BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(65), 2011, Pages 3–27
ISSN 1024–7696

# Serial rings and their generalizations

Vladimir Kirichenko, Makar Plakhotnyk

**Abstract.** We give a survey of results on the theory of semiprime semidistributive rings, in particular, serial rings. Besides this we prove that a serial ring is Artinian if and only if some power of its Jacobson radical is zero. Also we prove that a serial ring is Noetherian if and only if the intersection of all powers of Jacobson radical is zero. These two theorems hold for semiperfect semidistributive rings.

**Mathematics subject classification:** 16L30.
**Keywords and phrases:** Serial ring, SPSD-ring, quiver of ring.

*Dedicated to Professor Alexei I. Kashu*
*on the occasion of his seventieth birthday*

## 1 Introduction

Artinian uniserial, or primary decomposable serial rings were first introduced and studied by G. Köthe in the paper [9], where he proved that any module over such a ring is a direct sum of cyclic modules (he called such rings "Einreihige Ringen"). This result was generalized for Artinian serial rings by T. Nakayama, who called these rings "generalized uniserial rings" (see [16] and [17]). In these papers T. Nakayama proved that any module over such a ring is a direct sum of uniserial submodules each of which is a homomorphic image of an ideal generated by a primitive idempotent. T. Nakayama also showed that, conversely, these are the only rings whose indecomposable finitely generated modules (both left and right) are homomorphic images of ideals generated by primitive idempotents.

Artinian principal ideal rings were studied in papers of G. Köthe and K. Asano (see [1] and [2]), where it was proved that any Artinian principal right ideal ring is right uniserial. In fact, K. Asano proved that an Artinian ring is uniserial if and only if each ideal is a principal right ideal and a principal left ideal. The classical proof of this theorem is given in the book [7]. For such rings K. Asano also proved an analogue of the Wedderburn-Artin theorem, namely, he proved that any Artinian uniserial ring can be decomposed into a direct sum of full matrix rings of the form $M_n(A)$, where $A$ is a local Artinian ring with a cyclic radical. A one-sided characterization of Artinian principal ideal rings and its connection with primary decomposable serial rings is given in theorem 2.1 of the paper [4]

L. A. Skornyakov studied serial rings, which he called "semi-chain rings", in his paper [18]. There he proved that A is a right and left Artinian serial ring if and only if every left A-module is a direct sum of uniserial modules. His result generalizes a theorem proved by K. R. Fuller (see [5]), to the effect that if each left module over a ring A is a direct sum of uniserial modules, then A is a serial left Artinian ring.

The first serial non-Artinian rings were studied and described by R. B. Warfield and V. V. Kirichenko. In particular, they gave a full description of the structure of serial Noetherian rings. We follow the papers [12] and [10], where the technique of quivers was used systematically.

It is well known that many important classes of rings are naturally characterized by the properties of modules over them. As examples, we mention semisimple Artinian rings, uniserial rings, semiprime hereditary semiperfect rings and semidistributive rings.

There is the following chain of strict inclusions:

semisimple Artinian rings $\subset$ generalized uniserial rings $\subset$ serial rings $\subset$ semidistributive rings.

In this chain the first three classes of rings are semiperfect. The example of the ring of integers $\mathbb{Z}$ shows that a distributive ring is not necessarily semiperfect.

The reduction theorem for SPSD-rings and decomposition theorem for semiprime right Noetherian SPSD-rings were proved in the paper [14].

Quivers and prime quivers of SPSD-rings were studied in [13].

A semilocal ring $A$ is called **semiperfect** if idempotents of the ring $A$ can be lifted modulo $R$.

Semiperfect rings were introduced by H. Bass in 1960.

To understand the definition of a semilocal ring we need some additional definitions and propositions.

A nonzero ring $A$ is called **local** if it has the unique maximal right ideal.

The intersection of all maximal right ideals of a ring $A$ is called the **Jacobson radical** of $A$. The Jacobson radical is denoted $R = rad\ A$.

The following theorem contains the list of properties which are equivalent for any local ring.

**Theorem 1.1.** *The following properties of a ring $A$ with the Jacobson radical $R$ are equivalent:*

1. *$A$ is local;*
2. *$R$ is the unique maximal right ideal in $A$;*
3. *all non-invertible elements of $A$ form a proper ideal;*
4. *$R$ is the set of all non-invertible elements of $A$;*
5. *the quotient ring $A/R$ is a division ring.*

**Proposition 1.2.** *Let $e^2 = e \in A$. Then $rad(eAe) = eRe$, where $R$ is the radical of $A$.*

Recall that a module $M$ is called distributive if for any submodules $K, L, N$

$$K \cap (L + N) = K \cap L + K \cap N.$$

Clearly, a submodule and a quotient module of a distributive module are distributive. A module is called **semidistributive** if it is a direct sum of distributive modules. A ring $A$ is called **right (left) semidistributive** if the right (left) regular module $A_A$ (${}_AA$) is semidistributive. A right and left semidistributive ring is called **semidistributive**.

Obviously, every uniserial module is a distributive module and every serial module is a semidistributive module.

**Example 1.1.** *Let $S = \{\alpha_1, \ldots, \alpha_n\}$ be a finite poset with ordering relation $\leqslant$ and let $D$ be a division ring. Denote by $A(S, D)$ the following subring of $M_n(D)$:*

$$A(S, D) = \left\{ \sum_{\alpha_i \leq \alpha_j} d_{ij} e_{ij} \,|\, d_{ij} \in D \right\}.$$

*It is not difficult to check that $A(S, D)$ is a semidistributive Artinian ring.*

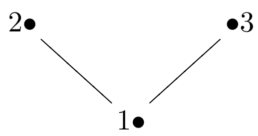
*In particular, the hereditary semidistributive ring*

$$A_3 = \left\{ \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix} \,|\, d_{ij} \in D \right\}$$

*is of the form:*

$$A_3 = A(P_3, D),$$

*where $P_3$ is the poset with the diagram*




*It is also clear that $A_3$ is a semidistributive ring which is left serial, but not right serial.*

**Proposition 1.3.** *Let $M$ be an $A$-module. Then $M$ is a distributive module if and only if all submodules of $M$ with two generators are distributive modules.*

*Proof.* Suppose that all two-generated submodules of $M$ are distributive modules. Let $K, L, N$ be submodules of $M$ and $k = l + n \in K \cap (L + M)$; $l \in L$, $n \in N$. Obviously, $kA \subset lA + nA$ and $KA = kA \cap (lA + nA) = KA \cap lA + kA \cap nA$. Therefore, $k \in K \cap L + K \cap N$, i.e. $K \cap (L + N) \subseteq K \cap L + K \cap N$. The inclusion $K \cap L + K \cap N \subseteq K \cap (L + N)$ is always valid. $\square$

**Lemma 1.4.** *Let $M$ be a distributive module over a ring $A$. Then for any $m$, $n \in M$ there exist $a, b \in A$ such that $1 = a + b$ and $maA + nbA \subset mA \cap nA$.*

*Proof.* Write $t = m + n$ and $H = mA \cap nA$. Obviously, $tA \subseteq mA + nA$ and $Ta \cap (mA + nA) = tA = (tA + mA) \cap (tA + nA)$. So there exist $b, d \in A$ such that $tb \in mA$, $td \in nA$ and $t = tb+td$. Then $nb = tb-mb \in H$ and $md = td-nd \in H$. Let $a = 1-b$ and $g = 1-b-d$. He have $tg = t-tb-td = 0$ and $ng = tg-mg = -mg \in H$. So $ma = md + mg \in H$ and $maA + nbA \subseteq mA \cap nA$. □

**Lemma 1.5.** *Let $M$ be an $A$-module. Then $M$ is a distributive module if and only if for any $m, n \in M$ there exist four elements $a, b, c, d$ of $A$ such that $1 = a + b$ and $ma = nc$, $nb = md$.*

*Proof.* Necessity follows from Lemma 1.4. Conversely, let $k \in K \cap (L + N)$, where $K, L, N$ are submodules of $M$. Then $k = m + n$, where $m \in L$ and $n \in N$. By assumption there exist $a, b \in A$ such that $1 = a+b$ and $ma \in mA \cap nA$, $nb \in mA \cap nA$. Consequently, $ka = ma + na \in kA \cap nA$ and $kb = mb + nb \in kA \cap mA$. Therefore, $k = ka + kb \in (kA \cap nA) + (kA \cap mA) \subset K \cap L + K \cap N$, i.e., $K \cap (L + N) = K \cap L + K \cap N$. □

Let $M$ be an $A$-module. Given two elements $m, n \in M$ we set

$$(m : n) = \{a \in A | na \in mA\}.$$

**Theorem 1.6 (W. Stephenson).** *A module $M$ is distributive if and only if*

$$(m : n) + (n : m) = A$$

*for all $m, n \in M$.*

*Proof.* This immediately follows from Lemma 1.5. □

A module $M$ has the **square-free socle** if its socle contains at most one copy of each simple module.

**Theorem 1.7 (V. Camillo).** *Let $M$ be an $A$-module. Then $M$ is a distributive module if and only if $M/N$ has the square-free socle for every submodule $N$.*

*Proof.* Necessity. Every quotient and submodule of a distributive module are distributive, so that if $M/N$ contains a submodule of the form $U \oplus U$, then $M$ is not a distributive module. Simply because $U \oplus U$ is not a distributive module. Indeed, for the diagonal $D(U \oplus U) = \{(u, u) | n \in U\}$ of $U \oplus U$ we have $D(U) \cap (U \oplus U) = D(U)$ and $D(U) \cap (U \oplus 0) = 0$ and $D(U) \cap (0 \oplus U) = 0$.

Conversely. Let $m, n \in M$. We show that $(m : n) + (n : m) = A$. Let $K$ be a maximal right ideal of $A$ and $U = A/K$. Consider the quotient module $mA + nA/mK + nK$. The socle of $mA + nA/mK + nK$ doesn't contain $U \oplus U$ if one of the following conditions holds:

(1) $m \in nA + mK + nK = nA + mK$;

(2) $m \in nA + mK + nK = nA + mK$;

In the case (1) we have $m = na+nK$ or $m(1 \oplus k) = na$. So $(1 \oplus k) \in (n : m)$. Since $(1 \oplus k) \notin K$, we have $(n : m) \not\subseteq K$. In the case (2) analogously $(m : n) \not\subseteq K$. □

**Theorem 1.8.** *A semiprimary right semidistributive ring $A$ is right Artinian.*

*Proof.* It is sufficient to show that each indecomposable projective $A$-module $P = eA$ is Artinian ($e$ is a nonzero idempotent of $A$). Let $m$ be the minimal natural number with $PR^m = 0$. Since the module $P$ is distributive, by Theorem 1.7, the quotient module $PR^i/PR^{i+1}$ decomposes into a finite direct sum of simple modules ($i = 1, \dots, m-1$). Thus, the module $P$ possesses a composition series and the module $P$ is Artinian. □

We write **SPSDR-ring** (**SPSDL-ring**) for a semiperfect right (left) semidistributive ring and **SPSD-ring** for a semiperfect semidistributive ring.

**Theorem 1.9** (**A. Tuganbaev**). *A semiperfect ring $A$ is right (left) semidistributive if and only if for any local idempotents $e$ and $f$ of the ring $A$ the set $eAf$ is a uniserial right $fAf$-module (uniserial left $eAe$-module) ([6], Theorem 14.2.1).*

## 2 Q-lemma and Annihilation lemma

Recall the definition of the Gabriel quiver for a finite dimensional algebra $A$ over a field $k$. We can restrict ourselves to **basic** split algebras. (An algebra A is called basic if $A/R$ is isomorphic to a product of division algebras, where R is the Jacobson radical of A. An algebra A over a field $k$ is called **split** if $A/R \simeq M_{n_1}(k) \times M_{n_2}(k) \times \ldots \times M_{n_s}(k)$.) All algebras over algebraically closed fields are split.

Let $P_1, \dots, P_s$ be all pairwise nonisomorphic principal right A-modules. Write $R_i = P_iR$ ($i = 1, \dots, s$) and $W_i = R_i/R_iR$. Since $W_i$ is a semisimple module, $W_i = \bigoplus\limits_{j=1}^{s} U_j^{t_{ij}}$, where $U_j = P_j/R_j$ are simple modules. It is equivalent to the isomorphism $P(R_i) \simeq \bigoplus\limits_{j=1}^{s} P_j^{t_{ij}}$. To each module $P_i$ assign a point $i$ in the plane and join the point $i$ with the point $j$ by $t_{ij}$ arrows. The so constructed graph is called the quiver of $A$ in the sense of P.Gabriel and denoted by $Q(A)$.

A semiperfect ring $A$ is called **reduced** if its quotient ring by the Jacobson radical $R$ is a direct sum of division rings.

Let $A$ be a semiperfect ring such that $A/R^2$ is a right Artinian ring. The quiver of the ring $A/R^2$ is called the **quiver** of the ring $A$ and is denoted by $Q(A)$.

**Theorem 2.1.** *Let $A$ be an arbitrary ring with an idempotent $e^2 = e \in A$. Set $f = 1 - e$, $eAf = X$, $fAe = Y$, and let*

$$A = \begin{pmatrix} eAe & X \\ Y & fAf \end{pmatrix}$$

*be the corresponding two-sided Peirce decomposition of the ring $A$. Then the ring $A$ is right Noetherian (Artinian) if and only if the rings $eAe$ and $fAf$ are right Noetherian (Artinian), $X$ is a finitely generated $fAf$-module and $Y$ is a finitely generated $eAe$-module.*

For further reasonings we will need the following proposition.

**Proposition 2.2.** *Let $A$ be a ring. For an $A$-module $P$ the following statements are equivalent:*

*1) $P$ is projective;*

*2) every short exact sequence $0 \to N \to M \to P \to 0$ splits;*

*3) $P$ is a direct summand of a free $A$-module $F$.*

Let $A_A = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$ be the decomposition of a semiperfect ring $A$ into a direct sum of principal right $A$-modules and let $1 = f_1 + \ldots + f_s$ be the corresponding decomposition of the identity of $A$ into a sum of pairwise orthogonal idempotents, i.e., $f_i A = P_i^{n_i}$ . Then ${}_A A = Af_1 \oplus \ldots \oplus Af_s = Q_1^{n_1} \oplus \ldots \oplus Q_s^{n_s}$ is the decomposition of the semiperfect ring $A$ into a direct sum of principal left $A$-modules, i.e. $Af_i = Q_i^{n_i}$ , where $Q_i$ is an indecomposable projective left $A$-module ($i = 1, \ldots, s$). Now consider the two-sided Peirce decomposition of the ring $A$

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}.$$

Consider also the two-sided Peirce decomposition of the Jacobson radical $R$ of $A$ : $R = \bigoplus\limits_{i,j} f_i R f_j$. Since $R$ is a two-sided ideal, $f_i R f_j \subset R$ for all $i, j$. By Proposition 1.2 we have $R_{ii} = f_i R f_i = rad(f_i A f_i)$ for $i = 1, \ldots, n$. We shall show that $f_i R f_j = f_i A f_j$ for $i \neq j$. Indeed, multiplying on the left elements from $f_j A$ by an element $f_i a f_j$ we obtain a homomorphism $\varphi_{ji}$ of the module $f_j A$ to $f_i A$. If $Im(\varphi_{ji}) = f_i A$, then $\varphi_{ji}$ is an epimorphism. Since $f_i A = P_i^{n_i}$, $f_j A = P_j^{n_j}$ are projective modules, by Proposition 2.2, and $P_i^{n_i}$ is isomorphic to a direct summand of the module $P_j^{n_j}$. But this is impossible, since the indecomposable modules $P_i$ and $P_j$ are non-isomorphic. Therefore $Im(\varphi_{ji}) \subset f_i A$. We can write the homomorphism $\varphi_{ji}$ in the form of a matrix $\varphi_{ji} = (\varphi_{ji}^{rs})$, where $\varphi_{ji}^{rs} : P_j \to P_i$ are homomorphisms of indecomposable non-isomorphic projective modules $P_j$ and $P_i$ for $r = 1, \ldots, n_i$, $s = 1, \ldots, n_j$. Since $Im(\varphi_{ji}^{rs}) \neq P_i$, we have $Im(\varphi_{ji}^{rs}) \subset P_i R$. Therefore $Im(\varphi_{ji}^{rs}) \subseteq f_i A R = f_i R$, i.e., $f_i A f_j \subseteq f_i R$. Hence $A_{ij} = f_i A f_j = f_i R f_j$ for $i \neq j$. Thus, we obtain the following result.

**Proposition 2.3.** *Let $A = P^{n_1} \oplus \ldots \oplus P_s^{n_s}$ be the decomposition of a semiperfect ring $A$ into a direct sum of principal right $A$-modules and let $1 = f_1 + \ldots + f_s$ be a corresponding decomposition of the identity of $A$ into a sum of pairwise orthogonal idempotents, i.e., $f_i A = P_i^{n_i}$. Then the Jacobson radical of the ring $A$ has a two-sided Peirce decomposition of the following form:*

$$R = \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ R_{21} & R_{22} & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n1} & R_{n2} & \cdots & R_{nn} \end{pmatrix}$$

*where* $R_{ii} = rad(f_iAf_i)$, $A_{ij} = f_iAf_j$ *for* $i, j = 1, \ldots, n$.

**Lemma 2.4. (Annihilation lemma)** *Let* $1 = f_1 + \ldots + f_s$ *be a canonical decomposition of* $1 \in A$. *For every simple right* $A$*-module* $U_i$ *and for each* $f_j$ *we have* $U_if_j = \delta_{ij}U_i$, $i, j = 1, \ldots, s$. *Similarly, for every simple left* $A$*-module* $V_i$ *and for each* $f_j$, $f_jV_i = \delta_{ij}V_i$, $i, j = 1, \ldots, s$.

*Proof.* We shall give the proof for the case of right modules. From the previous proposition we obtain that $f_iRf_j = f_iAf_j$ for $i \neq j$. Hence $P_i^{n_i}f_j \subset f_iR$. But $f_iA/f_iR \simeq U_i^{n_i}$. Therefore $U_i^{n_i}f_j = 0$ and so $U_if_j = 0$ for $i \neq j$.

We are going to show that $U_if_i = U_i$. Let $u \in U_i$. Then $u \cdot 1 = u(f_1 + \ldots + f_s) = uf_i$ since $uf_j = 0$ for $i \neq j$. The lemma is proved. □

Let A be a reduced semiperfect ring, and let $1 = e_1 + \ldots + e_s$ be a decomposition of $1 \in A$ into a sum of mutually orthogonal local idempotents.

Set $U_i = e_iA/e_iR$ and $V_i = Ae_i/Re_i$.

**Lemma 2.5. (Q-Lemma)** *The simple module* $U_k$ *(resp.* $V_k$*) appears in the direct sum decomposition of the module* $e_iR/e_iR^2$ *(resp.* $Re_i/R^2e_i$*) if and only if* $e_iR^2e_k$ *(resp.* $e_kR^2e_i$*) is strictly contained in* $e_iRe_k$ *(resp.* $e_kRe_i$*).*

*Proof.* If $U_k$ is a direct summand of the module $W_i = e_iR/e_iR^2$, then by Lemma 2.4, $Wie_k \neq 0$. Therefore $e_iRe_k$ does not equal $e_iR^2e_k$ and the inclusion $e_iRe_k \supset e_iR^2e_k$ is strict.

Conversely, suppose that $e_iR^2e_k$ is strictly contained in $e_iRe_k$. Consider a submodule $X_k$ contained in $e_iR$,

$$X_k = e_iRe_i \oplus \ldots \oplus e_iRe_{k-1} \oplus e_iR^2e_k \oplus e_iRe_{k+1} \oplus \ldots \oplus e_iRe_s$$

(here the direct sum sign denotes a direct sum of Abelian groups).

From the inclusions $e_iR \supset X_k \supset e_iR^2$ it follows that $e_iR/X_k$ is a semisimple module. We have the equalities $e_iR/X_k = e_iRe_k/e_iR^2e_k = (e_iR/X_k)e_k$. By Lemma 2.4 the module $e_iR/X_k$ decomposes into a direct sum of some copies of the module $U_k$. Since $e_iR/X_k$ is isomorphic to a direct summand $W_i$, the module $U_k$ is contained in $W_i$ as a direct summand.

For left modules $V_k$ the statement is proved analogously. The lemma is proved. □

**Lemma 2.6.** *Let* $A$ *be a semiperfect ring, and* $e, f$ *be nonzero idempotents of the ring* $A$ *such that* $\overline{e} = \overline{f} \in \overline{A}$. *Then there exists an invertible element* $a \in A$ *such that* $f = aea^{-1}$.

The quiver $Q(A)$ of a ring $A$ is called connected if it cannot be represented in the form of a union of two nonempty disjoint subsets $Q_1$ and $Q_2$ which are not connected by any arrows.

**Theorem 2.7.** *The following conditions are equivalent for a semiperfect Noetherian ring $A$:*

*(a) $A$ is an indecomposable ring;*

*(b) $A/R^2$ is an indecomposable ring;*

*(c) the quiver of $A$ is connected.*

*Proof.* Obviously, the conditions of the theorem are preserved by passing to the Morita equivalent rings. Therefore we can assume that the ring $A$ is reduced.

(a) $\Rightarrow$ (b). Let $\overline{A} = A/R^2 \simeq \overline{A}_1 \times \overline{A}_2$ and let $\overline{1} = \overline{P}+1+\overline{P}_2$ be the corresponding decomposition of the identity of the ring $A/R^2$ into a sum of orthogonal idempotents. Let $g_1, g_2 \in A$ be elements such that $g_1 + R^2 = \overline{f}_1$ and $g_2 + R^2 = \overline{f}_2$. There are idempotents $f_1, f_2 \in A$ such that $f_1 = g_1 + r_1$ and $f_2 = g_2 + r_2$, where $r_1, r_2 \in R^2$. Since $\overline{f}_1\overline{A}\overline{f}_2 = 0$ and $\overline{f}_2\overline{A}\overline{f}_1 = 0$, we have $g_1 a g_2 \in R^2$ and $g_2 a g_1 \in R^2$ for any $a \in A$. Clearly, $f_i = f_i g_i f_i + f_i r_i f_i$ $(i = 1, 2)$. Then the element $f_1 a f_2 = f_1 g_1 f_1 a f_2 g_2 f_2 + f_1 g_1 f_1 a f_2 r_2 f_2 + f_1 r_1 f_1 a f_2 g_2 f_2 + f_1 r_1 f_1 a f_2 r_2 f_2$ belongs to $R^2$ for any $a \in A$. This is immediate from Proposition 2.3. Exactly in the same way $f_2 A f_1 \in R^2$. Therefore $f_2 A f_1 = f_2 R_2 f_1$ and $f_1 A f_2 = f_1 R^2 f_2$. By Proposition 2.3, the two-sided Peirce decomposition of $R$ has the form: $R = \begin{pmatrix} R_1 & A_{12} \\ A_{21} & R_2 \end{pmatrix}$, where $R_i = Rad(f_i A f_i)$ $(i = 1, 2)$ and $A_{ij} = f_i A f_j$ for $i \neq j$. Calculating $R^2$ we obtain

$$R^2 = \begin{pmatrix} R_1^2 + A_{12}A_{21} & R_1 A_{12} + A_{12} R_2 \\ A_{21} R_1 + R_2 A_{21} & A_{21} A_{12} + R_2^2 \end{pmatrix}.$$

From the above we have: $A_{12} = R_1 A_{12} + A_{12} R_2$ and $A_{21} = R_2 A_{21} + A_{21} R_1$. By Theorem 2.1, taking into account Nakayama's lemma, we obtain that $A_{12} = 0$ and $A_{21} = 0$ and therefore $A = A_{11} \times A_{22}$, where $A_{ii} = f_i A f_i$ $(i = 1, 2)$.

(a) $\Rightarrow$ (c). Let the quiver of the ring $A$ be disconnected. Then $Q(A) = Q_1 \cup Q_2$ and $Q_1 \cap Q_2 = \varnothing$, and the points of the sets $Q_1$ and $Q_2$ are not connected by any arrows. Renumbering, if necessary, the principal right $A$-modules $P_1, \ldots, P_s$ one may assume that $Q_1 = \{1, \ldots, k\}$ and $Q_2 = \{k+1, \ldots, s\}$. Let $A = P_1 \oplus \ldots \oplus P_s$ be a decomposition of the ring $A$ into a direct sum of principal right $A$-modules (where $P_i = e_i A$, $e_i^2 = e_i \in A$, $1 = e_1 + \ldots + e_s$) and $1 = f_1 + f_2$, where $f_1 A = P_1 \oplus \ldots \oplus P_k$ and $f_2 A = P_{k+1} \oplus \ldots \oplus P_s$. We set $A_{ij} = f_i A f_j$, $R_i = radA_{ii}$ $(i = 1,\ 2)$. If $A_{12} \neq 0$, then by Theorem 2.1, taking into account Nakayama's lemma, we obtain that the inclusion $A_{12} \supset R_1 A_{12} + A_{12} R_2$ is strict. But $R_1 A_{12} + A_{12} R_2 = f_1 R^2 f_2$. Therefore there are local idempotents $e_i$ and $e_j$ such that $e_i$ is a summand of $f_1$ and $e_j$ is a summand of $f_2$ and $e_i R^2 e_j$ is strictly contained in $e_i R e_j$. By Lemma 2.5 we obtain that there is an arrow which connects the point $i$ with the point $j$. A contradiction. Analogously it can be proved that $A_{21} = 0$.

(c) $\Rightarrow$ (a). If the ring $A$ is decomposable then $A/R^2$ is also decomposable. Clearly, in this case $Q(A)$ is disconnected.

(b) $\Rightarrow$ (a) is trivial.

The theorem is proved. □

**Remark**. Theorem 2.7 is not true for semiperfect one-sided Noetherian rings. As an example one can consider the ring $A = \begin{pmatrix} \mathbb{Z}_{(p)} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$. The quiver of this ring consists of two points and one loop near one of them.

As $R^2 = \begin{pmatrix} p^2\mathbb{Z}_{(p)} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ then the ring $A/R^2$ decomposes into a direct product of rings:

$$A/R^2 \simeq \mathbb{Z}_{(p)}/p^2\mathbb{Z}_{(p)} \times \mathbb{Q}.$$

However, the ring A itself is indecomposable into a direct product of rings.

**Theorem 2.8.** *Let the ring $A$ be a serial ring such that the intersection of all powers of its radical $\cap_{n=1}^{\infty} R^n = 0$ is equal to zero. Then $A$ is right and left Noetherian ring.*

*Proof.* Let $M \in P$ and $\cap_{n=1}^{\infty} R^n = 0$.

Then the inclusion $MR \subset M$ is strong. If $M = MR$ then $M \subset R$ and the equality $M = MR^n$ gives that $M \subset R$ for all $n$ i.e. $M = 0$.

Let $e$ be an arbitrary idempotent of the ring $A$. Then $eRe = Rad\ eAe$ and $eAe \subset R$, $(eRe)^n \subset R^n$ and that is why $\cap(eRe)^n = 0$.

So for any local idempotent $e$ the ring $eAe$ is uniserial and the intersection of natural powers of the radical $R$ is equal to 0. That is why the ring $eAe$ is discrete valuated as it is Artinian. Assume that all rings $eAe$ are Artinian. Then $A$ is also Artinian. Let at least one ring of the form $e_iAe_i$ be discrete valuated. Then there exists a local idempotent $e_j$ such that the ring $(e_j + e_i)A(e_j + a_i)$ is of the form $\begin{pmatrix} A_j & X \\ Y & \mathcal{O}_i \end{pmatrix}$, where $X$ is an infinitely generated right $\mathcal{O}_i$-module. According to Lemma 3.28 $\begin{pmatrix} R_1X & X \\ Y & R_2 \end{pmatrix} = \begin{pmatrix} R_1^2 + XY & R_1X + XR_2 \\ YR_1 + R_2Y & YX + R_2 \end{pmatrix}$ and $XR_2 = X$. Consider the following module $M = (XY, X)$, which belongs to $(A_1X)$. It is obviousl that $(XY, X)\begin{pmatrix} R_1 & X \\ Y & R_2 \end{pmatrix} = (XY, X)$. This contradicts to the strong inclusion $MR \subset M$, whence $X$ is a finitely generated right $\mathcal{O}_2$-module, and in the same way $Y$ is a finitely generated left $\mathcal{O}_1$-module.

So according to Theorem 2.1 the ring $(e_i + e_j)A(e_i + e_j)$ is right Noetherian.

□

## 3 Semiperfect semidistributive rings

Theorem 1.9 has the following corollary.

**Corollary 3.1.** *Let $A$ be a semiperfect ring, and let $1 = e_1 + \ldots + e_n$ be a decomposition of $1 \in A$ into a sum of mutually orthogonal local idempotents. The ring $A$ is right (left) semidistributive if and only if for any idempotents $e_i$ and $e_j$ of the above decomposition, the set $e_iAe_j$ is a uniserial right $e_jAe_j$-module (left $e_iAe_i$-module).*

**Corollary 3.2. (Reduction Theorem for SPSD-rings)** *Let $A$ be a semiperfect ring, and let $1 = e_1 + \ldots + e_n$ be a decomposition of $1 \in A$ in a sum of mutually orthogonal local idempotents. The ring $A$ is right (left) semidistributive if and only if for any idempotents $e_i$ and $e_j$ $(i \neq j)$ of the above decomposition the ring $(e_i + e_j)A(e_i + e_j)$ is right (left) semidistributive.*

*Proof.* It is sufficient to prove the corollary for a reduced ring $A$. If $A$ is a right semidistributive, then $e_iAe_j$ is right uniserial $e_jAe_j$-module $(i \neq j)$ and the ring $e_iAe_i$ is right uniserial for $i = 1, \ldots, n$. By Corollary 3.1, the ring $(e_i + e_j)A(e_i + e_j)$ is right semidistributive. Conversely, if the ring $(e_i + e_j)A(e_i + e_j)$ is right semidistributive, then, by Theorem 1.9, the set $e_iAe_j$ is a uniserial right $A_{jj}$-module and, by Corollary 3.1, the ring $A$ is right semidistributive. □

**Corollary 3.3.** *Let $A$ be a Noetherian SPSD-ring, and let $1 = e_1 + \ldots + e_n$ be a decomposition of the identity $1 \in A$ into a sum of mutually orthogonal local idempotents, let $A_{ij} = e_iAe_j$ and let $R_i$ be the Jacobson radical of the ring $A_{ii}$. Then $R_iA_{ij} = A_{ij}R_{jj}$ for $i, j = 1, \ldots, n$.*

**Example 3.1.** *Consider*

$$A = \begin{pmatrix} \mathbb{R} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$$

*as an $\mathbb{R}$-algebra ($\mathbb{R}$ is the field of real numbers, $\mathbb{C}$ is the field of complex numbers). The Peirce decomposition of the Jacobson radical $R = R(A)$ has the form*

$$R = \begin{pmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{pmatrix}$$

*and the $\mathbb{R}$-algebra $A$ is right serial, i.e., right semidistributive.*

*The left indecomposable projective $Q_2 = \begin{pmatrix} \mathbb{C} \\ \mathbb{C} \end{pmatrix}$ has the socle $\begin{pmatrix} \mathbb{C} \\ 0 \end{pmatrix}$, which is a direct sum of two copies of the left simple module $\begin{pmatrix} \mathbb{R} \\ 0 \end{pmatrix}$. Consequently, by Proposition 1.3, the $\mathbb{R}$-algebra $A$ is an SPSDR-ring but it is not an SPSDL-ring.*

### 3.1 Quivers of SPSD-rings

Recall that a quiver without multiple arrows and multiple loops is called a simply laced quiver. Let $A$ be an SPSD-ring. By Theorem 1.8, the quotient ring $A/R^2$ is right Artinian and its quiver $Q(A)$ is defined by $Q(A) = Q(A/R^2)$.

**Theorem 3.4.** *The quiver $Q(A)$ of an SPSD-ring $A$ is simply laced. Conversely, for any simply laced quiver $Q$ there exists an SPSD-ring $A$ such that $Q(A) = Q$.*

*Proof.* We may assume that $A$ is reduced and $R^2 = 0$. Let $A_A = P_1 \oplus \ldots \oplus P_s$, where $P_1, \ldots, P_s$ are indecomposable. Then $P_iR$ is a semisimple A-module:

$$P_iR = \bigoplus_{j=1}^{s} U_j^{t_{ij}}$$

where $U_j = P_j/P_jR$ are simple. The $A$-module $P_iR$ is a submodule of a distributive A-module and, therefore, $P_iR$ is distributive. By the definition of $Q(A)$ we have $[Q(A)] = (t_{ij})$ and, by Theorem 1.7, $0 \leq t_{ij} \leq 1$. So $Q(A)$ is a simply laced quiver.

Conversely, let $kQ$ be the path $k$-algebra of a simply laced quiver $Q$ and $J$ be its fundamental ideal, i.e., the ideal generated by all arrows of $Q$. Write $B = kQ/J^2$ and $\pi : kQ \to B$ for the natural epimorphism. Let $\pi(\varepsilon_i) = e_i$, where $\varepsilon_1, \ldots, \varepsilon_s$ are all paths of length zero. Then $B = e_1B \oplus \ldots \oplus e_sB$, where $e_1B, \ldots, e_sB$ are indecomposable. Let $R$ be the Jacobson radical of $B$ and $AQ = \{\sigma_{ij}\}$ be the set of all arrows of $Q$. The elements $\pi(\sigma_{mp})$, where $\sigma_{mp} \in AQ$, form a basis of $e_mR$. Obviously, $e_mR^2 = 0$ for $m = 1, \ldots, s$. So, $e_mR$ is a semisimple module and $e_mR = \oplus_p U_p$ for all those $p$, where $\sigma_{mp} \in AQ$. Therefore $Q(B) = Q$ and $e_mR$ is a distributive module, by Theorem 3.27. Thus, B is a right semidistributive ring. The analogous arguments show that B is a left semidistributive ring.

So $B = kQ/J^2$ is an SPSD-algebra over a field $k$ and $Q(B) = Q$. □

**Corollary 3.5.** *The link graph $\mathcal{L}G(A)$ of an SPSD-ring A coincides with $Q(A)$.*

*Proof.* For any SPSD-ring A the following equalities hold: $\mathcal{L}G(A) = Q(A, R) = Q(A)$. □

**Theorem 3.6.** *For an Artinian ring A with $R^2 = 0$ the following conditions are equivalent:*

*(a) A is semidistributive;*

*(b) $Q(A)$ is simply laced and the left quiver $Q'(A)$ can be obtained from $Q(A)$ by reversing all arrows.*

*Proof.* $(a) \implies (b)$. By Theorem 3.4 it is sufficient to show that $Q'(A)$ can be obtained from $Q(A)$ by reversing all arrows. One can assume that $A$ is reduced. Write $A_A$ as a direct sum $A_A = P_1 \oplus \ldots \oplus P_s$, where the $P_i$ are indecomposable projective and let $1 = e_1 + \ldots + e_s$ be the corresponding decomposition of $1 \in A$ into a sum of mutually orthogonal local idempotents. If $A_{ij} = e_iAe_j \neq 0$, then, in view of Corollary 3.3,

$$A_{ij}R_j = R_iA_{ij} \text{ and } A_{ij} \subset R \text{ for } i \neq j.$$

Hence, $A_{ij}R_j = R_iA_{ij} = 0$ for $i \neq j$ and, in view of the $Q$-Lemma, it follows that there is a loop at the vertex $i$ both in $Q(A)$ and in $Q'(A)$. Thus the left quiver $Q'(A)$ can be obtained from $Q(A)$ by reversing all arrows.

$(b) \implies (a)$. By the Peirce decomposition for R we have: $R = \bigoplus_{i,j=1}^{s} e_iRe_j$, $e_iRe_i = R_i$ and $e_iRe_j = A$, $i \neq j$; $i, j = 1, \ldots, s$.

It follows that

$$P_iR = (A_{i1}, \ldots, A_{ii-1}, R_i, A_{ii+1}, \ldots, A_{is})$$

for $i = 1, \ldots, s$. If $A_{ij} \neq 0$, for $i \neq j$, then, in view of the Q-Lemma, $A_{ij}$ is a simple right $A_{jj}$-module and a simple left $A_{ii}$-module. If $R_i \neq 0$, then $R_i$ is a simple

$A_{ii}$-module and a simple left $A_{ii}$-module. Thus, in view of Theorem 1.9, the ring $A$ is semidistributive. □

**Remark**. The implication $(b) \Longrightarrow (a)$ isn't true even in the case of finite dimensional algebras as is shown by the following example.

