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The Lyapunov stability in restricted problems of cosmic dynamics

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Abstract. Majority of cosmic dynamical problems are described by Hamiltonian systems. In this case the Lyapunov stability problem is the toughest problem of qualitative theory, but for two freedom degrees KAM–theory (Kolmogorov–Arnold–Moser methods) allows for the complete study [1–3]. For application of Arnold–Moser theorem [4] it is necessary to make finite sequence of Poincaré–Birkhoff canonical transformations [5] for Hamiltonian normalization. With the help of Symbolic System "Mathematica" [6] we determine the conditions of Lyapunov stability and instability of equilibrium points of restricted n–body problems [7].

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1 Introduction

Let have the $2n$–dimensional Hamiltonian system

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} , \qquad \frac{dq}{dt} = \frac{\partial H}{\partial p} , \tag{1}$$

where the Hamiltonian $H(p, q)$ is of the type

$$H(p, q) \equiv H_0(p) + \mu H_1(p, q) \ , 0 \le \mu < 1 \ ,$$

where its perturbate part H_1 fulfils the condition

$$H_1(p, q) \equiv H_1(p, q + (2\pi)).$$

In addition we assume $H(p, q)$ to be 2π–periodical on $q_1, q_2, ..., q_n$ and analytical on $2n$–dimensional symplectic manifold

$$G_{2n} = \{p \in G_n, \|Imq\| < \rho < 1, \|Imq\| = \sum_{s=1}^{n} | \ Imq_s \ |\} \ ,$$

where G_n denotes a n–dimensional torus manifold in euclidean space. The variables (p, q) usually are referred to as "action – angle" coordinates [8]. The system of differential equations (1) describes the models of cosmic dynamics with the potential gravitational fields. The general and restricted newtonian n–body problems belong to this type.

According to H. Poincaré [9], it is necessary to do a full analytical and qualitative investigation of the system (1).

The problem of integration of the system (1) consists in finding a nondegenerate canonical mapping $G_{2n} \to G^*_{2n}, (p,q) \to (P,Q)$, that reduces the system (1) to the following one:

$$\frac{dP}{dt} = 0, \qquad \frac{dQ}{dt} = \frac{\partial H^*}{\partial P}\,.$$

It follows from this that in manifold G^*_{2n} one has

$$H^*(P) \equiv H(p,q).$$

On the base of this problem it is necessary to find effective methods of constructing periodical and quasi–periodical solution families of (1), and the investigation of the asymptotic evolution of trajectories of system (1) when $t \to \pm\infty$.

In KAM–theory the transformation $(p,q) \to (P,Q)$ is constructed with the help of an infinite sequence of convergent and nondegenerate canonical substitutions

$$(p,q) \leftrightarrow (p^{(1)}, q^{(1)}) \leftrightarrow (p^{(2)}, q^{(2)}) \leftrightarrow \ldots \leftrightarrow (p^{(\infty)}, q^{(\infty)}) \equiv (P,Q). \tag{2}$$

Convergence of the iterative process (2) is guaranteed by the method of accelerated convergence [10], in which the $k-th$ iteration has μ^{2^k}–order, i.e.

$$(p^{(k)}, \Delta q^{(k)}) = O(\mu^{2^k}),$$

where $\Delta q^{(k)}$ stands for the perturbation of the phase variable $q^{(k)}$.

In the classical methods, the $k-th$ iteration has μ^k–order, which means

$$(p^{(k)}, \Delta q^{(k)}) = O(\mu^k).$$

The process (2), constructed with the use of classical methods for the Hamiltonian systems of the dimensions $4, 6, 8, ...(n \geq 2)$, will be divergent. Therefore, H. Poincaré demonstrated [9] that in classic perturbation theory the sequence $(p,q) \to (P,Q)$ similar to (2) is divergent in G_{2n}.

Manifolds of convergence of canonical transformations (2) represent an infinite sequence of inclusions

$$G_{2n} \supset G^{(1)}_{2n} \supset G^{(2)}_{2n} \supset \ldots \supset G^{(\infty)}_{2n} \equiv G^*_{2n}\ ,$$

where $G^*_{2n} \neq \emptyset$ and

$$G^*_{2n} = \{P \in G^*_n, \|ImQ\| < \rho^* \leq \rho < 1\}\ .$$

V. Arnold demonstrated [11] that, unfortunately, the phase manifolds G^*_n and $\bar{G}_n = G_n \setminus G^*_n$ are everywhere dense in $G_n = G^*_n \cup \bar{G}_n$. C. Siegel has shown in [12], that in G^*_n the following inequality is true

$$|(k,\omega(p))| \geq \frac{K(\omega)}{\|k\|^{n+1}} \ ,$$

where $\omega(p) = \dfrac{\partial H_0}{\partial p}$, and the measures of manifolds $G_n^*, \bar{G_n}$ are

$$mesG_n^* = 1-\varepsilon, \quad mes\bar{G_n} = \varepsilon << 1 \ .$$

For study of Lyapunov stability problem it is not necessary to construct the infinite sequence (2), but it is sufficient to consider 4–8 iterations for and only for $n=2$. This fact is the main conclusion from the Arnold–Moser theorem [4].

In fact, if we represent the Hamiltonian $H(p,q)$ in neighbourhood of the equilibrium point $(0,0,0,0)$ in series form, we have

$$H(p,q) = H_2(p,q) + H_3(p,q) + H_4(p,q) + \dots,$$

where $H_k(p,q)$ are homogenous k–degree polynomials in $p=(p_1,p_2), q=(q_1,q_2)$.

The Arnold–Moser theorem's formulation is [4]:

If new (transformed) Hamiltonian has the form

$$\begin{gathered} W(\psi_1,\psi_2,T_1,T_2) = W_2(T_1,T_2) + W_4(T_1,T_2) + \dots, \\ W_2(T_1,T_2) = \sigma_1 T_1 - \sigma_2 T_2, \quad W_4(T_1,T_2) = c_{20}T_1^2 + c_{11}T_1T_2 + c_{02}T_2^2 \ , \end{gathered} \tag{3}$$

and is such that:

1) eigenvalues of linear system

$$\begin{aligned} \frac{dT_1}{dt} &= -\frac{\partial W_2}{\partial \psi_1} = 0, \frac{d\psi_1}{dt} = \frac{\partial W_2}{\partial T_1} = \sigma_1, \\ \frac{dT_2}{dt} &= -\frac{\partial W_2}{\partial \psi_2} = 0, \frac{d\psi_2}{dt} = \frac{\partial W_2}{dT_2} = -\sigma_2, \end{aligned}$$

are the numbers $\pm i\sigma_1, \pm i\sigma_2$*;*

$$2) \ \ n_1\sigma_1 + n_2\sigma_2 \neq 0, \ \textit{for } 0 < |n_1| + |n_2| \leq 4 \ ,$$

and

$$3) \ \ c_{20}\sigma_2^2 + c_{11}\sigma_1\sigma_2 + c_{02}\sigma_1^2 \neq 0;$$

then the equilibrium point

$$T_1 = T_2 = \psi_1 = \psi_2 = 0$$

of the Hamiltonian system with the Hamiltonian function W *(3) is stable in Lyapunov sense* [13].

While analyzing this theorem, we conclude that it is necessary to transform only expressions $H_2(p,q), H_3(p,q), H_4(p,q)$ to new forms $W_2, W_3 = 0, W_4$, in order to study the Lyapunov stability of the equilibrium point

$$p_1 = p_2 = q_1 = q_2 = 0$$

in the "nonresonant case".

2 Determination of equilibrium points

One application of the Arnold – Moser theorem is the study of stability in Lyapunov sense of Lagrange triangle in the famous, restricted circular problem of three bodies [4, 9, 14]. The other one is the study of equilibrium points stability in the restricted circular N–body problem [7, 15].

The differential equations of this dynamical problem in uniformly rotating coordinate system P_0xyz have the form [15]:

$$\begin{aligned} \frac{d^2x}{dt^2} - 2\omega_n \frac{dy}{dt} &= -\frac{m_0 x}{r^3} + \frac{\partial R}{\partial x} \ , \\ \frac{d^2y}{dt^2} + 2\omega_n \frac{dy}{dt} &= -\frac{m_0 y}{r^3} + \frac{\partial R}{\partial y} \ , \\ \frac{d^2z}{dt^2} &= -\frac{m_0 z}{r^3} + \frac{\partial R}{\partial z} \ , \end{aligned} \tag{4}$$

$$R(x,y,z) = \frac{\omega_n^2}{2}(x^2+y^2) + m\sum_{k=1}^{n}\left[\frac{1}{\Delta_k} - \frac{xx_k + yy_k + zz_k}{r_k^3}\right] \ ,$$

$$\Delta_k^2 = (x-x_k)^2 + (y-y_k)^2 + (z-z_k)^2 \ ,$$

$$r^2 = x^2+y^2+z^2, \ r_k^2 = x_k^2 + y_k^2 + z_k^2, \ k = 1, ..., n \ ,$$

$$x_k = a_0 \cos\frac{2\pi(k-1)}{n}, \ y_k = a_0 \sin\frac{2\pi(k-1)}{n}, \ z_k = 0, \ k = 1, .., n \ ,$$

$$\omega_n = \sqrt{\frac{1}{a_0^3}\left[m_0 + \frac{m}{4}\sum_{k=2}^{n}\left(\sin\frac{\pi(k-1)}{n}\right)^{-1}\right]} \ ,$$

$$n = N - 2 \ ,$$

ω_n is the angle speed of coordinate system P_0xyz relative to the original system, and also is the angle speed of regular polygon $P_1P_2...P_n$ in vertexes of which masses $m_1 = m_2 = ... = m_n \neq 0$ are situated round central body P_0 with mass m_0. If $m_0 = 0$ we have Lagrange–Wintner gravitational restricted models [15]. Of course it is always possible to write the equations (4) in the Hamiltonian form.

Determination of equilibrium positions of system (4) comes to solutions of the following nonlinear, functional equation system:

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0 \ ,$$

$$-\frac{m_0 x}{r^3} + \frac{\partial R}{\partial x} = -\frac{m_0 y}{r^3} + \frac{\partial R}{\partial y} = -\frac{m_0 z}{r^3} + \frac{\partial R}{\partial z} = 0 \ ,$$

or

$$\omega_n^2 x - \frac{m_0 x}{r^3} + m\sum_{k=1}^{n}\left[\frac{x_k - x}{\Delta_k^3} - \frac{1}{a_0^2}\cos\frac{2\pi(k-1)}{n}\right] = 0 \ ,$$

$$\omega_n^2 y - \frac{m_0 y}{r^3} + m\sum_{k=1}^{n}\left[\frac{y_k - y}{\Delta_k^3} - \frac{1}{a_0^2}\sin\frac{2\pi(k-1)}{n}\right] = 0\ , \tag{5}$$

$$-\frac{m_0 z}{r^3} + m\sum_{k=1}^{n}\frac{z}{\Delta_k^3} = 0\ .$$

In the system (5) the quantities x, y, z are unknowns.

Last equation from (5) for $z = 0$ is always realized. Then all equilibrium points of system (4) are located in the plane P_0xy. It can be shown that for any $n \geq 2$ the system (5) is equivalent to the system [15]:

$$\begin{aligned} &\omega_n^2 x - \frac{m_0 x}{r^3} + m\sum_{k=1}^{n}\frac{x_k - x}{\Delta_k^3} = 0\ ,\\ &\omega_n^2 y - \frac{m_0 y}{r^3} + m\sum_{k=1}^{n}\frac{y_k - y}{\Delta_k^3} = 0\ . \end{aligned} \tag{6}$$

For the famous restricted 3–body problem ($n = 1$) the equations (5) are of the form

$$\begin{aligned} &\omega_1^2 x - \frac{m_0 x}{r^3} + m\left(\frac{1 - x}{\Delta_1^3} - 1\right) = 0\ ,\\ &\omega_1^2 y - \frac{m_0 y}{r^3} - \frac{m y}{\Delta_1^3} = 0\ . \end{aligned} \tag{7}$$

For $y = 0$ the first equation from (7) has three solutions, which have been determined by Euler (collinear solutions). For $y \neq 0$ the system (7) has two solutions, which were determined by Lagrange (two equilateral triangles P_0P_1P). It is known that collinear points are unstable in first approximation for arbitrary values of parameter m.

Research of Lagrange triangle stability has a 200–year history. At first G. Gascheau, E. Routh and A. Lyapunov studied the triangle stability in first approximation [4]. The condition of this stability is

$$0 \leq m < \bar{m} = \frac{9 - \sqrt{69}}{18} = 0.0385209....$$

The stability in Lyapunov sense was studied by H. Poincaré, A. Lyapunov, G. Birkhoff, C. Siegel, V. Arnold, A. Deprit, J. Moser, A. Leontovich, A. Markeev, A. Sokolski, V. Sebehely and ultimate results were achieved on the base of KAM–theory [4, 16].

Using CSS "Mathematica", we have solved the equations (6) and counted the coordinates of equilibrium points in restricted 4, 5, 6, 7 – body problems. We found that all the "radial" [15] points are unstable in first approximation for all values $m \geq 0$, and the "bisectorial" [15] points are stable in first approximation for

$0 \leq m < m^*$, where the parameter m^* for different values of N is represented as follows:

N	$0 \leq m < m^*$
4	$0 \leq m < 0.085...$
5	$0 \leq m < 0.023...$
6	$0 \leq m < 0.0094...$
7	$0 \leq m < 0.0047...$

For all the values $0 \leq m < m^*$ all eigenvalues of the matrix of linear Hamiltonian equations in neighborhood of any bisectorial point S_i are the numbers $\pm\beta i, i = \sqrt{-1}$.

3 Research of Lyapunov stability

In order to use the Arnold–Moser theorem, one has to construct the operation of Birkhoff normalization of Hamiltonians with accuracy up to the fourth degree of local coordinates.

If we translate the origin of the coordinate system from point P_0 to any point S_i with coordinates x^*, y^* with the help of expressions

$$\begin{cases} X = x - x^*, \\ Y = y - y^*, \\ P_X = p_x - p_{x^*}, \\ P_Y = p_y - p_{y^*}, \end{cases}$$

and we pass to canonical variables (X, Y, P_X, P_Y), using classical transformations, we will receive, for example, the Hamiltonian $H(6)$ of the restricted problem of 6 bodies in the form:

$$\begin{aligned} H(6) = &- \left((X + x^*)^2 + (Y + y^*)^2\right)^{-1/2} - m\Bigg(\left((X + x^*)^2 + (Y + y^* - 1)^2\right)^{-1/2} + \\ &\left((X + x^* - 1)^2 + (Y + y^*)^2\right)^{-1/2} + \left((X + x^* + 1)^2 + (Y + y^*)^2\right)^{-1/2} + \\ &\left((X + x^*)^2 + (Y + y^* + 1)^2\right)^{-1/2} \Bigg) + \omega_4\Big((Y + y^*)(P_X + p_x^*) - \\ &(X + x^*)(P_Y + p_y^*)\Big) + 1/2\left((P_X + p_x^*)^2 + (P_Y + p_y^*)^2\right). \end{aligned} \tag{8}$$

Obviously Hamiltonian differential equations of restricted 6–body problem in the phase space (X, Y, P_X, P_Y) admit the solution

$$X = Y = P_X = P_Y = 0.$$

The performance of Birkhoff normalization of equations depends on the coordinates of concrete equilibrium point. In what follows, we will consider the bisectorial point S_1, stable in the first order approximation, with coordinates [17]

$$x^* = y^* = 0.709007,$$

calculated for $m = 0.009 < m^*$.

In small neighborhood of the point S_1, the Hamiltonian (8) is representable in the form of a convergent power series,

$$H = H_2(X, Y, P_X, P_Y) + H_3(X, Y) + H_4(X, Y) + ...,$$

where H_k are homogeneous of k–th degree polynomials and

$$\begin{aligned} H_2 &= -0.258702(X^2 + Y^2) + 0.5(P_X^2 + P_Y^2) - 1.44885XY + \omega_4(YP_X - XP_Y), \\ H_3 &= -0.148050513(X^3 + Y^3) + 1.5163341(X^2Y + XY^2), \\ H_4 &= 0.39066344(X^4 + Y^4) - 0.587145981(X^3Y + XY^3) - 3.53151X^2Y^2. \end{aligned} \tag{9}$$

The expressions (9) indicate that the quadratic form $H_2(X, Y, P_X, P_Y)$ contains the term $\omega_4(YP_X - XP_Y)$, which is the first obstacle on the way of Lyapunov stability investigation. Therefore, at first, we perform the nondegenerate canonical transformation

$$(X, Y, P_X, P_Y) \to (q_1, q_2, p_1, p_2),$$

$$[X, Y, P_X, P_Y]^T = A \cdot [q_1, q_2, p_1, p_2]^T, \tag{10}$$

where the matrix A is defined in such a way that the new transformed Hamiltonian K $\;(H(X, Y, P_X, P_Y) \to K(q_1, q_2, p_1, p_2))$ has the form

$$K(q_1, q_2, p_1, p_2) = K_2(q_1, q_2, p_1, p_2) + K_3(q_1, q_2, p_1, p_2) + K_4(q_1, q_2, p_1, p_2) + ...,$$

and its quadratic form K_2 does not contain the expressions $q_1p_2, q_2p_1, q_2p_2, q_1p_1, p_1p_2, q_1q_2$. The matrix A has the form

$$A = \begin{bmatrix} -2.74006 & 2.32275 & 2.27271 & 2.9743 \\ 0.204828 & 0.173633 & -3.55404 & -3.76981 \\ -1.96543 & 1.6661 & 1.44773 & 2.34872 \\ 0 & 0 & 2.44107 & 2.87964 \end{bmatrix}.$$

The matrix $A = [a_{ij}]$ is symplectic [4]. This means that, in the case of two freedom degrees, it fulfils the symplectic conditions represented by 6 equations:

$$\begin{aligned} a_{11}a_{32} - a_{12}a_{31} + a_{21}a_{42} - a_{22}a_{41} &= 0, \\ a_{11}a_{33} - a_{13}a_{31} + a_{21}a_{43} - a_{23}a_{41} &= 1, \\ a_{11}a_{34} - a_{14}a_{31} + a_{21}a_{44} - a_{24}a_{41} &= 0, \\ a_{12}a_{33} - a_{13}a_{32} + a_{22}a_{43} - a_{23}a_{42} &= 0, \\ a_{12}a_{34} - a_{14}a_{32} + a_{22}a_{44} - a_{24}a_{42} &= 1, \\ a_{13}a_{34} - a_{14}a_{33} + a_{23}a_{44} - a_{24}a_{43} &= 0. \end{aligned}$$

Finding transformation (10) is equivalent to determining the four–by–four matrix A with 16 unknown elements. Solution of the system of homogeneous linear algebraic equations of 16–th order turned out to be possible in practice only with the use of system of symbolic calculations.

Realization of the canonical transformation (10) gives the following expressions for the forms K_2, K_3 and K_4 [17]:

$$K_2 = 0.387142(p_1^2 + q_1^2) - 0.309396(p_2^2 + q_2^2), \tag{11}$$

$$\begin{gathered} K_3 = 20.6018p_1^3 + 17.5615p_2^3 + 5.202q_1^3 - 0.329434q_2^3 \\ -9.8733657q_1^2q_2 + 5.01276q_1q_2^2 + p_1^2(61.2536p_2 + 16.363q_1 \\ -21.60582q_2) + p_2^2(39.3824q_1 - 42.7265q_2) + p_1(58.3744p_2^2 \\ +54.0272p_2q_1 - 45.648q_1^2 - 62.9684p_2q_2 + 81.8039q_1q_2 \\ -35.9368q_2^2) - p_2(51.2223q_1^2 - 90.0176q_1q_2 + 38.754q_2^2)\ , \end{gathered} \tag{12}$$

$$\begin{gathered} K_4 = -73.253p_1^4 - 182.712p_2^4 + 23.3975q_1^4 + 9.51254q_2^4 \\ +p_1^3(-381.659p_2 + 344.902q_1 - 291.661q_2) + p_2^3(-600.158p_1 \\ +470.91q_1 - 377.686q_2) + p_1^2(-725.789p_2^2 + 1163.51p_2q_1 \\ -182.557q_1^2 - 967.849p_2q_2 + 234.407q_1q_2 - 64.4354q_2^2) \\ +p_2^2(1290.8p_1q_1 - 138.024q_1^2 - 1055.12p_1q_2 + 125.044q_1q_2 \\ -4.18566q_2^2) + p_1(-329.286p_2q_1^2 - 82.4767q_1^3 + 375.536p_2q_1q_2 \\ +268.272q_1^2q_2 - 75.8361p_2q_2^2 - 274.302q_1q_2^2 + 89.6987q_2^3) \\ +p_2(-106.674q_1^3 + 332.312q_1^2q_2 - 329.253q_1q_2^2 + 104.971q_2^3) \\ +q_1q_2(-78.8385q_1^2 + 96.5479q_1q_2 - 50.6582q_2^2). \end{gathered} \tag{13}$$

The new variables (q_1, q_2, p_1, p_2) are not variables of "action – angle" type, since K_2 depends not only on p_1, p_2, but also on the phase coordinates q_1, q_2. Therefore, one must further pass from the canonical variables (q_1, q_2, p_1, p_2) to the new canonical variables $(\theta_1, \theta_2, \tau_1, \tau_2)$ according to the Birkhoff formulas [5]

$$\begin{gathered} q_1 = \sqrt{2\tau_1}\sin\theta_1, \quad q_2 = \sqrt{2\tau_2}\sin\theta_2, \\ p_1 = \sqrt{2\tau_1}\cos\theta_1, \quad p_2 = \sqrt{2\tau_2}\cos\theta_2, \end{gathered} \tag{14}$$

The transformation (14) "eliminates" expressions with the new angle coordinates θ_1, θ_2 from the quadratic part of the new Hamiltonian F

$$K(q_1, q_2, p_1, p_2) \to F(\theta_1, \theta_2, \tau_1, \tau_2).$$

In other words, if one represents the new Hamiltonian F in the form

$$F(\theta_1, \theta_2, \tau_1, \tau_2) = F_2(\tau_1, \tau_2) + F_3(\theta_1, \theta_2, \tau_1, \tau_2) + F_4(\theta_1, \theta_2, \tau_1, \tau_2) + ...,$$

then its quadratic form F_2 should not depend on the phase angles θ_1, θ_2, but must depend only on the new momenta τ_1, τ_2. After the substitution (14) in expressions (11)–(13), we will have the following equalities for the forms F_2, F_3, and F_4:

$$F_2 = 0.774284\tau_1 - 0.618792\tau_2,$$

$$\begin{aligned}
&F_3 = (11.425\cos\theta_1 + 46.8457\cos 3\theta_1 + 22.6055\sin\theta_1 + 7.89204\sin 3\theta_1)\tau_1^{3/2}\\
&+ (21.6884\cos(2\theta_1+\theta_2) + 14.1864\cos\theta_2 + 137.377\cos(2\theta_1-\theta_2)\\
&+ 29.9068\sin(2\theta_1+\theta_2) - 44.5182\sin\theta_2 + 46.4992\sin(2\theta_1-\theta_2))\tau_1\tau_2^{1/2}\\
&+ (31.7316\cos\theta_1 + 3.03601\cos(\theta_1+2\theta_2) + 130.34\cos(\theta_1-2\theta_2)\\
&+ 62.7842\sin\theta_1 - 20.2224\sin(\theta_1+2\theta_2) + 68.8283\sin(\theta_1-2\theta_2))\tau_1^{1/2}\tau_2\\
&+ (9.85026\cos\theta_2 + 39.8211\cos 3\theta_2 - 30.911\sin\theta_2 - 29.9792\sin 3\theta_2)\tau_2^{3/2},
\end{aligned}$$

$$\begin{aligned}
&F_4 = (-166.062 - 193.301\cos 2\theta_1 + 66.3508\cos 4\theta_1 + 262.425\sin 2\theta_1\\
&+ 213.689\sin 4\theta_1)\tau_1^2 + (-736.077\cos(\theta_1+\theta_2) - 182.809\cos(3\theta_1+\theta_2)\\
&- 738.186\cos(\theta_1-\theta_2) + 130.436\cos(3\theta_1-\theta_2) + 118.39\sin(\theta_1+\theta_2)\\
&+ 355.128\sin(3\theta_1+\theta_2) + 725.101\sin(\theta_1-\theta_2) + 915.061\sin(3\theta_1-\theta_2))\tau_1^{3/2}\tau_2^{1/2}\\
&+ (-831.7 - 748.749\cos 2\theta_1 - 401.159\cos(2\theta_1+2\theta_2) - 895.925\cos 2\theta_2\\
&- 25.6232\cos(2\theta_1-2\theta_2) + 1016.5\sin 2\theta_1 + 132.471\sin(2\theta_1+2\theta_2)\\
&- 635.537\sin 2\theta_2 + 1432.63\sin(2\theta_1-2\theta_2))\tau_1\tau_2 + (-350.012\cos(\theta_1+3\theta_2)\\
&- 924.689\cos(\theta_1+\theta_2) - 951.62\cos(\theta_1-\theta_2) - 174.31\cos(\theta_1-3\theta_2)\\
&- 172.329\sin(\theta_1+3\theta_2) + 148.726\sin(\theta_1+\theta_2) + 934.752\sin(\theta_1-\theta_2)\\
&+ 972.492\sin(\theta_1-3\theta_2))\tau_1^{1/2}\tau_2^{3/2} - (261.892 + 384.449\cos 2\theta_2\\
&+ 84.507\cos 4\theta_2 + 272.715\sin 2\theta_2 + 241.328\sin 4\theta_2)\tau_2^2.
\end{aligned}$$

The canonical variables $(\theta_1, \theta_2, \tau_1, \tau_2)$ are variables of "action–angle" type, and the Hamiltonian equations in the neighborhood of the equilibrium point S_1 are expressed by equalities

$$\begin{aligned}
&\frac{d\tau_1}{dt} = -\frac{\partial F_3}{\partial\theta_1} - \frac{\partial F_4}{\partial\theta_1} + ..., \quad \frac{d\theta_1}{dt} = \frac{\partial F_2}{\partial\tau_1} + \frac{\partial F_3}{\partial\tau_1} + \frac{\partial F_4}{\partial\tau_1} + ...,\\
&\frac{d\tau_2}{dt} = -\frac{\partial F_3}{\partial\theta_2} - \frac{\partial F_4}{\partial\theta_2} + ..., \quad \frac{d\theta_2}{dt} = \frac{\partial F_2}{\partial\tau_2} + \frac{\partial F_3}{\partial\tau_2} + \frac{\partial F_4}{\partial\tau_2} + ...,
\end{aligned} \tag{15}$$

Unfortunately the Hamiltonian equations (15) still do not fulfil the conditions of the Arnold – Moser theorem. It is necessary to construct another canonical transformation $(\theta_1, \theta_2, \tau_1, \tau_2) \to (\psi_1, \psi_2, T_1, T_2)$ that will "annihilate" the form of order 3/2, i.e., transform $F_3(\theta_1, \theta_2, \tau_1, \tau_2)$ to $W_3(\psi_1, \psi_2, T_1, T_2) = 0$ and the second–order form $F_4(\theta_1, \theta_2, \tau_1, \tau_2)$ to $W_4(T_1, T_2)$.

We will search for the last canonical transformation (with the required accuracy) in the form

$$\begin{aligned}
\theta_1 &= \psi_1 + V_{13}(\psi_1, \psi_2, T_1, T_2) + V_{14}(\psi_1, \psi_2, T_1, T_2),\\
\theta_2 &= \psi_2 + V_{23}(\psi_1, \psi_2, T_1, T_2) + V_{24}(\psi_1, \psi_2, T_1, T_2),\\
\tau_1 &= T_1 + U_{13}(\psi_1, \psi_2, T_1, T_2) + U_{14}(\psi_1, \psi_2, T_1, T_2),
\end{aligned}$$

$$\tau_2 = T_2 + U_{23}(\psi_1, \psi_2, T_1, T_2) + U_{24}(\psi_1, \psi_2, T_1, T_2),$$

where $U_{13}, U_{23}, U_{14}, U_{24}, V_{13}, V_{23}, V_{14}, V_{24}$ are determined from some linear partial differential equations. For their solution, we apply the method of asymptotic integration of multifrequency systems of differential equations, developed in [18]. For example, the equation for the unknown function U_{13} has the form

$$\frac{\partial U_{13}}{\partial \psi_1}\sigma_1 - \frac{\partial U_{13}}{\partial \psi_2}\sigma_2 = A_{13}(\psi_1, \psi_2, T_1, T_2), \tag{16}$$

where
$A_{13} = (11.425 sin\psi_1 + 140.537 \sin 3\psi_1 - 22.6055 \cos \psi_1 - 23.6761 \cos 3\psi_1)T_1^{3/2}$
$+ (43.3768 \sin(2\psi_1 + \psi_2) + 274.753 \sin(2\psi_1 - \psi_2) - 59.8137 \cos(2\psi_1 + \psi_2)$
$- 92.9983 \cos(2\psi_1 - \psi_2))T_1 T_2^{1/2} + (31.7316 \sin \psi_1 + 3.03601 \sin(\psi_1 + 2\psi_2)$
$+ 130.34 \sin(\psi_1 - 2\psi_2) - 62.7842 \cos \psi_1 + 20.2224 \cos(\psi_1 + 2\psi_2)$
$- 68.8283 \cos(\psi_1 - 2\psi_2))T_1^{1/2} T_2.$

From all solutions of equation (16), it is necessary to choose one which ensures the form (3) of the new Hamiltonian,

$$W_2 = \sigma_1 T_1 - \sigma_2 T_2, \quad W_4 = c_{20} T_1^2 + c_{11} T_1 T_2 + c_{02} T_2^2,$$

where

$$\sigma_1 = 0.774284, \quad \sigma_2 = 0.618792,$$
$$c_{20} = 101.693, \quad c_{11} = 522.084, \quad c_{02} = 168.211.$$

Such solution exists and has the form
$U_{13} = -(14.7556 \cos \psi_1 + 60.5019 \cos 3\psi_1 + 29.1954 \sin \psi_1 + 10.1927 \sin 3\psi_1)T_1^{3/2}$
$- (126.769 \cos(2\psi_1 - \psi_2) + 42.9086 \sin(2\psi_1 - \psi_2) + 46.377 \cos(2\psi_1 + \psi_2)$
$+ 64.3313 \sin(2\psi_1 + \psi_2))T_1 T_2^{1/2} - (40.9819 \cos \psi_1 + 81.0867 \sin \psi_1$
$+ 64.7856 \cos(\psi_1 - 2\psi_2) + 34.2112 \sin(\psi_1 - 2\psi_2) - 6.55302 \cos(\psi_1 + 2\psi_2)$
$+ 43.6486 \sin(\psi_1 + 2\psi_2))T_1^{1/2} T_2.$

Thus, the expansion of the Hamiltonian of the restricted six–body problem in the neighborhood of the equilibrium S_1 with coordinates

$$x^* = y^* = 0.709007$$

presented finally in terms of the canonical variables $(\psi_1, \psi_2, T_1, T_2)$, fulfils all the conditions of the Arnold–Moser theorem, consequently, the equilibrium point S_1 is stable in Lyapunov sense.

In the interval $0 \le m \le 0.0094$, there exist two "resonant" values of the parameter m [15] ($m_1 \approx 0.005, m_2 \approx 0.0035$) for which the problem of the Lyapunov stability remains open.

Similarly, we have studied all equilibrium bisectorial points of restricted gravitational models, indicated in quote board.

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A Lie algebra of a differential generalized FitzHugh–Nagumo system

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Abstract. Some Lie algebra admissible for a generalized FitzHugh-Nagumo (F-N) system is constructed. Then this algebra is used to classify the dimension of the $Aff_3(2,R)$-orbits and to derive the four canonical systems corresponding to orbits of dimension equal to 1 or 2. The phase dynamics generated by these systems is studied and is found to differ qualitatively from the dynamics generated by the classical F-N system the $Aff_3(2,R)$-orbits of which are of dimension 3. A dynamic bifurcation diagram is also presented.

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1 The Lie algebra admissible for the generalized FitzHugh–Nagumo system

We investigate the generalized F-N system [1]

$$\dot{x} = a + cx + dy + px^3 \equiv P(x,y), \quad \dot{y} = b + ex + fy \equiv Q(x,y), \tag{1}$$

where the coefficients a, b, c, d, e, f, p are real and the phase functions x and y are real-valued and depend on the real time t; the dot over quantities stands for d/dt.

By [2], the Lie algebra consisting of the operators

$$X = \xi^1(x,y)\frac{\partial}{\partial x} + \xi^2(x,y)\frac{\partial}{\partial y} + D,$$

$$D = \eta^1\frac{\partial}{\partial a} + \eta^2\frac{\partial}{\partial b} + \eta^3\frac{\partial}{\partial c} + \eta^4\frac{\partial}{\partial d} + \eta^5\frac{\partial}{\partial e} + \eta^6\frac{\partial}{\partial f} + \eta^7\frac{\partial}{\partial p} \tag{2}$$

is admissible for (1) iff the coordinates of these operators satisfy the following system of partial differential equations

$$\xi^1_x P + \xi^1_y Q = \xi^1 P_x + \xi^2 P_y + DP,$$

$$\xi^2_x P + \xi^2_y Q = \xi^1 Q_x + \xi^2 Q_y + DQ, \tag{3}$$

where the coordinates ξ^1, ξ^2 and $\eta^i (i = \overline{1,7})$ are unknown functions of x and y and of the coefficients a, b, c, d, e, f, p, respectively.

Let us assume that ξ^1, ξ^2 are affine while η^i are linear functions, i.e.

$$\xi^1 = A + Bx + Cy, \quad \xi^2 = H + Kx + Ly,$$
$$\eta^i = \alpha_1^i a + \alpha_2^i b + \alpha_3^i c + \alpha_4^i d + \alpha_5^i e + \alpha_6^i f + \alpha_7^i p \quad (i = \overline{1,7}). \tag{4}$$

In this way, the determination of the unknown functions is reduced to the solution of an algebraic system in A, B, C, H, K, L, $\alpha_j^i (i, j = \overline{1,7})$. It is found

$$X = B\left(x\frac{\partial}{\partial x} + a\frac{\partial}{\partial a} + d\frac{\partial}{\partial d} - e\frac{\partial}{\partial e} - 2p\frac{\partial}{\partial p}\right) +$$
$$+H\left(\frac{\partial}{\partial y} - d\frac{\partial}{\partial a} - f\frac{\partial}{\partial b}\right) +$$
$$+L\left(y\frac{\partial}{\partial y} + b\frac{\partial}{\partial b} - d\frac{\partial}{\partial d} + e\frac{\partial}{\partial e}\right). \tag{5}$$

Since B, H and L are arbitrary, the expression (5) represents a family of operators, among which the operators corresponding to 1)$B = 0$, $H = 0$, $L = -1$; 2)$B = 0$, $H = -1$, $L = 0$; 3)$B = -1$, $H_3 = 0$, $L_3 = 0$ play a special role. Namely, we have

Theorem 1. *The maximum number of linearly independent Lie operators admissible for (1) and the coordinates of which are affine and linear functions of the phase functions and the parameters in (1), respectively, is equal to 3, and a triple of these operators reads*

$$X_1 = -y\frac{\partial}{\partial y} - b\frac{\partial}{\partial b} + d\frac{\partial}{\partial d} - e\frac{\partial}{\partial e},$$
$$X_2 = -\frac{\partial}{\partial y} + d\frac{\partial}{\partial a} + f\frac{\partial}{\partial b},$$
$$X_3 = -x\frac{\partial}{\partial x} - a\frac{\partial}{\partial a} - d\frac{\partial}{\partial d} + e\frac{\partial}{\partial e} + 2p\frac{\partial}{\partial p}. \tag{6}$$

It can be checked immediately that

Remark 1. *The Lie operators (6) form a three-dimensional Lie algebra the commutators of which are given in Table 1.*

Table 1

	X_1	X_2	X_3
X_1	0	X_2	0
X_2	$-X_2$	0	0
X_3	0	0	0

Denote this algebra by L_3.

Remark 2. *The Lie algebra L_3 is resolvable and has the operator X_3 as a nonnull central element.*

2 Functional basis of comitants and invariants

Let $Aff_3(2,R)$ be the group defined by the transformations q:

$$\overline{x} = \alpha x, \quad \overline{y} = \beta y + h, \quad \Delta = \alpha\beta \neq 0, \tag{7}$$

where $\alpha, \beta, h \in R$.

