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Pareto approximation of the tail by local exponential modeling

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Abstract. We give a new adaptive method for selecting the number of upper order statistics used in the estimation of the tail of a distribution function. Our approach is based on approximation by an exponential model. The selection procedure consists in consecutive testing for the hypothesis of homogeneity of the estimated parameter against the change-point alternative. The selected number of upper order statistics corresponds to the first detected change-point. Our main results are non-asymptotic.

Mathematics subject classification: primary 62G32, 62G08; secondary 62G05 .
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1 Introduction

This paper is concerned with the adaptive estimation of the tail of a distribution function (d.f.) F. A popular estimator for use in the extreme value theory was proposed by Hill (1975). Given a sample $X_1, ..., X_n$ from the d.f. F the Hill estimator is defined as

$$\widehat{\alpha}_{n,k} = \frac{1}{k} \sum_{i=1}^{k} \log \frac{X_{n,i}}{X_{n,k+1}},$$

where $X_{n,1} \geq ... \geq X_{n,n}$ are the order statistics pertaining to $X_1, ..., X_n$ and k is the number of upper order statistics used in the estimation. There is a vast literature on the asymptotic properties of the Hill estimator. Suppose that d.f. F is regularly varying with index of regular variation β [see for example Bingham, Goldie and Teugels (1987)]. Weak consistency for estimating β was established by Mason (1982), under the conditions that $k \to \infty$ and $k/n \to 0$ as $n \to \infty$. Asymptotic normality of the Hill estimator was proved by Hall (1982). A strong consistency result can be found in Deheuvels, Haeusler and Mason (1988). Further properties concerning the efficiency have been studied in Drees (2001). For extensions to dependent observations see, for instance, Resnik and Starica (1998) and the references therein. The asymptotic results mentioned above do not give any recipe about selecting the parameter k in practical applications, while the behavior of the error estimation depends essentially on it. Different approaches for data driven choices of k have been proposed in the literature, mainly based on the idea of balancing the bias and the asymptotic variance of the Hill estimator. We refer to Hall and Welsh

(1985), Danielson, de Haan, Peng, Vries (2001), Beirlant, Teugels and Vinysaker (1996), Resnik and Starica (1997), Drees and Kaufman (1998), among many others. However the bias of the Hill estimator for estimating the parameter of regular variation as a rule diminishes very slowly, which makes any choice of the parameter k not very efficient from the practical point of view. A striking example is the so called Hill Horror plot (see Figure 1, left).

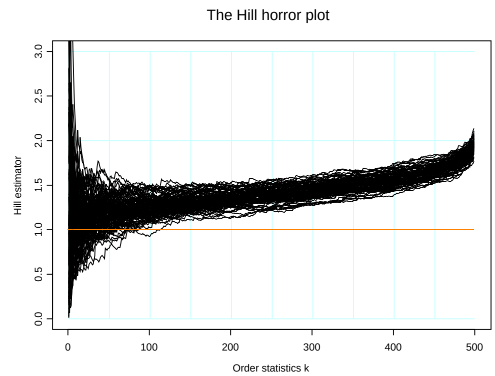

Figure 1. Left: 100 realizations of the Hill estimator for Pareto-log d.f. $F(x) = 1 - (x/e)^{-1/\beta} \log x,\ x \geq e$, where the parameter $\beta = 1$ is expected to be estimated. Right: 100 realizations of the Hill estimator for Pareto-log d.f. and the fitted Pareto parameter. Here the dark lines represent the fitted Pareto index computed from the approximation formulas (3.5), (3.1) and the light ones are the corresponding Hill plots.

For more insight on the problem the reader is referred to the book by Embrechts, Klüppelberg and Mikosch (1997), from which we cite on the page 351: "On various occasions we hinted at the fact that the determination of the number k of upper order statistics finally used remains a delicate point in the whole set-up. Various papers exist which offer a semi-automatic or automatic, so-called "optimal", choice of k. ... We personally prefer a rather pragmatic approach realizing that, whatever method one chooses, the "Hill horror plot" ... would fool most, if not all. It also serves to show how delicate a tail analysis in practice really is." An interesting exchange of opinions on this subject may be found in the survey paper by Resnik (1997) and in the supplied discussion.

The aim of the present paper is to give a natural resolution to the "Hill horror plot" paradox and to rehabilitate the Hill estimator, for finite sample sizes, by looking at the problem from the point of view of selecting an appropriate tail. In Section 3 we shall see that, for finite sample sizes, the Hill estimator is close to another quantity which can be interpreted as the parameter of the approximating Pareto distribution and which we shall call the fitted Pareto index [see (2.4) for the definition of this quantity]. In Figure 1, right, we give a simulation for the Pareto-log d.f.; other examples are presented in the Appendix 8. The importance of this interpretation, perhaps, is justified by the fact that it allows new approaches for selecting the number k of retained upper order statistics. For estimating the fitted Pareto index we propose a method based on successive testing of the hypothesis that the first k normed log-spacings follow exponential distributions with homogeneous parameters. The idea goes back to Spokoiny (1998). However our procedure is different in several aspects. First, our test is based on the likelihood ratio test statistic for testing homogeneity of the estimated parameters against the change-point alternative. Second, in our procedure the number k is selected to be the detected change-point. We also refer the reader to Picard and Tribouley (2002) where the change point Pareto model (see Pareto-CP d.f. in the Appendix) is used for estimation in the parametric context.

Our main results are non-asymptotic. We establish an "oracle" inequality for the adaptive estimator of the fitted index. The result claims that the risk of the adaptive estimator is only within some constant factor worse than the risk of the best possible estimator for the given model.

The paper is organized as follows. In Sections 2 and 3 we formulate the problem and give the approximation by the exponential model. The adaptive procedure is presented in Section 4. Section 5 illustrates the numerical performances of the method on some artificial data sets. The results and the proofs are given in Sections 6 and 7.

2 The model and the problem

Let $X_1, ..., X_n$ be i.i.d. observations with common d.f. $F(x)$ supported on (a, ∞), where $a > 0$ is a fixed real number. Assume that the function F is strictly increasing and has a continuous density f. Since $F(a) = 0$, the d.f. F can be represented as

$$F(x) = 1 - \exp\left(-\int_a^x \lambda(t)\, dt\right), \quad x \geq a, \tag{2.1}$$

where

$$\lambda(x) = \frac{f(x)}{1 - F(x)}, \quad x \geq a$$

is the hazard rate. Note that if $\lambda(x) = \frac{1}{\alpha x}$, then the d.f. F is Pareto with index $1/\alpha$, which is a typical fat tail distribution. To allow more general laws with heavy tails we shall assume that

$$\lambda(x) = \frac{1}{\alpha(x)\, x}, \tag{2.2}$$

where the function $\alpha(x)$, $x > a$, can be approximated by a constant for big values of x. For instance, this is the case when there exists an $\beta > 0$ such that

$$\lim_{x \to \infty} \alpha(x) = \beta. \tag{2.3}$$

Many regularly varying at infinity d.f.'s F satisfy the assumptions (2.1), (2.2) and (2.3), see representation theorems in Seneta (1976) or Bingham, Goldie and Teugels

(1987). If F is regularly varying at infinity, then the limit in (2.3) is nothing else but the index of regular variation.

Our problem can be formulated as follows. Let $X_{n,1} > ... > X_{n,n}$ be the order statistics pertaining to $X_1, ..., X_n$. The goal is to find a natural number k such that on the set $\{X_{n,1}, ..., X_{n,k}\}$ the function $\alpha(x)$, $x \geq a$, can be well approximated by the value $\alpha(X_{n,1})$ and to estimate this value. The intuitive meaning of this is to find a Pareto approximation for the tail of the d.f. F on the data set $\{X_{n,1}, ..., X_{n,k}\}$. Note that this problem is different from that of estimating the index of regular variation β defined by the limit (2.3). As it was stressed in the Introduction the main advantage of the present setting is, perhaps, the fact that it allows new algorithms for the choice of the nuisance parameter k. The approach adopted in this paper is based on the approximation by an exponential model which is presented in the next section.

Before to proceed with this, we shall point out the connection of the function $\alpha(\cdot)$ to the logarithmic mean excess of F :

$$\nu(t) = \int_t^\infty \log\frac{x}{t}\frac{F(dx)}{1-F(t)}, \quad t \geq a. \tag{2.4}$$

Integration by parts gives, for any $t \geq a$,

$$\int_t^\infty \alpha(x)\frac{F(dx)}{1-F(t)} = \nu(t). \tag{2.5}$$

By straightforward calculations it can be seen that the number $\nu(t)$ is the minimizer of the Kullback-Leibler distance between Pareto d.f. $P_\alpha(x) = 1 - x^{-1/\alpha}$, $x \geq 1$ and the excess d.f. $F(x|t) = 1 - (1 - F(xt)) / (1 - F(t))$, $x \geq 1$. Thus the number $\nu(t)$ can be interpreted as the parameter of the best Pareto fit to the tail of the d.f. F on the interval $[t, \infty)$. We shall call the function $\nu(t)$, $t \geq a$ the fitted Pareto index.

3 Approximation by exponential model

The function $\alpha(\cdot)$ will be estimated from the approximating exponential model. Our motivation is somewhat similar to that of Hill (1975) [see also Beirlant, Dierskx, Goegebeur et Matthys (2000) for another exponential approximation]. The construction of the approximating exponential model employs the following lemma, called Renyi representation of order statistics.

Lemma 3.1. *Let $X_1, ..., X_n$ be i.i.d. r.v.'s with common strictly increasing d.f. F and $X_{n,1} > ... > X_{n,n}$ be the order statistics pertaining to $X_1, ..., X_n$. Then the r.v.'s*

$$\xi_i = i\log\frac{1-F(X_{n,i+1})}{1-F(X_{n,i})}, \quad i = 1, ..., n-1.$$

are i.i.d. standard exponential.

Proof. See for instance Reiss (1989) or Example 4.1.5 in Embrechts, Klüppelberg and Mikosch (1997)]. □

Let $Y_i = i \log \frac{X_{n,i}}{X_{n,i+1}}$, $i = 1, ..., n-1$. Then $Y_i = \alpha_i \xi_i, \quad i = 1, ..., n-1$, where

$$\alpha_i = -\log \frac{X_{n,i}}{X_{n,i+1}} / \log \frac{1 - F(X_{n,i})}{1 - F(X_{n,i+1})}. \tag{3.1}$$

It is easy to see that the function $\alpha(x)$ is defined through the d.f. F by the equations

$$\frac{1}{\alpha(x)} = x\lambda(x) = \frac{xf(x)}{1 - F(x)} = -\frac{\frac{d}{dx}\log(1 - F(x))}{\frac{d}{dx}\log x}, \quad x \geq a. \tag{3.2}$$

By identity (3.2) the value α_i can be regarded as an approximation of the value of the function $\alpha(\cdot)$ at the point $X_{n,i+1}$. More precisely, the mean value theorem implies

$$\alpha_i = \alpha(X_{n,i+1} + \theta_{n,i+1}(X_{n,i} - X_{n,i+1})),$$

with some $\theta_{n,i+1} \in [0, 1]$, for $i = 1, ..., n-1$. These simple considerations reduce the original model to the following inhomogeneous exponential model

$$Y_i = \alpha_i \xi_i, \quad i = 1, ..., n-1, \tag{3.3}$$

where $\alpha = (\alpha_1, ..., \alpha_{n-1})$ is a vector of unknown parameters. We assume *local homogeneity* of this model which stipulates that the components α_i's nearly equal α_1 within some interval $I = [1, k]$. In the sequel finding the Pareto approximation for the tail of the d.f. F will be viewed as the problem of choosing the interval $I = [1, k]$ and of estimating the component α_1 from the observations (3.3).

Under the assumption that

$$\alpha_1 = ... = \alpha_k, \tag{3.4}$$

the maximum likelihood estimator of α_1 is the sample mean

$$\widehat{\alpha}_k = \frac{1}{k} \sum_{i=1}^{k} Y_i,$$

which is the well-known Hill estimator. Our main concern is to choose appropriately the number k of upper order statistics used in the estimation.

If the condition (3.4) is not satisfied, then from the definition of the model (3.3) it follows that the Hill estimator $\widehat{\alpha}_k$ approximates without bias the quantity

$$\overline{\alpha}_k = \frac{1}{k} \sum_{i=1}^{k} \alpha_i, \tag{3.5}$$

which, in turn, is an approximation of the fitted Pareto index (2.4): $\overline{\alpha}_k \approx \nu(X_{n,k+1})$, for k big enough. The assumption of local homogeneity implies that the quantities $\overline{\alpha}_k$, α_k and $\alpha_1 = \overline{\alpha}_1$ are close to each other and thus under this assumption the Hill estimator also approximates the fitted Pareto parameter $\nu(t)$ at the point $t =$

$X_{n,k+1}$. The simulations show a good concordance between the two latter quantities (see Figures 1, 4 and 5).

Although the above considerations shed some light on what does the Hill estimator estimate, the main problem, how to choose an appropriate value of k (even for the fitted Pareto index $\nu(X_{n,k+1})$ or equally for $\overline{\alpha}_k$) still remains open. Model selection based on the penalization terms [see Barron, Birge and Massart (1999)] could be a reasonable alternative for defining the optimal and adaptive values of k. In this paper we take another adaptive approach which is presented in the next section. To avoid difficult interpretations with the choice of the optimal value k for the parameter $\overline{\alpha}_k$ we shall consider that the Hill estimator estimates the value α_1, which may be regarded as a constant approximation of the values α_i, $i = 1, ..., k$.

4 Adaptive selection of the parameter k

This section presents a method of selecting the parameter k in a data driven way. Throughout the paper we shall denote by $|I|$ the number of elements of the set I.

4.1 The adaptive procedure

Let $\mathcal{I}$ be a family of intervals of the form $I = [1, k]$, where $k \in \{1, ..., n-1\}$, such that $|I| \geq 2m_0$, for a prescribed natural number m_0, where m_0 is much smaller than $(n-1)/2$. A special case of the family $\mathcal{I}$ is given by the set of all the intervals $I = [1, k]$, satisfying this condition. Another example used later on in the simulations, is the set $\mathcal{I} = \mathcal{I}_q$ of intervals $I = [1, k]$, with k approximately lying in the geometric grid $\left\{l : l \leq n,\ l = [m_0 + m_0 q^j],\ j = 1, 2, ...\right\}$, where $q > 1$. In the latter case the numbers m_0 and q will be parameters of the procedure.

The family $\mathcal{I}$ is naturally ordered by the length $|I|$ of $I \in \mathcal{I}$. The idea of our method is to test successively the hypothesis of no change-point within the interval I and to select k equal to the first detected change-point. The formal steps of the procedure for selecting the adaptive interval $\widehat{I}$ read as follows:

INITIALIZATION Start with the smallest interval $I = I_0 \in \mathcal{I}$.

STEP 1 Take the next interval $I \in \mathcal{I}$.

STEP 2 From observations (3.3) test on homogeneity the vector α within the interval I against the change-point alternative, as described in Section 4.2.

STEP 3 If the change point was detected for the interval I, then define $\widehat{I}$ as the interval from one to the detected change-point and stop the procedure, otherwise repeat the procedure from the Step 1. If there was no change-point for all $I \in \mathcal{I}$, then define $\widehat{I} = [1, n-1]$.

The adaptive estimator is defined as $\widehat{\alpha} = \widehat{\alpha}_{\widehat{I}}$, where

$$\widehat{\alpha}_I = \frac{1}{|I|} \sum_{i \in I} Y_i, \tag{4.1}$$

for any interval I. The essential point in the above procedure is the Step 2 which stipulates testing the hypothesis of homogeneity for the interval I. It consists in applying the classical change-point test which is described in the next section.

4.2 Test of homogeneity against the change-point alternative

The test of homogeneity against the change-point alternative is based on the likelihood ratio test statistic. For any interval $I \in \mathcal{I}$ denote by $\mathcal{J}_I$ the set of all subintervals $J \subset I$, $J \in \mathcal{I}$, such that $|I|/2 \le |J| \le |I| - m_0$. For every interval $J \in \mathcal{J}_I$ consider the problem of testing the hypothesis of homogeneity $\alpha_i = \theta$, $i \in I$ against the change-point alternative $\alpha_i = \theta_1$, $i \in J$ and $\alpha_i = \theta_2$, $i \in I \setminus J$ with $\theta_1 \ne \theta_2$. The likelihood ratio test statistic is defined by

$$\begin{aligned} T_{I,J} &= \sup_{\theta_1} L\left(Y_J, \theta_1\right) + \sup_{\theta_2} L\left(Y_{I\setminus J}, \theta_2\right) - \sup_{\theta} L\left(Y_I, \theta\right) \\ &= L\left(Y_J, \widehat{\alpha}_J\right) + L\left(Y_{I\setminus J}, \widehat{\alpha}_{I\setminus J}\right) - L\left(Y_I, \widehat{\alpha}_I\right), \end{aligned}$$

where $\widehat{\alpha}_I$ is the corresponding maximum likelihood estimator defined by (4.1) and

$$L\left(Y_I, \theta\right) = \sum_{i\in I} \log p\left(Y_i, \theta\right).$$

Since in the case under consideration $p\left(y, \theta\right) = \exp\left(-y/\theta\right)/\theta$, one gets

$$\begin{aligned} T_{I,J} &= -\sum_{i\in J}\left[\log\frac{\widehat{\alpha}_J}{\widehat{\alpha}_I} - Y_i\left(\frac{1}{\widehat{\alpha}_I} - \frac{1}{\widehat{\alpha}_J}\right)\right] + \sum_{i\in I\setminus J}\left[\log\frac{\widehat{\alpha}_{I\setminus J}}{\widehat{\alpha}_I} - Y_i\left(\frac{1}{\widehat{\alpha}_I} - \frac{1}{\widehat{\alpha}_{I\setminus J}}\right)\right] \\ &= |J|\, G\left(\frac{\widehat{\alpha}_J}{\widehat{\alpha}_I} - 1\right) + |I\setminus J|\, G\left(\frac{\widehat{\alpha}_{I\setminus J}}{\widehat{\alpha}_I} - 1\right), \end{aligned} \tag{4.2}$$

where $G\left(x\right) = x - \log\left(1 + x\right),\ x > -1$. The use of Taylor's expansion gives the approximating test statistic

$$\overline{T}_{I,J} = \frac{|J|}{2}\left(\frac{\widehat{\alpha}_J}{\widehat{\alpha}_I} - 1\right)^2 + \frac{|I\setminus J|}{2}\left(\frac{\widehat{\alpha}_{I\setminus J}}{\widehat{\alpha}_I} - 1\right)^2.$$

By simple algebra we can represent the latter statistic in the form

$$\overline{T}_{I,J} = \frac{|J|\cdot|I\setminus J|}{2\,|I|}\left(\frac{\widehat{\alpha}_J - \widehat{\alpha}_{I\setminus J}}{\widehat{\alpha}_I}\right)^2. \tag{4.3}$$

Now the test of homogeneity of α on the interval I can be based on the maximum of all such defined statistics $T_{I,J}$ or $\overline{T}_{I,J}$ over the set $\mathcal{J}_I$. The hypothesis of homogeneity on the interval I will be rejected if

$$T_I = \max_{J\in\mathcal{J}_I} T_{I,J} > t_\gamma, \quad \text{or} \quad \overline{T}_I = \max_{J\in\mathcal{J}_I} \overline{T}_{I,J} > \overline{t}_\gamma,$$

where the critical values t_γ and $\overline{t}_\gamma$ are defined to provide the prescribed rejection probability γ under the hypothesis of homogeneity within the interval I. These values can be computed by Monte-Carlo simulations from the homogeneous model with i.i.d. standard exponential observations Y_i, $i = 1, ..., n$. Here we utilize the fact that under the hypothesis of homogeneity the distributions of the test statistics T_I and $\overline{T}_I$ do not depend on α.

If the hypothesis of the homogeneity of α is rejected on the interval I then the detected change-point k^* corresponds to the length of the interval $J^* \in \mathcal{J}_I$ for which the statistic T_I attains its maximum, i.e.

$$k^* = |J^*|, \quad \text{where} \quad J^* = \arg\max_{J \in \mathcal{J}_I} T_{I,J}.$$

5 Simulation study

The aim of the present simulation study is to demonstrate the numerical performance of the proposed procedure. We focus on the quality of the selected interval I and of the corresponding adaptive estimator. The next figures present box-plots of the length of the selected interval $\widehat{I}$ and of the adaptive estimator $\widehat{\alpha}$ for different values of the parameter $\sqrt{t_\gamma}$ from 500 observations following Pareto and Pareto-log d.f.'s (see a list in the Appendix). The box-plots are obtained from 500 Monte-Carlo realizations. The set $\mathcal{I}$ is a geometric grid with parameters $m_0 = 25$, $q = 1.1$.

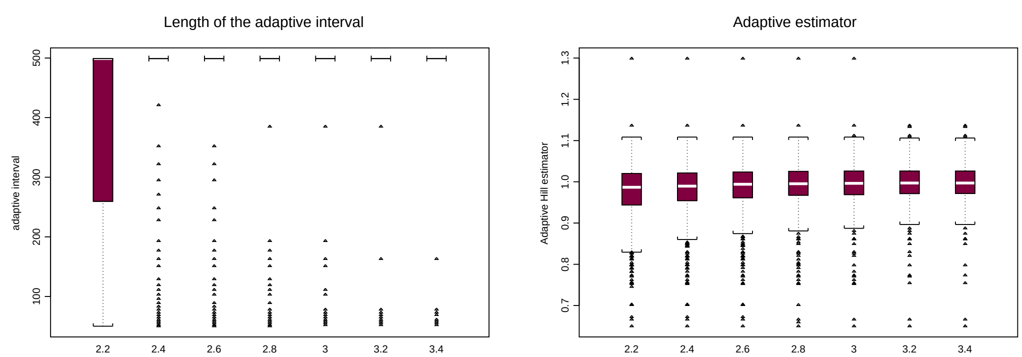

Figure 2. Box-plots of selected intervals and the adaptive estimators for Pareto d.f. from 500 realization.

In Table 1 the mean absolute error (MAE) of the adaptive estimator $\hat{\alpha}$ w.r.t. the value $\alpha_1 = \alpha(X_{n,1})$ is computed for the d.f.'s introduced above.

The results clearly indicate that the increase of the parameter t_γ results in a smaller variability of the estimator but in a larger bias (in case when the model is not Pareto). A reasonable compromise is attained for $\sqrt{t_\gamma}$ about 2.6 leading to a relatively stable behavior of the procedure in the Pareto case and to a moderate bias

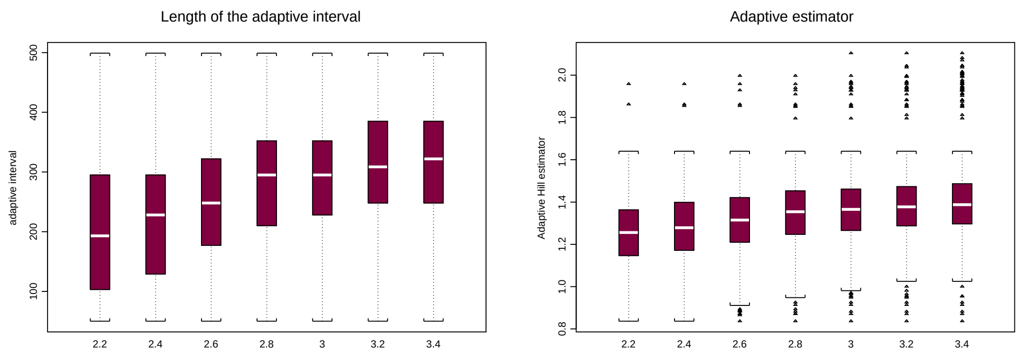

Figure 3. Box-plots of selected intervals and the adaptive estimators for Pareto-log d.f. from 500 realization.

Table 1. MAE computed for 500 realizations

	t_γ=2.2	t_γ=2.4	t_γ=2.6	t_γ=2.8	t_γ=3.0	t_γ=3.2	t_γ=3.4	
Pareto	0.0642	0.0583	0.0546	0.0487	0.0459	0.0433	0.0395	
Cauchy-plus	0.1036	0.1076	0.1116	0.1166	0.1204	0.1232	0.1275	
Pareto-log	0.1838	0.2039	0.2231	0.2388	0.2581	0.2854	0.3106	
Pareto-CP	0.0746	0.0704	0.0697	0.0658	0.0642	0.0626	0.0615	

in the non-Pareto case. The numerical simulation for the procedure with the parameter $\sqrt{t_\gamma} = 2.6$ for different values of the sample size n and different distributions (see a list in the Appendix 8) are summarized in Table 2. The other parameters are kept as in the previous case. In this table MAE is computed w.r.t. the value $\alpha_1 = \alpha(X_{n,1})$ for 500 simulations.

In the Appendix 8 we present the box-plots of the length (in %) of the selected interval $\widehat{I}$ and of the adaptive estimator $\widehat{\alpha}$ for different values of n from 500 simulations following different d.f.'s.

6 Theoretical results

This section discusses some theoretical properties of the procedure presented in Section 4. Let $t_\gamma > 0$ and $\bar{t}_\gamma > 0$ be the critical values entering the definition of the change point tests from Section 4.2.

6.1 Properties of the selected interval

We start with results concerning the choice of the interval of homogeneity. We will ensure that the following two properties hold:

Table 2. MAE computed for 500 realizations

	n=200	n=300	n=400	n=500	n=800	n=1000	2000	n=3000
Pareto	0.0573	0.0507	0.0473	0.0521	0.0456	0.0495	0.0453	0.0415
Cauchy-plus	0.1483	0.1210	0.1133	0.1155	0.0846	0.0943	0.0720	0.0577
Pareto-log	0.2544	0.2309	0.2274	0.2178	0.1895	0.1828	0.1783	0.1713
GPD	0.2563	0.1829	0.1770	0.1564	0.1488	0.1301	0.1171	0.1095
Hall model	0.2498	0.2448	0.2377	0.2439	0.2344	0.2222	0.1961	0.1699
Pareto-CP	0.1001	0.0881	0.0737	0.0669	0.0566	0.0558	0.0432	0.0321
Standard Normal tail	0.2273	0.1718	0.1438	0.1242	0.0983	0.0941	0.0689	0.0654
Standard Exponential	0.2989	0.2370	0.1913	0.1707	0.1432	0.1373	0.1133	0.1007

A. The intervals of homogeneity are accepted with high probabilities.

B. The intervals of non-homogeneity are rejected with high probabilities at least in some special cases, for instance, for the change-point model.

Consider first the property A. The assumption that the vector α is constant on some interval I can be quite restrictive for practical applications. Therefore the desirable property would be that the procedure accepts any interval $I \in \mathcal{I}$ for which α_i can be well approximated by a constant within the interval I. Let I be an interval and let α_I be the average of the α_i's over the interval I :

$$\alpha_I = \frac{1}{|I|} \sum_{i \in I} \alpha_i .$$

The non-homogeneity of the α_i's within the interval I can be naturally measured by the value

$$\Delta_I = \max_{i \in I} \left| \frac{\alpha_i}{\alpha_I} - 1 \right| .$$

We say that I is a "good" interval if the value Δ_I is small. The next result claims that a "good" interval I will be accepted by the procedure with a high probability provided that the critical value t_γ was taken sufficiently large.

For every interval $I \in \mathcal{I}$, denote

$$S_I = \frac{1}{|I|} \sum_{i \in I} \alpha_i \left(\xi_i - 1 \right) \quad \text{and} \quad V_I^2 = \sum_{i \in I} \alpha_i^2 .$$

For given intervals $I \in \mathcal{I}$ and $J \in \mathcal{J}_I$, denote $J^c = I \diagdown J$ and, with a real $\lambda > 0$, define the events

$$\Omega_{I,J} = \left\{ |S_I| \leq \frac{\lambda V_I}{|I|},\ |S_J| \leq \frac{\lambda V_J}{|J|},\ |S_{J^c}| \leq \frac{\lambda V_{J^c}}{|J^c|} \right\}$$

and

$$\Omega_I = \bigcap_{J \in \mathcal{J}_I} \Omega_{I,J} .$$

The function $G(x)$ is defined for all $x > -1$. We extend it to the whole real line by defining $G(x) = +\infty$ for $x \leq -1$.

Theorem 6.1. *A. Let $\gamma \in (0,1)$ and $I \in \mathcal{I}$. Let the numbers λ and m_0 be such that $\lambda \geq 2\sqrt{\log \frac{2|\mathcal{J}_I|+1}{\gamma}}$ and $\sqrt{m_0} > \frac{3}{2}\lambda\left(1+\Delta_I\right)$. Then $P\left(\Omega_I\right) \geq 1-\gamma$.*

B. Let $\gamma \in (0,1)$ and $I \in \mathcal{I}$. Let the numbers λ and m_0 be such that $\lambda \geq 2\sqrt{\log \frac{2|\mathcal{J}_I|+1}{\gamma}}$ and $\sqrt{m_0} > 3\lambda\left(1+\Delta_I\right)$. If Δ_I fulfills

$$G\left(-3\Delta_I - 3\lambda\left(1+\Delta_I\right) m_0^{-1/2}\right) \leq \frac{4t_\gamma}{|I|}, \tag{6.1}$$

then on the set Ω_I it holds $T_I \leq t_\gamma$.

C. Let $\gamma \in (0,1)$ and $I \in \mathcal{I}$. Let the numbers λ and m_0 be such that $\lambda \geq 2\sqrt{\log \frac{2|\mathcal{J}_I|+1}{\gamma}}$ and $\sqrt{m_0} > 3\lambda\left(1+\Delta_I\right)$. If Δ_I fulfills

$$\Delta_I \leq \frac{\frac{2\sqrt{2}}{3} t_\gamma^{1/2}\,|I|^{-1/2} - \lambda m_0^{-1/2}}{1+\lambda m_0^{-1/2}},$$

then on the set Ω_I it holds $\overline{T}_I \leq t_\gamma$.

Remark 6.2. *The condition on Δ_I from the part C of the theorem is similar to the condition (6.1) with the function $G(u)$ replaced by $u^2/2$. Moreover, the condition (6.1) follows from $\Delta_I \leq (C t_\gamma^{-1/2}\,|I|^{-1/2} - \lambda m_0^{-1/2})/(1+\lambda m_0^{-1/2})$ with some constant $C > 2\sqrt{2}/3$ provided that $3\Delta_I + 3\lambda\left(1+\Delta_I\right) m_0^{-1/2} < 1/2$, see Lemma 7.3.*

An immediate corollary of this result is an upper bound of the probability of rejecting a "good" interval I.

Corollary 6.3. *Under the conditions of the point B or C of Theorem 6.1 it holds respectively*

$$P\left(T_I > t_\gamma\right) < \gamma \quad \text{or} \quad P\left(\overline{T}_I > t_\gamma\right) < \gamma.$$

Now let us turn to the property B of the intervals of homogeneity. Consider the special case when the vector $\alpha = (\alpha_1, ..., \alpha_n)$ is piecewise constant. In this case an interval I is "good" if it does not contain a change point. The best choice of I can be defined as the interval $I^* = [1, k^*]$, where k^* is the first change point. Theorem 6.1 claims that the interval I^* will be accepted with high probability. The next result shows that all larger intervals will be rejected with high probability, thus implying that $\widehat{I}$ approximately equals I^*.

Theorem 6.4. *Let $\gamma \in (0,1)$ and $2\sqrt{\log \frac{3}{\gamma}} \leq \lambda \leq \sqrt{m}$. Assume that $\alpha_i = \alpha$, for $i \in I^*$, and $\alpha_i = \beta$, for $i \in I \setminus I^*$, where $I = [1, k^* + m]$ and $\alpha \neq \beta$. If m satisfies $m \leq k^*$ and*

$$\sqrt{m} \geq \max\left\{d^{-1}\left(3\sqrt{t_\gamma} + \lambda\right), 4t_\gamma\right\}, \tag{6.2}$$

where $d = |\alpha - \beta| / \left(2\alpha + |\alpha - \beta|\right)$, then

$$P\left(T_I \leq t_\gamma\right) \leq \gamma \quad \text{and} \quad P\left(\overline{T}_I \leq t_\gamma/2\right) \leq \gamma.$$

6.2 Properties of the adaptive estimator $\widehat{\alpha}$.

Let $\widehat{I}$ be the interval computed by the adaptive procedure described in Section 4.1 with the test statistic $T_{I,J}$. The next assertions describe the accuracy of the adaptive estimator $\widehat{\alpha} = \widehat{\alpha}_{\widehat{I}}$ under the condition that $\widehat{I} \supset I^*$, where $I^* \in \mathcal{I}$ is a "good" interval.

Theorem 6.5. *Let $\gamma \in (0,1)$ and $I \in \mathcal{I}$. Let the numbers λ and m_0 be such that $\lambda \geq 2\sqrt{\log \frac{2|\mathcal{J}_I|+1}{\gamma}}$ and $\sqrt{m_0} > \max\left\{\sqrt{4t_\gamma}, \frac{3}{2}\lambda\left(1+\Delta_I\right)\right\}$. Let the interval $I^* \in \mathcal{I}$ be such that $I^* \in \mathcal{J}_I$. If $T_I \leq t_\gamma$, then on the set Ω_I, it holds*

$$\left|\frac{\widehat{\alpha}_I - \widehat{\alpha}_{I^*}}{\widehat{\alpha}_{I^*}}\right| \leq \frac{\rho}{1-\rho},$$

where $\rho = 2\sqrt{t_\gamma \left|I^*\right|^{-1}}$.

From Theorem 6.5 it follows that if $\widehat{\alpha}_{I^*}$ provides a "good" estimate of α_{I^*}, then the adaptive estimator also provides a "good" estimate of α_{I^*}. A precise statement is given in the next corollary.

Corollary 6.6. *Let $\gamma \in (0,1)$ and $I \in \mathcal{I}$. Let the numbers λ and m_0 be such that $\lambda \geq 2\sqrt{\log \frac{2|\mathcal{J}_I|+1}{\gamma}}$ and $\sqrt{m_0} > \max\left\{\sqrt{4t_\gamma}, \frac{3}{2}\lambda\left(1+\Delta_I\right)\right\}$. Let the intervals $I^* \in \mathcal{I}$ and I be such that $I^* \in \mathcal{J}_{\widehat{I}(\omega)}$ and $\widehat{I}(\omega) \in \mathcal{J}_I$, for any $\omega \in \Omega_I$. Then on the set Ω_I the adaptive estimator $\widehat{\alpha}$ fulfills*

$$\frac{\left|\widehat{\alpha} - \alpha_{I^*}\right|}{\alpha_{I^*}} \leq \frac{1}{1-\rho}\frac{\lambda\left(1+\Delta_{I^*}\right)}{\sqrt{|I^*|}} + \frac{\rho}{1-\rho},$$

where $\rho = 2\sqrt{t_\gamma \left|I^*\right|^{-1}}$.

Similar properties can be established for the statistic $\overline{T}_{I,J}$.

7 Proofs of the main results

7.1 Auxiliary statements.

Lemma 7.1. *Let $\xi_1, ..., \xi_m$ be i.i.d. standard exponential r.v.'s and the numbers $\beta_1, ..., \beta_m$ satisfy the condition*

$$\left|\frac{\beta_i}{\overline{\beta}} - 1\right| \leq \Delta, \quad i = 1, ..., m,$$

where $\overline{\beta} = (\beta_1 + ... + \beta_m)/m$ and $\Delta \in [0,1]$. Then, for every $\lambda \leq \frac{2}{3}\sqrt{m}/(1+\Delta)$,

$$P\left(\left|\sum_{i=1}^m \beta_i(\xi_i - 1)\right| > \lambda V_m\right) \leq 2e^{-\lambda^2/4},$$

where $V_m^2 = \beta_1^2 + ... + \beta_m^2$.

Proof. By Chebyshev inequality, for any $u > 0$,

$$P\left(\left|\sum_{i=1}^{m}\beta_i\xi_i\right| > \lambda V_m\right) \leq \frac{E\exp\left(u\sum_{i=1}^{m}\beta_i\left(\xi_i-1\right)\right)}{\exp\left(u\lambda V_m\right)}.$$

Since $\xi_1, ..., \xi_n$ are independent, for any $u < \min\left\{\beta_i^{-1}\right\}$,

$$E\exp\left(u\sum_{i=1}^{m}\beta_i\left(\xi_i-1\right)\right) = \prod_{i=1}^{m}E\exp\left(u\beta_i\left(\xi_i-1\right)\right) = \prod_{i=1}^{m}\frac{\exp\left(-u\beta_i\right)}{1-u\beta_i}.$$

Therefore

$$P\left(\left|\sum_{i=1}^{m}\beta_i\xi_i\right| > \lambda V_m\right) \leq \exp\left(-u\lambda V_m - u\sum_{i=1}^{m}\beta_i - \sum_{i=1}^{m}\log\left(1-u\beta_i\right)\right).$$

This inequality with $u = \frac{\lambda}{2V_m}$ and the elementary inequality $-\log\left(1-x\right) \leq x + x^2$, for $x \leq 1/3$ yield

$$P\left(\left|\sum_{i=1}^{m}\beta_i\xi_i\right| > \lambda V_m\right) \leq \exp\left(-u\lambda V_m - u^2V_m^2\right) = \exp\left(-\frac{\lambda^2}{4}\right).$$

It remains to check that $\lambda \leq \frac{2\sqrt{m}}{3(1+\Delta)}$ implies that $u = \frac{\lambda}{2V_m} < \min\left\{\beta_i^{-1}\right\}$. Indeed $V_m^2 = \sum_{i=1}^{m}\beta_i^2 \geq m\overline{\beta}^2$ and therefore,

$$\beta_i u = \frac{\lambda\beta_i}{2V_m} \leq \frac{\lambda\beta_i}{2\overline{\beta}\sqrt{m}} \leq \frac{\lambda\left(1+\Delta\right)}{2\sqrt{m}} \leq \frac{1}{3},$$

which proves the lemma. $\square$

In the proofs we shall use the following bounds. Recall that $G\left(x\right) = +\infty$, for $x \leq -1$.

Lemma 7.2. *For any $\delta \in [0,1]$ and any real x, the function $G\left(\cdot\right)$ fulfills*

$$\delta\left(1-\delta\right)G\left(|x|\right) \leq \delta G\left((1-\delta)x\right) + \left(1-\delta\right)G\left(-\delta x\right) \leq \delta\left(1-\delta\right)G\left(-\left|x\right|\right). \quad (7.1)$$

Proof. The proof of these bounds is based on the simple fact that the function

$$H\left(x\right) = 2G\left(x\right)/x^2, \quad x > -1, \quad (7.2)$$

is monotonously decreasing. $\square$

Lemma 7.3. *Let $G_{+}^{-1}\left(x\right), x \geq 0$ be the inverse of the function $G\left(\cdot\right)$ on the interval $[0,\infty)$. Then*

$$G_{+}^{-1}\left(x\right) \leq 2\sqrt{x}, \quad 0 \leq x \leq 1/2.$$

Let $G_{-}^{-1}\left(x\right), x \geq 0$ be the inverse of the function $G\left(\cdot\right)$ on the interval $(-1,0]$. Then

$$-G_{-}^{-1}\left(x\right) \geq \sqrt{x}, \quad -1/2 \leq x \leq 0.$$

Proof. For any $a > 0$ and $x \in [0, G(a)]$ it holds $G_+^{-1}(x) \le \sqrt{\frac{2x}{H(a)}}$, where $H(\cdot)$ is defined by (7.2). Taking $a = 1.4$ one gets the first inequality. If $a \in (-1, 0]$ and $x \in [-G(a), 0]$ it holds $-G_-^{-1}(x) \ge \sqrt{\frac{2x}{H(a)}}$. The second inequality is obtained by putting $a = -0.7$. □

We shall also make use of the following bounds of the statistic $T_{I,J}$.

Lemma 7.4. *Let $\varepsilon = |J| / |I|$ and $R_{I,J} = \frac{\widehat{\alpha}_J - \widehat{\alpha}_{J^c}}{\widehat{\alpha}_I}$. Then the statistic $T_{I,J}$ satisfies*

$$\varepsilon (1-\varepsilon) |I| G(|R_{I,J}|) \le T_{I,J} \le \varepsilon (1-\varepsilon) |I| G(-|R_{I,J}|). \tag{7.3}$$

Proof. The trivial equality $|I| \widehat{\alpha}_I = |J| \widehat{\alpha}_J + |J^c| \widehat{\alpha}_{J^c}$ implies

$$\frac{\widehat{\alpha}_J}{\widehat{\alpha}_I} - 1 = (1-\varepsilon) R_{I,J} \quad \text{and} \quad \frac{\widehat{\alpha}_{J^c}}{\widehat{\alpha}_I} - 1 = -\varepsilon R_{I,J}. \tag{7.4}$$

Then the statistic $T_{I,J}$ can be written as

$$T_{I,J} = |I| \left[\varepsilon G((1-\varepsilon) R_{I,J}) + (1-\varepsilon) G(-\varepsilon R_{I,J}) \right]. \tag{7.5}$$

Using (7.1) one gets the required bounds. □

7.2 Proof of Theorem 6.1

Let $I \in \mathcal{I}$. For any $J \in \mathcal{J}_I$ denote $J^c = I \setminus J$. In the following J' denotes one of the intervals J, J^c or I. The definition of the sets $\mathcal{I}$ and $\mathcal{J}_I$ implies that $|J'| \ge m_0$.

Note that the estimator $\widehat{\alpha}_{J'}$ can be written as $\widehat{\alpha}_{J'} = \alpha_{J'} + S_{J'}$. Then, using Lemma 7.1, for any $\lambda \le \frac{2}{3}\sqrt{m_0}/(1+\Delta_I)$, one gets

$$P(\Omega_I) \ge 1 - \sum_{J \in \mathcal{J}_I} P\left(\Omega_{I,J}^c\right) \ge 1 - (2|\mathcal{J}_I| + 1) \exp\left(-\lambda^2/4\right).$$

With $\lambda \ge 2\sqrt{\log \frac{2|\mathcal{J}_I|+1}{\gamma}}$, it holds

$$P(\Omega_I) \ge 1 - \gamma,$$

thus proving the part A of the theorem.

For the part B we have to show that on the random set Ω_I the statistics $T_{I,J}$ and $\overline{T}_{I,J}$ obey $|T_{I,J}| \le t_\gamma$ and $\left|\overline{T}_{I,J}\right| \le \bar{t}_\gamma$, for any $J \in \mathcal{J}_I$.