Let $A = kQ_4$ be the path k-algebra of the quiver $Q_4$

$$Q_4 = \left\{ \begin{tikzcd} & \overset{2}{\bullet} \arrow[dr] & \\ 1\ \bullet \arrow[ur] \arrow[dr] & & \bullet\ 4 \\ & \underset{3}{\bullet} \arrow[ur] & \end{tikzcd} \right\}.$$

The basis of $kQ_4$ is $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$, $\sigma_{12}, \sigma_{13}, \sigma_{24}, \sigma_{34}, \sigma_{12}\sigma_{24}, \sigma_{13}\sigma_{34}$. The indecomposable projective $A$-modules are: $P_1 = \{\varepsilon_1, \sigma_{12}, \sigma_{13}, \sigma_{12}\sigma_{24}, \sigma_{13}\sigma_{34}\}$; $P_2 = \{\varepsilon_2, \sigma_{24}\}$; $P_3 = \{\varepsilon_3, \sigma_{34}\}$; $P_4 = \{\varepsilon_4\}$. Obviously, $socP_1 \simeq P_4 \oplus P_4$. By Theorem 1.7, $P_1$ is not distributive, but $Q(A) = Q_4$ and

$$Q'(A) = \left\{ \begin{tikzcd} & \overset{2}{\bullet} \arrow[dr] & \\ 4\ \bullet \arrow[ur] \arrow[dr] & & \bullet\ 1 \\ & \underset{3}{\bullet} \arrow[ur] & \end{tikzcd} \right\},$$

i.e., A satisfies condition (b) of Theorem 1.5.

Note that if we identify routes $\sigma_{12}\sigma_{24}$ and $\sigma_{13}\sigma_{34}$ then obtain the distributive algebra, which is isomorphic to the matrix algebra $M_4(k)$ of the following form

$$\begin{pmatrix} k & k & k & k \\ 0 & k & 0 & k \\ 0 & 0 & k & k \\ 0 & 0 & 0 & k \end{pmatrix}.$$

A semiperfect ring $A$ such that $A/R^2$ is Artinian will be called **$Q$-symmetric** if the left quiver $Q'(A)$ can be obtained from the right quiver $Q(A)$ by reversing all arrows.

**Corollary 3.7.** *Every SPSD-ring is Q-symmetric.*

**Remark.** Example 1.9 shows that an SPSDR-ring is not always $Q$-symmetric.

**Theorem 3.8.** *The intersection of all natural powers of Jacobson radical of SPSD-ring is equal to zero.*

*Proof.* Obviously we can consider the ring to be reducible. Denote

$$I_k = \begin{pmatrix} R_1^k & A_{12}R_2^k & \cdots & A_{1s}R_s^k \\ A_{21}R_1^k & R_2^k & \cdots & A_{2s}R_s^k \\ \cdots & \cdots & \cdots & \cdots \\ A_{s1}R_1^k & A_{s2}R_2^k & \cdots & R_s^k \end{pmatrix}.$$

Obviously $I_k$ is two-sided ideal of the ring $A$. It is easy to check that

$$I_k I_l = I_{k+1} \text{ and } R^2 \subset I_1.$$

So, $R^{sk} \subset I_k$ whence

$$\bigcap_{n=0}^{\infty} R^n \subset \bigcap_{k=0}^{\infty} R^n I_k.$$

As all rings $A_{ii}$ are Noetherian chain rings then [12] it follows that they are either discrete valuation rings or uniserial Artinian rings Kiote rings. The intersection of all powers of the Jacobson radical of such rings is equal to zero [12]. According to Theorem 1.9 the ring $A_{ij}$ is a cyclic chain $A_{jj}$-module and a cyclic left chain $A_{ii}$-module. But in this case

$$\bigcap_{k=0}^{\infty} A_{ij}R_j^k = 0, \quad i, j = 1, \ldots, s.$$

This means that the intersection of $I_k$ for all natural $k$ is equal to zero. Whence, the intersection of all natural powers of Jacobson radical is equal to zero. □

### 3.2 Semiprime semiperfect rings

In this section we shall describe the minors of first and second order of right Noetherian semiprime SPSD-rings.

The endomorphism ring of an indecomposable projective module over a semiperfect ring is called a **principal endomorphism ring**.

**Proposition 3.9.** *An Artinian principal endomorphism ring of a semiprime semiperfect ring is a division ring.*

*Proof.* This ring is an Artinian prime local ring and, consequently, is a division ring. □

**Lemma 3.10.** *Let $A_A = P_1^{n_1} \oplus P_2^{n_2} \oplus \ldots \oplus P_s^{n_s}$ be the decomposition of a semiprime semiperfect ring $A$ into principal modules and let $End_A(P_1) = D_1$ be a division ring. Then $A = M_{n_1}(D_1) \times End(P_2^{n_2} \oplus \ldots \oplus P_s^{n_s})$.*

*Proof.* Let $1 = f_1 + \ldots + f_s$ be a canonical decomposition of $1 \in A$ into a sum of pairwise orthogonal idempotents, i.e., $f_i A = P_i^{n_i}$ for $i = 1, \ldots, s$. Let $f_1 A f_1 = A_1$, $(1 - f_1)A(1 - f_1) = A_2$, $X = f_1 A(1 - f_1)$, $Y = (1 - f_1)Af_1$. If either $X \neq 0$ or $Y \neq 0$, then $K = \begin{pmatrix} 0 & X \\ Y & YX \end{pmatrix}$ is a nilpotent ideal and we have the contradiction. So $X = 0$, $Y = 0$, proving the lemma. □

**Theorem 3.11. (Decomposition theorem for semiprime semiperfect rings)** *A semiprime semiperfect ring is a finite direct product of indecomposable rings. An indecomposable semiprime semiperfect ring is either a simple Artinian ring or an indecomposable semiprime semiperfect ring such that all its principal endomorphism rings are non-Artinian.*

A proof immediately follows from Lemma 3.10.

Let $1 = g_1 + g_2$ be a decomposition of the identity of $A$ into a sum of the mutually orthogonal idempotents, and let $A = (A_{ij})$ be the corresponding Peirce decomposition of $A$, i.e., $A_{ij} = g_i A g_j$, $i, j = 1, 2$. Similarly, if M is a two-sided ideal of $A$, then $M = (M_{ij})$ is the Peirce decomposition of $M$, where $M_{ij} = g_i M g_j$, $i, j = 1, 2$.

**Lemma 3.12.** *Let $M = (M_{ij})$ be a two-sided ideal of a semiprime ring $A$. If $M_{ij} \neq 0$ for $i \neq j$, then $M_{ji} \neq 0$. Moreover, if $M_{ij} \neq 0$ for $i \neq j$, then $M_{ij}M_{ji} \neq 0$ and $M_{ji}M_{ij} \neq 0$.*

*Proof.* Let $M_{ij}M_{ji} = 0$. Clearly, $Z = M_{ij}A_{ji} + A_{ij}M_{ji} + M_{ij} + M_{ji}$ is a two-sided ideal and $Z^8 = 0$. The remaining cases are treated analogously. □

**Corollary 3.13.** *Let $1 = e_1 + \ldots + e_n$ be a decomposition of the identity of $A$ into a sum of the mutually orthogonal idempotents, $A_{ij} = e_i A e_j$, $i, j = 1, \ldots, n$, and let $M$ be a two sided ideal in $A$, $M_{ij} = e_i M e_j$, $i, j = 1, \ldots, n$. If $M_{ij} \neq 0$ for $i \neq j$, then $M_{ji} \neq 0$ and $M_{ij}M_{ji} \neq 0$, $M_{ji}M_{ij} \neq 0$. Moreover, from the equality $A_{ij}A_{ji} = 0$ it follows that $A_{ij} = 0$ and $A_{ji} = 0$.*

**Proposition 3.14.** *Let $A$ be a prime (resp. semiprime) ring, $e^2 = e \in A$. Then the ring $eAe$ is prime (resp. semiprime).*

**Theorem 3.15.** *For a semiprime semiperfect ring $A$ the following conditions are equivalent:*

*(1) $A$ is a finite direct product of prime rings;*

*(2) all principal endomorphism rings of $A$ are prime.*

*Proof.* (1) $\Longrightarrow$ (2) follows from Proposition 3.14.

(2) $\Longrightarrow$ (1) Obviously, we can assume that $A$ is indecomposable and reduced. Let $1 = e_1 + \ldots + e_n$ be a decomposition of $1 \in A$ into the sum of pairwise orthogonal local idempotents. We shall prove the theorem by induction on $n$. The case $n = 1$ is obvious. Suppose that $A$ is not prime. Then there exist two-sided nonzero ideals $M, N$ such that $MN = 0$. Let $h_1 = e_1 + \ldots + e_{n-1}$ and $h_2 = e_n$. We have

the equality $h_1Mh_1Nh_1 = 0$. By the induction hypothesis either $h_1Mh_1 = 0$ or $h_1Nh_1 = 0$. Let $h_1Mh_1 = 0$, then by Corollary 3.13 $h_1Mh_2 = 0$ and $h_2Mh_1 = 0$. If $h_2Mh_2 = 0$, then the theorem is proved, so $h_2Mh_2 \neq 0$ and $h_2Nh_2 = 0$. We have again $h_2Nh_1 = 0$ and $h_1Nh_2 = 0$. One can assume that $e_iNe_i \neq 0$ for $i = 1, \dots, t$ and $e_jNe_j = 0$ for $j = t+1, \dots, n$. So $N_{ii}A_{ij} = 0$ for $i = 1, \dots, t$ and $j = t+1, \dots, n$. Consequently, $N_{ii}A_{ij}A_{ji} = 0$ for the same $i$ and $j$. Since the $A_{ii}$ are prime, it follows that $A_{ij}A_{ji} = 0$. By Corollary 3.13, we obtain $A_{ij} = 0$ and $A_{ji} = 0$ for $i = 1, \dots, t$ and $j = t+1, \dots, n$. Hence, the ring $A$ is decomposable and we obtain a contradiction, which proves the theorem. □

Let $A$ be a ring, $P$ a finitely generated projective $A$-module which can be decomposed into a direct sum of $n$ indecomposable modules. The endomorphism ring $B = End_A(P)$ of the module P is called a **minor** of order $n$ of the ring $A$.

**Proposition 3.16.** *Every minor of an SPSD-ring is an SPSD-ring.*

The proof follows from Theorem 1.9 and Corollary 3.1.

**Corollary 3.17.** *Every minor of a right Noetherian semiprime SPSD-ring is a right Noetherian semiprime SPSD-ring.*

The proof follows from Theorem 2.1 and Proposition 3.14.

From Theorems 1.9 and 2.1 we obtain the following statement.

**Corollary 3.18.** *Every minor of a Noetherian SPSD-ring is a Noetherian SPSD-ring.*

**Proposition 3.19.** *A minor of the first order of a right Noetherian SPSD-ring is uniserial and it is either a discrete valuation ring or an Artinian uniserial ring.*

Let $\mathcal{O}$ be right local uniserial Noetherian ring with the unique maximal ideal $\mathfrak{M}$. Consider the following descending chain of two-sided ideals.

$$\mathcal{O} \supset \mathfrak{M} \supset \mathfrak{M}^2 \supset \dots \supset \mathfrak{M}^n \supset \dots$$

By Nakayama Lemma $\mathfrak{M}^k$ strictly contains $\mathfrak{M}^{k+1}$ for any $k \in \mathbb{N}$. As $\mathcal{O}$ is serial ring then right factor module $\mathfrak{M}^k/\mathfrak{M}^{k+1}$ is simple if $\mathfrak{M}^k \neq 0$.

Assume that $\mathfrak{M} \neq 0$. In this case if $\pi \in \mathfrak{M}\backslash\mathfrak{M}^2$ then $\pi\mathcal{O} + \mathfrak{M}^2 = \mathfrak{M}$ and according to Nakayama Lemma $\mathfrak{M} = \pi\mathcal{O}$.

Consider left ideals $\mathcal{O}\pi$ and $\mathfrak{M}$. The local property of the ring $\mathcal{O}$ gives that $\mathfrak{M} \supseteq \mathcal{O}\pi$.

The strong inclusion $\mathcal{O}\pi \supset \mathfrak{M}^2$ follows from that $\mathcal{O}$ is serial. Factor module $\mathfrak{M}/\mathfrak{M}^2$ is semisimple right $\mathcal{O}$-module and is left $\mathcal{O}$-module. As the ring $\mathcal{O}$ is serial then $\mathfrak{M}/\mathfrak{M}^2$ is simple from both sides. Whence $\mathcal{O}\pi = \pi\mathcal{O} = \mathfrak{M}$.

The next proposition immediately follows from this fact.

**Proposition 3.20.** *Let $\mathcal{O}$ be a right local Noetherian serial ring with the unique maximal ideal $\mathfrak{M} \neq 0$. Then $\mathfrak{M} = \pi\mathcal{O} = \mathcal{O}\pi$ and the ring $\mathcal{O}$ is both sides Artinian if and only if the element $\pi$ is nilpotent.*

That is why in future we will assume that the element $\pi$ is not nilpotent.

Consider the endomorphism $\pi$ of the right $\mathcal{O}$-module $\mathcal{O}_{\mathcal{O}}$ which multiplies $\alpha \in \mathcal{O}$ with the element $\pi$ from the left, i.e. $\pi(\alpha) = \pi\alpha$.

**Step** 1. $\ker \pi \subset \bigcap\limits_{n=1}^{\infty} \mathfrak{M}^n$.

*Proof.* Let $ker\pi = \{\alpha \in \mathcal{O} |\ \pi\alpha = 0\}$. It is obvious that $ker\pi$ is two-sided ideal. Really, if $\alpha \in ker\pi$ then $\pi(\alpha\alpha_1) = (\pi\alpha)\alpha_1 = 0$, i.e. $\alpha\alpha_1 \in ker\pi$.

Let $\alpha \in ker\pi$ and $\beta \in \mathcal{O}$. Consider $\pi(\beta\alpha) = (\pi\beta)\alpha = (\beta_1\pi)\alpha = \beta_1(\pi\alpha) = 0$. If $ker\pi = \mathfrak{M}^n$ for some $n$ then $\pi\mathfrak{M}^n = \pi\mathcal{O}\mathfrak{M}^n = \mathfrak{M}^{n+1} = 0$, whence $\pi^{n+1} = 0$. So, $ker\pi \subset \mathfrak{M}^n$ for any natural $n$, whence $ker\pi \subset Y = \bigcap\limits_{n=1}^{\infty} \mathfrak{M}^n$. □

**Step** 2. $\ker \pi = 0$.

*Proof.* Let $X = ker\pi \neq 0$. Then there exists ascending chain of ideals

$$ker\pi \subset ker\pi^2 \subset \ldots \subset ker\pi^n \subset \ldots.$$

Let us show that $\ker \pi^k \neq ker\pi^{k+1}$ for all $k$. Let $\ker \pi^k = ker\pi^{k+1}$ for some $k$ and $x \in X$, $x \neq 0$. So, $\pi x = 0$ and $x \in \bigcap\limits_{n=1}^{\infty} \mathfrak{M}^n$. This is followed by $x = \pi^k\alpha_k = \pi^{k+1}\alpha_{k+1}$. The equality $\pi x = 0$ implies $\pi^{k+1}\alpha_{k+1} = 0$ i.e. $\alpha_k \in ker\pi^{k+1}$ and this means that $\alpha_k \in ker\pi^k$ and $\pi^k\alpha_k = 0 = x$. That is why there exists strongly ascending chain of two-sided ideals

$$ker\pi \subset ker\pi^2 \subset \ldots \subset ker\pi^n \subset \ldots.$$

and this is a contradiction with the property of the ring $\mathcal{O}$ to be right Noetherian. The proposition is proved. □

**Step** 3. $\bigcap\limits_{n=1}^{\infty} \mathfrak{M}^n = 0$.

*Proof.* Let $Y = \bigcap\limits_{n=1}^{\infty} \mathfrak{M}^n \neq 0$. Consider two-sided ideal $Y\mathfrak{M}$ of the ring $\mathcal{O}$ which is the unique maximal submodule of $\mathfrak{M}$ as the ring $\mathcal{O}$ is right Noetherian.

Considering the factor ring $\mathcal{O}/Y\mathfrak{M}$ one may assume that the intersection $Y = \bigcap\limits_{n=1}^{\infty} \mathfrak{M}^n$ is a simple right $\mathcal{O}$-module in the former ring $\mathcal{O}$. The property of $Y$ to be a two-sided ideal and the equality $ker\pi = 0$ imply that $\pi Y = Y$.

Let $W = \{\alpha \in \mathcal{O} |\ \alpha\pi \in Y\}$. Obviously $W \neq 0$ because $y \in \mathfrak{M}$, $y \in Y$, $y \neq 0$ and $y = \alpha\pi$.

Let us show that $W$ is a two-sided ideal of the ring $\mathcal{O}$. Obviously $\alpha + \alpha_1 \in W$ if $\alpha, \alpha_1 \in W$.

Let $\alpha \in W$, i.e. $\alpha\pi \in Y$. Then $(\beta\alpha)\pi = \beta y_1 \in Y$, i.e. $\beta\alpha \in W$ for any $\beta \in \mathcal{O}$. Moreover, $(\alpha\beta)\pi = \alpha(\beta\pi) = \alpha(\pi\beta_1) = (\alpha\pi)\beta_1 \in Y$. If $W \not\subset Y$ then

$W = \pi^n \mathcal{O} = \mathcal{O}\pi^n$ and $W\pi \in Y$, i.e. $W\pi = \mathfrak{M}^{n+1} \subset Y$. The obtained contradiction shows that $W \subset Y$ and so $W = Y$. Let $\in Y$ and $y \neq 0$. Then $y = y_1\pi$ for some $y_1 \in Y$. But $Y\pi = 0$, whence $y = 0$. The obtained contradiction proves that $\mathcal{O}$ is a discrete valuation ring with the unique maximal ideal $\mathfrak{M} = \pi\mathcal{O} = \mathcal{O}\pi$. For more details see Warfield [1975]. □

**Corollary 3.21.** *A minor of the first order of a right Noetherian semiprime SPSD-ring is either a discrete valuation ring or a division ring.*

A ring $A$ is called **semimaximal** if it is a semiperfect semiprime right Noetherian ring such that for each local idempotent $e \in A$ the ring $eAe$ is a discrete valuation ring (not necessarily commutative), i.e., all principal endomorphism rings of $A$ are discrete valuation rings.

**Proposition 3.22.** *A semimaximal ring is a finite direct product of prime semimaximal rings.*

A proof follows from Theorem 3.15.

So, a semimaximal ring $A$ is indecomposable if and only if $A$ is prime.

**Proposition 3.23.** *A semiperfect reduced indecomposable ring $B$ is a second order minor of a right Noetherian semiprime SPSD-ring if and only if $B$ is semimaximal.*

*Proof.* Let $1 = e_1 + e_2$ be a decomposition of $1 \in B$ into a sum of local idempotents, let $B = \bigoplus\limits_{i,j=1}^{2} e_iBe_j$ be the corresponding two-sided Peirce decomposition, and let $B_{ij} = e_iBe_j$ $(i, j = 1, 2)$. The Jacobson radical $R$ of $B$ has the form: $R = \begin{pmatrix} R_1 & B_{12} \\ B_{21} & R_2 \end{pmatrix}$, where $R_i$ is the Jacobson radical of $B_{ii}$ $(i = 1, 2)$. Obviously,

$$R^2 = \begin{pmatrix} R_1^2 + B_{12}B_{21} & R_1B_{12} + B_{12}R_2 \\ R_2B_{21} + B_{21}R_1 & R_2^2 + B_{21}B_{12} \end{pmatrix}$$

By Corollary 3.19, $B_{ii}$ is either a discrete valuation ring or a division ring. If $B_{11} = D$ is a division ring, then $R = \begin{pmatrix} 0 & B_{12} \\ B_{21} & R_2 \end{pmatrix}$. Obviously, $J = \begin{pmatrix} 0 & B_{12} \\ B_{21} & B_{21}B_{12} \end{pmatrix}$ is a nonzero ideal in $B$ and $J_2 = 0$. So $B$ is semimaximal.

Let's now show that a semimaximal ring $B$ is semidistributive. We can assume that $B$ is prime. Let $R_i = \pi_iB_{ii} = B_{ii}\pi_i$ $(i = 1, 2)$. Now $b_{12}b_2 \neq 0$ for any $b12 \neq 0$ and $b_2 \neq 0$ $(b_{12} \in B_{12}, b_2 \in B_{22})$. Indeed, we can suppose that $b_2 = \pi_2^m$. Then $\begin{pmatrix} 0 & b_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix} \neq 0$ and, consequently, $b_{12}B_{22}p_2^m = b_{12}p_2^mB_{22} \neq 0$. So, $b_{12}p_2^m \neq 0$. Analogously, $b_{ij}b_j \neq 0$ and $b_ib_{ij} \neq 0$ for $i, j = 1, 2$. Further $b_{ij}b_{ji} \neq 0$ for $i \neq j$ and both factors are nonzero. We shall prove that $b_{21}b_{12} \neq 0$ for $b_{12} \neq 0$ and $b_{21} \neq 0$. Indeed, $\begin{pmatrix} 0 & b_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ b_{21} & 0 \end{pmatrix} \neq$

0. So, $b_{12}B_{22}b_{21} \neq 0$ and thus there exists $b_2 \in B_{22}$ such that $b_{12}b_{22}b_{21} \neq 0$. If $b_{21}b_{12} = 0$, then $b_{21}b_{12}b_{22}b_{21} = 0$ and we obtain a contradiction.

Next $B_{12}$ is a uniserial right $B_{22}$-module and a uniserial left $B_{11}$-module. By Theorem 2.1, $B_{12}$ is a finitely generated $B_{22}$-module. Consequently, if $B_{12}$ isn't uniserial, then $B_{12} = B_{12}^{(1)} \oplus B_{12}^{(2)}$, where $B_{12}^{(1)}$ and $B_{12}^{(2)}$ are nonzero $B_{22}$-submodules of $B_{12}$. Let $b_{21} \neq 0$. Then $b_{21}B_{12} = b_{21}B_{12}^{(1)} \oplus b_{21}B_{12}^{(2)}$, where $b_{21}B_{12}^{(1)}$ and $b_{21}B_{12}^{(1)}$ are nonzero right ideals in $\mathcal{O}_2$. This is a contradiction. Consequently, $B_{12}$ is a uniserial right $B_{22}$-module.

Finally $B_{12}$ is a uniserial left $B_{11}$-module. If this isn't true, then there exists a left $B_{11}$-submodule $N_{12}$ with two noncyclic generators in $B_{12}$. Consequently, $N_{12} = N_{12}^{(1)} \oplus N_{12}^{(2)}$ is a direct sum of two nonzero left $B_{11}$-submodules and so $N_{12}b_{21} = N_{12}^{(1)}b21 \oplus N_{12}^{(2)}b_{21}$ is a direct sum of two nonzero left ideals in $B_{11}$ for any nonzero $b_{21}$. This is a contradiction and so B12 is a uniserial left $B_{11}$-module. Analogously, $B_{21}$ is a uniserial right $B_{11}$-module and a uniserial left $B_{22}$-module. Thus, by Theorem 1.9 $B$ is semidistributive. The proposition is proved. □

**Corollary 3.24.** *An intersection of a finite number of nonzero submodules of an indecomposable projective module over a Noetherian prime SPSD- ring is nonzero.*

**Lemma 3.25.** *A local idempotent of a Noetherian prime SPSD-ring $A$ is a local idempotent of its classical ring of fractions.*

Note that an example of semimaximal rings is non-Artinian both sides Noetherian semiprime hereditary rings. They are exactly semimaximal hereditary rings. The article [19] contains a condition for the prime semimaximal ring $\Lambda$ to be of finite type. This condition is as follows. As an arbitrary prime semimaximal ring $\Lambda$ can be included into the prime ring of fractions $Q$, let $\mathfrak{M}(\Lambda)$ be the partially ordered set (in the sense of inclusion) of all projective $\Lambda$-modules which belong to some prime $Q$-module. So, the equivalent condition for the prime semimaximal ring to be of finite type is the nonexistence of critical subsets in the set $\mathfrak{M}(\Lambda)$. Here a subset of a partially ordered set is called critical if it is one of the following sets: $(1,1,1,1)$, $(2,2,2)$, $(1,3,3)$, $(1,2,5)$, $R = \{a_1 < a_2 > b_1 < b_2; c_1 < c_2 < c_3 < c_4\}$. Here we denote by $(l_1, \ldots, l_m)$ the cardinal sum of $m$ linearly ordered sets which contain $l_1, \ldots, l_m$ elements correspondingly.

For proving Lemma 3.25 we need the following proposition [6, Prop. 9.3.10].

**Proposition 3.26.** *Let $Q$ be a semisimple ring and $A$ be a right order in $Q$. Then $Q$ is a simple ring if and only if $A$ is prime.*

*Proof of Lemma 3.25.* By Proposition 3.26 $A$ is a right order in the simple Artinian ring $Q = M_n(D)$. One can assume that the local idempotent $e \in A$ is a sum of matrix idempotents $e = e_{i_1i_1} + \ldots + e_{i_ki_k}$. Let $k \geq 2$. Then there exist $q_1, \ldots, q_k \in Q$ such that $e_{i_1i_1}q_1, \ldots, e_{i_ki_k}q_k \in A$ and, consequently, $e_{i_1i_1}q_1A, \ldots, e_{i_ki_k}q_kA$ are nonzero right submodules of the right indecomposable projective module $eA$ and $e_{i_mi_m}q_mA \cap e_{i_pi_p}q_pA = 0$ for $m \neq p$. We obtain a contradiction with Corollary 3.24. □

### 3.3 Right Noetherian semiprime SPSD-rings

The following is a **decomposition theorem** for semiprime right Noetherian SPSD-rings.

**Theorem 3.27.** *The following conditions for a semiperfect semiprime right Noetherian ring $A$ are equivalent:*

*(a) the ring $A$ is semidistributive;*

*(b) the ring $A$ is a direct product of a semisimple Artinian ring and a semimaximal ring.*

*Proof.* $(a) \Longrightarrow (b)$. From Theorem 3.11 it follows that $A$ is a finite direct product of indecomposable semiprime rings. Every indecomposable ring is either a simple Artinian ring or a semiprime semiperfect ring such that all its principal endomorphism rings are non-Artinian. In the second case, by Corollary 3.21, such a ring is semimaximal.

$(b) \Longrightarrow (a)$. Obviously, a semiprime Artinian ring is a semiprime SPSD-ring. A semimaximal ring is an SPSD-ring, by Proposition 3.11 and the reduction theorem for SPSD-rings. □

**Lemma 3.28.** *The right uniserial modules over the ring $H_m(\mathcal{O})$ are exhausted by the $D^m$, all principal $H_m(\mathcal{O})$-modules and quotient modules of these modules.*

**Theorem 3.29.** *Each semimaximal ring is isomorphic to a finite direct product of prime rings of the following form:*

$$A = \begin{pmatrix} \mathcal{O} & \pi^{\alpha_{12}}\mathcal{O} & \dots & \pi^{\alpha_{1n}}\mathcal{O} \\ \pi^{\alpha_{21}}\mathcal{O} & \mathcal{O} & \dots & \pi^{\alpha_{2n}}\mathcal{O} \\ \dots & \dots & \dots & \dots \\ \pi^{\alpha_{n1}}\mathcal{O} & \pi^{\alpha_{n2}}\mathcal{O} & \dots & \mathcal{O} \end{pmatrix} \tag{1}$$

*where $n \geq 1$, $\mathcal{O}$ is a discrete valuation ring with a prime element $\pi$, and the $\alpha_{ij}$ are integers such that $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$ for all $i, j, k$ ($\alpha_{ii} = 0$ for any $i$).*

*Proof.* By Proposition 3.10 a semimaximal ring is a finite direct product of prime semimaximal rings. We shall show that a prime semimaximal ring is isomorphic to a ring of the form (1).

Let $1 = e_1 + \ldots + e_m$ be a decomposition of $1 \in A$ into a sum of pairwise orthogonal local idempotents, $A_{ij} = e_iAe_j$ for $i, j = 1, \ldots, m$. Denote by $B_{ij}$ $(i \neq j)$ the following second order minor: $B_{ij} = \begin{pmatrix} A_{ii} & A_{ij} \\ A_{ji} & A_{jj} \end{pmatrix}$. If $B_{ij}$ isn't reduced, then $B_{ij} \simeq M_2(A_{ii})$ and $B_{ij}$ is left Noetherian. If $B_{ij}$ is reduced, then $A_{ij}a_{ji} \subset A_{ij}$, $\varphi_{ji}: A_{ij} \to A_{ii}$ being the monomorphism of left $A_{ii}$-modules (for any nonzero $a_{ji}$) such that $\varphi_{ji}(a_{ij}) = a_{ij}a_{ji}$. If $A_{ij}$ isn't finitely generated, then $A_{ii}$ contains a non-finitely generated left $A_{ii}$-submodule $A_{ij}a_{ji}$, where $a_{ji} \neq 0$. This gives a contradiction. So, by Lemma 3.28, $A_{ij} \simeq A_{ii}$ and $B_{ij}$ is left Noetherian, by Theorem 2.1. Applying induction on $m$ and Theorem 2.1, we see that $A$ is left Noetherian. Consequently, $A$

is a prime Noetherian SPSD-ring. By Proposition 3.26, $A$ is a right order in a simple Artinian ring $Q = M_n(D)$. Suppose that every local idempotent $e_i$ from the above decomposition $1 = e_1 + \ldots + e_m$ is local in $M_n(D)$. Hence, the two decompositions: $1 = e_1 + \ldots + e_m$ and $1 = e_{11} + \ldots + e_{nn}$ are conjugate. Consequently, $m = n$ and we can assume that the matrix idempotents are the local idempotents of $A$.

Denote $A_{ii}$ by $A_i$. We have $Q = \sum\limits_{i,j=1}^{n} e_{ij}\mathcal{D}$ ($\mathcal{D}$ is a division ring, the $e_{ij}$, are matrix units commuting with the elements from $\mathcal{D}$) and $A = \sum\limits_{i,j=1}^{n} e_{ij}A_{ij}$, where $A_{ij} \subset \mathcal{D}$. All $A_i$ are discrete valuations rings, $A_{ij}A_{jk} \subset A_{ik}$ and $A_{ij} \neq 0$ for $i, j = 1, \ldots, n$ ($A$ is prime and $e_{ii}Ae_{jj} = A_{ij} \neq 0$).

We shall prove that $A_{ij} = d_{ij}A_j = A_i d_{ij}$, where $d_{ij} \in A_{ij} \subset \mathcal{D}$. Indeed, let $R_i$ be the Jacobson radical of $A_i$ and let $\pi_i A_i = A_i\pi_i = R_i$. By corollary 3.3, $R_iA_{ij} = A_{ij}R_j$. Take an element $0 \neq d_{ij} \in A_{ij}$ so that $A_i d_{ij} + R_iA_{ij} = A_{ij}$. By Nakayama's Lemma $A_{ij} = d_{ij}A_j = A_i d_{ij}$. Let $T = diag(d_{12}^{-1}, d_{23}^{1}, \ldots, d_{n-1n}^{-1}, 1)$. Consider $TAT{-}1$. One can assume that the following equalities $d_{12} = \ldots = d_{n-1n}$ hold in $A$, hence $A_1 = A_2 = \ldots = A_n$. Write $A_1 = \mathcal{O}$, where $\mathcal{O}$ is a discrete valuation ring (non-necessarily commutative). Consequently, $A_{ij} \supset \mathcal{O}$ for $i \leq j$. From $A_{ij}A_{ji} \subset \mathcal{O}$ we have $A_{ij}A_{ji} \supset A_{ji}$ and $A_{ji} \subset \mathcal{O}$ for $j \leq i$. So, one can assume that $d_{ji} = \pi^{\alpha_{ij}}$, where $\mathcal{M} = \pi\mathcal{O} = \mathcal{O}\pi$ is the unique maximal ideal of $\mathcal{O}$, $\alpha_{ji} \geq 0$ for $j \geq i$. Obviously, $d_{ij} = \pi^{\alpha_{ij}}$, where $\alpha_{ij} \geq -\alpha_{ji}$. Hence, we obtain a ring of the form 3.27. The converse assertion follows from the definition of a semimaximal ring. □

A ring A is called **a tiled order** if it is a prime Noetherian SPSD-ring with nonzero Jacobson radical.

**Remark**. Let $\mathcal{O}$ be a discrete valuation ring. Then from Theorem 3.29 it follows that each tiled order is of the form (1).

The ring $\mathcal{O}$ is embedded into a classical ring of fractions $\mathcal{D}$, which is a division ring. Therefore (14.5.1) denotes the set of all matrices $(a_{ij}) \in M_n(\mathcal{D})$ such that $a_{ij} \in \pi^{\alpha_{ij}}\mathcal{O} = e_{ii}Ae_{jj}$, where the $e_{11}, \ldots, e_{nn}$ are the matrix units of $M_n(\mathcal{D})$. It is clear that $M_n(\mathcal{D})$ is the classical ring of fractions of $A$.

According to the terminology of V. A. Jategaonkar and R. B. Tarsy, a ring $A \subset M_n(K)$, where $K$ is the quotient field of a commutative discrete valuation ring $\mathcal{O}$, is called a tiled order over $\mathcal{O}$ if $M_n(K)$ is the classical ring of fractions of $A$, $e_{ii} \in A$ and $e_{ii}Ae_{ii} = \mathcal{O}$ for $i = 1, \ldots, n$, where the $e_{11}, \ldots, e_{nn}$ are the matrix units of $M_n(K)$ (see [8]).

Denote by $M_n(\mathbb{Z})$ the ring of all square $n \times n$-matrices over the ring of integers $\mathbb{Z}$. Let $\mathcal{E} \in Mn(\mathbb{Z})$. We shall call a matrix $\mathcal{E} = (\alpha_{ij})$ an **exponent matrix** if $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$ for $i, j, k = 1, \ldots, n$ and $\alpha_{ii} = 0$ for $i = 1, \ldots, n$. A matrix $\mathcal{E}$ is called a **reduced exponent matrix** if $\alpha_{ij} + \alpha_{ji} > 0$ for $i, j = 1, \ldots, n$.

We shall use the following notation: $A = \{\mathcal{O}, \mathcal{E}(A)\}$, where $\mathcal{E}(A) = (\alpha_{ij})$ is the exponent matrix of a ring $A$, i.e., $A = \sum\limits_{i,j=1}^{n} e_{ij}\pi^{\alpha_{ij}}\mathcal{O}$, where the $e_{ij}$ are the matrix units. If a tiled order is reduced, then $\alpha_{ij} + \alpha_{ji} > 0$ for $i, j = 1, \ldots, n$, $i \neq j$, i.e., $\mathcal{E}(A)$ is reduced.

Let $\mathcal{O}$ be a discrete valuation ring. A right (resp. left) $A$-module $M$ (resp. $N$) is called a **right** (resp. **left**) $A$**-lattice** if $M$ (resp. $N$) is a finitely generated free $\mathcal{O}$-module.

For example, all finitely generated projective $A$-modules are $A$-lattices.

Given a tiled order $A$ we denote by $Lat_r(A)$ (resp. $Lat_l(A)$) the category of right (resp. left) $A$-lattices. We denote by $S_r(A)$ (resp. $S_l(A)$) the partially ordered set (by inclusion), formed by all $A$-lattices contained in a fixed simple $M_n(\mathcal{D})$-module $U$ (resp. in a left simple $M_n(\mathcal{D})$-module $V$). Such $A$-lattices are called **irreducible**.

Note that every simple right $M_n(\mathcal{D})$-module is isomorphic to a simple $M_n(\mathcal{D})$-module $U$ with $\mathcal{D}$-basis $e_1, \ldots, e_n$ such that $e_i e_{jk} = \delta_{ij} e_k$, where $e_{jk} \in M_n(\mathcal{D})$ are the matrix units. Respectively, every simple left $M_n(\mathcal{D})$-module is isomorphic to a left simple $M_n(\mathcal{D})$-module $V$ with $D$-basis $e_1, \ldots, e_n$ such that $e_{ij} e_k = \delta_{jk} e_i$.

Let $A = \{\mathcal{O}, E(A)\}$ be a tiled order, and let $U$ (resp. $V$) be a simple right (resp. left) $M_n(\mathcal{D})$-module as above.

