# Bieberbach-Auslander Theorem and Dynamics in Symmetric Spaces 

Boris N. Apanasov


#### Abstract

The aim of this paper (my extended contribution to Intern. Conf. on Discrete Geometry dedicated to A.M.Zamorzaev) is to study dynamics of a discrete isometry group action in a noncompact symmetric space of rank one nearby its parabolic fixed points. Due to Margulis Lemma, such an action on corresponding horospheres is virtually nilpotent, so our extension of the Bieberbach-Auslander theorem for discrete groups acting on connected nilpotent Lie groups can be applied. As result, we show that parabolic fixed points of a discrete group of isometries of such symmetric space cannot be conical limit points and that the fundamental groups of geometrically finite locally symmetric of rank one orbifolds are finitely presented, and the orbifolds themselves are topologically finite.


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## 1 Introduction

Here we apply our structural theorem on discrete actions on nilpotent groups [6,7] to study the dynamics of a discrete isometry group action nearby its parabolic fixed points at infinity of symmetric spaces of rank 1 with negative curvature. All those spaces $X$ are foliated by horospheres centered at a given point at infinity (i.e. by level surfaces of a Busemann function). The most important case is that of a discrete (parabolic) group $\Gamma \subset$ Isom $X$ that fixes a point at infinity and preserves setwise each of those horospheres. In this case, by applying the Margulis Lemma, it follows that this discrete parabolic group $\Gamma$ is virtually nilpotent. Furthermore, at least in symmetric spaces of rank 1 with negative curvature (that is the hyperbolic spaces either real, complex, quaternionic or octonionic ones), all those horospheres can be identified with a connected simply connected Lie group $\mathcal{N}$ and our discrete group $\Gamma$ isometrically acts on $\mathcal{N}$ as a subgroup $\Gamma \subset \mathcal{N} \rtimes C$ where $C$ is a compact group of automorphisms of $\mathcal{N}$. In the case of real hyperbolic spaces (of constant negative curvature), horospheres are flat, and discrete Euclidean isometry group actions are described by the Bieberbach theorem [2]. However in the other symmetric spaces of rank 1 this is no longer true. In these spaces horospheres may be represented as non-Abelian nilpotent Lie groups with a left invariant metric, and therefore they

[^0]have sectional curvatures of both signs. Here we can use our structural theorem, see $[6,7]$ :

Theorem 1. Let $\mathcal{N}$ be a connected, simply connected nilpotent Lie group, $C$ be a compact group of automorphisms of $\mathcal{N}$, and $\Gamma \subset \mathcal{N} \rtimes C$ be a discrete subgroup. Then there exist a connected Lie subgroup $\mathcal{N}_{\Gamma}$ of $\mathcal{N}$ and a finite index subgroup $\Gamma^{*}$ of $\Gamma$ with the following properties:

1. There exists $b \in \mathcal{N}$ such that $b \Gamma b^{-1}$ preserves $\mathcal{N}_{\Gamma}$;
2. $\mathcal{N}_{\Gamma} / b \Gamma b^{-1}$ is compact;
3. $b \Gamma^{*} b^{-1}$ acts on $\mathcal{N}_{\Gamma}$ by left translations, and this action is free.

Here the compactness condition on the group $C$ of automorphisms of $\mathcal{N}$ is essential. The situation when the group $C$ may be noncompact is completely different. For instance, G. Margulis [13] constructed discrete subgroups of $R^{3} \rtimes \mathrm{SO}(2,1)$ which are nonabelian free groups, whereas in the compact case any discrete subgroup of $\mathcal{N} \rtimes C$ must be virtually nilpotent, which resembles Gromov's almost flat manifolds [10]. On the other hand, when the group $C$ is compact, there exists a left invariant metric on $\mathcal{N}$ such that $\mathcal{N} \rtimes C$ acts on $\mathcal{N}$ as a group of isometries. So any discrete subgroup of $\mathcal{N} \rtimes C$ can be viewed as a discrete isometry group of $\mathcal{N}$ with respect to some left invariant metric. We remark that our Theorem advances a result by Louis Auslander [1] who proved its claims (1) and (2) only for a finite index subgroup of a given discrete group $\Gamma$. In the Euclidean case when $\mathcal{N}=R^{n}$, this is the Bieberbach theorem, see [2].

A motivation for our study comes from an attempt to understand parabolic (the so-called "thin") ends of negatively curved manifolds, as well as the geometry and topology of geometrically finite pinched negatively curved manifolds, see $[2,3,5,9]$. The concept of geometrical finiteness first arose in the context of (real) hyperbolic 3 -manifolds. Its original definition (due to L.Ahlfors) came from an assumption that such a geometrically finite real hyperbolic manifold $M$ may be decomposed into a cell by cutting along a finite number of its totally geodesic hypersurfaces. Since that time, other definitions of geometrical finiteness have been given by A.Marden, A.Beardon and B.Maskit, and W.Thurston, and the notion has become central to the study of real hyperbolic manifolds. Though other pinched Hadamard manifolds may not have totally geodesic hypersurfaces, the other definitions of geometrical finiteness work in the case of variable negative curvature as well, see $[4,7,9]$. Our previous paper [5] deals with geometrical finiteness in variable curvature in the case of complex hyperbolic manifolds, on the base of a structural theorem for discrete isometric actions on the Heisenberg groups, a predecessor of our Theorem 1. Our proof of Theorem 1 uses different algebraic ideas, see [6,7].

Here we apply our Theorem 1 in two directions. First we answer a question on dynamics of a discrete isometry group action nearby its limit points, which was left open for variable negative curvature spaces. Namely, it distinguishes two types of
limit points of a discrete group $G \subset$ Isom $X$ acting on a symmetric rank one space $X$ with negative curvature. Namely it shows that parabolic fixed points of such a discrete group $G$ cannot be its conical limit points, i.e. such points $z \in X(\infty)$ that for some (and hence every) geodesic ray $\ell$ in $X$ ending at $z$, there is a compact set $K \subset X$ such that the subset $\{g \in G: g(\ell) \cap K \neq \emptyset\}$ is infinite. Such a dichotomy has been recently proved only in the case of real hyperbolic spaces (of constant curvature) by Susskind and Swarup [16] and independently, from a dynamical point of view, by Starkov [15].

The second our result answers another open question (formulated as a conjecture in [9], p.230). Namely, it shows that discrete parabolic groups $\Gamma$ isometrically acting on a connected Lie groups $\mathcal{N}$ with a compact automorphism group, as well as geometrically finite discrete groups $G \subset$ Isom $X$ acting on the corresponding symmetric space of rank 1 are finitely presented, and the corresponding quotient orbifolds are topologically finite. Previously, it was known for constant negative curvature. For pinched Hadamard manifolds with various negative curvature, Bowditch [9] proved that such groups are finitely generated. The answer in the case of Heisenberg groups and complex hyperbolic manifolds has been earlier given by the author in $[5,6]$.

## 2 Preliminaries

The symmetric spaces of $\mathbb{R}$-rank one of non-compact type are the hyperbolic spaces $H_{\mathbb{F}}^{n}$, where $\mathbb{F}$ is either the real numbers $\mathbb{R}$, or the complex numbers $\mathbb{C}$, or the quaternions $\mathbb{H}$, or the Cayley numbers $\mathbb{O}$; in last case $n=2$. They are respectively called as real, complex, quaternionic and octonionic hyperbolic spaces (the latter one $H_{\oplus}^{2}$ is also known as the Cayley hyperbolic plane). Algebraically these spaces can be described as the corresponding quotients: $S O(n, 1) / S O(n), S U(n, 1) / S U(n)$, $S p(n, 1) / S p(n)$ and $F_{4}^{-20} / \operatorname{Spin}(9)$ where the latter group $F_{4}^{-20}$ of automorphisms of the Cayley plane $H_{\circlearrowleft}^{2}$ is the real form of $F_{4}$ of rank one. We normalize the metric so the (negative) sectional curvature of $H_{\mathbb{F}}^{n}$ is bounded from below by -1 .

Following Mostow [14] and using the standard involution (conjugation) in $\mathbb{F}$, $z \rightarrow \bar{z}$, one can define projective models of the hyperbolic spaces $H_{\mathbb{F}}^{n}$ as the set of negative lines in the Hermitian vector space $\mathbb{F}^{n, 1}$, with Hermitian structure given by the indefinite ( $n, 1$ )-form

$$
\langle\langle z, w\rangle\rangle=z_{1} \bar{w}_{1}+\cdots+z_{n} \bar{w}_{n}-z_{n+1} \bar{w}_{n+1} .
$$

Here, taking non-homogeneous coordinates, one can obtain unit ball models (in the unit ball $\left.B_{\mathbb{F}}^{n}(0,1) \subset \mathbb{F}^{n}\right)$ for the first three spaces. Since the multiplication by quaternions is not commutative, we specify that we use "left" vector space $\mathbb{H}^{n, 1}$ where the multiplication by quaternion numbers is on the left. However, it does not work for the Cayley plane since $\mathbb{O}$ is non-associative, and one should use a Jordan algebra of $3 \times 3$ Hermitian matrices with entries from $\mathbb{O}$ whose group of automorphisms is $F_{4}$, see [14].

Another models of $H_{\mathbb{F}}^{n}$ use the so called horospherical coordinates [6,11] based on foliations of $H_{\mathbb{F}}^{n}$ by horospheres centered at a fixed point $\infty$ at infinity $\partial H_{\mathbb{F}}^{n}$
which is homeomorphic to $\left(n \operatorname{dim}_{\mathbb{R}} \mathbb{F}-1\right)$-dimensional sphere. Such a horosphere can be identified with the nilpotent group $N$ in the Iwasawa decomposition $K A N$ of the automorphism group of $H_{\mathbb{F}}^{n}$. The nilpotent group $N$ can be identified with the product $\mathbb{F}^{n-1} \times \operatorname{Im} \mathbb{F}$ (see [14]) equipped with the operations:

$$
(\xi, v) \cdot\left(\xi^{\prime}, v^{\prime}\right)=\left(\xi+\xi^{\prime}, v+v^{\prime}+2 \operatorname{Im}\left\langle\xi, \xi^{\prime}\right\rangle\right) \quad \text { and } \quad(\xi, v)^{-1}=(-\xi,-v),
$$

where $\langle$,$\rangle is the standard Hermitian product in \mathbb{F}^{n-1},\langle z, w\rangle=\sum z_{i} \overline{w_{i}}$. The group $N$ is a 2-step nilpotent Carnot group with center $\{0\} \times \operatorname{Im} \mathbb{F} \subset \mathbb{F}^{n-1} \times \operatorname{Im} \mathbb{F}$, and acts on itself by the left translations $T_{h}(g)=h \cdot g, \quad h, g \in N$.

Now we may identify

$$
H_{\mathbb{F}}^{n} \cup \partial H_{\mathbb{F}}^{n} \backslash\{\infty\} \longrightarrow N \times[0, \infty)=\mathbb{F}^{n-1} \times \operatorname{Im} \mathbb{F} \times[0, \infty),
$$

and call this identification the "upper half-space model" for $H_{\mathbb{F}}^{n}$ with the natural horospherical coordinates $(\xi, v, u)$. In these coordinates, the above left action of $N$ on itself extends to an isometric action (Carnot translations) on the $\mathbb{F}$-hyperbolic space in the following form:

$$
T_{\left(\xi_{0}, v_{0}\right)}:(\xi, v, u) \longmapsto\left(\xi_{0}+\xi, v_{0}+v+2 \operatorname{Im}\left\langle\xi_{0}, \xi\right\rangle, u\right),
$$

where $(\xi, v, u) \in \mathbb{F}^{n-1} \times \operatorname{Im} \mathbb{F} \times[0, \infty)$.
There are a natural norm and an induced by this norm distance on the Carnot group $N=\mathbb{F}^{n-1} \times \operatorname{Im} \mathbb{F}$, which are known in the case of the Heisenberg group (when $\mathbb{F}=\mathbb{C}$ ) as the Cygan's norm and distance. Using horospherical coordinates, they can be extended to a norm on $H_{\mathbb{F}}^{n}$, see [6]:

$$
\begin{equation*}
|(\xi, v, u)|_{c}=\left|\left(|\xi|^{2}+u-v\right)\right|^{1 / 2}, \tag{1}
\end{equation*}
$$

where $|$.$| is the norm in \mathbb{F}$, and to a metric $\rho_{c}$ (still called the Cygan metric) on $\mathbb{F}^{n-1} \times \operatorname{Im} \mathbb{F} \times[0, \infty)=\bar{X} \backslash\{\infty\}:$

$$
\begin{equation*}
\rho_{c}\left((\xi, v, u),\left(\xi^{\prime}, v^{\prime}, u^{\prime}\right)\right)=\left|\left|\xi-\xi^{\prime}\right|^{2}+\left|u-u^{\prime}\right|-\left(v-v^{\prime}+2 \operatorname{Im}\left\langle\xi, \xi^{\prime}\right\rangle\right)\right|^{\frac{1}{2}} \tag{2}
\end{equation*}
$$

It follows directly from the definition that Carnot translations and rotations are isometries with respect to the Cygan metric $\rho_{c}$. Moreover, the restrictions of this metric to different horospheres centered at $\infty$ are the same, so Cygan metric plays the same role as Euclidean metric does on the upper half-space model for the real hyperbolic space $\mathbb{H}^{n}$.

The group of automorphisms of $H_{\mathbb{H}}^{n}$ is $\operatorname{PSp}(n, 1)$. The stabilizer $K$ of the origin is $S p(1) \times S p(n)$ which can be described in the matrix form as:

$$
\left[\begin{array}{cc}
M & 0 \\
0 & \nu
\end{array}\right]
$$

where $M \in S p(n)$ and $\nu \in S p(1)$. Note the matrix acts on the right, and the projectivization is given by multiplication on the left. So in the ball model, the action is:

$$
z \rightarrow \nu^{-1} z M
$$

The stabilizer of a real geodesic connecting two points $(0,1)$ and $(0,-1)$ is $M A=$ $S p(1) \times S p(n-1) \times \mathbb{R}$. This action can be described in the matrix form as:

$$
\left[\begin{array}{ccc}
M & 0 & 0 \\
0 & \nu \cosh r & \nu \sinh r \\
0 & \nu \sinh r & \nu \cosh r
\end{array}\right]
$$

where $M \in S p(n-1), \nu \in S p(1)$ and $r \in \mathbb{R}$. Specially $S p(1)$ acts as

$$
\left[\begin{array}{lll}
I & 0 & 0 \\
0 & \nu & 0 \\
0 & 0 & \nu
\end{array}\right]
$$

If $(0, \mathbb{H})$ is the $\mathbb{H}$-line containing the real geodesic joining $(0,1)$ and $(0,-1)$, the action of $S p(1)$ on this $\mathbb{H}$-line is:

$$
(0, \mathbb{H}) \rightarrow \nu^{-1}(0, \mathbb{H} \nu, \nu)=\left(0, \nu^{-1} \mathbb{H} \nu\right) .
$$

But in general, $\nu \in S p(1)$ maps $\left(z, z_{n}\right)$ to $\left(\nu^{-1} z, \nu^{-1} z_{n} \nu\right)$.
A Cayley number $z \in \mathbb{O}$ is a pair of quaternions, $z=\left(q_{1}, q_{2}\right)$, and the multiplication in $\mathbb{O}$ is given by

$$
\left(q_{1}, q_{2}\right)\left(p_{1}, p_{2}\right)=\left(q_{1} p_{1}-\overline{p_{2}} q_{2}, p_{2} q_{1}+q_{2} \overline{p_{1}}\right) .
$$

The standard involution (conjugation) in $\mathbb{O}$ is defined by $\overline{\left(q_{1}, q_{2}\right)}=\left(\overline{q_{1}},-q_{2}\right)$, so for $z=\left(q_{1}, q_{2}\right) \in \mathbb{H} \times \mathbb{H}=\mathbb{O}$, we have $\operatorname{Im} z=\left(\operatorname{Im} q_{1}, q_{2}\right)$ and $\operatorname{Re} z=\operatorname{Re} q_{1}$. Then Cayley numbers satisfy the usual properties like: $x \bar{x}=|x|^{2},|x y|=|x||y|, x^{-1}=$ $\bar{x} /|x|^{2}, \overline{x y}=\bar{y} \bar{x}$. Even though Cayley numbers are not commutative, nor associative, by Artin's lemma a subalgebra generated by two elements is associative. Cayley hyperbolic plane is made out of an exceptional Jordan algebra of $3 \times 3$ Hermitian matrices with entries from $\mathbb{O}$ whose group of automorphisms is $F_{4}$, see [14]. The group of automorphisms of the Cayley plane $H_{\mathbb{O}}^{2}$ is $F_{4}^{-20}$, the real form of $F_{4}$ of rank one. The stabilizer in $F_{4}^{-20}$ of the origin $(0,0) \in B_{\mathbb{O}}^{2}(0,1)=H_{\oplus}^{2}$ is $\operatorname{Spin}(9)$ operating on $\mathbb{O}^{2}=\mathbb{R}^{16}$ via the spinor representation. If $L_{1}=\mathbb{O} \times 0$ and $L_{2}=0 \times \mathbb{O}$ denote the coordinate $\mathbb{O}$-axes, then the stabilizer of $L_{1}$ acts on $L_{1}$ as $S O(8)$ via the even $\frac{1}{2}$-spin representation, and on $L_{2}$ as odd $\frac{1}{2}$-spin representation. The stabilizer of the real line through $(0,0)$ and $(1,0)$ is $\operatorname{Spin}(7)$.

## 3 Margulis region and parabolic cusps

One of the most important tools for studying negatively curved spaces is given by the Margulis Lemma which induces the thick-thin decomposition of corresponding
orbifolds, see $[8,9]$. Such orbifolds are quotients $M=X / G$ of symmetric spaces $X$ by discrete isometric actions of their fundamental groups $\pi_{1}^{\text {orb }} \cong G \subset$ Isom $X$. Adding the induced discrete action of $G$ in some domain at infinity $\partial X$, we obtain a partial closure $M(G)$ of that orbifold $M$. More precisely, let $\Lambda(G) \subset \partial X$ and $\omega(G)=\partial X \backslash \Lambda(G)$ be the limit and discontinuity sets of $G \subset$ Isom $X$. Then we set $M(G)=(X \cup \Omega(G)) / G$.

Let $\epsilon$ be a positive number less than $\epsilon(n)$, the Margulis constant for symmetric $n$-spaces of rank one. For a given discrete group $G \subset$ Isom $X$ and its orbifold $M=X / G$, we define the $\epsilon$-thin part $\operatorname{thin}_{\epsilon}(M)$ as

$$
\operatorname{thin}_{\epsilon}(M)=\left\{x \in X: G_{\epsilon}(x)=\langle g \in G: d(x, g(x))<\epsilon\rangle \text { is infinite }\right\} / G .
$$

The thick part thick $\epsilon_{\epsilon}(M)$ of $M$ is defined as the closure of the complement to the thin part, $\operatorname{thin}_{\epsilon}(M) \subset M$.

As a consequence of the Margulis Lemma, there is the following description of the thin part of $M[8,9]$ :

Theorem 2. Let $G \subset \operatorname{Isom} X$ be a discrete group and $\epsilon, 0<\epsilon<\epsilon(n)$, be chosen. Then the $\epsilon$-thin part $\operatorname{thin}_{\epsilon}(M)$ of $M=X / G$ is a disjoint union of its connected components, and each such component has the form $T_{\epsilon}(\Gamma) / \Gamma$ where $\Gamma$ is a maximal infinite elementary subgroup of $G$. Here, for each such elementary subgroup $\Gamma \subset G$, the connected component (Margulis region)

$$
T_{\epsilon}=T_{\epsilon}(\Gamma)=\left\{x \in X: \Gamma_{\epsilon}(x)=\langle g \in \Gamma: d(x, \gamma(x))<\epsilon\rangle \text { is infinite }\right\}
$$

is precisely invariant with respect to the subgroup $\Gamma$ in $G$ :

$$
\Gamma\left(T_{\epsilon}\right)=T_{\epsilon}, \quad g\left(T_{\epsilon}\right) \cap T_{\epsilon}=\emptyset \quad \text { for any } g \in G \backslash \Gamma .
$$

We note that in the real hyperbolic case of dimension 2 and 3, a Margulis region $T_{\epsilon}$ with parabolic stabilizer $\Gamma \subset G$ can be taken as a horoball neighborhood centered at the parabolic fixed point $p, \Gamma(p)=p$. It is not true in general due to Apanasov's construction in real hyperbolic spaces of dimension at least 4, see [2]. As we discussed it in [5], this construction works in complex hyperbolic spaces ch $n$ as well as in other rank one symmetric spaces $X$. However, we may apply our Theorem 1 to describe parabolic Margulis regions in all such spaces.

Namely, let $\Gamma \subset G$ be a discrete parabolic subgroup. We may view $X$ from the fixed point $p \in \partial X$ in the way we have used in $\S 2$ to define the upper half-space model for $X$. Then, by using the foliation of $X$ by horospheres $X_{t}$ centered at $p$, we identify $\bar{X} \backslash\{p\}$ and $\mathcal{N} \times[0, \infty)$, where $X_{t} \cong \mathcal{N}$ is a connected, simply connected Lie group with a compact automorphism group $C$. Since the parabolic group $\Gamma$ acts on each horosphere $X_{t}$ centered at the fixed point $p$ as a discrete subgroup of $\mathcal{N} \rtimes C$, we can apply Theorem 1 which implies that there exists a $\Gamma$-invariant connected subspace $\sigma \subset \partial X \backslash\{p\} \cong \mathcal{N}$ where $\Gamma$ acts co-compactly. Also we have a finite index subgroup $\Gamma^{*} \subset \Gamma$ which acts on $\sigma$ freely by left translations. In fact, $\sigma$ is a translate of a connected Lie subgroup $\mathcal{N}_{\Gamma}$ of $\partial X \backslash\{p\} \cong \mathcal{N}$. Now we define the subspace $\tau \subset X$
to be spanned by $\sigma$ and all geodesics $(z, p) \subset X$ connecting $z \in \sigma$ to the parabolic fixed point $p$. Let $\tau_{t}$ be the "half-plane" in $\tau$ of a height $t>0$, that is the part of $\tau$ whose last horospherical coordinate is at least $t$.

Lemma 3. Let $G \subset \operatorname{Isom} X$ be a discrete group in a rank one symmetric space $X$ and $p \in \partial X$ a parabolic fixed point of $G$. Let $T_{\epsilon}$ be a Margulis region for $p$ and let $\tau_{t}$ be the half-plane defined as above. Then for any $\delta, 0<\delta<\epsilon / 2$, there exists a positive number $t>0$ such that the Margulis region $T_{\epsilon}$ contains the $\delta$-neighborhood $N_{\delta}\left(\tau_{t}\right)$ of the half-plane $\tau_{t}$.

Proof: Let $\Gamma \subset G$ be the maximal parabolic subgroup fixing a given parabolic fixed point $p \in \partial X$. Since $\Gamma$ preserves the subspace $\sigma \subset \partial X \backslash\{p\}$, it preserves the boundary $\partial \tau_{t}$ of each half-plane $\tau_{t}$.

As it was shown in [12], the geometry of horospheres in the space $X$ with sectional curvatures $-1 \leq K \leq-1 / 4$ may be closely compared with that in the spaces of constant negatives curvature $-1 / 4$ and -1 , respectively. In particular, for two asymptotic geodesic rays $\ell$ and $\ell^{\prime}$ approaching $p \in \partial X$ from two points $x$ and $x^{\prime}$ on the same horosphere, with a horospherical distance $R_{0}$ between them, we have:

$$
\left(2 \operatorname{arcsinh}\left(2 R_{0}\right)\right) e^{-t} \leq d\left(\ell(t), \ell^{\prime}(t)\right) \leq R_{0} e^{-\frac{t}{2}} .
$$

This implies that distances on horospheres in $X$ of height $t$ exponentially decrease as $t$ goes to $+\infty$. On the other hand, due to Theorem 1, infinite order elements $\gamma \in \Gamma$ act on the boundary $\partial \tau_{t}, t>0$, as virtual translations, and the quotient $\partial \tau_{t} / \Gamma$ is compact. Therefore, for positive numbers $\delta$ and $\epsilon^{\prime}, 2 \delta+\epsilon^{\prime}<\epsilon$, there exist some height $t_{\epsilon^{\prime}}$ such that

$$
\partial \tau_{t} \subset T_{\epsilon^{\prime}}(\Gamma) \subset T_{\epsilon^{\prime}}(G)=T_{\epsilon^{\prime}} \quad \text { for all } t>t_{\epsilon^{\prime}} .
$$

Clearly, the same is true for the whole half-plane:

$$
\begin{equation*}
\tau_{t} \subset T_{\epsilon^{\prime}}(\Gamma) \subset T_{\epsilon^{\prime}} \tag{3}
\end{equation*}
$$

Now, for any $x \in N_{\delta}\left(\tau_{t}\right)$ with $t>t_{\epsilon^{\prime}}$, we have a $\delta$-close point $x_{0} \in \tau_{t}, d\left(x, x_{0}\right)<\delta$. Due to 3 , there is an infinite order element $\gamma \in \Gamma$ such that $d\left(x_{0}, \gamma\left(x_{0}\right)\right)<\epsilon^{\prime}$. It implies:

$$
d(x, \gamma(x)) \leq d\left(x, x_{0}\right)+d\left(x_{0}, \gamma\left(x_{0}\right)\right)+d\left(\gamma\left(x_{0}\right), \gamma(x)<2 \delta+\epsilon^{\prime}<\epsilon,\right.
$$

which shows that the point $x$ and thus the whole $\delta$-neighborhood $N_{\delta}\left(\tau_{t}\right)$ belong to the Margulis region $T_{\epsilon}$.

Now we can (negatively) answer the question of whether a parabolic fixed point of a discrete group $G \subset$ Isom $X$ may also be its conical limit point.

Here a limit point $z \in \Lambda(G)$ is called a conical limit point of a discrete group $G \subset \operatorname{Isom} X$ if, for some (and hence every) geodesic ray $\ell \subset X$ ending at $z$, there is a compact set $K \subset X$ such that $g(\ell) \cap K \neq \emptyset$ for infinitely many elements $g \in G$.

This definition is equivalent to a possibility to approximate the limit point $z \in$ $\Lambda(G)$ by a $G$-orbit $\left\{g_{i}(x)\right\}$ of a point $x \in X$ inside a tube (cone) in $X$ with vertex $z \in \partial X$. Applying an argument originally due to A.Beardon and B.Maskit, one can use the following equivalent definition of conical limit points [7]:

Lemma 4. A point $z \in \Lambda(G)$ is a conical limit point of a discrete group $G \subset \operatorname{Isom} X$ in a negatively curved space $X$ if and only if, for every geodesic ray $\ell \subset X$ ending at $z$ and for every $\delta>0$, there is a point $x \in X$ and a sequence of distinct elements $g_{i} \in G$ such that the orbit $\left\{g_{i}(x)\right\}$ approximates $z$ inside the $\delta$-neighborhood $N_{\delta}(\ell)$ of the ray $\ell$.

There are other (equivalent) definitions of conical limit points [9]. One of them is even intrinsic to the action of the group $G$ on the limit set $\Lambda(G)$. Namely, $z \in \Lambda(G)$ is a conical limit point if there is a sequence $\left\{g_{i}\right\}$ of distinct elements of $G$ such that, for any other limit point $y \in \Lambda(G) \backslash\{z\}$, the sequence of pairs $\left(g_{i}^{-1}(z), g_{i}^{-1}(y)\right)$ lies in a compact subset of $(\Lambda(G) \times \Lambda(G)) \backslash \Delta(\Lambda)$, where $\Delta(\Lambda)=\{(x, x): x \in \Lambda(G)\}$.

Theorem 5. Let $G \subset \operatorname{Isom} X$ be a discrete group in a rank one symmetric space $X$. Then any parabolic fixed point of $G$ cannot be its conical limit point.

Proof: Let $\Gamma \subset G$ be the maximal parabolic subgroup of given group $G$ fixing a parabolic fixed point $p \in \partial X$. As in Lemma 3, viewing $X$ from the point $p$ at infinity by using horospherical coordinates and applying Theorem 1, we again have a $\Gamma$-invariant connected subspace $\sigma \subset \partial X \backslash\{p\}$ where $\Gamma$ acts co-compactly, and on which a finite index subgroup $\Gamma^{*} \subset \Gamma$ acts freely by left translations. Applying Lemma 3 to the subspace $\tau \subset X$ spanned by $\sigma$ and $p$, we have positive numbers $\delta$ and $t$ so that the $\delta$-neighborhood $N_{\delta}\left(\tau_{t}\right)$ of the half-plane $\tau_{t}$ is contained in the parabolic Margulis region $T_{\epsilon}$ at $p$.

Now suppose that the point $p$ is also a conical limit point of $G$. Then for a geodesic ray $\ell \subset \tau_{t}$ tending to $p$, there must exist a point $x \in X$ and a sequence of distinct elements $g_{i} \in G$ such that the sequence $g_{i}(x)$ tends to $p$ inside of $\delta$ neighborhood $N_{\delta}(\ell)$ of the ray $\ell$, see Lemma 4. However, due to Lemma $3, N_{\delta}(\ell) \subset$ $N_{\delta}\left(\tau_{t}\right) \subset T_{\epsilon}$. Since the Margulis region $T_{\epsilon}$ is precisely invariant for the subgroup $\Gamma \subset$ $G$ (Theorem 2), it follows that all elements $g_{i}$ belong in fact to the parabolic subgroup $\Gamma$. Hence all $g_{i}$ preserve each horosphere $X_{t}$ centered at $p$. Using compactness of $\partial \tau_{t} / \Gamma$, we see then that all points $g_{i}(x)$ must lie in a compact part of $N_{\delta}\left(\tau_{t}\right)$ and hence cannot approach the limit point $p$. This contradiction completes the proof.

Now we shall apply our structural Theorem 1 to clarify the structure of cusp ends of geometrically finite locally symmetric rank one manifolds/orbifolds. This new geometric insight on dynamics of discrete isometry group actions near their parabolic fixed points will allow us to prove that fundamental groups of such manifolds/orbifolds are in fact finitely presented.

A parabolic fixed point $p \in \partial X$ of a discrete group $G \subset \operatorname{Isom} X$ in a pinched negatively curved space $X$ is called a cusp point if the quotient $(\Lambda(G) \backslash\{p\}) / G_{p}$ of
the limit set of $G$ by the action of the parabolic stabilizer $G_{p}=\{g \in G: g(p)=p\}$ is compact [9].

This leads to a definition (GF1, originally due to A.Beardon and B.Maskit, see [2]) of geometrically finite discrete groups $G \subset$ Isom $X$ (and their negatively curved orbifolds $M=X / G)$ as those whose limit set $\Lambda(G) \subset \partial X$ entirely consists of conical limit points and parabolic cusps.

Another definition of geometrical finiteness (GF2, originally due to Albert Marden, see [2]) is that the quotient $M(G)=X \cup \Omega(G) / G$ has only finitely many topological ends and each of these ends can be identified with the end of $M(\Gamma)$, where $\Gamma$ is a maximal parabolic subgroup of $G$.

Additional two definitions of geometrical finiteness are originally due to W.Thurston, see [2]:
(GF3): The thick part of the minimal convex retract (=convex core) $C(G)$ of $X / G$ is compact.
(GF4): For some $\epsilon>0$, the uniform $\epsilon$-neighborhood of the convex core $C(G) \subset$ $X / G$ has finite volume, and there is a universal bound on the orders of finite subgroups in $G$.
Theorem 6. [9] Let $X$ be a pinched Hadamard manifold. Then the four definitions GF1, GF2, GF3 and GF4 of geometrical finiteness for a discrete group $G \subset$ Isom $X$ are all equivalent.

We shall add that in contrast to the real hyperbolic geometry, our examples [5] of discrete parabolic groups acting in complex hyperbolic space suggest that there exists no elegant formulation of geometrical finiteness involving finite-sided polyhedra.

Now we shall give a new geometric definition of parabolic cusp points (cusp ends) for discrete isometry groups acting in non compact symmetric spaces $X$ of rank one. Let a point $p \in \partial X$ be a parabolic fixed point of a discrete group $G \subset \operatorname{Isom} X$ and let $\Gamma=G_{p}$ be the stabilizer of $p$ in $G$, that is a maximal parabolic subgroup in $G$ with fixed point $p$. As before, taking horospherical coordinates on $X$ with respect to $p \in \partial X$, we can regard this stabilizer as $\Gamma \subset \mathcal{N} \rtimes C$ where $C$ is a compact automorphism group of the connected Lie group $\mathcal{N}$ representing horospheres in $X$. Let $\rho_{c}$ be the ( $\mathcal{N} \rtimes C$-invariant) Cygan metric on $\mathcal{N} \times[0, \infty)=X \cup \partial X \backslash\{p\}$ defined in (2), and let $\mathcal{N}_{\Gamma} \subseteq \mathcal{N}=\partial X \backslash\{p\}$ be a minimal connected subgroup of the nilpotent group $\mathcal{N}$ given by Theorem 1 . The parabolic stabilizer $\Gamma$ preserves $\mathcal{N}_{\Gamma}$ and acts there cocompactly.
Definition 7. Given a positive number $\delta$ and a parabolic fixed point $p \in \partial X$ of a discrete group $G \subset \operatorname{Isom} X$ with stabilizer $\Gamma=G_{p} \subset G$, the set

$$
\begin{equation*}
U_{p, \delta}=\left\{x \in X \cup \partial X \backslash\{p\}: \rho_{c}\left(x, \mathcal{N}_{\Gamma}\right) \geq \frac{1}{\delta}\right\} \tag{4}
\end{equation*}
$$

is called a (closed) standard cusp neighborhood of radius $\delta>0$ at $p$, provided it is precisely invariant with respect to the stabilizer $\Gamma$ in $G$ :

$$
\gamma\left(U_{p, \delta}\right)=U_{p, \delta} \quad \text { for } \quad \gamma \in \Gamma=G_{p}
$$

$$
g\left(U_{p, \delta}\right) \cap U_{p, \delta}=\emptyset \quad \text { for } \quad g \in G \backslash G_{p}
$$

Lemma 8. Let $p \in \partial X$ be a parabolic fixed point of a discrete group $G \subset \operatorname{Isom} X$ in a rank one symmetric space $X$. Then $p$ is a parabolic cusp point if and only if it has a standard cusp neighborhood $U_{p, \delta}$.

Proof: As before, let $\Gamma \subset G$ be the parabolic stabilizer of a given parabolic fixed point $p$, and $\mathcal{N}_{\Gamma} \subseteq \mathcal{N}=\partial X \backslash\{p\}$ be the minimal connected $\Gamma$-invariant subspace of the nilpotent group $\mathcal{N}$ given by Theorem 1. If $p$ has a standard cusp neighborhood $U_{p, \delta} \subset \bar{X} \backslash\{p\}$ then the limit set $\Lambda(G)$ must lie in its complement $\partial X \backslash U_{p, \delta}$ due to the condition of its precise $\Gamma$-invariantness. Hence $\Lambda(G) \backslash\{p\} / \Gamma$ is compact because of compactness of $\mathcal{N}_{\Gamma} / \Gamma$ (due to Theorem 1). The converse statement follows from Bowditch's arguments in the proof [9] of Theorem 6.

For a given discrete group $\Gamma \subset \mathcal{N} \rtimes C \subset$ Isom $X$, the quotient space $M(\Gamma)=$ $(X \cup \partial X \backslash\{\infty\}) / \Gamma$ has a unique end. We call this end a standard parabolic end with $(X$, Isom $X)$-geometry. It is clear that (closed) neighborhoods of a standard parabolic end may be taken as $U_{\infty, \delta} / \Gamma, \delta>0$.

Applying the above definitions of cusp points and ends, Lemma 8 and Theorem 6 , we see that for a cusp point $p \in \partial X$ of a geometrically finite discrete group $G \subset \operatorname{Isom} X$, the family $E_{p}=\left\{U_{p, \delta} / G_{p}\right\}$ of closed subspaces in $M(G)$ naturally defines the cusp end of $M(G)$ identified by the $G$-orbit of the parabolic cusp point $p$. It is isometric to a standard cusp end, actually to the end of $M\left(G_{p}\right)$.

We may represent a standard cusp neighborhood $U_{p, \delta_{0}}$ at a cusp point $p$ of a discrete group $G \subset$ Isom $X$ as the product

$$
\begin{equation*}
U_{p, \delta_{0}}=S_{p, \delta_{0}} \times\left(0, r_{0}\right], \tag{5}
\end{equation*}
$$

if we foliate $U_{p, \delta_{0}}$ by subsets $S_{p, \delta}, 0<\delta \leq \delta_{0}$, of the form:

$$
\begin{equation*}
S_{p, \delta}=\left\{x \in X \cup \partial X \backslash\{p\}: \rho_{c}\left(x, \mathcal{N}_{G_{p}}\right)=1 / \delta\right\} . \tag{6}
\end{equation*}
$$

Since each set $S_{p, \delta}$ is $G_{p}$-invariant, we see that the standard cusp neighborhood $U_{p, \delta_{0}} / G_{p} \subset M(G)$ of the cusp end $E_{p}$ in the orbifold $M(G)$ is the product $\left(S_{p, \delta_{0}} / G_{p}\right) \times(0,1]$. Furthermore, due to compactness of the automorphism group $C$ of the nilpotent group $\mathcal{N}$, this foliation of a standard cusp neighborhood $U_{p, \delta_{0}}$ by $G_{p}$-invariant sets $S_{p, \delta}$ defines a $G_{p}$-equivariant retraction

$$
\begin{equation*}
R_{p}: \quad U_{p, \delta_{0}} \longrightarrow \mathcal{N}_{G_{p}} . \tag{7}
\end{equation*}
$$

This retraction shows topological finiteness of ends of noncompact orbifolds $\mathcal{N} / \Gamma$ for discrete parabolic groups $\Gamma \subset \mathcal{N} \rtimes C$ (and, with a little bit more work, existence of a vector bundle structure on them, compare our Theorem 4.1 in [5]), as well as topological finiteness of cusp ends of ( $X$, Isom $X$ )-orbifolds. So, due to Theorem 1, all those ends have the homotopy type of closed virtually nilpotent orbifolds $\mathcal{N}_{\Gamma} / \Gamma$. This, together with Theorem 6, completes the proof of the following fact:

Theorem 9. For any geometrically finite discrete group $G \subset \operatorname{Isom} X$ in a symmetric rank one non-compact space $X$, the orbifold $M=X / G$ is topologically finite. In other words, $M$ is orbifold-homeomorphic to the interior of a compact orbifold with boundary obtained from $M(G)$ by gluing to its ends closed virtually nilpotent orbifolds of the form $\mathcal{N}_{\Gamma} / \Gamma$ where $\Gamma \subset \mathcal{N} \rtimes C$ is a parabolic discrete group in the corresponding nilpotent group $\mathcal{N}$ representing horospheres in $X$.

It immediately implies:
Corollary 10. Let $\mathcal{N}$ be a nilpotent group representing horospheres in a symmetric rank one non-compact space $X$ and $C$ its compact group of automorphisms. Then all discrete parabolic groups $\Gamma \subset \mathcal{N} \rtimes C$ as well as geometrically finite groups $G \subset$ Isom $X$ are finitely presented.

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Boris N. Apanasov
Received October 25, 2002
Department of Mathematics
University of Oklahoma
Norman, OK 73019-0351, USA
E-mail:apanasov@ou.edu

# On some Hypergroups and their Hyperlattice Structures 

G.A. Moghani, A.R. Ashrafi


#### Abstract

Let $G$ be a hypergroup and $\mathcal{L}(G)$ be the set of all subhypergroups of $G$. In this survey article, we introduce some hypergroups $G$ from combinatorial structures and study the structure of the set $\mathcal{L}(G)$. We prove that in some cases $\mathcal{L}(G)$ has a lattice or hyperlattice structure.


