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Automodel solution for dynamic problem of two-component media

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Abstract. In this paper behavior of two-phase elastic medium at the movement in it of some concentrate load with supersonic speed was examined. Was obtained automodel solution for space in two-component problem at symmetrical axis. Analysis of obtained analytic solution demonstrates that major tensions and displacements are situated in the domain of the longitudinal and transversal wave action. As consequence, energy at the load movement is consumed preponderant to compressing and removing in the domain of superposition of all waves.

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Homogeneous elastic two-component medium is considered. Let this medium is not deformed and concentrated load with speed υ_0 parallel to z axes is moved forward in it. It is necessary to investigate wave movement, turned up in this medium, satisfying equations of spatial axisymmetrical movement of two-component elastic medium.

Behavior of this medium is described by the following equations of movement [1–4]:

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{\partial\sigma_{rz}}{\partial z}-\frac{\partial\pi_0}{\partial r}+\frac{1}{r}\left(\sigma_{rr}-\sigma_{\theta\theta}\right)=\rho_{11}\frac{\partial^2U_1}{\partial t^2}+\rho_{12}\frac{\partial^2U_2}{\partial t^2}+b\left(\frac{\partial U_1}{\partial t}-\frac{\partial U_2}{\partial t}\right);$$

$$\frac{\partial\sigma_{zr}}{\partial r}+\frac{\partial\sigma_{zz}}{\partial z}-\frac{\partial\pi_0}{\partial z}+\frac{\sigma_{rz}}{r}=\rho_{11}\frac{\partial^2V_1}{\partial t^2}+\rho_{12}\frac{\partial^2V_2}{\partial t^2}+b\left(\frac{\partial V_1}{\partial t}-\frac{\partial V_2}{\partial t}\right);$$

$$\frac{\partial\pi_{rr}}{\partial r}+\frac{\partial\pi_{rz}}{\partial z}+\frac{\partial\pi_0}{\partial r}+\frac{1}{r}\left(\pi_{rr}-\pi_{\theta\theta}\right)=\rho_{12}\frac{\partial^2U_1}{\partial t^2}+\rho_{22}\frac{\partial^2U_2}{\partial t^2}-b\left(\frac{\partial U_1}{\partial t}-\frac{\partial U_2}{\partial t}\right);\quad(1)$$

$$\frac{\partial\pi_{zr}}{\partial r}+\frac{\partial\pi_{zz}}{\partial z}+\frac{\partial\pi_0}{\partial z}+\frac{\pi_{rz}}{r}=\rho_{12}\frac{\partial^2V_1}{\partial t^2}+\rho_{22}\frac{\partial^2V_2}{\partial t^2}-b\left(\frac{\partial V_1}{\partial t}-\frac{\partial V_2}{\partial t}\right);$$

$$\pi_0=\rho_1/\rho\,\alpha_2\left(q_x+q_y\right)+\rho_2/\rho\alpha_2\left(\varepsilon_x+\varepsilon_y\right),$$

where $U_1,\ U_2,\ V_1,\ V_2$ – vector components of solid phases displacement; $\sigma_{rr},\ \sigma_{rz}$, $\sigma_{zr},\ \sigma_{zz},\ \sigma_{\theta\theta},\ \pi_{rr},\ \pi_{rz},\ \pi_{zr},\ \pi_{zz},\ \pi_{\theta\theta}$ – tensor tension components; $\varepsilon_{rr},\ \varepsilon_{rz},\ h_{zr}$, $\varepsilon_{zz},\ \varepsilon_{\theta\theta},\ q_{rr},\ q_{rz},\ h_{zr},\ q_{zz},\ q_{\theta\theta}$ – deformation components; $\rho_{11},\ \rho_{22}$ – effective weights components at their relative movement; $\rho_{11}+\rho_{12}=\rho_1,\ \ \rho_{22}+\rho_{12}=\rho_2$, ρ_{12} – "connecting parameter" between components of a mixture having dimension of weight or complementary apparent weight in relation to component motion; ρ_1, ρ_2 – density of phases; b – diffusion coefficient.

In roz coordinate system, in conditions of flat deformed state, relation between component of tension and deformation become:

$$\sigma_{rr} = -\alpha_2 + (\lambda_1 + 2\mu_1)\,\varepsilon_{rr} + \lambda_1(\varepsilon_{zz} + \varepsilon_{\theta\theta}) + (\lambda_3 + 2\mu_3)\,q_{rr} + \lambda_3(q_{zz} + q_{\theta\theta});$$

$$\begin{gathered}
\sigma_{\theta\theta} = -\alpha_2 + (\lambda_1 + 2\mu_1)\,\varepsilon_{\theta\theta} + \lambda_1(\varepsilon_{zz} + \varepsilon_{rr}) + (\lambda_3 + 2\mu_3)\,q_{\theta\theta} + \lambda_3(q_{zz} + q_{rr});\\
\sigma_{zz} = -\alpha_2 + (\lambda_1 + 2\mu_1)\,\varepsilon_{zz} + \lambda_1(\varepsilon_{rr} + \varepsilon_{\theta\theta}) + (\lambda_3 + 2\mu_3)\,q_{zz} + \lambda_3(q_{rr} + q_{\theta\theta});\\
\pi_{rr} = -\alpha_2 + (\lambda_2 + 2\mu_2)\,q_{rr} + \lambda_2(q_{zz} + q_{\theta\theta}) + (\lambda_4 + 2\mu_3)\,\varepsilon_{rr} + \lambda_4(\varepsilon_{zz} + \varepsilon_{\theta\theta});\\
\pi_{\theta\theta} = \alpha_2 + (\lambda_2 + 2\mu_2)\,q_{\theta\theta} + \lambda_2(q_{zz} + q_{rr}) + (\lambda_4 + 2\mu_3)\,\varepsilon_{\theta\theta} + \lambda_4(\varepsilon_{zz} + \varepsilon_{rr}); \qquad (2)\\
\pi_{zz} = \alpha_2 + (\lambda_2 + 2\mu_2)\,q_{zz} + \lambda_2(q_{rr} + q_{\theta\theta}) + (\lambda_4 + 2\mu_3)\,\varepsilon_{zz} + \lambda_4(\varepsilon_{rr} + \varepsilon_{\theta\theta});\\
\sigma_{rz} = 2\left(\mu_1\varepsilon_{rz} + \mu_3 q_{rz}\right) - \lambda_5\left(h_{rz} - h_{zr}\right);\\
\sigma_{zr} = 2\left(\mu_1\varepsilon_{rz} + \mu_3 q_{rz}\right) + \lambda_5\left(h_{rz} - h_{zr}\right);\\
\pi_{rz} = 2\left(\mu_2 q_{rz} + \mu_3\varepsilon_{rz}\right) + \lambda_5\left(h_{rz} - h_{zr}\right);\\
\pi_{zr} = 2\left(\mu_2 q_{rz} + \mu_3\varepsilon_{rz}\right) - \lambda_5\left(h_{rz} - h_{zr}\right),
\end{gathered}$$

here $\alpha_2 = \lambda_3 - \lambda_4$ – constant having dimension of tension; λ_j, μ_j, $(j = \overline{1,\,5})$ – Lame coefficients;

Following ratio connects the components of deformations and displacement:

$$\begin{gathered}
\varepsilon_{rr} = \partial U_1/\partial r, \quad \varepsilon_{\theta\theta} = U_1/r, \quad \varepsilon_{rz} = \partial U_1/\partial z + \partial V_1/\partial r, \quad \varepsilon_{zz} = \partial V_1/\partial z;\\
q_{rr} = \partial U_2/\partial r, \quad q_{\theta\theta} = U_2/r, \quad q_{rz} = \partial U_2/\partial z, +\partial V_2/\partial r, \quad q_{zz} = \partial V_2/\partial z; \qquad (3)\\
h_{rz} = \partial V_1/\partial r + \partial U_2/\partial z, \quad h_{zr} = \partial U_1/\partial z + \partial V_2/\partial r.
\end{gathered}$$

Equations of motion (1) after simple transformations in the base of formulas (2), (3) can be represented in such a way:

$$\begin{gathered}
A_{11}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{U_1}{r}\right)\right) + A_{12}\frac{\partial^2 U_1}{\partial z^2} + (A_{11} - A_{12})\frac{\partial^2 V_1}{\partial r\partial z} + B_{11}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{U_2}{r}\right)\right) +\\
+B_{12}\frac{\partial^2 U_2}{\partial z^2} + (B_{11} - B_{12})\frac{\partial^2 V_2}{\partial r\partial z} = \rho_{11}\frac{\partial^2 U_1}{\partial t^2} + \rho_{12}\frac{\partial^2 U_2}{\partial t^2} + b\left(\frac{\partial U_1}{\partial t} - \frac{\partial U_2}{\partial t}\right);\\
A_{21}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{U_2}{r}\right)\right) + A_{22}\frac{\partial^2 U_2}{\partial z^2} + (A_{21} - A_{22})\frac{\partial^2 V_2}{\partial r\partial z} + B_{21}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{U_1}{r}\right)\right) +\\
+B_{22}\frac{\partial^2 U_1}{\partial z^2} + (B_{21} - B_{22})\frac{\partial^2 V_2}{\partial r\partial z} = \rho_{12}\frac{\partial^2 U_1}{\partial t^2} + \rho_{22}\frac{\partial^2 U_2}{\partial t^2} - b\left(\frac{\partial U_1}{\partial t} - \frac{\partial U_2}{\partial t}\right);\\
A_{11}\frac{\partial^2 V_1}{\partial z^2} + A_{12}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_1}{\partial r}\right) + (A_{11} - A_{12})\frac{\partial}{\partial z}\left(\frac{\partial U_1}{\partial r} + \frac{U_1}{r}\right) + B_{11}\frac{\partial^2 V_2}{\partial z^2} +
\end{gathered}$$

$$+B_{12}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_2}{\partial r}\right)+\left(B_{11}-B_{12}\right)\frac{\partial}{\partial z}\left(\frac{\partial U_2}{\partial r}+\frac{U_2}{r}\right)=$$

$$=\rho_{11}\frac{\partial^2 V_1}{\partial t^2}+\rho_{12}\frac{\partial^2 V_2}{\partial t^2}+b\left(\frac{\partial V_1}{\partial t}-\frac{\partial V_2}{\partial t}\right); \tag{4}$$

$$A_{21}\frac{\partial^2 V_2}{\partial y^2}+A_{22}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_2}{\partial r}\right)+\left(A_{21}-A_{22}\right)\frac{\partial}{\partial z}\left(\frac{\partial U_2}{\partial r}+\frac{U_2}{r}\right)+B_{21}\frac{\partial^2 V_1}{\partial y^2}+$$

$$+B_{22}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_1}{\partial r}\right)+\left(B_{21}-B_{22}\right)\frac{\partial}{\partial z}\left(\frac{\partial U_1}{\partial r}+\frac{U_1}{r}\right)=\rho_{12}\frac{\partial^2 V_1}{\partial t^2}+\rho_{22}\frac{\partial^2 V_2}{\partial t^2}-$$

$$-b\left(\frac{\partial V_1}{\partial t}-\frac{\partial V_2}{\partial t}\right),$$

where $A_{j1}=\lambda_j+2\mu_j+(-1)^j\,\dfrac{\rho_{3-j}\,\alpha_2}{\rho}$; $A_{j2}=\mu_j-\lambda_5$; $B_{j1}=\lambda_{2+j}+2\mu_3+(-1)^j\,\dfrac{\rho_j\alpha_2}{\rho}$; $B_{j2}=\lambda_5+\mu_3$.

Entering potential functions Φ_j and Ψ_j as follows:

$$U_j=\frac{\partial\Phi_j}{\partial r}-\frac{\partial\Psi_j}{\partial z};\quad V_j=\frac{\partial\Phi_j}{\partial z}+\frac{\partial\Psi_j}{\partial r}+\frac{\Psi_j}{r}\quad (j=1,2)\,. \tag{5}$$

equations (4) can be reduced to four wave equations by equating to zero diffusion coefficient ($b=0$.)

Really, if to put $\Phi_1=\varphi$, $\Phi_2=\beta\,\varphi$, $\Psi_1=\psi$, $\Psi_2=\gamma\psi$, where the parameters β, γ are determined from algebraic equations:

$$a_1^*\beta^2+b_1^*\beta+c_1^*=0; \tag{6}$$

$$a_2^*\gamma^2+b_2^*\gamma+c_2^*=0; \tag{7}$$

and $a_1^*=B_{11}\rho_{22}-A_{21}\rho_{12}$; $b_1^*=\rho_{12}\left(B_{11}-A_{21}\right)+\rho_{22}\left(A_{11}-B_{21}\right)$; $c_1^*=A_{11}\rho_{12}-A_{21}\rho_{11}$; $a_2^*=B_{12}\rho_{22}-A_{22}\rho_{12}$; $b_2^*=\rho_{12}\left(B_2-A_{22}\right)+\rho_{22}\left(A_{21}-B_{22}\right)$; $c_2^*=A_{21}\rho_{12}-A_{22}\rho_{11}$; $\dfrac{\rho_{11}+\rho_{12}\beta}{A_{11}+B_{11}\beta}=\dfrac{\rho_{12}+\rho_{22}\beta}{A_{21}\beta+B_{21}}$; $\dfrac{\rho_{11}+\rho_{12}\gamma}{A_{12}+B_{12}\gamma}=\dfrac{\rho_{12}+\rho_{22}\gamma}{A_{22}\gamma+B_{22}}$ as the equations (6), (7) have till two radicals, $\Phi_1=\varphi_1+\varphi_2$, $\Phi_2=\beta_1\varphi_1+\beta_2\varphi_2$, $\Psi_1=\psi_1+\psi_2$, $\Psi_2=\gamma_1\psi_1+\gamma_2\psi_2$, the system (4) is disintegrated, by virtue of its linearity, on following wave equations:

$$\frac{\partial^2\varphi_j}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi_j}{\partial r}+\frac{\partial^2\varphi_j}{\partial z^2}=\frac{1}{a_j^2}\frac{\partial^2\varphi_j}{\partial t^2};$$

$$\frac{\partial^2\psi_j}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_j}{\partial r}-\frac{\psi_j}{r^2}+\frac{\partial^2\psi_j}{\partial z^2}=\frac{1}{b_j^2}\frac{\partial^2\psi_j}{\partial t^2}, \tag{8}$$

where

$$a_j^2=\frac{A_{11}+\beta_jB_{11}}{\rho_{11}+\beta_j\rho_{12}}=\frac{A_{21}+\beta_jA_{21}}{\rho_{12}+\beta_j\,\rho_{22}} \tag{9}$$

$$b_j^2 = \frac{A_{12} + \gamma_j B_{12}}{\rho_{11} + \gamma_j \rho_{12}} = \frac{B_{22} + \gamma_j A_{22}}{\rho_{12} + \gamma_j \rho_{22}} \quad (j = 1, 2)\,. \tag{10}$$

Note through a_j – speed of longitudinal wave propagation, and through b_j – speed of transversal waves in two-component medium. By the virtue of hyperbolic type of the initial system and expressions (9), (10) for definition of speeds, the elastic constants $\lambda_k \mu_k$ must be subjected to additional restrictions, given by the following inequalities: $A_{11}B_{21} - A_{21}B_{11} \neq 0$; $\mu_1\mu_2 - \mu_3^2 \neq 0$; $p_1p_2 - p_3p_4 \neq 0$; $\vartheta \neq 1$, where $p_j = \lambda_j + \mu_j$; $\vartheta = \vartheta_1 - \vartheta_2$; $\vartheta_1 = \dfrac{\alpha_2\,(\rho_2 p_2 - \rho_1 p_4)}{\rho\,(p_1p_2 - p_3p_4)}$; $\vartheta_2 = \dfrac{\alpha_2\,(\rho_1 p_1 - \rho_2 p_3)}{\rho\,(p_1p_2 - p_3p_4)}$.

Passing to mobile coordinate system, connected to driving load (transformation of Galilee) $\overline{z} = z + v_0 t$, $\overline{r} = r$, — $\overline{t} = t$, and for convenience omitting dashes, the system of equation (8) can be reduced to the form:

$$\frac{\partial^2 \varphi_j}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi_j}{\partial r} = \delta_j^2 \frac{\partial^2 \varphi_j}{\partial t^2}; \qquad \frac{\partial^2 \psi_j}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_j}{\partial r} - \frac{\psi_j}{r^2} = \varepsilon_j^2 \frac{\partial^2 \psi_j}{\partial t^2}, \tag{11}$$

where $\delta_j^2 = \dfrac{v_0^2}{a_j^2} - 1$, $\varepsilon_j^2 = \dfrac{v_0^2}{b_j^2} - 1$.

The equation (11) describe axisymmetrical motion of elastic two-component medium relative to mobile coordinate system under impact of concentrated source. And it demonstrates, that in this medium the disturbance are diffused in form of two longitudinal waves and two transversal waves.

Let's search the automodel solutions of equation (11) in the form:

$$\varphi_j = z^2 f_j(\xi), \quad \psi_j = z^2 g_j(\xi), \tag{12}$$

where $\xi = r/z$ – dimensionless coordinate. In the base of (12) equations (11) are reduced to ordinary second-order differential equations concerning unknowns functions $f_j(\xi)$, and $g_j(\xi)$:

$$(1 - \delta_j^2\xi^2)\xi f_j'' + (1 + 2\delta_j^2\xi^2) f_j' - 2\delta_j^2 \xi f_j = 0; \tag{13}$$

$$(1 - \varepsilon_j^2\xi^2)\xi^2 g_j'' + (1 + 2\varepsilon_j^2\xi^2)\xi f_j' - (1 + 2\varepsilon_j^2\xi^2) f_j = 0 \quad (j = 1, 2)\,. \tag{14}$$

The particular solutions of equation (13) are $f_j = \delta_j^2\xi^2 + 2$ and the general solutions take a form:

$$f_j = (\delta_j^2\xi^2 + 2)\left\{ C_{1j} + C_{2j}\left[\frac{3}{4}\frac{\sqrt{1 - \delta_j^2\xi^2}}{\delta_j^2\xi^2 + 2} + \frac{1}{8}\ln\frac{1 - \sqrt{1 - \delta_j^2\xi^2}}{1 + \sqrt{1 - \delta_j^2\xi^2}} \right] \right\}. \tag{15}$$

On fronts of longitudinal waves ($\xi_{1j} = 1/\delta_j$) functions $f_j(\xi) = 0$, therefore constant of integration $C_{1j} = 0$.

The particular solutions of equation (14) are $g_j = \varepsilon_j^2\xi$, and general solutions take form:

$$g_j = \varepsilon_j^2\xi^2\left\{ C_{3j} + C_{4j}\left[-\sqrt{1 - \varepsilon_j^2\xi^2} - \frac{1}{2}\frac{\sqrt{1 - \varepsilon_j^2\xi^2}}{\varepsilon_j^2\xi^2} - \frac{3}{4}\ln\frac{1 - \sqrt{1 - \varepsilon_j^2\xi^2}}{1 + \sqrt{1 - \varepsilon_j^2\xi^2}} \right] \right\}. \tag{16}$$

The functions $g_j(\xi)$ are determined in areas, $0 \le \xi \le 1/\varepsilon_j$ and they are equal to zero on transversal wave fronts of $(\xi_{2j} = 1/\varepsilon_j)$, therefore, $C_{3j} = 0$.

Thus, the tensioned and cinematic state of medium is determined with consideration of superposition of the corresponding waves. In the field of volume dilatation searched functions are determined only through the functions $f_j(\xi)$, but in the area compression-displacement through the functions $f_j(\xi)$and $g_j(\xi)$, in dependence of the corresponding waves degree's superposition.

Under general solutions (15) and (16), according to equality (12), the potential functions φ_j and ψ_j, necessary for finding of displacement components U_j and V_j from (5) are determined; by means of the last, components of deformation from (3) are received, finally giving from (2) components of tension influencing on medium.

The components of displacement U_j and V_j in the field of volume expansion are determined by the following formulas (we shall suppose $C_{3j} = C_{4j} = C_j$):

$$\frac{U_i}{z} = \sum_{j=1}^{2} C_j \beta_j^{i-1} \left(\frac{\eta_j}{2\xi} + \frac{1}{4}\delta_j^2 \xi \ln \frac{1-\eta_j}{1+\eta_j} \right);$$

$$\frac{V_i}{z} = \sum_{j=1}^{2} C_j \beta_j^{i-1} \left(\eta_j + \frac{1}{2} \ln \frac{1-\eta_j}{1+\eta_j} \right) \quad (i = 1, 2), \tag{17}$$

where $\eta_j = \sqrt{1 - \delta_j^2 \xi^2}$.

The components of tension in these areas look like:

$$\sigma_{rr} = -\alpha_2 + \sum_{j=1}^{2}\sum_{i=1}^{2} C_j \beta_j^{i-1} \left\{ \frac{1}{2} \left[(\lambda_{2i-1} + \mu_{2i-1})\, \delta_j^2 + \lambda_{2i-1} \right] \ln \frac{1-\eta_j}{1+\eta_j} - \mu_{2i-1} \frac{\eta_j}{\xi^2} \right\};$$

$$\sigma_{\theta\theta} = -\alpha_2 + \sum_{j=1}^{2}\sum_{i=1}^{2} C_j \beta_j^{i-1} \left\{ \frac{1}{2} \left[(\lambda_{2i-1} + \mu_{2i-1})\, \delta_j^2 + \lambda_{2i-1} \right] \ln \frac{1-\eta_j}{1+\eta_j} + \mu_{2i-1} \frac{\eta_j}{\xi^2} \right\};$$

$$\sigma_{zz} = -\alpha_2 + \sum_{j=1}^{2}\sum_{i=1}^{2} C_j \beta_j^{i-1} \left\{ \frac{1}{2} \left[\lambda_{2i-1}\delta_j^2 + (\lambda_{2i-1} + 2\mu_{2i-1}) \right] \ln \frac{1-\eta_j}{1+\eta_j} \right\};$$

$$\sigma_{rz} + \sigma_{zr} = 8 \sum_{j=1}^{2}\sum_{i=1}^{2} C_j \beta_j^{i-1} \mu_{2i-1} \frac{\eta_j}{\xi};$$

$$\sigma_{rz} - \sigma_{zr} = \frac{\lambda_5}{2} \sum_{j=1}^{2} C_j (-1)^{j-1} \beta_j^{j-1} \delta_j^2 \xi \ln \frac{1-\eta_j}{1+\eta_j}; \tag{18}$$

$$\pi_{rr} = \alpha_2 + \sum_{j=1}^{2}\sum_{i=1}^{2} C_{3-j} \beta_j^{i-1} \left\{ \frac{1}{2} \left[(\lambda_{2i} + \mu_{2i})\, \delta_{3-i}^2 + \lambda_{2i} \right] \ln \frac{1-\eta_{3-j}}{1+\eta_{3-j}} - \mu_{2i} \frac{\eta_{3-j}}{\xi^2} \right\};$$

$$\pi_{\theta\theta} = \alpha_2 + \sum_{j=1}^{2}\sum_{i=1}^{2} C_{3-j} \beta_j^{i-1} \left\{ \frac{1}{2} \left[(\lambda_{2i} + \mu_{2i})\, \delta_{3-j}^2 + \lambda_{2i} \right] \ln \frac{1-\eta_{3-j}}{1+\eta_{3-j}} + \mu_{2i} \frac{\eta_{3-j}}{\xi^2} \right\};$$

$$\pi_{zz} = \alpha_2 + \sum_{j=1}^{2}\sum_{i=1}^{2} C_{3-j}\beta_j^{i-1}\left\{\frac{1}{2}\left[\lambda_{2i}\delta_{3-j}^2 + (\lambda_{2i}+2\mu_{2i})\right]\ln\frac{1-\eta_{3-j}}{1+\eta_{3-j}}\right\};$$

$$\pi_{rz} + \pi_{zr} = 8\sum_{j=1}^{2}\sum_{i=1}^{2} C_{3-j}\beta_j^{i-1}\mu_{2i}\frac{\eta_{3-j}}{\xi};$$

$$\pi_{zr} - \pi_{rz} = \frac{\lambda_5}{2}\sum_{j=1}^{2} C_j(-1)^{j-1}\beta_j^{j-1}\delta_j^2\xi\ln\frac{1-\eta_j}{1+\eta_j}.$$

In remaining areas, where medium is affected by both longitudinal and transversal waves, components of displacement and tension expressed through $f_j(\xi)$ and $g_j(\xi)$, receive such a form:

$$\frac{U_i}{z} = \sum_{j=1}^{2} C_j\left\{\beta_j^{i-1}\left[\frac{\eta_j}{2\xi} + \frac{\delta_j^2\xi}{4}\ln\frac{1-\eta_j}{1+\eta_j}\right] + \frac{3\gamma_j^{i-1}}{2}\left[\zeta_j + \frac{\varepsilon_j\xi}{2}\ln\frac{1-\zeta_j}{1+\zeta_j}\right]\right\};$$

$$\frac{V_i}{z} = \sum_{j=1}^{2} C_j\left\{\beta_j^{i-1}\left[\eta_j + \frac{1}{2}\ln\frac{1-\eta_j}{1+\eta_j}\right] - 3\gamma_j^{i-1}\left[\varepsilon_j\zeta_j + \frac{\varepsilon_j}{2}\ln\frac{1-\zeta_j}{1+\zeta_j}\right]\right\};$$

$$\sigma_{rr} = -\alpha_2 + \sum_{j=1}^{2}\sum_{i=1}^{2} C_j\left\{\beta_j^{i-1}\left[\frac{1}{2}\left[(\lambda_{2i-1}+\mu_{2i-1})\,\delta_j^2 + \lambda_{2i-1}\right]\ln\frac{1-\eta_j}{1+\eta_j} - \mu_{2i-1}\frac{\eta_j}{\xi^2}\right]-\right.$$
$$\left.-3\gamma_j^{i-1}\left[\frac{\zeta_j}{\varepsilon_j^2\xi^2} - \frac{\varepsilon_j}{2}\ln\frac{1-\zeta_j}{1+\zeta_j}\right]\right\};$$

$$\sigma_{\theta\theta} = -\alpha_2 + \sum_{j=1}^{2}\sum_{i=1}^{2} C_j\left\{\beta_j^{i-1}\left[\frac{1}{2}\left[(\lambda_{2i-1}+\mu_{2i-1})\,\delta_j^2 + \lambda_{2i-1}\right]\ln\frac{1-\eta_j}{1+\eta_j} + \mu_{2i-1}\frac{\eta_j}{\xi^2}\right]-\right.$$
$$\left.-3\gamma_j^{i-1}\left[\frac{\zeta_j}{\varepsilon_j^2\xi^2} - \frac{\varepsilon_j}{2}\ln\frac{1-\zeta_j}{1+\zeta_j}\right]\right\};$$

$$\sigma_{zz} = -\alpha_2 + \sum_{j=1}^{2}\sum_{i=1}^{2} C_j\left\{\beta_j^{i-1}\left[\frac{1}{2}\left[\lambda_{2i-1}\delta_j^2 + (\lambda_{2i-1}+2\mu_{2i-1})\right]\ln\frac{1-\eta_j}{1+\eta_j}\right]+\right.$$
$$\left.+\frac{3\gamma_j^{i-1}}{2}\left[\lambda_{2i-1}\frac{1+\varepsilon_j\xi-\varepsilon_j^2\xi^2}{\xi\zeta_j} - 2\mu_{2i-1}\varepsilon_j\ln\frac{1-\zeta_j}{1+\zeta_j}\right]\right\}; \qquad (19)$$

$$\sigma_{rz} + \sigma_{zr} = 8\sum_{j=1}^{2}\sum_{i=1}^{2} C_j\mu_{2i-1}\left\{\beta_j^{i-1}\left[\frac{2\eta_j}{\xi} + \frac{3\left(1-\delta_j^2\right)\eta_j}{\delta_j\xi}\right]+\right.$$
$$\left.+3\gamma_j^{i-1}\left[\frac{\varepsilon_j\zeta_j}{\xi} + \frac{\left(1-\varepsilon_j\xi\right)}{2\zeta_j}\right]\right\};$$

$$\sigma_{rz}-\sigma_{zr}=\frac{\lambda_5}{2}\sum_{j=1}^{2}C_j(-1)^{j+1}\left\{\beta_j^{j-1}\delta_j^2\xi\ln\frac{1-\eta_j}{1+\eta_j}+3\gamma_j^{j-1}\left[\frac{\varepsilon_j\zeta_j}{\xi}+\frac{\left(1-\varepsilon_j\xi\right)}{2\zeta_j}\right]\right\};$$

$$\pi_{rr}=\alpha_2+\sum_{j=1}^{2}\sum_{i=1}^{2}C_{3-j}\left\{\beta_j^{i-1}\left[\frac{1}{2}\left[(\lambda_{2i}+\mu_{2i})\,\delta_{3-j}^2+\lambda_{2i}\right]\ln\frac{1-\eta_{3-j}}{1+\eta_{3-j}}-\mu_{2i}\frac{\eta_{3-j}}{\xi^2}\right]-\right.$$

$$\left.-3\gamma_j^{i-1}\left[\frac{\zeta_{3-j}}{\varepsilon_{3-j}^2\xi^2}-\frac{\varepsilon_{3-j}}{2}\ln\frac{1-\zeta_{3-j}}{1+\zeta_{3-j}}\right]\right\};$$

$$\pi_{\theta\theta}=\alpha_2+\sum_{j=1}^{2}\sum_{i=1}^{2}C_{3-j}\left\{\beta_j^{i-1}\left[\frac{1}{2}\left[(\lambda_{2i}+\mu_{2i})\,\delta_{3-j}^2+\lambda_{2i}\right]\ln\frac{1-\eta_{3-j}}{1+\eta_{3-j}}+\mu_{2i}\frac{\eta_{3-j}}{\xi^2}\right]-\right.$$

$$\left.-3\gamma_j^{i-1}\left[\frac{\zeta_{3-j}}{\varepsilon_{3-j}^2\xi^2}-\frac{\varepsilon_{3-j}}{2}\ln\frac{1-\zeta_{3-j}}{1+\zeta_{3-j}}\right]\right\};$$

$$\pi_{zz}=\alpha_2+\sum_{j=1}^{2}\sum_{i=1}^{2}C_{3-j}\left\{\beta_j^{i-1}\left[\frac{1}{2}\left[\lambda_{2i}\delta_j^2+(\lambda_{2i}+2\mu_{2i})\right]\ln\frac{1-\eta_{3-j}}{1+\eta_{3-j}}\right]+\right.$$

$$\left.+\frac{3\gamma_j^{i-1}}{2}\left[\lambda_{2i}\frac{1+\varepsilon_{3-j}\xi-\varepsilon_{3-j}^2\xi^2}{\xi\zeta_{3-j}}-2\mu_{2i}\varepsilon_{3-j}\ln\frac{1-\zeta_{3-j}}{1+\zeta_{3-j}}\right]\right\};$$

$$\pi_{rz}+\pi_{zr}=8\sum_{j=1}^{2}\sum_{i=1}^{2}C_{3-j}\mu_{2i}\left\{\beta_j^{j-1}\frac{2\eta_{3-j}}{\xi}+\gamma_j^{j-1}\frac{3\left(1-\varepsilon_{3-j}^2\right)\zeta_{3-j}}{\varepsilon_{3-j}\xi}\right\};$$

$$\pi_{zr}-\pi_{rz}=\frac{\lambda_5}{2}\sum_{j=1}^{2}C_j(-1)^{j-1}\left\{\beta_j^{j-1}\delta_j^2\xi\ln\frac{1-\eta_j}{1+\eta_j}+3\gamma_j^{j-1}\left[\frac{\varepsilon_j\zeta_j}{\xi}+\frac{\left(1-\varepsilon_j\xi\right)}{2\zeta_j}\right]\right\},$$

here $\zeta_j=\sqrt{1-\varepsilon_j^2\xi^2}$.