Then any right (resp. left) irreducible $A$-lattice $M$ (resp. $N$) lying in $U$ (resp. in $V$) is an $A$-module with $\mathcal{O}$-basis $(\pi_1^{\alpha} e_1, \ldots, \pi^{\alpha_n} e_n)$, while

$$\begin{cases} \alpha_i + \alpha_{ij} \geq \alpha_j, & \text{for the right case;} \\ \alpha_{ij} + \alpha_j \geq \alpha_i, & \text{for the left case.} \end{cases} \tag{2}$$

Thus, irreducible $A$-lattices $M$ can be identified with an integer-valued vector $(\alpha_1, \ldots, \alpha_n)$ satisfying (3.29). We shall write $[M] = (\alpha_1, \ldots, \alpha_n)$ or $M = (\alpha_1, \ldots, \alpha_n)$.

The order relation on the set of such vectors and the operations on them corresponding to sum and intersection of irreducible lattices are obvious.

**Remark.** Obviously, two irreducible $A$-lattices $M_1 = (\alpha_1, \ldots, \alpha_n)$ and $M_2 = (\beta_1, \ldots, \beta_n)$ are isomorphic if and only if $\alpha_i = \beta_i + z$ for $i = 1, \ldots, n$ and (a fixed) $z \in \mathbb{Z}$. We shall denote by $(\alpha_1, \ldots, \alpha_n)^T$ the column vector with coordinates $\alpha_1, \ldots, \alpha_n$.

Note that the posets $S_r(A)$ and $S_l(A)$ do not depend on the choice of simple $M_n(\mathcal{D})$-modules $U$ and $V$.

**Proposition 3.30.** *The posets $S_r(A)$ and $S_l(A)$ are anti-isomorphic distributive lattices.*

*Proof.* Since A is a semidistributive ring, $S_r(A)$ (resp. $S_l(A)$) is a distributive lattice with respect to the sum and intersection of submodules.

Let $M = (\alpha_1, \ldots, \alpha_n) \in S_r(A)$. We put $M^* = (-\alpha_1, \ldots, -\alpha_n)^T \in S_l(A)$. If $N = (\beta_1, \ldots, \beta_n)^T \in S_l(A)$, then $N^* = (-\beta_1, \ldots, -\beta_n) \in S_r(A)$.

Obviously, the operation * satisfies the following conditions:

1. $M^{**} = M$; 2. $(M_1+M_2)^* = M_1^* \cap M_2^*$; 3. $(M_1 \cap M_2)* = M_1^* + M_2^*$ in the right case and there are analogous rules in the left case. Thus, the map $*: S_r(A) \to S_l(A)$ is the anti-isomorphism. □

**Remark**. The map * defines a duality for irreducible A-lattices.

If $M_1 \subset M_2$, $(M_1, M_2 \in S_r(A))$, then $M_2^* \subset M_1^*$. In this case, the $A$-lattice $M_2$ is called an **overmodule** of the $A$-lattice $M_1$ (resp. $M_1^*$ is an overmodule of $M_2^*$).

### 3.4 Quivers of tiled orders

Recall that a quiver is called **strongly connected** if there is a path between any two vertices. By convention, a one-point graph without arrows will be considered a strongly connected quiver. A quiver $Q$ without multiple arrows and multiple loops is called **simply laced**, i.e., $Q$ is a simply laced quiver if and only if its adjacency matrix $[Q]$ is a $(0,1)$-matrix.

**Theorem 3.31.** *Let $A$ be a semiperfect two-sided Noetherian ring with the quiver $Q(A)$. Suppose the matrix $[Q]$ is block upper triangular with permutationally irreducible matrices $B_1, \dots, B_t$ on the main diagonal of the Peirce quiver of $A$. Then there exists a decomposition of $1 \in A$ into a sum of mutually orthogonal idempotents: $1 = g_1 + \dots g_t$ such that*

$$A = \bigoplus_{i,j=1}^{t} giAg_j$$

*is the two-sided Peirce decomposition with $g_iAg_j = 0$ for $j < i$, moreover, the adjacency matrices of the quivers $Q(A_i)$ of the rings $A_i = g_iAg_i$ coincide with $B_i$, $i = 1, \dots, t$.*

**Theorem 3.32.** *The quiver $Q(A)$ of a right and left Noetherian indecomposable semiprime semiperfect ring $A$ is strongly connected.*

A proof follows from Theorem 3.31 and Proposition 3.14. We use notations from Theorem 3.31. If $Q(A)$ isn't strongly connected, then the ring $(g_1 + g_2)A(g_1 + g_2)$ isn't semiprime. Indeed, for the nonzero ideal $J = \begin{pmatrix} 0 & g_1Ag_2 \\ 0 & 0 \end{pmatrix}$ we have $J^2 = 0$.

Let I be a two-sided ideal of a tiled order A. Obviously,

$$U = \sum_{i,j=1}^{n} e_{ij}\pi^{\mu_{ij}}\mathcal{O},$$

where the $e_{ij}$ are matrix units. Denote by $E(I) = (\mu_{ij})$ the exponent matrix of the ideal $I$. Suppose that $I$ and $J$ are two-sided ideals of the ring $A$, $\mathcal{E}(I) = (\mu_{ij})$, and $\mathcal{E}(J) = (\nu_{ij})$. It follows easily that $\mathcal{E}(IJ) = (\delta_{ij})$, where $\delta_{ij} = min_k\{\mu_{ik} + \nu_{kj}\}$.

**Theorem 3.33.** *The quiver $Q(A)$ of a tiled order $A$ over a discrete valuation ring $\mathcal{O}$ is strongly connected and simply laced. If $A$ is reduced, then $Q(A) = \mathcal{E}(R^2) - \mathcal{E}(R)$.*

*Proof.* Taking into account that $A$ is a prime Noetherian semiperfect ring, it follows from Theorem 3.32 that $Q(A)$ is a strongly connected quiver. Let A be a reduced order. Then $[Q(A)]$ is a reduced matrix. We shall use the following notation: $\mathcal{E}(A) = (\alpha_{ij})$; $\mathcal{E}(R) = (\beta_{ij})$, where $\beta_{ii} = 1$ for $i = 1, \dots, n$ and $\beta_{ij} = \alpha_{ij}$ for $i \neq j$ $(i, j = 1, \dots, n)$; $\mathcal{E}(R^2) = (\gamma_{ij})$, where $\gamma_{ij} = \min_{1 \leqslant k \leqslant n}\{\alpha_{ik} + \beta_{kj}\}$ for $i, j = 1, \dots, n$. Since, $\mathcal{E}(A)$ is reduced, we have $\alpha_{ij} + \alpha_{ji} \geqslant 1$ for $i, j = 1, \dots, n$, i.e., $\gamma_{ii} = \min_{1 \leqslant k \leqslant n;\, k \neq i,j}\{\alpha_{ik} + \alpha_{ki}\} = min_{1 \leqslant k \leqslant n,\, k \neq = i}\{\alpha_{ik} + \alpha_{ki}\}$. Hence $\gamma_{ii}$ is equal to 1 or 2. If $i \neq j$, then $\beta_{ij} = \alpha_{ij}$ and $\gamma_{ij} = \min\{\min_{1 \leqslant k \leqslant n, k \neq i,j}\{\alpha_{ik} + \alpha_{kj}\}, \alpha_{ij} + 1\}$, i.e., $\gamma_{ij}$ equals $\alpha_{ij}$ or $\alpha_{ij} + 1$.

To any irreducible $A$-lattice $M$ with $\mathcal{O}$-basis $(\pi_{\alpha_1} e_1, \dots, \pi^{\alpha_n} e_n)$ associate the $n$-tuple $[M] = (\alpha_1, \dots, \alpha_n)$. Let us consider

$$[P_i] = (\alpha_{i1}, \dots, 0, \dots, \alpha_{in}),$$

$$[P_i R] = (\alpha_{i1}, \dots, 1, \dots, \alpha_{in}) = (\beta_{i1}, \dots, \beta_{in}).$$

Set $[P_i R^2] = (\gamma_{i1}, \dots, \gamma_{in})$. Then $\overrightarrow{q}_i = [P_i R^2] - [P_i R]$ is a (0, 1)-vector. Suppose that the positions of the units of $\overrightarrow{q}_j$ are $j_1, \dots, j_m$. In view of the annihilation lemma, this means that $P_i R / P_i R^2 = U_{j_1} \oplus \ldots \oplus U_{j_m}$. By the definition of $Q(A)$ we have exactly one arrow from the vertex $i$ to each of $j_1, \dots, j_m$. Thus, the adjacency matrix $[Q(A)]$ is:

$$[Q(A)] = \mathcal{E}(R^2) - \mathcal{E}(R).$$

The theorem is proved. □

A tiled order $A = \{\mathcal{O}, \mathcal{E}(A)\}$ is called a $(0, 1)$**-order** if $\mathcal{E}(A)$ is a $(0, 1)$-matrix.

Henceforth a (0, 1)-order will always mean a tiled (0, 1)-order over a discrete valuation ring $\mathcal{O}$.

With a reduced $(0, 1)$-order $A$ we associate the partially ordered set

$$P_A = \{1, \dots, n\}$$

with the relation $\leqslant$ defined by $i \leqslant j \Leftrightarrow \alpha_{ij} = 0$.

Obviously, $(P, \leqslant)$ is a partially ordered set (poset).

Conversely, to any finite poset $P = \{1, \dots, n\}$ assign a reduced $(0, 1)$-matrix $\mathcal{E}_p = (A_{ij})$ in the following way: $A_{ij} = 0 \Leftrightarrow i \leqslant j$, otherwise $A_{ij} = 1$. Then $A(P) = \{\mathcal{O}, \mathcal{E}_P\}$ is a reduced (0, 1)-order.

We give a construction which for a given finite partially ordered set $P = \{p_1, \dots, p_n\}$ yields a strongly connected quiver without multiple arrows and multiple loops.

Denote by $P_{max}$ (respectively $P_{min}$) the set of the maximal (respectively minimal) elements of P and by $P_{max} \times P_{min}$ their Cartesian product.

The quiver $\widetilde{Q}(P)$ obtained from the diagram $Q(P)$ by adding the arrows $\sigma_{ij}$ : $i \to j$ for all $(p_i, p_j) \in P_{max} \otimes P_{min}$ is called **the quiver associated with the partially ordered set** $P$.

Obviously, $\widetilde{Q}(P)$ is a strongly connected simply laced quiver.

**Theorem 3.34.** *The quiver $Q(A(P))$ coincides with the quiver $\widetilde{Q}(P)$.*

*Proof.* Recall that $[Q(A(P))] = \mathcal{E}(R^2) - \mathcal{E}(R)$. Suppose that in $Q(P)$ there is an arrow from $s$ to $t$. This means that $\alpha_{st} = 0$ and there is no positive integer $k$ ($k \neq s, t$) such that $\alpha_{sk} = 0$ and $\alpha_{kt} = 0$. The elements $\beta_{ss}$ and $\beta_{tt}$ of the exponent matrix $\mathcal{E}(R) = (\beta_{ij})$ are equal to 1. We have that $\mathcal{E}(R^2) = (\gamma_{ij})$, where $\gamma_{ij} = min_{1 \leqslant k \leqslant n}(\beta_{sk} + \beta_{kt}) = 1$. Thus, in $[Q(A(P))]$ at the $(s,t)$-th position we have $\gamma_{st} - \beta_{st} = 1 - \alpha_{st} = 1 - 0 = 1$. Consequently, $Q(A(P))$ has an arrow from $s$ to $t$.

Suppose that $p \in P_{max}$. This means that $\alpha_{pk} = 1$ for $k \neq p$. Therefore the entries of the $p$-th row of $\mathcal{E}(R)$ are all 1, i.e., $(\beta_{p1}, \ldots, \beta_{pp}, \ldots, \beta_{pn}) = (1, \ldots, 1, \ldots, 1)$.

Similarly, if $q \in P_{min}$, then the $q$-th column $(\beta_{1q}, \ldots, \beta_{qq}, \ldots, \beta_{nq})^T$ of $\mathcal{E}(R)$ is $(1, \ldots, 1, \ldots, 1)^T$. Hence, $\gamma_{pq} = 2$, $\beta_{pq} = 1$, and $Q(A(P))$ has an arrow from $p$ to $q$. Consequently, we proved that $\widetilde{Q}(P)$ is a subquiver of $Q(A(P))$.

We show now the converse inclusion. Suppose that $\gamma_{pq} = 2$. Then obviously

$$(\beta_{p1}, \ldots, \beta_{pp}, \ldots, \beta_{pq}) = (1, \ldots, 1, \ldots, 1)$$

and

$$(\beta_{1q}, \ldots, \beta_{qq}, \ldots, \beta_{nq})^T = (1, \ldots, 1, \ldots, 1)^T.$$

Therefore $p \in P_{max}, q \in P_{min}$ and there is an arrow, which goes from $p$ to $q$.

Suppose $\gamma_{pq} = 1$ and $\beta_{pq} = 0$. Consequently, $p \neq q$, $\beta_{pq} = \alpha_{pq} = 0$ and $p < q$. Since $\gamma_{pq} = min_{1 \leqslant k \leqslant n}(\beta_{pk} + \beta_{kp})$, then $\beta_{pk} + \beta_{kq} \geqslant 1$ for $k = 1, \ldots, n$. Thus, for $k \neq p, q$ we have $\beta_{pk} + \beta_{kq} \geqslant 1$, whence we obtain $\alpha_{pk} + \alpha_{kp} \geqslant 1$. Therefore, there is no positive integer $k$ ($k \neq p, q$) such that $\alpha_{pk} = \alpha_{kq} = 0$. This means that there is an arrow from $p$ to $q$ in $\widetilde{Q}(P)$, and this proves the opposite inclusion. $\square$

## References

[1] Asano K. *Über verallgemeinerte Abelsche Gruppen mit hyperkomplexen Operatorenring und ihre Anwendungen.* Japan J. Math., 1939, **15**, 231–253.

[2] Asano K. *Über Hauptidealringe mit Kettensatz.* Osaka Math. J., 1949, **1**, 52–61.

[3] Camillo V. P. *Distributive modules.* J. Algebra, 1975. **36**, No. 1, 16–25.

[4] Eisenbud D., Griffith P. *The structure of serial rings.* Pacific J. Math., 1971, **36**, No. 1, 109–121.

[5] Eisenbud D., Griffith P. *On indecomposable injectives over artinian rings.* Pacific J. Math., 1969, **29**, 115–135.

[6] Hazewinkel M., Gubareni N. Kirichenko V. V. *Algebras, Rings and Modules, Vol. I*, Mathematics and Its Applications. Vol. 575, Kluwer Academic Publishers, 2004, 380 p.

[7] Jacobson N. *The theory of rings.* Amer. Math. Soc., v.2, Surveys, New Jork, 1943

[8] Jategaonkar V. A. *Global dimension of tiled orders over a discrete valuation ring.* Trans. Amer. Math. Soc., 1974, **196**, 313–330.

[9] Këothe G. *Verallgemeinerte Abelsche Gruppen mit Hyperkomplexen Operatorenring.* Math. Z., 1935, Vol. .39, 31–44.

[10] Kirichenko V. V. *Generalized uniserial rings.* Mat. sb., 1976, **99(141)**, No. 4, 559–581.

[11] Kirichenko V. V., Yaremenko Yu. V. *Noetherian biserial rings.* Ukr. Math. Journal, 1988, **40**, No. 4, 435–440.

[12] Kirichenko V. V. *Generalized uniserial rings.* Preprint IM-75-1, Kiev, 1975.

[13] Kirichenko V. V. *Semi-perfect semi-distributive rings.* Algebras and Representation theory, 2000, **3**, 81–98.

[14] Kirichenko V. V., Khibina M. A. *Semi-perfect semi-distributive rings.* Infinite Groups and Related Algebraic Topics, Institute of Mathematics NAS Ukraine, 1993, 457–480 (in Russian).

[15] Muller B. J. *On semi-peerfect rings*, Illionns J. Math., 1970, **14**, No. 3, 464–467.

[16] Nakayama T. *On Frobeniusean algebras I, II*, Ann. of Math., 1939, **40**, 611–633; 1941, **42**, 1–21.

[17] Nakayama T. *Note on uniserial and generalized uniserial rings*, Proc. Imp. Acad. Tokyo, 1940, **16**, 285–289.

[18] Skornyakov L. A. *When are all modules semi-chained?* Mat. Zametki, 1969, **5**, 173–182.

[19] Zavadskiy A. G., Kirichenko V. V. *Semimaximal rings af finite type.* Mat. Sbornik, 1977, **103(145)**, No. 3(7), 323–345.

Vladimir Kirichenko, Makar Plakhotnyk
Mech-Math. Dept., Kyiv Taras Shevchenko Univ.
Volodymyrska Str. 64, Kyiv, Ukraine, 01601

E-mail: *VV.Kirichenko@gmail.com;*
*Makar_Plakhotnyk@ukr.net*

*Received December 6, 2010*

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(65), 2011, Pages 28–32
ISSN 1024–7696

# Reconstruction of centrally symmetric convex bodies in $\boldsymbol{R}^n$

R. H. Aramyan

**Abstract.** The article considers the problem of existence and uniqueness of a centrally symmetric convex body in $\mathbb{R}^n$ for which the projection curvature radius function coincides with given function. A necessary and sufficient condition is found that ensures a positive answer. Also we find a representation for the support function of a centrally symmetric convex body.

**Mathematics subject classification:** 52A20, 53C45, 53C65.
**Keywords and phrases:** convex geometry, integral geometry, projection curvature.

## 1 Introduction

Let $F$ be a function defined on 2-dimensional unit sphere $\mathrm{S}^2$. The existence and uniqueness of convex body $\mathbf{B} \subset \mathbb{R}^3$ for which the mean curvature radius at the point on $\partial\mathbf{B}$ with outer normal direction $\omega$ coincides with $F(\omega)$ was posed by Christoffel (see [4]). The corresponding problem for Gauss curvature was posed and solved by Minkowski. A. D. Aleksandrov and A. V. Pogorelov generalized these problems for a class of symmetric functions $G(R_1(\omega), R_2(\omega))$ of principal radii of curvatures [4].

Let $\mathbf{B} \subset \mathbb{R}^n$ be a convex body with sufficiently smooth boundary and let $R_1(\omega), \ldots, R_{n-1}(\omega)$ signify the principal radii of curvature of the boundary of $\mathbf{B}$ at the point with outer normal direction $\omega \in \mathrm{S}^{n-1}$. In $n$-dimensional case a Christoffel–Minkowski problem was posed and solved by Firay (see [6]) and Berg (see [8]): what are necessary and sufficient conditions for a function $F$, defined on $\mathrm{S}^{n-1}$ to be the function $\sum R_{i_1}(\omega) \cdots R_{i_p}(\omega)$ for a convex body, where $1 \le p \le n-1$ and the sum is extended over all increasing sequences $i_1, \cdots, i_p$ of indices chosen from the set $i = 1, \ldots, n-1$.

In this paper we consider a similar problem posed for the 2-dimensional projection curvature radii of centrally symmetric convex bodies in $\mathbb{R}^n$. We use the following notation. By $\mathcal{B}_o$ we denote the class of convex bodies $\mathbf{B} \subset \mathbb{R}^n$ that have a center of symmetry at the origin $O \in \mathbb{R}^n$. For two different directions $\omega, \xi \in \mathrm{S}^{n-1}$, $\omega \neq \xi$ we denote by $B(\omega, \xi)$ the projection of $\mathbf{B} \in \mathcal{B}_o$ onto the 2-dimensional plane $e(\omega, \xi)$ containing the origin and the directions $\omega$ and $\xi$.

We define $R(\omega, \xi) =$ curvature radius of $\partial B(\omega, \xi)$ at the point whose outer normal direction is $\omega$, and call it 2-dimensional projection curvature radius of the body. Since $R(\omega, \xi_1) = R(\omega, \xi_2)$, where $\omega, \xi_1, \xi_2 \in \mathrm{S}^{n-1}$, are linearly dependent vectors, we assume where necessary that $\xi$ is orthogonal to $\omega$.

Let $F$ be a function defined on the space of "flags" an ordered pairs of orthogonal unit vectors $\{(\omega, \psi) : \omega \in \mathrm{S}^{n-1}, \psi \in \mathrm{S}_\omega\}$. By $\mathrm{S}_\omega$ we denote the great $(n-2)$-subsphere of $\mathrm{S}^{n-1}$ with pole $\omega \in \mathrm{S}^{n-1}$. In integral geometry the concept of a flag was first systematically employed by R. V. Ambartzumian in [1]. In this paper we study:

**Problem 1**: what are necessary and sufficient conditions for a function $F$ to be the 2-dimensional projection curvature radius function of a centrally symmetric convex body and

**Problem 2**: reconstruction of that centrally symmetric convex body.

In this paper we find a necessary and sufficient condition on $F(\omega, \psi)$ that ensures a positive answer. Note that the uniqueness (up to parallel shifts) follows from the classical uniqueness result on Christoffel problem.

Also we find a simple representation for the support function of a 2-smooth centrally symmetrical convex body in $\mathbb{R}^n$ in terms of 2-dimensional projection curvature radius function.

Now we describe the main result. Let $F$ be a nonnegative function defined on the ordered pairs of orthogonal unit vectors $\mathfrak{F} = \{(\omega, \psi) : \omega \in \mathrm{S}^{n-1}, \psi \in \mathrm{S}_\omega\}$.

**Theorem 1.** *A nonnegative $n$ times continuously differentiable function $F$ defined on $\mathfrak{F}$ is the 2-dimensional projection curvature radius function of a centrally symmetric convex body if and only if there is an even continuous function $f$ defined on $\mathrm{S}^{n-1}$ such that*

$$F(\omega, \psi) = 2 \int_{\mathrm{S}_\omega} |< \psi, u >|^2 f(u)\, \lambda_{n-2}(du), \tag{1}$$

*for all $\omega \in \mathrm{S}^{n-1}$ and all $\psi \in \mathrm{S}_\omega$, here $\lambda_{n-2}$ is the Lebesgue measure on $\mathrm{S}^{n-2}$, $< \cdot, \cdot >$ denotes the Euclidean scalar product.*

Note that in [3] the same problem was considered in $\mathbb{R}^3$ and a different necessary and sufficient condition was found.

Radon transform provide a technique for studying the Christoffel problem for centrally symmetric convex bodies. The solution of that problem is of different nature for even and odd values of $n$ (see [8]).

To reconstruct the convex body we find (by means of another method) a simple representation for the support function of a centrally symmetric convex body in terms of 2-dimensional projection curvature radius function.

**Theorem 2.** *The support function of 2-smooth centrally symmetric convex body $\mathbf{B} \subset \mathbb{R}^n$ has the following representation*

$$H(\xi) = \frac{1}{2\sigma_{n-2}} \int_{\mathrm{S}^{n-1}} \frac{R(\omega, \xi)}{\sin^{n-3}(\widehat{\omega, \xi})}\, \lambda_{n-1}(d\omega), \quad \xi \in \mathrm{S}^{n-1}. \tag{2}$$

*Here $\widehat{\omega, \xi}$ is the angle between $\omega$ and $\xi$, $\sigma_{n-2} = \lambda_{n-2}(\mathrm{S}^{n-2})$.*

We need the following results from the convexity theory.

## 2 Preliminaries

It is well known (see [7]) that *the support function of every sufficiently smooth convex body* $\mathbf{B} \in \mathcal{B}_o$ *has the unique representation*

$$H(\xi) = \int_{\mathrm{S}^{n-1}} |<\xi, \Omega>| \, h(\Omega) \, \lambda_{n-1}(d\Omega), \quad \xi \in \mathrm{S}^{n-1} \tag{3}$$

*with some unique even continuous function* $h(\Omega)$ *defined on* $\mathrm{S}^{n-1}$*, not necessarily nonnegative, called the generating density of the body.*

R. Schneider (see [8]) has showed that the smoothness of order $n$ yields the representation with a continuous generating density.

Below we use the following result of N. F. Lindquist (see [8]).

*An even continuous function* $h$ *defined on* $\mathrm{S}^{n-1}$ *is the generating density of a convex body* $\mathbf{B} \in \mathcal{B}_o$ *if and only if*

$$\int_{\mathrm{S}_\omega} |<\psi, u>|^2 \; h(u) \, \lambda_{n-2}(du) \geq 0, \tag{4}$$

*for all* $\omega \in \mathrm{S}^{n-1}$ *and all* $\psi \in \mathrm{S}_\omega$.

The author of the present paper gave a clear geometrical interpretation for integral (4). In [2] has proved the following theorem (here we present a short version of the proof for completeness).

**Theorem 3.** *For any sufficiently smooth convex body* $\mathbf{B} \in \mathcal{B}_o$

$$R(\omega, \xi) = 2 \int_{\mathrm{S}_\omega} |<\xi, u>|^2 \; h(u) \, \lambda_{n-2}(du), \tag{5}$$

*where* $\xi, \omega \in \mathrm{S}^{n-1}$, $\xi \perp \omega$, $h(u)$ *is the generating density of* $\mathbf{B}$.

*Proof.* We need some special representation for the element of Lebesgue measure on $\mathrm{S}^{n-1}$. Given an orthonormal system of unit vectors $z_1, z_2, x_1, x_2, ..., x_{n-2}$ in $\mathbb{R}^n$, we represent $\omega \in \mathrm{S}^{n-1}$ as $\omega = (\nu, \varphi, u)$, where $\nu$ is the angle between $\omega$ and $e(z_1, z_2)$, $\varphi$ is the angle between $z_1$ and the projection of $\omega$ onto $e(z_1, z_2)$, while $u$ is the direction of the projection of $\omega$ onto the $(n-2)$-dimensional subspace containing $x_1, x_2, ..., x_{n-2}$. The corresponding Jacobian for $n \geq 4$ is (see [6])

$$\lambda_{n-1}(d\omega) = \sin^{n-3} \nu \, \cos \nu \, d\nu \, d\varphi \, \lambda_{n-3}(du). \tag{6}$$

The support function of $B(\omega, \xi)$ is the restriction of $H(\xi)$ (the support function of the body) onto the circle $\mathrm{S}^{n-1} \cap e(\omega, \xi)$. We consider some orthonormal system of unit vectors $z_1, z_2, x_1, ..., x_{n-2}$, where $z_1 = \omega$, $z_2 = \xi$. Let $\phi$ be the angle between direction $\omega(\phi)$ in $e(\omega, \xi)$ and $\omega$. We have $\omega(\phi) = (\cos \phi, \sin \phi, 0, ..., 0)$. According to the formula for curvature radius in 2-dimensional case (see [5]) we have

$$R(\omega, \xi) = H(0) + H''(\phi) \mid_{\phi=0}, \tag{7}$$

where $H(\phi) = H(\omega(\phi))$. Using (3) we get

$$H(\phi) = \int_{S^{n-1}} |<\omega(\phi), \Omega>| \; h(\Omega)\, d\Omega \; = 2\int_{\{\Omega\,(\omega,\Omega)\geq 0\}} (\Omega_1 \cos\phi + \Omega_2 \sin\phi)\, h(\Omega)\, d\Omega, \tag{8}$$

where $\Omega = (\Omega_1, \Omega_2, ..., \Omega_n)$. Now we represent $\Omega$ as $\Omega = (\nu, \varphi, \delta)$, where $\delta \in S^{n-3}$, $\nu$ is the angle between $\Omega$ and $e(\omega, \xi)$, and $\varphi$ is a direction in $e(\varphi, \xi)$. Using (6) for the second derivative we get

$$H''(\phi) = 2\int_{\{\Omega\,(\omega,\Omega)\geq 0\}} (-\Omega_1 \cos\phi - \Omega_2 \sin\phi)\, h(\Omega)\, d\Omega + \tag{9}$$

$$+2\int_{S_{\omega(\phi)}} (-\Omega_1 \sin\phi + \Omega_2 \cos\phi)\, h(\nu, \phi + \frac{\pi}{2}, \delta)\; \sin^{n-3}\nu\; \cos\nu\, d\nu \lambda_{n-3}(d\delta).$$

Substituting (9) into (7) and taking into account that $\sin^{n-3}\nu\, d\nu\, \lambda_{n-3}(d\delta) = \lambda_{n-2}(du)$ where $u = (\nu, \delta)$, $u \in S_\omega$ and $\Omega_2 = \cos\nu = \cos(u, \xi)$ we get (5). □

## 3 Proofs of Theorems 1 and 2

**Proof of Theorem 1.** *Necessity:* let $R(\omega, \psi)$ be the projection curvature radius of a convex body $\mathbf{B} \in \mathcal{B}_o$. We have to prove that there is an even function $f$ defined on $S^{n-1}$ such that condition (1) satisfies for $R(\omega, \psi)$. It follows from (3) that for a sufficiently smooth convex body the generating density exists. As a function $f$, we take the generating density of centrally symmetric convex body $\mathbf{B}$. It follows from Theorem 3 that equation (1) is satisfied.

*Sufficiency:* let $F$ be a nonnegative function defined on $\mathfrak{F}$ for which there is an even continuous function $f$ defined on $S^{n-1}$ such that

$$F(\omega, \psi) = 2\int_{S_\omega} |<\psi, u>|^2 f(u)\, \lambda_{n-2}(du), \tag{10}$$

for all $\omega \in S^{n-1}$ and all $\psi \in S_\omega$. Since $F$ is nonnegative the right hand side of (10) is nonnegative. Hence according to Theorem 3 there exists a centrally symmetric convex body $\mathbf{B}$ for which even function $f$ is the generating density of $\mathbf{B}$. It follows from Theorem 3 that the right hand side of (10) is the 2-dimensional projection curvature radius function of $\mathbf{B}$. Hence $F$ is the 2-dimensional projection curvature radius of $\mathbf{B}$. □

**Proof of Theorem 2.** Let $u \in S_\xi$ be a direction perpendicular to $\xi \in S^{n-1}$. We approximate $B(u, \xi) \subset e(\omega, \xi)$ from inside by polygons that have their vertices on $\partial B(u, \xi)$. We denote by $a_i$ sides of the approximation polygon, by $\nu_i$ ($\nu_i$ is the angle between the normal direction and $\xi$) the direction normal to $a_i$ within $e(u, \xi)$. Let $H_{B(u,\xi)}$ be the support function of $B(u, \xi)$. We have

$$4H(\xi) = 4H_{B(u,\xi)}(\xi) = \lim \sum_i |a_i|\; \sin(\widehat{\xi, \nu_i}) = \tag{11}$$

$$= \lim \sum_i R_u(\nu_i)|\nu_{i+1} - \nu_i| \sin(\widehat{\xi, \nu_i}) = 2 \int_0^\pi R_u(\nu) \sin\nu \, d\nu,$$

$R_u(\nu)$ is the radius of curvature of $\partial B(u, \xi)$ at the point with normal $\nu$. Integrating both sides of (11) in $\lambda_{n-2}(du)$ over $\mathrm{S}_\xi$, and using standard formula $\lambda_{n-1}(d\omega) = \sin^{n-2}\nu \, d\nu \, \lambda_{n-2}(du)$, where $\omega = (\nu, u)$ we obtain

$$2\sigma_{n-2} H(\xi) = \int_{\mathrm{S}^\xi} \int_0^\pi R_u(\nu) \sin\nu \, d\nu \, \lambda_{n-2}(du) =$$

$$= \int_{\mathrm{S}_\xi} \int_0^\pi \frac{R_u(\nu)}{\sin^{n-3}\nu} \sin^{n-2}\nu \, d\nu \, \lambda_{n-2}(du) = \int_{\mathrm{S}^{n-1}} \frac{R(\omega, \xi)}{\sin^{n-3}(\widehat{\omega, \xi})} \lambda_{n-1}(d\omega). \qquad \square$$

Note that replacing $2H(\cdot)$ by the width function $W(\cdot)$ in (2) we get a formula for the width function for all convex bodies (not only centrally symmetric).

I would like to express my gratitude to Professor R. V. Ambartzumian for helpful discussions.

## References

[1] Ambartzumian R. V. *Combinatorial integral geometry, metric and zonoids.* Acta Appl. Math., 1987, **9**, 3–27.

[2] Aramyan R. H. *Curvature radii of planar projections of convex bodies in $R^n$.* Izv. Akad. Nauk Armenii. Matematika, [English translation: Journal of Contemporary Math. Anal. (Armenian Academy of Sciences)], 2002, **37 (1)**, 2–14.

[3] Aramyan R. H. *Reconstruction of centrally symmetrical convex bodies by projection curvature radii.* Izv. Akad. Nauk Armenii. Matematika, [English translation: Journal of Contemporary Math. Anal. (Armenian Academy of Sciences)], 2004, **39 (5)**, 3–8.

[4] Bakelman I. Ya, Verner A. L., Kantor B. E. *Differential Geometry in the Large.* Moskow, Nauka, 1973.

[5] Blaschke W. *Kreis und Kugel.* (Veit, Leipzig), 2nd Ed. De Gruyter, Berlin, 1956.

[6] Firey W. J. *Intermediate christoffel-minkowski problems for figures of revolution.* Israel Journal of Mathematics, 1970, **8 (4)**, 384–390.

[7] Wiel W., Schneider R. *Zonoids and related Topics.* Convexity and its Applications, Ed. P. Gruber and J. Wills, Birkhauser, Basel, 1983.

[8] Goody P., Wiel W. *Zonoids and Generalisations.* Handbook of Convex Geometry, Ed. P. Gruber and J. Wills, Birkhauser, Basel, 1993.

R. H. Aramyan
Institute of Mathematics
Armenian Academy of Sciences
E-mail: *rafikaramyan@yahoo.com*

*Received October 27, 2009*

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(65), 2011, Pages 33–41
ISSN 1024–7696

# On the structure of maximal non-finitely generated ideals of ring and Cohen's theorem

S. I. Bilavska, B. V. Zabavsky

**Abstract.** In this paper we consider analogues of Cohen's theorem. We introduce new notions of almost prime left (right) submodule and $dr$-prime left (right) ideal, this allows us to extend Cohen's theorem for modular and non-commutative analogues. We prove that if every almost prime submodule of a finitely generated module is a finitely generated submodule, then any submodule of this module is finitely generated.

**Mathematics subject classification:** 16D25, 16D80.
**Keywords and phrases:** Almost prime ideal, completely prime ideal, $dr$-prime ideal, duo-element, finite element, finitely generated module, maximal non-finitely generated ideal.

## 1 Introduction

The aim of this paper is to generalize Cohen's theorem for wider class of rings. In 1950 studying the structure of a commutative ring I. Cohen showed that if an arbitrary prime ideal in a commutative ring with $1 \neq 0$ is finitely generated (principal), then any ideal in $R$ is finitely generated (principal) [1]. This theorem has a rich history. In particular, R. Chandran proved it for duo-ring [2]. G. Mihler showed that if an arbitrary left (right) prime ideal is finitely generated in an associative ring, then an arbitrary left (right) ideal in a ring is finitely generated [3]. Another non-commutative analogue of Cohen's theorem was proved by B. Zabavsky [6] using a weakly prime left (right) ideal. Also in $[4-5]$ some attempts were made to extend this theorem for module, but with some restrictions on this module.

In this paper we prove analogues of Cohen's theorem for module over arbitrary associative ring with $1 \neq 0$, for this we introduce a new notion of almost prime left (right) submodule. So if every almost prime submodule of a finitely generated module is a finitely generated, then any submodule of the module is finitely generated. Notice that in a commutative ring and duo-ring the notions of almost prime submodule and prime submodule coincide.

In the next section we consider the commutative ring with $1 \neq 0$ which is not a noetherian ring, the notion of a maximal non-finitely generated ideal and a finite element are investigated here. Also some important corollaries are considered in this section.

The last section deals with non-commutative analogue of Cohen's theorem. A new notion of $dr$-prime left (right) ideal is introduced. Thus if any $dr$-prime left (right) ideal of a ring $R$ is principal, then any left (right) ideal in $R$ is principal.

## 2 Preliminaries

Let $M$ be a finitely generated left module considered over an associative ring $R$ with $1 \neq 0$. Suppose that there is at least one submodule which is not finitely generated, we call it a non-finitely generated submodule. We denote by $S$ the set of all non-finitely generated submodules and by $\{N_i\}_{i\in\Lambda}$ any chain of submodules of a module $M$ which belong to the set $S$, moreover $N = \bigcup\limits_{i\in\Lambda} N_i$.

