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Towards Structural Network Analysis

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Abstract. Structural network analysis is an intricate problem. In fact, the majority of techniques that have been developed so far are only applicable to investigate deterministic network models. This gives rise to develop novel graph-theoretical methods for applying them to more complex graphs and especially to statistically inferred networks. In this regard, we review methods for analyzing complex networks structurally putting the special emphasis on network partitioning and quantifying network complexity. Both areas are of general importance in structural graph theory as well as useful for exploring biological networks.

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1 Introduction

Prominent areas in which graph-theoretical methods have been intensely used are, e.g., social network analysis [71, 110], biological network analysis [59], chemical graph theory [103] and investigating technological networks [82]. In terms of developing methods for exploring complex networks, random graph models have been frequently investigated [37, 41]. But besides merely exploring random graphs, it turned out that many real world phenomena can be modeled by using non-random network topologies and, hence, meaningful methods for their structural analysis are crucial [30]. From a mathematical point of view, either descriptive or quantitative methods could be used to explore graphs structurally. To name some well-known examples, we mention metrical properties of graphs [97], general graph measures [54], graph polynomials [50], graph decompositions [20], graph colorings [54], graph complexity [72] and the partitioning of graphs [22]. Importantly, we want to remark that most of the just mentioned approaches are only suitable to analyze deterministic graphs. But the observation that complex networks are often the result of a dynamical processes led to the insight that their analysis can not be adequately performed in a deterministic framework [40]. Thus, there is a strong need to design novel techniques to meet this challenge.

In this paper we provide a review about the structural analysis of complex networks. Here, we focus on such techniques which have been preferably used in computational and systems biology. Concretely, we will put the emphasis on approaches to partition complex networks and to quantify network complexity. Both problems are challenging and there is a future need to find novel approaches when considering networks which were inferred statistically. Altogether, the main goal of this review paper is to demonstrate the usefulness and potential of Structural Graph Analysis and to stimulate the interest of other researchers to observe graph theory as a tool for solving interdisciplinary problems.

2 Graph-Theoretical Applications in Bioinformatics and Computational Biology

Various examples in the scientific literature have been demonstrated that biological phenomena and processes can be tackled by applying graph theory, see, e.g., [38, 80, 84].

In this section, we provide a general overview about important areas dealing with graph-based approaches in computational biology:

- Phylogenetics: During the last thirty years, various graph-based techniques have been successfully applied for solving problems in phylogenetics, see, e.g., [42, 89, 99]. A prominent example, for instance, is phylogenetic tree reconstruction that has been a major research goal for biologists because it often serves as indispensable interpretive framework for the analysis of evolutionary processes by representing the interrelationships among biological entities as graphs [42, 59]. Further, distance-, character-, and likelihood-based methods are three important approaches which have been used for phylogenetic tree analysis [42, 99, 100]. Besides the problem of inferring phylogenetic trees from biological data sets, the structural analysis of such graphs has been found as crucial. In this context, various tree distance measures and metrics [89, 92, 93] were used to determine the structural similarity between phylogenetic trees.

- RNA-Structure Analysis: Graphs play an important role when analyzing secondary structures inferred from biological sequences [109, 111]. For example, Nussinov [79] did the first attempt to calculate secondary structures for simplified energy models based on base-pairing rules. After the model was elaborated, it turned out that there is a further need for considering loops in the RNA secondary structure and, consequently, Zuker and Stiegler [116] proposed a recursive algorithm to take the loop types [116] into account. Moreover, an important contribution when analyzing secondary structures comparatively was proposed by McCaskill [70]. In order to compare secondary structures structurally, it turned out to be useful encoding them as trees and to use existing tree distance measures [92, 101] for determining their similarity. Note that a recent survey on graph-based techniques to model and process A-structures has been contributed by Washietl et al. [109].

- Molecular Biology: For instance, regulatory, signal transduction, or metabolic networks are often represented by networks to analyze molecular biological processes [57]. For this, special graph classes like bipartite graphs, hypergraphs and directed acyclic graphs [58, 59, 67] were particularly used. Apart from applying existing graph classes [21] to represent networks, graph-theoretical techniques have been intensely used to analyze molecular biological pathways. Exemplarily, we here mention a contribution due to Rosselló et al.[90] who describe development pathways by using graph grammars.

- High-Throughput Analysis: A hype dealing with employing graphs in computational biology started after the development of high-throughput techniques [24, 38] because they allow a large-scale identification of genes, RNAs, proteins. In this context, a key problem is to identify and study functional components of a biological system meaningfully, based on their molecular interactions involving, e.g., genes, proteins or metabolites, instead of exploring these components in isolation. For example, a challenging problem in the above mentioned area is to investigate complex diseases by investigating underlying network representations [38]. To tackle these problems, methods from statistical data analysis and machine learning have been used [26, 38, 39].

- Drug Design and Bio-chemical Graph Analysis: A still challenging and ongoing problem is to predict physico-chemical or toxic properties of bio-chemical molecules using structural graph descriptors [33, 102]. Particularly entropy-based measures to perform such studies within QSPR (quantitative structure-property relationship) and QSAR (quantitative structure-activity relationship) have been found to be powerful, see, e.g., [8, 14, 33]. But because a large number of measures to quantify molecular complexity have been developed so far, there is a strong need to examine which kind of structural information the measures do detect. Contributions to shed light on this problem were recently made in [7, 32]. Similarly, Pathway Analysis [88] using graph-based techniques became a crucial field when analyzing bio-chemical processes and complex diseases [38]. In particular, it allows, e.g., the identification of gene networks and to study how genes are regulated [31].

3 Applied Graph Partitioning

The investigation of general graph partitioning methods for finding community structures is currently of considerable interest when analyzing complex networks quantitatively as well as descriptively [49, 78, 115]. In this section, we briefly describe such methods by using graph partitioning. Before we start outlining concrete techniques, we sketch some seminal work concerning classical graph partitioning [61].

3.1 Classical Methods

To understand the underlying idea of graph partitioning properly, we firstly state the following definition that describes the problem intuitively, see, e.g., [60].

Definition 1. *Let $G = (V, E)$ be a graph. Then, we define the k-way graph partitioning problem as follows: Partition the vertex set V into k subsets $V_1, V_2, \ldots, V_k$*

such that $V_i \bigcap V_j = \emptyset$, $i \neq j$, $|V_i| = \frac{n}{k}$, $\bigcup_i V_i = V$, *and the number of edges of* E *whose incident vertices belong to different subsets is minimized.*

An important contribution in this area is the algorithm due to Kernighan and Lin [61]. Apart from this work, other approaches to partition graphs based on spectral clustering and multilevel partitioning have been explored, see, e.g., [76]. For instance, spectral methods produce a partition based on the eigendecomposition [51] of the graph. Also, spectral approximations for a variety of partitioning criteria have been formulated including the minimum cut [83], ratio cut [25] and normalized cut [95]. Interestingly, most of the multilevel algorithms are based on the work we already mentioned above, see [61]. A strong point of this heuristic algorithm is the fact that its time complexity is $O(|V|^3)$ on sparse graphs [61]. Improvements possessing lower computational complexity can be found in [106]. Let's now describe the original method presented in [61] in more detail: Let $G = (V, E)$ be a graph with weighted edges (costs) and $|V| = 2n$. Let S be a set of $2n$ points with an associated cost matrix, $C = (c_{ij}), i, j = 1, \ldots, 2n$. Further, without loss of generality, it is assumed [61] that C is a symmetric matrix and $c_{ii} = 0, \forall\, i$. Then, the aim of the algorithm is to partition S into two sets A and B, $|A| = |B| = n$, such that the so-called external costs $T = \sum_{A \times B} c_{ab}$ will be minimized.

Note that the work initiated by Kernighan and Lin [61] has already been successfully improved, see, e.g., [43, 60, 61]. A well-known example of such a recently developed multilevel approach is the METIS algorithm [60] that aims to partition graphs from different application domains efficiently. In addition, we mention another fast multi-level algorithm developed by Dhillon et al. [34] that directly optimizes various weighted graph clustering objectives. In particular, Dhillon et al. [34] show that a general weighted k-means objective is mathematically equivalent to a weighted graph clustering objective. The main advantage of this method is that it approximates graph clustering objectives without requiring an eigendecomposition, which can be computationally intensive for large graphs [34]. Another advantage of this algorithm, compared to other multilevel approaches, is that it does not require the partitions to be of equal sizes [34].

3.2 Community Structure Detection

In this section, we sketch known approaches to detect community structures within biological networks. Generally, to find community structures in complex networks, classical and recent graph partitioning methods have often been applied [48, 59]. Until now, the concept of graph partitioning has been used for detecting community structures in social networks, WWW-graphs, and biological or biochemical networks [44, 48, 56, 59, 77]. Informally speaking, the community structure property of a network can be understood by considering a graph in which the vertices are joined together in tightly-knit groups and there are only looser connections between them, see [48]. It is important to mention that the traditional method for detecting community structures in networks is hierarchical clustering [48]. If we start with a weighted graph $G = (V, E)$, $|V| = n$, (i) we first have to calculate a weight w_{ij} for every pair i, j of vertices in the network, (ii) select all vertices in the network with no edges between them and (iii) add edges between pairs one by one in the order of their weights, starting with the pair with the strongest weight and progressing to the weakest, see [48]. If edges are added, the resulting graph shows a nested set of connected subsets of vertices, which are expected to be the communities [48]. Note that algorithms of this kind are called agglomerative, see, e.g., [12].

To overcome existing shortcomings of agglomerative methods, see, e.g., [35], Girvan and Newman [48] proposed an alternative approach for detecting communities that represents a so-called divisive algorithm [48]. The main procedure works as follows: Start with the entire graph and iteratively cut the edges, thus dividing the network progressively into smaller and smaller disconnected sub-networks finally identified as the communities. The crucial point of this algorithm is the selection of the edges to be cut, which has to be those connecting communities and not those within them. The main steps of algorithm proposed in [48] can be stated as follows:

1. Calculate the betweenness for all edges in the network.

2. Remove the edge with the highest betweenness.

3. Recalculate betweenness for all edges affected by the removal.

4. Repeat from the step 2 until no edges remain.

Until now, several improvements and extensions using shortest path versions of this algorithm have already been proposed [48]. For example, Holme et al. [56] modified this method and then applied this modification, based on global centrality measure (betweenness), to a number of metabolic networks from different organisms for finding communities that correspond to functional units within these networks. Also, Wilkinson and Huberman [113] have applied the approach to a network representing relationships between genes, as established by the co-occurrence of gene names found in published research articles. For finding communities in network they used a nonlocal process exploiting the concept of betweenness centrality.

For finalizing, we state two more contributions in this area. The CONGA (Cluster-Overlap Newman Girvan Algorithm) [52] is an extension of [48]. It can be also used with undirected, unweighted graphs and performs hierarchical clustering but it allows overlapping clusters. Finally, the CFinder algorithm [81], that is a bottom-up approach, provides a method to interpret communities as union of cliques. For more details refer to [48, 81].

4 Topological Complexity Measures for Graphs

The problem of determining the structural complexity of a network can be understood as characterizing the graphs taking structural features into account [11,17,62]. Clearly, this task is not uniquely defined because no complexity index can measure all structural features which contribute to the complexity of a graph. Before starting with describing concrete non-information-theoretic and information-theoretic complexity measures, we outline existing applications in computational biology and bioinformatics:

- To investigate the evolution of PPI domains and the impact on organismal complexity and the complexity of protein-protein interaction networks [114].
- General studies to examine how, e.g., biological and technological networks differ by calculating their structural complexity [108].
- To find interrelations between the structure and complexity of the pathways and the phylogeny of species by using non-information-theoretic and information-theoretic complexity measures [17, 69].
- To use entropy-based measures for problems in QSPR (quantitative structure-property relationship) and QSAR (quantitative structure-activity relationship), see, e.g., [8, 14, 33].
- To employ non-information-theoretic and information-theoretic measures in the field of chemoinformatics [47], e.g., to perform correlation analyses [7] and develop similarity/diversity measures [107].

4.1 Distance-Based Measures

A large number of complexity measures that have been developed so far are based on distances in a graph [102]. As a strong point, such distances are simple to calculate by any shortest path algorithm to be applied to the underlying adjacency matrix. Often, a weak point of such measures is that they do not capture structural information uniquely, that means, the measures are highly degenerated [64]. Let $G = (V, E)$ be a graph. We now start by expressing the well-known Wiener-index [112],

$$W(G) := \frac{1}{2} \sum_{i=1}^{|V|} \sum_{j=1}^{|V|} d(v_i, v_j). \tag{1}$$

Originally, it was developed to detect branching of chemical graphs [53]. $d(v_i, v_j)$ denotes the shortest distance between v_i and v_j. Similarly, the Harary-index [4, 36],

$$H(G) := \frac{1}{2} \sum_{i=1}^{|V|} \sum_{j=1}^{|V|} (d(v_i, v_j))^{-1}, \quad i \neq j, \tag{2}$$

is based on reciprocal distances. A more complex example of such a measure is the Balaban J-index [2],

$$J(G) := \frac{|E|}{\mu + 1} \sum_{(v_i, v_j) \in E} [DS_i DS_j]^{-\frac{1}{2}}. \tag{3}$$

Note that DS_i denotes the distance sum (row sum) of $v_i \in V$ and $\mu := |E|+1-|V|$ is the cyclomatic number. Other important distance-based measures are, for instance, the mean distance deviation [97],

$$\Delta\mu(G) := \frac{1}{|V|} \sum_{i=1}^{|V|} |\mu(v_i) - \bar{\mu}|, \tag{4}$$

where

$$\mu(v_i) := \sum_{j=1}^{|V|} d(v_i, v_j) \quad \text{and} \quad \bar{\mu} := \frac{2W}{|V|}, \tag{5}$$

the product of row sums-index [91] given by

$$\log\big(\mathrm{PRS}(G)\big) := \log\left(\prod_{i=1}^{|V|} \mu(v_i)\right), \tag{6}$$

and, finally, the hyper-distance-path index [102],

$$D_P(G) := \frac{1}{2} \sum_{i=1}^{|V|} \sum_{j=1}^{|V|} d(v_i, v_j) + \frac{1}{2} \sum_{i=1}^{|V|} \sum_{j=1}^{|V|} \binom{d(v_i, v_j)}{2}. \tag{7}$$

For further distance-based measures, refer to [102].

4.2 Other and Related Complexity Indices

Besides distance-based measures, various other complexity indices for networks based on other graph invariants have been developed. To pursue outlining known graph complexity measures, we now state some important examples which have been used to measure molecular complexity [86]. Note that the purpose for deriving such indices was either to find measures with low computational complexity or with high discrimination power [64]. For example, the index of total adjacency [17] can be easily derived from the underlying adjacency matrix,

$$A(G) := \frac{1}{2} \sum_{i=1}^{|V|} \sum_{j=1}^{|V|} a_{ij}. \tag{8}$$

a_{ij} denotes the entry in the i-th row and j-th column of A. From this, it straightforwardly follows the normalized edge complexity [17],

$$E_{|V|}(G) := \frac{A}{|V|^2}. \tag{9}$$

Let k_{v_i} be the degree of the vertex $v_i \in V$. Interestingly, the vertex degrees seen as a graph invariant have also been used to define measures to quantify the structural complexity of graphs. In order to list some well-known examples, we now state the Zagreb group-indices [36],

$$Z_1(G) := \sum_{i=1}^{|V|} k_{v_i} \quad \text{or} \quad Z_2(G) := \sum_{(v_i,v_j)\in E} k_{v_i} k_{v_j}, \tag{10}$$

and the Randić connectivity-index [85],

$$R(G) := \sum_{(v_i,v_j)\in E} [k_{v_i} k_{v_j}]^{-\frac{1}{2}}. \tag{11}$$

An interesting generalization of such measures was developed by Bonchev [14] who developed the so-called Overall (OX) indices given by

$$OX(G) := \sum_{k=0}^{|E|} {}^{k}X; \quad \{OX(G)\} = \{{}^{0}X, {}^{1}X, \ldots, {}^{|E|}X\}. \tag{12}$$

OX is called the overall value of a certain graph invariant X by summing up its values in all subgraphs, and partitioning them into terms of increasing orders (increasing number of subgraph edges k). For instance, $OX = SC$ is equal to the subgraph count [14, 17].

More recent complexity measures were developed by Kim et al. [62]. To name an example, we here express the so-called Efficiency complexity C^e of a graph G that is based on calculating path lengths. Starting from

$$E'(G) := \frac{2}{|V|(|V|-1)} \sum_{i} \sum_{j>i} \frac{1}{d(v_i, v_j)}, \tag{13}$$

expressing the arithmetic mean of all inverse path lengths. Further, by defining

$$E_{\text{path}}(G) := \frac{2}{|V|(|V|-1)} \sum_{i=1}^{|V|-1} \frac{(|V|-i)}{i}, \tag{14}$$

the Efficiency complexity C^e yields to

$$Ce(G) := \left(\frac{E' - E_{\text{path}}}{1 - E_{\text{path}}}\right)\left(1 - \frac{E' - E_{\text{path}}}{1 - E_{\text{path}}}\right) \in [0, 1]. \tag{15}$$

Moreover, a measure that crucially relies on the largest eigenvalue of an undirected graph G was defined in [62]. If r stands for the largest eigenvalue calculated from the adjacency matrix of G, then, the graph index Cr is defined as

$$Cr := 4c_r(1 - c_r) \in [0, 1], \tag{16}$$

where

$$c_r := \frac{r - 2\cos\left(\frac{\pi}{N+1}\right)}{|V| - 1 - 2\cos\left(\frac{\pi}{|V|+1}\right)}. \tag{17}$$

Numerical examples to calculate these measures and details regarding their interpretation can be found in [62].

4.3 Information-theoretic Complexity Measures

The key concept for obtaining a further class of important graph complexity measures relies on Shannons's information theory [94]. Starting from inferred structural features of a network, the crucial step for quantifying its structural information is to infer a probability distribution and, then, to apply Shannons's entropy formula. As a result, one obtains topological entropies for characterizing networks [13,32,98]. Prior to start explaining concrete information measures, we emphasize that the main application domains of general information-theoretic methods to analyze networks have been biology [27,68,73,87], ecology [55,105], mathematical chemistry [6,8,13], software technology [1], and operations research [23,45].

4.3.1 Classical Information Measures for Graphs

The development of information measures represented by entropies to characterize the underlying topology of a given network was the starting point of applying information theory to investigate biological and chemical systems structurally [87,104]. These measures are based on the principle that by assuming a graph $G = (V, E)$, a graph invariant X and an equivalence criterion, distributions of X can be obtained. Particularly, this process can be understood by considering the following scheme [13]:

$$\begin{pmatrix} 1 & 2 & \cdots & k \\ |X_1| & |X_2| & \cdots & |X_k| \\ p_1 & p_2 & \cdots & p_k \end{pmatrix}. \tag{18}$$

The first row of this matrix contains the equivalency classes and the second row the cardinalities of the obtained partitions, respectively. The probability values, calculated by $p_i = \frac{|X_i|}{|X|}$, for each partition form the third row. Hence, $\mathcal{P}_G = (p_1, \ldots, p_k)$ represents a probability distribution. Then, the application of the well known Shannon-entropy [94]

$$H(X) := H\left(p(x_1), \ldots, p(x_k)\right) = -\sum_{i}^{k} p(x_i)\log(p(x_i)), \tag{19}$$

of a discrete random variable $X = (x_1, x_2, \ldots, x_k)$ leads to the following graph entropies [13],

$$I_t(G) := |X|\log(|X|) - \sum_{i=1}^{k} |X_i|\log(|X_i|), \tag{20}$$

$$I_m(G) := -\sum_{i=1}^{k} p_i \log(p_i) = -\sum_{i=1}^{k} \frac{|X_i|}{|X|} \log\left(\frac{|X_i|}{|X|}\right). \tag{21}$$

$I_t(G)$ is called the total structural information content of G and can be recalculated by using $I_m(G)$. The latter is called the mean structural information content of a graph.

Now, by using certain graph invariants X, special entropy measures can be obtained which serve as graph complexity measures. The starting point by developing concrete measures was done by Rashevsky [87] and Trucco [104]. Rashevsky's information measures to characterize G are concretely given by [87]

$$I_t^V(G) := |V| \log(|V|) - \sum_{i=1}^{k} N_i \log(N_i), \tag{22}$$

$$I_m^V(G) := -\sum_{i=1}^{k} \frac{|N_i|}{|V|} \log\left(\frac{|N_i|}{|V|}\right), \tag{23}$$

where $|N_i|$ denotes the number of topologically equivalent vertices in the i-th vertex orbit of G. k stands for the number of different vertex orbits. Trucco's measure [104] can be analogously obtained by using the edge automorphism group. After this seminal work, Mowshowitz [74] also investigated the measure I_m^V (see Equation (23)) in depth and additionally explored the chromatic information content of a graph [75]:

$$I_c(G) := \min_{\hat{V}} \left\{ -\sum_{i=1}^{h} \frac{n_i(\hat{V})}{|V|} \log\left(\frac{n_i(\hat{V})}{|V|}\right) \right\}. \tag{24}$$

$\hat{V} = \{V_i | 1 \leq i \leq h\}$ is an arbitrary chromatic decomposition of G, $|V_i| = n_i(\hat{V})$, and $h = \chi(G)$ is the chromatic number of G. Note that the computation of the chromatic number is a costly procedure for arbitrary graphs [54].

Apart from defining and calculating information measures for networks, there is also a strong need to understand the meaning of these measures in depth. This could be done by establishing their mathematical properties under certain theoretical assumptions (e.g., bounds and the behavior under certain graph operations etc.). Such a concrete result has been proven by Mowshowitz [74].

Theorem 1. *For graphs G and H*

$$I_m^V(G \times H) \leq I_m^V(G) + I_m^V(H), \tag{25}$$

and

$$I_m^V(G \circ H) \leq I_m^V(G) + I_m^V(H), \tag{26}$$

where $\times$ and $\circ$ represent the cartesian product and composition, respectively.

The assertion of this theorem is that the information measure is semi-additive on the cartesian product and on the composition of two graphs. Interestingly, we emphasize that formal properties like the just shown one or bounds for the entropies (by using important graph classes) are unknown for the majority of network information measures.

After the just outlined work, Bonchev [13,18] introduced the so-called magnitude-based information indices by defining a weighted probability scheme. These indices can be considered as generalization of the measures due to Rashevsky and Mowshowitz. It follows easily that such a scheme can be analogously applied to a system with $|N|$ elements to group these elements into k partitions according to the magnitude. Then, the modified scheme is [13]:

$$\begin{pmatrix} 1 & 2 & \cdots & k \\ |N_1| & |N_2| & \cdots & |N_k| \\ p_1 & p_2 & \cdots & p_k \\ w_1 & w_2 & \cdots & w_k \\ p_1^M & p_2^M & \cdots & p_k^M \end{pmatrix}. \tag{27}$$

In addition to the already existing rows of the introduced probability scheme (see Matrix (18)), the magnitudes representing weights $(w_1, w_2, \ldots, w_k)$ and the weighted probability values $(p_1^M, p_2^M, \ldots, p_k^M)$ were introduced [13,18]. Because it holds $M = \sum_{i=1}^{k} w_i N_i$ and $p_i = \frac{w_i}{M}$, the graph entropies represented by Equation (20) and Equation (21) can be rewritten as

$$I_t^M(G) := M \log(M) - \sum_{i=1}^{k} N_i w_i \log(w_i), \tag{28}$$

$$I_m^M(G) := -\sum_{i=1}^{k} \frac{N_i w_i}{M} \log\left(\frac{w_i}{M}\right). \tag{29}$$

By using this approach, concrete magnitude-based information were defined, for instance [13],

$$I_D(G) := -\frac{1}{N} \log\left(\frac{1}{N}\right) - \sum_{i=1}^{\rho(G)} \frac{2k_i}{N^2} \log\left(\frac{2k_i}{N^2}\right), \tag{30}$$

$$I_D^W(G) := W(G) \log(W(G)) - \sum_{i=1}^{\rho(G)} i k_i \log(i). \tag{31}$$

k_i is the occurrence of a distance possessing value i in the distance matrix of G. A strong point of these measures is their low degeneracy [63] compared to the classical measures mentioned in the beginning of this section. In general, one calls such a measure degenerated if for more than one graph the measure possesses the same value [3]. By using chemical graphs, numerical results are reported in [13, 65].

Again Bonchev [16,17] developed a substructure-based approach to detect molecular complexity. Let $G = (V, E)$ be a graph and X be a graph invariant. Then, the following entropy-based complexity measure

$$I(G, OX) := OX \log(OX) - \sum_{k=0}^{|E|} {}^{k}X \log\left({}^{k}X\right), \tag{32}$$

relies on the overall value OX (see Equation (12)) by summing up its values in all subgraphs [16]. The values will be partitioned into terms of increasing orders (increasing number of subgraph edges k) [16]. As an example, one can set $OX = SC$, i.e., OX equals the subgraph count [16]. Starting from this construction, Bonchev [16, 17] obtained several overall information indices such as overall connectivity (the sum of total adjacency of all subgraphs) [14], overall Wiener-index (the sum of total distances of all subgraphs) [15], overall Zagreb-indices [19], and the overall Hosoya-index [16]. Known earlier and also substructure-based contributions to detect molecular complexity were developed by, e.g., Bertz et al. [9, 10]. As a further remark, note that many further information measures for graphs which are similar to the outlined ones or which are based on the same construction principle (e.g., simple finite probability scheme, weighted probability scheme, etc.) can be found in [8, 13, 17, 33, 102].

To finalize this section as well as to show a different paradigm to derive graph entropies, we state the well-known Körner entropy [66, 96] that has been applied in information theory. The measure is defined by

$$H(G, P) := \lim_{t \longrightarrow \infty} \min_{U \subseteq V^t, P^t(U) > 1-\epsilon} \frac{1}{t} \log(\chi(G^t(U))). \tag{33}$$

For $V' \subseteq V(G)$, the induced subgraph on V' is denoted by $G(V')$ and $\chi(G)$ is the chromatic number [5] of G, G^t the t-th co-normal power [66] of G and

$$P^t(U) := \sum_{x \in U} P^t(x). \tag{34}$$

Examples and an interpretation of this measure can be found in [66, 96].

4.3.2 Parametric Information Measures

To compute the structural information content of arbitrary large networks, one needs a method whose underlying algorithm is efficient, i.e., its time complexity is polynomial. From Section (4.3.1), it follows that classical network information measures are often rely on algebraic principles, e.g., determining automorphism groups of graphs or chromatic decompositions. However it is known that for arbitrary networks, the computational complexity of the corresponding algorithms is often very high [46].

In order to overcome this problem, we now present parametric entropy measures whose time complexity has been proven to be polynomial [28]. The key principle to construct such information measures is as follows: Let $G = (V, E)$ be an arbitrary graph and let S be a given set, e.g., a set of vertices or paths etc. The function $f : S \longrightarrow \mathbb{R}_+$ is called an abstract information functional of G. Instead of inducing partitions using an equivalence criterion (see Section (4.1)), we start from an abstract information functional f and define the quantity [29],

$$p^f(v_i) := \frac{f(v_i)}{\sum_{j=1}^{|V|} f(v_j)}, \quad \forall\, v_i \in V. \tag{35}$$

Because the following equation

$$p^f(v_1) + p^f(v_2) + \ldots + p^f(v_{|V|}) = 1, \tag{36}$$

holds by definition, these entities can be interpreted as vertex probabilities. Hence, $(p^f(v_1), \ldots, p^f(v_{|V|}))$ forms a probability distribution. From this, it is straightforward to obtain families of graph entropy measures like

$$I_f(G) := -\sum_{i=1}^{|V|} \frac{f(v_i)}{\sum_{j=1}^{|V|} f(v_j)} \log\left(\frac{f(v_i)}{\sum_{j=1}^{|V|} f(v_j)}\right), \tag{37}$$

or

$$I_f^{\lambda}(G) := \lambda\left(\log(|V|) + \sum_{i=1}^{|V|} \frac{f(v_i)}{\sum_{j=1}^{|V|} f(v_j)} \log\left(\frac{f(v_i)}{\sum_{j=1}^{|V|} f(v_j)}\right)\right), \tag{38}$$

$\lambda > 0$. By incorporating special information functionals, one clearly obtains special entropies. To give an example for a special information functional that is based on metrical properties, we express [29]

$$f^{V_1}(v_i) := \alpha^{c_1|S_1(v_i,G)|+c_2|S_2(v_i,G)|+\cdots+c_{\rho(G)}|S_{\rho(G)}(v_i,G)|}, \qquad c_k > 0, 1 \leq k \leq \rho(G), \alpha > 0. \tag{39}$$

c_k are arbitrary real positive coefficients. $\rho(G)$ denotes the diameter of G and

$$S_j(v_i, G) := \{v \in V \,|\, d(v_i, v) = j,\ j \geq 1\}, \tag{40}$$

the j-sphere of a vertex v_i of G, respectively. f^{V_1} is a parametric information functional that depends on both the parameter α and the vector $(c_1, c_2, \ldots, c_{\rho(G)})$. The meaning of these parameters has been explained in [29]. Then, the resulting (parametric) information measure representing the entropy of the underlying graph topology is

$$I_{f^{V_i}}(G) := -\sum_{i=1}^{|V|} \frac{f^{V_i}(v_i)}{\sum_{j=1}^{|V|} f^{V_i}(v_j)} \log\left(\frac{f^{V_i}(v_i)}{\sum_{j=1}^{|V|} f^{V_i}(v_j)}\right), \quad i = 1, 2. \tag{41}$$

Of course, it is also possible to define

$$f^{V_2}(v_i) := c_1|S_1(v_i, G)| + c_2|S_2(v_i, G)| + \cdots + c_{\rho(G)}|S_{\rho(G)}(v_i, G)|, \tag{42}$$

that does not depend on α. Importantly, the process to design an information functional and, thus, the resulting information measures strongly depends on the specific problem when characterizing a graph using an information measure.

5 Summary and Conclusion

In this paper, we reviewed some concepts known in structural graph analysis. We emphasized that we particularly put the underscore on such methods which have been used in bioinformatics and systems biology. After outlining graph-theoretical approaches in these areas, we firstly began to survey graph partitioning methods to find clusters or communities within complex networks. Due to the steadily increasing complexity of real-world networks, we believe that it will be fruitful to further develop this field to process statistically inferred networks.

As future work, we want to focus on approaches combining graph-theoretical and information-theoretic techniques. Secondly, we studied the challenging problem to determine the structural complexity of graphs and reviewed classical and recent methods. We want to emphasize that finding a meaningful complexity measure to quantify structural information of a graph is far from trivial and usually not unique. These facts give an idea about the complexity of such measures. Also, in consideration of the fact that a vast number of graph complexity measures have been developed so far, the problem to examine which kind of structural information the measures do detect is not solved properly. Therefore, we would like to shed light on this important aspect in the future by examining correlations and interrelations between graph complexity measures.

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References

[1] Allen E.B. *Measuring graph abstractions of software: An information-theory approach.* Proceedings of the 8-th International Symposium on Software Metrics, page 182. IEEE Computer Society, 2002.

[2] Balaban A.T. *Highly discriminating distance-based topological index.* Chem. Phys. Lett., 1982, **89**, 399–404.

[3] Balaban A.T., Balaban T.S. *New vertex invariants and topological indices of chemical graphs based on information on distances.* J. Math. Chem., 1991, **8**, 383–397.

[4] Balaban A.T., Ivanciuc O. *Historical development of topological indices.* In: Balaban A.T., Devillers J., editors, *Topological Indices and Related Descriptors in QSAR and QSPAR*, pages 21–57. Gordon and Breach Science Publishers, Amsterdam, The Netherlands, 1999.

[5] Bang-Jensen J., Gutin G. *Digraphs. Theory, Algorithms and Applications.* Springer, London, Berlin, Heidelberg, 2002.

[6] Basak S.C. *Information-theoretic indices of neighborhood complexity and their applications.* In: Balaban A.T., Devillers J., editors, *Topological Indices and Related Descriptors in QSAR and QSPAR*, pages 563–595. Gordon and Breach Science Publishers, Amsterdam, The Netherlands, 1999.

[7] Basak S.C., Balaban A.T., Grunwald G.D., Gute B.D. *Topological indices: Their nature and mutual relatedness.* J. Chem. Inf. Comput. Sci., 2000, **40**, 891–898.

[8] Basak S.C., Magnuson V.R. *Molecular topology and narcosis.* Arzeim.-Forsch./Drug Design, 1983, **33(I)**, 501–503.

[9] Bertz S.H., Herndon W.C. *The similarity of graphs and molecules.* In: Artificial Intelligence Applications in Chemistry, Chapter 15, 169–175. American Chemical Society, 1986.

[10] Bertz S.H., Wright W.F. *The graph theory approach to synthetic analysis: Definition and application of molecular complexity and synthetic complexity.* Graph Theory Notes of New York, 1998, **35**, 32–48.

[11] Boccaletti S., Latora V., Moreno Y., Chavez M., Hwang D.-U. *Complex networks: Structure and dynamics.* Physics Reports, 2006, **424(4-5)**, 175–308.

[12] Bock H.H. *Automatische Klassifikation. Theoretische und praktische Methoden zur Gruppierung und Strukturierung von Daten.* Studia Mathematica. Vandenhoeck & Ruprecht, Göttingen, 1974.

[13] Bonchev D. *Information Theoretic Indices for Characterization of Chemical Structures.* Research Studies Press, Chichester, 1983.

[14] Bonchev D. *Overall connectivities and topological complexities: A new powerful tool for QSPR/QSAR.* J. Chem. Inf. Comput. Sci., 2000, **40(4)**, 934–941.

[15] Bonchev D. *The overall Wiener index – a new tool for characterization of molecular topology.* J. Chem. Inf. Comput. Sci., 2001, **41**, 582–592.

[16] Bonchev D. *The overall topological complexity indices.* In: Simos T., Maroulis G., editors, *Advances in Computational Methods in Science and Engineering*, Vol. 4B, pages 1554–1557. VSP Publications, 2005.

[17] Bonchev D., Rouvray D.H. *Complexity in Chemistry, Biology, and Ecology.* Mathematical and Computational Chemistry. Springer, New York, NY, USA, 2005.

[18] Bonchev D., Trinajstić N. *Information theory, distance matrix and molecular branching.* J. Chem. Phys., 1977, **67**, 4517–4533.

[19] Bonchev D., Trinajstić N. *Overall molecular descriptors. 3. Overall Zagreb indices.* SAR QSAR Environ. Res., 2001, **12**, 213–236.

[20] Bosák J. *Decompositions of Graphs.* Mathematics and its Applications. Springer, 1990.

[21] Brandstädt A., Le V.B., Sprinrad J.P. *Graph Classes. A Survey.* SIAM Monographs on Discrete Mathematics and Applications, 1999.

[22] Broere I., Hajnal P., Mihók P. *Partition problems and kernels of graphs.* Discussiones Mathematicae Graph Theory, 1997, **17**, 311–313.

[23] Calinescu A., Efstathiou J., Sivadasan S., Schirn J., Huaccho Huatuco L. *Complexity in manufacturing: an information theoretic approach.* Proceedings of the International Conference on Complex Systems and Complexity in Manufacturing, pages 30–44, 2000.

[24] Causton H.C., Brazma A.,Quackenbush J. *Microarray Gene Expression Data Analysis.* Blackwell Publishers, 2003.

[25] Chan P., Schlag M., Zien J. *Spectral k-way ratio cut partitioning.* IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems, 1994, **13**, 1088–1096.

[26] Cristianini N., Shawe-Taylor J. *An Introduction to Support Vector Machines.* Cambridge University Press, Cambridge, UK, 2000.

[27] Dancoff S.M., Quastler H. *Information content and error rate of living things.* In: Quastler H., editor, *Essays on the Use of Information Theory in Biology*, pages 263–274. University of Illinois Press, 1953.

[28] Dehmer M. *Information-theoretic concepts for the analysis of complex networks.* Appl. Artif. Intell., 2008, **22(7 & 8)**, 684–706.

[29] Dehmer M. *A novel method for measuring the structural information content of networks.* Cybernetics and Systems, 2008, **39**, 825–843.

[30] Dehmer M. *Structural Analysis of Complex Networks.* Birkhäuser/Springer, Boston/Basel, 2010.

[31] Dehmer M., Emmert-Streib F. *Analysis of Complex Networks: From Biology to Linguistics.* Wiley VCH, Weinheim, Germany, 2009.

[32] Dehmer M., Varmuza K., Borgert S., Emmert-Streib F. *On entropy-based molecular descriptors: Statistical analysis of real and synthetic chemical structures.* J. Chem. Inf. Model., 2009, **49**, 1655–1663.

[33] Devillers J., Balaban A.T. *Topological Indices and Related Descriptors in QSAR and QSPR.* Gordon and Breach Science Publishers, Amsterdam, The Netherlands, 1999.

[34] Dhillon I.S., Guan Y., Kulis B. *Weighted graph cuts without eigenvectors: a multilevel approach.* IEEE Transactions on Pattern Analysis and Machine Intelligence, 2007, **29**, 1944–1957.

[35] Diao Y., Lia M., Fenga Z., Yina J., Pana Y. *The community structure of human cellular signaling network.* Journal of Theoretical Biology, 2007, **247**, 608–615.

[36] Diudea M.V., Gutman I., Jäntschi L. *Molecular Topology.* Nova Publishing, Ney York, NY, USA, 2001.

[37] Dorogovtsev S.N., Mendes J.F.F. *Evolution of Networks. From Biological Networks to the Internet and WWW.* Oxford University Press, 2003.

[38] Emmert-Streib F., Dehmer M. *Analysis of Microarray Data: A Network-Based Approach.* Wiley-VCH, 2008.

[39] Emmert-Streib F., Dehmer M. *Information Theory and Statistical Learning.* Springer, New York, USA, 2008.

[40] Emmert-Streib F., Dehmer M. *Information processing in the transcriptional regulatory network of yeast: Functional robustness.* BMC Syst. Biol., 2009, **3**:35.

[41] Erdös P., Rényi P. *On the evolution of random graphs.* Magyar Tud. Akad. Mat. Kutató Int. Közl, 1960, **5**, 17–61.

[42] Felsenstein J. *Coalescents, phylogenies, and likelihoods.* Biological Bulletin, 1999, **196**, 343–344.

[43] Fiduccia C.M., Mattheyses R.M. *A linear-time heuristic for improving network partitions.* Proceedings of the 19-th Design Automation Conference, pages 241–247, 1982.

[44] Flake G.W., Lawrence S.R., Giles C.L., Coetzee F.M. *Self-organization and identification of web communities.* IEEE Computer, 2002, **35**, 66–71.

[45] Frizelle G., Woodcock E. *Measuring complexity as an aid to developing operational strategy.* International Journal of Operations & Production Management, 1995, **15**, 26–39.

[46] Garey M.R., Johnson D.S. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* Series of Books in the Mathematical Sciences. W. H. Freeman, 1979.

[47] Gasteiger J., Engel T. *Chemoinformatics - A Textbook.* Wiley VCH, Weinheim, Germany, 2003.

[48] Girvan M., Newman M.E.J. *Community structure in social and biological networks.* PNAS, 2002, **99**, 7821–7826.

[49] Gleiser P.M., Danon L. *Community structure in jazz.* Advances in complex systems, 2003, **6(4)**, 565–574.

[50] Godsil C., Royle G. *Algebraic Graph Theory.* Graduate Texts in Mathematics. Academic Press, 2001.

[51] Golub G.H., Van Loan C.F. *Matrix Computations.* John Hopkins University Press, Baltimore, USA, 1996.

[52] Gregory S. *An algorithm to find overlapping community structure in networks.* Proceedings of 11th European conference on principles and practice of knowledge discovery in databases (PKDD'07), pages 91–102, 2007.

[53] Gutman I., Randić M. *Algebraic characterization of skeletal branching.* Chemical Physics Letters, 1977, **47(1)**, 15–17.

[54] Harary F. *Graph Theory.* Addison Wesley Publishing Company, Reading, MA, USA, 1969.

[55] Hirata H., Ulanowicz R.E. *Information theoretical analysis of ecological networks.* Int. J. Syst. Sci., 1984, **15**, 261–270.

[56] Holme P., Huss M., Jeong H. *Subnetwork hierarchies of biochemical pathways.* Bioinformatics, 2003, **19**, 532–538.

[57] Huber W., Carey V., Long L., Falcon S., Gentleman R. *Graphs in molecular biology.* BMC Bioinformatics, 2007, **8(Suppl 6):S8**.

[58] Jungnickel D. *Graphs, networks and algorithms.* Springer/Verlag, 2005.

[59] Junker B.H., Schreiber F. *Analysis of biological networks.* Wiley-Interscience, 2008.

[60] Karypis G., Kumar V. *A fast and high quality multilevel scheme for partitioning irregular graphs.* SIAM Journal on Scientific Computing, 1999, **20**, 359–392.

[61] Kernighan B.W., Lin S. *An efficient heuristic procedure for partitioning graphs.* Bell Systems Technical Journal, 1970, **49**, 291–307.

[62] Kim J., Wilhelm T. *What is a complex graph?* Physica A, 2008, **387**, 2637–2652.

[63] Konstantinova E.V. *On some applications of information indices in chemical graph theory.* In: Ahlswede R., Bäumer L., Cai N., Aydinian H., Blinovsky V., Deppe C., Mashurian H., editors, *General Theory of Information Transfer and Combinatorics*, Lecture Notes of Computer Science, pages 831–852. Springer, 2006.

[64] Konstantinova E.V., Paleev A.A. *Sensitivity of topological indices of polycyclic graphs.* Vychisl. Sistemy, 1990, **136**, 38–48 (in Russian).

[65] Konstantinova E.V., Skorobogatov V.A., Vidyuk M.V. *Applications of information theory in chemical graph theory.* Indian Journal of Chemistry, 2002, **42**, 1227–1240.

[66] Körner J. *Coding of an information source having ambiguous alphabet and the entropy of graphs.* Transactions of the 6-th Prague Conference on Information Theory, pages 411–425, 1973.

[67] Lauritzen S.L. *Graphical Models.* Oxford Statistical Science Series. Oxford University Press, 1996.

[68] Linshitz H. *The information content of a battery cell.* In: Quastler H., editor, *Essays on the Use of Information Theory in Biology*, University of Illinois Press, Urbana, IL, USA, 1953.

[69] Mazurie A., Bonchev D., Schwikowski B., Buck G.A. *Phylogenetic distances are encoded in networks of interacting pathways.* Bioinformatics, 2008, **24(22)**, 2579–2585.

[70] McCaskill J.S. *The equilibrium partition function and base pair binding probabilities for RNA secondary structure.* Biopolymers, 1990, **29**, 1105–1119.

[71] Mehler A. *A Quantitative Graph Model of Social Ontologies by Example of Wikipedia.* In: Dehmer H., Emmert-Streib F., editors, *Towards an Information Theory of Complex Networks: Statistical Methods and Applications*, Birkhäuser, Boston/Basel, 2010, to appear.

[72] Minoli D. *Combinatorial graph complexity.* Atti. Accad. Naz. Lincei, VIII. Ser., Rend., Cl. Sci. Fis. Mat. Nat., 1975, **59**, 651–661.

[73] Morowitz H. *Some order-disorder considerations in living systems.* Bull. Math. Biophys., 1953, **17**, 81–86.

[74] Mowshowitz A. *Entropy and the complexity of the graphs I: An index of the relative complexity of a graph.* Bull. Math. Biophys., 1968, **30**, 175–204.

[75] Mowshowitz A. *Entropy and the complexity of graphs IV: Entropy measures and graphical structure.* Bull. Math. Biophys., 1968, **30**, 533–546.

[76] Narasimhamurthy A., Greene D., Hurley N., Cunningham P. *Partitioning large networks without breaking communities.* Springer-Verlag, London, 2009.

[77] Newman M.E.J. *Detecting community structure in networks.* The European Physical Journal B, 2004, **38**, 321–330.

[78] Newman M.E.J., Girvan M. *Finding and evaluating community structure in networks.* Physical Review E, 2003, **69**, 026113(15).

[79] Nussinov R., Jacobson A.B. *Fast algorithm for predicting the secondary structure of single-stranded RNA.* PNAS, 1980, **77(11)**, 6309–6313.

[80] Ogata H., Fujibuchi W., Goto S., Kanehisa M. *A heuristic graph comparison algorithm and its application to detect functionally.* Nucleic Acids Res., 2000, **28(20)**, 4021–4028.

[81] Palla G., Derenyi I., Farkas I., Vicsek T. *Uncovering the overlapping community structure of complex networks in nature and society.* Nature, 2005, **435**, 814–818.

[82] Patil A., Esfahanian A.-H., Xiao L., Liu Y. *Resource allocation using multiple edge-sharing multicast trees.* IEEE Transactions on Vehicular Technology, 2007, **58**, 3178–3186.

[83] Pothen A., Simon H.D., Liou K.P. *Partitioning sparse matrices with eigenvectors of graphs.* SIAM Journal on Mathematical Analysis, 1990, **11**, 430–452.

[84] Pržulj N. *Network comparison using graphlet degree distribution.* Bioinformatics, 2007, **23**, e177–e183.