For the proof we need some inequalities. Note that each α_i satisfies $\alpha_i \le \alpha_I (1+\Delta_I)$, for $i \in I$, and by summing α_i^2 over $i \in J'$, it follows

$$V_{J'}^2 \le (1+\Delta_I)^2 \alpha_I^2 \left|J'\right|. \tag{7.6}$$

The latter inequality implies that, on the set Ω_I, it holds

$$|S_{J'}| \le \lambda V_{J'} / \left|J'\right| \le \lambda \alpha_I (1+\Delta_I) \left|J'\right|^{-1/2}. \tag{7.7}$$

The decomposition $\widehat{\alpha}_{J'} = \alpha_{J'} + S_{J'}$ and the inequality (7.7) imply that, on the set Ω_I,

$$\left|\frac{\widehat{\alpha}_{J'}}{\alpha_{J'}} - 1\right| \le \lambda\left(1+\Delta_I\right)\left|J'\right|^{-1/2}. \tag{7.8}$$

Note that $\left|\frac{\alpha_J - \alpha_{J^c}}{\alpha_I}\right| \le 2\Delta_I$ and $|J'| \ge m_0$. Then, under the assumption $\sqrt{m_0} \ge 3\lambda\left(1+\Delta_I\right)$, the inequality (7.8) implies

$$\begin{aligned}
|R_{I,J}| &\le \frac{2\Delta_I + \lambda\left(1+\Delta_I\right)\left(|J|^{-1/2} + |J^c|^{-1/2}\right)}{1-\lambda\left(1+\Delta_I\right)|I|^{-1/2}} \\
&\le \frac{2\Delta_I + 2\lambda\left(1+\Delta_I\right)m_0^{-1/2}}{1-\lambda\left(1+\Delta_I\right)m_0^{-1/2}} \\
&\le 3\Delta_I + 3\lambda\left(1+\Delta_I\right)m_0^{-1/2}.
\end{aligned} \tag{7.9}$$

We consider first the case of statistic T_I. The bounds (7.3) and (7.9) yield

$$T_{I,J} \le \varepsilon\left(1-\varepsilon\right)|I|\,G\left(-|R_{I,J}|\right) \le \frac{|I|}{4}G\left(-3\Delta_I - 3\lambda\left(1+\Delta_I\right)m_0^{-1/2}\right) \le t_\gamma,$$

and the assertion of Theorem 6.1 concerning T_I follows.

In the same way we prove the assertion concerning $\overline{T}_I$. The inequality $|J|\cdot|J^c| \le |I|^2/4$ implies, on the set Ω_I,

$$\overline{T}_{I,J} \le \frac{|I|}{4}\frac{\left[3\Delta_I + 3\lambda\left(1+\Delta_I\right)m_0^{-1/2}\right]^2}{2} \le \widehat{t}_\gamma.$$

Theorem 6.1 is proved.

7.3 Proof of Theorem 6.4

To keep the same notations as in Theorem 6.1 denote $J = I^*$, $J^c = I \setminus J = [k^*+1, k^*+m]$. Using Lemma 7.1, for any λ and m_0 satisfying $2\sqrt{\log\frac{1}{3\gamma}} \le \lambda \le \frac{2}{3}\sqrt{m_0}/(1+\Delta_I)$, one gets

$$P\left(\Omega_{I,J}\right) \ge 1 - 3e^{-\lambda^2/4} \ge 1-\gamma.$$

It suffices to show that the event $\Omega_{I,J}$ implies $T_{I,J} \ge t_\gamma$. The lower bound in Lemma 7.4 implies

$$T_{I,J} \ge \varepsilon\left(1-\varepsilon\right)|I|\,G\left(|R_{I,J}|\right),$$

with $\varepsilon = |J|/|I|$ and $R_{I,J} = \frac{\widehat{\alpha}_J - \widehat{\alpha}_{J^c}}{\widehat{\alpha}_I}$. Since $k^* \ge m$ it follows that $\varepsilon = k^*/(k^*+m) \ge 1/2$. This and $1-\varepsilon = m/|I|$ imply

$$T_{I,J} \ge \frac{1}{2}mG\left(|R_{I,J}|\right), \tag{7.10}$$

Note that $V_J^2 = k^*\alpha^2$, $V_{J^c}^2 = m\beta^2$ and $V_I \le V_J + V_{J^c}$. Then, similarly to the proof of Theorem 6.1, on the set $\Omega_{I,J}$, it holds

$$|R_{I,J}| \ge \frac{|\alpha_J - \alpha_{J^c}| - \lambda\left(\alpha/\sqrt{k^*} + \beta/\sqrt{m}\right)}{\alpha_I + \lambda\left(\alpha/\sqrt{k^*} + \beta/\sqrt{m}\right)}.$$

For the change point model $\alpha_J = \alpha$, $\alpha_{J^c} = \beta$ and $\alpha_I = \alpha k^*/(k^*+m) + \beta m/(k^*+m)$. This yields

$$|R_{I,J}| \ge \frac{b - \lambda\left(1/\sqrt{k^*} + (1+b)/\sqrt{m}\right)}{1 + b\frac{m}{k^*+m} + \lambda\left(1/\sqrt{k^*} + (1+b)/\sqrt{m}\right)},$$

where $b = \left|\frac{\beta}{\alpha} - 1\right|$. It is easy to see that, for a fixed m, the minimum over $k^* \ge m$ of the latter expression is attained for $k^* = m$. Therefore

$$|R_{I,J}| \ge \frac{b - \lambda(2+b)/\sqrt{m}}{1 + b/2 + \lambda(2+b)/\sqrt{m}} = \frac{d - \lambda/\sqrt{m}}{1/2 + \lambda/\sqrt{m}},$$

where $d = b/(2+b)$. Together with (7.10) this yields

$$T_{I,J} \ge \frac{1}{2} m G\left(\frac{d - \lambda/\sqrt{m}}{1/2 + \lambda/\sqrt{m}}\right).$$

Now the assertion of the theorem amounts to prove that the right hand side in the latter inequality is greater than t_γ. This is equivalent to

$$\frac{d - \lambda/\sqrt{m}}{1/2 + \lambda/\sqrt{m}} \ge G_+^{-1}\left(\frac{2t_\gamma}{m}\right).$$

Since $G_+^{-1}(x) \le 2\sqrt{x}$, for all $x \in [0, 1/2]$ and $m > 4t_\gamma$, it suffices to show that

$$\frac{d - \lambda/\sqrt{m}}{1/2 + \lambda/\sqrt{m}} \ge 2\sqrt{\frac{t_\gamma}{m}}.$$

The latter inequality is implied by the conditions (6.2) and $\lambda \le \sqrt{m}$ of the theorem. This concludes the proof.

7.4 Proof of Theorem 6.5

To keep the same notations as in the proof of Theorem 6.1 let $J = I^*$, $J^c = I \backslash I^*$, $\varepsilon = |J|/|I|$ and $R_{I,J} = (\widehat{\alpha}_J - \widehat{\alpha}_{J^c})/\widehat{\alpha}_I$. It is clear that $T_I \le t_\gamma$ implies $T_{I,J} \le t_\gamma$. The bounds (7.1) imply

$$|I|\,\varepsilon(1-\varepsilon)\,G(|R_{I,J}|) \le T_{I,J} \le t_\gamma,$$

from which it follows that

$$|R_{I,J}| \leq G_+^{-1}\left(\frac{t_\gamma}{\varepsilon(1-\varepsilon)|I|}\right),$$

where $G_+^{-1}(x)$, $x \geq 0$ is the inverse of the function $G(\cdot)$ on the interval $[0,\infty)$. Now by the definition of the set $\mathcal{J}_I$ one has $\varepsilon = |J|/|I| \geq 1/2$. Since $m_0 > 4t_\gamma$ it holds

$$\frac{t_\gamma}{\varepsilon(1-\varepsilon)|I|} \leq \frac{\frac{1}{4}m_0}{\frac{1}{2}|J|} \leq \frac{1}{2}.$$

An applications of the upper bound in Lemma 7.3 yields

$$|R_{I,J}| \leq 2\sqrt{\frac{t_\gamma}{\varepsilon(1-\varepsilon)|I|}}.$$

From the identities (7.4) it follows that $R_{I,J} = \left(\frac{\widehat{\alpha}_J}{\widehat{\alpha}_I} - 1\right)/(1-\varepsilon)$, which together with the previous inequality gives

$$\left|\frac{\widehat{\alpha}_J}{\widehat{\alpha}_I} - 1\right| \leq \frac{2\sqrt{(1-\varepsilon)t_\gamma}}{\sqrt{\varepsilon|I|}} \leq \frac{2\sqrt{t_\gamma}}{\sqrt{|J|}}.$$

This implies

$$\left|\frac{\delta}{1-\delta}\right| \leq 2\sqrt{t_\gamma |J|^{-1}},$$

where $\delta = (\widehat{\alpha}_J - \widehat{\alpha}_I)/\widehat{\alpha}_J$, which in turn implies $|\delta| \leq \rho/(1-\rho)$, where $\rho = 2\sqrt{t_\gamma |J|^{-1}}$, and the assertion concerning T_I follows. The case of the statistic $\overline{T}_I$ can be handled in the same way.

7.5 Proof of Corollary 6.6

Since $\Omega_{I'} \subset \Omega_I$, for any $I' \subset I$, Theorem 6.5 implies that on the set Ω_I,

$$|\widehat{\alpha}_I - \widehat{\alpha}_{I^*}| \leq \widehat{\alpha}_{I^*}\frac{\rho}{1-\rho}.$$

From this it follows that, on the set Ω_I,

$$|\widehat{\alpha} - \alpha_{I^*}| \leq |\widehat{\alpha} - \widehat{\alpha}_{I^*}| + |\widehat{\alpha}_{I^*} - \alpha_{I^*}| \leq \frac{\rho}{1-\rho}\alpha_{I^*} + \frac{1}{1-\rho}|\widehat{\alpha}_{I^*} - \alpha_{I^*}|.$$

Since, on the set Ω_I,

$$|\widehat{\alpha}_{I^*} - \alpha_{I^*}| = |S_{I^*}| \leq \frac{\lambda V_{I^*}}{|I^*|},$$

one gets

$$\frac{|\widehat{\alpha} - \alpha_{I^*}|}{\alpha_{I^*}} \leq \frac{1}{1-\rho}\frac{\lambda V_{I^*}}{\alpha_{I^*}|I^*|} + \frac{\rho}{1-\rho}.$$

The inequality $V_{I^*}^2 \leq (1+\Delta_{I^*})^2 \, \alpha_{I^*}^2 \, |I^*|$ (see (7.6)) implies

$$\frac{|\widehat{\alpha} - \alpha_{I^*}|}{\alpha_{I^*}} \leq \frac{1}{1-\rho} \frac{\lambda\,(1+\Delta_{I^*})}{\sqrt{|I^*|}} + \frac{\rho}{1-\rho}.$$

8 Appendix

Table 3. The list of distribution functions used in the simulations.

	$F(x)$	Parameters
Pareto	$1 - x^{-1/\alpha},\ x \geq 1$	$\alpha = 1$
Pareto-log	$F(x) = 1 - (x/e)^{-1/\alpha} \log x,\ x \geq e$	$\alpha = 1$
Pareto-CP	$1 - \left(\frac{x}{x_1}\right)^{-1/\alpha_1},\ \text{if } x_1 \leq x < x_2$ $1 - \left(\frac{x_2}{x_1}\right)^{-1/\alpha_1} \left(\frac{x}{x_2}\right)^{-1/\alpha_2},\ \text{if } x > x_2$	$\alpha_1 = 1/2,\ \alpha_2 = 1$ $x_1 = 1,\ x_2 = 5$
Cauchy-plus	$F(x) = \frac{2}{\pi} \arctan x,\ x \geq 0$	
GPD	$1 - (1 + \alpha \frac{x-a}{\sigma})^{-1/\alpha},\ x \geq a$	$a = 0, \sigma = 1, \alpha = 1$
Hall model	$1 - cx^{-1/\alpha}(1 + x^{-1/\beta}),\ x \geq 1$	$\alpha = 1, \beta = 1$

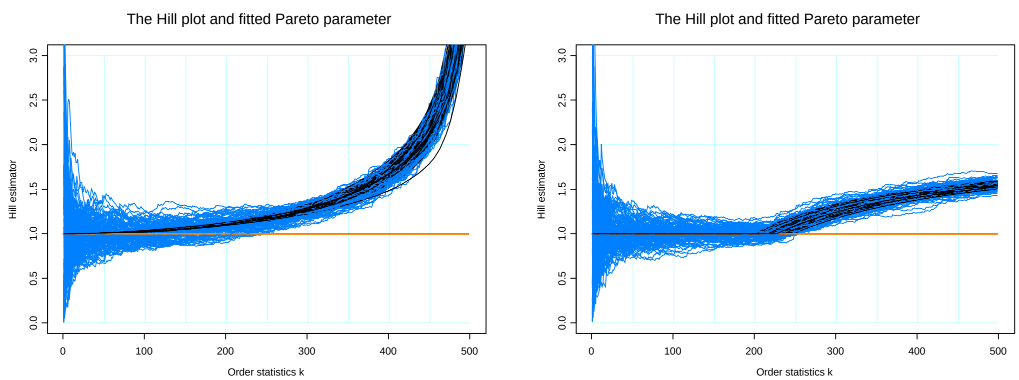

Figure 4. 100 realizations of the Hill estimator for Cauchy-plus (left) and Pareto-CP (right) d.f.'s and the corresponding fitted Pareto parameters. Here the dark lines represent the fitted Pareto parameter computed from the approximation formula (3.5) and the light ones are the corresponding Hill plots.

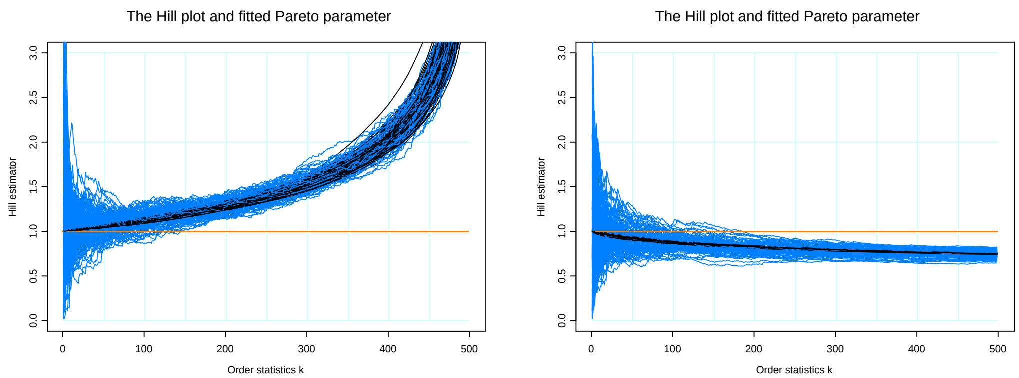

Figure 5. 100 realizations of the Hill estimator for GPD (left) d.f. and for the Hall model (right) and the corresponding fitted Pareto parameters. Here the dark lines represent the fitted Pareto parameter computed from the approximation formula (3.5) and the light ones are the corresponding Hill plots.

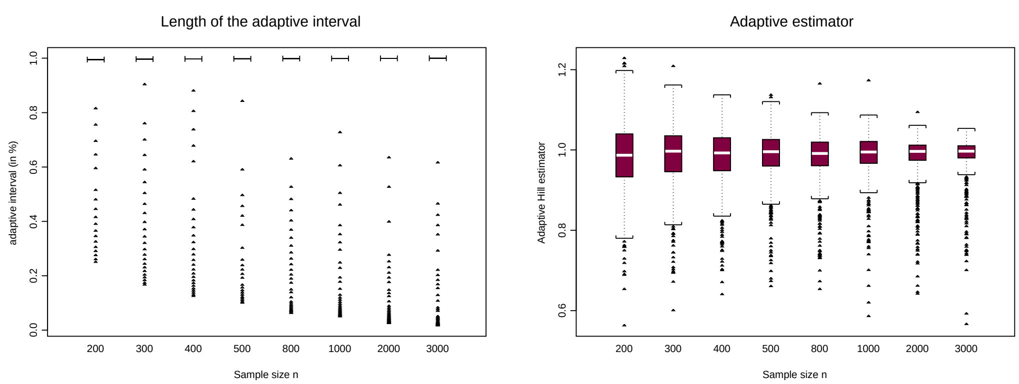

Figure 6. Box-plots of selected intervals (in %) and the adaptive estimators for Pareto d.f. from 500 realization for different sample sizes.

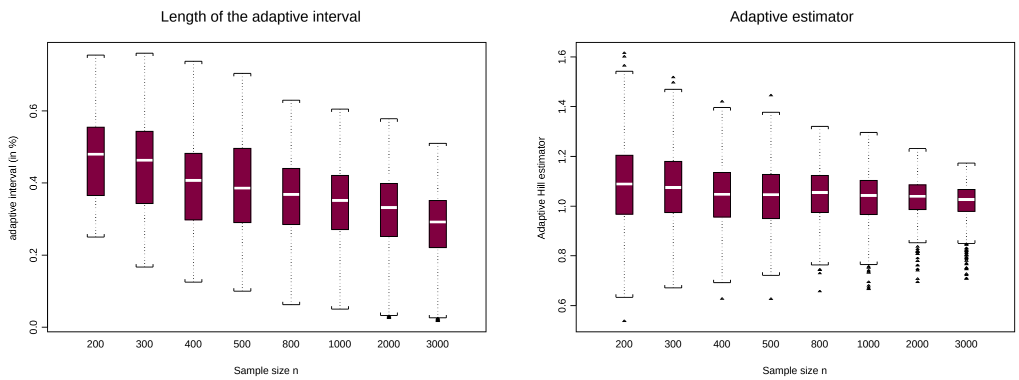

Figure 7. Box-plots of selected intervals (in %) and the adaptive estimators for Cauchy-plus d.f. from 500 realization for different sample sizes.

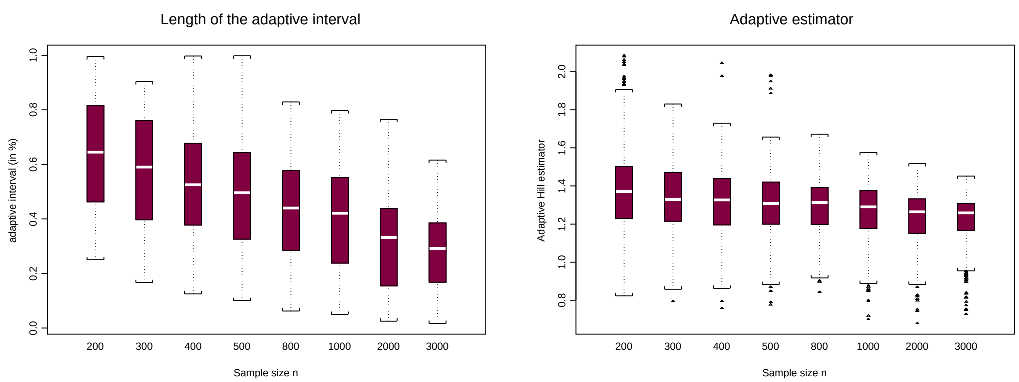

Figure 8. Box-plots of selected intervals (in %) and the adaptive estimators for Pareto-log d.f. from 500 realization for different sample sizes.

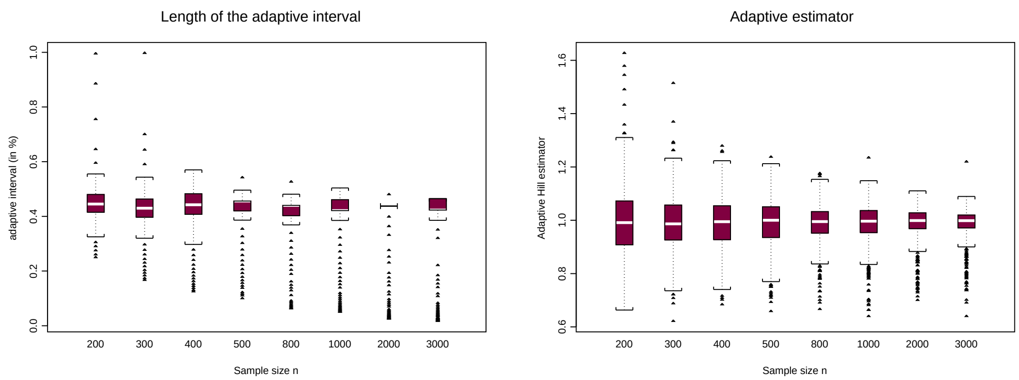

Figure 9. Box-plots of selected intervals (in %) and the adaptive estimators for Pareto-CP d.f. from 500 realization for different sample sizes.

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Classification of $GL(2,\mathbb{R})$-orbit's dimensions for the differential equations' system with homogeneities of the 4th order

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Abstract. Center-affine invariant conditions for $GL(2,\mathbb{R})$-orbit's dimensions are defined for two-dimensional autonomous system of differential polynomial equations with homogeneities of the 4th order.

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Consider two-dimensional differential system with homogeneities of the 4th order

$$\frac{dx^j}{dt} = a^j_{\alpha\beta\gamma\delta}x^\alpha x^\beta x^\gamma x^\delta \quad (j,\alpha,\beta,\gamma,\delta = \overline{1,2}), \tag{1}$$

where the coefficient tensor $a^j_{\alpha\beta\gamma\delta}$ is symmetrical in lower indices in which the complete convolution holds.

Consider also the group of center-affine transformations $GL(2,\mathbb{R})$ given by the equalities

$$\bar{x}^1 = \alpha x^1 + \beta x^2, \quad \bar{x}^2 = \gamma x^1 + \delta x^2, \quad \Delta = det\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \neq 0.$$

Further will use the notations

$$a^1_{1111} = a, \quad a^1_{1112} = b, \quad a^1_{1122} = c, \quad a^1_{1222} = d, \quad a^1_{2222} = e, \quad a^2_{1111} = f, \quad a^2_{1112} = g,$$
$$a^2_{1122} = h, \quad a^2_{1222} = k, \quad a^2_{2222} = l, \quad x^1 = x, \quad x^2 = y. \tag{2}$$

According to [1] and taking into consideration (2) the representation operators of the group $GL(2,\mathbb{R})$ in the space of coefficients and variables of the system (1) will take the form

$$X_1 = x\frac{\partial}{\partial x} - 3a\frac{\partial}{\partial a} - 2b\frac{\partial}{\partial b} - c\frac{\partial}{\partial c} + e\frac{\partial}{\partial e} - 4f\frac{\partial}{\partial f} - 3g\frac{\partial}{\partial g} - 2h\frac{\partial}{\partial h} - k\frac{\partial}{\partial k};$$

$$X_2 = y\frac{\partial}{\partial x} + f\frac{\partial}{\partial a} + (g-a)\frac{\partial}{\partial b} + (h-2b)\frac{\partial}{\partial c} + (k-3c)\frac{\partial}{\partial d} + (l-4d)\frac{\partial}{\partial e} -$$

$$-f\frac{\partial}{\partial g}-2g\frac{\partial}{\partial h}-3h\frac{\partial}{\partial k}-4k\frac{\partial}{\partial l};$$

$$X_3=x\frac{\partial}{\partial y}-4b\frac{\partial}{\partial a}-3c\frac{\partial}{\partial b}-2d\frac{\partial}{\partial c}-e\frac{\partial}{\partial d}+(a-4g)\frac{\partial}{\partial f}+$$

$$+(b-3h)\frac{\partial}{\partial g}+(c-2k)\frac{\partial}{\partial h}+(d-l)\frac{\partial}{\partial k}+e\frac{\partial}{\partial l};$$

$$X_4=y\frac{\partial}{\partial y}-b\frac{\partial}{\partial b}-2c\frac{\partial}{\partial c}-3d\frac{\partial}{\partial d}-4e\frac{\partial}{\partial e}+f\frac{\partial}{\partial f}-h\frac{\partial}{\partial h}-2k\frac{\partial}{\partial k}-3l\frac{\partial}{\partial l}. \quad (3)$$

The operators (3) form a four-dimensional reductive Lie algebra [1].

Let $\tilde{a}=(a,b,...,l)\in E^{10}(\tilde{a})$, where $E^{10}(\tilde{a})$ is Euclidean space of the coefficients of the right-hand sides of the system (1). Denote by $\tilde{a}(q)$ the point from $E^{10}(\tilde{a})$ that corresponds to the system, obtained from the system (1) with coefficients $\tilde{a}$ by a transformation $q\in GL(2,\mathbb{R})$.

Definition 1. Call the set $O(\tilde{a})=\{\tilde{a}(q)|q\in GL(2,\mathbb{R})\}$ the $GL(2,\mathbb{R})$-orbit of the point $\tilde{a}$ for the system (1).

Definition 2. Call the set $M\subseteq E^{10}(\tilde{a})$ the $GL(2,\mathbb{R})$-invariant if for any point $\tilde{a}\in M$ its orbit $O(\tilde{a})\subseteq M$.

It is known from [1] that

$$dim_{\mathbb{R}}O(\tilde{a})=rankM_1, \quad (4)$$

where M_1 is the following matrix

$$M_1=\begin{pmatrix} 3a & 2b & c & 0 & -e & 4f & 3g & 2h & k & 0\\ -f & a-g & 2b-h & 3c-k & 4d-l & 0 & f & 2g & 3h & 4k\\ 4b & 3c & 2d & e & 0 & 4g-a & 3h-b & 2k-c & l-d & -e\\ 0 & b & 2c & 3d & 4e & -f & 0 & h & 2k & 3l \end{pmatrix}, \quad (5)$$

constructed on coordinate vectors of operators (3).

Will use the following notations for the matrix M_1: denote by Δ_{ijkl} the minor of the 4th order constructed on columns i, j, k, l, $(i,j,k,l=\overline{1,10})$; denote by Δ^{ijk}_{lmn} the minor of the 3rd order constructed on lines i, j, k, $(i,j,k=\overline{1,4})$ and columns l, m, n, $(l,m,n=\overline{1,10})$; and by Δ^{ij}_{kl} will be denoted the minor of the 2nd order constructed on lines i, j, $(i,j=\overline{1,4})$ and columns k, l, $(k,l=\overline{1,10})$.

For the system (1) two comitants of the first order with respect to its coefficients are known from [2]

$$F_3=(a+g)x^3+3(b+h)x^2y+3(c+k)xy^2+(d+l)y^3,$$

$$F_5=-fx^5+(a-4g)x^4y+(4b-6h)x^3y^2+(6c-4k)x^2y^3+(4d-l)xy^4+ey^5. \quad (6)$$

According to [3], write a transvectant of index k for binary forms f and φ as follows

$$(f,\varphi)^{(k)} = \frac{(r-k)!(\rho-k)!}{r!\rho!}\sum_{h=0}^{k}(-1)^h \mathrm{C}_k^h \frac{\partial^k f}{\partial x^{k-h}\partial y^h}\frac{\partial^k \varphi}{\partial x^h \partial y^{k-h}}, \tag{7}$$

where r and ρ are degrees of these forms with respect to x and y correspondingly.

According to [4], the transvectant (7) on two comitants of the system (1) is a comitant (invariant) of this system too.

Taking into consideration the above mentioned, the following comitants and invariants of the system (1) were constructed in [2]:

$$\begin{gathered}
L_1=(F_5,F_5)^{(2)},\quad L_2=(F_5,F_5)^{(4)},\quad L_3=(F_3,F_3)^{(2)},\\
L_4=(F_3,F_5)^{(1)},\quad L_5=(F_3,F_5)^{(2)},\quad L_6=(F_3,F_5)^{(3)},\\
L_7=(L_2,F_5)^{(2)},\quad B_1=(L_3,L_3)^{(2)},\quad B_2=(L_1,L_1)^{(6)},\\
B_3=(L_1,L_4)^{(6)},\quad B_4=(L_3,L_6)^{(2)},\quad B_5=(L_5,L_5)^{(4)},\\
B_6=((L_3,L_5)^{(2)},L_3)^{(2)},\quad B_7=((L_7,L_7)^{(2)},L_2)^{(2)},\\
B_8=((((F_3,L_4)^{(2)},F_3)^{(2)},L_5)^{(2)},L_5)^{(4)},\\
B_9=(((L_7,L_7)^{(2)},L_7)^{(1)},L_7)^{(3)},\\
C_2=(L_1,L_1)^{(2)}.
\end{gathered} \tag{8}$$

Lemma 1. *For $F_5\equiv 0$ the rang of matrix M_1 is equal to four if and only if $B_1\neq 0$, where B_1 is from* (8).

Proof. Taking into consideration (6) from $F_5\equiv 0$ we obtain

$$e=f=0,\quad a=4g,\quad b=\frac{3}{2}h,\quad k=\frac{3}{2}c,\quad l=4d. \tag{9}$$

As conditions (9) hold the matrix M_1 takes the form

$$M_1^{(1)} = \begin{pmatrix}
12g & 3h & c & 0 & 0 & 0 & 3g & 2h & \frac{3}{2}c & 0\\
0 & 3g & 2h & \frac{3}{2}c & 0 & 0 & 0 & 2g & 3h & 6c\\
6h & 3c & 2d & 0 & 0 & 0 & \frac{3}{2}h & 2c & 3d & 0\\
0 & \frac{3}{2}h & 2c & 3d & 0 & 0 & 0 & h & 3c & 12d
\end{pmatrix}. \tag{10}$$

As conditions (9) hold the invariant B_1 takes the form

$$B_1=-\frac{625}{8}(S^2-4TR), \tag{11}$$

where

$$R=2cg-h^2,\quad S=4dg-ch,\quad T=2dh-c^2. \tag{12}$$

We note that all nonzero minors of the 4th order of matrix $M_1^{(1)}$ up to an constant factor coincide with $\Delta_{1234}=-\dfrac{108}{625}B_1$.

Hence, for $F_5 \equiv 0$, $B_1 \neq 0$ the rang of the matrix (5) is equal to four. Lemma 1 is proved.

Lemma 2. *For $F_5 \equiv 0$ the rang of matrix M_1 is equal to three if and only if holds*

$$B_1 = 0, \;\; L_3 \not\equiv 0, \tag{13}$$

where B_1, L_3 are from (8).

Proof. As conditions (9) hold, considering (1) and (2) the comitant L_3 takes the form

$$L_3 = \frac{25}{2}(Rx^2 + Sxy + Ty^2), \tag{14}$$

where R, S, T are from (12). Calculations yield that any nonzero third order minor of the matrix (10) up to a constant factor coincides with one of the following minors:

$$\Delta^{123}_{123} = -36hR + 18gS; \quad \Delta^{123}_{124} = -27cR; \quad \Delta^{123}_{134} = -9cS;$$

$$\Delta^{123}_{234} = -\frac{9}{2}cT; \quad \Delta^{124}_{123} = 36gR; \quad \Delta^{124}_{124} = 27gS;$$

$$\Delta^{124}_{134} = 36gT; \quad \Delta^{124}_{234} = -\frac{9}{4}(-3hT + 2dR);$$

$$\Delta^{134}_{123} = -72gT + 27hS; \quad \Delta^{134}_{124} = 54dR; \quad \Delta^{134}_{134} = 18dS;$$

$$\Delta^{134}_{234} = 9dT; \quad \Delta^{234}_{123} = -18hR; \quad \Delta^{234}_{124} = -\frac{27}{2}hS;$$

$$\Delta^{234}_{134} = -18hT; \quad \Delta^{234}_{234} = -9cT + \frac{9}{2}dS. \tag{15}$$

The necessity of the conditions (13) follows from Lemma 1, (14) and(15). It is evident that, according to (11) for $B_1 = 0$, $S^2 = 4TR$ holds and from $L_3 \not\equiv 0$ (see (14)) at least one of R, S and T will be nonzero. This fact with $c^2 + d^2 + g^2 + h^2 \neq 0$ ensure that at least one of minors (15) will be nonzero. Lemma 2 is proved.

Lemma 3. *For $F_5 \equiv 0$ the rang of matrix M_1 is equal to two if and only if*

$$L_3 \equiv 0, \;\; F_3 \not\equiv 0, \tag{16}$$

where F_3 is from (6) *and L_3 is from* (8).

Proof. According to (9) from (14) for $L_3 \equiv 0$ we obtain $T = R = S = 0$, where R, S, T are from (12). This implies $B_1 = 0$ and from Lemma 2 $rangM_1 < 3$. Will show that in this case $rangM_1 = 2$ if and only if

$$F_3 = 5gx^3 + \frac{15}{2}hx^2y + \frac{15}{2}cxy^2 + 5dy^3 \not\equiv 0. \tag{17}$$

And this is ensured by the existence of the following second order minors of the matrix $M_1^{(1)}$

$$\Delta_{12}^{12} = 36g^2, \quad \Delta_{12}^{34} = 9h^2, \quad \Delta_{34}^{12} = \frac{3}{2}c^2, \quad \Delta_{34}^{34} = 6d^2.$$

Lemma 3 is proved.

The next result is evident.

Lemma 4. *For $F_5 \equiv 0$ the rang of matrix M_1 is equal to zero if and only if $F_3 \equiv 0$, where F_3 is from* (6).

From Lemmas 1–4 and equality (4) follows

Theorem 5. *For $F_5 \equiv 0$ the dimension of $GL(2,\mathbb{R})$-orbit of the system* (1) *is equal to*

4 for $B_1 \neq 0$;

3 for $B_1 = 0, \quad L_3 \not\equiv 0$;

2 for $L_3 \equiv 0, \quad F_3 \not\equiv 0$;

0 for $F_3 \equiv 0$,

where F_3 and F_5 are from (6), *and B_1, L_3 are from* (8).

Lemma 6. *For $F_3 \equiv 0$ the rang of matrix M_1 is equal to four if and only if*

$$3L_1L_2 + 105C_2 + 26F_5L_7 \not\equiv 0, \tag{18}$$

where F_3 and F_5 are from (6), *and L_1, L_2, L_7, C_2 are from* (8).

Proof. Taking into consideration (6) from $F_3 \equiv 0$ we obtain

$$a = -g, \ b = -h, \ c = -k, \ d = -l. \tag{19}$$

On the other hand, according to [5] such $GL(2,\mathbb{R})$-transformation exists that the comitant F_5 will take the form $F_5 = y\tilde{F}_4$, where $\tilde{F}_4$ is the polynomial of the forth order on variables, corresponding to the system (1) after the transformation. Due to this we obtain that $f = 0$. As this holds from conditions (19) we obtain that the matrix M_1 takes the form

$$M_1^{(2)} = \begin{pmatrix} -3g & -2h & -k & 0 & -e & 0 & 3g & 2h & k & 0 \\ 0 & -2g & -3h & -4k & -5l & 0 & 0 & 2g & 3h & 4k \\ -4h & -3k & -2l & e & 0 & 5g & 4h & 3k & 2l & -e \\ 0 & -h & -2k & -3l & 4e & 0 & 0 & h & 2k & 3l \end{pmatrix}. \tag{20}$$

Nonzero 4th order minors of the matrix (20) will coincide up to the numerical factor with one of the following:

$$\Delta_{1234} = 12eg^2k - 9egh^2 + 36g^2l^2 - 129ghkl + 72gk^3 + 72h^3l - 48h^2k^2;$$

$$\Delta_{1235} = -48eg^2l + 156eghk - 108eh^3 - 30ghl^2 + 90gk^2l - 60h^2kl;$$
$$\Delta_{1236} = 60g^3k - 45g^2h^2;$$
$$\Delta_{1245} = 24e^2g^2 + 39eghl + 144egk^2 - 144eh^2k + 135gkl^2 - 120h^2l^2;$$
$$\Delta_{1246} = 90g^3l - 60g^2hk; \quad \Delta_{1256} = -120eg^3 - 75g^2hl;$$
$$\Delta_{1345} = 36e^2gh + 126egkl + 36eh^2l - 96ehk^2 + 90gl^3 - 60hkl^2;$$
$$\Delta_{1346} = 135g^2hl - 120g^2k^2; \quad \Delta_{1356} = -180eg^2h - 150g^2kl;$$
$$\Delta_{1456} = -240eg^2k - 225g^2l^2;$$
$$\Delta_{2345} = -12e^2gk + 27e^2h^2 - 12egl^2 + 114ehkl - 72ek^3 + 60hl^3 - 45k^2l^2;$$
$$\Delta_{2346} = -30g^2kl + 90gh^2l - 60ghk^2; \quad \Delta_{2356} = 60eg^2k - 135egh^2 - 75ghkl;$$
$$\Delta_{2456} = 30eg^2l - 180eghk - 150ghl^2; \quad \Delta_{3456} = 45eghl - 120egk^2 - 75gkl^2. \quad (21)$$

Also holds the equality

$$3L_1L_2 + 105C_2 + 26F_5L_7 = -2\Delta_{1236}x^8 - 4\Delta_{1246}x^7y - 2(\Delta_{1256} + 3\Delta_{1346})x^6y^2 -$$
$$-4(\Delta_{1356} + 2\Delta_{2346})x^5y^3 + 2(-5\Delta_{1234} + \Delta_{15610} - 3\Delta_{2356})x^4y^4 -$$
$$-4(2\Delta_{1235} + \Delta_{2456})x^3y^5 - 2(3\Delta_{1245} + \Delta_{3456})x^2y^6 - 4\Delta_{1345}xy^7 - 2\Delta_{2345}y^8. \quad (22)$$

Let prove the necessity of condition (18). Assume the contrary, i.e. for $3L_1L_2 + 105C_2 + 26F_5L_7 \equiv 0$ there exists at least one nonzero 4th order minor of the matrix M_1. Taking into consideration (6), (8), (19) , (21) and (22), from $3L_1L_2 + 105C_2 + 26F_5L_7 \equiv 0$ we obtain the following series of conditions for coefficients of the system (1):

$$I. \qquad g = h = k = 0; \qquad (23)$$
$$II. \qquad g = h = 0,\ e = -\frac{5l^2}{8k},\ k \neq 0; \qquad (24)$$
$$III. \qquad g = 0,\ l = \frac{2k^2}{3h},\ e = -\frac{10k^3}{27h^2},\ h \neq 0; \qquad (25)$$
$$IV. \qquad k = \frac{3h^2}{9g},\ l = \frac{h^3}{2g^2},\ e = -\frac{5h^4}{16g^3},\ g \neq 0. \qquad (26)$$

With the aid of (21) one can verify that while any of the series of the conditions (23)–(26) holds, all the 4th order minors of the matrix (20) will be equal to zero. Thus obtained contradiction proves the necessity of condition (18).

The sufficiency of condition (18) is ensured by equality (22). Lemma 6 is proved.

Lemma 7. *For $F_3 \equiv 0$ the rang of matrix M_1 is equal to three if and only if*

$$3L_1L_2 + 105C_2 + 26F_5L_7 \equiv 0,\ L_1 \not\equiv 0, \qquad (27)$$

where F_3 and F_5 are from (6), and L_1, L_2, L_7, C_2 are from (8).

Proof. In proof we will use the $GL(2,\mathbb{R})$-transformation from the proof of Lemma 6, and therefore, the equality $f = 0$ can be assumed. According to Lemma 6, as first condition from (27) holds for the coefficients of the system (1) besides (19) we obtain the values (23)–(26).

Consider the conditions (23). The matrix M_1 takes the form

$$M_1^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 & -e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5l & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2l & e & 0 & 0 & 0 & 0 & 2l & -e \\ 0 & 0 & 0 & -3l & 4e & 0 & 0 & 0 & 0 & 3l \end{pmatrix} \tag{28}$$

and the comitant L_1 takes the form

$$L_1 = -2l^2y^6. \tag{29}$$

Hence, it is evident that the condition $L_1 \not\equiv 0$ is sufficient and necessary.

Consider the conditions (24). The matrix M_1 takes the form (with $e = -\dfrac{5l^2}{8k}$)

$$M_1^{(4)} = \begin{pmatrix} 0 & 0 & -k & 0 & -e & 0 & 0 & 0 & k & 0 \\ 0 & 0 & 0 & -4k & -5l & 0 & 0 & 0 & 0 & 4k \\ 0 & -3k & -2l & e & 0 & 0 & 0 & 3k & 2l & -e \\ 0 & 0 & -2k & -3l & 4e & 0 & 0 & 0 & 2k & 3l \end{pmatrix} \tag{30}$$

and the comitant L_1 takes the form $L_1 = -\frac{3}{4}(4kx+ly)^2y^4$. Since $k \neq 0$, considering (30) and (24) we obtain $L_1 \not\equiv 0$ and $\Delta_{234}^{123} = -12k^3 \neq 0$, i.e. $rangM_1^{(4)} = 3$.

Consider the conditions (25). The matrix M_1 takes the form (with the values of the parameters l and e from (25))

$$M_1^{(5)} = \begin{pmatrix} 0 & -2h & -k & 0 & -e & 0 & 0 & 2h & k & 0 \\ 0 & 0 & -3h & -4k & -5l & 0 & 0 & 0 & 3h & 4k \\ -4h & -3k & -2l & e & 0 & 0 & 4h & 3k & 2l & -e \\ 0 & -h & -2k & -3l & 4e & 0 & 0 & h & 2k & 3l \end{pmatrix} \tag{31}$$

and the comitant L_1 takes the form

$$L_1 = -\frac{4}{27h^2}(3hx+ky)^4y^2.$$

So, as $h \neq 0$ we get $L_1 \not\equiv 0$ as well as $\Delta_{123}^{123} = -24h^3 \neq 0$.

Consider the conditions (26).

The matrix M_1 takes the form $M_1^{(2)}$ (with the values of the parameters k, l and e from (26)), and the comitant L_1 takes the form

$$L_1 = -\frac{1}{32g^4}(2gx+hy)^6.$$

Since $g \neq 0$ we obtain $L_1 \not\equiv 0$ and $\Delta_{126}^{123} = 30g^3 \neq 0$, i.e. $rangM_1^{(2)} = 3$.

Lemma 7 is proved.

Lemma 8. *For $F_3 \equiv 0$ the rang of matrix M_1 is equal to two if and only if*

$$L_1 \equiv 0, \;\; F_5 \not\equiv 0, \tag{32}$$

where F_3 and F_5 are from (6)*, and L_1 is from* (8)*.*

Proof. As $C_2 = (L_1, L_1)^{(2)}$ (see (8)) it is evident that from $L_1 \equiv 0$ follows $C_2 \equiv 0$. Moreover, as L_1 is the Hessian of the comitant F_5 then, for $L_1 \equiv 0$ it follows $F_5 = (\alpha x + \beta y)^5$, $a, b \in \mathbb{R}$ (see [3]). Considering (8) it is easy to verify that the transvectant $L_7 = (((\alpha x + \beta y)^5, (\alpha x + \beta y)^5)^4, (\alpha x + \beta y)^5)^2 = 0$. Hence, the condition $L_1 \equiv 0$ implies $3L_1L_2 + 105C_2 + 26F_5L_7 \equiv 0$ and then from Lemma 7 follows the necessity of the conditions (32).