From the general solution is evident, that in origin ($\xi=0$) components of displacement and tension tend to infinite and arbitrary constants, which must be determined from boundary conditions, remain unknown. Such automodel solutions can correspond to processes of burning and detonation, in which allocated energy grows on time.

The analysis of obtained above analytical solutions demonstrates, that the main predominant displacement and tension are watched in areas, where medium affected by longitudinal and transversal waves simultaneously, so the main part of concentrated load motion goes to compression and in the field of their superposition.

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On the structure of finite medial quasigroups

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Abstract. Some variants of Toyoda, Murdoch and Ježek-Kepka theorems on medial quasigroups are given. The structure of finite medial quasigroups is described.

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We shall use basic terms and concepts from books [1–3]. To economize time of readers we recall some known facts.

A quasigroup $(Q, \cdot)$ with the identity

$$xy \cdot uv = xu \cdot yv \tag{1}$$

is called medial. Crucial Toyoda theorem ([1, 2, 4–6]) says that every medial quasigroup $(Q, \cdot)$ can be presented in the form:

$$x \cdot y = \varphi x + \psi y + a, \tag{2}$$

where $(Q, +)$ is an abelian group, φ, ψ are automorphisms of $(Q, +)$ such that $\varphi\psi = \psi\varphi$, $x, y \in Q$, a is some fixed element from the set Q.

In view of Toyoda theorem the theory of medial quasigroups is very close to the theory of abelian groups but it is not exactly the theory of abelian groups. For example, a very simple for abelian groups fact, that every simple abelian group is finite, was proved for medial quasigroups only in 1977 [7].

Medial quasigroups as well as other classes of quasigroups isotopic to groups give us a possibility to construct quasigroups with preassigned properties. Often these properties can be expressed on the language of properties of groups and components of isotopy.

As usual, $L_a : L_a x = a \cdot x$, $R_a : R_a x = x \cdot a$ are respectively left and right translation of a quasigroup $(Q, \cdot)$. An element d such that $d \cdot d = d$ is called an idempotent element of a binary quasigroup $(Q, \cdot)$. By ε we mean the identity permutation.

A quasigroup $(Q, \circ)$ is called an *isotope of a quasigroup* $(Q, \cdot)$ if there exist permutations α, β, γ of the set Q such that $x \circ y = \gamma^{-1}(\alpha x \cdot \beta y)$ for all $x, y \in Q$. If $(Q, \circ) = (Q, \cdot)$, then the triple (α, β, γ) is an *autotopy* of the quasigroup $(Q, \cdot)$, the permutation γ is a *quasiautomorphism* of the quasigroup $(Q, \cdot)$. An isotopy of the form $(\varepsilon, \varepsilon, \gamma)$ is called a *principal isotopy* [1–3].

A quasigroup $(Q, \cdot)$ with the identity $x \cdot x = x$ is called an *idempotent quasigroup*. A quasigroup $(Q, \cdot)$ with the identity $x \cdot x = e$, where e is a fixed element of the set Q, is called an *unipotent quasigroup*.

Any quasiautomorphism γ of a group $(Q,+)$ has the form $R_a^+\beta$, where $a \in Q$, $\beta \in Aut(Q,+)$ ([1]; [2], p.24). Obviously $\beta 0 = 0$, where, as usual, 0 denotes the identity element of $(Q,+)$.

Medial quasigroups (as well as any other quasigroup class) can be divided into 2 classes: 1) quasigroups that have one or more idempotent elements; 2) quasigroups that have not any idempotent.

Theorem 1. *Conditions (i) and (ii) are equivalent:*

(i) $(Q,\cdot)$ *is a medial quasigroup that has idempotent element* 0*;*

(ii) there exist an abelian group $(Q,+)$ *with the identity element* 0*, two its commuting automorphisms* α, β *such that* $x \cdot y = \alpha x + \beta y + a$ *for all* $x, y \in Q$*, where* $-a \in (\alpha + \beta - \varepsilon)Q$*.*

Proof. (i) $\Longrightarrow$ (ii). LP-isotope $(R_0^{-1}, L_0^{-1}, \varepsilon)$ of quasigroup $(Q,\cdot)$ is a loop $(Q,+)$ with the identity element $0 \cdot 0 = 0$, i.e. $x + y = R_0^{-1}x \cdot L_0^{-1}y$ [1]. Then $x \cdot y = R_0 x + L_0 y, R_0 0 = 0, L_0 0 = 0$. Let $R_0 = \alpha, L_0 = \beta$. Therefore $x \cdot y = \alpha x + \beta y$. So we can rewrite medial identity in terms of the operation $+$ in the following way.

$$\alpha(\alpha x + \beta y) + \beta(\alpha u + \beta v) = \alpha(\alpha x + \beta u) + \beta(\alpha y + \beta v). \tag{3}$$

If we take $x = y = v = 0$ in (3), then we obtain $\alpha\beta y = \beta\alpha y$, i.e.

$$\alpha\beta = \beta\alpha. \tag{4}$$

By $u = v = 0$ in (3) we have $\alpha(\alpha x + \beta y) = \alpha^2 x + \beta\alpha y =^{(4)} \alpha^2 x + \alpha\beta y$. Therefore $\alpha \in Aut(Q,+)$.

If we substitute in (3) $x = y = 0$, then $\beta(\alpha u + \beta v) = \alpha\beta u + \beta^2 v =^{(4)} \beta\alpha u + \beta^2 v$, $\beta \in Aut(Q,+)$.

By $x = v = 0$ equality (3) takes the form $\alpha\beta y + \beta\alpha u = \alpha\beta u + \beta\alpha y$. Since $\alpha\beta = \beta\alpha$, we have $\alpha\beta y + \alpha\beta u = \alpha\beta u + \alpha\beta y$. Therefore $(Q,+)$ is a commutative loop.

Let $v = 0$ in relation (3). Since $\alpha, \beta \in Aut(Q,+), \alpha\beta = \beta\alpha$, further we obtain $(\alpha^2 x + \alpha\beta y) + \alpha\beta u = (\alpha^2 x + \alpha\beta u) + \alpha\beta y$. Then $(\alpha\beta y + \alpha^2 x) + \alpha\beta u = \alpha\beta y + (\alpha^2 x + \alpha\beta u)$, since $(Q,+)$ is a commutative loop. From the last equality we have that $(Q,+)$ is associative. Therefore $(Q,+)$ is an abelian group. It is easy to see that $0 \in (\alpha + \beta - \varepsilon)Q$.

(ii) $\Longrightarrow$ (i).

If conditions (ii) are fulfilled, then it is easy to check that the identity (1) holds. Indeed, $\alpha(\alpha x + \beta y + a) + \beta(\alpha u + \beta v + a) + a = \alpha(\alpha x + \beta u + a) + \beta(\alpha y + \beta v + a) + a$, $\alpha^2 x + \alpha\beta y + \alpha a + \beta\alpha u + \beta^2 v + \beta a + a = \alpha^2 x + \alpha\beta u + \alpha a + \beta\alpha y + \beta^2 v + \beta a + a$, $\alpha\beta y + \alpha\beta u = \alpha\beta u + \alpha\beta y$, $0 = 0$.

A quasigroup of such kind has at least one idempotent element. Indeed, let $-a = \alpha d + \beta d - d$, i.e. $\alpha d + \beta d = d - a$. Then $d \cdot d = \alpha d + \beta d + a = d - a + a = d$.

The theorem is proved.

From the proof of Theorem 1 follows the following

Corollary 1. *Any medial quasigroup $(Q,\cdot)$ with an idempotent element 0 can be presented in the form: $x\cdot y = \alpha x + \beta y$, where $(Q,+)$ is an abelian group with the identity element 0 and α,β are commuting automorphisms of the group $(Q,+)$.*

Remark. Equivalence of conditions (i) and (ii) of Theorem 1 it is possible to deduce from results of book [8] (3.1.4. Proposition).

Theorem 2. *Conditions (i) and (ii) are equivalent:*

(i) $(Q,\cdot)$ is a medial quasigroup that has not any idempotent element;

(ii) there exist an abelian group $(Q,+)$, its automorphisms α, φ, $\alpha\varphi=\varphi\alpha$, an element $a\in Q$, $-a\notin(\alpha+\varphi-\varepsilon)Q$ such that $x\cdot y=\alpha x+\varphi y+a$ for all $x,y\in Q$.

(i) $\Longrightarrow$ (ii). By proving this implication in the main we follow the book [2]. Let us consider a LP-isotope $(Q,+)$ of a medial quasigroup $(Q,\cdot)$ of the form: $x+y=R^{-1}_{r(0)}\cdot L^{-1}_0$ where $0\cdot r(0)=0$, i.e. $r(0)$ is a right local identity element of the element 0. This LP-isotope $(Q,+)$ is a loop with the identity element $0\cdot r(0)=0$. Denote $R_{r(0)}$ by α and L_0 by β. We remark that $R_{r(0)}0=0$, then $\alpha 0=0$.

Using our notations we can write medial identity in the following form:

$$\alpha(\alpha x+\beta y)+\beta(\alpha u+\beta v)=\alpha(\alpha x+\beta u)+\beta(\alpha y+\beta v). \tag{5}$$

By $x=0, y=\beta^{-1}0$ from (5) we have

$$\beta(\alpha u+\beta v)=\alpha\beta u+\beta(\alpha\beta^{-1}0+\beta v). \tag{6}$$

Therefore the permutation β is a quasiautomorphism of the loop $(Q,+)$.

By $u=0, v=\beta^{-2}0$ in (5) we have

$$\alpha(\alpha x+\beta y)=\alpha(\alpha x+\beta 0)+\beta(\alpha y+\beta^{-1}0) \tag{7}$$

and we obtain that the permutation α is a quasiautomorphism of the loop $(Q,+)$.

If we use equalities (6) and (7) in (5), then we have

$$(\alpha R_{\beta 0}\alpha x+\beta R_{\beta^{-1}0}\alpha y)+(\alpha\beta u+\beta L_{\alpha\beta^{-1}0}\beta v)=$$

$$(\alpha R_{\beta 0}\alpha x+\beta R_{\beta^{-1}0}\alpha u)+(\alpha\beta y+\beta L_{\alpha\beta^{-1}0}\beta v). \tag{8}$$

If we change in equality (8) the element x by the element $\alpha^{-1}R^{-1}_{\beta 0}\alpha^{-1}x$, the element y by $\alpha^{-1}R^{-1}_{\beta^{-1}0}\beta^{-1}y$, the element u by the element $\beta^{-1}\alpha^{-1}u$, the element v by the element $\beta^{-1}L^{-1}_{\alpha\beta^{-1}0}\beta^{-1}v$, then we have

$$(x+y)+(u+v)=(x+\beta R_{\beta^{-1}0}\alpha\beta^{-1}\alpha^{-1}u)+(\alpha\beta\alpha^{-1}R^{-1}_{\beta^{-1}0}\beta^{-1}y+v).$$

If we take $u=0$ in the last equality, then we have

$$(x+y)+v=(x+\beta R_{\beta^{-1}0}\alpha\beta^{-1}0)+(\alpha\beta\alpha^{-1}R^{-1}_{\beta^{-1}0}\beta^{-1}y+v). \tag{9}$$

If we take in (9) $v=0$, then we obtain $x+y=(x+r)+\alpha\beta\alpha^{-1}R^{-1}_{\beta^{-1}0}\beta^{-1}y$ where $r=\beta R_{\beta^{-1}0}\alpha\beta^{-1}0$ is a fixed element of the set Q.

If we change in equality (9) $x + y$ by the right side of the last equality, then we have

$$((x + r) + \alpha\beta\alpha^{-1}R^{-1}_{\beta^{-1}0}\beta^{-1}y) + v = (x + r) + (\alpha\beta\alpha^{-1}R^{-1}_{\beta^{-1}0}\beta^{-1}y + v).$$

From the last equality it follows that the loop $(Q, +)$ is associative, i.e. is a group.

Since α is quasiautomorphism of the group and $\alpha 0 = 0$, we have that the permutation α is an automorphism of the group $(Q, +)$. The permutation β has the form $\beta = R_a\varphi$ where $\varphi \in Aut(Q, +)$.

Then we can rewrite the medial identity in the form $\alpha^2x + \alpha\varphi y + \alpha a + \varphi\alpha u + \varphi^2v + \varphi a + a = \alpha^2x + \alpha\varphi u + \alpha a + \varphi\alpha y + \varphi^2v + \varphi a + a$ and, after the reduction in the last equality, we obtain

$$\alpha\varphi y + \alpha a + \varphi\alpha u = \alpha\varphi u + \alpha a + \varphi\alpha y. \tag{10}$$

From the last equality by $u = 0$ we have $\alpha\varphi y + \alpha a = \alpha a + \varphi\alpha y$ and by $y = 0$ we have $\alpha a + \varphi\alpha u = \alpha\varphi u + \alpha a$. Using these last equalities we can rewrite equality (10) in the form $\alpha a + \varphi\alpha y + \varphi\alpha u = \alpha a + \varphi\alpha u + \varphi\alpha y$. Hence $\varphi\alpha y + \varphi\alpha u = \varphi\alpha u + \varphi\alpha y$, $(Q, +)$ is an abelian group.

Then from equality $\alpha\varphi y + \alpha a = \alpha a + \varphi\alpha y$ it follows that $\alpha\varphi y = \varphi\alpha y$. Therefore $x \cdot y = \alpha x + \varphi y + a$, where $(Q, +)$ is an abelian group, α, φ are automorphisms of $(Q, +)$ such that $\alpha\varphi = \varphi\alpha$.

Now we must only demonstrate that the element $-a \notin (\alpha + \varphi - \varepsilon)Q$. Let us suppose the inverse. Let medial quasigroup $(Q, \cdot)$ have an idempotent element, for example, let $u{\cdot}u = u$. Then $\alpha u{+}\varphi u{+}a = u$, therefore $-a = \alpha u{+}\varphi u{-}u = (\alpha{+}\varphi{-}\varepsilon)u$ hence $-a \in (\alpha + \varphi - \varepsilon)Q$. We received a contradiction. Our assumption is not true. Hence, if medial quasigroup $(Q, \cdot)$ has not any idempotent element, then $-a \notin (\alpha + \varphi - \varepsilon)Q$.

(ii) $\Longrightarrow$ (i). This implication can be checked easy and we omit the proof of this implication. The theorem is proved.

The following theorem on the structure of finite medial quasigroups has been proved by D.C. Murdoch. We give Murdoch theorem in a slightly modernized form [9].

For a quasigroup $(Q, \cdot)$ we define the map s: $s(x) = x \cdot x$ for all $x \in Q$. As usual, $s^2(x) = s(s(x))$ and so on. For any medial quasigroup $(Q, \cdot)$ the map s is an endomorphism of this quasigroup, indeed, $s(xy) = xy \cdot xy = xx \cdot yy = s(x) \cdot (y)$.

Definition 1. *A quasigroup $(Q, \cdot)$ is called an unipotently-solvable quasigroup of degree m if there exists the following finite chain of unipotent quasigroups:*

$$Q/s(Q), s(Q)/s^2(Q), \ldots, s^m(Q)/s^{m+1}(Q),$$

where the number m is the minimal number with the property $|s^m(Q)/s^{m+1}(Q)| = 1$ [9].

Theorem 3. *Any finite medial quasigroup $(Q, \cdot)$ is isomorphic to the direct product of a medial unipotently-solvable quasigroup $(Q_1, \circ)$ and a principal isotope of the form $(\varepsilon, \varepsilon, \gamma)$ of a medial idempotent quasigroup $(Q_2, *)$, where $\gamma \in Aut(Q_2, *)$.*

It is clear that Theorem 3 reduces the study of the structure of finite medial quasigroups to the study of the structure of finite medial unipotent and idempotent quasigroups.

We notice, for any unipotent quasigroup $(Q, \cdot)$ with idempotent element e we have $s(Q) = e$, for any idempotent quasigroup $(Q, \cdot)$ we have $s = \varepsilon$. Therefore, in these cases we cannot say anything on the structure of medial unipotent and medial idempotent quasigroup using the endomorphism s.

As it has been mentioned above, simple medial quasigroups were described by J. Ježek and T. Kepka in [7]. We recall some definitions. As usual, a binary relation θ is an equivalence relation on Q if and only if θ is a reflexive, symmetric and transitive subset of Q^2. An equivalence θ is a *congruence* of a quasigroup $(Q, \cdot)$ if and only if the following implications are true: $a\theta b \Longrightarrow ac\theta bc$ and $a\theta b \Longrightarrow ca\theta cb$ for all $a, b, c \in Q$.

A congruence θ of a quasigroup $(Q, \cdot)$ is called *normal* if the following implications are true: $ac\theta bc \Longrightarrow a\theta b$, $ca\theta cb \Longrightarrow a\theta b$ for all $a, b, c \in Q$ [1,3].

A quasigroup $(Q, \cdot)$ is *simple* if its only normal congruences are the diagonal $\hat{Q} = \{(q, q) \mid q \in Q\}$ and $Q \times Q$ [1,3].

We give Ježek-Kepka Theorem in the following form [10].

Theorem 4. *If a medial quasigroup $(Q, \cdot)$ of the form $x \cdot y = \alpha x + \beta y + a$ over an abelian group $(Q, +)$ is simple, then*

1. *the group $(Q, +)$ is the additive group of a finite Galois field $GF(p^k)$;*

2. *the group $< \alpha, \beta >$ is the multiplicative group of the field $GF(p^k)$ in the case $k > 1$, the group $< \alpha, \beta >$ is any subgroup of the group $Aut(Z_p, +)$ in the case $k = 1$;*

3. *the quasigroup $(Q, \cdot)$ in the case $|Q| > 1$ can be quasigroup from one of the following disjoint quasigroup classes:*

   (a) *$\alpha + \beta = \varepsilon, a = 0$; in this case the quasigroup $(Q, \cdot)$ is an idempotent quasigroup;*

   (b) *$\alpha + \beta = \varepsilon$ and $a \neq 0$; in this case the quasigroup $(Q, \cdot)$ does not have any idempotent element, the quasigroup $(Q, \cdot)$ is isomorphic to the quasigroup $(Q, *)$ with the form $x * y = \alpha x + \beta y + 1$ over the same abelian group $(Q, +)$;*

   (c) *$\alpha + \beta \neq \varepsilon$; in this case the quasigroup $(Q, \cdot)$ has exactly one idempotent element, the quasigroup $(Q, \cdot)$ is isomorphic to the quasigroup $(Q, \circ)$ of the form $x \circ y = \alpha x + \beta y$ over the group $(Q, +)$.*

Proposition 1. *Any medial quasigroup $(Q, \circ)$ of the form $x \circ y = \alpha x + \beta y$ over an abelian group $(Q, +)$, where $\alpha + \beta \neq \varepsilon$, is either an unipotent quasigroup, or it is a principal isotope of the medial idempotent quasigroup $(Q, \cdot)$ of the form $x \cdot y = (\alpha + \beta)^{-1}(\alpha x + \beta y)$.*

Proof. If we suppose, that $(\alpha+\beta)x = 0$ for all $x \in Q$, where 0 denotes zero element of the group $(Q,+)$, then $x \circ x = \alpha x + \beta x = (\alpha+\beta)x = 0$ for all $x \in Q$.

If $\alpha + \beta \neq 0$, then there exists an element μ of the group $Aut(Q,+)$ such that $\mu(\alpha+\beta) = \varepsilon$, i.e. $\mu = (\alpha+\beta)^{-1}$. Therefore, $x \cdot x = (\alpha+\beta)^{-1}(\alpha x + \beta x) = (\alpha+\beta)^{-1}(\alpha+\beta)x = x$ for all $x \in Q$.

The quasigroup $(Q,\cdot)$ is medial ([12], Theorem 25). We repeat the proof of Theorem 25: since $\mu\alpha + \mu\beta = \varepsilon$, we have $\mu\alpha\mu\beta = \mu\alpha(\varepsilon - \mu\alpha) = \mu\alpha - (\mu\alpha)^2 = (\varepsilon - \mu\alpha)\mu\alpha = \mu\beta\mu\alpha$. The proposition is proved.

It is well known that the direct product of medial idempotent quasigroups is an idempotent quasigroup, a similar situation takes place for unipotent quasigroups.

Proposition 2. *If $(Q,\cdot)$ is a medial quasigroup such that $(Q,\cdot) = (Q_1,\cdot_1)\times(Q_2,\cdot_2)$ and the forms of quasigroups $(Q,\cdot)$, $(Q_1,\cdot_1)$ and $(Q_2,\cdot_2)$ are defined over groups $(Q,+)$, $(Q_1,+_1)$ and $(Q_2,+_2)$ respectively, then $(Q,+) \cong (Q_1,+_1)\times(Q_2,+_2)$* [9].

Example 1. *There exist directly irreducible finite idempotent medial quasigroups, finite unipotent medial quasigroups.*

Proof. We denote by $(Z_9,+)$ the additive group of residues modulo 9. The quasigroup $(Z_9,\circ)$ of the form $x \circ y = 2\cdot x + 8\cdot y$ is a medial idempotent quasigroup, quasigroup $(Z_9,*)$ of the form $x * y = 1\cdot x + 8\cdot y$ is a medial unipotent quasigroup.

These quasigroups are not simple. Indeed, if $Q = \{0,3,6\}$, then $(Q,\circ) \lhd (Z_9,\circ)$ and $(Q,*) \lhd (Z_9,*)$.

These quasigroups are directly irreducible. Indeed, if we suppose, that these quasigroups are directly reducible, then by Proposition 2 the group $(Z_9,+)$ is reducible into the direct product of subgroups of order 3. As it is well known [11], it is not true.

Proposition 3. *Any subquasigroup $(H,\cdot)$ of a medial quasigroup $(Q,\cdot)$ is normal, i.e. the set H coincides with an equivalence class of a normal congruence θ of the quasigroup $(Q,\cdot)$* ([12], Theorem 43).

Taking into consideration Proposition 3 we can say that in a simple medial quasigroup $(Q,\cdot)$ its only subquasigroups are one-element subquasigroups and the quasigroup $(Q,\cdot)$.

Remark. We notice, in general there exist non-simple medial quasigroups with only trivial subquasigroups. For example, the quasigroup $(Z_9,\diamond)$ with the form $x \diamond y = 2\cdot x + 8\cdot y + 1$, where $(Z_9,+)$ is the additive group of residues modulo 9, is a non-simple quasigroup without proper subquasigroups.

But situation is better for medial idempotent and medial unipotent quasigroups, since these quasigroups contain idempotent elements.

It is known ([1], p. 57; [2], p. 41), if θ is a normal congruence of a quasigroup $(Q,\cdot)$ and there exists an idempotent element e of the quasigroup $(Q,\cdot)$, then the equivalence class $\theta(e)$ forms a normal subquasigroup $(\theta(e),\cdot)$ of the quasigroup $(Q,\cdot)$.

We can summarize our remarks in the following

Proposition 4. *In an idempotent medial quasigroup or in an unipotent medial quasigroup* $(Q, \cdot)$ *any normal congruence* θ *contains at least one equivalence class* $\theta(e)$ *such that* $(\theta(e), \cdot)$ *is a normal subquasigroup of the quasigroup* $(Q, \cdot)$.

Proof. Any subquasigroup of an idempotent quasigroup is an idempotent subquasigroup, any subquasigroup of an unipotent quasigroup is an unipotent subquasigroup.

To reformulate Theorem 3 in more details we give the following

Definition 2. *We shall say that a quasigroup* $(Q, \cdot)$ *is solvable if there exists the following finite chain of quasigroups*

$$Q/Q_1, Q_1/Q_2, \dots, Q_m/Q_{m+1},$$

where the quasigroup Q_{i+1} *is a maximal normal subquasigroup of the quasigroup* Q_i *and* m *is the minimal number such that* $|Q_m/Q_{m+1}| = 1$.

Remark. Definition 2 differs from definition of solvability of groups [11].

Proposition 5. *Any finite medial idempotent quasigroup* $(Q, \cdot)$ *is solvable and any quasigroup* Q_i/Q_{i+1} *is a finite simple medial idempotent quasigroup.*

Proof. The proof it follows from Proposition 4 and the fact that the quasigroup $(Q, \cdot)$ is finite. The proposition is proved.

Proposition 6. *Any finite medial unipotent quasigroup* $(Q, \cdot)$ *is solvable and any quasigroup* Q_i/Q_{i+1} *is a finite simple medial unipotent quasigroup.*

Proof. The proof is similar to the proof of Proposition 5.

Taking into consideration Propositions 5 and 6 we can concretize Theorem 3.

Theorem 5. *Any finite medial quasigroup* $(Q, \cdot)$ *is isomorphic to the direct product of a medial unipotently-solvable quasigroup* $(Q_1, \circ)$ *and a principal isotope of a medial idempotent quasigroup* $(Q_2, *)$, *where the quasigroups* $(Q_i, \circ)/(Q_{i+1}, \circ)$ *and* $(Q_2, *)$ *are solvable for all admissible values of index* i, $\gamma \in Aut(Q_2, *)$.

Theorem 6. *A quasigroup* $(Q, \cdot)$ *of the form* $x \cdot y = \alpha x + \beta y$ *is isomorphic to a quasigroup* $(Q, *)$ *of the form* $x * y = \gamma x + \delta y$, *where* $\alpha, \beta, \gamma, \delta$ *are automorphisms of an abelian group* $(Q, +)$, *if and only if there exists an automorphism* ψ *of the group* $(Q, +)$ *such that* $\psi\alpha = \gamma\psi$, $\psi\beta = \delta\psi$.

Proof. The proof of this theorem, in fact, repeats the proof of the similar theorem from [14] and we omit it.

It is easy to see that Theorem 6 is true for medial idempotent quasigroups and for medial unipotent quasigroups.

Example 2. *We denote by* $(Z_{16}, +)$ *the additive group of residues modulo 16. The quasigroup* $(Z_{16}, \circ)$ *of the form* $x \circ y = 3 \cdot x + 15 \cdot y + 1$ *is isomorphic to the quasigroup* $(Q, *)$ *of the form* $x * y = 3 \cdot x + 15 \cdot y$.

This follows from Theorem 1. Furthermore, the quasigroup $(Q, *)$ is a quasigroup with the unique idempotent element 0, the quasigroup $(Q, *)$ is an unipotently-solvable quasigroup of degree 4, since $s^4(Q) = s^5(Q)$.

Example 3. *Let $(Z_{16}, +)$ be the additive group of residues modulo 16. The quasigroup $(Z_{16}, *)$ of the form $x * y = 1 \cdot x + 15 \cdot y$ is a solvable unipotent quasigroup of degree 3.*

Some results of this note were announced in [15].

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Optimal multicommodity flows in dynamic networks and algorithms for their finding

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Abstract. In this paper we study two basic problems related to dynamic flows: maximum multicommodity flow and the minimum cost multicommodity flow problems. We consider these problems on dynamic networks with time-varying capacities of edges. For minimum cost multicommodity flow problem we assume that cost functions, defined on edges, are nonlinear and depending on time and flow, and the demand function also depends on time. We propose algorithms for solving these dynamic problems, which are based on their reducing to static ones on a time-expanded network.

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1 Introduction

In this paper we study dynamic versions of the maximum multicommodity flow and the nonlinear minimum-cost multicommodity flow problems on networks. These problems generalize the classical static flow problems and extend some dynamic [10, 11] and control models on networks [12]. We propose algorithms for solving these dynamic problems, which are based on their reducing to static ones on a time-expanded network [7]. We also note some different methods for constructing time-expanded networks in the case of acyclic graphs.

For our problems the time is an essential component, either because the flows of some commodity take time to pass from one location to another, or because the structure of network changes over time. Classical static network flow models are known as valuable tools for different applications but they fail to capture the property of many real-life problems. To tackle this problem, we use dynamic network flow models instead of the static ones.

Dynamic flows are widely used to model network-structured, decision-making problems over time: problems in electronic communication, production and distribution, economic planning, cash flow, job scheduling, and transportation [1]) In considered dynamic models the flow passes an arc with time, it can be delayed at nodes, flow values on arcs and the network parameters can change with time. While very efficient solution methods exist for static flow problems, dynamic flow problems have proved to be more difficult to solve.

Dynamic multicommodity flows are among the most important and challenging problems in network optimization, due to the large size of these models in real world applications. Many product distribution, scheduling planning, telecommunication, transportation, communication, and management problems can be formulated and solved as multicommodity flow problems [2]. The multicommodity flow problem consists of shipping several different commodities from their respective sources to their sinks through a given network so that the total flow going through each edge does not exceed its capacity. No commodity ever transforms into another commodity, so that each one has its own flow conservation constraints, but they compete for the resources of the common network. Despite being closer to reality, dynamic multicommodity flow models, because of their complexity, have not been investigated as well as classical ones.

In this paper we study two basic problems related to dynamic flows: maximum multicommodity flow and the minimum cost multicommodity flow problems. We consider these problems on dynamic networks with time-varying capacities of edges. For minimum cost multicommodity flow problem we assume that cost functions, defined on edges, are nonlinear and depending on time and flow. Moreover, we assume that the demand function also depends on time. It is important to notice that if the edge costs do not depend on flow, then the dynamic multicommodity minimum-cost flow problem can be regarded as the network discrete optimal control problem or, equivalently, the problem of finding the shortest paths ([7]) in dynamic networks.

2 Static multicommodity flow problems

In order to study dynamic versions of multicommodity flow problems we will use the following static flow problems.

2.1 The maximum multicommodity flow problem

For the maximum multicommodity flow problem we consider the following static network $N = (V, V_-, V_+, E, K, w, u)$. A flow x on this network assigns every arc $e \in E$ for each commodity $k \in K$ a non-negative flow value x_e^k such that the following flow conservation constraints are obeyed:

$$\sum_{e \in E^+(v)} x_e^k - \sum_{e \in E^-(v)} x_e^k = \begin{cases} -y_v^k, \ v \in V_-^k, \\ 0, \ v \in V_0^k, \\ y_v^k, \ v \in V_+^k, \end{cases} \tag{1}$$

$$y_v^k \geq 0, \ \forall\, v \in V, \ \forall\, k \in K, \tag{2}$$

where $E^+(v) = \{(u,v) \,|\, (u,v) \in E\}$, $E^-(v) = \{(v,u) \,|\, (v,u) \in E\}$, V_-^k, V_+^k and V_0^k are sets of sources, sinks and intermediate nodes for commodity k of network N, $V_- = \cup_{k \in K} V_-^k$, $V_+ = \cup_{k \in K} V_+^k$, $V_0 = \cup_{k \in K} V_0^k$, $V = V_- \cup V_0 \cup V_+$.

The multicommodity flow x is called feasible if it obeys the mutual capacity constraints:

$$\sum_{k\in K} x_e^k \le u_e,\ \forall\, e \in E \tag{3}$$

and individual capacities of every arc for each commodity:

$$0 \le x_e^k \le w_e^k,\ \forall\, e \in E,\ \forall k \in K. \tag{4}$$

These constraints are called weak and strong forcing constraints, respectively.

The maximum multicommodity flow problem consists in maximizing the following objective function:

$$|x| = \sum_{k\in K} \sum_{v\in V_+^k} y_v^k$$

subject to (1)-(4).

2.2 The minimum cost multicommodity flow problem

For the minimum cost multicommodity flow problem we consider the following static network $N = (V, E, K, w, u, d, \varphi)$. A flow x on this network assigns every arc $e \in E$ for each commodity $k \in K = \{1, 2, \dots, k\}$ a non-negative flow value x_e^k such that the following flow conservation constraints are obeyed:

$$\sum_{e\in E^+(v)} x_e^k - \sum_{e\in E^-(v)} x_e^k = d_v^k,\ \forall\, v \in V,\ \forall\, k \in K, \tag{5}$$

where d: $V \times K \to R$ is a demand function and $\sum\limits_{v\in V} d_v^k = 0,\ \forall k \in K.$

The minimum cost multicommodity flow problem consists in minimizing the following objective function:

$$c(x) = \sum_{e\in E} \varphi_e(x_e^1, x_e^2, \dots, x_e^k),$$

subject to (5),(3),(4),

where φ: $E \times R_+ \to R_+$ is a cost function.

