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On Strong Stability of Linear Poisson Actions

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Abstract. Linear Poisson actions of the group R^m are considered. Conditions on the joint spectrum of the generators and on the centralizers assuring stability and strong stability of the action are given. We give also some examples of Poisson actions using CAS *"Mathematica"*.

Mathematics subject classification: 37J25.
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1 Introduction

The problem of stability and strong stability of the Hamiltonian systems is an old one and begins with the Poincare's and Lyapunov's classical results. Even the linear autonomous case represents an interesting problem and a series of papers has been devoted to these systems [1–7]. Some generalizations for dynamical systems with manydimensional time have been given in [8–10].

In the last decade some bihamiltonian systems as models of phisical problems appeared. In [11] a Poincare type classification of the fixed points of a bihamiltonian system in the dimension four has been purposed. In this connection the problem of stability and strong stability of fixed points, and more generally, of periodical orbits of these systems, arises. This problem is the main subject of the paper. More precisely, the linear parts of the Hamiltonian vector fields near fixed points give us a tuple of pairwise commuting linear Hamiltonian matrices, or, in other words, a linear Poisson action of the abelian group R^m in the vector space with a symplectic structure. We define stability and strong stability for such actions.

It is known that the linear differential equation

$$\dot{x} = Ax, \tag{1}$$

where $x \in R^{2n}$ and $A \in sp(2n, R)$, i.e. $A = JH$, $H^T = H$, $J^2 = -I$, is stable if and only if all the eigenvalues are purely imaginary and A is diagonalizable. Moreover, if the spectrum of A is simple and purely imaginary, then (1) is strongly stable [6,7]. M.G.Krein [5] has shown that strong stability holds even in the case when multiple eigenvalues occur, provided these eigenvalues are "positive definite".

Other criteria of strong stability has been stated (and proved using normal forms) by R.Cushman and R.Kelly ([2]). A geometrical proof of this result has been given by M.Levi ([3]).

Theorem 1. [2, 3] *An infinitesimally stable symplectic matrix A is strongly stable if and only if its centralizer $C(A)$ (in $sp(2n,R)$) consists of stable matrices.*

Another criterion in the language of first integrals has been stated by M.Wójtkowski ([4]). More precisely, remark that $h_k(x) = 1/2(JA^kx,x)$ is a quadratic first integral (if k is even, then $h = 0$); here $h(x) = 1/2(JAx,x)$ denotes the Hamiltonian of the system (1) $((\cdot,\cdot)$ is the standard scalar product in $R^{2n})$.

Theorem 2. [4] *A linear Hamiltonian system is strongly stable if and only if some linear combination of the quadratic first integrals h_k, $k = 1,\dots,n$, is a nondegenerate definite quadratic form.*

In what follows we generalize the above mentioned criteria to linear Poisson actions of the abelian group R^m. New problems arise in this context. Firstly, we have no kind of normal form of commuting m-tuples of linear operators, similar to the Jordan normal form of a matrix, or a normal form of Hamiltonian first integrals as those of Williamson [6]. We make use of results of L. Lerman and Ya. Umanskiy [11], who give normal forms of bihamiltonian systems in dimension four.

Another problem, an algebraic geometric one, is the question about the structure of the variety of commuting m-tuples of matrices in the direct product of Lie algebras $gl(n,C)$ or $sl(2n,R)$. For some related results see [12].

2 Basic notions

Let V be a real $2n$-dimensional vector space and let ω be a nondegenerate skew symmetric bilinear form on V. We call the pair (V,ω) a real *symplectic vector space.* The standard example of the *symplectic inner product* ω is $\omega(x,y) = [x,y] = x^TJy$, where the matrix J has the form:

$$J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

with I_n for the identity matrix. A *symplectic basis* for V is a basis $v_1,\dots,v_{2n}$ such that $\omega(v_i,v_j) = J_{ij}$, the i,jth entry of J.

A linear map $T : V \to V$ is called *symplectic* if $[Tx,Ty] = [x,y]$ for all $x,y \in V$. The group of all real symplectic operators on (V,ω) is denoted by $Sp(2n,R)$.

A linear operator $L : V \to V$ is called *Hamiltonian* if the condition

$$[Lx,y] + [x,Ly] = 0$$

holds for all $x,y \in V$. A matrix A is called Hamiltonian or *infinitesimally symplectic* if $A^TJ + JA = 0$. The Lie algebra of all Hamiltonian matrices is denoted by $sp(2n,R)$.

Let $\mathcal{T} = \{T_1,\dots,T_m\}$ be an m-tuple of bounded linear operators in a Hilbert space H. One says [13] that a point $\Lambda = \{\lambda_1,\dots,\lambda_m\} \in C^{m^*}$ belongs to the *left joint spectrum* $\sigma_l(\mathcal{T})$ (respectively, to the *right joint spectrum* $\sigma_r(\mathcal{T})$) if an m-tuple

$\mathcal{R} = \{R_1, \dots, R_m\}$ of linear bounded operators in H such that $\sum\limits_{k=1}^{m} R_k(T_k - \lambda_k I) = I$ (respectively, such that $\sum\limits_{k=1}^{m} (T_k - \lambda_k I)R_k = I$) does not exist. The *joint spectrum* of a polyoperator $\mathcal{T}$ is defined as a sum of its left and right joint spectra. It is denoted by $\sigma(\mathcal{T})$.

In the case of $n = 1$ the above mentioned definition is equivalent to the common definition of the operator's spectrum. (In the case of finite dimension, the notions of the left and the right joint spectra coincide.)

Another way to define the joint spectrum in the finite-dimensional case is based on the known fact from linear algebra that any family of commuting complex linear operators possesses a *joint eigenvector*, i.e. for any $\mathcal{A} = \{A_1, \dots, A_m\}$, $A_i \circ A_j = A_j \circ A_i$, there exists a vector $h \neq 0$ such that $A_j h = \lambda_j h$ for any $j = 1, \dots, m$ and some $\{\lambda_1, \dots, \lambda_m\} \in C^{m^*}$. Then $\Lambda = \{\lambda_1, \dots, \lambda_m\}$ is called the *eigenfunctional* corresponding to the joint eigenvector h. The set of all eigenfunctionals creates the joint spectrum $\sigma(\mathcal{A})$. Some details concerning the properties of joint spectra can be found in [8, 13].

We mention that for m-tuples of commuting hamiltonian matrices the joint spectrum has symmetry properties similar to those of a single hamiltonian matrix.

Let $\Phi : R^m \times V \to V$ be a continuous action of the group R^m on V such that for any fixed $t \in R^m$ the transformation $\Phi^t = \Phi(t, \cdot)$ is a linear symplectic transformation of the space V. An action of this type is called [11] a linear Poisson action.

Consider a Hamiltonian polyoperator $\mathcal{A} = \{A_1, \dots, A_m\}$. Remark that for the linear completely integrable system

$$\frac{\partial x}{\partial t_j} = A_j x \quad (x \in R^{2n},\ t_j \in R,\ j = 1, \dots, m) \tag{2}$$

the fundamental matrix is $\exp(\mathcal{A}, t) := \exp(A_1 t_1 + \dots + A_m t_m)$. The system (2) is called *stable* if $\exists M > 0$ such that $\|\exp(\mathcal{A}, t)\| < M$ for all $t \in R^m$. It is called *strongly stable* if there exists $\varepsilon > 0$ such that for any polyoperator $\mathcal{B} = \{B_1, \dots, B_m\} \in (sp(2n, R))^m$, $B_i \circ B_j = B_j \circ B_i$, $\|B_i - A_i\| < \varepsilon$ $(i, j = 1, \dots, m)$, the inequality $\|\exp(\mathcal{B}, t)\| < M$ holds for some $M > 0$ and all $t \in R^m$.

3 Results. Stability and strong stability of linear Poisson actions

Let $\mathcal{A} = \{A_1, \dots, A_m\}$ be a Hamiltonian polyoperator, i.e. the matrices A_j are Hamiltonian and pairwise commuting.

Theorem 3. *A linear constant completely integrable Hamiltonian system*

$$\frac{\partial x}{\partial t_j} = A_j x \quad (x \in R^{2n}, \quad t_j \in R \quad \forall j \in \{1, \dots, m\}) \tag{3}$$

is stable if and only if for any $j = 1, 2, \dots, m$ the Hamiltonian system

$$\frac{dx}{ds} = A_j x \quad (x \in R^{2n}, s \in R) \tag{4}$$

is stable.

Proof. Assume that the systems (4) are stable for $j = 1, \dots, m$. Hence, for each fixed j there exists $M_j > 0$ such that $\|\exp(A_j t_j)\| < M_j$ $(t_j \in R)$. Then, there is $M = M_1 \cdot \dots \cdot M_m > 0$ such that for all $(t_1, \dots, t_m) \in R^m$:

$$\|\exp(A_1 t_1 + \dots + A_m t_m)\| = \|\exp(A_1 t_1) \cdot \dots \cdot \exp(A_m t_m)\| \leq M.$$

Let (3) be stable. So, there exists $M > 0$, for which $\|\exp(A_1 t_1 + \dots + A_m t_m)\| < M$ $((t_1, \dots, t_m) \in R^m)$. In particular, the inequality holds for all $(t_1, 0, \dots, 0)$, $(0, t_2, 0, \dots, 0)$ and so on. So, one has $\|\exp(A_j t_j)\| < M$ for $j = 1, \dots, m$. Hence all systems (4) are stable.

The following result shows that this is not the case for strong stability.

Theorem 4. *Let (2) be stable and assume that there exists a strongly stable element* $\exp(\mathcal{A}, t_0)$ *for some $t_0 \in R^m$. Then the system (2) is strongly stable.*

Proof. Choose $\varepsilon > 0$ such that for each $B \in sp(2n, R)$ satisfying $\|B - (\mathcal{A}, t_0)\| < \varepsilon$, one has $\|\exp B\tau\| < \infty$ $(\tau \in R)$. Let $\mathcal{B} = \{B_1, B_2, \dots, B_m\}$ be a Hamiltonian polyoperator ε - close to $\mathcal{A}$, i.e. $\|B_i - A_i\| < \varepsilon$ and $B_i \circ B_j = B_j \circ B_i$ $(i, j = 1, 2, \dots, m)$. Then $\|(\mathcal{B}, t_0) - (\mathcal{A}, t_0)\| = \|(\mathcal{B} - \mathcal{A}, t_0)\| \leq \|\mathcal{B} - \mathcal{A}\| \cdot \|t_0\| < \varepsilon$ and $(\mathcal{B}, t_0)$ is strongly stable if $\|\mathcal{B}\| = \frac{\varepsilon}{\|t_0\|}$. On the other hand, $B_j \in C((\mathcal{B}, t_0))$, so, by Theorem 1 $\dot{x} = B_j x$ are stable (for every $j = 1, 2, \dots$), which implies that $\mathcal{B}$ is also stable.

Remark 1. *It is worth noting that at least formally, strong stability of the polyoperator is weaker than the condition of existence of a strongly stable element: a neighbourhood of a point in $sl(2n, R)$ is larger than a neighbourhood of a polyoperator in the subvariety of commuting m-tuples from $sl(2n, R)^m$. It is a problem whether this subvariety is irreducible or not.*

The following result reduces the problem of strong stability of a polyoperator on the whole phase space to the problem of such stability on the invariant symplectic subspaces. The main idea of the proof uses the fact that the centralizer of a block-diagonal matrix with spectrally separated blocks coincides with the direct sum of centralizers of the blocks.

Theorem 5. *Let $\mathcal{A} = \{A_1, \dots, A_m\}$ be a Hamiltonian polyoperator with multiple eigenfunctionals*

$$\Lambda_1 = \{i\lambda_1^1, \dots, i\lambda_m^1\}, -\Lambda_1, \dots, \Lambda_k = \{i\lambda_1^k, \dots, i\lambda_m^k\}, -\Lambda_k,$$

$m_1, \dots, m_k$ denoting corresponding multiplicities and V_r - the subspace of $(R^{2n})^m$ corresponding to the eigenfunctionals Λ_r and $-\Lambda_r$ with multiplicity m_r. Besides, let $\mathcal{A}/V_r$ stand for the polyoperator $\mathcal{A}$ restricted to this subspace. Then, $\mathcal{A}$ is strongly stable if and only if $\mathcal{A}/V_r$ is strongly stable for all r.

Proof. Assume that $\mathcal{A}$ is strongly stable, i.e. there is $\varepsilon > 0$ such that for any polyoperator $\mathcal{B} = \{B_1, \ldots, B_m\}$ such that $\|B_j - A_j\| < \varepsilon$ the inequality:

$$\|\exp(B_1 t_1 + \cdots + B_m t_m)\| < M$$

holds for some $M > 0$ and for all $(t_1, \ldots, t_m) \in R^m$. We shall show that $\mathcal{A}/V_r$ are strongly stable for $r = 1, \ldots, k$.

Recall that a subspace U of a symplectic space (V, ω) is called [1] symplectic if ω restricted to this subspace is nondegenerate. (Obviously, such U is of even dimension, hence (U, ω) is a symplectic space.) Choose a polyoperator $\mathcal{B}_r = \{B_1^r, \ldots, B_m^r\}$ on the symplectic subspace V_r such that $\|\mathcal{A}/V_r - \mathcal{B}_r\| \leq \varepsilon$ and consider a polyoperator $\mathcal{B} = \oplus_{s \neq r} \mathcal{A}/V_s \oplus \mathcal{B}_r$ on R^{2n} (here $\oplus$ stands for the direct sum of operators). Then $\|\mathcal{B} - \mathcal{A}\| = \|\mathcal{B}_r - \mathcal{A}/V_r\| \leq \varepsilon$, since $\mathcal{B}/V_s = \mathcal{A}/V_s$ for $s \neq r$.
Hence one has: $\exp(\mathcal{B}, t) = \oplus_{s \neq r} \exp(\mathcal{A}/V_s, t) \oplus \exp(\mathcal{B}_r, t)$ and

$$M \geq \|\exp(\mathcal{B}, t)\| = \prod_{s \neq r} \|\exp(\mathcal{A}/V_s, t)\| \, \|\exp(\mathcal{B}_r, t)\| \,. \tag{5}$$

Using the Banach-Steinhaus Theorem one can easily prove that

$$p = \inf_{t \in R^m} \prod_{s \neq r} \|\exp(\mathcal{A}/V_s, t)\| > 0.$$

From (5) we obtain

$$\|\exp(\mathcal{B}_r, t)\| \leq \frac{M}{p}.$$

So, $\mathcal{A}/V_r$ is strongly stable.

Assume now that $\mathcal{A}/V_s$ are strongly stable for $s = 1, \ldots, k$ and suppose that $\mathcal{A}$ is not strongly stable. That means that there exists a sequence $\{\mathcal{B}_k\}_{k=1}^{\infty} \to \mathcal{B}$ of nonstable polyoperators. Due to the upper semicontinuity of the joint spectrum, $\{\mathcal{B}_k\}$ have a spectral decomposition close to V_r and $\left\|\mathcal{B}_r/U_r^{(k)} - \mathcal{A}/V_r\right\| \to 0$ as $k \to \infty$. The latest implies that there is r such that $\mathcal{A}/V_r$ is not strongly stable. This contradiction proves the theorem.

Following [5, 7], we call an eigenfunctional $\Lambda \in C^{m*}$ definite if there exists an element $t_0 \in R^m$ such that $\exp(\Lambda, t_0)$ is a positive definite eigenvalue for the symplectic operator $\exp(\mathcal{A}, t_0)$.

Remark 2. *Mention that a simple purely imaginary eigenfunctional is definite and that, in this case, the system (2) is strongly stable.*

Theorem 6. *If the joint spectrum of the polyoperator $\mathcal{A}$ is purely imaginary and definite, then the differential system (2) is strongly stable.*

Proof. Due to Theorem 5, it is enough to consider the case when the polyoperator $\mathcal{A}$ has a single-point joint spectrum

$$\Lambda = \{i\omega_1, i\omega_2, \ldots, i\omega_n, -i\omega_1, -i\omega_2, \ldots, -i\omega_n\}$$

of some multiplicity s.

Let Λ be definite and let $t_0 \in R^m$ be such that $\exp(\Lambda, t_0)$ is definite. Then the element $(\mathcal{A}, t_0) \in sp(2n, R)$ is strongly stable because it has a positive definite first integral. From Theorem 4 it follows that the system (2) is strongly stable.

In what follows we give some generalizations of the strong stability criteria of Cushman-Kelly [2], M. Levi [3] and M. Wójtkowski [4].

Recall that for a given $A \in sp(2n, R)$, $C(A)$ denotes the center of A in $sp(2n, R)$, i.e. $C(A) = \{X \in sp(2n, R) : AX = XA\}$.

Theorem 7. *If* $\bigcup_{j=1}^{m} C(A_j)$ *consists of stable linear Hamiltonian operators, then* (2) *is strongly stable.*

Proof. Let $C(\mathcal{A})$ contain only stable operators and let $\{B_1, \dots, B_m\}$ be close enough to $\mathcal{A} = \{A_1, \dots, A_m\}$. By [3], each B_j can be written under the form $B_j = \exp(-T_j) \circ (A_j + D_j) \circ \exp(T_j)$ for some $D_j \in C(A_j)$ and $T_j \in sp(2n, R)$. Since A_j are stable and $D_j \in C(A_j)$, then D_j are stable, as well as $A_j + D_j$, and hence $\exists M > 0$ such that $\|\exp(B\tau)\| \leq M$ for all $\tau \in R$.

If, in addition, $B_i \circ B_j = B_j \circ B_i$, then $\|\exp(B_1 t_1 + B_2 t_2 + \dots + B_m t_m)\| < M^m$ for all $(t_1, t_2, \dots, t_m) \in R^m$.

Theorem 8. *If the system* (2) *is strongly stable, then* $\bigcap_{j=1}^{m} C(A_j)$ *consists of stable operators.*

Proof. Let $\mathcal{A} = \{A_1, A_2, \dots, A_m\}$ be strongly stable and let $B_1 \in \bigcap_{j=1}^{m} C(A_j)$. For $\mathcal{B} := \{B_1, 0, \dots, 0\} \in sp(2n, R)^m$ take $\varepsilon > 0$ small enough to assure stability of $\mathcal{A} + \varepsilon\mathcal{B}$. So we have: $\|\exp(\mathcal{A} + \varepsilon\mathcal{B}, t)\| < M$, $\|\exp(-\mathcal{A}, t)\| < M$ for some $M > 0$ and for all $t \in R^m$. Since $B_1 \in \bigcap_{j=1}^{m} C(A_j)$, one has: $\|\exp(\varepsilon B_1 t_1)\| = \|\exp(-\mathcal{A}, t)\exp(\mathcal{A} + \varepsilon\mathcal{B}, t)\| \leq M^2 \quad (t \in R^m)$.

