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# Optimal control for one complex dynamic system, I

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**Abstract.** The optimal control problem of the metal solidification in casting is considered. The process is modeled by a three-dimensional two-phase initial-boundary value problem of the Stefan type. A numerical algorithm is presented for solving the direct problem. The optimal control problem was solved numerically using the gradient method. The gradient of the cost function was found with the help of conjugate problem. The discreet conjugate problem was posed with the help of Fast Automatic Differentiation technique.

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## 1 Introduction

An important class of heat transfer problems is that describing processes in which the substance under study undergoes phase transitions accompanied by heat release or absorption (Stefan problems). A key feature of these problems is that they involve a moving interface between two phases (liquid and solid). The law of motion of the interface is unknown in advance and is to be determined. It is on this interface that heat release or absorption associated with phase transitions occurs. The thermal properties of the substance on the different sides of the moving interface can be different. Problems of this class are noticeably more complicated than those not involving phase transitions.

We consider an interesting problem of this class, namely, the optimal control of the process of solidification in metal casting. Figure 1 shows the experimental setup for metal solidification. It consists of upper and lower parts. The upper part consists of a furnace with a mold moving inside. The lower part is a cooling bath consisting of a large tank filled with liquid aluminum whose temperature is somewhat higher than the aluminum melting point. The cooling of liquid metal in the furnace proceeds as follows. On the one hand, the mold is slowly immersed in the low-temperature liquid aluminum, which causes the solidification of the metal. On the other hand, the mold gains heat from the walls of the furnace, which prevents the solidification process from proceeding too fast. The optimal control problem is to choose a regime of metal cooling and solidification at which the solidification front has a preset shape (or is close to it) and moves sufficiently slowly (at a speed close to the preset one).

An important part of the optimal control problem is the direct problem (of finding the temperature at each point of the metal and determining the solidification front). We describe the mathematical formulation of the direct problem, its finite-difference approximation, and a numerical algorithm for solving the direct problem. The problem was studied for an object of the simplest shape (a parallelepiped) and for an actual object of practice interest (Fig. 2). While discussing the numerical results, we give primary attention to the evolution of the solidification front and to how it is affected by the parameters of the problem.

The control function was approximated by a piecewise constant function. The minimum value of a cost function was finding numerically with use of gradient methods. The gradient of the cost function was found with the help of conjugate problem. The discreet conjugate problem was posed with the help of Fast Automatic Differentiation technique.

## 2 Mathematical formulation of the problem

The following optimal control problem of metal solidification in casting is considered.

A mold with specified outer and inner boundaries is filled with liquid metal (the longitudinal projections of an actual mold are presented in Fig. 2). The hatched area in the Fig. 2 depicts the mold wall, and the internal unhatched area shows the inside space filled with metal. The mold and the metal inside it are heated up to prescribed temperatures $T_{form}$ and $T_{met}$, respectively. Next, the mold filled with metal (which is hereafter referred to as the object) begins to cool gradually under varying surrounding conditions. The different parts of the mold's outer boundary are under different thermal conditions (the laws of heat transfer with the surroundings are different in these parts). Moreover, the thermal conditions affecting the parts vary with time.

The process of cooling the object is described by the equation:

$$\rho C \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(K \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(K \frac{\partial T}{\partial z}\right), \qquad (x, y, z) \in Q. \tag{1}$$

Here $x$, $y$, and $z$ are the Cartesian coordinates of a point; $t$ is time; $Q$ is a domain with a piecewise smooth boundary $\Gamma$; $T(x, y, z, t)$ is the substance temperature at the point with coordinates $(x, y, z)$ at time $t$; $\rho$, $C$ and $K$ are the density, heat capacity, and thermal conductivity of the substance respectively.

The conditions of heat transfer with the surrounding medium are set on the boundary $\Gamma$ of $Q$. As was mentioned above, these conditions depend on the given surface point and time. However, all the heat transfer conditions can be written in the general form:

$$\tilde{\alpha} T + \tilde{\beta} T_{\mathbf{n}} = \tilde{\gamma}. \tag{2}$$

Here $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\gamma}$ are given functions of the coordinates $(x, y, z)$ of a point on $\Gamma$ and the temperature $T(x, y, z, t)$, and $\dfrac{\partial T}{\partial \mathbf{n}} = T_{\mathbf{n}}$ is the derivative of $T$ in the direction $\mathbf{n}$ – the external normal to the surface $\Gamma$. It should be noted that the coefficients

$\rho$, $C$ and $K$ in (1) and (2) are different for the metal and the mold. They have the form:

$$K(T) = \begin{cases} K_1(T), & (x,y,z) \in \text{metal}, \\ K_2(T), & (x,y,z) \in \text{mold}, \end{cases}$$

$$K_1(T) = \begin{cases} k_S, & T < T_1, \\ \dfrac{k_L - k_S}{T_2 - T_1} T + \dfrac{k_S T_2 - k_L T_1}{T_2 - T_1}, & T_1 \le T < T_2, \\ k_L, & T \ge T_2, \end{cases}$$

$$K_2(T) = \begin{cases} k_{\Phi_1}, & T \le T_3, \\ k_{\Phi_2}, & T > T_3, \end{cases}$$

$$\rho(T) = \begin{cases} \rho_1(T), & (x,y,z) \in \text{metal}, \\ \rho_\Phi, & (x,y,z) \in \text{mold}, \end{cases} \qquad \rho_1(T) = \begin{cases} \rho_S, & T < T_1, \\ \rho_L, & T \ge T_2, \end{cases}$$

$$C(T) = \begin{cases} C_1(T), & (x,y,z) \in \text{metal}, \\ c_\Phi, & (x,y,z) \in \text{mold}, \end{cases} \qquad C_1(T) = \begin{cases} c_S, & T < T_1, \\ c_L, & T \ge T_2. \end{cases}$$

The constants $c_S$, $c_L$, $c_\Phi$, $\rho_S$, $\rho_L$, $\rho_\Phi$, $k_S$, $k_L$, $k_{\Phi_1}$, $k_{\Phi_2}$, $T_1$, $T_2$, and $T_3$ in these formulas are assumed to be known.

It should be noted that the thermodynamic coefficients have a jump at the metal-mold interface. Two conditions are set at this surface, namely, the temperature and the heat flux must be continuous.

Note also that the metal can be simultaneously in two phases: solid and liquid. The domain separating the phases is determined by a narrow range of temperatures $[T_1, T_2]$, in which $\rho$, $C$ and $K$ change very rapidly.

Thus, the solution to the direct problem consists in determining a function $T(x,y,z,t)$ that satisfies Eq. (1) in $Q$, conditions (2) on the outer boundary $\Gamma$ of $Q$, and the continuity conditions for the temperature and the heat flux at the metal-mold interface.

The optimal control problem is to choose a regime of metal cooling and solidification at which the solidification front has a preset shape or is close to it (namely, a plane orthogonal to the vertical axis of the object) and moves sufficiently slowly (at a speed close to the preset one). The evolution of the solidification front is affected by numerous parameters (for example, by the furnace temperature, the liquid aluminum temperature, the depth to which the object is immersed in the liquid aluminum, the speed at which the mold moves relative to the furnace, etc.). The solidification front as a function of the velocity of the object is of special interest in practice.

The speed $\widetilde{u}(t)$ of the displacement of foundry mold in the melting furnace was chosen as the control $U(t)$. The cost function is next:

$$I(U) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \iint_S [Z_{pl}(x,y,t) - z_*(t)]^2 \, dx dy dt. \qquad (3)$$

Here $t_1$ is the time, when the crystallization front is conceived; $t_2$ is the time, when the crystallization of metal completes; $(x, y, Z_{pl}(x, y, t))$ are the real coordinates of the interface at the time $t$; $(x, y, Z_*(t))$ are the desired coordinates of the interface at the time $t$; $S$ is the cross section of the mold which is filled by metal. The control function may be restricted by some prescribed functions $U_1(t)$ and $U_2(t)$: $U_1(t) \leq U(t) \leq U_2(t)$.

## 3 Numerical algorithm for solving the direct problem

The time's grid is introduced by relations: $\{t^j\}, \quad j = \overline{0, J}$, with the mesh sizes $\tau^j = t^j - t^{j-1}, \quad j = \overline{1, J}$.

The object being investigated is approximated by the body, which consists of a finite number of rectangular parallelepipeds. The approximating body is placed wholly into a certain large parallelepiped. For convenience in the further consideration let us introduce the coordinate system, connected with the moving foundry mold (see, Fig. 1). Axis $Oz$ let direct vertically upward, the axis $Ox$ will arrange in the horizontal plane and will direct from left to right, and the axis $Oy$ let select so the coordinate system $Oxyz$ would be right. The beginning $O$ of this coordinate system is compatible with the left, nearest to us vertex of the large parallelepiped, situated on its bottom. In this large parallelepiped a basic non-uniform rectangular grid is introduced:

$$\{x_n\},\ n = \overline{0, N}; \qquad \{y_i\},\ i = \overline{0, I}; \qquad \{z_l\},\ l = \overline{0, L};$$

with the mesh sizes: $h_n^x = x_{n+1} - x_n,\ \ n = \overline{0, N-1};\ \ \ h_i^y = y_{i+1} - y_i,\ \ i = \overline{0, I-1}$; $h_l^z = z_{l+1} - z_l,\ \ l = \overline{0, L-1}$. This grid is introduced in such a way that all external surfaces of the approximating body, and also surfaces which separate metal and form would coincide with the grid surfaces.

Besides the basic grid, the auxiliary grid is built whose surfaces are parallel to the surfaces of the basic grid and are displaced relative to it with a half-step in all directions:

$$\widetilde{x}_0 = x_0; \quad \widetilde{x}_n = x_{n-1} + h_{n-1}^x/2; \quad n = \overline{1, N}; \quad \widetilde{x}_{N+1} = x_N;$$

$$\widetilde{y}_0 = y_0; \quad \widetilde{y}_i = y_{i-1} + h_{i-1}^y/2; \quad i = \overline{1, I}; \quad \widetilde{y}_{I+1} = y_I;$$

$$\widetilde{z}_0 = z_0; \quad \widetilde{z}_l = z_{l-1} + h_{l-1}^z/2; \quad l = \overline{1, L}; \quad \widetilde{z}_{L+1} = z_L.$$

The planes $x = \widetilde{x}_n, y = \widetilde{y}_i$, and $z = \widetilde{z}_l$ split the object into elementary volumes, or elementary cells. An elementary cell is assigned by the indices $(n, i, l)$ if the cell's vertex nearest to the coordinates origin $O$ coincides with the nodal point $(\widetilde{x}_n, \widetilde{y}_i, \widetilde{z}_l)$. The volume of such an elementary cell is denoted by $V_{nil}$ and its outer surface is denoted by $S_{nil}$.

Let us assume that the temperature of the medium within an elementary cell is independent of the spatial coordinates but depends on time. Denote this temperature by $T_{nil}(t)$.

Any elementary cell is either completely filled with a single medium (metal or mold) or some part is filled with one medium and the remaining part with the other. Let $V^1_{nil}$ denote the part of $V_{nil}$ filled with the metal and $V^2_{nil}$ denote the part of $V_{nil}$ filled with the mold material. Similarly, $S^1_{nil}$ is the part of $S_{nil}$ that is adjacent to $V^1_{nil}$ and $S^2_{nil}$ is the part of $S_{nil}$ that is adjacent to $V^2_{nil}$.

The algorithm that solves the direct problem is based on the heat balance law and on the reformulation from the problem in terms of temperature to terms of enthalpy.

For any volume $V$ with outer boundary $S$, we have the heat balance law

$$\iiint\limits_V \left[H\left(T(x,y,z,t^{j+1})\right) - H\left(T(x,y,z,t^j)\right)\right] dV = \int\limits_{t^j}^{t^{j+1}} \iint\limits_S K(T) T_{\mathbf{n}} ds dt. \tag{4}$$

Here, $H\left(T(x,y,z,t)\right)$ is the enthalpy function defined as:

$$H\left(T(x,y,z,t)\right) = \begin{cases} H_1(T), & (x,y,z) \in \text{metal}, \\ H_2(T), & (x,y,z) \in \text{mold}, \end{cases}$$

$$H_1(T) = \begin{cases} \rho_S c_S T, & T < T_1, \\ \dfrac{\rho_S c_S (T_2 - T_1) + \rho_S \gamma}{T_2 - T_1} T - \dfrac{\rho_S \gamma T_1}{T_2 - T_1}, & T_1 \le T < T_2, \\ \rho_L c_L (T - T_2) + \rho_S c_S T_2 + \rho_S \gamma, & T \ge T_2, \end{cases} \tag{5}$$

$$H_2(T) = \rho_\Phi c_\Phi T. \tag{6}$$

Then relation (4) written for an elementary cell indexed by $(n,i,l)$ becomes:

$$\left[V^1_{nil} H_1\left(T^{j+1}_{nil}\right) + V^2_{nil} H_2\left(T^{j+1}_{nil}\right)\right] - \left[V^1_{nil} H_1\left(T^j_{nil}\right) + V^2_{nil} H_2\left(T^j_{nil}\right)\right] =$$
$$= \int\limits_{t^j}^{t^{j+1}} \left[\iint\limits_{S^1_{nil}} K_1(\widetilde{T}_{nil}(t))(\widetilde{T}_{\mathbf{n}}(t))_{nil} ds + \iint\limits_{S^2_{nil}} K_2(\widetilde{T}_{nil}(t))(\widetilde{T}_{\mathbf{n}}(t))_{nil} ds\right] dt. \tag{7}$$

Here $T^j_{nil} = T_{nil}(t^j)$, $K_1(\widetilde{T}_{nil}(t))(\widetilde{T}_{\mathbf{n}}(t))_{nil}$, and $K_2(\widetilde{T}_{nil}(t))(\widetilde{T}_{\mathbf{n}}(t))_{nil}$ are the heat flux densities through the cell surface.

The subsequent transformations of (7) are similar to those proposed in [1–3] and further developed in [4–7].

Let $M_{nil} = V^1_{nil}/V_{nil}$ be the metal volume fraction in the elementary cell indexed by $(n,i,l)$ and $\Phi_{nil} = V^2_{nil}/V_{nil}$ be the mold volume fraction in this elementary cell.

Define the aggregate enthalpy density in the cell indexed by $(n,i,l)$ at the time $t^j$ as $E^j_{nil} = M_{nil} H_1(T^j_{nil}) + \Phi_{nil} H_2(T^j_{nil})$. Taking into account (5) and (6), which define $H_1(T)$ and $H_2(T)$, we obtain an expression for $E(T^j_{nil})$:

$$E^j_{nil} \equiv E(T^j_{nil}) = \begin{cases} a_{nil} T^j_{nil}, & T^j_{nil} < T_1, \\ b^1_{nil} T^j_{nil} - b^2_{nil}, & T_1 \le T^j_{nil} < T_2, \\ d^1_{nil} T^j_{nil} + d^2_{nil}, & T^j_{nil} \ge T_2, \end{cases}$$

where

$$a_{nil} = M_{nil}\rho_S c_S + \Phi_{nil}\rho_\Phi c_\Phi,$$

$$b^1_{nil} = M_{nil}(\rho_S c_S + \rho_S\gamma/(T_2 - T_1)) + \Phi_{nil}\rho_\Phi c_\Phi, \quad b^2_{nil} = M_{nil}\rho_S\gamma T_1/(T_2 - T_1),$$

$$d^1_{nil} = M_{nil}\rho_L c_L + \Phi_{nil}\rho_\Phi c_\Phi, \qquad d^2_{nil} = M_{nil}(\rho_S\gamma + (\rho_S c_S - \rho_L c_L)T_2).$$

Now, the temperature is defined as a function of $E^j_{nil}$ (this function is the inverse of $E(T^j_{nil})$):

$$T^j_{nil} \equiv \beta(E^j_{nil}) = \beta^j_{nil} = \begin{cases} \dfrac{1}{a_{nil}} E^j_{nil}, & E^j_{nil} < a_{nil}T_1, \\ \dfrac{1}{b^1_{nil}} E^j_{nil} + \dfrac{b^2_{nil}}{b^1_{nil}}, & a_{nil}T_1 \le E^j_{nil} < d^1_{nil}T_2 + d^2_{nil}, \\ \dfrac{1}{d^1_{nil}} E^j_{nil} - \dfrac{d^2_{nil}}{d^1_{nil}}, & E^j_{nil} \ge d^1_{nil}T_2 + d^2_{nil}. \end{cases}$$

The functions $K_1(T^j_{nil})$ and $K_2(T^j_{nil})$ can be expressed in terms of enthalpy:

$$K_1(T^j_{nil}) \equiv \Omega_1(E^j_{nil}) = \begin{cases} k_S, & E^j_{nil} < \rho_S c_S T_1 \equiv E_1, \\ \dfrac{k_L - k_S}{E_2 - E_1} E^j_{nil} + \dfrac{k_S E_2 - k_L E_1}{E_2 - E_1}, & E_1 \le E^j_{nil} < \rho_S(c_S T_2 + \gamma) \equiv E_2, \\ k_L, & E^j_{nil} \ge E_2, \end{cases}$$

$$K_2(T^j_{nil}) \equiv \Omega_2(E^j_{nil}) = \begin{cases} k_{\Phi_1}, & E^j_{nil} < \rho_\Phi c_\Phi(T_3 - \delta) \equiv E_3, \\ \dfrac{k_{\Phi_2} - k_{\Phi_1}}{E_4 - E_3} E^j_{nil} + \dfrac{k_{\Phi_1} E_4 - k_{\Phi_2} E_3}{E_4 - E_3}, & E_3 \le E^j_{nil} < \rho_\Phi c_\Phi(T_3 + \delta) \equiv E_4, \\ k_{\Phi_2}, & E^j_{nil} \ge E_4, \end{cases}$$

where $\delta << T_3$.

In (7), we proceed from the variable $T_{nil}(t)$ to $E_{nil}(t)$ and obtain:

$$V_{nil} \cdot (E^{j+1}_{nil} - E^j_{nil}) = \int\limits_{t^j}^{t^{j+1}} \left[ \iint\limits_{S^1_{nil}} A_1(\widetilde{E}_{nil}(t))ds + \iint\limits_{S^2_{nil}} A_2(\widetilde{E}_{nil}(t))ds \right] dt, \tag{8}$$

where $A_1(\widetilde{E}_{nil}(t)) = \Omega_1(\widetilde{E}_{nil}(t))\beta_{\mathbf{n}}(\widetilde{E}_{nil}(t))$, $\quad A_2(\widetilde{E}_{nil}(t)) = \Omega_2(\widetilde{E}_{nil}(t))\beta_{\mathbf{n}}(\widetilde{E}_{nil}(t))$.

We introduce the notation

$$E^{j+1/3}_{nil} = E_{nil}\left(t^j + \frac{\tau^{j+1}}{3}\right), \quad E^{j+2/3}_{nil} = E_{nil}\left(t^j + \frac{2\tau^{j+1}}{3}\right).$$

The time discretization of Eq.(8) is based on the Peaceman-Rachford scheme (see [8]):

$$V_{nil} \cdot (E^{j+1}_{nil} - E^j_{nil}) = \frac{2\tau^{j+1}}{3} \left[ \iint\limits_{S^{1x+}_{nil} \bigcup S^{1x-}_{nil}} A_1(E^{j+\frac{1}{3}}_{nil})ds + \iint\limits_{S^{2x+}_{nil} \bigcup S^{2x-}_{nil}} A_2(E^{j+\frac{1}{3}}_{nil})ds \right] + \frac{\tau^{j+1}}{3} \times$$

$$\times\left[\iint\limits_{S_{nil}^{1x+}\bigcup S_{nil}^{1x-}} A_1(E_{nil}^{j+\frac{2}{3}})ds + \iint\limits_{S_{nil}^{2x+}\bigcup S_{nil}^{2x-}} A_2(E_{nil}^{j+\frac{2}{3}})ds + \iint\limits_{S_{nil}^{1y+}\bigcup S_{nil}^{1y-}} A_1(E_{nil}^{j})ds + \iint\limits_{S_{nil}^{2y+}\bigcup S_{nil}^{2y-}} A_2(E_{nil}^{j})ds\right] +$$

$$+\frac{2\tau^{j+1}}{3}\left[\iint\limits_{S_{nil}^{1y+}\bigcup S_{nil}^{1y-}} A_1(E_{nil}^{j+\frac{2}{3}})ds + \iint\limits_{S_{nil}^{2y+}\bigcup S_{nil}^{2y-}} A_2(E_{nil}^{j+\frac{2}{3}})ds\right] + \tag{9}$$

$$+\frac{\tau^{j+1}}{3}\left[\iint\limits_{S_{nil}^{1z+}\bigcup S_{nil}^{1z-}} A_1(E_{nil}^{j})ds + \iint\limits_{S_{nil}^{2z+}\bigcup S_{nil}^{2z-}} A_2(E_{nil}^{j})ds + \iint\limits_{S_{nil}^{1z+}\bigcup S_{nil}^{1z-}} A_1(E_{nil}^{j+\frac{1}{3}})ds + \right.$$

$$\left. + \int\limits_{S_{nil}^{2z+}\bigcup S_{nil}^{2z-}} A_2(E_{nil}^{j+\frac{1}{3}})ds + \int\limits_{S_{nil}^{1z+}\bigcup S_{nil}^{1z-}} A_1(E_{nil}^{j+1})ds + \int\limits_{S_{nil}^{2z+}\bigcup S_{nil}^{2z-}} A_2(E_{nil}^{j+1})ds\right].$$

Here $S_{nil}^{1x+}$ denotes the part of $S_{nil}^{1}$ that belongs to the plane $x = \widetilde{x}_{n+1}$ and $S_{nil}^{1x-}$ denotes the part of $S_{nil}^{1}$ that belongs to the plane $x = \widetilde{x}_n$. $S_{nil}^{1y+}, ..., S_{nil}^{1z-}, S_{nil}^{2x+}, ..., S_{nil}^{2z-}$ are defined in a similar fashion.

We simultaneously add and subtract $V_{nil}E_{nil}^{j+1/3}$ and $V_{nil}E_{nil}^{j+2/3}$ on the left-hand side of (9) and split this equation into three (with respect to the directions $x$, $y$ and $z$), thus, forming the following three subproblems:

$$\left(j=\overline{0,J-1},\ n=\overline{0,N},\ i=\overline{0,I},\ l=\overline{0,L}\right)$$

**x − direction**

$$V_{nil}\cdot(E_{nil}^{j+\frac{1}{3}} - E_{nil}^{j}) = \frac{\tau^{j+1}}{3}\left[\iint\limits_{S_{nil}^{1x+}\bigcup S_{nil}^{1x-}} A_1(E_{nil}^{j+\frac{1}{3}})ds + \iint\limits_{S_{nil}^{2x+}\bigcup S_{nil}^{2x-}} A_2(E_{nil}^{j+\frac{1}{3}})ds + \right.$$

$$\left. + \iint\limits_{S_{nil}^{1y+}\bigcup S_{nil}^{1y-}} A_1(E_{nil}^{j})ds + \iint\limits_{S_{nil}^{2y+}\bigcup S_{nil}^{2y-}} A_2(E_{nil}^{j})ds + \iint\limits_{S_{nil}^{1z+}\bigcup S_{nil}^{1z-}} A_1(E_{nil}^{j})ds + \iint\limits_{S_{nil}^{2z+}\bigcup S_{nil}^{2z-}} A_2(E_{nil}^{j})ds\right];$$

**y − direction**

$$V_{nil}\cdot(E_{nil}^{j+\frac{2}{3}} - E_{nil}^{j+\frac{1}{3}}) = \frac{\tau^{j+1}}{3}\left[\iint\limits_{S_{nil}^{1y+}\bigcup S_{nil}^{1y-}} A_1(E_{nil}^{j+\frac{2}{3}})ds + \iint\limits_{S_{nil}^{2y+}\bigcup S_{nil}^{2y-}} A_2(E_{nil}^{j+\frac{2}{3}})ds + \right.$$

$$\left. + \iint\limits_{S_{nil}^{1x+}\bigcup S_{nil}^{1x-}} A_1(E_{nil}^{j+\frac{1}{3}})ds + \iint\limits_{S_{nil}^{2x+}\bigcup S_{nil}^{2x-}} A_2(E_{nil}^{j+\frac{1}{3}})ds + \iint\limits_{S_{nil}^{1z+}\bigcup S_{nil}^{1z-}} A_1(E_{nil}^{j+\frac{1}{3}})ds + \iint\limits_{S_{nil}^{2z+}\bigcup S_{nil}^{2z-}} A_2(E_{nil}^{j+\frac{1}{3}})ds\right];$$

**z − direction**

$$V_{nil}\cdot(E_{nil}^{j+1}-E_{nil}^{j+\frac{2}{3}})=\frac{\tau^{j+1}}{3}\left[\iint\limits_{S_{nil}^{1z+}\bigcup S_{nil}^{1z-}}A_1(E_{nil}^{j+1})ds+\iint\limits_{S_{nil}^{2z+}\bigcup S_{nil}^{2z-}}A_2(E_{nil}^{j+1})ds+\right.$$

$$\left.+\iint\limits_{S_{nil}^{1x+}\bigcup S_{nil}^{1x-}}A_1(E_{nil}^{j+\frac{2}{3}})ds+\iint\limits_{S_{nil}^{2x+}\bigcup S_{nil}^{2x-}}A_2(E_{nil}^{j+\frac{2}{3}})ds+\iint\limits_{S_{nil}^{1y+}\bigcup S_{nil}^{1y-}}A_1(E_{nil}^{j+\frac{2}{3}})ds+\iint\limits_{S_{nil}^{2y+}\bigcup S_{nil}^{2y-}}A_2(E_{nil}^{j+\frac{2}{3}})ds\right].$$

The thermal conductivities $\Omega_1(\widetilde{E}_{nil}^j)$ and $\Omega_2(\widetilde{E}_{nil}^j)$ on internal surfaces of the elementary cell are approximated as:

$$\Omega_1(\widetilde{E}_{nil}^j)\Big|_{S_{nil}^{1x+}}=\frac{\Omega_1(E_{nil}^j)+\Omega_1(E_{n+1,il}^j)}{2}\equiv R_n^j,$$

$$\Omega_1(\widetilde{E}_{nil}^j)\Big|_{S_{nil}^{1x-}}=\frac{\Omega_1(E_{n-1,il}^j)+\Omega_1(E_{nil}^j)}{2}\equiv R_{n-1}^j,$$

$$\Omega_1(\widetilde{E}_{nil}^j)\Big|_{S_{nil}^{1y+}}=\frac{\Omega_1(E_{nil}^j)+\Omega_1(E_{n,i+1,l}^j)}{2}\equiv \widehat{R}_i^j,$$

$$\Omega_1(\widetilde{E}_{nil}^j)\Big|_{S_{nil}^{1y-}}=\frac{\Omega_1(E_{nil}^j)+\Omega_1(E_{n,i-1,l}^j)}{2}\equiv \widehat{R}_{i-1}^j.$$

The notations $\widetilde{R}_l^j$ and $\widetilde{R}_{l-1}^j$ for the surfaces $S_{nil}^{1z+}$ and $S_{nil}^{1z-}$ and the notations $B_n^j$, $B_{n-1}^j$, $\widehat{B}_i^j$, $\widehat{B}_{i-1}^j$, $\widetilde{B}_l^j$, and $\widetilde{B}_{l-1}^j$ for $\Omega_2(\widetilde{E}_{nil}^j)$ are introduced in a similar manner.

For simplicity, the subsequent presentation of the algorithm is given for the simplest domain – a rectangular parallelepiped.

The derivative $\beta_{\mathbf{n}}(E)$ in the outward normal direction $\mathbf{n}$ on $\Gamma$ are approximated by the formula: $\beta_{\mathbf{n}}(E)=(\nabla\beta,\mathbf{n})$, where, for example,

$$\beta_{\mathbf{n}}(E_{nil}^j)\Big|_{S_{nil}^{1x+}}=\frac{\beta_{n+1,il}^j-\beta_{nil}^j}{h_n^x},\quad \left(n=\overline{0,N-1};i=\overline{0,I};l=\overline{0,L}\right),$$

$$\beta_{\mathbf{n}}(E_{nil}^j)\Big|_{S_{nil}^{1x-}}=-\frac{\beta_{nil}^j-\beta_{n-1,il}^j}{h_{n-1}^x},\quad \left(n=\overline{1,N};i=\overline{0,I};l=\overline{0,L}\right).$$

With the notation introduced, the spatial approximation of the first subproblem inside the domain under consideration can be written as:

$$E_{nil}^{j+\frac{1}{3}}-E_{nil}^j=\omega_{nil}^{j+1}\left[\left(S_{nil}^{1x+}R_n^{j+\frac{1}{3}}+S_{nil}^{2x+}B_n^{j+\frac{1}{3}}\right)\frac{\beta_{n+1,il}^{j+\frac{1}{3}}-\beta_{nil}^{j+\frac{1}{3}}}{h_n^x}-\right.$$

$$-(S_{nil}^{1x-}R_{n-1}^{j+\frac{1}{3}}+S_{nil}^{2x-}B_{n-1}^{j+\frac{1}{3}})\frac{\beta_{nil}^{j+\frac{1}{3}}-\beta_{n-1,il}^{j+\frac{1}{3}}}{h_{n-1}^{x}}\Bigg]+\xi_{nil}^{j}, \tag{10}$$

$$\left(n=\overline{1,N-1};i=\overline{1,I-1};l=\overline{1,L-1}\right),$$

where

$$\omega_{nil}^{j+1}=\frac{\tau^{j+1}}{3V_{nil}},\qquad \xi_{nil}^{j}=\omega_{nil}^{j+1}\Bigg[\left(S_{nil}^{1y+}\widehat{R}_{i}^{j}+S_{nil}^{2y+}\widehat{B}_{i}^{j}\right)\frac{\beta_{n,i+1,l}^{j}-\beta_{nil}^{j}}{h_{i}^{y}}-$$

$$-\left(S_{nil}^{1y-}\widehat{R}_{i-1}^{j}+S_{nil}^{2y-}\widehat{B}_{i-1}^{j}\right)\frac{\beta_{nil}^{j}-\beta_{n,i-1,l}^{j}}{h_{i-1}^{y}}+$$

$$+\left(S_{nil}^{1z+}\widetilde{R}_{l}^{j}+S_{nil}^{2z+}\widetilde{B}_{l}^{j}\right)\frac{\beta_{ni,l+1}^{j}-\beta_{nil}^{j}}{h_{l}^{z}}-\left(S_{nil}^{1z-}\widetilde{R}_{l-1}^{j}+S_{nil}^{2z-}\widetilde{B}_{l-1}^{j}\right)\frac{\beta_{nil}^{j}-\beta_{ni,l-1}^{j}}{h_{l-1}^{z}}\Bigg].$$

The relation (10) is valid for internal cells of the domain $Q$. If any of the surfaces $S_{nil}^{1x+}, S_{nil}^{1x-}, ..., S_{nil}^{2z-}$ coincides with the outer boundary of the domain, then the corresponding term in the heat balance equation is approximated taking into account the boundary conditions. For this purpose, boundary conditions (2) on the outer boundary $\Gamma$ are rewritten in the general form:

$$K(T)T_{\mathbf{n}}|_{\Gamma}=\left(r(T)T+q(t)\right)|_{\Gamma}.$$

Since

$$K(T)=\begin{cases}K_1(T), & (x,y,z)\in S_{nil}^{1},\\ K_2(T), & (x,y,z)\in S_{nil}^{2},\end{cases}=\begin{cases}\Omega_1(E), & (x,y,z)\in S_{nil}^{1},\\ \Omega_2(E), & (x,y,z)\in S_{nil}^{2},\end{cases}$$

the last expression splits into two equalities:

$$\Omega_1(E)\beta_{\mathbf{n}}(E)|_{\Gamma}=\left(r_1(\beta(E))\beta(E)+q_1(t)\right)|_{\Gamma},\quad (x,y,z)\in S_{nil}^{1}, \tag{11}$$

$$\Omega_2(E)\beta_{\mathbf{n}}(E)|_{\Gamma}=\left(r_2(\beta(E))\beta(E)+q_2(t)\right)|_{\Gamma},\quad (x,y,z)\in S_{nil}^{2}. \tag{12}$$

These relations are used to derive a spatial approximation of the heat fluxes on the outer boundary of the domain. For example, for $n=0$, system (10) must be supplemented by the equality:

$$E_{0il}^{j+\frac{1}{3}}-E_{0il}^{j}=\omega_{0il}^{j+1}\Bigg[(S_{0il}^{1x+}R_{0}^{j+\frac{1}{3}}+S_{0il}^{2x+}B_{0}^{j+\frac{1}{3}})\frac{\beta_{1il}^{j+\frac{1}{3}}-\beta_{0il}^{j+\frac{1}{3}}}{h_{0}^{x}}\Bigg]+$$

$$+S_{0il}^{1x-}\left(r_1(\beta_{0il}^{j+\frac{1}{3}})\beta_{0il}^{j+\frac{1}{3}}+q_1^{j+\frac{1}{3}}\right)\Big|_{S_{0il}^{1x-}}+S_{0il}^{2x-}\left(r_2(\beta_{0il}^{j+\frac{1}{3}})\beta_{0il}^{j+\frac{1}{3}}+q_2^{j+\frac{1}{3}}\right)\Big|_{S_{0il}^{2x-}}+\xi_{0il}^{j}. \tag{13}$$

Let the function $\beta(E_{nil}^{j})$ be represented in the form:

$$\beta(E_{nil}^{j})=u_{nil}^{j}E_{nil}^{j}+v_{nil}^{j}, \tag{14}$$

where
$$u^j_{nil} = \begin{cases} \dfrac{1}{a_{nil}}, & E^j_{nil} < a_{nil}T_1, \\ \dfrac{1}{b^1_{nil}}, & a_{nil}T_1 \le E^j_{nil} < d^1_{nil}T_2 + d^2_{nil}, \\ \dfrac{1}{d^1_{nil}}, & E^j_{nil} \ge d^1_{nil}T_2 + d^2_{nil}, \end{cases}$$

$$v^j_{nil} = \begin{cases} 0, & E^j_{nil} < a_{nil}T_1, \\ \dfrac{b^2_{nil}}{b^1_{nil}}, & a_{nil}T_1 \le E^j_{nil} < d^1_{nil}T_2 + d^2_{nil}, \\ -\dfrac{d^2_{nil}}{d^1_{nil}}, & E^j_{nil} \ge d^1_{nil}T_2 + d^2_{nil}. \end{cases}$$

Then Eq. (10) can be rewritten as:

$$E^{j+\frac{1}{3}}_{nil} - E^j_{nil} = \omega^{j+1}_{nil} \frac{S^{1x+}_{nil} R^{j+\frac{1}{3}}_n + S^{2x+}_{nil} B^{j+\frac{1}{3}}_n}{h^x_n} \times$$
$$\times \left( u^{j+\frac{1}{3}}_{n+1,il} E^{j+\frac{1}{3}}_{n+1,il} + v^{j+\frac{1}{3}}_{n+1,il} - u^{j+\frac{1}{3}}_{nil} E^{j+\frac{1}{3}}_{nil} - v^{j+\frac{1}{3}}_{nil} \right) -$$
$$-\omega^{j+1}_{nil} \frac{S^{1x-}_{nil} R^{j+\frac{1}{3}}_{n-1} + S^{2x-}_{nil} B^{j+\frac{1}{3}}_{n-1}}{h^x_{n-1}} \left( u^{j+\frac{1}{3}}_{nil} E^{j+\frac{1}{3}}_{nil} + v^{j+\frac{1}{3}}_{nil} - u^{j+\frac{1}{3}}_{n-1,il} E^{j+\frac{1}{3}}_{n-1,il} - v^{j+\frac{1}{3}}_{n-1,il} \right) + \xi^j_{nil}.$$

The resulting system of nonlinear algebraic equations for $E^{j+\frac{1}{3}}_{nil}$ can be written as:

$$\hat{A}_n E^{j+\frac{1}{3}}_{n-1,il} - \hat{C}_n E^{j+\frac{1}{3}}_{nil} + \hat{B}_n E^{j+\frac{1}{3}}_{n+1,il} + \hat{D}_n = 0, \quad \left(n = \overline{1,N};\ i = \overline{0,I};\ l = \overline{0,L}\right). \tag{15}$$

Coefficients $\hat{A}_n, \hat{B}_n, \hat{C}_n$, and $\hat{D}_n$ are given in ([9]).

Taking into account (14), Eq. (13) for $n = 0$ is written as:

$$E^{j+\frac{1}{3}}_{0il} = n_0 E^{j+\frac{1}{3}}_{1il} + m_0, \quad i = \overline{0,I},\ l = \overline{0,L}. \tag{16}$$

The following relation for $n = N$ is derived by analogy with that for $n = 0$:

$$E^{j+\frac{1}{3}}_{Nil} = n_1 E^{j+\frac{1}{3}}_{N-1,il} + m_1, \quad i = \overline{0,I},\ l = \overline{0,L}. \tag{17}$$

Coefficients $n_0, m_0, n_1$, and $m_1$ are given in [9].

The resulting system of nonlinear algebraic equations (15)–(17) is divided into $(I+1)(L+1)$ subsystems. Each of them has the form of (15)–(17) with fixed indices $i \in \overline{0,I}$ and $l \in \overline{0,L}$ and is solved irrespective of the other subsystems by applying iteration and tridiagonal Gaussian elimination [8]. The coefficients $\hat{A}_n$, $\hat{B}_n$, $\hat{C}_n$, $\hat{D}_n$, $n_0$, $m_0$, $n_1$, and $m_1$ in the subsystems are determined by the calculated values of $E^{j+\frac{1}{3}}$ at the current iteration step. The value of $E^{j+\frac{1}{3}}$ at the next iteration step is determined by tridiagonal Gaussian elimination. The iteration halts after the required accuracy was achieved.

The spatial approximations of the second and third subproblems are performed in a similar manner with the use of the solution obtained for the previous subproblems.

If the considered domain is more complex and consists of a set of different parallelepipeds (see Fig. 2), minor modifications of the algorithm described must be done. It should only be taken into account that the ranges of $n$, $i$, and $l$ depend on the values of the pairs of numbers $(i, l)$, $(n, l)$, and $(n, i)$ respectively.

## 4 Approximation of boundary conditions

The mold and the metal are cooled via their interaction with the surroundings. On the one hand, the object is slowly immersed in a liquid medium (aluminum) of a low temperature, due to which the metal solidifies. On the other hand, the body receives heat from the walls of the furnace, which slows down the process of solidification.

Therefore, the individual parts of the outer boundary of the body are in different thermal conditions. The basic types of thermal conditions at a point of the outer boundary of the body can be described as follows.

1) The point is in the liquid aluminum. In this case, the following processes have to be taken into account:
(i) the heat lost by the body due to its own radiation;
(ii) the heat gained from the surrounding liquid aluminum due to its radiation;
(iii) the heat transfer due to conduction between the liquid aluminum and the body.

2) The point is outside the liquid aluminum. In this case, the following processes have to be taken into account:
(i) the heat lost by the body due to its own radiation;
(ii) the heat gained from the emitting walls of the furnace;
(iii) the heat gained from the emitting surface of the liquid aluminum.

One of the mechanisms of heat transfer in this problem is thermal radiation. It can be computed as follows.