[85] Randić M. *On characterization of molecular branching.* J. Amer. Chem. Soc., 1975, **97**, 6609–6615.

[86] Randić M., Plavšić D. *Characterization of molecular complexity.* International Journal of Quantum Chemistry, 2002, **91**, 20–31.

[87] Rashevsky N. *Life, information theory, and topology.* Bull. Math. Biophys., 1955, **17**, 229–235.

[88] Rison S.C.G., Thornton J.M. *Pathway evolution, structurally speaking.* Current Opinion in Structural Biology, 2002, **12**, 374–382.

[89] Robinson D.F., Foulds L.R. *Comparison of phylogenetic trees.* Mathematical Biosciences, 1981, **53**, 131–147.

[90] Rosselló F., Valiente G. *Graph transformation in molecular biology.* In: Kreowski H.J., Montanari U., Orejas F., Rozenberg G., Taentzer G., editors, *Formal Methods in Software and Systems Modeling*, pages 116–133, 2005.

[91] Schultz H.P., Schultz E.B., Schultz T.P. *Topological organic chemistry. 4. Graph Theory, Matrix Permanents, and Topological Indices of Alkanes.* Journal of Chemical Information and Computer Sciences, 1992, **32(1)**, 69–72.

[92] Selkow S.M. *The tree-to-tree editing problem.* Information Processing Letters, 1977, **6**, 184–186.

[93] Semple C., Steel M. *Phylogenetics.* Graduate Series Mathematics and its Applications. Oxford University Press, 2003.

[94] Shannon C.E., Weaver W. *The Mathematical Theory of Communication.* University of Illinois Press, Urbana, IL, USA, 1997.

[95] Shi J., Malik J. *Normalized cuts and image segmentation.* Proceedings of IEEE conference on computer vision and pattern recognition (CVPR '97), pages 731–737, 1997.

[96] Simonyi G. *Graph entropy: A survey.* In: Cook W., Lovász L., Seymour P., editors, *Combinatorial Optimization*, Vol. 20, pages 399–441. DIMACS Series in Discrete Mathematics and Theoretical Computer Science, 1995.

[97] Skorobogatov V.A., Dobrynin A.A. *Metrical analysis of graphs.* Commun. Math. Comp. Chem., 1988, **23**, 105–155.

[98] Solé R.V., Valverde S. *Information theory of complex networks: On evolution and architectural constraints.* Lecture Notes in Physics, Vol. 650, pages 189–207, 2004.

[99] Steel M., Penny D. *Parsimony, likelihood, and the role of models in molecular phylogenetics.* Molecular Biology and evolution, 2000, **17**, 839–850.

[100] Steel M., Székely L., Mossel E. *Phylogenetic information complexity: Is testing a tree easier than finding it?* Journal of Theoretical Biology, 2009, **258**, 95–102.

[101] Tai K.C. *The tree-to-tree correction problem.* Journal of the Association for Computing Machinery, 1979, **26**, 422–433.

[102] Todeschini R., Consonni V., Mannhold R. *Handbook of Molecular Descriptors.* Wiley-VCH, Weinheim, Germany, 2002.

[103] Trinajstić N. *Chemical Graph Theory.* CRC Press, Boca Raton, FL, USA, 1992.

[104] Trucco E. *A note on the information content of graphs.* Bull. Math. Biol., 1956, **18(2)**, 129–135.

[105] Ulanowicz R.E. *Information theory in ecology.* Computers and Chemistry, 2001, **25**, 393–399.

[106] Vahid F., Dm Le T. *Extending the kernighan/lin heuristic for hardware and software functional partitioning.* Design Automation for Embedded Systems, 1997, **2(2)**, 237–261.

[107] Varmuza K., Demuth W., Karlovits M., Scsibrany H. *Binary substructure descriptors for organic compounds.* Croat. Chem. Acta, 2005, **78**, 141–149.

[108] Wang J., Provan G. *Characterizing the structural complexity of real-world complex networks.* In: Zhou J., editor, *Complex Sciences*, Vol. 4 of *Lecture Notes of the Institute for Computer Sciences, Social Informatics and Telecommunications Engineering*, pages 1178–1189, Springer, Berlin/Heidelberg, Germany, 2009.

[109] Washietl S., Gesell T. *Graph representations and algorithms in computational biology of RNA secondary structure.* Structural Analysis of Complex Networks. Birkhäuser/Springer, 2010. To appear.

[110] Wasserman S., Faust K. *Social Network Analysis: Methods and Applications.* Structural Analysis in the Social Sciences. Cambridge University Press, 1994.

[111] Waterman M.S. *Secondary structure of single-stranded nucleic acids.* Studies on foundations and combinatorics, Advances in mathematics supplementary studies, Academic Press N.Y., 1978, **1**, 167–212.

[112] Wiener H. *Structural determination of paraffin boiling points.* Journal of the American Chemical Society, 1947, **69(17)**, 17–20.

[113] Wilkinson D., Huberman B.A. *A method for finding communities of related genes.* PNAS, 2004, **101(1)**, 5241–5248.

[114] Xia K., Fu Z., Hou L. *Impacts of protein-protein interaction domains on organism and network complexity.* Genome Res., 2008, **18**, 1500–1508.

[115] Zachary W. *An information flow model for conflict and fission in small groups.* Journal of Anthropological Research, 1977, **33**, 452–473.

[116] Zuker M., Stiegler P. *Optimal computer folding of large RNA sequences using thermodynamics and auxiliary information.* Nucleic Acids Research, 1981, **9(1)**, 133–148.

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About characteristics of graded algebras $S_{1,4}$ and $SI_{1,4}$

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Abstract. Hilbert series for the graded algebras of comitants $S_{1,4}$ and invariants $SI_{1,4}$ of differential system are constructed and with their help the Krull dimensions of these algebras are determined. The lower bounds for the number of the types of generators for the algebras $S_{1,4}$ and $SI_{1,4}$ are obtained

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1 Introduction

We consider differential system of the form

$$\dot{x}^j = a^j_\alpha x^\alpha + a^j_{\alpha\beta\gamma\delta} x^\alpha x^\beta x^\gamma x^\delta \ (j, \alpha, \beta, \gamma, \delta = 1, 2), \tag{1}$$

where the coefficient tensor $a^j_{\alpha\beta\gamma\delta}$ is symmetrical in lower indices in which the complete convolution holds.

In [1-7] the authors presented different methods of the study of the set of centro-affine invariants and comitants of the system (1), which later find an application for the qualitative study of these systems.

One of these methods is the method of generating functions and Hilbert series described in [5-7]. The method goes back to classical works [8-16] for invariants of binary forms, takes further investigation in works [17-28] for graded algebras of invariants of indicated forms and also for graded algebras of different abstract objects.

From [5-6] it is known that in order to construct a minimal polynomial base of invariants and comitants [3,4] of the system (1) it is enough to construct generators of algebra of unimodular comitants and invariants of indicated system.

Consider the system (1) with the group of unimodular transformations $SL(2, \mathbb{R})$, it is supposed this group acts in a natural way in $E^{16}(x, a)$, where $x = (x^1, x^2)$ is a vector of phase variables, and a is the set of coefficients of the system (1).

Let $\mathbb{R}[E^{16}(x, a)]$ be the algebra of polynomials on $E^{16}(x, a)$. The group $SL(2, \mathbb{R})$ acts also in $\mathbb{R}[E^{16}(x, a)]$.

Let $S_{1,4}$ be subalgebra of polynomials, depending only on x, a from $\mathbb{R}[E^{16}(x, a)]$, and it is formed from $SL(2, \mathbb{R})$ comitants [5-6] of the system (1).

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Following [5-6], we shall name $S_{1,4}$ the algebra of comitants, and its subalgebra $SI_{1,4}$ of polynomials depending only on x, will be called the algebra of invariants.

Let $\mathbb{R}[E^{16}(x,a)]^{(d)}$ be the set of polynomials of the type $(d) = (\delta, d_1, d_2)$, homogeneous of degree δ in variables $x = (x^1, x^2)$, of degree d_1 in coefficient tensor a^j_α, of degree d_2 in coefficient tensor $a^j_{\alpha\beta\gamma\delta}$. Let us assume

$$S^{(d)}_{1,4} = S_{1,4} \bigcap \mathbb{R}[E^{16}(x,a)]^{(d)}.$$

The algebra $S_{1,4}$ is graded through $S^{(d)}_{1,4}$ and

$$S_{1,4} = \bigoplus_{(d)} S^{(d)}_{1,4}\ (d \geq 0),\ S^{(d)}_{1,4} S^{(e)}_{1,4} \subseteq S^{(d+e)}_{1,4}$$

is considered finitely determined [5-6] for $S^{(0)}_{1,4} = \mathbb{R}$.

It is known [1-6], that $\mathbb{R}$-algebra $S_{1,4}$ is locally finite, i.e. $dim_{\mathbb{R}} S^{(d)}_{1,4} < \infty$ for all (d).

The arising here sequence is $\{dim_{\mathbb{R}} S^{(d)}_{1,4}\}$, and the corresponding generalized Hilbert series [5-6] is

$$H(S_{1,4}, u, b, e) = \sum_{(d)} dim_{\mathbb{R}} S^{(d)}_{1,4} u^\delta b^{d_1} e^{d_2}, \tag{2}$$

where $dim_{\mathbb{R}} S^{(0)}_{1,4} = 1$, and $(d) = (\delta, d_1, d_2)$.

The common Hilbert series is obtained from the generalized one as follows

$$H_{S_{1,4}}(u) = H(S_{1,4}, u, u, u). \tag{3}$$

Remark 1. The generalized (common) Hilbert series for the graded algebra of invariants $SI_{1,4} \subset S_{1,4}$ of the system (1) is formally obtained from (2) for $u = 0$ $(b = e = z)$.

Remark 2. Following [19], we remark that the transcendent degree over $\mathbb{R}$ of the field of quotients of algebra $S_{1,4}$ $(SI_{1,4})$ is called its dimension of Krull $\varrho(S_{1,4})$ $(\widetilde{\varrho}(SI_{1,4}))$. This dimension is equal to the maximum number of algebraically independent homogeneous elements in $S_{1,4}$ $(SI_{1,4})$, and also to the order of the pole of common Hilbert series at the unit.

2 Hilbert series and Krull dimensions for algebras $S_{1,4}$ and $SI_{1,4}$

We determine the lower bounds for the number and the totality of types of generators for algebras $S_{1,4}$ and $SI_{1,4}$. For that we construct Hilbert series for algebras $S_{1,4}$ and $SI_{1,4}$ for the system (1).

Following [5-6] we obtain that $dim_{\mathbb{R}} S_{1,4}^{(d)}$ is equal to the coefficient of $u^\delta b^{d_1} e^{d_2}$ in the expansion of initial function

$$\varphi_{1,4}^{(0)}(u) = \frac{1-u^{-2}}{(1-u^2 b)(1-b)^2(1-u^{-2}b)} \times$$
$$\times \frac{1}{(1-u^5 e)(1-u^3 e)^2(1-ue)^2(1-u^{-1}e)^2(1-u^{-3}e)^2(1-u^{-5}e)}. \tag{4}$$

From [5-6] it is known that the generalized Hilbert series (2) is the solution of the functional Cayley equation

$$H(S_{1,4}, u, b, e) - u^{-2} H(S_{1,4}, u^{-1}, b, e) = \varphi_{1,4}^{(0)}(u),$$

where $\varphi_{1,4}^{(0)}(u)$ is from (4).

Taking into consideration the last equality takes place

Theorem 1. *The generalized Hilbert series for the graded algebra $S_{1,4}$ of the system* (1) *is a rational function of u, b, e and has the form*

$$H(S_{1,4}, u, b, e) = \frac{N_{1,4}(u, b, e)}{D_{1,4}(u, b, e)}, \tag{5}$$

where

$$D_{1,4}(u, b, e) = (1-b)(1-b^2)(1-bu^2)(1-be^2)^2(1-b^3e^2)^2(1-b^5e^2)(1-e^4)^2 \times$$
$$\times (1-e^2)(1-e^6)^2(1-e^8)^2(1-eu)^2(1-eu^3)^2(1-eu^5), \tag{6}$$

$$N_{1,4}(u, b, e) = \sum_{k=0}^{13} R_k(b, e) u^k, \tag{7}$$

and

$$R_0(b,e) = 1 - e^2 + 4e^4 + e^6 + 18e^8 + 11e^{10} + 35e^{12} + 13e^{14} + 35e^{16} + 11e^{18} + 18e^{20} + e^{22} + 4e^{24} - e^{26} + e^{28} + b(e^2 + 5e^4 + 13e^6 + 26e^8 + 29e^{10} + 40e^{12} + 19e^{14} + 36e^{16} - 5e^{18} + 6e^{20} - 15e^{22} + e^{24} - 5e^{26} + 2e^{28} - 2e^{30}) + b^2(e^2 + 8e^4 + 16e^6 + 26e^8 + 27e^{10} + 20e^{12} + 12e^{14} - 11e^{16} - 29e^{18} - 31e^{20} - 22e^{22} - 11e^{24} - 4e^{26} - 2e^{28} - e^{30} + e^{32}) + b^3(e^2 + 10e^4 + 10e^6 + 24e^8 - 5e^{10} + 7e^{12} - 64e^{14} - 49e^{16} - 107e^{18} - 55e^{20} - 58e^{22} - 10e^{24} - 10e^{26} + 3e^{28} + e^{30}) + b^4(e^2 + 6e^4 + 9e^6 + 10e^8 - 7e^{10} - 29e^{12} - 87e^{14} - 75e^{16} - 117e^{18} - 29e^{20} - 30e^{22} + 26e^{24} + 2e^{26} + 17e^{28} - 3e^{30} + 4e^{32}) + b^5(5e^4 + 3e^6 + 10e^8 - 21e^{10} - 38e^{12} - 82e^{14} - 76e^{16} - 72e^{18} + e^{20} + 20e^{22} + 44e^{24} + 32e^{26} + 17e^{28} + 6e^{30} + 2e^{32} - 2e^{34}) + b^6(2e^4 + 3e^6 - 2e^8 - 29e^{10} - 36e^{12} - 84e^{14} - 41e^{16} - 48e^{18} + 48e^{20} + 41e^{22} + 84e^{24} + 36e^{26} + 29e^{28} + 2e^{30} - 3e^{32} - 2e^{34}) + b^7(2e^4 - 2e^6 - 6e^8 - 17e^{10} - 32e^{12} - 44e^{14} - 20e^{16} - e^{18} + 72e^{20} + 76e^{22} + 82e^{24} + 38e^{26} + 21e^{28} - 10e^{30} - 3e^{32} - 5e^{34}) + b^8(-4e^6 + 3e^8 - 17e^{10} - 2e^{12} - 26e^{14} + 30e^{16} + 29e^{18} + 117e^{20} + 75e^{22} + 87e^{24} + 29e^{26} + 7e^{28} - 10e^{30} - 9e^{32} - 6e^{34} - e^{36}) + b^9(-e^8 - 3e^{10} + 10e^{12} + 10e^{14} + 58e^{16} + 55e^{18} + 107e^{20} + 49e^{22} + 64e^{24} - 7e^{26} + 5e^{28} - 24e^{30} - 10e^{32} - 10e^{34} - e^{36}) + b^{10}(-e^6 + e^8 + 2e^{10} + 4e^{12} + 11e^{14} + 22e^{16} + 31e^{18} + 29e^{20} + 11e^{22} - 12e^{24} - 20e^{26} - 27e^{28} - 26e^{30} - 16e^{32} - 8e^{34} - e^{36}) + b^{11}(2e^8 - 2e^{10} + 5e^{12} - e^{14} + 15e^{16} - 6e^{18} + 5e^{20} - 36e^{22} - 19e^{24} - 40e^{26} - 29e^{28} - 26e^{30} - 13e^{32} - 5e^{34} - e^{36}) + b^{12}(-e^{10} + e^{12} - 4e^{14} - e^{16} - 18e^{18} - 11e^{20} - 35e^{22} - 13e^{24} - 35e^{26} - 11e^{28} - 18e^{30} - e^{32} - 4e^{34} + e^{36} - e^{38}),$$

$$R_1(b,e) = -2e + 4e^3 + e^5 + 15e^7 - 8e^9 + 21e^{11} - 18e^{13} + 26e^{15} - 27e^{17} + 6e^{19} - 19e^{21} + 7e^{23} - 6e^{25} + 2e^{27} - 2e^{29} + b(e + 5e^3 + 8e^5 + 5e^7 + e^9 + 15e^{11} - 8e^{13} + 16e^{15} - 47e^{17} + 11e^{19} - 19e^{21} + 18e^{23} - 13e^{25} + 7e^{27} - 4e^{29} + 4e^{31}) + b^2(2e + 6e^3 + 2e^5 + 6e^7 + 9e^9 + 7e^{11} - 7e^{13} - 34e^{15} - 24e^{17} + 2e^{19} + 11e^{21} + 6e^{23} + 5e^{25} + 5e^{27} + 4e^{29} + 2e^{31} - 2e^{33}) + b^3(e + 4e^3 + 16e^7 - 11e^9 + 14e^{11} - 77e^{13} - e^{15} - 63e^{17} + 59e^{19} - 6e^{21} + 52e^{23} - 3e^{25} + 19e^{27} - 3e^{29} - e^{31}) + b^4(4e^3 + 5e^5 + 3e^7 - 9e^9 - 26e^{11} - 60e^{13} + 4e^{15} - 35e^{17} + 88e^{19} - 11e^{21} + 51e^{23} - 24e^{25} + 30e^{27} - 20e^{29} + 8e^{31} - 8e^{33}) + b^5(4e^3 + 7e^7 - 23e^9 - 19e^{11} - 49e^{13} + e^{15} + e^{17} + 73e^{19} + 12e^{21} + 30e^{23} - e^{25} - 9e^{27} - 16e^{29} - 10e^{31} - 5e^{33} + 4e^{35}) + b^6(e^3 + e^5 - 3e^7 - 16e^9 - 10e^{11} - 51e^{13} + 26e^{15} - 11e^{17} + 96e^{19} - 3e^{21} + 60e^{23} - 45e^{25} - 4e^{27} - 38e^{29} - 7e^{31} + e^{33} + 3e^{35}) + b^7(-3e^5 + 2e^7 - 6e^9 - 21e^{11} - 23e^{13} + 18e^{15} + 26e^{17} + 73e^{19} + 7e^{21} + 11e^{23} - 39e^{25} - 15e^{27} - 39e^{29} + 7e^{31} - 4e^{33} + 6e^{35}) + b^8(6e^7 - 20e^9 - 24e^{13} + 61e^{15} + 9e^{17} + 88e^{19} - 49e^{21} + 20e^{23} - 56e^{25} - 18e^{27} - 25e^{29} - e^{31} + e^{33} + 6e^{35} + 2e^{37}) + b^9(-e^7 - e^9 + 7e^{11} + 3e^{13} + 44e^{15} + 45e^{19} - 53e^{21} + 31e^{23} - 65e^{25} + 10e^{27} - 47e^{29} + 12e^{31} + 14e^{35} + e^{37}) + b^{10}(2e^7 - 2e^9 - e^{11} + 9e^{13} + 16e^{15} + 7e^{17} - 6e^{19} - 12e^{21} - 12e^{23} - 9e^{25} - 21e^{27} - 7e^{29} + 14e^{31} + 14e^{33} + 8e^{35}) + b^{11}(-4e^9 + 7e^{11} + e^{13} + 14e^{15} - 23e^{17} + 11e^{19} - 35e^{21} + 18e^{23} - 34e^{25} + 7e^{27} + 5e^{29} + 21e^{31} + 8e^{33} + 3e^{35} + e^{37}) + b^{12}(2e^{11} - 4e^{13} - e^{15} - 15e^{17} + 8e^{19} - 21e^{21} + 18e^{23} - 26e^{25} + 27e^{27} - 6e^{29} + 19e^{31} - 7e^{33} + 6e^{35} - 2e^{37} + 2e^{39}),$$

$$R_2(b,e) = 4e^2 - 2e^4 + 7e^6 - 5e^8 + 27e^{10} - 13e^{12} + 20e^{14} - 29e^{16} + 14e^{18} - 17e^{20} + 5e^{22} - 14e^{24} + 3e^{26} - e^{28} + e^{30} + b(2e^2 - 3e^4 + 7e^6 + e^8 + 11e^{10} - 53e^{12} - 14e^{14} - 73e^{16} + 9e^{18} - 45e^{20} + 8e^{22} - 17e^{24} + 21e^{26} - 5e^{28} + 2e^{30} - 2e^{32}) + b^2(7e^6 - 12e^8 - 39e^{10} - 64e^{12} - 39e^{14} - 37e^{16} + 2e^{18} - 16e^{20} + 16e^{22} + 17e^{24} + 16e^{26} - 3e^{28} + e^{30} - e^{32} + e^{34}) + b^3(2e^2 - e^4 + e^6 - 42e^8 - 30e^{10} - 85e^{12} + 8e^{14} - 44e^{16} + 67e^{18} + e^{20} + 81e^{22} + 16e^{24} + 32e^{26} - 6e^{28} + e^{30} - e^{32}) + b^4(2e^2 - 5e^4 - 6e^6 - 30e^8 - 18e^{10} - 53e^{12} + 65e^{14} - 3e^{16} + 159e^{18} + 44e^{20} + 143e^{22} + 9e^{24} + 40e^{26} - 47e^{28} + 5e^{30} - 7e^{32} + 4e^{34}) + b^5(e^2 - 5e^4 - 28e^8 - 11e^{10} - 14e^{12} + 84e^{14} + 57e^{16} + 172e^{18} + 51e^{20} + 89e^{22} - 23e^{24} - 25e^{26} - 43e^{28} + e^{30} - 4e^{32} + 2e^{34} - 2e^{36}) + b^6(-e^4 - e^6 - 26e^8 + e^{10} - 12e^{12} + 103e^{14} + 67e^{16} + 162e^{18} + 6e^{20} + 51e^{22} - 98e^{24} - 37e^{26} - 59e^{28} - 4e^{30} - 4e^{32} + 3e^{34}) + b^7(-e^6 - 13e^8 + 7e^{10} + 22e^{12} + 103e^{14} + 51e^{16} + 104e^{18} - 20e^{20} - 9e^{22} - 119e^{24} - 60e^{26} - 79e^{28} + 8e^{30} - 5e^{32} + 10e^{34} + e^{36}) + b^8(-2e^4 + 2e^6 - 11e^8 + 26e^{10} + 23e^{12} + 87e^{14} + 79e^{18} - 92e^{20} - 24e^{22} - 141e^{24} - 68e^{26} - 61e^{28} + 12e^{30} + 3e^{32} + 14e^{34} + 3e^{36} - e^{38}) + b^9(3e^6 - 4e^8 + 22e^{10} + 5e^{12} + 55e^{14} - 30e^{16} + 30e^{18} - 137e^{20} - 41e^{22} - 183e^{24} - 39e^{26} - 53e^{28} + 43e^{30} + 10e^{32} + 18e^{34} - 2e^{36} + e^{38}) + b^{10}(-e^8 + 7e^{10} - 2e^{12} - 4e^{14} - 53e^{16} - 42e^{18} - 104e^{20} - 79e^{22} - 99e^{24} - 10e^{26} + 11e^{28} + 45e^{30} + 12e^{32} + 10e^{34} + 4e^{36} + 3e^{38}) + b^{11}(e^6 - e^8 + 5e^{10} - 9e^{12} - 6e^{14} - 36e^{16} - 10e^{18} - 67e^{20} - 7e^{22} - 19e^{24} + 46e^{26} + 31e^{28} + 35e^{30} + 13e^{32} + 17e^{34} + 7e^{36}) + b^{12}(-2e^8 + 2e^{10} - 9e^{12} + 3e^{14} - 22e^{16} + 11e^{18} - 32e^{20} + 49e^{22} - e^{24} + 69e^{26} + 15e^{28} + 43e^{30} + 8e^{32} + 19e^{34} - 2e^{36} + e^{38} - e^{40}) + b^{13}(e^{10} - e^{12} + 4e^{14} + e^{16} + 18e^{18} + 11e^{20} + 35e^{22} + 13e^{24} + 35e^{26} + 11e^{28} + 18e^{30} + e^{32} + 4e^{34} - e^{36} + e^{38}),$$

$$R_3(b,e) = -e + e^3 + e^5 - 3e^7 - 16e^9 - 38e^{11} - 45e^{13} - 46e^{15} - 56e^{17} - 50e^{19} - 32e^{21} - 14e^{23} - e^{27} - 2e^{29} + b(2e - e^3 - e^5 - 30e^7 - 35e^9 - 87e^{11} - 44e^{13} - 95e^{15} - 44e^{17} - 33e^{19} + 27e^{21} + 15e^{23} + 22e^{25} - 5e^{27} + 3e^{29} + 4e^{31}) + b^2(e - 2e^3 - 14e^5 - 32e^7 - 48e^9 - 64e^{11} - 46e^{13} - 73e^{15} + 41e^{17} + 49e^{19} + 101e^{21} + 45e^{23} + 36e^{25} - e^{27} + 12e^{29} - 3e^{31} - 2e^{33}) + b^3(-3e^3 - 13e^5 - 31e^7 - 43e^9 - 45e^{11} - 10e^{13} + 77e^{15} + 156e^{17} + 197e^{19} + 174e^{21} + 105e^{23} + 42e^{25} + 7e^{27} - 5e^{29} - 5e^{31} + e^{33}) + b^4(-2e^3 - 6e^5 - 31e^7 - 23e^9 - 46e^{11} + 110e^{13} + 113e^{15} + 271e^{17} + 189e^{19} + 148e^{21} - 16e^{23} - 19e^{25} - 61e^{27} - 10e^{29} - 6e^{31} - 7e^{33}) + b^5(-e^3 - 5e^5 - 20e^7 - 23e^9 + 20e^{11} + 115e^{13} + 132e^{15} + 252e^{17} + 92e^{19} + 61e^{21} - 97e^{23} - 75e^{25} - 110e^{27} - 17e^{29} - 33e^{31} + 5e^{33} + 6e^{35}) + b^6(-4e^5 - 23e^7 + 14e^9 + 33e^{11} + 122e^{13} + 144e^{15} + 165e^{17} + 40e^{19} - 26e^{21} - 159e^{23} - 158e^{25} - 114e^{27} - 48e^{29} - 5e^{31} + 17e^{33} + 3e^{35} - e^{37}) + b^7(-4e^5 + e^7 + 14e^9 + 19e^{11} + 109e^{13} + 69e^{15} + 126e^{17} - 49e^{19} - 116e^{21} - 252e^{23} - 138e^{25} - 122e^{27} + 4e^{29} + 13e^{31} + 16e^{33} + 8e^{35}) + b^8(4e^5 - e^7 + e^9 + 31e^{11} + 52e^{13} + 31e^{15} + 17e^{17} - 147e^{19} - 199e^{21} - 230e^{23} - 154e^{25} - 84e^{27} + 24e^{29} + 6e^{31} + 33e^{33} + 11e^{35} + e^{37}) + b^9(-6e^7 + 21e^9 + 4e^{11} + 28e^{13} - 63e^{15} - 61e^{17} - 224e^{19} - 145e^{21} - 207e^{23} - 58e^{25} - 18e^{27} + 35e^{29} + 35e^{31} + 38e^{33} + 16e^{35} + 2e^{37} - e^{39}) + b^{10}(2e^7 - 8e^{11} + e^{13} - 51e^{15} - 36e^{17} - 122e^{19} - 33e^{21} - 59e^{23} + 84e^{25} + 23e^{27} + 97e^{29} + 44e^{31} + 48e^{33} + 11e^{35} + e^{37} - 2e^{39}) + b^{11}(-2e^7 + 3e^{11} - 8e^{13} -$$

$-24e^{15}-26e^{17}-21e^{19}+20e^{21}+58e^{23}+56e^{25}+84e^{27}+82e^{29}+50e^{31}+28e^{33}+$
$+2e^{35}-e^{37}+e^{39})+b^{12}(4e^{9}-6e^{11}-2e^{13}-15e^{15}+26e^{17}+5e^{19}+73e^{21}+$
$+27e^{23}+80e^{25}+49e^{27}+45e^{29}+11e^{31}+6e^{33}-3e^{35}+2e^{39})+b^{13}(-2e^{11}+$
$+4e^{13}+e^{15}+15e^{17}-8e^{19}+21e^{21}-18e^{23}+26e^{25}-27e^{27}+6e^{29}-19e^{31}+$
$+7e^{33}-6e^{35}+2e^{37}-2e^{39}),$

$R_4(b,e)=3e^{2}-2e^{4}-2e^{6}-9e^{8}+8e^{10}-17e^{12}+11e^{14}-43e^{16}+26e^{18}-e^{20}+$
$+25e^{22}-6e^{24}+6e^{26}-3e^{28}+4e^{30}+b(-2e^{2}-e^{4}-12e^{6}+7e^{8}-32e^{10}-6e^{12}-$
$-36e^{14}+3e^{16}+54e^{18}+14e^{20}+12e^{22}-7e^{24}+10e^{26}-2e^{28}+6e^{30}-8e^{32})+$
$+b^{2}(-2e^{2}-4e^{4}-3e^{6}-22e^{8}-39e^{10}-13e^{12}-11e^{14}+89e^{16}+26e^{18}+25e^{20}-$
$-20e^{22}+9e^{24}-22e^{26}-14e^{30}-3e^{32}+4e^{34})+b^{3}(-4e^{4}-16e^{6}-30e^{8}-10e^{10}-$
$-12e^{12}+92e^{14}+27e^{16}+98e^{18}-37e^{20}+10e^{22}-75e^{24}-12e^{26}-33e^{28}+$
$+2e^{32})+b^{4}(e^{2}-8e^{4}-22e^{6}-2e^{8}-20e^{10}+92e^{12}+44e^{14}+88e^{16}+32e^{18}-$
$-72e^{20}-49e^{22}-54e^{24}-19e^{26}-32e^{28}+17e^{30}-12e^{32}+16e^{34})+b^{5}(-5e^{4}-$
$-10e^{6}-16e^{8}+27e^{10}+61e^{12}+59e^{14}+96e^{16}-31e^{18}-71e^{20}-89e^{22}-54e^{24}-$
$-54e^{26}+46e^{28}+7e^{30}+32e^{32}+10e^{34}-8e^{36})+b^{6}(-2e^{4}-13e^{6}+14e^{8}+10e^{10}+$
$+49e^{12}+73e^{14}+35e^{16}-14e^{18}-113e^{20}-84e^{22}-105e^{24}+48e^{26}+22e^{28}+$
$+62e^{30}+23e^{32}-e^{34}-4e^{36})+b^{7}(-2e^{4}+2e^{6}-e^{8}+8e^{10}+81e^{12}+25e^{14}+$
$+32e^{16}-99e^{18}-108e^{20}-101e^{22}+18e^{24}+17e^{26}+67e^{28}+58e^{30}+4e^{32}+$
$+11e^{34}-10e^{36}-2e^{38})+b^{8}(-5e^{6}+39e^{10}+28e^{12}+20e^{14}-39e^{16}-77e^{18}-$
$-133e^{20}+10e^{22}-29e^{24}+68e^{26}+77e^{28}+33e^{30}+30e^{32}-2e^{34}-15e^{36}-$
$-5e^{38})+b^{9}(17e^{8}-8e^{10}+13e^{12}-34e^{14}-13e^{16}-90e^{18}-33e^{20}-22e^{22}+$
$+2e^{24}+88e^{26}+38e^{28}+78e^{30}+4e^{32}-10e^{34}-28e^{36}-2e^{38})+b^{10}(e^{6}-$
$-2e^{8}+6e^{12}-39e^{14}-20e^{16}-69e^{18}+22e^{20}-23e^{22}+88e^{24}+7e^{26}+92e^{28}-$
$-2e^{30}-10e^{32}-34e^{34}-14e^{36}-3e^{38})+b^{11}(2e^{8}+2e^{10}-14e^{12}-13e^{14}-$
$-14e^{16}+5e^{18}-14e^{20}+55e^{22}-e^{24}+73e^{26}-20e^{30}-29e^{32}-17e^{34}-10e^{36}-$
$-4e^{38}-e^{40})+b^{12}(-7e^{10}+2e^{12}-3e^{14}+16e^{16}-12e^{18}+30e^{20}+13e^{22}+$
$+20e^{24}+17e^{26}-30e^{28}-8e^{30}-22e^{32}-3e^{34}-12e^{36}+3e^{38}-4e^{40})+$
$+b^{13}(4e^{12}-2e^{14}+7e^{16}-5e^{18}+27e^{20}-13e^{22}+20e^{24}-29e^{26}+14e^{28}-$
$-17e^{30}+5e^{32}-14e^{34}+3e^{36}-e^{38}+e^{40}),$

$R_5(b,e)=-3e^{3}-2e^{5}-13e^{9}-38e^{11}-30e^{13}-40e^{15}-3e^{17}-26e^{19}-5e^{21}-$
$-6e^{23}+16e^{25}-e^{27}+2e^{29}-2e^{31}+b(e-3e^{3}-8e^{7}-35e^{9}-20e^{11}+8e^{13}+$
$+26e^{15}+64e^{17}+26e^{19}+53e^{21}+36e^{23}+19e^{25}-18e^{27}+2e^{29}-4e^{31}+4e^{33})+$
$+b^{2}(-4e^{5}-22e^{7}-8e^{9}+37e^{11}+60e^{13}+81e^{15}+78e^{17}+56e^{19}+57e^{21}+$
$+15e^{23}-29e^{25}-13e^{27}-5e^{29}-e^{31}+2e^{33}-2e^{35})+b^{3}(-3e^{3}-8e^{5}-3e^{7}+$
$+19e^{9}+49e^{11}+103e^{13}+101e^{15}+118e^{17}+36e^{19}+31e^{21}-60e^{23}-29e^{25}-$
$-47e^{27}-5e^{29}-3e^{31}+3e^{33})+b^{4}(-5e^{3}+4e^{5}+4e^{7}+8e^{9}+63e^{11}+76e^{13}+$
$+54e^{15}+29e^{17}-109e^{19}-120e^{21}-169e^{23}-115e^{25}-64e^{27}+35e^{29}+13e^{33}-$
$-6e^{35})+b^{5}(-e^{3}+e^{5}-6e^{7}+26e^{9}+59e^{11}+38e^{13}+3e^{15}-82e^{17}-204e^{19}-$
$-143e^{21}-159e^{23}-68e^{25}+28e^{27}+28e^{29}+16e^{31}+11e^{33}-3e^{35}+3e^{37})+$
$+b^{6}(-3e^{5}+2e^{7}+26e^{9}+42e^{11}+42e^{13}-33e^{15}-130e^{17}-192e^{19}-134e^{21}-$
$-117e^{23}+42e^{25}+40e^{27}+75e^{29}+32e^{31}+12e^{33}-6e^{35})+b^{7}(17e^{9}+30e^{11}-$
$-16e^{13}-90e^{15}-123e^{17}-165e^{19}-110e^{21}-20e^{23}+70e^{25}+120e^{27}+119e^{29}+$
$+22e^{31}+11e^{33}-13e^{35}-3e^{37})+b^{8}(2e^{5}+15e^{9}+2e^{11}-49e^{13}-78e^{15}-$

$-109e^{17}-145e^{19}+3e^{21}-4e^{23}+131e^{25}+143e^{27}+92e^{29}+23e^{31}+e^{33}-$
$-23e^{35}-5e^{37}+e^{39})+b^{9}(-2e^{7}+5e^{9}-7e^{11}-33e^{13}-67e^{15}-94e^{17}-$
$-43e^{19}+62e^{21}+92e^{23}+222e^{25}+140e^{27}+88e^{29}-14e^{31}-22e^{33}-25e^{35}+$
$+4e^{37}-4e^{39})+b^{10}(2e^{9}-6e^{11}-17e^{13}-7e^{15}+32e^{17}+77e^{19}+139e^{21}+$
$+187e^{23}+172e^{25}+104e^{27}+9e^{29}-34e^{31}-25e^{33}-14e^{35}-10e^{37}-5e^{39})+$
$+b^{11}(-e^{7}+e^{9}-5e^{11}-6e^{13}+12e^{15}+25e^{17}+52e^{19}+110e^{21}+61e^{23}+55e^{25}-$
$-13e^{27}-30e^{29}-37e^{31}-28e^{33}-34e^{35}-11e^{37})+b^{12}(2e^{9}-2e^{11}+2e^{13}+$
$+10e^{15}+19e^{17}+40e^{19}+30e^{21}-16e^{23}-7e^{25}-60e^{27}-49e^{29}-59e^{31}-36e^{33}-$
$-26e^{35}+2e^{37}-3e^{39}+2e^{41})+b^{13}(-e^{11}+e^{13}+e^{15}-3e^{17}-16e^{19}-38e^{21}-$
$-45e^{23}-46e^{25}-56e^{27}-50e^{29}-32e^{31}-14e^{33}-e^{37}-2e^{39}),$

$R_6(b,e)=2e^{2}-2e^{4}+e^{6}-16e^{8}+6e^{10}+5e^{12}+26e^{14}+3e^{16}+40e^{18}+30e^{20}+$
$+38e^{22}+13e^{24}+2e^{28}+3e^{30}+b(-3e^{2}-7e^{6}+12e^{10}+36e^{12}+9e^{14}+63e^{16}+$
$+48e^{18}+34e^{20}-e^{22}-19e^{24}-17e^{26}+8e^{28}-6e^{30}-6e^{32})+b^{2}(-2e^{2}-3e^{4}-$
$-e^{6}+6e^{8}+19e^{10}+19e^{12}+35e^{14}+94e^{16}-10e^{18}+3e^{20}-72e^{22}-38e^{24}-$
$-33e^{26}-6e^{28}-20e^{30}+6e^{32}+3e^{34})+b^{3}(-3e^{4}-4e^{8}+29e^{10}+31e^{12}+82e^{14}-$
$-15e^{16}-22e^{18}-120e^{20}-114e^{22}-110e^{24}-50e^{26}-19e^{28}+7e^{30}+8e^{32}-$
$-2e^{34})+b^{4}(-e^{4}-11e^{6}+4e^{8}+39e^{10}+70e^{12}-13e^{14}-12e^{16}-152e^{18}-154e^{20}-$
$-122e^{22}-36e^{24}+10e^{26}+50e^{28}+5e^{30}+9e^{32}+12e^{34})+b^{5}(-3e^{4}-7e^{6}+12e^{8}+$
$+50e^{10}-5e^{14}-48e^{16}-179e^{18}-67e^{20}-114e^{22}+17e^{24}+46e^{26}+91e^{28}+20e^{30}+$
$+52e^{32}-8e^{34}-8e^{36})+b^{6}(-3e^{4}+22e^{8}+8e^{10}+7e^{12}-19e^{14}-93e^{16}-93e^{18}-$
$-96e^{20}-65e^{22}+58e^{24}+120e^{26}+98e^{28}+69e^{30}+8e^{32}-20e^{34}-e^{36})+b^{7}(2e^{6}-$
$-e^{8}+14e^{10}+14e^{12}-48e^{14}-36e^{16}-144e^{18}-47e^{20}+29e^{22}+161e^{24}+112e^{26}+$
$+120e^{28}-3e^{32}-10e^{34}-11e^{36}-e^{38})+b^{8}(-6e^{6}+7e^{8}+24e^{10}-32e^{12}-2e^{14}-$
$-73e^{16}-65e^{18}+33e^{20}+85e^{22}+136e^{24}+159e^{26}+78e^{28}-3e^{30}+13e^{32}-40e^{34}-$
$-11e^{36}-e^{38})+b^{9}(e^{6}+11e^{8}-16e^{10}-14e^{12}-26e^{14}-10e^{16}+9e^{18}+98e^{20}+$
$+61e^{22}+170e^{24}+78e^{26}+23e^{28}+4e^{30}-32e^{32}-36e^{34}-16e^{36}-5e^{38}+2e^{40})+$
$+b^{10}(-2e^{8}+e^{10}-e^{12}-13e^{14}+17e^{16}-e^{18}+90e^{20}+35e^{22}+86e^{24}-51e^{26}+$
$+14e^{28}-82e^{30}-33e^{32}-50e^{34}-12e^{36}-e^{38}+3e^{40})+b^{11}(3e^{8}-e^{10}-13e^{12}+$
$+10e^{14}+e^{16}+29e^{18}+20e^{20}+10e^{22}-16e^{24}+11e^{26}-66e^{28}-68e^{30}-49e^{32}-$
$-29e^{34}+6e^{36}+3e^{38}-2e^{40})+b^{12}(-6e^{10}+4e^{12}+9e^{14}+8e^{16}-2e^{18}+5e^{20}-$
$-30e^{22}+10e^{24}-59e^{26}-56e^{28}-31e^{30}-11e^{32}+4e^{34}+7e^{36}-e^{38}-2e^{40})+$
$+b^{13}(3e^{12}-2e^{14}-2e^{16}-9e^{18}+8e^{20}-17e^{22}+11e^{24}-43e^{26}+26e^{28}-e^{30}+$
$+25e^{32}-6e^{34}+6e^{36}-3e^{38}+4e^{40}),$

$$R_{13-k}(b,e)=-b^{13}e^{45}R_k(b^{-1},e^{-1}),\ (k=\overline{0,6}). \tag{8}$$

Taking into consideration the equality (3) from Theorem 1 we obtain

Corollary 1. *The common Hilbert series for graded algebra of comitants $S_{1,4}$ of the system* (1) *has the form*

$$H_{S_{1,4}}(u)=\frac{n_{1,4}(u)}{d_{1,4}(u)}, \tag{9}$$

where

$$d_{1,4}(u)=(1-u^2)(1-u^3)(1-u^4)^3(1-u^5)^2(1-u^6)^3(1-u^7)(1-u^8)^2,$$

$$n_{1,4}(u) = 1 + u + u^2 + 5u^3 + 17u^4 + 39u^5 + 100u^6 + 218u^7 + 467u^8 + \\
+865u^9 + 1586u^{10} + 2685u^{11} + 4467u^{12} + 6889u^{13} + 10423u^{14} + 14934u^{15} + \\
+20921u^{16} + 27849u^{17} + 36293u^{18} + 45278u^{19} + 55254u^{20} + 64697u^{21} + \\
+74134u^{22} + 81782u^{23} + 88328u^{24} + 91866u^{25} + 93539u^{26} + 91866u^{27} + \\
+88328u^{28} + 81782u^{29} + 74134u^{30} + 64697u^{31} + 55254u^{32} + 45278u^{33} + \\
+36293u^{34} + 27849u^{35} + 20921u^{36} + 14934u^{37} + 10423u^{38} + 6889u^{39} + \\
+4467u^{40} + 2685u^{41} + 1586u^{42} + 865u^{43} + 467u^{44} + 218u^{45} + 100u^{46} + \\
+39u^{47} + 17u^{48} + 5u^{49} + u^{50} + u^{51} + u^{52}. \tag{10}$$

With the help of Remark 2 and Corollary 1 we obtain

Theorem 2. *The Krull dimension $\varrho(S_{1,4})$ for graded algebra $S_{1,4}$ is equal to 13, i.e. $\varrho(S_{1,4}) = 13$.*

According to Remark 1 from Theorem 1 follows

Corollary 2. *The generalized Hilbert series for graded algebra of invariants $SI_{1,4}$ of the system* (1) *is a rational function of b, e and has the form*

$$H(SI_{1,4}, b, e) = \frac{N_{1,4}(b,e)}{D_{1,4}(b,e)}, \tag{11}$$

where

$$D_{1,4}(b,e) = (1-b)(1-b^2)(1-e^2)(1-be^2)^2(1-b^3e^2)^2(1-b^5e^2)(1-e^4)^2\times \\
\times(1-e^6)^2(1-e^8)^2, \\
N_{1,4}(b,e) = R_0(b,e), \tag{12}$$

and $R_0(b,e)$ is from (8).

