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The transvectants and the integrals for Darboux systems of differential equations

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Abstract. We apply the algebraic theory of invariants of differential equations to integrate the polynomial differential systems $dx/dt = P_1(x,y) + x\,C(x,y)$, $dy/dt = Q_1(x,y) + y\,C(x,y)$, where real homogeneous polynomials P_1 and Q_1 have the first degree and $C(x,y)$ is a real homogeneous polynomial of degree $r \geq 1$. In generic cases the invariant algebraic curves and the first integrals for these systems are constructed. The constructed invariant algebraic curves are expressed by comitants and invariants of investigated systems.

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1 Introduction

The problem of the integrability via invariant algebraic curves for planar polynomial differential systems has been investigated in many works. An ample survey on the Darboux integrability theory for planar complex and real polynomial systems can be found in [1]. In this book the authors mentioned that the detection of the integrable planar vector fields that are not Hamiltonian, in general, is a very difficult problem. In several works the problem of the integrability via invariant algebraic curves for some polynomial differential systems with degenerate infinity has been solved [2–11]. In works [12, 13] the invariant algebraic curve for Darboux differential systems with cubic nonlinearities has been expressed by invariants and comitants. In paper [14] the invariant algebraic curves, the integrating factors and some first integrals for Darboux differential systems with nonlinearities of degrees m $(2 \leq m \leq 7)$ has been constructed and expressed by invariants and comitants of investigated systems. In paper [15] a complete classification in the coefficient space $\mathbb{R}^{12}$ of quadratic systems with rational first integral of degree 2 has been obtained by using $Aff(2,\mathbb{R})$-invariants and comitants of these systems.

The main goal of this paper is to construct the invariant algebraic curves for integrable planar polynomial differential systems of Darboux type by using the $GL(2,\mathbb{R})$-invariants and the $GL(2,\mathbb{R})$-comitants of these systems [16] and classify the first integrals in generic cases. The generic cases include the systems with coprime right parts and exclude the linear systems.

In Section 2 we construct the main invariants and comitants for Darboux polynomial systems of differential equations. The definition of the transvectant of two polynomials and its properties are given. The construction part of the invariant algebraic curves of Darboux systems includes two subcases: the first one - for the polynomial $C(x,y)$ of odd degree and the second - for the polynomial $C(x,y)$ of even degree. Each subcase includes the results about forms of the invariant algebraic curves and the first integrals for investigated systems.

2 Darboux systems of differential equations

We consider the real systems of differential equations

$$\begin{aligned} \frac{dx}{dt} &= cx + dy + x\,C(x,y) = P(x,y), \\ \frac{dy}{dt} &= ex + fy + y\,C(x,y) = Q(x,y), \end{aligned} \tag{1}$$

where c, d, e, f are real coefficients and the polynomial $C(x,y)$ has real coefficients and degree $r \in \mathbb{N}^*$. This system can be written [17] in the following form

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{2}\frac{\partial R}{\partial y} + \frac{1}{2} S\,x + x\,C(x,y) = P(x,y), \\ \frac{dy}{dt} &= -\frac{1}{2}\frac{\partial R}{\partial x} + \frac{1}{2} S\,y + y\,C(x,y) = Q(x,y), \end{aligned} \tag{2}$$

where the $GL(2,\mathbb{R})$-invariant S and the $GL(2,\mathbb{R})$-comitants $R(x,y)$ and $C(x,y)$ of the system (1) have the form

$$S = c + f, \quad R(x,y) = -ex^2 + (c-f)x + dy^2, \quad C(x,y) = \sum_{k=0}^{r} a_k \binom{r}{k} x^{r-k} y^k. \tag{3}$$

From the classical invariant theory [18] the definition of the transvectant of two polynomials is well known.

Definition 1. Let $f(x,y)$ and $\varphi(x,y)$ be homogeneous polynomials in x and y with real coefficients of the degrees $\rho \in \mathbb{N}^*$ and $\theta \in \mathbb{N}^*$, respectively, and $k \in \mathbb{N}^*$. The polynomial

$$(f,\varphi)^{(k)} = \frac{(\rho-k)!(\theta-k)!}{\rho!\theta!} \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h}\partial y^h} \frac{\partial^k \varphi}{\partial x^h \partial y^{k-h}}$$

is called the *transvectant* of index k of polynomials f and φ.

Example 1. Hessian of the comitant $R(x,y)$ has the form

$$H = (R,R)^{(2)} = -\frac{1}{2}[4de + (c-f)^2] = -\frac{1}{2}\,\mathrm{Discr}\,R(x,y). \tag{4}$$

Remark 1. If the polynomials f and φ are $GL(2,\mathbb{R})$-comitants of the degrees $\rho \in \mathbb{N}^*$ and $\theta \in \mathbb{N}^*$, respectively, for the system (1), then the transvectant of the index $k \le \min(\rho,\theta)$ is a $GL(2,\mathbb{R})$-comitant of the degree $\rho+\theta-2k$ for the system (1) [19]. If $k > \min(\rho,\theta)$, then $(f,\varphi)^{(k)} = 0$.

For every homogeneous GL-comitant $K(x,y)$ with degree $s \in \mathbb{N}^*$ of the system (1) from (2) we obtain the total derivative of $K(x,y)$ with respect to t:

$$\frac{dK}{dt} = \frac{\partial K}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial K}{\partial y}\cdot\frac{dy}{dt} = \frac{\partial K}{\partial x}\left(\frac{1}{2}\frac{\partial R}{\partial y} + \frac{1}{2}xS + xC\right) + \\ + \frac{\partial K}{\partial y}\left(-\frac{1}{2}\frac{\partial R}{\partial x} + \frac{1}{2}yS + yC\right) = s(K,R)^{(1)} + \frac{s}{2}KS + sKC, \tag{5}$$

where $(K,R)^{(1)}$ is a Jacobian (the transvectant of the first index) of GL-comitants K and R. The representation (5) shows that the derivative with respect to t of every homogeneous $GL(2,\mathbb{R})$-comitant with the degree $s \ge 1$ of the system (1) is a $GL(2,\mathbb{R})$-comitant too.

Remark 2. If the homogeneous polynomials f, g, φ and ψ have the degrees m, n, μ and 0 (m, n, $\mu \in \mathbb{N}^*$), respectively, with respect to x and y and $l, q \in \mathbb{N}$, $\alpha \in \mathbb{R}$, then

$$(\alpha f, g)^{(k)} = (f, \alpha g)^{(k)} = \alpha(f,g)^{(k)}, \qquad (f^q, f)^{(2l+1)} = 0,$$
$$(f+g,\varphi)^{(k)} = (f,\varphi)^{(k)} + (g,\varphi)^{(k)}, \qquad (\psi, f)^{(k)} = 0,$$
$$(f\cdot g,\varphi)^{(1)} = \frac{m}{m+n}(f,\varphi)^{(1)}g + \frac{n}{m+n}(g,\varphi)^{(1)}f.$$

Remark 3. If the homogeneous polynomials f and φ have the degrees $m \in \mathbb{N}^*$ and 2, respectively, with respect to x and y, then

$$((f,\varphi)^{(1)},\varphi)^{(1)} = \frac{m-1}{m}(f,\varphi)^{(2)}\varphi - \frac{1}{2}f(\varphi,\varphi)^{(2)}.$$

We shall suppose that the polynomials $P(x,y)$ and $Q(x,y)$ are coprime and the polynomial $C(x,y)$ has a nonzero degree, i.e.

$$R(x,y) \not\equiv 0, \qquad C(x,y) \not\equiv 0, \qquad \deg C(x,y) \ge 1. \tag{6}$$

Remark 4. From (5) for $K = R(x,y)$ we obtain

$$\frac{dR}{dt} = R(S + 2\,C), \tag{7}$$

which shows that $R(x,y) = 0$ is an invariant algebraic curve of the system (1).

Let the polynomial $C(x,y)$ have the degree r ($r \in \mathbb{N}^*$) with respect to x and y. We denote by p the integer part of the number $\dfrac{r}{2}$, i.e. $p = \left[\dfrac{r}{2}\right]$. Now we suppose the following assumptions: if the lower index in the symbol of the sum $\sum$ is greater than upper index then the sum is equal to zero; in repeated using of the transvectants a set of the parenthesis of the type $((\ldots($ will be replaced by a single parenthesis of the form $[\![$.

2.1 The polynomial $C(x, y)$ has odd degree

Let $r = \deg C(x, y) = 2p + 1$, where $p \in \mathbb{N}$.

Theorem 1. *The system (1) with the conditions (6) has real invariant algebraic curve $F_r(x, y) = 0$ of the degree r, where the polynomial F_r is expressed by invariants and comitants of the system (1):*

$$F_r(x,y) = 2^{2p+1} \cdot r! \cdot R^p \left(\frac{2}{r} [\![C, \overbrace{R)^{(2)}, \dots, R)^{(2)}}^{p}, R)^{(1)} - [\![C, \overbrace{R)^{(2)}, \dots, R)^{(2)}}^{p} \cdot S \right) +$$

$$+ \sum_{i=0}^{p-1} \left[\frac{2^{2i+1} \cdot r!}{(r-2i)!} \cdot R^i \left(\frac{2(r-2i)}{r} [\![C, \overbrace{R)^{(2)}, \dots, R)^{(2)}}^{i}, R)^{(1)} - [\![C, \overbrace{R)^{(2)}, \dots, R)^{(2)}}^{i} \cdot S \right) \times \right.$$

$$\left. \times \prod_{j=i+1}^{p} \left(2(r-2j)^2 (R,R)^{(2)} + r^2 S^2 \right) \right] - \frac{1}{r^2} \prod_{j=0}^{p} \left(2(r-2j)^2 (R,R)^{(2)} + r^2 S^2 \right), \qquad (8)$$

Proof. The polynomial F_r is a sum of two polynomials $F_r(x,y) = \widehat{F}_r(x,y) + \widetilde{F}_r$, where $\widehat{F}_r(x,y)$ is a comitant of the degree r with respect to x and y and $\widetilde{F}_r = -\frac{1}{r^2} \prod_{j=0}^{p} \left(2(r-2j)^2 (R,R)^{(2)} + r^2 S^2 \right)$ is an invariant of the system (1).

By using the relation (5), Remarks 2 and 3, for the polynomial (8) we obtain:

$$\frac{dF_r}{dt} = \frac{d(\widehat{F}_r + \widetilde{F}_r)}{dt} = r(\widehat{F}_r, R)^{(1)} + \frac{r}{2} \widehat{F}_r S + r \widetilde{F}_r C =$$

$$= r 2^{2p+1} \cdot r! \left(\frac{2}{r^2} R^p [\![C, \overbrace{R)^{(2)}, \dots, R)^{(2)}}^{p}, R)^{(1)}, R)^{(1)} - \frac{1}{r} R^p [\![C, \overbrace{R)^{(2)}, \dots, R)^{(2)}}^{p}, R)^{(1)} \cdot S + \right.$$

$$\left. + \frac{1}{r} R^p [\![C, \overbrace{R)^{(2)}, \dots, R)^{(2)}}^{p}, R)^{(1)} \cdot S - \frac{1}{2} R^p [\![C, \overbrace{R)^{(2)}, \dots, R)^{(2)}}^{p} \cdot S^2 \right) +$$

$$+ r \sum_{i=0}^{p-1} \left[\frac{2^{2i+1} \cdot r!}{(r-2i)!} \left(\frac{2(r-2i)^2}{r^2} R^i [\![C, \overbrace{R)^{(2)}, \dots, R)^{(2)}}^{i}, R)^{(1)}, R)^{(1)} - \right. \right.$$

$$- \frac{(r-2i)}{r} R^i [\![C, \overbrace{R)^{(2)}, \dots, R)^{(2)}}^{i}, R)^{(1)} \cdot S + \frac{(r-2i)}{r} R^i [\![C, \overbrace{R)^{(2)}, \dots, R)^{(2)}}^{i}, R)^{(1)} \cdot S -$$

$$\left. \left. - \frac{1}{2} R^i [\![C, \overbrace{R)^{(2)}, \dots, R)^{(2)}}^{i} \cdot S^2 \right) \times \prod_{j=i+1}^{p} \left(2(r-2j)^2 (R,R)^{(2)} + r^2 S^2 \right) \right] + r \widetilde{F}_r C =$$

$$= r 2^{2p+1} \cdot r! \left(\frac{2}{r^2} \cdot \frac{r-2p-1}{(r-2p)} R^{p+1} [\![C, \overbrace{R)^{(2)}, \dots, R)^{(2)}}^{p+1} - \right.$$

$$-\frac{1}{r^2}R^p[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p}\cdot(R,R)^{(2)}-\frac{1}{2}R^p[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p}\cdot S^2\Bigg)+$$

$$+r\sum_{i=0}^{p-1}\left[\frac{2^{2i+1}\cdot r!}{(r-2i)!}\left(\frac{2(r-2i)^2}{r^2}\cdot\frac{r-2i-1}{(r-2i)}R^{i+1}[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i+1}-\right.\right.$$

$$\left.-\frac{(r-2i)^2}{r^2}R^i[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i}\cdot(R,R)^{(2)}-\frac{1}{2}R^i[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i}\cdot S^2\right)\times$$

$$\left.\times\prod_{j=i+1}^{p}\left(2(r-2j)^2(R,R)^{(2)}+r^2S^2\right)\right]+r\widehat{F}_rC=$$

$$=-r2^{2p+1}\cdot r!\cdot\frac{1}{2r^2}R^p[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p}\left(2(R,R)^{(2)}+r^2S^2\right)+$$

$$+r\sum_{i=0}^{p-1}\left[\frac{2^{2i+1}\cdot r!}{(r-2i)!}\left(\frac{2(r-2i)(r-2i-1)}{r^2}R^{i+1}[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i+1}-\right.\right.$$

$$\left.-\frac{1}{2r^2}R^i[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i}\left(2(r-2i)^2(R,R)^{(2)}+r^2S^2\right)\right)\times$$

$$\left.\times\prod_{j=i+1}^{p}\left(2(r-2j)^2(R,R)^{(2)}+r^2S^2\right)\right]+r\widehat{F}_rC=$$

$$=-r2^{2p}\cdot r!\cdot\frac{1}{r^2}R^p[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p}\left(2(R,R)^{(2)}+r^2S^2\right)+$$

$$+r\sum_{i=0}^{p-1}\left[\frac{2^{2(i+1)}\cdot r!}{(r-2(i+1))!}\cdot\frac{1}{r^2}R^{i+1}[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i+1}\times\prod_{j=i+1}^{p}\left(2(r-2j)^2(R,R)^{(2)}+r^2S^2\right)\right]-$$

$$-r\sum_{i=0}^{p-1}\left[\frac{2^{2i}\cdot r!}{(r-2i)!}\cdot\frac{1}{r^2}R^i[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i}\times\prod_{j=i}^{p}\left(2(r-2j)^2(R,R)^{(2)}+r^2S^2\right)\right]+r\widehat{F}_rC=$$

$$=r\sum_{i=1}^{p}\left[\frac{2^{2i}\cdot r!}{(r-2i)!}\cdot\frac{1}{r^2}R^i[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i}\times\prod_{j=i}^{p}\left(2(r-2j)^2(R,R)^{(2)}+r^2S^2\right)\right]-$$

$$-r\sum_{i=0}^{p}\left[\frac{2^{2i}\cdot r!}{(r-2i)!}\cdot\frac{1}{r^2}R^i[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i}\times\prod_{j=i}^{p}\left(2(r-2j)^2(R,R)^{(2)}+r^2S^2\right)\right]+r\widehat{F}_rC=$$

$$=-rC\frac{1}{r^2}\prod_{j=0}^{p}\left(2(r-2j)^2(R,R)^{(2)}+r^2S^2\right)+r\widehat{F}_rC=rC\widetilde{F}_r+r\widehat{F}_rC=rC(\widehat{F}_r+\widetilde{F}_r)=rCF_r.$$

So,

$$\frac{dF_r}{dt} = rCF_r \tag{9}$$

and $F_r(x,y) = 0$ is a real invariant algebraic curve for (1). Theorem 1 is proved.

Example 2. For $r \in \{1,3\}$ we obtain the invariant algebraic curves:

$$F_1(x,\, y) = 4(C,R)^{(1)} - 2C\cdot S - 2(R,R)^{(2)} - S^2 = 0,$$

$$F_3(x,\, y) = 32R((C,R)^{(2)},R)^{(1)} - 48R(C,R)^{(2)}\cdot S +$$

$$+\left(4[\![C,R)^{(1)} - 2C\cdot S\right)\left(2(R,R)^{(2)} + 9S^2\right) - \left(2(R,R)^{(2)} + 9S^2\right)\left(2(R,R)^{(2)} + S^2\right) = 0,$$

The next theorem classifies first integrals of the system (1) in this subcase.

Theorem 2. *The system (1) with the conditions (6) has the following real first integrals:*

a) for $S \neq 0,\ H > 0:$

$$\left|F_r\right|^{\frac{2}{r}}\cdot\left|R\right|^{-1}\cdot G_1 = c_1,\quad G_1 = \exp\left[\frac{2S}{\sqrt{2H}}\arctan\frac{\dfrac{\partial R}{\partial x} - y\cdot\sqrt{2H}}{\dfrac{\partial R}{\partial x} + y\cdot\sqrt{2H}}\right]; \tag{10}$$

b) for $S \neq 0,\ H < 0:$

$$\left|F_r\right|^{\frac{2}{r}}\cdot\left|R\right|^{-1}\cdot G_2 = c_2,\quad G_2 = \left|\frac{\dfrac{\partial R}{\partial x} - y\cdot\sqrt{-2H}}{\dfrac{\partial R}{\partial x} + y\cdot\sqrt{-2H}}\right|^{\frac{S}{\sqrt{-2H}}}; \tag{11}$$

c) for $S \neq 0,\ H = 0:$

$$\left|F_r\right|^{\frac{2}{r}}\cdot\left|R\right|^{-1}\cdot G_3 = c_3,\quad G_3 = \exp\left[\frac{S[(c-f)x^2 + 2(d+e)xy - (c-f)y^2]}{4(d-e)\,R}\right]; \tag{12}$$

d) for $S = 0:$

$$\left|F_r\right|^{\frac{2}{r}}\cdot\left|R\right|^{-1} = c_4, \tag{13}$$

where c_1, c_2, c_3 *and* c_4 *are real constants.*

Proof. If $S \neq 0$, then from (7) and (9) and after calculation of the derivatives with respect to t of the functions: G_1 for $H > 0$, G_2 for $H < 0$ and G_3 for $H = 0$, we obtain

$$\frac{dR}{dt} = R(S + 2\,C),\ \frac{dF_r}{dt} = rCF_r,\ \frac{dG_1}{dt} = SG_1,\ \frac{d\ln G_2}{dt} = S,\ \frac{dG_3}{dt} = SG_3. \tag{14}$$

From (14) we easy obtain first integrals (10), (11) and (12).

If $S = 0$, the relations (7) and (9) have the forms

$$\frac{dR}{dt} = 2\,R\,C,\quad \frac{dF_r}{dt} = rCF_r. \tag{15}$$

The relations (15) determine the first integral (13). Theorem 2 is proved.

2.2 The polynomial $C(x,y)$ has even degree

Let $r = \deg C(x,y) = 2p$, where $p \in \mathbb{N}^*$.

Theorem 3. *The system (1) with the conditions (6) has real invariant algebraic curve $F_r(x,y) = 0$ of the degree r, where the polynomial F_r is expressed by invariants and comitants of the system (1):*

$$F_r(x,y) = -2R^p I_r + rS \cdot \Phi_r(x,\, y), \tag{16}$$

where

$$I_r = 2^{2p} \cdot (r-1)! [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{p}, \tag{17}$$

$$\Phi_r(x,y) = \frac{2^{2p-1} \cdot r!}{2!} \cdot R^{p-1} \left(\frac{4}{r} [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{p-1}, R)^{(1)} - [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{p-1} \cdot S \right) +$$
$$+ \sum_{i=0}^{p-2} \left[\frac{2^{2i+1} \cdot r!}{(r-2i)!} \cdot R^i \left(\frac{2(r-2i)}{r} [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{i}, R)^{(1)} - [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{i} \cdot S \right) \times \right.$$
$$\left. \times \prod_{j=i+1}^{p-1} \Big(2(r-2j)^2 (R,R)^{(2)} + r^2 S^2 \Big) \right] - \frac{1}{r^2} \prod_{j=0}^{p-1} \Big(2(r-2j)^2 (R,R)^{(2)} + r^2 S^2 \Big). \tag{18}$$

Proof. The first step is to calculate the derivative $\dfrac{d\Phi_r(x,y)}{dt}$. The polynomial $\Phi_r(x,y)$ is a sum of two terms $\Phi_r(x,y) = \widehat{\Phi}_r(x,y) + \widetilde{\Phi}_r$, where $\widehat{\Phi}_r(x,y)$ is a comitant of the degree r with respect to x and y and $\widetilde{\Phi}_r = -\dfrac{1}{r^2} \prod\limits_{j=0}^{p-1} \big(2(r-2j)^2 (R,R)^{(2)} + r^2 S^2 \big)$ is an invariant of the system (1). By using the relation (5), Remarks 2 and 3, we obtain:

$$\frac{d\Phi}{dt} = \frac{d(\widehat{\Phi}_r + \widetilde{\Phi}_r)}{dt} = r(\widehat{\Phi}_r, R)^{(1)} + \frac{r}{2} \widehat{\Phi}_r S + r \widehat{\Phi}_r C =$$
$$= r \frac{2^{2p-1} \cdot r!}{2!} \left(\frac{8}{r^2} R^{p-1} [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{p-1}, R)^{(1)}, R)^{(1)} - \frac{2}{r} R^{p-1} [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{p-1}, R)^{(1)} \cdot S + \right.$$
$$\left. + \frac{2}{r} R^{p-1} [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{p-1}, R)^{(1)} \cdot S - \frac{1}{2} R^{p-1} [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{p-1} \cdot S^2 \right) +$$
$$+ r \sum_{i=0}^{p-2} \left[\frac{2^{2i+1} \cdot r!}{(r-2i)!} \left(\frac{2(r-2i)^2}{r^2} R^i [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{i}, R)^{(1)}, R)^{(1)} - \right. \right.$$
$$- \frac{(r-2i)}{r} R^i [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{i}, R)^{(1)} \cdot S + \frac{(r-2i)}{r} R^i [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{i}, R)^{(1)} \cdot S -$$

$$-\frac{1}{2}R^i[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i}\cdot S^2\Bigg)\times\prod_{j=i+1}^{p-1}\Big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\Big)\Bigg]+r\widehat{\Phi}_rC=$$

$$=r\frac{2^{2p-1}\cdot r!}{2!}\Bigg(\frac{8}{r^2}\cdot\frac{r-2p+1}{(r-2p+2)}R^p[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p}-$$

$$-\frac{4}{r^2}R^{p-1}[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-1}\cdot(R,R)^{(2)}-\frac{1}{2}R^{p-1}[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-1}\cdot S^2\Bigg)+$$

$$+r\sum_{i=0}^{p-2}\Bigg[\frac{2^{2i+1}\cdot r!}{(r-2i)!}\Bigg(\frac{2(r-2i)^2}{r^2}\cdot\frac{r-2i-1}{(r-2i)}R^{i+1}[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i+1}-$$

$$-\frac{(r-2i)^2}{r^2}R^i[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i}\cdot(R,R)^{(2)}-\frac{1}{2}R^i[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i}\cdot S^2\Bigg)\times$$

$$\times\prod_{j=i+1}^{p-1}\Big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\Big)\Bigg]+r\widehat{\Phi}_rC=$$

$$=2^{2p}\cdot(r-1)!\cdot R^p[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p}-$$

$$-r\frac{2^{2p-1}\cdot r!}{2!}\cdot\frac{1}{2r^2}R^{p-1}[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-1}\Big(8(R,R)^{(2)}+r^2S^2\Big)+$$

$$+r\sum_{i=0}^{p-2}\Bigg[\frac{2^{2i+1}\cdot r!}{(r-2i)!}\Bigg(\frac{2(r-2i)(r-2i-1)}{r^2}R^{i+1}[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i+1}-$$

$$-\frac{1}{2r^2}R^i[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i}\Big(2(r-2i)^2(R,R)^{(2)}+r^2S^2\Big)\Bigg)\times$$

$$\times\prod_{j=i+1}^{p-1}\Big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\Big)\Bigg]+r\widehat{\Phi}_rC=$$

$$=R^pI_r-r\frac{2^{2(p-1)}\cdot r!}{2!}\cdot\frac{1}{r^2}R^{p-1}[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-1}\Big(8(R,R)^{(2)}+r^2S^2\Big)+$$

$$+r\sum_{i=0}^{p-2}\Bigg[\frac{2^{2(i+1)}\cdot r!}{(r-2(i+1))!}\cdot\frac{1}{r^2}R^{i+1}[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i+1}\times\prod_{j=i+1}^{p-1}\Big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\Big)\Bigg]-$$

$$-r\sum_{i=0}^{p-2}\Bigg[\frac{2^{2i}\cdot r!}{(r-2i)!}\cdot\frac{1}{r^2}R^i[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i}\times\prod_{j=i}^{p-1}\Big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\Big)\Bigg]+r\widehat{\Phi}_rC=$$

$$= R^p I_r + r\sum_{i=1}^{p-1}\left[\frac{2^{2i}\cdot r!}{(r-2i)!}\cdot\frac{1}{r^2}R^i[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\times\prod_{j=i}^{p-1}\Big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\Big)\right]-$$

$$-r\sum_{i=0}^{p-1}\left[\frac{2^{2i}\cdot r!}{(r-2i)!}\cdot\frac{1}{r^2}R^i[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\times\prod_{j=i}^{p-1}\Big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\Big)\right]+r\widehat{\Phi}_rC=$$

$$= R^pI_r - rC\frac{1}{r^2}\prod_{j=0}^{p-1}\Big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\Big)+r\widehat{\Phi}_rC=$$

$$= R^pI_r + rC\widetilde{\Phi}_r + r\widehat{\Phi}_rC = R^pI_r + rC(\widehat{\Phi}_r+\widetilde{\Phi}_r) = R^pI_r + rC\Phi_r.$$

Thus,
$$\frac{d\Phi_r}{dt} = R^pI_r + rC\Phi_r. \tag{19}$$

By using the relation (5) and Remark 2 for polynomial R^p we have:

$$\frac{dR^p}{dt} = r(R^p,R)^{(1)} + \frac{r}{2}R^pS + rR^pC = \frac{r}{2}R^p(S+2C). \tag{20}$$

By virtue of the relations (19) and (20) for polynomial (16) we obtain:

$$\frac{dF_r}{dt} = \frac{d(-2R^pI_r + rS\Phi_r)}{dt} = -2I_r\frac{dR^p}{dt} + rS\frac{d\Phi_r}{dt} =$$
$$= -rI_rR^p(S+2C) + rS(R^pI_r + rC\Phi_r) = rC(-2R^pI_r + rS\Phi_r) = rCF_r.$$

So, $F_r(x,y)=0$ is an invariant algebraic curve for (1). Theorem 3 is proved.

Example 3. For $r\in\{2,4\}$ we obtain the invariant algebraic curves:

$$F_2(x,\,y) = -8R(C,R)^{(2)} + 2S\Big(4(C,R)^{(1)} - 2C\cdot S - 2(R,R)^{(2)} - S^2\Big) = 0,$$

$$F_4(x,\,y) = -192R^2((C,R)^{(2)},R)^{(2)} + 4S\Big[96R((C,R)^{(2)},R)^{(1)} - 96R(C,R)^{(2)}\cdot S +$$
$$+\Big(4(C,R)^{(1)} - 2C\cdot S\Big)\Big(8(R,R)^{(2)} + 16S^2\Big) - \Big(8(R,R)^{(2)} + 16S^2\Big)\Big(2(R,R)^{(2)} + S^2\Big)\Big] = 0.$$

The next theorem is similar to Theorem 2 and classifies first integrals of the system (1) in this subcase for $S\neq 0$.

Theorem 4. *The system (1) with the conditions (6) and $S\neq 0$ has the following real first integrals:*

a) for $H>0$:

$$\Big|F_r\Big|^{\frac{1}{p}}\cdot\Big|R\Big|^{-1}\cdot G_1 = c_5,\quad G_1 = \exp\left[\frac{2S}{\sqrt{2H}}\arctan\frac{\dfrac{\partial R}{\partial x} - y\cdot\sqrt{2H}}{\dfrac{\partial R}{\partial x} + y\cdot\sqrt{2H}}\right]; \tag{21}$$

b) for $H<0$:

$$\Big|F_r\Big|^{\frac{1}{p}}\cdot\Big|R\Big|^{-1}\cdot G_2 = c_6,\quad G_2 = \left|\frac{\dfrac{\partial R}{\partial x} - y\cdot\sqrt{-2H}}{\dfrac{\partial R}{\partial x} + y\cdot\sqrt{-2H}}\right|^{\frac{S}{\sqrt{-2H}}}; \tag{22}$$

c) for $H=0:$

$$\Big|F_r\Big|^{\frac{1}{p}}\cdot\Big|R\Big|^{-1}\cdot G_3=c_7,\quad G_3=\exp\left[\frac{S[(c-f)x^2+2(d+e)xy-(c-f)y^2]}{4(d-e)\,R}\right],\tag{23}$$

where c_5, c_6 *and* c_7 *are real constants.*

The proof of Theorem 4 is similar to the proof of Theorem 2.

Let $S=0$. The first result in this subcase for the system (1) with $S=0$ is the following theorem.

Theorem 5. *The system* (1) *with the conditions (6) and* $S=0,\ H=(R,R)^{(2)}=0$ *has the invariant algebraic curve* $\Psi_r(x,y)=0$ *of the form*

$$\Psi_r(x,y)=J_rV_rR+W_rQ_r,\tag{24}$$

where

$$J_r=\frac{I_r}{2^{2p}\cdot(2p-1)!}=[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p},\quad Q_r(x,y)=[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-1},R)^{(1)},$$

$$\begin{aligned}V_r(x,y)=\frac{r+1}{r}R\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p}+\\+\sum_{i=0}^{p-1}\Bigg(\binom{r}{2i+1}[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i-1},R)^{(1)}-\\-\binom{r}{2i+2}[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i},R)^{(1)}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i-1}\Bigg),\end{aligned}\tag{25}$$

$$\begin{aligned}W_r(x,y)=\sum_{i=0}^{p}\binom{r}{2i}R\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i}-\\-\sum_{i=0}^{p-1}\binom{r}{2i+1}[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i},R)^{(1)}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i-1},R)^{(1)}.\end{aligned}\tag{26}$$

Proof. From the first we calculate the derivative $\dfrac{dV_r(x,y)}{dt}$. The polynomial $V_r(x,y)$ is a sum of two terms $V_r(x,y)=\widetilde{V}_r(x,y)+\widehat{V}_r(x,y)$, where $\widehat{V}_r(x,y)$ is homogeneous polynomial of the degree $r+2$ with respect to x and y and the comitant $\widetilde{V}_r(x,y)=\dfrac{r+1}{r}R\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p}$ has the second degree with respect to x and y. By using the relation (5), Remarks 2 and 3, we obtain:

$$\frac{dV_r}{dt}=\frac{d(\widetilde{V}_r+\widehat{V}_r)}{dt}=2(\widetilde{V}_r,R)^{(1)}+2\widetilde{V}_rC+(r+2)(\widehat{V}_r,R)^{(1)}+(r+2)\widehat{V}_rC=$$

$$= (r+2)V_rC - r\widetilde{V}_rC + (r+2)(\widehat{V}_r, R)^{(1)} = (r+2)V_rC - r\widetilde{V}_rC +$$

$$+(r+2)\sum_{i=0}^{p-1}\left(\binom{r}{2i+1}\frac{r-2i}{r+2}[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i},R)^{(1)}\cdot[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-i-1},R)^{(1)}+\right.$$

$$+\binom{r}{2i+1}\frac{2i+2}{r+2}[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i}\cdot[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-i-1},R)^{(1)},R)^{(1)}-$$

$$-\binom{r}{2i+2}\frac{r-2i}{r+2}[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i},R)^{(1)},R)^{(1)}\cdot[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-i-1}-$$

$$\left.-\binom{r}{2i+2}\frac{2i+2}{r+2}[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i},R)^{(1)}\cdot[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-i-1},R)^{(1)}\right)=$$

$$= (r+2)V_rC - r\widetilde{V}_rC +$$

$$+\sum_{i=0}^{p-1}\left(\binom{r}{2i+1}(r-2i)[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i},R)^{(1)}\cdot[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-i-1},R)^{(1)}+\right.$$

$$+\binom{r}{2i+1}(2i+1)R\cdot[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i}\cdot[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-i}-$$

$$-\binom{r}{2i+2}(r-2i-1)R\cdot[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i+1}\cdot[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-i-1}-$$

$$\left.-\binom{r}{2i+1}(r-2i-1)[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i},R)^{(1)}\cdot[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-i-1},R)^{(1)}\right)=$$

$$= (r+2)V_rC - r\widetilde{V}_rC +$$

$$+\sum_{i=0}^{p-1}\binom{r}{2i+1}[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i},R)^{(1)}\cdot[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-i-1},R)^{(1)}+$$

$$+\sum_{i=0}^{p-1}\binom{r}{2i}(r-2i)R\cdot[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i}\cdot[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-i}-$$

$$-\sum_{i=1}^{p}\binom{r}{2i}(r-2i+1)R\cdot[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i}\cdot[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-i}=$$

$$= (r+2)V_rC - (r+1)RC\cdot[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p}+$$

$$+\sum_{i=0}^{p-1}\binom{r}{2i+1}[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i},R)^{(1)}\cdot[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-i-1},R)^{(1)}+$$

$$+rR\cdot C\cdot [\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p}-R\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p}\cdot C-$$

$$-\sum_{i=1}^{p-1}\binom{r}{2i}R\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i}=$$

$$=(r+2)V_rC-\sum_{i=0}^{p}\binom{r}{2i}R\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i}+$$

$$+\sum_{i=0}^{p-1}\binom{r}{2i+1}[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i},R)^{(1)}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i-1},R)^{(1)}=$$

$$=(r+2)V_rC-W_r.$$

Thus,
$$\frac{dV_r}{dt}=(r+2)V_rC-W_r. \tag{27}$$

Now we calculate the derivative $\dfrac{dW_r(x,y)}{dt}$, where $W_r(x,y)$ is a homogeneous comitant of the degree $r+2$ with respect to x and y.

$$\frac{dW_r}{dt}=(r+2)(W_r,R)^{(1)}+(r+2)W_rC=(r+2)W_rC+$$

$$+(r+2)\sum_{i=0}^{p}\binom{r}{2i}\frac{r-2i}{r+2}R\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i},R)^{(1)}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i}+$$

$$+(r+2)\sum_{i=0}^{p}\binom{r}{2i}\frac{2i}{r+2}R\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i},R)^{(1)}-$$

$$-(r+2)\sum_{i=0}^{p-1}\binom{r}{2i+1}\frac{r-2i}{r+2}[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i},R)^{(1)},R)^{(1)}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i-1},R)^{(1)}-$$

$$-(r+2)\sum_{i=0}^{p-1}\binom{r}{2i+1}\frac{2i+2}{r+2}[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i},R)^{(1)}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i-1},R)^{(1)},R)^{(1)}=$$

$$=(r+2)W_rC+\sum_{i=0}^{p}\binom{r}{2i}(r-2i)R\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i},R)^{(1)}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i}+$$

$$+\sum_{i=0}^{p}\binom{r}{2i}2iR\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i},R)^{(1)}-$$

$$-\sum_{i=0}^{p-1}\binom{r}{2i+1}(r-2i-1)R\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i+1}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i-1},R)^{(1)}-$$

$$-\sum_{i=0}^{p-1}\binom{r}{2i+1}(2i+1)R\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i},R)^{(1)}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i}=$$

$$= (r+2)W_rC + \sum_{i=0}^{p-1}\binom{r}{2i}(r-2i)R\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i},R)^{(1)}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i}+$$

$$+\sum_{i=1}^{p}\binom{r}{2i}2iR\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i},R)^{(1)}-$$

$$-\sum_{i=0}^{p-1}\binom{r}{2(i+1)}2(i+1)R\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i+1}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-(i+1)},R)^{(1)}-$$

$$-\sum_{i=0}^{p-1}\binom{r}{2i}(r-2i)R\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i},R)^{(1)}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i}=$$

$$=(r+2)W_rC+\sum_{i=1}^{p}\binom{r}{2i}2iR\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i},R)^{(1)}-$$

$$-\sum_{i=1}^{p}\binom{r}{2i}2iR\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i},R)^{(1)}=(r+2)W_rC.$$

Thus,

$$\frac{dW_r}{dt}=(r+2)W_rC. \tag{28}$$

In analogous way for polynomial $Q_r(x,y)$ we have:

$$\frac{dQ_r}{dt}=2(Q_r,R)^{(1)}+2Q_rC=2Q_rC+2[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-1},R)^{(1)},R)^{(1)}=$$

$$=2Q_rC+R\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p}=2Q_rC+RJ_r. \tag{29}$$

By using (24), (27), (28) and (29), we obtain:

$$\begin{gathered}\frac{d\Psi_r}{dt}=J_r\frac{dV_r}{dt}R+J_rV_r\frac{dR}{dt}+\frac{dW_r}{dt}Q_r+W_r\frac{dQ_r}{dt}=\\ =J_r\left((r+2)V_rC-W_r\right)R+2J_rV_rRC+(r+2)W_rCQ_r+W_r\left(2Q_rC+RJ_r\right)=\\ =(r+4)(J_rV_rR+W_rQ_r)C=(r+4)\Psi_rC.\end{gathered}$$

So,

$$\frac{d\Psi_r}{dt}=(r+4)\Psi_rC. \tag{30}$$

and $\Psi_r(x,y)=0$ is an invariant algebraic curve for (1). Theorem 5 is proved.

Remark 5. If $H=(R,R)^{(2)}=0$, then the following identity $W_r\equiv R^{p+1}\cdot(C,C)^{(r)}$ holds. By virtue of this identity the invariant algebraic curve $\Psi_r(x,y)=0$ can be written in the form $R\cdot\Psi_r^*(x,y)=0$, where $\Psi_r^*(x,y)=J_r\cdot V_r+Q_r\cdot R^p\cdot(C,C)^{(r)}$. So, $\Psi_r^*(x,y)=0$ is an invariant algebraic curve for the system (1) with $S=0,\ H=0$.

The next theorem classifies in this subcase first integrals of (1) for $S=0$.

Theorem 6. *The system (1) with the conditions (6) and $S=0$ has the following real first integrals:*

a) for $H>0,\ I_r\neq 0:$

$$\sqrt{\frac{H}{2}}\cdot\frac{1}{I_r}\cdot\frac{\Phi_r}{R^p}+G_4=c_8,\quad G_4=\arctan\frac{ex+fy}{y\cdot\sqrt{\dfrac{H}{2}}};\tag{31}$$

b) for $H<0,\ I_r\neq 0:$

$$G_5\cdot\exp\left(-\frac{\sqrt{-2H}}{I_r}\cdot\frac{\Phi_r}{R^p}\right)=c_9,\quad G_5=\left|\frac{\dfrac{\partial R}{\partial x}-y\cdot\sqrt{-2H}}{\dfrac{\partial R}{\partial x}+y\cdot\sqrt{-2H}}\right|;\tag{32}$$

c) for $H=0,\ I_r\neq 0:$

$$\Psi_r\cdot R^{-(p+2)}=c_{10};\tag{33}$$

d) for $I_r=0:$

$$\Phi_r\cdot R^{-p}=c_{11},\tag{34}$$

where c_8, c_9, c_{10} *and* c_{11} *are real constants.*

Proof. For $I_r\neq 0$ we have

$$\frac{d}{dt}\left(\frac{\Phi_r}{R^p}\right)=\frac{1}{R^{2p}}\cdot\left[R^p\cdot(R^p\cdot I_r+rC\Phi_r)-\Phi_r\cdot(p\cdot R^{p-1}\cdot 2CR)\right]=$$
$$\frac{1}{R^{2p}}\cdot\left[I_r\cdot R^{2p}+2pC\Phi_rR^p-2pC\Phi_rR^p\right]=I_r.\tag{35}$$

If $H>0$, then $\dfrac{dG_4}{dt}=-\sqrt{\dfrac{H}{2}}$ and we easy obtain first integral (31).

If $H<0$, then $\dfrac{d\ln G_5}{dt}=\sqrt{-2H}$ and we have first integral (32).

Let $H=0,\ I_r\neq 0$. In this case by virtue of (5) and (30) we have

$$\frac{d\left(\Psi_r/R^{p+2}\right)}{dt}=\frac{1}{R^{2(p+2)}}\left(\frac{d\Psi_r}{dt}R^{p+2}-\Psi_r\frac{dR^{p+2}}{dt}\right)=$$
$$=\frac{1}{R^{2(p+2)}}\left((r+4)\Psi_rCR^{p+2}-2(p+2)\Psi_rR^{p+2}C\right)=0.$$

So, the system (1) has first integral (33).

The first integral (34) for $I_r=0$ is given by (35). Theorem 6 is proved.

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Open problems on the algebraic limit cycles of planar polynomial vector fields

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Abstract. We collect several open problems that have appeared in the study of the algebraic limit cycles of the real planar polynomial vector fields.

Mathematics subject classification: 34C05.
Keywords and phrases: Algebraic limit cycle, polynomial vector field, Poincaré problem, invariant algebraic curve.

1 Introduction

We divide this brief presentation of several open problems on the algebraic limit cycles of the real planar polynomial vector fields into the following sections:

2. Invariant algebraic curves.
3. Algebraic limit cycles.
4. A unique irreducible invariant algebraic curve.
5. Quadratic polynomial vector fields.
6. Cubic polynomial vector fields.
7. Configurations of algebraic limit cycles.

2 Invariant algebraic curves

Since Darboux [12] has found in 1878 connections between algebraic curves and the existence of first integrals of planar polynomial vector fields, invariant algebraic curves are a central object in the theory of integrability of these vector fields. Today after more than one century of investigations the theory of invariant algebraic curves is still full of open questions.

A *real planar polynomial differential system* is a differential system of the form

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y), \tag{1}$$

where P and Q are real polynomials in the variables x and y. The dependent variables x and y, the independent variable t, and the coefficients of the polynomials P and Q are all real because in this paper we are interested in the real algebraic limit cycles of system (1). The *degree* n of the polynomial system (1) is the maximum of the degrees of the polynomials P and Q.

Associated to the (real) polynomial differential system (1) there is the *(real) polynomial vector field*

$$\mathcal{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y},$$

or simply $\mathcal{X} = (P, Q)$.

Let $f = f(x, y)$ be a (real) polynomial in the variables x and y. The algebraic curve $f(x, y) = 0$ of $\mathbb{R}^2$ is an *invariant algebraic curve* of the vector field $\mathcal{X}$ if for some polynomial $K \in \mathbb{R}[x, y]$ we have

$$\mathcal{X}f = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf. \tag{2}$$

The polynomial K is called the *cofactor* of the invariant algebraic curve $f = 0$. We note that since the polynomial system has degree n, then any cofactor has at most degree $n - 1$.

Since on the points of the algebraic curve $f = 0$ the gradient $(\partial f/\partial x,\ \partial f/\partial y)$ of the curve is orthogonal to the vector field $\mathcal{X} = (P, Q)$ (see (2)), the vector field $\mathcal{X}$ is tangent to the curve $f = 0$. Hence the curve $f = 0$ is formed by orbits of the vector field $\mathcal{X}$. This justifies the name of invariant algebraic curve given to the algebraic curve $f = 0$ satisfying (2) for some polynomial K, because it is *invariant* under the flow defined by $\mathcal{X}$.

The next result tell us that we can restrict our attention to the irreducible invariant algebraic curves.

Proposition 1. *We suppose that $f \in \mathbb{R}[x, y]$ and let $f = f_1^{n_1} \cdots f_r^{n_r}$ be its factorization in irreducible factors over $\mathbb{R}[x, y]$. Then for a polynomial vector field $\mathcal{X}$, $f = 0$ is an invariant algebraic curve with cofactor K_f if and only if $f_i = 0$ is an invariant algebraic curve for each $i = 1, \ldots, r$ with cofactor K_{f_i}. Moreover $K_f = n_1K_{f_1} + \ldots + n_rK_{f_r}$.*

For a proof of this proposition see for instance [25].

3 Algebraic limit cycles

We recall that a *limit cycle* of a polynomial vector field $\mathcal{X}$ is an isolated periodic orbit in the set of all periodic orbits of $\mathcal{X}$. An *algebraic limit cycle* of degree m of $\mathcal{X}$ is an oval of an irreducible invariant algebraic curve $f = 0$ of degree m which is a limit cycle of $\mathcal{X}$.

We must remark that when we are interested in the invariant algebraic curves for questions different from the algebraic limit cycles (like for instance integrability, multiplicity and others) it is important to consider complex invariant algebraic curves (i.e. $f = f(x, y)$ is a complex polynomial in the variables x and y), because the natural background for all the Darboux theory of integrability is the ring of the complex polynomials. But the limit cycles and in particular the algebraic limit cycles is a real phenomena, so in all this paper we restrict our attention to the real Darboux theory of integrability. In any case many of the results used here also hold inside the theory of complex polynomial differential systems.

A first question related with this subject is whether a polynomial vector field has or does not have invariant algebraic curves. The answer is not easy, see the large section in Jouanolou's book [21], or the long paper [31] devoted to show that one particular polynomial system has no invariant algebraic solutions. Even one of the more studied limit cycles, the limit cycle of the van der Pol system, until 1995 it was unknown that it is not algebraic [32].

One of the nice results in the theory of invariant algebraic curves is the following result.

Theorem 2 (Jouanolou's Theorem [21])**.** *A polynomial vector field of degree n has less than $[n(n+1)/2] + 2$ irreducible invariant algebraic curves, or it has a rational first integral.*

For a shorter proof of this result see [9] or [10].

Jouanolou's Theorem shows that for a given polynomial vector field $\mathcal{X}$ of degree n the maximum degree of its irreducible invariant algebraic curves is bounded, because either $\mathcal{X}$ has a finite number of invariant algebraic curves less than $[n(n+1)/2]+2$, or $\mathcal{X}$ has rational first integral $f(x, y)/g(x, y)$. In this last case all the orbits of $\mathcal{X}$ are contained in the invariant algebraic curves $af(x, y) + bg(x, y) = 0$ for some $a, b \in \mathbb{R}$.

Thus for each polynomial vector field there is a natural number N which bounds the degree of all its irreducible invariant algebraic curves. A natural question, going back to Poincaré [33] and which for some people in this area is now known as the *Poincaré problem*, is to give an effective procedure to find N. There are only partial answers to this question, see for instance [2–4, 36], ... We must mention here that the actual Poincaré problem is to determine when a polynomial differential system over the complex plane has a rational first integral, and that the previous called Poincaré problem is a main step according with Poincaré for solving the actual Poincaré problem.

Of course if we know for a polynomial vector field the maximum degree of its invariant algebraic curves, then it is possible (at least in theory) to compute its invariant algebraic curves.

We are interested in algebraic limit cycles of polynomial vector fields, and if a polynomial vector field has a rational first integral it cannot have limit cycles. Unfortunately for the class of polynomial vector fields with fixed degree n having finitely many invariant algebraic curves (i.e. having no rational first integrals), there does not exist a uniform upper bound $N(n)$ for N as it was shown in [11, 30]. This implies that there are polynomial vector fields with a fixed degree having irreducible invariant algebraic curve of arbitrary degree. Therefore a priori it is possible the existence of polynomial vector fields with a fixed degree having algebraic limit cycle of arbitrary degree. But it may be worse than that.

We shall need the next well known result.

Theorem 3 (Harnack's Theorem)**.** *The maximum number of ovals of a real algebraic curve of degree m is $[(m-1)(m-2)/2]+1$.*

Summarizing, a polynomial vector field of degree n with finitely many irreducible invariant algebraic curves has at most $[n(n+1)/2]+1$ of such curves, but we do not have a bound for the degree of these invariant algebraic curves. Consequently due to the Harnack's Theorem we do not have a uniform bound for the number of algebraic limit cycles that any polynomial vector field of degree n can have. So the second part of the 16–th Hilbert problem [20] (see also [19, 22]) which asks for finding a uniform bound for the number of limit cycles that any polynomial vector field of degree n can have, remains also open if we restrict our attention to the limit cycles which are algebraic.

Open problem 1. *Is there a uniform bound for the number of algebraic limit cycles that a polynomial vector field of degree n could have?*

From the previous paragraphs it is clear that a uniform positive answer to the Poincaré problem inside the class of all polynomial vector fields of degree n, i.e. to provide a uniform bound $N(n)$ for the degrees of the invariant algebraic curves of all polynomials vector fields of degree n, will provide also a uniform bound for the number of algebraic limit cycles of all polynomials vector fields of degree n.

4 A unique irreducible invariant algebraic curve

In this section we shall need the following result.

Theorem 4 (Bautin–Christopher–Dolov–Kuzmin Theorem)**.** *Let $f=0$ be a non–singular algebraic curve of degree m, and D a first degree polynomial, chosen so that the line $D=0$ lies outside all bounded components of $f=0$. Choose the constants a and b so that $aD_x+bD_y\neq 0$, then the polynomial differential system*

$$\dot{x}=af-Df_y, \qquad \dot{y}=bf+Df_x,$$

of degree m has all the bounded components of $f=0$ as hyperbolic limit cycles. Furthermore the vector field has no other limit cycles.

It seems that the main result in the paper of Bautin [1] is similar to the previous theorem. However the paper contains a mistake which was corrected in [13] and generalized in [14]. A proof of the statement of theorem like it is presented here appeared in [8].

The next proposition provides the maximum number of algebraic limit cycles that a polynomial vector field having a unique irreducible invariant algebraic curve can have in function of the degree of that curve. This proposition is well known in the area, we write it here for completeness.

Proposition 5. *Suppose that $f = 0$ of degree m is the unique irreducible invariant algebraic curve of a polynomial vector field X. Then X can have at most $[(m-1)(m-2)/2] + 1$ algebraic limit cycles. Moreover choosing that $f = 0$ has the maximal number of ovals for the irreducible algebraic curves of degree m, there exist vector fields X of degree m having exactly $[(m-1)(m-2)/2] + 1$ algebraic limit cycles.*

Proof. The first part of the proposition follows directly from the Harnack's Theorem, and the second part again from the Harnack's theorem and using Christopher's Theorem. □

5 Quadratic polynomial vector fields

In 1958 Qin Yuan–Xun [35] proved that quadratic (polynomial) vector fields can have algebraic limit cycles of degree 2, and when such a limit cycle exists then it is the unique limit cycle of the system.

Evdokimenco in [15–17] proved that quadratic vector fields do not have algebraic limit cycles of degree 3, for two different shorter proofs see [6, 25].

In 1966 Yablonskii [34] found the first class of algebraic limit cycles of degree 4 inside the quadratic vector fields. The second class was found in 1973 by Filiptsov [18]. More recently two new classes has been found and in [7] the authors proved that there are no other algebraic limit cycles of degree 4 for quadratic vector fields. The uniqueness of these limit cycles has been proved in [5]. Some other results on the algebraic limit cycles of quadratic vector fields can be found in [27, 28].

Doing convenient birational transformation of the plane to quadratic vector fields having algebraic limit cycles of degree 4 in [7] the authors obtained algebraic limit cycle of degrees 5 and 6 for quadratic vector fields. Of course in general a birational transformation does not preserve the degree of the polynomial vector field.

Open problems 2. *The following questions related with the algebraic limit cycles of quadratic polynomial vector fields remain open, see for instance[25].*

- (i) *What is the maximum degree of an algebraic limit cycle of a quadratic polynomial vector field?*
- (ii) *Does there exist a chain of rational transformations of the plane (as in [7]) which gives examples of quadratic systems with algebraic limits cycles of arbitrary degree, or at least of degree larger than 6?*
- (iii) *Is 1 the maximum number of algebraic limit cycles that a quadratic system can have?*

6 Cubic polynomial vector fields

It is known that there are cubic polynomial vector fields having algebraic limit cycles of degrees 2 and 3, see for instance [23, 24]. In [29] we provide cubic systems having algebraic limit cycles of degrees 4, 5 and 6 respectively, and an example of a cubic system having two algebraic limit cycles.

Open problems 3. *The following questions related with the algebraic limit cycles of cubic polynomial vector fields remain open, see for instance [29].*

(i) *What is the maximum degree of an algebraic limit cycle of a cubic polynomial vector field?*
(ii) *Does there exist a chain of rational transformations of the plane (as in [7]) which gives examples of cubic polynomial vector fields with algebraic limits cycles of arbitrary degree, or at least of degree larger than* 6*?*
(iii) *Is* 2 *the maximum number of algebraic limit cycles that a cubic polynomial vector fields can have?*

7 Configurations of algebraic limit cycles

In 1900 Hilbert not only proposed in the second part of his 16–th problem (see [20]) to estimate a uniform upper bound for the number of limit cycles of all polynomial vector fields of a given degree, but he also asked about the possible distributions or configurations of the limit cycles in the plane. This last question has been solved using algebraic limit cycles.

A *configuration of limit cycles* is a finite set $C = \{C_1, \ldots, C_n\}$ of disjoint simple closed curves of the plane such that $C_i \cap C_j = \emptyset$ for all $i \neq j$.

Two configurations of limit cycles $C = \{C_1, \ldots, C_n\}$ and $C' = \{C'_1, \ldots, C'_m\}$ are *(topologically) equivalent* if there is a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h\left(\cup_{i=1}^n C_i\right) = \left(\cup_{i=1}^m C'_i\right)$. Of course for equivalent configurations of limit cycles C and C' we have that $n = m$.

We say that a polynomial vector field $\mathcal{X}$ *realizes* the configuration of limit cycles C if the set of all limit cycles of X is equivalent to C.

Theorem 6. *Let $C = \{C_1, \ldots, C_n\}$ be an arbitrary configuration of limit cycles. Then the configuration C is realizable with algebraic limit cycles by a polynomial vector field.*

This theorem is proved in [26]. Looking at the way in which it is proved you can provide an alternative proof using the Bautin–Christopher–Dolov–Kuzmin Theorem.

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Integrals and phase portraits of planar quadratic differential systems with invariant lines of total multiplicity four

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Abstract. In this article we consider the class $\mathbf{QSL_4}$ of all real quadratic differential systems $\dfrac{dx}{dt} = p(x,y)$, $\dfrac{dy}{dt} = q(x,y)$ with $\gcd(p,q) = 1$, having invariant lines of total multiplicity four and a finite set of singularities at infinity. We first prove that all the systems in this class are integrable having integrating factors which are Darboux functions and we determine their first integrals. We also construct all the phase portraits for the systems belonging to this class. The group of affine transformations and homotheties on the time axis acts on this class. Our Main Theorem gives necessary and sufficient conditions, stated in terms of the twelve coefficients of the systems, for the realization of each one of the total of 69 topologically distinct phase portraits found in this class. We prove that these conditions are invariant under the group action.

Mathematics subject classification: 34A26, 34C40, 34C14.
Keywords and phrases: Quadratic differential system, Poincaré compactification, algebraic invariant curve, affine invariant polynomial, configuration of invariant lines, phase portrait.

1 Introduction

We consider here real planar differential systems of the form

$$(S) \qquad \frac{dx}{dt} = p(x,y), \qquad \frac{dy}{dt} = q(x,y), \tag{1}$$

where $p,\ q \in \mathbb{R}[x,y]$, i.e. $p,\ q$ are polynomials in $x,\ y$ over $\mathbb{R}$, their associated vector fields

$$\tilde{D} = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y} \tag{2}$$

and differential equations

$$q(x,y)dx - p(x,y)dy = 0. \tag{3}$$

We call *degree* of a system (1) (or of a vector field (2) or of a differential equation (3)) the integer $\deg(S) = \max(\deg p, \deg q)$. In particular we call *quadratic* a differential system (1) with $\deg(S) = 2$.

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A system (1.1) is said to be integrable on an open set U of $\mathbb{R}^2$ if there exists a C^1 function $F(x,y)$ defined on U which is a first integral of the system, i.e. such that $\tilde{D}F(x,y)=0$ on U and which is nonconstant on any open subset of U. The cases of integrable systems are rare but as Arnold said in [2, p. 405] "...these integrable cases allow us to collect a large amount of information about the motion in more important systems...". In particular we indicate below how integrable systems play a role in the second part of Hilbert's 16th problem for polynomial differential systems.

There are several hard open problems on the class of all quadratic differential systems (1). Among them the most famous one is the second part of Hilbert's 16th problem which asks for the determination of the so called Hilbert number $H(2)$ for this class where

$$H(n)=\max\{LC(S)\,\big|\,\deg(S)=n\}$$

and $LC(S)$ is the number of limit cycles of the system (S). It is known that for any polynomial system (S), $LC(S)$ is finite. This is the so called individual finiteness theorem which was proved independently by Ilyashenko and Ecalle (see [12, 15]).

The class of quadratic differential systems possessing a singularity which is a center is formed by integrable systems on open sets of $\mathbb{R}^2$ which are complements of real invariant algebraic curves. These systems do not possess limit cycles but they turn out to be very important in the determination of $H(2)$ as perturbations of such systems could produce limit cycles. Furthermore we have evidence indicating that $H(2)$ could be linked to the number of limit cycles occurring in perturbations of the most degenerate ones of all quadratic systems with a center (which happen to have a rational first integral) as we explain below.

In [3] the authors studied the class of all quadratic systems possessing a second order weak focus. It is known that the maximum number of limit cycles occurring in systems in this class is two (see [32, 33]). In the bifurcation diagram drawn in [3] for this three parameter family of systems, modulo the action of the affine group and time rescaling, the maximum number of two limit cycles which one has for this class, occurs in perturbations of an quadratic system (S_0) with a center, which has a rational first integral foliating the plane into conic curves. In addition this system (S_0) has three invariant affine lines and its line at infinity is filled up with singularities. Although other systems in this class having this maximum number of two limit cycles could be far away in the parameter space from the particular degenerate system (S_0), their phase portraits are topologically equivalent with a small perturbations of (S_0). This indicates the importance of integrable systems having invariant algebraic curves (see Definition 3), even with a rational first integral, in the second part of Hilbert's 16th problem and adds to the motivation for studying such systems. However, such a study is interesting for its own sake being at crossroads of differential equations and algebraic geometry.

The simplest class of integrable quadratic systems due to the presence of invariant algebraic curves is the class of integrable quadratic systems due to the presence of invariant lines. The study of this class was initiated in articles [25, 27–29]. In particular it was shown in [29] that the above mentioned system (S_0) possesses invariant affine lines of total multiplicity three.

In this article we study the class $\mathbf{QSL_4}$ of all quadratic differential systems possessing invariant lines of total multiplicity four (including the line at infinity and including multiplicities of the lines). The study of $\mathbf{QSL_4}$ was initiated in [27] where we proved a theorem of classification for this class. This classification, which is taken modulo the action of the group of real affine transformations and time rescaling, is given in terms of algebraic invariants and comitants and also geometrically, using cycles on the complex projective plane and on the line at infinity. An important ingredient in this classification is the notion of configuration of invariant lines of a polynomial differential system.

Definition 1. We call configuration of invariant lines of a system (1) the set of all its (complex) invariant lines (which may have real coefficients), each endowed with its own multiplicity [25] and together with all the real singular points of this system located on these lines, each one endowed with its own multiplicity.

The goal of this article is to complete the study we began in [27]. More precisely in this work we

- prove that all systems in this class $\mathbf{QSL_4}$ are integrable. We show this by using the geometric method of integration of Darboux. We construct explicit Darboux integrating factors and we give the list of first integrals for each system in this class;
- construct all topologically distinct phase portraits of the systems in this class;
- give invariant (under the action of the group $\mathrm{Aff}\,(2,\mathbb{R}) \times \mathbb{R}^*$)) necessary and sufficient conditions, in terms of the twelve coefficients of the systems, for the realization of each specific phase portrait.

This article is organized as follows:

In Section 2 we give the preliminary definitions and results needed in this article. These are mainly of a differential-algebraic nature.

In Section 3 we associate to each real quadratic system (1) possessing invariant lines with corresponding multiplicities, a divisor on the complex projective plane which encodes this information. We also define several integer-valued affine invariants of such systems using divisors on the line at infinity or zero-cycles on $\mathbb{P}_2(\mathbb{C})$ defined in [25] and [27], which encode the multiplicities of the singularities of the systems. We also state Theorem 5 which was proved in [27] illustrating how these cycles are useful for classification purposes. This theorem lists all possible configurations of invariant lines of total multiplicity four of the systems under study.

In Section 4 we prove the integrability of the systems in this class by using their invariant lines with their multiplicities. The main result in this Section states that all these systems have either a polynomial inverse integrating factor which splits into linear factors over $\mathbb{C}$ or a Darboux integrating factor which is a product of powers of the polynomials defining their invariant lines and an exponential factor $e^{G(x,y)}$ with

G a rational function over $\mathbb{C}$. The result is summed up in Table 1 where all these integrating factors are listed along with the first integrals, some of which but not all are Darboux functions.

In Section 5 we construct the phase portraits of the systems in this class and state our Main Theorem which gives necessary and sufficient conditions, invariant under the group action, for the realization of each one of the total of 69 topologically distinct phase portraits obtained for this class, in terms of the twelve coefficients of the systems.

2 Preliminaries

In this Section we give the basic notions and results needed in this paper. We are concerned here with the integrability in the sense of Darboux [10] of systems (1) possessing invariant straight lines of total multiplicity four. We work with the notion of multiplicity of an invariant line introduced by us in [25].

In [10] Darboux gave a geometric method of integration of planar complex differential equations (3) using invariant algebraic curves of the equations (see Definition 3). Each real differential system (1) generates a complex differential system when the variables range over $\mathbb{C}$. For this reason the method of Darboux can be applied also for real systems.

Poincaré was enthusiastic about the work of Darboux [10], which he called "admirable" in [19]. This method of integration was applied to give unified proofs of integrability for several families of systems (1). For example in [24] it was applied to show in a unified way (unlike previous proofs which used ad hoc methods) the integrability of planar quadratic systems possessing a center.

A brief and easily accessible exposition of the method of Darboux can be found in the survey article [23].

The topic of Darboux' paper [10] is best treated using the language of differential algebra, subject which started to be developed in the work of Ritt [1893–1951], long after Darboux wrote his paper [10]. The term *"Differential Algebra"* was introduced by Ellis Kolchin, who as Buium and Cassidy said in [6], "deepened and modernized differential algebra and developed differential algebraic geometry and differential algebraic groups". According to Ritt, differential algebra began to be developed in the 1930's (e.g.[21]) under the influence of Emmy Noether's work of the 1920's in algebra. (In his book [22] Ritt said: "the form in which the results of differential algebra are presented has been deeply influenced by her teachings".)

Whenever a definition below is given for a system (1) or equivalently for a vector field (2), this definition could also be given for an equation (3) and viceversa. For brevity we sometimes state only one of the possibilities.

An integrating factor of an equation (3) on an open subset U of $\mathbb{R}^2$ is usually defined as a C^1 function $R(x,y) \not\equiv 0$ such that the 1-form

$$\omega = R\,q(x,y)dx - R\,p(x,y)dy$$

is exact, i.e. there exist a C^1 function $F : U \longrightarrow \mathbb{K}$ on U such that

$$\omega = dF. \tag{4}$$

If R is an integrating factor on U of (3) then the function F such that $\omega = Rqdx - Rpdy = dF$ is a first integral of the equation $w = 0$ (or a system (1)). In this case we necessarily have on U:

$$\frac{\partial(Rq)}{\partial y} = -\frac{\partial(Rp)}{\partial x} \tag{5}$$

and developing the above equality we obtain $\dfrac{\partial R}{\partial x}p + \dfrac{\partial R}{\partial y}q = -R\Big(\dfrac{\partial p}{\partial x} + \dfrac{\partial q}{\partial y}\Big)$ or equivalently

$$\tilde{D}R = -R \operatorname{div} \tilde{D}. \tag{6}$$

In view of Poincaré's lemma (see for example [31]), if $R(x, y)$ is a C^1 function on a star-shaped open set U of $\mathbb{R}^2$, then $R(x, y)$ is an integrating factor of (3) if and only if (5) (or equivalently (6)) holds on U. So for star shaped open sets U (6) can be taken as a definition of an integrating factor on U. This is sufficient for our purpose. We note that this last definition is much simpler than the one usually used in textbooks as it no longer involves an existential quantifier.

In this work we shall apply to our real quadratic system (1) the method of integration of Darboux which was developed for complex differential equations (3). This method uses multiple-valued complex functions of the form:

$$F = e^{G(x,y)} f_1(x, y)^{\lambda_1} \cdots f_s(x, y)^{\lambda_s}, \quad G \in \mathbb{C}(x, y), \quad f_i \in \mathbb{C}[x, y], \quad \lambda_i \in \mathbb{C}, \tag{7}$$

$G = G_1/G_2$, $G_i \in \mathbb{C}[x, y]$, f_i irreducible over $\mathbb{C}$. It is clear that in general an expression (7) makes sense only for $G_2 \neq 0$ and for points $(x, y) \in \mathbb{C}^2 \setminus (\{G_2(x, y) = 0\} \cup \{f_1(x, y) = 0\} \cup \cdots \cup \{f_s(x, y) = 0\})$.

The above expression (7) yields a multiple-valued function on

$$\mathcal{U} = \mathbb{C}^2 \setminus (\{G_2(x, y) = 0\} \cup \{f_1(x, y) = 0\} \cup \cdots \cup \{f_s(x, y) = 0\}).$$

The function F in (7) belongs to a differential field extension of $\Big(\mathbb{C}(x, y), \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\Big)$ obtained by adjoining to $\mathbb{C}(x, y)$ a finite number of algebraic and of transcendental elements over $\mathbb{C}(x, y)$. For example $f(x, y)^{1/2}$ is an expression of the form (7), when $f \in \mathbb{C}[x, y] \setminus \{0\}$. This function belongs to the algebraic differential field extension $\Big(\mathbb{C}(x, y)[u], \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\Big)$ of $\Big(\mathbb{C}(x, y), \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\Big)$ obtained by adjoining to $\mathbb{C}(x, y)$ a root of the equation $u^2 - f(x, y) = 0$. In general, the expression (7) belongs to a differential field extension which is not necessarily algebraic. Indeed, for example this occurs if $G(x, y)$ is not a constant.

Definition 2. A function F in a differential field extension K of $\left(\mathbb{C}(x,y), \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\right)$ which is finite over $\mathbb{C}(x,y)$, is a first integral (integrating factor, respectively inverse integrating factor) of a complex differential system (1) or a vector field (2) or a differential equation (3) if $\tilde{D}F = 0$ ($\tilde{D}F = -F \operatorname{div} \tilde{D}$, respectively $\tilde{D}F = F \operatorname{div} \tilde{D}$).

In 1878 Darboux introduced the notion of invariant algebraic curve for differential equations on the complex projective plane. This notion can be adapted for equations (3) on $\mathbb{C}^2$ or equivalently for systems (1) or vector fields (2).

Definition 3 (Darboux [10])**.** An affine algebraic curve $f(x,y) = 0$, $f \in \mathbb{C}[x,y]$, $\deg f \geq 1$ is invariant for an equation (3) or for a system (1) if and only if $f \mid \tilde{D}f$ in $\mathbb{C}[x,y]$, i.e. $k = \dfrac{\tilde{D}f}{f} \in \mathbb{C}[x,y]$. In this case k is called the cofactor of f.

Definition 4 (Darboux [10])**.** An algebraic solution of an equation (3) (respectively (1), (2)) is an invariant algebraic curve $f(x,y) = 0$, $f \in \mathbb{C}[x,y]$ ($\deg f \geq 1$) with f an irreducible polynomial over $\mathbb{C}$.

Darboux showed that if an equation (3) or (1) or (2) possesses a sufficient number of such invariant algebraic solutions $f_i(x,y) = 0$, $f_i \in \mathbb{C}[x,y]$, $i = 1, 2, \ldots, s$ then the equation has a first integral of the form (7).

Definition 5. An expression of the form $F = e^{G(x,y)}$, $G(x,y) \in \mathbb{C}(x,y)$, i.e. G is a rational function over $\mathbb{C}$, is an exponential factor[1] for a system (1) or an equation (3) if and only if $k = \dfrac{\tilde{D}F}{F} \in \mathbb{C}[x,y]$. In this case k is called the cofactor of the exponential factor F.

Proposition 1 (Christopher [8])**.** *If an equation* (3) *admits an exponential factor* $e^{G(x,y)}$ *where* $G(x,y) = \dfrac{G_1(x,y)}{G_2(x,y)}$, $G_1, G_2 \in \mathbb{C}[x,y]$ *then* $G_2(x,y) = 0$ *is an invariant algebraic curve of* (3)*.*

Definition 6. We say that a system (1) or an equation (3) has a Darboux first integral (respectively Darboux integrating factor) if it admits a first integral (respectively integrating factor) of the form $e^{G(x,y)} \displaystyle\prod_{i=1}^{s} f_i(x,y)^{\lambda_i}$, where $G(x,y) \in \mathbb{C}(x,y)$ and $f_i \in \mathbb{C}[x,y]$, $\deg f_i \geq 1$, $i = 1, 2, \ldots, s$, f_i irreducible over $\mathbb{C}$ and $\lambda_i \in \mathbb{C}$. A system (1) or an equation (3) has a Liouvillian first integral (respectively a Liouvillian integrating factor) if it admits a first integral (respectively integrating factor) which is a Liouvillian function, i.e. a function which is built up from rational functions over $\mathbb{C}$ using exponentiation, integration and algebraic functions.

[1] Under the name *degenerate invariant algebraic curve* this notion was introduced by Christopher in [8].

Proposition 2 (Darboux [10]). *If an equation* (3) *(or* (1)*, or* (2)*) has an integrating factor (or first integral) of the form* $F = \prod_{i=1}^{s} f_i^{\lambda_i}$ *then* $\forall i \in \{1, \ldots, s\}$, $f_i = 0$ *is an algebraic invariant curve of* (3) *(*(1), (2)*).*

In [10] Darboux proved the following theorem of integrability using invariant algebraic solutions of differential equation (3):

Theorem 3 (Darboux [10]). *Consider a differential equation* (3) *with* $p, q \in \mathbb{C}[x, y]$. *Let us assume that* $m = \max(\deg p, \deg q)$ *and that the equation admits* s *algebraic solutions* $f_i(x, y) = 0$, $i = 1, 2, \ldots, s$ *(*$\deg f_i \geq 1$*). Then we have:*

I. If $s = m(m+1)/2$ *then there exists* $\lambda = (\lambda_1, \ldots, \lambda_s) \in \mathbb{C}^s \setminus \{0\}$ *such that* $R = \prod_{i=1}^{s} f_i(x, y)^{\lambda_i}$ *is an integrating factor of* (3).

II. If $s \geq m(m+1)/2 + 1$ *then there exists* $\lambda = (\lambda_1, \ldots, \lambda_s) \in \mathbb{C}^s \setminus \{0\}$ *such that* $F = \prod_{i=1}^{s} f_i(x, y)^{\lambda_i}$ *is a first integral of* (3).

Remark 1. We stated the theorem for the equation (3) but clearly we could have stated it for the vector field $\tilde{D}$ (2) or for the polynomial differential system (1). We recall that Darboux's work was done for differential equations in the complex projective plane. The above formulation is an adaptation of his theorem for the complex affine plane.

In [16] Jouanolou proved the following theorem which improves part II of Darboux's Theorem.

Theorem 4 (Jouanolou [16]). *Consider a polynomial differential equation* (3) *over* $\mathbb{C}$ *and assume that it has* s *algebraic solutions* $f_i(x, y) = 0$, $i = 1, 2, \ldots, s$ *(*$\deg f_i \geq 1$*). Suppose that* $s \geq m(m+1)/2 + 2$. *Then there exists* $(n_1, \ldots, n_s) \in \mathbb{Z}^s \setminus \{0\}$ *such that* $F = \prod_{i=1}^{s} f_i(x, y)^{n_i}$ *is a first integral of* (3). *In this case* $F \in \mathbb{C}(x, y)$, *i.e.* F *is rational function over* $\mathbb{C}$.

The above mentioned theorem of Darboux gives us sufficient conditions for integrability via the method of Darboux using algebraic solutions of systems (1). However these conditions are not necessary as it can be seen from the following example. The system

$$dx/dt = -y - x^2 - y^2, \quad dy/dt = x + xy$$

has two invariant algebraic curves: the invariant line $1 + y = 0$ and a conic invariant curve $f = 6x^2 + 3y^2 + 2y - 1 = 0$. This system is integrable having as first integral $F = (1 + y)^2 f$ but here $s = 2 < 3 = m(m+1)/2$.

Other sufficient conditions for Darboux integrability were obtained by Christopher and Kooij in [17] and Zoladek in [34]. Their theorems say that if a system has s invariant algebraic solutions in "generic position" (with "generic" as defined in their papers) such that $\sum_{i=1}^{s} \deg f_i = m + 1$ then the system has an inverse integrating factor of the form $\prod_{i=1}^{s} f_i$. But their theorem does not cover the above case as the two curves are not in "generic position". Indeed, the line $1 + y = 0$ is tangent to the curve $f = 0$ at $(0, -1)$. For similar reasons the above example is not covered by the more general result: Theorem 7.1 in [9]. Other sufficient conditions for integrability covering the example above were given in [7]. However we do not have necessary and sufficient conditions for Darboux integrability and the search is on for finding such conditions.

Problem resulting from the work [10] of Darboux: *Give necessary and sufficient conditions for a polynomial system (1.1) to have: (i) a polynomial inverse integrating factor; (ii) an integrating factor of the form* $\prod_{i=1}^{s} f_i(x,y)^{\lambda_i}$*; (iii) a Darboux integrating factor (or a Darboux first integral); (iv) a rational first integral.*

The last problem *(iv)* above, was stated in 1891 in the articles [19] and [20] of Poincaré where it was called the problem of algebraic integrability of the equations. In recent years there has been much activity in this area of research.

One of the goals of this work is to provide us with specific data to be used along with similar material for higher degree curves, for the purpose of dealing with questions regarding Darboux and algebraic integrability. We collect here in a systematic way information on quadratic systems having invariant lines of exactly four total multiplicity.

This material may also be used in studying quadratic systems which are small perturbations of integrable ones. In fact, as we have already indicated in the introduction, the maximum number of limit cycles of some subclasses of the quadratic class can be obtained by perturbing integrable systems having a rational first integral and invariant lines.

This article forms the basis for the study of some moduli spaces of quadratic systems, under the group action. One such moduli space which we intend to study in a following article is the moduli space of the closure within the quadratic class of the class $\mathbf{QSL_4}$.

3 Divisors associated to configurations of invariant lines

Consider real differential systems of the form:

$$(S)\qquad \begin{cases} \dfrac{dx}{dt} &= p_0 + p_1(x,y) + \ p_2(x,y) \equiv p(x,y),\\ \dfrac{dx}{dt} &= q_0 + q_1(x,y) + \ q_2(x,y) \equiv q(x,y) \end{cases} \tag{8}$$

with

$$\begin{aligned} &p_0 = a_{00}, \quad p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2,\\ &q_0 = b_{00}, \quad q_1(x,y) = b_{10}x + b_{01}y, \quad\ q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{aligned}$$

Let $a = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})$ be the 12-tuple of the coefficients of system (8) and denote $\mathbb{R}[a,x,y] = \mathbb{R}[a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02},\ b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}, x, y]$.