Remark 3. *So, if* $\bigcup_{j=1}^{m} C(A_j)$ *consists of stable linear Hamiltonian operators, then* $\bigcap_{j=1}^{m} C(A_j)$ *consists also of stable operators. A natural question if the inverse implication is true arises. In what follows we give a counterexample to this hypothesis.*

Proposition 1. *There exist polyoperators* $\mathcal{A}$ *such that* $\bigcap_{j=1}^{m} C(A_j)$ *consists of stable operators, but* $\bigcup_{j=1}^{m} C(A_j)$ *cointains unstable operators.*

Proof. The authors of [11] give (see Appendix A) the list of normal forms of all possible quadratic Hamilton functions in the case of two degrees of freedom and also of the quadratic functions that are additional integrals of the corresponding linear Hamiltonian system. There are 15 different possible cases. We use this classification to give the counterexample we need.

Consider the case 3 which is given through the following conditions:
the eigenvalues are $(\pm i\omega_1, \pm i\omega_2)$, $\omega_1 \neq \omega_2$, $\omega_1, \omega_2 \in R$, $\omega_1, \omega_2 \neq 0$,

$$H = \frac{\omega_1}{2}(p_1^2 + q_1^2) + \frac{\omega_2}{2}(p_2^2 + q_2^2),$$

$$K = \frac{\nu_1}{2}(p_1^2 + q_1^2) + \frac{\nu_2}{2}(p_2^2 + q_2^2).$$

The condition for the algebra to be two-dimensional is $\omega_1\nu_2 - \omega_2\nu_1 \neq 0$. In this case for fixed ν_1 and ν_2 the centralizer C coincides with the algebra generated by the pair H, K. Put $\omega_1 = 2$, $\omega_2 = 2$, $\nu_1 = 2$, $\nu_2 = 0$. Then we obtain the particular case where $H = p_1^2 + p_2^2 + q_1^2 + q_2^2$, $K = p_1^2 + q_1^2$ and the condition $\omega_1\nu_2 - \omega_2\nu_1 \neq 0$ is satisfied. It is obvious that for $\alpha_1 = 1$ and $\alpha_2 = 1$ the linear combination $\alpha_1 H + \alpha_2 K = 2p_1^2 + p_2^2 + 2q_1^2 + q_2^2$ is a positively definite quadratic form. So, the polyoperator $\{A_1, A_2\}$ is strongly stable (see [14]). In this case the matrices corresponding to the integrals H and K have the form:

$$A_1 = \begin{pmatrix} 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The following computations have been done with the help of CAS *"Mathematica"*. The centralizers of the matrices A_1 and A_2 are:

$$C(A_1) = \{C_1 = \begin{pmatrix} 0 & -k_3 & -n_1 & -n_2 \\ k_3 & 0 & -n_2 & -n_3 \\ n_1 & n_2 & 0 & -k_3 \\ n_2 & n_3 & k_3 & 0 \end{pmatrix} : k_3, n_1, n_2, n_3 \in R\},$$

$$C(A_2) = \{C_2 = \begin{pmatrix} 0 & 0 & -t_1 & 0 \\ 0 & r_4 & 0 & s_3 \\ t_1 & 0 & 0 & 0 \\ 0 & t_3 & 0 & -r_4 \end{pmatrix} : r_4, s_3, t_1, t_3 \in R\}.$$

So,

$$C(A_1) \cap C(A_2) = \{C_3 = \begin{pmatrix} 0 & 0 & -t_1 & 0 \\ 0 & 0 & 0 & s_3 \\ t_1 & 0 & 0 & 0 \\ 0 & -s_3 & 0 & 0 \end{pmatrix} : r_4, s_3, t_1, t_3 \in R\},$$

$$JordanForm(C_3) = \begin{pmatrix} -is_3 & 0 & 0 & 0 \\ 0 & is_3 & 0 & 0 \\ 0 & 0 & -it_1 & 0 \\ 0 & 0 & 0 & it_1 \end{pmatrix}.$$

Hence, $C(A_1) \cap C(A_2)$ consists of stable operators. Remark that some matrices in $C(A_2)$ possess real eigenvalues. Let, for example, $t_1 = 3$, $r_4 = 2\sqrt{2}$, $s_3 = 4$ and $t_3 = 2$. Then we get $C_2 \in C(A_2)$ with the eigenvalues ± 4, $\pm 3i$. So, $C(A_1) \cup C(A_2)$ cointains at least one unstable operator.

Remark 4. *The main results have been announced in [15].*

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Algebraic equations with invariant coefficients in qualitative study of the polynomial homogeneous differential systems*

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Abstract. For planar polynomial homogeneous real vector field $X = (P, Q)$ with $\deg(P) = \deg(Q) = n$ some algebraic equations of degree $n+1$ with $GL(2,\mathbb{R})$-invariant coefficients are constructed. A recurrent method for the construction of these coefficients is given. In the generic case each real or imaginary solution s_i $(i = 1, 2, \dots, n+1)$ of the main equation is a value of the derivative of the slope function, calculated for the corresponding invariant line. Other constructed equations have, respectively, the solutions $1/s_i$, $1 - s_i$, $s_i/(s_i - 1)$, $(s_i - 1)/s_i$, $1/(1 - s_i)$. The equation with the solutions $(n+1)s_i - 1$ is called residual equation. If X has real invariant lines, the values and signs of solutions of constructed equations determine the behavior of the orbits in a neighbourhood at infinity. If X has not real invariant lines, it is shown that the necessary and sufficient conditions for the center existence can be expressed through the coefficients of residual equation.

Mathematics subject classification: 34C05, 58F14.
Keywords and phrases: algebraic equation, invariant, differential homogeneous system, qualitative study, center problem.

1 The homogeneous differential system

Let $n \geq 1$ be a positive integer, $x, y : \mathbb{R} \to \mathbb{R}$ be some unknown functions of real variable t such that $x = x(t)$, $y = y(t)$, $(\forall)\ t \in \mathbb{R}$, $a_{i,j}$, $b_{i,j}$ be real numbers for all positive integers i and j with $i + j = n$, $C_n^k = \binom{n}{k}$ be the binomial coefficients for every positive integer k, $0 \leq k \leq n$.

Let us consider the polynomial homogeneous differential system

$$\frac{dx}{dt} = \sum_{k=0}^{n} C_n^k a_{n-k,k} x^{n-k} y^k = P_n(x, y),$$
$$\frac{dy}{dt} = \sum_{k=0}^{n} C_n^k b_{n-k,k} x^{n-k} y^k = Q_n(x, y). \tag{1}$$

Let $GL(2,\mathbb{R})$ be the group of non-degenerate linear homogeneous transformations. It is known that the homogeneous polynomials $P_n(x, y)$ and $Q_n(x, y)$ are relatively prime iff the resultant μ_n of these polynomials is not equal to zero.

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Remark 1. *The resultant* μ_n *is a* GL*-invariant of the degree* $2n$ *with respect to the system (1) coefficients and with the weight equal to* $n^2 - n$.

Remark 2. *The homogeneous polynomial* $F_{n+1}(x,y) = yP_n(x,y) - xQ_n(x,y)$ *is a* GL*-comitant of the degree* $n+1$ *with respect to variables* x *and* y, *of the degree* 1 *with respect to the system (1) coefficients and with the weight equal to* -1. *The nontrivial solutions of the equation* $F_{n+1}(x,y) = 0$ *determine the system (1) invariant straight lines (real or imaginary).*

We suppose that

$$\mu_n = Res\,(P_n, Q_n) \neq 0, \quad F_{n+1}(x,y) = yP_n(x,y) - xQ_n(x,y) \not\equiv 0 \tag{2}$$

and denote the following polynomials and functions:

$$\begin{gathered} G_{n+1}(x,y) = xP_n(x,y) + yQ_n(x,y), \quad T_{n-1}(x,y) = \frac{\partial P_n(x,y)}{\partial x} + \frac{\partial Q_n(x,y)}{\partial y}, \\ \varphi:\ \mathbb{C}\setminus E_\varphi \to \mathbb{C},\ \varphi(1,k) = \frac{Q_n(1,k)}{P_n(1,k)}, \quad \psi:\ \mathbb{C}\setminus E_\psi \to \mathbb{C},\ \psi(s,1) = \frac{P_n(s,1)}{Q_n(s,1)}, \end{gathered} \tag{3}$$

where $E_\varphi = \{k \mid k \in \mathbb{C},\ P_n(1,k) = 0\}$ and $E_\psi = \{s \mid s \in \mathbb{C},\ Q_n(s,1) = 0\}$. The functions φ and ψ are called the slope functions for the system (1).

Remark 3. *The homogeneous polynomial* $T_{n-1}(x,y)$ *is a* GL*-comitant of the degree* $n-1$ *with respect to variables* x *and* y, *of the degree* 1 *with respect to the system (1) coefficients and with the weight equal to* 0.

Because the GL-comitant $F_{n+1}(x,y)$ is not equal to zero identically, then there exist constants $u_i \in \mathbb{C}$ and $v_i \in \mathbb{C}$ such that $F_{n+1}(x,y)$ has the factorization

$$F_{n+1}(x,y) = \prod_{i=1}^{n+1}(u_i x + v_i y), \quad u_i^2 + v_i^2 \neq 0, \quad (\forall)\ i = 1,2,\ldots,n,n+1. \tag{4}$$

For $v_i \neq 0$ $(u_i \neq 0)$ we denote by $k_i = -u_i/v_i$ $(s_i = -v_i/u_i)$ the roots of the equation $F_{n+1}(1,k) = 0$ $(F_{n+1}(s,1) = 0)$.

The discriminant D_{n+1} of the homogeneous equation $F_{n+1}(x,y) = 0$ has the form

$$D_{n+1} = \prod_{1\le i<j\le n+1} d_{i,j}^2, \qquad d_{i,j} = u_i v_j - u_j v_i. \tag{5}$$

For $j \neq k$ $(k = 1,2,\ldots,n,n+1)$ we denote

$$f_k = (-1)^n \prod_{j=1}^{n+1} d_{k,j}. \tag{6}$$

From relations (5) and (6) follows

Proposition 1. *The discriminant* D_{n+1} *has the factorization*

$$\prod_{k=1}^{n+1} f_k = (-1)^{\frac{n(n+1)}{2}} D_{n+1}. \tag{7}$$

Remark 4. *Each* u_i *and* v_i *have the same degree* $1/(n+1)$ *and the weight equal, respectively, to* $-1/(n+1)$ *and* $n/(n+1)$. *Each* d_{ij} *has the degree* $2/(n+1)$ *and the weight equal to* $(n-1)/(n+1)$, *each* f_i *has the degree* $2n/(n+1)$ *and the weight equal to* $n(n-1)/(n+1)$. *The discriminant* D_{n+1} *is a* GL*-invariant of the degree* $2n$ *with respect to the system (1) coefficients and with the weight equal to* n^2-n.

Let $X_i = u_i x + v_i y$ be the factor i $(i = 1, 2, \dots, n+1)$ in the factorization (4) and $X_i = 0$ be the equation of the corresponding invariant line.

Let $p = (p_1, p_2, \dots, p_n, p_{n+1})$ and $q = (q_1, q_2, \dots, q_n, q_{n+1})$ be two symbolic $(n+1)$ - tuples of letters. Let us consider the symbolic differential operator

$$\Omega^1_{pq} \;=\; p_1\frac{\partial}{\partial q_1} + p_2\frac{\partial}{\partial q_2} + \ldots + p_n\frac{\partial}{\partial q_n} + p_{n+1}\frac{\partial}{\partial q_{n+1}}, \tag{8}$$

its powers $\Omega^m = \Omega^{m-1}(\Omega^1)$ for every positive integer $m \geq 2$ and $(n+1)$- tuples

$$\begin{aligned} u &= (u_1, u_2, \dots, u_n, u_{n+1}), \qquad & v &= (v_1, v_2, \dots, v_n, v_{n+1}),\\ f &= (f_1, f_2, \dots, f_n, f_{n+1}), \qquad & g &= (g_1, g_2, \dots, g_n, g_{n+1}). \end{aligned} \tag{9}$$

By using the differential operator (8) for $(n+1)$- tuples u and v from (9) by conditions (2) and (4) we obtain the following expressions for the system (1) coefficients:

$$\begin{gathered} b_{n,0} = -u_1 u_2 \dots u_n u_{n+1}, \quad a_{0,n} = v_1 v_2 \dots v_n v_{n+1},\\ C_n^k\, a_{n-k,k} = C_n^{k+1}\, b_{n-k-1,k+1} + \frac{1}{(k+1)!}\Omega^{k+1}_{vu}(-b_{n,0}), \quad 0 \leq k \leq n-1. \end{gathered} \tag{10}$$

Takes place

Lemma 1. *For every* $i = 1, 2, \dots, n, n+1$ *the relations*

$$\begin{gathered} F_{n+1}(v_i, -u_i) = 0, \quad Q_n(v_i, -u_i) = -u_i g_i, \quad P_n(v_i, -u_i) = v_i g_i,\\ \frac{\partial F_{n+1}}{\partial y}(v_i, -u_i) = v_i f_i, \quad \frac{\partial F_{n+1}}{\partial x}(v_i, -u_i) = u_i f_i,\\ G_{n+1}(v_i, -u_i) = (u_i^2 + v_i^2) g_i, \quad T_{n-1}(v_i, -u_i) = (n+1) g_i - f_i,\\ \mu_n = g_1 g_2 \cdot \ldots \cdot g_n g_{n+1}, \quad 1 - \varphi'(1, k_i) = 1 - \psi'(s_i, 1) = \frac{f_i}{g_i},\\ \frac{1}{n!}\Omega^n_{fg}(g_1 g_2 \cdot \ldots \cdot g_n g_{n+1}) = f_1 f_2 \cdot \ldots \cdot f_n f_{n+1} \end{gathered} \tag{11}$$

hold, where

$$g_i = \sum_{k=1}^{n} (-1)^{k+1} C_n^k\, b_{n-k,k} v_i^{n-k} u_i^{k-1} + \frac{\partial(-b_{n,0})}{\partial u_i}\, v_i^n. \tag{12}$$

Proof. The first two equalities from (11) are evident. From the identity

$$F_{n+1}(v_i,-u_i) = -u_iP_n(v_i,-u_i) - v_iQ_n(v_i,-u_i) = 0$$

we obtain $P_n(v_i,-u_i) = v_ig_i$. From the relations

$$\frac{\partial F_{n+1}(x,y)}{\partial x} = \sum_{i=1}^{n+1} u_i\frac{\partial F_{n+1}}{\partial X_i}, \quad \frac{\partial F_{n+1}(x,y)}{\partial y} = \sum_{i+1}^{n+1} v_i\frac{\partial F_{n+1}}{\partial X_i}$$

and (6) we obtain the identities

$$\frac{\partial F_{n+1}}{\partial y}(v_i,-u_i) = v_if_i, \quad \frac{\partial F_{n+1}}{\partial x}(v_i,-u_i) = u_if_i.$$

The relation for polynomial $G_{n+1}(x,y)$ results from the second and third equalities from (11). For polynomial $T_{n-1}(x,y)$ the following representation

$$(x^2+y^2)T_{n-1}(x,y) = (n+1)G_{n+1} - x\frac{\partial F_{n+1}}{\partial y} + y\frac{\partial F_{n+1}}{\partial x}$$

holds. From the last identity for $x = v_i$, $y = -u_i$ we obtain the required relation $T_{n-1}(v_i,-u_i) = (n+1)g_i - f_i$.

From Remark 1, the obtained relations (11) and $u_i^2+v_i^2 \neq 0$ it follows that each equality $g_i = 0$ implies the relation $\mu_n = 0$. From Remark 4, conditions (10) and (12) it results that each addendum from g_i has the weight and the degree equal, respectively, to $n(n-1)/(n+1)$ and $2n/(n+1)$. So, the product $g_1g_2\cdot\ldots\cdot g_ng_{n+1}$ has also the degree $2n$ with respect to the coefficients of the polynomials P_n and Q_n and the weight equal to n^2-n. Thus, $\mu_n = g_1g_2\cdot\ldots\cdot g_ng_{n+1}$.

Let $D_{n+1}\neq 0$. Because $\deg(F_{n+1}) = \deg(T_{n-1})+2$, then for $v_i\neq 0$ or $u_i\neq 0$ we obtain, respectively, the equalities:

$$\sum_{i=1}^{n+1}\frac{T_{n-1}(1,k_i)}{(F_{n+1})'_k(1,k_i)} = 0, \qquad \sum_{i=1}^{n+1}\frac{T_{n-1}(s_i,1)}{(F_{n+1})'_s(s_i,1)} = 0.$$

We have

$$\frac{T_{n-1}(1,k_i)}{(F_{n+1})'_k(1,k_i)} = \frac{T_{n-1}(1,k_i)}{v_i(F_{n+1})'_{X_i}(1,k_i)} = \frac{T_{n-1}(1,-u_i/v_i)}{v_i(F_{n+1})'_{X_i}(1,-u_i/v_i)} = \frac{T_{n-1}(v_i,-u_i)}{f_i} = \frac{(n+1)g_i-f_i}{f_i}. \tag{13}$$

Finally we obtain

$$\sum_{i=1}^{n+1}\frac{(n+1)g_i-f_i}{f_i} = 0 \quad\Leftrightarrow\quad \sum_{i=1}^{n+1}\frac{g_i}{f_i} = 1.$$

The last equality gives us the last relation from (11). If $D_{n+1} = 0$, then for some i and j $(i\neq j)$ we have $f_i = f_j = 0$ and the required equality is trivial.