Lemma 1. *Performing in (1) the transformations (7) we get the system*

$$\dot{\overline{x}} = \overline{a} + \overline{cx} + \overline{dy} + \overline{px}^3, \quad \dot{\overline{y}} = \overline{b} + \overline{ex} + \overline{fy},$$

where

$$\overline{a} = \alpha a - \frac{\alpha h d}{\beta}, \quad \overline{c} = c, \quad \overline{d} = \frac{\alpha d}{\beta}, \quad \overline{p} = \frac{p}{\alpha^2},$$
$$\overline{b} = \beta b - hf, \quad \overline{e} = \frac{\beta e}{\alpha}, \quad \overline{f} = f. \tag{8}$$

By [2], by solving the Lie equations for the operators (6) we have

Remark 3. *The Lie algebra L_3 is equivalent to the $Aff_3(2,R)$-group affine representation by formulae (7) in the space of coefficients of the system (1) given by (8).*

From Theorem 1 and Remarks 1 and 3 we obtain

Corollary 1. *The largest affine group admissible for system (1) is $Aff_3(2,R)$ defined by formulae (7).*

Definition 1. *A polynomial $k(x,y,a,b,c,d,e,f,p)$ of the coefficients of the system (1) and variables x and y is called a comitant of this system with respect to the $Aff_3(2,R)$-group if the identity*

$$k(\overline{x},\overline{y},\overline{a},\overline{b},\overline{c},\overline{d},\overline{e},\overline{f},\overline{p}) = \Delta^{-g} k(x,y,a,b,c,d,e,f,p) \tag{9}$$

holds for every coefficients and variables of the system (1) and every parameters α,β,$h \in R$ of the $Aff_3(2,R)$-group.

If the comitant k does not depend on the variables x and y then it is referred to as the invariant of the system (1) with respect to the $Aff_3(2,R)$-group and is denoted by j. The integer g in (9) is called the $k(j)$ comitant (invariant) weight. If $g \neq 0$ then the comitant (invariant) is said to be relative and otherwise it is absolute.

Let us introduce the following operators from (6)

$$D_1 = -b\frac{\partial}{\partial b} + d\frac{\partial}{\partial d} - e\frac{\partial}{\partial e}, \quad D_2 = d\frac{\partial}{\partial a} + f\frac{\partial}{\partial b},$$
$$D_3 = -a\frac{\partial}{\partial a} - d\frac{\partial}{\partial d} + e\frac{\partial}{\partial e} + 2p\frac{\partial}{\partial p}. \tag{10}$$

Theorem 2. *The polynomial $k(j)$ is the comitant (invariant) of the system (1) of weight g with respect to the $Aff_3(2,R)$ -group iff the equalities*

$$X_1(k) = X_3(k) = gk, \quad X_2(k) = 0$$

$$(D_1(j) = D_3(j) = gj, \quad D_2(j) = 0) \tag{11}$$

hold, where $X_1 - X_3$ are given by (6) while $D_1 - D_3$ by (10).

Proof. Let us examine the operators $\alpha\dfrac{\partial}{\partial\alpha}$, $\dfrac{\partial}{\partial h}$, $\beta\dfrac{\partial}{\partial\beta}$.

For them, from (7) we have

$$\alpha\frac{\partial\Delta}{\partial\alpha} = \Delta, \quad \frac{\partial\Delta}{\partial h} = 0, \quad \beta\frac{\partial\Delta}{\partial\beta} = \Delta. \tag{12}$$

Applying the operator $\alpha\frac{\partial}{\partial\alpha}$ to (9) and taking into account (12) we obtain

$$\alpha\frac{\partial}{\partial\alpha}\left[k(\overline{x},\overline{y},\overline{a},\overline{b},\overline{c},\overline{d},\overline{e},\overline{f},\overline{p})\right] = -g\Delta^{-g}k(x,y,a,b,c,d,e,f,p).$$

Differentiating the left-hand side of this equality as a compound function of α, we get

$$\frac{\partial k}{\partial\overline{x}}\left(\alpha\frac{\partial\overline{x}}{\partial\alpha}\right) + \frac{\partial k}{\partial\overline{y}}\left(\alpha\frac{\partial\overline{y}}{\partial\alpha}\right) + \frac{\partial k}{\partial\overline{a}}\left(\alpha\frac{\partial\overline{a}}{\partial\alpha}\right) + \frac{\partial k}{\partial\overline{b}}\left(\alpha\frac{\partial\overline{b}}{\partial\alpha}\right) + \frac{\partial k}{\partial\overline{c}}\left(\alpha\frac{\partial\overline{c}}{\partial\alpha}\right) +$$

$$+\frac{\partial k}{\partial\overline{d}}\left(\alpha\frac{\partial\overline{d}}{\partial\alpha}\right) + \frac{\partial k}{\partial\overline{e}}\left(\alpha\frac{\partial\overline{e}}{\partial\alpha}\right) + \frac{\partial k}{\partial\overline{f}}\left(\alpha\frac{\partial\overline{f}}{\partial\alpha}\right) + \frac{\partial k}{\partial\overline{p}}\left(\alpha\frac{\partial\overline{p}}{\partial\alpha}\right) = -g\Delta^{-g}k.$$

Taking into account (7) and (8) this equality implies

$$\frac{\partial k}{\partial\overline{x}}(\alpha x) + \frac{\partial k}{\partial\overline{a}}\left(\alpha a - \frac{\alpha h d}{\beta}\right) + \frac{\partial k}{\partial\overline{d}}\left(\frac{\alpha d}{\beta}\right) + \frac{\partial k}{\partial\overline{e}}\left(-\frac{\beta e}{\alpha}\right) +$$

$$+\frac{\partial k}{\partial\overline{p}}\left(-\frac{2p}{\alpha^2}\right) = -g\Delta^{-g}k. \tag{13}$$

This identity holds for every transformation (7) of the $Aff_3 3(2,R)$-group. In particular, the equality (13) holds also for the identity transformation given by $\alpha = \beta = 1$, $h = 0$. In this case (13) implies

$$x\frac{\partial k}{\partial x} + a\frac{\partial k}{\partial a} + d\frac{\partial k}{\partial d} - e\frac{\partial k}{\partial e} - 2p\frac{\partial k}{\partial p} = -gk,$$

whence $X_3(k) = gk$.

The other equations in (11) are proved in a similar way. Since the conclusions are invertible, Theorem 2 follows.

Definition 2. *A functional basis of the set of comitants of the system (1) with respect to the* $Aff_3(2,R)$*-group is the set of functionally independent invariants*

$$j_1, j_2, ..., j_m \tag{14}$$

and comitants

$$k_1, k_2, ..., k_n \tag{15}$$

such that every comitant of the system (1) with respect to the given group can be expressed as a function of the elements of (14), (15).

The functional basis of the set of invariants for the system (1) with respect to the $Aff_3(2,R)$-group is defined analogously.The relationships between relative and absolute comitants can be taken into account to prove, by using (11), the validity of the following result.

Theorem 3. *The number of elements of the functional basis of comitants (invariants) of the system (1) with respect to the elements of the* $Aff_3(2,R)$*-group is equal to* ***7(5)***.

Theorems 2 and 3 can be used to prove

Theorem 4. *The functional basis of the invariants of the system (1) with respect to the* $Aff_3(2,R)$*-group consists of the elements*

$$\begin{gathered} j_1 = c \ (g=0), \quad j_2 = f \ (g=0), \quad j_3 = dp \ (g=1), \\ j_4 = p(af-bd)^2 \ (g=0), \quad j_5 = de \ (g=0), \end{gathered} \tag{16}$$

where g *are the corresponding weights of the invariants* j_l $(l = \overline{1,5})$.

Proof. By Theorem 2 the relations $D_i(j_l) = 0 \quad (i = 1,2,3; l = 1,2,4,5)$ and $D_1(j_3) = D_3(j_3) = j_3, \quad D_2(j_3) = 0$ hold, which shows that the expressions (16) are the invariants of the system (1) with respect to the $Aff_3(2,R)$-group. On the other hand, by Theorem 3, these 5 invariants of (16) could form a functional basis for the system (1) with respect to this group. In order to prove the last assertion it is sufficient to show that the general rank of the Jacobi matrix constructed by means of the invariants of (16) is equal to 5.

Remark that the minor constructed on the last 5 columns of this matrix has the form

$$M = det \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & p & 0 & 0 & d \\ 0 & -2bp(af-bd) & 0 & 2ap(af-bd) & (af-bd)^2 \\ 0 & e & d & 0 & 0 \end{pmatrix},$$

whence $M = -dp(a^2f^2 - b^2d^2) \not\equiv 0$. The last inequality proves the assertion in Theorem 4.

Theorem 5. *The functional basis of comitants of the system (1) with respect to the $Aff_3(2,R)$-group consists of the elements*

$$j_1 = c\ (g=0),\quad j_2 = f\ (g=0),\quad j_3 = dp\ (g=1),\quad j_4 = p(af-bd)^2\ (g=0),$$

$$p_1 = px^2\ (g=0),\quad p_2 = ex^2\ (g=-1),\quad p_3 = bx + fxy\ (g=-1), \tag{17}$$

where g are the corresponding weights of the invariants j_l $(l=\overline{1,4})$ and comitants p_i $(i=\overline{1,3})$.

The proof of Theorem 5 is analogous to the proof of Theorem 4, where as the minor of the Jacobi matrix constructed on the functions (17) is taken the minor situated on its last 7 columns and which is written as $M = 2fp^2(a^2f^2 - b^2d^2)x^4 \not\equiv 0$. This shows that the expressions in (17) form the functional basis of the comitants of the system (11) with respect to the $Aff_3(2,R)$-group. This concludes the proof.

3 Dimension of the $Aff_3(2,R)$-orbits for $p_1 \not\equiv 0$

If $p_1 \equiv 0$ then p=0 and the system (1) becomes

$$\frac{dx}{dt} = a + cx + dy,\quad \frac{dy}{dt} = b + ex + fy$$

and admits the group $Aff_6(2,R)$, which needs a separate investigation.

Let $A = (a,b,c,d,e,f,p) \in E(A)$, where $E(A)$ is the Euclidean space of coefficients in (1). Denote by $A(q)$ the point of $E(A)$ corresponding to the system obtained from (1) by means of the transformation $q \in Aff_3(2,R)$ given by (7).

Definition 3. *The set $O(A) = \{A(q);\ q \in Aff_3(2,R)\}$ is referred to as the $Aff_3(2,R)$-orbit of the point A for the system (1).*

Let M_1 be the matrix the entries of which are the coordinates of operators (10), i.e.

$$M_1 = \begin{pmatrix} 0 & -b & 0 & d & -e & 0 & 0 \\ d & f & 0 & 0 & 0 & 0 & 0 \\ -a & 0 & 0 & -d & e & 0 & 2p \end{pmatrix}. \tag{18}$$

Remark 4. *In [3] it is proved that*

$$dim_R O(a) = rank M_1. \tag{19}$$

Lemma 2. *For $p_1 \not\equiv 0$ the $rank M_1 = 3$ iff*

$$j_2p_2 + j_3 \not\equiv 0. \tag{20}$$

Proof. Denote by Δ_{ijk} $(1 \leq i, j, k \leq 7)$ the third order minors of the matrix M_1 corresponding to the columns i, j, k. In this case the only possible nonnull such minors are

$$\Delta_{124} = d(af - bd), \quad \Delta_{125} = -e(af - bd), \quad \Delta_{127} = 2bdp,$$

$$\Delta_{147} = -2d^2p, \quad \Delta_{157} = 2dep, \quad \Delta_{247} = -2dfp, \quad \Delta_{257} = 2efp. \tag{21}$$

Therefore $rankM_1 = 3$ iff $d^2 + e^2f^2 \neq 0$. In this case $(\Delta_{147})^2 + (\Delta_{257})^2 \neq 0$, therefore $rankM_1 = 3$. Taking into account that $j_2p_2 + j_3 = efx^2 + dp$, Lemma 2 follows by replacing the conditions on d, e, f by those on $j_2p_2 + j_3$.

Relation (21) shows that

$$j_2p_2 + j_3 \equiv 0 \tag{22}$$

iff

$$d = ef = 0. \tag{23}$$

Whence we have

Corollary 2. *For $p_1 \not\equiv 0$ $rank\, M_1 < 3$ iff (23) holds.*

Lemma 3. *For $p_1 \not\equiv 0$ $rank\, M_1 = 2$ iff*

$$j_2p_2 + j_3 \equiv 0, \quad p_2 + p_3 \not\equiv 0. \tag{24}$$

Proof. Let Δ^{ij}_{kl} $(1 \leq i, j \leq 3; 1 \leq k, l \leq 7)$ denote the second order minors of M_1 corresponding to the rows i, j and columns k, l. By (23) we must consider only the following cases: 1) $d = e = f = 0$, 2) $d = e = 0, f \neq 0$, 3) $d = f = 0, e \neq 0$. In the case 1) the only nonnull minor is $\Delta^{13}_{27} = -2bp$, hence $rank\, M_1 = 2$ iff $b \neq 0$, or equivalently, iff $p_2 + p_3 \neq 0$ because $p_2 + p_3 = bx$. In the cases 2) and 3) we have $\Delta^{23}_{27} = 2fp \neq 0$ and $\Delta^{13}_{57} = -2ep \neq 0$, hence $rank\, M_1 = 2$. Since in these cases $p_2 + p_3 \not\equiv 0$, Lemma 3 follows.

It is immediate that the conditions

$$\Delta^{13}_{12} = -ae, \quad \Delta^{13}_{25} = -be, \quad \Delta^{13}_{27} = -2bp,$$

$$\Delta^{13}_{57} = -2ep, \quad \Delta^{23}_{12} = af, \quad \Delta^{23}_{27} = 2fp \tag{25}$$

hold iff

$$p_2 + p_3 \equiv 0. \tag{26}$$

Corollary 3. *If (26) holds then $j_2 = 0$, $p_2 \equiv 0$ holds, too.*

Lemma 4. *For $p_1 \not\equiv 0$ $rankM_1 = 1$ iff*

$$j_3 + p_2 + p_3 \equiv 0. \tag{27}$$

Proof. Lemma 3 shows that $rank\, M_1 < 2$ corresponds to

$$j_2 p_2 + j_3 = 0, \quad p_2 + p_3 \equiv 0, \tag{28}$$

or, equivalently,

$$b = d = e = f = 0. \tag{29}$$

For (29) M_1 becomes

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a & 0 & 0 & 0 & 0 & 0 & 2p \end{pmatrix}.$$

Since $p_1 = px^2 \not\equiv 0$ it follows that $rank M_1 = 1$, whence Lemma 4 holds.

The Lemmas 2–4 and relation (19) imply

Theorem 6. *For $p_1 \not\equiv 0$ the $Aff_3(2,R)$- orbits of the system (1) have the following dimensions*

$$3 \quad for \quad j_2 p_2 + j_3 \not\equiv 0;$$

$$2 \quad for \quad j_2 p_2 + j_3 \equiv 0, p_2 + p_3 \not\equiv 0;$$

$$1 \quad for \quad j_3 + p_2 + p_3 \equiv 0,$$

where j_2, j_3, p_2, p_3 are given by (17).

Definition 4. *The set $N \subseteq E(A)$ is called an $Aff_3(2,R)$-invariant if for every $A \in N$ we have $O(A) \subseteq N$.*

Let us denote by $N_1 \equiv N_1\,(j_3 + p_2 + p_3 \equiv 0)$ and $N_2 \equiv N_2\,(j_2p_2 + j_3 \equiv 0, p_2 + p_3 \not\equiv 0)$ the $Aff_3(2,R)$-invariant sets of Theorem 6 the orbits of which have the dimension 1 and 2, respectively. Let us remark that $N_2 = N_2' \bigcup N_2'' \bigcup N_2'''$, where $N_2' \equiv N_2'\,(j_2 = j_3 = 0, p_2 \equiv 0)$, $N_2'' \equiv N_2''\,(j_2 = j_3 = 0, p_2 \not\equiv 0)$, $N_2''' \equiv N_2'''\,(j_2 \neq 0, j_3 = 0,\, p_2 \equiv 0)$ are sets invariant with respect to the $Aff3(2,R)$-group. They are also mutually disjoint.

Remark 5. *The generalized F-N system (1) on the $Aff_3(2,R)$-invariant sets N_1 and N_2 have the following canonical forms*

$$\dot{x} = a + cx + px^3, \dot{y} = 0 \quad on \quad N_1, \; where \; p \neq 0; \tag{30}$$

$$\dot{x} = a + cx + px^3, \dot{y} = b \quad on \quad N_2', \; where \; pb \neq 0; \tag{31}$$

$$\dot{x} = a + cx + px^3, \dot{y} = b + ex \quad on \quad N_2'', \; where \; pe \neq 0; \tag{32}$$

$$\dot{x} = a + cx + px^3, \dot{y} = b + fy \quad on \quad N_2''', \; where \; pf \neq 0. \tag{33}$$

4 Phase dynamics for systems (30)-(33)

For an easier treatment we reduce the number of the parameters by the time rescaling $t = \dfrac{\tau}{p}$. Considering the new parameters

$$r = \frac{a}{p}, \quad q = \frac{c}{p}, \quad m = \frac{b}{p}, \quad n = \frac{e}{p}, \quad s = \frac{f}{p}, \tag{34}$$

systems (30)–(33) become

$$\dot{x} = r + qx + x^3, \quad \dot{y} = 0; \tag{35}$$

$$\dot{x} = r + qx + x^3, \quad \dot{y} = m,\ m \neq 0; \tag{36}$$

$$\dot{x} = r + qx + x^3, \quad \dot{y} = m + nx,\ n \neq 0; \tag{37}$$

$$\dot{x} = r + qx + x^3, \quad \dot{y} = m + sy,\ s \neq 0, \tag{38}$$

where the dot stands for the differentiation with respect to the new time τ and $x \neq 0$. The equilibrium points of these systems satisfy

$$\dot{x} = 0, \quad \dot{y} = 0. \tag{39}$$

That is why we consider first the equation

$$r + qx + x^3 = 0. \tag{40}$$

Its discriminant is $D = \left(\frac{q}{3}\right)^3 + \left(\frac{r}{2}\right)^2$.

Equation (40) has a single real solution x_0 if $D > 0$, three distinct real solutions x_1, x_2, x_3 if $D < 0$ and two distinct real solutions, one of them being double, if $D = 0$. It is convenient to consider the following expressions for these solutions [4]:

a) $D > 0$

a1) if $q = 0$, then

$$x_0 = \sqrt[3]{-r}; \tag{41}$$

a2) if $q > 0$, then

$$x_0 = -2\sqrt{\frac{q}{3}}\sinh\theta, \theta = \frac{1}{3}sinh^{-1}\left(\frac{r}{2\sqrt{\left(\frac{q}{3}\right)^3}}\right); \tag{42}$$

a3) if $q < 0$, then

$$x_0 = -2\sqrt{\frac{q}{3}}\frac{1}{sin(2\phi)}, \quad \phi \in (-\frac{\pi}{4}, \frac{\pi}{4}), \tag{43}$$

with $tg\,\phi = \sqrt[3]{tg\frac{\psi}{2}}, \quad sin\,\psi = \frac{2}{r}\sqrt{-\left(\frac{q}{3}\right)^3}, \quad \psi \in (-\frac{\pi}{2}, \frac{\pi}{2});$

b) $D < 0$ (so $q < 0$)

$$x_1 = -2\sqrt{-\frac{q}{3}} sin(\Phi + \frac{\pi}{3}), \tag{44}$$

$$x_2 = 2\sqrt{-\frac{q}{3}} sin\Phi, \tag{45}$$

$$x_3 = 2\sqrt{-\frac{q}{3}} sin(\frac{\pi}{3} - \Phi), \tag{46}$$

where $\Phi = \frac{1}{3} sin^{-1} \frac{r}{2\sqrt{\left(-\frac{q}{3}\right)^3}} \in (-\frac{\pi}{6}, \frac{\pi}{6})$. In addition, $x_1 < x_2 < x_3$.

c) $D = 0$ (so $q \leq 0$)

$$x_1 = -2\sqrt{-\frac{q}{3}}, \quad x_2 = x_3 = \sqrt{-\frac{q}{3}} \tag{47}$$

or

$$x_1 = x_2 = -\sqrt{-\frac{q}{3}}, \quad x_3 = 2\sqrt{-\frac{q}{3}}. \tag{48}$$

If $q = 0$ and $D = 0$, then $r = 0$ and $x_1 = x_2 = x_3 = 0$, but this situation will not be considered because $x \neq 0$.

In order to obtain the equilibria of system (35), system (39) must be solved. As the second equation (39) is always satisfied, system (35) has an infinity of equilibria, situated on one, three or two straight lines $x = x_i$ in the phase plane (x, y), as $D > 0$, $D < 0$ or $D = 0$, respectively. The matrix of the linearized system around an equilibrium point (x_i, k) is

$$A_1 = \begin{pmatrix} q + 3x_i^2 & 0 \\ 0 & 0 \end{pmatrix}$$

and the corresponding eigenvalues are $\lambda_1 = 0, \lambda_2 = q + 3x_i^2$. Although all equilibria are nonhyperbolic, their type can be deduced very easy, because the dynamics takes place on straight lines $y = k$. Thus, analyzing the variation of the function $F(x) = r + qx + x^3$, its sign can be found. It follows that x is increasing on the straight lines $y = k$ when $F > 0$ and is decreasing when $F < 0$. That is why, when $D > 0$, the equilibria (x_0, k) are repulsors, when $D < 0$, the equilibria (x_1, k) and (x_3, k) are repulsors and the equilibria (x_2, k) are attractors, while when $D = 0$, the simple equilibrium points are repulsors and the double equilibrium points are degenerated saddles. The bifurcation diagram of system (35) is given in Figure 1.

System (36) has no equilibrium points because the second equation (39) is never satisfied. With the transformation $\frac{y}{m} = y_1$, system (36) becomes

$$\dot{x} = r + qx + x^3, \quad \dot{y} = 1$$

and the equations of the phase trajectories can be found. Thus, if $D < 0$, then

$$\frac{dx}{dy_1} = (x - x_1)(x - x_2)(x - x_3),$$

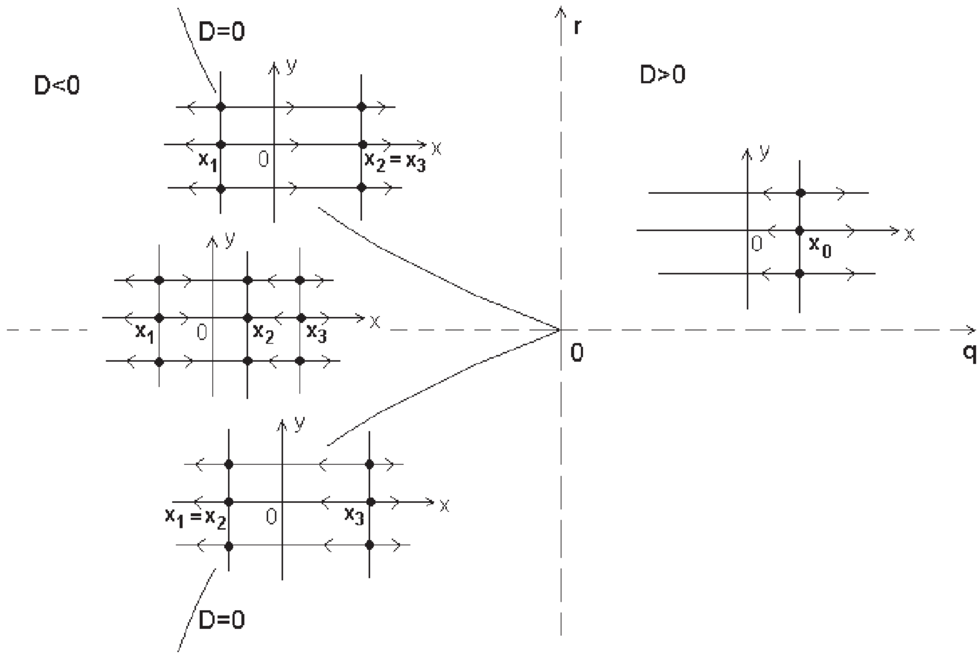

Fig. 1 Bifurcation diagram for system (35)

and the trajectory through (x_{10}, y_{10}) has the equation

$$y_1 = y_{10} + \alpha_1 \ln\left|\frac{x - x_1}{x_{10} - x_1}\right| + \alpha_2 \ln\left|\frac{x - x_2}{x_{10} - x_2}\right| + \alpha_3 \ln\left|\frac{x - x_3}{x_{10} - x_3}\right|$$

where

$$\alpha_1 = \frac{1}{(x_1 - x_2)(x_1 - x_3)},\ \alpha_2 = \frac{1}{(x_2 - x_1)(x_2 - x_3)},\ \alpha_3 = \frac{1}{(x_3 - x_1)(x_3 - x_2)}$$

and $y_1 = \tau + y_{10}$.

Similar formulas for $D \geq 0$ can be obtained.

The second equation (39) for system (37) gives $x = -\frac{m}{n}$. Consequently, system (37) has no equilibrium points if $-\frac{m}{n} \neq x_i$, $i = 0, 1, 2, 3$, where the expressions for x_i are given by (41)–(48) and it has an infinity of equilibria situated on the straightline $x = x_i$ of the phase plane if $-\frac{m}{n} = x_i$.

The linearized system around an equilibrium point (x_i, k), $k \in R$, has the matrix

$$A_2 = \begin{pmatrix} q + 3x_i^2 & 0 \\ n & 0 \end{pmatrix}$$

and the corresponding eigenvalues are the same as for the matrix A_1, namely $\lambda_1 = 0, \lambda_2 = q + 3x_i^2$. The equations of the phase trajectories can be obtained considering $\frac{dx}{dy} = \frac{r + qx + x^3}{m + nx}$.

Consequently $\dfrac{m+nx}{r+qx+x^3}dx = dy$ and $y = y(x)$ follow in every situation $D > 0$, $D < 0$, $D = 0$.

Consider now system (38). As from the second equation (39) it follows $y = -\frac{m}{s}$, system (38) can have one, three or two equilibria if $D > 0$, $D < 0$ or $D = 0$, respectively. The linearized system around an equilibrium point $(x_i, -\frac{m}{s})$, $i = 0, 1, 2, 3$, has the matrix

$$A_3 = \begin{pmatrix} q + 3x_i^2 & 0 \\ 0 & s \end{pmatrix}$$

and the corresponding eigenvalues are $\lambda_1 = s \neq 0, \lambda_2 = q + 3x_i^2$.

In order to find the type of the equilibrium points, the sign of λ_2 must be considered. Thus, if $D > 0$, $\lambda_2 = q + 3x_0^2$. Replacing the expression of x_0 given by (41), (42) or (43) into the expression of λ_2, it follows that $\lambda_2 > 0$. Consequently, if $s > 0$, then the equilibrium point $(x_0, -\frac{m}{s})$ is a repulsor and if $s < 0$, it is a saddle point.

If $D < 0$, using (44) we get $\lambda_2 = q + 3x_1^2 = q\left[1 - 4sin^2\left(\Phi + \frac{\pi}{3}\right)\right] > 0$. Consequently, the equilibrium point $(x_1, -\frac{m}{s})$ is a repulsor for $s > 0$ and a saddle for $s < 0$. Using (45), $\lambda_2 = q + 3x_2^2 = q(1 - 2sin\Phi)(1 + 2sin\Phi) < 0$ so $(x_2, -\frac{m}{s})$ is an attractor for $s < 0$ and a saddle for $s > 0$. Using (46), we get $\lambda_2 = q + 3x_3^2 = q(1 - 2sin\left(\frac{\pi}{3} - \Phi\right))(1 + 2sin\left(\frac{\pi}{3} - \Phi\right)) > 0$, so the equilibrium point $(x_3, -\frac{m}{s})$ is a repulsor for $s > 0$ and a saddle for $s < 0$.

If $D = 0$, using (47) and (48) it follows that $\lambda_2 = 0$ for the double equilibrium point and $\lambda_2 > 0$ for the simple equilibrium point. Thus, the simple equilibrium is a repulsor for $s > 0$ and a saddle for $s < 0$, while the double equilibrium point is nonhyperbolic. Its type will be deduced using the center manifold theory [5]. Using the transformation $u = x - x_i, \quad v = y + \frac{m}{s}$ where x_i is the abscissa of the double equilibrium point and taking into account that $\lambda_2 = 0$, system (38) becomes

$$\dot{u} = 3x_iu^2 + u^3, \quad \dot{v} = sv. \tag{49}$$

System (49) has the origin as an equilibrium point with the eigenvalues $\lambda_1 = s, \lambda_2 = 0$. The center manifold must be of the form

$$v = V(u) = \gamma_1 u^2 + \gamma_2 u^3 + ... \tag{50}$$

Replacing (50) into the second equation (49), we get

$$\frac{\partial V}{\partial u}\,\dot{u} = sV(u)$$

which is equivalent with

$$\left(2\gamma_1 u + 3\gamma_2 u^2 + ...\right)\left(3x_iu^2 + u^3\right) = s\left(\gamma_1 u^2 + \gamma_2 u^3 + ...\right)$$

It follows $\gamma_1 = \gamma_2 = ... = 0$. Consequently, the center manifold is $V(u) = 0$ and the flow on the center manifold is given by $\dot{u} = 3x_iu^2 + u^3$. As $x_i \neq 0$, it follows that the double equilibrium point is a nondegenerate saddle-node.

5 F-N system

The classical F-N system reads [1]

$$\dot{x} = c_{FN}(x + y - x^3/3), \quad \dot{y} = -(x - a_{FN} + yb_{FN})/c_{FN}$$

therefore it corresponds to the coefficients $a = 0$, $b = \frac{a_{FN}}{c_{FN}}$, $c = d = c_{FN}$, $e = -\frac{1}{c_{FN}}$, $f = -\frac{b_{FN}}{c_{FN}}$, $p = -c_{FN}$. Since $c_{FN} \neq 0$ it follows that the corresponding parameters belong to a set (manifold) of the $Aff_3(2, R)$-group. Let $N_3 = N_3(j_2p_2 + j_3 \neq 0)$. We have $N_3 = N_3'UN_3''$, where $N_3' = N_3'(j_3 \neq 0)$ and $N_3''(j_2p_2 \neq 0)$ are two disjoint sets. They are invariant with respect to the $Aff_3(2, R)$-group. On N_3' and N_3'' the system (1) has the following canonical forms

$$\dot{x} = a + cx + dy + px^3, \quad \dot{y} = b + ex + fy, \tag{51}$$

where $p \neq 0$, i.e. the given system (1),

$$\dot{x} = a + cx + px^3, \quad \dot{y} = b + ex + fy, \tag{52}$$

where $pe \neq 0$, $pf \neq 0$. Hence the classical F-N system is of the form (51). Its main characteristic is that, in general, it cannot be decoupled (i.e. it is not of separate variables). This is mainly due to the fact that the parameter d, which has a crucial role in the dynamics generated by (51), is nonzero. Since for the F-N system we have $c = d$, it follows that it never takes the forms (30)–(33), (52). This explains the big differences between the dynamics generated by the classical F-N system and the dynamics generated by (30)–(33), sketched in the previous section.

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Quadratic systems with limit cycles of normal size

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Abstract. In the class of planar autonomous quadratic polynomial differential systems we provide 6 different phase portraits having exactly 3 limit cycles surrounding a focus, 5 of them have a unique focus. We also provide 2 different phase portraits having exactly 3 limit cycles surrounding one focus and 1 limit cycle surrounding another focus. The existence of the exact given number of limit cycles is proved using the Dulac function. All limit cycles of the given systems can be detected through numerical methods; i.e. the limit cycles have "a normal size" using Perko's terminology.

Mathematics subject classification: 34C07, 34C08.
Keywords and phrases: quadratic systems, limit cycles.

1 Introduction

A planar autonomous quadratic polynomial differential system (or simply a *quadratic system*) in what follows is a system of the form

$$\frac{dx}{dt} = \sum_{i+j=0}^{2} a_{ij}x^i y^j \equiv P(x,y), \quad \frac{dy}{dt} = \sum_{i+j=0}^{2} b_{ij}x^i y^j \equiv Q(x,y), \tag{1}$$

with $a_{ij}, b_{ij} \in \mathbb{R}$. It is known (see, for instance [17]) that a quadratic system can have only limit cycles enclosing a unique singular point, which is a focus. As system (1) has no more than two foci [17], only the following distributions of limit cycles are allowed: n, (n_1, n_2), where $n \in \mathbb{N}$, and $n_1, n_2 \in \mathbb{N} \cup \{0\}$ with $n_1 + n_2 > 0$. Here n is the number of limit cycles surrounding a focus provided that system (1) has only one focus, and n_1 and n_2 are the number of limit cycles surrounding every one of the two foci provided that the system has exactly two foci. Recently, Zhang Pingguang [20, 21] has proved that if $n_i > 0$ for $i = 1, 2$, then either $n_1 = 1$, or $n_2 = 1$.

The following distributions of limit cycles for quadratic systems (1) are known:

(a) 1 and $(1,0)$;	(b) 2 and $(2,0)$;	(c) 3 and $(3,0)$;
(d) $(1,1)$;	(e) $(2,1)$;	(f) $(3,1)$.

With the help of distinct results on uniqueness of a limit cycle (see [15, 17, 22]), being the most effective result from Zhang Zhifen (see [14]), it has been proved for

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the quadratic systems (1) that the distributions of limit cycles (a) and (d) exist, see [5, 6, 14, 16, 18, 19]. The small class of systems with distribution (b) is obtained by bifurcation of limit cycles either from a focus and from a separatrix cycle, or from a focus, see [17].

The most complicated distributions of limit cycles are the distributions (c), (e) and (f). They are obtained also with the help of bifurcations, and by perturbing quadratic systems (1) having a center [2]. However using these methods it is only possible to obtain infinitesimal limit cycles which, in general, are very difficult to detect using numerical methods. Thus, Perko in the work [14] can exhibit quadratic systems with limit cycles "of normal size" using his terminology, i.e. limit cycles which can be detected easily by numerical methods. The main method used by him, consists in considering a set of systems with a rotating parameter, and in studying the bifurcations of limit cycles under the variation of this parameter. For more details on rotating families see [13, 14, 17, 22], and Section 2.

Perko in [14] provided examples of quadratic systems with the six distributions of limit cycles (a)–(f), but he did not consider all the possible phase portraits with these distributions of limit cycles. The purpose of this paper is: first, to systematize Perko's method; and second, to study different phase portraits with the distributions (c) and (f) of limit cycles.

For proving the existence of the exact given number of limit cycles we shall use Dulac functions, see [17] for more details on these functions. A key point for studying the distributions (c) and (f) of limit cycles are the works [1] and [12], where the qualitative phase portraits of all quadratic systems having a weak focus of third order are classified, and additionally, it is described the partition of the parameter space into domains associated to the different topological phase portraits.

By means of an affine transformation of the phase variables and a change of the time scale, a quadratic system (1) generically can be written as

$$\begin{aligned} \frac{dx}{dt} &= 1 + xy, \\ \frac{dy}{dt} &= a_{00} + a_{10}x + a_{20}x^2 + a_{01}y + a_{11}xy + ay^2, \end{aligned} \tag{2}$$

where $a_{00} = a_{01} + a_{11} - a_{10} - a_{20} - a$.

In Table 1 we summarize the main results of this paper, i.e. the different configurations of singular points compatible with the distributions (c) and (f) of limit cycles. The results of that table are for quadratic systems in the normal form (2). A focus, a node or a saddle is denoted by F, N and S, respectively. If they are at infinity in the Poincaré Compactification, then they have the subindex ∞. For more details on the Poincaré compactification of a planar polynomial differential system see [8].

N^0	Coefficients of system					Singular points	Cycle distr.
	a	a_{20}	a_{11}	a_{01}	a_{10}		
1	3	-12	-1.398	8.4	15.28	$1F+1N+2S_\infty+1N_\infty$	3
2	1.5	-15	0.79993	3.2	9.17	$2F+2S_\infty+1N_\infty$	$(3,0)$
3	-2	12	10.999	-14	-26.1	$1F+3S+3N_\infty$	3
4	-2	-1	9.49965	-12.5	6.955	$1F+1S+2N_\infty+1S_\infty$	3
5	-4	-1	13.9987	-21	12.4	$1F+1N+2S+2N_\infty+1S_\infty$	3
6	5	-50	-5.49995	16.5	76.45	$1F+2N+1S+1N_\infty+2S_\infty$	3
7	8/11	-12	2.1502	67/220	-26.5	$2F+1S_\infty$	(3,1)
8	1.04	-120	1.51997	1.56	-79.6	$2F+2S_\infty+1A_\infty$	(3,1)

Table 1. Different configurations of singular points compatible with the distributions 3 and $(3,0)$ for the limit cycles of the quadratic systems.

The paper is organized as follows. The results of Table 1 are proved in Section 3, but previously in Section 2, we present the main definitions and basic results which we shall use in the proofs of the results of Table 1.