Notation 1. *Whenever we refer to some specific point in* $\mathbb{R}^{12}$ *rather than a 12-tuple parameter we shall denote such a point in* $\mathbb{R}^{12}$ *by* $\boldsymbol{a} = (\boldsymbol{a}_{00}, \boldsymbol{a}_{10} \ldots, \boldsymbol{b}_{02})$*. Each particular system* (8) *yields an ordered 12-tuple* $\boldsymbol{a}$ *of its coefficients.*

Notation 2. *Let*

$$\begin{aligned} P(X,Y,Z) =& p_0(\boldsymbol{a})Z^2 + p_1(\boldsymbol{a},X,Y)Z + p_2(\boldsymbol{a},X,Y) = 0,\\ Q(X,Y,Z) =& q_0(\boldsymbol{a})Z^2 + q_1(\boldsymbol{a},X,Y)Z + q_2(\boldsymbol{a},X,Y) = 0. \end{aligned}$$

We denote $\quad \sigma(P,Q) = \{w \in \mathbb{P}_2(\mathbb{C}) \mid P(w) = Q(w) = 0\}$.

Definition 7. We consider formal expressions $\mathbf{D} = \sum n(w)w$ where $n(w)$ is an integer and only a finite number of $n(w)$ in $\mathbf{D}$ are nonzero. Such an expression is called: *i)* a zero-cycle of $\mathbb{P}_2(\mathbb{C})$ if all w appearing in $\mathbf{D}$ are points of $\mathbb{P}_2(\mathbb{C})$; *ii)* a divisor of $\mathbb{P}_2(\mathbb{C})$ if all w appearing in $\mathbf{D}$ are irreducible algebraic curves of $\mathbb{P}_2(\mathbb{C})$; *iii)* a divisor of an irreducible algebraic curve $\mathfrak{C}$ in $\mathbb{P}_2(\mathbb{C})$ if all w in $\mathbf{D}$ are points of the curve $\mathfrak{C}$. We call degree of the expression $\mathbf{D}$ the integer $\deg(\mathbf{D}) = \sum n(w)$. We call support of $\mathbf{D}$ the set $\mathrm{Supp}\,(\mathbf{D})$ of all w appearing in $\mathbf{D}$ such that $n(w) \neq 0$.

Definition 8. We say that an invariant affine straight line $\mathcal{L}(x,y) = ux+vy+w = 0$ (respectively the line at infinity $Z = 0$) for a quadratic vector field $\tilde{D}$ has multiplicity m if there exists a sequence of real quadratic vector fields $\tilde{D}_k$ converging to $\tilde{D}$, such that each $\tilde{D}_k$ has m (respectively $m-1$) distinct invariant affine straight lines $\mathcal{L}_i^j = u_i^j x + v_i^j y + w_i^j = 0$, $(u_i^j, v_i^j) \neq (0,0)$, $(u_i^j, v_i^j, w_i^j) \in \mathbb{C}^3$, converging to $\mathcal{L} = 0$ as $k \to \infty$ (with the topology of their coefficients), and this does not occur for $m+1$ (respectively m).

Notation 3. *Let us denote by*

$$\begin{aligned} \mathbf{QS} \;&=\; \left\{ (S) \;\middle|\; \begin{array}{l} (S) \textit{ is a real system } (1) \textit{ such that } \gcd(p(x,y),q(x,y)) = 1 \\ \textit{and} \quad \max\big(\deg(p(x,y)), \deg(q(x,y))\big) = 2 \end{array} \right\};\\ \mathbf{QSL} \;&=\; \left\{ (S) \in \mathbf{QS} \;\middle|\; \begin{array}{l} (S) \textit{ possesses at least one invariant affine line or} \\ \textit{the line at infinity has multiplicity at least two} \end{array} \right\}. \end{aligned}$$

In this section we shall assume that systems (8) belong to **QS**.

We define below the geometrical objects (divisors or zero-cycles) which play an important role in constructing the invariants of the systems.

Definition 9.

$$\begin{aligned} \mathbf{D}_{S}(P,Q) \;&=\; \sum_{w\in\sigma(P,Q)} I_w(P,Q)w;\\ \mathbf{D}_{S}(P,Q;Z) \;&=\; \sum_{w\in\{Z=0\}} I_w(P,Q)w;\\ \widehat{\mathbf{D}}_{S}(P,Q,Z) \;&=\; \sum_{w\in\{Z=0\}} \Big(I_w(C,Z),\, I_w(P,Q)\Big)w \quad \text{if} \quad Z \nmid C(X,Y,Z);\\ \mathbf{D}_{S}(C,Z) \;&=\; \sum_{w\in\{Z=0\}} I_w(C,Z)w \quad \text{if} \quad Z \nmid C(X,Y,Z), \end{aligned}$$

where $C(X,Y,Z) = YP(X,Y,Z) - XQ(X,Y,Z)$, $I_w(F,G)$ is the intersection number (see [11]) of the curves defined by homogeneous polynomials $F,\ G \in \mathbb{C}[X,Y,Z]$, $\deg(F), \deg(G) \geq 1$ and $\{Z=0\} = \{[X:Y:0] \mid (X,Y) \in \mathbb{C}^2 \setminus (0,0)\}$.

We denote by $\#A$ the number of points of a finite set A.

Notation 4.

$$n_{\mathbb{R}}^{\infty} = \#\{w \in Supp\, \mathbf{D}_S(C,Z) \,\big|\, w \in \mathbb{P}_2(\mathbb{R})\}.$$

A complex projective line $uX + vY + wZ = 0$ is invariant for the system (S) if either it coincides with $Z = 0$ or it is the projective completion of an invariant affine line $ux + vy + w = 0$.

Notation 5. *Let* $(S) \in \mathbf{QSL}$*. Let us denote*

$$\mathbf{IL}(S) = \left\{ \ l \ \left| \begin{array}{c} l \ \textit{is a line in } \mathbb{P}_2(\mathbb{C}) \textit{ such} \\ \textit{that } l \textit{ is invariant for } (S) \end{array} \right. \right\};$$

$$M(l) = \textit{the multiplicity of the invariant line } l \textit{ of } (S).$$

Remark 2. We note that the line $l_\infty : Z = 0$ is included in $\mathbf{IL}(S)$ for any $(S) \in \mathbf{QS}$.

Let $l_i\ :\ f_i(x,y) = 0$, $i = 1, \ldots, k$, be all the distinct invariant affine lines (real or complex) of a system $(S) \in \mathbf{QSL}$, in case they exist. Let $l_i'\ :\ \mathcal{F}_i(X,Y,Z) = 0$ be the complex projective completion of l_i.

Notation 6. *We denote*

$$\mathcal{G}\ :\quad \prod_i \mathcal{F}_i(X,Y,Z)\, Z = 0; \qquad Sing\, \mathcal{G} = \{w \in \mathcal{G} |\ w \textit{ is a singular point of } \mathcal{G}\};$$

$$\nu(w) = \textit{ the multiplicity of the point } w, \textit{ as a point of } \mathcal{G}.$$

Definition 10.

$$\begin{aligned} \mathbf{D}_{\mathbf{IL}}(S) &= \sum_{l \in \mathbf{IL}(S)} M(l) l, \quad (S) \in \mathbf{QSL}; \\ \operatorname{Supp} \mathbf{D}_{\mathbf{IL}}(S) &= \{\, l \mid l \in \mathbf{IL}(S)\}. \end{aligned}$$

Notation 7.

$$\begin{aligned} M_{\mathbf{IL}}(S) &= \deg \mathbf{D}_{\mathbf{IL}}(S); \\ N_{\mathbb{C}}(S) &= \#\operatorname{Supp} \mathbf{D}_{\mathbf{IL}}; \\ N_{\mathbb{R}}(S) &= \#\{l \in \operatorname{Supp} \mathbf{D}_{\mathbf{IL}} \,\big|\, l \in \mathbb{P}_2(\mathbb{R})\}; \\ n^{\mathbb{R}}_{\mathcal{G},\sigma}(S) &= \#\{\omega \in \operatorname{Supp} \mathbf{D}_S(P,Q) \mid \omega \in \mathcal{G}\big|_{\mathbb{R}^2}\}; \\ d^{\mathbb{R}}_{\mathcal{G},\sigma}(S) &= \sum_{\omega \in \mathcal{G}|_{\mathbb{R}^2}} I_\omega(P,Q); \\ m_{\mathcal{G}}(S) &= \max\{\nu(\omega) \mid \omega \in Sing\, \mathcal{G}\big|_{\mathbb{C}^2}\}; \\ m^{\mathbb{R}}_{\mathcal{G}}(S) &= \max\{\nu(\omega) \mid \omega \in Sing\, \mathcal{G}\big|_{\mathbb{R}^2}\}. \end{aligned} \tag{9}$$

For brevity we sometimes just write $M_{\mathbf{IL}}$, $N_{\mathbb{C}}$, ..., $m_{\mathcal{G}}^{\mathbb{R}}$.

In the following sections we shall prove the integrability of the quadratic differential systems having invariant lines of total multiplicity four, including the line at infinity and including possible multiplicities of the lines. Their possible configurations as well as invariant conditions with respect to the group action distinguishing these configurations were given in [27]. All possible such configurations for this class are found in Diagram 1 of Theorem 4.1 in [27]. This Theorem will be needed in the following sections so we reproduce it below. It also helps in illustrating how the concepts introduced in this section are used.

Notation 8. *We denote by* $\mathbf{QSL_4}$ *the class of all real quadratic differential systems* (8) *with* p, q *relatively prime* ($(p,q)=1$), $Z \nmid C$, *and possessing a configuration of invariant straight lines of total multiplicity* $M_{\mathbf{IL}} = 4$ *including the line at infinity and including possible multiplicities of the lines.*

Theorem 5. *(Schlomiuk and Vulpe [27]) The class* $\mathbf{QSL_4}$ *splits into 46 distinct subclasses indicated in Diagram 1 with the corresponding Configurations 4.1–4.46, where the complex invariant straight lines are indicated by dashed lines. If an invariant straight line has multiplicity* $k > 1$, *then the number* k *appears near the corresponding straight line and this line is in bold face. We indicate next to the real singular points of the systems, situated on the invariant lines, their multiplicities as follows:* $(I_\omega(p,q))$ *if* ω *is a finite singularity,* $(I_\omega(C,Z),\ I_\omega(P,Q))$ *if* ω *is an infinite singularity with* $I_w(P,Q) \neq 0$ *and* $(I_\omega(C,Z))$ *if* ω *is an infinite singularity with* $I_\omega(P,Q) = 0$.

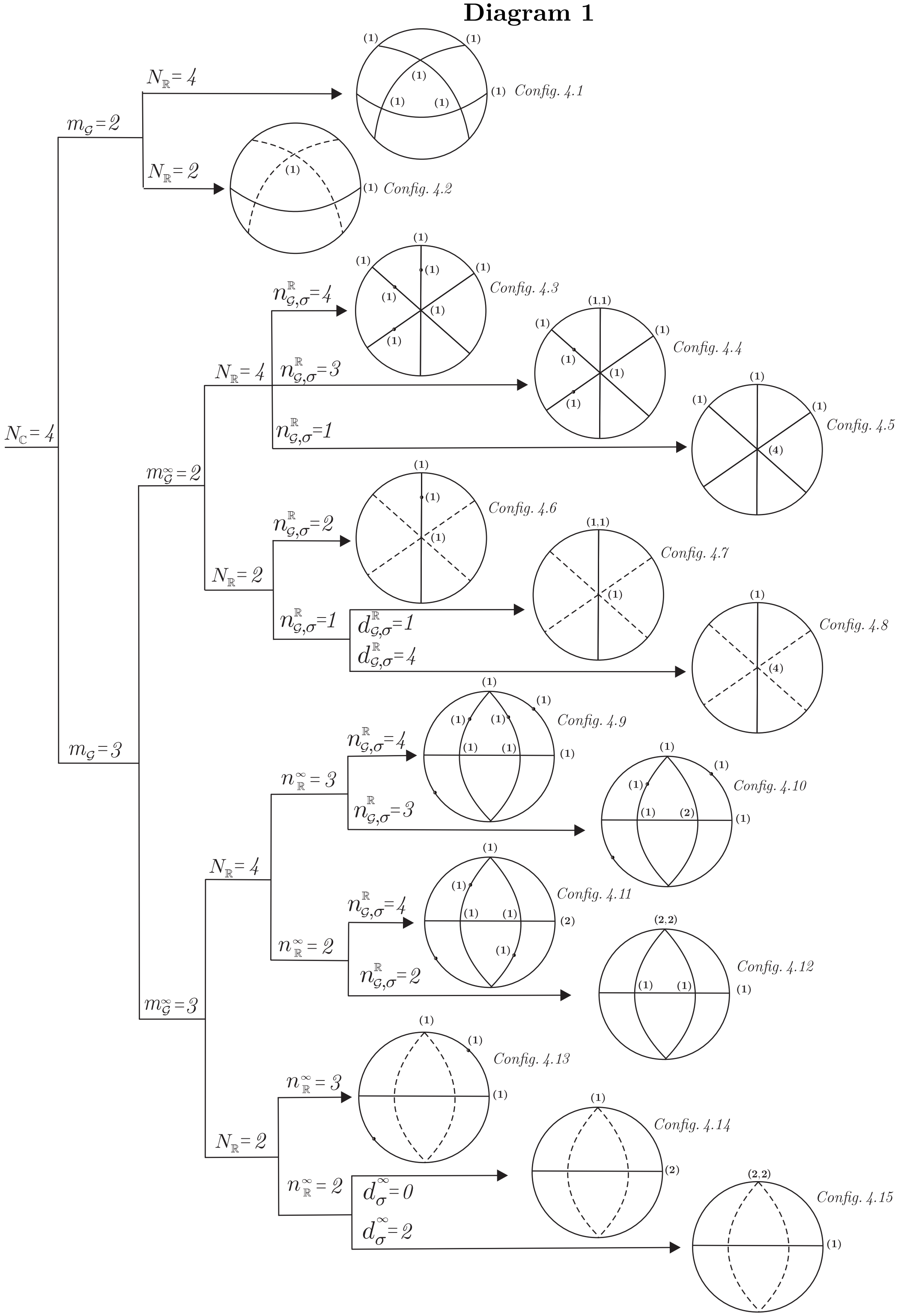
4 Integrability and phase portraits of the systems in the class of quadratic systems with total multiplicity four

4.1 Darboux integrating factors and first integrals

Theorem 6. *Consider a quadratic system* (8) *in* $\mathbf{QSL_4}$. *Then this system has either a polynomial inverse integrating factor which splits into linear factors over* $\mathbb{C}$ *or an integrating factor which is Darboux generating in the usual way a Liouvillian first integral. Out of 46 cases, 26 lead to Darboux integrals which produce, depending on the values of the parameters, 30 Darboux integrals. In the remaining cases the first integral involves special functions such as for example Hypergeometric functions, or Appell or Beta functions, etc. Furthermore the quotient set of* $\mathbf{QSL_4}$ *under the action of the affine group and time rescaling is formed by: (i) a set of 20 orbits; (ii) a set of twenty-three one-parameter families of orbits and (iii) a set of ten two-parameter families of orbits. A system of representatives of the quotient space is given in Table 1. This table also lists the corresponding cofactors of the lines as well as the inverse integrating factors and first integrals of the systems.*

Proof of Theorem 6. In [27] we obtained a total of 46 canonical forms for all the systems in the class $\mathbf{QSL_4}$. They correspond to the 46 possible configurations

Diagram 1



$N_{\mathbb{C}}=4$
$m_{\mathcal{G}}=2$
$N_{\mathbb{R}}=4$
$N_{\mathbb{R}}=2$
Config. 4.1
Config. 4.2
$m_{\mathcal{G}}^{\infty}=2$
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=4$
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=3$
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=1$
Config. 4.3
Config. 4.4
Config. 4.5
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=2$
$d_{\mathcal{G},\sigma}^{\mathbb{R}}=1$
$d_{\mathcal{G},\sigma}^{\mathbb{R}}=4$
Config. 4.6
Config. 4.7
Config. 4.8
$m_{\mathcal{G}}=3$
$m_{\mathcal{G}}^{\infty}=3$
$n_{\mathbb{R}}^{\infty}=3$
$n_{\mathbb{R}}^{\infty}=2$
Config. 4.9
Config. 4.10
Config. 4.11
Config. 4.12
Config. 4.13
$d_{\sigma}^{\infty}=0$
$d_{\sigma}^{\infty}=2$
Config. 4.14
Config. 4.15

Diagram 1 *(continued)*

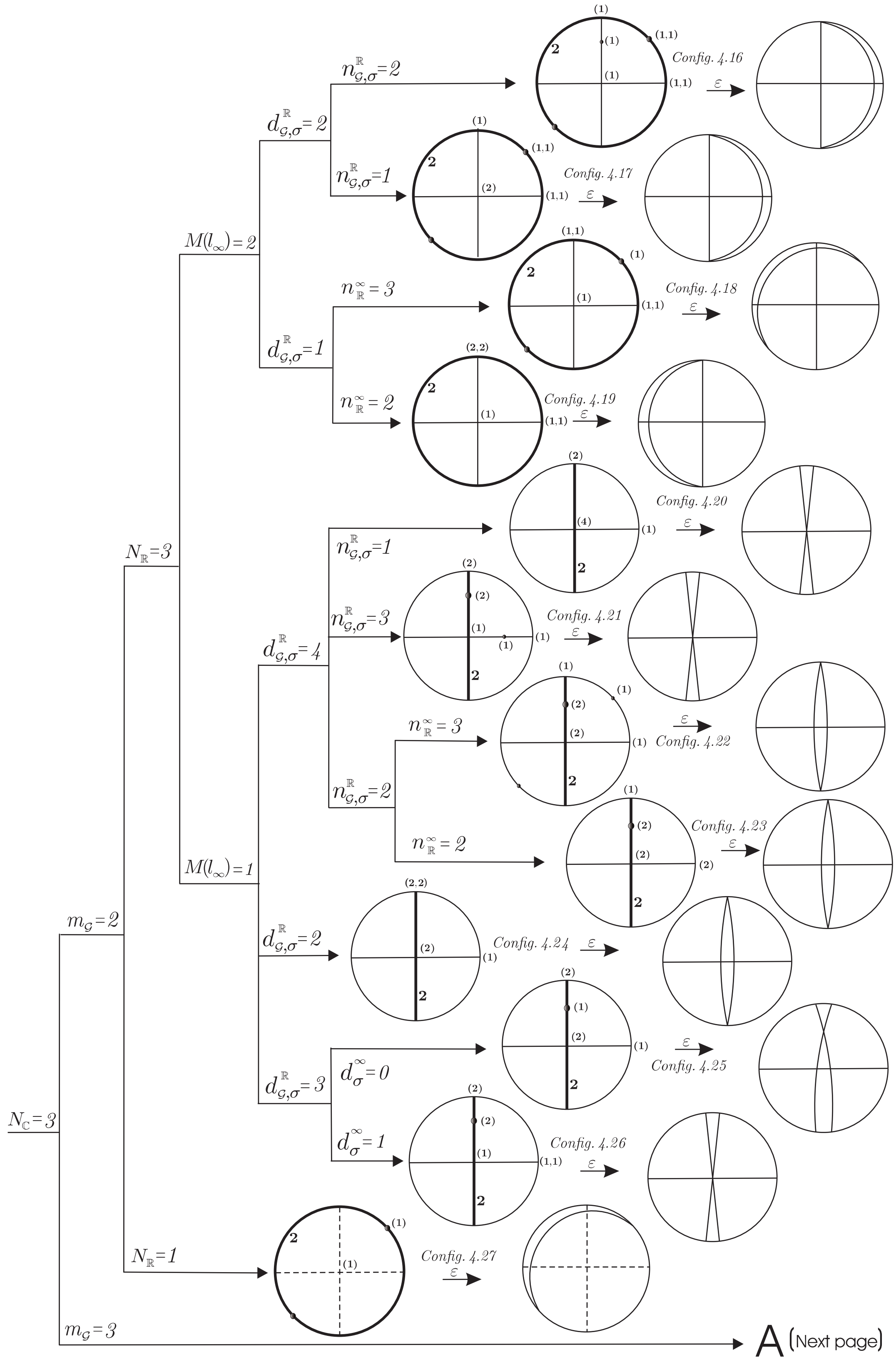


Diagram 1 *(continued)*

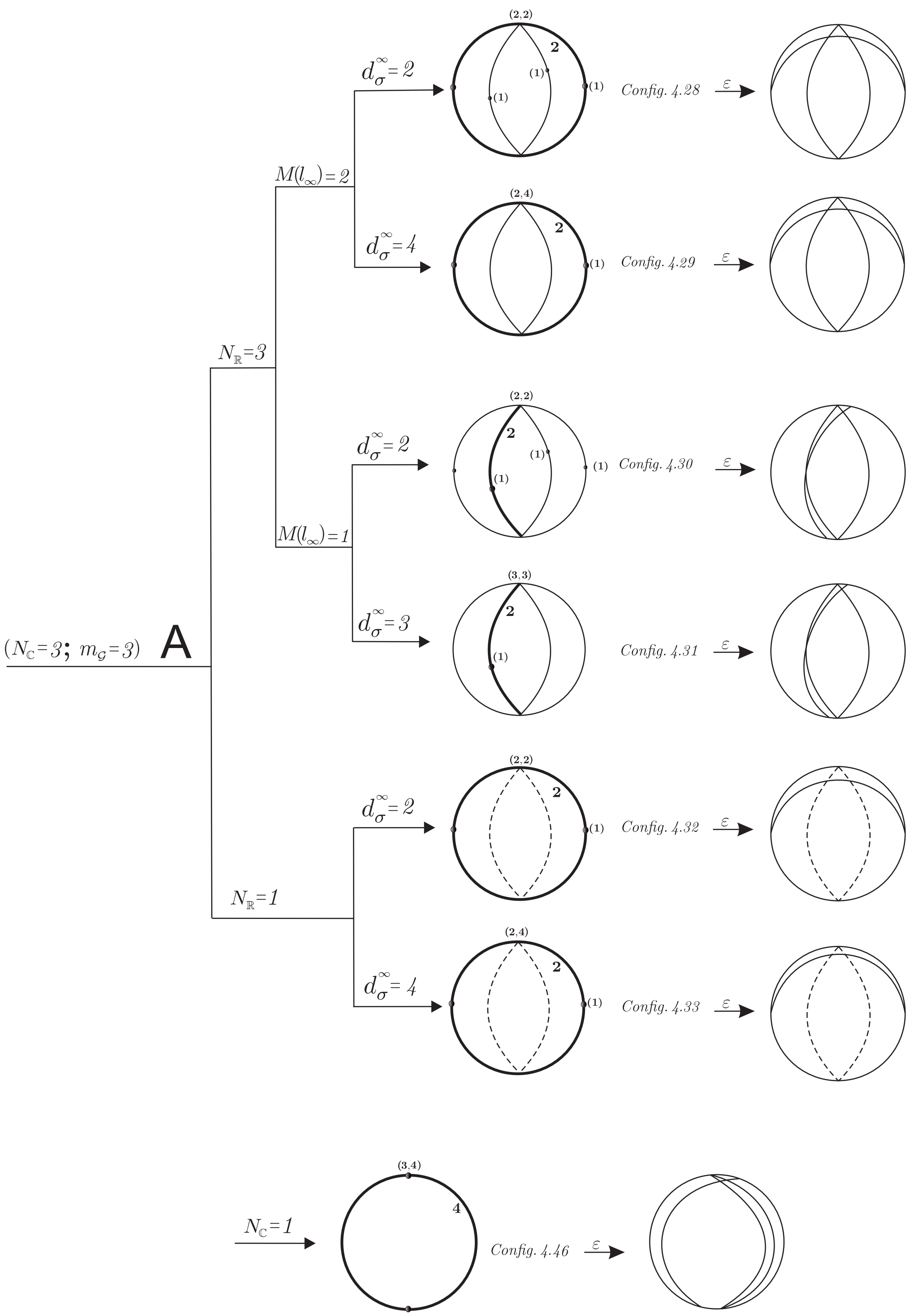
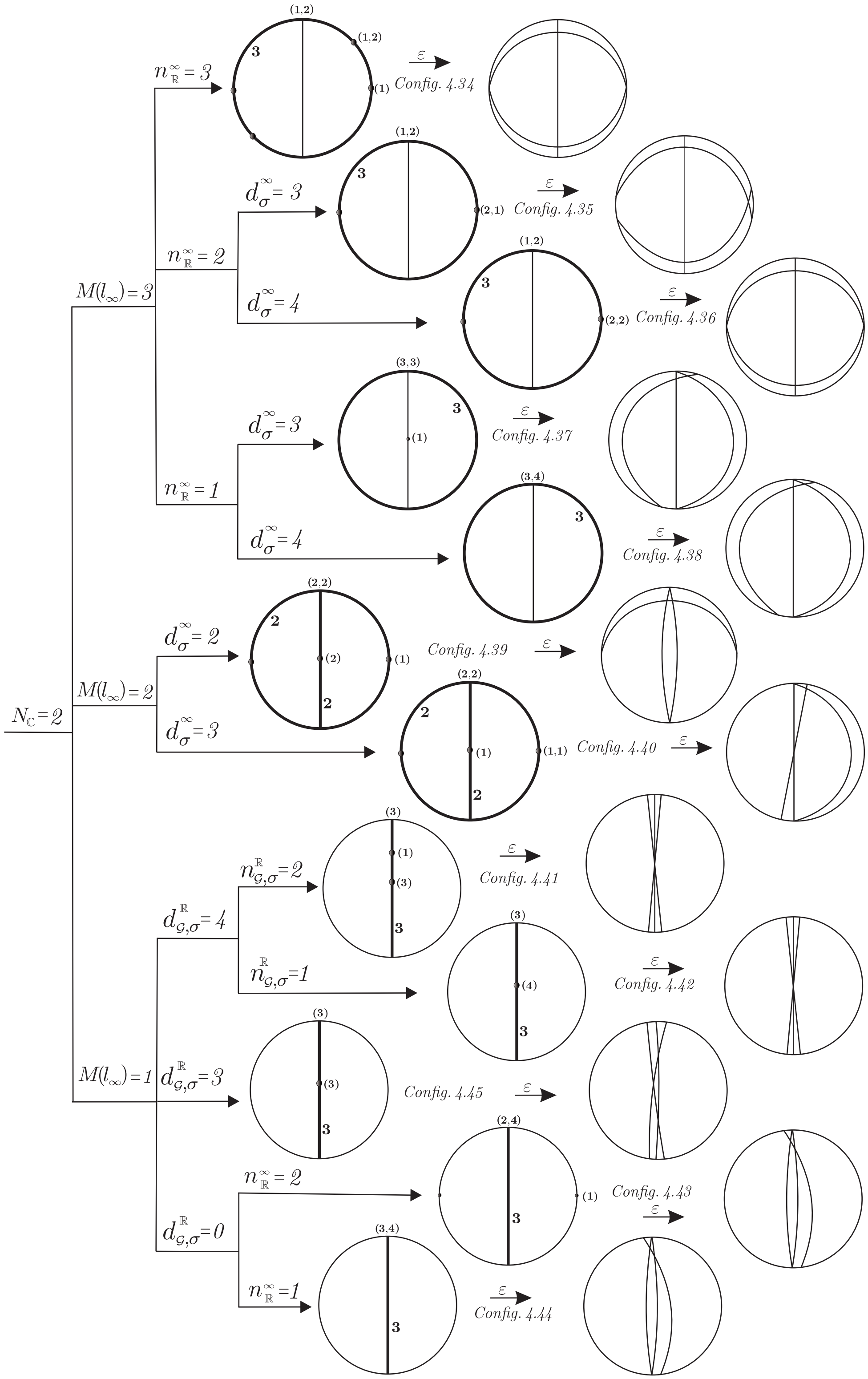


Diagram 1 *(continued)*



of invariant lines listed in Diagram 1. We take each one of these canonical forms, check their invariant lines with their respective multiplicities and determine their cofactors. As Darboux' work showed, these are instrumental in determining the integrating factors by showing linear dependence over $\mathbb{C}$ of the cofactors (of the invariant lines or of the exponential factors) together with the divergence of the vector field. Once the integrating factor is found one proceeds in the usual way to integrate the resulting differential equation (see Section 2). This integration can be done using MAPLE or MATHEMATICA. The calculations for the 46 cases considered yield the results given in **Table 1**.

$$\mathcal{F}_1 = x^h y^g (1+x-y)^{1-g-h};$$

$$\mathcal{F}_2 = x^{-2h}\Big[(x-h-1)+i(y+g)\Big]^{h+1-ig}\Big[(x-h-1)-i(y+g)\Big]^{h+1+ig};$$

$$\widetilde{\mathcal{F}}_2 = x^{-2h}\Big[(x-h-1)^2+(y+g)^2)\Big]^{h+1}\exp\Big[2g\,\mathrm{ArcTan}\,\frac{y+g}{x-h-1}\Big];$$

$$\mathcal{F}_3 = -x^h y^g (x-y)^{-(g+h)}\Big(\frac{y}{x}\Big)^{-g}\Big(1-\frac{y}{x}\Big)^{g+h}\Big[(1+gx)\,\mathrm{Beta}\Big[\frac{y}{x},\,g,\,1-g-h\Big]+$$
$$+(h-1)x\,\mathrm{Beta}\Big[\frac{y}{x},\,g+1,\,1-g-h\Big]+\int_{\omega_0}^{x}\Psi_3(\omega)d\omega,$$

$$\text{where}\quad \Psi_3(x) = x^{h-1}y^g(x-y)^{-(g+h)}\Bigg[y-x+x\Big(\frac{y}{x}\Big)^{-g}\Big(1-\frac{y}{x}\Big)^{g+h}$$
$$\Big[g\,\mathrm{Beta}\Big[\frac{y}{x},\,g,\,1-g-h\Big]+(h-1)\,\mathrm{Beta}\Big[\frac{y}{x},\,g+1,\,1-g-h\Big]\Big]\Bigg];\quad \frac{\partial}{\partial y}\Psi_3=0;$$

$$\mathcal{F}_4 = \Big(\frac{y}{x-y}\Big)^g\Big[g(x-y)+\Big(\frac{x-y}{x}\Big)^g\mathrm{Hypergeometric2F1}\Big[g,\,g,\,g+1,\,\frac{y}{x}\Big]\Big]\ (g\neq-1);$$

$$\widetilde{\mathcal{F}}_4 = xy^{-1}\exp\Big[\frac{(y-x)(y-x+1)}{y}\Big]\quad\text{for } g=-1;$$

$$\mathcal{F}_5 = x^h y^g (y-x)^{1-g-h};$$

$$\mathcal{F}_6 = -x^{-h}\int \mathcal{E}(x,y)\mathcal{H}(x,y)^{(h-1)/2}\big[gx+(h+1)y\big]dy+\int_{\omega_0}^{x}\Psi_6(\omega)d\omega,\ \text{where}$$
$$\Psi_6(x) = \mathcal{E}(x,y)\mathcal{H}(x,y)^{(h-1)/2}x^{-1-h}\big[(gx+hy-1)(y+1)-x^2\big]+$$
$$+x^{-h}\int \mathcal{E}(x,y)\mathcal{H}(x,y)^{(h-3)/2}\big[ghx^2+(h^2-1)xy-g(y+1)(gx+hy-1)\big]dy-$$
$$-hx^{-1-h}\int \mathcal{E}(x,y)\mathcal{H}(x,y)^{(h-1)/2}\big[gx+(h+1)y\big]dy,\qquad \frac{\partial}{\partial y}\Psi_6=0;$$
$$\mathcal{E}(x,y)=e^{-g\mathrm{ArcTan}[x/(1+y)]},\quad \mathcal{H}(x,y)=x^2+(y+1)^2;$$

$$\mathcal{F}_7 = \mathcal{F}_6\Big|_{h=0};$$

$$\mathcal{F}_8 = x^{-2h}(x+iy)^{h+1-ig}(x-iy)^{h+1+ig};$$

Table 1

Orbit representative	Invariant lines and their multiplicities	Inverse integrating factor $\mathcal{R}_i$
	Respective cofactors	First integral $\mathcal{F}_i$
1) $\begin{cases}\dot{x}=gx+gx^2+(h-1)xy,\\ \dot{y}=-hy+(g-1)xy+hy^2,\end{cases}$ $(g,h)\in\mathbb{R}^2$, $gh(g+h-1)\neq 0$, $(g-1)(h-1)(g+h)\neq 0$	$x\,(1)$, $y\,(1)$, $x-y+1\,(1)$	$\mathcal{R}_1=xy(x-y+1)$
	$g(x+1)+y(h-1)$, $x(g-1)+h(y-1)$, $gx+hy$	$\mathcal{F}_1$
2) $\begin{cases}\dot{x}=gx^2+(h+1)xy,\\ \dot{y}=h[g^2+(h+1)^2]+2ghy-x^2\\ \quad +(g^2+1-h^2)x+gxy+hy^2,\end{cases}$ $(g,h)\in\mathbb{R}^2$, $h(h+1)\neq 0$, $g^2+(h-1)^2\neq 0$	$x\,(1)$, $\pm i(y+g)+x-h-1\,(1)$	$\mathcal{R}_2=x\big[(y+g)^2+(x-h-1)^2\big]$
	$gx+(h+1)y$, $\mp i(x+h+h^2)+g(x+h)+hy$	$\mathcal{F}_2$, $\widetilde{\mathcal{F}}_2$
3) $\begin{cases}\dot{x}=x+gx^2+(h-1)xy,\\ \dot{y}=y+(g-1)xy+hy^2,\end{cases}$ $(g,h)\in\mathbb{R}^2$, $gh(g+h-1)\neq 0$, $(g-1)(h-1)(g+h)\neq 0$	$x\,(1)$, $y\,(1)$, $x-y\,(1)$	$\mathcal{R}_3=x^{1-h}y^{1-g}\times(x-y)^{g+h}$
	$gx+1+y(h-1)$, $x(g-1)+hy+1$, $gx+hy+1$	$\mathcal{F}_3$
4) $\begin{cases}\dot{x}=x+gx^2-xy,\\ \dot{y}=y+(g-1)xy,\end{cases}$ $g\in\mathbb{R}$, $g(g-1)\neq 0$	$x\,(1)$, $x-y\,(1)$, $y\,(1)$	$\mathcal{R}_4=xy^{1-g}(x-y)^g$
	$gx+1-y$, $gx+1$ $x(g-1)+1$	$\mathcal{F}_4$, $\widetilde{\mathcal{F}}_4$
5) $\begin{cases}\dot{x}=gx^2+(h-1)xy,\\ \dot{y}=(g-1)xy+hy^2,\end{cases}$ $(g,h)\in\mathbb{R}^2$, $gh(g+h-1)\neq 0$, $(g-1)(h-1)(g+h)\neq 0$	$x\,(1)$, $x-y\,(1)$, $y\,(1)$	$\mathcal{R}_5=xy(x-y)$
	$gx+y(h-1), gx+hy,$ $x(g-1)+hy$	$\mathcal{F}_5$
6) $\begin{cases}\dot{x}=gx^2+(h+1)xy,\\ \dot{y}=-1+gx+(h-1)y\\ \quad -x^2+gxy+hy^2,\ (g,h)\in\mathbb{R}^2,\end{cases}$ $h(h+1)\big[g^2+(h-1)^2\big]\neq 0$	$x\,(1)$, $\mathcal{I}_\pm=x\pm i(y+1)\,(1)$	$\mathcal{R}_6=\mathcal{I}_+^{(1-h-ig)/2}\times\mathcal{I}_-^{(1-h+ig)/2}x^{h+1}$
	$gx+(h+1)y$, $\pm ix+1+gx+hy$	$\mathcal{F}_6$
7) $\begin{cases}\dot{x}=gx^2+xy, \quad g\in\mathbb{R},\\ \dot{y}=-1+gx-y-x^2+gxy\end{cases}$	$x\,(1)$, $\mathcal{I}'_\pm=x\pm i(y+1)\,(1)$	$\mathcal{R}_7=\mathcal{I}'^{\,(1-ig)/2}_+\times\mathcal{I}'^{\,(1+ig)/2}_-\,x$
	$gx+y$, $\pm ix+1+gx$	$\mathcal{F}_7$
8) $\begin{cases}\dot{x}=gx^2+(h+1)xy,\ (g,h)\in\mathbb{R}^2,\\ \dot{y}=-x^2+gxy+hy^2,\end{cases}$ $h(h+1)\big[g^2+(h-1)^2\big]\neq 0$	$x\,(1)$, $x\pm iy\,(1)$	$\mathcal{R}_8=x(x^2+y^2)$
	$gx+(h+1)y$, $\pm ix+gx+hy$	$\mathcal{F}_8$, $\widetilde{\mathcal{F}}_8$

Table 1 (*continued*)

Orbit representative	Invariant lines and their multiplicities Respective cofactors	Inverse integrating factor $\mathcal{R}_i$ First integral $\mathcal{F}_i$
9) $\begin{cases}\dot{x}=x^2-1, \ (g,h)\in\mathbb{R}^2,\\ \dot{y}=(y+h)[y+(1-g)x-h],\end{cases}$ $g(g-1)[(g\pm1)^2-4h^2]\neq0$	$y+h\,(1)$, $\mathcal{I}''_{\pm}=x\pm1\,(1)$ $x(1-g)+y-h$, $x\mp1$	$\mathcal{R}_9=(y+h)^2\times \mathcal{I}''^{\,(g+1-2h)/2}_{+}\times \mathcal{I}''^{\,(g+1+2h)/2}_{-}$ $\mathcal{F}_9,\ \widetilde{\mathcal{F}}_9,\ \widehat{\mathcal{F}}_9,\ \mathcal{F}_9^*$
10) $\begin{cases}\dot{x}=x^2-1, \quad g\in\mathbb{R},\\ \dot{y}=(y+g)(y+2gx-g),\end{cases}$ $g(2g-1)\neq0$	$y+g\,(1)$, $x\pm1\,(1)$ $2gx+y-g$, $x\mp1$	$\mathcal{R}_{10}=(x+1)^{1-2g}\times (y+h)^2(x-1)$ $\mathcal{F}_{10},\ \widetilde{\mathcal{F}}_{10}$
11) $\begin{cases}\dot{x}=(x+g)^2-1, \quad g\in\mathbb{R},\\ \dot{y}=y(x+y), \qquad g\neq\pm1\end{cases}$	$y\,(1)$, $\mathcal{I}'''_{\pm}=x+ +g\pm1\,(1)$ $x+y$, $x+g\mp1$	$\mathcal{R}_{11}=\mathcal{I}'''^{\,(1-g)/2}_{+}\times \mathcal{I}'''^{\,(1+g)/2}_{-}y^2$ $\mathcal{F}_{11},\ \widetilde{\mathcal{F}}_{11}$
12) $\begin{cases}\dot{x}=(x+h)^2-1, \ (g,h)\in\mathbb{R}^2,\\ \dot{y}=(1-g)xy, \ g(g-1)\neq0,\end{cases}$ $(h^2-1)\big[\ h^2(g-1)^2-(g+1)^2\big]\neq0$	$y\,(1)$, $x+h\pm1\,(1)$ $x+y$, $x+h\mp1$	$\mathcal{R}_{12}=(x+h+1)\times (x+h-1)y$ $\mathcal{F}_{12}$
13) $\begin{cases}\dot{x}=x^2+1, \quad (g,h)\in\mathbb{R}^2,\\ \dot{y}=(y+h)[y+(1-g)x-h],\end{cases}$ $g(g-1)\big[(g+1)^2+h^2\big]\neq0$	$y+h\,(1)$, $x\pm i\,(1)$ $x(1-g)+y-h$, $x\mp i$	$\mathcal{R}_{13}=(y+h)^2\times (x+i)^{(1+g+2ih)/2}\times (x-i)^{(1+g-2ih)/2}$ $\mathcal{F}_{13},\ \widetilde{\mathcal{F}}_{13}$
14) $\begin{cases}\dot{x}=(x+g)^2+1,\\ \dot{y}=y(x+y), \quad g\in\mathbb{R}\end{cases}$	$y\,(1)$, $x+g\pm i\,(1)$ $x+y$, $x+g\mp i$	$\mathcal{R}_{14}=y^2\times (x+g+i)^{(1+ig)/2}\times (x+g-i)^{(1-ig)/2}$ $\mathcal{F}_{14},\ \widetilde{\mathcal{F}}_{14}$
15) $\begin{cases}\dot{x}=(x+h)^2+1,\\ \dot{y}=(1-g)xy, \quad (g,h)\in\mathbb{R}^2,\end{cases}$ $g(g-1)\big[(g+1)^2+h^2\big]\neq0$	$y\,(1)$, $x+h\pm i$, (1) x, $x+h\mp i$	$\mathcal{R}_{15}=y\times \big[(x+h)^2+1\big]$ $\mathcal{F}_{15},\ \widetilde{\mathcal{F}}_{15}$
16) $\begin{cases}\dot{x}=g+x, \qquad g\in\mathbb{R},\\ \dot{y}=y(y-x), \quad g(g-1)\neq0\end{cases}$	$x+g\,(1)$, $y\,(1)$ 1, $y-x$	$\mathcal{R}_{16}=e^xy^2\times (x+g)^{1-g}$ $\mathcal{F}_{16}$
17) $\begin{cases}\dot{x}=x,\\ \dot{y}=y(y-x)\end{cases}$	$x\,(1)$, $y\,(1)$ 1, $y-x$	$\mathcal{R}_{17}=x\,e^xy^2$ $\mathcal{F}_{17}$
18) $\begin{cases}\dot{x}=g(g+1)+gx+y, \ g\in\mathbb{R},\\ \dot{y}=y(y-x), \ g(g+1)\neq0\end{cases}$	$y\,(1)$, $x-y+g+1\,(1)$ $y-x$, $y+g$	$\mathcal{R}_{18}=y\times (x-y+g+1)$ $\mathcal{F}_{18}$

Table 1 (*continued*)

Orbit representative	Invariant lines and their multiplicities / Respective cofactors	Inverse integrating factor $\mathcal{R}_i$ / First integral $\mathcal{F}_i$
19) $\begin{cases}\dot{x}=g+x, & g\in\mathbb{R},\\ \dot{y}=-xy, & g(g-1)\neq 0\end{cases}$	$x+g\,(1),\quad y\,(1)$	$\mathcal{R}_{19}=y(x+g)$
	$1,\quad -x$	$\mathcal{F}_{19}$
20) $\begin{cases}\dot{x}=x(gx+y), & g\in\mathbb{R},\\ \dot{y}=(g-1)xy+y^2, & g(g-1)\neq 0\end{cases}$	$x\,(2),\quad y\,(1)$	$\mathcal{R}_{20}=x^2y$
	$gx+y,\ x(g-1)+y$	$\mathcal{F}_{20}$
21) $\begin{cases}\dot{x}=x(gx+y), & g(g-1)\neq 0,\\ \dot{y}=(y+1)(gx-x+y), & g\in\mathbb{R}\end{cases}$	$x\,(2),\quad y+1\,(1)$	$\mathcal{R}_{21}=x^{g+1}\times e^{-(gx+y+1)/x}\times (y+1)^{1-g}$
	$gx+y,\ x(g-1)+y$	$\mathcal{F}_{21},\ \widetilde{\mathcal{F}}_{21}$
22) $\begin{cases}\dot{x}=gx^2, & g\in\mathbb{R},\ g(g-1)\neq 0,\\ \dot{y}=(y+1)[y+(g-1)x-1]\end{cases}$	$x\,(2),\quad y+1\,(1)$	$\mathcal{R}_{22}=x^{(g+1)/g}\times (y+1)^2e^{-2/(gx)}$
	$x,\quad x(g-1)+y-1$	$\mathcal{F}_{22}$
23) $\begin{cases}\dot{x}=x^2+xy,\\ \dot{y}=(y+1)^2\end{cases}$	$x\,(1),\quad y+1\,(2)$	$\mathcal{R}_{23}=x^2(y+1)\times e^{-1/(y+1)}$
	$x+y,\ y+1$	$\mathcal{F}_{23}$
24) $\begin{cases}\dot{x}=(x+1)^2, & g\in\mathbb{R},\\ \dot{y}=(1-g)xy, & g(g-1)\neq 0\end{cases}$	$x+1\,(2),\quad y\,(1)$	$\mathcal{R}_{24}=(x+1)^2\,y$
	$x+1,\quad x$	$\mathcal{F}_{24}$
25) $\begin{cases}\dot{x}=gx^2+xy, & g(g-1)\neq 0,\\ \dot{y}=y+(g-1)xy+y^2, & g\in\mathbb{R}\end{cases}$	$x\,(2),\quad y\,(1)$	$\mathcal{R}_{25}=x^2\,y$
	$gx+y,\ x(g-1)+y+1$	$\mathcal{F}_{25}$
26) $\begin{cases}\dot{x}=xy,\\ \dot{y}=(y+1)(y-x)\end{cases}$	$x\,(2),\quad y\,(1)$	$\mathcal{R}_{26}=x(y+1)\times e^{-(y+1)/x}$
	$y,\quad y-x$	$\mathcal{F}_{26}$
27) $\begin{cases}\dot{x}=2gx+2y, & g\in\mathbb{R},\\ \dot{y}=g^2+1-x^2-y^2\end{cases}$	$y+g\pm i(x-1)\,(1)$	$\mathcal{R}_{27}=(x-1)^2+(y+g)^2$
	$g-y\pm i(x+1)$	$\mathcal{F}_{27},\ \widetilde{\mathcal{F}}_{27}$
28) $\begin{cases}\dot{x}=x^2-1, & g\in\mathbb{R},\\ \dot{y}=x+gy, & g(g^2-4)\neq 0\end{cases}$	$x+1\,(1),\ x-1\,(1)$	$\mathcal{R}_{28}=(x-1)^{1+g/2}\times (x+1)^{1-g/2}$
	$x-1,\ x+1$	$\mathcal{F}_{28}$
29) $\begin{cases}\dot{x}=x^2-1, & g\in\mathbb{R},\\ \dot{y}=g+x, & g\neq\pm 1\end{cases}$	$x+1\,(1),\ x-1\,(1)$	$\mathcal{R}_{29}=x^2-1$
	$x-1,\ x+1$	$\mathcal{F}_{29}$
30) $\begin{cases}\dot{x}=(x+1)(gx+1), & g\in\mathbb{R},\\ \dot{y}=1+(g-1)xy, & g(g^2-1)\neq 0\end{cases}$	$x+1\,(2),\ gx+1\,(1)$	$\mathcal{R}_{30}=(x+1)^2\times (gx+1)^{(g-1)/g}$
	$gx+1,\ x+1$	$\mathcal{F}_{30},\ \widetilde{\mathcal{F}}_{30},\ \widehat{\mathcal{F}}_{30}$

Table 1 (*continued*)

Orbit representative	Invariant lines and their multiplicities / Respective cofactors	Inverse integrating factor $\mathcal{R}_i$ / First integral $\mathcal{F}_i$
31) $\begin{cases}\dot{x}=x(x+1), & g\in\mathbb{R},\\ \dot{y}=g-x^2+xy, & g(g+1)\neq 0\end{cases}$	$x+1\,(2),\quad x\,(1)$	$\mathcal{R}_{31}=x(x+1)^2$
	$x,\quad x+1$	$\mathcal{F}_{31}$
32) $\begin{cases}\dot{x}=x^2+1, & g\in\mathbb{R},\\ \dot{y}=x+gy, & g\neq 0\end{cases}$	$x\pm i\,(1)$	$\mathcal{R}_{32}=(x+i)^{1+ig/2}\times(x-i)^{1-ig/2}$
	$x\mp i$	$\mathcal{F}_{32}$
33) $\begin{cases}\dot{x}=x^2+1, & g\in\mathbb{R},\\ \dot{y}=g+x\end{cases}$	$x\pm i\,(1)$	$\mathcal{R}_{33}=x^2+1$
	$x\mp i$	$\mathcal{F}_{33},\ \widetilde{\mathcal{F}}_{33}$
34) $\begin{cases}\dot{x}=g, & g\in\{-1,1\},\\ \dot{y}=y(y-x)\end{cases}$	$y\,(1)$	$\mathcal{R}_{34}=y^2\,e^{x^2/(2g)}$
	$y-x$	$\mathcal{F}_{34}$
35) $\dot{x}=g+y,\quad \dot{y}=xy,\quad g\in\{-1,1\}$	$y\,(1)$	$\mathcal{R}_{35}=y$
	x	$\mathcal{F}_{35}$
36) $\dot{x}=g,\quad \dot{y}=xy,\quad g\in\{-1,1\}$	$y\,(1)$	$\mathcal{R}_{36}=y$
	x	$\mathcal{F}_{36}$
37) $\begin{cases}\dot{x}=x, & g(g^2-1)\neq 0\\ \dot{y}=gy-x^2, & g\in\mathbb{R}\end{cases}$	$x\,(1)$	$\mathcal{R}_{37}=x^{g+1}$
	1	$\mathcal{F}_{37},\ \widetilde{\mathcal{F}}_{37}$
38) $\dot{x}=x,\quad \dot{y}=g-x^2,\quad 0\neq g\in\mathbb{R}$	$x\,(1)$	$\mathcal{R}_{38}=x$
	1	$\mathcal{F}_{38}$
39) $\dot{x}=x^2,\quad \dot{y}=x+y$	$x\,(2)$	$\mathcal{R}_{39}=x^2e^{-1/x}$
	x	$\mathcal{F}_{39}$
40) $\dot{x}=1+x,\quad \dot{y}=1-xy$	$x+1\,(2)$	$\mathcal{R}_{40}=(x+1)^2e^{-x}$
	1	$\mathcal{F}_{40}$
41) $\begin{cases}\dot{x}=gxy, & g\in\{-1,1\}\\ \dot{y}=y-x^2+gy^2\end{cases}$	$x\,(3)$	$\mathcal{R}_{41}=x^2e^{-g(y+g)^2/(2x^2)}$
	y	$\mathcal{F}_{41}$
42) $\begin{cases}\dot{x}=gxy, & g\in\{-1,1\}\\ \dot{y}=-x^2+gy^2\end{cases}$	$x\,(3)$	$\mathcal{R}_{42}=x^3$
	y	$\mathcal{F}_{42}$
43) $\begin{cases}\dot{x}=gx^2, & g(g^2-1)\neq 0\\ \dot{y}=1+(g-1)xy, & g\in\mathbb{R}\end{cases}$	$x\,(3)$	$g\neq\frac{1}{2}:\ \mathcal{R}_{43}=x^{2g}\times\big[1+(2g-1)xy\big]^{1-g};$ $g=\frac{1}{2}:\ \widetilde{\mathcal{R}}_{43}=x^3e^{-xy}$
	x	$\mathcal{F}_{43},\ \widetilde{\mathcal{F}}_{43}$

Table 1 (*continued*)

Orbit representative	Invariant lines and their multiplicities / Respective cofactors	Inverse integrating factor $\mathcal{R}_i$ / First integral $\mathcal{F}_i$
44) $\begin{cases}\dot x = x^2, \quad g\in\{-1,1\}\\ \dot y = g - x^2 + xy\end{cases}$	$x\ (3)$	$\mathcal{R}_{44} = x^3$
	x	$\mathcal{F}_{44}$
45) $\begin{cases}\dot x = gxy, \quad g\in\{-1,1\}\\ \dot y = x - x^2 + gy^2\end{cases}$	$x\ (3)$	$\mathcal{R}_{45} = x^3$
	y	$\mathcal{F}_{45}$
46) $\dot x = 1, \quad \dot y = y - x^2$	$-$	$\mathcal{R}_{46} = e^x$
	$-$	$\mathcal{F}_{46}$

$$\widetilde{\mathcal{F}}_8 = x^{-2h}(x^2+y^2)^{h+1}\exp\Big[2g\,\mathrm{ArcTan}\,\frac{y}{x}\Big];$$

$$\mathcal{F}_9 = (y+h)^{-1}(x^2-1)^{(1-g)/2}e^{2h\,\mathrm{ArcTanh}[x]} + \int_{\omega_0}^{x} e^{2h\,\mathrm{ArcTanh}[\omega]}(\omega^2-1)^{-(g+1)/2}d\omega,$$
$$\text{if}\quad h(g+1)\neq 0;$$

$$\widetilde{\mathcal{F}}_9 = \frac{(x+1)^h(x^2-1)}{(x-1)^h(y+h)} + 2\frac{(1-x)^h}{(x-1)^h}\,\mathrm{Beta}\Big[\frac{x+1}{2},\ h+1,\ 1-h\Big], \quad \text{for } \begin{cases} g=-1,\\ h\neq -1\end{cases};$$

$$\widehat{\mathcal{F}}_9 = (x+1)^{-2}\ \exp\Big[x + \frac{(x-1)^2}{y-1}\Big], \quad \text{for } g=h=-1;$$

$$\mathcal{F}_9^* = \frac{(x^2-1)^{(1-g)/2}}{y} + \frac{x(1-x^2)^{(1+g)/2}}{(x^2-1)^{(1+g)/2}}\,\mathrm{Hypergeometric2F1}\Big[\frac12,\ \frac{g+1}{2},\ \frac32,\ x^2\Big],$$
$$\text{for } h=0;$$

$$\mathcal{F}_{10} = (y+g)^{-1}(x^2-1)^g e^{2g\,\mathrm{ArcTanh}[x]} + \int_{\omega_0}^{x} e^{2g\,\mathrm{ArcTanh}[\omega]}(\omega^2-1)^{g-1}d\omega, \quad (g\neq 1);$$

$$\widetilde{\mathcal{F}}_{10} = (x-1)^2\ \exp\Big[x + \frac{(x+1)^2}{y+1}\Big], \quad \text{for } g=1;$$

$$\mathcal{F}_{11} = y^{-1}(x+g+1)^{1/2}(x+g-1)^{-1/2}e^{g\,\mathrm{ArcTanh}[x+g]}\Big[(g+1)(x+g-1)$$
$$+\,2y\,\mathrm{Hypergeometric2F1}\Big[1,\ \frac{g+1}{2},\ \frac{g+3}{2},\ \frac{x+g+1}{x+g-1}\Big]\Big], \quad (g\neq -3);$$

$$\widetilde{\mathcal{F}}_{11} = (x-2)\exp\Big[\frac{(x-4)^2+2y}{y(x-2)}\Big], \quad \text{for } g=-3;$$

$$\mathcal{F}_{12} = y^{2/(g-1)}(x+h+1)^{1+h}(x+h-1)^{1-h};$$

$$\mathcal{F}_{13} = (x^2+1)^{(1-g)/2}(y+h)^{-1}\exp\big[-2h\mathrm{ArcTan}\,x\big]+$$
$$\int_{\omega_0}^{x}(\omega^2{+}1)^{-(1+g)/2}\exp\big[-2h\,\mathrm{ArcTan}\,\omega\big]d\omega,\ \ (h{\neq}0);$$

$$\widetilde{\mathcal{F}}_{13} = y^{-1}\,(x^2+1)^{(1-g)/2} + x\ \mathrm{Hypergeometric2F1}\Big[\frac{1}{2},\ \frac{g+1}{2},\ \frac{3}{2}, -x^2\Big],\quad \text{for } h=0;$$

$$\mathcal{F}_{14} = y^{-1}\,\big[1+(x+g)^2\big]^{1/2}\ \exp\big[-g\,\mathrm{ArcTan}\,[g+x]\big]$$
$$+\int_{\omega_0}^{x}\big[1+(\omega+g)^2\big]^{-1/2}\ \exp\big[-g\,\mathrm{ArcTan}\,[g+\omega]\big]\,d\omega,\quad (g\neq 0);$$

$$\widetilde{\mathcal{F}}_{14} = y^{-1}\,(x^2+1)^{1/2} + \mathrm{ArcSinh}[x],\quad \text{for } g=0;$$

$$\mathcal{F}_{15} = y^{2/(g-1)}(x+h+i)^{1-ih}(x+h-i)^{1+ih};$$

$$\widetilde{\mathcal{F}}_{15} = y\big[(h+x)^2+1\big]^{(g-1)/2}\ \exp\big[h(1-g)\,\mathrm{ArcTan}[x+h]\big];$$

$$\mathcal{F}_{16} = -(g+x)^g y^{-1}e^{-x} + e^g\mathrm{Gamma}[g, g+x];$$

$$\mathcal{F}_{17} = y^{-1}e^{-x} + \mathrm{ExpIntegralEi}[-x];$$

$$\mathcal{F}_{18} = e^x y^g (x-y+g+1)^{-g-1};$$

$$\mathcal{F}_{19} = e^x y(x+g)^{-g};$$

$$\mathcal{F}_{20} = x^{1-g}\,y^g\,e^{y/x};$$

$$\mathcal{F}_{21} = xe^{(gx+y+1)/x}\Big(\frac{y+1}{-x}\Big)^g + \int_{\omega_0}^{-x/(gx+y+1)} e^{-1/\omega}\omega^{-(1+g)}(g\omega+1)^{g-1}\,d\omega\ \ (g\neq -1);$$

$$\widetilde{\mathcal{F}}_{21} = \frac{x(x+1)}{y+1}\,e^{(y-x+1)/x} + e^{-1}\mathrm{ExpIntegralEi}\Big[1,\ -\frac{y+1}{x}\Big]\ \ \text{for } g=-1;$$

$$\mathcal{F}_{22} = g(y+1)^{-1}x^{(g-1)/g}e^{2/(gx)} + x^{-1/g}\Big(-\frac{2}{gx}\Big)^{-1/g}\mathrm{Gamma}\Big[\frac{1}{g}, -\frac{2}{gx}\Big];$$

$$\mathcal{F}_{23} = (y+1)x^{-1}e^{1/(y+1)} - \mathrm{ExpIntegralEi}\Big[\frac{1}{y+1}\Big];$$

$$\mathcal{F}_{24} = (x+1)^{g-1}\,y\,e^{(g-1)/(x+1)};$$

$$\mathcal{F}_{25} = x^{1-g}\,y^g\,e^{(y+1)/x};$$

$$\mathcal{F}_{26} = e^{(y+1)/x}\,x + \mathrm{ExpIntegralE}\Big[1, -\frac{y+1}{x}\Big];$$

$$\mathcal{F}_{27} = e^x\,\big[y+g-i(x-1)\big]^{1-ig}\big[y+g+i(x-1)\big]^{1+ig};$$

$$\widetilde{\mathcal{F}}_{27} = \Big[(x-1)^2+(y+g)^2\Big]\exp\Big[x+2g\ \mathrm{arctg}\ \Big(\frac{y+g}{x-1}\Big)\Big];$$

$$\mathcal{F}_{28} = y\left(\frac{x-1}{x+1}\right)^{-g/2} + \frac{x^2}{4}(1-x)^{g/2}(x-1)^{-g/2}\times$$
$$\Big\{\mathrm{AppellF1}\big[2,\ 1+\frac{g}{2},\ -\frac{g}{2},\ 3,\ x,\ -x\big] + \mathrm{AppellF1}\big[2,\ \frac{g}{2},\ 1-\frac{g}{2},\ 3,\ x,\ -x\big]\Big\};$$

$$\mathcal{F}_{29} = e^{2y}\,(x-1)^{-1-g}\,(x+1)^{-1+g};$$

$$\mathcal{F}_{30} = y(x+1)^{-1}(gx+1)^{1/g} + \frac{1}{2g-1}(x+1)^{-2}(gx+g)^{1/g}\times$$
$$\text{Hypergeometric2F1}\Big[\frac{2g-1}{g},\frac{g-1}{g},\frac{3g-1}{g},\frac{g-1}{g(x+1)}\Big],\quad (g\neq 1/2, 1/3);$$

$$\widetilde{\mathcal{F}}_{30} = (x+1)^{-2}\exp\Big[\frac{(x+2)^2y+2}{x+1}\Big],\quad \text{for}\quad g=1/2;$$

$$\widehat{\mathcal{F}}_{30} = (x+1)^{-4}\exp\Big[-x+\frac{(x+3)^3y+12}{3(x+1)}\Big],\quad \text{for}\quad g=1/3;$$

$$\mathcal{F}_{31} = x^{-g}(x+1)^{1+g}\ \exp\Big[\frac{y-g+1}{x+1}\Big];$$

$$\mathcal{F}_{32} = y\,e^{-g\,\text{ArcTan}[x]} - \int_{\omega_0}^{x} e^{-g\,\text{ArcTan}[\omega]}\frac{\omega}{\omega^2+1}d\omega;$$

$$\mathcal{F}_{33} = e^{-2y}(x-i)^{1-ig}(x+i)^{1+ig};$$

$$\widetilde{\mathcal{F}}_{33} = (x^2+1)^{-1}\exp\Big[2(y+g\ \text{ArcTan}\frac{1}{x})\Big];$$

$$\mathcal{F}_{34} = y^{-1}\exp\big[-\frac{x^2}{2g}\big] + \frac{\sqrt{\pi}}{\sqrt{2g}}\ \text{Erf}\big[\frac{x}{\sqrt{2g}}\big];$$

$$\mathcal{F}_{35} = y^{-2g}\exp\big[x^2-2y\big];$$

$$\mathcal{F}_{36} = y\ \exp\big[-x^2/2g\big];$$

$$\mathcal{F}_{37} = x^{-g}\big[x^2+(2-g)y\big]\quad (g\neq 2);$$

$$\widetilde{\mathcal{F}}_{37} = y\ \exp\big[y/x^2\big]\quad \text{for}\quad g=2;$$

$$\mathcal{F}_{38} = x^{-2g}\exp\big[x^2+2y\big];$$

$$\mathcal{F}_{39} = e^{1/x}\,y + \text{ExpIntegralEi}\Big[\frac{1}{x}\Big];$$

$$\mathcal{F}_{40} = e^{x+1}(x+1)^{-1}\,(y+1) - \text{ExpIntegralEi}\big[1+x\big];$$

$$\mathcal{F}_{41} = x\sqrt{-g}\ \exp\Big[\frac{(gy+1)^2}{2gx^2}\Big] + \sqrt{\pi/2}\ \text{Erf}\Big[\frac{gy+1}{\sqrt{-2g}\,x}\Big];$$

$$\mathcal{F}_{42} = x^{2/g}\,e^{y^2/x^2};$$

$$\mathcal{F}_{43} = x^{1-2g}\big[1+(2g-1)xy\big]^g,\quad (g\neq 1/2);$$

$$\widetilde{\mathcal{F}}_{43} = x^{-2}\,e^{xy},\quad \text{for } g=1/2;$$

$$\mathcal{F}_{44} = x^2\,e^{(g+2xy)/x^2};$$

$$\mathcal{F}_{45} = x^2\,e^{(2x+gy^2)/x^2};$$

$$\mathcal{F}_{46} = e^{-x}\big[(x+1)^2-y+1\big].$$

5 Phase portraits

In order to construct the phase portraits corresponding to quadratic systems given by Table 1 we use the configurations of invariant straight lines already established in [27] as well as the CT-comitants constructed in [25] and [27] as follows.

Consider the polynomial $\Phi_{\alpha,\beta} = \alpha P + \beta Q \in \mathbb{R}[a, X, Y, Z, \alpha, \beta]$ where $P = Z^2 p(X/Z, Y/Z)$, $Q = Z^2 q(X/Z, Y/Z)$, $p,\ q \in \mathbb{R}[a, x, y]$ and $\max(\deg_{(x,y)} p, \deg_{(x,y)} q) = 2$. Then

$$\begin{aligned}
\Phi_{\alpha,\beta} = {} & c_{11}(\alpha,\beta)X^2 + 2c_{12}(\alpha,\beta)XY + c_{22}(\alpha,\beta)Y^2 + 2c_{13}(\alpha,\beta)XZ \\
& + 2c_{23}(\alpha,\beta)YZ + c_{33}(\alpha,\beta)Z^2, \\
\Delta_3(a,\alpha,\beta) = {} & \det ||c_{ij}(\alpha,\beta)||_{i,j\in\{1,2,3\}}, \quad \Delta_2(a,\alpha,\beta) = \det ||c_{ij}(\alpha,\beta)||_{i,j\in\{1,2\}}.
\end{aligned}$$

Using the differential operator $(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \dfrac{\partial^k f}{\partial x^{k-h} \partial y^h} \dfrac{\partial^k g}{\partial x^h \partial y^{k-h}}$ which is called *transvectant of index k of (f,g)*, $f,\ g \in \mathbb{R}[a, x, y]$ (cf.[13],[18]) we shall construct the following needed invariant polynomials:

$$\begin{aligned}
C_i(a,x,y) &= yp_i(a,x,y) - xq_i(a,x,y),\ \ i = 0,1,2; \\
D_i(a,x,y) &= \frac{\partial}{\partial x} p_i(a,x,y) + \frac{\partial}{\partial y} q_i(a,x,y),\ \ i = 1,2; \\
D(a,x,y) &= 4\Delta_3(a,-y,x); \\
B_3(a,x,y) &= (C_2, D)^{(1)} = Jacob\,(C_2, D), \\
B_2(a,x,y) &= (B_3, B_3)^{(2)} - 6B_3(C_2, D)^{(3)}, \\
B_1(a) &= \operatorname{Res}_x (C_2, D)/y^9 = -2^{-9}3^{-8}\,(B_2, B_3)^{(4)}, \\
M(a,x,y) &= (C_2, C_2)^{(2)} \ = \ 2\,\mathrm{Hess}\big(C_2(a,x,y)\big); \\
\eta(a) &= \mathrm{Discriminant}\,\big(C_2(a,x,y)\big); \\
K(a,x,y) &= \big(p_2, q_2\big)^{(1)} \ = \ \mathrm{Jacob}\,\big(p_2, q_2\big); \\
\mu(a) &= \operatorname{Res}_x(p_2, q_2)/y^4 \ = \ \mathrm{Discriminant}\,\big(K(a,x,y)\big)/16; \\
H(a,x,y) &= 4\Delta_2(a,-y,x); \\
N(a,x,y) &= K(a,x,y) + H(a,x,y); \\
\theta(a) &= \mathrm{Discriminant}\,\big(N(a,x,y)\big); \\
H_1(a) &= -\big(((C_2, C_2)^{(2)}, C_2)^{(1)}, D\big)^{(3)}; \\
H_2(a,x,y) &= (C_1,\ 2H - N)^{(1)} - 2D_1 N; \\
H_3(a,x,y) &= (C_2, D)^{(2)}; \\
H_4(a) &= \big((C_2, D)^{(2)}, (C_2, D_2)^{(1)}\big)^{(2)}; \\
H_5(a) &= \big((C_2, C_2)^{(2)}, (D, D)^{(2)}\big)^{(2)} + 8\big((C_2, D)^{(2)}, (D, D_2)^{(1)}\big)^{(2)}; \\
H_6(a,x,y) &= 16N^2(C_2, D)^{(2)} + H_2^2(C_2, C_2)^{(2)};
\end{aligned}$$

$$
\begin{aligned}
H_7(a) &= (N, C_1)^{(2)};\\
H_8(a) &= 9\big((C_2, D)^{(2)}, (D, D_2)^{(1)}\big)^{(2)} + 2\Big[(C_2, D)^{(3)}\Big]^2;\\
H_9(a) &= -\Big(\big((D, D)^{(2)}, D, \big)^{(1)} D\Big)^{(3)};\\
H_{10}(a) &= \big((N, D)^{(2)},\ D_2\big)^{(1)};\\
H_{11}(a, x, y) &= 8H\big[(C_2, D)^{(2)} + 8(D, D_2)^{(1)}\big] + 3H_2^2;\\
N_1(a, x, y) &= C_1(C_2, C_2)^{(2)} - 2C_2(C_1, C_2)^{(2)},\\
N_2(a, x, y) &= D_1(C_1, C_2)^{(2)} - \big((C_2, C_2)^{(2)}, C_0\big)^{(1)},\\
N_3(a, x, y) &= (C_2, C_1)^{(1)},\\
N_4(a, x, y) &= 4\,(C_2, C_0)^{(1)} - 3C_1 D_1,\\
N_5(a, x, y) &= \big[(D_2, C_1)^{(1)} + D_1 D_2\big]^2 - 4\big(C_2, C_2\big)^{(2)}\big(C_0, D_2\big)^{(1)},\\
N_6(a, x, y) &= 8D + C_2\Big[8(C_0, D_2)^{(1)} - 3(C_1, C_1)^{(2)} + 2D_1^2\Big].
\end{aligned}
$$

Remark 3. We note that by Discriminant (C_2) of the cubic form $C_2(a, x, y)$ we mean the expression given in Maple via the function "discrim$(C_2, x)/y^6$".

The CT-comitants indicated below (for detailed definitions of the notions involved see [26]) were constructed in [26] for the purpose of classifying the phase portraits in the vicinity of infinity of quadratic differential systems.

We consider the differential operator $\mathcal{L} = x \cdot \mathbf{L}_2 - y \cdot \mathbf{L}_1$ acting on $\mathbb{R}[a, x, y]$ constructed in [4], where

$$
\begin{aligned}
\mathbf{L}_1 &= 2a_{00}\frac{\partial}{\partial a_{10}} + a_{10}\frac{\partial}{\partial a_{20}} + \frac{1}{2}a_{01}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{10}} + b_{10}\frac{\partial}{\partial b_{20}} + \frac{1}{2}b_{01}\frac{\partial}{\partial b_{11}},\\
\mathbf{L}_2 &= 2a_{00}\frac{\partial}{\partial a_{01}} + a_{01}\frac{\partial}{\partial a_{02}} + \frac{1}{2}a_{10}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{01}} + b_{01}\frac{\partial}{\partial b_{02}} + \frac{1}{2}b_{10}\frac{\partial}{\partial b_{11}}.
\end{aligned}
$$

Then setting $\mu_0(a) = \mu(a) = \operatorname{Res}_x(p_2, q_2)/y^4$ we construct the following polynomials:

$$
\begin{aligned}
\mu_i(a, x, y) &= \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\ i = 1, .., 4;\\
\kappa(a) &= (M, K)^{(2)}/4;\\
\kappa_1(a) &= (M, C_1)^{(2)};\\
L(a, x, y) &= 4K(a, x, y) + 8H(a, x, y) - M(a, x, y);\\
R(a, x, y) &= L(a, x, y) + 8K(a, x, y);\\
K_1(a, x, y) &= p_1(x, y)q_2(x, y) - p_2(x, y)q_1(x, y);\\
K_2(a, x, y) &= 4\,Jacob(J_2, \xi) + 3\,Jacob(C_1, \xi)D_1 - \xi(16J_1 + 3J_3 + 3D_1^2);\\
K_3(a, x, y) &= 2C_2^2(2J_1 - 3J_3) + C_2(3C_0K - 2C_1J_4) + 2K_1(3K_1 - C_1D_2),
\end{aligned}
$$

where $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$ and $J_1 = Jacob(C_0, D_2)$, $J_2 = Jacob(C_0, C_2)$, $J_3 = Discrim(C_1)$, $J_4 = Jacob(C_1, D_2)$, $\xi = M - 2K$.

The local behavior of the trajectories in the neighborhood of a hyperbolic singular point (i.e. whose eigenvalues have non-zero real parts) is determined by the linearization of the system at this point (see for instance [14]). The simplest kind of singularities are: saddles, nodes, foci, centers and saddle–nodes. Their description can be found in most textbooks (see for example [1, Chapter IV]). We will call *anti–saddle* a singular point at which the linearization of the system has a matrix with positive determinant. In this case the singular point is either a node, or a focus or a center.

We shall use the following notations for a singular point $M_i(x_i, y_i)$:

$$\Delta_i = \begin{vmatrix} p'_x(x,y) & p'_y(x,y) \\ q'_x(x,y) & q'_y(x,y) \end{vmatrix}_{(x_i,y_i)}; \quad \rho_i = \big(p'_x(x,y) + q'_y(x,y)\big)\Big|_{(x_i,y_i)}; \quad \delta_i = \rho_i^2 - 4\Delta_i.$$

The following lemma is very useful for checking, in invariant form, conditions for existence of a center in terms of the coefficients of the systems (8) with $a_{00} = b_{00} = 0$, presented in the tensorial form:

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta, \quad (j, \alpha, \beta = 1, 2). \tag{10}$$

Here the notations $x^1 = x,\ x^2 = y,\ a^1_1 = a_{10}, \dots, a^2_{22} = b_{02}$ are used.

Lemma 7. [30] *The singular point* $(0,0)$ *of a quadratic system* (10) *is a center if and only if* $I_2 < 0$, $I_1 = I_6 = 0$ *and one of the following sets of conditions holds:*

$$1)\ I_3 = 0; \quad 2)\ I_{13} = 0; \quad 3)\ 5I_3 - 2I_4 = 13I_3 - 10I_5 = 0,$$

where

$$I_1 = a^\alpha_\alpha, \quad I_2 = a^\alpha_\beta a^\beta_\alpha, \quad I_3 = a^\alpha_p a^\beta_{\alpha q} a^\gamma_{\beta\gamma} \varepsilon^{pq}, \quad I_4 = a^\alpha_p a^\beta_{\beta q} a^\gamma_{\alpha\gamma} \varepsilon^{pq},$$
$$I_5 = a^\alpha_p a^\beta_{\gamma q} a^\gamma_{\alpha\beta} \varepsilon^{pq}, \quad I_6 = a^\alpha_p a^\beta_\gamma a^\gamma_{\alpha q} a^\delta_{\beta\delta} \varepsilon^{pq}, \quad I_{13} = a^\alpha_p a^\beta_{qr} a^\gamma_{\gamma s} a^\delta_{\alpha\beta} a^\mu_{\delta\mu} \varepsilon^{pq} \varepsilon^{rs}.$$

and the tensor ε *has the coordinates:* $\varepsilon^{12} = -\varepsilon^{21} = 1, \quad \varepsilon^{11} = \varepsilon^{22} = 0.$

To construct the phase portraits of quadratic systems possessing invariant lines of total multiplicity four we examine all the families, following step by step the canonical forms from **Table 1**. For the canonical systems corresponding to *Config. 4.i* we shall use the notation $(S_{4.i})$. To obtain the phase portraits we use the behavior of the vector fields on their invariant lines which can easily be established, as well as the behavior in the vicinity of infinity given by [26]. In general this information turns out to be sufficient. Whenever necessary we add extra arguments.

Theorem 8 (**Main Theorem**). *i) The total number of topologically distinct phase portraits in the class of quadratic differential systems with invariant lines of total multiplicity four is 69.*

ii) In Table 2 we give necessary and sufficient conditions, invariant with respect to the action of the affine group and time rescaling, for the realization of each one *of the phase portraits corresponding to the given configuration of invariant lines. More precisely the first column of Table 2 contains the list of all 46 configurations of invariant lines of total multiplicity four. In the second column we list the necessary and sufficient invariant conditions (obtained in [27]) for the realization of each configuration. The last column contains the names of the phase portraits. Whenever for a configuration Config. 4.i we have several phase portraits, we split the corresponding place in the last column into smaller boxes containing the names of these portraits. In the third column are listed the additional conditions needed for the realization of the corresponding phase portrait in the last column.*

Remark 4. Eleven of the 46 configurations from Diagram 1 produce each a unique phase portrait. Each one of the remaining 35 configurations produces several topologically distinct phase portraits. The total number of phase portraits thus obtained is 93 (see Tables 3(a)-3(d)). However only 69 of these phase portraits are topologically distinct. For example in the subclass with two real singularities at infinity (two pairs of opposite singularities on the Poincaré disk), the 38 cases of possible configurations of invariant lines lead to only 26 topologically distinct phase portraits.

Remark 5. a) In the subclass with one real and two complex singularities at infinity (two opposite singularities on the Poincaré disk), the 11 cases of possible configurations of invariant lines lead to 9 topologically distinct phase portraits.
b) In the subclass with only one singularity at infinity (real) (two opposite singularities on the Poincaré disk), the 16 cases of possible configurations of invariant lines lead to 15 topologically distinct phase portraits.
c) Some phase portraits in a) are topologically equivalent to portraits found in the case b) leading to a total of 18 topologically distinct phase portraits for the union of the two cases a) and b) (See Confrontation Table).

Proof of the Main Theorem. The first step in the proof is to construct the phase portrait *Picture 4.i* (or phase portraits *Picture 4.i(j)*, $j \in \{a,b,c,d,e\}$), $i \leq 46$, associated to a configuration *Config. 4.i.* This leads to 93 distinct such possibilities, with not all phase portraits topologically distinct. At the same time we also give necessary and sufficient conditions, invariant with respect to the action of the group for having each one of the 93 situations obtained. Here by situation we mean an ordered couple formed by a configuration and by one of the possible phase portraits associated to it. In the second part of the proof (see page 77) we look for topologically equivalent phase portraits appearing in the 93 cases and form the list of phase portraits which appear to be topologically distinct. Finally we show that the phase portraits in this list are indeed distinct.