From the obtained relations it follows that if $\mu_n \neq 0$, then

$$P_n^2(v_i,-u_i)+Q_n^2(v_i,-u_i)=(u_i^2+v_i^2)g_i^2\neq 0.$$

For derivatives of the defined slope functions we obtain:

If $v_i \neq 0$, then $F_{n+1}(1,k_i)=0$ and $P_n(1,k_i)\neq 0$. We calculate the derivative of the function $k-\varphi(1,k)$ and determine the value of this derivative for $k=k_i$:

$$1-\varphi'(1,k)=\left[k-\frac{Q_n(1,k)}{P_n(1,k)}\right]'=\left[\frac{kP_n(1,k)-Q_n(1,k)}{P_n(1,k)}\right]'=$$

$$\left[\frac{F_{n+1}(1,k)}{P_n(1,k)}\right]'=\frac{F'_{n+1}(1,k)P_n(1,k)-F_{n+1}(1,k)P'_n(1,k)}{P_n^2(1,k)},$$

$$1-\varphi'(1,k_i)=\frac{F'_{n+1}(1,k_i)}{P_n(1,k_i)}=\left(\frac{\partial F_{n+1}}{\partial y}(v_i,-u_i)/P_n(v_i,-u_i)\right)=\frac{v_if_i}{v_ig_i}=\frac{f_i}{g_i}.$$

If $u_i \neq 0$, then $F_{n+1}(s_i,1)=0$ and $Q_n(s_i,1)\neq 0$. We calculate the derivative of the function $s-\psi(s,1)$ and determine the value of this derivative for $s=s_i$:

$$1-\psi'(s,1)=\left[s-\frac{P_n(s,1)}{Q_n(s,1)}\right]'=\left[\frac{sQ_n(s,1)-P_n(s,1)}{Q_n(s,1)}\right]'=$$

$$-\left[\frac{F_{n+1}(s,1)}{Q_n(s,1)}\right]'=-\frac{F'_{n+1}(s,1)Q_n(s,1)-F_{n+1}(s,1)Q'_n(s,1)}{Q_n^2(s,1)},$$

$$1-\psi'(s_i,1)=-\frac{F'_{n+1}(s_i,1)}{Q_n(s_i,1)}=-\left(\frac{\partial F_{n+1}}{\partial x}(v_i,-u_i)/Q_n(v_i,-u_i)\right)=\frac{u_if_i}{u_ig_i}=\frac{f_i}{g_i}.$$

So, it follows that the values of derivatives of the functions $k-\varphi(1,k)$ and $s-\psi(s,1)$ for the invariant line $X_i=0$ are the same. Lemma 1 is proved.

Remark 5. *Each f_i and g_i have the same weight and the degree equal, respectively, to $n(n-1)/(n+1)$ and $2n/(n+1)$.*

From Lemma 1 and (10) we obtain the equality

$$\frac{\partial F_{n+1}}{\partial x}(v_i,-u_i)=-(n+1)b_{n,0}v_i^n+\sum_{k=1}^{n}\frac{(-1)^k}{k!}\,(n+1-k)\Omega_{vu}^k(-b_{n,0})v_i^{n-k}u^k. \quad (14)$$

2 Construction of algebraic equations with invariant coefficients

Methods of studying the behavior of the integral curves of the system (1) have been developed by many authors (see [1, 2, 5–9, 11, 14, 21, 23–26, 30, 31, 36]). Using Forster's method (in polar coordinates), Shilov's geometrical method or local charts method (traditional method) the systems (1) with $n=1,2,3$ were investigated (see [10,12,19,22,26,35,40,41,44,45]). A classification of the system (1) with $n=2$ by means of non-associative algebras was given in [16]. The algebraic and topological classifications of the system (1) with $n=2$ by means of quadratic transformations and invariants were established in [32] and [37]. The Poincaré index method for topological classification of system (1) was applied (see [17, 18]).

The GL-comitants of the system (1) with $n = 1,2,3$ and the polynomial basis of these comitants have been used for algebraic, topological and geometrical classifications (see [20, 28, 29, 33, 34, 38, 39, 43, 46]).

The problem under consideration is an important step in the qualitative investigation of behavior of integral curves: at infinity for planar polynomial differential systems with maximal degree equal to n; near critical point $(0,0)$ for planar polynomial differential systems with minimal degree equal to n. Because of this, much of the research in this area is dedicated to the investigation of the problem, usually in local charts. The simplest (but nontrivial) way of investigation is to find the algebraic classification of binary form $F_{n+1}(x,y)$ in coefficients terms (or invariant terms) and to use the results for classification of the system (1) (see [38],[44]).

Our first goal is to show that it is possible to express the conditions which delimit classes with different distributions of infinite singular points through affine invariants and comitants without knowing the basis of the affine invariants and comitants of the system (1). The second goal is to construct such invariants and comitants and to determine the geometrical significance of these objects.

In this work we develope the method of construction and show that the necessary and sufficient conditions for the center existence can be expressed through the coefficients of the residual equation. The contribution idea is due to P.Curtz paper's (see [42]) and Hilbert's symbolic operators (see [47]).

We verify our results by using Shilov's, Forster's and local charts methods for the system (1) with $n = 1,2,3$. The constructed invariants determine the values and the signs of the solutions and solve the problems of algebraical, topological and geometrical classifications of given systems.

For every $i = 1,2,\ldots,n+1$ we denote

$$\xi_i = \frac{f_i}{g_i} = 1 - \varphi'(1,k_i) = 1 - \psi'(s_i,1) \tag{15}$$

such that every ξ_i is a root of the algebraic equation

$$(g_1\xi - f_1)(g_2\xi - f_2)\cdot\ldots\cdot(g_{n+1}\xi - f_{n+1}) = 0.$$

By using the differential operator (8) for $(n+1)$-tuples f and g from (9) the last equation can be written in the form

$$t_0\,\xi^{n+1} - t_1\,\xi^n + t_2\,\xi^{n-1} - \ldots + (-1)^n\,t_n\,\xi + (-1)^{n+1}\,t_n = 0, \tag{16}$$

where

$$t_0 = \mu_n = g_1g_2\cdot\ldots\cdot g_{n+1}, \quad t_i = \frac{1}{i!}\,\Omega^i_{fg}\,(\mu_n) \text{ for } (\forall)\, i = 1,2,\ldots,n,$$
$$t_n = t_{n+1} = (-1)^{n(n+1)/2}\,D_{n+1} = f_1f_2\cdot\ldots\cdot f_{n+1}. \tag{17}$$

The equation (16) will be called the main equation of the system (1).

Remark 6. *For given solution ξ_i of the main equation the equations with solutions*

$$\frac{1}{\xi_i}, \quad 1-\xi_i, \quad \frac{\xi_i}{\xi_i-1}, \quad \frac{\xi_i-1}{\xi_i}, \quad \frac{1}{1-\xi_i} \tag{18}$$

can be constructed.

For example, if we put in (16) $\xi = 1-\eta$, then obtain the equation with solutions $\eta_i = \varphi'(1,k_i) = \psi'(s_i,1)$:

$$m_0\,\eta^{n+1} - m_1\,\eta^n + m_2\,\eta^{n-1} - \ldots + (-1)^n\,m_n\,\eta + (-1)^{n+1}\,m_{n+1} = 0 \tag{19}$$

such that for every $i=1,2,\ldots,n$ we have

$$m_0 = t_0 = \mu_n, \quad m_i = \sum_{r=0}^{i} (-1)^r\,C_{n+1-r}^{n+1-i}\,t_r, \quad m_{n+1} = \sum_{r=0}^{n-1} (-1)^r\,t_r. \tag{20}$$

Let us consider the following differential operator

$$\Theta^1 = \Omega^1_{uu} + \Omega^1_{vv}, \tag{21}$$

where u and v are from (9). It is very easy to verify that $\Theta^1(F_{n+1}) = (n+1)F_{n+1}$. So, the differential operator (21) does not change the invariant straight lines of the system (1).

From condition (2) and Euler's formulae we have two representations for the comitant $F_{n+1}(x,y)$:

$$\begin{gathered} F_{n+1}(x,y) = yP_n(x,y) - xQ_n(x,y), \\ (n+1)F_{n+1}(x,y) = y\frac{\partial F_{n+1}(x,y)}{\partial y} + x\frac{\partial F_{n+1}(x,y)}{\partial x}. \end{gathered} \tag{22}$$

It results from (22) that the differential operator (21) satisfies the relations

$$\Theta^1(P_n(x,y)) = \frac{\partial F_{n+1}(x,y)}{\partial y}, \qquad \Theta^1(Q_n(x,y)) = -\frac{\partial F_{n+1}(x,y)}{\partial x}.$$

From the last equalities we obtain the following coefficients relations:

$$\begin{gathered} \Theta^1(C_n^k a_{n-k,k}) = (k+1)(C_n^k a_{n-k,k} - C_n^{k+1} b_{n-k-1,k+1}), \\ k = 0,1,2,\ldots,n-2,n-1,\ \Theta^1(a_{0,n}) = (n+1)a_{0,n}, \\ \Theta^1(C_n^k b_{n-k,k}) = (n+1-k)(C_n^k b_{n-k,k} - C_n^{k-1} a_{n+1-k,k-1}), \\ k = 1,2,\ldots,n-1,n,\ \Theta^1(b_{n,0}) = (n+1)b_{n+1}. \end{gathered}$$

The equalities $(k+1)C_n^{k+1} = (n-k)C_n^k$ and $(n+1-k)C_n^{k-1} = kC_n^k$ imply the following rules of derivation for system's (1) coefficients:

$$\Theta^1(a_{n-k,k}) = (k+1)a_{n-k,k} - (n-k)b_{n-k-1,k+1},$$

$$k=0,1,2,\dots,n-2,n-1,\quad \Theta^1(a_{0,n})=(n+1)a_{0,n},$$
$$\Theta^1(b_{n-k,k})=(n+1-k)b_{n-k,k}-ka_{n+1-k,k-1},$$
$$k=1,2,\dots,n-1,n,\quad \Theta^1(b_{n,0})=(n+1)b_{n+1}.$$

Finally we obtain the expression of the differential operator (21) in system's (1) coefficients:

$$\Theta^1=\sum_{k=0}^{n-1}[(k+1)a_{n-k,k}-(n-k)b_{n-k-1,k+1}]\frac{\partial}{\partial a_{n-k,k}}+(n+1)a_{0,n}\frac{\partial}{\partial a_{0,n}}+$$
$$(n+1)b_{n,0}\frac{\partial}{\partial b_{n,0}}+\sum_{k=1}^{n}[(n+1-k)b_{n-k,k}-ka_{n+1-k,k-1}]\frac{\partial}{\partial b_{n-k,k}}. \tag{23}$$

Takes place

Theorem 1. *The coefficients* t_k $(k=0,1,2,\dots,n)$ *of the equation (16) are GL-invariants of the degree* $2n$ *with respect to the system (1) coefficients and with the weight equal to* n^2-n *such that*

$$t_0=\mu_n,\quad kt_k=\Theta^1(t_{k-1})-(n+1)(n+k-2)t_{k-1}. \tag{24}$$

Proof. From Remark 5 and (17) it follows that each coefficient t_k, $k=0,1,2,\dots,n$, is a homogeneous and isobaric polynomial of variables f_i and g_i (which are called irrational invariants). According to the results of invariant theory (see [3],[4]) every isobaric and homogeneous polynomial of the invariants f_i and g_i will be an invariant of the binary form $F_{n+1}(x,y)$. Because $F_{n+1}(x,y)$ is a comitant of the system (1) it results that each coefficient t_k is a GL-invariant of the system (1).

We express the operator (21) in the terms of f_i and g_i. Because $\Theta^1(u_i)=u_i$ and $\Theta^1(v_i)=v_i$ we easily obtain that $\Theta^1(d_{i,j})=2d_{i,j}$ and $\Theta^1(f_i)=2nf_i$. Now we shall prove that $\Theta^1(g_i)=(n-1)g_i+f_i$.

Let $u_i\neq 0$. From conditions (10), (12) and (14) we obtain

$$\Theta(g_i)=\sum_{k=1}^{n}(-1)^{k+1}\Theta(C_n^k\, b_{n-k,k}v_i^{n-k}u_i^{k-1})+\Theta(u_1\dots u_{i-1}u_{i+1}\dots u_{n+1}v_i^n)=$$

$$\sum_{k=1}^{n}(-1)^{k+1}[(n+1-k)(C_n^kb_{n-k,k}-C_n^{k-1}a_{n+1-k,k-1})v_i^{n-k}u_i^{k-1}+$$

$$+(n-1)\sum_{k=1}^{n}(-1)^{k+1}C_n^k\, b_{n-k,k}v_i^{n-k}u_i^{k-1}+2nu_1\dots u_{i-1}u_{i+1}\dots u_{n+1}v_i^n=$$

$$(n-1)g_i+(n+1)u_1\dots u_{i-1}u_{i+1}\dots u_{n+1}v_i^n+\sum_{k=1}^{n}\frac{(-1)^k}{k!}\,(n+1-k)\Omega_{vu}^k(b_{n,0})v_i^{n-k}u^k=$$

$$(n-1)g_i + \frac{1}{u_i} \cdot \frac{\partial F_{n+1}}{\partial x}(v_i, -u_i) = (n-1)g_i + f_i.$$

If $u_i = 0$, then $v_i \neq 0$ and from (12) $g_i = nb_{n-1,1}v_i^{n-1} + u_1 \dots u_{i-1}u_{i+1} \dots u_{n+1}v_i^n$. From (6) it results that $d_{i,j} = -v_i u_j$, $f_i = u_1 \dots u_{i-1}u_{i+1} \dots u_{n+1}v_i^n$ and $\Omega^1_{vu}(u_1 u_2 \dots u_{n+1}) = u_1 \dots u_{i-1}u_{i+1} \dots u_{n+1}v_i$. So, for $\Theta^1(g_i)$ we obtain

$$\Theta^1(g_i) = n(C_n^1 b_{n-1,1} - a_{n,0})v_i^{n-1} + (n-1)C_n^1 b_{n-1,1}v_i^{n-1} + 2nu_1 \dots u_{i-1}u_{i+1} \dots u_{n+1}v_i^n =$$

$$(n-1)g_i - nu_1 \dots u_{i-1}u_{i+1} \dots u_{n+1}v_i^n + (n+1)u_1 \dots u_{i-1}u_{i+1} \dots u_{n+1}v_i^n = (n-1)g_i + f_i.$$

So, the formula for $\Theta^1(g_i)$ is proved. Thus, the operator (21) can be written

$$\Theta^1 = \sum_{i=1}^{n+1} \Big\{ [(n-1)g_i + f_i] \frac{\partial}{\partial g_i} + 2nf_i \frac{\partial}{\partial f_i} \Big\}. \tag{25}$$

We show the recurrence (24) by induction. Let $t_0 = \mu_n = g_1 g_2 \cdot \ldots \cdot g_n g_{n+1}$. By using the operator (25) we have

$$\Theta^1(t_0) = [(n-1)g_1 + f_1]g_2 \cdot \ldots \cdot g_k \cdot \ldots \cdot g_{n+1} + g_1[(n-1)g_2 + f_2]g_3 \cdot \ldots \cdot g_k \cdot \ldots \cdot g_{n+1} +$$
$$\ldots + g_1 g_2 \cdot \ldots \cdot g_{k-1}[(n-1)g_k + f_k]g_{k+1} \cdot \ldots \cdot g_{n+1} +$$
$$g_1 g_2 \cdot \ldots \cdot g_k \cdot \ldots \cdot g_n[(n-1)g_{n+1} + f_{n+1}] = (n-1)(n+1)t_0 + t_1.$$

So, $t_1 = \Theta^1(t_0) - (n-1)(n+1)t_0$ and the recurrence (24) is true for $k = 1$. Now we suppose that the recurrence (24) is true for every positive integer $k = 1, 2, \dots, m$. We shall prove the relation

$$(m+1)t_{m+1} = \Theta^1(t_m) - (n+1)(n+m-1)t_m. \tag{26}$$

Every term of t_m is the product of m different factors from f and $n+1-m$ different factors from g such that the indexes of all factors of this term form a permutation of $\{1, 2, \dots, n+1\}$, for example $P = f_1 f_2 \cdot \ldots \cdot f_m g_{m+1} g_{m+2} \cdot \ldots \cdot g_{n+1}$. The action of the operator Θ^1 on the selected term generates $2nm + (n-1)(n+1-m) = (n+1)(n+m-1)$ terms equal with P and $n-m$ different terms from t_{m+1}. So, among all the generated terms of t_m there exist exactly $m+1$ equal terms from t_{m+1}. We obtain the equality (26). By the mathematical induction the recurrence (24) is true for all $k = 1, 2, 3, \dots, n$. Theorem 1 is proved.

Proposition 2. *If* $D_{n+1} \neq 0$, *then the values* $\theta_i = g_i/f_i$ *are the roots of the equation*

$$t_n \theta^{n+1} - t_n\, \theta^n + t_{n-1}\, \theta^{n-1} - \ldots + (-1)^n\, t_1\, \theta + (-1)^{n+1}\, t_0 = 0. \tag{27}$$

From Viette relations for the equation (27) and Lemma 1 results

Proposition 3. *If* $D_{n+1} \neq 0$, *then the roots* θ_i *of the equation (27) satisfy the equality*

$$\sum_{k=1}^{n+1} \theta_k = 1. \tag{28}$$

Lemma 2. *If* $D_{n+1} \neq 0$, *then the system (1) has the first integral of the form*

$$(u_1x + v_1y)^{\theta_1}(u_2x + v_2y)^{\theta_2} \cdot \ldots \cdot (u_nx + v_ny)^{\theta_n}(u_{n+1}x + v_{n+1}y)^{\theta_{n+1}} = c, \qquad (29)$$

where θ_i $(i = 1, 2, \ldots, n+1)$ *are the roots of the equation (27).*

Proof. Let $D_{n+1} \neq 0$. After substitution $y = xz$ the corresponding to the system (1) differential equation has the form

$$-\frac{dx}{x} = \frac{P_n(1,z)dz}{F_{n+1}(1,z)}.$$

For polynomial $P_n(x,y)$ Lagrange's interpolation formulae is applicable :

$$P_n(x,y) = P_n(v_1,-u_1)\frac{(u_2x+v_2y)(u_3x+v_3y)\ldots(u_{n+1}x+v_{n+1}y)}{(-d_{12})(-d_{13})\ldots(-d_{1,n+1})}$$

$$+P_n(v_2,-u_2)\frac{(u_1x+v_1y)(u_3x+v_3y)\ldots(u_{n+1}x+v_{n+1}y)}{(+d_{12})(-d_{23})\ldots(-d_{2,n+1})}+$$

$$\ldots + P_n(v_n,-u_n)\frac{(u_1x+v_1y)\ldots(u_{n-1}x+v_{n-1}y)(u_{n+1}x+v_{n+1}y)}{(d_{1,n})(d_{2,n})\ldots(d_{n-1,n})(-d_{n,n+1})}$$

$$+P_n(v_{n+1},-u_{n+1})\frac{(u_1x+v_1y)\ldots(u_{n-1}x+v_{n-1}y)(u_nx+v_ny)}{(d_{1,n+1})(d_{2,n+1})\ldots(d_{n-1,n+1})(d_{n,n+1})}.$$

From the last relation the polynomial $P_n(1,z)$ has the following representation

$$P_n(1,z) = \frac{g_1}{f_1}\frac{\partial F_{n+1}}{\partial X_1} + \frac{g_2}{f_2}\frac{\partial F_{n+1}}{\partial X_2} + \ldots + \frac{g_n}{f_n}\frac{\partial F_{n+1}}{\partial X_n} + \frac{g_{n+1}}{f_{n+1}}\frac{\partial F_{n+1}}{\partial X_{n+1}}.$$

Using the equality $\partial X_i(1,z)/\partial z = v_i$ and the factorization (4) of the polynomial $F_{n+1}(1,z)$ we obtain the following differential equation

$$-\frac{dx}{x} = \Big[\frac{g_1}{f_1}\frac{v_1}{u_1+v_1z} + \frac{g_2}{f_2}\frac{v_2}{u_2+v_2z} + \ldots + \frac{g_n}{f_n}\frac{v_n}{u_n+v_nz} + \frac{g_{n+1}}{f_{n+1}}\frac{v_{n+1}}{u_{n+1}+v_{n+1}z}\Big]dz.$$

After integration by using Proposition 3 we obtain the first integral (29). Lemma 2 is proved.