2 Main definitions and preliminary results for Lienard systems

The *surface of limit cycles* for the system

$$\frac{dx}{dt} = f(x,\alpha), \quad x \in \mathbb{R}^2, \quad \alpha \in \mathbb{R}, \quad f = (f_1, f_2)^T, \tag{3}$$

is the subset $SLC = \{(x,\alpha) \in \mathbb{R}^2 \times \mathbb{R} : x \in L(\alpha), \quad \alpha \in \mathbb{R}\}$, where $L(\alpha)$ is the subset of the phase plane $\mathbb{R}^2$ formed by limit cycles of system (3) with parameter α.

We remark that if all the limit cycles of system (3) surrounding the singular point $x = 0$ (i.e. $f(0,\alpha) = 0$), intersect the half–axis $x_2 = 0$, $x_1 > 0$ only in one point, then instead of working with the surface of limit cycles it is more convenient to consider the *curve of limit cycles*, denoted by CLC, and formed by the points (x_1, α), where x_1 is the abscissa of the point x belonging to a limit cycle and to the half–axis $x_2 = 0$, $x_1 > 0$ for system (3) with parameter α.

We say that the parameter α *rotates* the vector field $f(x,\alpha)$ associated to system (3), or that α is a *rotating parameter*, if one of the two inequalities

$$(f_1)'_\alpha f_2 - f_1 (f_2)'_\alpha \geq 0 \quad (\leq 0), \quad x \in \mathbb{R}^2, \quad \alpha \in \mathbb{R},$$

holds, and the inequality never becomes an identity equal to zero on any limit cycle of $L(\alpha)$. Here, $(f_i)'_\alpha$ denotes the derivative of $f_i(x,\alpha)$ with respect to α for $i = 1,2$. We remark that for the quadratic systems (2), a_{11} is a rotating parameter.

The condition that this inequality never becomes an identity equal to zero on any limit cycle of $L(\alpha)$ can be easily checked, and means that the limit cycles of system (3) really change their position under the variation of the parameter α. Moreover, if α is a rotating parameter, then $L(\alpha_1) \cap L(\alpha_2) = \emptyset$ if $\alpha_1 \neq \alpha_2$. For more details, see [22].

We assume that we have a system (3) and that α is a rotating parameter. Then, the surface of limit cycles SLC is an open subset of $\mathbb{R}^2 \times \mathbb{R}$. By definition the *Andronov–Hopf function* $F : \cup_{\alpha\in\mathbb{R}} L(\alpha) \to \mathbb{R}$ associates to the points of $L(\alpha)$ the value α. Therefore, the surface of limit cycles is determined by the equation $\alpha = F(x)$ running α in $\mathbb{R}$.

If the limit cycles surrounding the singular point $x = 0$ (i.e. $f(0, \alpha) = 0$), intersect the half–axis $x_2 = 0$, $x_1 > 0$ only in one point, instead of function $F(x)$ it is more convenient to consider the function $\alpha = \varphi(x_1) = F(x)|_{x_2=0,\, x_1>0}$, which provides a full information about the limit cycles of system (3) surrounding the point $x = 0$, and their bifurcations when the parameter α varies. Note that the function $\alpha = \varphi(x_1)$, running α in $\mathbb{R}$, defines a curve of limit cycles for system (3) surrounding the point $x = 0$.

For computing the number of limit cycles of quadratic systems (1) we shall use the following two theorems, see [4, 9]. See also [7].

Theorem 1. *Assume that system (1) is structurally stable in a connected region $\Omega \subset \mathbb{R}^2$. Then, there exist a function $\Psi(x, y) \in C^1(\Omega)$ and a constant $k < 0$, such that the inequality*

$$\Phi = k\,\Psi\, div\, f + \frac{\partial \Psi}{\partial x} P + \frac{\partial \Psi}{\partial y} Q > 0, \qquad f = (P, Q), \tag{4}$$

is satisfied in the region Ω. Moreover, the limit cycles of system (1) do not intersect the set $W = \{(x, y) \in \Omega : \Psi(x, y) = 0\}$, and in every two–dimensional connected subregion of Ω where either $\Psi(x, y) > 0$ or $\Psi(x, y) < 0$, system (1) has at most one limit cycle γ, and if exists, is hyperbolic and stable (respectively unstable) if $k\Psi|_\gamma < 0$ (respectively > 0).

If the function $\Psi(x, y)$ satisfies the condition (4), the function $B(x, y) = |\Psi(x, y)|^{1/k}$ is a Dulac function in each subregion $\Psi(x, y) > 0$ or $\Psi(x, y) < 0$, and we have that $\mathrm{div}(Bf) = \Phi|\Psi|^{1/k-1}(\mathrm{sign}\,\Psi)/k$.

Theorem 2. *Let Ω be a simple connected region where system (1) is defined and has a unique singular point, the antisaddle A with $divf(A) \neq 0$. Assume that there exist a function Ψ and a number $k < 0$ satisfying the assumptions of Theorem 1. Suppose that the equation $\Psi(x, y) = 0$ determines in the region Ω a nest of m of ovals surrounding the point A. Then, in each of the $m - 1$ annulus limited by two adjacent ovals, system (1) has exactly one limit cycle. Moreover, system (1) has in the region Ω at most m limit cycles.*

By Theorem 2 it follows that the ovals are transversal to the vector field associated to system (1), and that the annulus limited by two adjacent ovals satisfies the Bendixson principle, see [17] for more details on the Bendixson principle. An additional m-th limit cycle can exist between the most external oval and the boundary of the region Ω.

For the Lienard system

$$\frac{dx}{dt} = y - F(x), \quad \frac{dy}{dt} = -g(x), \tag{5}$$

the determination of the function $\Psi(x, y)$, satisfying the assumptions of Theorem 2, is easy. Thus, if we look for it in the form

$$\Psi = \sum_{i=1}^{n} \Psi_i(x) y^{n-i}, \tag{6}$$

then the appropriate function Φ of Theorem 1 depends only on x if and only if $\partial\Phi/\partial y \equiv 0$. Then, $\partial\Phi/\partial y \equiv 0$ implies

$$\begin{array}{ll} \Psi_1 = C_1, \quad \Psi_2' = kfC_1, & \Psi_2 = kFC_1 + C_2, \quad F'(x) = f(x) \\ \Psi_i' = kf\Psi_{i-1} + (n-i+2)g\Psi_{i-2} + F\Psi_{i-1}', & \Psi_i = \int \Psi_i'(t)dt + C_i, \quad i = 3, \ldots, n. \end{array} \tag{7}$$

where the C_i for $i = 1, \ldots, n$ are arbitrary constants of integration. Therefore, the function Φ has the form

$$\Phi = -kf\Psi_n - g\Psi_{n-1} - F\Psi_n'. \tag{8}$$

In general, the function Φ is a linear combination

$$\Phi = \sum_{j=1}^{n} C_j \Phi_j(x), \tag{9}$$

of convenient functions $\Phi_i(x)$, obtained from (7) and (8).

Theorem 3. *Suppose that the function $g(x)$ of the Lienard system (5) satisfies that $g(0) = 0$, and that its two nearest zeros at 0 are x_1 and x_2 with $x_1 < 0 < x_2$. Assume that there exist the constants $k < 0$ and C_i for $i = 1, \ldots, n$, such that the function Φ given in (9) is positive for $x \in (x_1, x_2)$. Then, system (5) has at most $n/2$ limit cycles surrounding the singular point $(0, 0)$.*

For the existence of the positive function Φ, given by (9), on an interval $[\alpha, \beta]$ with $x_1 < \alpha < 0 < \beta < x_2$, it is sufficient that the inequality

$$\max_{|C|\le 1} \min_{x\in[\alpha,\beta]} \Phi(x, C) = \frac{1}{R} > 0,$$

holds. This is equivalent to the existence of a solution for the following problem of optimization:

$$\Phi(x, C) \le 1, \quad |C| \le R, \quad |C| = \max|C_i|, \quad x_1 \le x \le x_2, \quad 0 < R \to \min. \tag{10}$$

We can obtain an approximate solution of problem (10) on the net points x_i solving the discretized problem

$$\Phi(x_i, C) \le 1, \quad |C| \le R, \quad x_i \in [x_1, x_2] \quad i = 1, \ldots, N, \quad 0 < R \to \min. \tag{11}$$

If the number N of net points is sufficiently large and problem (11) has a solution, we can expect that problem (10) has also a solution. Note that numerically it is easy to find a minimum of the function $\Phi(x, C^*)$ on $[x_1, x_2]$, where $C = C^*$ is a solution of the problem (11).

In the study of the limit cycles of system (5), the idea of using a function $V(x, y)$ such that its derivative dV/dt on the solutions of system (5) depends only on x, has been used successfully in [10] and also it was mentioned in [4].

3 Perturbed quadratic systems with a weak focus of order three

Since the straight line $x = 0$ is transversal for the vector field associated to system (2), its limit cycles do not intersect $x = 0$. Therefore, in the half–planes $x < 0$ and $x > 0$ its limit cycles can be studied separately. In the half–plane $x > 0$ the transformation $x = 1/\xi, \quad y = (\tilde{y} - F(\xi))\xi^{-a} - \xi$ writes system (2) into the Lienard system

$$\frac{d\xi}{dt} = \tilde{y} - F(\xi), \qquad \frac{d\tilde{y}}{dt} = -g(\xi), \tag{12}$$

where

$$\begin{aligned} f(\xi) &= (a_{11} + a_{01}\xi - (2a+1)\xi^2)\xi^{a-2}, \\ g(\xi) &= (a_{00} + a_{10}\xi + (a_{00} - a_{11})\xi^2 - a_{01}\xi^3 + a\xi^4)qzx\xi^{2a-3}, \\ F(\xi) &= \int_1^\xi f(t)dt = \tilde{P}_2(\xi)\xi^{a-1} - \tilde{P}_2(1). \end{aligned}$$

System (2) has a weak focus or a center at the point $A = (1, -1)$, if the conditions

$$L = 2a - a_{01} - a_{10} - 2a_{20} > 0, \qquad V_1 = a_{11} + a_{01} - 2a - 1 = 0, \tag{13}$$

hold. The last condition says that the divergence of system (2) at A is zero.

Clearly, for $a = 2, 3, \ldots$ system (12) is a Lienard polynomial differential system. Moreover, for $a = -2, -3, \ldots$ system (12), under the transformation $\xi = 1/x$, $\tilde{y} = -y$, goes over to

$$\frac{dx}{dt} = y + \hat{P}_2(x)x^{-a-1} - \hat{P}_2(1), \qquad \frac{dy}{dt} = \hat{P}_4(x)x^{-2a-3}, \tag{14}$$

where $\hat{P}_2(x)$, $\hat{P}_4(x)$ are polynomials. Thus, also system (2) for $a = -2, -3, \ldots$ is reduced to a Lienard polynomial differential system.

Under conditions (13) the multiplicity of the weak focus A of system (2) can be determined by its focal values (also called Lyapunov constants), see for instance [11]. For a integer and $|a| > 1$ these focal values can be calculated using the Lienard polynomial systems (12) or (14), or using [11] for an arbitrary value of a. Thus, for system (2) these focal values are

$$\begin{aligned} V_3 &= W_0 - a_{10}W, \\ V_5 &= (4 - 2a - a_{11})V/W, \\ V_7 &= -(a_{11} + 2a + 1)UV/W, \end{aligned} \tag{15}$$

where

$$\begin{aligned} W_0 &= a_{11}^2(a+1) + a_{11}(2a^2+a-1) - a_{20}(a_{11}(2a-1) + (2a+1)(2a-3)), \\ W &= -1 + 2a^2 + a(a_{11}-1), \\ V &= -a_{11}^2 a(a+1) + a_{20}(a-1)(2a+1)^2, \\ U &= (8-2a^2)(a_{11}+2a+1)^2 - 35(2a+1)(a_{11}+2a+1) + 35(2a+1)^2. \end{aligned}$$

In short, under conditions (13) system (2) has at A

(i) a focus of first order if $V_3 \neq 0$, it is stable if $V_3 < 0$, otherwise it is unstable;

(ii) a focus of second order if $V_3 = 0$, $V_5 \neq 0$, it is stable if $V_5 < 0$, otherwise it is unstable;

(iii) a focus of third order if $V_3 = V_5 = 0$, and $V_7 \neq 0$, it is stable if $V_7 < 0$, otherwise it is unstable;

(iv) a center if and only if $V_3 = V_5 = V_7 = 0$.

It is well known that perturbing a weak focus of order i inside the class of quadratic systems, we can obtain i infinitesimal limit cycles surrounding the perturbed focus. Therefore, to look for quadratic systems having three limit cycles surrounding a focus, it is natural to perturb systems (2) having a weak focus of order three.

We assume that $W \neq 0$. Then, the value a_{10} can be determined from $V_3 = 0$, that is $a_{10} = W_0/W$. In particular, we obtain that system (2) has a weak focus of order three at A if

$$\begin{array}{l} a_{11} = \tilde{a}_{11}^* = 4 - 2a, \quad a \neq 2, \quad a_{01} = \tilde{a}_{01}^* = 2a + 1 - \tilde{a}_{11}^*, \\ a_{10} = \tilde{a}_{10}^* = (6(a^2 - a - 2) + a_{20}(6a-7))/(1-3a), \\ (a - 3 - a_{20})/(1-3a) < 0. \end{array} \tag{16}$$

In short, we note that we have a 2–parameter family of quadratic systems (2) with a weak focus of order three at A, the two parameters are a and a_{20}.

We fix the parameters a and a_{20} of a system (2) having a weak focus of third order at A, and change the parameters a_{11}, a_{01} and a_{10} in order to obtain a quadratic system with one small limit cycle surrounding A, being A a weak focus of second order. We must change the parameters a_{11}, a_{01} and a_{10} in such a way that $V_1 = 0$ and $V_5 V_7 < 0$. We note that V_5 must be different from zero in order to have at A a weak focus of second order, and that the signs of V_5 and V_7 must be different, because the stability of A and of the limit cycle must be opposite. Thus, we can obtain a limit cycle passing through a point $(x, -1)$ with $x > 1$ but near 1 choosing adequately the functions $a_{11} = \tilde{a}_{11}(x)$, $a_{01} = \tilde{a}_{01}(x) = 2a + 1 - \tilde{a}_{11}(x)$ and $a_{10} = \tilde{a}_{10}(x) = W_0/W$. Of course, we have that $\tilde{a}_{11}(1) = \tilde{a}_{11}^*$, $\tilde{a}_{10}(1) = \tilde{a}_{10}^*$. The condition for the birth of such a limit cycle, if $a_{11} = \tilde{a}_{11}^* + \Delta a_{11}$, $a_{01} = \tilde{a}_{01}^* + \Delta a_{01}$ and $a_{10} = \tilde{a}_{10}^* + \Delta a_{10}$ must satisfy the inequality

$$(3a-1)(a-2)\Delta a_{11} > 0. \tag{17}$$

in order to have $V_5V_7 < 0$. Therefore, $\Delta a_{11} > 0$ if $a < 1/3$ or $a > 2$, and $\Delta a_{11} < 0$ if $1/3 < a < 2$.

Suppose that for $x = x_0 > 1$ and for the values $\tilde{a}_{11}(x_0)$, $\tilde{a}_{01}(x_0)$, $\tilde{a}_{10}(x_0)$ we have one limit cycle surrounding A and passing through $(x_0, -1)$ and that A is a weak focus of second order. Now we fix a_{11} and a_{01}, and change the parameter a_{10} in order to obtain a quadratic system having two limit cycles surrounding the focus A, being A a weak focus of first order. Such a system must satisfy $V_3 \neq 0$, $V_3V_5 < 0$ and $V_5V_7 < 0$. We denote by $a_{10} = a_{10}(x)$ with $a_{10}(1) = a_{10}(x_0)$, $a_{11} = \tilde{a}_{11}(x_0)$ and $a_{01} = \tilde{a}_{01}(x_0)$ the parameters of a quadratic system having two limit cycles around A such that the new second limit cycle passes through the point $(x, -1)$. The appropriate Andronov–Hopf function a_{10} will have one extremum. Now, we denote by a_{10}^* the value of a_{10} for which system (2) has two limit cycles surrounding A and being A a weak focus of first order.

Finally, we change the parameter a_{11} starting with value $\tilde{a}_{11}(x_0)$ and remaining fixed the other parameters, so that from the weak focus of first order A bifurcates a third limit cycle. Such perturbed quadratic system must satisfy $V_1 \neq 0$, $V_1V_3 < 0$, $V_3V_5 < 0$ and $V_5V_7 < 0$. Then, by changing a_{11} on some interval system (2) will have three limit cycle, and the appropriate Andronov–Hopf function $a_{11} = AH(x)$ will have two extrema.

We have described the general scheme for obtaining quadratic system (2) with three limit cycles around the focus A, and moving conveniently the parameters a_{11}, a_{01} and a_{10}. The limit cycles (which originally bifurcated from A) are not necessarily small.

In what follows, we shall consider quadratic systems (2) with different configuration of singular points and we shall look for distributions 3 and $(3, 1)$ of limit cycles. The functions $\tilde{a}_{11}(x)$, $a_{10}(x)$ and $AH(x)$ will be found with the help of numerical computations.

Example 1: *A quadratic system with* 1 *focus and* 1 *node, and* 3 *limit cycles surrounding the focus, having at infinity* 2 *saddles and* 1 *antisaddle.* We take $a = 3$, $a_{20} = -12$, $a_{11} = -1.398$, $a_{10} = 15.28$ and $a_{01} = 8.4$. Then system (2) has the focus A. Numerical computations show that the system has at least three limit cycles which pass through points $(x_i, -1)$ with $x_1 = 1.26$, $x_2 = 1.98$ and $x_3 = 3.95$. With the help of Bendixson annuli it is possible to prove these numerical results analytically, but here we shall not do it. It is much more interesting to provide the upper bound on the number of limit cycles. We shall show that this upper bound is 3. For that we shall work with the Lienard polynomial system (12). Doing the translation $\xi = x + 1$, we get again another Lienard polynomial system. For this system we shall search a function $\Psi(x, C)$ of the form (6) with $n = 10$ and $k = -1$ satisfying conditions (7). For the corresponding function $\tilde{\Phi}(x, C) = \Phi(x, C)/(1 + 4G^2)$ with $G = \int_0^x g(t)dt$, where $\Phi(x, C)$ is a function satisfying (8) and (9), we shall solve the problem of optimization (11) on a uniform net in the interval $[-0.8, 0.5]$ with $N = 320$ points. This problem has the solution C_i^* equal to -0.0594107, -0.343784,

-0.828227, -0.879519, 0.301152, 1, 0.0814624, -0.275238, -0.00639951, 0.00721968 for $i = 1, \ldots, 10$. All the real roots of the polynomial $\Phi(x, C^*)$ lie in interval $x \leq -1$. Therefore, this function is positive in $(-1, +\infty)$. The equation $\Psi(x, y, C^*) = 0$ determines in the half–plane $x > -1$ three annuli surrounding the focus $O = (0, 0)$ of the last Lienard polynomial system and Theorem 3 can be applied. Then, the considered quadratic system (2) have no more than three limit cycles enclosing the focus A and at least two limit cycles. Taking into consideration the numerical computations it is possible to check that the system has exactly 3 limit cycles around the focus A.

Example 2: *A quadratic system with* 2 *foci, and* 3 *limit cycles surrounding one focus and* 0 *limit cycles around the other focus, having at infinity* 2 *saddles and* 1 *antisaddle.* That is, this system has a distribution $(3, 0)$ for its limit cycles. We take $a = 1.5$, $a_{20} = -15$, $a_{11} = 0.79993$, $a_{10} = 9.17$ and $a_{01} = 3.2$. The corresponding system (2) has the foci A and $B = (x_0, -1/x_0)$ with $x_0 = -0.73$. In addition, there are at least 3 limit cycles around the focus A which pass through the points $(x_i, -1)$ for $x_1 = 1.4$, $x_2 = 1.9$ and $x_3 = 3.1$. Now, we show that this system has no more 3 limit cycles around the focus A. We consider a Lienard system (12) and a function $\Psi(\xi, \tilde{y}, C)$ as in (6) and (7) with $n = 11$, $k = -1$ and $\Psi_i = \int_1^{\xi} \Psi_i'(t)dt + C_i$. For the function $\tilde{\Phi}(\xi, C) = 10^3\Phi(\xi, C)/(1 + 4G^3)$ with $G = \int_1^{\xi} g(t)dt$, where $\Phi(\xi, C)$ is a function as in (8) and (9), we solve the problem of optimization (11) on a uniform net in the interval $[0.2, 1.7]$ with $N = 200$ points. This problem has the solution C_i^* equal to $6.77203 \cdot 10^{-6}$, 0.000127496, 0.00128263, 0.0189312, 0.0367929, -0.0316707, -0.41092, -0.0777118, 1, 0.0289165, -0.12485 for $i = 1, \ldots, 11$. The function $\Phi(\xi, C^*)$ is positive for $\xi > 0$, and the equation $\Psi(\xi, \tilde{y}, C^*) = 0$ determines in the region $\xi > 0$ three annuli surrounding the focus $\tilde{A} = (1, 0)$ of system (12). Then, we get the same conclusion than in Example 1. The absence of limit cycles around the focus A follows from works [18, 19].

Example 3: *A quadratic system with* 1 *focus,* 3 *saddles and* 3 *limit cycles surrounding the focus.* We take $a = -2$, $a_{20} = 12$, $a_{11} = 10.999$, $a_{10} = -26.1$ and $a_{01} = -14$. In this case system (2) has the focus A and the three saddles $S_i = (t_i, -1/t_i)$ with $t_1 = -0.67$, $t_2 = 0.15$ and $t_3 = 1.7$. In addition, there are at least 3 limit cycles around the focus A which pass through the points $(x_i, -1)$ with $x_1 = 0.32$, $x_2 = 0.66$ and $x_3 = 0.8$. We show that this system has at most 3 limit cycles. For that purpose we consider the Lienard polynomial system (14) associated to system (2) with

$$F(x) = -\frac{1001}{3000} - 3x + 7x^2 - \frac{10999}{3000}x^3, \quad g(x) = 2x - 14x^2 - 2.1x^3 + 26.1x^4 - 12x^5.$$

The function $\Psi(x, y, C)$ is as in (7) with $n = 10$, $k = -1$ and $\Psi_i = \int_0^{x} \Psi_i'(t)dt + C_i$. For the function $\tilde{\Phi}(x, C) = 100\Phi(x, C)/(1 + 8G^3)$ with $G = \int_1^{x} g(t)dt$, where $\Phi(x, C)$

is a function satisfying (8) and (9), we solve the problem of optimization (11) on a uniform net in the interval $[0.2, 1.72]$ with $N = 650$ points. The problem has the solution C_i^* equal to -0.00891837, -0.0884008, -0.322146, -0.448227, 0.240997, 1, 0.119569, -0.547241, -0.0178445, 0.0269709 for $i = 1, \ldots, 10$. The function $\Phi(x, C^*)$ is positive in the interval $(0, 1.705)$. The equation $\Psi(x, y, C^*) = 0$ determines for $x \in I$ three annuli surrounding the focus $\tilde{A} = (1, 0)$ of system (14). The limit cycles of system (14) are located in the strip $t_2 < x < t_3$ of the plane (x, y). The interval I contains the interval (t_2, t_3). Now, the conclusion follows in a similar way to the previous examples.

Example 4: *A quadratic system with* 1 *focus,* 1 *saddle and* 3 *limit cycles surrounding the focus.* We take $a = -2$, $a_{20} = -1$, $a_{11} = 9.49965$, $a_{10} = 6.955$ and $a_{01} = -12.5$. In this case system (2) has the focus A and the saddle $S = (x_0, -1/x_0)$ with $x_0 = 0.2$. In addition, there are at least three limit cycles around the focus A which pass through the points $(x_i, -1)$ with $x_1 = 0.56$, $x_2 = 0.75$ and $x_3 = 0.87$. We consider the Lienard polynomial system (14) associated to system (2) with

$$F(x) = -\frac{782}{9375} - 3x + \frac{25}{4}x^2 - \frac{118747}{375}x^3, \quad g(x) = 2x - \frac{25}{2}x^2 + \frac{3291}{200}x^3 + \frac{1391}{200}x^4 + x^5.$$

Moreover, the function $\Psi(x, y, C)$ is as in (7) with $n = 12$, $k = -1$ and $\Psi_i = \int_0^x \Psi_i'(t)dt + C_i$. For the function $\tilde{\Phi}(x, C) = 10^5 \Phi(x, C)/(1 + 4G^2)$ with $G = \int_1^x g(t)dt$, where $\Phi(x, C)$ satisfies (8) and (9), we solve the problem of optimization (11) on a uniform net in the interval $[0.3, 1.4]$ with $N = 750$ points. The problem has the solution C_i^* equal to -0.0257346, -0.141113, -0.371849, -0.602612, -0.602612, -0.281479, 0.102264, 0.157096, 0.0116869, -0.0191466, -0.00362004, 0.000197399 for $i = 1, \ldots, 12$. The function $\Phi(x, C^*)$ is positive on the interval $I = (0, 1.8)$, but not on interval $I = (x_0, +\infty)$. The equation $\Phi(x, y, C^*) = 0$ determines for $x \in I$ three ovals. For evaluating the number of limit cycles on the strip $x > x_0$ of the plane (x, y), we shall use the *method of reduction to the global uniqueness of a limit cycle.*

We consider the Andronov–Hopf function $AH(x) = a_{11}$ with $AH(1) = 9.5$ associated to our system (2). We recall that a_{11} is a rotating parameter for system (2). We fix all the parameters and we move only the parameter a_{11}. The function $AH(x)$ is considered on the interval $I_1 = [x_0, x_{max}]$ where the endpoints satisfy $x_0 < 1$ and $x_{max} > 1$, and x_{max} corresponds to the bifurcation of a limit cycle from a loop of the saddle S. If in a subinterval $I_0 = [x_1, x_2]$ of I_1 the number of zeros of the function $AH(x) = a_{11}^0$ is $2p$, then the number of limit cycles of system (2) in the strip $x_1 < x < x_2$ is p. Now, suppose that the equation $AH(x) = a_{11}^1$ with $a_{11}^1 < a_{11}^0$ provides a unique limit cycle which is localized in the strip $x_3 < x < x_4$ with $[x_3, x_4] \subset I_0 \subset I_1$, then the function $AH(x)$ cannot take the value a_{11}^0 outside the interval I_1. Consequently, for the value a_{11}^0 system (2) has exactly p limit cycles. This is the method of reduction to the global uniqueness of a limit cycle.

Now we go back to our particular system (2). Approximately $AH(x)$ is equal to

$$8.89863 + 4.39482x - 13.5991x^2 + 22.9703x^3 - 22.4248x^4 + 11.9886x^5 - 2.72941x^6,$$

on $I_0 = [0.6, 0.9]$. Of course $I_0 \subset I_1$. If we prove, for some a_{11} and remaining fixed the other parameters, that system (14) has a unique limit cycle on a strip $x \in (\check{x}, \hat{x})$ of the plane (x, y) with $(\check{x}, \hat{x}) \subset I_1$, then the function $AH(x)$ does not take the value $a_{11} = 9.49965$ outside the interval I_1, and $AH(x)$ has its complicated behavior only on I.

Now, we take $a_{11} = 9.4993$. Then, system (14) has a limit cycle, which is located on the strip $x_0 < x < 1.8$. We prove its uniqueness. For that we find functions $\Psi(x, y, C)$ and $\Phi(x, C)$ as before with $n = 5$ and $k = -2/3$. The problem (11) has the solution C_i^* equal to -0.126609, -0.0834262, -1, -0.253441, 0.207481 for $i = 1, \dots, 5$. The function $\Phi(x, C^*)$ is positive for $x > 0$. This means that systems (14), and the corresponding system (2) have for $a_{11} = 9.4993$ a unique limit cycle. Therefore, by applying the method of reduction to the global uniqueness of a limit cycle, the proof of the distribution of 3 limit cycles around the focus A for the considered system (2) follows.

Example 5: *A quadratic system with* 2 *saddles,* 1 *focus,* 1 *node and* 3 *limit cycles surrounding the focus.* We take $a = -4$, $a_{20} = -1$, $a_{11} = 13.9987$, $a_{10} = 12.4$ and $a_{01} = -21$. In this case system (2) has the focus A and the node $N = (t_0, -1/t_0)$ with $t_0 = 9.69$, and two saddles $S_i = (t_i, -1/t_i)$ with $t_1 = 0.29$ and $t_2 = 1.42$. In addition, there are at least three limit cycles which are located on the strip $t_1 < x < t_2$ of the plane (x, y) around the focus A and pass through the points $(x_i, -1)$ with $x_1 = 0.63$, $x_2 = 0.8$ and $x_3 = 0.88$. For computing the number of limit cycles we consider the Lienard polynomial system (14) associated to system (2) with

$$\begin{aligned} F(x) &= -\frac{17539}{150000} - \frac{7}{3}x^3 + \frac{21}{4}x^4 - \frac{139987}{50000}x^5, \\ g(x) &= x^5\left(4 - 21x + \frac{142}{5}x^2 - \frac{62}{5}x^3 + x^4\right), \end{aligned}$$

and we find a function $\Psi(x, y, C)$ as in (7) with $n = 11$, $k = -1$ and $\Psi_i = \int_1^x \Psi_i'(t)dt + C$. Now, for the function $\tilde{\Phi}(x, C) = 10^6\Phi(x, C)/(1 + 4G^2)$ with $G = \int_1^x g(t)dt$, where $\Phi(x, C)$ satisfies (8) and (9), we solve the problem (11) on a uniform net on the interval $[0.5, 1.33]$ with $N = 750$ points. The problem has the solution C_i^* equal to -0.206646, -0.701459, -1, -0.745283, -0.24893, 0.0331453, 0.0341755, 0.000943157, -0.00105898, $-5.55364 \cdot 10^{-6}$, $2.0652 4\dot{1}0^{-6}$ for $i = 1, \dots, 11$. The equation $\Psi(x, y, C^*) = 0$ defines in the strip $0.25 < x < 1.3$ only three ovals. The function $\Phi(x, C^*)$ is positive on $(0.25, 1.3)$, but not on $I = (t_1, t_2)$ where limit cycles are located. As in the previous example we can use the method of reduction to the global uniqueness of a limit cycle. We take $a_{11} = 13.998$ and fix the remaining parameters. The corresponding Lienard polynomial system (14) has a limit cycle which is located on the strip $0.25 < x < 1.3$. Then, we find functions $\Psi(x, y, C)$ and $\Phi(x, C)$ as before with $n = 7$, $k = -2/3$ and $a_{11} = 13.998$. The problem (11) has a solution C_i^* equal to -0.884833, -0.874942, -1, -0.158469, -1, -0.107391, 0.0599698 for $i = 1, \ldots, 7$. The function $\Phi(x, C^*)$ is positive for $x > 0$. This means that systems (14), and the corresponding system (2) have for $a_{11} = 13.998$ a unique limit cycle. Now, following with the method of reduction we can complete the proof of the distribution of 3 limit cycles surrounding the focus A of the considered system (2).

Example 6: *A quadratic system with* 1 *saddle,* 1 *focus,* 2 *nodes and* 3 *limit cycles surrounding the focus.* We take $a = 5$, $a_{20} = -50$, $a_{11} = -5.49995$, $a_{10} = 76.45$ and $a_{01} = 16.5$. Then, system (2) has the focus A, the nodes $N_1 = (t_1, -1/t_1)$, $N_2 = (t_2, -1/t_2)$ with $t_1 = -0.46$, $t_1 = 0.34$, and the saddle $S = (t_3, -1/t_3)$ with $t_3 = 0.65$. Also it has at least three limit cycles around the focus A, which pass through the points $(x_i, -1)$ with $x_1 = 1.05$, $x_2 = 1.16$ and $x_3 = 1.5$. For estimating the number of limit cycles we consider the Lienard polynomial system (12) with

$$\begin{aligned} F(\xi) &= -\frac{22003}{240000} - \frac{1099999}{80000}\xi^4 + \frac{33}{10}\xi^5 - \frac{11}{6}\xi^6, \\ g(\xi) &= \xi^7\left(-50 + \frac{1529}{20}\xi - \frac{299}{20}\xi^2 - \frac{33}{2}\xi^3 + 5\xi^4\right). \end{aligned}$$

As before we find functions $\Psi(\xi, \tilde{y}, C)$ and $\Phi(\xi, C)$ satisfying (7), (8) and (9) with $n = 10$, $k = -1$ and $\Psi_i = \int_1^\xi \Psi_i'(t)dt + C_i$ for $i = 1, ..., n$. For the function $\tilde{\Phi}(\xi, C) = 10^3\Phi(\xi, C)/\xi^3)$ we solve the problem (11) on a uniform net in the interval $[0.6, 1.21]$ with $N = 450$ points. For the computations it is better to do the change of variable $\xi \to \xi + 1$. The problem (11) has the solution C_i^* equal to -0.0104019, -0.0613161, -0.329415, -1, 0.0849137, 0.770697, 0.0133268, -0.124194, -0.000345956, 0.00107251 for $i = 1, \ldots, 10$. The equation $\Psi(\xi, \tilde{y}, C^*) = 0$ defines in the strip $\xi \in I = (0.1; 1.5)$ only three ovals. The function $\Phi(\xi, C^*)$ is positive on the interval I. Therefore, we use the reduction to a global uniqueness of a limit cycle in the half–plane $\xi > 0$. We take $a_{11} = -5.4997$ and suppose that remaining parameters are fixed. Then, the corresponding system (12) has a limit cycle which is located on the strip $\xi \in I$. Furthermore, we find functions $\Psi(\xi, \tilde{y}, C)$ and $\Phi(\xi, C)$ as before with $n = 5$, $k = -2/3$ and $a_{11} = -5.4997$. The problem (11) has the solution C_i^* equal to -0.029957, -0.00827985, -1, -0.104843, 0.487508 for $i = 1, \ldots, 5$. The function $\Phi(\xi, C^*)$ is positive on $(0, 1.8)$, and the equation $\Psi(\xi, \tilde{y}, C^*) = 0$ defines for $0 < \xi < 1.8$ only one oval. Hence, it follows the uniqueness of the limit cycle for the considered system (12). Finally, the original system (2) has exactly three limit cycles in the half–plane $x > 0$ around the focus A.