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## 1 Introduction

First of all we will recall some algebraic definitions used in the paper. A hyperstructure is a set $H$ together with a function $\cdot: H \times H \longrightarrow P^{\star}(H)$ called hyperoperation, where $P^{\star}(H)$ denotes the set of all non-empty subsets of $H$. F.Marty [18] defined a hypergroup as a hyperstructure ( $H,$. ) such that the following axioms hold: (i) $(x . y) . z=x .(y . z)$ for all $x, y, z$ in $H$, (ii) $x \cdot H=H \cdot x=H$ for all $x$ in $H$. The axiom (ii) is called the reproduction axiom. A commutative hypergroup $(H, o)$ is called a join space if for all $a, b, c, d \in H$, the implication $a / b \cap c / d \neq \emptyset \Longrightarrow a o d \cap b o c \neq \emptyset$ is valid, in which $a / b=\{x \mid a \in x o b\}$.

The concept of an $H_{v}$-group is introduced by T.Vougiouklis in [20] and it is a hyperstructure ( $H$,.) such that (i) $(x . y) . z \cap x .(y . z) \neq \emptyset$, for all $x, y, z$ in $H$, (ii) $x . H=H . x=H$ for all $x$ in $H$. The first axiom is called weak associativity.

Following Gionfriddo [12] and Vougiouklis [20], we define a generalized permutation on a non-empty set $X$ as a map $f: X \longrightarrow \mathcal{P}^{*}(X)$ such that the reproductive axiom is valid, i.e. $\cup_{x \in X} f(x)=f(X)=X$. The set of all generalized permutations on $X$ is denoted by $M_{X}$. We now assume that $(G, \cdot)$ is a hypergroup and $X$ is a set. The map $\odot: G \times X \longrightarrow \mathcal{P}(X)^{*}$ is called a generalized action of $G$ on $X$ if the following axioms hold:

1) For all $g, h \in G$ and $x \in X,(g h) \odot x \subseteq g \odot(h \odot x)$,
2) For all $g \in G, g \odot X=X$.

Here, for any $g \in G$ and $Y \subseteq X, g \odot Y$ is defined as $\cup_{x \in Y} g \odot x$, and for any $x \in X$ and $B \subseteq G, B \odot x$ is, by definition, equal to $\cup_{b \in B} b \odot x$. If the equality holds in the axiom 1) of definition, the generalized action is called strong (see [17]).

Following Konstantinidou and Mittas [15], we define a hyperlattice as a set $H$ on which a hyperoperation $\vee$ and an operation $\wedge$ are defined which satisfy the following

[^1]axioms:

1. $a \in a \vee a$ and $a \wedge a=a$,
2. $a \vee b=b \vee a$ and $a \wedge b=b \wedge a$,
3. $(a \vee b) \vee c=a \vee(b \vee c)$ and $(a \wedge b) \wedge c=a \wedge(b \wedge c)$,
4. $a \in[a \vee(a \wedge b)] \wedge[a \wedge(a \vee b)]$,
5. $a \in a \vee b$ implies that $b=a \wedge b$.

It is well known [8] that in a lattice the distributivity of the meet $(\wedge)$ with respect to the join $(\vee)$ implies the distributivity of the join with respect to the meet and vice versa, the lattice is then called distributive. But in a hyperlattice a distinction of several types of distributivity is needed. According to Konstantinidou [16], a hyperlattice $(H, \vee, \wedge)$ will be called distributive if and only if, $a \wedge(b \vee c)=$ $(a \wedge b) \vee(a \wedge c)$, for all $a, b, c \in H$. Also, the hyperlattice $(H, \vee, \wedge)$ is called modular if $a \leq b$, implies that $a \vee(b \wedge c)=b \wedge(a \vee c)$, for all $c \in H$.

The second author in $[2,3]$ and [5], studied the construction of join spaces from some combinatorial structures. In [4], he found a new closed formula for the partition function $p(n)$. We encourage reader to consult these papers for discussion and background material.

Our notation is standard and taken mainly from [1, 8-10] and [20].

## 2 The Structure of some Hypergroups

Let $G$ be a group, $\operatorname{Sym}(G)$ be the group of all permutations on $G$ and $\operatorname{Sym}_{e}(G)$ be the stabilizer of the identity $e \in G$ in $\operatorname{Sym}(G)$. Given two permutations $\phi$ and $\psi$ from $\operatorname{Sym}_{e}(G)$ and an element $g \in G$, we define a new permutation $\phi \odot_{g} \psi=$ $L_{\phi(g)^{-1}} \phi L_{g} \psi$, where $L_{\phi(g)^{-1}}, L_{g} \in \operatorname{Sym}(G)$ are left multiplications by the elements $\phi(g)^{-1}$ and $g$, respectively.

According to [13], a subgroup $H$ of $\operatorname{Sym}_{e}(G)$ closed under taking products of this form is called rotary closed, i.e. $H \leq \operatorname{Sym}_{e}(G)$ is called rotary closed provided that $\phi \odot_{g} \psi \in H$, for all $\phi, \psi \in H$ and $g \in G$. A nice family of rotary closed subgroups of $\operatorname{Sym}_{e}(G)$, for finite $G$ 's, comes from the theory of Cayley graphs and can be obained in the following way. Let $\Omega$ be a set of generators for a finite group $G$ not containing the identity element $e$ but containing $x^{-1}$ together with every $x$ contained in $\Omega$. The subgroup $\operatorname{Rot}_{\Omega}(G)$ of $\operatorname{Sym}_{e}(G)$ of all permutations preserving $e$ and satisfying the condition $\phi(a)^{-1} \phi(a x) \in \Omega$, for every $a \in G$ and $x \in \Omega$, is rotary closed (for details see [13] and [14]).

In this section, first we introduce a hyperoperation $\odot$ on $\operatorname{Sym}_{e}(G)$ and prove that $\left(\operatorname{Sym}_{e}(G), \odot\right)$ is a hypergroup. Next, we characterize the sub-hypergroups of this hypergroup. To do this, assume that $\phi, \psi \in \operatorname{Sym}_{e}(G)$, we define $\phi \odot \psi=$ $\left\{\phi \odot_{g} \psi \mid g \in G\right\}$.

Proposition 2.1. $\left(\operatorname{Sym}_{e}(G), \odot\right)$ is a hypergroup.
Proof. Suppose $\phi, \psi$ and $\eta$ are arbitrary permutations of $\operatorname{Sym}_{e}(G)$. Then we have

$$
\begin{aligned}
& (\phi \odot \psi) \odot \eta=\left\{\phi \odot_{g} \psi \mid g \in G\right\} \odot \eta=\bigcup_{g \in G}\left(\phi \odot_{g} \psi\right) \odot \eta= \\
= & \bigcup_{g \in G}\left\{\left(\phi \odot_{g} \psi\right) \odot_{h} \eta \mid h \in G\right\}=\left\{\left(\phi \odot_{g} \psi\right) \odot_{h} \eta \mid g, h \in G\right\}
\end{aligned}
$$

Using similar argument as in above, we can show that

$$
\phi \odot(\psi \odot \eta)=\left\{\phi \odot_{g}\left(\psi \odot_{h} \eta\right) \mid g, h \in G\right\}
$$

We now assume that $g, h \in G$, then we have

$$
\begin{gathered}
\left(\phi \odot_{g} \psi\right) \odot_{h} \eta=L_{\phi \odot_{g} \psi(h)^{-1}} \phi \odot_{g} \psi L_{h} \eta=L_{\phi(g \psi(h))^{-1} \phi(g)} \phi \odot_{g} \psi L_{h} \eta= \\
=L_{\phi(g \psi(h))^{-1} \phi(g)} L_{\phi(g)^{-1}} \phi L_{g} \psi L_{h} \eta=L_{\phi(g \psi(h))^{-1}} \phi L_{g} \psi L_{h} \eta= \\
=L_{\phi(g \psi(h))^{-1}} \phi L_{g \psi(h)} L_{\psi(h)^{-1}} \psi L_{h} \eta=\phi \odot_{g \psi(h)}\left(\psi \odot_{h} \eta\right) \in \phi \odot(\psi \odot \eta)
\end{gathered}
$$

Therefore, $\phi \odot(\psi \odot \eta) \subseteq(\phi \odot \psi) \odot \eta$. Using similar argument we have $\phi \odot(\psi \odot \eta) \subseteq$ $(\phi \odot \psi) \odot \eta$ and the associativity is valid. Next we assume that $\phi \in \operatorname{Sym}_{e}(G)$ and we have

$$
\phi \odot \operatorname{Sym}_{e}(G)=\bigcup_{\psi \in \operatorname{Sym}_{e}(G)} \phi \odot \psi=\bigcup_{\psi \in \operatorname{Sym}_{e}(G)}\left\{\phi \odot_{g} \psi \mid g \in G\right\}
$$

Suppose $\delta \in \operatorname{Sym}_{e}(G)$ is arbitrary and $\psi=L_{g^{-1}} \phi^{-1} L_{\phi(g)} \delta$. Then, $\phi \odot_{g} \psi=\delta$ and so $\operatorname{Sym}_{e}(G)=\phi \odot \operatorname{Sym}_{e}(G)$. Similarly, $\operatorname{Sym}_{e}(G) \odot \phi=\operatorname{Sym}_{e}(G)$, which completes the proof.

In what follows, we characterize the sub-hypergroups of the hypergroup $\left(\operatorname{Sym}_{e}(G), \odot\right)$.
Proposition 2.2. Let $G$ be a finite group and $H$ be a non-empty subset of $\operatorname{Sym}_{e}(G)$. $H$ is a sub-hypergroup of $\operatorname{Sym}_{e}(G)$ if and only if $H$ is a rotary closed subgroup of $\operatorname{Sym}_{e}(G)$.

Proof. $(\Rightarrow)$ Suppose $H$ is a sub-hypergroup of $\operatorname{Sym}_{e}(G)$. We first show that $H$ is a closed subset of $S^{\operatorname{Sm}} m_{e}(G)$. To do this, suppose $\phi$ and $\psi$ are elements of $H$. Then $\phi \psi=\phi \odot_{e} \psi \in \phi \odot \psi \subseteq H$ and so $\phi \psi \in H$. Next, for $\phi, \psi \in H$ and $g \in G$, $\phi \odot_{g} \psi \in \phi \odot \psi \subseteq H$, as desired.
$(\Leftarrow)$ Suppose $H \leq \operatorname{Sym}_{e}(G)$ is rotary closed and $\phi \in G$. Since $H$ is rotary closed $\phi \odot H \subseteq H$. Suppose $\psi \in H$. Put $\eta=\phi^{-1} \psi$ and $g=e$. Then, $\phi \odot_{g} \eta=\phi \phi^{-1} \psi=$ $\psi \in \phi \odot \eta$ and so $H=\phi \odot H$. Similar argument shows that $H \odot \phi=H$, proving the result.

It is a well-known fact that the set of all subgroups of a group $G$ has a lattice structure under the ordinary operations of meet and join. In general, it is far from
true that the set of all sub-hypergroups of a hypergroup has a lattice structure under these operations. In fact, the intersection of two sub-hypergroups of a hypergroup is not necessarily non-empty.

Let $\mathcal{L}(G)$ be the set of all sub-hypergroups of the hypergroup $G$. In what follows, we show that $\operatorname{Sym}_{e}(G)$ has a lattice structure under the ordinary operations of join and meet.

Proposition 2.3. $\mathcal{L}\left(\operatorname{Sym}_{e}(G)\right)$ has a lattice structure under the ordinary operations of meet and join.

Proof. It is an easy fact that $\left\{1_{G}\right\}$ and $\operatorname{Sym}_{e}(G)$ are rotary closed. Suppose that $H$ and $K$ are two rotary closed subgroups of $\operatorname{Sym}_{e}(G)$. It is clear that $H \cap K$ is rotary closed. We claim that $\langle H, K\rangle$ is also rotary closed. To do this, we assume that $\psi \in H, \phi \in K$ and $g \in G$. Then we have:

$$
\psi \odot_{g} \phi=L_{\psi(g)^{-1}} \psi L_{g} \phi=L_{\psi(g)^{-1}} \psi L_{g} \psi \psi^{-1} \phi=\psi \odot_{g} \psi \psi^{-1} \phi \in\langle H, K\rangle
$$

Also, for $\psi_{1}, \psi_{2} \in H, \phi_{1}, \phi_{2} \in K$ and $g \in G$, we have

$$
\begin{gathered}
\psi_{1} \phi_{1} \odot_{g} \psi_{2} \phi_{2}=L_{\psi_{1} \phi_{1}(g)^{-1}} \psi_{1} \phi_{1} L_{g} \psi_{2} \phi_{2}= \\
=L_{\left(\psi_{1}\left(\phi_{1}(g)\right)\right)^{-1}} \psi_{1} L_{\phi_{1}(g)} \psi_{1} \psi_{1}^{-1} L_{\phi_{1}(g)^{-1}} \phi_{1} L g \psi_{2} \phi_{2}= \\
=\left(\psi_{1} \odot_{\phi_{1}(g)} \psi_{1}\right) \psi_{1}^{-1}\left(\phi_{1} \odot_{g} \psi_{2} \phi_{2}\right) \in H K \subseteq\langle H, K\rangle .
\end{gathered}
$$

Using similar argument as in above, we can show that $\langle H, K\rangle$ is a rotary closed subgroup of $\operatorname{Sym}_{e}(G)$. This shows that $\mathcal{L}\left(\operatorname{Sym}_{e}(G)\right)$ has a lattice structure under ordinary operations of join and meet.

Let $G$ be a set, $B$ an algebraic Boolean algebra and $s$ a function from $G$ into $B$. We define the hyperoperation $\stackrel{s}{\star}$ as follows:

$$
a \stackrel{s}{\star} b=\{x \in G \mid s(x) \leq s(a) \vee s(b)\} .
$$

Since for all $x, y \in G\{x, y\} \subseteq x \stackrel{s}{\star} y,(G, \stackrel{s}{\star})$ is an $H_{v}$-group. It is also clear that the hyperoperation $\stackrel{s}{\star}$ is commutative.

In what follows, we study the sub-hypergroup structure of the hypergroup $(G, \stackrel{\Im}{\star})$. In some special cases we will show that the set $\mathcal{L}(G)$ has a hyperlattice structure. We also assume that $G_{a}=\{g \in G \mid s(g) \leq a\}$. It is easy to see that if $a \in B$ and $G_{a} \neq \emptyset$ then $G_{a}$ is an $H_{v}$-subgroup of $G$. In what follows, when we write $G_{a}$, we assume that $G_{a} \neq \emptyset$.

Proposition 2.4. Let $B$ be a complete Boolean algebra and $s: G \longrightarrow B$ be $a$ function such that $(G, \stackrel{\substack{*}}{ })$ constitute a hypergroup. Also, we assume that that

$$
a_{1} \stackrel{s}{\star} a_{2} \stackrel{s}{\star} \cdots \stackrel{s}{\star} a_{n}=\left\{g \in G \mid s(g) \leq s\left(a_{1}\right) \vee \cdots \vee s\left(a_{n}\right)\right\},
$$

and $H$ is a sub-hypergroup of $G$. Then there exists an element $a \in B$ such that $H=G_{a}$.

Proof. Let $H$ be a sub-hypergroup of $G$ and $a=\vee_{b \in H} s(b)$. We claim that $H=G_{a}$. To see this, assume $x \in H$. Then $s(x) \leq \vee_{b \in H} s(b)=a$ and so $x \in G_{a}$, i.e., $H \subseteq G_{a}$. We now assume that $x \in G_{a}$. Then $s(x) \leq a=\vee_{b \in H} s(b)$. Since $B$ is algebraic, there are the elements $b_{1}, b_{2}, \cdots, b_{r}$ of $H$ such that $s(x) \leq s\left(b_{1}\right) \vee \cdots \vee s\left(b_{r}\right)$. Now by assumption, $x \in\left\{g \in G \mid s(g) \leq s\left(b_{1}\right) \vee \cdots \vee s\left(b_{r}\right)\right\}=b_{1} \stackrel{s}{\star} b_{2} \stackrel{s}{\star} \cdots \stackrel{s}{\star} b_{r}$ and $H$ is a sub-hypergroup of $G$, so $x \in H$, proving the result.

It is clear that $G_{a \wedge b}=G_{a} \cap G_{b}$, for all $a, b \in B$. It is far from true that $G_{a \vee b}=G_{a} \cup G_{b}$. To see this, we construct an example as follows:

Example 2.5. Suppose $G=B=P(X), s$ is the identity function, $|X| \geq 3$ and $a, b, c$ distinct elements of $X$. Set $R=\{a, b\}$ and $S=\{c\}$. Then $G_{R}=P(R), G_{S}=$ $P(S)$ and $G_{R \cup S}=P(R \cup S)$. Now it is easy to see that $G_{R \cup S} \neq G_{R} \cup G_{S}$.

By the results of [3] and [4], if the image of $G$ is a $\vee$-sub-semilattice or constitutes a partition of 1 , then $\mathcal{L}(G)=\left\{G_{a} \mid a \in B \& G_{a} \neq \emptyset\right\}$. In this case, we define a hyperoperation $\vee$ on $\mathcal{L}(G)$ such that $(\mathcal{L}(G), \vee, \wedge)$ constitutes a hyperlattice. To do this, we assume that $G_{a} \vee G_{b}=\left\{G_{x} \mid a \vee b \leq x\right\}$.

In the following lemmas we investigate the conditions of a hyperlattice.
Lemma 2.6. $G_{a} \in G_{a} \vee G_{a}, G_{a} \wedge G_{a}=G_{a}, G_{a} \vee G_{b}=G_{b} \vee G_{a}$ and $G_{a} \wedge G_{b}=$ $G_{a \wedge b}=G_{b} \wedge G_{a}$.

Proof. Obvious.
Lemma 2.7. $\left(G_{a} \vee G_{b}\right) \vee G_{c}=G_{a} \vee\left(G_{b} \vee G_{c}\right)$ and $\left(G_{a} \wedge G_{b}\right) \wedge G_{c}=G_{a} \wedge\left(G_{b} \wedge G_{c}\right)$.
Proof. The associativity of $\wedge$ is obvious. We will show the associativity of $\vee$. Suppose $a, b, c \in B$. Then

$$
\begin{gathered}
\left(G_{a} \vee G_{b}\right) \vee G_{c}=\left\{G_{x} \mid a \vee b \leq x\right\} \vee G_{c}=\bigcup_{a \vee b \leq x} G_{x} \vee G_{c}= \\
=\bigcup_{a \vee b \leq x}\left\{G_{t} \mid x \vee c \leq t\right\}=\left\{G_{u} \mid a \vee b \vee c \leq u\right\}
\end{gathered}
$$

Similar argument shows that $G_{a} \vee\left(G_{b} \vee G_{c}\right)=\left\{G_{u} \mid a \vee b \vee c \leq u\right\}$, and the result follows.

Lemma 2.8. $G_{a} \in\left[G_{a} \vee\left(G_{a} \wedge G_{b}\right)\right] \cap\left[\left(G_{a} \wedge\left(G_{a} \vee G_{b}\right)\right]\right.$, for all $a, b \in B$.
Proof. Suppose $a, b$ are arbitrary elements of $B$, then we have

$$
\begin{gathered}
G_{a} \vee\left(G_{a} \wedge G_{b}\right)=G_{a} \vee G_{a \wedge b}= \\
=\left\{G_{t} \mid a \vee(a \wedge b) \leq t\right\}=\left\{G_{t} \mid a \leq t\right\} .
\end{gathered}
$$

Therefore, $G_{a} \in G_{a} \vee\left(G_{a} \wedge G_{b}\right)$. On the other hand, $G_{a}=G_{a \wedge(a \vee b)}=G_{a} \wedge G_{a \vee b} \in$ $G_{a} \wedge\left(G_{a} \vee G_{b}\right)$, as required.

Lemma 2.9. $G_{a} \in G_{a} \vee G_{b}$ implies that $G_{b}=G_{a} \wedge G_{b}$.
Proof. Suppose $G_{a} \in G_{a} \vee G_{b}$, then there exists $t \in B$ such that $G_{a}=G_{t}$ and $a \vee b \leq t$. Thus, $b=b \wedge(a \vee b) \leq b \wedge t$ and so $G_{b} \subseteq G_{b \wedge t}=G_{b} \wedge G_{t}=G_{a} \wedge G_{b}$. Therefore, $G_{b}=G_{a} \wedge G_{b}$ and the lemma is proved.

We summarize the above lemmas in the following theorem:
Theorem 2.10. Let $s: G \longrightarrow B$ be a function such that $(G, \stackrel{s}{\star})$ constitute a hypergroup. Also, we assume that for all positive integer $n$ and the elements $a_{1}, \cdots, a_{n}$ of $G$, we have

$$
a_{1} \stackrel{s}{\star} a_{2} \stackrel{s}{\star} \cdots \stackrel{s}{\star} a_{n}=\left\{g \in G \mid s(g) \leq s\left(a_{1}\right) \vee \cdots \vee s\left(a_{n}\right)\right\} .
$$

Then $(\mathcal{L}(G), \vee, \wedge)$ is a hyperlattice.
We now investigate the distributivity of $\mathcal{L}(G)$ and show this hyperlattice is not distributive, in general. In fact, we have the following example.

Example 2.11. There exists a function $s: G \longrightarrow B$ such that $(G, \stackrel{s}{\star})$ is a hypergroup which satisfies the conditions of Theorem 2.10, but $\mathcal{L}(G)$ is not distributive. To see this, we assume that $H$ is a finite group, $\Pi_{e}(H)=\{o(x) \mid x \in H\}$ and $s$ : $P(H) \longrightarrow P\left(\Pi_{e}(H)\right)$ defined by $s(A)=\{o(x) \mid x \in A\}$. It is easy to see that the function $s$ is onto, so by Theorem $3.6 \mathcal{L}(P(H))$ is a hyperlattice. Suppose, $H=Z_{4}=\left\{e, a, a^{2}, a^{3}\right\}$, the cyclic group of order four, and $G=P(H)$. Then $\Pi_{e}\left(Z_{4}\right)=\{1,2,4\}$. Set $A=\{1,2\}, B=\{1\}, C=\{2,4\}$ and $D=\{2\}$. It is clear that $G_{A} \wedge\left(G_{B} \vee G_{C}\right)=G_{A} \wedge G_{\Pi_{e}(G)}=G_{A}$ and $\left(G_{A} \wedge G_{B}\right) \vee\left(G_{A} \wedge G_{C}\right)=G_{B} \vee G_{D}=$ $\left\{G_{A}, G_{\Pi_{e}(G)}\right\}$. This shows that $G_{A} \wedge\left(G_{B} \vee G_{C}\right) \neq\left(G_{A} \wedge G_{B}\right) \vee\left(G_{A} \wedge G_{C}\right)$. Therefore, $\mathcal{L}\left(P\left(Z_{4}\right)\right)$ is a hyperlattice which is not distributive.

It is natural to ask about modularity of $\mathcal{L}(G)$. Here, we obtain an example such that its sub-hypergroup hyperlattice is not modular.

Example 2.12. Assume that $X=\{1,2,3,4,5\}, G=B=P(X)$ and $s$ is the identity function on $G$. Set $A=\{1,2\}, B=\{1,2,3\}$ and $C=\{4,5\}$. Then $G_{A} \subseteq$ $G_{B},\left|G_{A} \vee\left(G_{B} \wedge G_{C}\right)\right|=8$ and $\left|G_{B} \wedge\left(G_{A} \vee G_{C}\right)\right|=2$. This shows that the hyperlattice $\mathcal{L}(G)$ is not modular.

## 3 About some Generalized Action

Suppose $s: G \longrightarrow B$ is a function such that $(G, \stackrel{s}{\star})$ is a hypergroup and $A=$ Atom $(B)$. Define the map $\odot: G \times A \longrightarrow P^{\star}(A)$ by $g \odot x=\{a \in A \mid a \leq x \vee s(g)\}$. In this section, we obtain a condition on $s$ such that $\odot$ is a generalized action and prove that under this condition the hypergroup $(G, \stackrel{\substack{\star}}{\star}$ is isomorphic to a sub-hypergroup of $M_{A}$.

Finally, we define a generalized action of an $H_{v}$-group on a set $X$ as in hypergroups. We will apply the elementary properties of a generalized action and prove an inequality between the partition function $p o(n)$ and the order of the hypergroup $M_{I(n)}$.

Lemma 3.1. Let $B$ be a Boolean algebra and $A=\operatorname{Atom}(B)$. If $s: G \longrightarrow B$ is a function such that the image of $G$ is a partition of 1 , then the map $\odot: G \times A \longrightarrow$ $P^{\star}(A)$ defined by $g \odot a=\{x \in A \mid x \leq a \vee s(g)\}$ is a strong generalized action of $G$ on $A$.

Proof. Suppose $g \in G$. Then it is obvious that for all $x \in A$, we have $x \in$ $g \odot x \subseteq \bigcup_{a \in A} g \square a$, i.e., $A=\bigcup_{a \in A} g \odot a$. Thus, $g \odot A=A$ and the condition (i) is satisfied. We now assume that $T=\{a \in A \mid a \leq x \vee s(g) \vee s(h)\}$ and prove that $g h \odot x=g \odot(h \odot x)=T$. It is easy to see that $g h \odot x \cup g \odot(h \odot x) \subseteq T$. Suppose $a \in T$. Then $a \leq x \vee s(g) \vee s(h)$ and we have $a=(a \wedge s(h)) \vee[a \wedge(x \vee s(g)]$. We first assume that $a \neq s(h)$, then $a \wedge s(h)=0$ and so $a \leq x \vee s(g)$. This shows that $a \in g \square x \subseteq \bigcup_{t \in g h} t \square x=g h \square x$. Next we assume that $a=s(h)$. Then $a \in h \odot x \subseteq g h \odot x$ and so $T=g h \odot x$. Using similar argument as above, we have $T=g \odot(h \odot x)$, proving the lemma.

Lemma 3.2. Let $B$ be a Boolean algebra and $A=\operatorname{Atom}(B)$. If $s: G \longrightarrow B$ is a one-to-one function such that the image of $G$ is a partition of 1 and that for all $g \in G$, there exists an atom $x$ such that $|g \odot x| \leq 2$, then $(G, \stackrel{S}{\star})$ is isomorphic to a sub-hypergroup of the hypergroup $M_{A}$.

Proof. By Lemma 3.1, $\odot: G \times A \longrightarrow P^{\star}(A)$ is a strong generalized action of $G$ on $A$ and by Proposition 3.1 of [17] this action induced a good homomorphism $\xi: G \longrightarrow$ $M_{A}$ defined by $\xi(g)(a)=g \odot a$. It is enough to show that this homomorphism is one-to-one. To do this, suppose $g \odot x=h \odot x$, for all $x \in A$. By assumption, there exists an atom $x$ such that $|g \odot x| \geq 2$. If $a \neq x$ and $a \in g \odot x$ then $a \leq s(g) \vee x=s(h) \vee x$, and so $a \leq x \vee(s(g) \wedge s(h))$. Thus, $a=(a \wedge x) \vee(a \wedge s(g) \wedge s(h))=a \wedge s(g) \wedge s(h)$, i.e., $a \leq s(g) \wedge s(h)$. But, the image of $G$ is a partition of 1 , hence $s(g)=s(h)$ and by injectivity of $s, g=h$.

Suppose $s: \Pi_{d}(n) \longrightarrow P^{\star}(I(n))$ is defined by $s(\lambda)=\operatorname{Part}(\lambda)$. Then $\left(\Pi_{d}(n), \stackrel{s}{\star}\right)$ is an $H_{v}$-group. Define the map $\odot: \Pi_{d}(n) \times I(n) \longrightarrow P^{\star}(I(n))$ by $\lambda \odot k=\operatorname{Part}(\lambda) \cup$ $\{k\}$. In the following simple lemma we show that the map $\odot$ is a strong generalized action of $\Pi_{d}(n)$ on the set $I(n)$.

Lemma 3.3. The map $\odot: \Pi_{d}(n) \times I(n) \longrightarrow P^{\star}(I(n))$ defined by $\lambda \odot k=\operatorname{Part}(\lambda) \cup$ $\{k\}$ is a strong generalized action of $\Pi_{d}(n)$ on the set $I(n)$.

Proof. We first assume that $n$ is an odd positive integer, we define the partitions $\mu_{i}, 0 \leq i \leq\left[\frac{n}{2}\right]$, by the following table:

| $\mu_{0}$ | $\mu_{1}$ | $\mu_{2}$ | $\cdots$ | $m_{\left[\frac{n}{2}\right]}$ |
| :---: | :---: | :---: | :--- | :---: |
| $n=n$ | $n=1+(n-1)$ | $n=2+(n-2)$ | $\cdots$ | $n=\left[\frac{n}{2}\right]+\left(n-\left[\frac{n}{2}\right]\right)$ |

Next we assume that $n$ is even, then we define the partitions $\xi_{i}, 0 \leq i \leq \frac{n}{2}$, by $\xi_{i}=\mu_{i}$, for all $i<\frac{n}{2}$ and $\xi_{\frac{n}{2}}$ is the partition $n=1+\left(\frac{n}{2}-1\right)+\frac{n}{2}$. Then it is clear that $\bigcup_{i=1}^{\left[\frac{n}{2}\right]} \operatorname{Part}\left(\mu_{i}\right)=\bigcup_{i=1}^{\left[\frac{n}{2}\right]} \operatorname{Part}\left(\xi_{i}\right)=I(n)$, and so the reproduction axiom is valid. We now assume that $\lambda, \mu$ are arbitrary partitions and $m \in I(n)$. Then we have

$$
\begin{gathered}
(\lambda \stackrel{s}{\star} \mu) \odot m=\bigcup_{\delta \in \lambda_{\star}^{s} \mu} \delta \odot m= \\
=\bigcup_{\delta \in \lambda_{\star}^{s} \mu} \operatorname{Part}(\delta) \cup\{m\}=\operatorname{Part}(\lambda) \cup \operatorname{Part}(\mu) \cup\{m\}
\end{gathered}
$$

On the other hand,

$$
\begin{gathered}
\lambda \odot(\mu \odot m)=\bigcup_{k \in \lambda \odot m} \lambda \square k= \\
=\bigcup_{k \in \lambda \odot m}(\operatorname{Part}(\lambda) \cup\{k\})=\operatorname{Part}(\lambda) \cup \operatorname{Part}(\mu) \cup\{m\} .
\end{gathered}
$$

Therefore, the map $\odot$ is a strong generalized action of the $H_{v}$-group $\Pi_{d}(n)$ on the set $I(n)$.

Lemma 3.4. $p o(n) \leq \sum_{i=0}^{n-1}(-1)^{i}\binom{n}{i}\left(2^{n-i}-1\right)^{n}$.
Proof. By Euler's partition theorem [1], po $(n)=\left|\Pi_{d}(n)\right|$ and by Proposition 4.1 of [17], the right hand of this inequality is the order of $M_{I(n)}$. Therefore, it is enough to show that $\left|\Pi_{d}(n)\right| \leq\left|M_{I(n)}\right|$. To do this, we now prove that the induced homomorphism $\eta: \Pi_{d}(n) \longrightarrow M_{I(n)}$ by $\eta(\mu)(x)=\mu \odot x$ is one-to-one. Assume that $\eta(\mu)=\eta(\xi)$, then $\mu \odot x=\xi \odot x$, for all $x \in I(n)$. Thus, $\operatorname{Part}(\mu) \cup\{x\}=\operatorname{Part}(\xi) \cup\{x\}$, for all $x \in I(n)$. Choose $x \in \operatorname{Part}(\mu)$. We have $\operatorname{Part}(\xi) \subseteq \operatorname{Part}(\xi) \cup\{x\}=\operatorname{Part}(\mu)$. Similarly, $\operatorname{Part}(\mu) \subseteq \operatorname{Part}(\xi)$ and so $\operatorname{Part}(\mu)=\operatorname{Part}(\xi)$. Now since $\mu$ and $\xi$ have distinct parts, $\mu=\xi$.

In the end of this paper, we define a generalized action of $S y m_{e}(G)$ on the group $G$. Suppose $\square: \operatorname{Sym}_{e}(G) \times G \longrightarrow P^{\star}(G)$ sends $(\phi, g)$ to $\phi(\langle g\rangle)$. Then we have

$$
\begin{aligned}
& \phi \square(\psi \square g)=\phi \square \psi(\langle g\rangle)=\bigcup_{i \in Z} \phi \square \psi\left(g^{i}\right)= \\
& =\bigcup_{i \in Z} \phi\left(\left\langle\psi\left(g^{i}\right)\right\rangle\right)=\left\{\phi\left(\psi\left(g^{i}\right)\right)^{j} \mid i, j \in Z\right\}
\end{aligned}
$$

and $\phi \psi \square g=\phi \psi(\langle g\rangle)=\left\{\phi\left(\psi\left(g^{i}\right)\right) \mid i \in Z\right\}$. This shows that $\phi \psi \square g \subseteq \phi \square(\psi \square g)$. On the other hand, $\phi \square G=\bigcup_{g \in G} \phi \square g=\bigcup_{g \in G} \phi(\langle g\rangle)=G$, which shows that $\square$ is a generalized action of $\operatorname{Sym}_{e}(G)$ on the group $G$.

Question 3.5. When this generalized action is strong?

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G.A. Mogani

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Department of Mathematics
University of Mazandaran
Babolsar, Iran
E-mail:moghani@umz.ac.ir
A.R. Ashrafi

Department of Mathematics
University of Kashan, Kashan
E-mail:ashraf@kashanu.ac.ir

# The commutative Moufang loops with minimum conditions for subloops I 

N.I. Sandu


#### Abstract

The structure of the commutative Moufang loops (CML) with minimum condition for subloops is examined. In particular it is proved that such a CML $Q$ is a finite extension of a direct product of a finite number of the quasicyclic groups, lying in the centre of the CML $Q$. It is shown that the minimum conditions for subloops and for normal subloops are equivalent in a CML. Moreover, such CML also characterized by different conditions of finiteness of its multiplicative groups.


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The loop $Q$ satisfies minimum condition for subloops with the property $\alpha$ if any decreasing chain of its subloops with the property $\alpha H_{1} \supseteq H_{1} \supseteq \ldots$, i.e. $H_{n}=H_{n+1}=\ldots$ for a certain $n$. In this paper the construction of the commutative Moufang loops (abbreviated CMLs) with minimum condition for subloops is examined. In particular, it is shown that such a CML $Q$ decomposes into a direct product of finite number of quasicyclic groups which lies in the centre of $Q$, and a finite CML (Section 2). In the third Section these loops are described with the help of their multiplicative groups. Finally, it is shown in the fourth section that for the CML, the minimum conditions for subloops are equivalent to the minimum condition for normal subloops, and in the case of $Z A$-loops these conditions are equivalent to the minimum condition for normal associative subloops. It follows from the last statement that the infinite commutative Moufang $Z A$-loop $Q$ has an infinite centre and if the centre of the CML satisfies the minimum condition for the subloops, then $Q$ itself satisfies this condition.

We finally note that loops, in particular the CML, with different conditions of finiteness are examined in $[1-3]$. We remind that the condition of finiteness means such's property, that holds true for all finite loops, but there exist infinite loops that do not have this property.

## 1 Preliminaries

Let us bring some notions and results on the theory of the commutative Moufang loops from [4]. A commutative Moufang loop (abbreviated CML) is characterized by the identity
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$$
\begin{equation*}
x^{2} \cdot y z=x y \cdot x z \tag{1.1}
\end{equation*}
$$

The multiplicative group $\mathfrak{M}(Q)$ of the CML $Q$ is the group generated by all the translations $L(x)$, where $L(x) y=x y$. The subgroup $I(Q)$ of the group $\mathfrak{M}(Q)$ generated by all the inner mappings $L(x, y)=L^{-1}(x y) L(x) L(y)$ is called the inner mapping group of the CML $Q$. The subloop $H$ of the CML $Q$ is called normal (invariant) in $Q$ if $I(Q) H=H$.
Lemma 1.1 [4]. Let $Q$ be a commutative Moufang loop with the multiplicative group $\mathfrak{M}$. Then $\mathfrak{M} / Z(\mathfrak{M})$, where $Z(\mathfrak{M})$ is the centre of the group $\mathfrak{M}$, and $\mathfrak{M}^{\prime}=(\mathfrak{M}, \mathfrak{M})$ are locally finite 3 -groups and will be finite if $Q$ is finitely generated.

The associator $(a, b, c)$ of the elements $a, b, c$ of the CML $Q$ are defined by the equality $a b \cdot c=(a \cdot b c)(a, b, c)$. The identities

$$
\begin{gather*}
L(x, y) z=z(z, y, x)  \tag{1.2}\\
\left(x^{p}, y^{r}, z^{s}\right)=(x, y, z)^{p r s}  \tag{1.3}\\
(x, y, z)^{3}=1  \tag{1.4}\\
(x y, u, v)=(x, u, v)((x, u, v), x, y)(y, u, v)((y, u, v), y, x) \tag{1.5}
\end{gather*}
$$

hold in the CML [4].
The centre $Z(Q)$ of the CML $Q$ is a normal subloop $Z(Q)=\{x \in Q \mid(x, y, z)=$ $1 \forall y, z \in Q\}$.
Lemma 1.2 [4]. In a commutative Moufang loop $Q$ the following statements hold true:

1) for any $x \in Q x^{3} \in Z(Q)$;
2) the quotient loop $Q / Z(Q)$ has the index three.

Lemma 1.3 [4]. The periodic commutative Moufang loop is locally finite.
Lemma 1.4 [5]. The periodic commutative Moufang loop $Q$ decomposes into a direct product of its maximum p-subloops $Q_{p}$, in addition $Q_{p}$ belongs to the centre $Z(Q)$ under $p \neq 3$.

The system $\sigma$ of the normal subloops of the loop $Q$ is called normal if it:

1) contains the loop $Q$ and its identity subloop;
2) is linearly ordered by the inclusion;
3) the intersection and union of any non-empty set of elements of $\sigma$ is an element of $\sigma$ (fullness).
If $A \subseteq B$ are two members of the system $\sigma$ and between them there are no other members of this system then it is said that the subloops $A$ and $B$ form a jump in the system $\sigma$. The quotient loop $B / A$ is called the factor of this system. The normal system $\sigma$ is called central if for any jump $A$ and $B$ of the system $\sigma, B / A \subseteq Z(B / A)$.

The loop possessing a central system is called a $Z$-loop. This statement is proved in [4, Theorems 4.1, Chap. VI; 10.1, Chap. VIII].
Lemma 1.5. Any commutative Moufang loop is a Z-loop.
If the loop possesses a central system entirely ordered by the inclusion (the central series), then this loop is called $Z A$-loop.
Lemma 1.6 [3]. Any normal different from the identity element subloop $H$ of the commutative Moufang $Z A$-loop $Q$ has a different from identity element intersection with its centre.

If the upper central series of the $Z A$-loop have a finite length, then the loop is called centrally nilpotent. The least of such length is called the class of the central nilpotentcy.
Lemma 1.7 [3]. If a commutative Moufang $Z A$-loop $Q$ has an infinite associative normal subloop, then its centre $Z(Q)$ is infinite.
Lemma 1.8 (Bruck-Slaby Theorem) [4]. The finitely generated commutative Moufang loop is centrally nilpotent.
Lemma 1.9 [3]. If at least one maximal associative subloop of the commutative Moufang loop $Q$ satisfies the minimum conditions for subloops, then $Q$ satisfies these conditions itself.

The CML $Q$ will be called divisible it the equality $x^{n}=a$ has at least one solution in $Q$, for any number $n>0$ and any element $a \in Q$. If $n=3$, then $a=b^{3} \in Z(Q)$ by Lemma 1.2. Therefore it takes place.
Lemma 1.10. If a subloop of the commutative Moufang loop $Q$ is divisible, it belongs to the centre $Z(Q)$ and, consequently, is normal in $Q$.

The quasicyclic p-groups are some important examples of divisible CML. As abstract groups they have the set of generators $1=a_{0}, a_{1}, a_{2}, \ldots, a_{n}, \ldots$ and defining relations $a_{0}=a_{1}^{p}, a_{1}=a_{2}^{p}, \ldots, a_{n}=a_{n+1}^{p}, \ldots$.

A CML is called injective if there exists a homomorphism $\gamma: B \rightarrow Q$ such that $\alpha \gamma=\beta$, for any monomorphism $\alpha: A \rightarrow B$ and homomorphism $\beta: A \rightarrow Q$.
Lemma 1.11. The divisible commutative Moufang loops are injective.
Proof. By Lemma 1.10 a divisible CML is associative, but divisible abelian groups are injective [6].