Let prove the sufficiency. Assume $L_1 \equiv 0$, i.e. F_5 must be of the form $F_5 = (\alpha x + \beta y)^5$ (see above). On the other hand, as it was mentioned in the proof of Lemma 6, we assume $f = 0$ due to a $GL(2, \mathbb{R})$-transformation. Hence $\alpha = 0$ and considering (19) and (6) we obtain $g = h = k = l = 0$, $F_5 = ey^5$. In this case the matrix M_1 takes the form

$$M_1^{(6)} = \begin{pmatrix} 0 & 0 & 0 & 0 & -e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & -e \\ 0 & 0 & 0 & 0 & 4e & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{33}$$

It is evident that for $F_5 \equiv 0$ all 2nd order minors of the matrix $M_1^{(6)}$ will be equal to zero, and for $F_5 \not\equiv 0$ the 2nd order minor $\Delta_{45}^{13} = e^2$ will be nonzero.

Lemma 8 is proved.

With the aid of Lemmas 4-8 and equality (4) is proved

Theorem 9. *For $F_3 \equiv 0$ the dimension of $GL(2, \mathbb{R})$-orbit of the system* (1) *is equal to*

4 for $3L_1L_2 + 105C_2 + 26F_5L_7 \not\equiv 0$;

3 for $3L_1L_2 + 105C_2 + 26F_5L_7 \equiv 0, \quad L_1 \not\equiv 0$;

2 for $L_1 \equiv 0, \quad F_5 \not\equiv 0$;

0 for $F_5 \equiv 0$,

where F_3 and F_5 are from (6)*, and L_1, L_2, L_7, C_2 are from* (8)*.*

Lemma 10. *For $F_3F_5 \not\equiv 0$ the rang of matrix M_1 is equal to four if and only if*

$$12L_4^2 - 3L_3F_5^2 + 6L_1F_3^2 \not\equiv 0, \tag{34}$$

where F_3, F_5 are from (6) *and L_1, L_3, L_4 are from* (8)*.*

Proof. The sufficiency of the condition (34) follows from the equality

$$12L_4^2 - 3L_3F_5^2 + 6L_1F_3^2 = (\Delta_{1267} + \Delta_{1678})x^{12} + (4\Delta_{1268} + 2\Delta_{1367} + 2\Delta_{1679} + 4\Delta_{2678})x^{11}y +$$

$$+(\Delta_{1236} + 5\Delta_{1269} + 10\Delta_{1278} + 9\Delta_{1368} + \Delta_{1467} + \Delta_{16710} + 3\Delta_{1689} + 2\Delta_{2367} + 8\Delta_{2679} +$$

$$
\begin{aligned}
&+6\Delta_{3678})x^{10}y^2+(4\Delta_{1237}+2\Delta_{1246}+2\Delta_{12610}+16\Delta_{1279}+12\Delta_{1369}+24\Delta_{1378}+6\Delta_{1468}+\\
&+2\Delta_{16810}+4\Delta_{1789}+12\Delta_{2368}+4\Delta_{26710}+12\Delta_{2689}+12\Delta_{3679}+4\Delta_{4678})x^9y^3+(6\Delta_{1238}+\\
&+8\Delta_{1247}+\Delta_{1256}+7\Delta_{12710}+14\Delta_{1289}+3\Delta_{1346}+5\Delta_{13610}+40\Delta_{1379}+9\Delta_{1469}+18\Delta_{1478}+\\
&+\Delta_{1568}+\Delta_{16910}+3\Delta_{17810}+18\Delta_{2369}+36\Delta_{2378}+8\Delta_{2468}-\Delta_{2567}+8\Delta_{26810}+16\Delta_{2789}-\\
&-2\Delta_{3467}+6\Delta_{36710}+18\Delta_{3689}+8\Delta_{4679}+\Delta_{5678})x^8y^4+(4\Delta_{1239}+12\Delta_{1248}+4\Delta_{1257}+\\
&+8\Delta_{12810}+12\Delta_{1347}+2\Delta_{1356}+18\Delta_{13710}+36\Delta_{1389}+4\Delta_{14610}+32\Delta_{1479}+2\Delta_{1569}+\\
&+4\Delta_{1578}+2\Delta_{17910}+4\Delta_{2346}+8\Delta_{23610}+64\Delta_{2379}+16\Delta_{2469}+32\Delta_{2478}+4\Delta_{26910}+\\
&+12\Delta_{27810}+2\Delta_{3567}+12\Delta_{36810}+24\Delta_{3789}+4\Delta_{46710}+12\Delta_{4689}+2\Delta_{5679})x^7y^5+(\Delta_{12310}+\\
&+8\Delta_{1249}+6\Delta_{1258}+3\Delta_{12910}+18\Delta_{1348}+8\Delta_{1357}+21\Delta_{13810}+\Delta_{1456}+15\Delta_{14710}+\\
&+30\Delta_{1489}+\Delta_{15610}+8\Delta_{1579}+\Delta_{18910}+16\Delta_{2347}+3\Delta_{2356}+30\Delta_{23710}+60\Delta_{2389}+\\
&+8\Delta_{24610}+64\Delta_{2479}+3\Delta_{2569}+6\Delta_{2578}+8\Delta_{27910}+6\Delta_{3469}+12\Delta_{3478}-3\Delta_{3568}+6\Delta_{36910}+\\
&+18\Delta_{37810}-\Delta_{4567}+8\Delta_{46810}+16\Delta_{4789}+\Delta_{56710}+3\Delta_{5689})x^6y^6+(2\Delta_{12410}+4\Delta_{1259}+\\
&+12\Delta_{1349}+12\Delta_{1358}+8\Delta_{13910}+4\Delta_{1457}+18\Delta_{14810}+4\Delta_{15710}+8\Delta_{1589}+24\Delta_{2348}+\\
&+12\Delta_{2357}+36\Delta_{23810}+2\Delta_{2456}+32\Delta_{24710}+64\Delta_{2489}+2\Delta_{25610}+16\Delta_{2579}+4\Delta_{28910}+\\
&+4\Delta_{34610}+32\Delta_{3479}+12\Delta_{37910}-2\Delta_{4568}+4\Delta_{46910}+12\Delta_{47810}+2\Delta_{56810}+4\Delta_{5789})x^5y^7+\\
&+(\Delta_{12510}+3\Delta_{13410}+8\Delta_{1359}+6\Delta_{1458}+7\Delta_{14910}+5\Delta_{15810}+16\Delta_{2349}+18\Delta_{2358}+\\
&+14\Delta_{23910}+8\Delta_{2457}+40\Delta_{24810}+9\Delta_{25710}+18\Delta_{2589}+\Delta_{3456}+18\Delta_{34710}+36\Delta_{3489}+\\
&+\Delta_{35610}+8\Delta_{3579}+6\Delta_{38910}-\Delta_{4569}-2\Delta_{4578}+8\Delta_{47910}+\Delta_{56910}+3\Delta_{57810})x^4y^8+\\
&+(2\Delta_{13510}+4\Delta_{1459}+2\Delta_{15910}+4\Delta_{23410}+12\Delta_{2359}+12\Delta_{2459}+16\Delta_{24910}+12\Delta_{25810}+\\
&+4\Delta_{3457}+24\Delta_{34810}+6\Delta_{35710}+12\Delta_{3589}+4\Delta_{48910}+2\Delta_{57910})x^3y^9+(\Delta_{14510}+3\Delta_{23510}+\\
&+8\Delta_{2459}+5\Delta_{25910}+6\Delta_{3458}+10\Delta_{34910}+9\Delta_{35810}+\Delta_{45710}+2\Delta_{4589}+\\
&+\Delta_{58910})x^2y^{10}+(2\Delta_{24510}+4\Delta_{3459}+4\Delta_{35910}+2\Delta_{45810})xy^{11}+(\Delta_{34510}+\Delta_{45910})y^{12},
\end{aligned}
\tag{35}
$$

where Δ_{ijkl}, $(1 \le< j < k < l \le 10)$ are the minors of the matrix M_1.

Let us prove the necessity of the condition (34). Assume the contrary, i.e. suppose that the condition

$$12L_4^2 - 3L_3F_5^2 + 6L_1F_3^2 \equiv 0 \tag{36}$$

is satisfied. We claim that in this case all minors Δ_{ijkl} $(1 \le< j < k < l \le 10)$ of the fourth degree vanish. Indeed, since the comitant $F_3 \not\equiv 0$ is a cubic binary form in x and y, via a center-affine transformation [3] it can be brought to one of the following 3 canonical forms (depending on its factorization over $\mathbb{C}$):

$$(i)\quad Ax(x^2 \pm y^2); \qquad (ii)\quad Ax^2y; \qquad (iii)\quad Ax^3,$$

where $A \neq 0$ due to $F_3 \not\equiv 0$. According to [5], these canonical forms can be used in order to construct the transformed system (1) via the same center-affine transformation. We shall consider each case separately.

(i) $F_3 = Ax(x^2 \pm y^2)$. Taking into consideration (6) we obtain the following values for the coefficients of the system (1):

$$a = A - g,\ b = -h,\ c = \pm\frac{1}{3}A - k,\ d = -l.$$

Then considering (36) we get the following relations:

$$b = d = e = f = h = l = 0, \quad a = 4g = \frac{4}{5}A, \quad k = \frac{3}{2}c = \pm\frac{1}{5}A.$$

However for these values of the coefficients of system (1) we obtain $F_5 \equiv 0$ and this contradicts to lemma's condition $F_3F_5 \not\equiv 0$.

(ii) $F_3 = Ax^2y$. Considering (6) in this case we have

$$a = -g,\ b = \frac{1}{3}A - h,\ c = -k,\ d = -l.$$

Then from the identity (36) we calculate

$$a = c = d = e = f = g = k = l = 0, \quad b = \frac{1}{3}A - h. \tag{37}$$

In this case the matrix M_1 takes the form

$$M_1^{(7)} = \begin{pmatrix} 0 & \frac{2}{3}(A-3h) & 0 & 0 & 0 & 0 & 0 & 2h & 0 & 0 \\ 0 & 0 & \frac{2}{3}A - 3h & 0 & 0 & 0 & 0 & 0 & 3h & 0 \\ \frac{4}{3}(A-3h) & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3}A + 4h & 0 & 0 & 0 \\ 0 & \frac{1}{3}(A-3h) & 0 & 0 & 0 & 0 & 0 & h & 0 & 0 \end{pmatrix}$$

and $F_5 = \frac{2}{3}(2A - 15h)x^3y^2$. It is easy to observe that all 4^{th} order minors of the matrix $M_1^{(7)}$ are equal to zero.

(iii) $F_3 = Ax^3$. In the same manner as above in this case we obtain

$$a = A - g,\ b = -h,\ c = -k,\ d = -l.$$

and then from (36) we get

$$b = c = d = e = h = k = l = 0, \quad a = A - g. \tag{38}$$

For these values of the coefficients of system (1) the matrix M_1 takes the form

$$M_1^{(8)} = \begin{pmatrix} 3(A-g) & 0 & 0 & 0 & 0 & 4f & 3g & 0 & 0 & 0 \\ -f & A - 2g & 0 & 0 & 0 & 0 & f & 2g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -A + 5g & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -f & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $F_5 = -\big[fx + (5g - A)y\big]x^4$. And we observe again that all 4^{th} order minors of the matrix $M_1^{(8)}$ are equal to zero. As all possible cases are considered our claim is proved. This has completed the proof of Lemma 10.

Lemma 11. *For $F_3F_5 \not\equiv 0$ the rang of matrix M_1 is equal to three if and only if*

$$12L_4^2 - 3L_3F_5^2 + 6L_1F_3^2 \equiv 0, \tag{39}$$

where F_3, F_5 are from (6) *and L_1, L_3, L_4 are from* (8).

Proof. The necessity follows from Lemma 10.

Let prove the sufficiency. Assume $12L_4^2 - 3L_3F_5^2 + 6L_1F_3^2 \equiv 0$. Since $F_3F_5 \not\equiv 0$ from the proof of Lemma 10 it follows that we need to consider only two series of relations among the coefficients: (37) and (38).

If the relations (37) hold then the sufficiency is ensured by the equality

$$F_3F_5^2 = 3(\Delta_{123}^{123} + 4\Delta_{129}^{123} + 9\Delta_{138}^{123} - 6\Delta_{378}^{123})x^8y^5.$$

In the case when (38) holds then the sufficiency is ensured by the equality

$$3F_3F_5^2 = \Delta_{167}^{124}x^{13} + 2(\Delta_{126}^{124} + 3\Delta_{168}^{124} + 4\Delta_{267}^{124})x^{12}y - (\Delta_{126}^{123} + 3\Delta_{168}^{123} + 4\Delta_{267}^{123})x^{11}y^2.$$

Lemma 11 is proved.

From Theorems 5, 9 and Lemmas 10–11 follows

Theorem 12. *The dimension of $GL(2,\mathbb{R})$-orbit of the system* (1) *is equal to*

4 for $F_5 \equiv 0$, $B_1 \not\equiv 0$, or
$F_3 \equiv 0$, $3L_1L_2 + 105C_2 + 26F_5L_7 \not\equiv 0$, or
$F_3F_5(12L_4^2 - 3L_3F_5^2 + 6L_1F_3^2) \not\equiv 0$;

3 for $F_5 + B_1 \equiv 0$, $L_3 \not\equiv 0$, or
$F_3 + 3L_1L_2 + 105C_2 + 26F_5L_7 \equiv 0$, $L_1 \not\equiv 0$, or
$F_3F_5 \not\equiv 0$, $12L_4^2 - 3L_3F_5^2 + 6L_1F_3^2 \equiv 0$;

2 for $F_5 + L_3 \equiv 0$, $F_3 \not\equiv 0$, or
$F_3 \equiv L_1 \equiv 0$, $F_5 \not\equiv 0$;

0 for $F_3 \equiv F_5 \equiv 0$,

where F_3 and F_5 are from (6), *and B_1, L_1, L_2, L_3, L_4, L_7, C_2 are from* (8).

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Power sets of n-ary quasigroups

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Abstract. In the theory of latin squares and in the binary quasigroup theory the notion of a latin power set (a quasigroup power set) is known. These sets have a good property, and namely, they are orthogonal sets. Such sets were studied and methods of their construction were suggested in different articles (see, for example, [1–5]).
In this article we introduce (k)-powers of a k-invertible n-ary operation (with respect to the k-multiplication of n-ary operations) and (k)-power sets of n-ary quasigroups, $n \geq 2$, $1 \leq k \leq n$, prove pairwise orthogonality of such sets and consider distinct posibilities of their construction with the help of binary groups, in particular, using $n-T$-quasigroups and n-ary groups.

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1 Introduction

In the theory of latin squares the notion of a power set of latin squares or a latin power set is known. In the articles [1–4] some properties and different methods of constructing such sets , in particular, sets based and not based on groups, were considered. In [5] an algebraic approach to the study of latin power set was used and a new method of constructing quasigroup power sets based on cyclic S-systems (such systems correspond to a particular case of latin power sets [6]) and on pairwise balanced block designs of index one (BIB(v,b,r,k,1)) [7] was suggested.

Any power set of latin squares (of quasigroups) is an orthogonal set and can be used in applications, in particular, by the construction of some codes and ciphers. Such a ciphering device whose algorithm is based on a latin power set has been patented [8]. In [4] it was noticed, " It is obvious that latin power sets based on non-group tables are more preferable to those based on group tables because the greater irregularity makes the cipher safer".

In this article we introduce and study the power sets of n-ary quasigroups, in particular, prove pairwise orthogonality of such sets, consider distinct posibilities of their construction.

2 Necessary notions and results

We recall some notations, concepts and results which are used in the article. At first remember the following denations and notes from [9]. By x_i^j we will denote the sequence $x_i, x_{i+1}, \dots, x_j$, $i \le j$. If $j < i$, then x_i^j is the empty sequence, $\overline{1,n} = \{1, 2, \dots, n\}$. Let Q be a finite or an infinite set, $n \ge 2$ be a positive integer and let Q^n denote the Cartesian power of the set Q.

An *n-ary operation* A (briefly, an *n-operation*) on a set Q is a mapping $A : Q^n \to Q$ defined by $A(x_1^n) \to x_{n+1}$, and in this case we write $A(x_1^n) = x_{n+1}$.

A *finite n-groupoid* (Q, A) *of order* m is a set Q with one n-ary operation A defined on Q, where $|Q| = m$.

An *n-ary quasigroup (n-quasigroup)* is an n-groupoid such that in the equality

$$A(x_1^n) = x_{n+1}$$

each of n elements from x_1^{n+1} uniquely defines the $(n+1)$-th element. Usually a quasigroup n-operation A is itself considered as an n-quasigroup.

The n-operation E_i, $1 \le i \le n$, on Q with $E_i(x_1^n) = x_i$ is called the *i-th identity operation* (or the *i-th selector) of arity n*.

An n-operation A on Q is called *i-invertible* for some $i \in \overline{1,n}$ if the equation

$$A(a_1^{i-1}, x_i, a_{i+1}^n) = a_{n+1}$$

has a unique solution for each fixed n-tuple $(a_1^{i-1}, a_{i+1}^n, a_{n+1}) \in Q^n$.

For an i-invertible n-operation there exists the *i-inverse n-operation* ${}^{(i)}A$ defined in the following way:

$${}^{(i)}A(x_1^{i-1}, x_{n+1}, x_{i+1}^n) = x_i \Leftrightarrow A(x_1^n) = x_{n+1}$$

for all $x_1^{n+1} \in Q^{n+1}$.

It is evident that

$$A(x_1^{i-1}, {}^{(i)}A(x_1^n), x_{i+1}^n) = {}^{(i)}A(x_1^{i-1}, A(x_1^n), x_{i+1}^n) = x_i$$

and ${}^{(i)}[{}^{(i)}A] = A$ for $i \in \overline{1,n}$.

Let Ω_n be the set of all n-ary operations on a finite or infinite set Q. On Ω_n define a binary operation $\underset{i}{\oplus}$ (the i-multiplication) in the following way:

$$(A \underset{i}{\oplus} B)(x_1^n) = A(x_1^{i-1}, B(x_1^n), x_{i+1}^n),$$

$A, B \in \Omega_n, x_1^n \in Q^n$. Shortly this equality can be written as

$$A \underset{i}{\oplus} B = A(E_1^{i-1}, B, E_{i+1}^n)$$

where E_i is the i-th selector.

In [10] it was proved that $(\Omega_n; \underset{i}{\oplus})$ is a semigroup with the identity E_i. If Λ_i is the set of all i-invertible n-operations from Ω_n for some $i \in \overline{1,n}$, then $(\Lambda_i; \underset{i}{\oplus})$ is a group. In this group E_i is the identity, the inverse element of A is the operation ${}^{(i)}A \in \Lambda_i$, since $A \underset{i}{\oplus} E_i = E_i \underset{i}{\oplus} A$, $A \underset{i}{\oplus} {}^{(i)}A = {}^{(i)}A \underset{i}{\oplus} A = E_i$.

An n-ary quasigroup (Q, A) (or simply A), is an n-groupoid with an i-invertible n-operation for each $i \in \overline{1, n}$ [9].

Let $(x_1^n)_k$ denote the $(n-1)$-tuple $(x_1^{k-1}, x_{k+1}^n) \in Q^{n-1}$ and let A be an n-operation, then the $(n-1)$-operation A_a:

$$A_a(x_1^n)_k = A(x_1^{k-1}, a, x_{k+1}^n)$$

is called *the* $(n-1)$*-retract* of A, defined by position k, $k \in \overline{1, n}$, with the element a in this position (with $x_k = a$) [9].

An n-ary operation A on Q is called *complete* if there exists a permutation $\overline{\varphi}$ on Q^n such that $A = E_1\overline{\varphi}$ (that is $A(x_1^n) = E_1\overline{\varphi}(x_1^n)$). If a complete n-operation A is finite and has order m, then the equation $A(x_1^n) = a$ has exactly m^{n-1} solutions for any $a \in Q$ [10].

Any i-invertible (for some fixed i, $i \in \overline{1, n}$) n-operation A is complete, but there exist complete n-operations which are not i-invertible for each $i \in \overline{1, n}$ [10].

For $n \geq 2$, an *n-dimensional hypercube* (briefly, an *n-hypercube) of order m* is an $\underbrace{m \times m \times \cdots \times m}_{n}$ array with m^n points based upon m distinct symbols [11].

A hypercube is a generalization of a *latin square*, which in the case of squares of *order m*, is an $m \times m$ array in which m distinct symbols are arranged so that each symbol occurs once in each row and column. A latin square is a 2-dimensional hypercube of a special type.

In [12] the connection between n-hypercubes and (algebraic) n-ary operations was established. In addition we note that a k-invertible operation A_H corresponds to an n-hypercube H with the following property: whenever $n-1$ of the n coordinates, except the k-th coordinate, are fixed, each of the m symbols appears exactly one time in that subarray (in that k-th column). In this case the mapping $L_{(\overline{a})_k} = A_H(a_1^{k-1}, x, a_{k+1}^n)$ is a permutation on Q for each $(\overline{a})_k \in Q^{n-1}$ where $(\overline{a})_k = (a_1^{k-1}, a_{k+1}^n)$. In the theory of n-quasigroups this permutation is called the k-th translation of the n-quasigroup (Q, A_H) defined by the $(n-1)$-tuple $(\overline{a})_k$ [9].

In the case of n-ary operations for $n > 2$ it is possible to consider different versions of orthogonality. The weakest is the notion of the pairwise orthogonality.

Definition 1 [12]. *Two n-ary operations ($n \geq 2$) A and B given on a set Q of order m are called orthogonal (shortly, $A \perp B$) if the system $\{A(x_1^n) = a, B(x_1^n) = b\}$ has exactly m^{n-2} solutions for any $a, b \in Q$.*

This concept corresponds to two orthogonal n-dimensional hypercubes [12]. Two n-hypercubes H_1 and H_2 of order m are *orthogonal* if when superimposed, each of the m^2 ordered pairs appears m^{n-2} times [15],[11].

Definition 2 [12]. *A set $\Sigma = \{A_1^t\}$, $t \geq 2$, of n-operations is called pairwise orthogonal if any pair of distinct n-operations from Σ is orthogonal.*

In [13] the following criterion of orthogonality of two finite k-invertible n-operations was established.

Theorem 1 [13]. *Let k be a fixed number from $\overline{1,n}$. Two finite k-invertible n-operations A and B on a set Q are orthogonal if and only if the $(n-1)$-retract C_a of the n-operation $C = B \underset{k}{\oplus} {}^{(k)}A$, defined by $x_k = a$, is complete for every $a \in Q$.*

Definition 3. *We shall say that an n-operation C, given on a set Q, has the k-property if its $(n-1)$-retract C_a, defined by $x_k = a$, is complete for every $a \in Q$.*

Note that any n-quasigroup has the k-property for each $k \in \overline{1,n}$ since any its $(n-1)$-retract is an $(n-1)$-quasigroup.

3 Power sets of n-ary quasigroups and pairwise orthogonality

Let L be a latin square of order m, given on a set Q by its rows $\alpha_1, \alpha_2, \ldots, \alpha_m$ (which are permutations of Q). Then power l of L is defined as

$$L^l = (\alpha_1^l, \alpha_2^l, \ldots, \alpha_m^l).$$

If $L, L^2, \ldots, L^s$ are all latin squares, then the set $\{L, L^2, \ldots, L^s\}$ is called a *latin power set of size s*.

It is known that a binary quasigroup (Q, A) corresponds to every latin square L given on a set Q and if $\{L, L^2, \ldots, L^s\}$ is a latin power set, then $\{A, A^2, \ldots, A^s\}$ is the corresponding quasigroup power set where $A^l = A \cdot A \cdot \ldots \cdot A$ (l times), $1 \le l \le s$, $(A \cdot A)(x, y) = A^2(x, y) = A(x, A(x, y))$ [5].

Consider an analog of powers for n-operations. Let k be a fixed number of $\overline{1,n}$, A be a k-invertible n-operation.

The power $A^l = A \underset{k}{\oplus} A \underset{k}{\oplus} \ldots \underset{k}{\oplus} A$ (l times) with respect to the k-multiplication of n-ary operations is called the *(k)-power l of A*.

Note that if all (k)-powers $A, A^2, \ldots, A^s$ are n-quasigroups, then they are different, that is form a set, since the equality $A^t = A^r$, $t, r \in \overline{1,s}$, $t > r$, implies $A^{t-r} = E_k$ for $t - r < s$.

Definition 4. *A set $\Sigma_k = \{A, A^2, \ldots, A^s\}_k$, $s \ge 2$, is called a (k)-power set of n-quasigroups if all (k)-powers of A from Σ_k are n-quasigroups.*

Note that index k after a set shows additionally that the powers in this set are taken with respect to the k-multiplication of operations.

Using Theorem 1 it is easy to prove the following statement for any k-invertible n-operations, in particular, for n-quasigroups.

Theorem 2. *Let A be a finite k-invertible n-operation and the (k)-powers $A, A^2, \ldots, A^s$, $s \ge 2$, of A be different. Then the set $\Sigma_k = \{A, A^2, \ldots, A^s\}_k$ is a pairwise orthogonal if and only if each of the n-operations $A, A^2, \ldots, A^{s-1}$ has the k-property.*

Proof. At first we remember that all k-invertible n-operations, given on a set Q, form a group with the identity E_k with respect to the k-multiplication, ${}^{(k)}A$ is the inverse element of A in this group and $({}^{(k)}A)^l = {}^{(k)}(A^l)$. Let $1 \leq i \leq s-1$, $i < j \leq s$. By Theorem 1 $A^j \perp A^i$ if and only if the n-operation $A^j \underset{k}{\oplus} ({}^{(k)}A)^i = A^{j-i}$ has the k-property for any $1 \leq j-i \leq s-1$. □

For a binary operation A on a set Q 2-invertibility means that the equation $A(a, y) = b$ has a unique solution for any $a, b \in Q$. If A has the 2-property, then the equation $A(x, a) = b$ has a unique solution for any $a, b \in Q$, that is A is 1-invertible also. Thus, in Theorem 2 all (2)-powers $A, A^2, \ldots, A^{s-1}$ of a binary (2)-invertible operation A must be quasigroups, A^s can be only (2)-invertible and is true the following

Corollary 1. *Let A be a finite 2-invertible binary operation and the (2)-powers $A, A^2, \ldots, A^s$, $s \geq 2$, of A are different. Then the set $\Sigma_2 = \{A, A^2, \ldots, A^s\}_2$ is orthogonal if and only if $A, A^2, \ldots, A^{s-1}$ are quasigroups.*

In [1] the following result (Corollary 5a) with respect to latin power sets which we shall formulate in the language of quasigroups was proved, where $A^{-1} = {}^{(2)}A$ is the right inverse quasigroup for A ($A^{-1}(x, y) = z \Leftrightarrow A(x, z) = y$).

Proposition 1 [1]. *If $A, A^2, \ldots, A^s$, $s \geq 2$, are finite quasigroups, then any s successive quasigroups from $(A^{-1})^s, (A^{-1})^{s-1}, \ldots, A^{-1}, A, A^2, \ldots, A^s$ form an orthogonal set of quasigroups.*

Now we prove that for n-ary case, $n \geq 2$, an analogous situation takes place.

Theorem 3. *If a set $\Sigma_k = \{A, A^2, \ldots, A^s\}_k$ is a (k)-power set of finite quasigroups, then in the sequence*

$$({}^{(k)}A)^s, ({}^{(k)}A)^{s-1}, \ldots, ({}^{(k)}A)^2, {}^{(k)}A, A, A^2, \ldots, A^s$$

every s-tuple of successive n-quasigroups is a pairwise orthogonal set.

Proof. Let $1 \leq i, j \leq s$, $i < j$, then $A^j \underset{k}{\oplus} ({}^{(k)}A)^i = A^j \underset{k}{\oplus}{}^{(k)} (A^i) = A^{j-i} \in \Sigma_k$, so the n-operation A^{j-i} is an n-quasigroup, all its $(n-1)$-retracts are $(n-1)$-quasigroups too, so they have the k-property and by Theorem 2 we have $A^i \perp A^j$. On the other hand, by the same restrictions on i, j we obtain ${}^{(k)}(A^i) \underset{k}{\oplus} A^j = A^{j-i} \in \Sigma_k$, so ${}^{(k)}(A^i) \perp {}^{(k)}(A^j)$ by Theorem 2.

Let $1 \leq i \leq s-1$, $1 \leq j \leq s-i$, then $A^i \underset{k}{\oplus}{}^{(k)} ({}^{(k)}(A^j)) = A^i \underset{k}{\oplus} A^j = A^{i+j} \in \Sigma_k$, so $A^i \perp {}^{(k)}(A^j)$ by Theorem 2 (see the previous case). □

Corollary 2. *If in Theorem 3, in addition, $s+1$ is the smallest exponent such that $A^{s+1} = E_k$, then the sequence from the theorem is $A, A^2, \ldots, A^s$.*

Proof. Indeed, by these conditions $({}^{(k)}A)^i = A^{s+1-i}$ for all $i \in \overline{1, s}$. □

Theorem 4. *Let (Q, A) be a finite n-quasigroup of the form*

$$A(x_1^n) = \alpha_1 x_1 \cdot \ldots \cdot \alpha_{k-1} x_{k-1} \cdot x_k \cdot \alpha_{k+1} x_{k+1} \cdot \ldots \cdot \alpha_n x_n$$

for some fixed $k \in \overline{1,n}$, where α_i is a permutation of Q for every $i \in \overline{1,n}$, $i \neq k$, $(Q,\cdot)$ is a binary group. Then $\Sigma_k = \{A, A^2, \dots, A^s\}_k$ is a (k)-power set of n-quasigroups if and only if in the group $(Q,\cdot)$ the mapping $x \to x^l$ is a permutation for each $l \in \overline{2,s}$.

Proof. Let an n-quasigroup (Q,A) have the form of the theorem, then

$$A^2(x_1^n) = A(x_1^{k-1}, A(x_1^n), x_{k+1}^n) = \alpha_1 x_1 \cdot \ldots \cdot \alpha_{k-1}x_{k-1}\cdot$$
$$(\alpha_1 x_1 \cdot \ldots \cdot \alpha_{k-1}x_{k-1} \cdot x_k \cdot \alpha_{k+1}x_{k+1} \cdot \ldots \cdot \alpha_n x_n) \cdot \alpha_{k+1}x_{k+1} \cdot \ldots \cdot \alpha_n x_n =$$
$$(\alpha_1 x_1 \cdot \ldots \cdot \alpha_{k-1}x_{k-1})^2 \cdot x_k \cdot (\alpha_{k+1}x_{k+1} \cdot \ldots \cdot \alpha_n x_n)^2.$$

Taking into account that $A^l(x_1^n) = A^{l-1}(x_1^{k-1}, A(x_1^n), x_{k+1}^n)$ we shall obtain by the same way

$$A^l(x_1^n) = (\alpha_1 x_1 \cdot \ldots \cdot \alpha_{k-1}x_{k-1})^l \cdot x_k \cdot (\alpha_{k+1}x_{k+1} \cdot \ldots \cdot \alpha_n x_n)^l \tag{1}$$

for any $l \in \overline{1,s}$.

Let A^l be a finite n-quasigroup for some l, $2 \leq l \leq s$, then it is i-invertible for each $i \in \overline{1,n}$, that is for any $(n-1)$-tuple $(a_1^n)_i \in Q^{n-1}$,

$$A^l(a_1^{i-1}, x, a_{i+1}^n) = A^l(a_1^{i-1}, y, a_{i+1}^n) \Leftrightarrow x = y. \tag{2}$$

If $i \in \overline{1,k-1}$, then we have

$$(\alpha_1 a_1 \cdot \ldots \cdot \alpha_{i-1}a_{i-1} \cdot \alpha_i x \cdot \alpha_{i+1}a_{i+1} \cdot \ldots \cdot \alpha_{k-1}a_{k-1})^l \cdot a_k \cdot (\alpha_{k+1}a_{k+1} \cdot \ldots \cdot \alpha_n a_n)^l =$$
$$(\alpha_1 a_1 \cdot \ldots \cdot \alpha_{i-1}a_{i-1} \cdot \alpha_i y \cdot \alpha_{i+1}a_{i+1} \cdot \ldots \cdot \alpha_{k-1}a_{k-1})^l \cdot a_k \cdot (\alpha_{k+1}a_{k+1} \cdot \ldots \cdot \alpha_n a_n)^l$$

$\Leftrightarrow x = y$. Doing the respective cancelation we obtain

$$(a \cdot \alpha_i x \cdot b)^l = (a \cdot \alpha_i y \cdot b)^l \Leftrightarrow x = y, (L_a R_b \alpha_i x)^l = (L_a R_b \alpha_i y)^l \Leftrightarrow x = y,$$

where $a = \alpha_1 a_1 \cdot \ldots \cdot \alpha_{i-1}a_{i-1}$, $b = \alpha_{i+1}a_{i+1} \cdot \ldots \cdot \alpha_{k-1}a_{k-1}$, $L_a x = a \cdot x$, $R_b x = x \cdot b$. Changing x (y) with $\alpha_i^{-1}R_b^{-1}L_a^{-1}x$ $(\alpha_i^{-1}R_b^{-1}L_a^{-1}y)$, we obtained

$$x^l = y^l \Leftrightarrow \alpha_i^{-1}R_b^{-1}L_a^{-1}x = \alpha_i^{-1}R_b^{-1}L_a^{-1}y \Leftrightarrow x = y$$

by each $i \in \overline{1,k-1}$.

Let $i \in \overline{k+1,n}$. Then from (2) it follows

$$(\alpha_1 a_1 \cdot \ldots \cdot \alpha_{k-1}a_{k-1})^l \cdot a_k \cdot (\alpha_{k+1}a_{k+1} \cdot \ldots \cdot \alpha_{i-1}a_{i-1} \cdot \alpha_i x \cdot \alpha_{i+1}a_{i+1} \cdot \ldots \cdot \alpha_n a_n)^l =$$
$$(\alpha_1 a_1 \cdot \ldots \cdot \alpha_{k-1}a_{k-1})^l \cdot a_k \cdot (\alpha_{k+1}a_{k+1} \cdot \ldots \cdot \alpha_{i-1}a_{i-1} \cdot \alpha_i y \cdot \alpha_{i+1}a_{i+1} \cdot \ldots \cdot \alpha_n a_n)^l$$

$\Leftrightarrow x = y$,

$$(c \cdot \alpha_i x \cdot d)^l = (c \cdot \alpha_i y \cdot d)^l \Leftrightarrow x = y$$

where $c = \alpha_{k+1}a_{k+1} \cdot \ldots \cdot \alpha_{i-1}a_{i-1}, d = \alpha_{i+1}a_{i+1} \cdot \ldots \cdot \alpha_n a_n$. Then $x^l = y^l \Leftrightarrow x = y$ for all $l \in \overline{1,s}$ (see the previous case).

Thus, if all (k)-powers $A, A^2, A^3, \dots, A^s$ are n-quasigroups, then in the group $(Q,\cdot)$ the mapping $x \to x^l$ is a permutation for each $l \in \overline{1,s}$.

Conversely, let all mappings $x \to x^l$, $l \in \overline{2,s}$, be permutations in the group $(Q,\cdot)$. Then all k-powers $A, A^2, A^3, \dots, A^s$, defined by (1) are different, that is they form a set. Indeed, if $A^t(x_1^n) = A^r(x_1^n)$, $1 \le r, t \le s$ and $t > r$, then from (1) we have

$$x^t x_k y^t = x^r x_k y^r, \quad x^{t-r} x_k y^{t-r} = x_k$$

for any $x, y \in Q$. Setting $y = e$ (the identity of the group $(Q,\cdot)$) in the last equality we obtain that $x^{t-r}x_k = x_k$ and $x^{t-r} = e$ for $t - r < s$ and any $x \in Q$. But by the conditions all mappings $x \to x^l$, $l \in \overline{1,s}$, are permutations, so we have contradiction.

It remains to show that all (k)-powers are n-quasigroups. For that we can prove (2) fixing an arbitrary $(n-1)$-tuple $(a_1^n)_i$ of elements of Q and making the inverse transformations corresponding to the case $i \in \overline{1,k-1}$ $(i \in \overline{k+1,n})$. That is every (k)-power l of the finite n-quasigroup (Q,A) is i-invertible for any $i \in \overline{1,n}$, $i \neq k$. But the n-operation A^l is always k-invertible as a power with respect to the k-multiplication. Thus, (Q, A^l) is an n-quasigroup for each $l \in \overline{1,s}$ and the set $\Sigma_k = \{A, A^2, \dots A^s\}_k$ is a (k)-power set of n-ary quasigroups. □

Corollary 3. *Let $(Q,+)$ be an abelian group of order m, $m = p_1^{\beta_1} p_2^{\beta_1} \dots p_t^{\beta_t}$ be decomposition in prime multipliers, $p_1 < p_2 < \dots < p_t$, $p_1 \ge 3$, k be a fixed element, $1 \le k \le n$, (Q,A) be an n-quasigroup of the form:*

$$A(x_1^n) = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_{k-1} x_{k-1} + x_k + \alpha_{k+1} x_{k+1} + \dots + \alpha_n x_n,$$

where all α_i are permutations. Then $\Sigma_k = \{A, A^2, \dots, A^{p_1-1}\}_k$ is a (k)-power set of n-quasigroups.

Proof. In an abelian group of order m with the zero 0 all mappings $x \to 2x$, $x \to 3x, \dots, (p_1-1)x$ are permutations. Otherwise, $lx = ly \Rightarrow l(x-y) = 0$ if $x \neq y$, $2 \le l < p_1$, it means that in the group $(Q,+)$ there exists an element which has the order smaller than p_1. We have contradiction with Lagrange's Theorem stating that the order of any subgroup (and the order of any element) divides the order of a finite group [14]. Now use Theorem 4. □

For a finite elementary abelian group (that is a group which is a direct power of a group of a prime order [14]) from Corollary 3 immediately follows

Corollary 4. *If in Corollary 3 $(Q,+)$ is an elementary abelian group of order $m = p^t$, $p \ge 3$, then $\Sigma_k = \{A, A^2, \dots A^{p-1}\}_k$ is a (k)-power set of n-quasigroups.*

Corollary 5. *Let in Corollary 3 the order of an abelian group $(Q,+)$ be a prime number p, $p \ge 3$ and an n-quasigroup A have the form*

$$A(x_1^n) = x_1 + x_2 + \dots + x_n,$$

then $\Sigma_k = \{A, A^2, \dots, A^{p-1}\}_k$ is a (k)-power set of n-quasigroups for each $k \in \overline{1,n}$.

Remark. Note that in general $\Sigma_k \neq \Sigma_l$ if $k \neq l$, since powers of an n-quasigroup A, taken with respect to the k-multiplication and with respect to the l-multiplication of n-operations, can be different.

Corollary 6. *Let $\Sigma_k = \{A, A^2, \dots, A^{p-1}\}_k$ be a (k)-power set of n-quasigroups of Corollary 4 or 5, then $\Sigma'_k = \{E_k, A, A^2, \dots, A^{p-1}\}_k$ is a (cyclic) group with respect to the k-multiplication of n-operations.*

Proof. By Theorem 4 the (k)-powers of an n-quasigroup A in these sets have the form (1). By $l = p$ where p is a prime number in that case we obtain $A^p = E_k$, since $a^p = e$ in the group $(Q, \cdot)$ with the identity e for each $a \in Q$. □

Recall that an n-quasigroup (Q, A) is called an *$n-T$-quasigroup* if there exist a binary abelian group $(Q, +)$, its automorphisms $\alpha_1, \alpha_2, \dots, \alpha_n$ and an element $a \in Q$ such that

$$A(x_1^n) = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_{k-1} x_{k-1} + \alpha_k x_k + \alpha_{k+1} x_{k+1} + \dots + \alpha_n x_n + a \quad (3)$$

for all $x_1^n \in Q^n$ [16].

Corollary 7. *Let (Q, A) be an $n-T$-quasigroup of (3) with $\alpha_k = \epsilon$ (the identity permutation) for some fixed k, $k \in \overline{1, n}$, where (Q, A) is an abelian group of order $m = p_1^{\beta_1} p_2^{\beta_1} \dots p_t^{\beta_t}$, $p_1 < p_2 < \dots < p_t$, $p_1 \geq 3$. Then the set $\Sigma_k = \{A, A^2, \dots, A^{p_1 - 1}\}_k$ is a (k)-power set of n-quasigroups.*

Proof . Follows from Theorem 4 and Corollary 3, taking into account (with respect to the element a) that (Q, A) is an abelian group. □

Consider an *n-ary group* (Q, A) [9]. By Theorem of Gluskin-Hossu this n-group has the form

$$A(x_1^n) = x_1 \cdot \theta x_2 \cdot \theta^2 x_3 \cdot ... \cdot \theta^{n-1} x_n \cdot a, \quad (4)$$

where $(Q, \cdot)$ is a binary group, θ is its automorphism such that $\theta a = a$, $\theta^{n-1} x = axa^{-1}$. In this case we say that (Q, A) is an *n-group over the binary group $(Q, \cdot)$*.

For an n-group over an abelian group (it is a particular case of $n-T$-quasigroups) we have the following

Corollary 8. *Let (Q, A) be an n-group of (4) over an abelian group of order $m = p_1^{\beta_1} p_2^{\beta_1} \dots p_t^{\beta_t}$, $p_1 < p_2 < ... < p_t$, $p_1 \geq 3$. Then $\Sigma_1 = \{A, A^2, \dots, A^{p_1 - 1}\}_1$ is a (1)-power set of n-quasigroups, $\Sigma_n = \{A, A^2, \dots, A^{p_1 - 1}\}_n$ is an (n)-power set of n-quasigroups. Moreover, if the automorphism θ has order k, $2 \leq k \leq n-1$, then $\Sigma_{lk+1} = \{A, A^2, \dots, A^{p_1 - 1}\}_{lk+1}$ is also a $(lk+1)$-power set for each l such that $2 \leq lk \leq n-1$.*

Proof. In this case $\theta^{n-1} x_n = x_n$ and $\theta^{lk} = \epsilon$ for each l such that $1 \leq lk \leq n-1$ and all statements are true by Corollary 7. □

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Existence and uniqueness results for a class of nonlinear differential problems

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Abstract. We investigate the existence and uniqueness of the strong and weak solutions to a nonlinear differential system with boundary conditions and initial data.

Mathematics subject classification: 34G20, 35L50, 39A10, 47H05.

Keywords and phrases: Nonlinear differential system, boundary condition, maximal monotone operator, Cauchy problem, strong solution, weak solution.

1 Introduction

Let H be a real Hilbert space with the scalar product $< \cdot, \cdot >$ and the associated norm $\|\cdot\|$. We shall investigate the nonlinear differential system

$$\text{(S)} \qquad \begin{cases} \dfrac{du_n}{dt}(t) + \dfrac{v_n(t) - v_{n-1}(t)}{h} + c_n A(u_n(t)) \ni f_n(t), \\ \\ \dfrac{dv_n}{dt}(t) + \dfrac{u_{n+1}(t) - u_n(t)}{h} + d_n B(v_n(t)) \ni g_n(t), \end{cases}$$

$$n = 1, 2, \ldots, \quad 0 < t < T, \text{ in } H,$$

with the boundary condition

$$\text{(BC)} \qquad v_0(t) \in -\alpha(u_1(t)), \ \ 0 < t < T$$

and the initial data

$$\text{(IC)} \qquad u_n(0) = u_{n0}, \ \ v_n(0) = v_{n0}, \ \ n = 1, 2, \ldots,$$

where $c_n > 0$, $d_n > 0$, $\forall\, n = 1, 2, \ldots$, $h > 0$, and A, B, α are multivalued operators in H which satisfy some assumptions.