Example 7: *A quadratic system with* 2 *foci,* 1 *saddle at infinity, and the configuration* $(3, 1)$ *of limit cycles.* We take $a = 8/11$, $a_{20} = -12$, $a_{11} = 2.1502$, $a_{10} = -26.5$ and $a_{01} = 67/220$. The corresponding system (2) has the foci A and $B = (x_0, -1/x_0)$ with $x_0 = -3.2$, and a saddle at infinity. In addition, there are at least three limit cycles around A, which pass through the points $(x_i, -1)$ with $x_1 = 1.28$, $x_2 = 1.15$ and $x_3 = 4.43$; and there is at least one limit cycle around B. For studying the limit cycles surrounding the focus A we consider the associated system (12), which after the change of variable $y = 5\tilde{y}$ has the functions

$$\begin{aligned} F(\xi) &= \frac{10130461}{5700000} - \frac{118261}{75000\xi^{\frac{3}{11}}} + \frac{67}{800}\xi^{\frac{8}{11}} - \frac{7}{95}\xi^{\frac{19}{11}}, \\ g(\xi) &= -\frac{-12}{25\xi^{\frac{17}{11}}} - \frac{53}{5\xi^{\frac{6}{11}}} + \frac{8377}{5500}\xi^{\frac{5}{11}} - \frac{67}{5500}\xi^{\frac{6}{11}} + \frac{8}{275}\xi^{\frac{27}{11}}. \end{aligned}$$

For system (12) we find functions $\Psi(\xi,\tilde{y},C)$ and $\Phi(\xi,C)$ satisfying (7), (8) and (9) with $n = 11$, $k = -1$ and $\Psi_i = \int_1^{\xi} \Psi_i'(t)dt + C_i$ for $i = 1, ..., n$. Now, for the function $\tilde{\Phi}(\xi,C) = \Phi(\xi,C)\xi^4/(1+\xi^9)$ we solve the problem (11) on a uniform net in the interval $[0.001, 4]$ with $N = 790$ points. The problem has the solution C_i^* equal to -0.000309912, -0.00513088, -0.372386, -0.154282, -0.328544, 0.150592, 1, 0.0871286, -0.586201, -0.0121769, 0.0273162 for $i = 1, \ldots, 11$. The equation $\Psi(\xi,\tilde{y},C^*) = 0$ defines in the strip $0 < \xi < 4$ only three ovals. The function $\Phi(\xi,C^*)$ is positive only on $(0,4)$. Now we use again the method of reduction to the global uniqueness of a limit cycle for proving that there exist exactly three limit cycles surrounding the focus A. We take in system (2) $a_{11} = 2.156$ and the remaining parameters are fixed. Therefore, the corresponding system (12) has a limit cycle which is located on the strip $0 < \xi < 4$ of the phase plane $(\xi,\tilde{y})$. Now we find functions $\Psi(\xi,\tilde{y},C)$ and $\Phi(\xi,C)$ as before with $n = 3$, $k = -2/3$ and $a_{11} = 2.156$. The corresponding problem (11) has the solution C_i^* equal to -0.8033395, -0.299759, 1 for $i = 1, 2, 3$. The function $\Phi(\xi,C^*)$ is positive for $\xi > 0$. This means that systems (2) and (12) with $a_{11} = 2.156$ have a unique limit cycle, but they for $a_{11} = 2.1502$ have exactly three limit cycles around the focus A. Now, we shall prove the uniqueness of the limit cycle around the focus B for the original system (2) in the half–plane $x < 0$. In fact this uniqueness follows from the results of Zhang Pingguang [20, 21], but here we provide an independent proof. For doing that first, we translate the point B to the point A by means of the change of variables $x = x_0\hat{x}$, $y = \hat{y}/x_0$. System (2) becomes another quadratic system also in the form (2) and its parameters are $\tilde{a}_{00} = x_0^2 a_{00}$, $\tilde{a}_{10} = x_0^3 a_{10}$, $\tilde{a}_{20} = x_0^4 a_{20}$, $\tilde{a}_{01} = x_0 a_{01}$, $\tilde{a}_{11} = x_0^2 a_{11}$ and $\tilde{a} = a$. The uniqueness of the limit cycle is obtained in the half–plane $\tilde{x} > 0$. We can prove this with the help of the functions $\Psi(\xi,\tilde{y},C)$ and $\Phi(\xi,C)$ satisfying (7), (8) and (9), and the corresponding system (12) with $n = 3$, $k = -1$ and $\Psi_i = \int_1^{\xi} \Psi_i'(t)dt + C_i$ for $i = 1, 2, 3$. The problem (11) for the function $\tilde{\Phi}(\xi,C) = 10^{-2}\Phi(\xi,C)\xi^{20/11}$ has the solution C_i^* equal to $-8.0101 \cdot 10^{-5}$, $7.10538 \cdot 10^{-4}$, 1 for $i = 1, 2, 3$. The function $\Phi(\xi,C^*)$ is positive for $\xi > 0$, and the equation $\Psi(\xi,\tilde{y},C^*) = 0$ defines for $\xi > 0$ only one oval. By Theorem 3, the considered system has exactly one limit cycle in the half–plane $\xi > 0$, and the original system (2) has exactly one limit cycle around focus B. So, we have distribution $(3,1)$ of limit cycles for our system (2).

Example 8: *A quadratic system with* 2 *foci, and* 1 *saddle and* 1 *antisaddle at infinity, and the configuration* $(3,1)$ *of limit cycles.* In [1] the domain of parameters is found in order that a quadratic system has a weak focus of third order and a limit cycle around the other focus. Using these systems we can find a quadratic system (2) with the distribution of limit cycles $(3,1)$. We take $a = 1.04$, $a_{20} = -120$, $a_{11} = 1.51997$, $a_{10} = -79.6$ and $a_{01} = 1.56$. Then, system (2) has the foci A and $B = (x_0, -1/x_0)$ with $x_0 = -1.79$, one saddle and one node at infinity. In addition there are at least three limit cycles around A, which pass through the points $(x_i, -1)$ with $x_1 = 1.29$, $x_2 = 2.22$ and $x_3 = 4.63$; and there is at least one limit cycle around B. For studying the limit cycles surrounding the focus A we consider the associated system (12), which after the change of variable $y = 10\tilde{y}$ has the functions $F(\xi)$, $g(\xi)$:

$$\begin{aligned} F(\xi) &= -\frac{7749847}{2040000} + \frac{151997}{40000}\xi^{\frac{1}{25}} + \frac{3}{20}\xi^{\frac{26}{25}} - \frac{77}{510}\xi^{\frac{51}{25}}, \\ g(\xi) &= -\frac{-6}{5\xi^{\frac{23}{25}}} - \frac{119}{250}\xi^{\frac{2}{25}} + \frac{5003}{2500}\xi^{\frac{27}{25}} - \frac{39}{2500}\xi^{\frac{52}{25}} + \frac{13}{1250}\xi^{\frac{77}{25}}. \end{aligned}$$

As before we find the functions $\Psi(\xi, \tilde{y}, C)$ and $\Phi(\xi, C)$ satisfying (7), (8) and (9) with $n = 10$, $k = -1$ and $\Psi_i = \int_1^{\xi} \Psi_i'(t)dt + C_i$ for $i = 1, ..., n$. Now, for the function $\tilde{\Phi}(\xi, C) = 10^3\Phi(\xi, C)/(1 + 4G^2)$ with $G = \int_1^{\xi} g(t)dt$ we solve the problem (11) on a uniform net in the interval $[0.1, 2.2]$ with $N = 400$ points. The problem has the solution C_i^* equal to $9.774 \cdot 10^{-5}$, 0.00294242, 0.035928, 0.273929, -0.0477983, -1, -0.385115, 0.80362, 0.00645912, -0.00449499 for $i = 1, \ldots, 10$. The equation $\Psi(\xi, \tilde{y}, C^*) = 0$ defines in the strip $0 < \xi < 5$ only three ovals. The function $\Phi(\xi, C^*)$ is positive on the interval $I = (0; 5)$. Again we use the method of reduction to the global uniqueness of a limit cycle for proving that there exist exactly three limit cycles surrounding the focus A. We take $a_{11} = 1.5198$ and the remaining parameters are fixed. The corresponding system (12) has a limit cycle which is located in the strip $0 < \xi < 5$ of the phase plane $(\xi, \tilde{y})$. Now we find functions $\Psi(\xi, \tilde{y}, C)$ and $\Phi(\xi, C)$ as before with $n = 7$, $k = -1$ and $a_{11} = 1.5198$. The problem (11) has the solution C_i^* equal to 0.00132064, 0.0450009, 1, -0.00941069, 0.20056, 0.134724, -1 for $i = 1, \ldots, 7$. The function $\Phi(\xi, C^*)$ is positive for $\xi > 0$, and the equation $\Psi(\xi, \tilde{y}, C^*) = 0$ defines for $\xi > 0$ only one oval. By Theorem 3, the considered system has exactly one limit cycle in a half–plane $\xi > 0$, then the original system (2) has exactly three limit cycles in a half–plane $x > 0$ around the focus A. The uniqueness of the limit cycle around B for the original system (2) in the half–plane $x < 0$ is proved in the same way as in Example 7 if the function $\tilde{\Phi}(\xi, C^*)$ is equal to $\Phi(\xi, C)\xi^{24/25}/10^6$. The problem (11) has the solution C_i^* equal to $1.24736 \cdot 10^{-4}$, 0.00948753, 1 for $i = 1, 2, 3$. The function $\Phi(\xi, C^*)$ is positive for $\xi > 0$, and the equation $\Psi(\xi, \tilde{y}, C^*) = 0$ defines for $\xi > 0$ only one oval, which allows to show the uniqueness of the limit cycle of system (2) around the focus B.

Remark 1. *Kooij and Zegeling proved in [18, 19] that the distribution of limit cycles* $(3, 1)$ *is possible only for quadratic system of the type* $2A + 1S_\infty$, $2A + 2S_\infty + 1S_\infty$ *which we have considered.*

Remark 2. *For constructing the examples of quadratic system with the maximum number of limit cycles it is not necessary to use the function* $\tilde{a}_{11}(x)$*. It is enough to know that the function* $a_{10}(x)$ *has an extremum, then the function* $AH(x)$ *will have two extrema and provides the existence of an interval for the function* a_{11} *in which the system has three limit cycles. Also it is possible instead of using the normal form given by system (2), to use other canonical families of quadratic systems considered in [9, 17].*

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Global attractors for V-monotone nonautonomous dynamical systems*

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Abstract. This article is devoted to the study of the compact global atrractors of V-momotone nonautonomous dynamical systems.We give a description of the structure of compact global attractors of this class of systems. Several applications of general results for different classes of differential equations (ODEs, ODEs with impulse, some classes of evolutionary partial differential equations) are given.

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1 Introduction

The differential equations with monotone right-hand side are one of the most studied classes of nonlinear equations (see, for example,[4, 16, 20, 24, 25] and the literature quoted there).

Many authors studied the problem of the existence of almost periodic solutions of monotone nonlinear almost periodic equations (see [12, 13, 15, 18, 19, 24, 25] and others).

Purpose of our article is the study of global attractors of general V-monotone nonautonomous dynamical systems and their applications to different classes of differential equations (ODEs, ODEs with impulse, some classes of evolution partial differential equations).

For autonomous equations the analogous problem was studied before (see, for example,[2, 14, 23]), but for nonautonomous dynamical system this problem is considered in our paper for the first time.

2 Nonautonomous dynamical systems and skew-product flows

Definition 1. *Let $\Theta = \{\theta_t\}_{t\in\mathbb{R}}$ be a group of mappings of Ω into itself, that is a continuous time autonomous dynamical system on a metric space Ω, and let $\mathbb{B}$ be a*

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Banach space. Consider a continuous mapping $\varphi : \mathbb{R}^+ \times \Omega \times \mathbb{B} \to \mathbb{B}$ satisfying the properties

$$\varphi(0, \omega, \cdot) = id_{\mathbb{B}}, \ (\varphi(t + \tau, \omega, x) = \varphi(\tau, \theta_t \omega, \varphi(t, \omega, x))$$

for all $s, t \in \mathbb{R}^+$, $\omega \in \Omega$ and $x \in \mathbb{B}$. Such a mapping φ (or more explicit $\langle \mathbb{B}, \varphi, (\Omega, \mathbb{R}, \Theta) \rangle$) is called [1],[22] a continuous cocycle or nonautonomous dynamical system (NDS) on $\Omega \times \mathbb{B}$.

Example 1. *As an example, consider a parameterized differential equation*

$$\frac{dx}{dt} = F(\theta_t \omega, x) \quad (\omega \in \Omega)$$

on a Banach space $\mathbb{B}$ with $\Omega = C(\mathbb{R} \times \mathbb{B}, \mathbb{B})$. Define $\theta_t : \Omega \to \Omega$ by $\theta_t \omega(\cdot, \cdot) = \omega(t + \cdot, \cdot)$ for each $t \in \mathbb{R}$ and interpret $\varphi(t, \omega, x)$ as the solution of the initial value problem

$$\frac{d}{dt} x(t) = F(\theta_t \omega, x(t)), \quad x(0) = x. \tag{1}$$

Under appropriate assumptions on $F : \Omega \times \mathbb{B} \to \mathbb{B}$ (or even $F : \mathbb{R} \times \mathbb{B} \to \mathbb{B}$) with $\omega(t)$ instead of $\theta_t \omega$ in (1) to ensure forwards the existence and uniqueness, (Θ, φ) generates a nonautonomous dynamical system on $\Omega \times \mathbb{B}$.

The usual concept of a global attractor for the autonomous semi-dynamical system π on the state space $X = \Omega \times \mathbb{B}$ can be used here.

Definition 2. *The nonempty compact subset $\mathcal{A}$ of $X = \Omega \times \mathbb{B}$ is called maximal if it is π-invariant, that is*

$$\pi(t, \mathcal{A}) = \mathcal{A} \quad \textit{for all} \quad t \in \mathbb{R}^+,$$

and it attracts all compact subsets of $X = \Omega \times \mathbb{B}$, that is

$$\lim_{t \to \infty} \beta\left(\pi(t, \mathcal{D}), \mathcal{A}\right) = 0 \qquad \textit{for all} \quad \mathcal{D} \in \mathcal{K}(\mathbb{X}),$$

where $C(X)$ is the space of all nonempty compact subsets of X and β is the Hausdorff semi-metric on $C(X)$.

3 Global attractors of V- monotone NDS.

Let Ω be a compact topological space , (E, h, Ω) be a locally trivial Banach stratification [3] and $|\cdot|$ be a norm on (E, h, Ω) co-ordinated with the metric ρ on E (that is $\rho(x_1, x_2) = |x_1 - x_2|$ for any $x_1, x_2 \in X$ such that $h(x_1) = h(x_2)$).

Definition 3. *Let us remember [8],[5],[6] that the triplet $\langle (E, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \Theta), h \rangle$ is called a (general) nonautonomous dynamical system if $h : E \to \Omega$ is a homomorphism of the dynamical system $(E, \mathbb{T}_1, \pi)$ on $(\Omega, \mathbb{T}_2, \Theta)$, where $\mathbb{T}_1$ and $\mathbb{T}_2$ ($\mathbb{T}_1 \subseteq \mathbb{T}_2$) are two subsemigroups of the group $\mathbb{T}$.*

Example 2. *Let $\mathbb{T}_2$ be a subsemigroup of $\mathbb{T}$, $(\Omega, \mathbb{T}_2, \Theta)$ be a dynamical system on Ω and $\langle \mathbb{B}, \varphi, (\Omega, \mathbb{T}_2, \Theta)\rangle$ be a cocycle over $(\Omega, \mathbb{T}_2, \Theta)$ with the fiber $\mathbb{B}$, $X := \Omega \times \mathbb{B}$, $\mathbb{T}_1 \subseteq \mathbb{T}_2$ be a subsemigroup of $\mathbb{T}_2$, $(X, \mathbb{T}_1, \pi)$ be a semi-group dynamical system on X defined by the equality $\pi = (\varphi, \theta)$ (i.e. $\pi(t,(\omega,u)) := (\varphi(t,\omega,u), \theta_t\omega)$ for all $t \in \mathbb{T}_1$ and $(\omega, u) \in X$), then the triple $\langle (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \Theta), h\rangle$ $(h = pr_2)$ will be a nonautonomous dynamical system, generated by cocycle φ.*

Definition 4. *The cocycle $\langle \mathbb{B}, \varphi, (\Omega, \mathbb{T}, \Theta)\rangle$ is called compact dissipative if there is a nonempty compact $K \subseteq W$ such that*

$$\lim_{t\to+\infty} \sup\{\beta(\varphi(t,\omega)M, K) \mid \omega \in \Omega\} = 0 \qquad (2)$$

for any $M \in C(\mathbb{B})$, where $\varphi(t,\omega) := \varphi(t,\omega,\cdot)$.

If $M \subseteq \mathbb{B}$, then suppose

$$\Omega_\omega(M) = \bigcap_{t\geq 0} \overline{\bigcup_{\tau\geq t} \varphi(\tau, \theta_{-\tau}\omega, M)}$$

for every $\omega \in \Omega$.

Definition 5. *We will say that the space X possesses the (S)-property if for any compact $K \subseteq X$ there is a connected set $M \subseteq X$ such that $K \subseteq M$.*

Theorem 1. *[9] Let Ω be a compact metric space, $\langle \mathbb{B}, \varphi, (\Omega, \mathbb{T}, \Theta)\rangle$ be a compact dissipative cocycle and K be the nonempty compact appearing in the equality (2), then :*

1. $I_\omega = \Omega_\omega(K) \neq \emptyset$, is compact, $I_\omega \subseteq K$ and $\lim_{t\to+\infty} \beta(\varphi(t, \theta_{-t}\omega)K, I_\omega) = 0$ for every $\omega \in \Omega$;

2. $\varphi(t,\omega)I_\omega = I_{\theta_t\omega}$ for all $\omega \in \Omega$ and $t \in \mathbb{T}^+$;

3. $\lim_{t\to+\infty} \beta(\varphi(t,\theta_{-t})M, I_\omega) = 0$ for all $M \in C(\mathbb{B})$ and $\omega \in \Omega$;

4. $\lim_{t\to+\infty} \sup\{\beta(\varphi(t,\omega_{-t})M, I) \mid \omega \in \Omega\} = 0$ for any $M \in C(\mathbb{B})$, where $I = \cup\{I_\omega \mid \omega \in \Omega\}$;

5. $I_\omega = pr_1 I_\omega$ for all $\omega \in \Omega$, where J is a Levinson centre of $(X, \mathbb{T}^+, \pi)$, and, hence, $I = pr_1 J$;

6. the set I is compact;

7. the set I is connected if one of the following two conditions is fulfilled :

a. $\mathbb{T}^+ = \mathbb{R}^+$ and the spaces $\mathbb{B}$ and Ω are connected;

b. $\mathbb{T}^+ = \mathbb{Z}^+$ and the space $\Omega \times \mathbb{B}$ possesses the (S)-property or it is connected and locally connected.

Definition 6. *A nonautonomous dynamical system $\langle (X, \mathbb{T}^+, \pi), (\Omega, \mathbb{T}, \Theta), h\rangle$ is said to be uniformly stable in the positive direction on compacts of X [7] if, for arbitrary $\varepsilon > 0$ and $K \subseteq X$, there is $\delta = \delta(\varepsilon, K) > 0$ such that the inequality $\rho(x_1, x_2) < \delta$ $(h(x_1) = h(x_2))$ implies that $\rho(\pi^t x_1, \pi^t x_2) < \varepsilon$ for $t \in \mathbb{T}^+$.*

Definition 7. *A set $M \subset X$ is called minimal with respect to a dynamical system $(X, \mathbb{T}^+, \pi)$ if it is nonempty, closed and invariant and if no proper subset of M has these properties.*

Definition 8. *Denote by $X\dot{\times}X = \{(x_1, x_2) \in X \times X \mid h(x_1) = h(x_2)\ \}$. If there exists the function $V : X\dot{\times}X \to \mathbb{R}_+$ with the following properties:*

a. V is continuous.

b. V is positive defined, i.e. $V(x_1, x_2) = 0$ if and only if $x_1 = x_2$.

c. $V(x_1 t, x_2 t) \leq V(x_1, x_2)$ for all $(x_1, x_2) \in X\dot{\times}X$ and $t \in \mathbb{T}_+$,

then the nonautonomous dynamical system $\langle (X, \mathbb{T}^+, \pi), (\Omega, \mathbb{T}, \Theta), h\rangle$ is called (see [12],[13] and [19],[25]) V - monotone.

Theorem 2. *Every V - monotone compact dissipative nonautonomous dynamical system $\langle (X, \mathbb{T}^+, \pi), (\Omega, \mathbb{T}, \Theta), h\rangle$ is uniformly stable in the positive direction on compacts from X.*

Corollary 1. *Let $\langle (X, \mathbb{T}^+, \pi), (\Omega, \mathbb{T}, \Theta), h\rangle$ be a V - monotone compact dissipative nonautonomous dynamical system and Ω be minimal, then:*

1. J is uniformly orbitally stable in the positive direction, i.e., for $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that the inequality $\rho(x, J_{h(x)}) < \delta$ implies that $\rho(\pi^t x, J_{h(\pi^t x)}) < \varepsilon$ for $t \geq 0$;

2. J is an attractor of compact sets from X, i.e., for $\varepsilon > 0$ and a compact $K \subseteq X$, there is $L(\varepsilon, K) > 0$ such that $\pi^t K_\omega \subseteq \tilde{B}(J_{\theta_t \omega}, \varepsilon)$ for $\omega \in \Omega$ and $t \geq L(\varepsilon, K)$;

3. any motion on J can be continued to the left and J is bilaterally distal;

4. $J_\omega = X_\omega \bigcap J$ for $\omega \in \Omega$, is a connected set if X_ω is connected, and for distinct ω_1 and ω_2 the sets J_{ω_1} and J_{ω_2} are homeomorphic;

5. J is formed of recurrent trajectories, and two arbitrary points $x_1, x_2 \in J_\omega$ $(\omega \in \Omega)$ are mutually recurrent.

Theorem 3. *Let $\langle (X, \mathbb{T}^+, \pi), (\Omega, \mathbb{T}, \Theta), h\rangle$ be a V - monotone compact dissipative nonautonomous dynamical system, Ω be minimal and J be its Levinson center, then*

$$V(x_1 t, x_2 t) = V(x_1, x_2) \tag{3}$$

for all $x_1, x_2 \in J$ such that $h(x_1) = h(x_2)$.

Corollary 2. *Under the conditions of Theorem 3 if the nonautonomous dynamical system $\langle (X, \mathbb{T}^+, \pi), (\mathbb{B}, \mathbb{T}, \Theta), h\rangle$ is strictly monotone, i.e. $V(x_1 t, x_2 t) < V(x_1, x_2)$ for all $t > 0$ and $(x_1, x_2) \in X\dot{\times}X$ $(x_1 \neq x_2)$, then $J_\omega = J \bigcap X_\omega$ consists of a single point for all $\omega \in \Omega$.*

Theorem 4. *Let $\langle (X, \mathbb{T}^+, \pi), (\Omega, \mathbb{T}, \Theta), h\rangle$ be a $V-$monotone compact dissipative nonautonomous dynamical system with compact minimal base Ω and J be its Levinson center, then for every point $x \in X_y$ there exists a unique recurrent point $p \in J_\omega$ such that*

$$\lim_{t \to +\infty} \rho(xt, pt) = 0, \tag{4}$$

i.e. every trajectory of this system is asymptotically recurrent.

Corollary 3. *Under the conditions of Theorem 4 the following assertions hold:*

a. *$\omega-$limit set ω_x of every point $x \in X$ is a compact minimal set.*

b. *if $x_1, x_2 \in X_\omega$ ($\omega \in \Omega$) then $\omega_{x_1} = \omega_{x_2}$ or $\omega_{x_1} \bigcap \omega_{x_2} = \emptyset$.*

4 On the structure of Levinson center of V-monotone NDS with minimal base

Definition 9. *(X, ρ) is called [18] a metric space with segments if for any $x_1, x_2 \in X$ and $\alpha \in [0,1]$, the intersection of $B[x_1, \alpha r]$ (the closed ball centered at x with radius αr, where $r = \rho(x_1, x_2)$) and $B[x_2, (1-\alpha)r]$ has a unique element $S(\alpha, x_1, x_2)$.*

Definition 10. *The metric space (X, ρ) is called [18] strict-convex if (X, ρ) is a metric space with segments, and for any $x_1, x_2, x_3 \in X$, $x_2 \neq x_3$, and $\alpha \in (0,1)$, the inequality $\rho(x_1, S(a, x_2, x_3)) < \max\{\rho(x_1, x_2), \rho(x_1, x_3)\}$ holds.*

Definition 11. *Let X be a strict metric-convex space. A subset M of X is said to be metric-convex if $S(\alpha, x_1, x_2) \in M$ for any $\alpha \in (0,1)$ and $x_1, x_2 \in M$.*

We note that every convex closed subset X of the Hilbert space H equipped with the metric $\rho(x_1, x_2) = |x_1 - x_2|$ is strictly metric-convex.

Let $x \in X$, denote by Φ_x the family of all entire trajectories of dynamical system $(X, \mathbb{T}^+, \pi)$ passing through the point x for $t = 0$, i.e. $\gamma \in \Phi_x$ if and only if $\gamma : \mathbb{T} \to X$ is a continuous mapping with the properties: $\gamma(0) = x$ and $\pi^t \gamma(\tau) = \gamma(t + \tau)$ for all $t \in \mathbb{T}^+$ and $\tau \in \mathbb{T}$.

Theorem 5. *Let $\langle (X, \mathbb{T}^+, \pi), (\Omega, \mathbb{T}, \Theta), h \rangle$ be a V - monotone compact dissipative nonautonomous dynamical system, J is its Levinson center and the following conditions hold:*

1. $V(x_1, x_2) = V(x_2, x_1)$ for all $(x_1, x_2) \in X \dot{\times} X$.

2. $V(x_1, x_2) \leq V(x_1, x_3) + V(x_3, x_2)$ for all $x_1, x_2, x_3 \in X$ with the condition $h(x_1) = h(x_2) = h(x_3)$.

3. the space (X_ω, V_ω) is strict metric-convex for all $\omega \in \Omega$, where $X_\omega = h^{-1}(\omega) = \{x \in X | h(x) = \omega\}$ $(\omega \in \Omega)$ and $V_\omega = V|_{X_\omega \times X_\omega}$.

If $\gamma_{x_i} \in \Phi_{x_i}$ ($i = 1, 2$) and $x_1, x_2 \in I_\omega$ $(\omega \in \Omega)$, then the function $\gamma : \mathbb{T} \to X$ $(\gamma(t) = S(\alpha, \gamma_{x_1}(t), \gamma_{x_2}(t))$ for all $t \in \mathbb{T})$ defines an entire trajectory of dynamical system $(X, \mathbb{T}^+, \pi)$.

We denote by $\mathcal{K} = \{a \in C(\mathbb{T}_+, \mathbb{R}_+) \mid a(0) = 0,\ a \text{ is strictly increasing}\}$.

Theorem 6. *Under the conditions of Theorem 5 if in addition the nonautonomous dynamical system $\langle (X, \mathbb{T}^+, \pi), (\Omega, \mathbb{T}, \Theta), h \rangle$ is bounded k - dissipative and there exists a function $a \in \mathcal{K}$ with the property $\lim\limits_{t \to +\infty} a(t) = +\infty$ such that $a(\rho(x_1, x_2)) \leq V(x_1, x_2)$ for all $(x_1, x_2) \in X \dot{\times} X$, then J_ω will be metric-convex for all $\omega \in \Omega$, where $J_\omega = J \bigcap X_\omega$ and J is the Levinson center of $(X, \mathbb{T}^+, \pi)$.*

5 Almost periodic solutions of V - monotone almost periodic dissipative systems

Definition 12. *Let (X,ρ) be a metric space. A function $\phi : \mathbb{T} \to X$ is called almost periodic (in the sense of Bohr) if for every $\varepsilon > 0$ there exists a relatively dense subset A_ε of $\mathbb{T}$ such that*

$$\rho(\phi(t+\tau), \phi(t)) < \varepsilon$$

for all $t \in \mathbb{T}$ and $\tau \in A_\varepsilon$.

Definition 13. *A point $x \in X$ is said to be almost periodic if there is an entire trajectory $\gamma_x \in \Phi_x$ such that the function $\gamma_x : \mathbb{T} \to X$ is almost periodic.*

Definition 14. *The compact invariant set M of nonautonomous dynamical system $\langle (X,\mathbb{T}_+,\pi), (\Omega,\mathbb{T},\Theta), h\rangle$ is called [19],[5] distal on the invariant set M in the negative direction if $\inf\limits_{t\in\mathbb{T}_-} \rho(\gamma_{x_1}(t), \gamma_{x_2}(t)) > 0$ for all $x_1, x_2 \in M (h(x_1) = h(x_2)$ and $x_1 \neq x_2)$ and $\gamma_{x_i} \in \Phi_{x_i} (i = 1,2)$, where Φ_x is the set of all entire trajectories of $(X,\mathbb{T}_+,\pi)$ passing through the point $x \in X$.*

Lemma 1. *[19] Let Ω be a compact minimal set and $M \subseteq X$ be a compact invariant set of $(X,\mathbb{T}^+,\pi)$. If the nonautonomous dynamical system $\langle (X,\mathbb{T}^+,\pi), (\Omega,\mathbb{T},\Theta), h\rangle$ is distal on M in negative direction, then the mapping $\omega \longmapsto M_\omega := M \bigcap X_\omega$ is continuous with respect to Hausdorff metric.*

Lemma 2. *Let $M \subseteq X$ be a compact invariant set of $(X,\mathbb{T}^+,\pi)$. If the nonautonomous dynamical system $\langle (X,\mathbb{T}^+,\pi), (\Omega,\mathbb{T},\Theta), h\rangle$ is uniformly stable in the positive direction on compacts from X, then $\langle (X,\mathbb{T}^+,\pi), (\Omega,\mathbb{T},\Theta), h\rangle$ is distal on the invariant set M in the negative direction .*

Corollary 4. *Under the conditions of Lemma 2 if Ω is a compact minimal set, then the mapping $\omega \longmapsto J_\omega$ is continuous with respect to Hausdorff metric.*

Lemma 3. *Let (M,ρ) be a compact, strictly metric-convex space and E be a compact subsemigroup of isometries of semigroup M^M (i.e. $E \subseteq M^M$ and $\rho(\xi x_1, \xi x_2) = \rho(x_1,x_2)$ for all $x_1, x_2 \in M$). Then there exists a common fixed point $\bar{x} \in M$ of E, i.e. $\xi(\bar{x}) = \bar{x}$ for all $\xi \in E$.*

Theorem 7. *Let $\langle (X,\mathbb{T}^+,\pi), (\Omega,\mathbb{T},\Theta), h\rangle$ be a V - monotone bounded k - dissipative NDS, J be its Levinson center and the following conditions hold:*

1. $V(x_1,x_2) = V(x_2,x_1)$ for all $(x_1,x_2) \in X\dot{\times}X$.

2. $V(x_1,x_2) \leq V(x_1,x_3) + V(x_3,x_2)$ for all $x_1,x_2,x_3 \in X$ with the condition $h(x_1) = h(x_2) = h(x_3)$.

3. the space (X_ω, V_ω) is strictly metric-convex for all $\omega \in \Omega$, where $X_\omega = h^{-1}(\omega) = \{x \in X \mid h(x) = \omega\}$ $(\omega \in \Omega)$ and $V_\omega = V|_{X_\omega \times X_\omega}$.

Then the set-valued mapping $\omega \to J_\omega$ admits at least one continuous invariant section, i.e. there exists a continuous mapping $\nu : \Omega \to J$ with the properties: $h(\nu(\omega)) = \omega$ and $\nu(\theta(t,y)) = \pi(t,\nu(\omega))$ for all $t \in \mathbb{T}$ and $\omega \in \Omega$.

Corollary 5. *Under the conditions of Theorem 7 the Levinson center of dynamical system $(X, \mathbb{T}_+, \pi)$ contains at least one stationary (τ ($\tau > 0$) - periodic, quasiperiodic, almost periodic) point, if the minimal set Ω consists a stationary (τ ($\tau > 0$) - periodic, quasiperiodic, almost periodic) point.*

6 Applications

6.1 Finite-dimensional systems

Denote by $\mathbb{R}^n$ the real $n-$dimensional Euclidean space with the scalar product $\langle , \rangle$ and the norm $|\cdot|$, generated by the scalar product. Let $[\mathbb{R}^n]$ be the space of all the linear mappings $A : \mathbb{R}^n \to \mathbb{R}^n$, equipped with the operational norm.

Theorem 8. *Let Ω be a compact minimal set, $F \in C(\Omega \times \mathbb{R}^n, \mathbb{R}^n),\ W \in C(\Omega, [\mathbb{R}^n])$ and the following conditions hold:*

1. The matrix-function W is positively defined, i.e. $\langle W(\omega)u, u\rangle \in \mathbb{R}$ for all $\omega \in \Omega,\ u \in \mathbb{R}^n$ and there exists a positive constant a such that $\langle W(\omega)u, u\rangle \geq a|u|^2$ for all $\omega \in \Omega$ and $u \in \mathbb{R}^n$.

2. The function $t \to W(\theta_t\omega)$ is differentiable for every $\omega \in \Omega$ and $\dot{W}(\omega) \in C(\Omega, [\mathbb{R}^n])$, where $\dot{W}(\omega) = \frac{d}{dt}W(\theta_t\omega)|_{t=0}$.

3. $\langle \dot{W}(\omega)(u - v) + W(\omega)(F(\omega, u) - F(\omega, v)), u - v\rangle \leq 0$ for all $\omega \in \Omega$ and $u, v \in \mathbb{R}^n$.

4. There exist a positive constant r and the function $c : [r, +\infty) \to (0, +\infty)$ such that $\langle \dot{W}(\omega)u + W(\omega)F(\omega, u), u\rangle \leq -c(|u|)$ for all $|u| > r$.

Then the equation

$$u' = F(\theta_t\omega, u) \tag{5}$$

generates a cocycle φ on $\mathbb{R}^n$ which admits a compact global attractor $I = \{I_\omega \mid \omega \in \Omega\}$ with the following properties:

a. I_ω is a nonvoid, compact and convex subset of $\mathbb{R}^n$ for every $\omega \in \Omega$.

b. $I = \bigcup\{I_\omega \mid \omega \in \Omega\}$ is connected.

c. The mapping $\omega \to I_\omega$ is continuous with respect to Hausdorff metric.

d. $I = \{I_\omega \mid \omega \in \Omega\}$ is invariant, i.e. $\varphi(t, \omega, I_\omega) = I_{\theta_t\omega}$ for all $\omega \in \Omega$ and $t \in \mathbb{T}_+$.

e. $\lim\limits_{t\to+\infty} \beta(\varphi(t, \theta_t\omega)M, I_\omega) = 0$ for all $M \in C(\mathbb{R}^n)$ and $\omega \in \Omega$;

f. $\lim\limits_{t\to+\infty} \sup\{\beta(\varphi(t, \theta_t\omega)M, I) \mid \omega \in \Omega\ \} = 0$ for any $M \in C(\mathbb{R}^n)$, where $I = \bigcup\{I_\omega \mid \omega \in \Omega\ \}$.

g. $I = \{I_\omega \mid \omega \in \Omega\ \}$ is a uniform forward attractor, i.e.

$$\lim_{t\to+\infty} \sup_{\omega\in\Omega} \beta(\varphi(t, \omega)M, I_{\theta_t\omega}) = 0$$

for any $M \in C(\mathbb{R}^n)$.

h. The equation (5) admits at least one stationary (τ - periodic, quasiperiodic, almost periodic) solution if the point $\omega \in \Omega$ is stationary (τ - periodic, quasiperiodic, almost periodic).

Example 3. *As an example which illustrates this theorem we can consider the following equation*

$$u' = g(u) + f(\theta_t \omega),$$

where $f \in C(\Omega, \mathbb{R})$ *and*

$$g(u) = \begin{cases} (u+1)^2 & : \quad u < -1 \\ 0 & : \quad |u| \leq 1 \\ -(u-1)^2) & : \quad u > 1. \end{cases}$$

Example 4. *We consider the equation*

$$x'' + p(x)x' + ax = f(\theta_t \omega),$$

where $p \in C(\mathbb{R}, \mathbb{R})$, $f \in C(\Omega, \mathbb{R})$ *and* a *is a positive number. Denote by* $y = x' + F(x)$, *where* $F(x) = \int_0^x p(s)ds$, *then we obtain the system*

$$\begin{cases} x' = y - F(x) \\ y' = -ax + f(\theta_t \omega). \end{cases} \tag{6}$$

Theorem 9. *Suppose the following conditions hold:*

1. $p(x) \geq 0$ *for all* $x \in \mathbb{R}$.
2. *There exist positive numbers* r *and* k *such that* $p(x) \geq k$ *for all* $|x| \geq r$.

Then the nonautonomous dynamical system generated by (6) is compact dissipative and $V-$ *monotone.*

6.2 Evolution equations with monotone operators

Let H be a real Hilbert space with inner product $\langle , \rangle, |\cdot| = \sqrt{\langle , \rangle}$ and $\mathbb{B}$ be a reflexive Banach space contained in H algebraically and topologically. Furthermore, let $\mathbb{B}$ be dense in H in which case H can be identified with a subspace of the dual $\mathbb{B}'$ of $\mathbb{B}$ and $\langle , \rangle$ can be extended by continuity to $\mathbb{B}' \times \mathbb{B}$.

We consider the initial value problem

$$u'(t) + Au(t) = f(\theta_t \omega) \tag{7}$$

$$u(0) = u, \tag{8}$$

where $A : \mathbb{B} \to \mathbb{B}'$ is (generally nonlinear) bounded,

$$|Au|_{\mathbb{B}'} \leq C|u|_{\mathbb{B}}^{p-1} + K, u \in \mathbb{B}, p > 1,$$

coercive,

$$\langle Au, u \rangle \geq a|u|_{\mathbb{B}}^p, u \in \mathbb{B}, a > 0,$$

monotone,

$$\langle Au_1 - Au_2, u_1 - u_2 \rangle \geq 0, u_1, u_2 \in \mathbb{B},$$

and hemicontinuous (see [20]).

The nonlinear "elliptic" operator

$$Au = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \phi(\frac{\partial u}{\partial x_i}) \quad \text{in } D \subset \mathbb{R}^n,$$

$$u = 0 \text{ on } \partial D,$$

where D is a bounded domain in $\mathbb{R}^n$, $\phi(\cdot)$ is an increasing function satisfying

$$\phi|_{[-1,1]} = 0, \; c|\xi|^p \le \sum_{i=1}^{n} \xi_i \phi(\xi_i) \le C|\xi|^p \text{ (for all } |\xi| \ge 2 \text{)},$$

provides an example with $H = L^2(D), \mathbb{B} = W_0^{1,p}(D),\; \mathbb{B}' = W^{-1,p'}(D),\; p' = \frac{p}{p-1}$.