With the help of Remark 2 and Corollary 2 we obtain

Corollary 3. *The common Hilbert series for graded algebras of invariants $SI_{1,4}$ for the system* (1) *has the form*

$$H_{SI_{1,4}}(z) = \frac{N_{1,4}(z)}{D_{1,4}(z)}, \tag{13}$$

where

$$D_{1,4}(z) = (1-z^3)(1-z^4)^3(1-z^5)^2(1-z^6)^2(1-z^7)(1-z^8)^2, \\
N_{1,4}(z) = 1 + z + z^2 + 3z^3 + 8z^4 + 15z^5 + 32z^6 + 67z^7 + 129z^8 + 217z^9 + \\
+355z^{10} + 546z^{11} + 812z^{12} + 1122z^{13} + 1511z^{14} + 1948z^{15} + 2447z^{16} + \\
+2923z^{17} + 3410z^{18} + 3827z^{19} + 4183z^{20} + 4375z^{21} + 4461z^{22} + 4375z^{23} + \\
+4183z^{24} + 3827z^{25} + 3410z^{26} + 2923z^{27} + 2447z^{28} + 1948z^{29} + 1511z^{30} + \\
+1122z^{31} + 812z^{32} + 546z^{33} + 355z^{34} + 217z^{35} + 129z^{36} + 67z^{37} + 32z^{38} + \\
+15z^{39} + 8z^{40} + 3z^{41} + z^{42} + z^{43} + z^{44}. \tag{14}$$

With the of help Remark 2 and Corollary 3 we obtain

Theorem 3. *The Krull dimension $\varrho(SI_{1,4})$ for graded algebra $SI_{1,4}$ is equal to 11, i.e. $\varrho(SI_{1,4}) = 11$.*

Similarly [28] with the help of representative form of generating function, which is obtained from Hilbert series (5)-(8) by multiplication of the numerator and the denominator by expression $M_{1,4}(u, b, e) = (1+e^2)(1+ue)^2(1+u^3e)^2$ and taking into consideration the characteristics of algebras S_4 and SI_4 from [5-6] we have

Theorem 4. *The lower bound of the number of generators for the algebra $S_{1,4}(SI_{1,4})$ is not less than 311(138) irreducible comitants (invariants)[1-4], distributed in 58(20) types as follows:*

$$(0,1,0), (0,2,0), 6(0,0,4), 7(0,0,6), 15(0,0,8), 14(0,0,10), 3(0,1,2), 6(0,1,4), 15(0,1,6), 16(0,1,8), (0,2,2), 8(0,2,4), 15(0,2,6), 3(0,3,2), 10(0,3,4), 7(0,3,6), (0,4,2), 5(0,4,4), (0,5,2), 3(0,5,4), 2(1,0,3), 11(1,0,5), 20(1,0,7), 2(1,0,9), (1,1,1), 8(1,1,3), 20(1,1,5), 2(1,2,1), 9(1,2,3), 4(1,2,5), (1,3,1), 3(1,3,3), 3(1,4,3), (2,1,0), 3(2,0,2), 6(2,0,4), 12(2,0,6), 4(2,1,2), 9(2,1,4), 3(2,2,2), 2(2,3,2), (3,0,1), 6(3,0,3), 9(3,0,5), 2(3,1,1), 6(3,1,3), (3,2,1), (4,0,2), 6(4,0,4), 3(4,1,2), (5,0,1), 3(5,0,3), (5,1,1), 2(6,0,2), 2(6,0,4), (6,1,2), (7,0,3), (9,0,3).$$

The number of comitants and invariants of the given type is indicated before brackets, the omitted number means that it is equal to one.

References

[1] Sibirsky K. S. *Method of invariants in the qualitative theory of differential equations.* Kishinev, RIO AN Moldavian SSR, 1968 (in Russian).

[2] Sibirsky K. S. *Algebraic invariants of differential equations and matrices.* Kishinev, Shtiintsa, 1976 (in Russian).

[3] Sibirsky K. S. *Introduction to the Algebraic Theory of Invariants of Differential Equations.* Nonlinear Science:Theory and applications, Manchester University Press, Manchester, 1988.

[4] Vulpe N. I. *Polynomial bases of comitants of differential systems and their applications in qualitative theory.* Kishinev, Shtiintsa, 1986 (in Russian).

[5] Popa M. N. *Applications of algebras to differential systems.* Academy of Science of Moldova, 2001 (in Russian).

[6] Popa M. N. *Algebraic methods for differential systems.* Editura the Flower Power, Universitatea din Piteşti, Seria Matematică Aplicată şi Industrială, 2004, **15** (in Romanian).

[7] Popa M. N. *Algebre Lie şi sisteme diferenţiale.* Universitatea de Stat din Tiraspol cu sediul la Chişinău, Editura Academiei de Ştiinţe a Moldovei, 2008 (in Romanian).

[8] Cayley A. *9-th memoir upon quantics.* Philosophical Transactions, 1871, **161**.

[9] Sylvester J. *Determination d'une limite superieur au nombre total des invariants et covariants irréductibles des formes binaires.* Comptes Rendus, 1878, **86**, 1437–1441, 1491–1492.

[10] Sylvester J. *Sur le systéme complet des invariants et covariants irréductibles appartenant á la forme binaire du huitiéme degré.* Comptes Rendus, 1878, **86**, 1519–1521.

[11] Sylvester J. *Sur le covariants fondamentaux d'un systéme cubo-quadratique binaire.* Comptes Rendus, 1878, **87**, 242–244.

[12] Sylvester J., Franklin F. *Tables of the Generating Functions and Ground forms for Simultaneous Binary Quantics of the First Four Orders, taken two and two together.* American Journal of Mathematics Pure and Applied, 1879, **II**, No. 2, 293–306.

[13] Franklin F. *On the Calculation of the generating Function and Tables of Ground-forms for Binary Quantics.* American Journal of Mathematics, 1880, **3**, No. 1, 128–153.

[14] Alekseev V. G., *The theory of rational invariants of binary forms.* Iuriev, 1899 (in Russian).

[15] Weyl H. *The classical groups: Their invariants and representations.* Princeton Univ. Press, 1973.

[16] Hilbert D. *Gesammelte Adhandlungen. BD.2. Algebra. Invariantentheorie. Geometrie. Zweite Auflage.* Berlin-Heidelberg-New York., Springer, 1970.

[17] Stanley R. P. *Hilbert Function of Graded Algebras.* Advances in Mathematics, 1978, **28**, 57–83.

[18] Kobayashi Y. *The Hilbert series of some graded algebras and the Poincaré series of some local rings.* Math. scand. 1978, **42**, 19–33.

[19] Springer T. A. *Invariant theory.* Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[20] Anick D. J., Gulliksen T. H. *Rational dependence among Hilbert and Poincaré series.* Journal of Pure and Applied Algebra, 1985, **38**, 135–157.

[21] Babenko I. K. *Growth and rationality problems in algebra and topology.* Uspekhi Mat. Nauk, 1986, **41**, 95–142.

[22] Anick D. J. *Recent progress in Hilbert and Poincaré series.* Lect. Notes Math., 1988, **1318**, 1–25.

[23] Ufnarovskij V. A. *Combinatorial and Asymptotic Methods in Algebra.* Encyclopaedia of Mathematical Sciences VI, Springer, 1995.

[24] Broer A. *On the generating functions associated to a system of binary forms.* Indag. Math., N.S., 1990, **1**, 15–25.

[25] Broer A. *Hilbert series in invariant theory.* Utrecht, 1990, 1–100.

[26] Sturmfels B. *Algoritms in Invariant Theory.* Wien New York, Springer-Verlag, 1993.

[27] Broer A. *A new method for calcuating Hilbert series.* J. of Algebra, 1994, **168**, 43–70.

[28] Bicova E. V., Popa M. N. *Hilbert series for the graded algebra of comitants S_5 and its applications*, Bul. Acad. Ştiinţe Repub. Mold., Mat., 2000, No. 1(132), 55-66.

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A probabilistic method for solving minimax problems with general constraints

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Abstract. The method proposed in paper solves a convex minimax problem with a set of general constraints. It is based on a schema elaborated previously, but with constraints that can be projected on quite elementary. Such kind of problems are often encountered in technical, economical applied domains etc. It does not use penalty functions or Lagrange function – common toolkit for solving above mentioned problems. Movement directions have a stochastic nature and are built using estimators corresponding to target function and functions from constraints. At the same time every iteration admits some tolerance limits regarding non-compliance with constraints conditions.

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The following problem is considered:

$$\begin{cases} F(x) = \max\limits_{y \in Y_f} f(x,y) \to \min \\ \Phi(x) = \max\limits_{y \in Y_\varphi} \varphi(x,y) \le 0 \\ \quad x \in X \end{cases} \tag{1}$$

where X represents a compact and convex set in Euclidian space E^m, the sets Y_f, Y_φ are compact sets in E^{m_1} and E^{m_2}correspondingly. Suppose that the set of optimal solutions $X^* \neq \emptyset$.

Let us define:

$$\begin{aligned} & V(x,\varepsilon) = \{z \in E^n : \|x - z\| < \varepsilon\}, \\ & V(X,\varepsilon) = \bigcup_{x \in X} (x,\varepsilon), \\ & V_X(X^*,\varepsilon) = V(X^*,\varepsilon) \bigcap X, \\ & W_X(\tilde{x},r) = (V(\tilde{x},r) \bigcap X) \setminus V(X^*,\varepsilon), r > 0, \\ & W_Y(y,r) = V(y,r) \bigcap Y, r > 0. \end{aligned} \tag{2}$$

The functions $f(x,y_f)$ and $\varphi(x,y_\varphi)$ are supposed to be convex on $V(X,\varepsilon^*)$ for some $\varepsilon^* > 0$ and continuous on $V(X,\varepsilon^*) \times Y_f$ and $V(X,\varepsilon^*) \times Y_\varphi$ correspondingly.

Let's admit that on the sets Y_f, Y_φ probability repartitions $P_f(\cdot)$, $P_\varphi(\cdot)$ are defined that satisfy the conditions:

$$\int_{Y_f} P_f(dy) = 1, \int_{Y_\varphi} P_\varphi(dy) = 1. \tag{3}$$

For $\forall r > 0$ $\exists \gamma > 0$:

$$\begin{array}{l} \int\limits_{W_Y(y,r)} P_f(dz) \geq \gamma, \quad \text{if} \quad Y = Y_f \quad \text{for every} \quad y \in Y_f \,, \\ \int\limits_{W_Y(y,r)} P_\varphi(dz) \geq \gamma, \quad \text{if} \quad Y = Y_\varphi \quad \text{for every} \quad y \in Y_\varphi. \end{array} \tag{4}$$

1 Method description

Starting element $x^0 \in X$ is arbitrary taken. The sequence $\{x^k\}_{k\geq 1}$ is built. Let's admit that the approximate solution of order k – the element x^k – is already obtained. The approximation x^{k+1} is determined in the following way:

(A1) Two random variables $\xi \in Y_f$, $\psi \in Y_\varphi$ are simulated in series $m_k \geq 1$, $l_k \geq 1$ of independent probes with distribution laws P_f and P_φ correspondingly. More specifically, the sets $M_k = \{\xi_1, \xi_2, \ldots, \xi_{m_k}\}$, $L_k = \{\psi_1, \psi_2, \ldots, \psi_{l_k}\}$ are generated on each iteration k that contain independent realizations of random vectors $\xi(y_f) = y_f \in Y_f$, $\psi(y_\varphi) = y_\varphi \in Y_\varphi$.

(A2) The elements $y_f^k(x) = \xi_i \in M_k, 1 \leq i \leq m_k$, $y_\varphi^k(x) = \psi_j \in L_k, 1 \leq j \leq l_k$ are indicated:

$$\begin{array}{l} f\left(x^k, y_f^k(x)\right) = \max\limits_{y\in M_k} f\left(x^k, y\right), \\ \varphi\big(x^k, y_\varphi^k(x)\big) = \max\limits_{y\in L_k} \phi\left(x^k, y\right). \end{array} \tag{5}$$

(A3) $y_f^k \in \left\{y_f^{k-1}, y_f^k(x)\right\}$, $y_\varphi^k \in \left\{y_\varphi^{k-1}, y_\varphi^k(x)\right\}$ *aredetermined* where:

$$\begin{array}{l} f\left(x^k, y_f^k\right) = \max\left\{f\left(x^k, y_f^{k-1}\right), \, f\left(x^k, y_f^k(x)\right)\right\}, \quad \text{where } y_f^0 = y_f^0(x), \\ \varphi\big(x^k, y_\varphi^k\big) = \max\left\{\varphi\left(x^k, y_\varphi^{k-1}\right), \, \varphi\left(x^k, y_\varphi^k(x)\right)\right\}, \quad \text{where } y_\varphi^0 = y_\varphi^0(x). \end{array} \tag{6}$$

Definition 1. $f\left(x^k, y_f^k\right)$, $\varphi\left(x^k, y_\varphi^k\right)$ are called *estimators* of the functions $F(x)$ and $\Phi(x)$, correspondingly, for $x = x^k$.

(A4) The new element x^{k+1} is built using the relation:

$$x^{k+1} = \prod\nolimits_X \left(\tilde{x}^{k+1}\right), \ \tilde{x}^{k+1} = x^k - \rho_k \eta^k \tag{7}$$

where $\prod_X (\tilde{x})$ represents the projection of the element $\tilde{x} \in E^m$ on the set X, that is $\prod_X (\tilde{x})$ represents the closest element from X regarding $\tilde{x}$; ρ_k is the step value corresponding to iteration k.

(A5) The sequence of vectors $\{\eta^k\}$ is defined in the following way:

$$\eta^k = \begin{cases} \dfrac{g^k}{\|g^k\|}, & \text{if} \quad g^k \neq \bar{0}, \ k = 0, 1, 2, \ldots \\ \bar{0}, \ \text{for } g^k = \bar{0}. \end{cases} \tag{8}$$

(A6) The vector g^k is built as follows:

$$g^k = g^k\left(x^k\right) = \begin{cases} \partial f\left(x, y_f^k\right) & \text{for} \ \ x = x^k, \ \text{if} \ \ \varphi\left(x^k, y_\varphi^k\right) \leq \tau_k, \\ \partial \varphi\left(x, y_\varphi^k\right) & \text{for} \ \ x = x^k, \ \text{if} \ \ \varphi\left(x^k, y_\varphi^k\right) > \tau_k. \end{cases} \tag{9}$$

Here $\partial f\left(x^k, y_f^k\right)$ denotes the **subgradient** of the function $f\left(x, y_f^k\right)$ [2], and, respectively, $\partial \varphi\left(x^k, y_\varphi^k\right)$ is the **subgradient** of the function $\varphi\left(x, y_\varphi^k\right)$ for $x = x^k$. The vector g^0 is considered to be an arbitrary, but bounded vector.

At the same time we consider that the numerical sequence $\{\rho_k\}$ satisfies classical requirements that ensure the convergence of the methods with programmable modification of the step:

$$\rho_k > 0, \ \rho_k \to 0, \ \sum_{k=0}^{\infty} \rho_k = \infty. \tag{10}$$

Additionally, for any number $\tau \in (0, 1)$ we require the existence of a sequence $\{\bar{\varepsilon}_k\}$ with properties:

$$\bar{\varepsilon}_k \to 0, \ \ \frac{\bar{\varepsilon}_k}{\rho_k} \to \infty \tag{11}$$

so that for $\forall r_k \in \left[\dfrac{\bar{\varepsilon}_k}{2}, \bar{\varepsilon}_k\right]$ occurs the convergence of the series:

$$\sum_{kk=0}^{\infty} \tau^{L(k, r_k)} < \infty \tag{12}$$

where

$$L(k, r_k) = \begin{cases} 0, \ \text{if } \rho_k \geq r_k \quad \text{or} \quad k = 0, \\ s_k, \ \ \text{if} \ \ \sum\limits_{l=k-s_k}^{k} \rho_l < r_k \ \ \text{and} \ \ \sum\limits_{l=k-s_k-1}^{k} \rho_l \geq r_k. \end{cases} \tag{13}$$

In other words s_k is the biggest integer number among all numbers $j \geq 0$ that satisfies the relation $\sum\limits_{l=k-j}^{k} \rho_l < r_k$.

We will show that such numerical sequences $\{\rho_k\}$ and $\{\bar{\varepsilon}_k\}$ exist that conforms to the requirements (10)–(13). Above mentioned are justified by the following lemma:

Lemma 1. *The sequences of the form* $\rho_k = \dfrac{c}{k^\alpha + d}$, $c > 0$, $d \geq 0$, $\alpha \in (0,1]$ *and* $\bar{\varepsilon}_k = \dfrac{p}{k^\beta + q}$, $p > 0$, $q \geq 0$, $\beta \in (0,\ \alpha)$ *satisfy the (10)–(13) requirements.*

Proof. It is obvious that $\lim\limits_{k\to\infty} L(k, r_k) = \lim\limits_{k\to\infty} s_k = \infty$. For consecutive values of $k = 0, 1, 2, \dots$ the resulting values of $L(k, r_k)$ have the form:

$$\underbrace{0, \dots, 0}_{0 \leq C_0 \text{ times}},\ \underbrace{1, \dots, 1}_{0 \leq C_1 \text{ times}}, \dots, \underbrace{s_k, s_k, \dots, s_k}_{0 \leq C_i \text{ times}}, \tag{14}$$

$$\underbrace{(s_k + 1), \dots, (s_k + 1)}_{0 \leq C_{i+1} \text{ times}}, \underbrace{(s_k + 2), \dots, (s_k + 2)}_{0 \leq C_{i+2} \text{ times}}, \dots \tag{15}$$

In other words $L(k, r_k)$ takes the value 0 for C_0 times, the value 1 for C_1 times etc., the value s_k for C_i times, where $i = s_k$. We find out that the sequence $\{C_i\}$, $i = 0, 1, \dots$, is bounded. If we suppose the contrary, it means that exists a value $C_j \in \{C_i\}$ that can be however big. This implies that starting from some $k \geq k'$ all $L(k, r_k) = s_{k'}$. As a result, starting from k' all the values ρ_l from (13) contradict the requirement (11). Thus, there exists a number $C < \infty$ so that $C_i < C$, $\forall i = 0, 1, \dots$ So, we can conclude that the sequence $\{s_k\}$ can take values however big ($s_k \to \infty$).

Further we take an arbitrary, but fixed number $\tau \in (0, 1)$. The numerical series:

$$\begin{aligned}
&\Bigg(\underbrace{\tau^0 + \dots + \tau^0}_{C \text{ times}}\Bigg) + \Bigg(\underbrace{\tau^1 + \dots + \tau^1}_{C \text{ times}}\Bigg) + \dots \\
&+ \Bigg(\underbrace{\tau^{s_k} + \dots + \tau^{s_k}}_{C \text{ times}}\Bigg) + \Bigg(\underbrace{\tau^{s_k+1} + \dots + \tau^{s_k+1}}_{C \text{ times}}\Bigg) + \dots = \\
&= C\tau^0 + C\tau^1 + \dots + C\tau^{s_k} + C\tau^{s_k+1} + \dots = \\
&= C\left(\tau^0 + \tau^1 + \dots + \tau^{s_k} + \tau^{s_k+1} + \dots\right) = C\sum_{k=0}^{\infty} \tau^k = \frac{C}{1-\tau} < \infty.
\end{aligned} \tag{16}$$

But, on the other hand:

$$\sum_{k=0}^{\infty} \tau^{L(k, r_k)} \leq C \sum_{k=0}^{\infty} \tau^k. \tag{17}$$

That leads us to the satisfaction of the (12) requirement. Lemma is proved. $\square$

Now let's get back to the method of computation of the sequence $\{x^k\}$. It is the moment to remark that the iterative process can be modified, namely different distribution laws are applied for definition and simulation of random variables ξ, ψ for every new iteration. This can favor the increase of convergence speed in a certain sense of the sequence $\{x^k\}$.

The idea of using the subgradients of target function $F(x)$, in case that $\Phi\left(x^k\right) \leq 0$, and subgradients of the function $\Phi(x)$, if $\Phi\left(x^k\right) > 0$, for solving a convex model, is launched for the first time by B. Polyak in paper [1].

The stochastic subgradient method for solving a convex problem is defined in the following way:

$$\begin{cases} F(x) = \max\limits_{y \in Y_f} f(x,y) \to \min \\ \quad x \in X \end{cases}$$

is realized and argued in [5]. Paper [4] describes this method that is developed using the operation of normalization of subgradients and the convergence is established in the same probabilistic terms. The proof of the convergence is based on two principal stages. We will use and develop the mathematical mechanism used in [4] for arguing the method (A1)-(A6) when solving the problem (1). Thus, the following affirmation takes place:

Theorem 1. *Let's suppose that along with conditions mentioned above following take place:*

$$\tau_k > 0, \quad \tau_k \to 0, \quad \sum_{k=0}^{\infty} \rho_k \tau_k = \infty, \quad \frac{\tau_k}{\rho_k} \to \infty. \tag{18}$$

Then, for $\forall \varepsilon > 0$ fixed, all elements of the random sequence $\{x^k\}_{k \geq 0}$, obtained as a result of application of the described method (A1) -(A6) , are localized almost certain (with probability 1) in vicinity $V(X^*, 2\varepsilon)$, but excepting a finite number of elements. Formally this can be represented in the following way:

$$P\left\{ \lim_{k \to \infty} \min_{x^* \in X^*} \left\| x^k - x^* \right\| = 0 \right\} = 1,$$

where $x^k = x^k\left(\theta^0, \theta^1, \ldots, \theta^{k-1}\right)$, $\quad \theta^k \in \Theta^k = (M_k \times L_k)$.

Proof. If $X \subset V(X^*, 2\varepsilon)$ then the statement is obvious. Let's admit $X \backslash V(X^*, 2\varepsilon) \neq \emptyset$. We mention here that on every iteration k for the initial model (1) is associated the following problem:

$$\begin{cases} F(x) = \max\limits_{y \in Y_f} f(x,y) \to \min, \\ \quad \Phi(x) \leq \tau_k, \\ \quad x \in X \end{cases} \tag{19}$$

or, the group $\left\{f\left(x^k, y_f^k\right), \varphi\left(x^k, y_\varphi^k\right), \tau_k, X\right\}$ corresponds to the iteration k, in order to determine the direction η^k that will lead to obtaining the next element–x^{k+1}.

Two stages for proof development will be accentuated.

Stage 1. Firstly, the existence of a subsequence $\left\{x^{k_l}\right\} \subset \left\{x^k\right\}_{k\geq 0}$ that almost certain is contained in $V_X\left(X^*, \varepsilon\right)$ will be proved, i.e. $P\left\{\exists \left\{x^{k_l}\right\} \subset \left\{x^k\right\}_{k\geq 0} : x^{k_l} \in V_X\left(X^*, \varepsilon\right)\right\} = 1$.

Let's suppose the contrary. In this case for some $q \in (0,1)$ a natural number $K_q < \infty$ can be indicated such that the following event is produced

$$A_1 = \left\{\exists K_q : \forall k \geq K_q, \left\|x^k - x^*\right\| \geq \varepsilon, \text{ or } x^k \notin V_x\left(X^*, \varepsilon\right), \forall x^* \in X^*\right\} \tag{20}$$

with probability $P\left(A_1\right) \geq q$.

Let's denote $X_\varepsilon = X \backslash V\left(X^*, \varepsilon\right)$.

Since the functions $F\left(x\right)$, $\Phi\left(x\right)$ and their estimators $f\left(x, y_f^k\right)$, $\varphi\left(x, y_\varphi^k\right)$ are convex, the following inequalities are valid [2]:

$$\begin{aligned}
&F\left(x^*\right) - F\left(x^k\right) \geq \left(\partial F\left(x^k\right), x^* - x^k\right),\ f\left(x^{k+1}, y_f^k\right) - f\left(x^k, y_f^k\right) \geq \\
&\geq \left(\partial f\left(x^k, y_f^k\right), x^{k+1} - x^k\right), \\
&\Phi\left(x^*\right) - \Phi\left(x^k\right) \geq \left(\partial \Phi\left(x^k\right), x^* - x^k\right),\ \varphi\left(x^{k+1}, y_\varphi^k\right) - \varphi\left(x^k, y_\varphi^k\right) \geq \\
&\geq \left(\partial \varphi\left(x^k, y_\varphi^k\right), x^{k+1} - x^k\right)
\end{aligned} \tag{21}$$

for $\forall x^* \in X^*,\ \forall x^k, x^{k+1} \in X$.

Taking into consideration all properties enumerated above, two constants $C_1 > 0,\ C_2 > 0$ may be chosen, such that $\|x' - x''\| \leq C_1,\ \forall x', x'' \in X$ and $\|\partial F\left(x\right)\| \leq C_2$, $\|\partial \Phi\left(x\right)\| \leq C_2$, $\left\|\partial f\left(x, y_f^k\right)\right\| \leq C_2$, $\left\|\partial \varphi\left(x, y_\varphi^k\right)\right\| \leq C_2$, $\forall x \in X, \forall y_f \in Y_f, \forall y_\varphi \in Y_\varphi$.

Let's consider the case $\varphi\left(x^k, y_\varphi^k\right) \leq \tau_k$ and $x^k \in X_\varepsilon$. Since the function $F\left(x\right)$ is convex, results that exists the number $\Delta_F = \Delta\left(\varepsilon\right) > 0$ such that

$$\inf_{x \in X_\varepsilon, x^* \in X^*} \left(F\left(x\right) - F\left(x^*\right)\right) = 2\Delta_F \tag{22}$$

or, on basis of (21):

$$\left(\partial F\left(x^k\right), x^k - x^*\right) \geq 2\Delta_F, \tag{23}$$

$$\frac{\left(\partial F\left(x^k\right), x^k - x^*\right)}{\left\|\partial F\left(x^k\right)\right\| \cdot \left\|x^k - x^*\right\|} \geq \frac{\left(\partial F\left(x^k\right), x^k - x^*\right)}{C_2 \cdot C_1} \geq \frac{2\Delta_F}{C_1 \cdot C_2}.$$

From (22) it follows that for $\forall \tilde{x} \in X_\varepsilon$:

$$f(\tilde{x}, y_f(\tilde{x})) - F(x^*) \geq 2\Delta_F \tag{24}$$

where $y_f(\tilde{x})$ is **such an element from** Y_f **that** $f(\tilde{x}, y_f(\tilde{x})) = F(\tilde{x})$.

Taking into consideration the last inequality and the continuity of the function $f(x, y_f)$ regarding $(x, y_f) \in X \times Y_f$, we conclude that for $\forall \tilde{x} \in X_\varepsilon$ a number $r_0(\tilde{x}) > 0$ corresponds, so that:

$$f(x, y_f) \geq F(x^*) + \frac{3}{2}\Delta_f \tag{25}$$

as soon as $x \in W_X(\tilde{x}, r_0(\tilde{x}))$ and $y_f \in W_{Y_f}(y_f(\tilde{x}), r_0(\tilde{x}))$.

The set X_ε is compact. Therefore, there exists the number

$$r_0 = \min\left\{\min_{\tilde{x}\in X_\varepsilon} r_0(\tilde{x}),\ \varepsilon\right\} > 0. \tag{26}$$

Hence, the inequality (25) is satisfied for all $\forall \tilde{x} \in X_\varepsilon$, $x \in W_X(\tilde{x}, r_0)$, $y_f \in W_{Y_f}(y_f(\tilde{x}), r_0)$.

Similarly, in case that $\varphi\left(x^k, y_\varphi^k\right) > \tau_k$ and $x^k \in X_\varepsilon$, it follows

$$\Phi(x) - \Phi(x^*) \geq 2\tau_k \tag{27}$$

or, on basis of inequality from (21):

$$\left(\partial\Phi\left(x^k\right), x^k - x^*\right) \geq 2\tau_k, \tag{28}$$

$$\frac{\left(\partial\Phi\left(x^k\right), x^k - x^*\right)}{\|\partial\Phi(x^k)\| \cdot \|x^k - x^*\|} \geq \frac{\left(\partial\Phi\left(x^k\right), x^k - x^*\right)}{C_2 \cdot C_1} \geq \frac{2\tau_k}{C_1 \cdot C_2}.$$

From (27) it follows that for $\forall \tilde{x} \in X_\varepsilon$:

$$\varphi(\tilde{x}, y_\varphi(\tilde{x})) - \Phi(x^*) \geq 2\tau_k \tag{29}$$

where $y_\varphi(\tilde{x})$ **is such an element from** Y_φ **that** $\varphi(\tilde{x}, y_\varphi(\tilde{x})) = \Phi(\tilde{x})$.

Taking into consideration the last inequality and the continuity of the function $\varphi(x, y_\varphi)$ regarding $(x, y_\varphi) \in X \times Y_\varphi$, we conclude that for $\forall \tilde{x} \in X_\varepsilon$ a number $r_0(\tilde{x}) > 0$ corresponds so that:

$$\varphi(x, y_\varphi) \geq \Phi(x^*) + \frac{3}{2}\tau_k \tag{30}$$

as soon as $x \in W_X(\tilde{x}, r_0(\tilde{x}))$ and $y_\varphi \in W_{Y_\varphi}(y_\varphi(\tilde{x}), r_0(\tilde{x}))$.

As was specified previously, the set X_ε is compact. Therefore, there exists the number

$$r_0 = \min\left\{\min_{\tilde{x}\in X_\varepsilon} r_0(\tilde{x}),\ \varepsilon\right\} > 0. \tag{31}$$

Hence, the inequality (30) is satisfied for all $\forall \tilde{x} \in X_\varepsilon$, $x \in W_X(\tilde{x}, r_0)$, $y_\varphi \in W_{Y_\varphi}(y_\varphi(\tilde{x}), r_0)$.

Let's consider some numbers δ_F, δ_Φ^k from intervals $\left(0, \dfrac{2\Delta_F}{C_1 \cdot C_2}\right)$, $\left(0, \dfrac{2\tau_k}{C_1 \cdot C_2}\right)$ and label $\tilde{\delta}_k = \min\{\delta_F, \delta_\Phi^k\}$. Particularly, δ_F, δ_Φ^k can be taken as midpoints of the intervals $\left(0, \dfrac{2\Delta_F}{C_1 \cdot C_2}\right)$, $\left(0, \dfrac{2\tau_k}{C_1 \cdot C_2}\right)$:

$$\delta_F = \frac{\Delta_F}{C_1 \cdot C_2},\ \delta_\Phi^k = \frac{\tau_k}{C_1 \cdot C_2} \tag{32}$$

As a result the following is obtained:

$$\begin{aligned} &\left(\partial F\left(x^k\right), x^k - x^*\right) \geq 2\tilde{\delta}_k \left\|\partial F\left(x^k\right)\right\| \cdot \left\|x^k - x^*\right\|, \text{ if } \varphi\left(x^k, y_\varphi^k\right) \leq \tau_k, \\ &\left(\partial \Phi\left(x^k\right), x^k - x^*\right) \geq 2\tilde{\delta}_k \left\|\partial \Phi\left(x^k\right)\right\| \cdot \left\|x^k - x^*\right\|, \text{ if } \varphi\left(x^k, y_\varphi^k\right) > \tau_k. \end{aligned} \tag{33}$$

The following events are being considered:

1. $A_1^k = \left\{\left(\eta^k, x^k - x^*\right) \geq \tilde{\delta}_k \left\|x^k - x^*\right\|, \forall x^* \in X^*\right\}$. Obviously, the opposite event with regards to A_1^k has the following form:
   $\overline{A_1^k} = \left\{\exists x^* \in X^* : \left(\eta^k, x^k - x^*\right) < \tilde{\delta}_k \left\|x^k - x^*\right\|\right\}$;

2. $D_1 = \left\{\bigcup\limits_{k=K_\delta}^{\infty} \bigcap\limits_{i=k}^{\infty} A_1^i\right\}$, or, in other words, occurs all A_1^k $(k \geq K_q)$, without, perhaps, a finite number. It is obvious that $\overline{D_1} = \left\{\bigcap\limits_{k=K_\delta}^{\infty} \bigcup\limits_{i=k}^{\infty} \overline{A_1^i}\right\}$, or, in other words, an infinite number of events $\overline{A_1^k}$ are produced.

Let us evaluate $P(A_1)$. In order to do this let's represent

$$P(A_1) = P\left(A_1 \bigcap \left(D_1 \bigcup \overline{D_1}\right)\right) = P\left(A_1 \bigcap D_1\right) + P\left(A_1 \bigcap \overline{D_1}\right).$$

Both terms from the last expression will be estimated.

From the realization of event $A_1 \bigcap D_1$ follows the existence of such a natural number $K_\delta < \infty$ that for all $k \geq K_\delta$ and $\forall x^* \in X^*$ the following inequality occurs

$$\left(\eta^k, x^k - x_k^*\right) \geq \tilde{\delta}_k \left\|x^k - x_k^*\right\|. \tag{34}$$

Taking into consideration (34), for $k \geq K_\delta$ we have the following sequence of relations:

$$\left\|x^{k+1}-x^*\right\|^2 \le \left\|x^k-\rho_k\eta^k-x^*\right\|^2 = \left\|x^k-x^*\right\|^2 - 2\rho_k\left(x^k-x^*,\eta^k\right)+\rho_k^2\left\|\eta^k\right\|^2 \le$$
$$\le \left\|x^k-x^*\right\|^2 - 2\rho_k\tilde{\delta}_k\left\|x^k-x^*\right\| + \rho_k^2 \le \left\|x^k-x^*\right\|^2 - 2\rho_k\tilde{\delta}_k\varepsilon+\rho_k^2 =$$
$$= \left\|x^k-x^*\right\|^2 - \rho_k\left(2\tilde{\delta}_k\varepsilon-\rho_k\right).$$

Because $\rho_k \underset{k\to\infty}{\to} 0$, for some K_Φ: $\delta_F > \delta_\Phi^k$ or $\tilde{\delta}_k = \delta_\Phi^k$, as soon as $k \ge K_\Phi$. According to (18), (32) for some $K_\varepsilon \ge K_\Phi$: $\rho_k \le \tilde{\delta}_k\varepsilon$, as soon as $k \ge K_\varepsilon$. Evidently, for $k \ge \hat{k} = \max\{K_\delta, K_\varepsilon\}$:

$$\left\|x^{k+1}-x^*\right\|^2 \le \left\|x^k-x^*\right\|^2 - \rho_k\tilde{\delta}_k\varepsilon,$$
$$\left\|x^k-x^*\right\|^2 \le \left\|x^{k-1}-x^*\right\|^2 - \rho_{k-1}\tilde{\delta}_k\varepsilon \le \left\|x^{k-2}-x^*\right\|^2 -$$
$$-\varepsilon\left(\rho_{k-2}\tilde{\delta}_{k-2}+\rho_{k-1}\tilde{\delta}_{k-1}\right),\dots$$
$$\left\|x^{k+1}-x^*\right\|^2 \le \left\|x^{\hat{k}}-x^*\right\|^2 - \varepsilon\sum_{i=\hat{k}}^{k}\rho_i\tilde{\delta}_i,$$
$$\text{or}\quad \left\|x^{k+1}-x^*\right\|^2 \le \left\|x^{\hat{k}}-x^*\right\|^2 - \varepsilon\sum_{i=\hat{k}}^{k}\rho_i\delta_\varphi^i.$$

Due to imposed conditions on τ_k in (18), based on relation (32), we get:

$$\left\|x^{k+1}-x^*\right\|^2 \le \left\|x^{\hat{k}}-x^*\right\|^2 - \frac{\varepsilon}{C_1\cdot C_2}\sum_{i=\hat{k}}^{k}\rho_i\tau_i \to -\infty, \quad \text{for } \; k\to\infty. \tag{35}$$

We obtain a contradiction because the norm of any vector, moreover its square value, cannot be negative. Therefore, the realization of event $A_1\bigcap D_1$ implies realization of an event that is practically unrealizable, $F_1 = \left\{\left\|x^{k+1}-x^*\right\|^2 < 0,\ k\to\infty\right\}$. That is $P\left(A_1\bigcap D_1\right) \le P\left(F_1\right) = 0$. It means that $P\left(A_1\right) = P\left(A_1\bigcap \overline{D_1}\right)$.

Let us evaluate $P\left(A_1\bigcap\overline{D_1}\right)$. Let's take an arbitrary number r_k from the interval $\left[\frac{\bar{\varepsilon}_k}{2},\bar{\varepsilon}_k\right]$, where $\bar{\varepsilon}_k = \min\left\{r_0, \frac{\Delta_F}{2C_2}, \frac{\tau_k}{2C_2}\right\}$. The following events are defined:

1. B_f^k={at least one time among the iterations of the form $j = \overline{k-s_k,k}$ an element from the set $W_{Y_f}\left(y_f\left(x^j\right),r_k\right)$ is generated, where $x^j \in W_X\left(x^k,r_k\right)$, for s_k defined in (13)};
2. B_φ^k={at least one time among the iterations of the form $j = \overline{k-s_k,k}$ an element from the set $W_{Y_\varphi}\left(y_\varphi\left(x^j\right),r_k\right)$ is generated, where $x^j \in W_X\left(x^k,r_k\right)$, for s_k defined in (13)};
3. $B_1^k = B_f^k\bigcap B_\varphi^k$.

The simulation of the variables ξ and ψ on iteration k is executed in parallel and independently. Since the events B_f^k, B_φ^k are independent, it follows that $P\left(B_1^k\right) = P\left(B_f^k\right) \cdot P\left(B_\varphi^k\right)$.

The realization of the event B_f^k implies: for some iteration $j_k \in \overline{k - s_k, k}$ the generated element $y_f^{j_k}\left(x^{j_k}\right) = \xi_{t_k} \in M_{j_k}, 1 \le t_k \le m_{j_k}$ has the property $y_f^{j_k}\left(x^{j_k}\right) \in W_{Y_f}\left(y_f\left(x^{j_k}\right), r_k\right)$, that is, according to (25):

$$f\left(x^{j_k}, y_f^{j_k}\right) \ge f\left(x^{j_k}, y_f^{j_k}\left(x^{j_k}\right)\right) \ge F\left(x^*\right) + \frac{3}{2}\Delta_F. \tag{36}$$

Let's admit that j_k is an arbitrary element from the set of iterations $\{k - s_k, \ldots, k-1\}$. We will show that $f\left(x^k, y_f^k\right) \ge F\left(x^*\right) + \Delta_F$. Indeed, taking into consideration the convexity of the estimator $f\left(x, y_f\right)$ for $\forall y_f \in Y_f$ and the way of computation of the sequence $\left\{x^k\right\}$, we get:

$$\begin{aligned} & f\left(x^{k+1}, y_f\right) - f\left(x^k, y_f\right) \ge \left(\partial f(x^k, y_f), x^{k+1} - x^k\right) \ge \\ & \ge -\left\|\partial f(x^k, y_f)\right\| \cdot \left\|\prod\nolimits_X \left(x^k - \rho_k \eta^k\right) - x^k\right\| \ge -C_2 \rho_k. \end{aligned} \tag{37}$$

From (36) and (37) it follows:

$$\begin{aligned} & f\left(x^{j_k+1}, y_f^{j_k+1}\right) \ge f\left(x^{j_k+1}, y_f^{j_k}\right) \ge f\left(x^{j_k}, y_f^{j_k}\right) - C_2 \rho_{j_k}, \\ & \ldots \\ & f\left(x^{j_k+i}, y_f^{j_k+i}\right) \ge f\left(x^{j_k}, y_f^{j_k}\right) - C_2 \sum_{l=0}^{i-1} \rho_{j_k+l} \ge F\left(x^*\right) + \frac{3}{2}\Delta_F - C_2 r_k \ge \\ & \ge F\left(x^*\right) + \frac{3}{2}\Delta_F - C_2 \frac{\Delta_F}{2C_2} = F\left(x^*\right) + \Delta_F \end{aligned} \tag{38}$$

for all i that $\sum\limits_{l=0}^{i-1} \rho_{j_k+l} \le r_k$.

But, $\sum\limits_{l=0}^{k-j_k} \rho_{j_k+l} = \rho_{j_k} + \rho_{j_k+1} + \ldots + \rho_k \le \sum\limits_{l=k-s_k}^{k} \rho_l \le r_k$. Therefore,

$$f\left(x^k, y_f^k\right) \ge F\left(x^*\right) + \Delta_F. \tag{39}$$

But if $j_k = k$, then the last inequality is satisfied even more. As a consequence to (39) we have the following chain of inequalities

$$-\Delta_F \ge F\left(x^*\right) - f\left(x^k, y_f^k\right) \ge f\left(x^*, y_f^k\right) - f\left(x^k, y_f^k\right) \ge \left(\partial f\left(x^k, y_f^k\right), x^* - x^k\right)$$

or,

$$\left(\partial f\left(x^k, y_f^k\right), x^k - x^*\right) \ge \Delta_F. \tag{40}$$

Taking into consideration (40) and the way the number $\tilde{\delta}_k$ is chosen, we get:

$$\frac{\left(\partial f\left(x^k, y_f^k\right), x^k - x^*\right)}{\left\|\partial f\left(x^k, y_f^k\right)\right\| \cdot \|x^k - x^*\|} \geq \tilde{\delta}_k$$

or, in other words, the event A_1^k is realized.

The realization of the event B_φ^k implies: for some iteration $j_k \in \overline{k - s_k, k}$ the generated element $y_\varphi^{j_k}\left(x^{j_k}\right) = \psi_{t_k} \in L_{j_k}, 1 \leq t_k \leq l_{j_k}$ has the property $y_\varphi^{j_k}\left(x^{j_k}\right) \in W_{Y_\varphi}\left(y_\varphi\left(x^{j_k}\right), r_k\right)$, that is, according to (30):

$$\varphi\left(x^{j_k}, y_\varphi^{j_k}\right) \geq \varphi\left(x^{j_k}, y_\varphi^{j_k}\left(x^{j_k}\right)\right) \geq \Phi\left(x^*\right) + \frac{3}{2}\tau_k. \tag{41}$$

Let's admit that j_k is an arbitrary element from the set of iterations $\{k - s_k, \ldots, k-1\}$. We will show that $\varphi\left(x^k, y_\varphi^k\right) \geq \Phi\left(x^*\right) + \tau_k$. Indeed, taking into consideration the convexity of the estimator $\varphi\left(x, y_\varphi\right)$ for $\forall y_\varphi \in Y_\varphi$ and the way of computation of the sequence $\left\{x^k\right\}$, we get:

$$\begin{aligned} &\varphi\big(x^{k+1}, y_\varphi\big) - \varphi\left(x^k, y_\varphi\right) \geq \left(\partial\varphi(x^k, y_\varphi), x^{k+1} - x^k\right) \geq \\ &\geq -\left\|\partial\varphi(x^k, y_\varphi)\right\| \cdot \left\|\textstyle\prod_X \left(x^k - \rho_k \eta^k\right) - x^k\right\| \geq -C_2\rho_k. \end{aligned} \tag{42}$$

From (41) and (42) it follows:

$$\begin{aligned} &\varphi\Big(x^{j_k+1}, y_\varphi^{j_k+1}\Big) \geq \varphi\left(x^{j_k+1}, y_\varphi^{j_k}\right) \geq \varphi\left(x^{j_k}, y_\varphi^{j_k}\right) - C_2\rho_{j_k}, \\ &\ldots \\ &\varphi\Big(x^{j_k+i}, y_\varphi^{j_k+i}\Big) \geq \varphi\left(x^{j_k}, y_\varphi^{j_k}\right) - C_2\sum_{l=0}^{i-1}\rho_{j_k+l} \geq \Phi\left(x^*\right) + \frac{3}{2}\tau_k - C_2 r_k \geq \\ &\geq \Phi\left(x^*\right) + \frac{3}{2}\tau_k - C_2\frac{\tau_k}{2C_2} = \Phi\left(x^*\right) + \tau_k \end{aligned} \tag{43}$$

for all i that $\sum\limits_{l=0}^{i-1}\rho_{j_k+l} \leq r_k$.

But, $\sum\limits_{l=0}^{k-j_k}\rho_{j_k+l} = \rho_{j_k} + \rho_{j_k+1} + \ldots + \rho_k \leq \sum\limits_{l=k-s_k}^{k}\rho_l \leq r_k$. Therefore,

$$\varphi\left(x^k, y_\varphi^k\right) \geq \Phi\left(x^*\right) + \tau_k. \tag{44}$$

But if $j_k = k$, then the last inequality is satisfied even more. As a consequence to (44) we have the following chain of inequalities

$$-\tau_k \geq \Phi\left(x^*\right) - \varphi\left(x^k, y_\varphi^k\right) \geq \varphi\left(x^*, y_\varphi^k\right) - \varphi\left(x^k, y_\varphi^k\right) \geq \left(\partial\varphi\left(x^k, y_\varphi^k\right), x^* - x^k\right)$$

or,

$$\left(\partial\varphi\left(x^k, y_\varphi^k\right), x^k - x^*\right) \geq \tau_k. \tag{45}$$

Taking into consideration (45) and the way the number $\tilde{\delta}_k$ is chosen, we get:

$$\frac{\left(\partial\varphi\left(x^k, y_\varphi^k\right), x^k - x^*\right)}{\left\|\partial\varphi\left(x^k, y_\varphi^k\right)\right\| \cdot \left\|x^k - x^*\right\|} \geq \tilde{\delta}_k$$

or, in other words, the event A_1^k is realized.

The realization of the events B_f^k and B_φ^k implies the realization of the event B_1^k. At the same time the following implication takes place: $B_1^k \subset A_1^k$. Therefore, we get $P\left(B_1^k\right) \leq P\left(A_1^k\right)$, or, $P\left(\overline{A_1^k}\right) \leq P\left(\overline{B_1^k}\right)$. But, accordingly to (4), (13) follows: $P\left(\overline{B_1^k}\right) \leq \alpha^{L\left(k, r^k\right)}$ where $\alpha = 1 - \gamma$. We get following set of inequalities:

$$\sum_{k=0}^{\infty} P\left(\overline{A_1^k}\right) \leq \sum_{k=0}^{\infty} P\left(\overline{B_1^k}\right) \leq \sum_{k=0}^{\infty} \alpha^{L\left(k, r^k\right)} < \infty.$$

We are in the situation that the conditions of the Borel–Cantelli lemma are met [3]. It means that $P\left(\overline{D_1}\right) = 0$. Therefore,

$$q \leq P\left(A_1\right) = P\left(A_1 \bigcap \overline{D_1}\right) \leq P\left(\overline{D_1}\right) = 0.$$

Thus, $q = 0$.