Consider two small areas (hereafter called elementary) in space (see Fig. 3). Let $\triangle s$ and $\mathbf{n}$ denote the size of the first area and its normal vector and $\triangle S$ and $\mathbf{N}$ denote the same characteristics for the second area. Assume that the first area emits thermal energy diffusely and its emissivity is $\varepsilon$. According to [10], the radiation energy flux $\triangle q$ from the first area of temperature $T_{Sou}$ through the second area is calculated as

$$\triangle q = I\left(\mathbf{n}, \frac{\mathbf{r}}{|\mathbf{r}|}\right)\triangle s \triangle \omega,$$

where $I = \dfrac{1}{\pi}\varepsilon\sigma T_{Sou}^4$ is the intensity of the emission ($\sigma$ is the Stefan-Boltzmann constant), $\triangle\omega$ is the solid angle at which the second area is seen from the center of the first area, and $\mathbf{r}$ is the position vector beginning from the center of the first area and ending at the center of the second area (see Fig. 3). The solid angle $\triangle\omega$ is determined by the formula

$$\triangle\omega = \begin{cases} \frac{1}{|\mathbf{r}|^3}(\mathbf{N},\mathbf{r})\triangle S, & (\mathbf{N},\mathbf{r}) > 0, \\ 0, & (\mathbf{N},\mathbf{r}) \leq 0. \end{cases}$$

Thus,

$$\triangle q = \frac{I(\mathbf{n},\mathbf{r})(\mathbf{N},\mathbf{r})}{|\mathbf{r}|^4}\triangle s\triangle S.$$

If radiation is emitted by an extended body $s$, the radiation energy flux $q$ from it through the second elementary area is given by:

$$q = \triangle S \iint_S \left(\frac{1}{\pi}\varepsilon\sigma T_{Sou}^4\right)\frac{(\mathbf{n}(y_1,y_2),\mathbf{r}(y_1,y_2))}{|\mathbf{r}(y_1,y_2)|^4}(\mathbf{N},\mathbf{r}(y_1,y_2))ds_y, \tag{18}$$

where $y = (y_1, y_2)$ are local coordinates introduced on the source surface $s$.

In the space under study, we introduce a Cartesian coordinate system with its origin at the center of $\triangle S$ and with the basis $\{\mathbf{e_1},\mathbf{e_2},\mathbf{e_3}\}$. Then the vector $\mathbf{N}$ normal to the second area can be expressed as $\mathbf{N} = N_1\mathbf{e_1} + N_2\mathbf{e_2} + N_3\mathbf{e_3}$. The flux $q$ in (18) is represented by the sum of three fluxes:

$$q = [N_1q_1 + N_2q_2 + N_3q_3]\triangle S,$$

where

$$q_i = \iint_S \left(\frac{1}{\pi}\varepsilon\sigma T_{Sou}^4\right)\frac{(\mathbf{n}(y_1,y_2),\mathbf{r}(y_1,y_2))}{|\mathbf{r}(y_1,y_2)|^4}(\mathbf{e_i},\mathbf{r}(y_1,y_2))ds_y.$$

The expressions for $q_i$ $(i = 1, 2, 3)$ are derived assuming that the source surface is a rectangle. The basis vectors of the coordinate system are chosen so that they form a right-hand triple and $\mathbf{e_1}$ is orthogonal to the source plane and is directed toward it. Assume that the source has the size $l \times h$ and $\xi$ is the distance from this source to the second elementary area.

a) The second elementary area is orthogonal to $\mathbf{e_1}$ (see Fig. 4). In this case, $\mathbf{N} = (1, 0, 0)$, $\mathbf{n} = (-1, 0, 0)$. Then

$$q_1 = M_0\left[\frac{h}{\sqrt{\xi^2+h^2}}\arctan\left(\frac{l}{\sqrt{\xi^2+h^2}}\right) + \frac{l}{\sqrt{\xi^2+l^2}}\arctan\left(\frac{h}{\sqrt{\xi^2+l^2}}\right)\right],$$

where $M_0 = \frac{1}{2\pi}\varepsilon\sigma T_{Sou}^4$.

b) The second elementary area is orthogonal to $\mathbf{e_2}$ (see Fig. 5). In this case, $\mathbf{N} = (0, 1, 0)$, $\mathbf{n} = (-1, 0, 0)$. Then

$$q_2 = M_0\left[\arctan\left(\frac{h}{\xi}\right) - \frac{\xi}{\sqrt{\xi^2+l^2}}\arctan\left(\frac{h}{\sqrt{\xi^2+l^2}}\right)\right].$$

c) The second elementary area is orthogonal to $\mathbf{e_3}$ (see Fig. 6). In this case, $\mathbf{N} = (0, 0, 1)$, $\mathbf{n} = (-1, 0, 0)$. Then

$$q_3 = M_0 \left[\arctan\left(\frac{l}{\xi}\right) - \frac{\xi}{\sqrt{\xi^2 + h^2}} \arctan\left(\frac{l}{\sqrt{\xi^2 + h^2}}\right)\right].$$

Now, let's take a closer look at the description of the boundary conditions (11) and (12), i.e. at a more detailed description of the functions $r_1(\beta(E))$, $q_1(t)$, $r_2(\beta(E))$, and $q_2(t)$. For the sake of simplicity, the description of these functions is given for a rectangular parallelepiped.

Consider the face of the parallelepiped which is parallel to the plane $YOZ$ and is located closer to the right wall of the furnace (Fig. 1). As noted above, some areas of this face can be in different thermal conditions.

For the considered face of the object all the cells are filled with the material of the form. The considered face consists of this cell's surfaces that are designated as $S^{2x+}_{Nil}$, $(i = \overline{0, I},\quad l = \overline{0, L})$. For the time $t = t^j$, when the cell is located outside of the liquid aluminum we have:

$$\Omega_2(E^j_{Nil})\beta_{\mathbf{n}}(E^j_{Nil})\Big|_{S^{2x+}_{Nil}} = -\sigma \cdot \left(\beta^j_{Nil}\right)^4 + \varphi_s + \varphi_a, \tag{19}$$

where $\varphi_s$ is the radiation energy flux density of the whole right wall through the surface $S^{2x+}_{Nil}$ (let us note that in this case the radiation from the left wall of the furnace does not fall on the considered face of object), and $\varphi_a$ is the radiation energy flux density from the surface of liquid aluminum through this surface. The values $\varphi_s$ and $\varphi_a$ are calculated by the formulas:

$$\begin{aligned}\varphi_s = q_s(X_s, Y_{Sou} - y_i + L_{Sou}, Z_{Sou} - z_l + H_{Sou}) - q_s(X_s, Y_{Sou} - y_i, Z_{Sou} - z_l + H_{Sou}) + \\ + q_s(X_s, Y_{Sou} - y_i, Z_{Sou} - z_l) - q_s(X_s, Y_{Sou} - y_i + L_{Sou}, Z_{Sou} - z_l),\end{aligned} \tag{20}$$

$$\varphi_a = q_a(Z_a, Y_{al} - y_i + L_{al}, X_{al} - X_b + H_{al}) - q_a(Z_a, Y_{al} - y_i, X_{al} - X_b + H_{al}). \tag{21}$$

Here:
$X_s$ is the distance from the surface $S^{2x+}_{Nil}$ of the considered cell to the nearest wall of the furnace,
$(X_B, y_i, z_l)$ are the coordinates of center of the surface of the considered cell,
$Y_{Sou}$ is the ordinate of the lower vertex of the right wall of the furnace, nearest to the point of origin $O$ of the selected coordinate system,
$Z_{Sou}$ is the z-coordinate of the lower bound of the wall of the furnace at the moment $t = t^j$,
$X_{al}$, $Y_{al}$ are the abscess and the ordinate of the vertex of the surface of aluminum, nearest to the point of origin $O$ of the selected coordinate system,
$Z_a = z_l - U_{al}$ is the distance from the surface $S^{2x+}_{Nil}$ of the considered cell to the surface of the liquid aluminum,

$$U_{al} = \begin{cases} Z_{al}, & \text{object did not reach the surface of aluminum,} \\ Z_{al} + \dfrac{X_b \cdot Y_b \cdot Z_{al}}{L_{al} \cdot H_{al} - X_b \cdot Y_b}, & \text{object reached the surface of aluminum,} \end{cases}$$

$Z_{al} = Z_{Sou} - H_{air}$,
$X_b$ is the length of the parallelepiped along the $Ox$ axis,
$Y_b$ is the length of the parallelepiped along the $Oy$ axis,
$Z_b$ is the height of the parallelepiped along the $Oz$ axis,
$L_{Sou}$ is the length of the plate of the furnace along the $Oy$ axis,
$H_{Sou}$ is the height of the plate of the furnace along the $Oz$ axis,
$H_{air}$ is the distance from the furnace to the liquid aluminum,
$L_{al}$ is the length of the aluminum surface along the $Oy$ axis,
$H_{al}$ is the length of the aluminum surface along the $Ox$ axis.
Functions $q_s$ and $q_a$ are determined using the following formulas:

$$q_s(\xi,l,h)=M_s\cdot\left[\frac{h}{\sqrt{\xi^2+h^2}}\arctan\left(\frac{l}{\sqrt{\xi^2+h^2}}\right)+\frac{l}{\sqrt{\xi^2+l^2}}\arctan\left(\frac{h}{\sqrt{\xi^2+l^2}}\right)\right], \tag{22}$$

$$q_a(\xi,l,h) = M_a\cdot\left[\arctan\left(\frac{l}{\xi}\right) - \frac{\xi}{\sqrt{\xi^2+h^2}}\arctan\left(\frac{l}{\sqrt{\xi^2+h^2}}\right)\right], \tag{23}$$

where $M_s = \dfrac{1}{2\pi}\varepsilon_s\sigma T_{Sou}^4$, $M_a = \dfrac{1}{2\pi}\varepsilon_a\sigma T_{al}^4$, $T_{Sou}$ is the temperature of the plate of the furnace, $T_{al}$ is the temperature of the liquid aluminum, $\varepsilon_s$ is emissivity of the wall of the furnace, $\varepsilon_a$ is emissivity of the liquid aluminum.

When the cell is places outside the liquid aluminum then according to (12) and (19) we have:

$$r_2\left(\beta(E_{Nil}^j)\right)\Big|_{S_{Nil}^{2x+}} = -\sigma\cdot(\beta_{Nil}^j)^3, \qquad q_2(t)|_{S_{Nil}^{2x+}} = \varphi_s+\varphi_a.$$

When the cell is places inside the liquid aluminum we have:

$$\Omega_2(E_{Nil}^j)\beta_{\mathbf{n}}(E_{Nil}^j)\Big|_{S_{Nil}^{2x+}} = -\lambda\cdot(\beta_{Nil}^j - T_{al}) - \sigma\cdot(\beta_{Nil}^j)^4 + \sigma\cdot(T_{al})^4,$$

and accordingly

$$r_2\left(\beta(E_{Nil}^j)\right)\Big|_{S_{Nil}^{2x+}} = -\left(\sigma\cdot(\beta_{Nil}^j)^3+\lambda\right), \quad q_2(t)|_{S_{Nil}^{2x+}} = \lambda\cdot T_{al}+\sigma\cdot(T_{al})^4.$$

Here $\lambda$ is the coefficient of heat exchange between the object and the liquid aluminum. Boundary conditions for the remaining five faces of the parallelepiped are approximated analogously. The upper face of the parallelepiped is differed from the rest because the outer boundary consists of both the cells containing the material of the form and the cells containing the material of the metal, i.e. in this case both conditions (11) and (12) operate.

## 5 Numerical results of solving the direct problem

First, the direct problem was studied for an object of the simplest shape – a rectangular parallelepiped. This object was used to test and tune the algorithms proposed for solving the problem.

The direct problem was also solved for an actual object. Its longitudinal projections are displayed in Fig. 2. This object had two planes of symmetry and was located symmetrically in the furnace. It consisted of five parallelepipeds. The exterior view of its quarter is shown in Fig. 7. The object was immersed in the molten aluminum up to the fourth parallelepiped. The speed $\widetilde{u}(t)$ of the displacement of foundry mold was relied equal to zero, when it reached the maximum permissible depth.

The numerical results presented below were obtained for this object and for the following parameter values (given in SI units):

$$\begin{array}{llll}
\rho_S = 8200.0, & k_S = 23.3, & c_S = 670.0, & \rho_L = 7200.0, \\
k_L = 15.2, & c_L = 790.0, & \rho_\Phi = 2700.0, & k_{\Phi_1} = 4.7, \\
k_{\Phi_2} = 3.2, & c_\Phi = 780.0, & T_1 = 1493.15, & T_2 = 1633.15, \\
T_3 = 1100.15, & \delta = 20.0, & \gamma = 234000.0, & T_{met} = 1973.15, \\
T_{form} = 1853.15, & L_{al} = 0.500, & H_{al} = 0.300, & H_{air} = 0.070, \\
T_{Sou} = 1823.15, & T_{al} = 1003.15, & L_{Sou} = 0.450, & H_{Sou} = 0.535, \\
\lambda = 1.0, & \varepsilon_s = 0.8, & \varepsilon_a = 0.8, & T_{pl} = 0.5(T_1 + T_2),
\end{array}$$

$X_b = 0.07$ (the length of a quarter of the casting along the $Ox$ axis),
$Y_b = 0.1$ (the length of a quarter of the casting along the $Oy$ axis),
$Z_b = 0.435$ (the length of the casting along the $Oz$ axis).

The number $t^J$ which determines the length of the interval of time $[0, t^J]$ was selected so that the time during which the complete solidification of the metal in the foundry mold occurs would not exceed the value $t^J$ for all considered regimes of the process of crystallization.

In the computation of the direct problem, primary attention was given to the evolution of the solidification front. The dependance of this evolution as a function of the velocity of the object is illustrated in Figures 8–15, which show lines of constant temperature at different times in two cross sections ((a) and (b)) through the object's vertical axis of symmetry parallel to the object faces. Since the object is symmetric about the vertical axis, the figures present only halves of the cross sections. The light vertical and horizontal lines inside the object separate the metal and the mold. The light curves show lines of constant temperature, and the heavy curve depicts the contour line of $T = T_{pl}$. It separates the liquid and solid phases in the metal. The figures with different numbers correspond to different times. Figures 8–11 (first experiment) illustrate the process of metal solidification in a mold moving relative to the furnace at the constant speed $\widetilde{u}(t) = 2$ mm/min. In the second experiment (Figures 12–15), the speed was piecewise constant. More specifically, it remained constant in three time intervals. Over the first time interval, the first (narrowest) parallelepiped was immersed in the coolant at the speed 20 mm/min. Over the second and third time intervals, the second and third parallelepipeds were immersed in the coolant at the speeds 10 and 5 mm/min respectively. Poor results were obtained when the object moved at a constant low speed. The solidification of the metal proceeded from two sides (lower and upper). This led to the formation of bubbles of liquid metal that collapse inside the casting. It should be noted that the solidification front was nearly always far from a horizontal plane. In the second experiment, the solidification front always intersected the metal transversally only once and was noticeably more similar to a horizontal plane. No bubbles of liquid metal were observed inside the casting during the entire process.

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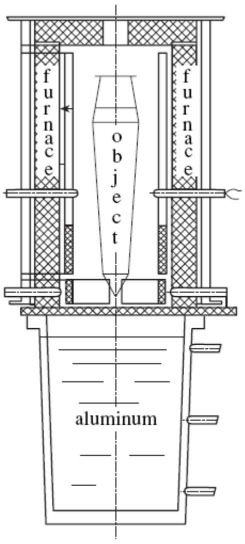


Fig. 1

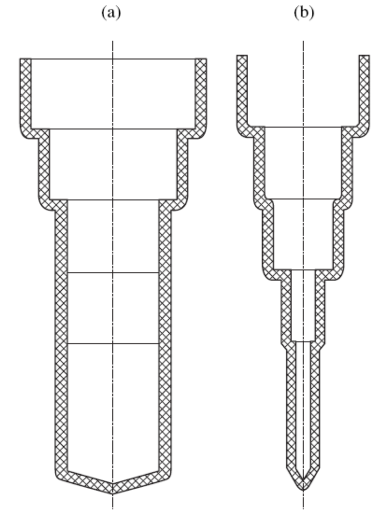


Fig. 2

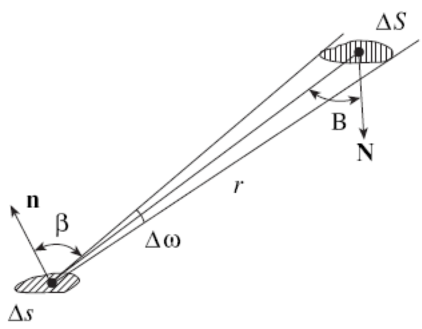


Fig. 3

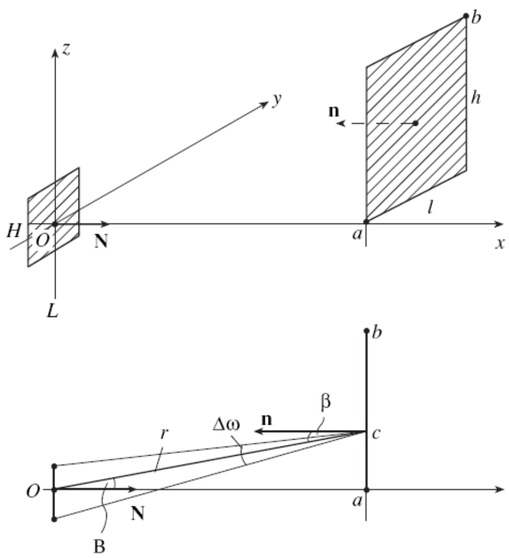


Fig. 4

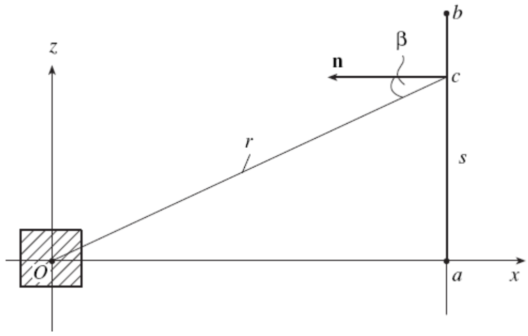


Fig. 5

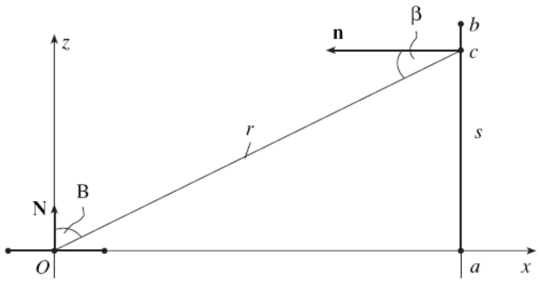


Fig. 6

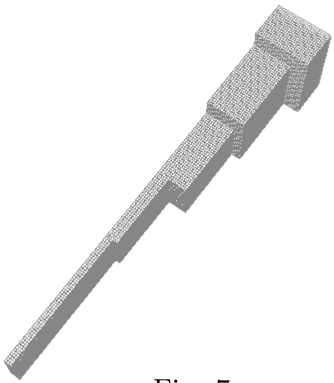

Fig. 7

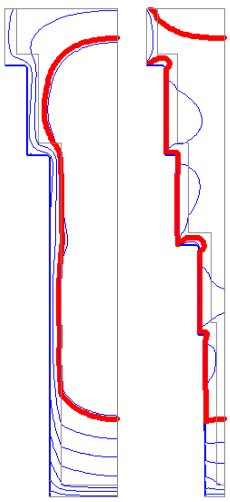

Fig. 8 a,b

Fig. 9 a,b

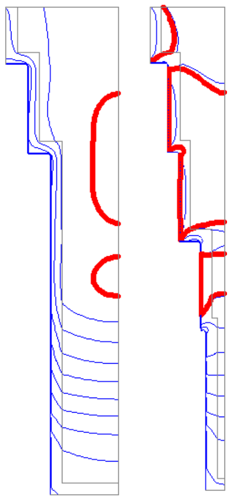

Fig. 10 a,b

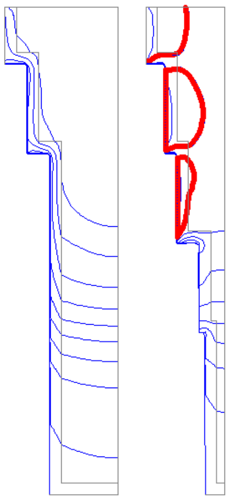

Fig. 11 a,b

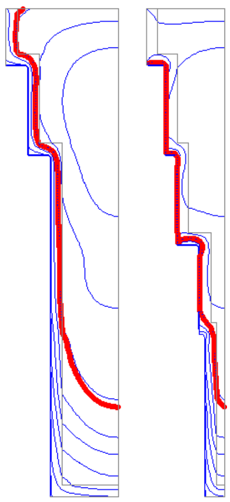

Fig. 12 a,b

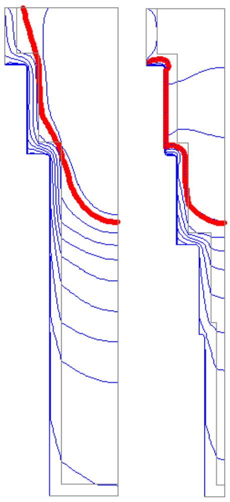

Fig. 13 a,b

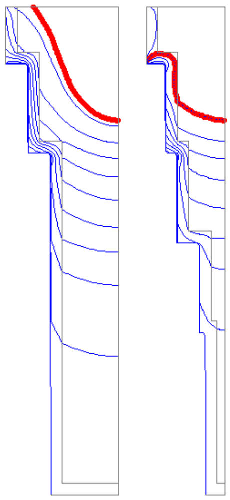

Fig. 14 a,b

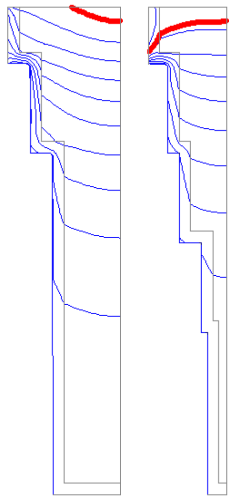

Fig. 15 a,b

## References


[1] Rose M.E. *A method for calculating solutions of parabolic equations with a free boundary.* Math. Comput., 1960, **14**, 249–256.

[2] White R.E. *An Enthalpy Formulation of the Stephan Problem.* SIAM J. Numer. Anal., 1982, **19**, No. 6, 1129–1157.

[3] White R.E. *A numerical solution of the enthalpy formulation of the Stephan problem.* SIAM J. Numer. Anal., 1982, **19**, No. 6, 1158–1172.

[4] Albu A.F., Gorbunov V.I., Zubov V.I. *Optimal Control of the Process of Melting.* Zh. Vychisl. Mat. Mat. Fiz., 2000, **40**, No. 4, 517–531 (Comp. Math. Math. Phys., 2000, **40**, 491–504).

[5] Albu A.F., Zubov V.I. *A Modified Scheme for Analyzing the Process of Melting.* Zh. Vychisl. Mat. Mat. Fiz., 2001, **41**, No. 9, 1434–1443 (Comp. Math. Math. Phys., 2001, **41**, 1363–137).

[6] Albu A.F., Zubov V.I. *Optimal Control of the Process of Solidification.* Zh. Vychisl. Mat. Mat. Fiz., 2004, **44**, No. 1, 38–50 (Comp. Math. Math. Phys., 2004, **44**, 34–44).

[7] Albu A.F. and Zubov V.I. *Modeling and optimization of melting and solidification process.* Buletinul Academiei de Ştiinţe a Republicii Moldova. Matematica, 2004, No. 3(46), 91–109.

[8] Samarskii A.A. *The Theory of Difference Schemes.* Marcel Dekker, New York, 2001.

[9] Albu A.F., Zubov V.I. *Mathematical modeling and study of the process of solidification in metal casting.* Zh. Vychisl. Mat. Mat. Fiz., 2007, **47**, No. 5, 882–902 (Comp. Math. Math. Phys., 2007, **47**, 843–862).

[10] Sparrow E.M., Cess R.D. *Radiation Heat Transfer.* Brooks-Cole, Belmont, Calif., 1972.


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# Conjugate-orthogonality and the complete multiplication group of a quasigroup

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**Abstract.** In this note we establish connections between the orthogonality of conjugates of a finite or infinite quasigroup and some strictly transitive subsets of the complete multiplication group of this quasigroup. These connections are used for the investigation of orthogonality of distinct pairs of conjugates for quasigroups (loops) from some classes. For finite quasigroups the quasi-identities corresponding to orthogonality of pairs of conjugates are given.

**Mathematics subject classification:** 20N05, 20N15.
**Keywords and phrases:** Quasigroup, loop, primitive quasigroup, quasi-identity, multiplication group, conjugate, parastrophe, conjugate-orthogonality.

## 1 Introduction

A quasigroup is an ordered pair $(Q, \cdot)$ (or $(Q, A)$) where $Q$ is a set and $(\cdot)$ (or $(A)$) is a binary operation on $Q$ such that each of the equations $ay = b$ and $xa = b$ is uniquely solvable for any pair of elements $a, b$ in $Q$. It is known that the multiplication table of a finite quasigroup defines a Latin square and six (not necessarily distinct) conjugates (or parastrophes) are associated with each quasigroup (Latin square) [1, 12].

Two quasigroups $(Q, A)$ and $(Q, B)$ defined on a set $Q$ are orthogonal if the system of equations $\{A(x, y) = a, B(x, y) = b\}$ is uniquely solvable for all $a, b \in Q$. The notion of orthogonality plays an important role in the theory of Latin squares, also in the quasigroup theory and in distinct applications.

There is significant interest in the investigation of quasigroups which are orthogonal to some their conjugates or two conjugates of which are orthogonal (so called conjugate-orthogonal or parastrophic-orthogonal quasigroups).

Many articles were devoted to the investigation of various aspects of conjugate-orthogonal quasigroups. Recall some of them. In [5, 7–9, 11, 16] the spectrum of conjugate-orthogonal quasigroups (Latin squares) was studied.

Different identities associated with the conjugate-orthogonality and related combinatorial designs were considered in [4, 6, 13]. In particular, F. E. Bennet in [6] investigated the spectrum of the varieties of quasigroups with every one of eight short conjugate-orthogonal identities (short two-variable identities).


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F. E. Bennet and H. Zhang [10] considered a problem related to the spectrum of Latin squares where each conjugate is required to be orthogonal to precisely its transpose from among the other five conjugates.

In [5,15] some quasi-identities of finite parastrophic-orthogonal quasigroups were established.

In this paper we study properties of multiplication groups of conjugate-orthogonal quasigroups. In particular, we prove that some strictly transitive subset of the complete multiplicative group of a quasigroup corresponds to orthogonality of any two from six conjugates of this quasigroup. We also give some quasi-identities related to the orthogonality of two conjugates of a finite quasigroup $(Q, A)$. The use of a criterion of conjugate-orthogonality in the strictly transitive subset language allows easily to obtain a number of useful statements with respect to the conjugate-orthogonality of quasigroups and loops from some classes.

## 2 Preliminaries

A quasigroup $(Q,\cdot)$ is finite of order $n$ if the set $Q$ is finite and $|Q| = n$.

A quasigroup with *the left (right) identity* $f$ $(e)$ is a quasigroup $(Q,\cdot)$ such that $fx = x$ $(xe = x)$ for every $x \in Q$. A *loop* is a quasigroup $(Q,\cdot)$ with the identity $e$: $xe = ex = x$ for each $x \in Q$ [1].

A loop $(Q,\cdot)$ is called *a Moufang loop* if it satisfies the identity $(zx\cdot y)x = z(x\cdot yx)$ [1].

A quasigroup is called *an IP-quasigroup* if there exist maps (permutations) $I_r$ and $I_l$ such that $(yx)\cdot I_r x = y$, $I_l x\cdot(xy) = y$ for any $x, y \in Q$ [1].

The permutations $L_a$, $R_a$ and $I_a$ defined by $L_a x = ax$, $R_a x = xa$ and $x \cdot I_a x = a$ for all $x \in Q$ are called the *left*, *right* and *middle translations* of a quasigroup $(Q,\cdot)$ respectively [1,3].

*The multiplication group or the group associated with* $M$ or $M_{(\cdot)}$ of a quasigroup $(Q,\cdot)$ (or the group associated with a quasigroup $(Q,\cdot)$) is the group generated by all left and all right translations of $(Q,\cdot)$: $M =< L_a, R_a \,|\, a \in Q >$ [1].

*The complete multiplication group* $\overline{M}$ (or the complete group associated with a quasigroup $(Q,\cdot)$ [3]) is the group generated by all left, right and middle translations of this quasigroup: $\overline{M} =< L_a, R_a, I_a \,|\, a \in Q >$. It is evident that $M \subseteq \overline{M}$.

With any quasigroup $(Q,\cdot)$ the system $\Sigma$ of six (not necessarily distinct) *conjugates (parastrophes)* is associated:

$$\Sigma = \left\{(\cdot), (\cdot)^{-1} = (\backslash), {}^{-1}(\cdot) = (/), {}^{-1}\left((\cdot)^{-1}\right), \left({}^{-1}(\cdot)\right)^{-1}, (*)\right\},$$

where $x \cdot y = z \Leftrightarrow x\backslash z = y \Leftrightarrow z/y = x \Leftrightarrow y * x = z$.

It is known [14] that the number of different conjugates in $\Sigma$ can be 1,2,3 or 6.

If a quasigroup operation is denoted by $A$, then a quasigroup $(Q, A)$ (or simply $A$) has the following system $\Sigma$ of conjugates:

$$\Sigma = \left\{A, {}^{r}A, {}^{l}A, {}^{lr}A, {}^{rl}A, {}^{s}A\right\}.$$

Here we use very suitable designation of conjugates of V. D. Belousov from [4], where

$$^{r}\!A = A^{-1}, \quad ^{l}\!A = ^{-1}\!A, \quad ^{lr}\!A = ^{-1}(A^{-1}), \quad ^{rl}\!A = (^{-1}\!A)^{-1}, \quad ^{s}\!A = A^{*},$$

$$A(x,y) = z \Leftrightarrow A^{-1}(x,z) = y \Leftrightarrow ^{-1}\!A(z,y) = x, \quad A^{*}(x,y) = A(y,x).$$

Note that

$$\left(^{-1}(A^{-1})\right)^{-1} = ^{rlr}\!A = ^{-1}\!\left((^{-1}\!A)^{-1}\right) = ^{lrl}\!A = ^{s}\!A$$

and $^{rr}\!A = ^{ll}\!A = A$.

In general $M_{(\cdot)} \neq M_{^{\sigma}(\cdot)}$, where $^{\sigma}(\cdot)$ is some conjugate of $(\cdot)$. But V. D. Belousov proved that the complete multiplication group $\overline{M}_{(\cdot)}$ is always invariant with respect to conjugacy as according to [3]

$$\overline{M}_{^{r}(\cdot)} =< L_a^{-1}, I_a, R_a >, \quad \overline{M}_{^{l}(\cdot)} =< I_a^{-1}, R_a^{-1}, L_a^{-1} >,$$

$$\overline{M}_{^{lr}(\cdot)} =< R_a^{-1}, I_a^{-1}, L_a >, \quad \overline{M}_{^{rl}(\cdot)} =< I_a, L_a^{-1}, R_a^{-1} >,$$

$$\overline{M}_{^{s}(\cdot)} =< R_a, L_a, I_a^{-1} > \text{ for all } a \in Q.$$

A quasigroup operation $(\cdot)$ and its inverse operations $(\backslash)$ and $(/)$ are connected by the identities:

$$x(x\backslash y) = y, \quad x\backslash xy = y, \quad (y/x)x = y, \quad yx/x = y.$$

The quasigroup $(Q, \cdot, \backslash, /)$ is called the *primitive quasigroup* corresponding to a quasigroup $(Q, \cdot)$ [1].

Let $Q$ be a finite or infinite set, $A, B$ be operations on $Q$, then the right, left multiplications $A \cdot B$, $A \circ B$ of Mann are defined in the following way [2]:

$$(A \cdot B)(x,y) = A(x, B(x,y)), \quad (A \circ B)(x,y) = A(B(x,y), y).$$

If $A$ and $B$ are quasigroups, then $A \cdot B$ $(A \circ B)$ is always *invertible from the right (from the left)*, that is the equation $(A \cdot B)(a,y) = b$ $(\,(A \circ B)(x,a) = b\,)$ has a unique solution.

According to *the criterion of Belousov* [2] two quasigroups $(Q, A)$ and $(Q, B)$ are orthogonal (shortly, $A \perp B$) if and only if the operation $A \cdot {}^{r}\!B$ $(A \circ {}^{l}\!B)$ is a quasigroup.

## 3 Orthogonality of a quasigroup to its conjugates and strictly transitive subsets of the multiplication group

Recall that the set $S$ of maps on a set $Q$ is called *strictly transitive* (more precisely, the set $S$ acts on $Q$ strictly transitively) if for any pair of elements $(a,b) \in Q^2$ there exists a unique map $\alpha$ of $S$ such that $\alpha a = b$.

Let $(Q, A)$ be a quasigroup and $(Q, {}^{\sigma}\!A)$ be its conjugate. It is evident that the sets $\{L_a \,|\, a \in Q\}$ and $\{R_a \,|\, a \in Q\}$, where $L_a, R_a$ are translations of $A$ $({}^{\sigma}\!A)$, form strictly transitive subsets in the multiplication group $M_A$ of the respective quasigroup.

We shall show that some strictly transitive subset of the multiplicative group $M_A$ corresponds to the orthogonality $A \perp^\sigma A$.

It is easy to see that if $A \perp B$, then ${}^sA \perp^s B$, so we have the following

**Proposition 1.** *Let $(Q, A)$ be a quasigroup. Then*

$$A \perp^r A \Leftrightarrow {}^s A \perp^{rl} A, \qquad A \perp^l A \Leftrightarrow {}^s A \perp^{lr} A, \qquad A \perp^{rl} A \Leftrightarrow {}^s A \perp^r A,$$

$$A \perp^{lr} A \Leftrightarrow {}^s A \perp^l A, \qquad {}^r A \perp^l A \Leftrightarrow {}^{rl} A \perp^{lr} A, \qquad {}^l A \perp^{rl} A \Leftrightarrow {}^{lr} A \perp^r A.$$

Define the following collection of elements of the multiplication group $M_A$ of a quasigroup $(Q, A)$:

$$\mathcal{L}^2 = \{L_x^2 \mid x \in Q\}, \qquad \mathcal{R}^2 = \{R_x^2 \mid x \in Q\}, \qquad \mathcal{RL} = \{R_x L_x \mid x \in Q\},$$

$$\mathcal{LR} = \{L_x R_x \mid x \in Q\}, \qquad \mathcal{RL}^{-1} = \{R_x L_x^{-1} \mid x \in Q\},$$

where $L_x y = A(x, y)$, $R_x y = A(y, x)$ and the permutations in the products act from the right to the left.

**Theorem 1.** *Let $(Q, A)$ be a quasigroup. Then*

$$A \perp^r A \ ({}^sA \perp^{rl} A) \Leftrightarrow \mathcal{L}^2 \quad \textit{is a strictly transitive subset (s.t.subset) of} \quad M_A;$$
$$A \perp^l A \ ({}^sA \perp^{lr} A) \Leftrightarrow \mathcal{R}^2 \quad \textit{is a s.t.subset of} \quad M_A;$$
$$A \perp^{rl} A \ ({}^sA \perp^{r} A) \Leftrightarrow \mathcal{RL} \quad \textit{is a s.t.subset of} \quad M_A;$$
$$A \perp^{lr} A \ ({}^sA \perp^{l} A) \Leftrightarrow \mathcal{LR} \quad \textit{is a s.t.subset of} \quad M_A;$$
$$A \perp^s A \Leftrightarrow \mathcal{RL}^{-1} \quad \textit{is a s.t.subset of} \quad M_A.$$

*Proof.* By the criterion of Belousov $A \perp^r A$ if and only if the operation $B(x, y) = A(x, A(x, y))$ is a quasigroup, that is the equation $A(x, A(x, a)) = b$ or $L_x^2 a = b$ has a unique solution $x$ for any pair $(a, b) \in Q^2$ as the operation $B$ is always invertible from the right. It means that $\mathcal{L}^2$ is a strictly transitive set.

$A \perp^l A$ if and only if the equation $A(A(a, y), y) = b$ or $R_y^2 a = b$ has a unique solution $y$ for any pair $(a, b) \in Q^2$.

By Proposition 1, $A \perp^{rl} A$ (${}^sA \perp^r A$) if and only if the equations ${}^sA(x, A(x, a)) = b$, $A(A(x, a), x) = b$ , $R_x L_x a = b$ have a unique solution $x$ for any $(a, b) \in Q^2$.

Analogously, $A \perp^{lr} A$ (${}^sA \perp^l A$ ) if and only if the equations $({}^sA \circ A)(a, y) = b$, ${}^sA(A(a, y), y) = A(y, A(a, y)) = b, L_y R_y a = b$ have a unique solution $y$ for any $(a, b) \in Q^2$.

$A \perp^s A$ if and only if the equation $A(x, {}^{lr}A(x, a)) = L_x R_x^{-1} a = b$ has a unique solution since if ${}^{lr}A(x, a) = t$, then ${}^rA(t, a) = x$, $A(t, x) = a$, $t = R_x^{-1} a$.

For the orthogonality ${}^sA \perp^{rl} A$, ${}^sA \perp^{lr} A$ the statements follow from Proposition 1. $\square$

Note that an analog of Theorem 1 for finite quasigroups was proved in [15, Theorem 9].

If $(Q,\cdot)$ is a finite quasigroup then the conditions of conjugate-orthogonality from Theorem 1 are equivalent to some quasi-identities in the primitive quasigroup $(Q,\cdot,\backslash,/)$.

**Corollary 1.** *Let $(Q,A)$ be a finite quasigroup. Then*

$$\begin{aligned}
(\cdot)\perp^r(\cdot) &\Leftrightarrow x\cdot xz = y\cdot yz \Rightarrow x=y;\\
(\cdot)\perp^l(\cdot) &\Leftrightarrow zx\cdot x = zy\cdot y \Rightarrow x=y;\\
(\cdot)\perp^{rl}(\cdot) &\Leftrightarrow xz\cdot x = yz\cdot y \Rightarrow x=y;\\
(\cdot)\perp^{lr}(\cdot) &\Leftrightarrow x\cdot zx = y\cdot zy \Rightarrow x=y;\\
(\cdot)\perp^{s}(\cdot) &\Leftrightarrow (x\backslash z)x = (y\backslash z)y \Rightarrow x=y;\\
&\quad \textit{or } x(z/x) = y(z/y) \Rightarrow x=y.
\end{aligned}$$

*Proof* follows from Theorem 1 if we take into account that

$$L_x^{-1}z = x\backslash z, \quad R_x^{-1}z = z/x \tag{1}$$

and that the strict transitivity of a set of maps $S=\{\alpha_1,\alpha_2,\dots,\alpha_n\}$ on a finite set $Q$ means that $\alpha_i x = \alpha_j x \Rightarrow i=j$ for any $x\in Q$. □

These quasi-identities for the finite case were also established in [15, Theorem 10] and [5, Theorem 1].

From the conditions of conjugate-orthogonality of Theorem 1 some properties of quasigroups (loops) of distinct classes easy follow.

At first we remind (see, for example, [1,12]) that a quasigroup $(Q,\cdot)$ is *diagonal* if the map $x\to xx=x^2$ is a permutation; the *left (right) alternative law* is $x\cdot xy = xx\cdot y$ ($yx\cdot x = y\cdot xx$); the *elastic law* is $xy\cdot x = x\cdot yx$; *a diassociative loop* is a loop any two elements of which generate a subgroup.

**Proposition 2.** *1) If a commutative quasigroup $(Q,A)$ is orthogonal to one of its conjugates different from ${}^sA$, then it is orthogonal to the rest ones (except ${}^sA$). If, in addition, $(Q,A)$ is a loop then it is diagonal.*

*2) If a quasigroup $(Q,A)$ has the right (left) identity $e$ ($f$) and $A\perp^r A$ or $A\perp^{rl}A$ ($A\perp^l A$ or $A\perp^{lr}A$), then it is diagonal.*

*3) If a quasigroup $(Q,A)$ satisfies the left (right) alternative law and $A\perp^r A$ ($A\perp^l A$) then it is diagonal. Conversely, for any diagonal quasigroup with the left (right) alternative law $A\perp^r A$ ($A\perp^l A$). For any diagonal and diassociative loop $A\perp^r A$ and $A\perp^l A$.*

*4) If in a quasigroup $(Q,A)$ the elastic law holds, then $A\perp^{rl}A \Leftrightarrow A\perp^{lr}A$. If $(Q,A)$ is a loop with the elastic low and $A\perp^{rl}A$, then $A$ is diagonal.*

*5) Any diagonal Moufang loop (in particular, a diagonal group) $(Q,A)$ is orthogonal to each of its conjugates, except ${}^sA$.*

*Proof.* 1) In a commutative quasigroup the equality $R_x = L_x$ holds for each $x \in Q$, so all collections $\mathcal{L}^2$, $\mathcal{R}^2$, $\mathcal{RL}$ and $\mathcal{LR}$ coincide. In a loop $(Q,\cdot)$ with the identity $e$ the equations $L_x^2 e = b$, $x \cdot xe = b$, $x^2 = b$ have a unique solution $x$ for any $b \in Q$ if $(\cdot) \perp^r (\cdot)$.