The following result is established in [20] (Ch.2 and Ch.4). If $x \in H$ and $f \in C(\Omega, \mathbb{B}'),\; p' = \frac{p}{p-1}$, then there exists a unique solution $\varphi \in C(\mathbb{R}_+, H)$ of (7) and (8).

We denote by $\varphi(\cdot, \omega, u)$ the unique solutions of (7) and (8). According to [21] $\varphi(\cdot, \omega, u)$ is a continuous cocycle on H.

Theorem 10. *Suppose that the operator A satisfies the conditions above and the cocycle φ, generated by equation (7), is asymptotically compact, then it admits a compact global attractor $I = \{I_\omega \mid \omega \in \Omega\}$ possessing the following properties:*

a. I_ω is a nonvoid, compact and convex subset of H for every $\omega \in \Omega$.

b. $I = \bigcup\{I_\omega \mid \omega \in \Omega\}$ is connected.

c. The mapping $\omega \to I_\omega$ is continuous with respect to Hausdorff metric.

d. $I = \{I_\omega \mid \omega \in \Omega\}$ is invariant, i.e. $\varphi(t, \omega, I_\omega) = I_{\sigma_t \omega}$ for all $\omega \in \Omega$ and $t \in \mathbb{T}^+$.

e. $\lim_{t\to+\infty} \beta(\varphi(t, \theta_{-t}\omega)M, I_\omega) = 0$ for all $M \in C(H)$ and $\omega \in \Omega$;

f. $\lim_{t\to+\infty} \sup\{\beta(\varphi(t, \theta_t\omega)M, I)| \; \omega \in \Omega\} = 0$ for any $M \in C(H)$, where $I = \bigcup\{I_\omega \mid \omega \in \Omega\}$.

g. $I = \{I_\omega \mid \omega \in \Omega\}$ is a uniform forward attractor, i.e.

$$\lim_{t\to+\infty} \sup_{\omega\in\Omega} \beta(\varphi(t, \omega)M, I_{\theta_t \omega}) = 0$$

for any $M \in C(H)$.

h. The equation (7) admits at least one stationary (τ - periodic, quasiperiodic, almost periodic) solution if the point $\omega \in \Omega$ is stationary (τ - periodic, quasiperiodic, almost periodic).

Remark 1. *If the injection of $\mathbb{B}$ into H is compact, then the cocycle φ generated by equation (7), evidently, is asymptotically compact.*

Example 5. *A typical example of equation of type (7) is the equation*

$$\frac{\partial}{\partial t} u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \varphi(\frac{\partial u}{\partial x_i}) + f(\theta_t \omega), \; u|_{\partial D} = 0 \tag{9}$$

with "nonlinear Laplacian" $Au = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\phi(\frac{\partial u}{\partial x_i})$ provides an example of equation of type (7) with $H = L^2(D), \mathbb{B} = W_0^{1,p}(D),\ \mathbb{B}' = W^{-1,p'}(D)$ and $p' = \frac{p}{p-1}$, where $\phi(\cdot)$ is an increasing function satisfying the condition

$$c|\xi|^p \leq \sum_{i=1}^{n} \xi_i \phi(\xi_i) \leq C|\xi|^p$$

for all $|\xi| \geq 2$ and $\phi|_{[-1,1]} = 0$. It is possible to verify (see, for example,[4, 20] and [2]) that the "nonlinear Laplacian" verifies all the conditions of Theorem 10 and, consequently, (9) admits a compact global attractor with the properties a.-h.. We note that the attractor of equation (9) is not trivial, i.e. the set I_ω is not a single point set at least for certain $\omega \in \Omega$.

Remark 2. *If the operator $A = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\phi(\frac{\partial u}{\partial x_i})$ is uniformly elliptic, i.e. $c|\xi|^p \leq \sum_{i=1}^{n} \xi_i \varphi(\xi_i) \leq C|\xi|^p$ (for all $\xi \in \mathbb{R}^n$), then the set I_ω is a single point set for all $\omega \in \Omega$ (for autonomous system see [23], Ch.III), because in this case the nonautonomous dynamical system generated by equation (9) is strictly monotone.*

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Wavelet analysis of nonlinear dynamical systems

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New constructive methods for analysis of resonant systems

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Abstract. The modern theory of perturbations, based on the Krylov–Bogolyubov method [1], has two essential advantages: the determination of the iterations does not require the preliminary solution of the generating equation and the choice of the initial conditions, which for every approximation minimizes the difference "exact solution minus asymptotic solution". The algorithm of constructing the perturbed solution may be realized with computer algebra methods.

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1 The classical perturbation theory

Consider an n–dimensional differential equation with small parameter

$$\frac{dz}{dt} = Z(z,t,\mu), \qquad z(0) = z_0, \tag{1}$$

where μ is the small parameter, while the vector function $Z(z,t,\mu)$ is the known and has properties which ensure the existence and uniqueness of solutions of the Cauchy problem (1) in the $(n+1)$–dimensional domain $G_{n+1} = \{(z,t) \in G_n \times R\}$ of the Euclidean space.

Our purpose is to construct this solution [2]. Along with (1) we consider the equivalent equation

$$\frac{dz}{dt} = \overline{Z}(z,t,\mu) + Z(z,t,\mu) - \overline{Z}(z,t,\mu), \qquad z(0) = z_0, \tag{2}$$

where $\overline{Z}(z,t,\mu)$ is an arbitrary function. We write the linear equality

$$z(t,\mu) = \overline{z}(t,\mu) + u(t,\mu), \tag{3}$$

where $\overline{z}$ and u are some new unknown functions. The solution of Cauchy problem for (1) can be found by solving the following two Cauchy problems:

$$\frac{d\overline{z}}{dt} = \overline{Z}(\overline{z},t,\mu), \qquad \overline{z}(0) = \overline{z}_0 \in G_n, \tag{4}$$

$$\frac{du}{dt} = Z(\overline{z} + u, t, \mu) - \overline{Z}(\overline{z}, t, \mu), \qquad u(0) = z_0 - \overline{z}_0, \tag{5}$$

where $\overline{z}_0$ is some new initial point. Equation (4) defines the choice of the initial approximation $\overline{z}(t, \mu)$ for the exact solution $z(t, \mu)$ of the problem (1), while equation (5) defines the total perturbation $u(t, \mu)$. From the problem (5) one can see that the perturbation $u(t, \mu)$ depends on the choice of the function $\overline{Z}(\overline{z}, t, \mu)$ and on the initial point $\overline{z}_0$, and, moreover, its finding is possible only after the solution of equation (4). Thus, for the Cauchy problem (1) it is possible to construct a set of variants of the perturbation theory with the parameters $\overline{Z}$ and $\overline{z}_0$. It does not mean at all that the function $\overline{Z}(\overline{z}, t, \mu)$ and the initial point z_0 may be chosen arbitrarily. It seems to be reasonable that the function $\overline{Z}(\overline{z}, t, \mu)$ would be chosen to have a possibly simpler analytic structure. On the other hand, the solutions of equation (5) must be "small" under the norm.

In classical works on nonlinear oscillations and cosmic dynamics there were commonly used three schemes:

$$\begin{cases} \frac{d\overline{z}}{dt} = A(t)\overline{z}, \quad \overline{z}(0) = z_0, \\ \frac{du}{dt} = Z(\overline{z} + u, t, \mu) - A(t)\overline{z}, \quad u(0) = 0, \end{cases} \tag{6}$$

$$\begin{cases} \frac{d\overline{z}}{dt} = Z(\overline{z}, t, 0), \quad \overline{z}(0) = z_0, \\ \frac{du}{dt} = Z(\overline{z} + u, t, \mu) - Z(\overline{z}, t, 0), \quad u(0) = 0, \end{cases} \tag{7}$$

$$\begin{cases} \frac{d\overline{z}}{dt} = \overline{Z}(\overline{z}, t, \mu), \quad \overline{z}(0) = z_0, \\ \frac{du}{dt} = Z(\overline{z} + u, t, \mu) - \overline{Z}(\overline{z}, t, \mu), \quad u(0) = 0. \end{cases} \tag{8}$$

Equations (6) represent the linearization method, equations (7) characterize the small parameter method, while equations (8) feature the averaging method, provided that the generator $\overline{Z}$ is constructed on the basis of some averaging operator.

The main idea of the classical perturbation theory (that is, the solution of the problems (4) and (5)) is that the solution of the generating equation (4) is being constructed by means of a finite number of analytic procedures or by numerical methods, after solving of the equation for perturbations (5) by means of any iterative method, symbolically designated by

$$\frac{du_k}{dt} = Z(\overline{z}(t, \mu) + u_{k-1}(t, \mu), t, \mu) - \overline{Z}(\overline{z}(t, \mu), \mu), \tag{9}$$

with $u_k(0) = z_0 - \overline{z}_0$ and $k = 1, 2, \ldots$.

2 New variants of the perturbation theory

Now we assume that the perturbation u depends on $\overline{z}, t$, and μ, that is instead of (3) we have the equality

$$z(t, \mu) = \overline{z}(t, \mu) + u(\overline{z}, t, \mu). \tag{10}$$

This equality represents the transformation from the phase space $\{z\}$ to the new phase $\{\overline{z}\}$ $(\{z\} \to \{\overline{z}\})$ and the inverse transformation $(\{\overline{z}\} \to \{z\})$ if the Jacobian matrix $\partial u/\partial \overline{z}$ is nonsingular.

The following differential equation holds

$$\frac{dz}{dt} = \frac{d\overline{z}}{dt} + \left(\frac{\partial u}{\partial \overline{z}}, \frac{d\overline{z}}{dt}\right) + \frac{\partial u}{\partial t}, \tag{11}$$

where $(\partial u/\partial \overline{z}, d\overline{z}/dt)$ is the product between the matrix $\partial u/\partial \overline{z}$ and the vector $d\overline{z}/dt$. Therefore, instead of equations (4) and (5) of the classical perturbation theory, we shall have the equations [2, 3]

$$\frac{d\overline{z}}{dt} = \overline{Z}(\overline{z}, t, \mu), \qquad \overline{z}(0) = \overline{z}_0, \tag{4}$$

$$\frac{\partial u}{\partial t} + \left(\frac{\partial u}{\partial \overline{z}}, \overline{Z}(z, t, \mu)\right) = Z(\overline{z} + u, t, \mu) - \overline{Z}(\overline{z}, t, \mu), \qquad u(0) = z_0 - \overline{z}_0. \tag{12}$$

The perturbation theory based on equations (4) and (12) differs from the classical perturbation theory in an essential point: the determination of the perturbation $u(\overline{z}, t, \mu)$ from equation (12) – called by us generalized Krylov–Bogolyubov equation [3, 4] – does not require the preliminary solving of the generating equation (4). This allows us to determine the perturbation and the initial approximation independently from each other, and therefore the accuracies of their determination are independent, too. This is impossible within the framework of the classical theory of perturbations.

The equation (12) constitutes the Cauchy problem for a quasilinear n–dimensional system of partial differential equations of first order with respect to the n–dimensional perturbation vector u. Its solution can be found by the methods of characteristics or by Cauchy's method. This equation was considered for the first time in a work of Bogolyubov [1] while tackling a question of applicability of the averaging method to a special class of ordinary differential equations.

The asymptotic theory of equation (12) for problems belonging to celestial mechanics was developed in the textbooks of Grebenikov [4] and Grebenikov and Mitropolsky [5]. We mean those problems of dynamics which are described by multifrequential systems of differential equations given on tori, and – in particular – by Hamiltonian systems with variables of the type action–angle and with the Hamiltonian periodic on the angular variable.

So, let a problem of celestial mechanics be described by a multifrequential system of the $(m + n)$–th order

$$\frac{dx}{dt} = \mu X(x, y), \tag{13.1}$$

$$\frac{dy}{dt} = \omega(x) + \mu Y(x, y), \tag{13.2}$$

where x and X are m–dimensional vectors, y, Y, and ω are n–dimensional vectors, $\omega(x)$ is the vector of frequencies, and we assume that $X(x,y)$ and $Y(x,y)$ are 2π–periodic functions with respect to y. Then they are represented by the n–multiple Fourier series

$$X(x,y) = \sum F_k(x,y), \tag{14.1}$$

$$Y(x,y) = \sum G_k(x,y), \tag{14.2}$$

where $F_k(x,y) = X_k(x)e^{i(k,y)}$, $G_k(x,y) = Y_k(x)e^{i(k,y)}$, $i = \sqrt{-1}, (k,y) = \sum\limits_{s=1}^{n} k_s y_s$, $\sum$ abridges $\sum\limits_{\|k\| \in I}$, $\|k\| = \sum\limits_{s=1}^{n} \mid k_s \mid$, $k_s = 0, \pm 1, ...,$ and $I = \{0, 1, 2, ...\}$.

Let us apply to system (13) the above stated idea of constructing a modern perturbation theory using asymptotic expansions with respect to the small parameter μ.

We choose for (13) a generating system of the form

$$\frac{d\overline{x}}{dt} = \mu \overline{X}(\overline{x}, \overline{y}) + \sum_{k \geq 2} \mu^k A_k(\overline{x}, \overline{y}), \tag{15.1}$$

$$\frac{d\overline{y}}{dt} = \omega(x) + \mu \overline{Y}(\overline{x}, \overline{y}) + \sum_{k \geq 2} \mu^k B_k(\overline{x}, \overline{y}), \tag{15.2}$$

where $\overline{X}, \overline{Y}, A_k, B_k$ are arbitrary functions of their arguments.

Let us look for the replacement of the variables (10) as formal series

$$x = \overline{x} + \sum_{k \geq 1} \mu^k u_k(\overline{x}, \overline{y}), \tag{16.1}$$

$$y = \overline{y} + \sum_{k \geq 2} \mu^k v_k(\overline{x}, \overline{y}), \tag{16.2}$$

with unknown functions $u_k(\overline{x}, \overline{y}), v_k(\overline{x}, \overline{y})$. After differentiating (16) and taking into account (13) and (15), to determine the transformation functions u_k and v_k, we have an infinite system of linear partial differential equations of first order

$$\left(\frac{\partial u_1}{\partial \overline{y}}, \omega(\overline{x}) \right) = X(\overline{x}, \overline{y}) - \overline{X}(\overline{x}, \overline{y}), \tag{17.1}$$

$$\left(\frac{\partial v_1}{\partial \overline{y}}, \omega(\overline{x}) \right) = \left(\frac{\partial \omega}{\partial \overline{x}}, u_1 \right) + Y(\overline{x}, \overline{y}) - \overline{Y}(\overline{x}, \overline{y}), \tag{17.2}$$

$$\left(\frac{\partial u_k}{\partial \overline{y}}, \omega(\overline{x}) \right) = \Phi_k(\overline{x}, \overline{y}, u_1, v_1, ..., v_{k-1}, u_{k-1} A_2, B_2, ..., A_k), \tag{17.3}$$

$$\left(\frac{\partial v_k}{\partial \overline{y}}, \omega(\overline{x}) \right) = \Psi_k(\overline{x}, \overline{y}, u_1, v_1, ..., v_{k-1}, u_k A_2, B_2, ..., A_k, B_k), \tag{17.4}$$

$$k = 2, 3, \dots .$$

The system (17) has a remarkable property: it is possible to integrate it analytically [3, 4] for any vector-index k if for the functions $\overline{X}$ and $\overline{Y}$ we choose some averages of the functions X and Y.

Indeed, let the generators $\overline{X}(\overline{x}, \overline{y}), \overline{Y}(\overline{x}, \overline{y})$ be the partial sums of series (14)

$$\overline{X}(\overline{x}, \overline{y}) = \sum_{1} F_k(\overline{x}, \overline{y}), \tag{18.1}$$

$$\overline{Y}(\overline{x}, \overline{y}) = \sum_{2} G_k(\overline{x}, \overline{y}), \tag{18.2}$$

where $\sum_{j}$ abridges $\sum_{\|k\| \in I_j}$, while I_1 and I_2 are subsets of integer nonnegative numbers from the set of all nonnegative integers I. In particular, I_1 or I_2 may consist of only one number, zero; that means

$$\overline{X}(\overline{x}, \overline{y}) = (2\pi)^{-n} \int_{0}^{2\pi} X(\overline{x}, \overline{y}) d\overline{y}_1, \dots, d\overline{y}_n. \tag{19}$$

Usually subsets I_1 and I_2 are "resonant": for $\|k\| \in I_j$

$$(k, \omega(x)) = 0.$$

If $\overline{X}$ and $\overline{Y}$ are chosen according to (18), then

$$X(\overline{x}, \overline{y}) - \overline{X}(\overline{x}, \overline{y}) = \sum_{*} F_k(\overline{x}, \overline{y}), \tag{20.1}$$

$$Y(\overline{x}, \overline{y}) - \overline{Y}(\overline{x}, \overline{y}) = \sum_{**} G_k(\overline{x}, \overline{y}), \tag{20.2}$$

where $\sum_{*}$ abridges $\sum_{\|k\| \in I - I_1}$, and $\sum_{**}$ abridges $\sum_{\|k\| \in I - I_2}$.

Using the method of characteristics, it is possible to find the exact solution of (17.1)–(17.2):

$$u_1(\overline{x}, \overline{y}) = \sum_{*} F_k^{*}(\overline{x}, \overline{y}) + \varphi_1(\overline{x}), \tag{21.1}$$

$$v_1(\overline{x}, \overline{y}) = \sum_{**} G_k^{*}(\overline{x}, \overline{y}) + \left(\frac{\partial \omega(\overline{x})}{\partial \overline{x}}, \sum_{*} F_k^{**}(\overline{x}, \overline{y}) \right) + \left(\left(\frac{\partial u_1}{\partial \overline{x}}, \varphi_1(\overline{x}) \right), \overline{y} \right) + \psi_1(\overline{x}), \tag{21.2}$$

where

$$F_k^*(\overline{x},\overline{y}) = \frac{F_k(\overline{x},\overline{y})}{f_k(\overline{x})}, \quad G_k^*(\overline{x},\overline{y}) = \frac{G_k(\overline{x},\overline{y})}{f_k(\overline{x})},$$

$$F_k^{**}(\overline{x},\overline{y}) = \frac{F_k(\overline{x},\overline{y})}{(f_k(\overline{x}))^2}, \quad f_k(\overline{x}) = i(k,\omega(\overline{x})),$$

while φ_1, ψ_1 are arbitrary differentiable functions of their arguments $\overline{x}_1, ..., \overline{x}_m$.

The integration of equations (17) for $k = 2, 3, ...$ is not very difficult, therefore the functions $u_2, v_2, ...$ are also presented by means of known analytic expressions [3, 4]. Rather important is the fact that while determining the functions u_2 and v_2 (those are perturbations of second order) we can use the functions $A_2, B_2, \varphi_1, \psi_1$.

By (21) one can see that if $\varphi_1 \neq 0$ then v_1 will be growing similary to the linear function t, because $\overline{y} \sim t$. Hence for the perturbations $u_1, v_1, u_2, v_2, ...$ to have an "oscillatory" but not a "rapidly growing" character it is necessary that

$$\varphi_k(\overline{x}) \equiv 0, \psi_k(\overline{x}) \equiv 0, \qquad k = 1, 2, ... \tag{22}$$

In their turn, these equalities show that the "best" perturbation theory is obtained when the generating equations and the perturbation equations are solved for other initial conditions in comparison with the initial equations. Indeed, if $\varphi_1(\overline{x}_0) = 0, \psi_1(\overline{x}_0) = 0$, it is easy to see, by (21), that $u_1(\overline{x}_0, \overline{y}_0) \neq 0, v_1(\overline{x}_0, \overline{y}_0) \neq 0$ and, with an accurate μ, the new initial conditions $(\overline{x}_0, \overline{y}_0)$ are connected with (x_0, y_0) by means of the functional equations

$$x_0 = \overline{x}_0 + \mu u_1(\overline{x}_0, \overline{y}_0), \tag{23.1}$$

$$y_0 = \overline{y}_0 + \mu v_1(\overline{x}_0, \overline{y}_0). \tag{23.2}$$

Similar equations for new initial conditions $(\overline{x}_0, \overline{y}_0)$ can be derived for the perturbation theory of any order k:

$$x_0 = \overline{x}_0 + \sum_{s=1}^{k} \mu^s u_s(\overline{x}_0, \overline{y}_0), \tag{24.1}$$

$$y_0 = \overline{y}_0 + \sum_{s=1}^{k} \mu^s v_s(\overline{x}_0, \overline{y}_0). \tag{24.2}$$

This is a second essential difference of the modern perturbation theory from the classical one, in which it is difficult to dispose by choice of the initial point z_0.

If we construct the perturbation theory of second order, that is, we write a system (17) for $k = 2$, we shall have

$$\left(\frac{\partial u_2}{\partial \overline{y}}, \omega(\overline{x})\right) = \Phi_2(\overline{x}, \overline{y}, u_1, v_1, A_2), \tag{25.1}$$

$$\left(\frac{\partial v_2}{\partial \overline{y}}, \omega(\overline{x})\right) = \Psi_2(\overline{x}, \overline{y}, u_1, v_1, u_2, A_2, B_2). \tag{25.2}$$

These equations include the arbitrary functions A_2, B_2, and the best way is to choose them such that

$$\int_0^{2\pi} \dots \int_0^{2\pi} \Phi_2 d\overline{y}_1, \dots, d\overline{y}_n = 0, \tag{26.1}$$

$$\int_0^{2\pi} \dots \int_0^{2\pi} \Psi_2 d\overline{y}_1, \dots, d\overline{y}_n = 0. \tag{26.2}$$

These conditions guarantee us a choice of solutions u_2 and v_2, which would also be of oscillatory character. This statement holds provided that the functions φ_2 and ψ_2 (by analogy with φ_1 and ψ_1) are chosen identically equal to zero.

The stated analytic algorithm means that we construct successively the replacement of variables

$$(x, y) \to (\overline{x}_1, \overline{y}_1) \to (\overline{x}_2, \overline{y}_2) \to \dots \to (\overline{x}_s, \overline{y}_s),$$

where

$$x_s = \overline{x} + \sum_{k=1}^{s} \mu^k u_k(\overline{x}, \overline{y}), \tag{27.1}$$

$$y_s = \overline{y} + \sum_{k=1}^{s} \mu^k v_k(\overline{x}, \overline{y}), \tag{27.2}$$

From the geometric point of view, the chain written above means the successive transformation of the initial phase space $\{x, y\}$ into the new phase space, in which the problem of perturbation determination of any order becomes analytically solvable.

Naturally, for the final construction of the solution of the initial equations (13), one has to solve the generating equation of the corresponding order s

$$\frac{d\overline{x}_s}{dt} = \mu \overline{X}(\overline{x}_s, \overline{y}_s) + \sum_{k=2}^{s} \mu^k A_k(\overline{x}_s), \tag{28.1}$$

$$\frac{d\overline{y}_s}{dt} = \omega(\overline{x}_s) + \mu \overline{Y}(\overline{x}_s, \overline{y}_s) + \sum_{k=2}^{s} \mu^k B_k(\overline{x}_s), \tag{28.2}$$

with the initial conditions $\overline{x}_s(0), \overline{y}_s(0)$ from equalities (24) and then, by means of (16), one can find an approximation s to

$$x_s(t, \mu) = \overline{x}_s(t, \mu) + \sum_{k=1}^{s} \mu^k u_k(\overline{x}_s, \overline{y}_s), \tag{29.1}$$

$$y_s(t,\mu) = \overline{y}_s(t,\mu) + \sum_{k=1}^{s} \mu^k v_k(\overline{x}_s, \overline{y}_s). \tag{29.2}$$

In conclusion, we want to note once again that in formulae (29) the functions u_k, v_k are found by analytic methods and, if the solution of the generating equation (28) can also be found through analytic methods, this is the best we can have in the nonlinear analysis. If this is not possible, then the combination of numerical methods with analytic ones applied to perturbation equations gives sometimes a large gain of economies of computer resources.

Finally, we will discuss the problems which can be solved by the contemporary methods of computer algebra.

1) The constructing of the averaging functions $\overline{X}(x,y), \overline{Y}(x,y)$.

First we calculate the initial frequencies $\omega_1(x_0), \omega_2(x_0), \ldots, \omega_n(x_0)$ and then we calculate the subsets of the integer numbers $I_1 \times I_2$, marking the proper k inequality vector

$$|(k, \omega(x_0))| < \varepsilon_1, \qquad |(k, \omega(x_0))| < \varepsilon_2.$$

If $\varepsilon_1 = \varepsilon_2$, $I_1 = I_2$. The ε_1 and ε_2 values are given apriori.

2) Afterwards, we calculate the perturbations of the first order $u_1(\overline{x}, \overline{y})$ and $v_1(\overline{x}, \overline{y})$ from equalities (21.1), (21.2).

3) The most arduous work is done while constructing Ψ_2 and Φ_2 functions, thanks to which we can calculate the functions of the second approximation u_2 and v_2 from equations (25.1), (25.2). It consists in multiplying Fourier series and assigning the resonant parts from the resulting products. Those resonant parts define the unknown functions A_2 and B_2.

4) If scientific researcher limits himself to the asymptotic theory of the second order, which is solving system (1) in the form

$$x(t,\mu) = \overline{x}(t,\mu) + \mu u_1(\overline{x}(t,\mu),\, \overline{y}(t,\mu)) + \mu^2 u_2(\overline{x}(t,\mu),\, \overline{y}(t,\mu)),$$

$$y(t,\mu) = \overline{y}(t,\mu) + \mu v_1(\overline{x}(t,\mu),\, \overline{y}(t,\mu)) + \mu^2 v_2(\overline{x}(t,\mu),\, \overline{y}(t,\mu)), \tag{30}$$

the initial conditions $\overline{x}(0,\mu)$ and $\overline{y}(0,\mu)$ for the solution of the generator system

$$\frac{d\overline{x}}{dt} = \mu \overline{X}(\overline{x}, \overline{y}) + \mu^2 A_2(\overline{x}), \quad \frac{d\overline{y}}{dt} = \omega(\overline{x}) + \mu \overline{Y}(\overline{x}, \overline{y}) + \mu^2 B_2(\overline{x}), \tag{31}$$

have to be calculated from nonlinear functional equations

$$\overline{x}(0,\mu) = x(0,\mu) - \mu u_1(\overline{x}(0,\mu),\, \overline{y}(0,\mu)) - \mu^2 u_2(\overline{x}(0,\mu),\, \overline{y}(0,\mu)),$$

$$\overline{y}(0,\mu) = y(0,\mu) - \mu v_1(\overline{x}(0,\mu),\, \overline{y}(0,\mu)) - \mu^2 v_2(\overline{x}(0,\mu),\, \overline{y}(0,\mu)). \tag{32}$$

The solutions of the system of equations (32) are to be found by means of iterative methods.

Finally, we will emphasize two extraordinary moments of the asymptotic theory based on averaging methods.

1. According to the super N.N. Bogolyubov's idea, the transformed equation (17.3), (17.4) is not given apriori at the beginning, but is constructed at every step of calculations. This is meant to minimalize the deviation of the asymptotic solution from the exact solution of the system (1). Such approach is not present in the classical perturbation theory.

2. The choice of the optimum initial conditions at every step of the constructing process improves the theory and the application practice of the resonant systems of differential equations.

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Note on multiple zeta–values*

Henryk Żołądek

Abstract. We introduce some generating functions $g(t;x)$ for multiple zeta values. They satisfy linear differential equations $Pg + x^a g = 0$ of the Fuch type. We find WKB-type expansions for g as $x \to \infty$. M41

1 Certain familiar generating function

D. Zagier had presented in [8] an 'ultra–simple' Calabi's proof of the Euler formula $\zeta(2) = \pi^2/6$. That proof uses the integral $\int_0^1 \int_0^1 (1-xy)^{-1}$ (equal to $\frac{3}{4}\zeta(2)$) and the substitution $(x,y) = \left(\frac{\sin u}{\cos v}, \frac{\sin v}{\cos u}\right)$. Below I present a proof which is even more simple (in my opinion).

The function $f_2(x) = \frac{\sin \pi x}{\pi x}$ has the Taylor expansion

$$f_2 = 1 - \frac{\pi^2}{3!}x^2 + \frac{\pi^4}{5!}x^4 - \frac{\pi^6}{7!}x^6 + \dots \tag{1}$$

and the infinite product representation

$$f_2 = \left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right)\left(1 - \frac{x^2}{3^2}\right)\dots. \tag{2}$$

Comparing the coefficients of x^2 we see immediately that $\sum \frac{1}{n^2} = \frac{\pi^2}{3!}$.

We recall that the *multiple ζ–values* are defined as follows:

$$\zeta(a_1, \dots, a_k) = \sum_{0<n_1<\dots<n_k} \frac{1}{n_1^{a_1} \dots n_k^{a_k}} \tag{3}$$

for integer $a_i \geq 1$, $a_k \geq 2$ (see [8]).

Therefore f_2 is the generating function for multiple zeta–values,

$$f_2(x) = 1 - \zeta(2)x^2 + \zeta(2,2)x^4 - \zeta(2,2,2)x^6 + \dots. \tag{4}$$

Since any $\zeta(2,\dots,2)$ (k arguments) is expressed via $\zeta(2l)$'s for $l \leq k$, one finds that

$$\zeta(2k) = \pi^{2k} \times \textit{rational number}.$$

For example, $\zeta(2,2) = \frac{1}{2}\sum_{m\neq n} m^{-2}n^{-2} = \frac{1}{2}\left(\sum_{m,n} - \sum_{m=n}\right) m^{-2}n^{-2}$, that gives $\zeta(4) = \zeta(2)^2 - 2\zeta(2,2) = \pi^4/36 - 2\pi^4/120 = \pi^4/90$; similarly, one finds $\zeta(6) = 3\zeta(2,2,2) + \frac{3}{2}\zeta(2)\zeta(4) - \frac{1}{2}\zeta(2)^3 = \pi^6/945$, etc.

Note also that instead of $\frac{\sin \pi x}{\pi x}$ one could use $\cos \pi x$ as a generating function for some quantities easily expressed via the multiple zeta–values.

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2 Irrationality of $\zeta(2)$

This result was firstly proved by A. Legendre [5]. The proof we present below is a modification of the proof of irrationality of π given in the book of A. Shidlovskiĭ [7].

One begins with the identities

$$\begin{aligned}\int_{-\pi/2}^{\pi/2}\varphi(y)\cos y &= \varphi(y)\sin y\,|_{-\pi/2}^{\pi/2} - \int \varphi'(y)\sin y \\ &= [\varphi(\frac{\pi}{2})+\varphi(-\frac{\pi}{2})] - \int \varphi''(y)\cos y \\ &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= \left[\varphi(\frac{\pi}{2})+\varphi(-\frac{\pi}{2})\right] - \left[\varphi''(\frac{\pi}{2})+\varphi''(-\frac{\pi}{2})\right] + \ldots\end{aligned} \tag{5}$$

Suppose that $\zeta(2)$ is rational, i.e. that $\frac{\pi^2}{4}=\frac{a}{b}$, $a,b\in\mathbb{Z}$. We take $\varphi(y)=\frac{b^n}{n!}(\frac{\pi^2}{4}-y^2)^n=\frac{(a-by^2)^n}{n!}$ in (5). The left-most (positive) integral in (5) behaves like $C^n/n!$ for large n and takes values between 0 and 1. Next, for $k<n$ we have $\varphi^{(k)}(\pm\pi/2)=0$ and for $k=2l\geq n$ and even, the polynomial $\varphi^{(k)}(y)$ is a sum of terms $\frac{1}{n!}b^m\left(y^{2m}\right)^{(2l)}\times$ integer $=\frac{(2l)!}{n!}\binom{2m}{2l}b^my^{2(m-l)}\times$ integer . Thus the right-most combination in (5) should represent an integer number (a contradiction).

Note that this proof relies essentially upon the fact that $(\cos x)''=-\cos x$, which follows from the ‘functional equation’ $\cos(\pi\pm y)=-\cos y$.

The proof of transcendency of $\zeta(2)$ was firstly given by F. Lindemann [6]. It is more complicated, so we do not present it here.

3 Other generating functions

Analogously to (3) one can define the functions

$$\begin{aligned}f_{a_1,\ldots,a_k}(x) &= 1-\zeta(a_1,\ldots,a_k)x^a+\zeta(a_1,\ldots,a_k,a_1,\ldots,a_k)x^{2a} \\ &\quad -\zeta(a_1,\ldots,a_k,a_1,\ldots,a_k,a_1,\ldots,a_k)x^{3a}+\ldots,\end{aligned}$$

$a=a_1+\ldots+a_k$.

It turns out that this function can be represented as $g(x;t)|_{t=1}$, where the function $g=g_{a_1,\ldots,a_k}(t;x)$ satisfies the following linear differential equation

$$Pg+x^ag=0. \tag{6}$$

Here $P=RQ^{a_1-1}RQ^{a_2-1}\ldots RQ^{a_1-1}$ is a differential operator defined via $Q=(1-t)\partial$, $R=t\partial$ and $\partial=\partial/\partial t$. Moreover $g(x;t)$ is analytic near $t=0$ and $g(x,t)=1+O(t)$.

To see this, following [8], introduce the functions

$$I(\varepsilon_1,\ldots,\varepsilon_m;t)=\int\cdots\int_{0<t_1<\ldots<t_m<t}\frac{dt_1}{A_{\varepsilon_1}(t_1)}\cdots\frac{dt_m}{A_{\varepsilon_m}(t_m)}$$

(indexed by $\varepsilon_1 = 0,1,\ \varepsilon_2 = 0,1,\ldots,\varepsilon_m = 0,1$) with

$$A_0(t) = t,\ A_1(t) = 1 - t.$$

Next, define

$$\tilde{\zeta}(a_1,\ldots,a_k;t) = I(\underbrace{1,0,\ldots,0}_{a_1},\ldots,\underbrace{1,0,\ldots,0}_{a_k};t);$$

one finds that $\zeta(a_1,\ldots,a_k) = \tilde{\zeta}(a_1,\ldots,a_k;1)$ (see [8]).

If $\mathbf{1}$ denotes the constant function $\mathbf{1}(t) \equiv 1$, then one has the formula

$$\begin{aligned}\tilde{\zeta}(a_1,\ldots,a_k;\cdot) &= \left[\partial^{-1}t^{-1}\right]^{a_k-1}\partial^{-1}(1-t)^{-1}\ldots\left[\partial^{-1}t^{-1}\right]^{a_1-1}\partial^{-1}(1-t)^{-1}\mathbf{1}\\ &= P^{-1}\mathbf{1}.\end{aligned}$$

Therefore the function

$$g = 1 - \tilde{\zeta}(a_1,\ldots,a_k;t)x^a + \tilde{\zeta}(a_1,\ldots,a_k,a_1,\ldots,a_k;t)x^{2a} - \ldots \tag{7}$$

equals

$$[(I - x^aP^{-1} + x^{2a}P^{-2} - \ldots)\mathbf{1}](t) = \left[(I + x^aP^{-1})^{-1}\mathbf{1}\right](t).$$

It implies that g satisfies the equation $(I + x^aP^{-1})g \equiv 1$ and, in consequence, the equation (6).

Example 1. In the case $k = 1$ and $a_1 = 2$ the equation (6) becomes the hypergeometric equation

$$(1-t)\partial(t\partial g) + x^2g = 0$$

(with singular points at $t = 0,1,\infty$). Its characteristic exponents (i.e. the powers α in the solutions $(t-t_0)^\alpha + \ldots$ as $t \to t_0$ or $t^\alpha + \ldots$ as $t \to \infty$) are the following: $\lambda = \lambda' = 0$ at $t = 0$; $\rho = 0, \rho' = 1$ at $t = 1$; $\tau = x, \tau' = -x$ at $t = \infty$. It follows (see [1]) that our distinguished solution is the hypergeometric function

$$g_2(x;t) = F(x,-x;1;t).$$

In [4] one can find the following interesting identities (proved by Broadhurst):

$$g_{1,3}(\sqrt{2}x;t) \equiv F(x,-x;1;t)F(ix,-ix;1;t), \quad f_{1,3}(\sqrt{2}x) = f_4(x).$$

Generally, the equation (6) is of the Fuchs type (i.e. with regular growth of solutions at singular points). Its characteristic equations (for the characteristic exponents) are the following: $\alpha^{a_k}(\alpha-1)^{a_{k-1}}\ldots(\alpha-k+1)^{a_1}$ at $t = 0$; $\alpha(\alpha-1)\ldots(\alpha-a+k)\cdot(\alpha-a_k+1)(\alpha-a_k-a_{k-1}+2)\ldots(\alpha-a_k-\ldots-a_2+k-1)$ at $t = 1$ and $(-1)^k\alpha^a + x^a = 0$ at $t = \infty$.

