the trees $GT^*_{y_0} = (Y^*,\ E^*_{y_0})$ and $GT^0_{y_f} = (Y^0,\ E^0_{y_f})$ are known then we can solve the following control problem:

To find an optimal trajectory from starting state $x_0 = x(0)$ to final state $x_f = x(T)$ such that the trajectory passes at the given moment of time $t \in [0,\ T]$ trough the state $x = x(t)$.

Finally we note that Algorithm can be simplified if we delete from $\overline{G}$ all vertices $y \in Y$ which are not attainable from y_0 and vertices $y \in Y$ for which does not exist a directed path from y to y_f. So, we should solve the auxiliary problem on a new graph $\overline{G}^0 = (\overline{Y}^0,\ \overline{E}^0)$ which is a subgraph of $\overline{G} = (Y,\ E)$.

3 The Discrete Control Problem with Cost Function of System's Passages that Depend on Transit-Time of States Transactions

In the control model from Section 1 the cost function

$$c_t(x(t),\ g_t(x(t),\ u(t))) = c_t(x(t),\ x(t+1))$$

of system's passage from the state $x = x(t)$ depends on the vector of control parameters $u(t)$. In general we may consider that the cost function of system's passage from the state $x(t)$ to state $x(t+1)$ depends also on transit-time $\tau(t)$, i.e. the cost function $c_t(x(t),\ g_t(x(t),\ u(t)),\ \tau(t)) = c_\tau(x(t),\ x(t+1),\ \tau(t))$ depends on $t,\ x(t),\ u(t)$ and $\tau(t)$.

It is easy to observe that the problem in such a general form can be solved in analogous way as the problem from Section 1 by using Algorithm with a simple modification. In the auxiliary time-expanded network the cost functions $\overline{c}_{\overline{e}}$ on edges $\overline{e}$ should be defined as follows:

$$\overline{c}_{\overline{e}} = c_{t_j}(x(t_j),\ x(t_{j+1}),\ \tau(u(t_j)))$$

So, the problem with cost functions of system's passage that depend on transit-time of states transactions can be solved by using Algorithm with the cost functions on time-expanded network defined above.

4 The Control Problem on Network with Transit-Time Functions on Edges

We extend the control problem on network from Section 4.1 by introducing the transit-time functions of states transaction on edges.

4.1 Problem Formulation

Let be given the dynamical system L with finite set of states X and given starting point $x_0 = x(0)$. Assume that system L should be transferred into the state x_f at the time moment T such that $T_1 \leq T(x_f) \leq T_2$, where T_1 and T_2 are given. We consider the control problem for which the dynamics of the system is described by directed graph $G = (X,\ E)$, where the vertices $x \in X$ correspond to the states and an arbitrary edge $e = (x,\ y) \in E$ means the possibility of the system to pass from the state x to the state y at every moment of time t. To each edge $e = (x,\ y) \in E$ is associated a transit function $\tau_e(t)$ of system's passage from the state $x = x(t)$ to the state y. This means that if at the time-moment t the system L starts to pass from the state $x = x(t)$ trough an edge $e = (x,\ y)$ then the state y is reached at the time-moment $t + \tau_e(t)$, i.e. $y = x(t + \tau_e(t))$. In addition to each edge $e(x,\ y) \in E$ is associated a cost function $c_e(t)$ that depends on time and which expresses the cost system's passage from the state $x = x(t)$ to the state $y = x(t + \tau_e(t))$.

The control on G with given transit-time functions τ_e on edges $e \in E$ is made in the following way.

For given starting state x_0 we fix $t_0 = 0$. Then select an directed edge $e_0 = (x_0,\ x_1)$ through which we transfer the system L from the state $x_0 = x(t_0)$ to the state $x_1 = x(t_1)$ at the moment of time t_1, where $t_1 = t_0 + \tau_{e_0}(0)$. If $x_1 = x_f$ then stop; otherwise we select an edge $e_1 = (x_1,\ x_2)$ and transfer the system L from the state $x_1 = x(t_1)$ at the moment of time t_1 to the state $x_2 = x(t_2)$ at the time moment $t_2 = t_1 + \tau_{e_1}(t_1)$. If $x_2 = x_f$ then stop; otherwise select an edge $e_2 = (x_2,\ x_3)$ and so on. In general, at the time moment t_{k-1} we select an edge $e_{k-1} = (x_{k-1},\ x_k)$ and transfer the system L from the state $x_{k-1} = x(t_{k-1})$ to the state $x_k = x(t_k)$ at the time-moment $t_k = t_{k-1} + \tau_{e_k}$. If $x_k = x_f$ then the integral-time cost of system passage from x_0 to x_f is

$$F_{x_0 x_k}(t_k) = \sum_{j=0}^{k-1} c_{(x(t_j),\ x(t_{j+1}))}(t_j).$$

So, at the time moment t_k the system L is transferred in the state $x_k = x_f$ with the integral-time cost $F_{x_0 x_f}(t_k)$. If $T \leq t_k \leq T_2$, we obtain an admissible control with $t_k = T(x_f)$ and integral-time cost $F_{x_0 x_f}(T(x_f))$.

We consider the following problem:

Problem 2. To find a sequence of system's transactions

$$(x_j,\ x_{j+1}) = (x(t_j),\ x(t_{j+1})),\quad t_{j+1} = t_j + \tau_{(x_j,\ x_{j+1})}(t_j),\quad j = 0,\ 1,\ 2,\ \ldots,\ k-1,$$

which transfer the system L from starting vertex (state) $x_0 = x(t_0),\ t_0 = 0$, to final vertex (state) $x_f = x_k = x(t_k)$ such that

$$T \le t_k \le T_2$$

and the integral-time cost

$$F_{x_0 x_f}(t_k) = \sum_{j=0}^{k-1} c_{(x_j,\ x_{j+1})}(t_j)$$

of system's transactions by a trajectory

$$x_0 = x(t_0),\ x(t_1),\ x(t_2),\ \dots,\ x(t_k) = x_f$$

is minimal.

4.2 Algorithm for Solving the Problem on Network with Transit-Time Functions on Edges

The algorithm from Section 2 can be specified for solving the control problem on the network with transit-time functions on the edges. Assume that $T_2 = T_1 = T$ and describe the details of the algorithm for the control problem on G.

We denote by $F^*_{x_0 x}(t_k)$ the minimal integral-time cost of system transactions from the starting state x_0 to the final state $x = x^*(t_k)$ by using t_k units of time, i.e.

$$F^*_{x_0 x}(t_k) = \sum_{j=0}^{k-1} c_{(x(t_j),\ x(t_{j+1}))}(t_j),$$

where

$$x_0 = x^*(0),\ x^*(t_1),\ x^*(t_2),\ \dots,\ x^*(t_k) = x_f$$

is an optimal trajectory from x_0 to x_f, where

$$t_{j+1} = t_j + \tau_{(x(t_j),\ x(t_{j+1}))}(t_j),\ j = 0,\ 1,\ 2,\ \dots,\ k-1.$$

It is easy to observe that the following recursive formula for $F^*_{x_0 x}(t_k)$ holds:

$$F^*_{x_0 x(t_j)}(t_j) = \min_{x(t_{j-1}) \in X^-_G(x(t_j))} \{F^*_{x_0 x(t_{j-1})}(t_{j-1}) + c_{(x(t_{j-1}),\ x(t_j))}(t_{j-1})\},$$

where

$$X^-_G(x(t_j)) = \{x = x(t_{j-1}) \mid (x(t_{j-1}),\ x(t_j)) \in E,\ t_j = t_{j-1} + \tau_{(x(t_{j-1}),\ x(t_j))}(t_{j-1})\}.$$

This means that if we start with $F^*_{x_0 x(0)}(0) = 0,\ F^*_{x_0 x(t)}(t) = \infty,\ t = 1,\ 2,\ \dots,\ t_k$, then on the basis of the recursive formula given above we can find $F^*_{x_0 x(t)}(t)$ for $t = 0,\ 1,\ 2,\ \dots,\ t_k$ for an arbitrary vertex $x = x(t)$. After that the optimal trajectory

$x_0 = x^*(0),\ x^*(t_1),\ x^*(t_2),\ \ldots,\ x^*(t_k) = x_f$ from x_0 to x_f can be found in the following way.

Fix the vertex $x^*_{k-1} = x^*(t_{k-1})$ for which

$$F^*_{x_0 x^*(t_{k-1})}(t_{k-1}) + c_{(x(t_{k-1}),\ x^*(t_k))}(t_{k-1}) =$$

$$= \min_{x(t_{k-1}) \in X^-_G(x^*(t_k))} \{F^*_{x_0 x(t_{k-1})}(t_{k-1}) + c_{(x(t_{k-1}),\ x^*(t_k))}(t_{k-1})\}.$$

Then we find the vertex $x^*(t_{k-2})$ for which

$$F^*_{x_0 x^*(t_{k-2})}(t_{k-2}) + c_{(x(t_{k-2}),\ x^*(t_{k-1}))}(t_{k-2}) =$$

$$= \min_{x(t_{k-2}) \in X^-_G(x^*(t_{k-1}))} \{F^*_{x_0 x(t_{k-2})}(t_{k-2}) + c_{(x(t_{k-2}),\ x^*(t_{k-1}))}(t_{k-2})\}.$$

After that we fix the vertex $x^*(t_{k-3})$ for which

$$F^*_{x_0 x^*(t_{k-3})}(t_{k-3}) + c_{(x(t_{k-3}),\ x^*(t_{k-2}))}(t_{k-3}) =$$

$$= \min_{x(t_{k-3}) \in X^-_G(x^*(t_{k-2}))} \{F^*_{x_0 x(t_{k-3})}(t_{k-3}) + c_{(x(t_{k-3}),\ x^*(t_{k-2}))}(t_{k-3})\}$$

and so on.

Finally we find the optimal trajectory

$$x_0 = x^*(0),\ x^*(t_1),\ x^*(t_2),\ \ldots,\ x^*(t_k) = x_f.$$

This algorithm also can be grounded on the basis of the time-expanded network method.

We give the construction which allows to reduce our problem to auxiliary one on time-expanded network. The structure of this time-expanded network corresponds to an directed graph $\overline{G} = (Y,\ \overline{E})$ without directed cycles. The set of vertices Y of $\overline{G}$ consists of $T+1$ copies of the set of vertices (states) X of the graph G corresponding to time-moments $t = 0,\ 1,\ 2,\ \ldots,\ T$, i.e.

$$Y = Y^0 \cup Y^1 \cup Y^2 \cup\ \ldots \cup Y^T \quad (Y^t \cap Y^l = \varnothing,\ t \neq l),$$

where $Y^t = (X,\ t)$. So, $Y^t = \{(x,\ t) \mid x \in X\}$, $\ t = 0,\ 1,\ 2,\ \ldots,\ T$.

We define the set of edges $\overline{E}$ of the graph $\overline{G}$ as follows: $\overline{e} = ((x,\ t_j),\ (z,\ t_{j+1})) \in \overline{E}$ if only if in G there exists a directed edge $e = (x,\ y) \in E$, where $x = x(t_j),\ z = x(t_{j+1})$, $t_{j+1} = t_j + \tau_e(t_j)$. So, in $\overline{G}$ we connect vertices $(x,\ t_j)$ and $(z,\ t_{j+1})$ with directed edge $(x(t_j),\ (z,\ t_{j+1})) \in \overline{G}$; to edge $\overline{e}$ we associate the cost $\overline{c}_{\overline{e}} = c_{(x,z)S}(t_j)$, i.e. $\overline{c}_{((x,\ t_j),\ (z,\ t_{j+1}))} = c_{(x,\ z)}(t_j)$.

On $\overline{G}$ we consider the problem of finding the directed path from $y_0 = (x_0,\ 0)$ to $y_f = (x_f,\ T)$ with minimum sum of edges cost. Basing on results from Section 2 we obtain the following result.

Lemma 2. *Let*

$$(x_j,\ x_{j+1}) = (x(t_j),\ x(t_{j+1})),\quad t_{j+1} = t_j + \tau_{(x_j,\ x_{j+1})}(t_j),\ j = 0,\ 1,\ 2,\ \ldots,\ k-1$$

be a sequence of system's transactions from the state $x_0 = x(t_0),\ t_0 = 0$, *to the state* $x_f = x_k = x(t_k),\ t_k = T$. *Then in* $\overline{G} = (Y,\ \overline{E})$ *there exists the directed path*

$$P_{\overline{G}}(y_0,\ y_f) = \{y_0 = (x_0,\ 0),\ (x_1,\ t_1),\ (x_2,\ t_2),\ \ldots,\ (x_k,\ T) = y_f\},$$

from y_0 *to* y_f, *where*

$$x_j = x(t_j),\quad j = 0,\ 1,\ 2,\ \ldots,\ k\quad (t_k = T).$$

So, between the set of vertices $\{x_0 = x(t_0),\ x(t_1),\ x(t_2),\ \ldots,\ x(t_k) = x_f\}$ *and the set of vertices of directed path* $P_{\overline{G}}(y_0,\ y_f)$ *there exists a bijective mapping*

$$(x_j,\ t_j) \Leftrightarrow x(t_j) = x_j,\quad j = 0,\ 1,\ 2,\ \ldots,\ k,$$

such that $x_j = x(t_j),\quad j = 0,\ 1,\ 2,\ \ldots,\ k$, *and*

$$\sum_{j=0}^{k-1} c_{(x_j,\ x_{j+1})}(t_j) = \sum_{j=0}^{k-1} \overline{c}_{((x_j,\ t_j),\ (x_{j+1},\ t_{j+1}))},$$

where $t_0 = 0,\ t_{j+1} = t_j + \tau_{(x_j,\ x_{j+1})}(t_j),\ j = 0,\ 1,\ 2,\ \ldots,\ k-1$.

This lemma follows as a corollary from Lemma 1.

The algorithm for solving the control problem on G is similar to Algorithm from Section 2. So, the control problem on G can be solved in the following way.

Algorithm

1. We construct the network consisting of an auxiliary graph $\overline{G} = (Y,\ \overline{E})$, cost function $\overline{c}:\ \overline{E} \to R$ and given starting and final states $y_0 = (x_0,\ 0),\ y_f = (x_f,\ t)$.

2. Find in G the directed path $P^*_{\overline{G}}(y_0,\ y_f)$ from y_0 to y_f with minimal sum of edges cost.

3. Determine the vertices $x_j = x(t_j),\ j = 0,\ 1,\ 2,\ \ldots,\ k$, which correspond to vertices $(x_j,\ t_j)$ of a directed path $P^*_G(y_0,\ y_f)$ from y_0 to y_f. Then $x_0 = x(0),\ x_1 = x(t_1),\ x_2 = x(t_2),\ \ldots,\ x_k = x(t_k) = x_f$ represent the optimal trajectory from x_0 to x_f in the control problem G.

Remark 1. Algorithm can be modified for solving the optimal control problem on a network when the cost function on edges $e \in E$ depends not only on time t but also depends on transit-time $\tau_e(t)$ of system's passage trough the edge $e = (x(t),\ x(t + \tau_e(t)))$. So, Algorithm can be used for solving the control problem when to each edge $e = (x,\ z) \in E$ is given a cost function $c_{(x,\ z)}(t,\ \tau_{(x,\ z)}(t))$ that depends on time t and on transit-time $\tau_{(x,\ z)}(t)$. The modification of the algorithm for solving the control problem on network in such general form can be made in the same way as the modification of Algorithm 1 for the problem from Section 3 This means that the cost functions $\overline{c}_e$ on the edges $\overline{e} = ((x,\ t_j),\ (z,\ t_{j+1}))$ of the graph $\overline{G}$ should be defined as follows:

$$\overline{c}_{((x,\ t_j),\ (z,\ t_{j+1}))} = c_{(x,\ z)}(t_j,\ \tau_{(x,\ z)}(t_j)).$$

Remark 2. Algorithm can be simplified if we delete from $\overline{G}$ all vertices $y \in Y$ which are not attainable from y_0 and all vertices for which there is no directed path from y to x_f. So, the problem may be solved on a simplified graph $\overline{G}^0 = (\overline{Y}^0,\ \overline{E}^0)$.

The proposed approach for studying and solving discrete optimal control problems can be developed also for the multi-objective control problems from [3–6].

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Measure of quasistability of a vector integer linear programming problem with generalized principle of optimality in the Helder metric

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Abstract. A vector integer linear programming problem is considered, principle of optimality of which is defined by a partitioning of partial criteria into groups with Pareto preference relation within each group and the lexicographic preference relation between them. Quasistability of the problem is investigated. This type of stability is a discrete analog of Hausdorff lower semicontinuity of the many-valued mapping that defines the choice function. A formula of quasistability radius is derived for the case of metric $l_p, 1 \leq p \leq \infty$, defined in the space of parameters of the vector criterion. Similar formulae had been obtained before only for combinatorial (boolean) problems with various kinds of parametrization of the principles of optimality in the cases of l_1 and l_∞ metrics [1–4], and for some game theory problems [5–7].

Mathematics subject classification: 90C10, 90C29, 90C31.
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1 Basic Definitions and Properties

Let us consider m-criteria integer linear programming problem with n variables:

$$Cx = (C_1x, C_2x, \dots, C_mx)^T \to \min_{x \in X},$$

where $C = [c_{ij}]_{m\times n} \in \mathbf{R}^{m\times n}$, $m, n \in \mathbf{N}$, C_i denotes the i-th row of matrix C, $i \in N_m = \{1, 2, \dots, m\}$, X is the finite set of solutions from $\mathbf{Z}^n$, $|X| > 1$, $x = (x_1, x_2, \dots, x_n)^T$.

We put for this problem parametric principle of optimality.

Let $s \in N_m$, $N_m = \bigcup_{r\in N_s} I_r$ be the partitioning of the set N_m into s nonempty disjoint subsets (groups), i. e. $I_r \neq \emptyset$, $r \in N_s$; $p \neq q \Rightarrow I_p \cap I_q = \emptyset$. In the criteria space $\mathbf{R}^m$ we put the binary relation of the strict preference $\Omega^m(I_1, I_2, \dots, I_s)$ between different vectors $y = (y_1, y_2, \dots, y_m)$ and $y' = (y'_1, y'_2, \dots, y'_m)$ in correspondence to any partition $(I_1, I_2, \dots, I_s)$, as follows:

$$y\ \Omega^m(I_1, I_2, \dots, I_s)\ y' \quad \Leftrightarrow \quad y_{I_k} \underset{P}{\succ} y'_{I_k},$$

where $k = \min\{i \in N_s : \ y_{I_i} \neq y'_{I_i}\}$; y_{I_k} and y'_{I_k} are projections of the vectors y and y' onto axises of the $\mathbf{R}^n$ space with indexes from group I_k; $\underset{P}{\succ}$ be the relation which generates Pareto principle of optimality:

$$y_{I_k} \underset{P}{\succ} y'_{I_k} \Leftrightarrow y_{I_k} \geq y'_{I_k} \ \& \ y_{I_k} \neq y'_{I_k}.$$

The introduced binary relation $\Omega^m(I_1, I_2, \ldots, I_s)$ sets such order of groups of criteria in which any previous group is more important than all the following ones. Consequently this relation generates one set of $(I_1, I_2, \ldots, I_s)$-effective (or, otherwise, generalized effective) solutions according to the rule

$$G^m(C, I_1, I_2, \ldots, I_s) = \{x \in X : \forall x' \in X \quad (Cx \ \overline{\Omega^m(I_1, I_2, \ldots, I_s)} \ Cx')\},$$

where $\overline{\Omega^m(I_1, I_2, \ldots, I_s)}$, as usual, means the negation of the binary relation $\Omega^m(I_1, I_2, \ldots, I_s)$.

It is obvious that the set $G^m(C, N_m)$ $(s = 1)$ of the N_m-effective solutions is Pareto set, i. e. the set of effective solutions

$$P^m(C) = \{x \in X : \ \forall x' \in X \ (Cx \ \underset{P}{\overline{\succ}} \ Cx')\}.$$

It is easy to understand that the set of $(\{1\}, \{2\}, \ldots, \{m\})$-effective solutions $G^m(C, \{1\}, \{2\}, \ldots, \{m\})$ $(s = m)$ is equal to the set of lexicographic optima

$$L^m(C) = \{x \in X : \ \forall x' \in X \ (Cx \ \underset{L}{\overline{\succ}} \ Cx')\}.$$

Here $\underset{L}{\succ}$ is a binary relation which sets lexicographic order:

$$y \underset{L}{\succ} y' \Leftrightarrow y_k > y'_k,$$

where $k = \min\{i \in N_m : \ y_i \neq y'_i\}$, $y = (y_1, y_2, \ldots, y_m)$, $y' = (y'_1, y'_2, \ldots, y'_m)$.

Thus in this case by the parametrization of the principle of optimality we mean introducing a characteristic of binary relation of preference that allows us to relate the well-known lexicographic and Pareto principles of optimality.

It is easy to show that the binary relation $\Omega^m(I_1, I_2, \ldots, I_s)$ is antireflexive, asymmetric, transitive, and hence it is cyclic. And since the set X is finite, the set $G^m(C, I_1, I_2, \ldots, I_s)$ is non-empty for any matrix C and any partitioning $(I_1, I_2, \ldots, I_s), s \in N_m$, of the set N_m.

By $Z^m(C, I_1, I_2, \ldots, I_s)$ problem we understand the problem of finding the set $G^m(C, I_1, I_2, \ldots, I_s)$.

The following properties directly follow from the introduced definitions.

Property 1. $G^m(C, I_1, I_2, \ldots, I_s) \subseteq P_1(C) \subseteq X$, *where*

$$P_1(C) = \{x \in X : \ \forall x' \in X \ (C_{I_1}x \ \underset{P}{\overline{\succ}} \ C_{I_1}x')\}.$$

Let us define C_{I_1} as a submatrix of the matrix C, consisting of the rows of matrix C with numbers from the group I_1.

Property 2. *If* $C_{I_1}x \underset{P}{\succ} C_{I_1}x'$, *then* $Cx\ \Omega^m(I_1, I_2, \ldots, I_s)\ Cx'$.

Property 3. *If* $Cx\ \Omega^m(I_1, I_2, \ldots, I_s)\ Cx'$, *then* $C_{I_1}x \geq C_{I_1}x'$.

Property 4. *The solution* $x \notin G^m(C, I_1, I_2, \ldots, I_s)$ *if and only if there exists such solution* x' *that* $Cx\ \Omega^m(I_1, I_2, \ldots, I_s)\ Cx'$.

Property 5. *The solution* $x \in G^m(C, I_1, I_2, \ldots, I_s)$ *if and only if for any solution* x' *the relation* $Cx\ \overline{\Omega^m(I_1, I_2, \ldots, I_s)}\ Cx'$ *is true.*

Property 6. $S_1(C) \subseteq G^m(C, I_1, I_2, \ldots, I_s)$, *where*

$$S_1(C) = \{x \in P_1(C) :\ \forall x' \in X \setminus \{x\} \quad \big(C_{I_1}x \neq C_{I_1}x'\big)\}.$$

Indeed, let $x \in S_1(C)$ and $x \notin G^m(C, I_1, I_2, \ldots, I_s)$. Then according to Property 4 there exists a solution x' such that

$$Cx\, \Omega^m(I_1, I_2, \ldots, I_s)\, Cx'.$$

Hence due to Property 3 we have $C_{I_1}x \geq C_{I_1}x'$. Taking into account the inclusion $x \in P_1(C)$, we obtain $C_{I_1}x = C_{I_1}x'$, i. e. $x \notin S_1(C)$, which contradicts the assumption.

It is obvious that the set $S_1(C)$ is nonempty. Directly from the definition of sets $P_1(C)$ and S_1 we obtain

Property 7. *For all* $x \in S_1(C)$ *for all* $x' \in X \setminus \{x\}$ *there exists such* $i \in I_1$ *that* $\big(C_ix' > C_ix\big)$.

For all natural number k in the real space $\mathbf{R}^k$ we define a Helder metric (l_p)

$$||y||_p = \Big(\sum_{j \in N_k} |y_j|^p\Big)^{\frac{1}{p}}, 1 \leq p < \infty.$$

Let us also use l_∞ metric:

$$||y||_\infty = \max\{|y_j| :\ j \in N_k\}.$$

It is known that l_p metric in the $\mathbf{R}^k$ and l_q metric in the conjugate space $(\mathbf{R}^k)^*$ are connected by the equality

$$\frac{1}{p} + \frac{1}{q} = 1,$$

where $1 < p < \infty$; in addition, $q = 1$ if $p = \infty$, and $q = \infty$ if $p = 1$. We suppose that the range of variation of p and q is $[1, \infty]$, and numbers p and q are connected by the above conditions. Then according to the Helder inequality for any index $i \in N_m$ is fair the inequality

$$C_ix \leq ||C_i||_p||x||_q. \tag{1}$$

Let us define an operator of projection of the vector $a = (a_1, a_2, \dots, a_n) \in \mathbf{R}^n$ onto nonnegative orthant:

$$a^+ = [a]^+ = (a_1^+, a_2^+, \dots, a_n^+),$$

where $a_i^+ = [a_i]^+ = \max\{0, a_i\}$. Then sign "+" over vector means a vector with positive coordinates and zero instead of negative coordinates.

Property 8. *If for some row $i \in N_m$ of the matrices $C, C' \in \mathbf{R}^{m\times n}$ the inequality*

$$(C_i + C'_i)(x' - x) \le 0, \tag{2}$$

is satisfied then for any number $p \in [1, \infty]$ the inequality

$$[C_i(x' - x)]^+ \le ||C'_i||_p ||x' - x||_q \tag{3}$$

is fair.

Really, with $C_i(x' - x) \le 0$ the inequality (3) is evident. If $C_i(x' - x) > 0$, then from the condition (2) and Helder inequality (1) it follows

$$[C_i(x' - x)]^+ = C_i(x' - x) \le -C'_i(x' - x) \le ||C'_i||_p ||x' - x||_q.$$

Property 9. *If $p > 1$ and vectors $y, y' \in \mathbf{R}^m$ are such that $y_j = {y'}_j^{q-1}$, $j \in N_m$, then*

$$||y||_p = ||y'||_q^{q-1}.$$

Indeed, according to $p = \infty$ ($q = 1$) Property 9 is trivial, taking into account $1 < p < \infty$ obtain

$$||y||_p = \Big(\sum_{j\in N_m} |y'_j|^{p(q-1)}\Big)^{\frac{1}{p}} = \Big(\sum_{j\in N_m} |y'_j|^q\Big)^{\frac{1}{p}} = ||y'||_q^{\frac{q}{p}} = ||y'||_q^{q-1}.$$

By analogy with [1–5] the problem $Z^m(C, I_1, I_2, \dots, I_s)$, $n \ge 1$, is called quasistabile if

$$\Xi_p := \{\varepsilon > 0 : \forall\, C' \in \Psi_p(\varepsilon)\ (G^m(C, I_1, I_2, \dots, I_s) \subseteq G^m(C+C', I_1, I_2, \dots, I_s))\} \ne \emptyset,$$

where

$$\Psi_p(\varepsilon) = \{C' \in \mathbf{R}^{m\times n} :\ ||C'||_p < \varepsilon\}$$

is perturbing matrices set. Under matrix metric we understand metric of the vector that consist from its elements.

Thereby, quasistability of the $Z^m(C, I_1, I_2, \dots, I_s)$ problem is the discrete analog of Hausdorff lower semicontinuity at the point C (for fixed p and partitioning method of the N_m into groups) of the many-valued mapping

$$G^m : \mathbf{R}^{m\times n} \to 2^X,$$

which puts into correspondence to any matrix C the $G^m(C, I_1, I_2, \dots, I_s)$ set. It is evident that the quasistability property is invariant relative to l_p metric, because all metrics in a finite linear space are equivalent ([6], p. 166).

According to above, let's define the quasistability radius of the problem $Z^m(C, I_1, I_2, \dots, I_s)$ by the next number in the space determined by the metric l_p:

$$\rho_p^m(C,\mathcal{I}) = \begin{cases} \sup \Xi_p(C,\mathcal{I}), & \text{if } \Xi_p(C,\mathcal{I}) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the quasistability radius of the problem $Z^m(C, I_1, I_2, \dots, I_s)$ specifies the limit of the element perturbations by matrix C in the space $\mathbf{R}^{m\times n}$ with l_p metric, such that the set of generalized effective solutions is preserved.

2 Lemmas

For any different solutions x and x' we define the fraction:

$$\gamma(x,x') = \frac{||[C_{I_1}(x'-x)]^+||_p}{||x'-x||_q}.$$

Lemma 1. *If*

$$\gamma(x,x') \geq \varphi > 0, \tag{4}$$

then the following relation holds for any perturbing matrix $C' \in \Psi_p(\varphi)$

$$(C+C')x \; \overline{\Omega^m(I_1,\, I_2, \dots,\, I_s)} \; (C+C')x'.$$

Proof. Let exist such matrix $C' \in \Psi_p(\varphi)$ that $(C+C')x\, \Omega^m(I_1, I_2, \dots, I_s)\, (C+C')x'$. Then by virtue of Property 3 for any index $i \in I_1$ inequality (2) holds. Therefore due to Property 8 for any index $i \in I_1$ and any $p \in [1,\infty]$ inequality (3) holds. Hence, when $1 \leq p < \infty$, we derive

$$||[C_{I_1}(x'-x)]^+||_p = \Big(\sum_{i\in I_1}([C_i(x'-x)]^+)^p\Big)^{\frac{1}{p}} \leq \Big(\sum_{i\in I_1}||C'_i||_p^p||x'-x||_q^p\Big)^{\frac{1}{p}} =$$

$$= \Big(\sum_{i\in I_1}||C'_i||_p^p\Big)^{\frac{1}{p}}||x'-x||_q = ||C'_{I_1}||_p||x'-x||_q < \varphi||x'-x||_q,$$

and when $p = \infty$, we derive

$$||[C_{I_1}(x'-x)]^+||_\infty = \max_{i\in I_1}[C_i(x'-x)]^+ \leq ||x'-x||_1 \max_{i\in I_1}||C'_i||_\infty =$$

$$= ||C'_{I_1}||_\infty ||x'-x||_1 < \varphi||x'-x||_1.$$

Inequalities from above contradict (4) and Lemma 1 holds.