2) If $(\cdot) \perp^r (\cdot)$ ( $(\cdot) \perp^{rl} (\cdot)$), then the equation $L_x^2 e = b$ or $x^2 = b$ ($R_x L_x e = b$ or $x^2 = b$) has a unique solution for any $b \in Q$. Analogously, if $(\cdot) \perp^l (\cdot)$ or $(\cdot) \perp^{lr} (\cdot)$.

3) If $(\cdot) \perp^r (\cdot)$ ($(\cdot) \perp^l (\cdot)$), then $L_x^2 a = x \cdot xa = x^2 \cdot a = b,\ x^2 = b/a$ ($R_x^2 a = ax \cdot x = a \cdot x^2 = b,\ x^2 = a \backslash b$). Conversely, if $(Q,\cdot)$ is diagonal and satisfies the left (right) alternative law then $x^2 = b \Rightarrow x^2 \cdot a = x \cdot xa = ba = c$ ($x^2 = b \Rightarrow a \cdot x^2 = ax \cdot x = ab = c$), where $c$ is any element of $Q$. Thus, the equation $L_x^2 a = c$ ($R_x^2 a = c$) has a unique solution for any $a, c \in Q$. If a loop is diassociative, then it satisfies the left and right alternative lows, so the last statement is true as well.

4) In a quasigroup with the elastic law $\mathcal{RL}$ is a strictly transitive set if and only if $\mathcal{LR}$ is a strictly transitive set, since $R_x L_x a = L_x R_x a$. In a loop with elastic low $R_x L_x e = b \Rightarrow x^2 = b$.

5) Any Moufang loop $(Q, A)$ is diassociative and satisfies the left and the right alternative laws and the elastic low, so $A \perp^r A$, $A \perp^l A$ and $A \perp^{rl} A$ by 3), and $A \perp^{lr} A$ by 4). It is known that any loop $A$ can not be selforthogonal ($A \not\perp^s A$). Indeed, the equation $R_x L_x^{-1} a = a,\ a \neq e$ has two solutions $x = a$ and $x = e$. □

Note that item 5) of Proposition 2 was proved in [5] for finite Moufang loops. It is known that a Moufang loop $(Q, A)$, just as a group, of odd order is diagonal, so by Proposition 2 it is orthogonal to each its conjugate, except $A^*$ (see also [1,5]).

## 4 Orthogonality of conjugates of a quasigroup and strictly transitive subsets of the complete multiplication group

Now we consider conditions for the orthogonality ${}^\sigma\!A \perp {}^\tau\!A$, where ${}^\sigma\!A,\ {}^\tau\!A \neq A$.

Denote ${}^r\!A = (\backslash),\ {}^l\!A = (/)$, then

$$R^{\backslash}_x y = y \backslash x = L_y^{-1} x = I_x y, \quad L^{/}_x y = x/y = R_y^{-1} x = I_x^{-1} y, \tag{2}$$

and

$$L_x^{-1} R^{\backslash}_x = L_x^{-1} I_x, \quad L^{/}_x L_x = I_x^{-1} L_x, \quad R_x^{-1} R^{\backslash}_x = R_x^{-1} I_x, \tag{3}$$

where $y \cdot I_x y = x$ for any $y \in Q$.

Consider the following collections of permutations of the complete multiplication group $\overline{M}_A$ of a quasigroup $(Q, A)$:

$$\mathcal{I}^{-1}\mathcal{L} = \{L^{/}_x L_x \,|\, x \in Q\} = \{I_x^{-1} L_x \,|\, x \in Q\},$$
$$\mathcal{I}^2 = \{(R^{\backslash}_x)^2 \,|\, x \in Q\} = \{I_x^2 \,|\, x \in Q\},$$
$$\mathcal{IL} = \{(L^{/}_x)^{-1} L_x \,|\, x \in Q\} = \{I_x L_x \,|\, x \in Q\},$$
$$\mathcal{R}^{-1}\mathcal{I} = \{R_x^{-1} R^{\backslash}_x \,|\, x \in Q\} = \{R_x^{-1} I_x \,|\, x \in Q\}.$$

**Theorem 2.** *Let $(Q, A)$ be a quasigroup. Then*

$$
\begin{gathered}
{}^{r}\!A \perp {}^{l}\!A \ ({}^{rl}\!A \perp {}^{lr}\!A) \Leftrightarrow \mathcal{I}^{-1}\mathcal{L} \quad \textit{is a s.t.subset of} \quad \overline{M}_A, \\
{}^{r}\!A \perp {}^{lr}\!A \ ({}^{l}\!A \perp {}^{rl}\!A) \Leftrightarrow \mathcal{I}^{2} \quad \textit{is a s.t.subset of} \quad \overline{M}_A, \\
{}^{r}\!A \perp {}^{rl}\!A \Leftrightarrow \mathcal{I}\mathcal{L} \quad \textit{is a s.t.subset of} \quad \overline{M}_A, \\
{}^{l}\!A \perp {}^{lr}\!A \Leftrightarrow \mathcal{R}^{-1}\mathcal{I} \quad \textit{is a s.t.subset of} \quad \overline{M}_A.
\end{gathered}
$$

*Proof.* By the Belousov criterion and Proposition 1:

${}^{l}\!A \perp^{r} A$ (${}^{rl}\!A \perp^{lr} A$) if and only if the equations ${}^{l}\!A(x, A(x,a)) = b$, $L^{/}_{x} L_x a = I_x^{-1} L_x a = b$ have a unique solution $x$ for any $(a,b) \in Q^2$. Thus, $\mathcal{I}^{-1}\mathcal{L}$ is a s.t.subset of $\overline{M}_A$.

${}^{r}\!A \perp^{lr} A$ (${}^{l}\!A \perp^{rl} A$) if and only if the equations ${}^{l}\!A(x, {}^{l}\!A(x,a)) = b$, $(L^{/}_{x})^2 a = b$, $I_x^2 b = a$ have a unique solution $x$ for any $(a,b) \in Q^2$. Thus, $\mathcal{I}^2$ is a s.t.subset of $\overline{M}_A$.

${}^{r}\!A \perp^{rl} A$ if and only if the equations ${}^{r}\!A(x, {}^{l}\!A(x,a)) = b$, $A(x,b) = {}^{l}\!A(x,a)$, $L_x b = L^{/}_{x} a = I_x^{-1} a$ , $I_x L_x b = a$ have a unique solution $x$ for any $(a,b) \in Q^2$. Hence, $\mathcal{I}\mathcal{L}$ is a s.t.subset of $\overline{M}_A$.

And finally, ${}^{l}\!A \perp^{lr} A$ if and only if the equations ${}^{l}\!A(x, {}^{s}\!A(x,a)) = b$, $A(b, A(a,x)) = x$, ${}^{r}\!A(b,x) = A(a,x)$, $R^{\backslash}_{x} b = R_x a$, $R_x^{-1} I_x b = a$ have a unique solution, that is $\mathcal{R}^{-1}\mathcal{I}$ is a s.t.subset of $\overline{M}_A$.

The rest four cases of possible orthogonality of conjugates were considered in Theorem 1. □

*Remark 1.* The conditions of Theorem 2 can be also obtained from Theorem 1 if instead of a quasigroup $A$ one takes the corresponding conjugate.

*Remark 2.* Note that there are quasigroups all subsets of Theorem 1 and Theorem 2 are strictly transitive. All conjugates of these quasigroups are distinct and pairwise orthogonal. An example of such quasigroup over the field of rational numbers: $xy = 2x + 3y$ is given by V.D. Belousov in [4, p. 66].

**Corollary 2.** *If $(Q, \cdot)$ is a finite quasigroup, then*

$$
\begin{gathered}
{}^{r}(\cdot) \perp^{l}(\cdot) \ ({}^{rl}(\cdot) \perp^{lr}(\cdot)) \Leftrightarrow x/(xz) = y/(yz) \Rightarrow x = y \text{ or } x\backslash(z\backslash x) = y\backslash(z\backslash y) \Rightarrow x = y, \\
{}^{r}(\cdot) \perp^{lr}(\cdot) \ ({}^{l}(\cdot) \perp^{rl}(\cdot)) \Leftrightarrow (z\backslash x)\backslash x = (z\backslash y)\backslash y \Rightarrow x = y \text{ or} \\
x/(x/z) = y/(y/z) \Rightarrow x = y, \\
{}^{r}(\cdot) \perp^{rl}(\cdot) \Leftrightarrow xz\backslash x = yz\backslash y \Rightarrow x = y \text{ or } x\backslash(x/z) = y\backslash(y/z) \Rightarrow x = y, \\
{}^{l}(\cdot) \perp^{lr}(\cdot) \Leftrightarrow (z\backslash x)/x = (z\backslash y)/y \Rightarrow x = y \text{ or } x/(zx) = y/(zy) \Rightarrow x = y.
\end{gathered}
$$

*Proof.* Prove the first quasi-identities in every pair of equivalent ones. The second quasi-identity can be obtained by change of the corresponding strictly transitive set by the set with inverse permutations and taking into account (2), (3).

$$
\begin{gathered}
\mathcal{I}^{-1}\mathcal{L}: \ L^{/}_{x} L_x z = L^{/}_{y} L_y z \Rightarrow x = y \text{ or } x/(xz) = y/(yz) \Rightarrow x = y, \\
\mathcal{I}^{2}: \ (R^{\backslash}_{x})^2 z = (R^{\backslash}_{y})^2 z \Rightarrow x = y \text{ or } (z\backslash x)\backslash x = (z\backslash y)\backslash y \Rightarrow x = y, \\
\mathcal{I}\mathcal{L}: \ R^{\backslash}_{x} L_x z = R^{\backslash}_{y} L_y z \Rightarrow x = y \text{ or } (xz)\backslash x = (yz)\backslash y \Rightarrow x = y, \\
\mathcal{R}^{-1}\mathcal{I}: \ R_x^{-1} R^{\backslash}_{x} z = R_y^{-1} R^{\backslash}_{y} z \Rightarrow x = y \text{ or } (z\backslash x)/x = (z\backslash y)/y \Rightarrow x = y.
\end{gathered}
$$

□

The following proposition eliminates the orthogonality of some conjugates (${}^{\sigma}\!A \not\perp^{\tau} A$) for quasigroups of some classes.

**Proposition 3.** *1) If $(Q, A)$ is a commutative quasigroup, then ${}^{r}\!A \perp^{l} A \Leftrightarrow {}^{r}\!A \perp^{rl} A \Leftrightarrow^{l} A \perp^{lr} A \Leftrightarrow^{rl} A \perp^{lr} A$ and $A \not\perp^{s} A$, ${}^{l}\!A \not\perp^{rl} A$, ${}^{r}\!A \not\perp^{lr} A$.*

*2) If a quasigroup $(Q, A)$ has the right (left) identity $e$ $(f)$, then ${}^{r}\!A \not\perp^{rl} A$ (${}^{l}\!A \not\perp^{lr} A$).*

*3) If $(Q, A)$ is an IP-quasigroup then ${}^{r}\!A \not\perp^{rl} A$ and ${}^{l}\!A \not\perp^{lr} A$.*

*4) For a loop $(Q, A)$ $A \not\perp^{s} A$ and the orthogonality of conjugates from Theorem 2 is impossible.*

*Proof.* 1) The first statements follows from Proposition 1 and Theorem 2 since in a commutative quasigroup $I_x = I_x^{-1}$, $R_x = L_x$, so $\mathcal{I}^{-1}\mathcal{L} = \mathcal{I}\mathcal{L}$ and $\mathcal{R}^{-1}\mathcal{I}$ is a s.t.subset of $\overline{M}_A$. if and only if $\mathcal{I}^{-1}\mathcal{R} = \mathcal{I}\mathcal{L}$ is a s.t.subset of $\overline{M}_A$. In a commutative quasigroup $R_a^{-1}x = L_a^{-1}x$, so $x/a = a\backslash x$, $(a\backslash x)a = x$, $(R_x^{\backslash})^2 a = (a\backslash x)\backslash x = a$ for any $x \in Q$. Hence, $I^2$ is not strictly transitive and so ${}^{r}\!A \not\perp^{lr} A$, ${}^{l}\!A \not\perp^{rl} A$ by Theorem 2. $A \not\perp^{s} A$ in view of Theorem 1 since in this case $\mathcal{R}^{-1}\mathcal{L} = \varepsilon$ (the identity permutation).

2) By Theorem 2 ${}^{r}(\cdot) \perp^{rl}(\cdot)$ if and only if the equations $R_x^{\backslash} L_x a = b$, $(xa)\backslash x = b$, $xa \cdot b = x$, $R_b R_a x = x$ have a unique solution $x$ for any $(a, b) \in Q^2$, ${}^{l}(\cdot) \perp^{lr} (\cdot)$ if and only if the equations $R_x^{-1} R_x^{\backslash} a = b$, $a\backslash x = bx$, $L_a^{-1}x = L_b x$, $L_a L_b x = x$ have a unique solution for any $(a, b) \in Q^2$. But by $a = b = e$ $(a = b = f)$ $R_e R_e x = x$ $(L_f L_f x = x)$ for any $x \in Q$, so ${}^{r}(\cdot) \not\perp^{rl}(\cdot)$ and ${}^{l}(\cdot) \not\perp^{lr}(\cdot)$.

3) Let $(Q, A)$ be an $IP$-quasigroup, then $R_a^{-1} = R_{I_r a}$, $L_a^{-1} = L_{I_l a}$ and $R_{I_r a} R_a x = R_a^{-1} R_a x = x$, $L_{I_l a} L_a x = L_a^{-1} L_a x = x$ for any $x \in Q$, so as above ${}^{r}\!A \not\perp^{rl} A$ and ${}^{l}\!A \not\perp^{lr} A$.

4) Let $(Q, \cdot)$ be a loop with the identity $e$. Then ${}^{r}(\cdot) \not\perp^{rl}(\cdot)$ and ${}^{l}(\cdot) \not\perp^{lr}(\cdot)$ by item 2).

${}^{r}(\cdot) \not\perp^{l}(\cdot)$ (${}^{rl}(\cdot) \not\perp^{lr}(\cdot)$) in view of Theorem 2 as $I_x^{-1} L_x e = L_x^{/} L_x e = x/(xe) = e$ for any $x \in Q$ and

${}^{r}(\cdot) \not\perp^{lr}(\cdot)$ (${}^{l}(\cdot) \not\perp^{rl}(\cdot)$) by Theorem 2 since $I_x^2 e = R_x^{\backslash} R_x^{\backslash} e = (e\backslash x)\backslash x = e$ for any $x \in Q$. □

## References


[1] Belousov V. D. *Foundations of the theory of quasigroups and loops.* Moscow, Nauka, 1967 (in Russian).

[2] Belousov V. D. *Systems of orthogonal operations.* Mat. sbornik, 1968, **77(119):1**, 38–58 (in Russian).

[3] Belousov V. D. *On the group associated with a quasigroup.* Mat. Issled., 1969, **4**, No. 3, 21–39 (in Russian).

[4] Belousov V. D. *Parastrophic-orthogonal quasigroups.* Quasigroups and related systems, 2005, **13**, No. 1, 25–72.

[5] Belyavskaya G. B., Diordiev A. D. *On some quasi-identities in finite quasigroups.* Buletinul Academiei de ştiinţe a Republicii Moldova, Matematica, 2005, No. 3(49), 19–32.

[6] Bennet F. E. *The spectra of a variety of quasigroups and related combinatorial designs.* Discrete Mathematics, 1989, **77**, 29–50.

[7] Bennet F. E. *Latin squares with pairwise orthogonal conjugates.* Discrete Mathematics, 1981, **36**, 117–137.

[8] Bennet F. E. *On conjugate orthogonal idempotent Latin squares.* Ars. Combinatorica, 1985, **19**, 37–50.

[9] Bennet F. E., Mendelsohn N. S. *Conjugate orthogonal Latin square graphs.* Congressus Numerantium, 1979, **23**, 179–192.

[10] Bennet F. E., Hantao Zhang. *Latin squares with self-orthogonal conjugates.* Discrete Mathematics, 2004, **284**, 45–55.

[11] Chaffer R. A., Lieberman D. J., Smith D. D. *The number of orthogonal conjugates of a quasigroup.* Congressus Numerantium, 1982, 169–180.

[12] Déneš J., Keedwell A. D. *Latin squares and their applications.* Académiai Kiado, Budapest and Academic Press, New York, 1974.

[13] Evans T. *Algebraic structures associated with Latin squares and orthogonal arrays.* Proc. Conf. Algebraic Aspects of Combinatorics, Congressus Numerantium, 1975, **13**, 31–52.

[14] Lindner C. C., Steedly D. *On the number of conjugates of a quasigroup.* Algebra Univ., 1975, **5**, 191–196.

[15] Mullen G., Shcherbacov V. *On orthogonality of binary operations and squares.* Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, 2005, No. 2(48), 3–42.

[16] Phelps K. T. *Conjugate orthogonal quasigroups.* J. Combin. Theory (A), 1978, **25**, 117–127.

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# The graded Jacobson radical of associative rings

B. J. Gardner, A. Plant

**Abstract.** We introduce a consistent definition for the graded Jacobson radical for group graded rings without unity. We compare the graded Jacobson radical for rings with unity and those without. We find that for group graded rings, the descriptions are equivalent.

**Mathematics subject classification:** 34C05, 58F14.
**Keywords and phrases:** Graded ring, Jacobson radical.

In the book of Năstăsescu and Van Oystaeyen [1] on group graded rings, two equivalent descriptions of the graded Jacobson radical for rings with unity are given. Several investigations of the graded Jacobson radical have appeared over the last two decades (see [2–6] or [7] for example) all for rings with unity. In [9] a comprehensive account of special radicals of graded rings without unity was presented. Unfortunately the descriptions given in the section for the Jacobson radical came (in the most part) from [1] on group graded rings with unity. After an extensive literature search, it seems that no actual definition of the graded Jacobson radical for rings without unity has appeared. The definition given here is the most natural one – the intersection of the annihilators of all simple graded modules – and it is meaningful more generally for semigroup graded rings, though for semigroups in general it may not be a graded ideal.

As an example of a consequence of this investigation, we show that a 1984 result of Năstăsescu [3] that $n\mathcal{J}(R) \subseteq \mathcal{J}_{gr}(R)$ (for a finite group $G$ of order $n \in \mathbb{Z}^+$ where $R$ is a $G$-graded ring with unity and $\mathcal{J}_{gr}$ is the $G$-graded Jacobson radical) can be extended to group graded rings without unity.

## 1 Unital Extensions of graded rings

By a unital extension of a ring, we mean an embedding of a ring $R$ without unity into a ring $R^u$ with unity. We do this in the standard way (see [10] for example). We reserve the use of $R^u$ to always mean the unital extension of $R$. Thus $R^u$ is made up of the additive group $R \bigoplus \mathbb{Z}$, where $\mathbb{Z}$ is the ring of integers. Elements in $R \bigoplus \mathbb{Z}$ are denoted by ordered pairs $\{(r,n) : r \in R, n \in \mathbb{Z}\}$ with componentwise addition and multiplication defined by

$$(r,n)(s,m) = (rs + mr + ns, nm)$$

where $r, s \in R$ with $n, m \in \mathbb{Z}$.

**Lemma 1** ([8], p.136)**.** *Let $R$ be a ring without identity, and let $R^u$ be the standard unital extension of $R$. Then $\mathcal{J}(R) = \mathcal{J}(R^u)$, where $\mathcal{J}$ is the Jacobson radical.*

For this investigation we require specifically that $R$ be group graded. This allows us to place the identity element carefully into our graded ring without causing major offence to the structure of our ring. So we begin with a ring $R$ graded by a group $G$ with group identity $e$. Any element $r \in R$ can be written uniquely as $r = \sum_{g\in G} r_g$, where $r_g \in R_g$ for each $g \in G$. We embed our $G$-graded ring $R$ into $R^u$ as above. We identify $R$ with its copy in $R^u$ and since $(0,1)R_g \subseteq R_g$ and $R_g(0,1) \subseteq R_g$ for all $g \in G$, we can grade $R^u$ by putting the identity element $(0,1)$ in the $e$ component, whence $R^u$ becomes $G$-graded with

$$R^u = R^u_e \oplus \bigoplus_{g\in G\setminus\{e\}} R_g.$$

For any $r \in R$ we have

$$(r,n) = (r_e, n) + \sum_{\substack{g \in G \\ g \neq e}} (r_g, 0).$$

(Recalling that $R_e$ is a subring of $R$, we can see that the $e$ component in $R^u$ is just given by the standard unital extension of $R_e$ in $R$.)

## 2 Graded ideals and modules

Let $G$ be a group or semigroup and suppose $I$ is an ideal (left, right or two-sided) of a $G$-graded ring $R$. Then $I$ is said to be a *$G$-graded ideal* if

$$I = \bigoplus_{g\in G} (I \cap R_g) = \bigoplus_{g\in G} I_g$$

(so that $I$ as a ring is $G$-graded). A $G$-graded left (right, two-sided) ideal $M$ of $R$ is a *$G$-graded-maximal* left (right, two-sided) ideal if $M \neq R$ and $M$ is not contained in any other proper $G$-graded left (right, two-sided) ideals of $R$.

Let $G$ be a group or semigroup. A left module $T$ over a $G$-graded ring $R$ is a *$G$-graded left module* if there exist additive subgroups $T_g$ of $T$ with

$$T = \bigoplus_{g\in G} T_g$$

and $R_x T_y \subseteq T_{xy}$ for all $x, y \in G$. We suppress the adjective "left" throughout, but note that a development based on right modules is also possible.

Let $G$ be a group or semigroup. A $G$-graded module $T$ over a $G$-graded ring $R$ is a *graded-simple module* if $T \neq 0$ and $0$ and $T$ are its only graded submodules.

The annihilator of any $G$-graded module $T$ is

$$\mathcal{A}(T) = \{a \in R : at = 0 \text{ for all } t \in T\}.$$

Annihilators of modules are ideals.

## 3 $\mathcal{J}_{gr}(R)$ for rings with unity

We describe the graded Jacobson radical here for group graded rings *with unity.* In Section 4 we give an equivalent description of the graded Jacobson radical for rings without unity.

Let $G$ be a group with identity element $e$ and $R$ a $G$-graded ring with unity. In this case the *graded Jacobson radical* $\mathcal{J}_{gr}(R)$ of $R$ is defined to be the intersection of all $S$-graded-maximal left ideals of $R$.

In [1] the equivalence of other definitions of the graded Jacobson radical of a group graded ring is shown. One of these defines the graded Jacobson radical as the intersection of all left annihilators of all $G$-graded-simple $R$-modules.

Recently, Abrams and Menini [7] considered the graded Jacobson radical of graded rings with unity, extending the definition to include semigroup-graded rings. In this case the graded Jacobson radical is defined to be the intersection of all left annihilators of all $G$-graded-simple $R$-modules.

## 4 $\mathcal{J}_{gr}(R)$ for rings without unity

The Jacobson radical of a ring without unity has a handful of equivalent descriptions (see [11] for example) including one as the intersection of modular maximal left ideals. For rings with unity, all ideals are modular and so the wording of the definition is altered slightly. In both cases, the equivalent definition of the Jacobson radical as the intersection of all the left annihilators of simple left modules is the same. It seems then that the natural choice for defining the graded Jacobson radical for group graded rings without unity is as the intersection of annihilators of simple modules, coincident with the definition of the graded Jacobson radical in the case the ring has unity.

For a ring $R$ graded by a semigroup $G$, we define the *graded Jacobson radical* of $R$ as the intersection of left annihilators of all $G$-graded-simple $R$-modules. We use the $_{gr}$ here to indicate a graded structure. It turns out that as long as $G$ is a group, $\mathcal{J}_{gr}(R)$ is always a graded ideal. For more general semigroups this need not be so.

**Example 1** ([9], Example 6)**.** Consider the set $S = \{(1,1),(1,2),(2,1),(2,2)\}$ under

$$(r,s)\cdot(t,u) = (r,u) \qquad (r,s,t,u \in \{1,2\}).$$

Then $(S,\cdot)$ forms a rectangular band. The semigroup ring $A = kS$ with coefficients in a field $k$ is $S$-graded in the usual way, that is

$$A = kS = k(1,1) \oplus k(1,2) \oplus k(2,1) \oplus k(2,2).$$

Then the element $(1,1)-(1,2)$ annihilates M, for any simple $A$-module $M$. (Note that $S \subseteq A$ as $k$ is unital.) This puts $(1,1)-(1,2)$ in $\mathcal{J}_{gr}(A)$. Now let $N = A(1,1)$. Then $N$ is a graded-simple $A$-module but $(1,1)$ doesn't annihilate $N$. The consequence is that $(1,1) \notin \mathcal{J}_{gr}(A)$. This means that $\mathcal{J}_{gr}(A)$ is actually an *ungraded* ideal of $A$.

**Theorem 1.** *Let $G$ be a group and let $R$ be a $G$-graded ring without unity. Then $\mathcal{J}_{gr}(R) = \mathcal{J}_{gr}(R^u)$ where $\mathcal{J}_{gr}$ is the $G$-graded Jacobson radical and $R^u$ is the unital extension of $R$.*

*Proof.* Any $R$-module $M$ becomes a unital $R^u$-module if we define

$$(r, n)m = rm + nm$$

for $(r, n) \in R^u$ and $m \in M$.

Let $Y$ be a $G$-graded $R$-module with $Y = \oplus_g Y_g$ and $R_h Y_g \subseteq RY_{hg}$. For $(r_e, n) \in R^u_e$ and any $y_g \in Y_g$ we have $(r_e, n)y_g = r_e y_g + ny_g \in Y_g$ and so $Y$ is a $G$-graded unital $R^u$-module. Any $G$-graded unital $R^u$-module is a $G$-graded $R$-module

Similarly, if $K$ is an $G$-graded $R$-submodule of a $G$-graded $R$-module $M$, then $K$ is a unital $G$-graded $R^u$-submodule of the $G$-graded unital $R^u$-module $M$, and *vice versa.*

So any $G$-graded-simple $R$-module is also a unital $G$-graded-simple $R^u$-module, and *vice versa.*

Suppose $M$ is any $G$-graded-simple $R$-module with left annihilator $\mathcal{A}(M)$. Take any $a \in \mathcal{A}(M)$. Then $a = \sum_{g\in G} a_g$. For any $h \in G$, pick an $m \in M_h$ (since $M$ is graded). Then

$$0 = \left(\sum_{g\in G} a_g\right) m = \sum_{g\in G} a_g m$$

where $a_g m \in M_{gh}$ for each $g \in G$. Since the sum runs over distinct $gh$ (here $G$ is a group), we have $a_g m = 0$ for all $g \in G$, and so all the homogeneous components $a_g \in \mathcal{A}(M)$. Thus the annihilator is a $G$-graded ideal:

$$\mathcal{A}(M) = \bigoplus_{g\in G} R_g \cap \mathcal{A}(M) = \bigoplus_{g\in G} \mathcal{A}(M)_g.$$

In the same way the set

$$\mathcal{A}(M)^u = \{r \in \mathbb{R}^u : rm = 0 \ \ \forall\, m \in M\}$$

is a graded ideal of $R^u$.

For any $a \in \mathcal{A}(M)$, the element $(a, 0) \in R^u$ is in $\mathcal{A}(M)^u$ since $(a, 0)m = am = 0$, so, $\mathcal{A}(M) \subseteq \mathcal{A}(M)^u$. It is clear to see that the elements $(a, 0) \in R^u$ behave exactly as the elements $a \in R$. So to compare $\mathcal{A}(M)$ with its unital extension, we need only consider the $e$-component.

Suppose there is an $(r_e, n) \in \mathcal{J}(R^u)\backslash\mathcal{J}(R)$. Then $(r_e, n) \in \mathcal{A}(M)^u_e = \mathcal{A}(M)^u \cap R^u_e$ with $n \neq 0$. Thus for all graded simple $R$-modules $M$ we have

$$(r_e, n)m = r_e m + nm = 0, \qquad \text{for all} \quad m \in M.$$

This means that multiplication of an element in any simple module by $r_e$ has the same effect as multiplying by $-n \in \mathbb{Z}$. For every prime $p$, $\mathbb{Z}_p$ is a simple $R$-module with trivial multiplication. It is graded-simple if we let $\mathbb{Z}_p$ be the $e$ component, with all other components being equal to zero. For all $x \in \mathbb{Z}_p$ we now have $0 = (r_e, n)x = r_e x + nx = nx$, so $n$ is divisible by $p$. This being so for every $p$, we conclude that $n = 0$.

This means that no extra killers are admitted by unital extension. Hence, if $R$ is a ring without unity, then $\mathcal{J}_{gr}(R) = \mathcal{J}_{gr}(R^u)$. $\square$

As an example of the potential application of Theorem 1 we extend a theorem of Năstăsescu for finite group graded rings with unity to include rings without unity.

**Theorem 2** ([12], [3], Theorem 5.4)**.** *Let $G$ be a finite group of order $n \in \mathbb{Z}^+$ and let $R$ be a $G$-graded ring with unity. Then $n\mathcal{J}(R) \subseteq \mathcal{J}_{gr}(R)$ where $\mathcal{J}(R)$ is the Jacobson radical of $R$ and $\mathcal{J}_{gr}$ is the $G$-graded Jacobson radical.*

**Corollary 1.** *Let $G$ be a finite group of order $n \in \mathbb{Z}^+$ and let $R$ be a $G$-graded ring with or without unity. Then $n\mathcal{J}(R) \subseteq \mathcal{J}_{gr}(R)$ where $\mathcal{J}_{gr}$ is the $G$-graded Jacobson radical.*

*Proof.* If $R$ is a ring graded by a finite group with unity, then this is just Theorem 2. Otherwise, applying Lemma 1 and Theorem 1 yields $n\mathcal{J}(R) = n\mathcal{J}(R^u) \subseteq \mathcal{J}_{gr}(R^u) = \mathcal{J}_{gr}(R)$ which completes the proof. $\square$

## References


[1] Năstăsescu C., Van Oystaeyen F. *Graded Ring Theory.* North-Holland, Amsterdam-New York, 1982.

[2] Cohen M., Rowen L. H. *Group graded rings.* Commun. Algebra, 1983, **11**, 1253–1270.

[3] Năstăsescu C. *Group rings of graded rings.* J. Pure Appl. Algebra, 1984, **33**, 313–335.

[4] Cohen M., Montgomery S. *Group-graded rings, smash products, and group actions.* Trans. Amer. Math. Soc., 1984, **282**, 237–258.

[5] Beattie M. A. *A generalization of the smash product of a graded ring.* J. Pure Appl. Algebra, 1988, **51**, 219–226.

[6] Karpilovsky G. *The Jacobson Radical of Classical Rings.* Wiley and Sons, New York, 1991.

[7] Abrams G., Menini C. *Embedding modules in graded modules over a semigroup-graded ring.* Comm. Algebra, 2001, **29**, 2611-2625.

[8] Kertész A. *Vorlesungen über artinsche Ringe.* Akadémiai Kiadó, Budapest, 1968.

[9] Balaba I. N. *Special radicals of graded rings.* Bul. Acad. Ştiinţe Repub. Mold., Mat., 2004, No. 1(44), 2004, 26–33.

[10] Hungerford T. W. *Algebra*. Springer, New York, 1974.

[11] Gardner B. J., Wiegandt R. *Radical Theory of Rings*. Marcel Dekker, New York, 2004.

[12] Bergman G. M. *On the Jacobson radical of graded rings*. Unpublished typescript.

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# Applications of the integral operator to the class of meromorphic functions

Camelia Mădălina Bălăeţi

**Abstract.** By using the Sălăgean integral operator $I^n f(z)$, $z \in U$, we introduce a class of holomorphic functions denoted by $\Sigma_k(\alpha, n)$ and we obtain an inclusion relation related to this class and some differential subordinations.

**Mathematics subject classification:** 30C45.
**Keywords and phrases:** Differential subordination, integral operator, meromorphic function.

## 1 Introduction and preliminaries

We denote the complex plane by $\mathbb{C}$ and the open unit disc by $U$

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

with $\dot{U} = U - \{0\}$.

Let $\mathcal{H}(U)$ denote the class of holomorphic functions in $U$.

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we have

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + \dots,\ z \in U\},$$
$$A_n = \left\{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \dots,\ z \in U\right\}$$

with $A_1 = A$.

For integer $k \geq 0$, denote by $\Sigma_k$ the class of meromorphic functions, defined in $\dot{U}$, which are of the form

$$f(z) = \frac{1}{z} + \sum_{n=k}^{\infty} a_n z^n.$$

A function $f \in \mathcal{H}(U)$ is said to be convex if it is univalent and $f(U)$ is a convex domain. The function $f$ is convex if and only if $f'(0) \neq 0$ and $\operatorname{Re}\left[\frac{zf''(z)}{f'(z)} + 1\right] > 0$, for $z \in U$ (see [2]).

We denote

$$K = \left\{f \in A,\ \operatorname{Re}\left[\frac{zf''(z)}{f'(z)} + 1\right] > 0,\ z \in U\right\}$$

the set of convex functions.

Let $f$ and $g$ be two analytic functions in $U$. The function $f$ is said to be subordinate to $g$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a function $w$ analytic in $U$, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = g(w(z))$, $z \in U$.

If $g$ is univalent, then $f \prec g$ if $f(0) = g(0)$ and $f(U) \subset g(U)$.

**Definition 1** ([2])**.** Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let $h$ be univalent in $U$. If $p$ is analytic in $U$ and satisfies the (second-order) differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \ z \in U \tag{1}$$

then $p$ is called a solution of the differential subordination. The univalent function $q$ is called a dominant of the solutions of the differential subordination, if $p \prec q$ for all $p$ satisfying (1). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec p$ for all dominants $q$ of (1) is said to be the best dominant of (1).

Note that the best dominant is unique up to a rotation of $U$.

If we require the more restrictive condition $p \in \mathcal{H}[a, n]$, then $p$ will be called an $(a, n)$ solution, $q$ an $(a, n)$ dominant and $\widetilde{q}$ the best $(a, n)$ dominant.

We will need the following lemma, which is due to D. J. Hallenbeck and St. Ruscheweyh.

**Lemma 1** ([1])**.** *Let $h$ be a convex in $U$, with $h(0) = a$, $\gamma \neq 0$ and* $\mathrm{Re}\gamma \geq 0$. *If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z), \ z \in U$$

*then*

$$p(z) \prec q(z) \prec h(z)$$

*where*

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t)t^{\frac{\gamma}{n}-1}dt.$$

*The function $q$ is convex and it is the best $(a, n)$ dominant.*

The following lemma is due to S. S. Miller and P. T. Mocanu.

**Lemma 2** ([3])**.** *Let $q$ be a convex function in $U$ and let*

$$h(z) = q(z) + n\beta zq'(z)$$

*where $\beta > 0$ and $n$ is a positive integer. If $p \in \mathcal{H}[q(0), n]$ and*

$$p(z) + \beta zp'(z) \prec h(z),$$

*then*

$$p(z) \prec q(z)$$

*and this result is sharp.*

**Lemma 3** ([2]). *Let $f \in A$, $\gamma > 1$ and $F$ is given by*

$$F(z) = \frac{1+\gamma}{z^{\frac{1}{\gamma}}} \int_0^z f(t) t^{\frac{1}{\gamma}-1} dt.$$

*If*

$$\mathrm{Re}\frac{zf''(z)}{f'(z)} + 1 > -\frac{1}{2},\ z \in U$$

*then $F \in K$.*

**Definition 2** ([5]). For $f \in \mathcal{H}(U)$, $f(0) = 0$ and $n \in \mathbb{N}$ we define the operator $I^n f$ by

$$I^0 f(z) = f(z),$$

$$I^1 f(z) = If(z) = \int_0^z f(t) t^{-1} dt,$$

$$I^n f(z) = I[I^{n-1} f(z)],\ z \in U.$$

*Remark* 1. For $n = 1$, $I^n f$ is the Alexander operator.

*Remark* 2. If we denote $l(z) = -\log(1-z)$, then

$$I^n f(z) = [\underbrace{(l * l * \cdots * l)}_{n-times} * f](z),\ f \in \mathcal{H}(U), f(0) = 0.$$

By " $*$ " we denote the Hadamard product or convolution (i.e. if $f(z) = \sum_{j=0}^{\infty} a_j z^j$, $g(z) = \sum_{j=0}^{\infty} b_j z^j$, then $(f * g)(z) = \sum_{j=0}^{\infty} a_j b_j z^j$).

*Remark* 3. $I^n f(z) = \int_0^z \int_0^{t_n} \dots \int_0^{t_2} \frac{f(t_1)}{t_1 t_2 \dots t_n} dt_1 dt_2 \dots dt_n$.

*Remark* 4. $D^n I^n f(z) = I^n D^n f(z) = f(z)$, $f \in \mathcal{H}(U)$, $f(0) = 0$, where $D^n f$ is the Sălăgean differential operator.

## 2 Main results

**Definition 3.** If $0 \le \alpha < 1$, $k$ positive integer and $n \in \mathbb{N}$, let $\Sigma_k(\alpha, n)$ denote the class of functions $f \in \Sigma_k$ which satisfy the inequality

$$\mathrm{Re}\left[I^n(z^2 f(z))\right]' > \alpha,\ z \in \dot{U}. \tag{2}$$

**Theorem 1.** *If $0 \le \alpha < 1$, $k$ positive integer and $n \in \mathbb{N}$, then*

$$\Sigma_k(\alpha, n) \subset \Sigma_k(\delta, n+1), \tag{3}$$

*where*

$$\delta = \delta(\alpha, n) = 2\alpha - 1 + 2(1-\alpha)\frac{1}{k+1}\beta\left(\frac{1}{k+1}\right)$$

*and*

$$\beta(x) = \int_0^z \frac{t^{x-1}}{1+t}dt.$$

*Proof.* Assume that $f \in \Sigma_k(\alpha, n)$. By using the properties of the operator $I^n f$ we have

$$I^n(z^2 f(z)) = z\left[I^{n+1}(z^2 f(z))\right]', \ z \in \dot{U}. \tag{4}$$

Differentiating this equality, we obtain

$$\left[I^n(z^2 f(z))\right]' = \left[I^{n+1}(z^2 f(z))\right]' + z\left[I^{n+1}(z^2 f(z))\right]''. \tag{5}$$

If we let

$$\left[I^{n+1}(z^2 f(z))\right]' = p(z)$$

with $p(z) \in \mathcal{H}[1, k+1], \ z \in \dot{U}$, then (5) becomes

$$\left[I^{n+1}(z^2 f(z))\right]' = p(z) + zp'(z), \ z \in \dot{U}.$$

Since $f \in \Sigma_k(\alpha, n)$, from Definition 3 we have

$$\operatorname{Re}[p(z) + zp'(z)] > \alpha, \ z \in \dot{U}$$

which is equivalent to

$$p(z) + zp'(z) \prec \frac{1 + (2\alpha - 1)z}{1+z} \equiv h(z), \ z \in \dot{U}.$$

Therefore, from Lemma 1 for $\gamma = 1$, it results that

$$p(z) \prec q(z) \prec h(z), \ z \in \dot{U},$$

where

$$q(z) = \frac{1}{(k+1)z^{\frac{1}{k+1}}}\int_0^z \frac{1+(2\alpha-1)t}{1+t}t^{\frac{1}{k+1}-1}dt$$

$$= (2\alpha - 1) + 2(1-\alpha)\frac{1}{k+1}\beta\left(\frac{1}{k+1}\right)\frac{1}{z^{\frac{1}{k+1}}}.$$

Moreover, the function $q$ is convex and is the best dominant.