Further we will denote by $\langle M\rangle$ the subloop of loop $Q$, generated by the set $M \subseteq Q$.
Proposition 1.12. The divisible subloop $D$ of the commutative Moufang loop $Q$ serves as a direct factor for $Q$, i.e. $Q=D \times C$ for a certain subloop $C$ of the loop $Q$. We can choose such a subloop that it possesses the given before subloop $B$ of the loop $Q$ for which $D \cap B=1$.
Proof. By Lemma 1.11 there exists such homomorphism $\beta: Q \rightarrow Q$, that $\beta \alpha=\varepsilon$ for the natural inclusion $\alpha: D \rightarrow Q$ and the identity mapping $\varepsilon: D \rightarrow D$. By Lemma 1.10 the subloop $D$ is normal in $Q$, therefore $Q=D \times \operatorname{ker} \beta$.

Let now the equality $B \cap D=1$ hold true for the subloop $B \subseteq Q$. We denote $H=<D, B>$. By Lemma $1.10 D \subseteq Z(Q)$ is the centre of the loop $H$, then it is easy to show that any element of the CML $H$ has the form $a u$, where $a \in B, u \in D$. By (1.2) and (1.5) we have $L(a u, b v) c=c(c, b v, a u)=c(c, b, a) \in B$, for any $a, b \in B$ and any $u, v \in D$. Consequently, the subloop $B$ is invariant in regard to the inner mapping group of the CML $H$, i.e. the subloop $B$ is normal in $H$. Then $\langle B, D\rangle=$ $B \times D$ and there is a homomorphism $\xi: B \times D \rightarrow D$ coinciding with the identity on $D$ and unitary on $B$. If we replace $\varepsilon$ by $\xi$ in the first part of this proof, then we obtain $Q=D \times \operatorname{ker} \beta$, where $B \subseteq \operatorname{ker} \beta$. This completes the proof of Proposition 1.12 .

The second part of this proposition states that a divisible CML is an absolute direct factor.

If the CML $Q$ is given, let us examine the subloop $D$ within it, generated by all divisible subloops of the CML $Q$. By Lemma 1.10 they all belong to the centre $Z(Q)$ of the CML $Q$, then it is easy to see that $D$ is a divisible CML. Thus it is the maximal divisible subloop of the CML $Q$. By Proposition $1.12 Q=D \times C$, where obviously $C$ is a reduced CML, meaning that it has no non-unitary divisible subloops. Consequently, we obtain
Proposition 1.13. Any commutative Moufanf loop $Q$ is a direct product of the divisible subloop $D$ that lies in the centre $Z(Q)$ of the loop $Q$, and the reduced subloop $C$. The subloop $D$ is unequivocally defined, the subloop $C$ is defined exactly till the isomorphism.

Proof. Let us prove the last statement. As $D$ is the maximal divisible subloop of the CML $Q$, it is entirely characteristic in $C$, i.e. it is invariant in regard to the endomorphisms of the CML $Q$. Let $Q=D^{\prime} \times C^{\prime}$, where $D^{\prime}$ is a divisible subloop, and $C^{\prime}$ is a reduced subloop of the CML $Q$. We denote by $\varphi, \psi$ the endomorphisms $\varphi: Q \rightarrow D^{\prime}, \psi: Q \rightarrow C^{\prime}$. As $D$ is an entirely characteristic subloop, $\varphi D$ and $\psi D$ are subloops of the loop $Q$. It follows from the inclusions $\varphi D \subseteq D^{\prime}$ and $\psi D \subseteq C^{\prime}$ that $\varphi D \cap \psi D=1$. By Lemma $1.10 D$ is an abelian group, therefore $\varphi D, \psi D$ are normal in $D$. Then $d=\varphi d \cdot \psi d(d \in D)$ gives $D=\varphi D \cdot \psi D$, so $D=\varphi D \times \psi D$. Obviously, $\varphi D \subseteq D \cap D^{\prime}, \psi D \subseteq D \cap C^{\prime}$, where from $D=\left(D \cap D^{\prime}\right) \times\left(D \cap C^{\prime}\right)$. But $D \cap C^{\prime}=1$ as a direct factor of the divisible CML, that is contained by the reduced CML. Therefore, $D \cap D^{\prime} \subseteq D, D \subseteq D^{\prime}$, i.e. $D=D^{\prime}$. This completes the proof of Proposition 1.13.

Let us finally prove
Proposition 1.14. The following conditions are equivalent for the commutative Moufang loop $D$ :

1) $D$ is a divisible loop;
2) $D$ is an injective loop;
3) D serves as a direct factor for any commutative Moufang loop that contains it.
Proof. The implication 1) $\longrightarrow 2$ ) is proved in Lemma 1.11.
$2) \longrightarrow 3)$. By the definition of the injective CML $D$ there is such a homomorphism $\beta: Q \rightarrow D$ that $\beta \alpha=\epsilon$ for the natural inclusion $\alpha: D \rightarrow Q$ and identity mapping $\epsilon: D \rightarrow D$. We denote ker $\beta=H$. Obviously $Q=<D, H>, H \cap D=1$ and if $a H=b H$, then $a=b$. Let $x \in Q, d \in D, h \in H$. The CML is an IP-loop, then $(L(x, h) d) H=\left((x h)^{-1}(x \cdot h d) H=\left(x^{-1}(x d)\right) H=d H\right.$, i.e. $L(x, h) d=d$. Any element from $Q$ has the form $d h$, where $d \in D, h \in H$. Using (1.2) and (1.5) it is easy to show then that the subloop $D$ is invariant in regard to the inner mapping group of the CML $Q$, i.e. $D$ is normal in $Q$. Consequently, $Q=D \times H$.
$3) \longrightarrow 1$ ). Let the CML $D$ satisfy the condition 3) and let there exist such generators $a, b, c$ of the CML $D$ that $(a, b, c) \neq 1$. Let us examine the CML $Q=<$ $D, x>$, where the element $x$ does not belong to $D$ and given by all the identity relations $(a, u, v)=(x, u, v)$ for any $u, v \in D$. Obviously, $D$ is a subloop of the CML $Q$, then it serves as a direct factor. Therefore the element $x$ associates with any two elements of the subloop $D$, in particular, $(x, b, c)=1$. But $(x, b, c)=(a, b, c) \neq 1$. Contradiction. A consequently, the CML $D$ is associative. By [6] any abelian group can be embedded as a subgroup into a divisible group. Therefore the CML $D$ is divisible. This completes the proof of Proposition 1.14.

## 2 Finitely cogenerated commutative Moufang loops

A subset $H$ of the CML $Q$ is called self-conjugate if $I(Q) H=H$, where $I(Q)$ is the inner mapping group of the CML $Q$. A self-conjugate set $L$ of elements of the loop $Q$ will be called a normal system of cogenerators if any homomorphism $\varphi: Q \rightarrow H$ for which $L \cap \operatorname{ker} \varphi \neq \emptyset$ or $\{1\}$ is a monomorphism, for any loop $H$. Obviously it is equivalent to the fact that any non-unitary normal subloop of the loop $Q$ contains an non-unitary element from $L$.

A loop $Q$ will be called finitely cogenerated if it possesses a finite normal system of cogenerators.
Theorem 2.1. The following conditions are equivalent for an arbitrary commutative Moufang loop $Q$ :

1) $Q$ is a finitely cogenerated loop;
2) the loop $Q$ possesses a finite normal subloop $B$ such that $B \cap H \neq\{1\}$, for any normal subloop $H$ of the loop $Q$;
3) the loop $Q$ is a direct product of a finite number of quasicyclic groups that lie in the centre $Z(Q)$ of the loop $Q$ and a finite loop;
4) the loop $Q$ satisfies the minimum conditions for subloops;
5) the loop $Q$ possesses a finite series of normal subloops any factor of which is either a group of a simple order, or a quasicyclic group.
Proof. 1) $\longrightarrow 2$ ). Let $L$ be a finite normal system of cogenerators of the CML $Q$ and $a \in Q$ be an element of an infinite order. By Lemma 1.2 the subloop $<a^{3^{n}}>$ is normal in the CML $Q$. The intersection $<a^{3^{n}}>\cap L$ is either null, or equal to $\{1\}$ for a certain large $n$, that contradicts the condition 1). Therefore there are no elements of an infinite order in the CML $Q$. Then, by Lemma 1.3, the subloop
$<L>$ is finite. The system of cogenerators is self-conjugate in the CML $Q$, then the subloop $<L>$ is normal in $Q$, as the inner mappings are automorphisms in the CML [4]. Consequently, the condition 2) holds in the CML $Q$.
$2) \longrightarrow 3$ ). It can be shown that the CML $Q$ is periodic, as it was done when proving the implication 1$) \longrightarrow 2$ ). Then, by Lemma 1.4 , it decomposes into a direct product of its maximal $p$-subgroups, therefore $Q$ contains a finite number of such $p$-subloops. In order to prove 3 ) we can suppose that $Q, B$ are 3 -loops.

Like in abelian groups [6] the non-negative number $n$ for which the equality $x^{3^{n}}=a$ has solutions in $Q$ will be called the 3-height $h(a)$ of the element $a$. If the equality $x^{3^{n}}=a$ has solutions for any $n$, then $a$ will be called the infinite 3 -height, $h(a)=\infty$.

We denote $Q[3]=\left\{x \in Q \mid x^{3}=1\right\}$ and let $a \in Q[3]$. Then $<\varphi a \mid \varphi \in I(Q)>$ is the minimal normal subloop containing the element $a$, where $I(Q)$ is the inner mapping group of the CML $Q$. By the condition 2) $a \in B$, and then $Q[3]$ will be a finite subloop. It follows from here that the equality $x^{3}=a$ can have not more than a finite number of solutions in CML $Q$, for a fixed element $a \in Q$. If $h(a)=\infty$, then the solutions $x_{1}, \ldots, x_{k}$ cannot have all finite heights, as if the equality $y^{3^{n}}=a$ holds for the element $y \in Q$, then $y^{3^{n-1}}$ is one of the elements $x_{1}, \ldots, x_{k}$.

Let now $a_{1} \in Q[3], h\left(a_{1}\right)=\infty$. We denote the solution of an infinite height of the equality $a_{1}=x^{3}$ by $a_{2}$, the solution of an infinite height of the equality $a_{2}=x^{3}$ by $a_{3}$ and so on. Consequently, we have constructed a quasicyclic group which lies in the centre of the CML $Q$, by Lemma 1.10 , i.e. it is normal in $Q$. The union $D$ of all quasicyclic groups of the CML $Q$ is a divisible group, therefore by Proposition $1.12 Q=D \times C, D \subseteq Z(Q)$. The subloop $C$ has no element of an infinite height, as if an element $a \in Q$ of the order $3^{n}(n \geq 1)$ has an infinite height, then $a^{3^{n-1}}=a^{3^{-1}}, a^{3^{-1}} \in Q[3]$ and the element $a^{3^{-1}}$ has an infinite height. We have shown that $C[3]$ is a finite subloop. If $a \in C[3], a^{3^{n}}=1, a=x^{3^{m}}$, then $x^{3^{n+m}}=1, x^{3^{m+n-1}}=x^{3^{-1}}, x^{3^{-1}} \in C[3]$, therefore there is an maximum of heights $k$ of the elements of subloops $C[3]$. But then $(C[3])^{k+1}=1$, but by Lemma 1.3 the subloop $C$ is finite. The finiteness of the quasicyclic groups of the CML $Q$ number follows from the finiteness of the subloop $D[3]$.
$3) \longrightarrow 4$ ). This statement follows from that the fact the quasicyclic groups and the direct product of their finite number satisfy the minimum conditions for subgroups.
4) $\longrightarrow 1$ ). The CML $Q$ has no elements of an infinite order, as if a is such an element, then $<a^{3^{n}}>(n=1,2, \ldots)$ is a strictly descending series of the subloops of the CML $Q$. Then, by Lemma 1.4, $Q$ decomposes into the direct product of a finite number of maximal $p$-subloops $Q_{p}$. The subloop $Q_{p}[p]$ is normal in $Q$ and it cannot be infinite. In such a case the subloop $\prod_{p} Q_{p}[p]$ will be a finite normal system of cogenerators.

The implication 3) $\longrightarrow 5$ ) follows from Lemma 1.8.
In order to prove the implication 5) $\longrightarrow 3$ ) we should first show that if $Q$ has a finite normal subloop $H$ such that the quotient loop $Q / H$ is a quasicyclic group, then $Q$ has a quasicyclic group of an finite index. First we suppose that the subloop $H$
is associative. By the definition of the quasicyclic group of the CML $Q$ is generated by the set $\left\{a_{0} H, \ldots, a_{i} H, \ldots,\right\}$, where $a_{i+1}^{p} H=a_{i} H, a_{0} \in H, i=1,2, \ldots$ We will show that $a_{i} \in Z_{Q}(H)$ is the centralizer of the subloop $H$ in $Q$. If $p=3$, then if follows from the equality $a_{i+1}^{3} h=a_{i}$, where $h \in H$, for $h_{1}, h_{2} \in H$ from (1.3)- (1.5), that $\left(a_{i}, h_{1}, h_{2}\right)=\left(a_{i+1}^{3} h, h_{1}, h_{2}\right)=1$, i.e. $a_{i} \in Z_{Q}(H)$. If $p \neq 3$, then by (1.3), (1.4) $\left(u^{p}, v, w\right)=(u, v, w)^{ \pm 1}$. Then we have $\left(a_{1}, h_{1}, h_{2}\right)=\left(a_{1}^{p}, h_{1}, h_{2}\right)^{ \pm 1}=\left(h, h_{1}, h_{2}\right)^{ \pm 1}=$ 1 from the relations $a_{1}^{p}=h \in H$. Further, if $a_{i} \in Z_{Q}(H)$ and $a_{i+1}^{p}=a_{i} h$, then $\left(a_{i+1}, h_{1}, h_{2}\right)=\left(a_{i+1}^{p}, h_{1}, h_{2}\right)^{ \pm 1}=\left(a_{i} h, h_{1}, h_{2}\right)=1$ by (1.5), i.e. $a_{i+1} \in Z_{Q}(H)$. Therefore $Q=H Z_{Q}(H)$. As the intersection $H \cap Z_{Q}(H)$ is contained in the centre of the CML $Z_{Q}(H)$, and the quotient loop $Z_{Q}(H) /\left(Z_{Q}(H) \cap H\right)$ is isomorphic to the quasicyclic group $Q / H=Z_{Q}(H) H / H$ the CML $Z_{Q}(H)$ is an infinite abelian group, and it satisfies the minimum condition for subgroups. Then it contains a quasicyclic group of finite index [6]. But by the relation $Q=H Z_{Q}(H)$, the latter has a finite index in the CML $Q$.

Let now $H$ be an arbitrary subloop. It is finite, then by Lemma 1.8 its upper central series has the form $1=Z_{0} \subset Z_{1} \subset \ldots \subset Z_{n-1} \subset Z_{n}=H$, where $Z_{i} / Z_{i-1}=Z\left(H / Z_{i-1}\right)$ or $Z_{i}=\left\{a \in H \mid\left(a, h_{1}, \ldots, h_{2 i}=1 \forall h_{1}, \ldots, h_{2 i}\right) \in H\right\}$. (Here $\left.\left(u_{1}, \ldots, u_{2 i-1}, u_{2 i}, u_{2 i+1}\right)=\left(\left(u_{1}, \ldots, u_{2 i-1}\right), u_{2 i}, u_{2 i+1}\right)\right)$. The inner mappings are automorphisms in CML [4], then it follows from the last equality that the subloop $Z_{i}$ is normal in $Q$, as the subloop $H$ is normal in $Q$. Further, if follows from the relations

$$
Q / H \cong\left(Q / Z_{n-1}\right) /\left(H / Z_{n-1}\right)=\left(Q / Z_{n-1}\right) /\left(Z_{n} / Z_{n-1}\right)=\left(Q / Z_{n-1}\right) / Z\left(H / Z_{n-1}\right)
$$

and according to the previous case that the CML $Q / Z_{n-1}$ ) contains a quasicyclic group of finite index. Without loss of generalitiy, we will consider that $Q / Z_{n-1}$ is a quasicyclic group, by Proposition 1.14. Let us now suppose that $Q / Z_{i}(i \leq n-1)$ is a quasicyclic group. Then it follows from the relations $Q / Z_{i} \cong\left(Q / Z_{i-1}\right) /\left(Z_{i} / Z_{i-1}\right)=$ $\left(Q / Z_{i-1}\right) / Z\left(Q / Z_{i-1}\right)$ that $Q / Z_{i-1}$ is a quasicyclic group. We obtain for $i=1$ that $Q$ contains a quasicyclic group of finite index.

It is obvious, that the implication 5) $\longrightarrow 3$ ) should be proved supposing that the CML $Q$ contains a series of normal subloops

$$
\begin{equation*}
1=H_{0} \subset H_{1} \subset \ldots \subset H_{m}=Q \tag{2.1}
\end{equation*}
$$

with $m \geq 2$ that have infinite factors and all are quasicyclic groups.
Let us show that the series (2.1) contains a member which has a quasicyclic group of finite index. If the subloop $H_{1}$ is infinite, then the statement is obvious. But if it is finite, then let $H_{k}$ be such a finite member of the series (2.1) that the next number $H_{k+1}$ is infinite. Then $H_{k+1}$ contains a quasicyclic group $L_{k+1}$ of finite index. If all factors of the series (2.1) which are after the factor $H_{k+1} / H_{k}$ are finite, then $L_{k+1}$ has a finite index in $Q$ and by Proposition 1.14 the statement 3) holds in the CML $Q$.

Let $H_{n+1} / H_{n}$ be the first infinite factor among those that are after $H_{k+1} / H_{k}$. By Lemma 1.10 the subloop $L_{k+1}$ is normal in $Q$. There exists a finite normal subloop
$H_{n} / L_{k+1}$ in the CML $H_{n+1} / L_{k+1}$ on which the quotient loop is a quasicyclic group. By the above proved, the CML $H_{n+1} / L_{k+1}$ contains a quasicyclic group $L_{n+1} / L_{k+1}$ of finite index. In the CML the quasicyclic groups lie in the centre (Lemma 1.10), then $L_{n+1}$ is a product of two quasicyclic groups. Continuing these reasonings, after a finite number of steps we will obviously obtain that the CML $Q$ contains a subloop that is the direct product of a finite number of quasicyclic groups of finite index. Then the CML $Q$ satisfies the condition 3). This completes the proof of Theorem 2.1.

Corollary 2.2. The commutative Moufang loops satisfying the minimum condition for subloops, compose a class closed in regard to the extension.

The statement follows from the equivalence of the conditions 4) and 5) of Theorem 2.1.

Corollary 2.3. The commutative Moufang loops, satisfying the minimum condition for subloop, are centrally nilpotent.

The statement follows from the equivalence of the conditions 3), 4) of the Theorem 2.1 and Lemma 1.8.
Corollary 2.4. The set of elements of any order is finite in the commutative Moufang loop satisfying the minimum condition for subloops.

## 3 The multiplicative groups of commutative Moufang loops with minimum condition for subloops

Let $Q$ be an arbitrary CML and let $H$ be a subset of the set $Q$. Let $\mathbf{M}(H)$ denote a subgroup of the multiplicative group $\mathfrak{M}(Q)$ of the CML $Q$, generated by the set $\{L(x) \mid \forall x \in H\}$. Takes place
Lemma 3.1. Let the commutative Moufang loop $Q$ with the multiplicative group $\mathfrak{M}, Z(\mathfrak{M})$, which is the centre of the group $\mathfrak{M}$ and the centre $Z(Q)$ decompose into the direct product $Q=D \times H$, moreover, $D \subseteq Z(Q)$. Then $\mathfrak{M}=\boldsymbol{M}(D) \times \boldsymbol{M}(H)$, and besides, $\boldsymbol{M}(D) \subseteq Z(\mathfrak{M}), \boldsymbol{M}(D) \cong D$.
Proof. It is obvious that any element $a \in Q$ has the form $a=d h$, where $d \in D, h \in$ $H$. As $d \in Z(Q)$, then $L(a)=L(d) L(h)$, therefore $\mathfrak{M}=<\mathbf{M}(D), \mathbf{M}(H)>$. It follows from the equality

$$
Z(\mathfrak{M})=\{\varphi \in \mathfrak{M} \mid \varphi=L(a) \forall a \in Z(Q)\}
$$

that $\mathbf{M}(D) \subseteq Z(\mathfrak{M})$, therefore it is easy to see that the subgroups $\mathbf{M}(D), \mathbf{M}(H)$ are normal in $\mathfrak{M}$ and $\mathbf{M}(D) \cong D$. Finally, if $\varphi \in \mathbf{M}(D) \cap \mathbf{M}(H)$, then $\varphi=L(u), L(u) 1 \in$ $D \cap H, \varphi$ is an inner mapping. Consequently, $\mathfrak{M}=\mathbf{D} \times \mathbf{H}$, as required.
Corollary 3.2. The multiplicative group $\mathfrak{M}$ of the periodic commutative Moufang loop $Q$ decomposes into the direct product of its maximal p-subgroups $\mathfrak{M}_{3}$, moreover, $\mathfrak{M}_{p} \subseteq Z(\mathfrak{M})$ for $p \neq 3$.
Proof. By Lemma 1.4 the CML $Q$ decomposes into the direct product of its maximal $p$-subgroups, moreover, $Q_{p} \subseteq Z(Q)$ for $p \neq 3$. Then it follows from lemma 3.1 that
the group $\mathfrak{M}$ decomposes into a direct product of the subgroups $\mathbf{M}\left(Q_{p}\right)$, moreover, $\mathbf{M}\left(Q_{p}\right) \subseteq Z(\mathfrak{M})$ and $\mathbf{M}\left(Q_{p}\right) \cong Q_{p}$ for $p \neq 3$. In order to finish the proof, it should be shown that $\mathbf{M}\left(Q_{p}\right)$ is a 3 -group. But this is shown in the next lemma.
Lemma 3.3. The multiplicative group $\mathfrak{M}$ of the commutative Moufang 3-loop $Q$ is a 3-group.
Proof. Let $\gamma$ be an arbitrary element from $\mathfrak{M}$. Then $\gamma$ can be presented as a product of a finite number of translation $\gamma=L\left(u_{1}\right) L\left(u_{2}\right) \ldots L\left(u_{n}\right)$, where $u_{1}, u_{2}, \ldots, u_{n} \in Q$. We denote $L=<u_{1}, u_{2}, \ldots, u_{n}>$. For any element $x \in Q$ we denote by $H(x)$ the subloop of CML $Q$, generated by set $x \cup L$, by $\mathfrak{N}(x)$ - the multiplicative group of CML $H(x)$, and by $\Gamma$ - the subgroup of group $\mathfrak{M}$ generated by the translations $L\left(u_{i}\right), i=1, \ldots, n$. By Lemmas 1.8 and $1.3 H(x)$ is a finite centrally nilpotent 3-loop. Let us show that $\mathfrak{N}(x)$ is a 3-loop. Indeed, we denote $H(x)=G$. By Lemma 1.3, Chap. IV from [4] $\mathfrak{M}(Z / Z(G)) \cong \mathfrak{M}(G) / Z^{*}$, where $Z^{*}=\{\alpha \in \mathfrak{M}(G) \mid \alpha x \cdot Z(G)=$ $x \cdot Z(G) \forall x \in G\}$. If $\theta \in Z^{*}$, then we define the function $f: G \longrightarrow Z(G)$ by the rule $\theta x=x f(x)$ for $\forall x \in G$. Obviously, $f(x) \in Z(G)$. If $\eta \in Z^{*}$ and $\eta x=x g(x)$, then $(\theta \eta) x=\theta(L(g(x)) x)=L(g(x)) \theta x=(g(x) f(x)) x$. Consequently, $Z^{*}$ is isomorphic to the group of one-to-one mappings of CML $Q$ on $Z(G)$. Therefore $Z^{*}$ is a 3-group. If CML $G$ is centrally nilpotent of the class $k$, then $G / Z(G)$ is centrally nilpotent of class $k-1$. Then by inductive assumption $\mathfrak{M}(G) / Z^{*}$ is a 3 -group, therefore $\mathfrak{M}(G)$ is also 3 -group.

The restriction $\Gamma$ on $H(x)$ is a homomorphism of $\Gamma$ on the subgroup of the group $\mathfrak{N}(x)$ which maps the element $\gamma \in \Gamma$ into the element $L\left(u_{1}\right) \ldots L\left(u_{n}\right)$ from $\mathfrak{N}(x)$ of the order $3^{t}$. Moreover, $\Gamma$ maps $H(x)$ into itself. Consequently, $\gamma^{3^{t}}$ induces an identity mapping on $H(x)$. In particular, $\gamma^{3^{t}}$ maps $x$ into itself for any $x$ from $Q$. Therefore $\gamma$ has the order $3^{t}$. This completes the proof of Lemma 3.3.
Lemma 3.4. The multiplicative group $\mathfrak{M}$ of an arbitrary commutative Moufang loop is locally nilpotent. But if group $\mathfrak{M}$ is periodic, then it is locally finite.

The proof of the first statement follows from Lemma 1.1. The second statement follows from the well-known fact of the group theory: a periodic locally nilpotent group is locally finite.

Now we can characterize CML, with the minimum conditions for subloops with the help of their multiplicative groups.
Theorem 3.5. For an arbitrary non-associative commutative Moufang loop $Q$ with a multiplicative group $\mathfrak{M}$ the following conditions are equivalent:

1) loop $Q$ satisfies the minimum condition for subloops;
2) group $\mathfrak{M}$ is a product of a finite number of quasicyclic groups lying in the centre of the group $\mathfrak{M}$ and a finite group;
3) group $\mathfrak{M}$ satisfies the minimum condition for subgroup;
4) group $\mathfrak{M}$ satisfies the minimum condition for normal subgroup;
5) group $\mathfrak{M}$ satisfies the minimum condition for non-abelian subgroup;
6) at least one maximal abelian subgroup of the group $\mathfrak{M}$ satisfies the minimum conditions for subgroups;
7) if group $\mathfrak{M}$ contains a solvable subgroup of the class $r$, then $\mathfrak{M}$ satisfies the minimum condition for solvable subgroups of the class $r$;
8) if group $\mathfrak{M}$ contains a nilpotent subgroup of the class $n$, then $\mathfrak{M}$ satisfies the minimum condition for nilpotent subgroups of the class $n$.
Proof. 1) $\longrightarrow 2$ ). If CML $Q$ satisfies the minimum condition for subloops, then by Theorem $2.1 Q=D \times H$, where $H$ is the direct product of a finite number of quasicyclic groups, besides, $D \subseteq Z(Q)$, and $H$ is a finite CML. Then by Lemma $3.1 \mathfrak{M}=\mathbf{M}(D) \times \mathbf{M}(H)$, and besides $\mathbf{D} \subseteq Z(\mathfrak{M}), \mathbf{M}(D) \cong D$. The group $\mathbf{M}(H)$ is finitely generated, then by Lemma $3.3 H$ is finite, as it follows from Corollary 3.2 that a multiplicative group of a periodic CML is periodic.

The implication 2$) \longrightarrow 3$ ) is obvious. Let now the group $\mathfrak{M}$ satisfy the condition $3)$, and the CML $Q$ do not satisfy the condition 1 ), and let $Q \supset H_{1} \supset H_{2} \supset$ $\ldots \supset H_{i} \supset \ldots$ be an infinite descending series of subloops of the CML $Q$. It is easy to see that $\mathbf{M}\left(H_{i}\right) \neq \mathbf{M}\left(H_{i+1}\right)$ follows from $H_{i} \neq H_{i+1}$, using the relation $\mathbf{M}\left(H_{i}\right) 1=H_{i}$, where $\mathbf{M}\left(H_{i}\right) 1=\left\{\alpha 1 \mid \alpha \in \mathbf{M}\left(H_{i}\right)\right\}$. But it contradicts the condition 3). Consequently, 3) $\longrightarrow 1$ ).

By Lemma 3.4 the group $\mathfrak{M}$ is locally nilpotent, then the implications 3) $\longleftrightarrow$ $4), 3) \longleftrightarrow 5$ ) follow, respectively, from Theorems 1.24 and Corollary 6.2 from [7].
$6) \longrightarrow 3)$. Let the maximal abelian subgroup $\mathfrak{N}$ of the group $\mathfrak{M}$ satisfy the minimum condition for subgroups. By Lemma 1.1 the quotient group $\mathfrak{M} / Z(\mathfrak{M})$ is a 3 -group, therefore by the periodicity of $\mathfrak{N}$, the group $\mathfrak{M}$ is also periodic. Thereof, and in view of Corollary 3.2 , we will consider $\mathfrak{M}$ a 3 -group. By Lemma 3.4 the group $\mathfrak{M}$ is locally nilpotent. Then the condition 6$) \longrightarrow 3$ ) follows from the statement that is proved using Lemma 1.6, analogous to Theorem 1.19 from [7]:
if at least one maximal abelian subgroup of the locally nilpotent p-group satisfies the minimum condition for subgroup, then the group satisfies this condition itself.

By Lemma 3.4 the group $\mathfrak{M}$ is locally nilpotent. It is proved in [8] that for such groups the conditions 3 ), 7 ), 8) are equivalent.

Finally, the implication 3$) \longrightarrow 6$ ) is obvious. This completes the proof of Theorem 3.5.

It is proved in [7] that if the locally finite $p$-group has a finite maximal elementary abelian subgroup (respect., a finite set of elements of any order different from unitary element), then it satisfces the minimum condition for subgroups (Theorem 1.21 (respect., Theorem 3.2)). Then from Lemmas 3.3, 3.4 and Theorem 3.6 follows the truth of the following statement.
Proposition 3.6. The following conditions are equivalent for an arbitrary commutative Moufang 3-loop with a multiplicative group $\mathfrak{M}$ :

1) the loop $Q$ satisfies the minimum condition for subloops;
2) the group $\mathfrak{M}$ contains only a finite set of elements of a certain order different from the unitary element.

Finally, let us prove the statement.
Proposition 3.7. The following conditions are equivalent for an arbitrary nonassociative commutative Moufang $Z$ A-loop $Q$ with a multiplicative group $\mathfrak{M}$ :

1) the loop $Q$ satisfies the minimum condition for subloops;
2) the group $\mathfrak{M}$ satisfies the minimum condition for noninvariant abelian subgroups.
Proof. Let us first observe that from Lemma 11.4, Chap. VIII from [4] it follows that CML $Q$ is a $Z A$-loop if and only if its multiplicative group is a $Z A$-group.

Let us suppose that the group $\mathfrak{M}$ satisfies the minimum condition for noninvariant abelian subgroups. It follows from the above-mentioned that it is a $Z A$-group. If $\mathfrak{M}$ does not contain noninvariant abelian subgroups, then, obviously, each subgroup is normal in it, i.e. it is hamiltonian. However, it is impossible that the multiplicative group of an arbitrary CML cannot contain a nonabelian gamiltonian subgroup. Indeed, arbitrary hamiltonian groups are described by the next theorem [7]:

A hamiltonian group can be decomposed into a direct product of the group of quaternions and abelian groups whose each element's order is not greater than 2. Conversely, a group that has such a decomposition is hamiltonian.

A group of quaternions is the group generated by the generators $a, b$ and that satisfies the identical relations $a^{4}=1, a^{2}=b^{2}, b^{-1} a b=a^{-1}$. Then it follows from Corollary 3.2 that in the case of a multiplicative group $a=b=1$. Consequently, the arbitrary hamiltonian group of the multiplicative group of CML is abelian.

Let now $\mathfrak{N}$ be a noninvariant abelian subgroup of the group $\mathfrak{M}$ and $\alpha$ be an element of infinite order from $\mathfrak{M}$. By Lemma 1.1 the quotient group $\mathfrak{M} / Z(\mathfrak{M})$ is a 3 -group, therefore $\alpha^{3^{k}} \in Z(\mathfrak{M})$ for a certain natural number $k$. This means that the descending series of noninvariant associative subgroups

$$
<\mathfrak{N}, \alpha^{3^{k}}>\supset<\mathfrak{N}, \alpha^{3^{k+1}}>\supset \ldots \supset<\mathfrak{N}, \alpha^{3^{k+i}}>\supset \ldots
$$

of the group $\mathfrak{M}$ does not break. But it contradicts the condition 5). Consequently, the group $\mathfrak{M}$ is periodic. In such a case, we will consider by Corollary 3.2 that $\mathfrak{M}$ is a 3-group.

Let us suppose that the group $\mathfrak{M}$ does not satisfy the minimum condition for subgroups. Then, by Lemma 3.4 and Theorem 1.21 from [7] the group $\mathfrak{M}$ contains the infinite direct product

$$
\mathfrak{N}=\mathfrak{N}_{1} \times \mathfrak{N}_{2} \times \ldots \times \mathfrak{N}_{n} \times \ldots
$$

of cyclic groups of the order three. If $\alpha$ is an arbitrary element from the centralizer $Z_{\mathfrak{M}}(\mathfrak{N})$ of the subgroup $\mathfrak{N}$ in $\mathfrak{M}$, then there exists such a number $n=n(\alpha)$ that

$$
<\alpha>\cap\left(\mathfrak{N}_{n+1} \times \mathfrak{N}_{n+2} \times \ldots\right)=1
$$

As the group $\mathfrak{M}$ satisfies the minimum condition for noninvariant abelian subgroups, the infinite descending series of abelian subgroups

$$
\Re^{k}(\alpha) \supset \Re^{k+1}(\alpha) \supset \ldots,
$$

where $\Re^{k}(\alpha)=<\alpha>\left(\mathfrak{N}_{k+1} \times \mathfrak{N}_{k+1} \times \ldots\right)$, contains an noninvariant subgroup $\Re^{k}(\alpha)$ $(r=r(\alpha))$, beginning with a certain natural number $k \geq n$. As the intersection of
all such noninvariant subgroups coincides with the subgroup $<\alpha>$, the latter is normal in $\mathfrak{M}$. But $\alpha$ is an arbitrary element from the centralizer $Z_{\mathfrak{M}}(\mathfrak{N})$, and it means that $Z_{\mathfrak{M}}(\mathfrak{N})$ is a hamiltonian group. From here follows that $Z_{\mathfrak{M}}(\mathfrak{N})$ is an abelian group. Obviously, $\mathfrak{N}_{i} \subseteq Z_{\mathfrak{M}}(\mathfrak{N})$, then the minimal subgroup $\mathfrak{N}_{i}$ is normal in $\mathfrak{M}$. By Proposition 1. 6 from [7], in a $Z A$-group the minimal normal subgroups are contained in its centre. Then $\mathfrak{N}_{i} \subseteq Z_{\mathfrak{M}}(\mathfrak{N})$, therefore $Z_{\mathfrak{M}}(\mathfrak{N})=\mathfrak{M}$. As $Z_{\mathfrak{M}}(\mathfrak{N})$ is an ablian group, the last equality contradicts the fact that $\mathfrak{M}$ is an noninvariant group. Consequently, the group $\mathfrak{M}$ satisfies the minimum condition for subgroups. Then the equivalence of the conditions 1) and 2) follows from the Theorem 3.5.

## 4 The commutative Moufang loops with the minimum condition for normal subloops

If it does not cause any misunderstandings, we will further omit the words "for subloops" in the expression "minimum condition for subloops".
Lemma 4.1. Let the series

$$
\begin{equation*}
1=Z_{0} \subset Z_{1} \subset \ldots \subset Z_{\alpha} \subset \ldots \subset Z_{\beta} \subset \ldots \subset Z_{\gamma}=Q \tag{4.1}
\end{equation*}
$$

be the upper central series of the commutative Moufang $Z A$-loop $Q, H$ be its arbitrary normal subloop. Then the non-emptiness of the intersection $H \cap\left(Z_{\beta} \backslash Z_{\alpha}\right.$ follows from the non-emptiness of the intersection $H \cap\left(Z_{\beta+1} \backslash Z_{\beta}\right.$ for any $\beta>\alpha$.
Proof. Let $h \in H \cap\left(Z_{\beta+1} \backslash Z_{\beta}\right)$. The existence of such elements $a, b \in Q$ that $(h, a, b) \in H \cap\left(Z_{\beta} \backslash Z_{\alpha}\right)$ follows from the normality of the subloop $H$ and the definition of the members of series (4.1). Indeed, if $(h, a, b) \in Z_{\alpha}$ for all $a, b \in Q$, then $h \in Z_{\alpha+1} \subset Z_{\beta}$. So, $h \notin Z_{\beta+1} \backslash Z_{\beta}$, and it contradicts the choice of the element $n$. This completes the proof of Lemma 4.1.
Lemma 4.2. Let the commutative Moufang $Z A$-loop $Q$ be the finite extension of the loop $Q$ satisfies the minimum condition if and only if the centre $Z(H)$ of the loop $H$ also satisfies this condition.
Proof. Let us suppose that the centre $Z(Q)$ satisfies the minimum condition for subloops, and let $a_{1}, \ldots, a_{n}$ be representations of cosets of $Q$ modulo $H$, taken by one from each coset. We denote $L=<Z(H), a_{1}, \ldots, a_{n}>$. Let us show that the centre $Z(L)$ of the CML $L$ satisfies the minimum condition. Indeed, the intersection $Z(L) \cap Z(H)$ is contained into $Z(Q)$, therefore it is a group with minimum condition. Obviously, the index $Z(H)$ in $<Z(H), Z(L)>$ is finite. We have

$$
<Z(H), Z(L)>/ Z(H) \cong Z(L) /(Z(L) \cap Z(H))
$$

It follows from this relation that $Z(L)$ which is a finite extension of the group $Z(H) \cap Z(L)$ satisfies the minimum condition, satisfies this condition itself.

Let $N_{k}$ be a subgroup of the group $Z(L)$ generated by all its elements whose orders are divisible by $p^{k}$. The group $N_{k}$ is finite, as the group $Z(L)$ satisfies the minimum condition. We denote by $Z_{k}$ the subgroup of the group $Z(H)$ generated
by all its elements whose orders are divisors of $p^{k}$. If $Z(H)$ does not satisfy the minimum condition, then $Z_{k}$ should be infinite. Let

$$
Z_{k}=Z_{k}^{(1)} \times \ldots \times Z_{k}^{(m)} \times Z_{k}^{(m+1)} \times \ldots
$$

be the decomposition of the group $Z_{k}$ into an infinite direct product of cyclic groups. If the intersection $Z_{k} \cap N_{k}$ is contained into the finite direct product $Z_{k}^{(1)} \times \ldots \times Z_{k}^{(m)}$, then the intersection of the groups $M_{k}=Z_{k}^{(m+1)} \ldots$ and $Z(L)$ should contain only the unitary element:

$$
\begin{equation*}
M_{k} \cap Z(L)=1 \tag{4.2}
\end{equation*}
$$

By Lemma 1.1 the subgroup $\Phi$ of the inner mapping group of the CML $Q$ generated by all the mappings of the form $L\left(a_{i}, a_{j}\right), i, j=1, \ldots, n$, is finite. The subgroups $\varphi M_{k}, \varphi \in \Phi$, is a (finite) the set of all conjugated subloops with $M_{k}$ in the CML $Q$, because the elements $a_{1}, \ldots, a_{n}$ present a full system of representations of cosets of CML $Q$ modulo $H$, and $M_{k} \subseteq Z(H)$. The intersection

$$
R_{k}=\cap_{\varphi \in \Phi} \varphi M_{k}
$$

is obviously an infinite normal subloop in $Q$. We remind that $R_{k} \subseteq L$, as $M_{k} \subseteq Z(L), \varphi M_{k} \subseteq L$. By Lemma 1.8 and Lemma $1.6 R_{k} \cap Z(L) \neq 1$, that contradicts (4.2). Consequently, the assumption that $Z(H)$ does not satisfy the minimum condition is not true.

Conversely, let $Z(Q)$ does not satisfy the minimum condition. As $H$ has a finite index in $Q$, then it follows from the relation

$$
Z(Q) H / H \cong Z(Q) /(Z(Q) \cap H)
$$

that $Z(Q) \cap H$ has a finite index in $Z(Q)$. Consequently, $Z(Q) \cap H$ does not satisfy the minimum condition. But $Z(Q) \cap H \subseteq Z(H)$, therefore $Z(H)$ does not satisfy the minimum condition as well. This completes the proof of Lemma 4.2.
Lemma 4.3. If the commutative Moufang $Z A$-loop, which is a finite extension of the loop $H$, possesses a normal subloop $K$, which lies in the centre $Z(H)$ of the loop $H$ and does not satisfy the minimum condition, then the intersection of $H$ with the centre $Z(Q)$ of the loop $Q$ does not satisfy the minimum condition as well.
Proof. By Lemma 1.4 we'll consider that the CML $Q$ is a 3 -loop. We denote by $L$ the lower layer of the abelian group $K$. As $K$ does not satisfy the minimum condition, $L$ is infinite.