The mathematical tool we are going to use for studying and solving dynamical versions of maximum and minimum cost multicommodity flow problems is based on special procedures of their reducing to static problems on auxiliary networks.

3 The dynamic maximum multicommodity flow problem

3.1 The problem formulation

The object of the maximum dynamic flow problem is to send a maximum amount of flow from sources to sinks within a given time bound without violating capacity constraints of any edge. The maximum multicommodity flow problem requires to find the maximum flow of a set of commodities through a network, where the arcs have an individual capacity for each commodity, and a mutual capacity for all the commodities.

We consider the discrete time model, in which all times are integral and bounded by horizon T. The time horizon (finite or infinite) is the time until which the flow can travel in the network and defines the makespan $\mathbb{T} = \{0, 1, \ldots, T\}$ of time moments we consider. Time is measured in discrete steps, so that if one unit of flow leaves node u at time t on arc $e = (u, v)$, one unit of flow arrives at node v at time $t + \tau_e$, where τ_e is the transit time of arc e.

Without loosing generality, we assume that no edges enter sources or exit sinks. We consider that all of the flow is dumped into the network at time 0. Accordingly the sources are nodes through which flow enters the network and the sinks are nodes through which flow leaves the network. The sources and sinks are sometimes called terminal nodes, while the intermediate nodes are called non-terminals. In the case of many sources and sinks the maximum flow problem can be reduced to the standard one by introducing one additional artificial source and one additional artificial sink and edges leading from the new source to true sources and from true sinks to the new sink. The transit times of these new edges are zero and the capacities of edges connecting the artificial source with all other sources are bounded by the capacities of these sources; the capacities of edges connecting all other sinks with the artificial sink are bounded by the capacities of these sinks.

We consider a network $N = (V, V_-, V_+, E, K, w, u, \tau)$ that contains a directed graph $G = (V, E)$ and a set of commodities K that must be routed through the same network. The graph G consists of set of vertices $V = V_- \cup V_+ \cup V_0$, where V_-, V_+ and V_0 are sets of sources, sinks and intermediate nodes, respectively, and set of edges E. Each edge $e \in E$ has a nonnegative time-varying capacity $w_e^k(t)$ which bounds the amount of flow of each commodity $k \in K$ allowed on arc e in every moment of time $t \in \mathbb{T}$. We also consider that every arc $e \in E$ has a nonnegative time-varying capacity for all commodities, which is known as the mutual capacity $u_e(t)$. Moreover, each edge $e \in E$ has an associated positive transit time τ_e which determines the amount of time it takes for flow to travel from the tail to the head of that edge.

A feasible dynamic flow on N is a function x: $E \times K \times \mathbb{T} \to R_+$ that satisfies the following conditions:

$$\sum_{\substack{e\in E^+(v)\\ t-\tau_e\geq 0}} x_e^k(t-\tau_e) - \sum_{e\in E^-(v)} x_e^k(t) = \begin{cases} -y_v^k(t), & v\in V_-^k,\\ 0, & v\in V_0^k,\\ y_v^k(t), & v\in V_+^k, \end{cases} \quad \forall\, t\in\mathbb{T},\ \forall\, v\in V,\ \forall k\in K; \tag{6}$$

$$y_v^k(t)\geq 0,\ \forall\, v\in V, \forall t\in\mathbb{T},\ \forall k\in K;$$

$$\sum_{k\in K} x_e^k(t)\leq u_e(t),\ \forall\, t\in\mathbb{T},\ \forall e\in E; \tag{7}$$

$$0\leq x_e^k(t)\leq w_e^k(t),\quad \forall\, t\in\mathbb{T},\ \forall\, e\in E,\ \forall k\in K; \tag{8}$$

$$x_e^k(t)=0,\ \forall\, e\in E,\ t=\overline{T-\tau_e+1,T},\ \forall k\in K. \tag{9}$$

Here the function x defines the value $x_e^k(t)$ of flow of commodity k entering edge e at time t. It is easy to observe that the flow of commodity k does not enter edge e at time t if it will have to leave the edge after time T; this is ensured by condition (9). Capacity constraints (8) mean that in a feasible dynamic flow, at most $w_e^k(t)$ units of flow of commodity k can enter arc e at time t. Mutual capacity constraints (7) mean that in a feasible dynamic flow, at most $u_e(t)$ units of flow can enter arc e at time t. Conditions (6) represent flow conservation constraints.

The value of the dynamic flow x is defined as follows:

$$|x| = \sum_{k\in K}\sum_{t\in\mathbb{T}}\sum_{v\in V_+^k} y_v^k(t).$$

The object of the maximum multicommodity flow problem is to find a feasible flow that maximizes this objective function.

It is easy to observe that if $\tau_e = 0, \forall\, e\in E$ and $T=0$ then the formulated problem becomes the static multicommodity flow problem.

3.2 The main results

In this paper we propose an approach for solving the formulated problem, which is based on reduction of this problem to a static well studied one. We show that the maximum multicommodity flow problem on network N can be reduced to a static problem on an auxiliary network N^T; we name this network as a time-expanded network. The advantage of this approach is that it turns the problem of determining a maximum dynamic flow into a classical static maximum flow problem on the time-expanded network.

The time-expanded network is a static representation of the dynamic network. The essence of the time-expanded network is that it contains a copy of the vertices of the dynamic network for each moment of time, and the transit times and flows are implicit in the edges linking those copies. In such a way, a dynamic flow problem in a given network with transit times on the arcs can be transformed into an equivalent static flow problem in the corresponding time-expanded network. A discrete dynamic flow in the given network can be interpreted as a static flow in the corresponding time-expanded network.

We define the time-expanded network as follows:

1. $V^T\colon\ = \{v(t)\,|\,v \in V,\ t \in \mathbb{T}\}$;
2. $E^T\colon\ = \{e(t) = (v(t), w(t+\tau_e))\,|\,e = (v,w) \in E,\ 0 \le t \le T - \tau_e\}$;
3. $u^T_{e(t)}\colon\ = u_e(t)$ for $e(t) \in E^T$;
4. ${w^k_{e(t)}}^T\colon\ = w^k_e(t)$ for $e(t) \in E^T,\ k \in K$.

Let $e(t) = (v(t), w(t+\tau_e)) \in E^T$ and let $x^k_e(t)$ be a flow of commodity $k \in K$ on the dynamic network N. The corresponding function on the time-expanded network N^T is defined as follows:

$$ {x^k_{e(t)}}^T = x^k_e(t),\ \forall k \in K. \tag{10} $$

Lemma 1. *The correspondence (10) is a bijection from the set of feasible flows on the dynamic network N onto the set of feasible flows on the time-expanded network N^T.*

Proof. It is obvious that the correspondence above is a bijection from the set of T-horizon functions on the dynamic network N onto the set of functions on the time-expanded network N^T. It is easy to observe that a feasible flow on the dynamic network N is a feasible flow on the time-expanded network N^T and vice-versa. Indeed, individual capacity constraints are obeyed:

$$ 0 \le {x^k_{e(t)}}^T = x^k_e(t) \le w^k_e(t) = {w^k_{e(t)}}^T,\ \forall\, e \in E,\ \ \forall\, t \in \mathbb{T},\ \forall\, k \in K $$

and mutual capacity constraints are also obeyed:

$$ \sum_{k \in K} {x^k_{e(t)}}^T = \sum_{k \in K} x^k_e(t) \le u_e(t) = u^T_{e(t)},\ \forall\, t \in \mathbb{T},\ \forall\, e \in E. $$

Therefore it is enough to show that each dynamic flow on the dynamic network N is put into the correspondence with a static flow on the time-expanded network N^T and vice-versa.

Henceforward we define

$$ d^k_v(t) = \begin{cases} -y^k_v(t),\ v \in V^k_-, \\ 0,\ v \in V^k_0, \\ y^k_v(t),\ v \in V^k_+, \end{cases} \quad y^k_v(t) \ge 0,\ \forall v \in V,\ \forall t \in \mathbb{T},\ \forall k \in K. $$

Let $x_e^k(t)$ be a dynamic flow of commodity k on N and let ${x_{e(t)}^k}^T$ be a corresponding function on N^T. Let's prove that ${x_{e(t)}^k}^T$ satisfies the conservation constraints on its static network. Let $v \in V$ be an arbitrary node in N and $t \in \mathbb{T}$ an arbitrary moment of time:

$$d_v^k(t) \overset{(i)}{=} \sum_{\substack{e \in E^+(v) \\ t-\tau_e \geq 0}} x_e^k(t-\tau_e) - \sum_{e \in E^-(v)} x_e^k(t) =$$

$$= \sum_{e(t-\tau_e) \in E^+(v(t))} {x_{e(t-\tau_e)}^k}^T - \sum_{e(t) \in E^-(v(t))} {x_{e(t)}^k}^T \overset{(ii)}{=} {d_{v(t)}^k}^T. \tag{11}$$

Note that according to the definition of the time-expanded network the set of edges $\{e(t-\tau_e) | e(t-\tau_e) \in E^+(v(t))\}$ consists of all edges that enter $v(t)$, while the set of edges $\{e(t) | e(t) \in E^-(v(t))\}$ consists of all edges that originate from $v(t)$. Therefore, all necessary conditions are satisfied for each node $v(t) \in V^T$. Hence, ${x_{e(t)}^k}^T$ is a flow on the time-expanded network N^T.

Let ${x_{e(t)}^k}^T$ be a static flow of commodity k on the time-expanded network N^T and let $x_e^k(t)$ be a corresponding function on the dynamic network N. Let $v(t) \in V^T$ be an arbitrary node in N^T. The conservation constraints for this node in the static network are expressed by equality (ii) from (11), which holds for all $v(t) \in V^T$ at all times $t \in \mathbb{T}$. Therefore, equality (i) holds for all $v \in V$ at all times $t \in \mathbb{T}$ and $x_e^k(t)$ is a flow on the dynamic network N. $\square$

In the following lemma we prove that values of any dynamic multicommodity flow and corresponding static multicommodity flow in the time-expanded network are equal.

Lemma 2. *If x is a flow on the dynamic network N and x^T is a corresponding flow on the time-expanded network N^T, then*

$$|x| = |x^T|.$$

Proof. The proof is straightforward:

$$|x| = \sum_{k \in K} \sum_{t \in \mathbb{T}} \sum_{v \in V_+^k} y_v^k(t) = \sum_{k \in K} \sum_{t \in \mathbb{T}} \sum_{v(t) \in {V_+^k}^T} {y_{v(t)}^k}^T = |x^T|. \qquad \square$$

The above lemmas imply the validity of the following theorem:

Theorem 3. *For each maximum multicommodity flow in the dynamic network there is a corresponding maximum multicommodity flow in the static time-expanded network.*

In such a way, we can solve the considered problem by reducing it to the maximum static multicommodity flow problem, solving the obtained problem in the corresponding time-expanded network and then reconstructing the solution to the solution of the initial problem. Therefore, the maximum multicommodity flow problem on dynamic networks can be solved by applying network flow optimization techniques for static flows directly to the expanded network.

3.3 The algorithm

Let the dynamic network N be given. Our object is to solve the maximum multicommodity flow problem on N. Proceedings are following:

1. Building the time-expanded network N^T for the given dynamic network N.
2. Solving the classical maximum multicommodity flow problem on the static network N^T, using one of the known algorithms [6, 8, 9, 13].
3. Reconstructing the solution of the static problem on N^T to the dynamic problem on N. $\square$

3.4 Some special cases

3.4.1 The case of limited time

In the above items we have discussed the problem of determining the maximum dynamic multicommodity flow from the zero time moment to the fixed time horizon T. In such problems we find the maximum amount of flow until the time T. In many practical cases it is necessary to know the maximum flow in the time period from t_1 to t_2, where $t_1 < t_2$. To obtain the solution of of this problem we have to construct a time-expanded network, the discrete time moments of which form the following makespan $\mathbb{T} = \{t_1, t_1+1, \ldots, t_2-1, t_2\}$. In that way, by constructing such a time-expanding network and finding the maximum flow in this network we can obtain the maximum flow in the dynamic network for the time period from t_1 to t_2.

3.4.2 The case of two-sided restrictions

The same argumentation as in the above items can be held to solve the maximum multicommodity flow problem on the dynamic networks in the case when, instead of the condition (8) in the definition of the feasible dynamic flow, the following condition takes place:

$$r_e^k(t) \le x_e^k(t) \le \overline{r}_e^k(t), \quad \forall\, t \in \mathbb{T},\ \forall\, e \in E,\ \forall\, k \in K$$

where $r_e^k(t)$ and $\overline{r}_e^k(t)$ are lower and upper boundaries of the capacity of the edge e correspondingly. In this case we introduce one additional artificial source b_1 and one additional artificial sink b_2. For every arc $e = (u, v)$, where $r_e^k(t) \neq 0$ we introduce arcs (b_1, v) and (u, b_2) with $r^k(t)$ and 0 as the upper and lower boundaries of the capacity of the edges. We reduce $\overline{r}^k(t)$ to $\overline{r}^k(t) - r^k(t)$, but $r^k(t)$ to 0. We also introduce the arc (b_2, b_1) with $\overline{r}^k_{(b_2,b_1)} = \infty$ and $r^k_{(b_2,b_1)} = 0$. The transit times of all introduced arcs are zero. In such a mode we obtain a new network, on which we can solve the standard maximum flow problem.

4 The dynamic minimum cost multicommodity flow problem

4.1 The problem formulation

The minimum cost flow problem is the problem of sending flows in a network from supply nodes to demand nodes at minimum total cost such that link capacities are not exceeded. The minimum cost multicommodity dynamic flow problem asks to find the flow of a set of commodities through a network with given time horizon, satisfying all supplies and demands with minimum cost.

As in the chapter 3 we consider the discrete time model, in which all times are integral and bounded by horizon T. Time is measured in discrete steps, the set of time moments we consider is $\mathbb{T} = \{0, 1, \dots, T\}$.

We consider a directed network $N = (V, E, K, w, u, \tau, d, \varphi)$ with set of vertices V, set of edges E and set of commodities K that must be routed through the same network. A dynamic network N consists of capacity function w: $E \times K \times \mathbb{T} \to R_+$, mutual capacity function u: $E \times \mathbb{T} \to R_+$, transit time function τ: $E \to R_+$, demand function d: $V \times K \times \mathbb{T} \to R$ and cost function φ: $E \times R_+ \times \mathbb{T} \to R_+$. The demand function $d^k_v(t)$ satisfies the following conditions:

a) there exists $v \in V$ for every $k \in K$ with $d^k_v(0) < 0$;

b) if $d^k_v(t) < 0$ for a node $v \in V$ for commodity $k \in K$ then $d^k_v(t) = 0$, $t = 1, 2, \dots, T$;

c) $\sum\limits_{t \in \mathbb{T}} \sum\limits_{v \in V} d^k_v(t) = 0, \forall k \in K.$

A feasible dynamic flow on N is a function x: $E \times K \times \mathbb{T} \to R_+$ that satisfies conditions (7)-(9) and the following conditions:

$$\sum_{\substack{e \in E^+(v) \\ t - \tau_e \geq 0}} x^k_e(t - \tau_e) - \sum_{e \in E^-(v)} x^k_e(t) = d^k_v(t),\ \forall\, t \in \mathbb{T},\ \forall\, v \in V,\ \forall k \in K.$$

To model transit costs, which may change over time, we define the cost function $\varphi_e(x^1_e(t), x^2_e(t), \dots, x^k_e(t), t)$ which indicates the cost of shipping flows over edge e entering the edge e at time t.

The total cost of the dynamic multicommodity flow x is defined as follows:

$$c(x) = \sum_{t \in \mathbb{T}} \sum_{e \in E} \varphi_e(x^1_e(t), x^2_e(t), \dots, x^k_e(t), t).$$

The object of the minimum cost multicommodity flow problem is to find a feasible flow that minimizes this objective function.

It is important to notice that in many practical cases cost functions are presented in the following form:

$$\varphi_e(x_e^1(t), x_e^2(t), \dots, x_e^k(t), t) = \sum_{k \in K} \varphi_e^k(x_e^k(t), t).$$

The separable case of cost functions represents the most important one from the practical standpoint. In the case when $\varphi_e^k(x_e^k(t), t)$ are linear functions the dynamic version of the considered problem is reduced to static linear programming problem on an auxiliary static network.

It is easy to observe that if $\tau_e = 0, \forall e \in E$ and $T = 0$ then the formulated problem becomes the static minimum cost multicommodity flow problem.

4.2 The main results

To solve the minimum cost multicommodity flow problem by its reduction to a static one we define the time-expanded network N^T as follows:

1. $V^T \colon = \{v(t) \,|\, v \in V,\ t \in \mathbb{T}\}$;
2. $E^T \colon = \{e(t) = (v(t), w(t + \tau_e)) \,|\, e = (v, w) \in E,\ 0 \le t \le T - \tau_e\}$;
3. $u_{e(t)}^T \colon = u_e(t)$ for $e(t) \in E^T$;
4. ${w_{e(t)}^k}^T \colon = w_e^k(t)$ for $e(t) \in E^T,\ k \in K$.
5. $\varphi_{e(t)}^T({x_{e(t)}^1}^T, {x_{e(t)}^2}^T, \dots, {x_{e(t)}^k}^T) \colon = \varphi_e(x_e^1(t), x_e^2(t), \dots, x_e^k(t), t)$ for $e(t) \in E^T$, $k \in K$;
6. ${d_{v(t)}^k}^T \colon = d_v^k(t)$ for $v(t) \in V^T,\ k \in K$.

If we define a flow correspondence by relation (10), it can be proved, using the same method as in Lemma 1, that the set of feasible flows on the dynamic network N corresponds to the set of feasible flows on the time-expanded network N^T.

In the following lemma we prove that costs of any dynamic multicommodity flow and corresponding static multicommodity flow in the time-expanded network are equal.

Lemma 4. *If x is a flow on the dynamic network N and x^T is a corresponding flow on the time-expanded network N^T, then*

$$c(x) = c^T(x^T).$$

Proof. The proof is straightforward:

$$c(x) = \sum_{t\in\mathbb{T}}\sum_{e\in E}\varphi_e(x_e^1(t), x_e^2(t), \ldots, x_e^k(t), t) =$$

$$= \sum_{t\in\mathbb{T}}\sum_{e(t)\in E^T}\varphi_{e(t)}^T({x_{e(t)}^1}^T, {x_{e(t)}^2}^T, \ldots, {x_{e(t)}^k}^T) = c^T(x^T). \qquad \square$$

Using the above results we obtain the following theorem:

Theorem 5. *For each minimum-cost multicommodity flow in the dynamic network there is a corresponding minimum-cost multicommodity flow in the static time-expanded network.*

In such a way, the minimum cost multicommodity flow problem on the dynamic network can be solved by static flow computations in the corresponding time-expanded network. The solution of the considered problem can be obtained by using the solution of the static minimum cost multicommodity flow problem on the time-expanded network.

4.3 The algorithm

Let the dynamic network N be given. The minimum-cost multicommodity flow problem is to be solved on N. Proceedings are following:

1. Building the time-expanded network N^T for the given dynamic network N.
2. Solving the classical minimum-cost multicommodity flow problem on the static network N^T ([3–6]).
3. Reconstructing the solution of the static problem on N^T to the dynamic problem on N. □

4.4 Generalization

Now let us study some general cases of the minimum cost dynamic multicommodity flow problems. First of all, we assume that only a part of the flow is dumped into the considered network at the time 0, i.e. the condition b) in the definition of the demand function $d_v^k(t)$ doesn't hold. Using the following, this case can be reduced to the one considered above.

Let the flow is dumped into the network from the node $v \in V$ at an arbitrary moment of time $\mathtt{t}$, different from the ordinary moment. We can reduce this problem to the problem, in which all of the flow is dumped into the network at the initial time by introducing loops in all nodes from V, except the node v, from which the flow is dumped into the network at the time $\mathtt{t}$. For such loops we attribute transit times which are equal to the time t. So, we can consider that all the flow is dumped in the network at the time t, which we define as the initial time.

The argumentation is the same, when the flow is dumped in the network from different nodes at different moments of time. Let t be the maximum of those moments. In this case we take t as the initial time and construct loops from all the nodes, except those that dump the flow in the network at time t. The transit times of these loops are equal to the difference between time t and the time when the flow from those nodes that generate loops is dumped in the network. So, we reduce this problem to the one, considered above, where the whole flow is dumped into the network at the initial moment of time.

Further we consider the variation of the dynamic network when the condition c) in the definition of the demand function $d_v^k(t)$ doesn't hold. We assume that after time $t = T$ there still is flow in the network, i.e. the following condition is true:

$$\sum_{t\in\mathbb{T}}\sum_{v\in V} d_v^k(t) \leq 0.$$

We can reduce this problem to the problem without flow in the network after an upper bound of time by introduction of the additional node v and additional edges. The rest of the flow in the network is sent to the node v through the arcs, which we just introduced. In such a way we obtain the initial model of the dynamic network.

The next model of the dynamic network is the one when we allow flow storage at the nodes. In this case we can reduce this dynamic network to the initial one by introducing the loops in those nodes in which there is flow storage. The flow which was stored at the nodes passes through these loops. Accordingly, we reduce this problem to the initial one.

The other variation of the dynamic network is the one when the cost functions also depend on the flow at the nodes. In this case we can reduce this model of the dynamic network to the initial one by introducing loops and attributing the cost functions, which were defined in the nodes, to these loops. Consequently, we obtain the initial model of the dynamic network.

The same reasoning to solve the minimum cost flow problem on the dynamic networks and its generalization can be held in the case when, instead of the condition (8) in the definition of the feasible dynamic flow, the following condition takes place:

$$r_e^k(t) \leq x_e^k(t) \leq \overline{r}_e^k(t), \quad \forall\, t \in \mathbb{T},\ \forall\, e \in E,\ \forall\, k \in k$$

where $r_e^k(t)$ and $\overline{r}_e^k(t)$ are lower and upper boundaries of the capacity of the edge e correspondingly.

5 The time-expanded network in the case of acyclic graphs

We will consider the dynamic network N, where the graph $G = (V, E)$ does not contain directed cycles. Let $T^* = \max\{|L|\} = \max\{\sum_{e\in L} \tau_e\}$, where L is a directed path in the graph G. In [10] it is shown that $x_e^k(t) = 0$ for $e \in E$, $k \in K$, $t \geq T^*$.

Using this result we can construct the time expanded-network that consists of $n(T^*+1)$ nodes and $m(T^*+1)$ edges, where n and m are numbers of nodes and edges in the initial network. Since the maximum number of edges a directed path can have in an acyclic network is $n-1$, it immediately results that the time-expanded network has not more than n^2 nodes and mn edges. In such a way, we have established a polynomial upper bound for the size of the time-expanded network.

It is easy to note that in many cases the large majority of intermediate nodes are not connected with a directed path both to a sink and a source. Removing such nodes from the considered network does not influence the set of flows on this network. We will call these nodes irrelevant to the flow problem. Intermediate nodes that are not irrelevant will be denoted relevant. The static network obtained by eliminating the irrelevant nodes and all edges adjacent to them from the time-expanded network will be called the reduced time-expanded network.

We propose the following algorithm for constructing the reduced network based on the process of elimination of irrelevant nodes from the time-expanded network.

Algorithm

1. Building the time-expanded network N^{T^*} for the given dynamic network N.

2. Performing a breadth-first parse of the nodes for each source from the time expanded-network. The result of this step is the set $V_-(V_-^{T^*})$ of the nodes that can be reached from at least a source in V^{T^*}.

3. Performing a breadth-first parse of the nodes beginning with the sink for each sink and parsing the edges in the direction opposite to their normal orientation. The result of this step is the set $V_+(V_+^{T^*})$ of nodes from which at least a sink in V^{T^*} can be reached.

4. The reduced network will consist of a subset of nodes V^{T^*} and edges from E^{T^*} determined in the following way
$$V'^{T^*} = V^{T^*} \cap V_-(V_-^{T^*}) \cap V_+(V_+^{T^*}), \qquad E'^{T^*} = E^{T^*} \cap (V'^{T^*} \times V'^{T^*}).$$

5. ${d'^{k}_{v(t)}}^{T^*} := d_v^k(t)$ for $v(t) \in V'^{T^*}$, $k \in K$.

6. ${u'_{e(t)}}^{T^*} := u_e(t)$ for $e(t) \in E'^{T^*}$.

7. ${w'^{k}_{e(t)}}^{T^*} := w_e^k(t)$ for $e(t) \in E'^{T^*}$, $k \in K$.

8. $\varphi^T_{e(t)}({x^1_{e(t)}}^T, {x^2_{e(t)}}^T, \dots, {x^k_{e(t)}}^T) := \varphi_e(x_e^1(t), x_e^2(t), \dots, x_e^k(t), t)$ for $e(t) \in E^T$, $k \in K$. $\square$

The complexity of this algorithm can be estimated to be the same as the complexity of constructing the time-expanded network. In [10] it is proved that the reduced network can be used in place of the time-expanded network. This algorithm begins with the dynamic network containing a small number of nodes, builds the time-expanded network with the largest number of nodes and then selects from it the reduced network with a smaller number of nodes.

Now we propose an algorithm for constructing the reduced network $N^{'T^*}$ directly from the dynamic network N.

Algorithm

1. Building the dynamic network N', which contains all the nodes in N except those that are not connected with a direct path with at least a sink and at least a source, employing the same method as used in the above algorithm for static networks.

2. Creating queue $C = \{v_1(0), v_2(0), \ldots, v_l(0)\}$, where $\{v_1, v_2, \ldots, v_l\} = V_-$. We consider only $v_i(0)$, $v_i \in V_-$, because all of the flow is dumped into the network at time 0.

3. Initializing sets:

$$V_-^{'T^*} = \emptyset, \qquad V_+^{'T^*} = \{v_i(t) | v_i \in V_+, t \in \mathbb{T}\}, \qquad V^{'T^*} = V_-^{'T^*} \cup V_+^{'T^*}.$$

4. While queue C is not empty execute for each node $v_1(t_1)$ at the head of the queue:

   a) If node $v_1(t_1)$ is already in $V^{'T^*}$, then jump to step (4d).

   b) For each $(v_1, v_i) \in E^-(v_1)$ in the dynamic network execute:

      i) If node $v_i \in V_0$ and node $v_i(t_1 + \tau_{(v_1,v_i)})$ is not already in $V^{'T^*}$ then add node $v_i(t_1 + \tau_{(v_1,v_i)})$ to queue C and add edge $(v_1(t_1), v_i(t_1 + \tau_{(v_1,v_i)}))$ to $E^{'T^*}$.

      ii) If node $v_i \in V_+$ and edge $(v_1(t_1), v_i(t_1 + \tau_{(v_1,v_i)}))$ is not already in $E^{'T^*}$, then add edge $(v_1(t_1), v_i(t_1 + \tau_{(v_1,v_i)}))$ to $E^{'T^*}$.

   c) Add node $v_1(t_1)$ to $V^{'T^*}$.

   d) Remove node $v_1(t_1)$ from queue C, all nodes moving one step closer to the head of the queue.

5. ${d'}^{k}_{v(t)}{}^{T^*} := d_v^k(t)$ for $v(t) \in V^{'T^*}$, $k \in K$.

6. ${u'}_{e(t)}^{T^*} := u_e(t)$ for $e(t) \in E^{'T^*}$.

7. ${w'}^{k}_{e(t)}{}^{T^*} := w_e^k(t)$ for $e(t) \in E^{'T^*}$, $k \in K$.

8. $\varphi_{e(t)}^{T}({x_{e(t)}^{1}}^{T}, {x_{e(t)}^{2}}^{T}, \ldots, {x_{e(t)}^{k}}^{T})\colon\ = \varphi_e(x_e^1(t), x_e^2(t), \ldots, x_e^k(t), t)$ for $e(t) \in E^T$, $k \in K$. $\square$

The network $N^{'T^*}$ built by this algorithm contains only intermediate nodes from the time-expanded network N^{T^*} that are relevant. Furthermore it contains all intermediate nodes with this property from N^{T^*}. The functions d, u, w, φ are the same as those on the reduced network and the built network is the reduced network.

6 Conclusions

In this paper we formulated and studied the maximum and minimum cost multicommodity dynamic flow problems on dynamic networks with time-varying capacities of edges. For minimum cost multicommodity flow problem we assumed that cost functions, defined on edges, are nonlinear and depending on time and flow, and the demand function also depends on time. To solve the proposed problems we reduced them to the static ones on auxiliary networks and proposed corresponding algorithms.

At the end we would like to note that the same argumentation and algorithms as were described for the multicommodity flow problems are evidently hold for one-commodity flow problems, one-commodity flow being a particular case of multicommodity flows. The difference consists in the fact that for one-commodity flow instead of individual and mutual capacity constraints only individual capacity constraints are considered. One-commodity flow can be regarded as multicommodity flow in the case of only one commodity.

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Some new exact solutions for the lineal flow of a non-Newtonian fluid

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Abstract. In this paper, the unsteady lineal flows of a second grade fluid and of a Maxwell one, between parallel plates, are investigated. The velocity fields corresponding to the flow induced by a constantly accelerating plate as well as those for the flow caused by the impulsive motion of the plate are determined. The solutions that have been obtained satisfy the associate partial differential equations and all imposed initial and boundary conditions. They also reduce to those for Newtonian fluids as limiting cases. Finally, some conclusions and illustrative comparisons are also presented.

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Keywords and phrases: Second grade fluid, Maxwell fluid, velocity field, impulsive motion, constantly accelerating plate.

1 Introduction

It is well known that, for real problems, a wide variety of fluids can be modelled by the Navier-Stokes equations. These are the Newtonian fluids. Water and air are classical examples. However, several experiments show that this is not the case for all fluids. Fluids that cannot be modelled by the Navier-Stokes equations are called non-Newtonian fluids. In the last years, the interest for fluids of this kind has considerably increased, mainly due to their connection with applied sciences.

Mechanics of non-linear fluids present a special challenge to engineers, physicists and mathematicians. The non-linearity can manifest itself in a variety of ways. One of the simplest ways in which the viscoelastic fluids have been classified is the methodology given by Rivlin and Ericksen [1] and Truesdell and Noll [2]. They present the stress tensor **T** as a function of the symmetric part of the velocity gradient **L** and his higher derivatives. Fluids of differential type, as the authors called them, have attracted much attention, as well as much controversy, in the last few decades. We refer the reader to Dunn and Rajagopal [3] for a complete and thorough discussion of all relevant issues.

Another class of models is that of the rate-type fluids, such as Maxwell fluids or more general Oldroyd fluids [4–6]. A recent and interesting review of the models of rate-type is given by Rajagopal and Srinivasa [7]. Among the many fluids of differential type and rate type, the fluids of second grade and those of Maxwell-B type enjoy a major attractiveness.

The goal of this paper is to establish the exact solutions corresponding to an unsteady lineal flow of such a fluid between two infinite parallel plates, one of them being subject to a constant acceleration A. The obtained solutions satisfy both the associate partial differential equations and all imposed initial and boundary conditions. They certainly reduce to those for a Newtonian fluid as a limiting case. Finally, some conclusions and illustrative comparisons are also presented.

