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Exponential inflationary economic growth

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Abstract. Some scenario of economic growth centered on the structural reforms of the Republic of Moldova is presented. Mathematical model elaborated in [1] was adopted to proposed scenario in order to obtain indicators of exponential inflationary growth taking into account production possibilities. Economy description was presented by the principal economic sectors restrictions and production function depending of capital; the labor was not considered. The effectiveness of growth programs is estimated by parameters of growth and inflation in concordance with exponential inflationary growth. This solution is a particular one admissible by the model.

Mathematics subject classification: 91B62.
Keywords and phrases: Mathematical modeling, scenario simulation, production function, differential equations in finite differences.

The small open economy is considered. It is supposed that the economy produces one aggregate product which is utilized domestically and exported. Four economic sectors are examined. The state sectors which collects taxes; pays off salaries, pensions, allowances, and stipends, and effects some social programs. The production sector that owns all production factors and, as a consequence, earns all real income. Households that receive salaries and take part in goods exchange buying it in the good market. The monetary sector, represented by the National Bank, which intervenes in the foreign currency market selling and buying international exchange. And the external sector, which buys back its liabilities from domestic state and production sectors and earns international reserves from National Bank. It is supposed that the goods and monetary markets are in the equilibrium for all time the of model action. This time period of the model action is sufficiently long for the economic agents' accommodation to structural reforms, but insufficiently long for the some cardinal changes in production efficiency to be done.

The balanced exponential inflationary growth characterizes such equilibrium and it is determined by constant coefficients which define production technology, consumer preferences and circulation of goods, resources and money. Since the equilibrated growth is mentioned,the production and consumption grows (decreases) by constant rate. The price indexes proportions are maintained and also can be increased (or decreased) with a constant inflation (deflation) rate. Therefore a macroeconomic model which described main production proportion can be utilized. In such a model a balanced growth rate is determined through the constant technology parameters average for entire economy, behavior parameters and circulation mechanism parameters. The economic state in discrete moments of time on the fixed time interval $[0, T]$ is examined by the model. The time interval $[t, t+1]$ is considered about one year. The economics growth scenario for Republic of Moldova is proposed for further examination:

1. The government functions will be reduced to the redistribution of the limited budget sources in the favor of vulnerable parts of population and will ensure the equal and fair competition between local and foreign economic agents.
2. The creation of formal and justice conditions for equal and fair competition will contribute to increasing the investment flows in the production and to the rational distribution of resources.
3. The fiscal reform will favor economic agents to reserve oneself profit and income by economic active population.
4. The internal and foreign credits will be mobilized in order to ensure economic growth.
5. Budget deficit will be reduced to zero and from the budget surplus the external debt will be paid off.
6. The monetary system will be based on the international currency reserves and on the internal credits.

The exposed scenario reflects main programs' characteristics of the internal resources mobilization and involves external resources in order to maintain the economic growth. Model [1] adapted to this scenario will be used for the economic effectiveness evaluation. If the economic growth will be examined for medium period, then it will be necessary to evaluate constant parameters which characterize economic efficiency in concordance with statistic data reflecting current state of national economy. The prudence in launching assumptions about future tendencies will be necessary.

The labor can be excluded from the principal production factor examination because sufficient reserves of unemployment exist. One of the most restricted production factors is raw materials . The fixed means of production which determine the production capacity are not restrictive production factor. Anyway, the potential economic growth is evaluated so that the fixed funds are considered as marginal and production is worked utilizing all the production capacity. But introducing new production capacities necessitates some additional investment in the production sector. So in the model output does not depend on the labor force but expenditure for it paying off will be considered.

Production sector

The output in year t is:

$$Y_t = \sum_{\tau=t-T_\tau-T_\mu}^{t-T_\tau} I_\tau (1-\mu)^{\tau+T-t}, \quad t = 0, 1, \dots, \tag{1}$$

and the current production expenditure V_t is equal:

$$V_t = a \sum_{\tau=t-T_\tau-T_\mu}^{t-T_\tau} I_\tau, \quad t = 0, 1, \ldots, \tag{2}$$

here $I_t = X_t^I/b$, b is the coefficient of the fund utilization for the one unity production capacity creation; a is the raw material consumption index; I_t is the production capacity in year t.

Price index is calculated in the following manner:

$$P_t = P_0(1+i)^t; \quad t = 0, 1, \ldots \tag{3}$$

Changes in money demand are presented as:

$$M_{t+1}^E = M_t^E + P_tY_t - P_tV - (n_1+n_2)(P_tY_t - P_tV_t), \quad t = 0, 1, \ldots \tag{4}$$

$$M_t^E = \theta_E(n_1+n_2)(P_tY_t - P_tV_t), \quad t = 0, 1, \ldots \tag{5}$$

Household revenue and expenditure balance:

$$M_{t+1}^H = M_t^H + (n_1+g_1)(1-n_3)(P_tY_t - P_tV_t), \quad t = 0, 1, \ldots \tag{6}$$

$$M_t^H = \theta_H P_t C_t, \quad t = 0, 1, \ldots, \tag{7}$$

State budget is represented as

The state taxes are collected in the volume of $(n_2+n_3(n_1+g_1))(P_tY_t - P_tV_t)$, the external borrowing F_t^D are evaluated at the current exchange rate ρ_t, and National Bank profit B_t^B, occurred at the reevaluation of currency reserves:

$$B_t^B = (\rho_{t+1}-)R_{t+1}^C, \quad t = 0, 1, \ldots \tag{8}$$

The main expenditure components are: the payment to population, the state program financing, the external debt payment, evaluated at the current exchange rate and the money reserves growth "freezed" in budget payment accounting. The overflow of expenditure over the revenue forms the budget deficit and this deficit increases internal debt. Therefore, the change in internal debt takes form:

$$L_{t+1} - L_t = (g_1+g_2-n_2-n_3(n_1+g_1))P_t(Y_t - V_t) + \rho_t F_t^R -$$
$$-\rho_t Ft^D - (\rho_{t+1}-\rho_t)R_{t+1}^C + M_{t+1}^G - M_t^G + \Delta D_t^G, \quad t = 0, 1, \ldots \tag{9}$$

here ΔD_t^G is the internal volume of credits, granted to the state sector, $\rho_t F_t^D$, $\rho_t F_t^R$ are the currency entered the country and leave the country:

$$M_t^G = \theta_G(g1+g_2)P_t(Y_t - V_t). \tag{10}$$

External currency reserves, export and import volumes

Let E_t^P be the volume of export; Z_t^P be the volume of import; F_t^D be the currency entered the country; F_t^R be the currency leave the country; $\rho_t R_{t+1}^C - \rho_t Rt^C$ be the change in international currency reserves. Taking into account that the exports and imports price indexes q_t, q_Z change slowly than domestic price index P_t,it will be considered that these price indexes are constant. In concordance with the proposed scenario, National Bank, protecting local producers, rules the currency rate in domestic market in such a way that the import operations give minimal earns. This is expressed by the following equalities

$$P_t - \rho q_Z = 0, \quad t = 0, 1, \dots \tag{11}$$

The export volume is expressed as a share of the output. Importers secure currency on the base of import sailing on the domestic goods market. So the currency reserves of the National Bank change on the following equation base:

$$\rho_t R_{t+1}^C - \rho_t Rt^C = Pt(E_t^P - Z_t^P) + \rho_t(F_t^D - F_t^R), \quad t = 0, 1, \dots. \tag{12}$$

National account:

$$Y_t + Z_t = C_t + V_t + bI_t + g_2(Y_t - V_t) + E_t^P, \quad t = 0, 1, \dots, \tag{13}$$

here C_t is the populations' consumption, $g_2(Y_t - V_t)$ is the state investment. On the other hand, from the monetary approach, change in currency reserves (balance of payments) is expressed as: $\rho_{t+1} R_{t+1}^C - \rho_t Rt^C = \Delta M - \Delta D$, here $\Delta M = (M_{t+1}^E - M_t^E) + (M_{t+1}^G - M_t^G) + (M_{t+1}^H - M_t^H)$

State debt servicing

Suppose that $\rho_t(F_t^D - F_t^R) = g_3 P_t(Y_t - V_t), \quad t = 0, 1, \dots,$ and $\Delta D_t^G + \Delta D_t^E = \Delta D_t = g_4 P_t * (Y_t - V_t)$. After some transformation on the base of given formulas it will be obtained:

$$\begin{aligned}(\rho_{t+1} - \rho_t)R_{t+1}^C &= (g_1 + g_2 + g_3 - n_2 - n_3(n_1 + g_1))\times \\ &\times P_t(Y_t - V_t) + (M_{t+1}^G - M_t^G), \quad t = 0, 1, \dots\end{aligned} \tag{14}$$

From equation (9) the Central Bank reserves reevaluation are expressed:

$$P_t E_t^P - P_t Z_t^P = \rho_t R_{t+1}^C - \rho_t R_t^C + g_3 P_t(Y_t - V_t), \quad t = 0, 1, \dots \tag{15}$$

Using equations (4),(6), (14) and (15) from the material balance equation (13) the variables values bI_t, C_t, $\rho_t R_{t+1}^C$ and $E_t^P - Z_t^P$ are excluded and the expression for reserves changes in national currency is obtained:

$$\rho_{t+1}R^C_{t+1} - \rho_t R^C_t = (M^E_{t+1} - M^E_t) + (M^H_{t+1} - M^H_t) +$$
$$+ (M^G_{t+1} - M^G_t) - \Delta D_t, \quad t = 0, 1, \dots \tag{16}$$

Model examination

Equations (1)–(16) represent a complete description of the growth economic model which reflects all conditions of the proposed scenario. Now some transformations are necessary in order to bring model to a form convenient for numerical analysis.

First, using the liquidity restriction (5) variables' values M^E_{t+1} and M^E_t are excluded from the financial balance equation (4). In result the real production investments are obtained:

$$bI_{t1} = (1 - (1 - \theta_E)(n_1 + n_2)(Y_t) - V_t) -$$
$$-\theta_E(n_1 + n_2)\frac{P_{t+1}}{P_t}(Y_{t+1}) - V_{t+1}), \quad t = 0, 1, \dots, \tag{17}$$

which are admitted by the production financial restrictions.

Second, from the material balance equation (13) C_t is excluded using equation (7), but $E^P_t - Z^P_t$ is excluded using equation (15) and another expression for real production investments is obtained:

$$bI_{t1} = (1 - (1 - (n_1 g_1)(1 - n_3) - g_2 - g_3)(Y_t) - V_t) +$$
$$+ \frac{1}{P_t}(M^H_{t+1}) - M^H_t) - \frac{1}{q_I}(R^C_{t+1} - R^C_t), \quad t = 0, 1, \dots, \tag{18}$$

which are admitted by the material balance and by monetary policy scenario.

The possibilities of economic growth will be evaluated through the balanced inflationary growth indicators. Let's:

$$Y_t = Y_0(1+\gamma)^t, \quad V_t = V_0(1+\gamma)^t, \quad I_t = I_0(1+\gamma)^t, \quad C_t = C_0(1+\gamma)^t, \tag{19}$$

where γ is the constant growth rate in real terms of the Y_0, V_0, I_0, C_0. Then from (3), (10), (12) and (14) it is obtained:

$$\rho_t = \rho_0(1+i)^t, \quad R^C_t = R^C_0(1+\gamma)^t, \tag{20}$$

where ρ_0 and R^C_0 are positive constants.

From equation (14) using expressions (10), (19) and (20)it is found:

$$R^C_t = \Big(\frac{g_1 + g_2 + g_3 - n_2 - n_3(n_1 + g_1)}{i(1+\gamma)} +$$
$$+ \frac{\theta(((1+\gamma)(1+i) - 1)(g_1 + g_2}{i(1+\gamma)})q_I(Y_t) - V_t\Big), \quad t = 0, 1, \dots \tag{21}$$

The difference $R^C_{t+1} - R^C_t$ is excluded from (18) using (21), the difference $M^H_{t+1} - M^H_t$ is excluded from (18) using (7)–(8), and the expression for real investments by the production side is found:

$$bI_t = \Big(1 - n_1 - n_2 - (f_E - d_E) - \frac{(1+\gamma)(1+i)-1}{i(1+\gamma)}\times$$
$$\times(g_1 + g_2 + f_G - d_G - n_2 - n_3(n_1+g_1)) - \frac{\gamma}{i(1+\gamma)}\theta_G\times$$
$$\times((1+\gamma)(1+i)-1)(g_1+g_2) + \frac{\theta_H((1+\gamma)(1+i)-1)}{\theta_H((1+\gamma)(1+i)-1)+1}\times$$
$$\times(n_1+g_1)(1-n_3)(Y_t - V_t)\Big), \quad t = 0,1,\dots, \tag{22}$$

Substituting (19) in (17) transforms it to:

$$bI_{t1} = (1 - (n_1 + n_2 - (f_E - d_E))\times$$
$$\times(\theta_E((1+\gamma)(1+i)-1)-1)) \times (Y_t - V_t), \quad t = 0,1,\dots, \tag{23}$$

Equating expressions (22) and (23) for the real production investments growth and inflation rate it will be obtained:

$$\theta_E(n_1+n_2) - (f_E - d_E)) = \frac{g_1 + g_2 + g_3 - n_2 - n_3(n_1+g_1)}{i(1+\gamma)} +$$
$$+\frac{\gamma\theta_G(g_1+g_2)}{i(1+\gamma)} - \frac{\theta_H(n_1+g_1)(1-n_3)}{\theta_H(\gamma + i(1+\gamma))+1}. \tag{24}$$

Finally, inserting expressions (1), (2) and (19) in (23), the sums are calculated and the second relation between the growth rate and the inflation rate is obtained:

$$b = \frac{1 - (d_E - f_E) - (n_1+n_2)(\theta_E((1+i)(1+\gamma)-1)+1)}{(1+\gamma)^T_I}\times$$
$$\times\Big(\frac{1-(1+\gamma)^{-T_\mu-1}(1+\mu)^{-T_\mu-1}}{1-(1+\gamma)^{-1}(1+\mu)^{-1}} - a\frac{1-(1+\gamma)^{-T_\mu-1}}{1-(1+\gamma)^{-1}}\Big). \tag{25}$$

The growth rate γ and the inflation rate i are determined by solving equations (24) and (25) in dependence on the model's parameters: a, b, μ, n_1 characterizing the economic effectiveness of production; g_1, g_2, f_G characterizing the state budget expenditures; n_2, n_3 characterizing the state budget revenue and the taxes pressure on production and households; $d = d_E + d_G$ which determine domestic credits rate in GDP; $g_3 = f_G + f_E$ is the total net foreign assets.

If the growth and inflation rates are determined then the external reserves in respect to GDP will be defined from (22) taking in account relation (25):

$$\frac{\rho_t R_t^C}{P_t(Y_t - V_t)} = \theta_E(n_1 + n_2) - (f_E - d_E) +$$

$$+\theta_G(g_1 + g_2) + \frac{\theta_H(n_1 + g_1)(1 - n_3)}{\theta_H(\gamma + i(1+\gamma)) + 1}, \tag{26}$$

and the net export in respect to GDP is defined from (15) using (20), (21), (1) and (2)

$$\frac{E_t^P - Z_t^P}{Y_t - V_t} = (f_E - f_G) + \gamma\Big(\theta_E(n_1 + n_2) +$$

$$+\theta_G(g_1 + g_2) + \frac{\theta_H(n_1 + g_1)(1 - n_3)}{\theta_H(\gamma + i(1+\gamma)) + 1}\Big), \tag{27}$$

here

$$\bar{a} = a\frac{1 - (1+\gamma)^{-T_m u - 1}}{1 - (1+\gamma)^{-T_{\mu-1}}(1+\mu)^{-T_{\mu-1}}} \cdot \frac{1 - (1+\gamma)^{-1}(1+\mu)^{-1}}{1 - (1+\gamma)^{-1}} \tag{28}$$

is the mean consumption index of materials V_t/Y_t.

On the base of historical data necessary constant coefficients were determined and the corresponding growth rate and inflationary rate were calculated.

References

[1] Petrov M.M., Shananin A.A. *Mathematical model for efficiency estimation of an economic growth scenario.* Mathematical Modeling, Russian Academy of Sciences, Department of Informatics, computational Technics and Automatisation. 2002, vol. 14, N 7, p. 27–52.

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The optimal flow in dynamic networks with nonlinear cost functions on edges

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Abstract. In this paper we study the dynamic version of the nonlinear minimum-cost flow problem on networks. We consider the problem on dynamic networks with nonlinear cost functions on edges that depend on time and flow. Moreover, we assume that the demand function and capacities of edges also depend on time. To solve the problem we propose an algorithm, which is based on reducing the dynamic problem to the classical minimum-cost problem on a time-expanded network. We also study some generalization of the proposed problem.

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Keywords and phrases: Dynamic networks, network flow, dynamic flows, flows over time, minimum cost flows.

1 Introduction

In this paper we study the dynamic version of the nonlinear minimum-cost flow problem on networks, in which flows from supply nodes should be sent, in minimum cost, to demand nodes such that the flows on used links do not exceed their capacities. This problem generalizes the well-known classical minimum-cost flow problem on static networks [1] and extends some dynamic models from [2–5].

Classical static network flow models have been well known as valuable tools for many applications. However, they fail to capture the property of many real-life problems. The static flow can not properly consider the evolution of the system in time. The time is an essential component, either because the flows of some commodity take time to pass from one location to another, or because the structure of network changes over time. To tackle this problem, we use dynamic network flow models instead of the static ones.

The minimum cost flow problem is the problem of sending flows in a network from supply nodes to demand nodes with minimum total cost such that link capacities are not exceeded. This problem has been studied extensively in the context of static networks. In this paper, we study the minimum cost flow problem in dynamic networks.

We consider the problem on dynamic networks with nonlinear cost functions on edges that depend on time and on flow. Moreover, we assume that the demand function and capacities of edges also depend on time. We propose an algorithm for solving the problem, which extends the algorithms from [2, 3] and is based on reducing the dynamic problem to the classical minimum-cost problem on a time-expanded network.

2 Problem formulation

A dynamic network $N = (V, E, u, \tau, d, \varphi)$ consists of directed graph $G = = (V, E)$ with the set of vertices V and the set of edges E, capacity function u: $E \times \mathbb{T} \to R$, transit time function τ_e: $E \to R_+$, demand function d: $V \times \mathbb{T} \to R$ and cost function φ: $E \times R_+ \times \mathbb{T} \to R_+$, where $\mathbb{T} = \{0, 1, 2, \ldots, T\}$. The demand function $d_v(t)$ satisfies the following conditions:

a) there exists $v \in V$ with $d_v(0) < 0$;

b) if $d_v(t) < 0$ for a node $v \in V$ then $d_v(t) = 0$, $t = 1, 2, \ldots, T$;

c) $\sum\limits_{t\in\mathbb{T}} \sum\limits_{v\in V} d_v(t) = 0$.

Nodes $v \in V$ with $\sum\limits_{t\in\mathbb{T}} d_v(t) < 0$ are called sources, nodes $v \in V$ with $\sum\limits_{t\in\mathbb{T}} d_v(t) > 0$ are called sinks and nodes $v \in V$ with $\sum\limits_{t\in\mathbb{T}} d_v(t) = 0$ are called intermediate.

A feasible dynamic flow on N is a function x: $E \times \mathbb{T} \to R_+$ that satisfies the following conditions:

$$\sum_{\substack{e\in E^+(v)\\ t-\tau_e\geq 0}} x_e(t-\tau_e) - \sum_{e\in E^-(v)} x_e(t) = d_v(t),\ \forall\, t \in \mathbb{T},\ \forall\, v \in V; \tag{1}$$

$$0 \leq x_e(t) \leq u_e(t), \quad \forall\, t \in \mathbb{T},\ \forall\, e \in E; \tag{2}$$

$$x_e(t) = 0,\ \forall\, e \in E,\ t = \overline{T - \tau_e + 1, T}; \tag{3}$$

where $E^+(v) = \{(u, v) \,|\, (u, v) \in E\}$, $\quad E^-(v) = \{(v, u) \,|\, (v, u) \in E\}$.

Here the function x defines the value $x_e(t)$ of flow entering edge e at time t. It is easy to observe that the flow does not enter edge e at time t if it will have to leave the edge after time T; this is ensured by condition (3).

To model transit costs, which may change over time, we define the cost function $\varphi_e(x_e(t), t)$ with the meaning that flow of value $\xi = x_e(t)$ entering edge e at time t will incur a transit cost of $\varphi_e(\xi, t)$. We consider the discrete time model, in which all times are integral and bounded by horizon T. The time horizon (finite or infinite) is the time until which the flow can travel in the network and defines the makespan $\mathbb{T} = \{0, 1, \ldots, T\}$ of time moments we consider.

The integral cost $F(x)$ of dynamic flow on N is defined as follows:

$$F(x) = \sum_{e\in E} \sum_{t\in\mathbb{T}} \varphi_e(x_e(t), t). \tag{4}$$

Our dynamic minimum-cost flow problem is to find a flow that minimizes the objective function (4).

It is easy to observe that if $\tau_e = 0, \forall\, e \in E$ and $T = 0$ then the formulated problem becomes the classical minimum-cost flow problem on a static network.

3 Main results

We have obtained a necessary and sufficient condition for the existence of admissible flow in dynamic network N, i.e. the condition when the set of solutions of the system (1)–(3) is not empty. In this paper we propose a new approach for solving the formulated problem, which is based on its reduction to a static minimum-cost flow problem. We show that our problem on network $N = (V, E, u, \tau, d, \varphi)$ can be reduced to a static problem on auxiliary static network $N^T = (V^T, E^T, u^T, d^T, \varphi^T)$; we name it the time-expanded network. We define this network as follows:

1. $V^T\colon\ = \{v(t)\,|\, v \in V,\ t \in \mathbb{T}\}$;
2. $E^T\colon\ = \{(v(t), w(t+\tau_e))\,|\, e = (v, w) \in E,\ 0 \le t \le T - \tau_e\}$;
3. $u^T_{e(t)}\colon\ = u_e(t)$ and $\varphi^T_{e(t)}(x_e(t))\colon\ = \varphi_e(x_e(t), t)$ for $e(t) \in E^T$;
4. $d^T_{v(t)}\colon\ = d_v(t)$ for $v(t) \in V^T$.

If we define a flow correspondence to be $x^T_{e(t)}\colon\ = x_e(t)$, the minimum-cost flow problem on dynamic networks can be solved by using the solution of the static minimum cost flow problem on the expanded network.

The essence of the time-expanded network is that it contains a copy of the vertices of the dynamic network for each time $t \in \mathbb{T}$, and the transit times and flows are implicit in the edges linking those copies.

Now let us define a correspondence between feasible dynamic flows on the dynamic network N and feasible static flows on the time-expanded network N^T. A feasible static flow on N^T is a function $x^T_{e(t)}$ that satisfies the following conditions:

$$\sum_{e(t)\in E^+(v(t))} x^T_{e(t)} - \sum_{e(t)\in E^-(v(t))} x^T_{e(t)} = d^T_{v(t)},\ \forall\, v(t) \in V^T;$$

$$0 \le x^T_{e(t)} \le u^T_{e(t)},\ \forall\, e(t) \in E^T;$$

$$x^T_{e(t)} = 0,\ \forall\, e(t) \in E^T,\ t = \overline{T - \tau_e + 1, T}.$$

Let $e(t) = (v(t), w(t+\tau_e)) \in E^T$ and let $x_e(t)$ be a flow on the dynamic network N. The corresponding function $x^T_{e(t)}$ on the time-expanded network N^T is defined as follows:

$$x^T_{e(t)} = x(v(t), w(t+\tau_e)) = x_e(t),\ \forall\, e(t) \in E^T. \tag{5}$$

Lemma 1. *The correspondence (5) is a bijection from the set of feasible flows on the dynamic network N onto the set of feasible flows on the time-expanded network N^T.*

Proof. It is obvious that the correspondence above is a bijection from the set of $\mathbb{T}$-horizon functions on the dynamic network N onto the set of functions on the time-expanded network N^T. It is also easy to observe that a feasible flow on the dynamic network N is a feasible flow on the time-expanded network N^T and vice-versa. Indeed,

$$0 \leq x^T_{e(t)} = x_e(t) \leq d_e(t) = d^T_{e(t)}, \ \forall\, e \in E, \quad 0 \leq t < T.$$

Therefore it is enough to show that each dynamic flow on the dynamic network N is put into the correspondence with a static flow on the time-expanded network N^T and vice-versa.

Let $x_e(t)$ be a dynamic flow on N and let $x^T_{e(t)}$ be a corresponding function on N^T. Let's prove that $x^T_{e(t)}$ satisfies the conservation constraints on its static network. Let $v \in V$ be an arbitrary node in N and t: $0 \leq t < T$ an arbitrary moment of time:

$$d_v(t) \overset{(i)}{=} \sum_{\substack{e \in E^+(v) \\ t-\tau_e \geq 0}} x_e(t-\tau_e) - \sum_{e \in E^-(v)} x_e(t) =$$

$$= \sum_{e(t-\tau_e) \in E^+(v(t))} x^T_{e(t-\tau(e))} - \sum_{e(t) \in E^-(v(t))} x^T_{e(t)} \overset{(ii)}{=} d^T_{v(t)}. \tag{6}$$

Note that according to the definition of the time-expanded network the set of edges $\{e(t-\tau_e) | e(t-\tau_e) \in E^+(v(t))\}$ consists of all edges that enter $v(t)$, while the set of edges $\{e(t) | e(t) \in E^-(v(t))\}$ consists of all edges that originate from $v(t)$. Therefore, all necessary conditions are satisfied for each node $v(t) \in V^T$. Hence, $x^T_{e(t)}$ is a flow on the time-expanded network N^T.

Let $x^T_{e(t)}$ be a static flow on the time-expanded network N^T and let $x_e(t)$ be a corresponding function on the dynamic network N. Let $v(t) \in V^T$ be an arbitrary node in N^T. The conservation constraints for this node in the static network are expressed by equality (ii) from (6), which holds for all $v(t) \in V^T$ at all times t: $0 \leq t < T$. Therefore, equality (i) holds for all $v \in V$ at all times t: $0 \leq t < T$ and $x_e(t)$ is a flow on the dynamic network N. □

The total cost of the static flow in the time-expanded network N^T is denoted as follows:

$$F^T(x) = \sum_{e(t) \in E} \sum_{t \in \mathbb{T}} \varphi^T_{e(t)}(x_e(t)).$$

Lemma 2. *If $x_e(t)$ is a flow on the dynamic network N and $x^T_{e(t)}$ is a corresponding flow on the time-expanded network N^T, then*

$$F(x_e(t)) = F^T(x_e(t)).$$

Proof. The proof is straightforward:

$$F(x_e(t)) = \sum_{e\in E}\sum_{t\in\mathbb{T}} \varphi_e(x_e(t),t) = \sum_{e(t)\in E}\sum_{t\in\mathbb{T}} \varphi^T_{e(t)}(x_e(t)) = F^T(x_e(t)). \qquad \square$$

The above lemmas imply the validity of the following theorem:

Theorem 1. *For each minimum-cost flow in the dynamic network there is a corresponding minimum-cost flow in the static network.*

Therefore, we can solve the dynamic minimum-cost flow problem by reducing it to the minimum-cost flow problem on static networks.

4 Algorithm

Let a dynamic network N be given. The minimum-cost flow problem is to be solved on N. Proceedings are following:

1. Building the time-expanded network N^T for the given dynamic network N.
2. Solving the classical minimum-cost flow problem on the static network N^T.
3. Reconstructing the solution of the static problem on N^T to the dynamic problem on N. $\square$

5 Generalization

Now let us study some general cases of the dynamic networks. First of all, we assume that only a part of the flow is dumped into the considered network at the time 0, i.e. the condition b) in the definition of the demand function $d_v(t)$ doesn't hold. Using the following, this case can be reduced to the one considered above.

Let us consider an arbitrary dynamic network N defined above and let the flow be dumped into the network from the node $v \in V$ at an arbitrary moment of time $\mathtt{t}$, different from the ordinary moment. We can reduce this problem to the problem in which all of the flow is dumped into the network at the initial time by introducing loops in all nodes from V, except the node v from which the flow is dumped into the network at the time $\mathtt{t}$. For such loops we attribute capacities $u_e(t)$ and transit times which are equal to the time $\mathtt{t}$. The cost functions are equal to 0 on these loops. So, we can consider that all the flow is dumped in the network at the time $\mathtt{t}$, which we define as the initial time.

The argumentation is the same when the flow is dumped in the network from different nodes at different moments of time. Let $\mathtt{t}$ be the maximum of those moments. In this case we take $\mathtt{t}$ as the initial time and attribute capacities $u_e(t)$ and transit times to loops constructed from all the nodes, except those that dump the flow in the network at time $\mathtt{t}$. The transit times are equal to the difference between time $\mathtt{t}$ and the time when the flow from those nodes that generate loops is dumped in the network. We consider the cost functions that are zero on such loops. So, we reduce this problem to the one considered above where the whole flow is dumped into the network at the initial moment of time.

Further we consider the variation of the dynamic network when the condition c) in the definition of the demand function $d_v(t)$ doesn't hold. We assume that after time $t = T$ there still is flow in the network, i.e. the following condition is true:

$$\sum_{t \in \mathbb{T}} \sum_{v \in V} d_v(t) \geq 0.$$

We also can reduce this case to the initial one, using the following argumentation.

Let us consider an arbitrary dynamic network N defined above and let the flow exist in the network after time $t = T$. We can reduce this problem to the problem without flow in the network after an upper bound of time by the introduction of an additional node $v \notin V$ and additional edges which are not contained in E. The rest of the flow in the network is sent to the node v through the arcs which we just introduced. We consider that these arcs have capacities $u_e(t)$ and specified limited transit times and that the cost functions on these loops are zero. In such a way we obtain the initial model of the dynamic network.

The next model of the dynamic network is the one when we allow flow storage at the nodes. In this case we can reduce this dynamic network to the initial one by introducing the loops in those nodes in which there is flow storage. For these loops we attribute capacities $u_e(t)$, specified limited transit times, and zero cost functions. The flow which was stored at the nodes passes through these loops. Accordingly, we reduce this problem to the initial one.

The other variation of the dynamic network is the one when the cost functions also depend on the flow at the nodes. In this case we can reduce this model of the dynamic network to the initial one by introducing new arcs and attributing the cost functions, which were defined in the nodes, capacities $u_e(t)$, and fixed transit times to these arcs. Consequently, we obtain the initial model of the dynamic network.

The same reasoning to solve the minimum-cost flow network problem on the dynamic networks and its generalization can be held in the case when, instead of the condition (2) in the definition of the feasible dynamic flow, the following condition takes place:

$$u_e^1(t) \leq x_e(t) \leq u_e^2(t), \quad \forall\, t \in \mathbb{T},\ \forall\, e \in E,$$

where $u_e^1(t)$ and $u_e^2(t)$ are lower and upper boundaries of the capacity of the edge e, respectively.

References

[1] Ford L., Fulkerson D. *Constructing maximal dynamic flows from static flows*. Operation Res., 1958, **6**, p. 419–433.

[2] Lozovanu D., Stratila D. *The minimum-cost flow problem on dynamic networks and algorithm for its solving*. Bul. Acad. Ştiinţe Repub. Mold., Matematica, 2001, N 3(37), p. 38–56.

[3] Lozovanu D., Stratila D. *Optimal flow in dynamic networks with nonlinear cost functions on edges*. Analysis and optimization of differential systems (Edited by V.Barbu, I.Lesiencko), ISBN 1–4020–7439–5. Kluwer Academic Publishers, 2003, p. 247–258.

[4] Fleisher L., Skutella M. *The quickest multicommodity flow problem*. Integer programming and combinatorial optimization. Springer, Berlin, 2002, p. 36–53.

[5] Klinz B., Woeginger C. *Minimum cost dynamic flows: the series parallel case*. Integer programming and combinatorial optimization (Copenhagen, 1995). Springer, Berlin, 1995, p. 329–343.

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On check character systems over groups

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Abstract. In this note we study check character systems (with one control symbol) over groups (over abelian groups) and the check formula $a_1 \cdot \delta a_2 \cdot \delta^2 a_3 \cdot \dots \cdot \delta^n a_{n+1} = e$, where e is the identity of a group, δ is an automorphism (a permutation) of a group. For a group we consider strongly regular automorphisms (anti-automorphisms), their connection with good automorphisms and establish necessary and sufficient conditions in order that a system to be able to detect all single errors, transpositions, jump transpositions, twin errors and jump twin errors simultaneously.

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1 Introduction

A check character (or digit) system with one check digit is an error detecting code over alphabet Q which arises by appending a check digit a_{n+1} to every word $a_1 a_2 \dots a_n \in Q^n$:

$$a_1 a_2 \dots a_n \to a_1 a_2 \dots a_n a_{n+1}$$

by some rule.

The aim of using such a system is to discover transmission errors of certain patterns. The examples used in praxis among others are the following:

the Universal Product Code (UPC),

the European Article Number (EAN) Code,

the International Book Number (ISBN) Code,

the system of the serial numbers of German banknotes.

Among the first publications with respect to these systems are articles of W. Friedman and C. J. Mendelsohn [5], based on code-tables, and by R. Schauffler [10] using algebraic structures. In his book [14] J. Verhoeff presented basic results which were in use up to 1970. Later the article of A. Ecker and G. Poch [4] was published where the group-theoretical background of the known methods was explained and new codes were presented that stem from the theory of quasigroups.

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Empirical investigations of J. Verhoeff [14] and Beckley [2] show that single errors $(\dots a \dots \to \dots b \dots)$, i.e. errors in only one component of a code word, (adjacent) transpositions $(\dots ab \dots \to \dots ba \dots)$, jump transpositions $(\dots acb \dots \to \dots bca \dots)$, twin errors $(\dots aa \dots \to \dots bb \dots)$ and jump twin errors $(\dots aca \dots \to \dots bcb \dots)$ are the most important errors made by human operators (see Table 8 in [8] of frequency of these error types).

The control digit a_{n+1} in a check character system can be calculated by different check formulas (check equations) in some algebraic structure (a group, a loop, a quasigroup). In the case of a group the most general check formula is the following

$$a_1 \cdot \delta_1 a_2 \cdot \delta_2 a_3 \cdot \dots \cdot \delta_n a_{n+1} = e, \tag{1}$$

where e is the identity of a group G, $\delta_1, \delta_2, \dots, \delta_n$ are some fixed permutations of G. Such a system is called a system over a group and always detects any single error. A survey of the known results concerning check character systems based on quasigroups (loops, groups) one can find in [1].

Often, one chooses a fixed permutation δ of G and puts $\delta_i = \delta^i$ for $i = 1, 2, \dots, n$. Equation (1) then becomes

$$a_1 \cdot \delta a_2 \cdot \delta^2 a_3 \cdot \dots \cdot \delta^n a_{n+1} = e. \tag{2}$$

There are many publications on check character systems over groups with check equation (2), detecting some error types or all of the pointed above error types.

We study check character systems over a finite group which detect all single errors, transpositions, jump transpositions, twin errors and jump twin errors simultaneously using such concepts as a complete mapping, an orthomorphism, a regular automorphism and a new concept of a strongly regular automorphism (anti-automorphism) of a group. For any group we consider the case when δ from (2) is an automorphism ($\delta \in AutG$) and reduce conditions for a good automorphism [3]. For an abelian group δ may be a permutation.

2 Good automorphisms and check character systems over groups

Denote by $S(G, \delta)$ a check character system over a group G with check formula (2), $n > 4$, where δ is a permutation on G.

According to the known results (see, for example, [11], Table 2) a system $S(G, \delta)$ detects all single errors and all

a) transpositions if and only if $x \cdot \delta y \neq y \cdot \delta x$ for all $x, y \in G$, $x \neq y$;

b) jump transpositions if and only if $xy \cdot \delta^2 z \neq zy \cdot \delta^2 x$ for all $x, y, z \in G$, $x \neq z$;

c) twin errors if and only if $x \cdot \delta x \neq y \cdot \delta y$ for all $x, y \in G$, $x \neq y$;

d) jump twin errors if and only if $xy \cdot \delta^2 x \neq zy \cdot \delta^2 z$ for all $x, y, z \in G$, $x \neq z$.

In Table of [3] sufficient (and necessary for $n > 4$) conditions on an automorphism δ of a group G with the identity e for error detection are given. These conditions we give in Table 1.

Table 1. Error detection for automorphism δ

Error types	Conditions on δ (for all $x, y \in G$, $x \neq e$)
single errors	none
transpositions	$\delta x \neq y^{-1}xy$
jump transpositions	$\delta^2 x \neq y^{-1}xy$
twin errors	$\delta x \neq y^{-1}x^{-1}y$
jump twin errors	$\delta^2 x \neq y^{-1}x^{-1}y$

If G is an abelian group, these conditions are, respectively, the following: $\delta x \neq x$, $\delta^2 x \neq x$, $\delta x \neq Ix$, $\delta^2 x \neq Ix$, if $x \neq e$, where $Ix = x^{-1}$: $x \cdot Ix = Ix \cdot x = e$.

A permutation δ satisfying the inequality $x \cdot \delta y \neq y \cdot \delta x$ for all $x, y \in G$, $x \neq y$ is called *anti-symmetric mapping* of a group G.

Groups with anti-symmetric mappings (check character systems over them detect all single errors and all transpositions according to condition a)) were studied in many articles (see, for example, [6–8] and [11–13]).

In [3] check character systems $S(G, \delta)$ over a finite group G with an automorphism δ, which detect all considered above error types simultaneously, were studied and the following concept of a good automorphism was introduced.

Definition 1 [3]. *Let G be a finite group. An automorphism δ of G is called good if δx is not conjugate to x or x^{-1} and $\delta^2 x$ is not conjugate to x or x^{-1} for all $x \in G$, $x \neq e$.*

In [3] it was also shown that there are many groups possessing a good automorphism. In particular, the following results were noted.

If G is abelian, then a good automorphism δ satisfies the conditions for detecting transpositions, jump transpositions and twin errors if δ^2 is regular (that is fixed point free on G, the same $\delta x \neq x$, if $x \neq e$) and δ is good if δ^4 is regular.

For any group G and an automorphism δ of odd order the condition $\delta x \neq y^{-1}xy$ (for all $x, y \in G$, $x \neq e$) implies that δ is good.

The following statement is also useful.

Lemma 1 [3]. *Let G be a p-group and $\delta \in \operatorname{Aut} G$. Suppose $gcd(\mathrm{o}(\delta), p(p-1)) = 1$ ($\mathrm{o}(\delta)$ is the order of δ). Then δ is good if and only if it is fixed point free.*

The conditions of Table 1 (the same the conditions of a good automorphism) are sufficient and necessary for detection of all single errors, transpositions, jump transpositions, twin errors and jump twin errors if $n > 4$ [3].

Thus, we have the following statement.

Proposition 1. *A system $S(G, \delta)$ over a group G where $\delta \in \operatorname{Aut} G$, detects all single errors, transpositions, jump transpositions, twin errors and jump twin errors if and only if the automorphism δ is good.*

3 Strongly regular automorphisms and check character systems over groups

Now we introduce the following useful concept.

Definition 2. *An automorphism (an anti-automorphism) δ of a group G is called strongly regular if*

$$\delta(xy) \neq yx$$

for all $x, y \in G$, $y \neq Ix$.

It is easy to see that a strongly regular automorphism (anti-automorphism) δ is regular and δ^{-1} is also strongly regular.

In abelian groups the concepts of a regular automorphism and a strongly regular automorphism coincide.

Recall that *a complete mapping* of a group G is a bijective mapping $x \to \theta x$ of G onto G such that the mapping $x \to \eta x$ defined by $\eta x = x \cdot \theta x$ is again a bijective mapping of G onto G.

A permutation α of G is called *an orthomorphism* of a group G, if the mapping $\beta: \ \beta x = x \cdot I\alpha x$ is also a permutation of G [9].

According to [9] an automorphism α is an orthomorphism if and only if the automorphism α is regular.

It is evident that if α is an orthomorphism, then $I\alpha$ is a complete mapping and conversely.

An automorphism is called complete if it is a complete mapping.

Proposition 2. *Let G be a group, $\delta \in \operatorname{Aut} G$. Then the following statements are equivalent:*

(i) $\delta x \neq y^{-1}xy$ *for all* $x, y \in G$, $x \neq e$;

(ii) δ *is strongly regular;*

(iii) δ *is anti-symmetric;*

(iv) δ *satisfies the inequality* $xy \cdot \delta z \neq zy \cdot \delta x$ *for all* $x, y, z \in G$, $x \neq z$.

Proof. (i)$\Leftrightarrow$(ii): let $x \neq e$, then $\delta x \neq y^{-1}xy \overset{x \rightleftarrows yx}{\Longleftrightarrow} \delta(yx) \neq y^{-1}(yx)y = xy$, if $y \neq Ix$.

(ii)$\Leftrightarrow$(iii): let $x \neq z$, then $x \cdot \delta z \neq z \cdot \delta x \Longleftrightarrow Iz \cdot x \neq \delta x \cdot I\delta z = \delta x \cdot \delta Iz \overset{z \rightleftarrows Iz}{\Longleftrightarrow}$ $zx \neq \delta(xz)$, if $x \neq Iz$, since $I\delta = \delta I$.

(iii)$\Leftrightarrow$(iv): let $x \neq z$, then $x \cdot \delta z \neq z \cdot \delta x \overset{x \rightleftarrows xy,\ z \rightleftarrows zy}{\Longleftrightarrow} xy \cdot \delta(zy) \neq zy \cdot \delta(xy) \Longleftrightarrow$ $xy \cdot \delta z \neq zy \cdot \delta x$, if $x \neq z$, since $\delta \in AutG$. $\square$

Proposition 3. *Let G be a finite group, $\delta \in \operatorname{Aut} G$. Then the following statements are equivalent:*

(i) $\delta x \neq y^{-1}x^{-1}y$ *for all* $x, y \in G$, $x \neq e$;

(ii) the anti-automorphism $I\delta$ is strongly regular;

(iii) δ is a complete mapping;

(iv) δ satisfies the inequality $xy \cdot \delta x \neq zy \cdot \delta z$ for all $x, y, z \in G$, $x \neq z$.

Proof. (i)⇔(ii): let $x \neq e$, then $\delta x \neq y^{-1}x^{-1}y \overset{x \rightleftarrows yx^{-1}}{\Longleftrightarrow} \delta(yx^{-1}) \neq y^{-1}(xy^{-1})y = y^{-1}x = I(x^{-1}y) \overset{x \rightleftarrows Ix}{\Longleftrightarrow} \delta(yx) \neq I(xy) \Longleftrightarrow I\delta(yx) \neq xy$, if $y \neq Ix$.

(ii)⇔(iii): let $x \neq Iy$, $I\delta(yx) \neq xy \Longleftrightarrow \delta(yx) \neq I(xy) \overset{x \rightleftarrows Ix}{\Longleftrightarrow} \delta y \cdot \delta Ix \neq Iy \cdot x \Longleftrightarrow y \cdot \delta y \neq x \cdot \delta x$, if $x \neq y$, since $\delta I = I\delta$. Thus, δ is a complete automorphism, since G is a finite group.

(iii)⇔(iv): let $x \neq z$, then $x \cdot \delta x \neq z \cdot \delta z \overset{x \rightleftarrows xy,\ z \rightleftarrows zy}{\Longleftrightarrow} xy \cdot \delta(xy) \neq zy \cdot \delta(zy) \Longleftrightarrow xy \cdot \delta x \neq zy \cdot \delta z$, since $x \neq z$ and $\delta \in AutG$. □

Proposition 4. *An automorphism δ (anti-automorphism $I\delta$) of a finite group G is strongly regular if and only if δ ($I\delta$) is regular on the conjugacy classes of G (that is it does not fix any conjugacy class of $G \backslash \{e\}$).*

Proof. By Proposition 2 an automorphism δ is strongly regular if and only if δ is anti-symmetric. But by Proposition 4.3 of [11] δ is anti-symmetric if and only if it does not fix any conjugacy class $H \neq \{e\}$ of G.