A contradiction has been obtained, because we have supposed that $q > 0$. Thus, there exists a subsequence $\left\{x^{k_l}\right\} \subset \left\{x^k\right\}_{k \geq 0}$ that almost certainly is contained in $V_X\left(X^*, \varepsilon\right)$.

Stage 2. Further will be proved that all elements of the sequence $\left\{x^k\right\}$, without just a finite number, belong to the set $V_X\left(X^*, 2\varepsilon\right)$ with probability 1.

The following events are defined:

$$\begin{aligned} A_2 &= \left\{\exists \left\{x^{k_l}\right\} \subset \left\{x^k\right\} : \ \left\{x^{k_l}\right\} \subset V_X\left(X^*, \varepsilon\right)\right\}, \\ B_2 &= \left\{\exists \left\{z^{k_m}\right\} \subset \left\{x^k\right\} : \ \left\{z^{k_m}\right\} \not\subset V_X\left(X^*, 2\varepsilon\right)\right\}. \end{aligned} \tag{46}$$

Next, $P\left(B_2\right)$ will be appreciated. We will find out that $P\left(B_2\right) = P\left(B_2 \bigcap A_2\right)$. Indeed, $P\left(B_2\right) = P\left(\left(B_2 \bigcap A_2\right) \bigcup \left(B_2 \bigcap \overline{A_2}\right)\right) = P\left(B_2 \bigcap A_2\right) + P\left(B_2 \bigcap \overline{A_2}\right) = P\left(B_2 \bigcap A_2\right)$, because $P\left(B_2 \bigcap \overline{A_2}\right) \leq P\left(\overline{A_2}\right) = 0$.

Further, the following event will be considered: $D_2 = A_2 \bigcap B_2$. Suppose that $P\left(D_2\right) > 0$. Realization of the event D_2 means that the transfer from $V_X\left(X^*, \varepsilon\right)$ to $X \backslash V_X\left(X^*, 2\varepsilon\right)$ and vice versa takes place infinitely.

Let us denote by:

1. K_1 – the number of first iteration when the event $\left\{x^{K_1} \in V_X\left(X^*, \varepsilon\right)\right\}$ is produced;

2. K_2 – the number of first iteration when the event $\left\{x^{K_2} \in V_X\left(X^*, \frac{3}{2}\varepsilon\right)\right\}$ is produced;

3. K_3 – the number of first iteration when the inequality $\rho_{K_3} \leq 2\varepsilon\tilde{\delta}_{K_3}$ is satisfied;
4. $\bar{K} = \max\{K_1, K_2, K_3\}$.

In case for some $k \geq \bar{K}$ and $x^k \notin V_X\left(X^*, \frac{3}{2}\varepsilon\right)$ the inequality that defines the event A_1^k is satisfied, then the following sequence of inequalities occurs: $\left\|x^{k+1} - x^*\right\|^2 \leq \left\|x^k - x^*\right\|^2 - \rho_k(2\varepsilon\tilde{\delta}_k - \rho_k) < \left\|x^k - x^*\right\|^2$, because $\left\|x^k - x^*\right\| > \varepsilon$.

That is, as soon as $k \geq \bar{K}$ and $x^k \notin V_X\left(X^*, \frac{3}{2}\varepsilon\right)$ it follows:

$$\left\|x^{k+1} - x^*\right\| < \left\|x^k - x^*\right\|. \tag{47}$$

Since $\rho_k \underset{k\to\infty}{\to} 0$, a number $K^* \geq \bar{K}$ will appear with the property: $x^{K^*} \in V_X\left(X^*, 2\varepsilon\right) \setminus V_X\left(X^*, \frac{3}{2}\varepsilon\right)$. This will happen certainly. Particularly, for $\rho_k < \frac{\varepsilon}{2}$:

$$\left\|x^{k+1} - x^k\right\| \leq \left\|x^k - \rho_k\eta^k - x^k\right\| \leq \rho_k < \frac{\varepsilon}{2}.$$

Therefore, there exists a number k that satisfies $x^k \in V_X\left(X^*, 2\varepsilon\right) \setminus V_X\left(X^*, \frac{3}{2}\varepsilon\right)$.

According to (47), $\left\|x^{K^*+1} - x^*\right\| < \left\|x^{K^*} - x^*\right\|$. In case $x^{K^*+1} \notin V_X\left(X^*, \frac{3}{2}\varepsilon\right)$, then $\left\|x^{K^*+2} - x^*\right\| < \left\|x^{K^*+1} - x^*\right\| < \left\|x^{K^*} - x^*\right\|$, and so forth, for all $j \geq 0$ that satisfy $x^{K^*+j} \notin V_X\left(X^*, \frac{3}{2}\varepsilon\right)$, takes place:

$$\min_{x^*\in X^*} \left\|x^{K^*+j} - x^*\right\| < \min_{x^*\in X^*} \left\|x^{K^*} - x^*\right\| < 2\varepsilon. \tag{48}$$

Let us denote $\left\{x^{k^l}\right\}_{l\geq 1}$ the sequence of all elements $\left\{x^k\right\}$ with the property that $k^l \geq K^*$, $x^{k^l} \in V_X\left(X^*, 2\varepsilon\right) \setminus V_X\left(X^*, \frac{3}{2}\varepsilon\right)$ and $x^{k^l-1} \in V_X\left(X^*, \frac{3}{2}\varepsilon\right)$. Then, for $l \geq 1$, $k^l < j < k^{l+1}$ and $x^j \notin V_X\left(X^*, \frac{3}{2}\varepsilon\right)$ the following inequality occurs:

$$\min_{x^*\in X^*} \left\|x^j - x^*\right\| < \min_{x^*\in X^*} \left\|x^{k^l} - x^*\right\| < 2\varepsilon. \tag{49}$$

Thus, in other words, admitting that for some K elements of type $x^k \notin V_X\left(X^*, \frac{3}{2}\varepsilon\right)$,

$k < \infty$, $k \geq K$ satisfy the inequality from the event A_1^k, then the event B_2 cannot occur with positive probability. The supposition that D_2 is realized means that beyond the layer $V_X\left(X^*, \frac{3}{2}\varepsilon\right)$ the penetration of the layer takes place only when infinitely the event $\overline{A_1^k}$ considered previously is produced. But, $P\left(\overline{D_1}\right) = 0$. So, the conclusion that can be drawn is that the transfer from the layer

$V_X(X^*, 2\varepsilon) \backslash V_X\left(X^*, \frac{3}{2}\varepsilon\right)$ into the layer $X \backslash V_X(X^*, 2\varepsilon)$ occurs only a finite number of times. That is, $P(D_2) = 0$, and it implies $P(B_2) = 0$. Theorem is proved. $\square$

Remark 1. In case the set of optimal solutions $X^* = \emptyset$, application of the above described method for solving the problem (1) leads us to the solution of the following problem:

$$\begin{cases} \Phi(x) = \max\limits_{y \in Y_\varphi} \varphi(x, y) \to \min \\ x \in X. \end{cases}$$

References

[1] Polyak B.T. *A general method for solving extreme problems.* Dokl. Akad. Nauk SSSR, 1967, **174**, No. 1, 33–336 (in Russian).

[2] Shor N.Z. *Nondifferentiable Optimization and Polynomial Problems.* Kluwer Academic Publishers, Netherlands, 1998.

[3] Shiryaev A.V. *Probability.* Moscow, Nauka, 1980 (in Russian).

[4] Godonoaga A. *Stohasticheskie shemy otyskanija minimaksa.* Sbornik nauchnyh trudov: Verojatnost' i ee prilozhenija, Kishinev, 1990, 3–13 (in Russian).

[5] Ermol'ev Yu. M., Gaivoronskii A.A. *Stochastic methods for solving minimax problems.* Kibernetika, 1983, No. 4, 92–97 (in Russian; English translation in: Cybernetics, **19**, No. 4, 550–559).

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About rings of continuous functions in the expanded field of numbers

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Abstract. In the present article the generalized rings $C_\infty(X)$ of all continuous functions on the expanded straight line are studied. The conditions under which $C_\infty(X)$ is a ring or a linear space are determined.

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Spaces of continuous maps on the expanded straight line play a leading role in the theory of topological semifields. In works [1–5] some applications of such maps have been specified. All spaces are assumed to be Tychonoff. Terminology is as in [7]. By $[A]$ or $[A]_X$ we denote the closure of a set A in a space X, $|Y|$ is the cardinality of a set Y, βX is the Stone-Čech compactification of the space X, on $N = \{0, 1, ...\}$ we consider only the discrete topology.

Let R be the field of real or complex numbers. By R_∞ we denote the one-point compactification of space R and $R_\infty = R \cup \{\infty\}$. We consider that $\infty + \infty = b + \infty = \infty$, $0 \cdot \infty = 0$, $\infty \cdot \infty = \infty$, $c \cdot \infty = \infty \cdot c = \infty$ for all $b \in R$, $c \in R \setminus \{0\}$.

Let $C_\infty(X)$ be the family of all continuous maps of space X in R_∞ in the topology of pointwise convergence and such that the set $H(f) = f^{-1}(\infty)$ is nowhere dense in X for all $f \in C_\infty(X)$. We suppose that $C(X) = \{f \in C_\infty(X) : H(f) = \emptyset\}$.

Let $f, g \in C_\infty(X)$. We say that the sum $f + g$ is defined if there exits a function $h \in C_\infty(X)$ such that $h(x) = f(x) + g(x)$ for all $x \in X \setminus (H(f) \cup H(g))$. The product $f \cdot g$ is defined if there exists a function $h \in C_\infty(X)$ such that $h(x) = f(x) \cdot g(x)$ for all $x \in X \setminus (H(f) \cup H(g))$.

For a map $\psi : C_\infty(X) \to C_\infty(Y)$ we consider the conditions:

a) if $f, g \in C_\infty(X)$, then the sum $f + g$ exists if and only if the sum $\psi(f) + \psi(g)$ exists and then $\psi(f + g) = \psi(f) + \psi(g)$;

b) $\psi(b \cdot f) = b \cdot y(f)$ for any $b \in R$, $f \in C_\infty(X)$;

c) if $f, g \in C_\infty(X)$, then the product $f \cdot g$ exists if and only if $\psi(f) \cdot \psi(g)$ exists and the product $\psi(f \cdot g) = \psi(f) \cdot \psi(g)$.

A one-to-one map $\psi : C_\infty(X) \to C_\infty(Y)$ is called

– *additive* if the condition a) is satisfied;
– *linear* if the conditions a) and b) are satisfied;
– *multiplicative* if the condition b) is satisfied;
– *an isomorphism* if the conditions a), b) and c) are satisfied.

Theorem 1. *If $\psi : C_\infty(X) \to C_\infty(Y)$ is a linear homeomorphism, then $\psi(C(X)) = C(Y)$.*

Proof. Let $f \in C_\infty(X)$. We put $f_n = 2^{-n} \cdot f$ and $O_X(x) = 0$ for all $x \in X$. The limit $\lim\limits_{n\to\infty} f_n = O_X$ exists in X only if $f \in C(X)$. If the limit $\lim\limits_{n\to\infty} f_n = O_X$ exists, then the limit $\lim\limits_{n\to\infty} \psi(f_n) = 2^{-n} \cdot \psi(f) = O_Y$ exists, too. Therefore, if $f \in C(X)$, then $\psi(f) \in C(Y)$ and $\psi(C(X)) \subseteq C(Y)$. Since ψ^{-1} is a linear homeomorphism, we have $\psi(C(X)) = C(Y)$. $\square$

Theorem 2. *If $\varphi : C_\infty C(X) \to C_\infty(Y)$ is an additive and multiplicative homeomorphism and R is the field of real numbers, then φ is an isomorphism and $\varphi(C(X)) = C(Y)$.*

Proof. We have $f = 1_X$ only if $g \cdot f = g$ for all $g \in C_\infty(X)$. Therefore $\varphi(1_X) = 1_Y$. Let $\lambda_X = \lambda \cdot 1_X$ for any $\lambda \in R$. Then $\varphi(n_X) = n_Y$ and $\varphi((1/n)_X) = (1/n)_Y$ for all $n \in N$ and $n \geq 1$. Hence $\varphi(\lambda_X) = \lambda_Y$ for all rational numbers $\lambda \in R$. Theorem 1 complete the proof. $\square$

Theorem 1 implies

Corollary 1. *If the homeomorphism $\varphi : C_\infty(X) \to C_\infty(Y)$ is an isomorphism, then the spaces X and Y are homeomorphic.*

A space X is called χ-sequential if for every nowhere dense closed set F there exist a point $x_0 \in F$ and a sequence $\{x_n \in X \backslash F : n = 1, 2, ...\}$ for which $x_0 = \lim x_n$. Each sequential space is χ-sequential.

The product of any number of metrizable compact spaces is χ-sequential. Let $X = \prod\{X_a : a \in A\}$, where $\{X_a : a \in A\}$ is a set of metrizable compact spaces. We fix nowhere dense in X set F and a point $x = \{x_a : a \in A\} \in F$. Let $Y = \{y = \{y_a : a \in A\} : |a : x_a \neq y_a\}| \leq \chi_0\}$. Then $x \in Y \cap F$ and Y is dense in X. The space Y is sequential. Therefore there is a sequence $\{x_n \in Y \setminus F\}$, converging to x.

Proposition 1. *If $f \in C(X)$, then $f + g$ exists for $g \in C_\infty(X)$ and the maps $u : R_\infty \to R_\infty$ and $v : C(X) \times C_\infty(X) \to C_\infty(X)$, where $u(x, y) = x + y$ and $v(f, g) = f + g$, are continuous.*

Proof. It is enough to prove the continuity of the map $u(x, y) = x + y$. If $x, y \in R$, then the function u is continuous at a point (x, y). Let $x_0 \in R$. For ∞ we consider the neighborhoods $U(n, \infty) = R_\infty \setminus \{x \in R : |x| \leq n\}$. Let $O_{x_0} = \{x \in X : |x - x_0| < 1$ and $m > n + |x_0| + 1\}$. Since $|x + y| \geq |x| - |y|$ we have $Ox_0 + U(m, \infty) \subset U(n, \infty)$. Therefore the map u is continuous. The assertion is proved. $\square$

Proposition 2. *Let X be a χ-sequential space. Then for any function $f \in C_\infty(X) \setminus C(X)$ there exists such a function $g \in C_\infty(X)$ that the sum $f + g$ is not defined.*

Proof. We have $H(f) \neq \emptyset$. Then there exist a point $x_0 \in H(f)$ and a sequence $\{x_n \in X \setminus H(f) : n = 1,\ 2, ...\}$ such that $\lim x_n = x_0$, $|f(x_1)| \geq 1$ and

$|f(x_{n+1})| > |f(x_n)| + 4$. Let $U_n = \{x \in R : |x - f(x_n)| < 1\}$. Then the system $\{f^{-1}U_n : n = 1, 2, ...\}$ is open and locally finite at $x \in X \setminus H(f)$. For any $n = 1, 2, ...$ we fix a continuous function $g_n : X \to [0; 1] \subset R$, where $g_n(x_n) = 1$ and $X \setminus f^{-1}U_n \subset g_n^{-1}(0)$. Let $g = -f + \sum \{(-1)^n \cdot g_n : n = 1, 2, ...\}$. Then $H(g) = H(f)$ and $|f(x) + g(x)| \leq 1$ for any $x \in X \setminus H(f)$. By construction, $f(x_{2n}) + g(x_{2n}) = 1$ and $f(x_{2n+1}) + g(x_{2n+1}) = -1$. Therefore the limit $\lim(f(x_n) + g(x_n))$ does not exist, therefore, the sum $f + g$ does not exist. □

Corollary 2. *Let X and Y be χ-sequential spaces and $\psi : C_\infty(X) \to C_\infty(Y)$ is a one-to-one additive map. Then $\psi/(C(X)) = C(Y)$.*

Proof. By virtue of Proposition 1, $f \in C(X)$ if and only if the sum $f + g$ is defined for any $g \in C_\infty(X)$. This fact follows from Proposition 2. Therefore the conditions and $\psi(f) \in C(Y)$ are equivalent. □

Proposition 3. *Let X be a χ-sequential space. Then for each function $f \in C_\infty(X) \setminus C(X)$ there exists such a function $g \in C_\infty(X)$ that the product $f \cdot g$ is not defined.*

Proof. We have that $H(f) \neq \emptyset$. We choose a point $x_0 \in H(f)$ and a sequence $\{x_n \in X \setminus H(f) : n = 1, 2, ...\}$ such that $\lim\limits_{n\to\infty} x_n = x_0$ and $|f(x_{n+1})| > |f(x_n)| + 4 > 4 + 2^{2n}$. Let $U_n = \{t \in R : |t - f(x_n)| < 1\}$. For any $n \in N$ we fix a continuous function $h_n : X \to [0; 1]$ such that $h_n(x_n) = 1$ and $X \setminus f^{-1}U_n \subset h_n^{-1}(0)$. Let $g_{2n} = 2^{-2n} \cdot h_{2n}$, and $g_{2n-1} = (f(x_{2n-1}))^{-1} \cdot h_{2n-1}$ for all $n = 1, 2, ...$ The function $g = \sum \{g_n \,|\, n = 1, 2, ...\}$ is continuous on X and $g \in C(X)$. We will prove that $f \cdot g$ does not exist. We notice that $|f(x_{2n}) \cdot g(x_{2n})| = 2^{-2n} \cdot |f(x_{2n})| > 2^{-2n} \cdot 2^{4n} = 2^{2n}$ and $|f(x_{2n-1}) \cdot g(x_{2n-1})| = |f(x_{2n-1})| \cdot |f(x_{2n-1})| = 1$. Then $\lim\limits_{n\to\infty} f(x) \cdot g(x_n)$ does not exist. The assertion is proved. □

Proposition 4. *For each space X there is a unique operator of extension $w : C_\infty(X) \to C_\infty(\beta X)$ which is linear, multiplicative and regular, i. e. $\|\omega(f)\| = \|f\|$ for all $f \in C_\infty(X)$.*

Proof. The space R_∞ is compact. Therefore for each continuous map $f : X \to R_\infty$ there exists a unique continuous map $w(f) : \beta X \to R_\infty$ such that $f = w(f)|X$. If the function is bounded, then the function $w(f)$ also is bounded and $\|\omega(f)\| = \|f\|$. Let $f, g \in C_\infty(X)$. If $\varphi = f + g$, then $w(\varphi) = w(f) + w(g)$. If $\varphi = f \cdot g$ then $w(\varphi) = w(f) \cdot w(g)$. Since $\omega(\lambda f) = \lambda \cdot \omega(f)$, the proof is complete. □

A set $H \subset X$ is functionally closed if $f^{-1}(0) = H$ for some function $f \in C(X)$. The complement to functionally closed sets are called the functionally open sets.

A space X is χ-normal if the set $[U]$ is functionally closed for any open in X set U.

A space X is extremely disconnected if the closure $[U]$ is open for any open set U.

Proposition 5. *Let X be an extremally disconnected space. Then:*

1) there exists a regular, linear and multiplicative extension operator $w : C_\infty(X) \to C_\infty(\beta X)$*;*

2) for any two functions $f, g \in C_\infty(X)$ *the sum* $f + g$ *and the product* $f \cdot g$ *are defined;*

3) $C_\infty(X)$ *is a ring and a vector space.*

Proof. Let $f, g \in C_\infty(X)$ and $Y = X \setminus (H(f) \cup H(g))$. Then the set Y is open in X. Let U be an open in Y set. Then the set U is open in X and the set $[U]_{\beta X}$ is open in βX. We have $\beta Y = \beta X$. Let $f_1 = f|Y,\ g_1 = g|Y$. Then $f_1,\ f_1 + g_1,\ f_1 \cdot g_1 \in C(Y)$ and by virtue of Proposition 4, there exist continuous extensions $w(f)$, $w(f + g)$, $w(f \cdot g)$ on βX. The proof is complete. □

Lemma 1. *Let* U *and* V *be open subsets of a space* X*,* $U \cap V \neq \emptyset$*,* $[U] \cup [V] = X$ *and* $F = [U] \cap [V]$ *is a non-empty functionally closed set. Then there exist such functions* $f, g \in C_\infty(X)$ *and* $h \in C(X)$ *that the sum* $f + g$ *and the product* $f \cdot h$ *do not exist.*

Proof. Clearly, $[U]$ and $[V]$ are functionally closed sets. Therefore there exist such continuous functions $\varphi_1, \varphi_2 : X \to [0; 1]$ that $\varphi_1^{-1}(0) = [U]$ and $\varphi_2^{-1}(0) = [V]$. We suppose that $\varphi = \varphi_1 + \varphi_2$ and $h = \varphi_1 - \varphi_2$. Then $\varphi^{-1}(0) = h^{-1}(0) = F$, the map $f = 1/\varphi : X \to R_\infty$ is continuous and $H(f) = F$.

The product $f \cdot h$ does not exist, since $(f \cdot g)(x) = 1$ if $x \in V$, and $(f \cdot g)(x) = -1$ if $x \in U$. The map $g : X \to R_\infty$, where $g(x) = 1 - f(x)$ if $x \in [U]$, and $g(x) = -1 - f(x)$ if $x \in V$, is continuous. The sum $f + g$ does not exist, since $(f + g)(x) = 1$ if $x \in U$, and $(f + g)(x) = -1$ if $x \in V$. The proof is complete. □

Proposition 6. *For a* χ*-normal space* X *the following statements are equivalent:*

– *the space* X *is extremally disconnected;*

– *for any functions* $f, g \in C_\infty$ *there exists the sum* $f + g$*;*

– *for any functions* $f, g \in C_\infty$ *there exists the product* $f \cdot g$*.*

Proof. Implications $1 \to 2$ and $1 \to 3$ follow from Proposition 5. Suppose that the space X is not extremally disconnected. Then there exists an open in X set U such that the set $[U]$ is not open. We put $V = X \setminus [U]$. We can consider that $U = X \setminus [V]$. Then $F = [U] \cap [V]$ is a nonempty functionally closed set. Therefore the implications $2 \to 1$ follow from Lemma 1. The proof is complete. □

Example 1. We consider the discrete sum $X = Y \oplus \beta N$, where Y is an infinite metrizable compact space. The space X is χ-normal and compact. However, the space X is not extremally disconnected. Therefore not for all pairs of functions $f, g \in C_\infty(X)$ the functions $f + g$ or $f \cdot g$ are defined. If $f \in C_\infty(X)$ and on $Y \subset X$ the function f is bounded, then the sum $f + g$ and the product $f \cdot g$ exist for all $g \in C_\infty(X)$. This fact follows from Proposition 5. If the function is not bounded on Y, i.e. $H(f) \cap Y \neq \emptyset$, then the sum $f + g$ and product $f \cdot \varphi$ are not defined for some $g, \varphi \in C_\infty(X)$. Therefore $C_\infty(X)$ is not a ring.

Lemma 2. *Let* $g \in C(X)$ *and the set* $g^{-1}(0)$ *is open. Then the product* $g \cdot f$ *exists for all* $f \in C_\infty(X)$*.*

Proof. The set $U = g^{-1}(0)$ is open-and-closed in X. Let $f \in C_\infty(X)$. If $H(f) = \emptyset$, then the assertion is obvious. We suppose that the set $H(f)$ is not empty. We put $h(x) = 0$ if $x \in U$ and $h(x) = g(x) \cdot f(x)$ if $x \in X \setminus U$. The function h is continuous at all points $x \in X$ for which $h(x) \neq \infty$. Let $x_0 \in X \setminus U$ and $h(x_0) = \infty$. Then $|g(x_0)| > 1/m$ for some $m \in N$. We fix $n \in N$. There exists a neighborhood Ox_0 of the point x_0 in X such that $|g(x)| > 1/m$ and $|f(x)| > nm$ for all $x \in Ox_0$. Then $|h(x)| > n$ for all $x \in Ox_0$. Therefore the function h is continuous at the point x_0 and $h = f \cdot g \in C_\infty(X)$. The proof is complete. □

Lemma 3. *Let X be a χ-normal χ-sequential space, $f \in C_\infty(X)$ and the set $f^{-1}(0)$ is not open in X. Then there exists a function $g \in C_\infty(X)$ such that the product $f \cdot g$ is not defined.*

Proof. As the set $F = f^{-1}(0)$ is not open. Then there exist a point $x_0 \in F$ and a sequence $\{x_n \in X : n = 1, 2, ...\}$ such that $\lim x_n = x_0$ and $0 < |f(x_n)| < 2^{-n}$ for all n. The set $P = F \cap [X \setminus F]$ is nowhere dense, functionally closed and $x_0 \in P$. There exists a continuous function $h \in C_\infty(X)$ such that $P = h^{-1}(0)$, $h(x_{2n}) = f(x_{2n})$ and $h(x_{2n+1}) = 2^{-1} f(x_{2n+1})$. Then $g = 1/h \in C_\infty(X)$, $g(x_{2n}) \cdot f(x_{2n}) = 1$ and $g(x_{2n+1}) \cdot f(x_{2n+1}) = 2$. The lemma is proved. □

Lemma 4. *Let $f \in C_\infty(X)$. The function $1/f$ exists if and only if the set $H(f) \cup f^{-1}(0)$ is nowhere dense.*

Proof. It is obvious. □

Theorem 3. *Let $\varphi : C_\infty(X) \to C_\infty(E)$ be a multiplicative homeomorphism with the property: if $f \in C_\infty(X)$ and $H(f) = f^{-1}(0) = \emptyset$, then $H(\varphi(f)) = \emptyset$. Then:*

1) if $f \in C(X)$ and $|f(x)| < 1$ for all $x \in X$, then $|\varphi(f)(y)| < 1$ for all $y \in Y$;

2) $\varphi(C(X)) \subseteq C(E)$.

Proof. The condition $|f(x)| < 1$ for all $x \in X$ is equivalent to $\lim f^n = 0_X$. The statement 1 of Theorem 3 is proved. Let $f \in C(X)$. We put $h(x) = 2 + |f(x)|$ and $g = 1/h$. Then $f_1 = f \cdot g \in C(X)$ and $|f(x)| < 1$ for all $x \in X$. By construction, $f = h \cdot f_1$ and $H(f_1) = f_1^{-1}(0) = \emptyset$. Considering that $\varphi(f_1), \varphi(h) \in C(Y)$ we receive $\varphi(f) = \varphi(f_1 \cdot h) = \varphi(f_1)\varphi(h) \in C(Y)$. The proof is complete. □

Corollary 3. *If $\varphi : C_\infty(X) \to C_\infty(Y)$ is a multiplicative homeomorphism and R is the field of real numbers, then:*

1) if $f \in C(X)$ and the set $f^{-1}(0)$ is open, then $g = \varphi(f) \in C(Y)$ and the set $g^{-1}(0)$ is open;

2) if $f \in C(X)$ and $f^{-1}(0) = \emptyset$, then $g = \varphi(f) \in C(Y)$ and $g^{-1}(0) = \emptyset$;

3) if $f \in C_\infty(X)$ and $g = 1/f \in C_\infty(X)$, then $\varphi(g) = 1/\varphi(f)$; $\varphi(1_X) = 1_Y$ and $\varphi(0_X) = 0_Y$;

4) if $|f(x)| = 1_X$, then $|\varphi(f)| = 1_Y$;

5) if $|f(x)| = 1_X$, then $|\varphi(f)| = 1_Y$;

6) if $f \geq 0$, then $\varphi(f) \geq 0$;

7) if $|f(x)| < 1$ for all $x \in X$, then $|\varphi(f)(y)| < 1$ for all $y \in Y$;

8) $\varphi(C(X)) = C(Y)$.

Proof. If $f \cdot g = f$ and $f \cdot h = h$ for all $f \in C_\infty(X)$, then $g = 1_X$ and $h = 0_X$. Therefore $\varphi(1_X) = 1_Y$ and $\varphi(0_X) = 0_Y$. The statement 4 of Corollary 3 is proved. The condition $|f| = 1_X$ is equivalent to the condition $f \cdot f = 1_X$. That proves the statement 5. The statement 3 is obvious. The condition $f \geq 0$ is equivalent to $f = g \cdot g$ and $g = (f)^{1/2}$. The statement 6 is proved. □

Let $f \in C_\infty(X)$. There exist such functions $g_n \in C_\infty(X)$ for which $(g_n)^n = f \cdot f$. The limit $\lim g_n$ exists in the pointwise convergensce topology if and only if $H(f) = H(g_n) = \emptyset$ and the set $f^{-1}(0) = g_n^{-1}(0)$ is open. Considering that $(\varphi(g_n))^n = \varphi(f)^2$ and $\lim \varphi(g_n) = \varphi(\lim g_n)$, we finish the proof of the statement 1. The statement 2 follows from the statements 1 and 3 and Lemma 4. The statements 7 and 8 follow from Theorem 3.

Proposition 7. *For a χ-normal χ-sequential space X the following statements are equivalent:*

1) $C(X) = C_\infty(X)$;

2) the space X is discrete.

Proof. Implication $2 \to 1$ is obvious. Assume that the space X is not discrete. Then there exists a non-isolated point $x_0 \in X$ and a sequence $\{x_n \in X \setminus \{x_0\} : n \in N\}$ such that $\lim x_n = x_0$. There exists two open in X sets U and V for which $U \cap V = \emptyset$, $\{x_{2n} : n \in N\} \subset U$ and $\{x_{2n+1} : n \in N\} \in V$. We put $F = [U]$ and $\Phi = [X \setminus F]$. Then $x_0 \in F \cap \Phi = H$, the set H is nowhere dense and there exists a continuous function $f : X \to [0;1]$ such that $H = f^{-1}(0)$. Then $g = 1/\varphi \in C_\infty(X) \setminus C(X)$. Implication $1 \to 2$ is proved. □

Example 2. Let $X = Y \cup \{b\}$ be the one-point compactification of the discrete space Y of uncountable cardinality. The neighborhoods of the point b have the form $O_b = X \setminus F$, where F is a finite subset of the set Y. The space X is χ-sequential, since X is a Frechet-Urysohn space. We will prove that $C(X) = C_\infty(X)$. Let $f \in C_\infty(X)$. Then $H(f) \subset \{b\}$. If $H(f) = \emptyset$, then $f \in C(X)$. Let $H(f) \neq \emptyset$. Then $H(f) = \{b\} = \cap\{f^{-1}((-\infty; -n) \cup (n; +\infty)) : n = 1, 2, ...\}$. This means that $H(f) = \{b\}$ is a G_δ-set. Then there exists a sequence of finite sets $F_n \subset Y : n = 1, 2, ...\}$ such that $X \setminus F_n \subset f^{-1}((-\infty; -n) \cup (n; +\infty))$, i.e. $\{b\} = \cap\{X \setminus F_n : n = 1, 2, \ldots\}$. Hence, $Y = \cup\{F_n : n = 1, 2, ...\}$, and the set Y is countable, a contradiction. Therefore $H(f) = \emptyset$.

A space X is called a P^*-space if for any monotone decreasing sequence $\{U_n : n \in N\}$ of open sets either $\cap\{U_n : n \in N\} = \emptyset$, or there exists a non-empty open set U such that $U \subset \{U_n : n \in N\}$. The space X from Example 2 is a P^*-space. The concepts of χ-normal spaces and of P^*-spaces are opposite. Only discrete spaces are simultaneously χ-normal and P^*-spaces.

Lemma 5. *If X is a P^*-space, then $C(X) = C_\infty(X)$.*

Proof. Let's suppose that there exists a function $f \in C_\infty(X)$. We put $U_n = f^{-1}([-\infty, n] \cap [n, +\infty])$ for all $n \in N$. Then $\cap\{U_n : n \in N\} = H(f)$. Let $H(f) \neq \emptyset$. There exists an open nonempty set U such that $U \subset \cap\{U_n : n \in N\} = H(f)$. Then the set $H(f)$ is not anywhere dense. Therefore $H(f) = \emptyset$ and $f \in C(X)$. The lemma is proved. $\square$

Lemma 6. *Suppose that in a space X there exists a sequence $\{U_n : n \in N\}$ of open in X sets such that $H = \cap\{U_n : n \in N\} \neq \emptyset$ and for any non-empty open in X set U we have $U \setminus H \neq \emptyset$. Then $C_\infty(X) \neq C(X)$.*

Proof. We fix $x_0 \in H$. We build such continuous functions $f_n : X \to [0; 1]$ for which $f_n(x_0) = 0$ and $f_n^{-1}(0) \subset U_n$. By construction, the function $f = \sum(2^{-n} \cdot f_n : n \in N)$ is continuous, $f(x_0) = 0$ and $f^{-1}(0) \subset H$. Thus the set $f^{-1}(0)$ is nowhere dense and it is not closed. We put $g = 1/f : X \to R_\infty$. Then $H(g) = f^{-1}(0)$, $x_0 \in H(g)$ and $g \in C_\infty(X) \setminus C(X)$. The lemma is proved. $\square$

Corollary 4. *For a space X the following statements are equivalent:*
1) X is a P^*-space;
2) $C_\infty(X) = C(X)$.

Example 3. Let Y be an infinite compact space, being P^*-space, and $Z = \beta N$. Then $X = Y \oplus Z$ is a compact space, the space X is not extremally disconnented, $C_\infty(X) \neq C(X)$ and $C_\infty(X)$ is a ring.

The space X is pseudocompact if all continuous real-valued functions are bounded on X.

Theorem 4. *Let X be a P^*-space. The following statements are equivalent:*
1) X is pseudocompact;
2) βX is a P^*-space;
3) $C(\beta X) = C_\infty(\beta X)$.

Proof. Implications $2 \to 3 \to 2$ are obvious. If the space X is not pseudocompact, there exists an unbounded function $f \in C(X)$. By virtue of the proposition from [6], there exists such a continuous map $g : \beta X \to R_\infty$ for which $f = g|X$. Clearly, $H(g) \neq \emptyset$. It proves the implication $2 \to 1$. Let the space X be pseudocompact. We consider a sequence $\{U_n : n \in N\}$ of open in βX sets such that $L = \cap\{U_n : n \in N\} \neq \emptyset$. We can consider that $[U_{n+1}] \subseteq U_n$. Then the set L is functionally closed. If $L \cap X = \emptyset$, then on X there exists some unbounded continuous function and X is not pseudocompact.

Therefore there exists such an open in βX set W for which $\emptyset \neq V = W \cap X \subseteq L$. By construction, $\emptyset \neq W \subseteq L$. Implication $1 \to 2$ is proved. The proof is finished.$\square$

Example 4. Let X be not a pseudocompact P^*-space. Then the map $\varphi : C_\infty(\beta X) \to C_\infty(X) = C(X)$ satisfies the following conditions:
1) φ is a continuous isomorphism;
2) φ is not a homeomorphism;
3) $\varphi(C(\beta X)) \neq C(X)$.

Example 5. Let $X = \beta Y$, where Y be an infinite discrete space. Then there exists a function $h \in C_\infty(X) \setminus C(X)$ such that $h^{-1}(0) = \emptyset$ and the mapping $\varphi : C_\infty(X) \to C_\infty(X)$, where $\varphi(f) = f \cdot h$, satisfies the following conditions:

1) is one-to-one;
2) is linear;
3) $C(x) \cap \varphi(C(X)) = \emptyset$.

From Examples 4 and 5 it follows that the condition that φ is a homeomorphism is essential in the conditions of Theorem 1: if $\varphi : C_\infty(X) \to C_\infty(X)$ is a linear homeomorphism, then $\varphi(C(X)) = C(Y)$.

References

[1] Antonovsky M. I., Boltyansky V. G., Sarymsavov T. A. *Survey of the theory of topological semifields.* Uspehi Mat. Nauk, 1966, **21**, 185–218 (in Russian).

[2] Ipate D. M. *Some properties of a ring of continuous functions in the extended straight line.* Topological algebra (theses of communications of the republican school). Kishinev, 1998, 30–31.

[3] Ipate D. M. *On the ring of the continuous functions in the extended straght line.* Izvestia AN SSR Moldova, Matematica, 1990, No. 3(3), 3–7 (in Russian).

[4] Ipate D. M. *About rings of continuous functions on the extended field of numbers.* Simpozionul al VI-lea Tiraspolian de topologie generală şi aplicaţiile ei. Chişinău, 1991, 104–105.

[5] Ipate D. M. *Rings of admissible mappings into a compactification of a topological field.* Lucrările confer. pentru congr. mat. rom. de pretutindeni, vol. II (19–21 noiembrie 1992), 93–95.

[6] Flaxmajer J. *Topological semifields and boolean algebras corresponding to them.* Trudy MIAN im. V. A. Steklova, 1983, **154**, 252–263 (in Russian).

[7] Shchepin E. V. *About χ-metrizable spaces.* Izvestia AN SSSR, Matematica, 1979, No. 2, **43**, 442–478 (in Russian).

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On preradicals associated to principal functors of module categories. III

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Abstract. The classes of modules and preradicals associated to the functor $Hom_R(\text{-}, U)$ are studied, continuing the investigations of parts I and II. The properties of classes of modules and of associated preradicals are shown, as well as the relations between preradicals. A similarity with the case of functor $T = U \otimes_S$ - is explained.

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Introduction

The preradicals associated to the functors $H = Hom_R(U, \text{-})$ and $T = U \otimes_S$ - are studied in parts I and II of this paper [1, 2], observing some duality between these cases. Now we will investigate the similar question for the contravariant functor $H' = H_U = Hom_R(\text{-}, U) : R\text{-}Mod \to \mathcal{A}b$, where ${}_RU \in R$-Mod. Preradicals of R-Mod defined by ${}_RU$ and H' are revealed, the properties of these preradicals and the relations between them are specified, the conditions of coincidence of some preradicals are shown. The correlation between the cases of functors T and H' is grounded, which explains the similarity of situations for these types of functors.

For Morita contexts and adjoint situations some facts are proved in [3]. For general theory of radicals and torsions the books [4–7] can be used.

1 Preradicals defined by functor H'

Let ${}_RU$ be an arbitrary left R-module. We consider the contravariant functor

$$H' = H_U = Hom_R(\text{-}, U) : R\text{-}Mod \to \mathcal{A}b.$$

Further, we denote by

$$Cog\,({}_RU) = \{M \in R\text{-}Mod \mid \exists \text{ mono } 0 \to M \xrightarrow{i} U^{(\mathfrak{A})}\}$$

the class of modules of R-Mod, cogenerated by ${}_RU$. The following statement is obvious.

Proposition 1.1. *The class of modules $Cog\,({}_RU)$ is pretorsionfree (i.e. is closed under submodules and direct products), therefore it defines a radical r_U in R-Mod such that $\mathcal{P}(r_U) \overset{def}{=\!=} Cog\,({}_RU)$. For every module $M \in R\text{-}Mod$ we have:*

$$r_U(M) = \cap\{Ker\, f \mid f : M \to U\}$$

(the reject of U in M). $\square$

For the functor $H' = Hom_R(\text{-}, U)$ we denote:

$$Ker\, H' = \{M \in R\text{-}Mod \mid H'(M) = 0\}.$$

Using the operator of Hom-orthogonality [1] we have:

$$Ker\, H' = \{{}_RU\}^{\uparrow}.$$

Proposition 1.2. *$Ker H$ is a torsionfree class (i.e. it is closed under homomorphic images, direct sums and extensions), thus it defines an idempotent radical $\overline{r}_U$ such that $\mathcal{R}(\overline{r}_U) \overset{def}{=\!=} Ker\, H'$ and the respective torsionfree class is:*

$$\mathcal{P}(\overline{r}_U) = (Ker\, H')^{\downarrow} = \{{}_RU\}^{\uparrow\downarrow}. \qquad \square$$

Since $\mathcal{P}(\overline{r}_U) = \{{}_RU\}^{\uparrow\downarrow}$ is the least torsionfree class containing ${}_RU$ (or: containing $Cog\,({}_RU) = \mathcal{P}(r_U)$), we obtain

Proposition 1.3. *For every module ${}_RU$ we have $r_U \geq \overline{r}_U$ and $\overline{r}_U$ is the greatest idempotent radical contained in the radical r_U.* $\square$

To establish when the relation $r_U = \overline{r}_U$ is true we need

Definition 1. The module ${}_RU$ will be called *weakly injective* if the functor $H' = Hom_R(\text{-}, U)$ preserves the short exact sequences of the form:

$$0 \to r_U(M) \xrightarrow[\subseteq]{i} M \xrightarrow[\text{nat}]{\pi} M \,/\, r_U(M) \to 0, \quad M \in R\text{-}Mod,$$

i.e. every morphism $f : r_U(M) \to U$ can be extended to a morphism $g : M \to U$ $(gi = f)$:

$$\begin{tikzcd} r_U(M) \arrow[rr, "i", "\subseteq"'] \arrow[dr, "f"'] & & M \arrow[dl, dashed, "g"] \\ & U & \end{tikzcd}$$

Fig. 1.

Proposition 1.4. *For the module ${}_RU$ the following conditions are equivalent*:

1) $r_U = \overline{r}_U$;
2) *radical* r_U *is idempotent*;
3) $Cog\,({}_RU) = (Ker\,H')^{\downarrow}\quad (= \{{}_RU\}^{\uparrow\downarrow})$;
4) ${}_RU$ *is weakly injective.*

Proof. 1) $\Longleftrightarrow$ 2) $\Longleftrightarrow$ 3) follow from Proposition 1.3.

2) $\Rightarrow$ 4). If r_U is idempotent, then $r_U\big(r_U(M)\big) = r_U(M)$ for every $M \in R$-Mod, therefore $r_U(M) \in \mathcal{R}(r_U) = \mathcal{R}(\overline{r}_U) = Ker\,H'$. This means that $Hom_R\big(r_U(M), U\big) = 0$, thus ${}_RU$ is weakly projective ($f = 0 \Rightarrow g = 0$).

4) $\Rightarrow$ 2). Let ${}_RU$ be weakly projective module. For any $f : r_U(M) \to U$ by definition there exists such $g : M \to U$ that $g\,i = f$. Now from the definition of $r_U(M)$ it follows $r_U(M) \subseteq Ker\,g$, so $g\,i = 0$ and $f = 0$. Thus $r_U(M) \subseteq Ker\,f$ for every $f : r_U(M) \to U$, i.e. $r_U(M) \subseteq r_U\big(r_U(M)\big)$ and r_U is idempotent. $\square$

The stronger condition on r_U is the requirement that the radical r_U is a torsion. The question when r_U is a torsion was studied earlier, see for example [6, 8]. The necessary and sufficient condition on ${}_RU$ is to be *pseudo-injective*, which is equivalent to the relation $E({}_RU) \in Cog\,({}_RU)$, where $E({}_RU)$ is the injective envelope of ${}_RU$. Now we will indicate another form of this condition.

Definition 2. Module ${}_RU$ is called *upper hereditary* if the class of modules $\{{}_RU\}^{\uparrow}$ is hereditary (i.e. from $Hom_R(M, U) = 0$ it follows $Hom_R(N, U) = 0$ for every submodule $N \subseteq M$).

From the above statements and definitions follows

Proposition 1.5. *For module ${}_RU$ the following conditions are equivalent*:

1) *radical* r_U *is a torsion*;
2) $r_U = \overline{r}_U$ *and the class* $Ker\,H' = \mathcal{R}(\overline{r}_U)$ *is hereditary*;
3) $r_U = \overline{r}_U$ *and the class* $Cog\,({}_RU)$ *is stable*;
4) ${}_RU$ *is weakly injective and upper hereditary.* $\square$

If the module ${}_RU$ is injective, then it is obvious that r_U is a torsion.

2 Preradicals defined by the ideal $I = (0 : {}_RU)$ and relations with $(r_U, \overline{r}_U)$

For a fixed module ${}_RU$ we apply the radical r_U to ${}_RR$ and obtain the ideal of R:

$$I = r_U({}_RR) = \cap\{Ker\,f \mid f : {}_RR \to {}_RU\}.$$

From the isomorphism $Hom_R(M, U) \cong {}_RU$ we have that every morphism $f : {}_RR \to {}_RU$ is of the form $f_u : {}_RR \to {}_RU$, where $u \in U$ and $f_u(r) = r\,u$ for every $r \in R$. It is obvious that

$$Ker\,f_u = (0 : u) = \{r \in R \mid r\,u = 0\},$$

therefore

$$I = \bigcap \{Ker\, f \mid f : {}_R R \to {}_R U\} = \bigcap_{u \in U} (0 : u) = (0 : {}_R U),$$

i.e. I is the *annihilator* of module ${}_R U$.

As in the previous cases we consider the classes of modules and preradicals defined in R-Mod by the ideal $I \lhd R$. We denote:

${}_I \mathcal{T} = \{M \in R\text{-}Mod \mid IM = M\}$;

${}_I \mathcal{F} = \{M \in R\text{-}Mod \mid m \in M, Im = 0 \Rightarrow m = 0\}$;

$\mathcal{A}(I) = \{M \in R\text{-}Mod \mid IM = 0\}$;

r^I is the *idempotent radical* defined by ${}_I \mathcal{T} : \mathcal{R}(r^I) \stackrel{\text{def}}{=\!=} {}_I \mathcal{T}$;

r_I is the *torsion* defined by ${}_I \mathcal{F} : \mathcal{P}(r_I) \stackrel{\text{def}}{=\!=} {}_I \mathcal{F}$;

$r^{(I)}$ is the *cohereditary radical* defined by $\mathcal{A}(I) : \mathcal{P}(r^{(I)}) \stackrel{\text{def}}{=\!=} \mathcal{A}(I)$;

$r_{(I)}$ is the *pretorsion* defined by $\mathcal{A}(I) : \mathcal{R}(r_{(I)}) \stackrel{\text{def}}{=\!=} \mathcal{A}(I)$.