Let us first examine the case when the quotient loop $Q / H$ is associative. Let

$$
1=g_{1}, g_{2}, \ldots, g_{n}
$$

be a full system of representations of cosets of CML $Q$ modulo $H$. We suppose by inductive considerations that the intersection of $L_{i-1}$ of the centre of the CML $<H, g_{1}, \ldots, g_{i-1}>$ with the subloop $L$ is infinite. As the quotient loop $Q / H$ is
associative, the subloop $<H, g_{1}, \ldots, g_{i-1}>$ inverse image of a normal subloop under the homomorphism $Q \longrightarrow Q / H$, is normal in $Q$. The subloop $L$ is invariant in regard to all automorphisms of the normal subloops $H$ of the CML $Q$. In the CML the inner mappings are its automorphisms [4]. Then the subloop $L$ is invariant in regard to the inner mapping group of the CML $Q$, i.e., it is normal in $Q$. Therefore the intersection $L_{i-1}$ is also a normal subloop in $Q$. Let us examine the CML $<L_{i-1}, g_{i}>$. By Lemma 4.2 this loop's center does not satisfy the minimum condition. Consequently, if the order of the element $g_{i}$ is $3^{k}$, then there exists such a number $r \leq 3^{k}$ that for the infinite set of elements $P$ of the order 3 from the CML $L_{i-1}$, the elements of the form $p g_{i}^{r}, p \in P$, belong to the centre of the CML $<L_{i-1}, g_{i}>$. Now, with the help of (1.1) we obtain for $p, q \in P$

$$
\begin{aligned}
g_{i}\left(g_{i}^{r} p \cdot g_{i}^{r} q\right) & =\left(g_{i} \cdot g_{i}^{r} p\right)\left(g_{i}^{r} q\right), \\
g_{i}\left(g_{i}^{2 r} \cdot p q\right) & =\left(g_{i}^{r} \cdot g_{i} p\right)\left(g_{i}^{r} q\right) . \\
g_{i}^{2 r}\left(g_{i} \cdot p q\right) & =g_{i}^{2 r}\left(g_{i} p \cdot q\right), \\
g_{i} \cdot p q & =g_{i} p \cdot q .
\end{aligned}
$$

The last equality shows that the infinite CML $P_{i}=<P>$ of the index three belongs to the centre of the CML $<H, g_{1}, \ldots, g_{i-1}>$. As $L_{i-1}$ belongs to the centre $<H, g_{1}, \ldots, g_{i-1}>$ and $P_{i} \subseteq P_{i-1}$, the CML $P_{i}$ belongs to the centre $<H, g_{1}, \ldots, g_{i-1}, g_{i}>$. So, the intersection of this CML's centre with $L_{i-1}$ is infinite, therefore it does not satisfy the minimum condition. But $L_{i-1} \subseteq L_{i} \subseteq H$, then the statement is proved in this case.

Let now $Q / H$ be an arbitrary finite CML and by Lemma 1.8 let

$$
\overline{1} \subset Z_{1} / H \subset \ldots \subset Z_{k} / H=\bar{Q}
$$

be the upper central series of the CML $Q / H$. By the first case, the intersection of the centre of the CML $Z_{1}$ with the subloop $L$ is infinite. As it has already been proved that the intersection of the centre of the CML $Z_{i}$ with the subloop $L$ is infinite, then applying the first case's results to the CML $Z_{i}$ and $Z_{i+1}$ we obtain that the intersection of the centre of the CML $Z_{i+1}$ with the subloop $L$ is also infinite. For $i+1=k$ follows the lemma's statement.

Lemma 4.4. If the periodic commutative Moufang ZA-loop contains an associative normal subloop $H$ that does not satisfy the minimum condition, then the latter contains a normal subloop of the loop $Q$ different from itself that does not satisfy the minimum condition as well.
Poof. By Lemma 1.4 we will consider that $Q$ is a 3 -loop. We denote by $L$ the lower layer of the group $H$. As $H$ does not satisfy the minimum condition, $L$ is infinite. Let

$$
1 \subset Z_{1} \subset \ldots \subset Z_{\gamma}=Q
$$

be the upper central series of the CML $Q$. If $L \subseteq Z_{1}$, then the lemma is proved.
Let us suppose that $L$ does not belong to $Z_{1}$. The product $L Z_{1}=Q_{1}$ does not satisfy the minimum condition. The subloop $L$ is contained in the centre of the CML $Q_{1}$ ( $Q_{1}$ is associative). By the assumption $L$ does not belong to the centre of the CML $Q$, so, there exists such an ordinal number $\alpha$ less than $\gamma$ that the centre of the CML $Z_{\alpha} L=Q_{\alpha+1}$ does not contain $L$ in its centre anymore. Consequently, there is such an element $a$ in $Z_{\alpha+1}$ that the centre $C$ of the finite extension $\left\langle Q_{\alpha}, a\right\rangle$ of the CML $Q$ does not contain the subloop $L$. By Lemma 4.2 the centre $C$ does not satisfy the minimum condition. The normality of the subloop $\left\langle Z_{\alpha}, a\right\rangle$ in the CML $Q$ follows from the relation $Z_{\alpha+1} / Z_{\alpha}=Z\left(Q / Z_{\alpha}\right)$, and hereof follows the normality of the subloop $\left.<Q_{\alpha}, a\right\rangle$. Consequently, the centre $C$ of the subloop $\left\langle Q_{\alpha}, a\right\rangle$ is normal in the CML $Q$. By Lemma 4.3 the intersection $C \cap L$ does not satisfy the minimum condition. It is different from the subloop $L$, as the latter does not belong to $C$. As this intersection is normal in $Q$, the statement is proved.
Corollary 4.5. In the periodic commutative Moufang $Z A$-loop $Q$ each associative normal subloop which satisfies the minimum condition for the normal subloops of the loop $Q$ satisfies the minimum condition for its subloops.

This statement follows from Lemma 4.4.
Theorem 4.6. If at least one maximal associative subloop of the commutative Moufang $Z A$-loop $Q$ satisfies the minimum condition for the normal subloops of the loop $Q$, then $Q$ satisfies the minimum condition for subloops.
Proof. By Lemma 1.2 we will consider that the CML $Q$ is periodic. Then the statement follows from Corollary 4.5 and Lemma 1.9.
Corollary 4.7. In the commutative Moufang ZA-loop the minimum condition for subloops and associative normal subloops are equivalent.
Corollary 4.8. If in a commutative Moufang $Z A$-loop at least one maximal associative normal subloop is finite, then the loop $Q$ is also finite.

The statement follows from the Theorems 4.6 and 2.1.
Corollary 4.9. The infinite commutative Moufang $Z A$-loop $Q$ has an infinite centre.
Proof. By Corollary 4.8 the CML $Q$ possesses an infinite associative normal subloop. Then the statement follows from Lemma 1.7.

We remark that in [4] an example of a CML with unitary centre is constructed.
Theorem 4.10. If the centre $Z(Q)$ of the commutative Moufang $Z A$-loop $Q$ satisfies the minimum condition for subloops, then the loop $Q$ satisfies the minimum condition for subloop itself.
Proof. By Theorem 2.1 the centre $Z(Q)$ decomposes into the direct product of a finite number of quasicyclic groups $D$ and a finite group $C$, and by Proposition 1.12 $Q=D \times L$. Obviously, the centre $Z(L)$ of the CML $L$ coincides with $C$. As $C$ is a
finite group, then by Corollary 4.9 the CML $L$ is finite. Then the CML $Q$ satisfies the minimum condition for subloops.
Theorem 4.11. If a commutative Moufang loop satisfies the minimum condition for normal subloops, it satisfies the minimum condition for subloops as well.
Proof. By Lemma 1.5 an arbitrary CML possesses a central system. It follows from the minimum condition for normal subloops that each central system of the CML $Q$ is an ascending central series, i.e. $Q$ is a $Z A$-loop. Now the statement follows from Corollary 4.7.

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State Agrarian University of Moldova
Received March 11, 2003
str. Mirceshti 44, Chişinău, MD-2028
Moldova
E-mail:sandumn@yahoo.com

# Weak convergence of the distributions of Markovian random evolutions in two and three dimensions 

A.D. Kolesnik


#### Abstract

We consider Markovian random evolutions performed by a particle moving in $R^{2}$ and $R^{3}$ with some finite constant speed $v$ randomly changing its directions at Poisson-paced time instants of intensity $\lambda>0$ uniformly on the $S_{2}$ and $S_{3}$-spheres, respectively. We prove that under the Kac condition


$$
v \rightarrow \infty, \quad \lambda \rightarrow \infty, \quad \frac{v^{2}}{\lambda} \rightarrow c, \quad c>0
$$

the transition laws of the motions weakly converge in an appropriate Banach space to the transition law of the two- and three-dimensional Wiener process, respectively, with explicitly given generators.

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## 1 Introduction

The processes of random evolution in some phase space are being described by the equality

$$
\begin{equation*}
\frac{\partial f}{\partial t}=\mathcal{A} f+\Lambda f \tag{1}
\end{equation*}
$$

which, in a certain sense, can be referred to as Kolmogorov equation for the random evolution. In this equation (1) $\mathcal{A}$ is some purely spacial operator of a special form acting in an appropriate Banach space, namely, some sort of diagonal matrix differential operator acting in the space of sufficiently smooth functions, and $\Lambda$ is the infinitesimal operator of a stochastic process governing the evolution. The particular form of equation (1) is determined by the type of the evolution space and kind of the controlling stochastic process. For instance, if the evolution is driven by a continuous-time Markov chain with a finite number of states $n, n \geq 2$, then equation (1) takes the form of a system of $n$ first-order PDEs, $\mathcal{A}$ is some diagonal $(n \times n)$ matrix differential operator acting in the space of differentiable vector-functions and $\Lambda$ is a scalar infinitesimal $(n \times n)$-matrix of the embedded Markov chain.

The operator $\mathcal{A}$ is responsible for the propagation velocity of the evolution and $\Lambda$ deals with the intensity of the switching stochastic process. Therefore, it is natural to represent these operators in the form $\mathcal{A}=\varepsilon_{1} A, \Lambda=\varepsilon_{2} Q$, where the parameters

[^2]$\varepsilon_{1}$ and $\varepsilon_{2}$ have the sense of the velocity of the evolution and the intensity of the governing stochastic process, respectively, and the operators $A$ and $Q$ do not depend on $\varepsilon_{1}$ and $\varepsilon_{2}$.

The systems of the form

$$
\begin{equation*}
\frac{\partial f}{\partial t}=\varepsilon_{1} A f+\varepsilon_{2} Q f \tag{2}
\end{equation*}
$$

have become the subject of a great deal of researches, among which the problem of diffusion approximation of random evolutions was of a special interest. It is clear that in order the evolution to have a diffusion limit, its velocity and rate of switches must satisfy some sort of equilibrium condition. In other words, the parameters $\varepsilon_{1}$ and $\varepsilon_{2}$ in (2) must be connected between themselves by that or another relationship.

In the case a random evolution is controlled by a continuous-time homogeneous Markov chain with $n$ states, the limit behaviour of a process governed by the system (2) with $\varepsilon_{1}=\varepsilon, \varepsilon_{2}=\varepsilon^{2}, A$ is some diagonal $(n \times n)$-matrix differential operator, $Q$ is a $(n \times n)$-matrix of infinitesimal parameters, has been examined by Pinsky [12], Griego and Hersh [1], Hersh and Papanicolaou [3], who have given the diffusion approximation theorems as $\varepsilon \rightarrow \infty$. A system of the form (2) has thoroughly been studied by Hersh and Pinsky [4] and a limit theorem has been given as the ratio $\left(\varepsilon_{1} / \varepsilon_{2}\right) \rightarrow 0$. An abstract version of these diffusion approximation theorems has been given by Kurtz [9] for arbitrary evolution space and kind of the controlling Markov process. The reader interested in more details on the subject should address to the survey article by Hersh [2] and, especially, to the monographs by Pinsky [14] and by Korolyuk and Swishchuk [8].

The most interesting case of random evolution performed by a particle moving in $R^{m}, m \geq 1$, at some finite constant speed $v$ subject to the control of a homogeneous Poisson process of rate $\lambda>0$ (so-called transport process), is being described by an equation of the form (2) with $\varepsilon_{1}=v$ and $\varepsilon_{2}=\lambda$. It is known that in many such cases the limiting diffusion process arises if $v$ and $\lambda$ satisfy the following Kac condition

$$
\begin{equation*}
v \rightarrow \infty, \quad \lambda \rightarrow \infty, \quad \frac{v^{2}}{\lambda} \rightarrow c, \quad c>0 \tag{3}
\end{equation*}
$$

and the transition functions of the evolution (as a two-parameter family of distributions depending on $v$ and $\lambda$ ) weakly converge to the transition function of a corresponding Brownian motion. Moreover, in all the cases (a very few ones) when the transition laws were obtained in an explicit form (see, for instance, Orsingher [10], theorem 1, for the transition law of the Goldstein-Kac telegraph process in $R^{1}$, and Orsingher [11], theorem 3.1, for the transition law of a random evolution with four directions in $R^{2}$ ), the condition (3) provided the pointwise convergence of the transition functions of the motion to the transition function of the Wiener process.

A Markovian random evolution with an arbitrary number of directions $n, n \geq 2$, in $R^{2}$ has been studied by Kolesnik and Turbin [7], and a $n$th order hyperbolic equation with constant coefficients governing the transition law of the motion has been obtained. It was also shown that under the Kac condition (3) the governing
hyperbolic operator turns into the classical parabolic diffusion operator in $R^{2}$ with the generator

$$
\begin{equation*}
G_{n}=\frac{c(n-1)}{2 n} \Delta, \quad n \geq 3 \tag{4}
\end{equation*}
$$

where $\Delta$ is the two-dimensional Laplacian. A diffusion approximation theorem proved in Kolesnik [6] also stated that under the Kac condition (3) the transition laws of the evolution weakly converge in a suitably chosen Banach space to the transition law of the Wiener process in $R^{2}$ with generator (4).

One should note that in both these works it was supposed that under each change of direction the particle took on any new one uniformly with probability $1 /(n-1)$, that is, it could not preserve its current direction. However, if we suppose that every new direction can be taken on uniformly with equal probabilities $1 / n$ (i.e. the transition probabilities of the embedded Markov chain are $p_{i j}=1 / n$ for any $i$ and $j$ ), then replacing everywhere $1 /(n-1)$ for $1 / n$ we obtain that the generator of the limiting Wiener process for any $n \geq 3$ is

$$
\begin{equation*}
G_{n}=\frac{c}{2} \Delta, \tag{5}
\end{equation*}
$$

and there is not the number of directions $n$ in the right-hand side of (5). In other words, under the full symmetry of the motion the limiting Wiener process does not depend on the number of directions $n$. It is worth to note that generator (5) also arises from (4) as $n \rightarrow \infty$.

This amazing fact allows us to expect that for an evolution with the continuum number of directions (i.e. when the particle chooses new directions uniformly on the unit circumference) the limiting Wiener process will have the same generator (5). Proof of this statement is one of the principal results of our paper.

Studying of random evolutions with the continuum number of directions in $R^{m}, m \geq 2$, is an extremely interesting, natural and practically useful problem. Although the equation governing such a motion is not obtained yet, nevertheless we are able to present the results concerning limiting behaviour of the transition laws of the evolutions in $R^{2}$ and $R^{3}$ under the Kac condition (3).

The main tool of our research is a diffusion approximation method given in Kurtz [9]. In Section 2, for the reader's convenience, we shall briefly remind the main points of this method in a form convenient for further applications. In Section 3 we shall apply it to the problem of studying the behaviour of the transition laws of a random evolution in $R^{2}$ governed by a jump Markov process on the $S_{2}$-sphere (unit circumference). We will show that under the Kac condition (3) the transition laws of the evolution weakly converge in an appropriate Banach space to the transition law of the two-dimensional Brownian motion with zero drift and the variance $\sqrt{c}$. In Section 4 we will give a similar result for a random evolution in $R^{3}$ driven by a jump Markov process on the $S_{3}$-sphere (surface of the unit 3D-ball) and will prove the weak convergence of the transition laws of the motion to the transition law of the Wiener process in $R^{3}$ with zero drift and the variance $\sqrt{2 c / 3}$.

## 2 Kurtz's Approximation Method

Let $U(t)$ and $S(t)$ be strongly continuous semigroups of linear contractions on a Banach space $L$ with infinitesimal operators $A$ and $B$, respectively. Let $\mathcal{D}(A)$ and $\mathcal{D}(B)$ be the domains of $A$ and $B$. Assume that for each sufficiently large $\alpha$, the closure of $A+\alpha B$ is the infinitesimal operator of a strongly continuous semigroup $T_{\alpha}(t)$ on $L$. Also suppose that $B$ is the closure of $B$ restricted to $\mathcal{D}(A) \cap \mathcal{D}(B)$. We are interested in the behaviour of $T_{\alpha}(t)$ as $\alpha$ goes to infinity.

Define the operator $P$ on $L$ by the equality

$$
\begin{equation*}
P f=\lim _{\gamma \rightarrow 0} \gamma \int_{0}^{\infty} e^{-\gamma t} S(t) f d t \tag{6}
\end{equation*}
$$

and suppose that the limit in the right-hand side of (6) exists for every $f \in L$. It is known (see Hille and Phillips [5], page 516) that operator $P$ defined by (6) is a bounded linear projection, i.e. $P^{2}=P$.

Denote by $\mathcal{R}(P)$ the image of the operator $P$. Let

$$
D=\{f \in \mathcal{R}(P): f \in \mathcal{D}(A)\}
$$

and for $f \in D$ define the operator $C$ by the equality $C f=P A f$. Kurtz's approximation method is given by the following theorem.

Theorem [Kurtz [9], theorem 2.2]. Let $U(t), S(t), T_{\alpha}(t), D, C$ be defined as above. Suppose that for all $f \in D$

$$
\begin{equation*}
C f=0 . \tag{7}
\end{equation*}
$$

Let

$$
\begin{equation*}
D_{0}=\{f \in D: \exists h \in \mathcal{D}(A) \cap \mathcal{D}(B) \text { such that } B h=-A f\} . \tag{8}
\end{equation*}
$$

For $f \in D_{0}$ define the operator $C_{0}$ by the equality

$$
\begin{equation*}
C_{0} f=P A h . \tag{9}
\end{equation*}
$$

Suppose that

$$
\begin{equation*}
\overline{\mathcal{R}\left(\mu-C_{0}\right)} \supset \overline{D_{0}} \tag{10}
\end{equation*}
$$

for some $\mu>0$.
Then the closure of $C_{0}$ restricted so that $C_{0} f \in \overline{D_{0}}$ is the infinitesimal operator of a strongly continuous contraction semigroup $T(t)$ defined on $\overline{D_{0}}$ and for all $f \in \overline{D_{0}}$

$$
T(t) f=\lim _{\alpha \rightarrow \infty} T_{\alpha}(\alpha t) f
$$

This theorem gives an effective method of obtaining approximation results for a wide class of stochastic processes. First of all one should note that the conditions of the theorem are not too burdensome. The equality (7) is some sort of symmetry condition which in practice can often be provided (if needed) by simple transformations.

Fulfilment of the condition (10) can be provided by the choice of an appropriate Banach space. Therefore, there are two crucial points in this method. The first one concerns finding of a solution $h \in \mathcal{D}(A) \cap \mathcal{D}(B)$ of the equation

$$
\begin{equation*}
B h=-A f \tag{11}
\end{equation*}
$$

for every element $f \in D_{0}$. The second point concerns the possibility of computing the projector $P$ defined by (6). The main question here is the existence of the limit in the right-hand side of (6).

In the case of Markovian random evolutions the projector $P$ can be found by means of a more explicit formula. Let $V(t)$ be a temporally homogeneous Markov process with measurable state space $(E, \mathcal{E})$ and transition function $P(t, x, \Gamma)$. Then the semigroup $S(t)$ in the Banach space of bounded strongly measurable functions $f: E \rightarrow L$ with the sup-norm is defined by

$$
S(t) f(x)=\int_{E} f(y) P(t, x, d y)
$$

and the projector $P$ is explicitly given by the formula

$$
\begin{equation*}
P f(x)=\int_{E} f(y) P(x, d y), \tag{12}
\end{equation*}
$$

where $P(x, \Gamma)$ is the limiting distribution, assumed to exist, of the process $V(t)$ starting from $x$, or the weak limit as $t \rightarrow \infty$ of the transition function $P(t, x, \Gamma)$. One should note that formula (12) takes an especially simple form if the limiting distribution $P(x, \Gamma)$ is uniform.

If $h$ and $P$ are found and the conditions (7) and (10) are fulfilled then, according to the conclusion of the Kurtz's theorem, one can assert that the transition laws of the random evolution weakly converge to the transition law of a process with generator given by the closure of $C_{0}$.

In the next sections we will apply this method to the Markovian random evolutions in $R^{2}$ and $R^{3}$ and prove that their transition functions weakly converge to the transition function of the two- and three-dimensional Brownian motion, respectively, with explicitly given generators.

## 3 Diffusion Approximation Theorem in $R^{2}$

Consider the following planar stochastic motion. A particle starts at the moment $t=0$ from the origin $x=y=0$ of the plane $R^{2}$ taking initial random direction uniformly on the $S_{2}$-sphere (unit circumference) and moves with some constant finite speed $v$. At every time instant $t>0$ it can have some random direction of motion $E_{\varphi}, \varphi \in[0,2 \pi)$ which forms the angle $\varphi$ with $x$-axis. In other words, the direction $E_{\varphi}$ is oriented like the vector $e_{\varphi}=(\cos \varphi, \sin \varphi), \varphi \in[0,2 \pi)$. The motion is controlled by a homogeneous Poisson process of rate $\lambda>0$ as follows. When a Poisson event
occurs, the particle instantly takes on a new random direction distributed uniformly on $S_{2}$ and continues its motion in the chosen direction with the same speed $v$ until the next Poisson event occurs, then it takes on a new random direction again, and so on. Thus, the evolution is controlled by the jump Markov process $\Phi_{t}$ on the unit circumference $S_{2}$.

Let $\Xi(t)=\left(X_{t}, Y_{t}\right)$ denote the particle's position in the plane at some instant $t>0$. Since the motion depends on $v$ and $\lambda$ then, in fact, we deal with a twoparameter family of stochastic processes $\Xi_{v}^{\lambda}(t)$. Bearing this in mind, we omit these indices in the sequel.

The main goal of this section is to study the behaviour of the transition laws of $\Xi(t)$ as the intensity of transitions $\lambda$ tends to infinity and, according to $\lambda$, the particle speed $v$ increases as well. The accordance between the growth rates of $\lambda$ and $v$ is determined by the Kac condition (3).

Since the sample paths of $\Xi(t)$ are continuous and differentiable almost everywhere and the velocity of the process is finite, the distribution of $\Xi(t)$ consists of the absolutely continuous component concentrated strictly inside the circle

$$
K_{t}=\left\{(x, y) \in R^{2}: x^{2}+y^{2}<v^{2} t^{2}\right\}, \quad t>0
$$

and the singular component on the boundary

$$
B_{t}=\left\{(x, y) \in R^{2}: x^{2}+y^{2}=v^{2} t^{2}\right\}, \quad t>0
$$

Therefore there exist the partial (with respect to directions) transition densities $f_{\varphi}=f_{\varphi}(x, y, t), \quad(x, y) \in K_{t}, t>0, \varphi \in[0,2 \pi)$ of the absolutely continuous component of $\Xi(t)$ defined by the equality

$$
f_{\varphi}(x, y, t) d x d y d \varphi=\operatorname{Prob}\left\{x \leq X_{t}<x+d x, y \leq Y_{t}<y+d y, \varphi \leq \Phi_{t}<\varphi+d \varphi\right\}
$$

Kolmogorov equation (1) written down for these transition densities has the form of the integro-differential equation

$$
\begin{equation*}
\frac{\partial f_{\varphi}}{\partial t}=-v \cos \varphi \frac{\partial f_{\varphi}}{\partial x}-v \sin \varphi \frac{\partial f_{\varphi}}{\partial y}-\lambda f_{\varphi}+\frac{\lambda}{2 \pi} \int_{0}^{2 \pi} f_{\theta} d \theta, \quad \varphi \in[0,2 \pi) \tag{13}
\end{equation*}
$$

Equation (13) is a particular case (for the uniform dissipation function identically equal to $1 /(2 \pi)$ ) of a some more general equation with an arbitrary dissipation function given in the monograph by Tolubinsky [15], page 40. One should note that the integral term in (13) appears due to the continuum number of directions. This is the main difference of the motion from the model with a finite number of directions studied in Kolesnik and Turbin [7] and Kolesnik [6] where only PDEs arose.

Consider the Banach space $\mathcal{B}$ of twice continuously differentiable functions on $R^{2} \times(0, \infty)$ vanishing at infinity. The transition densities $f_{\varphi}$ can be considered as the one-parameter family of functions $f=\left\{f_{\varphi}, \varphi \in[0,2 \pi)\right\}$ belonging to $\mathcal{B}$.

Introduce the one-parameter family $\mathcal{A}=\left\{A_{\theta}, \theta \in[0,2 \pi)\right\}$ of operators acting in $\mathcal{B}$ where

$$
A_{\theta}=-v \cos \theta \frac{\partial}{\partial x}-v \sin \theta \frac{\partial}{\partial y}
$$

Define the action of $\mathcal{A}$ on $f$ as

$$
\begin{equation*}
\mathcal{A} f=\left\{\delta(\theta, \varphi) A_{\theta} f_{\varphi}, \quad \theta, \varphi \in[0,2 \pi)\right\} \tag{14}
\end{equation*}
$$

where

$$
\delta(\theta, \varphi)= \begin{cases}1, & \text { if } \theta=\varphi \\ 0, & \text { otherwise }\end{cases}
$$

is the generalized Kronecker delta-symbol of rank 2. The operator $\mathcal{A}$ in (14) is an analogue of a diagonal matrix differential operator and the family $f$ is the continuum analogue of the vector-function of partial transition densities appearing in the finitestate case (see Kolesnik [6], formula (2)).

Introduce now the operator $\Lambda$ acting on $f$ by the following formula

$$
\begin{equation*}
\Lambda f=-\lambda f+\frac{\lambda}{2 \pi} \int_{0}^{2 \pi} f_{\theta} d \theta . \tag{15}
\end{equation*}
$$

Then equality (13) can be rewritten as follows

$$
\begin{equation*}
\frac{\partial f}{\partial t}=\mathcal{A} f+\Lambda f \tag{16}
\end{equation*}
$$

and it has exactly the form of equation (1).
The principal result of this section is given by the following theorem.
2D-Diffusion Approximation Theorem. Let the Kac condition (3) be fulfilled. Then in the Banach space $\mathcal{B}$ the semigroups generated by the transition functions of the process $\Xi(t)$ converge to the semigroup generated by the transition function of the Wiener process in $R^{2}$ with generator

$$
\begin{equation*}
G=\frac{c}{2} \Delta \tag{17}
\end{equation*}
$$

where $\Delta$ is the two-dimensional Laplace operator.
Remark. Note that generator (17) also formally appears from formula (13) of Kolesnik [6] and formula (4.3) of Kolesnik and Turbin [7] as $n \rightarrow \infty$.

Remark. One should also note that for the particular case when the limiting constant $c=1$, the generator (17) coincides with the evolutionary operator given in Proposition 4.8 of the paper by Pinsky [13] for the dimension $m=2$.

Proof. According to formulas (8) and (11) of the Kurtz's theorem above, we need to find a solution $h$ of the equation

$$
\begin{equation*}
\Lambda h=-\mathcal{A} f \tag{18}
\end{equation*}
$$

for arbitrary function (family) $f \in D_{0}$. As is easy to see, such a solution for any differentiable function $f$ is given by the formula

$$
\begin{equation*}
h=\frac{1}{\lambda} \mathcal{A} f+\frac{1}{2 \pi} \int_{0}^{2 \pi} f_{\theta} d \theta . \tag{19}
\end{equation*}
$$

Really, taking into account that for any $f \in \mathcal{B}$

$$
\begin{equation*}
\int_{0}^{2 \pi} \mathcal{A} f d \theta=\left(\int_{0}^{2 \pi} A_{\theta} d \theta\right) f=0 \tag{20}
\end{equation*}
$$

and using (15) and (19) we obtain

$$
\Lambda h=-\lambda\left(\frac{1}{\lambda} \mathcal{A} f+\frac{1}{2 \pi} \int_{0}^{2 \pi} f_{\varphi} d \varphi\right)+\frac{\lambda}{2 \pi} \int_{0}^{2 \pi}\left(\frac{1}{\lambda} \mathcal{A} f+\frac{1}{2 \pi} \int_{0}^{2 \pi} f_{\varphi} d \varphi\right) d \theta=-\mathcal{A} f
$$

and equality (18) is fulfilled.
Our next step is to compute the projector $P$ given by formula (12). Since the limiting distribution of the governing Markov process on $S_{2}$ is uniform with the density $1 /(2 \pi)$ then formula (12) simplifies, and the projector is given by

$$
\begin{equation*}
P f=\frac{1}{2 \pi} \int_{0}^{2 \pi} f_{\varphi} d \varphi \tag{21}
\end{equation*}
$$

Then, according to (9), (19) and (21), we obtain

$$
C_{0} f=P \mathcal{A} h=\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\frac{1}{\lambda} \mathcal{A}^{2} f+\frac{1}{2 \pi} \mathcal{A} \int_{0}^{2 \pi} f_{\varphi} d \varphi\right) d \theta .
$$

The well-known equalities

$$
\int_{0}^{2 \pi} \sin ^{2} \varphi d \varphi=\pi, \quad \int_{0}^{2 \pi} \cos ^{2} \varphi d \varphi=\pi, \quad \int_{0}^{2 \pi} \sin \varphi \cos \varphi d \varphi=0
$$

yield the formula

$$
\begin{equation*}
\int_{0}^{2 \pi} \mathcal{A}^{2} f d \theta=\left(\int_{0}^{2 \pi} A_{\theta}^{2} d \theta\right) f=\pi v^{2} \Delta f \tag{22}
\end{equation*}
$$

where $\Delta$ is the two-dimensional Laplacian and therefore, taking into account (20) and (22), for any $f \in \mathcal{B}$ we have

$$
C_{0} f=\left(\frac{1}{2 \pi \lambda} \int_{0}^{2 \pi} A_{\theta}^{2} d \theta\right) f+\frac{1}{4 \pi^{2}}\left(\int_{0}^{2 \pi} A_{\theta} d \theta\right)\left(\int_{0}^{2 \pi} f_{\varphi} d \varphi\right)=\frac{v^{2}}{2 \lambda} \Delta f .
$$

Thus, we obtain

$$
C_{0}=\frac{v^{2}}{2 \lambda} \Delta
$$

and therefore generator (17) under the Kac condition (3) is the limiting operator of the evolution.

It remains to check conditions (7) and (10) of the Kurtz's theorem. Taking into account equality (20) for any $f$ continuously differentiable we have

$$
C f=P \mathcal{A} f=\frac{1}{2 \pi}\left(\int_{0}^{2 \pi} A_{\theta} d \theta\right) f=0
$$

and thus condition (7) is fulfilled.
In order to check condition (10) it is sufficient to show that for any function $f$ twice continuously differentiable there exists a solution $g$ of the equation

$$
\begin{equation*}
\left(\mu-C_{0}\right) g=f \tag{23}
\end{equation*}
$$

for some $\mu>0$. One can easily see that for any $\mu>0$ equation (23) takes the form of an inhomogeneous Klein-Gordon equation (or Helmholtz equation with a purely imaginary constant) with a sufficiently smooth right-hand part, and existence of its solution is well-known from the general PDEs theory. Thus, condition (10) is also fulfilled.

Therefore, by the Kurtz's approximation theorem, one can assert that under the Kac condition (3) the semigroups generated by the transition laws of the process $\Xi(t)$ converge in $\mathcal{B}$ to the semigroup generated by the transition law of the Wiener process in $R^{2}$ with generator (17).

## 4 Diffusion Approximation Theorem in $R^{3}$

In this section we give a similar result concerning 3-dimensional random evolution. A particle starts at the moment $t=0$ from the origin $x=y=z=0$ of the space $R^{3}$ taking initial random direction uniformly on the $S_{3}$-sphere (surface of the unit 3D-ball) and moves with some constant finite speed $v$. At every time instant $t>0$ it can have some random direction of motion $\omega \in S_{3}$ where $\omega$ is a spacial (bodial) angle. The motion is driven by a Poisson process of rate $\lambda>0$ as follows. When a Poisson event occurs, the particle instantly takes on a new random direction distributed uniformly on $S_{3}$ and continues its motion in the chosen direction with the same speed $v$ until the next Poisson event occurs, then it takes on a new random direction again, and so on. Thus, the evolution is controlled by the jump Markov process $\Phi_{t}$ on $S_{3}$.

Let $\Xi(t)=\left(X_{t}, Y_{t}, Z_{t}\right)$ denote the particle's position in the space $R^{3}$ at some instant $t>0$. The main goal of this section is to study the behaviour of the transition laws of $\Xi(t)$ under the Kac condition (3).

Like in the planar case, the distribution of $\Xi(t)$ consists of the absolutely continuous component concentrated strictly inside the ball

$$
K_{t}=\left\{(x, y, z) \in R^{3}: x^{2}+y^{2}+z^{2}<v^{2} t^{2}\right\}, \quad t>0
$$

and the singular component on the boundary

$$
B_{t}=\left\{(x, y, z) \in R^{3}: x^{2}+y^{2}+z^{2}=v^{2} t^{2}\right\}, \quad t>0
$$

Therefore there exist the partial (with respect to directions) transition densities $f_{\omega}=f_{\omega}(x, y, z, t), \quad(x, y, z) \in K_{t}, \omega \in S_{3}, t>0$, of the absolutely continuous component of $\boldsymbol{\Xi}(t)$ defined by the equality

$$
\begin{gathered}
f_{\omega}(x, y, z, t) d x d y d z \mu(d \omega)= \\
\operatorname{Prob}\left\{x \leq X_{t}<x+d x, y \leq Y_{t}<y+d y, z \leq Z_{t}<z+d z, \Phi_{t} \in d \omega\right\}
\end{gathered}
$$

where $\mu(d \omega)$ is the measure of the elementary spacial angle $d \omega$.
Since in $R^{3}$ any direction $\omega$ is determined by the ordered pair of two planar angles $(\varphi, \psi), \varphi \in[0,2 \pi), \psi \in[0, \pi)$, and the measure of the elementary spacial angle $d \omega$ is equal to

$$
\begin{equation*}
\mu(d \omega)=\sin \psi d \psi d \varphi \tag{24}
\end{equation*}
$$

then Kolmogorov equation (1) written down for the transition densities $f_{\omega}=f_{\varphi, \psi}$ has the form of the integro-differential equation

$$
\begin{gather*}
\frac{\partial f_{\varphi, \psi}}{\partial t}=-v \sin \psi \cos \varphi \frac{\partial f_{\varphi, \psi}}{\partial x}-v \sin \psi \sin \varphi \frac{\partial f_{\varphi, \psi}}{\partial y}-v \cos \psi \frac{\partial f_{\varphi, \psi}}{\partial z} \\
-\lambda f_{\varphi, \psi}+\frac{\lambda}{4 \pi} \int_{S_{3}} f_{\omega} \mu(d \omega), \quad \varphi \in[0,2 \pi), \psi \in[0, \pi) . \tag{25}
\end{gather*}
$$

Equation (25) is a particular case (for the uniform dissipation function identically equal to $1 /(4 \pi)$ ) of a some more general equation with an arbitrary dissipation function given in the monograph by Tolubinsky [15], page 40.

Consider the Banach space $\mathcal{B}$ of twice continuously differentiable functions on $R^{3} \times(0, \infty)$ vanishing at infinity. The transition densities $f_{\varphi, \psi}$ can be considered as the two-parameter family of functions $f=\left\{f_{\varphi, \psi}, \varphi \in[0,2 \pi), \psi \in[0, \pi)\right\}$ belonging to $\mathcal{B}$.

Introduce the two-parameter family $\mathcal{A}=\left\{A_{\theta, \nu}, \quad \theta \in[0,2 \pi), \nu \in[0, \pi)\right\}$ of operators acting in $\mathcal{B}$ where

$$
A_{\theta, \nu}=-v \sin \psi \cos \varphi \frac{\partial}{\partial x}-v \sin \psi \sin \varphi \frac{\partial}{\partial y}-v \cos \psi \frac{\partial}{\partial z} .
$$

Define the action of $\mathcal{A}$ on $f$ as

$$
\begin{equation*}
\mathcal{A} f=\left\{\delta(\theta, \varphi) \delta(\nu, \psi) A_{\theta, \nu} f_{\varphi, \psi}, \quad \theta, \varphi \in[0,2 \pi), \nu, \psi \in[0, \pi)\right\} \tag{26}
\end{equation*}
$$

where $\delta(\cdot, \cdot)$ is the generalized Kronecker delta-symbol of rank 2 defined above.
Introduce now the operator $\Lambda$ acting on $f$ in the following way

$$
\begin{equation*}
\Lambda f=-\lambda f+\frac{\lambda}{4 \pi} \int_{S_{3}} f_{\omega} \mu(d \omega) . \tag{27}
\end{equation*}
$$

Then equality (25) can be rewritten as

$$
\frac{\partial f}{\partial t}=\mathcal{A} f+\Lambda f
$$

having the form of equation (1) and similar to (16).
The principal result of this section is given by the following theorem.

3D-Diffusion Approximation Theorem. Let the Kac condition (3) be fulfilled. Then in the Banach space $\mathcal{B}$ the semigroups generated by the transition functions of the process $\Xi(t)$ converge to the semigroup generated by the transition function of the Wiener process in $R^{3}$ with generator

$$
\begin{equation*}
G=\frac{c}{3} \Delta \tag{28}
\end{equation*}
$$

where $\Delta$ is the three-dimensional Laplace operator.
Remark. Note that for the particular case when the limiting constant $c=1$, the generator (28) coincides with the evolutionary operator given in Proposition 4.8 of the paper by Pinsky [13] for the dimension $m=3$.

Proof. The proof of the theorem is similar to that of the planar case. A solution $h$ of the equation

$$
\Lambda h=-\mathcal{A} f
$$

for any differentiable function (family) $f$ is

$$
\begin{equation*}
h=\frac{1}{\lambda} \mathcal{A} f+\frac{1}{4 \pi} \int_{S_{3}} f_{\omega} \mu(d \omega) . \tag{29}
\end{equation*}
$$

Since the limiting distribution of the governing Markov process on $S_{3}$ is uniform with the density $1 /(4 \pi)$ then, by formula (12), the projector $P$ is given by

$$
\begin{equation*}
P f=\frac{1}{4 \pi} \int_{S_{3}} f_{\omega} \mu(d \omega) \tag{30}
\end{equation*}
$$

Then, according to (9), (29) and (30), we obtain

$$
C_{0} f=P \mathcal{A} h=\frac{1}{4 \pi} \int_{S_{3}}\left(\frac{1}{\lambda} \mathcal{A}^{2} f+\frac{1}{4 \pi} \mathcal{A} \int_{S_{3}} f_{\omega} \mu(d \omega)\right) \mu(d \xi)
$$

Using the equality (24) one can easily show that for any $f \in \mathcal{B}$

$$
\begin{gather*}
\int_{S_{3}} \mathcal{A} f \mu(d \xi)=\left(\int_{0}^{2 \pi} d \theta \int_{0}^{\pi} A_{\theta, \nu} \sin \nu d \nu\right) f=0  \tag{31}\\
\int_{S_{3}} \mathcal{A}^{2} f \mu(d \xi)=\left(\int_{0}^{2 \pi} d \theta \int_{0}^{\pi} A_{\theta, \nu}^{2} \sin \nu d \nu\right) f=\frac{4 \pi v^{2}}{3} \Delta f \tag{32}
\end{gather*}
$$

where $\Delta$ is the three-dimensional Laplacian, and therefore we have

$$
C_{0} f=\frac{1}{4 \pi \lambda} \int_{S_{3}} \mathcal{A}^{2} f \mu(d \xi)+\frac{1}{16 \pi^{2}} \int_{S_{3}} \mathcal{A}\left(\int_{S_{3}} f_{\omega} \mu(d \omega)\right) \mu(d \xi)=\frac{v^{2}}{3 \lambda} \Delta f .
$$

Thus, we obtain

$$
C_{0}=\frac{v^{2}}{3 \lambda} \Delta
$$

and therefore generator (28) under the Kac condition (3) is the limiting operator of the evolution.