2 Governing equations

The constitutive equations of an incompressible fluid of second grade and of one of Maxwell-B type, as they were given in [1–4, 6, 7], are

$$\mathbf{T} = \text{-p}\mathbf{I} + \mathbf{S}, \quad \mathbf{S} = \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{2.1}$$

respectively,

$$\mathbf{T} = \text{-p}\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda(\dot{\mathbf{S}} - \mathbf{LS} - \mathbf{SL}^{\mathrm{T}}) = \mu\mathbf{A}_1, \tag{2.2}$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first two Rivlin-Ericksen tensors, $\mathbf{S}$ is the extra-stress tensor, -p$\mathbf{I}$ is the spherical stress due to the constraint of incompressibility, μ is the coefficient of viscosity, α_1 and α_2 are normal stress moduli, λ is the relaxation time and the superposed dot indicates the material time derivative. For $\alpha_1 = \alpha_2 = \lambda = 0$, both models reduce to the linearly viscous fluid model.

In the following we shall consider a lineal flow of the form [8, 9]

$$\mathbf{v} = v(y,t)\mathbf{i}, \tag{2.3}$$

where $\mathbf{i}$ denotes the unit vector along the x-direction of the system of Cartesian coordinates x, y and z. For these flows the constraint of incompressibility is automatically satisfied and the equations of motion, in the absence of a pressure gradient in the x-direction, reduce to [8, 9]

$$(\mu + \alpha_1\partial_t)\partial_y^2 v(y,t) = \rho\partial_t v(y,t), \tag{2.4}$$

respectively [6, 10],

$$\mu\partial_y^2 v(y,t) = \rho\partial_t v(y,t) + \rho\lambda\partial_t^2 v(y,t), \tag{2.5}$$

where ρ is the constant density of the fluid.

3 Couette flows of second grade fluids

Let us consider an incompressible second grade fluid at rest between two infinite parallel plates at a distance h apart. The upper plate is held fixed while the lower one is subject, after time zero, to the time-dependent velocity $V(t)$ in its plane. Due to the shear the fluid between the plates is gradually moved. The governing equation is (2.4) and the initial and boundary conditions are

$$v(y,0) = 0, \quad 0 \le y \le h \tag{3.1}$$

and

$$v(0,t) = V(t), \quad v(h,t) = 0; \ t > 0. \tag{3.2}$$

The solution of the partial differential equation (2.4), subject to the initial and boundary conditions (3.1) and (3.2), can be easily obtained by means of the finite Fourier sine transform [11]. Multiplying both sides of Eq. (2.4) by $\sin(\lambda_n y)$, integrating with respect to y from 0 to h and having in mind (3.1) and (3.2), we find that

$$\partial_t v_s(n,t) + \frac{\nu\lambda_n^2}{1+\alpha\lambda_n^2} v_s(n,t) = \frac{\lambda_n[\nu V(t) + \alpha V'(t)]}{1+\alpha\lambda_n^2}, \quad t > 0, \tag{3.3}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid, $\alpha = \alpha_1/\rho$, $\lambda_n = n\pi/h$ and the sine transforms $v_s(n,t)$ of $v(y,t)$ have to satisfy the conditions

$$v_s(n,0) = 0, \quad n = 1, 2, ... \tag{3.4}$$

Solving the ordinary differential equations (3.3) with the initial conditions (3.4) and using the Fourier's sine formula [11], we can write the velocity field $v(y,t)$ under the form

$$\begin{aligned} v(y,t) = \left(1 - \frac{y}{h}\right) V(t) - \frac{2V}{h} \sum_{n=1}^{\infty} \exp\left(-\frac{\nu\lambda_n^2}{1+\alpha\lambda_n^2} t\right) \frac{\sin(\lambda_n y)}{\lambda_n} - \\ -\frac{2}{h} \sum_{n=1}^{\infty} \frac{\sin(\lambda_n y)}{\lambda_n(1+\alpha\lambda_n^2)} \int_0^t V'(\tau) \exp\left[-\frac{\nu\lambda_n^2}{1+\alpha\lambda_n^2}(t-\tau)\right] d\tau, \end{aligned} \tag{3.5}$$

where $V = \lim_{t\to 0} V(t)$. For $\alpha \to 0$, this solution reduces to the similar solution

$$\begin{aligned} v(y,t) = \left(1 - \frac{y}{h}\right) V(t) - \frac{2V}{h} \sum_{n=1}^{\infty} e^{-\nu\lambda_n^2 t} \frac{\sin(\lambda_n y)}{\lambda_n} - \\ -\frac{2}{h} \sum_{n=1}^{\infty} \frac{\sin(\lambda_n y)}{\lambda_n} \int_0^t V'(\tau) e^{-\nu\lambda_n^2(t-\tau)} d\tau, \end{aligned} \tag{3.6}$$

corresponding to a Newtonian fluid.

3.1 Flow due to an impulsive motion of the plate

The solutions corresponding to the flow engendered due to the sudden imposition of a constant velocity V to the lower plate

$$v(y,t) = \left(1 - \frac{y}{h}\right) V - \frac{2V}{h} \sum_{n=1}^{\infty} \exp\left(-\frac{\nu\lambda_n^2}{1+\alpha\lambda_n^2} t\right) \frac{\sin(\lambda_n y)}{\lambda_n} \tag{3.7}$$

and

$$v(y,t) = \left(1 - \frac{y}{h}\right) V - \frac{2V}{h} \sum_{n=1}^{\infty} e^{-\nu\lambda_n^2 t} \frac{\sin(\lambda_n y)}{\lambda_n}, \tag{3.8}$$

are well known and available in the literature [8, 12]. They can be also obtained from (3.5) and (3.6) for $V(t) \equiv V$.

3.2 Flow induced by a constantly accelerating plate

Let us now suppose that the lower plate is subject, after time zero, to a constant acceleration A. The corresponding solutions

$$v(y,t) = \left(1 - \frac{y}{h}\right) At - \frac{2A}{\nu h} \sum_{n=1}^{\infty} \left[1 - \exp\left(-\frac{\nu\lambda_n^2}{1+\alpha\lambda_n^2} t\right)\right] \frac{\sin(\lambda_n y)}{\lambda_n^3}, \tag{3.9}$$

respectively,

$$v(y,t) = \left(1 - \frac{y}{h}\right) At - \frac{2A}{\nu h} \sum_{n=1}^{\infty} \left[1 - e^{-\nu\lambda_n^2 t}\right] \frac{\sin(\lambda_n y)}{\lambda_n^3}, \tag{3.10}$$

are immediately obtained from (3.5) and (3.6) by making $V(t) = At$.

4 Couette flows of Maxwell fluids

Assume that an incompressible Maxwell fluid fills the space between the same infinite parallel plates. The upper plate is again fixed while the lower one is subject, after time zero, to the velocity $At + V$ in its plane. The governing equation is (2.5) and the initial and boundary conditions are

$$v(y,0) = \partial_t v(y,0) = 0; \quad 0 \le y \le h \tag{4.1}$$

and

$$v(0,t) = At + V, \quad v(h,t) = 0; \quad t > 0. \tag{4.2}$$

Multiplying (2.5) by $\sin(\lambda_n y)$, integrating the result between the limits $y = 0$ and $y = h$ and taking into account (4.1) and (4.2) we get (see [11], Sec. 13)

$$\lambda \partial_t^2 v_s(n,t) + \partial_t v_s(n,t) + \nu\lambda_n^2 v_s(n,t) = \nu\lambda_n (At + V), \quad t > 0, \tag{4.3}$$

where

$$v_s(n,0) = \partial_t v_s(n,0) = 0; \quad n = 1, 2, ... \tag{4.4}$$

The solutions of the ordinary differential equations (4.3), subject to the initial conditions (4.4), are

$$v_s(n,t) = \frac{1}{\lambda_n} \left[\left(\frac{A}{\nu\lambda_n^2} - V\right) \frac{r_{2n}\exp(r_{1n}t) - r_{1n}\exp(r_{2n}t)}{r_{2n} - r_{1n}} + \right.$$

$$\left. + A\frac{\exp(r_{1n}t) - \exp(r_{2n}t)}{r_{2n} - r_{1n}} + At - \left(\frac{A}{\nu\lambda_n^2} - V\right)\right]$$

if $0 < \lambda_n \le \dfrac{1}{2\sqrt{\nu\lambda}}$, and

$$v_s(n,t) = \frac{1}{\lambda_n} \left\{ \exp\left(-\frac{t}{2\lambda}\right) \left[\left(\frac{A}{\nu\lambda_n^2} - V\right) \cos\left(\frac{\beta_n t}{2\lambda}\right) + \right.\right.$$

$$+\frac{A(1-2\nu\lambda\lambda_n^2)-\nu\lambda_n^2 V}{\nu\lambda_n^2\beta_n}\sin\left(\frac{\beta_n t}{2\lambda}\right)\Bigg] + At - \left(\frac{A}{\nu\lambda_n^2}-V\right)\Bigg\}$$

if $\lambda_n > \dfrac{1}{2\sqrt{\nu\lambda}}$. In the above relations

$$r_{1n,2n} = \frac{-1\pm\sqrt{1-4\nu\lambda\lambda_n^2}}{2\lambda} \text{ and } \beta_n = \sqrt{4\nu\lambda\lambda_n^2-1}.$$

Using again the Fourier's sine formula [11], we can write the velocity field under the form

$$\begin{aligned} v(y,t) &= \left(1-\frac{y}{h}\right)(At+V) - \frac{2A}{\nu h}\sum_{n=1}^{\infty}\frac{\sin(\lambda_n y)}{\lambda_n^3} + \\ &+\frac{2}{h}\sum_{n=1}^{p}\left[\left(\frac{A}{\nu\lambda_n^2}-V\right)\frac{r_{2n}\exp(r_{1n}t)-r_{1n}\exp(r_{2n}t)}{r_{2n}-r_{1n}} + \right. \\ &\left. + A\frac{\exp(r_{1n}t)-\exp(r_{2n}t)}{r_{2n}-r_{1n}}\right]\frac{\sin(\lambda_n y)}{\lambda_n} + \\ &+\frac{2}{h}\exp\left(-\frac{t}{2\lambda}\right)\sum_{n=p+1}^{\infty}\left[\left(\frac{A}{\nu\lambda_n^2}-V\right)\cos\left(\frac{\beta_n t}{2\lambda}\right) + \right. \\ &\left. +\frac{A(1-2\nu\lambda\lambda_n^2)-\nu\lambda_n^2 V}{\nu\lambda_n^2\beta_n}\sin\left(\frac{\beta_n t}{2\lambda}\right)\right]\frac{\sin(\lambda_n y)}{\lambda_n}, \end{aligned} \tag{4.5}$$

where p is chosen as to $\lambda_p \le 1/(2\sqrt{\nu\lambda})$.

In the special case when $\lambda \to 0$, corresponding to a Newtonian fluid, Eq. (4.5) reduces to

$$\begin{aligned} v(y,t) &= \left(1-\frac{y}{h}\right)(At+V) - \frac{2A}{\nu h}\sum_{n=1}^{\infty}\frac{\sin(\lambda_n y)}{\lambda_n^3} + \\ &+\frac{2}{h}\sum_{n=1}^{\infty}\left(\frac{A}{\nu\lambda_n^2}-V\right)e^{-\nu\lambda_n^2 t}\frac{\sin(\lambda_n y)}{\lambda_n}. \end{aligned} \tag{4.6}$$

4.1 Flow due to an impulsive motion of the plate

By letting $A \to 0$ in (4.5) we obtain

$$\begin{aligned} v(y,t) &= \left(1-\frac{y}{h}\right)V - \frac{2V}{h}\sum_{n=1}^{p}\frac{r_{2n}\exp(r_{1n}t)-r_{1n}\exp(r_{2n}t)}{r_{2n}-r_{1n}}\frac{\sin(\lambda_n y)}{\lambda_n} - \\ &-\frac{2V}{h}\exp\left(-\frac{t}{2\lambda}\right)\sum_{n=p+1}^{\infty}\left[\cos\left(\frac{\beta_n t}{2\lambda}\right)+\frac{1}{\beta_n}\sin\left(\frac{\beta_n t}{2\lambda}\right)\right]\frac{\sin(\lambda_n y)}{\lambda_n}, \end{aligned} \tag{4.7}$$

which represents the velocity field corresponding to the flow of a Maxwell fluid due to an impulsive motion of the lower plate. By making $\lambda \to 0$ in this last relation we again attain to the velocity field (3.8) for a Navier-Stokes fluid.

4.2 Flow induced by a constantly accelerating plate

The solutions corresponding to the flow induced by the lower plate that, after time zero, is subject to a constant acceleration A, are also obtained from (4.5) and (4.6) for $V \to 0$. While Eq. (4.6) reduces to (3.10), the other one becomes

$$v(y,t) = \left(1 - \frac{y}{h}\right) At - \frac{2A}{\nu h} \sum_{n=1}^{\infty} \frac{\sin(\lambda_n y)}{\lambda_n^3} +$$

$$+ \frac{2A\lambda}{\nu h} \sum_{n=1}^{p} \frac{r_{1n}^2 \exp(r_{2n} t) - r_{2n}^2 \exp(r_{1n} t)}{r_{2n} - r_{1n}} \frac{\sin(\lambda_n y)}{\lambda_n^3} +$$

$$+ \frac{2A}{\nu h} \exp\left(-\frac{t}{2\lambda}\right) \sum_{n=p+1}^{\infty} \left[\cos\left(\frac{\beta_n t}{2\lambda}\right) + \frac{1 - 2\nu\lambda\lambda_n^2}{\beta_n} \sin\left(\frac{\beta_n t}{2\lambda}\right)\right] \frac{\sin(\lambda_n y)}{\lambda_n}. \tag{4.8}$$

5 Conclusions and numerical results

In this paper, two unsteady lineal flows of non-Newtonian fluids of second grade and Maxwell-B are studied. In principle, our attention was fixed on the flow induced by a constantly accelerating plate. However, choosing more general boundary conditions, the solutions corresponding to the flow produced by an impulsive motion of the plate have been also obtained.

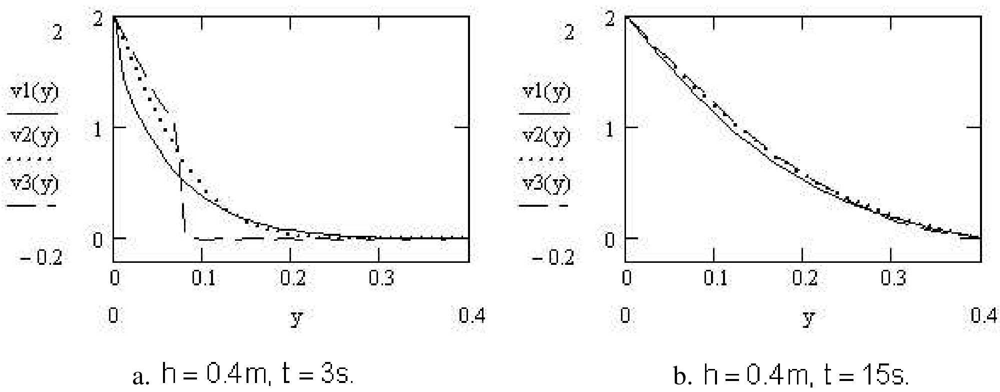
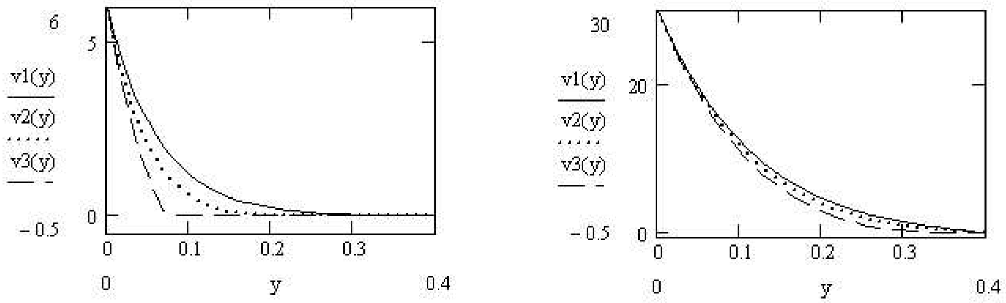


Fig.1. Velocity profiles corresponding to a second grade (curves v1), Newtonian (curves v2) and Maxwell fluid (curves v3), for $V = 2$, $\nu = 0.0011746$ (glycerin), $\alpha = 0.002$ and $\lambda = 2$

Direct computations show that $v(y,t)$ given by (3.5) and (4.5) satisfy both the associate partial differential equations (2.4) and (2.5) and all imposed initial and boundary conditions, the differentiation term by term in y and t being clearly permissible. In the special case when α_1 or $\lambda \to 0$ these solutions reduce to those for a Newtonian fluid. Furthermore, all solutions corresponding to a Maxwell fluid contain sine and cosine terms as functions of time t. This indicates that in contrast with the Newtonian and second grade fluids, whose solutions do not contain such terms, oscillations are set up in the fluid. The amplitudes of these oscillations decay exponentially in time, the damping being proportional to $\exp(-t/2\lambda)$.



a. $h = 0.4m$, $t = 3s$. b. $h = 0.4m$, $t = 15s$.

Fig.2. Velocity profiles corresponding to a second grade (curves v1), Newtonian (curves v2) and Maxwell fluid (curves v3), for $A = 2$, $\nu = 0.0011746$ (glycerin), $\alpha = 0.002$ and $\lambda = 2$

In Figs. 1 and 2, for comparison, the variations of the velocity fields (3.7), (3.8) and (4.7) respectively (3.9), (3.10) and (4.8) are plotted for various values of t and of the material constants. For small values of t, one can observe the differences between the three models. For large values of t the non-Newtonian effects become weak and the profiles of the velocity fields are close by. It can be also seen from figures that a second grade fluid induced by a constantly accelerating plate, unlike the flow due to an impulsive motion of the plate, flows faster than a Newtonian fluid. In its turn, the Newtonian fluid flows faster than a Maxwell fluid. This unexpected result is due to the initial condition.

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Absolute Asymptotic Stability of Discrete Linear Inclusions

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Abstract. The article is devoted to the study of absolute asymptotic stability of discrete linear inclusions in Banach (both finite and infinite dimensional) space. We establish the relation between absolute asymptotic stability, asymptotic stability, uniform asymptotic stability and uniform exponential stability. It is proved that for asymptotical compact (a sum of compact operator and contraction) discrete linear inclusions the notions of asymptotic stability and uniform exponential stability are equivalent. It is proved that finite-dimensional discrete linear inclusion, defined by matrices $\{A_1, A_2, ..., A_m\}$, is absolutely asymptotically stable if it does not admit nontrivial bounded full trajectories and at least one of the matrices $\{A_1, A_2, ..., A_m\}$ is asymptotically stable. We study this problem in the framework of non-autonomous dynamical systems (cocyles).

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1 Introduction

The aim of this paper is studying of the problem of the absolute asymptotic stability of discrete linear inclusion (see Gurvits [22] and the references therein)

$$x_{t+1} \in F(x_t), \tag{1}$$

where $F(x) = \{A_1x, A_2x, ..., A_mx\}$ for all $x \in E^d$ (E^d is a d-dimensional euclidian space) and A_i $(1 \le i \le m)$ is a $d \times d$-matrix.

The article is devoted to the study of absolute asymptotic stability of discrete linear inclusions in Banach space (both finite and infinite-dimensional case). The problem of asymptotic stability for the discrete linear inclusion arise in a number of different areas of mathematics: control theory – Molchanov [29]; linear algebra – Artzrouni [2], Beyn and Elsner [3], Bru, Elsner and Neumann [7], Daubechies and Lagarias [16], Elsner and Friedland [17], Elsner, Koltracht and Neumann [18], Gurvits [22], Vladimirov, Elsner and Beyn [40]; Markov Chains – Gurvits [19], Gurvits and Zaharin [20, 21]; iteration process – Bru, Elsner and Neumann [7], Opoitsev [30] and see also the bibliography therein.

We establish the relation between absolute asymptotic stability (AAS), asymptotic stability (AS), uniform asymptotic stability (UAS) and uniform exponential stability (UES). It is proved that for asymptotically compact (a sum of compact operator and contraction) discrete linear inclusions these notions of stability are equivalent. We study this problem in the framework of non-autonomous dynamical systems (cocyles). We show that the problem of absolute asymptotic stability for the discrete linear inclusions is related with the compact global attractors of non-autonomous dynamical systems (both ordinary dynamical systems (with uniqueness) and set-valued dynamical systems). We plan to continue the studying of discrete inclusions (both linear and nonlinear) in the framework of non-autonomous dynamical systems. In our future publications we will give the proofs of the followings results:

(i) infinite-dimensional discrete linear inclusion, defined by compact operators $\{A_1, A_2, ..., A_m\}$, is absolutely asymptotically stable if it does not admit non-trivial bounded full trajectories and at least one of the operators $\{A_1, A_2, ..., A_m\}$ is asymptotically stable;

(ii) discrete inclusion, defined by nonlinear (in particularly, affine) contractive mappings $\{A_1, A_2, ..., A_m\}$ admits a compact global chaotic attractor,

amongst others. We consider that this method of studying of discrete inclusions (both linear and nonlinear) is fruitful and it permits to obtain the new and nontrivial results.

This paper is organized as follows.

In Section 2 we give a new approach to the study of discrete linear inclusions (DLI) which is based on non-autonomus dynamical systems (cocycles).

Section 3 is devoted to the study of DLIs in arbitrary Banach spaces. We show that for an infinite-dimensional DLI the notions of asymptotic stability and uniform asymptotic stability are not equivalent (Example 3.1). We prove the equivalence of the uniform asymptotic stability and generalized contraction for DLIs (Theorem 3.5). If a discrete linear inclusion (DLI) is completely continuous (compact), then we prove that absolute asymptotic stability and uniform exponential stability are equivalent. We also give the description of absolute asymptotic stability in term of joint spectral radius.

Section 4 is dedicated to the study of asymptotically compact discrete linear inclusions. We establish the relation between different types of stability for this class of DLIs. The main results of this sections are Theorems 4.16, 4.17 and 4.18.

In Section 5 we study the problem of absolute asymptotic stability for finite-dimensional discrete linear inclusions. We establish some general properties of semi-group non-autonomous linear dynamical systems and we prove that finite-dimensional discrete linear inclusion, defined by matrices $\{A_1, A_2, ..., A_m\}$, is absolute asymptotic stable if it doesn't admit non-trivial bounded full trajectories and at least one of the matrices $\{A_1, A_2, ..., A_m\}$ is asymptotically stable (Theorem 5.24 - the main result of paper).

2 Discrete Linear Inclusions and Cocycles

Let E be a real or complex Banach space with norm $|\cdot|$, $\mathbb{S}$ be a group of real ($\mathbb{R}$) or integer ($\mathbb{Z}$) numbers, $\mathbb{T}$ ($\mathbb{S}_+ := \{s \in \mathbb{S} \ : \ s \geq 0\} \subseteq \mathbb{T}$) be a semi-group of the additive group $\mathbb{S}$. Denote by $[E]$ the space of all bounded operators $A : E \to E$. Consider a set of operators $\mathcal{M} \subseteq [E]$.

Definition 2.1. *A discrete linear (autonomous) inclusion $DLI(\mathcal{M})$ is called (see, for example,[22]) a set of all sequences $\{x(t)\}_{t\in\mathbb{Z}_+}$ of vectors in E such that*

$$x(t+1) = A(t)x(t) \tag{2}$$

for some $A(t) \in \mathcal{M}$, i.e.

$$x(t) = A(t)A(t-1)...A(1)A(0)x(0) \ \text{all } A(t) \in \mathcal{M},$$

where $A(0) := Id_E$.

Definition 2.2. *The bilateral sequence $\{x(t)\}_{t\in\mathbb{Z}}$ of vectors in E is called a full trajectory of $DLI(\mathcal{M})$ (entire trajectory or trajectory on $\mathbb{Z}$) if $x(t+s+1) = A(t)x(t+s)$ for all $s \in \mathbb{Z}$ and $t \in \mathbb{Z}_+$.*

We may consider this a discrete control problem, where at each moment of time t we may apply a control from the set $\mathcal{M}$, and $DLI(\mathcal{M})$ is the set of possible trajectories of the system. The basic issue for any control system concerns its stability. One of the most important types of stability is so-called absolute asymptotic stability (AAS).

Definition 2.3. *$DLI(\mathcal{M})$ is called absolutely asymptotically stable (AAS) (or convergent) if for any its trajectory $\{x(t)\}$ we have*

$$\lim_{t\to\infty} x(t) = 0.$$

Equivalently, all operator products

$$\lim_{t\to\infty} A(t)A(t-1)...A(1)A(0)x = 0 \quad (\text{all } A(t) \in \mathcal{M}) \tag{3}$$

for very $x \in E$.

Definition 2.4. *The set $\mathcal{M} \subseteq [E]$ of operators is called product bounded (or uniformly stable) if there exists a $M > 0$ such that $\|A(t)A(t-1)...A(1)A(0)\| \leq M$ for all finite sequence $\{A(t)\}_{t\in\mathbb{Z}_+}$ $(A(t) \in \mathcal{M})$.*

Definition 2.5. *$DLI(\mathcal{M})$ is said to be asymptotically stable (AS) if it is product bounded (or uniformly stable) and convergent.*

Let (X,ρ) be a complete metric space with the metric ρ. Denote by $C(X)$ the family of all compact subsets of X. Consider the set-valued function $F : E \to C(E)$ defined by the equality $F(x) := \{Ax \mid A \in \mathcal{M}\}$. Then the discrete linear inclusion $DLI(\mathcal{M})$ is equivalent to the difference inclusion

$$x(t+1) \in F(x(t)). \tag{4}$$

Denote by $\mathcal{F}_{x_0}$ the set of all trajectories of discrete inclusion (4) (or $DLI(\mathcal{M})$) issuing from the point $x_0 \in E$ and $\mathcal{F} := \bigcup\{\mathcal{F}_{x_0} \mid x_0 \in E\}$.

Below we will give a new approach concerning the study of discrete linear inclusions $DLI(\mathcal{M})$ (or difference inclusion (4)). Denote by $C(\mathbb{Z}_+, X)$ the space of all continuous mappings $f : \mathbb{Z}_+ \to X$ equipped with the compact-open topology. This topology can be metrized, for example, by the equality

$$d(f^1, f^2) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(f^1, f^2)}{1 + d_n(f^1, f^2)}$$

$$(d_n(f^1, f^2) := \max\{|f^1(k) - f^2(k)| \mid 0 \le k \le n\})$$

is defined a complete metric on $C(\mathbb{Z}_+, X)$ which generates compact-open topology. Denote by $(C(\mathbb{Z}_+, X), \mathbb{Z}_+, \sigma)$ a dynamical system of translations (shifts dynamical system or dynamical system of Bebutov [5,12,13,36–38]) on $C(\mathbb{Z}_+, X)$, i.e. $\sigma(k, f) := f_k$ and f_k is a $k \in \mathbb{Z}_+$ shift of f (i.e. $f_k(n) := f(n+k)$ for all $n \in \mathbb{Z}_+$).

Let now $Q \subseteq X$ be a compact. Denote by $C(\mathbb{Z}_+, Q) := \{f \in C(\mathbb{Z}_+, X) \mid f(\mathbb{Z}_+) \subseteq Q\}$. It is easy to see that $C(\mathbb{Z}_+, Q)$ is invariant (with respect to shifts) and closed subset of $(C(\mathbb{Z}_+, X), \mathbb{Z}_+, \sigma)$ and, consequently, on the space $C(\mathbb{Z}_+, Q)$ is defined a dynamical system of shifts $(C(\mathbb{Z}_+, Q), \mathbb{Z}_+, \sigma)$ (induced by the dynamical system of Bebutov $(C(\mathbb{Z}_+, X), \mathbb{Z}_+, \sigma)$). Note that by the theorem of Tikhonoff [26] the space $C(\mathbb{Z}_+, Q)$ is compact.

Let $\mathcal{M}$ be a compact subset of $[E]$ (for example, $\mathcal{M}$ may be a finite set, i.e. $\mathcal{M} = \{A_1, A_2, ..., A_m : A_i \in [E]\ (1 \le i \le m)\}$). Denote by $\Omega := \{f \in C(\mathbb{Z}_+, [E]) \mid f(\mathbb{Z}_+) \subseteq \mathcal{M}\}$. It is clear that Ω is an invariant (with respect to shifts) and closed subset of $C(\mathbb{Z}_+, [E])$ and, hence, on the space Ω is defined a dynamical system of shifts $(\Omega, \mathbb{Z}_+, \sigma)$ (induced by the dynamical system of Bebutov $(C(\mathbb{Z}_+, [E]), \mathbb{Z}_+, \sigma)$). Notice that by the Tikhonoff's theorem the space Ω is compact in $C(\mathbb{Z}_+, [E])$.

We may now rewrite equation (2) in the following way:

$$x(t+1) = \omega(t)x(t),\ \ (\omega \in \Omega) \tag{5}$$

where $\omega \in \Omega$ is the operator-function defined by the equality $\omega(t) := A(t)$ for all $t \in \mathbb{Z}_+$. Denote by $\varphi(t, x_0, \omega)$ a solution of equation (5) issuing from the point $x_0 \in E$ at the initial moment $t = 0$. Note that $\mathcal{F}_{x_0} = \{\varphi(\cdot, x_0, \omega) \mid \omega \in \Omega\}$ and $\mathcal{F} = \{\varphi(\cdot, x_0, \omega) \mid x_0 \in E^d, \omega \in \Omega\}$, i.e. $DLI(\mathcal{M})$ (or inclusion (4)) is equivalent to the family of linear non-autonomous equations (5) $(\omega \in \Omega)$.

From the general properties of linear difference equations it follows that the mapping $\varphi : \mathbb{Z}_+ \times E^d \times \Omega \to E$ satisfies the following conditions:

(i) $\varphi(0, x_0, \omega) = x_0$ for all $(x_0, \omega) \in E \times \Omega$;

(ii) $\varphi(t+\tau, x_0, \omega) = \varphi(t, \varphi(\tau, x_0, \omega), \sigma(\tau, \omega))$ for all $t, \tau \in \mathbb{Z}_+$ and $(x_0, \omega) \in E \times \Omega$;

(iii) the mapping φ is continuous;

(iv) $\varphi(t, \lambda x_1 + \mu x_2, \omega) = \lambda\varphi(t, x_1, \omega) + \mu\varphi(t, x_2, \omega)$ for all $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$), $x_1, x_2 \in E$ and $\omega \in \Omega$.

Let W, Ω be two complete metric spaces and $(\Omega, \mathbb{Z}_+, \sigma)$ be a discrete semi-group dynamical system on Ω.

Definition 2.6. *Recall [36] that a triplet $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ (or shortly φ) is called a cocycle over $(\Omega, \mathbb{Z}_+, \sigma)$ with the fiber W if φ is a mapping from $\mathbb{Z}_+ \times W \times \Omega$ to W satisfying the following conditions:*

1. *$\varphi(0, x, \omega) = x$ for all $(x, \omega) \in W \times \Omega$;*
2. *$\varphi(t+\tau, x, \omega) = \varphi(t, \varphi(\tau, x, \omega), \sigma(\tau, \omega))$ for all $t, \tau \in \mathbb{Z}_+$ and $(x, \omega) \in W \times \Omega$;*
3. *the mapping φ is continuous.*

If W is a real or complex Banach space and

4. *$\varphi(t, \lambda x_1 + \mu x_2, \omega) = \lambda\varphi(t, x_1, \omega) + \mu\varphi(t, x_2, \omega)$ for all $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$), $x_1, x_2 \in W$ and $\omega \in \Omega$,*

then the cocycle φ is called linear.

Let $X := W \times \Omega$, and define the mapping $\pi : X \times \mathbb{T}_1 \to X$ by the equality: $\pi((u, \omega), t) := (\varphi(t, u, \omega), \sigma(t, \omega))$ (i.e. $\pi = (\varphi, \sigma)$). Then it is easy to check that $(X, \mathbb{T}_1, \pi)$ is a dynamical system on X, which is called a skew-product dynamical system [1,36]; but $h = pr_2 : X \to \Omega$ is a homomorphism of $(X, \mathbb{T}_1, \pi)$ onto $(\Omega, \mathbb{T}_2, \sigma)$.