3 The center problem for the system (1)

Let $n = 2m+1$, $m \in \mathbb{N}$ and suppose that $u_i \in \mathbb{C} \setminus \mathbb{R}$ or $v_i \in \mathbb{C} \setminus \mathbb{R}$ for every $i = 1, 2, \ldots, 2m+1, 2m+2$. From [6] the singular point $(0,0)$ of the system (1) is a center if and only if the following condition

$$\int_0^{2\pi} \frac{G_{2m+2}(\cos\alpha,\sin\alpha)}{F_{2m+2}(\cos\alpha,\sin\alpha)}d\alpha = 0 \iff \int_0^{2\pi} \frac{T_{2m}(\cos\alpha,\sin\alpha)}{F_{2m+2}(\cos\alpha,\sin\alpha)}\,d\alpha = 0 \qquad (30)$$

holds. For each invariant line $X_i = 0$ determined by the equation $F_{2m+2}(x, y) = 0$ we denote by r_i the residue of the rational function $T_{2m}(x, y)/F_{2m+2}(x, y)$:

$$r_i \;=\; \mathrm{res}_{X_i=0}\, \frac{T_{2m}(x, y)}{F_{2m+2}(x, y)}.$$

The following lemma holds:

Lemma 3. *If the homogeneous equation* $F_{2m+2}(x, y) = 0$ *has no nontrivial real solutions and the discriminant* $D_{2m+2} \neq 0$, *then for every* $i = 1, 2, \ldots, 2m+2$ *the relation*

$$r_i = \frac{(2m+2)g_i}{f_i} - 1 = (2m+2)\theta_i - 1 \tag{31}$$

holds.

Proof. We will obtain the value of the residue r_i, corresponding to the invariant line $X_i = 0$, by using Lemma 1. Let us consider the following 2 cases:

1. Let $v_i \neq 0$. The substitution $z = \tan\alpha$ in the last integral from (30) implies the relation

$$\int_{-\infty}^{+\infty} \frac{T_{2m}(1, k)}{F_{2m+2}(1, k)} dk = 0.$$

For each root $k_i = -u_i/v_i$ of the equation $F_{2m+2}(1, k) = 0$ the residue of the rational function $T_{2m}(1, k)/F_{2m+2}(1, k)$ is equal to

$$r_i = \frac{T_{2m}(1, k_i)}{(F_{2m+2})'_k(1, k_i)} = \frac{T_{2m}(1, k_i)}{v_i(F_{2m+2})'_{X_i}(1, k_i)} = \frac{T_{2m}(1, -u_i/v_i)}{v_i(F_{2m+2})'_{X_i}(1, -u_i/v_i)} =$$

$$\frac{T_{2m}(v_i, -u_i)}{f_i} = \frac{(2m+2)g_i - f_i}{f_i} = (2m+2)\theta_i - 1.$$

2. Let $u_i \neq 0$. The substitution $z = \cot\alpha$ in the last integral from (30) implies the relation

$$\int_{-\infty}^{+\infty} \frac{T_{2m}(s, 1)}{F_{2m+2}(s, 1)} ds = 0.$$

For each root $s_i = -v_i/u_i$ of the equation $F_{2m+2}(s, 1) = 0$ the residue of the rational function $T_{2m}(s, 1)/F_{2m+2}(s, 1)$ is equal to

$$r_i = \frac{T_{2m}(s_i, 1)}{(F_{2m+2})'_s(s_i, 1)} = \frac{T_{2m}(s_i, 1)}{u_i(F_{2m+2})'_{X_i}(s_i, 1)} = \frac{T_{2m}(-v_i/u_i, 1)}{u_i(F_{2m+2})'_{X_i}(-v_i/u_i, 1)} =$$

$$\frac{T_{2m}(v_i, -u_i)}{f_i} = \frac{(2m+2)g_i - f_i}{f_i} = (2m+2)\theta_i - 1.$$

Lemma 3 is proved.

If we put $\theta = (r+1)/(n+1)$ in equation (27) then we obtain an equation of degree $n+1$, called the residual equation.

Proposition 4. *If* $D_{n+1} \neq 0$, *then the values* $r_i = (n+1)\theta_i - 1$ *are the roots of the equation*

$$X(r) = c_0\, r^{n+1} + c_2\, r^{n-1} + \ldots + c_n\, r + c_{n+1} = 0, \qquad (32)$$

where

$$c_k = \sum_{m=0}^{k} (-1)^m\, (n+1)^m\, C_{n+1-m}^{k-m}\, t_{n+1-m}, \quad (\forall)\ k = 0, 1, \ldots, n, n+1. \qquad (33)$$

Remark 7. *The equalities* $t_{n+1} = t_n = (-1)^{n(n+1)/2} D_{n+1}$ *imply the equality* $c_1 = 0$.

The discriminant of the equation (32) has the form

$$R_{n+1} = \text{Res}\,(X(r), X'(r)) = D_{n+1}^{2n} \Delta^2,$$

where

$$\Delta^2 = \prod_{1 \le i < j \le n+1} (r_j - r_i)^2$$

is a GL-invariant of the system (1).

Let us consider that the equation (32) has no real solutions and let $r_{i_1}, r_{i_2}, \ldots, r_{i_{m+1}}$ be the solutions with positive coefficients of the imaginary part. In this case it is known that

$$\int_{-\infty}^{+\infty} \frac{T_{2m}(1,k)}{F_{2m+2}(1,k)} dk = 2\,\pi\, i\, (r_{i_1} + r_{i_2} + \ldots + r_{i_{m+1}}).$$

We construct the polynomial of minimal degree $W(r_1, r_2, \ldots, r_{2m+2})$ such that it is simmetric with respect to variables r_i and has the form

$$W(r_1, r_2, \ldots, r_{2m+2}) = \prod (r_{i_1} + r_{i_2} + \ldots + r_{i_{m+1}}).$$

According to the theorem of the symmetric polynomials there exists some polynomial Φ such that the polynomial W can be expressed through the elementary symmetric polynomials of the variables r_i:

$$W(r_1, r_2, \ldots, r_{2m+2}) = \Phi(\frac{c_2}{c_0}, \frac{c_3}{c_0}, \ldots, \frac{c_n}{c_0}, \frac{c_{n+1}}{c_0}).$$

So, there exists positive integer l such that $V = c_0^l \Phi(\frac{c_2}{c_0}, \frac{c_3}{c_0}, \ldots, \frac{c_n}{c_0}, \frac{c_{n+1}}{c_0})$ is a polynomial of the variables $c_0, c_2, c_3, \ldots, c_{n+1}$.

Takes place

Proposition 5. *The system (1) with imaginary invariant straight lines has a center iff* $V = 0$ *and the residual equation (32) has no real solutions.*

Example 1. *For* $n = 3$ *the system (1) with imaginary invariant straight lines has a center iff at least one of the following two series of conditions is fulfilled:*

(i) $\quad V = c_3 = 0$ *and the inequalities* $c_0 c_2 < 0,\ c_2^2 - 4c_0 c_4 > 0$ *are not fulfilled simultaneously;*

(ii) $\quad V = c_3 = 0,\ c_2^2 - 4c_0 c_4 = 0,\ c_0 c_2 > 0.$

Example 2. *For $n = 5$ the system (1) with imaginary invariant straight lines has a center iff $V = -c_0c_5^2 + 4c_0c_4c_6 - c_3^2c_4 + c_2c_3c_5 = 0$ and the residual equation (32) has no real solutions.*

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Studying stability of the equilibrium solutions in the restricted Newton's problem of four bodies

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Abstract. Newton's restricted problem of four bodies is investigated. It has been shown that there are six equilibrium solutions of the equations of motion. Stability of these solutions is analyzed in linear approximation with computer algebra system *Mathematica*. It has been proved that four radial solutions are unstable while two bisector solutions are stable if the mass of the central body P_0 is large enough. There is also a domain of instability of the bisector solutions near the resonant point in the space of parameters and its boundaries are found in linear approximation.

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1 Introduction

The main problem of the dynamics is to investigate all possible motions of a system. In the case of the system of point particles moving under their mutual gravitational force this problem has been solved only for two particles. Although there are ten integrals for such systems, the general solution of the differential equations of motion in the case of three or more interacting particles can not be obtained. So further progress in this field seems to be connected with seeking and studying particular solutions of the equations of motion. In the case of three particles five particular solutions were found by L.Euler (1767) and J.L.Lagrange (1772) [1]. To study the stability of these solutions it turned out to be necessary to elaborate new qualitative, analytical and numerical methods for studying nonlinear Hamiltonian systems [2]. Nevertheless, the elaboration of the stability theory of Hamiltonian systems has not been completed yet and the investigations in this field are very topical.

In [3–5] it was proved that there is a new class of the exact particular solutions of the planar Newton's many-body problem. On this basis two new dynamical models were proposed that are known as Newton's restricted problems of $(n+2)$ bodies [6, 7]. Now it is necessary to find all equilibrium solutions in these problems and to investigate their stability. As in general case this problem is very complicated let us start with the case of four bodies. But even in this case the calculations are very complicated and can not be done without computer. So we have used here computer algebra system *Mathematica* that is a very powerful tool for doing both analytical and numerical calculations [8].

2 Equilibrium solutions of the equations of motion

Let two point particles P_1 and P_2 of equal masses m move in elliptical orbits about their common center of mass where the third particle P_0 of mass m_0 is resting. The particles attract each other according to Newton's law of gravitation. At any instant of time the particles P_1 and P_2 are symmetrical with respect to the particle P_0 and their orbits are situated in the xOy plane of the barycentric inertial frame of reference. Using cylindrical coordinates we can write a solution of the corresponding three-body problem in the form [5]

$$\rho_j(\nu) = \frac{p}{1 + e\cos\nu}, \;\; \varphi_j(t) = \nu(t) + \pi j, \;\; z_j(t) = 0 \quad (j = 1, 2), \tag{1}$$

where p and e are parameter and eccentricity of the elliptic orbit of the particles. The functions $\rho_j(t)$ and $\nu(t)$ are connected by the relation

$$\rho_j^2 \frac{d\nu}{dt} = \sqrt{fp(m_0 + m/4)} \equiv c, \tag{2}$$

where f is the constant of gravitation.

Let us consider the motion of the fourth particle P_3 of negligible mass m_1 in the gravitational field generated by the particles P_0, P_1 and P_2. Denoting its cylindrical coordinates as ρ, φ, z we can write Lagrangian of the system in the form

$$L = \frac{m_1}{2}(\dot\rho^2 + \rho^2\dot\varphi^2 + \dot z^2) + fm_1(\frac{m_0}{r} + \frac{m}{r_1} + \frac{m}{r_2}), \tag{3}$$

where

$$r = \sqrt{\rho^2 + z^2}, \quad r_j = \sqrt{\rho^2 + \rho_j^2 - 2\rho\rho_j\cos(\varphi - \varphi_j) + z^2} \quad (j = 1, 2)$$

are the distances between the particle P_3 and particles P_0, P_1, P_2, respectively, and the dot denotes the derivative $\frac{d}{dt}$. With Lagrangian (3) the equations of motion of the particle P_3 may be written as

$$\begin{gathered}
\ddot\rho - \rho\dot\varphi^2 + fm_0\frac{\rho}{r^3} + fm\frac{\rho - \rho_1\cos(\varphi - \varphi_1)}{r_1^3} + fm\frac{\rho - \rho_2\cos(\varphi - \varphi_2)}{r_2^3} = 0, \\
\rho\ddot\varphi + 2\dot\rho\dot\varphi + fm\frac{\rho_1\sin(\varphi - \varphi_1)}{r_1^3} + fm\frac{\rho_2\sin(\varphi - \varphi_2)}{r_2^3} = 0, \\
\ddot z + fm_0\frac{z}{r^3} + fm\frac{z}{r_1^3} + fm\frac{z}{r_2^3} = 0.
\end{gathered} \tag{4}$$

Taking into account (1) let us make a substitution in (4) according to the rule

$$\rho_j(t) \to \frac{p}{1 + e\cos\nu}, \qquad \rho(t) \to \frac{p}{1 + e\cos\nu}\,\rho(\nu),$$

$$z(t) \to \frac{p}{1 + e\cos\nu}\,z(\nu), \qquad \varphi(t) \to \nu + \varphi(\nu).$$

It means that we'll consider the motion of the particle in the frame of reference rotating about Oz axis where all distances are pulsating so that the particles P_1 and P_2 are resting on the Ox axis at the points $x = \mp 1$, respectively. Such frame is known as Nechvil's configurational space [1]. Besides, we use the polar angle ν determining the position of the particles P_1 and P_2 on the xOy plane of inertial frame of reference as a new independent variable. Then derivatives of the coordinates ρ and φ are transformed as

$$\frac{d\rho}{dt} \to \frac{c}{p}\ ((1 + e\cos\nu)\frac{d\rho}{d\nu} + e\sin\nu\ \rho),$$

$$\frac{d\varphi}{dt} \to \frac{c}{p^2}\ (1 + e\cos\nu)^2(1 + \frac{d\varphi}{d\nu}),$$

$$\frac{d^2\rho}{dt^2} \to \frac{c^2}{p^3}\ (1 + e\cos\nu)^2((1 + e\cos\nu)\frac{d^2\rho}{d\nu^2} + e\cos\nu\ \rho),$$

$$\frac{d^2\varphi}{dt^2} \to \frac{c^2}{p^4}\ (1 + e\cos\nu)^3((1 + e\cos\nu)\frac{d^2\varphi}{d\nu^2} - 2e\sin\nu\ \frac{d\varphi}{d\nu} - 2e\sin\nu).$$

Derivatives of the coordinate z are obtained from the corresponding derivatives of ρ with the substitution $\rho \to z$. Then equations of motion (4) become

$$\frac{d^2\rho}{d\nu^2} - \rho\left(\frac{d\varphi}{d\nu} + 1\right)^2 + \frac{e\cos\nu}{1 + e\cos\nu}\ \rho =$$

$$= -\frac{4}{(1 + 4\mu)(1 + e\cos\nu)}\left(\frac{\mu\ \rho}{(\rho^2 + z^2)^{3/2}} + \frac{\rho + \cos\varphi}{r_1^3} + \frac{\rho - \cos\varphi}{r_2^3}\right),$$

$$\rho\frac{d^2\varphi}{d\nu^2} + 2\frac{d\rho}{d\nu}\left(\frac{d\varphi}{d\nu} + 1\right) = \frac{4\sin\varphi}{(1 + 4\mu)(1 + e\cos\nu)}\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right), \qquad (5)$$

$$\frac{d^2 z}{d\nu^2} + \frac{e\cos\nu}{1 + e\cos\nu}\ z = -\frac{4z}{(1 + 4\mu)(1 + e\cos\nu)}\left(\frac{\mu}{(\rho^2 + z^2)^{3/2}} + \frac{1}{r_1^3} + \frac{1}{r_2^3}\right),$$

where

$$r_1 = \sqrt{\rho^2 + 1 + 2\rho\ \cos\varphi + z^2},\ r_2 = \sqrt{\rho^2 + 1 - 2\rho\ \cos\varphi + z^2}$$

and $\mu = \frac{m_0}{m}$.

The equilibrium solutions of the system (5) are determined from the condition that all derivatives are equal to zero. The only such solution satisfying the third equation of (5) is $z = 0$. In this case the second equation of (5) may be written as

$$\sin\varphi\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right) = 0.$$

It has four solutions

$$\varphi = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}.$$

For $\varphi = 0,\ \pi$ the equilibrium positions of the particle P_3 are on the straight line P_1P_2. In terms of [7] such solutions are called the radial equilibrium solutions. For $\varphi = \frac{\pi}{2},\ \frac{3\pi}{2}$ the equilibrium positions are on the straight line being perpendicular to the line P_1P_2 and are called the bisector equilibrium solutions. Substituting solutions $\varphi = 0,\ \pi$, $z = 0$ and $\rho = R = const$ into the first equation of (5) we obtain an equation determining the radial equilibrium positions

$$\mu\left(R - \frac{1}{R^2}\right) + \frac{R}{4} - \left(\frac{R-1}{|R-1|^3} + \frac{R+1}{|R+1|^3}\right) = 0. \tag{6}$$

The corresponding equation determining bisector equilibrium positions is

$$\mu\left(R - \frac{1}{R^2}\right) + \frac{R}{4} - \frac{2R}{(R^2+1)^{3/2}} = 0. \tag{7}$$

If $m_0 = 0$ then equations (6), (7) coincide with the corresponding equations determining positions of the points of libration in the restricted problem of three bodies [1, 2]. In this case equation (6) has two solutions. One solution is $R = 0$ and another one is such a root of the equation

$$R = \frac{8(R^2+1)}{(R^2-1)^2}$$

that satisfies the condition $R > 1$. Equation (7) has also two solutions: $R = 0$ and $R = \sqrt{3}$. The second solution determines two equilibrium positions of the particle P_3 being symmetrical with respect to the origin that correspond to the famous Lagrange's triangular solutions. It is known that all solutions above are unstable in the sense of Liapunov [2]. So further on we'll consider the case $m_0 \neq 0$. Analyzing equation (6) one can conclude that in the domain $0 \leq R < 1$ it can be rewritten as

$$\mu\left(R - \frac{1}{R^2}\right) + \frac{R}{4} + \frac{4R}{(1-R^2)^2} = 0 \tag{8}$$

and has only one root. For $R > 1$ equation (6) has the form

$$\mu\left(R - \frac{1}{R^2}\right) + \frac{R}{4} - \frac{2(R^2+1)}{(R^2-1)^2} = 0. \tag{9}$$

This equation also has one root. It should be noticed that the roots of equations (8), (9) tend to a limit $R = 1$ as $\mu \to \infty$. Equation (7) has only one root and its value decreases from $R = \sqrt{3}$ to $R = 1$ as parameter μ tends to infinity. Thus, in the case $m_0 \neq 0$ there are six equilibrium solutions of the restricted problem of four bodies in Nechvil's configurational space. Four of them are the radial equilibrium solutions $\varphi = 0,\ \pi$, $z = 0$ and the corresponding values of R are given as roots of equations (8), (9). The last two solutions form a couple of the bisector equilibrium solutions $\varphi = \frac{\pi}{2},\ \frac{3\pi}{2}$, $z = 0$ and R is given as a root of equation (7).