From $p(z) \prec q(z),\ z \in \dot{U}$ it results that

$$\mathrm{Re}p(z) > \mathrm{Re}q(1) = \delta = (2\alpha - 1) + 2(1-\alpha)\frac{1}{k+1}\beta\left(\frac{1}{k+1}\right).$$

But

$$\left[I^{n+1}(z^2 f(z))\right]' = p(z)$$

and

$$\mathrm{Re}\left[I^{n+1}(z^2 f(z))\right]' > \delta,$$

from Definition 3 we have $f \in \Sigma_k(\delta, n+1)$. □

**Theorem 2.** *Let $q$ be a convex function, $q(0) = 1$ and let $h$ be a function such that*

$$h(z) = q(z) + z(k+1)q'(z),\ z \in U.$$

*If $f \in \Sigma_k(\alpha, n)$ and satisfies the differential subordination*

$$\left[I^n(z^2 f(z))\right]' \prec h(z),\ z \in \dot{U} \tag{6}$$

*then*

$$\left[I^{n+1}(z^2 f(z))\right]' \prec q(z),\ z \in \dot{U}$$

*and this result is sharp.*

*Proof.* By using the properties of the operator $I^n f$ we have

$$I^n(z^2 f(z)) = z\left[I^{n+1}(z^2 f(z))\right]',\ z \in \dot{U}. \tag{7}$$

By differentiating (7), we obtain

$$\left[I^n(z^2 f(z))\right]' = \left[I^{n+1}(z^2 f(z))\right]' + z\left[I^{n+1}(z^2 f(z))\right]''. \tag{8}$$

If we let

$$\left[I^{n+1}(z^2 f(z))\right]' = p(z),$$

with $p(z) \in \mathcal{H}[1, k+1]$ then we obtain

$$p(z) + zp'(z) \prec h(z) = q(z) + z(k+1)q'(z),\ z \in \dot{U}.$$

By using Lemma 2 for $\beta = 1$, we have

$$p(z) \prec q(z),\ z \in \dot{U},$$

or

$$\left[I^{n+1}(z^2 f(z))\right]' \prec q(z),\ z \in \dot{U}$$

and this result is sharp. □

**Theorem 3.** *Let $q$ be a convex function with $q(0) = 1$ and*

$$h(z) = q(z) + z(k+1)q'(z),\ z \in U.$$

*If $f \in \Sigma_k(\alpha, n)$ and satisfies the differential subordination*

$$\left[I^n(z^2 f(z))\right]' \prec h(z),\ z \in \dot{U} \tag{9}$$

*then*

$$\frac{I^n(z^2 f(z))}{z} \prec q(z),\ z \in \dot{U}$$

*and this result is sharp.*

*Proof.* We let

$$p(z) = \frac{I^n(z^2 f(z))}{z},\ z \in \dot{U}. \tag{10}$$

By differentiating this relation, we obtain

$$\left[I^n(z^2 f(z))\right]' = p(z) + zp'(z),\ z \in \dot{U}.$$

Then (9) becomes

$$p(z) + zp'(z) \prec h(z) = q(z) + z(k+1)q'(z),\ z \in \dot{U}.$$

By using Lemma 2 we have

$$p(z) \prec q(z),\ z \in \dot{U}$$

i.e.

$$\frac{I^n(z^2 f(z))}{z} \prec q(z),\ z \in \dot{U}$$

and this result is sharp. □

**Theorem 4.** *Let $h \in \mathcal{H}(U)$, with $h(0) = 1$, and $h'(0) \neq 0$ which satisfies the inequality*

$$\operatorname{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2},\ z \in U.$$

*If $f \in \Sigma_k(\alpha, n)$ and satisfies the differential subordination*

$$\left[I^n(z^2 f(z))\right]' \prec h(z),\ z \in \dot{U} \tag{11}$$

*then*

$$\left[I^{n+1}(z^2 f(z)\right]' \prec g(z),\ z \in \dot{U}$$

*where*

$$q(z) = \frac{1}{(k+1)z^{\frac{1}{k+1}}} \int_0^z h(t) t^{\frac{1}{k+1}-1} dt,\ z \in U. \tag{12}$$

*The function $q$ is convex and it is the best $(1, k+1)$ dominant.*

*Proof.* By applying Lemma 3 for the function given by (12) and function $h$, for $\gamma = k+1$, we obtain that the function $q$ is convex.

By using the properties of the operator $I^n f$ we let

$$I^n(z^2 f(z)) = z\left[I^{n+1}(z^2 f(z))\right]', \ z \in \dot{U}. \tag{13}$$

If we let

$$\left[I^{n+1}(z^2 f(z))\right]' = p(z)$$

with

$$p(z) \in \mathcal{H}[1, k+1]$$

and differentiating (13) we obtain

$$\left[I^n(z^2 f(z))\right]' = p(z) + zp'(z), \ z \in \dot{U}$$

and (11) becomes

$$p(z) + zp'(z) \prec h(z), \ z \in \dot{U}.$$

By using Lemma 1 for $\gamma = 1$ and $n = k+1$ we have

$$p(z) \prec q(z) = \frac{1}{(k+1)z^{\frac{1}{k+1}}} \int_0^z h(t) t^{\frac{1}{k+1}-1} dt, \ z \in U,$$

i.e.

$$\left[I^n(z^2 f(z))\right]' \prec q(z) = \frac{1}{(k+1)z^{\frac{1}{k+1}}} \int_0^z h(t) t^{\frac{1}{k+1}-1} dt, \ z \in U.$$

Moreover the function $q$ is the best $(1, k+1)$ dominant. □

**Theorem 5.** *Let $h \in H(U)$ with $h(0) = 1$, $h'(0) \neq 0$, which verifies the inequality*

$$\mathrm{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}, \ z \in U.$$

*If $f \in \Sigma_k(\alpha, n)$ and satisfies the differential subordination*

$$\left[I^n(z^2 f(z))\right]' \prec h(z), \ z \in \dot{U} \tag{14}$$

*then*

$$\frac{I^n(z^2 f(z))}{z} \prec q(z), \ z \in \dot{U}$$

*where*

$$q(z) = \frac{1}{(k+1)z^{\frac{1}{k+1}}} \int_0^z h(t) t^{\frac{1}{k+1}-1} dt, \ z \in U.$$

*The function $q$ is convex and is the best $(1, k+1)$ dominant.*

*Proof.* We let
$$p(z) = \frac{I^n(z^2 f(z))}{z},\ z \in \dot{U} \tag{15}$$
with $p(z) \in \mathcal{H}[1, k+1]$.

By differentiating (15), we obtain
$$\left[I^n(z^2 f(z))\right]' = p(z) + zp'(z),\ z \in \dot{U}, \tag{16}$$
then (14) becomes
$$p(z) + zp'(z) \prec h(z),\ z \in \dot{U}.$$

By using Lemma 1, we have
$$p(z) \prec q(z) = \frac{1}{(k+1)z^{\frac{1}{k+1}}} \int_0^z h(t) t^{\frac{1}{k+1}-1} dt,\ z \in U,$$
i.e.
$$\left[I^n(z^2 f(z))\right]' \prec q(z) = \frac{1}{(k+1)z^{\frac{1}{k+1}}} \int_0^z h(t) t^{\frac{1}{k+1}-1} dt,\ z \in U.$$

Moreover the function $q$ is the best $(1, k+1)$ dominant. □

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## References

[1] Hallenbeck D.J., Ruschweyh S. *Subordination by convex functions.* Proc. Amer. Soc., 1975, **52**, 191–195.

[2] Miller S.S., Mocanu P.T. *Differential Subordinations. Theory and Applications.* Marcel Dekker Inc., New York, Basel, 2000.

[3] Miller S.S., Mocanu P.T., *On some classes of first-order differential subordinations.* Michigan Math. J., 1985, **32**, 185–195.

[4] Oros G.I. *A new application of Sălăgean differential operator at the class of meromorphic functions.* An. Univ. Oradea Fasc. Mat., 2004, **X**, 123–132.

[5] Sălăgean G.St. *Subclasses of univalent functions.* Lecture Notes in Math., Springer Verlag, 1983, **1013**, 362–372.

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# Global Attractors of Non-autonomous Difference Equations

D. Cheban, C. Mammana, E. Michetti

**Abstract.** The article is devoted to the study of global attractors of quasi-linear non-autonomous difference equations. The results obtained are applied to the study of a triangular economic growth model $T : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ recently developed in S. Brianzoni, C. Mammana and E. Michetti [1].

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**Keywords and phrases:** Triangular maps, non-autonomous dynamical systems with discrete time, skew-product flow, global attractors.

## 1 Introduction

Global attractors play a very important role in the qualitative study of difference equations (both autonomous and non-autonomous). The present work is dedicated to the study of global attractors of quasi-linear non-autonomous difference equations

$$u_{n+1} = A(\sigma^n\omega)u_k + F(u_k, \sigma^n\omega), \tag{1}$$

where $\Omega$ is a metric space (generally speaking non-compact), $(\Omega, \mathbb{Z}_+, \sigma)$ is a dynamical system with discrete time $\mathbb{Z}_+$, $A \in C(\Omega, [E])$ and the function $F \in C(E \times \Omega, E)$ satisfies "the condition of smallness". An analogous problem has been studied in D. Cheban and C. Mammana [5] when the space $\Omega$ is compact.

The results obtained are applied to the study of a class of triangular maps $T = (T_1, T_2)$ describing an economic growth model in capital accumulation and population growth rate as recently proposed by S. Brianzoni, C. Mammana and E. Michetti [1] [1].

## 2 Global attractors of dynamical systems

### 2.1 Triangular maps and non-autonomous dynamical systems

Let $W$ and $\Omega$ be two complete metric spaces and denote by $X := W \times \Omega$ its Cartesian product. Recall that a continuous map $F : X \to X$ is called triangular if



[1] The authors consider the neoclassical one–sector growth model with differential savings as in V. Bohm and L. Kaas [3], while assuming CES production function and the labour force dynamic described by the Beverton–Holt equation (see [2]), that has been largely studied in [7] and [8].

there are two continuous maps $f : W \times \Omega \to W$ and $g : \Omega \to \Omega$ such that $F = (f, g)$, i.e. $F(x) = F(u, \omega) = (f(u, \omega), g(\omega))$ for all $x =: (u, \omega) \in X$.

Consider a system of difference equations

$$\begin{cases} u_{n+1} = f(u_n, \omega_n), \\ \omega_{n+1} = g(\omega_n) \end{cases} \tag{2}$$

for all $n \in \mathbb{Z}_+$, where $\mathbb{Z}_+$ is the set of all non-negative integer numbers.

Along with system (2) we consider the family of equations

$$u_{n+1} = f(u_n, g^n\omega) \ (\omega \in \Omega), \tag{3}$$

which is equivalent to system (2). Let $\varphi(n, u, \omega)$ be a solution of equation (3) passing through the point $u \in W$ for $n = 0$. It is easy to verify that the map $\varphi : \mathbb{Z}_+ \times W \times \Omega \to W$ $((n, u, \omega) \mapsto \varphi(n, u, \omega)$ ) satisfies the following conditions:

(i) $\varphi(0, u, \omega) = u$ for all $u \in W$ and $\omega \in \Omega$;

(ii) $\varphi(n+m, u, \omega) = \varphi(n, \varphi(m, u, \omega), \sigma(m, \omega))$ for all $n, m \in \mathbb{Z}_+, u \in W$ and $\omega \in \Omega$, where $\sigma(n, \omega) := g^n\omega$;

(iii) the map $\varphi : \mathbb{Z}_+ \times W \times \Omega \to W$ is continuous.

Denote by $(\Omega, \mathbb{Z}_+, \sigma)$ the semi-group dynamical system generated by the positive powers of map $g : \Omega \to \Omega$, i.e. $\sigma(n, \omega) := g^n\omega$ for all $n \in \mathbb{Z}_+$ and $\omega \in \Omega$.

Recall [4, 9] that a triple $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ (or briefly $\varphi$) is called a cocycle over the dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$ with fiber $W$ if the mapping $\varphi : \mathbb{Z}_+ \times W \times \Omega \to \Omega$ possesses the properties (i)-(iii).

Let $X := W$ and $(X, \mathbb{Z}_+, \pi)$ be a dynamical system on $X$, where $\pi(n, (u, \omega)) := (\varphi(n, u, \omega), \sigma(n, \omega))$ for all $u \in W$ and $\omega \in \Omega$, then $(X, \mathbb{Z}_+, \pi)$ is called [9] a skew-product dynamical system, generated by the cocycle $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$.

Taking into consideration this fact, we can study triangular maps in the framework of cocycles with discrete time.

## 2.2 Global attractors of autonomous dynamical systems

A dynamical system $(X, \mathbb{T}, \pi)$ is called compact dissipative if there exists a nonempty compact subset $K \subseteq X$ such that

$$\lim_{t \to +\infty} \rho(xt, K) = 0; \tag{4}$$

for all $x \in X$ and the equality (4) takes place uniformly w.r.t. $x$ on the compact subsets from $X$.

For compact dissipative dynamical system $(X, \mathbb{T}, \pi)$ we denote by

$$J := \Omega(K) = \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \pi^\tau K},$$

then the set $J$ does not depend on the choice of the attractor $K$ and is characterized by the properties of the dynamical system $(X, \mathbb{T}, \pi)$ . The set $J$ is called a Levinson center of the dynamical system $(X, \mathbb{T}, \pi)$.

Let $E$ be a finite-dimensional Banach space and $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ be a cocycle over $(\Omega, \mathbb{Z}_+, \sigma)$ with the fiber $E$ (or shortly $\varphi$).

A cocycle $\varphi$ is called:

– dissipative, if there exists a number $r > 0$ such that

$$\limsup_{t\to+\infty} |\varphi(t, u, \omega)| \leq r \tag{5}$$

for all $\omega \in \Omega$ and $u \in E$;

– uniform dissipative, if there exists a number $r > 0$ such that

$$\limsup_{t\to+\infty} \sup_{\omega\in\Omega', |u|\leq R} |\varphi(t, u, \omega)| \leq r$$

for all compact subsets $\Omega' \subseteq \Omega$ and $R > 0$.

Let $(X, \mathbb{T}, \pi)$ be a dynamical system and $x \in X$. Denote by $\omega_x := \cap_{t\geq 0}\overline{\cup_{\tau\geq t}\pi(\tau, x)}$ the $\omega$-limit set of point $x$.

**Theorem 1** ([6])**.** *If the dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$ is compact dissipative and the cocycle $\varphi$ is uniform dissipative, then the skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$ is compact dissipative.*

## 2.3 Global attractors of quasi-linear triangular systems

Consider a difference equation

$$u_{n+1} = f(u_n, \sigma^n\omega) \ (\omega \in \Omega). \tag{6}$$

Denote by $\varphi(n, u, \omega)$ a unique solution of equation (6) with the initial condition $\varphi(0, u, \omega) = u$.

Equation (6) is said to be dissipative (respectively, uniformly dissipative), if a cocycle $\varphi$, generated by equation (6), is dissipative (respectively, uniformly dissipative), i.e. there exists a positive number $r$ such that

$$\limsup_{n\to+\infty} |\varphi(n, u, \omega)| \leq r \quad \text{(respectively, } \limsup_{n\to+\infty} \sup_{\omega\in\Omega', |u|\leq R} |\varphi(n, u, \omega)| \leq r)$$

for all $u \in E$ and $\omega \in \Omega$ (respectively, for all $R > 0$ and $\Omega' \in C(\Omega)$).

Consider a quasi-linear equation

$$u_{n+1} = A(\sigma^n\omega)u_k + F(u_k, \sigma^n\omega), \tag{7}$$

where $A \in C(\Omega, [E])$ and the function $F \in C(E \times \Omega, E)$ satisfies "the condition of smallness".

Denote by $U(k, \omega)$ the Cauchy matrix for the linear equation

$$u_{n+1} = A(\sigma^n\omega)u_k.$$

**Theorem 2.** *Suppose that the following conditions hold:*

1. *there are positive numbers $N$ and $q<1$ such that*

$$\|U(n,\omega)\| \leq Nq^n \qquad (n \in \mathbb{Z}_+); \tag{8}$$

2. $|F(u,\omega)| \leq C + D|u|$ $(C \geq 0,\ 0 \leq D < (1-q)N^{-1})$ *for all $u \in E$ and $\omega \in \Omega$.*

*Then equation (7) is uniform dissipative and*

$$|\varphi(n,u,\omega)| \leq (q+DN)^{n-1}qN|u| + \frac{CN}{q-1}(q^{n-1}-1). \tag{9}$$

*Proof.* This statement can be proved using the same type of arguments as in the proof of Theorem 5.2 from [5] (see also [6]) and we omit the details. □

**Theorem 3.** *Let $(\Omega, \mathbb{Z}_+, \sigma)$ be a compact dissipative dynamical system and $\varphi$ be a cocycle generated by equation (7). Under the conditions of Theorem 2 the skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$, generates by cocycle $\varphi$ admits a compact global attractor.*

*Proof.* This statement follows directly from Theorems 1 and 2. □

**Theorem 4** ([6]). *Let $A \in C(\Omega, [E])$ and $F \in C(E \times \Omega, E)$ and the following conditions be fulfilled:*

1. *the dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$ is compact dissipative and $J_\Omega$ its Levinson center;*

2. *positive numbers $N$ and $q<1$ exist such that inequality (8) holds;*

3. $C>0$ *exists such that $|F(0,\omega)| \leq C$ for all $\omega \in \Omega$;*

4. $|F(u_1,\omega) - F(u_2,\omega)| \leq L|u_1 - u_2|$ $(0 \leq L < N^{-1}(1-q))$ *for all $\omega \in \Omega$ and $u_1, u_2 \in E$.*

*Then*

1. *the equation (7) (the cocycle $\varphi$ generated by this equation) admits a compact global attractor;*

2. *there are two positive constants $\mathcal{N}$ and $\nu < 1$ such that*

$$|\varphi(n,u_1,\omega) - \varphi(n,u_2,\omega)| \leq \mathcal{N}\nu^n |u_1 - u_2| \tag{10}$$

*for all $u_1, u_2 \in E$ and $n \in \mathbb{Z}_+$.*

## 3 Non-Autonomous Dynamical Systems with Convergence

$\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is said to be convergent if the following conditions are valid:

1. the dynamical systems $(X,\mathbb{T}_1,\pi)$ and $(Y,\mathbb{T}_2,\sigma)$ are compactly dissipative;
2. the set $J_X \bigcap X_y$ contains at most one point for all $y \in J_Y$, where $X_y := h^{-1}(y) := \{x | x \in X, h(x) = y\}$ and $J_X$ (respectively, $J_Y$) is the Levinson center of the dynamical system $(X,\mathbb{T}_1,\pi)$ (respectively, $(Y,\mathbb{T}_2,\sigma)$).

**Theorem 5** ([4, Ch.II])**.** *Let $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ be a non-autonomous dynamical system and the following conditions be fulfilled:*

1. *the dynamical system $(Y,\mathbb{T}_2,\sigma)$ is compact dissipative and $J_Y$ its Levinson center;*
2. *there exists a homomorphism $\gamma$ from $(Y,\mathbb{T}_2,\sigma)$ to $(X,\mathbb{T}_1,\pi)$ such that $h \circ \gamma = Id_Y$;*
3. $\lim\limits_{t\to+\infty} \rho(\pi(t,x_1),\pi(t,x_2)) = 0$ *for all* $x_1,x_2 \in X$ $(h(x_1) = h(x_2))$*.*

*Then*

1. *the dynamical system $(X,\mathbb{T}_1,\pi)$ is compactly dissipative and $\gamma(J_Y) = J_X$;*
2. *$J_y$ consists a single point $\gamma(y)$ for all $y \in J_Y$.*

**Theorem 6.** *Let $A \in C(\Omega,[E])$ and $F \in C(E\times\Omega,E)$ and the following conditions be fulfilled:*

1. *the dynamical system $(\Omega,\mathbb{Z},\sigma)$ is compact dissipative and $J_\Omega$ its Levinson center;*
2. *there exist positive numbers $N$ and $q < 1$ such that inequality (8) holds;*
3. *there exists $C > 0$ such that $|F(0,\omega)| \le C$ for all $\omega \in \Omega$;*
4. $|F(u_1,\omega) - F(u_2,\omega)| \le L|u_1 - u_2|$ $(0 \le L < N^{-1}(1-q))$ *for all $\omega \in \Omega$ and $u_1,u_2 \in E$.*

*Then*

1. *the equation (7) (the cocycle $\varphi$ generated by this equation) admits a compact global attractor $\{I_\omega \mid \omega \in J_\Omega\}$ and $I_\omega$ consists of a single point $u_\omega$ (i.e. $I_\omega = \{u_\omega\}$) for all $\omega \in J_\Omega$;*
2. *the mapping $\omega \mapsto u_\omega$ is continuous and $\varphi(t,u_\omega,\omega) = u_{\sigma(t,\omega)}$ for all $\omega \in J_\Omega$ and $t \in \mathbb{Z}$;*

*3. there are two positive constants $\mathcal{N}$ and $\nu < 1$ such that*

$$|\varphi(n,u_1,\omega) - \varphi(n,u_2,\omega)| \le \mathcal{N}\nu^n |u_1 - u_2| \tag{11}$$

*for all $u_1, u_2 \in E$ and $n \in \mathbb{Z}_+$;*

*4.*

$$|\varphi(n,u,\omega) - u_{\sigma^n\omega}| \le \mathcal{N}\nu^n |u - u_\omega| \tag{12}$$

*for all $u \in E$, $\omega \in J_\Omega$ and $n \in \mathbb{Z}_+$.*

*Proof.* Let $\langle E, \varphi, (\Omega, \mathbb{Z}, \sigma)\rangle$ be the cocycle generated by equation (7) and $C_b(\Omega, E)$ be the space of all continuous and bounded functions $\mu : \Omega \mapsto E$ equipped with the sup-norm. For every $n \in \mathbb{Z}_+$ we define the mapping $S^n : C_b(\Omega, E) \mapsto C_b(\Omega, E)$ by equality $(S^n\mu)(\omega) := \varphi(n, \mu(\sigma(-n,\omega)), \sigma(-n,\omega))$ for all $\omega \in \Omega$. It easy to verify that the family of mappings $\{S^n \mid n \in \mathbb{Z}_+\}$ forms a commutative semigroup. From the inequality (9) it follows that $S^n\mu \in C_b(\Omega, E)$ for every $\mu \in C_b(\Omega, E)$ and $n \in \mathbb{Z}_+$. On the other hand from the inequality (10) we have

$$\|S^n\mu_1 - S^n\mu_2\| \le \mathcal{N}\nu^n \|\mu_1 - \mu_2\|$$

for all $\mu_1, \mu_2 \in C_b(\Omega, E)$ and $n \in \mathbb{Z}_+$, where $\mathcal{N} := \frac{qN}{q+LN}$ and $\nu := q + LN$. Under the conditions of Theorem $\nu = q + LN < q + 1 - q = 1$ and, consequently, the semi-group $\{S^n \mid n \in \mathbb{Z}_+\}$ is contracting. Thus there exists a unique fixed point $\mu \in C_b(\Omega, E)$ of the semi-group $\{S^n \mid n \in \mathbb{Z}_+\}$ and hence

$$\mu(\sigma(n,\omega)) = \varphi(n, \mu(\omega), \omega)$$

for all $n \in \mathbb{Z}_+$ and $\omega \in \Omega$.

Let $\langle (X, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}, \sigma), h\rangle$ be the non-autonomous dynamical system associated by cocycle $\varphi$ (i.e. $X := E \times \Omega$, $\pi := (\varphi, \sigma)$ and $h := pr_2 : X \mapsto \Omega$). Under the conditions of Theorem by Theorem 4 we have $\rho(x_1 t, x_2 t) \le \mathcal{N} e^{-\nu t}\rho(x_1, x_2)$ for all $x_1, x_2 \in X$ $(h(x_1) = h(x_2))$. Since $\gamma := (\mu, Id_\Omega)$ is an invariant section of the non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}, \sigma), h\rangle$, then according to Theorem 5 the dynamical system $(X, \mathbb{Z}_+, \pi)$ is compactly dissipative, its Levinson center $J_X = \gamma(J_\Omega)$ and $J_\omega := J \cap X_\omega$ $(X_\omega := h^{-1}(\omega))$ consists a single point $\gamma(\omega)$, i.e. $J_\omega = \{\gamma(\omega)\}$ for all $\omega \in \Omega$. Taking into consideration that the skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$ is compact dissipative, $J_\omega = I_\omega \times \omega$ and $\gamma = (\mu, Id_\Omega)$ we obtain $I_\omega = \mu(\omega)$ for all $\omega \in J_\Omega$. □

## 4 Economic Application

### 4.1 The model

Dynamic economic growth models have often considered the standard, one-sector neoclassical Solow model (see S. R. Solow [10]). V. Bohm and L. Kaas [3] considered the role of differential savings behavior between workers and shareholders and its effects with regard to the stability of stationary steady states within the framework of the discrete-time Solow growth model. More recently, S. Brianzoni, C. Mammana and E. Michetti [1] have proposed a discrete-time version of the Solow growth model with differential savings as formalized by V. Bohm and L. Kaas [3] while considering two different assumptions. Firstly they assume the CES production function. Secondly they assume the labor force growth rate not being constant, in particular they consider a model for density dependent population growth described by the Beverton-Holt equation (see [2]).

The resulting system $(T, \mathbb{R}^2_+)$ describing capital accumulation $k$ and population $n$ dynamics of the model studied in S. Brianzoni, C. Mammana and E. Michetti [1], where $T = (T_1, T_2)$, is given by

$$T_1(k,n) = \frac{(1-\delta)k + (k^\rho + 1)^{\frac{1-\rho}{\rho}}(s_w + s_r k^\rho)}{1+n}$$

and

$$T_2(n) = \frac{rhn}{h + (r-1)n}$$

for all $(k,n) \in \mathbb{R}^2_+$. In the model, $\delta \in (0,1)$ is the depreciation rate of capital, $s_w \in (0,1)$ and $s_r \in (0,1)$ are the constant saving rates for workers and shareholders respectively[2], $\rho \in (-\infty, 1), \rho \neq 0$ is a parameter related to the elasticity of substitution between the production factors given by $1/(1-\rho)$, $h > 0$ is the carrying capacity (for example resource availability) and $r > 1$ is the inherent growth rate (such a rate is determined by life cycle and demographic properties such as birth rates etc.). The Beverton-Holt $T_2$ have been studied extensively in J. V. Cushing and S. V. Henson [7, 8].

## 4.2 Existence of an attractor for $\rho \in (-\infty, 0)$

**Theorem 7.** *If $\rho < 0$, then the dynamical system $(\mathbb{R}^2_+, T)$ admits a compact global attractor.*

*Proof.* Assume $\rho \in (-\infty, 0)$ and let $\lambda = -\rho$, then $\lambda \in (0, +\infty)$. We write $T_1$ in terms of $\lambda$

$$T_1(k,n) = \frac{1}{1+n}\left[(1-\delta)k + (k^{-\lambda}+1)^{\frac{1+\lambda}{-\lambda}}(s_w + s_r k^{-\lambda})\right]$$

$$= \frac{1}{1+n}\left[(1-\delta)k + \frac{k}{(1+k^\lambda)^{\frac{1}{\lambda}}}\frac{s_r + s_w k^\lambda}{1+k^\lambda}\right]. \tag{13}$$

[2]The authors also assume $s_w \neq s_r$ since the standard growth model of R. V. Solow [10] is obtained if the two savings propensities are equal.

Note that $\dfrac{k}{(1+k^{\lambda})^{\frac{1}{\lambda}}} \longrightarrow 1$ as $k \longrightarrow +\infty$, $\dfrac{s_r + s_w k^{\lambda}}{1+k^{\lambda}} \longrightarrow s_w$ as $k \longrightarrow +\infty$ and, consequently, there exists $M > 0$ such that

$$\left| \frac{k}{(1+k^{\lambda})^{\frac{1}{\lambda}}} \frac{s_r + s_w k^{\lambda}}{1+k^{\lambda}} \right| \le M, \tag{14}$$

for all $k \in [0, +\infty)$.

Since $0 \le \dfrac{1}{1+n} \le 1$ for all $n \in \mathbb{R}_+$, then from (13) and (14) we obtain

$$0 \le T_1(k,n) \le \alpha k + M \tag{15}$$

for all $n, k \in \mathbb{R}_+$, where $\alpha := 1 - \delta > 0$.

Since the map $T$ is triangular, to prove this theorem it is sufficient to apply Theorem 3. Theorem is proved. □

*Remark* 1. 1. It is easy to see that the previous theorem is true also for $\delta = 1$ because in this case $\alpha = 1 - \delta = 0$ and from (15) we have $T_1(k,n) \le M$, $\forall k, n \in \mathbb{R}_+$. Now it is sufficient to refer to Theorem 1.

2. If $\delta = 0$ the problem is open.

According to Theorem 7, it is possible to conclude that if the elasticity of substitution between the two production factors (capital and labor) is positive and lesser than one (that is $\rho < 0$), capital and population dynamics cannot be explosive so economic patterns are bounded.

## 4.3 Existence of an attractor for $\rho \in (0,1)$ and $s_r < \delta$

The dynamical system $(X, \mathbb{T}, \pi)$ we will call:

– locally completely continuous if for every point $p \in X$ there exist $\delta = \delta(p) > 0$ and $l = l(p) > 0$ such that $\pi^l B(p, \delta)$ is relatively compact;

– weakly dissipative if a nonempty compact $K \subseteq X$ exists such that for every $\varepsilon > 0$ and $x \in X$ there is $\tau = \tau(\varepsilon, x) > 0$ for which $x\tau \in B(K, \varepsilon)$. In this case we will call $K$ a weak attractor.

Note that every dynamical system $(X, \mathbb{T}, \pi)$ defined on the locally compact metric space $X$ is locally completely continuous.

**Theorem 8** ([4])**.** *For the locally completely continuous dynamical systems the weak and compact dissipativity are equivalent.*

**Theorem 9.** *If $\rho \in (0,1)$ and $s_r < \delta$, then the mapping $T$ admits a compact global attractor.*

*Proof.* If $\rho \in (0,1)$ and $k > 0$ we have

$$T_1(k,n) = \frac{1}{1+n}\left[(1-\delta)k + (k^\rho+1)^{\frac{1-\rho}{\rho}}(s_w + s_r k^\rho)\right]$$
$$= \frac{1}{1+n}\left[(1-\delta)k + s_r k + \theta(k)k\right] \tag{16}$$

where $\theta(k) := \dfrac{(k^\rho+1)^{\frac{1}{\rho}}}{k(1+k^\rho)}(s_w + s_r k^\rho) - s_r \to 0$ as $k \to +\infty$. In fact $\dfrac{(k^\rho+1)^{\frac{1}{\rho}}}{k} \to 1$ as $k \to +\infty$ while $\dfrac{(s_w + s_r k^\rho)}{1+k^\rho} \to s_r$ as $k \to +\infty$ and, consequently,

$$\frac{\dfrac{(k^\rho+1)^{\frac{1}{\rho}}}{1+k^\rho}(s_w + s_r k^\rho)}{s_r k} = \frac{(k^\rho+1)^{\frac{1}{\rho}}}{k}\frac{(s_w + s_r k^\rho)}{s_r(k^\rho+1)} \to 1$$

as $k \to +\infty$, i.e. $\dfrac{(k^\rho+1)^{\frac{1}{\rho}}}{1+k^\rho}(s_w + s_r k^\rho) = s_r k + \theta(k)k$. From (16) we have

$$T_1(k,n) = \frac{1}{1+n}\left[(1-\delta+s_r)k + \theta(k)k\right]$$

for all $(k,n) \in \mathbb{R}^2_+$ with $k > 0$.

Since $s_r < \delta$ then $\alpha := 1 - \delta + s_r < 1$. Let $R_0 > 0$ be a positive number such that

$$|\theta(k)| < \frac{1-\alpha}{2}, \tag{17}$$

for all $k > R_0$. Note that for every $(k_0, n_0) \in \mathbb{R}^2_+$, with $k_0 > R_0$, the trajectory $\{T^t(k,n) \mid t \in \mathbb{Z}_+\}$ starting from point $(k_0,n_0)$ at the initial moment $t = 0$, at least one time intersects the compact $K_0 := [0,h_0] \times [0,R_0]$, $(h_0 > h)$. In fact, if we suppose that this statement is false, then exists a point $(k_0,n_0) \in \mathbb{R}^2_+ \setminus K_0$ exists such that

$$(k_t,n_t) := T^t(k_0,n_0) \in \mathbb{R}^2_+ \setminus K_0 \tag{18}$$

for all $t \in \mathbb{Z}_+$. Taking into consideration that $n_t \to h$ (or 0) as $t \to +\infty$, we obtain from (18) that $k_t > R_0$ for all $t \geq t_0$, where $t_0$ is a sufficiently large number from $\mathbb{Z}_+$. Without loss of generality, we may suppose that $t_0 = 0$ (if $t_0 > 0$ then we start from the initial point $(n_{t_0}, k_{t_0}) := T^{t_0}(n_0,k_0)$, where $T^{t_0} := T \circ T^{t_0-1}$ for all $t_0 \geq 2$). Thus we have

$$k_t > R_0 \tag{19}$$

for all $t \geq 0$ and

$$k_{t+1} = \frac{1}{1+n}\left[\alpha k_t + \theta(k_t)k_t\right] \tag{20}$$

From (17) and (20) we obtain

$$k_{t+1} \leq \alpha k_t + \frac{1-\alpha}{2}k_t = \frac{1+\alpha}{2}k_t \tag{21}$$

since $\dfrac{1}{1+n} \le 1$ for all $t \ge 0$. From (21) we have

$$k_t \le \left(\frac{1+\alpha}{2}\right)^t k_0 \to 0 \text{ as } t \to +\infty, \tag{22}$$

but (19) and (22) are contradictory. The obtained contradiction proves the statement. Let $(k_0, n_0) \in \mathbb{R}^2_+$ now be an arbitrary point.

(a) If $k_0 < R_0$ and $k_t \le R_0$ for all $t \in \mathbb{N}$, then $\limsup\limits_{t\to+\infty} k_t \le R_0$;

(b) If there exists $t_0 \in \mathbb{N}$ such that $k_{t_0} > R_0$, then there exists $\tau_0 \in \mathbb{N}$ ($\tau_0 > t_0$) such that $(k_{\tau_0}, n_{\tau_0}) \in K_0$ (see the proof above).

Thus we have proved that for all $(k_0, n_0) \in \mathbb{R}^2_+$ there exists $\tau_0 \in \mathbb{N}$ such that $(k_{\tau_0}, n_{\tau_0}) \in K_0$. According to Theorem 8 the dynamic system $(\mathbb{R}^2_+, T)$ admits a compact global attractor. The theorem is proved. $\square$

## 4.4 Structure of the attractor

A fixed point $p \in X$ of dynamical system $(X, \mathbb{T}, \pi)$ is called

– Lyapunov stable if for arbitrary positive number $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\rho(x, p) < \delta$ implies $\rho(\pi(t, x), p) < \varepsilon$ for all $t \ge 0$;

– attracting if there exists $\delta_0 > 0$ such that $\lim\limits_{t\to+\infty} \rho(\pi(t, x), p) = 0$ for all $x \in B(p, \delta_0) := \{x \in X \mid \rho(x, p) < \delta_0\}$;

– asymptotically stable if it is Lyapunov stable and attracting.

**Theorem 10.** *Suppose that $\rho < 0$ and one of the following conditions hold:*

1. $s_w < \min\{\delta, s_r\}$ *and* $0 < \lambda < \lambda_0$, *where* $\lambda_0$ *is a positive root of the quadratic equation* $(s_r - s_w)\lambda^2 + (s_r - 2\delta)\lambda - \delta = 0$;

2. $s_r < s_w < \delta$.

*Then*

1. *the dynamic system* $(\mathbb{R}^2_+, T)$ *admits a compact global attractor* $J = \{(0, n) \mid 0 \le n \le h\}$;

2. *for all point* $x := (k, n) \in \mathbb{R}^2_+$ *with* $n > 0$ *the* $\omega$*-limit set* $\omega_x$ *of* $x$ *consists a single fixed point* $(0, h)$ *of dynamical system* $(\mathbb{R}^2_+, T)$;

3. *the fixed point* $(0, h)$ *is asymptotically stable.*

*Proof.* Assume $\rho \in (-\infty, 0)$ and let $\lambda = -\rho$, then $\lambda \in (0, +\infty)$. We write $T_1$ in terms of $\lambda$ (see the proof of Theorem 9)

$$T_1(k,n) = \frac{1}{1+n}\left[(1-\delta)k + \frac{k}{(1+k^\lambda)^{\frac{1}{\lambda}}}\frac{s_w + s_r k^\lambda}{1+k^\lambda}\right].$$

Denote by

$$f(k) := \frac{k}{(1+k^\lambda)^{\frac{1}{\lambda}}}\frac{s_w + s_r k^\lambda}{1+k^\lambda},$$

then

$$f'(k) = \frac{s_w + (-s_w\lambda + (\lambda+1)s_r)k^\lambda}{(1+k^\lambda)^{2+1/\lambda}}.$$

It easy to verify that under the conditions of Theorem $f'(k) < s_w$ for all $k \geq 0$. Consider the non-autonomous difference equation

$$k_{t+1} = A(\sigma(t,n))k_t + F(k_t, \sigma(t,n)) \tag{23}$$

corresponding to triangular map $T = (T_1, T_2)$, where $A(n) := \dfrac{1}{n+1}$, $F(k,n) := \dfrac{1}{n+1} f(k)$ and $\sigma(t,n) := T_2^t(n)$ for all $t \in \mathbb{Z}_+$ and $n \in \mathbb{R}_+$. Under the conditions of the Theorem we can apply Theorem 6. By this Theorem the dynamical system $(\mathbb{R}^2_+, T)$ is compact dissipative with Levinson center $J$ and there exists a unique continuous bounded function $\mu : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $J = \{(\mu(n), n) \mid n \in [0,h]\}$. Since $F(n, 0) = 0$ for all $n \in \mathbb{R}_+$, then it easy to see that $\mu(n) = 0$ for all $n \in \mathbb{R}_+$.

Let $x = (k,n) \in \mathbb{R}^2_+$ and $n > 0$. Since the dynamical system $(\mathbb{R}^2_+, T)$ is compactly dissipative and its Levinson center $J = \cup\{J_n \mid 0 \leq n \leq h\}$, then $\omega_x \subseteq J$. Let $\tilde{x} = (\tilde{k}, \tilde{n}) \in \omega_x$, then there exists $t_m \to +\infty$ ($t_m \in \mathbb{Z}_+$) such that $T^{t_m}(k,n) \to (\tilde{k}, \tilde{n})$. It is evident that $\tilde{k} = 0$. Since $\lim\limits_{t\to+\infty} T_2^t n = h$ for all $n > 0$ we obtain $\tilde{n} = h$, i.e. $\tilde{x} = (0,h)$.

Now we will prove that the fixed point $(0,h)$ is stable. If we suppose that it is not true, then there are $\varepsilon_0 > 0$, $\delta_l \to 0$, $x_l := (k_l, n_l) \to (0,h)$ and $t_l \to +\infty$ (as $l \to +\infty$) such that $\rho(x_l, (0,h)) < \delta_l$ and

$$\rho(T^{t_l} x_l, (0,h)) \geq \varepsilon_0, \tag{24}$$

where $\rho(\cdot,\cdot)$ is the distance in $\mathbb{R}^2_+$. Since $T^{t_l} x_l = (\varphi(t_l, k_l, n_l), T_2^{t_l} n_l)$, where $\varphi(t,k,n)$ is the solution of equation (23) with initial condition $\varphi(0,k,n) = k$, and $n_l \to h$ by asymptotic stability of fixed point $h \in \mathbb{R}_+$ of dynamical system $(\mathbb{R}_+, T_2)$ we have $T_2^{t_l} n_l \to h$ as $l \to +\infty$. On the other hand by Theorem 6 we obtain

$$|\varphi(t_l, k_l, n_l) - \mu(T_2^{t_l})| \leq \mathcal{N}\nu^{t_l}|k_l - \mu(n_l)| = \mathcal{N}\nu^{t_l}|k_l| \to 0 \tag{25}$$

because $0 < \nu < 1$, $|k_l| \to 0$ and $t_l \to +\infty$. Taking into account that $\mu(n) = 0$ for all $n \geq 0$ we obtain $\mu(T_2^{t_l}) = 0$ for all $l \in \mathbb{N}$ and, consequently, $|\varphi(t_l, k_l, n_l)| \to 0$ as $l \to +\infty$, i.e.

$$\rho(T^{t_l} x_l, (0,h)) \to 0 \tag{26}$$

as $l \to +\infty$. The relations (24) and (26) are contradictory. The contradiction obtained proves our statement. $\square$

When considering Theorem 10 it is possible to conclude that if shareholders save less than workers and the depreciation rate of capital is big enough or, if workers save less than shareholders and the elasticity of substitution between the two factors is close to zero, then the economic system will converge to the steady state $(0, h)$ which is characterized by no capital accumulation.