Lemma 2. *Let $x \in G^m(C, I_1, I_2, \dots, I_s)$, $x' \in X \setminus x$ and components of vector $b = (b_1, b_2, \dots, b_m)$ satisfy the following:*

$$b_i||x'-x||_q > [C_i(x'-x)]^+,\ i \in I_1, \tag{5}$$

$$b_i = 0,\ i \in N_m \setminus I_1.$$

Then for any number $\varepsilon > ||b||_p$ there exists such matrix $C^* \in \Psi_p(\varepsilon)$ that

$$(C+C^*)x\ \Omega^m(I_1,\, I_2, \dots,\, I_s)\ (C+C^*)x'. \tag{6}$$

Proof. If $p > 1$, then let's define elements of matrix $C^* = [c^*_{ij} \in \mathbf{R}^{m\times n}]$ by the formula

$$c^*_{ij} = \begin{cases} b_i \ \mathrm{sign}(x_j - x'_j)\dfrac{|x'_j - x_j|^{q-1}}{||x' - x||_q^{q-1}}, & \text{if } \ i \in I_1, \quad j \in N_n, \\ 0, & \text{if } \ i \in N_m \setminus I_1, \quad j \in N_n, \end{cases}$$

Else if $p = 1$, then let's fix the index $s = \arg\max\{|x'_j - x_j| : \ j \in I_1\}$ and define elements of matrix C^* by formula

$$c^*_{ij} = \begin{cases} b_i \ \mathrm{sign}(x_j - x'_j), & \text{if } i \in I_1, j = s, \\ 0 & \text{otherwise.} \end{cases}$$

It is evident that $||C^*||_1 = ||b||_1$. In accordance with Property 9 it is easy to show that $||C^*||_p = ||b||_p$ when $p > 1$. Thus, $C^* \in \Psi_p(\varepsilon)$. By the construction of the matrix C^* for any index $i \in I_1$ the equality

$$C^*_i(x' - x) = \begin{cases} -b_i||x' - x||_\infty, & \text{if } p = 1, \\ -b_i||x' - x||_q^{1-q} \sum_{j\in I_1} |x'_j - x_j|^q, & \text{if } 1 < p \leq \infty, \end{cases}$$

holds. Then

$$C^*_i(x' - x) = -b_i||x' - x||_q, \qquad i \in I_1.$$

Therefore under (5) the relations

$$(C_i + C^*_i)(x' - x) = C_i(x' - x) + C^*_i(x' - x) =$$

$$= C_i(x' - x) - b_i||x' - x||_q \leq [C_i(x' - x)]^+ - b_i||x' - x||_q < 0, \qquad i \in I_1$$

are correct, and thereby

$$(C_{I_1} + C^*_{I_1})x \underset{P}{\succ} (C_{I_1} + C^*_{I_1})x'.$$

Hence in accordance with Property 2 relation (6) is true.

Lemma 2 is proved.

3 Theorem

Theorem 1. *For any $1 \leq s \leq m$, $1 \leq p \leq \infty$ and any partitioning of the set N_m into s subsets the quasistability radius $\rho^m_p(C, I_1, I_2, \dots, I_s)$ of a vector integer linear programming problem $Z^n(C, I_1, I_2, \dots, I_s)$ is expressed by the formula*

$$\rho^m_p(C, I_1, I_2, \dots, I_s) = \min_{x\in G^m(C, I_1, I_2,\dots, I_s)} \ \min_{x'\in X\setminus\{x\}} \frac{||[C_{I_1}(x' - x)]^+||_p}{||x' - x||_q}. \tag{7}$$

Proof. The right side of formula (7) we define by φ.

At first we prove the inequality

$$\rho_p^m(C, I_1, I_2, \dots, I_s) \ge \varphi. \tag{8}$$

Without loss of generality assume that $\varphi > 0$ (otherwise inequality (8) is obvious). From the definition of the number $\gamma(x, x')$ it follows that for any solutions $x \in G^m(C, I_1, I_2, \dots, I_s)$ and $x' \neq x$ the inequality (4) holds. Taking into account Lemma 1 we obtain $\forall C' \in \Psi_p(\varphi)\ \forall x \in G^m(C, I_1, I_2, \dots, I_s),\ \forall x' \in X$

$$(\,(C + C')x\ \overline{\Omega^m(I_1, I_2, \dots, I_s)}\ (C + C')x'\,),$$

Therefore by virtue of Property 5 any solution $x \in G^m(C, I_1, I_2, \dots, I_s)$ belongs to the set $G^m(C + C', I_1, I_2, \dots, I_s)$. Thus we conclude

$$\forall C' \in \Psi_p(\varphi)\ \ (G^m(C, I_1, I_2, \dots, I_s) \subseteq G^m(C + C', I_1, I_2, \dots, I_s)),$$

this formula proves (8).

It remained to prove that $\rho_p^m(C, I_1, I_2, \dots, I_s) \le \varphi$. Let $\varepsilon > \varphi$ and solutions $x \in G^m(C, I_1, I_2, \dots, I_s)$ and $x' \neq x$ would be in accordance with (7) such that

$$\varphi||x' - x||_q = ||[C_{I_1}(x' - x)]^+||_p.$$

Then, taking into account continuous dependence of vector metric on its components, we derive that there exists such vector $b = (b_1, b_2, \dots, b_m)$ with components

$$b_i > [C_i(x' - x)]^+||x' - x||_q^{-1},\ \ i \in I_1,$$

$$b_i = 0,\ \ i \in N_m \setminus I_1,$$

that $\varepsilon > ||b||_p > \varphi$. Then according to Lemma 2 the matrix $C^* \in \Psi_p(\varepsilon)$ exists and condition (6) holds. Hence taking into account (4) $x \notin G^m(C + C^*, I_1, I_2, \dots, I_s)$. Then

$$\forall \varepsilon > \varphi \quad \exists C^* \in \Psi_p(\varepsilon) \quad (\,G^m(C, I_1, I_2, \dots, I_s) \not\subseteq G^m(C + C^*, I_1, I_2, \dots, I_s)\,),$$

which proves $\rho_p^m(C, I_1, I_2, \dots, I_s) \le \varphi$.

Note that before similar to (7) formulae had been obtained only for combinatorial (boolean) problems with various kinds of parametrization of the optimality maxima in the cases of l_1 and l_∞ metrics [1–3, 5, 7], and for some game theory problems [8–11].

4 Corollaries

The theorem implies several of corollaries.

If $s = 1$, then the theorem transforms to the following corollary

Corollary 1. *The quasistability radius of a vector integer linear programming problem $Z^m(C, N_m)$, $m \geq 1$, of finding Pareto set $P^m(C)$ is expressed by the formula*

$$\rho_p^m(C, N_m) = \min_{x \in P^m(C)} \min_{x' \in X \setminus \{x\}} \frac{||[C(x'-x)]^+||_p}{||x'-x||_q}, \ \ 1 \leq p \leq \infty.$$

This formula easy transforms into quasistability radius formula of a vector integer linear programming problem in the metric l_∞ [12].

When $s = m$ the theorem transforms to the following corollary

Corollary 2. *For any $m \geq 1$ and $1 \leq p \leq \infty$ the quasistability radius of a vector integer linear programming problem of finding lexicographic optima set $L^m(C)$ is expressed by the formula*

$$\rho_p^m(C, \{1\}, \{2\}, \dots, \{m\}) = \min_{x \in L^m(C)} \min_{x' \in X \setminus \{x\}} \frac{C_1(x'-x)}{||x'-x||_q}.$$

Particular case of this formula is well-known formula of quasistability radius of a vector integer linear programming problem with lexicographic principle of optimality in the case of metric l_∞ [13].

Corollary 3. *For any partitioning $(I_1, I_2, \dots, I_s)$ of the set N_m, into s subsets, $s \in N_m$, the following statements are equivalent for a problem $Z^m(C, I_1, I_2, \dots, I_s))$, $m \geq 1$:*

(i) *the problem $Z^m(C, I_1, I_2, \dots, I_s)$ is quasistable,*

(ii) $\forall\ x \in G^m(C, I_1, I_2, \dots, I_s) \quad \forall\ x' \in X \setminus \{x\} \quad \exists\ i \in I_1 \quad \big(C_i x' > C_i x\big)$,

(iii) $G^m(C, I_1, I_2, \dots, I_s) = S_1(C)$.

Proof. The equivalence of statements (i) and (ii) follows directly from the theorem.

The implication (ii) $\Rightarrow$ (iii) is proved by contradiction. Suppose that (ii) holds but (iii) does not.

From Properties 1 and 6 we obtain

$$S_1(C) \subseteq G^m(C, I_1, I_2, \dots, I_s) \subseteq P_1(A).$$

Then (since $G^m(C, I_1, I_2, \dots, I_s) \neq S_1(C)$ is assumed) there exists such solution $x \in G^m(C, I_1, I_2, \dots, I_s)) \subseteq P_1(A)$ that $x \notin S_1(C)$. It indicates that there exists a solution $x' \in P_1(C)$ such that

$$x' \neq x, \quad C_{I_1} x = C_{I_1} x',$$

which contradicts the statement (ii).

The implication (iii) $\Rightarrow$ (i) is obvious by virtue of Property 7.

Corollary 3 is correct.

From Corollary 3 we easily obtain the following attendant results (see, for example, [14, 15]).

Corollary 4. *The problem $Z^m(C, N_m)$, $m \geq 1$ of finding Pareto set $P^m(C)$ is quasistable if and only if $S^m(C)$ and $P^m(C)$ coincide.*

Here $S^m(C)$ is Smale set [16], i. e. set of strictly efficient solutions:

$$S^m(C) = \{x \in P^m(C) : \forall x' \in X \setminus x \ \ (Cx \neq Cx')\}.$$

From Corollary 3 we also obtain

Corollary 5. *[13]. The problem $Z^m(C, \{1\}, \{2\}, \dots, \{m\})$, $m \geq 1$, of finding the lexicographically optimal solutions set $L^m(C)$ is quasistable if and only if*

$$|L^m(C)| = |Arg \min\{C_1 x : x \in X\}| = 1.$$

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Moments of the Markovian random evolutions in two and four dimensions

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Abstract. Closed-form expressions for the mixed moments of the Markovian random evolutions in the spaces $\mathbb{R}^2$ and $\mathbb{R}^4$, are obtained. The moments of the Euclidean distance from the origin at any time $t > 0$ are also presented.

Mathematics subject classification: 82C70, 60K35, 60K37, 82B41.
Keywords and phrases: Random motion, finite speed, random evolution, random flight, moments, transport process, explicit distribution, hypergeometric function.

1 Introduction

The symmetrical Markovian random evolution $\mathbf{X}(t)$ in the Euclidean spaces $\mathbb{R}^m$ of the lower dimensions $m = 2$, $m = 3$ and $m = 4$ have thoroughly been studied in [3–8]. In these works the distributions of $\mathbf{X}(t)$, $t \geq 0$, were explicitly obtained by different methods. The most difficult case $m = 3$ was examined in [8]. The distribution obtained has a very complicated integral form which seemingly cannot be expressed in terms of elementary functions.

In contrast to the three-dimensional case, the distributions of both the two- and four-dimensional random evolutions have fairly simple analytical forms (see [3, 5–7] for the planar random evolution, and [4] for the four-dimensional case). The reason of such a considerable difference in the forms of the distributions in different dimensions is not clear at all. A general method of studying the multidimensional random evolutions has recently been suggested in [2], however the closed-form expressions for the transition density of the motion cannot, apparently, been obtained in arbitrary higher dimension.

However, despite the fact that the distributions of random evolutions in the spaces $\mathbb{R}^2$ and $\mathbb{R}^4$ were obtained in the explicit forms, such an interesting and useful characteristic of the processes as their moments was not studied so far.

In this paper we obtain the closed-form expressions for the moments of the symmetrical Markovian random evolution $\mathbf{X}(t)$ in the dimensions $m = 2$ and $m = 4$. The moments of the Euclidean distance from the origin $\|\mathbf{X}(t)\|$ are also presented. We note that these moments are expressed in terms of special functions, namely, the Bessel and Struve functions in the planar case and the degenerated hypergeometric function and incomplete gamma-function for the four-dimensional random evolution.

2 Moments of the Planar Random Evolution

Consider the symmetrical planar random evolution performed by a particle that starts from the origin $\mathbf{0} = (0,0)$ of the plane $\mathbb{R}^2$ at time $t = 0$ and moves with constant finite speed c. The initial direction is a two-dimensional random vector with uniform distribution on the unit circumference

$$S_1^2 = \left\{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : \ x_1^2 + x_2^2 = 1 \right\}.$$

The particle changes its direction at random instants which form a homogeneous Poisson process of rate $\lambda > 0$. At these moments it instantaneously takes on the new direction with uniform distribution on S_1^2, independently of its previous motion.

Let $\mathbf{X}(t) = (X_1(t), X_2(t))$ denote the particle's position at an arbitrary time $t \geq 0$. At any time $t > 0$ the particle, with probability 1, is located in the planar disc of radius ct

$$\mathbf{B}_{ct}^2 = \left\{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : \ x_1^2 + x_2^2 \leq c^2 t^2 \right\}.$$

Consider the distribution $\Pr\left\{\mathbf{X}(t) \in d\mathbf{x}\right\}, \ \mathbf{x} \in \mathbf{B}_{ct}^2, \ t \geq 0$, where $d\mathbf{x}$ is the infinitesimal area in the plane $\mathbb{R}^2$. This distribution consists of two components. The singular component corresponds to the case when no Poisson event occurs in the interval $(0, t)$ and is concentrated on the circumference

$$S_{ct}^2 = \partial \mathbf{B}_{ct}^2 = \left\{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : \ x_1^2 + x_2^2 = c^2 t^2 \right\}.$$

In this case the particle is located on S_{ct}^2 and the probability of this event is

$$\Pr\left\{ \mathbf{X}(t) \in S_{ct}^2 \right\} = e^{-\lambda t}.$$

If at least one Poisson event occurs, the particle is located strictly inside the disc $\mathbf{B}_{ct}^2$, and the probability of this event is

$$\Pr\left\{ \mathbf{X}(t) \in \operatorname{int} \mathbf{B}_{ct}^2 \right\} = 1 - e^{-\lambda t}.$$

The part of the distribution $\Pr\left\{\mathbf{X}(t) \in d\mathbf{x}\right\}$ corresponding to this case is concentrated in the interior

$$\operatorname{int} \mathbf{B}_{ct}^2 = \left\{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : \ x_1^2 + x_2^2 < c^2 t^2 \right\},$$

and forms its absolutely continuous component. Therefore there exists the density of the absolutely continuous component of the distribution $\Pr\left\{\mathbf{X}(t) \in d\mathbf{x}\right\}$.

The principal known result states that the complete density $f(\mathbf{x}, t)$ of $\mathbf{X}(t)$ has the following form (see [5], formula (21)):

$$f(\mathbf{x}, t) = \frac{e^{-\lambda t}}{2\pi c t} \, \delta(c^2 t^2 - \|\mathbf{x}\|^2) + \frac{\lambda}{2\pi c} \frac{\exp\left(-\lambda t + \frac{\lambda}{c} \sqrt{c^2 t^2 - \|\mathbf{x}\|^2} \right)}{\sqrt{c^2 t^2 - \|\mathbf{x}\|^2}} \, \Theta(ct - \|\mathbf{x}\|), \quad (1)$$

$$\mathbf{x} = (x_1, x_2) \in \mathbf{B}^2_{ct}, \qquad \|\mathbf{x}\|^2 = x_1^2 + x_2^2, \qquad t \geq 0,$$

where $\delta(x)$ is the Dirac delta-function and $\Theta(x)$ is the Heaviside function. The first term in (1) represents the singular part, whereas the second term gives the absolutely continuous part of the density.

Let $\mathbf{q} = (q_1, q_2)$ be the two-multi-index. In this section we are interested in the mixed moments of the process $\mathbf{X}(t)$:

$$\mathrm{E}\mathbf{X}^{\mathbf{q}}(t) = \mathrm{E}X_1^{q_1}(t) X_2^{q_2}(t), \qquad q_1 \geq 1, \; q_2 \geq 1.$$

The mixed moments of $\mathbf{X}(t)$ are given by the following theorem.

Theorem 1. *For any $q_1, q_2 \geq 1$ the following formula holds*

$$E\mathbf{X}^{\mathbf{q}}(t) = \begin{cases} \dfrac{e^{-\lambda t}}{\pi} \, (ct)^{q_1+q_2} \, B\left(\dfrac{q_1+1}{2}, \dfrac{q_2+1}{2} \right) + \\ + \dfrac{e^{-\lambda t}}{\sqrt{\pi}} \left(\dfrac{2}{\lambda t} \right)^{(q_1+q_2-1)/2} (ct)^{q_1+q_2} \, \Gamma\left(\dfrac{q_1+1}{2} \right) \Gamma\left(\dfrac{q_2+1}{2} \right) \times \\ \times \left[I_{(q_1+q_2+1)/2}(\lambda t) + \mathbf{L}_{(q_1+q_2+1)/2}(\lambda t) \right], \\ \qquad\qquad\qquad \text{if } q_1 \text{ and } q_2 \text{ are even}, \\ 0, \qquad\qquad\qquad \text{otherwise}, \end{cases} \tag{2}$$

where

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k! \, \Gamma(\nu + k + 1)} \left(\frac{z}{2} \right)^{2k+\nu} \tag{3}$$

is the Bessel function of order ν with imaginary argument,

$$\mathbf{L}_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma\left(k + \frac{3}{2}\right) \, \Gamma\left(\nu + k + \frac{3}{2}\right)} \left(\frac{z}{2} \right)^{2k+\nu+1} \tag{4}$$

is the Struve function of order ν and $B(x, y)$ is the beta-function.

Proof. According to (1) we have

$$\begin{aligned}
\mathrm{E}\mathbf{X}^{\mathbf{q}}(t) = \frac{e^{-\lambda t}}{2\pi ct} & \iint\limits_{x_1^2+x_2^2=c^2t^2} x_1^{q_1}\ x_2^{q_2}\ dx_1\ dx_2 + \\
& + \frac{\lambda e^{-\lambda t}}{2\pi c} \iint\limits_{x_1^2+x_2^2\le c^2t^2} x_1^{q_1}\ x_2^{q_2}\ \frac{\exp\left(\frac{\lambda}{c}\sqrt{c^2t^2-(x_1^2+x_2^2)}\right)}{\sqrt{c^2t^2-(x_1^2+x_2^2)}}\ dx_1\ dx_2 = \\
& = \frac{e^{-\lambda t}}{2\pi}\ (ct)^{q_1+q_2} \int_0^{2\pi} (\cos\theta)^{q_1}\ (\sin\theta)^{q_2}\ d\theta + \\
& + \frac{\lambda e^{-\lambda t}}{2\pi c} \int_0^{ct}\int_0^{2\pi} (r\cos\theta)^{q_1}\ (r\sin\theta)^{q_2}\ \frac{\exp\left(\frac{\lambda}{c}\sqrt{c^2t^2-r^2}\right)}{\sqrt{c^2t^2-r^2}}\ r\ dr\ d\theta = \\
& = \frac{e^{-\lambda t}}{2\pi}\ (ct)^{q_1+q_2} \int_0^{2\pi} (\cos\theta)^{q_1}\ (\sin\theta)^{q_2}\ d\theta + \\
& + \frac{\lambda e^{-\lambda t}}{2\pi c} \int_0^{ct} r^{q_1+q_2+1}\ \frac{\exp\left(\frac{\lambda}{c}\sqrt{c^2t^2-r^2}\right)}{\sqrt{c^2t^2-r^2}}\ dr \int_0^{2\pi} (\cos\theta)^{q_1}\ (\sin\theta)^{q_2}\ d\theta.
\end{aligned}$$

Taking into account that

$$\int_0^{2\pi} (\cos\theta)^{q_1}\ (\sin\theta)^{q_2}\ d\theta = \begin{cases} 2B\left(\dfrac{q_1+1}{2}, \dfrac{q_2+1}{2}\right), & \text{if } q_1 \text{ and } q_2 \text{ are even,} \\ 0, & \text{otherwise,} \end{cases}$$

we obtain for even q_1 and q_2:

$$\begin{aligned}
\mathrm{E}\mathbf{X}^{\mathbf{q}}(t) = \frac{e^{-\lambda t}}{\pi}\ (ct)^{q_1+q_2}\ & B\left(\frac{q_1+1}{2}, \frac{q_2+1}{2}\right) + \\
& + \frac{\lambda e^{-\lambda t}}{\pi c}\ B\left(\frac{q_1+1}{2}, \frac{q_2+1}{2}\right) \int_0^{ct} r^{q_1+q_2+1}\ \frac{\exp\left(\frac{\lambda}{c}\sqrt{c^2t^2-r^2}\right)}{\sqrt{c^2t^2-r^2}}\ dr = \\
& = \frac{e^{-\lambda t}}{\pi}\ (ct)^{q_1+q_2}\ B\left(\frac{q_1+1}{2}, \frac{q_2+1}{2}\right) + \\
& + \frac{\lambda e^{-\lambda t}}{\pi c}\ B\left(\frac{q_1+1}{2}, \frac{q_2+1}{2}\right)\ (ct)^{q_1+q_2+1} \int_0^1 \xi^{q_1+q_2+1}\ \frac{e^{\lambda t\sqrt{1-\xi^2}}}{\sqrt{1-\xi^2}}\ d\xi.
\end{aligned}$$

The substitution $z = \sqrt{1-\xi^2}$ in the last integral reduces this expression to

$$\begin{aligned}
\mathrm{E}\mathbf{X}^{\mathbf{q}}(t) = \frac{e^{-\lambda t}}{\pi}\ (ct)^{q_1+q_2}\ & B\left(\frac{q_1+1}{2}, \frac{q_2+1}{2}\right) + \\
& + \frac{\lambda e^{-\lambda t}}{\pi c}\ B\left(\frac{q_1+1}{2}, \frac{q_2+1}{2}\right)\ (ct)^{q_1+q_2+1} \int_0^1 (1-z^2)^{(q_1+q_2)/2}\ e^{\lambda tz}\ dz.
\end{aligned}$$

Applying now [1], Formula 3.387(5), to the integral on the right-hand side of this equality we obtain

$$\begin{aligned}
\mathrm{E}\mathbf{X}^{\mathbf{q}}(t) &= \frac{e^{-\lambda t}}{\pi} \, (ct)^{q_1+q_2} \, B\left(\frac{q_1+1}{2}, \frac{q_2+1}{2}\right) + \\
&+ \frac{\lambda e^{-\lambda t}}{\pi c} \, B\left(\frac{q_1+1}{2}, \frac{q_2+1}{2}\right) \, (ct)^{q_1+q_2+1} \, \frac{\sqrt{\pi}}{2} \, \left(\frac{2}{\lambda t}\right)^{(q_1+q_2+1)/2} \times \\
&\times \Gamma\left(\frac{q_1+q_2}{2}+1\right) \, \left[I_{(q_1+q_2+1)/2}(\lambda t) + \mathbf{L}_{(q_1+q_2+1)/2}(\lambda t)\right] = \\
&= \frac{e^{-\lambda t}}{\pi} \, (ct)^{q_1+q_2} \, B\left(\frac{q_1+1}{2}, \frac{q_2+1}{2}\right) + \\
&+ \frac{e^{-\lambda t}}{\sqrt{\pi}} \left(\frac{2}{\lambda t}\right)^{(q_1+q_2-1)/2} (ct)^{q_1+q_2} \, \Gamma\left(\frac{q_1+1}{2}\right) \Gamma\left(\frac{q_2+1}{2}\right) \times \\
&\times \left[I_{(q_1+q_2+1)/2}(\lambda t) + \mathbf{L}_{(q_1+q_2+1)/2}(\lambda t)\right]
\end{aligned}$$

The theorem is proved. □

Consider now the one-dimensional stochastic process

$$R(t) = \|\mathbf{X}(t)\| = \sqrt{X_1^2(t) + X_2^2(t)},$$

representing the Euclidean distance of the moving particle from the origin $\mathbf{0}$. Clearly, $0 \le R(t) \le ct$ and, according to [5], Remark 2, the absolutely continuous part of the distribution of $R(t)$ has the form:

$$\begin{aligned}
\Pr\{R(t) < r\} &= \Pr\left\{\mathbf{X}(t) \in \mathbf{B}_r^2\right\} \\
&= 1 - \exp\left(-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2 - r^2}\right), \qquad 0 \le r < ct.
\end{aligned}$$

Therefore, the complete density of $R(t)$ in the interval $0 \le r \le ct$ is given by

$$f(r,t) = \frac{re^{-\lambda t}}{ct} \, \delta(ct-r) + \frac{\lambda}{c} \, \frac{r}{\sqrt{c^2t^2 - r^2}} \, \exp\left(-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2 - r^2}\right) \, \Theta(ct-r). \tag{5}$$

In the following theorem we present an explicit formula for the moments of the process $R(t)$.

Theorem 2. *For any $q \ge 1$ the following formula holds*

$$\begin{aligned}
ER^q(t) &= (ct)^q e^{-\lambda t} + \\
&+ e^{-\lambda t}\sqrt{\pi} \left(\frac{2}{\lambda t}\right)^{(q-1)/2} (ct)^q \, \Gamma\left(\frac{q}{2}+1\right) \left[I_{(q+1)/2}(\lambda t) + \mathbf{L}_{(q+1)/2}(\lambda t)\right],
\end{aligned} \tag{6}$$

where $I_\nu(x)$ and $\mathbf{L}_\nu(x)$ are given by (3) and (4), respectively.

Proof. According to (5) we have

$$\begin{aligned}
\mathrm{E}R^q(t) &= (ct)^q e^{-\lambda t} + \frac{\lambda e^{-\lambda t}}{c} \int_0^{ct} \frac{r^{q+1}}{\sqrt{c^2t^2-r^2}}\, e^{\frac{\lambda}{c}\sqrt{c^2t^2-r^2}}\, dr = \\
&= (ct)^q e^{-\lambda t} + \frac{\lambda e^{-\lambda t}}{c}\, (ct)^{q+1} \int_0^1 \xi^{q+1}\, (1-\xi^2)^{-1/2}\, e^{\lambda t\sqrt{1-\xi^2}}\, d\xi.
\end{aligned}$$

Making the substitution $z = \sqrt{1-\xi^2}$ in the last integral, we obtain

$$\begin{aligned}
\mathrm{E}R^q(t) &= (ct)^q e^{-\lambda t} + \frac{\lambda e^{-\lambda t}}{c}\, (ct)^{q+1} \int_0^1 (1-z^2)^{q/2}\, e^{\lambda t z}\, dz = \\
&= (ct)^q e^{-\lambda t} + \\
&+ \frac{\lambda e^{-\lambda t}\sqrt{\pi}}{2c} \left(\frac{2}{\lambda t}\right)^{(q+1)/2} (ct)^{q+1}\Gamma\left(\frac{q}{2}+1\right) \left[I_{(q+1)/2}(\lambda t) + \mathbf{L}_{(q+1)/2}(\lambda t)\right] = \\
&= (ct)^q e^{-\lambda t} + \\
&+ e^{-\lambda t}\sqrt{\pi} \left(\frac{2}{\lambda t}\right)^{(q-1)/2} (ct)^q\, \Gamma\left(\frac{q}{2}+1\right) \left[I_{(q+1)/2}(\lambda t) + \mathbf{L}_{(q+1)/2}(\lambda t)\right].
\end{aligned}$$

where we have used again [1], Formula 3.387(5). The theorem is proved. □

Remark 1. From (6) we can extract the formulae concerning two the most important moments, namely, the expectation and variance of the process $R(t)$:

$$\begin{aligned}
\mathrm{E}R(t) &= ct\, e^{-\lambda t} \left\{1 + \frac{\pi}{2}\, [I_1(\lambda t) + \mathbf{L}_1(\lambda t)]\right\}, \\
\mathrm{E}R^2(t) &= (ct)^2\, e^{-\lambda t} \left\{1 + \sqrt{\frac{2\pi}{\lambda t}}\, \left[I_{3/2}(\lambda t) + \mathbf{L}_{3/2}(\lambda t)\right]\right\}.
\end{aligned}$$

3 Moments of the Four-Dimensional Random Evolution

We consider now the similar symmetrical random evolution of a particle moving at constant finite speed c in the space $\mathbb{R}^4$ and subject to the control of a homogeneous Poisson process of rate $\lambda > 0$ in the manner described above.