Fulfilment of the condition (7) of the Kurtz's theorem is provided by equality (31), and condition (10) can be checked in the same manner as it was done in the planar case.

Therefore, by the Kurtz's approximation theorem, one can conclude that under the Kac condition (3) the distributions of the random evolution $\Xi(t)$ weakly converge in $\mathcal{B}$ to the distribution of the Wiener process in $R^{3}$ with generator (28).

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A.D. Kolesnik

Received March 28, 2003
Institute of Mathematics and Computer Science
5 Academiei str.
Chişinău, MD-2028, Moldova
E-mail: kolesnik@math.md

# On a nonlinear differential subordination I 

Georgia Irina Oros

Abstract. We find conditions on the complex-valued functions $A, B, C, D$ in the unit disc $U$ such that the differential inequality

$$
\left|A(z) z^{2} p^{\prime \prime}(z)+B(z) p^{2}(z)+C(z) p(z)+D(z)\right|<M
$$

implies $|p(z)|<N$, where $p$ is analytic in $U$, with $p(0)=0$.
Mathematics subject classification: 30C80.
Keywords and phrases: Differential subordination, dominant.

## 1 Introduction and preliminaries

We let $\mathcal{H}[U]$ denote the class of holomorphic functions in the unit disc

$$
U=\{z \in \mathbb{C}:|z|<1\}
$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^{*}$ we let

$$
\mathcal{H}[a, n]=\left\{f \in \mathcal{H}[U], f(z)=a+a_{n} z^{n}+a_{n+1} z^{n+1}+\ldots, z \in U\right\}
$$

and

$$
\mathcal{A}_{n}=\left\{f \in \mathcal{H}[U], f(z)=z+a_{n+1} z^{n+1}+a_{n+2} z^{n+2}+\ldots, z \in U\right\}
$$

with $\mathcal{A}_{1}=\mathcal{A}$.
In [1] chapter IV, the authors have analyzed a first-order linear differential subordination

$$
\begin{equation*}
A(z) z^{2} p^{\prime \prime}(z)+B(z) z p^{\prime}(z)+C(z) p(z)+D(z) \prec h(z), \tag{1}
\end{equation*}
$$

where $A, B, C, D$ and $h$ are complex-valued functions in the unit disc, where $p \in$ $\mathcal{H}[0, n]$. A more general version of (1) is given by:

$$
A(z) z^{2} p^{\prime \prime}(z)+B(z) z p^{\prime}(z)+C(z) p(z)+D(z) \in \Omega
$$

where $\Omega \subset \mathbb{C}$.
In [2] we found conditions on the complex-valued functions $A, B, C, D$ in the unit disc $U$ and the positive numbers $M$ and $N$ such that

$$
\left|A(z) z p^{\prime}(z)+B(z) p^{2}(z)+C(z) p(z)+D(z)\right|<M
$$

implies $|p(z)|<N$, where $p \in \mathcal{H}[0, n]$.
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In this paper we shall consider the following particular second-order nonlinear differential subordination given by the inequality

$$
\begin{equation*}
\left|A(z) z^{2} p^{\prime \prime}(z)+B(z) p^{2}(z)+C(z) p(z)+D(z)\right|<M \tag{2}
\end{equation*}
$$

where $p \in \mathcal{H}[0, n]$.
We find conditions on complex-valued functions $A, B, C, D$ and the positive numbers $M$ and $N$ such that (2) implies $|p(z)|<N$, where $p \in \mathcal{H}[0, n]$.

In order to prove the new results we shall use the following lemma, which is a particular form of Theorem 2.3h [1, p. 34].
Lemma A. [1, p. 34] Let $\psi: \mathbb{C}^{2} \times U \rightarrow \mathbb{C}$ and $M>0, N>0$ satisfy

$$
\begin{equation*}
\left|\psi\left(N e^{i \theta}, L ; z\right)\right| \geq M \tag{3}
\end{equation*}
$$

whenever $\operatorname{Re}\left[L e^{-i \theta}\right] \geq n(n-1) M, \quad z \in U$ and $\theta \in \mathbb{R}$, where $n$ is a positive integer.
If $p \in \mathcal{H}[0, n]$ and $\left|\psi\left(p(z), z^{2} p^{\prime \prime}(z) ; z\right)\right|<M$ then $|p(z)|<N$.

## 2 Main results

Theorem. Let $M>0, N>0$, and let $n$ be a positive integer. Suppose that the functions $A, B, C, D: U \rightarrow \mathbb{C}$ satisfy $A(z) \neq 0$,

$$
\begin{equation*}
\operatorname{Re} \frac{C(z)}{A(z)} \geq \frac{M+N^{2}|B(z)|+D(z) \mid}{N|A(z)|} . \tag{4}
\end{equation*}
$$

If $p \in \mathcal{H}[0, n]$ and

$$
\left|A(z) z^{2} p^{\prime \prime}(z)+B(z) p^{2}(z)+C(z) p(z)+D(z)\right|<M
$$

then $|p(z)|<N$.
Proof. Let $\psi: \mathbb{C}^{2} \times U \rightarrow \mathbb{C}$ be defined by

$$
\begin{equation*}
\psi\left(p(z), z^{2} p^{\prime \prime}(z) ; z\right)=A(z) z^{2} p^{\prime \prime}(z)+B(z) p^{2}(z)+C(z) p(z)+D(z) \tag{5}
\end{equation*}
$$

From (2) we have

$$
\begin{equation*}
\left|\psi\left(p(z), z^{2} p^{\prime \prime}(z) ; z\right)\right|<M, \text { for } z \in U . \tag{6}
\end{equation*}
$$

Using (4) and (5) we have

$$
\begin{aligned}
& \left|\psi\left(N e^{i \theta}, L, z\right)\right|=\left|A(z) L+B(z) N^{2} e^{2 i \theta}+C(z) N e^{i \theta}+D(z)\right|= \\
& =\left|A(z) L e^{-i \theta}+B(z) N^{2} e^{i \theta}+C(z) N+D(z) e^{-i \theta}\right|= \\
& \quad=|A(z)|\left|L e^{-i \theta}+\frac{B(z)}{A(z)} N^{2} e^{i \theta}+\frac{C(z)}{A(z)} N+\frac{D(z)}{A(z)} e^{-i \theta}\right| \geq
\end{aligned}
$$

$$
\begin{gathered}
\geq|A(z)|\left[\left|L e^{-i \theta}+\frac{B(z)}{A(z)} N^{2} e^{i \theta}+\frac{C(z)}{A(z)} N\right|-\left|\frac{D(z)}{A(z)}\right|\right] \geq \\
\geq|A(z)|\left[\left|L e^{-i \theta}+\frac{C(z)}{A(z)} N\right|-N^{2}\left|\frac{B(z)}{A(z)}\right|-\left|\frac{D(z)}{A(z)}\right|\right] \geq \\
\geq|A(z)|\left[\operatorname{Re} L e^{-i \theta}+N \operatorname{Re} \frac{C(z)}{A(z)}-N^{2}\left|\frac{B(z)}{A(z)}\right|-\left|\frac{D(z)}{A(z)}\right|\right] \geq \\
\geq|A(z)|\left[n(n-1) M+N \operatorname{Re} \frac{C(z)}{A(z)}-N^{2}\left|\frac{B(z)}{A(z)}\right|-\left|\frac{D(z)}{A(z)}\right|\right] \geq \\
\quad \geq|A(z)|\left[N \operatorname{Re} \frac{C(z)}{A(z)}-N^{2}\left|\frac{B(z)}{A(z)}\right|-\left|\frac{D(z)}{A(z)}\right|\right] \geq M .
\end{gathered}
$$

Hence condition (3) holds and by Lemma A we deduce that (6) implies $|p(z)|<N$.

Instead of prescribing the constant $N$ in Theorem, in some cases we can use (4) to determine an appropriate $N=N(M, n, A, B, C, D)$ so that (2) implies $|p(z)|<N$.

This can be accomplished by solving (4) for $N$ and by taking the supremum of the resulting function over $U$.

Condition (4) is equivalent to:

$$
\begin{equation*}
N^{2}|B(z)|-N|A(z)| \operatorname{Re} \frac{C(z)}{A(z)}+M+|D(z)| \leq 0 . \tag{7}
\end{equation*}
$$

If we suppose $B(z) \neq 0$, then the inequality (7) holds if

$$
\begin{equation*}
|A(z)| \operatorname{Re} \frac{C(z)}{A(z)} \geq 2 \sqrt{|B(z)|(M+|D(z)|)} \tag{8}
\end{equation*}
$$

If (8) holds, the roots of the trinomial in (7) are

$$
N_{1,2}=\frac{|A(z)| \operatorname{Re} \frac{C(z)}{A(z)} \pm \sqrt{\left[\left\lvert\, A(z) \operatorname{Re} \frac{C(z)}{A(z)}\right.\right]^{2}-4 M|B(z)|(M+|D(z)|)}}{2|B(z)|} .
$$

We let

$$
N=\frac{2(M+|D(z)|)}{|A(z)| \operatorname{Re} \frac{C(z)}{A(z)}+\sqrt{\left[|A(z)| \operatorname{Re} \frac{C(z)}{A(z)}\right]^{2}-4 M|C(z)|(M+|D(z)|)}} .
$$

If this supremum is finite, the Theorem can be rewritten as follows:
Corollary 1. Let $M>0$ and let $n$ be a positive integer. Suppose that $p \in \mathcal{H}[0, n]$ and let the functions $A, B, C, D: U \rightarrow \mathbb{C}$, with $A(z) \neq 0$.

If

$$
N=\sup _{|z|<1} \frac{2(M+|D(z)|)}{|A(z)| \operatorname{Re} \frac{C(z)}{A(z)}+\sqrt{\left[|A(z)| \operatorname{Re} \frac{C(z)}{A(z)}\right]^{2}-4 M|C(z)|(M+|D(z)|)}}<\infty
$$

then

$$
\left|A(z) z^{2} p^{\prime \prime}(z)+B(z) p^{2}(z)+C(z) p(z)+D(z)\right|<M
$$

implies $|p(z)|<N$.
Let $n=1, A(z)=-4, B(z)=12-5 i, C(z)=-20+7 i, D(z)=1-\sqrt{3} i, M=4$, we find $N=\frac{6}{10+\sqrt{22}}$.

In this case from Corollary 1 we deduce
Example 1. If $p \in \mathcal{H}[0,1]$, then

$$
\left|-4 z^{2} p^{\prime \prime}(z)+(12-5 i) p^{2}(z)+(-20+7 i) p(z)+(1-\sqrt{3} i)\right|<4
$$

implies

$$
|p(z)|<\frac{6}{10+\sqrt{22}}
$$

If $n=2, A(z)=6, B(z)=4+3 i, C(z)=18-5 i, D(z)=2 \sqrt{3}+2 i, M=5$, we find $N=\frac{6}{6+\sqrt{26}}$.

In this case from Corollary 1 we deduce
Example 2. If $p \in \mathcal{H}[0,2]$ then

$$
\left|6 z^{2} p^{\prime \prime}(z)+(4+3 i) p^{2}(z)+(18-5 i) p(z)+(2 \sqrt{3}+2 i)\right|<5
$$

implies

$$
|p(z)|<\frac{6}{10+\sqrt{22}}
$$

If $A(z)=A>0$ then the Theorem can be rewritten as follows:
Corollary 2. Let $M>0, N>0$ and let $n$ be a positive integer. Suppose that the functions $B, C, D: U \rightarrow \mathbb{C}$ satisfy

$$
\operatorname{Re} C(z) \geq \frac{M+N^{2}|B(z)|+|D(z)|}{N}
$$

If $p \in \mathcal{H}[0, n]$ and

$$
\left|A z^{2} p^{\prime \prime}(z)+B(z) p^{2}(z)+C(z) p(z)+D(z)\right|<M
$$

then $|p(z)|<N$.
If $n=1, A=8, B(z)=\sqrt{3}+i, C(z)=20-5 i, D(z)=3-4 i, M=8, N=2$.

In this case from Corollary 2 we deduce:
Example 3. If $p \in \mathcal{H}[0,1]$, and

$$
\left|8 z^{2} p^{\prime \prime}(z)+(\sqrt{3}+i) p^{2}(z)+(20-5 i) p(z)+(3-4 i)\right|<8
$$

then $|p(z)|<2$.
If $n=2, A=4, B(z)=-1-i \sqrt{3}, C(z)=16+4 i, D(z)=2+3 i, M=4, N=1$.

In this case from Corollary 2 we deduce:
Example 4. If $p \in \mathcal{H}[0,1]$, and

$$
\left|4 z^{2} p^{\prime \prime}(z)+(-1-i \sqrt{3}) p^{2}(z)+(16+4 i) p(z)+(2+3 i)\right|<4
$$

then $|p(z)|<1$.

## References

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# Invariant conditions for the dimensions of the $G L(2, R)$-orbits for one differential cubic system 

E.V. Starus


#### Abstract

A two-dimensional system of two autonomous polynomial equations with homogeneities of the zero and third orders is considered concerning to the group of center-affine transformations $G L(2, R)$. The problem of the classification of $G L(2, R)$ orbit's dimensions is solved completely for the given system with the help of Lie algebra of operators corresponding to the $G L(2, R)$ group, and algebra of invariants and comitants for the indicated system is built. The theorem on invariant division of all coefficient's set of the considered system to nonintersecting $G L(2, R)$-invariant sets is obtained.


Mathematics subject classification: 34C14, 34C05, 58F14.
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Consider the differential system

$$
\begin{equation*}
\frac{d x}{d \tau}=a+p x^{3}+3 q x^{2} y+3 r x y^{2}+s y^{3}, \frac{d y}{d \tau}=b+t x^{3}+3 u x^{2} y+3 v x y^{2}+w y^{3}, \tag{1}
\end{equation*}
$$

where the coefficients and variables take values from the field of real numbers $R$.
Let $A=(a, b, p, q, r, s, t, u, v, w) \in E(A)$, where $E(A)$ is the Euclidean space of the coefficients of right-hand sides of the system (1).

Will denote by $A(T)$ the point from $E(A)$ that belongs to the system, obtained from the system (1) with coefficients $A$ by transformation $T \in G L(2, R)$.

Definition 1. The set $O(A)=\{A(T) ; T \in G L(2, R)\}$ is called $G L(2, R)$-orbit of the point $A$ for the system (1).

Definition 2. Call the set $M \subseteq E(A) G L(2, R)$-invariant if for any point $A \in M$ its orbit $O(A) \subseteq M$.

It is known (see, for instance, [1]), that

$$
\operatorname{dim}_{R} O(A)=\operatorname{rank} M_{1}
$$

where $M_{1}$ is the matrix is constructed on the coordinate vectors of the Lie algebra operators obtained as a result of the representation of the $G L(2, R)$ group in the space $E(A)$ of the system (1).
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With the help of [1] it is possible to find that the matrix $M_{1}$ has the form

$$
M=\left(\begin{array}{cccccccccc}
-a & 0 & 2 p & q & 0 & -s & 3 t & 2 u & v & 0  \tag{2}\\
-b & 0 & -t & p-u & 2 q-v & 3 r-w & 0 & t & 2 u & 3 v \\
0 & -a & 3 q & 2 r & s & 0 & 3 u-p & 2 v-q & w-r & -s \\
0 & -b & 0 & q & 2 r & 3 s & -t & 0 & v & 2 w
\end{array}\right)
$$

Consider the invariants and comitants of the system (1) with respect to the group $G L(2, R)$, found in [2-3], which will be used further. With this purpose we rewrite the system (1) in the tensor form according to [4]

$$
\begin{equation*}
\frac{d x^{j}}{d t}=a^{j}+a_{\alpha \beta \gamma}^{j} x^{\alpha} x^{\beta} x^{\gamma}, \quad(j, \alpha, \beta, \gamma=1,2) \tag{3}
\end{equation*}
$$

where coefficient tensor $a_{\alpha \beta \gamma}^{j}$ is symmetrical in lower indexes, in which the complete convolution takes place. Note that among the coefficients and variables of the systems (1) and (3) there are equalities

$$
\begin{align*}
& x^{1}=x, a^{1}=a, a_{111}^{1}=p, a_{112}^{1}=q, a_{122}^{1}=r, a_{222}^{1}=s \\
& x^{2}=y, a^{2}=b, a_{111}^{2}=t, a_{112}^{2}=u, a_{122}^{2}=v, a_{222}^{2}=w \tag{4}
\end{align*}
$$

Then needed by us comitants and invariants of the system (3), and, consequently, of the system (1), take the form

$$
\begin{gather*}
P_{1}=a_{\alpha \beta \gamma}^{\alpha} x^{\beta} x^{\gamma}, P_{2}=a_{\alpha \beta \gamma}^{p} x^{\alpha} x^{\beta} x^{\gamma} x^{q} \epsilon_{p q}, P_{3}=a_{p \alpha \beta}^{\alpha} a_{q \gamma \delta}^{\beta} x^{\gamma} x^{\delta} \epsilon^{p q} \\
P_{4}=a_{\alpha \beta \gamma}^{\alpha} a_{\delta \mu \theta}^{\beta} x^{\gamma} x^{\delta} x^{\mu} x^{\theta}, P_{5}=a_{\beta \gamma \delta}^{\alpha} a_{\alpha \mu \theta}^{\beta} x^{\gamma} x^{\delta} x^{\mu} x^{\theta} \\
p_{2}=a_{\alpha \beta \gamma}^{p} a^{\alpha} x^{\beta} x^{\gamma} x^{q} \epsilon_{p q}, p_{9}=a_{\beta \gamma \delta}^{\alpha} a_{\alpha \mu \nu}^{\beta} a^{\gamma} x^{\delta} x^{\mu} x^{\nu}, p_{27}=a^{p} x^{q} \epsilon_{p q} \\
J_{1}=a_{\alpha p r}^{\alpha} a_{\beta q s}^{\beta} \epsilon^{p q} \epsilon^{r s}, J_{2}=a_{\beta p r}^{\alpha} a_{\alpha q s}^{\beta} \epsilon^{p q} \epsilon^{r s}, J_{4}=a_{p r u}^{\alpha} a_{\gamma q s}^{\beta} a_{\alpha \beta v}^{\gamma} \epsilon^{p q} \epsilon^{r s} \epsilon^{u v}, \tag{5}
\end{gather*}
$$

where $\varepsilon^{p q}\left(\varepsilon^{11}=\varepsilon^{22}=0, \quad \varepsilon^{12}=-\varepsilon^{21}=1\right.$ and $\varepsilon_{p q}\left(\varepsilon_{11}=\varepsilon_{22}=0, \varepsilon_{12}=-\varepsilon_{21}=1\right)$ are unit bivectors.

Considering (4) and (5) it is easy to establish the following
Remark 1. The condition $p_{27} \equiv 0$ for the system (1) is equivalent to the equalities

$$
\begin{equation*}
a=b=0 \tag{6}
\end{equation*}
$$

Taking into account Remark 1, Theorem 1.44 and Lemma 1.44 from [1] it is easy to obtain

Lemma 1. If $p_{27} \equiv 0$ the rank of matrix (2) is equal to

$$
\begin{aligned}
& 4 \quad \text { for } \quad P_{1} P_{2}\left(3 P_{1} P_{3}-2 J_{1} P_{2}\right) \not \equiv 0, \text { or } \\
& 50 P_{1} \equiv 0, P_{2}\left(J_{2} P_{5}-J_{4} P_{2}\right) \not \equiv 0 ; \\
& 3 \text { for } P_{1} P_{2} \not \equiv 0,3 P_{1} P_{3}-2 J_{1} P_{2} \equiv 0, \text { or } \\
& P_{2} P_{5} \not \equiv 0, P_{1} \equiv J_{2} P_{5}-J_{4} P_{2} \equiv 0, \text { or }
\end{aligned}
$$

$$
\begin{aligned}
& P_{2} \equiv 0, J_{1} \neq 0 ; \\
& \quad \text { for } P_{2} \not \equiv 0, P_{1} \equiv J_{2} P_{5}-J_{4} P_{2} \equiv P_{5} \equiv 0 \text {, or } \\
& P_{2} \equiv 0, J_{1}=0, P_{1} \not \equiv 0 ; \\
& 0 \quad \text { for } P_{1} \equiv P_{2} \equiv 0, \\
& \text { where } P_{1}, P_{2}, P_{3}, P_{5}, J_{1}, J_{2}, J_{4} \text { are taken from (5). }
\end{aligned}
$$

Let us prove
Lemma 2. If $P_{2} \equiv 0$ the rank of matrix (2) is equal to

$$
\begin{array}{ll}
4 & \text { for } \\
3 & J_{1} p_{27} \not \equiv 0 \\
3 & \text { for } \\
P_{1} \not \equiv 0, p_{27} J_{1} \equiv 0, p_{27}+J_{1} \not \equiv 0 ;  \tag{10}\\
2 & \text { for } \\
P_{1} \not \equiv 0, p_{27} \equiv 0, J_{1}=0, \text { or } P_{1} \equiv 0, p_{27} \not \equiv 0 ; \\
0 & \text { for } \\
P_{1} \equiv p_{27} \equiv 0 \\
\text { where } & P_{1}, P_{2}, p_{27}, J_{1} \text { are taken from (5). }
\end{array}
$$

Proof. Consider two cases: 1) If $p_{27} \equiv 0$, owing to the fact that $J_{1} \neq 0$ implies that $P_{1} \not \equiv 0$ (see [1]), we obtain that the corresponding cases of Lemma 2 coincide with the corresponding cases of Lemma 1, and, hence, its truth is evident.
$2)$ Let $p_{27} \not \equiv 0$, i.e., according to (4)-(5), we have

$$
\begin{equation*}
a^{2}+b^{2} \neq 0 . \tag{11}
\end{equation*}
$$

Since $P_{2} \equiv 0$ from (4)-(5) we obtain for the system (1)

$$
\begin{equation*}
t=0, \quad p=3 u, \quad q=v, \quad w=3 r, \quad s=0 . \tag{12}
\end{equation*}
$$

Because of (12), removing the zero columns matrix (2) takes the form

$$
M_{1}^{(1)}=\left(\begin{array}{ccccc}
-a & 0 & 0 & 2 u & v  \tag{13}\\
-b & 0 & v & 0 & 2 u \\
0 & -a & 0 & v & 2 r \\
0 & -b & 2 r & 0 & v
\end{array}\right)
$$

Consider the following subcases:
a) Denote by the $\Delta_{i j k l}(1 \leq i, j, k, l \leq 5)$ every possible minors of the fourth order of the matrix $M_{1}^{(1)}$ constructed on its columns with the numbers $i, j, k, l$. There is no difficulty to see that the following minors will be different from zero

$$
\begin{gather*}
8 \Delta_{1234}=-a b J_{1}, \quad 8 \Delta_{1235}=a^{2} J_{1}, \quad 8 \Delta_{1245}=-b^{2} J_{1}, \\
8 \Delta_{1345}=(a v+2 b r) J_{1}, \quad 8 \Delta_{2345}=-(b v+2 a u) J_{1}, \tag{14}
\end{gather*}
$$

where $J_{1}$ is invariant from (5), having in this case for the system (1) the form

$$
\begin{equation*}
J_{1}=-8\left(v^{2}-4 u r\right) . \tag{15}
\end{equation*}
$$

Taking into account (11) and (14) we note that the rank of matrix $M_{1}^{(1)}$ is equal to 4 if and only if the condition (7) takes place.
b) Due to (14) and (15) we can say that if with $p_{27} \not \equiv 0$ takes place the condition

$$
\begin{equation*}
J_{1}=0 \quad\left(v^{2}=4 u r\right) \tag{16}
\end{equation*}
$$

then all minors of the fourth order of the matrix (11) are equal to zero and, hence, $\operatorname{rank} M_{1}^{(1)}<4$.

There is no difficulty to check that from 40 different minors of the third order of the matrix (13) the following will be different from zero if (16) takes place:

$$
\begin{gather*}
\Delta_{123}^{123}=-a^{2} v, \Delta_{124}^{123}=2 a b u, \Delta_{125}^{123}=a b v-2 a^{2} u, \Delta_{245}^{123}=-4 a u^{2} \\
\Delta_{145}^{123}=\Delta_{234}^{123}=\Delta_{245}^{134}=2 a u v, \Delta_{124}^{124}=2 b^{2} u, \Delta_{125}^{124}=b^{2} v-2 a b u \\
\Delta_{134}^{124}=-4 b u r, \Delta_{135}^{124}=-\Delta_{134}^{234}=\Delta_{235}^{234}=-2 b v r, \Delta_{245}^{124}=-4 b u^{2} \\
\Delta_{123}^{134}=2 a^{2} r, \Delta_{125}^{134}=a^{2} v-2 a b r, \Delta_{135}^{134}=4 a r^{2}, \Delta_{234}^{134}=-4 a r u \\
\Delta_{123}^{234}=2 a b r, \Delta_{124}^{234}=-b^{2} v, \Delta_{125}^{234}=a b v-2 b^{2} r, \Delta_{135}^{234}=4 b r^{2} \\
\Delta_{134}^{123}=-\Delta_{235}^{123}=\Delta_{145}^{134}=-a v^{2}, \Delta_{135}^{123}=-\Delta_{134}^{134}=\Delta_{235}^{134}=-2 a r v \\
\Delta_{123}^{124}=\Delta_{124}^{134}=-a b v, \Delta_{235}^{124}=-\Delta_{145}^{234}=-\Delta_{234}^{234}=b v^{2} \\
\Delta_{145}^{124}=\Delta_{234}^{124}=\Delta_{245}^{234}=2 b u v \tag{17}
\end{gather*}
$$

where $\Delta_{l m n}^{i j k}(1 \leq i, j, k \leq 4 ; 1 \leq l, m, n \leq 5)$ are indicated minors of the matrix $M_{1}^{(1)}$ constructed on lines $i, j, k$ and columns $l, m, n$.

As with the help of (4)-(5) and (12) for $P_{1}$ we obtain

$$
\begin{equation*}
P_{1}=4 u x^{2}+4 v x y+4 r y^{2} \tag{18}
\end{equation*}
$$

then with (11) we have that at least one of minors (17) is different from zero if and only if $P_{1} p_{27} \not \equiv 0$ in this subcase. Therefore (8) is true. Let us note that $J_{1}=0$ does not contradict to $P_{1} \not \equiv 0$.
c) It follows from (17) and (18) that if with $p_{27} \not \equiv 0$ the equality

$$
\begin{equation*}
P_{1} \equiv 0 \quad(u=v=r=0) \tag{19}
\end{equation*}
$$

takes place then all minors of the third order of matrix (13) are equal to zero and hence $\operatorname{rank} M_{1}^{(1)}<3$.

Let form all possible unzero minors of the second order of matrix (13), which will denote by $\Delta_{k l}^{i j}(1 \leq i, j \leq 4 ; 1 \leq k, l \leq 2)$. It is not difficult to see that they are the following:

$$
\begin{equation*}
\Delta_{12}^{13}=a^{2}, \Delta_{12}^{14}=\Delta_{12}^{23}=a b, \Delta_{12}^{24}=b^{2} \tag{20}
\end{equation*}
$$

With (11) at least one of minors (20) is different from zero and hence the rank of matrix (13) is equal to 2 . And this provides the fulfillment of the second condition from (9).

The case (10) is evident. Lemma 2 is proved.

Lemma 3. If $P_{1} P_{2} \not \equiv 0$ the rank of matrix (2) is equal to 4 if and only if

$$
\begin{equation*}
2 P_{2}\left(3 P_{1} P_{3}-2 J_{1} P_{2}\right)+W_{1} \not \equiv 0 \tag{21}
\end{equation*}
$$

where

$$
\begin{equation*}
W_{1}=3 P_{2}\left(P_{3} p_{27}+2 p_{9}\right)+2 P_{5}\left(P_{1} p_{27}-4 p_{2}\right), \tag{22}
\end{equation*}
$$

and $P_{1}, P_{2}, P_{3}, P_{5}, p_{2}, p_{9}, p_{27}, J_{1}$ are taken from (5).
Proof. Necessity. Similarly to the proof of Lemma 2.44 from [1] we consider 3 cases.

1) The discriminant $D\left(P_{1}\right)>0$. Then, taking into consideration the expression for $P_{1}$ from (4)-(5) it is easy to check that by the center-affine transformation [1] we obtain

$$
\begin{equation*}
P_{1}=2(q+v) x y, p=-u, r=-w, \tag{23}
\end{equation*}
$$

where

$$
\begin{equation*}
q+v \neq 0 \tag{24}
\end{equation*}
$$

Let assume that the condition (21) is not necessary, i.e. that $2 P_{2}\left(3 P_{1} P_{3}-2 J_{1} P_{2}\right)+$ $W_{1} \equiv 0$. Since the expressions $2 P_{2}\left(3 P_{1} P_{3}-2 J_{1} P_{2}\right)$ and $W_{1}$ have 8 th and 7 th degrees, respectively, concerning variables $x, y$, then the last identity is equivalent to the system

$$
\begin{gather*}
2 P_{2}\left(3 P_{1} P_{3}-2 J_{1} P_{2}\right) \equiv 0,  \tag{25}\\
W_{1} \equiv 0 \tag{26}
\end{gather*}
$$

Taking into consideration the expression for $P_{1}$ from (4)-(5), conditions (23), (25), we obtain the following values for the coefficients of the system (1)

$$
\begin{equation*}
p=r=s=t=u=w=0 . \tag{27}
\end{equation*}
$$

With these coefficients the comitant $P_{2}$ and the identity (26) take the form

$$
\begin{gathered}
P_{2}=3(q-v) x^{2} y^{2} \\
W_{1}=-6 b(q-v)\left(-q^{2}+q v+5 v^{2}\right) x^{4} y^{3}+6 a(q-v)\left(-5 q^{2}-q v+v^{2}\right) x^{3} y^{4} \equiv 0 .
\end{gathered}
$$

From the last identity with $P_{2} \not \equiv 0$ we obtain for the coefficients of the system (1) as real values that $a=b=0$. In this case removing the zero columns matrix (2) takes the form

$$
M_{1}^{(2)}=\left(\begin{array}{cccccc}
0 & q & 0 & 0 & v & 0 \\
0 & 0 & 2 q-v & 0 & 0 & 3 v \\
3 q & 0 & 0 & 2 v-q & 0 & 0 \\
0 & q & 0 & 0 & v & 0
\end{array}\right) .
$$

There is no difficulty to see that all the minors of the fourth order of the matrix $M_{1}^{(2)}$ are equal to zero, i.e. $\operatorname{rank} M_{1}^{(2)}<4$.

Obtained contradictions prove the necessity of the condition (21).
2) The discriminant $D\left(P_{1}\right)=0$. Then, taking into consideration the expression for $P_{1}$ from (4)-(5) it is easy to check that by the center-affine transformation [1] we obtain

$$
\begin{equation*}
P_{1}=(p+u) x^{2}, q=-v, r=-w, \tag{28}
\end{equation*}
$$

where

$$
\begin{equation*}
p+u \neq 0 \tag{29}
\end{equation*}
$$

Let assume that the condition (21) is not necessary, i.e. let consider (25)-(26). Taking into account (25), (28), (29), we obtain the following values for the coefficients of the system (1):

$$
\begin{equation*}
q=r=s=v=w=0 \tag{30}
\end{equation*}
$$

With these coefficients the comitant $P_{2}$ and the identity (26) take the form

$$
\begin{gathered}
P_{2}=-t x^{4}+(p-3 u) x^{3} y \\
W_{1}=2\left(p^{2}+u^{2}\right)(a t-b p+3 b u) x^{7} \equiv 0 .
\end{gathered}
$$

From the last identity with (29) and $P_{2} \not \equiv 0$ we obtain the following real values for the coefficients of the system (1):
a) $a=\frac{b(p-3 u)}{t},(t \neq 0)$. In this case removing zero columns the matrix (2) takes the form

$$
M_{1}^{(3)}=\left(\begin{array}{ccccccc}
\frac{b(3 u-p)}{t} & 0 & 2 p & 0 & 3 t & 2 u & 0 \\
-b & 0 & -t & p-u & 0 & t & 2 u \\
0 & \frac{b(3 u-p)}{t} & 0 & 0 & 3 u-p & 0 & 0 \\
0 & -b & 0 & 0 & -t & 0 & 0
\end{array}\right)
$$

There is no difficulty to see that all the minors of the fourth order of the matrix $M_{1}^{(3)}$ are equal to zero, i.e. $\operatorname{rank} M_{1}^{(3)}<4$, that proves the necessity of the conditions (21).
b) $b=t=0$. In this case removing zero columns the matrix (2) takes the form

$$
M_{1}^{(4)}=\left(\begin{array}{ccccccc}
-a & 0 & 2 p & 0 & 0 & 2 u & 0 \\
0 & 0 & 0 & p-u & 0 & 0 & 2 u \\
0 & -a & 0 & 0 & 3 p-u & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) .
$$

There is no difficulty to see that all the minors of the fourth order of the matrix $M_{1}^{(4)}$ are equal to zero, i.e. $\operatorname{rank} M_{1}^{(3)}<4$, that proves the necessity of the conditions (21).
3) The discriminant $D\left(P_{1}\right)<0$. Then, taking into consideration the expression for $P_{1}$ from (4)-(5) it is easy to check that by the center-affine transformation [1] we obtain

$$
\begin{equation*}
P_{1}=A\left(x^{2}+y^{2}\right) \not \equiv 0, A=p+u=r+w . \tag{31}
\end{equation*}
$$

Let assume that the condition (21) is not necessary, i.e. let consider (25)-(26). Taking into account (25) and (31), we obtain the following values for the coefficients of the system (1):

$$
p=w=3 A / 4, q=-v, r=u=A / 4, s=-t=-3 v .
$$

With these coefficients the comitant $P_{2}$ and the identity (26) take the form

$$
\begin{gathered}
P_{2}=-3 v\left(x^{2}+y^{2}\right)^{2} \\
W_{1}=-\frac{3 v}{4}\left[\left(48 a v^{2}+48 b v A-5 a A^{2}\right) x+\left(48 b v^{2}-48 a v A-5 b A^{2}\right) y\right]\left(x^{2}+y^{2}\right)^{3} \equiv 0 .
\end{gathered}
$$

Taking into consideration the last identity with $P_{2} \not \equiv 0$ we obtain the following values for the coefficients of the system (1):

$$
a=b=0,(v \neq 0) .
$$

In this case removing zero columns the matrix (2) takes the form

$$
M_{1}^{(5)}=\left(\begin{array}{cccccccc}
\frac{3 A}{2} & -v & 0 & 3 v & 9 v & \frac{A}{2} & v & 0 \\
-3 v & \frac{A}{2} & -3 v & 0 & 0 & 3 v & \frac{A}{2} & 3 v \\
-3 v & \frac{A}{2} & -3 v & 0 & 0 & 3 v & \frac{A}{2} & 3 v \\
0 & -v & \frac{A}{2} & -9 v & -3 v & 0 & v & \frac{3 A}{2}
\end{array}\right) .
$$

Note that the second and third lines of the matrix $M_{1}^{(5)}$ coincide, hence $\operatorname{rank} M_{1}^{(5)}<$ 4 , that proves the necessity of the conditions (21).

The necessity of the conditions (21) is proved completely.
Sufficiency of the conditions (21) with $P_{1} P_{2} \not \equiv 0$ follows from the expression $2 P_{2}\left(3 P_{1} P_{3}-2 J_{1} P_{2}\right)$ written by the minors of the matrix (2), see [1], p.164; and the expression

$$
\begin{aligned}
& W_{1}=\left(-2 \Delta_{1378}+\Delta_{2347}-\Delta_{2379}-\Delta_{2478}\right) x^{7}+\left(6 \Delta_{1347}-3 \Delta_{1379}-14 \Delta_{1478}-18 \Delta_{1789}-\right. \\
& \left.\quad-7 \Delta_{2348}+10 \Delta_{2357}+\Delta_{23710}-12 \Delta_{2389}-11 \Delta_{2479}+4 \Delta_{2578}-5 \Delta_{27810}\right) x^{6} y+ \\
& +\left(-5 \Delta_{1348}-29 \Delta_{1357}-4 \Delta_{13710}+23 \Delta_{1479}-13 \Delta_{1578}+18 \Delta_{17810}+16 \Delta_{2349}+10 \Delta_{2358}-\right. \\
& \left.-21 \Delta_{2367}-15 \Delta_{2457}+\Delta_{24710}+33 \Delta_{2489}-7 \Delta_{2579}-22 \Delta_{2678}+10 \Delta_{27910}\right) x^{5} y^{2}+ \\
& +\left(-37 \Delta_{1349}-10 \Delta_{1358}+8 \Delta_{1367}+24 \Delta_{1457}-96 \Delta_{1489}+13 \Delta_{1579}+9 \Delta_{1678}-14 \Delta_{17910}-\right. \\
& -15 \Delta_{2345}+\Delta_{23410}-37 \Delta_{2359}+31 \Delta_{2368}-10 \Delta_{23910}-48 \Delta_{2458}+27 \Delta_{2467}-38 \Delta_{24810}- \\
& \left.-9 \Delta_{25710}-33 \Delta_{2589}+25 \Delta_{2679}-6 \Delta_{28910}\right) x^{4} y^{3}+\left(-6 \Delta_{1345}-10 \Delta_{13410}-38 \Delta_{1359-}-\right. \\
& -9 \Delta_{1368}+\Delta_{13910}-33 \Delta_{1458}+25 \Delta_{1467}-37 \Delta_{14810}+31 \Delta_{15710}-48 \Delta_{1589}+27 \Delta_{1679}- \\
& -15 \Delta_{18910}-14 \Delta_{2346}-96 \Delta_{2459}+13 \Delta_{2468}-37 \Delta_{24910}+9 \Delta_{2567}-10 \Delta_{25810}+8 \Delta_{26710}+ \\
& \left.+24 \Delta_{2689}\right) x^{3} y^{4}+\left(10 \Delta_{1346}+\Delta_{1369}+33 \Delta_{1459}-7 \Delta_{1468}+16 \Delta_{14910}-22 \Delta_{1567}+\right.
\end{aligned}
$$

$$
\begin{gather*}
+10 \Delta_{15810}-21 \Delta_{16710}-15 \Delta_{1689}+18 \Delta_{2356}-4 \Delta_{23610}+23 \Delta_{2469}-13 \Delta_{2568}-5 \Delta_{25910}- \\
\left.-29 \Delta_{26810}\right) x^{2} y^{5}+\left(-5 \Delta_{1356}+\Delta_{13610}-12 \Delta_{14510}-11 \Delta_{1469}+4 \Delta_{1568}-7 \Delta_{15910}+\right. \\
\left.+10 \Delta_{16810}-18 \Delta_{2456}-3 \Delta_{24610}-14 \Delta_{2569}+6 \Delta_{26910}\right) x y^{6}+\left(-\Delta_{14610}-\Delta_{1569}+\Delta_{16910}-\right. \\
\left.-2 \Delta_{25610}\right) y^{7}, \tag{32}
\end{gather*}
$$

where $\Delta_{i j k l}(1 \leq i, j, k, l \leq 10)$ is the minor of the fourth order of the matrix (2) constructed on its columns with the numbers $i, j, k, l$. Lemma 3 is proved.