This implies that the monodromy operators $\mathcal{M}_0$ and $\mathcal{M}_1$, induced by analytic prolongation of solutions to (6) along simple loops surrounding $t = 0$ and $t = 1$, are unipotent (with eigenvalues equal to 1). (Maybe this explains the fact that the multiple zeta–values generate the 'ring of periods of the pro-nilpotent completion of $\pi_1(\mathbb{C}P^1\backslash\{0,1,\infty\})$', see [3,8]). The monodromy operator $\mathcal{M}_\infty$ associated with a loop around $t=\infty$ is diagonalizable with different eigenvalues $e^{-2\pi i\alpha}$, $(-1)^k\alpha^a+x^a=0$.

The series in (7) defines g in the disc $|t|<1$, but $g(x;\cdot)$ can be prolonged to a multi-valued holomorphic function with ramifications at $t=1$ and $t=\infty$; (the further branches of g ramify also at $t=0$). Near $t=1$ one has the representation $g=h_0(t-1)+h_1(t-1)\log(t-1)+\ldots+h_r(t-1)\log^r(t-1)$ with analytic $h_j(z)$ near $z=0$. Note that $\zeta(a_1,\ldots,a_k)=h_0(0)$.

We refer the reader to the (very algebraic) paper of A. Goncharov [3] for further results about multiple zeta–values.

4 Asymptotic as $x\to\infty$

The equation (6) for large parameter x is solved using the WKB method. This means that one represents a solution as a finite sum of terms of the form

$$e^{xS(t)}[\varphi_\gamma(t)x^\gamma+\varphi_{\gamma-1}(t)x^{\gamma-1}+\ldots].$$

The 'action' S satisfies the 'Hamilton–Jacobi equation'

$$t^{a-k}(1-t)^k\left(S'\right)^a+1=0, \tag{8}$$

the coefficient φ_γ satisfies the 'transport equation' of the form

$$\varphi_\gamma'+W(t)\varphi_\gamma=0 \tag{9}$$

(with some rational function W) and the other coefficients $\varphi_{\gamma-m}$ satisfy some non-homogeneous equations (whose homogeneous parts are like in (9) and the rests depend on $S',S'',\ldots,\varphi_\gamma,\ldots,\varphi_{\gamma-m+1}$).

The Hamilton-Jacobi equation (8) has solutions of the form of Schwarz–Christoffel integral

$$S(t)=S_j(t)=\xi_j\cdot\int_0^t\tau^{k/a-1}(1-\tau)^{-k/a}d\tau, \tag{10}$$

where ξ_j is a root of (-1) of order a. The transport equation (9) is solved as follows:

$$\varphi_\gamma(t)=\varphi_{\gamma,j}(t)=C_j\cdot t^\mu(1-t)^\nu$$

for some exponents μ,ν depending on the situation. By the initial condition $g(x;0)=1$ the first exponent γ and the constants C_j must be chosen after expanding $e^{xS_j(t)}$ at $t=0$ and solving some further transport equations (we shall not do it). For the same reason the initial limit in the integral (10) is equal to 0.

From this the following expansion formula for the generating function $f_{a_1,\ldots,a_k}$ $=g_{a_1,\ldots,a_k}(x;1)$ follows:

$$f_{a_1,\ldots,a_k}\sim\sum_{j=1}^a e^{\beta\xi_j x}\left[\varphi_{\delta,j}(1)x^\delta+\varphi_{\delta-1,j}(1)x^{\delta-1}+\ldots\right], \tag{11}$$

where $\beta = B(\frac{k}{a}, 1-\frac{k}{a}) = \frac{\pi}{\sin \pi k/a}$ and the constants $\varphi_{\eta,j}(1)$, $\eta \leq \delta$ are (theoretically) calculable. In general, one cannot expect convergence in (11).

It seems that this method would give some insight into the nature of the coefficients of the generating functions $f_{a_1,\ldots,a_k}$.

Example 2. Consider the function $g_3(x;t)$. One finds that $\xi_1 = -1$, $\xi_{2,3} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$, $\beta = \frac{2\pi}{\sqrt{3}}$ and $\varphi_{\gamma,j}(t) = C_j \cdot t^{-1/3}(1-t)^{2/3}$. This suggests that the zeta–numbers $\zeta(3), \zeta(3,3), \zeta(3,3,3), \zeta(9), \zeta(15), \ldots$ have something common with the numbers π, i and $\sqrt{3}$. Maybe this is the way to show the transcendency of $\zeta(3)$. (Recall that the irrationality of $\zeta(3)$ was shown by R. Apéry [2], see also [4]).

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CMC–surfaces, φ–geodesics and the Carathéodory conjecture

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Abstract. A short proof of the Caratheodory conjecture about index of an isolated umbilic on the convex 2–dimensional sphere is suggested.

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1 Introduction

The constant mean curvature (CMC–) surfaces in E^3 are known to admit a continuous family of local, non–trivial, isometric deformations preserving mean curvature of the surface (H–deformations). In the case when surface is compact Umehara [11] showed that the converse is also true.

The maximal and minimal curvature lines of CMC–surface form an orthogonal net which is called *réseau de Bonnet*, cf Cartan [3]. The Bonnet Theorem says that if the CMC–surface is simply connected and umbilic–free, then under H–deformations the orthogonal net "rotates" through a constant angle which can be taken as a parameter of deformation.

If the CMC–surface is not simply connected or umbilic–free, Cartan seems to be the first to ask about possible scenario of evolution of *réseau de Bonnet* under the H–deformations.

In the present note we study [1] evolution of the orthogonal nets in the case when CMC–surface is simply connected with a single umbilic or, equivalently, doubly connected and umbilic–free. Namely, if we "pinch" the umbilic, the CMC–surface becomes an annulus whose points undergo H–deformations according to the Bonnet Theorem. In general, the rotation angle is no longer constant at all the points because annulus cannot be covered by a single chart.

However, the Bonnet Theorem implies that every curve of the orthogonal net is a φ–geodesic line whatever H–deformations are applied to a CMC–surface. Metric φ is given by the linear element $ds = |\varphi||dz|$, where φdz^2 is a holomorphic quadratic "differential" associated to the CMC–surface. Of course, $\varphi(0) = 0$ at the umbilical point.

This observation is crucial, because the φ–geodesics near n–th order zero of a holomorphic quadratic form are well–understood due to the works of Strebel [10].

[1] For the reasons which will be clear later.

Roughly speaking, the φ–geodesics fill–up the annulus either by "hyperbolas" or "radii". Therefore, possible configurations of *réseau de Bonnet* near the umbilic looks like a singularity with the finite number of hyperbolic and parabolic sectors.

Despite independent interest, the orthogonal nets are auxiliary for us. We postulate different fact here: H–deformations of orthogonal nets give an amazingly simple proof to the *Caratheodory'sche Vermutung (Conjecture)*:

Theorem 1. *Let S^2 be a C^∞ surface which bounds a convex compact body in the Euclidean space E^3. Then S^2 has at least two umbilical points. In other words, the Euler-Poincaré index of isolated umbilical point is at most $+1$.*

(A short overview of this conjecture can be found in [1]; see also [2],[5],[7].)

2 φ–geodesics

Until further indications, M is a simple domain of the complex parameter z. Let us consider the holomorphic functions $\varphi(z)$ vanishing at the unique point of M which we identify with 0. An order $n \geq 1$ is assigned to 0, if there exists a complex constant $a \neq 0$ such that $\varphi(z) = az^n + O(|z|^{n+1})$.

Flat metric φ with the cone singularity of angle $(n+2)\pi$ is given by the formula

$$|ds| = |\varphi||dz|,$$

provided $\varphi(z)dz^2$ is a quadratic form on M. By a *φ–geodesic line* in M one understands the line conisting of the shortest arcs relatively metric φ. Any two points in M (including 0) may be joined by the unique φ–geodesic line. Strebel classified the possible types of φ–geodesics in the neighborhood of n–th order zero by proving the following lemma.

Lemma 2. *([10]) Any two points in a neighborhood M of n–th order zero of holomorphic 2–form $\varphi(z)dz^2$ can be joined by a unique φ–geodesic. Moreover, each φ–geodesic is either an arc defined by the equation $Arg\ \varphi(z)dz^2 = Const$, or is composed of the two radii centered in 0 with the minimal angle $\geq 2\pi/(n+2)$.*

The foliation $\mathcal{F}$ on $M\backslash 0$ is said to be *geodesic* if every leaf of $\mathcal{F}$ is a φ–geodesic line. Before we state the general lemma on the structure of geodesic foliations, let us consider an example when all $\mathcal{F}$'s can be obtained by a "brute force".

If 0 is a double zero, then the φ–metric is given by the linear element $ds^2 = (u^2+v^2)(du^2+dv^2)$ where $u+iv$ is a natural parameter. The metric $|ds|$ is Liouville's and the geodesic lines in this metric are completely integrable. The general integral is known to be of the form

$$\int \frac{du}{\sqrt{u^2-a}} \pm \int \frac{dv}{\sqrt{v^2+a}} = a',$$

where a, a' are two independent constants. Easy calculations show that no information will be lost if we suppose $a = 0$. The integral takes the form $\ln|u| \pm \ln|v| = a'$. The geodesic foliation is described by two "families of curves": $v = Cu$ and $v = C/u$, where C is an arbitrary constant. Thus, $\mathcal{F}$ near a double zero is either the "node" with the geodesics radii tending to 0, or the "saddle" with four sectors filled–up by the geodesic "hyperbolas".

Let w be a finite "word" on the alphabet consisting of two symbols h and p. We introduce the elementary operations on w:

(i) a cyclic permutation of the symbols in w, and

(ii) a contraction of the p–symbol: $p^2 = p$.

Two words are *equivalent* $w_1 \sim w_2$ if and only if w_2 can be obtained from w_1 by the elementary operations. The equivalence class of word w is denoted by $[w]$.

Fix an integer number $n \geq 1$. To every symbol h in w we assign a *weight* $|h| = 2\pi/(n+2)$. To every symbol p we assign the weight $|p| = \alpha_i$, where α_i is a positive real. The weight of w is an additive function equal to the sum of weights of the symbols entering w. The equivalence class $[w]$ is called *normalized* if $|w| = 2\pi$ for all $w \in [w]$. (Note that the weight of w is one and the same for all $w \in [w]$.)

Lemma 3. *Let h and p stay for the hyperbolic and the parabolic sectors of the singularity w, respectively. We encode the singular point w by a sequence of symbols h and p in the order the h– and the p–sectors occur when turning clockwise around the singularity. Then:*

(i) each φ–geodesic foliation $\mathcal{F}$ is topologically equivalent to the singularity w of a normalized equivalence class $[w]$;

(ii) each normalized equivalence class $[w]$ can be realized as a φ–geodesic foliation $\mathcal{F}$ with the singularity $w \in [w]$ in a neighborhood of n–th order zero of φ for some $n \geq 1$.

Proof. Denote by M a neighborhood of the n–th order zero of φ. Let us introduce a partial order for the points $x, y \in M$: $x \leq y$ if and only if $Arg\ x \leq Arg\ y$. If $x \in M$ is an arbitrary point, then by Lemma 2 the φ–geodesic line through x is either (i) the hyperbola $Arg\ \varphi dz^2 = \ Const$ or (ii) the radius Ox. Let us consider the first possibility.

(i) The hyperbola $Arg\ \varphi dz^2 = \ Const$ must tend to the asymptotic rays Oz_1, Oz_2 with $z_1 < x < z_2$, enclosing the angle $2\pi/(n+2)$. Clearly, the only possibility to the geodesic foliation $\mathcal{F}$ is to form a hyperbolic sector z_1Oz_2. Of course, along Oz_1 and Oz_2 $Arg\ \varphi dz^2$ is constant.

(ii) Let Ox be the geodesic radius through x, distinct from the boundary radii of the hyperbolic sector. Then through the nearby points $|x - y| < \varepsilon$ one can draw the geodesic radii Oy's. Denote by y_1Oy_2 the maximal connected parabolic sector filled-up with the geodesic radii. Clearly, $y_1 < x < y_2$. The angle enclosed between two boundary radii, we denote by α. In general, $0 \leq \alpha \leq 2\pi$.

If the hyperbolic sector h is followed by another hyperbolic sector h, we write this as hh. If h is followed by a parabolic sector, we put it as hp. A parabolic sector p followed by the parabolic sector p, gives a larger parabolic sector $p = pp$ and the contraction rule (ii) follows. Of course, the "weights" of the sectors are equal to the angles swept by the sectors.

Finally, according to the definition of normalized equivalence class, each singularity consists of sequence of parabolic and hyperbolic sectors; every curve in these sectors is a geodesic arc.

The part (ii) of Lemma 3 is proved by the similar argument.

3 CMC–surfaces

Every smooth immersion $f : M \to E^3$ of an orientable surface M into the Euclidean space E^3 induces a Riemann structure on M; let $z = u + iv$ be the corresponding local parameter. With respect to z the first fundamental form can be written as $ds^2 = e^{2\lambda}|dz|^2$.

If $ldu^2 + 2mdudv + ndv^2$ is the second fundamental form, we consider a complex quadratic form φdz^2, such that $\varphi(z) = \frac{1}{2}(l - n) - im$. The Mainardi–Codazzi equations imply that φ is holomorphic on M if and only if $f(M)$ is a CMC–surface. Locally, along the lines of minimal and maximal curvature $Arg\ \varphi dz^2 = 0$ and $\varphi(0) = 0$ at the umbilic points.

A *continuous deformation* f_t of the immersion $f = f_0$ is the isometry of surface M such that $M \times [0,1] \to E^3$ is a continuous mapping. The continuous deformation f_t is called an *H–deformation* if $H_t = H$ for all $t \in [0,1]$, where $H : M \to \mathbb{R}$ is the mean curvature function.

The CMC–surfaces are known to admit a non–trivial H–deformations and in the case of compact surfaces, they are the only ones with such a property. Of course, there are known many examples of compact CMC–surfaces of genus $g > 0$.

What happens with the lines of principal (i.e. minimal or maximal) curvature of the CMC–surface during an H–deformation? If M is a local CMC–surface without umbilics, the principal curvature lines of $f_0(M)$ and $f_t(M)$ form two families of the parallel lines intersecting each other with the constant angle proportional to the parameter t (the Bonnet Theorem, see e.g.[4]). Note that if we fix the φ–metric on M corresponding to $f_0(M)$, then the principal curvature lines of $f_t(M)$ coincide with the φ–geodesic lines of the inclination t. If the umbilical points are allowed, then a law is given by the following lemma.

Lemma 4. *Suppose that $M_0 = f_0(M)$ is a canonical CMC–surface with the quadratic function $\varphi = z^n, n \geq 1$. Let φ be a metric on M corresponding to M_0. If $M_t = f_t(M)$ is an H–deformation of M_0, then one of the two principal curvature lines of M_t coincide with the φ–geodesic lines on M for any $t \geq 0$.*

Proof. In the polar coordinate system the coefficients of the second fundamental form of surface M_t are given by the equations:

$$\begin{aligned} l &= He^{\lambda} + |z|^n \cos(2t - n\ Arg\ z), \\ m &= |z|^n \sin(2t - n\ Arg\ z), \\ n &= He^{\lambda} - |z|^n \cos(2t - n\ Arg\ z), \end{aligned} \tag{1}$$

where t is a parameter of the H–deformation, cf [11]. The following two cases are possible.

(i) An H–deformation, such that t is constant on M. It can be immediately seen that in new coordinates $\tilde{u} = \cos t\ u + \sin t\ v$, $\tilde{v} = -\sin t\ u + \cos t\ v$ the first and the second forms of surfaces M_0 and M_t are the same. By the fundamental theorem, surfaces M_0 and M_t may differ only by a rigid motion in E^3. Thus, the H–deformation is trivial.

(ii) A non–trivial H–deformation. By item (i), t varies for the points of M. Thus far, associated to every $z \in M\backslash 0$, there is a chart in which the second fundamental form of surface $M_t(z)$ writes as

$$l = He^{\lambda} + \cos 2t, \quad m = \sin 2t, \quad n = He^{\lambda} - \cos 2t,$$

where t is the deformation parameter, cf [13]. A straightforward calculation shows that the principal curvature lines of the surface $M_t(z)$ coincide with the φ–geodesic lines of the slope t on M. (This fact follows also from the Bonnet Theorem.) Since every regular point $z \in M$ can be endowed with such a chart, Lemma 4 is proved.

4 Proof of Theorem 1

Take a convex C^∞ immersion $f_0 : S^2 \to E^3$ of the 2–sphere into the Euclidean space E^3 which is not totally umbilic (i.e. there are no $U \subseteq S^2$ such that $f_0(U)$ is a part of the round sphere). In other words, umbilics are supposed isolated and their number is finite. Denote by ds_0 a Riemann metric on S^2 induced by the immersion f_0 and by $H : S^2 \to \mathbb{R}$ the corresponding mean curvature function.

Definition 1. *By a Hopf spheroid in E^3 we understand a convex C^∞ immersion $f : S^2 \to E^3$ such that there exists at least one umbilical point p and a small closed disc $D \ni p$ such that $H(D) = Const$.*

Lemma 5. *There exist infinitely many Hopf spheroids in E^3.*

Proof. By the results of Wente and Kapouleas any compact orientable surface S_g of genus $g > 0$ admits an immersion into E^3 which is a CMC-surface with $H > 0$; cf.[6],[12]. Fix $g \geq 2$ and consider the lines of principal curvature of any such immersion. By the index argument, there exists an umbilic $p \in S_g$ and a small closed disc $D \ni p$ which is a convex local surface in E^3. We separate this local surface from S_g. To obtain a Hopf spheroid, it remains to complete this piece of CMC-surface to a C^∞ immersion $S^2 \to E^3$. By Urysohn's lemma this can be done in an infinite number of ways. □

Lemma 6. *For the Hopf spheroids the Caratheodory conjecture is true.*

Proof. Without loss of generality we can assume that the umbilic point p of Hopf spheroid is unique. (For otherwise, if there are more than one umbilic then we are done.) Since a Hopf spheroid is locally CMC, we apply Lemma 4 to identify the curvature lines in the disc $D \ni p$ with φ-geodesic lines in the vicinity of a singularity w.

Let $w \in [w]$ be a word of the minimal length in the normalized equivalence class $[w]$. According to Lemma 3, there exists a singularity of order n whose topological type is encoded by the sequence w of symbols h and p. Let w admit $\langle h \rangle$ symbols of type h and $\langle p \rangle$ symbols of type p. By the normalization axiom, $\langle h \rangle \le n+2$.

To estimate the Euler–Poincaré index of singularity w, note that the parabolic sectors make no contribution to the index value and the number $\langle p \rangle$ can be neglected. To the contrary, if there are no hyperbolic sectors (i.e. $w = p$) we necessarily have one parabolic sector. The general formula is true:

$$Ind\ w = \begin{cases} 1 - \frac{\langle h \rangle}{2} & \text{if} \quad w \ne p, \\ +1 & \text{if} \quad w = p. \end{cases}$$

In either case $Ind\ w \le 1$ and by the index argument the conjecture follows.

Now we are ready to finish the proof of Theorem 1. But first we wish to outline the main idea. To every convex C^∞ immersion $f_0 : S^2 \to E^3$ one can relate a Hopf spheroid. This spheroid is uniquely defined by f_0 and is a 'modification' of f_0 which has an interesting 'mechanical' interpretation.

Suppose that f_0 is a convex steel ball filled-up with a gas under a pressure. Let p be an isolated umbilic of f_0. We drill a small hole in p and glue-up a soap film D into this hole maintaining a pressure [2] inside the ball. We also 'deform' slightly the 'edges' of the cut in order to keep the modified surface $f : S^2 \to E^3$ in the class C^∞. We claim that f is a Hopf spheroid.

Indeed, $f(D)$ is a local CMC-surface with an umbilic point $p \in D$. Moreover, the index of umbilic on the Hopf spheroid is equal to the index of p on f_0. (This is because the foliation by principal curvature lines at the 'steel part' of ball remains intact.) In general, if $\mathcal{F}_0$ and $\mathcal{F}$ are foliations by the principal curvature lines on f_0 and f, respectively, then $\mathcal{F}$ is obtained from $\mathcal{F}_0$ by a *homotopy of opening of a leaf*; cf [9].

Let f_0 be as above. If p is an isolated umbilic of f_0 then we take a closed disc $|D| \le r$ centred at the point p. We are going to define a local CMC-surface $f(D)$. Let $z = u + iv$ be a local parameter which corresponds to a part of CMC-surface with an umbilic; see the beginning of this section. By the results of Umehara [11]

[2] The absolute value of the pressure depends on how 'flat' is the surface at the point p. Of course, by 'pressure' we understand difference of pressures inside and outside the steel ball.

(see also [3],[4]) there exists a family of isometric H-deformations depending on a real parameter t:

$$I = e^{2\lambda}|dz|^2, \qquad II_t = l du^2 + 2m\; dudv + n dv^2, \tag{2}$$

with l, m and n given by equations (1). The Mainardi-Codazzi and Gauss equations for I, II_t:

$$\frac{\partial \varphi}{\overline{\partial} z} = \frac{\partial H}{\partial z}, \qquad |\varphi|^2 = e^{4\lambda}(H^2 - K), \tag{3}$$

where $\varphi = e^{it} z^n$ is a complex quadratic form φdz^2, are satisfied for any real t. (Indeed, the first equation is true since $H = \; Const$ and φ is holomorphic; the second equation follows from $|\varphi e^{it}| = |\varphi|$ and the fact that H-deformation is an isometry.) Therefore, the fundamental forms (2) are realized by a concrete local CMC-surface for each real number t.

Let $f_t(D)$ be a family of local CMC-surfaces described above. Denote by A an annular region which surrounds disc D:

$$A = \{z = u + iv | \; r \le |z| \le r + \varepsilon\}. \tag{4}$$

To glue-up $f_t(D)$ properly, we fix the metric λ so that $\lambda|_{\partial A_{r+\varepsilon}} = \lambda|_{\partial A_r}$, where the left part denotes a metric on the exterior boundary of A which is induced by metric of the surface f_0. The boundary condition $\lambda|_{\partial A_r}$ gives a unique solution $f_{t=t^*}(D)$ to the Gauss equation, so that a representative in the family $f_t(D)$ is fixed.

To obtain a C^∞ Hopf spheroid it remains to conjugate $f_{t^*}(D)$ with the rest of the sphere:

$$f(S^2) = \begin{cases} f_{t^*}(D), & \text{if} \quad z \in D \subset Int\; D_{r+\varepsilon}, \\ f_0(S^2), & \text{if} \quad z \in S^2 \backslash D_{r+\varepsilon}. \end{cases} \tag{5}$$

By the Urysohn Lemma, function f in formula (5) can be chosen C^∞ for an arbitrary small ε, see formula (4). Moreover, taking r sufficiently small we can fix number n (see (1)) equal to the order of quadratic form φ at point p of the surface f_0. (Such an order is correctly defined for any φ, not necessary holomorphic.)

Thus, the surface f given by equation (5) is a Hopf spheroid. By the Lawson-Tribuzy theorem f is uniquely defined up to a rigid motion in E^3; see [8]. To finish the proof of Caratheodory conjecture, it remains to notice that passage from f_0 to f gives us a homotopy $h(\mathcal{F}_0) = \mathcal{F}$ between foliations induced by curvature lines. In particular, $Ind\; p_0 = Ind\; p$. By Lemma 6, the Caratheodory conjecture follows. $\square$

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Solution of the center problem for cubic systems with a bundle of three invariant straight lines*

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Abstract. For cubic differential system with three invariant straight lines which pass through the same point it is proved that a singular point with purely imaginary eigenvalues (weak focus) is a center if and only if the focal values g_{2j+1}, $j = \overline{1,5}$, vanish.

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1 Introduction

A cubic system with a singular point with pure imaginary eigenvalues ($\lambda_1 = \overline{\lambda}_2 = i$, $i^2 = -1$) by a nondegenerate transformation of variable and time rescaling can be brought to the form

$$\begin{aligned} \tfrac{dx}{dt} &= y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \tfrac{dy}{dt} &= -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv -Q(x,y). \end{aligned} \tag{1}$$

The variables x, y and coefficients $a, b, \ldots, r, s$ in (1) are assumed to be real. A singular point $(0,0)$ is a center or a focus for (1). The problem arises of distinguishing between a center and a focus, i.e. of finding the coefficient conditions on (1) under which $(0,0)$ is, for example, a center. These conditions are called the conditions for a center existence or the center conditions and the problem - the problem of the center.

Note that the singular point $(0,0)$ of the differential system (1) is called also weak focus (fine focus).

It is well known that the origin is a center for (1) if and only if all focal values g_{2j+1}, $j = \overline{1, \infty}$, vanish. The focal values are polynomials in coefficients of system (1). For example, the first of them looks as follows

$$g_3 = ac - bd + 2ag - 2bf + cf - dg - 3k + 3l - p + q. \tag{2}$$

If all the g_{2j+1} are zero up to $g_{2\tau+1}$, i.e. $g_{2j+1} = 0$, $j = \overline{1, \tau - 1}$, and $g_{2\tau+1} \neq 0$, then τ is called the order of the weak focus $(0,0)$.

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It is known also that the system of differential equations (1) has a center at $O(0,0)$ if and only if it has in some neighbourhood of the origin an independent of t holomorphic first integral $F(x,y) = C$ (an holomorphic integrating factor $\mu(x,y)$).

The problem of the center was solved for quadratic system ($k = l = m = n = p = q = r = s = 0$) by H. Dulac [10], and for symmetric cubic system ($a = b = c = d = f = g = 0$) by K.S. Sibirski [16].

If the cubic system (1) contains both quadratic and cubic nonlinearities, the problem of the center is solved only in some particular cases (see, for example, [2, 4, 6–9, 11–14]).

The quadratic system and symmetric cubic system with a singular point of center type are Darboux integrable, i.e. these systems have a first integral (integrating factor) of the form of product of invariant algebraic curves. Hence, the interest arose to study the center problem for polynomial differential systems with algebraic invariant curves. The problem of integrability for polynomial systems with invariant algebraic curves and, in particular, with invariant straight lines was considered in works [3, 5–8, 17, 20].

The straight line $C + Ax + By = 0$ is said to be invariant for (1) if there exists a polynomial $K(x,y)$ such that the identity holds

$$A \cdot P(x,y) - B \cdot Q(x,y) \equiv (C + Ax + By)K(x,y). \tag{3}$$

$K(x,y)$ is called the cofactor of the invariant straight line.

By [6] the cubic system (1) can not have more than four nonhomogeneous invariant straight lines, i.e. straight lines of the form

$$1 + Ax + By = 0 \ (|A| + |B| \neq 0). \tag{4}$$

As homogeneous straight lines $Ax + By = 0$ this system can have only the lines $x \pm iy = 0$, $i^2 = -1$. Hence, the cubic system (1) can not have more than six invariant straight lines. This case is realized. To solve the problem of the center in the case of system (1) with four nonhomogeneous invariant straight lines, it is enough to require the vanishing of the first focal value (Liapunov quantity) g_3 [6]. The vanishing of the first focal value in the case of system (1) with four invariant straight lines among which are also homogeneous ones is not enough for the existence of a center. Also the vanishing of the second focal value g_5 is necessary.

Thus, the cubic system (1) with four invariant straight lines (real, complex, real and complex) has at the origin a singular point of a center type if and only if the first two focal values vanish [7].

If (1) has three invariant straight lines two of which are homogeneous, then the presence of a center at $(0,0)$ is guaranteed by vanishing of the focal values $g_{2j+1} = 0$, $j = \overline{1,7}$ [19].

In this paper we study the center problem assuming that the cubic system (1) has three invariant straight lines which pass through the same point.

2 Conditions for the existence of a bundle of tree invariant straight lines

From (3) it results that (4) is an invariant straight line of (1) if and only if A and B are the solutions of the system

$$\begin{aligned}
F_1(A,B) &= AB^2 - fAB + bB^2 + rA - lB = 0,\\
F_2(A,B) &= A^2B + aA^2 - gAB - kA + sB = 0,\\
F_3(A,B) &= B^3 - 2A^2B + fA^2 + (c-b)AB - dB^2 - pA + nB = 0,\\
F_4(A,B) &= A^3 - 2AB^2 - cA^2 + (d-a)AB + gB^2 + mA - qB = 0.
\end{aligned} \tag{5}$$

The cofactor of (4) is

$$\begin{aligned}
K(x,y) \;=\; & -Bx + Ay + (aA - gB + AB)x^2 + \\
& (cA - dB + B^2 - A^2)xy + (fA - bB - AB)y^2.
\end{aligned} \tag{6}$$

Further, we shall assume that the cubic system (1) has three invariant straight lines which pass through the same point (x_0, y_0). By a rotation and rescaling coordinate axes we can make that $x_0 = 0$, $y_0 = 1$. Consequently, the equation of each invariant straight line of the bundle has the form

$$1 + Ax - y = 0. \tag{7}$$

It is evident that the point $(0,1)$ of the intersection of these straight lines is a singular point for (1), i.e. $P(0,1) = Q(0,1) = 0$. These equalities give $r = -f - 1$, $l = -b$. Substituting $B = -1$, $r = -f - 1$ and $l = -b$ in (5) we find that

$$\begin{aligned}
&F_1 \equiv 0,\ F_2 = (a-1)A^2 + (g-k)A - s = 0,\\
&F_3 = (f+2)A^2 + (b-c-p)A - d - n - 1 = 0,\\
&F_4 = A^3 - cA^2 + (a-d+m-2)A + g + q = 0.
\end{aligned}$$

From the above equalities we can see that the system (1) can have three distinct invariant straight lines of the form (7) iff the following conditions holds:

$$a = 1,\ f = -2,\ k = g,\ l = -b,\ n = -d - 1,\ p = b - c,\ r = 1,\ s = 0, \tag{8}$$

$$\begin{aligned}
&4(g+q)c^3 + (d-m+1)^2c^2 + 18(d-m+1)(g+q)c + 4d^3\\
&-12(m-1)d^2 + 12(m-1)^2d - 27(g+q)^2 - 4(m-1)^3 \neq 0.
\end{aligned} \tag{9}$$

In the conditions (8),(9) the straight line (7) is invariant for (1) iff A satisfies the equation

$$A^3 - cA^2 + (m - d - 1)A + g + q = 0. \tag{10}$$

The left-hand side of the inequality (9) coincides with the discriminant of the equation (10) and (9) gives that the roots A_1, A_2, A_3 of the (10) are not equal: $A_i \neq A_j\ \forall i \neq j$.

Using (5),(8) and (9) it is easy to show that along with three invariant straight lines of the form (4) the system (1) has also one more invariant nonhomogeneous straight line if and only if at least one of the following two series of conditions holds:

$$\begin{aligned} &a = r = 1,\ b = l = s = 0,\ f = -2,\ k = g, \\ &n = -d - 1,\ p = -c,\ q = g(d+1), \end{aligned} \tag{11}$$

$(d+1)(d+2) \neq 0$. The straight line $1 + (d+1)y = 0$;

$$\begin{aligned} &a = r = 1,\ f = -2,\ k = g,\ l = -b,\ n = -d - 1,\ p = b - c,\ s = 0, \\ &(m - gc + g^2)(b+g)^2 - (dg - q)(b+g) + bg = 0, \\ &2(b+g)^3 - (b+c)(b+g)^2 - (d+2)(b+g) + b = 0, \end{aligned} \tag{12}$$

$bg(b+g) \neq 0$. The straight line $1 + gx - g(b+g)^{-1}y = 0$.

3 Sufficient center conditions

a) Darboux integrability.

Lemma 1. *The conditions (11) are sufficient for the origin to be a center for the system (1).*

Proof. Assume that $(d+1)(d+2) \neq 0$ and that the inequality (9) holds.

Denote by $A_1,\ A_2, A_3$ the roots of the equation (10). Then

$$c = A_1 + A_2 + A_3,\ m = A_1A_2 + A_1A_3 + A_2A_3 + d + 1,\ g = -A_1A_2A_3/(d+2).$$

The straight lines $l_j \equiv 1 + A_j x - y = 0,\ j = 1, 2, 3,$ of the bundle and the straight line $l_4 \equiv 1 + (d+1)y = 0$ have, respectively, the cofactors (see (6)):

$$\begin{aligned} K_1(x,y) &= x + A_1 y - A_1A_2A_3(d+2)^{-1}x^2 + \\ &\quad (1 + d + A_1A_2 + A_1A_3)xy - A_1 y^2, \\ K_2(x,y) &= x + A_2 y - A_1A_2A_3(d+2)^{-1}x^2 + \\ &\quad (1 + d + A_1A_2 + A_2A_3)xy - A_2 y^2, \\ K_3(x,y) &= x + A_3 y - A_1A_2A_3(d+2)^{-1}x^2 + \\ &\quad (1 + d + A_1A_3 + A_2A_3)xy - A_3 y^2, \\ K_4(x,y) &= x(d+1)(y - 1 + A_1A_2A_3(d+2)^{-1}x). \end{aligned} \tag{13}$$

The system (1) has the first integral of the form $l_1^{\alpha_1} l_2^{\alpha_2} l_3^{\alpha_3} l_4^{\alpha_4} = const$, where $\alpha_j,\ j = \overline{1,4},\ \sum |\alpha_j| \neq 0$ are generally complex numbers if and only if the following identity holds

$$\sum_{j=1}^{4} \alpha_j K_j(x,y) \equiv 0. \tag{14}$$

Substituting (13) in (14) we obtain

$$\begin{aligned} &\alpha_1 = (A_2 - A_3)(A_2A_3 + d + 2), \\ &\alpha_2 = (A_3 - A_1)(A_1A_3 + d + 2), \\ &\alpha_3 = (A_1 - A_2)(A_1A_2 + d + 2), \\ &\alpha_4 = (A_1 - A_2)(A_1 - A_3)(A_2 - A_3)/(d+1). \end{aligned}$$

Therefore, in conditions (11), (9), $(d+1)(d+2) \neq 0$, the system (1) has in some neighborhood of the origin a holomorphic first integral of the form $F(x, y) = const$ and this means that $(0,0)$ is a center of (1).

Since the center variety is closed in the space of coefficients of the system (1), then $(0,0)$ will be a center also in the cases when one or both of the inequalities $(d+1)(d+2) \neq 0$ and (9) do not hold.

Lemma 2. *The conditions*

$$\begin{array}{l} a = r = 1, f = -2, k = g, l = -b, n = -d-1, p = b-c, \\ q = g + d(b+g),\ s = 0,\ (b+g)^4 - (2b+c)(b+g)^3 \\ +(b^2+bc+m+1)(b+g)^2 + bd(b+g) - b^2 = 0, \\ 2(b+g)^3 - (b+c)(b+g)^2 - d(b+g) - b - 2g = 0. \end{array} \tag{15}$$

are sufficient for the origin to be a center for system (1).

Proof. In the conditions (8) the equality $g_3 = 0$ (see (2)) looks $d(b+g)+g-q = 0$, from where we express $q:\ q = g+d(b+g)$. Note that the conditions (15) are included in (12) if in the last we put $q = g + d(b+g)$.