We now proceed to the first step mentioned above.

$$\text{Config. 4.1}: \quad \begin{cases} \dot{x} = gx + gx^2 + (h-1)xy, & (g-1)(h-1)(g+h) \neq 0, \\ \dot{y} = -hy + (g-1)xy + hy^2, & gh(g+h-1) \neq 0. \end{cases} \qquad (S_{4.1})$$

Finite singularities: $M_1(0,0)\big[\Delta_1 = -gh,\ \delta_1 = (g+h)^2\big]$; $M_2(0,1)\big[\Delta_2 = h(g+h-1),\ \delta_2 = (g-1)^2\big]$; $M_3(-1,0)\big[\Delta_3 = g(g+h-1),\ \delta_3 = (h-1)^2\big]$; $M_4(-h,g)\big[\Delta_4 = -gh(g+h-1),\ \delta_4 = 4gh(g+h-1)$.

Table 2

Configuration	*Necessary and sufficient conditions*	*Additional conditions for phase portraits*	*Phase portrait*
Config. 4.1	$\eta>0,\ B_3=0,\ \theta\neq 0,\ H_7\neq 0$	$\mu_0>0$	*Portrait 4.1(a)*
		$\mu_0<0,\ K<0$	*Portrait 4.1(b)*
		$\mu_0<0,\ K>0$	*Portrait 4.1(c)*
Config. 4.2	$\eta<0,\ B_3=0,\ \theta\neq 0,\ H_7\neq 0$	$\mu_0>0,\ \mathcal{G}_1\neq 0$	*Portrait 4.2(a)*
		$\mu_0>0,\ \mathcal{G}_1=0$	*Portrait 4.2(b)*
		$\mu_0<0,\ \mathcal{G}_1\neq 0$	*Portrait 4.2(c)*
		$\mu_0<0,\ \mathcal{G}_1=0$	*Portrait 4.2(d)*
Config. 4.3	$\eta>0,\ B_3=0,\ \theta\neq 0,$ $H_7=0,\ H_1\neq 0,\ \mu_0\neq 0$	$\mu_0>0$	*Portrait 4.3(a)*
		$\mu_0<0,\ K<0$	*Portrait 4.3(b)*
		$\mu_0<0,\ K>0$	*Portrait 4.3(c)*
Config. 4.4	$\eta>0,\ B_3=0,\ \theta\neq 0,$ $H_7=0,\ H_1\neq 0,\ \mu_0=0$	$K<0$	*Portrait 4.4(a)*
		$K>0$	*Portrait 4.4(b)*
Config. 4.5	$\eta>0,\ B_3=0,\ \theta\neq 0,$ $H_7=0,\ H_1=0$	$\mu_0>0$	*Portrait 4.5(a)*
		$\mu_0<0,\ K<0$	*Portrait 4.5(b)*
		$\mu_0<0,\ K>0$	*Portrait 4.5(c)*
Config. 4.6	$\eta<0,\ B_3=0,\ \theta\neq 0,$ $H_7=0,\ \mu_0\neq 0,\ H_9\neq 0$	$\mu_0>0$	*Portrait 4.6(a)*
		$\mu_0<0$	*Portrait 4.6(b)*
Config. 4.7	$\eta<0, B_3=0, \theta\neq 0,$ $H_7=\mu_0=0$	–	*Portrait 4.7*
Config. 4.8	$\eta<0,\ B_3=0,\ \theta\neq 0,$ $H_7=0,\ \mu_0\neq 0,\ H_9=0$	$\mu_0>0$	*Portrait 4.8(a)*
		$\mu_0<0$	*Portrait 4.8(b)*
Config. 4.9	$\eta>0,\ B_2=\theta=H_7=0,$ $\mu_0 B_3 H_4 H_9\neq 0$ *and either* $H_{10}N>0$ *or* $N=0,\ H_8>0$	$\mathcal{G}_2>0, H_4>0, \mathcal{G}_3<0$	*Portrait 4.9(a)*
		$\mathcal{G}_2<0$	*Portrait 4.9(b)*
		$\mathcal{G}_2>0, H_4<0$	
		$\mathcal{G}_2>0, H_4>0, \mathcal{G}_3>0$	*Portrait 4.9(c)*
Config. 4.10	$\eta>0,\ B_3\neq 0,\ B_2=\theta=0,$ $\mu_0\neq 0, H_7=H_9=0, H_{10}N>0$	$H_4>0, \mathcal{G}_3>0$	*Portrait 4.10(a)*
		$H_4<0$	*Portrait 4.10(b)*
		$H_4>0, \mathcal{G}_3<0$	*Portrait 4.10(c)*
	$\eta>0, B_3H_4\neq 0,$ $B_2=N=H_9=0, H_8>0$	–	
Config. 4.11	$\eta=0,\ MB_3\neq 0,\ B_2=\theta=0,$ $H_7=0,\ \mu_0\neq 0,\ H_{10}>0$	$H_4>0$	*Portrait 4.11(a)*
		$H_4<0$	*Portrait 4.11(b)*
Config. 4.12	$\eta=0,\ M\neq 0,\ B_3=\theta=0,$ $KH_6\neq 0, H_7=\mu_0=0, H_{11}>0$	$\mu_2>0,\ L>0$	*Portrait 4.12(a)*
		$\mu_2>0,\ L<0$	*Portrait 4.12(b)*
		$\mu_2<0,\ K<0$	*Portrait 4.12(c)*
		$\mu_2<0, K>0, L>0$	*Portrait 4.12(d)*
		$\mu_2<0, K>0, L<0$	*Portrait 4.12(e)*

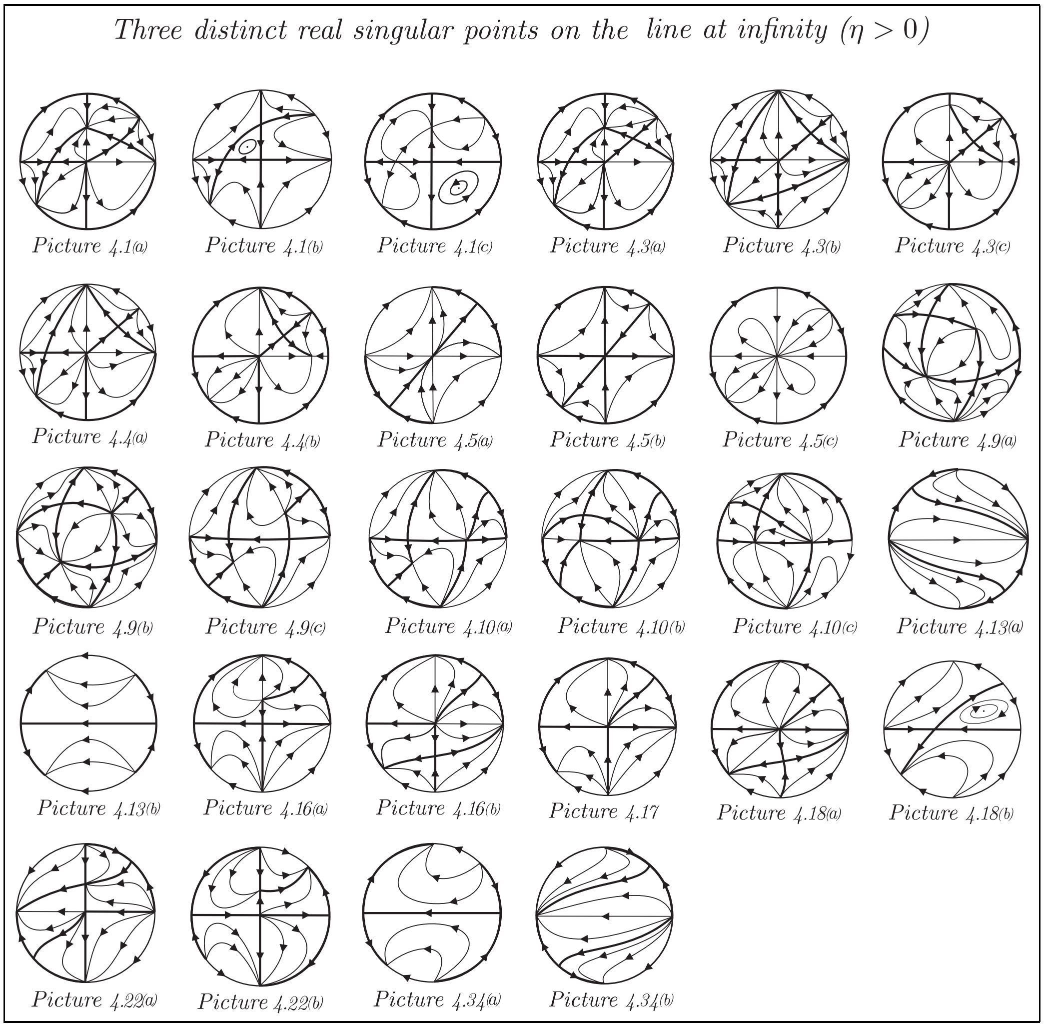
Table 2(*continued*)

Configuration	*Necessary and sufficient conditions*	*Additional conditions for phase portraits*	*Phase portrait*
Config. 4.13	$\eta>0, B_3\neq 0, B_2=\theta=0,$ $\mu_0\neq 0, H_7=0,$ $H_9\neq 0, NH_{10}<0$	$\mathcal{G}_2<0$	*Portrait 4.13(a)*
		$\mathcal{G}_2>0$	*Portrait 4.13(b)*
	$\eta>0, B_3H_4\neq 0,$ $B_2=N=0, H_8<0$	–	
Config. 4.14	$\eta=0,\ MB_3\neq 0,\ B_2=\theta=0,$ $H_7=0,\ \mu_0\neq 0,\ H_{10}<0$	–	*Portrait 4.14*
Config. 4.15	$\eta=0, M\neq 0, B_3=\theta=0,$ $KH_6\neq 0,\ \mu_0=H_7=0,\ H_{11}<0$	$L>0$	*Portrait 4.15(a)*
		$L<0$	*Portrait 4.15(b)*
Config. 4.16	$\eta>0,\ B_3\neq 0,\ B_2=\theta=0,$ $\mu_0=H_7=0,\ H_9\neq 0$	$\mathcal{G}_2>0$	*Portrait 4.16(a)*
		$\mathcal{G}_2<0$	*Portrait 4.16(b)*
Config. 4.17	$\eta>0,\ B_3\neq 0,\ B_2=\theta=0,$ $\mu_0=H_7=H_9=0,\ H_{10}\neq 0$	–	*Portrait 4.17*
Config. 4.18	$\eta>0,\ B_3=\theta=0,$ $\mu_0=0,\ H_7\neq 0$	$\mu_2L>0$	*Portrait 4.18(a)*
		$\mu_2L<0$	*Portrait 4.18(b)*
Config. 4.19	$\eta=0, M\neq 0, B_3=\theta=K=0,$ $NH_6\neq 0,\ \mu_0=H_7=0,\ H_{11}\neq 0$	$\mu_3K_1<0$	*Portrait 4.19(a)*
		$\mu_3K_1>0$	*Portrait 4.19(b)*
Config. 4.20	$\eta=0,\ M\neq 0,\ B_3=0,\ \theta\neq 0,$ $H_7=0,\ D=0$	$\mu_0>0$	*Portrait 4.20(a)*
		$\mu_0<0$	*Portrait 4.20(b)*
Config. 4.21	$\eta=0,\ M\neq 0,\ B_3=0,\ \theta\neq 0,$ $H_7=0,\ D\neq 0,\ \mu_0\neq 0$	$\mu_0>0$	*Portrait 4.21(a)*
		$\mu_0<0$	*Portrait 4.21(b)*
Config. 4.22	$\eta>0,\ B_3\neq 0,\ B_2=\theta=0,$ $\mu_0\neq 0,\ N\neq 0,\ H_7=H_{10}=0$	$H_1>0$	*Portrait 4.22(a)*
		$H_1<0$	*Portrait 4.22(b)*
	$\eta>0, B_3H_4\neq 0, B_2=\theta=N=H_8=0$	–	
Config. 4.23	$\eta=0,\ MB_3\neq 0,\ B_2=\theta=0,$ $\mu_0\neq 0,\ H_7=H_{10}=0$	–	*Portrait 4.23*
Config. 4.24	$\eta=0, M\neq 0, B_3=\theta=0,$ $KH_6\neq 0,\ \mu_0=H_7=H_{11}=0$	$L>0$	*Portrait 4.24(a)*
		$L<0$	*Portrait 4.24(b)*
Config. 4.25	$\eta=0,\ M\neq 0,\ B_3=0,\ \theta\neq 0,$ $H_7\neq 0$	$\mu_0>0$	*Portrait 4.25(a)*
		$\mu_0<0$	*Portrait 4.25(b)*
Config. 4.26	$\eta=0,\ M\neq 0,\ B_3=0,\ \theta\neq 0,$ $H_7=0,\ D\neq 0,\ \mu_0=0$	–	*Portrait 4.26*
Config. 4.27	$\eta<0,\ B_3=\theta=0,$ $N\neq 0,\ H_7\neq 0$	$\mathcal{G}_1\neq 0$	*Portrait 4.27(a)*
		$\mathcal{G}_1=0$	*Portrait 4.27(b)*
Config. 4.28	$\eta=0,\ M\neq 0,\ B_3=\theta=0,$ $\mu_0=N=K=0,\ N_1N_2\neq 0,$ $N_5>0,\ D\neq 0$	–	*Portrait 4.28*

Table 2(*continued*)

Configuration	*Necessary and sufficient conditions*	*Additional conditions for phase portraits*	*Phase portrait*
Config. 4.29	$\eta=0,\ M\neq0,\ B_3=\theta=\mu_0=0,$ $N=K=0,\ N_1N_2\neq0,\ N_5>0,\ D=0$	$\mu_4>0$	*Portrait 4.29(a)*
		$\mu_4<0$	*Portrait 4.29(b)*
Config. 4.30	$\eta=0,\ MB_3\neq0,\ B_2=\theta=\mu_0=0,$ $N\neq0,\ H_7=H_6=0,\ K\neq0,\ H_{11}\neq0$	$\mu_2>0$	*Portrait 4.30(a)*
		$\mu_2<0$	*Portrait 4.30(b)*
Config. 4.31	$\eta=M=0,\ B_3=\theta=0,$ $N\neq0,\ N_6\neq0,\ H_{11}\neq0$	$K_3>0$	*Portrait 4.31(a)*
		$K_3<0$	*Portrait 4.31(b)*
Config. 4.32	$\eta=0,\ M\neq0,\ B_3=\theta=\mu_0=0,$ $N=K=0,\ N_1N_2\neq0,\ N_5<0,\ D\neq0$	–	*Portrait 4.32*
Config. 4.33	$\eta=0,\ M\neq0,\ B_3=\theta=\mu_0=0,$ $N=K=0,\ N_1N_2\neq0,\ N_5<0,\ D=0$	–	*Portrait 4.33*
Config. 4.34	$\eta>0,\ B_3\neq0,\ B_2=\theta=0,$ $\mu_0=H_7=H_9=H_{10}=0$	$H_4<0$	*Portrait 4.34(a)*
		$H_4>0$	*Portrait 4.34(b)*
Config. 4.35	$\eta=0,\ M\neq0,\ B_3=\theta=0,$ $N\neq0,\ \mu_0=0,\ H_7\neq0$	$\mu_3K_1>0$	*Portrait 4.35(a)*
		$\mu_3K_1<0$	*Portrait 4.35(b)*
Config. 4.36	$\eta=0, M\neq0, B_3=\theta=K=0,$ $NH_6\neq0,\ \mu_0=H_7=0,\ H_{11}=0$	$\kappa_2<0$	*Portrait 4.36(a)*
		$\kappa_2>0$	*Portrait 4.36(b)*
Config. 4.37	$\eta=M=0,\ B_3=\theta=N=0,$ $N_3D_1\neq0,\ N_6\neq0,\ D\neq0$	$\mu_3K_1>0, K_3\geq0$	*Portrait 4.37(a)*
		$\mu_3K_1>0, K_3<0$	*Portrait 4.37(b)*
		$\mu_3K_1<0$	*Portrait 4.37(c)*
Config. 4.38	$\eta=M=0,\ B_3=\theta=N=0,$ $N_3D_1\neq0,\ N_6\neq0,\ D=0$	$\mu_4>0$	*Portrait 4.38(a)*
		$\mu_4<0$	*Portrait 4.38(b)*
Config. 4.39	$\eta=0,\ M\neq0,\ B_3=\theta=\mu_0=0,$ $N=K=0,\ N_1N_2\neq0,\ N_5=0$	–	*Portrait 4.39*
Config. 4.40	$\eta=0,\ MB_3\neq0,\ B_2=\theta=\mu_0=0,$ $N\neq0,\ H_7=H_6=0,\ K=0$	–	*Portrait 4.40*
Config. 4.41	$\eta=M=0,\ B_3=0,\ \theta\neq0,$ $H_7=0,\ D\neq0$	$\mu_0>0$	*Portrait 4.41(a)*
		$\mu_0<0$	*Portrait 4.41(b)*
Config. 4.42	$\eta=M=0,\ B_3=0,\ \theta\neq0,$ $H_7=0,\ D=0$	$\mu_0>0$	*Portrait 4.42(a)*
		$\mu_0<0$	*Portrait 4.42(b)*
Config. 4.43	$\eta=0,\ MB_3\neq0,\ B_2=\theta=\mu_0=0,$ $N\neq0,\ H_7=H_6=0,\ K\neq0,\ H_{11}=0$	$L<0$	*Portrait 4.43(a)*
		$L>0, R\geq0$	*Portrait 4.43(b)*
		$L>0, R<0$	*Portrait 4.43(c)*
Config. 4.44	$\eta=M=0,\ B_3=\theta=0,$ $N\neq0,\ N_6\neq0,\ H_{11}=0$	$K_3>0$	*Portrait 4.44(a)*
		$K_3<0$	*Portrait 4.44(b)*
Config. 4.45	$\eta=M=0,\ B_3=0,\ \theta\neq0,$ $H_7\neq0$	$\mu_0>0$	*Portrait 4.45(a)*
		$\mu_0<0$	*Portrait 4.45(b)*
Config. 4.46	$\eta=M=0,\ B_3=\theta=N=0,$ $N_3D_1\neq0,\ N_6=0$	–	*Portrait 4.46*

Table 3(a)



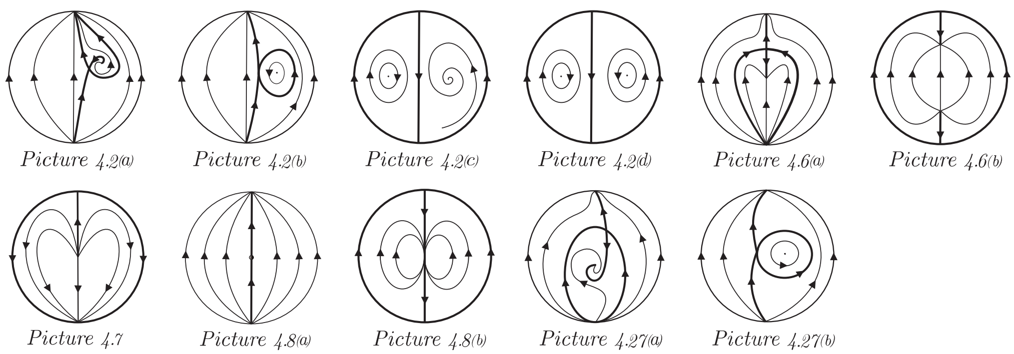
For systems $(S_{4.1})$ calculations yield: $K = 2\big[g(g-1)x^2 + 2ghxy + h(h-1)y^2\big]$, $\mu_0 = gh(g+h-1)$, $\quad \mathrm{sign}\,(\Delta_1\Delta_2\Delta_3\Delta_4) = \mathrm{sign}\,(\mu_0)$.

According to [5] a quadratic system cannot possess four anti–saddles, and neither could it possess four saddles. For this reason we obtain two saddles and two anti–saddles for $\mu_0 > 0$ and either (α) one saddle and three anti–saddles or (β) three saddles and one anti–saddle for $\mu_0 < 0$.

Assume $\mu_0 > 0$. As the singular points M_1, M_2 and M_3 are located on invariant lines and for M_4 we have $\mathrm{sign}\,(\delta_4) = \mathrm{sign}\,(\mu_0)$, we conclude that in this case a system $(S_{4.1})$ possesses two saddles and two nodes. Considering the existence of the invariant lines $x = 0$, $y = 0$ and $y = x + 1$ and the fact that the sum of Poincaré

Table 3(b)

One real and two imaginary singular points on the line at infinity ($\eta < 0$)



indices for finite singularities is zero, and at infinity we have 6 simple singularities (on the Poincaré disk), these must be: one couple of opposite saddles and two couples of opposite nodes and we get the phase portrait given by *Picture 4.1(a)*.

For $\mu_0 < 0$ we have $gh(g+h-1) < 0$ and then $\delta_4 < 0$, i.e. the singular point M_4 is either a focus or a center. We claim that it is a center. Indeed, via the translation of the origin of coordinates to this point we get the family of systems

$$\dot{x} = -ghx + h(1-h)y + gx^2 + (h-1)xy, \quad \dot{y} = g(g-1)x + ghy + (g-1)xy + hy^2.$$

Applying Lemma 7 to these systems we calculate: $I_1 = I_6 = I_3 = 0$, $I_2 = 2gh(g+h-1)$. Thus, $\operatorname{sign}(I_2) = \operatorname{sign}(\mu_0)$ and since $\mu_0 < 0$ (i.e. $I_2 < 0$) according to Lemma 7 we obtain that the singular point M_4 is a center, so our claim is proved.

On the other hand, for $\mu_0 < 0$ the T-comitant K becomes a binary form with well determined sign as $\mathrm{Discrim}(K) = \mu_0/16$.

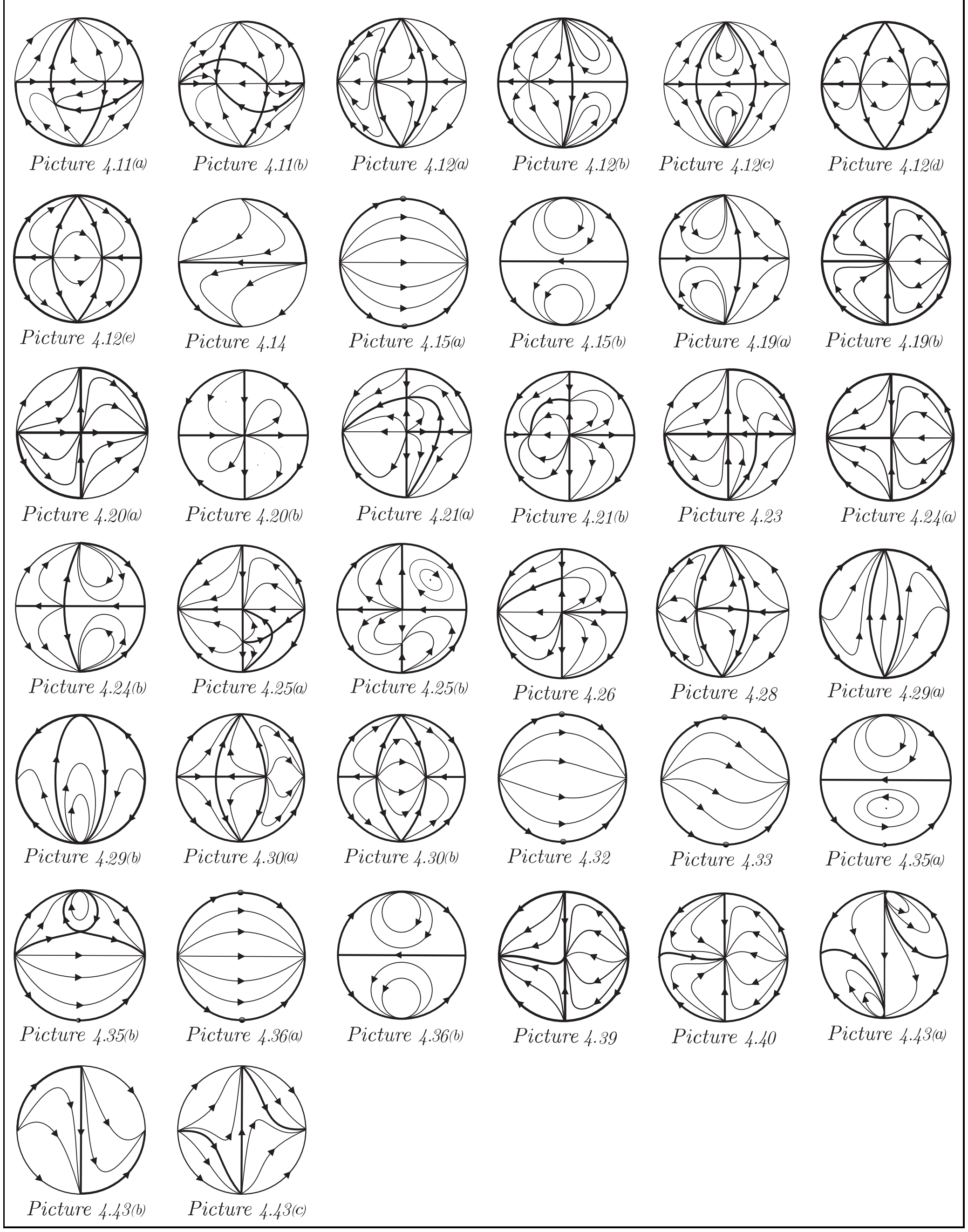
Assume $K < 0$. Then $0 < g < 1$, $0 < h < 1$ and from $\mu_0 < 0$ we obtain $g + h - 1 < 0$. In this case we have $\Delta_i < 0$ for all $i \in \{1, 2, 3\}$ and hence, besides a center systems $(S_{4.1})$ possess three saddles. Moreover, for these values of the parameters g and h the singular point $M_4(-h, g)$ is placed inside of the triangle $\triangle\ M_1M_2M_3$. So, considering the existence of the invariant lines $x = 0$, $y = 0$ and $y = x + 1$ and the fact that the sum of Poincaré indices for finite singularities is -2, we must have 6 nodes at infinity (3 in the projective plane) and we get the *Picture 4.1(b)*.

Suppose now that $K > 0$. Then $g(g-1) > 0$, $h(h-1) > 0$ and we claim that in this case besides the center M_4 systems $(S_{4.1})$ possess two nodes and one saddle. Indeed, supposing the contrary we obtain that all three M_i must be saddles, as $\Delta_1\Delta_2\Delta_3 = -(g^2h^2(g+h-1)^2 < 0$. Hence, $\Delta_i < 0$ for $i = 1, 2, 3$. From $\Delta_1 < 0$ we get $gh > 0$ and then the condition $\mu_0 < 0$ implies $g + h - 1 < 0$. Then the condition $\Delta_2 < 0$ yields $h > 0$ (and hence $g > 0$). Due to $K < 0$ we get the contradiction: $g > 1$, $h > 1$ and $g + h - 1 < 0$. This proves our claim.

So, systems $(S_{4.1})$ possess one saddle, two nodes and one center, and the last point is outside the triangle $\triangle\ M_1M_2M_3$. Clearly in this case at infinity we have

Table 3(c)

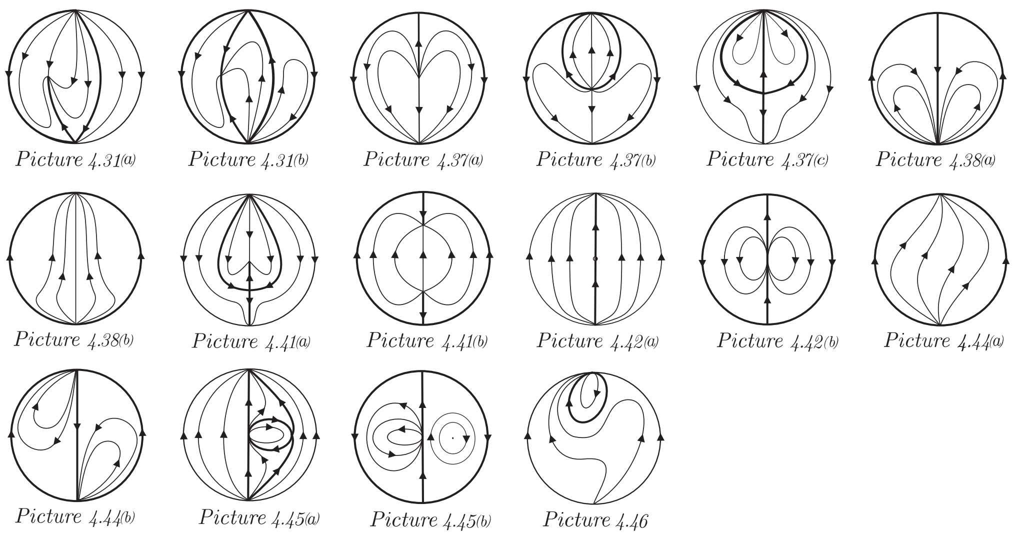
Two distinct real singular points on the line at infinity ($\eta = 0$, $M \neq 0$)



two saddles and one node (as the sum of Poincaré indeces for infinite singularities has to be -1). Considering the existence of the above indicated invariant lines we arrive at the phase portrait given by *Picture 4.1(c)*.

Table 3(d)

Only one singular point on the line at infinity, which is real ($\eta = 0 = M,\ C_2 \neq 0$)



Picture 4.31(a), Picture 4.31(b), Picture 4.37(a), Picture 4.37(b), Picture 4.37(c), Picture 4.38(a), Picture 4.38(b), Picture 4.41(a), Picture 4.41(b), Picture 4.42(a), Picture 4.42(b), Picture 4.44(a), Picture 4.44(b), Picture 4.45(a), Picture 4.45(b), Picture 4.46

So, systems ($S_{4.1}$) possess one saddle, two nodes and one center, and the last point is outside the triangle $\triangle\ M_1M_2M_3$. Clearly in this case at infinity we have two saddles and one node (as the sum of Poincaré indeces for infinite singularities has to be -1). Considering the existence of the above indicated invariant lines we arrive at the phase portrait given by *Picture 4.1(c)*.

$$\textit{Config. 4.2}: \begin{cases} \dot{x} = gx^2 + (h+1)xy, \qquad h(h+1)\Big[g^2 + (h-1)^2\Big] \neq 0, \\ \dot{y} = h\big[g^2+(h+1)^2\big]+(g^2+1-h^2)x+2ghy- x^2+gxy+hy^2. \end{cases} \qquad (S_{4.2})$$

Finite singularities: $M_1(0,0)\big[\Delta_1 = [g^2+(h+1)^2](h+1)^2 > 0,\ \delta_1 = -4(h+1)^4 < 0,$ $\rho_1 = 2g(h+1)\big]$; $M_2(-h(h+1), gh)\big[\Delta_2 = h\big[g^2 + (h+1)^2\big](h+1)^2,\ \delta_2 = -4h\big[g^2 + (h+1)^2\big](h+1)^2,\ \rho_2 = 0\big]$. Thus the singular point M_1 is either a focus or a center. To determine the conditions for M_1 to be a center, we make a translation and move this point to the origin of coordinates. We get the systems

$$\dot{x} = (1+h+x)(gx+y+hy), \quad \dot{y} = -(h+1)^2x + g(h+1)y - x^2 + gxy + hy^2,$$

for which calculations yield: $I_1 = 2g(h+1), \quad I_6 = g(h+1)^3(5+6h-3g^2-3h^2)/2,$

$$I_2 = 2g^2(h+1)^2 - 2(h+1)^4, \quad I_{13} = g(h+1)\Big[g^2(9h+8) + h(3h+1)^2\Big]/4.$$

Using Lemma 7 we see that M_1 is a center if and only if $g = 0$. If $g \neq 0$ this point is a strong focus. To distinguish between a focus and a center we define a new affine invariant as follows: $\mathcal{G}_1 = ((C_2, \tilde{E})^{(2)}, D_2)^{(1)}$, where $\tilde{E}(a,x,y) = \Big[D_1(2\omega_1 - \omega_2) - 3(C_1,\omega_1)^{(1)} - D_2(3\omega_3 + D_1D_2)\Big]/72$ and $\omega_1(a,x,y) = (C_2, D_2)^{(1)}$, $\omega_2(a,x,y) = (C_2, C_2)^{(2)}$, $\omega_3(a,x,y) = (C_1, D_2)^{(1)}$.

Since for the systems $(S_{4.2})$ calculation yields $\mathcal{G}_1 = 2g(h+1)\big[g^2 + (3h+1)^2\big]$, it is clear that the condition $g = 0$ is equivalent to $\mathcal{G}_1 = 0$.

Let us examine the point M_2. For systems $(S_{4.2})$ calculations yield: $\mu_0 = -h\big[g^2 + (h+1)^2\big]$. Hence $\operatorname{sign}(\delta_2) = \operatorname{sign}(\mu_0) = -\operatorname{sign}(\Delta_2)$. Therefore the point M_2 is a saddle if $\mu_0 > 0$ and it is either a focus or a center if $\mu_0 < 0$. Translating this point at the origin of coordinates we get the systems

$$\dot{x} = (x-h^2-h)[gx+(h+1)y], \quad \dot{y} = (h+1)(g^2+h+1)x+gh(h+1)y-x^2+gxy+hy^2,$$

for which $I_1 = I_6 = I_3 = 0$, $I_2 = -2h(h+1)^2\big[g^2 + (h+1)^2\big]$. Consequently, by Lemma 7 the point M_2 is a center if $\mu_0 < 0$.

We note, that the product of the abscissas of finite singularities equals $-h(h+1)^2$. This means that both points are on the same side (respectively on different sides) of the invariant line $x = 0$ if $\mu_0 > 0$ (respectively $\mu_0 < 0$).

It remains to observe that at infinity there are only two real simple singular points. When M_2 is a saddle, since M_1 is an anti-saddle (index $+1$), then the two infinite points must be nodes. When M_2 is a center, since M_1 is an anti-saddle, the two infinite points are saddles. In the last case the invariant line $x = 0$ is a separatrix of the saddle at infinity.

Thus, we obtain: *Picture 4.2(a)* if $\mu_0 > 0$ and $\mathcal{G}_1 \neq 0$; *Picture 4.2(b)* if $\mu_0 > 0$ and $\mathcal{G}_1 = 0$; *Picture 4.2(c)* if $\mu_0 < 0$ and $\mathcal{G}_1 \neq 0$; *Picture 4.2(d)* if $\mu_0 < 0$ and $\mathcal{G}_1 = 0$.

$$\textit{Config. 4.3}: \begin{cases} \dot{x} = \dot{x} = x + gx^2 + (h-1)xy, & gh(g+h-1) \neq 0, \\ \dot{y} = y + (g-1)xy + hy^2, & (g-1)(h-1)(g+h) \neq 0. \end{cases} \qquad (S_{4.3})$$

Finite singularities: $M_1(0,0)\big[\Delta_1 = 1,\ \delta_1 = 0\big]$; $M_2\Big(0, -\frac{1}{h}\Big)\Big[\Delta_2 = -\frac{1}{h}, \delta_2 = \frac{(h+1)^2}{h^2}\Big]$; $M_3\Big(-\frac{1}{g}, 0\Big)\Big[\Delta_3 = -\frac{1}{g}, \delta_3 = \frac{(g+1)^2}{g^2}\Big]$; $M_4\Big(-\frac{1}{g+h-1}, -\frac{1}{g+h-1}\Big)\Big[\Delta_4 = \frac{1}{g+h-1}, \delta_4 = \frac{(g+h-2)^2}{(g+h-1)^2}\Big]$. For systems $(S_{4.3})$ calculations yield: $\mu_0 = gh(g+h-1)$,

$$K = 2\big[g(g-1)x^2 + 2ghxy + h(h-1)y^2\big]; \quad \operatorname{sign}(\Delta_1\Delta_2\Delta_3\Delta_4) = \operatorname{sign}(\mu_0).$$

Since $\delta_i \geq 0$ for all points M_i we conclude that systems $(S_{4.3})$ possess two saddles and two nodes if $\mu_0 > 0$ and they possess either (α) one saddle and three nodes or (β) three saddles and one node if $\mu_0 < 0$.

Assume $\mu_0 > 0$. Then we have two nodes (one of them being the point M_1) and two saddles. Considering the existence of the invariant lines $x = 0$, $y = 0$ and $y = x$ and the fact that the sum of Poincaré indices for finite singularities is zero, then at infinity we have six simple singularities: two saddle and four nodes and we get the phase portrait given by *Picture 4.3(a)*.

If $\mu_0 < 0$ the T-comitant K becomes a sign defined binary form as $\operatorname{Discrim}(K) = \mu_0/16$.

Assume $K < 0$. Then $0 < g < 1$, $0 < h < 1$ and from $\mu_0 < 0$ we obtain $g + h - 1 < 0$. In this case $\Delta_i < 0$ for all $i \in \{2,3,4\}$ and hence, besides the star node M_1 systems $(S_{4.1})$ possess three saddles. Moreover, for these values of the parameters g and h the singular point $M_1(0,0)$ is placed inside of the triangle $\triangle\ M_2M_3M_4$. So, considering the existence of the invariant lines $x=0$, $y=0$ and $y=x$ and the fact that the sum of Poincaré indices for finite singularities is -2, we have six nodes at infinity and we get the *Picture 4.3(b)*.

Suppose now $K>0$. We claim that in this case besides the star node M_1 systems $(S_{4.3})$ possess two nodes and one saddle. Indeed, supposing the contrary, we obtain that all three M_i must be saddles, as $\operatorname{sign}(\Delta_2\Delta_3\Delta_4)=\operatorname{sign}(\mu_0)=-1$. Therefore, from $\Delta_2<0$ and $\Delta_3<0$ we get $h>0$ and $g>0$ respectively, and then the condition $\mu_0<0$ implies $g+h-1<0$. On the other the condition $K>0$ yields $g(g-1)>0$ and $h(h-1)>0$ and we get $g>1$ and $h>1$. This contradicts $g+h-1<0$ and hence proves our claim.

So, systems $(S_{4.3})$ possess one saddle and three nodes. Clearly in this case at infinity we have four saddles and two nodes (as the sum of Poincaré indices for infinite singularities has to be -2). Considering the presence of the above mentioned invariant lines we obtain the phase portrait given by *Picture 4.3(c)*.

$$\textit{Config. 4.4}:\quad \begin{cases} \dot{x}=x+gx^2-xy, & g\in\mathbb{R},\\ \dot{y}=y+(g-1)xy, & g(g-1)\neq 0. \end{cases} \qquad (S_{4.4})$$

Finite singularities: $M_1(0,0)\big[\Delta_1=1,\ \delta_1=0\big]$; $M_2\big(-\frac{1}{g},0\big)\big[\Delta_2=-\frac{1}{g},\ \delta_2=\frac{(g+1)^2}{g^2}\big]$; $M_3\big(\frac{1}{1-g},\frac{1}{1-g}\big)\big[\Delta_3=\frac{1}{g-1},\ \delta_3=\frac{(g-2)^2}{(g-1)^2}\big]$. For systems $(S_{4.3})$ calculations yield: $\mu_0=0$, $K=2g(g-1)x^2$, $\operatorname{sign}(\Delta_2\Delta_3)=-\operatorname{sign}(K)$. We observe, that the family of systems $(S_{4.4})$ is a subset of the family $(S_{4.3})$ defined by the condition $h=0$. So, since the singular point $M_2\big(0,-1/h\big)$ tends to infinity when $h\to 0$ we conclude that the infinite point $N(0,1,0)$ of systems $(S_{4.4})$ is a double point (a saddle-node).

On the other hand it is easy to determine that besides the star node M_1, systems $(S_{4.4})$ possess two saddles if $g(g-1)<0$ (i.e. $K<0$) and they possess one saddle and one node if $g(g-1)>0$ (i.e. $K>0$). Therefore, taking into consideration the invariant lines $x=0$, $y=0$ and $y=x$ and the sum of Poincaré indices we get the *Picture 4.4(a)* if $K<0$ and the *Picture 4.4(b)* when $K>0$.

$$\textit{Config. 4.5}:\quad \begin{cases} \dot{x}=gx^2+(h-1)xy, & (g-1)(h-1)(g+h)\neq 0,\\ \dot{y}=(g-1)xy+hy^2, & gh(g+h-1)\neq 0. \end{cases} \qquad (S_{4.5})$$

We observe that $(S_{4.5})$ is a family of homogenous systems, each having only the origin as finite singular point. These systems possess three invariant lines: $x=0$, $y=0$ and $y=x$. Hence $\eta>0$. We also have $\mu_0\neq 0$. Hence according to Table 4 in [26] we have the following possibilities for singular points at infinity: i) If $\mu_0>0$ we have two saddles and four nodes; ii) If $\mu_0<0$ and $\kappa<0$ we have four saddles and two nodes; If $\mu_0<0$ and $\kappa>0$ we have six nodes.

For systems $(S_{4.5})$ calculations yield:

$$\mu_0=gh(g+h-1),\quad \kappa=-16\big[g(g-1)+h(h-1)+gh\big],$$
$$K=2\big[g(g-1)x^2+2ghxy+h(h-1)y^2\big].$$

Mapping the sign of μ_0 in the plane h, g we determine that $\mu_0 < 0$ in the shaded areas of *Figure 1*. Hence in the shaded areas, K has well determined sign as indicated. On the same figure it is also easy to observe that for $\mu_0 < 0$ the following relation holds: $\operatorname{sign}(\kappa) = -\operatorname{sign}(K)$.

Thus, we obtain: *Picture 4.5(a)* if $\mu_0 > 0$; *Picture 4.5(b)* if $\mu_0 < 0$ and $K < 0$; *Picture 4.5(c)* if $\mu_0 < 0$ and $K > 0$.

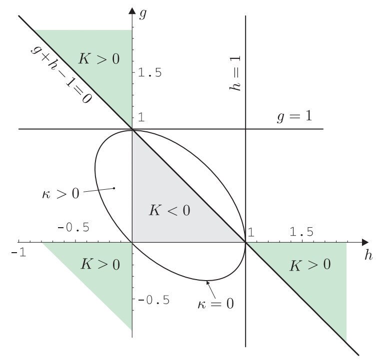


Figure 1

$$\textit{Config. 4.6}: \quad \begin{cases} \dot{x} = gx^2 + (h+1)xy, & h(h+1)\left[g^2 + (h-1)^2\right] \neq 0, \\ \dot{y} = -1 + gx + (h-1)y - x^2 + gxy + hy^2. \end{cases} \qquad (S_{4.6})$$

Finite singularities: $M_1(0,-1)\left[\Delta_1 = (h{+}1)^2,\ \delta_1 = 0\right]$ – a node; $M_2(0, 1/h)\left[\Delta_2 = (h+1)^2/h,\ \delta_2 = (h^2-1)^2/h^2\right]$ – a node if $h > 0$ and a saddle if $h < 0$. For systems $(S_{4.6})$ we calculate $\mu_0 = -h\left[g^2 + (h+1)^2\right]$, i.e. $\operatorname{sign}(\mu_0) = -\operatorname{sign}(h)$.

It remains to observe that at infinity there exist only one real singular point which is simple. Since M_1 is an anti-saddle (index $+1$), the infinite point is a node (index $+1$) when M_2 is a saddle and it is a saddle (index -1) when M_2 is a node. In the last case the invariant line $x = 0$ is a separatrix of the saddle at infinite. Hence we obtain *Picture 4.6(a)* if $\mu_0 > 0$ and *Picture 4.6(b)* if $\mu_0 < 0$.

$$\textit{Config. 4.7}: \quad \dot{x} = gx^2 + xy, \quad \dot{y} = -1 + gx - y - x^2 + gxy, \quad g \in \mathbb{R}. \qquad (S_{4.7})$$

Finite singularities: $M_1(0,-1)\left[\Delta_1 = 1,\ \delta_1 = 0\right]$ – a node. We observe that the family of systems $(S_{4.7})$ is a subset of the family $(S_{4.6})$ defined by the condition $h = 0$. So, since the singular point $M_2\big(0, -1/h\big)$ tends to infinite when $h \to 0$ we conclude that the infinite point $N(0,1,0)$ of systems $(S_{4.7})$ is a double point (a saddle-node). This leads to the *Picture 4.7*

$$\textit{Config. 4.8}: \quad \begin{cases} \dot{x} = gx^2 + (h+1)xy, \\ \dot{y} = -x^2 + gxy + hy^2, \quad h(h+1)\left[g^2 + (h-1)^2\right] \neq 0. \end{cases} \qquad (S_{4.8})$$

For systems $(S_{4.8})$ we calculate $\mu_0 = -h\left[g^2 + (h+1)^2\right]$, $\eta = -4 < 0$. According to [26] at infinity there exist two opposite nodes if $\mu_0 > 0$ and two opposite saddles if $\mu_0 < 0$.

Thus, taking into consideration the real invariant line $x = 0$ of systems $(S_{4.8})$ we obtain *Picture 4.8(a)* if $\mu_0 > 0$ and *Picture 4.8(b)* if $\mu_0 < 0$.

$$\textit{Config. 4.9}: \quad \begin{cases} \dot{x} = x^2 - 1, \quad g(g-1)[(g \pm 1)^2 - 4h^2] \neq 0, \\ \dot{y} = (y+h)[y + (1-g)x - h]. \end{cases} \qquad (S_{4.9})$$

Finite singularities: $M_1(-1,-h)\left[\Delta_1 = 2(2h - g + 1),\ \delta_1 = (2h - g - 1)^2\right]$;
$M_2(1,-h)\left[\Delta_2 = -2(2h + g - 1),\ \delta_2 = (2h + g + 1)^2\right]$;

$M_3(-1, h-g+1)\big[\Delta_3 = -2(2h-g+1),\ \delta_3 = (2h-g+3)^2\big];\ M_4(1, h+g-1)\big[\Delta_4 = 2(2h+g-1),\ \delta_4 = (2h+g-3)^2\big]$. For systems $(S_{4.9})$ calculations yield: $\mu_0 = 1 > 0$, $\eta = g^2 > 0$, $\operatorname{sign}(\Delta_1\Delta_2\Delta_3\Delta_4) = 1$.

Since $\delta_i \geq 0$ for all points M_i we conclude that systems $(S_{4.9})$ possess two saddles and two nodes in the finite part of its phase plane. From the behavior of trajectories at infinity, according to [26] we have four nodes and two opposite saddles. More concretely, we have the node $N_1(0,1,0)$ and the singular points $N_2(1,0,0)$ and $N_3(1,g,0)$ as well as their opposites. It is not hard to find out that the point $N_2(1,0,0)$ (respectively, $N_3(1,g,0)$) ia a saddle (respectively, a node) if $g < 0$ and it is a node (respectively, a saddle) if $g > 0$.

We note that the first equation depends only on x. $\dot{x} > 0$ for x outside $[-1,1]$ and $\dot{x} < 0$ for $x \in (-1,1)$. This yields the orientation of the vector field on the invariant line $x = -h$. The phase portrait on the invariant lines $x = \pm 1$ is easily obtained by replacing these values in the second equation which becomes $\dot{y} = (y + h)(y - (h+g-1))$ for $x = 1$ and $\dot{y} = (y+h)(y-(h-g+1))$ for $x = -1$. Hence $\dot{y} > 0$ for y outside the interval determined by the roots of the polynomials on the right hand sides and $\dot{y} < 0$ for y inside this interval. The sign of $\dot{y}$ thus depends on whether or not $-h$ is smaller or greater than $h+g-1$ (respectively $h-g+1$) which amounts to checking the sign of $2h+g-1$ (respectively $2h-g+1$). As the phase portrait around infinity depends on the sign of g the full discussion, which is elementary, depends on the sign of $g(2h-g+1)(2h+g-1)$.

Case 1) We first assume that $g(2h-g+1)(2h+g-1) > 0$. This could occur if either *i)* all three factors are positive or *ii)* two of the factors are negative and the third one is positive.

In the case *i)* we have that $-h < h-g+1$ and $-h < h+g-1$ so the points M_3 and M_4 lie above the line $y = -h$, M_3 being a saddle and M_4 being a node while M_1 is a node and M_2 is a saddle. This yields phase portrait *Picture 4.9(b).*

In the case *ii)* we observe that the case when the first two factors are negative and the third one is positive cannot occur. Indeed, in this case we would necessarily have $-(g-1) < 2h < g-1$ which yields a contradiction as $g > 0$. So we only need to consider the cases when only the first and last factors are negative or when only the second and last one are negative. In the first situation we have that $h+g-1 < -h < h-g+1$, so M_3 and M_4 are respectively above and below the invariant line $y = -h$. M_1 and M_2 are nodes and M_3 and M_4 are saddles. Considering the behavior at infinity we have that N_2 is a saddle and N_3 is a node located on the negative side of the u-axis and the phase portrait is *Picture 4.9(b).* If only $2h-g+1$ and $2h+g-1$ are negative and $g > 0$, the points M_3 and M_4 are both below the line $y = -h$, M_3 is a node and M_4 is a saddle while M_1 is a saddle and M_2 is a node. N_2 is a node and N_3 is on the positive side of the u-axis and it is a saddle. This yields again the phase portrait *Picture 4.9(b).*

Thus we conclude that in the case $g(2h-g+1)(2h+g-1) > 0$ the phase portrait of systems $(S_{4.9})$ corresponds to *Picture 4.9(b).*

Case 2) Suppose now that $g(2h-g+1)(2h+g-1) < 0$. This could occur if all three factors are negative or if only one is negative and the other two are positive.

In the first case M_3 and M_4 are both below the line $y = -h$ and M_1 and M_3 are nodes while M_2 and M_4 are saddles. N_2 is a saddle and N_3 lies on the negative side of the u-axis and it is a node. This yields picture *Picture 4.9(a)*.

It remains to consider the cases when only one of the three factors is negative. If $g < 0$ then M_3 and M_4 are both above the line $y = -h$ and M_3 and M_4 are both below the line $y = -h$ and M_1 and M_4 are nodes while M_3 and M_2 are saddles. N_2 is a saddle and N_3 is on the negative side of the u-axis and it is a node. So in this case we get *Picture 4.9(a)*. If only the second factor is negative, i.e. $2h - g + 1 < 0$ we have $h-g+1 < -h < h+g-1$ and hence M_3 and M_4 are nodes situated on the opposite sides of the line $y = -h$ and M_1 and M_2 are saddles. In this case N_2 is a node and N_3 is a saddle situated in the positive side of u. Hence the phase portrait is *Picture 4.9(c)*. If only the third factor is negative, i.e. $2h + g - 1 < 0$ then M_1 and M_2 are nodes and M_3 and M_4 are saddles located on the opposite sides of the line $y = -h$. In this case N_2 is a node and N_3 is located on the positive side of u and it is a saddle. The phase portrait is therefore *Picture 4.9(b)*.

For each phase portrait assembling together the above conditions we get:

- *Picture 4.9(b)* $\Leftrightarrow$ either $g\big[4h^2-(g-1)^2\big]>0$ or $g\big[4h^2-(g-1)^2\big]<0$ and $0<g<1$;
- *Picture 4.9(a)* $\Leftrightarrow$ $g\big[4h^2 - (g-1)^2\big] < 0$ and $g < 0$;
- *Picture 4.9(c)* $\Leftrightarrow$ $g\big[4h^2 - (g-1)^2\big] < 0$ and $g > 1$.

In order to determine the corresponding invariant conditions we construct the following affine invariants:

$$\mathcal{G}_2 = 8H_8 - 9H_5, \quad \mathcal{G}_3 = (\mu_0 - \eta)H_1 - 6\eta(H_4 + 12H_{10}).$$

Since for the systems $(S_{4.9})$ we have $G_2 = -2^9 3^3 g\big[4h^2 - (g-1)^2\big]$, $H_4 = 48(1-g)\big[4h^2 - (g+1)^2\big]$, $G_3 = 6gH_4$, we conclude that these three invariant polynomials distinguish the phase portraits of systems $(S_{4.9})$ for this configuration as it is indicated in the Table 2.

$$\textit{Config. 4.10}: \quad \dot{x} = x^2 - 1, \quad \dot{y} = (y+g)(y+2gx-g), \quad g(2g-1) \neq 0. \qquad (S_{4.10})$$

Finite singularities: $M_1(-1,-g)\big[\Delta_1 = 8g,\ \delta_1 = 4(2g-1)^2\big]$ – a node if $g > 0$ and a saddle if $g < 0$; $M_2(1,-g)\big[\Delta_2 = 0,\ \rho_2 = 2\big]$ – a saddle-node [1]; $M_3(-1,3g)\big[\Delta_3 = -8g,\ \delta_3 = 4(2g+1)^2\big]$ – a node if $g < 0$ and a saddle if $g > 0$. For systems $(S_{4.10})$ calculations yield: $\mu_0 = 1 > 0$, $\eta = (2g-1)^2 > 0$. Hence according to [26] at infinity we have six singularities: the node $N_1(0,1,0)$ with its opposite and the singular points $N_2(1,0,0)$ and $N_3(1,1-2g,0)$ with there opposites. It is not hard to find out that the point $N_2(1,0,0)$ (respectively, $N_3(1,1-2g,0)$) is a saddle (respectively, a node) if $1-2g < 0$ and it is a node (respectively, a saddle) if $1-2g > 0$.

a) Assume first $g < 0$, i.e. $M_1(-1,-g)$ is a saddle and $M_3(-1,3g)$ is a node. Since in this case $1-2g > 0$ we obtain that $N_2(1,0,0)$ is a node and $N_3(1,1-2g,0)$

is a saddle. Taking into consideration the location of these singularities we get the *Picture 4.10(a)*.

b) Assume now $g > 0$. Then $M_1(-1,-g)$ is a node and $M_3(-1,3g)$ is a saddle. Since the type of infinite singularities depends on sign $(1-2g)$ we shall consider two subcases: $1-2g > 0$ and $1-2g < 0$.

b_1) If $1-2g > 0$ then as in the previous case $N_2(1,0,0)$ is a node and $N_3(1,1-2g,0)$ is a saddle. Taking into consideration the relative location of the singularities of systems $(S_{4.10})$ for $0 < g < 1/2$, we obtain in this case the *Picture 4.10(b)*.

b_2) Supposing $1-2g < 0$ we have at infinity the saddle $N_2(1,0,0)$ and the node $N_3(1,1-2g,0)$. It is easy to observe that in this case we have a separatrix connection, between finite saddle-node M_2 and infinite saddle $N_2(1,0,0)$. So, in the same manner above we get the *Picture 4.10(c)*.

It remains to construct the respective affine invariant conditions. For systems $(S_{4.10})$ we have $H_4 = 384g(2g-1)$. Therefore, if $H_4 < 0$ (i.e. $0 < g < 1/2$) we obtain *Picture 4.10(b)*, whereas for $H_4 > 0$ we have either *Picture 4.10(a)* or *Picture 4.10(c)*. We observe that for systems $(S_{4.10})$ calculations yield: $\mathcal{G}_3 = -2304g(2g-1)^2$ and hence, for $H_4 > 0$ we get *Picture 4.10(a)* if $\mathcal{G}_3 > 0$ and *Picture 4.10(c)* if $\mathcal{G}_3 < 0$.

$$\textit{Config. 4.11}: \quad \dot{x} = (x+g)^2 - 1, \quad \dot{y} = y(x+y), \quad g^2 - 1 \neq 0. \qquad (S_{4.11})$$

Finite singularities: $M_1(-1-g,\,0)\big[\Delta_1 = 2(g+1),\ \delta_1 = (g-1)^2\big]$; $M_2(1-g,\,0)\big[\Delta_2 = -2(g-1),\ \delta_2 = (g+1)^2\big]$; $M_3(-1-g,\,g+1)\big[\Delta_3 = -2(g+1),\ \delta_3 = (g+3)^2\big]$; $M_4(1-g,\,g-1)\big[\Delta_4 = 2(g-1),\ \delta_4 = (g-3)^2\big]$.

Evidently, that we have two nodes and two saddles, and which singularities are nodes and which ones are saddles depends on sign (g^2-1).

For systems $(S_{4.11})$ calculations yield: $\mu_0 = 1 > 0$, $\eta = 0$, $M = -8y^2 \neq 0$, $C_2 = -xy^2$. So, according to [26] at infinity besides the node $N_1(0,1,0)$ systems $(S_{4.11})$ possess a double point $N_1(1,0,0)$, which is a saddle-node.

We shall examine three cases: $g < -1$, $-1 < g < 1$ and $g > 1$.

a) Case $g < -1$. Then the singular points M_1 and M_4 are saddles, whereas M_2 and M_3 are nodes. Moreover, M_3 and M_4 are on the same part of the invariant line $y = 0$. Thus we get the phase portrait given by *Picture 4.11(a)*.

b) Case $-1 < g < 1$. In this case the singular points $M_{1,2}$ are nodes and $M_{3,4}$ are saddles. And clearly M_3 and M_4 are on different sides of the invariant line $y = 0$. So we obtain *Picture 4.11(b)*.

c) Case $g > 1$. Then the singular points M_1 and M_4 are nodes, whereas M_2 and M_3 are saddles. In this case M_3 and M_4 are on the same part of the invariant line $y = 0$. Therefore, we get the phase portrait with is topologically equivalent to *Picture 4.11(a)*.

It remains to note that for the systems $(S_{4.11})$ we have $H_4 = 48(g^2-1)$ and evidently this invariant polynomial distinguishes *Picture 4.11(a)* $(H_4 > 0)$ from *Picture 4.11(b)* $(H_4 < 0)$.

$$\textit{Config. 4.12}: \begin{cases} \dot{x} = (x+h)^2 - 1, & g(g-1)(h^2-1) \neq 0, \\ \dot{y} = (1-g)xy, & h^2(g-1)^2 - (g+1)^2 \neq 0. \end{cases} \qquad (S_{4.12})$$

Finite singularities: $M_1(-1-h,\,0)\big[\Delta_1 = -2(h+1)(g-1),\ \delta_1 = [h(g-1)+(g+1)]^2\big]$; $M_2(1-h,\,0)\big[\Delta_2 = 2(h-1)(g-1),\ \delta_2 = [h(g-1)-(g+1)]^2\big]$. Since $\Delta_1\Delta_2 = -4(g-1)^2(h^2-1)$ and $\Delta_1+\Delta_2 = -4(g-1)$ we conclude that systems $(S_{4.12})$ possess a saddle and a node if $h^2-1>0$. For $h^2-1<0$ these systems possess two saddles if $g>1$ and they possess two nodes if $g<1$.

To determine the behavior of the trajectories at the infinity according to [26] for systems $(S_{4.12})$ we calculate:

$$\begin{aligned}&\eta = 0,\ M = -8g^2x^2 \neq 0,\ C_2 = gx^2y,\ \mu_0 = \mu_1 = \kappa = \kappa_1 = 0,\ L = 8gx^2,\\ &\mu_2 = (g-1)^2(h^2-1)x^2,\ K = 2(1-g)x^2,\ K_2 = 192(2g^2-g+1)x^2.\end{aligned}$$

We observe that by [26] the point $N_1(0,1,0)$ is of the multiplicity 4 (consisting of two finite and two infinite points which have coalesced). We also note that $K_2>0$ for any value of parameter g.

a) Case $\mu_2>0$. Then $h^2-1>0$ and systems $(S_{4.12})$ possess one saddle and one node. As $\operatorname{sign}(L) = \operatorname{sign}(g)$, following [26] we shall consider two subcases: $L>0$ and $L<0$.

a_1) *Assume* $L>0$. Since $K_2>0$ according to [26, Table 4] the behavior of the trajectories in the vicinity of infinity is given by *Figure 19*. Taking into consideration the invariant lines we get *Picture 4.12(a)*.

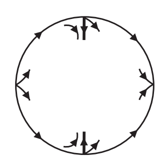


Figure 19

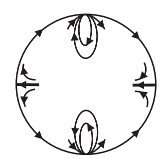


Figure 17

a_2) *Suppose* $L<0$. Then from [26, Table 4] we get *Figure 17*. So, in the same manner as above we obtain the phase portrait given by *Picture 4.12(b)*.

b) Case $\mu_2<0$. In this case we have $h^2-1<0$ and as it is determined above systems $(S_{4.12})$ possess two saddles if $g>1$ and they possess two nodes if $g<1$. As for these systems $K = 2(1-g)x^2$ we have $\operatorname{sign}(K) = -\operatorname{sign}(g-1)$ and we shall consider two subcases: $K<0$ and $K>0$.

b_1) *Assume first* $K<0$, i.e. $g>1$ and the finite singular points are both saddles. On the other hand for the infinite points the relation $L = 8gx^2>0$ holds and according to [26, Table 4] this leads to the *Figure 10*. So, taking into consideration the invariant lines of systems $(S_{4.12})$ we obtain the phase portrait given by *Picture 4.12(c)*.

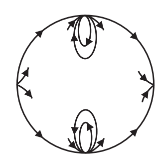


Figure 10

b_2) *Assume now* $K>0$. Then $g<1$ and the finite singular points are both nodes. According to [26, Table 4] the behavior of the trajectories in the vicinity of infinity in this case depends on the sign of the invariant polynomials $L = 8gx^2$.

If $L>0$ (i.e. $g>0$) we obtain *Figure 27*, whereas for $L<0$ (then $K<0$) we get *Figure 29*. Therefore, considering the existence of the invariant lines of systems $(S_{4.12})$ we obtain the *Picture 4.12(d)* if $L>0$ and the *Picture 4.12(e)* if $L<0$.

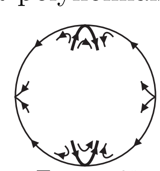


Figure 27

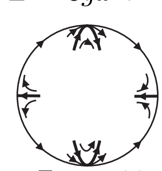


Figure 29

$$\textit{Config. 4.13}:\ \begin{cases}\dot{x} = x^2+1, \quad g(g-1)\big[(g+1)^2+h^2\big]\neq 0,\\ \dot{y} = (y+h)[y+(1-g)x-h].\end{cases} \qquad (S_{4.13})$$

No finite singularities. These systems have invariant lines $y=-h$ and $x=\pm i$. Calculations yield $\eta=g^2>0$, $\mu_0=1>0$ and according to [26] on the line at infinity there exist two nodes and one saddle. Due to the existence of the real invariant line $y=-h$ we have to distinguish when the point $N_1(1,0,0)$ is a saddle (having a saddle connection) and when it is a node. Constructing the respective to $(S_{4.13})$ family of systems at infinity we get

$$\dot u = gu + h(g-1)z - u^2 + h^2z^2 + uz^2,\quad \dot z = z + z^3.$$

Since the singularity $(0,0)$ of these systems corresponds to $N_1(1,0,0)$ we conclude that the singular point N_1 is a saddle if $g<0$ and it is a node if $g>0$. To distinguish these two possibilities we shall use the affine invariant G_2. For systems $(S_{4.13})$ we calculate $\mathcal{G}_2 = 13824g\big[4h^2+(g-1)^2\big]$. Thus, $\mathcal{G}_2\neq 0$ and $\operatorname{sign}(\mathcal{G}_2)=\operatorname{sign}(g)$. Hence we get *Picture 4.13(a)* if $\mathcal{G}_2>0$ and *Picture 4.13(b)* if $\mathcal{G}_2<0$.

$$\textit{Config. 4.14}:\quad \dot x=(x+g)^2+1,\quad \dot y = y(x+y),\quad g\in\mathbb{R}. \tag{$S_{4.14}$}$$

No finite singularities. Calculations yield $\eta=0$, $M=-8y^2\neq 0$, $C_2=-xy^2$, $\mu_0=1>0$. Thus the singular point $N_1(1,0,0)$ is a double point and according to [26] on the line at infinity at infinity there exist one node and one saddle-node (double). Hence, taking into account the real invariant line $y=0$ we get *Picture 4.14* for any value of the parameter g.

$$\textit{Config. 4.15}:\quad \begin{cases} \dot x=(x+h)^2+1, & g(g-1)\neq 0,\\ \dot y=(1-g)xy, & (g+1)^2+h^2\neq 0.\end{cases} \tag{$S_{4.15}$}$$

No finite singularities. Calculations yield

$$\eta=0,\ M=-8g^2x^2\neq 0,\ C_2=gx^2y,\ \mu_0=\mu_1=\kappa=\kappa_1=0,\ L=8gx^2,$$
$$\mu_2=(g-1)^2(h^2+1)x^2,\ K=2(1-g)x^2,\ K_2=-192(2g^2-g+1)x^2.$$

We observe [26] that the point $N_1(0,1,0)$ is of the multiplicity 4 (two finite and two infinite points have coalesced at this point). We also note that $\mu_2>0$ and $K_2<0$ for any value of parameters $(g,h)\in\mathbb{R}^2$.

Thus according to [26, Table 4] this leads to the *Figure 8* if $L>0$ and to the *Figure 17* (see above) if $L<0$. Taking into consideration the existence of the real invariant line $y=0$ we obtain *Picture 4.15(a)* if $L>0$ and *Picture 4.15(b)* if $L<0$.

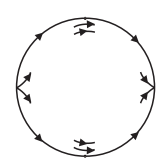


Figure 8

$$\textit{Config. 4.16}:\quad \dot x = g+x,\quad \dot y = y(y-x),\quad g(g-1)\neq 0. \tag{$S_{4.16}$}$$

Finite singularities: $M_1(-g,0)\big[\Delta_1=g,\ \delta_1=(g-1)^2\big]$; $M_2(-g,-g)\big[\Delta_2=-g,\ \delta_2=(g+1)^2\big]$. We observe that systems $(S_{4.16})$ possess a node and a saddle. To determine the behavior of the trajectories at the infinity according to [26] for these systems we calculate:

$$\eta=1>0,\ C_2=xy(x-y),\ \mu_0=\mu_1=\kappa=0,\quad L=8y(y-x),\ \mu_2=y(y-x).$$

Hence $\mu_2 L = 8y^2(y-x)^2 > 0$ and according to [26, Table 4] on the line at infinity there exist three real singular points, two of which are double and one simple. More precisely, the double points $N_1(1,0,0)$ and $N_2(1,1,0)$ are saddle-nodes, whereas the point $N_2(0,1,0)$ is a node and the geometric configuration corresponds to *Figure 4*.

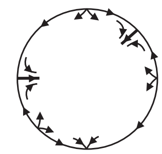


Figure 4

We observe that if the point $M_1(-g,\,0)$, located on the invariant line $y=0$ (as well as on the line $x=-g$) is a saddle (i.e. $g<0$), then we get a saddle connection with the saddle-node $N_1(1,0,0)$.

On the other hand for systems $(S_{4.16})$ we have $\mathcal{G}_2 = -3456g$. So, taking into consideration the invariant lines $x=-g$ and $y=0$ of systems $(S_{4.16})$ we obtain *Picture 4.16(a)* if $\mathcal{G}_2 > 0$ and *Picture 4.16(b)* if $\mathcal{G}_2 < 0$.

$$\textit{Config. 4.17:}\qquad \dot{x} = x, \quad \dot{y} = y(y-x). \qquad (S_{4.17})$$

We observe that this system can be obtained from the family $(S_{4.16})$ allowing the parameter g to vanish. In this case the points $M_1(-g,\,0)$ and $M_2(-g,\,-g)$ coalesced (at the origin of coordinates), yielding a saddle-node. So, as it can easily be determined, we get *Picture 4.17*.

$$\textit{Config. 4.18:}\qquad \dot{x} = g(g+1)+gx+y, \quad \dot{y} = y(y-x), \quad g(g+1)\neq 0. \qquad (S_{4.18})$$

Finite singularities: $M_1(-1-g,\,0)\big[\Delta_1 = g(g+1),\ \delta_1 = 1\big]$; $M_2(-g,\,-g)\big[\Delta_2 = -g(g+1),\ \delta_2 = 4g(g+1),\ \rho_2 = 0\big]$. We observe that systems $(S_{4.18})$ possess a saddle and a node if $g(g+1)>0$ and they possess a saddle and either a focus or a center if $g(g+1)<0$. We claim that in the second case the point M_2 is a center. Indeed, moving this point to the origin of coordinates we get the systems $\dot{x} = gx+y,\quad \dot{y} = (g-y)(x-y)$, for which considering Lemma 7 we calculate: $I_1 = I_6 = I_3 = 0,\ I_2 = 2g(g+1)$. Since $g(g+1)<0$ by Lemma 7 the point M_2 is a center and our claim is proved.

For systems $(S_{4.18})$ calculations yield:

$$\eta = 1,\ C_2 = xy(x-y),\ \mu_0=\mu_1=\kappa=0,\ L = 8y(y-x),\ \mu_2 = g(g+1)y(y-x).$$

Hence $\mu_2 L = 8g(g+1)y^2(y-x)^2 \neq 0$ and then $\operatorname{sign}(\mu_2 L) = \operatorname{sign}\big(g(g+1)\big)$. According to [26, Table 4] on the line at infinity there exist three real singular points, two of which are double and one simple. More precisely, the double points $N_1(1,0,0)$ and $N_2(1,1,0)$ are saddle-nodes, whereas the point $N_2(0,1,0)$ is a node.

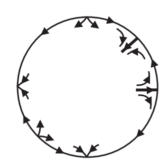


Figure 3

However, depending on the location of the saddle sectors of the saddle-nodes, at infinity there are two distinct configurations. As it was proved in [26] we have the *Figure 4* (see above) if $\mu_2 L > 0$ and the *Figure 3* if $\mu_2 L < 0$.

a) Case $\mu_2 L > 0$. Then $g(g+1)>0$ and systems $(S_{4.18})$ possess one saddle and one node. Taking into consideration the existence of the invariant lines $y=0$ and $x-y+g+1=0$ as well as *Figure 4* we get the *Picture 4.18(a)*.

a) Case $\mu_2 L < 0$. Then $g(g+1) < 0$ and systems $(S_{4.18})$ possess one saddle and one center. Moreover, the behavior of the trajectories at infinity corresponds to *Figure 3*. In this case we obtain the *Picture 4.18(b)*.

$$\textit{Config. 4.19}: \qquad \dot{x} = g + x, \quad \dot{y} = -xy, \quad g(g-1) \neq 0. \qquad (S_{4.19})$$

We observe that these systems possess one finite singular point $M_1(-g, 0)$ which is a saddle for $g < 0$ and it is a node if $g > 0$. We shall examine the infinite singularities. Considering [26] for systems $(S_{4.19})$ we calculate: $M = -8x^2 \neq 0$, $C_2 = x^2 y$, $\eta = \mu_0 = \mu_1 = \mu_2 = \kappa = \kappa_1 = L = 0$, $\mu_3 = -gx^2 y$, $K_1 = -x^2 y$.

According to [26] the point $N_1(0,1,0)$ is of the multiplicity 4 (consisting from two finite and two infinite points which have coalesced), while the singular point $N_2(1,0,0)$ is a double point which is a saddle-node (a finite and an infinite singular point being coalesced) Moreover, by [26, Table 4] the behavior of the trajectories at infinity corresponds to *Figure 12* if $\mu_3 K_1 < 0$ and to *Figure 21* if $\mu_3 K_1 > 0$.

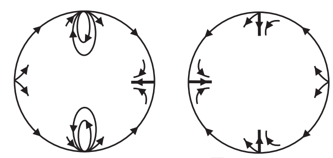


Figure 12 *Figure 21*

Since $\mu_3 K_1 = gx^4 y^2$ it follows $\operatorname{sign}(\mu_3 K_1) = \operatorname{sign}(g)$. Therefore, taking into account the existence of the invariant lines $y = 0$ and $x = -g$ and *Figures 12* and *21* we obtain *Picture 4.19(a)* if $\mu_3 K_1 < 0$ and *Picture 4.19(b)* if $\mu_3 K_1 > 0$.

$$\textit{Config. 4.20}: \qquad \dot{x} = x(gx + y), \quad \dot{y} = (g-1)xy + y^2, \quad g(g-1) \neq 0. \qquad (S_{4.20})$$

For systems $(S_{4.20})$ calculations yield: $\eta = 0$, $M = -8x^2$, $C_2 = x^2 y$, $\mu_0 = g \neq 0$. We observe that $(S_{4.20})$ is a family of homogenous systems, which possess two invariant lines: $x = 0$ (double) and $y = 0$. According to [26] on the line at infinity, besides the saddle-node $N_1(0,1,0)$ (corresponding to the double line), systems $(S_{4.20})$ have a node if $\mu_0 > 0$ (*Picture 4.20(a)*) and they have a saddle if $\mu_0 < 0$ (*Picture 4.20(b)*).

$$\textit{Config. 4.21}: \qquad \begin{cases} \dot{x} = x(gx + y), \quad g(g-1) \neq 0, \\ \dot{y} = (y+1)(gx - x + y). \end{cases} \qquad (S_{4.21})$$

Finite singularities: $M_1(0,0)\big[\Delta_1 = 0,\ \rho_1 = 1\big]$ – a saddle-node [1]; $M_2(0,-1)$ $\big[\Delta_2 = 1,\ \delta_2 = 0\big]$ – a node; $M_3(1/g, -1)\big[\Delta_3 = -1/g,\ \delta_3 = (g+1)^2/g^2\big]$ – a node if $g < 0$ and a saddle if $g > 0$. For systems $(S_{4.21})$ calculations yield: $\eta = 0$, $M = -8x^2$, $C_2 = x^2 y$, $\mu_0 = g$. Hence according to [26] on the line at infinity we have two singularities: the saddle-node $N_1(0,1,0)$ and the singular point $N_2(1,0,0)$, which is a node if $\mu_0 > 0$ and it is a saddle if $\mu_0 < 0$.

Thus, taking into account the invariant lines $x = 0$ (double) and $y = -1$ we get *Picture 4.21(a)* if $\mu_0 > 0$ and *Picture 4.21(b)* if $\mu_0 < 0$.

$$\textit{Config. 4.22}: \qquad \dot{x} = gx^2, \quad \dot{y} = (y+1)\big[y + (g-1)x - 1\big], \quad g(g-1) \neq 0. \qquad (S_{4.22})$$

Finite singularities: $M_1(0,0)\big[\Delta_1 = 0,\ \rho_1 = 2\big]$, $M_2(0,-1)\big[\Delta_2 = 0,\ \rho_2 = -2\big]$ – saddle-nodes [1].

To determine the behavior of the trajectories at the infinity for systems $(S_{4.22})$ we calculate: $\eta = 1 > 0$, $C_2 = xy(x-y)$, $\mu_0 = g^2 > 0$. Thus according to [26, Table 4] on the line at infinity there exists three real singular points: $N_1(1,0,0)$, and $N_2(1,1,0)$ and $N_3(0,1,0)$. More precisely, there are two nodes and one saddle. Using the transformation $x = 1/z$, $y = u/z$) we get the systems

$$\dot{u} = u + (1-g)z - u^2 + z^2, \quad \dot{z} = gz. \tag{11}$$

For the singular point $(0,0)$ (respectively $(1,0)$) of systems (11) corresponding to the point $N_1(1,0,0)$ (respectively $N_2(1,1,0)$) of systems $(S_{4.22})$ we have $\tilde{\Delta}_1 = g$ (respectively $\tilde{\Delta}_2 = -g$). Hence we conclude that besides the node $N_3(0,1,0)$ systems $(S_{4.22})$ possess at infinity the node $N_1(1,0,0)$ and the saddle $N_2(1,1,0)$ if $g > 0$ and they possess the saddle $N_1(1,0,0)$ and the node $N_2(1,1,0)$ if $g < 0$.