Both the initial and every new direction are taken on according to the uniform law on the unit sphere

$$S_1^4 = \left\{\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 :\ x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\right\}.$$

Let $\mathbf{X}(t) = (X_1(t), X_2(t), X_3(t), X_4(t)),\ t > 0$, be the position process. At any time $t > 0$ the particle, with probability 1, is located in the four-dimensional ball of radius ct

$$\mathbf{B}_{ct}^4 = \left\{\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^2 :\ x_1^2 + x_2^2 + x_3^2 + x_4^2 \le c^2t^2\right\}.$$

Similarly to the planar case, we consider the distribution $\Pr\{\mathbf{X}(t) \in d\mathbf{x}\}$, $\mathbf{x} \in \mathbf{B}_{ct}^4$, $t \geq 0$, where $d\mathbf{x}$ is the infinitesimal volume in the space $\mathbb{R}^4$. This distribution consists of two components. The singular component corresponds to the case when no Poisson event occurs in the interval $(0, t)$ and is concentrated on the sphere

$$S_{ct}^4 = \partial \mathbf{B}_{ct}^4 = \left\{ \mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \; x_1^2 + x_2^2 + x_3^2 + x_4^2 = c^2 t^2 \right\}.$$

In this case the particle is located on S_{ct}^4 and the probability of this event is

$$\Pr\left\{\mathbf{X}(t) \in S_{ct}^4\right\} = e^{-\lambda t}.$$

If at least one Poisson event occurs, the particle is located strictly inside the ball $\mathbf{B}_{ct}^4$, and the probability of this event is

$$\Pr\left\{\mathbf{X}(t) \in \operatorname{int} \mathbf{B}_{ct}^4\right\} = 1 - e^{-\lambda t}.$$

The part of the distribution $\Pr\{\mathbf{X}(t) \in d\mathbf{x}\}$ corresponding to this case is concentrated in the interior of the ball

$$\operatorname{int} \mathbf{B}_{ct}^4 = \left\{ \mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \; x_1^2 + x_2^2 + x_3^2 + x_4^2 < c^2 t^2 \right\},$$

and forms the absolutely continuous component of this distribution.

It is known that the density of $\mathbf{X}(t)$ has the form (see [4], formula (19) therein):

$$\begin{aligned} f(\mathbf{x}, t) = & \frac{e^{-\lambda t}}{2\pi^2 (ct)^3} \, \delta(c^2 t^2 - \|\mathbf{x}\|^2) + \\ & + \frac{\lambda t}{\pi^2 (ct)^4} \left[2 + \lambda t \left(1 - \frac{\|\mathbf{x}\|^2}{c^2 t^2} \right) \right] \exp\left(-\frac{\lambda}{c^2 t} \|\mathbf{x}\|^2 \right) \, \Theta(ct - \|\mathbf{x}\|), \end{aligned} \tag{7}$$

$$\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbf{B}_{ct}^4, \qquad \|\mathbf{x}\|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2, \qquad t \geq 0.$$

Let $\mathbf{q} = (q_1, q_2, q_3, q_4)$ be the four-multi-index. We are interested in the mixed moments of the process $\mathbf{X}(t)$:

$$\mathrm{E}\mathbf{X}^{\mathbf{q}}(t) = \mathrm{E} X_1^{q_1}(t) X_2^{q_2}(t) X_3^{q_3}(t) X_4^{q_4}(t), \qquad q_1 \geq 1, \; q_2 \geq 1, \; q_3 \geq 1, \; q_4 \geq 1.$$

The mixed moments of $\mathbf{X}(t)$ are given by the following theorem.

Theorem 3. *For any $q_1, q_2, q_3, q_4 \geq 1$ the following formula holds*

$$E\mathbf{X}^{\mathbf{q}}(t) = \begin{cases} \frac{e^{-\lambda t}}{\pi^2}(ct)^{q_1+q_2+q_3+q_4} \, B\left(\frac{q_1+1}{2}, \frac{q_2+q_3+q_4+1}{2}\right) \times \\ \times \, B\left(\frac{q_2+1}{2}, \frac{q_3+q_4+1}{2}\right) \, B\left(\frac{q_3+1}{2}, \frac{q_4+1}{2}\right) + \\ + \frac{2\lambda t}{\pi^2} \, (ct)^{q_1+q_2+q_3+q_4} \, \frac{\Gamma\left(\frac{q_1+1}{2}\right) \, \Gamma\left(\frac{q_2+1}{2}\right) \, \Gamma\left(\frac{q_3+1}{2}\right) \, \Gamma\left(\frac{q_4+1}{2}\right)}{\Gamma\left(\frac{q_1+q_2+q_3+q_4+4}{2}\right)} \times \\ \times \left[(\lambda t)^{-(q_1+q_2+q_3+q_4+4)/2} \, \gamma\left(\frac{q_1+q_2+q_3+q_4+4}{2}, \, \lambda t\right) + \right. \\ + \frac{\lambda t}{2} \, \frac{\Gamma\left(\frac{q_1+q_2+q_3+q_4+4}{2}\right)}{\Gamma\left(\frac{q_1+q_2+q_3+q_4+8}{2}\right)} \times \\ \left. \times \, {}_1F_1\left(\frac{q_1+q_2+q_3+q_4+4}{2}; \frac{q_1+q_2+q_3+q_4+8}{2}; -\lambda t\right) \right], \\ \qquad\qquad\qquad\qquad \text{if all } \; q_1, q_2, q_3, q_4 \; \text{ are even,} \\ 0, \qquad\qquad\qquad\qquad \text{otherwise,} \end{cases} \tag{8}$$

where

$$\gamma(\alpha, x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(\alpha+k)} \, x^{\alpha+k} \tag{9}$$

is the incomplete gamma-function, and

$${}_1F_1(\xi; \eta; z) = \Phi(\xi, \eta; z) = \sum_{k=0}^{\infty} \frac{(\xi)_k}{(\eta)_k} \frac{z^k}{k!} \tag{10}$$

is the degenerated hypergeometric function.

Proof. We consider separately the singular and the absolutely continuous parts of the density (7). According to (7), for the singular part of the distribution of the process we have:

$$\begin{aligned} \mathrm{E}\mathbf{X}_s^{\mathbf{q}}(t) &= \frac{e^{-\lambda t}}{2\pi^2 (ct)^3} \iiiint\limits_{x_1^2+x_2^2+x_3^2+x_4^2=c^2t^2} x_1^{q_1} \, x_2^{q_2} \, x_3^{q_3} \, x_4^{q_4} \, dx_1 \, dx_2 \, dx_3 \, dx_4 = \\ &= \frac{e^{-\lambda t}}{2\pi^2} \, (ct)^{q_1+q_2+q_3+q_4} \int_0^{\pi} (\cos\theta_1)^{q_1} \, (\sin\theta_1)^{q_2+q_3+q_4} \, d\theta_1 \times \\ &\times \int_0^{\pi} (\cos\theta_2)^{q_2} \, (\sin\theta_2)^{q_3+q_4} \, d\theta_2 \, \times \int_0^{2\pi} (\cos\theta_3)^{q_3} \, (\sin\theta_3)^{q_4} \, d\theta_3. \end{aligned} \tag{11}$$

Computing these integrals we have

$$\int_0^{2\pi} (\cos\theta_3)^{q_3} \, (\sin\theta_3)^{q_4} \, d\theta_3 = \begin{cases} 2B\left(\dfrac{q_3+1}{2}, \dfrac{q_4+1}{2}\right), & \text{if } q_3 \text{ and } q_4 \text{ are even,} \\ 0, & \text{otherwise,} \end{cases}$$

$$\int_0^{\pi} (\cos\theta_2)^{q_2} \, (\sin\theta_2)^{q_3+q_4} \, d\theta_2 = \begin{cases} B\left(\dfrac{q_2+1}{2}, \dfrac{q_3+q_4+1}{2}\right), & \text{if } q_2 \text{ is even,} \\ 0, & \text{otherwise,} \end{cases} \tag{12}$$

$$\int_0^{\pi} (\cos\theta_1)^{q_1} \, (\sin\theta_1)^{q_2+q_3+q_4} \, d\theta_1 = \begin{cases} B\left(\dfrac{q_1+1}{2}, \dfrac{q_2+q_3+q_4+1}{2}\right), & \text{if } q_1 \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Substituting these values (12) into (11) we obtain for even q_1, q_2, q_3, q_4:

$$\begin{aligned} \mathrm{E}\mathbf{X}_s^{\mathbf{q}}(t) = \frac{e^{-\lambda t}}{\pi^2} (ct)^{q_1+q_2+q_3+q_4} \, B\left(\frac{q_1+1}{2}, \frac{q_2+q_3+q_4+1}{2}\right) \times \\ \times B\left(\frac{q_2+1}{2}, \frac{q_3+q_4+1}{2}\right) \, B\left(\frac{q_3+1}{2}, \frac{q_4+1}{2}\right). \end{aligned} \tag{13}$$

Let us evaluate now the moments of the absolutely continuous part of the distribution of the process. By passing to four-dimensional polar coordinates, we have:

$$\begin{aligned} &\mathrm{E}\mathbf{X}_c^{\mathbf{q}}(t) = \\ &= \frac{\lambda t}{\pi^2 (ct)^4} \iiiint\limits_{x_1^2+x_2^2+x_3^2+x_4^2 \le c^2t^2} \prod_{i=1}^4 x_i^{q_i} \left[2 + \lambda t \left(1 - \frac{\|\mathbf{x}\|^2}{c^2t^2}\right)\right] \exp\left(-\frac{\lambda}{c^2 t}\|\mathbf{x}\|^2\right) \prod_{i=1}^4 dx_i = \\ &= \frac{\lambda t}{\pi^2 (ct)^4} \int_0^{ct} dr \int_0^{\pi} d\theta_1 \int_0^{\pi} d\theta_2 \int_0^{2\pi} d\theta_3 \times \\ &\times \Big\{ (r\cos\theta_1)^{q_1} \, (r\sin\theta_1\cos\theta_2)^{q_2} \, (r\sin\theta_1\sin\theta_2\cos\theta_3)^{q_3} \, (r\sin\theta_1\sin\theta_2\sin\theta_3)^{q_4} \times \\ &\times \left[2 + \lambda t\left(1 - \frac{r^2}{c^2t^2}\right)\right] \exp\left(-\frac{\lambda}{c^2t} r^2\right) \, r^3 \, (\sin\theta_1)^2 \sin\theta_2 \Big\} = \\ &= \frac{\lambda t}{\pi^2 (ct)^4} \int_0^{ct} r^{q_1+q_2+q_3+q_4+3} \left[2 + \lambda t\left(1 - \frac{r^2}{c^2t^2}\right)\right] \exp\left(-\frac{\lambda}{c^2t} r^2\right) \, dr \times \\ &\times \int_0^{\pi} (\cos\theta_1)^{q_1} \, (\sin\theta_1)^{q_2+q_3+q_4+2} d\theta_1 \, \int_0^{\pi} (\cos\theta_2)^{q_2} \, (\sin\theta_2)^{q_3+q_4+1} d\theta_2 \times \\ &\times \int_0^{2\pi} (\cos\theta_3)^{q_3} \, (\sin\theta_3)^{q_4} \, d\theta_3. \end{aligned}$$

Taking into account (12), we can rewrite this equality for even q_1, q_2, q_3, q_4 as follows:

$$\begin{aligned}
&\mathrm{E}\mathbf{X}_c^{\mathbf{q}}(t) = \\
&= \frac{\lambda t}{\pi^2 (ct)^4} \int_0^{ct} r^{q_1+q_2+q_3+q_4+3} \left[2 + \lambda t \left(1 - \frac{r^2}{c^2t^2} \right) \right] \exp\left(-\frac{\lambda}{c^2 t} r^2 \right) \, dr \times \\
&\times 2B\left(\frac{q_1+1}{2}, \frac{q_2+q_3+q_4+3}{2} \right) B\left(\frac{q_2+1}{2}, \frac{q_3+q_4+2}{2} \right) B\left(\frac{q_3+1}{2}, \frac{q_4+1}{2} \right) = \\
&= \frac{\lambda t}{\pi^2 (ct)^4} \frac{\Gamma\left(\frac{q_1+1}{2}\right) \Gamma\left(\frac{q_2+1}{2}\right) \Gamma\left(\frac{q_3+1}{2}\right) \Gamma\left(\frac{q_4+1}{2}\right)}{\Gamma\left(\frac{q_1+q_2+q_3+q_4+4}{2}\right)} \times \\
&\times \int_0^{ct} (r^2)^{(q_1+q_2+q_3+q_4+2)/2} \left[2 + \lambda t \left(1 - \frac{r^2}{c^2t^2} \right) \right] \exp\left(-\frac{\lambda}{c^2 t} r^2 \right) \, d(r^2) = \\
&= \frac{\lambda t}{\pi^2} (ct)^{q_1+q_2+q_3+q_4} \frac{\Gamma\left(\frac{q_1+1}{2}\right) \Gamma\left(\frac{q_2+1}{2}\right) \Gamma\left(\frac{q_3+1}{2}\right) \Gamma\left(\frac{q_4+1}{2}\right)}{\Gamma\left(\frac{q_1+q_2+q_3+q_4+4}{2}\right)} \times \\
&\times \int_0^1 z^{(q_1+q_2+q_3+q_4+2)/2} \; (2 + \lambda t (1-z)) \; e^{-\lambda t z} \, dz = \\
&= \frac{\lambda t}{\pi^2} (ct)^{q_1+q_2+q_3+q_4} \frac{\Gamma\left(\frac{q_1+1}{2}\right) \Gamma\left(\frac{q_2+1}{2}\right) \Gamma\left(\frac{q_3+1}{2}\right) \Gamma\left(\frac{q_4+1}{2}\right)}{\Gamma\left(\frac{q_1+q_2+q_3+q_4+4}{2}\right)} \times \\
&\times \left[2 \int_0^1 z^{(q_1+q_2+q_3+q_4+2)/2} \, e^{-\lambda t z} \, dz + \lambda t \int_0^1 z^{(q_1+q_2+q_3+q_4+2)/2} \, (1-z) \, e^{-\lambda t z} \, dz \right].
\end{aligned}$$

Applying now [1], Formula 3.381(1) and Formula 3.383(1) to the first and the second integrals of this equality, respectively, we obtain

$$\begin{aligned}
&\mathrm{E}\mathbf{X}_c^{\mathbf{q}}(t) = \\
&= \frac{2\lambda t}{\pi^2} (ct)^{q_1+q_2+q_3+q_4} \frac{\Gamma\left(\frac{q_1+1}{2}\right) \Gamma\left(\frac{q_2+1}{2}\right) \Gamma\left(\frac{q_3+1}{2}\right) \Gamma\left(\frac{q_4+1}{2}\right)}{\Gamma\left(\frac{q_1+q_2+q_3+q_4+4}{2}\right)} \times \\
&\times \left[(\lambda t)^{-(q_1+q_2+q_3+q_4+4)/2} \, \gamma\left(\frac{q_1+q_2+q_3+q_4+4}{2}, \; \lambda t \right) + \right. \\
&\left. + \frac{\lambda t}{2} \frac{\Gamma\left(\frac{q_1+q_2+q_3+q_4+4}{2}\right)}{\Gamma\left(\frac{q_1+q_2+q_3+q_4+8}{2}\right)} \, {}_1F_1\left(\frac{q_1+q_2+q_3+q_4+4}{2}; \frac{q_1+q_2+q_3+q_4+8}{2}; -\lambda t \right) \right].
\end{aligned}$$

Now, by adding to this expression the moments of the singular part of the distribution given by (13), we finally obtain (8). The theorem is thus completely proved. □

Consider now the following one-dimensional stochastic process

$$R(t) = \|\mathbf{X}(t)\| = \sqrt{X_1^2(t) + X_2^2(t) + X_3^2(t) + X_4^2(t)},$$

representing the Euclidean distance of the moving particle from the origin $\mathbf{0}$ of the space $\mathbb{R}^4$. Clearly, $0 \le R(t) \le ct$ and, according to [4], formula (18), the absolutely continuous part of the distribution of $R(t)$ has the form:

$$\begin{aligned}\Pr\{R(t) < r\} &= \Pr\left\{\mathbf{X}(t) \in \mathbf{B}_r^4\right\} = \\ &= 1 - \left(1 + \frac{\lambda}{c^2 t}\, r^2 - \frac{\lambda}{c^4 t^3}\, r^4\right) \exp\left(-\frac{\lambda}{c^2 t}\, r^2\right). \qquad 0 \le r < ct,\end{aligned}$$

Therefore, the complete density of $R(t)$ in the interval $0 \le r \le ct$ is given by

$$\begin{aligned}f(r,t) = &\frac{r^3 e^{-\lambda t}}{(ct)^3}\, \delta(ct - r) + \\ &+ \left[\left(\frac{4\lambda}{c^4 t^3} + \frac{2\lambda^2}{c^4 t^2}\right) r^3 - \frac{2\lambda^2}{c^6 t^4} r^5\right] \exp\left(-\frac{\lambda}{c^2 t}\, r^2\right) \Theta(ct - r).\end{aligned} \tag{14}$$

In the following theorem we present an explicit formula for the moments of $R(t)$.

Theorem 4. *For any $q \ge 1$ the following formula holds*

$$ER^q(t) = (ct)^q \left\{e^{-\lambda t} + (\lambda t)^{-(q+2)/2} \left[(2 + \lambda t)\, \gamma\left(\frac{q}{2} + 2, \lambda t\right) - \gamma\left(\frac{q}{2} + 3, \lambda t\right)\right]\right\}, \tag{15}$$

where $\gamma(\alpha, x)$ is the incomplete gamma-function given by (9).

Proof. According to (14) we have

$$\begin{aligned}\mathrm{E}R^q(t) = (ct)^q e^{-\lambda t} + &\left(\frac{4\lambda}{c^4 t^3} + \frac{2\lambda^2}{c^4 t^2}\right) \int_0^{ct} r^{q+3} \exp\left(-\frac{\lambda}{c^2 t}\, r^2\right) dr - \\ &- \frac{2\lambda^2}{c^6 t^4} \int_0^{ct} r^{q+5} \exp\left(-\frac{\lambda}{c^2 t}\, r^2\right) dr.\end{aligned}$$

Making the substitution $\xi = r^2$ in both integrals, we obtain

$$\begin{aligned}\mathrm{E}R^q(t) = (ct)^q e^{-\lambda t} + &\left(\frac{2\lambda}{c^4 t^3} + \frac{\lambda^2}{c^4 t^2}\right) \int_0^{c^2 t^2} \xi^{(q+2)/2} \exp\left(-\frac{\lambda}{c^2 t}\, r^2\right) d\xi - \\ &- \frac{\lambda^2}{c^6 t^4} \int_0^{c^2 t^2} \xi^{(q+4)/2} \exp\left(-\frac{\lambda}{c^2 t}\, r^2\right) d\xi =\end{aligned}$$

$$
\begin{aligned}
&= (ct)^q e^{-\lambda t} + \left(\frac{2\lambda}{c^4 t^3} + \frac{\lambda^2}{c^4 t^2} \right) \left(\frac{\lambda}{c^2 t} \right)^{-(q+4)/2} \gamma \left(\frac{q}{2} + 2, \lambda t \right) - \\
&- \frac{\lambda^2}{c^6 t^4} \left(\frac{\lambda}{c^2 t} \right)^{-(q+6)/2} \gamma \left(\frac{q}{2} + 3, \lambda t \right) = \\
&= (ct)^q e^{-\lambda t} + \frac{2+\lambda t}{c^2 t^2} \left(\frac{\lambda}{c^2 t} \right)^{-(q+2)/2} \gamma \left(\frac{q}{2} + 2, \lambda t \right) - \\
&- \frac{1}{c^2 t^2} \left(\frac{\lambda}{c^2 t} \right)^{-(q+2)/2} \gamma \left(\frac{q}{2} + 3, \lambda t \right) = \\
&= (ct)^q e^{-\lambda t} + \left(\frac{\lambda}{c^2 t} \right)^{-(q+2)/2} \left[\frac{2+\lambda t}{c^2 t^2} \gamma \left(\frac{q}{2} + 2, \lambda t \right) - \frac{1}{c^2 t^2} \gamma \left(\frac{q}{2} + 3, \lambda t \right) \right] = \\
&= (ct)^q \left\{ e^{-\lambda t} + (\lambda t)^{-(q+2)/2} \left[(2+\lambda t)\, \gamma \left(\frac{q}{2} + 2, \lambda t \right) - \gamma \left(\frac{q}{2} + 3, \lambda t \right) \right] \right\},
\end{aligned}
$$

where in the second step we have used [1], Formula 3.381(1). The theorem is proved. $\square$

Remark 2. From (15) we can extract the formulae for the mean value and variance of the process $R(t)$:

$$
\begin{aligned}
\mathrm{E}R(t) &= ct \left\{ e^{-\lambda t} + (\lambda t)^{-3/2} \left[(2+\lambda t)\, \gamma \left(\frac{5}{2}, \lambda t \right) - \gamma \left(\frac{7}{2}, \lambda t \right) \right] \right\}, \\
\mathrm{E}R^2(t) &= \frac{2c^2}{\lambda^2} \left(e^{-\lambda t} + \lambda t - 1 \right).
\end{aligned}
\tag{16}
$$

The first formula in (16) immediately follows from (15) for $q = 1$. Let us now prove the second formula in (16). For $q = 2$ formula (16) yields:

$$
\begin{aligned}
\mathrm{E}R^2(t) &= (ct)^2 \left\{ e^{-\lambda t} + (\lambda t)^{-2} \left[(2+\lambda t)\, \gamma(3, \lambda t) - \gamma(4, \lambda t) \right] \right\} = \\
&\qquad (\text{[1], Formula 8.356(1)}) \\
&= (ct)^2 \left\{ e^{-\lambda t} + (\lambda t)^{-2} \left[(2+\lambda t)\, \gamma(3, \lambda t) - 3\gamma(3, \lambda t) + (\lambda t)^3 e^{-\lambda t} \right] \right\} = \\
&= (ct)^2 \left\{ e^{-\lambda t} + (\lambda t)^{-2} \left[(\lambda t - 1)\, \gamma(3, \lambda t) + (\lambda t)^3 e^{-\lambda t} \right] \right\} = \\
&\qquad (\text{[1], Formula 8.352(1)}) \\
&= (ct)^2 \left\{ e^{-\lambda t} + (\lambda t)^{-2} \left[2(\lambda t - 1) \left(1 - e^{-\lambda t} \left(1 + \lambda t + \frac{(\lambda t)^2}{2!} \right) \right) + (\lambda t)^3 e^{-\lambda t} \right] \right\} = \\
&= (ct)^2 \left\{ e^{-\lambda t} + (\lambda t)^{-2} \left[2\lambda t - 2 - e^{-\lambda t} \left((\lambda t)^2 - 2 \right) \right] \right\} = \\
&= (ct)^2 \left\{ e^{-\lambda t} + \frac{2}{\lambda t} - \frac{2}{(\lambda t)^2} - \left(1 - \frac{2}{(\lambda t)^2} \right) e^{-\lambda t} \right\} = \\
&= (ct)^2 \left\{ \frac{2}{\lambda t} - \frac{2}{(\lambda t)^2} (1 - e^{-\lambda t}) \right\} = \frac{2c^2}{\lambda^2} \left(e^{-\lambda t} + \lambda t - 1 \right),
\end{aligned}
$$

and the second formula in (16) is proved.

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A virtual analog of Pollaczek-Khintchin transform equation *

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Abstract. The virtual queue length distribution for the queueing system $M|G|1$ is obtained. It is shown that these results can be viewed as generalization of Pollaczek-Khintchin transform equation.

Mathematics subject classification: 60K25, 68M20, 90B22.
Keywords and phrases: Laplace-Stieltjes transform, queue length, trafic intensity, generation function, transform equation.

1 Introduction

The queueing system $M|G|1$ plays an important role in Queueing Analysis. This system is studied and described in most standard textbooks and monographs on Queueing Theory (see, for example, Allen 1978 [1], Kleinrok 1975 [2], Cooper 1981 [3], Takagi 1991 [4], Gnedenko and Kovalenko 2005 [5]). Various methods and techniques necessary for the evaluation of its characteristics have been developed. Many characteristics were obtained by pioneers and founders of Queueing Theory. Among such characteristics one can mention the stationary distribution of the number of messages in the system, or, in other words, the queue length distribution, first obtained by Pollaczek in 1961 [6] and independently by Khintchin in 1963 [7]. It is necessary to mention that many outstanding researchers repeated this classical result using new elaborated methods and approaches and referred to it in their papers and books. Although the $M|G|1$ system is well studied the impetuous development of contemporary technologies puts forward new problems requesting new approaches and results. Thus, it turned out that some results for queueing system $M|G|1$ can be used for the analysis of polling systems: mathematical models used as a theoretical approach for broad band WLAN (Wireless Local Area Networks). In this paper the queue length distribution for an arbitrary time t ($t \in (0,\infty)$) is obtained for mentioned $M|G|1$ system. In other words, the nonstationary or virtual distribution of the number of the messages in the system is obtained. We show below that this virtual distribution contains as a particular case, namely in the steady state, the mentioned Pollaczek-Khintchin transform equation. In this context we can consider the distribution obtained below as an analog of a well-known Pollaczek-Khintchin equation. Results were obtained using the method of "catastrophes" (Gnedenco et

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all 1973 [8], Klimov and Mishkoy 1979 [9]) and the approach based on regenerative processes with embedded periods.

2 Preliminary results. Pollaczek-Khintchin transform equation

Let's consider the queueing system $M|G|1$ with exhaustive service (messages are served continuously until there is a nonmessage in the system). Denote by λ the parameter of input Poisson flow, by B the length of service and by $B(x) = P\{B < x\}$ the distribution function of service. Let's also denote by $\beta(s) = \int\limits_0^\infty e^{-sx} dB(x)$ the Laplace-Stieltjes transform of function $B(x)$, by $\beta_1 = \int\limits_0^\infty x dB(x)$ its first moment.

Let's consider the random variable X – the number of messages in the queue. Obviously, X is a discrete random variable. Denote by P_k distribution of this variable

$$P_k = \{X = k\}$$

and let $P(z)$ be the generating function of probabilities P_k,

$$P(z) = \sum_k z^k P_k$$

where $0 \le z \le 1$.

The traffic intensity ρ is defined as follows

$$\rho = \frac{E(B)}{E(z_k)}$$

where

$$E(B) = \beta_1 = \int\limits_0^\infty x dB(x), \quad B(x) = P\{B < x\},$$

$$E(z_k) = \int\limits_0^\infty x dA(x), \quad A(x) = P\{z_k < x\} = 1 - e^{-\lambda x}.$$

Here z_k is the interarrival interval between t_{k-1} and t_k time moments. Using the traffic intensity definition for $M|G|1$ system we get $\rho = \lambda\beta_1$.

The following result is known as Pollaczek-Khintchin transform equation [6,7].

Theorem 1. *If $\rho < 1$, then the steady state generating function of the queue length distribution is given by expresion*

$$P(z) = \frac{\beta(\lambda - \lambda z)(z-1)(1-\lambda\beta_1)}{z - \beta(\lambda - \lambda z)}. \tag{1}$$

The proof of the formula (1) can be obtained employing the method of the embedded Markov chain and notion of irreductibility, aperidiocity and ergodicity of the Markov chains (Asumussen 1987 [10], Cohen 1982 [11], Takagi 1991 [4]).

Remark 1. From expression (1) we easily get the mean value formula

$$N = \sum_k kP_k = P^{'}(1) = \lambda\beta_1 + \frac{\lambda^2\beta_1}{2(1-\lambda\beta_1)}. \tag{2}$$

Consider the busy period. By the busy period we shall understand the time interval beginning with the arrival message in the free $M|G|1$ system and finishing with the next moment when the system becomes free. Obviously, the busy period is a random variable. Let's denote by $\Pi(x)$ the distribution function of the busy period, by $\pi(s) = \int\limits_0^\infty e^{-sx}d\Pi(x)$ – the Laplace-Stieltjes transform of $\Pi(x)$ and by $\pi_1 = \int\limits_0^\infty xd\Pi(x)$ – the first moment.

The distribution function of the busy period is given (in terms of Laplace-Stieltjes transform) by the following theorem.

Theorem 2. *The Laplace-Stieltjes transform $\pi(s)$ of the busy period is determined in the unique way from functional equation*

$$\pi(s) = \beta(s + \lambda - \lambda\pi(s)). \tag{3}$$

If $\rho < 1$, then

$$\pi_1 = \frac{\beta_1}{1-\lambda\beta_1}. \tag{4}$$

The functional equation (3) is known as Kendall-Takacs functional equation (Kendall [12], Takacs [13]). The multidimensional analog of the mentioned equation is presented in (Mishkoy 2006 [14] and Mishkoy 2007 [15,16]).

3 Nonstationary (virtual) analog of transform equation

According to the $M|G|1$ system we assume that there is an infinite buffer to store waiting messages and exhaustive service, those messages are served continuously until there is no message in the system. By the service discipline we assume LIFO (last in, first out; or reverse order of arrival) or FIFO (first in, first out, or order of arrival). For these disciplines, the order of service is not affected by the service time of waiting messages. The first objective of our analysis in this section is to find the nonsteady state distribution of the queue size, that is the number of messages present in the system at the moment of time t.

Denote by $P_m(t)$ probability that at the instant t there are m messages in the system, by $P(z,t)$ the generating function of probabilities $P_m(t)$,

$$P(z,t) = \sum_{m \geq o} P_m(t)z^m, \quad 0 \leq z \leq 1$$

and by

$$p(z,s)=\int_0^\infty e^{-st}P(z,t)dt \tag{5}$$

its Laplace transform.