We show that $N \in S$. Suppose that $N \notin S$, then there exist elements $n_1, n_2, \dots n_k \in N$ such that

$$N = Rn_1 + Rn_2 + \ldots + Rn_k.$$

Since $N = \bigcup\limits_{i\in\Lambda} N_i$ for every $n_j$, $j = 1, 2, \dots, k$, there exists $s$ such that $n_j \in N_{i_s}$. Since $\{N_i\}_{i\in\Lambda}$ is a chain of submodules, there exists $t$ such that $n_1, n_2, \dots, n_k \in N_t$. Then $Rn_1 + Rn_2 + \ldots + Rn_k \subset N_t$. Since

$$N_t \subset N = \bigcup_{i\in\Lambda} N_i = Rn_1 + Rn_2 + \ldots + Rn_k,$$

this is a contradiction to $N_t \in S$. This contradiction shows that $N \in S$, therefore the set $S$ is inductive with respect to the order of submodules inclusion as a set.

According to Zorn's lemma there exists at least one maximal element in the set $S$. Therefore we have a submodule which is contained in the maximal element of $S$, we call it the maximal non-finitely generated submodule of the module $M$.

**Definition 1.** An element $a$ of a ring $R$ is called a duo-element if $aR = Ra$.

**Definition 2.** A left ideal $P$ of a ring $R$ is called an almost prime left ideal if from the condition $ab \in P$, where $a$ is a duo-element of the ring $R$ it follows that either $a \in P$ or $b \in P$.

**Definition 3.** A submodule $N$ of a module $M$ is called an almost prime left submodule if

$$(N : M) = \{r | r \in R, rM \subset N\}$$

is an almost prime left ideal of a ring $R$.

**Proposition 1.** *Any maximal left ideal of a ring $R$ is an almost prime left ideal.*

*Proof.* Let $M$ be an arbitrary maximal ideal of a ring $R$ and let $M$ be not almost prime. Then there exist elements $a \in R\backslash M$, $b \in R\backslash M$, where $a$ is a duo-element such that $ab \in M$. If $M$ is maximal then we have $M + bR = R$ and hence there exist elements $m \in M$, $r \in R$ such that $m + br = 1$. Hence $am + abr = a \in M$. But this is a contradiction with the choice of the element $a$. □

Remark that any maximal submodule of a module is an almost prime submodule [5]. We consider just a finitely generated submodule, so it is obvious that maximal submodules exist in it. It is easy to see that in module under consideration there always exist almost prime submodules.

## 3 Analogue of Cohen's theorem for modules

Suppose that for a module $M$ there exists at least one non-finitely generated submodule. Then according to what was proved above there exists a maximal non-finitely generated submodule. If there exists a maximal non-finitely generated submodule, then there exists an almost prime submodule.

**Theorem 1.** *Every maximal non-finitely generated submodule of a finitely generated module over a ring is an almost prime submodule.*

*Proof.* Let $M$ be a finitely generated module, $N$ be a maximal non-finitely generated submodule, $N \subset M$. According to restrictions on a ring $R$, $N$ could not be an almost prime ideal, that is there exist elements $a, b \in R$, where $a$ is a duo-element such that $abN \subset M$, but $a \notin N$ and $b \notin N$. Then we assume that $N + aM = \sum_{i=1}^{\alpha} Rx_i$ is a finitely generated submodule of the module $M$. Notice that $N + bM \subseteq (N : a)$, where $(N : a) = \{m | am \in N\}$. Thus $(N : a)$ is a finitely generated submodule of a module $M$, and let $(N : a) = \sum_{j=1}^{\beta} Ry_j$. Since $N \subset N + aM = \sum_{i=1}^{\alpha} Rx_i$, for any $n \in N$ there exist elements $r_i \in R$, $i = 1, 2, \ldots \alpha$, such that $n = r_1x_1 + \ldots + r_\alpha x_\alpha$. As $N + aM = \sum_{i=1}^{\alpha} Rx_i$, there exist $n_i^0 \in N$ and $s_i \in M$, where $i = 1, 2, \ldots \alpha$, such that $x_i = n_i^0 + as_i$. We show that

$$n = r_1n_1^0 + \ldots + r_\alpha n_\alpha^0 + r_1as_1 + \ldots + r_\alpha as_\alpha.$$

As $a$ is a duo-element, for every $r_i \in R$, $i = 1, 2, \ldots \alpha$, there exists $r_i' \in R$, $i = 1, 2, \ldots \alpha$, such that $r_ia = ar_i'$. Hence

$$n = r_1n_1^0 + \ldots + r_\alpha n_\alpha^0 + a(r_1's_1 + \ldots + r_\alpha's_1 + \ldots + r_\alpha's_\alpha).$$

Thus

$$n - r_1n_1^0 - \ldots - r_\alpha n_\alpha^0 = a(r_1's_1 + \ldots + r_\alpha's_\alpha) \in N.$$

If $(N : a) = \sum_{j=1}^{\beta} Ry_j$, we obtain $r_1's_1 + \ldots + r_\alpha's_\alpha = t_1y_1 + \ldots + t_\beta y_\beta$, for some $t_1, \ldots t_\beta \in R$ such that $at_1 = t_1'a, \ldots at_\beta = t_\beta'a$, and then

$$n = r_1n_1^0 + \ldots + r_\alpha n_\alpha^0 + t_1ay_1 + \ldots + t_\beta ay_\beta.$$

Since $n$ is an arbitrary element, we proved that

$$N \subseteq Rn_1^0 + \ldots + Rn_\alpha^0 + Ray_1 + \ldots + Ray_\beta.$$

If $y_i \in (N : a)$, $i = 1, 2, \ldots, \beta$, then $ay_1 \in N, \ldots, ay_\beta \in N$, and if $n_1^0, \ldots, n_\alpha^0 \in N$, then $Rn_1^0 + \ldots + Rn_\alpha^0 + Ray_1 + \ldots + Ray_\beta \subset N$. Thus

$$N = Rn_1^0 + \ldots + Rn_\alpha^0 + Ray_1 + \ldots + Ray_\beta$$

is a finitely generated submodule $N$, but this is a contradiction with $N \in S$. Thus $N$ is an almost prime submodule. □

Remind that if $R$ is a commutative ring or duo-ring, then the notion of almost prime submodule coincides with the notion of a prime submodule [4-5].

Also from Theorem 1, as a consequence, we obtain the modular analogue of Cohen's theorem. This theorem is the main result of the section.

**Theorem 2** (**Modular analogue of Cohen's theorem**). *If every almost prime submodule of a finitely generated module is a finitely generated submodule, then any submodule of this module is finitely generated.*

*Proof.* Let $M$ be a finitely generated module, and all almost prime submodules of the module $M$ are finitely generated. If $M$ does not contain non-finitely generated submodules, then everything is clear. In another case, for the module $M$ there exists at least one non-finitely generated submodule. Then, according to what was proved above for $M$ there exists at least one maximal non-finitely generated submodule. According to Theorem 1, $N$ is an almost prime submodule. But all almost prime submodules of the module $M$ are finitely generated, that is $N$ is finitely generated as an almost prime submodule and $N$ is not finitely generated as a maximal non-finitely generated submodule at the same time. But this is not possible, therefore $M$ does not contain non-finitely generated submodules. □

## 4 Maximal non-finitely generated ideals of commutative ring

Let $R$ be a commutative ring with $1 \neq 0$. Assume that $R$ is not a noetherian ring, that is there exist non-finitely generated ideals in $R$. Consider a ring $R$ as a module over itself, that is ${}_R R$.

**Definition 4.** An ideal $I$ in $R$ which is maximal in a set of non-finitely generated ideals is called maximal non-finitely generated ideal in $R$.

We can say that there exists at least one maximal non-finitely generated ideal in $R$. Moreover, using the theorem for the module ${}_R R$ we obtain that all maximal non-finitely generated ideals are prime ideals. Thus, the following theorem takes place.

**Theorem 3** (see [3]). *Let $R$ be a commutative ring which is not noetherian, then any maximal non-finitely generated ideal of the ring $R$ is a prime ideal.*

Hence, as an obvious corollary we obtain the known Cohen's theorem.

**Theorem 4** (see [1]). *If all prime ideals of a commutative ring $R$ are finitely generated, then $R$ is a noetherian ring.*

Consider the case when $R$ is a commutative ring but is not a noetherian ring. According to above there exists at least one maximal non-finitely generated ideal in $R$. Denote by $N(R)$ the intersection of all maximal non-finitely generated ideals of the ring $R$.

**Definition 5.** We say that a nonzero element $a$ of a ring $R$ is a finite element if any chain of ideals which contain the element $a$ is finite. That is for any chain of ideals $I_1 \subset I_2 \subset ...$ such that $a \in I_1$, there exists a number $n$ for which $I_n = I_{n+1} = ...$

All invertible elements and all factorial elements are examples of the finite element $a$ [6]. Thus we obtain the following corollary.

**Corollary 1.** *Let $R$ be a commutative ring and $a$ be an arbitrary element of the ring $R$. Then the following statements are equivalent:*

*1) $a$ is a finite element ;*

*2) any ideal which contains the element $a$ is finitely generated;*

*Proof.* 1) $\Longrightarrow$ 2). Let $I$ be any ideal of a ring $R$ which contains the element $a$. If $aR = I$, everything is clear, but otherwise there exists an element $i_1 \in I$ such that $i_1 \notin aR$. Consider the ideal $aR + i_1R$, it is obvious that $aR \subset aR + i_1R$. If $aR+i_1R \neq I$, then there exists an element $i_2 \in I$ such that $i_2 \notin aR+i_1R$. Consider the ideal $aR+i_1R+i_2R$, it is obvious, that $aR \subset aR+i_1R \subset aR+i_1R+i_2R$. This inclusion can be continued, but taking into account the definition of the element $a$, this chain can not be infinite. This means that there exist elements $i_1, i_2, \dots, i_n \in I$ such that

$$aR + i_1R + \ldots + i_nR = I,$$

that is $I$ is a finitely generated ideal.

2) $\Longrightarrow$ 1). Conversely, show that if any ideal which contains the element $a$ is finitely generated, then $a$ is a finite element of the ring $R$.

Let $\{I_\alpha\}_{\alpha\in\Lambda}$ be any chain of ideals, and all ideals of such type of this chain contain the element $a$. Show that this chain is finite. Let $I = \bigcup\limits_{\alpha\in\Lambda} I_\alpha$. Obviously, $a \in I$. As we assumed, $I$ is a finitely generated ideal, that is there exist elements $i_1, \dots, i_k \in I$ such that $I = i_1R + \ldots + i_kR$. Since $I = \bigcup\limits_{\alpha\in\Lambda} I_\alpha$, there exist numbers $\alpha_1, \dots, \alpha_k$ such that $i_1 \in I_{\alpha_1}, \dots, i_k \in I_{\alpha_k}$. If $\{I_\alpha\}_{\alpha\in\Lambda}$ is a chain of ideals, there exists a number $t$ such that $i_1, \dots, i_k \in I_{\alpha_t}$, that is $i_1R + \ldots + i_kR \subset I_{\alpha_t}$. As $\bigcup\limits_{\alpha\in\Lambda} I_\alpha = i_1R + \ldots + i_kR$, then $i_1R + \ldots + i_kR = I_{\alpha_t}$, that is the chain $\{I_\alpha\}_{\alpha\in\Lambda}$ is finite. □

**Corollary 2.** *Suppose that an element $a$ does not belong to any maximal non-finitely generated ideal of a ring $R$. Then $a$ is a finite element in $R$.*

*Proof.* Use Corollary 1 and the fact that the element $a$ is not contained in any non-finitely generated ideal. Thus, as it is proved above, the element $a$ is contained in at least one maximal non-finitely generated ideal. □

**Corollary 3.** *Let $n$ be an arbitrary element from $N(R)$. Then for any finite element $a \in R$ and any element $x \in R$, the element $a + nx$ is finite.*

*Proof.* We will prove it by contradiction. Let the element $a+nx$ be not finite, then it belongs to some maximal non-finitely generated ideal $N$ of the ring $R$. Since $n$ is an element from $N(R)$, we see that $n \in N$ and $x$ is an arbitrary element of the ring $R$, then $nx \in N$. From the definition of ideal we obtain $(a+nx)-nx \in N$, whence $a \in N$. However, the element $a$ is finite and as proved above, the element $a$ belongs to a maximal non-finitely generated ideal, which is impossible, according to Corollary 1. We obtain a contradiction. □

**Corollary 4.** *Let $R$ be a commutative ring with only one maximal non-finitely generated ideal $N = N(R)$. Then the following statements hold:*

*a) all non-finite elements from $R$ form an ideal which coincides with $N$;*

*b) an arbitrary divisor of a finite element is a finite element of the ring $R$;*

*c) for an arbitrary non-finite element $n$ and any finite element $a$, we obtain that $a+n$ is a finite element.*

*Proof.* a) From Corollary 1 it is known that any finite element does not belong to $N$ and every element which does not belong to $N$ is finite. Then all non-finite elements form the ideal which coincides with $N$.

b) Let $a$ be a finite element of the ring $R$ such that $a = bc$, $b \notin U(R)$ and $c \notin U(R)$, where $U(R)$ is the group of units of the ring $R$. If $b$ is not finite, then $b \in N$. Hence we see that $bc = a \in N$, but this is impossible, because the element $a$ is finite. Corollary 1 completes the proof.

c) If $n \in N$ and $a$ is a finite element, then obviously $a+n$ is not contained in $N$ (because there are only finite elements in $N$). Thus, $a+n$ is a finite element. □

## 5 Analogue of Cohen's theorem for principal ideals of noncommutative ring

In this section, we denote by $R$ an associative ring with $1 \neq 0$.

**Definition 6.** A left (right) ideal in $R$ which is maximal in the set of non-finitely generated left(fight) ideals is called maximal non-finitely generated left (right) ideal in the ring $R$.

**Definition 7.** A left (right) ideal in $R$ which is maximal in the set of non-principal left (right) ideals is called maximal non-principal left (right) ideal in the ring $R$.

**Corollary 5.** *Any left (right) non-finitely generated ideal of ring $R$ is contained in at least one maximal non-finitely generated left (right) ideal.*

*Proof.* Let $I$ be an arbitrary non-finitely generated left ideal of a ring $R$. Denote by $S$ the set of all non-finitely generated left ideals of the ring $R$ which contain the ideal $I$. We show that the set $S$ is inductive with respect to the order of ideals inclusion. If $\{I_\alpha\}_{\alpha\in\Lambda}$ is any chain of left ideals from the set $S$, denote $J = \bigcup\limits_{\alpha\in\Lambda} I_\alpha$. It is obvious that $J$ is an ideal of the ring $R$. Moreover, $J \in S$. Indeed, according to the definition of a left ideal, $I \in S$. If $J \notin S$, then there exist elements $j_1, \dots, j_k \in J$

such that $J = Rj_1 + \ldots + Rj_k$, and there exist elements $\alpha_1, \ldots, \alpha_k \in \Lambda$ such that $j_1 \in I_{\alpha_1}, \ldots, j_k \in I_{\alpha_k}$. Since $\{I_\alpha\}_{\alpha\in\Lambda}$ is a chain, there exists $t$ such that $j_1, \ldots, j_k \in I_{\alpha_t}$. As $I_{\alpha_t} \subset \bigcup\limits_{\alpha\in\Lambda} I_\alpha$, then $I_{\alpha_t} = Rj_1 + \ldots + Rj_k$, but this is impossible, because $I_{\alpha_t} \in S$. Zorn's lemma completes the proof of the corollary. □

In the same way we can consider the case of a right non-finitely generated ideal.

In the case of a principal left (right) ideals the following corollary takes place.

**Corollary 6.** *Every left (right) non-principal ideal of a ring $R$ is contained in at least one maximal non-principal left (right) ideal.*

*Proof.* Using the previous proof of Corollary 5 for any left (right) non-finitely generated ideal, we can prove in the same way for any left (right) non-principal ideal. □

**Definition 8.** Remind that an ideal $P$ of a ring $R$ is called prime left (right) ideal if the condition $aRb \subseteq P$ implies that either $a \in P$ or $b \in P$.

According to a result of [3] we have the following theorem.

**Theorem 5.** *Any maximal non-finitely generated left (right) ideal of a ring is a prime left (right) ideal.*

In [6] a noncommutative analogue of Cohen's theorem was proved, using weakly prime ideals.

**Definition 9.** We say that left (right) ideal $P$ of a rind $R$ is a weakly prime left (right) ideal, if from the condition $(a+P)R(b+P) \subseteq P$ if follows that either $a \in P$ or $b \in P$.

Using a result of the paper [7], the folowing theorem holds.

**Theorem 6.** *Any maximal non-principal left (right) ideal of a ring $R$ is a weakly prime left (right) ideal.*

**Definition 10.** Remind that an element of a ring $R$ is called an atom if it is non-inverse and non-zero and cannot be presented as the product of two noninvertible elements [8].

**Theorem 7.** *Let $N$ be an arbitrary maximal non-principal left ideal for which there exists a duo-element $c$ such that $N \subset Rc$. Then for any $n \in N$, from $n = cx$ it always follows $x \in N$.*

*Proof.* Consider the set $J = \{x | cx \in N\}$. Since $c$ is a duo-element, $J$ is a left ideal. Obviously $N \subset J$. If there exists an element $y$ such that $cy \in N$, but $y \notin N$, this means that $N \subset J$, but $N \neq J$. Using the definition of left ideal $N$ we see that $J = Rd$. We show that $N = Rcd$. Indeed, since $N \subset Rc = cR$, for any $n \in R$ there exists $t \in R$ such that $n = ct$. Since $t \in J$, we have $t = sd$. Hence $n = csd$. As $c$ is a duo-element, then there exists $s' \in R$ such that $cs = s'c$. Thus $n = s'cd$, that is $N \subset Rcd$. Since $1 \in R$ and $d \in J$, we obtain $cd \in N$. Hence $dcR \subset N$. Thus $N = dcR$. We obtain a contradiction to the choice of the left ideal. □

**Definition 11.** A left (right) ideal $P$ of an associative ring $R$ with $1 \neq 0$ is a *dr*-prime left (right) ideal if $P \subset Rc(P \subset cR)$, where $c$ is a duo-element, and for any $p \in P$, the condition $p = yc = cx(p = cx = yc)$ implies $x \in P$ $(y \in P)$.

**Proposition 2.** *Any maximal left (right) ideal $M$ of a ring $R$ is a dr-prime left (right) ideal.*

*Proof.* Let $M$ be a maximal left ideal. Since there is only one two-sided ideal in $R$ which contains $M$, for arbitrary $m \in M$ it always follows $m = 1m$.

A similar proof could be made for a maximal right ideal. □

**Theorem 8** (**Non-commutative analogue of Cohen's theorem**). *If any dr-prime left (right) ideal of a ring $R$ is principal, then any left (right) ideal from $R$ is principal.*

*Proof.* Let $R$ be a ring in which any *dr*-prime left ideal is principal, but $R$ is not a principal left ideal $I$. By Corollary 6, $I$ is contained in a maximal non-principal left ideal $N$. According to Theorem 9, $N$ is a *dr*-prime left ideal, since any *dr*-prime left ideal is principal. But this is a contradiction.

□

**Definition 12.** A two-sided ideal $P$ is called a completely prime ideal if the condition $ab \in P$, where $a, b \in R$, implies either $a \in P$ or $b \in P$ [8].

Notice that in the case of a commutative ring the notion of completely prime ideal coincides with the notion of prime ideal.

**Theorem 9.** *If a maximal non-finitely generated left (right) ideal of a ring $R$ is two-sided, then it is a completely prime ideal.*

*Proof.* Let $N$ be a maximal non-finitely generated left ideal of a ring $R$ which is two-sided. If $R/N$ is not a ring without zero divisors, then there exist elements $a \notin N$ and $b \notin N$ such that $ab \in N$ in $R$. Thus, the left ideal $J = \{x | x \in R, xb \in N\}$ contains the ideal $N$ and the element $a$. Hence, the inclusion $N \subset J$ is strict, and according to the restriction on $N$, the left ideal $J$ is finitely generated. Let $J = Rc_1 + \ldots + Rc_n$. Since $b \notin N$, according to the definition of the maximal non-finitely generated left ideal $N$, we obtain

$$N + Rb = Rd_1 + \ldots + Rd_k$$

for some elements $d_1, \ldots, d_k \in R$. Hence $d_i = n_i + r_i b$, where $n_i \in N$, $r_i \in R$, $i = 1, 2, \ldots, k$. As $N \subset Rd_1 + \ldots + Rd_k$, then any element $m \in N$ can be represented in the following form

$$m = s_1 d_1 + \ldots + s_k d_k,$$

where $s_1, \ldots, s_k \in R$.

Using what is written above, we obtain

$$m = s_1 d_1 + \ldots + s_k d_k = s_1(n_1 + r_1 b) + \ldots + s_k(n_k + r_k b) =$$

$$= s_1n_1 + \ldots + s_kn_k + s_1r_1b + \ldots + s_kr_kb.$$

Since $m \in N$ and $n_1, \ldots, n_k \in N$, we have

$$m - s_1n_1 - \ldots - s_kn_k = (s_1r_1 + \ldots + s_kr_k)b \in N,$$

then according to the definition of left ideal $J$ we have $s_1r_1 + \ldots + s_kr_k \in J$, that is there exist elements $t_1, \ldots, t_n \in R$ such that $s_1r_1 + \ldots + s_kr_k = t_1c_1 + \ldots + t_nc_n$, because $J = Rc_1 + \ldots + Rc_n$. Hence

$$m = s_1n_1 + \ldots + s_kn_k + t_1c_1b + \ldots + t_nc_nb.$$

Using the fact that element $m$ is arbitrary, we obtain

$$N \subset Rn_1 + \ldots + Rn_k + Rc_1b + \ldots + Rc_nb.$$

However, $n_1, \ldots, n_k \in N$ and $c_1, \ldots, c_k \in J$, so this means that $c_1b \in N_1, \ldots c_kb \in N$, that is $Rn_1 + \ldots + Rn_k + Rc_1b + \ldots + Rc_nb \subset N$. Thus

$$N = Rn_1 + \ldots + Rn_k + Rc_1b + \ldots + Rc_nb,$$

but this is a contradiction to the choice of $N$. □

## References


[1] Cohen I. *Commutative rings with restricted minimum conditions.* Duke Math. J., 1950, **17**, 24–42.

[2] Chandran R. *On two analogues of Cohen's theorem.* Indian. J. Pure and Appl. Math., 1977, **8**, 54–59.

[3] Mihler G. *Prime right ideals and right noetherian rings.* Proc. Conf. Ring Theory, Acad. Press, New York, London, 1972, 251–255.

[4] Behbodi M., Koohy H. *On minimal prime submodules.* Far East. J. Math. Sci., 2002, **6**, No. 1, 251–255.

[5] Gaur A., Maloo A. K., Parkash A. *Prime submodules in multiplication modules.* I. J. Alg., 2007, **8**, No. 8, 375–380.

[6] Zabavsky B. V. *Non-commutative analogue of Cohen's theorem.* Ukr. Mat. J., 1996, **48**, No. 5, 707–710 (in Ukrainian).

[7] Cohn P. M. *Free rings and their relations.* Acad Press, New York, London, 1971.

[8] Andrunakievich V. A. *About completely prime ideal of ring.* Mat. research, 1983, **3**, 121 (in Russian).

[9] Brungs H. H. *Rings with a distributive lattice of right ideals.* J. Algebra, 1976, **40**, 392–400.

[10] Andrunakievich V. A., Ryabukhin Yu. M. *Radicals of algebras and structure theory.* Moscow, Nauka, 1979 (in Russian).

[11] Van der Walt. *Weakly prime one-sided ideals.* J. Austral. Math. Soc., 1985, **38**, 84–91.



S. I. Bilavska, B. V. Zabavsky
Ivan Franko National University of Lviv
1 Universytetska str., Lviv, 79000
Ukraine
E-mail: *zosia_ meliss@yahoo.co.uk; b_zabava@ukr.net*


*Received March 2, 2010*

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(65), 2011, Pages 42–49
ISSN 1024–7696

# On 2-primal Ore extensions over Noetherian $\sigma(*)$-rings

Vijay Kumar Bhat

**Abstract.** In this article, we discuss the prime radical of skew polynomial rings over Noetherian rings. We recall $\sigma(*)$ property on a ring $R$ (i. e. $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$, where $P(R)$ is the prime radical of $R$, and $\sigma$ an automorphism of $R$). Let now $\delta$ be a $\sigma$-derivation of $R$ such that $\delta(\sigma(a)) = \sigma(\delta(a))$ for all $a \in R$. Then we show that for a Noetherian $\sigma(*)$-ring, which is also an algebra over $\mathbb{Q}$, the Ore extension $R[x; \sigma, \delta]$ is 2-primal Noetherian (i. e. the nil radical and the prime radical of $R[x; \sigma, \delta]$ coincide).

**Mathematics subject classification:** 16S36, 16N40, 16P40, 16S32, 16W20, 16W25.
**Keywords and phrases:** Minimal prime, 2-primal, prime radical, automorphism, derivation.

## 1 Introduction

A ring $R$ always means an associative ring with identity $1 \neq 0$. The fields of complex numbers, real numbers, rational numbers, the ring of integers and the set of natural numbers are denoted by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ respectively unless otherwise stated. The set of prime ideals of $R$ is denoted by $Spec(R)$. The set of minimal prime ideals of $R$ is denoted by $Min.Spec(R)$. The prime radical and the nil radical of $R$ are denoted by $P(R)$ and $N(R)$ respectively. Let $R$ be a ring and $\sigma$ an automorphism of $R$. Let $I$ be an ideal of $R$ such that $\sigma^m(I) = I$ for some $m \in \mathbb{N}$. We denote $\cap_{i=1}^{m}\sigma^i(I)$ by $I^0$. For any two ideals $I$, $J$ of $R$, $I \subset J$ means that $I$ is strictly contained in $J$.

This article concerns the study of skew polynomial rings (Ore extensions) in terms of 2-primal rings. Recall that the skew polynomial ring $R[x; \sigma, \delta]$ is the set of polynomials

$$\{\sum_{i=0}^{n} x^i a_i,\ a_i \in R,\ n \in \mathbb{N}\}$$

with usual addition of polynomials and multiplication subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We take any $f(x) \in R[x; \sigma, \delta]$ to be of the form $f(x) = \sum_{i=0}^{n} x^i a_i$, $a_i \in R$ as in McConnell and Robson [15]. We denote $R[x; \sigma, \delta]$ by $O(R)$. In case $\delta$ is the zero map, we denote $R[x; \sigma]$ by $S$ and in case $\sigma$ is the identity map, we denote $R[x; \delta]$ by $D$. The study of Ore-extension $O(R) = R[x; \sigma, \delta]$ and its special cases S and D have been of interest to many authors. For example [6–8, 10, 13, 14, 16].

2-primal rings have been studied in recent years and are being treated by authors for different structures. In [14], Greg Marks discusses the 2-primal property of $R[x;\sigma,\delta]$, where $R$ is a local ring, $\sigma$ an automorphism of R and $\delta$ a $\sigma$-derivation of $R$. In Greg Marks [14], it has been shown that for a local ring $R$ with a nilpotent maximal ideal, the Ore extension $R[x;\sigma,\delta]$ will or will not be 2-primal depending on the $\delta$-stability of the maximal ideal of $R$. In the case where $R[x;\sigma,\delta]$ is 2-primal, it will satisfy an even stronger condition; in the case where $R[x;\sigma,\delta]$ is not 2-primal, it will fail to satisfy an even weaker condition.

Minimal prime ideals of 2-primal rings have been discussed by Kim and Kwak in [11]. 2-primal near rings have been discussed by Argac and Groenewald in [1]. Recall that a ring $R$ is 2-primal if and only if $N(R) = P(R)$, i.e. if the prime radical is a completely semiprime ideal. An ideal $I$ of a ring $R$ is called completely semiprime if $a^2 \in I$ implies $a \in I$ for $a \in R$. We also note that a reduced ring is 2-primal and a commutative ring is also 2-primal. For further details on 2-primal rings, we refer the reader to [1–3, 11, 14].

Before proving the main result, we find a relation between the minimal prime ideals of $R$ and those of the Ore extension $R[x;\sigma,\delta]$, where $R$ is a Noetherian $\mathbb{Q}$-algebra, $\sigma$ an automorphism of R and $\delta$ a $\sigma$-derivation of $R$ such that $\delta(\sigma(a)) = \sigma(\delta(a))$ for all $a \in R$. This is proved in Theorem 3.

**$\sigma(*)$-rings**: Let $R$ be a ring and $\sigma$ an endomorphism of $R$. Then $\sigma$ is said to be a rigid endomorphism if $a\sigma(a) = 0$ implies that $a = 0$, for $a \in R$, and $R$ is said to be a $\sigma$-rigid ring (Krempa [12]).

For example let $R = \mathbb{C}$, and $\sigma : \mathbb{C} \to \mathbb{C}$ be the map defined by $\sigma(a+ib) = a - ib$, $a$, $b \in \mathbb{R}$. Then it can be seen that $\sigma$ is a rigid endomorphism of $R$.

In Theorem 3.3 of [12], Krempa has proved the following:
Let $R$ be a ring, let $\sigma$ be an endomorphism and $\delta$ a $\sigma$-derivation of $R$. If $\sigma$ is a monomorphism, then the skew polynomial ring $R[x;\sigma,\delta]$ is reduced if and only if $R$ is reduced and $\sigma$ is rigid. Under this conditions any minimal prime ideal (annihilator) of $R[x;\sigma;\delta]$ is of the form $P[x;\sigma;\delta]$ where $P$ is a minimal prime ideal (annihilator) in $R$.

In [13], Kwak defines a $\sigma(*)$-ring $R$ to be a ring in which $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$.

**Example 1.** Let R $= \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where $F$ is a field. Then $P(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ Let $\sigma : R \to R$ be defined by $\sigma\Big(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then it can be seen that $\sigma$ is an endomorphism of $R$ and $R$ is a $\sigma(*)$-ring.

We note that the above ring is not $\sigma$-rigid. For let $0 \neq a \in F$. Then
$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \sigma \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, but $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Kwak in [13] also establishes a relation between a 2-primal ring and a $\sigma(*)$-ring. The property is also extended to the skew-polynomial ring $R[x;\sigma]$. It has been proved in Theorem 5 of [13] that if $R$ is a 2-primal ring and $\sigma$ is an automorphism of $R$, then $R$ is a $\sigma(*)$-ring if and only if $\sigma(P) = P$ for all $P \in Min.Spec(R)$. In Theorem 12 of [13] it has been proved that if $R$ is a $\sigma(*)$-ring with $\sigma(P(R)) = P(R)$, then $R[x;\sigma]$ is 2-primal if and only if $P(R)[x;\sigma] = P(R[x;\sigma])$.

It is known that if $R$ is a 2-primal Noetherian $\mathbb{Q}$-algebra, and $\delta$ is a derivation of $R$, then $R[x;\delta]$ is 2-primal Noetherian. (Theorem 2.4 of Bhat [3]).

Let now $R$ be a ring, $\sigma$ an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. Recall from Bhat [2] that R is said to be a $\delta$-ring if $a\delta(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$. It is known that if $R$ is a $\delta$-Noetherian $\mathbb{Q}$-algebra, $\sigma$ an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$ such that $\delta(\sigma(a)) = \sigma(\delta(a))$ for all $a \in R$, $\sigma(P) = P$ for all $P \in Min.Spec(R)$ and $\delta(P(R)) \subseteq P(R)$, then $R[x;\sigma,\delta]$ is 2-primal Noetherian (Theorem 2.4 of Bhat [2]).

In a sense we generalize the above results of Bhat [2, 3] when $\sigma$ is an automorphism of $R$ and ultimately investigate the 2-primal property of $R[x;\sigma,\delta]$ when $R$ is a $\sigma(*)$-Noetherian $\mathbb{Q}$-algebra and prove the following, even without the hypothesis of $R$ being a $\delta$-ring:

Let $R$ be a Noetherian $\sigma(*)$-ring, which is also an algebra over $\mathbb{Q}$. Further $P \in Min.Spec(O(R))$ imply that $P \cap R \in Min.Spec(R)$. Then $R[x;\sigma,\delta]$ is 2-primal Noetherian, where $\delta(\sigma(a)) = \sigma(\delta(a))$ for all $a \in R$.

This result is proved in Theorem 5. We note that for a Noetherian $\sigma(*)$-ring, $\sigma(P) = P$ for all $P \in Min.Spec(R)$ (Theorem 2), and this is crucial in proving Theorem 4 and, therefore, the main result (Theorem 5).

We generalize Theorem 7 of [5] which states the following:

**Theorem 7 of [5]**. Let $R$ be a Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let $\sigma$ be an automorphism of $R$ such that $R$ is a $\sigma(*)$-ring and $\delta$ be a $\sigma$-derivation of $R$ such that $R$ is a $\delta$-ring and $\delta(\sigma(a)) = \sigma(\delta(a))$ for all $a \in R$. Further let $P \in Min.Spec(O(R))$ imply that $P \cap R \in Min.Spec(R)$. Then $R[x;\sigma,\delta]$ is 2-primal Noetherian.

## 2 Ore extensions

Recall that an ideal $I$ of a ring $R$ is called $\sigma$-invariant if $\sigma(I) = I$. Also $I$ is called completely prime if $ab \in I$ implies $a \in I$ or $b \in I$ for $a, b \in R$. ([13])

In commutative case completely prime and prime have the same meaning. We also note that every completely prime ideal of a ring $R$ is a prime ideal, but the converse need not be true.

The following example shows that a prime ideal need not be a completely prime ideal.

**Example 2.** Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} = M_2(\mathbb{Z})$. If $p$ is a prime number, then the ideal $P = M_2(p\mathbb{Z})$ is a prime ideal of $R$, but is not completely prime, since for $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we have $ab \in P$, even though $a \notin P$ and $b \notin P$.

We also recall that an ideal $J$ of a ring is called a $\sigma$-prime ideal of $R$ if $J$ is $\sigma$-invariant and for any $\sigma$-invariant ideals $K$ and $L$ with $KL \subseteq J$, we have $K \subseteq J$ or $L \subseteq J$.

We also note that if $R$ is a Noetherian ring, then $Min.Spec(R)$ is finite (Theorem 2.4 of Goodearl and Warfield [10]) and for any automorphism $\sigma$ of $R$ and for any $U \in Min.Spec(R)$, we have $\sigma^i(U) \in Min.Spec(R)$ for all $i \in \mathbb{N}$, therefore, it follows that there exists some $m \in N$ such that $\sigma^m(U) = U$ for all $U \in Min.Spec(R)$. As mentioned earlier we denote $\cap_{i=0}^{m} \sigma^i(U)$ by $U^0$.

We now prove the following Theorem. This Theorem has not been used to prove the main Theorem, but gives an idea to find a relation between $Min.Spec(R)$ and $Min.Spec(O(R))$ (namely Theorem 3) which is crucial in proving the main result (Theorem 5):

**Theorem 1.** *Let $R$ be a Noetherian ring and $\sigma$ an automorphism of $R$. Let $S = R[x;\sigma]$ be as usual. Then:*

1. *If $P \in Min.Spec(S)$, then $P = (P \cap R)S$ and there exists $U \in Min.Spec(R)$ such that $P \cap R = U^0$.*
2. *If $U \in Min.Spec(R)$, then $U^0 S \in Min.Spec(S)$.*

*Proof.* See Theorem 2.4 of Bhat [6]. □

**Proposition 1.** *Let $R$ be a ring and $\sigma$ an automorphism of $R$. Then $R$ is a $\sigma(*)$-ring implies $R$ is 2-primal.*

*Proof.* Let $a \in R$ be such that $a^2 \in P(R)$. Then $a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(P(R)) = P(R)$. Therefore $a\sigma(a) \in P(R)$ and hence $a \in P(R)$. □

A necessary and sufficient condition for a Noetherian ring to be a $\sigma(*)$-ring is given by Bhat in Theorem 2.4 of [4]:

**Theorem 2.** *Let $R$ be a Noetherian ring. Then $R$ is a $\sigma(*)$-ring if and only if for each minimal prime $U$ of $R$, $\sigma(U) = U$ and $U$ is completely prime ideal of $R$.*

*Proof.* Theorem 2.4 of [4]. □

We now give a relation between the minimal prime ideals of $R$ and those of $R[x;\sigma,\delta]$, where $R$ is a Noetherian $\mathbb{Q}$-algebra, $\sigma$ an automorphism of R and $\delta$ a $\sigma$-derivation of $R$. This is proved in Theorem 3. Towards this we have the following:

**Proposition 2.** *Let $R$ be a Noetherian $\mathbb{Q}$-algebra, $\sigma$ an automorphism and $\delta$ a $\sigma$-derivation of $R$ such that $\delta(\sigma(a)) = \sigma(\delta(a))$ for all $a \in R$. Then $e^{t\delta}$ is an automorphism of $T = R[[t, \sigma]]$, the skew power series ring.*

*Proof.* The proof is on the same lines as in Seidenberg [16] and in the non-commutative case on the same lines as provided by Blair and Small in [8]. □

Henceforth we denote $R[[t, \sigma]]$ by $T$. Let $I$ be an ideal of $R$ such that $\sigma(I) = I$. Then it is easy to see that $TI \subseteq IT$ and $IT \subseteq TI$. Hence $TI = IT$ is an ideal of $T$.