The relations between these classes (and respective preradicals) are indicated in part I [1]. In particular, we have:

${}_I \mathcal{T} = \mathcal{A}(I)^{\uparrow}, \quad {}_I \mathcal{F} = \mathcal{A}(I)^{\downarrow}$;

$r^I \leq r^{(I)}$ and r^I is the greatest idempotent radical contained in $r^{(I)}$;

$r_I \geq r_{(I)}$ and r_I is the least idempotent radical containing $r_{(I)}$;

$r^I = r^{(I)} \Leftrightarrow r_I = r_{(I)} \Leftrightarrow I = I^2$.

Further we will study the relations between the classes of modules defined by the ideal $I \lhd R$ and classes associated to preradicals r_U and $\overline{r}_U$.

Proposition 2.1. *$Cog\,({}_R U) \subseteq \mathcal{A}(I)$ (i.e. $\mathcal{P}(r_U) \subseteq \mathcal{P}(r^{(I)})$, so $r_U \geq r^{(I)}$).*

Proof. From the definition of I we have $U \in \mathcal{A}(I)$. Class $Cog\,({}_R U)$ is the least class containing ${}_R U$ and closed under submodules and direct products. Since the class $\mathcal{A}(I)$ also possesses these properties, we have $Cog\,({}_R U) \subseteq \mathcal{A}(I)$. $\square$

Proposition 2.2. *$\{{}_R U\}^{\uparrow} = \big(Cog\,({}_R U)\big)^{\uparrow}$.*

Proof. $(\supseteq)$ From ${}_R U \in Cog\,({}_R U)$ it follows $\{{}_R U\}^{\uparrow} \supseteq \big(Cog\,({}_R U)\big)^{\uparrow}$.

$(\subseteq)$ Let $M \in \{{}_R U\}^{\uparrow}$, i.e. $Hom_R(M, U) = 0$. If $N \in Cog\,({}_R U)$, then we have a monomorphism $0 \to N \xrightarrow{\varphi} U^{\mathfrak{A}}$, and every non-zero morphism $0 \neq f : M \to N$ leads to non-zero morphism

$$M \xrightarrow{f} N \xrightarrow{\varphi} U^{\mathfrak{A}} \xrightarrow{\pi_\alpha} U_\alpha = U,$$

a contradiction. Thus $Hom_R(M, N) = 0$ for every $N \in Cog\,({}_R U)$, i.e. $M \in \big(Cog\,({}_R U)\big)^{\uparrow}$. $\square$

Proposition 2.3. *${}_I \mathcal{T} \subseteq Ker\, H'$ (i.e. $\mathcal{R}(r^I) \subseteq \mathcal{R}(\overline{r}_U)$, so $r^I \leq \overline{r}_U$).*

Proof. Since $Cog\,({}_RU) \subseteq \mathcal{A}(I)$ (Proposition 2.1), we have $\big(Cog\,({}_RU)\big)^{\uparrow} \supseteq \mathcal{A}(I)^{\uparrow}$ and by Proposition 2.2 we obtain:

$$ {}_I\mathfrak{T} = \mathcal{A}(I)^{\uparrow} \subseteq \big(Cog\,({}_RU)\big)^{\uparrow} = \{{}_RU\}^{\uparrow} = Ker\,H'. \qquad \square $$

Totalizing we can give a review of relations between the studied classes of modules:

${}_I\mathfrak{T} \subseteq Ker\,H'$, where ${}_I\mathfrak{T} = \mathcal{A}(I)^{\uparrow} = \mathcal{R}(r^I)$ and

$Ker\,H' = \{{}_RU\}^{\uparrow} = \big(Cog\,({}_RU)\big)^{\uparrow} = \mathcal{R}(\overline{r}_U)$;

$Cog\,({}_RU) \subseteq \mathcal{A}(I)$, where $Cog\,({}_RU) = \mathcal{P}(r_U)$ and $\mathcal{A}(I) = \mathcal{R}(r_{(I)}) = \mathcal{P}(r^{(I)})$;

$Cog\,({}_RU) \subseteq \big(Cog\,({}_RU)\big)^{\uparrow\downarrow} = \{{}_RU\}^{\uparrow\downarrow} = \mathcal{P}(\overline{r}_U) \subseteq {}_I\mathfrak{T}^{\downarrow} = \mathcal{A}(I)^{\uparrow\downarrow} = \mathcal{P}(r^I)$;

$Cog\,({}_RU) \subseteq \mathcal{A}(I) \subseteq {}_I\mathfrak{T}^{\downarrow} = \mathcal{A}(I)^{\uparrow\downarrow} = \mathcal{P}(r^I)$;

${}_I\mathfrak{F} = \mathcal{A}(I)^{\downarrow} = \mathcal{P}(r_I)$;

$Cog\,({}_RU) \subseteq \mathcal{A}(I) \subseteq \mathcal{A}(I)^{\downarrow\uparrow} = {}_I\mathfrak{F}^{\uparrow} = \mathcal{R}(r_I)$.

For the corresponding preradicals in particular we have the following situation:

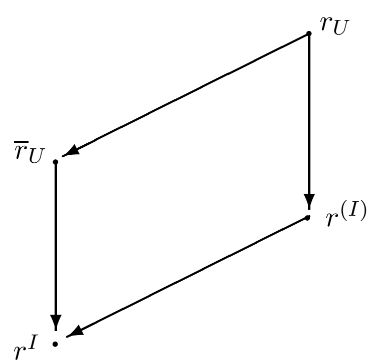


Fig. 2.

where $r_1 \leftarrow r_2$ means $r_1 \leq r_2$. The conditions when $r_U = \overline{r}_U$ or $r^I = r^{(I)}$ are mentioned above. Further we give some remarks on coincidence of other preradicals of Figure 2.

Definition 3. The module ${}_RU$ will be called *Ann-accessible* if from $Hom_R(M, U) = 0$ it follows $Hom_R\,(M, X) = 0$ for every ${}_RX$ with $IX = 0$, where $I = (0 : {}_RU)$.

If ${}_RU$ is Ann-accessible, then $\{{}_RU\}^{\uparrow} \subseteq \mathcal{A}(I)^{\uparrow}$, and the inverse inclusion is always true:

$$ \mathcal{A}(I)^{\uparrow} = {}_I\mathfrak{T} \subseteq Ker\,H' = \{{}_RU\}^{\uparrow}. $$

Thus we have ${}_I\mathfrak{T} = Ker\,H'$, i.e. $\mathcal{R}(r^I) = \mathcal{R}(\overline{r}_U)$, which means that $r^I = \overline{r}_U$.

From these considerations follows

Proposition 2.4. *The following conditions are equivalent*:

1) $r^I = \overline{r}_U$;
2) $Ker\, H' = {}_I\mathfrak{T}$;
3) ${}_I\mathfrak{T}^{\downarrow} = \{{}_RU\}^{\uparrow\downarrow}$;
4) ${}_RU$ *is Ann-accessible.* □

The following particular case is worth noting.

Corollary 2.5. *Let* ${}_RU$ *be a faithful module:* $I = (0 : {}_RU) = 0$. *The relation* $r^I = \overline{r}_U$ *is true if and only if* ${}_RU$ *is a cogenerator of R-Mod.*

Proof. If $I = 0$, then $\mathcal{A}(I) = R\text{-}Mod$, so $\mathcal{A}(I)^{\uparrow} = 0$ and we have ${}_I\mathfrak{T} = \mathcal{A}(I)^{\uparrow} = \mathfrak{R}(r^I) = 0$, i.e. $r^I = 0$. Thus $r^I = \overline{r}_U$ if and only if $\overline{r}_U = 0$.

($\Rightarrow$) If $r^I = \overline{r}_U$, then $Ker\, H' = {}_I\mathfrak{T} = 0$, so the relation $Hom_R(M, U) = 0$ implies $M = 0$. In particular, for every simple module $P \neq 0$ we have $Hom_R(P, U) \neq 0$. Therefore ${}_RU$ contains isomorphically every simple module, thus ${}_RU$ is a cogenerator of R-Mod.

($\Leftarrow$) If $Cog\,({}_RU) = R\text{-}Mod$, then $Cog\,({}_RU) = \mathcal{A}(I)$ and $\big(Cog\,({}_RU)\big)^{\uparrow} = \mathcal{A}(I)^{\uparrow} = 0$, i.e. $Ker\, H' = {}_I\mathfrak{T} = 0$ and this means that $r^I = \overline{r}_U = 0$. □

The relation $r^{(I)} = r_U$ is true if and only if $\mathcal{P}(r^{(I)}) = \mathcal{P}(r_U)$, i.e. $\mathcal{A}(I) = Cog\,({}_RU)$, what is reduced to the inclusion $\mathcal{A}(I) \subseteq Cog\,({}_RU)$.

The coincidence of all preradicals of Figure 2 is a strong condition can be expressed as follows.

Proposition 2.6. *The following conditions are equivalent*:

1) $r^I = r^U$ *(i.e.* $r^I = \overline{r}_U = r^{(I)} = r_U$;
2) $r^{(I)} = r^U$ *and* $I = I^2$;
3) ${}_RU$ *is Ann-accessible and weakly injective.* □

The general situation on relations between the classes of modules in the case of functor $H' = Hom_R(\text{-}, U)$ is illustrated in Figure 3.

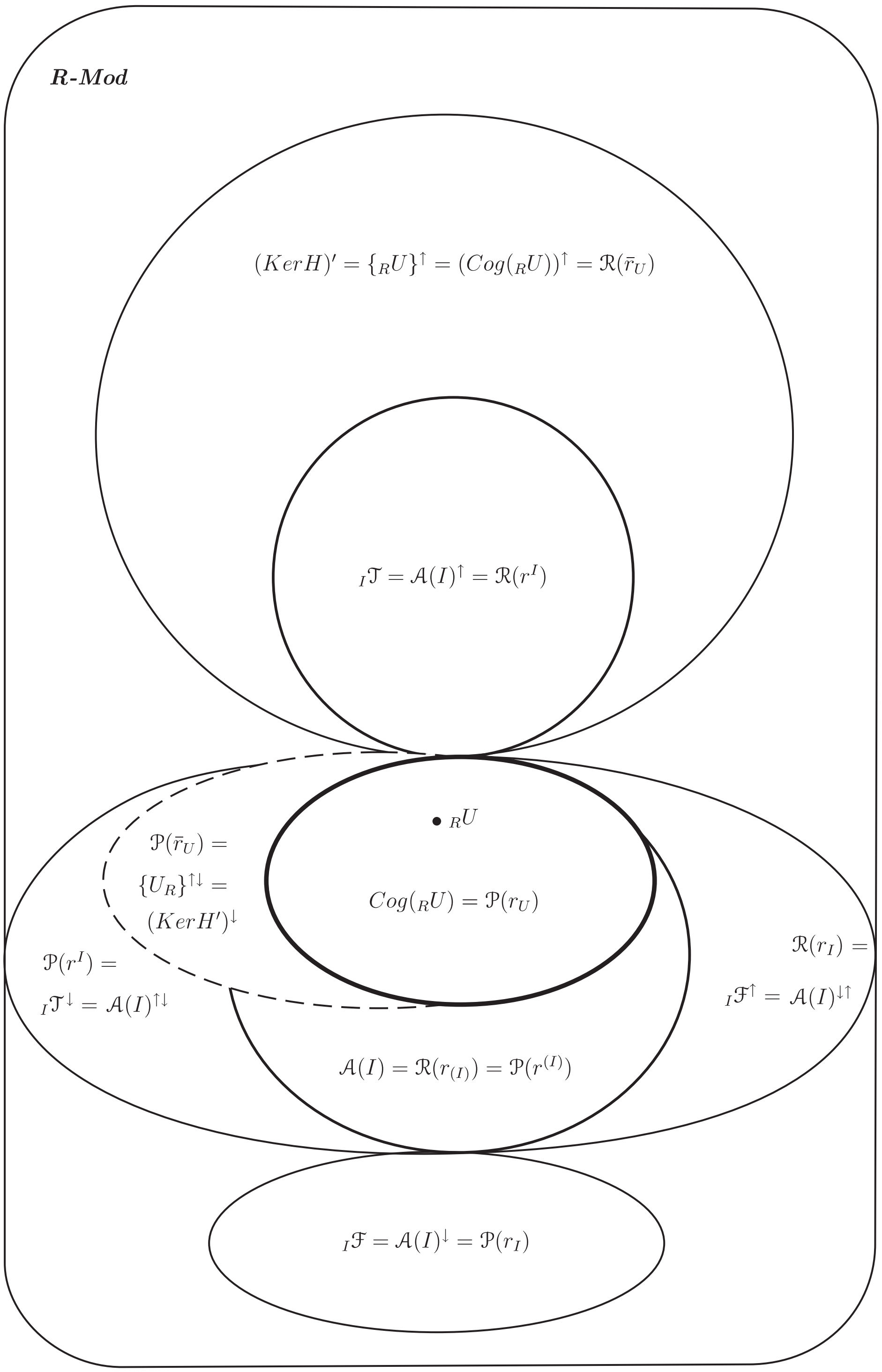
3 Comparing the situations for functors T and H'

Analizing the cases of functors T and H' one can observe an evident resemblance of the obtained situations on classes of modules and associated preradicals. Further we give an explanation of this similarity.

Let U_S be a fixed right S-module which defines the functor

$$T = T^U = U \otimes_S \text{-} : S\text{-}Mod \rightarrow \mathcal{A}b$$

and associated preradicals t_U and $\overline{t}_U$ with the respective classes of modules (see Part II, [2]). We will show that all classes of modules and all preradicals constructed



R-Mod
$(KerH)' = \{_RU\}^{\uparrow} = (Cog(_RU))^{\uparrow} = \mathcal{R}(\bar{r}_U)$
$_I\mathcal{T} = \mathcal{A}(I)^{\uparrow} = \mathcal{R}(r^I)$
$\bullet\ _RU$
$\mathcal{P}(\bar{r}_U) =$
$\{U_R\}^{\uparrow\downarrow} =$
$(KerH')^{\downarrow}$
$Cog(_RU) = \mathcal{P}(r_U)$
$\mathcal{R}(r_I) =$
$_I\mathcal{F}^{\uparrow} = \mathcal{A}(I)^{\downarrow\uparrow}$
$\mathcal{P}(r^I) =$
$_I\mathcal{T}^{\downarrow} = \mathcal{A}(I)^{\uparrow\downarrow}$
$\mathcal{A}(I) = \mathcal{R}(r_{(I)}) = \mathcal{P}(r^{(I)})$
$_I\mathcal{F} = \mathcal{A}(I)^{\downarrow} = \mathcal{P}(r_I)$

Fig. 3.

by U_S in category S-Mod can be obtained with the help of an associated module ${}_SU^*$ by the contravariant functor

$$H' = Hom_S(\text{-}, U^*) : S\text{-}Mod \to \mathcal{A}b$$

as in this part of work.

We fix an arbitrary *cogenerator* C of category $\mathcal{A}b$ of abelian groups (in particular, we can consider that $C = \mathbb{Q}/\mathbb{Z}$). We denote

$${}_SU^* = Hom_{\mathbb{Z}}\left({}_{\mathbb{Z}}U_S, C\right)$$

and consider the contravariant functor

$$H' = Hom_S\left(\text{-}, U^*\right) : S\text{-}Mod \to \mathcal{A}b.$$

The purpose of the following statements is to prove that the functors $T = U \otimes_S$- and $H' = Hom_S(\text{-}, U^*)$ define the same classes of modules, therefore they have the same associated preradicals.

For that we need some preliminary considerations. The fixed module U_S can be regarded as a bimodule ${}_{\mathbb{Z}}U_S$, so it defines the adjoint functors:

$$H = H^U = Hom_{\mathbb{Z}}(U, \text{-}) : \mathcal{A}b \to S\text{-}Mod,$$

$$T = T^U = U \otimes_S \text{-} : S\text{-}Mod \to \mathcal{A}b,$$

(where T is the left adjoint of H), with associated natural transformations $\Phi : TH \to \mathbf{1}_{\mathcal{A}b}$ and $\Psi : \mathbf{1}_{S\text{-}Mod} \to HT$, which satisfy the relations:

$$\Phi_{T(M)} \cdot T(\Psi_M) = 1_{T(M)}, \quad H(\Phi_N) \cdot \Psi_{H(N)} = 1_{H(N)} \tag{1}$$

for every $M \in S\text{-}Mod$ and $N \in \mathcal{A}b$.

In particular, the morphism $\Psi_M : {}_SM \to Hom_{\mathbb{Z}}(U, U \otimes_S M)$ is defined by the rule:

$$[\Psi_M(m)](u) \stackrel{\text{def}}{=} u \otimes_S m, \quad m \in M, \ u \in U.$$

Therefore, for every $M \in S\text{-}Mod$ we have:

$$Ker\,\Psi_M = \{m \in M \,|\, U \otimes_S m = 0\},$$

and Ψ_M is a monomorphism if and only if $U \otimes_S m = 0$ implies $m = 0$. From the definition of the class $\mathcal{F}(U_S)$ we have

Proposition 3.1. $\mathcal{F}(U_S) = \{M \in S\text{-}Mod \,|\, \Psi_M$ *is a monomorphism*$\}$. □

This permits us to prove the following essential relation.

Proposition 3.2. $\mathcal{F}(U_S) = Cog({}_SU^*)$.

Proof. ($\subseteq$) Let $M \in \mathcal{F}(U_S)$, i.e. from $U \otimes_S m = 0$ in $U \otimes_S M$ it follows $m = 0$. By Proposition 3.1 Ψ_M is a monomorphism. Since C is a cogenerator of $\mathcal{A}b$ and $U \otimes_S M \in \mathcal{A}b$, there exists a monomorphism of the form:

$$0 \to U \otimes_S M \xrightarrow{i} \prod_{\alpha \in \mathfrak{A}} C_\alpha, \quad C_\alpha = C.$$

Applying the functor $H = Hom_{\mathbb{Z}}(U, \text{-})$, which preserves monomorphisms and direct products, we obtain the exact sequence:

$$0 \to HT({}_SM) \xrightarrow{H(i)} H\Big(\prod_{\alpha \in \mathfrak{A}} C_\alpha\Big) \cong \prod_{\alpha \in \mathfrak{A}} H(C_\alpha) = \prod_{\alpha \in \mathfrak{A}} U^*_\alpha, \; U^*_\alpha = U^*.$$

Combining $H(i)$ with the monomorphism Ψ_M we obtain the monomorphism:

$$M \xrightarrow{\Psi_M} HT({}_SM) \xrightarrow{H(i)} H\Big(\prod_{\alpha \in \mathfrak{A}} C_\alpha\Big) \cong \prod_{\alpha \in \mathfrak{A}} U^*_\alpha,$$

which shows that $M \in Cog\,({}_SU^*)$.

($\supseteq$) Let $M \in Cog\,({}_SU^*)$. Then $r_{U^*}({}_SM) = \cap\{Ker\, f \mid f : M \to U^*\} = 0$. For every morphism $f : {}_SM \to {}_SU^*$ we have the following commutative diagram:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\Psi_M"'] & U^* \overset{\text{def}}{=\!=} H(C) \arrow[d, "\Psi_{H(C)}"] \\ HT(M) \arrow[r, "HT(f)"] & HTH(C) \end{tikzcd}$$

Fig. 4.

From the relation $H(\Phi_C) \cdot \Psi_{H(C)} = 1_{H(C)}$ (see (1)) it follows that $\Psi_{H(C)}$ is a monomorphism. If $m \in Ker\,\Psi_M$ then from the diagram it is obvious that $\Psi_{H(C)}\big(f(m)\big) = 0$ and, since $\Psi_{H(C)}$ is a monomorphism, it follows that $f(m) = 0$ for all $f : M \to U^*$. Therefore $m \in \cap\{Ker\, f \mid f : M \to U^*\} = 0$ and $Ker\,\Psi_M = 0$, i.e. $M \in \mathcal{F}(U_S)$ by Proposition 3.1. $\square$

Corollary 3.3. $t_U = r_{U^*}$ *and* $\overline{t}_U = \overline{r}_{U^*}$.

Proof. By definitions $\mathcal{F}(U_S) = \mathcal{P}(t_U)$ and $Cog\,({}_SU^*) = \mathcal{P}(r_{U^*})$, therefore by Proposition 3.2 we have $\mathcal{P}(t_U) = \mathcal{P}(r_{U^*})$, so $t_U = r_{U^*}$. But then the "nearest" idempotent radicals also coincide: $\overline{t}_U = \overline{r}_{U^*}$. $\square$

From the above results it follows that all constructions effected in S-Mod by the module U_S and the functor $T = U \otimes_S$ - coincide with the respective constructions by the module ${}_SU^*$ and the functor $H' = Hom_S(\text{-}, U^*)$. For example, the following classes of S-Mod coincide:

$$\mathcal{F}(U_S) = Cog\,({}_SU^*), \quad Ker\, T = Ker\, H', \quad \mathcal{A}(J) = \mathcal{A}(I),$$

$$(Ker\,T)^{\downarrow} = \{_S U^*\}^{\uparrow\downarrow}, \quad {}_J\mathfrak{T} = {}_I\mathfrak{T}, \quad {}_J\mathfrak{F} = {}_I\mathfrak{F}, \quad \text{etc.}$$

These facts completely explain the similarity of the situations for the functors T and H'.

From the conditions of coincidence of "near" preradicals ($t_U = \overline{t}_U$, Part II, Proposition 1.6; $r_U = \overline{r}_U$, Part III, Proposition 1.4) now follows

Corollary 3.4. *U_S is a weakly flat module if and only if $_S U^*$ is weakly injective.* □

References

[1] Kashu A. I., *On preradicals associated to principal functors of module categories. I.* Bul. A.Ş,R.M., Matematica, 2009, No. 2(60), 62–72.

[2] Kashu A. I., *On preradicals associated to principal functors of module categories. II.* Bul. A.Ş,R.M., Matematica, 2009, No. 3(61), 42–51.

[3] Kashu A. I., *Functors and torsions in categories of modules.* Acad. of Sciences of RM, Inst. of Math., Chişinău, 1997 (in Russian).

[4] Bican L., Kepka P., Nemec P. *Rings, modules and preradicals.* Marcel Dekker, New York, 1982.

[5] Golan J. S. *Torsion theories.* Longman Sci. Techn., New York, 1986.

[6] Kashu A. I. *Radicals and torsions in modules.* Ştiinţa, Chişinău, 1983 (In Russian).

[7] Stenström B. *Rings of quotients.* Springer Verlag, Berlin, 1975.

[8] Kashu A. I. *When the radical associated to module is a torsion.* Mat. Zametki, 1974, **6**, No. 1, 41–48 (In Russian).

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Invariant transformations of loop transversals. 1. The case of isomorphism

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Abstract. One special class of invariant transformations of loop transversals in groups is investigated. Transformations from this class correspond to arbitrary isomorphisms of transversal operations corresponding to the loop transversals mentioned above. Isomorphisms of loop transversal operations with the same unit 1 are investigated.

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1 Introduction

The notion of a transversal in a group to its own subgroup is well-known and has been studied during the last 70 years (since R. Baer's work [1]). Loop transversals (transversals whose transversal operations are loops) in some fixed groups to their own subgroups present special interest. Loop transversal may not exist in a given group G to its subgroup H (for example, if $G = S_6$, $H = St_{12}(S_6)$), but we will study such questions further. Let a group G and its proper subgroup H be set, and some loop transversal $T_0 = \{t_i\}_{i\in E}$ in G to H is given and fixed. How to describe all other loop transversals in G to H? In other words, what kind of transformations are admissible over loop transversal T_0 so that the obtained sets were loop transversals too? And how to describe the set of all such admissible transformations?

Generally speaking, such transformations are known, but not for transversals, only for operations – they are isomorphisms, isotopies, parastrophies (of a certain kind), isostrophies (of a certain kind) and crossed isotopies (of a certain kind). But firstly, they are transformations of operations (transversal operations, in particular) instead of transversals; and secondly, only isomorphisms, isotopies and isostrophies are well studied, but such a general transformation as crossed isotopy practically was not investigated.

These investigations are necessary and very important, since there is a number of important and known problems reduced to research of the set of all loop transversals in some given group G to its subgroup H. For example, when $G = S_n$ and $H = St_1(S_n)$, we obtain the set of all loops of some fixed order n. The calculation of their quantity for given natural number n is a well-known open problem (*enumeration problem*). Other known problem – about G-loops – also can be considered in terms of loop transversals transformations. In the present work we will investigate what transformations of loop transversals correspond to the first well-known transformation of transversal operations – to an isomorphism. We will limit ourselves only to those transformations which keep property to be loop transversals.

Let us begin with some necessary definitions and preliminary statements.

2 Necessary definitions and statements

2.1 Quasigroups, loops and transversals in groups

Definition 1. A system $< E, \cdot >$ is called a **left (right) quasigroup** if the equation $(a \cdot x = b)$ (the equation $(y \cdot a = b)$) has exactly one solution in the set E for any fixed $a, b \in E$. If for some element $e \in E$ we have

$$e \cdot x = x \cdot e = x \quad \forall x \in E,$$

then a left (right) quasigroup $< E, \cdot, e >$ is called a **left (right) loop** (the element $e \in E$ is called a **unit**). A left quasigroup $< E, \cdot >$ that is simultaneously a right quasigroup is called simply a **quasigroup**. Similarly, left loop which is simultaneously a right loop is called a **loop**.

Definition 2. Let G be a group and H be its subgroup. Let $\{H_i\}_{i \in E}$ be the set of all left (right) cosets in G to H, and we assume $H_1 = H$. A set $T = \{t_i\}_{i \in E}$ of representativities of the left (right) cosets (by one from each coset H_i and $t_1 = e \in H$) is called a **left (right) transversal** in G to H. If a left transversal T is simultaneously a right one, it is called a **two-side transversal**.

On any left transversal T in a group G to its subgroup H it is possible to define the following operation (*transversal operation*) :

$$x \overset{(T)}{\cdot} y = z \quad \overset{def}{\Longleftrightarrow} \quad t_x t_y = t_z h, \; h \in H,$$

and similarly for a right transversal:

$$x \overset{(T)}{\cdot} y = z \quad \overset{def}{\Longleftrightarrow} \quad t_x t_y = h t_z, \; h \in H.$$

Further we will do all researches only for the left transversals in G to H; for right transversals everything is similar.

Definition 3. If a system $< E, \overset{(T)}{\cdot}, 1 >$ is a loop, then such left transversal $T = \{t_x\}_{x \in E}$ is called a **loop transversal**.

The following statements are known (see [1,6]):

Lemma 1. *A system* $< E, \overset{(T)}{\cdot}, 1 >$ *is a left loop with the two-sided unit 1.*

Proof. See Lemma 1 in [6]. □

Lemma 2. *The following conditions are equivalent:*

1. *The set $T = \{t_x\}_{x \in E}$ is a loop transversal in G to H;*
2. *The set $T = \{t_x\}_{x \in E}$ is a left transversal in G to $\pi H \pi^{-1} \rightleftharpoons H^\pi$, $\forall \pi \in G$;*
3. *The set $\pi T \pi^{-1} \rightleftharpoons T^\pi$ is a left transversal in G to H, $\forall \pi \in G$.*

Proof. See [1]. □

Use further the following permutation representation $\widehat{G}$ of a group G by the left cosets of its subgroup H (see [5,6]):

$$\widehat{g}(x) = y \quad \stackrel{def}{\Longleftrightarrow} \quad g t_x H = t_y H.$$

For simplicity we consider

$$Core_G(H) = \bigcap_{g \in G} g H g^{-1} = \{e\};$$

then this representation is exact (see Lemma 6 in [6]), and we have $\widehat{G} \cong G$. Notice that $\widehat{H} = St_1(\widehat{G})$.

Lemma 3 (see [6])**.** *Let $T = \{t_x\}_{x \in E}$ be a left transversal in G to H. Then the following statements are true:*

1. $\widehat{h}(1) = 1 \ \forall h \epsilon H$;
2. $\forall x, y \in E$:

   $\widehat{t}_x(y) = x \overset{(T)}{\cdot} y = \widehat{L}_x(y), \qquad \widehat{t}_1(x) = \widehat{t}_x(1) = x,$

   $\widehat{t}_x^{-1}(y) = x \overset{(T)}{\diagdown} y = \widehat{L}_x^{-1}(y), \qquad \widehat{t}_x^{-1}(1) = x \overset{(T)}{\diagdown} 1, \qquad \widehat{t}_x^{-1}(x) = 1,$

   where "$\overset{(T)}{\diagdown}$" - is a left division for the operation $< E, \overset{(T)}{\cdot}, 1 >$ (i.e. $x \overset{(T)}{\diagdown} y = z \Longleftrightarrow x \overset{(T)}{\cdot} z = y$).

Proof. See Lemma 4 in [6]. □

Remark 1. The operation "$\overset{(T)}{\diagdown}$" is named a *left division* here – as an inverse operation to the *left multiplication* (multiplication at the left) "$\overset{(T)}{\cdot}$". Sometimes in the literature this operation may be named a *right division.*

Remark 2. As we can see from Lemma 3, item **2**), the elements of a left transversal T in G to H can be represented trought its transversal operation $< E, \overset{(T)}{\cdot}, 1 >$ as left translations $\{L_x\}_{x \in E}$. The similar holds for a right transversal.

At last, remind how any two left transversals T and P in a group G to its subgroup H are connected .

Lemma 4 (see [6]). *Let $T = \{t_x\}_{x\in E}$ and $P = \{p_x\}_{x\in E}$- be left transversals in G to H. Then there is a set of elements $\{h_{(x)}\}_{x\in E}$ from H such that:*

1. $p_x = t_x h_{(x)}$ $\forall x \in E$;

2. $x \overset{(P)}{\cdot} y = x \overset{(T)}{\cdot} \hat{h}_{(x)}(y)$.

Proof. See Lemma 7 in [6]. □

This set $\{h_{(x)}\}_{x\in E}$ is called (see [8]) a **derivation set** for transversal T (and for transversal operation $< E, \overset{(T)}{\cdot}, 1 >$).

Remind also the definitions of a left multiplicative group and of a left inner permutation group of a loop.

Definition 4. Let $< E, \cdot, e >$ be a loop. Then a group

$$LM(< E, \cdot, e >) \overset{def}{=} < L_a \mid a \in E >,$$

generated by all left translations L_a of loop $< E, \cdot, e >$, is called a **left multiplicative group** of the loop $< E, \cdot, e >$. Its subgroup

$$LI(< E, \cdot, e >) \overset{def}{=} < l_{a,b} \mid l_{a,b} = L^{-1}_{a\cdot b} L_a L_b, : a, b \in E >$$

generated by all permutations $l_{a,b}$, is called a **left inner permutation group** of the loop $< E, \cdot, e >$.

2.2 Morphisms of quasigroups and loops

Definition 5 (see [2]). A mapping $\Phi = (\alpha, \beta, \gamma)$ (α, β, γ are permutations on a set E) of the operation $< E, \cdot >$ on the operation $< E, \circ >$ is called an **isotopy** if

$$\gamma(x \cdot y) = \alpha(x) \circ \beta(y) \quad \forall x, y \in E.$$

If $\Phi = (\gamma, \gamma, \gamma)$, then such an isotopy is called an **isomorphism**. If $\Phi = (\alpha, \beta, id)$, then such an isotopy is called a **principal isotopy**.

Definition 6 (see [3]). A mapping $\Phi = (\alpha, B, \gamma)$, where α, γ are permutations on E and $B = B(x, y)$ is a right invertible operation on E ($B(x, y) = \varphi_x(y)$, φ_x is a permutation on E $\forall x \in E$), is called a **right crossed isotopy** (**RC-isotopy**) of operations $< E, \cdot >$ and $< E, \circ >$ if

$$\gamma(x \circ y) = \alpha(x) \cdot B(x, y) \quad \forall x, y \in E.$$

A **left crossed isotopy** (LC-**isotopy**) is defined similarly.

It is obvious that any isotopy is both RC-isotopy and LC-isotopy simultaneously.

Definition 7 (see [2]). The operations $A(x,y)$ and $B(x,y)$ on a set E are called **orthogonal**, if a system

$$\begin{cases} A(x,y) = a \\ B(x,y) = b \end{cases}$$

has an unique solution in a set $E \times E$ for any fixed pair $(a,b) \in E \times E$.

It is easy to show (see [4]) that the orthogonality of operations A and B is equivalent to the fact: the following mapping

$$\Theta = \begin{pmatrix} (1,1) & ... & (x,y) & ... \\ (A(1,1),B(1,1)) & ... & (A(x,y),B(x,y)) & ... \end{pmatrix}$$

is a permutation on the set $E \times E$. The following is true.

Lemma 5. *Let $< E, \cdot, e >$ be a left loop. Then RC-isotop $< E, \circ, e' >$ of the left loop $< E, \cdot, e >$ (by RC-isotopy $T = (\alpha, B, \gamma)$) is a loop $\Longleftrightarrow$ the operations $(\cdot)^{(\alpha,id,id)}$ and B^{-1} are orthogonal.*

Proof. See in [3, 8]. □

2.3 Communication between transformations of transversals, morphisms of transversal operations and transformations of derivation sets

Let G be some fixed group and H be its proper subgroup. Consider further the permutation representation $\widehat{G}$ of the group G (note that $\widehat{G} \cong G$, $\widehat{H} \cong St_1(\widehat{G})$).

According to Lemma 4, any two left transversals $T = \{t_x\}_{x \in E}$ and $P = \{p_x\}_{x \in E}$ in G to H are connected with the help of some RC-isotopy (id, B, id) of their transversal operations $< E, \overset{(T)}{\cdot}, 1 >$ and $< E, \overset{(P)}{\cdot}, 1 >$ (where $B(x,y) = \widehat{h}_{(x)}(y)$). It means that if we fix any *"good"* left transversal T_0 in G to H (for example, a *group* transversal if it exists), then we will receive all other left transversals in G to H from T_0 by the help of RC-isotopy. Moreover, any loop transversal P in G to H may be received from T_0 with the help of such RC-isotopy (id, B, id) (where $B(x,y) = \widehat{h}_{(x)}(y)$) that the operations $< E, \overset{(T_0)}{\cdot}, 1 >$ and $B^{-1}(x,y) = \widehat{h}^{-1}_{(x)}(y)$ are ortogonal (according to Lemma 5).

Remark 3. If we consider the case $G = S_n$ and $H = St_1(S_n)$, as it is described above, it is possible to express all loops of order n as the RC-isotopies (id, B, id) of some loop (group) $< E, \overset{(T_0)}{\cdot}, 1 >$ of order n, and the operation $< E, \overset{(T_0)}{\cdot}, 1 >$ is orthogonal to the operation $B^{-1}(x,y) = \widehat{h}^{-1}_{(x)}(y)$.

Further we will investigate only such special cases of RC-isotopy of a fixed loop transversal T_0 in G to H, which give as a result a loop transversal in G to H again. The research will be done by the following scheme:

$$< E, \overset{(T_0)}{\cdot}, 1 > \overset{\Phi}{\longleftrightarrow} < E, \overset{(P)}{\cdot}, 1 >$$
$$\updownarrow$$
$$T_0 = \{t_x\}_{x\in E} \xrightarrow{\Phi^*} P = \{p_x\}_{x\in E}$$
$$\updownarrow$$
$$p_x = t_x h^{(\Phi)}_{(x)},\ \{h^{(\Phi)}_{(x)}\}_{x\in E} \text{ is a derivation set, corresponding to transformation } \Phi$$
$$\updownarrow$$
$$\Theta_{(\Phi)} = \begin{pmatrix} --- & (x,y) & --- \\ --- & (x \overset{(T_0)}{\cdot} y, (\widehat{h}^{(\Phi)}_{(x)})^{-1}(y)) & --- \end{pmatrix},$$

where $\Theta_{(\Phi)}$ - is a permutation on a set $E\times E$, corresponding to orthogonal operations $< E, \overset{(T_0)}{\cdot}, 1 >$ and $B^{-1}(x,y) = \widehat{h}^{-1}_{(x)}(y)$.

Let us begin our investigation from an elementary invariant transformation on a set of loop transversals in G to H - from the transformation corresponding to isomorphism of transversal operations.

3 The transformations which correspond to isomorphisms of the transversal operations of loop transversals

Let $T = \{t_x\}_{x\in E}$ and $P = \{p_x\}_{x\in E}$ be two loop transversals in a group G to its subgroup H, and $< E, \overset{(T)}{\cdot}, 1 >$ and $< E, \overset{(P)}{\cdot}, 1 >$ are its transversal operations. Fix one of transversals, for example, $T = \{t_x\}_{x\in E}$. Consider the following group:

$$M_G(T) \overset{def}{=} < \alpha \mid \alpha \in St_1(S_E),\ LM(< E, \overset{(T)}{\cdot}, 1 >) \subseteq \alpha\widehat{G}\alpha^{-1} >,$$

it is generated by all permutations $\alpha \in St_1(S_E)$ which satisfy the condition

$$LM(< E, \overset{(T)}{\cdot}, 1 >) \subseteq \alpha\widehat{G}\alpha^{-1}.$$

Lemma 6. *The following propositions are true:*

1. $N_{St_1(S_E)}(\widehat{G}) \subseteq M_G(T) \subseteq St_1(S_E)$,

2. *$M_G(T)$ is maximal among subgroups $M \subseteq St_1(S_E)$ which satisfy the following property:*
$$LM(< E, \overset{(T)}{\cdot}, 1 >) = \bigcap_{\alpha\in M} (\alpha\widehat{G}\alpha^{-1}).$$

Proof. **1**. By definition $M_G(T) \subseteq St_1(S_E)$. Let $\alpha \in N_{St_1(S_E)}(\widehat{G})$, then

$$\begin{cases} \alpha \in St_1(S_E) \\ \alpha \widehat{G} \alpha^{-1} = \widehat{G} \end{cases}.$$

The following property is always fulfilled for any left transversal T in G to H,

$$LM(< E, \overset{(T)}{\cdot}, 1 >) \subseteq \widehat{G},$$

so

$$LM(< E, \overset{(T)}{\cdot}, 1 >) \subseteq \widehat{G} = \alpha \widehat{G} \alpha^{-1}.$$

Since $\alpha \in St_1(S_E)$ then $\alpha \in M_G(T)$, and

$$N_{St_1(S_E)}(\widehat{G}) \subseteq M_G(T).$$

2. It obviously follows from the definition of the group $M_G(T)$. □

Remark 4. Both bounds in the inclusion in item **1** of previous Lemma are reached:

a) Let $LM(< E, \overset{(T)}{\cdot}, 1 >) = \widehat{G}$, then

$$M_G(T) = < \alpha \mid \alpha \in St_1(S_E),\ \widehat{G} \subseteq \alpha \widehat{G} \alpha^{-1} > = N_{St_1(S_E)}(\widehat{G}).$$

b) Let $\widehat{G} = S_E$, $\widehat{H} = St_1(S_E)$, then

$$\begin{aligned} M_G(T) &= < \alpha \mid \alpha \in St_1(S_E),\ LM(< E, \overset{(T)}{\cdot}, 1 >) \subseteq \alpha S_E \alpha^{-1} > = \\ &= < \alpha \mid \alpha \in St_1(S_E) > = St_1(S_E). \end{aligned}$$

Lemma 7. *Let loops* $< E, \overset{(T)}{\cdot}, 1 >$ *and* $< E, \overset{(P)}{\cdot}, 1 >$ *be isomorphic, and* $\varphi : E \to E$ *be this isomorphism (note that* $\varphi(1) = 1$*). Then*

1. $\widehat{P} = h_0^{-1} \widehat{T} h_0$ *for some* $h_0 \in H^* = M_G(T)$*;*
2. $\varphi \equiv h_0$ *and* $LI(< E, \overset{(T)}{\cdot}, 1 >) \subseteq h_0 \widehat{H} h_0^{-1}$.

Proof. **1**. Let conditions of Lemma hold. We have:

$$\varphi(x \overset{(P)}{\cdot} y) = \varphi(x) \overset{(T)}{\cdot} \varphi(y) \forall x, y \in E.$$

According to Lemma 3,

$$\begin{aligned} \widehat{t}_x &= L_x, \quad \textit{where} \quad L_x(y) = x \overset{(T)}{\cdot} y, \\ \widehat{p}_x &= L_x, \quad \textit{where} \quad L_x(y) = x \overset{(P)}{\cdot} y. \end{aligned}$$

Since φ is a permutation on a set E and $\varphi(1)=1$, then $\varphi \in St_1(S_E)$. Further we have

$$\begin{aligned} \varphi L_x(y) &= L_{\varphi(x)}\varphi(y) \forall x,y \in E, \\ L_x(y) &= \varphi^{-1}L_{\varphi(x)}\varphi(y) \forall x,y \in E, \\ L_x &= \varphi^{-1}L_{\varphi(x)}\varphi \forall x \in E. \end{aligned} \tag{1}$$

It means that $\widehat{P} = \varphi^{-1}\widehat{T}\varphi$ and $\varphi \in St_1(S_E)$. Therefore we receive $\widehat{P} = h_0^{-1}\widehat{T}h_0$ for some $h_0 \in St_1(S_E)$ and $\varphi \equiv h_0$.

Moreover, since

$$LM(< E, \overset{(T)}{\cdot}, 1 >) = < L_a \mid a \in E >,$$

then from (1) it follows that

$$\begin{gathered} \varphi^{-1}(LM(< E, \overset{(T)}{\cdot}, 1 >))\varphi = \varphi^{-1} < L_a \mid a \in E > \varphi = \\ = < \varphi^{-1}L_a\varphi \mid a \in E > = < L_b \mid b \in E > = \\ = LM(< E, \overset{(P)}{\cdot}, 1 >) \subseteq \widehat{G}, \end{gathered}$$

and $h_0 = \varphi \in M_G(T)$.

2. Let $\alpha \in M_G(T)$, then we have

$$\begin{cases} \alpha \in St_1(S_E), \\ L_a \in \alpha\widehat{G}\alpha^{-1} \quad \forall a \in E. \end{cases}$$

$$\begin{cases} \alpha \in St_1(S_E), \\ \alpha^{-1}L_a\alpha \leftrightharpoons g_{a'} \in \widehat{G} \quad \forall a \in E. \end{cases}$$

$$a^{'} = g_{a'}(1) = \alpha^{-1}L_a\alpha(1) = \alpha^{-1}(1).$$

Then $\forall a,b \in E$

$$\begin{gathered} \alpha^{-1}l^{(T)}_{a,b}\alpha = \alpha^{-1}L^{-1}_{a \overset{(T)}{\cdot} b}L_aL_b\alpha = \\ = (\alpha^{-1}L^{-1}_{a \overset{(T)}{\cdot} b}\alpha)\cdot(\alpha^{-1}L_a\alpha)\cdot(\alpha^{-1}L_b\alpha) = \\ = g^{-1}_{\alpha^{-1}(a \overset{(T)}{\cdot} b)}g_{\alpha^{-1}(a)}g_{\alpha^{-1}(b)}. \end{gathered}$$

Assuming $a = \alpha(u)$ and $b = \alpha(v)$ (i.e. $u = \alpha^{-1}(a)$ and $v = \alpha^{-1}(b)$), we obtain

$$\alpha^{-1}l^{(T)}_{\alpha(u),\alpha(v)}\alpha = g^{-1}_{\alpha^{-1}(\alpha(u) \overset{(T)}{\cdot} \alpha(v))}g_ug_v.$$

Since α is an isomorphism of operations $(\overset{(T)}{\cdot})$ and $(\overset{(P)}{\cdot})$, then

$$\alpha(u \overset{(P)}{\cdot} v) = \alpha(u) \overset{(T)}{\cdot} \alpha(v),$$

and therefore

$$\alpha^{-1} l^{(T)}_{\alpha(u),\alpha(v)} \alpha = g^{-1}_{u \overset{(P)}{\cdot} v} g_u g_v = l^{(P)}_{u,v} \in LI(< E, \overset{(P)}{\cdot}, 1 >) \subseteq \widehat{H}.$$

It means that

$$\begin{aligned} \alpha^{-1} LI(&< E, \overset{(P)}{\cdot}, 1 >) \alpha \subseteq \widehat{H}, \\ LI(&< E, \overset{(T)}{\cdot}, 1 >) \subseteq \alpha \widehat{H} \alpha^{-1}. \end{aligned}$$

□

Lemma 8. *Let $T = \{t_x\}_{x \in E}$ be a fixed loop transversal in G to H and $h_0 \in N_{St_1(S_E)}(H)$. Define the set of permutations:*

$$p_{x'} \overset{def}{=} h_0^{-1} t_x h_0 \quad \forall x \in E.$$

Then

1. *$P = \{p_{x'}\}_{x' \in E}$ is a loop transversal in G to H;*
2. *The transversal operations $< E, \overset{(P)}{\cdot}, 1 >$ and $< E, \overset{(T)}{\cdot}, 1 >$ are isomorphic, and the isomorphism is set up by the mapping $\varphi(x) = h_0(x)$.*

Proof. **1**. Let the conditions of Lemma hold. At first we can see that $P = \{p_{x'}\}_{x' \in E}$ is a left transversal in G to H. It follows from Lemma 2 and the following calculation

$$x' = \widehat{p}_{x'}(1) = h_0^{-1} \widehat{t}_x h_0(1) = h_0^{-1}.$$

Any transversal conjugated with the transversal T will be conjugated with the transversal P. According to Lemma 2, the transversal $P = \{p_{x'}\}_{x' \in E}$ is a loop transversal in G to H.