According to Proposition 3 the anti-automorphism $I\delta$ is strongly regular if and only if $\delta x \neq y^{-1}x^{-1}y$ or $I\delta x \neq y^{-1}xy$ if $x \neq e$ for all $x, y \in G$. It means that $I\delta H \neq H$ for any conjugacy class H of G if $H \neq \{e\}$ (that is the anti-automorphism $I\delta$ is regular on the conjugacy classes, since it maps a class in a class). □

Proposition 5. *Let $\delta \in \operatorname{Aut} G$ and δ^2 be a strongly regular automorphism of a finite group G. Then the automorphism δ and the anti-automorphism $I\delta$ are also strongly regular.*

Proof. Let an automorphism δ^2 be strongly regular, then by Proposition 4 $\delta^2 H \neq H$ for any conjugacy class of G if $H \neq \{e\}$. From this it follows that $\delta H \neq H$ and $\delta H \neq IH$ (otherwise, $\delta^2 H = \delta(\delta H) = \delta(IH) = I\delta H = I^2 H = H$, contradiction) if $H \neq \{e\}$.

Thus, according to Proposition 4 δ and $I\delta$ are strongly regular. □

Note that this proposition means that from anti-symmetry of δ^2 anti-symmetry and completeness of δ follows (see Proposition 2 and Proposition 3).

Theorem 1. *An automorphism δ of a finite group G is good if and only if the automorphism δ^2 and the anti-automorphism $I\delta^2$ are strongly regular.*

Proof. The conditions of Definition 1 mean that an automorphism δ is good if and only if $\delta x \neq y^{-1}xy$, $\delta x \neq y^{-1}x^{-1}y$, $\delta^2 x \neq y^{-1}xy$ and $\delta^2 x \neq y^{-1}x^{-1}y$ for all $x, y \in G$, $x \neq e$ or $\delta H \neq H$, $\delta H \neq IH$, $\delta^2 H \neq H$ and $\delta^2 H \neq IH$ respectively for any conjugacy class H of G, $H \neq \{e\}$.

Taking into account Proposition 4 for the automorphisms δ and δ^2 (for the anti-automorphisms $I\delta$ and $I\delta^2$) we obtain that an automorphism δ of G is good if and only if δ, $I\delta$, δ^2 and $I\delta^2$ are strongly regular. Now use Proposition 5. □

Thus, the first two from four conditions of Definition 1 of a good automorphism are unnecessary.

From Proposition 1 and Theorem 1 it follows

Corollary 1. *A check character system $S(G, \delta)$ over a finite group G with $\delta \in \operatorname{Aut} G$ detects all single errors, transpositions, jump transpositions, twin errors and jump twin errors if and only if the automorphism δ^2 and the anti-automorphism $I\delta^2$ are strongly regular.*

By Proposition 2 (Proposition 3) δ^2 ($I\delta^2$) is a strongly regular automorphism (anti-automorphism) if and only if δ^2 is anti-symmetric (δ^2 is complete). So we obtain the following

Corollary 2. *A system $S(G, \delta)$ over a finite group G with $\delta \in \operatorname{Aut} G$ detects all five error types considered above if and only if δ^2 is an anti-symmetric and complete mapping.*

Corollary 3. *A system $S(G, \delta)$ over a finite abelian group with $\delta \in \operatorname{Aut} G$ detects all five error types considered above if and only if δ^2 is an orthomorphism and a complete mapping.*

Indeed, in this case the automorphism δ^2 is anti-symmetric if and only if it is regular (by Proposition 2 for δ^2), that is δ^2 is an orthomorphism.

As it was remarked after Definition 1 an automorphism δ of an abelian group admits to detect single errors, transpositions, jump transpositions and twin errors if δ^2 is fixed point free (that is regular).

Now consider check character systems $S(G, \delta)$ over a finite abelian group G where δ is a permutation on G ($\delta \in S_G$).

Theorem 2. *A check character system $S(G, \delta)$ over a finite abelian group G with $\delta \in S_G$ detects all single errors, transpositions, jump transpositions, twin errors and jump twin errors if and only if the permutations δ and δ^2 are orthomorphisms and complete mappings (that is all permutations δ, δ^2, $I\delta$ and $I\delta^2$ are complete mappings).*

Proof. In an abelian group G we have from conditions a) – b) in the beginning of section 2:

$$x \cdot \delta y \neq y \cdot \delta x \Longleftrightarrow x \cdot I\delta x \neq y \cdot I\delta y$$

for all $x \neq y$, that is δ is an orthomorphism;

$$xy \cdot \delta^2 z \neq zy \cdot \delta^2 x \Longleftrightarrow x \cdot \delta^2 z \neq z \cdot \delta^2 x \Longleftrightarrow x \cdot I\delta^2 x \neq z \cdot I\delta^2 z$$

for all $x \neq z$, that is δ^2 is an orthomorphism.

Condition c) means that δ is a complete mapping; for codition d) we have

$$xy \cdot \delta^2 x \neq zy\delta^2 z \Longleftrightarrow x \cdot \delta^2 x \neq z \cdot \delta^2 z$$

for all $x \neq z$, that is δ^2 is a complete mapping. □

According to Theorem 2.3 of [11] a finite abelian group G admits a complete mapping if and only if G has odd order or contains more than one involution (that is an element $a \in G$, $a \neq e$ such that $a^2 = e$), so we have from Theorem 2 the following

Corollary 4. *A check character system $S(G, \delta)$ over an abelian group (with one involution) and $\delta \in S_G$ is not able to detect all transpositions (jump transpositions, twin errors or jump twin errors).*

Example. Consider the abelian group $Z_2^3 = Z_2 \times Z_2 \times Z_2$ of order 8. Its Cayley Table is given in Table 2. In this group the permutation I is the identity permutation, so each complete mapping is an orthomorphism and conversely. According to [9] in Z_2^3 there are 48 regular automorphisms (that is orthomorphisms) which enter in eight subgroups of order 7. As computer research has shown one of such subgroups is the following:

$\varepsilon = (0\,1\,2\,3\,4\,5\,6\,7)$, $\delta_0 = (0\,2\,6\,5\,3\,7\,4\,1)$, $\delta_0^2 = (0\,6\,4\,7\,5\,1\,3\,2)$,
$\delta_0^3 = (0\,4\,3\,1\,7\,2\,5\,6)$, $\delta_0^4 = (0\,3\,5\,2\,1\,6\,7\,4)$, $\delta_0^5 = (0\,5\,7\,6\,2\,4\,1\,3)$, $\delta_0^6 = (0\,7\,1\,4\,6\,3\,2\,5)$.

We do not write the first row of permutations in the natural order.

Table 2. $Z_2^3 = Z_2 \times Z_2 \times Z_2$.

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	4	5	2	3	7	6
2	2	4	0	6	1	7	3	5
3	3	5	6	0	7	1	2	4
4	4	2	1	7	0	6	5	3
5	5	3	7	1	6	0	4	2
6	6	7	3	2	5	4	0	1
7	7	6	5	4	3	2	1	0

By Corollary 3 (or Theorem 2) each of six systems $S(Z_2^3, \delta)$, where δ is one of these automorphisms, $\delta \neq \varepsilon$, detects all single errors, transpositions, jump transpositions, twin errors and jump twin errors.

References

[1] Belyavskaya G.B., Izbash V.I., Shcherbacov V.A. *Check character systems over quasigroups and loops.* Quasigroups and related systems, 2003, **10**, p. 1–28.

[2] Beckley D.F. *An optimum system with modulus 11.* The Computer Bulletin, 1967, **11**, p. 213–215.

[3] Broecker C., Schulz P.-H., Stroth G. *Check character systems using Chevalley groups.* Design, Codes and Cryptography, DESI., 1997, **10**, p. 137–143.

[4] Ecker A., Poch G. *Check character systems.* Computing, 1986, N 37(4), p. 277–301.

[5] Friedman W., Mendelsohn C.J. *Notes on Codewords.* Am. Math. Monthly, 1932, p. 394–409.

[6] Gallian J.A., Mullin M.D. *Groups with anti-symmetric mappings.* Arch. Math., 1995, **65**, p. 273–280.

[7] Heiss S. *Antisymmetric mappings for finite solvable groups.* Arch. Math., 1997, **69**, p. 445–454.

[8] Heiss S. *Antisymmetric mappings for finite groups.* Preprint, 1999.

[9] Johnson D.M., Dulmage A.L., Mendelsohn N.S. *Orthomorphisms of groups and orthogonal Latin squares.* I. Canad. J. Math., 1961, **13**, p. 356–372.

[10] Schauffler R. *Über die Bilding von Codewörtern.* Arch. Electr. Übertragung, 1957, N 10(7), p. 303– 314.

[11] Schulz R.-H. *On check digit systems using anti-symmetric mappings.* In J. Althofer et al, editors. Numbers, Information and Complexity, Kluwer Acad. Dubl. Boston, 2000, p. 295–310.

[12] Schulz R.-H. *Equivalence of check digit systems over the dicyclic groups of order 8 and 12.* In J. Blankenagel & W. Spiegel editor, Matematikdidaktik aus Begeisterung für die Mathematik. Klett Verlag, Stuttgart, 2000, p. 227–237.

[13] Schulz R.-H. *Check Character Systems and Anti-symmetric Mappings.* H. Alt. (Ed): Computational Discrete Mathematics, LNCS 2122, 2001, p. 136–147.

[14] Verhoeff J. *Error detecting decimal codes.* Math. Centre Tracts, 1969, **29**, Math. Centrum Amsterdam.

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$GL(2,R)$-orbits of the polynomial sistems of differential equations

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Abstract. In this work we study the orbits of the polynomial systems $\dot{x} = P(x_1, x_2)$, $\dot{x} = Q(x_1, x_2)$ by the action of the group of linear transformations $GL(2,R)$. It is shown that there are not polynomial systems with the dimension of GL-orbits equal to one and there exist GL-orbits of the dimension zero only for linear systems. On the basis of the dimension of GL-orbits the classification of polynomial systems with a singular point $O(0,0)$ with real and distinct eigenvalues is obtained. It is proved that on GL-orbits of the dimension less than four these systems are Darboux integrable.

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1 Center-affine transformations

Consider the polynomial system

$$\dot{x}_1 = \sum_{k=0}^{n} P_k(x_1, x_2), \quad \dot{x}_2 = \sum_{k=0}^{n} Q_k(x_1, x_2), \tag{1}$$

where P_k, Q_k are homogeneous polynomial of degree k:

$$P_k = \sum_{i+j=k} a_{ij} x_1^i x_2^j, \ \ Q_k = \sum_{i+j=k} b_{ij} x_1^i x_2^j. \tag{2}$$

Denote by E the space of coefficients

$$a = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, a_{30}, ..., a_{0n}; b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}, b_{30}, ..., b_{0n})$$

of system (1) and by GL(2,R) the group of center-affine transformations of the phase space Ox, $x = (x_1, x_2)$. Applying in (1) the transformation $X = qx$, where $X = (X_1, X_2)$, $q \in GL(2,R)$, i.e.

$$q = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}; \ \alpha, \beta, \gamma, \delta \in R, \ \Delta = det(q) \neq 0, \ q^{-1} = \frac{1}{\Delta} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}, \tag{3}$$

we obtain the system

$$\dot{X}_1 = \sum_{k=0}^{n} P_k^*(X_1, X_2), \quad \dot{X}_2 = \sum_{k=0}^{n} Q_k^*(X_1, X_2), \tag{4}$$

where

$$P_k^* = \alpha \cdot P_k(q^{-1}x) + \beta \cdot Q_k(q^{-1}x) = \sum_{i+j=k} a_{ij}^* X_1^i X_2^j,$$
$$Q_k^* = \gamma \cdot P_k(q^{-1}x) + \delta \cdot Q_k(q^{-1}x) = \sum_{i+j=k} b_{ij}^* X_1^i X_2^j. \tag{5}$$

Remark 1. *It is easy to see from (5) that every transformation $q \in GL(2,R)$ acts separately on the homogeneities of the same order from (1).*

The coefficients a^* of system (4) can be expressed linearly by the coefficients of system (1): $a^* = L_{(q)}(a)$, $detL_{(q)} \neq 0$. The set $L = \{L_{(q)} | q \in GL(2,R)\}$ forms a 4-parameter group with the operation of composition. L is called the representation of the group $GL(2,R)$ of center-affine transformations of the phase space Ox in the space of coefficients E of system (1).

Let $a \in E$. A set $L(a) = \{L_{(q)}(a) | q \in GL(2,R)\}$ is called *the GL-orbit* of the point a or of the differential system (1) corresponding to this point.

2 Monoparametric transformations

Consider the function $g : R \times E \to E$ such that for every $\tau \in R$ the transformation $g^\tau : E \to E$, where $g^\tau(a) = g(\tau, a)$, $a \in E$, is a diffeomorphism. We say that $(E, \{g^\tau\})$ is a differentiable flow if:

1) $g^0 = id$;
2) $g^{\tau+s} = g^\tau g^s \quad \forall \tau, s \in R$;
3) $(g^\tau)^{-1} = g^{-\tau} \quad \forall \tau \in R$;
4) $g : R \times E \to E$ is a differentiable function.

By [1], [6] the 4-parameter transformation q (see(3)) can be represented as a product of four monoparametric transformations:

$$q^{\alpha_1^*} = \begin{pmatrix} \alpha_1^* & 0 \\ 0 & 1 \end{pmatrix}, q^{\alpha_2} = \begin{pmatrix} 1 & \alpha_2 \\ 0 & 1 \end{pmatrix}, q^{\alpha_3} = \begin{pmatrix} 1 & 0 \\ \alpha_3 & 1 \end{pmatrix}, q^{\alpha_4^*} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha_4^* \end{pmatrix},$$

where $\alpha_1^*, \alpha_4^* \in R \backslash \{0\}$; $\alpha_2, \alpha_3 \in R$. Denote

$$q^{\alpha_1} = \begin{pmatrix} e^{\alpha_1} & 0 \\ 0 & 1 \end{pmatrix}, q^{\alpha_4} = \begin{pmatrix} 1 & 0 \\ 0 & e^{\alpha_4} \end{pmatrix}, \alpha_1, \alpha_4 \in R;$$

$$L_l = L_{(q^{\alpha_l})},\ l = \overline{1,4};\ L_1^* = L_{\left(q^{\alpha_1^*}\right)},\ L_4^* = L_{\left(q^{\alpha_4^*}\right)}.$$

To every group of monoparametric transformations q^{α_l}, $l = \overline{1,4}$; $q^{\alpha_1^*}$, $q^{\alpha_4^*}$ of the phase space Ox corresponds a system of the form (4) with a_{ij}^*, b_{ij}^*, respectively.

It is easy to verify that $(E, \{L_{(q^{\alpha_l})}\}), l = \overline{1,4}$, are differential flows. They define in E the following systems of linear equations

$$\frac{da}{d\alpha_l} = \Big(\frac{dL_l(a)}{d\alpha_l}\Big)\Big|_{\alpha_l=0},\ l = \overline{1,4}, \tag{6}$$

or in coordinates

$$q^{\alpha_1}: \begin{cases} \frac{da_{ij}}{d\alpha_l} = \left(\frac{da^*_{ij}}{d\alpha_l}\right)\Big|_{\alpha_l=0} \equiv A^l_{ij}(a), \\ \frac{db_{ij}}{d\alpha_l} = \left(\frac{db^*_{ij}}{d\alpha_l}\right)\Big|_{\alpha_l=0} \equiv B^l_{ij}(a), \qquad l=\overline{1,4}. \\ i+j=\overline{0,n}; \end{cases} \tag{7}$$

In the cases $l=1$ and $l=4$ the matrix of coefficients of the system (7) is diagonal. Indeed, in these cases we have

$$\begin{aligned} A^1_{ij}(a) &= (1-i)a_{ij}, \quad B^1_{ij}(a) = -ib_{ij}, \\ A^4_{ij}(a) &= -ja_{ij}, \qquad B^4_{ij}(a) = (1-j)b_{ij}. \end{aligned} \tag{8}$$

Note that $(E,\{L_{(q^{\alpha^*_1})}\})$ and $(E,\{L_{(q^{\alpha^*_4})}\})$ are not flows.

Consider the systems

$$q^{\alpha^*_1}: \frac{da}{d\alpha^*_l} = \left(\frac{dL^*_l(a)}{d\alpha^*_l}\right)\Big|_{\alpha^*_l=1}, \quad l=1,4. \tag{9}$$

Remark 2. *The system* $((9), l=1)$ $(((9), l=4))$ *coincides with the system* $((6), l=1)$ $(((6), l=4))$.

The vector fields

$$V_l = \sum_{i+j=0}^{n} A^l_{ij}(a)\frac{\partial}{\partial a_{ij}} + B^l_{ij}(a)\frac{\partial}{\partial b_{ij}}, \quad l=\overline{1,4},$$

generate a Lie algebra. By [5], [7], [6] the dimension of orbit $O(a)$ is equal with the dimension of this algebra, i.e. with the rank of a matrix M composed from the coordinates of vectors V_l, $l=\overline{1,4}$.

3 The orbits of dimension zero

Consider the homogeneous system

$$\dot{x}_1 = P_k(x_1,x_2), \quad \dot{x}_2 = Q_k(x_1,x_2), \tag{10}$$

where $0 \le k \le n$ and P_k, Q_k are given in (2). For (10) we have the vector fields

$$W_l = \sum_{i+j=k} A^l_{ij}(a)\frac{\partial}{\partial a_{ij}} + B^l_{ij}(a)\frac{\partial}{\partial b_{ij}}, \ l=\overline{1,4}. \tag{11}$$

Denote by M_k the matrix of dimension $4\times(2k+2)$ composed from the coordinates of vectors (11). For example,

$$M_0 = \begin{pmatrix} a_{00} & 0 \\ b_{00} & 0 \\ 0 & a_{00} \\ 0 & b_{00} \end{pmatrix}, M_1 = \begin{pmatrix} 0 & a_{01} & -b_{10} & 0 \\ b_{10} & b_{01}-a_{10} & 0 & -b_{10} \\ -a_{01} & 0 & a_{10}-b_{01} & a_{01} \\ 0 & -a_{01} & b_{10} & 0 \end{pmatrix}. \tag{12}$$

We have $M = (M_0, M_1, \ldots, M_n)$ and therefore

$$rankM \geq rankM_k,\ \ k = \overline{0, n}. \tag{13}$$

Hence, the dimension of orbits of system (10) does not exceed the dimension of orbits of the corresponding system (1).

In the work [6], in each of the cases $k = 0, 1, 2, 3$ the systems (10) are classified in dependence of the dimension of orbits $O(a)$. So, it is shown that if $k = 0$, 2 or 3, then $dimO(a) = 0$ if and only if $P_k \equiv 0$, $Q_k \equiv 0$ and in the case $k = 1$ the dimension of $O(a)$ orbit is equal to zero if and only if the following conditions are satisfied

$$a_{10} - b_{01} = a_{01} = b_{10} = 0. \tag{14}$$

Lemma 1. *In the case $k \neq 1$ the dimension of $O(a)$ orbit of the system (10) is equal to zero if and only if $P_k \equiv 0$, $Q_k \equiv 0$.*

Proof. Assume $k \neq 1$. The orbit $O(a)$ of system (10) has the dimension zero if and only if a is at the same time a singular point for systems (7), $l = \overline{1, 4}$, i.e. $A^l_{ij}(a) = B^l_{ij}(a) = 0,\ \ \forall i + j = k,\ \ l = \overline{1, 4}$. From here, $j = k - i$ and (8) we have that

$$(1 - i)a_{i,k-i} = ib_{i,k-i} = 0,\ \ i = \overline{0, k}; \tag{15}$$

$$(k - i)a_{i,k-i} = (k - i - 1)b_{i,k-i} = 0, i = \overline{0, k}. \tag{16}$$

From (15) and $k \neq 1$ it follows that $a_{i,k-i} = 0,\ \ \forall i \neq 1$ and $b_{i,k-i} = 0,\ \ \forall i \neq 0$, but from (16) we also obtain that $a_{1,k-1} = b_{0k} = 0$. Therefore, $P_k \equiv 0$, $Q_k \equiv 0$. □

According to (13), Lemma 1 and (14) we have

Theorem 1. *The polynomial system (1) has the dimension of GL-orbit equal to zero if and only if it is of the form $\dot{x}_1 = bx_1,\ \ \dot{x}_2 = bx_2,\ \ b = const.$*

4 The absence of orbits of the dimension one

We consider system (10). In [6], it is shown that in the cases $k = 0, 1, 2, 3$, the orbits of system (10) have the dimensions not equal to one. We bring here our proof of this fact establishing simultaneously that every two-dimensional polynomial system possesses this property. By Theorem 1, we shall assume that $P_k \not\equiv 0$ or $Q_k \not\equiv 0$ and if $k = 1$, then $a_{10} \neq b_{01}$ or $|a_{01}| + |b_{10}| \neq 0$. From these conditions immediately follows that $rankM_0 = 2$ and $rankM_1 \geq 2$ (see(12)).

Next we consider $k \geq 2$ and $P_k \not\equiv 0$. Let, for example, $a_{\nu,k-\nu} \neq 0$, where ν is equal to one of the numbers $0, 1, 2, \ldots, k$. We will show that the matrix M_k has at least one non–zero minor of the second order. Let us assume the contrary, i.e. all the second order minors of M_k are equal to zero. For the beginning, we will examine the following minors constructed from the coordinates of vectors W_1 and W_4 (see(11),(8)):

$$\begin{aligned}\Delta^1_{\nu,i} &= \begin{vmatrix} (1-\nu)a_{\nu,k-\nu} & (1-i)a_{i,k-i} \\ (\nu-k)a_{\nu,k-\nu} & (i-k)a_{i,k-i} \end{vmatrix} = \\ &= (k-1)(\nu-i)a_{\nu,k-\nu}a_{i,k-i},\ i\neq\nu; \\ \Delta^2_{\nu,i} &= \begin{vmatrix} (1-\nu)a_{\nu,k-\nu} & -ib_{i,k-i} \\ (\nu-k)a_{\nu,k-\nu} & (1-k+i)b_{i,k-i} \end{vmatrix} = \\ &= (k-1)(\nu-i-1)a_{\nu,k-\nu}b_{i,k-i},\ i=\overline{0,k}.\end{aligned} \tag{17}$$

From $\Delta^1_{\nu,i}=0$ it follows that $a_{i,k-i}=0,\ \forall i\neq\nu$ and from $\Delta^2_{\nu,i}=0$ we have that $b_{i,k-i}=0,\ \forall i$ if $\nu=0$, and that $b_{i,k-i}=0,\ \forall i\neq\nu-1$ if $\nu\geq1$. Hence, the system (10) can have one of the forms

$$\dot{x}_1=a_{0,k}x_2^k,\quad \dot{x}_2=0,\quad a_{0,k}\neq0; \tag{18}$$

$$\dot{x}_1=a_{\nu,k-\nu}x_1^{\nu}x_2^{k-\nu},\quad \dot{x}_2=b_{\nu-1,k-\nu+1}x_1^{\nu-1}x_2^{k-\nu+1},\quad a_{\nu,k-\nu}\neq0. \tag{19}$$

For (18) we have $W_1=a_{0,k}\frac{\partial}{\partial a_{0,k}}$ and determine W_3. To this end we apply in (18) the transformation of coordinates q^{α_3}: $X_1=x_1,\ X_2=\alpha_3x_1+x_2$:

$$\dot{X}_1=\dot{x}_1=a_{0,k}x_2^k=a_{0,k}(X_2-\alpha_3X_1)^k=a_{0,k}X_2^k-k\alpha_3a_{0,k}X_1X_2^{k-1}+o(\alpha_3),$$

$$\dot{X}_2=\alpha_3\dot{x}_1+\dot{x}_2=\alpha_3a_{0,k}x_2^k=\alpha_3a_{0,k}(X_2-\alpha_3X_1)^k=\alpha_3a_{0,k}X_2^k+o(\alpha_3).$$

Hence, $W_3=-ka_{0,k}\frac{\partial}{\partial a_{1,k-1}}+a_{0,k}\frac{\partial}{\partial b_{0,k}}$ and the minor $\begin{vmatrix} a_{0,k} & 0 \\ 0 & a_{0,k}\end{vmatrix}\neq0$. The last inequality contradicts the assumption that all the second order minors of the matrix M_k are null.

We consider in (19) $\nu=1$. We have $W_4=(1-k)\Big(a_{1,k-1}\frac{\partial}{\partial a_{1,k-1}}+b_{0,k}\frac{\partial}{\partial b_{0,k}}\Big)$. Let us calculate W_3:

$$\dot{X}_1=\dot{x}_1=a_{1,k-1}x_1x_2^{k-1}=a_{1,k-1}X_1(X_2-\alpha_3X_1)^{k-1}=a_{1,k-1}X_1X_2^{k-1}+$$
$$+(1-k)\alpha_3a_{1,k-1}X_1^2X_2^{k-2}+o(\alpha_3),$$
$$\dot{X}_2=\alpha_3\dot{x}_1+\dot{x}_2=\alpha_3a_{1,k-1}x_1x_2^{k-1}+b_{0,k}x_2^k=\alpha_3a_{1,k-1}X_1(X_2-\alpha_3X_1)^{k-1}+$$
$$+b_{0,k}(X_2-\alpha_3X_1)^k=b_{0,k}X_2^k+\alpha_3(a_{1,k-1}-kb_{0,k})X_1X_2^{k-1}+o(\alpha_3).$$

Hence, $W_3=(1-k)a_{1,k-1}\frac{\partial}{\partial a_{2,k-2}}+(a_{1,k-1}-kb_{0,k})\frac{\partial}{\partial b_{1,k-1}}$ and

$\begin{vmatrix} (1-k)a_{1,k-1} & 0 \\ 0 & (1-k)a_{1,k-1}\end{vmatrix}\neq0$. We obtain contradiction.

Let us investigate now the case when in (19) $\nu\geq2$. We have

$$W_1=(1-\nu)a_{\nu,k-\nu}\frac{\partial}{\partial a_{\nu,k-\nu}}+(\nu-k-1)b_{\nu-1,k-\nu+1}\frac{\partial}{\partial b_{\nu-1,k-\nu+1}}. \tag{20}$$

Taking in (19) the transformation q^{α_2}: $X_1 = x_1 + \alpha_2 x_2$, $X_2 = x_2$ we obtain:

$$\dot{X}_1 = \dot{x}_1 + \alpha_2 \dot{x}_2 = a_{\nu,k-\nu} x_1^{\nu} x_2^{k-\nu} + \alpha_2 b_{\nu-1,k-\nu+1} x_1^{\nu-1} x_2^{k-\nu+1} =$$
$$= (X_1 - \alpha_2 X_2)^{\nu-1} X_2^{k-\nu} [a_{\nu,k-\nu} X_1 + \alpha_2 (b_{\nu-1,k-\nu+1} - a_{\nu,k-\nu}) X_2] =$$
$$= a_{\nu,k-\nu} X_1^{\nu} X_2^{k-\nu} + \alpha_2 (b_{\nu-1,k-\nu+1} - \nu a_{\nu,k-\nu}) X_1^{\nu-1} X_2^{k-\nu+1} + o(\alpha_2),$$
$$\dot{X}_2 = \dot{x}_2 = b_{\nu-1,k-\nu+1} x_1^{\nu-1} x_2^{k-\nu+1} = b_{\nu-1,k-\nu+1} (X_1 - \alpha_2 X_2)^{\nu-1} X_2^{k-\nu+1} =$$
$$= b_{\nu-1,k-\nu+1} X_1^{\nu-1} X_2^{k-\nu+1} + \alpha_2 (1-\nu) b_{\nu-1,k-\nu+1} X_1^{\nu-2} X_2^{k-\nu+2} + o(\alpha_2).$$

From here it follows that

$$W_2 = (b_{\nu-1,k-\nu+1} - \nu a_{\nu,k-\nu}) \frac{\partial}{\partial a_{\nu-1,k-\nu+1}} + (1-\nu) b_{\nu-1,k-\nu+1} \frac{\partial}{\partial b_{\nu-2,k-\nu+2}}.$$

Taking into account that $\nu \geq 2$ and that $a_{\nu,k-\nu} \neq 0$, the following two minors consisting of the coordinates of the vectors (20) and W_2:

$$\begin{vmatrix} (1-\nu) a_{\nu,k-\nu} & 0 \\ 0 & (1-\nu) b_{\nu-1,k-\nu+1} \end{vmatrix}, \quad \begin{vmatrix} (1-\nu) a_{\nu,k-\nu} & 0 \\ 0 & b_{\nu-1,k-\nu+1} - \nu a_{\nu,k-\nu} \end{vmatrix}$$

can not be equal to zero simultaneously.

Hence, we proved that when $P_k \not\equiv 0$ the dimension of every orbit of the system (10) can not be equal to one. The case $Q_k \not\equiv 0$ can be reduced to the case $P_k \not\equiv 0$ if we change in (10) the variables x_1 and x_2.

From Theorem 1, the inequality (13) and from what has been said above in this section, the following conclusion may be drawn

Theorem 2. *The dimension of GL-orbit of every polynomial system (1) is not equal to one.*

It is easy to check that the matrix M_1 from (12) can have the rank at most two. This fact, Theorems 1 and 2 lead to

Theorem 3. *The dimension of the GL-orbit of the linear system* $\dot{x}_1 = a_{10} x_1 + a_{01} x_2$, $\dot{x}_2 = b_{10} x_1 + b_{01} x_2$ *is equal to zero if and only if* $a_{10} - b_{01} = a_{01} = b_{10} = 0$ *and is two in other cases.*

Let us consider the system

$$\dot{x}_1 = a_{20} x_1^2 + a_{11} x_1 x_2 + a_{02} x_2^2, \quad \dot{x}_2 = b_{20} x_1^2 + b_{11} x_1 x_2 + b_{02} x_2^2.$$

Its matrix consists of the coordinates of vectors X_l, $\overline{1,4}$, and is of the form

$$M_2 = \begin{pmatrix} -a_{20} & 0 & a_{02} & -2b_{20} & -b_{11} & 0 \\ b_{20} & b_{11} - 2a_{20} & b_{02} - a_{11} & 0 & -2b_{20} & -b_{11} \\ -a_{11} & -2a_{02} & 0 & a_{20} - b_{11} & a_{11} - 2b_{02} & a_{02} \\ 0 & -a_{11} & -2a_{02} & b_{20} & 0 & -b_{02} \end{pmatrix}. \quad (21)$$

It is easy to see that for the system $\dot{x}_1 = 0$, $\dot{x}_2 = x_1x_2$ the rank of the matrix M_2 is equal to three, and for the system $\dot{x}_1 = x_2^2$, $\dot{x}_2 = x_1^2 + x_1x_2$ we have that $rank M_2 = 4$.

From here, Theorems 1, 2, 3 and the inequality (13), follows

Lemma 2. *If the right–hand sides of system (1) have at least one nonlinear term, then the dimension of the GL-orbit is equal to two, three or four.*

Next, this work is dedicated to the classification of systems (1) with a singular point $(0,0)$ with real and distinct eigenvalues λ_1 and λ_2, i.e.

$$\lambda_1, \lambda_2 \in R, \quad \lambda_1 \neq \lambda_2, \tag{22}$$

in dependence of the dimension of GL-orbits.

In this case $P_0 \equiv 0$, $Q_0 \equiv 0$ and according to [2] by transformation of coordinates $q \in GL(2,R)$, the system (1) can be brought to the form

$$\dot{x}_1 = \lambda_1 x_1 + \sum_{k=2}^{n} P_k(x_1,x_2), \quad \dot{x}_2 = \lambda_2 x_2 + \sum_{k=2}^{n} Q_k(x_1,x_2). \tag{23}$$

In (23) the notations (2) of the homogeneities P_k, Q_k, $k = \overline{2,n}$, were preserved. From (12) we have that for (23): $rank M_1 = 2$. From here and (13) it follows that the dimension of every GL-orbits of system (23) with conditions (22) can be equal to two, three or four.

5 The GL-orbits of system (23) of the dimension two

We consider the system

$$\dot{x}_1 = \lambda_1 x_1 + P_k(x_1,x_2), \quad \dot{x}_2 = \lambda_2 x_2 + Q_k(x_1,x_2), \tag{24}$$

where λ_1, λ_2 verify (22) and $2 \le k \le n$. In (24) the polynomials P_k, Q_k coincide with the polynomials P_k and Q_k, respectively, from (23). Evidently holds

Remark 3. *The dimension of every GL-orbit of system (23) is not smaller than the corresponding dimension of GL-orbit of system (24).*

From (12) and (8) we have that for (24) the matrix $M = (M_1, M_k)$ consisting of coordinates of vectors V_l, $l = \overline{1,4}$, after some elementary transformations takes the form

$$M \sim \begin{pmatrix} 0 & 0 & 0 & 0 & (1-k)a_{k,0} & (2-k)a_{k-1,1} & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & -a_{k-1,1} & \dots \end{pmatrix}$$

$$\begin{matrix} \dots & -b_{1,k-1} & 0 \\ \dots & 0 & 0 \\ \dots & 0 & 0 \\ \dots & (2-k)b_{1,k-1} & (1-k)b_{0,k} \end{matrix} \Bigg). \tag{25}$$

Consider the minors of the third order of the matrix (25):

$$\begin{vmatrix} 0 & 0 & (1-i)a_{ij} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 0 & -ib_{ij} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -ja_{ij} \end{vmatrix}, \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (1-j)b_{ij} \end{vmatrix},$$

$i+j=k$, we observe that they are simultaneously equal to zero if and only if $a_{ij}=b_{ij}=0,\ \forall i+j=k$. From here, Remark 3 and Theorem 3, follows

Lemma 3. *The dimension of the GL-orbit of system (23) with conditions (22) is equal to two if and only if* $P_k\equiv 0,\ \ Q_k\equiv 0,\ \ \forall k\geq 2$.

Next, taking into account this lemma and Remark 1, we obtain

Theorem 4. *Let the origin $O(0,0)$ be a singular point of (1) with real and distinct eigenvalues. Then the GL-orbit of system (1) has the dimension equal to two if and only if* $P_k\equiv 0,\ \ Q_k\equiv 0,\ \ \forall k\geq 2$.

6 The GL-orbits of system (23) of the dimension three

In this section we shall distinguish those systems of the form (23), (22) which have the dimension of the GL-orbit equal to three. Reasoning as above, we shall consider system (24). From (25) we have that $rankM = 2 + rank\tilde{M}_k$, where

$$\tilde{M}_k = \begin{pmatrix} (1-k)a_{k,0} & (2-k)a_{k-1,1} & \dots & -b_{1,k-1} & 0 \\ 0 & -a_{k-1,1} & \dots & (2-k)b_{1,k-1} & (1-k)b_{0k} \end{pmatrix}. \qquad (26)$$

The minors of the second order from (17) of the matrix $\tilde{M}_k$ are $\Delta^1_{\nu,i}$, $\Delta^2_{\nu,i}$ and

$$\Delta^3_{\nu,i} = \begin{vmatrix} -\nu b_{\nu,k-\nu} & -ib_{i,k-i} \\ (1+\nu-k)b_{\nu,k-\nu} & (1+i-k)b_{i,k-i} \end{vmatrix} = (k-1)(\nu-i)b_{\nu,k-\nu}b_{i,k-i}, i\neq\nu$$

(see (8)). If $a_{0,k}\neq 0$ ($b_{0,k}\neq 0$), then from $\Delta^1_{0,i}=0$, $i=\overline{1,k}$ ($\Delta^3_{k,i}=0$, $i=\overline{0,k-1}$) it follows that $a_{i,k-i}=0$ ($b_{i,k-i}=0$), and from $\Delta^2_{0,i}=0$ ($\Delta^2_{i,k}=0$), $i=\overline{0,k}$, we have that ($b_{i,k-i}=0$) ($a_{i,k-i}=0$). In these cases the system (24) looks as

$$S_n(k:1):\ \ \dot{x}_1=\lambda_1x_1+a_{0,k}x_2^k,\quad \dot{x}_2=\lambda_2x_2,\quad a_{0,k}\neq 0\quad (k\geq 2); \qquad (27)$$

$$S_n(1:k):\ \ \dot{x}_1=\lambda_1x_1,\quad \dot{x}_2=\lambda_2x_2+b_{k,0}x_1^k,\quad b_{k,0}\neq 0\quad (k\geq 2). \qquad (28)$$

We suppose now that $a_{\nu,k-\nu}\neq 0$ ($b_{\nu-1,k-\nu+1}\neq 0$) for a certain $\nu\in\{1,2,...,k\}$. From $\Delta^1_{\nu,i}=0$ ($\Delta^3_{\nu,i}=0$), $i\neq\nu$, and $\Delta^2_{\nu,i}=0$ ($\Delta^2_{i,\nu}=0$), $i\neq\nu-1$, it results that $a_{i,k-i}=0,\ \forall i\neq\nu$, and $b_{i,k-i}=0,\ \forall i\neq\nu-1$. These cases lead us to the systems

$$\begin{cases} \dot{x}_1 = x_1\Big(\lambda_1 + a_{\nu,k-\nu}x_1^{\nu-1}x_2^{k-\nu}\Big), \\ \dot{x}_2 = x_2\Big(\lambda_2 + b_{\nu-1,k-\nu+1}x_1^{\nu-1}x_2^{k-\nu}\Big), \qquad \nu=\overline{1,k}. \\ |a_{\nu,k-\nu}|+|b_{\nu-1,k-\nu+1}|\neq 0; \end{cases} \qquad (29)$$

Hence, is proved

Lemma 4. *The GL-orbit of system (24) has the dimension equal to three if and only if it has one of the forms (27)–(29).*

In passing, we will examine the system (23). As usual, by M we will denote the matrix consisting of coordinates of the vectors V_l, $j=\overline{1,4}$, corresponding to system (23), and by $\tilde{M}$ the matrix $(\tilde{M}_2, \tilde{M}_3, \ldots, \tilde{M}_n)$, where $\tilde{M}_k$, $k=\overline{2,n}$, are given in (26). Evidently,

$$rank M = 2 + rank\tilde{M} \geq 2 + rank\tilde{M}_k,\ k=\overline{2,n}. \tag{30}$$

If $rankM = 3$, then from (30) it follows that there exist $k : 2 \leq k \leq n$ such that $rank\tilde{M}_k = 1$. Hence

$$|P_k(x_1,x_2)| + |Q_k(x_1,x_2)| \not\equiv 0. \tag{31}$$

In the case if $P_j \equiv 0,\ \ Q_j \equiv 0,\ \ \forall j \neq k,\ \ 2 \leq j \leq n$, apply Lemma 4. Suppose that together with homogeneities of order k, the right–hand sides of system (23) contain also and homogeneities of other order, for example, of order l, where $l \neq k$, $2 \leq l \leq n$. Hence

$$|P_l(x_1,x_2)| + |Q_l(x_1,x_2)| \not\equiv 0. \tag{32}$$

The condition $rank\tilde{M}_k = rank\tilde{M}_l = 1$ implies that both P_k, Q_k and P_l, Q_l have the form like the right-hand sides of one of systems (27)–(29). In the case P_l, Q_l in (27)–(29) we substitute l for k.

Let $P_k = a_{0,k}x_2^k$, $a_{0,k} \neq 0$ and $Q_k \equiv 0$. The following minors of the matrix $\tilde{M}$:

$$\begin{vmatrix} a_{0,k} & (1-\mu)a_{\mu,l-\mu} \\ -ka_{0,k} & (\mu-l)a_{\mu,l-\mu} \end{vmatrix} = [1-l+(1-\mu)(k-1)]a_{0,k}a_{\mu,l-\mu},$$

$$\begin{vmatrix} a_{0,k} & -\mu b_{\mu,l-\mu} \\ -ka_{0,k} & (1+\mu-l)b_{\mu,l-\mu} \end{vmatrix} = [1-l+\mu(k-1)]a_{0,k}b_{\mu,l-\mu},$$

$0 \leq \mu \leq l$, are simultaneously equal to zero if and only if $a_{\mu,l-\mu} = b_{\mu,l-\mu} = 0$, $\mu = \overline{0,l}$, that is when $P_l \equiv 0,\ Q_l \equiv 0$, contradicting to (32).

Similarly, through examination of the minors

$$\begin{vmatrix} -kb_{k,0} & (1-\mu)a_{\mu,l-\mu} \\ b_{k,0} & (\mu-l)a_{\mu,l-\mu} \end{vmatrix}, \quad \begin{vmatrix} -kb_{k,0} & -\mu b_{\mu,l-\mu} \\ b_{k,0} & (1+\mu-l)b_{\mu,l-\mu} \end{vmatrix},$$

it is shown that the case $P_k \equiv 0,\ Q_k = b_{k,0}x_1^k,\ b_{k,0} \neq 0$ is not realized in the condition (32).

Taking into account Lemmas 3, 4 and the conditions (31), (32), it remains to investigate the case when

$$P_k = a_{\nu,k-\nu}x_1^{\nu}x_2^{k-\nu},\ Q_k = b_{\nu-1,k-\nu+1}x_1^{\nu-1}x_2^{k-\nu+1},\ P_l = a_{\mu,l-\mu}x_1^{\mu}x_2^{l-\mu},$$

$$Q_l = b_{\mu-1,l-\mu+1}x_1^{\mu-1}x_2^{l-\mu+1},\ 1 \leq \nu \leq k,\ 1 \leq \mu \leq l.$$

We consider the minors:

$$\Omega^1_{\nu,\mu}=\begin{vmatrix}(1-\nu)a_{\nu,k-\nu} & (1-\mu)a_{\mu,l-\mu}\\ (\nu-k)a_{\nu,k-\nu} & (\mu-l)a_{\mu,l-\mu}\end{vmatrix}=\omega_{\nu,\mu}a_{\nu,k-\nu}a_{\mu,l-\mu},$$

$$\Omega^2_{\nu,\mu}=\begin{vmatrix}(1-\nu)a_{\nu,k-\nu} & (1-\mu)b_{\mu-1,l-\mu+1}\\ (\nu-k)a_{\nu,k-\nu} & (\mu-l)b_{\mu-1,l-\mu+1}\end{vmatrix}=\omega_{\nu,\mu}a_{\nu,k-\nu}b_{\mu-1,l-\mu+1},$$

$$\Omega^3_{\nu,\mu}=\begin{vmatrix}(1-\nu)b_{\nu-1,k-\nu+1} & (1-\mu)b_{\mu-1,l-\mu+1}\\ (\nu-k)b_{\nu-1,k-\nu+1} & (\mu-l)b_{\mu-1,l-\mu+1}\end{vmatrix}=\omega_{\nu,\mu}b_{\nu-1,k-\nu+1}b_{\mu-1,l-\mu+1},$$

where $\omega_{\nu,\mu}=(\nu-1)(l-1)-(\mu-1)(k-1)$, $1\le\nu\le k$, and $1\le\mu\le l$. Evidently, $\omega_{1,1}=\omega_{k,l}=0$.

If $\nu=1$ $(\nu=k)$, then from (31) and (32) it follows that the equalities $\Omega^1_{1,\mu}=\Omega^2_{1,\mu}=\Omega^3_{1,\mu}=0$ hold if and only if $\mu=1$ $(\mu=l)$. Hence, the dimension of the GL-orbit of each of the systems

$$S_n(\lambda_1:0):\ \begin{cases}\dot{x}_1=x_1\Big(\lambda_1+\sum_{j=1}^{n-1}a_{1,j}x_2^j\Big),\\ \dot{x}_2=x_2\Big(\lambda_2+\sum_{j=1}^{n-1}b_{0,j+1}x_2^j\Big),\\ \sum_{j=1}^{n-1}|a_{1,j}|+|b_{0,j+1}|\neq0;\end{cases}\tag{33}$$

$$S_n(0:\lambda_2):\ \begin{cases}\dot{x}_1=x_1\Big(\lambda_1+\sum_{j=1}^{n-1}a_{j+1,0}x_1^j\Big),\\ \dot{x}_2=x_2\Big(\lambda_2+\sum_{j=1}^{n-1}b_{j,1}x_1^j\Big),\\ \sum_{j=1}^{n-1}|a_{j+1,0}|+|b_{j,1}|\neq0,\end{cases}\tag{34}$$

is equal to three.