**Lemma 1.** *Let $R$ be a Noetherian $\mathbb{Q}$-algebra, $\sigma$ an automorphism and $\delta$ a $\sigma$-derivation of $R$ such that $\delta(\sigma(a)) = \sigma(\delta(a))$ for all $a \in R$. Let $I$ be an ideal of $R$ such that $\sigma(I) = I$. Then $I$ is $\delta$-invariant if and only if $IT$ is $e^{t\delta}$-invariant.*

*Proof.* Let $IT$ be $e^{t\delta}$-invariant. Let $a \in I$. Then $a \in IT$. So $e^{t\delta}(a) \in IT$; i.e. $a + t\delta(a) + (t^2\delta^2/2!)(a) + ... \in IT$. Therefore $\delta(a) \in I$.

Conversely suppose that $\delta(I) \subseteq I$ and let $f = \sum t^i a_i \in IT$. Then $e^{t\delta}(f) = f + t\delta(f) + (t^2\delta^2/2!)(f) + ... \in IT$, as $\delta(a_i) \in I$. Therefore $e^{t\delta}(IT) \subseteq IT$. Replacing $e^{t\delta}$ by $e^{-t\delta}$, we get that $e^{t\delta}(IT) = IT$. □

**Assumption A**: Henceforth we assume that $R$ is a ring and $T$ as usual such that for any $U \in Min.Spec(R)$ with $\sigma(U) = U$, $UT \in Min.Spec(T)$.

**Proposition 3.** *Let $R$ be a Noetherian $\mathbb{Q}$-algebra. Let $\sigma$ be an automorphism of $R$ and $\delta$ be a $\sigma$-derivation of $R$ such that $\delta(\sigma(a)) = \sigma(\delta(a))$ for all $a \in R$. Then $P \in Min.Spec(R)$ with $\sigma(P) = P$ implies $\delta(P) \subseteq P$.*

*Proof.* Let $T$ be as usual. Now by Proposition (2) $e^{t\delta}$ is an automorphism of $T$. Let $P \in Min.Spec(R))$. Then by assumption $PT \in Min.Spec(T)$. Therefore there exists an integer $n \geq 1$ such that $(e^{t\delta})^n(PT) = PT$, i.e. $e^{nt\delta}(PT) = PT$. But $R$ is a $\mathbb{Q}$-algebra, therefore, $e^{t\delta}(PT) = PT$ and now Lemma 1 implies $\delta(P) \subseteq P$. □

**Proposition 4.** *Let $R$ be a $\sigma(*)$-ring, which is also an algebra over $\mathbb{Q}$ and $U \in Min.Spec(R)$. Then $U(O(R)) = U[x; \sigma, \delta]$ is a completely prime ideal of $O(R) = R[x; \sigma, \delta]$, where $\delta$ is a $\sigma$-derivation of $R$ such that $\delta(\sigma(a)) = \sigma(\delta(a))$ for all $a \in R$.*

*Proof.* Let $U \in Min.Spec(R)$. Then $\sigma(U) = U$ by Theorem 2, and $\delta(U) \subseteq U$ by Proposition 3). Now R is 2-primal by Proposition 1 and furthermore $U$ is completely prime by Theorem 2. Now we note that $\sigma$ can be extended to an automorphism $\overline{\sigma}$ of $R/U$ and $\delta$ can be extended to a $\overline{\sigma}$-derivation $\overline{\delta}$ of $R/U$. Now it is well known that $O(R)/U(O(R)) \simeq (R/U)[x; \overline{\sigma}, \overline{\delta}]$ and hence $U(O(R))$ is a completely prime ideal of $O(R)$. □

**Theorem 3.** *Let $R$ be a Noetherian $\mathbb{Q}$-algebra. Consider $O(R)$ as usual such that $R$ is a $\sigma(*)$-ring and $\delta(\sigma(a)) = \sigma(\delta(a))$ for all $a \in R$. Then $P_1 \in Min.Spec(R)$ with $\sigma(P_1) = P_1$ implies that $O(P_1) \in Min.Spec(O(R)$.*

*Proof.* Let $P_1 \in Min.Spec(R)$. Now by Theorem 2 $\sigma(P_1) = P_1$, and by Proposition 3 $\delta(P_1) \subseteq P_1$. Now Proposition (3.3) of [9] implies that $O(P_1) \in Spec(O(R))$. Suppose $O(P_1) \notin Min.Spec(O(R))$ and $P_2 \subset O(P_1)$ be a minimal prime ideal of $O(R)$. Then $P_2 = O(P_2 \cap R) \subset O(P_1) \subseteq Min.Spec(O(R))$. Therefore $(P_2 \cap R) \subset P_1$ which is a contradiction, as $(P_2 \cap R) \in Spec(R)$. Hence $O(P_1) \in Min.Spec(O(R))$. $\square$

We now prove the following Theorem, which is crucial in proving Theorem 5.

**Theorem 4.** *Let $R$ be a Noetherian $\sigma(*)$-ring, which is also an algebra over $\mathbb{Q}$, $\sigma$ an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$ such that $\delta(\sigma(a)) = \sigma(\delta(a))$ for all $a \in R$. Then $R[x;\sigma,\delta]$ is 2-primal if and only if $P(R)[x;\sigma,\delta] = P(R[x;\sigma,\delta])$.*

*Proof.* Let $R[x;\sigma,\delta]$ be 2-primal. Now by Proposition 4 $P(R[x;\sigma,\delta]) \subseteq P(R)[x;\sigma,\delta]$. Let

$$f(x) = \textstyle\sum_{j=0}^{n} x^j a_j \in P(R)[x;\sigma,\delta].$$

Now R is a 2-primal subring of $R[x;\sigma,\delta]$ by Proposition 1, which implies that $a_j$ is nilpotent and thus

$$a_j \in N(R[x;\sigma,\delta]) = P(R[x;\sigma,\delta]).$$

So we have $x^j a_j \in P(R[x;\sigma,\delta])$ for each j, $0 \leq j \leq n$, which implies that $f(x) \in P(R[x;\sigma,\delta])$. Hence $P(R)[x;\sigma,\delta] = P(R[x;\sigma,\delta])$.

Conversely suppose that $P(R)[x;\sigma,\delta] = P(R[x;\sigma,\delta])$. We will show that $R[x;\sigma,\delta]$ is 2-primal. Let

$$g(x) = \textstyle\sum_{i=0}^{n} x^i b_i \in R[x;\sigma,\delta],\ b_n \neq 0$$

be such that

$$(g(x))^2 \in P(R[x;\sigma,\delta]) = P(R)[x;\sigma,\delta].$$

We will show that $g(x) \in P(R[x;\sigma,\delta])$. Now leading coefficient $\sigma^{2n-1}(b_n)b_n \in P(R) \subseteq P$, for all $P \in Min.Spec(R)$. Also $\sigma(P) = P$ and $P$ is completely prime by Theorem 2. Therefore we have $b_n \in P$, for all $P \in Min.Spec(R)$, i.e. $b_n \in P(R)$. Now $\delta(P) \subseteq P$ for all $P \in Min.Spec(R)$ by Proposition 3, we get

$$(\textstyle\sum_{i=0}^{n-1} x^i b_i)^2 \in P(R[x;\sigma,\delta]) = P(R)[x;\sigma,\delta]$$

and as above we get $b_{n-1} \in P(R)$. With the same process in a finite number of steps we get $b_i \in P(R)$ for all $i$, $0 \leq i \leq n$. Thus we have $g(x) \in P(R)[x;\sigma,\delta]$, i.e. $g(x) \in P(R[x;\sigma,\delta])$. Therefore, $P(R[x;\sigma,\delta])$ is completely semiprime. Hence $R[x;\sigma,\delta]$ is 2-primal. $\square$

**Theorem 5.** *Let $R$ be a Noetherian, which is also an algebra over $\mathbb{Q}$. Let $\sigma$ be an automorphism of $R$ such that $R$ is a $\sigma(*)$-ring and $\delta$ a $\sigma$-derivation of $R$ such that $\delta(\sigma(a)) = \sigma(\delta(a))$ for all $a \in R$. Further let $P \in Min.Spec(O(R))$ imply that $P \cap R \in Min.Spec(R)$. Then $O(R) = R[x;\sigma,\delta]$ is 2-primal Noetherian.*

*Proof.* $R[x;\sigma,\delta]$ is Noetherian by Hilbert Basis Theorem (Theorem 1.12 of Goodearl and Warfield [10]). We now use Theorem 3 to get that $P(R)[x;\sigma,\delta] = P(R[x;\sigma,\delta])$, and the result now follows from Theorem 4. □

We note that the hypothesis that $R$ is a $\sigma(*)$-ring can not be deleted as can be seen below:

**Example 3.** Let $R = K \oplus K$, where $K$ is a field. Then the Ore extension $O(R) = R[x;\sigma,0]$, where $\sigma$ is an automorphism of R defined by $\sigma((a;b)) = (b;a)$, is a prime ring. Thus $P = 0$ is a minimal prime of $O(R)$. But $P \cap R = 0$ is not a prime ideal of $R$.

The following example shows that if $R$ is a Noetherian ring, then $R[x;\sigma,\delta]$ need not be 2-primal.

**Example 4.** Let $R = \mathbb{Q} \oplus \mathbb{Q}$ with $\sigma(a,b) = (b,a)$. Then the only $\sigma$-invariant ideals of $R$ are $\{0\}$ and $R$, and so $R$ is $\sigma$-prime. Let $\delta : R \to R$ be defined by $\delta(r) = ra - a\sigma(r)$, where $a = (0,\alpha) \in R$. Then $\delta$ is a $\sigma$-derivation of $R$ and $R[x;\sigma,\delta]$ is prime and $P(R[x;\sigma,\delta]) = 0$. But $(x(1,0))^2 = 0$ as $\delta(1,0) = -(0,\alpha)$. Therefore $R[x;\sigma,\delta]$ is not 2-primal. If $\delta$ is taken to be the zero map, then even $R[x;\sigma]$ is not 2-primal.

The following example shows that if $R$ is a Noetherian ring , then even $R[x]$ need not be 2-primal.

**Example 5.** Let $R = M_2(\mathbb{Q})$, the set of $2 \times 2$ matrices over $\mathbb{Q}$. Then $R[x]$ is a prime ring with non-zero nilpotent elements, and so can not be 2-primal.

From these examples we conclude that if $R$ is a Noetherian ring, then even $R[x]$ need not be 2-primal. But it is known that if $R$ is a 2-primal Noetherian $\mathbb{Q}$-algebra and $\delta$ is a derivation of $R$, then $R[x;\delta]$ is 2-primal Noetherian (Theorem 2.4 of Bhat [3]), and therefore, we have the following question:

**Question:** If $R$ is a 2-primal ring, is $R[x;\sigma,\delta]$ 2-primal (even if $R$ is commutative or the special case when $R$ is Noetherian)?

## References


[1] Argac N., Groenewald N. J. *A generalization of 2-primal near rings*. Quaest. Math., 2004, **27(4)**, 397–413.

[2] Bhat V. K. *On 2-primal Ore extensions*. Ukr. Math. Bull., 2007, **4(2)**, 173–179.

[3] Bhat V. K. *Differential operator rings over 2-primal rings*. Ukr. Math. Bull., 2008, **5(2)**, 153–158.

[4] Bhat V. K., Kumari Neetu, *Transparency of $\sigma(*)$-Rings and Their Extensions*. Int. J. Algebra, 2008, **2(19)**, 919–924.

[5] Bhat V. K. *Ore extensions over 2-primal Noetharian rings*. Bul. Acad. Stiinte Repub. Mold. Mat., 2008, **3(58)**, 34–43.

[6] Bhat V. K. *Associated prime ideals of skew polynomial rings*. Beitrage Algebra Geom., 2008, **49(1)**, 277–283.

[7] Bhat V. K. *On Near Pseudo-valuation rings and their extensions*. Int. Electron. J. Algebra, 2009, **5**, 70–77.

[8] Blair W. D. , Small L. W. *Embedding differential and skew-polynomial rings into artinain rings*. Proc. Amer. Math. Soc., 1990, **109(4)**, 881–886.

[9] Goodearl K. R. *Prime Ideals in Skew Polynomial Rings and Ouantized Weyl Algebras*. J. Algebra, 1992, **150**, 324–377.

[10] Goodearl K. R., Warfield Jr. R. B. *An introduction to non-commutative Noetherian rings*. Cambridge Uni. Press, 1989.

[11] Kim N. K., Kwak T. K. *Minimal prime ideals in 2-primal rings*. Math. Japon., 1999, **50(3)**, 415–420.

[12] Krempa J. *Some examples of reduced rings*. Algebra Colloq., 1996, **3(4)**, 289–300.

[13] Kwak T. K. *Prime radicals of skew-polynomial rings*. Int. J. Math. Sci., 2003, **2(2)**, 219–227.

[14] Marks G. *On 2-primal Ore extensions*. Comm. Algebra, 2001 **29(5)**, 2113–2123.

[15] McConnell J. C., Robson J. C. *Noncommutative Noetherian Rings*, Wiley 1987; Revised edition: Amer. Math. Soc. 2001.

[16] Seidenberg A. *Differential ideals in rings of finitely generated type*. Amer. J. Math., 1967, **89**, 22–42.

Vijay Kumar Bhat
School of Mathematics
SMVD University, Katra
India-182320

E-mail: *vijaykumarbhat2000@yahoo.com*

*Received June 2, 2010*

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(65), 2011, Pages 50–59
ISSN 1024–7696

# Free topological universal algebras and absolute neighborhood retracts

Taras Banakh, Olena Hryniv

**Abstract.** We prove that for a complete quasivariety $\mathcal{K}$ of topological $E$-algebras of countable discrete signature $E$ and each submetrizable $\mathsf{ANR}(k_\omega)$-space $X$ its free topological $E$-algebra $F_{\mathcal{K}}(X)$ in the class $\mathcal{K}$ is a submetrizable $\mathsf{ANR}(k_\omega)$-space.

**Mathematics subject classification:** 08B20; 22A30; 54C55.
**Keywords and phrases:** Topological universal algebra, free topological universal algebra, a quasivariety of topological algebras, absolute neighborhood retract, absolute neighborhood extensor, $k_\omega$-space.

## 1 Introduction

In this paper we study the construction of a free topological universal algebra and show that this construction preserves the class of submetrizable $\mathsf{ANR}(k_\omega)$-spaces.

To give a precise formulation of our main result, we need to recall some definitions related to topological universal algebras. For more detailed information, see [5–7].

**Definition 1.** Let $(E_n)_{n\in\omega}$ be a sequence of pairwise disjoint topological spaces. The topological sum $E = \bigoplus_{n\in\omega} E_n$ is called *a continuous signature*. The signature is called *discrete* (*countable*) if so is the space $E$.

A *topological universal algebra of signature* $E$ or briefly, a *topological $E$-algebra* is a topological space $X$ endowed with a family of continuous maps $e_{n,X} : E_n \times X^n \to X$, $n \in \omega$.

A topological $E$-algebra $(X, \{e_{n,X}\}_{n\in\omega})$ is called *Tychonoff* if the underlying topological space $X$ is Tychonoff.

Homomorphisms between $E$-algebras are defined as follows.

**Definition 2.** A function $h : X \to Y$ between two topological $E$-algebras $(X, \{e_{n,X}\}_{n\in\omega})$ and $(Y, \{e_{n,Y}\}_{n\in\omega})$ is called an *$E$-homomorphism* if

$$e_{n,Y}(z, h(x_1), \dots, h(x_n)) = h(e_{n,X}(z, x_1, \dots, x_n))$$

for any $n \in \omega$, $z \in E_n$, and $x_1, \dots, x_n \in X$.

Such a function $h$ is called an *algebraic isomorphism* (*topological isomorphism*) if $h$ is bijective and both functions $h$ and $h^{-1}$ are (continuous) $E$-homomorphisms of the $E$-algebras.

Next, we define some operations over $E$-algebras.

**Definition 3.** For topological $E$-algebras $X_\alpha$, $\alpha \in A$, the Tychonoff product $X = \prod_{\alpha\in A} X_\alpha$ is a topological $E$-algebra endowed with the structure mappings

$$e_{n,X}(z, x_1, \dots, x_n) = \big(e_{n,X_\alpha}(z, \mathrm{pr}_\alpha(x_1), \dots, \mathrm{pr}_\alpha(x_n))\big)_{\alpha\in A}$$

where $n \in \omega$, $z \in E_n$, $x_1, \dots, x_n \in X$, and $\mathrm{pr}_\alpha : \prod_{\alpha\in A} X_\alpha \to X_\alpha$ is the $\alpha$-coordinate projection.

**Definition 4.** A subset $A \subset X$ of a topological $E$-algebra $(X, \{e_n\}_{n\in\omega})$ is called a *subalgebra* if $e_n(E_n \times A^n) \subset A$ for all $n \in \omega$.

Since for any subalgebras $A_i \subset X$, $i \in \mathcal{I}$, of a topological $E$-algebra $X$ the intersection $A = \bigcap_{i\in\mathcal{I}} A_i$ is a subalgebra of $X$, for each subset $Z \subset X$ there is a minimal subalgebra $\langle Z\rangle$ of $X$ that contains $Z$. This is the subalgebra generated by the set $Z$. The structure of this subalgebra $\langle Z\rangle$ can be described as follows.

Given a subset $L \subset E$ and a subset $Z$ of a topological $E$-algebra $(X, \{e_n\}_{n\in\omega})$, let

$$\begin{aligned}
&\langle Z\rangle_0^L = Z,\\
&\langle Z\rangle_{n+1}^L = \langle Z\rangle_n^L \cup \bigcup_{k\in\omega} e_{k,X}\big((E_k \cap L) \times (\langle Z\rangle_n^L)^k\big) \text{ for } n \in \omega, \text{ and}\\
&\langle Z\rangle_\omega^L = \bigcup_{n\in\omega} \langle Z\rangle_n^L.
\end{aligned}$$

By induction, one can check that for compact subspaces $L \subset E$ and $Z \subset X$ the subset $\langle Z\rangle_n^L$ of $X$ is compact for every $n \in \omega$. Consequently, $\langle Z\rangle_\omega^L$ is a $\sigma$-compact subset of $X$.

Writing the signature $E$ and the space $Z$ as the unions $E = \bigcup_{n\in\omega} L_n$ and $Z = \bigcup_{n\in\omega} Z_n$ of non-decreasing sequences of subsets, we see that

$$\langle Z\rangle = \bigcup_{n\in\omega} \langle Z_n\rangle_n^{L_n}$$

is the subalgebra of $X$, generated by $Z$. If the spaces $Z_n$ and $L_n$, $n \in \omega$, are compact (finite), then each subset $\langle Z_n\rangle_n^{L_n}$, $n \in \omega$, of $X$ is compact (finite) and hence the algebraic hull $\langle Z\rangle$ of $Z$ in $X$ is $\sigma$-compact (at most countable).

**Definition 5.** A class $\mathcal{K}$ of topological $E$-algebras is called a *complete quasivariety* if

1) for each topological $E$-algebra $X \in \mathcal{K}$, each $E$-subalgebra of $X$ belongs to the class $\mathcal{K}$;

2) for any topological $E$-algebras $X_\alpha \in \mathcal{K}$, $\alpha \in A$, their Tychonoff product $\prod_{\alpha\in A} X_\alpha$ belongs to the class $\mathcal{K}$;

3) a Tychonoff $E$-algebra belongs to $\mathcal{K}$ if it is algebraically isomorphic to a topological $E$-algebra $Y \in \mathcal{K}$.

A complete quasivariety $\mathcal{K}$ is *non-trivial* if it contains a topological $E$-algebra $X$ that contains more than one point.

Finally, we recall the notion of a free topological $E$-algebra.

**Definition 6.** Let $\mathcal{K}$ be a complete quasivariety of topological $E$-algebras. A *free topological $E$-algebra in $\mathcal{K}$* over a topological space $X$ is a pair $(F_{\mathcal{K}}(X), \eta)$ consisting of a topological $E$-algebra $F_{\mathcal{K}}(X) \in \mathcal{K}$ and a continuous map $\eta : X \to F_{\mathcal{K}}(X)$ such that for any continuous map $f : X \to Y$ to a topological $E$-algebra $Y \in \mathcal{K}$ there is a unique continuous $E$-homomorphism $h : F_{\mathcal{K}}(X) \to Y$ such that $f = h \circ \eta$.

The construction $F_{\mathcal{K}}(X)$ of a free topological $E$-algebra has been intensively studied by M. M. Choban [6, 7]. In particular, he proved that for each complete quasivariety $\mathcal{K}$ of topological $E$-algebras and any topological space $X$ a free topological $E$-algebra $(F_{\mathcal{K}}(X), \eta)$ exists and is unique up to a topological isomorphism. Also he proved the following important result, see [6, 2.4]:

**Theorem 1** (Choban)**.** *If $\mathcal{K}$ is a non-trivial complete quasivariety of topological $E$-algebras, then for each Tychonoff space $X$ the canonical map $\eta : X \to F_{\mathcal{K}}(X)$ is a topological embedding and $F_{\mathcal{K}}(X)$ coincides with the subalgebra $\langle \eta(X) \rangle$ generated by the image $\eta(X)$ of $X$ in $F(X, \mathcal{K})$.*

Since $\eta : X \to F_{\mathcal{K}}(X)$ is a topological embedding, we can identify a Tychonoff space $X$ with its image $\eta(X)$ in $F_{\mathcal{K}}(X)$ and say that the free $E$-algebra $F_{\mathcal{K}}(X)$ is algebraically generated by $X$.

In fact, the construction of a free topological $E$-algebra $F_{\mathcal{K}}(X)$ determines a functor $F_{\mathcal{K}} : \mathbf{Top} \to \mathcal{K}$ from the category **Top** of topological spaces and their continuous maps to the category whose objects are topological $E$-algebras from the class $\mathcal{K}$ and morphisms are continuous $E$-homomorphisms.

In [5–7] a lot of attention was paid to the problem of preservation of various topological properties by the functor $F_{\mathcal{K}}$. In particular, it was shown that the functor $F_{\mathcal{K}}$ preserves (submetrizable) $k_\omega$-spaces provided the signature $E$ is a (submetrizable) $k_\omega$-space, see [7, 4.1.2].

A Hausdorff topological space $X$ is called *a $k_\omega$-space* if $X = \varinjlim X_n$ is the *direct limit* of a non-decreasing sequence of compact subsets $(X_n)_{n\in\omega}$ of $X$ in the sense that $X = \bigcup_{n\in\omega} X_n$ and a subset $U \subset X$ is open if and only if $U \cap X_n$ is open in $X_n$ for each $n \in \omega$. Such a sequence $(X_n)_{n\in\omega}$ is called a *$k_\omega$-sequence* for $X$.

An *$s_\omega$-space* is a direct limit $\varinjlim X_n$ of a $k_\omega$-sequence $(X_n)_{n\in\omega}$ consisting of second countable compact subspaces of $X$. It is easy to see that a $k_\omega$-space $X$ is an $s_\omega$-space if and only if it is *submetrizable* in the sense that $X$ admits a continuous metric.

**Theorem 2** (Choban)**.** *Let $\mathcal{K}$ be a complete quasivariety of topological $E$-algebras whose signature $E$ is a (submetrizable) $k_\omega$-space. Then for each (submetrizable) $k_\omega$-space $X$ the free topological $E$-algebra $F_{\mathcal{K}}(X)$ is a (submetrizable) $k_\omega$-space. Moreover, if $E = \varinjlim L_n$ and $X = \varinjlim X_n$ for some $k_\omega$-sequences $(L_n)_{n\in\omega}$ and $(X_n)_{n\in\omega}$, then $(\langle \eta(X_n) \rangle_n^{L_n})_{n\in\omega}$ is a $k_\omega$-sequence for $F_{\mathcal{K}}X$ and thus $F_{\mathcal{K}}X = \varinjlim \langle \eta(X_n) \rangle_n^{L_n}$.*

The principal result of this paper asserts that the functor $F_{\mathcal{K}}$ preserves $\mathsf{ANR}(k_\omega)$-spaces.

**Definition 7.** A $k_\omega$-space $X$ is called an *absolute neighborhood retract in the class of $k_\omega$-spaces* (briefly, an $\mathsf{ANR}(k_\omega)$) if $X$ is a neighborhood retract in each $k_\omega$-space that contains $X$ as a closed subspace.

In Theorem 10 we shall show that a submetrizable $k_\omega$-space $X$ is an $\mathsf{ANR}(k_\omega)$-space if and only if each map $f : B \to X$ defined on a closed subspace of a (metrizable) compact space $A$ extends to a continuous map $\bar{f} : N(B) \to X$ defined on a neighborhood $N(B)$ of $B$ in $A$.

A topological space $X$ is called *compactly finite-dimensional* if each compact subset of $X$ is finite-dimensional.

The following theorem is the main result of this paper.

**Theorem 3.** *If $\mathcal{K}$ is a complete quasivariety of topological $E$-algebras of countable discrete signature $E$, then for each submetrizable* (*compactly finite-dimensional*) *$\mathsf{ANR}(k_\omega)$-space $X$ so is its free topological $E$-algebra $F_{\mathcal{K}}X$ in the quasivariety $\mathcal{K}$.*

## 2 $\mathsf{ANR}(k_\omega)$-spaces

In this section we collect some information about $\mathsf{ANR}(k_\omega)$-spaces. Such spaces are tightly connected with ANE-spaces.

Following [11] we define a topological space $X$ to be an *absolute neighborhood extensor* for a class $\mathbf{C}$ of topological spaces (briefly, an $\mathsf{ANE}(\mathbf{C})$*-space*) if each map $f : B \to X$ defined on a closed subspace $B$ of a topological space $C \in \mathbf{C}$ has a continuous extension $\bar{f} : N(B) \to X$ defined on some neighborhood $N(B)$ of $B$ in $C$. If any such $f$ can be extended to the whole space $C$, then $X$ is called an *absolute extensor* for the class $\mathbf{C}$.

By the Dugundji-Borsuk Theorem [8],[4] each convex subset of a locally convex linear topological space, is an absolute extensor for the class of metrizable spaces. This theorem was generalized by Borges [3] who proved that a convex subset of a locally convex space is an absolute extensor for the class of stratifiable spaces. This class contains all metrizable spaces and all submetrizable $k_\omega$-spaces, and is closed with respect to many countable topological operations, see [3],[10].

An important example of an $\mathsf{ANR}(k_\omega)$-space is the space

$$Q^\infty = \{(x_i)_{i\in\omega} \in \mathbb{R}^\infty : \sup_{i\in\omega} |x_i| < \infty\}$$

of bounded sequences, endowed with the direct limit topology $\varinjlim [-n,n]^\omega$ generated by the $k_\omega$-sequence $([-n,n]^\omega)_{n\in\mathbb{N}}$ consisting of the Hilbert cubes. Being a locally convex linear topological space, $Q^\infty$ is an absolute extensor for the class of stratifiable spaces.

A topological space $X$ is called a $Q^\infty$*-manifold* if $X$ is Lindelöf and each point $x \in X$ has a neighborhood homeomorphic to an open subset of $Q^\infty$. The theory of $Q^\infty$-manifolds was developed by K.Sakai [12],[13] who established the following fundamental results:

**Theorem 4** (Characterization)**.** *A topological space $X$ is homeomorphic to (a manifold modeled on) the space $Q^\infty$ if and only if $X$ is a submetrizable $k_\omega$-space such that each embedding $f : B \to X$ of a closed subset $B$ of a compact metrizable space $A$ can be extended to a topological embedding of (an open neighborhood of $B$ in) the space $A$ into $X$.*

**Theorem 5** (Open Embedding)**.** *Each $Q^\infty$-manifold is homeomorphic to an open subset of $Q^\infty$.*

**Theorem 6** (Closed Embedding)**.** *Each submetrizable $k_\omega$-space is homeomorphic to a closed subspace of $Q^\infty$.*

**Theorem 7** (Classification)**.** *Two $Q^\infty$-manifolds are homeomorphic if and only if they are homotopically equivalent.*

**Theorem 8** (Triangulation)**.** *Each $Q^\infty$-manifold $X$ is homeomorphic to $K \times Q^\infty$ for some countable locally finite simplicial complex $K$.*

**Theorem 9** (ANR-Theorem)**.** *For each submetrizable $\mathsf{ANR}(k_\omega)$-space $X$ the product $X \times Q^\infty$ is a $Q^\infty$-manifold.*

We shall use these theorems in the proof of the following (probably known as a folklore) characterization of submetrizable $\mathsf{ANR}(k_\omega)$-spaces.

**Theorem 10.** *For a submetrizable $k_\omega$-space $X$ the following conditions are equivalent:*

1) *$X$ is an $\mathsf{ANR}(k_\omega)$-space;*
2) *$X$ is an ANE for the class of $k_\omega$-spaces;*
3) *$X$ is an ANE for the class of compact metrizable spaces;*
4) *$X$ is an ANE for the class of stratifiable spaces;*
5) *$X$ is a retract of a $Q^\infty$-manifold.*

*The equivalent conditions (1)–(5) hold if $X = \varinjlim X_n$ is the direct limit of a $k_\omega$-sequence consisting of compact ANR's.*

*Proof.* (1) ⇒ (5) Assume that $X$ is an $\mathsf{ANR}(k_\omega)$-space. By the Closed Embedding Theorem 6, we can identify the submetrizable $k_\omega$-space $X$ with a closed subspace of $Q^\infty$. Being an $\mathsf{ANR}(k_\omega)$, $X$ is a retract of an open neighborhood $N(X) \subset Q^\infty$. Since $N(X)$ is a $Q^\infty$-manifold, $X$ is a retract of a $Q^\infty$-manifold.

(5) ⇒ (4) Assume that $X$ is a retract of a $Q^\infty$-manifold $M$. By the Open Embedding Theorem 5, $M$ can be identified with an open subspace of $Q^\infty$. By the

Borges' Theorem [3], the locally convex space $Q^\infty$ is an absolute extensor for the class of stratifiable spaces. Then the open subspace $M$ of $Q^\infty$ is an ANE for this class and so is its retract $X$.

The implication (4) $\Rightarrow$ (3) is trivial since each metrizable space is stratifiable.

(3) $\Rightarrow$ (2) Assume that $X$ is an ANE for the class of compact metrizable spaces. First we prove that $X$ is an ANE for the class of compact Hausdorff spaces. Let $f : B \to X$ be a continuous map defined on a closed subspace $B$ of a compact Hausdorff space $A$. Embed the compact space $A$ into a Tychonoff cube $I^\kappa$. The image $f(B)$, being a compact subspace of the submetrizable space $X$, is metrizable. By [9, 2.7.12], the function $f$ depends on countably many coordinates, which means that there is a countable subset $C \subset \kappa$ such that $f = f_C \circ \mathrm{pr}_C$ where $\mathrm{pr}_C : I^\kappa \to I^C$ is the projection onto the face $I^C$ of the cube $I^\kappa$ and $f_C : \mathrm{pr}_C(B) \to f(B) \subset X$ is a suitable continuous map. Since $X$ is an ANE for compact metrizable spaces, the map $f_C$ has a continuous extension $\tilde{f}_C : U \to X$ defined on an open neighborhood $U$ of $\mathrm{pr}_C(B)$ in the cube $I^C$. It follows that $V = \mathrm{pr}_C^{-1}(U) \cap A$ is an open neighborhood of $B$ in $A$ and $\tilde{f} = \tilde{f}_C \circ \mathrm{pr}_C|V : V \to X$ is a continuous extension of the map $f$, witnessing that $X$ is an ANE for the class of compact Hausdorff spaces.

Next, we show that $X$ is an ANE for the class of $k_\omega$-spaces. Let $f : B \to X$ be a continuous map defined on a closed subset $B$ of a $k_\omega$-space $A$. Then $A = \varinjlim A_n$ for some $k_\omega$-sequence $(A_n)_{n\in\omega}$ of compact subsets of $A$. Let $A_{-1} = \emptyset$. By induction, for each $n \in \omega$ we can construct a continuous map $f_n : N_n(A_n \cap B) \to X$ defined on a closed neighborhood $N(B \cap A_n)$ of $B \cap A_n$ in $A_n$ and such that

- $N_n(B \cap A_n) \supset N_{n-1}(B \cap A_{n-1})$,
- $f_n|B \cap A_n = f|B \cap A_n$ and
- $f_n|N_{n-1}(B \cap A_n) = f_{n-1}$.

The inductive step can be done because $X$ is an ANE for the class of compact Hausdorff spaces. After completing the inductive construction, consider the set $N(B) = \bigcup_{n\in\omega} N_n(B \cap A_n)$ and the map $\tilde{f} = \bigcup_{n\in\omega} f_n : N(B) \to X$, which is a desired continuous extension of $f$ onto the open neighborhood $N(B)$ of $B$ in $A$.

The implication (2) $\Rightarrow$ (1) trivially follows from the definitions of $\mathsf{ANR}(k_\omega)$ and $\mathsf{ANE}(k_\omega)$-spaces.

Now assume that $X = \varinjlim X_n$ is the direct limit of a $k_\omega$-sequence $(X_n)_{n\in\omega}$ consisting of compact ANR's. We claim that $X$ is an ANE for the class of compact metrizable spaces. Let $f : B \to X$ be a continuous map defined on a closed subspace $B$ of a compact metrizable space $A$. Since $X$ carries the direct limit topology $\varinjlim X_n$, the compact subset $f(B)$ lies in some set $X_n$, $n \in \omega$. Since $X_n$ is an ANR, the map $f : B \to X_n$ has a continuous extension $\tilde{f} : N(B) \to X_n \subset X$ defined on a neighborhood $N(B)$ of $B$ in $A$. $\square$

## 3 Some subfunctors of the functor $F_{\mathcal{K}}$

In the proof of Theorem 3 we shall apply a deep Basmanov's result on the preservation of compact ANR's by monomorphic functors of finite degree in the category **Comp** of compact Hausdorff spaces and their continuous maps. Let **C** be a full subcategory of the category **Top**, containing all finite discrete spaces.

We say that a functor $F : \mathbf{C} \to \mathbf{Top}$

- is *monomorphic* if $F$ preserves monomorphisms (which coincide with injective continuous maps in the category **Top** and its full subcategory **C**);
- has *finite supports* (*degree* $\deg F \leq n$) if for each object $X$ of the category **C** and each element $a \in FX$ there is a map $f : A \to X$ of a finite discrete space $A$ (of cardinality $|A| \leq n$) such that $a \in Ff(FA)$;

The smallest number $n \in \omega$ such that $\deg F \leq n$ is called the *degree* of $F$ and is denoted by $\deg F$. If no such number $n \in \omega$ exists, then we put $\deg F = \infty$.

The following improvement of the classical Basmanov's theorem [2] was recently proved in [1].