3 Studying stability of the equilibrium solutions

The stability problem of the equilibrium solutions found above is connected with the investigation of nonlinear differential equations of the disturbed motion. Usually, the first step in solving this problem is an analysis of the corresponding linearized system. In order to investigate equations of motion (5) in the vicinity of the equilibrium solutions let us make in (5) the substitution

$$\rho(\nu) \to R + u(\nu), \ \varphi(\nu) \to \beta + \gamma(\nu).$$

Considering the functions $u(\nu)$, $\gamma(\nu)$, $z(\nu)$ as small perturbations of the equilibrium solutions we can expand equations (5) in Taylor series in powers of u, γ and z and neglect all terms of the second and higher orders. Then we obtain equations linearized in the vicinity of the radial equilibrium solutions in the form

$$\begin{aligned} &\frac{d^2u}{d\nu^2} - 2R\frac{d\gamma}{d\nu} = \frac{3+2a_j}{1+e\cos\nu}u, \\ &\frac{d^2\gamma}{d\nu^2} + \frac{2}{R}\frac{du}{d\nu} = -\frac{a_j}{1+e\cos\nu}\gamma, \\ &\frac{d^2z}{d\nu^2} + \frac{1+a_j+e\cos\nu}{1+e\cos\nu}z = 0, \end{aligned} \tag{10}$$

where

$$a_1 = \frac{8(R^2+3)}{(1+4\mu)(1-R^2)^3}, \quad a_2 = \frac{8(3R^2+1)}{(1+4\mu)(R^2-1)^3}.$$

The corresponding system of equations linearized in the vicinity of the bisector equilibrium solutions is

$$\begin{aligned} &\frac{d^2u}{d\nu^2} - 2R\frac{d\gamma}{d\nu} = \frac{3-b}{1+e\cos\nu}u, \\ &\frac{d^2\gamma}{d\nu^2} + \frac{2}{R}\frac{du}{d\nu} = \frac{b}{1+e\cos\nu}\gamma, \\ &\frac{d^2z}{d\nu^2} + z = 0, \end{aligned} \tag{11}$$

where

$$b = \frac{24}{(1+4\mu)(1+R^2)^{5/2}}.$$

Let us note that parameters R and μ are connected by the relations (7)-(9). So the constants a_1, a_2 and b depend only on the parameter μ.

Thus, we have obtained two systems of three linear differential equations of the second order with periodic coefficients. It is evident that coefficients of equations (10),(11) are analytic functions of parameter e in the domain $|e| < 1$. Consequently, the behavior of the solutions of these systems is determined by their characteristic exponents calculated for $e = 0$. If the system has at least one characteristic exponent with positive real part for $e = 0$, then it is unstable for sufficiently small $e \neq 0$. If all characteristic exponents of the system are complex numbers with unit magnitude but some of them are multiple, then the system is unstable, too. But if all characteristic exponents of the system are different and pure imaginary numbers, then the instabilities can arise only when the characteristic exponents λ_k satisfy the resonance conditions

$$\lambda_k \pm \lambda_l = iN \quad (k, l = 1,\ 2,\ 3,\ 4;\ N = 0,\ \pm 1,\ \pm 2,\ ...\ ). \tag{12}$$

So we should calculate the characteristic exponents of systems (10), (11) for $e = 0$.

The third equation in systems (10), (11) is independent of the first two. It means that in the linear approximation the disturbed motion of the particle P_3 in xOy plane does not depend on its motion along the $0z$ axis. So we may analyze these motions separately. The third equation of the system (10) is just a Hill's equation. For $e = 0$ it has two pure imaginary characteristic exponents $\pm i\sqrt{1 + a_j}$ because $a_j > 0$ for any μ. It was investigated in detail in [9] where it was shown that there are the domains of instability of this equation in the vicinity of the points $a_j = \frac{(2k-1)^2}{4} - 1$ $(k = 1,\ 2,\ ...)$. Using those results and relationships (8), (9) it is easy to construct the corresponding domains of instability of the third equation of (10) in the μOe plane. Characteristic exponents of the first two equations of system (10) for $e = 0$ are calculated very easy and may be written in the form

$$\lambda_k = \pm \frac{1}{\sqrt{2}} \left(-1 + a_j \pm \sqrt{1 + 10a_j + 9a_j^2} \right)^{1/2} \quad (k = 1,\ 2,\ 3,\ 4). \tag{13}$$

Numerical calculations show that one of the characteristic exponents (13) is a positive real number for any μ from the interval $0 \leq \mu < \infty$ and for both coefficients a_1 and a_2. So, according to Liapunov's theorem on linearized stability [2], we can conclude that radial equilibrium solutions of the restricted problem of four bodies are unstable.

The third equation of system (11) has two pure imaginary characteristic exponents $\pm i$ and is stable for any e. Characteristic exponents of the first two equations of system (11) for $e = 0$ can be written as

$$\lambda_k = \pm \frac{1}{\sqrt{2}} \left(-1 \pm \sqrt{1 - 12b + 4b^2} \right)^{1/2} \quad (k = 1,\ 2,\ 3,\ 4). \tag{14}$$

If the condition

$$0 < 1 - 12b + 4b^2 < 1 \tag{15}$$

is fulfilled, then characteristic exponents (14) are different and pure imaginary numbers $\lambda_k = \pm i\sigma_{1,2}$ where

$$\sigma_{1,2} = \frac{1}{\sqrt{2}} \left(1 \pm \sqrt{1 - 12b + 4b^2} \right)^{1/2}.$$

Numerical analysis shows that $1 - 12b + 4b^2 < 1$ for any μ. For $\mu = 11.7203$ the expression $1 - 12b + 4b^2$ becomes zero and the system (11) has two multiple characteristic exponents $\pm\frac{i}{\sqrt{2}}$. In this case the system is unstable for $e=0$ because its solution has linearly growing terms of the form $\nu\cos\frac{\nu}{\sqrt{2}}$, $\nu\sin\frac{\nu}{\sqrt{2}}$. The inequality (15) is true if $11.7203<\mu<\infty$. In this case

$$\frac{1}{\sqrt{2}}<\sigma_1<1,\quad 0<\sigma_2<\frac{1}{\sqrt{2}}.$$

So the resonance condition (12) can be fulfilled only for $\lambda_k=i/2$, $\lambda_l=-i/2$, $N=1$ when $\sigma_2=1/2$ and $\mu=\mu_R=15.9691$. For $e>0$ in the vicinity of the resonant value μ_R of parameter μ a domain of instability can exist. To find the boundaries of this domain it is necessary to calculate the fundamental matrix of the system (11). And to do this we'll use Liapunov-Poincare method of a small parameter.

The first two equations of system (11) can be written in the form

$$\frac{dx}{d\nu}=P(\nu,e)x, \tag{16}$$

where x is a vector with four components and $P(\nu,e)$ is an 4×4 matrix function that can be represented in the form

$$P(\nu,e)=P_0+\sum_{k=1}^{\infty}P_k(\nu)e^k, \tag{17}$$

and

$$P_0=\begin{pmatrix}0&0&1&0\\-2/R&0&0&1/R^2\\-(1+b)&0&0&2/R^2\\0&bR^2&0&0\end{pmatrix},\quad P_k(\nu)=(-\cos\nu)^k\begin{pmatrix}0&0&0&0\\0&0&0&0\\3-b&0&0&0\\0&bR^2&0&0\end{pmatrix}.$$

The series (17) converges for any ν in the domain $|e|<1$ and $P_k(\nu)$ are continuous finite functions. According to Liapunov theorem [10, 11], the fundamental matrix $X(\nu,e)$ for the system (16) normalized by the condition $X(0,e)=E_4$, where E_4 is an 4×4 identity matrix, may be represented in the form

$$X(\nu,e)=\exp(P_0\nu)Z(\nu,e)\exp(\nu W(e)), \tag{18}$$

where $Z(\nu,e)=Z(\nu+2\pi,e)$ is a periodic analytic matrix function and $W(e)$ is a constant matrix. The matrices $Z(\nu,e)$ and $W(e)$ may be also represented in the form of series in powers of e

$$Z(\nu,e)=\sum_{k=0}^{\infty}Z_k(\nu)e^k,\ Z_0(0)=E_4,\ Z_k(0)=0\ (k\geq1), \tag{19}$$

$$W(e)=\sum_{k=1}^{\infty}W_k\,e^k. \tag{20}$$

The series (19), (20) converge in the domain $|e| < 1$ for any ν and $Z_k(\nu)$ are continuous matrices satisfying the next recurrence relation

$$\frac{dZ_k}{d\nu} = \sum_{l=1}^{k} (\exp(-P_0\nu)P_l(\nu)\exp(P_0\nu)Z_{k-l} - Z_{k-l}W_l)\,. \tag{21}$$

Matrices W_k can be found from the condition that $Z_k(\nu)$ are periodic matrices. Actually, in the first order equation (21) has the form

$$\frac{dZ_1}{d\nu} = \exp(-P_0\nu)P_1(\nu)\exp(P_0\nu) - W_1. \tag{22}$$

Taking into account initial conditions (19) we can write a solution of equation (22) as

$$Z_1(\nu) = -W_1\nu + \int_0^{\nu} \exp(-P_0\tau)P_1(\tau)\exp(P_0\tau)d\tau.$$

Using periodicity of the matrix $Z_1(\nu)$ we obtain

$$W_1 = \frac{1}{2\pi}\int_0^{2\pi} \exp(-P_0\tau)P_1(\tau)\exp(P_0\tau)d\tau.$$

Calculations in the higher orders are done in a similar way. But with the k growth they become more and more cumbersome and can not be done without computer. Here we have calculated the fundamental matrix $X(\nu, e)$ in the vicinity of the resonant point $\mu_R = 15.9691$ with the accuracy $o(e^2)$. Then we can write the characteristic equation for the system (16) as

$$\begin{gathered}
\rho^4 + \rho^3(2 - 2\cos(\sqrt{3}\pi) - \frac{8}{\sqrt{3}}\, e\pi s_1 \sin(\sqrt{3}\pi) + \\
+\rho^2(2 - 4\cos(\sqrt{3}\pi) - \frac{16}{\sqrt{3}}\, e\pi s_1 \sin(\sqrt{3}\pi) + \frac{e^2}{48}(\pi^2(-99 + 1024 s_1^2) - \\
-6\cos^2(\sqrt{3}\pi/2)(297 + 215\cos(\sqrt{3}\pi)))) + \\
+\rho(2 - 2\cos(\sqrt{3}\pi) - \frac{8}{\sqrt{3}} e\pi s_1 \sin(\sqrt{3}\pi) + \frac{e^2}{24}(\pi^2(99\cos(\sqrt{3}\pi) - \\
-256 s_1^2(-1 + 3cos(\sqrt{3}\pi))) - 123\sin^2(\sqrt{3}\pi))) + \\
+1 + \frac{e^2}{96}(2181 + 2\pi^2(-99 + 1024 s_1^2) + 3072\cos(\sqrt{3}\pi) + 891\cos(2\sqrt{3}\pi)) = 0,
\end{gathered} \tag{23}$$

where s_1 is a small parameter determining deviation of σ_2 from its resonant value according to the relation $\sigma_2 = \frac{1}{2} + s_1 e$. Analysis of equation (23) shows that in the vicinity of the resonant value of parameter $\sigma_2 = \frac{1}{2}$ there is a domain in the $\sigma_2 e$ plane where the stability condition $|\rho| \leq 1$ is not fulfilled. This domain is bounded by the straight lines $\sigma_2 = \frac{1}{2} \pm \frac{\sqrt{33}}{16}\, e$. The corresponding domain of instability in the μe plane is bounded by the straight lines

$$\mu = 15.9691 \pm 16.2652\ e. \tag{24}$$

Thus, we can formulate the next theorems.

Theorem 1. *The radial equilibrium solutions of the restricted problem of four bodies are unstable for sufficiently small* e *and any values of parameter* μ*.*

Theorem 2. *The bisector equilibrium solutions of the restricted problem of four bodies are stable in linear approximation for* $e = 0$ *if parameter* μ *satisfies the next inequality:* $11.7203 < \mu < \infty$*. For sufficiently small values of* e *there is a domain of instability in the* μe *plane between the straight lines defined in (24).*

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The centre-focus problem for analytical systems of Lienard form in degenerate case

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Abstract. For analytical systems of Lienard form in the case of zero eigenvalues of its linear part is obtained the algebraic criterion of the centre existence, which is analogous to the Cherkas's criterion for systems with imaginary eigenvalues of linear part. We give the solution of centre-focus problem for one class of cubic systems.

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1 Introduction

For analytical systems of Lienard form in the case of pure imaginary eigenvalues of linear part L.A.Cherkas gives effective necessary and sufficient conditions of algebraic character for the centre existence [1–3]. For example, for the Lienard system

$$\dot{x} = y,\ \dot{y} = -xf(x) + xg(x)y, \tag{1}$$

where f, g are analytical in the neighborhood of $x = 0$ functions, $f(0) = 1$, he received the following result

Theorem 1. [1] *The origin of coordinate system (1) is a centre if and only if the system of equations*

$$F(x) = F(y),\ G(x) = G(y),$$

where $F(x) = \int_0^x t\,f(t)dt,\ G(x) = \int_0^x t\,g(t)dt$, *has an analytical in the neighborhood of* $x = 0$ *solution* $y = \varphi(x),\ \varphi(0) = 0,\ \varphi'(0) = -1$.

For the systems of type (1), where $f(x) = x^{2n} f_1(x),\ f_1(0) = 1$, the theorem analogous to Theorem 1 was proved in [4, 5].

In the present article we consider the system of differential equations

$$dx/dt = y,\ dy/dt = \sum_{i=0}^{3} p_i(x) y^i, \tag{2}$$

where $p_i(x)$ are analytical in the neighborhood of $x = 0$ functions of the form

$$p_0(x) = -x^{2n-1} + \sum_{k=2n}^{\infty} a_k x^k,\quad p_1(x) = Ax^{n-1} + \sum_{k=n}^{\infty} b_k x^k,$$

$$p_j(x) = \sum_{k=0}^{\infty} \alpha_{k,j} x^k,\ j = 2, 3. \tag{3}$$

If $4n - A^2 > 0$, then the critical point $O(0,0)$ of system (2) is either a centre or a focus [4,6]. We know [4,7] that there exists a formal transformation

$$x = u + \sum_{i+j=2}^{\infty} \alpha_{i,j} u^i v^j, \quad y = v + \sum_{i+j=2}^{\infty} \beta_{i,j} u^i v^j, \tag{4}$$
$$dt = (1 + \sum_{i+j=1}^{\infty} \gamma_{i,j} u^i v^j) d\tau$$

which transforms (1) to a formal system

$$du/d\tau = v + \sum_{k=n} A_k u^k,\ dv/d\tau = -u^{2n-1}, \tag{5}$$

where $A_n = A/n$.

Theorem 2. [7] *The critical point $O(0,0)$ of system (2) is a centre if and only if $A_{2i+1} = 0,\ i = [n/2], [n/2]+1, \ldots,$ in (5).*

Definition 1. *The critical point $O(0,0)$ of system (2), where p_i are analytical functions of (3) type with complex coefficients, is called a centre if there is a formal transformation (4) which transforms (2) to system (5), where $A_{2i+1} = 0,\ i = [n/2], [n/2]+1, \ldots$*

In the present paper we will show the algebraic criterion of the existence of the centre of the system (2) and will give the solution of centre-focus problem for the system

$$\dot{x} = y(1 + Dx + Px^2),\ \dot{y} = -x^3 + Axy + By^2 + Kx^2y + Lxy^2 + My^3, \tag{6}$$

where A, B, C, D, K, L, M are complex constants.

The solution of centre-focus problem for the system (6) where $D = P = 0$ is contained in [4, 8]. There are many works in which the centre-focus problem is solved for various classes of cubic systems in the case of imaginary eigenvalues of linear part (e.g.[9]-[23]).