Let $\gamma$ be a full trajectory of dynamical system $(X, \mathbb{T}, \pi)$. Denote by $\cap_{t\geq 0}\overline{\cup_{\tau\geq t}\gamma(\tau)} := \omega_\gamma$ (respectively, $\cap_{t\leq 0}\overline{\cup_{\tau\leq t}\gamma(\tau)} := \alpha_\gamma$).

**Theorem 11.** *Let $\rho \in (0,1)$, $s_r < \delta$ and $J$ be the Levinson center of dynamical system $(\mathbb{R}^2_+, T)$. Then the following statements hold:*

1. *$J$ is connected;*
2. *$J = \cup\{J_n \mid 0 \leq n \leq h\}$, where $J_n := I_n \times \{n\}$ and $I_n := [a_n, b_n]$ $(a_n, b_n \in \mathbb{R}_+)$;*
3. *dynamical systems $(\mathbb{R}_+, T_0)$ and $(\mathbb{R}_+, T_h)$ are compactly dissipative, where $T_0(k) := T(k, 0)$ and $T_h(k) := T(k, h)$ for all $k \in \mathbb{R}_+$;*
4. *$J_0 = [a_0, b_0] \times \{0\}$ (respectively, $J_h := [a_h, b_h] \times \{h\}$) is the Levinson center of dynamical system $(\mathbb{R}_+, T_0)$ (respectively, $(\mathbb{R}_+, T_h)$);*
5. *there exists at least one fixed point $p_0 \in J_0$ (respectively, $p_h \in J_h$) of the dynamical system $(\mathbb{R}_+, T_0)$ (respectively, $(\mathbb{R}_+, T_h)$);*
6. *for all point $x_0 := (k_0, n_0) \in J$ (with $0 < n_0 < h$) and $\gamma \in \Phi_{x_0}$ we have $\omega_\gamma \subseteq J_h$ and $\alpha_\gamma \subseteq J_0$.*

*Proof.* Let $\rho \in (0,1)$ and $s_r < \delta$, then by Theorem 9 the dynamical system $(\mathbb{R}^2_+, T)$ is compactly dissipative. Denote by $J$ the Levinson center of $(\mathbb{R}^2_+, T)$, then by Theorem 1.33 [4] the set $J$ is connected. Note that $J = \cup\{J_n \mid 0 \leq n \leq h\}$, where $J_n = I_n \times \{n\}$ and $I_n$ is a compact subset of $\mathbb{R}_+$. According to Theorem 2.25 [4] the set $I_n$ is connected and, consequently, there are $a_n, b_n \in \mathbb{R}_+$ such that $I_n = [a_n, b_n]$.

Since the set $\mathbb{R}_+ \times \{0\}$ (respectively, $\mathbb{R}_+ \times \{h\}$) is invariant with respect to dynamical system $(\mathbb{R}^2_+, T)$, then on the set $\mathbb{R}_+ \times \{0\}$ (respectively, on $\mathbb{R}_+ \times \{h\}$) is defined as a compactly dissipative dynamical system $(\mathbb{R}_+, T_0)$ (respectively, $(\mathbb{R}_+, T_h)$) and the set $J_0$ (respectively, $J_h$) is its Levinson center. Taking into account that $T_0$ (respectively, $T_h$) is a continuous mapping of $J_0 = [a_0, b_0] \times \{0\}$ (respectively, $J_h = [a_h, b_h] \times \{h\}$) on itself, then there exists at least one fixed point $p_0 \in J_0$ (respectively, $p_h \in J_h$) of dynamical system $(\mathbb{R}_+, T_0)$ (respectively, $(\mathbb{R}_+, T_h)$).

Let $x_0 := (k_0, n_0) \in J$ (with $0 < n_0 < h$), $\gamma \in \Phi_{x_0}$ and $x = (k, n) \in \omega_\gamma$ (respectively, $x \in \alpha_\gamma$). Then there exists a sequence $\{t_m\} \subseteq \mathbb{Z}$ such that $t_m \to +\infty$ (respectively, $t_m \to -\infty$) such that $\gamma(t_m) \to x$ as $m \to +\infty$. Since $x_0 = (k_0, n_0)$,

$0 < n_0 < h$ and $pr_2(\gamma(t_m)) = T_2^{t_m}(n_0)$, then $\{T_1^{t_m}(n_0)\} \to h$ (respectively, $\{T_2^{t_m}(n_0)\} \to 0$) as $m \to +\infty$. On the other hand $x \in J$ and, consequently, $p_2(x) = h$ (respectively, $pr_2(x) = 0$). Analogously we can prove that $\omega_{x_0} \subseteq J_h$ for all $x_0 = (k_0, n_0) \in \mathbb{R}^2_+$ with $n_0 > 0$, where $\omega_{x_0}$ is the $\omega$-limit set of point $x_0$. $\square$

## References


[1] Brianzoni S., Mammana C., Michetti E. *Complex dynamics in the neo-classical growth model with differential savings and non-constant labor force growth.* Studies in nonlinear dynamics and econometrics, Vol. **11**: No. **3**, Article 3, 2007. http://www.bepress.com/snde/vol11/iss3/art3.

[2] Beverton R. J. H., Holt S. J. *On the dynamics of exploited fish populations.* Fishery Investigations, 1957, **19**, 1–533.

[3] Bohm V., Kaas L. *Differential savings, factor shares, and endogenous growth cycles.* Journal of Economic Dynamics and Control, 2000, **24**, 965–980.

[4] Cheban D. N. *Global Attractors of Nonautonomous Dissipative Dynamical Systems. Interdisciplinary Mathematical Sciences 1.* River Edge, NJ: World Scientific, 2004.

[5] Cheban D. N., Mammana C. *Invariant Manifolds, Global Attractors and Almost Periodic Solutions of Non-autonomous Difference equations.* Nonlinear Analyses, serie A, 2004, **56**, No. 4, 465–484.

[6] Cheban D. N., Mammana C., Michetti E. *Global Attractors of Quasi-Linear Non-Autonomous Difference Equations.* Bulletinul Academiei de Stiinte a Republicii Moldova, Matematica, 2008, No 1(56), 84–104.

[7] Cushing J. M., Henson S. M. *Global dynamics of some periodically forced, monotone difference equations.* Journal of Difference Equations and Applications, 2001, **7**, 859–872.

[8] Cushing J. M., Henson S. M. *A periodically forced Beverton-Holt equation.* Journal of Difference Equations and Applications, 2002, **8(12)**, 119–1120.

[9] Sell G. R. *Topological Dynamics and Ordinary Differential Equations.* Van Nostrand-Reinhold, London, 1971.

[10] Solow R. M. *A contribution to the theory of economic growth.* Quarterly Journal of Economics, 1956, **70**, 65–94.



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# On free topological groups

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**Abstract.** In the present article the existence and unicity of almost $(\mathcal{U}, \mathcal{V})$-free group over given space, where $\mathcal{U}, \mathcal{V}$ are classes of topological groups is studied. If $\mathcal{V}$ is a quasivariety of compact topological groups and $\mathcal{V} \subseteq \mathcal{U}$, then these objects exist for any space. If $W$ is a quasivariety of compact groups, $\mathcal{U} = \mathcal{V}$ is the class of all pseudocompact subgroups of groups from $W$, then the almost $(\mathcal{U}, \mathcal{V})$-free groups exist only for some special spaces.

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All spaces considered are assumed to be completely regular pointed $T_1$-spaces. If $X$ is a space, then $p_X$ is the base point of $X$. If $G$ is a group, then the base point $p_G$ is the identity of $G$. We consider only mappings $f : X \to Y$ for which $f(p_X) = p_Y$.

For every space $X$ we denote by $\beta X$ the Stone-Čech compactification and by $|X|$, $w(X)$, $d(X)$ the cardinality, weight and density of the space $X$, respectively. If $indX = 0$, i.e. $X$ is zero-dimensional, then by $\beta_0 X$ we denote the maximal zero-dimensional compactification of $X$. If $Y$ is a subspace of a space $X$, then we consider that $p_Y = p_X$. In particular, $p_{bX} = p_X$ for any compactification $bX$ of $X$.

*Remark* 1. If $X$ is not a pointed space, then we put $\bar{X} = X \cup \{p_X\}$, where $p_X \notin X$ and $X$ is an open-and-closed subspace of the space $\bar{X}$. Thus every space may be completed to a pointed space.

## 1 A free topological group of a space

Let $\mathcal{U}$ and $\mathcal{V}$ be two classes of topological groups.

**Definition 1.** A pair $(F(X,\mathcal{U},\mathcal{V}), e_X)$ is said to be an almost $(\mathcal{U},\mathcal{V})$-free topological group over a space $X$ if $F(X,\mathcal{U},\mathcal{V}) \in \mathcal{U}$, $e_X : X \to F(X,\mathcal{U},\mathcal{V})$ is a continuous mapping, $e = e_X(p_X)$ is the identity of the group $F(X,\mathcal{U},\mathcal{V})$ and for every continuous mapping $f : X \to G$ with $G \in \mathcal{V}$ there exists a continuous homomorphism $\bar{f} : F(X,\mathcal{U},\mathcal{V}) \to G$ such that $f = \bar{f} \circ e_X$. If $\mathcal{U} = \mathcal{V}$, then we put $F(X,\mathcal{U},\mathcal{V}) = F(X,\mathcal{U})$ and $F(X,\mathcal{U})$ is called the almost $\mathcal{U}$-free group over $X$.

If for any continuous mapping $f : X \to G \in \mathcal{V}$ the homomorphism $\bar{f} : F(X,\mathcal{U},\mathcal{V}) \to G$ is unique, then $(F(X,\mathcal{U},\mathcal{V}), e_X)$ is called a $(\mathcal{U},\mathcal{V})$-free topological group of X.

*Remark* 2. Let $\mathcal{U}$ be a multiplicative class of topological groups, $G_0 \in \mathcal{U}$, $|G_0| \geq 2$, $X$ be a space and for $X$ there exists some almost $(\mathcal{U}, \mathcal{V})$-free topological group. Then the almost $(\mathcal{U}, \mathcal{V})$-free topological group over $X$ is not unique. Really, fix some almost $(\mathcal{U}, \mathcal{V})$-free topological group $(F(X,\mathcal{U}, \mathcal{V}), e_X)$ over $X$. Let $\tau \geq 1$ be a cardinal number, $e$ be the identity of $F(X,\mathcal{U}, \mathcal{V})$, $e_\tau$ be the identity of $G_0^\tau$, $F'(X,\mathcal{U}, \mathcal{V}) = F(X,\mathcal{U}, \mathcal{V}) \times G_0^\tau$, $e' = (e, e_\tau)$ and $\bar{e}_X(x) = (e_X(x), e_\tau)$ for any $x \in X$. Then $(F'(X,\mathcal{U}, \mathcal{V}), \bar{e}_X)$ is an almost $(\mathcal{U}, \mathcal{V})$-free topological group over $X$.

*Remark* 3. The concept of an almost $(\mathcal{U}, \mathcal{V})$-free topological group for non-pointed spaces was proposed by W.W.Comfort and J.van Mill (see [1], p.110). We consider that notion for pointed spaces. Moreover, our definition is more general for non-pointed spaces too. In the definition from [1] it is supposed that $e_X$ is an embedding, i.e. $X \subseteq F(X,\mathcal{U}, \mathcal{V})$.

We say that the topological group $G$ is complete if it is complete relative to the two-sided uniformity on $G$ (see [6]).

**Definition 2.** A class $\mathcal{U}$ of topological groups is called a quasivariety if the following properties hold:

– the class $\mathcal{U}$ is multiplicative;

– if a topological group $A$ is topologically isomorphic to a closed subgroup of some group $B \in \mathcal{U}$, then $A \in \mathcal{U}$.

We consider that any quasivariety $\mathcal{U}$ is non-trivial, i.e. there exists $G \in \mathcal{U}$ for which $|G| \geq 2$.

If $G$ is a topological group and $X$ is a subset of $G$, then $X$ is contained in the minimal subgroup $a(X, G) = \cap\{H : X \subseteq H \textit{ and } H \textit{ is a subgroup of } G\}$ of $G$. If $a(X, G) = G$, then we say that $X$ generated $G$. If $a(X, G)$ is dense in $G$, then we say that $X$ topologically generated $G$.

**Proposition 1.** *Let $\mathcal{U}$ be a quasivariety of topological groups. Then for any pointed space $X$ there exists a unique $\mathcal{U}$-free topological group $(F(X, \mathcal{U}), e_X)$ over $X$. Moreover, the set $e_X(X)$ topologically generated the group $F(X, \mathcal{U})$.*

*Proof.* Fix a space $X$. Let $m$ be an infinite cardinal and $|X| \leq m$. Let $\tau$ be an infinite cardinal and $\tau > 2^{2^m}$. We identify the isomorphic topological groups. Then $\mathcal{U}_\tau = \{G \in \mathcal{U} : |G| \leq \tau\}$ is a set. Therefore, the family $\{f_\alpha : X \to G_\alpha : \alpha \in A\}$ of all continuous mappings of $X$ into groups from $\mathcal{U}_\tau$ is a set too. Consider the mapping $e_X : X \to G = \prod\{G_\alpha : \alpha \in A\}$, where $e_X(x) = (f_\alpha(x) : \alpha \in A)$. By $F(X,\mathcal{U})$ we denote the closed subgroup of $G$ topologically generated by the set $e_X(X)$. Then $(F(X,\mathcal{U}), e_X)$ is a $\mathcal{U}$-free topological group over $X$.

The existence is proved.

Let $F \in \mathcal{U}$, $h : X \to F$ be a continuous mapping and for any continuous mapping $f : X \to G \in \mathcal{U}$ there exists a unique homomorphism $\hat{f} : F \to G$ such that $f = \hat{f} \circ h$. We mention that $h(X)$ topologically generated $F$. Suppose that $F_1$ is the closed subgroup of $F$ generated by $h(X)$ and $F_1 \neq F$. Then $F_1 \in \mathcal{U}$ and there exists a continuous homomorphism $h_1 : F \to F_1$ such that $h(x) = h_1(h(x))$ for any $x \in X$.

Then $h_1(y) = y$ for any $y \in F_1$. Now consider the homomorphism $h_2(y) = y$ for any $y \in F$. Then there exist two distinct homomorphisms $h_1, h_2 : F \to F$ such that $h(x) = h_1(h(x)) = h_2(h(x))$ for any $x \in X$, a contradiction. Thus $F = F_1$.

From definition, there exist two continuous homomorphisms $g_1 : F(X, \mathcal{U}) \to F$ and $g_2 : F \to F(X, \mathcal{U})$ such that $g_1(e_X(x)) = h(x)$ and $g_2(h(x)) = e_X(x)$ for any $x \in X$. Therefore $g_2(g_1(y)) = y$ and $g_1(g_2(z)) = z$ for all $y \in e_X(X)$ and $z \in h(X)$. Thus $g_3 = g_1|e_X(X)$ is a homeomorphism of $e_X(X)$ onto $h(X)$, $g_4 = g_2|h(X)$ is a homeomorphism of $h(X)$ onto $e_X(X)$ and $g_4 = {g_3}^{-1}$. Hence $g_1$, $g_2$ are isomorphisms and $g_1 = {g_2}^{-1}$. The proof is complete. □

**Example 1.** Let $\mathcal{U}_c$ be the class of all compact groups. Then $\mathcal{U}_c$ is a quasivariety of topological groups. For every space $X$ the mapping $e_X : X \to F(X,\mathcal{U}_c)$ is an embedding. We can consider that $X = e_X(X) \subseteq F(X,\mathcal{U}_c)$. Then $\beta X$ is the closure of $X$ in $F(X,\mathcal{U}_c)$.

**Example 2.** Let $\mathcal{U}_{ac}$ be the class of all commutative compact groups. Then $\mathcal{U}_{ac}$ is a quasivariety of topological groups. For every space $X$ the mapping $e_X : X \to F(X,\mathcal{U}_{ac})$ is an embedding. We can consider that $X = e_X(X) \subseteq F(X,\mathcal{U}_{ac})$. Then $\beta X$ is the closure of $X$ in $F(X,\mathcal{U}_{ac})$.

**Example 3.** Let $\mathcal{U}_{0c}$ be the class of all compact zero-dimensional groups. Then $\mathcal{U}_{0c}$ is a quasivariety. The mapping $e_X : X \to F(X,\mathcal{U}_{0c})$ is an embedding if and only if $indX = 0$. If $indX = 0$, then $\beta_0 X$ is the closure of $X = e_X(X)$ in $F(X,\mathcal{U}_{0c})$.

**Definition 3.** A class $\mathcal{U}$ of topological groups is said to be complete if the following properties hold:

– $(A, \mathcal{T}) \in \mathcal{U}$;

– if $\mathcal{T}'$ is a topology on $A$ and $(A, \mathcal{T}')$ is a topological group, then $(A, \mathcal{T}') \in \mathcal{U}$.

*Remark* 4. Let $\mathcal{U}$ be a complete quasivariety of topological groups. If $A$ is a subgroup of $B \in \mathcal{U}$, then $A \in \mathcal{U}$. In particular, the set $e_X(X)$ algebraically generated $F(X,\mathcal{U})$ provided $e_X(X)$ topologically generated $F(X,\mathcal{U})$ for any space X. In this case $e_X : X \to F(X,\mathcal{U})$ is an embedding for any space $X$.

*Remark* 5. Let $\mathcal{U}$ be a quasivariety, $F(X,\mathcal{U})$ be the almost $\mathcal{U}$-free topological group over a space $X$ and $F_0(X,\mathcal{U})$ be the closed subgroup of $F(X,\mathcal{U})$ generated by the set $e_X(X)$. Then:

1. $F_0(X,\mathcal{U})$ is the $\mathcal{U}$-free topological group over $X$.

2. There exists a continuous homomorphism $\varphi : F(X,\mathcal{U}) \to F_0(X,\mathcal{U})$ such that $\varphi(y) = y$ for any $y \in F_0(X,\mathcal{U})$. (It is obvious that $\varphi = \bar{e}_X$).

3. If $(F'(X,\mathcal{U}), e'_X)$ is another almost $\mathcal{U}$-free topological group over $X$ and $F'_0(X,\mathcal{U})$ is the closed subgroup of $F'(X,\mathcal{U})$ generated by $e'_X(X)$, then there exists a unique isomorphism $\psi : F_0(X,\mathcal{U}) \to F'_0(X,\mathcal{U})$ such that $\psi(e_X(x)) = e'_X(x)$ for any $x \in X$.

**Proposition 2.** *Let $\mathcal{U}$ be a quasivariety of topological groups and $\mathcal{V}$ be a class of topological groups. If $\mathcal{V} \subseteq \mathcal{U}$ and $(F(X,\mathcal{U}), e_X)$ is an almost $\mathcal{U}$-free topological group over $X$, then $(F(X,\mathcal{U}), e_X)$ is an almost $(\mathcal{U},\mathcal{V})$-free topological group over $X$.*

*Proof.* Obvious. □

If $\mathcal{V} \not\subseteq \mathcal{U}$, then Proposition 2 is not true. For example, if $\mathcal{U} = \mathcal{U}_{ac}$ and $\mathcal{V}$ is the class of all commutative groups, then Proposition 2 is not true.

## 2 On totally bounded groups

A topological group $A$ is totally bounded or precompact if $A$ is a subgroup of some compact group.

Let $\mathcal{U}_b$ be the class of all totally bounded groups. For any space $X$ there exists the $\mathcal{U}_b$-free group $(F(X,\mathcal{U}_b), e_X)$ and $e_X : X \to F(X,\mathcal{U}_b)$ is an embedding. Moreover, $F(X,\mathcal{U}_b)$ is the subgroup of $F(X,\mathcal{U}_c)$ generated by $X$. The quasivariety $\mathcal{U}_b$ is not complete.

**Example 4.** Consider the quasivariety $\mathcal{U}_c$. Let $X$ be the space of reals, $G$ be the topological group of reals and $f(x) = x$ for any $x \in X$. Let $F(X,\mathcal{U}_c)$ be the $\mathcal{U}_c$-free topological group over $X$ and $X$ topologically generated $F(X,\mathcal{U}_c)$. Denote by $A$ the subgroup of $F(X,\mathcal{U}_c)$ generated by $X$. There exists a unique homomorphism $g : A \to G$ for which $f = g|X$. The homomorphism $g$ is not continuous: the group $A$ is totally bounded, the group $G$ is not totally bounded and the continuous homomorphic image of totally bounded group is totally bounded.

## 3 On pseudocompact groups

A subset $L$ of a space $X$ is bounded in $X$ if for any real-valued continuous function $f$ on $X$ the set $f(L)$ is bounded. If the set $X$ is bounded in the space $X$, then we say that $X$ is a pseudocompact space.

Any pseudocompact group is totally bounded. For any totally bounded group $G$ there exists a unique compact group $bG$ such that $G$ is a dense subgroup of $bG$. If the group $G$ is pseudocompact, then $bG = \beta G$ (see [1], p.110).

The following assertion is a generalization of one theorem of M.Ursul (see [7]).

**Theorem 1.** *Let $A$ be a subgroup of a pseudocompact group $B$ and $\omega_1$ be the first uncountable cardinal. Then in $B^{\omega_1}$ there exist two subgroups $H$ and $G$ for which:*
1. *$G$ is a dense pseudocompact subgroup of $B^{\omega_1}$.*
2. *$H$ is a closed subgroup of $G$.*
3. *The groups $A$ and $H$ are topologically isomorphic.*

*Proof.* For any ordinal number $\alpha < \omega_1$ let $B_\alpha = B$, $A_\alpha = A$ and $e_\alpha$ be the identity in $B_\alpha$. It is wellknown that $B^{\omega_1} = \prod\{B_\alpha : \alpha < \omega_1\}$ is a pseudocompact group with the identity $e = (e_\alpha : \alpha < \omega_1)$. Let $G' = \{x = (x_\alpha : \alpha < \omega_1) \in B^{\omega_1}$ : *the set* $\{\alpha < \omega_1 : x_\alpha \neq e_\alpha\}$ *is countable* $\}$ and $G = \{x = (x_\alpha : \alpha < \omega_1) \in B^{\omega_1}$ : *the set* $\{\alpha < \omega_1 : x_\alpha \notin A_\alpha\}$ *is countable* $\}$. By construction, $G' \subseteq G \subseteq B^{\omega_1}$ and $G'$

is a pseudocompact subgroup of $B^{\omega_1}$. Since $G'$ is dense in $B^{\omega_1}$, the subspace $G$ is pseudocompact and dense in $B^{\omega_1}$. It is obvious that $G'$ and $G$ are subgroups of $B^{\omega_1}$. For any $x \in B$ we put $\bar{x} = (\bar{x}_\alpha : \alpha < \omega_1)$, where $x_\alpha = x$ for any $\alpha < \omega_1$. Then $\bar{B} = \{\bar{x} : x \in B\}$ is the diagonal of $B^{\omega_1}$. The diagonal $\bar{B}$ is a subgroup topologically isomorphic to $B$. The subspace $\bar{B}$ is closed in $B^{\omega_1}$. Let $H = \{\bar{x} : x \in A\}$. Then the topological groups $A$ and $H$ are topologically isomorphic. By construction, $H = G \cap \bar{B}$. Thus $H$ is a closed subgroup of the group $G$. $\square$

**Corollary 1.** *(see [7]) Every totally bounded subgroup is a closed subgroup of some pseudocompact group.*

**Theorem 2.** *Let $\mathcal{U}$, $\mathcal{V}$ be two classes of pseudocompact groups with properties:*

1. *$\mathcal{V} \subseteq \mathcal{U}$.*
2. *The classes $\mathcal{U}$ and $\mathcal{V}$ are multiplicative.*
3. *If $A \in \mathcal{U}$, then $A$ is a subgroup of some compact group $B \in \mathcal{V}$.*
4. *If $A \in \mathcal{U}$ and $B$ is a pseudocompact subgroup of $A$, then $B \in \mathcal{U}$.*
5. *If $A \in \mathcal{V}$ and $B$ is a compact group, then every closed subgroup of $B$ is an element of $\mathcal{V}$.*

*Denote by $\mathcal{U}_0$ the class of all subgroups of groups from $\mathcal{U}$.*

*The following assertions are true:*

*(A). $\mathcal{U}_0$ is a quasivariety of topological groups.*

*(B). If $X$ is a space, $F(X,\mathcal{U},\mathcal{V})$ is an almost $(\mathcal{U},\mathcal{V})$-free group over a space $X$ and $X = e_X(X) \subseteq F(X,\mathcal{U},\mathcal{V})$, then:*

*– the subgroup $F_0(X,\mathcal{U},\mathcal{V})$ generated by the space $X$ in $F(X,\mathcal{U},\mathcal{V})$ is the $\mathcal{U}_0$-free topological group over space $X$;*

*– the group $F_0(X,\mathcal{U},\mathcal{V})$ is finite or $F_0(X,\mathcal{U},\mathcal{V})$ is not pseudocompact.*

*Proof.* If $F_0(X,\mathcal{U},\mathcal{V})$ is a compact group, then $F_0(X,\mathcal{U},\mathcal{V}) \in \mathcal{V}$ and $F_0(X,\mathcal{U},\mathcal{V})$ is a $(\mathcal{U},\mathcal{V})$-free group over $X$.

Suppose that the space $F_0(X,\mathcal{U},\mathcal{V})$ is not compact. Consider the $\mathcal{U}_0$-free group $F(X,\mathcal{U}_0)$ over $X$. It is obvious that $X \subseteq F(X,\mathcal{U}_0)$ and $X$ generated $F(X,\mathcal{U}_0)$. By assumptions, $F(X,\mathcal{U}_0)$ is a dense subgroup of some compact group $G \in \mathcal{V}$. Thus there exists a continuous homomorphism $h : F(X,\mathcal{U},\mathcal{V}) \to G$ such that $h(x) = x$ for any $x \in X$. Therefore $h|F_0(X,\mathcal{U},\mathcal{V})$ is a topological isomorphism of $F_0(X,\mathcal{U},\mathcal{V})$ onto $F(X,\mathcal{U}_0)$. We can consider that $F_0(X,\mathcal{U},\mathcal{V}) = F(X,\mathcal{U}_0)$.

Let $Y = cl_G X$, $Y^{-1} = \{y^{-1} : y \in Y \subseteq G\}$ and $Y_n = \{y_1 \cdot y_2 \cdot \ldots \cdot y_n : y_1, y_2, \ldots, y_n \in Y \cup Y^{-1}\}$ for any $n \geq 1$. Then $H = \cup\{Y_n : n \in \mathbb{N}\}$ is a subgroup of $G$ and every set $Y_n$ is compact. The group $F(X,\mathcal{U}_0)$ is a dense subgroup of $H$ and $H$ is a dense subgroup of $G$. Let $X_n = Y_n \cap F(X,\mathcal{U}_0)$. By construction, $X_n$ is a closed subset of $F(X,\mathcal{U}_0)$, $X_1 = X \cup X^{-1}$ and $X_n = \{x_1 \cdot x_2 \cdot \ldots \cdot x_n : x_1, x_2, \ldots, x_n \in X_1\}$ for $n \geq 2$. If $F(X,\mathcal{U}_0)$ is infinite, then $F(X,\mathcal{U}_0)$ is not a discrete space.

It is obvious that $F(X,\mathcal{U}_0) = \cup\{X_n : n \in \mathbb{N}\}$.

*Case 1.* $X$ is a discrete space.

In this case every $X_n$ is a discrete closed subspace of $F(X,\mathcal{U}_0)$. Since $F(X,\mathcal{U}_0)$ is not discrete, $X_n$ is nowhere dense subset of $F(X,\mathcal{U}_0)$. Suppose that $F(X,\mathcal{U}_0)$

is pseudocompact. Then $W_n = F(X,\mathcal{U}_0) \setminus X_n$ is a dense open subset of $F(X,\mathcal{U}_0)$. From the Baire category theorem, the set $W = \cap\{W_n : n \in \mathbb{N}\}$ is dense in $F(X,\mathcal{U}_0)$. By construction, we have $W = \cap\{W_n : n \in \mathbb{N}\} = F(X,\mathcal{U}_0) \setminus \cup\{X_n : n \in \mathbb{N}\} = \emptyset$, a contradiction.

Thus $F(X,\mathcal{U}_0)$ is not pseudocompact.

*Case 2.* $X$ is not a discrete space.

Let $a$ be a non-isolated point of the space $X$. We put $a_{2i-1} = a$ and $a_{2i} = a^{-1}$ for any $i \geq 1$. Fix $n \in \mathbb{N}$, $b \in X_n$ and a neighbourhood $W$ of $b$ in $G$. Fix $m > n$. Then $b = b \cdot a_1 \cdot a_2 \cdot \ldots \cdot a_{2m-1} \cdot a_{2m} \in W$. There exist open subsets $W_0, W_1, \ldots, W_{2m}$ of $G$ such that $b \in W_0, a_1 \in W_1, \ldots, a_{2m} \in W_{2m}$ and $W_0 \cdot W_1 \cdot \ldots \cdot W_{2m} \subseteq W$. There exist distinct elements $x_1, x_2, \ldots, x_{2m} \in X$ such that $y_{2i-1} = x_{2i-1} \in W_{2i-1}$ and $y_{2i} = x_{2i}^{-1} \in W_{2i}$. Then $b' = b \cdot y_1 \cdot y_2 \cdot \ldots \cdot y_m \in W \setminus X_n$. Thus $X_n$ is nowhere dense in $F(X,\mathcal{U}_0)$. Suppose that $F(X,\mathcal{U}_0)$ is pseudocompact. Then $W_n = F(X,\mathcal{U}_0) \setminus X_n$ is a dense open subset of $F(X,\mathcal{U}_0)$. From the Baire category theorem, the set $W = \cap\{W_n : n \in \mathbb{N}\}$ is dense in $F(X,\mathcal{U}_0)$. By construction, we have $W = \cap\{W_n : n \in \mathbb{N}\} = F(X,\mathcal{U}_0) \setminus \cup\{X_n : n \in \mathbb{N}\} = \emptyset$, a contradiction.

Moreover, we prove that $F_0(X,\mathcal{U},\mathcal{V})$ is not pseudocompact provided it is infinite. Thus $F_0(X,\mathcal{U},\mathcal{V})$ is finite if and only if $F_0(X,\mathcal{U},\mathcal{V})$ is compact. □

**Theorem 3.** *Let $\mathcal{U}$, $\mathcal{V}$ be two classes of topological groups with the properties:*

1. $\mathcal{V} \subseteq \mathcal{U}$.
2. *The classes $\mathcal{U}$ and $\mathcal{V}$ are multiplicative.*
3. *Every group $A \in \mathcal{V}$ is compact.*
4. *If $A \in \mathcal{V}$ and $B$ is a closed subgroup of $A$, then $B \in \mathcal{V}$.*
5. *If $A \in \mathcal{U}$ and $B$ is a pseudocompact subgroup of $A$, then $B \in \mathcal{U}$.*
6. *If $A \in \mathcal{U}$, then $A$ is a subgroup of some compact group $B \in \mathcal{V}$.*

*Then for every space $X$ there exists some almost $(\mathcal{U}, \mathcal{V})$-free group $(F(X, \mathcal{U}, \mathcal{V}), e_X)$ over $X$ such that $F(X, \mathcal{U}, \mathcal{V})$ is pseudocompact and $e_X(X)$ is a closed subspace of $F(X,\mathcal{U},\mathcal{V})$.*

*Proof.* Denote by $\mathcal{U}_0$ the class of subgroups of groups from $\mathcal{U}$. Then $\mathcal{U}_0$ is a quasi-variety of topological groups.

Fix a space $X$. Then there exists the $\mathcal{U}_0$-free group $(F(X,\mathcal{U}_0), e_X)$ over $X$. The subspace $e_X(X)$ is closed in $F(X,\mathcal{U}_0)$. The group $F(X,\mathcal{U}_0)$ is a dense subgroup of some compact group $G \in \mathcal{V}$.

By construction, for every continuous mapping $f : X \to A \in \mathcal{V}$ there exists a unique continuous homomorphism $\bar{f} : G \to A$ such that $f = \bar{f} \circ e_X$. Thus $(G, e_X)$ is a $(\mathcal{U},\mathcal{V})$-free group over $X$.

Consider the projection $\pi : G^{\omega_1} \to G$ where $\pi((x_\alpha : \alpha < \omega_1)) = x_1$ for any $(x_\alpha : \alpha < \omega_1) \in G^{\omega_1}$.

Now we apply the construction from the proof of Theorem 1.

In $G^{\omega_1}$ there exists a dense pseudocompact subgroup $H$ and a closed subgroup $B$ of $H$ such that $\pi(H) \supseteq \pi(B) = F(X,\mathcal{U}_0)$ and $\pi|H : H \to F(X,\mathcal{U}_0)$ is a topological isomorphism. We consider that $F(X,\mathcal{U}_0) = B \subseteq H$. Let $e'_X(x) = e_X(x) \in B$ for any $x \in X$. Then $e_X(x) = \pi(e'_X(x)) \in e_X(X) \subseteq F(X,\mathcal{U}_0)$.

Fix a continuous mapping $f : X \to A \in V$. There exists a continuous homomorphism $f_1 : G \to A$ such that $f = f_1 \circ e_X$. Now we put $\bar{f}(y) = f_1(\pi(y))$ for any $y \in H$. Then $f = \bar{f} \circ e'_X$. By construction, the mapping $\bar{f}$ is a unique continuous homomorphism of $H$ into $A$ for which $f = \bar{f} \circ e'_X$. Therefore, $(H, e'_X)$ is a $(\mathcal{U}, \mathcal{V})$-free group over $X$. □

*Remark* 6. For $\mathcal{V} = \mathcal{U}_{ac}$ Theorem 3 was proved by Comfort and van Mill ( [1], Theorem 4.1.9b).

**Theorem 4.** *Let $\mathcal{U} = \mathcal{V}$ be a class of pseudocompact groups with the properties:*

1. *The class $\mathcal{U}$ is multiplicative.*
2. *If $A \in \mathcal{U}$ and $B$ is a pseudocompact subgroup of $A$, then $B \in \mathcal{U}$.*
3. *If $A \in \mathcal{U}$, then $A$ is a subgroup of some compact group $B \in \mathcal{U}$.*

*Denote by $\mathcal{U}_0$ the class of all subgroups of groups from $\mathcal{U}$.*

*If $X$ is a space and there exists an almost $(\mathcal{U}, \mathcal{V})$-free group over $X$ $(F(X,\mathcal{U}, \mathcal{V}), e_X)$, then $F(X,\mathcal{U}_0)$ is a finite group.*

*Proof.* Let $X$ be a space for which the $(\mathcal{U}, \mathcal{V})$-free group $(F(X,\mathcal{U}, \mathcal{V}), e_X)$ be given. We put $Y = e_X(X)$ and $G = F(X,\mathcal{U}, \mathcal{V})$. Then $Y \subseteq G$, $p_Y$ is the identity in $G$ and for any continuous mapping $f : Y \to A \in \mathcal{V}$ there exists a continuous homomorphism $\bar{f} : G \to A$ for which $f = \bar{f}|Y$. In particular, $G$ is the $(\mathcal{U}, \mathcal{V})$-free group over $Y$. Moreover, for any continuous mapping $g : X \to A \in \mathcal{V}$ there exists a unique continuous mapping $f : Y \to A$ such that $g = f \circ e_X$. Let $G$ be a dense subgroup of the compact group $\bar{G} \in \mathcal{U}$. Then there exists a compact subgroup $H$ of $\bar{G}$ such that $Y \subseteq H$ and $Y$ topologically generated $H$.

We can consider that $Y \subseteq F(Y,\mathcal{U}_0) \subseteq H$, i.e. the group $F(Y,\mathcal{U}_0)$ generated by $Y$ in $H$ is the $\mathcal{U}_0$-free group over $Y$. If $F(Y,\mathcal{U}_0)$ is finite, then $F(Y,\mathcal{U}_0)$ is the $(\mathcal{U}, \mathcal{V})$-free group over $X$ and over $Y$. Suppose that $F(Y,\mathcal{U}_0)$ is infinite. Then $F(Y,\mathcal{U}_0)$ is not pseudocompact.

Let $\tau = |\bar{G}|$.

There exist an uncountable cardinal $\lambda > \tau$, a set $\Lambda$ and a family $\{A_\alpha : \alpha \in \Lambda\}$ of pseudocompact groups such that:

1) any $A_\alpha$ is a subgroup of $H$ and $Y \subseteq A_\alpha$;

2) if $A$ is a pseudocompact subgroup of $H$ and $Y \subseteq A$, then $|\{\alpha \in \Lambda : A_\alpha \textit{ is isomorphic to } A\}| = \lambda$;

3) $|\Lambda| = \lambda$.

Denote by $H'$ the subgroup of $H$ generated by the set $Y$. Then $H'$ is a subgroup of $A_\alpha$ for any $\alpha \in \Lambda$.

Let $H_\alpha = H$ and $H'_\alpha = H'$ for any $\alpha \in \Lambda$. We put $B = \prod\{H_\alpha : \alpha \in \Lambda\}$ and $B' = \prod\{H'_\alpha : \alpha \in \Lambda\}$. If $x \in H$, then $\bar{x} = (x_\alpha : \alpha \in \Lambda)$, where $x_\alpha = x$ for any $\alpha \in \Lambda$. For any $L \subseteq H$ we put $\bar{L} = \{\bar{x} : x \in L\} \subseteq H$. Then $\bar{H}$ is the diagonal of $B = H^\lambda$. If $L$ is a subgroup of $H$, then $\bar{L}$ is a subgroup of $B$.

Consider $A = \prod\{A_\alpha : \alpha \in \Lambda\}$. Let $a \in H'$. Then $E_a = \{x = (x_\alpha : \alpha \in \Lambda) \in A :$ *the set* $\{\alpha \in \Lambda : x_\alpha \neq a\}$ *is countable* $\}$. Then $E_a$ is a dense pseudocompact subspace of the pseudocompact group $A$. If $a$ is the identity element of the group $H$, then $E_a$ is a subgroup of the group $A$.

Let $E = \cup\{E_a : a \in H'\}$. Then $E$ is a dense pseudocompact subgroup of $A$. Moreover, $A$ is a dense subgroup of the compact group $B$. By construction, $\bar{Y} \subseteq \bar{H}' \subseteq E$. The space $Y$ is homeomorphic to $\bar{Y}$. We consider that $y = \bar{y}$ for any $y \in Y$. Then $Y = \bar{Y} \subseteq E$. Since $E \in \mathcal{U} = \mathcal{V}$, there exists a continuous homomorphism $g : G \to E$ such that $g(y) = y = \bar{y}$ for any $y \in Y$.