This problem is a discrete version with respect to x (with $H = I\!R$) of the hyperbolic problem

$$
(\mathrm{S})_0 \qquad \begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \dfrac{\partial v}{\partial x}(t,x) + A(u(t,x)) \ni f(t,x), \\[2ex] \dfrac{\partial v}{\partial t}(t,x) + \dfrac{\partial u}{\partial x}(t,x) + B(v(t,x)) \ni g(t,x), \end{cases}
$$

$$x > 0, \;\; t > 0, \;\; \text{in } I\!R$$

with the boundary condition

$$
(\mathrm{BC})_0 \qquad v(t,0) \in -\overline{\alpha}(u(t,0)), \;\; t > 0
$$

and the initial data

$$
(\mathrm{IC})_0 \qquad u(0,x) = u_0(x), \;\; v(0,x) = v_0(x), \;\; x > 0.
$$

The above problem $(\mathrm{S})_0$+$(\mathrm{BC})_0$+$(\mathrm{IC})_0$ has applications in electrotechnics (the propagation phenomena in electrical networks) and mechanics (the variable flow of a fluid) [5, 6, 13]. The system $(\mathrm{S})_0$ for $x \in (0,1)$ or $x \in (0,\infty)$, subject to various boundary conditions has been studied by many authors: V. Barbu, V. Iftimie, G. Moroşanu, R. Luca, etc (see the papers [2, 3, 7, 9, 12]). The problem (S)+(BC)+(IC) is a generalization of the problem studied in [10], where the operator $\alpha : H \to H$ is everywhere defined and single-valued. The methods used in this paper to prove the maximal monotonicity of the operators $\mathcal{A}$ and $\mathcal{A}+\mathcal{B}$ (see below) are different than those used in [10]. We also mention the papers [10, 11] where we investigated the system (S) with $n = 1, 2, \ldots, N$ $(N \geq 1)$ with some boundary conditions and initial data. Although the proposed problem appeared by discretization of $(\mathrm{S})_0$+$(\mathrm{BC})_0$+$(\mathrm{IC})_0$, our problem also covers some nonlinear differential systems in Hilbert spaces. For the basic concepts and results in the theory of monotone operators and nonlinear evolution equations of monotone type in Hilbert spaces we refer the reader to [1, 4, 8, 13].

We present the assumptions that we shall use in the sequel

(H1) The operators $A : D(A) \subset H \to H$, $B : D(B) \subset H \to H$ are maximal monotone, possibly multivalued, $0 \in A(0)$, $0 \in B(0)$.

(H2) The operator $\alpha : D(\alpha) \subset H \to H$ is maximal monotone, possibly multivalued, with $D(\alpha) \neq \emptyset$.

(H3) $D(A) \cap D(\alpha) \neq \emptyset$.

(H4) i) The operator α is bounded on bounded sets.

ii) $(\operatorname{int} D(\alpha)) \cap D(A) \neq \emptyset$.

(H5) The constant $h > 0$.

(H6) The constants $c_n > 0$, $d_n > 0$, $\forall\, n \geq 1$.

2 The results

We shall write our problem (S)+(BC)+(IC) as a Cauchy problem in a certain Hilbert space, and we shall apply the theory of nonlinear evolution equations of monotone type.

We consider the Hilbert *space* $X = l_h^2(H) \times l_h^2(H)$, where $l_h^2(H) = \{(u_n)_n \subset H, \sum_{n=1}^{\infty} \|u_n\|^2 < \infty\}$ $(= l^2(H))$, with the scalar product

$$< ((u_n)_n, (v_n)_n), ((\overline{u}_n)_n, (\overline{v}_n)_n) >_X = < (u_n)_n, (\overline{u}_n)_n >_{l_h^2(H)} +$$

$$< (v_n)_n, (\overline{v}_n)_n >_{l_h^2(H)} = \sum_{n=1}^{\infty} h < u_n, \overline{u}_n > + \sum_{n=1}^{\infty} h < v_n, \overline{v}_n > .$$

We define the *operator* $\mathcal{A} : D(\mathcal{A}) \subset X \to X$, with

$$D(\mathcal{A}) = \{((u_n)_n, (v_n)_n) \in X,\ \ u_1 \in D(\alpha)\},$$

$$\mathcal{A}((u_n)_n, (v_n)_n) = \left\{ \left(\left(\frac{v_n - v_{n-1}}{h} \right)_n, \left(\frac{u_{n+1} - u_n}{h} \right)_n \right),\ \text{ with }\ v_0 \in -\alpha(u_1) \right\},$$

and the *operator* $\mathcal{B} : D(\mathcal{B}) \subset X \to X$, with $D(\mathcal{B}) = \big\{((u_n)_n, (v_n)_n) \in X$, $u_n \in D(A),\ v_n \in D(B),\ \forall\, n \geq 1,\ \{(c_n A(u_n))_n\} \subset l^2(H), \{(d_n B(v_n))_n\} \subset l^2(H)\big\}$, $\mathcal{B}((u_n)_n, (v_n)_n) = \big\{((c_n \gamma_n)_n, (d_n \delta_n)_n),\ \gamma_n \in A(u_n),\ \delta_n \in B(v_n),\ \forall\, n \geq 1\big\}$.

Theorem 1. *If the assumptions* (H2) *and* (H5) *hold, then the operator $\mathcal{A}$ is maximal monotone in X.*

Theorem 2. *If the assumptions* (H1), (H5) *and* (H6) *hold, then the operator $\mathcal{B}$ is maximal monotone in X.*

Theorem 3. *If the assumptions* (H1), (H2), (H3), [(H4)i) *or* (H4)ii)], (H5) *and* (H6) *hold, then the operator $\mathcal{A} + \mathcal{B}$ is maximal monotone.*

Using the operators $\mathcal{A}$ and $\mathcal{B}$ our problem (S)+(BC)+(IC) can be equivalently expressed as the following Cauchy problem in the space X

$$\text{(P)} \qquad \begin{cases} \dfrac{dU}{dt}(t) + \mathcal{A}(U(t)) + \mathcal{B}(U(t)) \ni F(t) \\ U(0) = U_0, \end{cases}$$

where $U = ((u_n)_n, (v_n)_n)$, $U_0 = ((u_{n0})_n, (v_{n0})_n)$, $F = ((f_n)_n, (g_n)_n)$.

The main result for our problem (S)+(BC)+(IC)$\Leftrightarrow$(P) is

Theorem 4. *Assume that the assumptions* (H1), (H2), (H3), [(H4)i) *or* (H4)ii)], (H5) *and* (H6) *hold. If* $u_{10} \in D(A) \cap D(\alpha)$, $u_{n0} \in D(A)$, $\forall\, n \geq 2$, $v_{n0} \in D(\mathcal{B})$, $\forall\, n \geq 1$ *with* $(u_{n0})_n$, $(v_{n0})_n \in l^2(H)$, $\{(c_n A(u_{n0}))_n\}$, $\{(d_n B(v_{n0}))_n\} \subset l^2(H)$, (*that is* $U_0 \in D(\mathcal{A}) \cap D(\mathcal{B})$), *and* $(f_n)_n$, $(g_n)_n \in W^{1,1}(0,T;l^2(H))$, *then there exist unique functions* u_n, v_n, $n \geq 1$, $(u_n)_n$, $(v_n)_n \in W^{1,\infty}(0,T;l^2(H))$, $u_1(t) \in D(A) \cap D(\alpha)$, $u_n(t) \in D(A)$, $\forall\, n \geq 2$, $v_n(t) \in D(\mathcal{B})$, $\forall\, n \geq 1$, $\forall\, t \in [0,T]$, *that verify the system* (S) *for all* $t \in [0,T)$, *the boundary condition* (BC) *for all* $t \in [0,T)$ *and the initial data* (IC). *Moreover* u_n, v_n, $n \geq 1$ *are everywhere differentiable from right in the topology of* H *and*

$$\frac{d^+u_n}{dt} = \left(f_n - c_n A(u_n) - \frac{v_n - v_{n-1}}{h}\right)^0, \;\; n \geq 1,$$

$$\frac{d^+v_n}{dt} = \left(g_n - d_n B(v_n) - \frac{u_{n+1} - u_n}{h}\right)^0, \;\; n \geq 1, \;\; t \in [0,T),$$

with $v_0(t) \in -\alpha(u_1(t))$, $\forall\, t \in [0,T)$.

Remark. If $U_0 \in \overline{D(\mathcal{A}) \cap D(\mathcal{B})}$ and $F \in L^1(0,T;X)$ then by [1, Corollary 2.2, Chapter III] the problem (P)$\Leftrightarrow$(S)+(BC)+(IC) has a unique weak solution $U \in C([0,T];X)$, that is there exist $(F_k)_k \subset W^{1,1}(0,T;\, X)$, $F_k \to F$, as $k \to \infty$, in $L^1(0,T;X)$ and $(U_k)_k \subset W^{1,\infty}(0,T;X)$, $U_k(0) = U_0$, $U_k \to U$ as $k \to \infty$ in $C([0,T];X)$, strong solutions for the problems

$$\frac{dU_k}{dt}(t) + (\mathcal{A} + \mathcal{B})(U_k(t)) \ni F_k(t), \;\text{ for a.a. } t \in (0,T), \;\; k = 1, 2, \ldots$$

3 The proofs

The proof of Theorem 1. The operator $\mathcal{A}$ has $D(\mathcal{A}) \neq \emptyset$ and it is well defined in X; if $((u_n)_n, (v_n)_n) \in D(\mathcal{A})$ then $\mathcal{A}((u_n)_n, (v_n)_n) \in X$. The operator $\mathcal{A}$ is monotone;

indeed

$$< Z - \overline{Z}, U - \overline{U} >_X = < \left(\frac{v_n - v_{n-1}}{h}\right)_n - \left(\frac{\overline{v}_n - \overline{v}_{n-1}}{h}\right)_n,$$

$$(u_n - \overline{u}_n)_n >_{l_h^2(H)} + < \left(\frac{u_{n+1} - u_n}{h}\right)_n - \left(\frac{\overline{u}_{n+1} - \overline{u}_n}{h}\right)_n,$$

$$(v_n - \overline{v}_n)_n >_{l_h^2(H)} = \sum_{n=1}^{\infty} h < \frac{v_n - v_{n-1} - \overline{v}_n + \overline{v}_{n-1}}{h},$$

$$u_n - \overline{u}_n > + \sum_{n=1}^{\infty} h < \frac{u_{n+1} - u_n - \overline{u}_{n+1} + \overline{u}_n}{h},$$

$$v_n - \overline{v}_n > = - < v_0 - \overline{v}_0, u_1 - \overline{u}_1 > \geq 0, \quad \forall U = ((u_n)_n, (v_n)_n),$$

$$\overline{U} = ((\overline{u}_n)_n, (\overline{v}_n)_n) \in D(\mathcal{A}), \quad Z \in \mathcal{A}(U), \quad \overline{Z} \in \mathcal{A}(\overline{U}),$$

$$u_1 \in D(\alpha), \quad \overline{u}_1 \in D(\alpha), \quad v_0 \in -\alpha(u_1), \quad \overline{v}_0 \in -\alpha(\overline{u}_1).$$

To prove that $\mathcal{A}$ is maximal monotone, it is sufficient (and necessary) to show that for any λ (equivalently there exists a $\lambda > 0$ such that) $R(I + \lambda\mathcal{A}) = X$ (see [4, Proposition 2.2]). We consider $\lambda = h$ and we shall prove that for any $Y = ((x_n)_n, (y_n)_n) \in X$, the equation

$$(I + h\mathcal{A})(U) \ni Y \tag{1}$$

has a solution $U = ((u_n)_n, (v_n)_n) \in D(\mathcal{A})$.

The equation (1) is equivalent to

$$\begin{cases} u_n + v_n - v_{n-1} = x_n \\ v_n + u_{n+1} - u_n = y_n, \quad n = 1, 2, \ldots, \end{cases} \tag{2}$$
$$\text{with } v_0 \in -\alpha(u_1).$$

We look for a solution for (2) in the form

$$\begin{cases} u_n = u_n^1 + u_n^2 \\ v_n = v_n^1 + v_n^2, \quad n = 1, 2, \ldots, \end{cases}$$

where $((u_n^1)_n, (v_n^1)_n)$ is a solution to

$$\begin{cases} u_n^1 + v_n^1 - v_{n-1}^1 = x_n \\ v_n^1 + u_{n+1}^1 - u_n^1 = y_n, \quad n = 1, 2, \ldots, \text{ in } H \end{cases} \tag{3}$$
$$\text{with } v_0^1 = 0,$$

and $((u_n^2)_n, (v_n^2)_n) = a((p_n)_n, (q_n)_n)$, where $a \in H$ will be determined below and $((p_n)_n, (q_n)_n) \in (l^2(I\!\!R))^2$ is solution of the system

$$\begin{cases} p_1 + q_1 = p \\ p_n + q_n - q_{n-1} = 0, \;\; n = 2, 3, \dots \\ q_n + p_{n+1} - p_n = 0, \;\; n = 1, 2, \dots, \;\; \text{with } \; p > 0, \;\; \text{in } \; I\!\!R. \end{cases} \tag{4}$$

The problem (3) has a solution. To prove this, we consider the *operator* $\mathcal{A}_0 : D(\mathcal{A}_0) = X \to X$, $\mathcal{A}_0((u_n)_n, (v_n)_n) = ((v_n - v_{n-1})_n, (u_{n+1} - u_n)_n)$, $v_0 = 0$. Then the problem (3) is equivalent to

$$U + \mathcal{A}_0(U) = Y. \tag{5}$$

The above equation (5) has solution, because the operator $\mathcal{A}_0$ is maximal monotone in X. Indeed, $\mathcal{A}_0$ is monotone

$$< \mathcal{A}_0(U) - \mathcal{A}_0(\overline{U}), U - \overline{U} >_X = \sum_{n=1}^{\infty} h < v_n - v_{n-1} - \overline{v}_n + \overline{v}_{n-1}, u_n - \overline{u}_n > +$$

$$+ \sum_{n=1}^{\infty} h < u_{n+1} - u_n - \overline{u}_{n+1} + \overline{u}_n, v_n - \overline{v}_n >= 0, \text{ where } v_0 = \overline{v}_0 = 0.$$

In addition, $\mathcal{A}_0$ is single-valued, everywhere defined and continuous. By [4, Proposition 2.4] we deduce that the operator $\mathcal{A}_0$ is maximal monotone and so the equation (5) ($\Leftrightarrow$ the problem (3)) has a (unique) solution.

Using the same argument used before, we deduce that the problem (4) (here $H = I\!\!R$) has a unique solution $((p_n)_n, (q_n)_n) \in l^2(I\!\!R) \times l^2(I\!\!R)$. We shall deduce in what follows the sequences $(p_n)_n$, $(q_n)_n$ by a direct computation (we shall need p_1, q_1).

We set $p_1 = r$. Then by (4) we have

$$\begin{aligned} &p_1 = r, \;\; q_1 = p - r, \\ &p_n = \Delta_{n-1} r - z_{n-2} p, \;\; n \geq 2, \\ &q_n = \Delta_{n-1} p - z_{n-1} r, \;\; n \geq 2, \end{aligned} \tag{6}$$

where $\Delta_0 = 1$, $\Delta_1 = 2$, $\Delta_2 = 5$, $\Delta_3 = 13$, $\Delta_4 = 34$, $\Delta_5 = 89, \dots$, $z_0 = 1$, $z_1 = 3$, $z_2 = 8$, $z_3 = 21$, $z_4 = 55, \dots$

The sequences $(\Delta_n)_n$, $(z_n)_n$ satisfy the recursive relations

$$\Delta_n = 3\Delta_{n-1} - \Delta_{n-2}, \;\; \Delta_0 = 1, \;\; \Delta_1 = 2,$$

$$z_n = 3z_{n-1} - z_{n-2}, \;\; z_0 = 1, \;\; z_1 = 3.$$

Using the characteristic equation $\lambda^2 - 3\lambda + 1 = 0$ with the solutions $\lambda_{1,2} = \dfrac{3 \pm \sqrt{5}}{2}$, we obtain for $(\Delta_n)_n$ and $(z_n)_n$ the formulas

$$\Delta_n = \frac{1}{\sqrt{5}} \left[\left(\frac{3+\sqrt{5}}{2} \right)^n \frac{\sqrt{5}+1}{2} + \left(\frac{3-\sqrt{5}}{2} \right)^n \frac{\sqrt{5}-1}{2} \right], \quad n = 0, 1, 2, \dots$$

$$z_n = \frac{1}{\sqrt{5}} \left[\left(\frac{3+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{3-\sqrt{5}}{2} \right)^{n+1} \right], \quad n = 0, 1, 2, \dots \tag{7}$$

Then by (6) we obtain

$$p_n = \frac{1}{\sqrt{5}} \left[\left(\frac{3+\sqrt{5}}{2} \right)^{n-1} \left(\frac{\sqrt{5}+1}{2} r - p \right) + \left(\frac{3-\sqrt{5}}{2} \right)^{n-1} \left(\frac{\sqrt{5}-1}{2} r + p \right) \right],$$

$$\begin{aligned} q_n = \frac{1}{\sqrt{5}} \Bigg[& \left(\frac{3+\sqrt{5}}{2} \right)^{n-1} \left(\frac{\sqrt{5}+1}{2} p - \frac{3+\sqrt{5}}{2} r \right) + \\ & + \left(\frac{3-\sqrt{5}}{2} \right)^{n-1} \left(\frac{\sqrt{5}-1}{2} p + \frac{3-\sqrt{5}}{2} r \right) \Bigg]. \end{aligned} \tag{8}$$

The only bounded sequences $(p_n)_n$, $(q_n)_n$ which satisfy the relations (6) (of the form (8)) are that in which the coefficient of $\left(\dfrac{3+\sqrt{5}}{2} \right)^{n-1}$ in (8) is 0. Therefore we obtain the condition $\dfrac{\sqrt{5}+1}{2} r - p = 0 \;\Rightarrow\; r = \dfrac{\sqrt{5}-1}{2} p$. Then the condition $\dfrac{\sqrt{5}+1}{2} p - \dfrac{3+\sqrt{5}}{2} r = 0$ is also satified. In this way we found the sequences $(p_n)_n$, $(q_n)_n \in l^2(I\!\!R)$, solutions for (4)

$$p_n = \left(\frac{3-\sqrt{5}}{2} \right)^{n-1} \frac{\sqrt{5}-1}{2} p, \quad q_n = \left(\frac{3-\sqrt{5}}{2} \right)^n p, \quad \forall\, n \geq 1.$$

Evidently $u_n = u_n^1 + u_n^2 = u_n^1 + a p_n$, $n \geq 2$ and $v_n = v_n^1 + v_n^2 = v_n^1 + a q_n$, $n \geq 2$ verify the relations $(2)_1$ for $n = 2, 3, \dots$ and $(2)_2$ for $n = 1, 2, \dots$ We shall determine $a \in H$ such that

$$u_1 + v_1 - v_0 = x_1, \;\; v_0 \in -\alpha(u_1) \;\;\Leftrightarrow\;\; u_1^1 + u_1^2 + v_1^1 + v_1^2 - v_0 = x_1, \;\; v_0 \in -\alpha(u_1)$$
$$\Leftrightarrow\;\; u_1^1 + a p_1 + v_1^1 + a q_1 - v_0 = x_1, \;\; v_0 \in -\alpha(u_1^1 + a p_1) \;\;\Leftrightarrow\;\; a p_1 + a q_1 \in -\alpha(u_1^1 + a p) \;\;\Leftrightarrow$$
$$\frac{\sqrt{5}-1}{2} a p + \frac{3-\sqrt{5}}{2} a p \in -\alpha \left(u_1^1 + \frac{\sqrt{5}-1}{2} a p \right) \;\;\Leftrightarrow\;\; a p + \alpha \left(u_1^1 + \frac{\sqrt{5}-1}{2} a p \right) \ni 0, \tag{9}$$

where u_1^1 is the solution for (3).

We denote $z = \dfrac{\sqrt{5}-1}{2}ap$; then the equation (9) is equivalent to

$$\frac{\sqrt{5}+1}{2}z + \alpha(u_1^1 + z) \ni 0.$$

We obtain the equation

$$\Lambda_1(z) + \Lambda_2(z) \ni 0, \tag{10}$$

where $\Lambda_1 : H \to H$, $\Lambda_1(z) = \dfrac{\sqrt{5}+1}{2}z$ and $\Lambda_2 : D(\Lambda_2) \subset H \to H$, $D(\Lambda_2) = \{z \in H,\ u_1^1 + z \in D(\alpha)\}$, $\Lambda_2(z) = \alpha(u_1^1 + z)$. The operator Λ_1 is single-valued, everywhere defined, strongly monotone and continuous (so maximal monotone) and the operator Λ_2 is maximal monotone. Then by [1, Corollary 1.3, Chapter II] we deduce that the operator $\Lambda_1+\Lambda_2$ is strongly maximal monotone in H, so the equation (10) has a (unique) solution $z \in D(\Lambda_2)$. Then $a = \dfrac{\sqrt{5}+1}{2p}z$ verifies the relation (9). So we proved the existence of solution $U = ((u_n)_n, (v_n)_n) \in D(\mathcal{A})$ of the system (2) or equation (1). Therefore the operator $\mathcal{A}$ is maximal monotone in X. Q.E.D.

The proof of Theorem 2. We suppose without loss of generality (for an easy writing) that A and B are single-valued. By (H1), $D(\mathcal{B}) \neq \emptyset$. Because $\mathcal{B}$ is defined by a standard product construction, this operator is evidently monotone. Moreover $\mathcal{B}$ is maximal monotone, that is $\forall\, \lambda > 0\ \ R(I+\lambda\mathcal{B}) = X \ \Leftrightarrow\ \forall\, Y = ((x_n)_n, (y_n)_n) \in X\ \exists\, U = ((u_n)_n, (v_n)_n) \in D(\mathcal{B})$ such that $U + \lambda\mathcal{B}(U) = Y$. The last relation is equivalent to

$$((u_n)_n, (v_n)_n) + \lambda((c_nA(u_n))_n, (d_nB(v_n))_n) = ((x_n)_n, (y_n)_n) \ \Leftrightarrow$$

$$\begin{cases} (u_n)_n + \lambda(c_nA(u_n))_n = (x_n)_n \\ (v_n)_n + \lambda(d_nB(v_n))_n = (y_n)_n \end{cases} \Rightarrow \begin{cases} u_n + \lambda c_nA(u_n) = x_n \\ v_n + \lambda d_nB(v_n) = y_n,\ \ n \geq 1 \end{cases} \Rightarrow$$

$$u_n = (I + \lambda c_nA)^{-1}(x_n) = J^A_{\lambda c_n}(x_n),\ \ v_n = (I + \lambda d_nB)^{-1}(y_n) = J^B_{\lambda d_n}(y_n),\ \ \forall\, n \geq 1.$$

Because $A(0) = 0$ we have $J^A_\mu(0) = 0$, $\forall\, \mu > 0$ and

$$\|J^A_\mu(x) - J^A_\mu(0)\| \leq \|x\| \ \Rightarrow\ \|J^A_\mu(x)\| \leq \|x\|,\ \ \forall\, x \in H,\ \ \forall\, \mu > 0.$$

Similarly by $B(0) = 0$ we deduce $J^B_\mu(0) = 0$, $\forall \mu > 0$ and $\|J^B_\mu(x)\| \leq \|x\|$, $\forall x \in H$, $\forall \mu > 0$. With this remark we have

$$\sum_{n=1}^{\infty} \|J^A_{\lambda c_n}(x_n)\|^2 \leq \sum_{n=1}^{\infty} \|x_n\|^2 < \infty,\ \ \sum_{n=1}^{\infty} \|J^B_{\lambda d_n}(y_n)\|^2 \leq \sum_{n=1}^{\infty} \|y_n\|^2 < \infty,$$

so $U = ((u_n)_n, (v_n)_n) \in D(\mathcal{B})$. Q.E.D.

The proof of Theorem 3. The operator $\mathcal{A} + \mathcal{B} : D(\mathcal{A}) \cap D(\mathcal{B}) \subset X \to X$ has $D(\mathcal{A}) \cap D(\mathcal{B}) = \{((u_n)_n, (v_n)_n),\ u_1 \in D(A) \cap D(\alpha),\ u_n \in D(A),\ \forall\, n \geq 2,\ v_n \in D(B),\ \forall\, n \geq 1,\ \text{with } \{(c_n A(u_n))_n\},\ \{(d_n B(v_n))_n\} \subset l^2(H)\} \neq \emptyset$, by (H1), (H3).

First, we suppose (H4)i) holds. The operator $\mathcal{A} + \mathcal{B}$ is monotone ($\mathcal{A}$, $\mathcal{B}$ are monotone). To prove that $\mathcal{A} + \mathcal{B}$ is maximal monotone, we shall show that for any $F_0 = ((f_n^0)_n, (g_n^0)_n) \in X$ the equation

$$U + \mathcal{A}(U) + \mathcal{B}(U) \ni F_0 \tag{11}$$

has at least a solution $U \in D(\mathcal{A}) \cap D(\mathcal{B})$.

For let $F_0 \in X$ be given. The equation (11) is equivalent to

$$\begin{cases} u_n + \dfrac{v_n - v_{n-1}}{h} + c_n A(u_n) \ni f_n^0 \\ v_n + \dfrac{u_{n+1} - u_n}{h} + d_n B(v_n) \ni g_n^0, \quad n = 1, 2, \ldots, \end{cases} \tag{12}$$

$$\text{with } v_0 \in -\alpha(u_1). \tag{13}$$

We consider the following approximate problem

$$\begin{cases} U^\lambda + \mathcal{A}(U^\lambda) + \mathcal{B}_\lambda(U^\lambda) \ni F_0 \\ U^\lambda \in D(\mathcal{A}), \quad \lambda > 0, \end{cases} \tag{14}$$

where $\mathcal{B}_\lambda((u_n)_n, (v_n)_n) = ((c_n A_\lambda(u_n))_n, (d_n B_\lambda(v_n))_n)$ with A_λ, B_λ the Yosida approximations of A, respectively B, ($A_\lambda = \frac{1}{\lambda}(I - J_\lambda^A)$, $B_\lambda = \frac{1}{\lambda}(I - J_\lambda^B)$).

Because A_λ, B_λ are everywhere defined ($D(A_\lambda) = D(B_\lambda) = H$), single-valued, monotone, continuous, we deduce that $\mathcal{B}_\lambda$ is also everywhere defined in X, single-valued, monotone and continuous, $\forall\, \lambda > 0$. As $\mathcal{A}$ is maximal monotone operator (Theorem 1), then it follows that $\mathcal{A} + \mathcal{B}_\lambda$ is maximal monotone, $\forall\, \lambda > 0$. Therefore, for any $\lambda > 0$ the problem (14) has a solution $U^\lambda = ((u_n^\lambda)_n, (v_n^\lambda)_n) \in D(\mathcal{A})$. The problem (15) is equivalent to

$$\begin{cases} u_n^\lambda + \dfrac{v_n^\lambda - v_{n-1}^\lambda}{h} + c_n A_\lambda(u_n^\lambda) \ni f_n^0 \\ v_n^\lambda + \dfrac{u_{n+1}^\lambda - u_n^\lambda}{h} + d_n B_\lambda(v_n^\lambda) \ni g_n^0, \quad n = 1, 2, \ldots, \end{cases} \tag{15}$$

$$v_0^\lambda \in -\alpha(u_1^\lambda), \quad (u_1^\lambda \in D(\alpha)). \tag{16}$$

Let $U^0 = ((u_n^0)_n, (v_n^0)_n) \in D(\mathcal{A})$, $u_1^0 \in D(\alpha)$. We denote

$$F_\lambda = ((f_n^\lambda)_n, (g_n^\lambda)_n) := U^0 + \mathcal{A}(U^0) + \mathcal{B}_\lambda(U^0), \ \ \lambda > 0. \tag{17}$$

The set $\{\mathcal{B}_\lambda(U^0); \ \ \lambda > 0\}$ is bounded in the space X; indeed

$$\|\mathcal{B}_\lambda(U^0)\|_X^2 = \sum_{n=1}^{\infty} h(c_n^2\|A_\lambda(u_n^0)\|^2 + d_n^2\|B_\lambda(v_n^0)\|^2) \leq$$

$$\leq \sum_{n=1}^{\infty} h(c_n^2\|A^0(u_n^0)\|^2 + d_n^2\|B^0(v_n^0)\|^2) = \|\mathcal{B}^0(U^0)\|_X^2, \ \ \forall\, \lambda > 0,$$

(where A^0 is the minimal section of A, that is $A^0(x) \in A(x)$, $\|A^0(x)\| = \inf\{\|y\|, \ y \in A(x)\}$, $\forall\, x \in D(A)$).

We deduce by the above inequality and (17) that $\|F_\lambda\|_X \leq$ const., $\forall\, \lambda > 0$, (const. is a positive constant independent of λ).

Using (14) and (17) (we substract them and we multiply the obtained relation by $U^\lambda - U^0$ in X), we get

$$\|U^\lambda - U^0\|_X \leq \|F_0 - F_\lambda\|_X \ \Rightarrow \ \|U^\lambda\|_X \leq \|U^0\|_X + \|F_0\|_X + \|F_\lambda\|_X \leq \text{const.}, \ \forall \lambda > 0.$$

We deduce that $\displaystyle\sum_{n=1}^{\infty} h(\|u_n^\lambda\|^2 + \|v_n^\lambda\|^2) \leq$ const. Because $\{u_1^\lambda; \ \lambda > 0\}$ is bounded in H, by (H4)i) we deduce that $\{v_0^\lambda; \ \lambda > 0\}$ is also bounded in H. So we obtain that $\{\mathcal{A}(U^\lambda); \ \lambda > 0\}$ is bounded in X. By (14) we get $\{\mathcal{B}_\lambda(U^\lambda); \ \lambda > 0\}$ is bounded in X, $\|\mathcal{B}_\lambda(U^\lambda)\|_X \leq$ const., $\forall\, \lambda > 0$, so

$$\sum_{n=1}^{\infty} h(\|c_n A_\lambda(u_n^\lambda)\|^2 + \|d_n B_\lambda(v_n^\lambda)\|^2) \leq \text{const.}, \ \ \forall\, \lambda > 0. \tag{18}$$

We shall prove in what follows that the sets $\{(u_n^\lambda)_n; \ \ \lambda > 0\}$, $\{(v_n^\lambda)_n; \ \ \lambda > 0\}$ are Cauchy sequences (in $l^2(H)$). For this, let $U^\lambda = ((u_n^\lambda)_n, (v_n^\lambda)_n)$, $U^\mu = ((u_n^\mu)_n, (v_n^\mu)_n)$, λ, $\mu > 0$, be solutions for (14), $u_1^\lambda \in D(\alpha)$, $v_0^\lambda \in -\alpha(u_1^\lambda)$, $u_1^\mu \in D(\alpha)$, $v_0^\mu \in -\alpha(u_1^\mu)$. Then by (14) we have $U^\lambda + Z^\lambda + \mathcal{B}_\lambda(U^\lambda) = F_0$, $U^\mu + Z^\mu + \mathcal{B}_\mu(U^\mu) = F_0$, $Z^\lambda \in \mathcal{A}(U^\lambda)$, $Z^\mu \in \mathcal{A}(U^\mu)$ and $U^\lambda - U^\mu + Z^\lambda - Z^\mu + \mathcal{B}_\lambda(U^\lambda) - \mathcal{B}_\mu(U^\mu) = 0$.

We multiply the above relation by $U^\lambda - U^\mu$ in X and after some computations we obtain

$$\sum_{n=1}^{\infty}(\|u_n^\lambda - u_n^\mu\|^2 + \|v_n^\lambda - v_n^\mu\|^2) \leq -\sum_{n=1}^{\infty}\{c_n[< A_\lambda(u_n^\lambda) - A_\mu(u_n^\mu), J_\lambda^A(u_n^\lambda) - J_\mu^A(u_n^\mu) > +$$

$$+ < A_\lambda(u_n^\lambda) - A_\mu(u_n^\mu), \lambda A_\lambda(u_n^\lambda) - \mu A_\mu(u_n^\mu) >] + d_n[< B_\lambda(v_n^\lambda) - B_\mu(v_n^\mu), J_\lambda^B(v_n^\lambda) -$$

$$-J_\mu^B(v_n^\mu)>+<B_\lambda(v_n^\lambda)-B_\mu(v_n^\mu),\lambda B_\lambda(v_n^\lambda)-\mu B_\mu(v_n^\mu)>]\}.$$

Because $A_\lambda(u_n^\lambda)\in A(J_\lambda^A(u_n^\lambda))$, $B_\lambda(v_n^\lambda)\in B(J_\lambda^B(v_n^\lambda))$, $\forall\,\lambda>0$,$\forall\, n\geq 1$, by the above inequality we deduce

$$\sum_{n=1}^{\infty}(\|u_n^\lambda-u_n^\mu\|^2+\|v_n^\lambda-v_n^\mu\|^2)\leq\sum_{n=1}^{\infty}[c_n(\lambda\|A_\lambda(u_n^\lambda)\|^2+\frac{\lambda}{2}\|A_\mu(u_n^\mu)\|^2+$$

$$+\frac{\lambda}{2}\|A_\lambda(u_n^\lambda)\|^2+\frac{\mu}{2}\|A_\lambda(u_n^\lambda)\|^2+\frac{\mu}{2}\|A_\mu(u_n^\mu)\|^2+\mu\|A_\mu(u_n^\mu)\|^2)+$$

$$+d_n(\lambda\|B_\lambda(v_n^\lambda)\|^2+\frac{\lambda}{2}\|B_\mu(v_n^\mu)\|^2+\frac{\lambda}{2}\|B_\lambda(v_n^\lambda)\|^2+\frac{\mu}{2}\|B_\lambda(v_n^\lambda)\|^2+\frac{\mu}{2}\|B_\mu(v_n^\mu)\|^2+$$

$$+\mu\|B_\mu(v_n^\mu)\|^2)]\leq\text{const.}(\lambda+\mu),\ \ \forall\,\lambda,\,\mu>0,\ \ (\text{by }\ (18)).$$

Therefore

$$\sum_{n=1}^{\infty}(\|u_n^\lambda-u_n^\mu\|^2+\|v_n^\lambda-v_n^\mu\|^2)\leq\text{const.}(\lambda+\mu),\ \ \forall\,\lambda,\,\mu>0.$$

We deduce that $\{(u_n^\lambda)_n;\ \lambda>0\}$ and $\{(v_n^\lambda)_n;\ \lambda>0\}$ are Cauchy sequences in $l^2(H)$. Then, there exist $\lim\limits_{\lambda\to0}(u_n^\lambda)_n=(u_n)_n$, $\lim\limits_{\lambda\to0}(v_n^\lambda)_n=(v_n)_n$, in $l^2(H)$, (evidently $u_n^\lambda\to u_n$, $v_n^\lambda\to v_n$ as $\lambda\to0$, in H, $\forall\,n\geq1$).

Because $u_1^\lambda\to u_1$, as $\lambda\to0$, in H, $\{v_0^\lambda;\ \lambda>0\}$ is bounded in H, so on a subsequence $v_0^\lambda\rightharpoonup v_0$, as $\lambda\to0$ ($v_0\in H$), $v_0^\lambda\in-\alpha(u_1^\lambda)$ and α is demiclosed, we deduce that $u_1\in D(\alpha)$ and $v_0\in-\alpha(u_1)$, that is (13).

Then, by $u_n^\lambda\to u_n$ and $v_n^\lambda\to v_n$ as $\lambda\to0$, we deduce that $J_\lambda^Au_n^\lambda\to u_n$, $J_\lambda^Bv_n^\lambda\to v_n$, as $\lambda\to0$, $\forall\,n\geq1$. Because $\{A_\lambda(u_n^\lambda);\ \lambda>0\}$, $\{B_\lambda(v_n^\lambda);\ \lambda>0\}$, $n\geq1$ are bounded (by (18), for any n fixed we have $\|c_nA_\lambda(u_n^\lambda)\|\leq$ const., $\forall\,\lambda>0$, so $\|A_\lambda(u_n^\lambda)\|\leq\frac{1}{c_n}$const., $\forall\,\lambda>0$) and A, B are demiclosed, we deduce that $u_n\in D(A)$ and $A_\lambda(u_n^\lambda)\rightharpoonup p_n$, as $\lambda\to0$ ($p_n\in H$), $p_n\in A(u_n)$, $v_n\in D(B)$ and $B_\lambda(v_n^\lambda)\rightharpoonup q_n$, as $\lambda\to0$ ($q_n\in H$), $q_n\in B(v_n)$, $\forall\,n\geq1$ (eventually on some sequences). By passing to $\lambda\to0$ in (15), (16) we deduce that $U=((u_n)_n,(v_n)_n)$ is a solution for (12) and (13). By (12) we obtain $(c_nA(u_n))_n$, $(d_nB(v_n))_n\in l^2(H)$, so $U\in D(\mathcal{A})\cap D(\mathcal{B})$ and $\mathcal{A}+\mathcal{B}$ is maximal monotone in X.

If (H4)ii) holds, by Theorem 1 and Theorem 2 we have that $\mathcal{A}$ and $\mathcal{B}$ are maximal monotone with $\operatorname{int}D(\mathcal{A})=\{((u_n)_n,(v_n)_n)\in X,\ u_1\in\operatorname{int}D(\alpha)\}$ and so $\operatorname{int}D(\mathcal{A})\cap D(\mathcal{B})\neq\emptyset$. Therefore, by [4, Corollaire 2.7] we deduce that $\mathcal{A}+\mathcal{B}$ is maximal monotone. Q.E.D.

The proof of Theorem 4. By Theorem 3, the operator $\mathcal{A}+\mathcal{B} : D(\mathcal{A}) \cap D(\mathcal{B}) \subset X \to X$ is maximal monotone in X. Using [1, Theorem 2.2, Corollary 2.1, Chapter III] we deduce that for $U_0 = ((u_{n0})_n, (v_{n0})_n) \in D(\mathcal{A}) \cap D(\mathcal{B})$ and $F = ((f_n)_n, (g_n)_n) \in W^{1,1}(0,T;X)$, the problem (P)⇔(S)+(BC)+(IC) has a unique strong solution $U = ((u_n)_n, (v_n)_n) \in W^{1,\infty}(0,T;X)$, $U(t) \in D(\mathcal{A}) \cap D(\mathcal{B})$, $\forall\, t \in [0,T)$. By considering the equation $(\mathrm{P})_1$ in the interval $[0, T+\varepsilon]$, with $\varepsilon > 0$ (by extending correspondingly the functions f_n, g_n, $n \geq 1$) we obtain $U(T) \in D(\mathcal{A}) \cap D(\mathcal{B})$. The solution U is everywhere differentiable from right and $\dfrac{d^+U}{dt}(t) = (F(t) - \mathcal{A}(U(t)) - \mathcal{B}(U(t)))^0$, $\forall\, t \in [0,T)$, that is the relations from the conclusion of the theorem are verified. In addition we have

$$\left\| \frac{d^+U}{dt}(t) \right\|_X \leq \left\| (F(0) - \mathcal{A}(U_0) - \mathcal{B}(U_0))^0 \right\|_X + \int_0^t \left\| \frac{dF}{ds}(s) \right\|_X ds, \ \ \forall\, t \in [0,T).$$

If U and V are the solutions of (P) corresponding to (U_0, F), $(V_0, G) \in (D(\mathcal{A}) \cap D(\mathcal{B})) \times W^{1,1}(0,T;X)$, then

$$\|U(t) - V(t)\|_X \leq \|U_0 - V_0\|_X + \int_0^t \|F(s) - G(s)\|_X ds, \ \ \forall\, t \in [0,T].$$

Q.E.D.

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Biharmonic curves in Cartan-Vranceanu (2n+1)-dimensional spaces

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Abstract. Biharmonic curves in Cartan-Vranceanu spaces of dimension 2n+1 are characterized and an example of such curve is given.

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Keywords and phrases: Biharmonic curves, Cartan-Vranceanu space.

1 Preliminaries

First we should recall some notions and results related to the biharmonic maps between Riemannian manifolds, as they are presented in [6] and in [7].

Harmonic maps $f:(M,g)\to(N,h)$ between a compact Riemannian manifold, (M,g), and a Riemannian manifold, (N,h), are the critical points of the energy functional $E(f)=\frac{1}{2}\int_M |df|^2\nu_g$ and it is proved (in [4]) that the corresponding Euler-Lagrange equation is $\tau(f)=trace\nabla df$, where $\tau(f)$ is called the tension field of f. If the manifold M is not compact f is said to be harmonic if $\tau(f)=0$. The critical points of the bienergy functional $E_2(f)=\frac{1}{2}\int_M |\tau(f)|^2\nu_g$ are called biharmonic maps. In [6] the Euler-Lagrange equation for E_2 is given

$$\tau_2(f)=-\Delta\tau(f)-traceR^N(df,\tau(f))df=0,$$

where $R^N(X,Y)=\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]}$. The equation $\tau_2(f)=0$ is called the biharmonic equation. Note that the harmonic maps are also biharmonic. Then the main interest is to find the non-harmonic biharmonic maps, which are called proper biharmonic maps.

2 Cartan-Vranceanu spaces

Let us consider the following two-parameter family of Riemannian metrics, called the Cartan-Vranceanu metrics,

$$ds^2_{l,m}=\sum_{i=1}^{n}\frac{dx_i^2+dy_i^2}{[1+m(x_i^2+y_i^2)]^2}+\Big[dz+\frac{l}{2}\sum_{i=1}^{n}\frac{y_idx_i-x_idy_i}{1+m(x_i^2+y_i^2)}\Big]^2 \tag{1}$$

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defined on (2n+1)-dimensional manifold M, where $M=\mathbb{R}^{2n+1}$ if $m\geq 0$, and

$$M=\left\{(x_1,y_1,x_2,y_2,...,x_n,y_n,z)\in\mathbb{R}^{2n+1}|x_i^2+y_i^2\lneqq-\frac{1}{m},i=\overline{1,n}\right\}$$

if $m\lneqq 0$. The biharmonic curves in 3-dimensional Cartan-Vranceanu spaces are characterized in [2] and, moreover, their explicit parametrizations is given in the cited paper. For the 3-dimensional case another results are obtained if $m=0$, $l\neq 0$ and if $l=1$, $m\neq 0$. Thus, if $m=0$, $l\neq 0$ then $(M,ds^2_{l,m})$ is the Heisenberg group, $\mathbb{H}_3$, and the biharmonic curves in this space are studied in [1]. If $l=1$, $m\neq 0$ the biharmonic curves are studied in [3]. In (2n+1)-dimensional case, if $m=0$, $l\neq 0$ then $(M,ds^2_{l,m})$ is the generalized Heisenberg group, $\mathbb{H}_{2n+1}$, and a study of biharmonic curves in this space was given in [5].