Definition 2.7. *Let $(X, \mathbb{T}, \pi)$ and $(Y, \mathbb{T}, \sigma)$ be two dynamical systems and $h : X \to Y$ be a homomorphism from $(X, \mathbb{T}, \pi)$ onto $(Y, \mathbb{T}, \sigma)$. A triplet $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is called a non-autonomous dynamical system.*

Thus, if we have a cocycle $\langle W, \varphi, (\Omega, \mathbb{T}_2, \sigma)\rangle$ over the dynamical system $(\Omega, \mathbb{T}_2, \sigma)$ with the fiber W, then there can be constructed a non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \sigma), h\rangle$ ($X := W \times \Omega$), which we will call a non-autonomous dynamical system generated (associated) by the cocycle $\langle W, \varphi, (\Omega, \mathbb{T}_2, \sigma)\rangle$ over $(\Omega, \mathbb{T}_2, \sigma)$.

From the presented above it follows that every $DLI(\mathcal{M})$ (respectively, inclusion (4)) in a natural way generates a linear cocycle $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$, where $\Omega = C(\mathbb{Z}_+, \mathcal{M})$, $(\Omega, \mathbb{Z}_+, \sigma)$ is a dynamical system of shifts on Ω and $\varphi(t, x, \omega)$ is a solution of equation (5) issuing from the point $x \in E$ at the initial moment $n = 0$. Thus, we can study inclusion (4) (respectively, $DLI(\mathcal{M})$) in the framework of the theory of linear cocycles with discrete time.

3 Absolute Asymptotic Stability of Discrete Linear Inclusions in Banach Spaces

In this section we will study $DLI(\mathcal{M})$ in an arbitrary Banach space. Let E be a real or complex Banach space with the norm $|\cdot|$ and $[E]$ be a Banach space of all linear bounded operators acting on the space E and equipped with the operational norm. Below we suppose that $\mathcal{M} := \{A_1, A_2, ..., A_m\}$ and $A_i \in [E]$.

Note that for infinite-dimensional discrete linear inclusions $DLI(\mathcal{M})$ ($\dim(E) < +\infty$) the notion of absolute asymptotic stability (AAS) and the equality

$$\lim_{t\to+\infty} \|A(t)A(t-1)...A(1)A(0)\| = 0 \tag{6}$$

are equivalent. It is easy to see that for infinite-dimensional $DLI(\mathcal{M})$ ($\dim(E) = +\infty$) it is not true. This fact is confirmed by the following example.

Example 3.1. *Let $E := c_0$, $A \in [c_0]$ be the operator defined by the equality*

$$A\xi := \{\xi_{k+1}\}$$

for all $\xi := \{\xi_k\} \in c_0$. It is easy to verify that the operator A possesses the following properties:

(i)

$$A^n\xi \to 0 \tag{7}$$

as $n \to \infty$ for each $\xi \in l_2$, where $A^n := A \circ A^{n-1}$ $(n \geq 1)$ and $A^0 := Id_E$;

(ii)

$$A^n e_{n+1} = e_1, \tag{8}$$

where $e_1 = (1, 0, 0, ...), e_2 = (0, 1, 0, ...), ...$ $(n = 1, 2, ...)$.

Let $\mathcal{M} := \{A\}$, i.e. $m = 1$. In this case $DLI(\mathcal{M})$ is equivalent to the linear autonomous difference equation

$$x(t + 1) = Ax(t).$$

From (7) it follows that $DLI(\mathcal{M})$ (with $\mathcal{M} = \{A\}$) is absolutely asymptotically stable. On the other hand, from equality (8) we have $\|A^n\| \geq 1$ and, consequently, equality (6) does not hold.

Let (X, h, Y) be a locally trivial Banach fiber bundle [4, 24].

Definition 3.2. *A non-autonomous dynamical system $\langle(X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is said to be linear, if the map $\pi^t : X_y \to X_{\sigma(t,y)}$ ($X_y := h^{-1}(y)$) is linear for every $t \in \mathbb{T}$ and $y \in Y$, where $\pi^t := \pi(t, \cdot)$.*

Let $\langle E, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a linear cocycle over $(Y, \mathbb{T}, \sigma)$ with the fiber E (or shortly φ). If $X := E \times Y$ and $(X, \mathbb{T}, \pi)$ is a skew-product dynamical system, then the triplet $\langle(X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h\rangle$, where $h := pr_2 : X \to Y$, is a linear non-autonomous dynamical system generated by cocycle φ.

Theorem 3.3. *[13] Let E be a Banach space, Ω be a compact metric space and $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ be a linear cocycle over $(\Omega, \mathbb{Z}_+, \sigma)\rangle$. Then the next conditions are equivalent:*

(i) the cocycle φ is uniformly asymptotically stable, i.e.

$$\lim_{n\to+\infty} \sup_{\omega\in\Omega} \|U(n,\omega)\| = 0;$$

(ii) the cocycle φ is uniformly exponentially stable, i.e. there are two positive constants $\mathcal{N}$ and ν such that $\|U(n,\omega)\| \le \mathcal{N}e^{-\nu t}$ for all $t \ge 0$ and $\omega \in \Omega$.

Let $\mathcal{M} \subseteq [E]$ be a nonempty bounded set of operators and denote by $\mathcal{S} = \mathcal{S}(\mathcal{M})$ the semigroup generated by $\mathcal{M}$ augmented with the identity operator $I := Id_E$, so that $\mathcal{S} = \bigcup_{n=0}^{\infty} \mathcal{M}^n$, where $\mathcal{M}^n := \{\prod_{t=1}^{n} A(t) \mid A(t) \in \mathcal{M}\}$.

Definition 3.4. *The number*

$$\rho(\mathcal{M}) := \limsup_{n\to\infty} \|\mathcal{M}^n\|^{\frac{1}{n}} \quad \textit{and} \quad \|\mathcal{M}\| := \sup\{\|A\| \ : \ A \in \mathcal{M}\}$$

is called [16, 22, 32] a joint spectral radius of bounded subset of linear operators $\mathcal{M}$.

Theorem 3.5. *Let $\mathcal{M} \subset [E]$ be a compact subset (in particular, the set $\mathcal{M}$ may consist of finite number of elements, i.e. $\mathcal{M} = \{A_1, A_2, ..., A_m\}$ with $A_i \in [E]$ ($1 \le i \le m$)). Then the following statements are equivalent:*

a) the discrete linear inclusion $DLI(\mathcal{M})$ is uniformly asymptotically stable;

b) $\rho(\mathcal{M}) < 1$.

Proof. Let $\Omega := C(\mathbb{Z}_+, \mathcal{M})$. Then Ω is a compact subset of $C(\mathbb{Z}_+, [E])$ and on Ω there is defined a dynamical system of translations $(\Omega, \mathbb{Z}_+, \sigma)$ induced by the Bebutov's dynamical system $(C(\mathbb{Z}_+, [E]), \mathbb{Z}_+, \sigma)$. Consider the cocycle $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ generated by $DLI(\mathcal{M})$, i.e. $\varphi(t, \omega, x) := A(t)A(t-1)...A(1)A(0)x$, where $A(t) \in \mathcal{M}$ and $\omega \in \Omega$ with $\omega(t) := A(t)$ (for all $t \in \mathbb{Z}_+$). According to Theorem 3.3 there exists two positive constants $\mathcal{N}$ and ν such that $|A(t)A(t-1)\ldots A(1)A(0)x| \le \mathcal{N}e^{-\nu t}|x|$ for all $A(t) \in \mathcal{M}$ ($t \in \mathbb{Z}_+$). From the last inequality we obtain $\rho(\mathcal{M}) \le e^{-nu} < 1$.

Let now $\alpha := \rho(\mathcal{M}) < 1$. Then for all $\varepsilon \in (0, 1 - \rho(\mathcal{M}))$ there exists a number $t(\varepsilon) \in \mathbb{N}$ such that

$$\|A(t)A(t-1)...A(1)A(0)\| \le (\alpha + \varepsilon)^t \tag{9}$$

for all $t \ge t(\varepsilon)$. Since $\beta := \alpha + \varepsilon < 1$, then from inequality (9) follows the condition a). The theorem is proved. □

Remark 3.6. *The statement close to Theorem 3.5 it was established before in [22] for infinite-dimensional DLIs and in [22, 30] for finite-dimensional DLIs.*

Lemma 3.7. *[15] Suppose that each operator A of $\mathcal{M}$ is compact, then for any bounded set $B \subset E$ and $t \in \mathbb{N}$ the set $U(t,\Omega)A$ is relatively compact, where $U(t,\omega) := \varphi(t,\cdot,\omega) = A(t)A(t-1)...A(1)A(0)$ ($\omega(t) := A(t)$ for all $t \in \mathbb{Z}_+$).*

Definition 3.8. *A cocycle $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)$ is called compact (completely continuous) if for each bounded subset $B \subset E$ there exists an integer number $t_0 = t_0(B) \in \mathbb{N}$ such that the set $U(t_0, \Omega)B$ is relatively compact.*

Theorem 3.9. *[13] Let Ω be a compact space and $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)$ be a compact cocycle. Then the following conditions are equivalent:*

(i) the cocycle φ is convergent, i.e.

$$\lim_{t \to \infty} |\varphi(t, x, \omega)| = 0 \tag{10}$$

for all $(x, \omega) \in E \times \Omega$;

(ii) the cocycle φ is uniformly exponentially stable.

Theorem 3.10. *Let $\mathcal{M} \subseteq [E]$ be a compact subset and suppose that each operator A from $\mathcal{M}$ is compact. Then the next statements are equivalent:*

(i) the discrete linear inclusion $DLI(\mathcal{M})$ is absolutely asymptotically stable;

(ii) $\rho(\mathcal{M}) < 1$.

Proof. Consider the cocycle $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)$ generated by $DLI(\mathcal{M})$. By Lemma 3.7, under the conditions of the theorem, this cocycle is compact. Now to finish the proof it is sufficient to apply Theorems 3.5 and 3.9. □

Remark 3.11. *In our paper [15] was established the equivalence of absolute asymptotic stability and uniform exponential stability for $DLI(\mathcal{M})$ in Banach spaces if $\mathcal{M} \subseteq [E]$ is compact and every operator A from $\mathcal{M}$ is compact.*

4 Asymptotically Compact Discrete Linear Inclusions

Definition 4.1. *The entire trajectory of the semigroup dynamical system $(X, \mathbb{T}, \pi)$ passing through the point $x \in X$ at $t = 0$ is defined as the continuous map $\gamma : \mathbb{S} \to X$ that satisfies the conditions $\gamma(0) = x$ and $\pi^t \gamma(s) = \gamma(s + t)$ for all $t \in \mathbb{T}$ and $s \in \mathbb{S}$, where $\pi^t := \pi(t, \cdot)$.*

Let $\Phi_x(\pi)$ be the set of all entire trajectories of $(X, \mathbb{T}, \pi)$ passing through x at $t = 0$ and $\Phi(\pi) = \cup\{\Phi_x(\pi) : x \in X\}$.

Definition 4.2. *A dynamical system $(X, \mathbb{T}, \pi)$ is said to be asymptotically compact [23, 28] if for all bounded positively invariant set $M \subset X$ there exists a nonempty compact subset K from X such that $\lim\limits_{t \to +\infty} \rho(\pi(t, M), K) = 0$.*

Definition 4.3. *A measure of non-compactness [23, 34] on a complete metric space X is a function β from the bounded sets of X to the nonnegative real numbers satisfying:*

(i) $\beta(A) = 0$ *for* $A \subset X$ *if and only if* A *is relatively compact;*

(ii) $\beta(A \cup B) = \max[\beta(A), \beta(B)]$;

(iii) $\beta(A + B) \leq \beta(A) + \beta(B)$ *for all* $A, B \subset X$ *if the space* X *is linear.*

Definition 4.4. *The Kuratowsky measure of non-compactness α is defined by*

$$\alpha(A) = \inf\{d : A \textit{ has a finite cover of diameter} < d\}.$$

Definition 4.5. *A dynamical system $(X, \mathbb{T}, \pi)$ (respectively, a cocycle φ) is said to be conditionally β-condensing [23] if there exists $t_0 > 0$ such that $\beta(\pi^{t_0} B) < \beta(B)$ for all bounded sets B in X with $\beta(B) > 0$ (respectively, for any bounded set $B \subseteq E$ the inequality $\alpha(\varphi(t_0, B, Y)) < \alpha(B)$ holds if $\alpha(B) > 0$.).*

Definition 4.6. *A dynamical system $(X, \mathbb{T}, \pi)$ (respectively, a cocycle φ) is said to be β-condensing if it is conditionally β-condensing and the set $\pi^{t_0} B$ is bounded for all bounded sets $B \subseteq X$ (respectively, the set $\varphi(t_0, B, Y) = \cup\{\varphi(t_0, u, Y) | u \in B, y \in Y\}$ is bounded for all bounded set $B \subseteq E$.)*

According to Lemma 2.3.5 in [23, p.15] and Lemma 3.3 in [9] the conditional condensing dynamical system $(X, \mathbb{T}, \pi)$ is asymptotically compact.

Let $X := E \times Y$, $A \subset X$, and $A_y := \{x \in A : pr_2 x = y\}$. Then $A = \cup\{A_y : y \in Y\}$. Let $\tilde{A}_y := pr_1 A_y$ and $\tilde{A} := \cup\{\tilde{A}_y : y \in Y\}$. Note that if the space Y is compact, then a set $A \subset X$ is bounded in X if and only if the set $\tilde{A}$ is bounded in E.

Lemma 4.7. *[11, 13] The equality $\alpha(A) = \alpha(\tilde{A})$ takes place for all bounded sets $A \subset X$, where $\alpha(A)$ and $\alpha(\tilde{A})$ are the Kuratowsky measure of non-compactness for the sets $A \subset X$ and $\tilde{A} \subset E$.*

Definition 4.8. *A cocycle φ is called conditional α-contraction of order $k \in [0, 1)$, if there exists $t_0 > 0$ such that for any bounded set $B \subseteq E$ for which $\varphi(t_0, B, Y) = \cup\{\varphi(t_0, u, Y) | u \in B, y \in Y\}$ is bounded the inequality $\alpha(\varphi(t_0, B, Y)) \leq k\alpha(B)$ holds.*

Definition 4.9. *The cocycle φ is called α-contraction if it is a conditional α-contraction cocycle and the set $\varphi(t_0, B, Y) = \cup\{\varphi(t_0, u, Y) | u \in B, y \in Y\}$ is bounded for all bounded sets $B \subseteq E$.*

Lemma 4.10. *[11, 13] Let Y be compact and the cocycle φ be α-condensing. Then the skew-product dynamical system $(X, \mathbb{T}, \pi)$, generated by the cocycle φ, is α-condensing.*

Denote by $\mathbb{B}(\pi) := \{x \in X \mid \exists \gamma \in \Phi_x(\pi)$, such that $\gamma(\mathbb{S})$ is bounded $\}$ and $\mathbb{B}_x(\pi) := \mathbb{B}(\pi) \bigcap \Phi_x(\pi)$.

Theorem 4.11. *[11, 13] Let $\langle (X,\mathbb{T},\pi),(Y,\mathbb{T},\sigma),h\rangle$ be a linear non-autonomous dynamical system, Y be compact and $(X,\mathbb{T},\pi)$ be conditionally α-condensing. Then the following assertions are equivalent:*

- *(i) (a) the non-autonomous dynamical system $\langle (X,\mathbb{T},\pi),(Y,\mathbb{T},\sigma),h\rangle$ is convergent, i.e. $\lim\limits_{t\to+\infty}|\pi(t,x)|=0$ for all $x\in X$;*
  - *(b) the dynamical system $(X,\mathbb{T},\pi)$ doesn't admit non-trivial bounded trajectories on $\mathbb{T}$, i.e. $\mathbb{B}(\pi)\subseteq\Theta=\{\theta_y : y\in Y,\theta_y\in X_y,|\theta_y|=0\}$.*
- *(ii) the non-autonomous dynamical system $\langle (X,\mathbb{T},\pi),(Y,\mathbb{T},\sigma),h\rangle$ is uniformly exponentially stable, i.e. there are positive constants $\mathcal{N}$ and ν such that $|\pi(t,x)|\le\mathcal{N}e^{-\nu t}|x|$ for all $x\in X$.*

Remark 4.12. *If the vector bundle fiber (X,h,Y) is finite dimensional (i.e. every fiber $X_y:=h^{-1}(y)$ is finite-dimensional) and Y is a compact metric space, then the non-autonomous dynamical system $\langle (X,\mathbb{T},\pi),(Y,\mathbb{T},\sigma),h\rangle$ is conditionally α-condensing. Thus Theorem 4.11 is true for the finite-dimensional linear non-autonomous dynamical system with compact base Y. This fact it was established before by Cheban [8].*

Theorem 4.13. *Let the following conditions be fulfilled:*

- *(i) the linear non-autonomous dynamical system $\langle (X,\mathbb{T},\pi),(Y,\mathbb{T},\sigma),h\rangle$ is convergent;*
- *(ii) Y is compact;*
- *(iii) the dynamical system $(X,\mathbb{T},\pi)$ is conditionally α-condensing;*
- *(iv) there exists a positive number M such that*

$$|\pi(t,x)|\le M|x| \tag{11}$$

for all $x\in X$.

Then the non-autonomous dynamical system $\langle (X,\mathbb{T},\pi),(Y,\mathbb{T},\sigma),h\rangle$ is uniformly exponentially stable.

Proof. Let $\langle (X,\mathbb{T},\pi),(Y,\mathbb{T},\sigma),h\rangle$ be a linear convergent non-autonomous dynamical system, Y be compact and there exists a positive number M such that the inequality (11) holds, then by Theorem 2.11.2 [12] (see also [10] or [13]) the trivial section $\Theta:=\{\theta_y \mid \theta_y\in X_y:=h^{-1}(y),|\theta_y|=0\}$ of the vectorial fiber (X,h,Y) is a maximal compact invariant set of dynamical system $(X,\mathbb{T},\pi)$). Thus this system doesn't admit non-trivial bounded trajectories on $\mathbb{S}$. To finish the proof of Theorem it is sufficient to apply Theorem 4.11. $\square$

Lemma 4.14. *Let $A', A'' \in [E]$, $A := A' + A''$ and the following conditions hold:*

(i) operator A' is contractive, i.e. $\|A'\| < 1$;

(ii) operator A'' is compact.

Then the operator A is α-contraction and $\alpha(A(B)) \le k\alpha(B)$ for all bounded subset $B \subseteq E$, where $k := \|A'\|$.

Proof. Since $A(B) \subseteq A'(B) + A''(B)$, then according to Lemma 2.2 [35] $\alpha(A(B)) \le \alpha(A'(B)) + \alpha(A''(B)) \le \|A'\|\alpha(B) + \alpha(A''(B))$. To finish the proof of Lemma it is sufficient to note that under the conditions of Lemma $\alpha(A''(B)) = 0$. □

Lemma 4.15. *Let $\mathcal{M}$ be a compact subset of $[E]$. Suppose that each operator A of $\mathcal{M}$ may be presented as a sum $A' + A''$, where A' is a contraction and A'' is a compact operator, then $\alpha(U(t,\Omega)B) \le k\alpha(B)$ for any bounded subset $B \subseteq E$ and $n \in \mathbb{N}$, where $U(t,\omega) := \varphi(t,\cdot,\omega) = A(t)A(t-1)...A(1)A(0)$ ($\omega(t) := A \in \mathcal{M}$ for all $t \in \mathbb{Z}_+$) and $k := \prod_{j=1}^{t} \|A(j)'\| < 1$.*

Proof. Since the set Ω is compact and $U(t,\omega) = \prod_{k=1}^{t} \omega(k)$ ($\omega \in \Omega$), then for each t the mapping $U(t,\cdot) : \Omega \to [E]$ is continuous. Note that $A(t) = A(t)' + A(t)''$ and, consequently, we have

$$U(t,\omega) := \prod_{j=1}^{t} A(j) = \prod_{j=1}^{t} (A(j)' + A(j)'') = \prod_{j=1}^{t} A(j)' + C,$$

where $C \in [E]$ is some compact operator. By Lemma 4.14 we have

$$\alpha(U(t,\omega)B) \le k_0\alpha(B)$$

for all bounded subset $B \subseteq E$, where $k_0 := \| \prod_{j=1}^{t} A(j)'\| \le \prod_{j=1}^{t} \|A(j)'\| := k < 1$. The lemma is proved. □

Theorem 4.16. *Let $\mathcal{M}$ be a compact subset of $[E]$. Suppose that each operator A of $\mathcal{M}$ may be presented as a sum $A' + A''$, here $A'$is a contraction and A'' is a compact operator. Then the following assertions are equivalent:*

(i) The discrete linear inclusion $DLI(\mathcal{M})$ is absolute asymptotic stable and $DLI(\mathcal{M})$ doesn't admit non-trivial bounded trajectories on $\mathbb{Z}$;

(ii) The discrete linear inclusion $DLI(\mathcal{M})$ is uniformly exponentially stable.

Proof. Consider the cocycle $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)$ generated by $DLI(\mathcal{M})$. By Lemma 4.15, under the conditions of the theorem, this cocycle is α-contraction. Now to finish the proof it is sufficient to apply Theorem 4.11, because every α-contraction cocycle φ is α-condensing. □

Theorem 4.17. *Let $\mathcal{M}$ be a compact subset of $[E]$. Suppose that the following conditions hold:*

(i) each operator A of $\mathcal{M}$ may be presented as a sum $A' + A''$, here $A'$is a contraction and A'' is a compact operator;

(ii) the discrete linear inclusion $DLI(\mathcal{M})$ doesn't admit non-trivial bounded trajectories on $\mathbb{Z}$;

(iii) the set $\mathcal{M} \subseteq [E]$ of operators is product bounded.

Then the discrete linear inclusion $DLI(\mathcal{M})$ is uniformly exponentially stable.

Proof. Consider the cocycle $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)$ generated by $DLI(\mathcal{M})$ and corresponding skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$, where $X := E \times Y$ and $\pi := (\varphi, \sigma)$. By Lemma 4.15, under the conditions of the theorem, this cocycle is α-contraction and, consequently, the dynamical system $(X, \mathbb{Z}_+, \pi)$ too. It is easy to verify that under the conditions of Theorem this non-autonomous dynamical system is uniformly stable, i.e. $|\pi(t,x)| \le M|x|$ for all $x := (u, y) \in X$ and $t \in \mathbb{Z}_+$ because $|\pi(t,x)| = |U(t,\omega)u| \le M|u| = M|x|$, where $U(t,\omega) := \varphi(t,\cdot,\omega) = A(t)A(t-1)\dots A(1)A(0)$ $(\omega(j) := A(j) \in \mathcal{M}$ for all $j \in \mathbb{Z}_+)$ and $\mathcal{M} \subset [E]$ is product bounded.

Now we will show that non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}_+, \sigma), h\rangle$ is convergent. In our case this means that $\lim\limits_{t\to+\infty} |\pi(t,x)| = 0$ for all $x \in X$. Indeed, the system $(X, \mathbb{Z}_+, \pi)$ is α-contraction and, consequently, it is asymptotically compact. The trajectory $\{\pi(t,x) \mid t \in \mathbb{Z}_+\}$ is bounded and consequently it is relatively compact. Denote by ω_x the ω-limit set of point x. This set is nonempty, compact and invariant. In particular ω_x consists of the full trajectories of $DLI(\mathcal{M})$ bounded on $\mathbb{Z}$. Under the conditions of our Theorem $\omega_x \subseteq \Theta := \{(0, y) \mid y \in Y\}$ and, consequently, $\lim\limits_{t\to+\infty} |\pi(t,x)| = 0$. Now to finish the proof it is sufficient to apply Lemma 4.15 and Theorem 4.13. □

Theorem 4.18. *Let $\mathcal{M}$ be a compact subset of $[E]$and each operator A of $\mathcal{M}$ may be presented as a sum $A' + A''$, here $A'$is a contraction and A'' is a compact operator.*

Then the following affirmations are equivalent:

(i) the discrete linear inclusion $DLI(\mathcal{M})$ is product bounded and absolutely asymptotically stable;

(ii) the set $\mathcal{M}$ is generalized contractive, i.e. there exist positive numbers $\mathcal{N}$ and ν such that $\|A(t)A(t-1)\dots A(1)\| \le \mathcal{N} e^{-\nu t}$ for all $t \in \mathbb{N}$ and $A(j) \in \mathcal{M}$ $(1 \le j \le t)$.

Proof. Consider the cocycle φ generated by $DLI(\mathcal{M})$. By Lemma 4.15 , under the conditions of theorem, this cocycle is α-condensing. Now to finish the proof it is sufficient to refer Theorem 4.13. □

Definition 4.19. *A dynamical system $(X,\mathbb{T},\pi)$ is called locally compact (locally completely continuous) if for any $x \in X$ there are $\delta_x > 0$ and $l_x > 0$ such that $\pi^t B(x,\delta_x)$ $(t \geq l_x)$ is relatively compact.*

Remark 4.20. *Note that the dynamical system $(X,\mathbb{T},\pi)$ is locally compact (completely continuous), if one of the following two conditions holds:*

(i) the phase space X of dynamical system $(X,\mathbb{T},\pi)$ is locally compact;

(ii) there exists a number $t_0 \in \mathbb{T}$ such that the operator π^t is completely continuous, where $\pi^t := \pi(t,\cdot)$.

Theorem 4.21. *[8] Let $(X,\mathbb{T},\pi)$ be locally compact and Y be compact. Then the following conditions are equivalent:*

1. $\lim\limits_{t\to+\infty} |xt| = 0$ *for all* $x \in X$;
2. *all the motions in $(X,\mathbb{T},\pi)$ are relatively compact and $(X,\mathbb{T},\pi)$ does not admit nontrivial compact motions defined on $\mathbb{S}$;*
3. *there are positive numbers $\mathcal{N}$ and ν such that $|xt| \leq \mathcal{N} e^{-\nu t}|x|$ for all $x \in X$ and $t \geq 0$.*

Theorem 4.22. *Let $\mathcal{M}$ be a compact subset of $[E]$. Suppose that each operator A of $\mathcal{M}$ is compact. Then the following assertions are equivalent:*

1. *the discrete linear inclusion $DLI(\mathcal{M})$ is absolute asymptotic stable;*
2. *every solution of $DLI(\mathcal{M})$ is relatively compact and $DLI(\mathcal{M})$ doesn't admit non-trivial bounded trajectories on $\mathbb{Z}$;*
3. *the discrete linear inclusion $DLI(\mathcal{M})$ is uniformly exponentially stable.*

Proof. Consider the cocycle $\langle E,\varphi,(\Omega,\mathbb{Z}_+,\sigma)$ generated by $DLI(\mathcal{M})$. By Lemma 4.15, under the conditions of Theorem, the cocycle φ is α-contraction and, consequently, the skew-product dynamical system $(X,\mathbb{T},\pi)$ ($X := E\times\Omega$, $\pi := (\varphi, \sigma)$), generated by cocycle φ, is completely continuous. Now to finish the proof it is sufficient to apply Theorem 4.21 to non-autonomous dynamical system $\langle(X,\mathbb{T},\pi), (\Omega,\mathbb{Z}_+,\sigma), h\rangle$ $(h := pr_2 : X \to \Omega)$. $\square$

Remark 4.23. *Note that for finite-dimensional Banach space E the equivalence of the statements 1. and 2. it was proved by Kozyakin [27].*

5 Absolute Asymptotic Stability of Finite-Dimensional Discrete Linear Inclusions

Let $(X, \mathbb{T}, \pi)$ be a dynamical system.

Definition 5.1. *Let $\mathbb{T}'$ be a subset of group $\mathbb{S}$ and $0 \in \mathbb{T}'$. The continuous mapping $\gamma : \mathbb{T}' \to X$ is called a trajectory of the point $x \in X$ on $\mathbb{T}'$ if $\pi^t\gamma(s) = \gamma(t+s)$ for all $t \in \mathbb{T}$ and $s \in \mathbb{T}'$ such that $t + s \in \mathbb{T}'$.*

Lemma 5.2. *[39] Let $\{\mathbb{T}_n\}$ be a family of subsets of $\mathbb{S}$ and the following conditions are fulfilled:*

(i) $\mathbb{T}_n \subseteq \mathbb{T}_{n+1}$ *for all* $n \in \mathbb{N}$;

(ii) γ_n *is the trajectory on* $\mathbb{T}_n$ *of the point* $x_n \in X$;

(iii) *the sequence* $\{x_n\} \subseteq X$ *converges to* $x \in X$.

Then there exists a trajectory on $\mathbb{T}' := \bigcup\{\mathbb{T}_n \ : n \in \mathbb{N}\}$ of the point $x \in X$ such that $\{\gamma_n\}$ converges to γ uniformly on the compacts from $\mathbb{T}'$, i.e. for every compact $K \subseteq \mathbb{T}'$ and positive number ε there exists a number $n_0 = n_0(\varepsilon, K) \in \mathbb{N}$ such that $K \subseteq \mathbb{T}_n$ and $\rho(\gamma_n(s), \gamma(s)) < \varepsilon$ for all $n \geq n_0$ and $s \in K$, where ρ is the distance on X.

Remark 5.3. *If $(X, \mathbb{T}, \pi)$ is a skew-product dynamical system generated by cocycle φ and $x := (u, y) \in X := E \times Y$, then $\gamma \in \Phi_x(\pi) \subseteq \Phi(\pi)$ if and only if there exist a continuous function $\nu : \mathbb{S} \to E$ and $\tilde{\gamma} \in \Phi_{h(x)}(\sigma)$such that $\varphi(t, \nu(s), y) = \nu(t+s)$ and $\gamma(s) = (\nu(s), \tilde{\gamma}(s))$ for all $t \in \mathbb{T}$ and $s \in \mathbb{S}$.*

Definition 5.4. *Let (X, h, Y) be a Banach fiber bundle with norm $|\cdot|$. The non-autonomous dynamical system $\langle(X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is said to be non-critical [25] (satisfying Favard's condition) if $B(\pi) = \Theta$, where $\Theta := \{\theta_y \mid \theta_y \in X_y,\ |\theta_y| = 0, y \in Y\}$.*

Remark 5.5. *Throughout the rest of this section we assume that the Banach fiber bundle (X, h, Y) is finite-dimensional, Y is compact and invariant (i.e. $\sigma^t Y = Y$ for all $t \in \mathbb{T}$, where $\sigma^t := \sigma(t, \cdot)$) and the non-autonomous linear dynamical system $\langle(X,\ \mathbb{T}, \pi),\ (Y, \mathbb{T}, \sigma),\ h\rangle$ is non-critical.*

Denote by $\omega_x := \bigcap\limits_{t \geq 0} \overline{\bigcup\{\pi(s, x) \ : \ s \geq t\}}$ and $\alpha_\gamma := \bigcap\limits_{t \leq 0} \overline{\bigcup\{\gamma(s) \ : \ s \leq t\}}$ if $\gamma \in \Phi(\pi)$.