We shall assume that independently of the evolution of the system some events, called "catastrophes", which form a Poisson flow with parameter $s>0$ happen. We also assume that an arbitrary message will be coloured either in red with probability z or in blue with probability $1-z$, independently of the colour of the other messages. We shall multiply the both parts of the expression (5) by s. Then

$$sp(z,s)=s\int_0^\infty e^{-st}P(z,t)dt$$

is the probability that the first "catastrophe" happens at the moment of time t when in the queueing system there are at least red messages. The profit from this probability means that we shall obtain the formulas to determine the function $p(z,s)$. We shall denote in addition by

$s\pi(z,s)$ – the probability that the first "catastrophe" happened during the busy period when in the queueing system there are at least red messages;

$s\beta(z,s)$ – the probability that the first "catastrophe" happened during a messages service time B when in the queueing system there are at least red messages.

Let's suppose that at any moment t there are n messages in the queue. The interval of time which starts with the service of one of the mentioned n messages and finishes as soon as the system becomes free, will be called Π^n–period. We shall denote the distribution function of this period by $\Pi^n(t)$ and its Laplace-Stieltjes transform by $\pi^n(s)$.

Obviously, $\pi^n(s)=[\pi(s)]^n$, where $\pi(s)$ is determined from the functional equation (3).

Let's consider a separate Π^n–period. Let's denote by $\overline{P}_m(t)$ the probability that at instant t in the queueing therre are m messages. Denote

$$\Pi^n(z,t)=\sum_{m\geq 0}\overline{P}_m(t)z^m,\quad 0\leq z\leq 1,$$

and let

$$\pi^n(z,s)=\int_0^\infty e^{-st}\Pi^n(z,t)dt$$

be the Laplace transform of generating function $\Pi^n(z,t)$.

Similarly as we did above we conclude

$s\pi^n(z,s)$ – is the probability that the first "catastrophe" happens during a Π^n–period when in the queueing system there are at least red messages.

Then the following auxiliary result holds.

Lemma 1. *Laplace transform of the generating function of the queue length distribution on Π^n–period is given by*

$$\pi^n(z,s) = \pi(z,s)\frac{z^n - [\pi(s)]^n}{z - \pi(s)}$$

where $\pi(z,s)$ is Laplace transform of the generating function of the queue length distribution on the busy period which will be obtained below; function $\pi(s)$ is determined from (3).

Proof. First let's prove that

$$s\pi^n(z,s) = s\pi(z,s)z^{n-1} + \pi(s)s\pi(z,s)z^{n-2} + \tag{6}$$

$$+ \cdots + [\pi(s)]^{n-1}s\pi(z,s)$$

Really, assume that the first "catastrophe" happens during a Π^n–period when in the queueing system there are at least red messages (the probability of this event is $s\pi^n(z,s)$). For this it is necessary and sufficient that either the first "catastrophe" happen during the associated busy period with the first n messages available in the system when in the queueing system there are at least red messages (the probability of this event is $s\pi^n(z,s)$), the remaining $n-1$ messages are red (the probability of this event is z^{n-1});

or the first "catastrophe" happen during the associated busy period with the second n messages available in the system when in the queueing system there are at least red messages (the probability of this event is $s\pi^n(z,s)$), during the associated busy period with the first message "catastrophe" does not happen (the probability of this event is $\pi(s)$), the remaining $n-2$ messages are red (the probability of this event is z^{n-2}), etc...,

or the first "catastrophe" happen during the associated busy period with the last n messages available in the system when in the queueing system there are at least red messages (the probability of this event is $s\pi^n(z,s)$), and the "catastrophe" does not happen during the associated busy period with $n-1$ initial messages (the probability of this event is $[\pi(s)]^{n-1}$).

We shall rewrite the expression (6) in the following way

$$s\pi^n(z,s) = s\pi(z,s)\{z^{n-1} + \pi(s)z^{n-2} + \cdots + [\pi(s)]^{n-1}\}$$

or

$$s\pi^n(z,s) = s\pi(z,s)\frac{z^n - [\pi(s)]^n}{z - \pi(s)}$$

and after reducing s we obtain the proof of Lemma 1. □

The following Lemma 2 gives the distribution of the queue size on the separate busy period.

Lemma 2. *The Laplace transform of the generating function of the queue size distribution on the busy period $\pi(z,s)$ is given by*

$$\pi(z,s) = \beta(z,s)\frac{z-\pi(s)}{z-\beta(s+\lambda-\lambda z)} \tag{7}$$

where $\pi(s)$ is determined from the functional equation (3), $\beta(s+\lambda-\lambda z)$ is the Laplace-Stieltjes transform of function $B(t)$ at point $s = s+\lambda-\lambda z$, $\beta(z,s)$ will be given below.

Proof. First we prove

$$s\pi(z,s) = s\beta(z,s) + \sum_{n\geq 1} s\pi^n(z,s)\int\limits_0^\infty e^{-st}\frac{(\lambda t)^n}{(n!)}e^{-\lambda t}dB(t). \tag{8}$$

Really, let's suppose that the first "catastrophe" happens on the separate busy period when in the system there are at least red messages. As we have mentioned the probability of this event is $s\pi(z,s)$.

For this it is necessary and sufficient that either the first "catastrophe" happen during the service of the message that opens the busy period, when in the system there are at least red messages (this probability is $s\beta(z,s)$); or "catastrophe" does not happen during the service time of this message (the probability is e^{-st}), messages arrive $n \geq 1$ (the probability of this event is $\frac{(\lambda t)^n}{(n!)}e^{-\lambda t}$) and the first "catastrophe" happens in the Π^n-period, when in the system there are at least red messages (the probability of this event is $s\pi^n(z,s)$).

Let's us denote by $\sum$ the second term in (8) and using Lemma 1 we have

$$\sum = \sum_{n\geq 1}\pi^n(z,s)\int\limits_0^\infty e^{-st}\frac{(\lambda t)^n}{(n!)}e^{-\lambda t}dB(t) =$$

$$= \frac{\pi(z,s)}{z-\pi(s)}\sum_{n\geq 1}\int\limits_0^\infty\left[\frac{(\lambda z t)^n}{n!} - \frac{(\lambda\pi(s)t)^n}{n!}\right]e^{-(-s+\lambda)t}dB(t).$$

Observe that for $n=0$ the sum is equal to 0, hense letting $n \geq 1$ we obtain

$$\sum = \frac{\pi(z,s)}{z-\pi(s)}\int\limits_0^\infty\left[e^{\lambda z t} - e^{\lambda\pi(s)t}\right]e^{-(-s+\lambda)t}dB(t) =$$

$$= \frac{\pi(z,s)}{z-\pi(s)}\left[\beta(s+\lambda-\lambda z) - \beta(s+\lambda-\lambda\pi(s))\right].$$

According to Theorem 3 since $\pi(s) = \beta(s+\lambda-\lambda\pi(s))$ we receive

$$\sum = \frac{\pi(z,s)}{z-\pi(s)}\left[\beta(s+\lambda-\lambda z) - \pi(s)\right].$$

Substituting the obtained formula in (8) and reducing s we have

$$\pi(z,s)=\beta(z,s)+\frac{\pi(z,s)}{z-\pi(s)}[\beta(s+\lambda-\lambda z)-\pi(s)]$$

hence

$$\pi(z,s)\{1-\frac{\beta(s+\lambda-\lambda z)-\pi(s)}{z-\pi(s)}\}=\beta(z,s)$$

or

$$\pi(z,s)=\frac{z-\beta(s+\lambda-\lambda z)}{z-\pi(s)}\beta(z,s).$$

This completes the proof of Lemma 2. □

Lemma 3. *The Laplace transform of generating function of the queue size distribution on service time B is given by*

$$\beta(z,s)=z\frac{1-\beta(s+\lambda-\lambda z)}{s+\lambda-\lambda z}. \tag{9}$$

Proof. First we give the proof of equality

$$s\beta(z,s)=z\int_0^\infty[1-B(x)]se^{-st}e^{-\lambda(1-z)t}dt. \tag{10}$$

Really, let's suppose that the first "catastrophe" happened during service time B. Probability of this event is $s\beta(z,s)$. For this it is necessary and sufficient that the first "catastrophe" happen at the moment t (the probability of this event is se^{-st}), when the service is not finished yet (the probability of this event is $1-B(t)$) until the happened "catastrophe" does not arrive no red messages into the system (the probability of this event is $e^{-\lambda(1-z)t}$), the given messages are red (the probability of this event is z).

In this way from (10) after reducing s we have

$$\beta(z,s)=z\frac{1-\beta(s+\lambda-\lambda z)}{s+\lambda-\lambda z}.$$

Thus, Lemma 3 is proved. □

Now we shall obtain the main result – the nonsteady state distribution of the queue length. The results are obtained in terms of Laplace transform, however we can easily get the moments of the size distribution and the stationary distribution. Besides, the results are well applicable for computer utilization.

Theorem 3. *The Laplace transform of the queue length distribution at an arbitrary time is given by*

$$p(z,s)=\frac{1+\lambda\pi(z,s)}{s+\lambda-\lambda\pi(s)} \tag{11}$$

where $\pi(z,s)$ is determined from Lemma 2, $\pi(s)$ – from functional equation (3).

Proof. We shall find the probability of the following event: "from two events: a) the "catastrophe" happened and b) the arrival of message, arrival of message occurs". The probability of the mentioned event, obviously, is

$$\frac{\lambda}{\lambda+s}.$$

Similarly, the probability of the event: "from two events: a) the "catastrophe" happened and b) the arrival or message – first "catastrophe" occurs", obviously, is

$$\frac{s}{\lambda+s}.$$

Now we show that the equality

$$sp(z,s)=\frac{s}{\lambda+s}+\frac{\lambda}{\lambda+s}sp(z,s)+\frac{\lambda}{\lambda+s}\pi(s)sp(z,s) \tag{12}$$

is fulfilled.

So, let's suppose that the first "catastrophe" happened at the moment when in the system there are at least red messages. The probability of this event is $sp(z,s)$. On the other hand, for this it is necessary and sufficient that
either the first "catastrophe" happen in the moment when the system is free (the probability of this event is $\frac{s}{\lambda+s}$);
or the first "catastrophe" happen during the busy period associated with the first message when in the system there are at least red messages (the probability of this event is $s\pi(z,s)$);
or during the busy period associated with the first message "catastrophe" do not happen, the first "catastrophe" happens after the completion of the busy period at the moment of time when in the system there are at least red messages (the probability of this event is $\frac{\lambda}{\lambda+s}\pi(s)sp(z,s)$).

Now from formula (12) we have

$$p(z,s)\left[1-\frac{\lambda\pi(s)}{\lambda+s}\right]=\frac{1+\lambda\pi(s)}{\lambda+s}$$

and Theorem 3 is proved. □

Note that the result of Theorem 3 allows us to receive the mean value of the queue size. Indeed, let's denote by $N(t)$ the mean value of the queue size at the moment t and by

$$n(s)=\int_0^\infty e^{-st}N(t)dt \tag{13}$$

the Laplace transform of function $N(t)$. Then from Theorem 3 we get

$$n(s)=\frac{\lambda}{s}\left[\frac{1}{s}-\frac{\beta(s)(1-\pi(s))}{(1-\beta(s))(s+\lambda-\lambda\pi(s))}\right]. \tag{14}$$

Formula (14) is obtained from (11) using the following algorithm

$$n(s) = \frac{\partial p(z,s)}{\partial z} \, |_{z=1} \, .$$

Remark 2. In the next section it will be shown that if $\rho < 1$, then Theorem 1 follows from Theorem 3, so the expression (11) given by Theorem 3 can be considered as virtual analog of formula (1). Also, the formula (14) can be considered to be a virtual analog of steady state mean value (2). Note that to get $N(t)$ it is necessary to invert $n(s)$, solving the integral equation (13).

4 Reduction of the virtual analog to Pollaczek-Khintchin equation

Using the method of embedded Markov chain we can prove that if the steady state condition $\rho = \lambda\beta_1 < 1$ is satisfied, then limits

$$\lim_{t\to\infty} P_m(t) = P_m, \quad \lim_{t\to\infty} P(z,t) = P(z)$$

exist.

Since $p(z,s)$ is the Laplace transform of generating function $P(z,t)$, applying the Tauber theorem we have

$$\lim_{t\to\infty} P(z,t) = \lim_{s\downarrow 0} sp(z,s)$$

or

$$P(z) = \lim_{s\downarrow 0} sp(z,s).$$

Thus, for $\lambda\beta_1 < 1$ substituing $p(z,s)$ given by Theorem 3 we obtain

$$P(z) = \lim_{s\downarrow 0} sp(z,s) = \lim_{s\downarrow 0} \frac{s(1+\lambda\pi(z,s))}{s+\lambda-\lambda\pi(s)}.$$

Since

$$\pi(0) = \int_0^\infty d\Pi(x) = 1, \tag{15}$$

applying the L'Hospital's rule we obtain

$$P(z) = \lim_{s\downarrow 0} \frac{[1+\lambda\pi(z,s)] + s\lambda\pi'(z,s)}{1-\lambda\pi'(s)},$$

or substituting $-\pi'(0)$ by

$$\pi_1 = \int_0^\infty x d\Pi(x) = -\pi'(0)$$

we obtain

$$P(z) = \frac{1 + \lambda\pi(z,0)}{1 - \lambda\pi_1}. \tag{16}$$

We get function $\pi(z,0)$ from Lemma 2 setting $s = 0$. We have

$$\pi(z,0) = \beta(z,0)\frac{z - \pi(0)}{z - \beta(\lambda - \lambda z)}. \tag{17}$$

We get function $\beta(z,0)$ from Lemma 3 setting $s = 0$. We have

$$\beta(z,0) = \frac{z[1 - \beta(\lambda - \lambda z)]}{\lambda - \lambda z}.$$

Find $\pi(z,0)$. According (17) and taking into consideration (15) we get

$$\pi(z,0) = \frac{z[1 - \beta(\lambda - \lambda z)]}{\lambda(1 - z)} \frac{z - 1}{z - \beta(\lambda - \lambda z)}$$

or

$$\pi(z,0) = \frac{z[1 - \beta(\lambda - \lambda z)]}{\lambda(\beta(\lambda - \lambda z)) - z}.$$

Substituting the obtained result in (16) and in accordance with Theorem 2

$$\pi_1 = \frac{\beta_1}{1 - \lambda\beta_1}$$

we finally obtain

$$P(z) = \frac{\beta(\lambda - \lambda z)(z - 1)(1 - \lambda\beta_1)}{z - \beta(\lambda - \lambda z)}$$

that exactly coincides with Pollaczek-Khintchin transform equation (Theorem 1).

Remark 3. From the above–presented we can conclude that Theorem 3 gives us the distribution of queue size for an arbitrary t, what can be considered as a virtual analog of the well–known Pollaczek-Khintchin equation.

Remark 4. Moreover, Lemmas 1-3 allows to get the steady state distribution of the queue size on a separate Π^n–period, a separate busy period Π, and a separate service time B. Indeed, let $P_1(z)$, $P_2(z)$, $P_3(z)$ be the generating functions of the mentioned intervals. If $\lambda\beta_1 < 1$, then the mentioned generating function can be obtained using the above procedure, namely

$$P_1(z) = \lim_{s\downarrow 0} s\pi^n(z,s),$$

$$P_2(z) = \lim_{s\downarrow 0} s\pi(z,s),$$

$$P_3(z) = \lim_{s\downarrow 0} s\beta(z,s).$$

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Resolvability of some special algebras with topologies

Liubomir Chiriac

Abstract. Let G be an infinite I_nP-n-groupoid. We construct a disjoint family $\{B_\mu : \mu \in M\}$ of non-empty subsets of G such that the sets $\{B_\mu\}$ are dense in all Choban's totally bounded topologies on G, $|M| = |G|$, $G = \cup\{B_\mu : \mu \in M\}$ and $\cup_{k=1}^{n} \Delta_\varphi \omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and every finite subsets K of G. In particular, we continue the line of research from [6, 9].

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1 Introductory notions

A space X is called resolvable if in X there exist two disjoint dense subsets. In [6] M. Choban and L. Chiriac has proved the following assertion.

Theorem. *Let G be an infinite group of cardinality τ. Then there exists a disjoint family $\{B_\mu : \mu \in M\}$ of subsets of G such that:*
1. $|M| = |G|$.
2. $G = \cup\{B_\mu : \mu \in M\}$.
3. $(G \setminus B\mu) \cdot K \neq G$ *for all* $\mu \in M$ *and every finite subset* K *of* G.
4. *The sets* $\{B_\mu : \mu \in M\}$ *are dense in all totally bounded topologies on* G.

This fact is a generalization of one Protasov's result [9]. In this paper the assertions of Theorem are proved for the special algebras – I_nP_k-n-groupoids. We shall use the notation and terminology from [1–4, 7, 8]. In particular, $|X|$ is the cardinality of a set $|X|$, $N = 0, 1, 2, ...$, R is the space of reals. By ω_0 we denote the first infinite cardinal. If τ is an infinite cardinal, then τ^+ is the first cardinal larger than τ. If $\tau \geq 1$ is a cardinal, then the space X is called τ-resolvable if there exists a family of pairwise disjoint dense subsets $\{B_\alpha : \alpha \in A\}$ of X such that $|A| = \tau$. Every space is 1-resolvable. If the space X is 2-resolvable, then we say that X is resolvable.

Denote by a_1^m a sequence $a_1, a_2, ..., a_m$. If $a_1 = a_2 = ... = a_m$, then we denote this sequence by a^m. For every space X we put

$$m(X) = min\{|U| : U \neq \emptyset, U \subseteq X, U \in \tau\}.$$

A space X is maximal resolvable if it is $m(X)$-resolvable. It is clear that if X is τ-resolvable then $\tau \leq m(X)$. If $m(X) = |X| > 1$ and X is maximal resolvable, then we say that X is superresolvable.

For every mapping $f : X \to X$ we put $f^{'} = f$ and $f^{n+1} = f \circ f^n$ for any $n \in N$. We can consider that $f^0 : X \to X$ is the identity mapping.

The problem of resolvability of totally bounded topological groups was solved by V.I. Malykhin, W.W. Comfort, S. Van Mill [5], I.V. Protasov [9] and M.M. Choban, L.L. Chiriac [6].

2 Groupoids with invertibility properties

Fix a sequence $\{E_n : n \in N\}$ of pairwise disjoint spaces. The discrete sum $E = \cup\{E_n : n \in N\}$ is called a signature or a set of fundamental operations. A universal algebra of signature E, or briefly, an E-algebra is a non-empty set G and a sequence of mappings $e_G = \{e_{nG} : E_n \times G^n \longrightarrow G : n \in N\}$. The set G is called a support of the E-algebra G and the mappings e_G are called the algebraical structure on G. Let G be an E-algebra. If $u \in E_0$, then the element $u_G = e_{0G}(\{u\} \times G^0)$ is called a constant of G and we put $u(x) = u_G$ for all $x \in G$. If $n \geq 1, u \in E_n$ and $x_1, ..., x_n \in G$, then we put $u(x_1, ..., x_n) = e_{nG}(u, x_1, ..., x_n)$. A pair (G, ω) is said to be a n-groupoid if G is a non-empty set and $\omega : G^n \to G$ is a mapping.

Definition 1. Let $k \leq n$. An n-groupoid (G, ω) is called:

1. an I_nP_k-n-groupoid if there exist the mappings $r_1, ..., r_{k-1}, r_{k+1}, ..., r_n : G \to G$ such that $\omega(r_1(x_1), ..., r_{k-1}(x_{k-1}), \omega(x_1, ..., x_{k-1}, y, x_{k+1}, ..., x_n), r_{k+1}(x_{k+1}), ..., r_n(x_n)) = y$ or $\omega(r_1^{k-1}(x_1^{k-1}), \omega(x_1^{k-1}, y, x_{k+1}^n), r_{k+1}^n(x_{k+1}^n)) = y$ for all $x_1, ..., x_{k-1}, x_{k+1}, ..., x_n, y \in G$. The mapping $r_i(x)$ is called k-involution, $i \in \{1, ..., k-1, k+1, ..., n\}$.

2. an I_nP-n-groupoid in the large sense if it is I_nP_k-n-groupoid for all $k = \overline{1, n}$. In this case the mapping $r_i(x)$ is called involution, $i \in \{1, ..., n\}$.

3. an I_nP-n-groupoid, or I_nP-n-groupoid in strong sense, if there exist the mappings $\{r_i : G \to G\text{: } i = \overline{1, n}\}$ such that $\{r_i : i \leqslant n, i \neq k\}$ is a family of k-involutions for any $k = \overline{1, n}$.

4. an I_0P_k-n-groupoid if there exist the mappings $r_1, ..., r_{k-1}, r_{k+1}, ..., r_n : G \to G$ such that $\omega(r_1(x), ..., r_{k-1}(x), \omega(x^{k-1}, y, x^{n-k}), r_{k-1}(x), ..., r_n(x)) = y$ for all $x, y \in G$.

5. an I_0P-n-groupoid if it is I_0P_k-n-groupoid for all $k = \overline{1, n}$.

Example 1. Let $(G, \cdot)$ be a topological non commutative group with the identity e. If we put $\omega(x, y, z, u) = y \cdot x \cdot u \cdot z$, then (G, ω) is an I_0P-4-quasigroup. Indeed:

1. (G, ω) is an I_0P_1-4-quasigroup for $r_2(y) = y^{-1}$, $r_3(z) = z^{-1}$, $r_4(u) = u^{-1}$. We have $\omega(\omega(x, t, t, t), r_2(t), r_3(t), r_4(t)) = r_2(t) \cdot t \cdot x \cdot t \cdot t \cdot r_4(t) \cdot r_3(t) = t^{-1} \cdot t \cdot x \cdot t \cdot t \cdot t^{-1} \cdot t^{-1} = e \cdot x \cdot t \cdot e \cdot t^{-1} = x \cdot t \cdot t^{-1} = x$.

2. (G, ω) is an I_0P_2-4-quasigroup for $r_1(x) = x^{-1}, r_3(z) = z^{-1}, r_4(u) = u^{-1}$. Really, $\omega(r_1(t), \omega(t, y, t, t), r_3(t), r_4(t)) = y \cdot t \cdot t \cdot t \cdot r_1(t) \cdot r_4(t) \cdot r_3(t) = y$.

3. (G, ω) is an I_0P_3-4-quasigroup for $r_1(x) = x^{-1}$, $r_2(y) = y^{-1}$, $r_4(u) = u^{-1}$. Really, $\omega(r_1(t), r_2(t), \omega(t, t, z, t), r_4(t)) = r_2(t) \cdot r_1(t) \cdot r_4(t) \cdot t \cdot t \cdot t \cdot z = z$.

4. (G, ω) is an I_0P_4-4-quasigroup for $r_1(x) = x^{-1}$, $r_2(y) = y^{-1}, r_3(z) = z^{-1}$. Really, $\omega(r_1(t), r_2(t), r_3(t), \omega(t, t, t, u)) = r_2(t) \cdot r_1(t) \cdot t \cdot t \cdot u \cdot t \cdot r_3(t) = u$.

In this case (G,ω) is an I_0P_i-4-quasigroup for every $i \in \{1,2,3,4\}$. Hence, (G,ω) is an I_0P-4-quasigroup.

Example 2. Let $(G,\cdot)$ be a topological group with the identity e. We put $\omega(x,y,z) = x \cdot y \cdot z$. In this case:

1. (G,ω) is a 3-groupoid;
2. (G,ω) is an I_0P_i-3-groupoid for every $i \in \{1,2,3\}$ and for $r_1(x) = r_2(x) = r_3(x) = x^{-1}$;
3. (G,ω) is an I_3P_2-3-groupoid for $r_1(x) = x^{-1}, r_3(x) = z^{-1}$. Indeed, $\omega(r_1(x), \omega(x,y,z), r_3(z)) = x^{-1} \cdot x \cdot y \cdot z \cdot z^{-1} = e \cdot y \cdot e = y$;
4. If the group G is non commutative, then (G,ω) is not an I_3P_i-3-groupoid for $i = \{1,3\}$.

Example 3. Let C be the field of the complex numbers, R be the field of the reals numbers. Let $A = C \setminus \{0\}$, $B = R \setminus \{0\}$ and $G = \{r \in R : r > 0\}$. Then $(A,\cdot)$, $(B,\cdot)$ and $(G,\cdot)$ are commutative multiplicative groups. We put $\omega(x,y,z) = x \cdot y^n \cdot z$, $n \geq 1$.

1. If $n = 1$, then (A,ω), (B,ω) and (G,ω) are I_3P-3 quasigroups.
2. If $n \geq 2$, then (A,ω), is a 3-groupoid with divisions. The equation $\omega(a,y,c) = d$ has n solutions.
3. If $n > 1$ and n is odd, then (B,ω) and (G,ω) are 3-quasigroups.
4. If $n \geq 2$ and n is even, then (B,ω) is not a 3-groupoid with divisions and (G,ω) is a 3-quasigroup.
5. (A,ω), (B,ω), (G,ω) are I_3P_1-3-groupoids and I_3P_3-3-groupoids. If $n \geq 2$, then (A,ω), (B,ω) and (G,ω) are not I_3P_2-3-groupoids.

Example 4. Let C be the field of the complex numbers and $A = C \setminus \{0\}$. We fix $k \in A$ and put $\omega_n(x_1, x_2, ..., x_n) = k \cdot x_1 \cdot x_2 \cdot ... \cdot x_n$, $(n \geq 2)$. In this case:

1. (A,ω_n) is a commutative quasigroup.
2. (A,ω_n) is an I_nP-n-groupoid in strong sense. Denote $r_i(x_i) = \sqrt[n-1]{\frac{1}{k^2}} \cdot x_i^{-1}$. Hence, $\omega_n(r_1(x_1), ..., r_{i-1}(x_{i-1}), \omega_n(x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_n), r_{i+1}(x_{i+1}), ..., r_n(x_n)) = k \cdot (\sqrt[n-1]{\frac{1}{k^2}})^{i-1} \cdot x_1^{-1} \cdot x_2^{-1} \cdot ... \cdot x_{i-1}^{-1} \cdot k \cdot x_1 \cdot x_2 \cdot ... \cdot x_{i-1} \cdot x_i \cdot x_{i+1} \cdot ... \cdot x_n \cdot (\sqrt[n-1]{\frac{1}{k^2}})^{n-i} \cdot x_{i+1}^{-1} \cdot ... \cdot x_n^{-1} = k^2 (\sqrt[n-1]{\frac{1}{k^2}})^{n-1} \cdot x_i = k^2 \cdot \frac{1}{k^2} \cdot x_i = x_i$. In strong sense there are $n-1$ complete involutions.
3. Let $n \geq 2$ and $m = 2+(n-1)$. There is $k \in A$ such that $k^m = 1$ and $k^i \neq 1$ for $i < m$. If $r_i(x_i) = k \cdot x_i^{-1}$ then $\{r_1, r_2, ..., r_n\}$ are involutions in strong sense. Hence, $\omega_n(r_1(x_1), ..., r_{i-1}(x_i), \omega_n(x_1, x_2, ..., x_n), r_{i+1}(x_{i+1}), ..., r_n(x_n)) = k^{n-1} \cdot k^2 \cdot x_1^{-1} \cdot ... \cdot x_{i-1}^{-1} \cdot x_1 \cdot ... x_{i-1} \cdot x_i \cdot x_{i+1} \cdot ... \cdot x_n \cdot x_{i+1}^{-1} \cdot ... \cdot x_n^{-1} = k^{2+n-1} \cdot x_i = k^m \cdot x_i = x_i$.
4. Let $n = 2$, $m \geq 3$, $k^m = 1$ and $k^i \neq 1$ for $i < m$. We put $\omega(x,y) = k \cdot x \cdot y$, $r_1(x) = k^{m-2}x^{-1}$, $r_2(y) = k^{m-2}y^{-1}$. In this case $\{r_1(x), r_2(x)\}$ are unique involutions in strong sense and $r_i^2(x_i) = k^{m-2}(r_i(x_i))^{-1} = k^{m-2}((k^{m-2} \cdot x^{-1})^{-1}) = k^{m-2} \cdot \frac{1}{k^{m-2}} \cdot x_i = x_i$.

Example 5. Let $(G,\cdot)$ be a topological group with the identity. If we put $\omega(x,y) = x \cdot y$, then:

1. (G,ω) is a 2-groupoid or, briefly, groupoid;
2. (G,ω) is an RIP-groupoid for $r_2(x) = x^{-1}$. Indeed, $\omega(\omega(y,x), r_2(x)) = (y \cdot x) \cdot x^{-1} = y$;
3. (G,ω) is an LIP-groupoid for $r_1(x) = x^{-1}$. Indeed, $\omega(r_1(x_1), \omega(x,y)) = x^{-1}(x \cdot y) = y$;
4. (G,ω) is an IP-groupoid if it is both an RIP-groupoid and an LIP-groupoid.

The notions LIP, RIP in the class of groupoids were introduced by R. H. Bruck [4].

Proposition 1. *Let (G,ω) be an I_nP_1-n-groupoid and $r_2, r_3, ..., r_n : G \to G$ be 1-involutions. Then the following assertions are equivalent:*

1. $\omega(\omega(y,x_2,...,x_n), r_2(x_2),...,r_n(x_n)) = y$;

2. $\omega(\omega(y,r_2(x_2),...,r_n(x_n)), x_2,...,x_n) = y$ for all $x_2^n \in G$.