On the other hand for systems $(S_{4.22})$ we have $H_1 = 1152g$. Hence, taking into consideration the invariant lines $x = 0$ (double) and $y = -1$ of systems $(S_{4.22})$ we get *Picture 4.22(a)* if $H_1 > 0$ and *Picture 4.22(b)* if $H_1 < 0$.

$$\textit{Config. 4.23}: \qquad \dot{x} = x(x+y), \quad \dot{y} = (y+1)^2. \tag{$S_{4.23}$}$$

Finite singularities: $M_1(0,-1)\big[\Delta_1 = 0,\ \rho_1 = -1\big]$, $M_2(1,-1)\big[\Delta_2 = 0,\ \rho_2 = 1\big]$ – saddle-nodes [1]. For these systems calculations yield: $\eta = 0$, $M = -8x^2$, $C_2 = x^2y$, $\mu_0 = 1 > 0$. We observe [26] that the point $N_1(0,1,0)$ is a double point and it is a saddle-node, whereas the second simple point $N_1(1,0,0)$ is a node. Thus, taking into account the invariant lines $x = 0$ and $y = -1$ (double) we get *Picture 4.23*.

$$\textit{Config. 4.24}: \qquad \dot{x} = (x+1)^2, \quad \dot{y} = (1-g)xy,\ g(g-1) \neq 0. \tag{$S_{4.24}$}$$

Finite singularities: $M_1(-1,0)\big[\Delta_1 = 0,\ \rho_1 = g-1\big]$ – saddle-node [1]. We calculate: $M = -8g^2x^2 \neq 0$, $C_2 = gx^2y$, $\eta = \mu_0 = \mu_1 = \kappa = \kappa_1 = K_2 = 0$, $\mu_2 = (g-1)^2x^2$, $L = 8gx^2$. Since $\mu_2 > 0$ and $K_2 = 0$ by [26, Table 4] the behavior of the trajectories at infinity corresponds to *Figure 19* if $L > 0$ and to *Figure 17* if $L < 0$ (see p. 67). Taking into consideration the existence of the real invariant lines $y = 0$ and $x = -1$ (double) we obtain *Picture 4.24(a)* if $L > 0$ and *Picture 4.24(b)* if $L < 0$.

$$\textit{Config. 4.25}: \qquad \dot{x} = gx^2 + xy, \quad \dot{y} = y + (g-1)xy + y^2, \quad g(g-1) \neq 0. \tag{$S_{4.25}$}$$

Finite singularities: $M_1(0,0)\big[\Delta_1 = 0,\ \rho_1 = 1\big]$ – a saddle-node [1]; $M_2(0,-1)$ $\big[\Delta_2 = 1,\ \delta_2 = 0\big]$ – a node; $M_3(1,-g)\big[\Delta_3 = -g,\ \delta_3 = 4g\big]$ – a saddle if $g > 0$ and either a focus or a center if $g < 0$.

We claim that the point M_3 is a center if $g < 0$. Indeed, translating this point to the origin of coordinates we get the systems $\dot{x} = (1+x)(gx+y)$, $\dot{y} = (g-y)(x-gx-y)$, for which considering Lemma 7 we calculate: $I_1 = I_6 = I_3 = 0$, $I_2 = 2g$. Since $g < 0$ by Lemma 7 the point M_3 is a center and our claim is proved.

On the other hand for systems $(S_{4.25})$ calculations yield: $\eta = 0$, $M = -8x^2$, $C_2 = x^2y$, $\mu_0 = g$. Hence according to [26] at infinity we have two singularities: the saddle-node $N_1(0,1,0)$ and the singular point $N_2(1,0,0)$, which is a node if $\mu_0>0$ and it is a saddle if $\mu_0<0$. Hence, taking into account the invariant lines $x=0$ (double) and $y=-1$ we get *Picture 4.25(a)* if $\mu_0>0$ and *Picture 4.25(b)* if $\mu_0<0$.

$$\textit{Config. 4.26}: \quad \dot{x}=xy, \quad \dot{y}=(y+1)(y-x). \qquad (S_{4.26})$$

Finite singularities: $M_1(0,0)\big[\Delta_1=0,\ \rho_1=1\big]$ – a saddle-node [1]; $M_2(0,-1)\big[\Delta_2=1,\ \delta_2=0\big]$ – a node. For systems $(S_{4.26})$ we calculate: $M=-8x^2\neq 0,\ C_2=x^2y,\ \eta=\mu_0=0,\ \mu_1=y,\ K=2y^2$.

According to [26, Table 4] in this case the behavior of the trajectories at infinity corresponds to *Figure 20.* So, taking into consideration the invariant lines $x=0$ (double) and $y=0$ of systems $(S_{4.26})$ we obtain *Picture 4.26*

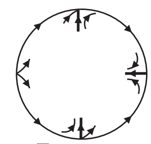


Figure 20

$$\textit{Config. 4.27}: \quad \dot{x}=2gx+2y, \quad \dot{y}=g^2+1-x^2-y^2, \quad g\in\mathbb{R}. \qquad (S_{4.27})$$

Finite singularities: $M_1(-1,\ g)\big[\Delta_1=-4(g^2+1)\big]$ – a saddle; $M_2(1,\ -g)\big[\Delta_2=4(g^2+1),\ \delta_2=-16\big]$. We observe that the singular point M_2 is a strong focus if $g\neq 0$ and it is a center if $g=0$. Indeed, translating this point to the origin of coordinates we get the systems $\dot{x}=2(gx+y)$, $\dot{y}=-2x+2gy-x^2-y^2$, for which we calculate: $I_1=4g,\ I_6=-8g,\ I_{13}=-2g,\ I_2=8(g^2-1)$ So, by Lemma 7 the point M_2 is a center if and only if $g=0$. To determine the behavior of the trajectories at the infinity for systems $(S_{4.27})$ calculations yield: $\eta=-4,\ C_2=x(x^2+y^2),\ \mu_0=\mu_1=\kappa=0,\ \mu_2=4(g^2+1)(x^2+y^2)$. So, according to [26] the unique real infinite singular point $N_1(0,1,0)$ of $(S_{4.27})$ is a node. Therefore, since for these systems we have $\mathcal{G}_1=16g$, we obtain *Picture 4.27(a)* if $\mathcal{G}_1\neq 0$ and *Picture 4.27(b)* if $\mathcal{G}_1=0$.

$$\textit{Config. 4.28}: \quad \dot{x}=x^2-1, \quad \dot{y}=x+gy, \quad g(g^2-4)\neq 0. \qquad (S_{4.28})$$

Finite singularities: $M_1(1,\ -1/g)\big[\Delta_1=2g,\ \delta_1=(g-2)^2\big]$; $M_2(-1,\ 1/g)\big[\Delta_2=-2g,\ \delta_2=(g+2)^2\big]$. We observe that systems $(S_{4.28})$ possess a node and a saddle. For these systems we calculate: $\eta=0,\ M=-8x^2,\ C_2=x^2y,\ \mu_0=\mu_1=\kappa=\kappa_1=0,\ L=8x^2,\ \mu_2=g^2x^2,\ K_2=384x^2$, and according to [26, Table 4] the behavior of the trajectories in the vicinity of infinity corresponds to *Figure 19* (see page 67). Taking into consideration the existence of the real invariant lines $x=\pm 1$ we obtain in both cases (i.e. either $g>0$ or $g<0$) the phase portraits topologically equivalent to *Picture 4.28.*

$$\textit{Config. 4.29}: \quad \dot{x}=x^2-1, \quad \dot{y}=g+x, \quad g^2-1\neq 0. \qquad (S_{4.29})$$

No finite singularities. For these systems calculations yield: $M=-8x^2,\ C_2=x^2y,$

$$\eta=\mu_0=\mu_1=\mu_2=\mu_3=\kappa=\kappa_1=0,\ L=8x^2,\ \mu_4=(g^2-1)x^4,\ K_2=384x^2.$$

We observe that $L > 0$, $K = 0$ and $K_2 > 0$ for any value of parameters $\pm 1 \neq g \in \mathbb{R}$. According to [26, Table 4] the behavior of the trajectories in the vicinity of infinity corresponds to *Figure 18* if $\mu_4 > 0$ and to *Figure 24* if $\mu_4 < 0$. Thus, taking into account the existence of the invariant lines $x = \pm 1$ we get *Picture 4.29(a)* if $\mu_4 > 0$ and *Picture 4.29(b)* if $\mu_4 < 0$.

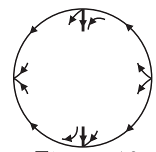


Figure 18

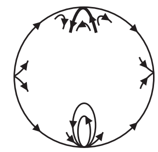


Figure 24

$$\textit{Config. 4.30}: \quad \dot{x} = (x+1)(gx+1), \quad \dot{y} = 1 + (g-1)xy, \quad g(g^2-1) \neq 0. \quad (S_{4.30})$$

Finite singularities: $M_1(-1, 1/(g-1))\big[\Delta_1 = (g-1)^2,\ \delta_1 = 0\big]$ – a node; $M_2(-1/g, g/(g-1))\big[\Delta_2 = -(g-1)^2/g,\ \delta_2 = (g^2-1)^2/g^2\big]$ – a node if $g < 0$ and a saddle if $g > 0$. For systems $(S_{4.30})$ calculations yield: $\eta = 0,\ M = -8x^2,\ C_2 = x^2y,\ \mu_0 = \mu_1 = \kappa = \kappa_1 = 0$ and

$$L = 8gx^2,\ \mu_2 = g(g-1)^2x^2,\ K_2 = 48(g-1)^2(g^2-g+2)x^2.$$

Since $\operatorname{sign}(\mu_2) = \operatorname{sign}(L) = \operatorname{sign}(g)$ and $K_2 > 0$, according to [26, Table 4] the behavior of the trajectories around the infinity corresponds to *Figure 19* (see p. 67) if $g > 0$ and to *Figure 29* (see p. 67) if $g < 0$. Taking into consideration the real invariant lines and $x + 1 = 0$ (double) and $(gx+1) = 0$ we obtain the phase portrait *Picture 4.30(a)* if $\mu_2 > 0$ and *Picture 4.30(b)* if $\mu_2 < 0$.

$$\textit{Config. 4.31}: \quad \dot{x} = x(x+1), \quad \dot{y} = g - x^2 + xy, \quad g(g+1) \neq 0. \quad (S_{4.31})$$

Finite singularities: $M_1(-1, g-1)\big[\Delta_1 = 1,\ \delta_1 = 0\big]$ – a node. We calculate:

$$\eta = M = 0,\ C_2 = x^3,\ \mu_0 = \mu_1 = \mu_2 = 0,\ \mu_3 = -gx^3, \quad K = 2x^2, \quad K_3 = -6gx^6.$$

Since $\mu_3 K \neq 0$ by [26, Table 4] the behavior of the trajectories in the neighborhood of infinity corresponds to *Figure 37* if $K_3 > 0$ (i.e. $g < 0$) and to *Figure 39* if $K_3 < 0$ (i.e. $g > 0$). Thus, taking into account the invariant lines $x = 0$ and $x = -1$ (double) of systems $(S_{4.31})$ we get *Picture 4.31(a)* if $K_3 > 0$ and *Picture 4.31(b)* if $K_3 < 0$.

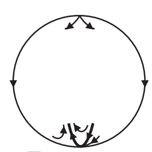


Figure 37

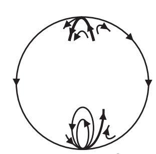


Figure 39

$$\textit{Config. 4.32}: \quad \dot{x} = x^2 + 1, \quad \dot{y} = x + gy, \quad g \neq 0. \quad (S_{4.32})$$

No finite singularities. For these systems calculations yield: $M = -x^2,\ C_2 = x^2y$,

$$\eta = \mu_0 = \mu_1 = \kappa = \kappa_1 = K = 0\ L = 8x^2,\ \mu_2 = g^2x^2,\ K_2 = -384x^2.$$

We note that $\mu_2 > 0$, $L > 0$ and $K_2 < 0$ for any value of parameter $0 \neq g \in \mathbb{R}$. According to [26, Table 4] the behavior of the trajectories at infinity corresponds to *Figure 8* (see p. 68). This leads to *Picture 4.32*.

$$\textit{Config. 4.33}: \quad \dot{x} = x^2 + 1, \quad \dot{y} = g + x, \quad g \in \mathbb{R}. \qquad (S_{4.33})$$

This family of systems does not possess real finite singularities and calculations yield:

$$M = -8x^2,\ C_2 = x^2y,\ \eta = \mu_0 = \mu_1 = \mu_2 = \mu_3 = \kappa = \kappa_1 = K = 0,\ L = 8x^2$$

and $\mu_4 = (g^2+1)x^4$, $K_2 = -384x^2$. We observe that $\mu_4 > 0$, $L > 0$ and $K_2 < 0$ for any value of the parameter $g \in \mathbb{R}$. According to [26, Table 4] the behavior of the trajectories around of infinity corresponds to *Figure 8* (see p. 68). Thus we obtain *Picture 4.33*.

$$\textit{Config. 4.34}: \quad \dot{x} = g, \quad \dot{y} = y(y - x), \quad g \in \{-1, 1\}. \qquad (S_{4.34})$$

No finite singularities. For these systems we calculate:

$$\eta = 1,\ C_2 = xy(x - y),\ \mu_0 = \mu_1 = \mu_2 = \mu_3 = \kappa = K_1 = 0,\ \mu_4 = g^2y^2(x - y)^2.$$

We note that $\mu_4 \neq 0$ and $K_1 = 0$ for any value of parameter $g \in \{-1, 1\}$. According to [26, Table 4] one of the triple points is a node and the other one is a saddle. However we need to distinguish when the point $N_1(1, 0, 0)$ is a saddle, as in this case the invariant line $y = 0$ will be a separatrix and this leads to a different phase portrait. So, we consider the corresponding systems (obtained via the transformation $x = 1/z$, $y = u/z$):

$$(S): \qquad \dot{u} = -u + u^2 - guz^2, \quad \dot{z} = -gz^3.$$

We observe that systems (S) has two invariant lines: $z = 0$ and $u = 0$. We consider the restrictions on (S) on these lines: $(S)|_{z=0}:\ \ \dot{u} = u(u - 1)$ and $(S)|_{u=0}:$ $\dot{z} = -gz^3$. On $z = 0$ and for $0 < u < 1$ we have $\dot{u} < 0$ while for $u < 0$, we have $\dot{u} > 0$. Hence on $z = 0$ the point $u = 0$ is an attractive singular point.

Now consider the restriction $(S)|_{u=0}$. We observe, that for $z > 0$, $\operatorname{sign}(\dot{z}) = -\operatorname{sign}(z)$. Hence on $u = 0$ the point $z = 0$ is an attractive singular point if $g > 0$ and it is a repulsing singular point if $g < 0$.

Thus we conclude that the triple singular point $N_1(1, 0, 0)$ of systems $(S_{4.34})$ is a node if $g > 0$ and it is a saddle if $g < 0$. On the other hand for systems $(S_{4.34})$) we have $H_4 = -48g$. So, considering invariant line $y = 0$ we get *Picture 4.34(a)* if $H_4 < 0$ and *Picture 4.34(b)* if $H_4 > 0$.

$$\textit{Config. 4.35}: \quad \dot{x} = g + y, \quad \dot{y} = xy, \quad g \in \{-1, 1\}. \qquad (S_{4.35})$$

Finite singularities: $M_1(0, -g)\big[\Delta_1 = g,\ \delta_1 = -4g\big]$. So, the point M_1 is a saddle if $g < 0$ and we claim that it is a center if $g > 0$. Indeed, translating this point to the origin of coordinates we get the systems $\dot{x} = y$, $\dot{y} = x(y - g)$, for which calculations yield: $I_1 = I_6 = I_3 = 0$, $I_2 = -2g$. So, by Lemma 7 the point M_1 is a center if and only if $g > 0$ and this proves our claim.

To determine the behavior of the trajectories around the infinity for systems $(S_{4.35})$ we calculate: $\eta = 0,\ M = -8x^2,\ C_2 = x^2y,\ \mu_0 = \mu_1 = \mu_2 = \kappa = L = 0,$ $\kappa_1 = -32,\ \mu_3 = gxy^2,\ K_1 = xy^2.$

Since $\kappa = L = 0$, $\kappa_1 \neq 0$ and $\mu_3 K_1 = gx^2y^4$ (i.e. *sign*$(g) = \mathrm{sign}\,(\mu_3 K_1)$), according to [26, Table 4] the behavior of the trajectories in the vicinity of infinity corresponds to *Figure 16* if $g < 0$ and to *Figure 9* if $g > 0$. So, considering invariant line $y = 0$ and the type of the singular point $M_1(0, -g)$ we get *Picture 4.35(a)* if $\mu_3 K_1 > 0$ and *Picture 4.35(b)* if $\mu_3 K_1 < 0$.

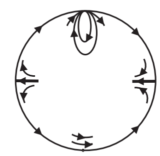


Figure 16

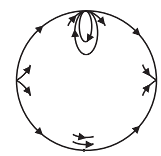


Figure 9

$$\textit{Config. 4.36}: \qquad \dot{x} = g, \quad \dot{y} = xy, g \in \{-1, 1\}. \qquad (S_{4.36})$$

No finite singularities. For these systems we calculate: $M = -8x^2,\ C_2 = -x^2y,$

$$\eta = \mu_0 = \mu_1 = \mu_2 = \mu_3 = \kappa = \kappa_1 = L = K_1 = 0,\ \kappa_2 = g,\ \mu_4 = g^2x^2y^2.$$

We note that $\mu_4 \neq 0$, $L = K_1 = 0$ and $\mathrm{sign}\,(g) = \mathrm{sign}\,(\kappa_2)$. According to [26, Table 4] the behavior of the trajectories around infinity corresponds to *Figure 8* (see p. 68) if $g < 0$ and it corresponds to *Figure 17* (see p. 67) if $g > 0$. Therefore, taking into account the invariant line $y = 0$ we obtain *Picture 4.36(a)* if $\kappa_2 < 0$ and *Picture 4.36(b)* if $\kappa_2 > 0$.

$$\textit{Config. 4.37}: \qquad \dot{x} = x, \quad \dot{y} = gy - x^2, \quad g(g^2 - 1) \neq 0. \qquad (S_{4.37})$$

Finite singularities: $M_1(0,0)\big[\Delta_1 = g,\ \delta_1 = (g-1)^2\big]$ – a saddle if $g < 0$ and a node if $g > 0$. For systems $(S_{4.37})$ calculations yield: $\eta = M = 0,\ C_2 = x^3,$ $\mu_0 = \mu_1 = \mu_2 = 0,\ \mu_3 = -gx^3,\ K = 0,\ K_1 = -x^3,\ K_3 = 6g(2-g)x^6.$

Since $K = 0$ and $\mu_3 K_1 \neq 0$ by [26, Table 4] if $\mu_3 K_1 > 0$ and $K_3 \geq 0$ then the singular point $N_1(0,1,0)$ is a saddle-node (with saddle sectors located on the same part of the line $Z = 0$). Otherwise the behavior of the trajectories around infinity corresponds to *Figure 38* if $\mu_3 K_1 > 0$ and $K_3 < 0$ and it corresponds to *Figure 33* if $\mu_3 K_1 < 0$.

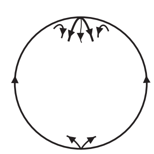


Figure 38

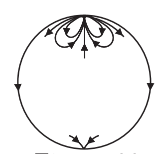


Figure 33

Thus considering the invariant line $x = 0$ and the type of the singularity $M_1(0,0)$ of $(S_{4.37})$ we obtain: *Picture 4.37(a)* when $\mu_3 K_1 > 0$ and $K_3 \geq 0$; *Picture 4.37(b)* when $\mu_3 K_1 > 0$ and $K_3 < 0$; *Picture 4.37(c)* when $\mu_3 K_1 < 0$.

$$\textit{Config. 4.38}: \qquad \dot{x} = x, \quad \dot{y} = g - x^2, \quad 0 \neq g \in \mathbb{R}. \qquad (S_{4.38})$$

No finite singularities. For these systems we calculate:

$$\eta = M = 0,\ C_2 = x^3,\ \mu_0 = \mu_1 = \mu_2 = \mu_3 = 0,\ \mu_4 = -gx^4,\ K = K_3 = 0.$$

Since $K = K_3 = 0$ by [26, Table 4] if $\mu_4 > 0$ then the point $N_1(0,1,0)$ is a node. In the case $\mu_4 < 0$ the behavior of the trajectories at infinity corresponds to *Figure 35.*

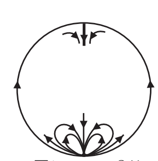


Figure 35

Thus taking into account the invariant line $x = 0$ we obtain *Picture 4.38(a)* if $\mu_4 > 0$ and *Picture 4.38(b)* if $\mu_4 < 0$.

$$\textit{Config. 4.39}: \qquad \dot{x} = x^2, \quad \dot{y} = x + y. \tag{$S_{4.39}$}$$

Finite singularities: $M_1(0,0)\big[\Delta_1 = 0,\ \rho_1 = 1\big]$ – a saddle-node [1]. Calculations yield: $\eta = 0,\ M = -8x^2,\ C_2 = x^2y,\ \mu_0 = \mu_1 = \kappa = \kappa_1 = 0,\ \mu_2 = x^2,\ L = 8x^2,$ $K_2 = 0$. As $\mu_2 > 0$, $L > 0$ and $K_2 = 0$, according to [26, Table 4] the behavior of the trajectories in the neighborhood of infinity corresponds to *Figure 19* (see p. 67).

Thus, taking into account the invariant line $x = 0$ we obtain *Picture 4.39.*

$$\textit{Config. 4.40}: \qquad \dot{x} = x + 1, \quad \dot{y} = 1 - xy. \tag{$S_{4.40}$}$$

Finite singularities: $M_1(-1,-1)\big[\Delta_1 = 1,\ \delta_1 = 0\big]$ – a node. For systems $(S_{4.40})$ we calculate: $\eta = 0,\ M = -8x^2,\ C_2 = x^2y,\ \mu_0 = \mu_1 = \mu_2 = \kappa = \kappa_1 = L = 0,$ $\mu_3 = -x^2y,\ K_1 = -x^2y.$ We observe that $L = 0$ and $\mu_3K_1 = x^4y^2 > 0$. So, according to [26, Table 4] the behavior of the trajectories at infinity corresponds to *Figure 21* (see p. 70). Considering the invariant line $x = 0$ we obtain *Picture 4.40.*

$$\textit{Config. 4.41}: \qquad \dot{x} = gxy, \quad \dot{y} = y - x^2 + gy^2, \quad g \in \{-1, 1\}. \tag{$S_{4.41}$}$$

Finite singularities: $M_1(0,0)\big[\Delta_1 = 0, \rho_1 = 1\big]$; $M_2(0, -1/g)\big[\Delta_2 = 1,\ \delta_2 = 0\big]$ – a node. We observe that M_1 is triple, as according to [1, §22] in its vicinity we obtain $\varphi(x) = \tilde{\Delta}_3\, x^3 + \ldots = gx^3 + \ldots,\ g \in \{-1, 1\}$. Moreover, since $\operatorname{sign}(\tilde{\Delta}_3) = \operatorname{sign}(g)$ by [1, §22] we conclude that the triple singular point $M_1(0,0)$ is a (topological) node if $g > 0$, and it is a (topological) saddle if $g < 0$.

We shall examine the infinite singularities. For systems $(S_{4.41})$ calculations yield: $\eta = 0 = M$, $C_2 = x^3$, $\mu_0 = -g^3 \neq 0$. Hence according to [26, Table 4] the triple singular point $N_1(0,1,0)$ is a node if $\mu_0 > 0$ (i.e. $g = -1$) and it is a saddle if $\mu_0 < 0$ (i.e. $g = 1$). So, in the first case we get *Picture 4.41(a)*, while in the second one we get *Picture 4.41(b).*

$$\textit{Config. 4.42}: \qquad \dot{x} = gxy, \quad \dot{y} = -x^2 + gy^2, \quad g \in \{-1, 1\}. \tag{$S_{4.42}$}$$

We observe that $(S_{4.42})$ are homogenous systems, which possess the triple invariant line $x = 0$. As for these systems $\eta = 0 = M,\ C_2 = x^3,\ \mu_0 = -g^3 \neq 0$, then according to [26] the infinite point $N_1(0,1,0)$ is a node if $\mu_0 > 0$ (*Picture 4.42(a)*) and it is a saddle if $\mu_0 < 0$ (*Picture 4.42(b)*).

$$\textit{Config. 4.43}: \qquad \dot{x} = gx^2, \quad \dot{y} = 1 + (g-1)xy, \quad g(g^2 - 1) \neq 0. \tag{$S_{4.43}$}$$

No finite singularities. For these systems we calculate: $\eta = \mu_0 = \mu_1 = \mu_2 = \mu_3 = \kappa = \kappa_1 = 0,\ M = -8x^2, C_2 = x^2y, \mu_4 = g^2x^4, L = 8gx^2, K = 2g(g-1)x^2, R = 8g(2g-1)x^2$.

As $\mu_4 > 0$ according to [26, Table 4] the behavior of the trajectories around infinity corresponds to *Figure 17* (see p. 67) if $L < 0$. And since $K \neq 0$, in the case $L > 0$ we have *Figure 18* (see p. 73) if $R \geq 0$ and *Figure 28* if $R < 0$. Thus, taking into account the triple invariant line $x = 0$ we obtain: *Picture 4.43(a)* if $L < 0$; *Picture 4.43(b)* if $L > 0$ and $R \geq 0$; *Picture 4.43(c)* if $L > 0$ and $R < 0$.

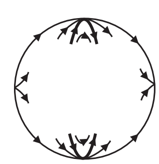


Figure 28

$$\textit{Config. 4.44}: \quad \dot{x} = x^2, \quad \dot{y} = g - x^2 + xy, \quad g \in \{-1, 1\}. \tag{$S_{4.44}$}$$

No finite singularities. For these systems we calculate: $\eta = M = 0,\ C_2 = x^3$,

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0,\ \mu_4 = g^2x^4,\ K = 2x^2,\ K_3 = -6gx^6.$$

Hence, by [26] the point $N_1(0,1,0)$ is of multiplicity seven (all finite and infinite singularities have coalesced at this point). As $\mu_4 > 0$ and $K \neq 0$, according to [26, Table 4] this point is a node if $K_3 > 0$ (i.e. $g = -1$, we get *Picture 4.44(a)*) and the behavior of the trajectories around infinity is as in *Figure 36* if $K_3 < 0$ (i.e. $g = 1$, we get *Picture 4.44(b)*).

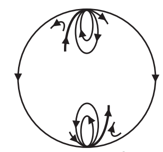


Figure 36

$$\textit{Config. 4.45}: \quad \dot{x} = gxy, \quad \dot{y} = x - x^2 + gy^2, \quad g \in \{-1, 1\}. \tag{$S_{4.45}$}$$

Finite singularities: $M_1(0,0)\big[\Delta_1 = 0, \rho_1 = 0\big]$; $M_2(1,0)\big[\Delta_2 = g,\ \delta_2 = -4g,\ \rho_2 = 0\big]$. The point M_2 is a saddle if $g < 0$ and we claim that it is a center if $g > 0$. Indeed, translating this point to the origin of coordinates we get the systems $\dot{x} = g(x+1)y,\ \dot{y} = -x - x^2 + gy^2$, for which calculations yield: $I_2 = -2g$, $I_1 = I_6 = I_3 = 0$. By Lemma 7 the point M_2 is a center if and only if $g > 0$ and this has proved our claim.

Let us examine the multiple point $M_1(0,0)$. We observe that M_1 is a nilpotent singular point. According to [1, §22] in its vicinity we calculate $\psi(x) = \tilde{a}_3x^3 + \ldots = -g^2x^3 + \ldots$, $\sigma(x) = \tilde{b}_1\, x + \ldots = 3gx$. Hence we obtain $a_3 = -g^2 < 0$ and for the quantity γ (see [1, §22]) in this case we obtain: $\gamma = \tilde{b}_1^2 + 8\tilde{a}_3 = g^2 > 0$. Therefore, the triple point is an "elliptic saddle" (i.e. a non–elementary singular point having one elliptic and one hyperbolic sectors [1]).

To determine the behavior of the trajectories at the infinity for systems ($S_{4.45}$) we calculate: $\eta = 0 = M$, $C_2 = x^3$, $\mu_0 = -g^3 \neq 0$. Hence according to [26, Table 4] the triple singular point $N_1(0,1,0)$ is a node if $\mu_0 > 0$ (*Picture 4.45(a)*) and it is a saddle if $\mu_0 < 0$ (*Picture 4.45(b)*).

$$\textit{Config. 4.46}: \quad \dot{x} = 1, \quad \dot{y} = y = y - x^2. \tag{$S_{4.46}$}$$

No finite singularities. For these systems calculations yield:

$$\eta = M = 0,\ C_2 = x^3,\ \mu_0 = \mu_1 = \mu_2 = \mu_3 = 0,\ \mu_4 = x^4,\ K = 0,\ K_3 = -6x^6.$$

Hence, by [26] the point $N_1(0,1,0)$ is of multiplicity seven seven (all finite and infinite singularities have coalesced at this point). As $\mu_4 > 0$, $K = 0$ and $K_3 < 0$, according to [26, Table 4] the behavior of the trajectories around infinity is as indicated in *Figure 32*. Thus, we obtain *Picture 4.46*.

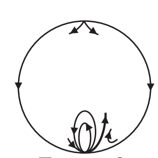


Figure 32

It remains to retain out of the 93 phase portraits *Picture 4.i(j)* in Tables 3(u), $u \in \{a,b,c,d,e\}$ only portraits which are topologically distinct. This is what we now do.

Three real singular points at infinity ($\eta > 0$)

Type of infinite singularities	Number and type of finite singularities; number of canonical regions and of separatrices					Total # of phase port-raits
	0	1	2	3	4	
(N,N,N)		4.5(b)			(S,S,S,C): 4.1(b); (S,S,S,N): 4.3(b)	3
(N,N,S)	$0SC$: 4.13(a)$\simeq$ 4.34(b); $1SC_{\infty}^{\infty}$: 4.13(b)$\simeq$ 4.34(a)	4.5(a)	$0SC_f^{\infty}$: 4.22(a); $1SC_f^{\infty}$: 4.22(b)	$0SC_f^{\infty}$,$1SC_f^f$: 4.10(a); $0SC$: 4.10(b) $1SC_f^{\infty}$,$0SC_f^f$: 4.10(c);	$0SC$: 4.1(a) $\simeq$ 4.3(a) $\simeq$ 4.9(b); $1SC_f^{\infty}$, $0SC_f^f$: 4.9(a); $0SC_f^{\infty}$, $1SC_f^f$: 4.9(c);	11
(N,S,S)		4.5(c)			(N,N,N,S): 4.3(c); (N,N,C,S): 4.1(c)	3
(N,S,S-N)				4.4(b)		1
(N,S-N,S-N)		4.17	(N,S): $0SC$: 4.18(a) $\simeq$ 4.16(b); $1SC_f^{\infty}$: 4.16(a); (S,C): 4.18(b)			4
(N,N,S-N)				4.4(a)		1
Total number of topologically distinct phase portraits						23

In order to distinguish topologically the phase portraits of the systems we obtained, we use the following invariants:

- The topological types of the infinite singularities. Whenever we have several sectors on the Poincaré disk we indicate the types of sectors, e.g. PEH means that we have three sectors (on the Poincaré disk): parabolic, elliptic, hyperbolic. In the case $\eta = 0 \neq M$ we place two opposite singularities at infinity at the north and south poles. Then for example in *Picture 4.29(b)* HHH-PEP means that the north pole has three hyperbolic sectors and the south pole has a parabolic sector followed by an elliptic sector and a parabolic one.
- Number and type of distinct finite real singular points.
- The total number SC (respectively the numbers SC_f^f, SC_f^{∞}, SC_{∞}^{∞}) of separatrix connections, i.e. of phase curves connecting two singularities which are

local separatrices of the two singular points (respectively of separatrix connections connecting two infinite singularities, a finite with an infinite singularity, two finite singularities).

One real and two complex singular points at infinity ($\eta < 0$)

Type of the infinite singularity	Number and type of finite singularities		Total number of phase portraits
	1	2	
(N)	4.8(a)	4.2(a)$\backsimeq$ 4.27(a)(S;F); 4.6(a)(S;N)	3
(S)	4.8(b)	4.2(c)(C,F); 4.2(d)(C,C); 4.6(b)(N;N)	4
(S-N)	4.7		1
(PHP-PHP)		4.2(b)$\backsimeq$ 4.27(b)	1
Total number of topologically distinct phase portraits			9

Only one singular point (real) at infinity ($\eta = 0 = M$, $C_2 \neq 0$)

Type of infinite singularity	Number and type of finite singularities			Total number of phase portraits
	0	1	2	
(N)	4.38(b)$\backsimeq$ 4.44(a)	4.31(a)(N) 4.42(a)(HH)	4.41(a)(S,N); 4.45(a) (S,HPEP)	5
(S)		4.42(b)	4.41(b)(N,N); 4.45(b)(EH,C)	3
(S-N)		4.37(a)		1
Existence of an elliptic sector	4.38(a)(HH-EE); 4.44(b)(EH-HE); 4.46(PEH-P)	4.31(b)(N); 4.37(c)(S)		5
(HPH-P)		4.37(b)		1
Total number of topologically distinct phase portraits				15

Two real singular points at infinity ($\eta = 0$, $M \neq 0$)

Type of infinite singularities	Number and type of finite singularities; number of canonical regions and of separatrices					Total # of phase portraits
	0	1	2	3	4	
(N,N)	4.43(c)				4.11(b)	2
(N,S-N)	$1SC_{\infty}^{\infty}$: 4.14 $\backsimeq$ 4.43(b); $0SC$: 4.29(a)	4.20(a)	4.23 ((N,S-N)	4.21(a)$\backsimeq$ 4.25(a)	4.11(a)	6
(N,PEP-PEP)			4.12(c)			1
(N,H-H)	4.15(a)$\backsimeq$ 4.32$\backsimeq$ 4.33$\backsimeq$ 4.36(a)					1
(N, PH-PH)		4.24(a)$\backsimeq$ 4.39	4.12(a)$\backsimeq$ 4.30(a)$\backsimeq$ 4.28			2
(N,HHH-HHH)			4.12(d)			1
(N,PEP-H)		4.35(b)				1
(N,HHH-PEP)	4.29(b)					1
(S,S-N)		4.20(b)		4.21(b) (N,N,S-N); 4.25(b) (N,C,S-N)		3
(S,EP-EP)		4.24(b)	4.12(b)			2
(S,PE-EP)	4.43(a)					1
(S,PHP-PHP)			4.30(b)$\backsimeq$ 4.12(e)			1
(S,E-H)		4.35(a)				1
(S,E-E)	4.15(b)$\backsimeq$ 4.36(b)					1
(S-N,S-N)			4.26			1
(S-N,PH-PH)		4.19(b)$\backsimeq$ 4.40				1
(S-N,EP-EP)		4.19(a)				1
Total number of topologically distinct phase portraits						27

Confrontation of phase portraits with $\eta < 0$ with those with $\eta = 0 = M$, $C_2 \neq 0$

<table>
<tr><th rowspan="2">Type of the infinite singularity in the two cases</th><th colspan="4">Number and type of finite singularities</th><th rowspan="2">Total # of phase port-raits</th></tr>
<tr><th colspan="2">1</th><th colspan="2">2</th></tr>
<tr><td rowspan="2">(N)</td><td>4.8(a)(HH)</td><td>4.31(a)(N);
4.42(a)((HH)</td><td>4.27(a)(S,F);
4.6(a)(S,N)</td><td>4.41(a)(S,N);
4.45(a)(S,HPEP)</td><td rowspan="2">5</td></tr>
<tr><td colspan="2">4.8(a)$\simeq$ 4.42(a)</td><td colspan="2">4.6(a)$\simeq$ 4.41(a)</td></tr>
<tr><td rowspan="2">(S)</td><td>4.8(b)(EE)</td><td>4.42(b)(EE)</td><td>4.2(c)(C,F);
4.2(d)(C,C);
4.6(b)(N,N)</td><td>4.41(b)(N,N);
4.45(b)(EH,C)</td><td rowspan="2">5</td></tr>
<tr><td colspan="2">4.8(b)$\simeq$ 4.42(b)</td><td colspan="2">4.6(b)$\simeq$ 4.41(b)</td></tr>
<tr><td rowspan="2">(S-N)</td><td>4.7(N)</td><td>4.37(a)(N)</td><td colspan="2" rowspan="2"></td><td rowspan="2">1</td></tr>
<tr><td colspan="2">4.7 $\simeq$ 4.37(a)</td></tr>
<tr><td colspan="5">Total number of topologically distinct phase portraits for
$\eta < 0$ or ($\eta = 0 = M$ and $C_2 \neq 0$)</td><td>24 - 5
=19</td></tr>
</table>

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Global Attractors of Quasi-Linear Non-Autonomous Difference Equations

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Abstract. The article is devoted to the study of global attractors of quasi-linear non-autonomous difference equations. We obtain the conditions for the existence of a compact global attractor. The obtained results are applied to the study of a special triangular map $T : R^2_+ \to R^2_+$ describing a growth model with logistic population growth rate.

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1 Introduction

The global attractors play a very important role in the qualitative study of difference equations (both autonomous and non-autonomous). The present work is dedicated to the study of global attractors of quasi-linear non-autonomous difference equations

$$u_{n+1} = A(\sigma(n,\omega))u_n + F(u_n, \sigma(n,\omega)), \tag{1}$$

where Ω is a metric space (generally speaking non-compact), (Ω, Z_+, σ) is a dynamical system with discrete time Z_+, $A \in C(\Omega, [E])$ and the function $F \in C(E \times \Omega, E)$ satisfies "the condition of smallness" (see condition (ii) in Theorem 4). An analogous problem was studied by Cheban D. and Mammana C. [6] when the space Ω is compact and Cheban D., Mammana C. and Michetti E. [8] in general case.

The obtained results are applied while studying a special class of triangular maps describing a discrete-time growth model of the Solow type where workers and shareholders have different but constant saving rates and the population growth rate dynamic is described by the logistic equation (see Brianzoni S., Mammana C. and Michetti E. [3]). The resulting system is given by $T = (T_2, T_1)$, where

$$T_2(u,\omega) = \frac{(1-\delta)u + (u^{\epsilon}+1)^{\frac{1-\epsilon}{\epsilon}}(s_w + s_r u^{\epsilon})}{1+\omega}$$

and

$$T_1(\omega) = \lambda\omega(1-\omega)$$

(for all $(u,\omega) \in R_+ \times [0,1]$), $\delta \in (0,1)$ is the depreciation rate of capital, $s_w \in (0,1)$ and $s_r \in (0,1)$ are the constant saving rates for workers and shareholders respectively, $\epsilon \in (-\infty, 1), \epsilon \neq 0$, is a parameter related to the elasticity of substitution between labor and capital.

This paper is organized as follows.

In Section 2 we establish the relation between triangular maps and non-autonomous dynamical systems with discrete time.

Section 3 is devoted to the study of the existence of compact global attractors of skew-product dynamical systems. The sufficient conditions of existence of compact global attractors for skew-product dynamical systems with non-compact base are given (Theorem 2).

In Section 4 we study the linear non-autonomous dynamical systems with discrete time and prove that they admit a unique compact invariant manifold and its description is given (Theorem 3).

In Section 5 we prove the existence of compact global attractors of quasi-linear dynamical systems (Theorem 5) and give the description of the structure of these attractors (Theorem 6).

In Section 6 we give some applications of general results from Sections 2–5 to the study of special class of the triangular maps $T : R^2_+ \to R^2_+$ describing a triangular growth model with logistic population growth rate as studied in Brianzoni S., Mammana C. and Michetti E. [3].

2 Triangular maps and non-autonomous dynamical systems

Let W and Ω be two complete metric spaces and denote by $X := W \times \Omega$ their Cartesian product. Recall (see, for example, [16–18]) that a continuous map $F : X \to X$ is called triangular if there are two continuous maps $f : W \times \Omega \to W$ and $g : \Omega \to \Omega$ such that $F = (f, g)$, i.e. $F(x) = F(u,\omega) = (f(u,\omega), g(\omega))$ for all $x =: (u,\omega) \in X$.

Consider a system of difference equations

$$\begin{cases} u_{n+1} = f(u_n, \omega_n) \\ \omega_{n+1} = g(\omega_n), \end{cases} \tag{2}$$

for all $n \in Z_+$, where Z_+ is the set of all non-negative integer numbers.

Along with system (2) we consider the family of equations

$$u_{n+1} = f(u_n, g^n\omega) \ (\omega \in \Omega), \tag{3}$$

which is equivalent to system (2). Let $\varphi(n, u, \omega)$ be a solution of equation (3) passing through the point $u \in W$ for $n = 0$. It is easy to verify that the map $\varphi : Z_+ \times W \times \Omega \to W$ $((n, u, \omega) \mapsto \varphi(n, u, \omega)$) satisfies the following conditions:

1. $\varphi(0, u, \omega) = u$ for all $u \in W$ and $\omega \in \Omega$;
2. $\varphi(n+m, u, \omega) = \varphi(n, \varphi(m, u, \omega), \sigma(m, \omega))$ for all $n, m \in Z_+, u \in W$ and $\omega \in \Omega$, where $\sigma(n, \omega) := g^n\omega$;

3. the map $\varphi : Z_+ \times W \times \Omega \to W$ is continuous.

Denote by (Ω, Z_+, σ) the semi-group dynamical system generated by positive powers of the map $g : \Omega \to \Omega$, i.e. $\sigma(n, \omega) := g^n\omega$ for all $n \in Z_+$ and $\omega \in \Omega$.

Recall [5, 19] that a triple $\langle W, \varphi, (\Omega, Z_+, \sigma)\rangle$ (or briefly φ) is called a cocycle over the semi-group dynamical system (Ω, Z_+, σ) with fiber W.

Let $X := W \times \Omega$ and (X, Z_+, π) be a semi-group dynamical system on X, where $\pi(n, (u, \omega)) := (\varphi(n, u, \omega), \sigma(n, \omega))$ for all $u \in W$ and $\omega \in \Omega$, then (X, Z_+, π) is called [19] a skew-product dynamical system, generated by the cocycle $\langle W, \varphi, (\Omega, Z_+, \sigma)\rangle$.

Remark 1. Thus, the reasoning above shows that every triangular map generates a cocycle and, obviously, vice versa, i.e. having a cocycle $\langle W, \varphi, (\Omega, Z_+, \sigma)\rangle$ we can define a triangular map $F : W \times \Omega \to W \times \Omega$ by the equality

$$F(u, \omega) := (f(u, \omega), g(\omega)),$$

where $f(u, \omega) := \varphi(1, u, \omega)$ and $g(\omega) := \sigma(1, \omega)$ for all $u \in W$ and $\omega \in \Omega$. The semi-group dynamical system defined by the positive powers of the map $F : X \to X$ $(X := W \times \Omega)$ coincides with the skew-product dynamical system, generated by cocycle $\langle W, \varphi, (\Omega, Z_+, \sigma)\rangle$

Taking into consideration this remark we can study triangular maps in the framework of cocycles with discrete time.

Let (X, Z_+, π) (respectively, $\langle W, \varphi, (\Omega, Z_+, \sigma)\rangle$) be a semi-group dynamical system (respectively, a cocycle).

A map $\gamma : Z \to X$ is called an entire trajectory of the semi-group dynamical system (X, Z_+, σ) passing through the point $x \in X$ (respectively, $u \in W$) if $\gamma(0) = x$ and $\gamma(n + m) = \pi(m, \gamma(n))$ for all $n \in Z$ and $m \in Z_+$.

Denote by $\Phi_\omega(\sigma)$ the set of all the entire trajectories of the semi-group dynamical system (Ω, Z_+, σ) passing through the point $\omega \in \Omega$ at the initial moment $n = 0$ and $\Phi(\sigma) := \bigcup\{\Phi_\omega(\sigma) \mid \omega \in \Omega\}$.

A map $\mu : Z \to W$ is called an entire trajectory of the cocycle $\langle W, \varphi, (\Omega, Z_+, \sigma)\rangle$ passing through the point $(u, \omega) \in W \times \Omega$ if $\mu(0) = u$ and there exists $\alpha \in \Phi_\omega(\sigma)$ such that $\mu(n + m) = \varphi(m, \mu(n), \alpha(n))$ for all $n \in Z$ and $m \in Z_+$.

Let Y be a complete metric space, (X, Z_+, π) (respectively, (Y, Z_+, σ)) be a semi-group dynamical system on X (respectively, Y), and $h : X \to Y$ be a homomorphism of (X, Z_+, π) onto (Y, Z_+, σ). Then the triple $\langle (X, Z_+, \pi), (Y, Z_+, \sigma), h\rangle$ is called a non-autonomous dynamical system.

Let W and Y be complete metric spaces, (Y, Z_+, σ) be a semi-group dynamical system on Y and $\langle W, \varphi, (Y, Z_+, \sigma)\rangle$ be a cocycle over (Y, Z_+, σ) with the fiber W (or, for short, φ), i.e. φ is a continuous mapping of $Z_+ \times W \times Y$ into W satisfying the following conditions: $\varphi(0, w, y) = w$ and $\varphi(t + \tau, w, y) = \varphi(t, \varphi(\tau, w, y), \sigma(\tau, y))$ for all $t, \tau \in Z_+, w \in W$ and $y \in Y$.

We denote $X := W \times Y$ and define on X a skew product dynamical system (X, Z_+, π) by the equality $\pi = (\varphi, \sigma)$, i.e. $\pi(t, (w, y)) = (\varphi(t, w, y), \sigma(t, y))$ for all

$t \in Z_+$ and $(w, y) \in W \times Y$. Then the triple $\langle (X, Z_+, \pi), ((Y, Z_+, \sigma), h\rangle$ is a non-autonomous dynamical system (generated by cocycle φ), where $h = pr_2 : X \mapsto Y$ is the projection on the second component.

3 Global attractors of dynamical systems

Let $\mathfrak{M}$ be a family of subsets from X.

A semi-group dynamical system (X, Z_+, π) will be called $\mathfrak{M}$-dissipative if for every $\varepsilon > 0$ and $M \in \mathfrak{M}$ there exists $L(\varepsilon, M) > 0$ such that $\pi(n, M) \subseteq B(K, \varepsilon)$ for any $n \geq L(\varepsilon, M)$, where K is a certain fixed subset from X depending only on $\mathfrak{M}$. In this case we will call K an attracting set for $\mathfrak{M}$.

For the applications the most important ones are the cases when K is bounded or compact and $\mathfrak{M} := \{\{x\} \mid x \in X\}$ or $\mathfrak{M} := C(X)$, or $\mathfrak{M} := \{B(x, \delta_x) \mid x \in X,\ \delta_x > 0\}$, or $\mathfrak{M} := B(X)$ where $C(X)$ (respectively, $B(X)$) is the family of all compact (respectively, bounded) subsets from X.

The system (X, Z_+, π) is called:

– point dissipative if there exists $K \subseteq X$ such that for every $x \in X$

$$\lim_{n \to +\infty} \rho(\pi(n, x), K) = 0; \tag{4}$$

– compactly dissipative if the equality (4) takes place uniformly w.r.t. x on the compact subsets from X.

Let (X, Z_+, π) be a compactly dissipative semi-group dynamical system and K be an attracting set for $C(X)$. We denote by

$$J := \Omega(K) = \bigcap_{n \geq 0} \overline{\bigcup_{m \geq n} \pi(m, K)},$$

then the set J does not depend of the choice of K and is characterized by the properties of the semi-group dynamical system (X, Z_+, π). The set J is called a Levinson center of the semi-group dynamical system (X, Z_+, π).

Theorem 1. *[5] Let (X, Z_+, π) be point dissipative. For (X, Z_+, π) to be compactly dissipative it is necessary and sufficient that Σ_K^+ be relatively compact for any compact $K \subseteq X$.*

Let E be a finite-dimensional Banach space and $\langle E, \varphi, (\Omega, Z_+, \sigma)\rangle$ be a cocycle over (Ω, Z_+, σ) with the fiber E (or shortly φ).

A cocycle φ is called:

- dissipative if there exists a number $r > 0$ such that

$$\limsup_{n \to +\infty} |\varphi(n, u, \omega)| \leq r \tag{5}$$

for all $\omega \in \Omega$ and $u \in E$;

- uniform dissipative on every compact subset from Ω if there exists a number $r > 0$ such that
$$\limsup_{n\to+\infty} \sup_{\omega\in\Omega', |u|\le R} |\varphi(n,u,\omega)| \le r$$
for all compact subset $\Omega' \subseteq \Omega$ and $R > 0$.

Let (X, Z_+, π) be a dynamical system and $x \in X$. Denote by ω_x the ω-limit set of point x.

Theorem 2. *The following statements hold:*

1. *if the semi-group dynamical system (Ω, Z_+, σ) and the cocycle φ are point dissipative, then the skew-product dynamical system (X, Z_+, π) is point dissipative;*
2. *if the semi-group dynamical system (Ω, Z_+, σ) is compactly dissipative and the cocycle φ is uniform dissipative on every compact subset from Ω, then the skew-product system (X, Z_+, π) is compactly dissipative.*

Proof. Let $x := (u, \omega) \in X := E \times \Omega$, then under the conditions of theorem the set $\Sigma_x^+ := \{\pi(n,x) : \ n \in Z_+\}$ is relatively compact and $\omega_x \subseteq B[0,r] \times K$, where $B[0,r] := \{u \in E : \ |u| \le r\}$, r is a number figuring in the inequality (5) and K is the compact appearing in (4). Thus the semi-group dynamical system (X, Z_+, π) is point dissipative.

According to first statement of theorem the skew-product dynamical system (X, Z_+, π) is point dissipative. Let M be an arbitrary compact subset from $X := E \times \Omega$, then there are $R > 0$ and a compact subset $\Omega' \subseteq \Omega$ such that $M \subseteq B[0,R] \times \Omega'$. Note that $\Sigma_M^+ := \{\pi(n,M) : \ n \in Z_+\} \subseteq \Sigma^+_{B[0,R]\times\Omega'} := \{(\varphi(n,u,\omega), \sigma(n,\omega)) : \ n \in Z_+, u \in B[0,R], \ \omega \in \Omega'\}$. We will show that the set Σ_M^+ is relatively compact. In fact, let $\{x_k\} \subseteq \Sigma_M^+$, then there are $\{u_k\} \subseteq B[0,R]$, $\{\omega_k\} \subseteq \Omega'$ and $\{n_k\} \subseteq Z_+$ such that $x_k = (\varphi(n_k, u_k, \omega_k), \sigma(n_k, \omega_k))$. By compact dissipativity of system (Ω, Z_+, σ) and uniform dissipativity of the cocycle φ the sequences $\{\varphi(n_k, u_k, \omega_k)\}$ and $\sigma(n_k, \omega_k))$ are relatively compact and, consequently, the sequence $\{x_k\}$ is so. Now to finish the proof it is sufficient to refer to Theorem 1. □

4 Linear non-autonomous dynamical systems

Let Ω be a complete metric space and (Ω, Z_+, σ) be a semi-group dynamical system on Ω with discrete time.

Recall that a subset $A \subseteq \Omega$ is called invariant (respectively, positively invariant, negatively invariant) if $\sigma(n, A) = A$ $(respectively, \sigma(n,A) \subseteq A, \ A \subseteq \sigma(n,A))$ for all $n \in Z_+$.

Below in this section we will suppose that the set Ω is invariant, i.e. $\sigma(n,\Omega) = \Omega$ for all $n \in Z_+$. Let E be a finite-dimensional Banach space with the norm $|\cdot|$ and W be a complete metric space. Denote by $[E]$ the space of all linear continuous operators on E and by $C(\Omega, W)$ the space of all the continuous functions $f : \Omega \to W$

endowed with the compact-open topology, i.e. the uniform convergence on compact subsets in Ω. The results of this section will be used in the next sections.

Consider a linear equation

$$u_{n+1} = A(\sigma(n,\omega))u_n \quad (\omega \in \Omega) \tag{6}$$

and an inhomogeneous equation

$$u_{n+1} = A(\sigma(n,\omega))u_n + f(\sigma(n,\omega)), \tag{7}$$

where $A \in C(\Omega, [E])$ and $f \in C(\Omega, E)$.

Recall that a linear bounded operator $P : E \to E$ is called a projection if $P^2 = P$, where $P^2 := P \circ P$.

Let $U(n,\omega)$ be the Cauchy operator of linear equation (6). Following [10] we will say that equation (6) has an exponential dichotomy on Ω if there exists a continuous projection valued function $P : \Omega \to [E]$ satisfying:

1. $P(\sigma(n,\omega))U(n,\omega) = U(n,\omega)P(\omega)$;
2. $U_Q(n,\omega)$ is invertible as an operator from $ImQ(\omega)$ to $ImQ(\sigma(n,\omega))$, where $Q(\omega) := I - P(\omega)$ and $U_Q(n,\omega) := U(n,\omega)Q(\omega)$;
3. there exist constants $0 < q < 1$ and $N > 0$ such that

   $$\|U_P(n,\omega)\| \le Nq^n \text{ and } \|U_Q(n,\omega)^{-1}\| \le Nq^n$$

   for all $\omega \in \Omega$ and $n \in Z_+$, where $U_P(n,\omega) := U(n,\omega)P(\omega)$.

Let $\omega \in \Omega$ and $\gamma_\omega \in \Phi_\omega(\sigma)$. Consider a difference equation

$$u_{n+1} = A(\gamma_\omega(n))u_n + f(\gamma_\omega(n)), \tag{8}$$

and the corresponding homogeneous linear equation

$$u_{n+1} = A(\gamma_\omega(n))u_n \quad (\omega \in \Omega). \tag{9}$$

Let (X, ρ) be a metric space with distance ρ. Denote by $C(Z, X)$ the space of all the functions $f : Z \to X$ equipped with a product topology. This topology can be metricised. For example, by the equality

$$d(f_1, f_2) := \sum_{1}^{+\infty} \frac{1}{2^n} \frac{d_n(f_1, d_2)}{1 + d_n(f_1, d_2)},$$

where $d_n(f_1, d_2) := \max\{\rho(f_1(k), f_2(k)) \mid k \in [-n, n]\}$, a distance is defined on $C(Z, X)$ which generates the pointwise topology.

If $x \in X$ and $A, B \subseteq X$, then denote by $\rho(x, A) := \inf\{\rho(x, a) \mid a \in A\}$ and $\beta(A, B) := \sup\{\rho(a, B) \mid a \in A\}$ the semi-distance of Hausdorff.

Denote by $C(X)$ (respectively, $B(X)$) the family of all compact (respectively, bounded) subsets from X, $C(\Omega, E)$ the space of all the continuous functions $f : \Omega \to E$, $C_b(\Omega, E) := \{f \in C(\Omega, E) : \ \|f\| := \sup_{\omega \in \Omega} |f(\omega)| < +\infty\}$. Note that the space $C_b(\Omega, E)$ equipped with the norm $\|\cdot\|$ is a Banach space.

Theorem 3. *Suppose that the linear equation* (6) *has an exponential dichotomy on* Ω*. Then for* $f \in C_b(\Omega, E)$ *the following statements hold:*

1. *the set* $I_\omega := \{u \in E \mid \exists \gamma_\omega \in \Phi_\omega$ *such that equation (8) admits a bounded solution* ψ_ω *defined on* Z *with the initial condition* $\psi_\omega(0) = u\}$ *is nonempty and compact;*

2. $\varphi(n, I_\omega, \omega) = I_{\sigma(n,\omega)}$ *for all* $n \in Z_+$ *and* $\omega \in \Omega$*, where* $\varphi(n, u, \omega)$ *is a solution of equation (7) with the condition* $\varphi(0, u, \omega) = u$ *and* $\varphi(n, M, \omega) := \{\varphi(n, u, \omega) \mid u \in M\}$*;*

3. *the map* $\omega \to I_\omega$ *is upper-semicontinuous, i.e.*

$$\lim_{\omega \to \omega_0} \beta(I_\omega, I_{\omega_0}) = 0$$

for every $\omega_0 \in \Omega$*, where* β *is the semi-distance of Hausdorff;*

4. *if* Ω *is compact, then the set* $I := \bigcup\{I_\omega \mid \omega \in \Omega\}$ *is also compact.*

Proof. Let $\omega \in \Omega$. Since Ω is invariant, the set $\Phi_\omega(\sigma) \neq \emptyset$. We fix $\gamma_\omega \in \Phi_\omega(\sigma)$. Under the conditions of Theorem 3 equation (9) has an exponential dichotomy on Ω with the same constants N and q that in equation (6). Then equation (8) admits the unique solution $\nu_{\gamma_\omega} : Z \to E$ with the condition

$$\|\nu_{\gamma_\omega}\|_\infty \leq N\frac{1+q}{1-q}\|f(\nu_{\gamma_\omega}(\cdot))\|_\infty \leq N\frac{1+q}{1-q}\|f\|, \tag{10}$$

where $\|b\| := \sup\{|f(\omega)| \mid \omega \in \Omega\}$ and $\|\nu_\omega\|_\infty := \sup\{|\nu_\omega(n)| \mid n \in Z\}$ (see, for example, [11, 15]). Thus, the set I_ω is not empty. From the continuity of the function $\varphi : Z_+ \times E \times \Omega \to E$ and inequality (10) it follows that the set I_ω is closed, bounded and

$$|u| \leq N\frac{1+q}{1-q}\|f\|$$

for all $u \in I_\omega$ and $\omega \in \Omega$.

The second statement of the theorem follows from the equality $S_h(\Phi_\omega(\sigma)) = \Phi_{\sigma(h,\omega)}(\sigma)$ $(h \in Z)$, where $S_h\gamma_\omega$ is an h-translation of the trajectory γ_ω, i.e. $S_h\gamma_\omega(n) := \gamma_\omega(n+h)$ for all $n \in Z$.

We will prove now the third statement. Let $\omega_0 \in \Omega$, $\omega_k \to \omega_0$, $u_k \in I_{\omega_k}$ and $u_k \to u$. To prove our statement it is sufficient to show that $u \in I_{\omega_0}$. Since $u_k \in I_{\omega_k}$, there is a trajectory $\gamma_{\omega_k} \in \Phi_{\omega_k}(\sigma)$ such that γ_{ω_k} converges to $\gamma_{\omega_0} \in \Phi_{\omega_0}(\sigma)$ in $C(Z, \Omega)$ and the equation

$$u_{n+1} = A(\gamma_{\omega_k}(n))u_n + f(\gamma_{\omega_k}(n)) \tag{11}$$

has a solution $\nu_{\gamma_{\omega_k}}$ with the initial condition $\nu_{\gamma_{\omega_k}}(0) = u_k$ and satisfying inequality (10), i.e.

$$|\nu_{\gamma_{\omega_k}}(n)| \leq N\frac{1+q}{1-q}\|f(\nu_{\gamma_{\omega_k}})\|_\infty \leq N\frac{1+q}{1-q}\|f\| \tag{12}$$

for all $n \in Z$ and $k = 1, 2, \dots$. We will show that the sequence $\{\nu_{\gamma_{\omega_k}}(n)\}$ converges for every $n \in Z$. In fact, by Tihonoff theorem the sequence $\{\nu_{\omega_k}\} \subset C(Z, E)$ is relatively compact. From equality (11) and inequality (12) it follows that every limit point of the sequence $\{\nu_{\gamma_{\omega_k}}\}$ is a (bounded on Z) solution of the equation

$$u_{n+1} = A(\gamma_{\omega_0}(n))u_n + f(\gamma_{\omega_0}(n)). \tag{13}$$

Taking into account that equation (13) admits a unique solution bounded on Z, we obtain the convergence of the sequence $\{\nu_{\gamma_{\omega_k}}\}$ in the space $C(Z, E)$. We put $\nu_0 := \lim\limits_{k \to +\infty} \nu_{\gamma_{\omega_k}}$. It is easy to see that $\nu_0(0) = u$ and, consequently, $u \in I_{\omega_0}$.

To prove the fourth assertion it is sufficient to remark that for every $\omega \in \Omega$ the set I_ω is compact, the map $\omega \to I_\omega$ is upper-semicontinuous and, consequently, the set $I := \bigcup\{I_\omega \mid \omega \in \Omega\}$ is compact. The theorem is completely proved. □

5 Global attractors of quasi-linear triangular systems

Consider a difference equation

$$u_{n+1} = \mathcal{F}(u_n, \sigma(n, \omega)) \ (\omega \in \Omega). \tag{14}$$

Denote by $\varphi(n, u, \omega)$ a unique solution of equation (14) with the initial condition $\varphi(0, u, \omega) = u$.

Equation (14) is said to be dissipative (respectively, uniform dissipative on every compact subset from Ω) if there exists a positive number r such that

$$\limsup_{n \to +\infty} |\varphi(n, u, \omega)| \le r \quad (\text{respectively, } \limsup_{n \to +\infty} \sup_{\omega \in \Omega', |u| \le R} |\varphi(n, u, \omega)| \le r)$$

for all $u \in E$ and $\omega \in \Omega$ (respectively, for all $R > 0$ and $\Omega' \in C(\Omega)$).

Consider a quasi-linear equation

$$u_{n+1} = A(\sigma(n, \omega))u_n + F(u_n, \sigma(n, \omega)), \tag{15}$$

where $A \in C(\Omega, [E])$ and the function $F \in C(E \times \Omega, E)$ satisfies "the condition of smallness" (condition (ii) in Theorem 4).

Denote by $U(k, \omega)$ the Cauchy matrix for the linear equation

$$u_{n+1} = A(\sigma(n, \omega))u_n.$$

Theorem 4. *Suppose that the following conditions hold:*

1. *there are positive numbers N and $q < 1$ such that*

$$\|U(n, \omega)\| \le Nq^n \qquad (n \in Z_+); \tag{16}$$

2. *$|F(u, \omega)| \le C + D|u|$ $(C \ge 0,\ 0 \le D < (1 - q)N^{-1})$ for all $u \in E$ and $\omega \in \Omega$.*

Then equation (15) is uniform dissipative on every compact subset from Ω*.*

Proof. Let $\varphi(\cdot,u,\omega)$ be the solution of equation (14) passing through the point $u \in E$ for $n = 0$. According to the formula of the variation of constants (see, for example,[14] and [15]) we have

$$\varphi(n,u,\omega) = U(k,\omega)u + \sum_{m=0}^{n-1} U(n-m-1,\omega)F(\varphi(m,u,\omega),\sigma(m,\omega)),$$

and, consequently,

$$|\varphi(n,u,\omega)| \leq Nq^n|u| + \sum_{m=0}^{n-1} q^{n-m-1}(C + D|\varphi(m,u,\omega)|). \tag{17}$$

We set $u(n) := q^{-n}|\varphi(n,u,\omega)|$ and, taking into account (17), obtain

$$u(n) \leq N|u| + CNq^{-1}\sum_{m=0}^{n-1} q^{-m} + DNq^{-1}\sum_{m=0}^{n-1} u(m). \tag{18}$$

Denote the right hand side of inequality (18) by $v(n)$. Note that

$$v(n+1) - v(n) = q^{-n}\frac{CN}{q} + \frac{DN}{q}u(n) \leq \frac{DN}{q}v(n) + \frac{CN}{q}q^{-n},$$

and, hence,

$$v(n+1) \leq \Big(1 + \frac{DN}{q}\Big)v(n) + \frac{CN}{q}q^{-n}.$$

From this inequality we obtain

$$v(n) \leq \Big(1 + \frac{DN}{q}\Big)^{n-1}v(1) + \frac{CN}{q}\frac{1-q^{n-1}}{1-q}.$$

Therefore,

$$|\varphi(n,u,\omega)| \leq (q + DN)^{n-1}qN|u| + \frac{CN}{q-1}(q^{n-1} - 1), \tag{19}$$

because $v(1) = N|u|$. From (19) it follows that

$$\limsup_{n\to+\infty} \sup_{\omega\in\Omega',|u|\leq R} |\varphi(n,u,\omega)| \leq \frac{CN}{1-q}$$

for all $R > 0$ and $\Omega' \in C(\Omega)$. The theorem is proved. □

Let $\langle E, \varphi, (\Omega, Z_+, \sigma)\rangle$ be a cocycle over (Ω, Z_+, σ) with the fiber E.

A family $\{I_\omega \mid \omega \in J_\Omega\}$ of nonempty compact subsets $I_\omega \subset E$ is called a compact global attractor of the cocycle φ if the following conditions are fulfilled:

1. the semi-group dynamical system (Ω, Z_+, σ) is compactly dissipative;

2. the set $I := \bigcup\{I_\omega \mid \omega \in J_\Omega\}$ is relatively compact, where J_Ω is the Levinson center of (Ω, Z_+, σ);

3. the family $I := \{I_\omega \mid \omega \in J_\Omega\}$ is invariant with respect to the cocycle φ, i.e. $\cup\{\varphi(n, I_q, q) \mid q \in (\sigma^n)^{-1}(\sigma(n,\omega))\} = I_{\sigma(n,\omega)}$ for all $n \in Z_+$ and $\omega \in J_\Omega$, where $\sigma^n := \sigma(n, \cdot)$;

4. the equality
$$\lim_{n\to+\infty} \sup_{\omega\in\Omega'} \beta(\varphi(n, K, \omega), I) = 0$$
takes place for every $K \in C(E)$ and $\Omega' \in C(\Omega)$, where $C(E)$ (respectively, $C(\Omega)$) is a family of compact subsets from E (respectively, Ω).

Lemma 1. *The cocycle φ is compactly dissipative if and only if the skew-product system (X, Z_+, π) ($X := E \times \Omega$ and $\pi := (\varphi, \sigma)$) is so.*

Proof. This statement follows directly from the correspondig definitions. □

Theorem 5. *Let (Ω, Z_+, σ) be a compactly dissipative system and φ be a cocycle generated by equation (15). Under the conditions of Theorem 4 the skew-product system (X, Z_+, π) ($X := E \times \Omega$ and $\pi := (\varphi, \sigma)$), generated by cocycle φ admits a compact global attractor.*

Proof. This statement follows directly from Theorems 4, 2 and Lemma 1. □

Remark 2. Simple examples show that under the conditions of Theorem 5 the compact global attractor $\{I_\omega \mid \omega \in \Omega\}$, generally speaking, is not trivial, i.e. the component set I_ω contains more than one point. This statement can be illustrated by the following example: $u_{n+1} = \dfrac{1}{2}u_n + \dfrac{2u_n}{1+u_n^2}$.

Theorem 6. *Let $A \in C(\Omega, [E])$ and $F \in C(E \times \Omega, E)$ and the following conditions be fulfilled:*

1. *the semi-group dynamical system (Ω, Z_+, σ) is compactly dissipative and J_Ω is its Levinson center;*

2. *there exist positive numbers N and $q < 1$ such that inequality (16) holds;*

3. *there exists $C > 0$ such that $|F(0, \omega)| \leq C$ for all $\omega \in \Omega$;*

4. *$|F(u_1, \omega) - F(u_2, \omega)| \leq L|u_1 - u_2|$ $(0 \leq L < N^{-1}(1 - q))$ for all $\omega \in \Omega$ and $u_1, u_2 \in E$.*

Then

1. *the equation (15) (the cocycle φ generated by this equation) admits a compact global attractor;*

2. there are two positive constants $\mathcal{N}$ and $\nu < 1$ such that

$$|\varphi(n, u_1, \omega) - \varphi(n, u_2, \omega)| \leq \mathcal{N}\nu^n |u_1 - u_2| \tag{20}$$

for all $u_1, u_2 \in E,\ \omega \in \Omega$ and $n \in Z_+$.

Proof. First step. We will prove that under the conditions of Theorem 6 equation (15) admits a compact global attractor $I = \{I_\omega \mid \omega \in J_\Omega\}$. In fact,

$$|F(u, \omega)| \leq |F(0, \omega)| + L|u| \leq C + L|u|$$

for all $u \in E$, where $C := \sup\{|F(0, \omega)| \mid \omega \in \Omega\}$. According to Theorems 2 and 4, equation (15) admits a compact global attractor $I = \{I_\omega \mid \omega \in J_\Omega\}$.

Second step. Let φ be the cocycle generated by equation (15). In virtue of the formula of the variation of constants, we have

$$\varphi(n, u, \omega) = U(n, \omega)u + \sum_{m=0}^{n-1} U(n - m - 1, \omega)F(\varphi(m, u, \omega), \sigma(m, \omega)).$$

Consequently,

$$\varphi(n, u_1, \omega) - \varphi(n, u_2, \omega) = U(n, \omega)(u_1 - u_2) +$$

$$\sum_{m=1}^{n-1} U(n - m - 1, A)[F(\varphi(m, u, \omega), \sigma(m, \omega)) - F(\varphi(m, u_2, \omega), \sigma(m, \omega))].$$

Thus,

$$\begin{aligned} &|\varphi(n, u_1, \omega) - \varphi(n, u_2, \omega)| \leq Nq^n(|u_1 - u_2| \\ &+ Lq^{-1} \sum_{m=0}^{n-1} q^{-m} |\varphi(m, u_1, \omega) - \varphi(m, u_2, \omega)|). \end{aligned} \tag{21}$$

Let $u(n) := |\varphi(n, u_1, \omega) - \varphi(n, u_2, \omega)|q^{-n}$. From (21) it follows that

$$u(n) \leq N\Big(|u_1 - u_2| + Lq^{-1} \sum_{m=0}^{n-1} u(m)\Big). \tag{22}$$

Denote by $v(n)$ the right hand side of (22). The following inequality

$$v(n + 1) - v(n) = LNq^{-1}u(n) \leq LNq^{-1}v(n). \tag{23}$$

holds. From (23) we obtain

$$v(n) \leq (1 + LNq^{-1})^{n-1} v(1)$$

and, since $v(1) = N|u_1 - u_2|$, we get

$$u(n) \leq (1 + LNq^{-1})N|u_1 - u_2|. \tag{24}$$

From (24) we have

$$|\varphi(n,u_1,\omega)-\varphi(n,u_2,\omega)|\le(q+LN)^{n-1}qN|u_1-u_2| \tag{25}$$

for all $u_1,u_2\in E$ and $\omega\in\Omega$.

To finish the proof of Theorem it is sufficient to put $\nu := q+LN$ and $\mathcal{N} := qN(q+LN)^{-1}$. The theorem is proved. □

Remark 3. It is possible to show that under the conditions of Theorems 3 and 6 the set I_ω contains a single point (for all $\omega\in J_\Omega$) if the mapping $\sigma(1,\cdot):\Omega\to\Omega$ is invertible. If the mapping $\sigma(1,\cdot)$ is not invertible, then the set I_ω may be very complicated (for example I_ω may be a Cantor set). Below we give an example which confirms this statement.

Example 1. Let $Y := [-1,1]$ and (Y,Z_+,σ) be a cascade generated by positive powers of the odd function g, defined on $[0,1]$ in the following way:

$$g(y)=\begin{cases} -2y & , \quad 0\le y\le \frac{1}{2} \\ 2(y-1) & , \quad \frac{1}{2}<y\le 1. \end{cases}$$

It is easy to check that $g(Y)=Y$. Let us put $X := R\times Y$ and denote by (X,Z_+,π) a semi-group dynamical system generated by the positive powers of the mapping $P:X\to X$

$$P\begin{pmatrix} u \\ y \end{pmatrix}=\begin{pmatrix} f(u,y) \\ g(y) \end{pmatrix}, \tag{26}$$

where $f(u,y) := \frac{1}{10}u+\frac{1}{2}y$. Finally, let $h=pr_2 : X\to Y$. From (26), it follows that h is a homomorphism of (X,Z_+,π) onto (Y,Z_+,σ) and, consequently, $\langle (X,Z_+,\pi),(Y,Z_+,\sigma),h\rangle$ is a non-autonomous dynamical system. Note that

$$|(u_1,y)-(u_2,y)|=|u_1-u_2|=10|P(u_1,y)-P(u_2,y)|. \tag{27}$$

From (27), it follows that

$$|P^n(u_1,y)-P^n(u_2,y)|\le \mathcal{N}e^{-\nu n}|(u_1,y)-(u_2,y)| \tag{28}$$

for all $n\in Z_+$, where $\mathcal{N}=1$ and $\nu=\ln 10$. By Theorem 6 the cocycle $\langle R,\varphi,(Y,Z_+,\sigma)\rangle$ admits a compact global attaror $I := \{I_y :\ y\in Y\}$ and φ is exponentially convergent, i.e. the inequality (20) takes place. According to [18, p.43] I_y is homeomorphic to the Cantor set for all $y\in[-1,1]$.

Remark 4. 1. If Ω is a compact metric space the close results (Sections 2–5) were established in [6].

2. The results of Sections 2–5 are true also in the case we replace the finite-dimensional Banach space E by its closed subset.

6 Applications

6.1 The model

The model we consider is a particular case of the growth model by Solow; it has been obtained while considering the standard, neoclassical one-sector growth model where the two types of agents, workers and shareholders, have different but constant saving rates as in Bohm V. and Kaas L. [4] and where the production function $F : R_+ \to R_+$, mapping capital per worker k into output per worker y, is of the CES type (as in Brianzoni S., Mammana C. and Michetti E. [1] and [2]), that is given by

$$F(u) = (1 + u^{\epsilon})^{\frac{1}{\epsilon}}. \tag{29}$$

However in the present work we add a further assumption, that is the population growth rate evolves according to the logistic law, as also considered in Brianzoni S., Mammana C. and Michetti E. [3].

The resulting system, $T = (\omega', u')$, describing capital per worker (u) and population growth rate (ω) dynamics, is given by:

$$T := \begin{cases} u' = \frac{1}{1+\omega}\left[(1-\delta)u + (u^{\epsilon}+1)^{\frac{1-\epsilon}{\epsilon}}(s_w + s_r u^{\epsilon})\right] \\ \\ \omega' = \lambda\omega(1-\omega) \end{cases} \tag{30}$$

where $\delta \in (0,1)$ is the depreciation rate of capital, $s_w \in (0,1)$ and $s_r \in (0,1)$ are the constant saving rates for workers and shareholders respectively, $\epsilon \in (-\infty, 1), \epsilon \neq 0$ is a parameter related to the elasticity of substitution between labor and capital (the elasticity of substitution between the two production factors is given by $\dfrac{1}{1-\epsilon}$) and, finally, $\lambda \in (0,4]$ for the dynamics generated by the logistic map not being explosive.

We get a dicrete-time dynamical system described by the iteration of a map of the plane of triangular type. In fact the second component of the previous system does not depend on k, therefore the map is characterized by the triangular structure:

$$T := \begin{cases} u' = g(u,\omega) \\ \\ \omega' = f(\omega) \end{cases} . \tag{31}$$

As a consequence, the dynamics of the map T are influenced by the dynamics of the one–dimensional map f, that is the well-known logistic map.

6.2 Dynamics of the logistic map $f_\lambda(x) = \lambda x(1-x)$

We recall some general results for map f_λ (see, for example, [20]). For $\lambda \in (0,4]$ the map f_λ acts from interval $[0,1]$ into itself and, consequently, it admits a compact global attractor $I_\lambda \subseteq [0,1]$. Since I_λ is connected (see, for example, Theorem 1.33 [5]) and $0 \in I_\lambda$, then $I_\lambda = [0, a_\lambda]$ $(a_\lambda \leq 1)$.

1. If $0 < \lambda \leq \lambda_0 := 1$, then $I_\lambda = \{0\}$.

2. If $\lambda_0 < \lambda < \lambda_1 := 3$, then the map f_λ has two fixed points: $x = 0$ is a repelling fixed point and $p_0 = 1 - 1/\lambda$ is an attracting fixed point. If $x \in I_\lambda \setminus \{0, p_0\}$, then $\alpha_x = 0$ and $\omega_x = p_0$.

3. If $\lambda_1 < \lambda \leq \lambda_2 := 1 + \sqrt{6}$, then the map f_λ has one repelling fixed point $x = 0$ and there is an attracting 2-periodic point p_1.

4. There exists a increasing sequence $\{\lambda_k\}_{k=0}^{\infty}$ such that
   (a) $\lambda_k \to \lambda_\infty$ as $k \to \infty$, where $\lambda_\infty \approx 3,569\ldots$.
   (b) If $\lambda_k < \lambda < \lambda_{k+1}$ $(k = 2, 3, \ldots)$, then the map f_λ has one repelling fixed point $x = 0$ and there is an attracting 2^k-periodic point p_k.