Lemma 5.6. *Let $\langle(X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ be a linear non-critical non-autonomous dynamical system and $x \in X$. Then the following statements hold:*

(i) *if* $\sup\{|\pi(t, x)| \ : \ t \in \mathbb{T}, t \geq 0\} < +\infty$, *then*

$$\lim_{t \to +\infty} |\pi(t, x)| = 0; \tag{12}$$

(ii) if $\gamma \in \Phi_x(\pi)$ *and* $\sup\{|\gamma(s)| \ : s \in \mathbb{S}, s \leq 0\} < +\infty$*, then*

$$\lim_{s \to -\infty} |\gamma(s)| = 0. \tag{13}$$

Proof. Let $\pi(\mathbb{T}_+, x)$ ($\mathbb{T}_+ := \{t \in \mathbb{T} \ : t \geq 0\}$) be bounded. Since (X, h, Y) is a locally trivial finite-dimensional fiber bundle and Y is compact, then the set $\pi(\mathbb{T}, x)$ is relatively compact and, consequently, $\omega_x \neq \emptyset$. Suppose that the equality (12) is not true. Then there exists a positive number ε_0 and strictly increasing sequence $t_n \to +\infty$ such that

$$|\pi(t_n, x)| \geq \varepsilon_0. \tag{14}$$

Without loss of generality we may suppose that the sequence $\{\pi(t_n, x)\}$ is convergent. Let $x_0 := \lim_{n \to +\infty} \pi(t_n, x)$. Denote by $\mathbb{T}_n := \{s \in \mathbb{S} \ : \ s \geq -t_n\}$ and $\gamma_n : \mathbb{T}_n \to X$ the continuous mapping defined by equality $\gamma_n(s) := \pi(s + t_n, x)$. It is easy to verify that γ_n is a trajectory of the point $x_n := \pi(t_n, x)$ on $\mathbb{T}_n$. By Lemma 5.2 the sequence $\{\gamma_n\}$ is convergent and its limit $\gamma \in \Phi_{x_0}(\pi)$. Note that $\gamma(s) \in \omega_x$ for all $s \in \mathbb{S}$ and $\gamma(0) = x_0$. According to (14) $x_0 \neq 0$. The obtained contradiction proves the first statement.

The second statement may be proved similarly. □

Denote by $X^s := \{x \in X \ : \ \lim_{t \to +\infty} |\pi(t, x)| = 0\}$ and $X^u := \{x \in X \ : \ \exists \gamma \in \Phi_x(\pi)$ such that $\lim_{t \to -\infty} |\gamma(t)| = 0\}$.

Lemma 5.7. *The following statement hold:*

(i) the set X^s *is vectorial, i.e. every fiber* $X^s_y := X^s \bigcap X_y$ *is a subspace of the linear space* X_y*;*

(ii) X^s *is a positively invariant subset of dynamical system* $(X, \mathbb{T}, \pi)$*, i.e.* $\pi^t X^s \subseteq X^s$ *for all* $t \in \mathbb{T}$*;*

(iii) the set X^s *is closed.*

Proof. The first and second statements are evident.

Let $a \in \overline{X^s} \setminus X^s$, then there exists $\tilde{x}_n \in E^s$ such that $\{\tilde{x}_n\} \to a$. Let $l_n := \sup\{|\pi(t, \tilde{x}_n)| \ : \ t \in \mathbb{T}\}$ and $\tau_n \in \mathbb{T}$ such that $l_n = |\pi(\tau_n, \tilde{x}_n)|$. Note that $\{l_n\} \to +\infty$ and $\{\tau_n\} \to +\infty$. We may suppose that the sequence $\{\tau_n\}$ is increasing. Assume that $x_n := l_n^{-1}\pi(\tau_n, \tilde{x}_n)$, then $|x_n| = 1$. Now we define the continuous mapping $\gamma_n : \mathbb{T}_n \to X$ by equality $\gamma_n(\tau) := l_n^{-1}\pi(\tau + \tau_n, \tilde{x}_n)$, where $\mathbb{T}_n := \{t \in \mathbb{S} \ : s \geq -\tau_n\}$. Then

(i) $\gamma_n(0) = x_n$;

(ii) γ_n is the trajectory on $\mathbb{T}_n$ of the point x_n;

(iii)
$$|\gamma_n(t)| \le 1 \tag{15}$$
for all $t \in \mathbb{T}_n$.

Without loss of generality we may suppose that the sequence $\{x_n\}$ is convergent. Let $x := \lim\limits_{n\to+\infty} x_n$, then $|x| = 1$. By Lemma 5.2 the sequence $\{\gamma_n\}$ is convergent uniformly on the compacts from $\mathbb{S}$ and if $\gamma := \lim\limits_{n\to+\infty} \gamma_n$, then $\gamma \in \Phi_x(\pi)$. From the inequality (15) follows that $|\gamma(t)| \le 1$ for all $t \in \mathbb{S}$, i.e. $\gamma \in \mathbb{B}(\pi)$, and $\gamma(0) = x \neq 0$. Under the conditions of the lemma we have $\mathbb{B}(\pi) = \Theta$. The obtained contradiction proves our statement. □

Lemma 5.8. *There exist positives numbers $\mathcal{N}$ and ν such that*
$$|\pi(t,x)| \le \mathcal{N} e^{-\nu t} \tag{16}$$
for all $x \in X^s$ and $t \ge 0$.

Proof. According to Lemma 5.7 the subset X^s is a positively invariant and closed subset of dynamical system $(X,\mathbb{T},\pi)$ and, consequently, on X^s is induced a dynamical system $(X^s,\mathbb{T},\pi)$. Now to finish the proof of Lemma it is sufficient to refer to Theorem 3.9. □

Lemma 5.9. *The following statement hold:*

1. *for any $x \in E^u$ the set $\Phi_x(\pi)$ contains a unique trajectory γ with condition $\lim\limits_{t\to-\infty} |\gamma(t)| = 0$;*
2. *the set X^u is vectorial;*
3. *X^u is a positively invariant subset of dynamical system $(X,\mathbb{T},\pi)$;*
4. *$E^s_y \bigcap E^u_y = \{\theta_y\}$ for all $y \in Y$, where $E^i_y := E^i \bigcap X_y$ $(i = s,u)$;*
5. *the set X^u is closed.*

Proof. Suppose that the first statement of Lemma is not true, then there exist $x_0 \in X^u$ and $\gamma_1,\gamma_2 \in \Phi_{x_0}$ such that $\gamma_1 \neq \gamma_2$ and $\lim\limits_{t\to-\infty} |\gamma_i(t)| = 0$ $(i = 1,2)$. Let $\gamma := \gamma_1 - \gamma_2$, then by linearity of non-autonomous dynamical system $\langle (X,\mathbb{T},\pi),(Y,\mathbb{T},\sigma),h\rangle$ we have $\gamma \in \Phi_{\theta_{y_0}}$, where θ_{y_0} is the zero in the space X_{y_0} and $y_0 := h(x_0)$. In addition we have

(i) $\lim\limits_{t\to-\infty} |\gamma(t)| = 0$;

(ii) $|\gamma(t)| = 0$ for all $t \ge 0$ since $\gamma_1(t) = \gamma_2(t) = \pi(t,x_0)$ for all $t \ge 0$.

Thus we found $\gamma \in \mathbb{B}(\pi) \setminus \Theta$. The obtained contradiction proves our affirmation.

The statements 2–4 are evident.

Let $b \in \overline{X^u} \setminus X^u$, then there exists $\tilde{x}_n \in E^u$ such that $\{\tilde{x}_n\} \to b$. Let $l_n := \sup\{|\pi(t, \tilde{x}_n)| \ : \ t \le 0\}$ and $\tau_n \le 0$ such that $l_n = |\pi(\tau_n, \tilde{x}_n)|$. Note that $\{l_n\} \to +\infty$ and $\{\tau_n\} \to -\infty$. We may suppose that the sequence $\{\tau_n\}$ is decreasing. Assume that $x_n := l_n^{-1}\pi(\tau_n, \tilde{x}_n)$, then $|x_n| = 1$. Now we define the continuous mapping $\gamma_n : \mathbb{S} \to X$ by equality $\gamma_n(\tau) := l_n^{-1}\pi(\tau + \tau_n, \tilde{x}_n)$. Then

(i) $\gamma_n(0) = x_n$;

(ii) γ_n is the full trajectory of the point x_n, i.e. $\gamma_n \in \Phi_{x_n}(\pi)$;

(iii)
$$|\gamma_n(t)| \le 1 \tag{17}$$
for all $t \in \mathbb{T}_n := \{t \in \mathbb{S} \ : \ t \le -\tau_n\}$.

Without loss of generality we may suppose that the sequence $\{x_n\}$ is convergent. Let $x := \lim\limits_{n\to+\infty} x_n$, then $|x| = 1$. By Lemma 5.2 the sequence $\{\gamma_n\}$ is convergent uniformly on the compacts from $\mathbb{S}$ and if $\gamma := \lim\limits_{n\to+\infty} \gamma_n$, then $\gamma \in \Phi_x(\pi)$. From the inequality (17) follows that $|\gamma(t)| \le 1$ for all $t \in \mathbb{S}$, i.e. $\gamma \in \mathbb{B}(\pi)$, and $\gamma(0) = x \ne 0$. The obtained contradiction prove our statement. □

Lemma 5.10. *On the set X^u is defined a group dynamical system $(X^u, \mathbb{S}, \tilde{\pi})$, where the mapping $\tilde{\pi} : \mathbb{S} \times X \to X$ is defined by equality $\tilde{\pi}(t, x) := \gamma_x(t)$ and γ_x is a unique trajectory from $\Phi_x(\pi)$ with condition $\lim\limits_{t\to-\infty} |\gamma_x(t)| = 0$.*

Proof. This statement directly follows from Lemmas 5.2 and 5.9 . □

Corollary 5.11. *The following statements hold:*

(i) there exist positive constants N and ν such that $|\tilde{\pi}(t, x)| \le Ne^{\nu t}|x|$ for all $x \in X^u$ and $t \le 0$ ($t \in \mathbb{S}$);

(ii) $\pi^t X^u_y = X^u_{\sigma(t,y)}$ for all $y \in Y$ and $t \ge 0$.

Proof. First statement of Corollary follows from Lemma 5.8 if we will change t by $-t$ in group dynamical system $(X^u, \mathbb{S}, \pi)$.

The second statement is evident. □

Lemma 5.12. *The following statements hold:*

(i)
$$\limsup_{t\to+\infty} |\pi(t, x)| = \liminf_{t\to+\infty} |\pi(t, x)| \tag{18}$$
for all $x \in X$ and also $\lim\limits_{t\to+\infty} |\pi(t, x)| = +\infty$ for all $x \notin X^s$;

(ii)

$$\limsup_{t\to-\infty}|\gamma(t)| = \liminf_{t\to-\infty}|\gamma(t)| \tag{19}$$

for all $x \in X$ and $\gamma \in \Phi_x(\pi)$ moreover $\lim\limits_{t\to-\infty}|\gamma(t)| = +\infty$ for all $x \notin X^u$.

Proof. Denote by

$$L := \limsup_{t\to+\infty}|\pi(t,x)|;\ \ l := \liminf_{t\to+\infty}|\pi(t,x)|. \tag{20}$$

Then $0 \le l \le L$. It is sufficient to consider the case $L \neq 0$. If $L < +\infty$, then $x \in \mathbb{B}^+(\pi) := \{x \in X \mid \sup\{|\pi(t,x)| \ : \ t \ge 0\} < +\infty\}$ and by Lemma 5.6 $L = l = 0$. Thus, if $L \neq 0$, then $L = +\infty$.

If $L = l = +\infty$, then Lemma is proved. If $l < +\infty$, then there exist sequences $\{\tau_n\}$ and $\{t_n\}$ ($\{t_n\} \subseteq \mathbb{T}$) such that $t_n \le \tau_n \le t_{n+1}$ ($\{t_n\} \to +\infty,\ t_{n+1} - t_n \ge n+1$) and $|\pi(\tau_n,x)| = \nu_n$, where $\nu_n := \max\{|pi(t,x)| \ : \ t \in [t_n, t_{n+1}]\}$. Since $L = +\infty$, then $\{\nu_n\} \to +\infty$. Assume $x_n := \nu_n^{-1}\pi(\tau_n,x)$ and $y_n := \sigma(\tau_n,y)$. Then $|x_n| = 1$

$$|\pi(x_n,t)| = \nu_n^{-1}|\pi(t+\tau_n,x)| \le 1 \tag{21}$$

for all $t \in [t_n, t_{n+1}]$. Let $\mathbb{T}_n := \{t \in \mathbb{S} \ : \ s \ge -\tau_n\}$ and $\gamma_n : \mathbb{T}_n \to X$ be a continuous function defined by equality

$$\gamma_n(t) := \nu_n^{-1}\pi(t+\tau_n,x). \tag{22}$$

Note that

$$|\gamma_n(t)| := \nu_n^{-1}|\pi(t+\tau_n,x)| \le 1 \tag{23}$$

for all $t \in [t_n - \tau_n, t_{n+1} - \tau_n]$. In addition

$$|\gamma_n(t)| \le \nu_n^{-1}(l+1) \tag{24}$$

when $t = t_n - \tau_n$ and $t_{n+1} - \tau_n$. From the inequality (24) it follows that $\{t_n - \tau_n\} \to -\infty$ and $\{t_{n+1} - \tau_n\} \to +\infty$. In fact, without loss of generality we may suppose that the sequence $\{x_n\}$ converges. Denote by $x* := \lim\limits_{n\to+\infty} x_n$. If we suppose, for example, that the sequence $\{s_n\} := \{t_{n+1} - \tau_n\}$ converges to s_0 then according to Lemma 5.2 the sequence $\{\gamma_n\}$ converges and its limit $\gamma \in \Phi_{x*}(\pi)$. From the inequality (24) we have $|\gamma(s_0)| = 0$ and, consequently, $|\gamma(t+s_0)| = |\pi^t\gamma(s_0)| = 0$ for all $t \ge 0$. On the other hand from (23) we have $|\gamma(t)| \le 1$ for all $t \le s_0$, since $t_{n+1} - t_n \ge n+1$ for all $n \in \mathbb{N}$. Thus we found $\gamma \in \mathbb{B}(\pi)$ with $\gamma(0) = x*$. Analogously we will obtain the contradiction if we suppose that the sequence $\{t_n - \tau_n\}$ does not converge to $-\infty$. The obtained contradiction proves required statement.

Thus $\{t_n - \tau_n\} \to -\infty$ and $\{t_{n+1} - \tau_n\} \to +\infty$, then according to inequality (23) we have $|\gamma(t)| \le 1$ for all $t \in \mathbb{S}$ and $\gamma(0) = x*$. This contradiction proves the first affirmation of Lemma.

Now we will prove the second statement. Note that $0 \le l \le L$ and $L \neq 0$. If $L < +\infty$, then $x := \gamma(0) \in X^u_y$ ($y := h(x)$) and, consequently, $0 = l = L$ and the required statement is proved. Let $L = +\infty$. We will show that $l = +\infty$. If we suppose that $l < +\infty$, then there exist sequences $\{\tau_n\}$ and $\{t_n\}$ ($t_n \le 0$, $t_n - t_{n+1} > n+1$) such that

(i) $\tau_n \in [t_{n+1}, t_n]$;

(ii) $|\gamma(t_n)| \le l + 1$;

(iii) $|\gamma(\tau_n)| = \nu_n := \max\limits_{t_{n+1} \le t \le t_n} |\gamma(t)|$.

Since $L = +\infty$, then $\{\nu_n\} \to +\infty$. Let $x_n := \nu_n^{-1}\gamma(\tau_n)$, then $|x_n| = 1$. We define $\gamma_n \in \Phi_{x_n}(\pi)$ by equality

$$\gamma_n(t) := \nu_n^{-1}\gamma(t + \tau_n) \ (t \in \mathbb{S}). \tag{25}$$

It is easy to see that

$$|\gamma_n(t)| = \nu_n^{-1}|\gamma(t + \tau_n)| \le 1 \ (\forall t \in [t_{n+1} - \tau_n, t_n - \tau_n]) \tag{26}$$

and

$$|\gamma_n(t)| \le \nu_n^{-1}(l + 1) \tag{27}$$

for $t = t_{n+1} - \tau_n$ and $t_n - \tau_n$. We may suppose that the sequence $\{x_n\}$ is convergent. Let $x* := \lim\limits_{n \to +\infty} x_n$, then $|x*| = 1$. Reasoning analogously as in the proof of the first statement we may prove that $\{t_{n+1} - \tau_n\} \to -\infty$ and $\{t_n - \tau_n\} \to +\infty$. By Lemma 5.2 the sequence $\{\gamma_n\}$ converges uniformly on the compacts from $\mathbb{S}$ and its limit $\gamma \in \Phi_{x*}(\pi)$. Therefore from (26) it follows that $|\gamma(t)| \le 1$ for all $t \in \mathbb{S}$. The obtained contradiction completes the proof of Lemma. $\square$

Denote by $k_y^s := dim(X_y^s)$ (respectively, $k_y^u := dim(X_y^u)$) the dimension of the space X_y^s (respectively, X_y^u).

Lemma 5.13. *[6] Let (X, h, y) be a finite-dimensional fiber bundle with compact base Y, $\{y_n\} \to y$ and E_n be a subspace of the fiber X_{y_n} with $dim(E_n) \ge l$. If*

$$L = \limsup E_n := \bigcap_{m=1}^{\infty} \overline{\bigcup \{E_n \ : \ n \ge m\}},$$

then the linear subspace span(L) *generated by L (i.e. the minimal space, containing L) is a subspace of the fiber X_y and* $\dim($span$(L)) \ge l$.

Lemma 5.14. *$k_y^u \le k_p^u$ for all $p \in \omega_y$.*

Proof. Let $p \in \omega_y$, then there exists a sequence $\{t_n\} \to +\infty$ such that $p = \lim\limits_{n \to +\infty} \sigma(t_n, y)$. Let $y_n := \sigma(t_n, y)$, $E_n := X_{y_n}^u$, $U := \limsup E_n$ and $l := k_y^u$. Now to finish the proof of Lemma, according to Lemma 5.13, it sufficient to show that $U \subseteq X_p^u$. Assume that it is not true, i.e. there exists $x \in U \setminus X_p^u$. From the definition of U it follows that $tx \in E$ for all $t \in \mathbb{R}$ and, consequently, we may suppose that $|x| = 1$. Then there exist $x_n \in X^u_{y_n}$ such that $|x_n| = 1$ and $\{x_n\} \to x$. By Lemma 5.12 $\lim\limits_{t \to -\infty} |\tilde{\pi}(t, x)| = +\infty$ and, consequently, for every $L > 0$ there exist $n = n(L) \in \mathbb{N}$ and $t_0 = t_0(L) < 0$ such that $|\tilde{\pi}(t_0, x_n)| \le L$ for all $n \ge n_0$. Since $x \in X^u_{y_n}$, then there exists a unique $\gamma_n \in \Phi_{x_n}(\pi)$ with condition $\lim\limits_{t \to -\infty} |\gamma_n(t)| = 0$, therefore there is $t_1 < t_0$ such that $||\gamma_n(t_1) \le 1$. Let $\nu_n := \max\{|\gamma_n(t)| \ : \ t \in [t_1, 0]\}$, then $\nu_n > L$. We will choose a number $s_n \in [t_1, 0]$ such that $\nu_n = |\gamma_n(s_n)|$. Denote by $z_n := \nu_n^{-1}\gamma_n(s_n)$ and consider the sequence $\{\tilde{\gamma}_n\}$ defined by equality

$$\tilde{\gamma}_n(t) := \nu_n^{-1}\gamma_n(t + s_n) \ (t \in \mathbb{S}).$$

It is clear that $\tilde{\gamma}_n \in \Phi_{z_n}(\pi)$ and $\lim\limits_{t \to -\infty} |\tilde{\gamma}_n(t)| = 0$. Note that

$$|\tilde{\gamma}_n(t)| = \nu_n^{-1}|\gamma_n(t + s_n)| \le 1 \tag{28}$$

for $t \in [t_1 - s_n, -s_n]$. If now $L \to +\infty$, then $s_n \to -\infty$ and $t_1 - s_n \to -\infty$. In fact, from the equality $\nu_n = |\gamma_n(s_n)|$ and inequality $\nu_n > L$ it follows that $s_n \to -\infty$.

Without loss of generality we may suppose that the sequence $\{z_n\}$ is convergent. Let $z := \lim\limits_{n \to +\infty} z_n$, then by Lemma 5.2 the sequence $\{\tilde{\gamma}_n\}$ is convergent too and its limit $\tilde{\gamma} \in \Phi_z(\pi)$. On the other hand by inequality (28) we have $|\tilde{\gamma}(t)| \le 1$ for all $t \in \mathbb{S}$ and $|\tilde{\gamma}(0)| = |z| = 1$. The obtained contradiction completes the proof of Lemma. $\square$

Lemma 5.15. *Let U_y be certain complementary subspace for subspace X^s_y, i.e. $X_y = X^s_y \dot{+} U_y$. Then there exists a positive number δ such that*

$$|\pi(s, x)| \ge \delta|\pi(t, x)| \tag{29}$$

for all $x \in U_y$ and $s \ge t \ge 0$.

Proof. Suppose that it is not true. Then there exist sequences $\{t_k\}$, $\{s_k\}$ and $\{x_k\}$ such that

$$s_k \ge t_k \ge 0, \ x_k \in U_y \ (|x_k| = 1) \tag{30}$$

and

$$|\pi(t_k, x_k)| \ge k|\pi(s_k, x_k)|. \tag{31}$$

Without loss of generality we may assume that the sequence $\{x_k\}$ is convergent and denote by x its limit, then $|x| = 1$.

Let τ_k be chosen so that $0 \le \tau_k \le s_k$ and $|\pi(\tau_k, x_k)| = \max\{|\pi(t, x_k)| \ : \ t \in [0, s_k]\}$. It is easy to see that $\{\tau_k\} \to +\infty$ as $k \to +\infty$. If we suppose that it is not so, then we will have $|\pi(s, x_k)| \le M$ for all $s \in [0, s_k]$ (M is a certain positive constant) and, consequently, $x \in E^s_y$. The obtained contradiction proves our statement.

Denote by $\xi_k := |\pi(\tau_k, x_k)|^{-1}\pi(\tau_k, x_k)$, then $|\xi_k| = 1$ and $\xi_k \in \pi^{\tau_k}U_y$. Let $\mathbb{T}_k := \{t \in \mathbb{S} \ : \ s \ge -\tau_k\}$ and define the mapping $\gamma_k : \mathbb{T}_k \to X$ by equality

$$\gamma_k(t) := |\pi(\tau_k, x_k)|^{-1}\pi(t + \tau_k, x_k), \tag{32}$$

then

$$|\gamma_k(t)| \le 1 \tag{33}$$

for all $t \in [-\tau_k, s_k - \tau_k]$. Logically there are two possibilities:

a) the sequence $\{s_k - \tau_k\} \to s \ge 0$ (or it contains a convergent subsequence), then

$$|\pi(\tau_k, x_k)| = \max_{0 \le s \le s_k} |\pi(s, x_k)| \ge |\pi(t_k, x_k)| \ge k|\pi(s_k, x_k)|$$

and, consequently,

$$\frac{|\pi(s_k, x_k)|}{|\pi(\tau_k, x_k)|} \le \frac{1}{k}. \tag{34}$$

From (32) and (34) we obtain

$$|\gamma_k(s_k - \tau_k)| = \frac{|\pi(s_k, x_k)|}{|\pi(\tau_k, x_k)|} \le \frac{1}{k}. \tag{35}$$

We may suppose that the sequence $\{\xi_k\} \to \xi$, then by Lemma 5.2 the sequence $\{\gamma_k\}$ is convergent (uniformly on the compacts from $\mathbb{S}$) too. Denote by $\gamma := \lim_{k \to +\infty} \gamma_k$, then from (35) we have $|\gamma(s)| = 0$ and, consequently, $|\gamma(t)| = 0$ for all $t \ge s$. On the other hand from (33) we obtain $|\gamma(t)| \le 1$ for all $t \le 0$ and, consequently, $\gamma \in \mathbb{B}(\pi)$ and $|\gamma(0)| = |x| = 1$.

b) $\{s_k - \tau_k\} \to +\infty$ and from (33) we have $|\gamma(t)| \le 1$ for all $t \in \mathbb{S}$.

Thus, if we suppose that the statement of Lemma is not true we obtain the contradiction. The lemma is proved. □

Lemma 5.16. *Let $y \in Y$ and $H^+(y) := \overline{\{\sigma(t, y) \ : \ t \in \mathbb{T}\}} = Y$. Then $k_y^u \ge k_y - k_p^s$ for all $p \in \omega_y$, where $k_y := \dim(X_y)$.*

Proof. Let U_y be certain subspace of X_y which is complementary subspace for X_y^s and $p \in \omega_y$. Then there exists a sequence $\{t_n\} \to +\infty$ such that $\{\sigma(t_n, y)\} \to p$. Denote by $U_n := \pi(t_n, U_y)$. By Lemma 5.15 $\dim(U_n) = \dim(U_y)$. Let $U := \limsup U_n$ and $x \in U$ ($|x| = 1$). Then there is $\{x_n\}$ ($x_n \in U_n$, $|x_n| = 1$) such that $x = \lim_{n \to +\infty} x_n$. We will prove that $x \in X_p^u$ (i.e. $U \subseteq E_p^u$). Let $\mathbb{T}_n := \{t \in \mathbb{S} \ : \ s \ge -t_n\}$ and we define the function $\gamma_n : \mathbb{T}_n \to X$ by following equality:

$$\gamma_n(t) := \pi(t, x_n) = \pi(t + t_n, \tilde{x}_n), \tag{36}$$

where $\tilde{x}_n \in U$ and $\pi(t_n, \tilde{x}_n) = x_n$. Since $\gamma_n(0) = x_n \to x$, then by Lemma 5.2 the sequence $\{\gamma_n\}$ is convergent too and its limit $\gamma \in \Phi_x(\pi)$. Note that

$$|x| = |\pi^t \gamma(-t)| = |\pi^t \lim_{n \to +\infty} \pi(-t + t_n, \tilde{x}_n)| = \tag{37}$$

$$\lim_{n \to +\infty} |\pi^t \pi(-t + t_n, \tilde{x}_n)| \ge \delta | \lim_{n \to +\infty} \pi(-t + t_n, \tilde{x}_n)| = \delta |\gamma(-t)|$$

for all $t \ge 0$. From the inequality (37) follows $|\gamma(t)| \le \delta^{-1}|x|$ for all $t \le 0$ and by Lemma 5.6 $x \in X_p^u$ ($p = h(x)$). Thus $\dim(X_p^u) \ge k_y - \dim(X_y^s)$. □

Corollary 5.17. *Under the conditions of Lemma 5.16, if additionally the point $y \in Y$ is stable in the sense of Poisson (i.e. $y \in \omega_y$), then $k^s_y + k^u_y = k_y$.*

Proof. In fact, by Lemma 5.16 we have $k^u_p \geq k_y - k^s_y$ for all $p \in \omega_y$. In particular $k^u_y \geq k_y - k^s_y$ because $y \in \omega_y$. On the other hand $k^s_y + k^u_y \leq k_y$ and, consequently, $k^s_y + k^u_y = k_y$. □

Corollary 5.18. *Under the conditions of Corollary 5.17 $k^s_{y\tau} + k^u_{y\tau} = k_y$ for all $\tau \in \mathbb{T}$, where $y\tau := \sigma(\tau, y)$.*

Proof. Note that the point $y\tau$ is stable in the sense of Poisson, $\omega_{y\tau} = \omega$ and $H^+(y\tau) = \omega_{y\tau} = \omega_y = H^+(y) = Y$. According to Corollary 5.17 we have $k^s_{y\tau} + k^u_{y\tau} = k_y$ for all $\tau \in \mathbb{T}$. □

Remark 5.19. *Note that the statements close to Lemmas 5.6 – 5.16 before were established for bilateral (i.e. when $\mathbb{T} = \mathbb{S}$) non-autonomous linear dynamical systems in the works [6, 25, 33].*

Lemma 5.20. *Under the conditions of Lemma 5.16, if additionally the point $y \in \omega_y$, then $k^s_p \geq k^s_y$ for all $p \in \omega_y$.*

Proof. Let $p \in \omega_y$, then there exists $\{t_n\} \to +\infty$ such that $\{yt_n\} \to p$. By Corollary 5.11 $k^u_{yt_n} = k^u y$ and according to Corollary 5.18 we have $k^s_{yt_n} = k_y - k^u_{yt_n} = k_y - k^u_y = k^s_y$ for all $n \in \mathbb{N}$. Let $V := \limsup X^s_{yt_n}$, then by Lemma 5.7 $V \subseteq X^s_p$. Since $\dim(V) = \lim\limits_{n\to+\infty} \dim(X^s_{yt_n}) = k_y - k^u_y = k^s_y$ and $\dim(X^s_p) \geq \dim(V)$, then $k^s_p \geq k^s_y$. □

Corollary 5.21. *Under the conditions of Lemma 5.20 $k^s_p + k^u_p = k_y$ for all $p \in \omega_y$.*

Proof. According to Lemma 5.20 we have

$$k^s_p \geq k^s_y \tag{38}$$

for all $p \in \omega_y$. On the other hand by Lemma 5.16

$$k^s_p \geq k_y - k^s_y \ (p \in \omega_y). \tag{39}$$

From (38) and (39) we obtain $k^s_p + k^u_p \geq k_y$ and, consequently, $k^s_p + k^u_p = k_y$ for all $p \in \omega_y$. □

Theorem 5.22. *Let $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ be a linear non-autonomous dynamical system and the following conditions be fulfilled:*

(i) the fiber bundle (X, h, Y) is finite-dimensional;

(ii) Y is compact and invariant ($\pi^t Y = Y$ for all $t \in \mathbb{T}$);

(iii) there exists a point $y \in Y$ such that $\omega_y = H^+(y) = Y$;

(iv) there exists at least one asymptotical stable fiber X_{p_0} (i.e. $k^s_{p_0} = k_{p_0}$ or equivalently $k^u_{p_0} = 0$).

Then $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is asymptotically stable, i.e. $X = X^s$.

Proof. According to Corollary 5.21 we have

$$k^s_p + k^u_p = k_y \tag{40}$$

for all $p \in \omega_y$. If $k^u_{p_0} = 0$ for certain $p_0 \in \omega_y$, then by Lemma 5.14 we obtain $k^u_y \le k^u_{p_0} = 0$, i.e. $k^u_y = 0$ and, consequently, $k^s_y = k_y$. From Lemma 5.20 we have $k^s_p \ge k^s_y = k_y$ for all $p \in \omega_y$ and, consequently, $k^s_p = k_y$ for all $p \in Y = \omega_y$. □

Definition 5.23. *Let E be a finite-dimensional ($k := \dim(E)$) Banach space. The linear operator $A \in [E]$ is called asymptotically stable if $|\lambda_j(A)| < 1$ ($j = 1, 2, \dots, k$), where $\sigma(A) := \{\lambda_1(A), \lambda_2(A), \dots, \lambda_k(A)\}$ is a spectrum of A.*

Theorem 5.24. *Let E be a finite-dimensional Banach space, $A_i \in [E]$ ($i = 1, 2, \dots, m$) and $\mathcal{M} := \{A_1, A_2, \dots, A_m\}$. Assume that the following conditions are fulfilled:*

(i) there exists $j \in \{1, 2, \dots, m\}$ such that the operator A_j is asymptotically stable;

(ii) the discrete linear inclusion $DLI(\mathcal{M})$ has not any nontrivial bounded on $\mathbb{Z}$ solutions.

Then the discrete linear inclusion $DLI(\mathcal{M})$ is absolutely asymptotically stable.

Proof. Let $Q := \mathcal{M}$, $Y = \Omega := C(\mathbb{Z}_+, Q)$ and $(Y, \mathbb{Z}_+, \sigma)$ be a semi-group dynamical system of shifts on Y (see Section 2). It is easy to see that $Y = C(\mathbb{Z}_+, Q)$ is topologically isomorphic to $\Sigma_m := \{0, 1, \dots, m-1\}^{\mathbb{Z}_+}$ and $(Y, \mathbb{Z}_+, \sigma)$ is dynamically isomorphic to shift dynamical system on Σ_m (see, for example,[31, 41]) and, consequently, it possesses the following properties:

(i) Y is compact;

(ii) $Y = \overline{Per(\sigma)}$, where $Per(\pi)$ the set of all periodic points of dynamical system $(Y, \mathbb{Z}_+, \sigma)$;

(iii) there exists a Poisson stable point $y \in Y$ such that $Y = H^+(y)$.