**Theorem 11.** *Let $F : \mathbf{Comp} \to \mathbf{Comp}$ be a monomorphic functor of finite degree $n = \deg F$ such that the space $Fn$ is finite. Then the functor $F$ preserves the class of compact finite-dimensional ANR-spaces.*

We shall apply this theorem to the subfunctors $\langle\cdot\rangle_n^L$ of the functor $F_{\mathcal{K}}$. We recall that $\mathcal{K}$ is a non-trivial complete quasivariety of topological $E$-algebras of countable discrete signature $E$. By Theorem 2, $F_{\mathcal{K}}$ can be thought as a functor $F_{\mathcal{K}} : \mathbf{K}_\omega \to \mathbf{K}_\omega$ in the category $\mathbf{K}_\omega$ of $k_\omega$-spaces and their continuous maps. By Theorem 2.4 of [6], for each Tychonoff space $X$ the free topological $E$-algebra $F_{\mathcal{K}}(X)$ is algebraically free in the sense that any bijective map $i : X_d \to X$ from a discrete topological space $X_d$ induces an algebraic isomorphism $F_{\mathcal{K}} i : F_{\mathcal{K}} X_d \to F_{\mathcal{K}} X$. This fact implies:

**Lemma 1.** *The functor $F_{\mathcal{K}} : \mathbf{Tych} \to \mathbf{Top}$ is monomorphic.*

*Proof.* Let $f : X \to Y$ be an injective continuous map between Tychonoff spaces and $f_d : X_d \to Y_d$ be the same map between these spaces endowed with the discrete topologies. Let $i_X : X_d \to X$ and $i_Y : Y_d \to Y$ be the identity maps. Let $r : Y_d \to X_d$ be any (automatically continuous) map such that $r \circ f_d = \mathrm{id}_{X_d}$. Thus we obtain the commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X_d \arrow[u, "i_X"] \arrow[r, shift left, "f_d"] & Y \arrow[u, "i_Y"'] \arrow[l, shift left, "r"] \end{tikzcd}$$

Applying the functor $F_{\mathcal{K}}$ to this diagram we get the diagram

$$\begin{tikzcd}
F_{\mathcal{K}}X \arrow[r, "F_{\mathcal{K}}f"] & F_{\mathcal{K}}Y \\
F_{\mathcal{K}}X_d \arrow[u, "F_{\mathcal{K}}i_X"] \arrow[r, shift left, "F_{\mathcal{K}}f_d"] & F_{\mathcal{K}}Y \arrow[u, "F_{\mathcal{K}}i_Y"'] \arrow[l, shift left, "F_{\mathcal{K}}r"]
\end{tikzcd}$$

The "vertical" maps $F_{\mathcal{K}}i_X : F_{\mathcal{K}}X_d \to F_{\mathcal{K}}X$ and $F_{\mathcal{K}}i_Y : F_{\mathcal{K}}Y_d \to F_{\mathcal{K}}Y$ in this diagram are bijective because the algebras $F_{\mathcal{K}}X$ and $F_{\mathcal{K}}Y$ are algebraically free. Taking into account that $F_{\mathcal{K}}r \circ F_{\mathcal{K}}f_d = F_{\mathcal{K}}(r \circ f_d) = F_{\mathcal{K}}\mathrm{id}_{X_d} = \mathrm{id}_{F_{\mathcal{K}}X_d}$, we conclude that the map $F_{\mathcal{K}}f_d$ is injective and so is the map $F_{\mathcal{K}}f : F_{\mathcal{K}}X \to F_{\mathcal{K}}Y$ because of the bijectivity of the maps $F_{\mathcal{K}}i_X$ and $F_{\mathcal{K}}i_Y$. $\square$

Now for every compact subset $L \subset E$ and every $n \in \omega$ consider the functor $\langle\cdot\rangle_n^L$ : **Comp** $\to$ **Comp** which assigns to each compact Hausdorff space $X$ the subspace $\langle X\rangle_n^L$ of $F_{\mathcal{K}}X$. The functor $\langle\cdot\rangle_n^L$ assigns to each continuous map $f : X \to Y$ between compact Hausdorff spaces the restriction $\langle f\rangle_n^L = F_{\mathcal{K}}f|\langle X\rangle_n^L$ of the homomorphism $F_{\mathcal{K}}f : F_{\mathcal{K}}X \to F_{\mathcal{K}}Y$.

**Lemma 2.** *For every $n \in \mathbb{N}$, $\langle\cdot\rangle_n^L$ :* **Comp** $\to$ **Comp** *is a well-defined monomorphic functor of finite degree in the category* **Comp**.

*Proof.* First we check that for each continuous map $f : X \to Y$ between compact Hausdorff spaces, the morphism $\langle f\rangle_n^L = F_{\mathcal{K}}f|\langle X\rangle_n^L$ is well-defined, which means that $F_{\mathcal{K}}f(\langle X\rangle_n^L) \subset \langle Y\rangle_n^L$. This will be done by induction on $n \in \omega$.

For $n = 0$ the inclusion $F_{\mathcal{K}}(\langle X\rangle_0^L) = F_{\mathcal{K}}(X) = f(X) \subset Y = \langle Y\rangle_0^L$ follows from the fact that the homomorphism $F_{\mathcal{K}}$ extends the map $f$ (here we identify $X$ and $Y$ with the subspaces $\eta(X)$ and $\eta(Y)$ in $F_{\mathcal{K}}(X)$ and $F_{\mathcal{K}}(Y)$, respectively).

Assume that the inclusion $F_{\mathcal{K}}f(\langle X\rangle_n^L) \subset \langle Y\rangle_n^L$ has been proved for some $n \in \omega$. By definition,

$$\langle X\rangle_{n+1}^L = \langle X\rangle_n^L \cup \bigcup_{k\in\omega} e_{k,X}((E_k \cap L) \times (\langle X\rangle_n^L)^k).$$

Fix any element $x \in \langle X\rangle_{n+1}^L$. If $x \in \langle X\rangle_n^L$, then

$$F_{\mathcal{K}}(x) \in F_{\mathcal{K}}(\langle X\rangle_n^L) \subset \langle Y\rangle_n^L \subset \langle Y\rangle_{n+1}^L$$

by the inductive assumption.

If $x \in \langle X\rangle_{n+1}^L \setminus \langle X\rangle_n^L$, then $x = e_{k,X}(z, x_1, \dots, x_k)$ for some $k \in \omega$, $z \in E_k \cap L$, and points $x_1, \dots, x_k \in \langle X\rangle_n^L$. Since $F_{\mathcal{K}}f$ is an $E$-homomorphism, we get

$$\begin{aligned} F_{\mathcal{K}}f(x) = F_{\mathcal{K}}f(e_{k,X}(z, x_1, \dots, x_k)) = e_{k,Y}(z, F_{\mathcal{K}}f(x_1), \dots, F_{\mathcal{K}}f(x_k)) \in \\ \in e_{k,Y}((E_k \cap L) \times (\langle Y\rangle_n^L)^k) \subset \langle Y\rangle_{n+1}^L. \end{aligned}$$

Thus for every $n \in \omega$ the functor $\langle\cdot\rangle_n^L$ is well-defined. It is monomorphic as a subfunctor of the monomorphic functor $F_{\mathcal{K}}$.

Next, we show that the functor $\langle\cdot\rangle$ has finite degree. This will be done by induction on $n \in \omega$. Since $\langle X\rangle_0^L = X$, $\deg\langle\cdot\rangle_0^L = 1$.

Assume that for some $n \in \omega$ the functor $\langle\cdot\rangle_n^L$ has finite degree $d$. Since $L$ is a compact subset of $E$, there is $m \in \omega$ such that $L \cap E_k = \emptyset$ for all $k \geq m$. We claim that $\deg\langle\cdot\rangle_{n+1}^L \leq m \cdot d$. Take any element $x \in \langle X\rangle_{n+1}^L$. If $x \in \langle X\rangle_n^L$, then by the inductive assumption there is a subset $A \subset X$ of cardinality $|A| \leq d$ such that $x \in \langle A\rangle_n^L$ and we are done. If $x \in \langle X\rangle_{n+1}^L \setminus \langle X\rangle_n^L$, then $x = e_{k,X}(z, x_1, \dots, x_k)$ for some $k \in \omega$, $z \in E_k \cap L$, and points $x_1, \dots, x_k \in \langle X\rangle_n^L$. Since $L \cap E_k \ni z$ is not empty, $k \leq m$. By the inductive assumption, for every $i \leq k$ there is a finite subset $A_i \subset X$ of cardinality $|A_i| \leq d$ such that $x_i \in \langle A_i\rangle_n^L$. Then the union $A = \bigcup_{i=1}^k A_i$ has cardinality $|A| \leq k \cdot d \leq m \cdot d$ and

$$x = e_{k,X}(z, x_1, \dots, x_k) \in e_{k,X}((L \cap E_k) \times (\langle A\rangle_n^L)^k) \subset \langle A\rangle_{n+1}^L$$

witnessing that the functor $\langle\cdot\rangle_{n+1}^L$ has finite degree $\deg\langle\cdot\rangle_{n+1}^L \leq m \cdot d$. □

**Lemma 3.** *If $L \subset E$ is finite, then for each $n \in \omega$ the functor $\langle\cdot\rangle_n^L$ preserves finite spaces.*

*Proof.* Let $X$ be a finite space. By induction on $n \in \omega$ we shall show that the space $\langle X\rangle_n^L$ is finite. This is clear for $n = 0$. Assume that for some $n \in \omega$ the space $\langle X\rangle_n^L$ is finite. Since $L \subset E$ is finite there is $m \in \omega$ such that $L \cap E_n = \emptyset$ for all $k > m$. Then

$$\langle X\rangle_{n+1}^L = \langle X\rangle_n^L \cup \bigcup_{k \leq m} e_{k,X}((E_k \cap L) \times (\langle X\rangle_n^L)^k)$$

is finite as the finite union of finite sets. □

Combining Lemmas 2, 3 with Theorem 11, we get

**Corollary 1.** *For any finite subset $L \subset E$ and every $n \in \omega$ the functor $\langle\cdot\rangle_n^L$ preserves (finite-dimensional) compact ANR's.*

## 4 Proof of Theorem 3

Without loss of generality, the quasivariety $\mathcal{K}$ is non-trivial (otherwise, $F_{\mathcal{K}}(X)$ is a singleton and hence is an $\mathsf{ANR}(k_\omega)$-space for each non-empty space $X$).

Let $X$ be a submetrizable $\mathsf{ANR}(k_\omega)$-space. By the ANR-Theorem 9, the product $X \times Q^\infty$ is a $Q^\infty$-manifold. By the Triangulation Theorem 8, $X \times Q^\infty$ is homeomorphic to $T \times Q^\infty$ for a countable locally finite simplicial complex $T$. This implies that $X \times Q^\infty$ can be written as the direct limit $X \times Q^\infty = \varinjlim X_n$ of a $k_\omega$-sequence $(X_n)_{n\in\omega}$ of compact ANR's.

Write the countable discrete space $E$ as the direct limit $E = \varinjlim L_n$ of a $k_\omega$-sequence $(L_n)_{n\in\omega}$ of finite subsets of $E$. By Choban's Theorem 2, the space $F_{\mathcal{K}}(X \times Q^\infty)$ is the direct limit $\varinjlim\langle X_n\rangle_n^{L_n}$ of the $k_\omega$-sequence $\langle X_n\rangle_n^{L_n}$. By Corollary 1, each space $\langle X_n\rangle_n^{L_n}$, $n \in \omega$, is a compact metrizable ANR. Consequently, $F_{\mathcal{K}}(X \times Q^\infty) =$

$\varinjlim\langle X_n\rangle_n^{L_n}$ is a submetrizable $\mathsf{ANR}(k_\omega)$-space by Theorem 10. Since $X$ is a retract of $X \times Q^\infty$, the space $F_{\mathcal{K}}X$ is a retract of $F_{\mathcal{K}}(X \times Q^\infty)$ and hence $F_{\mathcal{K}}X$ is a submetrizable $\mathsf{ANR}(k_\omega)$-space.

Now assume that $X$ is a compactly finite-dimensional $s_\omega$-space. Then $X = \varinjlim X_n$ is the direct limit of finite-dimensional compact metrizable spaces. By the Choban's Theorem 2, the space $F_{\mathcal{K}}(X\times Q^\infty)$ is the direct limit $\varinjlim\langle X_n\rangle_n^{L_n}$ of the $k_\omega$-sequence $\langle X_n\rangle_n^{L_n}$. Corollary 1 implies that each compact space $\langle X_n\rangle_n^{L_n}$ is metrizable and finite-dimensional. Then the space $F_{\mathcal{K}}X = \varinjlim\langle X_n\rangle_n^{L_n}$ is compactly finite-dimensional, being the direct limit of finite-dimensional compact spaces.

## References


[1] Banakh T., Klymenko M., Zarichnyi M. *On monomorphic topological functors with finite supports*, preprint (arXiv:1004.0457).

[2] Basmanov V. *Covariant functors, retracts, and dimension*, Dokl. Akad. Nauk SSSR, 1983, **271:5**, 1033–1036.

[3] Borges C. *On stratifiable spaces*, Pacific J. Math., 1966, **17**, 1–16.

[4] Borsuk K. *Über Isomorphie der Functionalräume*, Bull. Inst. Acad. Polon. Sci. Ser. A, 1933, 1–10.

[5] Choban M. M. *On the theory of topological algebraic systems*, Trudy Moskov. Mat. Obshch., 1985, **48**, 106–149.

[6] Choban M. M. *Some topics in topological algebra*, Topology Appl., 1993, **54:1-3**, 183–202.

[7] Choban M. M. *Algebraical equivalences of topological spaces*, Bul. Acad. Stiinte Repub. Mold., Mat., 2001, No. 1(35), 12–36.

[8] Dugundji J. *An extension of Tietze's theorem*, Pacific J. Math., 1951, **1**, 353–367.

[9] Engelking R. *General Topology*, Heldermann Verlag, Berlin, 1989.

[10] Gruenhage G. *Generalized metric spaces*, Handbook of set-theoretic topology, North-Holland, Amsterdam, 1984, 423–501.

[11] Hu S.-T. *Theory of Retracts.* Wayne State Univ. Press, Detroit, 1965.

[12] Sakai K. *On $\mathbb{R}^\infty$-manifolds and $Q^\infty$-manifolds*, Topology Appl., 1984, **18:1**, 69–79.

[13] Sakai K. *On $\mathbb{R}^\infty$-manifolds and $Q^\infty$-manifolds. II. Infinite deficiency*, Tsukuba J. Math., 1984, **8:1**, 101–118.


Taras Banakh
Ivan Franko National University of Lviv, Ukraine
Jan Kochanowski University in Kielce, Poland
E-mail: *t.o.banakh@gmail.com*

*Received April 6, 2010*

Olena Hryniv
Ivan Franko National University of Lviv, Ukraine
E-mail: *olena_hryniv@ukr.net*

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(65), 2011, Pages 60–65
ISSN 1024–7696

# On the instability of solutions of seventh order nonlinear delay differential equations

Cemil Tunç

**Abstract.** A kind of seventh order nonlinear delay differential equations is considered. By using the Lyapunov-Krasovskii functional approach [5], some sufficient conditions are established which guarantee that the zero solution of the equation considered is unstable. Our conditions are new and supplement previously known results.

**Mathematics subject classification:** 34K20.
**Keywords and phrases:** Instability, Lyapunov-Krasovskii functional, delay differential equation, seventh order.

## 1 Introduction

Since 1992 till now, by using the Lyapunov's direct method, the qualitative behaviors of solutions of the seventh order nonlinear differential equations without a deviating argument have been studied and are still being investigated in the literature. See, for example, the papers of Bereketoğlu [2], Sadek [6], Tejumola [7], Tunç [8, 9], Tunç and Tunç [10]. In the mentioned papers, [6–10], the Lyapunov's direct method was used to show the instability of the solutions of some seventh order nonlinear differential equations without a deviating argument. However, to the best of our knowledge, we did not find any paper relative to the instability of the solutions of the seventh order linear and nonlinear differential equations with a deviating argument in the literature. The basic reason related to the absence of any paper on this topic may be the difficulty of the construction or definition of appropriate Lyapunov functions or functionals for the instability problems relative to the seventh order linear and nonlinear differential equations with a deviating argument.

As regards our problem here, in 2000, Tejumola [7] studied the instability of the zero solution of the seventh order nonlinear differential equation without a deviating argument

$$x^{(7)} + a_1 x^{(6)} + a_2 x^{(5)} + a_3 x^{(4)} + \psi_4(x, x', ..., x^{(6)})x''' + \psi_5(x')x'' + \\ + \psi_6(x, x', ..., x^{(6)}) + \psi_7(x) = 0. \quad (1)$$

In this paper, instead of Eq. (1), we take into consideration the seventh order nonlinear differential equation with a constant deviating argument $r$:

$$x^{(7)} + a_1 x^{(6)} + a_2 x^{(5)} + a_3 x^{(4)} + \psi_4(x, x(t-r), x', x'(t-r), ..., x^{(6)}(t-r))x''' +$$

$$+\psi_5(x')x'' + \psi_6(x, x(t-r), x', x'(t-r), ..., x^{(6)}(t-r)) + \psi_7(x(t-r)) = 0. \quad (2)$$

We write Eq. (2) in system form as

$$x_1' = x_2,\ x_2' = x_3,\ x_3' = x_4,\ x_4' = x_5,\ x_5' = x_6,\ x_6' = x_7,$$
$$x_7' = -a_1 x_7 - a_2 x_6 - a_3 x_5 - \psi_4(x_1, x_1(t-r), ..., x_7(t-r))x_4 - \psi_5(x_2)x_3 -$$
$$-\psi_6(x_1, x_1(t-r), ..., x_7(t-r)) - \psi_7(x_1) + \int_{t-r}^{t} \psi_7'(x_1(s))x_2(s)ds, \quad (3)$$

which is obtained as usual by setting $x = x_1,\ x' = x_2,\ x'' = x_3,\ x''' = x_4,\ x^{(4)} = x_5,\ x^{(5)} = x_6$ and $x^{(6)} = x_7$ in (2), where $r$ is a positive constant, $a_1,\ a_2$ and $a_3$ are some constants, the primes in Eq. (2) denote differentiation with respect to $t$, $t \in \Re_+$, $\Re_+ = [0, \infty)$; the functions $\psi_4$, $\psi_5$, $\psi_6$ and $\psi_7$ are continuous on $\Re^{14}$, $\Re$, $\Re^{14}$ and $\Re$ with $\psi_6(x_1, x_1(t-r), 0, ..., x_4(t-r)) = \psi_7(0) = 0$, and satisfy a Lipschitz condition in their respective arguments. Hence, the existence and uniqueness of the solutions of Eq. (2) are guaranteed (see Elśgolt́s [1, p. 14, 15]). We assume in what follows that the function $\psi_7$ is differentiable, and $x_1(t),\ x_2(t), x_3(t), x_4(t), x_5(t),\ x_6(t)$ and $x_7(t)$ are abbreviated as $x_1,\ x_2, x_3, x_4, x_5,\ x_6$ and $x_7$, respectively.

Here, by defining an appropriate Lyapunov functional, we prove an instability theorem for Eq. (2). By this work, we improve an instability result obtained in the literature [7, Theorem 6] relative to a seventh order nonlinear differential equation without a deviating argument to the instability of the zero solution of a certain seventh order nonlinear differential equation with a deviating argument, Eq. (2). Our motivation comes from the papers contained in the references of this paper.

Let $r \geqslant 0$ be given, and let $C = C([-r, 0], \Re^n)$ with

$$\|\phi\| = \max_{-r \leqslant s \leqslant 0} |\phi(s)|\,,\ \phi \in C.$$

For $H > 0$ define $C_H \subset C$ by

$$C_H = \{\phi \in C : \|\phi\| < H\}.$$

If $x : [-r, a] \to \Re^n$ is continuous, $0 < A \leqslant \infty$, then, for each $t$ in $[0, A)$, $x_t$ in $C$ is defined by

$$x_t(s) = x(t+s), -r \leqslant s \leqslant 0, \quad t \geqslant 0.$$

Let $G$ be an open subset of $C$ and consider the general autonomous delay differential system

$$\dot{x} = F(x_t), \quad x_t = x(t+\theta), \quad -r \leqslant \theta \leqslant 0, \quad t \geqslant 0,$$

where $F : G \to \Re^n$ is continuous and maps closed and bounded sets into bounded sets. It follows from the conditions on $F$that each initial value problem

$$\dot{x} = F(x_t), \quad x_0 = \phi \in G$$

has a unique solution defined on some interval $[0, A), 0 < A \leqslant \infty$. This solution will be denoted by $x(\phi)(.)$ so that $x_0(\phi) = \phi$.

**Definition 1.** The zero solution, $x = 0$, of $\dot{x} = F(x_t)$ is stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\| < \delta$ implies that $|x(\phi)(t)| < \varepsilon$ for all $t \geqslant 0$. The zero solution is said to be unstable if it is not stable.

**Theorem 1.** *Suppose there exists a Lyapunov function $V : G \to \Re_+$ such that $V(0) = 0$ and $V(x) > 0$ if $x \neq 0$. If either*

***(i)*** *$\dot{V}(\phi) > 0$ for all $\phi$ in $G$ for which*

$$V[\phi(0)] = \max_{-s \leqslant t \leqslant 0} V[\phi(s)] > 0$$

*or*

***(ii)*** *$\dot{V}(\phi) > 0$ for all $\phi$ in $G$ for which*

$$V[\phi(0)] = \min_{-s \leqslant t \leqslant 0} V[\phi(s)] > 0,$$

*then the solution $x = 0$ of $\dot{x} = F(x_t)$ is unstable (see Haddock and Ko [3]).*

## 2 Main result

The following theorem is our main result.

**Theorem 2.** *Together with all the assumptions imposed on the functions $\psi_4$, $\psi_5, \psi_6$ and $\psi_7$ in Eq. (2), we assume that there exist constants $a_2 < 0$, $a_7 > 0$, $\delta_0 > 0$ and $\delta > 0$ such that the following conditions hold:*

$$\psi_7(x_1) \neq 0, \quad (x_1 \neq 0), \quad \frac{\psi_7(x_1)}{x_1} \geqslant \delta_0, \ (x_1 \neq 0), \ 0 < \psi_7'(x_1) \leqslant a_7,$$

$$\psi_6(x_1, ..., x_4(t-r)) \neq 0, \ (x_2 \neq 0), \quad \frac{1}{4a_2}\psi_4^2(.) - \frac{\psi_6(.)}{x_2} \geqslant \delta, \ (x_2 \neq 0).$$

*Then, the zero solution, $x = 0$, of Eq. (2) is unstable provided that $r < \dfrac{\delta}{a_7}$.*

*Remark* 1. For the proof of the theorem, under the conditions sated in the theorem, it suffices to find that there exists a continuous Lyapunov functional $V = V(x_{1t}, ..., x_{7t})$ which has the following three properties, Krasovskii properties [4], say $(K_1), (K_2)$ and $(K_3)$:

$(K_1)$ In every neighborhood of $(0, 0, 0, 0, 0, 0, 0)$ there exists a point $(\xi_1, ..., \xi_7)$ such that $V(\xi_1, ..., \xi_7) > 0$,

$(K_2)$ the time derivative $\dot{V} = \dfrac{d}{dt}V(x_{1t}, ..., x_{7t})$ along solution paths of (3) is positive semi-definite,

$(K_3)$ the only solution $(x_1, ..., x_7) = (x_1(t), ..., x_7(t))$ of (3) which satisfies $\dfrac{d}{dt}V(x_{1t}, ..., x_{7t}) = 0$ $(t \geqslant 0)$, is the trivial solution $(0, 0, 0, 0, 0, 0, 0)$.

*Proof.* Consider the Lyapunov functional $V = V(x_{1t}, ..., x_{7t})$ defined by

$$V = x_2x_7 + a_1x_2x_6 + a_2x_2x_5 + a_3x_2x_4 - x_3x_6 - a_1x_3x_5 - a_2x_3x_4 + x_4x_5 -$$
$$-\frac{1}{2}a_3x_3^2 + \frac{1}{2}a_1x_4^2 + \int_0^{x_1} \psi_7(s)ds + \int_0^{x_2} \psi_5(s)sds - \lambda \int_{-r}^{0}\int_{t+s}^{t} x_2^2(\theta)d\theta ds, \tag{4}$$

where $s$ is a real variable such that the integral $\int_{-r}^{0}\int_{t+s}^{t} x_2^2(\theta)d\theta ds$ is non-negative and $\lambda$ is a positive constant which will be determined later in the proof.

From (4) it follows that

$$V(0, 0, 0, 0, 0, 0, 0) = 0$$

and

$$V = x_2x_7 + a_1x_2x_6 + a_2x_2x_5 + a_3x_2x_4 - x_3x_6 - a_1x_3x_5 - a_2x_3x_4 + x_4x_5 -$$
$$-\frac{1}{2}a_3x_3^2 + \frac{1}{2}a_1x_4^2 + \int_0^{x_1} \frac{\psi_7(s)}{s}sds + \int_0^{x_2} \psi_5(s)sds - \lambda \int_{-r}^{0}\int_{t+s}^{t} x_2^2(\theta)d\theta ds \geqslant$$
$$\geqslant x_2x_7 + a_1x_2x_6 + a_2x_2x_5 + a_3x_2x_4 - x_3x_6 - a_1x_3x_5 - a_2x_3x_4 + x_4x_5 -$$
$$-\frac{1}{2}a_3x_3^2 + \frac{1}{2}a_1x_4^2 + \frac{1}{2}\delta_0x_1^2 + \int_0^{x_2} \psi_5(s)sds - \lambda \int_{-r}^{0}\int_{t+s}^{t} x_2^2(\theta)d\theta ds.$$

Hence, we get

$$V(\varepsilon, 0, 0, 0, 0, 0, 0) = \frac{1}{2}\delta_0\varepsilon^2 > 0$$

for all sufficiently small $\varepsilon$, $\varepsilon \in \Re$, so that every neighborhood of the origin in the $(x_1, ..., x_7)-$space contains points $(\xi_1, ..., \xi_7)$ such that $V(\xi_1, ..., \xi_7) > 0$.

Let

$$(x_1, ..., x_7) = (x_1(t), ..., x_7(t))$$

be an arbitrary solution of (3).

Differentiating the Lyapunov functional $V$ in (4) along this solution, we get

$$\dot{V} = x_5^2 - a_2x_4^2 - \psi_4(x_1, ..., x_7(t-r))x_2x_4 - \psi_6(x_1, ..., x_7(t-r))x_2 +$$

$$+x_2 \int\limits_{t-r}^{t} \psi_7'(x_1(s))x_2(s)ds - \lambda r x_2^2 + \lambda \int\limits_{t-r}^{t} x_2^2(s)ds.$$

The assumption $0 < \psi_7'(x_1) \leqslant a_7$ of the theorem and the estimate $2|mn| \leqslant m^2 + n^2$ imply that

$$x_2 \int\limits_{t-r}^{t} \psi_7'(x_1(s))x_2(s)ds \geqslant -|x_2| \int\limits_{t-r}^{t} \psi_7'(x_1(s))|x_2(s)|ds \geqslant$$

$$\geqslant -\frac{1}{2}a_7 r x_2^2 - \frac{1}{2}a_7 \int\limits_{t-r}^{t} x_2^2(s)ds.$$

Hence

$$\dot{V} \geqslant x_5^2 - a_2\left[x_4 + \frac{1}{2a_2}\psi_4(x_2, ..., x_4(t-r))x_2\right]^2 +$$

$$+\frac{1}{4a_2}\psi_4^2(x_1, ..., x_7(t-r))x_2^2 - \psi_6(x_1, ..., x_7(t-r))x_2 -$$

$$-\{(\lambda + \frac{1}{2}a_7)r\}x_2^2 + \left(\lambda - \frac{1}{2}a_7\right) \int\limits_{t-r}^{t} x_2^2(s)ds.$$

Let $\lambda = \frac{1}{2}a_7$. Then, we get

$$\dot{V} \geqslant x_5^2 - a_2\left[x_4 + \frac{1}{2a_2}\psi_4(x_1, ..., x_7(t-r))x_2\right]^2 +$$

$$+\left[\frac{1}{4a_2}\psi_4^2(x_1, ..., x_7(t-r)) - \frac{\psi_6(x_1, ..., x_7(t-r))}{x_2} - a_7 r\right]x_2^2 \geqslant$$

$$\geqslant (\delta - a_7 r)x_2^2 > 0$$

provided that $r < \dfrac{\delta}{a_7}$. Thus if the assumptions of the theorem hold then $\dot{V}$ is positive semi-definite.

Now observe that $\dot{V} = 0$ for all $t \geqslant 0$ necessarily implies that $x_2 = 0$ and therefore also that

$$x_2 = x' = 0,\ x_3 = x'' = 0,\ x_4 = x''' = 0,$$

$$x_5 = x^{(4)} = 0,\ x_6 = x^{(5)} = 0,\ x_7 = x^{(6)} = 0$$

for all $t \geqslant 0$. Hence

$$x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = 0\ (t \geqslant 0).$$

Moreover, in view of $\dot{V} = 0$ and the system (4), one can also easily obtain $x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = 0$, which verifies the property $(K_3)$ of

Krasovskii [4]. It now follows that the Lyapunov functional $V$ thus has all the requisite Krasovskii properties, $(K_1), (K_2)$ and $(K_3)$, subject to the conditions in the theorem. By the above discussion, we conclude that the zero solution of Eq. (2) is unstable. The proof of the theorem is completed. $\square$

## References


[1] Elśgolt́s L. E. *Introduction to the theory of differential equations with deviating arguments. Translated from the Russian by Robert J.* McLaughlin Holden-Day, Inc., San Francisco, Calif.-London-Amsterdam, 1966.

[2] Bereketoğlu H. *On the periodic solutions of certain class of seventh-order differential equations.* Period. Math. Hungar., 1992, **24**, 13–22.

[3] Haddock John R., Ko Y. *Liapunov-Razumikhin functions and an instability theorem for autonomous functional-differential equations with finite delay. Second Geoffrey J. Butler Memorial Conference in Differential Equations and Mathematical Biology.* Rocky Mountain J. Math. 1995, **25**, 261–267.

[4] Krasovskii N. N. *On conditions of inversion of A. M. Lyapunov's theorems on instability for stationary systems of differential equations.* Dokl. Akad. Nauk SSSR, 1955, **101**, 17–20 (in Russian).

[5] Krasovskii N. N. *Stability of motion. Applications of Lyapunov's second method to differential systems and equations with delay. Translated by J. L. Brenner.* Stanford University Press, Stanford, Calif. 1963.

[6] Sadek A. I. *An instability theorem for a certain seventh-order differential equation.* Ann. Differential Equations, 2003, **1**, 1–5.

[7] Tejumola H. O. *Instability and periodic solutions of certain nonlinear differential equations of orders six and seven. Ordinary differential equations (Abuja, 2000), 56-65.* Proc. Natl. Math. Cent. Abuja Niger., 2000.

[8] Tunç C. *A result on the instability of solutions to a class of non-autonomous differential equations of seventh-order.* Math. Sci. Res. J., 2007, **11**, 360–366.

[9] Tunç C. *New results about instability of nonlinear ordinary vector differential equations of sixth and seventh orders.* Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., 2007, **14**, 123–136.

[10] Tunç C., Tunç E. *Instability of solutions of certain nonlinear vector differential equations of order seven.* Iran J. Sci. Technol. Trans. A Sci., 2005, **29**, 515–521.


Cemil Tunç
Yüzüncü Yil University
65080, Van, Turkey
E-mail: *cemtunc@yahoo.com*

*Received September 10, 2010*

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(65), 2011, Pages 66–82
ISSN 1024–7696

# Algorithms for Determining the State-Time Probabilities and the Limit Matrix in Markov Chains

Dmitrii Lozovanu, Stefan Pickl

**Abstract.** New calculation procedures for finding the probabilities of state transitions of the system in Markov chains based on dynamic programming are developed and polynomial time algorithms for determining the limit state matrix in such processes are proposed. Computational complexity aspects and possible applications of the proposed algorithms for the stochastic optimization problems are characterized.

**Mathematics subject classification:** 93E20, 49L20.
**Keywords and phrases:** Discrete Markov Process, Probability of State Transition, Limit State Matrix, Dynamic Programming, Polynomial Time Algorithm.

## 1 Introduction and Preliminary Results

In this paper we develop a dynamic programming approach for finite Markov processes and propose polynomial time algorithms for determining the limit state matrix in Markov chains. A characterization of a simple Markov process and the basic definitions related to determining the probabilities of state transitions of the system in such processes can be found in [3–5, 9, 10]. Here, for the finite Markov processes, we consider the problem of determining the probability of system's transition from a starting state to a final one when the final state is reached at the time-moment which belongs to a given interval of time. For such a specific case, we develop dynamic programming algorithms. Furthermore, the asymptotic behavior of the proposed algorithms are analyzed. Such a characterization of the problem allows us to apply a new approach for studying Markov chains and to elaborate polynomial time algorithms for determining the limit state probabilities of the dynamical system in such processes. We show that for non-ergodic Markov chains the limit probability matrix can be found in polynomial time. Therefore, we propose two polynomial time algorithms. The computational complexity of the first algorithm is $O(n^4)$ and of the second one is $O(n^3)$. Note that the well-known algorithm from [9] (see also [4, 5, 10]) in the worst case uses $O(n^4)$ elementary operations. Comparing this algorithm with proposed ones we can conclude that the approach described below allows us to ground new efficient algorithms for determining the limit state matrix in Markov chains. Additionally, we develop dynamic programming procedures for the calculation of the state probability transitions in the non-stationary discrete Markov processes. The proposed calculation procedures and algorithms can be used for studying and solving the stochastic version of classical discrete optimal control problems [2–9].

In this paper we consider discrete Markov processes with a finite set of states [3, 5, 7]. We denote the set of states of the dynamical system in such processes by $X = \{x_1, x_2, \ldots, x_n\}$. At the moment of time $t = 0$ the state of the system is $x_{i_0}$. For an arbitrary state $x \in X$ the probabilities $p_{x,y}$ of system's transitions from $x$ to another states $y \in X$ such that $\sum\limits_{y \in X} p_{x,y} = 1$ are given. So, we assume that the Markov process is determined by the stochastic matrix of probabilities $P = (p_{x,y})$ and the starting state $x_{i_0}$ of the dynamical system. The probability $P_{x_{i_0}}(x,t)$ of system's transitions from the state $x_{i_0}$ to an arbitrary state $x \in X$ by using $t$ transitions is defined and calculated on the basis of the following recursive formula [3]

$$P_{x_{i_0}}(x, \tau+1) = \sum_{y \in X} P_{x_{i_0}}(y, \tau) p_{y,x}, \quad \tau = 0, 1, 2, \ldots, t-1,$$

where $P_{x_{i_0}}(x_{i_0}, 0) = 1$ and $P_{x_{i_0},0}(x, 0) = 0$ for $x \in X \setminus \{x_{i_0}\}$. This formula can be represented in the matrix form by

$$\pi(\tau+1) = \pi(\tau)P, \quad \tau = 0, 1, 2, \ldots, t-1. \tag{1}$$

Here $\pi(\tau) = (\pi_1(\tau), \pi_2(\tau), \ldots, \pi_n(\tau))$ is the vector, where the component $i$ expresses the probability of the system $L$ to reach from $x_{x_{i_0}}$ the state $x_i$ at the moment of time $\tau$, i.e. $\pi_i(\tau) = P_{x_{i_0}}(x_i, \tau)$. At the starting moment of time $\tau = 0$ the vector $\pi(\tau)$ is given and its components are defined by $\pi_{i_0}(0) = 1$ and $\pi_i(0) = 0$ for arbitrary $i \neq i_0$. If for given starting vector $\pi(0)$ we apply our formula for $t = 0, 1, 2, \ldots, t-1$, then we obtain

$$\pi(t) = \pi(0)P^{(t)}$$

where $P^{(t)} = P \times P \times \cdots \times P$. So, an arbitrary element $p^{(t)}_{x,y}$ of this matrix expresses the probability of system $L$ to reach the state $y$ from $x$ by using $t$ units of times.

Formula (1) can be applied for the calculation of the state probabilities of the system in finite Markov processes. In the case $\tau \to \infty$ this formula leads to the relation $\pi = \pi P$ which together with the condition $\sum\limits_{i=1}^{n} \pi_i = 1$ allows us to determine the limit state probabilities in ergodic Markov chains.

## 2 The Main Results

To solve our main problem we need to develop special calculation procedures for determining the probability of system's transitions from a starting state to a final one when the final state is reached at the time-moment from given interval of time. We describe such calculation procedures which will allow us to ground polynomial time algorithms for finding the limit state matrix in aperiodic Markov chains.