5. For all $0 < \lambda < \lambda_\infty$ the structure of the attractor I_λ is sufficiently simple. Every trajectory is asymptotically periodic. There exists a unique attracting 2^m-periodic point p (the number m depends on λ) which attracts all trajectory from $[0, 1]$, except for a countable set of points. For $\lambda \geq \lambda_\infty$ the attractor I_λ is more complicated, in particularly, it may be a strange attractor (see [20]).

Let (X, Z_+, π) be a semi-group dynamical system with discrete time.

A number m is called an ε-almost period of the point x if $\rho(\pi(m+n, x), \pi(n, x)) < \varepsilon$ for all $n \in Z_+$.

The point x is called almost periodic if for any $\varepsilon > 0$ there exists a positive number $l \in Z_+$ such that on every segment (in Z_+) of length l there may be found an ε-almost period of the point x.

(vi) Denote by $Per(f_\lambda)$ the set of all periodic points of f_λ. If $\lambda = \lambda_\infty$, then the map f_λ has the 2^i-periodic point p_i for all $i \in Z_+$ (all the points p_i are repelling). The boundary $K = \partial Per(f_\lambda)$ of set $P(f_\lambda)$ is a Cantor set. The set K is an almost periodic minimal and it does not contain periodic points. The set K attracts all trajectory from $[0, 1]$, except for a countable set of points $P = \cup_{i=0}^{\infty} f_\lambda^{-i}(Per(f_\lambda))$. If $x \in [0, 1] \setminus P$, then $\omega_x = K$ (see [20]).

6.3 Existence of an attractor for $\epsilon \in (-\infty, 0)$

Lemma 2. *Let $(R_+ \times [0, 1], T)$ be a triangular map admitting a compact global attractor $J \subset R_+ \times [0, 1]$. If $p \in [0, 1]$ is a m-periodic point of the map $T_1 : [0, 1] \mapsto [0, 1]$ ($T = (T_2, T_1)$), then*

1. *$J_p = I_p \times \{p\}$, where $I_p = [a_p, b_p]$ ($a_p, b_p \in R_+$ and $a_p \leq b_p$);*

2. *there exists $q \in I_p = [a_p, b_p]$ such that (q, p) is an m-periodic point of the map T.*

Proof. Let $p \in [0,1]$ be an m-periodic point of T_1, i.e. $T_1^m(p) = p$. Denote by $S := T^m$ the mapping from $X_p := R_+ \times \{p\}$ into itself. Then, the semi-group dynamical system (X_p, S) is compactly dissipative and its Levinson center coincides with $J_p = I_p \times \{p\}$. By Theorem 1.33 from [5] the compact set $I_p \subset R_+$ is connected and, consequently, there are $a_p, b_p \in R_+$ such that $a_p \le b_p$, $I_p = [a_p, b_p]$ and

$$U(m,p)[a_p, b_p] = [a_p, b_p], \tag{32}$$

where $T^m(q,p) = (U(m,p)q, T_1^m(p))$ for all $(q,p) \in R_+ \times [0,1]$. Since $U(m,p)$ is a continuous mapping from $[a_p, b_p]$ onto itself, then there exists at least one $q \in [a_p, b_q]$ such that $U(m,p)q = q$. It is evident that (q,p) is an m-periodic point of the mapping $T = (T_2, T_1)$. □

Theorem 7. *For all $\epsilon < 0$ the dynamical system $(R_+ \times [0,1], T)$ admits a compact global attractor $J \subset R_+ \times [0,1]$. If $p \in [0,1]$ is an m-periodic point of the map $T_1 : [0,1] \mapsto [0,1]$ ($T = (T_2, T_1)$), then*

1. *$J_p = I_p \times \{p\}$, where $I_p = [a_p, b_p]$ ($a_p, b_p \in R_+$ and $a_p \le b_p$);*
2. *there exists $q \in I_p = [a_p, b_p]$ such that (q,p) is an m-periodic point of the map T.*

Proof. Assume $\epsilon \in (-\infty, 0)$ and let $\lambda = -\epsilon$, then $\lambda \in (0, +\infty)$. We write T_1 in terms of λ

$$\begin{aligned} T_1(u,\omega) &= \frac{1}{1+\omega}\left[(1-\delta)u + (u^{-\lambda}+1)^{\frac{1+\lambda}{-\lambda}}(s_w + s_r u^{-\lambda})\right] = \\ &= \frac{1}{1+\omega}\left[(1-\delta)u + \left(\frac{1+u^\lambda}{u^\lambda}\right)^{-\frac{1+\lambda}{\lambda}}\left(\frac{s_r + s_w u^\lambda}{u^\lambda}\right)\right] = \\ &= \frac{1}{1+\omega}\left[(1-\delta)u + \left(\frac{u^\lambda}{1+u^\lambda}\right)^{\frac{1+\lambda}{\lambda}}\left(\frac{s_r + s_w u^\lambda}{u^\lambda}\right)\right] = \\ &= \frac{1}{1+\omega}\left[(1-\delta)u + \frac{u}{(1+u^\lambda)^{\frac{1+\lambda}{\lambda}}}(s_r + s_w u^\lambda)\right] = \\ &= \frac{1}{1+\omega}\left[(1-\delta)u + \frac{u}{(1+u^\lambda)^{\frac{1}{\lambda}}}\frac{s_r + s_w u^\lambda}{1+u^\lambda}\right]. \end{aligned} \tag{33}$$

Note that $\dfrac{u}{(1+u^\lambda)^{\frac{1}{\lambda}}} \longrightarrow 1$ as $u \longrightarrow +\infty$, $\dfrac{s_r + s_w u^\lambda}{1+k^\lambda} \longrightarrow s_w$ as $u \longrightarrow +\infty$ and, consequently, there exists $M > 0$ such that

$$\left|\frac{u}{(1+u^\lambda)^{\frac{1}{\lambda}}}\frac{s_r + s_w u^\lambda}{1+u^\lambda}\right| \le M, \tag{34}$$

for all $u \in [0, +\infty)$.

Since $0 \leq \frac{1}{1+\omega} \leq 1$ for all $\omega \in [0,1]$, then from (33) and (34) we obtain

$$0 \leq T_1(u,\omega) \leq \alpha u + M \tag{35}$$

for all $(u,\omega) \in R_+ \times [0,1]$, where $\alpha := 1 - \delta > 0$.

Since the map T is triangular, to prove the first statement of Theorem it is sufficient to apply Theorem 5. The second statement follows from Lemma 2. □

Remark 5. 1. It is easy to see that the previous theorem is true also for $\delta = 1$ because in this case $\alpha = 1 - \delta = 0$ and from (35) we have $T_1(u,\omega) \leq M$, $\forall (u,\omega) \in R_+ \times [0,1]$. Now it is sufficient to refer to Theorem 2.

2. If $\delta = 0$ the problem is open.

6.4 Existence of an attractor for $\epsilon \in (0,1)$ and $s_r < \delta$

The semi-group dynamical system (X, Z_+, π) is said to be:

- locally completely continuous if for every point $p \in X$ there exist $\delta = \delta(p) > 0$ and $l = l(p) > 0$ such that $\pi^l B(p,\delta)$ is relatively compact;
- weakly dissipative if there exists a nonempty compact $K \subseteq X$ such that for every $\varepsilon > 0$ and $x \in X$ there is $\tau = \tau(\varepsilon, x) > 0$ for which $\pi(\tau, x) \in B(K, \varepsilon)$. In this case we will call K a weak attractor.

Note that every semi-group dynamical system (X, Z_+, π) defined on the locally compact metric space X is locally completely continuous.

Theorem 8. *[5] For the locally completely continuous dynamical systems the weak, point and compact dissipativity are equivalent.*

Theorem 9. *For all $\epsilon \in (0,1)$ and $s_r < \delta$ the dynamical system $(R_+ \times [0,1], T)$ admits a compact global attractor $J \subset R_+ \times [0,1]$. If $p \in [0,1]$ is an m-periodic point of the map $T_1 : [0,1] \mapsto [0,1]$ $(T = (T_2, T_1))$, then*

1. *$J_p = I_p \times \{p\}$, where $I_p = [a_p, b_p]$ $(a_p, b_p \in R_+$ and $a_p \leq b_p)$;*
2. *there exists $q \in I_p = [a_p, b_p]$ such that (q,p) is an m-periodic point of the map T.*

Proof. If $\epsilon \in (0,1)$ we have

$$T_1(u,\omega) = \frac{1}{1+\omega}\left[(1-\delta)u + (u^\epsilon + 1)^{\frac{1-\epsilon}{\epsilon}}(s_w + s_r u^\epsilon)\right] =$$

$$= \frac{1}{1+\omega}\left[(1-\delta)u + \frac{(u^\epsilon+1)^{\frac{1}{\epsilon}}}{1+u^\epsilon}(s_w + s_r u^\epsilon)\right] =$$

$$= \frac{1}{1+\omega}\left[(1-\delta)u + s_r u + \theta(u)u\right] \tag{36}$$

where $\theta(u) \to 0$ as $u \to +\infty$. In fact $\dfrac{(u^\epsilon+1)^{\frac{1}{\epsilon}}}{u} \to 1$ as $u \to +\infty$ while $\dfrac{(s_w + s_r u^\epsilon)}{1+u^\epsilon} \to s_r$ as $u \to +\infty$ and, consequently,

$$\frac{\dfrac{(u^\epsilon+1)^{\frac{1}{\epsilon}}}{1+u^\epsilon}(s_w + s_r u^\epsilon)}{s_r u} = \frac{(u^\epsilon+1)^{\frac{1}{\epsilon}}}{u}\frac{(s_w + s_r u^\epsilon)}{s_r(u^\epsilon+1)} \to 1$$

as $u \to +\infty$, i.e. $\dfrac{(u^\epsilon+1)^{\frac{1}{\epsilon}}}{1+u^\epsilon}(s_w + s_r u^\epsilon) = s_r u + \theta(u)u$. From (36) we have

$$T_1(u,\omega) = \frac{1}{1+\omega}\left[(1-\delta+s_r)u + \theta(u)u\right]$$

for all $(u,\omega) \in R_+^2$.

Since $s_r < \delta$ then $\alpha := 1-\delta+s_r < 1$. Let $R_0 > 0$ be a positive number such that

$$|\theta(u)| < \frac{1-\alpha}{2}, \tag{37}$$

for all $u > R_0$. Note that for every $(u_0,\omega_0) \in R_+ \times [0,1]$, with $u_0 > R_0$, the trajectory $\{T^n(u,\omega) \mid n \in Z_+\}$ starting from point (u_0,ω_0) at the initial moment $n = 0$, at least one time intersects the compact $K_0 := [0,h_0] \times [0,R_0]$, $(h_0 > h)$. In fact, if we suppose that this statement is false, then there exists a point $(u_0,\omega_0) \in R_+ \times [0,1] \setminus K_0$ such that

$$(u_n,\omega_n) := T^n(u_0,\omega_0) \in R_+ \times [0,1] \setminus K_0 \tag{38}$$

for all $n \in Z_+$. Taking into consideration that $\omega_n \to h$ (or 0) as $n \to +\infty$, we obtain from (38) that $u_n > R_0$ for all $n \geq n_0$, where n_0 is a sufficiently large number from Z_+. Without loss of generality, we may suppose that $n_0 = 0$ (if $n_0 > 0$ then we start from the initial point $(u_{n_0},\omega_{n_0}) := T^{n_0}(u_0,\omega_0)$, where $T^{n_0} := T \circ T^{n_0-1}$ for all $n_0 \geq 2$). Thus we have

$$u_n > R_0 \tag{39}$$

for all $n \geq 0$ and

$$u_{n+1} = \frac{1}{1+\omega}\left[\alpha u_n + \theta(u_n)u_n\right] \tag{40}$$

From (37) and (40) we obtain

$$u_{n+1} \leq \alpha u_n + \frac{1-\alpha}{2}u_n = \frac{1+\alpha}{2}u_n \tag{41}$$

since $\dfrac{1}{1+\omega} \leq 1$ for all $\omega \geq 0$. From (41) we have

$$u_n \leq \left(\frac{1+\alpha}{2}\right)^n u_0 \to 0 \text{ as } n \to +\infty, \tag{42}$$

but (39) and (42) are contradictory. The obtained contradiction proves the statement. Let now $(u_0,\omega_0) \in R_+ \times [0,1]$ be an arbitrary point.

(a) If $u_0 < R_0$ and $u_n \leq R_0$ for all $n \in N$, then $\limsup_{n \to +\infty} u_n \leq R_0$;

(b) If there exists $n_0 \in N$ such that $u_{n_0} > R_0$, then there exists $m_0 \in N$ $(m_0 > n_0)$ such that $(u_{m_0}, \omega_{m_0}) \in K_0$ (see the proof above).

Thus we proved that for all $(u_0, \omega_0) \in R^2_+$ there exists $m_0 \in N$ such that $(u_{m_0}, \omega_{m_0}) \in K_0$. According to Theorem 8 the semi-group dynamical system $(R_+ \times [0,1], T)$ admits a compact global attractor.

The second statement follows from Lemma 2. The theorem is proved. □

6.5 Structure of the attractor

Lemma 3. *Suppose that the following conditions are fulfilled:*

1. *$(R_+ \times [0,1], T)$ is a triangular map admitting a compact global attractor $J \subset R_+ \times [0,1]$;*

2. *$p \in [0,1]$ is a periodic point of the map $T_1 : [0,1] \mapsto [0,1]$ $(T = (T_2, T_1))$;*

3. *there are two positive numbers $\mathcal{N}$ and $q < 1$ such that*

$$\rho(T^n(u_1, \omega), T^n(u_2, \omega)) \leq \mathcal{N} q^n \rho(u_1, u_2) \tag{43}$$

for all $(u_i, \omega) \in R_+ \times [0,1]$ $(i = 1, 2)$ and $n \in N$.

Then $J_p = I_p \times \{p\}$, where $I_p = [a_p, b_p]$ $(a_p, b_p \in R_+$ and $a_p = b_p$, i.e. I_p consists of a single point.

Proof. To prove this statement we note that from the conditions (43) and (32) we have

$$diam(J_p) = diam(T^{mk}(J_p)) \leq \mathcal{N} q^k diam(J_p) \tag{44}$$

for all $k \in N$. From the inequality (44) we obtain $diam(J_p) = 0$. Taking into consideration the equalities $J_p = I_p \times \{p\}$ and (32) we obtain $a_p = b_p$. □

Theorem 10. *[9] Let X be a compact metric space and $\langle (X, Z_+, \pi), (\Omega, Z_+, \sigma), h \rangle$ be a non-autonomous dynamical system. Suppose that the following conditions are fulfilled:*

1. *The point $\omega \in \Omega$ is almost periodic;*

2. *$\lim_{t \to +\infty} \rho(\pi(t, x_1), \pi(t, x_2)) = 0$ for all $x_1, x_2 \in X$ such that $h(x_1) = h(x_2)$.*

Then there exists a unique almost periodic point $x_\omega \in X_\omega$ such that

$$\lim_{t \to +\infty} \rho(\pi(t, x), \pi(t, x_\omega)) = 0$$

for all $x \in X_\omega$.

Theorem 11. *Suppose that $\epsilon < 0$ and one of the following conditions holds:*

1. $s_w < \min\{\delta, s_r\}$ *and* $0 < \lambda < \lambda_0$, *where* λ_0 *is a positive root of the quadratic equation* $(s_r - s_w)\lambda^2 + (s_r - 2\delta)\lambda - \delta = 0$;
2. $s_r < s_w < \delta$.

Then

1. *the semi-group dynamical system* $(R_+ \times [0,1], T)$ *admits a compact global attractor* $J \subset R_+ \times [0,1]$;
2. *if* $p \in [0,1]$ *is an* m*-periodic (respectively, almost periodic) point of the map* $T_1 : [0,1] \mapsto [0,1]$ $(T = (T_2, T_1))$, *then* $J_p = I_p \times \{p\}$, *where* $I_p = [a_p, b_p]$ $(a_p, b_p \in R_+$ *and* $a_p = b_p$, *i.e.* I_p *consists of a single* m*-periodic (respectively, almost periodic) point .*

Proof. Assume $\epsilon \in (-\infty, 0)$ and let $\lambda = -\epsilon$, then $\lambda \in (0, +\infty)$. We write T_1 in terms of λ (see the proof of Theorem 9)

$$T_1(u,\omega) = \frac{1}{1+\omega}\left[(1-\delta)u + \frac{u}{(1+u^\lambda)^{\frac{1}{\lambda}}}\frac{s_w + s_r u^\lambda}{1+u^\lambda}\right].$$

Denote by

$$f(u) := \frac{u}{(1+u^\lambda)^{\frac{1}{\lambda}}}\frac{s_w + s_r u^\lambda}{1+u^\lambda},$$

then

$$f'(u) = \frac{s_w + (-s_w\lambda + (\lambda+1)s_r)u^\lambda}{(1+u^\lambda)^{2+1/\lambda}}.$$

It is easy to verify that under the conditions of theorem $f'(u) < s_w$ for all $u \geq 0$. Consider the non-autonomous difference equation

$$u_{n+1} = A(\sigma(n,\omega))u_n + F(u_n, \sigma(n,\omega)) \tag{45}$$

corresponding to triangular map $T = (T_1, T_2)$, where $A(\omega) := \frac{1}{\omega+1}$, $F(u,\omega) := \frac{1}{\omega+1}f(u)$ and $\sigma(n,\omega) := T_2^n(\omega)$ for all $n \in Z_+$ and $\omega \in [0,1]$. Under the conditions of theorem we can apply Theorem 6. By this theorem the semi-group dynamical system $(R_+ \times [0,1], T)$ is compactly dissipative with Levinson center J and there are two positive numbers $\mathcal{N}$ and $q < 1$ such that

$$\rho(T^n(u_1,\omega), T^n(u_2,\omega)) \leq \mathcal{N} q^n \rho(u_1, u_2) \tag{46}$$

for all $(u_i, \omega) \in R_+ \times [0,1]$ $(i = 1,2)$. To finish the proof of theorem it is sufficient to apply Lemma 3 and Theorem 10. □

6.6 Conclusion

Under the conditions of Theorem 7 or 9 the mapping $T = (T_2, T_1)$ $(T_1 = f_\lambda)$ admits a compact global attractor $J_\lambda \subset R_+ \times [0,1]$. There exists an increasing sequence $\{\lambda_k\}_{k=0}^{\infty}$ such that

1. $\lambda_k \to \lambda_\infty$ as $k \to \infty$, where $\lambda_\infty \approx 3,569\ldots$.
2. If $\lambda_k < \lambda < \lambda_{k+1}$ $(k = 2, 3, \dots)$, then the map $T = (T_2, T_1)$ has at least one fixed point $(q_0, 0) \in J_\lambda$ and there is a 2^k-periodic point $(q_k, p_k) \in J_\lambda$.
3. For $\lambda \geq \lambda_\infty$ the set J_λ may be a strange attractor. For example, under the conditions of Theorem 11, for $\lambda = \lambda_\infty$ the attractor J_λ contains an almost periodic (but not periodic) minimal set.

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Determinantal Analysis of the Polynomial Integrability of Differential Systems

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Abstract. This work deals with the polynomial and formal (formal series) integrability of the polynomial differential systems around a singular point, namely the conditions which assure the start of the algorithmic process for computing the polynomial or the formal first integrals. When the linear part of the differential system is nonzero, we have established ([9]) the existence of the so called starting equations whose (integer) solutions are exactly the partition of the lower degree of the eventual formal first integrals.

In this work, we study some extensions of the starting equations to the case when the linear part is zero and, particularly, to the bidimensionnal homogeneous differential systems. The principal tool used here is the classical invariant theory.

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1 Introduction

Many works are devoted to the investigation of local (or global) formal first integrals of the differential system

$$\frac{dx^i}{dt} = P^i(x), \quad i = 1, 2, \ldots, n \tag{1}$$

where P^i are polynomials of degree m with coefficients in the field $\mathbb{C}$.

Definition 1. *A function $F \in \mathcal{C}^1(\mathcal{O})$ where $\mathcal{O}$ is an open set of $\mathbb{C}^n$, is a first integral of the differential system (1) if*

$$\forall\, x \in \mathcal{O}, \quad \Delta_P(F) = \sum_{j=1}^{n} \frac{\partial F}{\partial x^j}(x)\, P^j(x) = 0. \tag{2}$$

It is well known that around a regular point x_0, there are exactly $n-1$ functionally independent analytical first integrals. This result is just theoretic.

Around a singular point and for practical and computational reasons, many people [1, 3, 5] have oriented their investigations to special classes of integrals like the polynomial, rational, algebraic or exponential ones. It occurs that these different types of first integrals need the knowledge of the polynomial ones and so, the knowledge of the degree of the first terms.

The same problem interested other mathematicians ([6–8]) which studied the local or analytical integrability. In these works, the question of the resonance of the eigenvalues of the linear *nonzero* part plays a fundamental role.

In [9], when the lower degree of P is 1, was given a constructive method to get, for any dimension n, a so called starting equation satisfied by the lower degree of an eventual first integral F.

This work consists in three parts. In the first one (Subsection 2.4), we give an "extension" of the starting equation ([9]) to the general homogeneous polynomial differential systems. In the second part (Section 3), we study the bidimensional homogeneous systems : we explicitly calculated the matrices whose the kernel contain the polynomial first integrals and we have reduced the integrability problem (existence of a polynomial first integral) to the nullity of some determinant. Finally, in the third part, we use the classical invariant theory to present significiant simplifications for computing the above determinant.

2 Notations and matricial writing of the integrability problem

2.1 The total-lexicographic order

Let $\mathrm{Sym}(n\,,\,k)$ be the linear space of the homogeneous algebraic forms of degree k in n variables and $\mathrm{S}(n)$ the infinite dimensional space of the formal series (expansions) :

$$\mathrm{S}(n)\;=\;\bigoplus_{k=1}^{\infty}\mathrm{Sym}(n\,,\,k),$$

where ([14, p. 21])

$$\dim(\mathrm{Sym}(n\,,\,k))\;=\;\frac{n(n+1)(n+2)\,\cdots\,(n+k-1)}{k!}\;=\;\frac{(n+k-1)!}{(n-1)!k!}.$$

We identify the set of the multi-degrees of the multivariate polynomials with $\mathbb{N}^n$. The total degree of the monomial $(x^1)^{i_1}(x^2)^{i_n}\cdots(x^n)^{i_n}$ is, by definition, equal to $|i|\,=\,i_1\,+\,i_2\,+\,\cdots\,+\,i_n$.

The n-fold cartesian set $\mathbb{N}^n$ is provided by the total-lexicographical order :

$$i\,\geq\,j \Longleftrightarrow \begin{cases} |i|\,>\,|j| \\ \text{or} \\ |i|\,=\,|j| \text{ and the left-most nonzero entry of } i-j \quad \text{is} \quad \text{positive,} \end{cases}$$

which induces a total order over the monomials

$$\{(x^1)^{i_1}(x^2)^{i_2}\cdots(x^n)^{i_n};\quad |i|\,=\,k\}.$$

Any algebraic form of $\mathrm{Sym}(n\,,\,k)$, $\sum_{|i|=k} f_{i_1,i_2,\ldots,i_n}\,(x^1)^{i_1}(x^2)^{i_2}\cdots(x^n)^{i_n}$, can be written as follows :

$$[f_{k,0,0,\ldots,0},\ f_{k-1,1,0,\ldots,0},\ f_{k-1,0,1,\ldots,0},\ldots,\ f_{0,0,0,\ldots,k}]\begin{bmatrix}(x^1)^k\\(x^1)^{k-1}(x^2)^1\\(x^1)^{k-1}(x^3)^1\\\vdots\\(x^n)^k\end{bmatrix} = F_k\,X^k, \quad (3)$$

where F_k and X^k denote the corresponding row and column vectors. Using this notation, the formal series $\sum_{k=1}^{\infty}\sum_{|i|=k} f_{i_1,i_2,\ldots,i_n}\,(x^1)^{i_1}(x^2)^{i_2}\cdots(x^n)^{i_n}$ and the right side of (1) with vanishing linear part become respectively

$$F_1\,X^1 + F_2\,X^2 + \cdots + F_k\,X^k + \cdots\,, \qquad P_l\,X^l + P_{l+1}\,X^{l+1} + \cdots + P_m\,X^m, \quad (4)$$

where, this time, P_i $(i = l, l+1, \ldots, m)$ denotes a matrix with n rows and $\dfrac{(n+i-1)!}{(n-1)!i!}$ columns.

2.2 The integrability conditions

The formal series $F(x) = F_1\,X^1 + F_2\,X^2 + \cdots + F_k\,X^k + \cdots$ is a first integral of the system (1) if, by definition, $\Delta_P(F) = 0$ i.e.

$$\sum_{j=1}^{n}\frac{\partial(F_1\,X^1 + F_2\,X^2 + \cdots)}{\partial x^j}\Big[P_l^j X^l + P_{l+1}^j X^{l+1} + \cdots + P_m^j X^m\Big] = 0.$$

After collecting the terms w.r.t. the total degree, we obtain an infinite sequence of conditions :

$$\begin{aligned}
l:&\quad \sum_{j=1}^{n}\frac{\partial(F_1X^1)}{\partial x^j}P_l^jX^l = 0,\\
l{+}1:&\quad \sum_{j=1}^{n}\Big[\frac{\partial(F_1X^1)}{\partial x^j}P_{l+1}^jX^{l+1} + \frac{\partial(F_2X^2)}{\partial x^j}P_l^jX^l\Big] = 0,\\
l{+}2:&\quad \sum_{j=1}^{n}\Big[\frac{\partial(F_1X^1)}{\partial x^j}P_{l+2}^jX^{l+2} + \frac{\partial(F_2X^2)}{\partial x^j}P_{l+1}^jX^{l+1} + \frac{\partial(F_3X^3)}{\partial x^j}P_l^jX^l\Big] = 0,\\
\vdots&\qquad\qquad\vdots\\
m:&\quad \sum_{j=1}^{n}\Big[\frac{\partial(F_1X^1)}{\partial x^j}P_m^jX^m + \frac{\partial(F_2X^2)}{\partial x^j}P_{m-1}^jX^{m-1} + \cdots + \frac{\partial(F_{m-l+1}X^{m-l+1})}{\partial x^j}P_l^jX^l\Big] = 0,\\
m{+}1:&\quad \sum_{j=1}^{n}\Big[\frac{\partial(F_2X^2)}{\partial x^j}P_m^jX^m + \frac{\partial(F_3X^3)}{\partial x^j}P_{m-1}^jX^{m-1} + \cdots + \frac{\partial(F_{m-l+2}X^{m-l+2})}{\partial x^j}P_l^jX^l\Big] = 0,\\
\vdots&\qquad\qquad\vdots
\end{aligned}$$

Putting $p = \min(k, m)$, the equation corresponding to the total degree k in x is :

$$\sum_{j=1}^{n} \Big[\frac{\partial(F_{k+1-p}X^{k+1-p})}{\partial x^j} P_p^j X^p + \frac{\partial(F_{k+2-p}X^{k+2-p})}{\partial x^j} P_{p-1}^j X^{p-1} + \\ + \cdots + \frac{\partial(F_{k-l+1}X^{k-l+1})}{\partial x^j} P_l^j X^l \Big] = 0. \qquad (5)$$

Since the equation (5) is homogeneous of degree k in the coordinates of x, there are $N = p - l + 1$ matrices, denoted $M_{[i,k]}$ ($i = k-p+1, \ldots, k-l+1$), such that the previous equation can be rewritten in the form :

$$\Big[F_{k-p+1} M_{[k-p+1,k]} + F_{k-p+2} M_{[k-p+2,k]} + \cdots + F_{k-l+1} M_{[k-l+1,k]} \Big] X^k = 0. \qquad (6)$$

For the differential systems of lower degree l, $k = l,\ l+1,\ l+2, \ldots$, we get :

$$\begin{cases} F_1\, M_{[1,\, l]} = 0 \\ F_1\, M_{[1,\, l+1]} + F_2\, M_{[2,\, l+1]} = 0 \\ F_1\, M_{[1,\, l+2]} + F_2\, M_{[2, l+2]} + F_3\, M_{[3, l+2]} = 0 \\ \qquad\qquad \vdots \\ \qquad\qquad \vdots \end{cases} \qquad (7)$$

The matrix $M_{[i,k]}$ has exactely $\dfrac{(n+i-1)!}{(n-1)!i!}$ rows and $\dfrac{(n+k-1)!}{(n-1)!k!}$ columns.

We give some examples of the matrices $M_{[d, l+d-1]}$ when $n = 2$.

<u>l = 2</u>; $P^1(x) = \sum_{i=0}^{2} \binom{2}{i} a_i (x^1)^{2-i}(x^2)^i$, $P^2(x) = \sum_{i=0}^{2} \binom{2}{i} b_i (x^1)^{2-i}(x^2)^i$.

$$M_{[1,2]} = \begin{bmatrix} a_0 & 2a_1 & a_2 \\ b_0 & 2b_1 & b_2 \end{bmatrix}, \qquad M_{[2,3]} = 2 \begin{bmatrix} a_0 & 2a_1 & a_2 & 0 \\ b_0 & a_0 + 2b_1 & 2a_1 + b_2 & a_2 \\ 0 & b_0 & 2b_1 & b_2 \end{bmatrix},$$

$$M_{[3,4]} = 3 \begin{bmatrix} a_0 & 2\,a_1 & a_2 & 0 & 0 \\ b_0 & 2\,b_1 + 2\,a_0 & b_2 + 4\,a_1 & 2\,a_2 & 0 \\ 0 & 2\,b_0 & 4\,b_1 + a_0 & 2\,b_2 + 2\,a_1 & a_2 \\ 0 & 0 & b_0 & 2\,b_1 & b_2 \end{bmatrix},$$

$$M_{[4,5]} = 4 \begin{bmatrix} a_0 & 2\,a_1 & a_2 & 0 & 0 & 0 \\ b_0 & 3\,a_0 + 2\,b_1 & b_2 + 6\,a_1 & 3\,a_2 & 0 & 0 \\ 0 & 3\,b_0 & 6\,b_1 + 3\,a_0 & 3\,b_2 + 6\,a_1 & 3\,a_2 & 0 \\ 0 & 0 & 3\,b_0 & 6\,b_1 + a_0 & 2\,a_1 + 3\,b_2 & a_2 \\ 0 & 0 & 0 & b_0 & 2\,b_1 & b_2 \end{bmatrix}.$$

l = 3; $P^1(x) = \sum_{i=0}^{3}\binom{3}{i}a_i(x^1)^{3-i}(x^2)^i$, $P^3(x) = \sum_{i=0}^{3}\binom{3}{i}b_i(x^1)^{3-i}(x^2)^i$.

$$M_{[1,3]}=\begin{bmatrix} a_0 & 3\,a_1 & 3\,a_2 & a_3 \\ b_0 & 3\,b_1 & 3\,b_2 & b_3 \end{bmatrix},\ M_{[2,4]}=2\begin{bmatrix} a_0 & 3a_1 & 3a_2 & a_3 & 0 \\ b_0 & 3b_1+a_0 & 3b_2+3a_1 & b_3+3a_2 & a_3 \\ 0 & b_0 & 3\,b_1 & 3b_2 & b_3 \end{bmatrix},$$

$$M_{[3,5]} = 3\begin{bmatrix} a_0 & 3\,a_1 & 3\,a_2 & a_3 & 0 & 0 \\ b_0 & 2\,a_0+3\,b_1 & 3\,b_2+6\,a_1 & b_3+6\,a_2 & 2\,a_3 & 0 \\ 0 & 2\,b_0 & 6\,b_1+a_0 & 6\,b_2+3\,a_1 & 3\,a_2+2\,b_3 & a_3 \\ 0 & 0 & b_0 & 3\,b_1 & 3\,b_2 & b_3 \end{bmatrix},$$

$$M_{[4,6]} = 4\begin{bmatrix} a_0 & 3\,a_1 & 3\,a_2 & a_3 & 0 & 0 & 0 \\ b_0 & 3\,a_0+3\,b_1 & 9\,a_1+3\,b_2 & b_3+9\,a_2 & 3\,a_3 & 0 & 0 \\ 0 & 3\,b_0 & 9\,b_1+3\,a_0 & 9\,b_2+9\,a_1 & 3\,b_3+9\,a_2 & 3\,a_3 & 0 \\ 0 & 0 & 3\,b_0 & 9\,b_1+a_0 & 3\,a_1+9\,b_2 & 3\,a_2+3\,b_3 & a_3 \\ 0 & 0 & 0 & b_0 & 3\,b_1 & 3\,b_2 & b_3 \end{bmatrix}.$$

Proposition 1. *If the differential system (1) has a formal first integral of lower degree d, then :*

$$rank\,(M_{[d,l+d-1]}) < \frac{(d+n-1)!}{(n-1)!\,d!}.$$

The existence of the formal first integral of lower degree d implies that the linear system $F_d\,M_{[d,l+d-1]} = 0$ admits a non-vanishing solution. If for any d, $rank\,(M_{[d,l+d-1]}) = \dfrac{(d+n-1)!}{(n-1)!\,d!}$, the differential system (1) hasn't a formal first integral.

2.3 The case of the lower degree $l = 1$ ([9])

When $l = 1$, the matrices $M_{[d,l+d-1]} = M_{[d,d]}$ are square.
Let $A = (A^i_j)_{1\le i,j\le n}$ be the matrix of the linear part of the differential system (1).

Proposition 2. *[9] The matrix* $L = M_{[d,d]}$ *is defined by :*

$$\begin{cases} L^{j_1 j_2\ldots j_n}_{i_1 i_2\ldots i_n} = 0 & \text{if } |i-j| > 2, \\ L^{i_1 i_2\ldots i_n}_{i_1\ldots(i_l-1)\ldots(i_q+1)\ldots i_n} = i_l A^l_q\,, & \\ L^{i_1 i_2\ldots i_n}_{i_1\ldots(i_l+1)\ldots(i_q-1)\ldots i_n} = i_q A^q_l\,, & \\ L^{i_1 i_2\ldots i_n}_{i_1 i_2\ldots i_n} = (i_1A^1_1 + i_2A^2_2 + \cdots + i_nA^n_n). & \end{cases} \tag{8}$$

Corollary 3. *[9] The matrix* $L = M_{[d,d]}$ *is diagonal (respectively lower triangular, upper triangular) for any* $d = 1, 2, 3, \ldots$, *if and only if the matrix A is diagonal (respectively lower triangular, upper triangular).*

Corollary 4. *[9] If the eigenvalues of the matrix A are $\lambda_1, \lambda_2, \dots, \lambda_n$, then the eigenvalues of the matrix $L = M_{[d,d]}$ have the form :*

$$i_1\lambda_1 + i_2\lambda_2 + \cdots + i_n\lambda_n,$$

where $i_1, i_2, \dots i_n \in \mathbb{N}$ and $i_1 + i_2 + \dots + i_n = d$.

From Corollary 2, it follows

$$det(M_{[d,d]}) = \prod_{i_1+i_2+\cdots+i_n=d} \Big(i_1\lambda_1 + i_2\lambda_2 + \cdots + i_n\lambda_n\Big) \tag{9}$$

It is clear that the existence of a formal first integral (of lower degree d) of (1) with $A \neq 0$, implies the existence of a non-negative integer d such that $F_d\, M_{[d,d]} = 0$ has a nonzero solution F_d, i.e. $det(M_{[d,d]}) = 0$.

The factors $\Big(i_1\lambda_1 + i_2\lambda_2 + \cdots + i_n\lambda_n\Big)$ can be regrouped in orbits $\mathcal{O}(i_1, i_2, \dots, i_n)$ with respect to the action of the symmetric group over the multidegrees. These orbits are represented by the partitions of d in not more than n parts:

$$det(M_{[d,d]}) = \prod_{\substack{i_1+i_2+\cdots+i_n=d \\ i_1 \geq i_2 \geq \cdots \geq i_n \geq 0}} \left[\prod_{(j_1,j_2,\dots,j_n)\in\mathcal{O}(i_1,i_2,\dots,i_n)} \Big(j_1\lambda_1 + j_2\lambda_2 + \cdots + j_n\lambda_n\Big) \right].$$

As a symmetric function, the polynomial

$$\mathcal{R} = \prod_{(j_1,j_2,\dots,j_n)\in\mathcal{O}(i_1,i_2,\dots,i_n)} \Big(j_1\lambda_1 + j_2\lambda_2 + \cdots + j_n\lambda_n\Big)$$

belongs to the ring $\mathbb{Z}[\sigma_1, \sigma_2, \dots, \sigma_n]$ where $\sigma_i = \displaystyle\sum_{1\leq j_1\leq j_2\cdots\leq j_n\leq n} \lambda_{j_1} \dots \lambda_{j_i}$.

The equation $\mathcal{R} = 0$ is called the starting equation of the existence of the formal first integrals. Its integer solutions give the lower degree of the eventual first integral.

Some generic examples

1. The partition $(d, 0, \dots, 0)$ corresponds to the factor $\det(A)$.

2. When $d = kn$, the partition $(k, k, \dots, k)$ represents a one element orbit. So, the factor that corresponds to this partition is $k^n \mathrm{trace}(A)$.

3. When $n = 2$, the starting equation is

$$\Big[d_1 d_2 \Big(\mathrm{trace}(A)\Big)^2 + (d_1 - d_2)^2 \det(A)\Big] = 0.$$

In [9], the case of the dimension 3 is also detailed and for other dimensions, a procedure for obtaining the starting equation is given.

Remark 1. The above Diophantine equation (9) can be found under various aspects in many works ([4,7,8]). For example, in [7], when the linear part A is not zero, the author wrote : "*If system (1) has nontrivial integrals analytic in a neighbourhood of a trivial solution $x = 0$, then eigenvalues of the matrix A have to satisfy certain resonant conditions*".

The starting equation gives namely an achieved form of these resonant conditions.

2.4 A consequence (strong condition) for the homogeneous polynomial systems

Let's return to the systems (1) which we suppose homogeneous of degree m :

$$\frac{dx^i}{dt} = P^i(x), \quad i = 1, 2, \ldots, n. \tag{10}$$

Denote by $Jac(x)$ the Jacobian matrix of P and by $J(x)$ and $T(x)$ respectively the determinant and the trace of the matrix $Jac(x)$.
A polynomial first integral $F_d X^d$ satisfies the relation

$$\sum_{j=1}^{n} \frac{\partial(F_d\, X^d)}{\partial x^j} \Big[P^j(x)\Big] = \sum_{j=1}^{n} \sum_{|i|=k} i_j f_{i_1,\ldots,i_n}\, (x^1)^{i_1} \cdots (x^j)^{i_j-1} \ldots (x^n)^{i_n} = 0. \tag{11}$$

Using the Euler's formulae

$$P^j(x) = \frac{1}{m} \sum_{k=1}^{n} x^k \frac{\partial(P^j(x))}{\partial x^k} = \frac{1}{m} \sum_{k=1}^{n} [Jac(x)]^j_k\, x^k,$$

the relation (11) becomes :

$$F_d L_d(x) X^d = 0$$

where the matrix $L_d(x)$ has the same structure as the matrix L when $l = 1$ (see Proposition 2). This is due to the substitution $A^j_k := [Jac(x)]^j_k$ which leads to the matrix $M_{[d,d]}(x)$ and thus, to the starting equation $\mathcal{R}(x)$, depending on x.

Proposition 5. *Suppose that the starting equation $\mathcal{R}(x) = 0$ admits an integer n-tuple solution $(d_1, d_2, \ldots, d_n)$ and suppose that the linear equation $F_d \cdot L_d(x) = 0$ has a constant solution F (not depending on x). Then, the system (10) has a polynomial first integral of total degree d.*

Corollary 6. *When $n = 2$, the strong condition is given by the equation*

$$\Big[d_1 d_2 \Big(T(x)\Big)^2 + (d_1 - d_2)^2\, J(x)\Big] = 0. \tag{12}$$

There are two particular cases of the above Corollary 6.

First case: $n = 2$ and $T(x) = 0$. Hence, $d_1 = d_2$.
We see later that in this case, the polynomial $x^2 P^1(x) - x^1 P^2(x)$ is a first integral.

This case can be extended to the $2n$-dimensional differential systems

$$\frac{dx^i}{dt} = P^i(x,y), \quad \frac{dy^i}{dt} = Q^i(x,y), \qquad i = 1,2,\dots,n$$

satisfying the conditions

$$\frac{\partial P^i(x,y)}{\partial x^i} + \frac{\partial Q^i(x,y)}{\partial y^i} = 0, \quad (i = 1,2,\dots n).$$

The polynomial function $\sum_{i=1}^{n}\Big(y^iP^i(x,y) - x^iQ^i(x,y)\Big)$ is a first integral. Among these systems we find (of course) the homogeneous Hamiltonian ones

$$\frac{dx^i}{dt} = -\frac{\partial H(x,y)}{\partial y^i}, \quad \frac{dy^i}{dt} = \frac{\partial H(x,y)}{\partial x^i}.$$

Remark 2. In general (see Proposition 1.16 from [2]), the condition $J(x) = 0$ implies the algebraic dependence $W(P^1, P^2, \dots, P^n) = 0$ where W is a multivariate polynomial with coefficients in $\mathbb{C}$.

Furthemore, if the polynomials $P^1, P^2, \dots, P^n$ are homogeneous of the same degree, the polynomial W is necessary homogeneous.

Second case: $n = 2$ and $J(x) = 0$.

From the above remark and the homogeneity of W, we have : $W(x) = \prod_{i=1}^{k}(\alpha_i x^1 + \beta_i x^2)$ and so,

$$W(P^1, P^2) = 0 \Longrightarrow \exists\, \alpha, \beta \in \mathbb{C};\, \alpha P^1(x) + \beta P^2(x) = 0.$$

Thus, the linear form $\alpha x^1 + \beta x^2$ is a first integral.

3 The bidimensional homogeneous systems

We have seen that the starting equations of (1) depend only on the homogeneous part of lower degree of (1). Let's consider the homogeneous differential systems of (total) degree l.

$$\frac{dx}{dt} = A(x,y) = \sum_{i=0}^{l}\binom{l}{i} a_i x^{l-i} y^i, \quad \frac{dy}{dt} = B(x,y) = \sum_{i=0}^{l}\binom{l}{i} b_i x^{l-i} y^i, \quad (13)$$

where $A(x,y)\,,\, B(x,y) \in \mathbb{C}_l[x\,,\, y]$.
Because of the homogeneity of the polynomials A and B, a formal first integral is necessarily a polynomial one which satisfies the equation $F_d M_{[d,d+l-1]} X^{l+l-1} = 0$.

By the proposition 1, a necessairy and sufficient condition for finding a nontrivial solution F_d is :

$$rank\Big(M_{[d,l+d-1]}\Big) < d+1.$$

In the following section we compute concretely the matrices $M_{[d,l+d-1]}$, we establish the equivalence between the rank condition and the nullity of some determinant and finally, by using the classical theory, we reduce the computation of this determinant to that of its leading term.

3.1 Some basic facts about the integrals of homogeneous planar differential systems

The following results are wellknown.

Lemma 7. *Let $H(x,y) = H_1(x,y)H_2(x,y)$ be a factorisation of the polynomial H into two coprime polynomials H_1 and H_2. The polynomial H is a partial integral of (13) if and only if H_1 and H_2 are also partial integrals of (13).*

Proof. [4] (Lemma 2.2, p. 8).

Proposition 8. *Let $K(x,y)$ be the polynomial $yA(x,y) - xB(x,y)$. Then*

$$\frac{\partial K(x,y)}{\partial x}A(x,y) + \frac{\partial K(x,y)}{\partial y}B(x,y) = Div(A,B)\,K(x,y)$$

where $Div(A,B)$ is the divergence of the vector field (A,B).

Proof. Putting $A_x = \frac{\partial A}{\partial x}$, $A_y = \frac{\partial A}{\partial y}$, $B_x = \frac{\partial B}{\partial x}$, $B_y = \frac{\partial B}{\partial y}$ and using the Euler's formulae, we get

$$\frac{\partial K(x,y)}{\partial x}A(x,y) + \frac{\partial K(x,y)}{\partial y}B(x,y) = (yA_x - xB_x - B)A + (yA_y + A - xB_y)B$$

$$= (A_x + B_y)K(x,y) + B(xA_x + yA_y) - A(xB_x + yB_y) = Div(A,B)K(x,y).$$

Remark 3. Each factor of $K(x,y)$ is a partial integral.

Proposition 9. *The line $\alpha x + \beta y = 0$ is an invariant curve for the system (13) if and only if $K(\beta,-\alpha) = 0$.*

Proof. The sufficient condition follows immediately from the fact that

$$K(x,y) = \prod_{i=1}^{m+1}(\alpha_i x + \beta_i y)$$

and the previous results.
Let $\alpha x + \beta y = 0$ be the equation of the line. The point $(\beta,-\alpha)$ belongs to the line and so, $\alpha A(\beta,-\alpha) + \beta B(\beta,-\alpha) = 0$.

Corollary 10. *If $Div(A,B) = 0$, then $K(x,y)$ is a first integral,*

Proposition 11. *If $K(x,y) \neq 0$, then $\frac{1}{K(x,y)}$ is an integrating factor for the system (13).*

Proof. Directly from the definition of the integrating factor.

3.2 Computation of the matrices $M_{[d,l+d-1]}$

Proposition 12. *Let $M_{[d,l+d-1]}$ be the matrix defined in Section 2.2. With the assumption $a_{-1} = b_{l+1} = 0$ and $\binom{i}{k} = 0$ for any $j = 0,1,2,\dots d$, $r = 0,1,2,\dots,l+d-1$ such that $i < k$, we have:*

$$\Big[M_{[d,l+d-1]}\Big]_r^j = d\Big[\binom{d-1}{j}\binom{l}{r-j}a_{r-j} + \binom{d-1}{j-1}\binom{l}{r+1-j}b_{r+1-j}\Big]. \quad (14)$$

Proof. The polynomial $F(x,y) = \sum_{j=0}^{d}\binom{d}{j} f_j x^{d-j}y^j$ is a first integral if and only if

$$F_d M_{[d,l+d-1]} X^{l+d-1} = \frac{\partial F(x,y)}{\partial x}A(x,y) + \frac{\partial F(x,y)}{\partial y}B(x,y) =$$

$$= d\sum_{i=0}^{l}\binom{l}{i}\Big[a_i\sum_{j=0}^{d-1}\binom{d-1}{j}x^{l+d-i-j-1}f_j\,y^j + b_i\sum_{j=1}^{d}\binom{d-1}{j-1}x^{l+d-i-j}f_j y^{j-1}\Big]y^i =$$

$$= dx^{d-1}f_0\sum_{r=0}^{l}\binom{l}{r}a_r x^{l-r}y^r + d\sum_{j=1}^{d-1}f_j\Big[\sum_{r=j}^{l+j}\binom{d-1}{j}\binom{l}{r-j}a_{r-j}\,x^{l+d-r-1}y^r$$

$$+\sum_{r=j-1}^{l+j-1}\binom{d-1}{j-1}\binom{l}{r-j+1}b_{r-j+1}\,x^{l+d-r-1}y^r\Big] + d\,y^{d-1}f_d\sum_{r=0}^{l}\binom{l}{r}b_r x^{l-r}y^r = 0.$$

The element $\Big[M_{[d,l+d-1]}\Big]_r^j$ is the coefficient of $f_j\,y^r$ ($j = 0,1,2,\dots d$ and $r = 0,1,2,\dots,l+d-1$) in the last expression. More precisely :

$$\Big[M_{[d,l+d-1]}\Big]_r^0 = d\binom{l}{r}a_r \text{ if } 0 \le r \le l,$$

$$\Big[M_{[d,l+d-1]}\Big]_{j-1}^j = d\binom{d-1}{j-1}b_0 \text{ if } 1 \le j \le d-1,$$

$$\Big[M_{[d,l+d-1]}\Big]_r^j = d\Big[\binom{d-1}{j}\binom{l}{r-j}a_{r-j} + \binom{d-1}{j-1}\binom{l}{r+1-j}b_{r+1-j}\Big]$$
$$\text{if } 1 \le j \le d-1 \text{ and } j \le r \le l+j-1,$$

$$\Big[M_{[d,l+d-1]}\Big]_{l+j}^j = d\binom{d-1}{j}a_l \text{ if } 1 \le j \le d-1,$$

$$\Big[M_{[d,l+d-1]}\Big]_r^d = d\binom{l}{r}b_r \text{ if } d-1 \le r \le l+d-1,$$

$$\Big[M_{[d,l+d-1]}\Big]_r^j = 0 \quad \text{elsewhere}$$

where $j = 0, 1, 2, \ldots d$ and $r = 0, 1, 2, \ldots, l+d-1$. These expressions can be rewritten using the copact form given in the proposition.

Corollary 13. *For all* $d \in \{1, 2, \ldots, \}$, $i \in \{1, 2, \ldots, d\}$, *and* $j \in \{1, 2, \ldots, l + d - 1\}$, *we have :*

$$d\left[M_{[d+1,l+d]}\right]_j^i = (d+1)\Big(\left[M_{[d+1,l+d]}\right]_j^i + \left[M_{[d+1,l+d]}\right]_{j-1}^{i-1}\Big).$$

By Proposition 1, the condition on the rank requires the computation of $\begin{pmatrix} d+l \\ d+1 \end{pmatrix}$ minors. The aim of the following subsection is to reduce the computation of these minors to that of one and only one determinant of some matrix.

3.3 Reduction to a square matrix

Proposition 14. *The polynomial* $F_d M_{[d,l+d-1]} X^{l+d-1} \in \mathbb{C}[x, y]$ *vanishes identically if and only if the polynomial*

$$\sum_{k=0}^{l-1} \begin{pmatrix} l-1-k \\ k \end{pmatrix} \frac{\partial^{l-1} F_d\, M_{[d,l+d-1]}\, X^{l+d-1}}{\partial x^{l-1-k}\, \partial y^k}\, u^{l-1-k} v^k \tag{15}$$

vanishes in $\mathbb{C}[x, y, u, v]$.

Proof. With the help of the Euler's formulae,

$$\frac{\partial Q(x,y)}{\partial x} x + \frac{\partial Q(x,y)}{\partial y} y = (l + d - 1)\, Q(x,y)$$

where $Q = F_d\, M_{[d,l+d-1]}\, X^{l+d-1}$, we remark that the homogeneous polynomial Q vanishes if and only if $\dfrac{\partial Q(x,y)}{\partial x} = \dfrac{\partial Q(x,y)}{\partial y} = 0$. By the same way, we claim that the polynomial Q vanishes if and only if all its derivatives of order $l - 1$ vanish.

In the following, we denote by $S_{d,k}$ the $(d+1) \times (d+1)$-matrix such that

$$F_d\, S_{d,k}\, X^d = \frac{\partial^{l-1}\Big(F_d\, M_{[d,l+d-1]}\, X^{l+d-1}\Big)}{\partial x^{l-1-k}\, \partial y^k}$$

and by $S_{[d]}(u\,,\, v)$ the matrix $\sum_{k=0}^{l-1} \begin{pmatrix} l-1-k \\ k \end{pmatrix} S_{d,k}\, u^{l-1-k} v^k$.

Proposition 15. *The differential system (13) has a polynomial first integral of degree d if and only if*

$$\det S_{[d]}(u, v) = 0$$

Examples of matrices $S_{[d]}(u,v)$.

<u>**l = 2 :**</u>

$$S_{[1]}(u,v)=\begin{bmatrix} 2a_0 & 2a_1 \\ 2b_0 & 2b_1 \end{bmatrix}u+\begin{bmatrix} 2a_1 & 2a_2 \\ 2b_1 & 2b_2 \end{bmatrix}v,$$

$$S_{[2]}(u,v)=\begin{bmatrix} 3a_0 & 4a_1 & a_2 \\ 3b_0 & 2a_0+4b_1 & 2a_1+b_2 \\ 0 & 2\,b_0 & 2b_1 \end{bmatrix}u+\begin{bmatrix} 2\,a_1 & 2a_2 & 0 \\ a_0+2b_1 & 4a_1+2b_2 & 3a_2 \\ b_0 & 4b_1 & 3b_2 \end{bmatrix}v,$$

$S_{[3]}(u,v)=$

$$\begin{bmatrix} 4a_0 & 6a_1 & 2a_2 & 0 \\ 4b_0 & 6b_1+6a_0 & 8a_1+2b_2 & 2a_2 \\ 0 & 6b_0 & 2a_0+8b_1 & 2b_2+2a_1 \\ 0 & 0 & 2\,b_0 & 2b_1 \end{bmatrix}u+\begin{bmatrix} 2a_1 & 2a_2 & 0 & 0 \\ 2b_1+2a_0 & 8a_1+2b_2 & 6a_2 & 0 \\ 2b_0 & 2a_0+8b_1 & 6b_2+6a_1 & 4a_2 \\ 0 & 2b_0 & 6b_1 & 4b_2 \end{bmatrix}v.$$

<u>**l = 3 :**</u>

$$S_{[1]}(u,v)\begin{bmatrix} 3\,a_0 & 3\,a_1 \\ 3\,b_0 & 3\,b_1 \end{bmatrix}u^2+\begin{bmatrix} 6\,a_1 & 6\,a_2 \\ 6\,b_1 & 6\,b_2 \end{bmatrix}uv+\begin{bmatrix} 3\,a_2 & 3\,a_3 \\ 3\,b_2 & 3\,b_3 \end{bmatrix}v^2,$$

$S_{[2]}(u,v)=$

$$\begin{bmatrix} 6\,a_0 & 9\,a_1 & 3\,a_2 \\ 6\,b_0 & 3\,a_0+9\,b_1 & 3\,b_2+3\,a_1 \\ 0 & 3\,b_0 & 3\,b_1 \end{bmatrix}u^2+\begin{bmatrix} 9\,a_1 & 12\,a_2 & 3\,a_3 \\ 3\,a_0+9\,b_1 & 12\,b_2+12\,a_1 & 9\,a_2+3\,b_3 \\ 3\,b_0 & 12\,b_1 & 9\,b_2 \end{bmatrix}uv+$$

$$+\begin{bmatrix} 3\,a_2 & 3\,a_3 & 0 \\ 3\,b_2+3\,a_1 & 9\,a_2+3\,b_3 & 6\,a_3 \\ 3\,b_1 & 9\,b_2 & 6\,b_3 \end{bmatrix}v^2.$$

3.4 Computation of the matrices $S_{d,k}$

Starting from the polynomial

$$F_d\,M_{[d,l+d-1]}\;X^{l+d-1}\;=\sum_{i=0}^{d}\sum_{j=0}^{l+d-1}f_i\Big(M_{[d,l+d-1]}\Big)_j^i x^{l+d-1-j}y^j,$$

we get

$$\frac{\partial^{l-1}F_d\,M_{[d,l+d-1]}X^{l+d-1}}{\partial^{l-1-k}x\partial^k y}=\sum_{i=0}^{d}\sum_{j=k}^{k+d}f_i\Big(M_{[d,l+d-1]}\Big)_j^i\frac{(l+d-1-j)!j!}{(d+k-j)!(j-k)!}x^{d+k-j}y^{j-k}$$

$$=\sum_{i=0}^{d}\sum_{p=0}^{d}f_i\Big(M_{[d,l+d-1]}\Big)_{k+p}^i\frac{(l+d-1-k-p)!}{(d-p)!}\,\frac{(k+p)!}{p!}\,x^{d-p}y^p.$$

Proposition 16. *The* $(d+1)\times(d+1)$*-matrix* $S_{d,k}(u\,,\,v)$ *(for* $k\in\{0,1,2,...,l-1\}$*) is defined by :*

$$\Big(S_{d,k}\Big)^i_p = \Big(M_{[d,l+d-1]}\Big)^i_{k+p} \frac{(l+d-1-k-p)!}{(d-p)!}\,\frac{(k+p)!}{p!}. \tag{16}$$

Proof. The coefficient of $f_i\,y^p$ corresponds to the coefficient $(S_{d,k})^i_p$.

Consequently, by Proposition 15, the differential system (13) admits a polynomial first integral if and only if there exists a positive integer d such that

$$det(S_{[d]}(u,v))=s^d_0 u^{(l-1)(d+1)}+s^d_1 u^{(l-1)(d+1)-1}v+\cdots+s^d_{(l-1)(d+1)}v^{(l-1)(d+1)}=0. \tag{17}$$

For verifying the condition $rank\Big(M_{[d,l+d-1]}\Big)<d+1$, we must compute $N_1=\binom{d+l}{d+1}$ minors, but for verifying the condition $det(S_{[d]}(u,v))=0$, we need the computation of $N_2=(d+1)(l-1)+1$ expressions. The next table shows how the difference N_1-N_2 increases w.r.t. the degrees l and d.

l \ d	1	2	3	4	5	6	7	8	9	10
2	0	0	0	0	0	0	0	0	0	0
3	3	6	10	15	21	28	36	45	55	66
4	7	16	30	50	77	112	156	210	275	352
5	12	31	65	120	203	322	486	705	990	1353
6	18	52	121	246	455	784	1278	1992	2992	4356
7	25	82	205	456	917	1708	2994	4995	7997	12364
8	33	116	325	786	1709	3424	6424	11430	19437	31812

Remark 4.

1. When $l=2$, the coefficients of the determinant (17) correspond to the minors of the matrix $M_{[d,d+1]}=[C_1,C_2,\ldots,C_{d+2}]$, rewritten with the columns $C_i\,;\,i=1,2,\ldots,d+2$:

   $$s^d_k=\lambda_i\,det([C_1,C_2,\ldots,C_{d+1-k},\check{C}_{d+2-k},C_{d+3-k},\ldots,C_{d+2}])$$

   for $k=0,\ldots,d+1$ and $\lambda_i\in\mathbb{N}$.
   Here, the symbol "ˇ" means the removing of the corresponding column.

2. When $l>2$, we have :

   $$s^d_0=\lambda_0\,det([C_1,C_2,\ldots,C_{d+1}])$$
   $$s^d_1=\lambda_1 det([C_1,C_2,\ldots,\hat{C}_{d+1},C_{d+2}])$$

It is obvious that when the degree d increases, the computation of the determinant $\det(S_{[d]}(u,v))$ becomes more and more complicated. However, by using the classical invariant theory, we will show that from the knowledge of the leading term s^d_0, we can deduce that of the other terms $s^d_1,\,s^d_2,\ldots,s^d_{d+1}$.

4 The computation of the determinant $\det(S_{[d]}(u,v))$ by using the classical invariant theory

4.1 Introduction to the classical invariant theory ([10, 12, 14])

Let $(\mathbb{C}^n)^*$ be the dual of the vector space $\mathbb{C}^n$. The linear space of the differential systems (13) can be looked upon as the tensorial product $S(n\,,\,m)\otimes(\mathbb{C}^n)^*$ denoted by $\mathcal{S}^1_m$. For example, $\mathcal{S}^1_1$ is the linear differential systems. In tensorial language, $\mathcal{S}^1_m$ is the space of tensors once contravariant and m times covariant which are symmetric with respect to the lower indices.

Let G be the linear group acting rationally on a finite-dimensional vector space $\mathcal{W}$, $GL(\mathcal{W})$ the group of automorphisms of $\mathcal{W}$ and

$$\rho\ : G \longmapsto GL(\mathcal{W})$$

the corresponding rational representation. Let $\mathbb{C}[\mathcal{W}]$ be the algebra of polynomials whose indeterminates are the coordinates of a generic vector of $\mathcal{W}$.

Definition 2. *A polynomial function $K \in \mathbb{C}[\mathcal{W}]$ is said to be a G-invariant of $\mathcal{W}$ if there exists a character of the group G, denoted λ, such that*

$$\forall g \in G, \quad K \circ \rho(g) = \lambda(g).K.$$

Here, the character of the group G is a rational (commutative) morphism of group G into $\mathbb{C}_m$ where $\mathbb{C}_m$ is the multiplicative group of $\mathbb{C}$.
If $\lambda(g) \equiv 1$, the invariant is said absolute. Otherwise, it is relative.

Definition 3. *A $GL(n,\mathbb{C})$-invariant of $\mathcal{S}^1_m$ is a $GL(n,\mathbb{C})$-invariant of the linear space*
$\mathcal{W} = \mathcal{S}^1_m$.

A $GL(n,\mathbb{C})$-covariant of $\mathcal{S}^1_m$ is a $GL(n,\mathbb{C})$-invariant of the linear space $\mathcal{W} = \mathcal{S}^1_m \times (\mathbb{C}^n)$.

When G is a subgroup of $GL(n,\mathbb{C})$, $\lambda(g) = (\det\, g)^{-\kappa}$ with the so called weight $\kappa \in \mathbb{Z}$.

Remark 5. If $G = SL(n,\mathbb{C})$, all the covariants are absolute.

Remark 6. A polynomial $K \in \mathbb{C}[\mathcal{S}^1_m]$ is a $GL(n,\mathbb{C})$ covariant if and only if it is a $SL(n,\mathbb{C})$ covariant.

Examples

1. Concerning the $GL(n,\mathbb{C})$-invariants, take for example the trace and the determinant of the linear part of (1).

2. The divergence and the jacobian determinant of the vector field P are the simplest $GL(n,\mathbb{C})$-covariants of the space of the differential systems (1).

For more details, see [10–12, 14, 15].

The following theorem gives a procedure for calculating the generators of the the $GL(n,\mathbb{C})$-covariants, step by step increasing the degree.

Theorem 1 (Fundamental Theorem of the classical invariant theory)**.** *The expressions obtained with the help of succesive alternations and complete contraction over the tensorial products*

$$(\mathcal{S}_m^1)^{\otimes p} \otimes (\mathcal{V})^{\otimes r}$$

form a system of generators of the algebra of $GL(2,\mathbb{C})$*-covariants of* $\mathcal{S}_m^1$ *.*
Such polynomials are called basic covariants.

From now on, we will be ineterested in the bidimensional case.
A $GL(2,\mathbb{C})$-covariant K of degree k (with respect to (x,y)) is a polynomial

$$C_0 x^k + \binom{k}{1} C_1 x^{k-1} y + \cdots + \binom{k}{i} C_i x^{k-i} y^i + \cdots + C_k y^k$$

where the coefficients C_i are homogeneous polynomial functions of degree d depending on coefficients a_j and b_j. The integers k and d satisfy the relation

$$d(m-1) - k = 2\,\kappa$$

where κ is the weight of the covariant K.

To apply previous Theorem 1, it is useful to introduce the tensorial writing of the algebraic forms and the polynomial differential systems.

Putting $x = x^1$, $y = x^2$ and using Einstein's notation : $\gamma_i \delta^i = \sum_{i=1}^2 \gamma_i \delta^i$ the polynomials F, A and B, once symmetrised ([10]), become :

$$F(x,y) = \varphi(x) = \varphi_{i_1 i_2 \ldots i_d}\, x^{i_1} x^{i_2} \cdots x^{i_d},$$

$$A(x,y) = \Lambda^1(x) = \alpha^1_{i_1 i_2 \ldots i_d}\, x^{i_1} x^{i_2} \cdots x^{i_m},$$

$$B(x,y) = \Lambda^2(x) = \alpha^2_{i_1 i_2 \ldots i_d}\, x^{i_1} x^{i_2} \cdots x^{i_m},$$

where $\alpha^1_{11..122..2} = a_i$ and $\alpha^2_{11..122..2} = b_i$ (with i "2") and $i_1, i_2, \ldots \in \{1,2\}$. Consequently,

$$\begin{aligned} D_{A,B} F(x,y) &= \frac{\partial\varphi(x)}{\partial x^1}\Lambda^1(x) + \frac{\partial\varphi(x)}{\partial x^2}\Lambda^2(x) \\ &= d\varphi_{i_1 i_2 \ldots i_d} \alpha^{i_d}_{j_1 j_2 \ldots j_m} x^{i_1} x^{i_2} \cdots x^{i_{d-1}} x^{j_1} x^{j_2} \cdots x^{j_m} \end{aligned}$$

is a $GL(2,\mathbb{C})$-covariant because it is a total contraction (see Theorem 1).

Proposition 17. *The polynomial* $F_d S_{[d]}(u,v) X^d$ *is a* $GL(2,\mathbb{C})$*-covariant.*

Proof. It is wellknown ([14]) that for any covariant $K(x,y)$, the polynomial

$$\frac{\partial K(x,y)}{\partial x}u + \frac{\partial K(x,y)}{\partial y}v$$

is a covariant. This is the polarization process. Repeating this process k times, we obtain again a covariant :

$$\sum_{i=0}^{k}\binom{k}{i}\frac{\partial^k K(x,y)}{\partial x^{k-i}\partial y^i}u^{k-i}v^i.$$

Consequently, since $D_{A,B}F$ is a covariant, the polynomial

$$F_d S_{[d]}(u,v)X^d = \sum_{i=0}^{l-1}\binom{l-1}{i}\frac{\partial^k D_{A,B}F}{\partial x^{k-i}\partial y^i}u^{k-i}v^i$$

is also a covariant.

Corollary 18. *The polynomial $det(S_{[d]}(u,v))$ is a $GL(2,\mathbb{C})$-covariant.*

Proof. Let $Sym(n,d)^*$ be the dual of the linear space $Sym(n,d)$ and $GL(Sym(n,d))$ (resp. $GL(Sym(n,d))^*$) the group of the automorphisms of $Sym(n,d)$ (resp. $GL(Sym(n,d))^*$). We denote the elements of $Sym(n,d)$ by F_d and those of $Sym(n,d)^*$ by X^d.

The change of coordinates $(x,y)^T \leftrightarrow (\overline{x},\overline{y}) = [g^{-1}(x,y)]^T$, where $(x,y)^T$ is the transpose vector of (x,y), induces the linear representations :

$$\Phi\ : GL(2,\mathbb{C}) \to GL(Sym(n,d)) \quad \Psi\ : GL(2,\mathbb{C}) \to GL((Sym(n,d))^*).$$

Since the polynomial $F_d X^d$ is an absolute $GL(2,\mathbb{C})$-covariant, we have :

$$\forall\, g \in GL(2,\mathbb{C}), \quad \Phi(g)\Psi(g) = 1.$$

Hence, the matrix $S_{[d]}(u,v)$ is transformed into $\overline{S}_{[d,d]}(\overline{u},\overline{v}) = \Phi(g)S_{[d]}(u,v)\Phi(g)^{-1}$. Consequently,

$$det(\overline{S}_{[d,d]}(\overline{u},\overline{v})) = det(S_{[d]}(u,v)).$$

4.2 Differential operators

Let's consider the connected component of the identity of the subgroups H_l, H_u and H_d of the lower-triangular, upper-triangular, diagonal matrices parametrized by $\tau \in \mathbb{C}$:

$$H_l = \left\{\begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix}, \tau \in \mathbb{C}\right\}, H_u = \left\{\begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}, \tau \in \mathbb{C}\right\}, H_d = \left\{\begin{pmatrix} e^{\tau} & 0 \\ 0 & e^{-\tau} \end{pmatrix}, \tau \in \mathbb{C}\right\}.$$

The family H_l, H_u, H_d generates the unimodular group $SL(2,\mathbb{C})$. H_l (resp H_u, H_d) is homeomorphic to the additive group $(\mathbb{C}\,,+)$. Each element of H_l (resp H_u, H_d) can be identified with $\tau \in \mathbb{C}$. These groups induce linear actions over the space $(\mathcal{S}_m^1)$

$$\Phi\ : H_l \to Aut(\mathcal{S}_m), \qquad \Psi\ : H_u \to Aut(\mathcal{S}_m), \qquad \Gamma\ : H_d \to Aut(\mathcal{S}_m)$$

defined by $\Phi(t)(a,b)=(\underline{a}(t),\underline{b}(t))$, $\Psi(t)(a,b)=(\overline{a}(t),\overline{b}(t))$ and $\Gamma(t)(a,b)=(\hat{a}(t),\hat{b}(t))$ where

$$\begin{array}{ll}
\underline{a}_k(\tau) = \sum_{i=k}^{m} \binom{m-k}{i-k} \tau^{i-k} a_i & \underline{b}_k(\tau) = \sum_{i=k}^{m} \binom{m-k}{i-k} \tau^{i-k} (b_i - \tau\, a_i) \\
\overline{a}_k(\tau) = \sum_{i=0}^{k} \binom{k}{i} \tau^{k-i} (a_i - \tau\, b_i) & \overline{b}_k(\tau) = \sum_{i=0}^{k} \binom{k}{i} \tau^{k-i} b_i \\
\hat{a}_k(\tau) = e^{\tau(m-2k-1)} a_k & \hat{b}_k(\tau) = e^{\tau(m-2k+1)} b_k.
\end{array} \tag{18}$$

To each subgroup H_l, H_u, H_d, we associate a differential operator acting over the algebra of the polynomials $\mathbb{C}[a\,,\,b]$

$$\Omega_l \;=\; \sum_{k=1}^{m} (m-k+1)\Big(a_k \frac{\partial}{\partial a_{k-1}} \;+\; b_k \frac{\partial}{\partial b_{k-1}}\Big) \;-\; \sum_{k=0}^{m} a_k \frac{\partial}{\partial b_k}, \tag{19}$$

$$\Omega_u \;=\; \sum_{k=1}^{m} (k)\Big(a_{k-1} \frac{\partial}{\partial a_k} \;+\; b_{k-1} \frac{\partial}{\partial b_k}\Big) \;-\; \sum_{k=0}^{m} b_k \frac{\partial}{\partial a_k}, \tag{20}$$

and

$$\Omega_d \;=\; \sum_{k=0}^{m} \Big(\, (m-2k-1)\, a_k \frac{\partial}{\partial a_k} \;+\; (m-2k+1)\, b_k \frac{\partial}{\partial b_k}\Big). \tag{21}$$

They are obtained by derivating respectively the expressions $\Phi(\tau)(a,b)$, $\Psi(\tau)(a,b)$ and $\Gamma(\tau)(a,b)$ with respect to τ and setting $\tau=0$.

These operators play an important role in the description of the algebra of the $GL(2,\mathbb{C})$-covariants. Indeed, the next result gives a relation between any covariant

$$U{=}A_0u^p{+}A_1\binom{p}{1}u^{p-1}v{+}\cdots{+}A_i\binom{p}{i}u^{p-i}v^i{+}\cdots{+}A_{p-1}\binom{p}{p-1}uv^{p-1}{+}A_p\,v^p. \tag{22}$$

and its leading term A_0.

Definition 4. *The weight of the coefficient a_k (resp. b_k) is the number k (resp. $k-1$). The weight of any monomial $\lambda a_0^{i_0}\cdots a_l^{i_l} b_0^{j_0}\cdots b_l^{j_l}$ is the number $\sum_{k=0}^{l}(ki_k + (k-1)j_k)$. A polynomial $K \in \mathbb{C}[a_0,\ldots,a_l,b_0\ldots b_l]$ is isobaric if all its monomials have the same weight.*

Proposition 19. *For any $GL(2,\mathbb{C})$-covariant (22), with coefficients $A_0\,, A_1,\,\ldots, A_p$, homogeneous polynomial functions of $a_0,\,a_1,\,\ldots\,,a_m\,,\,b_0,\,b_1,\,\ldots\,,b_m$, we have :*

$$\binom{p}{k} A_k \;=\; \frac{1}{k!}\Omega_l^{(k)}(A_0) \qquad \forall k=0,\,1,\,2,\,\ldots,p$$

Corollary 20. *If the homogeneous polynomial I depending on $a_0, a_1, \ldots, a_m$ and $b_0, b_1, \ldots, b_m$, is a $GL(2,\mathbb{C})$-invariant , then $\Omega_l(I) = 0$.*

Proposition 21. *For any $GL(2,\mathbb{C})$-covariant (22), where coefficients $A_0, A_1, \ldots, A_p$ are homogeneous polynomial functions, depending on $a_0, a_1, \ldots, a_m$, $b_0, b_1, \ldots, b_m$, we have :*

$$\Omega_u(A_0) = 0.$$

Theorem 2 ([15])**.** *An algebraic form*

$$U = A_0 u^p + A_1 \binom{p}{1} u^{p-1} v + \cdots + A_{p-1} \binom{p}{k-1} u v^{p-1} + A_p v^p$$

with an isobaric polynomial $A_0 \in \mathbb{C}[a_0, a_1, \ldots, a_l, b_0, b_1, \ldots, b_l]$, is a $GL(2,\mathbb{C})$-covariant if and only if $\Omega_u(A_0) = 0$ and

$$U = \sum_{k=0}^{d} \frac{1}{k!} \Omega_l^k(A_0)\, u^{d-k}\, v^k.$$

Let's return to the determinant of the matrix $S_{[d]}(u,v)$ which is a covariant (18).

Corollary 22. *The covariant (17) is determined by its leading term s_0^d :*

$$det(S_{[d]}(u,v) = \sum_{k=0}^{d} \frac{1}{k!} \Omega_l^k(s_0^d)\, u^{d-k}\, v^k.$$

4.3 The computation of the leading term s_0^d: an open question

Following the previous subsection, the computation of $det(S_{[d,d]})$ can be reduced to that of its leading term, s_0^d. This is somewhat difficult.

Examples of s_0^d: for any $i, j \in \{0, 1, 2, \ldots, l\}$ such that $i < j$ and $k \in \{0, 1, 2, \ldots, l-1\}$, we put

$$\delta_{i,j} = \binom{l}{i}\binom{l}{j}\Big(a_i b_j - a_j b_i\Big), \qquad \tau_k = \binom{l}{k} a_k + \binom{l}{k+1} b_{k+1}.$$

Then, with the help of Groebner basis [13], we get :
<u>for $l = 2$</u> :

$$\begin{aligned}
s_0^1 &:= \delta_{0,1};\\
s_0^2 &:= -b_0\delta_{0,2} + \delta_{0,1}\tau_0\\
s_0^3 &:= \delta_{0,1}\left(2\,{\tau_0}^2 + \delta_{0,1}\right) - 4\, b_0\delta_{0,2}\tau_0 + 2\,{b_0}^2\delta_{1,2}\\
s_0^4 &:= 3\,\delta_{0,1}\tau_0\left(3\,{\tau_0}^2 + 4\,\delta_{0,1}\right) + 9\,\left(-\delta_{0,1}a_2 + 2\,\tau_0\delta_{1,2} + \tau_1\delta_{0,2}\right){b_0}^2\\
&\quad -3\,\delta_{0,2}\left(4\,\delta_{0,1} + 9\,{\tau_0}^2\right) b_0\\
s_0^5 &:= 4\,\delta_{0,1}\left(\delta_{0,1} + 6\,{\tau_0}^2\right)\left(9\,\delta_{0,1} + 4\,{\tau_0}^2\right) + 8\left(-36\,a_2\delta_{0,1}\tau_0 + 9\,\delta_{1,2}\delta_{0,1} + 36\,\delta_{0,2}\tau_0\tau_1\right.\\
&\quad \left. +36\,\delta_{1,2}{\tau_0}^2 + 20\,{\delta_{0,2}}^2\right){b_0}^2 - 96\,\left(-a_2\delta_{0,2} + \tau_1\delta_{1,2}\right){b_0}^3 - 16\,\delta_{0,2}\tau_0(24\,{\tau_0}^2\\
&\quad +29\,\delta_{0,1})b_0.
\end{aligned}$$

for $l = 3$:

$$\begin{array}{lcl} s_0^1 & := & \delta_{0,1}; \\ s_0^2 & := & -\delta_{0,2} b_0 + \delta_{0,1}\tau_0; \\ s_0^3 & := & 2\,(\delta_{0,3} + \delta_{1,2}) {b_0}^2 - 4\,\delta_{0,2}\tau_0 b_0 + \delta_{0,1}(2\,{\tau_0}^2 + \delta_{0,1}) \\ s_0^4 & := & -18\,\delta_{1,3}{b_0}^3 + 9\,(3\,\delta_{0,3}\tau_0 - \tau_2\delta_{0,1} + 2\,\delta_{1,2}\tau_0 + \delta_{0,2}\tau_1){b_0}^2 - 3\,\delta_{0,2}(4\,\delta_{0,1} \\ & & +9\,{\tau_0}^2) b_0 + 3\,\delta_{0,1}\tau_0(3\,{\tau_0}^2 + 4\,\delta_{0,1}); \\ s_0^5 & := & 192\,\delta_{2,3}{b_0}^4 + 96\,(2\,\delta_{0,1}a_3 + \tau_2\delta_{0,2} - 3\,\tau_1\delta_{0,3} - \tau_1\delta_{1,2} - 5\,\tau_0\delta_{1,3}){b_0}^3 \\ & & +8\,(9\,\delta_{0,1}\delta_{1,2} + 72\,\delta_{0,3}{\tau_0}^2 + 36\,\delta_{1,2}{\tau_0}^2 - 36\,\delta_{0,1}\tau_2\tau_0 + 36\,\tau_1\delta_{0,2}\tau_0 + 9\,\delta_{0,1}\delta_{0,3} \\ & & +20\,{\delta_{0,2}}^2){b_0}^2 - 16\tau_0\delta_{0,2}(29\delta_{0,1} + 24{\tau_0}^2)b_0 + 4\delta_{0,1}(\delta_{0,1} + 6\,{\tau_0}^2)(9\delta_{0,1} + 4{\tau_0}^2). \end{array}$$

We remark that if $b_0 = 0$ and for $l = 2, 3$,

$$\begin{array}{lclcl} Lt_1 & = & s_0^1 & = & \delta_{0,1} \\ Lt_2 & = & s_0^2 & = & \delta_{0,1}\tau_0 \\ Lt_3 & = & s_0^3 & = & \delta_{0,1}(\delta_{0,1} + 2{\tau_0}^2) \\ Lt_4 & = & s_0^4 & = & 3\,\tau_0\delta_{0,1}(3\,{\tau_0}^2 + 4\,\delta_{0,1}) \\ Lt_5 & = & s_0^5 & = & 4\,\delta_{0,1}(\delta_{0,1} + 6\,{\tau_0}^2)(9\,\delta_{0,1} + 4\,{\tau_0}^2). \end{array}$$

It is easy to recognize the type of the factors that are present in these expressions. Indeed, it coincides with the starting equation , when the linear part is nonzero. In fact, this result is more general.