In the following let us consider a (2n+1)-dimensional Cartan-Vranceanu space $(M,ds^2_{l,m})$, with $m\neq 0$, and the elements of M are of the form $X=(x_1,y_1,x_2,y_2,...$ $...,x_n,y_n,z)$. We can define a global orthonormal frame field on M by

$$E_{2i-1}=F_i\frac{\partial}{\partial x_i}-\frac{ly_i}{2}\frac{\partial}{\partial z},\quad E_{2i}=F_i\frac{\partial}{\partial y_i}+\frac{lx_i}{2}\frac{\partial}{\partial z},\quad E_{2n+1}=\frac{\partial}{\partial z},$$

for $i=\overline{1,n}$, where $F_i=1+m(x_i^2+y_i^2)$. The Levi-Civita connection of the metric $ds^2_{l,m}$ is given by,

$$\left\{\begin{array}{l}
\nabla_{E_{2i-1}}E_{2j-1}=2\delta_{ij}my_iE_{2i},\\
\nabla_{E_{2i}}E_{2j}=2\delta_{ij}mx_iE_{2i-1},\\
\nabla_{E_{2i-1}}E_{2j}=\delta_{ij}(-2my_iE_{2i-1}+\frac{l}{2}E_{2n+1}),\\
\nabla_{E_{2i}}E_{2j-1}=\delta_{ij}(-2mx_iE_{2i-1}-\frac{l}{2}E_{2n+1}),\\
\nabla_{E_{2n+1}}E_{2i-1}=\nabla_{E_{2i-1}}E_{2n+1}=-\frac{l}{2}E_{2i},\\
\nabla_{E_{2n+1}}E_{2i}=\nabla_{E_{2i}}E_{2n+1}=\frac{l}{2}E_{2i-1},\\
\nabla_{E_{2n+1}}E_{2n+1}=0,
\end{array}\right. \tag{2}$$

for $i,j=\overline{1,n}$. Also, one obtains

$$\left\{\begin{array}{l}
[E_{2i-1},E_{2j-1}]=0,[E_{2i},E_{2j}]=0,\\
[E_{2i-1},E_{2n+1}]=0,[E_{2i},E_{2n+1}]=0,\\
[E_{2i-1},E_{2j}]=\delta_{ij}(2mx_iE_{2i}-2my_iE_{2i-1}+lE_{2n+1}),
\end{array}\right.$$

The curvature tensor field of ∇ is

$$R(X,Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z,$$

and the Riemann–Christoffel tensor field is

$$R(X,Y,Z,W)=g(R(X,Y)W,Z),$$

where $X,Y,Z,W\in\chi(\mathbb{R}^{2n+1})$. We will use the notations

$$R_{abc}=R(E_a,E_b)E_c,\quad R_{abcd}=R(E_a,E_b,E_c,E_d),$$

where $a, b, c, d = \overline{1, 2n+1}$. Then the non-zero components of the curvature tensor field and of the Riemann-Christoffel tensor field are, respectively

$$\begin{cases} R_{(2i-1)(2j-1)(2k)} = -\frac{l^2}{4}\delta_{jk}E_{2i} + \frac{l^2}{4}\delta_{ik}E_{2j}, \\ R_{(2i-1)(2j)(2j-1)} = \frac{l^2}{4}E_{2i},\ i \neq j, \\ R_{(2i-1)(2i)(2k-1)} = \delta_{ik}(\frac{l^2}{4} - 4m)E_{2i} + \frac{l^2}{2}E_{2k}, \\ R_{(2i-1)(2j)(2i)} = -\frac{l^2}{4}E_{2j-1},\ i \neq j, \\ R_{(2i-1)(2i)(2k)} = -\delta_{ik}(\frac{l^2}{4} - 4m)E_{2i-1} - \frac{l^2}{2}E_{2k-1}, \\ R_{(2i-1)(2n+1)(2i-1)} = -\frac{l^2}{4}E_{2n+1}, \\ R_{(2i-1)(2n+1)(2n+1)} = \frac{l^2}{4}E_{2i-1}, \\ R_{(2i)(2j)(2k-1)} = -\frac{l^2}{4}\delta_{jk}E_{2i-1} + \frac{l^2}{4}\delta_{ik}E_{2j-1}, \\ R_{(2i)(2n+1)(2i)} = -\frac{l^2}{4}E_{2n+1}, \\ R_{(2i)(2n+1)(2n+1)} = \frac{l^2}{4}E_{2i}, \end{cases} \tag{3}$$

$$\begin{cases} R_{(2i-1)(2j-1)(2i)(2j)} = -\frac{l^2}{4},\ i \neq j, \\ R_{(2i-1)(2j)(2j-1)(2i)} = -\frac{l^2}{4},\ i \neq j, \\ R_{(2i)(2i-1)(2i-1)(2i)} = \frac{3l^2}{4} - 4m, \\ R_{(2i)(2i-1)(2j-1)(2j)} = \frac{l^2}{2},\ i \neq j, \\ R_{(2n+1)(2i-1)(2i-1)(2n+1)} = -\frac{l^2}{4}, \\ R_{(2n+1)(2i)(2i)(2n+1)} = -\frac{l^2}{4}, \end{cases} \tag{4}$$

for $i, j, k = \overline{1, n}$.

3 Biharmonic curves in (2n+1)-dimensional Cartan–Vranceanu spaces

Let $\gamma : I \subset \mathbb{R} \to (M, ds^2_{l,m})$ be a non-inflexionar curve, parametrized by its arc length. Let $\{T, N_1, ..., N_{2n}\}$ be the Frenet frame in $(M, ds^2_{l,m})$ defined along γ, where $T = \gamma'$ is the unit tangent vector field of γ, N_1 is the unit normal vector field of γ, with the same direction as $\nabla_T T$ and the vectors $N_1, ..., N_{2n}$ are the unit vectors obtained from the following Frenet equations for γ.

$$\begin{cases} \nabla_T T & = & \chi_1 N_1 \\ \nabla_T N_1 & = & -\chi_1 T + \chi_2 N_2 \\ \dots & \dots & \dots \\ \nabla_T N_{2n-1} & = & -\chi_{2n-2}N_{2n-2} + \chi_{2n-1}N_{2n} \\ \nabla_T N_{2n} & = & -\chi_{2n-1}N_{2n-1} \end{cases} \tag{5}$$

where $\chi_1 = \|\nabla_T T\| = \|\tau(\gamma)\|$, and $\chi_2 = \chi_2(s), ..., \chi_{2n} = \chi_{2n}(s)$ are real valued functions, named the curvatures of γ, where s is the arc length of γ.

In [2] is proved the following result

Proposition 3.1. *Let $\gamma : I \subset \mathbb{R} \to (N^n, h)$, $n \geq 2$, be a curve parametrized by arc length from an open interval of $\mathbb{R}$ into a Riemannian manifold (N, g). Then γ is* *biharmonic if and only if*

$$\begin{cases} \chi_1\chi_1' = 0 \\ \chi_1'' - \chi_1^3 - \chi_1\chi_2^2 + \chi_1 R(T, N_1, T, N_1) = 0 \\ 2\chi_1'\chi_2 + \chi_1\chi_2' + \chi_1 R(T, N_1, T, N_2) = 0 \\ \chi_1\chi_2\chi_3 + \chi_1 R(T, N_1, T, N_3) = 0 \\ \chi_1 R(T, N_1, T, N_k) = 0, \ k = \overline{4, n}. \end{cases} \tag{6}$$

Using Proposition 3.1 and equations (4), after a straightforward computation, one obtains

Theorem 3.2. *Let $\gamma : I \subset \mathbb{R} \to (M, ds^2_{l,m})$ be a curve parametrized by its arc length. Then γ is a proper biharmonic curve if and only if*

$$\begin{cases} \chi_1 \in \mathbb{R} \setminus \{0\}, \\ \chi_1^2 + \chi_2^2 = -\eta_1, \\ \chi_2' = \eta_2, \\ \chi_2\chi_3 = \eta_3, \\ \eta_k = 0, \ k = \overline{4, 2n}, \end{cases} \tag{7}$$

with η_k, $k = \overline{1, 2n}$, given by

$$\eta_1 = -R(T, N_1, T, N_1) = -4m \sum_{i=1}^{n} (T_{2i-1}N_1^{2i} - T_{2i}N_1^{2i-1})^2 + \tag{8}$$

$$+\frac{3l^2}{4}\Big[\sum_{i=1}^{n}(T_{2i}N_1^{2i-1} - T_{2i-1}N_1^{2i})\Big]^2 - \frac{l^2}{4}(T_{2n+1}^2 + (N_1^{2n+1})^2)$$

and

$$\eta_k = -R(T, N_1, T, N_k) = \tag{9}$$

$$= -4m \sum_{i=1}^{n} (T_{2i-1}N_1^{2i} - T_{2i}N_1^{2i-1})(T_{2i-1}N_k^{2i} - T_{2i}N_k^{2i-1}) +$$

$$+\frac{3l^2}{4}\sum_{i=1}^{n}(T_{2i}N_1^{2i-1} - T_{2i-1}N_1^{2i})\sum_{i=1}^{n}(T_{2i}N_k^{2i-1} - T_{2i-1}N_k^{2i}) - \frac{l^2}{4}N_1^{2n+1}N_k^{2n+1},$$

for $k = \overline{2, 2n}$, where $T = \sum_{a=1}^{2n+1} T_a E_a$ and $N_k = \sum_{a=1}^{2n+1} N_k^a E_a$.

From the second equation of (7) follows immediately

Corollary 3.3. *If $l = 0$ and $m \leq 0$ then all biharmonic curves of $(M, ds^2_{l,m})$ are geodesics.*

In order to find a proper biharmonic curve $\gamma : I \subset \mathbb{R} \to (M, ds^2_{l,m})$, $\gamma = (x_1(s), y_1(s), x_2(s), y_2(s), ..., x_n(s), y_n(s), z(s))$, let us suppose that the components of its tangent vector $T(s) = \gamma'(s)$ are $T_{2i-1}(s) = \frac{\cos\beta_i(s)\sin\alpha}{\sqrt{n}}$, $T_{2i}(s) = \frac{\sin\beta_i(s)\sin\alpha}{\sqrt{n}}$, $T_{2n+1}(s) = \cos\alpha$, for $i = \overline{1, n}$, where β_i are smooth functions, s being the arc length of γ, and $\alpha \in (0, \pi)$ is a constant. Working this way is suggested by the fact that T is a unitary vector field and by the paper [2], where it is proved that, in dimension 3, the tangent vector is of this form for all proper biharmonic curves of Cartan–Vranceanu spaces.

The covariant derivative of the vector field T is given by

$$\nabla_T T = \sum_{i=1}^{n} [(T'_{2i-1} - 2my_i T_{2i} T_{2i-1} + 2mx_i T_{2i}^2 + lT_{2i} T_{2n+1}) E_{2i-1} +$$

$$+(T'_{2i} - 2mx_i T_{2i} T_{2i-1} + 2my_i T_{2i-1}^2 - lT_{2i-1} T_{2n+1}) E_{2i}] + T'_{2n+1} E_{2n+1} =$$

$$= \sum_{i=1}^{n} \frac{\sin\alpha}{\sqrt{n}} (-A_i \sin\beta_i E_{2i-1} + A_i \cos\beta_i E_{2i}),$$

where

$$A_i = \beta'_i - 2mx_i \frac{\sin\beta_i \sin\alpha}{\sqrt{n}} + 2my_i \frac{\cos\beta_i \sin\alpha}{\sqrt{n}} - l\cos\alpha.$$

Next, assume that $A_i = A$, for any $i = \overline{1,n}$ (that is the values of A_i's are the same for all indices). It follows, from the first Frenet equation, that χ_1 is given by

$$\chi_1 = \|\nabla_T T\| = |A \sin\alpha|.$$

Suppose that $A \sin\alpha \gneq 0$. Then

$$\chi_1 = A \sin\alpha \tag{10}$$

and

$$N_1 = \sum_{i=1}^{n} \frac{1}{\sqrt{n}} (-\sin\beta_i E_{2i-1} + \cos\beta_i E_{2i}).$$

The system $T = \gamma'$ is equivalent with

$$\begin{cases} \dfrac{x'_i}{1 + m(x_i^2 + y_i^2)} = \dfrac{\cos\beta_i \sin\alpha}{\sqrt{n}}, \\ \dfrac{y'_i}{1 + m(x_i^2 + y_i^2)} = \dfrac{\sin\beta_i \sin\alpha}{\sqrt{n}}, \quad i = \overline{1,n}, \\ z' = \cos\alpha + \frac{l}{2} \dfrac{\sin\alpha}{\sqrt{n}} \sum_1^n (x_i \sin\beta_i - y_i \cos\beta_i). \end{cases} \tag{11}$$

Assume that $\beta'_i \neq 0$, for any $i = \overline{1,n}$. By derivation of (10), taking into account that χ_1 must to be constant, one obtains

$$\beta''_i = \frac{2mx_i x'_i + 2my_i y'_i}{1 + m(x_i^2 + y_i^2)} \cdot \beta'_i, \quad i = \overline{1,n}.$$

From the last equations we have

$$b_i \beta'_i = 1 + m(x_i^2 + y_i^2), \quad i = \overline{1,n},$$

where b_i are constants. If we take $b_i = b$ to be independent of i, one obtains

$$\begin{cases} x_i(s) = \dfrac{b\cos\beta_i\sin\alpha}{\sqrt{n}}, \\ y_i(s) = -\dfrac{b\sin\beta_i\sin\alpha}{\sqrt{n}}, \quad i=\overline{1,n}, \\ z(s) = (\cos\alpha)s + \dfrac{lb}{2n}(\sin^2\alpha)s. \end{cases} \tag{12}$$

Again using the facts that χ_1 is a constant and the terms A_i do not depend on i it follows that β_i' must be constants which values are the same for all indices. Hence

$$\beta_i' = C = \frac{1+m(x_i^2+y_i^2)}{b} = \frac{n+mb^2\sin^2\alpha}{bn}, \ i=\overline{1,n}.$$

Thus

$$\beta_i(s) = \frac{n+mb^2\sin^2\alpha}{bn}\cdot s + d_i, \tag{13}$$

where d_i are constants.

From expressions of χ_1, T and N_1 we have, after a straightforward computation,

$$\nabla_T N_1 + \chi_1 T = \sum_{i=1}^{n} \frac{B\cos\alpha}{\sqrt{n}}(\cos\beta_i E_{2i-1} + \sin\beta_i E_{2i}) +$$

$$+\Big(\frac{l}{2}\sin\alpha + A\sin\alpha\cos\alpha\Big)E_{2n+1},$$

where $B = (-\frac{1}{b} + \frac{mb}{n}\sin^2\alpha + l\cos\alpha)\cos\alpha - \frac{l}{2} = -A\cos\alpha - \frac{l}{2}$. From the second Frenet equation we have $\chi_2^2 = \|\nabla_T N_1 + \chi_1 T\|^2 = B^2\cos^2\alpha + (\frac{l}{2}\sin\alpha + A\sin\alpha\cos\alpha)^2$. It follows that χ_2 is a constant. Now, since $-\eta_1 = (\frac{4m}{n} - l^2)\sin^2\alpha + \frac{l^2}{4}$ and $\eta_k = 0$, $k \geq 2$, the curve γ is biharmonic and non-geodesic if and only if the second and the fourth equations of (7) hold. From the second equation, after a straightforward computation, one obtains that

$$A^2 + Al\cos\alpha - (\frac{4m}{n} - l^2)\sin^2\alpha = 0. \tag{14}$$

Assume that $m \gneq 0$. If $l^2 + (\frac{16m}{n} - 5l^2)\sin^2\alpha \gneq 0$ then, solving equation (14), one obtains

$$A = \frac{-l\cos\alpha \pm \sqrt{l^2 + (\frac{16m}{n} - 5l^2)\sin^2\alpha}}{2},$$

and

$$b = \frac{-\Big(nl \pm \sqrt{l^2 + (\frac{16m}{n} - 5l^2)\sin^2\alpha}\Big)}{4m\sin^2\alpha} \pm \tag{15}$$

$$\pm \frac{\sqrt{\Big(nl\cos\alpha \pm \sqrt{l^2 + (\frac{16m}{n} - 5l^2)\sin^2\alpha}\Big)^2 + 16nm\sin^2\alpha}}{4m\sin^2\alpha}.$$

Since $\chi_1 \neq 0$ one obtains $A \neq 0$. Then $\frac{4m}{n} - l^2 \neq 0$.

For the values founded for b, from the third Frenet equation, it follows that $\chi_3 = 0$, and then the fourth equation of (7) holds.

We obtained

Proposition 3.4 *Let $(M, ds^2_{l,m})$ be a (2n+1)-dimensional Cartan-Vranceanu space such that $m \gneq 0$ and $\frac{4m}{n} - l^2 \neq 0$. Let $\gamma : I \subset \mathbb{R} \to (M, ds^2_{l,m})$,*

$$\gamma = (x_1(s), y_1(s), x_2(s), y_2(s), ..., x_n(s), y_n(s), z(s)),$$

be a curve parametrized by its arc length, given by

$$\begin{cases} x_i(s) = \dfrac{b \cos \beta_i \sin \alpha}{\sqrt{n}}, \\ y_i(s) = -\dfrac{b \sin \beta_i \sin \alpha}{\sqrt{n}}, \quad i = \overline{1, n}, \\ z(s) = (\cos \alpha)s + \dfrac{lb}{2n}(\sin \alpha)^2 s. \end{cases}$$

where $\alpha \in (0, \pi)$, β_i are given by (13), b is given by (15) and $l^2 + (\frac{16m}{n} - 5l^2) \sin^2 \alpha \geq 0$. Then γ is a biharmonic curve and it is not a geodesic.

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Infinitely many functional pre-complete classes of formulas in the propositional provability intuitionistic logic

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Abstract. We consider the propositional provability intuitionistic logic I^{Δ}, introduced by A.V. Kuznetsov [2]. We prove that there are infinitely many classes of formulas in the calculus of I^{Δ}, which are pre-complete with respect to functional expressibility in I^{Δ}. This result is stronger than an ealier one stated by the author in [1].

Mathematics subject classification: 03F45, 03B45, 03B55.
Keywords and phrases: Provability-intuitionistic logic, functional expressibility, pre-complete classes of formulas.

In the present paper we extend the result established in [1] to a wider class of logics, which includes the provability-intuitionistic logic I^{Δ} itself. The last one is the logic of propositional provability-intuitionistic calculus I^{Δ} proposed and formalized by A.V. Kuznetsov [2, 3]. It is based on formulas built in an usual way from propositional variables $p, q, r, \dots$ (may be indexed) by means of logical connectives $\&, \vee, \supset, \neg, \Delta$. The axioms of I^{Δ} consist of well-known axioms of the intuitionistic propositional calculus and three additional Δ-axioms:

$$(p \supset \Delta p), \tag{1}$$
$$((\Delta p \supset p) \supset p), \tag{2}$$
$$(((p \supset q) \supset p) \supset (\Delta q \supset p)). \tag{3}$$

Its rules of inference consist of traditional rules of *substitution*, and *modus ponens* and *the rule of necessitation* $\frac{A}{\Delta A}$. An extension L of the logic I^{Δ} is defined as usual as any set of formulas which contains the axioms of I^{Δ} and is closed with respect to the rules of inference of the calculus I^{Δ}. In the following let us denote by L any extension of I^{Δ} if other things are not stated. By equivalence of formulas A and B, denoted $A \sim B$, in the logic L we understand the formula $((A \supset B)\&(B \supset A))$.

Let us remind the notion of Δ-pseudo-Boolean algebra [2, 3] as a system of type $\mathfrak{A} =< E; \&, \vee, \supset, \neg, \Delta >$, where $< E; \&, \vee, \supset, \neg >$ is a pseudo-Boolean algebra and operation Δ satisfies the relations

$$x \leq \Delta x, \qquad (\Delta x \supset x) = x, \qquad \Delta x \leq y \vee (y \supset x).$$

These algebras serve as algebraic models for logic I^{Δ} [2, 3]. Valid formulas on the algebra $\mathfrak{A}$ are defined as usual. It is also known that the set of valid formulas on

$\mathfrak{A}$ constitutes an extension of I^Δ [3], it is called *the logic of the algebra* $\mathfrak{A}$, and it is denoted by $L\mathfrak{A}$.

We consider the Δ-pseudo-Boolean algebra $\mathfrak{C} =< E; \&, \vee, \supset, \neg, \Delta >$, where E is the chain of elements $0 = \tau_0 < \tau_1 < \cdots < 1$, and for any elements α and β of $\mathfrak{C}$ we assume that $\alpha\&\beta = min(\alpha, \beta)$, $\alpha \vee \beta = max(\alpha, \beta)$, $\alpha \supset \beta = 1$ when $\alpha \leq \beta$, $\alpha \supset \beta = \beta$ when $\alpha > \beta$, $\neg\alpha = \alpha \supset 0$, $\Delta\tau_i = \tau_{i+1}$ for $i = 0, 1, \dots,$ and $\Delta 1 = 1$.

By a formula realization [5, 6] of the Δ-pseudo-Boolean algebra $\mathfrak{A}$ into the proof (provability) logic L we undestand a mapping f from the algebra $\mathfrak{A}$ into the set of formulas such that if we examine the formulas up to equivalent ones, then f is an isomorphism between the algebra $\mathfrak{A}$ and some subalgebra of the Lindenbaum algebra of the logic L. Let us build a formula realization of the algebra $\mathfrak{C}$ into the logic I^Δ. So, we have first of all to map each element of $\mathfrak{C}$ into the set of formulas. Consider the mapping f is as follows:

$$f(0) = 0, \qquad f(1) = 1,$$

$$f(\tau_i) = \Delta^i 0, \qquad i = 1, 2, \dots$$

Let us prove the following

Lemma 1. *For any elements β and γ of the algebra $\mathfrak{C}$ the next deductions in the I^Δ logic take place*

$$\vdash (f(\beta\&\gamma) \sim (f(\beta)\&f(\gamma))), \tag{4}$$

$$\vdash (f(\beta \vee \gamma) \sim (f(\beta) \vee f(\gamma))), \tag{5}$$

$$\vdash (f(\beta \supset \gamma) \sim (f(\beta) \supset f(\gamma))), \tag{6}$$

$$\vdash (f(\neg\beta) \sim \neg f(\beta)), \tag{7}$$

$$\vdash (f(\Delta\beta) \sim \Delta f(\beta)). \tag{8}$$

Proof. Let us consider arbitrary elements β and γ of the algebra $\mathfrak{C}$. Let prove first the relation (4). If $\beta = 1$ or $\gamma = 1$ then the statement (4) is obvious. Let $\beta \neq 1$ and $\gamma \neq 1$. Since elements β and γ are arbitrary we can consider that $\beta = \tau_i$ and $\gamma = \tau_j$, where $i < j$. Then $\beta\&\gamma = \tau_i\&\tau_j = \tau_i$. So,

$$f(\beta\&\gamma) = f(\beta) = f(\tau_i) = \Delta^i 0. \tag{9}$$

The axiom (1) admits the following generalization

$$\vdash p \supset \Delta^k p, \text{ where } k = 0, 1, 2, \dots \tag{10}$$

Taking into consideration the last relation (10) we have the following sequence of equalities

$$(f(\beta)\&f(\gamma)) = (f(\tau_i)\&f(\tau_j)) = \Delta^i 0\&\Delta^j 0 = \Delta^i 0. \tag{11}$$

So the relation (4) follows from (10) and (11).

The proof of the statement (5) is analogous to the proof of the deduction (4).

Let us prove the relation (6). We will consider two possible cases for elements β and γ, when a) $\beta \leq \gamma$, and b) $\beta > \gamma$.

Let $\beta \leq \gamma$. Then obviously $\beta \supset \gamma = 1$, and $f(\beta \supset \gamma) = f(1) = 1 = (p \supset p)$. If $\beta \neq 1$ and $\gamma \neq 1$, then we can consider as above that there exist i and j, where $i, j = 0, 1, 2, \dots$ and $i \leq j$, such that $\beta = \Delta^i 0$ and $\gamma = \Delta^j 0$. Thus we have that $f(\beta) = \Delta^i 0$, $f(\gamma) = \Delta^j 0$, $(f(\beta) \supset f(\gamma)) = \Delta^i 0 \supset \Delta^j 0 = 1$.

Now let $\beta > \gamma$. Then $\beta \supset \gamma = \gamma$, and $f(\beta \supset \gamma) = f(\gamma)$. Obviously, $\gamma \neq 1$. So, there exists an integer positive j such that $\gamma = \Delta^j 0$, and thus $f(\beta \supset \gamma) = \Delta^j 0$. Now, let us evaluate $f(\beta) \supset f(\gamma)$. If $\beta = 1$, then, obviously, $f(\beta) \supset f(\gamma) = 1 \supset \Delta^j 0 = \Delta^j 0$. Suppose $\beta \neq 1$. Then there exists an $i > j$ such that $\beta = \Delta^i 0$, and $f(\beta) \supset f(\gamma) = \Delta^i 0 \supset \Delta^j 0 = \Delta^j 0$.

Now let us to prove the relation (7). If $\beta = 1$ then $f(\beta) = 1$, $\neg\beta = 0$, $\neg f(\beta) = \neg 1 = 0$. Suppose $\beta \neq 1$. Then there is an i such that $\beta = \Delta^i 0$. Obviously, $\neg\beta = 0$, and $\neg f(\beta) = \neg\Delta^i 0 = 0$.

Finally, let us look at the last statement (8) of the lemma. If $\beta = 1$ then (8) follows obviously. Suppose $\beta \neq 1$. Then there exists an i such that $\beta = \Delta^i 0$, and $f(\Delta\beta) = f(\Delta\Delta^i 0) = f(\Delta^{i+1} 0) = \Delta^{i+1} 0$, $\Delta f(\beta) = \Delta f(\Delta^i 0) = \Delta\Delta^i 0 = \Delta^{i+1} 0$. Comparing the last two sequences of statements we conclude (8). Lemma is proved.

Next lemma is a generalization of the previous lemma.

Lemma 2. *For any formula $F(p_1, \dots, p_n)$ of the provability-intuitionistic logic I^Δ and for any elements $\beta_1, \dots, \beta_n$ of the algebra $\mathfrak{C}$ the following relation is true in I^Δ*

$$\vdash f(F[p_1/\beta_1, \dots, p_n/\beta_n]) \sim F[p_1/f(\beta_1), \dots, p_n/f(\beta_n)].$$

The proof can be easily done by induction over the structure of the formula F and using the relations proved in Lemma 1.

On the basis of Lemma 2 we conclude that examining formulas up to equivalent ones in the logic I^Δ the mapping f is an isomorphism between the algebra $\mathfrak{C}$ and some subalgebra of the Lindenbaum's algebra of the logic I^Δ. This fact means that f is a formula realization of the algebra $\mathfrak{C}$ into the logic I^Δ. We see from the definition of this formula realization that it puts into correspondence only unary formulas to elements of $\mathfrak{C}$. So, we get next theorem as a consequence from Lemma 1 and Lemma 2.

Theorem 1. *The mapping f defined above is a formula realization of the algebra $\mathfrak{C}$ into the provability-intuitionistic logic I^Δ.*

Next lemma ilustrates a usefull property of the formula realization f defined above.

Lemma 3. *The formula realization f of the algebra $\mathfrak{C}$ into the logic I^Δ puts into correspondence to any element β of $\mathfrak{C}$ such unary formula $f(\beta)$ that the equality holds*

$$f(\beta)[\gamma] = \beta. \tag{12}$$

Proof. Really, let element β be someone from $\mathfrak{C}$. Then, obviously, it is either equal to the unit 1 of the algebra, or there exists an index k such that $\beta = \tau_k$. Recalling that $f(\tau_k) = \Delta^k(p\&\neg p)$ we get that the result of substitution $\Delta^k(p\&\neg p)[\gamma]$ does not depend on the element γ, and, moreover,

$$\Delta^k(p\&\neg p)[\gamma] = \tau_k.$$

The last relation together with the fact that element β is an arbitrary element of $\mathfrak{C}$ ensure us the validness of (12) from the lemma. The lemma is proved.

Definitions regarding expressibility in logics were proposed by A.V. Kuznetsov [7, 8, 9]. A system of formulas Σ is said to be *complete (with respect to functional expessibility) in the logic* L if any formula of the language of logic L can be obtained from variables and formulas of Σ applying a finite number of times the following two rules: the weak rule of substitution and the rule of replacement by equivalent formula in L. The system of formulas Σ is called *pre-complete (with respect to functional expressibility) in the logic* L if it is incomplete in L, and for any formula F, which is not expressible in L via Σ the system $\Sigma \cup \{F\}$ is complete in L. They say the formula $F(p_1, \dots, p_n)$ *conserves on the algebra* $\mathfrak{A}$ *the predicate* $R(x_1, \dots, x_m)$ if for any elements $\alpha_{ij} \in \mathfrak{A}$ $(i = 1, \dots, m; j = 1, \dots, n)$ the relations

$$R[\alpha_{11}, \dots, \alpha_{m1}], \dots, R[\alpha_{1n}, \dots, \alpha_{mn}]$$

imply the truth of the predicate

$$R[F[\alpha_{11}, \dots, \alpha_{1n}], \dots, F[\alpha_{m1}, \dots, \alpha_{mn}]].$$

In the following let L be any extension of the provability-intuitionistic logic I^Δ, which satisfies the relation $I^\Delta \subseteq L \subseteq L\mathfrak{C}$. Let us denote by K_i the class of all formulas that preserves on the algebra $\mathfrak{C}$ the predicate $x \leq \tau_i$ $(i = 0, 1, \dots)$.

Lemma 4. *The classes* $K_0, K_1, \dots$ *are two by two distinct with respect to set inclusion.*

Really, suppose $r < s$ and let us consider classes K_r and K_s. Can be checked the relation

$$(\Delta^{r+1}0 \supset p) \in K_r \setminus K_s, \quad \Delta^s 0 \in K_s \setminus K_r.$$

So, the classes $K_0, K_1, \dots$ are two by two distinct with respect to the set inclusion.

Lemma 5. *Let* f *be the formula realization of the algebra* $\mathfrak{C}$ *into the logic* I^Δ *that was defined ealier. Then for any element* β *of this algebra and for any* $j = 0, 1, 2, \dots$ *the formula* $f(\beta)$ *belongs to the class* K_j *if and only if the folowing relation holds*

$$\beta \leq \Delta^j 0. \tag{13}$$

Proof. Let element β satisfy the relation (13) from the lemma. We have to show that the formula $f(\beta)$ conserves the relation R_j on the algebra $\mathfrak{C}$. Let an arbitrary element $\gamma \in \mathfrak{C}$ conserve the predicate R_j on $\mathfrak{C}$, i.e. $R_j(\gamma)$ holds. Then, using (13) and the equality $f(\beta)[\gamma] = \beta$ already proved ealier in Lemma 3, we obtain the relation $f(\beta)[\gamma] \leq \Delta^j 0$, i.e. the relation $R_j(f(\beta)[\gamma]$. So, we get that $f(\beta) \in K_j$.

Conversely, let element β do not satisfy the condition (13). Then, since elements of $\mathfrak{C}$ form a chain, we get

$$\beta > \Delta^j 0.$$

After that, using again the above mentioned equality (12) we get the inequality

$$f(\beta)[\gamma] > \Delta^j 0,$$

which is equivalent to the fact that $R_j(f(\beta)[\gamma])$ is false. Subsequently, the formula $f(\beta)$ does not belong to the class K_j. The lemma is proved.

Lemma 6. *Let L be any logic such that $I^\Delta \subseteq L \subseteq L\mathfrak{C}$. Then the classes of formulas $K_0, K_1, \ldots$ are pre-complete with respect to expressibility in the logic L.*

Really, according to Lemma 4, no one of these classes is complete with respect to expressibility in the logic L. Let us prove that for any $j = 0, 1, \ldots$ the class K_j is pre-complete in L. Let $B(p_1, \ldots, p_n)$ be any formula which does not belong to the class K_j. Then, according to the definition of the class K_j there exist elements $\beta_1, \ldots, \beta_n$ of the algebra $\mathfrak{C}$ such that for any $i = 1, \ldots, n$ we have

$$\beta_i \leq \Delta^j 0,$$

but

$$B[\beta_1, \ldots, \beta_n] \leq \Delta^j 0$$

is false. Since all elements of $\mathfrak{C}$ form a chain we get the strict inequality on the algebra $\mathfrak{C}$

$$\Delta^j 0 < B[\beta_1, \ldots, \beta_n],$$

which implies the relation

$$\Delta^{j+1} 0 \leq B[\beta_1, \ldots, \beta_n].$$

The last statement implies also

$$(\Delta^{j+1} 0 \supset B[\beta_1, \ldots, \beta_n]) = 1.$$

In view of the above defined formula realization f of the algebra $\mathfrak{C}$ in the logic I^Δ, the last formula conducts us to the relation

$$\vdash f(\Delta^{j+1} 0 \supset B[\beta_1, \ldots, \beta_n]) \sim f(1).$$

Applying Lemma 2 to the left part of the above equivalence, we get that

$$\vdash (f(\Delta^{j+1} 0) \supset f(B[\beta_1, \ldots, \beta_n])) \sim f(1).$$

Reminding ourselves that $f(\Delta^{j+1}0)$ is the formula $\Delta^{j+1}0$ and $f(1)$ is 1, and applying once again Lemma 2 to formula B, we get the deduction

$$\vdash (\Delta^{j+1}0 \supset B[f(\beta_1), \ldots, f(\beta_n)]) \sim 1.$$

The left hand side of the last equivalence can be represented as

$$\vdash (\Delta^{j+1}0 \supset \pi)[\pi/B[f(\beta_1), \ldots, f(\beta_n)]] \sim 1.$$

Let us note that the formula $(\Delta^{j+1}0 \supset \pi)$ belongs to the class K_j. Apart from this, since elements β_i, when $i = 1, \ldots, n$, satisfy the condition $\beta_i \leq \Delta^{j+1}0$, the formulas $f(\beta_i)$ also belong, according to Lemma 5, to the class K_j. That is why the last deduction shows that the formula 1 is expressible in the logic I^Δ by means of the formula B and of the formulas of the class K_j. We shall show that any formula F is expressible in I^Δ via system $K_j \cup \{1\}$. Let F do not contain the variable π. Then it is sufficient to take the formula $F\&\pi$, which belongs to the class K_j, and to use the following fact

$$F \sim (F\&\pi)[\pi/1].$$

Thus it is proved that the system $K_j \cup \{B\}$ is complete in I^Δ, and the more so, it is also complete in the logic L. So, lemma is proved.

Now we can state the folowing results.

Theorem 2. *Let L be any logic such that $I^\Delta \subseteq L \subseteq L\mathfrak{C}$. The classes $K_0, K_1, \ldots$ constitute a numerable collection of distinct two by two pre-complete in the logic L classes of formulas.*

In 1956 A.V. Kuznetsov [10, 11] have proved that for any finite-valued logic L there exists an algorithm which permits to recognize whether a system of formulas is complete with respect to functional expressibility in L. He have shown that there exists theoretically a finite collection of pre-complete classes of formulas in that logic. Unfortunately, the proposed algorithm is very computationally inefficient. Next theorem establishes that even such algorithm is impossible for any logic L that satisfies the condition $I^\Delta \subseteq L \subseteq L\mathfrak{C}$.

Theorem 3. *Let L be any logic such that $I^\Delta \subseteq L \subseteq L\mathfrak{C}$. The traditional formulation of the theorem of completeness with respect to functional expressibility in terms of a finite collection of pre-complete classes of formulas in the logic L does not exist.*

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Linear convolution of criteria in the vector p-center problem *

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Abstract. We investigate a linear convolution of criteria and possibility of its application for finding Pareto set in the vector variant of the well-known combinatorial p-center problem. The polynomial algorithm which transforms any vector p-center problem to a solvable problem with the same Pareto set is proposed. An example which illustrates the work of algorithm is performed.

Mathematics subject classification: 90C27, 90C29, 90C47.
Keywords and phrases: p-center problem, Pareto set, algorithm of linear convolution (ALC), solvability by ALC.

1 Introduction

Wide and important class of the best choice problems is the vector (multicriteria) problems in which the quality of decision making is estimated by several criteria at once. One of the main method of vector optimization is scalaring. Scalaring is a process of transforming vector problem of finding best alternatives to a scalar problem with aggregated (generalized) criterion which is a convolution of criteria. Such convolution of criteria usually depends on parameters. The central concept in vector optimization is a Pareto principle of optimality. And an important method of finding Pareto-optimal solutions (efficient alternatives) is based on linear convolution of criteria. But this approach cannot always guarantees to find the whole Pareto set. In these cases we say that the vector problem is unsolvable by ALC. Many classes of the vector problems which are solvable by ALC were found by Koopmans, Karlin, Geoffrion, Kuhn, Tucker, Saaty and others. The history review of this question was presented in [1] (see also [2]).

We investigate the possibility of application of ALC to finding all Pareto set in vector variant of the well-known combinatorial p-center problem, i. e. the problem of best locating p facilities (see, for example, [3–5]). In this paper it is shown that there exist (see, theorem 1) vector p-center problems which are unsolvable by ALC. Analogous results for various kinds of vector discrete optimization problems (salesman problem, optimal spanning tree, perfect matching and others) were obtained earlier in [6–11].

Using the well-known sufficient condition of solvability [12] we build an algorithm, which transforms any vector p-center problem to an equivalent solvable problem.

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Earlier in the works [12–15] similar algorithms were built for the vector trajectory problems with another kinds of partial criteria.

2 Basic definitions and notations

We consider the vector (s-criteria) variant of the p-center problem. Let us use the following definitions:

$N_m = \{1, 2, \dots, m\}$ is the set of possible locating points of facilities (equipment, warehouses, providers etc.),

N_n is the set of clients,

$D_k = [d_{ij}^k] \in \mathbf{R}^{m\times n}$ is a matrix of costs connected with delivery of product from point $i \in N_m$ to point $j \in N_n$ by criterion $k \in N_s$.

The vector $D = (D_1, D_2, \dots, D_s)$, composed from matrices of costs, is called a system of costs.

Let $1 \le p \le m-1$ and T be some system of nonempty subsets (p-centers) of the set N_m such that

$$|t| = p, \quad \forall t \in T.$$

As usual (see, for example, [10–14]), all the elements of set T are called trajectories.

We define the vector function on T:

$$f(t, D) = (f_1(t, D_1), f_2(t, D_2), \dots, f_s(t, D_s)),$$

where

$$f_k(t, D_k) = \max_{j\in N_n} \min_{i\in t} d_{ij}^k \to \min_{t\in T}, \quad k \in N_s.$$

The s-criteria (vector) $m \times n$-dimensional p-center problem is understood as the problem of finding Pareto set (the set of efficient trajectories)

$$P^s(T, D) = \{t \in T : \ \forall\, t' \in T \ \ (t \underset{D}{\not\succ} t')\},$$

where $\underset{D}{\not\succ}$ is the negation of binary relation $\underset{D}{\succ}$, which specifies the Pareto principle of optimality:

$$t \underset{D}{\succ} t' \ \Leftrightarrow\ f(t, D) \ge f(t', D) \ \&\ f(t, D) \ne f(t', D).$$

This problem is denoted by $Z^s_{m\times n}(T, D)$. Scalar (single-criterion) problem $Z^1_{m\times n}(T, D)$, $D \in \mathbf{R}^{m\times n}$, can be interpreted as an extremal problem on graphs or on networks. It consists in locating p facilities and assigning clients to them in order to minimize the maximum distance between a client and the facility to which it is assigned. If we want to optimize the location of p facilities by several criteria then it leads us to the above multicriteria variant of the p-center problem.

Following [7–9], the problem $Z^s_{m\times n}(T, D)$, $s \ge 2$, is called solvable by ALC if

$$P^s(T, D) = \Xi^s(T, D),$$

where

$$\Xi^s(T, D) = \bigcup_{\lambda \in \Lambda^s} T(\lambda),$$

$$\Lambda^s = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s) : \sum_{k=1}^{s} \lambda_k = 1, \quad \lambda_k > 0, \quad k \in N_s\},$$

$$T(\lambda) = \text{Argmin}\{\langle \lambda, f(t, D)\rangle : t \in T\}$$

and $\langle \lambda, f(t, D)\rangle = \sum\limits_{k=1}^{s} \lambda_k f_k(t, D_k)$ is a linear convolution of criteria $f_k(t, D_k),\ k \in N_s$.

Thus, the problem $Z^s_{m\times n}(T, D)$ is solvable if for any efficient trajectory $t^* \in P^s(T, D)$ there exists a vector $\lambda^* \in \Lambda^s$ such that

$$\langle \lambda^*, f(t^*, D)\rangle = \min\{\langle \lambda^*, f(t, D)\rangle : t \in T\},$$

i. e. any trajectory $t^* \in P^s(T, D)$ can be found as a solution of a scalar minimization problem with function which is a linear convolution of partial criteria with an appropriate vector $\lambda^* \in \Lambda^s$.

Otherwise, if there exists a trajectory $t^* \in P^s(T, D)$ such that for any vector $\lambda \in \Lambda^s$ the inequality

$$\langle \lambda, f(t^*, D)\rangle > \min\{\langle \lambda, f(t, D)\rangle : t \in T\}$$

holds, then the problem $Z^s_{m\times n}(T, D)$ is called unsolvable by ALC. It is evident, that in this case we have

$$\Xi^s(T, D) \subset P^s(T, D).$$

3 Insolubility

The set of trajectories T is called primitive if the following two conditions hold:

1) there exist three pairwise different trajectories $t_1,\ t_2$ and t_3 such that

$$i \in t_i \setminus t^0, \quad i = 1, 2, 3,$$

where

$$t^0 = \bigcup_{1 \le r_1 < r_2 \le 3} (t_{r_1} \cap t_{r_2}),$$

2) for any trajectory $t \in T \setminus \{t_1, t_2, t_3\}$ the equality

$$t \cap N_3 = \emptyset$$

holds.

Thus, in the case, where the set T is primitive, the number of possible locating points of facilities $m \ge 4$.