Assume that the inequalities (9) and $bg(b+g)(1+(b+g)(A_2+A_3)-A_2A_3) \neq 0$, where A_2, A_3 are the roots of the equations (10), hold. Denote $\nu = b+g$. The last two equalities from (15) give us

$$d = 2\nu^2 - (b+c)\nu - 2 + b\nu^{-1},\ m = c\nu - \nu^2 - 1 + 2b\nu^{-1}.$$

In this case the equation (10) looks

$$(A-\nu)(A^2 - (c-\nu)A - 2\nu^2 + (b+c)\nu + b\nu^{-1} = 0.$$

We put $A_1 = \nu$ and let A_2, A_3 be the roots of the quadratic equation $A^2 - (c-\nu)A - 2\nu^2 + (b+c)\nu + b\nu^{-1} = 0$. Then

$$b = \nu(A_2-\nu)(A_3-\nu)/(\nu^2+1),\ c = A_2 + A_3 + \nu.$$

The invariant straight lines

$$l_j = 1 + A_jx - y,\ j = \overline{1,3},\ l_4 = 1 + \nu^2 + (1 + A_2\nu + A_3\nu - A_2A_3)(\nu x - y)$$

of the system (1) have, respectively, the cofactors:

$$\begin{aligned} K_1(x,y) &= x + \nu y + \big[(\nu(-A_2A_3 + A_2\nu + A_3\nu + 1))x^2 \\ &\quad +((A_2+A_3)\nu^3 + (1-A_2A_3)\nu^2 - (A_2+A_3)\nu + A_2A_3 - 1)xy \\ &\quad +(\nu(-\nu A_2 - \nu A_3 + A_2A_3 - 1))y^2\big]/(\nu^2+1), \\ K_2(x,y) &= x + A_2y + \big[(\nu(A_2\nu + A_3\nu - A_2A_3 + 1))x^2 \\ &\quad +(\nu^3A_2 + \nu^2 - \nu A_2 - 2\nu A_3 + 2A_2A_3 - 1)xy \\ &\quad +(\nu^3 - 2\nu^2A_2 - \nu^2A_3 + \nu A_2A_3 - A_2)y^2\big]/(\nu^2+1), \\ K_3(x,y) &= x + A_3y + \big[(\nu(A_2\nu + A_3\nu - A_2A_3 + 1))x^2 \end{aligned}$$

$$
\begin{aligned}
& +(\nu^3 A_3 + \nu^2 - \nu A_3 - 2\nu A_2 + 2A_2A_3 - 1)xy \\
& +(\nu^3 - 2\nu^2 A_3 - \nu^2 A_2 + \nu A_2 A_3 - A_3)y^2\big]/(\nu^2+1), \\
K_4(x,y) \quad = \quad & (1 + (A_2 + A_3)\nu - A_2A_3)(\nu x - y + 1)(\nu y + x)/(\nu^2+1).
\end{aligned}
$$

The system (1) has an integrating factor of the Darboux form $\mu(x,y) = l_1^{\beta_1} l_2^{\beta_2} l_3^{\beta_3} l_4^{\beta_4}$ (this means that $(0,0)$ is a center) if and only if the numbers β_1, β_2, β_3, β_4 satisfy the identity

$$\sum_{j=1}^{4} \beta_j K_j(x,y) \equiv \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}.$$

Substituting in this identity the expressions of the cofactors and identifying the coefficients of x, y, x^2, xy and y^2, we obtain that

$$
\begin{aligned}
\beta_1 &= 1, \\
\beta_2 &= (A_2A_3\nu - A_2 + 2A_3)/(A_2 - A_3), \\
\beta_3 &= (A_2A_3\nu + 2A_2 - A_3)/(A_3 - A_2), \\
\beta_4 &= (A_2A_3\nu^2 - A_2A_3 + 2\nu A_2 + 2\nu A_3 + 2)/(A_2A_3 - \nu A_2 - \nu A_3 - 1).
\end{aligned}
$$

Lemma 3. *The conditions*

$$a = -n = r = 1,\ d = s = 0,\ f = -2,\ k = q = g,$$
$$l = -b,\ p = b - c,\ m = 3 + (b+g)(3c - 3b - 5g)$$

are sufficient for the origin to be a center for the system (1).

Proof. In the conditions of lemma 3 the equation (10) looks

$$A^3 - (2b + \beta)A^2 + (8b\nu + 3\nu\beta - 5\nu^2 + 2)A - 2b + 2\nu = 0, \tag{16}$$

where $\nu = b + g$, $\beta = c - 2b$. Suposse that (16) has three diferent roots A_1, A_2, A_3. The straight line $l_j \equiv 1 + A_j x - y = 0$ of the bundle has the cofactor $K_j(x,y) = x + A_j y + (\nu - b)x^2 + (1 - A_j^2 + 2bA_j + \beta A_j)xy + (b - A_j)y^2$ $(j = 1,2,3)$.

The identity

$$\sum_{j=1}^{3} \beta_j K_j(x,y) \equiv \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}$$

holds if

$$
\begin{aligned}
\beta_1 = \ & (-2A_2A_3 + \beta A_2 + \beta A_3 - 2b\beta + 16b\nu - \beta^2 + \\
& 6\beta\nu - 10\nu^2 + 6)/((A_1 - A_2)(A_1 - A_3)), \\
\beta_2 = \ & (2A_1A_3 - \beta A_1 - \beta A_3 + 2b\beta - 16b\nu + \beta^2 - \\
& 6\beta\nu + 10\nu^2 - 6)/((A_1 - A_2)(A_1 - A_3)), \\
\beta_3 = \ & (-2A_1A_2 + \beta A_1 + \beta A_2 - 2b\beta + 16b\nu - \beta^2 + \\
& 6\beta\nu - 10\nu^2 + 6)/((A_1 - A_2)(A_1 - A_3)).
\end{aligned}
$$

Therefore, $\mu(x,y) = l_1^{\beta_1} l_2^{\beta_2} l_3^{\beta_3}$ is an integrating factor of the system (1) and, consequently, $(0,0)$ is a center.

By the closedness of the center variety in the space of coefficients of (1) the singular point $(0,0)$ will be the center type also in the cases when the equation (16) has multiple roots.

b) Symmetry.

Let

$$\begin{array}{l} a = r = 1,\ c = 6b + 5g,\ f = -2,\ k = g,\ l = -b,\ p = -5(b+g),\ s = 0, \\ d = -5(b+g)(4 + (3b+2g)(4b+3g))/(13b+10g), \\ m = (5(b+g)(3b+2g)(5b+4g) - b - 10g)/(13b+10g), \\ n = (5(b+g)(3b+2g)(4b+3g) + 7b + 10g)/(13b+10g), \\ q = -(5(b+g)^2(3b+2g)(4b+3g) + 20b^2 + 27bg + 10g^2)/(13b+10g). \end{array} \tag{17}$$

The system (1) with (17), after the change of coordinates

$$X = \frac{x}{1-y}, \quad Z = \frac{y}{1-y},$$

defines the following equation of nonlinear oscillations:

$$P_4(X) Z Z' = -X P_0(X) - 3X P_1(X) Z - P_2(X) Z^2 - P_3(X) Z^3, \tag{18}$$

where

$$\begin{array}{rl} P_0(X) = & 1 + gX, \\ P_1(X) = & (19b + 10g - 5(4b+3g)(3b+2g)(b+g) - (20b^2 + bg - \\ & 10g^2 + 5(4b+3g)(3b+2g)(b+g)^2)X)/(3(13b+10g)), \\ P_2(X) = & (13b^2 + 10bg + (6b - 5(4b+3g)(3b+2g)(b+g))X - \\ & (20b^2 + 14bg + 5(4b+3g)(3b+2g)(b+g)^2)X^2)/(13b+10g), \\ P_3(X) = & b, \\ P_4(X) = & (13b + 10g + (13b+10g)(6b+5g)X + \\ & (6b + 5(9b+7g)(3b+2g)(b+g))X^2 + \\ & (20b^2 + 14bg + 5(4b+3g)(3b+2g)(b+g)^2)X^3)/(13b+10g). \end{array}$$

The substitution $Z = \frac{P_0(X)Y}{1-P_1(X)Y}$ [15] reduces the equation (18) to the form

$$Q_4(X) Y Y' = -X - Q_2(X) Y^2 - Q_3(X) Y^3,$$

where

$$\begin{array}{rcl} Q_2(X) & \equiv & P_0(X)P_2(X) - 3XP_1^2(X) + P_0'(X)P_4(X), \\ Q_3(X) & \equiv & 2XP_1^3(X) - P_0(X)P_1(X)P_2(X) + P_0^2(X)P_3(X) + \\ & & P_0(X)P_1'(X)P_4(X) - P_0'(X)P_1(X)P_4(X), \\ Q_4(X) & \equiv & P_0(X)P_4(X). \end{array}$$

By Theorem 9.4 of [1] in the case $Q_3(X) = X^{2j+1}\widetilde{P}(X)$, $\widetilde{P}(0) \neq 0$ the origin is a center for the equation (18) if and only if the system of equations

$$\begin{aligned} &y^4R^3(x)Q_3^5(y) - x^4R^3(y)Q_3^5(x) = 0,\\ &xQ(x)R^2(y) - yQ(y)R^2(x) = 0, \end{aligned} \tag{19}$$

where

$$R(X) \equiv Q_4(X)[Q_3(X) - XQ_3'(X)] + 3XQ_2(X)Q_3(X),$$
$$Q(X) \equiv Q_4(X)[R'(X)Q_3(X) - 3R(X)Q_3'(X)] + 4Q_2(X)Q_3(X)R(X)$$

has in some neighborhood of $X = 0$ a holomorphic solution

$$Y = \phi(X), \quad \phi(0) = 0,\ \phi'(0) = -1. \tag{20}$$

Let us consider the following two series of conditions on the coefficients of system (1):

$$\begin{aligned} &a = r = 1,\ c = g = k = -3b/2,\ d = -5,\ f = -2,\\ &l = -b,\ m = -7,\ n = 4,\ p = 5b/2,\ q = b; \end{aligned} \tag{21}$$

$$\begin{aligned} &a = r = 1,\ c = 6b + 5g,\ f = -2,\ k = g,\ p = -5(b + g),\\ &l = -b,\ d = -5(b + g)(4 + (3b + 2g)(4b + 3g))/(13b + 10g),\\ &m = (5b(b + g)(3b + 2g) - 21b + 30g)/(13b + 10g),\\ &n = (5b(b + g)(3b + 2g) - 8b + 40g)/(13b + 10g),\\ &q = -4g,\ r = 1,\ s = 0,\ (3b + 2g)(b + g)^2 + b - 2g = 0. \end{aligned} \tag{22}$$

Remark. *The conditions (21) (respectively, (22)) can be obtained from conditions (17) if to the last we add the equality* $g = -3b/2$ *(respectively,* $(3b + 2g)(b + g)^2 + b - 2g = 0$*).*

Lemma 4. *Each of conditions (21), (22) are sufficient conditions for the system (1) to have a center at the origin.*

Proof. Assume first that the conditions (21) hold. Then the equalities (19) have a solution in the form of (20):

$$Y = \frac{3b^2X^2 - 20bX + 12 + (bX - 2)\sqrt{3(3b^2X^2 - 20bX + 12)}}{2b(2 - 3bX)}.$$

Now, assume that the conditions (22) hold. From $(3b + 2g)(b + g)^2 + b - 2g = 0$ we find that

$$b = 2\nu(1 - \nu^2)/(\nu^2 + 3),\ g = \nu(1 + 3\nu^2)/(\nu^2 + 3),$$

where ν is a parameter. The conditions (22) look:

$$\begin{aligned} &a = r = 1,\ b = 2\nu(1 - \nu^2)/(\nu^2 + 3),\ c = \nu(3\nu^2 + 17)/(\nu^2 + 3),\\ &d = -5(3\nu^2 + 1)/(\nu^2 + 3),\ f = -2,\ g = \nu(1 + r\nu^2)/(\nu^2 + 3),\\ &k = \nu(1 + r\nu^2)/(\nu^2 + 3),\ l = 2\nu(\nu^2 - 1)/(\nu^2 + 3),\\ &m = (13\nu^2 - 1)/(\nu^2 + 3),\ n = 2(7\nu^2 + 1)/(\nu^2 + 3),\\ &p = -5\nu,\ q = -4\nu(3\nu^2 + 1)/(\nu^2 + 3),\ s = 0. \end{aligned}$$

Finally, it is easy to verify that ecuations (19) have a solution in the form of (20):

$$Y = -\frac{3\nu^2X^2 + 10\nu X + 3 - (\nu X + 1)\sqrt{3(3\nu^2X^2 + 10\nu X + 3)}}{2(3\nu X + 1)}.$$

4 The problem of the center

In this section by "$\Longrightarrow$" we will understand "further it is used".

Theorem. *The order of a weak focus for cubic differential systems with a bundle of three invariant straight lines is at most five.*

Proof. Without loss of generality, we shall consider the cubic system (1) with conditions (8),(9). In the same conditions we shall calculate the first five focal values using the algorithms described in ([18]). The first one looks: $g_3 = q - g - d(b + g)$ (see (2), (8)). From $g_3 = 0$ we find q:

$$q = g + d(b + g)$$

and substitute into the expression for g_5. We have $g_5 = bd(m - (b + g)(3c - 3b - 5g) - 2d - 3)$. If $b = 0$ then $\Longrightarrow$ Lemma 1, if $d = 0$ then $\Longrightarrow$ Lemma 3.

Let

$$bd \neq 0 \tag{23}$$

and

$$m = (b + g)(3c - 3b - 5g) + 2d + 3.$$

The third focal value being cancelled by non-zero factors is of the form $g_7 = f_1f_2$, where

$$f_1 = 2(b + g)^3 - (b + c)(b + g)^2 - d(b + g) - b - 2g,$$
$$f_2 = 6b + 5g - c.$$

If $f_1 = 0 \Longrightarrow$ Lemma 2. Further, we shall consider that $bdf_1 \neq 0$. Simplify the focal values g_9 and g_{11} by bdf_1.

From $f_2 = 0$ we express c:

$$c = 6b + 5g$$

and substitute it in g_9, g_{11}. The g_9 looks as

$$g_9 = (13b + 10g)d + 5(b + g)(4 + (3b + 2g)(4b + 3g)). \tag{24}$$

If the coefficient d in g_9 is equal to zero, i.e. $g = -13b/10$, then $g_9 = -3b(b^2 + 100)/50 \neq 0$ (see (23)). We require that $13b + 10g \neq 0$. From $g_9 = 0$ (see (24)) express d :

$$d = -5(b + g)(4 + (3b + 2g)(4b + 3g))/(13b + 10g) \tag{25}$$

and substitute it in g_{11}. For g_{11}, after corresponding simplifications, i.e. after elimination of a denominator and non-zero factors, including numerical one, we have

$$g_{11} = (b + g)(3b + 2g)((3b + 2g)(b + g)^2 + b - 2g).$$

If $b+g=0$, then from (25) $d=0$. That is in contradiction with assumption (23). If $(3b+2g)((3b+2g)(b+g)^2+b-2g)=0 \Longrightarrow$ Lemma 4 (in the case of $3b+2g=0$ we have the series (21) of conditions on the coefficients of the system (1) and in the case $(3b+2g)(b+g)^2+b-2g=0$, respectively, the series (22)).

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Classification of quadratic systems with a symmetry center and simple infinite singular points

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Abstract. We classify the family of planar quadratic differential systems with a center of symmetry and two invariant straight lines according to the topology of their phase portraits. The case of the existence of simple infinite singular points is only considered. For each of the obtained distinct topological classes we give necessary and sufficient conditions in terms of algebraic invariants and comitants. The program was implemented for computer calculations.

Mathematics subject classification: 34C14, 34C05, 58F14.
Keywords and phrases: quadratic differential system, invariant line, phase portrait, limit cycle.

1 Introduction and the statement of main results

Consider generic quadratic systems of the form:

$$\begin{aligned} \frac{dx}{dt} &= p_0 + p_1(x,y) + p_2(x,y) \equiv P(x,y), \\ \frac{dx}{dt} &= q_0 + q_1(x,y) + q_2(x,y) \equiv Q(x,y) \end{aligned} \tag{1}$$

with real homogeneous polynomials $p_i,\ q_i \in \mathbb{R}[\mathbf{a}, x, y]$ $(i = 0, 1, 2)$ of degree i in x, y.

In paper [10] the notion of a *dicritical* (not necessarily singular) point of a quadratic differential system is introduced. As particular cases, it comprises *symmetry* point of the corresponding integral curves, *dicritical nodal singular* point and *homogeneity* point (i.e., such point that system (1) becomes homogeneous after shifting the point to the origin). The class of quadratic system with homogeneity point was studied in [2, 7, 11–13, 16, 19, 21–23]. In papers [4, 20] the topological classification of system (1) having a dicritical nodal singular point is obtained. Some classes of the quadratic systems (1) possessing a symmetry point were examined in papers [3, 17, 18].

The purpose of our article is the study of quadratic system (1) with a symmetry point and two parallel invariant straight lines which can be: (a) real distinct; (b) imaginary; (c) coincided in the finite part of the phase plane; (c) coincided at infinity. For this class of system (1) all possible topological distinct phase portraits

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will be constructed and the respective necessary and sufficient conditions for their realization will be established.

We introduce the following polynomials:

$$C_i = yp_i(x,y) - xq_i(x,y)\ (i=0,1,2), \qquad D_i = \frac{\partial p_i}{\partial x} + \frac{\partial q_i}{\partial y}\ (i=1,2),$$

which in fact are GL-comitants [5, 16]. To formulate the statement of the Main Theorem we shall construct T-comitants and CT-comitants (see [15] for detailed definitions) which distinguish phase portraits of the class of system (1) possessing a center of symmetry and two parallel invariant straight lines. All of them will be constructed only by using polynomials C_i and D_i via the differential operator $(f,g)^{(k)}$ called *transvectant of the index* k [8, 14] which acts on $\mathbb{R}[\mathbf{a},x,y]$ as follows:

$$(f,g)^{(k)} = \sum_{h=0}^{k}(-1)^h\binom{k}{h}\frac{\partial^k f}{\partial x^{k-h}\partial y^h}\ \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

Here $f(x,y)$ and $g(x,y)$ are polynomials in x, y of the degree r and ρ, respectively, and $\mathbf{a}\in\mathbb{R}^{12}$ is 12-tuple of the coefficients of system (1).

First we construct the following comitants of the second degree with respect to coefficients of initial system (1):

$$\begin{array}{lll} T_1 = (C_0,C_1)^{(1)}, & T_2 = (C_0,C_2)^{(1)}, & T_3 = (C_0,D_2)^{(1)}, \\ T_4 = (C_1,C_1)^{(2)}, & T_5 = (C_1,C_2)^{(1)}, & T_6 = (C_1,C_2)^{(2)}, \\ T_7 = (C_1,D_2)^{(1)}, & T_8 = (C_2,C_2)^{(2)}, & T_9 = (C_2,D_2)^{(1)}. \end{array}$$

By using the initial T-comitants: $\tilde{A}$, $\tilde{B}$, $\tilde{C}\equiv C_2$, $\tilde{D}$, $\tilde{E}$, $\tilde{F}$, $\tilde{G}\equiv D_2$, $\tilde{H}$, $\tilde{K}$ written in tensorial form in paper [5] was constructed a minimal polynomial basis of T-comitants of system (1) up to degree 12.

We shall use here some of these T-comitants, expressed through C_i and D_j:

$$\begin{aligned}
\tilde{A}(\mathbf{a}) = {} & \big(C_1, T_8 - 2T_9 + D_2^2\big)^{(2)}/144,\\
\widetilde{D}(\mathbf{a},x,y) = {} & \big[2C_0(T_8 - 8T_9 - 2D_2^2) + C_1(T_7 - T_6) - (C_1,T_5)^{(1)}\\
& +6D_1(C_1D_2 - T_5) - 9D_1^2C_2\big]/36,\\
\widetilde{E}(\mathbf{a},x,y) = {} & \big[D_1(2T_9 - T_8) - 3\,(C_1,T_9)^{(1)} - D_2(3T_7 + D_1D_2)\big]/72,\\
\widetilde{F}(\mathbf{a},x,y) = {} & \big[6D_1^2(D_2^2 - 4T_9) + 4D_1D_2(T_6 + 6T_7) + 48C_0\,(D_2,T_9)^{(1)} - D_2^2T_4 + 288D_1\widetilde{E}\\
& -24\big(C_2,\widetilde{D}\big)^{(2)} + 120\big(D_2,\widetilde{D}\big)^{(1)} - 36C_1\,(D_2,T_7)^{(1)} + 8D_1\,(D_2,T_5)^{(1)}\big]/144,\\
\widetilde{K}(\mathbf{a},x,y) = {} & (T_8 + 4T_9 + 4D_2^2)/72,\\
\widetilde{H}(\mathbf{a},x,y) = {} & (-T_8 + 8T_9 + 2D_2^2)/72.
\end{aligned}$$

Now the needed T-comitants expressed only through the polynomials C_i $(i=0,1,2)$

and D_j $(j=1,2)$ via differential operator $(*,*)^{(k)}$ can be constructed:

$$
\begin{aligned}
M(\mathbf{a},x,y) &= T_8/8 \equiv \text{Hessian}(C_2)/4,\\
K(\mathbf{a},x,y) &= \widetilde{K}(\mathbf{a},x,y) \equiv \big(p_2(x,y),q_2(x,y)\big)^{(1)}/4,\\
N_1(\mathbf{a},x,y) &= (T_8-2T_9+D_2^2)/36,\\
N_2(\mathbf{a},x,y) &= \big[D_1(2T_9-T_8-3D_2^2)-3D_2T_7-3\,(C_1,T_9)^{(1)}\big]/72,\\
N_5(\mathbf{a},x,y) &= (T_5-3C_2D_1+2C_1D_2)/6,\\
V(\mathbf{a},x,y) &= \big[4(T_2+C_0D_2)^2-3(T_5-3C_2D_1+2C_1D_2)(T_1+C_0D_1)\big]/36,\\
W_1(\mathbf{a},x,y) &= 2\big(C_2,\widetilde{D}\big)^{(2)}-7\big(D_2,N_1\big)^{(2)}+18\widetilde{F},\\
W_2(\mathbf{a},x,y) &= 15\,C_2\big[23\big((\widetilde{D},\widetilde{D})^{(2)},D_2\big)^{(1)}+7\big((C_2,\widetilde{D})^{(2)},\widetilde{D}\big)^{(2)}\big]-11\big[\big(C_2,\widetilde{D}\big)^{(2)}\big]^2+\\
&\quad 36\widetilde{D}\big[42\big(C_2,\widetilde{F}\big)^{(2)}-197\big(\widetilde{D},\widetilde{K}\big)^{(2)}+184\big(\widetilde{D},\widetilde{H}\big)^{(2)}\big]+\\
&\quad 6D_2\big[168\big(\widetilde{D},\widetilde{F}\big)^{(1)}-19\big((C_2,\widetilde{D})^{(2)},\widetilde{D}\big)^{(1)}\big]+\\
&\quad 288\widetilde{F}\big[2\big(C_2,\widetilde{D}\big)^{(2)}+9\widetilde{F}\big]+172\big(C_2,\widetilde{D}\big)^{(3)}\big(C_2,\widetilde{D}\big)^{(1)}+\\
&\quad 12\big(49\widetilde{K}-197\widetilde{H}\big)\big(\widetilde{D},\widetilde{D}\big)^{(2)}-194\big(C_2,\widetilde{D}\big)^{(2)}\big(D_2,\widetilde{D}\big)^{(1)},\\
W_3(\mathbf{a},x,y) &= \big(\big(C_2,\widetilde{D}\big)^{(1)},\big(C_2,\widetilde{D}\big)^{(1)}\big)^{(2)}-6\big(C_2,\widetilde{D}\big)^{(1)}\big(C_2,\widetilde{D}\big)^{(3)},\\
\eta(\mathbf{a}) &= (M,M)^{(2)}/6 \equiv \text{Discrim}(C_2),\\
\mu(\mathbf{a}) &= -(K,K)^{(2)}/2 \equiv \text{Discrim}(K),\\
\kappa(\mathbf{a}) &= -(N_1,N_1)^{(2)}/8 \equiv \text{Discrim}(N_1)/4,\\
G_1(\mathbf{a}) &= \big(C_1,T_8-2T_9+D_2^2\big)^{(1)}/144,\\
H_1(\mathbf{a}) &= 9\big(\big(\big(\widetilde{D},\widetilde{D}\big)^{(2)},D_2\big)^{(1)},D_2\big)^{(1)}+270\big((\widetilde{D},\widetilde{D})^{(2)},(6\widetilde{K}+N_1)\big)^{(2)}+\\
&\quad 576\big((\widetilde{D},\widetilde{F})^{(2)},D_2\big)^{(1)}+396\big((C_2,\widetilde{D})^{(2)},\widetilde{F}\big)^{(2)}-86\big[\big(C_2,\widetilde{D}\big)^{(3)}\big]^2,\\
H_2(\mathbf{a}) &= \big(\widetilde{H},\widetilde{K}\big)^{(2)}-3\big(\widetilde{H},\widetilde{H}\big)^{(2)},\\
H_3(\mathbf{a}) &= -6\big(\widetilde{F},\widetilde{K}\big)^{(2)}-4\big((\widetilde{D},\widetilde{H})^{(2)},D_2\big)^{(1)}-\big((\widetilde{D},\widetilde{K})^{(2)},D_2\big)^{(1)},\\
F_1(\mathbf{a}) &= 10\big[\big(C_2,\widetilde{D}\big)^{(3)}\big]^2-99\big(\big(\big(\widetilde{D},\widetilde{D}\big)^{(2)},D_2\big)^{(1)},D_2\big)^{(1)}-36\big((C_2,\widetilde{D})^{(2)},\widetilde{F}\big)^{(2)}+\\
&\quad 54\big((\widetilde{D},\widetilde{D})^{(2)},(7\widetilde{H}-\widetilde{K})\big)^{(2)}-288\big((\widetilde{D},\widetilde{F})^{(2)},D_2\big)^{(1)},\\
F_2(\mathbf{a}) &= \big(\widetilde{H},\widetilde{K}\big)^{(2)}+\big(\widetilde{H},\widetilde{H}\big)^{(2)},\\
F_3(\mathbf{a}) &= \big(C_2,\widetilde{D}\big)^{(3)},\\
E_1(\mathbf{a}) &= 4\big((\widetilde{D},\widetilde{F})^{(2)},D_2\big)^{(1)}+3\big((\widetilde{D},\widetilde{D})^{(2)},(\widetilde{K}+3\widetilde{H})\big)^{(2)}-4\big((C_2,\widetilde{D})^{(2)},\widetilde{F}\big)^{(2)},\\
E_2(\mathbf{a}) &= \big((\widetilde{D},N_1)^{(2)},D_2\big)^{(1)},\\
E_3(\mathbf{a}) &= \big(((\widetilde{D},D_2)^{(2)},D_2)^{(1)},D_2\big)^{(1)}.
\end{aligned}
$$

In order to formulate the statement of the Main Theorem we note that the geometrical meaning of the condition $\kappa=0$ is given by Lemma 1.

Main Theorem. *For $\kappa=0$ the phase portraits of the non-degenerate quadratic system (1) with a point of symmetry and such that polynomial $C_2=yp_2(x,y)-xq_2(x,y)\neq 0$ has only simple roots (i.e.,$\eta\neq 0$), are determined by the respective affine invariant conditions given in Table 1. Here by r_i (respectively, c_i) the real (respectively, imaginary) singular point of multiplicity i is denoted.*

Table 1

<table>
<tr><th>Infinite singular points</th><th>Condi-tions</th><th>Finite singular points</th><th>Conditions</th><th>Phase portrait</th><th colspan="2">Additional conditions for determining phase portraits</th></tr>
<tr><td rowspan="18">$r_1r_1r_1$</td><td rowspan="18">$\eta > 0$</td><td rowspan="5">$r_1r_1r_1r_1$</td><td rowspan="5">$W_2 > 0,$
$W_1 > 0$</td><td rowspan="3">Figure 1</td><td colspan="2">$N_1 \geq 0$</td></tr>
<tr><td rowspan="2">$N_1 < 0$</td><td>$W_3 < 0$</td></tr>
<tr><td>$W_3 \geq 0, E_1 > 0$</td></tr>
<tr><td>Figure 2</td><td colspan="2">$N_1 < 0, W_3 = 0, E_1 < 0$</td></tr>
<tr><td>Figure 3</td><td colspan="2">$N_1 < 0, W_3 > 0, E_1 < 0$</td></tr>
<tr><td rowspan="4">$c_1c_1c_1c_1$</td><td rowspan="4">$W_2 < 0$ or
$W_2 > 0$ &
$W_1 < 0$</td><td rowspan="3">Figure 4</td><td colspan="2">$N_1 = 0$</td></tr>
<tr><td rowspan="2">$N_1 \neq 0$</td><td>$W_3 \neq 0$</td></tr>
<tr><td>$W_3 = 0, N_1E_1 > 0$</td></tr>
<tr><td>Figure 5</td><td colspan="2">$W_3 = 0, N_1E_1 < 0$</td></tr>
<tr><td rowspan="4">$r_2\,r_2$</td><td rowspan="4">$W_2 = 0$
$W_1 > 0$</td><td rowspan="2">Figure 6</td><td colspan="2">$N_1 = 0$</td></tr>
<tr><td colspan="2">$N_1 \neq 0,\ E_2 = 0$</td></tr>
<tr><td>Figure 7</td><td colspan="2">$E_2 \neq 0,\ N_1 > 0$</td></tr>
<tr><td>Figure 8</td><td colspan="2">$E_2 \neq 0,\ N_1 < 0$</td></tr>
<tr><td>$c_2\,c_2$</td><td>$W_2 = 0, W_1 < 0$</td><td>Figure 4</td><td colspan="2">–</td></tr>
<tr><td>r_4</td><td>$\mu \neq 0, W_1 = 0$</td><td>Figure 9</td><td colspan="2">–</td></tr>
<tr><td rowspan="3">–</td><td rowspan="3">$\mu = 0, V \neq 0$</td><td rowspan="2">Figure 4</td><td colspan="2">$W_3 \neq 0$</td></tr>
<tr><td colspan="2">$W_3 = 0, E_3 < 0$</td></tr>
<tr><td>Figure 5</td><td colspan="2">$W_3 = 0, E_3 > 0$</td></tr>
<tr><td rowspan="10">$r_1c_1c_1$</td><td rowspan="10">$\eta < 0$</td><td>$r_1r_1r_1r_1$</td><td>$\mu > 0, W_2 > 0$
$W_1 > 0$</td><td>Figure 10</td><td colspan="2">–</td></tr>
<tr><td rowspan="3">$r_1r_1c_1c_1$</td><td rowspan="3">$\mu < 0, W_2 \neq 0$</td><td>Figure 15</td><td colspan="2">$H_1 \neq 0$</td></tr>
<tr><td>Figure 16</td><td colspan="2">$H_1 = 0,\ H_3 > 0$</td></tr>
<tr><td>Figure 17</td><td colspan="2">$H_1 = 0,\ H_3 < 0$</td></tr>
<tr><td>$c_1c_1c_1c_1$</td><td>$W_2 < 0$ or
$\mu > 0, W_2 > 0$
& $W_1 < 0$</td><td>Figure 11</td><td colspan="2">–</td></tr>
<tr><td rowspan="2">$r_2\,r_2$</td><td rowspan="2">$W_2 = 0, W_1 > 0$</td><td>Figure 12</td><td colspan="2">$E_2 = 0$</td></tr>
<tr><td>Figure 13</td><td colspan="2">$E_2 \neq 0$</td></tr>
<tr><td>$c_2\,c_2$</td><td>$W_2 = 0, W_1 < 0$</td><td>Figure 11</td><td colspan="2">–</td></tr>
<tr><td rowspan="2">r_4</td><td rowspan="2">$\mu \neq 0, W_1 = 0$</td><td>Figure 14</td><td colspan="2">$\mu > 0$</td></tr>
<tr><td>Figure 18</td><td colspan="2">$\mu < 0$</td></tr>
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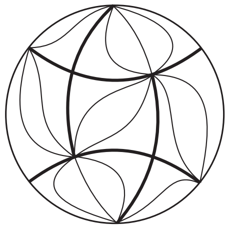
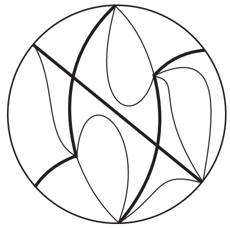


Figure 1



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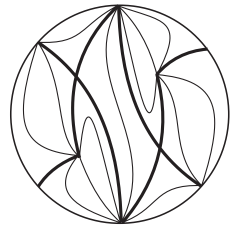

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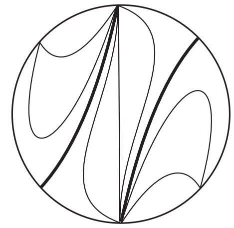

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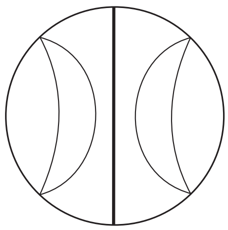

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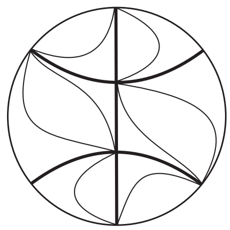

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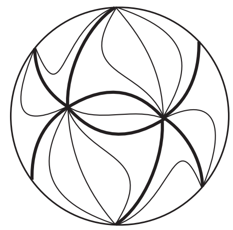

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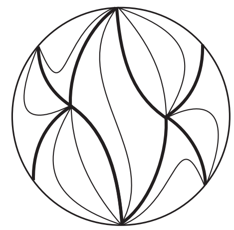

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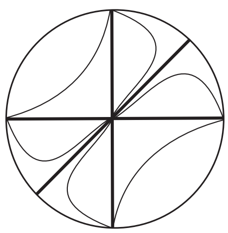

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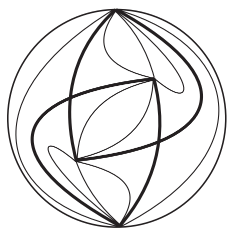

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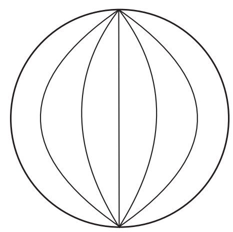

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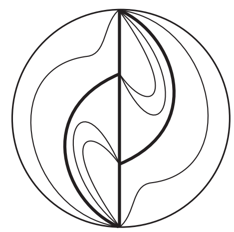

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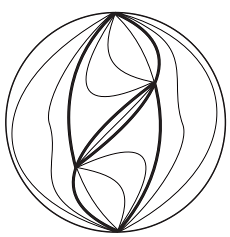

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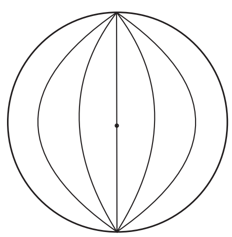

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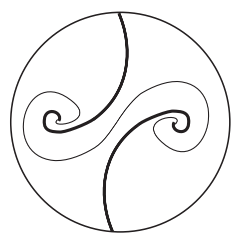
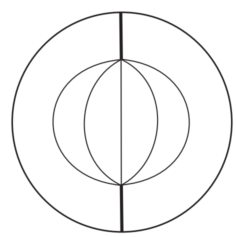


Figure 15



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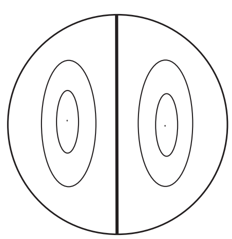

Figure 17

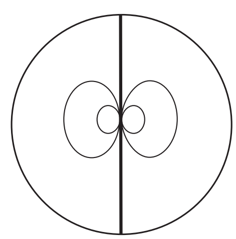

Figure 18

2 Some preliminary results

Proposition 1. [10] *System 1 has a single symmetry point if and only if either* $N_1(\mathbf{a},x,y) \neq 0$ *and* $G_1(\mathbf{a}) = N_2(\mathbf{a},x,y) = 0$ *or* $N_1(\mathbf{a},x,y) = N_3(\mathbf{a},x,y) = 0$,

$N_4(\mathbf{a}, x, y) \neq 0$; and it has an infinite number of such points if and only if $N_1(\mathbf{a}, x, y) = N_3(\mathbf{a}, x, y) = N_4(\mathbf{a}, x, y) = N_5(\mathbf{a}, x, y) = 0$.

Proposition 2. [16] *The number of distinct roots (real and imaginary) of the polynomial $C_2 = yp_2(x, y) - xq_2(x, y) \not\equiv 0$ is determined by the following conditions:*

- *3 real for $\eta > 0$;*
- *1 real and 2 imaginary for $\eta < 0$;*
- *2 real (one double and one simple) for $\eta = 0,\ M \neq 0$;*
- *1 real (triple) for $\eta = M = 0$.*

Proposition 3. [9] *The number and the types of the finite singular points of the non-homogeneous system (1) with a point of symmetry are determined in Table 1. The notations 'sdl', 'nod', 'sdl-nod', 'foc' and 'cnt' are used to denote saddle, node, saddle-node, focus, and center, respectively, and by $(\mathfrak{A}_1)$ we denote the following set of conditions:*

$$F_1 = F_3 = 0,\ F_2 \geq 0. \qquad (\mathfrak{A}_1)$$

The geometrical meaning of the condition $\kappa = 0$ is given by the next lemma.

Lemma 1. *Assume that for the quadratic system (1) with a point of symmetry and $C_2 \neq 0$ the condition $\kappa = 0$ holds. Then this system possesses two parallel invariant straight lines which can be: (a) real distinct; (b) imaginary; (c) coincided in the finite part of the phase plane; (c) coincided at infinity.*

Proof: We shall consider all the cases given by Proposition 2.

Case $\eta > 0$. Applying an affine transformation system (1) with a point of symmetry can be brought [16] to the canonical form

$$\dot{x} = a + gx^2 + (h-1)xy, \quad \dot{y} = b + (g-1)xy + hy^2.$$

For this system we have $\kappa = (1-g)(h-1)(g+h)/8$. So, the condition $\kappa = 0$ yields $(g-1)(h-1)(g+h) = 0$ and without loss of generality we may assume $h = 1$. Indeed, if $g = 1$ (respectively, $g + h = 0$) we can apply the linear transformation $x = y_1$, $y = x_1$ (respectively, $x = -y_1$, $y = x_1 - y_1$). Thus, $h = 1$ and we obtain the system

$$\dot{x} = a + gx^2, \quad \dot{y} = b + (g-1)xy + y^2 \qquad (2)$$

which, evidently, possesses two parallel invariant straight lines: $gx^2 + a = 0$ ($g^2 + a^2 \neq 0$). Clearly we obtain the case (a) (respectively, (b); (c); (d)) indicated in the statement of Lemma 1 when $ag < 0$ (respectively, $ag > 0$; $a = 0$; $g = 0$).