Next, suppose that $2\le\nu\le k-1$, $2\le\mu\le l-1$. From (31), (32) and $\Omega^j_{\nu,\mu}=0$, $j=\overline{1,3}$, it follows that $\omega_{\nu,\mu}=0$, Therefore, we have that $\frac{l-1}{\mu-1}=\frac{k-1}{\nu-1}>1$. Hence, there exist integer positive numbers p,q,i,j such that

$$(p,q)=1,\ k=(p+q)i+1,\ \nu=qi+1,\ l=(p+q)j+1,\ \mu=qj+1.$$

Hence, for any natural reciprocal prim numbers p and q, the system

$$S_n(p:-q):\ \begin{cases}\dot{x}_1=x_1\left[\lambda_1+\sum\limits_{i=1}^{n^*}a_{qi+1,pi}\Big(x_1^qx_2^p\Big)^i\right],\\ \dot{x}_2=x_2\left[\lambda_2+\sum\limits_{i=1}^{n^*}b_{qi,pi+1}\Big(x_1^qx_2^p\Big)^i\right],\\ \sum\limits_{i=1}^{n^*}|a_{qi+1,pi}|+|b_{qi,pi+1}|\neq0,\quad(p,q)=1,\end{cases}\tag{35}$$

where $n^*=\left[\frac{n-1}{p+q}\right]$, has the dimension of the GL-orbit equal to three.

Hence, is proved

Theorem 5. *The dimension of the GL-orbit of system (23) with the conditions (22) is equal to three if and only if it has one of the following forms (27), (28), (33), (34) or (35).*

Corollary 1. *The cubic system* ($n = 3$) *of the form (22), (23) has the dimension of the GL-orbit equal to three if and only if it has one of the forms* $S_3(2:1)$, $S_3(3:1)$, $S_3(1:2)$, $S_3(1:3)$, $S_3(\lambda_1:0)$, $S_3(0:\lambda_2)$, $S_3(1:-1)$, *that is*

$$\dot{x}_1 = \lambda_1 x_1 + a_{02}x_2^2, \quad \dot{x}_2 = \lambda_2 x_2, \quad a_{02} \neq 0; \tag{36}$$

$$\dot{x}_1 = \lambda_1 x_1 + a_{03}x_2^3, \quad \dot{x}_2 = \lambda_2 x_2, \quad a_{03} \neq 0; \tag{37}$$

$$\dot{x}_1 = \lambda_1 x_1, \quad \dot{x}_2 = \lambda_2 x_2 + b_{20}x_1^2, \quad b_{20} \neq 0; \tag{38}$$

$$\dot{x}_1 = \lambda_1 x_1, \quad \dot{x}_2 = \lambda_2 x_2 + b_{30}x_1^3, \quad b_{30} \neq 0; \tag{39}$$

$$\begin{cases} \dot{x}_1 = x_1\Big(\lambda_1 + a_{11}x_2 + a_{12}x_2^2\Big), \\ \dot{x}_2 = x_2\Big(\lambda_2 + b_{02}x_2 + b_{03}x_2^2\Big), \ |a_{11}| + |a_{12}| + |b_{02}| + |b_{03}| \neq 0; \end{cases} \tag{40}$$

$$\begin{cases} \dot{x}_1 = x_1\Big(\lambda_1 + a_{20}x_1 + a_{30}x_1^2\Big), \\ \dot{x}_2 = x_2\Big(\lambda_2 + b_{11}x_1 + b_{21}x_1^2\Big), \ |a_{20}| + |a_{30}| + |b_{11}| + |b_{21}| \neq 0; \end{cases} \tag{41}$$

$$\dot{x}_1 = x_1\Big(\lambda_1 + a_{21}x_1x_2\Big), \dot{x}_2 = x_2\Big(\lambda_2 + b_{12}x_1x_2\Big), |a_{21}| + |b_{12}| \neq 0. \tag{42}$$

The assertion of Corollary 1 can be obtained and by direct method, that is if we equate to zero all the minors of the order four of the matrix $M = (M_1, M_2, M_3)$ with condition that at least one of the minors of the order three is not equal to zero. Here, M_1 coincides with the matrix M_1 from (12) if in the last matrix we put $a_{01} = b_{10} = 0$, $a_{10} = \lambda_1$, $b_{01} = \lambda_2$; the matrix M_2 is given in (21) and

$$M_3 = \left(\begin{array}{cccc} -2a_{30} & -a_{21} & 0 & a_{03} \\ b_{30} & b_{21} - 3a_{30} & b_{12} - 2a_{21} & b_{03} - a_{12} \\ -a_{21} & -2a_{12} & -3a_{03} & 0 \\ 0 & -a_{21} & -2a_{12} & -3a_{03} \end{array} \right.$$

$$\left. \begin{array}{cccc} -3b_{30} & -2b_{21} & -b_{12} & 0 \\ 0 & -3b_{30} & -2b_{21} & -b_{12} \\ a_{30} - b_{21} & a_{21} - 2b_{12} & a_{12} - 3b_{03} & a_{03} \\ b_{30} & 0 & -b_{12} & -2b_{03} \end{array} \right).$$

7 The resonance

By $\varphi(x_1, x_2)$ and $\psi(x_1, x_2)$ we shall denote, respectively, the nonlinearities from the right–hand side of each equation of system (23), i.e.

$$\varphi(x_1, x_2) = \sum_{k=2}^{n} P_k(x_1, x_2), \quad \psi(x_1, x_2) = \sum_{k=2}^{n} Q_k(x_1, x_2), \tag{43}$$

where the polynomials P_k and Q_k, $k = \overline{2, n}$, are shown in (2).

Let λ_1 and λ_2 be two real and distinct numbers. If there exist integer nonnegative numbers $m_1, m_2;\ m_1 + m_2 \geq 2$ $(n_1, n_2;\ n_1 + n_2 \geq 2)$ such that

$$\lambda_1 = m_1\lambda_1 + m_2\lambda_2 \tag{44}$$

or

$$\lambda_2 = n_1\lambda_1 + n_2\lambda_2, \tag{45}$$

then the couple of numbers (λ_1, λ_2) is called *resonant*.

Taking into account (44) ((45)), we say that $a_{m_1,m_2}x_1^{m_1}x_2^{m_2}$ $(b_{n_1,n_2}x_1^{n_1}x_2^{n_2})$ is *a resonant term* of the polynomial $\varphi(x_1, x_2)$ $(\psi(x_1, x_2))$ corresponding to the resonant couple (λ_1, λ_2).

A *couple* of polynomials (φ, ψ) is call *resonant* if they contain only resonant terms corresponding to the same resonant couple of the numbers (λ_1, λ_2), considering $\psi \equiv 0$ $(\varphi \equiv 0)$ if λ_1 and λ_2 verify (44) ((45)) and do not verify (45) ((44)) for any integer numbers $n_1, n_2 \geq 0$, $n_1 + n_2 \geq 2$ $(m_1, m_2 \geq 0, m_1 + m_2 \geq 2)$.

In passing, in this section, we will describe a couple of resonant polynomials. Suppose that (λ_1, λ_2) is a resonant couple. We will distinguish the following four possible cases: 1) $\lambda_1 \cdot \lambda_2 > 0,\ \lambda_1 \neq \lambda_2$; 2) $\lambda_1 \neq 0,\ \lambda_2 = 0$; 3) $\lambda_1 = 0,\ \lambda_2 \neq 0$ and 4) $\lambda_1 \cdot \lambda_2 < 0$.

1) $\lambda_1 \cdot \lambda_2 > 0$, $\lambda_1 \neq \lambda_2$. In this case the equalities (44) and (45) do not hold simultaneously. If we consider the equality (44), then it looks as:

$$\lambda_1 = 0 \cdot \lambda_1 + k \cdot \lambda_2, \tag{46}$$

where k is one of the numbers $2, 3, \ldots$. To the couple (λ_1, λ_2) which verifies (46) the resonant couple of polynomials

$$\varphi(x_1, x_2) = a_{0,k}x_2^k, \quad \psi(x_1, x_2) \equiv 0$$

corresponds.

Similarly, if we have the equality (45), then it looks as: $\lambda_2 = k \cdot \lambda_1 + 0 \cdot \lambda_2$ and leads to the resonant couple of polynomials

$$\varphi(x_1, x_2) \equiv 0, \quad \psi(x_1, x_2) = b_{k,0}x_1^k.$$

2) $\lambda_1 \neq 0$, $\lambda_2 = 0$. In these condition the relation (44) holds for $m_1 = 1$ and any $m_2 \in \{1, 2, 3, \ldots\}$ and the relation (45) holds for $n_1 = 0$ and $n_2 \in \{2, 3, \ldots\}$. To the resonant couple (λ_1, λ_2) the couple of resonant polynomials

$$\varphi(x_1, x_2) = x_1 \sum_{j=1}^{n-1} a_{1,j} x_2^j, \quad \psi(x_1, x_2) = x_2 \sum_{j=1}^{n-1} b_{0,j+1} x_2^j$$

corresponds.

3) $\lambda_1 = 0$, $\lambda_2 \neq 0$. The equality (44) holds for $m_1 \in \{2, 3, \ldots\}$ and $m_2 = 0$, and (45) for $n_1 \in \{1, 2, 3, \ldots\}$ and $n_2 = 1$. Hence, we come to the resonant couple of polynomials

$$\varphi(x_1, x_2) = x_1 \sum_{j=1}^{n-1} a_{j+1,0} x_1^j, \quad \psi(x_1, x_2) = x_2 \sum_{j=1}^{n-1} b_{j,1} x_1^j.$$

4) $\lambda_1 \cdot \lambda_2 < 0$. Every of the relations (44) and (45) can hold only in the case when λ_1/λ_2 is a rational number. Let $\lambda_1 : \lambda_2 = p : (-q)$, where p and q are integer positive reciprocal prime numbers, i.e. $(p, q) = 1$. Denote by n^* the integer part of the number $(n-1)/(p+q)$. In this case, the equality (44) holds for $m_1 = qi + 1$, $m_2 = pi$, and (45) for $n_1 = qi$, $n_2 = pi + 1$, $i = \overline{1, n^*}$. The resonant couple of polynomials (φ, ψ) corresponding to (λ_1, λ_2) is

$$\varphi(x_1, x_2) = x_1 \sum_{i=1}^{n^*} a_{qi+1,pi} \left(x_1^q x_2^p\right)^i, \quad \psi(x_1, x_2) = x_2 \sum_{i=1}^{n^*} b_{qi,pi+1} \left(x_1^q x_2^p\right)^i.$$

From what have been said above and Theorem 5, follows

Theorem 6. *The dimension of GL-orbit of system (23) with conditions (22) is equal to three if and only if the polynomials φ and ψ from (43) are not simultaneously equal to zero and the pair (φ, ψ) is resonant.*

Taking into account Theorems 1, 2, 4 and 6, we obtain the following characteristic of systems (23) with the dimension of orbit equal to four:

Theorem 7. *The dimension of GL-orbit of system (23) with the conditions (22) is equal to four if and only if $|\varphi(x_1, x_2)| + |\psi(x_1, x_2)| \not\equiv 0$ and the pair of polynomials (φ, ψ) is not resonant.*

8 The integrability on the GL-orbits of the dimension three of system (23)

We consider the polynomial system

$$\dot{x}_1 = P(x_1, x_2), \quad \dot{x}_2 = Q(x_1, x_2). \tag{47}$$

Let $n = max\{degP, degQ\}$ and $D = P\partial/\partial x_1 + Q\partial/\partial x_2$. A curve $f(x_1, x_2) = 0$, $f \in C[x_1, x_2]$, (an expression $f = exp[h(x_1, x_2)/g(x_1, x_2)]$, where $h, g \in C[x_1, x_2]$), is called an *algebraic invariant curve* (an *exponential invariant curve*) for (47) if there exists a polynomial $K \in C[x_1, x_2]$ of the order at most $n-1$ such that the following identity $D(f) \equiv f \cdot K$ holds. The polynomial $K(x_1, x_2)$ is called *the cofactor* of the invariant curve f. By [4], if $f = exp(h/g)$ is an exponential invariant curve for a system (47), then $g(x_1, x_2) = 0$ is an algebraic invariant curve for the same system.

Let $f_1, \dots, f_s$ be a collection of algebraic invariant curves and exponential invariant curves of system (47) and, respectively, $K_1, \dots, K_s$ their cofactors. If there exist such numbers $\beta_1, \beta_2, \dots, \beta_s \in C$ that $F \equiv f_1^{\beta_1} f_2^{\beta_2} \dots f_s^{\beta_s} = const$ ($\mu = f_1^{\beta_1} f_2^{\beta_2} \dots f_s^{\beta_s}$) is a first integral (an integrating factor) for (47), that is $D(F) \equiv 0$ ($D(\mu) + \mu(P'_{x_1} + Q'_{x_2}) \equiv 0$), then we say that the system of differential equations (47) is *Darboux integrable in the generalized sense.* If among $f_1, \dots, f_s$ there are not an exponential invariant curve, then we shall speak on Darboux integrability of (47).

It easy to show that F (μ) is a first integral (an integrating factor) of the Darboux type for (47) if and only if the following identity

$$\sum_{i=1}^{s} \beta_i K_i(x_1, x_2) \equiv 0 \quad \left(\sum_{i=1}^{s} \beta_i K_i(x_1, x_2) \equiv -(P'_{x_1} + Q'_{x_2}) \right)$$

is verified.

Next, we will examine on integrability the systems of the form (23), (22) which have the dimension of GL-orbit equal to three, i.e. systems (27), (28), (33)–(35). Because the system (28) ((34)) can be reduced to the system (27) ((33)) by a substitution $x_1 \to x_2$, $x_2 \to x_1$, we shall consider only the problem of integrability of systems (27), (33) and (35).

By [3], the systems of normal form are integrable in quadratures. The aim of this section is to show that the given systems are Darboux integrable in the generalized sense.

The system (27). a) Let $\lambda_1 \neq k\lambda_2$. It is easy to check that the curves $f_1 = x_2$ and $f_2 = (\lambda_1 - k\lambda_2)x_1 + a_{0,k}x_2^k$ are algebraic invariant curves for (27) and have the cofactors $K_1(x_1, x_2) = \lambda_2$ and $K_2(x_1, x_2) = \lambda_1$, respectively. Evidently, the identity $\beta_1 \cdot K_1 + \beta_2 \cdot K_2 \equiv 0$ holds for $\beta_1 = \lambda_1$, $\beta_2 = -\lambda_2$ and therefore $F = f_1^{\lambda_1} f_2^{-\lambda_2}$ is a first integral of system (27).

b) $\lambda_1 = k\lambda_2$. In this case besides the invariant curve $f_1 = x_2$ with $K_1 = \lambda_2$, we have also an exponential invariant curve $f_2 = exp(x_1/x_2^k)$ with $K_2 = a_{0,k}$. The first integral is $F = f_1^{a_{0,k}} f_2^{-\lambda_2}$.

The system (33). Let

$$\tilde{\varphi} = \lambda_1 + \sum_{j=1}^{n-1} a_{1,j} x_2^j, \quad \tilde{\psi} = x_2 \left(\lambda_2 + \sum_{j=1}^{n-1} b_{0,j+1} x_2^j \right).$$

If $\tilde{\varphi} \equiv 0$ ($\tilde{\psi} \equiv 0$), then $F = x_1$ ($F = x_2$) is a first integral of (33) and if $\tilde{\psi} \not\equiv 0$, this integral looks

$$F = x_1 exp[-\int (\tilde{\varphi}/\tilde{\psi}) dx_2].$$

Let $\tilde{\varphi} \not\equiv 0$, $\tilde{\psi} \not\equiv 0$, $r = deg\tilde{\psi}$, $s = max\{0, deg\tilde{\psi} - deg\tilde{\varphi} + 1\}$, $\tilde{\psi} = b_{0,r}(x_2 - b_1)^{r_1} \dots (x_2 - b_m)^{r_m}$, where $b_1 = 0$, $b_j \in C \setminus \{0\}$, $j = \overline{2,m}$, $r_1 + \dots + r_m = r$. For system (33) $f_0 \equiv x_1 = 0$, $f_i \equiv x_2 - b_i = 0$, $i = \overline{1,m}$, are invariant lines, and

$$f_{m+1} = exp\frac{1}{x_2 - b_1}, \dots, f_{m+r_1-1} = exp\frac{1}{(x_2 - b_1)^{r_1-1}}, \dots,$$

$$f_r = exp\frac{1}{(x_2 - b_m)^{r_m-1}},\ f_{r+1} = exp(x_2), \dots, f_{r+s} = exp(x_2^s)$$

are exponential invariant curves. Because

$$\int \frac{\tilde{\varphi}}{\tilde{\psi}} dx_2 = -\left[\beta_1 ln\,|x_2 - b_1| + \dots + \beta_m ln\,|x_2 - b_m| + \frac{\beta_{m+1}}{x_2 - b_1} + \dots \right.$$

$$\left. + \frac{\beta_r}{(x_2 - b_m)^{r_m-1}} + \beta_{r+1}x_2 + \dots + \beta_{r+s}x_2^s\right],$$

the integral F of (33) can be written in the Darboux form: $F = \prod\limits_{i=0}^{r+s} f_i^{\beta_i}$.

In the investigated case it is more easy to find an integrating factor which looks $\mu = 1/(x_1\tilde{\psi})$.

The system (35). Because p and q are reciprocal prime numbers, for them such integer positive numbers u and v can be found that $pu - qv = 1$. The transformation $z_1 = x_1^u x_2^v$, $z_2 = x_1^q x_2^p$ [3] reduces (35) to a system similar with (33):

$$\dot{z}_1 = z_1\left[u\lambda_1 + v\lambda_2 + \sum_{i=1}^{n^*}(ua_{qi+1,pi} + vb_{qi,pi+1})z_2^i\right],$$

$$\dot{z}_2 = z_2\left[q\lambda_1 + p\lambda_2 + \sum_{i=1}^{n^*}(qa_{qi+1,pi} + pb_{qi,pi+1})z_2^i\right].$$

Thus, we shall integrate directly system (35). If

$$\lambda_1 : \lambda_2 = a_{qi+1,pi} : b_{qi,pi+1} = -p : q, \quad i = \overline{1,n^*}, \tag{48}$$

then the right–hand sides of (35) have a common factor $\lambda_1 + \sum\limits_{i=1}^{n^*} a_{qi+1,pi}(x_1^q x_2^p)^i$. After their cancelation by this factor, we obtain the system $\dot{x}_1 = x_1$, $\dot{x}_2 = \frac{\lambda_2}{\lambda_1}x_2$ which has a general integral $x_1^{\lambda_2}x_2^{-\lambda_1} = const$. In the case when (48) is not satisfied we have an integrating factor

$$\mu = \left[x_1x_2\Big(q\lambda_1 + p\lambda_2 + \sum_{i=1}^{n^*}(qa_{qi+1,pi} + pb_{qi,pi+1})(x_1^q x_2^p)^i\Big)\right]^{-1}.$$

From what has been said above, follows

Theorem 8. *On GL-orbits of dimension three the system (23) with the conditions (22) has a generalized Darboux first integral (a Darboux integrating factor).*

In the case of cubic systems (36) and (37) we have the first integrals

$$x_2^{\lambda_1}\left[(\lambda_1 - j\lambda_2)x_1 + a_{0,j}x_2^j\right]^{-\lambda_2} \quad if \quad \lambda_1 \neq j\lambda_2,$$

and

$$x_2^{a_{0,j}} exp(-\lambda_2 x_1/x_2^j) \quad if \quad \lambda_1 = j\lambda_2, \quad j = \overline{2,3}.$$

The system (40) has a first integral $x_2 = c$ if $\lambda_2 = b_{02} = b_{03} = 0$ and an integrating factor $\mu = \left[x_1x_2(\lambda_2 + b_{02}x_2 + b_{03}x_2^2)\right]^{-1}$ if $|\lambda_2| + |b_{02}| + |b_{03}| \neq 0$. At the same time, the system (42) has a first integral $x_1^{\lambda_2}x_2^{-\lambda_1} = const$ if $\lambda_1 + \lambda_2 = a_{21} + b_{12} = 0$ and an integrating factor $\mu = [x_1x_2(\lambda_1 + \lambda_2 + (a_{21} + b_{12})x_1x_2)]^{-1}$ in other cases. The cubic systems (38), (39) and (41) can be reduced to the systems investigated above by substitution $x_1 \to x_2$, $x_2 \to x_1$.

References

[1] Alekseev V.G. *The theory of rational binary forms.* Yuriev, 1899 (in Russian).

[2] Bautin N.N., Leontovich E.A. *Methods and ways (examples) of the qualitative analysis of dynamical systems in a plane.* Moscow, Nauka, 1990 (in Russian).

[3] Bruno A.D. *Local method of nonlinear analysis of differential equations.* Moscow, Nauka, 1979 (in Russian).

[4] Christopher C., Llibre J. *Algebraic aspects of integrability for polynomial systems.* Qualitative theory of dynamical systems, 1999, **1**, N 1, p. 71–95.

[5] Ovsyanikov L.V. *Group analysis of differential equations.* Moscow, Nauka, 1978 (English transl. by Academic press, 1982.)

[6] Popa M.N. *Applications of algebras to differential systems.* Academy of Sciences of Moldova, Chişinău, 2001 (in Russian).

[7] Winberg E.B., Popov V.L. *The theory of invariants.* The results of Science and Technics. Ser. "The Contemporary Problems in Mathematics", vol. 55. Moscow, 1989, p. 137–313 (in Russian).

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Asymptotic Stability of autonomous and Non-Autonomous Discrete Linear Inclusions

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Abstract. The article is devoted to the study of absolute asymptotic stability of discrete linear inclusions (both autonomous and non-autonomous) in Banach space. We establish the relation between absolute asymptotic stability, uniform asymptotic stability and uniform exponential stability. It is proved that for compact (completely continuous) discrete linear inclusions these notions of stability are equivalent. We study this problem in the framework of non-autonomous dynamical systems (cocyles).

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1 Introduction

The aim of this paper is studying the problem of absolute asymptotic stability of the discrete linear inclusion (see, for example,[2, 18] and the references therein)

$$x_{t+1} \in F(x_t), \tag{1}$$

where $F(x) = \{A_1x, A_2x, ..., A_mx\}$ for all $x \in E$ (E is a Banach space) and A_i ($1 \le i \le m$) is a linear bounded operator acting on E.

The problem of asymptotic stability for the discrete linear inclusion arises in a number of different areas of mathematics: control theory – Molchanov [23]; linear algebra – Artzrouni [1], Beyn and Elsner [3], Bru, Elsner and Neumann [5], Daubechies and Lagarias [12], Elsner and Friedland [13], Elsner, Koltracht and Neumann [14], Gurvits [18], Vladimirov, Elsner and Beyn [31], Wirth [33, 34]; Markov Chains – Gurvits [15], Gurvits and Zaharin [16, 17]; iteration process – Bru, Elsner and Neumann [5], Opoitsev [24] and see also the bibliography therein.

Along with inclusion (1) we consider also the more general inclusions (non-autonomous case)

$$x_{t+1} \in F(t, x_t), \tag{2}$$

with $F(t, x) := \{A_1(t)x, A_2(t)x, ..., A_m(t)x\}$ and the operator-functions $A_i : \mathbb{Z}_+ \to [E]$ ($[E]$ is the space of all linear bounded operators $A : E \to E$).

We establish the relation between absolute asymptotic stability, uniform asymptotic stability and uniform exponential stability. It is proved that for compact (completely continuous) discrete linear inclusions these notions of stability are equivalent.

We study this problem in the framework of non-autonomous dynamical systems (cocycles). We show that the problem of absolute asymptotic stability for the discrete linear inclusions is related with the compact global attractors of non-autonomous dynamical systems (both ordinary dynamical systems (with uniqueness) and set-valued dynamical systems). We plan to continue the studying of discrete inclusions (both linear and nonlinear) in the framework of non-autonomous dynamical systems. In our future publications we will give the proofs of the followings results:

(i) finite-dimensional discrete linear inclusion, defined by matrices $\{A_1, A_2, ..., A_m\}$, is absolutely asymptotically stable if it does not admit nontrivial bounded full trajectories and at least one of the matrices $\{A_1, A_2, ..., A_m\}$ is asymptotically stable;

(ii) discrete inclusion, defined by nonlinear (in particular, affine) contractive mappings $\{A_1, A_2, ..., A_m\}$ admits a compact global chaotic attractor,

amongst others. We consider that this method of studying discrete inclusions (both linear and nonlinear) is fruitful and it permits to obtain the new and nontrivial results.

This paper is organized as follows.

In Section 2 we give a new approach to the study of discrete linear inclusions (DLI) which is based on the non-autonomous dynamical systems (cocycles). The main result of this section is Theorem 2.6 which gives conditions for the asymptotic stability for finite-dimensional DLI.

In Section 3 we introduce the shift dynamical system on the space of continuous set-valued functions, set-valued cocycles and set-valued non-autonomous dynamical systems. They play a very important role in the study of of discrete linear inclusions. We show that every discrete linear inclusion generates a cocycle (Example 3.2).

Section 4 is dedicated to the study of non-autonomous discrete linear inclusions (Example 4.1). The main result of this section is Theorem 4.12 which establishes the equivalence between absolute asymptotic stability and uniform exponential stabilty for the compact (completely continuous) non-autonomous discrete linear inclusions on the arbitrary Banach space.

2 Autonomous discrete linear inclusions and cocycles

Let E be a real or complex Banach space, $\mathbb{S}$ be a group of real ($\mathbb{R}$) or integer ($\mathbb{Z}$) numbers, $\mathbb{T}$ ($\mathbb{S}_+ \subseteq \mathbb{T}$) be a semigroup of additive group $\mathbb{S}$. Consider a finite set of operators $: = \{A_i \mid 1 \leq i \leq m\}$, where $A_i \in [E]$.

Definition 2.1. *The discrete linear (autonomous) inclusion $DLI(\mathcal{M})$ is called (see, for example,[18]) the set of all sequences $\{\{x_j\} \mid j \geq 0\}$ of vectors in E such that*

$$x_j = A_{i_j} x_{j-1} \tag{3}$$

for some $A_{i_j} \in \mathcal{M}$, i.e. $x_j = A_{i_j} A_{i_{j-1}} ... A_{i_1} x_0$ all $A_{i_k} \in \mathcal{M}$.

We may consider this as a discrete control problem, where at each time j we may apply a control from the set $\mathcal{M}$, and $DLI(\mathcal{M})$ is the set of possible trajectories of the system. A basic issue for any control system concerns its stability. One of the more important type of stability is so called absolute asymptotic stability (AAS).

Definition 2.2. *$DLI(\mathcal{M})$ is called absolute asymptotic stable if for any of its trajectories $\{x_j\}$ we have $\lim\limits_{j\to\infty} x_j = 0$.*

Let (X,ρ) be a complete metric space with metric ρ. Denote by $K(X)$ the family of all compact subsets of X. Consider the set-valued function $F : E \to K(E)$ defined by $F(x) := \{A_1x, A_2x, ..., A_mx\}$, then the discrete linear inclusion $DLI(\mathcal{M})$ is equivalent to difference inclusion

$$x_j \in F(x_{j-1}). \tag{4}$$

Denote by Φ_{x_0} the set of all trajectories of discrete inclusion (4) (or $DLI(\mathcal{M})$) issuing from the point $x_0 \in E$ and $\Phi := \bigcup\{\Phi_{x_0} \mid x_0 \in E^d\}$ the set of all trajectories of (4).

Below we will give a new approach to the study of discrete linear inclusions $DLI(\mathcal{M})$ (or difference inclusion (4)). Denote by $C(\mathbb{T}, X)$ the space of all continuous mappings $f : \mathbb{T} \to X$ equipped with the compact-open topology. This topology may be metrizied, for example, by the equality

$$d(f^1, f^2) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(f^1, f^2)}{1 + d_n(f^1, f^2)},$$

where $d_n(f^1, f^2) := \max\{|f^1(t) - f^2(t)| \mid |t| \le n,\ t \in \mathbb{T}\}$, a complete metric is defined on $C(\mathbb{T}, X)$ which generates compact-open topology. Denote by $(C(\mathbb{T}, X), \mathbb{T}, \sigma)$ a dynamical system of translations (shifts dynamical system or dynamical system of Bebutov [29, 30]) on $C(\mathbb{T}, X)$, i.e. $\sigma(t, f) := f_t$ and f_t is a $t \in \mathbb{T}$ shift of f ($f_t(s) := f(t+s)$ for all $s \in \mathbb{T}$).

Denote by $\Omega := \{f \in C(\mathbb{Z}_+, [E]) \mid f(\mathbb{Z}_+) \subseteq \mathcal{M}\}$. It is clear that Ω is an invariant (with respect to shifts) and closed subset of $C(\mathbb{Z}_+, [E])$ and, consequently, on the space Ω a dynamical system of shifts $(\Omega, \mathbb{Z}_+, \sigma)$ (induced by dynamical system of Bebutov $(C(\mathbb{Z}_+, [E]), \mathbb{Z}_+, \sigma)$) is defined.

Notice that by Tihonoff's theorem (see, for example,[21]) the space Ω is compact in $C(\mathbb{Z}_+, [E])$.

We may now rewrite the equation (3) in the following way

$$x_{j+1} = \omega(j)x_j,\ \ (\omega \in \Omega) \tag{5}$$

where $\omega \in \Omega$ is an operator-function defined by the equality $\omega(j) := A_{i_{j+1}}$ for all $j \in \mathbb{Z}_+$.

Denote by $\varphi(n, x_0, \omega)$ a solution of equation (5) issuing from the point $x_0 \in E$ at the initial moment $n = 0$. Notice that $\Phi_{x_0} = \{\varphi(\cdot, x_0, \omega) \mid \omega \in \Omega\}$ and

$\Phi = \{\varphi(\cdot, x_0, \omega) \mid x_0 \in E, \omega \in \Omega\}$, i.e. the $DLI(\mathcal{M})$ (or inclusion (4)) is equivalent to the family of linear non-autonomous equations (5) ($\omega \in \Omega$).

From the general properties of linear difference equations it follows that the mapping $\varphi : \mathbb{Z}_+ \times E \times \Omega \to E$ satisfies the following conditions:

(i) $\varphi(0, x_0, \omega) = x_0$ for all $(x_0, \omega) \in E \times \Omega$;

(ii) $\varphi(n + \tau, x_0, \omega) = \varphi(n, \varphi(\tau, x_0, \omega), \sigma(\tau, \omega))$ for all $n, \tau \in \mathbb{Z}_+$ and $(x_0, \omega) \in E \times \Omega$;

(iii) the mapping φ is continuous;

(iv) $\varphi(n, \lambda x_1 + \mu x_2, \omega) = \lambda\varphi(n, x_1, \omega) + \mu\varphi(n, x_2, \omega)$ for all $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$), $x_1, x_2 \in E$ and $\omega \in \Omega$.

Let W, Ω be two complete metric spaces and $(\Omega, \mathbb{Z}_+, \sigma)$ be a discrete semi-group dynamical system on Ω.

Definition 2.3. *Recall [29] that the triplet $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ (or shortly φ) is called a cocycle over $(\Omega, \mathbb{Z}_+, \sigma)$ with fiber W if φ is a mapping from $\mathbb{Z}_+ \times W \times \Omega$ to W satisfying the following conditions:*

1) $\varphi(0, x, \omega) = x$ for all $(x, \omega) \in W \times \Omega$;

2) $\varphi(n+\tau, x, \omega) = \varphi(n, \varphi(\tau, x, \omega), \sigma(\tau, \omega))$ for all $n, \tau \in \mathbb{Z}_+$ and $(x, \omega) \in W \times \Omega$;

3) the mapping φ is continuous.

If W is a real or complex Banach space and

4) $\varphi(n, \lambda x_1 + \mu x_2, \omega) = \lambda\varphi(n, x_1, \omega) + \mu\varphi(n, x_2, \omega)$ for all $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$), $x_1, x_2 \in W$ and $\omega \in \Omega$, then the cocycle φ is called linear.

Definition 2.4. *Let $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle (respectively, linear cocycle) over $(Y, \mathbb{T}, \sigma)$ with the fiber W (or shortly φ). If $X := W \times Y, \pi := (\varphi, \sigma)$, i.e. $\pi((u, y), t) := (\varphi(t, x, y), \sigma(t, y))$ for all $(u, y) \in W \times Y$ and $t \in T$, then the dynamical system $(X, \mathbb{T}, \pi)$ is called [29] a skew product over $(Y, \mathbb{S}, \sigma)$ with the fiber W.*

Let $(X, \mathbb{T}, \pi)$ be a dynamical system. Denote by $\omega_x := \bigcap_{t\geq 0} \overline{\bigcup\{\pi(s, x) \ : \ s \geq t\}}$ and $\alpha_x := \bigcap_{t\leq 0} \overline{\bigcup\{\pi(s, x) \ : \ s \leq t\}}$ if $\mathbb{T} = \mathbb{S}$.

Let $\mathbb{T} = \mathbb{S}$, $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a linear cocycle (respectively, linear cocycle) over $(Y, \mathbb{T}, \sigma)$ with the fiber W and $(X, \mathbb{T}, \pi)$ be a skew-product dynamical system, generated by cocycle φ. Denote by $X^s := \{x \in X : \lim_{t\to+\infty} |\pi(t, x)| = 0\}$, $X^u := \{x \in X : \lim_{t\to-\infty} |\pi(t, x)| = 0\}$, $X^s_y := X^s \bigcap X_y$ and $X^u_y := X^u \bigcap X_y$, where $X_y := W \times \{y\}$.

From the above it follows that every $DLI(\mathcal{M})$ (respectively, inclusion (4)) generates in the natural way a linear cocycle $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$, where $\Omega = C(\mathbb{Z}_+, \mathcal{M})$, $(\Omega, \mathbb{Z}_+, \sigma)$ is a dynamical system of shifts on Ω and $\varphi(n, x, \omega)$ is a solution of the equation (5) issuing from the point $x \in E$ at the initial moment $n = 0$. Thus we may study the inclusion (4) (respectively, $DLI(\mathcal{M})$) in the framework of the theory of linear cocycles with discrete time.

Definition 2.5. *A linear operator $A \in [E]$ is called asymptotically stable if $\sigma(A) \subseteq \mathbb{D}$, where $\sigma(A)$ is the spectrum of A and $\mathbb{D} := \{z|in\mathbb{C} \ : \ |z| < 1\}$ is a unit disk in $\mathbb{C}$.*

Theorem 2.6. *Let E be a finite-dimensional Banach space, $Dim(E) = n$, $A_i \in [E]$ $(i = 1, 2, ..., m)$ and $\mathcal{M} := \{A_1, A_2, ..., A_m\}$. Assume that the following conditions are fulfilled:*

1) every operator $A_j \in \mathcal{M}$ is invertible;

2) there exists $j \in \{1, 2, ..., m\}$ such that the operator A_j is asymptotically stable;

3) the discrete linear inclusion $DLI(\mathcal{M})$ has no nontrivial bounded on $\mathbb{Z}$ solutions.

Then the discrete linear inclusion $DLI(\mathcal{M})$ is absolutely asymptotically stable.

Proof. Let $Q := \mathcal{M} \bigcup \mathcal{M}^{-1}$ (where $\mathcal{M}^{-1} := \{A^{-1} \ : A \in \mathcal{M}\}$, $Y = \Omega := C(\mathbb{Z}, Q)$ and $(Y, \mathbb{Z}, \sigma)$ be a group dynamical system of shifts on Y (see Section 2). It is easy to see that $Y = C(\mathbb{Z}, Q)$ is topologically isomorphic to $\Sigma_m := \{0, 1, ..., m-1\}^{\mathbb{Z}}$ and $(Y, \mathbb{Z}, \sigma)$ is dynamically isomorphic to the shift dynamical system on Σ_m (see, for example,[25, 32]) and, consequently, it possesses the following properties:

(i) Y is compact;

(ii) $Y = \overline{Per(\sigma)}$, where $Per(\sigma)$ is the set of all periodic points of the dynamical system $(Y, \mathbb{Z}, \sigma)$;

(iii) there exists a Poisson stable point $y \in Y$ (i.e. $y \in \omega_y = \alpha_y$) such that $Y = H(y) := \overline{\{\sigma(t, y) \ : \ t \in \mathbb{Z}\}}$.

Let $\langle E, \varphi, (Y, \mathbb{Z}, \sigma)\rangle$ be a cocycle generated by $DLI(\mathcal{M})$ (i.e. $\varphi(n, u, \omega) := U(n, \omega)u$, where $U(n, \omega) = \prod_{k=1}^{n} \omega(k)$ $(\omega \in \Omega)$), $(X, \mathbb{Z}, \pi)$ be a skew-product system associated with the cocycle φ (i.e. $X := E \times Y$ and $\pi := (\varphi, \sigma)$) and $\langle (X, \mathbb{Z}, \pi), (Y, \mathbb{Z}, \sigma), h\rangle$ $(h := pr_2 : X \to Y)$ be a linear non-autonomous dynamical system generated by the cocycle φ. According to Theorem D from [28] (see also [4, 27]) X^s and X^u are two fiber sub-bundles of fiber bundle (X, h, Y). In particular there exists a number $k \in \mathbb{Z}_+$ $(0 \le k \le dim(E) = n$, where $dim(E)$ is the dimension of the space $E)$ such that $dim(X^s_y) = k$ for all $y \in Y$. Denote by $\omega_0 : \mathbb{Z} \to \mathcal{M}$ the mapping defined by the quality $\omega_0(i) = A^i_j$ for all $i \in \mathbb{Z}$, where $A^i_j := A_j \circ A^{i-1}_j$ $(i \in \mathbb{Z})$. Since the operator A_j is asymptotically stable, then the fiber X_{ω_0} $(\omega_0 \in Y)$ is asymptotically stable, i.e. $X_{\omega_0} = X^s_{\omega_0}$. Now to finish the proof of the theorem it is sufficient to note that $k = dim(X^s_y) = dim(X^s_{\omega_0}) = dim(X_{\omega_0}) = n$ for all $y \in Y$. □

Remark 2.7. *This statement is true also without assumption 1), but the proof in this case is much more complicated. We will present it in a future publication.*

3 Dynamical system of translations, set-valued cocycles and non-autonomous dynamical systems

Let $\mathcal{E}$ be a real or complex Banach space with norm $|\cdot|$ and ρ be a distance on $\mathcal{E}$ generated by norm $|\cdot|$. We denote by $K(E)$ the family of all compacts of E, by $\rho(a, B) := \inf\{\rho(a, b) \ |b \in B\}$ $(a \in E$ and $B \in K(E))$ and by α the Hausdorff's distance distance on $K(E)$, i.e. $\alpha(A, B) := \max\{\beta(A, B), \beta(B, A)\}$ and $\beta(A, B) := \sup\limits_{a \in A} \rho(a, B)$. Let $C(\mathbb{Z}_+ \times E, K(E))$ be the set of all continuous in Hausdorff's metric and bounded on every bounded set from $\mathbb{Z}_+ \times E$ mappings $F : \mathbb{R} \times E \to K(E)$ equipped with the distance

$$d(F_1, F_2) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(F_1, F_2)}{1 + d_k(F_1, F_2)}, \tag{6}$$

where $d_k(F_1, F_2) := \sup\{\alpha(F_1(t,x), F_2(t,x)) \ : 0 \le t \le k, |x| \le k, (t,x) \in \mathbb{Z}_+ \times E\}$. The distance (6) defines on the space $C(\mathbb{Z}_+ \times E, K(E))$ the topology of convergence uniform on every bounded subset of $\mathbb{Z}_+ \times E$.

Denote by $(C(\mathbb{Z}_+ \times E, K(E)), \mathbb{Z}_+, \sigma)$ a dynamical system of translations on $C(\mathbb{Z}_+ \times E, K(E))$ (see, for example,[29,30]), where $\sigma(n, F)$ is an n-shift of function F with respect to variable $t \in \mathbb{Z}_+$, i.e $\sigma(n, F)(t,x) := F(t+n, x)$ for all $(t,x) \in \mathbb{Z}_+ \times E$.

Definition 3.1. *The triplet $\langle W, \varphi, (Y, \mathbb{Z}_+, \sigma)\rangle$ is said to be a set-valued cocycle over $(Y, \mathbb{Z}_+, \sigma)$ with the fiber W, where φ is a mapping of $\mathbb{Z}_+ \times W \times Y$ onto $K(W)$ and possesses the properties:*

(i) $\varphi(0, u, y) = u$ *for all* $u \in W$ *and* $y \in Y$;

(ii) $\varphi(t + \tau, u, y) = \varphi(t, \varphi(\tau, u, y), yt)$ *for all* $t, \tau \in \mathbb{Z}_+$ *and* $(u, y) \in W \times Y$, *where* $yt := \sigma(t, y)$ *and* $\varphi(t, A, y) := \bigcup\{\varphi(t, u, y) \ : \ u \in A\}$;

(iii) $\lim_{t \to t_0, u \to u_0, y \to y_0} \beta(\varphi(t, u, y), \varphi(t_0, u_0, y_0)) = 0$ *for all* $(t_0, u_0, y_0) \in \mathbb{Z}_+ \times W \times Y$.

Let $X := W \times Y$. We denote by $(X, \mathbb{Z}_+, \pi)$ the set-valued dynamical system on X defined by the equality $\pi := (\varphi, \sigma)$, i.e. $\pi^t x := \{(v, q) : v \in \varphi(t, u, y), \ q \in \sigma(t, y)\}$ for every $t \in \mathbb{Z}_+$ and $x = (u, y) \in X = W \times Y$. Then the triplet $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}_+, \sigma), h\rangle$ is a set-valued non-autonomous dynamical system (a skew-product system), where $h = pr_2 : X \mapsto Y$.

Thus, if we have a set-valued cocycle $\langle W, \varphi, (Y, \mathbb{Z}_+, \sigma)\rangle$ over the dynamical system $(Y, \mathbb{Z}_+, \sigma)$ with the fiber W, then it generates a set-valued non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}_+, \sigma), h\rangle$ $(X := W \times Y)$, which is called a non-autonomous dynamical system generated by the cocycle $\langle W, \varphi, (Y, \mathbb{Z}_+, \sigma)\rangle$ over $(Y, \mathbb{Z}_+, \sigma)$.

Example 3.2. *(Difference inclusions).* Denote by $K(E)$ the family of all compact subsets of E. Let us consider the difference inclusion

$$u(t+1) \in F(t, u(t)), \tag{7}$$

where $F \in C(\mathbb{Z}_+ \times E, K(E))$. Along with difference inclusion (7) we will consider the family of difference inclusions

$$v(t+1) \in G(t, v(t)), \tag{8}$$

where $G \in H(F) = \overline{\{F_\tau : \tau \in \mathbb{Z}_+\}}, F_\tau(t, u) = F(t + \tau, u)$ and by bar the closure in $C(\mathbb{Z} \times E, C(E))$ is denoted.

We denote by $\varphi_{(v,G)}(n)$ a solution of inclusion (8) passing through the point v for $t = 0$ and defined for all $t \ge 0$. We set $\varphi(t, v, G) := \{\varphi_{(v,G)}(t) : \varphi_{(v,G)} \in \Phi_{(v,G)}\}$, where $\Phi_{(v,G)}$ is the set of all solutions of inclusion (8) , passing through the point v for $t = 0$. From the general properties of difference inclusions it follows that the mapping $\varphi : \mathbb{Z}_+ \times E \times H(F) \to K(E)$ possesses the next properties :

1) $\varphi(0, v, G) = v$ for all $v \in E, G \in H(F)$;
2) $\varphi(t + \tau, v, G) = \varphi(t, \varphi(\tau, v, G), G_\tau)$ for all $v \in E, G \in H(F)$ and $t, \tau \in \mathbb{Z}_+$;
3) the mapping $\varphi : \mathbb{Z}_+ \times E \times H(F) \to K(E)$ is β-continuous.

Assume $Y = H(F)$ and denote by $(Y, \mathbb{Z}_+, \sigma)$ the disperse dynamical system of translations on Y. Then the triplet $\langle E, \varphi, (Y, \mathbb{Z}_+, \sigma)\rangle$ is a set-valued cocycle over $(Y, \mathbb{Z}_+, \sigma)$ with the fiber E. Thus, non-autonomous difference inclusion (7) in a natural way generates a non-autonomous set-valued dynamical system $\langle(X, \mathbb{Z}_+, \pi)$, $(Y, \mathbb{Z}_+, \sigma)$, $h\rangle$, where $X = E \times Y, \pi = (\varphi, \sigma)$ and $h = pr_2 : X \to Y$.