Theorem 1. *For any primitive set of trajectories T there exists a system of costs D such that the p-center problem $Z^s_{m\times n}(T, D),\ p \ge 1,\ s \ge 2,\ m \ge 4,\ n \ge 1,$ is unsolvable by ALC.*

Proof. First we consider the case where $s = 2$. Let $t_1,\ t_2,\ t_3$ be the three trajectories of set T described above. Let the matrices $D_k = [d_{ij}^k] \in \mathbf{R}^{m\times n}$, $k = 1,\ 2$, have the following form

$$D_1 = \begin{pmatrix} 0 & 0 & \dots & 0 \\ a & a & \dots & a \\ b & b & \dots & b \\ c & c & \dots & c \\ \dots & \dots & \dots & \dots \\ c & c & \dots & c \end{pmatrix}, \quad D_2 = \begin{pmatrix} a & a & \dots & a \\ 0 & 0 & \dots & 0 \\ b & b & \dots & b \\ c & c & \dots & c \\ \dots & \dots & \dots & \dots \\ c & c & \dots & c \end{pmatrix},$$

where $c > a > b > a/2 > 0$.

Then, taking into account the primitivity of set T, we obtain the vector evaluations of trajectories of T:

$$f(t_1, D) = (0, a),$$

$$f(t_2, D) = (a, 0),$$

$$f(t_3, D) = (b, b),$$

$$f(t, D) = (c, c) \quad \forall\ t \in T \setminus \{t_1, t_2, t_3\}.$$

Therefore we have $P^2(T, D) = \{t_1, t_2, t_3\}$ and for any vector $\lambda \in \Lambda^2$ the following relations hold

$$\langle\lambda, f(t_3, D)\rangle = b > a/2 \geq \min\{\langle\lambda, f(t_i, D)\rangle : i = 1, 2\} \geq \min\{\langle\lambda, f(t, D)\rangle : t \in T\}. \tag{1}$$

It follows that the theorem is valid in the case $s = 2$.

Finally, the theorem ($s > 2$) can be proved if the matrices D_1 and D_2 are the same as before, and

$$D_k = D_2, \quad k \in\ 3, 4, \dots, s.$$

As a result we obtain the following vector evaluations of trajectories:

$$f(t_1, D) = (0, a, a, \dots, a) \in \mathbf{R}^s,$$

$$f(t_2, D) = (a, 0, 0, \dots, 0) \in \mathbf{R}^s,$$

$$f(t_3, D) = (b, b, b, \dots, b) \in \mathbf{R}^s,$$

$$f(t, D) = (c, c, \dots, c) \in \mathbf{R}^s \quad \forall\ t \in T \setminus \{t_1, t_2, t_3\}.$$

Therefore we have $P^s(T, D) = \{t_1, t_2, t_3\}$ and any vector $\lambda \in \Lambda^s$ satisfies relations (1).

Theorem 1 has been proved.

4 Algorithm

Each of five stages of algorithm Ψ, which builds a transformed system of costs $\tilde{D}$ consists of s steps ($s \geq 2$).

Stage 1. Step $k \in N_s$. For any number $j \in N_n$ we sort all the elements d_{ij}^k, $i \in N_m$, of j-th column of matrix D_k:

$$d_{i_1 j}^k \geq d_{i_2 j}^k \geq \ldots \geq d_{i_m j}^k.$$

Stage 2. Step $k \in N_s$. We delete all the elements of the following sequence from matrix D_k

$$d_{i_1 j}^k,\ d_{i_2 j}^k, \ldots,\ d_{i_q j}^k, \quad j \in N_n, \tag{2}$$

where $q = p - 1$.

Stage 3. Step $k \in N_s$. We sort the rest of elements of matrix D_k in ascending order:

$$b_1^k \leq b_2^k \leq \ldots \leq b_u^k, \tag{3}$$

where $u = n(m - p + 1)$. Of course, all the elements stay on the same places in matrix D_k.

Stage 4. Step $k \in N_s$. We transform the elements of (3) by the following recurring formula

$$\tilde{b}_r^k = \begin{cases} b_r^k, & \text{if } r = 1, 2, \\ \tilde{b}_{r-1}^k, & \text{if } \Delta^k(r, r-1) = 0,\ r = 3, 4, \ldots, u, \\ b_r^k + s(\tilde{b}_{r-1}^k - \tilde{b}_1^k) + \tilde{b}_1^k, & \text{if } \Delta^k(r, r-1) > 0,\ r = 3, 4, \ldots, u, \end{cases} \tag{4}$$

where $\Delta^k(v, w) = b_v^k - b_w^k$. As a result we obtain $\tilde{b}_1^k$, $\tilde{b}_2^k, \ldots,$ $\tilde{b}_u^k$.

Stage 5. Step $k \in N_s$. Instead of elements of sequence (2), deleted on the step 2, we write the number $\tilde{b}_u^k + 1$, i. e. for any $j \in N_n$ we set

$$\tilde{d}_{i_1 j}^k = \tilde{d}_{i_2 j}^k = \ldots = \tilde{d}_{i_q j}^k = \tilde{b}_u^k + 1.$$

As a result of the work of the algorithm Ψ the system of costs D is replaced by $\tilde{D} = (\tilde{D}_1, \tilde{D}_2, \ldots, \tilde{D}_s)$, where $\tilde{D}_k = [\tilde{d}_{ij}^k]_{m \times n}$.

Remark 1. It is easy to see that for any number $k \in N_s$ and for any trajectory t, the inequality $f_k(t, \tilde{D}_k) < \tilde{b}_u^k + 1$ holds.

5 Substantiation of algorithm

First we prove that the obtained vector problem $Z_{m \times n}^s(T, \tilde{D})$ is solvable by ALC. To prove this we use the known sufficient condition of solvability for the vector discrete problems [12] and formulate it in the form convenient for us. Let us introduce a new definition.

For any natural numbers $s \geq 2$ and $h \geq 1$ the set composed of h pairwise different numbers is called (s, h)-regular if after sorting these numbers in ascending order

$$a_1 < a_2 < \ldots < a_h$$

under $h \geq 3$ the inequalities

$$s \cdot \delta(r+1, 1) \leq \delta(r+2, 1), \ r \in N_{h-2}$$

hold, where $\delta(u, v) = a_u - a_v$.

Remark 2. It is evident that for any $s \geq 2$ the set composed of one or two different numbers is $(s, 1)$- or $(s, 2)$-regular, respectively.

In these terms for any s-criteria discrete problem Z^s

$$f_k(t) \to \min_{t \in T}, \quad k \in N_s,$$

where $s \geq 2$, $f_k(t) \in \mathbf{R}$, $|T| < \infty$, the following known sufficient condition of solvability is valid.

Theorem 2. *([12]) If for any number $k \in N_s$ the set, composed of $h(k)$ different values of k-th partial criterion $f_k(t)$ on the set T, is $(s, h(k))$-regular, then the problem Z^s is solvable by ALC.*

It is easy to see that for any number $k \in N_s$ the set of $h(k)$ different numbers of sequence

$$\tilde{b}_1^k, \ \tilde{b}_2^k, \ldots, \ \tilde{b}_u^k,$$

obtained as a result of the Stage 4 of algorithm Ψ, is $(s, h(k))$-regular. Hence, taking into account Remark 1, we conclude that the set of $h'(k)$ $(h'(k) \leq h(k))$ different values of the k-th criterion $f_k(t, \tilde{D}_k)$ on the set T is $(s, h'(k))$-regular, and therefore in view of Theorem 2 the problem $Z^s_{m\times n}(T, \tilde{D})$ is solvable by ALC.

Taking into account algorithm Ψ, we conclude that for any two trajectories t and t' the following formula

$$t \underset{D}{\succ} t' \Leftrightarrow t \underset{\tilde{D}}{\succ} t',$$

holds, which implies that the vector problems $Z^s_{m\times n}(T, D)$ and $Z^s_{m\times n}(T, \tilde{D})$ are equivalent, i. e. the equality $P^s(T, D) = P^s(T, \tilde{D})$ is valid.

It is easy to see that the complexity of Stages 1 and 2 of algorithm Ψ is $O(snm \log_2 m)$. Since the Stage 3 is a multi-way merging of ordered numerical sequences then the complexity of Stage 3 is $O(sn(m-p+1)\log_2 n)$ (see, for example [16]). The complexity of Stages 4 and 5 of algorithm Ψ are $O(sn(m-p+1))$ and $O(sn(p-1))$ respectively.

Summarizing the said above, we conclude that the following theorem holds.

Theorem 3. *The algorithm Ψ transforms any vector p-center problem $Z^s_{m\times n}(T, D)$, $s \geq 2$, to the equivalent p-center problem $Z^s_{m\times n}(T, \tilde{D})$, which is solvable by ALC, moreover the complexity of algorithm Ψ is $O(sn(m \log_2 m + (m-p+1)\log_2 n))$.*

In view of Remark 2 Theorem 3 implies

Corollary 4. *The problem $Z^s_{m\times n}(T,D)$ is solvable by ALC if for any $k \in N_s$ the inequality $h(k) \le 2$ holds.*

In the partial case, where $p = 1$, the complexity of algorithm Ψ is $O(smn \log_2 mn)$.

6 Example

We consider 2-criteria 2-center problem with 5×2-dimension, i. e. $s = 2,\ p = 2,\ m = 5,\ n = 2$. Let $T = \{t_1, t_2, t_3, t_4\},\ t_1 = \{1,2\},\ t_2 = \{2,3\},\ t_3 = \{3,4\},\ t_4 = \{4,5\},\ D = (D_1, D_2)$,

$$D_1 = \begin{pmatrix} 4 & 0 \\ 0 & 6 \\ 6 & 4 \\ 10 & 2 \\ 8 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 2 & 6 \\ 1 & 10 \\ 0 & 4 \\ 8 & 0 \\ 10 & 3 \end{pmatrix}.$$

Then the problem $Z^2_{5\times 2}(T,D)$ has the following vector evaluations of trajectories

$$f(t_1, D) = (0,6),$$

$$f(t_2, D) = (4,4),$$

$$f(t_3, D) = (6,0),$$

$$f(t_4, D) = (8,8).$$

It is easy to see that $P^2(T,D) = \{t_1, t_2, t_3\}$, and the problem $Z^2_{5\times 2}(T,D)$ is unsolvable, because for any vector $\lambda \in \Lambda^2$ we have

$$\langle \lambda, f(t_2, D)\rangle = 4 > 3 \ge \min\{\langle \lambda, f(t_i, D)\rangle : i \in N_4\},$$

i. e. $t_2 \notin \Xi^2(T,D)$.

Using algorithm Ψ, we transform the system of costs D to $\tilde{D} = \{\tilde{D_1}, \tilde{D_2}\}$.

Stage 1. Let us sort all the elements of each column for each matrix D_1 and D_2 in ascending order

$$D_1 = \begin{pmatrix} 4^{\mathbf{2}} & 0^{\mathbf{1}} \\ 0^{\mathbf{1}} & 6^{\mathbf{5}} \\ 6^{\mathbf{3}} & 4^{\mathbf{4}} \\ 10^{\mathbf{5}} & 2^{\mathbf{3}} \\ 8^{\mathbf{4}} & 1^{\mathbf{2}} \end{pmatrix}, \quad D_2 = \begin{pmatrix} 2^{\mathbf{3}} & 6^{\mathbf{4}} \\ 1^{\mathbf{2}} & 10^{\mathbf{5}} \\ 0^{\mathbf{1}} & 4^{\mathbf{3}} \\ 8^{\mathbf{4}} & 0^{\mathbf{1}} \\ 10^{\mathbf{5}} & 3^{\mathbf{2}} \end{pmatrix}.$$

Stage 2. Delete from each column of matrices D_1 and D_2 one $(q = p - 1 = 1)$ maximal number:

$$D_1 = \begin{pmatrix} 4^{\mathbf{2}} & 0^{\mathbf{1}} \\ 0^{\mathbf{1}} & \\ 6^{\mathbf{3}} & 4^{\mathbf{4}} \\ & 2^{\mathbf{3}} \\ 8^{\mathbf{4}} & 1^{\mathbf{2}} \end{pmatrix}, \quad D_2 = \begin{pmatrix} 2^{\mathbf{3}} & 6^{\mathbf{4}} \\ 1^{\mathbf{2}} & \\ 0^{\mathbf{1}} & 4^{\mathbf{3}} \\ 8^{\mathbf{4}} & 0^{\mathbf{1}} \\ & 3^{\mathbf{2}} \end{pmatrix}.$$

Stage 3. Sort the rest of elements of each matrices D_1 and D_2 in ascending order, and put them in matrices B_1 and B_2, respectively:

$$D_1 = \begin{pmatrix} 4^{\mathbf{2}} & 0^{\mathbf{1}} \\ 0^{\mathbf{1}} & \\ 6^{\mathbf{3}} & 4^{\mathbf{4}} \\ & 2^{\mathbf{3}} \\ 8^{\mathbf{4}} & 1^{\mathbf{2}} \end{pmatrix}, \quad D_2 = \begin{pmatrix} 2^{\mathbf{3}} & 6^{\mathbf{4}} \\ 1^{\mathbf{2}} & \\ 0^{\mathbf{1}} & 4^{\mathbf{3}} \\ 8^{\mathbf{4}} & 0^{\mathbf{1}} \\ & 3^{\mathbf{2}} \end{pmatrix},$$

$$B^1 = (b_1^1, b_2^1, \dots, b_8^1) = (0, 0, 1, 2, 4, 4, 6, 8),$$

$$B^2 = (b_1^2, b_2^2, \dots, b_8^2) = (0, 0, 1, 2, 3, 4, 6, 8).$$

Stage 4. Transform the numbers of matrices B_1 and B_2 according to the formula (4):

$$\tilde{B}^1 = (\tilde{b}_1^1, \tilde{b}_2^1, \dots, \tilde{b}_8^1) = (0, 0, 1, 4, 12, 12, 30, 68),$$

$$\tilde{B}^2 = (\tilde{b}_1^2, \tilde{b}_2^2, \dots, \tilde{b}_8^2) = (0, 0, 1, 4, 11, 26, 58, 124).$$

Stage 5. In each column of matrix D_1 on the deleted number place (Stage 2) we put the number

$$\tilde{b}_u^1 + 1 = \tilde{b}_8^1 + 1 = 69,$$

and in each column of matrix D_2 we put the number

$$\tilde{b}_u^2 + 1 = \tilde{b}_8^2 + 1 = 125.$$

As a result we obtain the problem $Z^2_{5\times 2}(T, \tilde{D})$, where

$$\tilde{D}_1 = \begin{pmatrix} 12 & 0 \\ 0 & 69 \\ 30 & 12 \\ 69 & 4 \\ 68 & 1 \end{pmatrix}, \quad \tilde{D}^2 = \begin{pmatrix} 4 & 58 \\ 1 & 125 \\ 0 & 26 \\ 124 & 0 \\ 125 & 11 \end{pmatrix}.$$

Since

$$f(t_1, \tilde{D}) = (0, 58),$$

$$f(t_2, \tilde{D}) = (12, 26),$$

$$f(t_3, \tilde{D}) = (30, 0),$$

$$f(t_4, \tilde{D}) = (68, 124),$$

we have $P^2(T,D) = P^2(T,\tilde{D})$, i. e. the problems $Z^2_{5\times 2}(T,D)$ and $Z^2_{5\times 2}(T,\tilde{D})$ are equivalent. Moreover, suppose

$$\lambda^1 = (0.9, 0.1),$$
$$\lambda^2 = (0.6, 0.4),$$
$$\lambda^3 = (0.1, 0.9),$$

we obtain

$$\langle \lambda^1, f(t_1,\tilde{D})\rangle = \min\{\langle \lambda^1, f(t_i,\tilde{D})\rangle : i \in N_4\} = 5.8,$$
$$\langle \lambda^2, f(t_2,\tilde{D})\rangle = \min\{\langle \lambda^2, f(t_i,\tilde{D})\rangle : i \in N_4\} = 17.6,$$
$$\langle \lambda^3, f(t_3,\tilde{D})\rangle = \min\{\langle \lambda^3, f(t_i,\tilde{D})\rangle : i \in N_4\} = 3.$$

Hence, $P^2(T,\tilde{D}) = \Xi^2(T,\tilde{D})$, i. e. the problem $Z^2_{5\times 2}(T,\tilde{D})$ is solvable by ALC.

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On LCA groups in which some closed subgroups have commutative rings of continuous endomorphisms

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Abstract. We describe here the locally compact abelian (LCA) groups all of whose closed polythetic, respectively, copolythetic subgroups have commutative rings of continuous endomorphisms. We also determine the LCA groups all of whose polythetic, respectively, copolythetic quotients by closed subgroups have commutative rings of continuous endomorphisms.

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1 Introduction

The problem of characterizing the abelian groups with commutative endomorphism ring was first considered by T. Szele and J. Szendrei in [11], where, among other things, certain important special cases were completly solved. Later L. C. A. van Leeuwen [7] noted that, if X is an abelian group, then every finitely generated subgroup of X has a commutative endomorphism ring if and only if X is isomorphic to a subgroup of $\mathbb{Q}$ or of $\mathbb{Q}/\mathbb{Z}$.

Inspired by the above mentioned paper of T. Szele and J. Szendrei, we initiated in [10] the study of LCA groups with commutative ring of continuous endomorphisms. In the present paper, we continue in the same direction by extending the L. C. A. van Leeuwen's result to this more general setting. Some other results of this nature will also be established. To be more explicit, we need a couple of definitions.

Definition 1.1. *An LCA group X is said to be*

(i) *polythetic if it contains a dense finitely generated subgroup.*

(ii) *copolythetic if there exists a continuous injective homomorphism from X into a group of the form $\mathbb{T}^n$ for some $n \in \mathbb{N}$.*

Let $\mathcal{L}$ be the class of LCA groups. For $X \in \mathcal{L}$, we let $E(X)$ denote the ring of continuous endomorphisms of X.

Our aim here is to determine the groups $X \in \mathcal{L}$ such that every closed polythetic, respectively, copolythetic subgroup G of X has a commutative ring $E(G)$. We also determine the groups $X \in \mathcal{L}$ such that every polythetic, respectively, copolythetic quotient of X by a closed subgroup G, indicated as usual by X/G, has a commutative ring $E(X/G)$.

2 Notation

Since this paper continues the work of [10], we employ here the notation and terminology introduced there.

In addition, $\mathcal{L}_0$ will denote the subclass of $\mathcal{L}$ consisting of those groups which have a compact open subgroup, and $\dot{+}$ will stand for the algebraic direct sum.

For any $p \in \mathbb{P}$ and $X \in \mathcal{L}$, we let $t_p(X)$ denote the p-primary component of X, and

$$S_0(X) = \{q \in \mathbb{P} \mid t_q(X) \neq \{0\}\}.$$

Given a family $(A_i)_{i\in I}$ of subgroups of X, $\sum_{i\in I} A_i$ will designate the minimal subgroup of X containing $\bigcup_{i\in I} A_i$.

If M is a set, $|M|$ will stand for the cardinality of M.

We shall also use the groups of integers, $\mathbb{Z}$, and of rationals, $\mathbb{Q}$, both taken discrete, and the groups of reals, $\mathbb{R}$, and of reals modulo one, $\mathbb{T}$, both taken with their usual topologies.

3 Polythetic subgroups

In this section we characterize completly the groups $X \in \mathcal{L}$ such that every closed polythetic subgroup G of X has a commutative ring $E(G)$. By use of duality, we obtain also the characterization of those groups $X \in \mathcal{L}$ which have the property that for every closed subgroup G of X such that X/G is copolythetic, the ring $E(X/G)$ is commutative.

We start with some preparatory lemmas.

Lemma 3.1. *Let $X \in \mathcal{L}$. For any $a, b \in X$ such that $a \in k(X)$ and $\overline{\langle a\rangle} \cap \overline{\langle b\rangle} = \{0\}$, we have $\overline{\langle a, b\rangle} \cong \overline{\langle a\rangle} \times \overline{\langle b\rangle}$.*

Proof. Since $\overline{\langle a\rangle}$ is compact, $\overline{\langle a\rangle} + \overline{\langle b\rangle}$ is closed in X [6, (4.4)]. It is then easy to see that $\overline{\langle a, b\rangle} = \overline{\langle a\rangle} + \overline{\langle b\rangle}$, so that $\overline{\langle a, b\rangle} = \overline{\langle a\rangle} \dot{+} \overline{\langle b\rangle}$, and hence $\overline{\langle a, b\rangle} = \overline{\langle a\rangle} \oplus \overline{\langle b\rangle}$ by [1, Proposition 6.5]. □

Lemma 3.2. *Any group $X \in \mathcal{L}$ with $\{0\} \neq k(X) \neq X$ has a closed polythetic subgroup G such that $E(G)$ is not commutative.*

Proof. Pick any $a \in k(G)$ and $b \in X \setminus k(G)$. Since $\overline{\langle b\rangle} \cong \mathbb{Z}$ [6, (9.1)], we have $\overline{\langle a\rangle} \cap \overline{\langle b\rangle} = \{0\}$, so that $\overline{\langle a, b\rangle} = \overline{\langle a\rangle} \times \overline{\langle b\rangle}$ by Lemma 3.1. We can take $G = \overline{\langle a, b\rangle}$. □

Lemma 3.3. *Let $X \in \mathcal{L}$ and $S = \{p \in \mathbb{P} \mid X_p \neq \{0\}\}$. If every closed polythetic subgroup of X has a commutative ring of continuous endomorphisms, then, for each $p \in S$, either X_p is torsion and $X[p]$ is isomorphic to $\mathbb{Z}(p)$ or X_p is torsionfree and every its compact subgroup is topologically isomorphic to $\mathbb{Z}_p$.*

Proof. Fix any $p \in S$. If X_p were mixed, we could find two elements $a, b \in X_p$ such that $1 < o(a) < \infty$ and $o(b) = \infty$, i.e. such that $\overline{\langle a\rangle} \cong \mathbb{Z}(p^n)$ for some $n \in \mathbb{N}_0$ and $\overline{\langle b\rangle} \cong \mathbb{Z}_p$ [1, Lemma 2.11]. It would then follow from Lemma 3.1 that

$$\overline{\langle a, b\rangle} \cong \mathbb{Z}(p^n) \times \mathbb{Z}_p,$$

which would contradict the hypothesis because $\mathbb{Z}(p^n) \times \mathbb{Z}_p$ is polythetic and $E(\mathbb{Z}(p^n) \times \mathbb{Z}_p)$ is not commutative. Consequently, X_p must be either torsion or torsionfree. In the former case, we conclude from [4, Ch. 2, §4, Théorème 2] that

$$X[p] \cong \mathbb{Z}(p)^{(\alpha)} \times \mathbb{Z}(p)^{\beta}$$

for some cardinal numbers α, β with $\alpha + \beta \geq 1$. But, in view of the hypothesis, X can contain no copy of $\mathbb{Z}(p) \times \mathbb{Z}(p)$. Therefore we must have $\alpha + \beta = 1$, and so $X[p] \cong \mathbb{Z}(p)$. In the second case, let G be a nonzero compact subgroup of X_p. Then $G \cong \mathbb{Z}_p^{\gamma}$ for some nonzero cardinal number γ [6, (24.25)]. Since our hypothesis ensures that X contains no copy of $\mathbb{Z}_p \times \mathbb{Z}_p$, we must have $\gamma = 1$, so $G \cong \mathbb{Z}_p$. □

We now can dispose of the important case of topological primary groups in $\mathcal{L}$.

Theorem 3.4. *Let $p \in \mathbb{P}$. For a topological p-primary group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) *Every closed polythetic subgroup of X has a commutative ring of continuous endomorphisms.*

(ii) *X is topologically isomorphic with one of the groups $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\mathbb{Z}(p^\infty)$ or $\mathbb{Z}(p^n)$ for some $n \in \mathbb{N}$.*

Proof. Assume X is nonzero and satisfies condition (i). By Lemma 3.3, we know that either X is torsion and $X[p]$ is isomorphic to $\mathbb{Z}(p)$ or X is torsionfree and every its compact subgroup is topologically isomorphic to $\mathbb{Z}_p$.

If the former case occurs, we claim that X is isomorphic with either $\mathbb{Z}(p^\infty)$ or $\mathbb{Z}(p^n)$ for some $n \in \mathbb{N}$. To see this, it will be enough, in view of L.C.A. van Leeuwen's result mentioned in the introduction, to show that X is discrete. Pick a compact open subgroup V of X. By the structure theorem for torsion compact abelian groups [6, (25.9)], V is topologically isomorphic to a group of the form $\prod_{i\in I} \mathbb{Z}(p^{n_i})$, where the set $\{n_i \mid i \in I\}$ is finite. As

$$V[p] \subset X[p] \cong \mathbb{Z}(p),$$

it follows that I cannot contain more than one element, so V is finite, and hense X is discrete. This establishes the claim.

In the latter case, fix a compact open subgroup W of X and a topological group isomorphism f from W onto $\mathbb{Z}_p$. Also let η denote the canonical injection of $\mathbb{Z}_p$ into $\mathbb{Q}_p$. Since $\mathbb{Q}_p$ is divisible and W is open in X, $\eta \circ f$ extends to a continuous open homomorphism h from X into $\mathbb{Q}_p$ [6, (A.7)]. Pick any $x \in \ker(h)$. Since X is a topological p-primary group, $p^m x \in W$ for sufficiently large $m \in \mathbb{N}$. We then have

$$p^m x \in \ker(f) = \{0\},$$

so $x = 0$ because X is torsionfree. It follows that h induces a topological isomorphism of X onto an open subgroup of $\mathbb{Q}_p$, and hence X is topologically isomorphic with either $\mathbb{Q}_p$ or $\mathbb{Z}_p$. This proves that (i) implies (ii).

The converse implication is clear in view of [7] and the fact that every nontrivial closed subgroup in the groups $\mathbb{Z}_p$ and $\mathbb{Q}_p$ is topologically isomorphic with $\mathbb{Z}_p$. □

As a consequence, we obtain the solution to the considered problem in the case of topological torsion groups in $\mathcal{L}$.

Corollary 3.5. *For a topological torsion group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) *Every closed polythetic subgroup of X has a commutative ring of continuous endomorphisms.*

(ii) *For each $p \in S(X)$, X_p is topologically isomorphic with one of the groups $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\mathbb{Z}(p^\infty)$ or $\mathbb{Z}(p^{n_p})$ for some $n_p \in \mathbb{N}$.*

Proof. Assume (i). Since $X \in \mathcal{L}$ is a topological torsion group, we have

$$X \cong \prod_{p \in S(X)} (X_p; U_p),$$

where, for each $p \in S(X)$, X_p is the topological p-primary component of X and U_p is a compact open subgroup of X_p [1, Theorem 3.13]. Pick any $s \in S(X)$, and let G be a closed polythetic subgroup of X_s. Further, letting

$$\eta_s : X_s \to \prod_{p \in S(X)} (X_p; U_p)$$

denote the canonical injection, put $G' = \eta_s(G)$. Then G' is a closed polythetic subgroup of $\prod_{p \in S(X)} (X_p; U_p)$, so that $E(G')$ must be commutative. Since $E(G) \cong E(G')$, $E(G)$ is commutative as well. It follows that every closed polythetic subgroup of X_s has a commutative ring of continuous endomorphisms, so that, by Theorem 3.4, X_s is topologically isomorphic to one of the groups $\mathbb{Q}_s$, $\mathbb{Z}_s$, $\mathbb{Z}(s^\infty)$ or $\mathbb{Z}(s^{n_s})$ for some $n_s \in \mathbb{N}$.

Assume (ii), and let G be a closed polythetic subgroup of X. Since $k(G) = G$, G is compact [1, (5.40)(c)]. It follows that for each $p \in S(G)$, G_p is a compact subgroup of X_p, so that G_p is topologically isomorphic with either $\mathbb{Z}_p$ or $\mathbb{Z}(p^{n_p})$ for some $n_p \in \mathbb{N}$. Since, by [10, Lemma 3.4],

$$E(G) \cong \prod_{p \in S(G)} E(G_p),$$

we conclude that $E(G)$ is commutative. □

The next lemma shows that the case of groups in $\mathcal{L}$ consisting entirely of compact elements and having a nonzero connected component reduces to the case of compact and connected groups.

Lemma 3.6. *Let X be a group in $\mathcal{L}$ such that $c(X) \neq \{0\}$ and $k(X) = X$. If every closed polythetic subgroup of X has a commutative ring of continuous endomorphisms, then X is compact and connected.*

Proof. We shall show first that $X_p \subset c(X)$ for all $p \in \mathbb{P}$. For this purpose, fix any $p \in \mathbb{P}$ and let C_p denote the topological p-primary component of $c(X)$. As is well known, C_p is dense in $c(X)$ [1, Corollary 4.18(a)], so that C_p and hence X_p is nonzero. In view of Lemma 3.3, we distinguish two cases.

Case (a): X_p is torsion and $X[p]$ is isomorphic to $\mathbb{Z}(p)$. Then $X[p]$ has no nontrivial subgroups, so that

$$X[p] = C_p[p] \subset c(X).$$

To apply induction, assume $X[p^k] \subset c(X)$ for some $k \in \mathbb{N}_0$, and choose any $a \in X[p^{k+1}]$. Since $pa \in X[p^k] \subset c(X)$ and since $c(X)$ is divisible, there exists $c \in c(X)$ such that $pa = pc$, and so

$$b = a - c \in X[p] \subset c(X),$$

whence $a = b + c \in c(X)$. As $a \in X[p^{k+1}]$ was arbitrarily chosen, $X[p^{k+1}] \subset c(X)$. Consequently, $X[p^i] \subset c(X)$ for all $i \in \mathbb{N}_0$, and hence

$$X_p = \bigcup_{i \in \mathbb{N}_0} X[p^i] \subset c(X).$$

Case (b): X_p is torsionfree and every its nonzero compact subgroup is topologically isomorphic to $\mathbb{Z}_p$. Pick an arbitrary nonzero $x \in X_p$. We assert that $\overline{\langle x \rangle} \cap c(X) \neq \{0\}$. For if not, then choosing any nonzero $x' \in C_p$ we would certainly have $\overline{\langle x \rangle} \cap \overline{\langle x' \rangle} = \{0\}$. By Lemma 3.1, this would imply that

$$\overline{\langle x, x' \rangle} \cong \overline{\langle x \rangle} \times \overline{\langle x' \rangle},$$

which contradicts the hypothesis because $\overline{\langle x \rangle} \cong \mathbb{Z}_p \cong \overline{\langle x' \rangle}$. Consequently, we must have $\overline{\langle x \rangle} \cap c(X) \neq \{0\}$, and hence

$$\overline{\langle x \rangle} \cap c(X) = p^m \overline{\langle x \rangle}$$

for some $m \in \mathbb{N}$. In particular $p^m x \in c(X)$, and so $p^m x = p^m z$ for some $z \in c(X)$, because of the divisibility of $c(X)$. Then $x - z \in t(X_p) = \{0\}$, and hence $x = z \in c(X)$. Since $x \in X_p$ was arbitrary, we have $X_p \subset c(X)$. Thus either case leads us to the conclusion that $X_p \subset c(X)$.

We now are ready to show that X is compact and connected. Let

$$\mathcal{V} = \{V \mid V \quad \text{is a compact open subgroup of} \quad X\}.$$

Since $k(X) = X$, it is clear that $X = \bigcup_{V \in \mathcal{V}} V$. We will be done if we show that every $V \in \mathcal{V}$ coincides with $c(X)$. To this end, pick an arbitrary $V \in \mathcal{V}$ and let r_0 denote the torsionfree rank of the discrete group V^*. We have $r_0 \neq 0$, since otherwise it would follow that V^* is torsion, and so V would be totally desconnected [6, (24.26)], which is however impossible because $c(X) \subset V$. By [5, §16] or [9, §3], there exists an injective homomorphism $f : \mathbb{Z}^{(r_0)} \to V^*$ such that $V^*/\operatorname{im}(f)$ is torsion. The adjoint mapping f^* is then a continuous open homomorphism from V onto $\mathbb{T}^{r_0}$ [6, (24.40)]. Letting $K = \ker(f^*)$, it follows that $V/K \cong \mathbb{T}^{r_0}$ [3, Ch. 3, §2, Proposition 24], and hence V/K is connected. Also, since

$$K = A(V, \operatorname{im}(f)) \cong (V^*/\operatorname{im}(f))^*$$

[6, (24.38) and (23.25)] and $V^*/\operatorname{im}(f)$ is torsion, K is totally disconnected [6, (24.26)]. It follows that $K \cong \prod_{p \in S(K)} K_p$ [1, Proposition 3.10], and thus

$$K = \overline{\sum_{p \in S(K)} K_p}$$

[6, (6.2)], whence $K \subset c(X)$ because by the above every K_p is contained in $c(X)$. Taking account of [6, (5.34)], we then have

$$V/c(X) \cong (V/K)/(c(X)/K),$$

so that $V/c(X)$ must be connected. Since $V/c(X)$ is certainly totally disconnected [6, (7.3)], this can occur only when $V = c(X)$, and the proof is complete. □

We now consider the case of compact and connected groups in $\mathcal{L}$.

Theorem 3.7. *Let $X \in \mathcal{L}$ be compact and connected. The following statements are equivalent:*

(i) *Every closed polythetic subgroup of X has a commutative ring of continuous endomorphisms.*

(ii) *X is topologically isomorphic to a quotient of $\mathbb{Q}^*$ by a closed subgroup.*

Proof. Assume X is nonzero and satisfies (i). If $t(X) = \{0\}$, then $X \cong (\mathbb{Q}^*)^\alpha$ for some cardinal number $\alpha \geq 1$ [6, (25.8)]. We must have $\alpha = 1$, since otherwise X would contain a copy of $\mathbb{Q}^* \times \mathbb{Q}^*$, which is a contradiction because $\mathbb{Q}^* \times \mathbb{Q}^*$ is polythetic [1, (5.40)(b)] but $E(\mathbb{Q}^* \times \mathbb{Q}^*)$ is not commutative.

Now suppose $t(X) \neq \{0\}$, and pick any $p \in \mathbb{P}$ with $t_p(X) \neq \{0\}$. We know from Lemma 3.3 that $X_p = t_p(X)$ and $X[p] \cong \mathbb{Z}(p)$. Since X_p is dense in X [1, Corollary 4.18(a)], it follows that $X = \overline{t_p(X)}$. Also, since X is divisible [6, (24.25)], $t_p(X)$ is divisible too, so that $t_p(X)$ is algebraically isomorphic to $\mathbb{Z}(p^\infty)$ [9, Lemma, p. 33]. To see that X is topologically isomorphic to a quotient of $\mathbb{Q}^*$ by a closed subgroup, it will be enough in view of [6, (24.11)] and Pontryagin duality theorem to show that X^* is isomorphic to a subgroup of $\mathbb{Q}$, i. e. that X^* is of rank 1. Pick any nonzero $\gamma \in X^*$. Since $o(\gamma) = \infty$, we will be done if we show that $X^*/\langle\gamma\rangle$ is torsion [9, Proposizione 1, p. 23]. But

$$(X^*/\langle\gamma\rangle)^* \cong A(X, \langle\gamma\rangle)$$

[6, (23.25)], so that in order to show that X is topologically isomorphic to a quotient of $\mathbb{Q}^*$ by a closed subgroup, it will suffice to show that $A(X, \langle\gamma\rangle)$ is totally disconnected [6, (24.26)]. Assume not, and let $C = c(A(X, \langle\gamma\rangle))$. By [1, Corollary 4.18(a)], C_p is then a nonzero subgroup of $X_p = t_p(X)$, so that $C_p = t_p(C)$, and hence C_p is divisible. As $t_p(X)$ is algebraically isomorphic to $\mathbb{Z}(p^\infty)$, we must have $C_p = t_p(X)$, and so

$$X = \overline{t_p(X)} = \overline{C_p} = c(A(X, \langle\gamma\rangle)),$$

whence $X = A(X, \langle\gamma\rangle)$, which contradicts our assumption that $\gamma \neq 0$. Consequently, X must be topologically isomorphic to a quotient of $\mathbb{Q}^*$ by a closed subgroup.

For the converse, assume (ii). It follows from [6, (24.11)] that X^* is topologically isomorphic to a subgroup of $\mathbb{Q}$. Let G be a closed polythetic subgroup of X. Since $G^* \cong X^*/A(X^*, G)$ and since every quotient of $\mathbb{Q}$ by a nonzero subgroup is isomorphic to a divisible subgroup of $\mathbb{Q}/\mathbb{Z}$, we conclude that G^* is isomorphic either to a subgroup of $\mathbb{Q}$ or to a subgroup of $\mathbb{Q}/\mathbb{Z}$. By the result of [7] mentioned in the introduction, it follows that in either case $E(G^*)$ is commutative, so that in view of [10, Lemma 3.1] $E(G)$ is commutative as well. The proof is complete. □

We are now ready to prove the main theorem of this section, which describes the groups $X \in \mathcal{L}$ such that every closed polythetic subgroup G of X has a commutative ring $E(G)$.

Theorem 3.8. *For a group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) *Every closed polythetic subgroup of X has a commutative ring of continuous endomorphisms.*

(ii) *X is topologically isomorphic to one of the groups:*
(1) $\mathbb{R}$, (2) *a subgroup of* $\mathbb{Q}$, (3) *a quotient of* $\mathbb{Q}^*$ *by a closed subgroup,* (4) $\prod_{p \in S(X)}(X_p; U_p)$, *where, for each* $p \in S(X)$, X_p *is topologically isomorphic to either* $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\mathbb{Z}(p^\infty)$, *or* $\mathbb{Z}(p^{n_p})$ *for some* $n_p \in \mathbb{N}$, *and* U_p *is a compact open subgroup of* X_p.

Proof. Assume (i). If $X \notin \mathcal{L}_0$, we can write $X = V \oplus Y$, where V, Y are closed subgroups of X such that $V \cong \mathbb{R}^d$ for some $d \in \mathbb{N}_0$ and $Y \in \mathcal{L}_0$ [6, (24.30)]. Since V

is of course polythetic [1, (5.40)(e)], $E(V)$ must be commutative, so that we must have $d = 1$. Also, since $X \neq k(X)$, we deduce from Lemma 3.2 that $k(X) = \{0\}$, so Y is discrete and V is open in X. If Y were not the zero group, it would follow that, for every nonzero $a \in Y$, $V + \langle a \rangle$ is an open and hence closed subgroup of X satisfying $V + \langle a \rangle \cong \mathbb{R} \times \mathbb{Z}$. This is a contradiction because $\mathbb{R} \times \mathbb{Z}$ is polythetic [1, (5.40)(f)] and $E(R \times \mathbb{Z})$ is not commutative. Consequently, in this case $X \cong \mathbb{R}$.

Now let $X \in \mathcal{L}_0$. In view of Lemma 3.2, we must have either $k(X) = \{0\}$ or $k(X) = X$. If the former case occurs, X is discrete and torsionfree, and hence it is isomorphic to a subgroup of $\mathbb{Q}$, by the result of [7] mentioned in the introduction. Assume the latter. If $c(X) = \{0\}$, it follows from Corollary 3.5 that

$$X \cong \prod_{p \in S(X)} (X_p; U_p),$$

where, for each $p \in S(X)$, X_p is topologically isomorphic to one of the groups $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\mathbb{Z}(p^\infty)$ or $\mathbb{Z}(p^{n_p})$ for some $n_p \in \mathbb{N}$, and U_p is a compact open subgroup of X_p. Finally, if $c(X) \neq \{0\}$, we conclude from Lemma 3.6 that X is compact and connected, and hence X is topologically isomorphic to a quotient of $\mathbb{Q}^*$ by a closed subgroup, according to Theorem 3.7.

The converse is clear. □

Dualizing the preceding theorem, we obtain the description of groups in $\mathcal{L}$ all of whose copolythetic quotients by closed subgroups have commutative rings of continuous endomorphisms.

Corollary 3.9. *For a group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) *Every copolythetic quotient of X by a closed subgroup has a commutative ring of continuous endomorphisms.*

(ii) *X is topologically isomorphic to one of the groups:*
(1) $\mathbb{R}$, (2) *a subgroup of* $\mathbb{Q}$, (3) *a quotient of* $\mathbb{Q}^*$ *by a closed subgroup,* (4) $\prod_{p \in S(X)} (X_p; U_p)$, *where, for each* $p \in S(X)$, X_p *is topologically isomorphic to either* $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\mathbb{Z}(p^\infty)$, *or* $\mathbb{Z}(p^{n_p})$ *for some* $n_p \in \mathbb{N}$, *and* U_p *is a compact open subgroup of* X_p.

4 Copolythetic subgroups

In the present section, we characterize the groups $X \in \mathcal{L}$ such that every closed copolythetic subgroup G of X has a commutative ring $E(G)$. By utilizing duality, we also get the description of those groups $X \in \mathcal{L}$ which have the property that for each closed subgroup G of X such that X/G is polythetic, the ring $E(X/G)$ is commutative.

We will obtain these results as a consequence of a number of lemmas. The first two of these establish, for certain particular types of groups, some necessary conditions.

Lemma 4.1. *Let X be a group in $\mathcal{L}$ such that every its closed copolythetic subgroup has a commutative ring of continuous endomorphisms. If $X = A \oplus Y$, where A is topologically isomorphic with either $\mathbb{R}$ or $\mathbb{T}$, then Y is torsionfree.*

Proof. If $t(Y)$ were nonzero, it would contain a copy of $\mathbb{Z}(p)$ for some $p \in \mathbb{P}$. Since A is topologically isomorphic with either $\mathbb{R}$ or $\mathbb{T}$, it would then follow from Lemma 3.1 that X contains a copy of either $\mathbb{Z} \times \mathbb{Z}(p)$ or $\mathbb{Z}(p) \times \mathbb{Z}(p)$, a contradiction. □

Lemma 4.2. *Let X be a group in $\mathcal{L}$ with $t(X) \neq \{0\}$. If every closed copolythetic subgroup of X has a commutative ring of continuous endomorphisms, then $k(X) = X$ and $X[p] \cong \mathbb{Z}(p)$ for all $p \in S_0(X)$.*

Proof. Pick any $p \in S_0(X)$. By [4, Ch. 2, §4, Théorème 2], we have

$$X[p] \cong \mathbb{Z}(p)^{(\alpha)} \times \mathbb{Z}(p)^{\beta}$$

for some cardinal numbers α, β with $\alpha + \beta \geq 1$. Since X cannot contain copies of $\mathbb{Z}(p) \times \mathbb{Z}(p)$, we must have $\alpha + \beta = 1$, and so $X[p] \cong \mathbb{Z}(p)$. If there existed $x \in X \setminus k(X)$, it would then follow from Lemma 3.1 that X contains a copy of $\mathbb{Z} \times \mathbb{Z}(p)$, contradicting the hypothesis. □

We continue with two simple lemmas that will be usefull in the sequel.