2. Consider the transversal operation $< E, \overset{(P)}{\cdot}, 1 >$ which corresponds to the transversal P. We have

$$x \overset{(P)}{\cdot} y = z \quad \Longleftrightarrow \quad p_x p_y = p_z h, \quad h \in H, \quad \forall x, y \in E. \tag{2}$$

Since

$$h_0^{-1} t_x h_0 = p_{x'} = p_{h_0^{-1}(x)},$$

then after replacing $x \to h_0(u)$ we have

$$p_u = h_0^{-1} t_{h_0(u)} h_0 \quad \forall u \in E.$$

From (2) we obtain

$$p_x p_y = p_z h, \quad h \in H, \quad (where \quad z = x \overset{(P)}{\cdot} y),$$

$$h_0^{-1}t_{h_0(x)}h_0 \cdot h_0^{-1}t_{h_0(y)}h_0 = h_0^{-1}t_{h_0(x \overset{(P)}{\cdot} y)}h_0 \cdot h, \quad h \in H,$$
$$t_{h_0(x)}t_{h_0(y)} = t_{h_0(x \overset{(P)}{\cdot} y)} \cdot (h_0 h h_0^{-1}).$$

Since $h_0 \in N_{St_1(S_E)}(\widehat{H})$ then $(h_0 h h_0^{-1}) = h' \in \widehat{H}$. Therefore we obtain

$$h_0(x) \overset{(T)}{\cdot} h_0(y) = h_0(x \overset{(P)}{\cdot} y) \quad \forall x, y \in E,$$

i.e. $\varphi = h_0$ is an isomorphism of the operations $< E, \overset{(P)}{\cdot}, 1 >$ and $< E, \overset{(T)}{\cdot}, 1 >$. $\square$

It means that conjugated loop transversals in G to H correspond to isomorphic loop transversal operations and vice versa.

Further according to the scheme from Section 2, we will find out the form of derivation sets $\{h_{(x)}\}_{x\in E}$ which correspond to isomorphic transformations.

Lemma 9. *Let $T = \{t_x\}_{x\in E}$ and $P = \{p_x\}_{x\in E}$ be two loop transversals in G to H which correspond to isomorphic transversal operations. Let $p_x = t_x h_{(x)}$ and $\{h_{(x)}\}_{x\in E}$ be a derivation set. Then*

$$h_{(x)} = t_x^{-1} h_0^{-1} t_{h_0(x)} h_0, \quad \forall x \in E$$

for some $h_0 \in M_G(T)$.

Proof. Let conditions of Lemma hold. According to Lemma 7 $\forall x \in E$:

$$p_x = h_0^{-1} t_{h_0(x)} h_0,$$

for some $h_0 \in M_G(T)$. From the other hand

$$p_x = t_x h_{(x)} \quad \forall x \in E.$$

Therefore we have

$$t_x h_{(x)} = h_0^{-1} t_{h_0(x)} h_0 \quad \forall x \in E,$$
$$h_{(x)} = t_x^{-1} h_0^{-1} t_{h_0(x)} h_0 \quad \forall x \in E,$$

as it had to be shown. $\square$

At last according to the scheme from Section 2 we will express the form of permutations Θ which correspond to isomorphic transformations of transversals.

Lemma 10. *Let $T = \{t_x\}_{x\in E}$ and $P = \{p_x\}_{x\in E}$ be loop transversals in G to H, and its transversal operations $< E, \overset{(P)}{\cdot}, 1 >$ and $< E, \overset{(T)}{\cdot}, 1 >$ are isomorphic. A permutation Θ on $E \times E$ corresponds to ortogonal operations " $\overset{(T)}{\cdot}$ " and $B^{-1}(x, y)$ (see in Section 2), can be expressed in the following form (for some $h_0 \in M_G(T)$): $\forall x, y \in E$*

$$\Theta = \begin{pmatrix} ... & (x, y) & ... \\ ... & (x \overset{(T)}{\cdot} y, h_0^{-1}(h_0(x) \overset{(T)}{\backslash} h_0(x \overset{(T)}{\cdot} y))) & ... \end{pmatrix}.$$

Proof. According to the previous lemma we have: it is true for some $h_0 \in M_G(T)$:

$$h_{(x)} = t_x^{-1} h_0^{-1} t_{h_0(x)} h_0 \quad \forall x \in E.$$

Then

$$h_{(x)}^{-1} = h_0^{-1} t_{h_0(x)}^{-1} h_0 t_x \quad \forall x \in E.$$

According to the definition, the permutation Θ can be expressed in the following form

$$\Theta = \begin{pmatrix} \dots & (x, y) & \dots \\ \dots & (x \overset{(T)}{\cdot} y,\, h_{(x)}^{-1}(y)) & \dots \end{pmatrix}.$$

We have $\forall x \in E$:

$$\begin{aligned} h_{(x)}^{-1}(y) &= h_0^{-1} \widehat{t}_{h_0(x)}^{-1} h_0 \widehat{t}_x(y) = h_0^{-1} \widehat{t}_{h_0(x)}^{-1} h_0(x \overset{(T)}{\cdot} y) = \\ &= h_0^{-1}(h_0(x) \overset{(T)}{\backslash} h_0(x \overset{(T)}{\cdot} y)), \end{aligned}$$

and finally we obtain

$$\Theta = \begin{pmatrix} \dots & (x, y) & \dots \\ \dots & (x \overset{(T)}{\cdot} y,\, h_0^{-1}(h_0(x) \overset{(T)}{\backslash} h_0(x \overset{(T)}{\cdot} y))) & \dots \end{pmatrix}.$$

□

Remark 5. A permutation $h_0 = id$, the derivation set $\{h_{(x)}\} = id \quad \forall x \in E$ and the permutation

$$\Theta_0 = \begin{pmatrix} \dots & (x, y) & \dots \\ \dots & (x \overset{(T)}{\cdot} y,\, y) & \dots \end{pmatrix}$$

correspond to the trivial isomorphism $\varphi = id$.

Consider the product (composition) $\Theta_0^{-1}\Theta$ as a composition of two permutations from $S_{E\times E}$. We have

$$\begin{aligned} \Theta_0^{-1}\Theta &= \begin{pmatrix} \dots & (x \overset{(T)}{\cdot} y,\, y) & \dots \\ \dots & (x,\, y) & \dots \end{pmatrix} \circ \begin{pmatrix} \dots & (x,\, y) & \dots \\ \dots & (x \overset{(T)}{\cdot} y,\, h_{(x)}^{-1}(y)) & \dots \end{pmatrix} = \\ &= \begin{pmatrix} \dots & (x \overset{(T)}{\cdot} y,\, y) & \dots \\ \dots & (x \overset{(T)}{\cdot} y,\, h_0^{-1}(h_0(x) \overset{(T)}{\backslash} h_0(x \overset{(T)}{\cdot} y))) & \dots \end{pmatrix} = \\ &\overset{x \overset{(T)}{\cdot} y = z}{=} \begin{pmatrix} \dots & (z,\, y) & \dots \\ \dots & (z,\, h_0^{-1}(h_0(z \overset{(T)}{/} y) \overset{(T)}{\backslash} h_0(z))) & \dots \end{pmatrix} \leftrightharpoons \Theta^*. \end{aligned}$$

As a corollary we received two interesting particular cases:

$$\Theta_0^{-1}\Theta(z,\, z) = (z,\, h_0^{-1}(\underset{\overset{\shortparallel}{1}}{h_0(1)} \overset{(T)}{\backslash} h_0(z))) = (z,\, h_0^{-1}(h_0(z))) = (z,\, z) \quad \forall z \in E.$$

$$\Theta_0^{-1}\Theta(z,\,1) = (z,\,h_0^{-1}(h_0(z)\overset{(T)}{\diagdown}h_0(z))) = (z,\,h_0^{-1}(1)) = (z,\,1) \quad \forall z \in E,$$

i.e. $\Theta^* \in St_{(a,\,a),\,(a,\,1)}(S_{E\times E}) \quad \forall a \in E.$

References

[1] Baer R. *Nets and groups.* Trans. Amer. Math. Soc., 1939, **46**, 110–141.

[2] Belousov V. *Foundations of quasigroup and loop theory.* Moscow, Nauka, 1967 (in Russian).

[3] Belousov V. *Cross isotopies of quasigroups.* Quasigroups and their systems: Mat. issled., vyp. 113. Kishinev, Shtiintsa, 1990, 14–20 (in Russian).

[4] Belousov V. *On properties of binary operations.* Uch. zapiski Beltskogo gospedinstituta im. A. Russo, 1960, **5**, 9–28.

[5] Hall M. *Group theory.* Moscow, IL, 1962 (in Russian).

[6] Kuznetsov E. *Transversals in groups. 1. Elementary properties.* Quasigroups and related systems, 1994, **1**, No. 1, 22–42.

[7] Kuznetsov E. *Transversals in groups. 2. Loop transversals in a group by the same subgroup.* Quasigroups and related systems, 1999, **6**, 1–12.

[8] Kuznetsov E. *Transversals in groups. 4. Derivation construction.* Quasigroups and related systems, 2002, **9** 67–84.

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An Approach for Determining the Matrix of Limiting State Probabilities in Discrete Markov Processes

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Abstract. A new approach for determining the matrix of limiting state probabilities in Markov processes is proposed and a polynomial time algorithm for calculating this matrix is grounded. The computational complexity of the algorithm is $O(n^4)$, where n is the number of the states of the discrete system.

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1 Introduction and Problem Formulation

Consider a stochastic discrete system L with finite set of states

$$X = \{x_1,\ x_2,\ \dots,\ x_n\}.$$

Assume that the dynamics of the system is modeled by a Markov process with given stochastic matrix of probabilities transitions $P = (p_{ij})_{i,j=\overline{1,n}}$ where

$$\sum_{j=1}^{n} p_{i,j} = 1,\ i = \overline{1,n}; \quad 0 \le p_{i,j} \le 1,\ i,j = \overline{1,n}.$$

The probability $P_{x_{i_0}}(x,t)$ of system's passage from the state x_{i_0} to an arbitrary state $x \in X$ by using t transitions is defined and calculated on the basis of the following recursive formula [2]

$$P_{x_{i_0}}(x,\tau+1) = \sum_{y\in X} P_{x_{i_0}}(y,\tau)p_{y,x}, \quad \tau = \overline{0,t-1}, \tag{1}$$

where $P_{x_{i_0}}(x_{i_0},0) = 1$ and $P_{x_{i_0}}(x,0) = 0,\ \forall x \in X\setminus\{x_{i_0}\}$. We call these probabilities

state-time probabilities of system L. Formula (1) can be represented in the matrix form as follow

$$\pi(\tau+1) = \pi(\tau)P, \quad \tau = \overline{0,t-1}. \tag{2}$$

Here $\pi(\tau) = (\pi_1(\tau), \pi_2(\tau), \dots, \pi_n(\tau))$ is the vector, where an arbitrary component i expresses the probability of the system L to reach the state x_i from x_{i_0} at the moment of time τ, i.e. $\pi_i(\tau) = P_{x_{i_0}}(x_i, \tau)$. At the starting moment of time $\tau = 0$ the vector $\pi(\tau)$ is given and its components are defined as follows: $\pi_{i_0}(0) = 1$ and $\pi_i(0) = 0$ for arbitrary $i \neq i_0$. It is easy to observe that if for given starting vector $\pi(0)$ we apply formula (2) for $\tau = 0, 1, 2, \ldots, t-1$, then we obtain

$$\pi(t) = \pi(0)P^t,$$

where $P^t = P \times P \times \cdots \times P$. So, an arbitrary element $p^{(t)}_{x_i,x_j}$ of this matrix expresses the probability of system L to reach the state x_j from x_i by using t units of times. It is easy to see that for given starting representation of the vector $\pi(0)$ the following properties holds

$$\sum_{i=1}^{n} \pi_i(\tau) = 1, \quad \tau = 0, 1, 2, \ldots. \tag{3}$$

The correctness of this property can be easy proved using induction principle with respect to τ. Indeed, for $\tau = 0$ the equality (3) holds according to the definition. If we assume that (3) holds for every $\tau \leq t$ then we obtain the correctness of this formula for $\tau = t + 1$ as follows

$$\sum_{i=1}^{n} \pi_i(t+1) = \sum_{i=1}^{n}\sum_{j=1}^{n} p_{x_j,x_i}\pi_j(t) = \sum_{j=1}^{n} \pi_j(t) \sum_{i=1}^{n} p_{x_j,x_i} = \sum_{j=1}^{n} \pi_j(t) = 1.$$

So, formula (3) holds. In order to analyze the asymptotic behavior of the state-time probabilities of the system using formula (3) we will assume that there exists the limit

$$\lim_{t\to\infty} P^t = Q.$$

If this limit exists then there exists the following limit

$$\pi = \lim_{t\to\infty} \pi(t) = \pi(0) \lim_{t\to\infty} P^t = \pi(0)Q,$$

where an arbitrary component π_j of the vector $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ expresses the probability that the system L will occupy the state x_j after a large number of transitions when it starts transitions in the state x_{i_0}. The vector π will be called the vector of limiting state probabilities. Based on the mentioned above property we may conclude that

$$\sum_{j=1}^{n} \pi_j = 1$$

for an arbitrary given starting vector $\pi(0)$. This means that the matrix $Q = (q_{x,y})$ satisfies the condition

$$\sum_{y\in X} q_{x,y} = 1, \quad \forall x \in X,$$

where $q_{x,y} \geq 0, \quad \forall x, y \in X$, i.e. $Q = (q_{x,y})$ is a stochastic matrix. An arbitrary element $q_{x,y}$ of this matrix expresses the probability that the system will occupy the state y after a large number of transitions if it starts transitions in the state x. The matrix Q is called the matrix of limiting states probabilities of the Markov process.

An important class of discrete Markov process represents ergodic Markov chain. For this class all rows of the matrix of limiting states probabilities Q are the same, i.e. $q_{x,y} = q_{v,y}, \quad \forall x, y, v \in X$. In this case the limiting state probabilities $\pi_j,\ j = \overline{1,n}$, does not depend on the state in which the system starts transitions. The vector π of limiting state probabilities can be found by solving the system of linear equations

$$\begin{cases} \pi = \pi P \\ \sum\limits_{j=1}^{n} \pi_j = 1. \end{cases} \tag{4}$$

The first condition $\pi = \pi P$ in this system is obtained from (2) when $\tau \to \infty$ and the second one reflects the property that after a large number of transitions the dynamical system will be in one of the states $x_j \in X$. It is well known that for ergodic Markov chains the system (4) has a unique solution [2, 4]. The necessary and sufficient conditions for the ergodicity of Markov processes are given in [2, 4]. In general system (4) may have a unique solution also when the limit $\lim\limits_{t\to\infty} P^t$ does not exist. This case may correspond to periodic Markov process and a component π_j of vector π that satisfies (4) can be treated as the probability of the system L to occupy the state x_j at the random moment of times during a large number of transitions. In the following we can see that the definition of the matrix of limiting-state probabilities Q can be extended for an arbitrary Markov process, however in the case when $\lim\limits_{t\to\infty} P^t$ does not exist the elements of the matrix Q have another interpretation.

In this paper we describe an approach for determining the matrix of limiting state probabilities in Markov processes and propose a polynomial time algorithm for calculating of this matrix. We show that the running time the algorithm is $O(n^4)$, where n is the number of the states of the discrete system.

2 The main results

The aim of this section is to ground a polynomial time algorithm for determining the limit matrix Q for an arbitrary discrete Markov process with given stochastic matrix P. We describe such an algorithm which is based on the idea of z-transform and classical numerical methods.

2.1 The Main Approach and the General Scheme of the Algorithm

Let $\mathbb{C}$ be the complex space and denote by $\mathbb{M}(\mathbb{C})$ the set of complex matrices with n rows and n columns. We consider the function $A : \mathbb{C} \to \mathbb{M}(\mathbb{C})$, where

$$A(z) = I - zP, \quad z \in \mathbb{C}.$$

We denote the elements of the matrix $A(z)$ by $a_{i,j}(z),\ i, j = \overline{1,n}$, i.e.

$$a_{i,j}(z) = \delta_{i,j} - zp_{ij} \in \mathbb{C}[z]$$

where

$$\delta_{i,j} = \begin{cases} 1 & if\ i = j \\ 0 & if\ i \neq j \end{cases} \quad i, j = \overline{1, n}.$$

It is evident that the determinant $\Delta(z)$ of the matrix $A(z)$ is a polynomial of degree less or equal to n, $(deg(\Delta(z)) \leq n,\ \Delta(z) \in \mathbb{C}[z])$. Therefore if we denote $\mathcal{D} = \{z \in \mathbb{C} \mid \Delta_P(z) \neq 0\}$ then we obtain that $|\mathbb{C}\backslash\mathcal{D}| \leq deg(\Delta(z)) \leq n$ and for an arbitrary $z \in \mathcal{D}$ there exists the inverse matrix of $A(z)$. So, we can define the function $F : \mathcal{D} \to \mathbb{M}(\mathbb{C})$ where

$$F(z) = (A(z))^{-1}.$$

Then the elements $F_{i,j}(z),\ i, j = \overline{1, n}$ of $F(z)$ can be found as follows:

$$F_{i,j}(z) = \frac{M_{j,i}(z)}{\Delta(z)},\ i, j = \overline{1, n},$$

where

$$M_{i,j}(z) = (-1)^{i+j} A_{i,j}(z)$$

and $A_{i,j}(z)$ is the determinant of the matrix obtained from $A(z)$ by deleting the row i and the column j, $i, j = \overline{1, n}$. Therefore

$$M_{j,i}(z) \in \mathbb{C}[z],\ deg(M_{j,i}(z)) \leq n - 1,\ i, j = \overline{1, n}.$$

Note that $\Delta(1) = 0$ because for the matrix $A(1)$ holds the property

$$\sum_{j=1}^{n}(\delta_{ij} - p_{ij}) = \sum_{j=1}^{n}\delta_{ij} - \sum_{j=1}^{n}p_{ij} = \delta_{ii} - 1 = 0,\ i = \overline{1, n}.$$

This means that $1 \in \mathbb{C}\backslash\mathcal{D}$ and therefore $\Delta(z)$ can be factored by $(z - 1)$. Taking into account that $F_{i,j}(z)$ is a rational fraction with the denominator $\Delta(z)$ we can represent $F_{i,j}(z)$ uniquely in the following form

$$F_{i,j}(z) = B_{ij}(z) + \sum_{y \in \mathbb{C}\backslash\mathcal{D}_P} \sum_{k=1}^{m(y)} \frac{\alpha_{i,j,k}(y)}{(z - y)^k},\ i, j = \overline{1, n}, \tag{5}$$

where $m(z)$ is the order of the root z of the polynomial $\Delta(z)$, $z \in \mathbb{C}\backslash\mathcal{D}$, and $\alpha_{ijk}(y) \in \mathbb{C},\ \forall y \in \mathbb{C}\backslash\mathcal{D},\ k = \overline{1, m(y)},\ i, j = \overline{1, n}$. In this representation of $F_{i,j}(z)$ the degree of the polynomial $B_{ij}(z) \in \mathbb{C}[z]$ satisfies the condition

$$deg(B_{i,j}(z)) = deg(M_{j,i}(z)) - deg(\Delta(z)),$$

where $deg(M_{j,i}(z)) \geq deg(\Delta(z))$, otherwise $B_{i,j}(z) = 0$.

To represent (5) in a more convenient form we shall use some elementary properties of the function $\nu_k(z) = \dfrac{1}{(1 - z)^k}$, $k = 1, 2, \ldots$. It is well known that in the case $k = 1$ the function $\nu_1(z)$ admits the series expansion $\nu_1(z) = \sum\limits_{t=0}^{\infty} z^t$. In general case (for an arbitrary $k > 1$) the following recursive relation holds $\nu_{k+1}(z) = \dfrac{d\nu_k(z)}{kdz}$, $k = 1, 2, \ldots$. Using these properties and induction principle we can obtain the series expansion of the function $\nu_k(z)$, $\forall k \geq 1$: $\nu_k(z) = \sum\limits_{t=0}^{\infty} T_{k-1}(t)z^t$, where $T_{k-1}(t)$ is a polynomial of degree less or equal to $(k-1)$.

Based on mentioned above properties we can make the following transformation in (5) we can make the following transformation:

$$F_{i,j}(z) = B_{i,j}(z) + \sum_{y \in \mathbb{C} \setminus \mathcal{D}} \sum_{k=1}^{m(y)} \frac{\left(-\frac{1}{y}\right)^k \alpha_{i,j,k}(y)}{\left(1 - \frac{1}{y}z\right)^k} =$$

$$= B_{i,j}(z) + \sum_{y \in \mathbb{C} \setminus \mathcal{D}} \sum_{k=1}^{m(y)} \left(-\frac{1}{y}\right)^k \alpha_{i,j,k}(y) \nu_k \left(\frac{z}{y}\right) =$$

$$= B_{i,j}(z) + \sum_{y \in \mathbb{C} \setminus \mathcal{D}} \sum_{k=1}^{m(y)} \left(-\frac{1}{y}\right)^k \alpha_{i,j,k}(y) \sum_{t=0}^{\infty} T_{k-1}(t) \left(\frac{z}{y}\right)^t =$$

$$= B_{i,j}(z) + \sum_{t=0}^{\infty} \left(\frac{z}{y}\right)^t \sum_{y \in \mathbb{C} \setminus \mathcal{D}} \sum_{k=0}^{m(y)-1} \left(-\frac{1}{y}\right)^{k+1} \alpha_{i,j,k+1}(y) T_k(t).$$

We can observe that in the relation above the expression

$$\sum_{k=0}^{m(y)-1} \left(-\frac{1}{y}\right)^{k+1} \alpha_{i,j,k+1}(y) T_k(t)$$

represents a polynomial of degree less or equal to $m(y) - 1$ and we can write it in the form $\sum\limits_{k=0}^{m(y)-1} \beta_{i,j,k}(y) t^k$, where $\beta_{i,j,k}$ represent the corresponding coefficients of this $\sum\limits_{k=0}^{m(y)-1} \beta_{i,j,k}(y) t^k$ for polynomial. Therefore if in the expression above we substitute $\sum\limits_{k=0}^{m(y)-1} \left(-\frac{1}{y}\right)^{k+1} \alpha_{i,j,k+1}(y) T_k(t)$ then we obtain

$$F_{i,j}(z) = B_{i,j}(z) + \sum_{t=0}^{\infty} z^t \sum_{y \in \mathbb{C} \setminus \mathcal{D}} \sum_{k=0}^{m(y)-1} \frac{t^k}{y^t} \beta_{i,j,k}(y) =$$

$$= W_{i,j}(z) + \sum_{t=1+deg(B_{i,j}(z))}^{\infty} z^t \sum_{y \in \mathbb{C} \backslash \mathcal{D}} \sum_{k=0}^{m(y)-1} \frac{t^k}{y^t} \beta_{i,j,k}(y), \ i,j = \overline{1,n}, \tag{6}$$

where $\beta_{i,j,k}(y) \in \mathbb{C}, \ \forall y \in \mathbb{C} \backslash \mathcal{D}, \ k = \overline{0, m(y)-1}, \ i,j = \overline{1,n}$, and $W_{ij}(z) \in \mathbb{C}[z]$ is a polynomial of degree that satisfies the condition $deg(W_{i,j}(z)) = deg(B_{i,j}(z))$, $i,j = \overline{1,n}$.

Note that for the norm of the matrix P we have $\|P\| = \max\limits_{i=\overline{1,n}} \sum\limits_{j=1}^{n} p_{i,j} = 1$, and therefore $\|zP\| = |z|\|P\| = |z|$. Let $|z| < 1$. Then for $F(z)$ we have

$$F(z) = (I - zP)^{-1} = \sum_{t=0}^{\infty} P^t z^t.$$

This means that

$$F_{i,j}(z) = \sum_{t=0}^{\infty} p_{i,j}(t) z^t, \ i,j = \overline{1,n}. \tag{7}$$

From definition of z-transform and from (6) – (7) we obtain

$$p_{i,j}(t) = \sum_{y \in \mathbb{C} \backslash \mathcal{D}} \sum_{k=0}^{m(y)-1} \frac{t^k}{y^t} \beta_{i,j,k}(y), \ \forall t > deg(B_{i,j}(z)), \ i,j = \overline{1,n}.$$

Since $0 \le p_{i,j}(t) \le 1, \ i,j = \overline{1,n}, \ \forall t \ge 0$, we have

$$|y| \ge 1, \ \forall y \in \mathbb{C} \backslash \mathcal{D}, \ \beta_{i,j,k}(1) = 0, \ \forall k \ge 1.$$

This involves $\alpha_{i,j,k}(1) = 0, \ \forall k \ge 2$.

Now let us assume that $\Delta(z) = (z-1)^{m(1)} T(z), \ T(1) \neq 0$. Then the relation (5) is represented as follows:

$$F_{i,j}(z) = \frac{\alpha_{i,j,1}(1)}{z-1} + B_{i,j}(z) + \sum_{y \in (\mathbb{C} \backslash \mathcal{D}) \backslash \{1\}} \sum_{k=1}^{m(y)} \frac{\alpha_{i,j,k}(y)}{(z-y)^k} =$$

$$= \frac{\alpha_{i,j,1}(1)}{z-1} + \frac{Y_{i,j}(z)}{T(z)}, \ i,j = \overline{1,n},$$

where $Y_{i,j}(z) \in \mathbb{C}[z]$ and

$$deg(Y_{i,j}(z)) = deg(B_{i,j}(z)) + deg(T(z)) = deg(B_{i,j}(z)) + deg(\Delta(z)) - m(1) =$$

$$= deg(M_{j,i}(z)) - m(1) \le n - 1 - m(1) \le n - 2, \ i,j = \overline{1,n}.$$

In the following we will denote

$$Y(z) = (Y_{i,j}(z))_{i,j=\overline{1,n}},\ \alpha_1(1) = (\alpha_{i,j,1}(1))_{i,j=\overline{1,n}}.$$

Then the matrix $F(z)$ can be represented as follows:

$$F(z) = \frac{1}{z-1}\,\alpha_1(1) + \frac{1}{T(z)}\,Y(z). \tag{8}$$

From this formula and from definition of the limiting-state matrix Q we have

$$Q = -\alpha_1(1), \tag{9}$$

i.e Q in the inverse matrix of $(I - zP)$ corresponds to the term with the coefficient $\dfrac{1}{1-z}$.

From (8) and (9) we obtain formula

$$Q = \lim_{z\to 1}(1-z)(I-zP)^{-1}.$$

In the following we show how to determine the polynomial $\Delta(z)$ and the function $F(z)$ in the matrix form.

2.2 Algorithm for Determining the Polynomial $\Delta(z)$

Let us consider the characteristic polynomial

$$K(z) = |P - zI| = \sum_{k=0}^{n} \nu_k z^k.$$

In this polynomial the coefficient of the term with maximal degree of variable z is $\nu_n = |-I_n| = (-1)^n \neq 0$. This means that $deg(K(z)) = n$ and we can represent $K(z)$ in the form

$$K(z) = (-1)^n(z^n - \alpha_1 z^{n-1} - \alpha_2 z^{n-2} - \ldots - \alpha_n).$$

If we denote $\alpha_0 = -1$, then it is easy to see that the coefficients ν_k can be represented as follows:

$$\nu_k = (-1)^{n+1}\alpha_{n-k},\ k = \overline{0,n}.$$

In $[1, 5]$ it is shown that the coefficients α_k can be calculated basing on Leverrier's method using $O(n^3)$ elementary operations. This method can be applied for determining the coefficients α_k in the following way:

1) We determine the matrices

$$P^{(k)} = (p_{i,j}(k))_{i,j=\overline{1,n}},\ k = \overline{1,n},$$

where $P^{(k)} = P \times P \times \cdots \times P$;

2) Then we determine the traces of these matrices:

$$s_k = trP^{(k)} = \sum_{j=1}^{n} p_{j,j}(k),\ k = \overline{1,n};$$

3) Finally we calculate the coefficients

$$\alpha_k = \frac{1}{k}\left(s_k - \sum_{j=1}^{k-1} \alpha_j s_{k-j}\right),\ k = \overline{1,n}.$$

If the coefficients α_k are known then we can determine the coefficients of the polynomial $\Delta(z) = \sum\limits_{k=0}^{n} \beta_k z^k$. Indeed, if $z \in \mathbb{C}\backslash\{0\}$ then

$$\Delta(z) = |I - zP| = (-z)^n \left|P - \frac{1}{z}I\right| = (-1)^n z^n K\left(\frac{1}{z}\right) =$$

$$= (-1)^n z^n \sum_{k=0}^{n} \nu_k \frac{1}{z^k} = (-1)^n \sum_{k=0}^{n} \nu_k z^{n-k} = \sum_{k=0}^{n} (-1)^n \nu_{n-k} z^k =$$

$$= \sum_{k=0}^{n} (-1)^n (-1)^{n+1} \alpha_k z^k = \sum_{k=0}^{n} (-\alpha_k) z^k.$$

For $z = 0$ we have

$$\Delta(0) = |I| = 1 = -\alpha_0.$$

Therefore finally we obtain

$$\Delta(z) = \sum_{k=0}^{n} (-\alpha_k) z^k,\ \forall z \in \mathbb{C}.$$

This means $\beta_k = -\alpha_k,\ k = \overline{0,n}$. So, the coefficients $\beta_k,\ k = \overline{0,n}$, can be calculated using a similar recursive formula

$$\beta_k = -\alpha_k = -\frac{1}{k}\left(s_k - \sum_{j=1}^{k-1} \alpha_j s_{k-j}\right) = -\frac{1}{k}\left(s_k + \sum_{j=1}^{k-1} \beta_j s_{k-j}\right),\ k = \overline{1,n},$$

$$\beta_0 = -\alpha_0 = 1,$$

We can use the following algorithm for determining the coefficients β_k.

Algorithm 1.1: Determining the coefficients of the polynomial $\boldsymbol{\Delta(z)}$

1) Calculate the matrices $P^{(k)} = (p_{i,j}(k))_{i,j=\overline{1,n}},\ k = \overline{1,n}$;

2) Determine the traces of the matrices $P^{(k)}$:

$$s_k = trP^{(k)} = \sum_{j=1}^{n} p_{j,j}(k),\ k = \overline{1,n};$$

3) Find the coefficients

$$\beta_0 = 1,\ \beta_k = -\frac{1}{k}\left(s_k + \sum_{j=1}^{k-1} \beta_j s_{k-j}\right),\ k = \overline{1,n}.$$

2.3 Polynomial Time Algorithm for Determining the Function $\boldsymbol{F(z)}$

Consider

$$T'(z) = (z-1)T(z)$$

and denote $N = deg(T'(z)) = n - (m(1) - 1)$. We have already shown that the function $F(z)$ can be represented in the following matrix form:

$$F(z) = \frac{1}{T'(z)} \sum_{k=0}^{N-1} R^{(k)} z^k,$$

where

$$(z-1)^{m(1)-1} \sum_{k=0}^{N-1} R_{i,j}^{(k)} z^k = M_{j,i},\ i,j = \overline{1,n}.$$

We will make some transformation using the identity $I = (I - zP)(I - zP)^{-1}$. We have

$$T'(z)I = (I - zP) \sum_{k=0}^{N-1} z^k R^{(k)} = \sum_{k=0}^{N-1} z^k R^{(k)} - \sum_{k=0}^{N-1} z^{k+1}(PR^{(k)}) =$$

$$= R^{(0)} + \sum_{k=1}^{N-1} z^k (R^{(k)} - PR^{(k-1)}) - z^N(PR^{(N-1)}).$$

Let $T'(z) = \sum\limits_{k=0}^{N} \beta_k^* z^k$ and substitute this expression in obtained above relation. Then we obtain the following formula for determining the matrices $R^{(k)},\ \ k = \overline{0, N-1}$:

$$R^{(0)} = \beta_0^* I; \quad R^{(k)} = \beta_k^* I + PR^{(k-1)},\ k = \overline{1, N-1}. \tag{10}$$

So, we have

$$F(z) = \left(\frac{V_{ij}(z)}{T'(z)}\right)_{i,j=\overline{1,n}},$$

where

$$V_{i,j}(z) = \sum_{k=0}^{N-1} R_{ij}^{(k)} z^k, \ i,j = \overline{1,n}.$$

Based on these formula we can develop algorithm for determining the matrix Q.

2.4 Polynomial Time Algorithm for Determining the Matrix of Limiting-State Probabilities Q

Consider

$$T(z) = \sum_{k=0}^{N-1} \gamma_k z^k; \ Y(z) = \sum_{k=0}^{N-2} y^{(k)} z^k; \ y^* = \alpha_1(1).$$

Then according to relation (8) we obtain

$$\frac{V_{i,j}(z)}{T'(z)} = F_{i,j}(z) = \frac{y_{i,j}^*}{z-1} + \frac{\sum\limits_{k=0}^{N-2} y_{ij}^{(k)} z^k}{T(z)}, \ i,j = \overline{1,n}.$$

This involve

$$\sum_{k=0}^{N-1} R_{i,j}^{(k)} z^k = V_{i,j}(z) = y_{i,j}^* T(z) + (z-1) \sum_{k=0}^{N-2} y_{i,j}^{(k)} z^k = y_{i,j}^* \sum_{k=0}^{N-1} \gamma_k z^k +$$

$$+ \sum_{k=0}^{N-2} y_{i,j}^{(k)} z^{k+1} - \sum_{k=0}^{N-2} y_{i,j}^{(k)} z^k = \sum_{k=0}^{N-1} \gamma_k y_{i,j}^* z^k + \sum_{k=1}^{N-1} y_{i,j}^{(k-1)} z^k - \sum_{k=0}^{N-2} y_{i,j}^{(k)} z^k =$$

$$= (\gamma_0 y_{i,j}^* - y_{i,j}^{(0)}) + \sum_{k=1}^{N-2} (\gamma_k y_{i,j}^* + y_{i,j}^{(k-1)} - y_{i,j}^{(k)}) z^k + (\gamma_{N-1} y_{i,j}^* + y_{i,j}^{(N-2)}) z^{N-1}, \ i,j = \overline{1,n}.$$

If we equate the corresponding coefficients of the variable z with the same exponents then we obtain the following system of linear equations:

$$\begin{cases} R_{i,j}^{(0)} = \gamma_0 y_{i,j}^* - y_{i,j}^{(0)}, \\ R_{i,j}^{(k)} = \gamma_k y_{i,j}^* + y_{i,j}^{(k-1)} - y_{i,j}^{(k)}, \ k = \overline{1, N-2}, \ \ i,j = \overline{1,n}. \\ R_{i,j}^{(N-1)} = \gamma_{N-1} y_{i,j}^* + y_{i,j}^{(N-2)}, \end{cases}$$

This system is equivalent to the following system:

$$\begin{cases} y_{i,j}^{(0)} = \gamma_0 y_{i,j}^* - R_{i,j}^{(0)}, \\ y_{ij}^{(k)} = \gamma_k y_{i,j}^* + y_{i,j}^{(k-1)} - R_{i,j}^{(k)}, \ k = \overline{1, N-2}, \quad i,j = \overline{1,n}. \\ y_{i,j}^{(N-2)} = -\gamma_{N-1} y_{i,j}^* + R_{i,j}^{(N-1)}. \end{cases}$$

Here we can see that there exist the coefficients $u_{i,j}^{(k)}, \ v_{i,j}^{(k)} \in \mathbb{C}, \ k = \overline{0, N-2}, \ \ i,j = \overline{1,n}$, such that

$$y_{i,j}^{(k)} = u_{i,j}^{(k)} y_{i,j}^* + v_{i,j}^{(k)}, \ k = \overline{0, N-2}, \ i,j = \overline{1,n}.$$

From the first equation we obtain

$$u_{i,j}^{(0)} = \gamma_0, \ v_{i,j}^{(0)} = -R_{i,j}^{(0)}, \ i,j = \overline{1,n}.$$

From the next $N-2$ equations we obtain

$$y_{i,j}^{(k)} = \gamma_k y_{i,j}^* + y_{i,j}^{(k-1)} - R_{i,j}^{(k)} = \gamma_k y_{i,j}^* + u_{i,j}^{(k-1)} y_{i,j}^* + v_{i,j}^{(k-1)} - R_{i,j}^{(k)} =$$

$$= (\gamma_k + u_{i,j}^{(k-1)}) y_{i,j}^* + (v_{i,j}^{(k-1)} - R_{ij}^{(k)}), \ k = \overline{1, N-2}, \ i,j = \overline{1,n},$$

which involve the recursive equations

$$u_{i,j}^{(k)} = u_{i,j}^{(k-1)} + \gamma_k, \ v_{ij}^{(k)} = v_{i,j}^{(k-1)} - R_{i,j}^{(k)}, \ k = \overline{1, N-2}, \ i,j = \overline{1,n}.$$

In a such way we obtain the direct formula for calculation of the coefficients:

$$u_{i,j}^{(k)} = \sum_{r=0}^{k} \gamma_r, \ v_{i,j}^{(k)} = -\sum_{r=0}^{k} R_{i,j}^{(r)}, \ k = \overline{0, N-2}, \ i,j = \overline{1,n}.$$

If we introduce these coefficients in the last equation of the system then we obtain

$$u_{i,j}^{(N-2)} y_{ij}^* + v_{i,j}^{(N-2)} = -\gamma_{N-1} y_{ij}^* + R_{i,j}^{(N-1)}, \ i,j = \overline{1,n} \ \Leftrightarrow$$

$$\Leftrightarrow \ y_{i,j}^* \sum_{r=0}^{N-1} \gamma_r = \sum_{r=0}^{N-1} R_{i,j}^{(r)}, \ i,j = \overline{1,n} \ \Leftrightarrow$$

$$\Leftrightarrow \ y_{i,j}^* = \frac{\sum\limits_{r=0}^{N-1} R_{i,j}^{(r)}}{\sum\limits_{r=0}^{N-1} \gamma_r} = \frac{R_{i,j}}{T(1)}, \ i,j = \overline{1,n},$$

where $R_{ij} = \sum\limits_{r=0}^{N-1} R_{i,j}^{(r)}, \ i,j = \overline{1,n}$. Finally, if we denote $R = (R_{ij})_{i,j=\overline{1,n}}$ then

$$Q = -\frac{1}{T(1)} R. \tag{11}$$

Based on result described above we can describe the algorithm for determining the matrix Q.

Algorithm 1.2: Determining the Limiting-State Matrix Q

1) Find the coefficients of the polynomial $\Delta(z) = \sum_{k=0}^{n} \beta_k z^k$ using Algorithm 1.1;

2) Divide $m(1)$ times the polynomial $\Delta(z)$ by $z-1$, using Horner scheme and find the polynomial $T(z)$ that satisfies the condition $T(1) \neq 0$. At the same time we preserve the coefficients β_k^*, $k = \overline{0, N}$, of the polynomial $T'(z) = (z-1)T(z)$ obtained at the previous step of the Horner's scheme;

3) Determine $T(1)$ according to the rule described above;

4) Find the matrices $R^{(k)}$, $k = \overline{0, N-1}$, according to (10);

5) Find the matrix $R = \sum_{k=0}^{N-1} R^{(k)}$;

6) Calculate the matrix Q according to formula (11);

It is easy to check that the running time of Algorithm 1.2 is $O(|X|^4)$. Indeed, step 1) and step 4) of the algorithm use $O(|X|^4)$ elementary operations and each of remainder steps 2) - 3) and 5) - 6) use in the worst case $O(|X|^3)$ elementary operations.

3 Numerical examples

In this section we give some numerical examples which illustrate the main details of the algorithms from previous section.

Example 1. Consider the discrete Markov process with the stochastic matrix of probability transactions $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We can see that $P^{n)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $P^{2n+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\forall n \geq 0$, i.e. the Markov chain is 2-periodic.

So, in this case the limit $\lim_{n\to\infty} P^n$ does not exist, but there exists the matrix Q which can be found by using algorithm described above. If we apply this algorithm then we obtain:

1) $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $P^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $s_1 = trP = 0$, $s_2 = trP^2 = 2$;

$$\beta_0 = 1,\ \beta_1 = -s_1 = 0,\ \beta_2 = -\frac{1}{2}(s_2 + \beta_1 s_1) = -1;$$

2) We divide the polynomial $\beta_2 z^2 + \beta_1 z + \beta_0$ by $z-1$ using Horner's scheme

	-1	0	1
1	-1	-1	0
1	-1	-2	

and obtain $m(1) = 1,\ N = 2;\ \beta_0^* = 1,\ \beta_1^* = 0,\ \beta_2^* = -1;\ \gamma_0 = -1,\ \gamma_1 = -1;$

3) $T(1) = \gamma_0 + \gamma_1 = -2;$

4) $R^{(0)} = \beta_0^* I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ R^{(1)} = \beta_1^* I + PR^{(0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$

5) $R = R^{(0)} + R^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix};$

6) $Q = -\dfrac{1}{T(1)}R = \dfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}.$

In such a way we obtain the limit matrix $Q = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$, however the considered process is not ergodic because the matrix $P^{(n)}$ contains zero elements $\forall n \geq 0$. The rows of this matrix are the same and the vector of limiting probabilities $\pi^* = (0.5, 0.5)$ can be found also by solving the system of linear equation (4).

Example 2. Consider the Markov process with the stochastic matrix $P = \begin{pmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{pmatrix}$. We can see that in this case the Markov process is ergodic. We can find the matrix Q using our algorithm:

$$P = \begin{pmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{pmatrix},\ P^2 = \begin{pmatrix} 0.45 & 0.55 \\ 0.44 & 0.56 \end{pmatrix};$$

$$s_1 = trP = 0.5 + 0.6 = 1.1,\ s_2 = trP^2 = 0.45 + 0.56 = 1.01;$$

$$\beta_0 = 1,\ \beta_1 = -s_1 = -1.1,\ \beta_2 = -\frac{1}{2}(s_2 + \beta_1 s_1) = -\frac{1}{2}(1.01 - 1.1 \cdot 1.1) = 0.1;$$

	0.1	-1.1	1
1	0.1	-1	0
1	0.1	-0.9	

$$\beta_0^* = 1,\ \beta_1^* = -1.1,\ \beta_2^* = 0.1;\ \gamma_0 = -1,\ \gamma_1 = 0.1;\ T(1) = \gamma_0 + \gamma_1 = -0.9;$$

$$R^{(0)} = \beta_0^* I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ R^{(1)} = \beta_1^* I + PR^{(0)} = \begin{pmatrix} -0.6 & 0.5 \\ 0.4 & -0.5 \end{pmatrix};$$

$$R = R^{(0)} + R^{(1)} = \begin{pmatrix} 0.4 & 0.5 \\ 0.4 & 0.5 \end{pmatrix};\ Q = -\frac{1}{T(1)}R = \frac{1}{9}\begin{pmatrix} 4 & 5 \\ 4 & 5 \end{pmatrix}.$$

We have $Q = \begin{pmatrix} 4/9 & 5/9 \\ 4/9 & 5/9 \end{pmatrix}$. The rows of this matrix are the same and all elements of the matrix $P^{(n)}$ are non zero when $t \to \infty$. So, this is ergodoc Markov process with the vector of limiting probabilities $\pi_1^* = \dfrac{4}{9}$. As we have shown this vector can be found by solving system (4).

Example 3. We consider a non ergodic Markov process with the stochastic matrix of probabilities transactions

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}.$$

In this case the solution of the system of linear equations (4) is not unique. If we apply the proposed algorithm we can determine the matrix Q. According to this algorithm we obtain:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}, \; P^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \end{pmatrix}, \; P^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{13}{27} & \frac{13}{27} & \frac{1}{27} \end{pmatrix};$$

$$s_1 = trP = 7/3, \; s_2 = trP^2 = 19/9, \; s_3 = trP^3 = 55/27; \; \beta_0 = 1,$$

$$\beta_1 = -s_1 = -7/3, \; \beta_2 = -(s_2 + \beta_1 s_1)/2 = 5/3, \; \beta_3 = -(s_3 + \beta_1 s_2 + \beta_2 s_1)/3 = -1/3;$$

	-1/3	5/3	-7/3	1
1	-1/3	4/3	-1	0
1	-1/3	1	0	
1	-1/3	2/3		

$$\beta_0^* = -1, \; \beta_1^* = 4/3, \; \beta_2^* = -1/3; \; \gamma_0 = 1, \; \gamma_1 = -1/3; \; T(1) = \gamma_0 + \gamma_1 = 2/3;$$

$$R^{(0)} = \beta_0^* I = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; R^{(1)} = \beta_1^* I + PR^{(0)} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ -1/3 & -1/3 & 1 \end{pmatrix};$$

$$R = R^{(0)} + R^{(1)} = \begin{pmatrix} -2/3 & 0 & 0 \\ 0 & -2/3 & 0 \\ -1/3 & -1/3 & 0 \end{pmatrix}; \; Q = -\frac{1}{T(1)} R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/2 & 0 \end{pmatrix}.$$

So, finally we have

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/2 & 0 \end{pmatrix}.$$

In this case all rows of the matrix Q are different. It is easy to observe that for the considered example there exits $\lim_{n \to \infty} P^{(n)} = Q$.