Let $\beta \in \Lambda$ and $\pi_\beta : B \to H_\beta$ be the projection $\pi_\beta(x_\alpha : \alpha \in \Lambda) = x_\beta$ for any $(x_\alpha : \alpha \in \Lambda) \in B$. Then $\pi_\beta(g(G))$ is a pseudocompact subgroup of the pseudocompact group $A_\beta$ for any $\beta \in \Lambda$. Since $H'$ is not a pseudocompact group, $\pi_\beta(g(G)) \setminus H'_\beta \neq \emptyset$ for any $\beta \in \Lambda$. Thus, for any $\beta \in \Lambda$ there exists $z_\beta \in G$ for which $\pi_\beta(\pi_\beta(z_\beta)) \notin H'_\beta = H'$. By assertion, $|G| \leq |\bar{G}| = \tau < \lambda$. Thus, there exists $b \in G$ such that $|\{\beta \in \Lambda : z_\beta = b\}| = \lambda$. Let $\Lambda_b = \{\beta \in \Lambda : z_\beta = b\}$ and $c = g(b) \in g(G) \subseteq E$. There exists $a \in H'$ such that $c = (c_\alpha : \alpha \in \Lambda) \in E_a$. Thus, the set $\Lambda'_c = \{\alpha : c_\alpha \neq a\}$ is countable. Then $|\Lambda_b \setminus \Lambda'_c| = \lambda$ and $c_\alpha = a$ for any $\alpha \in \Lambda_b \setminus \Lambda'_c$, a contradiction. The theorem is proved. □

**Corollary 2.** *Let* $\mathcal{U} = \mathcal{V}$ *be a class of pseudocompact groups with the properties:*

1. *The class* $\mathcal{U}$ *is multiplicative.*
2. *If* $A \in \mathcal{U}$ *and* $B$ *is a pseudocompact subgroup of* $A$*, then* $B \in \mathcal{U}$*.*
3. *If* $A \in \mathcal{U}$*, then* $A$ *is a subgroup of some compact group* $B \in \mathcal{U}$*.*
4. *There exist* $A \in \mathcal{U}$ *and* $a \in A$ *such that* $a^m \neq a^n$ *for any distinct* $m, n \in \mathbb{N}$*.*

*If* $F(X,\mathcal{U},\mathcal{V})$ *is an almost* $(\mathcal{U},\mathcal{V})$*-free topological group over pointed space* $X$*, then either* $|X| = 1$ *or* $X$ *is connected and every connected subset of any* $H \in \mathcal{U}$ *is a singleton set.*

**Example 5.** Let $A$ be a finite non-trivial group. Denote by $\mathcal{U} = \mathcal{V}$ the family of all pseudocompact subgroups of the groups $A^\tau$, where $\tau$ is an arbitrary cardinal number. The class $\mathcal{U}$ is multiplicative and every group $A \in \mathcal{U}$ is a subgroup of some compact group $B \in \mathcal{U}$. Denote by $\mathcal{U}_0$ the class of all subgroups of groups from $\mathcal{U}$. Then $\mathcal{U}_0$ is a quasivariety of topological groups. If the space $X$ is finite, then $F(X,\mathcal{U}_0)$ is a finite group. If $X_1$ is a finite space and $X_2$ is a connected space, then $F(X_1 \times X_2,\mathcal{U}_0) \equiv F(X_1,\mathcal{U}_0)$ is a finite group.

*Remark* 7. Corollary 2 improved the results obtain by M. Tkachenco and R. Fokkink (see [1], p. 110).

## References

[1] Comfort W.W., Hofmann K.-H., Remus D. *Topological groups and semigroups.* Recent Progress in General Topology. Editors Hušek M., van Mill J. Elssevire Sciences Publishers B.V., 1992, 57–144.

[2] Comfort W.W., van Mill J. *On the existence of free topological groups.* Topology Appl., 1988, **29**, 245–269.

[3] Comfort W.W., van Mill J. *Concerning connected, pseudocompact Abelian groups.* Topology Appl., 1989, **33**, 21–45.

[4] Engelking R. *General Topology.* Warszawa, PWN, 1977.

[5] Fokkink R. *A note on pseudocompact groups.* Proc. Amer. Math. Soc., 1989, **107**, 569–571.

[6] Roelke W., Dierolf S. *Uniform structures of topological groups and their quotients.* McGraw-Hill, New York, 1981.

[7] Ursul M.I. *Imbeddings of locally precompact groups in locally pseudocompact ones.* Izvestia Akad. Nauk Mold. SSR, Fizika–Tehnica–Matematica, 1989, N 3, 54–56 (in Russian).

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# On Topological Groupoids and Multiple Identities

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**Abstract.** This paper studies some properties of $(n, m)$-homogeneous isotopies of medial topological groupoids. It also examines the relationship between paramediality and associativity. We extended some affirmations of the theory of topological groups on the class of topological $(n, m)$-homogeneous primitive goupoids with divisions.

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## 1 Introduction

In this article we study the $(n, m)$-homogeneous isotopies of topological groupoids with multiple identities and relation between paramediality and associativity. In Section 3 we expand on the notions of multiple identities and homogeneous isotopies introduced in [2]. This concept facilitates the study of topological groupoids with $(n, m)$-identities and homogeneous quasigroups, which are obtained by using isotopies of topological groups.

The results established in Section 4 are related to the results of M. Choban and L. Kiriyak [2] and to the research papers [5−8, 11]. We prove that if $(G, +)$ is a medial topological groupoid and $e$ is a $(k, p)$-zero, then every $(n, m)$-homogeneous isotope $(G, \cdot)$ of $(G, +)$ is medial, with $(mk, np)$-identity $e$ in $(G, \cdot)$. We present some interesting properties of a class of $(n, m)$-homogeneous quasigroups.

K. Sigmon, continuing the work of Professor A. D. Wallace, has shown that whenever a medial topological groupoid contains a bijective idempotent, it can be obtained from some commutative topological semigroup [12]. In Section 5, we obtain these and some other results in the case of paramedial topological groupoids. The relationship between mediality, paramediality and associativity was also studied in [9,10]. In Section 6 we extended one well-known statement of the theory of topological groups on the class of topological $(n, m)$-homogeneous primitive groupoids with divisions.

## 2 Basic notions

A non-empty set $G$ is said to be a groupoid relative to a binary operation denoted by $\{\cdot\}$ if for every ordered pair $(a, b)$ of elements of $G$ there is a unique element $ab \in G$.

If the groupoid $G$ is a topological space and the multiplication operation $(a, b) \to a \cdot b$ is continuous, then $G$ is called a topological groupoid.

A groupoid $G$ is called a groupoid with division if for every $a, b \in G$ the equations $ax = b$ and $ya = b$ have, not necessarily unique, solutions.

A groupoid $G$ is called reducible or cancellative if for any equality $xy = uv$ the equality $x = u$ is equivalent to the equality $y = v$.

A groupoid $G$ is called a primitive groupoid with divisions if there exist two binary operations $l : G \times G \to G$, $r : G \times G \to G$, such that $l(a, b) \cdot a = b$, $a \cdot r(a, b) = b$ for all $a, b \in G$. Thus a primitive groupoid with divisions is a universal algebra with three binary operations.

If in a topological groupoid $G$ the primitive divisions $l$ and $r$ are continuous, then we can say that $G$ is a topological primitive groupoid with continuous divisions.

A primitive groupoid $G$ with divisions is called a quasigroup if both equations $ax = b$ and $ya = b$ have unique solutions. In the quasigroup $G$ the divisions $l, r$ are unique.

An element $e \in G$ is called an identity if $ex = xe = x$ for every $x \in X$.

A quasigroup with an identity is called a loop.

If a multiplication operation in a quasigroup $(G, \cdot)$ endowed with a topology is continuous, then $G$ is called a semitopological quasigroup. If in a semitopological quasigroup $G$ the divisions $l$ and $r$ are continuous, then $G$ is called a topological quasigroup.

A groupoid $G$ is called medial if it satisfies the law $xy \cdot zt = xz \cdot yt$ for all $x, y, z, t \in G$. A groupoid $G$ is called paramedial if it satisfies the law $xy \cdot zt = ty \cdot zx$ for all $x, y, z, t \in G$.

If a medial (paramedial) quasigroup $G$ contains an element $e$ such that $e \cdot x = x (x \cdot e = x)$ for all $x$ in $G$, then $e$ is called a left (right) identity element of $G$ and $G$ is called a left (right) medial (paramedial) loop.

A groupoid $G$ is said to be hexagonal if it is idempotent, medial and semisymmetric, i.e. the equalities $x \cdot x = x$, $xy \cdot zt = xz \cdot yt$, $x \cdot zx = xz \cdot x = z$ hold for all of its elements.

Let $N = \{1, 2, ...\}$ and $Z = \{..., -2, -1, 0, 1, 2, ...\}$. Furthermore, we shall use the terminology from [1−4].

## 3 Multiple identities and homogeneous isotopies

Consider a groupoid $(G, +)$. For every two elements $a, b$ from $(G, +)$ we denote:

$$1(a, b, +) = (a, b, +)1 = a + b, \quad \text{and} \quad n(a, b, +) = a + (n - 1)(a, b, +),$$

$$(a, b, +)n = (a, b, +)(n - 1) + b$$

for all $n \geq 2$.

If a binary operation $(+)$ is given on a set $G$, then we shall use the symbols $n(a, b)$ and $(a, b)n$ instead of $n(a, b, +)$ and $(a, b, +)n$.

**Definition 1.** Let $(G, +)$ be a groupoid and let $n, m \geq 1$. The element $e$ of the groupoid $(G, +)$ is called:

– an $(n, m)$-zero of $G$ if $e + e = e$ and $n(e, x) = (x, e)m = x$ for every $x \in G$;

– an $(n, \propto)$-zero if $e + e = e$ and $n(e, x) = x$ for every $x \in G$;

– an $(\propto, m)$-zero if $e + e = e$ and $(x, e)m = x$ for every $x \in G$.

Clearly, if $e \in G$ is both an $(n, \propto)$-zero and an $(\propto, m)$-zero, then it is also an $(n, m)$-zero. If $(G, \cdot)$ is a multiplicative groupoid, then the element $e$ is called an $(n, m)$-identity. The notion of $(n, m)$-identity was introduced in [6].

**Example 1.** Let $(G, \cdot)$ be a paramedial groupoid, $e \in G$ and $xe = x$ for every $x \in G$. Then $(G, \cdot)$ is paramedial groupoid with $(2, 1)$-identity $e$ in $G$. Actually, if $x \in G$, then $e \cdot ex = ee \cdot ex = xe \cdot ee = xe \cdot e = xe = x$.

**Example 2.** Let $G = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. We define the binary operation $\{\cdot\}$.

| $(\cdot)$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 8 | 6 | 2 | 9 | 4 | 3 | 7 | 5 |
| 2 | 4 | 2 | 9 | 5 | 3 | 7 | 6 | 1 | 8 |
| 3 | 7 | 5 | 3 | 8 | 6 | 1 | 9 | 4 | 2 |
| 4 | 6 | 1 | 8 | 4 | 2 | 9 | 5 | 3 | 7 |
| 5 | 9 | 4 | 2 | 7 | 5 | 3 | 8 | 6 | 1 |
| 6 | 3 | 7 | 5 | 1 | 8 | 6 | 2 | 9 | 4 |
| 7 | 8 | 6 | 1 | 9 | 4 | 2 | 7 | 5 | 3 |
| 8 | 2 | 9 | 4 | 3 | 7 | 5 | 1 | 8 | 6 |
| 9 | 5 | 3 | 7 | 6 | 1 | 8 | 4 | 2 | 9 |

Then $(G, \cdot)$ is a non-commutative hexagonal quasigroup and each element from $(G, \cdot)$ is a $(6, 6)$-identity in $G$.

**Definition 2.** Let $(G, +)$ be a topological groupoid. A groupoid $(G, \cdot)$ is called a homogeneous isotope of the topological groupoid $(G, +)$ if there exist two topological automorphisms $\varphi, \psi : (G, +) \to (G, +)$ such that $x \cdot y = \varphi(x) + \psi(y)$, for all $x, y \in G$.

For every mapping $f : X \to X$ we denote $f^1(x) = f(x)$ and $f^{n+1}(x) = f(f^n(x))$ for any $n \geq 1$.

**Definition 3.** Let $n, m \leq \infty$. A groupoid $(G, \cdot)$ is called an $(n, m)$-homogeneous isotope of a topological groupoid $(G, +)$ if there exist two topological automorphisms $\varphi, \psi : (G, +) \to (G, +)$ such that:

1. $x \cdot y = \varphi(x) + \psi(y)$ for all $x, y \in G$;
2. $\varphi\psi = \psi\varphi$;
3. If $n < \infty$, then $\varphi^n(x) = x$ for all $x \in G$;
4. If $m < \infty$, then $\psi^m(x) = x$ for all $x \in G$.

**Definition 4.** A groupoid $(G, \cdot)$ is called an isotope of a topological groupoid $(G, +)$, if there exist two homeomorphisms $\varphi, \psi : (G, +) \to (G, +)$ such that

$$x \cdot y = \varphi(x) + \psi(y) \text{ for all } x, y \in G.$$

Under the conditions of Definition 4 we shall say that the isotope $(G, \cdot)$ is generated by the homeomorphisms $\varphi, \psi$ of the topological groupoids $(G, +)$ and write $(G, \cdot) = g(G, +, \varphi, \psi)$.

**Example 3.** Let $(G, +)$ be a topological commutative additive group with a zero.

1. If $\varphi(x) = x$, $\psi(x) = -x$ and $x \cdot y = x - y$, then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a topological medial quasigroup with a $(2, 1)$-identity 0.

2. If $\varphi(x) = -x$, $\psi(x) = x$ and $x \cdot y = y - x$, then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a topological medial quasigroup with a $(1, 2)$-identity 0.

**Example 4.** Let $(R, +)$ be a topological Abelian group of real numbers.

1. If $\varphi(x) = x$, $\psi(x) = 2x$ and $x \cdot y = x + 2y$, then $(R, \cdot) = g(R, +, \varphi, \psi)$ is a commutative locally compact medial quasigroup. By virtue of Theorem 7 from [2], there exists a right invariant Haar measure on $(R, \cdot)$.

2. If $\varphi(x) = x$, $\psi(x) = x + 7$ and $x \cdot y = x + y + 7$, then $(R, \cdot) = g(R, +, \varphi, \psi)$ is a commutative locally compact medial quasigroup and $(R, \cdot)$ does not contain $(n, m)$-identities. As above, by virtue of Theorem 7 from [2] there exists an invariant Haar measure on $(R, \cdot)$.

**Example 5.** Denote by $Z_p = Z/pZ = \{0, 1, ..., p - 1\}$ the cyclic Abelian group of order $p$. Consider the commutative group $(G, +) = (Z_7, +)$, $\varphi(x) = 3x$, $\psi(x) = 4x$ and $x \cdot y = 3x + 4y$. Then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a medial quasigroup with $(3, 6)$-identity in $(G, \cdot)$, which coincides with the zero element in $(G, +)$.

**Example 6.** Consider the commutative group $(G, +) = (Z_5, +)$, $\varphi(x) = 2x$, $\psi(x) = 3x$ and $x \cdot y = 2x + 3y$. Then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a medial quasigroup and the zero from $(G, \cdot)$ is a $(4, 4)$-identity in $G$.

**Example 7.** Consider the Abelian group $(G, +) = (Z_5, +)$, $\varphi(x) = 4x$, $\psi(x) = 2x$ and $x \cdot y = 4x + 2y$. Then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a medial quasigroup and each element from $(G, \cdot)$ is a $(4, 2)$-identity in $G$.

## 4 Some properties of $(n, m)$-homogeneous isotopies

**Proposition 1.** *If $(G, +)$ is a medial topological groupoid and $e$ is a $(k, p)$-zero, then every $(n, m)$-homogeneous isotope $(G, \cdot)$ of the topological groupoid $(G, +)$ is medial with $(mk, np)$-identity $e$ in $(G, \cdot)$ and $(x \cdot y) + (u \cdot v) = (x + u) \cdot (y + v)$ for all $x, y, u, v \in G$ and $n, m, p, k \in N$.*

*Proof.* The mediality of the $(n,m)$-homogeneous isotope $(G,\cdot)$ follows from [12]. We will prove that $e$ is an $(mk,np)$-identity in $(G,\cdot)$ by the method described in [2].

Let $(G,\cdot)$ be an $(n,m)$-homogeneous isotope of the groupoid $(G,+)$ and $e$ be a $(k,p)$-zero in $(G,+)$. We mention that $\varphi^q(e)=\psi^q(e)=e$ for every $q\in N$. If $k<+\infty$, then in $(G,+)$ we have $qk(e,x,+)=x$ for each $x\in G$ and for every $q\in N$. Let $m<+\infty$ and $\psi^m(x)=x$ for all $x\in G$. Then $1(e,x,\cdot)=1(e,\psi(x),+)$ and $q(e,x,\cdot)=q(e,\psi^q(x),+)$ for every $q\geq 1$. Therefore

$$mk(e,x,\cdot)=mk(e,\psi^{mk}(x),+)=mk(e,x,+)=x.$$

Analogously we obtain that

$$(e,x,\cdot)np=(e,\varphi^{np}(x),+)np=(e,x,+)np=x.$$

Hence $e$ is an $(mk,np)$-identity in $(G,\cdot)$.

Using the algorithm from [12] we will show that $(x\cdot y)+(u\cdot v)=(x+u)\cdot(y+v)$. Let $x\cdot y=\varphi(x)+\psi(y)$ and $u\cdot v=\varphi(u)+\psi(v)$. Then

$$(x\cdot y)+(u\cdot v)=[\varphi(x)+\psi(y)]+[\varphi(u)+\psi(v)]=$$

$$=[\varphi(x)+\varphi(u)]+[\psi(y)+\psi(v)]=\varphi(x+u)+\psi(y+v)=(x+u)\cdot(y+v).$$

In this way we have that $(x\cdot y)+(u\cdot v)=(x+u)\cdot(y+v)$. The proof is complete. □

**Corollary 1.** *If $(G,+)$ is a medial topological groupoid, then every homogeneous isotope $(G,\cdot)$ of the topological groupoid $(G,+)$ such that $\varphi\psi=\psi\varphi$ is medial and $(x\cdot y)+(u\cdot v)=(x+u)\cdot(y+v)$.*

**Definition 5.** A topological quasigroup $(G,\cdot)$ is called:

– homogeneous if $(G,\cdot)$ is a homogeneous isotope of the topological group $(G,+)$.

– $(n,m)$-homogeneous if $(G,\cdot)$ is a $(n,m)$-homogeneous isotope of the topological group $(G,+)$.

We denote by:

– $T$ the class of all medial quasigroups.

– $Q(n,m)$ the class of all $(n,m)$-homogeneous quasigroups.

We consider the class: $M(n,m)=T\cap Q(n,m)$.

The class $M(1,1)$ coincides with the class of topological abelian groups.

**Example 8.** Let $(G,\cdot)$ be a topological medial quasigroup.

1. If $e\in G$, such that $ex=x$ and $xx=e$ for each $x\in G$, then $(G,\cdot)\in M(1,2)$ and $(G,\cdot)$ is a topological medial quasigroup with $(1,2)$-identity $e$ in $G$.

2. If $e\in G$, such that $xe=x$ and $xx=e$ for each $x\in G$, then $(G,\cdot)\in M(2,1)$ and $(G,\cdot)$ is a topological medial quasigroup with $(2,1)$-identity $e$ in $G$.

**Theorem 1.** *Let $Q(n,m)$ be a class of $(n,m)$-homogeneous quasigroups. Then:*

*1. For each $G \in Q(n,m)$ there exists an $(n,m)$-identity $e \in G$ with the following properties:*

*1.1 $e \cdot e = e$;*

*1.2 $n(e,x) = x$;*

*1.3 $(x,e)m = x$;*

*1.4 $ex \cdot e = e \cdot xe$;*

*2. If $\varphi(x) = ex$ and $\varphi^n(x) = n(e,x) = x$ then $\varphi^{-1}(x) = (n-1)(e,x)$;*

*3. If $\varphi^{-1}(x) = (n-1)(e,x)$ and $\varphi^n(x) = n(e,x) = x$ then $(n-1)(e,ex) = x$;*

*4. If $\psi(x) = xe$ and $\psi^m(x) = (x,e)m = x$ then $\psi^{-1}(x) = (x,e)(m-1)$;*

*5. If $\psi^{-1}(x) = (x,e)(m-1)$ and $\psi^m(x) = (x,e)m = x$ then $(xe,e)(m-1) = x$.*

*Proof.* **1.** Let $(G,+)$ be a topological group and $\varphi, \psi : G \longrightarrow G$ be topological automorphisms of this group, such that $\varphi^n(x) = \psi^m(x) = x, \varphi \cdot \psi = \psi \cdot \varphi$, for each $x \in G$ and $(G,\cdot) = g(G,+,\varphi,\psi)$. Let $e$ be a zero in $(G,+)$. According to Theorem 3 from [2], $e$ is an $(n,m)$-identity in $(G,\cdot)$. Hence, $e \cdot e = e$, $n(e,x) = x$ and $(x,e)m = x$. Thus, assertions 1.1, 1.2 and 1.3 are proved.

It is easy to see that $\varphi(x) = ex$ and $\psi(x) = xe$. From the equality $\varphi\psi = \psi\varphi$ we have $\varphi\psi = \varphi(xe) = e \cdot xe$ and $\psi\varphi = \psi(ex) = ex \cdot e$. Therefore $e \cdot xe = ex \cdot e$. Assertion 1 is proved.

**2.** We will show that if $\varphi(x) = ex$ and $\varphi^n(x) = n(e,x) = x$, then

$$\varphi^{-1}(x) = (n-1)(e,x).$$

We have $\varphi(x) = ex$, hence $\varphi(\varphi^{-1}(x)) = e \cdot \varphi^{-1}(x)$. But $\varphi(\varphi^{-1}(x)) = x$. Therefore, $e \cdot \varphi^{-1}(x) = x$. Since $n(e,x) = x$, we obtain that

$$e \cdot (\varphi^{-1}(x)) = n(e,x). \tag{1}$$

By the definition of multiple identities we have

$$e \cdot (n-1)(e,x) = n(e,x). \tag{2}$$

From (1) and (2) we infer that $\varphi^{-1}(x) = (n-1)(e,x)$, which proves assertion 2.

**3.** We will prove that if $\varphi^{-1}(x) = (n-1)(e,x)$ and $\varphi^n(x) = n(e,x) = x$ then

$$(n-1)(e,ex) = x.$$

Let be $(n-1)(e,ex) = t$. Then

$$e \cdot (n-1)(e,ex) = et. \tag{3}$$

By the definition of multiple identities

$$e \cdot (n-1)(e,ex) = n(e,ex) = ex. \tag{4}$$

From (3) and (4) it follows $ex = et$ and $t = x$. Hence $(n-1)(e, ex) = x$, as desired.

**4.** By following the same guidelines as in property 2 we obtain that if $\psi(x) = xe$ and $\psi^m(x) = (x, e)m$, then $\psi^{-1}(x) = (x, e)(m-1)$.

**5.** Similarly to properties 3 we prove that if $\psi^{-1}(x) = (x, e)(m-1)$ and $\psi^m(x) = (x, e)m = x$, then $(xe, e)(m-1) = x$.
The proof of the theorem is now complete. □

**Corollary 2.** *A class $Q(n, m)$ of $(n, m)$-homogeneous quasigroups forms a variety.*

**Corollary 3.** *A class $M(n, m)$ of topological medial quasigroups with $(n, m)$-identities forms a variety.*

## 5 Paramedial topological groupoids

We provide an example of a paramedial groupoid which is not medial.

**Example 9.** Let $G = \{1, 2, 3, 4\}$. We define the binary operation $\{\cdot\}$.

| $(\cdot)$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 2 | 4 | 3 |
| 2 | 3 | 4 | 2 | 1 |
| 3 | 2 | 1 | 3 | 4 |
| 4 | 4 | 3 | 1 | 2 |

Then $(G, \cdot)$ is a paramedial quasigroup but it is not medial. For example

$$(2 \cdot 3) \cdot (1 \cdot 4) \neq (2 \cdot 1) \cdot (3 \cdot 4).$$

An element $e$ is called idempotent if $ee = e$. If the maps $x \to xe$ and $x \to ex$ are homeomorphisms then $e$ is also called bijective.

**Theorem 2.** *Let $(G, \cdot)$ be a paramedial topological groupoid and let $e, e_1, e_2$ be elements of $G$ for which:*

1. *$ee_1 = e_1$ and $e_2e = e_2$;*
2. *The maps $x \to e_1x$ and $x \to xe_2$ are homeomorphisms of $G$ onto itself;*
3. *The map $x \to xe$ is surjective.*

*If there exists a binary operation $\{\circ\}$ on $G$ such that $(e_1x) \circ (ye_2) = yx$, then $(G, \circ)$ is a commutative topological semigroup having $e_1e_2$ as identity.*

*Proof.* Since $x \to e_1x$ and $x \to xe_2$ are homeomorphisms it is clear that $\{\circ\}$ is continuous.

Using surjectivity and the fact that $(e_1e_2)\circ(ye_2)=ye_2$ and $(e_1x)\circ(e_1e_2)=e_1x$ we see that $e_1e_2$ is an identity for $(G,\circ)$. Observe that $xe_1\cdot e_2=xe_1\cdot e_2e=ee_1\cdot e_2x=e_1\cdot e_2x$.

One can see that

$$xe_1\cdot zt=(e_1\cdot zt)\circ(xe_1\cdot e_2)=(ee_1\cdot zt)\circ(xe_1\cdot e_2)=$$
$$=(te_1\cdot ze)\circ(xe_1\cdot e_2)=[(e_1\cdot ze)\circ(te_1\cdot e_2)]\circ(xe_1\cdot e_2);$$

$$te_1\cdot zx=(e_1\cdot zx)\circ(te_1\cdot e_2)=(ee_1\cdot zx)\circ(te_1\cdot e_2)=$$
$$=(xe_1\cdot ze)\circ(te_1\cdot e_2)=[(e_1\cdot ze)\circ(xe_1\cdot e_2)]\circ(te_1\cdot e_2).$$

From paramediality we have

$$[(e_1\cdot ze)\circ(te_1\cdot e_2)]\circ(xe_1\cdot e_2)=[(e_1\cdot ze)\circ(xe_1\cdot e_2)]\circ(te_1\cdot e_2).$$

Setting $z=e_2$ then since $e_2e=e_2$ and $e_1e_2$ is an identity it follows that:

$$[e_1e_2\circ(te_1\cdot e_2)]\circ(xe_1\cdot e_2)=[e_1e_2\circ(xe_1\cdot e_2)]\circ(te_1\cdot e_2)$$

and

$$(te_1\cdot e_2)\circ(xe_1\cdot e_2)=(xe_1\cdot e_2)\circ(te_1\cdot e_2).$$

Hence, $(G,\circ)$ is a commutative topological groupoid and then the associativity is immediate. Indeed

$$[(te_1\cdot e_2)\circ(e_1\cdot ze)]\circ(xe_1\cdot e_2)=(te_1\cdot e_2)\circ[(e_1\cdot ze)\circ(xe_1\cdot e_2)].$$

The proof is complete. □

**Theorem 3.** *Let $(G,\cdot)$ be a paramedial topological groupoid satisfying the following conditions:*

*1. It contains an idempotent $e$;*

*2. The maps $x\to xe$ and $x\to ex$ are homeomorphisms of $G$ onto itself;*

*3. There exists a binary operation $\{\circ\}$ on $G$ such that $(ex)\circ(ye)=yx$.*

*Then $(G,\circ)$ is a commutative topological semigroup having $e$ as identity. Moreover, the maps $x\to xe$ and $x\to ex$ are antihomomorphisms of $(G,\circ)$ and $xe\cdot e=e\cdot ex$.*

*Proof.* The first part of Theorem 3 follows from Theorem 2 with $e=e_1=e_2$. Indeed, we have $xe\cdot e=xe\cdot ee=ee\cdot ex=e\cdot ex$. Since

$$(ex\circ ye)\,e=yx\cdot e=yx\cdot ee=ex\cdot ey=(e\cdot ey)\circ(ex\cdot e)=(ye\cdot e)\circ(ex\cdot e),$$

we see that $x\to xe$ is an antihomorphism of $(G,\circ)$. Similarly

$$e\,(ex\circ ye)=e\cdot yx=ee\cdot yx=xe\cdot ye=(e\cdot ye)\circ(xe\cdot e)=(e\cdot ye)\circ(e\cdot ex).$$

Consequently, we obtain that $x\to ex$ is an antihomorphism of $(G,\circ)$. The proof is complete. □

A topological groupoid $(G,\circ)$ is called radical if the map $s : G \to G$, defined by $s(x) = x \circ x$, is a homeomorphism.

If $(G,\circ)$ is paramedial and radical then $s$, and hence $s^{-1}$, is an antihomomorphism of $(G,\circ)$.

A topological groupoid $(G,\cdot)$ where $\{\cdot\}$ is defined by

$$x \cdot y = s^{-1}(x) \circ s^{-1}(y) = s^{-1}(y \circ x)$$

is called the radical isotope of $(G,\circ)$.

A radical isotope $(G,\cdot)$ of $(G,\circ)$ is idempotent since

$$x \cdot x = s^{-1}(x \circ x) = s^{-1}(s(x)) = x$$

for each $x \in G$.

**Theorem 4.** *If $(G,\circ)$ is a topological groupoid with unit $e$, $(G,\cdot)$ is a commutative, idempotent topological groupoid and*

$$(x \circ y) \cdot (z \circ t) = (ty) \circ (zx),$$

*then $(G,\circ)$ is a commutative radical semigroup.*

*Proof.* If we define $t : G \to G$ by $t(x) = ex$ then $t$ is an antihomomorphism of $(G,\circ)$. Indeed, for all $x, y \in G$ we have,

$$t(x \circ y) = e(x \circ y) = (e \circ e)(x \circ y) = (ye) \circ (xe) = (ey) \circ (ex) = t(y) \circ t(x).$$

In particular, we obtain

$$t(s(x)) = t(x \circ x) = t(x) \circ t(x) = s(t(x));$$

where $s : G \to G$ is defined by $s(x) = x \circ x$.

Also, for each $x, y \in G$ and each unit $e$ in $(G,\circ)$

$$xy = (e \circ x) \cdot (e \circ y) = (e \circ x) \cdot (y \circ e) = (ex) \circ (ye) =$$
$$= (ex) \circ (ey) = t(x) \circ t(y) = t(y \circ x).$$

Hence $t(s(x)) = t(x \circ x) = xx = x$.

It follows that $t$ is a continuous inverse for $s$ so that $(G,\circ)$ is radical. Since $(G,\cdot)$ is commutative and $x \circ y = s(yx) = s(xy) = y \circ x$ then $\{\circ\}$ is commutative. Since $xy = t(y \circ x)$ and $t = s^{-1}$ then $(G,\cdot)$ is the radical isotope of $(G,\circ)$. It only remains to show that $\{\circ\}$ is associative.

Since $t$ is bijective and

$$t[(x \circ y) \circ z] = z \cdot (x \circ y) = (e \circ z) \cdot (x \circ y) = (yz) \circ (xe) =$$
$$= (yz) \circ (ex) = t(z \circ y) \circ t(x) = t[x \circ (z \circ y)] = t[x \circ (y \circ z)].$$

we conclude that $(G,\circ)$ is a commutative radical semigroup. The proof is complete. $\square$

## 6 On topological primitive groupoid with divisions

The following fundamental Theorem was proved in [2].

**Theorem.** *Let $(G,+)$ be a topological groupoid, $\varphi, \psi : (G,+) \longrightarrow (G,+)$ be homeomorphisms and $(G,\cdot) = g(G,+,\varphi,\psi)$. Then:*

*1. $(G,+) = g(G,\cdot,\varphi^{-1},\psi^{-1})$;*

*2. $(G,\cdot)$ is a topological groupoid;*

*3. If $(G,+)$ is a reducible groupoid , then $(G,\cdot)$ is a reducible groupoid too;*

*4. If $(G,+)$ is a groupoid with divisions, then $(G,\cdot)$ is a groupoid with divisions too;*

*5. If $(G,+)$ is a topological primitive groupoid with divisions, then $(G,\cdot)$ is a topological primitive groupoid with divisions too;*

*6. If $(G,+)$ is a topological quasigroup, then $(G,\cdot)$ is a topological quasigroup;*

*7. If $n,m,p,k \in N$ and $(G,\cdot)$ is an $(n,m)$-homogeneous isotope of the groupoid $(G,+)$ and $e$ is an $(k,p)$-zero in $(G,+)$, then $e$ is an $(mk,np)$-identity in $(G,\cdot)$.*

We consider a topological groupoid $(G,+)$. If $\alpha$ is a binary relation on $G$, then $\alpha(x) = \{y \in G : x\alpha y\}$ for every $x \in G$.

An equivalence relation $\alpha$ on $G$ is called a congruence on $(G,+)$ if from $(x\alpha u)$ and $(y\alpha v)$ it follows $(x+y)\alpha\ (u+v)$ for all $x,y,u,v \in G$.

If $(G,+)$ is a primitive groupoid with divisions $l$ and $r$, then we consider that $l(x,y)\alpha\ l(u,v)$, and $r(x,y)\alpha\ r(u,v)$ provided $(x\alpha u)$ and $(y\alpha v)$.

Let $(G,+,r,l)$ be a topological primitive groupoid with divisions $r,l$ and $(k,p)$-zero. Let $(G,\cdot) = g(G,+,\varphi,\psi)$ be an $(n,m)$-homogeneous isotope. Then, by virtue of the aforementioned Theorem, $e$ is an $(mk,np)$-identity of the topological primitive groupoid with divisions $(G,\cdot)$.

**Definition 6.** A primitive subgroupoid with divisions $H$ of the primitive groupoid with divisions $(G,+,r,l)$ is called a normal primitive subgroupoid with divisions if $e \in H$ and $H = G(\alpha)$, for some congruence $\alpha$.

**Lemma 1.** *Let $\alpha$ be a congruence of the topological primitive groupoid with divisions $(G,+,r,l)$. Then there exists a unique normal primitive subgroupoid with divisions $G(\alpha)$, which is called the primitive subgroupoid with divisions defined by congruence $\alpha$ such that $e \in G$.*

*Proof.* The set $G(\alpha) = \alpha(e) = \{y \in G : e\alpha y\}$ is the desired primitive subgroupoid with divisions. The proof is complete. $\square$

**Definition 7.** The primitive subgroupoids with divisions $(H_1,+,r,l)$ and $(H_2,+,r,l)$ of the topological primitive groupoid with divisions $(G,+,r,l)$ are called conjugate if $H_2 = h(H_1)$ for some topological automorphism $h : G \to G$.

**Theorem 5.** *Let $H$ be a primitive subgroupoid with divisions of the topological primitive groupoid with divisions $(G, +, r, l)$ and let $e \in H$. Then there exists such a primitive subgroupoid with divisions $Q$ of the topological primitive groupoids with divisions $(G, +, r, l)$ and $(G, \cdot, r_1, l_1)$ for which:*

*1. $e \in Q \subseteq H$.*

*2. $Q$ is the intersection of a finite number of the primitive subgroupoids with divisions conjugate to $H$ of the $(G, +, r, l)$.*

*3. If $H$ is a closed set, then $Q$ is closed too.*

*4. If $H$ is a $G_\delta$ set, then $Q$ is a $G_\delta$ set too.*

*5. If $H$ is an open set, then $Q$ is open too.*

*6. If $H$ is a normal primitive subgroupoid with divisions, then $Q$ is a normal primitive subgroupoid with divisions of $(G, +, r, l)$ and $(G, \cdot, r_1, l_1)$.*

*Proof.* We put $\{h_p : p \leq n \cdot m\} = \{\varphi^i \circ \psi^j : i \leq n, j \leq m\}, H_p = h_p(H)$ and $Q = \cap\{H_p : p \leq n \cdot m\}$.

We consider that $h_1(x) = x$ for each $x \in H$. Fix $i \leq n$ and $j \leq n$. Let $h_p = \varphi^i \circ \psi^j$. It is clear that $h_p$ is an automorphism of $(G, +, r, l)$. Thus $H_p = h_p(H)$ is a primitive subgroupoid with divisions of $(G, +, r, l)$ conjugate to $H$ in $(G, +, r, l)$. Therefore $Q$ is a primitive subgroupoid with divisions of $(G, +, r, l)$. This establishes assertions 1–5.

Firstly we prove that $Q$ is a primitive sugroupoid with divisions of $(G, \cdot, r_1, l_1)$. Let $x, y, b \in Q$. Then $xy = \varphi(x) + \psi(y)$ and $\varphi(x), \psi(x) \in H_i$ for any $i$. Thus $xy \in Q$. If $ax = b$, then $a = l_1(x, b) \in H_i$ for every $i$ and $a \in Q$. Similarly, if $xa = b$, then $a = r_1(x, b) \in H_i$ for any $i$ and $a \in Q$. Hence $Q$ is a primitive subgroupoid with divisions of $(G, \cdot, r_1, l_1)$.

Let $\alpha$ be a congruence of $(G, +, r, l)$. Then, by virtue of Lemma 1, there exists a unique normal primitive subgroupoid with divisions $H = G(\alpha)$ and $e \in H$. Because $h_p$ is a topological automorphism of $(G, +, r, l)$, then $H_p = h_p(H)$ is a normal primitive subgroupoid with divisions of $(G, +, r, l)$ conjugate to the normal primitive subgroupoid with divisions $H$. Therefore $Q$ is a normal primitive subgroupoid with divisions of $(G, +, r, l)$ and $Q$ is a normal primitive subgroupoid of $(G, \cdot, r, l)$. This proves assertion 6 and completes the proof of the theorem. □

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## References

[1] Belousov V.D. *Foundation of the theory of quasigroups and loops.* Moscow, Nauka, 1967 (in Russian).

[2] Choban M.M., Kiriyak L.L. *The topological quasigroups with multiple identities.* Quasigroups and Related Systems, 2002, **9**, 19–31.

[3] Engelking R. *General topology.* Polish Scientific Publishers, Warszawa, 1977.

[4] Hewitt E., Ross K.A. *Abstract harmonic analysis. Vol. 1: Structure of topological groups. Integration theory. Group representation.* Berlin, 1963.

[5] Kyriyak L.L., Choban M.M. *About homogeneous isotopies and congruences.* Învaţamîntul universitar din Moldova la 70 ani, Vol. 3, Chişinău, 2000, p. 33.

[6] Choban M.M., Kiriyak L.L. *The Medial Topological Quasigroup with Multiple identities.* The 4 th Conference on Applied and Industrial Mathematics, Oradea – CAIM, 1996.

[7] Chiriac L.L. *On Topological Quasigroups and Homogeneous Isotopies.* Analele Universităţii din Piteşti, Buletin Ştiinţific, Seria Matematică şi informatica, 2003, Nr. 9, 191–196.

[8] Chiriac L.L., Bobeica N. *Some properties of the homogeneous isotopies.* Acta et Commentationes, Vol. III, Universitatea de Stat Tiraspol, Chişinău, 2006, 107–112.

[9] Chiriac L.L. *On some generalization of commutativity in topological groupoids.* $6^{th}$ Congress of Romanian Mathematicians, June 28 – July 4, 2007, Bucharest, Romania, p. 45.

[10] Chiriac L.L., Bobeica N. *Paramedial topological groupoids.* $6^{th}$ Congress of Romanian Mathematicians, June 28 – July 4, 2007, Bucharest, Romania, p. 25–26.

[11] Chiriac L.L. *On topological primitive groupoid with divisions.* Acta et Commentationes, Vol. III, Universitatea de Stat Tiraspol, Chişinău, 2003, p. 175–178.

[12] Kermit Sigmon *Medial topological groupoids.* Aeq. Math., Vol. 1, 1968, p. 217–234.

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# The distribution of a planar random evolution with random start point

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**Abstract.** We consider the symmetric Markovian random evolution $\mathbf{X}(t)$ in the Euclidean plane $\mathbb{R}^2$ starting from a random point whose coordinates are the independent standard Gaussian random variables. The integral and series representations of the transition density of $\mathbf{X}(t)$ are obtained.