4 Non-stationary discrete linear inclusions

Example 4.1. Let $\mathcal{M} \subset [E]$ be a compact set and $F : \mathbb{Z}_+ \times E \to K(E)$ be the set-valued mapping defined by the equality $F(t, x) := \{A(t)x \ : \ A \in C(\mathbb{Z}_+, \mathcal{M})\}$ for all $t \in \mathbb{Z}_+$ and $x \in E$. It is easy to verify that the function $F : \mathbb{Z}_+ \times E \to K(E)$ is continuous, i.e. $F \in C(\mathbb{Z}_+ \times E, K(E))$. Consider the difference inclusion

$$x(t + 1) \in F(t, x(t)). \tag{9}$$

Note that the solution of inclusion (9) is a sequence $\{\{x(t)\} \mid t \in \mathbb{Z}_+\}$ of vectors in E such that $x(t + 1) = A_{i_t}(t)x(t)$ for some $A_{i_t}(t) \in \mathcal{M}$, i.e.

$$x(t) = A_{i_t}(t)A_{i_{t-1}}(t-1)...A_{i_1}(1)x(0) \ (A_{i_t}(t) \in \mathcal{M}).$$

Along with inclusion (9) we consider its H-class (see Example 3.2), i.e. the family of inclusions

$$x(t + 1) \in G(t, x(t)), \tag{10}$$

where $G \in H(F) := \overline{\{F_s \mid s \in \mathbb{Z}_+\}}$ and $F_s(t, x) := F(t+s, x)$ for all $(t, x) \in \mathbb{Z}_+ \times E$.

Let Y be a compact metric space and (X, h, Y) be a locally trivial fiber bundle [20] with the fiber E, (X, ρ) be a complete metric space.

Definition 4.2. *$\langle(X, \mathbb{Z}_+, \pi)$,$(Y, \mathbb{Z}_+, \sigma), h\rangle$ is said to be homogeneous if for any $x \in X$ and any $\gamma_x \in \Phi_x$ the function $\tilde{\gamma} : D(\gamma_x) \to X$ ($D(\gamma_x) := [r_x, +\infty)$ is the domain of the definition of γ_x, where $r_x \in \mathbb{Z}$) defined by $\tilde{\gamma}(t) := \lambda\gamma_x(t)$ is the motion of (X, Z_+, π) issuing from the point $\lambda x \in X$, i.e. $\tilde{\gamma} \in \Phi_{\lambda x}$.*

Remark 4.3. *1. Note that non-autonomous dynamical system from Example 3.2 is homogeneous if the set-valued mapping F which figures in this example is homogeneous, i.e. $F(t, \lambda x) = \lambda F(t, x)$ for all $(t, x) \in \mathbb{Z}_+ \times E$.*

2. The non-autonomous dynamical system generated by discrete linear inclusion (9) is homogeneous, because the function $F(t, x) := \{A(t)x \ : \ A \in C(\mathbb{Z}_+, \mathcal{M})\}$ (for all $(t, x) \in \mathbb{Z}_+ \times E$) is homogeneous with respect to $x \in E$.

If $x \in X$, then we put $|x| := \rho(x, \theta_{h(x)})$, where θ_y $(y \in Y)$ is the null (trivial) element of the linear space X_y and $\Theta := \{\theta_y \mid y \in Y\}$ is the null (trivial) section of the vectorial bundle (X, h, Y). Let $A \in K(X)$, then we denote by $|A| := \max\{|a| \ : \ a \in A\}$. Denote by X^s a stable manifold of $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}_+, \sigma)\rangle$, i.e. $X^s := \{x \mid x \in X, \ \lim\limits_{t\to+\infty} |\pi(t, x)| = 0\}$.

Definition 4.4. *Let W be a Banach space. The cocycle $\langle W, \varphi, (Y, \mathbb{Z}_+, \sigma), h\rangle$ is said to be homogeneous if the skew-product set-valued dynamical system $(X, \mathbb{Z}_+, \pi)$ ($X := W \times Y, \pi := (\varphi, \sigma)$) also is homogeneous.*

Theorem 4.5. *[8] Let Y be a compact metric space and $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}_+, \sigma), h\rangle$ be a homogeneous set-valued non-autonomous dynamical system. Then the following assertions are equivalent:*

(i) the trivial section Θ of fibering (X, h, Y) is uniformly asymptotically stable, i.e. $\lim\limits_{t\to\infty} \|\pi^t\| = 0$, where $\pi^t := \pi(t, \cdot) : X \to K(X)$, $\|\pi^t\| := \sup\{|\pi^t x| \ : x \in X, |x| \leq 1\}$ and $|A| := \sup\{|a| \ : a \in A\}$;

(ii) the trivial section Θ of fibering (X, h, Y) is uniformly exponentially stable, i.e. there are two positive constants $\mathcal{N}$ and ν such that $|\pi(t, x)| \leq \mathcal{N} e^{-\nu t}$ for all $x \in X$ and $t \in \mathbb{Z}_+$.

Definition 4.6. *A set-valued dynamical system $(X, \mathbb{Z}_+, \pi)$ is called compact (completely continuous) if for any bounded set $A \subseteq X$ there exists a positive number $l \in \mathbb{Z}_+$ such that the set $\pi^t A$ is relatively compact. A non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}_+, \sigma), h\rangle$ is called compact if the system $(X, \mathbb{T}_1, \pi)$ is so.*

Denote by $K(X)$ $(B(X))$ the family of all compact (bounded) subsets of X and $B(M, \delta) := \{x \in X \ |\rho(x, M) < \delta\}$.

Definition 4.7. *A system $(X, \mathbb{Z}_+, \pi)$ is called [6]:*

– pointwise dissipative if there exists $K_0 \in C(X)$ such that for all $x \in X$

$$\lim_{t\to\infty} \beta(xt, K_0) = 0; \tag{11}$$

– compactly dissipative if equality (11) holds uniformly w.r.t. x on compacts from X;

– locally dissipative if for any point $p \in X$ there exists $\delta_p > 0$ such that equality (11) holds uniformly w.r.t. $x \in B(p, \delta_p)$.

Theorem 4.8. *[6] Let $(X, \mathbb{Z}_+, \pi)$ be a pointwise dissipative compact dynamical system, then it is locally dissipative.*

Theorem 4.9. *Let Y be a compact metric space and $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}_+, \sigma), h\rangle$ be a compact, homogeneous set-valued non-autonomous dynamical system. Then the following assertions are equivalent:*

1) the non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}_+, \sigma), h\rangle$ is convergent, i.e.

$$\lim_{t\to\infty} |\pi(t, x)| = 0, \tag{12}$$

for all $x \in X$;

2) the trivial section Θ of fibering (X, h, Y) is uniformly exponentially stable.

Proof. To prove this affirmation obviously it is sufficient to show that 1) implies 2), because the implication 2) $\to$ 1) is trivial. Since the space Y is compact and the fibering (X, h, Y) is locally trivial, then the trivial section Θ of (X, h, Y) is compact. Taking into account this fact and the equality (12) we obtain the pointwise dissipativity of dynamical system $(X, \mathbb{Z}_+, \pi)$. Now to finish the proof it is sufficient to apply Theorem 4.8. $\square$

Definition 4.10. *Following [18] the inclusion (9) is said to be absolute asymptotic stable (AAS) if for any trajectory $\{x(t)|\ t \in \mathbb{Z}_+\}$ of any inclusion (10) $\lim\limits_{t\to+\infty} x(t) = 0$.*

Theorem 4.11. *[6] Let $(X, \mathbb{Z}_+, \pi)$ be a completely continuous (compact) and trajectory dissipative set-valued dynamical system, then it is locally dissipative.*

Theorem 4.12. *Let $\mathcal{M} \subset [E]$ be compact and every operator $A \in \mathcal{M}$ be compact too. Then the following two statements are equivalent:*

1) the inclusion (9) is absolute asymptotic stable;

2) the inclusion (9) is uniformly exponentially stable, i.e. there are positive numbers N and ν such that $|x(t)| \le Ne^{-\nu t}|x(0)|$ for all $t \in \mathbb{Z}_+$, where $\{x(t)\ |t \in \mathbb{Z}_+\}$ is an arbitrary solution (trajectory) of arbitrary inclusion (10).

Proof. Denote by $\Omega := H(F)$ the closure (in the space $C(\mathbb{Z}_+ \times E, C(E))$) of family of translations $\{F_s \mid s \in \mathbb{Z}_+\}$ of function $F(t,x) := \{At)x \ : \ A \in C(\mathbb{Z}_+, \mathcal{M})\}$, $(\Omega, \mathbb{Z}_+, \sigma)$ the dynamical system of translations on Ω, $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ the cocycle generated by non-stationary discrete linear inclusion (10). Finally, by $\langle (X, \mathbb{Z}_+, \pi),$ $(\Omega, \mathbb{Z}_+, \sigma), h\rangle$ we denote the non-autonomous dynamical system system, generated by cocycle φ ($X := E \times \Omega$, $\pi := (\varphi, \sigma)$ and $h := pr_2 : X \to \Omega$). Note that this dynamical system possesses the following properties:

1) the set $\Omega = H(A)$ is compact, according to theorem of Tihonoff;

2) the non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}_+, \sigma), h\rangle$ is homogeneous;

3) the non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}_+, \sigma), h\rangle$ is compact. In fact, let $A \subset E$ be a bounded subset of E , $c := \sup\{|a| \ : a \in A\}$ and $l \in \mathbb{N}$. We will show that the set $\varphi(l, A, \Omega)$ is relatively compact. Consider a sequence $\{y_k\} \subseteq \varphi(l, A, \Omega)$, then there are $\{x_k\} \subseteq A$, $\{\omega_k\} = \{G_k\} \subseteq \Omega$ ($G_k \in H(F)$) and $B^k_i \in H(A_i)$ ($H(A_i)$ is a closure of the set of translations $\{A_i(t+s) \mid s \in \mathbb{Z}_+\}$ in the space $C(\mathbb{Z}_+, [E])$ of all continuous mappings $f : \mathbb{Z}_+ \to [E]$ equipped with compact-open topology) such that

$$y_k = B^k_{i_l}(l)B^k_{i_{l-1}}(l-1)...B^k_{i_1}(1)x_k.$$

Under the conditions of Theorem the operators $\{B^k_{i_s}(s)\}$ ($1 \le s \le l$ and $k \in \mathbb{N}$) are compact. Without loss of generality we may suppose that the sequences

$\{B^k_{i_s}\} \subset C(\mathbb{Z}_+, [E])$ are convergent as $k \to \infty$ (in the space $C(\mathbb{Z}_+, [E])$). Let $B_{i_s}(t) := \lim_{k\to\infty} B^k_{i_s}(t)$ (for any $t \in \mathbb{Z}_+$), then this operator will be compact and, consequently, the operator $B(t) := B_{i_l}(t)B_{i_{l-1}}(t)...B_{i_1}(t)$ will be so too. Since the sequence $\{x_k\} \subseteq A$ is bounded, then the sequence $\{B(l)x_k\}$ is relatively compact. For simplicity we may suppose that this sequence converges and denote by $y := \lim_{k\to\infty} B(l)x_k$, then we have

$$\begin{aligned} &|y_k - y| \le |B^k_{i_l}(l)B^k_{i_{l-1}}(l-1)...B^k_{i_1}(1)x_k - B_{i_l}(l)B_{i_{l-1}}(l-1)...B_{i_1}(1)x_k| + \\ &|B_{i_l}(l)B_{i_{l-1}}(l-1)...B_{i_1}(1)x_k - y| \le \|B^k_{i_l}(l)B^k_{i_{l-1}}(l-1)...B^k_{i_1}(1) - \\ &B_{i_l}(l)B_{i_{l-1}}(l-1)...B_{i_1}(1)\| \cdot c + |B_{i_l}(l)B_{i_{l-1}}(l-1)...B_{i_1}(1)x_k - y| \end{aligned} \tag{13}$$

for all $k \in \mathbb{N}$. Passing to limit in the relation (13) we obtain $y = \lim_{n\to\infty} y_k$ and the required statement is proved.

4) the dynamical system $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{T}_+, \sigma), h\rangle$ is convergent.

In fact, from condition 1. it follows that the skew-product set-valued dynamical system $(X, \mathbb{Z}_+, \pi)$ ($X := E^n \times Y$ and $\pi := (\varphi, \sigma)$) is trajectory dissipative and by Theorem 4.8 the skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$ is locally dissipative and, in particular, we have $\lim_{t\to+\infty} \sup_{|x|\le 1} |\pi(t, x)| = 0$.

Note that the non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}_+, \sigma), h\rangle$, generated by cocycle φ is homogeneous and compact. Now to finish the proof of Theorem it is sufficient to apply Theorem 4.9. □

Remark 4.13. *1. Note that a similar result has been proved for reflexive Banach spaces in [33, 34] for arbitrary bounded sets of bounded operators.*

2. Theorem 4.12 is true also for non-autonomous nonlinear homogeneous inclusions, i.e. if the operators $A \in \mathcal{M}$ are continuous and homogeneous, but in general not linear.

References

[1] Artzrouni M. *On the Convergence of Infinite Products of Matrices.* Linear Algebra and its Applications, 1986, **74**, p. 11–21.

[2] Barabanov N.E. *Lyapunov indicator of discrete inclusions. I–III.* Autom. Remote Control, 1988, **49:152-157**, p. 283-287, 558-565.

[3] Beyn W.-J., Elsner L. *Infinite Products and Paracontracting Matrices.* The Electronic Journal of Linear Algebra, 1997, **2**, p. 1–8.

[4] Bronstein I.U. *Nonautonomous Dynamical Systems.* Chişinău, Stiintsa, 1984 (in Russian)

[5] Bru R., Elsner L., Neumann M. *Convergence of Infinite Products of Matrices and Inner-Outer Iteration Schemes.* Electronic Transactions on Numerical Analysis, 1994, **2**, p. 183–194.

[6] Cheban D.N., Fakeeh D.S. Global Attractors of Disperse Dynamical Systems. Chişinău, Sigma, 1994.

[7] Cheban D.N., Fakeeh D.S. *Global Attractors of Infinite-Dimensional Dynamical Systems, III.* Bulletin of Academy of Sciences of Republic of Moldova, Mathematics, 1995, N 2-3(18-19), p. 3–13.

[8] Cheban D.N. *The Asymptotics of Solutions of Infinite Dimensional Homogeneous Dynamical Systems.* Mat. Zametki, 1998, **63**, N 1, p. 115–126; Translation in Mathematical Notes, 1998, **63**, N 1, p. 115–126.

[9] Cheban D.N. *Global Attractors of Nonautonomous Dynamical Systems.* Chişinău, State University of Moldova, 2002 (in Russia).

[10] Cheban D.N. *Global Attractors of Nonautonomous Dissipative Dynamical Systems.* World Scientific (to appear).

[11] Clement Ph., Heijman J., Angement S., C. Van Dujin C., De Pagter P. *One-Parameter Semigroups.* CWI Monograph 5, North-Holland, Amsterdam-New York-Oxford-Tokyo, 1987.

[12] Daubechies I., Lagarias J.C. *Sets of Matrices All Infnite Products of Which Converge.* Linear Algebra Appl., 1992, **161**, p. 227–263.

[13] Elsner L., Friedland S. *Norm Conditions for Convergence of Infinite Products.* Linear Algebra Appl., 1997, **250**, p. 133–142.

[14] Elsner L., Koltracht I., Neumann M. *On the Convergence of Asynchronous Paracontractions with Applications to Tomographic Reconstruction from Incomplete Data.* Linear Algebra Appl., 1990, **130**, p. 65–82.

[15] Gurvits L. *Geometrical Approach to Some Problems of Control, Observation and Lumping.* PhD. Thesis, Dept. of Mathematics, Gorky State Univ., USSR, 1985.

[16] Gurvits L.N., Zaharin A.M. *System Theoretic and Graph Theoretic Aspects of Markov Chains Lumping.* 5th International Conference on Mathematical Statistics, Vilnius, 1989.

[17] Gurvits L.N., Zaharin A.M. *Lumpability of Markov Processes and Systems Theory.* MTNS, Amsterdam, 1989.

[18] Gurvits L. *Stability of Discrete Linear Inclusion.* Linear Algebra and its Applications, 1995, **231**, p. 47–85.

[19] Hale J. K. *Asymptotic Behaviour of Dissipative Systems.* Amer. Math. Soc., Providence, RI, 1988.

[20] Husemoller D. *Fibre Bundles.* Springer–Verlag, Berlin–Heidelberg–New York, 1994.

[21] Kelley J. *General Topology.* D. Van Norstand Company Inc., Toronto–New York–London, 1957.

[22] Kozyakin V.S. *Algebric insolvability of problem of absolute stability of desynchronized systems.* Autom. Rem. Control, 1990, N 51(6), p. 754–759.

[23] Molchanov A.P. *Lyapunov function for nonlinear discrete-time control systems.* Avtomat. i Telemekh., 1987, **6**, p. 26–35.

[24] Opoitsev V.I. *Stability of nonautonomous systems.* Avtomat. i Telemekh., 1986, N 10, p. 56–63.

[25] Robinson C. *Dynamical Systems: Stabilty, Symbolic Dynamics and Chaos (Studies in Advanced Mathematics).* Boca Raton Florida: CRC Press, 1995.

[26] Rota G.C., Strang W.G. *A Note on the Joint Spectral Radius.* Indag. Math., 1960, **22**, p. 379–381.

[27] Sacker R.J., Sell G.R. *Existence of Dichotomies and Invariant Splittings for Linear Differential Systems, I.* Journal of Differential Equations, 1974, **15**, p. 429–458.

[28] Sacker R.J., Sell G.R. *Dichotomies for Linear Evoltionary Equations in Banach Spaces.* Journal of Differential Equations, 1994, **113**, p. 17–67.

[29] Sell G.R. *Lectures on Topological Dynamics and Differential Equations, Vol. 2.* Van Nostrand Reinhold math. studies. Van Nostrand–Reinbold, London, 1971.

[30] Shcherbakov B.A. *Topologic Dynamics and Poisson Stability of Solutions of Differential Equations.* Chişinău, Ştiinţa, 1972.

[31] Vladimirov A., Elsner L., Beyn W.-J. *Stability and paracontractivity of discrete linear inclusion.* Linear Algebra and its Applications, 2000, **312**, p. 125–134.

[32] De Vries J. *Elements of Topological Dynamics.* Mathematics and Its Applications, Vol. 257. Kluwer Academic Publishers, Dordecht-Boston-London, 1993.

[33] Wirth F. *On Stability of Infinite-Dimensional Discrete Inclusions.* Journal of Mathematical Systems, Estimations and Control, 1998, **8**, N 4, p. 507–510.

[34] Wirth F. *The generalized spectral radius and external norms.* Linear Algebra end its Applications, 2002, **342**, p. 17–40.

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Stability and fold bifurcation in a system of two coupled demand-supply models

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Abstract. A model of two coupled demand-supply systems, depending on 4 parameters is considered. We found that the dynamical system associated with this model may have at most two symmetric and at most two nonsymmetric equilibria as the parameters vary.

The topological type of equilibria is established and the locus in the parameter space of the values corresponding to nonhyperbolic equilibria is determined.

We found that the nonhyperbolic singularities can be of fold, Hopf, double-zero (Bogdanov-Takens) or fold-Hopf type.

In addition, the fold bifurcation is studied using the normal form method and the center manifold theory.

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1 The mathematical model

The demand-supply model describes the way in which the price p and the quantity q reacts one to another. This model was proposed by Beckmann and Ryder [1] (1969) and Collel (1986). It is based on the economic principles of Walras and Marshall [2]. According to their hypothesis, the variation of the price is function of the difference between the demanded quantity of the product $D(p)$ and the offered quantity $S(p)$ at the price p, while the variation of the quantity is function of the difference between the price $p_d(q)$ demanded for the quantity q and the price $p_s(q)$ offered for this quantity. In addition, these two functions keep constant the sign of their argument. Thus, the mathematical model has the form [4]:

$$\begin{cases} \dot{p} = f(D(p) - S(p)), \\ \dot{q} = g(p_d(q) - p_s(q)). \end{cases} \tag{1}$$

with $f(0) = g(0) = 0$, $f'(0) > 0$, $g'(0) > 0$.

If $f(x) = x, g(x) = x$, $S(p) = q$, $p_d(q) = p$, $D(p) = ap + \beta$, $p_s(q) = cq^2 + \delta$, system (1) becomes:

$$\begin{cases} \dot{p} = ap + \beta - q, \\ \dot{q} = p - cq^2 - \delta. \end{cases} \tag{2}$$

In economy, the laws of demand and offer are available. According to them [3], as the price of the product increases, the demanded quantity decreases, so the function $D(p)$

is decreasing and we must have $a < 0$. Similarly, the function $p_s(q)$ is increasing, so we have $c > 0$.

The economic interest is to reach an equilibrium between the price and the quantity.

With the transformation $u = p - \delta$ and denoting $b = a\delta + \beta$, system (2) is written as:

$$\begin{cases} \dot{u} = au - q + b, \\ \dot{q} = u - cq^2. \end{cases} \tag{3}$$

A study of dynamics and bifurcation of this system is developed in [5]. The coordinates of equilibria of system (3) satisfy

$$\begin{cases} au - q + b = 0, \\ u - cq^2 = 0. \end{cases}$$

Denote $\Delta = 1 - 4abc$. Since $ac \neq 0$ there are two equilibria $\left(c\alpha^2, \alpha\right)$, with $\alpha = \frac{1 \pm \sqrt{\Delta}}{2ac}$, as $\Delta > 0$, a single equilibrium $\left(\frac{1}{4a^2c}, \frac{1}{2ac}\right)$ as $\Delta = 0$ and no equilibria as $\Delta < 0$. The equilibrium $\left(\frac{1}{4a^2c}, \frac{1}{2ac}\right)$ is always nonhyperbolic, namely of saddle-node type as $a \neq -1$ and of double zero type as $a = -1$. The equilibrium $\left(c\alpha^2, \alpha\right)$, with $\alpha = \frac{1-\sqrt{\Delta}}{2ac}$, is nonhyperbolic of Hopf type iff $\sqrt{\Delta} = 1 - a^2$, $a \in (-1, 0)$. Otherwise, it is a repulsor as $a^2 - 1 + \sqrt{\Delta} > 0$ and an attractor as $a^2 - 1 + \sqrt{\Delta} < 0$. In [5] it is shown that crossing the parameter stratum $\sqrt{\Delta} = 1 - a^2$, $a \in (-1, 0)$, a subcritical Hopf bifurcation takes place. Finally, the equilibrium $\left(c\alpha^2, \alpha\right)$, with $\alpha = \frac{1+\sqrt{\Delta}}{2ac}$, is always hyperbolic of saddle type.

In our study, a model of two identical demand-supply dynamical systems (3), symmetrically coupled via the quantity flow is considered. It reads

$$\begin{cases} \dot{x}_1 = ax_1 - x_2 + b, \\ \dot{x}_2 = x_1 - cx_2^2 + d\left(x_2 - x_4\right), \\ \dot{x}_3 = ax_3 - x_4 + b, \\ \dot{x}_4 = x_3 - cx_4^2 + d\left(x_4 - x_2\right). \end{cases} \tag{4}$$

This system models the interaction between two identical demand-supply models. Thus we shall focus on parameter values such that the system (4) display either a steady stable state or periodic behavior. Systems coupled in the form (5) are often used in the literature. As a result of the couplage, some characteristics of the behavior around the equilibria are preserved, but new kind of dynamics arise [6–9].

A consequence of this form of coupling and of the assumption that the models are identical is the invariance of (4) under the transformation $(x_1, x_2, x_3, x_4) \to (x_3, x_4, x_1, x_2)$. The same symmetry leads to the existence of an invariant subspace

$$I = \left\{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4, x_1 = x_3, x_2 = x_4\right\}.$$

A solution of (4) lying in I will be referred to as symmetric solution, while one which does not lie in I as nonsymmetric solution. By economic reasons, we shall investigate only the case $a < 0$, $c > 0$. We also assume $d > 0$. Thus we consider the set of parameters of interest from application point of view as

$$D = \left\{(a, b, c, d) \in \mathbf{R}^4, a < 0, c > 0, d > 0\right\}.$$

2 Equilibria and nonhyperbolic singularities

System (4) possesses at most two symmetric equilibria of the form

$$e_s = \left(c\alpha^2, \alpha, c\alpha^2, \alpha\right), \tag{5}$$

where $\alpha \in \mathbf{R}$ satisfies the equation

$$ac\alpha^2 - \alpha + b = 0, \tag{6}$$

whose discriminant is Δ already introduced. As $ac \neq 0$, for $\Delta = 0$, there exists a unique equilibrium e_{0s}, with $\alpha = \frac{1}{2ac}$; and for $\Delta > 0$, system (4) has two symmetric equilibria e_{1s}, e_{2s}, corresponding to $\alpha_1 = \frac{1+\sqrt{\Delta}}{2ac}$ and $\alpha_2 = \frac{1-\sqrt{\Delta}}{2ac}$, respectively; while for $\Delta < 0$ there are no symmetric equilibria.

As $ac \neq 0$, system (4) may also possess at most two nonsymmetric equilibrium points, of the form

$$e_a = \left(\frac{\alpha' - b}{a}, \alpha', \frac{1 + 2ad}{a^2 c} - \frac{\alpha' + b}{a}, \frac{1 + 2ad}{ac} - \alpha'\right), \tag{7}$$

where α' satisfies the equation

$$c\alpha^2 - \frac{1 + 2ad}{a}\alpha + \frac{d + 2ad^2 + bc}{ac} = 0. \tag{8}$$

Denote by $\Delta' = 1 - 4abc - 4a^2d^2$ the discriminant of (8). Note that if $\Delta' = 0$, we have $\alpha' = \frac{1+2ad}{2ac}$ and the corresponding equilibrium e_a coincides with e_{2s}. Thus we obtain the following result:

Lemma 1. *Assume $a < 0$, $c > 0$.*

(i) *If $\Delta < 0$, system (4) has no equilibria;*

(ii) *if $\Delta = 0$, system (4) has a unique equilibrium point e_{0s}, given by (5) with $\alpha = \frac{1}{2ac}$;*

(iii) *if $\Delta > 0$ and $\Delta' \leq 0$ system (4) has two equilibria e_{1s}, e_{2s};*

(iv) *if $\Delta' > 0$, system (4) has four equilibrium points e_{1s}, e_{2s}, e_{1a}, e_{2a}.*

As a consequence, the static bifurcation diagram of the dynamical system (4) in D is the set

$$S = \left\{(a, b, c, d) \in D, (1 - 4abc)\left(1 - 4abc - 4a^2d^2\right) = 0\right\}.$$

Sections in the static bifurcation set S with a plane $b = b_0$, $c = c_0$, are plotted in Fig. 1, for different values of b_0, c_0, and the number of equilibrium points corresponding to each stratum is shown.

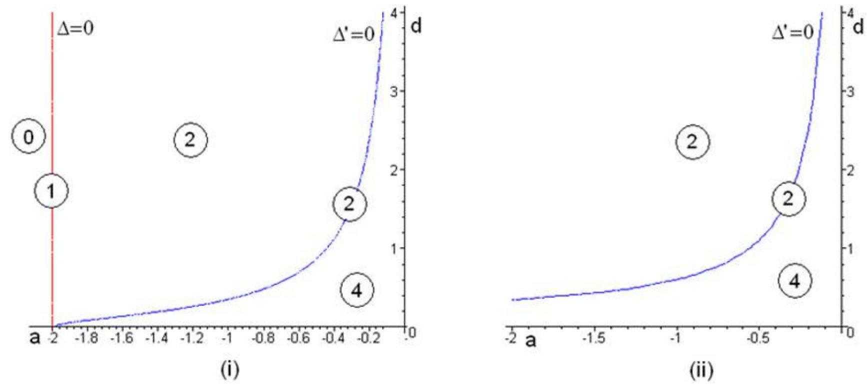

Figure 1. *Section with a plane $b = b_0$, $c = c_0$ in the static bifurcation diagram: i) $b = -0.5$, $c = 0.25$; ii) $b = 0.5$, $c = 0.25$. The number of equilibria corresponding to each stratum is shown*

3 The topological type of equilibria

In this section we determine the topological type of the four equilibrium points of system (4), analyzing the variation of the eigenvalues of the Jacobi matrix of the linearized system associated with (4) around each of the four equilibria.

Let $e = (e_1, e_2, e_3, e_4) \in \mathbf{R}^4$ be an equilibrium point of system (4). The Jacobi matrix of (4) around e reads

$$J(e) = \begin{pmatrix} a & -1 & 0 & 0 \\ 1 & d - 2ce_2 & 0 & -d \\ 0 & 0 & a & -1 \\ 0 & -d & 1 & d - 2ce_4 \end{pmatrix}.$$

Denote by T^s, T^u, T^c the stable, unstable and critical eigenspaces of $J(e)$, respectively, and by s, u, c the dimension of these subspaces of $\mathbf{R}^4$.

As the characteristic equation for the equilibrium e_{0s} is

$$\lambda\left(\lambda - \frac{a^2-1}{a}\right)\left[\lambda^2 - \lambda\left(a + 2d - \frac{1}{a}\right) + 2ad\right] = 0,$$

we obtain the following result:

Lemma 2. *If $\Delta = 0$, for parameters in D, the unique equilibrium point of system (4) is nonhyperbolic, with one zero eigenvalue as $a \neq -1$ or two zero eigenvalues as $a = -1$.*

If $\Delta > 0$, the characteristic equation for the symmetric equilibria e_{1s}, e_{2s} reads [11]:

$$\left[\lambda^2 - \left(a - \frac{1 \pm \sqrt{\Delta}}{a}\right)\lambda \mp \sqrt{\Delta}\right]\left[\lambda^2 - \left(a + 2d - \frac{1 \pm \sqrt{\Delta}}{a}\right)\lambda + 2ad \mp \sqrt{\Delta}\right] = 0. \tag{9}$$

Denote by λ_1, λ_2 the roots of the first bracket in (9) and by λ_3, λ_4 the roots of the second one.

As for e_{1s} we have $\lambda_1\lambda_2 = -\sqrt{\Delta} < 0$, $\lambda_3\lambda_4 = 2ad - \sqrt{\Delta} < 0$, we may conclude:

Lemma 3. *If $\Delta > 0$, for parameters in D, the symmetric equilibrium e_{1s} of system (4) is hyperbolic, namely it is a saddle of type $(s, u) = (2, 2)$.*

In order to establish the topological type of e_{2s}, let us introduce the following notations:

$$\begin{aligned} SN_1 &= \{(a,b,c,d) \in D, \quad \Delta = 0\}, \\ SN_2 &= \left\{(a,b,c,d) \in D, \quad \Delta > 0, \quad 2ad + \sqrt{\Delta} = 0\right\}, \\ H_1 &= \left\{(a,b,c,d) \in D, \quad \Delta > 0, \quad a^2 - 1 + \sqrt{\Delta} = 0\right\}, \\ H_2 &= \left\{(a,b,c,d) \in D, \quad \Delta > 0, 2ad + \sqrt{\Delta} \geq 0, \quad a^2 - 1 + 2ad + \sqrt{\Delta} = 0\right\}. \end{aligned}$$

Lemma 4. *For $\Delta > 0$ and $(a,b,c,d) \in D - (SN_2 \cup H_1 \cup H_2)$ the symmetric equilibrium e_{2s} of system (4) is hyperbolic, namely:*

(i) *if $2ad + \sqrt{\Delta} < 0$ and $a^2 - 1 + \sqrt{\Delta} < 0$, then e_{2s} is a saddle of type $(3, 1)$;*

(ii) *if $2ad + \sqrt{\Delta} < 0$ and $a^2 - 1 + \sqrt{\Delta} > 0$, then e_{2s} is a saddle of type $(1, 3)$;*

(iii) *if $2ad + \sqrt{\Delta} > 0$ and $a^2 - 1 + \sqrt{\Delta} > 0$, then e_{2s} is a repulsor;*

(iv) *if $2ad + \sqrt{\Delta} > 0$, $a^2 - 1 + \sqrt{\Delta} < 0$ and $a^2 - 1 + 2ad + \sqrt{\Delta} > 0$, then e_{2s} is a saddle of type $(2, 2)$;*

(v) *if $2ad + \sqrt{\Delta} > 0$, $a^2 - 1 + \sqrt{\Delta} < 0$ and $a^2 - 1 + 2ad + \sqrt{\Delta} < 0$, then e_{2s} is an attractor.*

In addition, if $(a,b,c,d) \in SN_2 \cup H_1 \cup H_2$, then e_{2s} is a nonhyperbolic equilibrium, namely of Hopf type as $(a,b,c,d) \in (H_1 \cup H_2) - SN_2$, of fold type as $(a,b,c,d) \in SN_2 - (H_1 \cup H_2)$, of double zero type as $(a,b,c,d) \in SN_2 \cap H_2$ or of fold-Hopf type as $(a,b,c,d) \in SN_2 \cap H_1$.

In Fig. 2 is represented a section with a plane $b = b_0$, $c = c_0$ in the bifurcation diagram of system (4) around the equilibrium e_{2s}. Inside each region (s, u) is given.

As $\Delta' > 0$, for the nonsymmetric equilibria e_{1a}, e_{2a}, the corresponding characteristic equation is written as

$$\lambda^4 - \Delta_1\lambda^3 + \Delta_2\lambda^2 - \Delta_3\lambda + \Delta_4 = 0, \tag{10}$$

where:

$$\Delta_1 = 2\left(a - d - \frac{1}{a}\right); \quad \Delta_2 = \frac{1 + 2ad - \Delta'}{a^2} + a^2 - 4ad - 2;$$

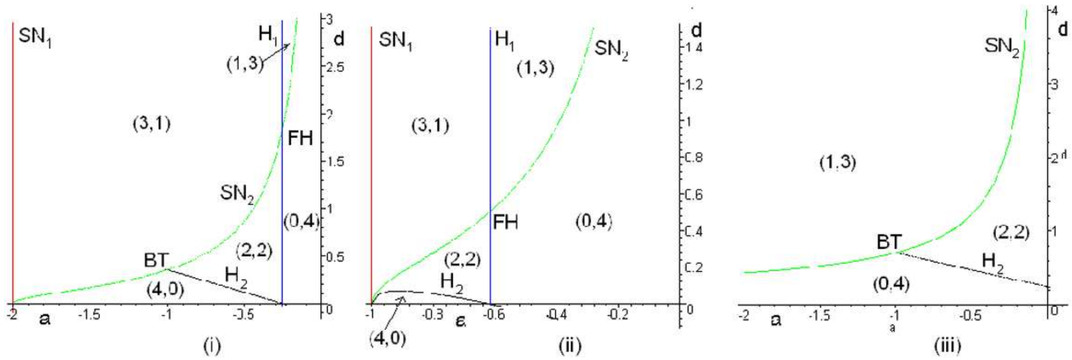

Figure 2. *Section with a plane $b = b_0$, $c = c_0$ in the local parameter portrait around e_{2s}: i) $b = -0.5$, $c = 0.25$; ii) $b = -0.5$, $c = 0.5$; iii) $b = 0.5$, $c = 0.5$*

$$\Delta_3 = 2\left[d - a^2 d - \frac{\Delta'}{a}\right]; \quad \Delta_4 = -\Delta'.$$

Since $\Delta_4 < 0$, it follows $\lambda_i \neq 0$, $i = \overline{1,4}$. Therefore, the equilibrium $e_{1,2a}$ may be nonhyperbolic only if (10) has a pair of purely imaginary solutions. This situation arises if the following conditions are fulfilled [8]

$$\Delta_1 \neq 0, \frac{\Delta_3}{\Delta_1} > 0, \frac{\Delta_3}{\Delta_1} + \Delta_4 \frac{\Delta_1}{\Delta_3} = \Delta_2 \tag{11}$$

or

$$\Delta_1 = 0, \Delta_3 = 0, \Delta_4 < 0. \tag{12}$$

Consequently, we obtained:

Lemma 5. *If $\Delta' > 0$, then the nonsymmetric equilibria $e_{1,2a}$ of system (4) are*

(i) *hyperbolic saddles, of type (1,3) or (3,1), as the conditions (11), (12) do not hold;*

(ii) *nonhyperbolic of Hopf type, as (11) or (12) holds.*

In Fig. 3 is represented a section with a plane $b = b_0$, $c = c_0$ in the bifurcation diagram of system (4) around the equilibria $e_{1,2a}$. The parameter strata for which (11) or (12) holds are denoted by H.

4 Fold bifurcation

Let $e = (e_1, e_2, e_3, e_4)$ be an equilibrium of system (4). Performing the translation $y = x - e$, system (4) reads

$$\dot{y} = J(e)\, y + F(y), \quad y \in \mathbf{R}^4, \tag{13}$$

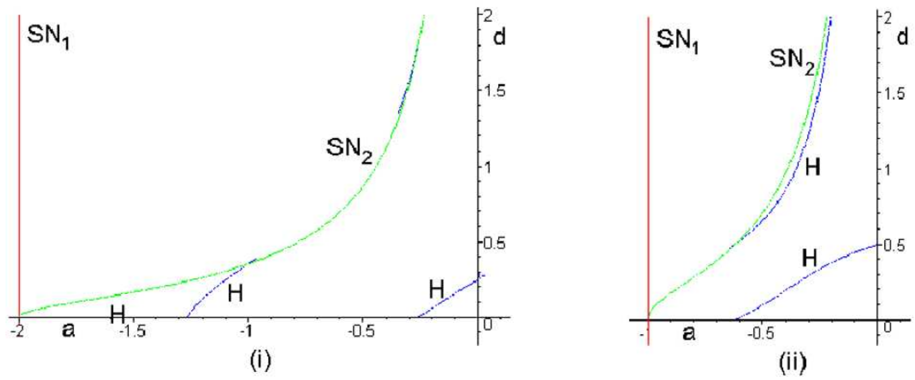

Figure 3. *Section with a plane* $b = b_0,\ c = c_0$ *in the local parameter portrait around* $e_{1,2a}$*: i)* $b = -0.5,\ c = 0.25$*; ii)* $b = -0.5,\ c = 0.5$

with $F(y) = \left(0, -cy_2^2, 0, -cy_4^2\right)^t$, and the corresponding equilibrium is the origin $0 \in \mathbf{R}^4$.

Using the normal form and the center manifold theory [10], we establish the topological type of the nonhyperbolic equilibria of saddle-node type determined in Section 3 and the local bifurcation generated by them.

Case 1. For parameters situated in the set SN_1 we have $\Delta = 0$ and the Jacobi matrix associated with the unique equilibrium point of (4) $e_{0s} = \left(\frac{1}{4a^2c}, \frac{1}{2ac}, \frac{1}{4a^2c}, \frac{1}{2ac}\right)$, has the eigenvalues $\lambda_1 = 0$, $\lambda_2 = a - \frac{1}{a}$, $\lambda_3 \lambda_4 = 2ad < 0$. Assume $a^2 - 1 \neq 0$. Thus $J(e_{0s})$ has a simple zero eigenvalue and the corresponding critical eigenspace is spanned by the eigenvector $q = (1, a, 1, a) \in \mathbf{R}^4$. Let $p = \frac{1}{2(1-a^2)}(1, -a, 1, -a) \in \mathbf{R}^4$ be the normalized adjoint vector, i.e. $J(e_{0s})^t p = 0$ and $\langle p, q \rangle = 1$. We discompose any vector $y \in \mathbf{R}^4$ as $y = uq + z$, where $uq \in T^c$, $z \in T^{su}$. Here T^{su} is the 3-dimensional eigenspace of $J(e_{0s})$ corresponding to all eigenvalues, other than 0. The explicit expressions for u and z are:

$$\begin{cases} u = \langle p, y \rangle, \\ z = y - \langle p, y \rangle q. \end{cases} \tag{14}$$

The scalar u and the vector z can be considered as new coordinates on $\mathbf{R}^4$. By the Fredholm alternative [10], the components of z always satisfy the orthogonality condition $\langle p, z \rangle = 0$.

In these new coordinates, system (13) with $e = e_{0s}$ can be written as [10]

$$\begin{cases} \dot{u} = \langle p, F(uq + z) \rangle, \\ z = J(e_{0s}) z + F(uq + z) - \langle p, F(uq + z) \rangle q, \end{cases} \tag{15}$$

that is

$$\begin{cases} \dot{u} = \frac{a^3 c}{1-a^2} u^3 + \frac{a^2 c}{1-a^2} u \left(z_2 + z_4\right) + \frac{ac}{1-a^2} \left(z_2^2 + z_4^2\right), \\ z = J\left(e_{0s}\right) z + \begin{pmatrix} 0 \\ -c\left(au + z_2\right)^2 \\ 0 \\ -c\left(au + z_4\right)^2 \end{pmatrix} - \frac{ac}{2(1-a^2)} \left(2a^2 u^2 + 2au\left(z_2 + z_4\right) + \left(z_2^2 + z_4^2\right)\right) q. \end{cases} \tag{16}$$

The center manifold has the representation

$$z = V\left(u\right) = \frac{1}{2} w_2 u^2 + O\left(u^3\right), \tag{17}$$

where $w_2 \in T^{su}$, that is $\langle p, w_2 \rangle = 0$. The vector w_2 also satisfies the equation $J\left(e_{0s}\right) w_2 + A = 0$, where $A = -\frac{2a^2 c}{1-a^2}\left(a, 1, a, 1\right) \in \mathbf{R}^4$. From the above conditions we obtain

$$w_2 = -\frac{a^3 c}{\left(1-a^2\right)^2}\left(a, 2-a^2, a, 2-a^2\right).$$

Substituting in (16) and (17) the expression of w_2 we obtain:

Proposition 1. *The restriction of (16) to the center manifold has the form*

$$\dot{u} = \frac{a^3 c}{1-a^2} u^2 + O\left(u^3\right).$$

In addition, since $\frac{a^3 c}{1-a^2} \neq 0$, the equilibrium e_{0s} is a nondegenerated saddle-node and around it a nondegenerated fold bifurcation takes place.

Returning to the y coordinates, we get the following result.

Proposition 2. *For $\Delta = 0$, $a^2 - 1 \neq 0$, the center manifold corresponding to e_{0s} can be written as*

$$y_1 = y_3, \quad y_2 = y_4, \quad y_1 - \frac{1}{1-a^2}\left(y_1 - a y_2\right) + \frac{a^4 c}{2\left(1-a^2\right)^3}\left(y_1 - a y_2\right)^2 = 0. \tag{18}$$

Case 2. For parameters situated in the set SN_2 we have $\Delta > 0$ and $2ad + \sqrt{\Delta} = 0$. The Jacobi matrix $J\left(e_{2s}\right)$ of the equilibrium point $e_{2s} = \left(c\alpha^2, \alpha, c\alpha^2, \alpha\right)$ of (4), with $\alpha = \frac{1+2ad}{2ac}$, has the eigenvalues $\lambda_3 = 0$, $\lambda_4 = a - \frac{1}{a}$, $\lambda_1 \lambda_2 = -2ad > 0$, $\lambda_1 + \lambda_2 = a - \frac{1+2ad}{a}$.

Consider that $a^2 - 1 \neq 0$ and $a^2 - 1 - 2ad \neq 0$. This means that $\lambda_4 \neq 0$, and the parameters are not situated in H_1 or H_2. Thus $J\left(e_{2s}\right)$ has a simple zero eigenvalue and the corresponding critical eigenspace T^c is spanned by the eigenvector $q = (1, a, -1, -a) \in \mathbf{R}^4$. Let $p = \frac{1}{2(1-a^2)}\left(1, -a, -1, a\right) \in \mathbf{R}^4$ be the normalized adjoint vector.

Performing the change (14), system (13) with $e = e_{2s}$ reads

$$\begin{cases} \dot{u} = \frac{a^2 c}{1-a^2} u \left(z_2 + z_4\right) + \frac{ac}{2(1-a^2)} \left(z_2^2 - z_4^2\right), \\ z = J\left(e_{2s}\right) z + \begin{pmatrix} 0 \\ -c\left(au + z_2\right)^2 \\ 0 \\ -c\left(au + z_4\right)^2 \end{pmatrix} - \frac{ac}{2(1-a^2)} \left(2au\left(z_2 + z_4\right) + z_2^2 - z_4^2\right) q. \end{cases} \tag{19}$$

As for the previous case, we obtain the following result.