References

[1] Helmberg G., Voltkamp G. *On Fadeev-Leverrier's Method for the Computation of the Characteristic Polynomial of the Matrix and of Eigenvectors.* Linear Algebra and its Application, 1993, No. 185, 219–233.

[2] Howard R.A. *Dynamic Programming and Markov Processes.* Wiley, 1960.

[3] Lozovanu D., Pickl S. *Dynamic Programming Algorithms for Solving Stochastic Discrete Control Problems.* Bulletin of the Academy of Science of RM, Mathematics, 2009, No. 2(60), 73–90.

[4] Puterman M. *Markov Decision Processes.* Wiley, 1993.

[5] Paţiuc V., Râbacova G., Secrieru I., Zolotarevschi V. *Metode numerice în probleme diferenţiale, integrale şi de valori proprii.* Chişinău, 2001.

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Numerical modeling of multidimensional problems of gravitational gas dynamics with high resolution schemes

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Abstract. The aim of this paper is to implement and analyze a nonoscillatory high-resolution scheme for multudimensional hyperbolic conservation laws. Using methods of Nessyahu and Tadmor for solving three–dimensional equations of gravitational gas dynamics we provide a central two-step (predictor and corrector) scheme.

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1 Introduction

High resolution numerical schemes are used to solve multidimensional problems of gravitational gas dynamics. Most of modern cosmological models assume existence of two matter types in the Universe – baryonic matter and another one known as a dark matter. The first may be straight examined and includes atoms of any sort. The second one is undetectable by its emitted radiation, but its presence can be inferred from gravitational effects on visible matter. Gaseous nebula is considered to be a formation of gas, dust and other materials that ”clump” together to form larger masses, which attract further matter, and eventually become big enough to form stars. The remaining materials are then believed to form planets, and other planetary system objects [1, 2]. For a sufficiently accurate description of these problems we need to apply high-resolution difference schemes which use high-order schemes. A stable calculation in presence of shock waves requires a certain amount of numerical dissipation, in order to avoid the formation of unphysical numerical oscillations [3].

A three–dimensional difference scheme of the type TVD and some other related results are presented in the paper [4].

2 Governing Equations

The equations of a self-gravitational ideal hydrodynamics may be expressed in a conservative form with a source term:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F_x}}{\partial x} + \frac{\partial \mathbf{F_y}}{\partial y} + \frac{\partial \mathbf{F_z}}{\partial z} = 0, \tag{1}$$

together with Poissons equation

$$\nabla^2 \Phi = 4\pi G \rho, \tag{2}$$

here $\mathbf{U}$ is a vector of conservative variables; $\mathbf{F_x}$, $\mathbf{F_y}$ and $\mathbf{F_z}$ are numerical fluxes. In equation (2) Φ, G and ρ denote respectively the gravitational potential, the gravitational constant and the density.

Equation (1) for ideal gas with self-gravity $\mathbf{U}$ is expressed in terms of

$$\mathbf{U} = (\rho, \rho\mathbf{v_x}, \rho\mathbf{v_y}, \rho\mathbf{v_z}, \rho\mathbf{E})^{\mathbf{T}}, \tag{3}$$

$$\mathbf{F_x} = \begin{pmatrix} \rho v_x \\ \rho v_x^2 + p + \rho g_x \\ \rho v_x v_y \\ \rho v_x v_z \\ \rho E + p + \rho \mathbf{g} \end{pmatrix}, \tag{4}$$

here $\mathbf{v} = (v_x, v_y, v_z)^T$ are the speed components, $\mathbf{g} = (g_x, g_y, g_z)^T = -\nabla\Phi$ is the gravity, $E = \frac{|\mathbf{v}|^2}{2} + \frac{p}{(\gamma-1)\rho}$ is the total energy, and p is the pressure. Components $\mathbf{F_y}$ and $\mathbf{F_z}$ are obtained similarly [1]. The pressure is presented by barotropic and isothermal equations of state with $\gamma = 5/3$.

3 Discretization

Many modern high-resolution numerical schemes for gasdynamics conservation laws use the Godunov approach. These methods are also called finite volume methods. They, as a rule, use two-step-by-step methods of type predictor-corrector. Many of them use a uniform grid: cubic or parallelepiped. These schemes utilize the sliding average of the solution $u(x, y, z, t)$ in x direction:

$$\overline{u}(x,t) \equiv \frac{1}{|I_x|} \int_{I_x} u(s,t) ds, \quad I_x \equiv \{s : |s - x| \leq \frac{\Delta x}{2}\}$$

so that the integration of the conservation laws (1) over the rectangle $I_x \times [t, t + \Delta t]$ gives the equivalent formulation:

$$\overline{u}(x, t+\Delta t) = \overline{u}(x,t) - \frac{1}{\Delta x}\Big[\int_{I_x}^{t+\Delta x} f(u(x + \frac{\Delta x}{2}, \tau))d\tau - \tag{5}$$
$$- \int_{I_x}^{t+\Delta x} f(u(x - \frac{\Delta x}{2}, \tau))d\tau\Big].$$

Central schemes of the type Lax-Wendroff denote a class of difference methods for solving hyperbolic partial differential equations. The original one involves a strong viscosity and low resolution. Nessyahu and Tadmor [5] proposed a second order accurate scheme with a piecewise constant approximation replaced by linear interpolation. Thus the resolution of Nessyahu and Tadmor scheme is better than the resolution of upwind schemes, and are much more easier to implement than the schemes that use Riemann invariants.

The average value $\overline{w}_j^n$ may be calculated at a time t^n in the mesh cell $I_j \equiv \{x : x_{j-1/2} \leq x \leq x_{j+1/2}\}$. It is necessary to form a piecewise linear interpolation polynomial with respect to mean values $\overline{w}_j^n$ at a time t^n in order to calculate the mean value in the cell $I_{j+\frac{1}{2}} \equiv \{x : x_j \leq x \leq x_{j+1}\}$ at the time level t^{n+1} .

A 1-D piece-wise linear approximation may be written as follows

$$w(x,t^n) = \sum \left[\overline{w}_j^n + {w_j}'(\frac{x-x_j}{\Delta x})\right]\chi_j(x).$$

Here $\chi_p(x)$ is a characteristic cell function, but ${w_j}'$ is a first order limiter built on mean values of neighbourhood cells $\{\overline{w}_j^n\}$. If $\{\overline{w}_j^n,\ t \geq t^n\}$ is a conservation laws exact solution $w_t + f(w)_x = 0$, then a central difference scheme is obtained versus Godunov's upwind scheme. Let $\overline{w}_{j+1/2}^n(t) = \frac{1}{\Delta x}\int_{I_{j+1/2}} w(\xi,t)d\xi$ be a mean value shifted to the cell center. Then the control value (5) integrating gives:

$$\overline{w}_{j+1/2}^n(t^{n+1}) = \overline{w}_{j+1/2}^n(t^n)- \tag{6}$$
$$-\lambda\Big[\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(w_{j+1}(\tau))d\tau - \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(w_j(\tau))d\tau\Big].$$

Here $\lambda = \frac{\Delta t}{\Delta x}$ is a common restriction to the time step

Piece-wise linear mean values constructed at time-step $t = t^n$ give $\overline{w}_{j+1/2}^n(t^{n+1}) = 1/2(w_{j+1}^n + w_j^n) + 1/8(w_j' - w_{j-1}')$. It follows easily that $\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(w_j(\tau))d\tau \sim f(w_j(t^{n+1/2}))$. The values

$$w_j^{n+\frac{1}{2}} = \overline{w}_j^n - \frac{\lambda}{2}(f(w_j))' \tag{7}$$

are calculated in the end of the predictor step.

The expression

$$\overline{w}_{j+\frac{1}{2}}^{n+1} = \frac{1}{2}(\overline{w}_j^n + \overline{w}_{j+1}^n) + \frac{1}{8}(w_j' + w_{j+1}') - \lambda\Big[f(w_{j+1}^{n+\frac{1}{2}}) - f(w_j^{n+\frac{1}{2}})\Big] \tag{8}$$

gives a possibility to obtain the values on the corrector step. Here w_j' and $f(w_j)'$ are the spatial discrete slopes for the corresponding mesh functions described in [4, 9].

Let the piecewise linear scheme (8) be modified in order to avoid the shift by 1/2

$$\overline{w}_j^{n+1} = \frac{1}{4}(\overline{w}_{j-1}^n + 2\overline{w}_j^n + \overline{w}_{j+1}^n) - \frac{1}{16}((w_x)_{j+1} - (w_x)_{j-1})- \tag{9}$$

$$-\frac{\lambda}{2}\Big[f(w_{j+1}^{n+\frac{1}{2}})-f(w_{j-1}^{n+\frac{1}{2}})\Big]-\frac{1}{8}((w_x)_{j+\frac{1}{2}}-(w_x)_{j-\frac{1}{2}}).$$

Consider (9) so that $(w_x)_j$ and $(w_x)_{j+\frac{1}{2}}$ are the discrete time derivatives for the t^n and t^{n+1} time steps. The value $w_j^{n+\frac{1}{2}}$ is defined on the predictor step by (7). The Courant-Friedrichs-Levi condition must be fulfilled for the given central difference scheme.

Consider a two-dimensional case, then a piecewise linear approximation $\overline{w}_{i,j}^n$ is obtained for the mean values corresponding to the cell center C_{ij}:

$$C_{ij}=\left\{(\xi,\eta):|\xi-x_i|\le\frac{\Delta x}{2},|\eta-y_j|\le\frac{\Delta y}{2}\right\}$$

For the predictor step we have the following:

$$w(x,y,t^n)=\sum\left[\overline{w}_{ij}^n+w_{ij}'(\frac{x-x_i}{\Delta x})+\grave{w}_{ij}(\frac{y-y_j}{\Delta y})\right]\chi_{ij}(x,y) \tag{10}$$

here w_{ij}' and $\grave{w}_{ij}$ are the limiters along x and y axes.

4 Numerical Experiments for High Resolution Schemes. Numerical tests in 2D

Implementing any differrence scheme includes a quite important stage – testing. Our code was tested using three test problems in a two-dimensional setting thus the accuracy and robustness could be examined. The first test to implement was a Sedov-Taylor problem. It is a well known and rather severe spherically symmetric shock wave propagation problem. We complicated it by considering an interaction of two spherically symmetric shock waves propagating from two explosion sources of equal power. Thus the oscillationns beyond shocks and steep gradients common to this difference scheme may be analyzed.

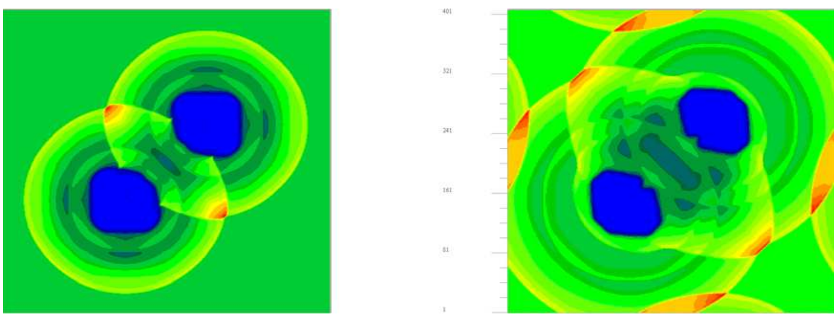

Figure 1. Sedov-Teylor test for interacting shock waves. On the left figure t=2.2631, on the right t=4.6978

The second test is a shock wave and gas buble interacting problem. The buble is considered to be filled in with the gas of low density [6]. And the last test problem, considered in this paper, was taken from [7].

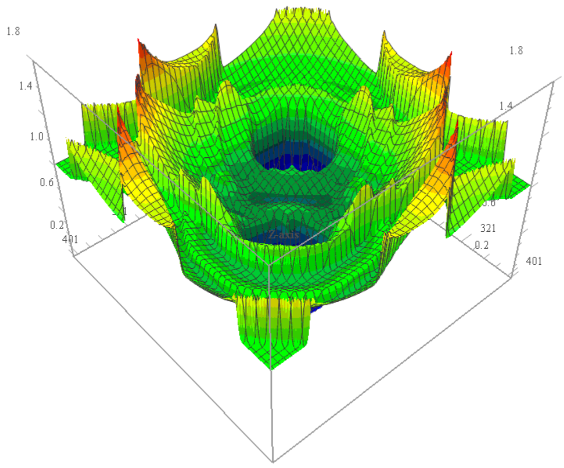

Figure 2. 3D figures for Sedov-Teylor test. Time t=4.6978

4.1 Sedov-Taylor test

Consider a rectangular 400 x 400 cell computational domain. Two power sources are situated on its diagonal and equally distanted from the center. Spacial steps are dx=0.05 and dy=0.05, specific heat ratio is $\gamma = 1.4$. The initial values of density and pressure are 1.0 in the whole domain, velocity components are equal to 0. Notice that rectangular grids are noninvariant with respect to rotation. So the difference scheme "quality" can be estimated by obtaining a spherically symmetric shock waves.

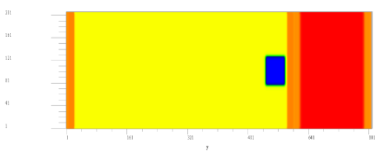

Figure 3. Shock waves interacting with gas bubble at time t=0.12

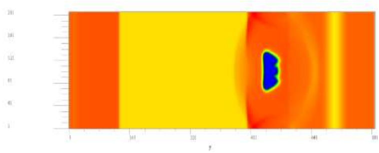

Figure 4. Shock waves interacting with gas bubble at time t=2.335

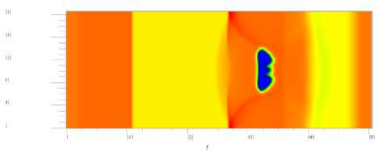

Figure 5. Shock waves interacting with gas bubble at time t=6.586

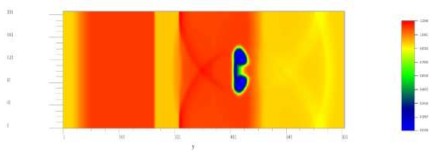

Figure 6. Shock waves interacting with gas bubble at time t=10.0

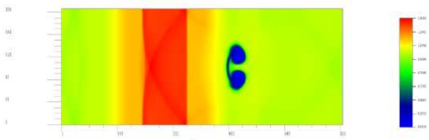

Figure 7. Shock waves interacting with gas bubble at time t=14.058

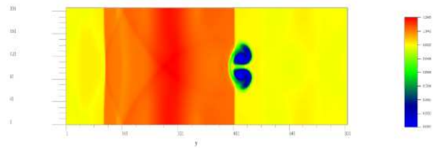

Figure 8. Shock waves interacting with gas bubble at time t=18.988

4.2 Bubble test

We simulate the interaction of a low density gas bubble of radius r = 0.2, centered at (0.5, 0) with a shock wave. The shock is initially at x = 0.2, and the initial conditions to the right of the shock and outside the bubble are $(\rho, u, v, p)^T = (1, 0, 0, 1)^T$, inside the bubble the pressure and density are p = 1 and $\rho = 0.1$, and to the left of the shock, they are determined by the Rankine-Hugoniot conditions [3].

We consider the 2-D Euler equation of gas dynamics in the strip R:(-0.5, 0.5) with the solid wall boundary conditions prescribed at $y = \pm 0.5$. The initial data correspond to a vertical left-moving shock, initially located at x = 0.75, and a circular bubble with radius 0.25, initially located at the origin. Notice that as the problem was considered for the rectangular grid, then the low density gas domain should be defined for the corresponding rectangular domain. See on the right-hand side in figure 3. These results demonstrate the robustness and stability of the proposed central scheme to evolve the solution of hyperbolic conservation laws. In (3) – (8) the interaction of gas cloud and shock wave at various times is presented.

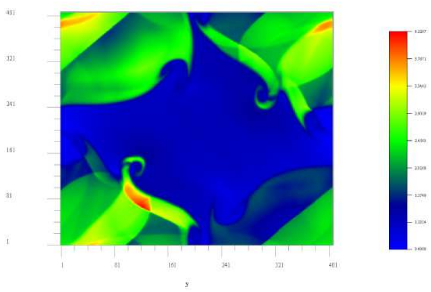

Figure 9. 2D shock tube t=0.4564

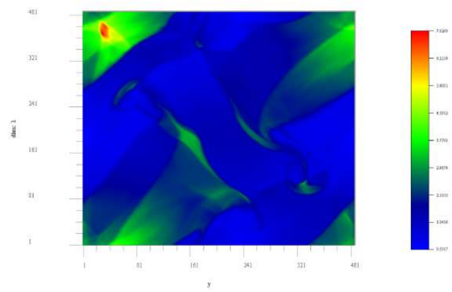

Figure 10. 2D shock tube t=0.90947

4.3 Two-dimension shock tube test

Consider (1) 2-D shock tube problem [7, 10]. Computational domain is a square R:{0:1 x 0:1}, divided into four quadrants by lines x = 1/2, y = 1/2. Spatial steps are dx=0.0025 and dy=0.0025, specific heat ratio is $\gamma = 1.4$. We denote the quadrants [7]: left lower – 1.1, right lower – 1.2, left top – 2.1, right top – 2.2 and set a Riemann problem initial data in these quadrants as follows: $(\rho, u, v, p)^T = (2.0, 0.75, 0.5, 1.0)^T$; $(\rho, u, v, p)^T = (1.0, 0.75, -0.5, 1.0)^T$; $(\rho, u, v, p)^T = (1.0, -0.75, 0.5, 1.0)^T$; $(\rho, u, v, p)^T = (3.0, -0.75, -0.5, 1.0)^T$.

In Figures (11) – (12) ρ is the density, u and v are the velocity components,

$$E = \rho e + \frac{\rho(u^2 + v^2)}{2}$$

is the total energy per unit volume and e is the internal energy. Ideal gas law $p = \rho e(\gamma - 1)$ is used to solve the system of equations that is under consideration.

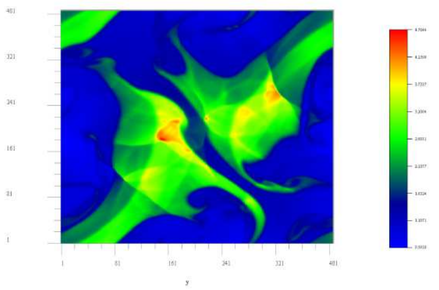

Figure 11. t=1.3686

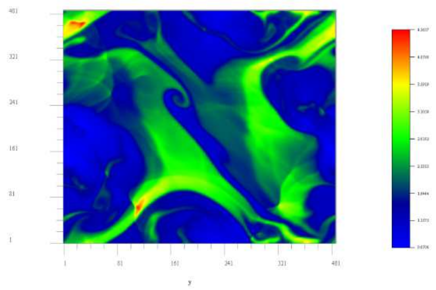

Figure 12. t=1.9106

Figures 9 - 12 present the disttribution graphs of the density in the different time steps. Here the number of Courant-Friedrichs-Levi is CFL=0.45. The results almoust coincide with the data obtained in [7]. One may observe that shock fronts are enough sharp and there are not any considerable oscillations beyond them.

4.4 Conclusions

We have presented a difference scheme for solving multidimensional gas-dynamics equations. In particular it has been shown that the scheme and code are able to model the processes goverened by conservation laws robustly and accurately. The main purpose of this article was to develop high resolution schemes and to illustrate their potential. Our numerical experiments suggest that these schemes have a good resolution and may be applied for solving various astrophysical problems.

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References

[1] MATSUMOTO T. *Self-gravitational Magnetohydrodynamics with Adaptive Mesh Refinement for Protospellar Collapse*. Publ. Astron. Soc. Japan, **59**, 2007, October 25.

[2] GHELLER C., PANTANO O., MOSCARDINI L. *A cosmological hydrodynamic code based on the Piecewise Parabolic Method*. Mon. Not. R. Astron. Soc. 5 October 2006.

[3] Jiang G.-S., Levy D., Lin C.-T., Osher S., Tadmor E. *High-Resolution Nonoscillatory Central Schemes With Nonstaggered Grids For Hyperbolic Conservation Laws.* SIAM J. NUMER. ANAL. Vol. 35, No. 6, p. 2147–2168, December 1998.

[4] Rybakin B., ShiderN. *Computer modeling of multidimensional problems of gravitational gas dynamics on multiprocessor computers.* Computer Science Journal of Moldova, 2009, **17**, No. 1(49), 3–13.

[5] Nessyahu H., Tadmor E. *Non-oscillatory central differencing for hyperbolic conservation laws.* J. Comp. Phys., 1990, **87**, 408–448.

[6] Balbras J., Qian Xin. *Non-oscillatory Central Schemes for 3D Hyperbolic Conservation Laws.* "Hyperbolic Partial Differential Equations, Theory, Numerics and Applications", Proceedings of the 12th international conference, University of Maryland, 2009.

[7] Levy D., Puppo G., Russo G. *A Fourty-Order Weno Scheme for Multidimensional Hyperbolic System of Conservation Laws.* Mathematical Modelling and Numerical Analysis, 1999, **33**, No. 3, 547–571

[8] Kurganov A., Chi-Tien Lin. *On the Reduction of Numerical Dissipation in Central-Upwind Schemes.* Communications in Computational Physics, 2007, **2**, No. 1, 141–163.

[9] Toth G., Odstrcil D. *Comparision of Some Flux Corrected Transport and Total Variation Diminishing Numerical Schemes for Hydrodynamic and Magnetohydrodynamic Problems.* Journal of Comp. Phys., 1996, **128**, 82–100.

[10] Liska R., Wendroff B. *Comparison of Several Difference Schemes on 1d and 2d Test Problems for the Euler Equations.* SIAM J. SCI. COMPUT, 2003, **25**, No. 3, 995–1017.

[11] Kulikov I., Lazareva G., Snytnikov A., Vshivkov V. *Supercomputer Simulation of an Astrophysical Object Collapse by the Fluids-in-Cell Method.* Springer-Verlag, Berlin-Heidelberg. V. Malyshkin (Ed.): PaCT 2009, LNCS 5698, p. 414–422.

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On the action of differentiation operator in some classes of Nevanlinna-Djrbashian type in the unit disk and polydisk

Romi Shamoyan, Haiying Li

Abstract. We introduce new area Nevanlinna type spaces in the unit disk and polydisk and study the action of classical operator of differentiation on them. We substantially supplement the list of previously known assertions of this type.

Mathematics subject classification: 30D45.
Keywords and phrases: Holomorphic functions, differentiation operator, Nevanlinna-Djrbashian type analytic classes.

1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in $\mathbb{C}$, $\mathrm{T} = \{|z| = 1\}$ be the unit circle, $I^n = (0,1]^n, \mathrm{T}^n = \mathrm{T}\cdots\mathrm{T}$, $\mathbb{D}^n = \{z = (z_1, z_2, \cdots, z_n) : |z_j| < 1, j = 1, 2, \cdots, n\}$ be the unit polydisk, $H(\mathbb{D})$ be the space of all holomorphic functions in the unit disk, and let $H(\mathbb{D}^n)$ be the space of all holomorphic functions in the polydisk. Let $T(f,\tau)$ be the Nevanlinna characteristic of f, $f \in H(\mathbb{D})$ [1]. Let below always w be a function from the set of all positive slowly growing functions, $w \in L^1(0,1)$ such that there are two numbers $m_w > 0, M_w > 0$ and a number $q_w \in (0,1)$ such that $m_w \le \dfrac{w(\lambda\tau)}{w(\tau)} < M_w$, $\tau \in (0,1)$, $\lambda \in [q_w, 1]$ (see [7]). We define several subspaces of $H(\mathbb{D})$ for fixed function $w \in L^1(0,1]$, $w > 0$.

$$N^1_{p,w,\beta} = \left\{ f \in H(\mathbb{D}) : \sup_{0<R\le 1} \int_0^R (T(f,\tau))^p w(1-\tau)d\tau(1-R)^\beta < +\infty \right\},$$

$$N^2_{p,w,\alpha} = \left\{ f \in H(\mathbb{D}) : \int_0^1 \left[\sup_{\tau\in(0,R]} (T(f,\tau))^p w(1-\tau) \right] (1-R)^\alpha dR < +\infty \right\},$$

$$N^3_{p,q,w,\alpha} = \left\{ f \in H(\mathbb{D}) : \int_0^1 \left(\int_0^R (T(f,\tau))^p w(1-\tau)d\tau \right)^{\frac{q}{p}} (1-R)^\alpha dR < +\infty \right\},$$

$$N^4_{p,q,w} = \left\{ f \in H(\mathbb{D}) : \int_0^1 \left(\int_{-\pi}^{\pi} \ln^+ |f(\tau\xi)|^p d\xi \right)^{\frac{q}{p}} w(1-\tau)d\tau < +\infty \right\},$$

$$N^5_{p,q,w} = \left\{ f \in H(\mathbb{D}) : \int_{-\pi}^{\pi} \left(\int_0^1 \ln^+ |f(\tau\xi)|^p w(1-\tau) d\tau \right)^{\frac{q}{p}} d\xi < +\infty \right\},$$

$$N^p = \left\{ f \in H(\mathbb{D}) : \sup_{\tau<1} \int_{-\pi}^{\pi} (\ln^+ |f(\tau\xi)|)^p d\xi < \infty \right\},$$

where $0 < p, q < \infty$, $\alpha > -1$, $\beta \geq 0$.

Note that these are complete metric spaces which can be checked without difficulties.

It is obvious that for $q = \infty$, $w = 1$ the $N^4_{p,q,w}$ coincides with the well-known N^p spaces of holomorphic functions with bounded characteristic [5].

In recent papers [4, 5] it was noted that the following assertions concerning the action of differentiation $\mathcal{D}(f)(z) = f'(z)$ and integration $I(f)(z) = \int_0^z f(t)dt$ are valid in mentioned analytic classes. $N^4_{q,q,\alpha}$ is closed under differentiation and integration operator (if $w(|z|) = (1-|z|)^\alpha$ we denote $N^4_{p,q,w}$ by $N^4_{p,q,\alpha}$), $N^4_{q,q,w}$ and $N^4_{1,q,w}$ are closed under differentiation operator $\mathcal{D}(f)$ if and only if $\int_0^1 w(t)(\ln \frac{1}{t})^p dt < +\infty$. The study $I(f), \mathcal{D}(f)$ in Smirnov N^+ class were studied also earlier (see [6] and references there).

We note that much earlier in [2] Frostman then W. K. Hayman [3] established that the N^p class is not invariant under differentiation operator, but $N^p, p > 1$ are closed under integration operator, but not N^1.

The natural question is to study differentiation operator in $N^i_{p,w,\alpha}$, $i = 1, 2, 3, 4, 5$. The goal of this paper is to provide several new sharp results in this direction. Finally we would like to indicate that all assertions of this note were obtained by modification of approaches and arguments provided recently in [4]. All our results in higher dimension were obtained for $n = 1$ in [4]. Throughout the paper, we write C (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

2 Main results

Motivated by the mentioned above results in this section we provide new assertions concerning differentiation operator $\mathcal{D}(f)$ in new Nevanlinna-Djrbashian type spaces that were defined above. In the following assertion, we provide several sharp results on the action of the differentiation operator in Nevanlinna type analytic spaces in the unit disk complementing previously known propositions of this type obtained earlier by various authors (see, for example, [2–6] and references there).

Theorem 1. *1) $N^1_{p,w,\alpha}$ is closed under differentiation operator $\mathcal{D}(f)$ if and only if*

$$\sup_{R \in (0,1)} (1-R)^\alpha \int_0^R \left(\ln \frac{1}{1-\tau} \right)^p w(1-\tau) d\tau < \infty, \; 0 < p < \infty, \; \alpha \geq 0.$$

2) $N^2_{p,w,\alpha}$ is closed under differentiation operator $\mathcal{D}(f)$ if and only if

$$\int_0^1 \sup_{R<\tau} w(1-R)\left(\ln\frac{1}{1-R}\right)^p (1-\tau)^\alpha d\tau < \infty,\ 0<p<\infty,\ \alpha>-1.$$

3) $N^3_{p,q,w,\alpha}$ is closed under differentiation operator $\mathcal{D}(f)$ if and only if

$$\int_0^1 \left(\int_0^R w(1-\tau)\left(\ln\frac{1}{1-\tau}\right)^p d\tau\right)^{\frac{q}{p}} (1-R)^\alpha dR < \infty,\ 0<p,q<\infty,\ \alpha>-1.$$

In the following theorem we provide sharp assertions concerning the operator of Differentiation in $N^4_{p,q,\widetilde{w}}$ and $N^5_{p,q,\widetilde{w}}$.

Theorem 2. *$\mathcal{D}(f)$ is acts from $N^4_{p,q,\widetilde{w}}$ and $N^5_{p,q,\widetilde{w}}$ to $N^1_{s,s,w}$,*

$$\widetilde{w}(1-|z|) = w(1-|z|)^{\frac{q}{s}}(1-|z|)^{\frac{2q}{s}-\frac{q}{p}-1},\ \frac{2}{s}-\frac{1}{p}>0,\ s\ge 1,\ s\ge \max\{q,p\}$$

if and only if

$$\int_0^1 \left(\ln\frac{1}{t}\right)^s w(t)dt < \infty.$$

Now we formulate some new sharp results in higher dimensions. Let always below for any function $f \in H(\mathbb{D}^n)$,

$$\mathcal{D}f(z) = \frac{\partial f(z_1, z_2, \ldots, z_n)}{\partial z_1, \ldots, \partial z_n}.$$

Note that Nevanlinna type classes in higher dimension were studied also earlier see for example [10] and references there.

Theorem 3. *Let $0<p<\infty$, $\int_0^1 w_j(t)dt < +\infty$, $j=1,2,\ldots,n$. Then*

$$\int_{I^n}\left(\int_{\mathrm{T}^n} \ln^+ \left|\mathcal{D}f(\tau_1\xi_1,\ldots,\tau_n\xi_n)\right| d\xi_1\ldots d\xi_n\right)^p \Pi_{j=1}^n w_j(1-\tau_i)d\tau_1\ldots d\tau_n \le$$
$$\le C\int_{I^n}\left(\int_{\mathrm{T}^n} \ln^+ |f(\tau_1\xi_1,\ldots,\tau_n\xi_n)| d\xi_1\ldots d\xi_n\right)^p \Pi_{j=1}^n w_j(1-\tau_i)d\tau_1\ldots d\tau_n,$$
$$\overrightarrow{\tau} = (\tau_1,\ldots,\tau_n), \tau_i \in (0,1).$$

if and only if

$$\int_0^1 w_j(t)\left(\ln\frac{1}{t}\right)^p dt < +\infty,\ j=1,2,\ldots,n.$$

Theorem 4. *Let* $s \geq 1, s \geq \max\{q, p\}, w = \Pi_{j=1}^{n} w_j$. *Let*

$$\frac{2}{s} - \frac{1}{p} > 0,\ \widetilde{w_j}(1-|z_j|) = w_j(1-|z_j|)^{\frac{q}{s}}(1-|z_j|)^{\frac{2q}{s}-\frac{q}{p}-1}.$$

Then $\mathcal{D}f$ *is acts from* $N^4_{p,q,\widetilde{w}}(N^5_{p,q,\widetilde{w}})$ *to* $N^1_{s,s,w}$ *if and only if*

$$\int_0^1 w_j(1-\tau)\left(\ln\frac{1}{1-\tau}\right)^s d\tau_1 \dots d\tau_n < +\infty,\ j = 1, 2, \dots, n,$$

where

$$N^4_{p,q,w}(\mathbb{D}^n) = \left\{ f \in H(\mathbb{D}^n) : \int_{\mathrm{T}^n} \left(\int_{I^n} \ln^+ |f(\tau\xi)|^p \Pi_{k=1}^n w(1-\tau_k) d\tau \right)^{\frac{q}{p}} d\xi < +\infty \right\},$$

$$N^5_{p,q,w}(\mathbb{D}^n) = \left\{ f \in H(\mathbb{D}^n) : \int_{I^n} \left(\int_{\mathrm{T}^n} \ln^+ |f(\tau\xi)|^p d\xi \right)^{\frac{q}{p}} \times \right.$$

$$\left. \times \Pi_{k=1}^n w(1-\tau_k) d\tau_1 \dots d\tau_n < +\infty \right\}.$$

Let us mention some lemmas that are needed for the proofs.

Lemma 1. *The following estimates are true.*

1)

$$\int_{\mathrm{T}^n} \ln^+ |\mathcal{D}f(\tau_1\varphi_1, \dots, \tau_n\varphi_n)| d\varphi_1 \dots d\varphi_n \leq$$

$$\leq C\left(\left(\sum_{j=1}^n \ln\frac{1}{1-\tau_j} \right) + \int_{\mathrm{T}^n} \ln^+ |f(\overrightarrow{\tau}\xi)| dm_n(\xi) \right),\ \overrightarrow{\tau} = \left(\frac{1+\tau_1}{2}, \dots, \frac{1+\tau_n}{2} \right),$$

$$\tau_i \in (0,1), i = 1, \dots, n;$$

2)

$$\ln^+ T\left(\frac{1+\tau}{2}, f\right) \leq CT\left(\frac{1+\tau}{2}, f\right), \tau \in (0,1),$$

$$T(f, R) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln^+ |f(R\xi)| d\xi, R \in (0,1).$$

Lemma 2. *Let* $\lambda_k = 2^{\lambda_k}$, $\lambda > 0$, $\tau_n = \exp\left(-\frac{1}{2^{n\lambda}}\right)$. *Then for* $\varphi \in [0, 2\pi]$, *there exist a function* $f, f \in H(\mathbb{D})$,

$$\ln^+ |f'(\tau_n e^{i\varphi})| \geq C \ln\frac{1}{1-\tau_n}, f(z) = \sum_{k=0}^{\infty} \lambda_k^{\alpha-1} z^{\lambda_k}, 0 < \alpha < 1, \lambda > 0.$$

Lemma 3. *1) Let* $R_{m_j} = \exp\left(-\dfrac{1}{2^{\lambda m_j}}\right) \in (0,1]$, $t \in (0,+\infty)$, $\lambda > 0$, $j = 1,2,\dots,n$. *Then there exists a function* f, $f \in H(\mathbb{D}^n)$,

$$\left(\ln^+ |\mathcal{D}f(R_{m_1}e^{i\varphi_1},\dots,R_{m_n}e^{i\varphi_n})|\right)^t \geq C\sum_{j=0}^{n}\left(\ln\frac{1}{1-R_{m_j}}\right)^t, \ \varphi_i \in (0,2\pi].$$

2)

$$\int_{\mathrm{T}^n}\left(\ln^+ |\mathcal{D}f(\tau_1\xi_1,\dots,\tau_n\xi_n)|\right)^s d\xi_1\dots d\xi_n$$

is growing as a function of $\tau_1,\dots,\tau_n$ *for every* $s \geq 1$, $f \in H(\mathbb{D}^n)$.

Remark 1. The statements of Theorem 2 for $q = p = s$ were established in [4].

Remark 2. As W. Hayman shows in the unit disk there is a function so that $T(\tau, I(f)) > C\ln\dfrac{1}{1-\tau}$, $T(\tau,f) < C$, $\tau \in (0,1)$. Let X be any normed class $X \subset H(\mathbb{D})$ so that $\|f\|_{X(w)} \leq C\sup_\tau T(\tau,f)$. If for $f \in X(w)$, $I(f) \in X(w)$, then $\|\ln\dfrac{1}{1-\tau}\|_{X(w)} < +\infty$. As $X(w)$ we can obviously take any space $N^i_{p,q,w}, i = 1,2,3,4,5$ under some natural additional assumption on w.

Remark 3. It is not difficult to see that the statements of Theorem 1 and Theorem 2 remain true if we replace $\mathcal{D}$ operator by $\bigwedge(f)(z) = \sum_{k=0}^n f_k(z)\mathcal{D}^k(f)(z)$, where f_k are functions from $N^i_{p,q,w}$, $i = 1,2,3,4,5$. The same statement is true for $\widetilde{I}^k$.

Note that with the help of so-called slice functions technique in [4,9], some results of this paper can be even expanded to the unit ball.

References

[1] Nevanlinna R. *Analytic functions GITTL*. Moscow, 1991.

[2] Frostman O. *Sur les produits de Blaschke*. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar, 1942, **12(15)**, 169–182.

[3] Hayman W. K. *On the characteristic of functions meromorphic in the unit disk and of their integrals*. Acta Math., 1964, **112**, 181–214.

[4] Shamoyan F. A., Kursina I. N. *On the invariance of some classes of holomorphic functions under integral and differential operators, Investigations on linear operators and function theory*. Part 26, Zap. Nauchn. Sem. POMI, 255, POMI, St. Petersburg, 1998, 184–197.

[5] Choe B. R., Koo H., Smith W. *Carleson measures for the area Nevanlinna spaces and applications*. J. Anal. Math., 2008, **104**, 207–233.

[6] Yanagihara N., Kawase S. *On the characteristic of some classes of functions and of their integrals*. Proc. London Math. Soc., 1972, **25(3)**, 577–585.

[7] Seneta E. Regularly varying functions, Nauka, Moscow, 1985.

[8] Golˊdberg A. A. Ostrovskiĭ I. V. *The distribution of values of meromorphic functions*, Nauka, Moscow, 1970.

[9] Zhu K. *Spaces of holomorphic functions in the unit ball.* Graduate Texts in Mathematics, 226, Springer-Verlag, New York, 2005.

[10] Rudin W. *Function theory in polydisks*, W. A. Benjamin, New York, 1969.

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On a Product of Classes of Algebraic Systems

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Abstract. This paper defines a product of classes of algebraic systems and proves that it is a universal class, a quasi-variety or variety if these classes are universal classes, quasi-varieties or varieties, respectively.

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A class of algebraic systems of signature σ is any set (possibly empty) of systems of signature σ which contains together with every its system all isomorphic to it systems. A class K of algebraic systems of signature σ is called a universal class if there exists such a set Σ of universal formulae ($\forall$-formulae) that K be formed of all systems of signature σ, with the formulae from Σ holding true within it. If all formulae from Σ are identities or quasi-identities, then the class K is called a variety or quasi-variety, respectively.

In [1] W. Taylor defined the product of two varieties of algebras of different signatures as a variety of non-indexed products of algebras from these varieties. The non-indexed product of two algebras A and B is defined as an algebra with the basic set equal to the Cartesian product of the basic sets of these algebras and with the set of operations consisting of all pairs of terms of the same number of variables. It is difficult to investigate this product, which we will name hereafter a Taylor product, due to the fact that the signature of the Taylor product of two (or more) varieties is complex, and it is difficult to investigate the operations of the algebras from this variety.

This paper presents a new concept for the product of two or more classes of algebraic systems of different signatures and shows that the product of universal classes is a universal class, the product of quasi-varieties is a quasi-variety, and the product of varieties is a variety.

1 Preliminary notions and results

For any class K of algebraic systems of signature σ we will denote by
$S(K)$ – the class of all subsystems of K-systems,
$P(K)$ – the class of all Cartesian products of K-systems,
$F(K)$ – the class of all filtered products of K-systems,
$H(K)$ – the class of all homomorphic images of K-systems.

If $S(K) = K$, $P(K) = K$, or $H(K) = K$ holds for a class K, then one class K is called hereditary, multiplicatively closed, filteredly closed, or homomorphically closed, respectively.

Next we will need the following main results from the variety and quasi-variety theory, obtained by Birkhoff G. and Mal'cev A.I.

Theorem 1 (Birkhoff [2]). *A class K of algebraic systems is a variety if and only if the class K satisfies the following conditions:*

(a) *is hereditary,*
(b) *is multiplicatively closed,*
(c) *is homomoprhically closed.*

Theorem 2 (Mal'cev [3]). *A class K of algebraic systems is a quasi-variety if and only if the class K satisfies the following conditions:*

(a) *is hereditary,*
(b) *is filteredly closed,*
(c) *contains the unitary system.*

It is worth reminding that a filter over a non-empty set I is a set D of subsets of I that satisfies the following conditions:

1) $A \in D \ \& \ B \in D \Rightarrow A \cap B \in D$,
2) $A \in D \ \& \ A \subseteq B \subseteq I \Rightarrow B \in D$,
3) $\emptyset \notin D$.

It is obvious that the set of all filters over I is a partially ordered set relative to the inclusion. A maximal filter over I is called a ultrafilter.

A filtered product of algebraic systems is defined as follows. Let A_i, $i \in I$, be a set of algebraic systems of the same signature σ and D a filter over I. We define the basic relation $\equiv$ on the Cartesian product $A = \prod_{i \in I} A_i$, putting $a \equiv b$ $(a, b \in A)$ if and only if the set of indices $\{i \in I | a(i) = b(i)\}$ belongs to the filter D. The binary relation $\equiv$ is an equivalence; moreover, it is stable relative to any operation of system A, that is, if f^A is an n operation of the system A, then

$$a_1 \equiv b_1 \& \ldots \& a_n \equiv b_n \Rightarrow f(a_1, \ldots, a_n) \equiv f(b_1, \ldots, b_n)$$

for any elements a_i, b_i, $i = 1, \ldots, n$, from A. This means that the operations $f^{A/\equiv}$, $f \in \sigma$, and predicates r^A, $r \in \sigma$, can be naturally defined on the set $A/\equiv$ of classes of equivalences $a/\equiv$, $(a \in A)$ in such a way that:

$$f^{A/\equiv}(a_1/\equiv, \ldots, a_n/\equiv) = f^A(a_1, \ldots, a_n)/\equiv,$$

where n is the arity of the functional symbol f; in $A/\equiv$ the following relation

$$r^{A/\equiv}(a_1/\equiv, \ldots, a_m/\equiv),$$

holds, where m is the arity of predicate r if and only if the set $\{i \in I | A_i \models r^{A_i}(a_1(i), \ldots, a_m(i))\}$ belongs to the filter D. The algebraic system, built in such a way, is denoted by A/D and is called the filtered product of systems A_i, $i \in I$, and if D is an ultrafilter, it is called an ultraproduct.

2 Products of universal classes and quasi-varieties

Let A_i, $i \in I$, be a set of algebraic systems of arbitrary signatures σ_i, $i \in I$. We complete the signature of every system A_i with the functional symbols p_j, $j \in I$, that correspond to the operations of projections $p_j^{A_i}$, $j \in I$, defined on the Cartesian power A_i^I with values from A_i : $p_j^{A_i}(a) = a_j$, $j \in I$, for any element $a = a_i, i \in I) \in A_i^I$. If not all systems from this set are algebraic, then we also complete the signature of every system A_i with the predicative symbol e that corresponds to the real identical predicate e^{A_i}, defined on the Cartesian power A_i^I with real values: $A_i \models e^{A_i}(a)$ ($e^{A_i}(a)$ holds in A_i) for any $a = (a_i, i \in I) \in A_i^I$. The system we obtain in such a way will be called an *enriched algebraic system* and will be also denoted by A_i.

The enriched Cartesian product of the enriched algebraic systems A_i, $i \in I$, is an algebraic system $\otimes_i A_i$ with the basic set $A = \prod_{i \in I} A_i$, which for each family of basic n-operations $(f_i^{A_i}, i \in I)$ and each family of basic m-predicates $(r_i^{A_i}, i \in I)$ of the enriched systems $A_i, i \in I$, has a basic n-operation f^A and a basic m-predicate, defined by

$$f^A(a_1, \ldots, a_m) = (f_i^{A_i}(a_1(i), \ldots, a_n(i)), i \in I),$$

$$A \models r^A(a_1, \ldots, a_m) \Leftrightarrow \&_{i \in I} A_i \models r_i^{A_i}(a_1(i), \ldots, a_m(i)),$$

where $a_1, a_2, \ldots$ are elements from A and it doesn't have any other basic operations and predicates.

We notice that if $A_i, i \in I$, are algebras, then the system $\otimes_i A_i$ is an algebra.

Let now $Q_i, i \in I$ be a set of classes of algebraic systems. The signatures of these classes may be different. We will define the *product of classes* $Q_i, i \in I$, as the class of algebraic systems, consisting of all isomorphisms of algebraic systems of the form $\otimes_i A_i$, where $A_i \in Q_i$, $i \in I$. We will denote the product of classes $Q_i, i \in I$, by $\otimes_i Q_i$ and by $Q_1 \otimes \ldots \otimes Q_n$ if $I = \{1, \ldots, n\}$.