Proof. Suppose that

$$\omega(\omega(y,x_2,...,x_n), r_2(x_2),...,r_n(x_n)) = y \tag{1}$$

for all $x_2^n, y \in G$. From (1) we have

$$\omega(\omega(\omega(y,x_2,...,x_n), r_2(x_2),...,r_n(x_n)), r_2^2(x_2),...,r_n^2(x_n)) = \omega(y,x_2,...,x_n) \tag{2}$$

and

$$\omega(\omega(\omega(y,x_2,...,x_n), r_2(x_2),...,r_n(x_n)), r_2^2(x_2),...,r_n^2(x_n)) = \tag{3}$$

$$= \omega(y, r_2^2(x_2),...,r_n^2(x_n)).$$

Using (2) and (3) we obtain

$$\omega(y,x_2,...,x_n) = \omega(y, r_2^2(x_2),...,r_n^2(x_n)). \tag{4}$$

Let in (4) $y = \omega(y, r_2(x_2),...,r_n(x_n))$. Therefore from (4)

$$\omega(\omega(y,r_2(x_2),...,r_n(x_n)), x_2,...,x_n)) =$$

$$= \omega(\omega(y,r_2(x_2),...,r_n(x_n)), r_2^2(x_2),...,r_n^2(x_n)).$$

The implication $1 \to 2$ is proved. Suppose that

$$\omega(\omega(y,r_2(x_2),...,r_n(x_n)), x_2,...,x_n) = y. \tag{5}$$

From (5) it follows that

$$\omega(\omega[y, r_2^2(x_2),...,r_n^2(x_n)], r_2(x_2),...,r_n(x_n)) = y. \tag{6}$$

It is clear that

$$\omega(\omega[\omega[y, r_2^2(x_2), ..., r_n^2(x_n)], r_2(x_2), ..., r_n(x_n)], x_2, ..., x_n) = \tag{7}$$

$$= \omega(y, r_2^2(x_2), ..., r_n^2(x_n)).$$

From (6) we obtain

$$\omega(\omega[\omega[y, r_2^2(x_2), ..., r_n^2(x_n)], r_2(x_2), ..., r_n(x_n)], x_2, ..., x_n) = \omega(y, x_2, ..., x_n). \tag{8}$$

Using (7) and (8) we have

$$\omega(y, r_2^2(x_2), ..., r_n^2(x_n)) = \omega(y, x_2, ..., x_n). \tag{9}$$

Therefore

$$\omega(\omega(y, x_2, ..., x_n), r_2(x_2), ..., r_n(x_n)) =$$

$$= \omega(\omega[y, r_2^2(x_2), ..., r_n^2(x_n)], r_2(x_2), ..., r_n(x_n)) = y.$$

The implication $2 \to 1$ is proved. The proof is complete. □

Definition 2. An n-groupoid (G, ω) is called:

1. a k-cancellative n-groupoid if for every $a, b, x_1, ..., x_{k-1}, x_{k+1}, ..., x_n \in G$ we have $\omega(x_1, ..., x_{k-1}, a, x_{k+1}, ..., x_n) = \omega(x_1, ..., x_{k-1}, b, x_{k+1}, ..., x_n)$ if and only if $a = b$.
2. a cancellative n-groupoid if it is k-cancellative groupoid for all $k = \overline{1, n}$
3. an n-quasigroup if the equation $\omega(a_1^{i-1}, x, a_{i+1}^n) = b$ has unique solution for every a_i^n, b and each $i = \overline{1, n}$.

Definition 3. An element e from (G, ω) is called:

1. a k-identity of n-groupoid (G, ω) if $\omega(e^{k-1}, x, e^{n-k}) = x$ for every $x \in G$.
2. an identity of n-groupoid (G, ω) if $\omega(e^{i-1}, x, e^{n-i}) = x$ for every $x \in G$ and each $i = \overline{1, n}$.

If n-quasigroup (G, ω) contains at least one identity, then (G, ω) is called n-loop.

Proposition 2. *Let (G, ω) be an I_nP_1-n-groupoid and $r_2, r_3, ..., r_n : G \to G$ be 1-involutions. Then:*

1. $\omega(x_1, x_2, ..., x_n) = \omega(x_1, r_2^2(x_2), ..., r_n^2(x_n))$ *for all* $x_1^n \in G$.
2. $\omega(\omega(y, r_2(x_2), ..., r_n(x_n)), x_2, ..., x_n) = y$ *for all* $x_2^n, y \in G$.
3. (G, ω) *is* 1-*cancellative* .
4. *For every* $b, a_2^n \in G$, *the equation* $\omega(y, a_2, ..., a_n) = b$ *has a unique solution.*

Proof. The proof of the assertion 1 is contained in the proof of Proposition 1. The assertion 2 follows from Proposition 1. Let $a, b, x_2^n \in G$ and $\omega(a, x_2, ..., x_n) = \omega(b, x_2, ..., x_n)$. Then $a = \omega(\omega(a, x_2, ..., x_n), r_2(x_2), ..., r_n(x_n)) = \omega(\omega(b, x_2, ..., x_n), r_2(x_2), ..., r_n(x_n)) = b$. The assertion 3 is proved. We consider the equation $\omega(y, a_2, ..., a_n) = b$. Then from Proposition 1 we have $y = \omega(b, r_2(x_2), ..., r_n(x_n))$. Hence the equation $\omega(y, a_2, ..., a_n) = b$ has a unique solution. The proof is complete. □

Corollary 1. *Let (G, ω) be an I_nP-n-groupoid in the large sense and $r_i : G \to G, i = \overline{1, n}$, are the involutions on G. Then (G, ω) is cancellative.*

Proof. The assertion follows from Proposition 2. □

Academician M.M. Choban observed the following interesting fact.

Proposition 3. *Let (G, ω) be an I_nP-n-groupoid in the large sense and $r_i : G \to G, i = \overline{1, n}$, are the involutions on G. Then $x_i = r_i^{2(n-1)}(x_i)$, for every $i = \overline{1, n}$ and $n \geq 2$.*

Proof. It is sufficient to prove that $x_1 = r_1^{2(n-1)}(x_1)$ for any $x_1 \in G$. Fix $x_1, x_2, ..., x_n \in G$. From Proposition 2 we have $\omega(x_1, x_2, ..., x_n) = \omega(x_1, r_2^2(x_2), ..., r_n^2(x_n)) = \omega(r_1^2(x_1), r_2^2(x_2), r_3^4(x_3), ..., r_n^4(x_n)) = ... = \omega(r_1^{2i}(x_1), ..., r_{i+1}^{2i}(x_{i+1}), r_{i+2}^{2(i+1)}(x_{i+2}), ..., r_n^{2(i+1)}(x_n)) = ... = \omega(r_1^{2(n-1)}(x_1), r_2^{2(n-1)}(x_2), ..., r_n^{2(n-1)}(x_n))$, i.e. It is obvious that $\omega(x_1, x_2, ..., x_n) = \omega(x_1, r_2^{2m}(x_2), ..., r_n^{2m}(x_n))$ for any $m \geq 1$. Hence for $m = n - 1$, we have $\omega(x_1, r_2^{2(n-1}(x_2), ..., r_n^{2(n-1)}(x_n)) = \omega(r_1^{2(n-1)}(x_1), r_2^{2(n-1)}(x_2), ..., r_n^{2(n-1)}(x_n))$. Therefore $x_1 = r_1^{2(n-1)}(x_1)$ for any $x_1 \in G$ and $x_i = r_i^{2(n-1)}(x_i)$, for every $i = \overline{1, n}$ and $n \geq 2$. The proof is complete. □

Proposition 4. *Let (G, ω) be an I_nP-n-groupoid in the large sense and $r_i : G \to G, i = \overline{2, n}$, are the involutions on G. If $e_1, e_2, ..., e_n \in G$, $e_i = r_i^{2m}(e_i)$, for all $i = \overline{2, n}$, then $x_i = r_i^{2m}(x_i)$, for every $x_i \in G$ and $n \geq 2$.*

Proof. From Proposition 2 it follows that $\omega(x_1, x_2, ..., x_n) = \omega(x_1, r_2^{2m}(x_2), ..., r_n^{2m}(x_n))$. Fix $i = \overline{2, n}$. Then $\omega(e_1, e_2, ..., e_{i-1}, x_i, e_{i+1}, ..., e_n) = \omega(e_1, e_2, ..., e_{i-1}, r_i^{2m}(x_i), e_{i+1}, ..., e_n)$. Hence, $x_i = r_i^{2m}(x_i)$, for every $x_i \in G$, $i = \overline{2, n}$ and $n \geq 2$. The proof is complete. □

3 Topologies on algebras

We consider arbitrary topologies on universal algebras. There are a lot of types of bounded topology. We fix $n \geq 2$ and $k \leq n$. Consider a mapping $\varphi : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$. We will use Choban's bounded topology.

Definition 4. Let (G,ω) be an n-groupoid and $L_1, L_2, ..., L_n$ be a family of subsets of G. Then:

1. The sets $L_1, L_2, ..., L_n$ are k-α-associated with the mapping φ and denote $(L_1, L_2, ..., L_n)\alpha(k)\varphi$ if $L_i = L_j$ provided $\varphi(i) = \varphi(j)$ and $i \neq k$, $j \neq k$.

2. If $x_1, x_2, ..., x_n \in G$ and $(\{x_1\}, \{x_2\}, ..., \{x_n\})\alpha(k)\varphi$, then we put $(x_1, x_2, ..., x_n)\alpha(k)\varphi$.

3. We put $\Delta_{\varphi(k)}\omega(L_1, L_2, ..., L_n) = \{\omega(x_1, x_2, ..., x_n) : x_1 \in L_1, x_2 \in L_2, ..., x_n \in L_n$ and $(x_1, x_2, ..., x_n)\alpha(k)\varphi\}$.

Remark 1. Let $L_1, L_2, ..., L_n$ be subsets of G, and $L'_k = L_k$ and $L'_i = \bigcap\{L_j : j \leq n, \varphi(j) = \varphi(i)\}$ for any $i \neq k$. Then $(L'_1, L'_2, ..., L'_n)\alpha(k)\varphi$ and $\Delta_{\varphi(k)}\omega(L'_1, L'_2, ..., L'_n) = \Delta_{\varphi(k)}\omega(L_1, L_2, ..., L_n)$.

Definition 5. Let $k \leq n$. An n-groupoid (G,ω) is called an $I_\varphi P_k$-n-groupoid if there exist the mappings $r_i : G \to G$, $i \in \{1, ..., k-1, k+1, ..., n\}$ such that $\omega(r_1(x_1), ..., r_{k-1}(x_{k-1}), \omega(x_1, ..., x_{k-1}, y, x_{k+1}, ..., x_n), r_{k+1}(x_{k+1}), ..., r_n(x_n)) = y$ provided $(x_1, ..., x_{k-1}, x_{k+1}, ..., x_n)\ \alpha(k)\varphi$ for all $x_1, ..., x_{k-1}, x_{k+1}, ..., x_n, y \in G$.

We say that the mapping $r_i : G \to G$, $i \in \{1, ..., k-1, k+1, ..., n\}$ is called k-φ-involution.

If $\varphi(i) = \varphi(j)$ for all $i, j \leq n$, then $I_\varphi P_k$-n-groupoid is an $I_0 P_k$-n-groupoid.

Definition 6. Let (G,ω) be an n-groupoid and λ be an infinite cardinal. A topology $\mathcal{T}$ on G is called:

– a λ-k-φ-bounded topology if for every non-empty open set $U \in \mathcal{T}$ there exists a subset $K \subseteq G$ such that $|K| < \lambda$ and $\Delta_{\varphi(k)}\omega(K^{k-1}, U, K^{n-k}) = G$.

– a λ-φ-bounded topology if it is λ-k-φ-bounded topology for every $k = \overline{1,n}$. An ω_0-k-φ-bounded topology is called a k-φ-totally bounded topology. The topology is said to be φ-totally bounded if it is a k-φ-totally bounded topology for every $k = \overline{1,n}$.

Remark 2. If in Definition 6 the mapping φ is one-to-one, then a topology $\mathcal{T}$ on G is called respectively: a λ-k-bounded topology, a λ-bounded topology, a ω_0-k-bounded topology, a k-totally bounded topology and totally bounded topology, for every $k = \overline{1,n}$.

Proposition 5. *Let $\varphi : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ be a mapping, (G,ω) be an n-groupoid with the properties:*

1. The equation $\omega(a^{k-1}, x, a^{n-k}) = b$ is solvable for every $a, b \in G$.

2. For every $a, b \in G$ there exist $a_1, a_2, ..., a_n \in G$ such that $a_k = a$, $(a_1, a_2, ..., a_n)\alpha(k)\varphi$ and $\omega(a_1, a_2, ..., a_n) = b$.

Then the minimal compact T_1-topology $\mathcal{T} = \{\varnothing\} \cup \{G \setminus F : F$ is a finite subset of $G\}$ is a k-φ-totally bounded topology on G.

Proof. Let $U \in \mathcal{T}$ and $U \neq \varnothing$. Then the set $F = G \setminus U$ is finite. Fix $a \in U$. Then $h_a : G \to G$, where $h_a(x) = \omega(a^{k-1}, x, a^{n-k})$ for any $x \in G$ is a mapping of G onto G. Thus $F' = G \setminus h_a(U) \subseteq h_a(F)$ is a finite set. For any $x \in G$ there exist

$y_1(x), y_2(x), ..., y_n(x) \in G$ such that $y_k(x) = a$, $(y_1(x), y_2(x), ..., y_n(x))\alpha(k)\varphi$ and $\omega(y_1(x), y_2(x), ..., y_n(x)) = x$. We put $\Phi = \{a\} \cup \{\{y_1(x), y_2(x), ..., y_n(x)\} : x \in F'\}$. The set Φ is finite. By construction, $\Delta_{\varphi(k)}\omega(\Phi^{k-1}, U, \Phi^{n-k}) = G$. The proof is complete. $\square$

Proposition 6. *Let $\varphi : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ be a mapping, (G, ω) be an n-groupoid with the properties:*

1. For every $a, b \in G$ there exist $a_1, a_2, ..., a_n \in G$ such that $a_k = a$, $(a_1, a_2, ..., a_n)\alpha(k)\varphi$ and $\omega(a_1, a_2, ..., a_n) = b$.

2. There exists $e \in G$ such that $G \setminus \omega(e^{k-1}, G, e^{n-k})$ is a finite set (in particular, $\omega(e^{k-1}, x, e^{n-k}) = x$ for every $x \in G$).

Then the minimal compact T_1-topology $\mathcal{T} = \{\varnothing\} \cup \{G \setminus F : F$ is a finite subset of $G\}$ is a k-φ-totally bounded topology on G.

Proof. Let $U \in \mathcal{T}$ and $U \neq \varnothing$. Then the set $F = G \setminus U$ is finite. Fix $a \in U$. Consider the mapping $h_e : G \to G$, where $h_e(x) = \omega(e^{k-1}, x, e^{n-k})$ for any $x \in G$. The set $G \setminus h_e(G)$ is finite. Thus the set $F' = G \setminus h_e(U) \subseteq (G \setminus h_e(G)) \bigcup h_e(F)$ is a finite set. For any $x \in F'$ fix $\{y_1(x), y_2(x), ..., y_n(x)\} \subseteq G$ such that $y_k(x) = a$, $(y_1(x), y_2(x), ..., y_n(x))\alpha(k)\varphi$ and $\omega(y_1(x), y_2(x), ..., y_n(x)) = x$. Let $\Phi = \{e\} \cup \cup \{\{y_1(x), y_2(x), ..., y_n(x)\} : x \in F'\}$. The set Φ is finite. By construction, $\Delta_{\varphi(k)}\omega(\Phi^{k-1}, U, \Phi^{n-k}) = G$. The proof is complete. $\square$

Proposition 7. *Let $\varphi : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ be a mapping, (G, ω) be an infinite I_nP_k-n-groupoid, $B \subseteq G$, m be an infinite cardinal and $\Delta_{\varphi(k)}\omega(K^{k-1}, G \setminus B, K^{n-k}) \neq G$ for every subset K of cardinality $|K| < m$. Then the set B is dense in every m-k-φ-bounded topology $\mathcal{T}$ on G.*

Proof. Suppose that $\mathcal{T}$ is an m-k-φ-bounded topology on G and $U = G \setminus cl_G B \neq \varnothing$. Then $U \in \mathcal{T}$ and $U \subseteq G \setminus B$. By assumption there exists a subset K of G such that $\Delta_{\varphi(k)}\omega(K^{k-1}, U, K^{n-k}) = G$ and $|K| < m$. Since $U \subseteq G \setminus B$, we have $G \supseteq \Delta_{\varphi(k)}\omega(K^{k-1}, G \setminus B, K^{n-k}) \supseteq \Delta_{\varphi(k)}\omega(K^{k-1}, U, K^{n-k}) = G$, a contradiction. The proof is complete. $\square$

4 Decomposition of I_nP_k-n-groupoids

We fix $n \geq 2$ and $k \leq n$. Consider a mapping $\varphi : \{1, 2, ..., n\} \to \{1, 2, ...n\}$.

Lemma 1. *Let G be an infinite I_nP_k-n-groupoid, $r_1, ..., r_{k-1}, r_{k+1}, ..., r_n : G \to G$ be k-involutions, L and M be subsets of G and $|L \cup M| < |G|$. Then there exists an element $a \in G$ such that $\omega(L^{k-1}, a, L^{n-k}) \cap M = \varnothing$ and $\Delta_{\varphi(k)}\omega(L^{k-1}, a, L^{n-k}) \cap M = \varnothing$.*

Proof. Let $H = \{\omega(r_1(y_1), ..., r_{k-1}(y_{k-1}), x, r_{k+1}(y_{k+1}), ..., r_n(y_n)) : x \in M, y_1, ..., y_{k-1}, y_{k+1}, ..., y_n \in L\}$. Thus $|H| < |G|$ and there exists an element $a \in G \setminus H$.

Suppose that $\omega(L^{k-1}, a, L^{n-k}) \cap M \neq \varnothing$. Fix $\omega(L^{k-1}, a, L^{n-k}) \cap M$. Then $x = \omega(y_1, ..., y_{k-1}, a, y_{k+1}, ..., y_n)$ for some $y_1, ..., y_{k-1}, y_{k+1}, ..., y_n \in L$. Hence

$$a = \omega(r_1(y_1), ..., r_{k-1}(y_{k-1}), \omega(y_1^{k-1}, a, y_{k+1}^n), r_{k+1}(y_{k+1}), ..., r_n(y_n)) =$$

$$= \omega(r_1(y_1), ..., r_{k-1}(y_{k-1}), x, r_{k+1}(y_{k+1}), ..., r_n(y_n)) \in$$

$$\in \omega(r_1(y_1), ..., r_{k-1}(y_{k-1}), M, r_{k+1}(y_{k+1}), ..., r_n(y_n)) \subseteq H,$$

a contradiction. By construction, $\Delta_{\varphi(k)}\omega(L^{k-1}, M, L^{n-k}) \subseteq \omega(L^{k-1}, M, L^{n-k})$. Hence, $\Delta_{\varphi(k)}\omega(L^{k-1}, a, L^{n-k}) \cap M = \varnothing$. The proof is complete. □

Theorem 1. *Let G be an infinite I_nP_k-n-groupoid, $\mathcal{L}$ be a a non-empty family of non-empty subsets of G, $|\mathcal{L}| \leq |G|$ and for every set A and mapping $\Psi : A \to \mathcal{L}$ we have $| \cup\{\Psi(\alpha) : \alpha \in A\} |<| G |$ provided $| A |<| G |$. Then there exists a family $\{B_\mu : \mu \in M\}$ of non-empty subsets of G such that:*

1. *$| M |=| G |$.*
2. *$B_\mu \cap B_\eta = \varnothing$ for all $\alpha, \beta \in M$ and $\alpha \neq \beta$.*
3. *$G = \cup\{B_\mu : \mu \in M\}$.*
4. *$\omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and $K \in \mathcal{L}$.*
5. *$\Delta_{\varphi(k)}\omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and $K \in \mathcal{L}$.*

Proof. Consider on G some k-involutions, $r_1, ..., r_{k-1}, r_{k+1}, ..., r_n : G \to G$. Let $\tau = |G|$. Denote by $|\alpha|$ the cardinality of the ordinal number α. We put $\Omega_\tau = \{\alpha : 1 \leq |\alpha| < \tau\}$. If $K \subseteq G$, then $K_i^{-1} = \{r_i(x_i) : x_i \in K\}, i = 1, ..., k-1, k+1, ..., n$, and $K^{-1} = \cup\{K_i^{-1} : i = 1, 2, ..., k-1, k+1, ...n\}$. Let $\mathcal{L}_\infty = \{K^{-1} : K \in \mathcal{L}\} \cup \mathcal{L}$. It is clear that $|\mathcal{L}_1| \leq \tau$. Moreover, if A is a set, $|A| < \tau$ and $\Psi : A \to \mathcal{L}_1$ is a mapping, then $| \cup \{\Psi(\alpha) : \alpha \in A\}| < \tau$. Fix a set M of the cardinality τ. Since $|\Omega_\tau| = |M \times \mathcal{L}_1| = \tau$ then there exists a bijection $h : \Omega_\tau \to M \times \mathcal{L}_1$. If $\alpha \in \Omega_\tau$, then we consider that $h(\alpha) = (\mu_\alpha, K_\alpha) \in M \times \mathcal{L}_1$. If $\mu \in M$, then we put $A_\mu = h^{-1}(\{\mu\} \times \mathcal{L}_1)$. It is obvious that $A_\mu = \{\alpha \in \Omega_\tau : \mu_\alpha = \mu\}$ and $\{K_\alpha : \alpha \in A_\mu\} = \mathcal{L}_1$. Now we affirm that there exists a transfinite sequence $\{a_\alpha : \alpha \in \Omega_\tau\} \subseteq G$ such that $\omega(K_\alpha^{k-1}, a_\alpha, K_\alpha^{n-k}) \cap \omega(K_\beta^{k-1}, a_\beta, K_\beta^{n-k}) = \varnothing$ for all $\alpha, \beta \in \Omega_\tau$ and $\alpha \neq \beta$. We fix $a_1 \in G$. Let $1 < \beta, \beta \in \Omega_\tau$ and the elements $\{a_\alpha : \alpha < \beta\}$ are constructed. We put now $H_\beta = \cup\{\omega(K_\alpha^{k-1}, a_\alpha, K_\alpha^{n-k}) : \alpha < \beta\}$. Since $|\alpha \in \Omega_\tau : \alpha < \beta| \leq |\beta| < |G|$, then $|H_\beta| < |G|$. From Lemma 1 it follows that there exists $a_\beta \in G$ such that $\omega(K_\beta^{k-1}, a_\beta, K_\beta^{n-k}) \cap H_\beta = \varnothing$. By the transfinite induction if follows that the set $\{a_\alpha : \alpha \in \Omega_\tau\}$ is constructed. We put $P_\mu = \cup\{\omega(K_\alpha^{k-1}, a_\alpha, K_\alpha^{n-k}) : \alpha \in A_\mu\}$ for every $\mu \in H$. Fix $\mu, \eta \in M$ and $\mu \neq \eta$. Then $A_\mu \cap A_\eta = \varnothing$. Since $\omega(K_\alpha^{k-1}, a_\alpha, K_\alpha^{n-k}) \cap \omega(K_\beta^{k-1}, a_\beta, K_\beta^{n-k}) = \varnothing$ for all $\alpha \in A_\alpha$ and $\beta \in A_\eta$, then $P_\mu \cap P_\eta = \varnothing$. Fix $\mu \in M$ and $K \in \mathcal{L}$. Then $K^{-1} \in \mathcal{L}_1$ and $(\mu, K^{-1}) = (\mu_\alpha, K_\alpha)$ for some $\alpha \in A_\mu$. Suppose that $\omega(K^{k-1}, G \setminus P_\mu, K^{n-k}) = G$. Then $a_\alpha \in \omega(K^{k-1}, G \setminus P_\mu, K^{n-k})$, i.e. $a_\alpha = \omega(y_1^{k-1}, x, y_{k+1}^n)$ for some $x \in G \setminus P_\mu$ and $y_1, ..., y_{k-1}, y_{k+1}, ..., y_n \in K$. By construction, we have $r_1(y_1), ..., r_{k-1}(y_{k-1}), r_{k+1}(y_{k+1}), ..., r_n(x_n) \in K_\alpha$ and

$\omega(r_1(y_1),...,r_{k-1}(y_{k-1}),a_\alpha,r_{k+1}(y_{k+1}),...,r_n(x_n)) \in \omega(K_\alpha^{k-1},a_\alpha,K_\alpha^{n-k}) \subseteq P_\mu$. By assumption, we have that

$$\omega(r_1(y_1),...,r_{k-1}(y_{k-1}),a_\alpha,r_{k+1}(y_{k+1}),...,r_n(x_n)) =$$

$$= \omega(r_1(y_1),...,r_{k-1}(y_{k-1}),\omega(y_1^{k-1},x,y_{k+1}^{n-k}),r_{k+1}(y_{k+1}),..,r_n(y_n)) = x \in G \setminus P_\mu,$$

a contradiction. Hence $\omega(K^{k-1}, G\setminus P_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and $K \in \mathcal{L}$. Now we fix $\mu_0 \in M$. We put $B_\mu = P_\mu$ for all $\mu \in M\setminus\{\mu_0\}$ and $B_{\mu_0} = G\setminus \cup\{P_\mu : \mu \in M\setminus\{\mu_0\}\}$. By construction, we have $P_\mu \subseteq B_\mu$ for all $\mu \in M$ and $G = \cup\{B_\mu : \mu \in H\}$. If $\mu \in M$, then $G\setminus B_\mu \subseteq G\setminus P\mu$ and $\omega(K^{k-1}, G\setminus B_\mu, K^{n-k}) \neq G$ for all $K \in \mathcal{L}$. The proof is complete. □

Theorem 2. *Let (G,ω) be an infinite I_nP_k-n-groupoid, $\tau = |G|$, m be an infinite cardinal, $\tau = \sum\{\tau^q : q < m\}$ and either $m < \tau$, or τ be a regular cardinal. If $\mathcal{L}_m = \{K \subseteq G : |K| < m\}$, then there exists a family $\{B_\mu : \mu \in M\}$ of non-empty subsets of G such that:*

1. *$|M| = \tau$.*
2. *$B_\mu \cap B_\eta = \varnothing$ for all $\mu,\eta \in M$ and $\mu \neq \eta$.*
3. *$G = \cup\{B_\mu : \mu \in M\}$.*
4. *$\omega(K^{k-1}, G\setminus B_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and $K \in \mathcal{L}_m$.*
5. *$\Delta_{\varphi(k)}\omega(K^{k-1}, G\setminus B_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and $K \in \mathcal{L}_m$*
6. *The sets B_μ are dense in every m-k-φ-bounded topology on G.*
7. *Relative to every m-k-φ-bounded topology G is super-resolvable.*
8. *The sets B_μ are dense in every m-k-bounded topology on G.*
9. *Relative to every m-k-bounded topology G is super-resolvable.*

Proof. Since $\tau = \sum\{\tau^q : q < m\}$, we have $m \leq \tau$. Let A be a set, $|A| < \tau, \Psi : A \to L_m$ be a mapping and $H = \cup\{\Psi(\alpha) : \alpha \in A\}$. If $m < \tau$, then $|H| \leq \omega(m,...m,|A|,m,...,m) = \omega(m^{k-1},|A|,m^{n-k}) < \tau$. If $m = \tau$ and $|H| = \tau$, then $cf(\tau) \leq |A| < \tau$ and the cardinal τ is not regular. Hence $|H| < \tau$. Theorem 1 and Proposition 7 complete the proof. □

Corollary 2. *Let G be an infinite I_nP_k-n-groupoid. Then there exists a family $\{B_\mu : \mu \in M\}$ of non-empty subsets of G such that:*

1. *$|M| = |G|$.*
2. *$B_\mu \cap B_\eta = \varnothing$ for all $\mu,\eta \in M$ and $\mu \neq \eta$.*
3. *$G = \cup\{B_\mu : \mu \in M\}$.*
4. *$\omega(K^{k-1}, G\setminus B_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and every finte subset K of G.*
5. *$\Delta_{\varphi(k)}\omega(K^{k-1}, G\setminus B_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and every finte subset K of G.*
6. *The sets $\{B_\mu : \mu \in M\}$ are dense in every k-φ-totally bounded topology on G.*
7. *Relative to every k-φ-totally bounded topology G is super-resolvable.*
8. *The sets $\{B_\mu : \mu \in M\}$ are dense in every k-totally bounded topology on G.*
9. *Relative to every k-totally bounded topology G is super-resolvable.*

Corollary 3. *Let G be an infinite I_nP_k-n-groupoid, $\tau = |G|$, m be an infinite cardinal and $\tau^m = \tau$. Then there exists a family $\{B_\mu : \mu \in M\}$ of non-empty subsets of G such that:*

1. *$|M| = |G|$.*
2. *$B_\mu \cap B_\eta = \varnothing$ for all $\mu, \eta \in M$ and $\mu \neq \eta$.*
3. *$G = \cup\{B_\mu : \mu \in M\}$.*
4. *If $\mu \in M$, $K \subseteq G$ and $|K| < m$ then $\omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$.*
5. *If $\mu \in M$, $K \subseteq G$ and $|K| < m$ then $\Delta_{\varphi(k)}\omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$.*
6. *The sets $\{B_\mu : \mu \in M\}$ are dense in every m^+-k-φ-bounded topology on G.*
7. *Relative to every m^+-k-φ-bounded topology G is super-resolvable.*
8. *The sets $\{B_\mu : \mu \in M\}$ are dense in every m^+-k-bounded topology on G.*
9. *Relative to every m^+-k-bounded topology G is super-resolvable.*

5 Decomposition of I_nP-n-groupoids

We fix $n \geq 2$ and $k \leq n$. Consider a mapping $\varphi : \{1, 2, ..., n\} \to \{1, 2, ...n\}$.