Proposition 3. *If $\Delta > 0$, $a^2 - 1 \neq 0$, $a^2 - 1 - 2ad \neq 0$, the center manifold corresponding to e_{2s} can be written as*

$$y_2 = ay_1, \quad y_4 = ay_2, \quad y_1 + y_3 + \frac{ac}{4d}\left(y_1 - y_3\right)^2 = 0. \tag{20}$$

Taking into account (20), from (19) we obtain.

Proposition 4. *The restriction of (19) to the center manifold (20) is*

$$\dot{u} = \frac{a^4 c^2}{d\left(1 - a^2\right)} u^3. \tag{21}$$

In addition, since in D we have $\frac{a^4c^2}{d(1-a^2)} \neq 0$, the equilibrium e_{2s} is a degenerated saddle-node of order two. On the center manifold a degenerated fold bifurcation takes place around e_{2s}.

Remark also that as $a \in (-1, 0)$ the coefficient of u^3 is positive, therefore the solution $u = 0$ of (21) is weakly repulsive and so is e_{2s} on the center manifold. Similarly, as $a < -1$, e_{2s} is weakly attractive on the center manifold.

The bifurcation corresponding to the other nonhyperbolic singularities, namely of Hopf, double-zero of fold-Hopf type, will be treated elsewhere.

References

[1] Beckmann M., Ryder H. *Simultaneous price and quantity adjustment in a single market.* Econometrica, 1969, **37**, p. 470–484.

[2] Marshall A. *Principles of Economics.* London, Mac Millan, 1890.

[3] Samuelson P.A., Nordhaus W. *Economie politica.* Teora, 2000.

[4] Tu P. *Dynamical systems. An introduction with applications in economics and biolog.* Berlin, Springer, 1994.

[5] Georgescu C. *Contributions to the study of some o.d.e's from economy.* PhD Thesis, Univ. of Piteşti, 2004 (in Romanian).

[6] Campbell Sue Ann, Waite M. *Multistability in coupled FitzHugh-Nagumo oscillators.* Nonlinear Analysis, 2001, **47**, p. 1093–1104.

[7] Volkov E., Romanov V. *Bifurcations in the system of two identical diffusively coupled Brusselators.* Physica Scripta, 1995, **51**, p. 19–28.

[8] Rocşoreanu C., Sterpu M. *Nonhyperbolic singularities in a system of two coupled oscillators* Proc. of 11th Conf. Appl. Ind. Math., CAIM 2003, University of Oradea, Romania, 2003, p. 195–201.

[9] Sterpu M., Rocşoreanu C. *Hopf bifurcation in a system of two symmetrically coupled advertising oscillators*. Nonlinear Analysis: Real World Application, 2005, **6**, p. 1–2.

[10] Kuznetsov Yu. *Elements of applied bifurcation theory.* New York, Springer, 1995.

[11] Wolfram S. *Mathematica-A system for doing mathematics by computer*. Wolfram Research Inc., 1994.

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Some hyperbolic manifolds

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In article [1] the authors construct and classify all the hyperbolic space-forms H_n/Γ where Γ is a torsion-free subgroup of minimal index in the congruence two subgroup Γ_n^2 for $n = 3, 4$. In the present paper some hyperbolic 3- and 4-manifolds are constructed that are absent in [1].

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Variety of the center and limit cycles of a cubic system, which is reduced to Lienard form

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Abstract. In the present work for the system $\dot{x} = y(1 + Dx + Px^2)$, $\dot{y} = -x + Ax^2 + 3Bxy + Cy^2 + Kx^3 + 3Lx^2y + Mxy^2 + Ny^3$ 25 cases are given when the point $O(0,0)$ is a center. We also consider a system of the form $\dot{x} = yP_0(x)$, $\dot{y} = -x + P_2(x)y^2 + P_3(x)y^3$, for which 35 cases of a center are shown. We prove the existence of systems of the form $\dot{x} = y(1 + Dx + Px^2)$, $\dot{y} = -x + \lambda y + Ax^2 + Cy^2 + Kx^3 + 3Lx^2y + Mxy^2 + Ny^3$ with eight limit cycles in the neighborhood of the origin of coordinates.

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1. We will consider the system of differential equations

$$\dot{x} = y(1+Dx+Px^2),\ \dot{y} = -x+Ax^2+3Bxy+Cy^2+Kx^3+3Lx^2y+Mxy^2+Ny^3, \quad (1)$$

where A, B, C, D, K, L, M, N, P are real constants. The origin of coordinates of system (1) is a critical point of the center or focus type. The center-focus problem for (1) in the case of $D = P = 0$ was first investigated by I.S. Kukles in [1]. In [2, 3] for the system (1) for $D = P = 0$ necessary and sufficient center conditions of algebraic nature were given. For $B = D = P = 0$ the solution of the center-focus problem for (1) is in [4–7]. In the case of $N = 0$ the center-focus problem for (1) was solved in [8]. In [9] all the cases of the center for system (1) for $D = P = 0$ were found, although their necessity was not established completely. Using Cherkas method [10; 11, p.70] the center-focus problem for $D = P = 0$ was solved in [12]; on the basis of investigation of focal values the solution of this problem was reduced in [13]. In [13] the existence of cubic systems of nonlinear oscillations with seven limit cycles was also proved. In [14] it was shown that in the case of the existence of invariant straight line the necessary and sufficient center condition is the equality to zero of the first five focal values. The case of reversible system of the type (1) from the class $C\mathbb{R}_3^{10}$ was shown in [15].

Together with the system (1) we consider a system of the form

$$\dot{x} = yP_0(x),\ \dot{y} = -x + P_2(x)y^2 + P_3(x)y^3, \quad (2)$$

where $P_0(x) = 1 + \sum_{k=1}^{4} c_k x^k$, $P_2(x) = \sum_{k=0}^{3} a_k x^k$, $P_3(x) = \sum_{k=0}^{4} b_k x^k$, a_i, b_j, $c_k \in \mathbb{C}$, $i = \overline{0,3}$, $j = \overline{0,4}$, $k = \overline{1,4}$. System (1) by change $y = (1 - Ax - Kx^2)Y/[1 + (B + Lx)Y]$ and change of time [3] is transformed to the system (2), where

$$
\begin{gathered}
a_0 = A + C, \qquad a_1 = 3B^2 + A(D - C) + 2K + M, \\
a_2 = K(2D - C) + 6BL + A(P - M), \qquad a_3 = 3L^2 + K(2P - M), \\
c_1 = D - A, \qquad c_2 = P - K - AD, \qquad c_3 = -DK - AP, \\
c_4 = -KP, \qquad b_0 = B(A + C) + L + N, \\
b_1 = B[2B^2 + A(D - C) + 2K + M] + L(C + D) - 2AN, \qquad (3) \\
b_2 = B[K(2D - C) + 6BL + A(P - M)] + L(K + P - AC) + N(A^2 - 2K), \\
b_3 = B[6L^2 + K(2P - M)] + L[K(D - C) - AM] + 2AKN, \\
b_4 = L[2L^2 + K(P - M)] + K^2N.
\end{gathered}
$$

There exists a formal series for system (1)

$$
U = x^2 + y^2 + \sum_{i+j=3}^{\infty} q_{i,j} x^i y^j, \qquad (4)
$$

for which on account of (1)

$$
\dot{U} = \sum_{i=1}^{\infty} f_i (x^2 + y^2)^{i+1},
$$

where f_i, $i = 1, 2, \ldots$, are the focal values of system (1). If in (4) $q_{0,2i} = 0$, $i = 2, 3, \ldots$, then the function U and focal values f_i, $i = 1, 2, \ldots$, are defined in a unique way.

Let us form the ideal [16, p. 46] $J = \langle f_1, \ldots, f_9, \ldots \rangle$, where f_i, $i = 1, 2, \ldots$, are the focal values of system (1). Together with the ideal J we will use the ideals $\overline{J}_i = \langle f_1, \ldots, f_i \rangle$, $i = 1, 2, \ldots$. The first focal value of system (1) has the form: $f_1 = B(A + C) + L + N$, the second focal value f_2 has 38 summands, the third – 192, the 4th – 702, the 5th – 2093, the 6th – 5406, the 7th – 12538, the 8th - 26726, the 9th – 53212. To compute the focal values we use computer package MATHEMATICA 5.0. The program for the computing of the focal values is in the paper [13].

The focal values f_i, $i = 1, 2, \ldots$, are the polynomials from the ring $\mathbb{C}[q]$, where $q = (A, B, C, D, K, L, M, N, P)$, that's why $J, \overline{J}_i \subset \mathbb{C}[q]$, $i = 1, 2, \ldots$. The variety of ideal J is the set [16, p. 108] $\mathbb{V}(J) = \{q \in \mathbb{C}^9 : \forall f \in J \;\; f(q) = 0\}$, which we name a variety of the center of system (1). For all i, $i = 1, 2, \ldots$, $\mathbb{V}(\overline{J}_i) \supset \mathbb{V}(J)$. It is obvious that the critical point $O(0, 0)$ of system (1) is a center if and only if $q \in \mathbb{V}(J)$. Thus a solution of the center-focus problem for system (1) is reduced to finding the variety $\mathbb{V}(J)$.

The next result takes place [8]:

Theorem 1. *The next equality is true:* $\mathbb{V}(N)\bigcap\mathbb{V}(J)=\bigcup\limits_{k=1}^{11}\mathbb{V}(J_k)$, *where*

$J_1=\langle B,\,L,\,N\rangle$, $J_2=\langle A,\,C,\,D,\,L,\,N\rangle$, $J_3=\langle A+C,\,A-D,\,N,\,2K-M,\,K+P,\,L\rangle$, $J_4=\langle A+C,\,N,\,2K(A+2D)-AM,\,M-2P,\,L\rangle$, $J_5=\langle A+2C,\,3A+2D,\,N,\,A^2-2P,\,AB+2L\rangle$, $J_6=\langle 2A+3C,\,N,\,2A^2(A+D)+(7A+6D)K,\,2(A+D)(A+2D)+M,\,(A+D)(A+2D)+P,\,AB+3L\rangle$, $J_7=\langle 4A+5C+D,\,N,\,2(A+C)(A+2C)-K,\,2(A+C)(3A+4C)-M,\,(A+C)(3A+4C)-P,\,B(A+C)+L\rangle$, $J_8=\langle 5A+6C+D,\,N,\,A(A+C)(2A+3C)+(5A+8C)K,(A+C)(2A+3C)+M,\,3(A+C)(2A+3C)-P,\,B(A+C)+L\rangle$, $J_9=\langle 7A+9C+2D,\,N,\,(A+C)(A+3C)^2-(2A+5C)K,\,(A+C)(2A+3C)-2M,\,3(A+C)(2A+3C)-2P,\,B(A+C)+L\rangle$, $J_{10}=\langle N,\,C(A+C)-K,\,C(A+C)(C-D)+(A+2C)M-CP,\,B(A+C)+L\rangle$, $J_{11}=\langle N,\,A(A+C)(2A+C+D)+(5A+4C+2D)K,\,(A+C)(2A+C+D)-M,\,(A+C)(A+C+D)+P,\,B(A+C)+L\rangle$.

For system (2) we can construct the series (4), for which $\dot{U}=\sum\limits_{i=1}^{\infty}g_i(x^2+y^2)^{i+1}$, where g_i, $i=1,2,\ldots$, are the focal values of system (2). The first focal value of system (2) has the form $g_1=b_0$, the second - $g_2=3a_0b_1+b_2$, the third - $g_3=3b_4+b_3(13a_0+2c_1)-3b_1(15a_0^3-2a_0a_1-a_2+5a_0^2c_1+a_0c_2)$, g_4 contains 32 summands, g_5 - 98, g_6 - 241, g_7 - 540, g_8 - 1084, g_9 - 2024, g_{10} - 3581, g_{11} - 6039, g_{12} - 9772, g_{13} - 15325. Let's introduce $h=(a_0,\,a_1,\,a_2,\,a_3,\,b_0,\,b_1,\,b_2,\,b_3,\,b_4,\,c_1,\,c_2,\,c_3,\,c_4)$. The focal values g_i, i=1, 2, ..., are the polynomials from the ring $\mathbb{C}[h]$. Now we form the ideal I=$\langle g_1,\,g_2,\ldots\rangle\subset\mathbb{C}[h]$. The variety of the center of system (2) is a set $\mathbb{V}(I)$=$\{h\in\mathbb{C}^{13}:\forall\, g\in I\;\; g(h)=0\}$. Together with ideal I we will consider the ideals $\overline{I}_j$=$\langle g_1,\ldots,g_j\rangle$, $j=1,2,\ldots$.

The first necessary center condition for system (2) has the form $b_0=0$, then the polynomial $P_3(x)$ can be represented as $P_3(x)=xQ(x)$, where $Q(x)$ is the polynomial of the 3rd degree. Let's denote

$$R_1(x)=Q'(x)P_0(x)+3Q(x)P_2(x)$$

then the second necessary center condition is $R_1(0)=0$. Taking into account this condition we have $R_1(x)\equiv xQ_1(x)$, where $Q_1(x)$ is the polynomial of the 5th degree. The next statement is correct [12]:

Theorem 2. *The origin of system (2) is a center if and only if*

$$b_0=0,\;\; R_k(0)=0,\;\; k=1,2,\ldots, \tag{5}$$

where $R_1(x)$ *is expressed by the formula* (4), $R_k(x)\equiv Q'_{k-1}(x)P_0(x)+(2k+1)Q_{k-1}(x)P_2(x)$, $Q_k(x)\equiv R_{k-1}(x)/x$, $\quad k=2,3,\ldots$.

Let for system (2) the first four necessary conditions from (5) be held. Then the next theorem takes place:

Theorem 3 [11, p. 70]. *The origin of system (2) is a center if and only if the system of equations*

$$Q^5(x)R_1^3(y)=R_1^3(x)Q^5(y),\;\; P_0(x)S(x)R_1^2(y)=R_1^2(x)P_0(y)S(y), \tag{6}$$

where the polynomials $R_1(x)$, $S(x)$ *and coefficients* r_i, s_i, $i=\overline{0,5}$, *have the form:*

$$R_1(x)=[Q'(x)P_0(x)+3Q(x)P_2(x)]/x\equiv\sum_{k=0}^{5}r_kx^k,$$

$$S(x)=[3R_1'(x)Q(x)-5Q'(x)R_1(x)]/x\equiv\sum_{k=0}^{5}s_kx^k,$$

$r_0=2b_3-3b_1(3a_0^2-a_1+a_0c_1)$, $r_1=3b_4+b_3(3a_0+2c_1)-3b_1(3a_0a_1-a_2+a_0c_2)$, $r_2=3b_4(a_0+c_1)+b_3(3a_1+2c_2)-3b_1(3a_0a_2-a_3+a_0c_3)$, $r_3=3b_4(a_1+c_2)+b_3(3a_2+2c_3)-3a_0b_1(3a_3+c_4)$, $r_4=3b_4(a_2+c_3)+b_3(3a_3+2c_4)$, $r_5=3b_4(a_3+c_4)$, $s_0=-2[5(3a_0^2b_1+b_3)r_0-3b_1r_2]$, $s_1=5(7a_0b_3-3b_4)r_0-3b_1(a_0r_2-3r_3)$, $s_2=4[15a_0b_4r_0-b_3r_2-3b_1(a_0r_3-r_4)]$, $s_3=-9b_4r_2-b_3r_3-3b_1(7a_0r_4-5r_5)$, $s_4=-2(3b_4r_3-b_3r_4+15a_0b_1r_5)$, $s_5=-3b_4r_4+5b_3r_5$, *has an analytical in the neighborhood of* $x=0$ *solution* $y=\psi(x)$, $\psi(0)=0$, $\psi'(0)=-1$, *or at least one of the equations of system (6) is an identity.*

2. We will consider the solution of the center-focus problem for system (2) under various assumptions for the coefficients a_i, $i=\overline{0,3}$, b_j, $j=\overline{0,4}$, c_k, $k=\overline{1,4}$.

To formulate a theorem we introduce the ideals $E_k\subset\mathbb{C}[h]$, $k=\overline{1,9}$:

$E_1=\langle b_3,\,b_4\rangle$, $E_2=\langle 9a_0^2b_1-4b_3,\,b_4\rangle$, $E_3=\langle b_4+a_0^3b_1,\,b_3-3a_0^2b_1\rangle$, $E_4=\langle a_2-a_0(3a_0^2-2a_1+a_0c_1+c_2),\,2(135a_0^3+81a_0^2c_1+17a_0c_1^2+c_1^3)-(4a_0-c_1)c_2-3c_3\rangle$, $E_5=\langle a_3-t(a_2-a_1t+a_0t^2),\,b_3+b_1t(3a_0+t),\,b_4,\,c_3-t(2c_2-3c_1t+4t^2),\,c_4-t^2(c_2-2c_1t+3t^2)\rangle\bigcap\mathbb{C}[h]$, $E_6=\langle a_2-t(a_1-c_2-a_0t+2c_1t-3t^2),\,a_3+t^2(c_2-2c_1t+3t^2),\,b_4-t(b_3+3a_0b_1t+b_1t^2),\,c_3-t(2c_2-3c_1t+4t^2),\,c_4-t^2(c_2-2c_1t+3t^2)\rangle\bigcap\mathbb{C}[h]$, $E_7=\langle a_1,\,a_2,\,a_3,\,c_2,\,c_3,\,c_4\rangle$, $E_8=\langle r_0,\,r_1,\,r_2,\,r_3,\,r_4,\,r_5\rangle$, $E_9=\langle s_0,\,s_1,\,s_2,\,s_3,\,s_4,\,s_5\rangle$.

Let $W=\bigcup\limits_{k=1}^{9}\mathbb{V}(E_k)$, $W\subset\mathbb{C}^{13}$.

Further we will use the next notations: $\alpha=a_1-a_0(a_0+c_1)$, $\beta=a_0(3a_0+c_1)-a_1$, $\gamma=3a_0+c_1$, $\delta=3a_0+2c_1$, $\sigma=5a_0+c_1$, $\mu=3a_0b_1+b_2$, $\nu=9a_0^2b_1-4b_3$, $\tau=a_0(a_0^2+a_0c_1+c_2)-a_2$, $\xi=a_0(3a_0+c_1)+a_1$. We will denote by I_j, $j=\overline{1,13}$, the following ideals:

$I_1=\langle b_0,\,b_1,\,b_2,\,b_3,\,b_4\rangle$, $I_2=\langle a_0,\,a_2,\,a_3,\,b_0,\,b_2,\,b_3,\,b_4\rangle$,

$I_3=\langle a_0,\,a_2,\,c_1,\,c_3,\,b_0,\,b_2,\,b_4\rangle$, $I_4=\langle\beta,\,a_2-a_0(3a_0\gamma+c_2),\,3a_0(3a_2+c_3)+c_4,\,a_3-a_0(3a_2+c_3),\,b_4,\,b_3,\,\mu,\,b_0\rangle$, $I_5=\langle 2a_1-a_0\delta,\,4a_2-a_0(3a_0\delta+4c_2),\,3a_0(3a_2+2c_3)+4c_4,\,2a_3-a_0(3a_2+2c_3),\,b_4,\,\nu,\,\mu,\,b_0\rangle$, $I_6=\langle 3\alpha(6a_0+c_1)+6a_0(3a_0\delta+c_2)+2c_3,\,3\beta(3\xi+2c_2)-4c_4,\,2a_2-\alpha(9a_0+2c_1)-2a_0(a_0c_1+c_2),\,2a_3+\beta(3\xi+2c_2),\,b_4,\,2b_3-3\alpha b_1,\,\mu,\,b_0\rangle$, $I_7=\langle 3\xi+2c_2,\,2a_2-\gamma\beta+2c_3,\,\gamma(a_2+\gamma\beta)+c_4,\,a_3-\gamma(a_2+\gamma\beta),\,b_4+b_1(a_2+\gamma\beta),\,2b_3-3b_1\beta,\,\mu,\,b_0\rangle$,

$I_8=\langle\alpha^2(2\gamma^2-\alpha)+2\tau(2\alpha\gamma+\tau),\,\delta\alpha^2+2\tau[3a_0(2a_0+c_1)+c_2]-\alpha[5a_2-a_0\gamma(11a_0+6c_1)-c_2(7a_0+2c_1)+2c_3],\,5\alpha^3-2\alpha^2[13a_0\delta+c_1(a_0+5c_1)-c_2]-2\alpha[a_0(135a_0^3+161a_0^2c_1+56a_0c_1^2+6c_1^3)-a_2(27a_0+8c_1)+c_2(45a_0^2+20a_0c_1+2c_1^2)]-4\tau[a_0(22a_0^2+18a_0c_1+3c_1^2)+\sigma c_2+c_3],\,3\alpha^2-2\alpha(c_1^2+2a_0\delta-c_2)-2[a_0^2\gamma(a_0+c_1)-a_2(4a_0+c_1)+a_0(\gamma c_2-c_3)-c_4],\,3\alpha^2-2\alpha(c_1^2+2a_0\delta-c_2)-2[a_3+a_0^2\gamma(a_0+c_1)-a_2(4a_0+c_1)+a_0(\gamma c_2-c_3)],\,2b_4-b_1[\alpha(9a_0+2c_1)-2(a_0^3-\tau)],\,2b_3-3b_1\beta,\,\mu,\,b_0\rangle$,

$I_9=\langle a_2-a_0(a_1+c_2-3\alpha),\,a_0[a_0(a_1+c_2-\alpha)+c_3]+c_4,\,a_3-a_0[a_0(a_1+c_2)+$

$c_3]$, $a_0^3b_1+b_4$, $3a_0^2b_1-b_3$, μ, $b_0\rangle$, $I_{10}=\langle 2a_0^2(8a_0^2-c_2)+c_4$, $a_0(20a_0^2-3c_2)-c_3$, $a_2+a_0(12a_0^2+2a_1-c_2)$, σ, $a_0(2a_0^2b_1-b_3)-b_4$, μ, $b_0\rangle$, $I_{11}=\langle a_2+a_0(15a_0^2+2a_1-c_2)$, $10a_0(9a_0^2-c_2)-3c_3$, $6a_0+c_1$, $a_0^2(9a_0^2-c_2)+c_4$, $3a_3-a_0^2(18a_0^2+3a_1-c_2)$, $a_0b_3+3b_4$, μ, $b_0\rangle$, $I_{12}=\langle 2a_2+a_0(21a_0^2+4a_1-2c_2)$, $a_0(189a_0^2-34c_2)-12c_3$, $a_0^2(27a_0^2-4c_2)+2c_4$, $3a_3-a_0^2(36a_0^2+3a_1-4c_2)$, $9a_0+2c_1$, $a_0\nu-3b_4$, μ, $b_0\rangle$,

$I_{13}=\langle(13a_0+2c_1)[2(30a_0^2+12a_0c_1+c_1^2)+c_2]+3a_0^3$, $2c_3+a_0(81a_0^2+36a_0c_1+2c_1^2+7c_2)$, $a_3-a_0^2(a_1+2c_2+219a_0^2+87a_0c_1+8c_1^2)$, $a_2+a_0(3\alpha-a_1-c_2)$, $2a_0^2(135a_0^2+54a_0c_1+5c_1^2+c_2)+c_4$, $3b_4-6a_0^2b_1(6a_0+c_1)+b_3(13a_0+2c_1)$, μ, $b_0\rangle$.

Notice that the bases of ideals I_j, $j=\overline{1,13}$, have no more than nine elements. Further we introduce the ideals I_j, $j=\overline{14,35}$, which have the form:

$I_{14}=\langle 5a_0^2+2a_1$, a_2, a_3, b_0, μ, $5a_0^2b_1-4b_3$, b_4, σ, $25a_0^2-4c_2$, c_3, $c_4\rangle$, $I_{15}=\langle 15a_0^2+4a_1$, $75a_0^3-16a_2$, $125a_0^4+64a_3$, b_0, μ, $35a_0^2b_1-16b_3$, b_4, σ, $75a_0^2-8c_2$, $125a_0^3+16c_3$, $625a_0^4-256c_4\rangle$, $I_{16}=\langle 10a_0^2+3a_1$, $25a_0^3-9a_2$, a_3, b_0, μ, $20a_0^2b_1-9b_3$, b_4, σ, $25a_0^2-3c_2$, $125a_0^3+27c_3$, $c_4\rangle$, $I_{17}=\langle a_1$, a_2, a_3, b_0, μ, $10a_0^2b_1+b_3$, b_4, σ, c_2, c_3, $c_4\rangle$,

$I_{18}=\langle a_0(7a_0^2+2a_1)+a_2$, $3a_0^2(2a_0^2+a_1)+a_3$, b_0, μ, b_3, b_4, $4a_0+c_1$, $2a_0^2+c_2$, $12a_0^3-c_3$, $9a_0^4-c_4\rangle$, $I_{19}=\langle 2a_0(2a_0^2+a_1)+a_2$, a_3, b_0, μ, b_3, b_4, $7a_0+c_1$, $16a_0^2-c_2$, $12a_0^3+c_3$, $c_4\rangle$, $I_{20}=\langle 5a_0^2+a_1$, a_2, a_3, b_0, μ, b_3, b_4, $10a_0+c_1$, $25a_0^2-c_2$, c_3, $c_4\rangle$, $I_{21}=\langle 5a_0^2+a_1$, $25a_0^3-4a_2$, a_3, b_0, μ, b_3, b_4, $15a_0+2c_1$, $75a_0^2-4c_2$, $125a_0^3+8c_3$, $c_4\rangle$, $I_{22}=\langle 5a_0^2+a_1$, $25a_0^3-3a_2$, $125a_0^4+27a_3$, b_0, μ, b_3, b_4, $20a_0+3c_1$, $50a_0^2-3c_2$, $500a_0^3+27c_3$, $625a_0^4-81c_4\rangle$,

$I_{23}=\langle 7a_0^2+2a_1$, a_2, a_3, b_0, μ, ν, b_4, $7a_0+c_1$, $49a_0^2-4c_2$, c_3, $c_4\rangle$, $I_{24}=\langle 2a_0(2a_0^2+a_1)+a_2$, a_3, b_0, μ, ν, b_4, $11a_0+2c_1$, $10a_0^2-c_2$, $6a_0^3+c_3$, $c_4\rangle$, $I_{25}=\langle 7a_0^2+2a_1$, $49a_0^3-16a_2$, a_3, b_0, μ, ν, b_4, $21a_0+4c_1$, $147a_0^2-16c_2$, $343a_0^3+64c_3$, $c_4\rangle$, $I_{26}=\langle a_0(13a_0^2+8a_1)+4a_2$, $3a_0^2(2a_0^2+a_1)-4a_3$, b_0, μ, ν, b_4, $4a_0+c_1$, $11a_0^2-2c_2$, $3a_0^3+c_3$, $9a_0^4-16c_4\rangle$, $I_{27}=\langle 7a_0^2+2a_1$, $49a_0^3-12a_2$, $343a_0^4+216a_3$, b_0, μ, ν, b_4, $14a_0+3c_1$, $49a_0^2-6c_2$, $343a_0^3+54c_3$, $2401a_0^4-1296c_4\rangle$,

$I_{28}=\langle 2a_0(2a_0^2+a_1)+a_2$, a_3, b_0, μ, b_3, $4a_0^3b_1-b_4$, γ, c_2, $4a_0^3-c_3$, $c_4\rangle$, $I_{29}=\langle 2a_0(2a_0^2+a_1)+a_2$, a_3, b_0, μ, ν, $a_0^3b_1+2b_4$, $9a_0+2c_1$, $6a_0^2-c_2$, $2a_0^3+c_3$, $c_4\rangle$, $I_{30}=\langle 5a_0^2+2a_1$, $25a_0^3-16a_2$, a_3, b_0, μ, $45a_0^2b_1-16b_3$, $25a_0^3b_1+32b_4$, $15a_0+4c_1$, $75a_0^2-16c_2$, $125a_0^3+64c_3$, $c_4\rangle$,

$I_{31}=\langle 14a_0^2+5a_1$, $49a_0^3-25a_2$, a_3, b_0, μ, $72a_0^2b_1-25b_3$, $112a_0^3b_1+125b_4$, $21a_0+5c_1$, $147a_0^2-25c_2$, $343a_0^3+125c_3$, $c_4\rangle$, $I_{32}=\langle 5a_0^2+3a_1$, a_2, a_3, b_0, μ, $5a_0^2b_1-3b_3$, $25a_0^3b_1-27b_4$, $10a_0+3c_1$, $25a_0^2-9c_2$, c_3, $c_4\rangle$, $I_{33}=\langle 7a_0^2+3a_1$, a_2, a_3, b_0, μ, $5a_0^2b_1-3b_3$, $7a_0^3b_1+27b_4$, $14a_0+3c_1$, $49a_0^2-9c_2$, c_3, $c_4\rangle$, $I_{34}=\langle a_1$, a_2, a_3, b_0, μ, $24a_0^2b_1+b_3$, $28a_0^3+b_4$, $7a_0+c_1$, c_2, c_3, $c_4\rangle$, $I_{35}=\langle a_1$, a_2, a_3, b_0, μ, $15a_0^2b_1+4b_3$, $25a_0^3-2b_4$, $5a_0+2c_1$, c_2, c_3, $c_4\rangle$.

Notice that the bases of ideals $I_{14},...,I_{35}$ contain 10 or 11 elements.

It is significant that $\mathbb{V}(I_k)$, $k=\overline{1,35}$, are irreducible varieties. Let $V=\bigcup\limits_{k=1}^{35}\mathbb{V}(I_k)$.

Theorem 4. *The next equality takes place:* $V=W\bigcap\mathbb{V}(I)$.

The proof of Theorem 4 is given in p. 3.

Theorem 4 gives the solution of the center-focus problem for system (2) in the case of $h \in W$. It is obvious that $V \subset \mathbb{V}(I)$. Question: is it true that $W \supset \mathbb{V}(I)$? If $W \supset \mathbb{V}(I)$ then $V = \mathbb{V}(I)$, i.e. in that case Theorem 4 gives the solution of the center-focus problem for system (2).

We will point out further the solution of the center-focus problem for system (1) under different assumptions for the coefficients A, B, C, D, K, L, M, N, P. Let's construct the ideals $G_i \subset \mathbb{C}[q]$, $i = \overline{1,17}$, in which $a_i\,(i = \overline{0,3})$, $b_j\,(j = \overline{0,4})$, c_k $(k = \overline{1,4})$ are expressed by the formulas (3):

$G_1 = \langle b_3,\, b_4\rangle$, $G_2 = \langle 4b_3 - 9a_0^2 b_1,\, b_4\rangle$, $G_3 = \langle b_4 + a_0^3 b_1,\, b_3 - 3a_0^2 b_1\rangle$, $G_4 = \langle a_2 - a_0(3a_0^2 - 2a_1 + a_0c_1 + c_2),\, 2(135a_0^3 + 81a_0^2c_1 + 17a_0c_1^2 + c_1^3) - (4a_0 - c_1)c_2 - 3c_3\rangle$, $G_5 = \langle a_3 - t(a_2 - a_1t + a_0t^2),\, b_3 + b_1t(3a_0 + t),\, b_4,\, c_3 - t(2c_2 - 3c_1t + 4t^2),\, c_4 - t^2(c_2 - 2c_1t + 3t^2)\rangle \bigcap \mathbb{C}[q]$, $G_6 = \langle a_2 - t(a_1 - c_2 - a_0t + 2c_1t - 3t^2),\, a_3 + t^2(c_2 - 2c_1t + 3t^2),\, b_4 - t(b_3 + 3a_0b_1t + b_1t^2),\, c_3 - t(2c_2 - 3c_1t + 4t^2),\, c_4 - t^2(c_2 - 2c_1t + 3t^2)\rangle \bigcap \mathbb{C}[q]$, $G_7 = \langle a_1,\, a_2,\, a_3,\, c_2,\, c_3,\, c_4\rangle$, $G_8 = \langle r_0,\, r_1,\, r_2,\, r_3,\, r_4,\, r_5\rangle$, $G_9 = \langle s_0,\, s_1,\, s_2,\, s_3,\, s_4,\, s_5\rangle$,

$G_{10} = \langle N\rangle$, $G_{11} = \langle B\rangle$, $G_{12} = \langle K,\, L\rangle$, $G_{13} = \langle A + C,\, 3K + M + P\rangle$, $G_{14} = \langle B(3A+3C+D)+L,\, (2A+3C+D)(3A+3C+D)-K\rangle$, $G_{15} = \langle 2(A+C)(2A+C+D)+3K+M,\, (A+C)(A+C+D)+P\rangle$, $G_{16} = \langle 3(A+C)(A+3C)-4K,\, 3B(A+C)+2L\rangle$, $G_{17} = \langle Kt-(N-Ct)(N-At-Ct),\, 2N^2-Nt(3C-D)+t^2[C(C-D)+M-t(t+3B)],\, 3Lt^4+N^2(N-2Ct)+Nt^2(C^2+M-P+t^2)-t^3[C(M-P)+t^2(A+C)]\rangle \bigcap \mathbb{C}[q]$.

Let $\widetilde{G} = \bigcup\limits_{k=1}^{17} \mathbb{V}(G_i)$ and $T = \bigcup\limits_{k=1}^{25} \mathbb{V}(J_k)$, where the ideals $J_i \subset \mathbb{C}[q]$, $i = \overline{12,25}$, have the form:

$J_{12} = \langle A+C,\, K(A-D)+AM+3BN,\, K(2K+M)-N^2,\, 3K+M+P,\, L+N\rangle$,

$J_{13} = \langle 3B^2-(2A+3C)(4A+3C+D),\, B(A+C)+N,\, K,\, 2(A+C)(3A+3C+D)+M,\, 3(A+C)(3A+3C+D)+P,\, L\rangle$, $J_{14} = \langle 6B^2-(A+3C)(A+D),\, B(A+C)-2N,\, 3(A+C)(A+3C)-4K,\, (A+C)(3A+6C+D)+2M,\, 3(A+C)(3A+3C+2D)+4P,\, 3B(A+C)+2L\rangle$, $J_{15} = \langle 3B^2-(A+D)(2A+3C+D),\, B(2A+2C+D)-N,\, (2A+3C+D)(3A+3C+D)-K,\, (2A+3C+D)(3A+3C+D)+M,\, (3A+3C+D)(3A+3C+2D)+P,\, B(3A+3C+D)+L\rangle$,

$J_{16} = \langle 3(A+C)+D,\, B(A+2C)+2N,\, 3AB^2+4K(2A+3C),\, 6B^2-A(2A+3C)+4K+3M,\, 3B^2-2A(2A+3C)+2K+2P,\, AB+2L\rangle$, $J_{17} = \langle 3(A+C)+D,\, 4(A+3C)(2A+3C)^2+9B^2(5A+7C),\, 3B(A+2C)-N,\, (A+3C)(2A+3C)-K,\, 27B^2-4(2A+3C)(A+4C)-4M,\, 27B^2+12C(2A+3C)+4P,\, B(4A+7C)+L\rangle$,

$J_{18} = \langle 7A+9C+2D,\, 36B^2(11A+17C)+(A+3C)(17A+27C)^2,\, 3B(A+C)(3A+5C)-2N(17A+27C),\, 3(A+C)(A+3C)-4K,\, (A-3C)(171A+139C)-4(57A^2+27B^2+16M),\, 3(A+3C)(67A+73C)+4(16A^2+27B^2-16P),\, B(A+C)+L+N,\, t(17A+27C)-1\rangle \bigcap \mathbb{C}[q]$,

$J_{19} = \langle (A+C)(A+3C)(2A+C+D)-K(A-C+D)+3BN,\, (A+C)^2(A+2C)(2A+C+D)(2A+2C+D)+K[(A+C)^2+K]+N[3B(A+C)+N],\, 2(A+C)(2A+C+D)+3K+M,\, (A+C)(A+C+D)+P,\, B(A+C)+L+N\rangle$,

$J_{20} = \langle 2B^2(A+2C)-(2A+C+D)(A^2+4K)+4BN,\, A^2(2A+C+D)+2(2A+C)(B^2+K)+AM+2BN,\, B(2B^2+2K+M)-(2A+C+D)(BC+N),\, A(4A+$

$C+3D)+2(3B^2+3K+M+P)$, $B(A+C)+L+N$, $t(A(2A+C+D)^2+2B^2(3A+2C+D))-1\rangle \bigcap \mathbb{C}[q]$,

$J_{21} = \langle B^2K-(BC+N)[B(A+C)+N]$, $B(2B^2+2K+M)-(2A+C+D)(BC+N)$, $B[2AB^2-K(A+D)]+(BC+N)[4B^2+A(A+D)+K+P]$, $B(A+C)+L+N$, $tB(BC+N)-1\rangle \bigcap \mathbb{C}[q]$,

$J_{22} = \langle(6A+8C+D)(7A+9C+D)(7A+9C+2D)+2(A+C)(2A+3C)(11A+15C+2D)$, $27B^2-12(2A+3C)(10A+11C)+(7A+21C-23D)(7A+9C+D)$, $3B(4A+5C+D)(5A+7C+D)-(33A+45C+7D)N$, $(4A+6C+D)(5A+6C+D)-K$, $(A+C)(2A+3C)+(2A+3C+D)(7A+9C+D)+M$, $(7A+9C+D)(7A+9C+2D)+P$, $B(A+C)+L+N$, $t(33A+45C+7D)-1\rangle \bigcap \mathbb{C}[q]$,

$J_{23} = \langle(5A+3C+2D)[(A+3C)(11A+18C)+4(2A+3C)(A-3C+D)]-12B^2(4A+3C+2D)+4K(7A+6C+2D)$, $B[(A+3C)(4A+7C)+2(A+2C)(A-3C+D)-4K]-6B^3-2N(2A+D)$, $6B^2(3A+C+2D)-(2A+3C)[(5A+3C+2D)^2+4K]-12BN$, $6B^2-A(A+6C)-C(3C+2D)+2(2K+M)$, $3(A+C)(3A+3C+2D)+4P$, $B(A+C)+L+N$, $t(2A+D)(7A+6C+2D)-1\rangle \bigcap \mathbb{C}[q]$,

$J_{24} = \langle 3B^2(7A+6C+2D)+A[(2A+C+D)(2A+C+2D)-2(A+C)(3A+5C)]-K(A+2D)$, $B[(A+C)(7A+10C)+(A+2C)(2A+C+D)-2K]-3B^3+N(A+3C-D)$, $3B^2(3A+2C+D)-A[(A+C)(5A+8C)+(C-D)(2A+C+D)]+AK-3BN$, $3B^2-2A(A+3C)-C(3C+D)+2K+M$, $3(A+C)(3A+3C+D)+P$, $B(A+C)+L+N$, $t(A+2D)(A+3C+D)-1\rangle \bigcap \mathbb{C}[q]$,

$J_{25} = \langle A[(4A+3C+2D)(8A+9C+3D)-3B^2]+K(13A+12C+6D)$, $B[C(A+6C)+6(2A+C+D)(2A+3C+D)-2K]-3B^3-N(5A+3C+3D)$, $A(4A+3C+2D)(7A+9C+3D)+(A+3C)(3B^2+2K)+9BN$, $3B^2-(2A+3C+D)(4A+3C+2D)+2K+M$, $(3A+3C+D)(3A+3C+2D)+P$, $B(A+C)+L+N$, $t(5A+3C+3D)(13A+12C+6D)-1\rangle \bigcap \mathbb{C}[q]$;

and the ideals J_i, $i=\overline{1,11}$ are from Theorem 1.

Theorem 5. *The next equality takes place:* $T = \mathbb{V}(J)\bigcap \widetilde{G}$.

The proof of Theorem 5 is given in p. 4.

Theorem 6. *There exist systems of the form*

$$\dot{x} = y(1+Dx+Px^2),\ \dot{y} = -x+\lambda y+Ax^2+Cy^2+Kx^3+3Lx^2y+Mxy^2+Ny^3, \quad (7)$$

having eight limit cycles in any infinitely small neighborhood of the origin.

The proof is given in p. 5.

3. Now we will examine the focal values of system (2). From $g_1=0$ we have $b_0=0$, from $g_2=0$ we find $b_2=-3a_0b_1$. Taking into account b_0, b_2 from g_3 we obtain $b_4 = b_1(15a_0^3-2a_0a_1-a_2+5a_0^2c_1+a_0c_2)-b_3(13a_0+2c_1)/3$. Using the quantities of b_0, b_2, b_4 we can present g_k, $k=4,5,\ldots$, in the form: $g_k = v_kb_3+w_kb_1$, where $v_k, w_k \in \mathbb{C}[a_0, a_1, a_2, a_3, c_1, c_2, c_3, c_4]$. Construct the ideal $X = \langle v_4, w_4, v_5, w_5, \ldots\rangle + \langle g_1, g_2, g_3\rangle$. It is obvious that $X \supset I$.

Statement 1. *The next formula takes place:* $\mathbb{V}(X) = \mathbb{V}(I_3)\bigcup(\bigcup_{k=10}^{13} \mathbb{V}(I_k))$. *Here the ideals* I_3, I_k, $k=\overline{10,13}$, *are prime.*

Proof. Computing the Groebner basis [16, p. 105] for the ideal $X_7 = \langle v_4,\, w_4, \ldots, v_7,\, w_7\rangle + \langle g_1,\, g_2,\, g_3\rangle$ with the order

$$b_0 > b_2 > b_4 > c_3 > c_4 > a_3 > a_2 > a_1 > c_2 > c_1 > b_3 > b_1 > a_0$$

we get $X_7 = \langle a_0^4(5a_0+c_1)^7(6a_0+c_1)^2(9a_0+2c_1)^2[783a_0^3+432a_0^2c_1+74a_0c_1^2+4c_1^3+c_2(13a_0+2c_1)],\, h_2, \ldots, h_{78}\rangle$, where $h_i \in \mathbb{C}[h]$, $i=\overline{2,78}$. Further we find the ideal $\widetilde{X}_7 = \langle a_0(5a_0+c_1)(6a_0+c_1)(9a_0+2c_1)[783a_0^3+432a_0^2c_1+74a_0c_1^2+4c_1^3+c_2(13a_0+2c_1)],\, \widetilde{h}_2, \ldots, \widetilde{h}_{25}\rangle$; at the same time the radicals of ideals [16, p. 230] X_7 and $\widetilde{X}_7$ are equal, i.e. $\sqrt{X_7} = \sqrt{\widetilde{X}_7}$. Using for $\widetilde{X}_7$ the operations of intersection and division of ideals we find the radical $\sqrt{\widetilde{X}_7} = I_3 \bigcap (\bigcap\limits_{k=10}^{13} I_k)$. In that case $\sqrt{X_7} = \langle(5a_0+c_1)(6a_0+c_1)(9a_0+2c_1)[783a_0^3+432a_0^2c_1+74a_0c_1^2+4c_1^3+c_2(13a_0+2c_1)],\, (5a_0+c_1)[3a_0^2a_1-3a_3+2(4635a_0^4+3681a_0^3c_1+1067a_0^2c_1^2+133a_0c_1^3+6c_1^4)+(149a_0^2+61a_0c_1+6c_1^2)c_2],\, a_2-a_0[3a_0(a_0+c_1)-2a_1+c_2],\, 29727a_0^5+29835a_0^4c_1+11766a_0^3c_1^2+2278a_0^2c_1^3+216a_0c_1^4+8c_1^5+(501a_0^3+299a_0^2c_1+60a_0c_1^2+4c_1^3)c_2-3a_0c_4,\, 2(105705a_0^5+105705a_0^4c_1+41553a_0^3c_1^2+8019a_0^2c_1^3+758a_0c_1^4+28c_1^5)+2(1755a_0^3+1053a_0^2c_1+211a_0c_1^2+14c_1^3)c_2+3c_1c_4,\, 2(135a_0^3+81a_0^2c_1+17a_0c_1^2+c_1^3)-(4a_0-c_1)c_2-3c_3,\, 3b_4-6a_0^2b_1(6a_0+c_1)+b_3(13a_0+2c_1),\, 3a_0b_1+b_2,\, b_0\rangle$.