Lemma 4.3. *If $G \in \mathcal{L}_0$ is copolythetic and contains no copy of $\mathbb{T}$, then G is discrete.*

Proof. By the definition of copolythetic groups, there exists for some $n \in \mathbb{N}_0$ an injective $h \in H(G, \mathbb{T}^n)$. Let K be a compact open subgroup of G. Then $h(K)$ is closed in $\mathbb{T}^n$, and hence $h(K)$ is topologically isomorphic to a group of the form $\mathbb{T}^m \times F$, where m is an integer satisfying $0 \leq m \leq n$ and F is a direct sum of at most $n - m$ finite cyclic groups [3, Ch. VII, §1, Proposition 11]. Since h is injective, its restriction to K establishes a topological isomorphism of K onto $h(K)$ [2, Ch. I, §9, Théorème 2, Corollaire 2], and so

$$K \cong \mathbb{T}^m \times F.$$

As G does not contain copies of $\mathbb{T}$, we must have $m = 0$, so that K is finite, and hence G is discrete. □

Lemma 4.4. *Let K be a closed subgroup of an abelian topological group Y such that $K = A \oplus B$ for some subgroups A, B of K. For any closed subset C of A, $C + B$ is closed in Y.*

Proof. Let φ denote the canonical projection of K onto A. We have $C + B = \varphi^{-1}(C)$, so that $C + B$ is closed in K and hence in Y. □

The following two lemmas establish, for certain particular types of groups, some sufficient conditions.

Lemma 4.5. *If X is a torsionfree group in $\mathcal{L}$ all of whose nonzero discrete subgroups are of rank one, then every closed copolythetic subgroup of X has a commutative ring of continuous endomorphisms.*

Proof. Let G be a nonzero closed copolythetic subgroup of X. If $G \notin \mathcal{L}_0$, we can write $G = A \oplus B$, where $A \cong \mathbb{R}^d$ for some $d \in \mathbb{N}_0$ and $B \in \mathcal{L}_0$. We must have $d = 1$, since otherwise G would contain discrete subgroups of rank greater than one. Further, being torsionfree, X contains no copy of $\mathbb{T}$. As B is clearly copolythetic, it then follows from Lemma 4.3 that B is discrete, so that either $B = \{0\}$ or B is of rank one. But the latter is impossible. To see this, assume the contrary and pick any nonzero $a \in A$. Since $\langle a \rangle$ is closed in A, it follows from Lemma 4.4 that $\langle a \rangle + B$ is closed in X. As $\langle a \rangle + B$ is countable, we deduce from [8, Corollary, p. 23] that $\langle a \rangle + B$ is discrete, a contradiction because $\langle a \rangle + B$ is, in our case, of rank two. Thus $B = \{0\}$, and hence $E(G)$ is commutative.

In case $G \in \mathcal{L}_0$, it follows again from Lemma 4.3 that G is discrete, and hence of rank one, so that $E(G)$ is commutative.

Lemma 4.6. *If $X = A \times Y$, where $A \cong \mathbb{T}$ and Y is a torsionfree group in $\mathcal{L}$ with $k(Y) = Y$, then every closed copolythetic subgroup of X has a commutative ring of continuous endomorphisms.*

Proof. Since $\mathbb{T}$ is compact, it is clear that $X = k(X)$. We also have

$$X[p] \cong A[p] \times Y[p] \cong \mathbb{T}[p] \cong \mathbb{Z}(p)$$

for all $p \in \mathbb{P}$. It follows in particular that X cannot contain copies of $\mathbb{T} \times \mathbb{T}$. Indeed, if there existed a closed subgroup K of X satisfying $K \cong \mathbb{T} \times \mathbb{T}$, then, picking any $p \in \mathbb{P}$, we would have

$$K[p] \cong \mathbb{T}[p] \times \mathbb{T}[p] \cong \mathbb{Z}(p) \times \mathbb{Z}(p),$$

contardicting the fact that $X[p] \cong \mathbb{Z}(p)$.

Now, fix an arbitrary nonzero closed copolythetic subgroup G of X. If G contains no copy of $\mathbb{T}$, it follows from Lemma 4.3 that G is discrete, so $G = k(G) = t(G)$. Since

$$t(X) \cong t(A) \times t(Y) \cong t(\mathbb{T}),$$

we conclude that G is isomorphic to a subgroup of $\mathbb{Q}/\mathbb{Z}$, and so $E(G)$ is commutative by [11, Theorem 1].

Suppose next that G contains closed subgroups topologically isomorphic to $\mathbb{T}$. Since for any cardinal number ν the group $\mathbb{T}^\nu$ is splitting in $\mathcal{L}$ [6, (25.31)(b)], we can write $G = B \oplus C$ for some closed subgroups B, C of X with $C \cong \mathbb{T}$. Now, B must be torsionfree since otherwise X would contain a copy of $\mathbb{Z}(p) \times \mathbb{Z}(p)$ for some $p \in \mathbb{P}$, in contradiction with the fact that $X[p] \cong \mathbb{Z}(p)$. Moreover, since X cannot contain copies of $\mathbb{T} \times \mathbb{T}$, B contains no copy of $\mathbb{T}$. As B is clearly copolythetic, it follows from Lemma 4.3 that B is discrete. Therefore

$$B = k(B) = t(B) = \{0\},$$

so $G \cong \mathbb{T}$, and hence $E(G)$ is commutative. □

We now combine the above results to obtain the desired description of groups in $\mathcal{L}$ all of whose copolythetic subgroups have a commutative ring of continuous endomorphisms.

Theorem 4.7. *For a group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) *Every closed copolythetic subgroup of X has a commutative ring of continuous endomorphisms.*

(ii) *X satisfies one of the following three conditions:*

(1) *X is torsionfree and every its nonzero discrete subgroup is of rank one.*

(2) *$X \cong \mathbb{T} \times Y$, where Y is a torsionfree group in $\mathcal{L}$ with $k(Y) = Y$.*

(3) *X contains no copy of $\mathbb{T}$, $k(X) = X$, $t(X) \neq \{0\}$, and $X[p] \cong \mathbb{Z}(p)$ for all $p \in S_0(X)$.*

Proof. Assume (i). We consider first the case when X is torsionfree. Pick an arbitrary nonzero discrete subgroup G of X, and let M be a maximal free subset of G. Then

$$\langle M \rangle \cong \bigoplus_{x \in M} \langle x \rangle$$

[9, Proposizione 1, p. 23]. Note also that every subgroup of G, being discrete, is closed in X [6, (5.10)]. If G were not of rank one, we would have $|M| > 1$, so that X would contain a copy of $\mathbb{Z} \times \mathbb{Z}$, contradicting the hypothesis. Thus G must be of rank one. As G was chosen arbitrarily among the nonzero discrete subgroups of X, we conclude that in this case X satisfies (1).

Next suppose that $t(X) \neq \{0\}$. It follows from Lemma 4.2 that $k(X) = X$ and $X[p] \cong \mathbb{Z}(p)$ for all $p \in S_0(X)$. Therefore, if X contains no copy of $\mathbb{T}$, we are led to (3). If, on the other hand, X contains a closed subgroup $A \cong \mathbb{T}$, we can write $X = A \oplus Y$ for some closed subgroup Y of X. Then, for each $p \in S_0(X)$, we have

$$X[p] \cong A[p] \oplus Y[p] \quad \text{and} \quad A[p] \cong \mathbb{T}[p] \cong \mathbb{Z}(p),$$

so that, in view of the above mentioned fact that $X[p]$ is simple, $Y[p] = \{0\}$. It follows that $t(Y) = \{0\}$, and hence X satisfies (2).

Now assume (ii). If X satisfies (1), then Lemma 4.5 shows that (i) holds. In case X satisfies (2), the validity of (i) follows from Lemma 4.6. Finally, suppose that X satisfies (3), and let G be a closed copolythetic subgroup of X. It follows from Lemma 4.3 that G is discrete, so $G = k(G) = t(G)$, and hence

$$G \cong \bigoplus_{p \in S_0(G)} G_p.$$

Since, clearly, $G_p[p] = X[p] \cong \mathbb{Z}(p)$ for all $p \in S_0(G)$, we conclude that G is isomorphic to a subgroup of $\bigoplus_{p \in S_0(G)} \mathbb{Z}(p^\infty)$, so that $E(G)$ is commutative by [11, Theorem 1]. □

By use of duality, we obtain the description of groups in $\mathcal{L}$ all of whose polythetic quotients by closed subgroups have commutative rings of continuous endomorphisms.

Corollary 4.8. *For a group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) *Every polythetic quotient of X by a closed subgroup has a commutative ring of continuous endomorphisms.*

(ii) *X satisfies one of the following three conditions:*

(1) *X is densely divisible and every its compact quotient by a proper closed subgroup is of dimension one.*

(2) *$X \cong \mathbb{Z} \times Y$, where Y is a densely divisible and totally disconnected group in $\mathcal{L}$.*

(3) *X is a totally disconnected group with no quotients by closed subgroups topologically isomorphic to $\mathbb{Z}$, $\bigcap_{p\in\mathbb{P}} \overline{pX} \neq X$, and $X/\overline{pX} \cong \mathbb{Z}(p)$ for all $p \in \mathbb{P}$ such that $\overline{pX} \neq X$.*

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Multi-dimensional Darboux type differential systems with quadratic nonlinearities

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Abstract. In the article the n-dimensional autonomous Darboux type differential systems with nonlinearities of the 2^{nd} degree are considered. With the aid of theorem on integrating factor the particular invariant $GL(n,\mathbb{R})$-integrals are constructed as well as the first integrals of Darboux type for considered systems. These integrals represent the algebraic curves of the 1^{st} degree. The recurrence formula of particular invariant $GL(n,\mathbb{R})$-integrals of the Darboux type differential system is found.

Mathematics subject classification: 34C05,34C14.
Keywords and phrases: The Darboux type differential system, comitant, invariant $GL(n,\mathbb{R})$-integrating factor, invariant $GL(n,\mathbb{R})$-integral.

Consider the system of differential equations

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta \equiv P^j(x,a) \quad (j,\alpha,\beta = \overline{1,n};\ n \geq 2), \tag{1}$$

where coefficient tensor $a^j_{\alpha\beta}$ is symmetrical in lower indices, in which the complete convolution holds. The system (1) is considered with the action of the group $GL(n,\mathbb{R})$ of center-affine transformations [1], and $x = (x^1, x^2, ..., x^n)$ is a phase variable vector of the system.

Suppose that system (1) admits $(n-1)$-dimensional commutative Lie algebra with operators

$$X_\alpha = \xi^j_\alpha(x)\frac{\partial}{\partial x^j} \qquad (j = \overline{1,n};\ \alpha = \overline{1,n-1}) \tag{2}$$

and

$$\Lambda = P^j(x,a)\frac{\partial}{\partial x^j} \qquad (j = \overline{1,n}). \tag{3}$$

Consider the determinant constructed on coordinates of operators (2)-(3) as follows

$$\Delta = \begin{vmatrix} \xi^1_1 & \xi^2_1 & \xi^3_1 & ... & \xi^n_1 \\ \xi^1_2 & \xi^2_2 & \xi^3_2 & ... & \xi^n_2 \\ ... & ... & ... & ... & ... \\ \xi^1_{n-1} & \xi^2_{n-1} & \xi^3_{n-1} & ... & \xi^n_{n-1} \\ P^1 & P^2 & P^3 & ... & P^n \end{vmatrix}. \tag{4}$$

From [2] it follows that holds

Theorem 1. *If n-dimensional differential system* (1) *admits* $(n-1)$*-dimensional commutative Lie algebra of operators* (2)*, then the function $\mu = \frac{1}{\Delta}$, where $\Delta \neq 0$ from* (4)*, is the integrating factor for Pfaff equations*

$$\sum_{i=1}(-1)^{i+j}\begin{vmatrix} \xi_1^1 & ... & \xi_1^{i-1} & \xi_1^{i+1} & ... & \xi_1^n \\ ... & ... & ... & ... & ... & ... \\ \xi_{j-1}^1 & ... & \xi_{j-1}^{i-1} & \xi_{j-1}^{i+1} & ... & \xi_{j-1}^n \\ \xi_{j+1}^1 & ... & \xi_{j+1}^{i-1} & \xi_{j+1}^{i+1} & ... & \xi_{j+1}^n \\ ... & ... & ... & ... & ... & ... \\ P^1 & ... & P^{i-1} & P^{i+1} & ... & P^n \end{vmatrix} dx^i = 0 \quad (i=\overline{1,n};\ j=\overline{1,n-1}), \tag{5}$$

defining a general integral of the system (1)*.*

Following [3], consider system (1) in a "Darboux" like case, i.e. system (1) written in the form

$$\frac{dx^j}{dt} = a_\alpha^j x^\alpha + 2x^j R(x) \equiv P^j(x,a) \quad (j,\alpha = \overline{1,n};\ n \geq 2), \tag{6}$$

where $R(x) \neq 0$ is a homogeneous linear polynomial with constant coefficients in coordinates of the vector x.

According to [4] will treat invariant $GL(n,\mathbb{R})$-integrating factors and invariant $GL(n,\mathbb{R})$-integrals of the system (6) with $n = 2,3,4,5,...$

1. Case $n = 2$**.** Will denote the invariants and comitants of the system (1) as follows

$$\begin{aligned} &I_{1,2} = a_{\alpha_1}^{\alpha_1}, \quad I_{2,2} = a_{\alpha_2}^{\alpha_1} a_{\alpha_1}^{\alpha_2}, \quad K_{1,2} = a_\alpha^{\alpha_1} x^\alpha x^{\alpha_2} \varepsilon_{\alpha_1\alpha_2}, \\ &P_{1,2} = a_{\alpha_1\beta}^{\alpha_1} x^\beta, \quad P_{2,2} = a_{\alpha_2}^{\alpha_1} a_{\alpha_1\beta}^{\alpha_2} x^\beta, \ \widetilde{K}_{1,2} = a_{\beta\gamma}^{\alpha_1} x^\beta x^\gamma x^{\alpha_2} \varepsilon_{\alpha_1\alpha_2}, \end{aligned} \tag{7}$$

where the first of lower indices for I, K, P and $\widetilde{K}$ from (7) shows the degree of invariant or comitant with respect to coefficients of the system (1), and the second lower index shows the dimension of the system ($n = 2$). In [4] it is shown that invariant condition which differs the system (6) from (1) is the following: $\widetilde{K}_{1,2} \equiv 0$. In the same paper with the aid of Theorem 1 and expressions (7) is proved

Theorem 2. *System* (1) *with $\widetilde{K}_{1,2} \equiv 0$ and $n = 2$ has the invariant $GL(2,\mathbb{R})$-integrating factor μ of the form $\mu^{-1} = K_{1,2}\Phi_{2,2}$, where $K_{1,2} = 0$ and*

$$\Phi_{2,2} \equiv 8I_{1,2}P_{1,2} - 12P_{2,2} + 3(I_{1,2}^2 - I_{2,2}) = 0$$

are invariant particular $GL(2,\mathbb{R})$-integrals of this system.

2. Case $n = 3$**.** Following [3] will denote the invariants, comitants and covariants of the system (1) as follows

$$\begin{aligned}
&I_{1,3} = a^{\alpha_1}_{\alpha_1}, \quad I_{2,3} = a^{\alpha_1}_{\alpha_2} a^{\alpha_2}_{\alpha_1}, \quad I_{3,3} = a^{\alpha_1}_{\alpha_3} a^{\alpha_2}_{\alpha_1} a^{\alpha_3}_{\alpha_2},\\
&K_{3,3} = a^{\beta_1}_{\alpha_1} a^{\beta_2}_{\alpha_2} a^{\alpha_2}_{\alpha_3} x^{\alpha_1} x^{\alpha_3} x^{\beta_3} \varepsilon_{\beta_1\beta_2\beta_3},\\
&P_{1,3} = a^{\alpha_1}_{\alpha_1\beta} x^{\beta}, \quad P_{2,3} = a^{\alpha_1}_{\alpha_2} a^{\alpha_2}_{\alpha_1\beta} x^{\beta}, \; P_{3,3} = a^{\alpha_1}_{\alpha_3} a^{\alpha_2}_{\alpha_1} a^{\alpha_3}_{\alpha_2\beta} x^{\beta},\\
&\widetilde{K}_{1,3} = a^{\alpha_1}_{\beta\gamma} x^{\beta} x^{\gamma} x^{\alpha_2} x^{\alpha_3}_1 \varepsilon_{\alpha_1\alpha_2\alpha_3},
\end{aligned} \tag{8}$$

where the meaning of the lower indices for I, K, P and $\widetilde{K}$ is the same, and the vector $x_1 = (x^1_1, x^2_1, x^3_1)$ is cogradient [5] to the phase variable vector $x = (x^1, x^2, x^3)$. The vectors x and x_1 are independent. In [3] it is shown that invariant condition which differs the system (6) from (1) is the following: $\widetilde{K}_{1,3} \equiv 0$. In the same paper with the aid of Theorem 1 and expressions (8) is proved

Theorem 3. *System* (1) *with* $\widetilde{K}_{1,3} \equiv 0$ *and* $n = 3$ *has the invariant* $GL(3, \mathbb{R})$*-integrating factor* μ *of the form* $\mu^{-1} = K_{3,3}\Phi_{3,3}$*, where* $K_{3,3} = 0$ *and*

$$\Phi_{3,3} \equiv 1/3(I^2_{1,3} - 3I_{1,3}I_{2,3} + 2I_{3,3}) - 3/2(I_{2,3} - I^2_{1,3})P_{1,3} - 4I_{1,3}P_{2,3} + 4P_{3,3} = 0$$

are invariant particular $GL(3, \mathbb{R})$*-integrals of this system.*

3. Case $n = 4$**.** Consider the next invariants, comitants and covariants of the system (1)

$$\begin{aligned}
&I_{1,4} = a^{\alpha_1}_{\alpha_1}, \quad I_{2,4} = a^{\alpha_1}_{\alpha_2} a^{\alpha_2}_{\alpha_1}, \quad I_{3,4} = a^{\alpha_1}_{\alpha_3} a^{\alpha_2}_{\alpha_1} a^{\alpha_3}_{\alpha_2}, \; I_{4,4} = a^{\alpha_1}_{\alpha_4} a^{\alpha_2}_{\alpha_1} a^{\alpha_3}_{\alpha_2} a^{\alpha_4}_{\alpha_3},\\
&K_{6,4} = a^{\beta_1}_{\alpha_1} a^{\beta_2}_{\alpha_2} a^{\alpha_2}_{\alpha_3} a^{\beta_3}_{\alpha_4} a^{\alpha_4}_{\alpha_5} a^{\alpha_5}_{\alpha_6} x^{\alpha_1} x^{\alpha_3} x^{\alpha_6} x^{\beta_4} \varepsilon_{\beta_1\beta_2\beta_3\beta_4}, \quad P_{1,4} = a^{\alpha_1}_{\alpha_1\beta} x^{\beta},\\
&P_{2,4} = a^{\alpha_1}_{\alpha_2} a^{\alpha_2}_{\alpha_1\beta} x^{\beta}, \; P_{3,4} = a^{\alpha_1}_{\alpha_3} a^{\alpha_2}_{\alpha_1} a^{\alpha_3}_{\alpha_2\beta} x^{\beta}, \; P_{4,4} = a^{\alpha_1}_{\alpha_4} a^{\alpha_2}_{\alpha_1} a^{\alpha_3}_{\alpha_2} a^{\alpha_4}_{\alpha_3\beta} x^{\beta},\\
&\widetilde{K}_{1,4} = a^{\alpha_1}_{\beta\gamma} x^{\beta} x^{\gamma} x^{\alpha_2} x^{\alpha_3}_1 x^{\alpha_4}_2 \varepsilon_{\alpha_1\alpha_2\alpha_3\alpha_4},
\end{aligned} \tag{9}$$

where the meaning of the lower indices for I, K, P and $\widetilde{K}$ is the same, and the vectors $x_1 = (x^1_1, x^2_1, x^3_1, x^4_1)$ and $x_2 = (x^1_2, x^2_2, x^3_2, x^4_2)$ are cogradient to the phase variable vector x. One can verify easily that invariant condition which differs the system (6) from (1) is the following: $\widetilde{K}_{1,4} \equiv 0$. With the aid of Theorem 1 and expressions (9) it is proved the following

Theorem 4. *System* (1) *with* $\widetilde{K}_{1,4} \equiv 0$ *and* $n = 4$ *has the invariant* $GL(4, \mathbb{R})$*-integrating factor* μ *of the form* $\mu^{-1} = K_{6,4}\Phi_{4,4}$*, where* $K_{6,4} = 0$ *and*

$$\Phi_{4,4} \equiv L_{4,4} - 2(4/5L_{3,4}P_{1,4} + L_{2,4}P_{2,4} + L_{1,4}P_{3,4} + P_{4,4}) = 0 \tag{10}$$

are invariant particular $GL(4, \mathbb{R})$*-integrals of this system. In* (10) *we have*

$$L_{1,4} = -I_{1,4}, \quad L_{2,4} = 1/2(I^2_{1,4} - I_{2,4}), \quad L_{3,4} = 1/6(3I_{1,4}I_{2,4} - 2I_{3,4} - I^3_{1,4}),$$

$$L_{4,4} = 1/24(8I_{1,4}I_{3,4} - 6I_{4,4} - 6I^2_{1,4}I_{2,4} + 3I^2_{2,4} + I^4_{1,4}),$$

where $I_{k,4}$ $(k = \overline{1,4})$ *are from* (9).

4. Case $n = 5$**.** Consider the next invariants, comitants and covariants of the system (1)

$$\begin{aligned}
&I_{1,5} = a^{\alpha_1}_{\alpha_1}, \quad I_{2,5} = a^{\alpha_1}_{\alpha_2}a^{\alpha_2}_{\alpha_1}, \quad I_{3,5} = a^{\alpha_1}_{\alpha_3}a^{\alpha_2}_{\alpha_1}a^{\alpha_3}_{\alpha_2},\\
&I_{4,5} = a^{\alpha_1}_{\alpha_4}a^{\alpha_2}_{\alpha_1}a^{\alpha_3}_{\alpha_2}a^{\alpha_4}_{\alpha_3}, \; I_{5,5} = a^{\alpha_1}_{\alpha_5}a^{\alpha_2}_{\alpha_1}a^{\alpha_3}_{\alpha_2}a^{\alpha_4}_{\alpha_3}a^{\alpha_5}_{\alpha_4},\\
&K_{10,5} = a^{\beta_1}_{\alpha_1}a^{\beta_2}_{\alpha_2}a^{\alpha_2}_{\alpha_3}a^{\beta_3}_{\alpha_4}a^{\alpha_4}_{\alpha_5}a^{\alpha_5}_{\alpha_6}a^{\beta_4}_{\alpha_7}a^{\alpha_7}_{\alpha_8}a^{\alpha_8}_{\alpha_9}a^{\alpha_9}_{\alpha_{10}}x^{\alpha_1}x^{\alpha_3}x^{\alpha_6}x^{\alpha_{10}}x^{\beta_5}\varepsilon_{\beta_1\beta_2\beta_3\beta_4\beta_5},\\
&P_{1,5} = a^{\alpha_1}_{\alpha_1\beta}x^{\beta}, \quad P_{2,5} = a^{\alpha_1}_{\alpha_2}a^{\alpha_2}_{\alpha_1\beta}x^{\beta}, \; P_{3,5} = a^{\alpha_1}_{\alpha_3}a^{\alpha_2}_{\alpha_1}a^{\alpha_3}_{\alpha_2\beta}x^{\beta},\\
&P_{4,5} = a^{\alpha_1}_{\alpha_4}a^{\alpha_2}_{\alpha_1}a^{\alpha_3}_{\alpha_2}a^{\alpha_4}_{\alpha_3\beta}x^{\beta}, P_{5,5} = a^{\alpha_1}_{\alpha_5}a^{\alpha_2}_{\alpha_1}a^{\alpha_3}_{\alpha_2}a^{\alpha_4}_{\alpha_3}a^{\alpha_5}_{\alpha_4\beta}x^{\beta},\\
&\widetilde{K}_{1,5} = a^{\alpha_1}_{\beta\gamma}x^{\beta}x^{\gamma}x^{\alpha_2}x^{\alpha_3}_1x^{\alpha_4}_2x^{\alpha_5}_3\varepsilon_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5},
\end{aligned} \tag{11}$$

where the meaning of lower indices for I, K, P and $\widetilde{K}$ is the same, and the vectors $x_i = (x^1_i, x^2_i, x^3_i, x^4_i, x^5_i)$, $(i = \overline{1,3})$ are cogradient to the phase variable vector x. As it is easy to see the invariant condition which differs the system (6) from (1) is the following: $\widetilde{K}_{1,5} \equiv 0$. With the aid of Theorem 1 and expressions (11) is proved the following

Theorem 5. *System* (1) *with* $\widetilde{K}_{1,5} \equiv 0$ *and* $n = 5$ *has the invariant* $GL(5,\mathbb{R})$-*integrating factor* μ *of the form* $\mu^{-1} = K_{10,5}\Phi_{5,5}$, *where* $K_{10,5} = 0$ *and*

$$\Phi_{5,5} \equiv L_{5,5} - 2(5/6L_{4,5}P_{1,5} + L_{3,5}P_{2,5} + L_{2,5}P_{3,5} + L_{1,5}P_{4,5} + P_{5,5}) = 0 \tag{12}$$

are invariant particular $GL(5,\mathbb{R})$-*integrals of this system. In* (12) *we have*

$$L_{1,5} = -I_{1,5}, \quad L_{2,5} = 1/2(I^2_{1,5} - I_{2,5}), \quad L_{3,5} = 1/6(3I_{1,5}I_{2,5} - 2I_{3,5} - I^3_{1,5}),$$

$$L_{4,5} = 1/24(8I_{1,5}I_{3,5} - 6I_{4,5} - 6I^2_{1,5}I_{2,5} + 3I^2_{2,5} + I^4_{1,5}),$$

$$L_{5,5} = -1/120(I^5_{1,5} - 10I^3_{1,5}I_{2,5} + 20I^2_{1,5}I_{3,5} + 15I_{1,5}I^2_{2,5} - 30I_{1,5}I_{4,5} - 20I_{2,5}I_{3,5} + 24I_{5,5}),$$

where $I_{k,5}$ $(k = \overline{1,5})$ *are from* (11).

5. The general case $n \geq 2$**.**

Write the center-affine invariants, comitants and covariants in general case of system (1) as follows

$$\begin{aligned}
&I_{1,n} = a^{\alpha_1}_{\alpha_1}, \quad I_{2,n} = a^{\alpha_1}_{\alpha_2}a^{\alpha_2}_{\alpha_1}, \quad I_{3,n} = a^{\alpha_1}_{\alpha_3}a^{\alpha_2}_{\alpha_1}a^{\alpha_3}_{\alpha_2}, ..., I_{n,n} = a^{\alpha_1}_{\alpha_n}a^{\alpha_2}_{\alpha_1}a^{\alpha_3}_{\alpha_2}...a^{\alpha_n}_{\alpha_{n-1}},\\
&K_{m,n} = a^{\beta_1}_{\alpha_1}a^{\beta_2}_{\alpha_2}a^{\alpha_2}_{\alpha_3}a^{\beta_3}_{\alpha_4}a^{\alpha_4}_{\alpha_5}a^{\alpha_5}_{\alpha_6}a^{\beta_4}_{\alpha_7}a^{\alpha_7}_{\alpha_8}a^{\alpha_8}_{\alpha_9}a^{\alpha_9}_{\alpha_{10}}...a^{\alpha_{m-1}}_{\alpha_m}x^{\alpha_1}x^{\alpha_3}x^{\alpha_6}x^{\alpha_{10}}...x^{\alpha_m}x^{\beta_n}\varepsilon_{\beta_1...\beta_n},\\
&P_{1,n} = a^{\alpha_1}_{\alpha_1\beta}x^{\beta}, \quad P_{2,n} = a^{\alpha_1}_{\alpha_2}a^{\alpha_2}_{\alpha_1\beta}x^{\beta}, \; P_{3,n} = a^{\alpha_1}_{\alpha_3}a^{\alpha_2}_{\alpha_1}a^{\alpha_3}_{\alpha_2\beta}x^{\beta}, ...,\\
&P_{n,n} = a^{\alpha_1}_{\alpha_n}a^{\alpha_2}_{\alpha_1}a^{\alpha_3}_{\alpha_2}...a^{\alpha_n}_{\alpha_{n-1}\beta}x^{\beta},\\
&\widetilde{K}_{1,n} = a^{\beta_1}_{\alpha\beta}x^{\alpha}x^{\beta}x^{\beta_2}x^{\beta_3}_1x^{\beta_4}_2...x^{\beta_n}_{n-2}\varepsilon_{\beta_1\beta_2...\beta_n},\\
&(\alpha, \alpha_1, \alpha_2, ..., \alpha_m, ..., \alpha_n, \beta, \beta_1, \beta_2, ..., \beta_n = \overline{1,n}; \quad m = \tfrac{n(n-1)}{2}; \quad n \geq 2)
\end{aligned} \tag{13}$$

where $\varepsilon_{\beta_1\beta_2\beta_3...\beta_n}$ is a unit n-vector, and the vectors $x_i = (x_i^1, x_i^2, ..., x_i^n)$, $(i = \overline{1, n-2})$ are independent cogradient vectors [5] to x .

Remark 1. System (1) with $\widetilde{K}_{1,n} \equiv 0$ has the form (6), where $R(x) = \frac{1}{n+1} P_{1,n}$.

Will call the systems written in the form (6) a Darboux type differential system (analogically to the case when $n = 2$ in [4]).

As it is easy to see the center-affine invariant condition differ the system (6) from (1). Indeed, it is true that for system (6) with $\widetilde{K}_{1,n} \equiv 0$, we have $P_{1,n} = (n+1)R(x)$.

One can verify that the next theorem generalizes cases 1-4

Theorem 6. *System* (1) *with* $\widetilde{K}_{1,n} \equiv 0$ *and* $n = 2, 3, 4, 5$ *has the invariant* $GL(n, \mathbb{R})$*-integrating factor* μ *of the form*

$$\mu^{-1} = K_{m,n}\Phi_{n,n},$$

where $K_{m,n} = 0$ *and*

$$\Phi_{n,n} \equiv L_{n,n} - 2\left(\frac{n}{n+1} L_{n-1,n}P_{1,n} + L_{n-2,n}P_{2,n} + L_{n-3,n}P_{3,n} + ... + L_{1,n}P_{n-1,n} + P_{n,n}\right) = 0 \tag{14}$$

are invariant particular $GL(n, \mathbb{R})$*-integrals of this system, and* $L_{i,n}$ $(i = \overline{1, n})$ *are the coefficients of characteristic equation of the system* (1) *as follows*

$$\lambda^n + L_{1,n}\lambda^{n-1} + L_{2,n}\lambda^{n-2} + ... + L_{n-1,n}\lambda + L_{n,n} = 0 \tag{15}$$

and they can be expressed though the invariants from (13) *by the recurrence formula*

$$L_{i,n} = -\frac{1}{i}(I_{i,n} + I_{i-1,n}L_{1,n} + I_{i-2,n}L_{2,n} + ... + I_{1,n}L_{i-1,n}) \quad (i = \overline{1, n}). \tag{16}$$

With the aid of the cases 1-4 it is easy to verify that holds the next

Theorem 7. *System (1) with* $\widetilde{K}_{1,n} \equiv 0$ *and* $n = 2, 3, 4, 5$ *has the first invariant* $GL(n, \mathbb{R})$*-integral of Darboux type [6] as follows*

$$K_{m,n}^{-1}\Phi_{n,n}^n = C \tag{17}$$

if and only if $I_{1,n} = 0$*, where* $K_{m,n},\ \widetilde{K}_{1,n}, I_{1,n}$ *are from (13), and* $\Phi_{n,n}$ *is from (14).*

The proof of Theorem 7 for system (6) follows from the equation

$$\Lambda(K_{m,n}^{-1}\Phi_{n,n}^n) = -I_1 K_{m,n}^{-1}\Phi_{n,n}^n,$$

where Λ is from (3).

Remark 2. There exists the assumption that Theorems 6 and 7 hold for $n \geq 6$.

One can verify that holds

Remark 3. Expression $K_{m,n} = 0$ from (13) is the invariant particular $GL(n, \mathbb{R})$-integral for linear system $\frac{dx^j}{dt} = a^j_\alpha x^\alpha \ \ (\alpha = \overline{1, n})$.

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$GL(2, R)$-orbits in a competing species model

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Abstract. A particular model with two parameters describing the dynamics of two competing species is analyzed from algebraic viewpoint involving the $GL(2, R)$-orbits.

Mathematics subject classification: 34C14.
Keywords and phrases: Differential system, parametric portrait, Lie algebra of the operators, $GL(2, \mathbb{R})$-orbit, first integral.

1 Introduction

In this paper we study a particular family of planar vector fields modelling the dynamics of two competing populations and corresponding to a couple of similar species of animals which compete with each other for a common food supply.

The competition between two species is modelled by the competitive Lotka-Volterra system

$$\begin{cases} \dot{x}_1 = x_1(r_1 - a_{11}x_1 - a_{12}x_2), \\ \\ \dot{x}_2 = x_2(r_2 - a_{21}x_1 - a_{22}x_2), \end{cases} \tag{1}$$

where $x_1,\ x_2$ represent the number of the populations of the two species, $r_1,\ r_2$, represent the growth rate of the species, and $a_{ij} > 0,\ i, j = 1, 2$, represent the competitive impacts of species j on the growth of species i.

The model we study in this paper is proposed as an application by M. W. Hirsch, S. Smale and R. L. Devaney in [2] and has the form

$$\begin{cases} \dot{x}_1 = x_1(a - x_1 - ax_2), \\ \\ \dot{x}_2 = x_2(b - bx_1 - x_2), \end{cases} \tag{2}$$

where $x_1,\ x_2$ represent the number of the populations of the two species, and a and b are positive parameters. The system (2) is a particular case of (1).

In [3] the equilibrium points are found and the phase portrait and the parameter portraits are determined. Herein we determine the $GL(2, R)$-orbits of the system (2) and construct the corresponding Lie algebra. Then we determine the first integrals of the system (2) for particular values of the parameters a and b.

2 $GL(2,R)$-orbits of the system (2)

In the tensorial form the system (2) reads

$$\dot{x}^j = a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta, \quad (j,\alpha,\beta = 1,2). \tag{3}$$

The dimensions of the $GL(2,R)$-orbits of the system (3) are given by

Theorem 1. *[1]. The $GL(2,R)$-orbits of the system (3) has the dimension:*

4, if $K_5(K_9+\beta) \neq 0$,
or $K_5 \not\equiv 0,\ K_9+\beta \equiv 0,\ K_1 \not\equiv 0,\ W_1 \not\equiv 0$,
or $K_5 \not\equiv 0,\ K_9+\beta \equiv 0,\ K_1 \equiv 0,\ K_2 \not\equiv 0,\ W_2 \neq 0$,
or $K_5 \equiv 0,\ K_1 \not\equiv 0,\ I_4 \neq 0$;

3, if $K_5 \not\equiv 0,\ K_9+\beta \equiv 0,\ K_1 \not\equiv 0,\ W_1 \equiv 0$,
or $K_5 \not\equiv 0,\ K_9+\beta \equiv 0,\ K_1 \equiv 0,\ K_2 \not\equiv 0, W_2 = 0$,
or $K_5 \not\equiv 0,\ K_9+\beta \equiv 0,\ K_1 \equiv 0,\ K_2 \equiv 0, K_7 \not\equiv 0$,
or $K_5 \equiv 0,\ K_1 \not\equiv 0,\ I_4 = 0,\ K_2 \not\equiv 0$;

2, if $K_5 \not\equiv 0,\ K_9+\beta \equiv 0,\ K_1 \equiv 0,\ K_2 \equiv 0,\ K_7 \equiv 0$,
or $K_5 \equiv 0,\ K_1 \not\equiv 0,\ I_4 = 0,\ K_2 \equiv 0$,
or $K_5 \equiv 0,\ K_1 \equiv 0,\ K_2 \not\equiv 0$;

0, if $K_5 \equiv 0,\ K_1 \equiv 0,\ K_2 \equiv 0$,

where $\beta = 27I_8 - I_9 - 18I_7$, $W_1 = K_1(2K_{11} - I_1K_5 - 2K_1K_2) + K_2K_6$, $W_2 = 3K_2K_7 - 2K_3K_5$, *the invariants* $I_1,\ I_4,\ I_7,\ I_8,\ I_9$ *and the comitants* $K_1,\ K_2,\ K_5,\ K_6,\ K_7,\ K_9,\ K_{11}$ *having the forms [6]:*

$$I_1 = a^\alpha_\alpha,\ I_4 = a^\alpha_p a^\beta_{\beta q} a^\gamma_{\alpha\gamma}\varepsilon^{pq},\ I_7 = a^\alpha_{pr} a^\beta_{\alpha q} a^\delta_{\gamma\delta}\varepsilon^{pq}\varepsilon^{rs},\ I_8 = a^\alpha_{pr} a^\beta_{\alpha q} a^\gamma_{\delta s} a^\delta_{\beta\gamma}\varepsilon^{pq}\varepsilon^{rs},$$

$$I_9 = a^\alpha_{pr} a^\beta_{\beta q} a^\gamma_{\gamma s} a^\delta_{\alpha\delta}\varepsilon^{pq}\varepsilon^{rs},\ K_1 = a^\alpha_{\alpha\beta},\ K_2 = a^p_\alpha x^\alpha \varepsilon^{pq},\ K_3 = a^\alpha_\beta a^\beta_{\alpha\gamma} x^\gamma,$$

$$K_5 = a^p_{\alpha\beta} x^\alpha x^\beta x^p \varepsilon^{pq},\ K_6 = a^\alpha_{\alpha\beta} a^\beta_{\gamma\delta} x^\gamma x^\delta,\ K_7 = a^\alpha_{\beta\gamma} a^\beta_{\alpha\delta} x^\gamma x^\delta,\ K_9 = a^\alpha_{\alpha p} a^\beta_{\gamma q} a^\gamma_{\alpha\beta} x^\delta,$$

$$K_{11} = a^p_\alpha a^\alpha_{\beta\gamma} x^\beta x^\gamma x^q \varepsilon_{pq}.$$

For system (2) we have $a^1_1 = a$, $a^1_2 = 0$, $a^2_1 = 0$, $a^2_2 = b$, $a^1_{11} = -1$, $a^1_{12} = -a/2$, $a^1_{22} = 0$, $a^2_{11} = 0$, $a^2_{12} = -b/2$, $a^2_{22} = -1$. Therefore, in our particular case, we obtain

$$I_1 = a+b,\ I_4 = \frac{1}{4}(2+a)(a-b)(2+b),$$

$$I_7 = -a^2/4 + a/2 - 3a^2b/8 + ab/2 + b/2 - a^2b^2/4 - 3ab^2/8 - b^2/4,$$

$$I_8 = -a^2/4 + a/2 - a^2b/8 + b/2 - a^2b^2/4 - ab^2/8 - b^2/4,$$

$$I_9 = -(a+2)(b+2)(a+b+2ab-4)/8,\ K_1 = -(1+b/2)x - (1+a/2)y,$$

$$K_2 = (a-b)xy,\ K_3 = -(a+b^2/2)x - (b+a^2/2)y,$$

$$K_5 = (b-1)x^2y - (a-1)xy^2,\ K_6 = (1+b/2)x^2 + (a+ab+b)xy + (1+a/2)y^2,$$

$$K_7 = (1 + b^2/4)x^2 + (a + ab/2 + b)xy + (a^2/4 + 1)y^2,$$
$$K_9 = (1 - ab/2 - b/2)x - (1 - a/2 - ab/2)y,\ K_{11} = (b^2 - a)x^2y + (b - a^2)xy^2,$$
$$\beta = -2(b-1)^2(a-1)^2,\ W_1 = x^2y^2(a + b + 2ab - 4)(a - b)/2,$$
$$W_2 = (a + 2ab + b^3/4 + 3ab^2/4 - b^2 - 3b)x^3y + (5a^2b/2 - 5ab^2/2 - 2b + 2a)x^2y^2 -$$
$$-(3a^2b/4 - a^2 + b - 3a + a^3/4 + 2ab)xy^3,$$

whence, the theorem holds

Theorem 2. *$GL(2,R)$-orbits of the system (2) has the dimension:*
4, *if* $a \neq 1$ *or* $b \neq 1$,
2, *if* $a = b = 1$.

3 The parametric portrait and the phase portraits for the system (2)

The number and the nature of the equilibrium points of the system (2) are studied in [3]. Namely, from the biological viewpoint $(x, y > 0)$ and for $a, b \geq 0$, in the parametric portrait there are 12 strata (Fig. 1), i.e. there are 12 topological nonequivalent corresponding phase portraits (Fig. 2).

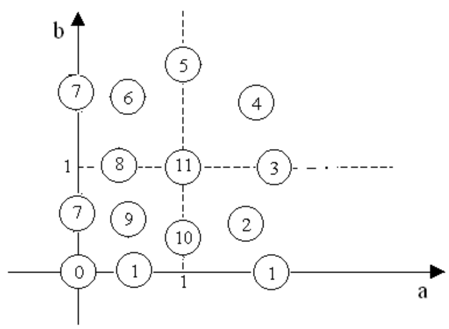
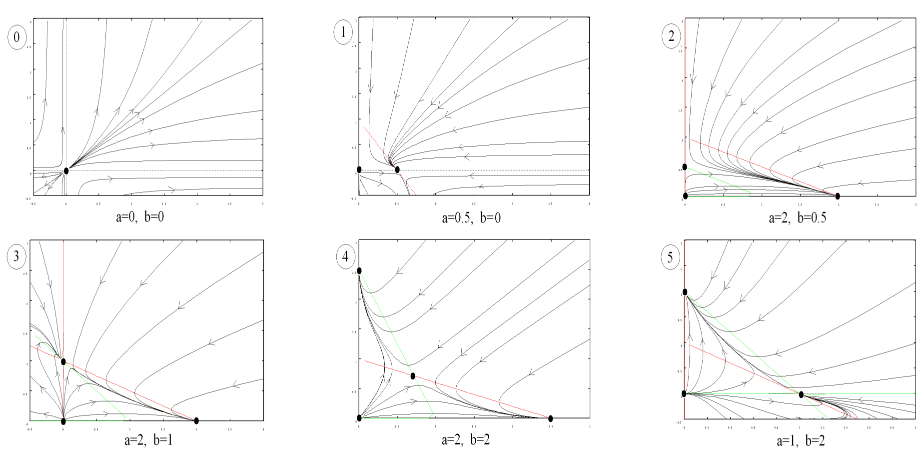


Fig. 1 The parametric portrait



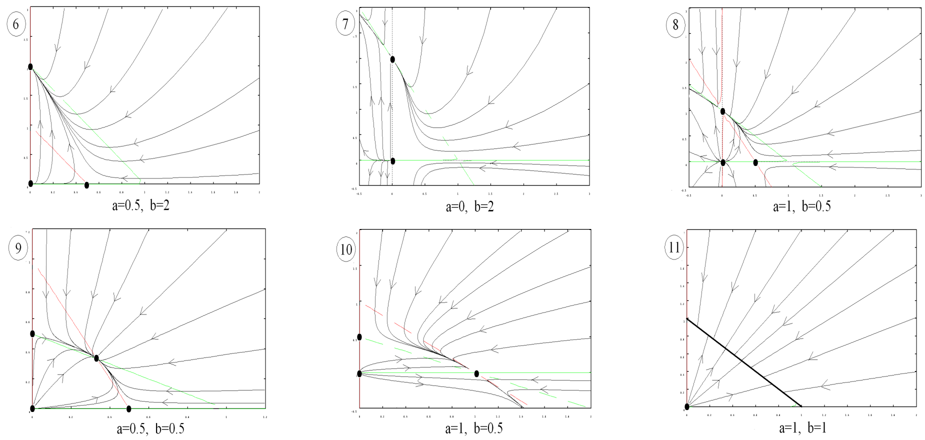


Fig. 2. Phase portraits for (2)

Remark 1. The case 11 in Fig.2 corresponds to the orbit of dimension 2, and the others to the orbit of dimension 4.