2 The algebraic criterion for the existence of a centre

Theorem 3. *The critical point $O(0,0)$ of system (2) is a centre if and only if the system of equations*

$$F_1(x) = F_1(y),\ F_2(x) = F_2(y) \tag{7}$$

or the system

$$F_1(x) = F_1(y),\ F_3(x) = F_3(y), \tag{8}$$

where $F_1 = Q_2^3/Q_1^5,\ F_2 = Q_3^3/Q_1^7,\ F_3 = Q_4/Q_1^3,$

$$Q_1 = 2p_1^3 - 9p_0p_1p_2 + 27p_0^2p_3 + 9p_1p_0' - 9p_0p_1',$$
$$Q_2 = Q_1R - p_0Q_1', \quad Q_3 = 5Q_2R - 3p_0Q_2',$$
$$Q_4 = 7Q_3R - 3p_0Q_3', \quad R = p_1^2 - 3p_0p_2 + 3p_0',$$

has a solution $y = \varphi(x)$*, where* $\varphi(x)$ *is an analytical in the neighborhood of* $x = 0$ *function such that* $\varphi(0) = 0,\ \varphi'(0) = -1$ *(we do not exclude the case when one or both equations of systems (7), (8) turn to the identity).*

Proof. Necessity. Suppose that the critical point $O(0,0)$ is a centre for system (2). The change

$$y = z/[v(x)(1+z)], \tag{9}$$

where $v(x)$ is the solution of the differential equation

$$v' = -p_3(x) - p_2(x)v - p_1(x)v^2 - p_0(x)v^3 \tag{10}$$

with the initial condition $v(0) = 1$, and the elimination of the time transform the system (2) to the equation

$$\begin{aligned} v(x)zz' = p_0(x)v^3(x) + [p_1(x)v^2(x) + 3p_0(x)v^3(x)]z+ \\ +[p_1(x)v^2(x) + 2p_0(x)v^3(x) - p_3(x)]z^2. \end{aligned} \tag{11}$$

Then, the change $z = \alpha(x)w$, where $\alpha(x)$ is the solution of the differential equation

$$\alpha'v(x) = \alpha[p_1(x)v^2(x) + 2p_0(x)v^3(x) - p_3(x)], \tag{12}$$

with $\alpha'(0) = 1$, transforms (11) into the equation

$$ww' = f(x) + g(x)w, \tag{13}$$

where $f(x) = p_0(x)[v(x)/\alpha(x)]^2,\ g(x) = [p_1(x) + 3p_0(x)v(x)]v(x)/\alpha(x)$. From the theorem 19.7 from [4] we conclude that $O(0,0)$ of the equation (13) is a centre if and only if the system of equations

$$F(x) = F(y),\ G(x) = G(y),$$

where $F(x) = \int_0^x f(t)dt,\ G(x) = \int_0^x g(t)dt$, has an analytical in the neighbourhood of $x = 0$ solution $y = \varphi(x),\ \varphi(0) = 0,\ \varphi'(0) = -1$. Thus, in the examined case we have

$$F(x) = F[\varphi(x)],\ G(x) = G[\varphi(x)]. \tag{14}$$

From (14) it follows that

$$f(x) = f[\varphi(x)]\varphi'(x),\ g(x) = g[\varphi(x)]\varphi'(x). \tag{15}$$

From (15) we get that $\omega_0(x) = \omega_0[\varphi(x)]$, where

$$\omega_0(x) = f(x)/g(x) = p_0(x)v(x)/[\alpha(x)(p_1(x) + 3p_0(x)v(x))].$$

The differentiation of $\omega_0(x)$ taking into account (10), (12) gives

$$\omega_0'(x)/g(x) + 2/9 = Q_1(x)/[9(p_1(x) + 3p_0(x)v(x))^3]. \qquad (16)$$

From (16) we have $\omega_1(x) = \omega_1[\phi(x)]$, where

$$\omega_1(x) = Q_1(x)/[p_1(x) + 3p_0(x)v(x)]^3. \qquad (17)$$

The derivation of (17) gives us

$$\omega_1'(x)\omega_0(x)/g(x) - \omega_1^2(x)/3 - \omega_1(x)/3 = -Q_2(x)/[p_1(x) + 3p_0(x)v(x)]^5.$$

Consequently, $\omega_2(x) = \omega_2[\varphi(x)]$, where

$$\omega_2(x) = Q_2(x)/[p_1(x) + 3p_0(x)v(x)]^5. \qquad (18)$$

Then (18) gives

$$\omega_2'(x)\omega_0(x)/g(x) - 5\omega_1(x)\omega_2(x)/9 - 5\omega_2(x)/9 = -Q_3(x)/[3(p_1(x) + 3p_0(x)v(x))^7].$$

Thus, $\omega_3(x) = \omega_3[\varphi(x)]$, where

$$\omega_3(x) = Q_3(x)/[p_1(x) + 3p_0(x)v(x)]^7. \qquad (19)$$

The derivation of (19) gives

$$\omega_3'(x)\omega_0(x)/g(x) - 7\omega_1(x)\omega_3(x)/9 - 7\omega_3(x)/9 = -Q_4(x)/[3(p_1(x) + 3p_0(x)v(x))^9].$$

Hence, $\omega_4(x) = \omega_4[\varphi(x)]$, where

$$\omega_4(x) = Q_4(x)/[p_1(x) + 3p_0(x)v(x)]^9. \qquad (20)$$

From (17), (18) we have $F_1(x) = F_1[\varphi(x)]$, from (17), (19) we have $F_2(x) = F_2[\varphi(x)]$, and from (17), (20) we have $F_3(x) = F_3[\varphi(x)]$. The necessity is proved. The sufficiency is proved in the same way [2].

For system (2), where $p_3(x) = 0$, we have the following result.

Theorem 4. *The critical point $O(0,0)$ of system (2) in the case of $p_3(x) = 0$ is a centre if and only if the system of equations*

$$W_1(x) = W_1(y),\ W_2(x) = W_2(y), \qquad (21)$$

where $W_1 = (p_0p_1p_2 - p_1p_0' + p_0p_1')/p_1^3,\ W_2 = W_1'p_0/p_1^2$, has a solution $y = \varphi(x)$, where $\varphi(x)$ is an analytical in the neighbourhood of $x = 0$ function, $\varphi(0) = 0$, $\varphi'(0) = -1$ (we do not exclude the case when one or both equations of system (21) turn into the identities).

3 The solution of centre-focus problem for system (6)

Together with system (6) we will examine the equation

$$yy' = \sum_{i=0}^{3} p_i(x) y^i, \tag{22}$$

where

$$p_0(x) = -x^3/(1+Dx+Px^2), \qquad p_1(x) = (Ax+Kx^2)/(1+Dx+Px^2),$$
$$p_2(x) = (B+Lx)/(1+Dx+Px^2), \quad p_3(x) = M/(1+Dx+Px^2).$$

By the method [4, 7] we find a formal change for system (6)

$$x = u + \sum_{i+j=2}^{\infty} \alpha_{i,j} u^i v^j, \quad y = v + \sum_{i+j=2}^{\infty} \beta_{i,j} u^i v^j, \tag{23}$$

which transforms (6) to the system

$$du/dt = v + \sum_{i=2}^{\infty} d_i u^i, \quad dv/dt = -u^3 + \sum_{i=4}^{\infty} h_i u^i. \tag{24}$$

If in (23) $\alpha_{0,j} = \beta_{0,j} = 0,\ j = 2, 3, \dots$, then all $d_i,\ h_i$ in (24) are defined uniquely. In this case in (22)

$$d_2 = A/2, \qquad d_3 = A(B+D)/6 + K/3,$$
$$d_4 = A(B+D)(2B+D)/24 + K(B+D)/4 + A(L+2P)/24,$$
$$d_5 = A(B+D)(2B+D)(3B+D)/120 + K(B+D)(11B+7D)/60+$$
$$AL(7B+5D)/120 - M(A^2-18)/30 + AP(3B+2D)/30 + K(L+2P)/15;$$
$$h_4 = -(B+3D)/2, \quad h_5 = -(B+D)(B+5D)/4 - P,$$
$$h_6 = -(B+D)(B^2+9BD+6D^2)/8 + L(B-5D)/24 - AM/6 - P(3B+4D)/2;$$

$d_i,\ i = \overline{6,15}$, are polynomials of A, B, D, K, L, M, P, which consist accordingly of 27, 47, 75, 117, 172, 251, 350, 485, 651, 869 addends; $h_i,\ i = \overline{7,16}$, are polynomials, which consist accordingly of 17, 27, 45, 67, 102, 145, 208, 284, 391, 518 addends.

The change of $u_1 = \varphi(u) = u(1 - \sum_{k=4}^{\infty} h_k u^{k-3})^{1/4},\ d\tau = (1 - \sum_{k=4}^{\infty} h_k u^{k-3})dt$ reduces system (24) to the form

$$du_1/d\tau = v + \sum_{k=2}^{\infty} d_k[\varphi^{-1}(u_1)]^k = v + \sum_{k=2}^{\infty} A_k u_1^k,\ dv/d\tau = -u_1^3. \tag{25}$$

The values $f_i = A_{2i+1},\ i = 1, 2, \dots$, where A_{2i+1} is from (25) will be called the focus values of system (6). Focus values $f_k,\ k = 1, 2, \dots$, are the polynomials from the ring $\mathbb{C}[K, M, L, P, D, B, A]$; $f_i,\ i = \overline{1,7}$, contain accordingly 3, 15, 47, 117, 251, 485, 869 addends.

Let's generate the ideal [24] $I = \langle f_1, f_2, ..., f_k, ...\rangle \subset \mathbb{C}[K, M, L, P, D, B, A]$. Let us denote by $\mathbb{V}(I)$ the variety of ideal I [24], i.e. $\mathbb{V}(I) = \{a = (K, M, L, P, D, B, A) \in \mathbb{C}^7 : \text{for any} f \in I,\ f(a) = 0\}$.

Definition 2. *The set $W = \mathbb{V}(I)$ is called the variety of the centre of system (6).*

It is obvious that $O(0,0)$ of system (6) is a centre if and only if $a \in W$.

The focus values $f_k,\ k = \overline{1,7}$, can be found with the help of computer system Mathematica 4.1. Instead of focus values $f_k,\ k = \overline{1,7}$, we will examine

$$g_1 = 15f_1 = A(B-2D)+5K,\ g_k = f_k(mod\langle g_1,...,g_{k-1}\rangle),\ k=\overline{2,7}.$$

The values g_k can be found with the help of the division algorithm [24]. We have

$$g_2 = A(B-2D)(B^2-9BD+4D^2)+10AL(3B-D)-25M(2A^2-21)- 5AP(13B-6D),$$
$$g_3 = A(B-2D)^2(2B+D)(19B^2+389BD-204D^2)-1250AL^2(3B-D)- 125AL(B-2D)(17B^2-3BD-2D^2)+5625M(-7B^2-2BD+12D^2+ 20L-45P)+125AP(B-2D)(53B^2+16BD-24D^2)+ 625AP[P(29B-18D)-L(B-2D)].$$

Let us note that $g_k,\ k=\overline{4,7}$, contains accordingly 51, 90, 143, 211 addends. In so doing $I = \langle f_1,...,f_k,...\rangle = \langle g_1,...,g_k,...\rangle$. We put $I_k = \langle f_1,...,f_k\rangle$. Then $I_k = \langle g_1,...,g_k\rangle$.

Theorem 5. *The variety of the centre of system (6) can be represented in the form $W = \mathbb{V}(J_1)\bigcup\mathbb{V}(J_2)\bigcup...\bigcup\mathbb{V}(J_{14})$, where*

$J_1 = \langle A,\ M,\ K\rangle,\ \ J_2 = \langle B,\ D,\ M,\ K\rangle\ ,\ \ J_3 = \langle B-2D,\ L-2P,\ M,\ K\rangle,$
$J_4 = \langle 3B-D,\ P-2B^2,\ M,\ AB-K\rangle,$
$J_5 = \langle (B-2D)(B+3D)+25P,\ (B-2D)(3B-D)+25L,\ M,\ A(B-2D)+5K\rangle,$
$J_6 = \langle 2B-D, 9B^2-25P, 3B^2+25L, M, 3AB-5K\rangle,$
$J_7 = \langle 17B-4D,\ 9B^2-2P,\ 3B^2-2L,\ M,\ 3AB-2K\rangle,$
$J_8 = \langle 7B-4D,\ B^2+2P,\ B^2+L,\ M,\ AB-2K\rangle,$
$J_9 = \langle A^2-6,\ 3(B-2D)(3B+4D)+100P,\ A(17B-4D)(B-2D)^2+4500M,$
$B(B-2D)+5L,\ A(B-2D)+5K\rangle,$
$J_{10} = \langle A^2-6,\ 3(B-2D)(3B-D)+25P,\ (17B-9D)(B-2D)+25L,$
$2A(B-2D)^2(2B-D)+225M,\ A(B-2D)+5K\rangle,$
$J_{11} = \langle A^2-6,\ (3B-D)(3B+4D)+25P,\ 11B^2+BD+4D^2+25L,$
$2A(7B-4D)(2B+D)^2+1125M,\ A(B-2D)+5K\rangle,$
$J_{12} = \langle A^2-6,\ 3B-D,\ 3(2B^2+L)-4P, AB(2B^2-P)-9M,\ AB-K\rangle,$
$J_{13} = \langle A-3, (B-7D)(B-2D)+25(L-2P), 3(B-2D)+5K,$
$-(B-2D)^2(B+3D)-25P(B-2D)+125M\rangle,$
$J_{14} = \langle A+3,\ (B-7D)(B-2D)+25(L-2P),\ -3(B-2D)+5K,$
$(B-2D)^2(B+3D)+25P(B-2D)+125M\rangle$

and $\mathbb{V}(J_i),\ i=\overline{1,14}$, are irreducible.

Proposition 1. *If $2A^2-7=0,\ 3B-D=0,\ 3B^2-P=0,\ 2B^2-L=0,\ AB^3+14M=0,\ AB-K=0,\ B\neq 0$, then $O(0,0)$ of system (6) is a focus of 8^{th} order.*

Proof. In the examined case the system (6) looks as

$$\dot{x} = y(1+3Bx+3B^2x^2),\ \dot{y} = -x^3+Axy+By^2+By(Ax^2+2Bxy-AB^2y^2/14),\ (26)$$

where $A^2 = 7/2,\ B \neq 0$. There exists a change (4) which reduces (26) to the system

$$\frac{du}{d\tau} = v + A(u^2/2 - 55B^6u^8/25088 - 5445B^{12}u^{14}/314703872 - 55B^{15}u^{17}/161308784$$
$$-28655B^{18}u^{20}/281974669312 - 3267B^{21}u^{23}/374283822640 + ...), \quad \frac{dv}{d\tau} = -u^3.$$

Hence, the focus values $f_k = 0,\ k = \overline{1,7}$, but $f_8 \neq 0$, i.e. $O(0,0)$ is a focus of 8^{th} order of system (26).

Lemma 1. *Consider $M = 0$. Then the variety of the centre of the system (6) is shown as $V_1 = \mathbb{V}(J_1) \bigcup \mathbb{V}(J_2) \bigcup ... \bigcup \mathbb{V}(J_8)$.*

Proof. Let us make the ideal $J_0 = I_7 + \langle M \rangle$. We compute the Groebner basis of J_0 with lex-ordering with the order $K > M > L > P > D > B > A$ and get

$$J_0 = \langle A(7B-4D)(17B-4D)(B-2D)(2B-D)(3B-D)^2[(B-2D)(B+3D)+25P],\ -A(B-2D)[(B-2D)(B+3D)+25P](4427B^4-5798B^3D+2805B^2D^2-608BD^3+48D^4+125B^2P),\ A(B-2D)\,[\,(B-2D)(B+3D)+25P]\,[2(157B^3-157B^2D+69BD^2-16D^3)-25P(4B-3D)],\ A[(B-2D)(B^2-9BD+4D^2)+10L(3B-D)-5P(13B-6D)],\ A\,[2\,(B-2D)(156B^4-1823B^3D+569B^2D^2-142BD^3+96D^4)+6250\,BL\,(2B^2-P)-125\,P\,(239B^3-101B^2D+56BD^2-20D^3)+625P^2(23B-6D)],\ M,\ A(B-2D)+5K\rangle.$$

Hence, $\mathbb{V}(J_0) = \mathbb{V}(J_1) \bigcup \mathbb{V}(J_2) \bigcup ... \bigcup \mathbb{V}(J_8)$. Let us show then that on the set $\mathbb{V}(J_0)$ the equation (22) and so the system (6) have a centre in $O(0,0)$. Indeed, on the sets $\mathbb{V}(J_1),\ \mathbb{V}(J_2)$ we find the cases of symmetry and therefore the equation (22) has a centre in $O(0,0)$. To prove the existence of the centre on the sets $\mathbb{V}(J_k),\ k = \overline{3,8}$, we will use Theorem 4. On the set $\mathbb{V}(J_4)$ for equation (22) the functions $W_1,\ W_2$ from (21) look like

$$W_1(x) = 2/A^2 - Lu(x)/A^2, \quad W_2(x) = -2B^2Lu^2(x)/A^4 + 2Lu(x)/A^4,$$

where $u(x) = x^2/(1+Bx)^2$. Consequently, the equation (22) in this case has a centre in $O(0.0)$. On the sets $\mathbb{V}(J_3),\ \mathbb{V}(J_5)$ the existence of the centre follows from the fact that $W_1 = 2/A^2$. On the set $\mathbb{V}(J_6)$

$$W_1(x) = 2/A^2 - 18B^2u(x)/A^2, \quad W_2(x) = 36B^2u(x)/A^4 - 972B^4u^2(x)/A^4,$$

where $u(x) = x^2(5+Bx)/(5+3Bx)^3$; $O(0,0)$ of the equation (22) is a centre. On the set $\mathbb{V}(J_7)$ the equation (22) has a centre in $O(0,0)$ because

$$W_1(x) = 2/A^2 - 9B^2u(x)/A^2, \quad W_2(x) = 18B^2u(x)/A^4 - 243B^4u^2(x)/A^4,$$

where $u(x) = x^2(1+2Bx)/(2+3Bx)^3$. On the variety $\mathbb{V}(J_8)$

$$W_1(x) = 2/A^2 - 9B^2u(x)/A^2, \quad W_2(x) = 18B^2u(x)/A^4 - 243B^4u^2(x)/A^4,$$

where $u(x) = x^2/(2+Bx)^3$; $O(0,0)$ of the equation (22) is a centre.