Lemma 2. *Let G be an infinite I_nP-gruopoid, $r_1, ..., r_n : G \to G$ be involutions, L and M be subsets of G and $|L \cup M| < |G|$. Then there exists an element $a \in G$ such that:*

1. $\bigcup_{k=1}^{n} \omega(L^{k-1}, a, L^{n-k}) \cap M = \varnothing$, where $\bigcup_{k=1}^{n} \omega(L^{k-1}, a, L^{n-k}) = \omega(a, L^{n-1}) \cup \omega(L^1, a, L^{n-2}) \cup ... \cup \omega(L^{n-1}, a)$.

2. $\bigcup_{k=1}^{n} \Delta_{\varphi(k)}\omega(L^{k-1}, a, L^{n-k}) \cap M = \varnothing$, where $\bigcup_{k=1}^{n} \Delta_{\varphi(k)}\omega(L^{k-1}, a, L^{n-k}) = \Delta_{\varphi(k)}\omega(a, L^{n-1}) \cup \Delta_{\varphi(k)}\omega(L^1, a, L^{n-2}) \cup ... \cup \Delta_{\varphi(k)}\omega(L^{n-1}, a)$.

Proof. Let $H = \{\omega(x, r_2(y_2), ..., r_n(y_n)) : x \in M, y_2, ...y_n \in L\} \cup \{\omega(r_1(y_1), x, r_3(y_3), ..., r_n(y_n)) : x \in M, y_1, y_3, ..., y_n \in L\} \cup ... \cup \{\omega(r_1(y_1), ..., r_{n-1}(y_{n-1}), x) : x \in M, y_1, ...y_{n-1} \in L\}$. Since $|H| < |G|$, then there exists an element $a \in G \setminus H$. Let $\omega(a, L, ...L) \cap M \neq \varnothing$. Fix $x \in \omega(a, L, ...L) \cap M$. Then $x = \omega(a, y_2, ..., y_n)$ for some $y_2, ..., y_n \in L$. Hence $a = \omega(\omega(a, y_2, ..., y_n), r_2(y_2), ..., r_n(y_n)) = \omega(x, r_2(y_2), ..., r_n(y_n)) \in \omega(M, r_2(y_2), ..., r_n(y_n)) \leq H$, a contradiction. In similar way we prove that $\omega(L^{k-1}, a, L^{n-k}) \cap M$ for all $k = \overline{1, n}$. Hence $\bigcup_{k=1}^{n} \omega(L^{k-1}, a, L^{n-k}) \cap M = \varnothing$. By construction, $\Delta_{\varphi(k)}\omega(L^{k-1}, a, L^{n-k}) \subseteq \omega(L^{k-1}, a, L^{n-k})$. Hence, $\bigcup_{k=1}^{n} \Delta_{\varphi(k)}\omega(L^{k-1}, a, L^{n-k}) \cap M = \varnothing$. The proof is complete. □

Theorem 3. *Let G be an infinite I_nP-n-groupoid, $\mathcal{L}$ be a non-empty family of non-empty subsets of G, $|\mathcal{L}| \leq |G|$ and for every set A and mapping $\Psi : A \to \mathcal{L}$ we have $| \cup\{\Psi(\alpha) : \alpha \in A\} |<| G |$ provided $| A |<| G |$. Then there exists a family $\{B_\mu : \mu \in M\}$ of non-empty subsets of G such that:*

1. *$| M |=| G |$.*
2. *$B_\mu \cap B_\eta = \varnothing$ for all $\alpha, \beta \in M$ and $\alpha \neq \beta$.*
3. *$G = \cup\{B_\mu : \mu \in M\}$.*
4. *$\bigcup_{k=1}^{n} \omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and $K \in \mathcal{L}$*
5. *$\bigcup_{k=1}^{n} \Delta_{\varphi(k)}\omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and $K \in \mathcal{L}$.*

Proof. Consider on G involutions, $r_1, ..., r_n : G \to G$. Let $\tau = |G|$. Denote by $|\alpha|$ the cardinality of the ordinal number α. We put $\Omega_\tau = \{\alpha : 1 \leq |\alpha| < \tau\}$. If $K \subseteq G$, then $K_i^{-1} = \{r_i(x_i) : i = \overline{1,n}, x_i \in K\}$. We put $K^{-1} = \cup K_i^{-1}$ and $\mathcal{L}_1 = \{K^{-1} : K \in \mathcal{L}\} \cup \mathcal{L}$. It is clear that $|\mathcal{L}_1| \leq \tau$. Moreover, if A is a set, $|A| < \tau$ and $\Psi : A \to \mathcal{L}_1$ is a mapping, then $|\cup\{\Psi(\alpha) : \alpha \in A\}| < \tau$. Fix a set M of the cardinality τ. Since $|\Omega_\tau| = |M \times \mathcal{L}_1| = \tau$, then there exists a bijection $h : \Omega_\tau \to M \times \mathcal{L}_1$. Let $A_\mu = h^{-1}(\{\mu\} \times \mathcal{L}_1) = \alpha \in \Omega_\tau : \mu_\alpha = \mu\}$. If $\alpha \in \Omega_\tau$, then we consider that $h(\alpha) = (\mu_\alpha, K_\alpha) \in M \times \mathcal{L}_1$. It is obvious that $A_\mu = \{\alpha \in \Omega_\tau : \mu_\alpha = \mu\}$ and $\{K_\alpha : \alpha \in A_\mu\} = \mathcal{L}_1$. As in the proof of Theorem 1 from Lemma 2 it follows that there exists a transfinite sequence $\{a_\alpha \in G : \alpha \in \Omega_\tau\}$ such that $(\bigcup_{k=1}^n \omega(K_\alpha^{k-1}, a_\alpha, K_\alpha^{n-k})) \cap (\bigcup_{k=1}^n \omega(K_\beta^{k-1}, a_\beta, K_\beta^{n-k})) = \varnothing$ for all $\alpha, \beta \in \Omega_\tau$ and $\alpha \neq \beta$. Now we put $P_\mu = \cup\{\bigcup_{k=1}^n \omega(K_\alpha^{k-1}, a_\alpha, K_\alpha^{n-k}) : \alpha \in A_\mu\}$ for every $\mu \in M$. If $P_\mu^k = \bigcup_{k=1}^n \omega\{(K_\alpha^{k-1}, a_\alpha, K_\alpha^{n-k}) : \alpha \in A_\mu\}$ for all $k = \overline{1,n}$, then $P_\mu = \bigcup_{k=1}^n P_\mu^k$ and $\omega(K^{k-1}, G \setminus P_\mu^k, K^{n-k}) \neq G$ for every $K \in \mathcal{L}$. Suppose that $K \in \mathcal{L}$, $\mu \in M$ and $G = \bigcup_{k=1}^n \omega(K^{k-1}, G \setminus P_\mu^k, K^{n-k})$. For some $\alpha \in A_\mu$ we have $K_\alpha = \bigcup_{i=1}^n K_i^{-1} = K^{-1}$. Then $\bigcup_{k=1}^n \omega(K_\alpha^{k-1}, a_\alpha, K_\alpha^{n-k}) \subseteq P_\mu$ and $a_\alpha \in G$. Suppose that $a_\alpha \in \omega(K^{k-1}, G \setminus P_\mu^k, K^{n-k})$. Then $a_\alpha = \omega(y_1, ..., y_{k-1}, x, y_{k+1}, ..., y_n)$ for some $y_1, ..., y_{k-1}, y_{k+1}, ..., y_n \in K$ and $x \in G \setminus P_\mu$. Therefore $\omega(r_1(y_1), ..., r_{k-1}(y_{k-1}), a_\alpha, r_{k+1}(y_{k+1}), ..., r_n(y_n)) = \omega(r_1(y_1), ..., r_{k-1}(y_{k-1}), \omega(y_1^{k-1}, x, y_{k+1}^{n-k}), r_{k+1}(y_{k+1}), .., r_n(y_n)) = x \in G \setminus P_\mu$. Since $r_i(y_i \in K_\alpha)$, $i = \overline{1,n}$, we have $x = \omega(r_1(y_1), ..., r_{k-1}(y_{k-1}), a_\alpha, r_{k+1}(y_{k+1}), ..., r_n(y_n)) \in \omega(K_\alpha^{k-1}, a_\alpha, K_\alpha^{n-k}) \subseteq P_\mu$, a contradiction. Hence $\bigcup_{k=1}^n \omega(K^{k-1}, G \setminus P_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and $K \in \mathcal{L}$. Now we fix $\mu_0 \in M$. We put $B_\mu = P_\mu$ for all $\mu \in M \setminus \{\mu_0\}$ and $B_{\mu_0} = G \setminus \cup\{P_\mu : \mu \in M \setminus \{\mu_0\}\}$. By construction, we have $P_\mu \subseteq B_\mu$ for all $\mu \in M$ and $G = \cup\{B_\mu : \mu \in H\}$. If $\mu \in M$, then $G \setminus B_\mu \subseteq G \setminus P\mu$ and $\bigcup_{k-1}^n \omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$ for all $K \in \mathcal{L}$. The proof is complete. □

Theorem 4. *Let (G) be an infinite I_nP-n-groupoid, $\tau = |G|$, m be an infinite cardinal, $\tau = \sum\{\tau^q : q < m\}$ and either $m < \tau$, or τ be a regular cardinal. If $\mathcal{L}_m = \{K \subseteq G : |K| < m\}$, then there exists a family $\{B_\mu : \mu \in M\}$ of non-empty subsets of G such that:*

1. *$|M| = \tau$.*
2. *$B_\mu \cap B_\eta = \varnothing$ for all $\mu, \eta \in M$ and $\mu \neq \eta$.*
3. *$G = \cup\{B_\mu : \mu \in M\}$.*
4. *$\bigcup_{k=1}^n \omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and $K \in \mathcal{L}_m$.*
5. *$\bigcup_{k=1}^n \Delta_{\varphi(k)}\omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and $K \in \mathcal{L}_m$.*
6. *The sets B_μ are dense in every m-φ-bounded topology on G.*
7. *Relative to every m-φ-bounded topology T on G the space (G, T) is superresolvable.*
8. *The sets B_μ are dense in every m-bounded topology on G.*
9. *Relative to every m-bounded topology T on G the space (G, T) is superresolvable.*

Proof. Is similar to the proof of Theorem 2. □

Corollary 4. *Let G be an infinite I_nP-n-groupoid. Then there exists a family $\{B_\mu : \mu \in M\}$ of non-empty subsets of G such that:*

1. $|M| = |G|$.
2. $B_\mu \cap B_\eta = \varnothing$ *for all* $\mu, \eta \in M$ *and* $\mu \neq \eta$.
3. $G = \cup\{B_\mu : \mu \in M\}$.
4. $\bigcup_{k=1}^{n} \omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$ *for all* $\mu \in M$ *and every finte subset* K *of* G.
5. $\bigcup_{k=1}^{n} \Delta_{\varphi(k)}\omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$ *for all* $\mu \in M$ *and every finte subset* K *of* G.
6. *The sets* $\{B_\mu : \mu \in M\}$ *are dense in every* φ*-totally bounded topology on* G.
7. *Relative to every* φ*-totally bounded topology* G *is super-resolvable.*
8. *The sets* $\{B_\mu : \mu \in M\}$ *are dense in every totally bounded topology on* G.
9. *Relative to every totally bounded topology* G *is super-resolvable.*

Corollary 5. *Let G be an infinite I_nP-n-groupoid, $\tau = |G|$, m be an infinite cardinal and $\tau^m = \tau$. Then there exists a family $\{B_\mu : \mu \in M\}$ of non-empty subsets of G such that:*

1. $|M| = |G|$.
2. $B_\mu \cap B_\eta = \varnothing$ *for all* $\mu, \eta \in M$ *and* $\mu \neq \eta$.
3. $G = \cup\{B_\mu : \mu \in M\}$.
4. *If* $\mu \in M$, $K \subseteq G$ *and* $|K| \leq m$ *then* $\bigcup_{k=1}^{n} \omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$.
5. *If* $\mu \in M$, $K \subseteq G$ *and* $|K| \leq m$ *then* $\bigcup_{k=1}^{n} \Delta_{\varphi(k)}\omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$.
6. *The sets* $\{B_\mu : \mu \in M\}$ *are dense in every* m^+*-*k*-*φ*-bounded topology on* G.
7. *Relative to every* m^+*-*k*-*φ*-bounded topology* G *is super-resolvable.*
8. *The sets* $\{B_\mu : \mu \in M\}$ *are dense in every* m^+*-*k*-bounded topology on* G.
9. *Relative to every* m^+*-*k*-bounded topology* G *is super-resolvable.*

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Groups with many hypercentral subgroups

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Abstract. We obtain a characterization of solvable groups with the minimal condition on non-hypercentral (respectively non-nilpotent) subgroups.

Mathematics subject classification: 20F16, 20F19, 20F22.
Keywords and phrases: Hypercentral group, nilpotent group, HM^*-group, minimal condition.

Introduction. We say that a group G satisfies the minimal condition on non-hypercentral subgroups Min-$\overline{ZA}$ (respectively, the minimal condition on non-nilpotent subgroups Min-$\overline{N}$) if for any properly descending chain $G_1 \geq G_2 \geq \cdots \geq G_n \geq \cdots$ of subgroups G_n in G there exists a number $m \in \mathbb{N}$ such that G_n is hypercentral (respectively, nilpotent) for each $n \geq m$. The minimal non-hypercentral (respectively, non-nilpotent) group satisfies Min-$\overline{ZA}$ (respectively, Min-$\overline{N}$). Recall if χ is a property pertaining to subgroups and an infinite group G is not a χ-group but all its proper subgroups have χ, then G is called a minimal non-χ group. Any Heineken-Mohamed type group (i.e. non-nilpotent groups with nilpotent and subnormal proper subgroups) is a minimal non-nilpotent group [1] (see e.g. [2]) and satisfies Min-$\overline{N}$ and Min-$\overline{ZA}$. In this paper we study the groups that satisfy Min-$\overline{ZA}$ (respectively Min-$\overline{N}$) and prove the following

Theorem. *Let G be a solvable group. Then G satisfies the minimal condition on non-hypercentral (respectively, non-nilpotent) subgroups if and only if one of the following holds:*

(1) G is a hypercentral (respectively, nilpotent) group;

(2) G is a Černikov group;

(3) $G = P \times Q$ is a group direct product of a hypercentral (respectively, nilpotent) Černikov p'-group Q and a non-hypercentral (respectively, non-nilpotent) p-group P which contains a normal HM^*-subgroup H of finite index (respectively, HM^*-subgroup H of finite index with the nilpotent commutator subgroup H') with the normalizer condition.

Note that earlier groups with the minimal condition on non-abelian subgroups have been studied by S.N. Černikov (see e.g. [3]) and V.P. Šunkov [4].

Throughout this paper p will always denote a prime. Most of the standard notations may by found in [2, 5] and [6]. Recall only that a group G is called an HM^*-group if its commutator subgroup G' is hypercentral and the quotient group

G/G' is a divisible Černikov p-group (see [7] and [8]). Every Heineken-Mohamed group is an HM^*-group. Other examples of HM^*-groups are contained in [8–10].

1. In the next we shall need some properfies of groups with Max-$\overline{N}$ (respectively, Max-$\overline{ZA}$).

Lemma 1. *Let G be a locally nilpotent group. If G satisfies Min-$\overline{N}$ (respectively, Min-$\overline{ZA}$), then G satisfies the normalizer condition and every minimal non-nilpotent (respectively, non-hypercentral) subgroup of G is subnormal.*

Proof. Let H be a proper subgroup of G. If H is either a non-nilpotent (respectively, non-hypercentral) or maximal nilpotent (respectively, hypercentral) subgroup of G, then the set $\{S|H < S \leq G\}$ has a minimal element, say M. Since H is a maximal subgroup of M, we conclude that H is normal in M. Moreover, every nilpotent (respectively, hypercentral) subgroup of G satisfies the normalizer condition and so G has also this property.

Let H be a minimal non-nilpotent (respectively, non-hypercentral) subgroup of G. Then the quotient group G/G' is quasicyclic (respectively, quasicyclic or trivial). If the derived subgroup H' is not normal in G, then $N_G(H')$ is a proper subgroup of $N_G(N_G(H'))$. Since any radicable abelian ascendant subgroup is subnormal [6, p.136], the subgroup H/H' is subnormal in M/H'. As a consequence, H is subnormal in M. The quotient group M/H' has a finite series whose quotients satisfy the minimal condition on subgroups and so it is a Černikov group.

Let $t \in N_G(M) \setminus M$. Then $(H')^t$ is normal in M and therefore $M/(H' \cap (H')^t)$ is Černikov. Hence $H' = H' \cap (H')^t$. Now it is not difficult to prove that $H' = (H')^t$, a contrary with the choice of t. Thus a subgroup H' is normal in G. Since H/H' is ascendant in G/H', we conclude by the same argument as above that H/H' is subnormal in G/H' and consequently H is subnormal in G. □

Corollary 1. *Let G be a non-nilpotent (respectively, non-hypercentral) locally nilpotent group satisfying Max-$\overline{N}$ (respectively, Max-$\overline{ZA}$). If all proper normal subgroups of G are nilpotent (respectively, hypercentral), then G is minimal non-nilpotent (respectively, non-hypercentral) group.*

Proof. Let H be a proper subgroup of G and H be a minimal non-nilpotent (respectively, non-hypercentral) group. By Lemma 1 H is subnormal in G. Then the normal closure H^G of H in G is a proper normal subgroup of G and, moreover, H^G is non-nilpotent (respectively, non-hypercentral), a contradiction. Hence G is a minimal non-nilpotent (respectively, non-hypercentral) group. □

2. Proof of Theorem. ($\Leftarrow$) is obvious.

($\Rightarrow$) Let G be a solvable group satisfying Min-$\overline{N}$ (respectively, Min-$\overline{ZA}$). We assume that G is neither nilpotent (respectively, hypercentral) nor a Černikov group.

Then G contains a subnormal non-nilpotent (respectively, non-hypercentral) subgroup H in which any normal subgroup is nilpotent (respectively, hypercentral). Furthermore, if $H = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ is a finite subnormal series connecting H to G, then every quotient G_{i+1}/G_i satisfies the minimal condition on subgroups and so it is Černikov.

If the subgroup H is not locally nilpotent, then it contains a finitely generated non-nilpotent subgroup F. Then $H = H'F$ and the quotient H/H' is cyclic of prime power order, because in other case H is nilpotent (respectively, hypercentral) as product of two nilpotent (respectively, hypercentral) normal subgroups. So the intersection $H' \cap F$ is a nilpotent subgroup. Now it is easy to see that F is finite. Since the set of all subgroups containing F satisfies the minimal condition, G is a Černikov group. This is a contradiction. Hence H is a locally nilpotent group and therefore by Corollary 1 H is a minimal non-nilpotent (respectively, non-hypercentral) p-group for some prime p. As a consequence, G is a locally finite group. By the above argument G is a locally nilpotent group.

To complete the proof it is enough to suppose that G is a p-group and to prove that in this case it contains an HM^*-subgroup of finite index satisfying the normalizer condition. This is obvious if $n = 0$ because $G = H$ is a minimal non-nilpotent (respectively, non-hypercentral) group and so G/G' is quasicyclic.

By induction on n, we may suppose that G_{n-1} contains an HM^*-group T of finite index. Since G_{n-1} is normal in G, without loss of generality we can assume that T is normal in G. Then the quotient group G/T is Černikov. If D is a preimage of the finite residual D/T of G/T, then D is an HM^*-subgroup of finite index in G. In view of Lemma 1 D satisfies the normalizer condition. The proof is complete. □

Corollary 2. *Let G be a non-hypercentral (respectively, non-nilpotent) group. Then G satisfies Min-$\overline{ZA}$ (respectively, Min-$\overline{N}$) if and only if G satisfies the normalizer condition (respectively, G satisfies the normalizer condition and G' is nilpotent).*

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The Euler Tour of n-Dimensional Manifold with Positive Genus

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Abstract. In the paper [1] it is proved that abstract cubic n-dimensional torus [2] possesses a directed Euler tour of the same dimension. The result prompts to a new (virtual) device for transmission and reception of information. In the present paper it is shown that every abstract cubic n-dimensional manifold without borders, of positive genus possesses a n-dimensional directed Euler tour. This result has practical application.

Mathematics subject classification: 18F15, 32Q60, 32C10.
Keywords and phrases: Abstract manifold, abstract cube, vacuum, directed Euler tour.

Via a complex of multi-ary relations K^n and their groups of the direct homologies on the groups of integer numbers Z [5] we had defined an abstract cube and a cubic oriented manifold, without borders [3, 4].

Definition 1 [2]. *A cubic complex $\mathcal{I}^n$ is called an* ***abstract cubic n-dimensional manifold without borders*** *if the following proprieties are satisfied:*

A. *any $I^{n-1} \subset \mathcal{I}^n$ is a joint face exactly of two n-dimensional cubes from $\mathcal{I}^n$;*

B. *for $\forall I_i^n, I_j^n \in \mathcal{I}^n$, $i \neq j$, there exists a sequence of n-dimensional cubes $I_{i_1}^n = I_i^n, I_{i_2}^n \ldots, I_{i_q}^n = I_j^n$, where $I_r^n \cap I_{r+1}^n = I_{r,r+1}^{n-1}$, $r \in \{i_1, i_2, \ldots, i_{q-1}\}$;*

C. *for $\forall I^p \in \mathcal{I}^n$, $0 \leq p \leq n-1$, $\exists I^n \in \mathcal{I}^n$, where I^p is a face of I^n;*

D. *for disjoint cubes $\forall I_i^n, I_j^n \in \mathcal{I}^n$, $I_i^n \cap I_j^n = I^p$, $2 \leq p < n$, there exists a sequence of abstract cubes, $I_{i_1}^n = I_i^n, I_{i_2}^n, \ldots, I_{i_q}^n = I_j^n$, such that $\bigcap\limits_{j=1}^{q} I_{i_j}^n = I^p$.*

Definition 2 [3]. *The property of n-dimensional abstract cubic manifold without borders $V_p^n = K^n$, that* ***every*** *m-dimensional cube $I^m \subset K^n$, $0 \leq m \leq n$, belongs to 2^{n-m} n-dimensional cubes, is called a* ***normal cubiliaj****.*

The set of abstract multidimensional oriented manifolds without borders defined by abstract cubes can be classified in the same way as the classification of the abstract multidimensional orientated manifolds without borders defined by abstract simplexes is done [4]. By the groups of direct homologies on the integer numbers Z this is done for the complexes of multi-ary relations [5] and for the complexes of abstract cubes similarly [3]. As a result, manifolds of the sequence from Figure 1, denoted by $V_0^n(\square)$, $V_1^n(\square)$, $\ldots$ $V_p^n(\square)$, $\ldots$, represents as an element of every class of manifolds defined by abstract cubes.

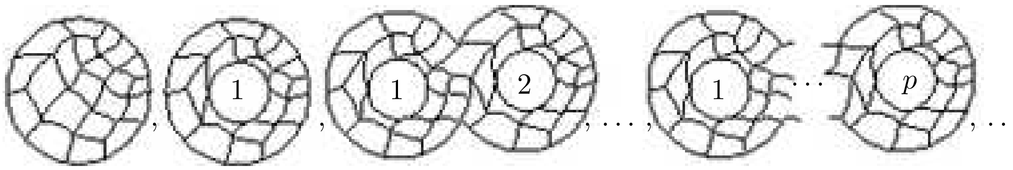

Figure 1

Let $K^n = \{\mathcal{I}^0, \mathcal{I}^1, \ldots, \mathcal{I}^n\}$ be an abstract cubic complex, where $\mathcal{I}^m$ is the set of all abstract m-dimensional cubes of K^n, $0 \le m \le n$ [2].

Definition 3 [2]. *A **linear** n-**dimensional chain** of the complex K^n is a sequence of oriented cubes, $I_1^n, I_2^n, \ldots, I_i^n, I_{i+1}^n, \ldots, I_k^n$, $k > 1$, where $I_i^n \cap I_{i+1}^n \in \in \mathcal{I}^{n-1}$, $I_i^n \cap I_{i+2}^n = \emptyset, \forall i \in \{1, 2, \ldots, k-1\}$. If I_i^n, I_{i+1}^n are coherences cubs for all $i = 1, 2, .., k-1$, then this chain is called a **linear** n-**dimensional oriented chain** of the complex K^n.*

Any abstract cubic oriented manifold without borders V^n can be transformed into a manifold in which all n-dimensional cubes are coherences [2, 5].

Definition 4 [2]. *If an abstract cubic oriented manifold without borders V^n possesses a linear directed n-dimensional chain $I_1^n, I_2^n, \ldots, I_i^n, I_{i+1}^n, \ldots, I_{\alpha_n}^n$ that contains every cube $I^n \in V^n$ exactly once, where $I_{i-1}^n \cap I_i^n = I_{i-1,i}^{n-1}$ and $I_i^n \cap I_{i+1}^n = I_{i,i+1}^n$ are common faces of the cube I_i^n, $2 \le i \le \alpha_{n-1}$, then this directed chain is called a **linear directed Euler chain** of dimension n of the manifold V^n, where I_i^n and I_{i+1}^n are coherences cubs for all $i = 1, 2, .., k-1$. If additionally it is verified that I_1^n and I_k^n are the same, then it is said that this directed chain is a **linear directed Euler tour** of dimension n of the manifold V^n.*

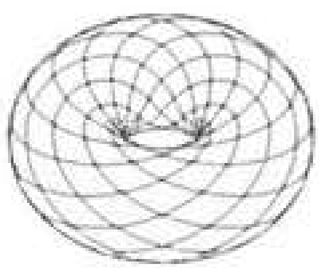

Figure 2

In the paper [1] is proved the theorem which affirms that the *unique* abstract oriented manifold without borders V_1^n that satisfies the property of normal cubiliaj is torus. In the paper [1] it is defined as an abstract oriented cartesian product. It is indicated that an abstract n-dimensional torus V_1^n is represented as a cartesian product of n abstract circumferences (abstract manifolds of dimension 1). This product and the property of normal cubiliaj, which is proper for torus, permitted us to indicate the conditions of existence of directed Euler tour of dimension n for an abstract n-dimensional torus.

Was proved

Theorem 1 [1]. *The abstract cubic n-dimensional concordant oriented torus V_1^n possesses a directed Euler tour of dimension n (see Figure 2).*

This theorem permits us to indicate a (virtual) device which is more efficient in solving Posthumus' problem of reception and transmission of information and in other applications [1].

In this paper we indicate the conditions of existence of n-dimensional directed Euler tour on any abstract cubic oriented n-dimensional manifold without borders, with positive genus (more holes than on the torus), and which is defined by abstract cubes.

Theorem 2. *The abstract cubic oriented manifold without borders V_p^n, where $p \geq 1$ is its genus, possesses a directed Euler tour of dimension n.*

Proof. To prove the theorem we use the induction on manifold's genus q.

For $q = 1$ the main theorem results by Theorem 1.

Let $q = p - 1$. We admit that the theorem is true for the manifold V_{p-1}^n.

Let $q = p$. We consider the abstract manifolds V_{p-1}^n and V_1^n, which possess normal cubiliaj and V_1^n has the same orientation as V_{p-1}^n. By induction, V_{p-1}^n and V_1^n possess directed Euler tours of dimension n, $C_{(p-1)}^n$ and $C_{(p)}^n$ respectively. Let $I_{i_1}^n$ be an arbitrary abstract cube of the torus V_{p-1}^n and $\overset{\circ}{I}{}_{i_{p-1}}^n$ its vacuum [2]. We take out this vacuum from the manifold V_{p-1}^n. Similarly we do with the torus V_1^n, taking out the vacuum $\overset{\circ}{I}{}_{j_1}^n$, which corresponds to the concordance given on an arbitrary n-dimensional cube $I_{j_1}^n$ of this torus. Therefore we obtain the manifolds $V_{p-1}^n \backslash \overset{\circ}{I}{}_{i_{p-1}}^n$ and $V_1^n \backslash \overset{\circ}{I}{}_{j_1}^n$, which possess the borders $I_{i_{p-1}}^n \backslash \overset{\circ}{I}{}_{i_{p-1}}^n$ and $I_{j_1}^n \backslash \overset{\circ}{I}{}_{j_1}^n$ respectively. We glue together these borders on the same concordant orientation, and we obtain the abstract oriented manifold V_p^n.