Let us show that $\mathbb{V}(X) = \mathbb{V}(X_7)$. For that it is enough to show that for $h \in \mathbb{V}(I_3) \bigcup (\bigcup\limits_{k=10}^{13} \mathbb{V}(I_k))$ $O(0,0)$ is a center. Let at first $h \in \mathbb{V}(I_{13})$. In that case $P_0(x) = (1-a_0x)^6\widetilde{P}_0(z)/[1-(6a_0+c_1)x]$, $Q(x) = (1-a_0x)^3Q_0(z)$, $R_1(x) = (1-a_0x)^5R_0(z)$, where

$$\begin{gathered}
\widetilde{P}_0(z) = 1-[3z(326a_0^3+180a_0^2c_1+33a_0c_1^2+2c_1^3)-9a_0^3z^2(5a_0+c_1)^2]/(13a_0+2c_1),\\
R_0(z) = [2b_3-3b_1(3a_0^2-a_1+a_0c_1)+3z(3a_0^2b_1(615a_0^3-13a_0a_1+355a_0^2c_1-2a_1c_1+\\
66a_0c_1^2+4c_1^3)-b_3(665a_0^3-13a_0a_1+375a_0^2c_1-2a_1c_1+68a_0c_1^2+4c_1^3))]/(13a_0+2c_1),\\
Q_0(z) = b_1+z(b_3-3a_0^2b_1),\ z = x^2(1-(13a_0+2c_1)x/3]/(1-a_0x)^3.
\end{gathered} \tag{8}$$

The change

$$y = YQ^{-1/3}(x) \tag{9}$$

reduces the system (2) after the excluding of time to the equation:

$$P_0(x)YY' = -x(1-Y^3)Q^{2/3}(x) + xR_1(x)Y^2/(3Q(x)). \tag{10}$$

Further the change (8) reduces the equation (10) to the form:

$$2\widetilde{P}_0(z)Y\frac{dY}{dz} = Q_0^{2/3}(z)(Y^3-1) + R_0(z)Y^2/(3Q_0(z)). \tag{11}$$

So in that case for system (2) there exists an analytical in the neighborhood of $O(0,0)$ integral and the critical point $O(0,0)$ is a center.

Let now $h \in \mathbb{V}(I_{10})$. Then $P_0(x) = (1-a_0x)^5\widetilde{P}_0(z)$, $Q(x) = (1-a_0x)^3Q_0(z)$, $R_1(x) = (1-a_0x)^5R_0(z)$, where

$$\begin{gathered}\widetilde{P}_0(z) = 1 - z(10a_0^2 - c_2) + a_0^2z^2(9a_0^2 - c_2),\ Q_0(z) = b_1 + z(b_3 - 3a_0^2b_1),\\ R_0(z) = 3b_1(2a_0^2 + a_1) + 2b_3 - z[3b_1(12a_0^4 + 4a_0^2a_1 - a_3) + b_3(8a_0^2 - 3a_1 - 2c_2)]- \\ z^2(3a_0b_1 - b_3)[3a_3 - a_0^2(42a_0^2 + 3a_1 - 4c_2)],\ z = x^2/(1-a_0x)^2.\end{gathered} \tag{12}$$

The change (12) transforms (10) to the form (11), i.e. in that case for system (2) there also exists an analytical in the neighborhood of $O(0,0)$ integral and $O(0,0)$ is a center.

If $h \in \mathbb{V}(I_{11})$ then $P_0(x) = (1-a_0x)^6\widetilde{P}_0(z)$, $Q(x) = (1-a_0x)^3Q_0(z)$, $R_1(x) = (1-a_0x)^5R_0(z)$, where

$$\begin{gathered}\widetilde{P}_0(z) = 1 - z(15a_0^2 - c_2) + 3a_0^2z^2(12a_0^2 - c_2),\ Q_0(z) = b_1 + z(b_3 - 3a_0^2b_1),\\ R_0(z) = 3b_1(3a_0^2 + a_1) + 2b_3 - z[9a_0^2b_1(3a_0^2 + a_1) + b_3(15a_0^2 - 3a_1 - 2c_2)],\\ z = x^2(1 - a_0x/3)/(1-a_0x)^3,\end{gathered} \tag{13}$$

and using the change (13) equation (10) is reduced to (11).

If $h \in \mathbb{V}(I_{12})$ then $P_0(x) = (1-a_0x)^6\widetilde{P}_0(z)/(2-3a_0x)$, $Q(x) = (1-a_0x)^3Q_0(z)$, $R_1(x) = (1-a_0x)^5R_0(z)$, where

$$\begin{gathered}\widetilde{P}_0(z) = 2[1 + z(4c_2 - 33a_0^2)/4 + 3a_0^2z^2(15a_0^2 - c_2)/8],\ Q_0(z) = b_1 + z(b_3 - 3a_0^2b_1),\\ R_0(z) = [6b_1(3a_0^2 + 2a_1) + 8b_3 + z(9a_0^2b_1(3a_0^2 - 4a_1 - 2c_2) - 4b_3(6a_0^2 - 3a_1 - 2c_2))]/4,\\ z = x^2(1 - 4a_0x/3)/(1-a_0x)^3,\end{gathered}$$

i.e. in that case (10) also is transformed to (11).

Under $h \in \mathbb{V}(I_3)$ the presence of a center at $O(0,0)$ is obvious. □

Remark. From the proof of Statement 1 it follows that $\langle a_2 - a_0[3a_0(a_0+c_1) - 2a_1 + c_2],\ 2(135a_0^3 + 81a_0^2c_1 + 17a_0c_1^2 + c_1^3) - (4a_0 - c_1)c_2 - 3c_3\rangle \subset X$.

Investigating the first ten focal values with the help of Statement 1, we get

Statement 2. *For the ideal* $\widetilde{I} = I + \langle a_2 - a_0[3a_0(a_0+c_1) - 2a_1 + c_2],\ 2(135a_0^3 + 81a_0^2c_1 + 170a_0c_1^2 + c_1^3) - (4a_0 - c_1)c_2 - 3c_3\rangle$ *the next formula takes place:* $\sqrt{\widetilde{I}} = I_3 \bigcap (\bigcap_{k=10}^{13} I_k) \bigcap (\bigcap_{j=1}^{3} \widehat{I}_j)$, *where* $\widehat{I}_1 = \langle 3a_0^2b_1 - b_3,\ a_2 - a_0[3a_0(a_0+c_1) - 2a_1 + c_2],\ 3(a_0^2a_1 - a_3) + 2a_0(135a_0^3 + 81a_0^2c_1 + 179a_0c_1^2 + c_1^3) - a_0c_2(a_0 - c_1),\ a_0(273a_0^3 + 165a_0^2c_1 + 34a_0c_1^2 + 2c_1^3) - a_0c_2(a_0 - c_1) + 3c_4,\ 2(135a_0^3 + 81a_0^2c_1 + 17a_0c_1^2 + c_1^3) - (4a_0 - c_1)c_2 - 3c_3,\ a_0^3b_1 + b_4,\ 3a_0b_1 + b_2,\ b_0\rangle$, $\widehat{I}_2 = \langle b_1,\ b_3,\ a_2 - a_0[3a_0(a_0+c_1) - 2a_1 + c_2],\ 2(135a_0^3 + 81a_0^2c_1 + 17a_0c_1^2 + c_1^3) - (4a_0 - c_1)c_2 - 3c_3,\ b_4,\ b_2,\ b_0\rangle$, $\widehat{I}_3 = \langle a_0, b_3, a_2, a_3, 3c_3 - c_1(2c_1^2 + c_2), b_4, b_2, b_0\rangle$.

Statement 3. *Let* $X = I + \langle b_3,\ b_4\rangle$. *Then the next equality takes place:* $\sqrt{X} = I_1 \bigcap I_2 \bigcap I_4 \bigcap (\bigcap_{k=18}^{22} I_k) \bigcap (\bigcap_{j=1}^{3} \widehat{I}_j)$, *where* $\widehat{I}_1 = \langle a_0,\ a_2,\ c_1,\ c_3,\ b_0,\ b_2,\ b_3,\ b_4\rangle$, $\widehat{I}_2 =$

$\langle 6a_0 + c_1,\ 10a_0(9a_0^2 - c_2) - 3c_3,\ a_2 + a_0(15a_0^2 + 2a_1 - c_2),\ a_0^2(9a_0^2 - c_2) + c_4,\ 3a_3 - a_0^2(18a_0^2 + 3a_1 - c_2),\ b_4,\ b_3,\ 3a_0b_1 + b_2,\ b_0\rangle$, $\widehat{I}_3 = \langle 6a_0 + c_1,\ 12a_0^2 - c_2,\ 2a_0^2 + a_1,\ 10a_0^3 + c_3,\ a_0^3 - a_2,\ 3a_0^4 - c_4,\ a_3,\ b_4,\ b_3,\ 3a_0b_1 + b_2,\ b_0\rangle$.

Proof. With the help of operations of division and intersection of ideals one finds the radical of the ideal $X_9 = \langle g_1, \dots, g_9\rangle + \langle b_3, b_4\rangle$. We have $\sqrt{X_9} = I_1 \bigcap I_2 \bigcap I_4 \bigcap (\bigcap_{k=18}^{22} I_k) \bigcap (\bigcap_{j=1}^{3} \widehat{I}_j)$.

Let us show that for $h \in \mathbb{V}(X_9)$ $O(0,0)$ is a center. Let $h \in \mathbb{V}(I_1) \bigcap \mathbb{V}(I_2) \bigcap \mathbb{V}(I_4)$, then $R_1(h) \equiv 0$, i.e. $h \in \mathbb{V}(I)$. In that case the equation (2) by change (9) after excluding time is transformed to

$$P_0(x)YY' = -x(1 - Y^3)Q^{2/3}(x).$$

Let now $h \in \bigcup_{k=18}^{22} \mathbb{V}(I_k)$. Then $P_0(h)S(h)/R_1^2(h) \equiv const$, so $h \in \mathbb{V}(I)$. Further we have $\mathbb{V}(\widehat{I}_1) \subset \mathbb{V}(I_3)$; $\mathbb{V}(\widehat{I}_2)$, $\mathbb{V}(\widehat{I}_3) \subset \mathbb{V}(I_{11})$, therefore for $h \in \bigcup_{j=1}^{3} \mathbb{V}(\widehat{I}_j)$ the critical point $O(0,0)$ is a center. Thus $\sqrt{X} = \sqrt{X_9}$. □

Statement 4. *Let* $X = I + \langle 9a_0^2b_1 - 4b_3,\ b_4\rangle$. *Then the next formula takes place:* $\sqrt{X} = I_1 \bigcap I_2 \bigcap I_5 \bigcap (\bigcap_{k=23}^{27} I_k) \bigcap (\bigcap_{j=1}^{3} \widehat{I}_j)$, *where* $\widehat{I}_1 = \langle a_0,\ a_2,\ c_1,\ c_3,\ b_0,\ b_2,\ b_3,\ b_4\rangle$, $\widehat{I}_2 = \langle 9a_0 + 2c_1,\ a_0(189a_0^2 - 34c_2) - 12c_3,\ 2a_2 + a_0(21a_0^2 + 4a_1 - 2c_2),\ a_0^2(27a_0^2 - 4c_2) + 2c_4,\ 3a_3 - a_0^2(36a_0^2 + 3a_1 - 4c_2),\ b_4\ 9a_0^2b_1 - 4b_3,\ 3a_0b_1 + b_2,\ b_0\rangle$, $\widehat{I}_3 = \langle 9a_0 + 2c_1,\ 15a_0^2 - 2c_2,\ 25a_0^2 + 8a_1,\ 11a_0^3 + 2c_3,\ 13a_0^3 - 4a_2,\ 3a_0^4 - 2c_4,\ 9a_0^4 + 8a_3,\ b_4,\ 9a_0^2b_1 - 4b_3,\ 3a_0b_1 + b_2,\ b_0\rangle$.

Proof. By means of division and intersection operations we find the radical of ideal $X_9 = \langle g_1, \dots, g_9\rangle + \langle 9a_0^2b_1 - 4b_3,\ b_4\rangle$. Then we have $\sqrt{X_9} = I_1 \bigcap I_2 \bigcap I_5 \bigcap (\bigcap_{k=23}^{27} I_k) \bigcap (\bigcap_{j=1}^{3} \widehat{I}_j)$. The further is analogous to the proof of Statement 3. □

Statement 5. *Let* $X = I + \langle 3a_0^2b_1 - b_3,\ a_0^3b_1 + b_4\rangle$. *Then the next equality takes place:* $\sqrt{X} = I_1 \bigcap I_2 \bigcap I_9 \bigcap (\bigcap_{j=1}^{5} \widehat{I}_j)$, *where* $\widehat{I}_1 = \langle a_0,\ a_2,\ c_1,\ c_3,\ b_0,\ b_2,\ b_3,\ b_4\rangle$, $\widehat{I}_2 = \langle 5a_0 + c_1,\ a_0(20a_0^2 - 3c_2) - c_3,\ a_2 + a_0(12a_0^2 + 2a_1 - c_2),\ 2a_0^2(8a_0^2 - c_2) + c_4,\ a_0^3b_1 + b_4,\ 3a_0^2b_1 - b_3,\ 3a_0b_1 + b_2,\ b_0\rangle$, $\widehat{I}_3 = \langle 6a_0 + c_1,\ 10a_0(9a_0^2 - c_2) - 3c_3,\ a_2 + a_0(15a_0^2 + 2a_1 - c_2),\ a_0^2(9a_0^2 - c_2) + c_4,\ 3a_3 - a_0^2(18a_0^2 + 3a_1 - c_2),\ a_0^3b_1 + b_4,\ 3a_0^2b_1 - b_3,\ 3a_0b_1 + b_2,\ b_0\rangle$, $\widehat{I}_4 = \langle 9a_0 + 2c_1,\ a_0(189a_0^2 - 34c_2) - 12c_3,\ 2a_2 + a_0(21a_0^2 + 4a_1 - 2c_2),\ a_0^2(27a_0^2 - 4c_2) + 2c_4,\ 3a_3 - a_0^2(36a_0^2 + 3a_1 - 4c_2),\ a_0^3b_1 + b_4,\ 3a_0^2b_1 - b_3,\ 3a_0b_1 + b_2,\ b_0\rangle$, $\widehat{I}_5 = \langle 783a_0^3 + 432a_0^2c_1 + 74a_0c_1^2 + 4c_1^3 + c_2(13a_0 + 2c_1),\ a_0[(9a_0 + 2c_1)^2 - 2c_1^2 + 7c_2] + 2c_3,\ a_2 - a_0[3a_0(a_0 + c_1) - 2a_1 + c_2],\ 2a_0^2(135a_0^2 + 54a_0c_1 + 5c_1^2 + c_2) + c_4,\ a_3 - a_0^2[3a_0(73a_0 + 29c_1) + a_1 + 2c_2],\ a_0^3b_1 + b_4,\ 3a_0^2b_1 - b_3,\ 3a_0b_1 + b_2,\ b_0\rangle$.

The proof follows from Statement 1 and Theorem 3.

By direct examination we become sure that the next statement is true.

Statement 6. *The next equalities are right:* $\sqrt{I+E_8} = I_1 \bigcap (\bigcap_{k=4}^{8} I_k) \bigcap \langle a_0,\, a_1,\, a_2,\, a_3,\, b_0,\, b_2,\, b_3,\, b_4\rangle$, $\quad \sqrt{I+E_9} = I_1 \bigcap I_2 \bigcap (\bigcap_{k=4}^{9} I_k)$.

Proof of Theorem 4. Computing the radicals of ideals $E_k \bigcap I$, $k=\overline{5,7}$, we get $\bigcup_{k=5}^{7} \mathbb{V}(E_k \bigcap I) \subset V$. If $h \in \bigcup_{k=14}^{35} \mathbb{V}(I_k)$ then $O(0,0)$ is a center since in that case $P_0(h)S(h)/R_1^2(h) \equiv const$. Further taking into account Statements 1-6 we become sure in the correctness of Theorem 4. □

4. Now we will examine system (1).

Statement 7. *The next formula is true:* $\sqrt{J+\langle B\rangle} = \bigcap_{k=1}^{9} \widetilde{J}_k$, *where the radical ideals* $\widetilde{J}_k$ *have the form:*

$\widetilde{J}_1 = \langle B,\, L,\, N\rangle$, $\widetilde{J}_2 = \langle A+C,\, B,\, (A-D)K+AM,\, K(2K+M)-N^2,\, 3K+M+P,\, L+N\rangle$, $\widetilde{J}_3 = \langle 17A+27C,\, 2A+3D,\, B,\, 100A^4-177147N^2,\, 20A^2+81K,\, 50A^2-243M,\, 10A^2-81P,\, L+N\rangle$, $\widetilde{J}_4 = \langle 2A+D,\, B,\, C(A+3C)^2(2A+3C)+4N^2,\, (A+3C)^2+4K,\, 2C^2+(A+2C)^2-M,\, 3(A-3C)(A+C)-4P,\, L+N\rangle$, $\widetilde{J}_5 = \langle A+3C-D,\, B,\, A(2A+3C)^2(3A+4C)-N^2,\, A(2A+3C)-K,\, (A-2C)(2A+3C)+M,\, 6(A+C)(2A+3C)+P,\, L+N\rangle$, $\widetilde{J}_6 = \langle 2A+C+D,\, B,\, 2K(2A+C)+AM,\, K[A(A+C)-2(2K+M)]+2N^2,\, A(A+C)-3K-M-P,\, L+N\rangle$, $\widetilde{J}_7 = \langle 5A+3C+3D,\, B,\, A(2A+3C)^2(5A+12C)+81N^2,\, A(2A+3C)+3K,\, (2A+3C)(7A+6C)-9M,\, 2(A-3C)(2A+3C)-9P,\, L+N\rangle$, $\widetilde{J}_8 = \langle 73A^2+180AC+117C^2,\, 3(11A+15C)+7D,\, B,\, 100A^3(341A+360C)+415233N^2,\, A(29A+9C)+91K,\, A(157A+237C)-273M,\, 2A(29A+9C)+91P,\, L+N\rangle$, $\widetilde{J}_9 = \langle B,\, (A+C)(A+3C)(2A+C+D)-(A-C+D)K,\, (A+C)^2[(A+2C)(2A+C+D)+K]+K^2+N^2,\, 2(A+C)(2A+C+D)+3K+M,\, (A+C)(A+C+D)+P,\, L+N\rangle$.

Proof. Let's generate the ideal $\widetilde{J} = (J+\langle 2u-v+w-A,\, u-v+w+C,\, 3u-2v+w+D\rangle) \bigcap \mathbb{C}[B,\, K,\, L,\, M,\, N,\, P,\, u,\, v,\, w]$. Using Groebner bases one obtains that the radical of ideal $\widetilde{J}_0 = \widetilde{J}+\langle Nuvw(P-2u^2+2uv-uw)(u+w)(2u-w)(2u-4v+w)(u-2v+w)(v-w)(2v-w)(2v+w)(3v-2w)(4v-3w)(5v-4w)(6v-7w)(6v-5w)(7v-5w)(8v-5w)(8v-3w)(26v-19w)\rangle$ has the form: $\sqrt{\widetilde{J}_0} = \bigcap_{k=1}^{9} \widetilde{\widetilde{J}}_k$, where $\widetilde{\widetilde{J}}_k = (\widetilde{J}_k + \langle 2u-v+w-A,\, u-v+w+C,\, 3u-2v+w+D\rangle) \bigcap \mathbb{C}[B,\, K,\, L,\, M,\, N,\, P,\, u,\, v,\, w]$, $k=\overline{1,9}$. In that way it is proved that $\sqrt{J+\langle B\rangle} \subset \bigcap_{k=1}^{9} \widetilde{J}_k$. Let's show that if $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$ then $O(0,0)$ is a focus.

The first focal value for system (1) in the case $B=0$ has the form: $\widetilde{f}_1 = L+N$. Focal values $\widetilde{f}_i$, $i=\overline{2,8}$, have accordingly 18, 82, 274, 750, 1790, 3854, 7662 summands. Denote by $\widehat{I}$ the ideal $\widehat{I} = \langle \widetilde{f}_1, \ldots, \widetilde{f}_8, \ldots\rangle$. Notice that $\widetilde{f}_i$, $i=$

$1, 2, \ldots$ are the polynomials from the ring $\mathbb{C}[A,\ C,\ D,\ K,\ L,\ M,\ N,\ P]$, so $\widehat{I} \subset \mathbb{C}[A,\ C,\ D,\ K,\ L,\ M,\ N,\ P]$.

From the condition $\widetilde{f}_1 = 0$ we get

$$L = -N. \tag{14}$$

Considering the condition (14) for system (7) to be held, exclude from $\widetilde{f}_i$, $i = \overline{2,8}$, the variable L and get $\widetilde{f}_i = \alpha_i N F_i$, $i = \overline{2,8}$, where $\alpha_i \neq 0$, $F_i \in \mathbb{C}[A,\ C,\ D,\ K,\ M,\ N,\ P]$, $i = \overline{2,8}$. As $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$, then $N \neq 0$. Construct the ideal $\widehat{\widehat{I}} = (\langle L+N,\, F_2, \ldots, F_8\rangle + \langle 2u - v + w - A,\ u - v + w + C,\ 3u - 2v + w + D\rangle) \bigcap \mathbb{C}[K,\ L,\ M,\ N,\ P,\ u,\ v,\ w]$. The ideal $\widehat{\widehat{I}}$ has the form: $\widehat{\widehat{I}} = \langle L+N,\, \widetilde{F}_2, \ldots, \widetilde{F}_8\rangle$, where $\widetilde{F}_i \in \mathbb{C}[K,\ M,\ N,\ P,\ u,\ v,\ w]$. From the condition $\widetilde{F}_2 = 0$ we get:

$$K = [u(2u - 4v + w) - M - P]/3. \tag{15}$$

Taking into account the conditions (14) and (15) we have: $\widetilde{F}_i = \beta_i \widetilde{g}_i$, $i = \overline{3,8}$, where $\beta_i \neq 0$, $\widetilde{g}_i \in \mathbb{C}[M,\ N,\ P,\ u,\ v,\ w]$. Notice that $\widetilde{g}_i = \widetilde{g}_3 \gamma_i + \widetilde{G}_i$, $i = \overline{4,8}$, where $\gamma_i \in \mathbb{C}[M,\ N,\ P,\ u,\ v,\ w]$, $\widetilde{G}_i \in \mathbb{C}[M,\ P,\ u,\ v,\ w]$, $i = \overline{4,8}$. The polynomials $\widetilde{G}_i$, $i = \overline{4,8}$, can be written in the form: $\widetilde{G}_i = \delta_i M^{i-2} w^2 + M^{i-3}\omega_{i,1} + \ldots + \omega_{i,i-2}$, where $\delta_i \neq 0$, $\omega_{i,1}$, $\omega_{i,i-2}$, $i = \overline{4,8}$, are the polynomials in $P,\ u,\ v,\ w$.

Since $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$ then $w \neq 0$. And the next relations will be true: $\widetilde{G}_i = \theta_i \widetilde{G}_4 + T_i$, $i = \overline{5,8}$, where $\theta_i \neq 0$, and the variable M has the 1st degree in T_5. Solving the equation $T_5 = 0$ we have:

$$M = T_{5,1}/(wT_{5,2}), \tag{16}$$

where $T_{5,1}$ and $T_{5,2}$ are coprime polynomials in variables $P,\ u,\ v,\ w$. Taking into consideration the condition (16) we get $\widetilde{G}_4 = Y_5/T_{5,2}^2$, $T_i = Y_i/T_{5,2}$, $i = \overline{6,8}$, where the polynomials Y_i, $i = \overline{5,8}$, can be represented in the form: $Y_i = \chi_i uv(P - 2u^2 + 2uv - uw)\widetilde{Y}_i$, at the same time $\chi_i \neq 0$, $i = \overline{5,8}$, $\widetilde{Y}_5, \ldots, \widetilde{Y}_8$ are the polynomials from the ring $\mathbb{C}[P,\ u,\ v,\ w]$, containing accordingly 314, 314, 541 and 853 summands.

Since $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$ then $uv(P - 2u^2 + 2uv - uw) \neq 0$. Assume that $T_{5,2} \neq 0$. Then the critical point $O(0,0)$ can be a center if $\widetilde{Y}_i = 0$, $i = \overline{5,8}$. Further we will denote by $R_x(F_1,\ F_2)$ the resultant [16, p. 209] of the polynomials F_1 and F_2 in a variable x. Let's compute two resultants: $\widetilde{R}_1 = R_P(O_5,\ sO_6 + O_7)$ and $\widetilde{R}_2 = R_P(O_5,\ sO_6 + O_8)$, where $O_i = \widetilde{Y}_i|_{w=1}$, $i = \overline{5,8}$. We have $\widetilde{R}_1 = \sum\limits_{i=1}^{6} s^{i-1} S_i$, where S_i, $i = \overline{1,6}$, are the polynomials in variables $u,\ v$ of the form $S_i = \varepsilon_i u^6 (1+u)(-1+2u)(1+2u-4v)(1+u-2v)^3 S_0^2 Z_i$, $i = \overline{1,5}$, $S_6 = \varepsilon_6 u^6 (1+u)(-1+2u)(1+2u-4v)(1+u-2v)^3 S_0^2 \widetilde{S}_0 Z_6$, at that $\varepsilon_i \neq 0$, $i = \overline{1,6}$, S_0 is a polynomial in variables $u,\ v$, $\widetilde{S}_0 = -2736 - 11424u - 15885u^2 - 9625u^3 + 16224v + 46596uv + 42165u^2v + 5250u^3v - 33360v^2 - 60792uv^2 - 18180u^2v^2 + 28736v^3 + 21060uv^3 - 8160v^4$; $Z_1, \ldots, Z_6$ are coprime polynomials in $u,\ v$, including accordingly 1792, 1671, 1554, 1404, 1250 and 925 summands.

The resultant $\widetilde{R}_2$ can be represented in the next form: $\widetilde{R}_2 = \sum\limits_{i=1}^{6} s^{i-1} W_i$, where $W_i = \tau_i u^6 (1+u)(-1+2u)(1+2u-4v)(1+u-2v)^3 S_0^2 \xi_i$, $i = \overline{1,5}$, $W_6 = \varepsilon_6 u^6 (1+$

$u)(-1+2u)(1+2u-4v)(1+u-2v)^3 S_0^2 \widetilde{S}_0 \xi_6$, at that $\tau_i \neq 0, i = \overline{1,6}$, and $\xi_1, \dots, \xi_6$ are coprime polynomials in u, v, including accordingly 2940, 2634, 2344, 1981, 1623 and 925 summands.

As far as $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$ then $u^6(1+u)(-1+2u)(1+2u-4v)(1+u-2v)^3 \neq 0$. Let $S_0 \neq 0$, then the critical point $O(0,0)$ can be a center if the following conditions are held: $Z_i = 0$, $\xi_i = 0$, $i = \overline{1,6}$. Let's compute the next resultants: $\widetilde{r}_i = R_u(Z_6,\ Z_{6-i})$, $i = \overline{1,5}$, and also $\widetilde{r}_0 = R_u(\xi_6,\ \xi_5)$. Here r_i, $i = \overline{1,5}$, are the polynomials in v having accordingly 1986th, 2115th, 2230th, 2341th and 2427th degrees, the coefficients of which are coprime integer numbers of the orders $10^{2128} - 10^{3619}$, $10^{2141} - 10^{2793}$, $10^{2299} - 10^{3003}$, $10^{2654} - 10^{4405}$, $10^{2616} - 10^{3401}$ accordingly; $\widetilde{r}_0$ is a polynomial in v of 2247th degree, and its coefficients are coprime integer numbers of the order $10^{2252} - 10^{2948}$.

The greatest common divisor of the polynomials $\widetilde{r}_1, \dots, \widetilde{r}_5$ is $(v-1)^3(2v-1)^{39}(2v+1)^3(3v-2)^{39}(4v-3)^{54}(5v-4)^{17}(6v-7)^6(6v-5)^5(7v-5)^5(8v-5)^5(8v-3)^3(26v-19)^3\widehat{P}^3$, where $\widehat{P}$ is a polynomial in v of 77th degree, and its coefficients are coprime integer numbers of the order $10^{78} - 10^{103}$.

As $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$ and the greatest common divisor of the polynomials $\widehat{P}$ and $\widetilde{r}_0$ is 1 then in that case the origin is a focus.

Let $S_0 = 0$. Notice that $R_P(T_{5,1}, T_{5,2}) = \alpha_0 u^2 v^2 T_0 \widehat{S}_0$, where $\alpha_0 \neq 0$, T_0 is the polynomial in variables u, v, w, $\widehat{S}_0|_{w=1} = S_0$. Denote by $\widetilde{T}_i$, $i = \overline{5,7}$, the following resultants: $\widetilde{T}_i = R_M(T_4, T_i)$, $i = \overline{5,7}$. Here $\widetilde{T}_i$, $i = \overline{5,7}$, are the polynomials in variables P, u, v, w, containing 314, 846 and 1756 summands accordingly. For any $i = 6, 7$, the equalities are true: $\widetilde{T}_i = \widetilde{S}_i \widehat{S}_0 + \widetilde{\widetilde{T}}_i$, where $\widetilde{\widetilde{T}}_i$, $i = 6, 7$, are the polynomials in P, u, v, w, including 824 and 1571 summands accordingly. Further using resultants we exclude the variable P: $H_1 = R_P(\widetilde{O}_5, \widetilde{O}_6)$, $H_2 = R_P(\widetilde{O}_5, \widetilde{O}_7)$, where $\widetilde{O}_5 = \widetilde{T}_5|_{w=1}$, $\widetilde{O}_i = \widetilde{\widetilde{T}}_i|_{w=1}$, $i = 6, 7$. H_1 and H_2 are the polynomials in u, v having 7652 and 12987 summands accordingly. For H_i, $i = 1, 2$, the next equalities are true: $H_i = \widehat{H}_i S_0 + \widetilde{H}_i$, where $\widetilde{H}_1$ and $\widetilde{H}_2$ are the polynomials in u, v, having accordingly 1474 and 1978 summands. Further we have $\widetilde{Z}_i = R_v(S_0, \widetilde{H}_i)$, $i = 1, 2$. Here $\widetilde{Z}_1$, $\widetilde{Z}_2$ are the polynomials in one variable u of 1505th and 1988th degrees accordingly, containing 1448 and 1931 summands. The greatest common divisor of $\widetilde{Z}_1$ and $\widetilde{Z}_2$ has the form:

$$u^{58}(1+u)^5(-1+2u)^{18}(1+4u)^9(46-103u-563u^2+60u^3)^6(1540+5011u-35614u^2+$$
$$+51479u^3-24216u^4+1920u^5)^2(-2885120+48860768u-338183580u^2+1252033136u^3-$$
$$-2176161807u^4+3494962821u^5-2544493968u^6+920349000u^7-118729800u^8+$$
$$+4860000u^9)^3(155605184-2227701700u+2040477985u^2+22348142299u^3-$$
$$-64132349961u^4+70372499301u^5-40878190008u^6+14935630500u^7-1932076800u^8+$$
$$+77760000u^9).$$

Since $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$ then $u(1+u)(-1+2u) \neq 0$. Consider the case $1+4u=0$. The next equality is right: $\widetilde{\widetilde{H}}_1 = \widehat{S}_0 \widetilde{X}_0 + \widetilde{V}_1$, where $\widetilde{\widetilde{H}}_1|_{w=1} = \widetilde{H}_1$; $\widetilde{X}_0$, $\widetilde{V}_1 \in \mathbb{C}[u,\ v,\ w]$, the polynomial $\widetilde{V}_1$ has 1474 summands. Let's generate the ideal $\widetilde{U}_1 = \langle 4u + w,\, \widehat{S}_0,\, \widetilde{V}_1\rangle$. The Groebner basis of this ideal is $\widetilde{U}_1 = \langle (8v - 3w)w^{123},\, \widetilde{h}_2, \ldots,\, \widetilde{h}_{15}\rangle$, where $\widetilde{h}_i \in \mathbb{C}[u,\, v,\, w]$, $i = \overline{2,15}$. But according to the condition $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$, so $(8v - 3w)w \neq 0$, i.e. in that case $O(0,0)$ is a focus.

Denote by $e_0 = 60u^3 - 563u^2w - 103uw^2 + 46w^3$. Then we have $\widehat{S}_0 = e_0\widetilde{X}_1 + \widehat{T}_0$, where $\widetilde{X}_1,\, \widehat{T}_0 \in \mathbb{C}[u,\, v,\, w]$, $\widetilde{\widetilde{H}}_1$ can be written in the form: $\widetilde{\widetilde{H}}_1 = \widehat{T}_0\widetilde{X}_2 + e_0\widetilde{X}_3 + \widetilde{V}_2$, where $\widetilde{X}_2,\, \widetilde{X}_3,\, \widetilde{V}_2 \in \mathbb{C}[u,\, v,\, w]$. Let's generate the ideal $\widetilde{U}_2 = \langle e_0,\, \widehat{T}_0,\, \widetilde{V}_2\rangle$ and compute its Groebner basis. We get $\widetilde{U}_2 = \langle w^{26}u_0,\, \widetilde{h}_2, \ldots,\, \widetilde{h}_{52}\rangle$, where $u_0 = 240v^3 - 1486v^2w + 1203vw^2 - 237w^3$, $\widetilde{h}_i \in \mathbb{C}[u,\, v,\, w]$, $i = \overline{2,52}$. Since $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$ then $w \neq 0$. Computing for the ideal $\widetilde{U}_2 + \langle u_0\rangle$ its Groebner basis we get $\widetilde{U}_2 + \langle u_0\rangle = \langle w^{15}(u - 2v + w),\, \widetilde{h}_2, \ldots,\, \widetilde{h}_8\rangle$, where $\widetilde{h}_i \in \mathbb{C}[u,\, v,\, w]$, $i = \overline{2,8}$. But $w(u - 2v + w) \neq 0$ as far as $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$ so $O(0,0)$ is a focus.

Let now $e_0 = 1920u^5 - 24216u^4w + 51479u^3w^2 - 35u^2w^3 + 5011uw^4 + 1540w^5$. In that case $\widehat{S}_0 = e_0\widetilde{X}_4 + \widehat{T}_1$, where $\widetilde{X}_4,\, \widehat{T}_1 \in \mathbb{C}[u,\, v,\, w]$, then $\widetilde{\widetilde{H}}_1$ can be represented as $\widetilde{\widetilde{H}}_1 = \widehat{T}_1\widetilde{X}_5 + e_0\widetilde{X}_6 + \widetilde{V}_3$, where $\widetilde{X}_5,\, \widetilde{X}_6,\, \widetilde{V}_3 \in \mathbb{C}[u,\, v,\, w]$. The Groebner basis for the ideal $\widetilde{U}_3 = \langle e_0,\, \widehat{T}_1,\, \widetilde{V}_3\rangle$ is $\widetilde{U}_3 = \langle w^{28}u_0,\, \widetilde{h}_2, \ldots,\, \widetilde{h}_{94}\rangle$, where $u_0 = 491520v^5 - 3714048v^4w + 6701504v^3w^2 - 4849808v^2w^3 + 1471076vw^4 - 143019w^5$, $\widetilde{h}_i \in \mathbb{C}[u,\, v,\, w]$, $i = \overline{2,94}$. Further we have $\widetilde{U}_3 + \langle u_0\rangle = \langle w^{18}(2u - 4v + w), \widetilde{h}_2, \ldots,\, \widetilde{h}_{22}\rangle$, where $\widetilde{h}_i \in \mathbb{C}[u,\, v,\, w]$, $i = \overline{2,22}$. As $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$ then $w(2u - 4v + w) \neq 0$. The critical point $O(0,0)$ is a focus.

Let

$$e_0 = 4860000u^9 - 118729800u^8w + 920349000u^7w^2 - 2544493968u^6w^3 +$$
$$+3494962821u^5w^4 - 2716161807u^4w^5 + 1252033136u^3w^6 - 338183580u^2w^7 +$$
$$+48860768uw^8 - 2885120w^9.$$

Then $\widehat{S}_0$ and $\widetilde{\widetilde{H}}_1$ can be represented as $\widehat{S}_0 = e_0\widetilde{X}_7 + \widehat{T}_2$; $\widetilde{\widetilde{H}}_1 = \widehat{T}_2\widetilde{X}_8 + e_0\widetilde{X}_9 + \widetilde{V}_4$, where $\widehat{T}_2,\, \widetilde{X}_7,\, \widetilde{X}_8,\, \widetilde{X}_9,\, \widetilde{V}_4 \in \mathbb{C}[u,\, v,\, w]$. The greatest common divisor of resultants $R_u(e_0,\, \widehat{T}_2)$ and $R_u(e_0,\, \widetilde{V}_4)$ equals $\widetilde{\gamma}u_0$, where $\widetilde{\gamma} \neq 0$, $u_0 = 1244160000v^9 - 20796134400v^8w + 130889433600v^7w^2 - 40702563776v^6w^3 + 725607750864v^5w^4 - 795952371456v^4w^5 + 548902046936v^3w^6 - 232670029920v^2w^7 + 55523773877vw^8 - 5720760000w^9$. Using Groebner basis the ideal $\widetilde{U}_4 = \langle e_0,\, u_0,\, \widehat{T}_2,\, \widetilde{V}_4\rangle$ can be represented in the next form: $\widetilde{U}_4 = \langle w^{24}(u - 2v + w), \widetilde{h}_2, \ldots,\, \widetilde{h}_{72}\rangle$, where $\widetilde{h}_i \in \mathbb{C}[u,\, v,\, w]$, $i = \overline{2,72}$. Since $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$ then $w(u - 2v + w) \neq 0$, so in that case $O(0,0)$ is a focus.

Consider now the last possible case of the center for $S_0 = 0$. Let's denote $e_0 = 77760000u^9 - 1932076800u^8w + 14935630500u^7w^2 - 40878190008u^6w^3 + 70372499301u^5w^4 - 64132349961u^4w^5 + 22348142299u^3w^6 + 2040477985u^2w^7 - 2227701700uw^8 + 155605184w^9$. Then $\widehat{S}_0$ and $\widetilde{\widetilde{H}}_1$ are represented in the form: $\widehat{S}_0 = e_0\widetilde{X}_{10} + \widehat{T}_3$; $\widetilde{\widetilde{H}}_1 = \widehat{T}_3\widetilde{X}_{11} + e_0\widetilde{X}_{12} + \widetilde{V}_5$, where $\widehat{T}_3,\, \widetilde{V}_5,\, \widetilde{X}_{10},\, \widetilde{X}_{11},\, \widetilde{X}_{12}, \in \mathbb{C}[u,\, v,\, w]$. Finding the greatest common divisor of the resultants $R_u(e_0,\, \widehat{T}_3)$ and $R_u(e_0,\, \widetilde{V}_5)$ we have $\widetilde{\mu}u_0$, where $\widetilde{\mu} \neq 0$,

$$u_0 = 79626240000v^9 - 1168382361600v^8w + 5981127091200v^7w^2 -$$

$$-137592210378246v^6w^3 + 18275585731200v^5w^4 - 14958694725792v^4w^5 +$$
$$+7438311225796v^3w^6 - 2117102790235v^2w^7 + 303888967255vw^8 - 15968920230w^9.$$

The Groebner basis of ideal $\widetilde{U}_4 = \langle e_0,\, u_0,\, \widehat{T}_3,\, \widetilde{V}_5\rangle$ has the form: $\widetilde{U}_4 = \langle w^{24}(2u - 4v + w), \widetilde{h}_2, \ldots, \widetilde{h}_{72}\rangle$, where $\widetilde{h}_i \in \mathbb{C}[u,\, v,\, w]$, $i = \overline{2, 72}$. As far as $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$ then $w(2u - 4v + w) \neq 0$; hence, the critical point $O(0, 0)$ is a focus. □

Further through $\widetilde{I}_k$, $k = \overline{1, 35}$, we will denote the ideals obtained from I_k, $k = \overline{1, 35}$, if the coefficients $a_i\,(i = \overline{0, 3})$, $b_j\,(j = \overline{0, 4})$, $c_m\,(m = \overline{1, 4})$ are expressed by the formulas (). Notice that $\widetilde{I}_k \subset \mathbb{C}[q]$, $k = \overline{1, 35}$. Using Groebner basis we become sure in the truth of the next statement.

Statement 8. *The next equalities take place:*

$$\sqrt{\widetilde{I}_2} = J_4 \bigcap J_{12} \bigcap \langle A,\, C,\, N,\, K,\, L\rangle \bigcap \langle A + C,\, N,\, K,\, M - P,\, L\rangle,$$
$$\sqrt{\widetilde{I}_9} = J_{10} \bigcap J_{11} \bigcap J_{19},$$
$$\sqrt{\widetilde{I}_3} = J_2 \bigcap J_3 \bigcap \langle A,\, C,\, D,\, B,\, K(2K + M) - N^2,\, 3K + M + P,\, L + N\rangle,$$

and at the same time the next inclusions are true:

$$\mathbb{V}(\langle A,\, C,\, N,\, K,\, L\rangle) \subset \mathbb{V}(J_{10}),\ \mathbb{V}(\langle A + C,\, N,\, K,\, M - P,\, L\rangle) \subset \mathbb{V}(J_{10}),$$
$$\mathbb{V}(\langle A,\, C,\, D,\, B,\, K(2K + M) - N^2,\, 3K + M + P,\, L + N\rangle) \subset \mathbb{V}(J_{12}).$$

Statement 9. *Denote by $\widehat{J}_1$ and $\widehat{J}_2$ the following ideals: $\widehat{J}_1 = \langle A,\, C,\, D,\, N,\, K,\, L\rangle$, $\widehat{J}_2 = \langle A,\, C,\, D,\, B,\, K^2 + N^2,\, 3K + M,\, P,\, L + N\rangle$. Then the radical of ideal $\widetilde{I}_{10}$ can be written in the next form:*

$$\sqrt{\widetilde{I}_{10}} = J_5 \bigcap J_7 \bigcap \widehat{J}_1 \bigcap \langle 2A + 3C,\, 2A + 3D,\, N,\, 2A^2 + 9K,\, M - 2P,\, AB + 3L\rangle \bigcap \langle 4A + 5C + D,\, N,\, C(A + C) - K,\, M,\, 2(A + C)(2A + 3C) - P,\, B(A + C) + L\rangle \bigcap \langle A + 2C,\, 3A + 2D,\, B,\, K(A^2 + 4K) + 4N^2,\, 3K + M,\, A^2 - 2P,\, L + N\rangle \bigcap \langle 2A + 3C,\, 2A + 3D,\, AB - 3N,\, 2A^2 + 9K,\, 2(A - 3B)(A + 3B) - 9M,\, A^2 + 18B^2 - 9P,\, 2AB + 3L\rangle \bigcap \langle 4A + 5C + D,\, 2B^2 + (A + C)^2,\, B(A + 2C) + N,\, C(A + C) - K,\, 3(A + C)^2 - M,\, 2(A + C)(2A + 3C) - P,\, BC - L\rangle,$$

and at the same time the next inclusions are correct:

$$\mathbb{V}(\langle 2A + 3C,\, 2A + 3D,\, N,\, 2A^2 + 9K,\, M - 2P,\, AB + 3L\rangle) \bigcup \mathbb{V}(\langle 4A + 5C + D,\, N,\, C(A + C) - K,\, M,\, 2(A + C)(2A + 3C) - P,\, B(A + C) + L\rangle) \subset \mathbb{V}(J_{10}),$$
$$\mathbb{V}(\langle A + 2C,\, 3A + 2D,\, B,\, K(A^2 + 4K) + 4N^2,\, 3K + M,\, A^2 - 2P,\, L + N\rangle) \subset \mathbb{V}(J_{19}),$$
$$\mathbb{V}(\langle 2A + 3C,\, 2A + 3D,\, AB - 3N,\, 2A^2 + 9K,\, 2(A - 3B)(A + 3B) - 9M,\, A^2 + 18B^2 - 9P,\, 2AB + 3L\rangle) \bigcup \mathbb{V}(\langle 4A + 5C + D,\, 2B^2 + (A + C)^2,\, B(A + 2C) + N,\, C(A + C) - K,\, 3(A + C)^2 - M,\, 2(A + C)(2A + 3C) - P,\, BC - L\rangle) \subset \mathbb{V}(J_{21}),$$

and $\mathbb{V}(\widehat{J}_1) \subset \mathbb{V}(J_{10})$, $\mathbb{V}(\widehat{J}_2) \subset \mathbb{V}(J_{12})$.