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The planar random motion at finite speed was dealt with in a series of works [2–4]. In these works the following planar stochastic motion was studied. A particle starts from the origin $\mathbf{0} = (0, 0)$ of the plane $\mathbb{R}^2$ at time $t = 0$ and moves with constant finite speed $c$. The initial direction is a two-dimensional random vector with uniform distribution on the unit circumference

$$S(\mathbf{0}, 1) = \left\{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : \ \|\mathbf{x}\|^2 = x_1^2 + x_2^2 = 1 \right\}.$$

The particle changes its direction at random instants that form a homogeneous Poisson process of rate $\lambda > 0$. At these moments it instantaneously takes on the new direction with uniform distribution on $S(\mathbf{0}, 1)$, independently of its previous motion.

Let $\mathbf{X}(t) = (X_1(t), X_2(t))$ denote the particle's position at an arbitrary instant $t > 0$. At any time $t > 0$ the particle, with probability 1, is located in the planar disc of radius $ct$

$$\mathbf{B}(\mathbf{0}, ct) = \left\{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : \ \|\mathbf{x}\|^2 = x_1^2 + x_2^2 \le c^2t^2 \right\}.$$

Let $d\mathbf{x}$ be the infinitesimal element of the plane $\mathbb{R}^2$ with the Lebesgue measure $\mu(d\mathbf{x}) = dx_1 dx_2$. The distribution $Pr\left\{\mathbf{X}(t) \in d\mathbf{x}\right\}, \ \mathbf{x} \in \mathbf{B}(\mathbf{0}, ct), \ t \ge 0$, consists of two components. The singular component corresponds to the case when no Poisson event occurs in the interval $(0, t)$ and is concentrated on the circumference

$$S(\mathbf{0}, ct) = \partial\mathbf{B}(\mathbf{0}, ct) = \left\{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : \ \|\mathbf{x}\|^2 = x_1^2 + x_2^2 = c^2t^2 \right\}.$$

In this case, in the moment $t$, the particle is located on the sphere $S(\mathbf{0}, ct)$ and the probability of this event is

$$Pr\left\{\mathbf{X}(t) \in S(\mathbf{0}, ct)\right\} = e^{-\lambda t}.$$

If at least one Poisson event occurs, the particle is located strictly inside the disc $\mathbf{B}(\mathbf{0}, ct)$, and the probability of this event is

$$Pr\,\{\mathbf{X}(t) \in \text{int}\,\mathbf{B}(\mathbf{0}, ct)\} = 1 - e^{-\lambda t}.$$

The part of the distribution $Pr\,\{\mathbf{X}(t) \in d\mathbf{x}\}$ corresponding to this case is concentrated in the interior

$$\text{int}\,\mathbf{B}(\mathbf{0}, ct) = \left\{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : \ \|\mathbf{x}\|^2 = x_1^2 + x_2^2 < c^2t^2\right\},$$

and forms its absolutely continuous component. Therefore there exists the density of the absolutely continuous component of the distribution $Pr\,\{\mathbf{X}(t) \in d\mathbf{x}\}$.

The principal known result states that the complete density $f(\mathbf{x}, t)$ of the process $\mathbf{X}(t)$ (starting from the origin $\mathbf{0}$), has the form

$$f(\mathbf{x}, t) = \frac{e^{-\lambda t}}{2\pi ct}\,\delta(c^2t^2 - \|\mathbf{x}\|^2) + \frac{\lambda}{2\pi c}\frac{\exp\left(-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2 - \|\mathbf{x}\|^2}\right)}{\sqrt{c^2t^2 - \|\mathbf{x}\|^2}}\Theta(ct - \|\mathbf{x}\|), \quad (1)$$

$$\mathbf{x} = (x_1, x_2) \in \mathbf{B}(\mathbf{0}, ct), \quad \|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}, \quad t \geq 0,$$

where $\delta(x)$ is the Dirac delta-function and $\Theta(x)$ is the Heaviside step function. The first term in (1) represents the density of the singular part of the distribution concentrated on the sphere $S(\mathbf{0}, ct)$, while the second term is the density of the absolutely continuous part of the distribution concentrated in int $\mathbf{B}(\mathbf{0}, ct)$.

If the process $\mathbf{X}(t)$ starts from some arbitrary fixed point $\mathbf{x}^0 = (x_1^0, x_2^0) \in \mathbb{R}^2$, then, given the phase space $\mathbb{R}^2$ is isotropic and homogeneous, the density of $\mathbf{X}(t)$ has the form

$$\begin{aligned} f(\mathbf{x} - \mathbf{x}_0, t) = {} & \frac{e^{-\lambda t}}{2\pi ct}\,\delta(c^2t^2 - \|\mathbf{x} - \mathbf{x}^0\|^2) + \\ & + \frac{\lambda}{2\pi c}\frac{\exp\left(-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2 - \|\mathbf{x} - \mathbf{x}^0\|^2}\right)}{\sqrt{c^2t^2 - \|\mathbf{x} - \mathbf{x}^0\|^2}}\Theta(ct - \|\mathbf{x} - \mathbf{x}^0\|), \end{aligned} \quad (2)$$

$$\mathbf{x} = (x_1, x_2) \in \mathbf{B}(\mathbf{x}^0, ct), \quad \|\mathbf{x} - \mathbf{x}^0\| = \sqrt{(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2}, \quad t \geq 0.$$

Suppose that the start point $\mathbf{x}^0 = (x_1^0, x_2^0)$ is a two-dimensional random variable (random vector) with given density $p(\mathbf{x})$ on the plane $\mathbb{R}^2$. If the random vectors $\mathbf{X}(t)$ and $\mathbf{x}^0$ are independent for any $t > 0$, then the density of $\mathbf{X}(t)$ is given by the convolution

$$\varphi(\mathbf{x}, t) = f(\mathbf{x}, t) * p(\mathbf{x}) = \int\limits_{\mathbb{R}^2} f(\mathbf{x} - \boldsymbol{\xi}, t)\, p(\boldsymbol{\xi})\, \mu(d\boldsymbol{\xi}). \quad (3)$$

In this paper we obtain a closed-form expression for density (3) when the initial point $\mathbf{x}^0 = (x_1^0, x_2^0)$ is a two-dimensional standard Gaussian vector with independent coordinates. In this case the density $p(\mathbf{x})$ has the form

$$p(\mathbf{x}) = p(x_1, x_2) = \frac{1}{2\pi}\,\exp\left(-\frac{x_1^2 + x_2^2}{2}\right). \quad (4)$$

Due to the fairly simple form of function (4) we are able to obtain the density of the process $\mathbf{X}(t)$ starting from a Gaussian random point of the Euclidean plane $\mathbb{R}^2$.

First, we will prove two auxiliary lemmas.

**Lemma 1.** *For arbitrary $q > 0$ and any integer $n \geq 0$ the following formula holds*

$$\int_0^1 x^n \, I_0(q\sqrt{1-x^2}) \, dx = 2^{(n-1)/2} \, \Gamma\left(\frac{n+1}{2}\right) \frac{I_{(n+1)/2}(q)}{q^{(n+1)/2}}, \tag{5}$$

*where $I_\nu(x)$ is the Bessel function of order $\nu$ with imaginary argument given by*

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{k! \, \Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k+\nu}. \tag{6}$$

*Proof.* Making the substitution $z = \sqrt{1-x^2}$ in the integral on the left-hand side of (5), we obtain

$$\begin{aligned}
\int_0^1 x^n \, I_0(q\sqrt{1-x^2}) \, dx &= \int_0^1 z \, (1-z^2)^{(n-1)/2} \, I_0(qz) \, dz = \\
&= \frac{1}{2} \int_0^1 (1-\xi)^{(n-1)/2} \, I_0(q\sqrt{\xi}) \, d\xi = \\
&= \frac{1}{2} \int_0^1 (1-\xi)^{(n-1)/2} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{q\sqrt{\xi}}{2}\right)^{2k} d\xi = \\
&= \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{q}{2}\right)^{2k} \int_0^1 \xi^k \, (1-\xi)^{(n-1)/2} \, d\xi = \\
&= \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{q}{2}\right)^{2k} B\left(\frac{n+1}{2}, k+1\right) = \\
&= \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{q}{2}\right)^{2k} \frac{\Gamma\left(\frac{n+1}{2}\right) \Gamma(k+1)}{\Gamma\left(\frac{n+1}{2}+k+1\right)} = \\
&= \frac{1}{2} \, \Gamma\left(\frac{n+1}{2}\right) \sum_{k=0}^{\infty} \frac{1}{k! \, \Gamma\left(\frac{n+1}{2}+k+1\right)} \left(\frac{q}{2}\right)^{2k} = \\
&= \frac{1}{2} \, \Gamma\left(\frac{n+1}{2}\right) \left(\frac{2}{q}\right)^{(n+1)/2} \sum_{k=0}^{\infty} \frac{1}{k! \, \Gamma\left(\frac{n+1}{2}+k+1\right)} \left(\frac{q}{2}\right)^{2k+(n+1)/2} = \\
&= \frac{1}{2} \, \Gamma\left(\frac{n+1}{2}\right) \left(\frac{2}{q}\right)^{(n+1)/2} I_{(n+1)/2}(q) = \\
&= 2^{(n-1)/2} \, \Gamma\left(\frac{n+1}{2}\right) \frac{I_{(n+1)/2}(q)}{q^{(n+1)/2}}.
\end{aligned}$$

The lemma is proved. □

**Lemma 2.** *For arbitrary $a > 0,\ b > 0$ and $q > 0$ the following formula holds*

$$\begin{aligned}\int_0^1 & e^{ax^2+bx}\ I_0(q\sqrt{1-x^2})\ dx = \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{a^{n-k}\ b^k}{k!\ (n-k)!}\ 2^{(2n-k-1)/2}\ \Gamma\left(\frac{2n-k+1}{2}\right)\ \frac{I_{(2n-k+1)/2}(q)}{q^{(2n-k+1)/2}},\end{aligned} \tag{7}$$

*where $I_\nu(x)$ is the Bessel function of order $\nu$ with imaginary argument given by (6).*

*Proof.* By expanding the exponential and applying formula (5) of Lemma 1, we obtain

$$\begin{aligned}\int_0^1 e^{ax^2+bx}\ I_0(q\sqrt{1-x^2})\ dx &= \sum_{n=0}^{\infty}\frac{1}{n!}\ \int_0^1 (ax^2+bx)^n\ I_0(q\sqrt{1-x^2})\ dx = \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n} C_n^k\ a^{n-k}\ b^k\ \int_0^1 x^{2n-k}\ I_0(q\sqrt{1-x^2})\ dx = \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{a^{n-k}\ b^k}{k!\ (n-k)!}\ 2^{(2n-k-1)/2}\ \Gamma\left(\frac{2n-k+1}{2}\right)\ \frac{I_{(2n-k+1)/2}(q)}{q^{(2n-k+1)/2}},\end{aligned}$$

proving (7). The lemma is proved. □

The series on the right-hand side of (7) has a fairly complicated form and seemingly cannot be reduced to a more elegant expression. Nevertheless, it enables us to obtain a series representation of the transition density of $\mathbf{X}(t)$.

Now we are able to establish our main result. It is given by the following theorem.

**Theorem 1.** *The transition density of the planar random evolution $\mathbf{X}(t)$ started from a random point $\mathbf{x}^0$ with Gaussian density (4) is given by the formula*

$$\begin{aligned}\varphi(\mathbf{x},t) = \frac{e^{-\lambda t}}{2\pi}\ & e^{-(\|\mathbf{x}\|^2+c^2t^2)/2}\ I_0(ct\|\mathbf{x}\|)+ \\ &+ \frac{\lambda t\ e^{-\lambda t}}{2\pi}\ e^{-(\|\mathbf{x}\|^2+c^2t^2)/2}\ \int_0^1 e^{(c^2t^2/2)\xi^2+\lambda t\xi}\ I_0(ct\|\mathbf{x}\|\sqrt{1-\xi^2})\ d\xi.\end{aligned} \tag{8}$$

*The density (8) has the following series representation*

$$\begin{aligned}\varphi(\mathbf{x},t) = \frac{e^{-\lambda t}}{2\pi}\ & e^{-(\|\mathbf{x}\|^2+c^2t^2)/2}\ I_0(ct\|\mathbf{x}\|) + \frac{\lambda t\ e^{-\lambda t}}{2\pi}\ e^{-\|\mathbf{x}\|^2/2}\times \\ &\times \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(\lambda t)^k\ 2^{(k+1)/2}}{k!\ (n-k)!}\ (c^2t^2)^{n-k}\ \Gamma\left(\frac{2n-k+1}{2}\right)\ \frac{I_{(2n-k+1)/2}(ct\|\mathbf{x}\|)}{(ct\|\mathbf{x}\|)^{(2n-k+1)/2}}.\end{aligned} \tag{9}$$

*Proof.* According to (3) and taking into account (2) and (4), we have

$$\varphi(\mathbf{x},t) = \varphi(x_1,x_2,t) =$$
$$= \frac{e^{-\lambda t}}{4\pi^2 ct} \iint\limits_{\mathbb{R}^2} \exp\left(-\frac{\xi_1^2+\xi_2^2}{2}\right) \delta(c^2t^2 - (x_1-\xi_1)^2 - (x_2-\xi_2)^2)\, d\xi_1\, d\xi_2 +$$
$$+ \frac{\lambda e^{-\lambda t}}{4\pi^2 c} \iint\limits_{\mathbb{R}^2} \frac{\exp\left(\frac{\lambda}{c}\sqrt{c^2t^2 - (x_1-\xi_1)^2 - (x_2-\xi_2)^2}\right)}{\sqrt{c^2t^2 - (x_1-\xi_1)^2 - (x_2-\xi_2)^2}} \exp\left(-\frac{\xi_1^2+\xi_2^2}{2}\right) \times$$
$$\times \Theta\left(ct - \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2}\right) d\xi_1\, d\xi_2 =$$
$$= \frac{e^{-\lambda t}}{4\pi^2 ct} \iint\limits_{\mathbb{R}^2} \exp\left(-\frac{(x_1-\xi_1)^2 + (x_2-\xi_2)^2}{2}\right) \delta(c^2t^2 - (\xi_1^2+\xi_2^2))\, d\xi_1\, d\xi_2 +$$
$$+ \frac{\lambda e^{-\lambda t}}{4\pi^2 c} \iint\limits_{\mathbb{R}^2} \frac{\exp\left(\frac{\lambda}{c}\sqrt{c^2t^2 - (\xi_1^2+\xi_2^2)}\right)}{\sqrt{c^2t^2 - (\xi_1^2+\xi_2^2)}} \exp\left(-\frac{(x_1-\xi_1)^2 + (x_2-\xi_2)^2}{2}\right) \times$$
$$\times \Theta\left(ct - \sqrt{\xi_1^2+\xi_2^2}\right) d\xi_1\, d\xi_2.$$

By changing to the polar coordinates $\xi_1 = \rho\cos\alpha,\ \xi_2 = \rho\sin\alpha$, in both integrals, we obtain

$$\begin{aligned} \varphi(\mathbf{x},t) = \frac{e^{-\lambda t}}{4\pi^2 ct} \int_0^\infty d\rho \Big\{ &\rho\, \delta(c^2t^2-\rho^2) \times \\ &\times \int_0^{2\pi} \exp\left(-\frac{(x_1-\rho\cos\alpha)^2 + (x_2-\rho\sin\alpha)^2}{2}\right) d\alpha \Big\} + \\ + \frac{\lambda e^{-\lambda t}}{4\pi^2 c} \int_0^\infty d\rho \Big\{ &\frac{\rho\, \exp\left(\frac{\lambda}{c}\sqrt{c^2t^2-\rho^2}\right)}{\sqrt{c^2t^2-\rho^2}}\, \Theta(ct-\rho) \times \\ &\times \int_0^{2\pi} \exp\left(-\frac{(x_1-\rho\cos\alpha)^2 + (x_2-\rho\sin\alpha)^2}{2}\right) d\alpha \Big\}. \end{aligned} \tag{10}$$

Let's evaluate separately the interior integral in (10):

$$\int_0^{2\pi} \exp\left(-\frac{(x_1-\rho\cos\alpha)^2 + (x_2-\rho\sin\alpha)^2}{2}\right) d\alpha =$$
$$= \int_0^{2\pi} \exp\left(-\frac{1}{2}\left[x_1^2 + x_2^2 + \rho^2 - 2\rho(x_1\cos\alpha + x_2\sin\alpha)\right]\right) d\alpha =$$
$$= e^{-(x_1^2+x_2^2+\rho^2)/2} \int_0^{2\pi} e^{\rho(x_1\cos\alpha + x_2\sin\alpha)}\, d\alpha =$$
$$= 2\pi\, e^{-(\|\mathbf{x}\|^2+\rho^2)/2}\, I_0(\rho\|\mathbf{x}\|).$$

Substituting this into (10) we obtain

$$\begin{aligned}
\varphi(\mathbf{x},t) = {} & \frac{e^{-\lambda t}}{2\pi ct} \int_0^\infty \rho \, \delta(c^2t^2-\rho^2) \, e^{-(\|\mathbf{x}\|^2+\rho^2)/2} \, I_0(\rho\|\mathbf{x}\|) d\rho + \\
& + \frac{\lambda e^{-\lambda t}}{2\pi c} \int_0^\infty \rho \frac{\exp\left(\frac{\lambda}{c}\sqrt{c^2t^2-\rho^2}\right)}{\sqrt{c^2t^2-\rho^2}} \Theta(ct-\rho) e^{-(\|\mathbf{x}\|^2+\rho^2)/2} I_0(\rho\|\mathbf{x}\|) d\rho = \\
& = \frac{e^{-\lambda t}}{2\pi} \, e^{-(\|\mathbf{x}\|^2+c^2t^2)/2} \, I_0(ct\|\mathbf{x}\|) + \\
& + \frac{\lambda e^{-\lambda t}}{2\pi c} \, e^{-\|\mathbf{x}\|^2/2} \int_0^{ct} \rho \, \frac{\exp\left(\frac{\lambda}{c}\sqrt{c^2t^2-\rho^2}\right)}{\sqrt{c^2t^2-\rho^2}} \, e^{-\rho^2/2} \, I_0(\rho\|\mathbf{x}\|) \, d\rho
\end{aligned}$$

Making the substitution $z = \sqrt{c^2t^2-\rho^2}$ in the last integral, we obtain

$$\begin{aligned}
\varphi(\mathbf{x},t) = {} & \frac{e^{-\lambda t}}{2\pi} \, e^{-(\|\mathbf{x}\|^2+c^2t^2)/2} \, I_0(ct\|\mathbf{x}\|) + \\
& + \frac{\lambda e^{-\lambda t}}{2\pi c} \, e^{-(\|\mathbf{x}\|^2+c^2t^2)/2} \int_0^{ct} e^{(\lambda/c)z} \, e^{z^2/2} \, I_0(\|\mathbf{x}\|\sqrt{c^2t^2-z^2}) \, dz = \\
= {} & \frac{e^{-\lambda t}}{2\pi} \, e^{-(\|\mathbf{x}\|^2+c^2t^2)/2} \, I_0(ct\|\mathbf{x}\|) + \\
& + \frac{\lambda t \, e^{-\lambda t}}{2\pi} \, e^{-(\|\mathbf{x}\|^2+c^2t^2)/2} \int_0^1 e^{(c^2t^2/2)\xi^2+\lambda t\xi} \, I_0(ct\|\mathbf{x}\|\sqrt{1-\xi^2}) \, d\xi,
\end{aligned} \tag{11}$$

proving (8).

According to Lemma 2, the last integral in (11) is

$$\begin{aligned}
& \int_0^1 e^{(c^2t^2/2)\xi^2+\lambda t\xi} \, I_0(ct\|\mathbf{x}\|\sqrt{1-\xi^2}) \, d\xi = \\
& = \sum_{n=0}^\infty \sum_{k=0}^n \frac{2^{(2n-k-1)/2}}{k!(n-k)!} \left(\frac{c^2t^2}{2}\right)^{n-k} (\lambda t)^k \, \Gamma\left(\frac{2n-k+1}{2}\right) \frac{I_{(2n-k+1)/2}(ct\|\mathbf{x}\|)}{(ct\|\mathbf{x}\|)^{(2n-k+1)/2}} = \\
& = \sum_{n=0}^\infty \sum_{k=0}^n \frac{(\lambda t)^k \, 2^{(k+1)/2}}{k! \, (n-k)!} \, (c^2t^2)^{n-k} \, \Gamma\left(\frac{2n-k+1}{2}\right) \frac{I_{(2n-k+1)/2}(ct\|\mathbf{x}\|)}{(ct\|\mathbf{x}\|)^{(2n-k+1)/2}}.
\end{aligned}$$

Substituting this into (11) we obtain (9).

It remains to check that the (non-negative) function $\varphi(\mathbf{x},t)$ given by (8) is really the density of the process. For this we should show that for any $t>0$

$$\int_{\mathbb{R}^2} \varphi(\mathbf{x},t) \, \mu(d\mathbf{x}) = 1. \tag{12}$$

We have

$$\int_{\mathbb{R}^2} e^{-\|\mathbf{x}\|^2/2}\, I_0(ct\|\mathbf{x}\|)\, \mu(d\mathbf{x}) = \iint_{\mathbb{R}^2} e^{-(x_1^2+x_2^2)/2}\, I_0(ct\sqrt{x_1^2+x_2^2})\, dx_1\, dx_2 =$$

$$= \int_0^\infty dr \int_0^{2\pi} d\theta \{r\, e^{-r^2/2}\, I_0(ctr)\} = 2\pi \int_0^\infty r\, e^{-r^2/2}\, I_0(ctr)\, dr =$$

$$= \pi \int_0^\infty e^{-z/2}\, I_0(ct\sqrt{z})\, dz = \quad (\text{see}[1], \text{Formula } 6.643(2))$$

$$= \frac{2\pi\sqrt{2}}{ct}\, e^{c^2t^2/4}\, M_{-1/2,0}\left(\frac{c^2t^2}{2}\right),$$

where $M_{\xi,\eta}(z)$ is the Whittaker function. By applying now [1], Formula 9.220(2), we reduce the Whittaker function on the right-hand side of the last equality to the degenerated hypergeometric function and obtain

$$\int_{\mathbb{R}^2} e^{-\|\mathbf{x}\|^2/2}\, I_0(ct\|\mathbf{x}\|)\, \mu(d\mathbf{x}) = 2\pi\Phi\left(1; 1; \frac{c^2t^2}{2}\right) = 2\pi\, e^{c^2t^2/2}. \tag{13}$$

From (13) it also follows that

$$\int_{\mathbb{R}^2} e^{-\|\mathbf{x}\|^2/2}\, I_0(ct\sqrt{1-\xi^2}\, \|\mathbf{x}\|)\, \mu(d\mathbf{x}) = 2\pi\, e^{c^2t^2(1-\xi^2)/2}. \tag{14}$$

Therefore, by taking into account (13) and (14), we obtain

$$\int_{\mathbb{R}^2} \varphi(\mathbf{x}, t)\, \mu(d\mathbf{x}) = \frac{e^{-\lambda t}}{2\pi}\, e^{-c^2t^2/2} \int_{\mathbb{R}^2} e^{-\|\mathbf{x}\|^2/2}\, I_0(ct\|\mathbf{x}\|)\, \mu(d\mathbf{x}) +$$

$$+ \frac{\lambda t e^{-\lambda t}}{2\pi} e^{-c^2t^2/2} \int_0^1 e^{(c^2t^2/2)\xi^2 + \lambda t\xi} \left\{ \int_{\mathbb{R}^2} e^{-\|\mathbf{x}\|^2/2} I_0(ct\sqrt{1-\xi^2}\|\mathbf{x}\|)\, \mu(d\mathbf{x}) \right\} d\xi =$$

$$= \frac{e^{-\lambda t}}{2\pi} e^{-c^2t^2/2}\, 2\pi e^{c^2t^2/2} + \frac{\lambda t e^{-\lambda t}}{2\pi} e^{-c^2t^2/2} \int_0^1 e^{(c^2t^2/2)\xi^2 + \lambda t\xi}\, 2\pi e^{c^2t^2(1-\xi^2)/2} d\xi =$$

$$= e^{-\lambda t} + \lambda t\, e^{-\lambda t} \int_0^1 e^{\lambda t\xi}\, d\xi = e^{-\lambda t} + e^{-\lambda t}\left(e^{\lambda t} - 1\right) = 1,$$

proving (12). The theorem is completely proved. □

*Remark 1.* We have supposed that the start point $\mathbf{x}^0$ was a two-dimensional random vector whose coordinates are the independent standard random variables with Gaussian density (4). However, we can consider in the same manner the case when the coordinates of the start point $\mathbf{x}^0$ are some dependent Gaussian random variables with given characteristics $(a_1, \sigma_1)$ and $(a_2, \sigma_2)$, respectively. In this case the density of $\mathbf{x}^0$ has the form

$$p(\mathbf{x}) = p(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \times$$
$$\times \exp\left[ -\frac{1}{2(1-r^2)} \left\{ \frac{(x_1-a_1)^2}{\sigma_1^2} - 2r\frac{(x_1-a_1)(x_2-a_2)}{\sigma_1\sigma_2} + \frac{(x_2-a_2)^2}{\sigma_2^2} \right\} \right], \tag{15}$$
$$-1 < r < 1.$$

The similar analysis can be done to evaluate the convolution (3) of the transition density (2) with Gaussian density (15), however the computations will be much more difficult and tedious.

## References

[1] Gradshteyn I. S., Ryzhik I. M. *Tables of Integrals, Sums, Series and Products.* Academic Press, NY, 1980.

[2] Kolesnik A. D., Orsingher E. *A planar random motion with an infinite number of directions controlled by the damped wave equation.* J. Appl. Probab., 2005, **42**, 1168–1182.

[3] Masoliver J., Porrá J. M., Weiss G. H. *Some two and three-dimensional persistent random walks.* Physica A, 1993, **193**, 469–482.

[4] Stadje W. *The exact probability distribution of a two-dimensional random walk.* J. Stat. Phys., 1987, **46**, 207–216.

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# About the solvability of systems of integral equations with different degrees of differences in kernels

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**Abstract.** The work defines the conditions of solvability of one system of integral convolutional equations with different degrees of differences in kernels. Such the system of the integral convolutional equations has not been studied earlier, and it turned out that all the methods used for the investigation of such a system with the help of Riemann boundary problem at the real axis can not be applied there. The investigation of such a type of the system of equations is based on the investigation of the equivalent system of singular integral equations with the Cauchy type kernels at the real axis. It is determined that the system of the equations is not a Noetherian one. Besides, we have shown the number of the linear independent solutions of the homogeneous system of equations and the number of conditions of solvability for the system of heterogeneous equations. The general form of these conditions is also shown and the spaces of solutions of that system of equations are determined. Thus the system of the convolutional equations that hasn't been studied earlier is presented in that work and the theory of its solvability is built here. So some new and interesting theoretical results are got in the paper.

**Mathematics subject classification:** 45E05, 45E10.
**Keywords and phrases:** The system of integral convolutional equations, singular integral equations, Cauchy type kernel, a Noetherian system of equations, conditions of solvability, index, the number of the linear independent solutions, spaces of solutions.

The present work is devoted to determining conditions of solvability and some properties of solutions of the next system of Winer-Hoph's type integral equations

$$P_1(x)\varphi(x) + \frac{1}{\sqrt{2\pi}} \int\limits_0^{+\infty} k(x-t)P_2(t)\varphi(t)\,dt = h(x), \quad x > 0, \tag{1}$$

where $h(x) \in \mathbf{L}_2$ is a known vector-function which is an $n$-dimensional one, $k(x) \in \mathbf{L}$, is a known matrix-function, which is an $n$-dimensional one, too. $\varphi(x)$ is an unknown vector-function and it is an $n$-dimensional one.

$$P_1(x) = \sum_{k=0}^{m} a_k x^k, \quad P_2(x) = \sum_{\nu=0}^{s} b_\nu x^\nu$$

are the known polynomials with the degrees $m, s$ respectively. We will note that the belonging of vector-functions and matrix-functions to any space means their elements' belonging to it. The norms of vector-functions and matrix-functions are compatible with each other.

Let $D^+ = \{z \in \mathbf{C} : \mathbf{Imz} > \mathbf{0}\}$ be an upper half plane and $D^- = \{z \in \mathbf{C} : \mathbf{Imz} < \mathbf{0}\}$ be a lower half plane of the complex plane $\mathbf{C}$; $\mathbf{R}$ is the real axis. According to the properties of Fourier transformation [2, p. 16] the investigation of the system of equations (1) reduces to the investigation of the following matrix differential boundary problem

$$\sum_{k=0}^{m}(-1)^k A_k \Phi^{+(k)}(x) + \sum_{\nu=0}^{s}(-1)^{\nu} B_{\nu} K(x)\Phi^{+(\nu)}(x) = H(x) + \Phi^-(x), \quad x \in \mathbf{R}.$$

Here $K(x), H(x)$ are the Fourier transformations of the matrix-function $k(x)$ and the vector-function $h(x)$ accordingly. $\Phi^{+(p)}(x)$, $\Phi^-(x)$ are the boundary values at $\mathbf{R}$ of the unknown vector-functions $\Phi^{+(p)}(z)$ and $\Phi^-(z)$ accordingly, where $\Phi^{+(p)}(z)$, $\Phi^-(z)$ are unknown vector-functions, which are analytical in the domains $D^+$ and $D^-$ accordingly. Let's rewrite this differential boundary problem as the following one

$$\left[\sum_{k=0}^{m}(-1)^k A_k \Phi^{+(k)}(x) + \sum_{\nu=0}^{s}(-1)^{\nu} B_{\nu} K(x)\Phi^{+(\nu)}(x)\right] - \Phi^-(x) = H(x), \quad x \in \mathbf{R}. \tag{2}$$

As all the transformations of the system (2) and the system (1) are identical, then they are equivalent in such a sense that they are solvable or unsolvable at the same time, and there is one and only one solution $\Phi^{\pm}(x)$ of the system (2) for every solution $\varphi(x)$ of the system (1) and vice versa. Further the systems with these properties we will name by the equivalent systems. The solutions of the system of equations (1) are expressed via solutions of the system (2) according to the formula

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \Phi^+(t) e^{-ixt}\, dt, \quad x > 0. \tag{3}$$

Later on we will consider that $K(x) \in \mathbf{H}_\alpha^{(r)}$ , $r \geq 0, 0 < \alpha \leq 1$, where $\mathbf{H}_\alpha^{(r)}$ is a space of functions $f(x) \in C^{(r)}$, the derivatives with the order $r$ of which satisfy the next condition at the real axis $\mathbf{R}$:

$$\left| f^{(r)}(x+h) - f^{(r)}(x) \right| (1+|x|)^{\alpha}(1+|x+h|)^{\alpha} \leq A_r h^{\alpha}, x \in \mathbb{R}, \quad h > 0,$$

where $A_r$ is the Holder's constant of the function $f^{(r)}(x)$ and $\alpha$ is its Holder's exponent; $H(x) \in \mathbf{L}_2^{(r)}$, $r \geq 0$. As the matrix function $k(x) \in \mathbf{L}$, then according to Riemann-Lebesgue theorem $\lim\limits_{x\to\infty} K_{ij}(x) = 0$, $i, j = \overline{1, n}$, thus $\det K(x) = 0$ when $x \to \infty$, where $K_{ij}(x)$ is the Fourier transformation of the elements $k_{ij}(x)$ of the matrix function $k(x)$.

The theory of the solvability of systems of Winer-Hoph type equations with different degrees of differences in kernels such as

$$\varphi(x) + \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} k(x-t)\varphi(t)\, dt = h(x), \quad x > 0,$$

was built in the papers [1, 5, 6] with rather wide assumptions concerning their kernels and right parts. The investigation of systems of such a type of equations was based on the investigation of the corresponding Riemann boundary problem at the real axis, which appears after the Fourier transformation of the every system. But the methods used in the papers [2, 7, 10] can't be applied to the investigation of the system of equations (1), as this system is transformed into the corresponding system of differential boundary problems at the real axis (2) with the help of the properties of Fourier transformation. It is necessary to mention that the attempt of studying the case of an equation as (1) was made in papers [5, 6], where integral representations for functions and their derivatives analytical in domains $D^+$, $D^-$ were applied for the investigation of the corresponding differential boundary problem. But the kernels of these integral representations had the additional branch points in these domains and it led to appearing multivalued unknown functions which were analytical in domains $D^+$, $D^-$. We must also mention that the integral Winer-Hoph equation (or the scalar case) was studied in details in the paper [12]. Thus we will study the system of equations (1) basing on the investigation of the system of differential boundary problems (2). We will transform the system of differential boundary problems (2) into the system of singular integral equations with the kernel of Cauchy using integral representations for the vector functions and derivatives of them which are analytical in domains $D^+$, $D^-$. Let construct vector functions $\Phi^+(z)$ and $\Phi^-(z)$ such that they are analytical in the domains $D^+$, $D^-$ accordingly and disappear at infinity. Besides, the boundary values at $\mathbf{R}$ of vector functions $\Phi^{+(p)}(z)$ and $\Phi^-(z)$ satisfy the following condition $\Phi^{+(p)}(x) \in \mathbf{L}_2^{(r)}$, $\Phi^-(x) \in \mathbf{L}_2^{(r)}$, $r \geq 0, p \geq 0$. According to the papers [4, 11] such vector functions as:

$$\Phi^\pm(z) = (2\pi\imath)^{-1} \int\limits_{\mathbf{R}} P^\pm(x,z)\rho(x)\,dx, \quad z \in D^\pm, \tag{4}$$

where

$$P^-(x,z) = \frac{1}{x-z}, \quad z \in D^-;$$

$$P^+(x,z) = \frac{(-1)^p (x+\imath)^{-p}}{(p-1)!} \times$$

$$\times \left[ (x-z)^{p-1} \ln\left(1 - \frac{x+\imath}{z+\imath}\right) - \sum_{k=0}^{p-2} d_{p-k-2}(x+\imath)^{k+1}(z+\imath)^{p-k-2} \right], \quad z \in D^+;$$

$$d_{p-k-2} = (-1)^{k+1} \sum_{j=0}^{k} C_{p-1}^{p-1-j}(k-j+1)^{-1}, \quad k = \overline{0, m-2}$$

satisfy these conditions, and here $C_n^m$ are binomial coefficients; the function $\ln\left[1 - \dfrac{x+\imath}{z+\imath}\right]$ is the main branch ($\ln 1 = 0$) of the logarithmic function in the complex plane with the cut connecting such points as $z = -\imath$ and $z = \infty$, following the negative direction of the axis of ordinate.

It's easy to verify that defined by (4) vector functions $\Phi^+(z)$ and $\Phi^-(z)$ according to the structure of $P^{\pm}(x,z)$ and due to the papers [4, 11] are unique analytical functions in the domains $D^+$, $D^-$ accordingly. The next vector function $\rho(x) \in \mathbf{L}_2$ or the density of the integral representations (4), is defined uniquely by the vector functions $\Phi^+(z)$ and $\Phi^-(z)$ and vice versa, so with the help of the given vector function $\rho(x) \in \mathbf{L}_2$ both vector functions $\Phi^+(z)$ and $\Phi^-(z)$ can be constructed uniquely. The following representations take place at the same time:

$$\Phi^{+(p)}(z) = (2\pi\imath)^{-1} \int\limits_{\mathbf{R}} (z+\imath)^{-p}(x-z)^{-1}\rho(x)\,dx, \quad z \in D^+,$$

$$\Phi^-(z) = (2\pi\imath)^{-1} \int\limits_{\mathbf{R}} (x-z)^{-1}\rho(x)\,dx, \quad z \in D^-. \tag{5}$$

We consider the case when $m = s$. Using the properties of partial derivatives of function $P^+(x,z)$ with respect to $z$ and Sohotski formulas for derivatives from [7, p. 42], with the help of the representations (4), (5), we will transform the system of differential boundary problems (2) into the following system of singular integral equations and later on investigate it. The system of singular integral equations is

$$A(x)\rho(x) + B(x)(\pi\imath)^{-1} \int\limits_{\mathbf{R}} (t-x)^{-1}\rho(t)\,dt + (T\rho)(x) = H(x), \quad x \in \mathbf{R}, \tag{6}$$

where

$$A(x) = 0,5\{(-1)^m\,[A_m + B_m K(x)]\,(x+\imath)^{-m} + E\},$$
$$B(x) = 0,5\{(-1)^m\,[A_m + B_m K(x)]\,(x+\imath)^{-m} - E\}, \tag{7}$$

$$(T\rho)(x) = \int\limits_{\mathbf{R}} K(x,t)\rho(t)\,dt, \tag{8}$$

$$K(x,t) = \frac{1}{2\pi\imath}\sum_{k=0}^{m-1}(-1)^k\,[A_k + B_k K(x)]\,\frac{\partial^k P^+(t,x)}{\partial x^k},$$

where $E$ is unity matrix and $\dfrac{\partial^k P^+(t,x)}{\partial x^k}$ is a limiting value at $\mathbf{R}$ of the function $\dfrac{\partial^k P^+(t,z)}{\partial z^k}$, $k = \overline{0, m-1}$.

**Lemma 1.** *If the matrix function* $K(x) \in \mathbf{H}_\alpha^{(r)}$, $r \geq 0$, $0 < \alpha \leq 1$, *then the operator*

$$T : \mathbf{L}_2^{(r)} \to \mathbf{L}_2^{(r)},$$

$r \geq 0$, *defined by the formula (8), is a compact one.*

The proof of this lemma follows from Frechet-Kolmogorov-Riesz criterion of compactness vector functions' sets in the spaces $\mathbf{L}_p$, $p > 1$ and integral operators at the real axis in the spaces $\mathbf{L}_p,\ p > 1$, the properties of function $P^+(x, z)$ and the results of the work [8].

According to the work [10, p. 406], the system of differential boundary problems (2) and the system of singular integral equations (6) are equivalent in such a sense that they are solvable or unsolvable at the same time, and for every solution $\rho(x)$ of the system (6) there exists maybe a nonunique solution $\Phi^{\pm}(x)$ of the system (2) and vice versa. In order to make this correspondence unique it is necessary to set initial conditions for the system (2). As its solutions $\Phi^{\pm}(x)$ are found in spaces of functions that disappear at infinity, then according to the properties of Cauchy type integral the solutions of the system (2) are such that $\Phi^{+(k)}(\infty) = 0$, $k = \overline{0, m-1}$, it means that the initial conditions of the system (2) are trivial and set automatically. Thus it follows that the system of differential boundary problems (2) and the system of singular integral equations (6) are equivalent in such a sense that they are solvable or unsolvable at the same time, and there is one and only one solution $\rho(x)$ of the system (6) for every solution $\Phi^{\pm}(x)$ of the system (2) and vice versa. By the force of formula (4), the solutions of the system (2) are expressed via solutions of the system (6) according to the formula

$$\Phi^+(x) = (2\pi\imath)^{-1} \int\limits_{\mathbf{R}} P^+(t, x)\rho(t)\, dt, x \in \mathbf{R}, \tag{9}$$

where $p = m;\ P^+(t, x)$ are the boundary values at $\mathbf{R}$ of the vector functions $P^+(t, z)$, and the vector function $\rho(x)$ is the solution of the system (6). As the system of the equations (1) and the system (2) are equivalent, the system (2) and the system of singular integral equations (6) are equivalent, too, it follows that the system (1) and the system (6) are equivalent in such a sense that they are solvable or unsolvable at the same time, and there is one and only one solution $\varphi(x)$ of the system of equations (1) for every solution $\rho(x)$ of the system of the equations (6) and vice versa. Thus the solutions of the system (1) are expressed via solutions of the system (6) according to the formulas (10), (3). That is why we will call the system of the equations (1) a Noetherian if the system of the equations (6) is a Noetherian one.

**Theorem 1.** *The system (1) is not a Noetherian one.*

*Proof.* According to the papers [2, 3, 10] the system of the singular integral equations (6) is a Noetherian one if and only if when the following conditions take place:

$$\det[A(x) + B(x)] \neq 0, \quad \det[A(x) - B(x)] \neq 0$$

at $\mathbf{R}$. As $A(x) - B(x) = E;\quad A(x) + B(x) = (-1)^m(x + \imath)^{-m}[A_m + B_m K(x)]$, then $\det[A(x) + B(x)]$ has a null at least with order $m$ at infinity. It means that the system of the equations (6) is not a Noetherian one. Then as the systems (1) and (6) are equivalent, the system of the equations (1) is not a Noetherian one, too.