A directed n-dimensional torus is constructed on V_{p-1}^n beginning with an arbitrary n-dimensional cube, for example with $I_{i_{p-1}}^n$. By the $q = p - 1$ step of induction, V_{p-1}^n possesses a directed n-dimensional Euler tour, $C_{(p-1)}^n$, constructed of $I_{i_{p-1}}^n = I_{i_1}^n, I_{i_2}^n, \ldots, I_{i_{\beta_{p-1}^n}}^n$, where β_{p-1}^n is the number of all n-dimensional cubes on V_{p-1}^n. The last cube of the directed Euler tour is $I_{i_1}^n$, because this tour begins with $I_{i_{\beta_{p-1}^n}}^n$. This is why we take out the vacuum of the cube $I_{i_{\beta_{p-1}^n}}^n$, and so we have the subcomplex $C_{(p-1)}^n \backslash \overset{\circ}{I}{}_{i_{\beta_{p-1}^n}}^n$ of V_{p-1}^n. Similarly by $q = 1$ step of induction, the torus V_1^n possesses the directed Euler tour of dimension n, $C_{(1)}^n$, constructed of cubes $I_{j_1}^n, I_{j_2}^n, \ldots, I_{j_{\beta_1^n}}^n$. Taking out the vacuums $\overset{\circ}{I}{}_{i_{p-1}}^n$ and $\overset{\circ}{I}{}_{i_1}^n$ from $C_{(p-1)}^n$ and $C_{(1)}^n$ respectively, we glue together the borders $I_{i_{p-1}}^n \backslash \overset{\circ}{I}{}_{i_{p-1}}^n$ and $I_{i_1}^n \backslash \overset{\circ}{I}{}_{i_1}^n$ in the same concordance of orientation. This is possible because of the same orientation of V_{p-1}^n and V_1^n. Therefore the directed Euler tour on the manifold of genus p, V_p^n, is determined by the sequence of abstract n-dimensional cubes $I_{i_1}^n, I_{i_2}^n, \ldots, I_{i_{\beta_{p-1}^n}-1}^n, I_{j_1}^n, I_{j_2}^n, \ldots, I_{j_{\beta_1^n}-1}^n$. This sequence determines the directed Euler tour $C_{(p)}^n$ (see Figure 3, where $n = 2$ and the genus is p).

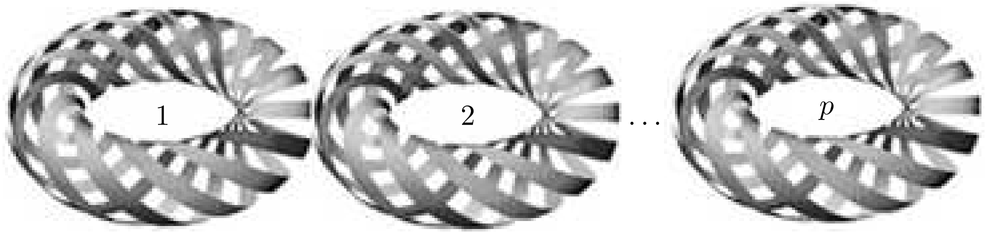

Figure 3

The theorem is proved.

Corollary 1. *Every manifold $V_p^n(\square)$, $p \geq 2$, possesses a cubiliaj, but this one is not a normal cubiliaj as it results from the proof of Theorem 2.*

Hypothesis. For every abstract spherical manifold V_0^n, defined by cubes, there exists a cube $I^m \subset V_0^n$, $0 \leq m \leq n-1$, which is incident to less than 2^{n-m} number of n-dimensional cubes.

The hypothesis is verified for $n = 2$ [6].

Remark 1. It is possible that the device from paper [1] can be done more efficiently using any manifold $V_p^n(\square)$, $p \geq 2$, instead of torus.

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Symmetric random evolution in the space $\mathbb{R}^6$

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Abstract. A closed-form expression for the transition density of a symmetric Markovian random evolution in the Euclidean space $\mathbb{R}^6$ is presented.

Mathematics subject classification: 82C70, 60K35, 60K37, 82B41.
Keywords and phrases: Random motion, finite speed, random evolution, random flight, transport process, explicit distribution, hypergeometric function.

This note is motivated by the recent works on random motions at finite speed (also called the random flight, isotropic transport process or, in a more general sense, random evolution) in the Euclidean space $\mathbb{R}^m$. Such processes in the Euclidean spaces of different dimensions have thoroughly been examined in a series of works. In the study of such processes the most desirable goal is undoubtedly their explicit distributions in the cases (very few indeed) when such distributions can be obtained. The explicit form of the distribution of a two-dimensional symmetric random motion at finite speed was derived (by different methods) by Stadje [9], Masoliver *et al.* [7], Kolesnik and Orsingher [6], Kolesnik [2]. The distribution of a random flight in $\mathbb{R}^3$ was given by Tolubinsky [11] and Stadje [10] in fairly complicated integral forms. Finally, the explicit form of the distribution of a random flight in the space $\mathbb{R}^4$ was obtained by Kolesnik [4] and by Orsingher and De Gregorio [8]. The random flights in arbitrary higher dimensions were examined by Kolesnik [1, 3, 5] and by Orsingher and De Gregorio [8], however no new distributions were obtained in these works for higher dimensions $m \geq 5$.

Since the exact probability laws of random flights in lower dimensions were derived by fairly complicated and sometimes tricky methods, the possibility of obtaining the explicit form of the distributions seemed very doubtful in higher dimensions $m \geq 5$.

However, a general and unified method of studying the random flights in arbitrary dimension was suggested in the works by Kolesnik [1, 3, 5] based on the analysis of the integral transforms of their distributions. This method, applied to the six-dimensional random motion, enables us to obtain the explicit probability law of the process and this result is presented here. While this method works in any dimension, the derivation of the explicit probability law in such a fairly high dimension $m = 6$ looks like a "lucky accident" which, apparently, cannot be extended in higher dimensions.

The distribution derived has a considerably more complicated form in comparison with those obtained for the dimensions 2 and 4. It is presented as a series of the finite sums of the Gauss hypergeometric functions which seemingly cannot be reduced to a more elegant formula. Nevertheless, this formula is of a certain interest because it gives the *explicit* form of the distribution which can be directly used for practical calculations and, on the other hand, it is a new step toward the most desirable goal, namely, constructing a general theory of distributions for random flights in the Euclidean spaces $\mathbb{R}^m$ of arbitrary dimension $m \geq 2$.

We consider the stochastic motion performed by a particle starting its motion from the origin $\mathbf{0} = (0,0,0,0,0,0)$ of the six-dimensional Euclidean space $\mathbb{R}^6$ at time $t = 0$. The particle is endowed with constant, finite speed c (note that c is treated as the constant norm of the velocity). The initial direction is a six-dimensional random vector with uniform distribution (Lebesgue probability measure) on the unit sphere

$$S_1 = \left\{ \mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 : \ \|\mathbf{x}\|^2 = \sum_{i=1}^{6} x_i^2 = 1 \right\}.$$

The particle changes direction at random instants which form a homogeneous Poisson process of rate $\lambda > 0$. At these moments it instantaneously takes on the new direction with uniform distribution on S_1, independently of its previous motion.

Let $\mathbf{X}(t) = (X_1(t), X_2(t), X_3(t), X_4(t), X_5(t), X_6(t))$ be the position of the particle at an arbitrary time $t > 0$ and denote by $d\mathbf{x}$ the infinitesimal element in the space $\mathbb{R}^6$.

At any time $t > 0$ the particle, with probability 1, is located in the six-dimensional ball of radius ct

$$\mathbf{B}_{ct} = \left\{ \mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 : \ \|\mathbf{x}\|^2 = \sum_{i=1}^{6} x_i^2 \leq c^2 t^2 \right\}.$$

The distribution $\Pr\{\mathbf{X}(t) \in d\mathbf{x}\}, \ \mathbf{x} \in B_{ct}, \ t \geq 0$, consists of two components. The singular component corresponds to the case when no Poisson event occurs in the interval $(0, t)$ and is concentrated on the sphere

$$S_{ct} = \partial \mathbf{B}_{ct} = \left\{ \mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6) \in R^6 : \ \|\mathbf{x}\|^2 = \sum_{i=1}^{6} x_i^2 = c^2 t^2 \right\}.$$

In this case the particle is located on the sphere S_{ct} and the probability of this event is

$$\Pr\{\mathbf{X}(t) \in S_{ct}\} = e^{-\lambda t}.$$

If one or more than one Poisson events occur, the particle is located strictly inside the ball $\mathbf{B}_{ct}$, and the probability of this event is

$$\Pr\{\mathbf{X}(t) \in \operatorname{int} \mathbf{B}_{ct}\} = 1 - e^{-\lambda t}.$$

The part of the distribution $\Pr\{\mathbf{X}(t) \in d\mathbf{x}\}$ corresponding to this case is concentrated in the interior

$$\operatorname{int} \mathbf{B}_{ct} = \left\{ \mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 : \ \|\mathbf{x}\|^2 = \sum_{i=1}^{6} x_i^2 < c^2 t^2 \right\},$$

and forms its absolutely continuous component.

Therefore there exists the density $p(\mathbf{x},t) = p(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6;\ t)$, $\mathbf{x} \in \operatorname{int} \mathbf{B}_{ct},\ t > 0$, of the absolutely continuous component of the distribution function $\Pr\{\mathbf{X}(t) \in d\mathbf{x}\}$. Our principal result is given by the following theorem.

Theorem. *For any $t > 0$ the density $p(\mathbf{x},t)$ has the form*

$$p(\mathbf{x},t) = \frac{16\lambda t e^{-\lambda t}}{\pi^3 (ct)^6} \left(1 - \frac{5}{6} \frac{\|\mathbf{x}\|^2}{c^2t^2} \right) +$$

$$+ \frac{e^{-\lambda t}}{2\pi^3 (ct)^6} \sum_{n=2}^{\infty} (\lambda t)^n (n+1)! \sum_{k=0}^{n+1} \frac{(k+1)(k+2)(n+2k+1)}{3^k (n-k+1)!(n+k-2)!} \times$$

$$\times F\left(-(n+k-2), k+3; 3; \frac{\|\mathbf{x}\|^2}{c^2t^2} \right) \tag{1}$$

where $\|\mathbf{x}\|^2 = \sum\limits_{i=1}^{6} x_i^2$,

$$F(\xi, \eta; \zeta; z) = {}_2F_1(\xi, \eta; \zeta; z) = \sum_{k=0}^{\infty} \frac{(\xi)_k (\eta)_k}{(\zeta)_k} \frac{z^k}{k!}$$

is the Gauss hypergeometric function and

$$(a)_k = a(a+1)\dots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}$$

is the standard Pochgammer symbol.

It's interesting to note that, since the first coefficient of the hypergeometric function in formula (1) is always negative for arbitrary n and k, the hypergeometric function itself represents, in fact, some polynomial. This is a characteristic feature of random flights in even-dimensional spaces.

The proof of the theorem is substantially based on the ideas and methods developed in the works by Kolesnik [1, 3, 5]. In particular, the applications of formulae (2.13) of [1] or (9) of [5] yield an explicit form of the Laplace transforms of the conditional characteristic functions corresponding to an arbitrary number of changes of directions, which then can be easily inverted. This gives the easy-treatable formulas for the conditional characteristic functions of the process from which the closed-form expressions for the conditional densities can be easily obtained by applying the classical Hankel inversion formula. By using then the total probability formula we immediately obtain our main result (1).

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On numerical algorithms for solving multidimensional analogs of the Kendall functional equation*

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Abstract. The numerical algorithms for solution of multidimensional analogs of the Kendall (Kendall-Takacs) equation, including effective fast algorithm are discussed.

Mathematics subject classification: 60K25, 68M20, 90B22.
Keywords and phrases: Priority classes, busy period distribution, fast algorithm.

The mathematical models of Queueing Systems, in particular Priority Systems, play an important role in the analysis, modelling and design of various modern networks and their components (Vishnevskii 2003 [1]). Thus, Priority models with switchover time represent a special class of Polling Systems and are widely used in broadband WLAN (Wireless Local Area Networks) (Vishnevskii and Semenova 2007 [2]). However, the obtained theoretical results for mentioned systems have a rather complicated mathematical structure. For example, the presented below multidimensional analogs of the famous Kendall equation represent a system of r recurrent functional equations expressed in terms of Laplace-Stieltjes transform (LST) which has no exact analytical solution. Besides that, this system is involved in the core of many important characteristics of the evolution of priority system, such as traffic intensity, probabilities of states, etc. In what follows we discuss some numerical algorithms of the solution of mentioned recurrent functional equation.

Let us consider the queueing systems $M|G|1$ with an exhaustive service. Denote by λ the parameter of input Poisson flow, by $B(x) = P\{B < x\}$ the distribution function of service, by $\beta(s) = \int\limits_0^\infty e^{-sx} dB(x)$ and by $\beta_1 = \int\limits_0^\infty x dB(x)$ its LST and first moment, respectively. Denote by $\Pi(x)$ the distribution function of the busy period, by $\pi(s)$ and π_1 the LST of $\Pi(x)$ and its first moment, respectively.

The following result is known as Kendall (Kendall-Takacs) functional equation.

The LST of the busy period is determined in the unique way from the functional equation

$$\pi(s) = \beta(s + \lambda - \lambda\pi(s)). \tag{1}$$

If $\lambda\beta_1 < 1$, *then*

$$\pi_1 = \frac{\beta_1}{1 - \lambda\beta_1}. \tag{2}$$

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Formulae (1) and (2) are referred to in most standard textbooks on Queueing Theory (see, for example Gnedenko 2005 [3], Takagi 1991 [4]).

Let us consider the queueing system $M_r|G_r|1$ with r priority classes of messages. The moments of i-messages' appearance represent a Poisson flow with parameter λ_i and the service times are random variables with the distribution functions $B_i(x)$, $i = 1, \ldots, r$. The priority classes are numbered in the decreasing order of priorities, namely, it is assumed that i-messages have a higher priority than j-messages if $1 \leq i, j \leq r$. It is also assumed that the server needs some time C_{ij} to switch the service process from the queue i to queue j. The length of the ij – switching C_{ij} is considered to be a random variable with the distribution function $C_{ij}(t)$, $1 \leq i, j \leq r$, $i \neq j$.

Also suppose that the switching C_{ij} depends only on index j, $C_{ij} = C_j$. The strategy in the free state is considered "reset". Denote by $\Pi_k(x)$ the distribution function of the busy period with the messages of the priority not less than k, $\sigma_k = \lambda_1 + \ldots + \lambda_k$, $\sigma = \sigma_r$, $\sigma_0 = 0$, $\beta_i(s)$, $c_j(s)$, $\pi(s)$, $\ldots$, $\pi_k(s)$ are the LST of the distribution functions $B_i(x)$, $C_j(x)$, $\Pi(x)$,$\ldots$, $\Pi_k(x)$, respectively. More details regarding the priority queueing systems with switchover times are presented in Mishkoy 2007 [5], Mishkoy et al. 2008 [6].

In what follows we suppose that the interrupted switching and interrupted service of message will be continued from the time point it was interrupted at. As it is shown in previous studies (see for example Mishkoy et. al 2008 [6]) busy periods distribution and traffic intensity can be determined solving the following recurrent system of functional equations:

$$\begin{aligned}\pi_k(s) = \frac{\sigma_{k-1}}{\sigma_k}\pi_{k-1}(s+\lambda_k) + \frac{\sigma_{k-1}}{\sigma_k}(\pi_{k-1}(s+\lambda_k[1-\pi_{kk}(s)])-\\ -\pi_{k-1}(s+\lambda_k))\nu_k(s+\lambda_k[1-\pi_{kk}(s)])+\\ +\frac{\lambda_k}{\sigma_k}\nu_k(s+\lambda_k[1-\pi_{kk}(s)])\pi_{kk}(s),\end{aligned} \tag{3}$$

$$\pi_{kk}(s) = h_k(s+\lambda_k[1-\pi_{kk}(s)]), \tag{4}$$

$$\nu_k(s) = c_k(s+\sigma_{k-1}[1-\pi_{k-1}(s)]), \tag{5}$$

$$h_k(s) = \beta_k(s+\sigma_{k-1}[1-\pi_{k-1}(s)\nu_k(s)]). \tag{6}$$

The system of functional equations presented above can be viewed as the generalization of the Kendall-Takacs equation (1). Namely, in Mishkoy 2007 [5] it is shown that for $C_j = 0$ and the number of priority classes $r = 1$, equation (1) follows from the mentioned system.

Note that equation (4) appear as a key equation of mentioned system of recurrent functional equations. Namely, to obtain $\pi_k(s)$, $h_k(s)$, $\nu_k(s)$ it is necessary to solve the functional equation (4) first. The numerical algorithms for solving (4) can be elaborated using the classical scheme of successive approximation (see for example Gnedenko et al. 1973 [7]):

...
$\pi_{kk}^{(0)}(s^*) := 0$; $n := 1$;
Repeat
$\pi_{kk}^{(n)}(s^*) := h_k(s^* + \lambda_k - \lambda_k \pi_{kk}^{(n-1)})$;
$inc(n)$;
Until $|\pi_{kk}^{(n)}(s^*) - \pi_{kk}^{(n-1)}(s^*)| < \varepsilon$;
$\pi_{kk}(s^*) := \pi_{kk}^{(n)}(s^*)$
...
or using the improved algorithms elaborated in Bejan 2006 [8]:
...
$\underline{\pi}_{kk}^{(n)}(0) := 0$; $\overline{\pi}_{kk}^{(n)}(0) = 1$;
Repeat
$\overline{\pi}_{kk}^{(n)}(s^*) = h_k(s^* + \lambda_k - \lambda_k \overline{\pi}_{kk}^{(n-1)}(s^*))$;
$\underline{\pi}_{kk}^{(n)}(s^*) = h_k(s^* + \lambda_k - \lambda_k \underline{\pi}_{kk}^{(n-1)}(s^*))$;
$inc(n)$;
Until $\frac{|\overline{\pi}_{kk}^{(n)}(s^*) - \underline{\pi}_{kk}^{(n-1)}(s^*)|}{2} < \varepsilon$;

$\pi_{kk}(s^*) := \frac{\overline{\pi}_{kk}^{(n)}(s^*) + \underline{\pi}_{kk}^{(n-1)}(s^*)}{2}$;
...

Unfortunately the procedure based on such type of algorithms is consuming very much time especially for large r ($r \geq 10$). A new effective algorithm for evaluation the characteristics can be elaborated using the method elaborated in Mishkoy, Grama 1994 [9] based on the binary tree data structure. The main idea of this procedure is to solve the system of equations for fixed r and s with respect to the collection of the unknown values $\pi_{kk}(s(k,j))$, were $k = r, \dots, 1, j = 1, \dots, 2^{r-k}$. To this end we define an iteration process each iteration m of which represents a recurrent procedure constructed by means of the binary tree in the following manner:

$$\sigma_k \pi_k(s(k,j)^{(m)}) = \sigma_{k-1}\pi_{k-1}(s(k-1,2j-1)^{(m)})+$$
$$+\sigma_{k-1}\{\pi_{k-1}(s(k-1,2j)^{(m)}) - \pi_{k-1}(s(k-1,2j-1)^{(m)})\}\times$$
$$\times\nu_k(s(k-1,2j)^{(m)}) + \lambda_k \nu_k(s(k-1,2j)^{(m)})\pi_{kk}^{(m)}(s(k,j)^{(m)}),$$
$$\pi_{kk}^{(m)} s(k,j)^{(m)} = h_{k-1}(s(k-1,2j)^{(m)}), s(r,1)^{(m)} = s,$$
$$s(k-1,2j-1)^{(m)} = s_{k1}(s(k,j)^{(m)}), s(k-1,2j)^{(m)} = s_{k2}(s(k,j)^{(m)}),$$

where $k = r, \dots, 1, j = 1, \dots, 2^{r-k}$, $s_{ki}(s(k,j)^{(m-1)})$, $i = 1,2$, are determined by $s_{k1}(s) = s + \lambda_k$, $s_{k2}(s) = s + \lambda_k(1 - \pi_{kk}(s))$ with $\pi_{kk}^{(m-1)}(s(k,j)^{(m-1)})$ instead of $\pi_{kk}(s)$ and $\nu_k(s)$, $h_k(s)$, are determined by (5), (6), $m = 1, 2, \dots$, the initial values for $\pi_{kk}^{(0)}(s(k,j)^{(0)})$, $k = r, \dots, 1, j = 1, \dots, 2^j$, may be taken arbitrary in the interval [0,1].

The iteration process may be easily performed by the computer and turns out to provide very rapid calculations for the above mentioned characteristics even for enough large value of k.

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Classification of $\mathit{Aff}\,(2,\mathbb{R})$-orbit's dimensions for quadratic differential system

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Abstract. Affine invariant conditions for $\mathit{Aff}\,(2,\mathbb{R})-$orbit's dimensions are defined for two-dimensional autonomous quadratic differential system.

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Keywords and phrases: Differential system, Lie algebra of the operators, $\mathit{Aff}\,(2,\mathbb{R})-$orbit.

Consider two-dimensional quadratic differential system

$$\frac{dx^j}{dt} = a^j + a^j_\alpha x^\alpha + a^j_{\alpha\beta}x^\alpha x^\beta \quad (j,\alpha,\beta=\overline{1,2}), \tag{1}$$

where the coefficient tensor $a^j_{\alpha\beta}$ is symmetrical in lower indices in which the complete convolution holds.

Consider also the group $\mathit{Aff}\,(2,\mathbb{R})$ of affine transformations given by the equalities:

$$\bar{x}^1 = \alpha x^1 + \beta x^2 + h^1, \quad \bar{x}^2 = \gamma x^1 + \delta x^2 + h^2, \quad \Delta = det\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \neq 0,$$

where $\alpha,\beta,\gamma,\delta,h^1,h^2$ take real values.

Further will use the notations

$$\begin{gathered} a^1=a,\quad a^2=b,\quad a^1_1=c,\quad a^1_2=d,\quad a^2_1=e,\quad a^2_2=f,\quad a^1_{11}=g,\quad a^1_{12}=h,\\ a^1_{22}=k,\quad a^2_{11}=l,\quad a^2_{12}=m,\quad a^2_{22}=n,\quad x^1=x,\quad x^2=y. \end{gathered} \tag{2}$$

According to [1] and taking into consideration (2), the representation operators of the group $\mathit{Aff}\,(2,\mathbb{R})$ in the space of coefficients and variables of the system (1) will take the form

$$\begin{aligned}
X_1 &= x\frac{\partial}{\partial x} + a\frac{\partial}{\partial a} + d\frac{\partial}{\partial d} - e\frac{\partial}{\partial e} - g\frac{\partial}{\partial g} + k\frac{\partial}{\partial k} - 2l\frac{\partial}{\partial l} - m\frac{\partial}{\partial m},\\
X_2 &= y\frac{\partial}{\partial x} + b\frac{\partial}{\partial a} + e\frac{\partial}{\partial c} + (f-c)\frac{\partial}{\partial d} - e\frac{\partial}{\partial f} + l\frac{\partial}{\partial g} +\\
&+(m-g)\frac{\partial}{\partial h} + (n-2h)\frac{\partial}{\partial k} - l\frac{\partial}{\partial m} - 2m\frac{\partial}{\partial n},\\
X_3 &= x\frac{\partial}{\partial y} + a\frac{\partial}{\partial b} - d\frac{\partial}{\partial c} + (c-f)\frac{\partial}{\partial e} + d\frac{\partial}{\partial f} - 2h\frac{\partial}{\partial g} - k\frac{\partial}{\partial h} +\\
&+(g-2m)\frac{\partial}{\partial l} + (h-n)\frac{\partial}{\partial m} + k\frac{\partial}{\partial n},\\
X_4 &= y\frac{\partial}{\partial y} + b\frac{\partial}{\partial b} - d\frac{\partial}{\partial d} + e\frac{\partial}{\partial e} - h\frac{\partial}{\partial h} - 2k\frac{\partial}{\partial k} + l\frac{\partial}{\partial l} - n\frac{\partial}{\partial n},\\
X_5 &= \frac{\partial}{\partial x} - c\frac{\partial}{\partial a} - e\frac{\partial}{\partial b} - 2g\frac{\partial}{\partial c} - 2h\frac{\partial}{\partial d} - 2l\frac{\partial}{\partial e} - 2m\frac{\partial}{\partial f},\\
X_6 &= \frac{\partial}{\partial y} - d\frac{\partial}{\partial a} - f\frac{\partial}{\partial b} - 2h\frac{\partial}{\partial c} - 2k\frac{\partial}{\partial d} - 2m\frac{\partial}{\partial e} - 2n\frac{\partial}{\partial f}.
\end{aligned} \tag{3}$$

The operators (3) form a six-dimensional Lie algebra [1]. Let $\tilde{a} = (a, b, ..., n) \in E^{12}(\tilde{a})$, where $E^{12}(\tilde{a})$ is the Euclidean space of the coefficients of the right-hand sides of the system (1). Denote by $\tilde{a}(q)$ the point from $E^{12}(\tilde{a})$ that corresponds to the system, obtained from the system (1) with coefficients $\tilde{a}$ by a transformation $q \in \mathit{Aff}(2,\mathbb{R})$.

Definition 1. *Call the set $O(\tilde{a}) = \{\tilde{a}(q) | q \in \mathit{Aff}(2,\mathbb{R})\}$ the $\mathit{Aff}(2,\mathbb{R})$ – orbit of the point $\tilde{a}$ for the system* (1).

It is known from [1] that

$$dim_{\mathbb{R}} O(\tilde{a}) = rank M_1, \tag{4}$$

where M_1 is the following matrix

$$M_1=\begin{pmatrix}
a & 0 & 0 & d & -e & 0 & -g & 0 & k & -2l & -m & 0\\
b & 0 & e & -c+f & 0 & -e & l & -g+m & -2h+n & 0 & -l & -2m\\
0 & a & -d & 0 & c-f & d & -2h & -k & 0 & g-2m & h-n & k\\
0 & b & 0 & -d & e & 0 & 0 & -h & -2k & l & 0 & -n\\
-c & -e & 2g & -2h & -2l & -2m & 0 & 0 & 0 & 0 & 0 & 0\\
-d & -f & 2h & -2k & -2m & -2n & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \tag{5}$$

constructed on coordinate vectors of operators (3). Denote by $\Delta_{i,j,k,l,m,n}$ the minor of the 6th order of the matrix M_1, constructed on columns i, j, k, l, m, n ($i, j, k, l, m, n \in \{1, \ldots, 12\}$).

For the system (1) from [2], [4] are known the following center-affine comitants and invariants

$$
\begin{gathered}
I_1=a^{\alpha}_{\alpha},\ I_2=a^{\alpha}_{\beta}a^{\beta}_{\alpha},\ I_3=a^{\alpha}_{p}a^{\beta}_{\alpha q}a^{\gamma}_{\beta\gamma}\varepsilon^{pq},\ I_4=a^{\alpha}_{p}a^{\beta}_{\beta q}a^{\gamma}_{\alpha\gamma}\varepsilon^{pq},\\
I_5=a^{\alpha}_{p}a^{\beta}_{\gamma q}a^{\gamma}_{\alpha\beta}\varepsilon^{pq}, I_7=a^{\alpha}_{pr}a^{\beta}_{\alpha q}a^{\gamma}_{\beta s}a^{\delta}_{\gamma\delta}\varepsilon^{pq}\varepsilon^{rs},\ I_8=a^{\alpha}_{pr}a^{\beta}_{\alpha q}a^{\gamma}_{\delta s}a^{\delta}_{\beta\gamma}\varepsilon^{pq}\varepsilon^{rs},\\
I_9=a^{\alpha}_{pr}a^{\beta}_{\beta q}a^{\gamma}_{\gamma s}a^{\delta}_{\alpha\delta}\varepsilon^{pq}\varepsilon^{rs},\ K_3=a^{\alpha}_{\beta}a^{\beta}_{\alpha\gamma}x^{\gamma},\ K_4=a^{\alpha}_{\gamma}a^{\beta}_{\alpha\beta}x^{\gamma},\\
K_5=a^{p}_{\alpha\beta}x^{\alpha}x^{\beta}x^{q}\varepsilon^{pq},\ K_6=a^{\alpha}_{\alpha\beta}a^{\beta}_{\gamma\delta}x^{\gamma}x^{\delta},\ K_7=a^{\alpha}_{\beta\gamma}a^{\beta}_{\alpha\delta}x^{\gamma}x^{\delta},\\
K_{11}=a^{p}_{\alpha}a^{\alpha}_{\beta\gamma}x^{\beta}x^{\gamma}x^{q}\varepsilon_{pq},\ K_{12}=a^{\alpha}_{\beta}a^{\beta}_{\alpha\gamma}a^{\gamma}_{\delta\mu}x^{\delta}x^{\mu},\ K_{13}=a^{\alpha}_{\gamma}a^{\beta}_{\alpha\beta}a^{\gamma}_{\delta\mu}x^{\delta}x^{\mu},\\
K_{17}=a^{\alpha}_{\beta\nu}a^{\beta}_{\alpha\gamma}a^{\gamma}_{\delta\mu}x^{\delta}x^{\mu}x^{\nu},\ K_{21}=a^{p}x^{q}\varepsilon_{pq},\ K_{23}=a^{p}a^{q}_{\alpha\beta}x^{\alpha}x^{\beta}\varepsilon_{pq}.
\end{gathered}
\tag{6}
$$

According to [3], write a transvectant of index k for binary forms f and φ as follows

$$
(f,\varphi)^{(k)}=\sum_{h=0}^{k}(-1)^h\left(\begin{array}{c}k\\h\end{array}\right)\frac{\partial^k f}{\partial x^{k-h}\partial y^h}\frac{\partial^k\varphi}{\partial x^h\partial y^{k-h}}. \tag{7}
$$

According to [4], the transvectant (7) on two comitants of the system (1) is a comitant (invariant) of this system too.