Statement 10. *Let $\widehat{J}_1$ and $\widehat{J}_2$ be the ideals from Statement 9, then the radicals of ideals $\widetilde{I}_{11}$, $\widetilde{I}_{12}$ and $\widetilde{I}_{13}$ can be written in the form:*

$$\sqrt{\widetilde{I}_{11}} = J_8 \bigcap \widehat{J}_1 \bigcap \widehat{J}_2 \bigcap \langle A + 2C,\, 2A + D,\, N,\, A^2 + 4K,\, 3A^2 - 4P,\, AB + 2L\rangle \bigcap \langle 3A + 4C,\, A + 2D,\, N,\, 3A^2 + 16K,\, 3A^2 - 16M,\, A^2 - 16P,\, AB + 4L\rangle \bigcap \langle A,\, 6C + D,\, B^2 +$$

$3C^2,\ BC+N,\ K,\ 6C^2-M,\ 9C^2-P,\ L\rangle$, $\sqrt{\widetilde{I}_{12}} = J_9 \bigcap J_{18} \bigcap \widehat{J}_1 \bigcap \widehat{J}_2 \bigcap \langle A+2C,\ 5A+4D,\ N,\ A^2+4K,\ 3A^2-8P,\ AB+2L\rangle \bigcap \langle 3A+5C,\ 4A+5D,\ N,\ 6A^2+25K,\ 6A^2-25M,\ 4A^2-25P,\ 2AB+5L\rangle$,

$\sqrt{\widetilde{I}_{13}} = J_6 \bigcap J_{17} \bigcap J_{22} \bigcap \widehat{J}_1 \bigcap \widehat{J}_2 \bigcap \langle A,\ C,\ N,\ K,\ 2D^2+P,\ L\rangle \bigcap \langle A+2C,\ 3A+2D,\ N,\ A^2+4K,\ A^2-2P,\ AB+2L\rangle \bigcap \langle A,\ 6C+D,\ N,\ K,\ 3C^2+M,\ 9C^2-P,\ BC+L\rangle \bigcap \langle A+C,\ A-2D,\ N,\ K,\ M,\ P,\ L\rangle \bigcap \langle A+C,\ A-D,\ N,\ K,\ M,\ P,\ L\rangle \bigcap \langle A+C,\ 2A-D,\ N,\ K,\ M,\ P,\ L\rangle \bigcap \langle A+3C,\ 2A+D,\ N,\ K,\ A^2-3M,\ A^2-P,\ 2AB+3L\rangle$,

and at the same time the following inclusions are held:

$\mathbb{V}(\langle A+2C,\ 2A+D,\ N,\ A^2+4K,\ 3A^2-4P,\ AB+2L\rangle) \bigcup \mathbb{V}(\langle 3A+4C,\ A+2D,\ N,\ 3A^2+16K,\ 3A^2-16M,\ A^2-16P,\ AB+4L\rangle) \bigcup \mathbb{V}(\langle A+2C,\ 5A+4D,\ N,\ A^2+4K,\ 3A^2-8P,\ AB+2L\rangle) \bigcup \mathbb{V}(\langle 3A+5C,\ 4A+5D,\ N,\ 6A^2+25K,\ 6A^2-25M,\ 4A^2-25P,\ 2AB+5L\rangle) \bigcup \mathbb{V}(\langle A,\ C,\ N,\ K,\ 2D^2+P,\ L\rangle) \bigcup \mathbb{V}(\langle A+2C,\ 3A+2D,\ N,\ A^2+4K,\ A^2-2P,\ AB+2L\rangle) \bigcup \mathbb{V}(\langle A+C,\ A-2D,\ N,\ K,\ M,\ P,\ L\rangle) \bigcup \mathbb{V}(\langle A+C,\ A-D,\ N,\ K,\ M,\ P,\ L\rangle) \bigcup \mathbb{V}(\langle A+C,\ 2A-D,\ N,\ K,\ M,\ P,\ L\rangle) \subset \mathbb{V}(J_{10})$, $\mathbb{V}(\langle A,\ 6C+D,\ N,\ K,\ 3C^2+M,\ 9C^2-P,\ BC+L\rangle) \subset \mathbb{V}(J_8)$, $\mathbb{V}(\langle A+3C,\ 2A+D,\ N,\ K,\ A^2-3M,\ A^2-P,\ 2AB+3L\rangle) \subset \mathbb{V}(J_9)$, $\mathbb{V}(\langle A,\ 6C+D,\ B^2+3C^2,\ BC+N,\ K,\ 6C^2-M,\ 9C^2-P,\ L\rangle) \subset \mathbb{V}(J_{13})$.

To formulate the next statements we denote by $\widehat{J}_1, ..., \widehat{J}_6$ the ideals of the form:

$\widehat{J}_1 = \langle A,\ C,\ N,\ K,\ 3B^2+M,\ L\rangle$, $\widehat{J}_2 = \langle A+C,\ N,\ K,\ 3B^2+A(A+D)+M,\ 3B^2+A(A+D)+P,\ L\rangle$,

$\widehat{J}_3 = \langle A,\ B,\ N,\ K,\ C(3C+D)-M,\ C(3C+D)+P,\ L\rangle$, $\widehat{J}_4 = \langle A,\ 3C+D,\ B,\ N,\ K,\ 3M+2P,\ L\rangle$,

$\widehat{J}_5 = \langle A+2C,\ N,\ A^2+4K,\ A(A+2D)+4(3B^2+M),\ A(A+2D)+4P,\ AB+2L\rangle$,

$\widehat{J}_6 = \langle A+3C,\ B,\ N,\ K,\ A(A+D)+3M,\ A(A+D)+P,\ L\rangle$.

For the ideals $\widehat{J}_1, ..., \widehat{J}_6$ the inclusions are held: $\mathbb{V}(\widehat{J}_1) \bigcup \mathbb{V}(\widehat{J}_2) \bigcup \mathbb{V}(\widehat{J}_5) \subset \mathbb{V}(J_{10})$, $\mathbb{V}(\widehat{J}_3) \bigcup \mathbb{V}(\widehat{J}_4) \bigcup \mathbb{V}(\widehat{J}_6) \subset \mathbb{V}(J_1)$.

Statement 11. *For the radicals of ideals $\widetilde{I}_1$, $\widetilde{I}_4$, $\widetilde{I}_5$, $\widetilde{I}_7$ the next equalities are true:*

$\sqrt{\widetilde{I}_1} = J_1 \bigcap J_{20} \bigcap J_{21}$, $\sqrt{\widetilde{I}_4} = J_{13} \bigcap J_{24} \bigcap \widehat{J}_1 \bigcap \widehat{J}_2 \bigcap \langle 5A+6C,\ B,\ N,\ A^2+4K,\ 5A(A+2D)+12M,\ A(A+2D)+4P,\ L\rangle$,

$\sqrt{\widetilde{I}_5} = J_{14} \bigcap J_{23} \bigcap \widehat{J}_1 \bigcap \widehat{J}_2 \bigcap \langle A,\ B,\ N,\ K,\ C(3C+2D)-2M,\ 3C(3C+2D)+4P,\ L\rangle$,

$\sqrt{\widetilde{I}_7} = J_{15} \bigcap (\bigcap_{k=3}^{6} \widehat{J}_k) \bigcap \langle A,\ 3C+D,\ BC+N,\ K,\ 2B^2+M,\ 3B^2+2P,\ L\rangle \bigcap \langle A,\ B^2-C(3C+D), DC+N, K,\ 2C(3C+D)+M,\ 3C(3C+D)+P,\ L\rangle \bigcap \langle 3(A+C)+D,\ B,\ N,\ K,\ A(2A+3C)-3M,\ A(2A+3C)-P,\ L\rangle \bigcap \langle 5A+6C+2D,\ B,\ N,\ A^2+4K,\ A(2A+3C)-2M,\ A(2A+3C)-2P,\ L\rangle$,

and at the same time the inclusions take place:

$\mathbb{V}(\langle 5A+6C,\, B,\, N,\, A^2+4K,\, 5A(A+2D)+12M,\, A(A+2D)+4P,\, L\rangle)\bigcup \mathbb{V}(\langle A,\, B,\, N,\, K,\, C(3C+2D)-2M,\, 3C(3C+2D)+4P,\, L\rangle)\bigcup\mathbb{V}(\langle 3(A+C)+D,\, B,\, N,\, K,\, A(2A+3C)-3M,\, A(2A+3C)-P,\, L\rangle)\bigcup\mathbb{V}(\langle 5A+6C+2D,\, B,\, N,\, A^2+4K,\, A(2A+3C)-2M,\, A(2A+3C)-2P,\, L\rangle)\subset\mathbb{V}(J_1)$, and also $\mathbb{V}(\langle A,\, 3C+D,\, BC+N,\, K,\, 2B^2+M,\, 3B^2+2P,\, L\rangle)\bigcup\mathbb{V}(\langle A,\, B^2-C(3C+D),\, DC+N,\, K,\, 2C(3C+D)+M,\, 3C(3C+D)+P,\, L\rangle)\subset\mathbb{V}(J_{21})$.

Statement 12. *The radicals of ideals* $\widetilde{I}_6$ *and* $\widetilde{I}_8$ *can be represented in the form:*

$\sqrt{\widetilde{I}_6}=J_{13}\bigcap J_{14}\bigcap J_{16}\bigcap \widehat{J}_1\bigcap\widehat{J}_3\bigcap\widehat{J}_4\bigcap\langle 2A+3C,\, B,\, N,\, A^2+4K,\, A(A+2D)+3M,\, A(A+2D)+4P,\, L\rangle\bigcap\langle 3(A+C)+D,\, B,\, N,\, A^2+4K,\, A(5A+6C)-3M,\, A(5A+6C)-4P,\, L\rangle$,

$\sqrt{\widetilde{I}_8}=J_{13}\bigcap J_{15}\bigcap J_{25}\bigcap\widehat{J}_1\bigcap\widehat{J}_3\bigcap\widehat{J}_5\bigcap\widehat{J}_6\bigcap\langle 3B^2-C(A+D),\, N,\, C(A+C)-K,\, C(A+C)-M,\, (A+C)(A+C+D)+P,\, B(A+C)+L\rangle\bigcap\langle A,\, B,\, N,\, K,\, C(3C+2D)-2M,\, 3C(3C+2D)+4P,\, L\rangle\bigcap\langle A,\, 2B^2-C(3C+2D),\, BC+N,\, K,\, C(3C+2D)+M,\, 3C(3C+2D)+4P,\, L\rangle\bigcap\langle 5A+3C+2D,\, B,\, N,\, K,\, A(A+C)-2M,\, 3A(A+C)-2P,\, L\rangle\bigcap\langle 7A+6C+4D,\, B,\, N,\, A^2+4K,\, A(5A+6C)-8M,\, A(5A+6C)-8P,\, L\rangle$,

and the inclusions are true:

$\mathbb{V}(\langle A,\, 2B^2-C(3C+2D),\, BC+N,\, K,\, C(3C+2D)+M,\, 3C(3C+2D)+4P,\, L\rangle)\subset\mathbb{V}(J_{21})$, $\mathbb{V}(\langle 2A+3C,\, B,\, N,\, A^2+4K,\, A(A+2D)+3M,\, A(A+2D)+4P,\, L\rangle)\bigcup\mathbb{V}(\langle 3(A+C)+D,\, B,\, N,\, A^2+4K,\, A(5A+6C)-3M,\, A(5A+6C)-4P,\, L\rangle)\bigcup\mathbb{V}(\langle A,\, B,\, N,\, K,\, C(3C+2D)-2M,\, 3C(3C+2D)+4P,\, L\rangle)\bigcup\mathbb{V}(\langle 5A+3C+2D,\, B,\, N,\, K,\, A(A+C)-2M,\, 3A(A+C)-2P,\, L\rangle)\bigcup\mathbb{V}(\langle 7A+6C+4D,\, B,\, N,\, A^2+4K,\, A(5A+6C)-8M,\, A(5A+6C)-8P,\, L\rangle)\subset\mathbb{V}(J_1)$, $\mathbb{V}(\langle 3B^2-C(A+D),\, N,\, C(A+C)-K,\, C(A+C)-M,\, (A+C)(A+C+D)+P,\, B(A+C)+L\rangle)\subset\mathbb{V}(J_{10})$.

Proof. To find the radicals $\sqrt{\widetilde{I}_6}$ and $\sqrt{\widetilde{I}_8}$ we will consider the ideals $\widetilde{\widetilde{I}}_6=\widetilde{I}_6+\langle 3(3a_0^2-2a_1+2a_0c_1)+\widetilde{u}^2\rangle$ and $\widetilde{\widetilde{I}}_8=\widetilde{I}_8+\langle 2(a_0^2-a_1+a_0c_1)+\widetilde{u}^2\rangle$. Using Groebner bases we find the radicals $\sqrt{\widetilde{\widetilde{I}}_6}$ and $\sqrt{\widetilde{\widetilde{I}}_8}$ and get $\sqrt{\widetilde{I}_6}=\sqrt{\widetilde{\widetilde{I}}_6}\bigcap\mathbb{C}[q]$, $\sqrt{\widetilde{I}_8}=\sqrt{\widetilde{\widetilde{I}}_8}\bigcap\mathbb{C}[q]$. □

Statement 13. *The radicals of ideals* $J+G_{12}$, $J+G_{13}$, $J+G_{14}$ *have the form:*

$\sqrt{J+G_{12}}=J_{13}\bigcap\langle B(A+C)+N,\, K,\, 2B^2+M+A(2A+C+D),\, 2B^2+P+A(A+D),\, L\rangle\bigcap\langle A+C,\, N,\, K,\, M-P,\, L\rangle\bigcap\langle B,\, N,\, K,\, L\rangle\bigcap\langle A,\, BC+N,\, K,\, 2B^2+M,\, L\rangle\bigcap\langle A,\, C,\, N,\, K,\, L\rangle$,

$\sqrt{J+G_{13}}=J_{12}\bigcap\langle A+C,\, B,\, N,\, 3K+M+P,\, L\rangle\bigcap\langle A+C,\, 2A+3D,\, B,\, 7A^4-81N^2,\, A^2-3K,\, A^2+9M,\, 8A^2+9P,\, L+N\rangle\bigcap\langle A,\, C,\, 3B^2-D^2,\, BD-N,\, D^2-K,\, D^2+M,\, 2D^2+P,\, BD+L\rangle\bigcap\langle A,\, C,\, D,\, N,\, 3K+M+P,\, L\rangle$,

$\sqrt{J+G_{14}}=J_{15}\bigcap\langle B(2A+2C+D)-N,\, (2A+3C+D)(3A+3C+D)-K,\, 2B^2+(2A+3C+D)(4A+5C+D)+M,\, (2A+3C)(2A+3C+D)^2+2B^2(5A+6C+2D)+(2A+3C+D)P,\, B(3A+3C+D)+L\rangle\bigcap\langle 2A+2C+D,\, N,\, C(A+C)-K,\, C(A+C)(2A+3C)+(A+2C)M-CP,\, B(A+C)+L\rangle\bigcap\langle B,\, N,\, (2A+3C+D)(3A+3C+D)-K,\, L\rangle$, *and the next inclusions are true:*

$\mathbb{V}(\langle B(A+C)+N,\, K,\, 2B^2+M+A(2A+C+D),\, 2B^2+P+A(A+D),\, L\rangle)\bigcup\bigcup\mathbb{V}(\langle A,\, BC+N,\, K,\, 2B^2+M,\, L\rangle)\bigcup\mathbb{V}(\langle B(2A+2C+D)-N,\, (2A+3C+D)(3A+3C+D)-K,\, 2B^2+(2A+3C+D)(4A+5C+D)+M,\, (2A+3C)(2A+3C+D)^2+2B^2(5A+6C+2D)+(2A+3C+D)P,\, B(3A+3C+D)+L\rangle)\subset\mathbb{V}(J_{21})$, $\mathbb{V}(\langle A+C,\, N,\, K,\, M-P,\, L\rangle)\bigcup\mathbb{V}(\langle A,\, C,\, N,\, K,\, L\rangle)\bigcup\mathbb{V}(\langle 2A+2C+D,\, N,\, C(A+C)-K,\, C(A+C)(2A+3C)+(A+2C)M-CP,\, B(A+C)+L\rangle)$, $\mathbb{V}(\langle B,\, N,\, K,\, L\rangle)\bigcup\mathbb{V}(\langle A+C,\, B,\, N,\, 3K+M+P,\, L\rangle)\bigcup\mathbb{V}(\langle B,\, N,\, (2A+3C+D)(3A+3C+D)-K,\, L\rangle)\subset\mathbb{V}(J_1)$, $\mathbb{V}(\langle A+C,\, 2A+3D,\, B,\, 7A^4-81N^2,\, A^2-3K,\, A^2+9M,\, 8A^2+9P,\, L+N\rangle)\subset\mathbb{V}(J_{25})$, $\mathbb{V}(\langle A,\, C,\, 3B^2-D^2,\, BD-N,\, D^2-K,\, D^2+M,\, 2D^2+P,\, BD+L\rangle)\subset\mathbb{V}(J_{15})$, $\mathbb{V}(\langle A,\, C,\, D,\, N,\, 3K+M+P,\, L\rangle)\subset\mathbb{V}(J_2)$.

Statement 14. *The radicals of ideals $J+G_{15}$, $J+G_{16}$, $J+G_{17}$ can be written in the next form:*

$\sqrt{J+G_{15}}=J_{19}\bigcap\langle A+2C,\, 3A+2D,\, N,\, 3K+M,\, A^2-2P,\, AB+2L\rangle\bigcap\langle A+2C,\, 5A+4D,\, A^2-48B^2,\, AB-4N,\, 3A^2+16K,\, 5A^2-16M,\, 3A^2-8P,\, 3AB+4L\rangle\bigcap\langle A+2C,\, 2A+D,\, A^2-12B^2,\, AB+2N,\, K,\, A^2-2M,\, 3A^2-4P,\, L\rangle\bigcap\langle B,\, N,\, 2(A+C)(2A+C+D)+3K+M,\, (A+C)(A+C+D)+P,\, L\rangle\bigcap\langle 5A+7C,\, 8A+7D,\, B,\, A^4-343N^2,\, A^2+7K,\, 17A^2-49M,\, 12A^2-49P,\, L+N\rangle\bigcap\langle A+5C,\, 2A+D,\, B,\, 7A^4-625N^2,\, A^2+25K,\, 11A^2-25M,\, 24A^2-25P,\, L+N\rangle$,

$\sqrt{J+G_{16}}=J_{14}\bigcap\langle B(A+C)-2N,\, 3(A+C)(A+3C)-4K,\, 4B^2+(A+3C)(A+2C-D)+2M,\, 16B^2(2A+3C)+(A+3C)^2(A+3C-2D)+4(A+3C)P,\, 3B(A+C)+2L\rangle\bigcap\langle A+C,\, N,\, K,\, M-P,\, L\rangle\bigcap\langle B,\, N,\, 3(A+C)(A+3C)-4K,\, L\rangle\bigcap\langle A,\, C,\, N,\, K,\, L\rangle$,

$\sqrt{J+G_{17}}=J_{10}\bigcap J_{12}\bigcap J_{17}\bigcap J_{18}\bigcap J_{19}\bigcap J_{21}\bigcap J_{22}\bigcap\langle A+3C,\, 2A+D, N, K, A^2-3M,\, A^2-P,\, 2AB+3L\rangle$,

and the following inclusions take place:

$\mathbb{V}(\langle A+2C,\, 3A+2D,\, N,\, 3K+M,\, A^2-2P,\, AB+2L\rangle)\subset\mathbb{V}(J_5)$, $\mathbb{V}(\langle A+2C,\, 5A+4D,\, A^2-48B^2,\, AB-4N,\, 3A^2+16K,\, 5A^2-16M,\, 3A^2-8P,\, 3AB+4L\rangle)\subset\mathbb{V}(J_{14})$, $\mathbb{V}(\langle A+2C,\, 2A+D,\, A^2-12B^2,\, AB+2N,\, K,\, A^2-2M,\, 3A^2-4P,\, L\rangle)\subset\mathbb{V}(J_{13})$, $\mathbb{V}(\langle 5A+7C,\, 8A+7D,\, B,\, A^4-343N^2,\, A^2+7K,\, 17A^2-49M,\, 12A^2-49P,\, L+N\rangle)\subset\mathbb{V}(J_{24})$, $\mathbb{V}(\langle A+5C,\, 2A+D,\, B,\, 7A^4-625N^2,\, A^2+25K,\, 11A^2-25M,\, 24A^2-25P,\, L+N\rangle)\subset\mathbb{V}(J_{23})$, $\mathbb{V}(\langle B(A+C)-2N,\, 3(A+C)(A+3C)-4K,\, 4B^2+(A+3C)(A+2C-D)+2M,\, 16B^2(2A+3C)+(A+3C)^2(A+3C-2D)+4(A+3C)P,\, 3B(A+C)+2L\rangle)\subset\mathbb{V}(J_{21})$, $\mathbb{V}(\langle A+C,\, N,\, K,\, M-P,\, L\rangle)\bigcup\mathbb{V}(\langle A,\, C,\, N,\, K,\, L\rangle)\bigcup\mathbb{V}(\langle A+3C,\, 2A+D,\, N,\, K,\, A^2-3M,\, A^2-P,\, 2AB+3L\rangle)\subset\mathbb{V}(J_{10})$.

The proof of Theorem 5. The proof follows directly from Theorem 3 and Statements 8–14.

5. The polynomial $\widehat{P}$ has 23 real roots. Let's introduce a vector $p(u,v)$. The system of equations $Z_i=0$, $i=\overline{1,6}$, has 45 real solutions $p=p_k$, $k=\overline{1,45}$. Here

$p_1 = (-2.98291\ldots, 0.61354\ldots)$, $p_2 = (0.28767\ldots, 0.61354\ldots)$,
$p_3 = (2.92233\ldots, 0.61354\ldots)$, $p_4 = (-2.24140\ldots, 1.80824\ldots)$,
$p_5 = (0.70470\ldots, 1,80824\ldots)$, $p_6 = (4.15318\ldots, 1.80824\ldots)$,
$p_7 = (-0.84828\ldots, 0.59506\ldots)$, $p_8 = (0.11443\ldots, 0.64371\ldots)$,
$p_9 = (0.38246\ldots, 0.70488\ldots)$, $p_{10} = (0.13270\ldots, 0.71955\ldots)$,
$p_{11} = (0.50158\ldots, 0.86047\ldots)$, $p_{12} = (1.23899\ldots, 1.37056\ldots)$,
$p_{13} = (1.83858\ldots, 1.79718\ldots)$.

All values $p = p_k$, $k = \overline{1,45}$, were computed to within 300 digits after the decimal point. Notice that this system has no other real solutions. We substitute p_k in the system $O_5 = 0$, $O_6 = 0$ and find $P = P_k$, $k = \overline{1,45}$. Further replacing w by 1 and computed values $p = p_k$, $P = P_k$, $k = \overline{1,45}$, in $T_{5,1}$ and $T_{5,2}$, one gets accordingly $M = M_k$, $k = \overline{1,45}$. From $\widetilde{g}_3 = 0$ after substitution $w = 1$, $p = p_k$, $P = P_k$, $M = M_k$, $k = \overline{1,45}$, we obtain the next equalities: $\widetilde{\alpha}_k N^2 + \widetilde{\beta}_k = 0$ ($\widetilde{\alpha}_k, \widetilde{\beta}_k \in \mathbb{R}$, $k = \overline{1,45}$), which for $k = \overline{1,13}$ have real roots $N_{2k-1} = -N_{2k}$, and for $k = \overline{14,45}$ - complex roots. Thus we obtain 26 real solutions of system $\widetilde{g}_i = 0$, $i = \overline{3,8}$, of the type $(M_j, N_j, P_j, u_j, v_j)$, $j = \overline{1,26}$. Let $n = (A, D, K, L, M, N, P)$. As a result we get 26 real solutions $n = n_j$ of system $\widetilde{f}_i = 0$, $i = \overline{1,8}$. As a case in point we give one of this solutions: $n_1 = (-2.1488396095\ldots C,\ 3.5339790637\ldots C,\ -0.6697945306\ldots C^2,\ -0.6271954771\ldots C^2, 2.4276250887\ldots C^2,\ 0.627195477147\ldots C^2,\ 2.86484242\ldots C^2)$, where $C \neq 0$.

As $\widehat{P}$ and $\widetilde{r}_0$ are coprime polynomials in v then $\widetilde{f}_8|_{n=n_j} \neq 0$, $j = \overline{1,26}$. Direct computations give $\widetilde{f}_8|_{n=n_j} = \widetilde{f}_{8,j}$, $j = \overline{1,26}$, where $\widetilde{f}_{8,1} = 3.3665\ldots \cdot 10^{14} C^{16}$. For $n = n_j$, $j = \overline{1,13}$, $\widetilde{f}_i = 0$, $i = \overline{1,7}$, but $\widetilde{f}_8 \neq 0$. So we get

Statement 15. *When $n = n_j$, $j = \overline{1,13}$, the critical point $O(0,0)$ of system (7), where $\lambda = 0$ is a focus of 8th order.*

Statement 16. *For any $\widetilde{\varepsilon}, \widetilde{\delta} > 0$, $j = \overline{1,26}$, there exist $n \in V_{\widetilde{\delta}}(n_j)$ and $\lambda \in V_{\widetilde{\delta}}(0)$, where $V_{\widetilde{\delta}}(n_j)$ is $\widetilde{\delta}$-neighborhood n_j, $V_{\widetilde{\delta}}(0)$ - $\widetilde{\delta}$-neighborhood of zero, at which system (7) has in $\widetilde{\varepsilon}$-neighborhood of the point $O(0,0)$ 8 limit cycles.*

Proof. Let $e = (\lambda, n) = (\lambda, A, D, K, L, M, N, P)$. Denote $e_k = (0, n_k)$, $k = \overline{1,26}$. For system (7) there exists the only polynomial $\widetilde{W} = (1 + \lambda^2/2)x^2 - \lambda xy + y^2 + \sum_{i+j=3}^{18} P_{i,j} x^i y^j$, where $P_{0,2k} = 0$, $k = \overline{2,9}$, for which on account of system (7) $\dot{\widetilde{W}} = \sum_{i=0}^{8} \widehat{f}_i(e)(x^2+y^2)^{i+1} + m_{19}(x,y) + m_{20}(x,y)$, where $\widehat{f}_0(e) = \lambda$, $\widehat{f}_i(e)|_{\lambda=0} = \widetilde{f}_i$, $i = \overline{1,8}$, m_i, $i = 19, 20$, are homogeneous polynomials of i^{th} degree. Let's generate $\widehat{f}(e) = (\widehat{f}_0(e), \widehat{f}_1(e), \ldots, \widehat{f}_7(e))$. Then $det\ \partial\widehat{f}(e_k)/\partial e = \rho_k$, where $\rho_k \neq 0$, $k = \overline{1,26}$. The further is analogous to the proof of Theorem 3 from [13].

Proof of Theorem 6. The proof directly follows from Statements 15–16.

References

[1] Kukles I.S. *About some cases of difference between focal and center.* Rep. of A.S. USSR, 1944, **42**, N 5, p. 208–211 (in Russian).

[2] Cherkas L.A. *Center conditions for the equation* $yy' = \sum_{i=0}^{3} p_i(x)y^i$. Diff.Equations, 1978, **14**, N 9, p. 1594–1600 (in Russian).

[3] Sadovskii A.P. *To the conditions of center and focal for the equations of nonlinear oscillations.* Diff.Equations, 1979, **15**, N 9, p. 1716–1719 (in Russian).

[4] Christopher C.J., Lloyd N.G. *On the paper of Jin and Wang concerning the conditions for a centre in certain cubic systems.* Bull. London Math Soc., 1990, **22**, N 1, p. 5–12.

[5] Suba A.S. *On center conditions of Kukles and Cherkas for one cubic system.* Diff.Equations, 1993, **29**, N 4, p. 728–730 (in Russian).

[6] Lloyd N.G., Pearson J.M. *Conditions for a center and the bifurcation of limit cycles in a class of cubic system.* Lecture Notes in Math. Bifurcations of Planar vector fields. Berlin, Springer, 1990, **1455**, p. 230-242.

[7] Sadovskii A.P. *On limit cicles for cubic system of nonlinear oscillations.* Diff. Equations, 1993, **29**, N 2, p. 226–231 (in Russian).

[8] Sadovskii A.P. *On center conditions for one cubic system of differential equations.* Diff. Equations, 2000, **36**, N 1, p. 98–102 (in Russian).

[9] Lloyd N.G., Pearson J.M. *Computing centre condition for certain cubic systems.* J. Comp & Appl. Math., 1992, **40**, N 2, p. 323–336.

[10] Cherkas L.A. *On one sign of difference between center and focal for Lienard equation.* Rep. of A.S. BSSR, 1978, **22**, N 11, p. 969–970 (in Russian).

[11] Amelkin V.V., Lukashevichi N.A. *Nonlinear oscillations in the systems of second order.* Minsk, 1982 (in Russian).

[12] Sadovskii A.P. *Solution of the problem of center and focal for the cubic system on nonlinear oscillations.* Diff. Equations, 1997, **33**, N 2, p. 236–244 (in Russian).

[13] Sadovskii A.P. *Cubic systems of nonlinear oscillations with seven limit cicles.* Diff. Equations, 2003, **39**, N 4, p. 472–481 (in Russian).

[14] Suba A., Cosma D. *Solution of the problem of the centre for cubic differential system with three invariant straight lines two of which are parallel.* Buletinul A.S. a R.M., Matematica, 2001, N 2(36), p. 75–86.

[15] Zoladek H. *The classification of reversible cubic systems with center.* Topological methods in nonlinear analysis, Journal of the Juliusz Schauder Center, 1994, **4**, N 1, p. 79–136.

[16] Kox D., Little J., O'Shi D. *Ideals, manifolds and algorithms. Introduction in computer aspects of algebraic geometry and commutative algebra.* Moscow, 2000 (in Russian)

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Modeling and optimization of melting and solidification process

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Abstract. An optimal control problem is considered for two-phase Stefan problem describing the process of melting and solidification. The problem is solved numerically by variation and finite-difference methods. The results are described and analyzed in detail. Some of them are presented as tables and plots.

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1 Introduction

Heat transfer in various media has a great effect on many practically important processes. For this reason, many studies in both physics and mathematics have been devoted to this subject. Mathematically, heat transfer is described by boundary value problems for a heat equation. These boundary value problems have been thoroughly described and investigated in both handbooks and specialized literature.

Since few boundary value problems for the heat equation have analytical solutions, much effort has been focused on the development of numerical methods for problems of this kind.

Practically interesting problems concern not only the description and analysis of heat transfer processes but also the optimal control of them. As a result, the theory of optimal control of thermal processes has been created, which includes the existence and uniqueness of optimal solution, finite-difference approximation and regularization of optimal control problems, and solutions to specific practically important problems. Relevant results in this direction can be found, for example, in [1,4].

An important class of heat transfer problems is that describing processes in which the substance under study undergoes phase changes accompanied by heat release or absorption. Problems of this kind (known as Stefan problems) arise in many situations, of which the most important and widespread are melting and solidification processes. An important feature of these problems is that they involve a moving interface between two phases (liquid and solid). The law of motion of the interface is unknown in advance and is to be determined. It is on this interface that heat release or absorption associated with phase changes occurs. The thermal properties of the substance on the different sides of the moving interface can be different. Problems of this class are noticeably more complicated than those not involving phase changes. An analysis of direct Stefan problems and methods for their solution are broadly presented in scientific literature.

Studies concerning optimal control of processes with phase changes are relatively few. Interesting and important (in our view) studies in this area can be found in [5,6].

In this paper, we consider the following optimal control problem for the process of melting and solidification. Given a heat source with a time-varying strength (which is treated as a control function), the problem is to find a source strength temporal distribution such that no less than a prescribed portion of the sample is melted, solidification proceeds at a rate not exceeding a prescribed magnitude, and the total heat supplied by the source is minimal.

This problem is analyzed here in a one-dimensional (radially symmetric) time-dependent setting. The heat source is located along the axis of symmetry. We analyze the case of a distributed and a point source. The control function is subject to inequality constrains, which simulate requirements imposed on the process of melting and solidification.

2 The mathematical formulation of the problem

In the plane of independent variables (r,t) we consider a rectangular domain $Q = \{(r,t) : 0 < r < R, 0 < t \leq \Theta\}$ (see Fig. 1). a smooth curve AB with the equation $r = \xi(t)$ divides Q into two subdomains: L (liquid domain) and S (solid domain). The curve AB is the trajectory of the front of melting and solidification. Let $t_0 \geq 0$ be the time at which AB originates. Then L and S are defined by

$$L = \{(r,t) : \quad 0 < r < \xi(t), \quad t_0 < t \leq \Theta\},$$

$$S = \{(r,t) : \quad \xi(t) < r < R, \quad 0 < t \leq \Theta\}.$$

In Q we consider the two-phase Stefan problem

$$M_L \equiv \rho_L C_L \frac{\partial T_L}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(r k_L \frac{\partial T_L}{\partial r}\right) - F(r,t) = 0, \qquad (r,t) \in L, \tag{1}$$

$$M_S \equiv \rho_S C_S \frac{\partial T_S}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(r k_S \frac{\partial T_S}{\partial r}\right) - F(r,t) = 0, \qquad (r,t) \in S, \tag{2}$$

$$T_S(r,0) = T_{in}(r), \qquad 0 < r < R, \tag{3}$$

$$T_L\left(\xi(t),t\right) = T_S\left(\xi(t),t\right) = T_{pl}, \qquad t_0 \leq t \leq \Theta, \tag{4}$$

$$\left[k_S \frac{\partial T_S}{\partial r}\right]\Bigg|_{(\xi(t)+0,t)} - \left[k_L \frac{\partial T_L}{\partial r}\right]\Bigg|_{(\xi(t)-0,t)} = \rho_S \lambda \xi'(t), \qquad t_0 \leq t \leq \Theta, \tag{5}$$

$$k_S \frac{\partial T_S}{\partial r}\Bigg|_R = \alpha\left[T_{ex} - T_S(R,t)\right], \qquad 0 < t \leq \Theta, \tag{6}$$

$$\frac{\partial T_L}{\partial r}(0,t) = 0, \qquad t_0 < t \leq \Theta, \tag{7}$$

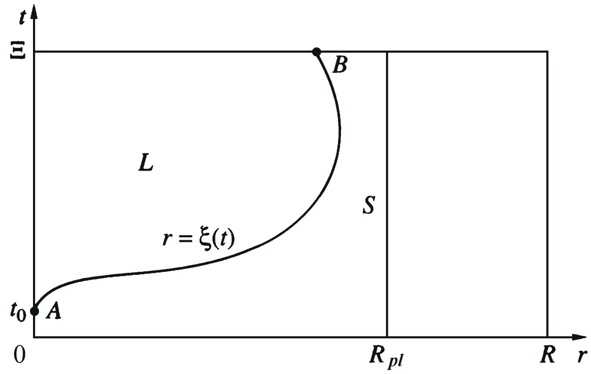

Fig. 1

$$\frac{\partial T_S}{\partial r}(0, t) = 0, \qquad 0 < t < t_0. \tag{8}$$

Here, $T(r, t)$ is the substance temperature at the point with coordinates (r, t); ρ, C, and k are the substance density, specific heat capacity, and thermal conductivity, respectively; λ is the heat of fusion of the substance; the subscripts L and S denote the liquid and solid phases, respectively; T_{pl} is the temperature of fusion; $T_{in}(r)$ is the initial temperature of the substance, $T_{in}(r) \leq T_{pl}$; α is the heat exchange coefficient with the surrounding medium; and T_{ex} is the ambient temperature.

The source $F(r, t)$ of input heat can be represented as $F(r, t) = \varphi(r)f(t)$, where $\varphi(r)$ is a given function describing the spatial distribution of supplied heat. Along with $\varphi(r)$ and $f(t)$, the source of input heat will also be characterized by the function

$$f_w(t) = \int_0^\infty 2\pi r F(r, t) dr = f(t) \int_0^\infty 2\pi r \varphi(r) dr.$$

Especially worth noting is the particular case where $\varphi(r) = \delta(r)$ is the delta function (a point source). Overall, the statement of direct problem (1)—(8) then remains the same, except that we set $F(r, t) \equiv 0$ and $t_0 = 0$ in Eqs. (1), (2) and conditions (7) and (8) are replaced by

$$\lim_{r \to 0} \left(-2\pi k_L r \frac{\partial T_L}{\partial r} \right) = f(t), \qquad 0 < t \leq \Theta. \tag{9}$$

Note that $f(t)$ coincides with $f_w(t)$ for a point source.

Problem (1)—(8) (or (1)—(6), (9)) with a given $f(t)$ is referred to as the direct problem.

Let $\xi(t)$ be the interface corresponding to the source $f(t)$, $t \in [0, \Theta]$, and let ξ_f be the maximum of $\xi(t)$ over $t_0 \leq t \leq \Theta$. The function $f(t)$ is said to belong to $K(\Theta)$ if it satisfies the following conditions:

(i) it is defined and piecewise continuous on $[0, \Theta]$;

(ii) it has a piecewise continuous derivative;

(iii) it satisfies $0 \leq f(t) \leq f_{max}$ for all $t \in [0, \Theta]$;
(iv) the corresponding $\xi_f \geq R_{pl}$, where R_{pl} is given and satisfies $R_{pl} < R$;
(v) it holds that for all $t \in [0, \Theta - \beta^2]$

$$\xi^{'}(t) \geq -d^2. \tag{10}$$

Note that the value of f_{max} can be infinitely large, i.e., unbounded from above. Note also that, for a given finite f_{max}, Θ cannot be less than a certain value, because otherwise the class $K(\Theta)$ will be empty.

The variation problem to be solved is stated as follows: among the functions $f(t)$ in $K(\Theta)$, find $f_{opt}(t)$ that minimizes the functional

$$J = \int_0^{\Theta} f(t)dt. \tag{11}$$

The objective functional J is proportional to the total heat J_w supplied by the source over the observation time and equal to

$$J_w = \int_0^{\Theta} f_w(t)dt. \tag{12}$$

For mathematical modeling of the direct problem (determination of temperature distribution and interface separating the phases when control function – supplied heat – is given) the numerical algorithm was worked out and realized.

3 The algorithm of solving the direct problem

The algorithm that solves the direct problem is designed to deal with a distributed source, when $\varphi(r) \neq \delta(r)$. Essentially, it is a non front-capturing algorithm. The main idea of the algorithm was proposed by M. Rose in [7] and was developed by R.E. White in [8,9]. Here the path of the interface is not regarded as an explicitly imposed interior boundary condition. M.E.Rose suggested a generalized formulation of the problem and shows that genuine solution of the problem is its weak solution. On the other hand two genuine solutions whose domains of definition are separated by a smooth curve will constitute a weak solution if and only if the Stefan conditions (4), (5) connecting solid and liquid phases on the line take place.

In accordance with [7] we change from the unknown temperature $T(r,t)$ to the enthalpy function $E(r,t)$ defined in terms of temperature as

$$E(T) = \begin{cases} \rho_S C_S T, & T < T_{pl}, \\ \rho_L C_L (T - T_{pl}) + \rho_S C_S T_{pl} + \rho_S \lambda, & T \geq T_{pl}. \end{cases}$$

Note that the function $E(T)$ has a jump at the melting point T_{pl}. Treating the enthalpy $E(r,t)$ as a basic variable and the temperature $T(E)$ as defined by the relation

$$T(E)=\begin{cases} E\rho_S^{-1}C_S^{-1}, & E<E_-=\rho_S C_S T_{pl},\\ T_{pl}, & E_-\le E\le E_+=E_-+\rho_S\lambda,\\ \left[E+(\rho_L C_L-\rho_S C_S)T_{pl}-\rho_S\lambda\right]\rho_L^{-1}C_L^{-1}, & E_+<E \end{cases}$$

one can consider temperature as a continuous function of enthalpy.

In the general case, the heat conductivity depends on temperature and has a jump at the melting point, which corresponds to a transition from solid to liquid phase. In the proposed algorithm, the heat conductivity is a function of enthalpy defined as

$$\Omega(E)=k(T(E))=\begin{cases} k_S, & E<E_-,\\ k_S+(E-E_-)(k_L-k_S)(E_+-E_-)^{-1}, & E_-\le E\le E_+,\\ k_L, & E>E_+. \end{cases}$$

Problem (1)–(8) is reformulated in terms of the enthalpy function $E(r,t)$ as

$$\begin{aligned} &\frac{\partial E}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\Omega(E)\frac{\partial T(E)}{\partial r}\right)+F(r,t), && (r,t)\in Q,\\ &E(r,0)=E(T_{in}(r)), && 0<r<R,\\ &\left.\frac{\partial E}{\partial r}\right|_{r=0}=0, && 0\le t\le\Theta,\\ &\left.\Omega(E)\frac{\partial T(E)}{\partial r}\right|_{r=R}=\alpha\left[T_{ex}-T(E(R,t))\right], && 0\le t\le\Theta. \end{aligned} \tag{13}$$

To approximate the boundary value problem (13) in the domain Q, we introduce a nonuniform grid $\omega=\{r_i,t^j\}$, where

$$r_0=t^0=0,\quad r_i=r_{i-1}+h_{i-1},\quad t^j=t^{j-1}+\tau^j,\quad (i=1,\dots,K;\, j=1,\dots,M).$$

Using an implicit approximation with respect to time and an integro-interpolation method, we obtain the following system of finite-difference equations:

$$\begin{aligned} &E_0^j+a_0\hat\Omega(E_0^j)T(E_0^j)-a_0\hat\Omega(E_0^j)T(E_1^j)=E_0^{j-1}+\tau^jF_0^j,\\ &E_i^j+[a_i\hat\Omega(E_i^j)+b_i\hat\Omega(E_{i-1}^j)]T(E_i^j)-b_i\hat\Omega(E_{i-1}^j)T(E_{i-1}^j)-\\ &-a_i\hat\Omega(E_i^j)T(E_{i+1}^j)=E_i^{j-1}+\tau^jF_i^j, \qquad (1\le i\le K-1),\\ &E_K^j+[a_K\alpha+b_K\hat\Omega(E_{K-1}^j)]T(E_K^j)-b_K\hat\Omega(E_{K-1}^j)T(E_{K-1}^j)=\\ &=a_K\alpha T_{ex}+E_K^{j-1}+\tau^jF_K^j,\\ &\qquad j=1,\dots,M. \end{aligned} \tag{14}$$

Here, we introduce the following notation:

$$a_0 = \frac{4\tau^j}{h_0^2}, \qquad a_K = \frac{8\tau^j r_K}{4r_K h_{K-1}(1-h_{K-1})}, \qquad b_K = \frac{4\tau^j(2r_k - h_{K-1})}{h_{K-1}^2(4r_k - h_{K-1})},$$

$$a_i = \frac{4\tau^j(2r_i + h_i)}{4r_i h_i(h_i + h_{i-1}) + h_i^3 - h_{i-1}^2 h_i}, \quad b_i = \frac{4\tau^j(2r_i - h_{i-1})}{4r_i h_{i-1}(h_i + h_{i-1}) - h_{i-1}^3 + h_i^2 h_{i-1}},$$

$$(i = 1, \dots, K-1),$$

$$E_i^j = E(r_i, t^j), \qquad F_i^j = F(r_i, t^j), \qquad \hat{\Omega}(E_i^j) = \Omega[(E_i^j + E_{i+1}^j)/2].$$

The system of finite-difference equations (14) is an implicit approximation of the boundary value problem (13) restricted to $O(\tau, h^2)$ terms, where $\tau = \max_j \tau^j$, $h = \max_i h_i$.