### 2.1 Calculation of the Probabilities of States Transition of the System with a Given Restriction on the Number of Stages

In this subsection we show how to calculate the probability of system's transitions from the state $x_{i_0}$ to the state $x$ when $x$ is reached at the time moment $T(x)$ such that $T_1 \leq T(x) \leq T_2$ where $T_1$ and $T_2$ are given. So, we consider the problem of determining the probability of the system $L$ to reach the state $x$ at least at one of the moments of time $T_1, T_1+1, \ldots, T_2$. We denote this probability by $P_{x_{i_0}}(x, T_1 \leq T(x) \leq T_2)$. Some reflections on this definition allow us to write the following formula

$$P_{x_{i_0}}(x, T_1 \leq T(x) \leq T_2) =$$
$$= P_{x_{i_0}}(x, 0 \leq T(x) \leq T_2) - P_{x_{i_0}}(x, 0 \leq T(x) \leq T_1 - 1).$$

Further we describe some results which allow to calculate the probability $P_x(y, 0 \leq T(y) \leq t)$ for $x, y \in X$ and $t = 1, 2, \ldots$. For this reason we shall give the graphical interpretation of the Markov processes using the graph of state transitions $GR = (X, ER)$ [1, 3, 7, 10]. In this graph each vertex $x \in X$ corresponds to a state of the dynamical system and a possible system passage from one state $x$ to another state $y$ with positive probability $p_{x,y}$ is represented by the directed edge $e = (x, y) \in ER$ from $x$ to $y$; to directed edges $(x, y) \in ER$ in $GR$ the corresponding probabilities $p_{x,y}$ are associated. It is evident that in the graph $GR$ each vertex $x$ contains at least one leaving edge $(x, y)$ and $\sum\limits_{y \in X} p_{x,y} = 1$. As an example the graph of state transitions $GR = (X, ER)$ for the Markov process with the stochastic matrix of probabilities

$$P = \begin{pmatrix} 0.3 & 0.3 & 0.4 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.3 & 0.5 & 0.2 \end{pmatrix}$$

is represented in Fig. 1.

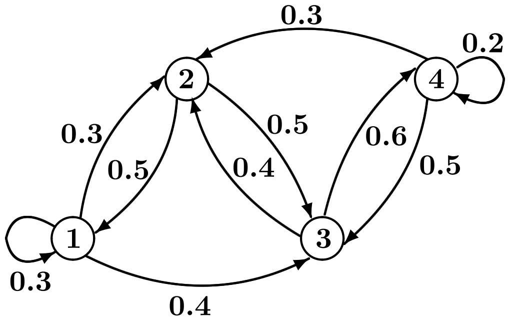



**Fig. 1**

In general we will consider also the stochastic process which may stop if one of the states from a given subset of states of dynamical system is reached. This means that the graph of such a process may contain the so-called deadlock vertices. So, we consider the stochastic process for which the graph of transition probabilities may contain the deadlock vertices $y \in X$ and $\sum_{z \in X} p_{x,z} = 1$ for the vertices $x \in X$ which contain at least one leaving directed edge. As an example in Fig. 2 a graph $GR = (X, ER)$ which contains a deadlock vertex is represented.

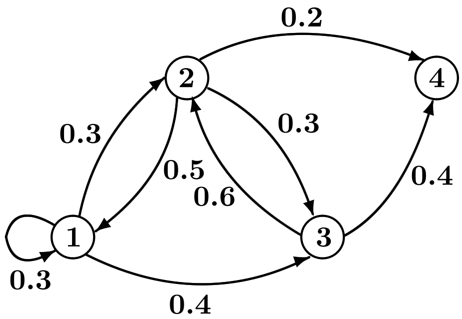



**Fig. 2**

This graph corresponds to the stochastic process with the following matrix of state transitions

$$P = \begin{pmatrix} 0.3 & 0.3 & 0.4 & 0 \\ 0.5 & 0 & 0.3 & 0.2 \\ 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Such graphs do not correspond to a Markov process and the matrix of probability $P$ contains a row with zero components. Nevertheless the probabilities $P_{x_0}(x, t)$ in this case can be calculated on the basis of the recursive formula given above. Note that the matrix $P$ can be easily transformed into a stochastic matrix changing the probabilities $p_{y,y} = 0$ for deadlock states $y \in X$ by the probabilities $p_{y,y} = 1$. This transformation leads to a new graph which corresponds to a Markov process because the obtained graph contains a new directed edge $e = (y, y)$ with $p_e = 1$ for $y \in X$. We call the vertices $y \in X$ in this graph the absorbing vertices and the corresponding states of the dynamical system in Markov process the absorbing states. So, the stochastic process which may stop in a given set of states can be represented either by a graph with deadlock vertices or by a graph with absorbing vertices. In Fig. 3 represents the graph with absorbing vertex $y = 4$ for the Markov process defined by the matrix $P$ given below.

$$P = \begin{pmatrix} 0.3 & 0.3 & 0.4 & 0 \\ 0.5 & 0 & 0.3 & 0.2 \\ 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 1.0 \end{pmatrix}.$$

It is easy to see that the stochastic matrix $P$ in this example is obtained from the previous one by changing $p_{4,4} = 0$ with $p_{4,4} = 1$. The corresponding graph with

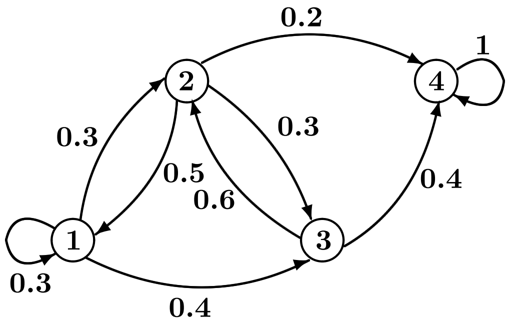



**Fig. 3**

the absorbing vertex $y = 4$ in this case is obtained from the graph on Fig. 2 by adding the directed edge $e = (4, 4)$ with $p_{4,4} = 1$.

We shall calculate the probabilities $P_x(y, 0 \leq T(y) \leq t)$ by using the graph with absorbing vertices.

**Lemma 1.** *Let a Markov process be given for which the graph $GR = (X, ER)$ contains an absorbing vertex $y \in X$. Then for an arbitrary state $x \in X$ the following recursive formula holds:*

$$P_x(y, 0 \leq T(y) \leq \tau + 1) = \sum_{z \in X} p_{x,z} P_z(y, 0 \leq T(z) \leq \tau), \quad \tau = 0, 1, 2, \ldots,$$

*where $P_x(y, 0 \leq T(y) \leq 0) = 0$ if $x \neq y$ and $P_y(y, 0 \leq T(y) \leq 0) = 1$.*

*Proof.* It is easy to observe that for $\tau = 0$ the theorem holds. Moreover, we can see that here the condition that $y$ is an absorbing state is essential; otherwise for $x = y$ the recursive formula from lemma fails to hold. For $\tau \geq 1$ the correctness of this formula follows from the definition of the probabilities $P_x(y, 0 \leq T(y) \leq \tau + 1), P_z(y, 0 \leq T(z) \leq \tau)$ and from the induction principle on $\tau$. $\square$

The recursive formula from this lemma can be written in matrix form by

$$\pi'(\tau + 1) = P\pi'(\tau), \quad \tau = 0, 1, 2, \ldots.$$

Here $P$ is the stochastic matrix of the Markov process with the absorbing state $y \in X$ and

$$\pi'(\tau) = \begin{pmatrix} \pi'_1(\tau) \\ \pi'_2(\tau) \\ \vdots \\ \pi'_n(\tau) \end{pmatrix}, \quad \tau = 0, 1, 2, \ldots$$

are the column vectors, where an arbitrary component $\pi'_i(\tau)$ expresses the probability of the dynamical system to reach the state $y$ from $x_i$ by using not more than $\tau$ unites of times, i.e. $\pi'_i(\tau) = P_{x_i}(y, 0 \leq T(y) \leq \tau)$. At the starting moment of time

$\tau = 0$ the vector $\pi'(0)$ is given: All components are equal to zero except for the component corresponding to the absorbing vertex which is equal to one, i.e.

$$\pi'_i(0) = \begin{cases} 0, & \text{if } x_i \neq y; \\ 1, & \text{if } x_i = y. \end{cases}$$

If we apply this formula for $\tau = 0, 1, 2, \dots, t-1$, then we obtain

$$\pi'(t) = P^{(t)}\pi'(0), \quad t = 1, 2, \dots.$$

So, if we denote by $j_y$ the column of the matrix $P^{(t)}$ which corresponds to the absorbing state $y$ then an arbitrary element $p^{(t)}_{i,j_y}$ of this column expresses the probability of the system $L$ to reach the state $y$ from $x_i$ by using not more than $t$ units of time, i.e. $p^{(t)}_{i,j_y} = P_{x_i}(y, 0 \le T(x) \le t)$. This allows us to formulate the following lemma:

**Lemma 2.** *Let a finite Markov process with the absorbing state $y \in X$ be given. Then:*

$$a) \quad P_{x_i}(y,\tau) = P_{x_i}(y, 0 \le T(y) \le \tau), \quad for \quad x_i \in X \setminus \{y\}, \quad \tau = 1, 2, \dots;$$

$$b) \quad P_{x_i}(y, T_1 \le T(y) \le T_2) = p^{(T_2)}_{i,j_y} - p^{(T_1-1)}_{i,j_y}.$$

*Proof.* The condition $a)$ in this lemma holds because

$$P_{x_i}(y,\tau) = p^{(\tau)}_{i,j_y} = P_{x_i}(y, 0 \le T(y) \le \tau).$$

The condition $b)$ we obtain from Lemma 1 and the following properties

$$P_{x_i}(y, 0 \le T(y) \le T_2) = p^{(T_2)}_{i,j_y}, \quad P_{x_i}(y, 0 \le T(y) \le T_1 - 1) = p^{(T_1-1)}_{i,j_y}.$$

□

So, to calculate $P_{x_i}(y, T_1 \le T(y) \le T_2)$ it is sufficient to find the matrices $P^{(T_1-1)}$, $P^{(T_2)}$ and then to apply the formula from Lemma 2.

The procedure of the calculation of the probabilities $P_x(y, 0 \le T(y) \le t)$ in the case of the Markov process without absorbing states can be easily reduced to the procedure of the calculation of the probabilities in the Markov process with the absorbing state $y$ by using the following transformation of the stochastic matrix $P$. We put $p_{i_y,j} = 0$ if $j \neq i_y$ and $p_{i_y,i_y} = 1$. It is easy to see that such a transformation of the matrix $P$ does not change the probabilities $P_x(y, 0 \le T(y) \le t)$. After such a transformation we obtain a new stochastic matrix for which the recursive formula from the Lemma 21 can be applied. In general for the Markov processes with absorbing state these probabilities can be calculated by using the algorithm which works with the original matrix $P$ without changing its elements. Below such an algorithm is described.

**Algorithm 1: Determining the state probabilities of the system with a restriction on number of transitions (stationary case)**

*Preliminary step (Step 0):* Put $P_x(y, 0 \leq T(y) \leq 0) = 0$ for every $x \in X \setminus \{y\}$ and $P_y(y, 0 \leq T(x) \leq 0) = 1$.

*General step (Step $\tau + 1, \tau \geq 0$):* For every $x \in X \setminus \{y\}$ calculate

$$P_x(y, 0 \leq T(x) \leq \tau + 1) = \sum_{z \in X} p_{x,z} P_z(y, 0 \leq T(y) \leq \tau) \tag{2}$$

and then put

$$P_y(y, 0 \leq T(y) \leq \tau + 1) = 1. \tag{3}$$

If $\tau < t - 1$ then go to next step; otherwise STOP.

**Theorem 1.** *Algorithm 1 correctly finds the probabilities $P_x(y, 0 \leq T(x) \leq \tau)$ for $x \in X, \quad \tau = 0, 1, 2, \ldots, t - 1$.*

*Proof.* It is easy to see that the probabilities $P_x(y, 0 \leq T(x) \leq \tau + 1)$ at the general step of the algorithm are calculated on the basis of formula (2) which takes into account condition (3). This calculation procedure is equivalent with the calculation of the probabilities $P_x(y, 0 \leq T(x) \leq \tau + 1)$ with the condition that the state $y$ is an absorbing state. So, the algorithm is correct. $\square$

If in Algorithm 1 we use the notation $\pi'_i(\tau) = P_{x_i}(y, 0 \leq T(y) \leq \tau)$, $\pi_{i_y}(\tau) = P_y(y, 0 \leq T(y) \leq \tau)$ then we obtain the following description of the algorithm in a matrix form:

**Algorithm 2: Calculation of the state probabilities of the system in the matrix form (stationary case)**

*Preliminary step (Step 0):* Fix the vector $\pi'(0) = (\pi'_1(0), \pi'_2(0), \ldots, \pi'_n(0))$, where $\pi'_i(0) = 0$ for $i \neq i_y$ and $\pi'_{i_y}(0) = 1$.

*General step (Step $\tau + 1, \tau \geq 0$):* For given $\tau$ calculate

$$\pi'(\tau + 1) = P\pi'(\tau)$$

and then put

$$\pi'_{i_y}(\tau + 1) = 1.$$

If $\tau < t - 1$ then go to next step; otherwise STOP.

Note that the condition $\pi'_{i_y}(\tau + 1) = 1$ in the algorithm allows us to preserve the value $\pi'_{i_y}(t) = 1$ at every moment of time $t$ in the calculation process. This condition reflects the property that the system remains in the state $y$ at every time-step $t$ if the state $y$ is reached. We can modify this algorithm for determining the probability $P_x(y, 0 \leq T(y) \leq 0)$ in a more general case if we assume that the system will remain at every time step $t$ in the state $y$ with the probability $\pi'_{i_y}(t) = q(y)$, where $q(y)$ may differ from 1, i.e, $q(y) \leq 1$. In the following we can see that this modification is very important for determining the matrix of limit probabilities in finite Markov processes. So, $q(y) \leq 1$, and we can use the following algorithm:

**Algorithm 3: Calculation of the state probabilities of the system with a given probability of its remaining in the final state (stationary case)**

*Preliminary step (Step 0):* Fix the vector $\pi'(0) = (\pi'_1(0),\ \pi'_2(0),\ \ldots,\ \pi'_n(0))$, where $\pi'_i(0) = 0$ for $i \neq i_y$ and $\pi'_{i_y}(0) = q(y)$.

*Genrral step (Step $\tau + 1, \tau \geq 0$):* For given $\tau$ calculate

$$\pi'(\tau + 1) = P\pi'(\tau)$$

and then put

$$\pi'_{i_y}(\tau + 1) = q(y).$$

If $\tau < t - 1$ then go to next step; otherwise STOP.

*Remark* 1. All results and algorithms described above are also valid for the stochastic processes in the case when $\sum\limits_{z \in X} p_{x,z} = r(x) \leq 1$ for $x \in X$.

## 2.2 Polynomial time algorithms for determining the limit state matrix in Markov chains

Denote by $S = (s_{i,j})$ the limit matrix of probabilities for the Markov chain induced by stochastic matrix $P = (p_{x,y})$. We denote the vector columns of the matrix $S$ by

$$S^j = \begin{pmatrix} s_{1,j} \\ s_{2,j} \\ \vdots \\ s_{n,j} \end{pmatrix}, \quad j = 0, 1, 2, \ldots, n,$$

and the row vectors of the matrix $S$ we denote by $S_i = (s_{i,1}, s_{i,2}, \ldots, s_{i,n}), \quad i = 1, 2, \ldots, n$. To describe the algorithms for finding the limit matrix $S$ for non-ergodic Markov process we need to analyze the structure of the graph of transition probabilities and to study the behavior of the algorithms from the previous subsection in the case $t \to \infty$. First of all we note that for the ergodic Markov chain the graph $GR$ is strongly connected and all vector rows $S_i, \quad i = 1, 2, \ldots, n$, are the same. In this case the limit state probabilities can be found by solving the system of linear equations

$$\pi = \pi P, \quad \sum_{J=1}^{n} \pi_j = 1,$$

i.e. $S_i = \pi, \quad i = 1, 2, \ldots, n$. In general, such an approach can be used for an arbitrary ergodic Markov process if the limit state probabilities exist.

In the multichain Markov processes the graph $GR = (X, ER)$ consists of several strongly connected components $G^1 = (X^1, E^1)$, $G^2 = (X^2, E^2)$, $\ldots$, $G^k = (X^k, E^k)$ where $\bigcup\limits_{i=1}^{k} X^i = X$. Additionally, among these components, there are such strongly connected components $G^{i_p} = (X^{i_p}, E^{i_p})$, $p = 1, 2, \dots, q$, which do not contain a leaving directed edge $e = (x, y)$ where $x \in X^{i_p}$ and $y \in X \setminus X^{i_p}$. We call such components $G^{i_p}$ *deadlock* components in $GR$. A characterization of the ergodic classes (recurrence classes) in the Markov process can be made in terms of a graph of transition probabilities using the deadlock components.

**Lemma 3.** *If $G^{i_p} = (X^{i_p}, E^{i_p})$ is a strongly connected deadlock component in $GR$ then $X^{i_p}$ is an ergodic class (recurrence chain) of the Markov process; if $x \in X \setminus \bigcup_{p=1}^{q} X^{i_p}$ then $x$ is a transient state of the system in the Markov process.*

Lemma [3] reflects the well known properties of the Markov chains from [3–5, 10] in the terms of graphs of transition probabilities. The proof of the lemma follows from [3–5, 10].

Below we give some auxiliary results which can be obtained from the algorithmic procedure from the previous subsection in the case $t \to \infty$. Let a Markov process with a finite set of states $X$ be given. For an arbitrary state $x_j \in X$ we denote by $X_j$ the subset of states $x_k \in X$ for which in $GR$ there exists at least a directed path from $x_k$ to $x_j$. Additionally, we denote $N = \{1, 2, \dots, n\}$, $I(X_j) = \{k | x_k \in X_j\}$.

**Lemma 4.** *Let a Markov process with a finite set of states $X$ be given and assume that $x_j$ is an absorbing state. Let $\pi^j$ be a solution of the following system of linear equations*

$$\pi^j = P\pi^j; \quad \pi_{j,j} = 1; \quad \pi_{i,j} = 0 \quad for \quad i \in N \setminus I(X_j), \tag{4}$$

*where*

$$\pi^j = \begin{pmatrix} \pi_{1,j} \\ \pi_{2,j} \\ \vdots \\ \pi_{n,j} \end{pmatrix}.$$

*Then $\pi^j = S^j$, i.e. $\pi_{i,j} = s_{i,j}$, $i = 1, 2, \dots, n$. If $x_j$ is a unique absorbing state of the Markov process and if $x_j$ in $GR$ is attainable from every $x_i \in X$ (i.e. $I(X_j) = N$) then $\pi_{i,j} = s_{i,j} = 1$, $i = 1, 2, \dots, n$.*

*Proof.* We apply Algorithm 2 with respect to a given absorbing state $x_j$ ($y_j = x_j$) when $t \to \infty$. Then $\pi(t)' \to \pi^j$ and therefore we obtain $\pi^j = P\pi^j$ where $\pi_{j,j} = 1$ and $\pi_{i,j} = 0$ for $i \in N \setminus I(Y_j^+)$. The correctness of the second part of the lemma corresponds to the case when $I(X_j) = N$ and therefore we obtain that the vector $\pi^j$ with the components $\pi_{i,j} = 1$, $i = 1, 2, \dots, n$ is the solution of the system $\pi^j = P\pi^j, \pi_{j,j} = 1$. So, Lemma 4 holds. $\square$

*Remark* 2. If $x_j$ is not an absorbing state then Lemma 4 may fail to hold.

*Remark* 3. Lemma 4 can be extended for the case when $\sum_{y \in X} p_{x_i,y} = r(x_i) \le 1$ for some states $x_i \in X$. The solution of the system (4) in this case also coincides with the vector of limit probabilities $S^j$ if such a vector of limit probabilities exists.

However, if $N = I(X_j)$ then some components $\pi_{i,j}$ of the solution $\pi_{ij}$ may be less than 1.

Let us show that the result formulated above allows us to find the vector of limit probabilities $S^j$ of the matrix $S$ if the diagonal element $s_{j,j}$ of $S$ is known. We consider the subset of the states $Y^+ = \{x_j | s_{j,j} \geq 0\}$. It is easy to observe that $Y^+ = \bigcup\limits_{p=1}^{q} X^{i_p}$ ; we denote the corresponding set of indexes of this set by $I(Y^+)$. For each $j \in I(Y^+)$ we define the set $X_j$ in the same way as we introduced it above.

**Lemma 5.** *If a non-zero diagonal element $s_{j,j}$ of the limit matrix $S$ in the non-ergodic Markov process is known, i.e. $s_{j,j} = q(x_j)$, then the corresponding vector $S^j$ of the matrix $S$ can be found by solving the following systems of linear equations:*

$$S^j = PS^j; \quad s_{j,j} = q(x_j); \quad s_{i,j} = 0 \quad for \quad i \in N \setminus I(X_j)$$

*Proof.* We apply Algorithm 3 with respect to the fixed final state $y_j = x_j \in X$ with $q(y_j) = s_{j,j}$ when $t \to \infty$. Then for a given $y_j = x$ we have $\pi(t)' \to S^j$ and therefore we obtain $S^j = PS^j$ where $q(y_j) = s_{j,j}$ and $s_{i,j} = 0$ for $i \in N \setminus I(X_j)$. So, Lemma 5 holds. □

Basing on this lemma and Algorithm 3 we can prove the following result.

**Theorem 2.** *The limit state matrix $S$ for aperiodic Markov chains can be found by using the following algorithm:*

*1) For each ergodic class $X^{i_p}$ solve the system of linear equations*

$$\pi^{i_p} = \pi^{i_p} P^{i_p}, \qquad \sum_{j \in I(X^{i_p})} \pi_j^{i_p} = 1,$$

*where $\pi^{i_p}$ is the row vector with components $\pi_j^{i_p}$ for $j \in I(X^{i_p})$ and $P^{i_p}$ is the submatrix of $P$ induced by the class $X^{i_p}$. Then for every $j \in I(X^{i_p})$ put $s_{j,j} = \pi_j^{i_p}$; for each $j \in I(X \setminus \bigcup\limits_{p=1}^{q} X^{i_p})$ set $s_{j,j} = 0$;*

*2) For every $j \in I(Y^+)$, $\quad Y^+ = \bigcup\limits_{p=1}^{q} X^{i_p}$ solve the system of linear equations*

$$S^j = PS^j; \quad s_{j,j} = \pi_{j,j}; \quad s_{i,j} = 0 \quad for \quad i \in N \setminus I(X_j)$$

*and determine the vector $S^j$. For every $j \in I(X \setminus Y^+)$ set $S^j = \mathbf{0}$, where $\mathbf{0}$ is the vector row with zero components.*

*The algorithm finds the matrix $S$ using $O(n^4)$ elementary operations.*

*Proof.* Let us show that the algorithm finds correctly the limit matrix $S$. Item 1) of the algorithm finds the limit probabilities $s_{j,j}$. This item is based on Lemma 3 and on the conditions which each ergodic class $X^{i_p}$ and each transient state $x_\in X \setminus Y^+$

should satisfy. So, item 1) correctly finds the limit probabilities $s_{i,j}$ for $j \in N$. Item 2) of the algorithm is based on Lemma 5 and therefore determines correctly the vectors $S^j$ of the matrix $S$ when the diagonal elements $s_{j,j}$ are known. So, the algorithm finds correctly the limit matrix $S$ of the non-ergodic Markov processes if such a limit matrix exists. The computational complexity of the algorithm is determined by the computational complexity of solving $q \leq n$ equations for each ergodic class $X^{i_p}$(item 1) and the computational complexity of solving not more than $n$ systems of linear equations for determining the vectors $S^j$ (item 2). So, the running time of the algorithm is $O(n^4)$. □

Basing on this theorem we can find the limit matrix $S$ using algorithm from Theorem 2. In the worst case the running time of the algorithm is $O(n^4)$ however intuitively it is clear that the upper bound of this estimation couldn't be reached. Practically this algorithm efficiently finds the limit matrix $S$. In the following we show that for determining the limit matrix in aperiodoc Markov chains there exists an algorithm with computational complexity $O(n^3)$.

## 2.3 An algorithm for the calculation of the limit matrix in aperiodic Markov chains with running time $O(n^3)$

We describe another algorithm for finding the limit matrix for aperiodic Marcov chains which in the most part takes into account the structure properties of the random graph of the Markov process. We can see that such an approach allows us to ground an algorithm with computational complexity $O(n^3)$.

**Algorithm 4: Determining the limit state matrix for non-ergodic Markov processes**

The algorithm consists of two parts: The first part determines the limit probabilities $s_{x,y}$ for $x \in \bigcup_{p=1}^{q} X^{i_p}$ and $y \in X$. The second procedure calculates the limit probabilities $s_{x,y}$ for $x \in X \setminus \bigcup_{p=1}^{q} X^{i_p}$ and $y \in X$.

*Procedure 1:*

1. For each ergodic class $X^{i_p}$ we solve the system of linear equations:

$$\pi^{i_p} = \pi^{i_p} P^{i_p}, \quad \sum_{y \in X^{i_p}} \pi_y^{i_p} = 1,$$

   where $P^{i_p}$ is the matrix of probability transitions corresponding to the ergodic class $X^{i_p}$, i.e. $P^{i_p}$ is a submatrix of $P$, and $\pi^{i_p}$ is a row vector with the components $\pi_y^{i_p}$ for $y \in X^{i_p}$. If $\pi_y^{i_p}$ are known then $s_{x,y}$ for $x \in X^{i_p}$ and $y \in X$ can be calculated as follows:

   Set $s_{x,y} = \pi_y^{i_p}$ if $x, y \in X^{i_p}$ and $s_{x,y} = 0$ if $x \in X^{i_p}$, $y \in X \setminus X^{i_p}$.

*Procedure 2:*

1. We construct an auxiliary acyclic directed graph $GA = (XA, EA)$ which is obtained from the graph $GR = (X, ER)$ by using the following transformations:

   We contract each set of vertices $X^{i_p}$ into one vertex $z^{i_p}$ where $X^{i_p}$ is a set of vertices of a strongly connected deadlock component $G^{i_p} = (X^{i_p}, E^{i_p})$ in $GR$. If the obtained graph contains parallel directed edges $e^1 = (x, z)$, $e^2 = (x, z)$, ..., $e^r = (x, z)$ with the corresponding probabilities $p^1_{x,z}$, $p^2_{x,z}$, ..., $p^r_{x,z}$ then we change them by one directed edge $e = (x, z)$ with the probability $p_{x,z} = \sum_{i=1}^{r} p^i_{x,z}$; after this transformation of each vertex $z^i_p$ we put equivalently a directed edge of the form $e = (z^p, z^p)$ with the probability $p'_{z^p,z^p} = 1$.

2. We fix the directed graph $GA = (XA, EA)$ obtained by the construction principle from step 1 where $XA = \Big(X \setminus (\bigcup_{p=1}^{q} X^{i_p})\Big) \cup Z^p$, $\quad Z^p = (z^1, z^2, \dots, z^q)$. Additionally, we fix the new probability matrix $P' = (p'_{x,y})$ which corresponds to this random graph $GA$.

3. For each $x \in XA$ and every $z^i \in Z^p$ we find the probability $\pi'_x(z^i)$ of the system transaction from the state $x$ to the state $z^p$. The probabilities $\pi'_x(z^i)$ can be found by solving the following $p$ systems of linear equations:

$$P'\pi'(z^1) = \pi'(z^1), \quad \pi'_{z^1}(z^1) = 1, \quad \pi'_{z^2}(z^1) = 0, \quad \dots, \quad \pi'_{z^q}(z^1) = 0;$$

$$P'\pi'(z^2) = \pi'(z^2), \quad \pi'_{z^1}(z^2) = 0, \quad \pi'_{z^2}(z^2) = 1, \quad \dots, \quad \pi'_{z^q}(z^2) = 0;$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$P'\pi'(z^q) = \pi'(z^q), \quad \pi'_{z^1}(z^q) = 0, \quad \pi'_{z^2}(z^q) = 0, \quad \dots, \quad \pi'_{z^q}(z^q) = 1,$$

   where $\pi'(z^i)$, $\quad i = 1, 2 \dots, p$ are the column vectors with components $\pi'_x(z^i)$ for $x \in XA$. So, each vector $\pi'_x(z^i)$ defines probabilities of system transitions from states $x \in XA$ to the ergodic class $X^i$.

4. We put $s_{x,y} = 0$ for every $x, y \in X \setminus \bigcup_{p=1}^{q} X^{i_p}$ and $s_{x,y} = \pi'_x(z^p)\pi^{i_p}_y$ for every $x \in X \setminus \bigcup_{p=1}^{q} X^{i_p}$ and $y \in X^{i_p}$, $X^{i_p} \subset X$. If $x \in X^{i_p}$ and $y \in X \setminus X^{i_p}$ then we fix $s_{x,y} = 0$.

**Theorem 3.** *The algorithm correctly finds the limit state matrix $S$ and the running time of the algorithm is $O(|X|^3)$.*

*Proof.* The correctness of Procedure 1 of the algorithm follows from the definition of the ergodic Markov class (recurrence chain). So, Procedure 1 finds the probabilities $s_{x,y}$ for $x \in \bigcup_{p=1}^{q} X^{i_p}$ and $y \in X$. Let us show that Procedure 2 correctly finds the rest elements $s_{x,y}$ of the matrix $S$. Indeed, each vertex $x \in X \setminus \bigcup_{p=1}^{q} X^{i_p}$ in $GA$ corresponds to a transient state of the Markov chain and therefore we have $s_{x,y} = 0$ for every $x, y \in X \setminus \bigcup_{p=1}^{q} X^{i_p}$. If $x \in X^{i_p}$ then the system couldn't reach a state $y \in X \setminus X^{i_p}$ and therefore for arbitrary two states $x, y$ we have $s_{x,y} = 0$. Finally, we show that the algorithm correctly determines the limit probability $s_{x,y}$ if $x \in X \setminus \bigcup_{p=1}^{q} X^{i_p}$ and $y \in X^{i_p}$. In this case the limit probability $s_{x,y}$ is equal to the limit probability of the system to reach the ergodic class $X^{i_p}$ multiplied by the limit probability of the system to remain in the state $y \in X^{i_p}$, i.e. $s_{x,y} = \pi'_x(z^p)\pi_y^{i_p}$. Here $\pi_y^{i_p}$ is the probability of the system to remain in the state $y \in X^{i_p}$ and $\pi_x(z^{i_p})$ is the limit probability of the system to reach the absorbing state $z_{i_p}$ in $GA$. The value $\pi_x(z^{i_p})$ according to the construction of auxiliary graph $GA$ coincides with the limit probability of the system to reach the ergodic class $X^{i_p}$. The correctness of this fact can easily be obtained from Lemma 3 and Theorem 2. According to Lemma 3 the probabilities $\pi_x(z^p)$ for $x \in X \setminus \bigcup_{p=1}^{q} X^{i_p}$ can be found by solving the following system of linear equations

$$P'\pi'(z^p) = \pi'(z^p), \quad \pi'_{z^1}(z^p) = 0, \quad \pi'_{z^2}(z^p) = 0, \quad \dots, \quad \pi'_{z^p}(z^p) = 1,$$

which determined them correctly. So, the algorithm correctly finds the limit state matrix $S$.

Now let us show that the running time of the algorithm is $O(n^3)$. We obtain this estimation in the item 4 solving $q \leq n$ systems of linear equations. Each of these systems contains no more than $n$ variables. All these systems have the same left part and therefore they can be solved simultaneously applying Gaussian method. The simultaneous solution of these $q$ systems with the same left part by using Gaussian method uses $O(n^3)$ elementary operations. □

## 3 Determining the State Probabilities of the Dynamical System in Non-Stationary Markov Processes

In the case when the probabilities of system's transitions from one state to another depend on time we have a non-stationary process defined by a dynamic matrix $P(t) = (p_{x,y}(t))$ which describes this process. If this matrix is stochastic for every moment of time $t = 1, 2, \dots,$ then the state probabilities $P_{x_{i_0}}(x, t)$ can be defined and calculated by using a similar formula obtained from Section 1 changing $p_{x,y}$ by

$p_{x,y}(\tau)$, i.e.

$$P_{x_{i_0}}(x, \tau+1) = \sum_{y \in X} P_{x_{i_0}}(y, \tau) p_{y,x}(\tau), \quad \tau = 0, 1, 2, \dots, t-1$$

where $P_{x_{i_0}}(x_{i_0}, 0) = 1$ and $P_{x_{i_0}}(x, 0) = 0$ for $x \in X \setminus \{x_{i_0}\}$. In the matrix form this formula can be represented as follows

$$\pi(\tau+1) = \pi(\tau)P, \quad \tau = 0, 1, 2, \dots, t-1$$

where $\pi(\tau) = (\pi_1(\tau), \pi(\tau), \dots, \pi_n(\tau))$ is the vector with the components $\pi_i(\tau) = P_{x_{i_0}}(x_i, \tau)$. At the starting moment of time $\tau = 0$ the vector $\pi(\tau)$ is given in the same way as for the stationary process, i.e. $\pi_{i_0}(0) = 1$ and $\pi_i(0) = 0$ for arbitrary $i \neq i_0$. If for a given starting vector $\pi(0)$ and $\tau = 0, 1, 2, \dots, t-1$ we apply this formula then we obtain

$$\pi(t) = \pi(0)P(0)P(1)P(2)\dots P(t-1).$$

So, an arbitrary element $q_{x_i,x_j}(t)$ of the matrix

$$Q(t) = P(0)P(1)P(2)\dots P(t-1)$$

expresses the probability of system $L$ to reach the state $x_j$ from $x_i$ by using $t$ units of times.

Now let us show how to calculate the probability $P_{x_{i_0}}(y, T_1 \leq T(y) \leq T_2)$ in the case of non-stationary Markov processes. In the same way as for the stationary case we consider the non-stationary Markov process with given absorbing state $y \in X$. So, we assume that the dynamic matrix $P(t)$ is given which is stochastic for every $t = 0, 1, 2, \dots$ and $p_{y,y}(t) = 1$ for arbitrary $t$ is given. Then the probabilities $P_x(y, 0 \leq T(y) \leq t)$ for $x \in X$ can be determined if we tabulate the values $P_x(y,\, t-\tau \leq T(y) \leq t), \quad \tau = 0, 1, 2 \dots, t$, using the following recursive formula:

$$P_x(y,\, t-\tau-1 \leq T(y) \leq t) = p_{x,z}(t-\tau-1)P_z(y,\, t-\tau \leq T(y) \leq t)$$

where for $\tau = 0$ we fix

$$P_x(y,\, t \leq T(y) \leq t) = 0 \text{ if } x \neq y \text{ and } P_y(y,\, t \leq T(y) \leq t) = 1.$$

This recursive formula can be represented in the following matrix form

$$\pi''(t-\tau-1) = P(t-\tau-1)\pi''(\tau), \quad t = 0, 1, 2, \dots t-1.$$

At the starting moment of time $t = 0$ the vector $\pi''(0)$ is given: All components are equal to zero except the component corresponding to the absorbing vertex which is equal to one, i.e.

$$\pi''_i(0) = \begin{cases} 0, & \text{if } x_i \neq y; \\ 1, & \text{if } x_i = y. \end{cases}$$

If we apply this formula for $\tau = 0, 1, 2, \dots, t-1$ then we obtain

$$\pi''(t) = P(0)P(1)P(2)\cdots P(t-1)\pi''(0), \quad t = 1, 2, \dots .$$

So, if we consider the matrix $Q = P(0)P(1)P(2)\dots P(t-1)$ then an arbitrary element $q_{i,j_y}$ of the column $j_y$ in the matrix $Q$ expresses the probability of the system $L$ to reach the state $y$ from $x_i$ by using not more than $t$ units of time, i.e. $q_{i,j_y} = P_{x_i}(y,\, 0 \le T(x) \le t)$.

Here the matrix $P(t)$ is stochastic matrix for $t = 0, 1, 2, \dots$ where $p_{y,y}(t) = 1$ for every $t$ and

$$\pi''(\tau) = \begin{pmatrix} \pi''_1(\tau) \\ \pi''_2(\tau) \\ \vdots \\ \pi''_n(\tau) \end{pmatrix}, \quad \tau = 0, 1, 2, \dots$$

is the column vector, where an arbitrary component $\pi''_i(\tau)$ expresses the probability of the dynamical system to reach the state $y$ from $x_i$ by using not more than $\tau$ unites of times when the system start transitions in the sate $x$ at the moment of time $t-\tau$, i.e. $\pi''_i(\tau) = P_{x_i}(y,\, t-\tau \le T(y) \le t)$. This means that in the case when $y$ is an absorbing state the probability $P_x(y,\, T_1 \le T(y) \le T_2)$ can be found in the following way:

$a)$ find the matrices

$$Q^1 = P(0)P(1)P(2)\cdots P(T_1 - 1) \text{ and } Q^2 = P(0)P(1)P(2)\cdots P(T_2 - 1);$$

$b)$ calculate

$$P_x(y,\, T_1 \le T(y) \le T_2) =$$
$$= P_x(y,\, 0 \le T(y) \le T_2) - P_x(y,\, 0 \le T(y) \le T_1 - 1) = q^2_{i_x j_y} - q^2_{i_x j_y}.$$

The results described above allows to develop algorithms for calculation the probabilities $P_x(y,\, 0 \le T(y) \le t)$ for an arbitrary non-stationary Markov process. Such algorithms can be obtained if in the general steps of the algorithms we change the matrix $P$ by the matrix $P(t-\tau-1)$ and $\pi'(\tau)$ by $\pi''(\tau)$.