For the system (1) from [4] will use the following transvectants

$$
T_5=(C,K)^{(2)},\ T_{16}=(C,D)^{(1)}, \tag{8}
$$

where $C=K_5,\ D=I_1K_1K_2+K_2K_3-K_2K_4+\frac{1}{2}I_1^2K_5-\frac{1}{2}I_2K_5-I_1K_{11}-2K_1^2K_{21}+4K_6K_{21}-2K_7K_{21},\ K=\frac{1}{2}(K_1^2-K_7)$.

Denote by $\tilde{a}(\tau)$ the point from the space $E^{12}(\tilde{a})$, that corresponds to the system , obtained from the system (1) with coefficients $\tilde{a}$ by the transformation of translation $\tau: x=\bar{x}+x_0,\ y=\bar{y}+y_0$. It is evident $\tilde{a}(\tau)=\tilde{a}(x_0,y_0)$. According to [6], if $I(\tilde{a})$ is the center-affine invariant of the system (1), then the polynomial

$$
K(\tilde{a},x,y)=I(\tilde{a}(x_0,y_0))|_{\{x_0=x,y_0=y\}}
$$

is a affine comitant of this system. Then considering (6), we construct the following affine comitants:

$$
Af_i(\tilde{a},x,y)=I_i(\tilde{a}(x_0,y_0))|_{\{x_0=x,y_0=y\}}\quad (i=3,4,5). \tag{9}
$$

Lemma 1. *For $\beta\neq 0$, the rang of matrix M_1 is equal to six, where*

$$
\beta=27I_8-I_9-18I_7 \tag{10}
$$

and $I_8,\ I_9,\ I_7$ are from (6).

Proof. We suppose the contrary $\beta\neq 0$, and all minors of the sixth order of the matrix M_1 are zero. We observe that β is the discriminant of the cubic form K_5. It is known [2] that for $\beta\neq 0$ there exists a linear transformation of the system (1). Such that the comitant K_5 from (6) evaluated for the transformed system takes the form:

$$K_5 = x(x^2+\delta y^2),\ (\delta=\pm 1,\ l=-1, k=0,\ g=2m,\ n=2h-\delta). \tag{11}$$

In case of (10) we obtain $\Delta_{3,4,7,8,9,10} = -16\delta h^4 = 0$, $\delta \neq 0$, this implies $h=0$. Taking into consideration $h=0$, we obtain $\Delta_{3,6,7,9,11,12} = -16\delta^4 m^2 = 0$, this implies $m=0$. Taking into consideration $m=0$, we obtain $\Delta_{5,6,9,10,11,12} = 8\delta^4 \neq 0$. We obtain a contradiction. Lemma 1 is proved. □

Lemma 2. *For $K_5 \not\equiv 0$, $\beta = 0$, the rang of matrix M_1 is equal to six if and only if*

$$Af_5^2 + T_5^2 + T_{16}^2 + Kom^2 \not\equiv 0, \tag{12}$$

where

$$\begin{aligned} Kom \equiv\ & I_1K_1^2(2K_1^2K_2 - 2K_2K_6 - K_1K_{11}) - 2K_1^3(K_2K_4 + 2K_7K_{21}) + \\ & + 4K_1K_6^2K_{21} + K_1^2[2K_4K_{11} + K_2K_{13} + 2K_{23}(K_6+K_7)] - 4K_6^2K_{23}, \end{aligned} \tag{13}$$

and $I_1, K_1, K_2, K_4, K_5, K_6, K_7, K_{11}, K_{12}, K_{13}, K_{16}, K_{17}, K_{21}, K_{23}$ are from (6), T_5, T_{16} are from (8), β is from (9), Af_5 is affine comitant of the system (1).

Proof. It is known from [2] that for $K_5 \not\equiv 0$ and $\beta = 0$ there exists such affine transformation exists that the comitant K_5 will take the form:

$$K_5 = x^2(x+\delta y),\ (\delta = 0,1;\ l=-1, k=0,\ g=2m+\delta,\ n=2h). \tag{14}$$

Let (12) be not true. It is easy to verify that in case (14) $T_5 \equiv 8h^2\delta y + 8h(3h - 2m\delta - \delta^2)x$, from $T_5 \equiv 0$ we have $h=0$. Taking into consideration $h=0$ in the case (14), we obtain $Af_5 \equiv -d(\delta^2 + 4m\delta + 5m^2)$. From $Af_5 \equiv 0$ it follows

$$d[\delta + (1-\delta)m] = 0,\ (\delta = 0,1) \tag{15}$$

In the case $\delta = 1$ from (15) we obtain $d=0$, $T_{16} \equiv 6m(cf+ef-f^2-2am-2bm)x^4$, and Kom is the CT comitant [5], $Kom \equiv (1+3m)^3(cf+ef-f^2-2am-2bm)x^6$. It is evident if $T_{16} \equiv Kom \equiv 0$, we obtain:

$$h = d = cf + ef - f^2 - 2am - 2bm = 0. \tag{16}$$

In the case when $\delta = 0$ from (15) we obtain: $m=0$, $Kom \equiv 0$, $T_{16} \equiv 6dfx^4 - 6d^2x^3y$ or $d=0$, $Kom \equiv 27m^3(cf-f^2-2am)x^6$, $T_{16} \equiv 6m(cf-f^2-2am)x^4$. It is evident if $T_{16} \equiv Kom \equiv 0$ it follows:

$$h = d = m(cf - f^2 - 2am) = 0. \tag{17}$$

It is easy to verify that in cases (16), (17) all 6^{th} order minors of the matrix M_1 are equal to zero. The necessity of Lemma 2 is proved.

Prove the sufficiency. In the case when (14) holds for $\delta = 1$ we obtain: $\Delta_{3,4,7,8,11,12} = -8h^4$, for $h=0$ $\Delta_{1,3,4,6,7,11} = 2d^3(1+3m)^2$, $\Delta_{4,5,6,10,11,12} = -8dm^4$;

for $h = d = 0$ $\Delta_{2,5,6,10,11,12} = 4m^3(cf + ef - f^2 - 2am - 2bm)$, $\Delta_{2,3,5,7,10,11} = 2(1 + 2m)(1 + 3m)(cf + ef - f^2 - 2am - 2bm)$.

In the case when (14) holds for $\delta = 0$ we obtain: $\Delta_{3,4,7,8,10,11} = -24h^4$, for $h = 0$ $\Delta_{1,3,4,6,7,11} = 18d^3m^2$; for $h = m = 0$, $\Delta_{1,3,4,5,7,10} = 2d^3$. If $h = d = 0$, we obtain $\Delta_{2,3,5,7,10,11} = 12m^2(cf - f^2 - 2am)$. Lemma 2 is proved. □

Lemma 3. *For $K_5 \equiv 0$, the rang of matrix M_1 is equal to six if and only if $Af_4(Af_4 - Af_3) \not\equiv 0$, where Af_3, Af_4 are affine comitants of the system* (1).

Proof. For $K_5 \equiv 0$, according to (2), (6) ($l = k = 0,\ g = 2m,\ n = 2h$) we obtain that $Af_4(Af_4 - Af_3) \equiv 27(eh^2 + chm + fhm - dm^2)^2$, and all 6^{th} order minors of the matrix M_1 consist of the factor expession $eh^2 + chm - fhm - dm^2$. Taking into consideration the above mentioned proof of the Lemma 3 follows. □

From Lemmas 1,2,3 follows:

Theorem. *The dimension of $Aff(2,\mathbb{R})$-orbit of the system (1) is equal to six if and only if*

$$\beta \neq 0, \textit{ or}$$
$$\beta = 0,\ K_5(Af_5^2 + T_5^2 + T_{16}^2 + Kom^2) \not\equiv 0, \textit{ or}$$
$$\beta = 0,\ K_5 \equiv 0,\ Af_4(Af_4 - Af_3) \not\equiv 0,$$

where K_5 from (6)*, T_5, T_{16} from* (8)*, β from* (9)*, Kom from* (13)*, Af_3, Af_4, Af_5 from* (9) *are affine comitants of the system* (1).

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Professor Mihail Popa – 60th anniversary

On May 15th 2008 the university professor Mihail Popa will be 60 years old. Professor Mihail Popa is Habilitated Doctor in Mathematical and Physical Sciences and Director of the Institute of Mathematics and Informatics of the Academy of Sciences of Moldova. He is a well known scientist for his work on differential equations and for his rich scientific and didactic professional activity he was awarded the Prize of the Presidium of the Academy of Sciences of Moldova and the Constantin Sibirschi Prize.

Mihail Popa was born in the village of Vălcineţ, in the district of Călăraş, in the Republic of Moldova. In 1963 he graduated from the elementary school of the village of Temeleuţi, the district of Călăraşi; in 1966 he finished the secondary school nr.1 of the city of Călăraş and in 1971 he graduated from the Faculty of Physics and Mathematics of the State University of Chişinău. In 1978 he was enrolled as a student in the Ph. D. program of the Institute of Mathematics and Computer Sciences of the Academy of Sciences of Moldova (in the specialty 01.01.02 – Differential Equations).

In 1979 professor Popa defended his Ph.D. Thesis in Mathematical and Physical Sciences at Gorki University. His thesis was supervised by the well known academician Constantin Sibirschi. In 1992 he defended his Habilitation Doctor's thesis in Kiev at the Ukrainian Academy of Sciences.

The professional activity of professor Popa took place at the I.M.I of the Academy of Sciences of Moldova and it evolved as follows: upper rank member of the Laboratory (1975 – 1977), lower rank Scientific Collaborator (1977 – 1980), Scientific Secretary of I.M.I. (1980 - 1999), Assistant-Director of I.M.I. (1999 – 2005), Director of I.M.I. (2005 up to now).

The scientific interests of professor Popa involve the use of invariant processes in the qualitative study of differential equations. A new viewpoint was established in the qualitative theory of differential equations based on the method of algebraic invariants founded by the academician C. Sibirschi. This new viewpoint consists in the application of the Lie algebras of operators of representations of the linear groups in the space of coefficients of systems of polynomial differential equations and of the graduate algebras of invariants and comitants to the geometry of these systems. This new viewpoint extended the scientific domain where it was applied, so as to involve methods of group analysis. This brought forth the study of the graduate algebras of invariants of differential equations with the help of generating functions and of Hilbert series. A sequence of generating series and of Hilbert series for diverse graduate algebras of comitants and invariants of differential systems was obtained for which it is possible to evaluate their Krull dimension. A substantial part of the results are about the study of the Lie algebra of operators L_4 for the center-affine group and its representations in the space of coefficients of autonomous systems of polynomial ordinary differential equations (S.O.D.E) of first order. Another category of results is connected to the classification of the dimensions of orbits of polynomial S.O.D.E with respect to the admissible groups. A new direction in the use of Lie algebras and of algebras of invariants is the extension to autonomous multidimensional systems of first order differential equations with polynomial right-hand sides, which have constant coefficients.

Professor Popa is the author of over 80 scientific publications, among them two monographs about applications of algebras to systems of differential equations and a textbook for Master's Degree students about Lie algebras and about systems of differential equations.

The research of professor Popa drew the attention of many scientific researchers in differential equations and in algebra. His collaboration with specialists of the Université de Limoges (France), University of Minsk (Belarus), University of Piteşti (România), The Center of Research in Mathematics of Montreal (Canada), the University of Lund (Sweden), the Institute of Mathematics of the Romanian Academy (Bucureşti), the State University M.Lomonosov of Moscow indicate the importance of his research.

From February to June 2001 M. Popa was Invited Professor at the Université de Limoges (France), where he gave courses and seminars for students and professors. M. Popa was the director for 6 Ph. D. theses and he presently has two Ph.D. students whose theses are in the final stages. From the year 1996 professor Popa lectured for students in their fourth and fifth year at the State University of Tiraspol (in Chisţinău), where he won by competition the position of Full Professor. Under his direction The Seminar on Differential Equations and Algebras was organized at the Tiraspol State University and since 2002 this Seminar for Master and Ph. D. students and professors meets regularly.

Professor Popa is the President of the Scientific Committee of the Institute of Mathematics and Informatics of the Academy of Sciences of Moldova, he is a member of the Committee of Experts of the CNAA, a member of the Editorial Board of the Bulletin of the Academy of Sciences in Mathematics (Moldova) and of ROMAI Journal (Romania).

He was the project director for the organization of the Workshop "Qualitative Study of Differential Equations" (Chişinău, February 14-15, 2003), of the Second Conference of the Mathematical Society of Moldova, (Chişinău, August 17-19, 2004), and of the International Conference "Algebraic Systems and their Applications to Differential Equations and to other mathematical domains" (Chişinău, August 21-23, 2007).

We congratulate professor Popa on the occasion of his 60th anniversary and we wish him good health, prosperity and new successes in his scientific and didactic activity.

Academician Radu Miron – Eighty Years of Life and Sixty Years of Efforts

Now, when Academician Professor Doctor Radu Miron is 80 years old, he can proudly look at the accomplishments that he has made during his scientific life. His vast and constructive scholarly work has brought him international recognition and has established him as an irrefutable leader of the Romanian school of geometry. A remarkably gifted professor, endowed with the grace of speaking, he has left an indelible mark upon numerous generations of mathematicians. Being highly concerned with the teaching of geometry at all levels, he wrote books for pupils and students, as well as monographs having a high scientific level meant for researchers.

Academician Radu Miron was born at Codăeşti, in Vaslui County, on October 3, 1927, Romania. He attends the courses of the primary school in his native village. Then he finished his high school studies at the technical school in Bârlad. In 1948 he enrolled at the Faculty of Mathematics and Physics of the "Al. I. Cuza" University of Iaşi. Here he quickly attracted the attention of the famous and exigent teaching staff, so that at the beginning of the 3rd year he was appointed an instructor and shortly afterwards he was promoted to the assistantship. A year after his graduation from the faculty he has begun preparing his Doctor's Degree - the chosen speciality being Mechanics - at the Mathematical Institute of the Romanian Academy, the

Iaşi branch, his main adviser being Academician Mendel Haimovici. Thus he begins the research activity by which he will make his mark as an extremely valuable mathematician. Steady and full of energy, he has gone through the stages of a remarkable scientific and didactic career. In 1956 he became Assistant Professor at the Faculty of Mathematics and Physics and senior researcher at the Mathematical Institute of the Academy. The following year, he received his Ph.D. in the field of Physics and Mathematics, and in 1963 he became Associated Professor at the Faculty of Mathematics and Mechanics of the "Al.I.Cuza" University of Iaşi and head of a department at the Mathematical Institute of the Academy. The same year he was awarded the Ministry of Education prize for a series of papers published in 1962. The year 1968 brought him the *Gh. Tzitzeica Award* of the Romanian Academy for his monograph *"The Geometry of the Myller configurations".* In 1965 he became Full Professor at the Faculty of Mathematics. Between 1972 and 1976 he was the dean of the Faculty of Mathematics. In 1973 he receives the title of Doctor Docent. As Head of the Department of Geometry for several years and member of the faculty executive council, he has brought a decisive contribution to the proper carrying out of the activity in the faculty, as well as to the progress of the Romanian school of Geometry. In 1991 he was elected Member of the Romanian Academy, the highest forum of Romanian spirituality and the highest recognition which a scholar may receive. He retired from the Faculty of Mathematics in 1998 receiving the homage of his colleagues and former students and he continued to act as Consulting Professor at the Faculty of Mathematics and as Full Professor at the private University "Petre Andrei" of Iaşi.

Since 1972, Professor R.Miron has been scholarly adviser for Ph.D.Thesis. Hi was scientific adviser of more than 30 doctors of sciences from Romania, Japan, Italy, Hungary, Vietnam and more than two hundreds of master theses.

Professor Radu Miron began his scientific activity, as he himself confessed, as an *apprentice at the great school of the Iaşi Mathematics Seminar* in a period when its founders, the Academicians A1. Myller and O. Mayer, were still active. The genesis of research directions of Academician Radu Miron due to the scientefic activity of the remarkable members of the Romanian Academy A1. Myller, O. Mayer, S. Stoilow, S. Procopiu, V. Volcovici, Gh. Vranceanu, G. Moisil, D. Mangeron, M. Haimovici, D. Barbilian, N. Teodorescu and of Professors A. Climescu, I. Creanga, A. Haimovici, Gh. Gheorghiev and I. Popa, the brilliant representatives of the previous generations of mathematicians of Iaşi.

Reviewing almost two hundred fifty titles that have been published among them being 30 textbooks, books and monographs, one can clearly perceive the coordinates of evolution of the scientific and didactic thinking of Professor Radu Miron. All these reflect his inventive spirit and his special concern for the innovative and for the comprehensible. They are well known, often cited and used as basic references.

His original scholarly work falls within three main fields: the differential geometry, the applications of differential geometry, the basic elements of geometry and algebra. Professor's Radu Miron preoccupations in the three fields are closely interwoven, they have influenced, conditioned and augmented each other. In general, his research has a theoretical character. But some of it was related to the applications in Analytical Mechanics, Theoretical Physics, Optimal Control, Biology, etc.

The first three papers in 1955 are devoted to the differential geometry of the surfaces from the tree-dimensional Euclidean space. They already had showed his interest in the geometry of the nonholonomic manifolds begining from 1953 under the influence of the works of the geometers E. Cartan, Gh. Vranceanu and M. Haimovici concerning the geometrization of the nonholonomic mechanical systems. The problem of the geometrization of the nonholonomic mechanical systems with scleronomous links that have derivative systems was still unsolved. The solving required a complicated analytic apparatus and this had made E. Cartan's assert that the problem is either impossible to be solved or it must be treated from case to case, losing thus its theoretical interest. By generalizing a method due to M. Haimovici and by inventing a special technique the young researcher Radu Miron succeeds in building a faithful geometrization of the above mentioned mechanical systems. It was expounded in detail in the thesis for his doctor's degree entitled *"The Problem of the Geometrization of the Nonholonomic Mechanical Systems"*.

The framework of the problem his Ph.D. Thesis dealt with was greatly enlarged afterwards and the methods he used were extended to the study of nonholonomic manifolds for the Riemannian spaces with nondefinite metric. Thus, after a first success in the field of applications, Professor Radu Miron returns to the differential geometry with six sizable papers devoted to the nonholonomic manifolds in Riemannian spaces. The results he has obtained are so varied and profound that one can safely assert that the theory of the nonholonomic manifolds in the Riemannian spaces is a definite Romanian creation due mainly to Gh. Vranceanu, M. Haimovici and R. Miron.

In 1960 Professor's Radu Miron research turns to a new trend that will lead him to a numerous new and interesting results and that will give him the opportunity of bringing a substantial homage to his Professors Al. Myller and O. Mayer by the continuation and the brilliant development of their work. He studied the so-called Myller configurations. The results were presented in his specific manner, that is geometrically, clearly, concisely, using an elevated language in the monograph *"The Geometry of the Myller Configurations"* published in 1966. For this monograph the Romanian Academy awarded him the *Gh. Tzitzeica Prize* in 1968.

The notion of the Myller configuration was then extended to spaces with the affine connection. The study of the Myller configurations led him to the problem of the local existence of the manifolds immersed in spaces with the affine connection. In order to solve this problem he cooperated with Prof. Dr. Dan Papuc. On this occasion they have developed a theory of the distributions in space with affine connections.

Professor Radu Miron was interested in the theory of the connections on several occasions. Starting from Norden's results concerning the conjugated connection, he introduced the notion of Norden space. Professor Radu Miron also brings a significant contribution to the theory of the Weyl spaces by the study of the conformal movements in these spaces. In the theory of G-structures, he particularly studied the properties of the connection compatible with these. Dealing with the almost conformal symplectic structure, he determined the set of all linear connections compatible with such structures and established the main properties of this set.

The research activity in Finslerian geometry and its generalization that Professor Radu Miron has introduced have brought him great satisfaction and accomplishments both in the theoretical aspects and in their applications. This topic aroused his interest as early as the 1960's when he published two papers dealing with Finsler spaces with indefinite metric, both cited in the monograph devoted to Finsler spaces by Prof. Dr. M. Matsumoto from Japan. In 1974 he firmly comes back to Finsler spaces with an outstanding contribution by the building of a field of orthonormal frames intrinsically associated to an n-dimensional Finsler space. This field has as special cases the field of frames used by L. Berwald and A. Moor in the 2 and respectively 3 dimensions. It was called the *Miron frame* by M. Matsumoto in his monograph devoted to Finsler spaces.

The relation of friendship with Professor M. Matsumoto led Professor Radu Miron to getting deeply into the theory developed by the Japanese school, theory which he would soon include in a larger one and provide with the modern and efficient methods.

Simultaneously he has been preoccupied with the stage of the research in the Finsler geometry in Romania. At the National Conference on Geometry and Topology in Timişoara, in 1977, he gave a talk *"Finsler Geometry. Romanian Mathematicians' Contributions"*.

His talent of revealing the beauty of a subject matter stimulated the interest in the study of this geometry of a great number of participants that began studying various problems taking advantage of the advice and help of Professor Radu Miron. He himself has determined the set of metrical Finsler connections and has proved, in a simple way, the existence and the uniqueness of the Cartan connection.

Professor Radu Miron initiated *"The First National Seminar on Finsler Geometry"* at the University of Braşov in 1980. In a four-hour lecture, a text that comprised 53 pages, he offerd an original introduction to the geometry of the Finsler spaces, fusing together the influence of earlier Romanian, Japanese and French researches. In this lecture, the Finslers connection appear, for the first time, as linear connections in the total space TM tangent to a manifold M, compatible to the almost complex structure naturally associated to nonlinear connections on TM. Here he pointed out the decisive role of the nonlinear connections, as well as that of the geometric objects of Finsler type, called later distinguished geometric objects. The most important novelty in this lecture was the introduction and study of the space with metrical Finsler structures, later named as generalized Lagrange spaces.

The introduction and the study of metrical Finsler structures is Professor's Radu Miron second major original contribution to the theory of Finsler spaces. This contribution has actually modified the framework of the Finsler geometry and led to new generalizations and new points of view. The most interesting point of view which offers wide prospects, also belongs to Professor Radu Miron, who has noticed that the advanced techniques in the geometries of generalized Finsler spaces can be used in the study of the geometry of the total space of any vector bundle. He developed, by using such techniques, an elegant theory of geometric structures and of connections compatible with these, on the total space of a vector bundle, geometrical, with easy-to-follow calculus. His co-workers from Romania, developed, applied and adapted the idea to different situations, so that the geometry of the total space of a vector bundle can be said to be Professor's Radu Miron and his co-workers' creation.

The theory of subspace in the Finsler spaces less developed because of the great volume of calculus, stimulated Professor Radu Miron to apply in this field also the new points of view and the techniques he has discovered.

His theory enjoyed a large interest at the Romanian-Japanese Colloquium on Finsler Geometry, organized by R. Miron and M. Hashiguchi, which held at the universities from Iaşi and Braşov in August 1984.

After 1970, in the Theoretical Physics there has appeared an interest in developing a Finslerian Theory of Relativity that should offer the possibility of describing anisotropy properties of space. As to this matter Professor Radu Miron had a simple idea, as all great ideas: to consider the Einstein equations in Lagrange spaces as the Einstein equations associated to the canonical metrical connection from the almost Kählerian model. By decomposing the Einstein equations from the model in the adapted frames to nonlinear connection, he obtained two sets of Einstein equations. Prof. Dr. S. Ikeda from the University of Sciences of Tokyo explained the physical foundations of the entire theory in the work published as the last chapter of the monograph *"Vector Bundles. Lagrange Spaces. Applications to Relativity"*, written together with M. Anastasiei and published by the Romanian Academy in 1987. For this monograph, the second author received the *Gh. Lazar Prize* of the Romanian Academy.

Then Professor Radu Miron has defined for the first time the Hamilton spaces which he researches as dual of Lagrange spaces. By using the Legendre transformation he determines a nonlinear connection which depends only on the Hamiltonian.

Since 1988 Professor Radu Miron has focussed more on applications of the theory of Lagrange spaces and of generalized Lagrange spaces to Theoretical Physics. He developed the theory of electromagnetism and studied the geometrical optics based on a generalization of a metric due to J.L. Synge. These researches were included in the book *"The Geometry of Lagrange Spaces: Theory and Applications"* (R. Miron, M. Anastasiei, 1994) published in the series *The Fundamental Theories of Physics (FTPH)* of Kluwer Academic Publishers. He continues entireness with a deep study of the higher order Lagrange spaces. Thus he has solved the famous problem of the prolongation of order $k > 1$ of the Riemannian space, which brings a solid contribution to the foundation of the Mechanics of the Lagrangians which depends on the higher order accelerations and creates some new geometrical models for the theory of physical fields. This study was extended to two monographs *"The Geometry of Higher Order Lagrange Spaces. Applications to Mechanics and Physics"* (R. Miron, 1997) published in the same *FTPH* series of the Kluwer Academic Publishers and *"The Geometry of Higher Order Finsler Spaces"*, published by Hadronic Press in 1998. In 2001 he published also in Kluwer the monograph *"The geometry of La-*

grange and Hamilton spaces" in collaboration with D. Hrimiuc, H. Shimada and V.S. Sabau. In 2007 his book titled *"Finsler - Lagrange Geometry. Applications to Dynamycal Systems"* joint with I. Bucataru was published by the Romanian Academy.

Professor Radu Miron published some of his works in cooperation with Japanese geometers M. Matsumoto, M. Hashiguchi, Y. Ichijio, H. Izumi, S. Kikuchi, S. Watanabe, S. Ikeda, and with members of the National Seminar on Finsler and Lagrange Geometry.

It is well-known that a profound physical theory can be constructed only on the adequate perfect geometry. New geometrical theores, proposed by Professor Radu Miron, from the logical and philosophical points of view are based on fundamental concepts of the Analytical Mechanics, Euler-Lagrange equations, Hamilton-Jacobi equations, conservation laws, Nöther symmetries, non-linear connections and its structure equations, relation of duality estabilished by the Legendre transformations, etc. Therefore the geometrical theores of Professor Radu Miron correspond to the modern spirit of Mathematics and Phisics, embrace both classical branches of the Riemann-Cayley-Cartan-Finsler geometries and new arised conceptions and problems, as well as offer the richest arsenal of well-elaborated methods of investigations.

Having a great prestige in the world of Mathematics, Professor Radu Miron has been invited to lecture by the well-known institutions of France, Great Britain, the former Soviet Union, Italy, Greece, Germany, Hungary, Yugoslavia, Japan and so on. He passionately and skillfully organized several national and internatonal conferences on geometry and topology. Professor Radu Miron is often invited as an active participant (as member of the Organizing Committee with pleanary talks) in many conferences organized in the Republic of Moldova.

The "Al.I. Cuza" University of Iaşi and the Iaşi Branch of the Romanian Academy organized the international *"Conference on Differential Geometry : Lagrange and Hamilton Spaces"* which held in Iaşi, on September 3-8, 2007. This Conference was devoted to the geometry of Lagrange and Hamilton spaces, concepts introduced and studied by Academician Radu Miron. In the opening of the Conference it was scheduled an Anniversary Symposium with the aim to evoke the achievements of the Academician Radu Miron during the almost 60 years he devoted to the mathematical education. At the same time it was put into relief the impact of his scientific creation in the today Mathematics.

Professor Radu Miron received the title of Honorific Member of the Academy of Sciences of Moldova and the title of "Doctor Honoris Causa" from the Romanian univerities of Constantza, Bacău, Oradea, Craiova, and from the Tiraspol State University of the Republic of Moldova. He also received Diplomas of Excelence from the Romanian Ministry of Education and Culture, the Academy of Sciences of Moldova, Council of Iaşi County, Council of Vaslui County, Mathematical Society of the Republic of Moldova, "Al.I. Cuza" University of Iaşi and "P. Andrei" University of Iaşi. In 2002 Professor Radu Miron has received the *Award "V. Pogor"* of the Iaşi city. In 2003 National Council CNSIT awarded him the *"Opera Omnia" Prize*, one of the distinnguished honoring awards for Romanian research activity. Repeatedly was awarded by prestige prizes of Romanian Academy (1966, 1974, 1987) and of the Romanian Ministry of Education (1963). He has recently become Emeritus Professor of the "Al.I. Cuza" University of Iaşi.

Professor Radu Miron is a member-founder and the first President of the Balkanic Society of Geometry, a member of the American Mathematical Society, of Japan Tensor Society, Finsler Geometry Society, etc.

At the age of 80, full of vigor and optimism, the Academician Radu Miron is a prominent personality and continues to be an active presence in the academic community. As author of many discoveries that belong to Mathematics in the world and leader of a scientific school fully recognized everywhere he is a model for many generations of students and researchers.

The more complete description of the life of Academician Radu Miron and his creative scientific work is included in the following publications: [1]. M. Anastasiei. *The Mathematicean Radu Miron. His Work and Life.* – Geometry Balkan Press. Bucharest, 1998. [2]. M. Anastasiei. *The Mathematicean Radu Miron. His Work and Life at 75 Anniversary.* – "Al.I.Cuza" University Press, Iaşi, 2003. [3]. I. Ivanici and P. Marcu (ed.). *Academicieni români.* – Bucharest, 1995. [4]. *Membrii Academiei de Ştiinţe a Moldovei. Dicţionar 1961–2006.* – Ştiinţa, Chişinău, 2006.

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