4 Lie algebras and some first integrals of the system (2)

To complete our algebraic investigation of the system (2), we attempted to construct Lie algebras for each system from Section 3. We supposed that system (2) admits the Lie algebra corresponding to the linear group of transformations having as generator the operator [5]:

$$X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y}, \tag{4}$$

where

$$\xi_1 = Ax + By + C, \quad \xi_2 = Dx + Ey + F.$$

It was found that the only system which admits such an algebra correspond to the orbit of dimension 2. One can verify using CSA Mathematica or Maple that the system (2) on the orbit with dimension 4 does not admit such algebra. Similarly we have found that this systems does not admit Lie algebra having as generator the operator (4) with coefficient vectors as follows:

1. $\xi_1 = A_1x^2 + A_2xy + A_3y^2 + A_4x + A_5y + A_6,$

$$\xi_2 = B_1x^2 + B_2xy + B_3y^2 + B_4x + B_5y + B_6;$$

2. $\xi_1 = A_1x^3 + A_2x^2y + A_3xy^2 + A_4y^3 + A_5x + A_6y + A_7,$

$$\xi_2 = B_1x^3 + B_2x^2y + B_3xy^2 + B_4y^3 + B_5x + B_6y + B_7;$$

3. $\xi_1 = A_1x^4 + A_2x^3y + A_3x^2y^2 + A_4xy^3 + A_5y^4 + A_6x + A_7y,$

$$\xi_1 = B_1x^4 + B_2x^3y + B_3x^2y^2 + B_4xy^3 + B_5y^4 + B_6x + B_7y;$$

$$4.\ \ \xi_1 = \frac{A_1x + B_1y + C_1}{D_1x + E_1y + F_1}, \quad \xi_2 = \frac{A_2x + B_2y + C_2}{D_2x + E_2y + F_2}.$$

Assume that $x^1 = x,\ x^2 = y$. So, following [3] we have considered the cases:

1. The system (2) for $a = b = 1$ corresponds to the case 11 (Fig.2). As this system is on the orbits with dimension 2, it admits the one-dimensional Lie algebra with operator $X = -x\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. By means of this operator the first integral $\mathcal{F}_1 \equiv \frac{y}{x} = C_1$ was found.

2. The system (2) for $a = b = 0$ corresponds to the case 0 (Fig.2). It admits the first integral $\mathcal{F}_2 \equiv -\frac{1}{y} + \frac{1}{x} + C_2 = 0$.

3. The system (2) for $a \neq 0,\ b = 0$ corresponds to the case 1 (Fig.2). It admits the first integral

$$\mathcal{F}_3 \equiv \left(y^a \left(-\frac{a}{y}\right)^a \Gamma\left(-a, -\frac{a}{y}\right) ax - y^a \left(-\frac{a}{y}\right)^a \Gamma\left(-a\right) ax + e^{\frac{a}{y}} ay^a + C_3\, ax - \right.$$

$$\left. - e^{\frac{a}{y}} y^a x \right) a^{-1} x^{-1} = 0.$$

4. The system (2) for $b \neq 0,\ a = 0$ corresponds to the case 7 (Fig. 2). It admits the first integral

$$\mathcal{F}_4 \equiv \left(x^b \left(-\frac{b}{x}\right)^b \Gamma\left(-b\right) by - x^b \left(-\frac{b}{x}\right)^b \Gamma\left(-b, -\frac{b}{x}\right) by - be^{\frac{b}{x}} x^b + C_4\, by + \right.$$

$$\left. + e^{\frac{b}{x}} x^b y \right) b^{-1} y^{-1} = 0.$$

Remark 2. The integrals $\mathcal{F}_1$ - $\mathcal{F}_4$ can not be expressed by center-affine invariants and comitants of the system (2). Moreover, integrals $\mathcal{F}_3$, $\mathcal{F}_4$ contain Gamma-functions.

Remark 3. For the system with $ab \neq 0,\ a \neq 1$ and $b \neq 1$ the authors were not able to find the first integral.

Throughout the paper the Computer Algebra Systems Maple 9.5 and Mathematica 5 were widely used.

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Minimum Cost Multicommodity Flows in Dynamic Networks and Algorithms for their Finding

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Abstract. We consider the minimum cost multicommodity flow problem in dynamic networks with time-varying capacities of arcs and transit times on arcs that depend on the sort of commodity entering them. We assume that cost functions, defined on arcs, are nonlinear and depend on time and flow, and the demand function also depends on time. Moreover, we study the problem in the case when transit time functions depend on time and flow. The modification of the time-expanded network method and new algorithms for solving the considered classes of problems are proposed.

Mathematics subject classification: 90B10, 90C35, 90C27.
Keywords and phrases: Network flows, dynamic networks, multicommodity flows, dynamic minimum cost flow problem.

1 Introduction and Problem Formulation

In this paper we study the dynamic version of the nonlinear minimum cost multicommodity network flow problem, which generalizes the classical static flow problem and extends some dynamic problems considered in [1,3,4]. We consider this problem on dynamic networks with time-varying capacities of arcs and transit times on arcs that depend on the sort of commodity entering them, what means that the transit time functions on the set of arcs for different commodities can be different. We assume that cost functions, defined on arcs, are nonlinear and depend on time and flow. Moreover, we assume that the demand function also depends on time. To solve the considered dynamic problem, we reduce it to the static one on a special time-expanded network, the structure of which differs from the standard one introduced by Ford and Fulkerson in [3]. We propose algorithms for solving the general minimum cost multicommodity flow problem and its variants with different forms of restrictions by parameters of network and time. We also consider dynamic networks with transit time functions that depend on flow and time and elaborate methods for solving problems on such networks.

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The minimum cost multicommodity dynamic flow problem asks to find the flow of a set of commodities through a network with given time horizon, satisfying all supplies and demands with minimum cost such that link capacities are not exceeded. We consider the discrete time model, in which all times are integral and bounded by horizon T. The time horizon is the time until which the flow can travel in the network and defines the makespan $\mathcal{T} = \{0, 1, \dots, T\}$ of time moments we consider. Time is measured in discrete steps, so that if one unit of flow of commodity k leaves node u at time t on arc $e = (u, v)$, one unit of flow arrives at node v at time $t + \tau_e^k$, where τ_e^k is the transit time on arc e for commodity k. Without loosing generality, we assume that no arcs enter sources or exit sinks. Accordingly the sources are nodes through which flow enters the network and the sinks are nodes through which flow leaves the network.

We consider a directed network $N = (V, E, K, w, u, \tau, d, \varphi)$ with set of vertices V, set of arcs E and set of commodities $K = \{1, 2, \dots, q\}$ that must be routed through the same network. A dynamic network N consists of capacity function w: $E \times K \times \mathcal{T} \to \mathbb{R}_+$, mutual capacity function u: $E \times \mathcal{T} \to \mathbb{R}_+$, transit time function τ: $E \times K \to \mathbb{R}_+$, demand function d: $V \times K \times \mathcal{T} \to \mathbb{R}$ and cost function φ: $E \times \mathbb{R}_+ \times \mathcal{T} \to \mathbb{R}_+$. So, $\tau_e = (\tau_e^1, \tau_e^2, \dots, \tau_e^q)$ is a vector, each component of which reflects the transit time on arc e for commodity $k \in K$. Such formulation of the problem extends models studied in [1, 2, 4, 5]. The demand function $d_v^k(t)$ satisfies the following conditions:

a) there exists $v \in V$ for every $k \in K$ with $d_v^k(0) < 0$;

b) if $d_v^k(t) < 0$ for a node $v \in V$ for commodity $k \in K$ then $d_v^k(t) = 0$, $t = 1, 2, \dots, T$;

c) $\displaystyle\sum_{t \in \mathcal{T}} \sum_{v \in V} d_v^k(t) = 0, \forall k \in K.$

Nodes $v \in V$ with $\sum_{t \in \mathcal{T}} d_v^k(t) < 0,\ k \in K$, are called sources for commodity k, nodes $v \in V$ with $\sum_{t \in \mathcal{T}} d_v^k(t) > 0,\ k \in K$, are called sinks for commodity k and nodes $v \in V$ with $\sum_{t \in \mathcal{T}} d_v^k(t) = 0,\ k \in K$, are called intermediate for commodity k.

A multicommodity dynamic flow in N is a function x: $E \times \mathcal{T} \to \mathbb{R}_+$ that satisfies the following conditions:

$$\sum_{\substack{e \in E^+(v) \\ t - \tau_e^k \geq 0}} x_e^k(t - \tau_e^k) - \sum_{e \in E^-(v)} x_e^k(t) = d_v^k(t),\ \forall\, t \in \mathcal{T},\ \forall\, v \in V,\ \forall k \in K; \tag{1}$$

$$\sum_{k \in K} x_e^k(t) \leq u_e(t),\ \forall\, t \in \mathcal{T},\ \forall e \in E; \tag{2}$$

$$0 \leq x_e^k(t) \leq w_e^k(t),\quad \forall\, t \in \mathcal{T},\ \forall\, e \in E,\ \forall k \in K; \tag{3}$$

$$x_e^k(t) = 0,\ \forall\, e \in E,\ t = \overline{T - \tau_e^k + 1, T},\ \forall k \in K, \tag{4}$$

where $E^-(v) = \{(v, z) \,|\, (v, z) \in E\}$, $\quad E^+(v) = \{(z, v) \,|\, (z, v) \in E\}$.

Here the function x defines the value $x_e^k(t)$ of flow of commodity k entering arc e at time t. It is easy to observe that the flow of commodity k does not enter arc

e at time t if it will have to leave the arc after time T; this is ensured by condition (4). Capacity constraints (3) mean that at most $w_e^k(t)$ units of flow of commodity k can enter arc e at time t. Mutual capacity constraints (2) mean that at most $u_e(t)$ units of flow can enter arc e at time t. Conditions (1) represent flow conservation constraints.

To model transit costs, which may change over time, we define the cost function $\varphi_e(x_e^1(t), x_e^2(t), \ldots, x_e^q(t), t)$ which indicates the cost of shipping flows over arc e entering arc e at time t. The total cost of the dynamic multicommodity flow x is defined as follows:

$$c(x) = \sum_{t \in \mathcal{T}} \sum_{e \in E} \varphi_e(x_e^1(t), x_e^2(t), \ldots, x_e^q(t), t).$$

The object of the minimum cost multicommodity dynamic flow problem is to find a flow that minimizes this objective function.

It is important to notice that in many practical cases the cost functions are presented in the following form:

$$\varphi_e(x_e^1(t), x_e^2(t), \ldots, x_e^q(t), t) = \sum_{k \in K} \varphi_e^k(x_e^k(t), t). \tag{5}$$

The case when $\tau_e^k = 0$, $\forall\, e \in E$, $\forall\, k \in K$ and $T = 0$ can be considered as the static minimum cost multicommodity flow problem.

2 The Main Results

We show that the minimum cost multicommodity flow problem on network N can be reduced to a static problem on an auxiliary network N^T, which we name the time-expanded network. The advantage of this approach is that it turns the problem of determining a minimum cost dynamic flow problem into a classical static minimum cost flow problem on the time-expanded network, which we regard as a static representation of the dynamic network.

2.1 Constructing the Time-Expanded Network for the General Case of the Problem

So, we study the general case of the considered minimum cost flow problem when transit times on an arc are different for different commodities. We define the time-expanded network $N^T = (V^T, E^T, K, d^T, w^T, u^T, \varphi^T)$ as follows:

1. $\overline{V}^T \colon = \{v(t) \,|\, v \in V,\ t \in \mathcal{T}\}$;
2. $\widetilde{V}^T \colon = \{e(v(t)) \,|\, v(t) \in \overline{V}^T,\ e \in E^-(v),\ t \in \mathcal{T} \setminus T\}$;
3. $V^T \colon = \overline{V}^T \cup \widetilde{V}^T$;

4. $\widetilde{E}^T:=\{\widetilde{e}(t)=(v(t),e(v(t)))\,|\,v(t)\in\overline{V}^T$ and corresponding $e(v(t))\in\widetilde{V}^T,\ t\in\mathcal{T}\setminus T\}$;

5. $\overline{E}^T:=\{e^k(t)=(e(v(t)),z(t+\tau_e^k))\,|\,e(v(t))\in\widetilde{V}^T,\ z(t+\tau_e^k)\in\overline{V}^T,\ e=(v,z)\in E,\ 0\le t\le T-\tau_e^k,\ k\in K\}$;

6. $E^T:=\overline{E}^T\cup\widetilde{E}^T$;

7. ${d_{v(t)}^k}^T:=d_v^k(t)$ for $v(t)\in\overline{V}^T,\ k\in K$;
   ${d_{e(v(t))}^k}^T:=0$ for $e(v(t))\in\widetilde{V}^T,\ k\in K$;

8. ${w_{e^k(t)}^l}^T:=\begin{cases} w_e^k(t), & \text{if } l=k \text{ for } e^k(t)\in\overline{E}^T,\ k\in K;\\ 0, & \text{if } l\ne k \ \text{ for } e^k(t)\in\overline{E}^T,\ k\in K\end{cases}$
   and ${w_{\widetilde{e}(t)}^l}^T=\infty$ for $\widetilde{e}(t)\in\widetilde{E}^T,\ l\in K$;

9. ${u_{\widetilde{e}(t)}}^T:=u_e(t)$ for $\widetilde{e}(t)=(v(t),e(v(t)))\in\widetilde{E}^T$;
   ${u_{e^k(t)}}^T:=\infty$ for $e^k(t)\in\overline{E}^T,\ k\in K$;

10. $\varphi_{\widetilde{e}(t)}^T({x_{\widetilde{e}(t)}^1}^T,{x_{\widetilde{e}(t)}^2}^T,\dots,{x_{\widetilde{e}(t)}^q}^T):=\varphi_e(x_e^1(t),x_e^2(t),\dots,x_e^q(t),t)$ for $\widetilde{e}(t)=(v(t),e(v(t)))\in\widetilde{E}^T$;
    $\varphi_{e^k(t)}^T({x_{e^k(t)}^1}^T,{x_{e^k(t)}^2}^T,\dots,{x_{e^k(t)}^q}^T):=0$ for $e^k(t)\in\overline{E}^T,\ k\in K$.

The correspondence between flows in the dynamic network and the static time-expanded network is presented by the following lemma.

Lemma 1. *Let $x^T\colon E^T\to\mathbb{R}_+$ be a multicommodity flow in the static network N^T. Then the multicommodity flow $x\colon E\times\mathcal{T}\to\mathbb{R}_+$ in the dynamic network N can be defined in the following way. Let $e^k(t)=(e(v(t)),z(t+\tau_e^k))\in\overline{E}^T$, $\widetilde{e}(t)=(v(t),e(v(t)))\in\widetilde{E}^T$. Then the dynamic flow $x_e(t)$ on arc $e=(v,z)$ is determined as follows: $x_e^k(t)={x_{e^k(t)}^k}^T={x_{\widetilde{e}(t)}^k}^T,\ \forall k\in K,\ \forall t\in\mathcal{T}$.*

If $x\colon E\times\mathcal{T}\to\mathbb{R}_+$ is a multicommodity flow in the dynamic network N, then the multicommodity flow $x^T\colon E^T\to\mathbb{R}_+$ in the static network N^T can be determined as follows. Let $x_e(t)$ be a dynamic multicommodity flow on arc $e=(v,z)\in E$. Then the tuple $({x_{\widetilde{e}(t)}}^T,{\overline{x}_{\widetilde{e}(t)}}^T)={x_{e(t)}}^T$ is a corresponding static multicommodity flow, where ${x_{\widetilde{e}(t)}}^T$ is a static multicommodity flow on additional arc $\widetilde{e}(t)=(v(t),e(v(t)))\in\widetilde{E}^T$, at that ${x_{\widetilde{e}(t)}^k}^T=x_e^k(t),\ \forall k\in K;$ ${\overline{x}_{\widetilde{e}(t)}}^T=({x_{e^1(t)}}^T,{x_{e^2(t)}}^T,\dots,{x_{e^q(t)}}^T)$ is a q dimension vector of static multicommodity flows on arcs $e^k(t)=(e(v(t)),z(t+\tau_e^k))\in\overline{E}^T,\ k\in K$, at that ${x_{e^k(t)}^k}^T=x_e^k(t);\ {x_{e^k(t)}^l}^T=0,\ l\ne k$.

Proof. To prove the first part of the lemma we have to show that conditions (1)–(4) for defined above x in the dynamic network N are true. These conditions evidently result from the following definition of multicommodity flows in the static network N^T:

$$\sum_{e(t-\tau_e^k)\in E^+(v(t))} {x^k_{e(t-\tau_e^k)}}^T - \sum_{e(t)\in E^-(v(t))} {x^k_{e(t)}}^T = {d^k_{v(t)}}^T, \quad \forall\, t \in \mathcal{T},\ \forall\, v(t) \in V^T,\ \forall k \in K; \tag{6}$$

$$\sum_{k\in K} {x^k_{e(t)}}^T \le {u_{e(t)}}^T,\ \forall e(t) \in E^T,\ \forall\, t \in \mathcal{T}; \tag{7}$$

$$0 \le {x^k_{e(t)}}^T \le {w^k_{e(t)}}^T,\ \forall\, e(t) \in E^T,\ \ \forall\, t \in \mathcal{T},\ \forall k \in K; \tag{8}$$

$${x^k_{e(t)}}^T = 0,\ \forall\, e(t) \in E,\ t = \overline{T - \tau_e^k + 1, T},\ \forall k \in K. \tag{9}$$

In order to prove the second part of the lemma it is sufficient to show that conditions (6)–(9) hold. Correctness of these conditions results from the procedure of constructing the time-expanded network, correspondence between flows in static and dynamic networks and the satisfied conditions (1)–(4). □

The following theorem holds.

Theorem 2. *If x^{*T} is a static minimum cost multicommodity flow in the static network N^T, then the corresponding according to Lemma 1 dynamic multicommodity flow x^* in the dynamic network N is also a minimum cost one and vice-versa.*

Proof. Taking into account the correspondence between static and dynamic multicommodity flows on the basis of Lemma 1, we obtain that costs of multicommodity flow in the time-expanded network N^T and multicommodity flow in the dynamic network N are equal. Indeed, to solve the minimum cost multicommodity flow problem in the static time-expanded network N^T, we have to solve the following problem:

$$c^T(x) = \sum_{t\in\mathcal{T}} \sum_{e(t)\in E^T} \varphi^T_{e(t)}(x^1_{e(t)}, x^2_{e(t)}, \ldots, x^q_{e(t)}) \to \min$$

subject to (6)–(9). □

2.2 The Case of the Problem with Separable Cost Functions and without Mutual Capacity of Arcs

The minimum cost flow problem with separable cost functions (5) and without mutual capacity constraints for arcs allows us to simplify the procedure of constructing the time-expanded network. In this case we don't have to add a new set of vertexes $\widetilde{V}^T$ and a new set of arcs $\widetilde{E}^T$. In that way the time-expanded network N^T is defined as follows:

1. $V^T\colon = \{v(t) \mid v \in V,\ t \in \mathcal{T}\}$;
2. $E^T\colon = \{e^k(t) = (v(t), z(t+\tau_e^k)) \mid e = (v,z) \in E,\ 0 \le t \le T - \tau_e^k,\ k \in K\}$;

3. ${d^k_{v(t)}}^T := d^k_v(t)$ for $v(t) \in V^T,\ k \in K$;

4. ${w^l_{e^k(t)}}^T := \begin{cases} w^k_e(t), & \text{if } l = k \text{ for } e^k(t) \in E^T,\ k \in K; \\ 0, & \text{if } l \neq k \ \text{ for } e^k(t) \in E^T,\ k \in K; \end{cases}$

5. ${\varphi^l_{e^k(t)}}^T({x^l_{e^k(t)}}^T) := \begin{cases} \varphi^k_e(x^k_e(t), t), & \text{if } l = k \text{ for } e^k(t) \in E^T,\ k \in K; \\ 0, & \text{if } l \neq k \ \text{ for } e^k(t) \in E^T,\ k \in K. \end{cases}$

The correspondence between flows in the dynamic network N and the static network N^T is defined as follows. Let $x^T\colon E^T \to \mathbb{R}_+$ be a multicommodity flow in the static network N^T. Then the following flow $x\colon E \times \mathcal{T} \to \mathbb{R}_+$, where $x^k_e(t) = {x^k_{e^k(t)}}^T$, $\forall e \in E,\ \forall k \in K,\ \forall t \in \mathcal{T}$, represents the multicommodity flow in the dynamic network N. If $x\colon E \times \mathcal{T} \to \mathbb{R}_+$ is a multicommodity flow in the dynamic network N, then the flow $x^T\colon E^T \to \mathbb{R}_+$, where ${x^k_{e^k(t)}}^T = x^k_e(t),\ {x^l_{e^k(t)}}^T = 0,\ \forall e^k(t) \in E^T,\ \forall k \in K,\ l \neq k$, represents the multicommodity flow in the static network N^T.

As above, it can be proved that if x^{*T} is a static minimum cost multicommodity flow in the static network N^T, then the corresponding dynamic multicommodity flow x^* in the dynamic network N is also a minimum cost flow and vice-versa.

2.3 The Case of the Problem with Common Transit Times on Arcs for Commodities

In the case of the minimum cost flow problem with common transit times for each commodity the time-expanded network also can be constructed in more simple way without adding a new set of vertexes $\widetilde{V}^T$ and a new set of arcs $\widetilde{E}^T$. Thus the time-expanded network N^T is defined as follows:

1. $V^T := \{v(t) \,|\, v \in V,\ t \in \mathcal{T}\}$;
2. $E^T := \{e(t) = (v(t), z(t + \tau_e)) \,|\, e = (v, z) \in E,\ 0 \le t \le T - \tau_e\}$;
3. ${d^k_{v(t)}}^T := d^k_v(t)$ for $v(t) \in V^T,\ k \in K$;
4. $u^T_{e(t)} := u_e(t)$ for $e(t) \in E^T$;
5. ${w^k_{e(t)}}^T := w^k_e(t)$ for $e(t) \in E^T,\ k \in K$;
6. $\varphi^T_{e(t)}({x^1_{e(t)}}^T, {x^2_{e(t)}}^T, \ldots, {x^q_{e(t)}}^T) := \varphi_e(x^1_e(t), x^2_e(t), \ldots, x^q_e(t), t)$ for $e(t) \in E^T$.

In this case the correspondence between flows in the dynamic network N and the static network N^T is defined in the following way. Let $x^T\colon E^T \to \mathbb{R}_+$ be a multicommodity flow in the static network N^T. Then the following flow $x\colon E \times \mathcal{T} \to$

$\mathbb{R}_+$, where $x_e^k(t) = {x_{e(t)}^k}^T$, $\forall e \in E$, $\forall k \in K$, $\forall t \in \mathcal{T}$, represents the multicommodity flow in the dynamic network N. If x: $E \times \mathcal{T} \to \mathbb{R}_+$ is a multicommodity flow in the dynamic network N, then the flow x^T: $E^T \to \mathbb{R}_+$, where ${x_{e(t)}^k}^T = x_e^k(t)$, $\forall e(t) \in E^T$, $\forall k \in K$, $\forall t \in \mathcal{T}$, represents the multicommodity flow in the static network N^T.

As above, it can be proved that if x^{*T} is a static minimum cost multicommodity flow in the static network N^T, then the corresponding dynamic multicommodity flow x^* in the dynamic network N is also a minimum cost flow and vice-versa.

3 The Algorithm and Examples

On the basis of results from the previous section we can propose the following algorithm for solving the minimum cost multicommodity dynamic flow problem. In such a way, to solve the minimum cost multicommodity flow problem in N we have to build the time-expanded network N^T for the given dynamic network N, after what to solve the classical minimum cost multicommodity flow problem in the static network N^T and then to reconstruct according to Lemma 1 and Theorem 2 the solution of the static problem in N^T to the dynamic problem in N.

In the following we construct in different cases the time-expanded network N^T for the dynamic network N given on Fig. 1 with two commodities.

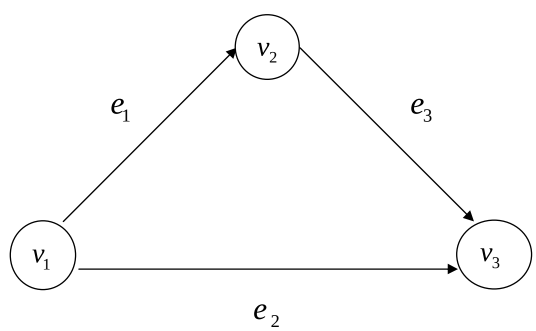

Figure 1. The dynamic network

The set of time moments we consider is $\mathcal{T} = \{0, 1, 2, 3\}$. The transit times on each arc for each commodity are defined in the following way: $\tau_{e_1}^1 = 2$, $\tau_{e_1}^2 = 1$, $\tau_{e_2}^1 = 1$, $\tau_{e_2}^2 = 3$, $\tau_{e_3}^1 = 1$, $\tau_{e_3}^2 = 2$. The mutual capacity, individual capacity, demand and cost functions are considered to be known.

The time-expanded network N^T for the dynamic network N in the general case is represented on Fig. 2. The time-expanded network N^T for the dynamic network N in the case of separable cost functions and without mutual capacity of arcs is represented on Fig. 3. The time-expanded network N^T for the dynamic network N in the case of common transit times for each commodity with $\tau_{e_1} = 1$, $\tau_{e_2} = 1$, $\tau_{e_3} = 2$ is represented on Fig. 4.

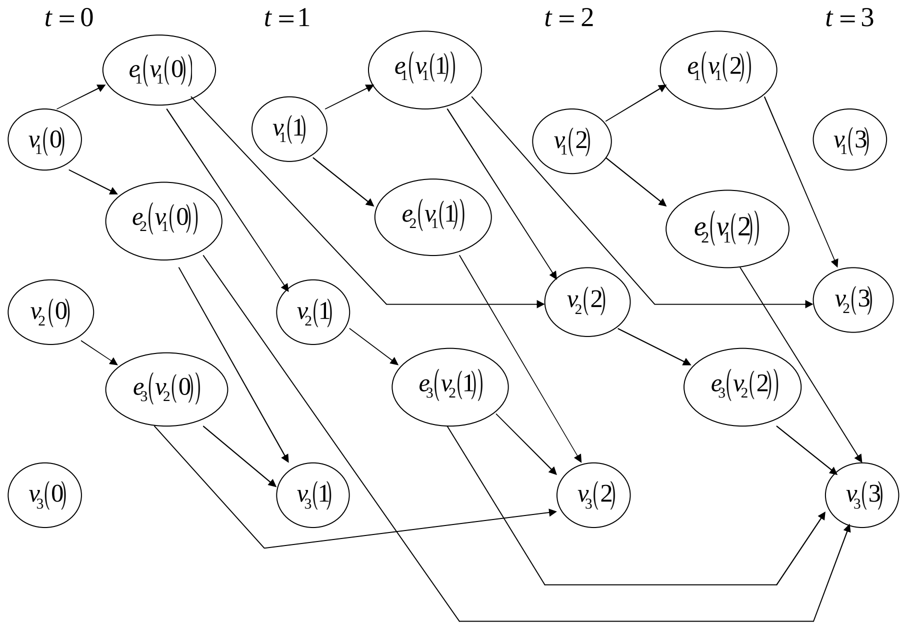

Figure 2. The time-expanded network

Remark 1. The proposed above approach can be used to solve some more general cases of the minimum cost dynamic multicommodity flow problem such as the problem when only a part of the flow is dumped into the considered network at the time 0, when flow storage at nodes is allowed and when the cost functions also depend on the flow at the nodes. The same reasoning to solve the minimum cost flow problem in the dynamic networks and its generalization can be held in the case when, instead of the condition (3) in the definition of the multicommodity dynamic flow, the following condition takes place: $w_e'^k(t) \le x_e^k(t) \le w_e''^k(t)$, $\forall\, t \in \mathcal{T}$, $\forall\, e \in E$, $\forall\, k \in K$, where $w_e'^k(t)$ and $w_e''^k(t)$ are lower and upper bounds of the capacity of the arc e respectively.

Remark 2. The maximum multicommodity dynamic flow problem also can be solved by reduction to a static problem in an auxiliary time-expanded network N^T, which is defined as above but without demand and cost functions.

4 Determining the Minimum Cost Flows in Dynamic Networks with Transit Time Functions that Depend on Flow and Time

In the problems studied in the previous sections the transit time functions are assumed to be constant at every moment of time for each arc of the network. A more general class of dynamic multicommodity flow problems can be obtained if the transit time functions τ_e^k, $\forall e \in E$, $\forall k \in K$, depend on flows and on time. From the practical point of view we can state that the transit time function possesses the

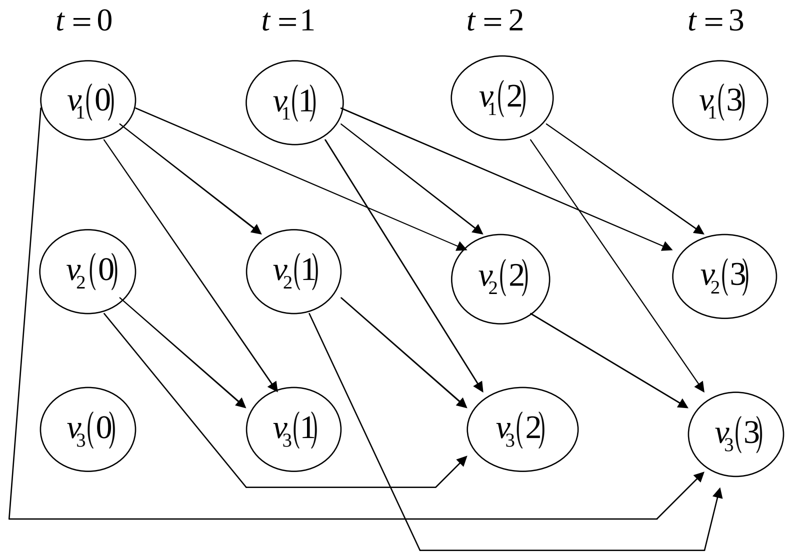

Figure 3. The time-expanded network (case of separable cost functions and without mutual capacity of arcs)

property of being a non-negative and non-decreasing function. So, we will assume that the transit time function is a non-decreasing non-negative step function. First we will describe the method for solving the minimum cost single-commodity flow problem in dynamic networks with transit time functions that depend on flow and time. Then the dynamic multicommodity flow problem with transit time functions that depend on flows and time can be solved by using the similar approach extended to the multicommodity case of the problem. The detailed elaboration of the time-expanded network method for such class of the problem can be obtained for the case of separable transit-time functions $\tau_e^k(x_e^1, x_e^2, \dots, x_e^q, t) = \sum_{p=1}^q \tau_e^k(x_e^p, t)$, $\forall e \in E$, $\forall t \in \mathcal{T}$, $\forall k \in K$, where the functions $\tau_e^k(x_e^p, t)$ satisfy the conditions described below.

4.1 The Minimum Cost Dynamic Flow Problem with Transit Time Functions that Depend on Flow and Time

Let us formulate the minimum cost single-commodity flow problem in dynamic networks with transit time functions that depend on flow and time. Let be given a directed network $N = (V, E, u', u'', \tau, d, \varphi)$ with set of vertices V and set of arcs E, lower and upper capacity functions u', u'': $E \times \mathcal{T} \to \mathbb{R}_+$, transit time function τ: $E \times \mathcal{T} \times \mathbb{R}_+ \to \mathbb{R}_+$, demand function d: $V \times \mathcal{T} \to \mathbb{R}$ and cost function φ: $E \times \mathbb{R}_+ \times \mathcal{T} \to \mathbb{R}_+$. As above, we consider the discrete time model, in which all times are integral and bounded by a time horizon T, which defines the set $\mathcal{T} = \{0, 1, \dots, T\}$ of time moments we consider. We suppose that all flow is dumped into the network at

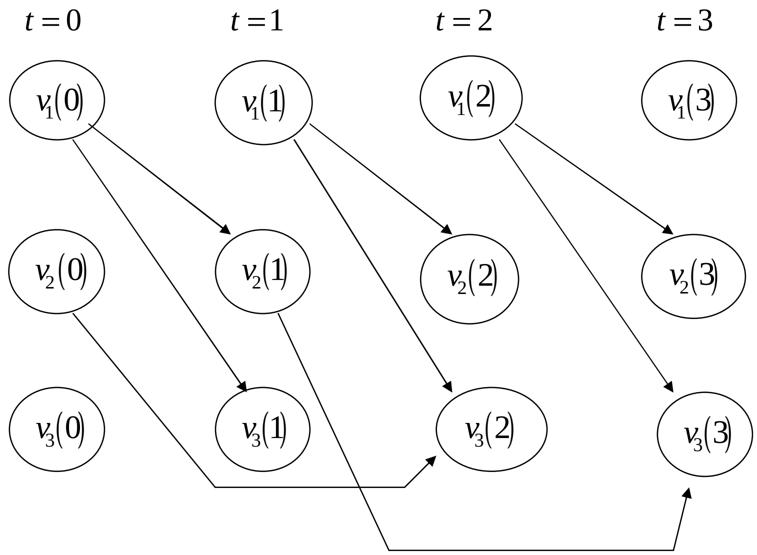

Figure 4. The time-expanded network (case of common transit times for each commodity)

time 0 and the supply is equal to the demand, i.e. $\sum_{t\in\mathcal{T}}\sum_{v\in V} d_v(t) = 0$. Without losing generality, we assume that no arcs enter sources or exit sinks.

A dynamic flow in N is represented by a function x: $E \times \mathcal{T} \to \mathbb{R}_+$, which defines the value $x_e(t)$ of flow entering arc e at time t. Since we require that all arcs must be empty after time horizon T, the following implication must hold for all $e \in E$ and $t \in \mathcal{T}$: if $x_e(t) > 0$, then $t + \tau_e(x_e(t), t) \le T$. The dynamic flow x must satisfy the flow conservation constraints, which mean that at any time moment $t \in \mathcal{T}$ for every vertex $v \in V$ the difference between the total amount of flow that leaves node v and the total amount of flow that enters node v, is equal to $d_v(t)$. The dynamic flow x is called feasible if it satisfies the following capacity constraints: $u'_e(t) \le x_e(t) \le u''_e(t),\ \forall\, t \in \mathcal{T},\ \forall\, e \in E$.

The total cost of the dynamic flow x is defined as follows:

$$F(x) = \sum_{t\in\mathcal{T}}\sum_{e\in E} \varphi_e(x_e(t), t).$$

The object of the minimum cost dynamic flow problem is to find a feasible flow that minimizes this objective function.

4.2 The Method for Solving the Problem

We propose an approach for solving the formulated problem, which is based on reduction of this problem to a static one on a special auxiliary time-expanded network N^T. We define the network $N^T = (V^T, E^T, d^T, u'^T, u''^T, \varphi^T)$ as follows:

1. $\overline{V}^T\colon = \{v(t) \,|\, v \in V,\ t \in \mathcal{T}\}$;

2. $\widetilde{V}^T\colon = \{e(v(t)) \,|\, v(t) \in \overline{V}^T,\ e \in E^-(v),\ t \in \mathcal{T} \setminus T\}$;

3. $V^T\colon = \overline{V}^T \cup \widetilde{V}^T$;

4. $\widetilde{E}^T\colon = \{\widetilde{e}(t) = (v(t), e(v(t))) \,|\, v(t) \in \overline{V}^T$ and corresponding $e(v(t)) \in \widetilde{V}^T,\ t \in \mathcal{T} \setminus T\}$;

5. $\overline{E}^T\colon = \{e^p(t) = (e(v(t)), z(t+\tau_e^p)) \,|\, e(v(t)) \in \widetilde{V}^T,\ z(t+\tau_e^p) \in \overline{V}^T,\ e = (v,z) \in E,\ 0 \le t \le T - \tau_e^p,\ p \in P\}$;

6. $E^T\colon = \overline{E}^T \cup \widetilde{E}^T$;

7. $d^T_{v(t)}\colon = d_v(t)$ for $v(t) \in \overline{V}^T,\ k \in K$;
   $d^T_{e(v(t))}\colon = 0$ for $e(v(t)) \in \widetilde{V}^T$;

8. ${u'}_{\widetilde{e}(t)}^{\,T}\colon = u'_e(t)$ for $\widetilde{e}(t) = (v(t), e(v(t))) \in \widetilde{E}^T$;
   ${u''}_{\widetilde{e}(t)}^{\,T}\colon = u''_e(t)$ for $\widetilde{e}(t) = (v(t), e(v(t))) \in \widetilde{E}^T$;
   ${u'}_{e^p(t)}^{\,T}\colon = \overline{x_e^{p-1}}(t)$ for $e^p(t) \in \overline{E}^T,\ p \in P$, where $\overline{x_e^0}(t) = 0$;
   ${u''}_{e^p(t)}^{\,T}\colon = \overline{x_e^p}(t)$ for $e^p(t) \in \overline{E}^T,\ p \in P$;

9. $\varphi^T_{\widetilde{e}(t)}({x_{\widetilde{e}(t)}}^T)\colon = \varphi_e(x_e(t), t)$ for $\widetilde{e}(t) = (v(t), e(v(t))) \in \widetilde{E}^T$;
   $\varphi^T_{e^p(t)}({x_{e^p(t)}}^T)\colon = \varepsilon_p$ for $e^p(t) \in \overline{E}^T,\ p \in P$, where $\varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_{|P|}$ are small numbers.

To make the notations more clear we construct a part of the time-expanded network for the fixed moment of time t for the given arc $e = (v,z)$ with the transit time function presented by Fig. 5.

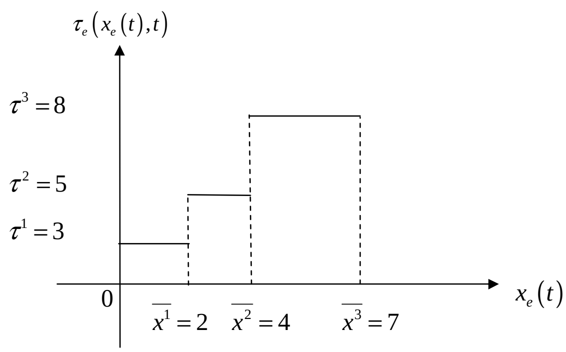

Figure 5. The transit time function

The constructed part of the time-expanded network is given on Fig. 6, where lower and upper capacities are written above each arc and the cost is written below each arc.

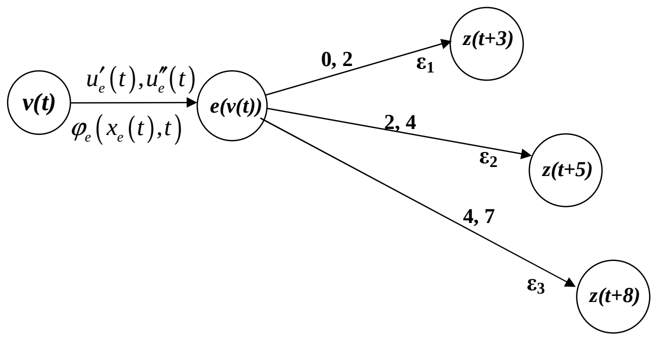

Figure 6. The constructed part of the time-expanded network

The following lemma gives the correspondence between flows in the dynamic network and the time-expanded network.

Lemma 3. *Let $x^T\colon E^T \to \mathbb{R}_+$ be a flow in the static network N^T. Then the flow $x\colon E \times \mathcal{T} \to \mathbb{R}_+$ in the dynamic network N can be defined in the following way. Let $e^p(t) = (e(v(t)), z(t+\tau_e^p)) \in \overline{E}^T$, $\widetilde{e}(t) = (v(t), e(v(t))) \in \widetilde{E}^T$. Then the dynamic flow $x_e(t)$ on arc $e = (v,z)$ is determined as follows: $x_e(t) = x^T_{\widetilde{e}(t)} = x^T_{e^p(t)}$, $\forall t \in \mathcal{T}$, where $p \in P$ is such that $x^T_{\widetilde{e}(t)} \in (\overline{x_e^{p-1}}(t), \overline{x_e^p}(t)]$.*

If $x\colon E \times \mathcal{T} \to \mathbb{R}_+$ is a flow in the dynamic network N, then the flow $x^T\colon E^T \to \mathbb{R}_+$ in the static network N^T can be determined as follows. Let $x_e(t)$ be a dynamic flow on arc $e = (v,z) \in E$. Then the tuple $({x_{\widetilde{e}(t)}}^T, {\overline{x}_{\widetilde{e}(t)}}^T) = {x_{e(t)}}^T$ is a corresponding static flow, where ${x_{\widetilde{e}(t)}}^T$ is a static flow on additional arc $\widetilde{e}(t) = (v(t), e(v(t))) \in \widetilde{E}^T$, at that $x^T_{\widetilde{e}(t)} = x_e(t)$; ${\overline{x}_{\widetilde{e}(t)}}^T = ({x_{e^1(t)}}^T, {x_{e^2(t)}}^T, \ldots, {x_{e^{|P|}(t)}}^T)$ is a $|P|$ dimensional vector of static flows on arcs $e^p(t) = (e(v(t)), z(t+\tau_e^p)) \in \overline{E}^T$, $p \in P$, at that $x^T_{e^p(t)} = x_e(t)$ if $x_e(t) \in (\overline{x_e^{p-1}}(t), \overline{x_e^p}(t)]$ or $x^T_{e^p(t)} = 0$ otherwise.

The proof of this lemma is similar to the proof of Lemma 1.

The following theorem holds.

Theorem 4. *If x^{*T} is a static minimum cost flow in the static network N^T, then the corresponding according to Lemma 3 dynamic flow x^* in the dynamic network N is also a minimum cost flow and vice-versa.*

In such a way, the minimum cost multicommodity flow problem in the dynamic network can be solved by static flow computations in the corresponding time-expanded network. To solve the minimum cost flow problem in dynamic networks with transit time functions that depend on flow and time we construct the time-expanded network, then solve the static minimum cost flow problem and reconstruct the solution of the static problem to the dynamic problem.

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