The theorem is proved. □

Let's determine conditions when the system of equations (1) is a Notherian one and it is a solvable one due to it. First we consider the case when $\det[A_m + B_m K(x)] \neq 0$ at the finite points of the real axis $\mathbf{R}$. The following representation [3, p. 329] for the matrix function $A(x) + B(x)$ takes place:

$$A(x) + B(x) = M(x) \cdot D(x) \cdot R(x). \tag{10}$$

Here $M(x)$ is a matrix function of size measure $n$ and $\det M(x) \neq 0$ at $\mathbf{R}$;

$R(x)$ is a matrix function with such a determinant which is constant and different from zero with polynomials of degrees $\dfrac{1}{x+\imath}$ as its elements;

$D(x)$ is a diagonal matrix function such as:

$$D(x) = diag\left\{\frac{1}{(x+\imath)^{\nu_0^{(1)}}}, \cdots, \frac{1}{(x+\imath)^{\nu_0^{(n)}}}\right\},$$

where $\nu_0^{(1)}, \ldots, \nu_0^{(n)}$ are integer non-negative numbers such that

$$\sum_{j=0}^{n} \nu_0^{(j)} = \nu_0 = m. \tag{11}$$

Let denote

$$r_0 = \max\{\nu_0^{(1)}, \ldots, \nu_0^{(n)}\}. \tag{12}$$

We will investigate the matrix function $M(x) \in \mathbf{H}_\alpha^{(r)}$, $r \geq r_0$, $0 < \alpha \leq 1$, where the number $r_0$ is defined by the formula (12). According to the paper [9, p. 53] it allows the following factorisation

$$M(x) = X^+(x) \cdot \Lambda(x) \cdot X^-(x), \tag{13}$$

where $\det X^\pm(x) \neq 0$ at $\mathbf{R}$ and

$$\Lambda(x) = diag\left\{\left(\frac{x-\imath}{x+\imath}\right)^{\chi_1}, \cdots, \left(\frac{x-\imath}{x+\imath}\right)^{\chi_n}\right\}, \tag{14}$$

and $\chi_j$, $j = \overline{1,n}$ are the partial indexes of the matrix function $M(x)$.

As there can be positive and negative partial indexes at the same time among all of them, we will define them by the next equalities:

$$\omega = \sum_{\chi_j \geq 0} \chi_j, \quad q = -\sum_{\chi_j < 0} \chi_j, \tag{15}$$

then the summarized index of the matrix function $M(x)$ is defined by the formula

$$\chi = \omega - q. \tag{16}$$

The next theorem takes place.

**Theorem 2.** *Let the matrix function* $k(x) \in \mathbf{L}$, *vector function* $h(x) \in \mathbf{L}_2$; *the matrix function* $K(x) \in \mathbf{H}_\alpha^{(r)}$, $r \geq r_0$, $0 < \alpha \leq 1$, *the vector function* $H(x) \in \mathbf{L}_2^{(r)}$, $r \geq r_o$, *where the number* $r_0$ *is defined by the formula (12);* $\det[A_m + B_m K(x)] \neq 0$ *at the finite points of the real axis* $\mathbf{R}$, *the numbers* $\omega, q$ *are defined by the formula (15), the number* $\chi$ *is defined by the formula (16) and the representation (13) takes place.*

*If* $q - 2m \geq 0$, *then the homogeneous system (1) has not less than* $q - 2m$ *linear independent solutions; the heterogeneous system(1) is a solvable one if not less than* $\omega$ *conditions of solvability*

$$\int_{\mathbf{R}} H(x)\psi_j(x)\,dt = 0, \tag{17}$$

*are executed. Here in (17) the vector function* $H(x)$ *is a right part of the system of the singular integral equations (6) and the vector functions* $\psi_j(x)$ *are linear independent solutions of the system of homogeneous singular integral equations*

$$\tilde{A}(x)\psi(x) - (\pi\imath)^{-1}\int_{\mathbf{R}}(t-x)^{-1}\tilde{B}(t)\psi(t)\,dt + \int_{\mathbf{R}}\tilde{K}(t,x)\psi(t)\,dt = 0, \tag{18}$$

*allied to the equation (6), where the matrices* $\tilde{A}(x), \tilde{B}(x), \tilde{K}(x,t)$ *are transposed with respect to matrices* $A(x), B(x), K(x,t)$ *which are the coefficients and the regular kernel in the system of equations (6) respectively.*

*If* $q - 2m < 0$ *then the heterogeneous system(1) is an unsolvable one. It will become a solvable one if* $\omega + 2m$ *conditions (17) are executed.*

*The summarized index of the system (1) is* $-(\chi + 2m)$.

According to the paper [2, p. 262] let's denote by $\mathbf{L}_2[-\mu; 0]$ the space of functions $\varphi(x) \in \mathbf{L}_2$ which satisfy the condition $(x+\imath)^\mu \varphi(x) \in \mathbf{L}_2$.

**Theorem 3.** *Let the matrix function* $k(x) \in \mathbf{L}$, *the vector function* $h(x) \in \mathbf{L}_2$; *the matrix function* $K(x) \in \mathbf{H}_\alpha^{(r)}$, $r \geq r_0$, $0 < \alpha \leq 1$, *where the number* $r_0$ *is defined by the formula (12), the vector function* $H(x) \in \mathbf{L}_2^{(r)}$, $r \geq r_0$; $\det[A_m + B_m K(x)] \neq 0$ *at the finite points of the real axis* $\mathbf{R}$ *and the system (1) is a solvable one. Then its solutions belong to the space* $\mathbf{L}_2[-r - m + r_0; 0]$, $r \geq r_0$.

Now we will study the singular case.

Let the condition $\det[A_m + B_m K(x)] \neq 0$ at the finite points of the real axis $\mathbf{R}$ is not executed. Then we suppose that $\det[A_m + B_m K(x)]$ has zeroes at the real axis $\mathbf{R}$ in finite points $a_1, a_2, \ldots, a_u$ with integer orders $\nu_1, \nu_2, \ldots, \nu_u$ respectively. Then in virtue of the work [3, p. 328] the representation (10) for the matrix function $A(x) + B(x)$ takes place. Here the matrix functions $M(x)$, $R(x)$ are the same as in the previous case, and $D(x)$ is the following diagonal matrix

$$D(x) = diag\left\{\frac{1}{(x+\imath)^{\nu_0^{(1)}}}\prod_{j=1}^{u}\left(\frac{x-a_j}{x+\imath}\right)^{\nu_j^{(1)}}, \cdots, \frac{1}{(x+\imath)^{\nu_0^{(n)}}}\prod_{j=1}^{u}\left(\frac{x-a_j}{x+\imath}\right)^{\nu_j^{(n)}}\right\}, \tag{19}$$

where $\nu_0^{(1)}, \ldots, \nu_0^{(n)}, \nu_1^{(1)}, \ldots, \nu_1^{(n)}, \ldots, \nu_u^{(1)}, \ldots, \nu_u^{(n)}$ are integer nonnegative numbers such that

$$\sum_{j=1}^{n} \nu_0^{(j)} = \nu_0 = m,$$

$$\nu_k = \sum_{j=1}^{n} \nu_k^{(j)}, \quad k = \overline{1, u}, \quad \nu = \sum_{k=1}^{u} \nu_k. \tag{20}$$

Let

$$r_0 = \max\{\nu_0^{(1)}, \ldots, \nu_0^{(n)}, \nu_1^{(1)}, \ldots, \nu_1^{(n)}, \ldots, \nu_u^{(1)}, \ldots, \nu_u^{(n)}\}. \tag{21}$$

Analogously as in the previous case the matrix function $M(x) \in \mathbf{H}_\alpha^{(r)}$, $r \geq r_0$, admits the factorization (13), where the matrix function $\Lambda(x)$ is defined by the formula (14) and $\chi_j$, $j = \overline{1, n}$ are the partial indexes of the matrix function $M(x)$ that are defined by the formula (15). The summarizing index of the system of singular integral equations (6) is also given by the formula (16).

The next theorem takes place.

**Theorem 4.** *Let the matrix function $k(x) \in \mathbf{L}$, vector function $h(x) \in \mathbf{L}_2$; the matrix function $K(x) \in \mathbf{H}_\alpha^{(r)}$, $r \geq r_0$, $0 < \alpha \leq 1$, the vector function $H(x) \in \mathbf{L}_2^{(r)}$, $r \geq r_o$, where the number $r_0$ is defined by the formula (21), $\det[A_m + B_m K(x)]$ has zeroes at the real axis $\mathbf{R}$ at such finite points as $a_1, a_2, \ldots, a_u$ with integer orders $\nu_1, \nu_2, \ldots, \nu_u$ respectively and the representation (10) takes place.*

*Here in that representation the matrix function $D(x)$ is defined by the formula (19); $\det M(x) \neq 0$ at $\mathbf{R}$; the numbers $\omega, q$ are defined by the formula (15), the number $\nu$ is defined by the formula (20) and the number $\chi$ is defined by the formula (16) and the representation (13) takes place.*

*If $q - 2m - 2\nu \geq 0$, then the homogeneous system (1) has not less than $q - 2m - 2\nu$ linear independent solutions; the heterogeneous system(1) is a solvable one if not less than $\omega$ conditions of solvability (17) are executed.*

*If $q - 2m - 2\nu < 0$ then the heterogeneous system(1) is an unsolvable one. It will become a solvable one if $\omega + 2m + 2\nu$ conditions (17) are executed.*

*The summarized index of the system (1) is $-(\chi + 2m + 2\nu)$.*

**Theorem 5.** *Let the matrix function $k(x) \in \mathbf{L}$, the vector function $h(x) \in \mathbf{L}_2$; the matrix function $K(x) \in \mathbf{H}_\alpha^{(r)}$, $r \geq r_0$, $0 < \alpha \leq 1$, where the number $r_0$ is defined by the formula (21), the vector function $H(x) \in \mathbf{L}_2^{(r)}$, $r \geq r_0$; $\det[A_m + B_m K(x)]$ has zeroes at the real axis $\mathbf{R}$ at finite points $a_1, a_2, \ldots, a_u$ with integer orders $\nu_1, \nu_2, \ldots, \nu_u$ respectively, the representation (10) takes place and the system (1) is a solvable one. Then its solutions belong to the space $\mathbf{L}_2[-r - m + r_0; 0]$, $r \geq r_0$.*

## References


[1] Gakhov F.D. *Boundary problems.* Moskva, Nauka, 1977 (in Russian).

[2] Gakhov F.D., Cherski Y.I. *Convolutional type equations.* Moskva, Nauka, 1978 (in Russian).

[3] Gohberg I.Ts., Feldman I.A. *The equations in convolutions and the projectional methods of their resolving.* Moskva, Nauka, 1971 (in Russian).

[4] Litvinchyuk G.S. *Boundary problems and singular integral equations with displacement.* Moskva, Nauka, 1977 (in Russian).

[5] Muskhelishvili N.I. *Singular integral equations.* Moskva, Nauka, 1968 (in Russian).

[6] Presdorf Z. Some classes of singular integral equations. Moskva, Mir, 1979 (in Russian).

[7] Tikhonenko N.Y., Melnik A.S. Different. Equations, 2002, **38**, No. 9, 1218–1224 (in Russian).

[8] Tikhonenko N.Y., Svyagina N.N. Izv. Vuzov. Math., 1996, No. 9, 83–87 (in Russian).

[9] Tikhonenko N.Y., Scherbakova A.G. Different. Equations, 2004, **40**, No. 9, 1218–1224 (in Russian).

[10] Scherbakova A.G. Boundary problems, 2004, **11**, 204–211 (in Russian).


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# Nash equilibria in the noncooperative informational extended games

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**Abstract.** In this article* we will analyse informational extended games, i.e. games in which the players choose their actions simultaneously, with assumption that they have some information about the future strategies which will be chosen by other players. All informational extended games of this type will assume that players' payoff functions are common knowledge. Under these assumptions the last section will define the informational extended games and analyse Nash equilibrium and conditions of its existence. The essential result of this article is a theorem of Nash equilibrium existence in informational extended games with $n$ players. Our treatment is based on a standard fixed point theorem which will be stated without proof in the first section.

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## 1 Preliminary facts

### 1.1 Fixed points and contraction mappings

Consider the function $f : X \to X$. An element $x \in X$ is called a fixed point of $f$ if $f(x) = x$.

The fixed points of the function $f$ are the intersection points of the graph of $f$ with the product $X \times X$.

Properties of fixed points.

1. If there are two functions $f$ and $g$ from $X$ into $Y$, then the point $x^* \in X$ for which $f(x^*) = g(x^*)$, is called [2] point of coincidence for the functions $f$ and $g$.

2. Sometimes it is convenient to use the cyclic points of the function $f$ together with the fixed points, especially in the case when fixed points do not exist. Cyclic points are the points which are images of the iterative function $f^n$, where $n$ is a natural number. These are cyclic points of the $n-$th order. Often such points do not exist and in these cases we can use boundary cycles. Also we can speak about the invariant sets, i.e. subsets $Y \subset X$, for which $f(Y) = Y$. In such cases the minimal invariant subsets are very important.

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Next the notation $F : X \rightrightarrows 2^Y$ will denote a point-to-set mapping, were $2^Y$ denotes the set of all subsets of $Y$. A fixed point of the point-to-set mapping $F : X \rightrightarrows 2^Y$ is a point $x^* \in X$ such that $x^* \in F(x^*)$.

The graph for the application $F$ is the set
$gr(F) = \{(x, y) \in X \times Y \,|x \in X, y \in F(x)\}$. This set can contain some points or can be the empty set.

## 1.2 The Kakutani fixed point theorem

The existence of the fixed points is considered an important problem. The existence (and other properties) of the fixed point for the function $f : X \to X$ depends on the properties of $f$ and on the properties of the space $X$. Often it is considered that $f$ is a continuous function.

**Definition 1.1.** *The function $f$ of the metric space into itself is called [2] contraction mapping if there exists a constant $K < 1$ such that for each two points $x$ and $y$ the inequality $\rho(f(x), f(y)) \leq K\rho(x, y)$ holds, where $\rho$ is the metrics of the space.*

There are some important properties for the fixed points.

**Proposition 1.1.** *If $f$ is a contraction mapping, then there exists not more than a single fixed point [1, 2].*

**Theorem 1.1.** *(Principle of the contraction mapping). Consider that $f$ is a contraction mapping of the complete metric space $X$ into itself. Then for each point $x \in X$ the sequence $x, f(x), f^2(x) = f(f(x)), f^3(x), \ldots$ converges to a fixed point. So $f$ has a single fixed point [1, 2].*

The points $x, f(x), f^2(x), \ldots$ are called consequent approximations of the fixed point.

In the case of the contraction mapping we can consider as a start element every element $x$ and the consecutive approximations converge to the fixed point.

The Kakutani fixed point theorem is a fixed-point theorem for point-to-set mapping. It provides sufficient conditions for a point-to-set mapping defined on a convex, compact subset of a Euclidean space to have a fixed point, i.e. a point which is mapped to a set containing it. The Kakutani fixed point theorem is a generalization of Brouwer fixed point theorem. The Brouwer fixed point theorem is a fundamental result in topology which proves the existence of fixed points for continuous functions defined on compact, convex subsets of Euclidean spaces. Kakutani theorem extends this to point-to-set mapping.

The theorem was developed by Shizuo Kakutani in 1941 and was famously used by John Nash in his description of Nash equilibrium. It has subsequently found widespread application in game theory and economics. Many problems in economy appear as problems of maximization and usually the solution of such problems is many-valued.

Before giving this theorem we need to recall some definitions and theorems.

**Definition 1.2.** *Consider topological spaces $X$ and $Y$. A point-to-set mapping $F : X \rightrightarrows 2^Y$ is said to be closed if the graph of $F$ is closed as a subset into the product of the spaces $X \times Y$.*

That is if the sequence of points $(x_n, y_n)$ from $gr(F)$ converges to a point $(x, y) \in X \times Y$, then the limit point $(x, y) \in gr(F)$ [2].

**Theorem 1.2 (Kakutani, 1941).** *Let $X$ be a Banach space and $K$ a non-empty, compact and convex subset of $X$. Let $F : K \rightrightarrows 2^K$ be a point-to-set mapping on $K$ with a closed graph and the property that the set $F(x)$ is non-empty and convex for all $x \in K$. Then $F$ has a fixed point.*

For proof see [1].

Before giving the applications of the fixed points in the game theory we will recall some other important theorems.

Let $C(K)$ be the space of all continuous functions defined on the compactum $K$.

**Theorem 1.3 (Arzelà-Ascoli).** *(Compactness criterion). A set of continuous functions $E \subseteq C(K)$ is compact if and and only if the set $E$ is uniformly bounded: ($|x(t)| \leqslant M, \forall t \in K$, for $\forall x \in E$) and the functions from the set $E$ are equicontinuous (i.e. for $\forall \varepsilon, \exists \delta$ so that if $\rho(t_1, t_2) < \delta$ then $|x(t_1) - x(t_2)| < \varepsilon$ for $\forall x \in E$).*

**Theorem 1.4 (Tikhonov).** *A product of a family of compact topological spaces $X = \prod\limits_{\alpha \in A} X_\alpha$ is compact.*

**Lemma 1.1.** *1) If $X$ and $Y$ are two compacta with the same metric, $f : X \to Y$ is a continuous function, then the set $Arg \max\limits_{x \in X} f(x) = \left\{ x \in X \,\middle|\, f(x) = \max\limits_{z \in X} f(z) \right\}$ is compact too (see [3]).*

*2) If $X$ and $Y$ are two compacta with the same metric, and $K(x, y)$ is a continuous function on $X \times Y$, then $\varphi(y) = \max\limits_{x \in X} K(x, y)$ and $\psi(x) = \min\limits_{y \in Y} K(x, y)$ are continuous functions on $Y$ and $X$ respectively [3].*

## 2 Strategic form games and Nash equilibria

In this part we will analyse games in which the players choose their actions simultaneously (without the knowledge of other player choices). The game will assume that players' payoff functions are common knowledge.

**Definition 2.1.** *A strategic form of the game consists of: a finite set of players $I = \{1, 2, \ldots, n\}$, action spaces (set of strategies) of players, denoted by $X_i, i \in I$; and payoff functions of players $H_i : X \to R, i \in I$, where $X = X_1 \times \cdots \times X_n$. We refer to such a game as the tuple $< I, (X_i)_{i \in I}, (H_i)_{i \in I} >$ denoted by $\Gamma$.*

An outcome is an action profile $(x_1, x_2, \ldots, x_n)$, and the outcome space is $X = \times_{i \in I} X_i$. The game is common knowledge among the players.

One of the most common interpretations of Nash equilibrium (introduced by John Nash in 1950) is that it is a steady state in the sense that no rational player has an incentive to unilaterally deviate from it. Let $x_{-i} \equiv (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and $(x_{-i}, y_i) \equiv (x_1, x_2, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$.

**Definition 2.2.** *A Nash equilibrium of the game* $\Gamma$ *is an action profile* $x^* \in X$ *such that for every* $i \in I$

$$H_i(x^*) \geqslant H_i\left(x^*_{-i}, x_i\right) \text{ for all } x_i \in X_i.$$

Another and sometimes more convenient way of defining Nash equilibrium is via the best response correspondences $Br_i : \underset{j \in I \setminus \{i\}}{\times} X_j \rightrightarrows X_i$
such that

$$Br_i(x_{-i}) = \left\{x_i \in X_i : H_i(x) \geqslant H_i\left(x_{-i}, x_i'\right) \text{ for } \forall x_i' \in X_i\right\}. \qquad (*)$$

**Definition 2.3.** *A Nash equilibrium is an action profile* $x^*$ *such that* $x_i^* \in Br_i\left(x^*_{-i}\right)$ *for all* $i \in I$.

If the sets $X_i$ are compacts and the functions $H_i$ are continuous, then the best response set (*) for the player $i$ can be represented by:

$$Br_i(x_{-i}) = Arg \max_{x_i \in X_i} H_i(x_{-i}, x_i).$$

Given a strategic form of the game $\Gamma \equiv < I, (X_i)_{i \in I}, (H_i)_{i \in I} >$, the set of Nash equilibria is denoted by $NE(\Gamma)$.

Using the best response sets of the players we consider the point-to-set mapping $Br : \underset{i \in I}{\times} X_i \rightrightarrows 2^X$ such that $Br = (Br_1, Br_2, \ldots, Br_n)$.

Then we can easily prove that $x^* \in NE(\Gamma) \rightleftarrows x^*$ is a fixed point of the set-valued mapping $Br$, i.e. $x^* \in Br(x^*)$.

## 3 Nash equilibria in the noncooperative informational extended games with $n$ players

We analyse a static game with $n$ players:

$$\Gamma = \left\langle I, X_i, i = \overline{1,n}, H_i, i = \overline{1,n} \right\rangle \qquad (1)$$

where $I = \{1, 2, \ldots, n\}$ is the set of the players, the set of strategies for the $i$-th player is denoted by $X_i$, $\left(i = \overline{1,n}\right)$, and the payoff functions are defined by: $H_i : \prod_{i \in I} X_i \to R$, $\left(i = \overline{1,n}\right)$.

Next we will analyse a static informational extended game with $n$ players. In this informational extended game we will consider that each player is informed of the strategies of the other players which will be chosen. In this case the sets of the strategies for each player will be a set of functions defined on the product of the sets of strategies of the rest players from the initial game (1).

The game is realised as follows: the strategies are chosen simultaneously by players (with assumption that each of them knows which strategies will be chosen by all other players), after that each of players determines his payoff and the game is over.

This informational extended game can be described in the normal form by:

$$_n\Gamma = \left\langle I, \overline{X}_i, i = \overline{1,n}, \overline{H}_i, i = \overline{1,n} \right\rangle,$$

where the sets of the strategies for the players are defined by:

$$\overline{X}_i = \left\{ \varphi_i : \prod_{j \in I, j \neq i} X_j \to X_i \right\}, i = \overline{1,n},$$

where $\prod\limits_{j \in I, j \neq i} X_j = X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n$.

The payoff functions are defined on the product of the extended sets of strategies: $\overline{H}_i : \prod\limits_{i \in I} \overline{X}_i \to R, \ \left(i = \overline{1,n}\right)$.

In this case we analyse the informational extended game in which we consider that all players know the chosen strategies of all other players and each player $i \in I$ chooses his strategy from the set $\overline{X}_i$.

If some players do not know which strategies other players will choose, then those players $j \in I$ will choose their strategies from their initial sets $X_j$. Thus we can define some different informational extended games in which the outcome will consist of strategies $x_j \in X_j, j \in J$ and $\varphi_k \in \overline{X}_k, k \in I \backslash J$, where $J$ is the set of players which do not have some information about chosen strategies of other players.

We denote by $C\left(\prod\limits_{j \in I, j \neq i} X_j, X_i\right), \left(i = \overline{1,n}\right)$ the space of all continuous functions from $\prod\limits_{j \in I, j \neq i} X_j$ into $X_i$, were $\prod\limits_{j \in I, j \neq i} X_j$ and $X_i$ are compacta.

Next we will apply the fixed point theorem to prove the following theorem of the Nash equilibrium existence for the informational extended game $_n\Gamma$ with $n$ players.

**Theorem 3.1.** *Let us consider that for the game* $_n\Gamma$ *the next conditions hold:*

*1) the sets* $X_i \neq \emptyset, \left(i = \overline{1,n}\right)$ *are compacta of Banach spaces,*

*2) the sets of functions* $\overline{X}_i \subset C\left(\prod\limits_{j \in I, j \neq i} X_j, X_i\right), \left(i = \overline{1,n}\right)$ *are uniformly bounded and the functions from the sets* $\overline{X}_i$ *are equicontinuous;*

*3) the payoff functions* $H_i(\cdot), \ \left(i = \overline{1,n}\right)$ *are continuous on the compactum* $\prod\limits_{i \in I} X_i$ *and the functions* $\overline{H}_i(\cdot), \left(i = \overline{1,n}\right)$ *are concave on* $\overline{X}_i$ *for* $\forall \varphi_{-i}$*, respectively.*

*Then* $NE\left(_n\Gamma\right) \neq \emptyset$*.*

*Proof.* Let $\overline{X} = \prod_{i\in I} \overline{X}_i$ be the outcome space. According to Arzelà-Ascoli theorem the sets $\overline{X}_i, (i \in I)$ are compact, and according to Tikhonov theorem the outcome space $\overline{X}$ is a compactum too.

Let us denote an outcome of the extended game by $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \in \overline{X} = \prod_{i\in I} \overline{X}_i$, where $\varphi_i = \varphi_i (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \overline{X}_i$.

Later we will use the next notations: $\varphi_{-i} = (\varphi_1, \varphi_2, \ldots, \varphi_{i-1}, \varphi_{i+1}, \ldots, \varphi_n)$, $\overline{X}_{-i} = \prod_{\substack{j\in I\\ j\neq i}} \overline{X}_j$.

Since the payoff functions $H_i(\cdot)$, $\left(i = \overline{1,n}\right)$ are continuous on the compact $\prod_{i\in I} X_i$ (from the third condition of the theorem) and because the functions $\varphi_i \in \overline{X}_i$ are continuous on the compact $\prod_{j\in I, j\neq i} X_j$, then the functions $\overline{H}_i, i = \overline{1,n}$ are continuous on the compact $\prod_{i\in I} \overline{X}_i$ as compound functions of continuous functions $\overline{H}_i(\varphi) = H_i(\varphi(x))$.

We define the point-to-set mapping $B : \overline{X} \rightrightarrows 2^{\overline{X}}$, such that $B(\varphi) = (B_1(\varphi_{-1}), B_2(\varphi_{-2}), \ldots, B_n(\varphi_{-n}))$, where $B_i(\varphi_{-i}), (i \in I)$ represents the best response set for the player $i$ for the chosen strategies of all players $j \in I \backslash \{i\}$.

Because the sets $\overline{X}_i, (i \in I)$ are compacts and $\overline{H}_i$, for $i = \overline{1,n}$ are continuous functions, then according to the Weierstrass theorem we can write:

$$B_i(\varphi_{-i}) = Arg \max_{\varphi_i \in \overline{X}_i} \overline{H}_i(\varphi_1, \varphi_2, \ldots, \varphi_n),$$

i. e.:

$$B_i(\varphi_{-i}) = \left\{ \varphi_i \in \overline{X}_i : \overline{H}_i(\varphi_1, \varphi_2, \ldots, \varphi_n) = \max_{\varphi_i' \in \overline{X}_i} \overline{H}_i(\varphi_1, \varphi_2, \ldots, \varphi_n) \right\}, \left(i = \overline{1,n}\right).$$

In order to use the Kakutani theorem we need to prove that:

1) $\overline{X} = \prod_{i\in I} \overline{X}_i \neq \emptyset$ is a non-empty convex compact set;

2) for the point-to-set mapping $B : \overline{X} \rightrightarrows 2^{\overline{X}}$ the next conditions hold:

a) for $\forall \varphi_i \in \overline{X}_i, \left(i = \overline{1,n}\right)$ the set $B(\varphi) \neq \emptyset$ is a convex subset of $\overline{X}$;

b) the point-to-set mapping $B$ is closed.

Firstly we will prove that $\overline{X}$ is convex and compact.

The set $\overline{X}_i, (i \in I)$ is convex if: for $\forall \varphi_i', \varphi_i'' \in \overline{X}_i$, and $\lambda \in [0,1]$ the function $\lambda\varphi_i' + (1-\lambda)\varphi_i''$ is bounded by the same constant $N$ (see Arzelà-Ascoli theorem) and the function $\lambda\varphi_i' + (1-\lambda)\varphi_i''$ is equicontinuous.

It is easy to prove that the function $\lambda\varphi_i' + (1-\lambda)\varphi_i''$ is bounded by the same constant $N$:
$|\lambda\varphi_i'(x_{-i}) + (1-\lambda)\varphi_i''(x_{-i})| \leqslant \lambda|\varphi_i'(x_{-i})| + (1-\lambda)|\varphi_i''(x_{-i})| \leqslant \lambda N + (1-\lambda)N = N$ for all $\varphi_i', \varphi_i'' \in \overline{X}_i$, and $\lambda \in [0,1]$.

Evidently the function $\lambda\varphi_i' + (1-\lambda)\varphi_i''$ is equicontinuous. So the set $\overline{X}_i, (i \in I)$ is convex. Then the set $\overline{X}$ is convex and compact too.

Next we need to prove that for the point-to-set mapping $B : \overline{X} \rightrightarrows 2^{\overline{X}}$ the conditions a) and b) hold.

Firstly we will prove the condition a). For $\forall \varphi_i \in \overline{X}_i, \left(i = \overline{1,n}\right)$ the set $B(\varphi)$ is non-empty, this follows from the Weierstrass theorem, because $B_i(\varphi_{-i})$, $\forall i \in I$ are non-empty sets.

Next we need to prove that the set $B(\varphi)$ is convex for $\forall \varphi_i \in \overline{X}_i, \left(i = \overline{1,n}\right)$.

So we will prove that the sets $B_i(\varphi_{-i}), \forall i = \overline{1,n}$ are convex.

The function $\overline{H}_i(\varphi_1, \varphi_2, \ldots, \varphi_n) = \overline{H}_i(\varphi_i, \varphi_{-i})$ is concave on the compact set $\overline{X}_i \subset C\left(\prod\limits_{j\in I, j\neq i} X_j, X_i\right)$, $(i \in I)$, then by definition for $\forall \lambda \in [0,1]$, and $\forall \varphi_i', \varphi_i'' \in \overline{X}_i$ the relation $\overline{H}_i(\lambda\varphi_i' + (1-\lambda)\varphi_i'', \varphi_{-i}) \geqslant \lambda\overline{H}_i(\varphi_i', \varphi_{-i}) + (1-\lambda)\overline{H}_i(\varphi_i'', \varphi_{-i})$ holds.

For $\forall \varphi_{-i}$ the set $B_i(\varphi_{-i})$ will be convex since the function $\overline{H}_i(\cdot)$ is continuous on $\overline{X}_i$ and $\overline{H}_i(\cdot)$ is concave by $\varphi_i$, for $\forall i = \overline{1,n}$.

From what was proved it follows that for $\forall \varphi_i \in \overline{X}_i$, $\left(i = \overline{1,n}\right)$ we will have a convex subset $B(\varphi) = (B_1(\varphi_{-1}), B_2(\varphi_{-2}), \ldots, B_n(\varphi_{-n})) \neq \emptyset$ from $\overline{X} = \prod\limits_{i\in I} \overline{X}_i$.

Next we will prove the condition b). We need to prove that the point-to-set mapping $B$ is closed.

The point-to-set mapping $B$ is closed if its graph is a closed set [4]. Since $B_i(\varphi_{-i})$ is a subset from the compactum $\overline{X}_i$ for all $i = \overline{1,n}$, then $grB_i(\varphi_{-i})$, $\left(i = \overline{1,n}\right)$ are compact sets. Here the graph for $B_i(\varphi_{-i})$ is defined by:

$$\begin{aligned} grB_i(\varphi_{-i}) &= \left\{ (\varphi_1, \varphi_2, \ldots, \varphi_n) \in \overline{X} \,\middle|\, \varphi_i \in Arg \max_{\varphi_i' \in \overline{X}_i} \overline{H}_i\left(\varphi_i', \varphi_{-i}\right), \varphi_{-i} \in \overline{X}_{-i} \right\} = \\ &= \left\{ (\varphi_1, \ldots, \varphi_n) \in \overline{X} \,\middle|\, \varphi_i \in B_i(\varphi_{-i}), \varphi_j \in \overline{X}_j, j \in I, j \neq i \right\}. \end{aligned}$$

We will prove that for the chosen strategies $\varphi_{-i}$ the sets $B_i(\varphi_{-i}), i = \overline{1,n}$, are closed.

The set $B_i(\varphi_{-i})$ can be rewritten as follows:

$$B_i(\varphi_{-i}) = \left\{ \varphi_i \in \overline{X}_i : \overline{H}_i(\varphi_i, \varphi_{-i}) - \max_{\varphi_i' \in \overline{X}_i} \overline{H}_i\left(\varphi_i', \varphi_{-i}\right) = 0 \right\}.$$

Because the set $\overline{X}_i$ is compact and the function $\overline{H}_i$ is continuous on $X$, then the function $\overline{H}_i(\varphi_i, \varphi_{-i}) - \max\limits_{\varphi_i' \in \overline{X}_i} \overline{H}_i(\varphi_i', \varphi_{-i})$ is continuous on $\overline{X}_i$ too. So for $\forall \varphi_{-i}$, the set $B_i(\varphi_{-i}) \subset \overline{X}_i$ is closed (and compact).

Then according to the Tikhonov theorem, because $grB_i(\varphi_{-i})$ is a closed set for all $i = \overline{1,n}$, so it follows that

$$grB = \left\{ (\varphi_1, \ldots, \varphi_n) \in \overline{X} \,\middle|\, \varphi_i \in B_i(\varphi_{-i}), \forall i = \overline{1,n} \right\}$$

is a closed set too.

Thus the point-to-set mapping $B$ is closed.

Therefore we can apply the Kakutani theorem.

Let $\varphi^* = (\varphi_1^*, \varphi_2^*, \ldots, \varphi_n^*) \in \overline{X} = \prod_{i \in I} \overline{X}_i$ be a fixed point for the point-to-set mapping $B$, i.e. $(\varphi_1^*, \varphi_2^*, \ldots, \varphi_n^*) \in B(\varphi_1^*, \varphi_2^*, \ldots, \varphi_n^*) = \prod_{i \in I} B_i(\varphi_{-i})$, so the relation

$$\overline{H}_i(\varphi_1^*, \ldots, \varphi_i^*, \ldots, \varphi_n^*) = \max_{\varphi_i \in \overline{X}_i} \overline{H}_i(\varphi_1^*, \ldots, \varphi_i, \ldots, \varphi_n^*)$$

holds for all $i = \overline{1,n}$, thus by definition of the Nash equilibrium it follows that $(\varphi_1^*, \ldots, \varphi_i^*, \ldots, \varphi_n^*) \in NE({}_n\Gamma) \neq \emptyset$. $\square$

## References


[1] Kantorovici L. V., Akilov G. P. *Funktsionalinyi analiz.* Moskva, Nauka, 1984.

[2] Danilov V. I. Lectures on fixed points, 1998 (in Russian); http://www.nes.ru/russian/research/pdf/1998/Danilov.pdf

[3] Hâncu Boris. *Probleme de optimizare pe multe nivele, Partea I*, CE USM, Chişinău, 2002.



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# Professor Nicolae Vulpe – 60th anniversary

The mathematical community of the Republic of Moldova congratulates Nicolae Vulpe, University Professor and Doctor Habilitat, on the occasion of the 60th anniversary of his birthday. This is an opportunity to acknowledge his exceptional contributions to the development of the mathematical school of Moldova.

Nicolae Vulpe was born on the 22nd of February 1949 in the village of Brinza in the area of Vulcanesti. In 1963 he graduated from the primary school in his village and continued his studies at a secondary school in Cahul. In 1966 he became a student in the Faculty of Physics and Mathematics of the Pedagogical Institute in Tiraspol where he later graduated with honours and became Assistant at the Chair of Mathematical Analysis of the institute. He also ended his military service about the same time. He was recommended to work in the section of Differential Equations and Methods of Computations, section headed by Constantin Sibirschi, the founder of the school in differential equations of the Republic of Moldova and later on, member of the Academy.

In 1972 Nicolae Vulpe became a assistant of the Institute of Mathematics and Computer Science of the Academy of Sciences, where he remains to this day. He married his former class colleague, both in the school and at the university, and together they raised and educated three children. Constantin Sibirschi initiated the work on algebraic invariants of polynomial ordinary differential equations and he published many articles in prestigious mathematical journals on this subject. He also trained a group of scientists in this direction of research, among them the young Nicolae Vulpe. These scientists continued the work of Constantin Sibirschi after his death and obtained many very valuable results which largely extend his contributions.

The whole mathematical career of Nicolae Vulpe was dedicated to this area of research to which he made important contributions. His sustained work was appreciated and he was promoted first as Junior Scientific Researcher (1975), then

Senior Scientific Researcher (1981), then Head of the Section in Differential Equations (1984) of the Institute, then Principal Scientific Collaborator (1993).

Nicolae Vulpe obtained his Doctorate in Mathematical and Physical Sciences in 1976, in 1985 he brilliantly defended his Habilitation Thesis and in 1999 he became Full Professor. Professor Nicolae Vulpe is author or coauthor of 120 scientific works. All these works are devoted to the qualitative study of polynomial differential systems using the theory of algebraic invariants introduced by Constantin Sibirschi.

The mathematical results of Nicolae Vulpe extend this theory and built applications of this theory by using algebraic comitants (polynomials depending on the coefficients of the systems involved which are invariant under the action of various transformation groups), the method of $T$-comitants (algebraic invariant polynomials with coefficients invariant under translations) and the method of differenial operators (of Hilbert type or transvectant). Although his investigations spread over a larger area, the focus of his interest was the theory of planar quadratic differential systems, that is systems of two differential equations defined by polynomials of maximum degree two. This area offered an excellent testing ground for proving the power of the method of algebraic invariants for obtaining qualitative results for these equations. In his first articles he studied topological and geometric structures of the homogeneous differential systems and obtained invariant partitions expressed in terms of algebraic invariants.

Afterwards he solved a problem which had a long history of unsuccessful attempts for obtaining its solution: the problem of finding all phase portraits of quadratic differential systems with a singular point which is a center. This is a highly cited paper by numerous authors. This success reinforced his interest in this area in which he since obtained alone or with collaborators many other interesting results such as: the stratification in $R^{12}$ of the class of quadratic differential systems according to their global scheme of finite singularities; a formula which relates the degree of freedom in the class of all quadratic systems subject to having a given configuration of finite singularities (real or complex, simple or multiple): the sum of the degree of freedom and of the number of distinct finite singularities is 4; the determination of the affine invariant criteria for polynomial integrability within the class of quadratic systems; the connection between the existence of a polynomial first integral and the rationality of the solutions of a certain algebraic equation whose coefficients are absolute affine invariants of the systems studied; the determination of affine invariant conditions of quadratic systems possessing rational first integrals of degree two and the construction of all phase portraits of this class on the Poincaré disk as well as of the affine invariant conditions for the determination of each one of these phase portraits; the proof of Darboux integrability of quadratic systems having invariant straight lines of total multiplicities 5 and 6; the classification of all possible configurations of quadratic systems possessing invariant straight lines of total multiplicity 4 and the proof of Liouvillian integrability of all such systems; the topological classification of all quadratic systems possessing invariant straight lines of total multiplicity at least 4 and the construction of the moduli space, under the action of the group of affine transformations and homotheties of time of this whole family; the topological classification of the class of quadratic systems according to their behavior around their infinite singularities.

The contributions of professor Vulpe to the development of the school in differential equations of Chisinau were very much appreciated by the mathematical community of the Republic of Moldova. Professor Vulpe received many prizes such as: “Republican Premium” (1978) for young researchers, awarded to him for a series of articles on the theory of algebraic invariants of differential equations; “Diploma of Recognition” (1999) of the Academy of Sciences of the Republic of Moldova; the “C. Sibirschi Prize” for the series of works on the application of invariant polynomials in the qualitative study of differential equations. We believe that Constantin Sibirschi would have been proud of his former student and disciple, whose work is now internationally known.

Professor Vulpe is regularly invited to attend international conferences abroad and he has been invited to lecture in several universities (the Technical University of Delft, Holland; York University, Great Britain; Université de Montréal, Canada; Universitat Autonoma de Barcelona, Spain). In November of 2008 he participated in a Workshop at BIRS (Banff International Research Station for Mathematical Innovation and Discovery) in Canada where his results were appreciated by some of the best experts in the world in the qualitative theory of differential equations.

Since 1998 Dr. habilitat Nicolae Vulpe has been Editor-in-Chief of the journal ”Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica”.

We wish Professor Nicolae Vulpe much happiness and joy from his children and grandchildren and good health and vigor for many years to come as well as much success in his scientific research.