Lemma 4. If $P_{1} \equiv 0, P_{2} \not \equiv 0$ the rank of the matrix (2) is equal to 4 if and only if

$$
\begin{equation*}
J_{2} P_{5}-J_{4} P_{2}+W_{1}+W_{2} \not \equiv 0 \tag{33}
\end{equation*}
$$

where $W_{1}$ is taken from (22), and

$$
\begin{equation*}
W_{2}=p_{27}^{2}\left(P_{1}^{2}+6 P_{4}-9 P_{5}\right)+2 p_{2}^{2} \tag{34}
\end{equation*}
$$

where $P_{1}, P_{2}, P_{4}, P_{5}, p_{2}, p_{27}, J_{2}, J_{4}$ are taken from (5).
Proof. Necessity. From $P_{1} \equiv 0$ we obtain the following values for the coefficients of the system (1):

$$
\begin{equation*}
p=-u, q=-v, r=-w . \tag{35}
\end{equation*}
$$

With coefficients (35) comitants $P_{2}, W_{1}$ and $W_{2}$ take the form

$$
\begin{align*}
& P_{2}=-t x^{4}-4 u x^{3} y-6 v x^{2} y^{2}-4 w x y^{3}+s y^{4},  \tag{36}\\
& W_{1}=\left(-4 a t^{2} v+4 a t u^{2}+6 b t^{2} w-22 b t u v+16 b u^{3}\right) x^{7}+\left(-14 a t^{2} w+14 a t u v-6 b s t^{2}+\right. \\
& \left.+4 b t u w-66 b t v^{2}+56 b u^{2} v\right) x^{6} y+\left(10 a s t^{2}-44 a t u w+54 a t v^{2}-26 b s t u-126 b t v w+\right. \\
& \left.+64 b u^{2} w+36 b u v^{2}\right) x^{5} y^{2}+\left(50 a s t u+70 a t v w-80 a u^{2} w+60 a u v^{2}+16 b s t v-56 b s u^{2}-\right. \\
& \left.-84 b t w^{2}+8 b u v w+36 b v^{3}\right) x^{4} y^{3}+\left(16 a s t v+84 a s u^{2}+56 a t w^{2}+8 a u v w+36 a v^{3}+\right. \\
& \left.+50 b s t w-70 b s u v-80 b u w^{2}+60 b v^{2} w\right) x^{3} y^{4}+\left(-26 a s t w+126 a s u v+64 a u w^{2}+\right. \\
& \left.+36 a v^{2} w-10 b s^{2} t+44 b s u w-54 b s v^{2}\right) x^{2} y^{5}+\left(6 a s^{2} t-4 a s u w+66 a s v^{2}+56 a v w^{2}-\right. \\
& \left.-14 b s^{2} u-14 b s v w\right) x y^{6}+\left(6 a s^{2} u+22 a s v w+16 a w^{3}-4 b s^{2} v-4 b s w^{2}\right) y^{7},
\end{align*}
$$

and

$$
\begin{align*}
& W_{2}=\left(2 a^{2} t^{2}+4 a b t u+18 b^{2} t v-16 b^{2} u^{2}\right) x^{6}+\left(12 a^{2} t u-24 a b t v+48 a b u^{2}+36 b^{2} t w-\right. \\
& \left.-24 b^{2} u v\right) x^{5} y+\left(30 a^{2} t v-60 a b t w+120 a b u v-18 b^{2} s t+48 b^{2} u w-36 b^{2} v^{2}\right) x^{4} y^{2}+ \\
& +\left(40 a^{2} t w+32 a b s t-32 a b u w+144 a b v^{2}-40 b^{2} s u\right) x^{3} y^{3}+\left(-18 a^{2} s t+48 a^{2} u w-\right. \\
& \left.-36 a^{2} v^{2}+60 a b s u+120 a b v w-30 b^{2} s v\right) x^{2} y^{4}+\left(-36 a^{2} s u-24 a^{2} v w+24 a b s v+\right. \\
& \left.\quad+48 a b w^{2}-12 b^{2} s w\right) x y^{5}+\left(-18 a^{2} s v-16 a^{2} w^{2}-4 a b s w+2 b^{2} s^{2}\right) y^{6} . \tag{38}
\end{align*}
$$

Let assume that the condition (33) is not necessary i.e. that if $J_{2} P_{5}-J_{4} P_{2}+W_{1}+$ $W_{2} \equiv 0$, than there are some minors of the fourth order of the matrix (2) which are different from zero. As the expressions $J_{2} P_{5}-J_{4} P_{2}, W_{1}$ and $W_{2}$ have 4th, 7 th and 6 th degrees, respectively, concerning variables $x, y$, then the last identity is equivalent to the system

$$
\begin{equation*}
J_{2} P_{5}-J_{4} P_{2} \equiv W_{1} \equiv W_{2} \equiv 0 . \tag{39}
\end{equation*}
$$

With (35) from the first identity from (39) we obtain the system of the polynomial equations of the fourth degree concerning coefficients of the system (1). Solving indicated system (see [1], ? 171), we obtain the following four real solutions:

$$
\begin{equation*}
\text { 1) } p=q=r=s=t=u=v=w=0 \text {. } \tag{40}
\end{equation*}
$$

With these coefficients $P_{2} \equiv 0$, what contradicts to the condition of Lemma 4.

$$
\begin{equation*}
\text { 2) } p=q=t=u=v=0, r=-w \text {. } \tag{41}
\end{equation*}
$$

With these equalities unzero minors of the fourth order of the matrix (2) will be the following

$$
\begin{gather*}
\Delta_{16910}=\Delta_{1569}=\Delta_{14610}=\Delta_{1456}=-2 \Delta_{1269}=-2 \Delta_{1246}=4 w^{2}(4 a w-b s), \\
\Delta_{12610}=\Delta_{1256}=8 a^{2} w^{2}+2 a b s w-b^{2} s^{2} . \tag{42}
\end{gather*}
$$

And expressions $W_{1}$ and $W_{2}$ take the form

$$
\begin{gather*}
W_{1}=4 w^{2}(4 a w-b s) y^{7} \\
W_{2}=12 b w(4 a w-b s) x y^{5}+2\left(-8 a^{2} w^{2}-2 a b s w+b^{2} s^{2}\right) y^{6} . \tag{43}
\end{gather*}
$$

Taking into consideration the last equalities from (39) and the obtained with the help (42)-(43) contradiction we find the necessity of the conditions (33) in this case, too.

$$
\begin{equation*}
\text { 3) } p=t=u=0, q=-v, r=-w, 3 s v=-2 w^{2} \text {. } \tag{44}
\end{equation*}
$$

Let substitute these coefficients into $W_{1}$ and $W_{2}$ :

$$
\begin{gathered}
W_{1}=36 b v^{3} x^{4} y^{3}+\left(60 b v^{2} w+36 a v^{3}\right) x^{3} y^{4}+\left(36 a v^{2} w+36 b v w^{2}\right) x^{2} y^{5}+\left(12 a v w^{2}+\right. \\
\left.\quad+\frac{28}{3} b w^{3}\right) x y^{6}+\left(\frac{4}{3} a w^{3}-\frac{4}{3} b s w^{2}\right) y^{7} ; \\
W_{2}=36 b^{2} v^{2} x^{4} y^{2}+144 a b v^{2} x^{3} y^{3}+\left(-36 a^{2} v^{2}+120 a b v w+20 b^{2} w^{2}\right) x^{2} y^{4}+ \\
+\left(-24 a^{2} v w+32 a b w^{2}-12 b^{2} s w\right) x y^{5}+\left(-4 a^{2} w^{2}-4 a b s w+2 b^{2} s^{2}\right) y^{6} .
\end{gathered}
$$

From $W_{1} \equiv W_{2} \equiv 0$ with the last equalities we obtain:
a) $a=b=0,(v \neq 0)$. Matrix (2) takes the form

$$
M_{1}^{(6)}=\left(\begin{array}{cccccccccc}
0 & 0 & 0 & -v & 0 & -s & 0 & 0 & v & 0 \\
0 & 0 & 0 & 0 & -3 v & -4 w & 0 & 0 & 0 & 3 v \\
0 & 0 & -3 v & -2 w & s & 0 & 0 & 3 v & 2 w & -s \\
0 & 0 & 0 & -v & -2 w & 3 s & 0 & 0 & v & 2 w
\end{array}\right) .
$$

There is no difficulty to see that all the minors of the fourth order of this matrix are equal to zero, i.e. $\operatorname{rank} M_{1}^{(6)}<4$. Therefore the conditions (33) are necessary in this case.
b) With $v=0$ from (44) we obtain $p=q=r=t=u=v=w=0$. With these equalities the following minors of the fourth order of the matrix (2) will be different from zero

$$
\begin{equation*}
\Delta_{1256}=\Delta_{12610}=-b^{2} s^{2} \tag{45}
\end{equation*}
$$

and the expressions $W_{1}$ and $W_{2}$ take the form

$$
\begin{equation*}
W_{1} \equiv 0, W_{2}=2 b^{2} s^{2} y^{6} \tag{46}
\end{equation*}
$$

If we demand that the equality $W_{2} \equiv 0$ from (46) takes place, then with the help of (45) we obtain that in this subcase all the minors of the fourth order of the matrix (2) are equal to zero. This contradiction proves the necessity of the condition (33).

$$
\begin{equation*}
\text { 4) } t=0,4 u w=3 v^{2}, 2 s u^{2}=-v^{3} \tag{47}
\end{equation*}
$$

Let substitute these coefficients into $W_{1}$ and $W_{2}$ :

$$
\begin{aligned}
& W_{1}=16 b u^{3} x^{7}+56 b u^{2} v x^{6} y+112 b u^{2} w x^{5} y^{2}+\left(-128 b s u^{2}+8 b u v w\right) x^{4} y^{3}+\left(12 a s u^{2}+\right. \\
& +8 a u v w-70 b s u v) x^{3} y^{4}+\left(126 a s u v+84 a v^{2} w-28 b s u w\right) x^{2} y^{5}+\left(63 a s v^{2}+56 a v w^{2}-\right. \\
& \left.\quad-14 b s^{2} u-14 b s v w\right) x y^{6}+\left(6 a s^{2} u+22 a s v w+16 a w^{3}-4 b s^{2} v-4 b s w^{2}\right) y^{7} ; \\
& \left.\begin{array}{c}
W_{2}= \\
+\left(60 a b s u b^{2} u^{2} x^{6}+\left(48 a b u^{2}-24 b^{2} u v\right) x^{5} y+120 a b u v x^{4} y^{2}+\left(160 a b u w-40 b^{2} s u\right) x^{3} y^{3}+\right. \\
\left.\quad-12 b^{2} s w\right) x y^{5}+\left(-18 a^{2} s v\right) x^{2} y^{4}+\left(-36 a^{2} s u-24 a^{2} v w+24 a b s v+48 a b w^{2}-\right. \\
+
\end{array} w^{2}-4 a b s w+2 b^{2} s^{2}\right) y^{6} .
\end{aligned}
$$

From $W_{1} \equiv W_{2} \equiv 0$ with the last equalities we obtain $b u=b v=0$.
If $b \neq 0$, then we come to the case 1) from the proof of the necessity in Lemma 4.

If $b=0$ then we have:

$$
\begin{gathered}
\begin{array}{c}
W_{1}=4 a u(2 v w+3 s u) x^{3} y^{4}+42 a v(2 v w+3 s u) x^{2} y^{5}+7 a v\left(9 s v+8 w^{2}\right) x y^{6}+2 a\left(3 s^{2} u+\right. \\
\left.+11 s v w+8 w^{3}\right) y^{7} ; \\
W_{2}=-12 a^{2}(2 v w+3 s u) x y^{5}-2 a^{2}\left(8 w^{2}+9 s v\right) y^{6} .
\end{array} .
\end{gathered}
$$

From $W_{1} \equiv W_{2} \equiv 0$ we obtain the following subcases:
a) $a=0$. This subcase is considered in ([1], ?.173).
b) $a \neq 0, s \neq 0, p=q=r=u=v=w=0$. Taking into account the case 3) b) from the proof of the necessity in Lemma 4 we conclude that all the minors of the fourth order are equal to zero here, that proves the necessity of the condition (33).
c) $a \neq 0, u \neq 0, q=r=s=v=w=0$. The matrix (2) in this case takes the form

$$
M_{1}^{(7)}=\left(\begin{array}{cccccccccc}
-a & 0 & -2 u & 0 & 0 & 0 & 0 & 2 u & 0 & 0 \\
0 & 0 & 0 & -2 u & 0 & 0 & 0 & 0 & 2 u & 0 \\
0 & -a & 0 & 0 & 0 & 0 & 4 u & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) .
$$

There is no difficulty to see that all the minors of the fourth order of the matrix $M_{1}^{(7)}$ are equal to zero, i.a. $\operatorname{rank} M_{1}(7)<4$. Hence the condition (33) is necessary in this subcase.
d) $a \neq 0, v \neq 0, w \neq 0, u=\frac{3 v^{2}}{4 w}, s=-\frac{8 w^{2}}{9 v}$. Matrix (2) takes the form in this case

$$
M_{1}^{(8)}=\left(\begin{array}{cccccccccc}
-a & 0 & -\frac{3 v^{2}}{2 w} & -v & 0 & \frac{8 w^{2}}{9 v} & 0 & \frac{3 v^{2}}{2 w} & v & 0 \\
0 & 0 & 0 & -\frac{3 v^{2}}{2 w} & -3 v & -4 w & 0 & 0 & \frac{3 v^{2}}{2 w} & 3 v \\
0 & -a & -3 v & -2 w & -\frac{8 w^{2}}{9 v} & 0 & \frac{3 v^{2}}{w} & 3 v & 2 w & \frac{8 w^{2}}{9 v} \\
0 & 0 & 0 & -v & -2 w & -\frac{8 w^{2}}{3 v} & 0 & 0 & v & 2 w
\end{array}\right) .
$$

There is no difficulty to see that all the minors of the fourth order of the matrix $M_{1}^{(8)}$ are equal to zero, i.a. $\operatorname{rank} M_{1}(8)<4$. Hence the condition (33) is necessary in this subcase.

The necessity of the condition (33) is proved completely.
Sufficiency of the condition (33) with $P_{1} \equiv 0, P_{2} \not \equiv 0$ follows from the expression $J_{2} P_{5}-J_{4} P_{2}$ written by the minors of the matrix (2), see [1], p.169, and the expression

$$
\begin{aligned}
& W_{2}=\left(-\Delta_{1237}-\Delta_{1278}\right) x^{6}+\left(-2 \Delta_{1238}-4 \Delta_{1247}-2 \Delta_{1279}\right) x^{5} y+\left(-\Delta_{1234}-\Delta_{1239}-\right. \\
& \left.-9 \Delta_{1248}-5 \Delta_{1257}-\Delta_{12710}-3 \Delta_{1289}\right) x^{4} y^{2}+\left(-2 \Delta_{1235}-6 \Delta_{1249}-12 \Delta_{1258}-2 \Delta_{1267}-\right. \\
& \left.-2 \Delta_{12810}\right) x^{3} y^{3}+\left(-\Delta_{1236}-3 \Delta_{1245}-\Delta_{12410}-9 \Delta_{1259}-5 \Delta_{1268}-\Delta_{12910}\right) x^{2} y^{4}+ \\
& +\left(-2 \Delta_{1246}-\Delta_{12510}-4 \Delta_{1269}\right) x y^{5}+\left(-\Delta_{1256}-\Delta_{12610}\right) y^{6},
\end{aligned}
$$

where $\Delta_{i j k l},(1 \leq i, j, k, l \leq 10)$ - is the minor of the fourth order of the matrix (2), constructed on its columns with the numbers $i, j, k, l$. Lemma 4 is proved.
Lemma 5. If $P_{1} P_{2} \not \equiv 0$ the rank of the matrix (2) is equal to 3 if and only if

$$
\begin{equation*}
2 P_{2}\left(3 P_{1} P_{3}-2 J_{1} P_{2}\right)+W_{1} \equiv 0, \tag{48}
\end{equation*}
$$

where $P_{1}, P_{2}, P_{3}, J_{1}$ are taken from (5), and $W_{1}$ from (22).
Proof. Necessity of the conditions (48) follows from Lemma 3.
Sufficiency of the conditions (48) can be proved similarly to the first part of the proof of the sufficiency of Lemma 3.44 from [1]. Lemma 5 is proved.

Lemma 6. If $P_{1} \equiv 0, P_{2} \not \equiv 0$ the rank of the matrix (2) is equal to 3 if and only if

$$
\begin{equation*}
J_{2} P_{5}-J_{4} P_{2}+W_{1}+W_{2} \equiv 0, P_{5} \not \equiv 0, \tag{49}
\end{equation*}
$$

where $P_{1}, P_{2}, P_{5}, J_{2}, J_{4}$ are taken from (5), $W_{1}$ from (22), and $W_{2}$ from (34).
Proof. The necessity of the identity (49) follows from Lemma 4. The necessity of the inequality from (49) can be proved similarly to the second part of the proof of the necessity and sufficiency of Lemma 3.44 from [1].

Sufficiency of the conditions (49) can be proved similarly to the second part of the proof of the necessity and sufficiency of Lemma 3.44 from [1]. Lemma 6 is proved.

Lemma 7. If $P_{2} \not \equiv 0$ the rank of the matrix (2) is equal to 2 if and only if

$$
\begin{equation*}
P_{1} \equiv P_{5} \equiv J_{2} P_{5}-J_{4} P_{2}+W_{1}+W_{2} \equiv 0, \tag{50}
\end{equation*}
$$

where $P_{1}, P_{2}, P_{5}, J_{2}, J_{4}$ are taken from (5), $W_{1}$ from (22), and $W_{2}$ from (34).
Proof. Necessity of the condition (50) follows from Lemma 6.
Let prove the sufficiency. If the conditions of Lemma 7 take place, than in every case 1)-10) from the proof of the sufficiency of Lemma 6, where we do not have any contradictions, we obtain $P_{2}=s y^{4}$, and $a=b=p=q=r=t=u=v=w=0$. With these equalities from $P_{2} \not \equiv 0$ follows $s \neq 0$ and the rank of matrix (2) is equal to 2 , since the minors of the second order $3 \Delta_{56}^{13}=3 \Delta_{610}^{13}=\Delta_{56}^{34}=\Delta_{610}^{34}=3 s^{2}$ are different from zero. Lemma 7 is proved.

Theorem 1. GL(2, R)-orbit of the system (1) has the dimension

$$
\begin{aligned}
& 4 \quad \text { for } \quad P_{1} P_{2} \not \equiv 0,3 P_{1} P_{3}-2 J_{1} P_{2}+W_{1} \not \equiv 0 \text {, or } \\
& p_{27} \equiv 0, P_{1} P_{2}\left(3 P_{1} P_{3}-2 J_{1} P_{2}\right) \not \equiv 0, \text { or } \\
& 50 p_{27} \equiv 0, P_{1} \equiv 0, P_{2}\left(J_{2} P_{5}-J_{4} P_{2}\right) \not \equiv 0, \text { or } \\
& P_{1} \equiv 0, P_{2} \not \equiv 0, J_{2} P_{5}-J_{4} P_{2}+W_{1}+W_{2} \not \equiv 0 \text {, or } \\
& 50 P_{2} \equiv 0, J_{1} p_{27} \equiv 0 ; \\
& 3 \quad \text { for } P_{1} P_{2} \not \equiv 0,3 P_{1} P_{3}-2 J_{1} P_{2}+W_{1} \equiv 0 \text {, or } \\
& P_{1} P_{2} \neq 0, p_{27} \equiv 0,3 P_{1} P_{3}-2 J_{1} P_{2} \equiv 0, \text { or } \\
& 50 P_{2} P_{5} \not \equiv 0, P_{1} \equiv J_{2} P_{5}-J_{4} P_{2}+W_{1}+W_{2} \equiv 0 \text {, or } \\
& 50 P_{2} \equiv 0, P_{1} \not \equiv 0, J_{1}+p_{27} \not \equiv 0, J_{1} p_{27} \equiv 0 ; \\
& 2 \quad \text { for } P_{2} \not \equiv 0, P_{1} \equiv P_{5} \equiv J_{2} P_{5}-J_{4} P_{2}+W_{1}+W_{2} \equiv 0 \text {, or } \\
& P_{2} \equiv p_{27} \equiv 0, P_{1} \not \equiv 0, J_{1}=0, \text { or } \\
& P_{1} \equiv P_{2} \equiv 0, p_{27} \equiv \equiv 0 ; \\
& 0 \quad \text { for } P_{1} \equiv P_{2} \equiv p_{27} \equiv 0, \\
& \text { where } P_{1}, P_{2}, P_{3}, P_{5}, p_{27}, J_{1}, J_{2}, J_{4} \text { are taken from (5), } W_{1} \text { is taken from (22), } \\
& \text { and } W_{2} \text { - from (34). }
\end{aligned}
$$

Let introduce the following designations:

$$
\begin{gathered}
M_{1}=M_{1}\left(P_{1} P_{2} \not \equiv 0,3 P_{1} P_{3}-2 J_{1} P_{2}+W_{1} \not \equiv 0\right) \\
M_{2}=M_{2}\left(p_{27} \equiv 0, P_{1} P_{2}\left(3 P_{1} P_{3}-2 J_{1} P_{2}\right) \not \equiv 0\right)
\end{gathered}
$$

$$
\begin{gather*}
M_{3}=M_{3}\left(p_{27} \equiv 0, P_{1} \equiv 0, P_{2}\left(J_{2} P_{5}-J_{4} P_{2}\right) \not \equiv 0\right) ; \\
M_{4}=M_{4}\left(P_{1} \equiv 0, P_{2} \not \equiv 0, J_{2} P_{5}-J_{4} P_{2}+W_{1}+W_{2} \not \equiv 0\right) ; \\
M_{5}=M_{5}\left(P_{2} \equiv 0, J_{1} p_{27} \not \equiv 0\right) ; \\
M_{6}=M_{6}\left(P_{1} P_{2} \not \equiv 0,3 P_{1} P_{3}-2 J_{1} P_{2}+W_{1} \equiv 0\right) ; \\
M_{7}=M_{7}\left(P_{1} P_{2} \not \equiv 0, p_{27} \equiv 0,3 P_{1} P_{3}-2 J_{1} P_{2} \equiv 0\right) ; \\
M_{8}=M_{8}\left(P_{2} P_{5} \not \equiv 0, P_{1} \equiv J_{2} P_{5}-J_{4} P_{2}+W_{1}+W_{2} \equiv 0\right) ; \\
M_{9}=M_{9}\left(P_{2} \equiv 0, P_{1} \not \equiv 0, J_{1}+p_{27} \not \equiv 0, J_{1} p_{27} \equiv 0\right) ; \\
M_{10}=M_{10}\left(P_{2} \not \equiv 0, P_{1} \equiv P_{5} \equiv J_{2} P_{5}-J_{4} P_{2}+W_{1}+W_{2} \equiv 0\right) ; \\
M_{11}=M_{11}\left(P_{2} \equiv p_{27} \equiv 0, P_{1} \not \equiv 0, J_{1}=0\right) ; \\
M_{12}=M_{12}\left(P_{1} \equiv P_{2} \equiv 0, p_{27} \not \equiv 0\right) ; \\
M_{13}=M_{13}\left(P_{1} \equiv P_{2} \equiv p_{27} \equiv 0\right) . \tag{51}
\end{gather*}
$$

According to Definitions 1 and 2 from Theorem 1 follows
Theorem 2. Sets $M_{i}(1 \leq i \leq 13)$ from (51) form $G L(2, R)$-invariant division of the set $E(A)$ of the coefficient of the system (1), i.a.

$$
\bigcup_{i=1}^{13} M_{i}=E(A), \quad M_{i} \cap_{i \neq j} M_{j}=\oslash
$$

where each $M_{i}$ is $G L(2, R)$-invariant.

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E.V. Starus

Received July 22, 2003
Institute of Mathematics and Computer Science
5 Academiei str., Chişinău, MD-2028
Moldova
E-mail: helen@from.md

# $X$-normal mappings 

P.V. Dovbush


#### Abstract

This is a survey of achievements in the theory of normal holomorphic mappings. We systematize and present all the results on the subject that are obtained by the author from the beginning of the theory until the date of writing the paper.


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## 1 Introduction

The idea to connect with a meromorphic function $f$ in the unit disc $U=\{z \in$ $\mathbb{C}:|z|<1\}$ a family $\mathcal{F}=\{f \circ g\}$, where $g$ ranges over all automorphisms of $U$ (one-to-one holomorphic mappings of $U$ onto itself) and study those functions $f$ whose family $\mathcal{F}$ is normal was arise apparently of K. Yosida [36] in 1934 and considered by K. Noshiro [27] in 1937. O. Lehto and K.I. Virtanen [26] call "normal functions" those meromorphic functions $f$ whose family $\mathcal{F}$ is normal.

The results obtained by O. Lehto and K.I. Virtanen [26] in 1957 motivated further study of normal meromorphic function. In the period between 1957 and 1965, a significant contribution to the theory was madden by F. Bagemihl, W. Seidel, V.I. Gavrilov, P. Lappan.

A systematic study of normal functions in $\mathbb{C}^{n}$ was begun by the author in 1981 in the papers [9-11]. In 1983 the first dissertation on normal functions was defended at Moscow State University by the author and J.A. Cima and S.G. Krantz have published, in the USA, the article [6] in which they have developed the ideas contained in [8], [9]. In such a way appeared the theory of normal mappings.

The first application of this theory was obtained by V.I. Gavrilov and the author in the paper [18]. The first survey on the theory of normal mappings was published by V.I. Gavrilov and P.V. Dovbush [19] in 2001.

The fact that the new subsection Several Complex Variables $32 A 18$ Bloch functions, normal functions was created in the Mathematics Subject Classification Scheme of the AMS journal Mathematical Review in 2000 emphasizes the actuality of this theme.

All results of this work belong to the author, are published in [9-20], and the author reported about them at:

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(c) P.V. Dovbush, 2003

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The 5th Congress of Romanian mathematicians. Pitesti. Romania. June 22-28, 2003.

## $2 X$-normal mappings

Let $M$ and $N$ be complex manifolds. We denote the set of all holomorphic mappings from $M$ into $N$ by $H(M, N)$.

A subset $\mathcal{F}$ of $H(M, N)$ is said to be a normal family in the sense of H.Wu [35] iff every sequence $\left\{f_{j}\right\}$ of $\mathcal{F}$ has a subsequence which ether converges uniformly on compact subsets of $M$ (i.e. converges normally on $M$ ) or, given any compact $K$ in $M$, and a compact $K^{\prime}$ in $N$, there exists an $j_{0}$ such that $f_{j}(K) \cap K^{\prime}=\emptyset$ for all $j \geq j_{0}$.

For the complex manifolds $M$, which have a transitive group of biholomorphic automorphisms ${ }^{1}$ (i.e., if given $p, q \in M$ there exists a biholomorphic automorphism $\phi: M \rightarrow M$ with $\phi(p)=q)$ the definition of normal mapping can be introduced by analogy with the one dimensional case.

Definition 1. Let $M$ be a homogeneous manifold and $N$ be a connected Hermitian manifold. We say that a holomorphic mapping $f: M \rightarrow N$ is normal if the family $\mathcal{F}=\{f \circ g\}$, where $g$ ranges over all automorphisms (one-to-one holomorphic mappings of $M$ onto itself), forms a normal family in the sense of H.Wu.

The normality of a complex function imposes a restriction on the growth of function. Our first result is the following.

Theorem 1. Let $f$ be a normal meromorphic function on the unit ball $B=\{z \in$ $\left.\mathbb{C}^{n}:|z|<1\right\}$ and let $\Omega=\left\{z \in B:\left|2 z_{n}-1\right|<1,\left.\right|^{\prime} z\left|<\left|1-z_{n}\right|\right\}\right.$. If for all $z \in \Omega$

$$
|f(z)|<\exp \left(\frac{-1}{(1-|z|)^{1+\varepsilon}}\right)
$$

for some $\varepsilon>0$. Then $f \equiv 0$.
It is important to note that:
(a) The unit disc in $\mathbb{C}$ is a canonical domain, because Riemann mapping theorem says that every proper simply connected open subset $D$ of $\mathbb{C}$ is biholomorphic to the disc. Poincare's theorem that the ball and the polydisc are biholomorphically inequivalent, shows that there is no

[^3]Riemann mapping theorem in several complex variables. This implies that there is no canonical domain in $\mathbb{C}^{n}$ for $n>1$.
(b) It has long been known that in $\mathbb{C}^{2}$ there exist simply connected domains whose only holomorphic automorphism is the identity (cf. Benke, Tullen [3, p. 169]). And, what is more smoothly, bounded domains in $\mathbb{C}^{n}$ generally have no biholomorphic self mappings different from the identity. A result due to Burns, Shnider and Wells [5] clarifies how truly dismal the situation is.
(c) Every domain in the complex plane with $C^{2}$-boundary is strongly pseudoconvex. The result due to Bun Wong [34] and Rosay [30] states that the only strongly pseudoconvex domain in $\mathbb{C}^{n}$ with transitive automorphism group is the ball.
(d) E. Cartan proved that any bounded homogeneous domain in $\mathbb{C}^{2}$, is biholomorphic to either the ball $B^{2}=\left\{\left(z_{1}, z_{2}\right):\left|z_{1}\right|^{2}+\left|z_{2}\right|^{2}<1\right\}$ or the polydisc $U^{2}=\left\{\left(z_{1}, z_{2}\right):\left|z_{1}\right|<1,\left|z_{2}\right|<1\right\}$. In $\mathbb{C}^{3}$, E. Cartan's result is that any bounded homogeneous domain is biholomorphic to either the ball, the polydisc, or (writing $z_{j}=x_{j}+y_{j}$ ) the tube domain

$$
\left\{\left(z_{1}, z_{2}, z_{3}\right): y_{3}>\left[\left(y_{1}\right)^{2}+\left(y_{2}\right)^{2}\right]^{1 / 2}\right\} .
$$

(While the third of these domains is unbounded, it has a bounded realization.) In any $\mathbb{C}^{n}$, the set of equivalence classes of bounded symmetric domains ${ }^{2}$ is finite, as shown by E. Cartan.

Since general domains in $\mathbb{C}^{n}, n>1$, have trivial automorphism groups it is natural to try to generalize the notion of normal mappings to general complex manifolds and to extend classical results to more general settings.

Let $N$ be a connected paracompact hermitian manifold with hermitian metric $d s_{N}$ which induces the standard topology on $N$. By $s_{N}$ we denote the distance function associated with $d s_{N}$. Let $Y$ be a relatively compact complex subspace of a hermitian manifold $N$. We shall denote by $H(M, Y)$ the space of all holomorphic mappings $f: M \rightarrow N$ with $f(M) \subset Y$.

The classical hyperbolic metric on the unit disk can be extended to the higher dimension at least by three different ways.

Let $T_{p}(M)$ be a complex tangent space to $M$ at $p \in M$ and vector $v \in T_{p}(M)$. The Kobayashi norm is given by

$$
\begin{aligned}
K_{M}(p, v)= & \inf \{1 / r: r>0 \text { and there exists } \\
& \left.h \in H(U, M), h(0)=p, h^{\prime}(0)=r \cdot v\right\} .
\end{aligned}
$$

With $v \in T_{p}(M)$ as above, the Caratheodory norm is defined by

$$
\left.C_{M}(p, v)=\sup \left\{\mid d g_{p}(v)\right) \mid: g \in H(M, U)\right\} .
$$

[^4]The Bergman norm, denoted by $B_{M}(p, v)$, is defined by the relation ${ }^{3}$

$$
\left(B_{M}(p, v)\right)^{2}=\sum_{j, k=1}^{n} g_{j, k}(p) v_{j} \bar{v}_{k}
$$

In following let $X_{M}$ denote the Caratheodory, Kobayashi or Bergman norms on M.

First we need the following generalization of Marty's Criterion for normality of holomorphic mappings.

Lemma 1. Let $M$ be a complex manifold such that for each $p \in M$, there exists a neighborhood $U$ and a constant $c=c(U)>0$ such that $X_{M}(p, v) \geq c \cdot|v|$ for $(p, v) \in T(U)=U \times T_{p}(M)$, and let $Y$ be a relatively compact complex subspace of an Hermitian manifold $N$ with the Hermitian metric $d s_{N}$. The family $F \subset H(M, Y)$ is normal in the sense of $H$. Wu if for each compact subset $K \subset M$ there exists a positive constant $L=L(K)$ such that

$$
d s_{N}^{2}(f(p), d f(p) v) \leq L(K) \cdot X_{M}(p, v)^{2}
$$

for all $p \in K, v \in T_{p}(M)$ and all $f \in F$.
Marty's Criterion was first proved by the author in $[8]^{4}$ for $M=$ domain in $\mathbb{C}^{n}$ and $N=\mathbb{C}$. See also $[33]^{5},[6],[21]$.

Marty's Criterion plays a fundamental role in the theory of $\mathcal{X}$-normal mappings. Using Marty's Criterion, we can prove the following elegant geometric characterization of normal mappings.

Theorem 2. Let $M$ be a homogenious complex manifold such that for each $p \in M$, there exists a neighborhood $U$ and a constant $c=c(U)>0$ such that $X_{D}(p, v) \geq c \cdot|v|$ for $(p, v) \in U \times T_{p}(M)$ and let $Y$ be a relatively compact complex subspace of an Hermitian manifold $N$ with the Hermitian metric $d s_{N}$. A holomorphic mapping $f \in H(D, Y)$ is normal iff there exists a finite positive constant $L$ such that

$$
f^{*} d s_{N}^{2} \leq L \cdot\left(X_{D}\right)^{2}
$$

Results related to Theorem 2 can be found in [9, 20, 21, 22].
Using Shwarz-Pick lemma it is easy to check that if $D=U$ then $X_{U}(z, v)$ coincides with the Poincaré metric in $U \rho(z)|v|=\left(1-|z|^{2}\right)^{-1}|v|$. Hence Theorem 1 is the full generalization of the Lexto-Virtanen Criterion to a higher-dimensional domain:

A meromorphic function $f: U \rightarrow \overline{\mathbb{C}}$ is normal iff there exist a finite constant $L$ such that

$$
f^{*} d s_{\overline{\mathbb{C}}}^{2}(z, v) \leq L \cdot(\rho(z)|v|)^{2}
$$

for all $z \in U, v \in \mathbb{C}$.
The characterization of normal mappings given in Theorem 2 leads to a natural generalization of the concept of $\mathcal{X}$-normal mappings. We give the following:

[^5]Definition 2. Let $M$ be a complex manifold and let $N$ be a Hermitian manifold $N$ with the Hermitian metric $d s_{N}$. We say that a holomorphic mapping $f: M \rightarrow N$ is $\mathcal{X}$-normal if there exists a finite positive constant $L$ such that

$$
f^{*} d s_{N}^{2} \leq L \cdot\left(X_{M}\right)^{2} .
$$

One sees at once that all Bloch functions (see [33]) of several complex variables are normal functions.

It is easy to check using the definition of the Caratheodory norm that all bounded holomorphic function are $C$-normal. On the other hand, from the Lindelof's theorem follows that there are normal functions which do not belong to any Hardy space $H^{p}(D)$, and functions in $H^{p}$ which are not normal.

In the case of strongly pseudoconvex domains, the classes of normal mappings defined in terms of the Bergman, Carathéodory, and Kobayashi norms are the same. This assertion follows from well-known estimates on the asymptotic behavior of these norms.

Since $X_{D} \geq C_{D}$, the class of $\mathcal{K}$-normal or $\mathcal{B}$-normal mappings contains the class of $\mathcal{C}$-normal mappings. But in what follows, we will show that these classes, generally speaking, are different.

## 3 Extension properties of $\mathcal{X}$-normal mappings

First we prove that in the one dimensional case the following result holds.
Proposition 1. Let $f$ be a meromorphic function in punctured unit disk $U^{*}=$ $U \backslash\{0\}$. If $f$ has an isolated singularity at the origin and there exists a monotone increasing function $h$ such that

$$
\left|z f^{\prime}(z)\right| \leq h(|f(z)|)
$$

for all $z \in U^{*}$, then $f$ has a meromorphic extension at the origin.
If $D \subset \mathbb{C}$ is multiply connected, $f$ is said to be normal on $D$ if $f$ is normal on the universal cover of $D$.

From Proposition 1 immediately follows the extension of the big Picard theorem due to O. Lehto and K.I. Virtanen [26, Theorem 9, p. 92]:

Isolated singularities are removable for normal meromorphic functions of the one complex variable.

Since norms $K_{U}$ and $K_{U^{*}}$ are comparable near $\partial U$, the extended function is normal in $U$.

It is of interest to have analogues of this theorem in several variables.
Generalizing O. Lehto and K.I. Virtanen's result to the case of several complex variables P.Järvi [24] proved that $\mathcal{K}$-normal mappings can be extended to holomorphic mappings through analytic subvarieties of codimension 1 provided the singularities are normal crossings.
J. Riihentaus [29] generalized P.Järvy result and proved that $\mathcal{K}$-normal mappings can be extended to holomorphic mappings through closed in $D$ subsets of locally finite ( $2 n-2$ )-dimensional Hausdorff measure.

The principal our result is the following:
Theorem 3. Suppose that $D$ is a bounded domain in $\mathbb{C}^{n}, n>1$, such that $X_{D}$ is a continuous function on $D \times \mathbb{C}^{n}$, and suppose that $E \subset D$ is closed in $D$ and has the zero $(2 n-1)$-dimensional Hausdorff measure and such that $X_{D \backslash E} \equiv X_{D}$ on $(D \backslash E) \times \mathbb{C}^{n}$. If $f: D \backslash E \rightarrow \overline{\mathbb{C}}$ is $\mathcal{X}$-normal, then $f$ extends to a holomorphic mapping $F: D \rightarrow \overline{\mathbb{C}}$ which is $\mathcal{X}$-normal on $D$.

If $E \subset D$ is closed in $D$ and has the zero $(2 n-1)$-dimensional Hausdorff measure then $C_{D \backslash A}(z, v) \equiv C_{D}(z, v)$ on $(D \backslash A) \times \mathbb{C}^{n}$. Since Caratheodory norm $C_{D}(z, v)$ is a continuous function on $D \times \mathbb{C}^{n}$ as a consequence of Theorem 3 we have

Theorem 4. Let $D \subset \mathbb{C}^{n}, n>1$, be a domain and let $E \subset D$ be closed in $D$ and have the zero $(2 n-1)$-dimensional Hausdorff measure. If $f: D \backslash A \rightarrow \overline{\mathbb{C}}$ is $\mathcal{C}$-normal mapping, then $f$ has a $\mathcal{C}$-normal extension $F: D \rightarrow \overline{\mathbb{C}}$.

If $A$ is an analytic subset of $D$ of codimension at least one, then $B_{D \backslash A}(z, v) \equiv$ $B_{D}(z, v)$ on $(D \backslash A) \times \mathbb{C}^{n}$ (see [4]). Bergman norm $B_{D}(z, v)$ is a continuous function on $D \times \mathbb{C}^{n}$. It follows that Theorem 3 has the following consequence:

Theorem 5. Let $D \subset \mathbb{C}^{n}$, $n>1$, be a domain and let $A \subset D$ be an analytic subvariety of codimension at least one. If $f: D \backslash A \rightarrow \overline{\mathbb{C}}$ is $\mathcal{B}$-normal mapping, then $f$ has a $\mathcal{B}$-normal extension $F: D \rightarrow \overline{\mathbb{C}}$.

Since we can consider any $a \in U$ as an analytic subset of $U$ of codimension one we can interpret Theorem 5 as the full generalization of classical Lehto-Virtanen's theorem [26, Theorem 9, p. 92] to a higher-dimensional domain.

Using the notion of $\mathcal{P}$-sequence (a sequence $\left\{z_{j}\right\} \subset D$ is a $\mathcal{P}$-sequence for an holomorphic mapping $f: D \rightarrow \overline{\mathbb{C}}$ if $\lim _{j \rightarrow \infty} k_{D}\left(z_{j}, w_{j}\right)=0$ but $\overline{\lim }_{j \rightarrow \infty} s_{\mathbb{C}}\left(f\left(z_{j}\right), f\left(w_{j}\right)\right) \geq$ $\epsilon$ for some $\epsilon>0$ and some $\left.\left\{w_{j}\right\} \subset D\right)$ we prove the following result.

Theorem 6. Let $D \subset \mathbb{C}^{n}, n>1$, be a domain and let $E \subset D$ be closed in $D$ and have the zero ( $2 n-2$ )-dimensional Hausdorff measure. If $f: D \backslash A \rightarrow \overline{\mathbb{C}}$ is $\mathcal{K}$-normal mapping, then $f$ has a $\mathcal{K}$-normal extension $F: D \rightarrow \overline{\mathbb{C}}$.

It is shown in [22] that all mappings whose range omits at least three values belong to the class of all $K$-normal mappings. The following simple example shows that this is not true for the rest two classes mentioned above.