Case $\eta < 0$. According to [16] via an affine transformation system (1) can be brought to the canonical form

$$\dot{x} = a + gx^2 + (h+1)xy, \quad \dot{y} = b - x^2 + gxy + hy^2. \qquad (3)$$

Table 1

Singular points	*Affine invariant conditions*	*Characters*	*Additional conditions for determining characters*	
$r_1r_1r_1r_1$	$\mu>0, W_1>0, W_2>0$	*sdl, sdl, nod, nod*	$H_1 \geq 0$	
		sdl, sdl, foc, foc	$H_1 < 0,\ \neg(\mathfrak{A}_1)$	
		sdl, sdl, foc, cnt	$H_1 < 0,\quad (\mathfrak{A}_1)$	
$r_1r_1c_1c_1$	$\mu < 0,\ W_2 \neq 0$	*sdl, sdl*	$K < 0$	
		nod, nod	$K>0$	$H_1 < 0,\ H_3 > 0$
				$H_1=0, H_2<0, H_3>0$
				$H_1 = 0,\ H_2 \geq 0$
				$H_1 > 0,\ H_2 > 0$
		foc, foc	$K>0,\ \neg(\mathfrak{A}_1)$	$H_1 < 0,\ H_3 < 0$
				$H_1=0, H_2<0, H_3<0$
				$H_1 > 0,\ H_2 < 0$
		cnt, cnt	$K>0,\ (\mathfrak{A}_1)$	$H_1 < 0,\ H_3 < 0$
				$H_1=0, H_2<0, H_3<0$
				$H_1 > 0,\ H_2 < 0$
$c_1c_1c_1c_1$	$\mu > 0$ *and* $W_2 < 0$ *or* $W_2 > 0,\ W_1 \leq 0$	—	—	
r_2r_2	$\mu>0, W_1>0, W_2=0$	*sdl-nod, sdl-nod*	$F_2 \neq 0$	
		cusp, cusp	$F_2 = 0$	
c_2c_2	$\mu>0, W_1<0, W_2=0$	—	—	
r_4	$\mu\neq 0, W_1=0, W_2=0$	—	*Homogeneous system ([16])*	
r_1r_1	$\mu = 0,\ W_1 > 0$	*sdl, sdl*	$K < 0$	
		nod, nod	$K>0$	$H_1 \geq 0$
		foc, foc	$K>0$	$H_1 < 0,\ \neg(\mathfrak{A}_1)$
		cnt, cnt	$K>0$	$H_1 < 0,\ (\mathfrak{A}_1)$
c_1c_1	$\mu = 0,\ W_1 < 0$	—	—	
—	$\mu=0, W_1=0, V\neq 0$	—	*There are no singular points*	
—	$\mu=0, W_1=0, V=0$	—	*System is degenerate*	

For system (3) we have $\kappa = (h+1)\big[(h-1)^2 + g^2\big]/8$, and the condition $\kappa = 0$ yields two subcases: $h+1=0$ and $h-1=g=0$.

Subcase $h=-1$. The system (3) becomes $\dot{x} = a + gx^2,\quad \dot{y} = b - x^2 + gxy - y^2$, which has the parallel lines $a + gx^2 = 0$.

Subcase $h-1=g=0$. We obtain the system $\dot{x} = a + 2xy,\quad \dot{y} = b - x^2 + y^2$, which possesses the following two couples of imaginary invariant straight lines:

$$(x-iy)^2 = b + ia, \quad (x+iy)^2 = b - ia.$$

Case $\eta = 0,\ M \neq 0$**.** System (1) by means of an affine transformation can be brought [16] to the canonical form

$$\dot{x} = a + gx^2 + hxy, \quad \dot{y} = b + (g-1)xy + hy^2. \tag{4}$$

For this system we have $\kappa = h^2(1-g)/8$ and, hence, the condition $\kappa = 0$ implies either $g = 1$ or $h = 0$.

Subcase $g = 1$. Evidently, in this case system (4) possesses two parallel invariant straight lines $hy^2 + b = 0$ types of which are governed by parameters h and b.

Subcase $h = 0$. The system (4) becomes $\dot{x} = a + gx^2, \quad \dot{y} = b + (g-1)xy$, and again possesses the invariant straight lines $gx^2 + a = 0$.

Case $M = 0,\ C_2 \neq 0$. Via an affine transformation system (1) with a point of symmetry can be brought [16] to the canonical form

$$\dot{x} = a + gx^2 + hxy, \quad \dot{y} = b - x^2 + gxy + hy^2.$$

For this system we have $\kappa = h^3/8$ and the condition $\kappa = 0$ yields $h = 0$. This leads to the system $\dot{x} = a + gx^2, \quad \dot{y} = b - x^2 + gxy$, which possesses two parallel invariant straight lines $gx^2 + a = 0$. Lemma 1 is proved.

3 The proof of the Main Theorem

In what follows we assume that the condition $\kappa = 0$ is fulfilled.

3.1 Systems with 3 real roots of C_2

According to Proposition 2 the condition $\eta > 0$ holds. It was shown in the proof of Lemma 1 that in this case the quadratic system can be brought to the canonical form

$$\dot{x} = a + gx^2, \quad \dot{y} = b + (g-1)xy + y^2 \tag{5}$$

for which we have:

$$C_2 \equiv yp_2(x,y) - xq_2(x,y) = xy(x-y), \quad \mu = g^2.$$

Then we conclude that the intersection point of the line $x = 0$ (respectively, $y = 0$; $y = x$) with Poincaré's circumference is a real infinite singular point of system (5), which we will denote by $\widetilde{N}_1(0,1,0)$ (respectively, $\widetilde{N}_2(1,0,0)$; $\widetilde{N}_3(1,1,0)$). Since the conditions $\eta > 0$ and $\mu \neq 0$ are fulfilled in accordance with the paper [15] at infinity there exist one saddle and two nodes on the Poincaré circumference. In what follows we need to know where exactly the saddle is placed. So, by using the transformation $x = v/z,\ y = 1/x,\ dt = zd\tau$ system (5) will be brought to the system

$$\frac{dv}{d\tau} = -v + v^2 + az^2 - bvz^2, \quad \frac{dz}{d\tau} = -z + (1-g)vz - bz^3, \tag{6}$$

whereas applying the transformation $x = 1/z,\ y = u/z,\ dt = zd\tau$ we obtain the system

$$\frac{du}{d\tau} = -u + u^2 + bz^2 - auz^2, \quad \frac{dz}{d\tau} = -gz - az^3. \tag{7}$$

Clearly, the point $\widetilde{N}_1(0,1,0)$ corresponds to the singular point $(0,0)$ of system (6) and the point $\widetilde{N}_2(1,0,0)$ (respectively, $\widetilde{N}_3(1,1,0)$) corresponds to the singular point $(0,0)$ (respectively, $(1,0)$) of system (7).

Considering the eigenvalues of the corresponding linear matrix for each of these singular points we obtain, respectively:

$$\widetilde{N}_1(0,1,0): \ \lambda_1\lambda_2 = 1; \quad \widetilde{N}_2(1,0,0): \ \lambda_1\lambda_2 = g; \quad \widetilde{N}_3(0,1,0): \ \lambda_1\lambda_2 = -g.$$

Hence, we have the next affirmation:

Remark 1. *For system (5) with $\mu \neq 0$ the infinite singular point $N_1(0,1,0)$ is a node and the point $N_2(1,0,0)$ (respectively, $N_3(1,1,0)$) is a node (respectively, a saddle) for $g > 0$ and a saddle (respectively, a node) for $g < 0$.*

Let us emphasize some useful geometrical proprieties of system (5).

Remark 2. *For $g^2 - 1 = 0$ system (5) possesses two couples of parallel invariant straight lines. Moreover, one couple of parallel lines is directed to the node $N_1(0,1,0)$ and the second one is directed to the node $N_2(1,0,0)$ (respectively, node $N_3(1,1,0)$) for $g = 1$ (respectively, $g = -1$).*

Remark 3. *For $b = 0$ (respectively, $b = a$) system (5) possesses one invariant straight line which passes through the infinite singular point $N_2(1,0,0)$ (respectively, $N_3(1,1,0)$).*

For system (5) one can calculate

$$\begin{array}{l}
W_1 = -24g\left[a(g-1)^2 + 2bg\right]x^2 - 48ag\,(g-1)\,xy - 48\,agy^2,\\
W_2 = 2^7 3^3 ag^2\left[a(g-1)^2 + 4bg\right]\left[(g-1)\,x + 2\,y\right]^2 x^2,\\
H_1 = 2^5 3^4\left[a(g-1)(3g-1) - 4bg\right]^2, \quad F_2 = -4g^2,\\
Discrim(W_1) = -2^8 3^2 ag^2\left[a(g-1)^2 + 4bg\right], \quad \mu = g^2.
\end{array} \tag{8}$$

Case $W_2 > 0$. Then $a\left[a(g-1)^2 + 4bg\right] > 0$, and we obtain $Discrim(W_1) < 0$. Hence, the quadratic form $W_1(x,y)$ became sign definite. Moreover, by (8) we obtain $\text{sign}(W_1) = -\text{sign}(ag)$. Since $ag \neq 0$ by applying the transformation

$$x = \alpha x_1, \quad y = \alpha y_1, \quad t = \alpha^{-1} t_1, \quad (\alpha = \sqrt{|ag^{-1}|}), \tag{9}$$

system (5) can be brought to the following canonical form (we keep the previous notations):

$$\dot{x} = g\left(x^2 + Sign(ag)\right), \quad \dot{y} = b + (g-1)xy + y^2. \tag{10}$$

<u>*Subcase*</u> $W_1 > 0$. Then $ag < 0$ and system (10) becomes

$$\dot{x} = g\left(x^2 - 1\right), \quad \dot{y} = b + (g-1)xy + y^2. \tag{11}$$

This system possesses two parallel real invariant straight lines: $x = \pm 1$. Since by (8) we have $H_1 \geq 0$, according to Proposition 3 for $W_2 > 0$, $W_1 > 0$ and $H_1 \geq 0$

system (5) has 4 real singular points placed on the invariant straight lines $x = \pm 1$ and namely, two saddles and two nodes: $M_1^{\pm}\left(1, y_1^{\pm}\right)$, $\quad M_2^{\pm}\left(-1, y_2^{\pm}\right)$, where

$$y_1^{\pm} = \frac{1 - g \pm \sqrt{\Delta}}{2}, \quad y_2^{\pm} = \frac{g - 1 \pm \sqrt{\Delta}}{2}, \quad \Delta = (g-1)^2 - 4b.$$

We note that $\Delta > 0$ because of $W_2 > 0$. The symmetry of the vector field of system (11) implies the symmetry of the point M_1^+ with M_2^- as well as the symmetry of the point M_1^- with M_2^+. Thus, it is sufficient to determine only the types of the points $M_1^{\pm}$. It is not difficult to calculate the corresponding eigenvalues and to find out for each point:

$$M_1^+: \quad \lambda_1\lambda_2 = 2g\sqrt{\Delta}; \qquad M_1^-: \quad \lambda_1\lambda_2 = -2g\sqrt{\Delta}.$$

1) $g < 0$. Then the singular point M_1^+ (respectively, M_1^-) is a saddle (respectively, a node), and $y_1^+ > y_1^-$. Taking into account the coordinates of the singular points we observe that the straight line which connects the saddles M_1^+ and M_2^- will be

$$y = K_l x, \qquad K_l = \frac{1 - g + \sqrt{\Delta}}{2} = y_1^+ > 0.$$

Remark 4. *It is known ([24], Lemma 11.4) that if the line passing through two singular points of quadratic system is not an invariant straight line, then it must be a line without contact except singular points.*

In order to determine the position of the separatrices of the saddle M_1^+ with respect to the line $y = y_1^+ x$, we shall determine the direction of the proper vectors of the linear matrix corresponding to this singular point. So, besides the evident direction $x = 1$ we obtain the direction: $y = K_s x$, $\quad K_s = (1-g)y_1^+/(\sqrt{\Delta} - 2g) > 0$. It is not difficult to determine that for $g < 0$ the following relations hold:

$$\begin{array}{llll} K_s < K_l & iff \quad g \le -1 \ or \ -1 < g < 0,\ b < -g & \Leftrightarrow & Figure\ 1; \\ K_s = K_l & iff \quad -1 < g < 0,\ b = -g & \Leftrightarrow & Figure\ 2; \\ K_s > K_l & iff \quad -1 < g < 0,\ b > -g & \Leftrightarrow & Figure\ 3. \end{array} \tag{12}$$

We observe that for $b = -g$ we obtain $y_1^+ = 1$ and then the line $y = x$ becomes invariant straight line of system (11) which connects two saddles M_1^+ and M_2^-. Hence we obtain Figure 2.

Taking into consideration Remark 4 we conclude that inside the domain bounded by the invariant straight lines $x = \pm 1$ the separatrix will connect the saddle M_1^+ with the node M_1^+ for $K_s < K_l$ (Figure 1) and with the infinite node $N_1(0, 1, 0)$ for $K_s > K_l$ (Figure 3).

2) For $g > 0$ we obtain that the singular point M_1^+ (respectively, M_1^-) is a node (respectively, a saddle), and $y_1^+ > y_1^-$. In the same manner as above we can examine the directions of the separatrices for the saddle M_1^-. And it is not too hard to determine that for $(g-1)^2 - 2b > 0$ and $g > 0$ the corresponding phase portraits for the canonical system (11) will be realized if and only if the following conditions are fulfilled, respectively:

$$\begin{array}{lll} Figure\ 1 & iff & g \geq 1\ or\ 0 < g < 1\ and\ b < 0; \\ Figure\ 2 & iff & 0 < g < 1,\ b = 0; \\ Figure\ 3 & iff & 0 < g < 1,\ b > 0. \end{array} \tag{13}$$

It remains to find out the corresponding affine invariant conditions. For the system (11) we have

$$\begin{array}{l} E_1 = 384(2b+g)g^2(g^2-1), \quad N_1 = (g^2-1)x^2/4, \\ W_3 = -648b(b+g)(g^2-1)^2x^4. \end{array} \tag{14}$$

Taking into consideration (12), (13) and (14) it is not too difficult to obtain the following correspondence between Figures 1-3 and respective affine invariant conditions:

$$\begin{array}{lll} Figure\ 1 & iff & N_1 \geq 0\ or\ N_1 < 0\ and\ either\ W_3 < 0\ or\ W_3 \geq 0\ and\ E_1 > 0; \\ Figure\ 2 & iff & N_1 < 0,\ W_3 = 0,\ E_1 < 0; \\ Figure\ 3 & iff & N_1 < 0,\ W_3 > 0,\ E_1 < 0. \end{array}$$

<u>Subcase</u> $W_1 < 0$. Then $ag > 0$ and system (10) becomes

$$\dot{x} = g\left(x^2+1\right), \quad \dot{y} = b + (g-1)xy + y^2. \tag{15}$$

This system possesses two parallel imaginary invariant straight lines: $x = \pm i$ and it has no real singular points. For system (15) we have

$$\begin{array}{l} W_3 = 648b(g-b)(g^2-1)^2x^4, \\ E_1 = 384(g-2b)g^2(g^2-1), \\ N_1 = (g^2-1)x^2/4. \end{array} \tag{16}$$

1) We assume that the condition $N_1 \neq 0$ holds. By Remark 3 system (15) has one real invariant line for $b(b-g)$=0. Moreover, considering Remark 1 we obtain that this line will be a separatrix of infinite saddle if either $g < 0$ and $b = 0$ or $g > 0$ and $b = g$. By $N_1 \neq 0$ from (16) we obtain Figure 4 if either $W_3 \neq 0$ or $W_3 = 0$ and $N_1E_1 > 0$ and we obtain Figure 5 for $W_3 = 0$ and $N_1E_1 < 0$.

2) If $N_1 = 0$ then $g^2 - 1 = 0$. Since system (15) has a center of symmetry that a separatrix connection can be only if this separatrix is an invariant straight line. So, by Remark 2 we conclude that for $N_1 = 0$ the phase portrait of system (15) is given by Figure 4.

Case $W_2 < 0$**.** According to Proposition 3 system (5) has not real singular points and by (8) the condition $a\left[a(g-1)^2 + 4bg\right] < 0$ holds. Then system (5) has 2 parallel invariant straight lines $a + gx^2 = 0$ which connect two infinite nodes. So, we obtain the phase portrait given by Figure 5 (respectively, Figure 4) if there exists (respectively, does not exist) a separatrix connection of the infinite saddles. As it was mentioned above since system (5) has a center of symmetry then a separatrix connection can be only if this separatrix is an invariant straight line. For this system we have

$$\begin{array}{l} W_3 = 648b(a-b)(g^2-1)^2x^4, \quad N_1 = (g^2-1)x^2/4, \\ E_1 = 384ag(a-2b)(g^2-1), \end{array} \tag{17}$$

1) If $N_1 = 0$ then $g^2 - 1 = 0$ and by Remark 2 we obtain that there can not exist a separatrix connection. Therefore we get Figure 4.

2) We assume that the condition $N_1 \neq 0$ holds. According to Remark 1 for $g < 0$ (respectively, $g > 0$) the infinite saddle is located at the point $N_2(1,0,0)$ (respectively, $N_3(1,1,0)$). Therefore, by Remark 2 we obtain a separatrix connection if and only if either $b = 0$ and $g < 0$ or $b = a$ and $g > 0$. Taking into account (17) we conclude that the phase portrait of system (5) is given by Figure 5 for $W_3 = 0$ and $N_1E_1 < 0$ and it is given by Figure 5 if either $W_3 \neq 0$ or $W_3 = 0$ and $N_1E_1 > 0$.

Case $W_2 = 0$**.** Then $a\left[a(g-1)^2 + 4bg\right] = 0$, and according to (8) we obtain $Discrim(W_1) = 0$. Therefore, $W_1(x,y)$ became sign definite quadratic form and we shall consider three subcases: $W_1 > 0$, $W_1 < 0$ and $W_1 = 0$.

<u>*Subcase*</u> $W_1 > 0$. From (8) we obtain $g \neq 0$ and then $\mu > 0$ and $F_2 \neq 0$. By Proposition 3 system (5) has 2 double singular points which are saddle-nodes. For this system we have $N_1 = (g^2-1)x^2/4, \quad E_2 = -8ag(g^2-1)$.

1) If $N_1 = 0$ then $g^2 - 1 = 0$ and without loss of generality we can assume $g = 1$, otherwise the transformation $x_1 = -x$, $y_1 = y - x$ and $g \to -g$ which keeps canonical system (5) can be applied. Then we obtain the system

$$\dot{x} = a + x^2, \quad \dot{y} = b + y^2 \tag{18}$$

for which $W_2 = 2^{11}3^3abx^2y^2, \quad W_1 = -48(bx^2 + ay^2)$. Therefore, the conditions $W_2 = 0$ and $W_1 \neq 0$ yield $ab = 0$ and $a^2 + b^2 \neq 0$. We can assume $b = 0$ (via changing $x \leftrightarrow y$) and from $W_1 > 0$ we get $a < 0$. Thus, system (5) possesses 3 invariant lines $x = \pm\sqrt{-a}$ and $y = 0$ as well as 2 saddle-nodes $(\pm\sqrt{-a}, 0)$. So, we get the phase portrait given by Figure 6.

2) We assume now that the condition $N_1 \neq 0$ holds. Then $g^2 - 1 \neq 0$ and we shall consider two subcases: $E_2 = 0$ and $E_2 \neq 0$.

a) If $E_2 = 0$ then by $W_1N_1 \neq 0$ we obtain $a = 0$ (then $W_2 = 0$) and from (8) the condition $W_1 > 0$ yields $b < 0$. Then the saddle-nodes $(0, \pm\sqrt{-b})$ of system (5) are placed on the double invariant straight line $x = 0$. So, we get again Figure 6.

b) For $E_2 \neq 0$ we have $a \neq 0$ and the condition $W_2 = 0$ yields $a(g-1)^2 + 4bg = 0$. Since $g - 1 \neq 0$ we can substitute for b a new parameter u by setting $b = u(g-1)^2$ and then we have $a = -4gu$. Thus, we obtain the system

$$\dot{x} = -4gu + gx^2, \quad \dot{y} = u(g-1)^2 + (g-1)xy + y^2 \tag{19}$$

for which we have: $W_2 = 0, \quad W_1 = 48ug^2\left[(g-1)x + 2y\right]^2$. Hence, the condition $W_1 > 0$ yields $u > 0$. System (19) has 2 real invariant straight lines $x = \pm 2\sqrt{u}$ and two singular points which are saddle-nodes: $M_{1,2}(\pm 2\sqrt{u},\ \pm(1-g)\sqrt{u})$. We shall examine more detailed the singular point M_1. After the transformation

$$x_1 = x + \frac{4g}{(g-1)^2}y + \frac{2(g+1)\sqrt{u}}{g-1}, \quad y_1 = x - 2\sqrt{u} \quad \text{and} \quad t_1 = 4g\sqrt{u}\,t \tag{20}$$

which removes this point to the origin of coordinates, we obtain the standard [1] canonical system

$$\dot{x}_1 = \frac{1}{16g^2\sqrt{u}}[(g-1)x_1 + (g+1)y_1]^2, \quad \dot{y} = y_1 + \frac{1}{4\sqrt{u}}y_1^2. \tag{21}$$

Following [1] we obtain $\psi(x) = \tilde{\Delta}_2\, x^2 + \ldots\; = \dfrac{(g-1)^2}{16g^2\sqrt{u}}\, x^2 + \ldots$, and, hence, the semi-axis $y_1 = 0, x_1 < 0$ is one of the separatrices of the saddle-node $M_1(0,0)$ and other two separatrices are tangent to the axis $x_1 = 0$ at this point.

On the other hand the second saddle-node $M_2(x_0,\, y_0)$ of system (21) with coordinates $x_0 = 4(g+1)\sqrt{u}(g-1)$, $y_0 = -4\sqrt{u}$ is placed on the invariant line $y_1 = -4\sqrt{u}$ and $x_0 > 0$ for $g^2-1 > 0$ and $x_0 < 0$ for $g^2 - 1 < 0$. We observe that the transformation (20) removed infinite singular point as following:

$$\widetilde{N}_1(0,1,0) \to \widehat{N}_1(1,0,0);\ \widetilde{N}_2(1,0,0) \to \widehat{N}_2(1,1,0);\ \widetilde{N}_3(1,1,0) \to \widehat{N}_3\Big(1, \frac{(g-1)^2}{(g+1)^2}, 0\Big).$$

Thus, taking into consideration Remark 1 and the fact that according to Remark 4 M_0M_1 is a segment without contact, we obtain Figure 7 for $N_1 > 0$ and Figure 8 for $N_1 < 0$.

<u>*Subcase*</u> $W_1 < 0$. From (8) we obtain $g \neq 0$ and then $\mu > 0$. Then by Proposition 3 system (5) has 2 double imaginary singular points. Since system (5) has a center of symmetry then a separatrix connection can be only if this separatrix is an invariant straight line. We claim that this system can not possesses an invariant straight line as a separatrix. Indeed, by Remark 3 the condition $b(b-a) = 0$ must be satisfied. By (8) the condition $W_2 = 0$ yields $a\left[a(g-1)^2 + 4bg\right] = 0$. Then $a \neq 0$, otherwise for $a = 0$ the condition $b(b-a) = 0$ contradicts $W_1 = -48bg^2x^2 < 0$. Therefore, we obtain $a(g-1)^2 + 4bg = 0$.

If $b = 0$ we obtain $g = 1$ and by Remark 3 the invariant straight line $y = 0$ of system (5) connect two nodes. For $b = a$ we have $(g-1)^2 + 4g = (g+1)^2 = 0$, i.e. $g = -1$ and we again obtain that the invariant line $y = x$ connects two nodes. The claim is proved. Consequently, we get Figure 4.

<u>*Subcase*</u> $W_1 = 0$. From (8) we obtain $g\left[a(g-1)^2 + 2bg\right] = ag(g-1) = ag = 0$.

1) Assume $\mu \neq 0$. Then by (8) we have $g \neq 0$ and, hence, $a = b = 0$. Consequently, system (5) becomes quadratic homogeneous system, which according to Remark 1 has at infinity two nodes and one saddle. So, we get Figure 9.

1) For $\mu = 0$ from (8) we obtain $g = 0$ and system (5) becomes

$$\dot{x} = a, \quad \dot{y} = b - xy + y^2 \tag{22}$$

for which we have:

$$\begin{array}{l} \mu = W_1 = W_2 = 0, \quad V = a^2y^2(x-y)^2 \neq 0, \\ W_3 = 648b(a-b)x^4, \quad E_3 = 24(2b-a). \end{array} \tag{23}$$

Taking into consideration systems (6) and (7) (for $g = 0$) we conclude that the singular point $N_1(0,1,0)$ is a node, and according to [1] the triple singular point

$N_2(1,0,0)$ (respectively, $N_3(1,1,0)$) is a node (respectively, a saddle) for $a>0$ and a saddle (respectively, a node) for $a<0$.

By Remark 3 we conclude that system (22) has an invariant straight line which connects two infinite saddles if and only if either $b=0$ and $a<0$ or $b=a$ and $a>0$. So, considering (23) we obtain Figure 5 if $W_3=0$, $E_3>0$ and Figure 4 if either $W_3\neq 0$ or $W_3=0$ and $E_3<0$.

3.2 Systems with 1 real and 2 imaginary roots of C_2

According to Proposition 2 the condition $\eta>0$ holds and according to [16] the system can be brought to the canonical form

$$\dot{x}=a+gx^2+(h+1)xy,\quad \dot{y}=b-x^2+gxy+hy^2. \tag{24}$$

For this system we have

$$\begin{aligned}&\kappa=(h+1)\Big[(h-1)^2+g^2\Big]/8,\quad C_2\equiv yp_2(x,y)-xq_2(x,y)=x(x^2+y^2),\\&N_1=[(g^2-2h+2)x^2+2g(h+1)xy+(h^2-1)y^2]/4,\end{aligned} \tag{25}$$

and, hence, $N_1(0,1,0)$ is a real infinite singular point of this system. On the other hand the condition $\kappa=0$ yields two cases: $h+1=0$ and $h-1=g=0$ which are equivalent to $N_1\neq 0$ and $N_1=0$, respectively.

Case $N_1\neq 0$**.** Then $h=-1$ and we obtain the system

$$\dot{x}=a+gx^2,\quad \dot{y}=b-x^2+gxy-y^2, \tag{26}$$

for which

$$\begin{aligned}&W_1=-24g\left[a(g^2-2)-2bg\right]x^2+48ag^2xy-48\,agy^2,\\&W_2=2^7 3^3 ag^2\left[a(g^2-4)-4bg\right]\left[g\,x-2\,y\right]^2x^2,\\&H_1=2^5 3^4\left[3ag^2+4bg+4a\right]^2,\qquad F_2=-4g^2,\\&Discrim(W_1)=-2^8 3^2 ag^2\left[a(g^2-4)-4bg\right],\quad \mu=g^2.\end{aligned} \tag{27}$$

If $\mu\neq 0$ then from (27) it follows $\mu>0$ and since $\eta<0$ the singular point $N_1(0,1,0)$ is a node [15].

<u>*Subcase*</u> $W_2>0$. Then $a\left[a(g^2-4)-4bg\right]>0$ and by (27) we obtain $Discrim(W_1)<0$, and, hence, $\text{sign}(W_1)=-\text{sign}(ag)$. Since $ag\neq 0$ by applying the transformation (9) we get the system:

$$\dot{x}=g\left(x^2+Sign(ag)\right),\quad \dot{y}=b-x^2+gxy-y^2. \tag{28}$$

1) If $W_1>0$ then $ag<0$ and system (28) becomes

$$\dot{x}=g\left(x^2-1\right),\quad \dot{y}=b-x^2+gxy-y^2. \tag{29}$$

This system possesses two parallel real invariant straight lines: $x=\pm 1$. Since by (27) we have $H_1\geq 0$ according to Proposition 3 for $W_2>0$, $W_1>0$ and $H_1\geq 0$

system (26) has 4 real singular points located on the invariant straight lines $x = \pm 1$ and namely, two saddles and two nodes: $M_1^{\pm}\left(1, y_1^{\pm}\right)$, $M_2^{\pm}\left(-1, y_2^{\pm}\right)$, where

$$y_1^{\pm} = (g \pm \sqrt{\Delta})/2, \quad y_2^{\pm} = (-g \pm \sqrt{\Delta})/2, \quad \Delta = g^2 + 4b - 4 > 0.$$

For the singular points $M_1^{\pm}$ we have $\lambda_1\lambda_2 = \mp 2g\sqrt{\Delta}$. We can assume $g > 0$ via the transformation $y \leftrightarrow -y$ and $t \leftrightarrow -t$. In this case M_1^+ is a saddle and M_1^- is a node and $y_1^+ > y_1^-$. Taking into account the coordinates of the singular points we observe that the straight line $y = y_1^+ x$ connects the saddles M_1^+ and M_2^-.

On the other hand the directions of the separatrices of the saddle M_1^+ are $x = 1$ and $y = K_s x$, where $K_s = \dfrac{gy_1^+ - 2}{g + 2y_1^+}$. Therefore, $K_s - y_1^+ = -\left[\left(y_1^+\right)^2 + 2\right]/(g + 2y_1^+) < 0$ by $g > 0$. Thus, the located inside the domain $-1 < x < 1$ separatrix of the saddle M_1^+ by Remark 4 must connect this saddle with the node M_2^-. So, we get Figure 10.

2) Condition $W_1 < 0$ implies $ag > 0$ and system (28) has no real singular points. Taking into account the infinite node we obtain Figure 11.

<u>*Subcase*</u> $W_2 < 0$. According to Proposition 3 system (28) has no real singular points and we again get Figure 11.

<u>*Subcase*</u> $W_2 = 0$. Then $a\left[a(g^2 - 4) - 4bg\right] = 0$ and by (27) we obtain $Discrim(W_1) = 0$. Therefore, $W_1(x, y)$ became sign definite quadratic form and we shall consider three subcases: $W_1 > 0$, $W_1 < 0$ and $W_1 = 0$.

1) If $W_1 > 0$ then from (27) we obtain $g \neq 0$ and then $\mu > 0$ and $F_2 \neq 0$. By Proposition 3 system (26) has 2 double singular points which are saddle-nodes. For this system we have $E_2 = -8ag(g^2 + 4)$.

a) If $E_2 = 0$ then $a = 0$ and the saddle-nodes are located on the invariant straight line $x = 0$ of system (26). So, we obtain Figure 12.

b) For $E_2 \neq 0$ we have $ag \neq 0$ and the condition $W_1 > 0$ by (27) yields $ag < 0$. Then we obtain system (29) for which the condition $W_2 = 0$ yields $g^2 + 4b - 4 = 0$. Therefore, we get the system

$$\dot{x} = g\left(x^2 - 1\right), \quad \dot{y} = 1 - g^2/4 - x^2 + gxy - y^2 \tag{30}$$

with two real invariant straight lines $x = \pm 1$ and two saddle-nodes $M_1(1, g/2)$ and $M_2(-1, -g/2)$. We can assume $g > 0$, otherwise the substitution $y \leftrightarrow -y$, $t \leftrightarrow -t$ and $g \leftrightarrow -g$ can by applied. On the line $y = gx/2$ which connects singular points M_1 and M_2 we have $dy/dx = (g^2 - 4)/(4g)$ and by $g > 0$ we obtain $(g^2 - 4)/(4g) - g/2 = -(g^2 + 4)/(4g) < 0$. Consequently, we get Figure 13.

2) For $W_1 < 0$ according to Proposition 3 system (28) has no real singular points and we obtain Figure 11.

3) Assume $W_1 = 0$. From (27) we obtain $g\left[a(g^2 - 2) - 2bg\right] = ag = 0$.

a) For $\mu \neq 0$ we have $g \neq 0$ and, hence, $a = b = 0$. Consequently, system (26) becomes a quadratic homogeneous system which has a unique real infinite singular point (a node). So, we get Figure 14.

b) If $\mu = 0$ then from (27) we obtain $g = 0$ and system (26) becomes

$$\dot{x} = a, \quad \dot{y} = b - x^2 - y^2$$

which has not finite singular points and has one real simple infinite point (a node). Therefore we obtain Figure 11.

Case $N_1 = 0$**.** Then by (25) we have $h-1 = g = 0$ and we obtain the system

$$\dot{x} = a + 2xy, \quad \dot{y} = b - x^2 + y^2, \tag{31}$$

for which

$$\begin{array}{l} W_1 = -96(bx^2 - 2axy + by^2), \quad W_2 = 2^{11}3^3\left(a^2+b^2\right)\left(x^2+y^2\right)^2, \\ \mu = -4, \quad K = x^2+y^2, \quad F_1 = 2^{13}3^4a^2 = H_1, \\ F_2 = 16 = -H_2, \quad H_3 = -2^9 b, \quad F_3 = -192a. \end{array} \tag{32}$$

Since $\mu < 0$ and $\eta < 0$ the singular point $N_1(0,1,0)$ is a saddle (see,[15]).

<u>*Subcase*</u> $W_2 \neq 0$. Then according to Proposition 3 system (31) has 2 real and 2 imaginary singular points.

1) If $H_1 \neq 0$ by (32) we have $H_1 > 0$ and since $K > 0$ and $H_2 < 0$ according to Proposition 3 the real points of system (31) are foci. We claim that this system can not possess limit cycles. Indeed, condition $H_1 \neq 0$ yields $a \neq 0$ and via the transformation

$$x_1 = sign(a)\,|a|^{-1/2}x, \quad y_1 = |a|^{-1/2}y, \quad t_1 = |a|^{1/2}x, \quad b = c^2 - \frac{1}{4c^2}$$

system (31) becomes

$$\dot{x} = 1 + 2xy, \quad \dot{y} = c^2 - \frac{1}{4c^2} - x^2 + y^2. \tag{33}$$

This system possesses the following two couples of parallel imaginary invariant straight lines:

$$x - iy = \pm\frac{2c^2+i}{2c}, \quad x + iy = \pm\frac{2c^2-i}{2c}.$$

Following [6] we construct the first integral of system (33) in the complex form:

$$\big(x-iy-c-\frac{i}{2c}\big)^{i-2c^2}\big(x-iy+c+\frac{i}{2c}\big)^{2c^2-i}\big(x+iy-c+\frac{i}{2c}\big)^{-i-2c^2}\big(x+iy+c-\frac{i}{2c}\big)^{i+2c^2}.$$

Then the corresponding real first integral of system (33) can be constructed:

$$\exp\Big[-2arctg\Big(\frac{4cx+8c^3y}{1+4c^4-4c^2(x^2+y^2)}\Big)\Big]\left(\frac{1+4c^4+8c^3x-4cy+4c^2(x^2+y^2)}{1+4c^4-8c^3x+4cy+4c^2(x^2+y^2)}\right)^{2c^2}.$$

Since the curve $1+4c^4-4c^2(x^2+y^2) = 0$ is not a particular solution of system (33) and the identity

$$1+4c^2-8c^3x+4cy+4c^2(x^2+y^2) = 4c^2\big(x-iy-c-\frac{i}{2c}\big)\big(x+iy-c+\frac{i}{2c}\big)$$

holds, we conclude that our claim is proved.

Taking into account that the line $x = 0$ is not an invariant straight line of system (33) we obtain Figure 15.

2) Assume $H_1 = 0$. Then $a = 0$ and by Proposition 3 system (31) has two nodes located on the invariant straight line $x = 0$ for $H_3 > 0$ (Figure 16) and it has two centers for $H_3 < 0$ (Figure 17).

Subcase $W_2 = 0$. By (32) we have $a = b = 0$ and system (31) becomes a homogeneous system with one real invariant straight line which is a separatrix of the saddle $N_1(0, 1, 0)$. Therefore, we obtain Figure 18.

In order to obtain the respective to the case $\eta < 0$ conditions from Table 1 the following Remark has to be taking into account:

Remark 5. *For system (26) with $\kappa = 0$ from (27) and (32) we obtain:*
- *condition $\mu < 0$ is equivalent to $N_1 = 0$;*
- *conditions $W_2 = 0,\ W_1 \neq 0$ implies $\mu > 0$;*
- *condition $W_1 = 0$ implies $W_2 = 0$.*

The Main Theorem is proved.

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