Lemma 1. *The product of a finite number of filteredly closed classes is a filteredly closed class.*

Proof. Let $K_i, i = 1, \ldots, n$, be a set of closed classes relative to filtered products and $K = K_1 \otimes \ldots \otimes K_n$ be their product, $A^i = A_1^i \otimes \ldots A_n^i, i \in I$, be a set of algebraic systems with $A_j^i \in K_j, j = 1, \ldots, n$, and D be a filter over I. Then

$$\prod_{i \in I} A_1^i/D \in K_1, \ldots, \prod_{i \in I} A_n^i/D \in K_n.$$

We will show that the following isomorphism holds

$$\prod_{i \in I} A^i/D \cong (\prod_{i \in I} A_1^i/D) \otimes \ldots \otimes (\prod_{i \in I} A_n^i/D)$$

and then we will get

$$\prod_{i \in I} A^i/D \in K.$$

Let φ be a mapping from $\prod_{i\in I} A^i/D$ in $(\prod_{i\in I} A_1^i/D)\otimes\ldots\otimes(\prod_{i\in I} A_n^i/D)$ defined by the relation

$$\varphi(a) = \varphi((a_1^i,\ldots,a_n^i), i\in I)D = ((a_1^i, i\in I)D,\ldots,(a_n^i, i\in I)D)$$

for any $a = ((a_1^i,\ldots,a_n^i), i \in I)D \in \prod_{i\in I} A^i/D$, where $a_1^i \in A_1^i,\ldots,a_n^i \in A_n^i, i\in I$, is an epimorphism. We denote $A = \prod_{i\in I} A^i/D, A_1 = \prod_{i\in I} A_1^i/D$, $\ldots, A_n = \prod_{i\in I} A_n^i/D$.

We consider a basic operation f^A of arity k of the algebraic system A. By the definition, we have

$$f^A(a_1,\ldots,a_k) =$$
$$= f^A(((a_{11}^i,\ldots,a_{1n}^i), i\in I)D,\ldots,((a_{k1}^i,\ldots,a_{kn}^i), i\in I)D) =$$
$$= ((f^{A_1^i}(a_{11}^i,\ldots,a_{k1}^i), i\in I),\ldots,(f^{A_n^i}(a_{1n}^i,\ldots,a_{kn}^i), i\in I))D$$

for all $a_1 = ((a_{11}^i,\ldots,a_{1n}^i), i\in I)D,\ldots,a_k = ((a_{k1}^i,\ldots,a_{kn}^i), i\in I)D$ from A. It follows from here that

$$\varphi(f^A(a_1,\ldots,a_k)) = ((f^{A_1^i}(a_{11}^i,\ldots,a_{k1}^i), i\in I)D,\ldots,(f^{A_n^i}(a_{1n}^i,\ldots,a_{kn}^i), i\in I)D) =$$
$$= (f^{A_1}((a_{11}^i, i\in I)D,\ldots,(a_{k1}^i, i\in I)D),\ldots,f^{A_n}((a_{1n}^i, i\in I)D,\ldots,(a_{kn}^i, i\in I)D)) =$$
$$= f^{A_1\otimes\ldots\otimes A_n}(\varphi(((a_{11}^i,\ldots,a_{k1}^i), i\in I)D),\ldots,\varphi(((a_{1n}^i,\ldots,a_{kn}^i), i\in I)D)) =$$
$$= f^{\varphi(A)}(\varphi(a_1),\ldots,\varphi(a_k)).$$

Let now r^A be a basic m-relation of the algebraic system A and for elements $a_1 = ((a_{11}^i,\ldots,a_{1n}^i), i\in I)D,\ldots,a_k = ((a_{k1}^i,\ldots,a_{kn}^i), i\in I)D$ let

$$A \models r^A(((a_{11}^i,\ldots,a_{1n}^i), i\in I)D,\ldots,((a_{m1}^i,\ldots,a_{mn}^i), i\in I)D).$$

Then we have

$$I_0 = \{i\in I | A_1^i\otimes\ldots\otimes A_n^i \models r^{A_1^i\otimes\ldots\otimes A_n^i}((a_{11}^i,\ldots,a_{1n}^i),\ldots,(a_{m1}^i,\ldots,a_{mn}^i))\} \in D.$$

But

$$A_1^i\otimes\ldots\otimes A_n^i \models r^{A_1^i\otimes\ldots\otimes A_n^i}((a_{11}^i,\ldots,a_{1n}^i),\ldots,(a_{m1}^i,\ldots,a_{mn}^i))$$

implies $A_j^i \models r^{A_j^i}(a_{1j}^i,\ldots,a_{mj}^i), j = 1,\ldots,n$ for any $i\in I_0$. Hence

$$I_1 = \{i\in I | A_1^i \models r^{A_1^i}(a_{1j}^i,\ldots,a_{mj}^i)\} \supseteq I_0,\ldots$$

$$\ldots, I_n = \{i\in I | A_n^i \models r^{A_n^i}(a_{1n}^i,\ldots,a_{mn}^i)\} \supseteq I_0,$$

therefore $I_1\in D,\ldots,I_n\in D$, thus

$$A_1 \models r^{A_1}((a_{11}^i, i\in I)D,\ldots,(a_{m1}^i, i\in I)D),\ldots,A_n \models$$
$$r^{A_n}((a_{1n}^i, i\in I)D,\ldots,(a_{mn}^i, i\in I)D).$$

It follows from here that

$$A_1 \otimes \ldots \otimes A_n \models r^{A_1 \otimes \ldots \otimes A_n}(((a^i_{11}, i \in I), \ldots, (a^i_{1n}, i \in I))D, \ldots$$

$$\ldots, ((a^i_{mi}, i \in I), \ldots, (a^i_{mn}, i \in I))D)$$

or, as φ is obviously a surjective mapping, $\varphi(A) \models r^{\varphi(A)}(\varphi(a_1), \ldots, \varphi(a_m))$. Thus, φ is an epimorphism. Let us show that it is an isomorphism.

Let r^A be a basic m-relation of the algebraic system $A_1 \otimes \ldots \otimes A_n$, and for the images by φ of elements $a_1, \ldots, a_n \in A$ let $\varphi(A) \models r^{\varphi(A)}(\varphi(a_1), \ldots, \varphi(a_n))$, that is

$$A_1 \otimes \ldots \otimes A_n \models r^{A_1 \otimes \ldots \otimes A_n}(((a^i_{11}, i \in I)D, \ldots, (a^i_{1n}, i \in I)D), \ldots$$

$$\ldots, ((a^i_{m1}, i \in I)D, \ldots, (a^i_{mn}, i \in I)D)),$$

therefore

$$A_1 \models r^{A_1}((a^i_{11}, i \in I)D, \ldots, (a^i_{m1}, i \in I)D), \ldots, A_n \models$$

$$r^{A_n}((a^i_{1n}, i \in I)D, \ldots, (a^i_{mn}, i \in I)D).$$

Hence, we will get

$$I_1 = \{i \in I | A^i_1 \models r^{A_1}(a^i_{11}, \ldots, a^i_{m1})\} \in D, \ldots$$

$$\ldots, I_n = \{i \in I | A^i_n \models r^{A_1}(a^i_{1n}, \ldots, a^i_{mn})\} \in D.$$

As $I_1 \cap \ldots, \cap I_n \in D$ and

$$I_0 \supseteq I_1 \cap \ldots, \cap I_n = \{i \in I | A^i_1 \otimes \ldots \otimes A^i_n \models$$

$$r^{A^i_1 \otimes \ldots \otimes A^i_n}((a^i_{11}, \ldots, a^i_{1n}), \ldots, (a^i_{m1}, \ldots, a^i_{mn})),$$

it follows that $I_0 \in D$ and we get $A \models r^A(((a^i_{11}, \ldots, a^i_{1n}), i \in I)D, \ldots, ((a^i_{m1}, \ldots, a^i_{mn}), i \in I)D)$, that is, $A \models r^A(a_1, \ldots, a_n)$. □

Corollary. *The product of a finite number of multiplicatively closed classes is a multiplicatively closed class.*

Indeed, it follows from the proof of Lemma 1 that

$$\prod_{i \in I} A^i/D \cong (\prod_{i \in I} A^i_1/D) \otimes \ldots \otimes (\prod_{i \in I} A^i_n/D).$$

In particular, if filter D over I is a maximal filter, consisting only of the set I, then we have

$$\prod_{i \in I} A^i \cong \prod_{i \in I} A^i/D \cong (\prod_{i \in I} A^i_1/D) \otimes \ldots (\prod_{i \in I} A^i_n/D) \cong (\prod_{i \in I} A^i_1) \otimes \ldots \otimes (\prod_{i \in I} A^i_n).$$

Thus $\prod_{i \in I}(A^i_1 \otimes \ldots \otimes A^i_n) \cong \prod_{i \in I} A^i_1 \otimes \ldots \otimes \prod_{i \in I} A^i_n$ and therefore $\prod_{i \in I}(A^i_1 \otimes \ldots \ldots \otimes A^i_n) \in K$.

Lemma 2. *The product of a finite number of hereditary classes is a hereditary class.*

Proof. Let $K_i, i \in I$ be a set of hereditary classes and $K = K_1 \otimes \ldots \otimes K_n$ be their product, $A = A_1 \otimes \ldots \otimes A_n$ be an algebraic system from Q with $A_i \in K_i, i = 1, \ldots n$, and B be a subsystem of system A. Then $B \subseteq \prod_{i=1}^n A_i$. Let $\pi_i : B \to A_i, i = 1, \ldots, n$, be the projective mappings from B on its components. We denote $B_i = \pi_i(B), i = 1, \ldots, n$. Then we have $B \subseteq \prod_{i=1}^n B_i$. Conversely, let $(b_1, \ldots, b_n) \in \prod_{i=1}^n B_i$, then for any $i = 1, \ldots, n$, there exists such an element $b'_i = (b_{i1}, \ldots, b_{in})$ in B that $b_{ii} = b_i$. We consider the projection operations $p_i^{A_i}, i = 1, \ldots, n$, defined by the following relations

$$p_i^{A_i}(a_1, \ldots, a_n) = a_i, \quad i = 1, \ldots, n.$$

Then the operation $p^A = (p_i^{A_i}, i = 1, \ldots, n)$, defined on the algebraic system A, corresponds to the family of operations $(p_i^{A_i}, i = 1, \ldots, n)$. As

$$p^A(b'_1, \ldots, b'_n) = (p_i^{A_i}(b_{i1}, \ldots, b_{in}), i = 1, \ldots, n) =$$

$$(b_{11}, \ldots, b_{nn}) = (b_1, \ldots, b_n) \in B,$$

finally we get $B = \prod_{i=1}^n B_i$ with $B_i \in K_i, i \in I$, and therefore $B \in K$. □

By Tarski-Los Theorem (see [4], a class K is a universal class if and only if the class K satisfies the following conditions:

(1) is hereditary,

(2) is ultrafilteredly closed.

Thus, by Lemmas 1 and 2 we get

Theorem 3. *The product of a finite number of universal classes is a universal class.*

Theorem 4. *If the classes of algebraic systems $K_1, \ldots, K_n$ are quasi-varieties, then their product $K_1 \otimes \ldots \otimes K_n$ is a quasi-variety.*

Proof. By Lemma 1

$$F(K_1 \otimes \ldots \otimes K_n) = (F(K_1), \ldots, (F(K_n)) = K_1 \otimes \ldots \otimes K_n,$$

and by Lemma 2

$$S(K_1 \otimes \ldots \otimes K_n) = (S(K_1), \ldots, (S(K_n)) = K_1 \otimes \ldots \otimes K_n.$$

As any quasi-variety K_i contains a unitary algebraic system E_i ($i = 1, \ldots, n$), we get that the class $K_1 \otimes \ldots, \otimes K_n$ contains also the unitary algebraic system $E_1 \otimes \ldots, \otimes E_n$. Then, by Theorem 2, the product of quasi-varieties $K_1 \otimes \ldots, \otimes K_n$ is a quasi-variety. □

3 Product of varieties

Let $A = (A, \sigma)$ be an algebraic system of signature σ. It is worth reminding that the signature σ consists of a set of functional symbols σ^F, a set of predicative symbols σ^P and a function $\nu : \sigma^F Y \sigma^P \to \omega = \{0, 1, 2, \ldots\}$ that defines the arity of these symbols.

A subset $\theta \subseteq \prod_{r\in\sigma^P} A^{\nu(r)}$ is called a *congruence* on the algebraic system A (see [5]) if it satisfies the following properties:

(i) $\theta(\approx)$ is congruent on algebra (A, σ^F);

(ii) $A \models r(a_1, \ldots, a_{\nu(r)}) \Rightarrow (a_1, \ldots, a_{\nu(r)}) \in \theta(r)$;

(iii) $(a_1, \ldots, a_{\nu(r)}) \in \theta(r) \& (a_1, b_1) \in \theta(\approx) \& \ldots$
$\ldots \& (a_{\nu(r)}, b_{\nu(r)}) \in \theta(\approx) \Rightarrow (b_1, \ldots, b_{\nu(r)}) \in \theta(r), \forall r \in \sigma^P.$

The component $\theta(\approx)$ will be called an algebraic congruence on system A. The set of all congruences on algebraic system A will be denoted by $Con(A)$, and for a certain predicative symbol $r \in \sigma$ by $Con(A)(r)$ we will denote the set, consisting only of the components $\theta(r), \theta \in Con(A)$.

Relative to the inclusion $\subseteq$ the set $Con(A)$ is partially ordered and has the greatest element $\nabla = (\nabla(r) = A^{\nu(r)} | r \in \sigma^P)$. The intersection of any non-empty set $\{\theta_i, i \in I\}$ of congruences θ_i of system A is also a congruence of this system. Therefore, $Con(A)$ is a complete lattice, wherein the congruence $\triangle = (\triangle(r) = A^{\nu(r)} | r \in \sigma^P)$ is the lowest element, where $\triangle(r) = \{(a_1, \ldots, a_{\nu(r)}) \in A^{\nu(r)} | A \models r^A(a_1, \ldots, a_{\nu(r)})\}$, and for any two elements α and β from $Con(A)$, the operations $\wedge, \vee$ are defined as follows:

$$\alpha \wedge \beta = (\alpha \cap \beta)(r) = (\alpha(r) \cap \beta(r) | r \in \sigma^P),$$

$$\alpha \vee \beta = \cap\{\gamma \in Con(A) | \alpha \subseteq \beta, \beta \subseteq \gamma\}.$$

We notice that the set $\theta \subseteq \prod_{r\in\sigma^P} A^{\nu(r)}$, with the components defined via the formula

$$\theta(r) = \{(a_1, \ldots, a_{\nu(r)}) \in A^{\nu(r)} | (\exists b_1, \ldots, b_{\nu(r)} \in A)((a_1, b_1) \in (a \vee \beta)(\approx) \&$$

$$\ldots \& (a_{\nu(r)}, b_{\nu(r)}) \in (\alpha \vee) \beta)(\approx) \& (b_1, \ldots, b_{\nu(r)}) \in \alpha(r) \cup \beta(r)$$

is a congruence on system A. At the same time $\alpha \subseteq \theta$, $\beta \subseteq \theta$ and any other arbitrary congruence γ of system A, which contains α and β, also contains θ. Thus, $\theta = \alpha \vee \beta$ and therefore for any $r \in \sigma^P, r \not\approx$ we have

$$(\alpha \vee \beta)(r) = \{(a_1, \ldots, a_{\nu(r)}) \in A^{\nu(r)} | (\exists b_1, \ldots, b_{\nu(r)} \in A)((a_1, b_1) \in$$

$$(a \vee \beta)(\approx) \& (a_{\nu(r)}, b_{\nu(r)}) \in (\alpha \vee \beta)(\approx) \& (b_1, \ldots, b_{\nu(r)}) \in \alpha(r) \cup \beta(r)\}.$$

Lemma 3. *Let $K = K_1 \otimes \ldots \otimes K_n$ be a product of classes, $A = A_1 \otimes \ldots \otimes A_n$ be an algebraic system of signature σ from K with $A_i \in K_i, i = 1, \ldots, n$. If $\theta_i \in Con(A_i)$, $i = 1, \ldots, n$, and we denote by θ a subset from $\prod_{r\in\sigma}(A_1 \otimes \ldots \otimes A_n)^{\nu(r)}$, whose components are defined as follows $\theta(r) = \{((a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn})) \in (A_1 \otimes \ldots \otimes A_n)^m | (a_{11}, \ldots, a_{m1}) \in \theta_1(r), \ldots, (a_{1n}, \ldots, a_{mn}) \in \theta_n(r)\}$, where r is any predicative m-symbol of signature σ of the algebraic system $A = A_1 \otimes \ldots \otimes A_n$, then θ is a congruence on system A.*

Proof. Indeed, the relation $\theta(\approx)$ is reflexive, symmetrical and transitive. Let now $((a_{11}, \ldots, a_{1n}), (b_{11}, \ldots, b_{1n}) \in \theta(\approx), \ldots, ((a_{m1}, \ldots, a_{mn}), (b_{m1}, \ldots, b_{mn})) \in \theta(\approx)$ and $f^A = (f^{A_i}, i = 1, \ldots, n)$ be an m-operation of the algebraic system A. As $(a_{1i}, b_{1i}) \in \theta(\approx), \ldots, (a_{mi}, b_{mi}) \in \theta(\approx), i = 1, \ldots, n$, we get

$$(f^{A_i}(a_{1i}, \ldots, a_{mi}), f^{A_i}(b_{1i}, \ldots, b_{mi})) \in \theta(\approx), i = 1, \ldots, n.$$

Then we have

$$((f^{A_1}(a_{11}, \ldots, a_{m1}), \ldots, f^{A_n}(a_{1n}, \ldots, a_{mn})), (f^{A_1}(b_{11}, \ldots, b_{m1}), \ldots$$
$$\ldots, f^{A_n}(b_{1n}, \ldots, b_{mn}))) = (f^A((a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn})),$$
$$f^A((b_{11}, \ldots, b_{1n}), \ldots, (b_{m1}, \ldots, b_{mn}))) \in \theta(\approx)$$

and, thus, $\theta(\approx)$ is closed relative to the operations of the system A. Hence, property (i) from the definition of congruence holds for θ. Let now the relation $r^A((a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mm}))$ hold in the algebraic system A, then we have

$$A_1 \models r^{A_1}(a_{11}, \ldots, a_{m1}), \ldots, A_n \models r^{A_n}(a_{1n}, \ldots, a_{mn}),$$

which implies $((a_{11}, \ldots, a_{1n}), \ldots, (a_{n1}, \ldots, a_{mm})) \in \theta(r)$. It follows from here that $((a_{11}, \ldots, a_{1n}), \ldots, (a_{n1}, \ldots, a_{mm})) \in \theta(r)$, therefore the property (ii) from the definition of congruence holds for θ. Finally, let

$$((a_{11}, \ldots, a_{1n}), \ldots, (a_{n1}, \ldots, a_{mm})) \in \theta(r),$$

$$((a_{11}, \ldots, a_{1n}), (b_{11}, \ldots, b_{1n})) \in \theta(\approx), \ldots, ((a_{n1}, \ldots, a_{nm}), (b_{n1}, \ldots, b_{nm})) \in \theta(\approx).$$

Then we will have

$$(a_{11}, \ldots, a_{m1}) \in \theta_1(r), \ldots, (a_{1n}, \ldots, a_{mn}) \in \theta_n(r),$$
$$a_{11} \equiv b_{11} mod(\theta_1), \ldots, a_{1n} \equiv b_{1n} mod(\theta_n), \ldots$$
$$\ldots, a_{n1} \equiv b_{n1} mod(\theta_1), \ldots, a_{nm} \equiv b_{nm} mod(\theta_n),$$

which implies $(b_{11}, \ldots, b_{m1}) \in \theta_1(r), \ldots, (b_{1n}, \ldots, b_{mn}) \in \theta_n(r)$. It follows from here that

$$((b_{11}, \ldots, b_{1n}), \ldots, (b_{n1}, \ldots, b_{nm}) \in \theta(r),$$

hence, property (iii) from the definition of congruence holds for θ. □

Lemma 4. *Let $K = K_1 \otimes \ldots \otimes K_n$ be a product of classes, $A = A_1 \otimes \ldots \otimes A_n$ be an algebraic system from K with $A_i \subseteq K_i, i = 1, \ldots, n$. Then the following isomorphism holds*

$$Con(A_1 \otimes \ldots \otimes A_n) \cong Con(A_1) \times \ldots \times Con(A_n).$$

Proof. According to Lemma 3, an element θ of the set $Con(A)$ corresponds to every element $(\theta_1, \ldots, \theta_n)$ of the set $Con(A_1) \times \ldots \times Con(A_n)$ and we will denote it by $\theta = \theta_1 \times \ldots \times \theta_n$. Conversely, let $\theta \in Con(A)$. Then we will show that such congruences $\theta_i \in Con(A_i), i = 1, \ldots, n$, can be found that $\theta = \theta_1 \times \ldots \times \theta_n$, it means that for any predicative m-symbol r of the signature of system A we will have

$$((a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn})) \in \theta(r) \Leftrightarrow \&_{i=1}^{n}(a_{1i}, \ldots, a_{mi}) \in \theta_i(r) \Leftrightarrow$$

$$((a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn})) \in \theta_1(r) \times \ldots, \times\theta_n(r).$$

To prove this we define $\theta_1(r), \ldots, \theta_n(r)$, so that

$$\theta_1(r) = \{(a_{11}, \ldots, a_{m1}) \in A_i^m | (\exists b_{12}^1, \ldots, b_{m2}^1 \in A_2, \ldots, \exists b_{1n}^1, \ldots, b_{mn}^1 \in A_n)$$
$$((a_{11}, b_{12}^1, \ldots, b_{1n}^1), \ldots, (a_{m1}, b_{m2}^1, \ldots, b_{mn}^1)) \in \theta(r)\},$$

$$\theta_2(r) = \{(a_{12}, \ldots, a_{m2}) \in A_2^m | (\exists b_{11}^2, \ldots, b_{m1}^2 \in A_1,$$
$$\exists b_{13}^2, \ldots, b_{m3}^2 \in A_3, \ldots, \exists b_{1n}^2, \ldots, b_{mn}^2 \in A_n)$$
$$((b_{11}^2, a_{12}, b_{13}^2, \ldots, b_{1n}^2), \ldots, (b_{m1}^2, a_{m2}, b_{m3}^2, \ldots, b_{mn}^2)) \in \theta(r)\}, \ldots,$$

$$\theta_n(r) = \{(a_{1n}, \ldots, a_{mn}) \in A_n^m | (\exists b_{11}^n, \ldots,$$
$$b_{m1}^n \in A_1, \ldots, \exists b_{1n-1}^n, \ldots, b_{mn-1}^n \in A_n)$$
$$((b_{11}^n, \ldots, b_{1n-1}^n, a_{1n}), \ldots, (b_{m1}^n, \ldots, b_{mn-1}^n, a_{mn})) \in \theta(r)\}.$$

First, if r coincides with the binary predicative symbol $\approx$, we notice that the relation $\theta_1(\approx)$ is reflexive and symmetrical. Let us show that it is also transitive. Let $(a, a') \in \theta_1(\approx)$ and $(a', a'') \in \theta_1(\approx)$. Then for any $i = 1, \ldots, n$, in A there exist such elements $b_i^1, b_i^2, c_i^1, c_i^2$ that $((a, b_2^1, \ldots, b_n^1),\ (a', b_2^2, \ldots, b_n^2)) \in \theta_1(\approx)$ and $((a', c_2^1, \ldots, c_n^1), (a'', c_2^2, \ldots, c_n^2)) \in \theta_1(\approx)$. As $\theta_1(\approx)$ is stable relative to the operation $p = (p_1^{2A}, p_2^{2A})$, where $p_i^{2A} = (p_i^{2A_1}, \ldots, p_i^{2A_n}),\ i = 1, 2$, we will have

$$p(((a, b_2^1, \ldots, b_n^1), (a', b_2^2, \ldots, b_n^2)), (a', c_2^1, \ldots, c_n^1), (a'', c_2^2, \ldots, c_n^2))) =$$
$$= (p_1^{2A}((a, b_2^1, \ldots, b_n^1), (a', c_2^1, \ldots, c_n^1)), p_2^{2A}((a', b_2^2, \ldots, b_n^2), (a'', c_2^2, \ldots, c_n^2))) =$$
$$= ((p_1^{2A_1}(a, a'), p_1^{2A_2}(b_2^1, c_2^1), \ldots, p_1^{2A_n}(b_n^1, c_n^1)), (p_2^{2A_1}(a', a''), p_2^{2A_2}(b_2^2, c_2^2), \ldots$$
$$\ldots, p_2^{2A_n}(b_n^2, c_n^2))) = ((a, b_2^1, \ldots, b_n^1), (a'', c_2^2, \ldots, c_n^2)) \in \theta(\approx),$$

resulting that $(a, a') \in \theta(\approx)$. If f^{A_1} is a certain m-operation of the system A_1 and $(a_1, a''_1) \in \theta_1(\approx), \ldots, (a_m, a''_m) \in \theta_1(\approx)$, then we will take the basic operation $f^A = (f^{A_1}, p_1^{mA_2}, \ldots, p_1^{mA_n})$ of the algebraic system A. As for certain $b_2^1, \ldots, b_2^m, b_2'^1, \ldots, b_2'^m \in A_2, \ldots, b_n^1, \ldots \ldots, b_n^m, b_n'^1, \ldots, b_n'^m \in A_n$

$$(a_1, b_2^1, \ldots, b_n^1) \equiv (a'_1, b_2'^1, \ldots, b_n'^1) \mathrm{mod} \theta(\approx), \ldots$$

$$\ldots, (a_m, b_2^m, \ldots, b_n^m) \equiv (a'_1, b_2'^m, \ldots, b_n'^m) \mathrm{mod} \theta(\approx),$$

we will have

$$f^A((a_1, b_2^1, \ldots, b_n^1), \ldots, (a_m, b_2^m, \ldots, b_n^m)) \equiv$$
$$f^A((a'_1, b_2'^1, \ldots, b_n'^1), \ldots, (a'_m, b_2'^m, \ldots, b_n'^m)) \mathrm{mod} \theta(\approx),$$

that is

$$(f^{A_1}(a_1, \ldots, a_m), b_2^1, \ldots, b_n^1) \equiv (f^{A_1}(a'_1, \ldots, a'_m), b_2'^1, \ldots, b_n'^1) \mathrm{mod} \theta(\approx),$$

resulting that $(f^{A_1}(a_1, \ldots, a_m), f^{A_1}(a'_1, \ldots, a'_m) \in \theta_1(\approx)$. Hence $\theta_1(\approx)$ is stable relative to the operations of system A_1, that is, $\theta_1(\approx)$ is a congruence on algebra A.

Let now r be a predicative symbol of arity m of the signature of system A_1 and $x_{11}, \ldots, x_{m1}$ be such elements from A_1 that the real relation $A_1 \models r^{A_1}(x_{11}, \ldots, x_{m1})$ or $(x_{11}, \ldots, x_{m1}) \in \theta_1(\approx)$ holds, and $(x_{11}, y_1) \in \theta_1(\approx), \ldots, (x_{m1}, y_m) \in \theta_1(\approx)$ holds for certain elements $y_1, \ldots, y_m \in A_1$. We consider the basic predicate $r^A = (r^{A_1}, r^{A_2}, \ldots, r^{A_n})$ of the algebraic system A, where $r^{A_2}, \ldots, r^{A_n}$ are real basic predicates on the systems $A_2, \ldots, A_n$. Then for any elements $x_{12}, \ldots, x_{m2} \in A_2, \ldots, x_{1n}, \ldots, x_{mn} \in A_n$ we have $A_2 \models r^{A_2}(x_{12}, \ldots, x_{m2}) \Rightarrow (x_{12}, \ldots, x_{m2}) \in \theta_1(r), \ldots, A_n \models r^{A_n}(x_{1n}, \ldots, x_{mn}) \Rightarrow (x_{1n}, \ldots, x_{mn}) \in \theta_n(r)$, and if we have $A_1 \models r^{A_1}(x_{11}, \ldots, x_{m1})$ then we get $A \models r^A((x_{11}, \ldots, x_{1n}), \ldots, (x_{m1}, \ldots, x_{mn}))$, which implies that $((x_{11}, \ldots, x_{1n}), \ldots, (x_{m1}, \ldots, x_{mn})) \in \theta(\approx)$; if we have $(x_{11}, \ldots, x_{m1}) \in \theta_1(r)$ then obviously we will also get $((x_{11}, \ldots, x_{1n}), \ldots, (x_{m1}, \ldots, x_{mn}) \in \theta(r)$, for certain elements $x_{12}, \ldots, x_{m2} \in A_2, \ldots, x_{1n}, \ldots, x_{mn} \in A_n$. As $(x_{11}, y_1) \in \theta_1(\approx), \ldots, (x_{m1}, y_m) \in \theta_1(\approx)$ we have $((x_{11}, x_{12} \ldots, x_{1n}), (y_1, x_{12} \ldots, x_{1n})) \in \theta(\approx), \ldots, ((x_{m1}, x_{m2}, \ldots, x_{mn}), (y_m, x_{m2}, \ldots, x_{mn})) \in \theta(\approx)$, then from $((x_{11}, \ldots \ldots, x_{1n}), \ldots, (x_{m1}, \ldots, x_{mn})) \in \theta(r)$, follows $((y_1, x_{12}, \ldots, x_{1n}), \ldots, (y_m, x_{m2}, \ldots \ldots, x_{mn})) \in \theta(r)$, and from here $(y_1, \ldots, y_m) \in \theta_1(r)$. Therefore, θ_1 is a congruence on the algebraic system A_1. It can be shown by analogy that $\theta_2, \ldots, \theta_n$ are congruences on the systems $A_2, \ldots, A_n$, respectively. At the same time we notice that $\theta \subseteq \theta_1 \times \ldots \times \theta_n$, that is $\theta(r) \subseteq \theta_1(r) \times \ldots \times \theta_n(r)$ for any predicative symbol r of the signature of system A.

The inverse inclusion also takes place. We will also show that $(\theta_1 \times \ldots \ldots \times \theta_n)(\approx) \subseteq \theta(\approx)$. Indeed, let

$$((a_{11}, \ldots, a_{1n}), (a_{21}, \ldots, a_{2n})) \in (\theta_1 \times \ldots \times \theta_n)(\approx).$$

Then $(a_{11}, a_{21}) \in \theta_1(\approx), \ldots, (a_{1n}, a_{2n}) \in \theta_n(\approx)$. It follows from here that according to the constructions of $\theta_1(\approx), \ldots, \theta_n(\approx)$, we will have

$$((a_{11}, b_{12}^1, \ldots, b_{1n}^1), (a_{21}, b_{22}^1, \ldots, b_{2n}^1)) \in \theta(\approx),$$

$$((b_{11}^2, a_{12}, b_{13}^2, \ldots, b_{1n}^2), (b_{m1}^2, a_{22}, b_{m3}^2, \ldots, b_{mn}^2)) \in \theta(\approx), \ldots$$
$$\ldots, ((b_{11}^n, \ldots, b_{1n-1}^n, a_{1n}), (b_{21}^n, \ldots, b_{2n-1}^n, a_{2n})) \in \theta(\approx),$$

for certain elements

$$b_{12}^1, b_{22}^1 \in A_2, \ldots, b_{1n}^1, b_{2n}^1 \in A_n, b_{11}^2, b_{21}^2 \in A_1, b_{13}^2, b_{23}^2 \in A_3, \ldots$$
$$\ldots, b_{1n}^2, b_{2n}^2 \in A_n, \ldots, b_{11}^n, b_{21}^n \in A_1, \ldots, b_{1n-1}^n, b_{2n-1}^n \in A_{n-1}.$$

Applying the projection operation $p = p_1^{nA}, p_2^{nA}, \ldots, p_n^{nA})$ for the last elements from $\theta(\approx)$ we get $((a_{11}, a_{12}, \ldots, a_{1n}), (a_{21}, \ldots, a_{2n})) \in \theta(\approx)$. Hence, $\theta(\approx) \supseteq (\theta_1 \times \ldots \ldots \times \theta_n)(\approx)$ and thus $\theta(\approx) = (\theta_1 \times \ldots \times \theta_n)(\approx)$.

Let now $r \neq\approx$ be a predicative m-symbol of the signature of the algebraic system A and let

$$((x_{11}, \ldots, x_{1n}), \ldots, (x_{m1}, \ldots, x_{mn})) \in (\theta_1 \times \ldots \times \theta_n)(r),$$

thus

$$(x_{11}, \ldots, x_{m1}) \in \theta_1(r), \ldots, (x_{1n}, \ldots, x_{mn}) \in \theta_n(r).$$

By the constructions of the components $\theta_1(r), \ldots, \theta_n(r)$, there exist such elements $b_{12}^1, \ldots, b_{m2}^1 \in A_2, \ldots, b_{1n}^1, \ldots, b_{m2}^1 \in A_n$ that

$$((a_{11}, b_{12}^1, \ldots, b_{1n}^1), (a_{11}, \ldots, a_{1n})) \in \theta(\approx), \ldots$$
$$\ldots, ((a_{m1}, b_{m2}^1, \ldots, b_{mn}^1), (a_{m1}, \ldots, a_{mn})) \in \theta(\approx)$$

and $((a_{11}, b_{12}^1, \ldots, b_{1n}^1), \ldots, (a_{m1}, b_{m2}^1, \ldots, b_{mn}^1)) \in \theta(r)$, at the same time. From here, by the definition of congruence we get $((a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn})) \in \theta(r)$. Hence, $\theta_1 \times \ldots \times \theta_n \subseteq \theta$ and we get the equality $\theta_1 \times \ldots \times \theta_n = \theta$.

Now it is easy to see that the bijective correspondence, defined by $\theta = \theta_1 \times \ldots \ldots \times \theta_n \to (\theta_1 \ldots \theta_n)$, defines the isomorphism we are looking for, meaning that it satisfies the following properties:

$$(\theta \wedge \theta')(r) \to ((\theta_1 \wedge \theta_1')(r), \ldots, (\theta_n \wedge \theta_n')(r)),$$
$$(\theta \vee \theta')(r) \to ((\theta_1 \vee \theta_1')(r), \ldots, (\theta_n \vee \theta_n')(r))$$

for any congruences $\theta = \theta_1 \times \ldots \times \theta_n$ and $\theta = \theta_1' \times \ldots \times \theta_n'$ from $Con(A)$. □

Lemma 5. *The product of a finite number of homomorphically closed classes is a homomorphically closed class.*

Proof. Let $K_i, i = 1, \ldots, n$, be a family of homomorphically closed classes and $K = K_1 \otimes \ldots \otimes K_n$ be their product, $A = A_1 \otimes \ldots \otimes A_n$ – an algebraic system with $A_1 \subseteq K_i, i = 1, \ldots, n$, and C – an algebraic system from Q. We will show that an epimorphism $\varphi : A_1 \otimes \ldots \otimes A_n \to C$ exists if and only if there exist epimorphisms $\varphi_i : A_i \to C_i, i = 1, \ldots, n$ and the isomorphism $C \cong C_1 \otimes \ldots \otimes C_n$ takes place. From here we will get that $H(K) = H(K_1) \otimes \ldots \otimes H(K_n)$ and hence $H(K) = H$.

Let epimorphism $\varphi : A_1 \otimes \ldots \otimes A_n \to C$ exist and let $\theta = \ker(\varphi)$ be its kernel. By Lemma 4

$$\theta = \theta_1 \otimes \ldots \otimes \theta_n,$$

where $\theta_i \in Con(A_i), i = 1, \ldots, n$. We consider the canonical isomorphisms $\varphi_i : A_i \to A_i/\theta_i, i = 1, \ldots, n$. The mapping

$$\varphi^*(a_1, \ldots, a_n) = (\varphi_1(a_1), \ldots, \varphi_n(a_n))(a_i \in A_i, i = 1, \ldots, n)$$

is a homomorphism from $A_1 \otimes \ldots \otimes A_n$ to $(A_1/\theta_1) \otimes \ldots \otimes (A_n/\theta_n)$. Let $\theta^* = \ker(\varphi^*)$. We show that $\theta = \theta^*$ and then we will get

$$C \cong (A_1 \otimes \ldots \otimes A_n)/\theta \cong (A_1 \otimes \ldots \otimes A_n)/\theta^* \cong (A_1/\theta_1) \otimes \ldots \otimes (A_n/\theta_n).$$

If r is an arbitrary predicative m-symbol of the signature of system A and

$$r^{\varphi^*(A_1 \otimes \ldots \otimes A_n)}(\varphi^*(a_{11}, \ldots, a_{1n}), \ldots, \varphi^*(a_{m1}, \ldots, a_{mn})) =$$

$$= r^{\varphi_1(A_1) \otimes \ldots \otimes \varphi_n(A_n)}((\varphi_1(a_{11}), \ldots, \varphi_n(a_{1n})), \ldots, (\varphi_1(a_{m1}), \ldots, \varphi_n(a_{mn})) \Rightarrow$$

$$\Rightarrow r^{\varphi_1(A_1)}(\varphi_1(a_{11}), \ldots, \varphi_1(a_{1n}))\& \ldots \& r^{\varphi_n(A_n)}(\varphi_n(a_{1n}), \ldots, \varphi_n(a_{mn})),$$

then $(a_{11}, \ldots, a_{1n}) \in \theta_1(r), \ldots, (a_{1n}, \ldots, a_{mn}) \in \theta_n(r)$, resulting that

$$((a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn})) \in (\theta_1 \times \ldots \times \theta_n)(r) = \theta(r).$$

Conversely, if

$$((a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn})) \in \theta(r),$$

then

$$(a_{11}, \ldots, a_{m1}) \in \theta_1(r), \ldots, (a_{1n}, \ldots, a_{mn}) \in \theta_n(r),$$

resulting that

$$\varphi_1(A_1) \models r^{\varphi_1(A_1)}(\varphi_1(a_{11}), \ldots, \varphi_1(a_{m1})), \ldots$$

$$\ldots, \varphi_n(A_n) \models r^{\varphi_n(A_n)}(\varphi_n(a_{1n}), \ldots, \varphi_n(a_{mn})),$$

therefore

$$\varphi^*(A) \models r^{\varphi^*(A)}(\varphi^*(a_{11}, \ldots, a_{1n}), \ldots, \varphi^*(a_{1n}, \ldots, a_{mn})).$$

It follows from here that $\theta(r) \subseteq \theta^*(r)$. Hence, $\theta = \theta^*$.

Conversely, let us have the r-epimorphisms $\varphi_i : A_i \to C_i, i = 1, \ldots, n$. We show that the mapping

$$\varphi(a_1, \ldots, a_n) = (\varphi_1(a_1), \ldots, \varphi_n(a_n))(a_1 \in A_1, \ldots, a_n \in A_n)$$

is homomorphism from $A_1 \otimes \ldots \otimes A_n$ on $C_1 \otimes \ldots \otimes C_n$.

We consider a basic operation f^A of arity k of the algebraic system $A = A_1 \otimes \ldots \otimes A_n$. By its definition, we have

$$f^A((a_{11}, \ldots, a_{1n}), \ldots, (a_{k1}, \ldots, a_{kn})) =$$

$$= (f^{A_1}(a_{11}, \ldots, a_{1n}), \ldots, f^{A_n}(a_{k1}, \ldots, a_{kn}))$$

for all $a_{1i}, \ldots, a_{ki}$ from A_i $(i = 1, \ldots, n)$. It follows from here that

$$\varphi(f^A((a_{11}, \ldots, a_{1n}), \ldots, (a_{k1}, \ldots, a_{kn}))) =$$

$$= (\varphi_1(f^{A_1}(a_{11}, \ldots, a_{1n})), \ldots, \varphi_n(f^{A_n}(a_{k1}, \ldots, a_{kn}))) =$$

$$= (f^{\varphi_1(A_1)}(\varphi_1(a_{11}), \ldots, \varphi_1(a_{k1})), \ldots, f^{\varphi_n(A_n)}(\varphi_n(a_{1n}), \ldots, \varphi_n(a_{kn})) =$$

$$= f^{\varphi(A)}(\varphi(a_{11}, \ldots, a_{n1}), \ldots, \varphi(a_{k1}, \ldots, a_{kn})).$$

Let now $r^A = (r^{A_1}, \ldots, r^{A_n})$ be a basic m-relation of the algebraic system A and let

$$A \models r^A((a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn})).$$

Then we have

$$A_i \models r^{A_i}(a_{1i}, \ldots, a_{mi}), \ i = 1, \ldots, n.$$

Hence, we will get

$$\varphi_i(A_i) \models r^{\varphi_i(A_i)}(\varphi_i(a_{1i}), \ldots, \varphi_i(a_{mi})), \ i = 1, \ldots, n.$$

Therefore

$$\varphi(A) \models r^{\varphi(A)}(\varphi(a_{1i}, \ldots, a_{1n}), \ldots, \varphi(a_{m1}, \ldots, a_{mn})). \qquad \square$$

Theorem 5. *If the classes of algebraic systems $K_1, \ldots, K_n$ are varieties, then their product $K_1 \otimes \ldots \otimes K_n$ is a variety.*

Proof. By Gorollary of Lemma 1

$$P(K_1 \otimes \ldots \otimes K_n) = (P(K_1), \ldots, P(K_n)) = K_1 \otimes \ldots \otimes K_n.$$

By Lemma 2

$$S(K_1 \otimes \ldots \otimes K_n) = (S(K_1), \ldots, S(K_n)) = K_1 \otimes \ldots \otimes K_n,$$

and by Lemma 5

$$H(K_1 \otimes \ldots \otimes K_n) = (H(K_1), \ldots, H(K_n)) = K_1 \otimes \ldots \otimes K_n.$$

Hence, by Theorem 2, the product of varieties $K_1 \otimes \ldots \otimes K_n$ is a variety. $\square$

4 Observations

a) We will say that the algebraic system A with signature σ is *isomorphically embedded* in the algebraic system A' with the signature σ' if there exists a mapping $\varphi : A \to A'$ and an injective mapping $\alpha : \sigma \to \sigma'$ that makes a single function n-symbol $f \in \sigma$ and a single predicative m-symbol $r \in \sigma$ *to correspond to each functional n-symbol* $f' \in \sigma'$ and each predicative m-symbol $r' \in \sigma'$, so that

$$\varphi(f^A(a_1, \ldots, a_n)) = f'^{A'}(\varphi(a_1), \ldots, \varphi(a_n))$$

and

$$r^A(a_1, \ldots, a_m) = r'^{A'}(\varphi(a_1), \ldots, \varphi(a_m))$$

for any elements $a_1, a_2, \ldots$ from A.

If K and K' are two classes of algebraic systems of signatures σ and σ' respectively, then we will say that the class K is isomorphically embedded in the class K' if any algebraic system from the class K is isomorphically embedded in one of the systems of class K'.

It is easy to realize that: *any product of classes of algebras is isomorphically embedded in the Taylor product of the same classes of algebras.*

It is also easy to show that: *if any class $Q_i (i \in I)$ of algebraic systems contains a unitary algebraic system E_i, then any class Q_i is isomorphically embedded in the product of classes $Q = \otimes_{i \in I} Q_i$. In particular, if all classes $Q_i, i \in I$, are quasi-varieties (or varieties), then any quasi-variety Q_i is isomorphically embedded in the product of quasi-varieties $Q = \otimes_{i \in I} Q_i$.*

Indeed, if A_i is an arbitrary algebraic system from class Q_i $(i \in I)$, then obviously the system A_i is isomorphically embedded in the algebraic system $\otimes_{j \in J} A_j \in Q$, where $A_j = E_j$ for any $j \in I \backslash \{i\}$.

b) The product of classes of algebraic systems can be also extended for the case of an infinite number of classes, obtaining the same results as for the finite number of classes, which are proved analogously.

c) Let $K_i, i \in I = \{1, 2, \ldots\}$ be an infinite totality of such classes of algebras that each class $K_i (i \in I)$ contains an algebra L_i that strictly contains a unitary subalgebra $E_i = \{e_i\}$. Then the algebra $L = \otimes_{i \in I} L_i$ belongs to the class $\otimes_{i \in I} K_i$. We consider the set M, consisting of such elements $a = (a(i), o \in I)$ from algebra L for which the sets $\{i \in I | a(i) \neq e_i\}$ are finite. It is obvious that M is closed relative to any operation of finite arity of algebra L, hence it is isomorphically embedded in L. But M is not a subalgebra of algebra L, because any subalgebra from L has the form $M_1 \otimes M_2 \ldots$, where M_i is a subalgebra of L_i. Therefore, the product of an infinite number of classes of algebraic systems cannot be defined as a class of algebraic systems only with finite operations and predicates.

References

[1] Taylor W. *Caracterizing Mal'cev conditions.* Ibid, 1973, **3**, No. 3, 351–397.

[2] Birkhoff G. *Universal algebra,* Proc. First. Canad. Math. Congr., Montreal, 1945, 310–326.

[3] Mal'cev A. I. *Nauka publishing: Classical algebra.* Moskva, Nauka, 1976, vol. 2, 331–335 (in Russian).

[4] Mal'cev A. I. *Algebraic systems.* Moskva, Nauka, 1970 (in Russian).

[5] Gorbunov V. A. *Theory of quasivarieties of algebraic systems.* Siberian school of algebra and logic, Novosibirsk, 1999 (in Russian).

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