Proposition 23. *When $b_0 = 0$, the leading term of the polynomial $det(S_{[d]}(u, v)$ is defined, up to a numeric constant, by :*

$$Lt_d = c_d(\tau_0)^{(d)} \prod_{\substack{d_1 + d_2 = d \\ d_1 > d_2}} \left[d_1 d_2\, \tau_0^2 + (d_1 - d_2)^2\, \delta_{0,1} \right], \tag{23}$$

where c_d is a numerical coefficient, $(d) = 0$ if d is odd and $(d) = 1$ if d is even.

Proof. It is obvious that $s_0^d = det(S_{[d]}(1, 0))$. Since $b_0 = 0$, from (16) and (14), the matrix $S_{[d]}(1, 0) = S_{d,0}$ is upper triangular and so, its determinant is equal to the product of the diagonal elements which are regrouped two by two (the j^{th} term with the $(d + 1 - j)^{th}$ term, for $j = 1, \ldots, d + 1)$:

1. when $d_1 > d_2$, we get $(d_1 a_0 + d_2 l b_1)(d_2 a_0 + d_1 l b_1) = (d_1^2 + d_2^2)\tau_0^2 + (d_1 - d_2)^2 \delta_{0,1}$,
2. when $d_1 = d_2 = \frac{d}{2}$, the coefficient $(S_{d,0})_{d_1}^{d_1} = \lambda(a_0 + l b_1) = \lambda\tau_0$.

Remark 7. By analogy, when $a_m = 0$, the relation

$$Mt_d = c_d(\tau_{l-1})^{(d)} \prod_{\substack{d_1 + d_2 = d \\ d_1 > d_2}} \left[d_1 d_2\, \tau_{l-1}^2 + (d_1 - d_2)^2\, \delta_{l-1,l} \right]$$

is verified.

At this step, we arrive at the question: how to deduce the leading coefficient s_0^d from the above expression (23)? Does there exist some operator which transforms Lt_d to the leading term s_0^d? Up to now, this question is open.

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Orthogonal Solutions for a Hyperbolic System

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Abstract. We consider the hyperbolic system

$$\begin{cases} u_t = a\nabla v + f_1(u,v) \\ v_t = a\nabla u + f_2(u,v) \\ u(0,x) = \xi(x) \\ v(0,x) = \eta(x), \end{cases}$$

and we are looking for necessary and sufficient conditions on the forcing terms f_i, $i = 1,2$, in order that the semigroup solutions, u and v, starting from orthogonal data $\xi, \eta \in L^2(\mathbb{R}^n)$, remain orthogonal on $\mathbb{R}_+$.

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1 The main result

Let us consider the hyperbolic system

$$\begin{cases} u_t = a\nabla v + f_1(u,v) \\ v_t = a\nabla u + f_2(u,v) \\ u(0,x) = \xi(x) \\ v(0,x) = \eta(x), \end{cases} \tag{1}$$

where $a \in \mathbb{R}^n$, $f_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\xi, \eta \in L^2(\mathbb{R}^n)$. We are looking for necessary and sufficient conditions on the forcing terms f_i, $i = 1,2$, in order that the mild solutions, u and v, of (1), starting from orthogonal data $\xi, \eta \in L^2(\mathbb{R}^n)$, remain orthogonal on $\mathbb{R}_+$, i.e.,

$$\langle u(t,\cdot), v(t,\cdot)\rangle = 0 \tag{2}$$

for each $t \in \mathbb{R}_+$, whenever $\xi, \eta \in L^2(\mathbb{R}^n)$ satisfy

$$\langle \xi, \eta\rangle = 0. \tag{3}$$

The main result of this paper concerning the problem above is

Theorem 1. *Let us assume that $f_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are globally Lipschitz. Then, a necessary and sufficient condition in order that for each $\xi, \eta \in L^2(\mathbb{R}^n)$, satisfying* (3)*, to exist a unique mild solution $(u, v) : \mathbb{R}_+ \to L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ of* (1)*, satisfying* (2) *for each $t \in \mathbb{R}_+$, is*

$$\langle \xi, f_2(\xi, \eta)\rangle + \langle \eta, f_1(\xi, \eta)\rangle = 0, \tag{4}$$

for each $\xi, \eta \in L^2(\mathbb{R}^n)$ satisfying (3).

The proof of Theorem 1 is based on a combination of C_0-semigroup techniques developed in Vrabie [7] and viability arguments which we recall in the next section.

2 Introduction to mild viability

Let X be a Banach space, let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup, $\{S(t) : X \to X;\ t \geq 0\}$, and let $f : K \to X$ be a continuous function. Let us consider the Cauchy problem

$$\begin{cases} u'(t) \in Au(t) + f(u(t)) \\ u(0) = \xi. \end{cases} \tag{5}$$

Definition 1. We say that K is *mild viable with respect to $A + f$* if for each $\xi \in K$ there exist $T > 0$ and a continuous function $u : [\,0, T\,] \to K$ satisfying

$$u(t) = S(t)\xi + \int_0^t S(t-s)f(u(s))ds$$

for each $t \in [\,0, T\,]$.

In order to get a necessary and sufficient condition for mild viability, some preliminaries are needed.

Definition 2. We say that $\eta \in X$ is *A-tangent to K at $\xi \in K$* if

$$\liminf_{h \downarrow 0} \frac{1}{h} \operatorname{dist}\,(S(h)\xi + h\eta; K) = 0.$$

In other words, $\eta \in X$ is A-tangent to K at $\xi \in K$ if for each $\delta > 0$ and each neighborhood V of 0 there exist $h \in (0, \delta)$ and $p \in V$ such that

$$S(h)\xi + h(\eta + p) \in K. \tag{6}$$

The set of all A-tangent elements to K at $\xi \in K$ is denoted by $\mathcal{T}_K^A(\xi)$. We notice that if $A \equiv 0$, then $\mathcal{T}_K^A(\xi)$ is the *contingent cone* at $\xi \in K$ in the sense of Bouligand [1] and Severi [6], i.e.

$$\mathcal{T}_K^0(\xi) = \mathcal{T}_K(\xi).$$

Proposition 1. *If $\eta \in \mathcal{T}_K^A(\xi)$ then, for every function $h \mapsto \eta_h$ from $(0,1)$ to X satisfying $\lim_{h\downarrow 0} \eta_h = \eta$, we have*

$$\liminf_{h\downarrow 0} \frac{1}{h} \operatorname{dist}(S(h)\xi + h\eta_h; K) = 0. \tag{7}$$

If there exists a function $h \mapsto \eta_h$ from $(0,1)$ to X satisfying both

$$\lim_{h\downarrow 0} \eta_h = \eta$$

and (7), then $\eta \in \mathcal{T}_K^A(\xi)$.

The next result is a necessary and sufficient condition for mild viability due to Cârjă and Motreanu [3]. For a more general theorem extending both Nagumo's [4] and Pavel's [5] main viability results, see Burlică and Roşu [2].

Theorem 2. *Let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup, $K \subseteq X$ a nonempty and locally closed subset in X and let $f : K \to X$ be a locally Lipschitz function. Then, a necessary and sufficient condition in order that K be mild viable with respect to $A + f$ is the generalized tangency condition*

$$f(\xi) \in \mathcal{T}_K^A(\xi) \tag{8}$$

for each $\xi \in K$.

3 The abstract Banach space setting

Let K be a nonempty subset in X, invariant with respect to A, in the sense that $S(t)K \subseteq K$ for each $t \in \mathbb{R}_+$, and let $f : K \to X$ be a continuous function. Next, we prove some appropriate sufficient conditions on f in order that K be mild viable with respect to $A + f$.

Lemma 1. *Let X be a Banach space, $A : D(A) \subseteq X \to X$ the infinitesimal generator of a C_0-semigroup, $\{S(t) : X \to X;\ t \geq 0\}$, and K a nonempty subset in X. Assume that K is invariant with respect to A, i.e., $S(t)K \subseteq K$ for each $t \in \mathbb{R}_+$. Then $\mathcal{T}_K(\xi) \subseteq \mathcal{T}_K^A(\xi)$ for each $\xi \in K$. If, instead of a C_0-semigroup, A generates a C_0-group of isometries, $\{G(t) : X \to X;\ t \in \mathbb{R}\}$, satisfying $G(t)K \subseteq K$ (or, equivalently, $G(t)K = K$) for each $t \in \mathbb{R}$, then $\mathcal{T}_K(\xi) = \mathcal{T}_K^A(\xi)$ for each $\xi \in K$.*

Proof. Let $\eta \in \mathcal{T}_K(\xi)$. By Proposition 1, it suffices to check that

$$\liminf_{h\downarrow 0} \frac{1}{h} \operatorname{dist}(S(h)\xi + hS(h)\eta; K) = 0.$$

Let $M \geq 1$ and $a \in \mathbb{R}$ be such that $\|S(t)\| \leq Me^{at}$ for each $t \geq 0$. Since $S(t)K \subseteq K$ for each $t \geq 0$, we have

$$\operatorname{dist}(S(h)\xi + hS(h)\eta; K) \leq$$
$$\leq \operatorname{dist}(S(h)\xi + hS(h)\eta; S(h)K) \leq Me^{ah} \operatorname{dist}(\xi + h\eta; K).$$

Thus

$$\liminf_{h\downarrow 0}\frac{1}{h}\mathrm{dist}\,(S(h)\xi+hS(h)\eta;K)\le\liminf_{h\downarrow 0}\frac{1}{h}Me^{ah}\mathrm{dist}\,(\xi+h\eta;K)=0.$$

Since the conclusion in the case of a C_0-group of isometries follows from the preceding one, this completes the proof. □

Theorem 3. *Let X be a Banach space, $A : D(A) \subseteq X \to X$ the infinitesimal generator of a C_0-semigroup, $\{S(t) : X \to X;\ t \ge 0\}$, K a nonempty and locally closed subset in X, and $f : K \to X$ a locally Lipschitz function. If $S(t)K \subseteq K$ for each $t \ge 0$ and*

$$f(\xi)\in\mathcal{T}_K(\xi) \tag{9}$$

for each $\xi \in K$, then K is mild viable with respect to $A + f$.

Proof. The conclusion follows from Lemma 1 and Theorem 2. □

Theorem 4. *Let X be a Banach space, $A : D(A) \subseteq X \to X$ the infinitesimal generator of a C_0-group of isometries, $\{G(t) : X \to X;\ t \in \mathbb{R}\}$, K a nonempty and locally closed subset in X, and $f : K \to X$ a locally Lipschitz function. If $G(t)K \subseteq K$ (or, equivalently, $G(t)K = K$) for each $t \in \mathbb{R}$, then a necessary and sufficient condition in order that K be mild viable with respect to $A + f$ is (9).*

Proof. The conclusion follows from Lemma 1 and Theorem 2. □

4 Proof of the main result

We can now pass to the proof of the main result which rests heavily on Theorem 4.
Proof. First, let us observe that the problem (1) can be rewritten as an abstract evolution equation of the form (5), where $X = L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$, $A : D(A) \subseteq X \to X$ is defined by

$$\begin{cases} D(A)=\{(u,v)\in X;\ (a\nabla v,a\nabla u)\in X\}\\ A(u,v)=(a\nabla v,a\nabla u)\ \text{ for all }\ (u,v)\in D(A),\end{cases} \tag{10}$$

and $f : X \to X$ is given by

$$f(u,v)(x)=(f_1(u(x),v(x)),f_2(u(x),v(x))), \tag{11}$$

for each $(u, v) \in X$ and a.e. for $x \in \mathbb{R}^n$.

On X we consider the usual Hilbert space norm

$$\|(u,v)\|=\sqrt{\langle u,u\rangle+\langle v,v\rangle},$$

for each $(u, v) \in X$, where $\langle\cdot,\cdot\rangle$ is the usual inner product on $L^2(\mathbb{R}^n)$.

It is well-known that the linear operator A, defined by (10), generates a C_0-group of isometries, $\{G(t) : X \to X;\ t \in \mathbb{R}\}$, given by

$$[G(t)(u,v)]\,(x)=\frac{1}{2}\begin{pmatrix} u(x+ta)+u(x-ta)+v(x+ta)-v(x-ta)\\ u(x+ta)-u(x-ta)+v(x+ta)+v(x-ta)\end{pmatrix}^{\mathcal{T}},$$

where $B^{\mathcal{T}}$ denotes the transpose of the matrix B. See Vrabie [7]. Second, since f_i, $i = 1, 2$, are globally Lipschitz, the function $f : X \to X$, given by (11), is well-defined and globally Lipschitz on X.

Next, let us define

$$K = \{(\xi, \eta) \in X;\ \xi \text{ and } \eta \text{ satisfy (3), i.e., } \langle \xi, \eta \rangle = 0\}$$

which is nonempty and closed in X. Let us observe that $G(t)K \subseteq K$ for each $t \in \mathbb{R}$. Indeed, let $(\xi, \eta) \in K$ and let us denote by

$$G(t)(\xi, \eta) = (G_1(t)(\xi, \eta), G_2(t)(\xi, \eta)),$$

where

$$G_1(t)(\xi, \eta) = \frac{1}{2}\left(\xi(\cdot + ta) + \xi(\cdot - ta) + \eta(\cdot + ta) - \eta(\cdot - ta)\right)$$

$$G_2(t)(\xi, \eta) = \frac{1}{2}\left(\xi(\cdot + ta) - \xi(\cdot - ta) + \eta(\cdot + ta) + \eta(\cdot - ta)\right)$$

for each $(t, x) \in \mathbb{R} \times \mathbb{R}^n$.

We have

$$\begin{aligned}\langle G_1(t)(\xi, \eta), G_2(t)(\xi, \eta)\rangle &= \|\xi(\cdot + ta)\|_{L^2(\mathbb{R}^n)} - \langle \xi(\cdot + ta), \xi(\cdot - ta)\rangle + \\ &+\langle \xi(\cdot + ta), \eta(\cdot + ta)\rangle + \langle \xi(\cdot + ta), \eta(\cdot - ta)\rangle + \langle \xi(\cdot - ta), \xi(\cdot + ta)\rangle - \\ &-\|\xi(\cdot - ta)\|_{L^2(\mathbb{R}^n)} + \langle \xi(\cdot - ta), \eta(\cdot + ta)\rangle + \langle \xi(\cdot - ta), \eta(\cdot - ta)\rangle + \\ &+\langle \eta(\cdot + ta), \xi(\cdot + ta)\rangle - \langle \eta(\cdot + ta), \xi(\cdot - ta)\rangle + \|\eta(\cdot + ta)\|_{L^2(\mathbb{R}^n)} + \\ &+\langle \eta(\cdot + ta), \eta(\cdot - ta)\rangle - \langle \eta(\cdot - ta), \xi(\cdot + ta)\rangle + \langle \eta(\cdot - ta), \xi(\cdot - ta)\rangle - \\ &-\langle \eta(\cdot + ta), \eta(\cdot - ta)\rangle - \|\eta(\cdot - ta)\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Since the Lebesgue measure on $\mathbb{R}^n$ is translation invariant, we deduce that the right hand side vanishes which proves that $G(t)K \subseteq K$.

Thanks to Theorem 4, K is mild viable with respect to $A + f$ if and only if $f(\xi, \eta) \in \mathcal{T}_K(\xi, \eta)$ for each $(\xi, \eta) \in K$. The last condition is equivalent to the existence of two sequences, $(h_n)_n$ in $\mathbb{R}_+$ and $((p_n, q_n))_n$ in X, with $h_n \downarrow 0$, $\lim_n (p_n, q_n) = (0, 0)$ and such that

$$(\xi, \eta) + h_n(f_1(\xi, \eta), f_2(\xi, \eta)) + h_n(p_n, q_n) \in K$$

for $n = 1, 2, \dots$. Equivalently,

$$\langle \xi + h_n f_1(\xi, \eta) + h_n p_n, \eta + h_n f_2(\xi, \eta) + h_n q_n\rangle = 0$$

for $n = 1, 2, \dots$. A simple calculation using the fact that $\langle \xi, \eta\rangle = 0$, $h_n \downarrow 0$ and $\lim_n p_n = \lim_n q_n = 0$, shows that the last relation is equivalent to (4), and this shows that K is mild viable with respect to $A + f$.

Finally, since f_i, $i = 1, 2$, are globally Lipschitz, it follows that f inherits the very same property and thus it has linear growth. A classical argument involving Gronwall's Lemma and the fact that K is closed and mild viable with respect to $A + f$, shows that each mild solution of (5) can be continued to a global one and this completes the proof. □

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Две проблемы, относящиеся к качественной теории гамильтоновых систем

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Abstract. Two problems of KAM-theory, which are not solved still, are discussed

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Существенное развитие математики в XIX столетии и, прежде всего, математического анализа, аналитической, качественной и численной теорий дифференциальных уравнений, разработка Максвеллом математических основ электродинамики вместе с выдающимися астрономическими открытиями (наряду с великим открытием планеты Нептун, приведем, например, открытие Фраунгофером спектрального анализа) предопределили появление знаменитого трехтомного сочинения А.Пуанкаре "Новые методы небесной механики" [1]. Гениальный трактат обусловил развитие многих направлений математики XX столетия и, в частности, способствовал появлению и развитию ее раздела, который ныне хорошо известен под названием КАМ-теория. К 1967 году ее создатели А.Н. Колмогоров, В.И. Арнольд и Ю. Мозер сформулировали и доказали основные теоремы, составляющие ее фундамент [2–5]. Сегодня "КАМ-теория" (сама аббревиатура появилась позднее) объединяет проблемы и фундаментальные результаты А.Н. Колмогорова, В.И. Арнольда, Ю. Мозера, а также их учеников и последователей по аналитической и качественной теории условно-периодических решений гамильтоновых систем, задаваемых на многомерных торах. Математическая теория нелинейных колебаний, трактуемая нами как теория периодических и почти периодических решений нелинейных систем обыкновенных дифференциальных уравнений безусловно относится к тому разделу, для которого методы КАМ-теории представляются наиболее эффективными. В "Новых Методах" Пуанкаре, сформулировал следующую проблему.

Пусть заданы две функции $2n$ переменных $x_1, \ldots, x_n, y_1, \ldots, y_n$:

$$F_1(x_1, \ldots, x_n; y_1, \ldots, y_n) \in C^{(1)}(G_{2n}), \quad F_2(x_1, \ldots, x_n; y_1, \ldots, y_n) \in C^{(1)}(G_{2n}),$$

где G_{2n} – область $2n$-мерного евклидова пространства специальной структуры, типичной для пространств периодических функций "с многими периодами" [4]. Говорят, что функции F_1 и F_2 *находятся в инволюции* в G_{2n}, если их скобка

Пуассона [1, 3] в G_{2n} равна нулю, т.е. если

$$(F_1, F_2) \equiv \sum_{k=1}^{n} \left(\frac{\partial F_1}{\partial x_k} \cdot \frac{\partial F_2}{\partial y_k} - \frac{\partial F_1}{\partial y_k} \cdot \frac{\partial F_2}{\partial x_k} \right) = 0. \tag{1}$$

Рассмотрим автономную гамильтонову систему дифференциальных уравнений

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \qquad \frac{dq}{dt} = \frac{\partial H}{\partial p}, \tag{2}$$

с гамильтонианом

$$\begin{gathered} H(p,q) \equiv H_0(p) + \mu H_1(p,q), \quad 0 \le \mu \ll 1, \\ H_1(p,q) \equiv H_1(p, q + (2\pi)), \end{gathered} \tag{3}$$

аналитическим в $2n$-мерной области

$$G_{2n} = \left\{ p \in G_n, \|\mathrm{Im}\, q\| < \rho < 1, \|\mathrm{Im}\, q\| = \sum_{s=1}^{n} \|\mathrm{Im}\, q_s\| \right\}, \tag{4}$$

где n – мерное многообразие G_n состоит из n-мерных торов, а Фурье-представление гамильтониана является n-кратным сходящимся рядом по переменным q (нами использована символика В.И. Арнольда [3, 4]).

Теорема Лиувилля [1, 4]. *Если известны n независимых первых интегралов системы (2), удовлетворяющих инволюционному равенству (1), тогда гамильтонова система (2) интегрируема в квадратурах.*

Формулировка теоремы Лиувилля не содержит утверждений, связанных с преобразованиями гамильтоновой системы (2), но Пуанкаре сформулировал проблему интегрируемости основных уравнений классической динамики, т.е. гамильтоновых систем вида (2), в терминах их преобразования к следующему виду:

$$\frac{d\widetilde{P}}{dt} = -\frac{\partial \widetilde{H}}{\partial Q}, \qquad \frac{d\widetilde{Q}}{dt} = \frac{\partial \widetilde{H}}{\partial P} = \omega(P), \tag{5}$$

где новый гамильтониан зависит только от "медленных" переменных P, но не зависит от "быстрых" переменных Q.

Иными словами, требуется доказать теорему существования такого невырожденного канонического преобразования $(p, q) \longrightarrow (P, Q)$, которое преобразует гамильтонову систему (2) в систему (5) и обратно, систему (5) в систему (2). Именно Пуанкаре сформулировал и фактически наметил путь к решению следующей **проблемы**:

Решить проблему существования и построить такое невырожденное каноническое преобразование [3–5]

$$p = \varphi(P, Q), \quad q = \psi(P, Q), \tag{6}$$

которое преобразует систему (2) с гамильтонианом, удовлетворяющим условиям (3) – (4), в систему (5), в которой новый гамильтониан зависит только от новых "медленных" переменных (новых "импульсов" P) и не зависит от новых "быстрых" переменных (фазовых углов Q) [6].

Эта проблема является центральной проблемой в КАМ-теории и, помимо корректного подхода, который подразумевает обязательное исследование сходимости встречающихся преобразований, она была объектом исследования и в интерпретации "асимптотической" [1]. Здесь фундаментальные пионерские работы принадлежат Н.Н. Боголюбову, Ю.А. Митропольскому, А.М. Самойленко [7, 8]. Асимптотическая теория многочастотных систем, построенная на базе специальных схем усреднения, учитывающих наличие частотных резонансов, которые являются основным препятствием при фактическом построении указанных преобразований, была предложена в работах [9,10]. Гамильтониан (2) удовлетворяет теореме Дирихле о существовании его Фурье-разложения [11] в области G_{2n}, т.е. его можно представить n-кратным рядом Фурье по "угловым" переменным q, который в комплексной форме имеет вид

$$H(p,q) = \sum_{\|k\|\geq 0} h_k(p) e^{i(k,q)},$$
$$(k,q) = \sum_{s=1}^{n} k_s q_s, \quad \|k\| = \sum_{s=1}^{n} |k_s|, \quad k_s = 0, \pm 1, \pm 2, \dots, \quad i = \sqrt{-1}, \tag{7}$$

$$h_k(p) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} H(p,q) e^{-i(k,q)} dq_1 \dots dq_n. \tag{8}$$

Коэффициенты Фурье $h_k(p)$ разложения (7) являются быстро убывающими величинами с ростом нормы целочисленного вектора , т.е. $\|k\|$, и удовлетворяют "экспоненциальной" оценке В.И.Арнольда [4]

$$\|h_k(p)\| \leq M e^{-\|k\|_\rho}, \tag{9}$$

где M и ρ – некоторые положительные константы.

В силу этого, представляется целесообразным поиск преобразования (6) в виде n-кратных тригонометрических рядов [4, 6]

$$\begin{cases} p = \varphi(P,Q) = \displaystyle\sum_{\|k\|\geq 0}^{\infty} \varphi_k(P) e^{i(k,\,Q)}, \\ q = \psi(P,Q) = \displaystyle\sum_{\|k\|\geq 0}^{\infty} \psi_k(P) e^{i(k,\,Q)}, \end{cases} \tag{10}$$

где коэффициенты $\varphi_k(P)$, $\psi_k(P)$, в свою очередь, представляются степенными рядами вида

$$\begin{cases} \varphi_k(P) = \sum\limits_{\|s\| \geq \|k\|} a_s P^s \equiv \sum\limits_{s_1+s_2+\ldots+s_n \geq \|k\|}^{\infty} a_{s_1,s_2,\ldots,s_n} P_1^{s_1} P_2^{s_2} \ldots P_n^{s_n}, \\ \psi_k(P) = \sum\limits_{\|s\| \geq \|k\|} b_s P^s, \end{cases} \tag{11}$$

$$s_j = 0, 1, 2, \ldots, \quad k_j = 0, \pm 1, \pm 2, \ldots, \quad j = 1, 2, \ldots, n,$$

с неизвестными коэффициентами a_s, b_s, которые подлежат определению.

Если продифференцировать выражения (10) по t и подставить их в левые части основной гамильтоновой системы (2), а в правых частях осуществить замену (10) с коэффициентами из (11), то после выполнения соответствующих операций (среди них алгебраические операции над коэффициентами Фурье и операции дифференцирования) в принципе может быть выписана *бесконечная система нелинейных алгебраических уравнений* с неизвестными a_s, b_s, вида

$$F_r(a_0, b_0, \ldots, a_s, b_s) = 0, \quad r = 1, 2, 3\ldots . \tag{12}$$

Решение алгебраической задачи (12) *эквивалентно* задаче о построении замены переменных (10) тогда и только тогда, когда ряды (10) и (11) являются сходящимися в соответствующих областях.

Фундаментальный результат Пуанкаре о расходимости рядов небесной механики [1] применительно к поиску преобразования (6) состоит в том, что ряды вида (10) являются расходящимися из-за присутствия в структуре коэффициентов $\varphi_k(P)$ и $\psi_k(P)$ *малых знаменателей типа* $(k, \widetilde{\omega}(P))$, которые при $\|k\| \to \infty$ могут быть сколь угодно малыми, в том числе и нулевыми величинами, причем *"случайность"* такого поведения величин $(k, \widetilde{\omega}(P))$ как функций вектора k можно считать основным препятствием при исследовании их сходимости.

Рассмотрим условие

$$|(k, \widetilde{\omega}(P))| = \alpha \ll 1, \tag{13}$$

которое принято называть *α-резонансом частот* $\widetilde{\omega}_1(P), \widetilde{\omega}_2(P), \ldots, \widetilde{\omega}_n(P)$ [12]. При $\alpha = 0$ мы имеем точный резонанс частот. Очевидно, что резонансы могут возникнуть только при $n \geq 2$, т.е. могут иметь место в двух, трех и так далее многочастотных динамических системах. Если переменные Q в гамильтоновой системе (5) трактовать как фазовые углы, то тогда в физическом смысле величины $\widetilde{\omega}_1(P), \widetilde{\omega}_2(P), \ldots, \widetilde{\omega}_n(P)$ – это угловые скорости, или, как принято их называть в теории колебаний, *частоты.* Именно появление в процессе интегрирования гамильтоновой системы (2) частотных резонансов вида (13), неизбежно приводит к расходимости рядов, участвующих в вышеуказанных преобразованиях.

Это побудило Пуанкаре предложить теорию *асимптотических представлений* (синонимом этого понятия является *асимптотический ряд*), на основе которой была разработана теория асимптотической интегрируемости дифференциальных уравнений динамики.

Комментарием к этим рассуждениям может служить отрывок из знаменитого сочинения Пуанкаре "Новые методы небесной механики" [1]: "Геометры и астрономы по-разному понимают слово сходимость. Геометры, всецело озабоченные достижением безукоризненной строгости и зачастую совершенно безразличные к продолжительности сложных вычислений, говорят, что некоторый ряд сходится, если сумма его членов стремится к какому-то определенному пределу, даже в том случае, когда первые члены ряда убывают чрезвычайно медленно. В противоположность этому астрономы обычно говорят, что некоторый ряд сходится, если, например, первые двадцать членов этого ряда убывают очень быстро, не смотря на то, что последующие его члены неограниченно возрастают ... . Обе точки зрения законны: первая – в теоретических исследованиях, вторая – в численных приложениях. Обе господствуют безраздельно, но в различных областях, и границы этих областей необходимо четко различать".

Отметим также, что последние десятилетия получила также развитие топологическая трактовка проблемы интегрируемости гамильтоновых систем [13,14], состоящая в построении глобальной классификации траекторий динамических систем по их различным признакам и связанное с этим разделение фазового пространства на области, включающие в себя траектории различных классов. Так или иначе, исследования Пуанкаре показали, что основным препятствием к интегрированию гамильтоновых систем (2) являются малые знаменатели типа (13), неизбежно появляющиеся в искомых рядах. Что касается построения бесконечной последовательности невырожденных преобразований

$$(p,q) \leftrightarrow \left(p^{(1)},q^{(1)}\right) \leftrightarrow \left(p^{(2)},q^{(2)}\right) \leftrightarrow \ldots \leftrightarrow \left(p^{(n)},q^{(n)}\right) \leftrightarrow \ldots \equiv (P,Q), \qquad (14)$$

которая преобразует систему (2) в интегрируемую систему (5), такое решение может быть достигнуто лишь в рамках КАМ-теории. Фазовые многообразия, в которых существуют сходящиеся невырожденные канонические преобразования, составляют бесконечную последовательность включений

$$G_{2n} \supset G_{2n}^{(1)} \supset G_{2n}^{(2)} \supset \ldots \supset G_{2n}^{(m)} \supset \ldots \equiv G_{2n}^{*},$$

причем предельное многообразие G_{2n}^{*} не должно быть пустым ($G_{2n}^{*} \neq \emptyset$) и его можно записать в виде

$$G_{2n}^{*} = \left\{P \in G_{n}^{*},\ \|\mathrm{Im}\, Q\| < \rho^{*} \leq \rho < 1\right\}.$$

Сходимость итерационного процесса (14) гарантируется методом ускоренной сходимости [4], в котором k-я итерация имеет порядок малости

$$\left(p^{(k)}, \Delta q^{(k)}\right) = O(\mu^{2^k})$$

где $\Delta q^{(k)}$ – "возмущение", т.е. "малый добавок" к фазовой угловой переменной $q^{(k)}$.

Если строить итерации классическим методом, для которого порядок малости k-й итерации имеет "степенную" оценку

$$\left(p^{(k)}, \Delta q^{(k)}\right) = O(\mu^{k}),$$

тогда для гамильтоновых систем (2) размерности $4, 6, 8, \ldots$ $(n \geq 2)$ итерационный процесс (14) будет расходящимся.

Именно на основе этого факта Пуанкаре сделал заключение, что последовательность простых "степенных" преобразований $(p, q) \to (P, Q)$ является расходящейся в области G_{2n}. Кроме того, фазовые n-мерные многообразия G_n^* и $\bar{G}_n = G_n \setminus G_n^*$ всюду плотны в $G_n = G_n^* \cup \bar{G}_n$, из чего следует, что проблема сходимости "степенных" преобразований $(p, q) \to (P, Q)$ фактически становится неразрешимой.

Вместе с тем К. Зигель показал [15], что на полном многообразии G "малые знаменатели" (k, ω) удовлетворяют неравенству

$$|(k, \omega(p))| \geq \frac{K(p)}{\|k\|^{n+1}}, \tag{15}$$

где $\omega(p) = \dfrac{\partial H_0}{\partial p}$, и лебеговы меры многообразий G_n^*, $\bar{G}_n$ соответственно равны

$$0 < mes\, \bar{G}_n = \varepsilon \ll 1, \quad mes\, G_n^* = 1 - \varepsilon,$$

где ε – малое положительное число.

Идеология методов "ускоренной сходимости", развитых в КАМ-теории для "борьбы" с малыми знаменателями вида (15), оказывается весьма эффективной [2, 4] и состоит в том, что с учетом условия (15) можно построить сходящиеся процедуры (14) и, следовательно, можно привести первоначальную систему дифференциальных уравнений (2) к системе (5). Проблема малых знаменателей присутствует не только в гамильтоновой динамике. Она фактически появляется всюду, где имеются много периодические процессы (например, в задаче об отображении окружности на себя, в задаче об устойчивости особой точки типа "центр" [14–17], в теории движения планет и спутников, в космической динамике).

На основании изложенного можно сформулировать как минимум две фундаментальные проблемы качественной и аналитической теории обыкновенных дифференциальных уравнений, которые я имел честь и удовольствие неоднократно обсуждать с выдающимся математиком, академиком Константином Сергеевичем Сибирским. И сегодня они весьма важны и для теории нелинейных колебаний, и для качественной теории обыкновенных дифференциальных уравнений.

Формулировка первой проблемы. Выше написано, что построение бесконечной цепочки преобразований (14) возможно лишь на базе КАМ-теории с учетом топологической структуры n-мерных многообразий G_n^* и $\bar{G}_n = G_n \setminus G_n^*$ [3, 4].

Проблема состоит в том, чтобы выделить и исследовать аналитические свойства такого подмножества функций n переменных из множества $\{F(x)\}$ всевозможных 2π-периодических аналитических функций вида $f(x_1, x_2, \ldots, x_n) = f(x_1 + 2\pi, x_2 + 2\pi, \ldots, x_n + 2\pi)$, определенных на n-мерных торах G_n, для которого его мера $mes\,\bar{G}_n = 0$, и соответственно определить второе подмножество функций, для которого мера $mes\,G_n^* = 1$.

Применительно к гамильтоновым системам вида (2) это означает, что необходимо найти и изучить свойства такого подмножества гамильтоновых систем из множества (2), для которого оценки, построенные В.И. Арнольдом [4], справедливы при $\varepsilon = 0$, т.е. $mes\,\bar{G}_n = 0$, а $mes\,G_n^* = 1$.

Конкретные примеры таких гамильтонианов существуют (например, гамильтонианы, аналитическая структура которых суть два-периодические тригонометрические полиномы с заданными постоянными частотами). Решение этой проблемы могло бы способствовать исследованию динамической эволюции на больших, космогонических интервалах времени, конкретных планетных систем (каковой, например, является наша Солнечная система), для которых частоты, определенные из длительных наблюдений, можно считать заданными.

Формулировка второй проблемы. Известен пример В.И. Арнольда о неустойчивости положения равновесия для гамильтоновой системы с тремя импульсами и тремя фазовыми углами [16]. Этот пример стимулирует поиск таких, по-видимому, достаточно жестких условий, которым должны удовлетворять гамильтонианы, зависящие от 6-, 8-, 10-ти и т.д. канонических переменных, чтобы качество устойчивости положения равновесия имело место не только в первом приближении.

Эта проблема стала весьма актуальной в последнее десятилетие, в связи с новыми фундаментальными результатами по динамической эволюции реальной (а не гипотетической) Солнечной системы, полученными американскими и российскими исследователями [17–20]. Современные суперкомпьютеры позволили вычислить эволюцию динамики реальных планет на интервалах времени в миллиарды лет с гарантированной (проверенной независимыми методами) точностью. Выводы вытекают однозначные: Солнечная система имеет огромный запас конфигурационной прочности в интервалах времени в миллиарды лет (и "вперед", и "назад"), если учитывать при этом только внутренние (интерпланетарные и планетарно-солнечные) гравитационные связи, т.е. такие связи, которые гарантируют корректность формализма гамильтоновой динамики.

В заключение хочу отметить, что я имел огромное счастье в жизни обсуждать эти и многие другие научные проблемы с замечательным, скромнейшим человеком, выдающимся ученым Константином Сергеевичем Сибирским. Мне безмерно жаль, что сегодня я лишен возможности общаться с ним.

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n-Homogeneous dynamical systems and n-ary algebras

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Abstract. A bijective correspondence between the classes of center-affinely equivalent n-homogeneous equations ($n \geq 2$) and the classes of isomorphic commutative n-ary algebras is established. It generates a correspondence between the properties of these equations and the structural properties of the associated n-ary algebras.

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1 Introduction

Let $E(\|\cdot\|)$ be a Banach space over the field K (here K is $\mathbb{R}$ or $\mathbb{C}$). A *n-homogeneous dynamical system* comes from a n-homogeneous differential equation (shortly, nHDE) of the form (S) $\dfrac{dX}{dt} = F(X)$, where $F : E \to E$ is a continuous n-monomial vector form. The polar form for F is a symmetric n-linear vector form which allows us to define a n-ary algebra on E; this algebra is commutative and nonassociative (more exactly, it is not necessarily an associative algebra).

The nHDE (S) is said to be *center-affinely equivalent* (CA-equivalent) with another nHDE (S_1) defined on a Banach space E' if and only if there exists an invertible continuous linear mapping $h : E' \to E$ such that $X = h(Y)$ is a solution for (S) as long as Y is a solution for (S_1). Then, the following result holds: *(S) is CA-equivalent with (S_1) if and only if their associated n-ary algebras are continuously isomorphic.* According to this result, one gets: *there exists a bijection between the set of all classes of CA-equivalent nHDEs and the set of all classes of isomorphic commutative n-ary K-algebras.* It means that a classification up to an isomorphism of commutative n-ary K-algebras gives the classification up to a CA-equivalence of all nHDEs.

Actually, the structure of the associated algebra allows us to determine some features of the analyzed nHDE as well as of its space of solutions. As examples, we quote the following results:

1. semisimple algebras give a decoupling of the initial equation into equations occurring in simple algebras,
2. solvable algebras give solutions via a subset of linear differential equations,
3. the n-degree nilpotents N (i.e., with $N^n = [N, N, ..., N] = 0$) are *steady-state points* or *equilibria*, i.e., they are the *constant* solutions,

4. the n-degree idempotents e (i.e., with $e^n = [e, e, ..., e] = e$) give the *ray* solutions,

5. the origin is never asymptotically stable and the existence of an idempotent implies that the origin is actually an instable steady state.

Recall that the automorphisms of the associated algebra keep invariant all equilibria, periodic orbits, and domains of attraction.

Some quantitative results can be obtained, too. Firstly, it must be remarked that, F being an analytic function then the solution of every CAUCHY problem for (S) is an analytical one. Besides, if the n-ary algebra associated with the nHDE (S) is power-associative, then there exists a formula giving the solution of every CAUCHY problem for (S).

Several new results can be obtained in the particular case of the nHDEs defined on finite dimensional spaces. This time, any nHDE becomes really an n-homogeneous differential system of equations (shortly, nHSDE).

NOTE. We have preferred to work in a BANACH space not only to generalize some known results but, mainly, because in this frame a good understanding of the facts is necessary (facts which - in the finite dimensional case - are hidden behind of some bushy computations).

2 Preliminaries

Polynomial mappings. Throughout this paper the following notations will be used:

E – a BANACH space,

$C_n(E)$ – the space of all continuous n-homogeneous functions from E to E,

$L(E^n, E)$ – the BANACH space of all continuous n-multilinear forms from E^n to E (endowed with the norm induced by the one of E),

$L_s(E^n, E) \subset L(E^n, E)$ – the BANACH space of all continuous n-multilinear symmetric forms from E^n to E,

$\Delta : E \to E^n$ – the *n-diagonal mapping* on E (i.e., $\Delta(x) = (\underbrace{x, x, ..., x}_{n\,times})$, $\forall x \in E$).

The mapping $\mathcal{P} : L_s(E^n, E) \to C_n(E)$ defined by

$$\mathcal{P}(G) = G \circ \Delta, \quad \forall G \in L_s(E^n, E)$$

is the so-called *polynomial projection*, while $\mathcal{P}(G)$ is called the *n-homogeneous polynomial* associated with G (or a *monomial* of degreee n, or a *n-monomial*). Any (finite) linear combination of monomials on E is called a *polynomial*; the *degree of a polynomial* is the maximum among the degrees of its monomial components.

Actually, the polynomial projection $\mathcal{P}$ is a bijection between $L_s(E^n, E)$ and the space $\mathcal{P}_n(E)$ of all n-monomials on E. Indeed, for any $F \in \mathcal{P}_n(E)$, the mapping

$G \in L_s(E^n, E)$ defined by

$$G(x_1, x_2, ..., x_n) = \frac{1}{n!} \left[F\left(\sum_{i=1}^{n} x_i\right) - \sum_{j=1}^{n} F\left(\sum_{\substack{i=1 \\ i \neq j}}^{n} x_i\right) + \right.$$
$$\left. + \sum_{\substack{j,k=1 \\ j<k}}^{n} F\left(\sum_{\substack{i=1 \\ i \notin \{j,k\}}}^{n} x_i\right) + ... + (-1)^n \sum_{i=1}^{n} F(x_i) \right], \quad \forall x_1, x_2, ..., x_n \in E$$

satisfies $F = G \circ \Delta$; G is called the *polar form* of F.

***n*-ary Algebras.** Let E be a K-vector space (finite dimensional or not).

Definition 1. *A n-ary algebra is any K-vector space E endowed with a n-multilinear vector mapping* $[., ..., .] : E^n \to E$

$$(x_1, x_2, ..., x_n) \to [x_1, x_2, ..., x_n], \quad \forall (x_1, x_2, ..., x_n) \in E^n.$$

We denote it by $E([., ..., .])$.

In this case, the mapping $G : E^n \to E$ defined by

$$G(x_1, x_2, ..., x_n) = [x_1, x_2, ..., x_n], \quad \forall x_1, x_2, ..., x_n \in E$$

is a $(1, n)$-tensor, i.e., $G \in E^{*\otimes n} \otimes_K E$. Actually, the set of all n-ary K-algebras on E is identifiable with the tensor product $E^{*\otimes n} \otimes_K E$.

Recall that, for any algebra $E([., ..., .])$ and any $x_1, x_2, ..., x_{n-1} \in E$ we can define the i-multiplication $M^i_{x_1, x_2, ..., x_{n-1}} : E \to E$ $(i \in \{1, 2, ..., n\})$ by

$$x \to M^i_{x_1, x_2, ..., x_{n-1}}(x) = [x_1, x_2, ..., x_{i-1}, \underset{i}{x}, x_i, ..., x_{n-1}], \quad \forall x \in E.$$

$L_{x_1, x_2, ..., x_{n-1}} = M^n_{x_1, x_2, ..., x_{n-1}}$ is called the *left multiplication* by $(x_1, x_2, ..., x_{n-1})$, and $R_{x_1, x_2, ..., x_{n-1}} = M^1_{x_1, x_2, ..., x_{n-1}}$ is called the *right multiplication* by $(x_1, x_2, ..., x_{n-1})$.

The algebra $E([., ..., .])$ is said to be *associative* if

$$[[x_1, x_2, ..., x_n], y_2, ..., y_n] = [x_1, [x_2, ..., x_n, y_2], y_3, ..., y_n] = ... =$$
$$= [x_1, x_2, ..., x_{n-1}, [x_n, y_2, ..., y_n]], \quad \forall x_1, x_2, ..., x_n, y_2, ..., y_n \in E.$$

It results that $E([., ..., .])$ is associative if and only if the following equations hold

$$L_{x_1, x_2, ..., x_{n-1}} \circ L_{y_1, ..., y_{n-1}} = L_{[x_1, x_2, ..., x_{n-1}, y_1], y_2, y_3, ..., y_{n-1}} =$$
$$= L_{x_1, [x_2, ..., x_{n-1}, y_1, y_2], y_3, ..., y_{n-1}} = ... = L_{x_1, x_2, ..., x_{n-2}, [x_{n-1}, y_1, y_2, ..., y_{n-1}]},$$
$$\forall x_1, x_2, ..., x_n, y_1, ..., y_{n-1} \in E.$$

The associativity of a n-ary algebra can be similarly characterized by means of the right multiplications.

The n-ary algebra $E([.,...,.])$ is said to be *commutative* or *symmetric* if

$$[x_1, x_2, ..., x_n] = [x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)}], \quad \forall x_1, x_2, ..., x_n \in E,\ \forall \sigma \in S_n$$

(here S_n denotes the symmetric group of n elements).

$e \in E$ is said to be an *identity element* for $E([.,...,.])$ if

$$[e, e, ..., e, x] = [e, ..., e, x, e] = [e, x, e, ..., e] = [x, e, ..., e] = x,\ \ \forall x \in E.$$

An identity element for $E([.,...,.])$, if it exists, is not necessarily unique (in contrast with the case of binary algebras).

Any n-ary algebra $E([.,...,.])$ having no identity element can be naturally embedded into a n-ary algebra with identity element, namely $\overline{E} = E \oplus K$ endowed with n-ary composition

$$[x_1 \oplus \lambda_1, x_2 \oplus \lambda_2, ..., x_n \oplus \lambda_n] = ([x_1, x_2, ..., x_n] + \lambda_2 ... \lambda_n x_1 + ... + \\ +\lambda_1 ... \lambda_{n-1} x_n, \lambda_1 \lambda_2 ... \lambda_n), \quad \forall x_1 \oplus \lambda_1, x_2 \oplus \lambda_2, ..., x_n \oplus \lambda_n \in \overline{E};$$

obviously, $0 \oplus 1$ is an identity element for $\overline{E}$.

$e \in E \setminus 0$ is said to be an *idempotent element* for $E([.,...,.])$ if $[e, e, ..., e] = e$.

$e \in E \setminus 0$ is said to be a *nilpotent element* for $E([.,...,.])$ if $[e, e, ..., e] = 0$.

For any fixed $x \in E$ one considers the left/right powers defined recurrently by:
left powers: $x^n = [x, x, ..., x],\ \ x^{n+k(n-1)} = [x^{n+(k-1)(n-1)}, x, ..., x], n \geq 2$,
right powers: $x^{[n]} = [x, x, ..., x],\ \ x^{[n+k(n-1)]} = [x, ..., x, x^{[n+(k-1)(n-1)]}], n \geq 2.$

In a commutative algebra, left and right powers of any element are coincident.

$E([.,...,.])$ is named *power-associative* or *mono-associative* if

$$[x^{n+m_1(n-1)}, x^{n+m_2(n-1)}, ..., x^{n+m_n(n-1)}] = x^{n+(m_1+m_2+...+m_n+n)(n-1)}, \\ \forall m_1, m_2, ..., m_n \in \mathbb{N}^*,\ \forall x \in E.$$

$E([.,...,.])$ is power-associative if and only if

$$L^k_{\underbrace{x, x, ..., x}_{n-1\,times}}(x) = L_{x^{n+(k-2)(n-1)}, x, ..., x}(x) = L_{x, x^{n+(k-2)(n-1)}, ..., x}(x) = ... = \\ = L_{x, ..., x, x^{n+(k-2)(n-1)}}(x), \qquad \forall x \in E,\ k \geq 2.$$

In any power-associative n-ary algebra both left and right powers of any element are coincident.

3 Polynomial equations

Definition 2. *a) A n-polynomial differential equation ($nPoDE$) on E is every differential equation of the form*

$$\frac{dX}{dt} = P_n(X) \tag{1}$$

where P_n is a polynomial of degree n and $X : U \to E$, with U an interval of $\mathbb{R}$, is the unknown vector function.

b) A n-homogeneous differential equation (nHDE) on E is every differential equation of the form

$$\frac{dX}{dt} = F(X) \tag{2}$$

where F is a n-degree monomial on E.

Every polynomial P_n on E can be considered as the restriction on $E(\equiv E \times 1)$ of a n-homogeneous polynomial P_n^h defined on $E \times K$ by

$$P_n^h(X, Y) = Y^n P_n(X/Y)$$

(here Y has nonzero values, only). Thus, (1) can be always transformed in a n-homogeneous equation on $E \times K$ namely

$$\begin{cases} \dfrac{dX}{dt} = P_n^h(X, Y), \\ \dfrac{dY}{dt} = 0, \end{cases} \tag{3}$$

for which the only solutions of interest will be the ones satisfying the condition $Y(t_0) = 1$. Consequently, the study of any nPoDEs can be reduced to that of nHDEs, i.e. it is enough to study the nHDEs.

Let us consider the n-homogeneous equation

$$\frac{dY}{dt} = F_1(Y) \tag{4}$$

on the Banach space E'.

Definition 3. *It is said that (2) is (center-) affinely equivalent (shortly, CA-equivalent) with (4) if there exists a continuous invertible linear mapping $h : E' \to E$ such that $X = h(Y)$ is a solution for (2) as long as Y is a solution for (4); h is called a CA-equivalence. A CA-equivalence of (2) with itself is called an automorphism for (2).*

Theorem 1. *The nHDE (2) is CA-equivalent with (4) if and only if there exists a continuously invertible linear mapping $h : E' \to E$ such that*

$$h \circ F_1 = F \circ h. \tag{5}$$

Proof. Let $y_o \in E'$ be an arbitrarily chosen (but fixed) element and $Y(t)$ be the solution of the Cauchy problem

$$\frac{dY}{dt} = F_1(Y), \qquad Y(t_0) = y_0. \tag{6}$$

Since (2) is equivalent to (4) there exists an invertible continuous linear transformation $h : E' \to E$ such that $X(t) = h(Y(t))$ is the solution of (2) with the initial condition $x_o = h(y_o)$. Consequently, the following equations hold:

$$\begin{aligned} &\frac{dX(t)}{dt} = F(X(t)) = \\ &= F(h(Y(t))) = h\left(\frac{dY(t)}{dt}\right) = h(F_1(Y(t)), \ \ \forall t \in I_1, \end{aligned} \tag{7}$$

i.e., $(F \circ h)(y_o) = (h \circ F_1)(y_o)$ (here $I_1 \subset \mathbb{R}$ is the domain of $Y(t)$). As y_o was arbitrarily chosen in E it follows that (5) holds. Conversely, if $Y(t)$ is a solution of problem (4) and (5) holds, then the equations

$$\frac{dX(t)}{dt} = h\left(\frac{dY(t)}{dt}\right) = (h \circ F_1)(Y(t)) = (F \circ h)(Y(t))) = F(X(t)), \tag{8}$$

also hold, i.e., $X(t)$ is a solution of equation (2) with the same domain as $Y(t)$.

As (5) is equivalent to $F_1 \circ h^{-1} = h^{-1} \circ F$ (and h^{-1} is continuous), it follows

Corollary 1. *If (2) is equivalent to (4), then (4) is also equivalent to (2).*

Obviously, E' can be identified, via h, with E so that it is enough to analyze the set of all nHDEs given on a fixed Banach space E, only. Thus, any CA-equivalence is really an equivalence on the set of all nHDEs on a fixed BANACH space (i.e. it is a reflexive, symmetric and transitive binary relation).

4 The algebra associated with a nHDE

The polar form $G : E^n \to E$ for F in (2) is a continuous and symmetric n-linear vector form on E. The n-ary algebra $E([., ..., .])$ defined by the n-ary operation

$$[x_1, x_2, ..., x_n] = G(x_1, x_2, ..., x_n), \quad \forall x_1, x_2, ..., x_n \in E$$

is a *commutative* (or, *symmetric*) n-ary algebra; it is a non-associative algebra, i.e. it is not necessarily an associative one.

Recall that there exists $\|G\|_1$ such that $\|G(x_1, x_2, ..., x_n)\|_1 \le \|G\|_1 \|x_1\|_1 \|x_2\|_1 ... \|x_n\|_1$. Then, the norm $\|\cdot\|$ on E defined by $\|x\| = \sqrt[n-1]{\|G\|_1} \cdot \|x\|_1$ has the property

$$\|[x_1, x_2, ..., x_n]\| \le \|x_1\| \cdot \|x_2\| ... \|x_n\|.$$

The n-ary algebra $E([., ..., .])$ endowed with norm $\|\cdot\|$ is called the B-*algebra* associated with (2).

Theorem 2. *The nHDE (2) is CA-equivalent with (4) if and only if the n-ary algebras associated with them as before are continuously isomorphic.*

Proof. If (2) and (4) are equivalent equations then there exists, according to Theorem 1, an invertible continuous linear mapping $h : E' \to E$ which satisfies (5). By passing to the polar forms for F and F_1, it follows that h is an algebra isomorphism between $E'(\{\cdot, \cdot, ..., \cdot\})$ and $E([\cdot, \cdot, ..., \cdot])$. Conversely, if $h : E'(\{\cdot, ..., \cdot\}) \to E([\cdot, ..., \cdot])$ is a continuous algebra isomorphism, then (5) holds, i.e. h is an equivalence of the nHDEs (2) and (4).

Remark 1. Theorem 2 ensures that there exists a bijection between the classes of affinely equivalent nHDEs on E and the classes of isomorphic commutative n-ary algebras on E. Consequently, there exists a correspondence between certain qualitative properties of a nHDE (2) and the invariant properties under an isomorphism of its associated n-ary commutative algebra.

5 Solving nHDEs

Let us suppose that the n-ary algebra $E([\cdot, ..., \cdot])$ associated with (2) has no identity element. Then, $E([\cdot, ..., \cdot])$ can be embedded into the n-ary algebra $\overline{E}([\cdot, ..., \cdot])$ whose operation has the n-monomial form

$$\overline{F}(x \oplus \lambda) = (F(x) + n\lambda^{n-1}x) \oplus \lambda^n, \quad \forall x \oplus \lambda \in \overline{E},$$

which suggests us to consider the following nHDE on $\overline{E}$

$$\begin{cases} \dfrac{dX}{dt} = n\lambda^{n-1}X + F(X), \\ \dfrac{d\lambda}{dt} = \lambda^n. \end{cases} \tag{9}$$

Then, with every Cauchy problem

$$\frac{dX}{dt} = F(X), \qquad X(t_0) = x_0 \tag{10}$$

we associate the following Cauchy problem

$$\begin{cases} \dfrac{dX}{dt} = n\lambda^{n-1}X + F(X), \\ \dfrac{d\lambda}{dt} = \lambda^n, \end{cases} \qquad \begin{cases} X(t_0) = x_0, \\ \lambda(t_0) = 0, \end{cases} \tag{11}$$

for (9). Consequently, there exists the 1-1 correspondence

$$X(t) \Leftrightarrow X(t) \oplus \{0\}$$

between the sets of solutions for (2) and the set of solutions of (9) with $\lambda \equiv 0$, respectively. That is why, in what follows, we shall study only nHDEqs for which the associated (B-)algebra has at least an identity element.

In order to imply the n-ary algebra in the study of nHDE (2), it is suitable to use the equality $F(x) = x^n$, and then (2) becomes

$$\frac{dX}{dt} = X^n. \tag{12}$$

CAUCHY-KOWALEWSKAIA Theorem assures us that there exists a unique analytic solution for CAUCHY problem (10).

Let $X(t)$ be the saturated solution for (10). Then we get

$$X(t_0) = x_0, \qquad \frac{dX(t_0)}{dt} = x_0^n$$

Further, we get recurrently:

$$\left\| \frac{d^i X(t_0)}{dt^i} \right\| \leq ((i-1)(n-1)+1)\underbrace{!...!}_{n-1\,times} \cdot \|x_0\|^{i(n-1)+1}, \quad \forall i \geq 1,$$

where $((i-1)(n-1)+1)\underbrace{!...!}_{n-1\,times} = 1 \cdot (1(n-1)+1) \cdot (2(n-1)+1) \cdot ... \cdot ((i-1)(n-1)+1)$. Consequently, the series

$$\|x_0\| + \frac{|t-t_0|}{1!}\left\|\frac{dX(t_0)}{dt}\right\| + \frac{|t-t_0|^2}{2!}\left\|\frac{d^2X(t_0)}{dt^2}\right\| + ... + \frac{|t-t_0|^k}{k!}\left\|\frac{d^kX(t_0)}{dt^k}\right\| + ...$$

is upper bounded by the numerical series for

$$\frac{\|x_0\|}{\sqrt[n-1]{1-(n-1)\|x_0\| \cdot |t-t_0|}}.$$

Thus, the TAYLOR series for $X(t)$, around t_0, is absolutely and uniformly convergent for $(n-1)\|x_0\| \cdot |t-t_0| < 1$.

In the next section we shall use these computations to find a formula for solving (10) in case when the associated algebra satisfies a "weak" associativity axiom (e.g., the monoassociativity). Obviously, these computations allow also to prove again the analyticity of the solution of (10).

6 The case of nHDEs with power-associative algebras

Let us consider the CAUCHY problem (10). Suppose the corresponding n-ary B-algebra is a power-associative algebra. Then, its solution $X(t)$ satisfies the conditions

$$\begin{aligned} &\frac{dX(t_0)}{dt} = x_0^n = L_{x_0,...,x_0}(x_0), \\ &\frac{d^kX(t_0)}{dt^k}\| = ((k-1)(n-1)+1)\underbrace{!...!}_{n-1\,times} \cdot L^k_{x_0,...,x_0}x_0, \quad k > 1 \end{aligned}$$

(here $L_{x_0,...,x_0}$ is instead of $L\underbrace{x_0,...,x_0}_{n-1\ times}$). Consequently, one gets

$$\begin{aligned} X(t) = &\Bigg(I + \frac{t-t_0}{1!}L_{x_0,...,x_0} + \frac{n!...!(t-t_0)^2}{2!}L^2_{x_0,...,x_0} + ... + \\ &+ \frac{((k-1)(n-1)+1)!...!(t-t_0)^k}{k!}L^k_{x_0,...,x_0} + ...\Bigg)(x_0) = \\ &= (I-(n-1)(t-t_0)L_{x_0,...,x_0})^{-\frac{1}{n-1}}(x_0) \end{aligned}$$

where I denotes the identity mapping on E and !...! is used instead of $\underbrace{!...!}_{n-1\,times}$. By a direct checking one gets that the analytic mapping

$$Y(t) = (I - (n-1)(t-t_0)L_{x_0,...,x_0})^{-\frac{1}{n-1}}(x_0),$$

defined for all t such that $(n-1)^{-1}(t-t_0)^{-1}$ does not belong to the spectrum of $L_{x_0,...,x_0}$, satisfies the equation

$$\frac{dY(t)}{dt} = Y^n(t)(= G(Y(t), Y(t), ..., Y(t))).$$

Theorem 3. *If the B-algebra $E([.,...,.])$ associated with (2) is power-associative, then the saturated solution for (10) is*

$$X(t) = (I - (n-1)(t-t_0)L_{x_0,...,x_0})^{-\frac{1}{n-1}}(x_0). \tag{13}$$

Remark 2. The computations performed for proving that $Y(t)$ is the solution of (10) can be similarly performed in the case when x_0 has associative powers, only. Consequently, (13) is the solution for (10) as long as x_0 has associative powers.

7 Properties of n-ary commutative algebras reflected in those of nHDE

The correspondence between the classes of CA-equivalent nHDEs and the classes of continuously isomorphic n-ary algebras induces the existence of a correspondence between the qualitative properties of nHDEs and those of n-ary algebras.

We shall present below some results on this line.

1) $x_0 \in E$ is a critical point (or, a steady state point) for (2) if and only if $x_0^n = 0$, i.e. if and only if it is a nilpotent for $E([.,.,.])$. If x_0 is a critical point for (2), then λx_0 is also a critical point for (2), for every $\lambda \in K$. $0 \in E$ is always a critical point for (2); it is an isolated critical point if and only if $E([.,.,.])$ has no nilpotent element.

2) If $e \in E$ is an idempotent element for $E([.,.,.])$, then it has associative powers and

$$X(t) = (1 - (n-1)(t-t_0))^{-\frac{1}{n-1}} \cdot e \tag{14}$$

is the unique solution of the Cauchy problem (10) with $x_0 = e$. The idempotent elements identify unbounded solutions of (2).

Let us consider the Cauchy problem (10) with $x_0 = P$, where $P^n = aP$, $a \in \mathbb{R}$. Then P has associative powers and

$$X(t) = a(1 - (n-1)(t-t_0))^{-\frac{1}{n-1}} \cdot P \tag{15}$$

is the solution of (2)+($X(t_0) = P$).

Following step by step the proof of Proposition 3.4 [2] and using (15) one gets the result:

Proposition 1. *If $E([.,\ldots,.])$ has an idempotent, then the origin $0 \in E$ is an unstable critical point for (2). Consequently, if the origin is stable for (2) then $E([.,\ldots,.])$ has no idempotent and, in particular, no identity element.*

As an immediate consequence of the fact that $N_0 \neq 0$ is a nilpotent element ($N_0^n = 0$) implies $N = \lambda N_0$ is also a nilpotent (for every $\lambda \in K$), we can readily prove the following proposition.

Proposition 2. *If $E([.,\ldots,.])$ has a (nonzero) nilpotent, then the origin $0 \in E$ is not asymptotically stable steady state point for (2).*

Following the idea of proving Theorem 1 [1] it can be proved

Proposition 3. *If $E([.,\ldots,.])$ is a real commutative finite dimensional n-ary algebra, with n an even number, it has a (nonzero) idempotent or a (nonzero) nilpotent.*

Proof. Suppose $E = \mathbb{R}^m$. Let us consider the mapping $F : \mathbb{R}^m \to \mathbb{R}^m$ defined by $F(x) = x^n$. If $E([.,...,.])$ has an idempotent e, then it is a fixed point for F which still keeps invariant the axe $\{\lambda e \mid \lambda \in \mathbb{R}\}$. Moreover, if $F(x_0) = cx_0$ and c is positive, then $e = \dfrac{1}{\sqrt[n-1]{c}} \cdot x_0$ is an idempotent (in case when c is negative and n is even number, then $e = -\dfrac{1}{\sqrt[n-1]{-c}} \cdot x_0$ is also an idempotent), i.e. F has necessarily a fixed point. Supposing that F has no fixed points and n is an even number, it results

$$(1-\lambda)F(x) \neq \lambda x, \ \ \forall x \neq 0, \ \forall \lambda \in \mathbb{R}.$$

If in addition $E([.,...,.])$ has no nilpotent, then $F(x) \neq 0$ for all $x \neq 0$, and it induces a function $g : S^{m-1} \to S^{m-1}$ defined by $g(x) = \dfrac{F(x)}{\|F(x)\|}$ for all $x \in S^{m-1}$. Let us define also the uniparametric family of functions $G(\cdot,\lambda) : S^{m-1} \to S^{m-1}$ by

$$G(x,\lambda) = \frac{(1-h)F(x) + \lambda(-x)}{\|(1-h)F(x) + \lambda(-x)\|}, \ 0 \leq \lambda \leq 1, \ \forall x \in S^{m-1}.$$

G is an homotopic mapping on S^{m-1} of g with the antipodal mapping $\mathbf{a} : S^{m-1} \to S^{m-1}$ (defined by $\mathbf{a}(x) = -x$). Consequently, these mappings must have the same degree, what is not possible because the degree of g is an even number, while the degree of $\mathbf{a}$ is 1.

Corollary 2. *The origin $0 \in E$ is not asymptotically stable for (2) if n is an even number.*

Theorem 4. *Let (2), given on a finite dimensional real vector space E, be such that its associated algebra $E([.,\ldots,.])$ has a symmetric positive definite bilinear form $H : E \times E \to \mathbb{R}$. If H satisfies*

$$H(X, X^n) = 0, \quad \forall X \in E,$$

(or $H(X, X^n) \leq 0$) then the origin $0 \in E$ is a stable point.

Proof. The function $V : E \to \mathbb{R}$ defined by $V(x) = H(x,x)$ is a LIAPUNOV function. Indeed, V is a positive definite quadratic form and its derivative $\dot{V}(X(t))$ vanishes identically along any trajectory $X(t)$ of (2).

Consequently, the existence of H implies the nonexistence of an idempotent.

Similar arguments as for Theorem 3.10 [2] allow us to prove the following result.

Theorem 5. *Let (2) be a nHDE and $E([\cdot,...,\cdot])$ be its associated n-ary algebra.*

(1) *The trajectory through $P \in E$ does not pass through aP for any $a \leq 0$. If P lies on a periodic trajectory, the trajectory through P does not pass through aP for any $a \neq 1$.*

(2) *If $\gamma \subset E$ is a periodic orbit with the least period τ, then $a\gamma = \{aP | P \in \gamma\}$ is a periodic trajectory with the least period $\tau/|a|$ for $a \neq 0$. Thus, scalar multiples of periodic orbits are periodic, and solutions of any period exist, provided that one periodic orbit exists.*

(3) *The periodic trajectories lie on cones.*

Theorem 6. *Let (2) be a nHDE and $E([\cdot,...,\cdot])$ be its associated n-ary algebra. Then no periodic orbit is an attractor.*

Proof. If $\gamma(t)$, $\gamma(t_0) = P$ is a periodic solution and $\mathcal{U}$ is an open neighborhood of it, then there exists $a \in \mathbb{R}$ such that $aP \in \mathcal{U}$. Then $a\gamma(t)$ is also a periodic solution contained in $\mathcal{U}$. Consequently, $\underset{n\to\infty}{lim} \|a\gamma - \gamma\| \neq 0$ and γ is not an attractor.

3) Let E_0 be a closed ideal of $E([\cdot,...,\cdot])$ and E_1 – a closed vector subspace which is its complement in E, i.e. $E = E_0 \oplus E_1$. We denote by $p_i : E \to E_i$ $(i = 0,1)$ the two projectors associated with the direct sum decomposition of E, $X_i = p_i(X)$, $i = 0,1$, then (2) becomes

$$\begin{cases} \dfrac{dX_0}{dt} = F(X_0) + \sum\limits_{i=1}^{n-1} \begin{pmatrix} n \\ i \end{pmatrix} G(\underbrace{X_0,...,X_0}_{i\ times}, \underbrace{X_1,...,X_1}_{n-i\ times}) + (p_0 \circ F)(X_1) \\ \dfrac{dX_1}{dt} = (p_1 \circ F)(X_1). \end{cases}$$

In the particular case when E_1 is also a closed ideal for $E([.,.,.])$, then

$$\begin{cases} \dfrac{dX_0}{dt} = F(X_0) \\ \dfrac{dX_1}{dt} = F(X_1). \end{cases}$$

Further, if E_1 is only a subalgebra of $E([\cdot,...,\cdot])$, then

$$\begin{cases} \dfrac{dX_0}{dt} = F(X_0) + \sum\limits_{i=1}^{n-1} \begin{pmatrix} n \\ i \end{pmatrix} G(\underbrace{X_0,...,X_0}_{i\ times}, \underbrace{X_1,...,X_1}_{n-i\ times}) \\ \dfrac{dX_1}{dt} = (p_1 \circ F)(X_1). \end{cases}$$

4) If only a finite number of the powers of x_0 are linearly independent, then the subalgebra $K(x_0)$ spanned by all these powers will be a finite-dimensional one. Let $(x_0, x_1, ..., x_s)$ be the basis of $K(x_0)$ consisting of the first smallest independent powers of x_0. In this case, the solution of (2) gets the form

$$X(t) = f_0(t)x_0 + f_1(t)x_1 + ... + f_s(t)x_s$$

where the mappings $f_i(t)$, $i = 0, 1, ..., s$, satisfy an n-homogeneous differential system of the form

$$\frac{df_i}{dt} = \sum_{j_1, j_2, ..., j_n = 0}^{s} C_{ij_1 j_2 ... j_n} f_{j_1} f_{j_2} ... f_{j_n}, \quad i, j_1, j_2, ..., j_n = 0, 1, ..., s$$

with the initial conditions $f_0(t_0) = 1, f_1(t_0) = 0, ..., f_s(t_0) = 0$; here, $C_{ij_1 j_2 ... j_n}$ are the structure constants of the subalgebra $K(x_0)$ in basis $(x_0, x_1, ..., x_s)$ defined by

$$[x_{j_1}, x_{j_2}, ..., x_{j_n}] = \sum_{i=0}^{s} C_{ij_1 j_2 ... j_n} x_i, \quad i, j_1, j_2, ..., j_n = 0, 1, ..., s.$$

This situation is usually met in the case of n-homogeneous differential systems on finite-dimensional spaces. It warns us of the necessity to pay a special attention to the algebras with a single generator.

8 nHDEs on real finite dimensional spaces

If $E = \mathbb{R}^p$, $\mathcal{B} = (e_1, e_2, ..., e_p) \subset \mathbb{R}^p$ is its natural basis and $X = X^i e_i$, then (2) becomes

$$\frac{dX^i}{dt} = C^i_{j_1 j_2 ... j_n} X^{j_1} X^{j_2} ... X^{j_n}, \quad i, j_1, j_2, ..., j_n = 1, ..., p \tag{16}$$

where the EINSTEIN's convention on summation is used.

The associated n-ary algebra $E([\cdot, ..., \cdot])$ has $C^i_{j_1 j_2 ... j_n}$ as its structure constants in basis $\mathcal{B}$. It is suitable to denote the left multiplication $L_{\underbrace{x_0, ..., x_0}_{n-1\, times}}$ by G_{x_0}, for any $x_0 \in E$ (i.e., $G_{x_0} = L_{\underbrace{x_0, ..., x_0}_{n-1\, times}}$). G_{x_0} is an endomorphism of E having, in basis $\mathcal{B}$, the matrix

$$[G_{x_0}]_{\mathcal{B}} = [C^i_{j_1 j_2 ... j_{n-1} j} x_0^{j_1} x_0^{j_2} ... x_0^{j_{n-1}}].$$

The solution of the CAUCHY problem (10) is an analytical function; more exactly, there exists an analytical function f such that its solution has the form:

$$X(t) = f((t - t_0) \cdot G_{x_0})(x_0).$$

For example, recall that in the case when $E([\cdot, ..., \cdot])$ is a power-associative algebra we have

$$f(\lambda) = (1 - (n-1)\lambda)^{-\frac{1}{n-1}}$$

and the solution of the CAUCHY problem for (10) is

$$[X(t)]_{\mathcal{B}} = f((t-t_0)[G_{x_0}]_{\mathcal{B}})[x_0]_{\mathcal{B}}. \tag{17}$$

For every analytical function h we can obtain $h(G_{x_0})$ using the JORDAN (e.g., upper) form for G_{x_0}. Indeed, there exists a basis $\mathcal{B}_J$ in E such that the matrix of G_{x_0} is

$$[G_{x_0}]_J = diag\ \ (J_1, J_2, ..., J_s, R_1, R_2, ..., R_q),$$

where J_i are superior Jordan cells corresponding to the real eigenvalues of G_{x_0}, while R_i are the superior Jordan blocs corresponding to the complex eigenvalues of G_{x_0} (see, [3]). If S denotes the transformation matrix from $\mathcal{B}$ to $\mathcal{B}_J$ one gets

$$h([G_{x_0}]_{\mathcal{B}}) = S \cdot h([G_{x_0}]_J) \cdot S^{-1},$$

where $h([G_{x_0}]_J) = diag\ \ (h(J_1), h(J_2), ..., h(J_s), h(R_1), h(R_2), ..., h(R_q))$.

9 Example

Let us consider, in $\mathbb{R}^4$ with the natural basis (e_1, e_2, e_3, e_4), the cubic homogeneous differential system

$$\begin{cases} \dfrac{dx^1}{dt} = (x^1)^3 - 3x^1(x^2)^2 \\ \dfrac{dx^2}{dt} = 3(x^1)^2x^2 - (x^2)^3 \\ \dfrac{dx^3}{dt} = 3\left[(x^1)^2x^3 - (x^2)^2x^3 - 2x^1x^2x^4\right] \\ \dfrac{dx^4}{dt} = 3\left[(x^1)^2x^4 - (x^2)^2x^4 + 2x^1x^2x^3\right]. \end{cases}$$

The left multiplication $L_{X,X}$ has the matrix

$$L_{X,X} = \begin{bmatrix} (x^1)^2-(x^2)^2 & -2x^1x^2 & 0 & 0 \\ 2x^1x^2 & (x^1)^2-(x^2)^2 & 0 & 0 \\ 2(x^1x^3-x^2x^4) & -2(x^1x^4+x^2x^3) & (x^1)^2-(x^2)^2 & -2x^1x^2 \\ 2(x^1x^4+x^2x^3) & 2(x^1x^3-x^2x^4) & 2x^1x^2 & (x^1)^2-(x^2)^2 \end{bmatrix}.$$

The ternary algebra associated with this system is associative. If $x_0 = ae_1 + be_2 + ce_3 + de_4$, then L_{x_0,x_0} has the eigenvalues $\lambda_1, \lambda_2 = a^2 - b^2 \pm 2iab = (a \pm ib)^2$ with multiplicity 2. Thus, the solution $X(t)$ with $X(t_0) = x_0$ is

$$X(t) = (f(\alpha+i\beta)A_1 + f(\alpha-i\beta)A_2 + f'(\alpha+i\beta)A_3 + f'(\alpha-i\beta)A_4)(x_0),$$

where $\lambda_1 = \alpha + i\beta$, $f(\lambda) = (1-2(t-t_0)\lambda)^{-\frac{1}{2}}$ and A_1, A_2, A_3, A_4 can be defined by the identities

$$\begin{cases} A_1 + A_2 = I_4, \\ 2i\beta(A_3 - A_4) = A^2 - 2\alpha A + (\alpha^2 + \beta^2)I_4, \\ i\beta(A_1 - A_2) + (A_3 + A_4) = A - \alpha I_2, \\ i\beta(3\alpha^2 - \beta^2)(A_1 - A_2) + 3(\alpha^2 - \beta^2)(A_3 + A_4) + 6i\alpha\beta(A_3 - A_4) = \\ \qquad = A^3 - \alpha(\alpha^2 - 3\beta^2)I_4, \end{cases}$$

where A denotes the matrix of the left multiplication L_{x_0,x_0}.

The solution of the Cauchy problem with $X(t_0) = x_0$ is

$$\begin{cases} x^1 = \Re e \dfrac{\rho}{\sqrt{cos\ 2\alpha - isin\ 2\alpha - 2\rho^2(t - t_0)}}, \\ x^2 = \Im m \dfrac{\rho}{\sqrt{cos\ 2\alpha - isin\ 2\alpha - 2\rho^2(t - t_0)}}, \\ x^3 = \Re e \dfrac{\kappa\rho}{\sqrt{cos\ 2\alpha - isin\ 2\alpha - 2\rho^2(t - t_0)}}, \\ x^4 = \Im m \dfrac{\kappa\rho}{\sqrt{cos\ 2\alpha - isin\ 2\alpha - 2\rho^2(t - t_0)}}, \end{cases}$$

where $a + ib = \rho(cos\ \alpha + i\ sin\ \alpha)$, $\kappa = \dfrac{c + id}{a + ib}$.

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Attractors in affine differential systems with impulsive control *

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Abstract. In this paper we prove that an asymptotic equilibrium of an affine system of differential equations can become a strange attractor under affine impulsive control. The linear oscillator is studied as example.