Let $f\left(z_{1}, z_{2}\right)=z_{1} / z_{2}$ and let $A=\left\{z_{1} z_{2}\left(z_{1}-z_{2}\right)=0\right\}$. Because $f(D \backslash A) \subset$ $\mathbb{C} \backslash\{0,1\}$, it follows that $f$ is $\mathcal{K}$-normal in $D \backslash A$. The function $f$ can not be $\mathcal{C}$ normal or $\mathcal{B}$-normal in $D \backslash A$. Otherwise $f$ would have a holomorphic extension $F: D \rightarrow \overline{\mathbb{C}}$.

Therefore, we have the following proposition.
Proposition 2. Let $D$ be a domain in $\mathbb{C}^{2}$ and let $A=\left\{z_{1} z_{2}\left(z_{1}-z_{2}\right)=0\right\}$.
(a) The class of $\mathcal{C}$-normal mappings defined on $D \backslash A$ is a proper subclass of $\mathcal{K}$-normal mappings defined on $D \backslash A$.
(b) The class of $\mathcal{B}$-normal mappings defined on $D \backslash A$ is different from the class of $\mathcal{K}$-normal mappings defined on $D \backslash A$.
(c) The Kobayashi norm on $D \backslash A$ is not compatible with the Bergman or the Caratheodory norm on $D \backslash A$.

## 4 Boundary behavior of holomorphic mappings

In the case of one complex variable, O. Lehto and K.I. Virtanen [26] showed that the notion of normal meromorphic functions is closely related to some important problems from the theory of boundary behavior of holomorphic mappings.

Let $D$ be a bounded domain in $\mathbb{C}^{n}$ with $C^{2}$-boundary and let $\delta(z)=\inf \{|z-\zeta|:$ $\zeta \in \partial D\}$. If $\xi \in \partial D$, let $\nu_{\xi}$ denote the unit outward normal at $\xi$.

We say that $f \in H(D, N)$ has radial limit $l \in N$ at $\xi \in \partial D$ if

$$
\lim _{t \rightarrow 0^{+}} s_{N}\left(f\left(\xi-t \nu_{\xi}\right), l\right)=0
$$

An admissible approach region $\mathcal{A}_{\alpha}(\xi)$ with the vertex at $\xi \in \partial D$ and of the aperture $\alpha>0$ is defined as follows ([32]):

$$
\mathcal{A}_{\alpha}(\xi)=\left\{z \in D:\left|\left(z-\xi, \nu_{\xi}\right)\right|<(1+\alpha) \delta_{\xi}(z),|z-\xi|^{2}<\alpha \delta_{\xi}(z)\right\}
$$

where $($,$) is the usual Hermitian product in \mathbb{C}^{n}$, and $\delta_{\xi}(z)=\min \left\{\delta(z), \operatorname{dist}\left(z, T_{\xi}(\partial D)\right)\right\}$.
We say that $f \in H(D, N)$ has an admissible limit $l \in N$ at $\xi \in \partial D$ if

$$
\lim _{\mathcal{A}_{\alpha}(\xi) \ni z \rightarrow \xi} s_{N}(f(z), l)=0
$$

for every $\alpha \geq 1$.
E. Stein [32] proves that admissible domains give a Fatou-type theorem on any smoothly bounded domain in $\mathbb{C}^{n}$, but his result is only optimal for strongly pseudoconvex domains (see [23]). In Stein's theory the aperture $\alpha$ of the approach regions is fixed once and for all.

We prove the following theorems.
Theorem 7. Let $D$ be a bounded domain in $\mathbb{C}^{n}$, $n>1$, with $C^{2}$-boundary. If $f \in H(D, \mathbb{C})$ has the radial limit $l \in \mathbb{C}$ at $\xi \in \partial D$ and Ref has admissible limit at $\xi$, then $f$ has an admissible limit at $\xi$.
Theorem 8. Let $D$ be a bounded domain in $\mathbb{C}^{n}$ with $C^{2}$-boundary. If $f: D \rightarrow \overline{\mathbb{C}}$ is $\mathcal{K}$-normal in $D$, and

$$
\lim _{\mathcal{A}_{\beta}(\xi) \ni z \rightarrow \xi} s_{\overline{\mathbb{C}}}(f(z), l)=0 \text { exists for some } \beta>0,
$$

then $f$ has an admissible limit $l \in \overline{\mathbb{C}}$ at $\xi$.

If $\alpha \geq 0$, define the $\mathcal{K}$-admissible approach region of aperture $\alpha$ at $\xi$ to be (see [25])

$$
\mathcal{K}_{\alpha}(\xi)=\left\{z \in D: k_{D}\left(z, N_{\xi}\right)<\alpha\right\} .
$$

Here $k_{D}\left(z, N_{\xi}\right)$ represents the Kobayashi distance from $z$ to $N_{\xi}$.
If $D \subset \subset \mathbb{C}^{n}$ is strongly pseudoconvex domain then there are constants $c_{1}, c_{2}>0$ depending on $D$ and an open set $W \supseteq \partial D$ such that

$$
U \cap \mathcal{A}_{c_{1} \alpha}(\xi) \supseteq \mathcal{K}_{\alpha}(\xi) \cap W \supseteq U \cap \mathcal{A}_{c_{2} \alpha}(\xi)
$$

for any $\xi \in \partial D$ and $\alpha>1$.
We say that a mapping $g: D \rightarrow N$ has the $\mathcal{K}$-limit $l \in N$ at $\xi \in \partial D$ if

$$
\lim _{\mathcal{K}_{\alpha}(\xi) \ni z \rightarrow \xi} s_{N}(g(z), l)=0,
$$

for every $\alpha \geq 1$.
Denote by

$$
Q_{f}(z)=\sup _{v \in \mathbb{C}^{n} \backslash\{0\}}\left\{\frac{d s_{N}(f(z), d f(z)(v))}{K_{D}(z, v)}\right\}
$$

In [26], O. Lehto and K.I. Virtanen showed that if a meromorphic function $f$ in the unit disk $U$ has the radial limit at the point $\mathbf{1} \in \partial U$, then $f$ has the angular limit at $\mathbf{1}$ iff $Q_{f}$ is bounded on every Stolz regions at $\mathbf{1}$.

This is not longer true for several variables. The function $f\left(z_{1}, z_{2}\right)=z_{2}^{2 m} /\left(1-z_{1}\right)$ is bounded and holomorphic on the Tullen domain $\left\{\left(z_{1}, z_{2}\right) \in \mathbb{C}^{2}:\left|z_{1}\right|^{2}+\left|z_{2}\right|^{2 m}<1\right\}$ and $f$ has the radial limit 0 at $\mathbf{1}=(1,0)$ but it does not have a $\mathcal{K}$-limit at $\mathbf{1}$.

We prove the following criterion for existence of $\mathcal{K}$-limits.
Theorem 9. Let $D$ be a complete hyperbolic domain in $\mathbb{C}^{n}$ and let $Y$ be a relatively compact complex subspace of an Hermitian manifold $N$ with the Hermitian metric $d s_{N}$. If $f \in H(D, Y)$ has the radial limit at $\xi \in \partial D$, then $f$ has the $\mathcal{K}$-limit at $\xi$ iff $Q_{f}$ has the $\mathcal{K}$-limit zero at $\xi$.

In [2], F. Bagemihl and W. Seidel posed the following question:
Given a sequence $\left\{z_{j}\right\} \subset U$ converging to same $\xi \in \partial U$ and a holomorphic mapping $f \in H(U, \overline{\mathbb{C}})$ such that $\lim _{j \rightarrow \infty} s_{\overline{\mathbb{C}}}(f(z), l)=0$ for same $l \in \overline{\mathbb{C}}$, under what condition on $f$ and $\left\{z_{j}\right\}$ can $f$ have the limit l along some continuum in $U$ which is asymptotic at $\xi$.

They answer this question with two interesting sufficient condition on $f$ and $\left\{z_{j}\right\}$. We extend their results to the higher dimensional case.

Theorem 10. Let $D$ be a strongly pseudoconvex domain in $\mathbb{C}^{n}(n \geq 1), \xi \in \partial D$, and let $Y$ be a relatively compact complex space of an Hermitian manifold $N$ with the Hermitian metric $d s_{N}$. Let $f \in H(D, Y)$ be a normal mapping which omits $l \in \bar{Y}$ in $D$. Let $\left\{a^{m}\right\}$ and $\left\{b^{m}\right\}$ be sequences in $D$ such that

$$
\lim _{m \rightarrow \infty} a^{m}=\xi \in \partial D \quad \text { and } \quad \lim _{m \rightarrow \infty} b^{m}=\xi
$$

If $k_{D}\left(a^{m}, b^{m}\right)<\epsilon<\infty$ for all $m \geq 1$ and

$$
\lim _{m \rightarrow \infty} s_{N}\left(f\left(a^{m}\right), l\right)=0, \text { then } \lim _{m \rightarrow \infty} s_{N}\left(f\left(b^{m}\right), l\right)=0 .
$$

The same results holds when we replace "strongly pseudoconvex" by "convex".
From Theorem 10 immediately follows the following strengthening of the Lin-delöf-Lehto-Virtanen's theorem:

Theorem 11. Let $D$ be a strongly pseudoconvex domain in $\mathbb{C}^{n}(n \geq 1), \xi \in \partial D$, and let $Y$ be a relatively compact complex space of an Hermitian manifold $N$ with the Hermitian metric $d s_{N}$. If $f \in H(D, N)$ is a normal mapping which omits $l \in \bar{Y}$ in $D$ and $f$ has radial limit $l$ at $\xi$, then $f$ has the admissible limit at $\xi$.

The hypothesis of "radial limit" may be replaced by "limit along some nontangential curve $\gamma$ " as proved by the author in $[11]^{6}$. In $[28]^{7}$ this theorem was proved for $D=B$ and $f \in H^{\infty}(B)$.

Theorem 10 also holds when we replace "strongly pseudoconvex" by "convex".
The following theorem illustrates more precisely the Lindelöf principle:
Theorem 12. Let $D$ be a convex domain in $\mathbb{C}^{n}(n \geq 1), \xi \in \partial D$, and let $Y$ be a relatively compact complex space of an Hermitian manifold $N$ with the Hermitian metric ds $s_{N}$. If $f \in H(D, N)$ is a normal mapping which omits $l \in \bar{Y}$ in $D$ and $f$ has radial limit $l$ at $\xi$, then $f$ has the $\mathcal{K}$-limits at $\xi$.

Again the hypothesis of "radial limit" may be replaced by "limit along some non-tangential curve $\gamma$."

It should be noted that, in general, $\mathcal{K}$-admissible domains are strongly larger than admissible domains.

A hypoadmissible approach region $\mathcal{A}_{\alpha}^{\epsilon}(\xi), 0<\epsilon<1$, with vertex $\xi \in \partial D$ and aperture $\alpha>0$ is defined as follows ([7]):

$$
\mathcal{A}_{\alpha}^{\epsilon}(\xi)=\left\{z \in D:\left|\left(z-\xi, \nu_{\xi}\right)\right|<(1+\alpha) \delta_{\xi}(z),|z-\xi|^{2}<\alpha \delta_{\xi}^{1+\epsilon}(z)\right\} .
$$

We say that a mapping $f \in H(D, Y)$ has the hypoadmissible limit $l \in \bar{Y}$ at $\xi \in \partial D$, if for every $\alpha>0$ and $\epsilon, 0<\epsilon<1$,

$$
\lim _{\mathcal{A}_{\alpha}^{\epsilon}(\xi) \ni z \rightarrow \xi} s_{N}(f(z), l)=0 .
$$

Theorem 13. Let $D$ be a strongly pseudoconvex domain in $\mathbb{C}^{n}(n \geq 1), \xi \in \partial D$, and let $Y$ be a relatively compact complex space of an Hermitian manifold $N$ with the Hermitian metric $d s_{N}$. Let $\left\{a^{m}\right\}$ be a sequence of points in $D$ that tends to

[^6]a boundary point $\xi \in \partial D$ at which the unit outward normal $\nu_{\xi}$ exists and let $\lim _{m \rightarrow \infty} k_{D}\left(a^{m}, a^{m+1}\right)=0$. If $f \in H(D, Y)$ is a normal mapping such that
$$
\lim _{m \rightarrow \infty} s_{N}\left(f\left(a^{m}\right), l\right)=0
$$
for some $l \in \bar{Y}$, then
$$
\lim _{\mathcal{A}_{\alpha}^{\epsilon}(\xi) \ni z \rightarrow \xi} s_{N}(f(z), l)=0 \text { for all } \alpha>0 .
$$

From this we immediately obtain the generalization of the Lindelöf-LehtoVirtanen's theorem proved by the author [9] in 1982.

Theorem 14. Let $D$ be a strongly pseudoconvex domain in $\mathbb{C}^{n}$ with $C^{2}$-boundary and let $Y$ be a relatively compact complex space of an Hermitian manifold $N$ with the Hermitian metric $d s_{N}$. Let $f \in H(D, N)$ be normal in $D$ and $\xi \in \partial D$. Let $l \in \bar{Y}$ and suppose that $f$ has the radial limit $l$ at $\xi$. Then $f$ has the hypoadmissible limit $l$ at $\xi$.

Results related to Theorem 14 can be found in [1],[6],[21],[22].
Example (Rudin,[31, 8.4.7]) Fix $c>1 / 2$. The holomorphic function $f\left(z_{1}, z_{2}\right)=$ $\left(1-z_{1}\right)^{-c} z_{2} \in H^{p}(B)$ for all $p<4 /(2 c-1)$. The function $f$ has the radial limit at the point $\mathbf{1}=(1,0) \in \partial B$, but does not have a hypoadmissible limit at $\mathbf{1}$. It follows, generally speaking, that the class of normal functions differs from the Hardy $H^{p_{-}}$ classes even in the case of only holomorphic normal functions.

## 5 Polynomiality criterion for entire functions

A function $f: \mathbb{C}^{n} \rightarrow \mathbb{C}, n \geq 1$, holomorphic on the whole $n$-dimensional space $\mathbb{C}^{n}$ is called an entire function.

If an entire function $f$ has the homogenious polynomial expansion

$$
f(z)=\sum_{j=0}^{\infty} P_{j}(z),
$$

where $P_{j}$ are homogenious polynomials in $\mathbb{C}^{n}$ of degree $j$, then the radial derivative $R f$ is defined as ([31]):

$$
\mathcal{R} f(z)=\sum_{j=1}^{\infty} j P_{j}(z) .
$$

We prove the "radial" polynomiality criterion for entire functions of several complex variables.

Theorem 15. An entire function $f, \mathbb{C}^{n}, n \geq 1$, is a polynomial if and only if for any complex line $l \subseteq \mathbb{C}^{n}$ passing through the origin we have

$$
\varlimsup_{|\lambda| \rightarrow \infty} \frac{|\mathcal{R} f(l(\lambda))|}{1+|f(l(\lambda))|^{2}}<\infty .
$$

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P.V. Dovbush

Received October 17, 2003
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
5 Academiei str.,
Chişinău, MD-2028, Moldova
E-mail: dovbush@mail333.com

# Approximate solution of the Dirichlet problem in a circle 

Alexander Kouleshoff


#### Abstract

The approaches to the solution of Dirichlet problem in a unit radius circle are constructed in the manner of rational functions. There were found the estimates of approaches' inaccuracies. Assuming that the boundary condition is to be a measurable bounded function with the finite number of discontinuities. Constructions use the solution of trigonometric problem of moments.


Mathematics subject classification: 30C80.
Keywords and phrases: Dirichlet problem, rational functions, trigonometric problem of moments, orthonormalized polynomials, rational approximation.

## 1 Introduction

Consider the Dirichlet problem in the unit circle $\Omega$ :

$$
\begin{align*}
\Delta U=0, & (x, y) \in \Omega  \tag{1}\\
U(\cos \theta, \sin \theta)=\varphi(\theta), & \theta \in[0,2 \pi] \tag{2}
\end{align*}
$$

The problem (1), (2) can be resolved in a polar coordinate system using the variables separation method. In this case the solution is written in the following way:

$$
\begin{equation*}
u(r, \phi)=C+\sum_{n=1}^{\infty} r^{n}\left(A_{n} \cos (n \phi)+B_{n} \sin (n \phi)\right) \tag{3}
\end{equation*}
$$

where the coefficients $C,\left\{A_{n}\right\}_{n=1}^{\infty},\left\{B_{n}\right\}_{n=1}^{\infty}$ are calculated according to well-known formulas. The right-side formula (3) is summed up to the Poisson integral. In some cases the Poisson integral is simply calculated with the help of the residue theory. It happens, for example [1], if the function

$$
\varphi(\phi)=\Phi(\cos \phi, \sin \phi),
$$

where $\Phi\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2 i}\right)$ is a function, regular in the $\Gamma$ points and meromorphic in the $\Omega$ circle, and having there only a finite number of poles. In general cases we make use of approximate solution methods. The choice of various approximate problem solution methods (1), (2) depends on a boundary condition smoothness $\varphi(\phi)$. It is assumed in the given work, that the real valued function $\varphi(\phi)$ is measurable,

[^7]bounded and having a finite number of discontinuities. In such situation polynomial approximations on the basis of formula (3) are not valid. For getting some rational approximations to find the problem's solution (1), (2) we should use the following construction. Any real valued, measurable on the closed interval $[0,2 \pi]$ and bounded function $\psi(\phi)$ can be represented as the following:
\[

$$
\begin{equation*}
\psi(\phi)=\psi_{+}(\phi)-\psi_{-}(\phi), \quad \phi \in[0,2 \pi], \tag{4}
\end{equation*}
$$

\]

where

$$
\begin{gathered}
\psi_{+}(\phi)=\{\psi(\phi), \text { if } \psi(\phi) \geq 0 \text { and } 0, \text { if } \psi(\phi)<0\}, \\
\psi_{-}(\phi)=\{0, \text { if } \psi(\phi) \geq 0 \text { and }-\psi(\phi), \text { if } \psi(\phi)<0\} .
\end{gathered}
$$

Moreover, if $\psi(\phi)$ is a function of bounded variation, then it can be represented in the form

$$
\begin{equation*}
\psi(\phi)=\int_{0}^{\theta}\left|\psi^{\prime}(\xi)\right| d \xi-\left(\int_{0}^{\theta}\left|\psi^{\prime}(\xi)\right| d \xi-\psi(\phi)\right), \tag{5}
\end{equation*}
$$

that is in the form of two monotone non-decreasing functions.
It follows from the formula (4) (or (5)) that functions $\psi_{+}(\phi)$ and $\psi_{-}(\phi)$ are nonnegative, measurable, bounded and with a finite number of discontinuities.

Consider two auxiliary problems:

$$
\begin{align*}
\Delta U_{+}=0, & z \in \Omega,  \tag{6}\\
U_{+}\left(e^{i \phi}\right)=\varphi_{+}(\phi), & \phi \in[0,2 \pi] \tag{7}
\end{align*}
$$

and

$$
\begin{align*}
\Delta U_{-}=0, & z \in \Omega,  \tag{8}\\
U_{-}\left(e^{i \phi}\right)=\varphi_{-}(\phi), & \phi \in[0,2 \pi] . \tag{9}
\end{align*}
$$

It is known, that the problem (1), (2) has a unique solution concerning the function $\varphi$ with the help of made suggestions. So the problem (6), (7) and the problem (8), (9) also has a unique solution. Therefore,

$$
U=U_{+}-U_{-}
$$

That is enough to consider the case of $\varphi(\phi) \geq 0$ with almost all $\phi \in[0,2 \pi]$. Further it is supposed unconditionally, that a boundary condition is a nonnegative function. In conclusion of the item we should remember the fact, that the suggestion of boundary condition's insufficiency cannot be omitted, as the theorem of the problem's unique solution (1), (2) would be false. The following function can serve as an example:

$$
V(x, y)=\frac{1-x^{2}-y^{2}}{(x-1)^{2}+y^{2}} .
$$

This function satisfies Laplace equation in the radius 1 circle. It is continuous up to the circle's boundary except for point (1.0). The function is identically equal to zero and it also satisfies all these conditions.

## 2 Rational approximations' building process

We write the solution to the problem (1), (2) according to Shwarz formula:

$$
U(x, y)=\operatorname{Re}\left[\frac{1}{2 \pi i} \oint_{|\xi|=1} \varphi(\xi) \frac{(\xi+z)}{(\xi-z)} \frac{d \xi}{\xi}\right]
$$

where $z=x+i y$ or

$$
U(x, y)=\operatorname{Re}\left[\frac{1}{2 \pi i} \oint_{0}^{2 \pi} U\left(e^{i \theta}\right) \frac{\left(e^{i \theta}+z\right)}{\left(e^{i \theta}-z\right)} d \theta\right]
$$

This formula will be used below.
Denote:

$$
\sigma(\phi)=\frac{1}{\int_{0}^{2 \pi} \varphi(\xi) d \xi} \int_{0}^{\phi} \varphi(\xi) d \xi
$$

For an easier narration we suggest, that the function $\sigma$ is continuous. Then the following formula is true:

$$
\int_{0}^{2 \pi} \chi(\phi) d \sigma(\phi)=\int_{0}^{2 \pi} \chi(\phi) \varphi(\phi) d \phi
$$

where $\chi(\phi)$ is a continuous function, and there is Riehmann common integral at the right side of the formula.

Denote $\left\{P_{n}(z)\right\}_{n=0}^{\infty}$ as the sequence of orthonormalized polynomials relative to the positive measure $d \sigma$ on a radius 1 circle. These polynomials satisfy the following conditions:

$$
\frac{1}{2 \pi} \int_{0}^{2 \pi} P_{n}\left(e^{i \phi}\right) \overline{P_{m}\left(e^{i \phi}\right)} d \sigma(\phi)= \begin{cases}0, & n \neq m \\ 1, & n=m\end{cases}
$$

For making such polynomials we can use the Hilbert-Schmidt orthogonalization process for a sequence $\left\{e^{i n \phi}\right\}_{0}^{\infty}$ in the Hilbert space $L_{2}^{\sigma}(0,2 \pi)$ with the inner product:

$$
\langle f, g\rangle=\frac{1}{2 \pi} \int_{0}^{2 \pi} f(\phi) \overline{g(\phi)} d \sigma(\phi)
$$

It is known, that a sequence of orthonormalized polynomials can be built using the following sequence of moments $d \sigma$ :

$$
C_{k}=\frac{1}{2 \pi} \int_{0}^{2 \pi} e^{-i k \theta} d \sigma(\phi), \quad( \pm k=0,1,2, \ldots)
$$

The solution of the $d \sigma$ measure finding problem with a given set of numbers $C_{k}$ is called the solution of moments' trigonometric problem and it comes essentially towards the spectral theory of the second order's finite differences equation (P.L. Chebyshev (1858), A.A. Markov (1884), T.J. Stieltjes (1894), H.L. Hamburger (1920) and others). Extensive reading materials are devoted to the discussion of the problem field (see, for example, $[2,3]$ ).

Let $\left\{\alpha_{n}\right\}_{n=0}^{\infty}$ be the coefficients at orthogonal polynomials' highest degrees $\left\{P_{n}(z)\right\}_{n=0}^{\infty}$. The numbers

$$
a_{n}=-\frac{\overline{P_{n+1}(0)}}{\overline{\alpha_{n+1}}}, \quad(n=0,1,2, \ldots)
$$

play an important role in the theory of orthogonal polynomials in a circle. They are called circular parameters.

Consider the second order differences equation.

$$
\begin{equation*}
a_{n} y_{n+2}-\left(a_{n}+a_{n+1} z\right) y_{n+1}+a_{n+1} z\left(1-\left|a_{n}\right|^{2}\right) y_{n}=0 . \tag{10}
\end{equation*}
$$

We join some boundary conditions to the equation (10):

$$
\begin{equation*}
y_{0}=1, y_{1}=1+a_{0} z \tag{11}
\end{equation*}
$$

or boundary conditions:

$$
\begin{equation*}
y_{0}=1, y_{1}=1-a_{0} z . \tag{12}
\end{equation*}
$$

We denote the problem's solution (10), (11) as $\stackrel{*}{\psi}(z)$, and the problem's solution (10), (12) as $\stackrel{*}{\Phi}_{n}(z)$.

Suppose

$$
\nu=\int_{0}^{2 \pi} \varphi(\xi) d \xi
$$

It is said, that the Stieltjes positive measure $d \sigma$ on closed interval [ $0,2 \pi$ ], having an integrable density $\varphi(\phi)$ at the Lebesgue measure, satisfies the condition of Szegö, if

$$
\int_{0}^{2 \pi} \ln \left(\frac{\varphi(\phi)}{\nu}\right) d \phi>-\infty
$$

We will use standard designations below:

$$
\begin{gathered}
\ln ^{+}(x)=\{\ln (x), \text { if } x \geq 1 \text { and } 0, \text { if } 0<x<1\}, \\
\ln ^{-}(x)=\{0, \text { if } x \geq 1 \text { and }-\ln (x), \text { if } 0<x<1\} .
\end{gathered}
$$

Theorem 1. Assume, that the $d \sigma$ measure satisfies the condition of Szeg̈, then uniformly on the compacts $|z| \leq \nu<1$

$$
\lim _{n \rightarrow \infty} \frac{\nu}{\left|\Phi_{n}(z)\right|^{2} \alpha_{n}^{2}}=\operatorname{Re}(\hat{U}(z)),
$$

where

$$
\hat{U}(z)=\frac{1}{2 \pi i} \oint_{|\xi|=1} \varphi(\xi) \frac{(\xi+z)}{(\xi-z)} \frac{d \xi}{\xi}+i \operatorname{Im} \hat{U}(0)
$$

Proof. A sequence of polynomials $\stackrel{*}{\Phi}_{n}(z)$ converges uniformly on the compact subsets of the unite circle (see, for example, [2, p. 141]) towards the function $\frac{D(0)}{D(z)}$, where

$$
D(z)=\exp \left(\frac{1}{4 \pi} \int_{0}^{2 \pi} \ln \left(\frac{\varphi(\phi)}{\nu}\right) \frac{\left(e^{i \theta}+z\right)}{\left(e^{i \theta}-z\right)} d \phi\right)
$$

Moreover,

$$
\ln \left(\frac{\varphi(\phi)}{\nu}\right) \in L_{1}(0,2 \pi)
$$

and the function

$$
D(z) \in H^{2}
$$

where $H^{2}$ is Hardy space, and is the only thing of the space to satisfy the equality

$$
\begin{equation*}
\left|D\left(e^{i \theta}\right)\right|^{2}=\frac{\varphi(\phi)}{\nu} . \tag{13}
\end{equation*}
$$

According to the logarithm definition we get

$$
2 \ln (D(z))=\frac{1}{2 \pi} \int_{0}^{2 \pi} \ln \left(\frac{\varphi(\phi)}{\nu}\right) \frac{\left(e^{i \theta}+z\right)}{\left(e^{i \theta}-z\right)} d \phi,
$$

where the logarithm branch is allocated with a condition $\operatorname{Im}(\ln (D(0)))=\phi$.
Consider the function

$$
\nu e^{2 \ln (D(z))}=\nu D^{2}(z) .
$$

The equality is true for

$$
\begin{equation*}
\operatorname{Re}\left(\nu e^{2 \ln (D(z))}\right)=\operatorname{Re}\left(\nu D^{2}(z)\right)=\nu \operatorname{Re}\left(D^{2}(z)\right)=\nu|D(z)|^{2} . \tag{14}
\end{equation*}
$$

It follows from the formulas (13), (14), that the unit circle harmonic function $\operatorname{Re}\left(\nu D^{2}(z)\right)$ takes a meaning which is equal to $\varphi(\phi)$ at the circle's boundary, and therefore has the sought solution of Dirichlet problem (1), (2). In particular,

$$
\begin{equation*}
\hat{U}(z)=\nu e^{2 \ln (D(z))}=\nu D^{2}(z) . \tag{15}
\end{equation*}
$$

Note, that the function $D(z)$ does not have zeros inside the unit circle and

$$
\sup _{0 \leq r<1} \frac{1}{2 \pi} \int_{0}^{2 \pi} \ln ^{+}\left(\left|D\left(r e^{i \phi}\right)\right|\right) d \varphi<+\infty .
$$

Furthermore, the following equality $(|z|<1)$ is true

$$
\frac{1}{\left|\stackrel{*}{\Phi}_{n}(z)\right|^{2}} \leq \frac{\alpha_{n}^{2}}{\alpha_{0}^{2}\left(1-|z|^{2}\right)}
$$

and it is uniform at $|z| \leq r<1$ (see [3], p. 14 and p. 26)

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{\nu}{\left|\Phi_{n}(z)\right|^{2} \alpha_{n}^{2}}=\operatorname{Re}(\hat{U}(z)) . \tag{16}
\end{equation*}
$$

Equality (16) shows, that the sequence of rational functions

$$
\frac{\nu}{\left|\stackrel{*}{\Phi}_{n}(z)\right|^{2} \alpha_{n}^{2}}
$$

is an approximation to the solution of Dirichlet problem (1), (2). Notice that we consider the case of nonnegative boundary condition. The theorem has been proved.

We find the convergence speed rating of rational approximations shown in the theorem 1. Denote:

$$
\delta_{n}=\left(\frac{1}{2 \pi} \int_{0}^{2 \pi}\left|\frac{\chi_{E}(\phi)}{D\left(e^{i \phi}\right)}-\alpha_{n}^{2} \stackrel{*}{\Phi}_{n}\left(e^{i \phi}\right) D(0)\right|^{2} d \sigma(\phi)\right)^{\frac{1}{2}}
$$

where $\chi_{E}(\phi)$ is the characteristic function of the set of points $E$, where exists a finite and positive derivative $\frac{d \sigma(\phi)}{d \phi}$. It is known, that almost everywhere $\frac{d \sigma(\phi)}{d \phi}=\varphi(\phi)$. If for measure the condition of Szegö is met, then $\delta_{n} \rightarrow 0$ with $n \rightarrow \infty$. In the next theorem the condition of Szegö is supposed to be met.
Theorem 2. The inequalities are true for all $z,|z|<1$ :

$$
\begin{aligned}
& \left|\frac{\nu}{\left|\left.\right|_{\Phi_{n}} ^{*}(z)\right|^{2} \alpha_{n}^{2}}-\operatorname{Re}[\hat{U}(z)]\right| \leq \frac{D(0)^{2} \nu \delta_{n}}{\left(1-|z|^{2} \alpha_{0}^{4}\right.}\left(\frac{1}{\sqrt{1-|z|}}+\frac{D(0)^{2} \delta_{n}}{D(0)^{2}+\alpha_{0}}\right) \times \\
& \quad \times\left(2 \alpha_{0}+D(0)^{2} \delta_{n}\left(\frac{1}{\sqrt{1-|z|}}+\frac{D(0)^{2} \delta_{n}}{D(0)^{2}+\alpha_{0}}\right)\right) .
\end{aligned}
$$

Proof. For points $z$ lying inside the unit circle we have:

$$
\begin{align*}
& \nu D^{2}(z)-\frac{\nu}{\left(\stackrel{*}{\Phi}_{n}(z)\right)^{2} \alpha_{n}^{2}}=\frac{\nu}{\left(\stackrel{*}{\Phi}_{n}(z)\right)^{2}}\left(\alpha_{n}^{2}\left(\stackrel{*}{\Phi}_{n}(z)\right)^{2} D^{2}(z)-1\right)= \\
& =\frac{\nu}{\left(\stackrel{*}{\left.\Phi_{n}(z)\right)^{2}}\right.}\left(2\left(\alpha_{n} \stackrel{*}{\Phi}_{n}(z) D(z)-1\right)+\left(\alpha_{n} \stackrel{*}{\Phi}_{n}(z) D(z)-1\right)^{2}\right) \tag{17}
\end{align*}
$$

From the formula (17) we get:

$$
\begin{gather*}
\left|\left|\nu D^{2}(z)\right|-\left|\frac{\nu}{\stackrel{*}{\Phi_{n}(z)^{2} \alpha_{n}^{2}}}\right|\right| \leq\left|\nu D^{2}(z)-\frac{\nu}{\stackrel{*}{\Phi}_{n}(z)^{2} \alpha_{n}^{2}}\right| \leq \\
\leq \frac{\nu}{\alpha_{n}^{2}\left|\stackrel{*}{\Phi}_{n}(z)^{2}\right|}\left(2\left|\alpha_{n} \stackrel{*}{\Phi}_{n}(z) D(z)-1\right|+\left|\alpha_{n} \stackrel{*}{\Phi}_{n}(z) D(z)-1\right|^{2}\right) . \tag{18}
\end{gather*}
$$

Estimate the right side of the formula (18). First of all we note, that

$$
\begin{equation*}
\frac{\nu}{\alpha_{n}^{2}\left|\stackrel{*}{\Phi}_{n}(z)^{2}\right|}=\frac{\nu}{\alpha_{n}^{2}\left|\stackrel{*}{\Phi}_{n}(z)\right|^{2}} \leq \frac{\nu}{\alpha_{0}\left(1-|z|^{2}\right)} . \tag{19}
\end{equation*}
$$

Secondly, this estimation is true (see [3, p. 108-109]):

$$
\begin{equation*}
\left|\alpha_{n} \stackrel{*}{\Phi}(z) D(z)-1\right| \leq \frac{\delta_{n} D(0)^{2}}{\alpha_{0}}\left(\frac{1}{\sqrt{1-|z|}}+\frac{\delta_{n} D(0)^{2}}{D(0)^{2}+\alpha_{0}}\right) . \tag{20}
\end{equation*}
$$

It follows from (18) - (20):

$$
\begin{align*}
& \left|\frac{\nu}{\left|\stackrel{*}{\Phi}_{n}(z)\right|^{2} \alpha_{n}^{2}}-\operatorname{Re}[\hat{U}(z)]\right|=\left|\left|\nu D^{2}(z)\right|-\left|\frac{\nu}{\Psi_{n}(z)^{2} \alpha_{n}^{2}}\right|\right| \leq \\
& \leq \frac{\nu}{\alpha_{0}^{2}\left(1-|z|^{2}\right)}\left(\frac{2 \delta_{n} D(0)^{2}}{\alpha_{0}}\left(\frac{1}{\sqrt{1-|z|}}+\frac{\delta_{n} D(0)^{2}}{D(0)^{2}+\alpha_{0}}\right)+\right.  \tag{21}\\
& \left.\quad+\frac{\delta_{n}^{2} D(0)^{4}}{\alpha_{0}^{2}}\left(\frac{1}{\sqrt{1-|z|}}+\frac{\delta_{n} D(0)^{2}}{D(0)^{2}+\alpha_{0}}\right)^{2}\right) .
\end{align*}
$$

Transforming the right side of formula (21) we get the sought estimation:

$$
\left|\frac{\nu}{\left|\stackrel{*}{\Phi}_{n}(z)\right|^{2} \alpha_{n}^{2}}-\operatorname{Re}[\hat{U}(z)]\right| \leq \frac{D(0)^{2} \nu \delta_{n}}{\left(1-|z|^{2}\right) \alpha_{0}^{4}}\left(\frac{1}{\sqrt{1-|z|}}+\frac{D(0)^{2} \delta_{n}}{D(0)^{2}+\alpha_{0}}\right) \times
$$

$$
\times\left(2 \alpha_{0}+D(0)^{2} \delta_{n}\left(\frac{1}{\sqrt{1-|z|}}+\frac{D(0)^{2} \delta_{n}}{D(0)^{2}+\alpha_{0}}\right)\right)
$$

The theorem has been proved.
The rate at which sequence $\delta_{n}$ decreases with $n \rightarrow \infty$ depends on the function $\varphi(\phi)$ properties (see [3, p. 199, table I]).

Consider another more general case, when the function $\varphi(\phi)$ cannot satisfy the term of Szegö.

Theorem 3. Uniformly on the compacts $|z| \leq r<1$

$$
\hat{U}(z)=C_{0} \nu \lim _{n \rightarrow \infty} \frac{\stackrel{*}{\Psi_{n}(z)}}{\Phi_{n}(z)}
$$

and for the solution of Dirichlet problem $(\operatorname{Re}(\hat{U}(z)))$ the following estimation of approximations convergence rate is true:

$$
\left|\operatorname{Re}(\hat{U}(z))-\frac{\nu}{2 \pi} \operatorname{Re}\left(\frac{\stackrel{*}{\Psi}(z)}{{\underset{\Phi}{\Phi}}_{n}(z)}\right)\right| \leq \frac{\sqrt{2} r^{n} \nu}{2 \pi(1-r)^{\frac{3}{2}}} .
$$

Proof. It follows directly from [3, p. 16 and p. 160]. The theorem has been proved. We pay attention to the circumstance, that the equalities:

$$
\frac{1}{2 \pi} \operatorname{Re}\left(\frac{\stackrel{*}{\Psi_{n}}(z)}{\stackrel{*}{\Phi_{n}}(z)}\right)=\frac{1}{\left|\stackrel{*}{\Phi}_{n}(z)\right|^{2} \alpha_{n}^{2}}
$$

are true, in general, at the unit circle boundary only [3, p. 17].

## 3 Final remarks

So, the rational functions sequence $R_{n}(z)=\operatorname{Re}\left(\frac{\stackrel{*}{\Psi_{n}(z)}}{{\underset{\Phi}{n}}(z)}\right)$ (in case of the condition of Szegö fulfillment $R_{n}(z)=\frac{\nu}{\left|\stackrel{*}{\Phi}_{n}(z)\right|^{2} \alpha_{n}^{2}}$ ) converges uniformly inside the unit circle to the solution of Dirichlet problem (1), (2) with nonnegative boundary condition $\varphi$. In general case we denote the sequence of rational approximations for the problem (6), (7) via $R_{n}^{+}(z)$, and for the problem (8), (9) via $R_{n}^{-}(z)$. The sequence of rational functions $R_{n}^{+}(z)=R_{n}^{+}(z)-R_{n}^{-}(z)$ will be converging uniformly on the compacts inside the unit circle to the solution of the problem (1), (2). The approximations ratings can be made from the above-proved theorems.

The calculations of polynomials $\stackrel{*}{\Psi}_{n}(z)$ and $\stackrel{*}{\Phi}_{n}(z)$ are achieved easily. For this we should use the recurrent equation (10) and notice, that circular parameters can be found according to formulas:

$$
a_{n}=\frac{(-1)^{n}}{\bar{\Delta}_{n-1}}\left|\begin{array}{cccc}
\bar{C}_{-1} & \bar{C}_{-2} & \ldots, & \bar{C}_{-n} \\
\bar{C}_{-0} & \bar{C}_{-1} & \ldots, & \bar{C}_{-n+1} \\
\vdots & \vdots & & \vdots \\
\bar{C}_{n-2} & \bar{C}_{n-3} & \ldots, & \bar{C}_{-1}
\end{array}\right| \text {, }
$$

where $\Delta_{k}$ is Toeplitz matrix determinants

$$
\begin{equation*}
\left\{C_{p-q}\right\}_{p, q=0}^{k} . \tag{22}
\end{equation*}
$$

There exist some special methods for matrix determinant calculation in the view of (22).

The author has realized the algorithms of rational approximations construction within the computer technical system Mathematica.

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# On Riemann extension of the Schwarzschild metric 

V. Dryuma


#### Abstract

The properties of the Riemann extension of the Schwarzschild metric are studied.


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## 1 Introduction

The notice of the Riemann extension of nonriemannian spaces was first introduced in ([1]). Main idea of this theory is application of the methods of Riemann geometry for studying of the properties of nonriemannian spaces.

For example the system differential equations in form

$$
\begin{equation*}
\frac{d^{2} x^{k}}{d s^{2}}+\Pi_{i j}^{k} \frac{d x^{i}}{d s} \frac{d x^{j}}{d s}=0 \tag{1}
\end{equation*}
$$

with arbitrary coefficients $\Pi_{i j}^{k}\left(x^{l}\right)$ can be considered as the system of geodesic equations of affinely connected space with local coordinates $x^{k}$.

For the n-dimensional Riemannian spaces with the metrics

$$
{ }^{n} d s^{2}=g_{i j} d x^{i} d x^{j}
$$

the system of geodesic equations looks same but the coefficients $\Pi_{i j}^{k}\left(x^{l}\right)$ now have very special form and depends from the choice of the metric $g_{i j}$.

$$
\Pi_{k l}^{i}=\Gamma_{k l}^{i}=\frac{1}{2} g^{i m}\left(g_{m k, l}+g_{m l, k}-g_{k l, m}\right)
$$

In order that the methods of Riemann geometry can be applied for studying of the properties of the spaces with equations (1) the construction of 2 n -dimensional extension of the space with local coordinates $x^{i}$ was introduced.