Let $\langle E, \varphi, (Y, \mathbb{Z}_+, \sigma)\rangle$ be a cocycle, generated by $DLI(\mathcal{M})$ (i.e. $\varphi(n, u, \omega) := U(n, \omega)u$, where $U(n, \omega) = \prod_{k=1}^{n} \omega(k)$ ($\omega \in \Omega$)), $(X, \mathbb{Z}_+, \pi)$ be a skew-product system associated with cocycle φ (i.e. $X := E \times Y$ and $\pi := (\varphi, \sigma)$) and $\langle (X, \mathbb{Z}_+, \pi)$, $(Y, \mathbb{Z}_+, \sigma)$, $h\rangle$ ($h := pr_2 : X \to Y$) be a linear non-autonomous dynamical system, generated by cocycle φ. Denote by $\omega_0 : \mathbb{Z}_+ \to \mathcal{M}$ the mapping defined by equality $\omega_0(i) = A^i_j$ for all $i \in \mathbb{N}$, where $A^i_j := A_j \circ A^{i-1}_j$ ($i = 2, \dots$). Since the operator A_j is asymptotically stable, then the fiber X_{p_0} ($p_0 := \omega_0 \in Y$) is asymptotically stable. Now to finish the proof of Theorem it is sufficient to refer to Theorem 5.22. □

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On geometrical properties of the spaces defined by the Pfaff equations

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Abstract. Geometrical properties of holonomic and non holonomic varieties defined by the Pfaff equations connected with the first order system of equations are studied. The Riemann extensions of affine connected spaces for investigations of geodesics and asymptotic lines are used.

Mathematics subject classification: 53Bxx.
Keywords and phrases: Non holonomic variety, Pfaff equation, Riemann extensions, geodesics.

1 Introduction

There is the connection between of the Pfaff equation

$$P(x,y,z)\frac{dx}{ds}+Q(x,y,z)\frac{dy}{ds}+R(x,y,z)\frac{dz}{ds}=0 \tag{1}$$

and the first order system differential equations in form

$$\frac{dx}{P(x,y,z)}=\frac{dy}{Q(x,y,z)}=\frac{dz}{R(x,y,z)}. \tag{2}$$

For example the equation (1) is exactly integrable at the conditions

$$\frac{\partial P(x,y,z)}{\partial y}-\frac{\partial Q(x,y,z)}{\partial x}=0, \quad \frac{\partial Q(x,y,z)}{\partial z}-\frac{\partial R(x,y,z)}{\partial y},$$

$$\frac{\partial R(x,y,z)}{\partial x}-\frac{\partial P(x,y,z)}{\partial z}=0$$

and its general integral determines the family of the surfaces in R^3 space

$$V(x,y,z)=constant,$$

which are orthogonal to the lines of the vector field

$$\vec{N}=(P(x,y,z),Q(x,y,z),R(x,y,z)). \tag{3}$$

In more general case

$$P(x,y,z)\left(\frac{\partial Q(x,y,z)}{\partial z}-\frac{\partial R(x,y,z)}{\partial y}\right)+Q(x,y,z)\left(\frac{\partial R(x,y,z)}{\partial x}-\frac{\partial P(x,y,z)}{\partial z}\right)+$$

$$+R(x,y,z)\left(\frac{\partial R(x,y,z)}{\partial y}-\frac{\partial Q(x,y,z)}{\partial x}\right)=0$$

the equation (1)

$$\mu\left(P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz\right)=dU(x,y,z)$$

is also integrable by means of integrating multiplier μ and determines the family of the surfaces $U(x,y,z)=const$ passing through the each point of the space and orthogonal to the vector field (3).

The vector field (3) in the space R^3 with conditions

$$\left(\vec{N},rot\vec{N}\right)=0$$

is called holonomic.

In general case the Pfaff equation (1) is not integrable and them corresponds the system of the integral curves (Pfaff variety) passing through each point (x,y,z) with tangent lines lying on the plane

$$P(x,y,z)(X-x)+Q(x,y,z)(Y-y)+R(x,y,z)(Z-z)=0. \tag{4}$$

The set of the planes and points (4) defined by the equation (1) in general case forms two-dimensional non holonomic variety M^2 which is generalization of the surface.

For the variety M^2 may be extended many of the results of classical differential geometry of surfaces. For example the notion of the asymptotic lines, curvature lines and geodesic has analog for the variety M^2 [1–3].

In fact the solutions of the system

$$Pdx+Qdy+Rdz=0,\quad dPdx+dQdy+dRdz=0$$

or

$$P\dot{x}+\dot{y}+R\dot{z}=0,$$
$$P_x(\dot{x})^2+Q_y(\dot{y})^2+R_z(\dot{z})^2+(P_y+Q_x)\dot{x}\dot{y}+(P_z+R_x)\dot{x}\dot{z}+(Q_z+R_y)\dot{y}\dot{z}=0$$

give us the curve lines of the variety M^2 which are the analog of the of asymptotic lines on the holonomic surface.

The notion of the curvature lines also can be generalized on the variety M^2.

They may be of the two kinds and one of them is defined by the solutions of the system of equations

$$Pdx+Qdy+Rdz=0,$$
$$\begin{vmatrix} 2P_xdx+(Q_x+P_y)dy+(P_x+R_x)dz & P & dx\\ (Q_x+P_y)dx+2Q_ydy+(Q_z+R_y)dz & Q & dy\\ (P_z+R_x)dx+(Q_z+R_y)dy+2R_zdz & R & dz \end{vmatrix}=0.$$

The notion of the geodesics also can be extended on the variety V^2 (1) and they may be of two types.

The first type is determined from the condition

$$\begin{bmatrix} P(x,y,z) & Q(x,y,z) & R(x,y,z) \\ \frac{d}{ds}x(s) & \frac{d}{ds}y(s) & \frac{d}{ds}z(s) \\ \frac{d^2}{ds^2}x(s) & \frac{d^2}{ds^2}y(s) & \frac{d^2}{ds^2}z(s) \end{bmatrix} = 0,$$

or

$$\left(P(x,y,z)\frac{d}{ds}y(s) - \left(\frac{d}{ds}x(s)\right)Q(x,y,z)\right)\frac{d^2}{ds^2}z(s)+$$

$$+\left(-P(x,y,z)\frac{d}{ds}z(s) + \left(\frac{d}{ds}x(s)\right)R(x,y,z)\right)\frac{d^2}{ds^2}y(s)+$$

$$+\left(\frac{d^2}{ds^2}x(s)\right)\left(Q(x,y,z)\frac{d}{ds}z(s) - R(x,y,z)\frac{d}{ds}y(s)\right) = 0.$$

This relation is equivalent the system of equations

$$\frac{d^2x}{ds^2} + \frac{P}{P^2+Q^2+R^2}\left(\frac{dx}{ds}\frac{dP}{ds} + \frac{dy}{ds}\frac{dQ}{ds} + \frac{dz}{ds}\frac{dR}{ds}\right) = 0, \tag{5}$$

$$\frac{d^2y}{ds^2} + \frac{Q}{P^2+Q^2+R^2}\left(\frac{dx}{ds}\frac{dP}{ds} + \frac{dy}{ds}\frac{dQ}{ds} + \frac{dz}{ds}\frac{dR}{ds}\right) = 0, \tag{6}$$

$$\frac{d^2z}{ds^2} + \frac{R}{P^2+Q^2+R^2}\left(\frac{dx}{ds}\frac{dP}{ds} + \frac{dy}{ds}\frac{dQ}{ds} + \frac{dz}{ds}\frac{dR}{ds}\right) = 0. \tag{7}$$

Remark that after substitution of the corresponding expressions for the second derivatives on coordinates from the (5)–(7) into the relations

$$\frac{dP(x,y,z)}{ds}\frac{dx}{ds} + \frac{dQ(x,y,z)}{ds}\frac{dy}{ds} + \frac{dR(x,y,z)}{ds}\frac{dz}{ds} + P(x,y,z)\frac{d^2x}{ds^2}+$$

$$+Q(x,y,z)\frac{d^2y}{ds^2} + R(x,y,z)\frac{d^2z}{ds^2} = 0$$

one get the identity.

The definition of the second type of geodesic in the system of integral curves of the Pfaff equation (1) is more complicated and does not be used hereinafter.

2 The Riemann extension of the affine connected space

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The formulas (5)–(7) can be rewritten in form of geodesic of the R^3-space equipped with symmetrical affine connection $\Pi^i_{jk}(x^l) = \Pi^i_{kj}(x^l)$

$$\frac{d^2x^i}{ds^2} + \Pi^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0.$$

In our case we get the system of equations

$$\frac{d^2x}{ds^2} + \frac{P}{\Delta}\left(P_x\left(\frac{dx}{ds}\right)^2 + (P_y+Q_x)\frac{dx}{ds}\frac{dy}{ds} + (P_z+R_x)\frac{dx}{ds}\frac{dz}{ds} + Q_y\left(\frac{dy}{ds}\right)^2 + \right.$$

$$\left. + (Q_z+R_y)\frac{dy}{ds}\frac{dz}{ds} + R_z\left(\frac{dz}{ds}\right)^2\right) = 0,$$

$$\frac{d^2y}{ds^2} + \frac{Q}{\Delta}\left(P_x\left(\frac{dx}{ds}\right)^2 + (P_y+Q_x)\frac{dx}{ds}\frac{dy}{ds} + (P_z+R_x)\frac{dx}{ds}\frac{dz}{ds} + Q_y\left(\frac{dy}{ds}\right)^2 + \right.$$

$$\left. + (Q_z+R_y)\frac{dy}{ds}\frac{dz}{ds} + R_z\left(\frac{dz}{ds}\right)^2\right) = 0,$$

$$\frac{d^2z}{ds^2} + \frac{R}{\Delta}\left(P_x\left(\frac{dx}{ds}\right)^2 + (P_y+Q_x)\frac{dx}{ds}\frac{dy}{ds} + (P_z+R_x)\frac{dx}{ds}\frac{dz}{ds} + Q_y\left(\frac{dy}{ds}\right)^2 + \right.$$

$$\left. + (Q_z+R_y)\frac{dy}{ds}\frac{dz}{ds} + R_z\left(\frac{dz}{ds}\right)^2\right) = 0,$$

where

$$\Delta = P^2 + Q^2 + R^2,$$

from which we get the expressions for the coefficients of affine connection. They are

$$\Pi^1_{11} = \frac{PP_x}{\Delta}, \qquad \Pi^1_{22} = \frac{PQ_y}{\Delta}, \qquad \Pi^1_{33} = \frac{PR_z}{\Delta},$$

$$\Pi^1_{12} = \frac{P(P_y+Q_x)}{2\Delta}, \qquad \Pi^1_{13} = \frac{P(P_z+R_x)}{2\Delta}, \qquad \Pi^1_{23} = \frac{P(Q_z+R_y)}{2\Delta},$$

$$\Pi^2_{11} = \frac{QP_x}{\Delta}, \qquad \Pi^2_{22} = \frac{QQ_y}{\Delta}, \qquad \Pi^2_{33} = \frac{QR_z}{\Delta},$$

$$\Pi^2_{12} = \frac{Q(P_y + Q_x)}{2\Delta}, \qquad \Pi^2_{13} = \frac{Q(P_z + R_x)}{2\Delta}, \qquad \Pi^2_{23} = \frac{Q(Q_z + R_y)}{2\Delta},$$

$$\Pi^3_{11} = \frac{RP_x}{\Delta}, \qquad \Pi^3_{22} = \frac{RQ_y}{\Delta}, \qquad \Pi^3_{33} = \frac{RR_z}{\Delta},$$

$$\Pi^3_{12} = \frac{R(P_y + Q_x)}{2\Delta}, \qquad \Pi^3_{13} = \frac{R(P_z + R_x)}{2\Delta}, \qquad \Pi^3_{23} = \frac{R(Q_z + R_y)}{2\Delta}.$$

So with any equation (1) can be associated 3-dimensional affine connected space and its properties will be dependent from the coefficients of connection Π^k_{ij} which are determined by the functions P, Q, R.

In general case such type of connection is not metrizable and corresponding space is not a Riemannian.

Further we apply the notion of the Riemann extension of nonriemannian space which were used earlier in [4–6].

Remind basic properties of this construction.

With help of the coefficients of affine connection of a given n-dimensional space can be introduced a 2n-dimensional Riemann space D^{2n} with local coordinates (x^i, Ψ_i) having the metric in form

$$^6ds^2 = -2\Pi^k_{ij}(x^l)\Psi_k dx^i dx^j + 2d\Psi_k dx^k \tag{8}$$

where Ψ_k are the additional coordinates.

The important property of such type metric is that the geodesic equations of metric (8) decomposes into two parts

$$\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0, \tag{9}$$

and

$$\frac{\delta^2\Psi_k}{ds^2} + R^l_{kji}\dot{x}^j\dot{x}^i\Psi_l = 0, \tag{10}$$

where

$$\frac{\delta\Psi_k}{ds} = \frac{d\Psi_k}{ds} - \Pi^l_{jk}\Psi_l\frac{dx^j}{ds}$$

and R^l_{kji} are the curvature tensor of 3-dimensional space with a given affine connection.

The first part (9) is the system of equations for geodesic of basic space with local coordinates x^i and they does not contains the supplementary coordinates Ψ_k.

The second part (10) of the full system of geodesics has the form of linear 3×3 matrix system of second order ODE's for supplementary coordinates Ψ_k

$$\frac{d^2\vec{\Psi}}{ds^2} + A(s)\frac{d\vec{\Psi}}{ds} + B(s)\vec{\Psi} = 0. \tag{11}$$

It is important to note that the geometry of extended space connects with geometry of basic space. For example the property of the space to be Ricci-flat keeps also for the extended space.

This fact give us the possibility to use the linear system of equation (11) for the studying of geometrical properties of the basic space.

In particular the invariants of $k \times k$ matrix-function

$$E = B - \frac{1}{2}\frac{dA}{ds} - \frac{1}{4}A^2$$

under change of the coordinates Ψ_k can be of used for that.

The first applications the notion of extended spaces for the studying of nonlinear second order differential equations connected with nonlinear dynamical systems have been considered in articles of author [4–6].

Here we consider some properties of the space defined by the Pfaff equations connected with a nonlinear dynamical systems.

3 The Lorenz dynamical system

The equations of Lorenz dynamical system are

$$\frac{dx}{ds} = \sigma(y-x), \quad \frac{dy}{ds} = rx - y - zx, \quad \frac{dz}{ds} = xy - bz, \tag{12}$$

where σ, b, r are the parameters.

These equations describe the behaviour of the flow lines along the vector field

$$\vec{N} = [\sigma(y-x), rx - y - zx, xy - bz]$$

depending from the parameters σ, b, r.

The properties of non holonomic variety L^2 in this case are determined by the following Pfaff equation

$$\sigma(y-x)\frac{dx}{ds} + (rx - y - zx)\frac{dy}{ds} + (xy - bz)\frac{dz}{ds} = 0. \tag{13}$$

The object of holonomicity for the Lorenz vector field is

$$\left(\vec{N}, rot\vec{N}\right) = \sigma\, xy - 2\,\sigma\, x^2 + y^2 - bzr + bz^2 + bz\sigma.$$

The properties of asymptotic lines of the corresponding variety M^2 are defined by the system of equations

$$\left(\frac{\partial}{\partial x}P(x,y,z)\right)\left(\frac{d}{ds}x(s)\right)^2 + \left(\frac{\partial}{\partial y}Q(x,y,z)\right)\left(\frac{d}{ds}y(s)\right)^2 + \left(\frac{\partial}{\partial z}R(x,y,z)\right) \times$$

$$\times \left(\frac{d}{ds}z(s)\right)^2 + \left(\frac{\partial}{\partial y}P(x,y,z) + \frac{\partial}{\partial x}Q(x,y,z)\right)\left(\frac{d}{ds}x(s)\right)\frac{d}{ds}y(s) +$$

$$+\left(\frac{\partial}{\partial z}P(x,y,z)+\frac{\partial}{\partial x}R(x,y,z)\right)\left(\frac{d}{ds}x(s)\right)\frac{d}{ds}z(s)+$$
$$+\left(\frac{\partial}{\partial y}R(x,y,z)+\frac{\partial}{\partial z}Q(x,y,z)\right)\left(\frac{d}{ds}z(s)\right)\frac{d}{ds}y(s)=0, \tag{14}$$

$$P(x,y,z)\frac{d}{ds}x(s)+Q(x,y,z)\frac{d}{ds}y(s)+R(x,y,z)\frac{d}{ds}z(s)=0, \tag{15}$$

which take the form

$$-\left(\frac{d}{ds}y(s)\right)^2+(r+\sigma-z)\left(\frac{d}{ds}x(s)\right)\frac{d}{ds}y(s)-\sigma\left(\frac{d}{ds}x(s)\right)^2+$$
$$+y\left(\frac{d}{ds}x(s)\right)\frac{d}{ds}z(s)-b\left(\frac{d}{ds}z(s)\right)^2=0, \tag{16}$$

$$(-bz+xy)\frac{d}{ds}z(s)+(rx-y-zx)\frac{d}{ds}y(s)+(\sigma\, y-\sigma\, x)\frac{d}{ds}x(s)=0. \tag{17}$$

We find from these equations the

$$\frac{d}{ds}x(s)=-\frac{1}{\sigma\,(-y+x)}\times$$
$$\times\left\{-\left(\frac{d}{ds}y(s)\right)rx+\left(\frac{d}{ds}y(s)\right)y+\left(\frac{d}{ds}y(s)\right)zx-\left(\frac{d}{ds}z(s)\right)xy+\left(\frac{d}{ds}z(s)\right)bz\right\}, \tag{18}$$

and

$$A\,(x(s))^2+Bx(s)+C=0, \tag{19}$$

where

$$A=(-\sigma\, z(s)+\sigma\, r-\sigma)\left(\frac{d}{ds}y(s)\right)^2+\left(\frac{d}{ds}y(s)\right)\sigma\left(\frac{d}{ds}z(s)\right)y(s)-b\left(\frac{d}{ds}z(s)\right)^2\sigma,$$

$$B=\left(-(y(s))^3+2\,b\sigma\, y(s)+y(s)bz(s)\right)\left(\frac{d}{ds}z(s)\right)^2+$$
$$+\left(rbz(s)+2\,z(s)\,(y(s))^2-\sigma\, bz(s)-(z(s))^2\,b-2\,r\,(y(s))^2-\sigma\,(y(s))^2+(y(s))^2\right)\times$$

$$\times\left(\frac{d}{ds}y(s)\right)\frac{d}{ds}z(s)+\left(-z(s)y(s)+\sigma\, z(s)y(s)-r^2y(s)-\sigma\, ry(s)+2\, rz(s)y(s)+\right.$$

$$\left.+\sigma\, y(s)+ry(s)-\left(z(s)\right)^2y(s)\right)\left(\frac{d}{ds}y(s)\right)^2,$$

$$C=\left(-b\sigma\,\left(y(s)\right)^2-b^2\left(z(s)\right)^2+\left(y(s)\right)^2bz(s)\right)\left(\frac{d}{ds}z(s)\right)^2+$$

$$+\left(-\left(z(s)\right)^2by(s)+rbz(s)y(s)-2\, y(s)bz(s)+\left(y(s)\right)^3+\sigma\, bz(s)y(s)\right)\times$$

$$\times\left(\frac{d}{ds}y(s)\right)\frac{d}{ds}z(s)+\left(r\left(y(s)\right)^2-\left(y(s)\right)^2-z(s)\left(y(s)\right)^2\right)\left(\frac{d}{ds}y(s)\right)^2.$$

After differentiating the equation (19) on the variable s and taking into account the expression (18) for $\dfrac{dx(s)}{ds}$ we get the relation

$$E\left(x(s)\right)^3+F\left(x(s)\right)^2+Hx(s)+K=0 \tag{20}$$

with some functions E, F, H, K which does not contains the variable $x(s)$.

The resultant of the equations (19) and (20) with regard of the variable

$$\frac{d}{ds}z(s)=\frac{\frac{d}{ds}y(s)}{\frac{d}{dz}y(z)}$$

give us a following conditions

$$\left(r-1-z\right)y(z)\frac{d}{dz}y(z)-bz+\left(y(z)\right)^2=0, \tag{21}$$

$$L\left(\frac{d}{dz}y(z)\right)^2+M\frac{d}{dz}y(z)+N=0, \tag{22}$$

where the coefficients of equation are

$$\frac{L}{1-r+z}=-z^4b^2+\left(2\,\sigma\, b^2+b\left(y(z)\right)^2+2\, rb^2\right)z^3+$$

$$+\left(\left(-2\, rb-2\,\sigma\, b\right)\left(y(z)\right)^2+4\,\sigma\, b^2-r^2b^2-\sigma^2b^2-2\,\sigma\, b^2r\right)z^2+$$

$$+\left(\left(y(z)\right)^4+\left(\sigma^2b+2\,\sigma\, rb+r^2b-4\,\sigma\, b\right)\left(y(z)\right)^2\right)z+\left(1-r\right)\left(y(z)\right)^4,$$

$$-1/2\,\frac{M}{y(z)\,(1-r+z)} = -2\,z^3b^2 + \left(3\,\sigma\,b^2 + 3\,rb^2 + b\,(y(z))^2\right)z^2 +$$
$$+\left((-2\,\sigma\,b - rb - 2\,b)\,(y(z))^2 - 2\,\sigma\,b^2r - \sigma^2b^2 + 4\,\sigma\,b^2 - r^2b^2\right)z +$$
$$+\,(y(z))^4 + \left(-3\,\sigma\,b + \sigma\,rb + rb + \sigma^2b\right)(y(z))^2\,,$$

$$N = b^3z^4 + \left(-3\,(y(z))^2\,b^2 - 2\,rb^3 + 2\,\sigma\,b^3\right)z^3 +$$
$$+\left(b\,(y(z))^4 + \left(5\,rb^2 + 4\,\sigma\,b^2\right)(y(z))^2 + \sigma^2b^3 + r^2b^3 - 2\,\sigma\,b^3r\right)z^2 +$$
$$+\left((-rb - 2\,\sigma\,b - 3\,b)\,(y(z))^4 + \left(-8\,\sigma\,b^2r - 2\,r^2b^2 + 12\,\sigma\,b^2 - 2\,\sigma^2b^2\right)(y(z))^2\right)z +$$
$$+\,(y(z))^6 + \left(2\,rb + \sigma^2b + 2\,\sigma\,rb - b - 4\,\sigma\,b\right)(y(z))^4 + \left(4\,\sigma\,b^2 + 4\,\sigma\,r^2b^2 - 8\,\sigma\,b^2r\right)(y(z))^2$$

The solutions of the first order differential equations (21), (22) together with conditions (16)-(17) allow us to get the expressions for asymptotic line of the variety L^2.

Let us consider some examples.

The equation (22) has a set of singular solutions $y(z)$.

One of them determines from the relation

$$z^2b + (-2\,\sigma\,b - 2\,rb)\,z + y^2 + 2\,br\sigma + br^2 - 4\,\sigma\,b + b\sigma^2 = 0$$

which presents the second order curve in the plane (z,y).

Remark that the equation (22) presents the algebraic curve

$$\Phi(y', y, z) = 0$$

of genus $g = 1$ with respect to variables y', y in case $r \neq 1$ and genus $g = 0$ when $r = 1$.

According with general theory some of such type equations can be integrated with help of elliptic functions or can be brought to integration of the Rikkati equation.

In both cases the properties of asymptotic lines should be dependent from the parameters of model.

4 The Rössler dynamical system

Differential equations of the Rössler dynamical model are

$$\frac{dx}{ds} = -y - z, \quad \frac{dy}{ds} = x + ay, \quad \frac{dz}{ds} = b + xz - cz, \tag{23}$$

where a, b, c are the parameters.

For corresponding non holonomic variety V^2 we have a following Pfaff equation

$$(-y-z)\frac{dx}{ds}+(x+ay)\frac{dy}{ds}+(b+xz-cz)\frac{dz}{ds}=0. \tag{24}$$

The object of holonomicity for the Rössler vector field is

$$\left(\vec{N}, rot\vec{N}\right)=-x+xz-ay-ayz+2\,b-2\,cz.$$

In this case the properties of the system (16)-(17) for asymptotic lines are determined by the equation

$$A\frac{d^2}{dz^2}y(z)+B=0 \tag{25}$$

where

$$A=\left(-y(z)az+az-az^2+y(z)a\right)\left(\frac{d}{dz}y(z)\right)^2+$$

$$+\left(-2\,az^3-2\,y(z)az-2\,az^2y(z)-2\,\left(y(z)\right)^2a\right)\frac{d}{dz}y(z)-$$

$$-ay(z)z^3-2\,bz-z^3c+cz^2+y(z)c+bz^2-cy(z)z-a\left(y(z)\right)^2z+$$

$$+\left(y(z)\right)^2a+b+az^2y(z)$$

and

$$B=2\left(z+y(z)\right)\left(\left(\frac{d}{dz}y(z)\right)ay(z)+\left(\frac{d}{dz}y(z)\right)c-\right.$$

$$\left.-\left(\frac{d}{dz}y(z)\right)^3a-az\left(\frac{d}{dz}y(z)\right)^2+b\right).$$

This is equation of the second range at the condition $a\neq 0$.

In the case $a=0$ it takes a form of the first range equation

$$\left(2\,bz-y\left(z\right)c+z^3c-cz^2-b+cy\left(z\right)z-bz^2\right)\frac{d^2}{dz^2}y\left(z\right)+2\,cz\frac{d}{dz}y\left(z\right)+$$

$$+2\,y\left(z\right)c\frac{d}{dz}y\left(z\right)+2\,bz+2\,y\left(z\right)b=0.$$

At the conditions $a=0,\quad b=0,\ c\neq$ the properties of asymptotic lines of corresponding non holonomic variety are dependent from the solutions of the equation

$$\left(z^3-z^2-y\left(z\right)+y\left(z\right)z\right)\frac{d^2}{dz^2}y\left(z\right)+2\,z\frac{d}{dz}y\left(z\right)+2\,y\left(z\right)\frac{d}{dz}y\left(z\right)=0.$$

5 Quadratic systems

Here we consider the properties of the Pfaff equations connected with a polynomial differential systems in R^2 defined by

$$\frac{dx}{ds} = p(x,y), \quad \frac{dy}{ds} = q(x,y), \tag{26}$$

where $p(x,y)$ and $q(x,y)$ are polinomials of degree 2.

The system (26) takes the form of the Pfaff equation after extension on a projective plane

$$(x\,Q\,(x,y,z) - y\,P\,(x,y,z))\,dz - z\,Q\,(x,y,z)\,dx + z\,P\,(x,y,z)\,dy = 0, \tag{27}$$

where

$$P(x,y,z), \quad Q(x,y,z), \quad R(x,y,z)$$

are the homogeneous functions constructed from the functions $p(x,y)$ and $q(x,y)$.

As example for the system

$$\frac{dx}{ds} = kx + ly + ax^2 + bxy + cy^2, \quad \frac{dy}{ds} == mx + ny + ex^2 + fxy + hy^2,$$

with a ten parameters one get a Pfaff equation after a projective extension

$$P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0,$$

or

$$\begin{gathered}\left((-ny - mx)\,z^2 + \left(-hy^2 - fxy - ex^2\right)z\right)dx + \left((ly + kx)\,z^2 + \right.\\ \left. + \left(cy^2 + bxy + ax^2\right)z\right)dy + \left(\left(-ly^2 + (n-k)\,xy + mx^2\right)z - cy^3 + \right.\\ \left. + (h-b)\,xy^2 + (f-a)\,x^2y + ex^3\right)dz = 0. \end{gathered} \tag{28}$$

This equation corresponds the system

$$\frac{dx}{ds} = P(x,y,z), \quad \frac{dy}{ds} = Q(x,y,z), \quad \frac{dz}{ds} = R(x,y,z), \tag{29}$$

where

$$\begin{gathered}P(x,y,z) = (-ny - mx)\,z^2 + \left(-hy^2 - fxy - ex^2\right)z,\\ Q(x,y,z) = (ly + kx)\,z^2 + \left(cy^2 + bxy + ax^2\right)z,\\ R(x,y,z) = \left(-ly^2 + (n-k)\,xy + mx^2\right)z - cy^3 + (h-b)\,xy^2 + (f-a)\,x^2y + ex^3\end{gathered}$$

It is important to note that the vector field $\vec{N} = (P,Q,R)$ connected with a system (32) is holonomic due the condition

$$(\vec{N}, rot\vec{N}) = 0.$$

Corresponding Pfaff (31) equation is integrable and determines the family of developing surfaces $U(x,y,z) = C$ in the R^3-space.

The investigation of the asymptotic lines of the surfaces which corresponds the equation (31) may be useful in applications.

Let us consider some of examples.

The system of equations

$$\frac{dx}{ds} = 5x + 6x^2 + 4(1+\mu)xy + \mu y^2, \quad \frac{dy}{ds} == x + 2y + 4xy + (2+3\mu)y^2,$$

with $(-71 + 17\sqrt{(}17)/32 < \mu < 0$ posseses the invariant algebraic curve

$$x^2 + x^3 + x^2 y + 2\mu xy^2 + 2\mu xy^3 + mu^2 y^4 = 0.$$

The conditions of compatibility of equations for asymptotic lines of the variety defined by the corresponding Pfaff equation lead to a following conditions on the functions

$$\left(3\,\mu^2 + 4\,\mu + 2\right)(y(s))^2 + (9\,\mu + 6)\,x(s)\,y(s) + 6\,(x(s))^2 = 0,$$

$$(y(s))^2\,\mu + (-2\,\mu - 1)\,x(s)\,y(s) - 3\,(x(s))^2 = 0.$$

The values of parameter μ determined by the conditions

$$(3\mu + 10)(\mu + 4)(\mu - 2) = 0, \quad \mu = 0$$

are special.

Next example is the system with at least a four limit cycles.

$$\frac{dx}{ds} = kx - y - 10x^2 + bxy + y^2, \quad \frac{dy}{ds} == x + x^2 + fxy,$$

The studying of asymptotic lines for this system give a following conditions on parameters

$$10\,k^2 + 11\,fk^2 + f^2k^2 - 7\,k + 2\,fk - 9\,b + k^3 - 80 - 81\,f + b^2k + 2\,bk^2 + 9\,bk + 9\,fbk = 0$$

and functions

$$x^3 + 10\,y(x)\,x^2 - (y(x))^3 + fx^2y(x) - bx\,(y(x))^2 = 0,$$

$$f\,(y(x))^2 - fy(x)\,kx + (y(x))^2 + y(x)\,bx + y(x)\,x - kx^2 - 10\,x^2 = 0.$$

Remark that these equations equivalent the equations of direct lines.

6 Cubic systems

By analogy can be investigated the properties of the asymptotic lines of the surfaces connected with the system

$$\frac{dx}{ds} = p(x,y), \quad \frac{dy}{ds} = q(x,y), \tag{30}$$

where $p(x,y)$ and $q(x,y)$ are polinomials of degree 3.

Let us consider the examples.

The system

$$\frac{dx}{ds} = y, \quad \frac{dy}{ds} = -x - x^2y + mu^2y \tag{31}$$

connects with the Van der Pol equation.

After extension on the projective plane we get an integrable Pfaff equation

$$\left(-x^2z^2 - x^3y + x\mu^2yz^2 - y^2z^2\right) dz + \left(z^3x + zx^2y - z^3\mu^2y\right) dx + yz^3\, dy = 0.$$

The equations for the asymptotic lines of corresponding surface give the conditions

$$x^2 + (y\,(x))^2 - x\mu^2 y\,(x) = 0$$

and

$$-x^2 + 2\;(y\,(x))^2 + 4\,x\mu^2 y\,(x) = 0.$$

From the first condition we find

$$y(x) = \left(1/2\,\mu^2 + (-)1/2\,\sqrt{\mu^4 - 4}\right) x,$$

and the second gives us

$$y(x) = \left(-\mu^2 + (-)1/2\,\sqrt{4\,\mu^4 + 2}\right) x.$$

For the system

$$\frac{dx}{ds} = -y + ax(x^2 + y^2 - 1), \quad \frac{dy}{ds} = x + by(x^2 + y^2 - 1) \tag{32}$$

the point $(0,0$ is node at the condition $ab > -1$ and $(a-b)^2 >= 4$ and it has the limit cycle around this point.