The system of algebraic equations (14) can be split into M subsystems relating the enthalpy dependent quantities calculated on the j-th time layer with those calculated on the $(j-1)$-th time layer, $j = 1, \dots, M$. To facilitate further analysis, we represent the dependence of temperature T on E as $T(E) = \mu E + \nu$, where the functions μ and ν are defined as follows:

$$\mu(E) = \begin{cases} \rho_S^{-1} C_S^{-1}, & E < E_-, \\ 0, & E_- \le E \le E_+, \\ \rho_L^{-1} C_L^{-1}, & E_+ < E, \end{cases}$$

$$\nu(E) = \begin{cases} 0, & E < E_-, \\ T_{pl}, & E_- \le E \le E_+, \\ ((\rho_L C_L - \rho_S C_S) T_{pl} - \rho_S \lambda)\, \rho_L^{-1} C_L^{-1}, & E_+ < E. \end{cases}$$

We also introduce $(K+1)$-dimensional vectors $\mathbf{D}(\mathbf{E}^j)$, $\mathbf{L}(\mathbf{E}^j)$, $\mathbf{U}(\mathbf{E}^j)$ and $\boldsymbol{\eta}^j$, defined in terms of the components of the $(K+1)$-dimensional vector $\mathbf{E}^j = \|E_0^j\ E_1^j \dots E_K^j\|^T$ by the relations

$$\begin{array}{ll} D_0(\mathbf{E}^j) = a_0 \hat{\Omega}(E_0^j), \quad D_K(\mathbf{E}^j) = \alpha a_K + b_K \hat{\Omega}(E_{K-1}^j), & \\ D_i(\mathbf{E}^j) = a_i \hat{\Omega}(E_i^j) + b_i \hat{\Omega}(E_{i-1}^j), & (i = 1, \dots, K-1), \\ L_0(\mathbf{E}^j) = 0, \qquad L_i(\mathbf{E}^j) = b_i \hat{\Omega}(E_{i-1}^j), & (i = 1, \dots, K), \\ U_K(\mathbf{E}^j) = 0, \qquad U_i(\mathbf{E}^j) = a_i \hat{\Omega}(E_i^j), & (i = 0, \dots, K-1), \\ \eta_K^j = a_K \alpha T_{ex} + E_K^{j-1} + \tau^j F_K^j, \qquad \eta_i^j = E_i^{j-1} + \tau^j F_i^j, & (i = 0, \dots, K-1). \end{array}$$

Now, the j-th subsystem of (14) $(j = 1, \dots, M)$ can be written as

$$E_i^j + D_i(\mathbf{E}^j) T(E_i^j) - L_i(\mathbf{E}^j) T(E_{i-1}^j) - U_i(\mathbf{E}^j) T(E_{i+1}^j) = \eta_i^j, \tag{15}$$

$$(i = 0, \dots, K).$$

In [8], two iterative algorithms were proposed for solving the nonlinear system of equations (15). One of them is based on a modified Jacobi method. Defining the

$(K+1)$-dimensional vector $\mathbf{V}^n = \|V_0^n\, V_1^n \ldots V_K^n\|^T$ obtained as the approximation of $\mathbf{E}^j$ at the n-th iteration step (the initial approximation $\mathbf{V}^0$ is the vector $\mathbf{E}$ calculated on the preceding time layer, i.e., $\mathbf{E}^{j-1}$), we formulate the modified Jacobi method as the iterative calculation of the vector $\mathbf{V}^{n+1}$ given by the relation

$$V_i^{n+1} = (1-\omega)V_i^n + \omega \frac{L_i(\mathbf{V}^n)T(V_{i-1}^n) + U_i(\mathbf{V}^n)T(V_{i+1}^n) - D_i(\mathbf{V}^n)\nu(V_i^n) + \eta_i^j}{1 + D_i(\mathbf{V}^n)\mu(V_i^n)}, \qquad (16)$$

$$(i = 0, \ldots, K).$$

The iteration is continued until the relative difference in the values of the desired function between consecutive iteration steps,

$$\varepsilon = \max_{i=0,\ldots,K} \frac{V_i^{n+1} - V_i^n}{V_i^n}$$

(i.e., the iteration error), becomes less than a required value.

The other algorithm proposed in [8] is based on a modified Gauss-Seidel method. In this algorithm, the vector $\mathbf{V}^{n+1}$ is calculated as

$$V_i^{n+1} = (1-\omega)V_i^n + \omega \frac{L_i(\mathbf{V}^n)T(V_{i-1}^{n+1}) + U_i(\mathbf{V}^n)T(V_{i+1}^n) - D_i(\mathbf{V}^n)\nu(V_i^n) + \eta_i^j}{1 + D_i(\mathbf{V}^n)\mu(V_i^n)}, \qquad (17)$$

$$(i = 0, \ldots, K).$$

In both algorithms, the parameter ω is introduced to improve convergence. We recommend to define this parameter as follows:

$$\omega(E) = \begin{cases} \omega_0, & E < E_-, \\ (1-\omega_0)(E - \rho_S C_S T_{pl})\rho_S^{-1}\lambda^{-1} + \omega_0, & E_- \le E \le E_+, \\ 1, & E_+ < E, \end{cases}$$

where ω_0 is an arbitrary parameter (referred to as the accelerating parameter); $1 \le \omega_0 < 2$. In both (16) and (17), ω is calculated by using the values found at the preceding iteration step. In [8, 9] it was proved that the proposed iterative processes are convergent under certain conditions, and various examples of Stefan-type problems solved by applying algorithms (16) and (17) to the corresponding systems of algebraic equations were presented.

Previously, we used both the modified Jacobi algorithm and the modified Gauss-Seidel algorithm to solve problem (13). In the course of our computations, we found that the rate of convergence of the iterative processes (16) and (17) executed to solve the actual systems of algebraic equations was low. It was also found that the iterative processes could be substantially accelerated by using a new procedure [10]. Let us define the vector $\mathbf{V}^{n+1}$ at each iteration step as a solution to the following system of equations:

$$-L_i(\mathbf{V}^n)\mu(V_{i-1}^n)V_{i-1}^{n+1} + [1 + D_i(\mathbf{V}^n)\mu(V_i^n)]V_i^{n+1} - U_i(\mathbf{V}^n)\mu(V_{i+1}^n)V_{i+1}^{n+1} =$$

$$= L_i(\mathbf{V}^n)\nu(V_{i-1}^n) - D_i(\mathbf{V}^n)\nu(V_i^n) + U_i(\mathbf{V}^n)\nu(V_{i+1}^n) + \eta_i^j, \qquad (18)$$

$$(i = 0, \ldots, K).$$

System (18) has a tridiagonal matrix. If the time step τ^j is not too large, then this matrix has a diagonal dominance, and system (18) can be solved by means of the efficient tridiagonal algorithm. The new iterative process (18), combined with the tridiagonal algorithm for determining a solution at the $(n+1)$-th iteration step, is the essence of the proposed modification of the approaches developed in [8,9].

The process of solving (13) is terminated by determination of the melting front. Define $E_{pl} = (E_- + E_+)/2$. If the conditions $E_z^j \geq E_{pl}$ and $E_{z+1}^j < E_{pl}$ are satisfied for some $0 \leq z \leq K$ at $t = t^j$, then the melting radius is calculated as

$$\xi^j = \frac{(E_{pl} - E_{z+1}^j)(r_z - r_{z+1})}{E_z^j - E_{z+1}^j} + r_{z+1}. \qquad (19)$$

4 The solution of the variation problem

The variation problem formulated in Chapter 1 was solved numerically by gradient methods . For calculation the gradient of function the Fast Automatic Differentiation methodology was used [11]. To pick comparison functions from the set of class $K(\Theta)$ piecewise continuous functions, we used the method of external penalty functions. In this approach, the set of admissible comparison functions is much broader than $K(\Theta)$, but the cost functional is minimized by an element of the class $K(\Theta)$. This reduces the constraint minimization of the cost functional J in (11) to the unconstraint minimization of the generalized functional $I = J + g(\xi_f) + \Xi$, were $g(r) = A_0(r - R_{pl})^2$ (with a constant A_0) is the penalty functional responsible for the fulfillment of the condition $\xi_f = R_{pl}$, and

$$\Xi = \int_0^\Theta A(t)\left(\frac{d\xi}{dt} + d^2\right)dt, \qquad A(t) = \begin{cases} 0, & \left(\frac{d\xi}{dt} + d^2\right) \geq 0, \\ A_0(t), & \left(\frac{d\xi}{dt} + d^2\right) < 0, \end{cases}$$

is the penalty functional ensuring an admissible cooling rate. Here $\xi_f = \max\limits_{1 \leq j \leq M} \xi^j$, were ξ^j is given by (19). If this maximum is reached at $j = n$ $(1 \leq n \leq M)$, then

$$\xi_f = \frac{(E_{pl} - E_{z+1}^n)(r_z - r_{z+1})}{E_z^n - E_{z+1}^n} + r_{z+1}.$$

Using the rectangles method to approximate the functional J in (11), we obtain the following approximate expression for the generalized functional I:

$$I \approx \tilde{I} = \sum_{j=1}^{M} \tau^j f^j + \tilde{I}^1 + \tilde{I}^2, \quad \tilde{I}^1 = A_0\left[\frac{(E_{pl} - E_{z+1}^n)(r_z - r_{z+1})}{E_z^n - E_{z+1}^n} + r_{z+1} - R_{pl}\right]^2,$$

$$\tilde{I}^2 = \sum_{j=1}^{M} \tau^j A^j(\sigma^j + d^2), \qquad A^j = A(t^j), \qquad \sigma^j = \left(\frac{d\xi}{dt}\right)^j.$$

The Fast Automatic Differentiation methodology allows us to deduce next formula for calculation the components of the gradient of the generalized functional I:

$$\frac{d\tilde{I}}{df^j} = \tau^j + \sum_{i=0}^{w} \tau^j p_i^j \varphi_i, \qquad (1 \le j \le M),$$

were w is the vertex number defined by the condition

$$\varphi(r) = \begin{cases} \varphi_w(r) \ne 0, & 0 \le r \le r_w, \\ 0, & r > r_w. \end{cases}$$

In this expression p_i^j denote the values of conjugate variables (impulses). The impulses are determined by the next linear system of algebraic equations:

$$p_i^{M+1} = 0, \qquad (i = 0, \ldots, K),$$

$$p_0^j = -a_0 Y_1^j p_0^j + b_1 Y_1^j p_1^j + p_0^{j+1} + \tilde{I}^1_{E_0^j} + \tilde{I}^2_{E_0^j},$$

$$p_1^j = a_0 X_1^j p_0^j - a_1 Y_2^j p_1^j - b_1 X_1^j p_1^j + b_2 Y_2^j p_2^j + p_1^{j+1} + \tilde{I}^1_{E_1^j} + \tilde{I}^2_{E_1^j},$$

$$p_i^j = a_{i-1} X_i^j p_{i-1}^j - a_i Y_{i+1}^j p_i^j - b_i X_i^j p_i^j + b_{i+1} Y_{i+1}^j p_{i+1}^j + p_i^{j+1} + \tilde{I}^1_{E_i^j} + \tilde{I}^2_{E_i^j},$$

$$(2 \le i \le K-2),$$

$$p_{K-1}^j = a_{K-2} X_{K-1}^j p_{K-2}^j - b_{K-1} X_{K-1}^j p_{K-1}^j - a_{K-1} Y_K^j p_{K-1}^j + b_K Y_K^j p_K^j +$$

$$+ p_{K-1}^{j+1} + \tilde{I}^1_{E_{K-1}^j} + \tilde{I}^2_{E_{K-1}^j},$$

$$p_K^j = a_{K-1} X_K^j p_{K-1}^j - b_K X_K^j p_K^j - a_K \alpha T'_{E_K^j}(E_K^j) p_K^j + p_K^{j+1} + \tilde{I}^1_{E_K^j} + \tilde{I}^2_{E_K^j},$$

$$(j = M, M-1, \ldots, 1).$$

Here X_i^j and Y_i^j denote the following derivatives:

$$X_i^j = \frac{\partial}{\partial E_i^j}\left(\hat{\Omega}(E_{i-1}^j) T(E_i^j)\right) - \frac{\partial}{\partial E_i^j}\left(\hat{\Omega}(E_{i-1}^j) T(E_{i-1}^j)\right),$$

$$Y_i^j = \frac{\partial}{\partial E_{i-1}^j}\left(\hat{\Omega}(E_{i-1}^j) T(E_{i-1}^j)\right) - \frac{\partial}{\partial E_{i-1}^j}\left(\hat{\Omega}(E_{i-1}^j) T(E_i^j)\right),$$

and $\tilde{I}^1_{E_i^j}$, $\tilde{I}^2_{E_i^j}$ represent the partial derivatives of the functions $\tilde{I}^1$, $\tilde{I}^2$ with respect to E_i^j:

$$\tilde{I}^1_{E_i^j} = \begin{cases} \Lambda(E_{pl} - E_{z+1}^n), & i = z, \quad j = n, \\ \Lambda(E_z^n - E_{pl}), & i = z+1, \quad j = n, \\ 0, & \text{in other case}, \end{cases}$$

$$\tilde{I}^2_{E_i^j} = \sum_{j=1}^{M} \left(A^j \tau^j \frac{\partial \sigma^j}{E_i^j} \right),$$

$$\Lambda = 2A\frac{(r_{z+1}-r_z)}{(E_z^n - E_{z+1}^n)^2}\left[\frac{(E_{pl}-E_{z+1}^n)(r_z - r_{z+1})}{E_z^n - E_{z+1}^n} + r_{z+1} - R_{pl}\right].$$

To find $\tilde{I}^2_{E_i^j}$, we have to evaluate $\partial\sigma^j/\partial E_i^j$. So first we describe an algorithm for determining $\{\sigma^j\}$ (see Fig.2).

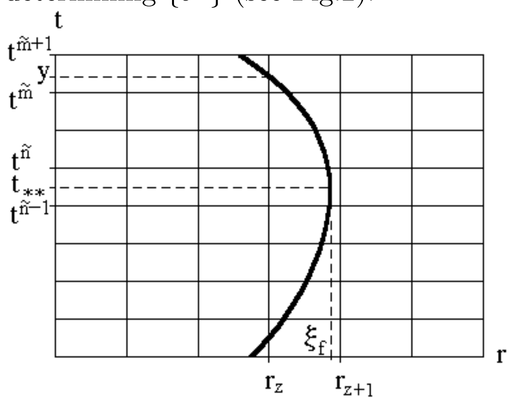

Fig. 2a

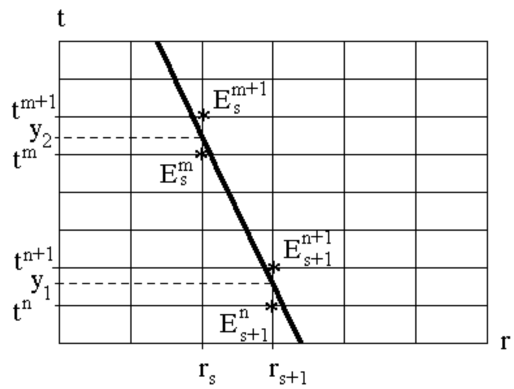

Fig. 2b

Suppose that $r = \xi(t)$, describing interface motion has its maximum value ξ_f at $t = t_{**}$, where ξ_f is defined as

$$\xi_f = \frac{(E_{pl}-E_{z+1}^{\tilde{n}})(r_z - r_{z+1})}{E_z^{\tilde{n}} - E_{z+1}^{\tilde{n}}} + r_{z+1}.$$

Here the index z indicates a spatial interval containing the maximum of $r = \xi(t)$, i. e. $r_z < \xi_f \le r_{z+1}$ (see Fig. 2a), and the index $\tilde{n}$ separates the time intervals before and after $t = t_{**}$, i. e. $t^{\tilde{n}-1} < t_{**} \le t^{\tilde{n}}$. The index $\tilde{m}$ can be used to determine a time interval containing the intersection point (r_z, y) of the curve $r = \xi(t)$ of the coordinate line $r = r_z$, i. e. $t^{\tilde{m}} \le y \le t^{\tilde{m}+1}$. Now all components of $\{\sigma^j\}$ can be divided into two groups: regular and singular. The coordinates of $\{\sigma^j\}$ and their partial derivatives for each group are calculated by somewhat different formulas. While deriving these formulas, we assumed in both cases that the slope of $r = \xi(t)$ (i. e., the component σ^j) within a single spatial cell is a constant and the energy E_i^j is a linear function on $[t^j, t^{j+1}]$.

1. Let us consider the singular group first. It consists of those components of $\{\sigma^j\}$ for which the corresponding $\xi^j = \xi(t^j)$ belong to the interval $r_z < \xi^j \le \xi_f$ or, equivalently, for which $\tilde{n} \le j \le \tilde{m}$ (see Fig. 2a). The component σ^j of $\{\sigma^j\}$ can be calculated by the formula $\sigma^j = (r_z - \xi_f)/(y - t_{**})$. Introducting the notation $b_1^* = E_z^{\tilde{m}+1} - E_z^{\tilde{m}}$, $b_2^* = E_{z+1}^{\tilde{n}} - E_z^{\tilde{n}}$, $\nu^* = (E_{pl} - E_z^{\tilde{m}})\tau^{\tilde{m}+1} + b_1^*(t^{\tilde{m}} - t^{\tilde{n}})$ and taking into account that the time $t = y$ can be found by linear interpolation as

$$y = \frac{(E_{pl} - E_z^{\tilde{m}+1})\tau^{\tilde{m}+1}}{E_z^{\tilde{m}+1} - E_z^{\tilde{m}}} + t^{\tilde{m}+1},$$

we can represent the component σ^j of $\{\sigma^j\}$ as $\sigma^j = h_z b_1^*(E_z^{\tilde{n}} - E_{pl})/b_2^*\nu^*$. Each σ^j depends on the point energy values $E_z^{\tilde{m}}$, $E_z^{\tilde{m}+1}$, $E_z^{\tilde{n}}$, and $E_{z+1}^{\tilde{n}}$. Consequently, the expression for $\tilde{I}^2_{E_i^j}$ contains the derivatives of σ^j only with respect to these components of the energy vector. These derivatives are calculated by the formulas:

$$\frac{\partial\sigma^j}{\partial E_z^{\tilde{n}}} = \frac{c_1^*(E_{z+1}^{\tilde{n}} - E_{pl})}{(b_2^*)^2}, \qquad \frac{\partial\sigma^j}{\partial E_{z+1}^{\tilde{n}}} = \frac{c_1^*(E_{pl} - E_z^{\tilde{n}})}{(b_2^*)^2},$$

$$\frac{\partial\sigma^j}{\partial E_z^{\tilde{m}}} = \frac{c_2^*\tau^{\tilde{m}+1}(E_z^{\tilde{m}+1} - E_{pl})}{(\nu^*)^2}, \quad \frac{\partial\sigma^j}{\partial E_z^{\tilde{m}+1}} = \frac{c_2^*\tau^{\tilde{m}+1}(E_{pl} - E_z^{\tilde{m}})}{(\nu^*)^2},$$

where $c_1^* = h_z b_1^*/\nu^*$, $\quad c_2^* = h_z(E_z^{\tilde{n}} - E_{pl})/b_2^*$.

2. Now consider the regular group of components. It consists of all components of $\{\sigma^j\}$ not included in the singular group. A characteristic feature of this group is that $\xi^j \le r_z$. Suppose that the interface $r = \xi(t)$ intersects the coordinate line $r = r_{s+1}$ at the point $t = y_1$ lying on the time interval $(t^n, t^{n+1}]$ and intersects the coordinate line $r = r_s \quad (r_s < r_{s+1})$ at the point $t = y_2$, $y_2 \in (t^m, t^{m+1}]$ (see Fig.2b). Then all σ^j whose index j satisfies $n < j \le m$ can be calculated by the formula $\sigma^j = (r_s - r_{s+1})/(y_2 - y_1) = h_s/(y_1 - y_2)$. The times $t = y_1$ and $t = y_2$ can be found by linear interpolation:

$$y_1 = \frac{(E_{pl} - E_{s+1}^{n+1})\tau^{n+1}}{E_{s+1}^{n+1} - E_{s+1}^n} + t^{n+1}, \qquad y_2 = \frac{(E_{pl} - E_s^{m+1})\tau^{m+1}}{E_s^{m+1} - E_s^m} + t^{m+1}.$$

As a result, σ^j is expressed as $\sigma^j = a_1^* a_2^* h_s/z^*$, where

$$a_1^* = E_s^{m+1} - E_s^m, \qquad a_2^* = E_{s+1}^{n+1} - E_{s+1}^n,$$

$$z^* = a_1^*(E_{pl} - E_{s+1}^n)\tau^{n+1} - a_2^*(E_{pl} - E_s^m)\tau^{m+1} + a_1^* a_2^*(t^n - t^m).$$

Each σ^j depends on the point energy values E_s^m, E_s^{m+1}, E_{s+1}^{n+1}, and E_{s+1}^n. Consequently, the expression for $\tilde{I}^2_{E_i^j}$ contains the derivatives of σ^j only with respect to these components of the energy vector. These derivatives are calculated by the formulas:

$$\frac{\partial\sigma^j}{\partial E_s^m} = h_s(a_2^*)^2\tau^{m+1}(E_{pl} - E_s^m - a_1^*)/(z^*)^2,$$

$$\frac{\partial\sigma^j}{\partial E_s^{m+1}} = h_s(a_2^*)^2\tau^{m+1}(E_s^m - E_{pl})/(z^*)^2,$$

$$\frac{\partial\sigma^j}{\partial E_{s+1}^n} = h_s(a_1^*)^2\tau^{n+1}(E_{s+1}^n - E_{pl} + a_2^*)/(z^*)^2,$$

$$\frac{\partial\sigma^j}{\partial E_{s+1}^{n+1}} = h_s(a_1^*)^2\tau^{n+1}(E_{pl} - E_{s+1}^n)/(z^*)^2.$$

In the other cases, $\partial\sigma^j/\partial E_i^j$ was set equal to zero.

5 The results of solution of variation problem

The variation problem, with the input parameters varying in wide ranges, was solved numerically in numerous runs. The qualitative behavior of the optimal control of melting and solidification and its structure were found to depend weakly on the input parameters of the problem.

All results presented below were obtained for the following thermophysical parameters given in SI units:

$$\rho_S = 7700, \quad k_S = 22, \quad C_S = 730, \quad \rho_L = 7700, \quad k_L = 22,$$

$$C_L = 730, \quad T_{pl} = 1773.15, \quad T_{ex} = 293.15, \quad T_{in}(r) \equiv 293.15,$$

$$\alpha = 1, \quad \lambda = 1291666.615.$$

Previously, the equations and the boundary conditions were nondimensionalized. Changing to dimensionless variables, we divided all the lengths by R_{pl}; all the temperatures by T_{pl}; the density, heat conductivity, and specific heat capacity by their respective means ρ_*, k_*, and C_*; the time by $R_{pl}^2\rho_* C_*/k_*$; and the source strength F by $k_* T_{pl}/R_{pl}^2$. In what follows, all the quantities are dimensionless.

The computations were performed on a nonuniform spatial grid containing 400 nodes. The grid was finer toward the axis $r = 0$ and the line $r = R_{pl}$. The time step was constant and was chosen so as to ensure the required accuracy of numerical results. The source was nearly a point ($r_w = 0.003$).

An analysis of the numerical results obtained suggests the following conclusions about the structure of the optimal control:

(i) The optimal control consists of two basic components.

(ii) The first optimal-control component (responsible primarily for melting) coincides with the upper bound $f(t) \equiv f_{max}$.

(iii) The second optimal-control component (responsible for solidification) is smaller than the first (if we compare their averages) and is separated from the latter by a short interval with $f(t) \equiv 0$.

(iv) The time t_{on} for which the source is turned on at the phase of solidification depends on both f_{max} and the limit cooling rate d^2. Depending on these parameters, t_{on} either precedes the time t_{**} at which the extent of the melted domain reaches its maximum possible value $\xi(t_{**}) = R_{pl}$ (for small values of d^2), succeeds t_{**} (for large values of d^2), or coincides with it.

To illustrate the general structure of the optimal control, Fig.3 shows its temporal dependencies obtained by solving the variation problem. The plots correspond to $f_{max} = 10.0$ and $d^2 = 0.3$ (Fig.3a), $f_{max} = 10.0$ and $d^2 = 0.4$ (Fig.3b), and $f_{max} = 5.0$ and $d^2 = 0.5$ (Fig.3c).

In the article the influence of different parameters of the problem on to optimal control was investigated.

a) Influence of d^2 on the First Optimal-Control Component

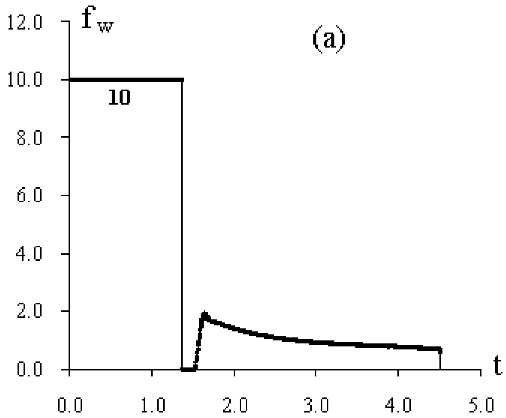

Fig. 3a

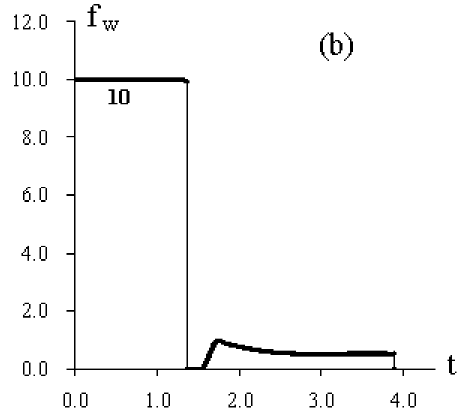

Fig. 3b

In all the regimes, the first optimal-control component is given by

$$f_{opt}(t) = \begin{cases} f_{max}, & 0 \leq t \leq t_*, \\ 0, & t_* < t. \end{cases} \tag{20}$$

Here, t_* (the time for which the source is turned on at the melting stage) depends on the regime.

If d^2 is such that $t_{on} \geq t_{**}$, then t_* is determined by the condition that the maximum radius of the melted domain is R_{pl} for the source defined by (20); i.e. $\xi_f = R_{pl}$. In this case, the first optimal-control component is not affected by the second one.

If d^2 is such that $t_{on} < t_{**}$, then the first optimal-control component is affected by the second. The value of t_* somewhat decreases in this case. However, our numerical computations have shown that this effect is small and can be neglected within the accuracy of the numerical results.

Hence, the first optimal-control component can be determined regardless of the second component by applying the algorithm described in [12].

b) Influence of d^2 on the Second Optimal-Control Component

We examined how the second optimal-control component depends on d^2 for a fixed f_{max}. Both the first and the second optimal-control components were determined by solving the variation problem. Figure 4 shows the optimal distributions of the source strength f_w vs. time for various values of d^2. The number near a curve indicates the value of d^2 used for obtaining this optimal control. An analysis of the numerical results presented in Fig. 4a,b,c shows that the optimal controls corresponding to different values of d^2 behave likewise, and all characteristic parameters (the length of the interval $t \in [t_{**}, \Theta_* - \beta^2]$, the maximum and minimum values of $f_w(t)$, etc.) decrease with increasing d^2 (for details see [13]).

c) Influence of f_{max} on the Second Optimal-Control Component

We also examined how the second optimal-control component depends on f_{max} for a fixed d^2. The numerical computations revealed that f_{max} has a large effect on the second optimal-control component. Figure 5 displays the location of the liquid — solid interface vs. time. The source was defined by (20) at the stage of melting and was not turned on at the stage of solidification (which corresponds to $d^2 = \infty$). The digits near the curves indicate the value of f_{max} used in the determination of the corresponding front. The value of R_{pl} is reached more rapidly when f_{max} is higher. The segments of the curves corresponding to the motion of the solidification front seem parallel, but this is not the case. For smaller values of f_{max}, the trajectory is steeper and the solidification rate is lower. Inspection of the plots suggests that small values of f_{max} are preferable. However, numerous computations have shown that it is preferable to increase f_{max}, thus increasing the violation of (10) and, accordingly,

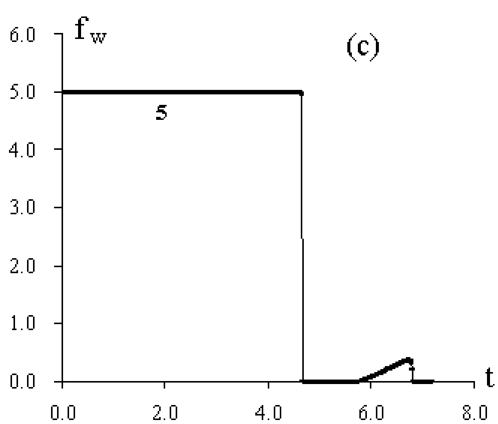

Fig. 3c

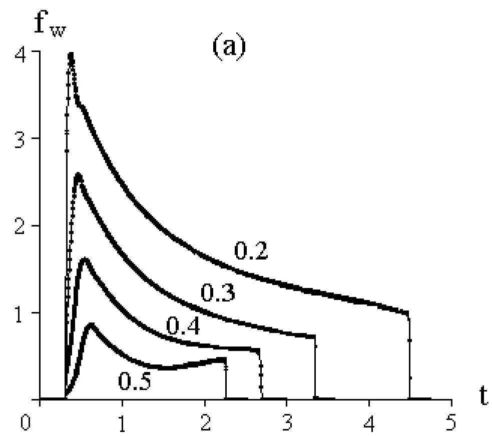

Fig. 4a

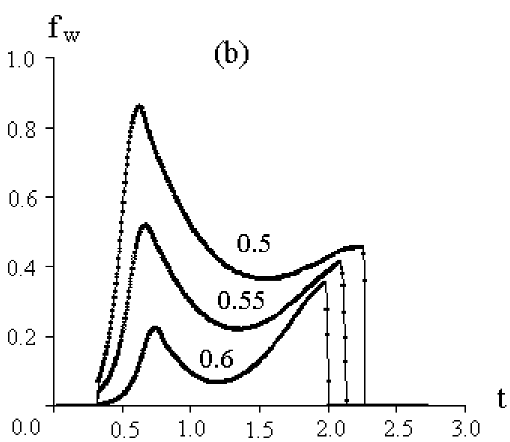

Fig. 4b

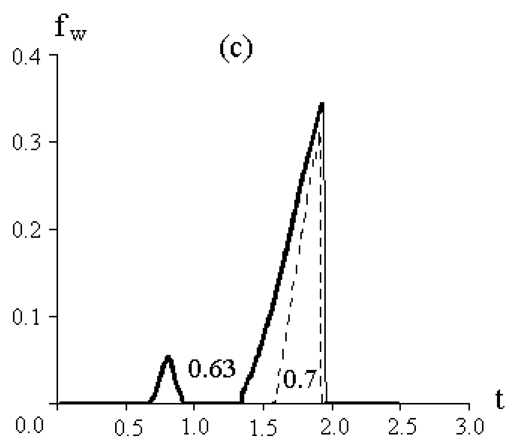

Fig. 4c

increasing the source strength at the solidification stage.

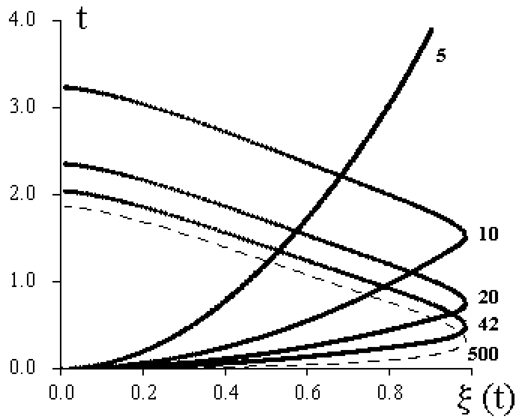

Fig. 5

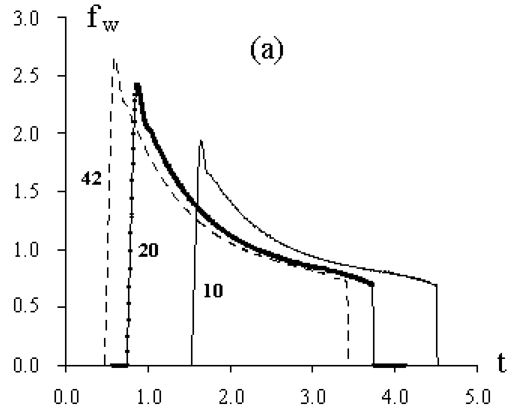

Fig. 6a

To confirm this conclusion, the full variation problem was solved numerically with $f_{max} = 9.98$ and $f_{max} = 10.0$. The resulting values of the cost functional were found to be $J_w = 15.160$ and $J_w = 15.152$, respectively. Figure 6 shows the optimal control at the stage of solidification (the second optimal-control component) for various values of f_{max}. Figures 6a and 6b correspond to $d^2 = 0.3$, and Figs.6c and 6d, to $d^2 = 0.4$. The digits near the curves indicate the corresponding values of f_{max}. The zero time corresponds to the beginning of the process. It should be noted that the second optimal-control components qualitatively resemble each other for various parameter values. The larger the value of f_{max}, the earlier is the turn-on time at the stage of solidification. The smaller the value of d^2, the higher the source strength at the respective instants and the longer the turn-on time. It should be noted that the second optimal-control components approach each other as f_{max} increases. The curves for which $f_{max} > 80.0$ are virtually indistinguishable. The functional values virtually do not differ from those corresponding to $f_{max} = 500$ (see Table 1).

d) Influence of f_{max} and d^2 on the Functional

To determine the effect of these parameters on the optimal control, we carried out a large amount of computations. Some of the results are presented in the Table 1, which lists the values of J_w, $J_w^{(1)}$, and $J_w^{(2)}$. As mentioned above, the heat source was turned on twice: at the stage of melting and at the stage of solidification. The regimes under study were chosen so that the time intervals with the source turned on were not overlapped. In this case, J_w in (12) can be represented as the sum $J_w = J_w^{(1)} + J_w^{(2)}$, where $J_w^{(1)}$ is the heat supplied during the first turn-on (melting) and $J_w^{(2)}$ is the heat supplied during the second turn-on (solidification). Based on numerous numerical results, the following conclusions can be made about the influence of f_{max} and d^2 on the cost functional value corresponding to the optimal control (see Table 1).

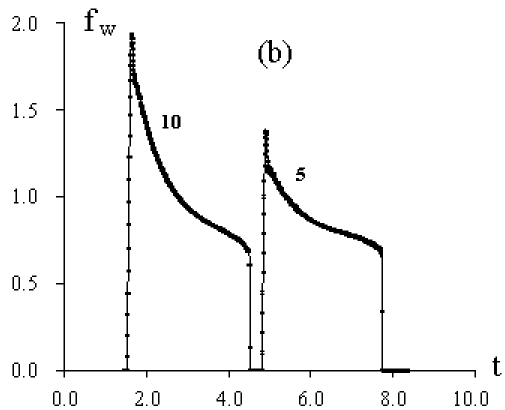

Fig. 6b

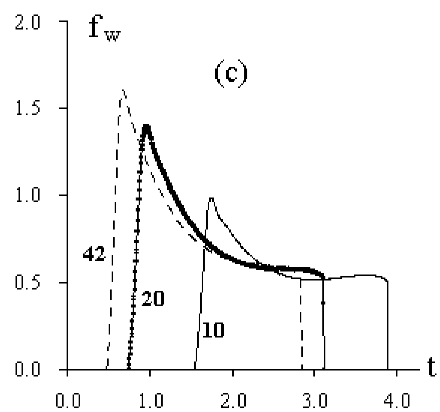

Fig. 6c

(i) The larger f_{max}, the smaller the values of J_w and $J_w^{(1)}$.

(ii) The larger d^2, the smaller the values of J_w and $J_w^{(2)}$.

(iii) The larger f_{max}, the larger the value of $J_w^{(2)}$.

(iv) The contribution of $J_w^{(2)}$ to J_w (the ratio $J_w^{(2)}/J_w^{(1)}$) increases with f_{max}.

These conclusions are supported by the plots of J_w, $J_w^{(1)}$, and $J_w^{(2)}$ vs. f_{max} for $d^2 = 0.4$ displayed in Fig. 7. Note that the qualitative behavior of J_w and $J_w^{(1)}$ is similar.

Importantly, the smaller d^2, the greater the number of iterations and CPU time required for obtaining the optimal control, although qualitative changes in the optimal control vs. time were not observed. For small values of d^2, small variations in this parameter lead to substantial quantitative changes in the optimal control. For example, for $f_{max} = 10$, we have $\max f_w(t) \approx 1$ for $d^2 = 0.4$, $\max f_w(t) \approx 2$ for $d^2 = 0.3$, and $\max f_w(t) \approx 4$ for $d^2 = 0.2$. a similar dependence is observed for the duration ΔT of the source operation: $\Delta T \approx 2.3$ for $d^2 = 0.4$, $\Delta T \approx 3.0$ for $d^2 = 0.3$, and $\Delta T \approx 4.5$ for $d^2 = 0.2$.

Note that the contribution of $J_w^{(2)}$ to J_w is not small and increases noticeably with decreasing d^2 for a fixed f_{max}. For example, at $f_{max} = 10$, we have $J_w^{(2)}/J_w \approx 0.03$ for $d^2 = 0.5$, $J_w^{(2)}/J_w \approx 0.09$ for $d^2 = 0.4$, $J_w^{(2)}/J_w \approx 0.18$ for $d^2 = 0.3$, and $J_w^{(2)}/J_w \approx 0.32$ for $d^2 = 0.2$ (see Table 1).

e) Alongside with the problem posed above two supplementary subproblems were studied: the problem of melting at absence of limitations (10) on speed of crystallization [12] and task of crystallization at given control at a stage of melting [13].

The first part of optimal control (responsible for melting process) has next structure [12] . If there were no restrictions on source power from top then the optimal control represents the injection all necessary heat at initial time moment; if there are restrictions from the top then the optimal control consists of two parts coincide

Table 1

d^2	f_{max}	5	10	20	42	500
0.5	J_w	23.4906	14.2022	12.0533	11.3994	11.1772
	$J_w^{(1)}$	23.2843	13.7659	11.3264	10.5337	10.2544
	$J_w^{(2)}$	0.2063	0.4363	0.7269	0.8657	0.9228
0.4	J_w	24.2699	15.1520	13.0855	12.4537	12.2483
	$J_w^{(1)}$	23.2841	13.7630	11.3264	10.5337	10.2507
	$J_w^{(2)}$	0.9858	1.3890	1.7591	1.9200	1.9976
0.3	J_w	25.8284	16.8314	14.8389	14.2136	14.0131
	$J_w^{(1)}$	23.2850	13.7659	11.3260	10.5336	10.2540
	$J_w^{(2)}$	2.5434	3.0655	3.5129	3.6800	3.7591
0.2	J_w		20.2064			
	$J_w^{(1)}$		13.7640			
	$J_w^{(2)}$		6.4424			

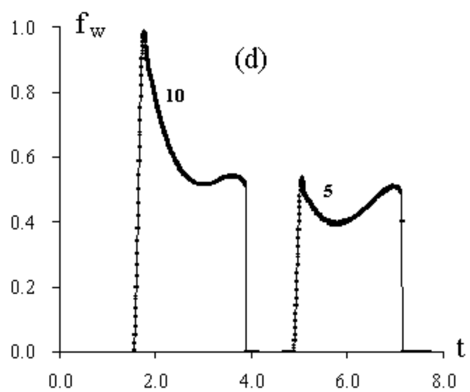

Fig. 6d

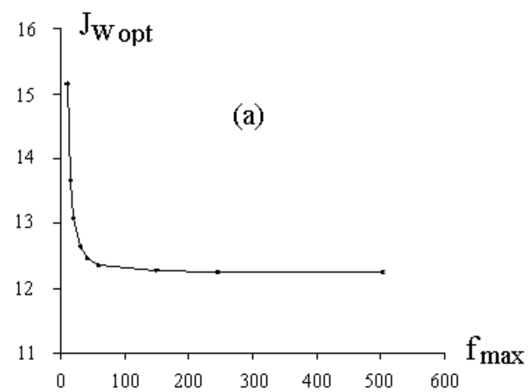

Fig. 7a

with the boundary. If the heat source is distributed in space the structure of optimal control is the same.

As the optimal control at a stage of melting coincides the upper boundary restriction, for its determination it is necessary to find only moment of switchover of a source from the upper limitation on lower.

The optimal control on the stage of substance crystallization consists also of two parts [13] . First, it coincides with lower boundary of the source power constraint and then changes over (continuously or stepwise) to the second part. This second

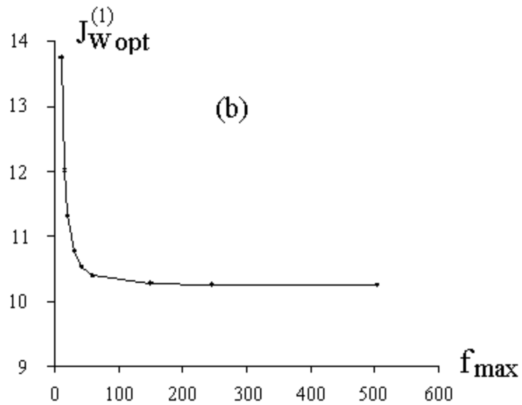

Fig. 7b

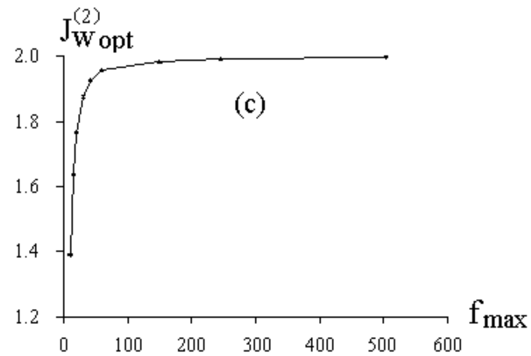

Fig. 7c

part is determined by requirements that the rate of the crystallization front should be not more than given amount and that emerged energy of the source should be minimal.

The numerous results of the solution of the supplementary problems have coincided with large accuracy with the applicable results, which one were obtained at the solution of the problem in full posing. It is no wonder: as is marked in post a), with accuracy of spent calculations the first part of optimal control is instituted irrespective of the second part.

The investigations of the problem permit to make following conclusions. In the parameter range that was used while investigations took part, the optimal control could be determined from the solution of two successive problems. First, we solve the melting problem and then, using its results as the initial data for the second one, we examine the crystallization problem. Usage of such splitting at the solution of the full variation problem essentially economizes expenditures on deriving of the optimal solution.

References

[1] Lions J.-L. *Controle optimal de systemes gouvernes par des equations au derivees partielles.* Paris, Gauthier-Villars, 1968; Moscow, Mir, 1972.

[2] Lur'e K.A. *Optimal Control in Problems of Mathematical Physics.* Moscow, Nauka, 1975 (in Russian).

[3] Ahmed N.U., Teo K.L. *Optimal Control of Distributed Parameter Systems.* New York, North Holland, 1981.

[4] Vasil'ev F.P. *Optimization Methods.* Moscow, Faktorial, 2002 (in Russian).

[5] Yurii A.D. *An Optimal Stefan-Type Problem.* Dokl. Akad. Nauk SSSR, 1980, **251**, p. 1317–1321.

[6] Gol'dman N.L. *Inverse Stefan Problems: Theory and Solution Methods.* Moscow, Mosk. Gos. Univ., 1999 (in Russian).

[7] Rose M.E. *A Method for Calculating Solutions of Parabolic Equations with a Free Boundary*. Math. Comput., 1960, **14**, p. 249–256.

[8] White R.E. *An Enthalpy Formulation of the Stephan Problem*. SIAM J. Numer. Anal., 1982, **19**, p. 1129–1157.

[9] White R.E. *A Numerical Solution of the Enthalpy Formulation of the Stephan Problem*. SIAM J. Numer. Anal., 1982, **19**, p. 1158–1172.

[10] Albu A.F., Zubov V.I. *Modified Scheme for Analyzing the Process of Melting*. Zh. Vychisl. Mat. Mat. Fiz., 2001, **41**, p. 1434–1443 (Comp. Math. Math. Phys., 2001, **41**, p. 1363–1371).

[11] Evtushenko Y.G. *Computation of Exact Gradients in Distributed Dynamic Systems*. Optimizat. Methods and Software, 1998, **9**, p. 45–75.

[12] Albu A.F., Gorbunov V.I., Zubov V.I. *Optimal Control of the Process of Melting*. Zh. Vychisl. Mat. Mat. Fiz., 2000, **40**, p. 517–531 (Comp. Math. Math. Phys., 2000, **40**, p. 491–504).

[13] Albu A.F., Zubov V.I. *Optimal Control of the Process of Solidification*. Zh. Vychisl. Mat. Mat. Fiz., 2004, **44**, p. 38–50 (Comp. Math. Math. Phys., 2004, **44**, p. 34–44).

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