Lemma 2. *Consider $A^2-6=0$. Then the variety of the centre of system (6) can be shown in the following way:*

$$V_2=\mathbb{V}(J_2+\langle A^2-6\rangle)\bigcup\mathbb{V}(J_3+\langle A^2-6\rangle)\bigcup\mathbb{V}(J_4+\langle A^2-6\rangle)\bigcup \mathbb{V}(J_5+\langle A^2-6\rangle)\bigcup\mathbb{V}(J_9)\bigcup\mathbb{V}(J_{10})\bigcup\mathbb{V}(J_{11})\bigcup\mathbb{V}(J_{12}).$$

Proof. When we compute the Groebner basis of the ideal $S=I_7+\langle A^2-6\rangle$ we have $S=\langle h_1,...,h_{27}\rangle$, where

$h_1=A^2-6,\dots,h_4=(B-2D)(3B-D)^4[(B-2D)(B+3D)+25P][(3B-D)\times(3B+4D)+25P][3(B-2D)(3B-D)+25P][3(B-2D)(3B+4D)+100P],\dots,$
$h_7=-B^5(B-2D)(3B-D)[(B-2D)(3B-D)(1701B^6-78732B^5D+538335B^4D^2-584060B^3D^3-7860B^2D^4+285408BD^5-69696D^6)-5000BDL(9B-8D)\times(4B-3D)(3B-D)(9B+2D)+625P(3B-D)(135B^5+4617B^4D-6230B^3D^2-2508B^2D^3+7848\,BD^4-2016\,D^5)+625P^2(B-2D)\,(6489B^3+22071B^2D-14852BD^2+8D^3)+390625P^3(63B^2-6BD-56D^2+108P)],\dots,$
$h_{24}=-(B-2D)(213B^4-804B^3D-1663B^2D^2+2734BD^3-792D^4)-125L(21B^3+79B^2D-168BD^2+52D^3)+6250L^2(3B-D)+125P(29B^3+173B^2D-310BD^2+96D^3)-3125LP(21B-8D)+2500P^2(22B-9D),\dots,$
$h_{26}=A(B-2D)(B^2-9BD+4D^2)+10AL(3B-D)+225M-5AP(13B-6D),$
$h_{27}=A(B-2D)+5K.$

Hence, $\mathbb{V}(S)=V_2$. From Lemma 1 it follows that on the set $\mathbb{V}(J_k+\langle A^2-6\rangle)$, $k=\overline{2,5}$, $O(0,0)$ of the equation (22) is a centre. On the sets $\mathbb{V}(J_k)$, $k=\overline{9,12}$, the presence in $O(0,0)$ of the centre of the equation (22) follows from the fact that here $F_2=0$, where F_2 is from (7).

Remark 1. *On the set $\mathbb{V}(J_9)$ the system (6) has the integrating factor of Darboux form* $R_1(x,y)=[1-3(B-2D)x/10]^{1/3}[1+(3B+4D)x/10]^{-1}/[x^4-Ax^2y+2y^2+(B-2D)(Ax^2-4y)xy/5+2A(B-2D)^3xy^3/1125-(B-2D)^2(Ay-12x^2)y^2/150]$, *since* $\frac{\partial}{\partial x}\left[y(1+Dx+Px^2)R_1(x,y)\right]+\frac{\partial}{\partial y}\left[(-x^3+Axy+By^2+Kx^2y+Lxy^2+My^3)R_1(x,y)\right]=0$. *On the sets $\mathbb{V}(J_{10})$, $\mathbb{V}(J_{11})$, $\mathbb{V}(J_{12})$ the integrating factors of the system (6) are, accordingly, the functions $R_2(x,y)$, $R_3(x,y)$, $R_4(x,y)$, where*

$R_2(x,y)=[1-3(B-2D)x/5]^{-1/3}[1+(3B-D)x/5]^{-1}/[x^4-Ax^2y+2y^2+(B-2D)\times(Ax^2-4y)xy/5+2A(B-2D)^3xy^3/1125-2(B-2D)^2(Ay-3x^2)y^2/75],$
$R_3(x,y)=[1-(3B-D)x/5]^{1/3}[1+(3B+4D)x/5]^{-1/3}/[x^4-Ax^2y+2y^2+(B-2D)\times(Ax^2-4y)xy/5+2(B-2D)^2x^2y^2/25-2A(2B+D)^2y^3/75+2A(7B-4D)\times(2B+D)^2xy^3/1125],$
$R_4(x,y)=[1+3Bx+3(2B^2+L)x^2/4]^{-1/3}/[x^4-Ax^2y+2y^2-Bxy\,(Ax^2-4y)+A(3L-4B^3x)y^3/18-B^2(Ay-6x^2)y^2/3].$

Remark 2. *On the sets $\mathbb{V}(J_k)$, $k=\overline{9,12}$, the change (9), where $v(x)\,(v(x)\neq 0)$ is, accordingly, the function of the type*

$v(x) = A[1 - (B - 2D)x/5 - (1 - 3(B - 2D)x/10)^{2/3}]/(3x^2)$,
$v(x) = A[1 - (B - 2D)x/5 - (1 - 3(B - 2D)x/5)^{1/3}]/(3x^2)$,
$v(x) = A[1 - (B - 2D)x/5 - (1 - (3B - D)x/5)^{2/3}(1 + (3B + 4D)x/5)^{1/3}]/(3x^2)$,
$v(x) = A[1 + Bx - (1 + 3x(4B + (2B^2 + L)x)/4)^{1/3}]/(3x^2)$,
transforms the equation (22) to equation (11).

Lemma 3. *Consider $A^2 - 9 = 0$. Then the variety of the centre of system (6) is shown as $V_3 = \mathbb{V}(J_{13}) \bigcup \mathbb{V}(J_{14})$.*

Proof. The ideal $S_0 = I_7 + \langle A - 3\rangle$ is represented through Groebner basis in the following way: $S_0 = \langle q_1, ..., q_{25}\rangle$, where

$$q_1 = A - 3, \quad q_2 = -B^7(7B - 4D)(17B - 4D)(2B - D)(3B - D)^3[(B - 7D)\times (B - 2D) + 25(L - 2P)], \dots,$$
$$q_{22} = [(B - 7D)(B - 2D) + 25(L - 2P)][563B^3 + 87B^2D - 1154BD^2 + 456D^3 - 650L(3B - D) + 25P(161B - 72D)],$$
$$q_{23} = [(B - 7D)(B - 2D) + 25(L - 2P)][2(14661862B^5 - 23476145B^4D + 12603805B^3D^2 - 2621310B^2D^3 - 155240BD^4 + 154016D^5) - 1543750BL(2B^2 - P) + 125P(160257B^3 - 135683B^2D + 65508BD^2 - 17240D^3) - 625P^2(8969B - 3948D)],$$
$$q_{24} = A(B - 2D)(B^2 - 9BD + 4D^2) + 10L(3B - D) + 25M - 5P(13B - 6D),$$
$$q_{25} = 3(B - 2D) + 5K.$$

In this case $\mathbb{V}(S_0) = \mathbb{V}(J_{13})$. On the set $\mathbb{V}(J_{13})$ the equation (22) has a centre in $O(0, 0)$ because here $Q_1 = 0$, and therefore, systems (7), (8) turn into the identities. The case $\mathbb{V}(J_{14})$ is examined in the same way.

Remark 3. *On the set $\mathbb{V}(J_{13})$ the system (6) has the integrating factor of Darboux form $R_5(x, y) = f_2^{S_2} f_3^{S_3}/f_1^3$, where*

$$f_1 = x^2 - [1 - (B - 2D)x/5]y, \quad f_2 = 1 + (D + g)x/2,$$
$$f_3 = 1 + (D - g)x/2, \quad g^2 = D^2 - 4P,$$
$$S_2 = (2B + D)[(2D - B)(D - g) - Pg(3D - 4B)/(D^2 - 4P)]/(25DP),$$
$$S_3 = (2B + D)[(2D - B)(D + g) + Pg(3D - 4B)/(D^2 - 4P)]/(25DP).$$

On the set $\mathbb{V}(J_{14})$ the system (6) has the integrating factor $R_6(x, y) = f_2^{S_2} f_3^{S_3}/f_0^3$, where $f_0 = x^2 + [1 - (B - 2D)x/5]y$.

Lemma 4. *If*

$$M(A^2 - 6)(A^2 - 9) \neq 0, \tag{27}$$

then $O(0, 0)$ of system (6) is a focus.

Proof. Finding Groebner basis of the ideal $I_7 + \langle A\rangle$ we have $I_7 + \langle A\rangle = \langle A, M, K\rangle$, i.e. when (27) holds in the case $A = 0$, $O(0, 0)$ of system (6) is a focus. The ideal $I_7 + \langle B\rangle$ via Groebner basis looks as

$$I_7 + \langle B\rangle = \langle B, -A(A^2 - 6)(A^2 - 9)D^{11}(6D^2 - 25P), \dots, 5K - 2AD\rangle.$$

After that we have

$$I_7 + \langle B, D(6D^2 - 25P)\rangle = \langle B, D(6D^2 - 25P), -AD^7(A^2 - 6)(2D^2 + 25L),$$
$$64AD^7(2D^2 + 25L) - 474609375M^3, \dots, 2AD + 5K\rangle.$$

Consequently, when $B = 0$ and (27) holds, system (6) has a focus in $O(0, 0)$. We find the ideal $I_7 + \langle 3B - D\rangle$ in the form

$$I_7 + \langle 3B - D\rangle = \langle 3B - D, AB^{13}(A^2 - 6)(A^2 - 9)(2A^2 - 7)(P - 2B^2), \dots, AB - K\rangle.$$

Moreover,

$$I_7 + \langle 3B - D, B(2B^2 - P)\rangle = \langle 3B - D, B(2B^2 - P), M^3, \dots, AB - K\rangle$$

and

$$I_7 + \langle 3B - D, 2A^2 - 7\rangle = \langle 2A^2 - 7, 3B - D, B^7(2B^2 - P)(3B^2 - P), AB - K,$$
$$-B(2B^2 - P)(3B^2 - P)^2, -B(2B^2 - P)(B^2 - 2L + P), AB(2B^2 - P) - 14M\rangle.$$

Hence, taking into account Proposition 1, we conclude that when $3B - D = 0$ and (27), $O(0, 0)$ of system (6) is a focus. Since

$$I_7 + \langle B - 2D\rangle = \langle B - 2D, -AB^9(A^2 - 6)(L - 2P), \dots, K\rangle,$$
$$I_7 + \langle B - 2D, B(L - 2P)\rangle = \langle B - 2D, B(L - 2P), B^8M, M(13B^6 - 800M^2), \dots, K\rangle,$$

then when $B - 2D = 0$ together with the condition (27) the system (6) also has a focus in $O(0, 0)$. For the ideal $I_7 + \langle 4B - 3D\rangle$ the Groebner basis gives

$$I_7 + \langle 4B - 3D\rangle = \langle 4B - 3D, A(A^2 - 6)(A^2 - 9)B^{11}(3P - B^2), \dots, AB - 3K\rangle.$$

In this case

$$I_7 + \langle 4B - 3D, B(B^2 - 3P)\rangle = \langle 4B - 3D, B(B^2 - 3P),$$
$$-A(A^2 - 6)B^7(B^2 - 9L), 64AB^7(B^2 - 9L) - 4782969M^3, \dots, AB - 3K\rangle,$$

i.e. when $4B - 3D = 0$ and (27) holds, $O(0, 0)$ of system (6) is a focus. While examining the ideal $I_7 + \langle 2B + D\rangle$, we have

$$I_7 + \langle 2B + D\rangle = \langle 2B + D, AB^9(A^2 - 6)(A^2 - 9)(B^2 - P)^2, \dots, K + AB\rangle.$$

Here

$$I_7 + \langle 2B + D, B(B^2 - P)\rangle = \langle 2B + D, B(B^2 - P), M^3, \dots, AB + K\rangle.$$

Consequently, when $2B + D = 0$, $O(0, 0)$ of system (6) is a focus. For the ideal $I_7 + \langle 2B - D\rangle$ we find a representation in the form

$$I_7 + \langle 2B - D\rangle = \langle 2B - D, AB^9(A^2 - 6)(A^2 - 9)(9B^2 - 25P)(21B^2 - 25P), \dots, 3AB - 5K\rangle.$$

In this case $I_7 + \langle 2B - D, B(9B^2 - 25P)(21B^2 - 25P)\rangle = \langle 2B - D, B(9B^2 - 25P)(21B^2 - 25P), -AB^7(A^2 - 9)(3B^2 + 10L - 5P), -432AB^7(3B^2 + 10L - 5P) - 390625M^3, \dots, 3AB - 5K\rangle$. So, in the case when relations (27) and $2B - D = 0$ are fulfilled, $O(0, 0)$ of system (6) is a focus. So, when (27) holds and $AB(3B - D)(B - 2D)(4B - 3D)(2B + D)(2B - D) = 0$, $O(0, 0)$ is a focus. We will assume that the condition

$$AB(3B - D)(B - 2D)(4B - 3D)(2B + D)(2B - D) \neq 0$$

holds. Applying the change $x = x_1/B,\ y = y_1/B^2,\ dt = Bd\tau$ we will transform system (6) to the form

$$dx_1/d\tau = y_1(1 + Dx_1/B + Px_1^2/B^2),$$
$$dy_1/d\tau = -x_1^3 + Ax_1y_1 + y_1^2 + Kx_1^2y_1/B + Lx_1y_1^2/B^2 + My_1^3/B^3. \qquad (28)$$

System (28) shows that it is enough to study the system (6) when $B = 1$, and then to change D, P, K, L, M for $D/B, P/B^2, K/B, L/B^2, M/B^3$. Let us show that when $B = 1$ $A(A^2-6)(A^2-9)(D-3)(2D-1)(3D-4)(D+2)(D-2) \neq 0$, $O(0,0)$ of system (6) is a focus.

Suppose the contrary, that $O(0,0)$ of system (6) is a centre. From $g_1 = 0$ we find $K = A(2D-1)/5$. Taking into account $A(D-3) \neq 0$ we get from $g_2 = 0$ $L = [-A(2D-1)(4D^2-9D+1) - 25M(2A^2-21) + 5AP(6D-13)]/[10A(D-3)]$. Considering L we have $g_i = \alpha_i h_i/[A(D-3)]^{i-2},\ i = \overline{3,7}$, where $\alpha_i \neq 0$,

$$h_3 = 4A^2(2D-1)^2(3D+1)(16D^3-69D^2+157D-157) + 625AM[10A^2(2D-1)\times (2D^2-3D-3) - 3(164D^3-316D^2+33D-33)] + 15625M^2(2A^2-21)\times (2A^2-57) - 250A^2P(2D-1)(10D^3-33D^2+63D-62) - 3125AMP\times (28A^2D-54A^2-348D+549) + 1250A^2P^2(2D-1)(3D-4),$$

$h_i,\ i = \overline{4,7}$, are polynomials in A, D, P, M. Taking into account $h_3 = 0,\ h_i,\ i = \overline{4,7}$, we show that $h_i = \beta_i v_i/[(2D-1)(3D-4)]^{i-3}$, where $p_i \neq 0,\ v_i,\ i = \overline{4,7}$, are polynomials in A, D, P, M of the first degree with respect to P. We shall denote by $R_x(u,v)$ the resultant of polynomials u, v with respect to x. We have

$$R_p(v_4, h_3) = \gamma_4 A^2(A^2-9)(D-3)^2(2D-1)(3D-4)Mr_4,$$
$$R_p(v_4, v_i) = \gamma_i A^2(A^2-9)(D-3)^2(2D-1)(3D-4)Mr_i,\ i = \overline{5,7},$$

where $\gamma_i \neq 0,\ r_i,\ i = \overline{4,7}$, are polynomials in A, D, M with integer coefficients. As far as $A(A^2-9)(D-2)(2D-1)(3D-4)M \neq 0$, then $r_i = 0,\ i = \overline{4,7}$. In the same way $R_p(h_3, v_i) = \delta_i A^2(A^2-9)(D-3)[(D-3)(2D-1)(3D-4)]^{i-3}Ms_i,\ i = \overline{5,7}$, where $\delta_i \neq 0, s_i,\ i = \overline{5,7}$, are polynomials in A, D, M with integer coefficients. Here is $s_i = 0, i = \overline{5,7}$, too. Let us notice that r_4, r_5 are polynomials of 5^{th} degree relative to M, s_5, r_7 of 7^{th} degree, r_6, s_6, s_7 are of $6^{th}, 9^{th}, 11^{th}$ degree, respectively. While computing the resultant of polynomials $r_4, Sr_5 + r_6$ relative to M we get

$$R_M(r_4, Sr_5 + r_6) = \alpha A^{25}(A^2-6)^3(2A^2-21)^4(D-3)^{13}(D-2)(D+2)^4(2D-1)^{19}\times (3D-4)^4(4D-17)(4D-7)\, H_0^2\,[4405854208A^5(D-3)^6(2D-1)^5(3D-4)^4T_0 + 196689920A^4(D-3)^5(2D-1)^4(3D-4)^3]T_1S - 26342400A^3(D-3)^3(2D-1)^3\times (3D-4)^2T_2s^2 + 3528000A^2(D-3)^2(2D-1)^2(3D-4)T_3s^3 - 472500A(D-3)\times (2D-1)T_4s^4 - 253125H_1T_5s^5],$$

where $\alpha \neq 0$, $H_1 = (A^2-6)[28A^4(2D-1)(3D-4) - 6A^2(2596D^2-9316D+9459) + 9(13298D^2-62623D+80687)] - 81(D-3)(622D-1481)$, $H_0, T_i,\ i = \overline{0,5}$, are polynomials in A, D with integer coefficients. If $2A^2 - 21 = 0$, then from $r_4 = 0, s_5 = 0, s_6 = 0$ we have

$$A^2(D-3)^{14}(D-2)(D+2)^4(2D-1)^{18}(3D+1)^2(4D-17)(4D-7) = 0.$$

Hence, $(3D+1)(4D-17)(4D-7 = 0)$. Since

$I_7+\langle 2A^2-21, B+3D\rangle = \langle 2A^2-21, B+3D, B^{11}P, P(104763912686092943360AB^9-$
$1315546762295588998887953MP^3), 186824475M(413P^3+13056M^2), ..., AB+3K\rangle,$
$I_7 + \langle 2A^2 - 21, 17B - 4D\rangle = \langle 2A^2 - 21, 17B - 4D, B^7(33B^2 - 8P)(9B^2 - 2P),$
$-71876935680M^3 - 7AB(33B^2 - 8P)(9B^2 - 2P)(102952372707B^4-$
$47325572912B^2P + 5362662080P^2), \dots, 3AB - 2K\rangle,$
$I_7 + \langle 2A^2 - 21, 7B - 4D\rangle = \langle 2A^2 - 21, 7B - 4D, B^7(5B^2 - 8P)(B^2 + 2P),$
$AB(5B^2 - 8P)(B^2 + 2P)(56056383B^4 - 124509840B^2P + 66718400P^2)-$
$98014003200M^3, \dots, AB - 2K\rangle,$

then when $2A^2 - 21 = 0$ and (27) holds, $O(0,0)$ of system (6) is a focus. If $AB \neq 0$, $(17B - 4D)(7B - 4D) = 0$, then the study of the system of equations $g_i = 0$, $i = \overline{1,7}$, shows that if (27) is fulfilled, $O(0,0)$ of system (6) can be a centre only in the case $2A^2 - 21 = 0$. So, when $(17B - 4D)(7B - 4D) = 0$ we have the case of focus.