The metric of extended space constructs with help of coefficients of equation (1) and looks as follows

$$
\begin{equation*}
{ }^{2 n} d s^{2}=-2 \Pi_{i j}^{k}\left(x^{l}\right) \Psi_{k} d x^{i} d x^{j}+2 d \Psi_{k} d x^{k} \tag{2}
\end{equation*}
$$

where $\Psi_{k}$ are the coordinates of additional space.
(c) Valery Dryuma, 2003

The important property of such type metric is that the geodesic equations of metric (2) consist from the two parts

$$
\begin{equation*}
\ddot{x}^{k}+\Gamma_{i j}^{k} \dot{x}^{i} \dot{x}^{j}=0 \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\delta^{2} \Psi_{k}}{d s^{2}}+R_{k j i}^{l} \dot{x}^{j} \dot{x}^{i} \Psi_{l}=0 \tag{4}
\end{equation*}
$$

where

$$
\frac{\delta \Psi_{k}}{d s}=\frac{d \Psi_{k}}{d s}-\Pi_{j k}^{l} \Psi_{l} \frac{d x^{j}}{d s}
$$

The first part (3) of complete system is the system of equations for geodesics of basic space with local coordinates $x^{i}$ and they does not contains the coordinates $\Psi_{k}$.

The second part (4) of system of geodesic equations has the form of linear $4 \times 4$ matrix system of second order ODE's for coordinates $\Psi_{k}$

$$
\begin{equation*}
\frac{d^{2} \vec{\Psi}}{d s^{2}}+A(s) \frac{d \vec{\Psi}}{d s}+B(s) \vec{\Psi}=0 \tag{5}
\end{equation*}
$$

From this point of view we get the case of geodesical extension of basic space in local coordinates $\left(x^{i}\right)$.

It is important to note that the geometry of extended space is connected with geometry of basic space. For example the property of this space to be Ricci-flat keeps also for the extended space.

This fact give us the possibility to use the linear system of equation (5) for studying of the properties of basic space.

In particular the invariants of the $4 \times 4$ matrix-function

$$
E=B-\frac{1}{2} \frac{d A}{d s}-\frac{1}{4} A^{2}
$$

under change of the coordinates $\Psi_{k}$ can be used for that.
The first applications of the notice of extended spaces the studying of nonlinear second order differential equations connected with nonlinear dynamical systems was done in works of author ([2-4]).

Here we consider the properties of extended spaces for the Einstein-spaces in General Relativity.

## 2 The Schwarzschild space-time and geodesic equation

The line element of standard metric of the Schwarzschild space-time in coordinate system $x, \theta, \phi, t$ has the form

$$
\begin{equation*}
d s^{2}=-\frac{1}{(1-2 M / x)} d x^{2}-x^{2}\left(d \theta^{2}+\sin ^{2} \theta d \phi^{2}\right)+(1-2 M / x) d t^{2} \tag{6}
\end{equation*}
$$

The geodesic equations of this type of the metric are

$$
\begin{gather*}
\frac{d^{2}}{d s^{2}} x(s)-\frac{M\left(\frac{d}{d s} x(s)\right)^{2}}{(x-2 M) x}+(-x+2 M)\left(\frac{d}{d s} \theta(s)\right)^{2}-  \tag{7}\\
-(x-2 M)(\sin (\theta))^{2}\left(\frac{d}{d s} \phi(s)\right)^{2}+\frac{(x-2 M) M\left(\frac{d}{d s} t(s)\right)^{2}}{x^{3}}=0, \\
\frac{d^{2}}{d s^{2}} \theta(s)+2 \frac{\left(\frac{d}{d s} x(s)\right) \frac{d}{d s} \theta(s)}{x}-\sin (\theta) \cos (\theta)\left(\frac{d}{d s} \phi(s)\right)^{2}=0,  \tag{8}\\
\frac{d^{2}}{d s^{2}} \phi(s)+2 \frac{\left(\frac{d}{d s} x(s)\right) \frac{d}{d s} \phi(s)}{x}+2 \frac{\cos (\theta)\left(\frac{d}{d s} \theta(s)\right) \frac{d}{d s} \phi(s)}{\sin (\theta)}=0,  \tag{9}\\
\frac{d^{2}}{d s^{2}} t(s)-2 \frac{M\left(\frac{d}{d s} x(s)\right) \frac{d}{d s} t(s)}{x(2 M-x)}=0 \tag{10}
\end{gather*}
$$

The symbols of Christoffel of the metric (6) looks as

$$
\begin{gathered}
\Gamma_{11}^{1}=\frac{M}{x(2 M-x)}, \quad \Gamma_{22}^{1}=(2 M-x), \quad \Gamma_{33}^{1}=(2 M-x) \sin ^{2} \theta \\
\Gamma_{44}^{1}=-\frac{M(2 M-x)}{x^{3}}, \quad \Gamma_{12}^{2}=\frac{1}{x}, \quad \Gamma_{33}^{2}=-\sin \theta \cos \theta \\
\Gamma_{23}^{3}=\frac{\cos \theta}{\sin \theta}, \quad \Gamma_{14}^{4}=-\frac{M}{x(2 M-x)}, \quad \Gamma_{13}^{3}=\frac{1}{x}
\end{gathered}
$$

The equations of geodesic (7)-(10) have the first integrals

$$
\begin{gather*}
\frac{d}{d s} x(s)=h \sqrt{1-\left(1-2 \frac{M}{x(s)}\right)\left(h^{-2}+\frac{C^{2}}{(x(s))^{2}}\right)}  \tag{11}\\
\left(\frac{d}{d s} \theta(s)\right)^{2}=h^{2}\left(B^{2}-\frac{C^{2}}{(\sin (\theta))^{2}}\right) x^{-4} \\
\frac{d}{d s} \phi(s)=\frac{h C}{x^{2}(\sin (\theta))^{2}}  \tag{12}\\
\frac{d}{d s} t(s)=h\left(1-2 \frac{M}{x(s)}\right)^{-1}
\end{gather*}
$$

where a dot denotes differentiation with respect to parameter $s$ and $(C, B, h)$ are the constants of motion.

## 3 The Riemann extension of the Schwarzschild metric

Now with help of the formulae (2) we construct the eight-dimensional extension of basic metric (6)

$$
\begin{gather*}
d s^{2}=-\frac{2 M}{x(2 M-x)} P d x^{2}-\frac{2}{x} Q d x d \theta-2(2 M-x) P d \theta^{2}-\frac{2}{x} U d x d \phi+ \\
+2 \frac{M}{x(2 M-x)} V d x d t-2 \frac{\cos \theta}{\sin \theta} U d \phi d \theta-2\left((2 M-x) \sin ^{2} \theta P-\sin \theta \cos \theta Q\right) d \phi^{2}+ \\
\quad+2 \frac{M(2 M-x)}{x^{3}} P d t^{2}+2 d x d P+2 d \theta d Q+2 d \phi d U+2 d t d V \tag{13}
\end{gather*}
$$

where $(P, Q, U, V)$ are the additional coordinates of extension.
The metrics of a given type are the metrics with vanishing curvature invariants. They play an important role in general theory of Riemannian spaces. In particular the metrics for pp -waves in General Relativity belong to this class.

The eight-dimensional space in local coordinates $(x, \theta, \phi, t, P, Q, U, V)$ with this type of metric is also the Einstein space with condition on the Ricci tensor

$$
{ }^{8} R_{i k}=0 .
$$

The complete system of geodesic equations for the metric (??) decomposes into two groups of equations.

The first group coincides with the equations (7-10) on the coordinates $(x, \theta, \phi, t)$ and second part forms the linear system of equations for coordinates $P, Q, U, V$.

They are defined as

$$
\begin{gathered}
\ddot{P}+\frac{2 M}{x(x-2 M)} \dot{x} \dot{P}-\frac{2}{x} \dot{\theta} \dot{Q}-\frac{2}{x} \dot{\phi} \dot{U}-\frac{2 M}{x(x-2 M)} \dot{t} \dot{V}- \\
-\left(\frac{2 M}{x^{2}(x-2 M)} \dot{x}^{2}+\frac{(x-2 M)}{x} \dot{\theta}^{2}+\frac{\sin ^{2} \theta(x-2 M)}{x} \dot{\phi}^{2}+\frac{2 M(x-2 M)}{x^{4}} \dot{t}^{2}\right) P+ \\
+\left(\frac{4}{x^{2}} \dot{x} \dot{\theta}-\frac{2 \cos \theta}{x} \dot{\phi}^{2}\right) Q+\left(\frac{4}{x^{2}} \dot{x} \dot{\phi}+\frac{4 \cos \theta}{x \sin \theta} \dot{\theta} \dot{\phi}\right) U+\left(\frac{4 M^{2}}{x^{2}(x-2 m)^{2}} \dot{x} \dot{t}\right) V=0, \\
\ddot{Q}+2(x-2 m) \dot{\theta} \dot{P}-\frac{2}{x} \dot{x} \dot{Q}-\frac{2 \cos \theta}{\sin \theta} \dot{\phi} \dot{U}-\frac{2(x-4 M)}{x} \dot{x} \dot{\theta} P+ \\
+\left(\frac{2(x-3 M)}{x^{2}(x-2 M)} \dot{x}^{2}-\frac{2(x-2 M)}{x} \dot{\theta}^{2}-\frac{\left(x-4 M \sin ^{2} \theta\right)}{x} \dot{\phi}^{2}+\frac{2 M(x-2 M)}{x^{4}} \dot{t}^{2}\right) Q+ \\
+\left(\frac{4 \cos \theta}{x \sin \theta} \dot{x} \dot{\phi}+\frac{4 \cos ^{2} \theta}{\sin ^{2} \theta} \dot{\theta} \dot{\phi}\right) U=0, \\
\ddot{U}+2 \sin ^{2} \theta(x-2 M) \dot{\phi} \dot{P}+2 \sin \theta \cos \theta \dot{\phi} \dot{Q}-\left(\frac{2 \cos \theta}{\sin \theta} \dot{\theta}+\frac{2}{x} \dot{x}\right) \dot{U}-\frac{2 \sin ^{2} \theta(x-4 M)}{x} \dot{x} \dot{\phi} P-
\end{gathered}
$$

$$
\begin{gathered}
-\left(\frac{4 \sin \theta \cos \theta}{x} \dot{x} \dot{\phi}+2 \dot{\theta} \dot{\phi}\right) Q+ \\
+\left(\frac{2(x-3 M)}{x^{2}(x-2 m)} \dot{x}^{2}+\frac{4 \cos \theta}{x \sin \theta} \dot{x} \dot{\theta}+\frac{2\left(x \cos ^{2} \theta+2 M \sin ^{2} \theta\right)}{x \sin ^{2} \theta} \dot{\theta}^{2}\right)- \\
-\left(\frac{2\left(x-2 M \sin ^{2} \theta\right)}{x} \dot{\phi}^{2}\right) U=0 \\
\ddot{V}-\frac{2 M(x-2 m)}{x^{3}} \dot{t} \dot{P}-\frac{2 M}{x(x-2 M)} \dot{x} \dot{V}+\frac{4 M(x-2 M)}{x^{4}} \dot{x} \dot{t} P+ \\
+\left(\frac{2 M(2 x-3 M)}{x^{2}(x-2 M)^{2}} \dot{x}^{2}-\frac{2 M}{x} \dot{\theta}^{2}-\frac{2 M \sin ^{2} \theta}{x} \dot{\phi}^{2}+\frac{2 M^{2}}{x^{4}} \dot{t}^{2}\right) V=0 .
\end{gathered}
$$

So we get the linear matrix-second order ODE for the coordinates $U, V, P, Q$

$$
\begin{equation*}
\frac{d^{2} \Psi}{d s^{2}}+A(x, \theta, \phi, t) \frac{d \Psi}{d s}+B(x, \theta, \phi, t) \Psi=0 \tag{14}
\end{equation*}
$$

where

$$
\Psi(s)=\left(\begin{array}{c}
P(s) \\
Q(s) \\
U(s) \\
V(s)
\end{array}\right)
$$

and $A, B$ are some $4 \times 4$ matrix-functions depending from the coordinates $x^{i}(s)$ and their derivatives.

We shall study this system of equations at the condition $\theta=\pi / 2$.
In this case we get the system for the coordinates of basic space

$$
\begin{aligned}
& \frac{d^{2}}{d s^{2}} x(s)+\frac{M\left(\frac{d}{d s} x(s)\right)^{2}}{x(s)(2 M-x(s))}+(2 M-x(s))\left(\frac{d}{d s} \phi(s)\right)^{2}- \\
&-\frac{M(2 M-x(s))\left(\frac{d}{d s} t(s)\right)^{2}}{(x(s))^{3}}=0 \\
& \frac{d^{2}}{d s^{2}} \phi(s)+2 \frac{\left(\frac{d}{d s} x(s)\right) \frac{d}{d s} \phi(s)}{x(s)}=0 \\
& \frac{d^{2}}{d s^{2}} t(s)-\frac{M\left(\frac{d}{d s} x(s)\right) \frac{d}{d s} t(s)}{x(s)(2 M-x(s))}=0
\end{aligned}
$$

and the system of equations for the supplementary coordinates

$$
\begin{gathered}
\left(2 \frac{M\left(\frac{d}{d s} x(s)\right)^{2}}{(2 M-x(s))(x(s))^{2}}-\frac{(-2 M+x(s))\left(\frac{d}{d s} \phi(s)\right)^{2}}{x(s)}\right) P(s)+ \\
+\left(2 \frac{M(2 M-x(s))\left(\frac{d}{d s} t(s)\right)^{2}}{(x(s))^{4}}\right) P(s)+
\end{gathered}
$$

$$
\begin{aligned}
& +\frac{d^{2}}{d s^{2}} P(s)+4 \frac{U(s)\left(\frac{d}{d s} x(s)\right) \frac{d}{d s} \phi(s)}{(x(s))^{2}}+4 \frac{M^{2} V(s)\left(\frac{d}{d s} x(s)\right) \frac{d}{d s} t(s)}{(x(s))^{2}(2 M-x(s))^{2}}-2 \frac{M\left(\frac{d}{d s} x(s)\right) \frac{d}{d s} P(s)}{x(s)(2 M-x(s))}- \\
& -2 \frac{\left(\frac{d}{d s} \phi(s)\right) \frac{d}{d s} U(s)}{x(s)}+2 \frac{M\left(\frac{d}{d s} t(s)\right) \frac{d}{d s} V(s)}{x(s)(2 M-x(s))}=0, \\
& \left(2 \frac{(-x(s)+3 M)\left(\frac{d}{d s} x(s)\right)^{2}}{(2 M-x(s))(x(s))^{2}}-\frac{(x(s)-4 M)\left(\frac{d}{d s} \phi(s)\right)^{2}}{x(s)}\right) Q(s)- \\
& -\left(2 \frac{M(2 M-x(s))\left(\frac{d}{d s} t(s)\right)^{2}}{(x(s))^{4}}\right) Q(s)+\frac{d^{2}}{d s^{2}} Q(s)-2 \frac{\left(\frac{d}{d s} x(s)\right) \frac{d}{d s} Q(s)}{x(s)}=0, \\
& -2 \frac{(x(s)-4 M)\left(\frac{d}{d s} x(s)\right)\left(\frac{d}{d s} \phi(s)\right) P(s)}{x(s)}+ \\
& +2\left(-\frac{(-2 M+x(s))\left(\frac{d}{d s} \phi(s)\right)^{2}}{x(s)}+\frac{(-x(s)+3 M)\left(\frac{d}{d s} x(s)\right)^{2}}{(2 M-x(s))(x(s))^{2}}\right) U(s)- \\
& -\left(\frac{M(2 M-x(s))\left(\frac{d}{d s} t(s)\right)^{2}}{(x(s))^{4}}\right) U(s)+\frac{d^{2}}{d s^{2}} U(s)-2(2 M-x(s))\left(\frac{d}{d s} \phi(s)\right) \frac{d}{d s} P(s)- \\
& -2 \frac{\left(\frac{d}{d s} x(s)\right) \frac{d}{d s} U(s)}{x(s)}=0, \\
& \left(-2 \frac{M(3 M-2 x(s))\left(\frac{d}{d s} x(s)\right)^{2}}{(x(s))^{2}(2 M-x(s))^{2}}-2 \frac{M\left(\frac{d}{d s} \phi(s)\right)^{2}}{x(s)}+2 \frac{M^{2}\left(\frac{d}{d s} t(s)\right)^{2}}{(x(s))^{4}}\right) V(s)+ \\
& +\frac{d^{2}}{d s^{2}} V(s)-4 \frac{M P(s)(2 M-x(s))\left(\frac{d}{d s} x(s)\right) \frac{d}{d s} t(s)}{(x(s))^{4}}+2 \frac{M\left(\frac{d}{d s} x(s)\right) \frac{d}{d s} V(s)}{x(s)(2 M-x(s))}+ \\
& +2 \frac{M(2 M-x(s))\left(\frac{d}{d s} t(s)\right) \frac{d}{d s} P(s)}{(x(s))^{3}}=0 .
\end{aligned}
$$

In this case the matrix $A$ takes the form

$$
A=-\left[\begin{array}{cccc}
-2 \frac{\left(\frac{d}{d s} x(s)\right) M}{x(s)(-2 M+x(s))} & 0 & 2 \frac{\frac{d}{d s} \phi(s)}{x(s)} & 2 \frac{M \frac{d}{d s} t(s)}{x(s)(-2 M+x(s))} \\
0 & 2 \frac{\frac{d}{d s} x(s)}{x(s)} & 0 & 0 \\
-2(-2 M+x(s)) \frac{d}{d s} \phi(s) & 0 & 2 \frac{\frac{d}{d s} x(s)}{x(s)} & 0 \\
2 \frac{(-2 M+x(s))\left(\frac{d}{d s} t(s)\right) M}{(x(s))^{3}} & 0 & 0 & 2 \frac{\left(\frac{d}{d s} x(s)\right) M}{x(s)(-2 M+x(s))}
\end{array}\right]
$$

and matrix $B$ has an elements

$$
B_{11}=
$$

$$
\begin{aligned}
& =\frac{\left(\frac{d}{d s} \phi(s)\right)^{2}(x(s))^{4}(x(s)-4 M)+4 M^{2}\left(\frac{d}{d s} \phi(s)\right)^{2}(x(s))^{3}+2 M\left(\frac{d}{d s} t(s)\right)^{2}(x(s))^{2}}{(-2 M+x(s))(x(s))^{4}}+ \\
& +\frac{2 M\left(\frac{d}{d s} x(s)\right)^{2}(x(s))^{2}(-2 M+x(s))(x(s))^{4}-8 M^{2}\left(\frac{d}{d s} t(s)\right)^{2} x(s)+8 M^{3}\left(\frac{d}{d s} t(s)\right)^{2}}{(-2 M+x(s))(x(s))^{4}}, \\
& B_{12}=0, B_{13}=-4 \frac{\left(\frac{d}{d s} x(s)\right) \frac{d}{d s} \phi(s)}{(x(s))^{2}}, B_{14}=-4 \frac{M^{2}\left(\frac{d}{d s} x(s)\right) \frac{d}{d s} t(s)}{(x(s))^{2}\left(4 M^{2}-4 x(s) M+(x(s))^{2}\right)}, \\
& B_{21}=0, \quad B_{22}= \\
& \frac{-8 M^{3}\left(\frac{d}{d s} t(s)\right)^{2}+\left(\frac{d}{d s} \phi(s)\right)^{2}(x(s))^{5}+8 M^{2}\left(\frac{d}{d s} \phi(s)\right)^{2}(x(s))^{3}-2\left(\frac{d}{d s} x(s)\right)^{2}(x(s))^{3}}{(-2 M+x(s))(x(s))^{4}}- \\
& -\frac{2 M\left(\frac{d}{d s} t(s)\right)^{2}(x(s))(x(s)+4 M)+6 M\left(\frac{d}{d s} x(s)\right)^{2}(x(s))^{2}-6 M\left(\frac{d}{d s} \phi(s)\right)^{2}(x(s))^{4}}{(-2 M+x(s))(x(s))^{4}}, \\
& B_{23}=0, \quad B_{24}=0, \\
& B_{31}=2 \frac{(x(s)-4 M)\left(\frac{d}{d s} x(s)\right) \frac{d}{d s} \phi(s)}{x(s)}, \quad B_{32}=0, \\
& B_{33}= \\
& -2 \frac{-\left(\frac{d}{d s} \phi(s)\right)^{2}(x(s))^{3}\left(x(s)^{2}+4 M^{2}\right)+4 M^{3}\left(\frac{d}{d s} t(s)\right)^{2}-3 M\left(\frac{d}{d s} x(s)\right)^{2}(x(s))^{2}}{(-2 M+x(s))(x(s))^{4}}+ \\
& +\frac{\left(\frac{d}{d s} x(s)\right)^{2}(x(s))^{3}+4 M\left(\frac{d}{d s} \phi(s)\right)^{2}(x(s))^{4}-4 M^{2}\left(\frac{d}{d s} t(s)\right)^{2} x(s)+M\left(\frac{d}{d s} t(s)\right)^{2}(x(s))^{2}}{(-2 M+x(s))(x(s))^{4}}, \\
& B_{34}=0, \quad B_{41}=-4 \frac{(-2 M+x(s))\left(\frac{d}{d s} t(s)\right)\left(\frac{d}{d s} x(s)\right) M}{(x(s))^{4}}, \quad B_{42}=0, \quad B_{43}=0, \\
& B_{44}=-2 \frac{M\left(-3 M\left(\frac{d}{d s} x(s)\right)^{2}(x(s))^{2}-\left(\frac{d}{d s} \phi(s)\right)^{2}(x(s))^{5}-4 M^{2}\left(\frac{d}{d s} \phi(s)\right)^{2}(x(s))^{3}\right)}{(x(s))^{4}\left(4 M^{2}-4 x(s) M+(x(s))^{2}\right)}- \\
& -2 \frac{M\left(4 M\left(\frac{d}{d s} \phi(s)\right)^{2}(x(s))^{4}+2\left(\frac{d}{d s} x(s)\right)^{2}(x(s))^{3}+M\left(\frac{d}{d s} t(s)\right)^{2}(2 M-x(s))^{2}\right)}{(x(s))^{4}\left(4 M^{2}-4 x(s) M+(x(s))^{2}\right)} .
\end{aligned}
$$

Now we will integrate our system.
Remark that the equation for the coordinate $Q(s)$ is independent from others equations and can be reduced after the substitution

$$
Q(s)=x(s) F(s)
$$

to the equation for the function $F(s)$

$$
\begin{equation*}
\frac{d^{2}}{d s^{2}} F(s)+\left(\frac{M}{(x(s))^{3}}+3 \frac{C^{2} h^{2} M}{(x(s))^{5}}\right) F(s)=0 \tag{15}
\end{equation*}
$$

To integrate the equations for the coordinates $P(s), U(s), V(s)$ we use the relation

$$
\begin{align*}
&\left(\frac{d}{d s} x(s)\right) P(s)+\left(\frac{d}{d s} \theta(s)\right) Q(s)+\left(\frac{d}{d s} \phi(s)\right) U(s)+\left(\frac{d}{d s} t(s)\right) V(s)- \\
&-1 / 2 s-\mu=0 \tag{16}
\end{align*}
$$

which is consequence of the well known first integral of geodesic equations of arbitrary Riemann space

$$
g_{i k} \frac{d x^{i}(s)}{d s} \frac{d x^{k}(s)}{d s}=\text { const. }
$$

In our case it takes the form

$$
\left(\frac{d}{d s} x(s)\right) P(s)+\left(\frac{d}{d s} \phi(s)\right) U(s)+\left(\frac{d}{d s} t(s)\right) V(s)-1 / 2 s-\mu=0
$$

Solving this equation with respect the function $V(s)$

$$
\begin{equation*}
V(s)=-1 / 2 \frac{2\left(\frac{d}{d s} x(s)\right) P(s)+2\left(\frac{d}{d s} \phi(s)\right) U(s)-s-2 \mu}{\frac{d}{d s} t(s)} \tag{17}
\end{equation*}
$$

and substituting this expression into the last two equations of the system we get the following two equations for coordinates $U(s)$

$$
\begin{gathered}
\frac{d^{2}}{d s^{2}} U(s)=-2 \frac{\left(-2 C h(x(s))^{3} M+C h(x(s))^{4}\right) \frac{d}{d s} P(s)}{(x(s))^{5}}+ \\
+2 \sqrt{\frac{(x(s))^{3} h^{2}-(x(s))^{3}-C^{2} h^{2} x(s)+2 M(x(s))^{2}+2 C^{2} h^{2} M}{(x(s))^{3} h^{2}}} \\
\left(h \frac{d}{d s} U(s)(x(s))^{-1}-h^{2} C(x(s))^{2}(4 M-x(s)) P(s)(x(s))^{-5}\right)- \\
-2 \frac{\left(3 M(x(s))^{2}-(x(s))^{3}+(x(s))^{3} h^{2}-2 C^{2} h^{2} x(s)+5 C^{2} h^{2} M\right) U(s)}{(x(s))^{5}}
\end{gathered}
$$

and $P(s)$
$\frac{d^{2}}{d s^{2}} P(s)+\frac{4 M h}{x(s)(x(s)-2 M)} \sqrt{\frac{(x(s))^{3}\left(h^{2}-1\right)-C^{2} h^{2} x(s)+2 M(x(s))^{2}+2 C^{2} h^{2} M}{(x(s))^{3} h^{2}}} \times$

$$
\begin{gathered}
\times \frac{d}{d s} P(s)=-\frac{\left(-2(x(s))^{3} C h+6(x(s))^{2} C h M\right) \frac{d}{d s} U(s)}{(x(s))^{5}(-2 M+x(s))}- \\
-\frac{\left(2(x(s))^{3} M\left(1-2 h^{2}\right)-(x(s)) h^{2} C^{2}(x(s)-8 M)-6(x(s))^{2} M^{2}\right) P(s)}{(x(s))^{5}(-2 M+x(s))}- \\
-\frac{14 h^{2} C^{2} M^{2} P(s)}{(x(s))^{5}(-2 M+x(s))}- \\
-\sqrt{\frac{(x(s))^{3} h^{2}-(x(s))^{3}-C^{2} h^{2} x(s)+2 M(x(s))^{2}+2 C^{2} h^{2} M}{(x(s))^{3} h^{2}}} \times \\
\times \frac{\left(4(x(s))^{2} h^{2} C-12 x(s) h^{2} C M\right) U(s)}{(x(s))^{5}(-2 M+x(s))}+\frac{M}{x(s)(-2 M+x(s))}
\end{gathered}
$$

So we have showed that every motion on orbit in usual space corresponds the motion in additional space.

Let us consider some examples.
According to ([5]) in the Schwarzshild space-time exists the cyclic orbit

$$
x(s)=6 M
$$

which is the solution of the geodesic equations at the condition

$$
h=1, \quad C=3 \sqrt{(2)} M .
$$

In this case our system takes the form

$$
\begin{equation*}
\frac{d^{2}}{d s^{2}} P(s)-1 / 48 \frac{\sqrt{2} \frac{d}{d s} U(s)}{M^{2}}-\frac{19}{864} \frac{P(s)}{M^{2}}-1 / 24 M^{-1}=0 \tag{18}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{d^{2}}{d s^{2}} U(s)+2 / 3 \sqrt{2} \frac{d}{d s} P(s)-\frac{1}{216} \frac{U(s)}{M^{2}}=0 . \tag{19}
\end{equation*}
$$

The simplest solution of this system looks as

$$
\begin{equation*}
P(s)=-\frac{36}{19} M+A \sin \left(\frac{1}{72} \frac{\sqrt{3+3 i \sqrt{303}} s}{M}\right) \tag{20}
\end{equation*}
$$

and

$$
\begin{equation*}
U(s)=A(39+i \sqrt{303}) \sqrt{2} M \cos \left(\frac{1}{72} \frac{\sqrt{3+3 i \sqrt{303}} s}{M}\right) \frac{1}{\sqrt{3+3 i \sqrt{303}}} \tag{21}
\end{equation*}
$$

where $A$ is arbitrary parameter.

The equation for coordinate $Q(s)$ after substitution

$$
x(s)=6 M, \quad h=1, \quad C=3 \sqrt{(2)} M
$$

takes a form

$$
\frac{d^{2}}{d s^{2}} Q(s)+\frac{1}{108} \frac{Q(s)}{M^{2}}=0
$$

and its solution is

$$
\begin{equation*}
Q(s)=C_{1} \cos \left(1 / 18 \frac{\sqrt{3} s}{M}\right)+C_{2} \sin \left(1 / 18 \frac{\sqrt{3} s}{M}\right) \tag{22}
\end{equation*}
$$

At last the expression for coordinate $V(s)$ in considered case can be found from the relation (??).

It has the form

$$
\begin{align*}
V(s)=-1 / 9 A(39+i \sqrt{303}) \cos \left(\frac{1}{72} \frac{\sqrt{3+3 i \sqrt{303}} s}{M}\right) \frac{1}{\sqrt{3+3 i \sqrt{303}}}+ \\
+1 / 3 s+2 / 3 \mu \tag{23}
\end{align*}
$$

So the formulaes (??,??,??,??) describe the relation between the properties of motion of the test particle on the orbit $x(s)=6 M$ in basic physical space with coordinates $(x, \theta, \phi, t)$ and its map into additional space with coordinates $(P, Q, U, V)$

The solution of the equation (??) relatively parameter $s$ is

$$
s=72 M \arccos \left(\frac{U \sqrt{3+3 i \sqrt{303}} \sqrt{2}}{2 A M((39+i \sqrt{303})}\right) \frac{1}{\sqrt{3+3 i \sqrt{303}}}
$$

The substitution of this value into the formulae for the coordinate $P(s)$ give us the quadric
$95 i U^{2} \sqrt{303}+2071 U^{2}-184832 A^{2} M^{2}+184832 P^{2} M^{2}+700416 P M^{3}+663552 M^{4}=0$.
In the case of radial motion $C=0, \quad h=1$ we get

$$
\begin{equation*}
x(s)=1 / 22^{2 / 3} 3^{2 / 3} \sqrt[3]{M} s^{2 / 3} \tag{24}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{d}{d s} t(s)=3 \frac{s^{2 / 3}}{3 s^{2 / 3}-2 M^{2 / 3} \sqrt[3]{2} \sqrt[3]{3}} \tag{25}
\end{equation*}
$$

The system of equations for additional coordinates takes the form

$$
\frac{d^{2}}{d s^{2}} P(s)+4 M^{2} \sqrt{2} \frac{d}{d s} P(s)(x(s))^{-2} \frac{1}{\sqrt{\frac{M}{x(s)}}}(-2 M+x(s))^{-1}-
$$

$$
\begin{gathered}
-\left(2 x(s) \sqrt{\frac{M}{x(s)}}+6 M \sqrt{\frac{M}{x(s)}}\right) M P(s) \frac{1}{\sqrt{\frac{M}{x(s)}}}(-2 M+x(s))^{-1}(x(s))^{-3}- \\
-\frac{M}{x(s)(-2 M+x(s))}=0 \\
-2 \frac{\left(-M\left(\frac{d}{d s} t(s)\right)^{2}(x(s)-2 M)^{2}-\left(\frac{d}{d s} x(s)\right)^{2}(x(s))^{2}(x(s)-3 M)\right) Q(s)}{(-2 M+x(s))(x(s))^{4}}=0 \\
-2 \frac{\left(\left(\frac{d}{d s} x(s)\right)(x(s))^{4}-2 M\left(\frac{d}{d s} x(s)\right)(x(s))^{3}\right) \frac{d}{d s} Q(s)}{(-2 M+x(s))(x(s))^{4}} Q(s)-2 \\
\frac{(-2}{d s^{2}} U(s)+4 / 3 \frac{U(s)}{s^{2}}-4 / 3 \frac{\frac{d}{d s} U(s)}{s}=0
\end{gathered}
$$

and

$$
V(s)+1 / 2 \frac{2\left(\frac{d}{d s} x(s)\right) P(s)-s-2 \mu}{\frac{d}{d s} t(s)}=0
$$

After substitution here the relations (??,??) we find the solutions

$$
\begin{gathered}
P(s)=1 / 2 \frac{\sqrt[3]{s} \sqrt[3]{2} \sqrt[3]{3}\left(3 C_{5}+s \sqrt[3]{2} \sqrt[3]{3} M^{2 / 3}\right)}{-3 \sqrt[3]{M} s^{2 / 3}+2 M \sqrt[3]{2} \sqrt[3]{3}} \\
Q(s)=C_{3} s+C_{4} s^{4 / 3} \\
U(s)=C_{1} s+C_{2} s^{4 / 3}
\end{gathered}
$$

and

$$
V(s)=s / 2+\frac{C_{5}}{s^{2 / 3}}
$$

The linear system of geodesics for additional coordinates (??) may be used for the studying of the properties of a basic space. In particular the sequence of the matrixes

$$
E(s), \quad E_{; s}, \quad E_{; s s}, \ldots
$$

where

$$
E_{s}=\frac{d E(s)}{d s}+\frac{1}{2}[A(s), E(s)]
$$

and their invariants are important characteristic of a basic space.
Remark that for a given example the matrix $E(s)$ has a property

$$
\operatorname{Det}(E(s))=0, \quad \operatorname{Trace}(E(s))=0
$$

More detail consideration leads to conclusion that in general case for the matrix $E(s)$ the condition

$$
\operatorname{Trace}(E(s))=R_{i j} \dot{x}^{i} \dot{x}^{j}
$$

is obeyed, where $R_{i j}$ is the Ricci tensor of the basic space.
The generalization and the interpretation of these results will be done later.

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V. Dryuma

Received November 28, 2003
Institute of Mathematics and Computer Science
5 Academiei str.
Chişinău, MD-2028 Moldova
E-mail:valery@dryuma.com; cainar@mail.md

# On regularization of singular operators with Carleman shift 

Galina Vornicescu


#### Abstract

The paper is devoted to regularization of some singular integral operators. The necessary and sufficient conditions by which the singular operators with Carleman shift and conjugation admits the equivalent regularization are found.

Mathematics subject classification: 45E05. Keywords and phrases: Regularization, noetherian singular operators, singular operators with conjugation, singular operators with shift.


Let $A \in L(\mathfrak{L})$, where $\mathfrak{L}$ is the Banach space. An operator $M \in L(\mathfrak{L})$ is said to be regularizing for $A$ in a space $\mathfrak{L}$, if the operators $M A-I$ and $A M-I$ are compact in $\mathfrak{L}$. The class of linear bounded operators admitting regularization is attracted by that, for the operators of this class and only for them hold the following properties (F.Neother theorems):

1) the equations $A x=0$ and $A^{*} \varphi=0$ have a finite number of linear independent solutions;
2) the equations $A x=y$ is solvable if and only if its right -hand side is ortogonal to each of solutions of the equation $A^{*} \varphi=0$;

The operator $A \in L(\mathfrak{L})$ satisfying conditions 1) and 2) is called the Fredholm operator and the number

$$
\begin{equation*}
\operatorname{Ind} A=\operatorname{dimker} A-\operatorname{dimker} A^{*} \tag{1}
\end{equation*}
$$

is called its index.
If it is known the regularizing operator $M$ for $A$, then the solution of the equation

$$
\begin{equation*}
A x=y \tag{2}
\end{equation*}
$$

can be reduced to solving the equation

$$
\begin{equation*}
M A x=M y \tag{3}
\end{equation*}
$$

in which the operator $M A-I$ is compact.
For investigation of equation (3) can be applied many methods developed for inversion of operators $I+T$, where $T$ is compact.

Of course a special interest represent the case when the equations (2) and (3) are equivalent by any vector $y$. This means that equations (2) and (3) are solvable
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simultaneously and have the same solutions. This happens to be if and only if Ker $M=\{0\}$.

We say that the operator $A$ admits an equivalent regularization if it has regularizing operator $M$ for which the equations (2) and (3) are equivalent for all $y \in \mathfrak{L}$. In this case the operator $M$ is called equivalent regularizing operator for $A$.

From what has been said above it follows that the operator $M$ is equivalent regularizing operator for $A$, if it is regularizing for $A$ and reversible from the left.
Theorem 1 (see [1]). The operator $A \in L(\mathfrak{L})$ admits an equivalent regularizing if and only if

$$
\begin{equation*}
\operatorname{Ind} A \geq 0 . \tag{4}
\end{equation*}
$$

Next we shall consider singular integral operators with shift

$$
\begin{equation*}
A=a P+b Q+(c P+d Q) V \tag{5}
\end{equation*}
$$

where $a, b, c, d \in C(\Gamma), \Gamma$ is a closed Liapunov contour, $P$ and $Q$ are Riesz operators and $V$ is the operator of Carleman shift: $(V \varphi)(t)=\varphi(\alpha(t)), \alpha^{\prime} \in H_{\mu}(\Gamma)$ and $\alpha(\alpha(t)) \equiv t$.
Theorem 2. Let $\alpha$ preserve the orientation on $\Gamma$

$$
\Delta_{1}=c(t) \tilde{c}(t)+a(t) \tilde{a}(t), \quad \Delta_{2}=d(t) \tilde{d}(t)+b(t) \tilde{b}(t)
$$

where $\tilde{f}(t)=f(\alpha(t)) \quad(\alpha(t) \not \equiv t)$. The operator $A$ assume the regularization in a space $L_{p}(\Gamma)$ if and only if

$$
\begin{equation*}
\Delta_{1}(t) \neq 0, \quad \Delta_{2}(t) \neq 0 \quad \forall t \in \Gamma \tag{6}
\end{equation*}
$$

Theorem 3. The operator $A$ admits an equivalent regularizing in $L_{p}(\Gamma)$ if and only if the conditions (6) hold and

$$
\begin{equation*}
\operatorname{ind} \frac{\Delta_{1}(t)}{\Delta_{2}(t)} \leq 0 \tag{7}
\end{equation*}
$$

Theorem 4. Let $\alpha$ change the orientation on $\Gamma$. The operator $A$ admits regularization in $L_{p}(\Gamma)$ if and only if

$$
\begin{equation*}
\Delta(t)=a(t) \tilde{b}(t)-c(t) \tilde{d}(t) \neq 0 \quad \forall t \in \Gamma \tag{8}
\end{equation*}
$$

Theorem 5. The operator $A$ admits equivalent regularization if and only if the conditions (8) are satisfied and

$$
\begin{equation*}
i n d \Delta(t) \leq 0 \tag{9}
\end{equation*}
$$

Let us consider the singular operator of the following form

$$
\begin{equation*}
B=a P+b Q+(c P+d Q) W \tag{10}
\end{equation*}
$$

with complex conjugation $W \varphi(t)=\bar{\varphi}(t)$. For the operator $B$ are valid theorems 4 and 5 in which a function $\tilde{f}(t)$ is replaced by $\bar{f}(t)$.

By proving theorems 2 and 4 the theory of normalized rings is applied (see [2]) and local principle [3]. At the end we mention that enough part of theorems 2 and 3 are contained in monograph [4].

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G. Vornicescu

Received December 2, 2003
Tiraspol State University
5 Iablocikin str.
Chişinău, MD-2069 Moldova


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[^2]:    © A.D. Kolesnik, 2003

[^3]:    ${ }^{1}$ These manifolds are called homogeneous manifolds.

[^4]:    ${ }^{2}$ A domain $D \subseteq \mathbb{C}^{n}$ is called symmetric if for each $z_{0} \in D$ there exists a biholomorphic automorphism $\phi_{z_{0}}: D \rightarrow D$ with $\phi_{z_{0}} \circ \phi_{z_{0}}=i d$ so that $z_{0}$ is an isolated fixed point of $\phi_{z_{0}}$.

[^5]:    ${ }^{3}\left(B_{M}(p, v)\right)^{2}$ is called Bergman's form of $M$.
    ${ }^{4}$ Received by the Editor on 18 July, 1979.
    ${ }^{5}$ Received by the Editor on 17 October, 1979; revised form on 30 January, 1980.

[^6]:    ${ }^{6}$ The article [11] was appeared on March 1986
    ${ }^{7}$ Received by the Editor on June 16, 1986

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