Mathematics subject classification: 34A37, 37C70.
Keywords and phrases: Attractor, impulsive affine systems, linear oscillator.

1 Introduction

We are concerned with the system of affine differential equations

$$\dot{x} = Ax + b \quad (x \in \mathbb{R}^m), \tag{1}$$

where A is a nonsingular matrix.

Suppose that at the moments $t = n \in \mathbb{N}^*$ instantaneous control actions occur, which change the state of system as follows:

$$\Delta x\big|_{t=n} := x(n+0) - x(n-0) = C_{i_n} x(n-0) + d_{i_n} \quad (n \in \mathbb{N}^*), \tag{2}$$

where the matrices C_{i_n} and vectors d_{i_n} belong to given sets (finite or infinite).

Between any two consecutive kicks the motion of the system obeys (1). At the moment $t = n$ the elements C_{i_n} and d_{i_n}, which determine the jump by (2), are chosen, say randomly. For convenience, we will consider that all solutions of system (1)–(2) are right continuous at the moments $t \in \mathbb{N}$.

Let $\mathcal{F}$ be the set of all affine maps $\{F_{i_n} : x \mapsto C_{i_n} x + d_{i_n}\}_{n \in \mathbb{N}^*}$. For simplicity, we assume that the set $\mathcal{F}$ is finite and contains only r distinct elements, say $\mathcal{F} = \{F_1, F_2, \ldots, F_r\}$. Denote $\tilde{F_n} = E + F_n$ $(1 \le n \le r)$, where E is the identity operator.

Assume that the spectra of the operators $(E + C_n)e^A$ $(1 \le n \le r)$ are located strictly inside of the unit circle. Sometimes, if necessary, it is required that all operators $E + C_n$ $(1 \le n \le r)$ are invertible.

It is known (see, e.g., [1]) that if the sequence $\{F_{i_n}\}_{n \in \mathbb{N}^*}$ is periodic, then the behavior of the system is quite simple: there exists a globally attracting periodic cycle, corresponding to a periodic motion. This situation may, however, be changed essentially in the general case.

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In what follows we show that there exists a global attractor in the extended phase space and for "typical" sequence $\{F_{i_n}\}_{n\in\mathbb{N}^*}$ this invariant set has a non-integer Hausdorff dimension (see, e.g., [2]) and represents a fractal. Moreover, the motions on the attractor are chaotic by Li-Yorke (in the meaning that every trajectory on the attractor is dense) and, as a consequence, every solution of the impulsive system (1)–(2) tends to a chaotic one.

This paper represents an application of general results from [3, 4].

2 Invariant sets

If one denotes by $x(n)$ the state of the system immediately after the n-th kick, then by a straightforward calculation we end up with a sequence of affine maps $\Phi_{i_n} : x(n) \mapsto x(n+1)$, $n \in \mathbb{N}^*$. More precisely,

$$\Phi_{i_n} : x \mapsto (E + C_{i_{n+1}})e^{A}x + (E + C_{i_{n+1}})(e^{A} - E)A^{-1}b + d_{i_{n+1}}.$$

We call the sequence $\{\Phi_{i_n}\}_{n\in\mathbb{N}}$ the *Poincaré system* associated with the impulsive system (1)–(2). Under the above assumptions, this Poincaré system is generated by r affine maps $\{\Phi_1, \Phi_2, \dots, \Phi_r\}$.

Let $\mathcal{P}_{cp}(\mathbb{R}^m)$ be the set of all nonempty compact subsets of $\mathbb{R}^m$, endowed with the Hausdorff-Pompeiu metrics (see, e.g.,[2]). Denote by Φ the *Nadler-Hutchinson* operator [6] on the space $\mathcal{P}_{cp}(\mathbb{R}^m)$, defined by $\Phi(M) := \bigcup\limits_{n=1}^{r} \Phi_n[M]$, $M \in \mathcal{P}_{cp}(\mathbb{R}^m)$.

Let $\psi(\cdot, \tau, x_0)$ stand for the solution of the system (1)–(2) with the initial condition $x(\tau) = x_0$. Since the system (1)–(2) is affine, there is a unique such solution, defined on $\mathbb{R}$ (see, e.g., [1]).

Lemma 1. *For every $x \in \mathbb{R}^m$ and $t_1, t_2, t_3 \in \mathbb{R}$ one has*

$$\psi(t_1, t_2, \psi(t_2, t_3, x)) = \psi(t_1, t_3, x). \tag{3}$$

Proof. This follows immediately from the uniqueness of the solution of respective Cauchy problem. □

Lemma 2. *Every solution of the system (1)–(2) can be written as follows:*

$$\psi(t, \tau, x) = e^{(t-\tau)A}x, \;\; \textit{if } [t] = [\tau], \;\; \textit{or}$$

$$\psi(t, \tau, x) = e^{(t-[t])A}\tilde{F}_{i_{[t]}}e^{([t]-\tau)A}x, \;\; \textit{if } [t] = [\tau] + 1, \;\; \textit{or}$$

$$\psi(t, \tau, x) = e^{(t-[t])A}\left(\prod_{j=[\tau]+2}^{[t]} \Phi_{i_j}\right)\tilde{F}_{i_{[\tau]+1}}e^{([\tau]+1-\tau)A}x, \;\; \textit{if } [t] > [\tau] + 1, \;\; \textit{or}$$

$$\psi(t, \tau, x) = e^{(t-[\tau])A}\tilde{F}^{-1}_{i_{[\tau]}}e^{([\tau]-\tau)A}x, \;\; \textit{if } [t] = [\tau] - 1, \;\; \textit{or}$$

$$\psi(t, \tau, x) = e^{(t-[t]-1)A}\tilde{F}^{-1}_{i_{[t]+1}}\left(\prod_{j=[\tau]}^{[t]+2} \Phi^{-1}_{i_j}\right)e^{([\tau]-\tau)A}x, \;\; \textit{if } [t] < [\tau] - 1,$$

where $[\cdot]$ denotes the integral part.

Proof. The proof is straightforward. □

Remark 1. In fact, the impulsive system (1)–(2) is nonautonomous and we can consider only its integral curves. Even in the case of periodic sequence $\{F_{i_n}\}_{n\in\mathbb{N}^*}$ we cannot factorize the system to obtain a (autonomous) system on the direct product $S^1 \times \mathbb{R}^m$. However, since the impulse actions occur at the moments $t \in \mathbb{N}$, we can obtain a foliation on the cylinder $S^1 \times \mathbb{R}^m$ by factorization on time. This foliation consists of pieces of integral curves of the system (1)–(2).

Project the system (1)-(2) to the cylinder $S^1 \times \mathbb{R}^m$, using the projection $\pi : (t, y) \mapsto (t \pmod 1, y)$.

We will say that a set $V \subset S^1 \times \mathbb{R}^m$ is *positive invariant* (*invariant*) with respect to the system (1)-(2), if for every point $(\tau, x) \in V$ and every natural k one has

$$(t \pmod 1, \psi(t, \tau + k, x)) \in V \quad \text{for} \quad t \geq \tau + k \quad (\text{for} \quad t \in \mathbb{R}). \tag{4}$$

In other words, V consists of pieces of integral curves.

By definition, such a set $V \subset S^1 \times \mathbb{R}^m$ covers the whole base S^1 by projection. Denote by $(t, V_t) := \{(t, x) \in S^1 \times \mathbb{R}^m \mid (t, x) \in V\}$ the *fiber* over the point $t \in [0, 1)$. For convenience, we will identify this fiber with V_t. Moreover, in the sequel the notation V_t for $t \in \mathbb{R}$ will mean $V_{t \pmod 1}$.

Theorem 3. *The set $V \subset S^1 \times \mathbb{R}^m$ is positive invariant (invariant) with respect to the system (1)-(2) if and only if it satisfies the following conditions:*

1. $e^{(t-\tau)A} V_\tau \subset V_t$ *for* $0 \leq \tau \leq t < 1$ $(e^{(t-\tau)A} V_\tau = V_t$ *for* $\tau, t \in [0, 1))$*;*
2. $\bigcup_{n=1}^{r} \tilde{F}_n e^{(1-\tau)A} V_\tau \subset V_0$ $(\bigcup_{n=1}^{r} \tilde{F}_n e^{(1-\tau)A} V_\tau = V_0)$ *for* $0 \leq \tau < 1$.

Proof. Necessity. Assume that the set $V \subset S^1 \times \mathbb{R}^m$ is positive invariant (invariant). If $x \in V_\tau$, then $(\tau, x) \in V$, and by Lemma 2 one has for $0 \leq \tau \leq t < 1$

$$e^{(t-\tau)A} x = \psi(t, \tau, x) \in V_t. \tag{5}$$

At the same time, for every natural n there is a natural k_n such that

$$\tilde{F}_n e^{(1-\tau)A} x = \psi([\tau] + k_n + 1, \tau + k_n, x) \in V_{[t]+k_n+1} = V_0.$$

In the case of invariance the relation (5) holds for $\tau, t \in [0, 1)$. Moreover, for every $z \in V_t$ there is $y = \psi(\tau, t, z) \in V_\tau$ which verifies $e^{(t-\tau)A} y = z$.

Analogously, for every $0 \leq \tau < 1$, $z \in V_0$ and $1 \leq n \leq r$ there is a natural k_n such that $\tilde{F}_{k_n} = \tilde{F}_n$ and there is $y = \psi(\tau + k_n - 1, k_n, z) \in V_{\tau+k_n-1} = V_\tau$, verifying $\tilde{F}_n e^{(1-\tau)A} y = \psi(k_n, \tau + k_n - 1, y) = z \in V_0$.

Sufficiency. Assume that conditions 1)–2) hold. Let $(\tau, x) \in V$. To proof (5) take firstly $t \geq \tau + k$, $k \in \mathbb{N}^*$.

If $[t] = [\tau] + k$ or $[t] = [\tau] + k + 1$, then (4) is a consequence of the conditions 1)–2) and Lemma 2.

If $[t]=q, [\tau]+k=p$, $q>p+1$, then by Lemmas 1 and 2 the conditions 1)–2) imply for every $x \in V_\tau$:

$$\psi(t,\tau+k,x) = e^{(t-q)A} \prod_{j=p+2}^{q} \Phi_{i_j} \tilde{F}_{i_{p+1}} e^{(p+1-\tau-k)A} x \in V_t,$$

$$\psi(t,\tau+k,x) = \psi(t,q,\psi(q,p+1,\psi(p+1,\tau+k,x))) \in V_t.$$

In the case of invariance we consider in addition $t<[\tau]+k$. In this case there is $y \in V_0$ such that $x \in V_\tau = V_{\tau+k}$ may be represented as $x=\psi(\tau+k,[\tau]+k,y)$. In turn, there is $z \in V_{[t]} = V_0$ such that $y = \psi([\tau]+k,[t],z)$. By Lemma 1, $\psi(t,\tau+k,x) = \psi(t,[\tau]+k,y) = \psi(t,[t],z) \in V_t$. This completes the proof. □

Corollary 4. *If $V \subset S^1 \times \mathbb{R}^m$ is positive invariant (invariant), then V_0 is positive invariant (invariant) with respect to the Nadler-Hutchinson operator Φ, i.e. $\Phi(V_0) = \bigcup\limits_{n=1}^{r} \Phi_n[V_0] \subset V_0 \quad (\Phi(V_0)=V_0)$.*

3 IFS and attractors

Since the eigenvalues of all matrices $(E+C_n)e^A$ $(1 \le n \le r)$ are located strictly inside the unit circle, all operators Φ_n $(1 \le n \le r)$ are contracting.

We associate to the system (1)-(2) a hyperbolic *Iterated Function System* (IFS) $\{\mathbb{R}^m; \Phi_1, \Phi_2, \dots, \Phi_r\}$ (see, e.g., [2]), consisting of affine contractions. This IFS determines in $\mathbb{R}^m$ a global compact attractor K, which is the unique fixed point of the corresponding contractive Nadler-Hutchinson operator Φ.

Given the natural k we say that the sequence $\{F_{i_n}\}_{n\in\mathbb{N}^*}$ is *k-universal* if it contains every word of the length k from the alphabet of $\mathcal{F} = \{F_1, F_2, \dots, F_r\}$. We say that the sequence $\{F_{i_n}\}_{n\in\mathbb{N}^*}$ is *universal* if it is *k-universal* for every natural k.

By Lemma Borel-Cantelli [5], if the sequence $\{F_{i_n}\}_{n\in\mathbb{N}^*}$ is chosen randomly with a uniform distribution, then with probability 1 it is universal.

If the sequence $\{F_{i_n}\}_{n\in\mathbb{N}}$ is universal, then the orbit of each point in K is dense on K (is chaotic by Li-Yorke).

Recall some notions (see, e.g., [2]). A set is called *totally disconnected* if for every its point the connected component, containing this point, is the point itself. A set S is called *perfect* if it is closed and every point $p \in S$ is the limit of points $q_n \in S \setminus \{p\}$. A set is called a *Cantor set* if it is totally disconnected, perfect and compact.

Theorem 5. *If the spectra of the operators $(E+C_n)e^A$ $(1 \le n \le r)$ are located strictly inside the disk of radius $\frac{1}{r}$, then the attractor K is totally disconnected.*

Proof. It is known [7] that if an IFS consists of r contractions, each of them with the contraction coefficient s, and $rs<1$, then the attractor K of this IFS is totally disconnected. It is sufficient to say that in our case $s = \max\limits_{1\le n\le r} \|(E+C_n)e^A\| < \frac{1}{r}$. □

Remark 2. The hypotheses of Theorem 5 are far from being also necessary conditions for the attractor K to be totally disconnected.

Denote the distance from the point $x \in \mathbb{R}^m$ to the compact $M \subset \mathbb{R}^m$ by $\varrho(x, M) := \min\{d(x, y) \,|\, y \in M\}$.

A bounded subset $V \subset S^1 \times \mathbb{R}^m$ is called an *attractor* of the system (1)-(2) if it is positive invariant and for every solution $\psi(\cdot, \tau, x)$ one has $\varrho(\psi(t, \tau, x), V_t) \to 0$ as $t \to +\infty$.

Theorem 6. *There exists an attractor of the system (1)-(2)*

$$K^* = \big\{(t, e^{tA}x + (e^A - E)A^{-1}b) \,|\, t \in [0, 1), x \in K\big\} \subset S^1 \times \mathbb{R}^m$$

with the Hausdorff dimension $DH(K^*)$, verifying the inequalities:

$$1 < DH(K^*) \le 1 - \frac{\ln r}{\ln s}, \tag{6}$$

where s is the smallest radius of a disc centered at the origin of coordinates, which contains the spectra of the operators $(E + C_n)e^A$ $(1 \le n \le r)$.

Proof. By Theorem 3, the set K^* is positive invariant. Since K is compact, every fiber $K_t = e^{tA}K + (e^A - E)A^{-1}b$ $(0 \le t < 1)$ is compact as well. The compact K attracts every compact M in the fiber $0 \times \mathbb{R}^m$ under the actions of the Nadler-Hutchinson operator Φ. As a result every fiber $K_t = e^{\nu A}K + (e^A - E)A^{-1}b$, where $\nu = t \pmod 1$, attracts the image $e^{(t-[t])A}\Phi^n e^{([t]+1-t)A}M$ as $n \to +\infty$ uniformly on $t \in \mathbb{R}$.

It is known (see, e.g.,[2]) that the Hausdorff dimension of K verifies the inequality $DH(K) \le -\frac{\ln r}{\ln s}$. This implies that $DH(K^*) = 1 + DH(K) \le 1 - \frac{\ln r}{\ln s}$. □

Theorem 7. *Let the spectra of the operators $(E + C_n)e^A$ $(1 \le n \le r)$ be located strictly inside the disk of radius $\frac{1}{r}$ and let the sequence $\{F_{i_n}\}_{n\in\mathbb{N}}$ be universal. Then the attractor K^* is homeomorphic to the direct product of a half-open interval and a Cantor set, its Hausdorff dimension verifying the inequalities: $1 < DH(K^*) < 2$.*

Proof. It is easily seen from Theorem 6 that the attractor K^* is homeomorphic to the direct product $[0, 1) \times K$. Under the given hypothesis, the compact K is perfect and by Theorem 5 is totally disconnected, and, as a consequence, it is a Cantor set. In this case the inequalities (6) become: $1 < DH(K^*) < 2$. □

Lemma 8. *There exist $L > 0, \gamma > 0$ such that for any $y \in \mathbb{R}^m$ there exists $x \in K$, satisfying*

$$\|\psi(t, 0, y) - \psi(t, 0, x)\| \le Le^{-\gamma t}\|y - x\| \quad (t \ge 0).$$

Proof. This follows immediately from Lemma 2 and Theorem 6. □

Theorem 9. *If the sequence $\{F_{i_n}\}_{n\in\mathbb{N}}$ is universal, then every integral curve of the system (1)–(2), starting in K^*, is chaotic by Li-Yorke, i.e. is dense in K^*.*

Proof. If the sequence $\{F_{i_n}\}_{n\in\mathbb{N}}$ is universal, then for every solution $\psi(\cdot,\tau,x)$ with $(\tau,x)\in K^*$, the sequence $\{\psi(j,\tau,x)\}_{j\geq[\tau]+1}$ is the orbit of the point $\psi([\tau]+1,\tau,x)\in K$ under the IFS $\{\mathbb{R}^m;\Phi_1,\ldots,\Phi_r\}$. Since this orbit is dense on K, the integral curve of the solution $\psi(\cdot,\tau,x)$ is dense on K^*. $\square$

Corollary 10. *If the sequence $\{F_{i_n}\}_{n\in\mathbb{N}}$ is universal, then every integral curve of the system (1)-(2) is chaotic by Li-Yorke or tends to a chaotic one.*

Proof. This follows from Lemma 8 and Theorem 9. $\square$

Remark 3. If the impulses occur only in some integer moments, i.e.

$$\Delta x\big|_{t=\tau_n} := x(\tau_n+0)-x(\tau_n-0)=C_{i_n}x(\tau_n-0)+d_{i_n}\quad(\tau_n\in\mathbb{N}^*), \tag{7}$$

then the system (1),(7) may be considered as a particular case of the system (1)-(2) by supplementing the set $\mathcal{F}$ with the null operator $F=0$ for other integer moments.

Remark 4. Analogously, if the spectra of all operators $(E+C_n)e^A$ $(1\leq n\leq r)$ are located strictly outside the unit circle, we can say about the repeller of the system (1)–(2).

Remark 5. Many classical fractals may be represented as attractors of affine IFS on $\mathbb{R}^2$. Fig. 1 shows some of them as attractors K of impulsive differential equations on $\mathbb{C}$, for example:

- the *Sierpinski triangle* in $\dot z=-z\cdot\ln 2, \Delta z|_{t=n}=i\exp\frac{2\pi ni}{3}$ $(n\in\mathbb{N})$;
- the *pentagasket* in $\dot z=-z\cdot\ln(\frac{3+\sqrt5}{2}), \Delta z|_{t=n}=i\exp\frac{2\pi ni}{5}$ $(n\in\mathbb{N})$;
- the *hexagasket* in $\dot z=-z\cdot\ln 3, \Delta z|_{t=n}=\exp\frac{\pi ni}{3}$ $(n\in\mathbb{N})$.

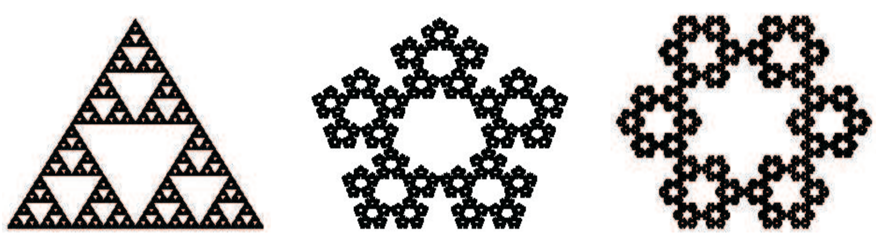

Figure 1. Fractals: the Sierpinski triangle, pentagasket and hexagasket as attractors K of impulsive affine systems

4 Linear oscillator

Let us consider, as an example, the linear oscillator with impulsive actions

$$\ddot{x} + c\dot{x} + kx = 0 \quad (c > 0, k > 0), \tag{8}$$

$$\Delta\dot{x}\big|_{t=n} := \dot{x}(n+0) - \dot{x}(n-0) = \xi_{i_n} \quad (n \geq 1). \tag{9}$$

Assume that the range of the sequence $\{\xi_{i_n}\}_{n\geq 1}$ contains only r distinct elements.

We can reduce the equations (8)–(9) to an affine system of impulsive differential equations (1)–(2) in the phase space $x_1 = x, x_2 = \dot{x}$. In this case we obtain some analogues of the previous theorems.

Theorem 11. *[8] There exists an attractor $K^* \subset S^1 \times \mathbb{R}^2_{(x,\dot{x})}$ of the system (8)-(9) with the Hausdorff dimension*

$$DH(K^*) = 1 + \frac{2\sqrt{2}\ln r}{\sqrt{2}c - \sqrt{c^2 - 4k + |c^2 - 4k|}}.$$

Theorem 12. *[8] If*

$$2\ln r < c < \frac{k}{\ln r} + \ln r$$

and the sequence $\{\xi_{i_n}\}_{n\in\mathbb{N}}$ is universal, then the attractor K^* is homeomorphic to the direct product of a half-open interval and a Cantor set, its Hausdorff dimension verifying the inequalities: $1 < DH(K^*) < 2$.

Fig. 2 represents the respective attractors $K \subset \mathbb{R}^2_{(x,\dot{x})}$ for distinct values of parameters for two impulsive differential equations (8)–(9): on the left for $c = 5/2$, $k = 2$ and $r = 3$ (K is totally disconnected), on the right for $c = 1$, $k = 5/4$ and $r = 3$ (K is connected).

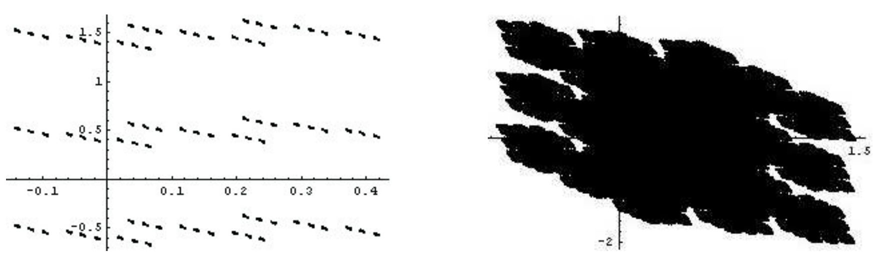

Figure 2. Attractor K for: $c = 5/2$, $k = 2$, $r = 3$ (left) and $c = 1$, $k = 5/4$, $r = 3$ (right)

Remark 6. If the range of sequence $\{\xi_{i_n}\}_{n\geq 1}$ is infinite but bounded, then the system (8)–(9) admits an attractor as well.

Fig. 3 represents the attractor K (of an infinite IFS) for an impulsive differential equation (8)–(9) with $c = 2$, $k = 2$, where the values $\{\xi_{i_n}\}_{n\geq 1}$ are randomly chosen from $[0, 1]$.

All calculations and graphic objects have been done using the Computer Algebra System *Mathematica*.

Figure 3. Attractor K for $c = 2$, $k = 2$, $\xi \in [0, 1]$

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On theory of surfaces defined by the first order systems of equations

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Abstract. The properties of surfaces defined by spatial systems of differential equations are studied. The Monge equations connected with the first order nonlinear p.d.e. are investigated. The properties of Riemannian metrics defined by the systems of differential equations having applications in theory of nonlinear dynamical systems with regular and chaotic behaviour are considered.

Mathematics subject classification: 34C14, 53B21.
Keywords and phrases: Differential systems, Monge equations, Riemann spaces, translation surfaces.

1 Introduction

An investigation of the properties of spatial systems of differential equations having three degrees of freedom represented by the dynamical variables x, y and z

$$\frac{dx}{ds} = P(x, y, z), \quad \frac{dy}{ds} = Q(x, y, z), \quad \frac{dz}{ds} = R(x, y, z) \tag{1}$$

is an important task of modern mathematics.

The Lorenz

$$\frac{dx}{ds} = \sigma(y - x), \quad \frac{dy}{ds} = rx - y - xz, \quad \frac{dz}{ds} = xy - bz, \tag{2}$$

and the Rössler

$$\frac{dx}{ds} = -(y + z), \quad \frac{dy}{ds} = x + ay, \quad \frac{dz}{ds} = b + xz - cz \tag{3}$$

are the most famous examples of the systems of equations having regular and chaotic behavior of trajectories at some values of parameters.

To study the properties of the systems (1) we propose geometrical approach founded on consideration of the surfaces of the form $z = z(x, y)$, $x = x(y, z)$ or $y = y(x, z)$ in R^3-space which are connected naturally with such type of systems.

In result we get from the system (1) a set of nonlinear of the first order partial differential equation for every pair of variables.

As example, in the case $z = z(x, y)$ we can write the equation

$$\frac{\partial z(x,y)}{\partial x} P(x, y, z(x, y)) + \frac{\partial z(x,y)}{\partial x} Q(x, y, z(x, y)) - R(x, y, z(x, y)) = 0. \tag{4}$$

Its solutions the surfaces in the R^3-space are locally presented.

A studies of surfaces defined by spatial systems of equations like (1) can be useful for understanding of their properties.

2 The examples of surfaces corresponding to the Lorenz system

For the system (2) we consider the variable z as a function of variables x and y i.e. $z = z(x, y)$.

As result we get the partial first order differential equation

$$\sigma(y-x)\frac{\partial z(x,y)}{\partial x} + (rx - y - xz(x,y))\frac{\partial z(x,y)}{\partial x} - xy + bz(x,y) = 0 \tag{5}$$

determines the surface $z = z(x, y, \sigma, b, r)$ depending on parameters.

To the integration of this equation we present it in the equivalent form

$$\begin{gathered}\sigma\, y \left(\frac{\partial}{\partial x} z(x,y)\right) + ((r-1)\,x - (\sigma+1)\,y - xz(x,y))\left(\frac{\partial}{\partial y} z(x,y)\right) - \\ -yx - x^2 + bz(x,y) = 0\end{gathered} \tag{6}$$

which is connected with the previous form by the change of variable

$$y = Y + x.$$

In fact after such a substitution the Lorenz system looks as

$$\frac{dx}{ds} = \sigma Y, \quad \frac{dY}{ds} = rx - Y - x - \sigma Y - xz, \quad \frac{dz}{ds} = x^2 + xY - bz \tag{7}$$

and the corresponding equation for the function $z = z(x, Y)$ takes the form (6), where we conserve old name of variable $Y = y$.

2.1 Simplest solutions

1. The substitution

$$z(x,y) = A(x)$$

into the equation (6) leads to the conditions

$$A(x) = \frac{x^2}{2\sigma}, \quad b = 2\sigma.$$

From the system (7) in the case $z = z(x)$ we get the equation

$$y(x)\frac{d}{dx}y(x) + \frac{(\sigma+1)}{\sigma}y(x) - \frac{(r-1)}{\sigma}x + \frac{1}{2\sigma^2}x^3 = 0.$$

2. The substitution of more general form

$$z(x,y) = A(x)(1 + B(x)y + C(x)y^2)$$

gives rise to the expression

$$z(x,y) = A(x)\left(1 + \left(2\,\frac{\sigma}{x} - 2\,x^{-1}\right) y\,(A(x))^{-1} - \frac{\sigma\,y^2}{x^2 A(x)}\right), \quad r = 2\sigma - 1,$$

$$b = 6\sigma - 2,$$

where

$$A(x) = -1/4\,\frac{4 - x^2 + 4\,\sigma^2 - 8\,\sigma}{\sigma}.$$

In an explicit form we have

$$z(x,y) = -\frac{\sigma\,y^2}{x^2} + 2\,\frac{(\sigma-1)\,y}{x} + 1/4\,\frac{x^2}{\sigma} - \frac{(\sigma-1)^2}{\sigma}. \tag{8}$$

Returning to the system (6) we get from the system (7) the Abel equation

$$\frac{d}{dx}y(x) - 1/4\,\frac{x^4 + \left(-12\,\sigma^2 - 4 + 12\,\sigma\right)x^2 + \left(-4\,\sigma + 8\,\sigma^2\right)y(x)x - 4\,\sigma^2\,(y(x))^2}{x\,(-y(x)+x)\,\sigma^2} = 0 \tag{9}$$

for the function $y = y(x)$.

3. The next example is the solution of equation (6) in the form

$$z(x,y) = 1/4\,\frac{x^4 + 4\,x^2 - 4\,rx^2 - 4\,y^2}{x^2 + 4 - 4\,r},$$

where

$$\sigma = 1, \quad b = 4, \quad r \quad is \quad arbitrary.$$

Remark 1. To integrate the partial nonlinear first order differential equation

$$F(x, y, z(x,y), z_x, z_y) = 0 \tag{10}$$

a following method can be applied.

We use the change of variables

$$z(x,y) \to u(x,t), \quad y \to v(x,t), \quad z_x \to u_x - \frac{v_x}{v_t}u_t, \quad v_y \to \frac{u_t}{v_t}. \tag{11}$$

In result instead of the equation (10) one gets the relation between the new variables $u(x,t)$ and $v(x,t)$ and their partial derivatives

$$\Phi(u, v, u_x, u_t, v_x, v_t) = 0. \tag{12}$$

In some cases to solve the last equation is more simple problem than to solve the equation (10).

To illustrate this method let us consider an example.

The equation

$$\frac{\partial}{\partial x}z(x,y) - \left(\frac{\partial}{\partial y}z(x,y)\right)^2 = 0 \tag{13}$$

is transformed into the following form

$$\frac{\partial}{\partial x}u(x,t) - \frac{\left(\frac{\partial}{\partial t}u(x,t)\right)\frac{\partial}{\partial x}v(x,t)}{\frac{\partial}{\partial t}v(x,t)} - \frac{\left(\frac{\partial}{\partial t}u(x,t)\right)^2}{\left(\frac{\partial}{\partial t}v(x,t)\right)^2} = 0.$$

Using the substitution

$$u(x,t) = t\frac{\partial}{\partial t}\omega(x,t) - \omega(x,t), \quad v(x,t) = \frac{\partial}{\partial t}\omega(x,t)$$

we find the equation for $\omega(x,t)$

$$\frac{\partial}{\partial x}\omega(x,t) + t^2 = 0.$$

Its integration leads to

$$\omega(x,t) = -t^2x + F_1(t)$$

where $F_1(t)$ is an arbitrary function.

Now with the help of $\omega(x,t)$ we find the functions $u(x,t)$ and $v(x,t)$

$$u(x,t) = -t^2x + t\frac{d}{dt}F_1(t) - F_1(t), \quad v(x,t) = -2\,tx + \frac{d}{dt}F_1(t)$$

or

$$u(x,t) = ty + t^2x - F_1(t), \quad y = -2\,tx + \frac{d}{dt}F_1(t).$$

After the choice of arbitrary function $F_1(t)$ and the elimination of the parameter t from these relations we get the function $z(x,y)$, satisfying the equation (13).

We apply this method for the study of the surfaces connected with the Lorenz model (7) in the case $y = y(x,z)$.

The corresponding partial differential equation is

$$\begin{gathered}\left(\frac{\partial}{\partial x}y(x,z)\right)\sigma\, y(x,z) + \left(\frac{\partial}{\partial z}y(x,z)\right)\left(xy(x,z) + x^2 - bz\right) + \\ +\sigma\, y(x,z) - rx + y(x,z) + x + xz = 0.\end{gathered} \tag{14}$$

In new variables it looks as

$$\begin{gathered}\left(\frac{\partial}{\partial x}u(x,t) - \frac{\left(\frac{\partial}{\partial t}u(x,t)\right)\frac{\partial}{\partial x}v(x,t)}{\frac{\partial}{\partial t}v(x,t)}\right)\sigma\, u(x,t) + \frac{\left(\frac{\partial}{\partial t}u(x,t)\right)\left(xu(x,t) + x^2 - bv(x,t)\right)}{\frac{\partial}{\partial t}v(x,t)} + \\ +\sigma\, u(x,t) - rx + u(x,t) + x + xv(x,t) = 0.\end{gathered} \tag{15}$$

After the substitution

$$u(x,t) = t\frac{\partial}{\partial t}\omega(x,t) - \omega(x,t), \quad v(x,t) = \frac{\partial}{\partial t}\omega(x,t)$$

we get the equation for the function $\omega(x,t)$

$$-\left(\frac{\partial}{\partial x}\omega(x,t)\right)\sigma\, t\frac{\partial}{\partial t}\omega(x,t)+\left(\frac{\partial}{\partial x}\omega(x,t)\right)\sigma\,\omega(x,t)+xt^2\frac{\partial}{\partial t}\omega(x,t)-tx\omega(x,t)-tb\frac{\partial}{\partial t}\omega(x,t)+$$

$$+tx^2+\sigma\, t\frac{\partial}{\partial t}\omega(x,t)-\sigma\,\omega(x,t)-rx+t\frac{\partial}{\partial t}\omega(x,t)-\omega(x,t)+x+x\frac{\partial}{\partial t}\omega(x,t)=0.$$

The simplest solution of this equation has the form

$$\omega(x,t)=A(t)x,$$

where

$$A(t)=1+\sqrt{t^2+1}C_1,\quad b=1,\sigma=1,r=C_1^2.$$

With the help of this solution we find the functions $u(x,t)$ and $v(x,t)$

$$u(x,t)=-\frac{x\sqrt{r}}{\sqrt{t^2+1}}-x,\quad v(x,t)=\frac{\sqrt{r}tx}{\sqrt{t^2+1}}.$$

Elimination the parameter t from these relations we find the corresponding solution of the equation (14)

$$y(x,z)=-\sqrt{-z^2+x^2r}-x,$$

with

$$b=1,\quad \sigma=1,\quad \textit{and}\quad r\quad \textit{is}\quad \textit{arbitrary}.$$

For the surfaces in the form

$$x=x(y,z)$$

we get the equation

$$\begin{aligned}&\left(\frac{\partial}{\partial y}x(y,z)\right)(-\sigma\, y+rx(y,z)-y-x(y,z)-x(y,z)z)+\\&+\left(\frac{\partial}{\partial z}x(y,z)\right)\left((x(y,z))^2-bz+x(y,z)y\right)-\sigma\, y=0.\end{aligned}\tag{16}$$

The simplest solution of this equation is

$$x(y,z)=\sqrt{2\,rz-z^2}-y,\quad b=1,\quad r=0,\quad \sigma\quad \textit{is}\quad \textit{arbitrary}.$$

Another type of solution is

$$x(y,z)=-1/2\,\frac{z^2}{y}-1/2\,y,\quad b=1,\quad \sigma=1,\quad r=1.$$

More general solutions in the case $x=x(y,z)$ can be obtained with the help of transformation like (11).

On the first step with the help of relations

$$\frac{\partial}{\partial z}x(y,z)=\frac{\frac{\partial}{\partial t}u(y,t)}{\frac{\partial}{\partial t}v(y,t)},\quad \frac{\partial}{\partial y}x(y,z)=\frac{\partial}{\partial y}u(y,t)-\frac{\left(\frac{\partial}{\partial t}u(y,t)\right)\frac{\partial}{\partial y}v(y,t)}{\frac{\partial}{\partial t}v(y,t)},$$

$$x(y,z)=u(y,t),\quad z=v(y,t)$$

we get the equation

$$\left(\frac{\partial}{\partial y}u(y,t)-\frac{\left(\frac{\partial}{\partial t}u(y,t)\right)\frac{\partial}{\partial y}v(y,t)}{\frac{\partial}{\partial t}v(y,t)}\right)(-\sigma\, y+ru(y,t)-y-u(y,t)-u(y,t)v(y,t))+$$

$$+\frac{\left(\frac{\partial}{\partial t}u(y,t)\right)\left((u(y,t))^2-bv(y,t)+u(y,t)y\right)}{\frac{\partial}{\partial t}v(y,t)}-\sigma\, y=0$$

and then with the help of substitutions

$$u(y,t)=t\frac{\partial}{\partial t}\omega(y,t)-\omega(y,t),\quad v(y,t)=\frac{\partial}{\partial t}\omega(y,t)$$

we find the equation for the function $\omega(y,t)$

$$\left(\frac{\partial}{\partial y}\omega(y,t)\right)\sigma\, y-\left(\frac{\partial}{\partial y}\omega(y,t)\right)rt\frac{\partial}{\partial t}\omega(y,t)+\left(\frac{\partial}{\partial y}\omega(y,t)\right)r\omega(y,t)+\left(\frac{\partial}{\partial y}\omega(y,t)\right)y+$$

$$+\left(\frac{\partial}{\partial y}\omega(y,t)\right)t\frac{\partial}{\partial t}\omega(y,t)-\left(\frac{\partial}{\partial y}\omega(y,t)\right)\omega(y,t)+\left(\frac{\partial}{\partial y}\omega(y,t)\right)t\left(\frac{\partial}{\partial t}\omega(y,t)\right)^2-$$

$$-\left(\frac{\partial}{\partial y}\omega(y,t)\right)\left(\frac{\partial}{\partial t}\omega(y,t)\right)\omega(y,t)+t^3\left(\frac{\partial}{\partial t}\omega(y,t)\right)^2-2\,t^2\left(\frac{\partial}{\partial t}\omega(y,t)\right)\omega(y,t)+$$

$$+t\,(\omega(y,t))^2-tb\frac{\partial}{\partial t}\omega(y,t)+yt^2\frac{\partial}{\partial t}\omega(y,t)-ty\omega(y,t)-\sigma\, y=0.$$

The separation of variables in this equation leads to the solution

$$\omega(y,t)=1/4\,\frac{y\left(2+4\,C_1+2\,\sqrt{1+2\,C_1+2\,t^2C_1}\right)}{C_1},$$

where

$$r=(1+2\,C_1)^{-1},\quad b=1,\quad \sigma=1.$$

Using this solution we get the functions $u(y,t)$ and $v(y,t)$ and then after the elimination of the parameter t from the relations

$$x=u(y,t),\quad z=v(y,t)$$

obtain the family of solutions $x(y,z)$ of equation (16)

$$x(y,z)=1/4\,\frac{-4\,yC-2\,y+2\,\sqrt{2\,y^2C+y^2-2\,C\,z^2-4\,z^2C^2}}{C},$$

$$r = (1 + 2\,C)^{-1}\,, \quad b = 1, \quad \sigma = 1.$$

The next example.

With the help of substitution

$$v(x,t) = t\frac{\partial}{\partial t}\omega(x,t) - \omega(x,t), \quad u(x,t) = \frac{\partial}{\partial t}\omega(x,t)$$

into the relation

$$\sigma\ (y - x)\,\frac{\partial}{\partial x}z(x,y) + (rx - y - xz(x,y))\,\frac{\partial}{\partial y}z(x,y) - yx + bz(x,y) = 0$$

we get the equation

$$\left(-\sigma\,\frac{\partial}{\partial x}\omega(x,t) + tx + 1\right)\omega(x,t) + \left(-x\sigma + \sigma\,t\frac{\partial}{\partial t}\omega(x,t)\right)\frac{\partial}{\partial x}\omega(x,t) +$$

$$\left(bt - x - xt^2 - t\right)\frac{\partial}{\partial t}\omega(x,t) + rx = 0.$$

It has the solution

$$\omega(x,t) = 1/2\,\frac{x^2\left(1 + t^2\right)}{t}$$

by the conditions

$$\sigma = 1/2, \quad b = 1, \quad r = 0.$$

From here we get the equation of the corresponding surface

$$z(x,y) = 1/2\,\frac{x^4 - y^2}{x^2}.$$

3 Spatial homogeneous quadratic first order systems of equations

The system of equations

$$\begin{aligned} \frac{dx}{ds} &= a_0 + a_1x + a_2y + a_{11}x^2 + a_{12}xy + a_{22}y^2, \\ \frac{dy}{ds} &= b_0 + b_1x + b_2y + b_{11}x^2 + b_{12}xy + b_{22}y^2 \end{aligned} \tag{17}$$

where a_i, a_{ij} and b_i, b_{ij} are parameters after the extension on the projective plane takes the form of Pfaff equation

$$\left(x\tilde{Q} - \tilde{P}y\right)dz - dx\,z\tilde{Q} + \tilde{P}\,dy\,z = 0 \tag{18}$$

where the functions $\tilde{P}, \tilde{Q}$ are homogeneous polinomials.

In the explicit form we get the expression

$$\left(xb\_0\,z^2 + b\_1\,x^2z + xb\_2\,yz + b\_\{11\}\,x^3 + b\_\{12\}\,x^2y + xb\_\{22\}\,y^2 - ya\_0\,z^2\right)dz -$$

$$-\left(ya\_1\,xz - a\_2\,y^2z - ya\_\{11\}\,x^2 - a\_\{12\}\,xy^2 - a\_\{22\}\,y^3\right)dz+$$
$$+\left(z^2a\_2\,y + za\_\{11\}\,x^2 + z^3a\_0 + z^2a\_1\,x + za\_\{12\}\,xy + za\_\{22\}\,y^2\right)dy+$$
$$+\left(-z^3b\_0 - z^2b\_1\,x - zb\_\{12\}\,xy - zb\_\{22\}\,y^2 - z^2b\_2\,y - zb\_\{11\}\,x^2\right)dx = 0.$$

The spatial first order system of equations

$$\frac{dx}{ds} = P(x,y,z), \quad \frac{dy}{ds} = Q(x,y,z), \quad \frac{dz}{ds} = R(x,y,z), \tag{19}$$

connected with a given Pfaff equation has the following form

$$\frac{dx}{ds} = Q_z - R_y, \quad \frac{dy}{ds} = R_x - P_z, \quad \frac{dz}{ds} = P_y - Q_x,$$

and in our case looks as

$$\frac{d}{ds}x(s) = 4\,a\_0\,z^2 + (4\,a\_2\,y + (3\,a\_1 - b\_2)\,x)\,z + 4\,a\_\{22\}\,y^2+$$
$$+(3\,a\_\{12\} - 2\,b\_\{22\})\,xy + (2\,a\_\{11\} - b\_\{12\})\,x^2,$$
$$\begin{aligned}\frac{d}{ds}y(s) = 4\,b\_0\,z^2 + ((3\,b\_2 - a\_1)\,y + 4\,b\_1\,x)\,z + (2\,b\_\{22\} - a\_\{12\})\,y^2+\\ +(-2\,a\_\{11\} + 3\,b\_\{12\})\,xy + 4\,b\_\{11\}\,x^2,\end{aligned} \tag{20}$$
$$\frac{d}{ds}z(s) = (-b\_2 - a\_1)\,z^2 + ((-2\,b\_\{22\} - a\_\{12\})\,y - b\_\{12\}\,x - 2\,a\_\{11\}\,x)\,z.$$

For such a system of equations the condition

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0$$

is fulfilled.

The system of equations (20) is equivalent to the first order equation connected with the system (17) in coordinates $\chi(s)$ and $\eta(s)$

$$\xi(s) = \frac{x(s)}{z(s)}, \quad \eta(s) = \frac{y(s)}{z(s)}$$

or

$$\frac{d\xi}{d\eta} = \frac{a_0 + a_1\xi + a_2\eta + a_{11}\xi^2 + a_{12}\xi\eta + a_{22}\eta^2}{b_0 + b_1\xi + b_2\eta + b_{11}\xi^2 + b_{12}\xi\eta + b_{22}\eta^2}.$$

The equation of the surfaces defined by the system (20) has the form

$$\left(\frac{\partial}{\partial x}z(x,y)\right)P(x,y,z) + \left(\frac{\partial}{\partial y}z(x,y)\right)Q(x,y,z) - R(x,y,z) = 0.$$

Let us consider some examples.

For the system of equations

$$\frac{d}{dt}x(t)+4\,z(t)y(t)-3\,z(t)lx(t)-4\,\left(y(t)\right)^2-3\,mx(t)y(t)+20\,\left(x(t)\right)^2+\left(x(t)\right)^2n=0,$$

$$\frac{d}{dt}y(t)-4\,z(t)x(t)+z(t)ly(t)-4\,\left(x(t)\right)^2-3\,x(t)y(t)n-20\,x(t)y(t)+m\left(y(t)\right)^2=0,$$

$$\frac{d}{dt}z(t)+l\left(z(t)\right)^2+z(t)my(t)+z(t)x(t)n-20\,z(t)x(t)=0,$$

which is connected with the projective extension of the planar system that cab be found in the theory of limit cycles

$$\frac{d}{dt}x(t)-lx+y+10\,x^2-mxy-y^2=0,$$

$$\frac{d}{dt}y(t)-x-x^2-nxy=0,$$

the equation of surface $z=z(x,y)$ takes the form

$$\left(4\,z(x,y)x-z(x,y)ly+4\,x^2+3\,xyn+20\,xy-my^2\right)\frac{\partial}{\partial y}z(x,y)+$$

$$+\left(-4\,yz(x,y)+3\,z(x,y)lx+4\,y^2+3\,mxy-20\,x^2-x^2n\right)\frac{\partial}{\partial x}z(x,y)+$$

$$+z(x,y)my+z(x,y)xn-20\,z(x,y)x+l\left(z(x,y)\right)^2=0.$$

A simplest solution of this equation can be obtained with the help of u,v-transformation with the conditions

$$u(x,t)=t\frac{\partial}{\partial t}\omega(x,t)-\omega(x,t),\quad v(x,t)=\frac{\partial}{\partial t}\omega(x,t). \tag{21}$$

As result we get the equation

$$\left(\left(-3\,xlt-4\,\omega(x,t)-3\,xm\right)\frac{\partial}{\partial t}\omega(x,t)+\left(4\,t-4\right)\left(\frac{\partial}{\partial t}\omega(x,t)\right)^2\right)\frac{\partial}{\partial x}\omega(x,t)+$$

$$+\left(3\,xl\omega(x,t)+(n+20)\,x^2\right)\frac{\partial}{\partial x}\omega(x,t)+$$

$$+\left(-tl\omega(x,t)+4\,xt^2-m\omega(x,t)+4\,txn\right)\frac{\partial}{\partial t}\omega(x,t)-$$

$$-xn\omega(x,t)-4\,tx\omega(x,t)+l\left(\omega(x,t)\right)^2+20\,x\omega(x,t)+4\,tx^2=0.$$

A simplest solution of this equation can be presented in the form

$$\omega(x,t)=A(t)+kxt$$

with parameter k.

After the substitution of this expression we find possible values of parameters

$$k=-1/9,\quad m=\frac{85}{36},\quad n=-5/4,\quad l=0$$

and the expression for the function $A(t)$

$$A(t)=t^{\frac{85}{16}}\_C1\,(-1+t)^{-\frac{69}{16}}.$$

After the elimination of variable t from the relations (21) we get the expression for the function $z(x,y)$

$$4477456\,(x+9\,y-9\,z(x,y))^5\left(-23\,\frac{x+9\,y-9\,z(x,y)}{23\,x+207\,y+48\,z(x,y)}\right)^{\frac{5}{16}}-$$

$$-3234611728125\,\_C1\,255^{\frac{5}{16}}\,(z(x,y))^4\left(\frac{z(x,y)}{23\,x+207\,y+48\,z(x,y)}\right)^{\frac{5}{16}}=0.$$

In general case the equation of the surfaces looks as

$$z_x\left(4\,a_0z^2+(4\,a_2\,y+(3\,a_1-b_2)\,x)\,z+4\,a_{22}\,y^2+(3\,a_{12}-2\,b_{22})\,xy+(2\,a_{11}-b_{12})\,x^2\right)+$$

$$+z_y\left(4\,b_0\,z^2+((3\,b_2-a_1)\,y+4\,b_1\,x)\,z+(2\,b_{22}-a_{12})\,y^2+(-2\,a_{11}+3\,b_{12})\,xy+4\,b_{11}\,x^2\right)+$$

$$+(b_2+a_1)\,(z)^2+((2\,b_{22}+a_{12})\,y+(2\,a_{11}+b_{12})\,x)\,z=0.$$

After the u,v-transformation (21) with the function $\omega(x,t)=A(t)x$ we find from here the equation on the function $A(t)$

$$\left(a_0\,t^2A\,(t)+a_2\,tA\,(t)-b_2\,t^2-b_{22}\,t+a_{22}\,A\,(t)-t^3b_0\right)\left(\frac{d}{dt}A\,(t)\right)^2+$$

$$+\left(-(2\,a_0\,t+a_2)\,(A\,(t))^2+(a_{12}-t^2b_1+a_1\,t-b_{12}\,t+b_2\,tA+2\,t^2b_0)\,A\,(t)\right)\frac{d}{dt}A\,(t)-$$

$$-a_1\,(A\,(t))^2+a_{11}\,A\,(t)+a_0\,(A\,(t))^3+tb_1\,A\,(t)-tb_{11}-tb_0\,(A\,(t))^2.$$

A genus of this equation depends on the values of the parameters and can be $g=1$ or $g=0$.

In the first case to integration of the equation on $A(t)$ may be used parametrization by elliptic functions an by rational functions in the second case.

Another way of investigation of the properties of this equation follows from its geometrical interpretation as equation of asymptotical lines on the some surface.

4 Monge equations in the theory of the first order nonlinear p.d.e.'s

The Monge equations

$$\Phi(x, y, z, dx, dy, dz) = 0 \tag{22}$$

are equations homogeneous with respect to the differentials dx, dy, dz.

They are naturally connected with equations of the form

$$F(x, y, z, z_x, z_y) = F(x, y, z, p, q) = 0 \tag{23}$$

and can be used for the study of their properties.

To construct the equation (22) for the equation (23) it is necessary to eliminate the variables p and q from the system of equations

$$F(x, y, z, p, q) = 0, \quad \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{pP + qQ},$$

where

$$P = \frac{\partial F(x, y, z, p, q)}{\partial q}, \quad Q = \frac{\partial F(x, y, z, p, q)}{\partial q}.$$

Let us consider some example.

For the equation

$$F(x, y, z, p, q) = p^2 + qy^2 - 1 \tag{24}$$

the system

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{pP + qQ},$$

leads to

$$dx\, y^2 - 2\, dy\, p = 0, \quad 2\, dy\, p^2 + dy\, qy^2 - dz\, y^2 = 0. \tag{25}$$

After the elimination of the variables p and q from the equations (24-25) we find the corresponding Monge equation

$$-\left(\frac{d}{ds}x(s)\right)^2 (y(s))^4 + 4\left(\frac{d}{ds}z(s)\right)(y(s))^2 \frac{d}{ds}y(s) - 4\left(\frac{d}{ds}y(s)\right)^2 = 0. \tag{26}$$

Its general solution has the form

$$z(s) = -s\frac{d}{ds}B(s) + 1/2\, s^2 \frac{d^2}{ds^2}B(s) + 1/2\, \frac{d^2}{ds^2}B(s) + B(s),$$

$$y(s) = -2\left(\frac{d^2}{ds^2}B(s)\right)^{-1}, \; x(s) = -\frac{d}{ds}B(s) + s\frac{d^2}{ds^2}B(s)$$

where $B(s)$ is an arbitrary function and it can be obtained with the help of complete integral of equation (24)

$$z(x, y) = Ax + \frac{(A^2 - 1)}{y} + B,$$

where A and B are parameters.

5 From Monge equations to the Riemann geometry

Riemann space with the metric

$$ds^2 = g_{ij}dx^i dx^j \tag{27}$$

has geodesic satisfying the system of equations

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0,$$

where Γ^i_{jk} are the Christoffel symbols of the metrics (27).

These equations have the first integral of the form

$$g_{ij}\frac{dx^j}{ds}\frac{dx^k}{ds} = \mu$$

or

$$g_{ij}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

by the condition $\mu = 0$.

In the three-dimensional case such integral has the form of the Monge equation

$$g_{11}(x,y,z)\left(\frac{dx}{ds}\right)^2 + 2g_{12}(x,y,z)\frac{dx}{ds}\frac{dy}{ds} + g_{22}(x,y,z)\left(\frac{dy}{ds}\right)^2 + 2g_{13}(x,y,z)\frac{dx}{ds}\frac{dz}{ds} +$$
$$+2g_{23}(x,y,z)\frac{dy}{ds}\frac{dz}{ds} + g_{33}(x,y,z)\left(\frac{dz}{ds}\right)^2 = 0. \tag{28}$$

and can be considered as quadratic first integral of null-geodesic of some three-dimensional space endowed with the metric

$$ds^2 = g_{11}(x,y,z)dx^2 + 2g_{12}(x,y,z)dxdy + g_{22}(x,y,z)dy^2 + 2g_{13}(x,y,z)dxdz +$$
$$+2g_{23}(x,y,z)dydz + g_{33}(x,y,z)dz^2.$$

From this point of view the methods of Riemann geometry can be used for the investigation of the properties of Monge equations and the corresponding first order nonlinear p.d.e.

In particular, scalar invariants and the theory of surfaces of such type of spaces can be used with this aim.

We consider two-dimensional surfaces of the translation $y^\nu = [x(u,v),\ y(u,v),\ z(u,v)]$ in a three-dimensional Riemann space which corresponds to the Monge equation of the form (26) as it was described above.

The equations for determination of translation surfaces have the form

$$\frac{\partial y^\alpha}{\partial u \partial v} + \Gamma^\alpha_{\beta\gamma}\frac{\partial y^\beta}{\partial u}\frac{\partial y^\gamma}{\partial v} = 0. \tag{29}$$

Let us consider some examples.

For the equation (24) the Monge equation is (26).

The metric of corresponding Riemann space is

$$ds^2 = -y^4\,dx^2 - 4\,dy^2 + 4\,y^2 dy\,dz. \tag{30}$$

The geodesic equations of this space are of the form

$$\left(\frac{d^2}{ds^2}x(s)\right)y(s) + 4\left(\frac{d}{ds}y(s)\right)\frac{d}{ds}x(s) = 0,$$

$$\left(\frac{d^2}{ds^2}y(s)\right)y(s) + 2\left(\frac{d}{ds}y(s)\right)^2 = 0,$$

$$\left(\frac{d^2}{ds^2}z(s)\right)(y(s))^3 + (y(s))^4\left(\frac{d}{ds}x(s)\right)^2 + 4\left(\frac{d}{ds}y(s)\right)^2 = 0.$$

Taking in consideration the equation (26) we get the solutions of geodesic equations

$$x(s) = \_C3 + \frac{\_C4}{\sqrt[3]{3\,\_C1\,s + 3\,\_C2}}, \quad y(s) = \sqrt[3]{3\,\_C1\,s + 3\,\_C2},$$

and

$$z(s) = -1/4\,\frac{4 + \_C4^2}{\sqrt[3]{3\,\_C1\,s + 3\,\_C2}} + \_C6.$$

The equations (29) for the surfaces of translation of the space with metric (30) looks as

$$\left(\frac{\partial^2}{\partial u \partial v}x(u,v)\right)y(u,v) + 2\left(\frac{\partial}{\partial u}x(u,v)\right)\frac{\partial}{\partial v}y(u,v) + 2\left(\frac{\partial}{\partial u}y(u,v)\right)\frac{\partial}{\partial v}x(u,v) = 0,$$

$$\left(\frac{\partial^2}{\partial u \partial v}y(u,v)\right)y(u,v) + 2\left(\frac{\partial}{\partial u}y(u,v)\right)\frac{\partial}{\partial v}y(u,v) = 0,$$

$$\left(\frac{\partial^2}{\partial u \partial v}z(u,v)\right)(y(u,v))^3 + (y(u,v))^4\left(\frac{\partial}{\partial u}x(u,v)\right)\frac{\partial}{\partial v}x(u,v) +$$

$$+4\left(\frac{\partial}{\partial u}y(u,v)\right)\frac{\partial}{\partial v}y(u,v) = 0.$$

They can be integrated without problems.

The simplest solution is

$$x(u,v) = u - v, \quad y(u,v) = \sqrt[3]{u+v},$$

$$z(u,v) = \_F2(u) + \_F1(v) + \frac{9}{28}\,(u+v)^{7/3} - \frac{1}{\sqrt[3]{u+v}}.$$

In particular case $\_F2(u) = u$, $\_F1(v) = 0$ with the help of relations

$$u = 1/2\,x + 1/2\,y^3, \quad v = -1/2\,x + 1/2\,y^3$$

we get the surface with equation

$$z(x,y) = \frac{9}{28}\,y^7 + 1/2\,y^3 + 1/2\,x - y^{-1}.$$

6 On the surfaces connected with the Rössler system

Let us consider examples of surfaces corresponding to the Rössler system of equations (3)

One of them has the form

$$\left(\frac{\partial}{\partial y}x(y,z)\right)(x(y,z)+ay)+\left(\frac{\partial}{\partial z}x(y,z)\right)(b+(x(y,z)-c)\,z)+y+z. \tag{31}$$

In the case $a=0,\ b=0,\ c=0$ we find the solution

$$x(y,z)=\sqrt{-y^2-2\,z+\_F1(\frac{z}{e^y})}.$$

where $\_F1(\frac{z}{e^y})$ is an arbitrary function.

To construct the Monge equation corresponding the equation (31) we present its in a new notations as

$$\left(\frac{\partial}{\partial u}y(u,v)\right)(y(u,v)+au)+\left(\frac{\partial}{\partial v}y(u,v)\right)(b+(y(u,v)-c)\,v)+u+v=0.$$

After the change of variables

$$\frac{\partial}{\partial u}y(u,v)=\frac{\frac{\partial}{\partial x}\omega(x,v)}{\frac{\partial}{\partial x}\lambda(x,v)},\quad \frac{\partial}{\partial v}y(u,v)=\frac{\partial}{\partial v}\omega(x,v)-\frac{\left(\frac{\partial}{\partial x}\omega(x,v)\right)\frac{\partial}{\partial v}\lambda(x,v)}{\frac{\partial}{\partial x}\lambda(x,v)},$$

$$y(u,v)=\omega(x,v),\quad u=\lambda(x,v)$$

with conditions

$$\lambda(x,v)=x\frac{\partial}{\partial x}\rho(x,v)-\rho(x,v)\omega(x,v)=\frac{\partial}{\partial x}\rho(x,v)$$

one gets the equation equivalent to (31)

$$\left(1+ax+x^2+v\frac{\partial}{\partial v}\rho(x,v)\right)\frac{\partial}{\partial x}\rho(x,v)+(-cv+b)\frac{\partial}{\partial v}\rho(x,v)-x\rho(x,v)+$$
$$+vx-a\rho(x,v)=0.$$

For this p.d.e. the Monge equation is defined by the condition

$$dz^2y^2+\left(\left(2\,yx^2+2\,yax+2\,y\right)dy+\left(2\,by-2\,cy^2\right)dx\right)dz+$$
$$+\left(2\,x^2+2\,ax+x^4+1+2\,ax^3+a^2x^2\right)dy^2+(cy-b)^2\,dx^2$$
$$+\left(2\,axcy-2\,b-2\,axb+4\,xy^2+2\,x^2cy-2\,x^2b-4\,xzy-4\,azy+2\,cy\right)dx\,dy=0$$

and it may be considered as the first integral of geodesic of the corresponding three-dimensional Riemann space.

Another aproach to the study of the Rössler system gives us the investigation of the integral manifolds of corresponding Pfaff equation

$$-(y+z)dx+(x+ay)dy+(b+(x-c)zdz=0.$$

It is reduced to the form

$$-(y+z)dx+(x+ay)dy+(b+(x-c)zdz=dU+VdW$$

and so determines one-dimensional integral manifolds.

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Regular maps and quasilinear total differential equations

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Abstract. The bounded and S-concordant solutions of the quasilinear total differential equations with a real parameter by the nonlinearity and with a regular homogeneous part are investigated.

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1 Introduction

In the paper the quasilinear total differential equations [3] of the form

$$y'h = a(t)hy + (b(t) + \lambda g(t,y))h \quad (h \in E) \tag{1}$$

with a real parameter λ by the nonlinearity and with a regular homogeneous part are investigated. Here: $a \in C(P, L(E, L(T,T)))$, $b \in C(P, L(E,T))$, $g \in C(P \times T, L(E,T))$, $y \in C(P,T)$ is an unknown map; E is a normed real space; T is a Banach space (real or complex); $P \subset E$ is an open set; the prime $'$ means the operation of taking a bounded derivative (derivative Frechet). By $C(X,Y)$ we designate the space of all continuous maps of the space X in the space Y endowed with the uniform structure of compact convergence (compact open topology); by $L(X,Y)$ we designate the space of all linear continuous maps of the normed space X in the normed space Y with the natural operator norm.

For such equations some sufficient conditions of the existence of bounded, compact, Lagrange stable, concordant and uniformly concordant solutions are established. Earlier similar problems for ordinary differential equations ($E = P = R$) and multidimensional differential equations, for $E = P = R^n$ and $T = R^m$, were considered in [1,4,8]. The Lagrange stability, concordance and uniform concordance are considertd relative to some semigroup $S \subset P$, in contrast to [1, 4, 8], where $S = E$. The peculiarity of our researches is that dynamical systems (transformation groups or semigroups) are not used.

We propose also some general approach to the research of quasilinear equations, based on the concept of a regular, generally speaking, many-valued map. We consider regular maps in the first section of this paper. In the second section we indicate some applications of results obtained in the first section to the quasilinear equations (1).

2 Regular maps

The concept of a regular map naturally arises when abstracting from the concrete type of the regular equation. Under a regular ordinary differential equation we understand such an ordinary differential equation $y' = a(t)y$ that for an arbitrary bounded function $f : R \to R^n$ there exists a unique bounded solution $\varphi : R \to R^n$ of the equation $y' = a(t)y + f(t)$. It is known [7] (Theorem 51. A) that for the regular equation $y' = a(x)y$ there is a constant $r > 0$ such that $\sup_{t\in R} \|\varphi(t)\| \leq r \sup_{t\in R} \|f(t)\|$, where φ is a bounded solution of the equation $y' = a(t)y + f(t)$ with the bounded function f. Close connection between a regular and an exponential dichotomy is known, too [2, 7].

By the research of quasilinear equations $y' = a(t)y + b(t) + f(t, y)$ sometimes one of crucial is the following property of linear equations: if φ_i is a solution of the equation $y' = a(t)y + b_i(t)$ $(i = 1, 2)$, then $\varphi_1 - \varphi_2$ is a solution of the equation $y' = a(t)y + (b_1 - b_2)(t)$. Besides with homogeneous equations of the form $y' = a(t)y$, generally speaking, a many-valued map naturally associates that to each function b puts in correspondence the set of solutions of the equation $y' = a(t)y + b(t)$. The last two facts in combination with definition of a regular homogeneous equation will be taken as a basis in the definition of a regular map.

Definition 1. *Let X, Y be normed real spaces, 2^Y be a family of all subsets of Y, $r > 0$. A map $q : X \to 2^Y$ is called weakly r-regular if:*

1) $\forall x \in X\ q(x) \neq \emptyset$;

2) $\forall x, y \in X\ q(x) - q(y) \subset q(x - y)$;

3) $\forall x \in X\ \forall y \in q(x) \quad \|y\| \leq r \cdot \|x\|$.

A weakly r-regular map is called r-regular if it is a one-valued map.

Let's give some examples of regular maps.

Example 1. Let $X = Y$ be the space of bounded maps from $C(R,\ R^m)$ with the norm *sup* and the map $a \in C(R,\ L(R^m,\ R^m))$ be such that the differential equation $y' = a(t)y$ is regular. Then there is a positive number r such that the map $q : X \to 2^Y$ defined by the rule

$$\varphi \in q(f) \Longleftrightarrow \varphi'(t) = a(t)\varphi(t) + f(t) \quad (t \in R)$$

is r-regular.

Example 2. Let E and T be Banach spaces, $a \in L(E, L(T, T))$ be a permutable operator (i.e. $ahak = akah$ for $\forall h, k \in E$) such that (Sp a)e does not intersect with the imaginary axis of the complex plane for some vector $e \in E$ of the unit norm; X be the space of all continuously differentiable bounded maps $f : E \to L(E, T)$ with the norm *sup* which satisfy the condition $\wedge\{ahf(t)k - f'(t)kh\} = 0$ for $\forall h, k, t \in E$; Y be the space of all continuous bounded maps $E \to T$ with the norm *sup*. And let

$r = 2c/\alpha$, where $c > 0$ and $\alpha > 0$ are constants for which $\|G_e(t)\| \le c \cdot \exp(-\alpha|t|)$ $(t \in R)$; G_e be the main Green function of the operator ae [3]. Then, as it follows from the theorem 12.2 of [3], the map $q : X \to 2^Y$ defined by the rule

$$\varphi \in q(f) \Longleftrightarrow \varphi'(t)h = a(t)h\varphi(t) + f(t)h \quad (t,\ h \in E)$$

is r-regular (by the symbol $\wedge$ everywhere in the paper we designate the operation of taking the skew-symmetric part of bilinear operator: $\wedge\{Chk\} = 1/2(Chk - Ckh)$ $(C \in L_2(E, F))$).

Example 3. Let $1 \le p,\ q < \infty,\ 1/p + 1/q = 1$; $D \subset R^n$ be a bounded closed set, $M_0 = (mesD)^{\frac{1}{p}}$; $X = Y = L_p(D)$; $K : D \times D \to R$ be a measurable function such that for some number $M > 0$ and for almost all $t \in D$,

$$\Big(\int_D |K(t,s)|^q\ ds\Big)^{\frac{1}{q}} \le M;$$

the number λ is such that $|\lambda| < 1/(MM_0)$; $r = 1/(1 - |\lambda|MM_0)$. Then the map $q : X \to 2^Y$ defined by the rule

$$\varphi \in q(f) \Longleftrightarrow \varphi(t) = f(t) + \lambda \int_D K(t,s)\varphi(s)ds \quad (t \in D)$$

is r-regular.

Theorem 1. *Let Y be a complete space; $q : X \to 2^Y$ be a weakly r-regular map; $b \in X$ and the map $B : Y \to X$ satisfies the Lipschitz condition with the constant L. Then for $\forall\lambda, |\lambda| < 1/(rL)$, in Y there is an element x_λ for which $x_\lambda \in q(b+\lambda B(x_\lambda))$; in addition $\|x_\lambda\| \le r \cdot \|b + \lambda B(x_\lambda)\|$. The element x_λ is determined uniquely if q is an r-regular map. In the last case x_λ may be found as the limit of the sequence*

$$y_1, y_2, \cdots, y_n, \cdots \tag{2}$$

for $\forall y_1 \in Y$ and for any $n > 1$ $y_n = q(b + \lambda B(y_{n-1}))$.

Proof. Let λ be such that $|\lambda| < 1/(rL)$. Let's designate by $H_\lambda : Y \to Y$ the choice function for the composition $f_\lambda \circ q$ where $f_\lambda : Y \to X$ is defined by the rule: $f_\lambda(x) = b + \lambda B(x)$. We shall prove that H_λ is a contraction map. Let $x_1, x_2 \in Y$. Then

$$H_\lambda(x_1) - H_\lambda(x_2) \in q(b + \lambda B(x_1)) - q(b + \lambda B(x_2)) \subset q(\lambda(B(x_1) - B(x_2))),$$

i.e. $H_\lambda(x_1) - H_\lambda(x_2) \in q(\lambda B(x_1) - B(x_2))$. Then

$$\|H_\lambda(x_1) - H_\lambda(x_2)\| \le r\|\lambda(B(x_1) - B(x_2))\| \le r|\lambda|L\|x_1 - x_2\|,$$

i.e. $\|H_\lambda(x_1) - H_\lambda(x_2)\| \le r|\lambda|L\|x_1 - x_2\|$. Since $r|\lambda|L < 1$, then the map H_λ is contracting. By virtue of the completeness of the space Y, according to the Banach contracting principle, there exists a unique x_λ such that $H_\lambda(x_\lambda) = x_\lambda$. Therefore $x_\lambda \in q(b + \lambda B(x_\lambda))$. It is clear that x_λ does not depend on the choice function H_λ, hence, it is determined uniquely as the limit of the sequence (2) if q is a r-regular map.

Theorem 2. *Let Y be a complete space; $q : X \to 2^Y$ be a weakly r-regular map; $b \in X$; $x \in q(b)$; $\delta > 0$; V_x^δ be a closed δ-neighbourhood of x and the map $B : Y \to X$ satisfies the Lipschitz condition on V_x^δ with the constant L. Then for $\forall\lambda, |\lambda| < \lambda_0 = \delta/(rL\delta + rl_1)$, where $l_1 = \|B(x)\|$, in V_x^δ there is an element x_λ for which $x_\lambda \in q(b + \lambda B(x_\lambda))$; in addition $\|x_\lambda\| \le r \cdot \|b + \lambda B(x_\lambda)\|$. The element x_λ is determined uniquely if q is an r-regular map. In the last case x_λ may be found as the limit of the sequence* (2) *which begins at an arbitrary point $y_1 \in V_x^\delta$.*

Proof. Let $\lambda \in]-\lambda_0,\ \lambda_0[,\ \ z \in V_x^\delta$ and $y \in q(b + \lambda B(z))$. Since $x \in q(b)$, then $y - x \in q(\lambda B(z))$. Hence

$$\|y - x\| \le r|\lambda|\|B(z)\| \le r|\lambda|(\|B(z) - B(x)\| + \|B(x)\|) \le$$
$$\le r|\lambda|(L\|z - x\| + l_1) \le r|\lambda|(L\delta + l_1) < \delta,$$

i.e. $\|y - x\| < \delta$, therefore $y \in V_x^\delta$. We have proved that for $\forall z \in V_x^\delta$, $q(b + \lambda B(z)) \subset V_x^\delta$. Therefore the composition $f_\lambda \circ q$, where $f_\lambda : V_x^\delta \to X$ is defined by the rule: $f_\lambda(x) = b + \lambda B(x)$, is a map $V_x^\delta \to 2^{V_x^\delta}$. Let's designate by $H_\lambda : V_x^\delta \to V_x^\delta$ the choice function for the composition $f_\lambda \circ q$. We shall also prove that H_λ is a contraction map. Let $x_1, x_2 \in V_x^\delta$. Then

$$H_\lambda(x_1) - H_\lambda(x_2) \in q(b + \lambda B(x_1)) - q(b + \lambda B(x_2)) \subset q(\lambda(B(x_1) - B(x_2))),$$

i.e. $H_\lambda(x_1) - H_\lambda(x_2) \in q(\lambda(B(x_1) - B(x_2)))$. Then

$$\|H_\lambda(x_1) - H_\lambda(x_2)\| \le r\|\lambda(B(x_1) - B(x_2))\| \le r|\lambda|L\|x_1 - x_2\|,$$

i.e. $\|H_\lambda(x_1) - H_\lambda(x_2)\| \le r|\lambda|L\|x_1 - x_2\|$. Since $r|\lambda|L < 1$, then the map H_λ is contracting. According to the Banach contracting principle there is a unique $x_\lambda \in V_x^\delta$ such that $H_\lambda(x_\lambda) = x_\lambda$. It is clear that x_λ is the required element.

Definition 2. *Let $X \subset Z^P$ and $F : Y \times P \to W$ be a map. The map $f : P \times W \to Z$ is called F-admissible if for $\forall x \in Y$, the map f^x, where $f^x(t) = f(t, F(x,t))$ $(t \in P)$, belongs to X.*

From Theorems 1 and 2 for $X \subset Z^P$ and $B(y) = f^y$ $(y \in Y)$ as a corollary we obtain the following two theorems.

Theorem 3. *Let $X \subset Z^P$, $q : X \to 2^Y$ be a weakly r-regular map, $b \in X$, $f : P \times W \to Z$ be a F-admissible map, $L \in R$ and the following conditions be satisfied:*

1) *Y is a complete space;*

2) $\forall y, z \in Y$ $\|f^y - f^z\| \le L \cdot \|y - z\|$.

Then for $\forall\lambda, |\lambda| < 1/(rL)$, in Y there is an element x_λ such that $x_\lambda \in q(b + \lambda f^{x_\lambda})$; in addition $\|x_\lambda\| \le r \cdot \|b + \lambda f^{x_\lambda}\|$. The element x_λ is determined uniquely if q is a r-regular map. In the last case x_λ may be found as the limit of the sequence $y_1, y_2, \cdots, y_n, \cdots$, where y_1 is an arbitrary element from Y and for any $n > 1$, $y_n = q(b + \lambda f^{y_{n-1}}))$.

Theorem 4. *Let $X \subset Z^P$, $q : X \to 2^Y$ be a weakly r-regular map, $b \in X$, $f : P \times W \to Z$ be an F-admissible map, $L \in R$, $\delta > 0$ and the following conditions are satisfied:*

1) *Y is a complete space;*
2) *V_x^δ is a closed δ-neighbourhood of $x \in q(b)$;*
3) *$\forall y, z \in V_x^\delta$ $\|f^y - f^z\| \leq L \cdot \|y - z\|$.*

Then for $\forall \lambda, |\lambda| < \delta/(r(L\delta + l_1))$, where $l_1 = \|f^x\|$, in V_x^δ there is an element x_λ for which $x_\lambda \in q(b + \lambda f^{x_\lambda})$; in addition $\|x_\lambda\| \leq r \cdot \|b + \lambda f^{x_\lambda}\|$. The element x_λ is determined uniquely if q is an r-regular map. In the last case x_λ may be found as the limit of the sequence $y_1, y_2, \cdots, y_n, \cdots$, where y_1 is an arbitrary element from V_x^δ and for any $n > 1$, $y_n = q(b + \lambda f^{y_{n-1}}))$.

Theorem 5. *Let Z, W be normed spaces, $X \subset Z^P$, $q : X \to 2^Y$ be a weakly r-regular map, $f : P \times W \to Z$ be an F-admissible map, $b \in X$, $x \in q(b)$, $V_{F(x,P)}^\delta$ be a closed δ-neighbourhood of $F(x, P)$; $A_x^\delta = \{y \mid y \in Y$ and $F(y, P) \subset V_{F(x,P)}^\delta\}$. If the following conditions are valid:*

1) *A_x^δ is a complete subset of the space Y;*
2) *$\forall g \in X$ $\|g\| \leq \sup_{t \in P} \|g(t)\|$;*
3) *f satisfies the Lipschitz condition in the second argument with the constant L_1 on the set $V_{F(x,P)}^\delta$ and F satisfies the Lipschitz condition in the first argument with the constant L_2;*
4) *$\sup_{(t,s) \in P \times P} \|f(t, F(x, s))\| \leq l_1 \in R$,*

then for $\forall \lambda, |\lambda| < \lambda_0 = \delta/(rL_2(L_1\delta + l_1))$, in the set A_x^δ there is an element x_λ for which $x_\lambda \in q(b + \lambda f^{x_\lambda})$; in addition $\|x_\lambda\| \leq r \cdot \|b + \lambda f^{x_\lambda}\|$. The element x_λ is determined uniquely if q is an r-regular map. In the last case x_λ may be found as the limit of the sequence $y_1, y_2, \cdots, y_n, \cdots$, where y_1 is an arbitrary element from A_x^δ and for any $n > 1$, $y_n = q(b + \lambda f^{y_{n-1}}))$.