Below we describe these algorithms which can be grounded in an analogues way as the algorithms in Section 2.

**Calculation of the state probabilities of the system in the matrix form (non-stationary case)**

*Preliminary step (Step 0):* Fix the vector $\pi''(0) = (\pi''_1(0),\ \pi''_2(0),\ \dots,\ \pi''_n(0))$, where $\pi''_i(0) = 0$ for $i \ne i_y$ and $\pi''_{i_y}(0) = 1$.

*General step (Step $\tau+1,\ \tau \ge 0$):* For given $\tau$ calculate

$$\pi''(\tau+1) = P(t-\tau-1)\pi''(\tau)$$

and then put

$$\pi''_{i_y}(\tau+1) = 1.$$

If $\tau < t-1$ then go to next step; otherwise STOP.

**Calculation of the state probabilities of the system with given probability of its remaining in the final state (non-stationary case)**

*Preliminary step (Step 0):* Fix the vector $\pi''(0) = (\pi''_1(0),\ \pi''_2(0),\ \dots,\ \pi''_n(0))$, where $\pi''_i(0) = 0$ for $i \neq i_y$ and $\pi''_{i_y}(0) = 1$.

*General step (Step $\tau+1,\ \tau \geq 0$):* For given $\tau$ calculate

$$\pi''(\tau+1) = P(t-\tau-1)\pi''(\tau)$$

and then put

$$\pi''_{i_y}(\tau+1) = q(y).$$

If $\tau < t-1$ then go to next step; otherwise STOP.

Note that the algorithm finds the probabilities $P_x(y,\, 0 \leq T(y) \leq t)$ when the value $q(y)$ is given. We treat this value as the probability of the system to remain in the state $y$; for the case $q(y) = 1$ this algorithm coincides with previous one.

## 4 Conclusion

A new approach for studying finite Marcov processes and determining the limit matrix of probability transitions in Markov chains is proposed. The proposed approach allows us to develop new algorithms for determining the states probability in the considered Markov processes. Polynomial time algorithms for finding the limit matrix of probability transitions of the system in Markov chains are elaborated. These algorithms can be used for determining the average cost per transaction of dynamical system in decision Markov processes (Markov processes with rewards) [3, 5, 9] and and stochastic discrete optimal control problems [2, 6–8].

## References


[1] Bollobas B. *Random Graphs.* Second Edition, Cambridge University Press, 2001.

[2] Bellman R., Kalaba R. *Dynamic programming and modern control theory.* Academic Press, New York and London, 1965.

[3] Howard R. A. *Dynamic Programming and Markov Processes.* Wiley, 1960.

[4] Karlin S., Howard E. M. *A First Course in Stochastic Processes.* Academic Press, New York, 1975.

[5] Kemeny J. G, Snell J. L., Knapp A. W. *Denumarable Markov Chains.* New York, Springer Verlag, 1976.

[6] Lozovanu D., Pickl. S. *Optimization and Multiobjective Control of Time-Discrete Systems.* Springer Verlag, 2009.

[7] Lozovanu D., Pickl. S. *Dynamic Programming Algorithms for Solving Stochastic Discrete Control Problems.* Bulletin of the Academy of Sciences of Moldova. Mathematics, 2009, No. 2(60), 73–90.

[8] Lozovanu D., Pickl S. *Algorithmic Solutions of Discrete Control Problems on Stochastic Networks.* Procceedings of CTW09 Workshop on Graphs and Combinatorial Optimization, Paris, 2009, 221–224.

[9] Puterman M. *Markov Decision Processes: Stochastic Dynamic Programming.* John Wiley, New Jersey, 2005.

[10] Stewart W. J. *Introduction to the Numerical Solution of Markov Chains.* Princeton University Press, 1995.

Dmitrii Lozovanu
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
5 Academiei str., Chişinău, MD−2028
Moldova
E-mail: *lozovanu@math.md*

*Received October 6, 2010*

Stefan Pickl
Institut fur Angewandte Systemwissenschaften
und Wirtschaftsinformatik
Fakultät fur Informatik, Universität der Bundeswehr
München
E-mail: *stefan.pickl@unibw.de*

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(65), 2011, Pages 83–94
ISSN 1024–7696

# On stability radius of the multicriteria variant of Markowitz's investment portfolio problem

Vladimir Emelichev, Vladimir Korotkov

**Abstract.** Basing on Markowitz's classical theory we formulate a multicriteria Boolean portfolio optimization problem with Savage's minimax (bottleneck) risk criteria. We obtain lower and upper attainable bounds for stability radius of the problem of finding the Pareto set, consisting of efficient portfolios in the case of Chebyshev metric $l_\infty$ in the risk and state spaces, and linear metric $l_1$ in the portfolios space.

**Mathematics subject classification:** 90C09, 90C29, 90C31, 90C47.
**Keywords and phrases:** multicriteria optimization, investment portfolio, Markowitz's problem, Pareto set, efficient portfolio, Savage's risk criteria, bottleneck criteria, stability radius of the problem.

## 1 Introduction

In paper [1] we obtained lower and upper attainable bounds for the stability radius of a Pareto optimal portfolio of the multicriteria Boolean investment problem with Savage's minimax (bottleneck) risk criteria in the case of Chebyshev metric $l_\infty$ in every operation factors space of the problem. In present paper we obtain results of similar nature for stability radius of the Markowitz's investment problem of finding Pareto set in the case of linear metric $l_1$ in the portfolios space, and Chebyshev metric $l_\infty$ in the risk and market state spaces.

## 2 Problem statement and basic definitions

Basing on the Markowitz's portfolio theory [2] we consider the vector variant of the financial managing problem. To this end, we introduce the following notations:

$N_n = \{1, 2, ..., n\}$ be a set of assets (stocks, bonds, real estate, etc.);

$N_m$ be a set of market states;

$N_s$ be a set of risks (financial, environmental, industrial etc.);

$R$ be a three-dimensional $m \times n \times s$ risk matrix (a matrix of missed opportunities) with elements $r_{ijk} \in \mathbf{R}$,

$r_{ijk}$ be a risk quantity of assets $j \in N_n$ chosen by the investor under criterion (type of risk) $k \in N_s$ in the situation when the market was in state $i \in N_m$;

$x = (x_1, x_2, \ldots, x_n)^T \in X \subseteq \mathbf{E}^n$ be an investment portfolio, where $\mathbf{E} = \{0, 1\}$,

$$x_j = \begin{cases} 1 & \text{if the investor chooses asset } j, \\ 0 & \text{otherwise.} \end{cases}$$

Along with the three-dimensional matrix $R = [r_{ijk}] \in \mathbf{R}^{m\times n\times s}$ we use its two-dimensional cuts $R_k \in \mathbf{R}^{m\times n},\ k \in N_s$.

Let the following vector objective function

$$f(x, R) = (f_1(x, R_1), f_2(x, R_2), ..., f_s(x, R_s))$$

estimate efficiency of the choosing portfolio (boolean vector) $x \in X$, $|X| \geq 2$ with *Savage's minimax risk* (extreme pessimism) *criteria* [3]

$$f_k(x, R_k) = \max_{i\in N_m} \sum_{j\in N_n} r_{ijk}x_j \to \min_{x\in X}, \qquad k \in N_s.$$

Thus, Savage's bottleneck criteria under uncertainty of the market state recommends choosing the portfolio in which the total risk value takes the smallest value in the most unfavorable situation, namely, when the risk is the greatest.

Note that minimax (maximin) problems are quite typical for optimization theory. For example, monographs [4–6] are devoted to similar types of optimization problems. These problems include also the problem on the best uniform approximation of functions originally posed by Chebyshev.

A problem of finding *the Pareto set* $P^s(R)$ which contains all efficient (Pareto optimal) portfolios will be viewed as a multicriteria *investment portfolio problem* $Z^s(R)$:

$$P^s(R) = \{x \in X :\ P^s(x, R) = \emptyset\},$$

where

$$P^s(x, R) = \{x' \in X :\ x \ \succ\ x'\}.$$

Here the symbol $\succ$ is a binary relation over the set $X$ which is defined as follows:

$$x \ \succ\ x' \quad \Leftrightarrow \quad g(x, x', R) \geq \mathbf{0} \ \ \& \ \ g(x, x', R) \neq \mathbf{0},$$

where

$$\mathbf{0} = (0, 0, ..., 0) \in \mathbf{R}^s,$$

$$g(x, x', R) = (g_1(x, x', R_1), g_2(x, x', R_2), ..., g_s(x, x', R_s)),$$

$$g_k(x, x', R_k) = f_k(x, R_k) - f_k(x', R_k) = \max_{i\in N_m} R_{ik}x - \max_{i\in N_m} R_{ik}x' =$$

$$= \min_{i'\in N_m} \max_{i\in N_m} (R_{ik}x - R_{i'k}x'),\ k \in N_s,$$

and $R_{ik} = (r_{i1k}, r_{i2k}, ..., r_{ink})$ is $i$-th row of cut $R_k \in \mathbf{R}^{m\times n}$.

It is obvious that $P^s(R) \neq \emptyset$ for any matrix $R \in \mathbf{R}^{m\times n\times s}$.

We define linear metric $l_1$ in the portfolios space $\mathbf{R}^n$. We also define metric $l_\infty$ (Chebyshev metric) in the state space $\mathbf{R}^m$ and risk (criteria) space $\mathbf{R}^s$, i.e. we assume that

$$\|R_{ik}\|_1 = \sum_{j\in N_n} |r_{ijk}|, \quad i \in N_m, \quad k \in N_s,$$

$$\|R_k\| = \max_{i\in N_m} \|R_{ik}\|_1 = \max_{i\in N_m} \sum_{j\in N_n} |r_{ijk}|, \quad k \in N_s,$$

$$\|R\| = \max_{k\in N_s} \|R_k\| = \max_{k\in N_s} \max_{i\in N_m} \|R_{ik}\|_1 = \max_{k\in N_s} \max_{i\in N_m} \sum_{j\in N_n} |r_{ijk}|.$$

In this notation for any indexes $i \in N_m$ and $k \in N_s$ the following inequalities are obvious:

$$\|R_{ik}\|_1 \le \|R_k\| \le \|R\|.$$

Moreover for any cut $R_k \in \mathbf{R}^{m\times n}$ and any vectors $x, x' \in X$ the following inequalities are true:

$$R_{ik}x - R_{i'k}x' \ge -\|R_{ik}\|_1 - \|R_{i'k}\|_1 \ge -2\|R_k\|, \quad i, i' \in N_m, \quad k \in N_s. \tag{1}$$

Similar to [9–13], *the stability radius of problem* $Z^s(R)$, $s \ge 1$, the perturbing parameters of the vector criteria, i.e. the perturbing elements of the risk matrix $R$ is defined as:

$$\rho = \rho(n, m, s) = \begin{cases} \sup\ \Xi & \text{if } \Xi \ne \emptyset, \\ 0 & \text{if } \Xi = \emptyset, \end{cases}$$

where

$$\Xi = \{\varepsilon > 0 :\ \forall R' \in \Omega(\varepsilon) \quad (P^s(R + R') \subseteq P^s(R))\},$$

$$\Omega(\varepsilon) = \{R' \in \mathbf{R}^{m\times n\times s} :\ \|R'\| < \varepsilon\}.$$

Thus, the problem stability radius is the supremum level of risk matrix perturbations such that new efficient portfolios do not appear. In other words, $\rho(n, m, s)$ is the radius of such type of stability, which is a discrete analogue of the upper Hausdorff semicontinuity of point-set mapping [14] which puts in correspondence the set of efficient portfolios to each point of the space of problem parameters.

Here $\Omega(\varepsilon)$ is *the set of perturbing matrixes*, and $Z^s(R + R')$ is *the perturbed problem.*

Obviously, the stability radius $\rho = \rho(n, m, s)$ is infinite as the equality $P^s(R) = X$ holds. Further we exclude this case from our consideration. And if the set $X \setminus P^s(R)$ is non-empty we say that the problem $Z^s(R)$ is non-trivial.

## 3 Problem stability radius bounds

Let

$$\varphi = \varphi(n, m, s) = \min_{x\notin P^s(R)} \max_{x'\in P^s(x,R)} \min_{k\in N_s} \min_{i'\in N_m} \max_{i\in N_m} (R_{ik}x - R_{i'k}x').$$

Considering that for any portfolio $x \notin P^s(R)$ the set $P^s(x, R)$ is non-empty, then the following formula is true:

$$\forall x \notin P^s(R) \quad \forall x' \in P^s(x, R) \qquad (x \succ x').$$

Hence, this results in $\varphi \geq 0$.

**Theorem 1.** *For the stability radius $\rho(n, m, s)$, $s \geq 1$, of a multicriteria non-trivial problem $Z^s(R)$ the following bounds are true:*

$$\varphi(n, m, s)/2 \leq \rho(n, m, s) \leq n\varphi(n, m, s).$$

*Proof.* To prove Theorem 1 it is necessary first to prove that $\rho \geq \varphi/2$. It is evident if $\varphi = 0$. Let $\varphi > 0$. According to the definition of $\varphi$ for any portfolio $x \notin P^s(R)$ (if problem $Z^s(R)$ is non-trivial then there exist such portfolios) there exists portfolio $x^0 \in P^s(x, R)$ such that

$$\min_{i' \in N_m} \max_{i \in N_m} (R_{ik}x - R_{i'k}x^0) \geq \varphi, \quad k \in N_s. \tag{2}$$

Further, taking into account (1) and (2), for any matrix $R' \in \mathbf{R}^{m \times n \times s}$ and any index $k \in N_s$ we have:

$$g_k(x, x^0, R_k + R'_k) = \max_{i \in N_m}(R_{ik} + R'_{ik})x - \max_{i' \in N_m}(R_{i'k} + R'_{i'k})x^0 =$$

$$= \min_{i' \in N_m} \max_{i \in N_m} (R_{ik}x - R_{i'k}x^0 + R'_{ik}x - R'_{i'k}x^0) \geq \min_{i' \in N_m} \max_{i \in N_m} (R_{ik}x - R_{i'k}x^0) - 2\|R'_k\| \geq$$

$$\geq \varphi - 2\|R'_k\|.$$

Hence, in view of $R' \in \Omega(\varphi/2)$, i.e. $\|R'_k\| < \varphi/2$, $k \in N_s$, we obtain

$$g_k(x, x^0, R_k + R'_k) > 0, \quad k \in N_s.$$

Thus $x$ is not efficient portfolio of the perturbed problem $Z^s(R + R')$.

Summarizing and considering $x \notin P^s(R)$ we conclude that

$$\forall R' \in \Omega(\varphi/2) \quad (P^s(R + R') \subseteq P^s(R)).$$

Hence, we have the inequality $\rho \geq \varphi/2$.

Further, we prove the inequality $\rho \leq n\varphi$. Accordance to the definition of $\varphi$ there exists $x^0 \notin P^s(R)$ such that for any $x \in P^s(x^0, R)$ there exists $q = q(x) \in N_s$ such that

$$\min_{i \in N_m} \max_{i^0 \in N_m} (R_{i^0q}x^0 - R_{iq}x) \leq \varphi. \tag{3}$$

Now, setting $\varepsilon > n\varphi$, we consider a perturbing matrix $R^0 = [r^0_{ijk}] \in \mathbf{R}^{m \times n \times s}$ whose elements are defined as follows:

$$r^0_{ijk} = \begin{cases} -\delta & \text{if } i \in N_m,\ x^0_j = 1,\ k \in N_s, \\ \delta & \text{otherwise,} \end{cases}$$

where $\varphi < \delta < \varepsilon/n$. Then

$$\|R^0\| = \|R^0_k\| = \|R^0_{ik}\|_1 = n\delta,\ i \in N_m,\ k \in N_s.$$

Therefore $R^0 \in \Omega(\varepsilon)$. In addition, all rows $R^0_{ik}$, $i \in N_m$, of cut $R^0_k$ are equal and consist of components $\delta$ and $-\delta$ for any index $k \in N_s$. Thus, denoting this row as $B$ (it only depends on $x^0$), we have

$$\|B\|_1 = n\delta,$$

$$B(x^0 - x) = -\delta\|x^0 - x\|_1 \leq -\delta < -\varphi \leq 0. \tag{4}$$

Hence, in view of (3) and the structure of perturbing matrix $R^0$, we deduce that for any portfolio $x \in P^s(x^0, R)$ the relations

$$g_q(x^0, x, R_q + R^0_q) = \max_{i \in N_m}(R_{iq} + B)x^0 - \max_{i \in N_m}(R_{iq} + B)x =$$

$$= \min_{i^0 \in N_m} \max_{i \in N_m}(R_{iq}x^0 - R_{i^0q}x) + B(x^0 - x) < 0$$

hold.

As a result we obtain that

$$\forall x \in P^s(x^0, R) \quad (x \notin P^s(x^0, R + R^0)). \tag{5}$$

Let portfolio $x \notin P^s(x^0, R)$, i.e. the binary relation

$$x^0 \ \succ \ x$$

is not satisfied. Let us show, that $x \notin P^s(x^0, R + R^0)$. To do it we consider two possible case:

**Case 1.** $g(x^0, x, R) = \mathbf{0}$. Then for any index $k \in N_s$ equalities (4) imply

$$g_k(x^0, x, R_k + R^0_k) = \max_{i \in N_m}\ (R_{ik} + B)x^0 - \max_{i \in N_m}\ (R_{ik} + B)x =$$

$$= g_k(x^0, x, R_k) + B(x^0 - x) < 0.$$

**Case 2.** There exists index $l \in N_s$ such that $g_l(x^0, x, R_l) < 0$. Then, using (4) again, we have $g_l(x^0, x, R_l + R^0_l) < 0$.

Thus, if $x \notin P^s(x^0, R)$ then $x \notin P^s(x^0, R + R^0)$. Hence, in view of (5) we obtain $P^s(x^0, R + R^0) = \emptyset$. This means that $x^0$ is the efficient portfolio of the perturbed problem $Z^s(R + R^0)$. Then, taking into account $x^0 \notin P^s(R)$, we conclude that

$$\forall \varepsilon > n\varphi \quad \exists R^0 \in \Omega(\varepsilon) \qquad (P^s(R + R^0) \not\subseteq P^s(R)).$$

Therefore the inequality $\rho \leq n\varphi$ is true. □

The next statement directly follows from Theorem 1.

**Corollary 1.** *The stability radius $\rho(n,m,s)$ of problem $Z^s(R)$ is zero if and only if $\varphi(n,m,s)=0$.*

For $m=1$ our problem $Z^s(R)$ is transformed into a multicriteria Boolean programming problem with linear criteria

$$R_k x \to \min_{x\in X}, \quad k\in N_s, \tag{6}$$

where $R_k$ is $k$-th row of the matrix $R\in \mathbf{R}^{s\times n}$, $X\subseteq \mathbf{E}^n$, while metric $l_1$ in the solutions space $\mathbf{R}^n$ and metric $l_\infty$ in the criteria space $\mathbf{R}^s$.

In this case for $m=1$ the known theorem on the stability radius of an efficient solution of the vector integer problem, where the Pareto set has a unique efficient solution, holds.

**Theorem 2 [13].** *If $X\subset \mathbf{Z}^n$, $|X|<\infty$, $x^0$ is a unique efficient solution of problem* (6)*, then for stability radius the following formula is true:*

$$\rho(n,1,s)=\varphi(n,1,s)=\min_{x\in X\setminus\{x^0\}}\min_{k\in N_s} R_k(x-x^0).$$

## 4 Attainability of the lower bound

Let us show that lower bound for the problem stability radius, indicated in Theorem 1, is attainable.

**Theorem 3.** *For $m\geq 2$ and for $n\geq 2$ there exists a class of problems $Z^s(R)$, $s\geq 1$, such that for stability radius of every problem of this class the following formula is true:*

$$\rho(n,m,s)=\varphi(n,m,s)/2. \tag{7}$$

*Proof.* To prove the equality $\rho=\varphi/2$ in accordance with Theorem 1, it is sufficient to identify the class of non-trivial problems for which the inequality $\rho\leq\varphi/2$ is true. In what follows, we scrutinize this.

We consider the class of problems $Z^s(R)$ such that the following terms are right

$$X=\{x^0,\widehat{x}\}, \quad P^s(x^0,R)=\{\widehat{x}\}, \tag{8}$$

$$(R_{i(x^0)q}-R_{i(\widehat{x})q})x^0>\varphi/2, \tag{9}$$

where

$$i(\widehat{x})=\arg\max\{R_{iq}\widehat{x}:\ i\in N_m\},$$

$$i(x^0)=\arg\max\{R_{iq}x^0:\ i\in N_m\}.$$

Then the definition of $\varphi$ entails that

$$0 \le g_q(x^0, \widehat{x}, R_q) = \varphi, \tag{10}$$

and from inequality (9) we obtain $i(x^0) \neq i(\widehat{x})$. Further we suppose that there exist two indexes $p \neq l$ over the set $N_n$ such that $x_p^0 > \widehat{x}_p$, $x_l^0 < \widehat{x}_l$, i.e. the inequalities

$$x_p^0 = \widehat{x}_l = 1, \ x_l^0 = \widehat{x}_p = 0 \tag{11}$$

hold.

For any number $\varepsilon > \varphi/2$ we define the elements of the perturbing matrix $R^0 = [r_{ijk}^0] \in \mathbf{R}^{m\times n\times s}$ by the rule

$$r_{ijk}^0 = \begin{cases} \delta & \text{if } i = i(\widehat{x}), \ j = l, \ k = q, \\ -\delta & \text{if } i \in N_m \setminus \{i(\widehat{x})\}, \ j = p, \ k = q, \\ 0 & \text{otherwise}, \end{cases} \tag{12}$$

where

$$\varphi/2 < \delta < \min\left\{\varepsilon, (R_{i(x^0)q} - R_{i(\widehat{x})q})x^0\right\}. \tag{13}$$

It is worth to note that the last inequalities are correct because of (9).

Due to the structure of the perturbing matrix $R^0$ the next equalities are obvious:

$$R^0_{i(\widehat{x})q}\widehat{x} = \delta, \tag{14}$$

$$R^0_{iq}\widehat{x} = 0, \quad i \in N_m \setminus \{i(\widehat{x})\}, \tag{15}$$

$$R^0_{iq}x^0 = -\delta, \quad i \in N_m \setminus \{i(\widehat{x})\}, \tag{16}$$

$$R^0_{i(\widehat{x})q}x^0 = 0, \tag{17}$$

$$\|R^0_{iq}\|_1 = \|R^0_q\| = \|R^0\| = \delta, \quad i \in N_m, \ R^0 \in \Omega(\varepsilon).$$

The remainder of the proof will be dedicated to proving that portfolio $x^0 \in P^s(R + R^0)$. To this end, in view of (8) it suffices to show that $\widehat{x} \notin P^s(x^0, R + R^0)$. In line with (14) and (15), we have

$$f_q(\widehat{x}, R_q + R_q^0) = \max\Big\{(R_{i(\widehat{x})q} + R^0_{i(\widehat{x})q})\widehat{x}, \ \max_{i\neq i(\widehat{x})}(R_{iq} + R^0_{iq})\widehat{x}\Big\} =$$

$$= \max\Big\{f_q(\widehat{x}, R_q) + \delta, \ \max_{i\neq i(\widehat{x})} R_{iq}\widehat{x}\Big\} = f_q(\widehat{x}, R_q) + \delta. \tag{18}$$

It is easy to see the equality is hold:

$$f_q(x^0, R_q + R_q^0) = f_q(x^0, R_q) - \delta. \tag{19}$$

Indeed from (16) the following relations are true:

$$f_q(x^0, R_q + R_q^0) = \max\Big\{(R_{i(x^0)q} + R^0_{i(x^0)q})x^0,\ \max_{i \neq i(x^0)} (R_{iq} + R^0_{iq})x^0\Big\} =$$

$$= \max\Big\{f_q(x^0, R_q) - \delta,\ \max_{i \neq i(x^0)} (R_{iq} + R^0_{iq})x^0\Big\}.$$

Thus, taking into account obvious due to (16) relations

$$f_q(x^0, R_q) - \delta \geq (R_{iq} + R^0_{iq})x^0, \quad i \in N_m \setminus \{i(\widehat{x})\},$$

it remains to prove that

$$f_q(x^0, R_q) - \delta = \max_{i \in N_m} R_{iq}x^0 - \delta \geq (R_{i(\widehat{x})q} + R^0_{i(\widehat{x})q})x^0.$$

Using (13) and (17), we have

$$f_q(x^0, R_q) - \delta - (R_{i(\widehat{x})q} + R^0_{i(\widehat{x})q})x^0 = (R_{i(x^0)q} - R_{i(\widehat{x})q})x^0 - \delta > 0.$$

So, the equality (19) is true.

Finally, consistently applying (18), (19), (10) and (13), we obtain

$$g_q(x^0, \widehat{x}, R_q + R_q^0) = g_q(x^0, \widehat{x}, R_q) - 2\delta = \varphi - 2\delta < 0.$$

Therefore $\widehat{x} \notin P^s(x^0, R + R^0)$. It proves that $x^0$ is efficient portfolio of the perturbed problem $Z^s(R + R^0)$. Hence, because of $x^0 \notin P^s(R)$ we derive

$$\forall \varepsilon > \varphi/2 \quad \exists R^0 \in \Omega(\varepsilon) \qquad (P^s(R + R^0) \not\subseteq P^s(R)).$$

Thus $\rho \leq \varphi/2$. In summary, we have just proven that (7) is valid. □

We give a numeric example proving existence of scalar problem $Z^1(R)$, $R \in \mathbf{R}^{m \times n}$, which stability radius $\rho(n, m, 1)$ is $\varphi(n, m, 1)/2$.

**Example.** Let $m = 3$, $n = 3$, $s = 1$; $X = \{\widehat{x}, x^0\}$, $\widehat{x} = (1, 1, 0)^T$, $x^0 = (0, 1, 1)^T$;

$$R = \begin{pmatrix} 1 & -1 & 2 \\ -4 & 0 & 4 \\ 1 & -1 & 2 \end{pmatrix}$$

is risk matrix with rows $R_i$, $i \in N_m$. Then

$$i(\widehat{x}) = 1, \quad i(x^0) = 2,$$

$$f(\widehat{x}, R) = 0, \quad f(x^0, R) = 4.$$

Hence, $\widehat{x} \in P^1(R)$, $x^0 \notin P^1(R)$, $\varphi = 4$ and inequality (9) implies

$$(R_2 - R_1)x^0 = 3 > 2 = \varphi/2.$$

The perturbing matrix

$$R^0 = \begin{pmatrix} \delta & 0 & 0 \\ 0 & 0 & -\delta \\ 0 & 0 & -\delta \end{pmatrix}, \quad 2 < \delta < 3,$$

with rows $R_i$, $i \in N_m$, is constructed according (12).

Thus, taking into account the equality

$$R + R^0 = \begin{pmatrix} 1+\delta & -1 & 2 \\ -4 & 0 & 4-\delta \\ 1 & -1 & 2-\delta \end{pmatrix},$$

we have

$$f(\widehat{x}, R + R^0) = \max_{i \in N_3}(R_i + R_i^0)\widehat{x} = \delta,$$

$$f(x^0, R + R^0) = \max_{i \in N_3}(R_i + R_i^0)x^0 = 4 - \delta.$$

Hence, in view of the inequality $2 < \delta < 3$, we conclude that

$$f(\widehat{x}, R + R^0) > f(x^0, R + R^0).$$

Thus, $x^0 \in P^1(R + R^0)$. From this inclusion and relations

$$\|R^0\| = \delta > \varphi/2 = 2,$$

$$x^0 \notin P^1(R),$$

we have that $\rho \leq \varphi/2 = 2$.

Therefore, by Theorem 1 we have $\rho = 2$.

## 5 Attainability of upper bound

Let us show the upper bound $n\varphi(n, m, s)$ for the stability radius of problem $Z^s(R)$ is attainable for $m = s = 1$.

**Theorem 4.** *If $m = s = 1$ there exists a class of problems $Z^1(R)$ such that for stability radius of every problem of this class the following formula is true:*

$$\rho(n, 1, 1) = n\varphi(n, 1, 1).$$

*Proof.* According to Theorem 1 to prove the equality $\rho = n\varphi$ it is sufficient to identify the class of non-trivial problems, for which the inequality $\rho \geq n\varphi$, where $\varphi > 0$ is true.

Let us show that there exists such class, when $X = \{x^0, x^1, x^2, \ldots, x^n\} \subset \mathbf{E}^n$, $n \geq 2$, where $x^0 = \mathbf{0}$, $x^j = e^j$, $j \in N_n$. Here $e^j$ is a unit column vector of $\mathbf{R}^n$, i.e.

$e^j$ is $j$-th column of the unit matrix $E \in \mathbf{R}^{n\times n}$. It is obvious the following equations are true:

$$x^0 - x = -e^j, \quad j \in N_n. \tag{20}$$

Further, in view of $m = s = 1$, let $R = (-r, -r, \ldots, -r) \in \mathbf{R}^n$, where $r > 0$. Hence, we have

$$f_1(x^0) = 0,$$

$$f_1(x^j) = -r, \quad j \in N_n,$$

i.e. $x^0 \notin P^1(R)$, $x^j \in P^1(R) = P^1(x^0, R)$, $j \in N_n$. From here, according to the definition of $\varphi$, the equality

$$\varphi = r \tag{21}$$

is true.

Now, in view of $m = s = 1$, let $R' = (r'_1, r'_2, \ldots, r'_n)$ be any perturbing row of $\Omega(n\varphi)$. Then we have

$$R + R' = (-r + r'_1, -r + r'_2, \ldots, -r + r'_n). \tag{22}$$

Since $\|R'\| < n\varphi$, then by contradiction it is easy to prove that there exists a unique index $p$ such that $|r'_p| < \varphi$. Hence, in view of (20), (21) and (22), we obtain

$$g_1(x^0, x^p, R + R') = (R + R')(x^0 - x^p) = r - r'_p = \varphi - r'_p > 0.$$

Therefore we conclude that for any perturbing row vector $R' \in \Omega(n\varphi)$ portfolio $x^0 \notin P^1(R + R')$. And we get $\rho \geq n\varphi$.

Thus, $\rho = n\varphi$. $\square$

## 6 Stability problem terms

The multicriteria investment portfolio problem $Z^s(R)$, $s \geq 1$, is called *stable* if its stability radius $\rho$ is greater than zero. We consider *the Slater set* [15] consisting of weakly efficient (Slater optimality) portfolio

$$Sl^s(R) = \{x \in X : \quad Sl^s(x, R) = \emptyset\},$$

where $Sl^s(x, R) = \{x' \in X : \quad \forall k \in N_s \quad (g_k(x, x', R_k) > 0)\}$.

Obviously $P^s(R) \subseteq Sl^s(R)$ for any matrix $R \in \mathbf{R}^{m\times n\times s}$.

**Theorem 5.** *For the multicriteria non-trivial problem $Z^s(R)$, $s \geq 1$, the following statements are equivalent:*

*$(i)$ problem $Z^s(R)$ is stable,*

*$(ii)$ $P^s(R) = Sl^s(R)$,*

*$(iii)$ $\varphi(n, m, s) > 0$.*

*Proof.* $(i) \Rightarrow (ii)$. We suppose that problem $Z^s(R)$ is stable, but $P^s(R) \neq Sl^s(R)$. Then there exists weakly efficient portfolio $x^0 \in Sl^s(R) \setminus P^s(R)$. Therefore $Sl^s(x^0, R) = \emptyset$ and $P^s(x^0, R) \neq \emptyset$. It means that

$$\exists x^0 \notin P^s(R) \quad \forall x \in P^s(x^0, R) \quad \exists q \in N_s \qquad (g_q(x^0, x, R_q) = 0).$$

Hence, $\varphi = 0$ and by Theorem 1 value $\rho = 0$. It contradicts the fact that problem $Z^s(R)$ is stable.

$(ii) \Rightarrow (iii)$. If $P^s(R) = Sl^s(R)$, then form any portfolio $x \notin P^s(R)$ set $Sl^s(x, R) \neq \emptyset$. Hence, there exists $x^0 \in X$ such that the inequalities

$$g_k(x, x^0, R_k) > 0, \quad k \in N_s,$$

are true, i.e. $x^0 \in P^s(x, R)$. Thus, the following formula is true:

$$\forall x \notin P^s(R) \quad \exists x^0 \in P^s(x, R) \quad \forall k \in N_s \qquad (g_k(x, x^0, R_k) > 0).$$

Hence, $\varphi > 0$.

$(iii) \Rightarrow (i)$. By Theorem 1 this implication is obvious. □

Since $P^1(R) = Sl^1(R)$, then from Theorem 5 the following corollary impliesa.

**Corollary 2.** *Scalar problem $Z^1(R)$ is stable for any matrix $R \in \mathbf{R}^{m\times n}$.*

Finally, we note that by the equivalence of any two norms in finite linear space (see, for example, [16, 17]) the result of Theorem 5 is correct for any norm in $\mathbf{R}^{m\times n\times s}$.

## References


[1] Emelichev V., Korotkov V., Kuzmin K. *On stability of a Pareto-optimal solution of a portfolio optimization problem with Savage's minimax risk criteria.* Bulletin of the Academy of Sciences of Moldova. Mathematics, 2010, No. 3(64), 35–44.

[2] Markowitz H. M. *Portfolio selection: efficient diversification of investments.* Oxford, Blackwell Publ., 1991.

[3] Savage L. J. *The foundations of statistics.* New York, Dover Publ., 1972.

[4] Dem'Yanov V. F., Malozemov V. N. *Introduction to minimax.* New York, Dover Publ., 1990.

[5] Fedorov V. V. *Numerical maximin methods.* Moscow, Nauka, 1979 (in Russian).

[6] *Minimax and applications.* Edited by Du D.-Z., Pardalos P. M. Dordrecht, Kluwer Acad. Publ., 1995.

[7] Pareto V. *Manuel d'economie politique.* Paris, Giard, 1909.

[8] Emelichev V. A., Leonovich A. M. *A sensitivity measure of the Pareto set in a vector $l_\infty$-extreme combinatorial problem.* Computer Science Journal of Moldova, 2001, **9**, No. 3, 291–304.

[9] Emelichev V. A., Girlich E., Nikulin Yu. V., Podkopaev D. P. *Stability and regularization of vector problems of integer linear programming.* Optimization, 2002, **51**, No. 4, 645–676.

[10] Emelichev V. A., Karelkina O. V., Kuzmin K. G. *Measure of stability and quasistability to a vector integer programming problem in the $l_1$ metric.* Bulletin of the Academy of Sciences of Moldova. Mathematics, 2006, No. 1(50), 39–50.

[11] Emelichev V. A., Gurevsky E. E., Platonov A. A. *Measure of stability for a finite cooperative game with a generalized concept of equilibrium.* Bulletin of the Academy of Sciences of Moldova. Mathematics, 2006, No. 3(52), 17–26.

[12] Emelichev V. A., Gurevsky E. E., Platonov A. A. *On stability and quasi-stability radii for a vector combinatorial problem with a parametric optimality principle.* Bulletin of the Academy of Sciences of Moldova. Mathematics, 2009, No. 2(60), 55–61.

[13] Emelichev V. A., Kuzmin K. G. *Stability radius of a vector integer linear programming problem: case of a regular norm in the space of criteria.* Cybernetics and Systems Analysis, 2010, **46**, No. 1, 72–79.

[14] Tanino T., Sawaragi Y. *Stability of nondominated solutions in multicriteria decision-making.* Journal of Optimimization Theory and Applications, 1980, **30**, No. 2, 229–253.

[15] Slater M. *Lagrange multipliers revisited.* Cowles foundation discussion papers 80, Cowles foundation for research in economics, Yale University, 1959.

[16] Kolmogorov A. N., Fomin S. V. *Elements of the theory of functions and functional analysis.* New York, Dover Publ., 1999.

[17] Suhubi E. *Functional analysis.* Berlin, Springer, 2003.

Vladimir Emelichev, Vladimir Korotkov
Belarusian State University
av. Nezavisimosti, 4 220030 Minsk, Belarus
E-mail: *emelichev@bsu.by; wladko@tut.by*

*Received January 10, 2011*