After extension we get an integrable Pfaff equation

$$\left(x^2z^2 + byx^3 + xby^3 - xbyz^2 + y^2z^2 - yax^3 - axy^3 + yaxz^2\right) dz +$$

$$+ \left(-z^3y + zax^3 + zaxy^2 - z^3ax\right) dy + \left(-z^3x - zbyx^2 - zby^3 + z^3by\right) dx = 0$$

with a developing surfaces as general integral.

The conditions on parameters for existence of asymptotic lines are

$$ab + 1 = 0, \qquad -a + b = 0.$$

For the corresponding functions one get the equations of direct lines

$$xby\left(x\right)-x^{2}-y\left(x\right)ax-\left(y\left(x\right)\right)^{2}=0,$$

or

$$y(x)=xby\left(x\right)-x^{2}-y\left(x\right)ax-\left(y\left(x\right)\right)^{2},$$

and

$$\begin{gathered}\left(30\,ab+8\,a^{3}b-25\,a^{2}-9\,b^{2}-12\,a^{4}\right)x^{6}+\\+\left(36\,b^{3}+76\,ba^{2}-108\,ab^{2}-4\,a^{3}-32\,a^{3}b^{2}+32\,ba^{4}\right)yx^{5}+\\+\left(69\,b^{2}+53\,a^{2}+36\,a^{4}-64\,a^{3}b-134\,ab+24\,ab^{3}-8\,a^{2}b^{2}\right)y^{2}x^{4}+\\+\left(-24\,b^{3}-8\,ba^{2}+64\,a^{3}b^{2}+8\,ab^{2}+24\,a^{3}-64\,a^{2}b^{3}\right)y^{3}x^{3}+\\+\left(-8\,a^{2}b^{2}-134\,ab+24\,a^{3}b+36\,b^{4}-64\,ab^{3}+69\,a^{2}+53\,b^{2}\right)y^{4}x^{2}+\\+\left(-32\,b^{4}a-36\,a^{3}-76\,ab^{2}+32\,a^{2}b^{3}+4\,b^{3}+108\,ba^{2}\right)y^{5}x+\\+\left(-25\,b^{2}-9\,a^{2}+30\,ab+8\,ab^{3}-12\,b^{4}\right)y^{6}=0\end{gathered}$$

with a very complicate relations between the parameters a, b.

7 Quatric systems

The system

$$\frac{dx}{ds}=-Ay+yx^{2}-x^{4},\quad \frac{dy}{ds}=ax-x^{3} \tag{33}$$

has a center and the limit cycle at the conditions $a = 2A + A^2$, $0 < A < 5.10^{-5}$.

After extension we get an integrable Pfaff equation

$$\left(z^{2}x^{3}-z^{4}ax\right)\frac{d}{ds}x\left(s\right)+\left(-x^{4}z-z^{4}Ay+z^{2}x^{2}y\right)\frac{d}{ds}y\left(s\right)+$$

$$+\left(ax^{2}z^{3}-x^{4}z+Ay^{2}z^{3}-zx^{2}y^{2}+yx^{4}\right)\frac{d}{ds}z\left(s\right)=0$$

with a developing surfaces as general integral.

For the functions one get the equations of direct lines

$$x^{10}+\left(3\,a-9\,A\right)z^{2}x^{8}+\left(24\,A^{2}-3\,Aa\right)z^{4}x^{6}+\left(A^{2}-36\,A^{3}-2\,Aa+a^{2}\right)z^{6}x^{4}+$$

$$\left(6\,A^{3}+36\,A^{3}a-12\,A^{2}a+6\,a^{2}A\right)z^{8}x^{2}+\left(9\,a^{2}A^{2}-18\,A^{3}a+9\,A^{4}\right)z^{10}=0$$

and the family of conics

$$\left(z\left(s\right)\right)^{2}a-\left(x\left(s\right)\right)^{2}=0$$

$$\left(x\left(s\right)\right)^{2}-A\left(z\left(s\right)\right)^{2}=0.$$

8 Geodesics of the first kind

A geodesics of the first kind on non holonomic variety are defined by the system of equations (5)-(7) .

Let us consider the solutions of this system of equations for the variety V^2 defined by the vector field

$$\vec{F} = [ayz, bxz, cxy],$$

where a, b, c are parameters.

In this case the condition $\vec{N} = 0$ holds and variety is holonomic.

The system of equations for geodesics is

$$\frac{d^2}{ds^2}x(s) + \frac{ayzx\,(c+b)\left(\frac{d}{ds}y(s)\right)\frac{d}{ds}z(s)}{a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2} + \frac{ay^2z\,(a+c)\left(\frac{d}{ds}x(s)\right)\frac{d}{ds}z(s)}{a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2} + \\ + \frac{ayz^2\,(a+b)\left(\frac{d}{ds}x(s)\right)\frac{d}{ds}y(s)}{a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2} = 0,$$

$$\frac{d^2}{ds^2}y(s) + \frac{bx^2z\,(c+b)\left(\frac{d}{ds}y(s)\right)\frac{d}{ds}z(s)}{a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2} + \frac{bxzy\,(a+c)\left(\frac{d}{ds}x(s)\right)\frac{d}{ds}z(s)}{a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2} + \\ + \frac{bxz^2\,(a+b)\left(\frac{d}{ds}x(s)\right)\frac{d}{ds}y(s)}{a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2} = 0,$$

$$\frac{d^2}{ds^2}z(s) + \frac{cx^2y\,(c+b)\left(\frac{d}{ds}y(s)\right)\frac{d}{ds}z(s)}{a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2} + \frac{cxy^2\,(a+c)\left(\frac{d}{ds}x(s)\right)\frac{d}{ds}z(s)}{a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2} + \\ + \frac{cxyz\,(a+b)\left(\frac{d}{ds}x(s)\right)\frac{d}{ds}y(s)}{a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2} = 0.$$

This system of equations has an integral

$$ay(s)z(s)\frac{d}{ds}x(s) + bx(s)z(s)\frac{d}{ds}y(s) + cx(s)y(s)\frac{d}{ds}z(s) = 0$$

and can be integrated.

In fact after substitution of the expression $\frac{d}{ds}z(s)$ into the above equations one get the algebraic relations with respect the variable $z(s)$ which are compatible at the conditions

$$\left(ab + b^2\right)(z(y))^2 + 2\,bz(y)cy\frac{d}{dy}z(y) + \left(ay^2c + c^2y^2\right)\left(\frac{d}{dy}z(y)\right)^2 = 0$$

and lead to the solution

$$z(y) = y^K\_C1$$

where

$$K = -\frac{cb}{c^2 + ac} - \frac{\sqrt{-cab\,(a + c + b)}}{c^2 + ac}.$$

Finally we present the expression for the Chern-Simons invariant of affine connection defined by the equations of geodesics (5-7) in case of the Lorenz system of equations.

It has been obtained with the help of a six dimensional Riemann extension of corresponding space and has the form

$$CS(\Gamma) = \int \frac{L}{2M^2} dxdydz,$$

where

$$L = (2\,b + 2 - 2\,\sigma)\,x^2y^2 + \left(3\,\sigma^2 + 4\,\sigma\,r - 4\,rb - 2\,b\sigma - 4\,r\right) zx^2 + (2\,b + 2 - 2\,\sigma)\,z^2x^2 +$$
$$+\left(-3\,r\sigma^2 + 4\,\sigma^2 b - 5\,\sigma^3 + 2\,b\sigma\,r - 2\,\sigma\,r^2 + 4\,\sigma^2 + 2\,br^2 + 2\,r^2\right) x^2 +$$
$$+\left(-2\,rb - 4\,r + 9\,\sigma^3 + 2\,r\sigma^2\right) yx + \left(-2\,\sigma + 4 - 2\,\sigma^2 - 2\,\sigma\,r + 2\,b\sigma - 4\,b^2\right) zyx -$$
$$-\left(\left(b\sigma\,r + 2\,\sigma^2 + \sigma^2 b - \sigma\,z^2 - 2\,\sigma\,r - \sigma\,r^2\right) y - \sigma\,y^3\right) x + \left(-2\,\sigma\,b^2 + 2\,b^3 - 2\,b^2\right) z^2 +$$
$$+\left(-4\,\sigma^3 - \sigma^2 b + 2 - 2\,b - \sigma^2 + \sigma\,r + (-\sigma + b\sigma)\,z - b\sigma\,r - 2\,\sigma\right) y^2 +$$
$$+\left(-2\,\sigma\,b^2 + 2\,rb^2 + 2\,b^2 r\sigma - 2\,b^2\sigma^2\right) z,$$

and

$$M = \left(x^2 + b^2\right) z^2 + \left((-2\,b + 2)\,xy - 2\,rx^2\right) z +$$
$$+\left(\sigma^2 + 1 + x^2\right) y^2 + \left(-2\,\sigma^2 - 2\,r\right) xy + \left(\sigma^2 + r^2\right) x^2.$$

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A nonlinear hydrodynamic stability criterion derived by a generalized energy method

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Abstract. By applying a new variant of the A. Georgescu – L. Palese – A. Redaelli (G-P-R) method [8], based on the symmetrization of a linear operator, we deduce a nonlinear stability criterion of a state of thermal conduction of a horizontal fluid layer subject to a vertical upwards uniform magnetic field and a vertical upwards constant temperature gradient. The Boussinesq approximation is used. The upper and lower surfaces of the layer are two rigid walls. It is assumed that the magnetic Prandtl number is strictly greater than unity.

Mathematics subject classification: 76-XX Fluid Dynamics.
Keywords and phrases: Nonlinear stability, Hydrodynamic, Magnetic Benard problem.

1 The perturbation problem

Consider an infinite horizontal layer of a homogeneous viscous electrically conducting fluid at rest ($\mathbf{V}=0$) subject to the influence of a uniform vertical upwards magnetic field $\mathbf{H}$ and of an adverse constant vertical temperature gradient $\beta>0$. Let $Oxyz$ be a Cartesian coordinate system, with $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ the unit vectors of the axes, where the vertical axis Oz has the direction opposite to the gravity. Suppose that the fluid is confined between the planes $z=0$ and $z=1$, on which the temperatures $T|_{z=0}=T_0$ and $T|_{z=1}=-\beta+T_0$ respectively are kept constant.

In the Oberbeck-Boussinesq approximation, the stability of the basic state m_0 ($\mathbf{V}=0$, $\mathbf{H}=H\mathbf{k}$, $T=-\beta z+T_0, P$) is governed [1] by the following dimensionless equations for the perturbation fields $(\mathbf{u},\mathbf{h},\theta,p_1)$ of the state m_0

$$\begin{aligned}&\partial\mathbf{u}/\partial t+(\mathbf{u}\cdot\mathbf{grad})\mathbf{u}-P_m(\mathbf{h}\cdot\mathbf{grad})\mathbf{h}=\\&=-\mathbf{grad}\,p_1+R\theta\mathbf{k}+\triangle\mathbf{u}+Q\partial\mathbf{h}/\partial z,\end{aligned}\tag{1.1}$$

$$div\ \mathbf{u}=0,\tag{1.2}$$

$$P_m(\partial\mathbf{h}/\partial t+(\mathbf{u}\cdot\mathbf{grad})\mathbf{h}-(\mathbf{h}\cdot\mathbf{grad})\mathbf{u})=\triangle\mathbf{h}+Q\partial\mathbf{u}/\partial z,\tag{1.3}$$

$$div\ \mathbf{h}=0,\tag{1.4}$$

$$P_r(\partial\theta/\partial t+(\mathbf{u}\cdot\mathbf{grad})\theta)=Rw+\triangle\theta,\tag{1.5}$$

where $(t,\mathbf{x})\in(0,\infty)\times\mathbb{R}^2\times(0,1)$, $\mathbf{x}=(x,y,z)$, and by the conditions

$$\mathbf{u}(0,\mathbf{x})=\mathbf{u}_0(\mathbf{x}),\quad \mathbf{h}(0,\mathbf{x})=\mathbf{h}_0(\mathbf{x}),\quad \theta(0,\mathbf{x})=\theta_0(\mathbf{x}),\quad \mathbf{x}\in\mathbb{R}^2\times(0,1),\tag{1.6}$$

$$div\ \mathbf{u}_0 = div\ \mathbf{h}_0 = 0, \tag{1.7}$$

$$\mathbf{u}(t, \mathbf{x}) = \mathbf{h}(t, \mathbf{x}) = \mathbf{0}, \quad \theta(t, \mathbf{x}) = 0 \quad \text{at} \quad z = 0,\ z = 1,\ t \geq 0. \tag{1.8}$$

Here $\mathbf{u} = (u, v, w) = (u_1, u_2, u_3)$, $w = \mathbf{u} \cdot \mathbf{k}$, $\mathbf{h} = (h_1, h_2, h_3)$, θ, p_1 are the perturbations of the velocity, magnetic, temperature and pressure (including the magnetic pressure) fields respectively. The dimensionless numbers are the Prandtl number $P_r = \nu/\kappa$, the Rayleigh number $R^2 = g\alpha\beta d^4/(\kappa\nu)$, the magnetic Prandtl number $P_m = \nu/\eta$, and the Chandrasekhar number $Q^2 = \mu H^2 d^2/(4\pi\rho\nu\eta)$, where ν is the coefficient of kinematic viscosity, κ is the coefficient of thermometric conductivity, $-g\mathbf{k}$ is the gravitational acceleration, α is the coefficient of volume expansion, ρ is the density, $\eta = 1/(4\pi\mu\sigma)$ is the resistivity, μ is the magnetic permeability, and σ is the coefficient of electrical conductivity. Assume that the perturbation fields are periodic functions of x and y, of periods $2\pi/a_x$ and $2\pi/a_y$ respectively, where $a_x, a_y > 0$. Denote by V the periodicity cell, $V = [0, 2\pi/a_x] \times [0, 2\pi/a_y] \times [0, 1]$ and let ∂V_h be the horizontal boundary. We have $\partial V_h = \partial V_1 \cup \partial V_0$, where ∂V_1 and ∂V_0 are the upper and lower boundary respectively. In the sequel, the brackets $\langle \cdot \rangle$ stand for the integration over V, i.e. $\langle \cdot \rangle = \int\limits_V \cdot\, dV$. We impose the extra conditions

$$\langle u \rangle = \langle v \rangle = 0. \tag{1.9}$$

2 Energy relation

In order to obtain nonlinear stability criteria, let us apply the G-P-R method [8] to the perturbation problem (1.1) – (1.8). To this aim, first we write the system (1.1) – (1.5) as the equivalent system consisting of the equations (1.1), (1.3) and (1.5), in the space

$$\begin{aligned}\mathcal{N}_1 = \{(\theta, \mathbf{u}, \mathbf{h}) \in H^2(V)^7 \mid\ & div\ \mathbf{u} = div\ \mathbf{h} = 0; \\ & \mathbf{u} = \mathbf{h} = \mathbf{0},\ \theta = 0 \text{ on } \partial V_h\}.\end{aligned}$$

In turn, this system is equivalent to the modified system in $\mathcal{N}_1$

$$\partial\theta/\partial t + (\mathbf{u} \cdot \mathbf{grad})\theta = P_r^{-1}\triangle\theta + P_r^{-1}R\mathbf{u} \cdot \mathbf{k}, \tag{2.1}$$

$$\begin{aligned}& a(\partial\mathbf{u}/\partial t + (\mathbf{u} \cdot \mathbf{grad})\mathbf{u}) + ag_3P_m(\partial\mathbf{h}/\partial t + (\mathbf{u} \cdot \mathbf{grad})\mathbf{h}) = \\ & = -a\ \mathbf{grad}\ p_1 + aR\theta\mathbf{k} + a\triangle\mathbf{u} + aQ\partial\mathbf{h}/\partial z + aP_m(\mathbf{h} \cdot \mathbf{grad})\mathbf{h} + \\ & + ag_3Q\partial\mathbf{u}/\partial z + ag_3\triangle\mathbf{h} + ag_3P_m(\mathbf{h} \cdot \mathbf{grad})\mathbf{u},\end{aligned} \tag{2.2}$$

$$\begin{aligned}& bP_m(\partial\mathbf{h}/\partial t + (\mathbf{u} \cdot \mathbf{grad})\mathbf{h}) + bg_2(\partial\mathbf{u}/\partial t + (\mathbf{u} \cdot \mathbf{grad})\mathbf{u}) = \\ & = bQ\partial\mathbf{u}/\partial z + b\triangle\mathbf{h} + bP_m(\mathbf{h} \cdot \mathbf{grad})\mathbf{u} - bg_2\mathbf{grad}\ p_1 + \\ & + bg_2R\theta\mathbf{k} + bg_2\triangle\mathbf{u} + bg_2Q\partial\mathbf{h}/\partial z + bg_2P_m(\mathbf{h} \cdot \mathbf{grad})\mathbf{h},\end{aligned} \tag{2.3}$$

obtained by the following algebraic operations: (2.1) = (1.5)P_r^{-1}, (2.2) = a(1.1) + ag_3 (1.3), (2.3) = b(1.3) + bg_2(1.1), where a, b, g_2 and g_3 are, so far, undetermined nonnull constants.

Consider on $\mathcal{N}_1$ the scalar product $(\cdot, \cdot)$ of $\mathbf{L}^2(V)$ ($\equiv L^2(V)^7$). Introduce two linear operators $L_1 \in L(\mathcal{N}_1, \mathbf{L}^2(V))$, $L_2 \in L(\mathcal{N}_1, \mathcal{N}_1)$ and use the notation $\mathbf{U} =$

$(\theta, \mathbf{u}, \mathbf{h})^T \in \mathcal{N}_1$, $\mathbf{U}_1 = L_2\mathbf{U} = (\theta, a\mathbf{u} + ag_3P_m\mathbf{h}, bg_2\mathbf{u} + bP_m\mathbf{h})^T$, where L_1 and L_2 are defined by

$$L_1 = \begin{bmatrix} P_r^{-1}\triangle & P_r^{-1}R\mathbf{k} & 0 \\ aR\mathbf{k} & a\triangle + ag_3Q\partial/\partial z & ag_3\triangle + aQ\partial/\partial z \\ bg_2R\mathbf{k} & bg_2\triangle + bQ\partial/\partial z & b\triangle + bg_2Q\partial/\partial z \end{bmatrix},$$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & ag_3P_m \\ 0 & bg_2 & bP_m \end{bmatrix}.$$

In addition, we define the nonlinear mapping

$$T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & ag_3P_m(\mathbf{h}\cdot\mathbf{grad}) & aP_m(\mathbf{h}\cdot\mathbf{grad}) \\ 0 & bP_m(\mathbf{h}\cdot\mathbf{grad}) & bg_2P_m(\mathbf{h}\cdot\mathbf{grad}) \end{bmatrix}.$$

It follows that the system (2.1) - (2.3) in $\mathbf{U} \in \mathcal{N}_1$ reads

$$(\partial/\partial t + \mathbf{u}\cdot\mathbf{grad})\mathbf{U}_1 = L_1\mathbf{U} + (0, -a\ \mathbf{grad}\ p_1, -bg_2\ \mathbf{grad}\ p_1)^T + T(\mathbf{U})$$

or, equivalently,

$$\partial\mathbf{U}_1/\partial t = L_1\mathbf{U} + N(\mathbf{U}) + T(\mathbf{U}), \tag{2.4}$$

where the mapping $N(\mathbf{U})$ corresponds to the advective and pressure terms, i.e.

$$N(\mathbf{U}) = -(\mathbf{u}\cdot\mathbf{grad})\mathbf{U}_1 + (0, -a\ \mathbf{grad}\ p_1, -bg_2\ \mathbf{grad}\ p_1)^T.$$

According to the Weyl decomposition lemma, a vector from $\mathbf{L}^2(V)$ is uniquely written as a sum of a solenoidal vector and a gradient of a scalar function. Then a projection of $\mathbf{L}^2(V)$ to $\mathcal{N}_1$ can be defined. If $(\cdot,\cdot)$ stands for the inner product in $\mathbf{L}^2(V)$, this projection of the system (2.1)-(2.3) to $\mathcal{N}_1$ is defined by the inner product of (2.4) by $\mathbf{U}$. As a result, from (2.4), we obtain the energy relation

$$(\partial\mathbf{U}_1/\partial t, \mathbf{U}) = (L_1\mathbf{U}, \mathbf{U}) + (N(\mathbf{U}), \mathbf{U}) + (T(\mathbf{U}), \mathbf{U}). \tag{2.5}$$

If $a = b$, then $(T(\mathbf{U}), \mathbf{U}) = 0$ because $\int_V (\mathbf{h}\cdot\mathbf{grad})\mathbf{u}\cdot\mathbf{u}\,dV = \int_V (\mathbf{h}\cdot\mathbf{grad})\mathbf{h}\cdot\mathbf{h}\,dV = 0$ and $\int_V (\mathbf{h}\cdot\mathbf{grad})\mathbf{h}\cdot\mathbf{u}\,dV = -\int_V (\mathbf{h}\cdot\mathbf{grad})\mathbf{u}\cdot\mathbf{h}\,dV$. Moreover, in order for the coefficients of $(\partial\mathbf{u}/\partial t)\cdot\mathbf{h}$ and $(\partial\mathbf{h}/\partial t)\cdot\mathbf{u}$ in the left - hand side of (2.5) be equal, we must have $g_3P_m = g_2$. Then (2.5) becomes

$$\frac{1}{2}\,d(\mathbf{U}_1, \mathbf{U})/dt = (L_1\mathbf{U}, \mathbf{U}) + (N(\mathbf{U}), \mathbf{U}). \tag{2.6}$$

Using Green identities, the relation $g_3P_m = g_2$ and the fact that $\mathbf{grad}\ p_1$ is orthogonal to the solenoidal vectors $\mathbf{u}$ and $\mathbf{h}$, it follows that $(N(\mathbf{U}), \mathbf{U}) = 0$. Consequently, the energy relation (2.6) becomes

$$\frac{1}{2}\,d(\mathbf{U}_1, \mathbf{U})/dt = (L_1\mathbf{U}, \mathbf{U}). \tag{2.7}$$

The symmetric part of L_1 reads

$$L_{1s} = \begin{bmatrix} P_r^{-1}\triangle & \delta_1\mathbf{k} & \delta_2\mathbf{k} \\ \delta_1\mathbf{k} & a\triangle & \delta_3\triangle \\ \delta_2\mathbf{k} & \delta_3\triangle & b\triangle \end{bmatrix},$$

where $\delta_1 = 0.5(a + P_r^{-1})R$, $\delta_2 = 0.5ag_2R$, $\delta_3 = 0.5a(g_3 + g_2)$. Since $(L_1\mathbf{U}, \mathbf{U}) = (L_{1s}\mathbf{U}, \mathbf{U})$, (2.7) becomes

$$\begin{aligned} \frac{1}{2} \cdot d(\mathbf{U}_1, \mathbf{U})/dt = P_r^{-1}[-\langle|\mathbf{grad}\ \theta|^2\rangle - P_r a\langle|\mathbf{grad}\ \mathbf{u}|^2\rangle - \\ - P_r a\langle|\mathbf{grad}\ \mathbf{h}|^2\rangle - P_r a(g_3 + g_2)\langle\mathbf{grad}\ \mathbf{u} \cdot \mathbf{grad}\ \mathbf{h}\rangle + \\ + P_r R(a + P_r^{-1})\langle\theta w\rangle + P_r R a g_2\langle\theta h_3\rangle]. \end{aligned} \tag{2.8}$$

3 The algebraic associated system

Introduce the functions

$$\phi_1 = a_1\mathbf{u} + a_2\mathbf{h}, \quad \phi_2 = b_1\mathbf{u} + b_2\mathbf{h},$$

where the constants $a_1, a_2, b_1, b_2 \in \mathbb{R}$ are to be determined and $\phi_1 = \phi_1(t, \mathbf{x})$, $\phi_2 = \phi_2(t, \mathbf{x})$. Remark that this choice represents an extension of the G-P-R method, because here ϕ_1 and ϕ_2 are vector functions. Thus, the expression

$$(\mathbf{U}_1, \mathbf{U}) = \langle\ |\theta|^2\rangle + a\langle|\mathbf{u}|^2 + (g_3P_m + g_2)\ \mathbf{u} \cdot \mathbf{h} + P_m|\mathbf{h}|^2\rangle$$

must read, equivalently,

$$(\mathbf{U}_1, \mathbf{U}) = \langle\ |\theta|^2\rangle + d_1\langle\ |\phi_1|^2\rangle + d_2\langle\ |\phi_2|^2\rangle,$$

where $d_1, d_2 \in \mathbb{R}^+$, implying

$$d_1a_1^2 + d_2b_1^2 = a, \quad d_1a_1a_2 + d_2b_1b_2 = ag_2, \quad d_1a_2^2 + d_2b_2^2 = aP_m, \tag{3.1}$$

where d_1, d_2, b_1, b_2 are determined up to some factor. Eliminating d_1 and d_2 between these equalities, we obtain the relationship between b_1 and b_2

$$a_2b_2 + P_ma_1b_1 - g_2(a_2b_1 + a_1b_2) = 0, \quad a_2^2b_1^2 - a_1^2b_2^2 \neq 0, \tag{3.2}$$

defining ϕ_2 up to a factor.

Let us find a_1 and a_2 such that (2.8) has the simple form

$$\frac{1}{2} \cdot d(\mathbf{U}_1, \mathbf{U})/dt = P_r^{-1}[-\langle|\mathbf{grad}\ \theta|^2 + |\mathbf{grad}\ \phi_1|^2\rangle + P_rRk'\langle\theta\phi_1 \cdot \mathbf{k}\rangle], \tag{3.3}$$

where k' is an undetermined factor. By identifying (2.8) and (3.3), it follows

$$\begin{aligned} -P_r a\langle|\ \mathbf{grad}\ \mathbf{u}|^2\rangle - P_r a\langle|\ \mathbf{grad}\ \mathbf{h}|^2\rangle - \\ -P_r a(g_3 + g_2)\langle\mathbf{grad}\ \mathbf{u} \cdot \mathbf{grad}\ \mathbf{h}\rangle = -\langle|\mathbf{grad}\ \phi_1|^2\rangle, \end{aligned}$$

$$P_r R(a + P_r^{-1})\langle\theta w\rangle + P_r R a g_2\langle\theta h_3\rangle = P_r R k'\langle\theta\phi_1\cdot\mathbf{k}\rangle.$$

If $P_m > 1$, we obtain

$$a = (P_m+1)[P_r(P_m-1)]^{-1}, \quad a_1 = \pm a_2,$$
$$g_2 = \pm 2P_m(P_m+1)^{-1}, \quad g_3 = P_m^{-1}g_2,$$

where the signs $+$ and $-$ correspond, and

$$a_1 = \pm\sqrt{(P_m+1)(P_m-1)^{-1}}, \quad k' = \pm 2P_m(P_r\sqrt{P_m^2-1})^{-1},$$

where the signs $+$ and $-$ correspond.

From $(3.1_{1,3})$ it follows that

$$d_1 = (b_2^2 - P_m b_1^2)/[P_r(b_2^2-b_1^2)], \quad d_2 = a(P_m-1)/(b_2^2-b_1^2).$$

Then, for $a_1 = a_2$, (3.2) implies $b_2/b_1 = P_m$, while, for $a_1 = -a_2$, (3.2) implies $b_2/b_1 = -P_m$. In both these cases, we have

$$d_1 = P_m/[P_r(P_m+1)], \quad d_2 = aP_m^2/[b_2^2(P_m+1)].$$

Therefore all these four solutions a, b, g_2, g_3, a_1, a_2, b_2/b_1, k' are convenient. In the next Section, we show that they lead to the same stability criterion.

4 The stability criterion

Introduce the functions

$$E(t) = \langle\ |\theta|^2 + d_1\ |\phi_1|^2\rangle/2, \quad \Psi(t) = d_2\langle\ |\phi_2|^2\rangle/2$$

and the notation

$$\xi^2 = \min_{\theta,\phi_1}\frac{2\langle\ |\mathbf{grad}\ \theta|^2 + |\mathbf{grad}\ \phi_1|^2\rangle}{\langle\ |\theta|^2 +\ |\phi_1|^2\rangle}\ ,$$
$$\frac{1}{\sqrt{R_a^*}} = \max_{\theta,\phi_1}\frac{2\langle\theta\phi_1\cdot\mathbf{k}\rangle}{\langle\ |\mathbf{grad}\ \theta|^2 + |\mathbf{grad}\ \phi_1|^2\rangle}\ . \tag{4.1}$$

Then, due to the fact that $\phi_1 = 0$ on ∂V_h, for $k' > 0$, the energy relation (2.7) becomes successively

$$\begin{aligned}\frac{dE}{dt} + \frac{d\Psi}{dt} &= P_r^{-1}[-\langle|\mathbf{grad}\ \theta|^2 + |\mathbf{grad}\ \phi_1|^2\rangle + P_r R k'\langle\theta\phi_1\cdot\mathbf{k}\rangle] = \\ &= -P_r^{-1}\langle|\mathbf{grad}\ \theta|^2 + |\mathbf{grad}\ \phi_1|^2\rangle\cdot[1 - \frac{P_r R k'\langle\theta\phi_1\cdot\mathbf{k}\rangle}{\langle|\mathbf{grad}\ \theta|^2 + |\mathbf{grad}\ \phi_1|^2\rangle}],\end{aligned} \tag{4.2}$$

implying

$$\frac{dE}{dt} + \frac{d\Psi}{dt} \le -P_r^{-1}\xi^2\frac{1}{max\{1,d_1\}}\cdot[1 - P_r R k'\ \frac{1}{2\sqrt{R_a^*}}]E, \tag{4.3}$$

whence *the stability criterion*

Theorem 1. *Suppose that* $P_m > 1$. *If* $R < \sqrt{P_m^2 - 1}\sqrt{R_a^*}/P_m$, *then the basic state* m_0 *is nonlinearly stable.*

Let $k' < 0$ and remark that, from the definition of $\mathcal{N}_1$, it follows that if $(\theta, \mathbf{u}, \mathbf{h}) \in \mathcal{N}_1$ than $(\theta, \phi_1, \phi_2) \in \mathcal{N}_1$, $(-\theta, \phi_1, \phi_2) \in \mathcal{N}_1$.

Introduce the space

$$\widetilde{\mathcal{N}}_1 = \{(\theta, \phi_1) \in H^2(V)^4 \mid div\ \phi_1 = 0;\ \phi_1 = \mathbf{0}, \quad \theta = 0 \text{ on } \partial V_h\}.$$

Obviously, $\widetilde{\mathcal{N}}_1$ is imbedded in $\mathcal{N}_1$, i.e. $\widetilde{\mathcal{N}}_1 \subset \mathcal{N}_1$. In addition, if (θ, ϕ_1) runs over $\widetilde{\mathcal{N}}_1$, than $(-\theta, \phi_1)$ runs over $\widetilde{\mathcal{N}}_1$ too.

We have $k'\langle\theta\phi_1 \cdot \mathbf{k}\rangle = |k'|\langle-\theta\phi_1 \cdot \mathbf{k}\rangle$ and $|\mathbf{grad}\ \theta|^2 = |\mathbf{grad}\ (-\theta)|^2$. Therefore (4.1) holds also for θ replaced by $-\theta$. Consequently, (4.3) hold for k' replaced by $|k'|$. In this way, for the case $k' < 0$, Theorem 1 holds too.

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