Proof. Let $z \in A_x^\delta$, $|\lambda| < \lambda_0$ and $y \in q(b + \lambda f^z)$. Then $y - x \in q(\lambda f^z)$. Therefore $\|y - x\| \leq r|\lambda|\|f^z\|$. Since $z \in A_x^\delta$, for arbitrary $t \in P$ there exists $p_t \in P$ such that $\|F(z, t) - F(x, p_t)\| \leq \delta$. Then for $\forall s \in P$

$$\begin{aligned}
&\|F(y, s) - F(x, s)\| \leq L_2\|y - x\| \leq r|\lambda|L_2\|f^z\| \leq rL_2|\lambda| \sup_{t \in P} \|f(t, F(z, t))\| \leq \\
&\leq rL_2|\lambda| \sup_{t \in P}(\|f(t, F(z, t)) - f(t, F(x, p_t))\| + \|f(t, F(x, p_t))\|) \leq \\
&\leq rL_2|\lambda|(L_1 \sup_{t \in P} \|F(z, t) - F(x, p_t)\| + l_1) \leq rL_2|\lambda|(L_1\delta + l_1) < \delta,
\end{aligned}$$

i.e. $\|F(y, s) - F(x, s)\| \leq \delta$. Hence $y \in A_x^\delta$. We have proved that $q(b + \lambda f^z) \subset A_x^\delta$ for $\forall z \in A_x^\delta$. Therefore the composition $f_\lambda \circ q$, where $f_\lambda : A_x^\delta \to X$ is defined by the rule: $f_\lambda(z) = b + \lambda f^z$, is a map $A_x^\delta \to 2^{A_x^\delta}$. Let's designate by $H_\lambda : A_x^\delta \to A_x^\delta$ choice function of this map. We shall also prove that H_λ is a map of contraction. Let $x_1, x_2 \in A_x^\delta$. Then

$$H_\lambda(x_1) - H_\lambda(x_2) \in q(b + \lambda f^{x_1}) - q(b + \lambda f^{x_2}) \subset q(\lambda(f^{x_1} - f^{x_2})),$$

i.e. $H_\lambda(x_1) - H_\lambda(x_2) \in q(\lambda(f^{x_1} - f^{x_2}))$. Therefore

$$\begin{aligned} &\|H_\lambda(x_1) - H_\lambda(x_2)\| \le r\|\lambda(f^{x_1} - f^{x_2})\| \le \\ &\le r|\lambda| \sup_{t\in P} \|f(t, F(x_1, t)) - f(t, F(x_2, t))\| \le \\ &\le r|\lambda| L_1 \sup_{t\in P} \|F(x_1, t) - F(x_2, t\| \le r|\lambda| L_1 L_2 \|x_1 - x_2\|, \end{aligned}$$

i.e. $\|H_\lambda(x_1) - H_\lambda(x_2)\| \le rL_1L_2|\lambda|\|x_1 - x_2\|$. Since $rL_1L_2|\lambda| < 1$ then the map H_λ is contracting. By virtue of the completeness of the set A_x^δ there exists unique $x_\lambda \in A_x^\delta$ for which $H_\lambda(x_\lambda) = x_\lambda$. It is clear that x_λ is the required element.

3 Quasilinear equations

Lemma 1. *Let the space E be quasicomplete; P be a connected convex set if the space E is finite-dimensional and $P = E$ if the space E is infinite-dimensional; K be a compact set from T; $\lambda \in R$; I be a directional set and for $\forall i \in I$, φ_i is a solution of the total differential equation*

$$y'h = a_i(X)hy + (b_i(x) + \lambda g_i(x, y))h \ (h \in E), \tag{3}$$

and $\overline{\varphi_i(P)} \subset K$ and $\lim_i(a_i, b_i, g_i) = (a, b, g)$ in $C(P, L(E, L(T, T))) \times C(P, L(E, T))$ $\times C(P \times T, L(E, T))$. Then:

1) *the set $\overline{\{\varphi_i \,|\, i \in I\}}$ is compact;*
2) *the limit φ of an arbitrary subnet of the net $\{\varphi_i\}$ is a solution of the total differential equation*

$$y'h = a(X)hy + (b(x) + \lambda g(x, y))h \ (h \in E) \tag{4}$$

and $\overline{\varphi(P)} \subset K$.

Proof. We shall define for $\forall i \in I$ the maps $f_i : P \times T \to L(E, T)$ and $f : P \times T \to L(E, T)$ by the rules: for $\forall(x, y) \in P \times T$, $\forall h \in E$, $f_i(x, y)h = a_i(x)hy + (b_i(x) + \lambda g_i(x, y))h$ and $f(x, y)h = a(x)hy + (b(x) + \lambda g(x, y))h$. It is clear that for $\forall i \in I$, the equation (3) is equivalent to the equation

$$y' = f_i(x, y) \tag{5}$$

and the equation (4) is equivalent to the equation

$$y' = f(x, y). \tag{6}$$

Let's prove that $\lim_i f_i = f$ in $C(P\times T, L(E,T))$. Let ε be an arbitrary positive number, $Q\times M$ be an arbitrary compact set from $P\times T$, $m^* = \sup_{m\in M}\|m\|$, $\varepsilon_1 = \varepsilon/(m^*+|\lambda|+1)$. Since $\lim_i(a_i,b_i,g_i)=(a,b,g)$ then there exists $i_0\in I$ such that for arbitrary $i>i_0$, $(t,m)\in Q\times M$ and $h\in E$, $\|h\|\le 1$, the following relations are fulfilled

$$\|(a_i(t)-a(t))h\|<\varepsilon_1,\ \|(b_i(t)-b(t))h\|<\varepsilon_1 \text{ and } \|(g_i(t,m)-g(t,m))h\|<\varepsilon_1.$$

From these relations for $i>i_0$, $(t,m)\in Q\times M$ and $h\in E$, $\|h\|\le 1$, we shall receive

$$\begin{aligned}&\|(f_i(t,m)-f(t,m))h\| =\\ &= \|a_i(t)hm+(b_i(t)+\lambda g_i(t,m))h-a(t)hm-(b(t)+\lambda g(t,m))h\|\le\\ &\le \|(a_i(t)-a(t))hm\|+\|(b_i(t)-b(t))h\|+|\lambda|\|(g_i(t,m)-g(t,m))h\|<\\ &< \varepsilon_1 m^*+\varepsilon_1+|\lambda|\varepsilon_1=\varepsilon,\end{aligned}$$

i.e. $\|(f_i(t,m)-f(t,m))h\|<\varepsilon$. The proof also means that $\lim_i f_i = f$. At this point, since the equations (3) is equivalent to the equation (5) and the equation (4) is equivalent to the equation (6), then our lemma follows from Lemma 2 [5].

Further by S we designate a subsemigroup of the group E, $0\in S\subset P$ and $S+P\subset P$. To each map f from the spaces of maps under consideration and to every $s\in S$ with the help of the shift σ in the argument from P we put in correspondence some map f_s which is defined as follows. If $f: P\to Y$ then $f_s(p)=f(s+p)$ $(p\in P)$. If $f: P\times T\to Y$ then $f_s(p,t)=f(s+p,t)$ $(p\in P,\ t\in T)$. By fS we designate the set $\{f_s \mid s\in S\}$.

Definition 3. *The solution φ of the equation* (1) *is called to compact if the set $\overline{\varphi(P)}$ is compact.*

If X is the set of maps on which the operation of a semigroup S is defined with the help of the shift σ, then the map $f\in X$ is called Lagrange S-stable if the set $\overline{fS}$ is compact.

The following proposition contains some sufficient conditions of Lagrange S-stable solutions of the equation (1).

Proposition 1. *Let the map f be defined by the rule: $f(x,y)h=a(x)hy+(b(x)+\lambda g(x,y))h$ $((x,y)\in P\times T,\ h\in E)$. A compact solution φ of the equation* (1) *is Lagrange S-stable if one of the conditions is valid:*

1) *the map φ is uniformly continuous.*

2) *the set $f(P,\overline{\varphi(P)})$ is bounded.*

3) *f is Lagrange S-stable, the space E is quasicomplete, P is a connected convex set if the space E is finite-dimensional and $P=E$ if the space E is infinite-dimensional.*

Lemma 2. *Let $K \subset T$ be a compact set, $g : P \times T \to L(E, T)$. If for $\forall k \in K$ the set $\overline{g(P, k)}$ is compact and the map g satisfies the Lipschitz condition in the second argument on K, then the set $\overline{g(P, K)}$ is compact. The set $\overline{G(P, K)}$ for $\forall G \in \overline{gS}$ is compact, too.*

Proof. Let L be the Lipschitz constant of g. For the proof of the compactness of the set $\overline{g(P, K)}$ it is sufficient to prove that from every sequence $\{g(t_i, k_i)\}$, where $(t_i, k_i) \in P \times K$, it is possible to single out a subsequence of Cauchy. Let $\varepsilon > 0$ be an arbitrary number, $\{(t_i, k_i)\} \subset P \times K$. By virtue of the compactness of K there is a subsequence $\{k_{i_l}\} \subset K$ such that $\lim_l k_{i_l} = k \in K$. Then for number $\varepsilon/4L$ there is a number $l_1 \in N$ such that for $\forall l > l_1$,

$$\|k_{i_l} - k\| < \varepsilon/4L. \tag{7}$$

By virtue of the compactness of the set $\overline{g(P, k)}$ we consider that $\exists \lim_l g(t_{i_l}, k) = g_0$. Then for the number $\varepsilon/4$ there is $l_2 \in N$ such that for $\forall l > l_2$,

$$\|g(t_{i_l}, k) - g_0\| < \varepsilon/4. \tag{8}$$

Let $l_0 = \max(l_1, l_2)$ and $l > l_0\, p > l_0$. With the account of relations (7) and (8) we obtain

$$\begin{aligned}
&\|g(t_{i_l}, k_{i_l}) - g(t_{i_p}, k_{i_p})\| \leq \|g(t_{i_l}, k_{i_l}) - g(t_{i_l}, k)\| + \\
&+\|g(t_{i_l}, k) - g_0\| + \|g(t_{i_p}, k) - g_0\| + \|g(t_{i_p}, k) - g(t_{i_p}, k_{i_p})\| < \\
&< L \cdot \|k_{i_l} - k\| + \varepsilon/4 + \varepsilon/4 + L \cdot \|k_{i_p} - k\| < \varepsilon,
\end{aligned}$$

i.e. $\|g(t_{i_l}, k_{i_l}) - g(t_{i_p}, k_{i_p})\| < \varepsilon$ for $\forall l, p > l_0$. The proof means that $\{g(t_{i_l}, k_{i_l})\}$ is a Cauchy sequence. So, the set $\overline{g(P, K)}$ is compact. If $G \in \overline{gS}$, then $G = \lim g_{t_i}$ for some net $\{t_i\} \subset S$. Therefore for $\forall (t, k) \in P \times K$, $G(t, k) = \lim g_{t_i}(t, k) \in \overline{g(P, K)}$, hence, the set $\overline{G(P, K)}$ is compact.

Lemma 3. *Let $W \subset T$ and $g|_W$ be the contraction of the map $g : P \times T \to L(E, T)$ on the set $P \times W$. If:*

1) *for $\forall y \in W$, the map g is uniformly continuous on $P \times \{y\}$ and the set $\overline{g(P, y)}$ is compact;*

2) *g satisfies the Lipschitz condition in the second argument on W,*

then the map $g|_W$ is Lagrange S-stable.

Proof. By virtue of Ascoli theorem it is sufficient to prove equicontinuity of the family of maps $\{g_t|_W \mid t \in S\}$ on each compact set from $P \times W$. Beforehand we shall prove that for an arbitrary compact set $K \subset W$ the map g is uniformly continuous on the set $P \times K$. Let $K \subset W$ be a compact set and the map g be non-uniformly continuous on the set $P \times K$. Then there is an $\varepsilon_0 > 0$ such that for an arbitrary natural i there are elements (t_1^i, k_1^i) and (t_2^i, k_2^i) in $P \times K$ for which the following relations are fulfilled

$$\|(t_1^i, k_1^i) - (t_2^i, k_2^i)\| < 1/i \tag{9}$$

and

$$\|g(t_1^i, k_1^i) - g(t_2^i, k_2^i)\| \geq \varepsilon_0. \tag{10}$$

Let L be a Lipschitz constant of g. From the relation (9) the relation $\|k_1^i - k_2^i\| < 1/i$ follows. Therefore by virtue of the compactness of K we may consider that $\lim_i k_1^i = \lim_i k_2^i = k \in K$. In that case for number $\varepsilon_0/(3L)$ there will be a natural number i_1 such that for arbitrary $i > i_1$ the following relations are fulfilled

$$\|k_1^i - k\| < \varepsilon_0/(3L), \quad \|k_2^i - k\| < \varepsilon_0/(3L). \tag{11}$$

From the relation (9) also follows the relation

$$\|t_1^i - t_2^i\| < 1/i. \tag{12}$$

Since the map g is uniformly continuous on $P \times \{k\}$, then for $\varepsilon_0/3$ there is a number $\delta > 0$ such that for $\forall (t_1, k), (t_2, k) \in P \times \{k\}$ from the relation $\|(t_1, k) - (t_2, k)\| < \delta$ the following relation follows

$$\|g(t_1, k) - g(t_2, k)\| < \varepsilon_0/3. \tag{13}$$

Let a natural number i_2 be such that $1/i_2 < \delta$. Then for an arbitrary $i > i_2$, by virtue of the relations (12) and (13), the following relation is fulfilled

$$\|g(t_1^i, k) - g(t_2^i, k)\| < \varepsilon_0/3. \tag{14}$$

Let $i_0 = \max(i_1, i_2)$. For $i > i_0$, the relations (11) and (14) are simultaneously fulfilled. Therefore for $i > i_0$

$$\begin{aligned}
&\|g(t_1^i, k_1^i) - g(t_2^i, k_2^i)\| \leq \|g(t_1^i, k_1^i) - g(t_1^i, k)\| + \|g(t_1^i, k) - g(t_2^i, k)\| + \\
&+\|g(t_2^i, k) - g(t_2^i, k_2^i)\| < L\|k_1^i - k\| + \varepsilon_0/3 + L\|k_2^i - k\| < \\
&< L\varepsilon_0/(3L) + \varepsilon_0/3 + L\varepsilon_0/(3L) = \varepsilon_0,
\end{aligned}$$

i.e. $\|g(t_1^i, k_1^i) - g(t_2^i, k_2^i)\| < \varepsilon_0$. The obtained relation contradicts the relation (10). So, the map g is uniformly continuous on the set $P \times K$ for an arbitrary compact set $K \subset W$. Let D be an arbitrary compact set from $P \times W$ and the compact set $M \times K \subset P \times W$ is such that $D \subset M \times K$. And let $\varepsilon > 0$, $t \in S$, $(m, k) \in D$ and δ be a number corresponding to the number ε by virtue of an uniform continuity of g on the set $P \times K$. We shall assume that $(m_1, k_1) \in D$ and $\|(m_1, k_1) - (m, k)\| < \delta$. Then $\|(t + m_1, k_1) - (t + m, k)\| < \delta$. Therefore $\|g_t(m_1, k_1) - g_t(m, k)\| < \varepsilon$. The proof means equicontinuity of the family of maps $\overline{\{g_t|_W \mid t \in S\}}$ at the point $(m, k) \in D$. In that case the family of maps $\overline{\{g_t|_W \mid t \in S\}}$ is equicontinuous on D.

Lemma 4. *If the map $g : P \times T \to L(E, T)$ satisfies the Lipschitz condition on the second argument in a set W from T with the Lipschitz constant L, then any map from $\overline{g|_W S}$ satisfies the Lipschitz condition in the second argument on W with the Lipschitz constant L.*

Proof. The proof is obvious.

Let's introduce the concept of concordance of maps and we shall describe shortly its purpose.

Let X and Y be some spaces of maps on which the operation of the semigroup S is defined with the help of shift σ, $\mathcal{U}[X]$ and $\mathcal{U}[Y]$ be uniform structures of spaces X and Y respectively; $\varphi \in X$, $f \in Y$.

Definition 4. *We say that φ is S-concordant with f if for every index $\alpha \in \mathcal{U}[X]$ there is an index $\gamma \in \mathcal{U}[Y]$ such that $s \in S$ and $(f, f_s) \in \gamma$ implies $(\varphi, \varphi_s) \in \alpha$. We say that φ is uniformly S-concordant with f if for every index $\alpha \in \mathcal{U}[X]$ there is an index $\gamma \in \mathcal{U}[Y]$ such that $s, t \in S$ and $(f_t, f_s) \in \gamma$ imply $(\varphi_t, \varphi_s) \in \alpha$.*

The essence of the concept of S-concordance is that if φ is S-concordant with f and f has certain property of the recursiveness, then φ has this property of the recursiveness, too. Let's explain this in more details.

Let $[S]$ be some class of subsets from S, $f \in X$ (or $f \in Y$).

The map f is called $[S]$-recursive if for an arbitrary index $\alpha \in \mathcal{U}[X]$ there is a set $A \in [S]$ for which $(f, f_a) \in \alpha$, for all $a \in A$. The set fS is called $[S]$-recursive if for an arbitrary index $\alpha \in \mathcal{U}[X]$ there is a set $A \in [S]$ for which $(f_s, f_{s+a}) \in \alpha$, for all $s \in S$ and $a \in A$.

And let $\varphi \in X$, $f \in Y$. Then: 1) If φ is S-concordant with f and the map f is $[S]$-recursive, then the map φ is $[S]$-recursive, too. 2) If φ is uniformly S-concordant with f and the set fS is $[S]$-recursive, then the set φS is $[S]$-recursive, too.

The last definitions and proposition are well concordant with the facts known for dynamic systems [1].

As concrete definitions of $[S]$-recursivenesses various types of Poisson stability of maps, in particular, Poisson SQ-stability, Poisson $S\mathcal{P}$-stability, SQ-recurrentness in sense of Birkhoff, SQ-almost periodicity in sense of Bohr (here Q is a subset from S, $\mathcal{P}$ is some family of subset of S). We shall give corresponding definitions, for $\varphi \in C(P, T)$ (for more details see [1, 6]).

A map φ is Poisson SQ-stable if for arbitrary $\varepsilon > 0$, a compact set A from P and arbitrary $q \in Q$ there is $p \in Q$ for which $\|\varphi(a) - \varphi(a+q+p)\| < \varepsilon$, for all $a \in A$.

If a map is Poisson SQ-stable for arbitrary $Q \in \mathcal{P}$, then it is called as Poisson $S\mathcal{P}$-stable.

A map φ is SQ-recurrent in sense of Birkhoff if for arbitrary $\varepsilon > 0$ and a compact set A from P there is a compact set $K \subset Q$ such that for $\forall q \in Q$ $\exists k \in K$ for which $\|\varphi(a) - \varphi(a+q+k)\| < \varepsilon$, for all $a \in A$.

A map φ is SQ-almost periodic in sense of Bohr if for arbitrary $\varepsilon > 0$ and a compact set A from P there is a compact set $K \subset Q$ such that for $\forall q \in Q \;\; \exists k \in K$ for which $\|\varphi(s+a) - \varphi(s+a+q+k)\| < \varepsilon$, for all $s \in S$ and $a \in A$.

Thus, if it is established that some solution φ of the equations is S-concordant (uniformly S-concordant) with the right-hand side f of this equation and the map f is Poisson SQ-stable, or Poisson $S\mathcal{P}$-stable, or SQ-recurrent in sense of Birkhoff (SQ-almost periodic in sense of Bohr), then the solution φ is respectively Poisson SQ-stable, or Poisson $S\mathcal{P}$-stable, or SQ-recurrent in sense of Birkhoff (SQ-almost periodic in sense of Bohr), too.

Definition 5. *Let $a \in C(P, L(E, L(T,T)))$. The total differential equation*

$$y'h = a(t)hy \quad (h \in E) \tag{15}$$

is called weakly regular (regular) of type 1 with the constant $r > 0$ if for an arbitrary bounded map $b \in C(P, L(E,T))$ the equation

$$y'h = a(t)hy + b(t)h \quad (h \in E) \tag{16}$$

has a compact (unique compact) solution $x \in C(P,T)$. In addition, for an arbitrary compact solution x of the equation (16) is valid the estimation

$$\sup_{t \in P} \|x(t)\| \leq r \cdot \sup_{t \in P} \|b(t)\|. \tag{17}$$

Theorem 6. *Let for the equation (1) the following conditions be fulfilled:*

1) *the equation (15) is weakly regularly of type 1 with the constant r;*
2) *the map b is bounded;*
3) *there is $t_0 \in T$ for which the set $g(P, t_0)$ is bounded;*
4) *the map g satisfies the Lipschitz condition in the second argument with the Lipschitz constant L.*

Then for an arbitrary λ, $|\lambda| < 1/(rL)$, the equation (1) has a compact solution $x_\lambda \in C(P,T)$ and for it the estimation is valid

$$\sup_{t \in P} \|x_\lambda(t)\| \leq r \cdot \sup_{t \in P} \|b(t) + \lambda g(t, x_\lambda(t))\|. \tag{18}$$

If in addition to the conditions 1) – 4) of our theorem the following condition is fulfilled:

5) *the map a is bounded and the set $\overline{g(P,y)}$ is compact for $\forall y \in T$,*

then the solution x_λ is Lagrange S-stable.

If in addition to conditions 1) – 4) of our theorem the following conditions are fulfilled:

6) *the equation* (15) *is regular of type of* 1 *with the constant* r*;*

7) *the space* E *is quasicomplete,* P *is a connected convex set if the space* E *is finite-dimensional and* $P = E$ *if the space* E *is infinite-dimensional,*

then the solution x_λ *is* S*-concordant with* (a, b, g)*.*

Proof. Let the conditions 1) – 4) of our theorem be fulfilled. We shall use Theorem 3 for: X is the space of bounded maps from $C(P, L(E, T))$ with the norm *sup*, Y is the space of compact maps from $C(P, T)$ with the norm *sup*, q is the map that to every $p \in X$ puts in correspondence the set of all solutions of the equation

$$y'h = a(t)hy + p(t)h \quad (h \in E)$$

contained in Y. As F we shall take the map $Y \times P \to T$ according to the rule $F(\varphi, t) = \varphi(t)$, and as f we shall take the map g. Since the equation (15) is weakly regular of type 1 with the constant r, then the map q is weakly r-regular. It is directly checked that Y is a complete space.

Since for $\forall x \in Y$ and $\forall t \in P$

$$\begin{aligned} &\|g(t, x(t))\| \le \|g(t, x(t)) - g(t, t_0)\| + \|g(t, t_0)\| \le \\ &\le L(\sup_{s \in P} \|x(s)\| + \|t_0\|) + \sup_{s \in P} \|g(s, t_0)\| \equiv l \in R, \end{aligned}$$

i.e. $\|g(t, x(t))\| \le l$, then the map g^x is bounded. Therefore the map g is F-admissible.

From the condition 4) of our theorem the condition 2) of Theorem 3 follows.

According to Theorem 3 for an arbitrary λ, $|\lambda| < 1/(rL)$, there is an $x_\lambda \in q(b + \lambda g^{x_\lambda})$. By the definition of the maps q and g^{x_λ} the map x_λ is compact and satisfies the equality $x'_\lambda(t)h = a(t)hx_\lambda(t) + (b(t) + \lambda g(t, x_\lambda(t))h$ for an arbitrary $t \in P$, $h \in E$. It also means that x_λ is a compact solution of the equation (1). The estimation (18) follows from the estimation for x_λ from Theorem 3.

Let's assume that the condition 5) of our theorem is also fulfilled, and we shall prove the Lagrange S-stability of the solution x_λ. Let's designate by g^* the map $P \times T \to L(E, T)$ according to the rule: $g^*(t, y)h = a(t)hy + (b(t) + \lambda g(t, y))h$ $((t, y) \in P \times T,\ h \in E)$. And let $K \subset T$ be a compact set. By Lemma 2 the set $\overline{g(P, K)}$ is compact. Since for $\forall (t, y) \in P \times K$

$$\|g^*(t, y)\| = \sup_{\|h\|=1} \|g^*(t, y)h\| \le \|a(t)\|\|y\| + \|b(t)\| + |\lambda|\|g(t, y)\|,$$

then the map g^* is bounded on the set $P \times K$. In that case the solution x_λ is Lagrange S-stable by Proposition 1.

Let's assume that the conditions 1) – 4) and 6) – 7) of our theorem are fulfilled. Then x_λ is a unique compact solution of the equation (1). Suppose that x_λ is not S-concordant with (a, b, g). Then there is an index α of the uniform structure of the space $C(P, T)$ such that for an arbitrary index γ of the uniform structure of the space $C(P, L(E, L(T, T))) \times C(P, L(E, T)) \times C(P \times T, L(E, T))$ there is an element $s_\gamma \in S$ such that

$$((a, b, g), (a_{s_\gamma}, b_{s_\gamma}, g_{s_\gamma})) \in \gamma \tag{19}$$

and

$$(x_\lambda, (x_\lambda)_{s_\gamma}) \notin \alpha. \tag{20}$$

From the relation (19) it follows that $\lim_\gamma (a_{s_\gamma}, b_{s_\gamma}, g_{s_\gamma}) = (a, b, g)$. By virtue of Lemma 1 from the net $\{(x_\lambda)_{s_\gamma}\}$ it is possible to single out a subnet $\{(x_\lambda)_{s_\beta}\}$ converging to some compact solution ψ of the equation (1). According to the proved above $\psi = x_\lambda$, therefore $\lim_\beta (x_\lambda)_{s_\beta} = x_\lambda$. The obtained relation contradicts (20). The contradiction says that the solution x_λ is S-concordant with (a, b, g).

Theorem 7. *Let E be a Banach space, $P = E$ and for the equation* (1) *the following conditions are fulfilled:*

1) *for $\forall t \in E$, $a(t) = a$ is a permutable operator such that $(Sp\ a)e$ does not intersect the imaginary axis of the complex plane for some vector $e \in E$ of the unit norm;*

2) *the map b is bounded, continuously differentiable and $\wedge\{ahb(t)k - b'(t)kh\} = 0$ for $\forall h, k, t \in E$;*

3) *for an arbitrary bounded map $y \in C(E, T)$ the map g^y is continuously differentiable and $\wedge\{ahg^y(t)k - (g^y)'(t)kh\} = 0$ for $\forall h, k, t \in E$ (here and further, g^y is the map according to the rule $g^y(t) = g(t, y(t))$ for $\forall t \in E$);*

4) *the map g satisfies the Lipschitz condition in the second argument with the Lipschitz constant L;*

5) *there is $t_0 \in T$ for which the set $g(E, t_0)$ is bounded.*

Then for an arbitrary λ, $|\lambda| < 1/(rL)$, where r is the constant from the Example 2 of regular maps, the equation (1) *has a unique bounded solution $x_\lambda \in C(E, T)$ and the estimation also is valid*

$$\sup_{t \in E} \|x_\lambda(t)\| \leq r \cdot \sup_{t \in E} \|b(t) + \lambda g(t, x_\lambda(t))\|. \tag{21}$$

If in addition to the conditions 1) – 5) *of our theorem the set $\overline{g(E, y)}$ is compact for $\forall y \in T$, then the solution x_λ is Lagrange E-stable.*

If in addition to the conditions 1) – 5) *of our theorem $T = R^m$, then the solution x_λ is E-concordant with (a, b, g).*

Proof. We relate the equation (1) to the r-regular map from the Example 2 of regular maps, and the proof of the theorem is done by the proof scheme of Theorem 6 taking into account that for $T = R^m$, the solution x_λ is compact.

Theorem 8. *Let for the equation* (1) *the following conditions be fulfilled:*

1) *the equation* (15) *is weakly regular of type* 1 *with the constant* r*;*

2) *the map* b *is bounded;*

3) x *is a compact solution of the equation* (16) *and* $V^{\delta}_{x(P)}$ *is a closed* δ*-neighbourhood of the set* $x(P)$ $(\delta > 0)$*;*

4) *the map* g *satisfies the Lipschitz condition in the second argument on* $V^{\delta}_{x(P)}$ *with the Lipschitz constant* L*;*

5) *for some* $t_0 \in P$, $\sup_{t\in P} \|g(t, x(t_0))\| = l \in R$.

Then for an arbitrary λ, $|\lambda| < \lambda_0 = \delta/(r(L(d+\delta)+l))$, *where* d *is the diameter of the set* $x(P)$, *the equation* (1) *has a compact solution* $x_\lambda : P \to V^{\delta}_{x(P)}$ *and for it the estimation* (18) *is valid. In addition, if* $\lim_{i\to\infty} \lambda_i = 0$, *then* $\lim_{i\to\infty} x_{\lambda_i} = x$ *uniformly on* P.

If in addition to the conditions 1) – 5) *of our theorem the map* a *is bounded, then the solution* x_λ *is Lagrange* S*-stable.*

If in addition to the conditions 1) – 5) *of our theorem the following conditions are fulfilled:*

6) *the equation* (15) *is regular of type* 1 *with the constant* r*;*

7) *the space* E *is quasicomplete,* P *is a connected convex set if the space* E *is finite-dimensional and* $P = E$ *if the space* E *is infinite-dimensional.*

Then the solution x_λ *is* S*-concordant with* (a, b, g).

Proof. Let the conditions 1) – 5) of our theorem be fulfilled. We shall use Theorem 5 for: X is the space of bounded maps from $C(P, L(E,T))$ with the norm *sup*, Y is the space of compact maps from $C(P,T)$ with the norm *sup*, q is the map that to every $p \in X$ puts in correspondence the set of all solutions of the equation

$$y'h = a(t)hy + p(t)h \quad (h \in E)$$

contained in Y. As F we shall take the map $Y \times P \to T$ according to the rule $F(\varphi, t) = \varphi(t)$ and as f the map g. Since the equation (15) is weakly regular of type 1 with the constant r, then the map q is weakly r-regular. Since for $\forall y \in Y$

$$\begin{aligned}&\sup_{t\in P} \|g(t, y(t))\| \le \sup_{t\in P} \|g(t, y(t)) - g(t, x(t_0))\| + \sup_{t\in P} \|g(t, x(t_0))\| \le \\ &\le L(\sup_{t\in P} \|y(t)\| + \|x(t_0)\|) + \sup_{t\in P} \|g(t, x(t_0))\| \in R,\end{aligned}$$

i.s. $\sup_{t\in P} \|g(t, y(t))\| \in R$, then the map g^y is bounded. Therefore the map g is F-admissible.

Let $A_x^\delta = \{y \mid y \in Y \text{ and } y(P) \subset V_{x(P)}^\delta\}$. It is clear that the set A_x^δ is closed, hence, it is complete, as closed subset of a complete space Y. Since

$$\sup_{(t,s)\in P\times P} \|g(t,x(s))\| \leq \sup_{(t,s)\in P\times P} \|g(t,x(s)) - g(t,x(t_0))\| +$$
$$+ \sup_{t\in P} \|g(t,x(t_0))\| \leq L \sup_{t\in P} \|x(t) - x(t_0)\| + l \leq Ld + l,$$

i.e. $\sup_{t\in P} \|g(t,x(t))\| \leq Ld + l$, then taking as l_1 in the condition 4) of Theorem 5 the number $Ld + l$, we shall receive that our number λ_0 coincides with λ_0 from Theorem 5. According to Theorem 5 for an arbitrary λ, $|\lambda| < \lambda_0$, there exists $x_\lambda \in q(b + \lambda g^{x_\lambda}) \cap A_x^\delta$. By the definition of maps q, g^{x_λ} and of set A_x^δ the map $x_\lambda : P \to V_{x(P)}^\delta$ is compact and satisfies the equality $x'_\lambda(t)h = a(t)hx_\lambda(t) + (b(t) + \lambda g(t,x_\lambda(t))h$ for an arbitrary $t \in P$, $h \in E$. It also means that $x_\lambda : P \to V_{x(P)}^\delta$ is a compact solution of the equation (1). The estimation (18) follows from the estimation of x_λ from Theorem 5.

Let's prove that if $\lim_{i\to\infty} \lambda_i = 0$, then $\lim_{i\to\infty} x_{\lambda_i} = x$ is uniform on P. Let $\varepsilon > 0$ be an arbitrary number. Since $\lim_{i\to\infty} \lambda_i = 0$, then there is a number i_0 such that for all $i > i_0$ $|\lambda_i| < \varepsilon/(r(Ld + l))$. Let $i > i_0$. Because $x_{\lambda_i} - x$ is a compact solution of the equation

$$y'h = a(t)hy + \lambda_i g(t, x_{\lambda_i}(t))h \quad (h \in E)$$

with a bounded map $\lambda_i g(t, x_{\lambda_i}(t))$ $(t \in P)$, then using the conditions 1) and 5) of our theorem, we have

$$\sup_{t\in P} \|x_{\lambda_i}(t) - x(t)\| \leq r|\lambda_i| \sup_{t\in P} \|g(t, x_{\lambda_i}(t))\| \leq r|\lambda_i|(Ld + l) < \varepsilon,$$

i.e. $\sup_{t\in P} \|x_{\lambda_i}(t) - x(t)\| < \varepsilon$. The proof also means that $\lim_{i\to\infty} x_{\lambda_i} = x$ is uniform on P.

Let's assume that the map a is bounded and we shall prove that x_λ is Lagrange S-stable. Let's designate by g^* the map $P \times T \to L(E,T)$ by the rule: $g^*(t,y)h = a(t)hy + (b(t) + \lambda g(t,y))h$ $((t,y) \in P \times T,\ h \in E)$. Since $\overline{x_\lambda(P)} \subset V_{x(P)}^\delta$ and for $\forall y \in \overline{x_\lambda(P)}$

$$\sup_{t\in P} \|g^*(t,y)\| = \sup_{t\in P} \sup_{\|h\|=1} \|g^*(t,y)h\| \leq \sup_{t\in P}(\|a(t)\|\|y\| + \|b(t)\| +$$
$$+ |\lambda|\|g(t,y)\|) \leq \sup_{t\in P}(\|a(t)\|\|y\| + \|b(t)\|) + |\lambda|(Ld + l) = m \in R,$$

i.e. $\sup_{t\in P} \|g^*(t,y)\| \leq m \in R$, then the map g^* is bounded on the set $P \times \overline{x_\lambda(P)}$. In that case the solution x_λ is Lagrange S-stable according to Proposition 1.

If the conditions 1) – 7) of our theorem are fulfilled, then the S-concordance of x_λ with (a, b, g) is proved as in Theorem 6.

Theorem 9. *Let E be a Banach space, $P = E$ and for the equation* (1) *the following conditions are fulfilled:*

1) *for* $\forall t \in E$ $a(t) = a$ *is an operator such that* $(Sp\ a)e$ *does not intersect the imaginary axis of the complex plane for some vector* $e \in E$ *of the unit norm;*

2) *the map* b *is bounded, continuously differentiable and* $\wedge\{ahb(t)k - b'(t)kh\} = 0$ *for arbitrary* $h, k, t \in E$*;*

3) *for an arbitrary bounded map* $y \in C(E,T)$*, the map* g^y *is continuously differentiable and* $\wedge\{ahg^y(t)k - (g^y)'(t)kh\} = 0$ *for arbitrary* $h, k, t \in E$*;*

4) x *is a bounded solution of the equation* (16) *and* $V^{\delta}_{x(E)}$ *is a closed* δ*-neighbourhood of the set* $x(E)$ $(\delta > 0)$*;*

5) *the map* g *satisfies the Lipschitz condition in the second argument on* $V^{\delta}_{x(E)}$ *with the Lipschitz constant* L*;*

6) *for some* $t_0 \in E$ $\sup_{t\in E} \|g(t, x(t_0))\| = l \in R$*;*

7) $\lambda_0 = \delta/(r(L(d+\delta)+l))$*, where* r *is the constant from the Example 2 of regular maps and* d *is the diameter of the set* $x(E)$*.*

Then for an arbitrary λ, $|\lambda| < \lambda_0$*, the equation* (1) *has a unique bounded solution* $x_\lambda : E \to V^{\delta}_{x(E)}$*. This solution is Lagrange* E*-stable and for it the estimation* (21) *is valid. Besides, if* $\lim_{i\to\infty} \lambda_i = 0$*, then* $\lim_{i\to\infty} x_{\lambda_i} = x$ *is uniform on* E*.*

If in addition to the conditions 1) – 7) *of our theorem* $T = R^m$*, then the solution* x_λ *is* E*-concordant with* (a, b, g)*.*

Proof. We connect with the equation (1) the r-regular map from the Example 2 of regular maps, and the proof of the theorem is done by the proof scheme of Theorem 8 taking into account that for $T = R^m$ the solution x_λ is compact.

Alongside with the equation (1) we also consider the limiting equations

$$y'h = A(t)hy + (B(t) + \lambda G(t, y))h \quad (h \in E), \tag{22}$$

where $A \in \overline{aS}$, $B \in \overline{bS}$, $G \in \overline{gS}$.

Definition 6. *The equation* (15) *is called weakly regular* (*regular*) *of type* 2 *with the constant* $r > 0$ *if for an arbitrary* $A \in \overline{aS}$ *and an arbitrary bounded map* $b \in C(P, L(E,T))$ *the equation*

$$y'h = A(t)hy + b(t)h \quad (h \in E). \tag{23}$$

has a compact (*unique compact*) *solution* $x \in C(P,T)$*. In addition, for an arbitrary compact solution* x *of the equation* (23) *the estimation* (17) *takes place.*

Theorem 10. *Let for the equation* (1) *the following conditions be fulfilled:*

1) *the equation* (15) *is weakly regular of type* 2 *with the constant* r*;*

2) *the map* b *is bounded;*

3) *there exists* $t_0 \in T$ *for which the set* $g(P, t_0)$ *is bounded;*

4) *the map* g *satisfies the Lipschitz condition in the second argument with the Lipschitz constant* L.

Then for an arbitrary λ, $|\lambda| < 1/(rL)$, *and an arbitrary triple* $(A, B, G) \in \overline{(a, b, g)S}$, *the equation* (22) *has a compact solution* x_λ *and for it the estimation is valid*

$$\sup_{t \in P} \|x_\lambda(t)\| \leq r \cdot \sup_{t \in P} \|B(t) + \lambda G(t, x_\lambda(t))\|. \tag{24}$$

If in addition to the conditions 1) – 4) *of our theorem the following condition is fulfilled:*

5) *the map* a *is bounded and the set* $\overline{g(P, y)}$ *is compact for* $\forall y \in T$,

then the solution x_λ *is Lagrange* S*-stable.*

If in addition to the conditions 1) – 4) *of our theorem the following conditions are fulfilled:*

6) *the equation* (15) *is regular of type* 2 *with the constant* r*;*

7) *the space* E *is quasicomplete,* P *is a connected convex set if the space* E *is finite-dimensional and* $P = E$ *if the space* E *is infinite-dimensional,*

then the solution x_λ *is* S*-concordant with* (A, B, G).

If in addition to the conditions 1) – 4) *and* 6) – 7) *of our theorem the following conditions are fulfilled:*

8) *the map* (a, b) *is Lagrange* S*-stable;*

9) *for* $\forall y \in T$, *the map* g *is uniformly continuous on the set* $P \times \{y\}$ *and the set* $\overline{g(P, y)}$ *is compact,*

then the solution x_λ *is uniformly* S*-concordant with* (A, B, G).

Proof. Let the conditions 1) – 4) of theorem be fulfilled and $(A, B, G) \in \overline{(a, b, g)S}$. Since in the conditions of our theorem the map B is bounded, then the set $G(P, t_0)$ is bounded and the map G satisfies, according to Lemma 4, the Lipschitz condition in the second argument with the constant L, then the conclusion of our theorem follows from Theorem 6. If the conditions 1) – 5) of the theorem are valid, then the conclusion of our theorem follows from Theorem 6, so in our case A is a bounded map and the set $\overline{G(P, y)}$ is compact for an arbitrary $y \in T$.

If the conditions 1) – 4) and 6) – 7) of our theorem are fulfilled, then the conclusion of our theorem follows from Theorem 6.

Let the conditions 1) – 4) and 6) – 9) of our theorem be fulfilled. According to Lemma 3 the map g is Lagrange S-stable. Therefore the maps (a, b, g) and $(A, B, G) \in \overline{(a, b, g)S}$ are Lagrange S-stable, too. If the equation (15) is regular of type 2 with the constant r, then each equation (22) for an arbitrary λ, $|\lambda| <$

$1/(rL)$, has a unique compact solution x_λ. Suppose that x_λ is not uniformly S-concordant with (A,B,G). Then there exists an index α of the uniform structure of the space $C(P,T)$ such that for an arbitrary index β the uniform structure of the space $C(P,L(E,L(T,T)))\times C(P,L(E,T))\times C(P\times T,L(E,T))$ there are elements $s_\beta, t_\beta \in S$ such that

$$((A_{t_\beta},B_{t_\beta},G_{t_\beta}),(A_{s_\beta},B_{s_\beta},G_{s_\beta}))\in\beta \tag{25}$$

and

$$((x_\lambda)_{t_\beta},(x_\lambda)_{s_\beta})\notin\alpha. \tag{26}$$

By virtue of the compactness of the set $\overline{(A,B,G)S}$ we may consider that

$$\lim_\beta (A_{t_\beta},B_{t_\beta},G_{t_\beta})=(A_1,B_1,G_1)$$

and

$$\lim_\beta (A_{s_\beta},B_{s_\beta},G_{s_\beta})=(A_2,B_2,G_2).$$

In this case from the relation (25) we obtain that $(A_1,B_1,G_1)=(A_2,B_2,G_2)=(A_0,B_0,G_0)$. By virtue of Lemma 1 we suppose that $\lim_\beta (x_\lambda)_{t_\beta}=\psi_1$ and $\lim_\beta (x_\lambda)_{s_\beta}=\psi_2$, in addition, ψ_1 and ψ_2 are solutions of the equation

$$y'h=A_0(t)hy+(B_0(t)+\lambda G_0(t,y))h \quad (h\in E).$$

Since this equation has a unique compact solution then $\psi_1=\psi_2$, that contradicts (26). The contradiction indicates that the solution x_λ is uniformly S-concordant with (A,B,G).

Theorem 11. *Let for the equation* (1) *the following conditions be fulfilled:*

1) *the equation* (15) *is weakly regular of type* 2 *with the constant* r*;*
2) *the map* b *is bounded;*
3) x *is a compact solution of the equation* (16) *and* $V^\delta_{x(P)}$ *is a closed* δ*-neighbourhood of* $x(P)$ $(\delta>0)$*;*
4) *the map* g *satisfies the Lipschitz condition in the second argument from* $V^\delta_{x(P)}$ *with the Lipschitz constant* L*;*
5) *for some* $t_0\in P$, $\sup_{t\in P}\|g(t,x(t_0))\|=l\in R$.

Then for an arbitrary λ, $|\lambda|<\lambda_0=\delta/r(L(d+\delta)+l)$, *where* d *is the diameter of the set* $x(P)$, *and for* $\forall(A,B,G)\in\overline{(a,b,g|_{V^\delta_{x(P)}})S}$, *where* $g|_{V^\delta_{x(P)}}$ *is the contraction of the map* g *on the set* $P\times V^\delta_{x(P)}$, *the equation* (22) *has a compact solution* $x_\lambda : P\to V^\delta_{x(P)}$ *and for it the estimation* (24) *is valid. Besides if* $\lim_{i\to\infty}\lambda_i=0$, *then* $\lim_{i\to\infty}x_{\lambda_i}=z$ *is uniform on* P *for some compact solution* z *of the equation*

$$y'h=A(t)hy+B(t)h \quad (h\in E)$$

(z exists by virtue of the condition 1) *of our theorem).*

If in addition to the conditions 1) – 5) *of our theorem the map a is bounded, then the solution x_λ is Lagrange S-stable.*

If in addition to the conditions 1) – 5) *of our theorem the following conditions are fulfilled:*

6) *the equation* (15) *is regular of type* 2 *with the constant r;*

7) *the space E is quasicomplete, P is a connected convex set if the space E is finite-dimensional and $P = E$ if the space E is infinite-dimensional,*

then the solution x_λ is S-concordant with (A, B, G).

If in addition to the conditions 1) – 7) *of our theorem the map (a, b) is Lagrange S-stable, for an arbitrary $y \in V^{\delta}_{x(P)}$ the map g is uniformly continuous on the set $P \times \{y\}$ and the set $\overline{g(P, y)}$ is compact, then the solution x_λ is uniformly S-concordant with (A, B, G).*

Proof. The proof is similar to the proof of Theorem 10 using Theorem 8 instead of Theorem 6.

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Singularly perturbed Cauchy problem for abstract linear differential equations of second order in Hilbert spaces

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Abstract. We study the behavior of solutions to the problem

$$\begin{cases} \varepsilon\Big(u_\varepsilon''(t) + A_1 u_\varepsilon(t)\Big) + u_\varepsilon'(t) + A_0 u_\varepsilon(t) = f(t), \quad t > 0, \\ u_\varepsilon(0) = u_0, \quad u_\varepsilon'(0) = u_1, \end{cases}$$

in the Hilbert space H as $\varepsilon \mapsto 0$, where A_1 and A_0 are two linear selfadjoint operators.

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1 Introduction

Let H be a real Hilbert space endowed with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$. Let $V \subset H$ be a real Hilbert space which is endowed with norm $||\cdot||$ such that the inclusion is dense and continuous. Let $V = V_0 \cap V_1$, and $A_i : D(A_i) = V_i \to H$, $i = 0, 1$, be two linear selfadjoint operators such that

$$\Big((A_0 + \varepsilon A_1)u, u\Big) \geq \gamma ||u||^2, \quad u \in V, \quad \gamma > 0, \tag{1}$$

for some $\varepsilon \ll 1$ and εA_1 generates a C_0- semigroup $\{S(t,\varepsilon), t \geq 0\}$ with the following two properties:

$$A_0 S(t,\varepsilon)u = S(t,\varepsilon)A_0 u, \forall u \in V. \tag{2}$$

$$\exists \delta > 0 : |S(t,\varepsilon)u| \geq \delta |u|, u \in V. \tag{3}$$

Consider the following Cauchy problem, which will be called (P_ε):

$$\begin{cases} \varepsilon\Big(u_\varepsilon''(t) + A_1 u_\varepsilon(t)\Big) + u_\varepsilon'(t) + A_0 u_\varepsilon(t) = f(t), \quad t > 0, \\ u_\varepsilon(0) = u_0, \quad u_\varepsilon'(0) = u_1, \end{cases}$$

where $\varepsilon > 0$ is a small parameter, $u, f : [0,\infty) \to H$. We will investigate the behavior of solutions $u_\varepsilon(t)$ to the perturbed system (P_ε) as $\varepsilon \to 0$. We will establish a relationship between solutions to the problem (P_ε) and the corresponding solutions to the following unperturbed system, which will be called (P_0):

$$\begin{cases} v'(t) + A_0 v(t) = f(t), \quad t > 0, \\ v(0) = u_0. \end{cases}$$

2 A *priori* estimates for solutions to the problems (P_ε) and (P_0)

In this section we remind the existence theorems for the solutions to the problems (P_ε) and (P_0) and give some a *priori* estimations for them.

Definition 1. *We say a function $u \in L^2(0,T,V)$, with $u' \in L^2(0,T,V')$ is a solution of (P_0) if*

$$\langle u', v\rangle + (A_0 u, v) = (f, v)$$

for each $v\in V$ and a.e. time $0 < t < T$, and

$$u(0) = u_0.$$

Definition 2. *We say a function $u \in L^2(0,T,V)$, with $u' \in L^2(0,T,H)$ and $u'' \in L^2(0,T,V')$ is a solution of (P_ε) if*

$$\varepsilon\langle u'', v\rangle + \varepsilon(A_1 u, v) + (u', v) + (A_0 u, v) = (f, v)$$

for each $v\in V$ and a.e. time $0 < t < T$, and

$$u(0) = u_0, \quad u'(0) = u_1,$$

where $\langle,\rangle$ express the pairing between H and H'.

Theorem A [1]. *Let $T > 0$. If condition (1) is fulfilled, $f \in W^{1,1}(0,T;H)$, $u_0 \in V$, then there exists a unique solution $v \in W^{1,\infty}(0,T;H)$ of the problem (P_0) such that*

$$|v(t)| + |v'(t)| \le C(T, u_0, f, \gamma), \quad t \in [0,T].$$

Theorem B [1, 2]. *Let $T > 0$. If condition (1) is fulfilled, $f \in W^{1,1}(0,T;H)$, $u_0 \in V, u_1 \in H$, then there exists a unique solution of the problem (P_ε) such that $u_\varepsilon \in C(0,T;V)$, $u'_\varepsilon \in C(0,T;H) \cap L^\infty(0,T;V)$, $u''_\varepsilon \in L^\infty(0,T;H)$. Moreover, for u the following estimate*

$$|u_\varepsilon(t)| + |u'_\varepsilon(t)| \le C(T, u_0, u_1, f, \gamma), \quad t \in [0,T],$$

is true.

3 Relation between solution to the problems (P_ε) and (P_0)

Now we are going to establish the relationship between the solution of the problem (P_ε) and the corresponding solutions of the problem (P_0). This relationship was inspired by the work [2]. To this end we defined the kernel of transformation which realizes this relationship.

For $\varepsilon > 0$ denote

$$K(t,\tau,\varepsilon) = \frac{1}{2\sqrt{\pi}\varepsilon}\Big(K_1(t,\tau,\varepsilon) + 3K_2(t,\tau,\varepsilon) - 2K_3(t,\tau,\varepsilon)\Big),$$

where

$$K_1(t,\tau,\varepsilon) = \exp\Big\{\frac{3t-2\tau}{4\varepsilon}\Big\}\lambda\Big(\frac{2t-\tau}{2\sqrt{\varepsilon t}}\Big),$$

$$K_2(t,\tau,\varepsilon) = \exp\Big\{\frac{3t+6\tau}{4\varepsilon}\Big\}\lambda\Big(\frac{2t+\tau}{2\sqrt{\varepsilon t}}\Big),$$

$$K_3(t,\tau,\varepsilon) = \exp\Big\{\frac{\tau}{\varepsilon}\Big\}\lambda\Big(\frac{t+\tau}{2\sqrt{\varepsilon t}}\Big), \quad \lambda(s) = \int_s^\infty e^{-\eta^2}\,d\eta.$$

The properties of kernel $K(t,\tau,\varepsilon)$ are collected in the following lemma.

Lemma 1 [2]. *The function $K(t,\tau,\varepsilon)$ possesses the following properties*:

- **(i)** *For any fixed $\varepsilon > 0$ $K \in C(\{t \geq 0\} \times \{\tau \geq 0\}) \cap C^\infty(R_+ \times R_+)$;*
- **(ii)** $K_t(t,\tau,\varepsilon) = \varepsilon K_{\tau\tau}(t,\tau,\varepsilon) - K_\tau(t,\tau,\varepsilon), \quad t > 0, \tau > 0;$
- **(iii)** $K(0,\tau,\varepsilon) = \dfrac{1}{2\varepsilon}\exp\Big\{-\dfrac{\tau}{2\varepsilon}\Big\}$, $\tau \geq 0$; $\quad \varepsilon K_\tau(t,0,\varepsilon) - K(t,0,\varepsilon) = 0$, $t \geq 0$;
- **(iv)** *For each fixed $t > 0$, $s,q \in \mathbb{N}$ there exist constants $C_1(s,q,t,\varepsilon) > 0$ and $C_2(s,q,t) > 0$ such that*

  $$|\partial_t^s \partial_\tau^q K(t,\tau,\varepsilon)| \leq C_1(s,q,t,\varepsilon)\exp\{-C_2(s,q,t)\tau/\varepsilon\}, \quad \tau > 0;$$

- **(v)** *Let ε be fixed, $0 < \varepsilon \ll 1$ and H is a Hilbert space. For any $\varphi : [0,\infty) \to H$ continuous on $[0,\infty)$ such that $|\varphi(t)| \leq M\exp\{Ct\}$, $t \geq 0$, the relation*

  $$\lim_{t\to 0}\int_0^\infty K(t,\tau,\varepsilon)\varphi(\tau)d\tau = \int_0^\infty e^{-\tau}\varphi(2\varepsilon\tau)d\tau,$$

  is valid in H;
- **(vi)** $\int_0^\infty K(t,\tau,\varepsilon)d\tau = 1, \quad t \geq 0; \quad K(t,\tau,\varepsilon) > 0, \quad t \geq 0, \quad \tau \geq 0;$
- **(vii)** *Let $f \in W^{1,\infty}(0,\infty;H)$. Then there exists a positive constant C such that*

  $$\Big|\Big|f(t) - \int_0^\infty K(t,\tau,\varepsilon)f(\tau)d\tau\Big|\Big|_H \leq C\sqrt{\varepsilon}(1+\sqrt{t})\|f'\|_{L^\infty(0,\infty;H)}, \quad t \geq 0;$$

- **(viii)** *There exists $C > 0$ such that*

  $$\int_0^t\int_0^\infty K(\tau,\theta,\varepsilon)\exp\Big\{-\frac{\theta}{\varepsilon}\Big\}d\theta d\tau \leq C\varepsilon, \quad t \geq 0, \quad \varepsilon > 0.$$

Denote by $\mathcal{K}(t,\tau,\varepsilon) = K(t,\tau,\varepsilon)S(t,\varepsilon)$.

Theorem 1. *Suppose that A_1 satisfies condition (2). If $f \in L^\infty(0,\infty;H)$ and $u_\varepsilon \in W^{2,\infty}(0,\infty;H) \cap L^\infty(0,\infty;V)$, is the solution to the problem (P_ε), then the function $v_{0\varepsilon}$ which is defined as*

$$v_{0\varepsilon}(t) = \int_0^\infty \mathcal{K}(t,\tau,\varepsilon)u_\varepsilon(\tau)d\tau$$

is the solution to the problem:

$$\begin{cases} v_{0\varepsilon}'(t) + A_0 v_{0\varepsilon}(t) = f_0(t,\varepsilon), \quad t > 0, \\ v_{0\varepsilon}(0) = \varphi_\varepsilon, \end{cases}$$

where

$$f_0(t,\varepsilon) = F_0(t,\varepsilon) + \int_0^\infty \mathcal{K}(t,\tau,\varepsilon) f(\tau) d\tau,$$

$$F_0(t,\varepsilon) = \frac{1}{\sqrt{\pi}}\Big[2\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big) - \lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big] S(t,\varepsilon) u_1,$$

$$\varphi_\varepsilon = \int_0^\infty e^{-\tau} u_\varepsilon(2\varepsilon\tau) d\tau.$$

Moreover, $v_{0\varepsilon} \in W^{2,\infty}(0,\infty;H) \cap L^\infty(0,\infty;V)$.

Proof. Integrating by parts,using the properties of C_0- semigroups, (ii), (iii) from Lemma 1 and (2) we get:

$$v_{0\varepsilon}'(t) = \Big(\int_0^\infty \mathcal{K}(t,\tau,\varepsilon) u_\varepsilon(\tau) d\tau\Big)' = \int_0^\infty K_t(t,\tau,\varepsilon) S(t,\varepsilon) u_\varepsilon(\tau) d\tau +$$

$$+ \int_0^\infty K(t,\tau,\varepsilon) S'(t,\varepsilon) u_\varepsilon(\tau) d\tau =$$

$$= \int_0^\infty [\varepsilon K_{\tau\tau}(t,\tau,\varepsilon) - K_\tau(t,\tau,\varepsilon)] S(t,\varepsilon) u_\varepsilon(\tau) d\tau +$$

$$+ \int_0^\infty K(t,\tau,\varepsilon) S'(t,\varepsilon) u_\varepsilon(\tau) d\tau = \varepsilon K_\tau(t,\tau,\varepsilon) S(t,\varepsilon) u_\varepsilon(\tau)|_0^\infty -$$

$$- \int_0^\infty \varepsilon K_\tau(t,\tau,\varepsilon) S(t,\varepsilon) u_\varepsilon'(\tau) d\tau - K(t,\tau,\varepsilon) S(t,\varepsilon) u_\varepsilon(\tau)|_0^\infty +$$

$$+ \int_0^\infty K(t,\tau,\varepsilon) S(t,\varepsilon) u_\varepsilon'(\tau) d\tau +$$

$$+ \int_0^\infty K(t,\tau,\varepsilon) S'(t,\varepsilon) u_\varepsilon(\tau) d\tau = [\varepsilon K_\tau(t,\tau,\varepsilon) - K(t,\tau,\varepsilon)] S(t,\varepsilon) u_\varepsilon(\tau)|_0^\infty -$$

$$- \varepsilon K(t,\tau,\varepsilon) S(t,\varepsilon) u_\varepsilon'(\tau)|_0^\infty + \int_0^\infty \varepsilon K(t,\tau,\varepsilon) S(t,\varepsilon) u_\varepsilon''(\tau) d\tau +$$

$$+ \int_0^\infty K(t,\tau,\varepsilon) S(t,\varepsilon) u_\varepsilon'(\tau) d\tau + \int_0^\infty K(t,\tau,\varepsilon) S'(t,\varepsilon) u_\varepsilon(\tau) d\tau =$$

$$= [\varepsilon K_\tau(t,0,\varepsilon) - K(t,0,\varepsilon)] S(t,\varepsilon) u_\varepsilon(0) + \varepsilon K(t,0,\varepsilon) S(t,\varepsilon) u_1 +$$

$$+ \int_0^\infty K(t,\tau,\varepsilon) S(t,\varepsilon)(\varepsilon u_\varepsilon''(\tau) + u'(\tau)) d\tau + \int_0^\infty K(t,\tau,\varepsilon) S'(t,\varepsilon) u_\varepsilon(\tau) d\tau =$$

$$= \varepsilon K(t,0,\varepsilon) S(t,\varepsilon) u_1 +$$

$$+\int_0^\infty K(t,\tau,\varepsilon)S(t,\varepsilon)(f(\tau)-A_0u_\varepsilon(\tau)-\varepsilon A_1u_\varepsilon(\tau))d\tau+$$

$$+\int_0^\infty K(t,\tau,\varepsilon)S'(t,\varepsilon)u_\varepsilon(\tau)d\tau=$$

$$=\varepsilon K(t,0,\varepsilon)S(t,\varepsilon)u_1+\int_0^\infty K(t,\tau,\varepsilon)S(t,\varepsilon)f(\tau)d\tau-A_0v_{0\varepsilon}(t)+$$

$$+\int_0^\infty K(t,\tau,\varepsilon)[S'(t,\varepsilon)u_\varepsilon(\tau)-\varepsilon A_1S(t,\varepsilon)u_\varepsilon(\tau)]d\tau=$$

$$=\varepsilon K(t,0,\varepsilon)S(t,\varepsilon)u_1+\int_0^\infty K(t,\tau,\varepsilon)S(t,\varepsilon)f(\tau)d\tau-A_0v_{0\varepsilon}(t)=$$

$$=F_0(t,\varepsilon)+\int_0^\infty K(t,\tau,\varepsilon)S(t,\varepsilon)f(\tau)d\tau-A_0v_{0\varepsilon}(t).$$

Thus $v_{0\varepsilon}(t)$ satisfies the equation from Theorem 1.

The initial condition is a simple consequence of property (iii) from Lemma 1.

Theorem 1 is proved.

4 The limit of solutions to the problem (P_ε) as $\varepsilon \mapsto 0$

In this section we will study the behavior of solutions to the problem (P_ε) as $\varepsilon \mapsto 0$.

Lemma 2. *Let A_0 and A_1 satisfy the conditions (1) and (2). If $u_0 \in V, u_1, f \in W^{1,\infty}(0,T;H)$ then the estimate:*

$$|S(t,\varepsilon)u_\varepsilon(t)-v_{0\varepsilon}(t)| \le C(T,u_0,u_1,f,\gamma,\gamma_1)\sqrt{\varepsilon}, \quad t\in[0,T],$$

is true.

Proof. According to the C_0-semigroup theory there exists a constant $\gamma_1>0$ such that

$$|S(t,\varepsilon)| \le \gamma_1(T,\varepsilon). \tag{4}$$

Using the last mentioned property of $S(t,\varepsilon)$, Theorem B and the **property(vii)** of Lemma 1 we can easy obtain:

$$|S(t,\varepsilon)u_\varepsilon(t)-\int_0^\infty \mathcal{K}(t,\tau,\varepsilon)u_\varepsilon(\tau)d\tau| \le |S(t,\varepsilon)||u_\varepsilon(t)-\int_0^\infty K(t,\tau,\varepsilon)u_\varepsilon(\tau)d\tau| \le$$

$$\le \gamma_1|u_\varepsilon(t)-\int_0^\infty K(t,\tau,\varepsilon)u_\varepsilon(\tau)d\tau| \le \widetilde{c}(1+\sqrt{t}) \parallel f' \parallel_{L^\infty(0,T:H)} \le C(T,u_0,u_1,f,\gamma,\gamma_1).$$

Lemma 2 is proved.

To prove the following result we need to remember an important inequality:

Lemma A [4]. *Let $\psi \in L^1(a,b)(-\infty < a < b < \infty)$ with $\psi \geq 0$ a. e. on (a,b) and c be a fixed real constant. If $h \in C[a,b]$ verifies*

$$\frac{1}{2}h^2(t) \leq \frac{1}{2}c^2 + \int_a^t \psi(s)h(s)ds, \forall t \in [a,b],$$

then

$$h(t) \leq |c| + \int_a^t \psi(s)ds, \forall t \in [a,b]$$

also holds.

Lemma 3. *Let the operators A_0, A_1 satisfy conditions (1)- (3).*
If $u_0, A_1 u_0 \in V, u_1 \in H, f, A_1 f \in W^{1,\infty}(0,T;H)$ then the estimate:

$$|S(t,\varepsilon)v(t) - v_{0\varepsilon}(t)| \leq C(T, u_0, u_1, f, \gamma, \gamma_1,)\sqrt{\varepsilon}, \quad t \in [0,T]$$

is true.

Proof. Let $v(t)$ be the solution to the problem (P_0). We will denote by $w(t) = S(t,\varepsilon)v(t)$. Thus

$$w'(t) = S'(t,\varepsilon)v(t) + S(t,\varepsilon)v'(t) = \varepsilon A_1 S(t,\varepsilon)v(t) +$$

$$+ S(t,\varepsilon)v'(t) = \varepsilon A_1 w(t) + S(t,\varepsilon)[f(t) - A_0 v(t)] =$$

$$= \varepsilon A_1 w(t) + S(t,\varepsilon)f(t) - A_0 S(t,\varepsilon)v(t) = \varepsilon A_1 w(t) + S(t,\varepsilon)f(t) - A_0 w(t),$$

and

$w(0) = S(0,\varepsilon)v(0) = v(0) = u_0$.

So we obtained the following Cauchy problem for $w(t)$:

$$\begin{cases} w'(t) + (A_0 - \varepsilon A_1)w(t) = S(t,\varepsilon)f(t), \\ w(0) = u_0. \end{cases}$$

To estimate $|S(t,\varepsilon)v(t) - v_{0\varepsilon}(t)|$ we denote by $R_\varepsilon(t)$=$w(t) - v_{0\varepsilon}(t)$. Then for $R_\varepsilon(t)$ we get the following Cauchy problem:

$$\begin{cases} R'_\varepsilon(t) + A_0 R_\varepsilon(t) = \varepsilon A_1 w(t) + S(t,\varepsilon)f(t) - f_0(t), \quad t > 0 \\ R_\varepsilon(0) = u_0 - \varphi_\varepsilon \end{cases}$$

Then taking scalar product of last equation with $R_\varepsilon(t)$ and integrating on [0,t], by Lemma A we get:

$$|R_\varepsilon(t)| \leq C(T)\Big[|u_0 - \varphi_\varepsilon| + 1/C(T)\int_0^t \Big|\varepsilon A_1 w(\tau) + S(\tau,\varepsilon)f(\tau) - f_0(\tau)\Big|d\tau\Big] \leq$$

$$\leq C(T)\Big[|u_0 - \varphi_\varepsilon| + 1/C(T)\int_0^t |\varepsilon A_1 w(\tau)|d\tau + 1/C(T)\int_0^t |F_0(\tau,\varepsilon)|d\tau +$$

$$+1/C(T)\int_0^t \Big|S(\tau,\varepsilon)f(\tau)-\int_0^\infty K(\tau,\mu,\varepsilon)S(\tau,\varepsilon)f(\mu)d\mu\Big|d\tau\Big],\quad 0\le t\le T. \tag{5}$$

Now step by step we will estimate all terms in the right of inequality (5).

In what follows we will denote by C all constants depending on $T, u_0, u_1, f, \gamma, \gamma_1$.

In conditions of Theorem B we can estimate the difference

$$|u_0-\varphi_\varepsilon| = \Big|\int_0^\infty e^{-\tau}(u_\varepsilon(2\varepsilon\tau)-u_0)d\tau\Big| \le \int_0^\infty e^{-\tau}\int_0^{2\varepsilon\tau}|u'_\varepsilon(\mu)|d\mu \le C\varepsilon. \tag{6}$$

Using the property (vii) from Lemma 1 we have

$$\Big|S(\tau,\varepsilon)f(\tau)-\int_0^\infty K(\tau,\mu,\varepsilon)S(\tau,\varepsilon)f(\mu)d\mu\Big| \le$$

$$\le |S(\tau,\varepsilon)|\Big|f(\tau)-\int_0^\infty K(\tau,\mu,\varepsilon)f(\mu)d\mu\Big| \le$$

$$\le \gamma_1\sqrt{\varepsilon}(1+\sqrt{t})\|f'\|_{L^\infty(0,\infty;H} = C\sqrt{\varepsilon}. \tag{7}$$

In [2] it is also shown that

$$\int_0^t e^{\gamma\tau}|F_0(\tau,\varepsilon)|d\tau \le \widetilde{C}\varepsilon|u_1| \le C\varepsilon. \tag{8}$$

To estimate $|A_1 w(t)|$ we will consider now the (P_0) problem and will apply to it the operator A_1 to obtain:

$$\begin{cases} A_1 v'(t) + A_1 A_0 v(t) = A_1 f(t), t>0 \\ A_1 v(0) = A_1 u_0. \end{cases} \tag{9}$$

In condition (2) we can observe that

$$\varepsilon A_1 A_0 v(t) = \lim_{h\mapsto 0}\frac{S(h,\varepsilon)A_0 v(t) - A_0 v(t)}{h} = \lim_{h\mapsto 0}\frac{A_0 S(h,\varepsilon)v(t)-A_0 v(t)}{h} =$$

$$= \lim_{h\mapsto 0} A_0\frac{S(h,\varepsilon)v(t)-v(t)}{h} = \varepsilon A_0 A_1 v(t)$$

Thus, denoting by $y(t) = A_1 v(t)$ we can write the problem for y

$$\begin{cases} y'(t) + A_0 y(t) = A_1 f(t), t>0 \\ y(0) = A_1 u_0. \end{cases}$$

If $A_1 u_0 \in V, A_1 f \in W^{1,1}(0,T,H)$, then by Theorem B we obtain the estimate

$$|y(t)| \le C(T, u_0, f, \gamma).$$

But

$$|A_1 w(t)| = |A_1 S(t,\varepsilon)v(t)| \le \gamma_1 |A_1 v(t)| = \gamma_1 |y(t)| \le C. \tag{10}$$

From estimates (5)-(10) we finally obtain the estimate

$$|R_\varepsilon(t)| \le \sqrt{\varepsilon} C(T, u_0, u_1, f, \gamma, \gamma_1).$$

Lemma 3 is proved.

Theorem 2. *Let $T > 0$. If $u_0, A_1 u_0 \in V, u_1 \in H, f, A_1 f \in W^{1,\infty}(0,T;H)$ and A_0, A_1 satisfies conditions (1)-(3) then the estimate*:

$$|u_\varepsilon(t) - v(t)| \le C(u_0, u_1, f, \gamma, \gamma_1, \delta)\sqrt{\varepsilon}, \quad t \in [0,T], \quad 0 < \varepsilon \ll 1$$

is true.

The proof of this theorem is a simple consequence of Lemmas 1 and 2. Indeed

$$|u_\varepsilon(t) - v(t)| \le \frac{1}{\delta} |S(t,\varepsilon) u_\varepsilon(t) - S(t,\varepsilon) v(t)| \le$$

$$\le \frac{1}{\delta}[|S(t,\varepsilon)u_\varepsilon(t) - v_{0\varepsilon}(t)| + |S(t,\varepsilon)v(t) - v_{0\varepsilon}(t)|] \le \sqrt{\varepsilon} C(T, u_0, u_1, f, \gamma, \gamma_1, \delta).$$

Theorem 3. *Let $T > 0$. If*

$$u_0, A_0 u_0, A_1 u_0, A_1 A_0 u_0, u_1, f(0), A_1 f(0) \in V, \quad f, A_1 f \in W^{2,\infty}(0,T;H)$$

and A_0, A_1 satisfies conditions (1)-(3), then the estimate

$$|u'_\varepsilon(t) - v'(t) + h e^{-\frac{t}{\varepsilon}}| \le \sqrt{\varepsilon} C(u_0, u_1, f, \gamma, \gamma_1, \delta),$$

is true, where $h = f(0) - u_1 - A_0 u_0$.

Proof. Denote by $z_\varepsilon(t) = u'_\varepsilon(t) + h e^{-\frac{t}{\varepsilon}}$. Then for $z_\varepsilon(t)$ we get the following Cauchy problem:

$$\begin{cases} \varepsilon z''_\varepsilon(t) + z'_\varepsilon(t) + (A_0 + \varepsilon A_1) z_\varepsilon(t) = f'(t) + e^{-\frac{t}{\varepsilon}}(A_0 + \varepsilon A_1)h, \quad t > 0 \\ z(0) = f(0) - A_0 u_0, \quad z'(0) = -A_1 u_0. \end{cases} \tag{11}$$

As $A_0 u_0, f(0) \in V, \quad f \in W^{2,\infty}(0,T;H)$, according to Theorem 1 the function

$$w_{1\varepsilon}(t) = \int_0^\infty \mathcal{K}(t,\tau,\varepsilon) z_\varepsilon(\tau) d\tau$$

is the solution to the problem:

$$\begin{cases} w'_{1\varepsilon}(t) + A_0 w_{1\varepsilon}(t) = F_1(t,\varepsilon), \quad t > 0, \\ w_{1\varepsilon}(0) = \int_0^\infty e^{-\tau} z_\varepsilon(2\varepsilon\tau) d\tau, \end{cases} \tag{12}$$

where

$$F_1(t,\varepsilon) = \int_0^\infty \mathcal{K}(t,\tau,\varepsilon)[f'(\tau)d\tau + e^{-\frac{\tau}{\varepsilon}}(A_0 + \varepsilon A_1)h] d\tau -$$

$$-\frac{1}{\sqrt{\pi}}\Big[2\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big)-\lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big]S(t,\varepsilon)A_1u_0.$$

Denoting by $v_1(t)=v'(t),(P_0)$, for v_1 we have the problem (Pv_1):

$$\begin{cases} v_1'(t)+A_0v_1(t)=f'(t), \quad t>0,\\ v_1(0)=f(0)-A_0u_0.\end{cases}$$

If $w_{2\varepsilon}(t)=S(t,\varepsilon)v_1(t)$, then (Pv_1) becomes

$$\begin{cases} w_{2\varepsilon}'(t)+[A_0-\varepsilon A_1]w_{2\varepsilon}(t)=S(t,\varepsilon)f'(t), \quad t>0,\\ w_{2\varepsilon}(0)=f(0)-A_0u_0.\end{cases} \tag{13}$$

Let $R_{1\varepsilon}(t)=w_{1\varepsilon}(t)-w_{2\varepsilon}(t)$. Then, using (12) and (13) we get the following Cauchy problem for it:

$$\begin{cases} R_{1\varepsilon}'(t)+A_0R_{1\varepsilon}(t)=F_1(t,\varepsilon)-S(t,\varepsilon)f'(t)-\varepsilon A_1w_{2\varepsilon}(t), \quad t>0,\\ R_{1\varepsilon}(0)=\int_0^\infty e^{-\tau}\int_0^{2\varepsilon\tau}z_\varepsilon'(\theta)d\theta d\tau.\end{cases} \tag{14}$$

Taking scalar product of (14) with $R_{1\varepsilon}(t)$, integrating on $[0,t]$ and using Lemma A we get the estimate

$$|R_{1\varepsilon}(t)|\le e^{-\gamma t}\Big(|R_{1\varepsilon}(0)|+\int_0^t e^{\gamma\tau}\Big|F_1(\tau,\varepsilon)-S(\tau,\varepsilon)f'(\tau)-\varepsilon A_1w_{2\varepsilon}(\tau)\Big|d\tau\Big). \tag{15}$$

As we can see in (11) $z_\varepsilon(t)$ is the solution to a second order Cauchy problem which is similar to (P_ε). So, in conditions of this theorem, using Theorem B, the following estimate is true:

$$|z_\varepsilon(t)|\le C(|f|_{W^{2,\infty}(0,T;H)},|A_0u_0|,|A_1u_0|,\gamma)=C.$$

Then

$$|R_{1\varepsilon}(0)|=\Big|\int_0^\infty e^{-\tau}\int_0^{2\varepsilon\tau}z_\varepsilon'(\theta)d\theta d\tau\Big|\le C\varepsilon.$$

From properties (vii), (viii) and (4) it follows:

$$\int_0^t e^{\gamma\tau}\Big|F_1(\tau,\varepsilon)-S(\tau,\varepsilon)f'(\tau)\Big|d\tau\le\int_0^t e^{\gamma\tau}\Big|\int_0^\infty\mathcal{K}(\tau,\mu,\varepsilon)f'(\mu)d\mu-S(\tau,\varepsilon)f'(\tau)\Big|d\tau+$$

$$+\int_0^t\int_0^\infty\mathcal{K}(\tau,\mu,\varepsilon)e^{-\frac{\mu}{\varepsilon}}|(A_0+\varepsilon A_1)h|d\mu d\tau+$$

$$+\int_0^t\frac{1}{\sqrt{\pi}}\Big|2\exp\Big\{\frac{3\tau}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{\tau}{\varepsilon}}\Big)-\lambda\Big(\frac{1}{2}\sqrt{\frac{\tau}{\varepsilon}}\Big)\Big|\Big|S(\tau,\varepsilon)A_1u_0\Big|d\tau\le$$

$$\le C\gamma_1\Big[\sqrt{\varepsilon}(1+\sqrt{t})\|f''\|_{L^\infty(0,T;H)}+\varepsilon+\sqrt{\varepsilon}|A_0u_0|\Big]\le C\sqrt{\varepsilon}.$$

To estimate $|\varepsilon A_1 w_{2\varepsilon}(t)|$ we will apply A_1 to (Pv_1) and denote by $y_1(t) = A_1 v_1(t)$ to get

$$\begin{cases} y_1'(t) + A_0 y_1(t) = A_1 f'(t), \quad t > 0 \\ y_1(0) = A_1 f(0) + A_1 A_0 u_0. \end{cases}$$

As $A_1 A_0 u_0, A_1 f(0) \in V, A_1 f \in W^{2,\infty}(0,T;H)$, Theorem A implies the estimate

$$|y_1(t)| \leq C(T, \gamma, A_1 A_0 u_0, A_1 f).$$

Consequently,

$$|\varepsilon A_1 w_{2\varepsilon}(t)| = \varepsilon |A_1 S(t,\varepsilon) v_1(t)| \leq \varepsilon \gamma_1 |A_1 v_1(t)| = \varepsilon \gamma_1 |y_1(t)| \leq \varepsilon C.$$

Using the last three inequalities from (15) follows the estimate

$$|R_{1\varepsilon}(t)| \leq C\sqrt{\varepsilon}, \quad 0 \leq t \leq T. \tag{16}$$

From property (vii) from Lemma 1 and (4) it follows:

$$|S(t,\varepsilon) z_\varepsilon(t) - w_{1\varepsilon}(t)| = |S(t,\varepsilon) z_\varepsilon(t) - \int_0^\infty \mathcal{K}(t,\tau,\varepsilon) z_\varepsilon(\tau) d\tau| \leq$$

$$\leq \gamma_1 C(1+\sqrt{t}) \parallel z' \parallel_{L^\infty(0,T:H)} \leq \sqrt{\varepsilon} C. \tag{17}$$

Finally, using condition (3) and estimates (16), (17) we get

$$|u_\varepsilon'(t) - v'(t) - he^{-\frac{t}{\varepsilon}}| = |z_\varepsilon(t) - v_1(t)| \leq \frac{1}{\delta} |S(t,\varepsilon) z_\varepsilon(t) - S(t,\varepsilon) v_1(t)| \leq$$

$$\leq \frac{1}{\delta} \Big[|S(t,\varepsilon) z_\varepsilon(t) - w_{1\varepsilon}(t)| + |w_{1\varepsilon}(t) - S(t,\varepsilon) v_1(t)| \Big] \leq \sqrt{\varepsilon} C(u_0, u_1, f, \gamma, \gamma_1, \delta).$$

Theorem 3 is proved.

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