Let us examine now the case $H_0 = 0$. To do this we find

$$R_M(h_3, v_i) = \mu_i A^{30}(A^2 - 6)^3(2D - 1)^{17}(D - 3)^{10}(D - 2)(D + 2)^4(4D - 17)\times$$
$$(4D - 7)[(2A^2 - 21)^2(D - 3)^4(2D - 1)^3]^{i-4}B_i,\ i = \overline{5,6},$$

where $\mu_i \neq 0,\ B_5 = T_5C_1,\ B_6 = T_0C_2,\ C_1$ and C_2 are polynomials in A, D, which consist of 1273 and 2088 factors, respectively. Then we find

$$R_A(H_0, B_i) = \lambda_i(D - 3)^{14}(2D - 1)^{59}(3D-4)^6(3D+1)^2(4D-7)^2(4D^2 + 36D-$$
$$69)^2(14D^2 - 19D - 24)^4(16D^2 - 21D - 6)^2(632D^3 - 3408D^2 + 6159D - 3994)\times$$
$$(1456D^3-11244D^2+28752D-25247)(88D^4-410D^3+993D^2-1792D+1256)\times$$
$$(2184D^5 - 7700D^4 - 19135D^3 + 102085D^2 - 98789D - 3402)E_i,\quad i = 5, 6,$$

where $\lambda_i \neq 0,\ E_i,\ i = 5, 6,$ are coprime polynomials in D. Since $(D-3)(2D-1)(3D-4)(3D + 1)(4D - 7) \neq 0$ only in the case $(2A^2 - 21)(A^2 - 6) = 0$, $R_A(H_0, B_i) = 0$, $i = 5, 6$, we can conclude that also when $H_0 = 0$, $O(0,0)$ of system (6) cannot be a centre. In the case when $H_1 = 0$ the study of system $H_1 = 0,\ T_i = 0,\ i = \overline{0,4}$, shows that $O(0,0)$ of system (6) is a focus. So, in the case when $B = 1$ and (27) holds, $O(0,0)$ of system (6) can be a centre only when

$$T_i = 0,\ i = \overline{0,5}. \tag{29}$$

Let us find the real solutions of system (29), where $A^2 < 8$. We have

$$R_z(T_s, T_i) = \gamma_{i,0}(D - 3)^{24}(D + 2)^3(2D - 1)^{22}(3D - 4)^2\Theta_0 B_{i,0},\ i = \overline{0,4},$$

where $z = A^2$, $\gamma_{i,0} \neq 0$, Θ_0 are polynomials in D of 66^{th} degree whose coefficients are coprime integer numbers of the order from 10^{77} to 10^{114}, $B_{i,0}$, $i = \overline{0,4}$, are coprime polynomials in D of degree 690, 668, 657, 635, 613, respectively. Notice that the polynomial Θ_0 has 20 real roots.

On the other hand,

$$R_D(T_5, T_i) = \gamma_{i,1}(A^2 - 9)^3(A^2 - 6)^2(7A^2 - 30)^{12}\Theta^3 B_{i,1},\ i\overline{= 0,4},$$

where $\gamma_{i,1} \neq 0$, $B_{i,1}$, $i = \overline{0,4}$, are coprime polynomials in A, Θ is a polynomial in A of 44^{th} degree which consists of terms in even degrees and whose coefficients are coprime integer numbers of the order from 10^{32} to 10^{54}.

Let us introduce the vector $q = (A, D)$. The system (29) has 18 real solutions $q = q_i$, where $q_i = (A_i, D_i),\ i = \overline{1,9},\ q_{i+9} = (-A_i, D_i),\ i = \overline{1,9}$, and

$$A_1 = 2.48741...,\ A_2 = A_3 = A_4 = 2.495479\dots,\ A_5 = A_6 = A_7 = 2.189944\dots,$$
$$A_8 = 2.072126\dots,\ A_9 = 1.916074\dots,\ D_1 = 2.027444\dots,\ D_2 = 2.9540003\dots,$$
$$D_3 = 2.990356\dots,\ D_4 = 3.008036\dots,\ D_5 = 0.617227\dots,\ D_6 = 1.954363\dots,$$
$$D_7 = 5.659333\dots,\ D_8 = 4.479633\dots,\ D_9 = 3.057486\dots.$$

Replacing $q_i,\ i = \overline{1,18}$, by $r_i,\ i = \overline{4,6}$, which was found from the system of equations $r_i = 0,\ i = \overline{4,6}$, find $M = M_i,\ i = \overline{1,18}$.

Then from $v_4 = 0$ we find $p = p_i,\ i = \overline{1,18}$. Consider $r = (A, D, K, L, M, P)$. Taking into account K, L which were found before, when $B = 1$, we have 18 real solutions $r = r_k = (A_k, D_k, K_k, L_k, M_k, P_k),\ k = \overline{1,18}$, of the system of equations $g_i = 0,\ i = \overline{1,6}$. Here $r_{i+9} = (-A_i, D_i, -K_i, L_i, -M_i, P_i),\ i = \overline{1,9}$. Notice that A_i are roots of the polynomial Θ, D_i are roots of the polynomial θ_0.

Let us show that $g_7|_{r=r_k} \neq 0,\ k = \overline{1,18}$. We have

$$R_M(r_4, r_7) = \alpha_0 A^{35}(A^2-6)^3(2A^2-21)^5(D-3)^{24}(D-2)(D+2)^4(2D-1)^{29}\times$$
$$(3D-4)^{12}(4D-17)(4D-7)H_0^2T_6,$$

where $\alpha_0 \neq 0$. Then we find $R_{A^2}(T_5, T_6) = \gamma_{5,6}(D-3)^{30}(D+2)(2D-1)^{26}(3D-4)^2C_0$, where $\gamma_{5,6} \neq 0,\ C_0$, is a polynomial in D of 997^{th} degree whose coefficients are coprime integer numbers of the order from 10^{3009} to 10^{3580}. Since $\Theta_0,\ C_0$ are coprime polynomials in D, then $v_7|_{r=r_k} \neq 0,\ k = \overline{1,18}$. So, when (29) is fulfilled, $O(0,0)$ cannot be a centre.

Proof of Theorem 5. The proof follows directly from Lemmas 1–4.

Proposition 2. *When* $r = r_k,\ k = \overline{1,18}$, $B = 1$, *the critical point* $O(0,0)$ *of system (6) is a focus of* 7^{th} *order.*

Proof. The proof follows from Lemma 4.

Theorem 6. *For any* $\varepsilon > 0,\ \delta > 0,\ k,\ (k = \overline{1,18})$ *there exists* $r \in U_\delta(r_k)$, *where* $U_\delta(r_k)$ *is a* δ*-neighbourhood of* r_k*, such that system (6) with* $B = 1$ *has in* ε*-neighbourhood* $U_\varepsilon(0)$ *of the point* $O(0,0)$ *6 limit cycles.*

Proof. The proof is analogous to the proof of Theorem 3 from [25], using Lemma 1 from [25].

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On initial value problem in theory of the second order differential equations

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Abstract. We consider the properties of the second order nonlinear differential equations $b'' = g(a,b,b')$ with the function $g(a,b,b'=c)$ satisfying the following nonlinear partial differential equation

$$g_{aacc} + 2cg_{abcc} + 2gg_{accc} + c^2 g_{bbcc} + 2cgg_{bccc} + g^2 g_{cccc} + (g_a + cg_b)g_{ccc} -$$
$$4g_{abc} - 4cg_{bbc} - cg_c g_{bcc} - 3gg_{bcc} - g_c g_{acc} + 4g_c g_{bc} - 3g_b g_{cc} + 6g_{bb} = 0\,.$$

Any equation $b'' = g(a,b,b')$ with this condition on the function $g(a,b,b')$ has the General Integral $F(a,b,x,y) = 0$ shared with General Integral of the second order ODE's $y'' = f(x,y,y')$ with the condition $\frac{\partial^4 f}{\partial y'^4} = 0$ on the function $f(x,y,y')$ or $y''+a_1(x,y)y'^3+3a_2(x,y)y'^2+3a_3(x,y)y'+a_4(x,y) = 0$ with some coefficients $a_i(x,y)$.

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1 Introduction

The relation between the equations in the form

$$y'' + a_1(x,y)y^2 + 3a_3(x,y)y' + a_4(x,y) = 0 \tag{1}$$

and

$$b'' = g(a,b,b') \tag{2}$$

with the function $g(a,b,b')$ satisfying the p.d.e

$$\begin{gathered} g_{aacc} + 2cg_{abcc} + 2gg_{accc} + c^2 g_{bbcc} + 2cgg_{bccc} + g^2 g_{cccc} + (g_a + cg_b)g_{ccc} - \\ 4g_{abc} - 4cg_{bbc} - cg_c g_{bcc} - 3gg_{bcc} - g_c g_{acc} + 4g_c g_{bc} - 3g_b g_{cc} + 6g_{bb} = 0. \end{gathered} \tag{3}$$

from geometrical point of view was studied by E. Cartan [1].

In fact, according to the expressions for curvature of the space of linear elements $(x,\, y,\, y')$ connected with equation (1)

$$\Omega^1_2 = a[\omega^2 \wedge \omega^2_1]\,, \quad \Omega^0_1 = b[\omega^1 \wedge \omega^2]\,, \quad \Omega^0_2 = h[\omega^1 \wedge \omega^2] + k[\omega^2 \wedge \omega^2_1]\,,$$

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where
$$a=-\frac{1}{6}\frac{\partial^4 f}{\partial y'^4}\,,\quad h=\frac{\partial b}{\partial y'}\,,\quad k=-\frac{\partial \mu}{\partial y'}-\frac{1}{6}\frac{\partial^2 f}{\partial^2 y'}\frac{\partial^3 f}{\partial^3 y'}\,,$$

and
$$\begin{aligned} 6b &= f_{xxy'y'}+2y'f_{xyy'y'}+2ff_{xy'y'y'}+y'^2f_{yyy'y'}+2y'ff_{yy'y'y'}\\ &+ f^2f_{y'y'y'y'}+(f_x+y'f_y)f_{y'y'y'}-4f_{xyy'}-4y'f_{yyy'}-y'f_{y'}f_{yy'y'}\\ &- 3ff_{yy'y'}-f_{y'}f_{xy'y'}+4f_{y'}f_{yy'}-3f_yf_{y'y'}+6f_{yy}\end{aligned}$$

two types of equations by a natural way are evolved: the first type from the condition $a=0$ and the second type from the condition $b=0$.

The first condition $a=0$ determines the equation in form (1) and the second condition leads to the equation (2) where the function $g(a,b,b')$ satisfies the above p.d.e. (3).

From the elementary point of view the relation between both equations (1) and (2) is a result of special properties of their General Integral $F(x,y,a,b)=0$. So we have the following fundamental diagram:

$$\begin{array}{ccccc} & & F(x,y,a,b)=0 & & \\ & \swarrow\nearrow & & \searrow\nwarrow & \\ y''=f(x,y,y') & & & & b''=g(a,b,b') \\ \Updownarrow & & & & \Updownarrow \\ M^3(x,y,y') & & \Longleftrightarrow & & N^3(a,b,b') \end{array}$$

which presents the General Integral $F(x,y,a,b)=0$ (as some 3-dim orbifold) in the form of the twice nontrivial fibre bundles on circles over corresponding surfaces:

$$M^3(x,y,y')=U^2(x,y)\times S^1 \quad \text{and} \quad N^3(a,b,b')=V^2(a,b)\times S^1\,.$$

2 Examples of solutions of dual equation

Let us consider the solutions of equation (3).

It has many types of reductions and the simplest of them are

$$g=c^\alpha\omega[ac^{\alpha-1}],\quad g=c^\alpha\omega[bc^{\alpha-2}],\quad g=c^\alpha\omega[ac^{\alpha-1},bc^{\alpha-2}],$$
$$g=a^{-\alpha}\omega[ca^{\alpha-1}],\quad g=b^{1-2\alpha}\omega[cb^{\alpha-1}],\quad g=a^{-1}\omega(c-b/a),$$
$$g=a^{-3}\omega[b/a,b-ac],\quad g=a^{\beta/\alpha-2}\omega[b^\alpha/a^\beta,c^\alpha/a^{\beta-\alpha}].$$

For any type of reduction we can write the corresponding equation (2) and then integrate it.

For example, for the function $g=a^{-\gamma}A(ca^{\gamma-1})$ we get the equation

$$[A+(\gamma-1)\xi]^2A^{IV}+3(\gamma-2)[A+(\gamma-1)\xi]A^{III}+(2-\gamma)A^IA^{II}+(\gamma^2-5\gamma+6)A^{II}=0.$$

One solution of this equation is

$$A = (2-\gamma)[\xi(1+\xi^2) + (1+\xi^2)^{3/2}] + (1-\gamma)\xi.$$

This solution corresponds to the equation

$$b'' = \frac{1}{a}[b'(1+b'^2) + (1+b'^2)^{3/2}]$$

with the General Integral

$$F(x,y,a,b) = (y+b)^2 + a^2 - 2ax = 0.$$

The dual equation has the form

$$y'' = -\frac{1}{2x}(y'^3 + y').$$

Remark that the first examples of solutions of equation (3) were obtained in [3-6].

Proposition 1. *The equation (3) can be represented in the form*

$$g_{ac} + g g_{cc} - g_c^2/2 + c g_{bc} - 2 g_b = h(a,b,c), \qquad (4)$$
$$h_{ac} + g h_{cc} - g_c h_c + c h_{bc} - 3 h_b = 0.$$

From this it follows that there exists the class of equations (2) with the function $g(a,b,c)$ satisfying the condition

$$g_{ac} + g g_{cc} - g_c^2/2 + c g_{bc} - 2 g_b = 0 \qquad (5)$$

which is easier solved than equation (3).

Here we present some solutions of the equation (5) as functions depending on two variables $g = g(a,c)$

In the case when $g = g(a,c)$ and $h = 0$ we have the equation

$$g_{ac} + g g_{cc} - \frac{1}{2} g_c^2 = 0\,.$$

To integrate this equation we can transform it into a more convenient form using the variable $g_c = f(a,c)$. Then one obtains:

$$2 f_c f_{ac} + (f^2 - 2 f_a) f_{cc} = 0\,.$$

After the Legendre transformation we obtain the equation:

$$[(\xi\omega_\xi + \eta\omega_\eta - \omega)^2 - 2\xi]\omega_{\xi\xi} - 2\eta\omega_{\xi\eta} = 0\,.$$

Using the new variable $\xi\omega_\xi + \eta\omega_\eta - \omega = R$ we have the new equation for R:

$$R_\xi - \frac{1}{2}R^2\omega_{\xi\xi} = 0$$

and the following relations:

$$\omega_\eta = \frac{\omega}{\eta} + \frac{R}{\eta} + \frac{2\xi}{\eta R} - \frac{\xi A(\eta)}{\eta}\,, \qquad \omega_\xi = -\frac{2}{R} + A(\eta)$$

with an arbitrary function $A(\eta)$. From the conditions of compatibility it follows:

$$2\eta R_\eta + R_\xi(2\xi - R^2) + \eta A_\eta R^2 = 0\,.$$

Integrating this equation we can obtain general integral.

In the particular case $A = \frac{1}{\eta}$ we have:

$$\frac{R^2}{R - 2\eta} = -\frac{\xi}{\eta} + \Phi\Big(\frac{1}{\eta} - \frac{2}{R}\Big)\,.$$

By the condition $A = 0$ we obtain the equation $2\eta R_\eta + (2\xi - R^2)R_\xi = 0$, which has the solution:

$$R^2 = 2\xi + 2\eta\Phi(R)\,,$$

were $\Phi(R)$ is an arbitrary function.

After choosing the function $\Phi(R)$ we can find the function ω and then using the inverse Legendre transformation, the function g which determines dual equation $b'' = g(a, c)$.

Remark 1. *The solutions of the equations of type*

$$u_{xy} = uu_{xx} + \varepsilon u_x^2 \tag{6}$$

were constructed in [7]. In the article [8] it was showed that they can be presented in the form

$$u = B'(y) + \int [A(z) - \varepsilon y]^{(1-\varepsilon)/\varepsilon} dz,$$

$$x = -B(y) + \int [A(z) - \varepsilon y]^{1/\varepsilon} dz.$$

To integrate the above equations we can apply the parametric representation

$$u = A(a) + U(a, \tau), \quad y = B(a) + V(a, \tau). \tag{7}$$

Using the formulas

$$u_y = \frac{u_\tau}{y_\tau}, \quad u_x = u_x + u_\tau \tau_x$$

we get after the substitution in (6) the conditions

$$A(x) = \frac{dB}{dx} \qquad \text{and} \qquad U_{x\tau} - \left(\frac{V_x U_\tau}{V_\tau}\right)_\tau + U\left(\frac{U_\tau}{V_\tau}\right)_\tau - \frac{1}{2}\frac{U_\tau^2}{V_\tau} = 0.$$

So we get one equation for two functions $U(x,\tau)$ and $V(x,\tau)$. Any solution of this equation determines the solution of equation (6).

Let us consider some examples.

$$A = B = 0, \quad U = 2\tau - \frac{x\tau^2}{2}, \quad V = x\tau - 2\ln(\tau).$$

Using the representation $U = \tau\omega_\tau - \omega, \quad V = \omega_\tau$ it is possible to obtain other solutions of this equation.

The equation $g_{ac} = gg_{cc} - g_c^2/2$ can be integrated in explicit form and the solutions are

$$g = -H'(a) + \int \frac{dz}{[A(z) + \frac{1}{2}a]^3}\,, \qquad c = H(a) + \int \frac{dz}{[A(z) + \frac{1}{2}a]^2}\,,$$

with arbitrary functions $H(a)$ and $A(z)$.

In fact, for $A(z) = z$ we have

$$g = -H'(a) + \int \frac{dz}{[z + \frac{1}{2}a]^3} = -H'(a) - \frac{1}{2}\frac{1}{[z + \frac{1}{2}a]^2}$$

and

$$c = H(a) + \int \frac{dz}{[z + \frac{1}{2}a]^2} = H(a) - \frac{1}{[z + \frac{1}{2}a]^3}\,.$$

As result we get the solution.

Remark 2. *In general case the equation* $g_{acc} + gg_{ccc} = 0$ *is equivalent to the equation*

$$g_{ac} + gg_{cc} - \frac{1}{2}{g_c}^2 = B(a)\,.$$

It can be integrated with the help of Legendre transformation as in the previous case.

Really, we get

$$[(\xi\omega_\xi + \eta\omega_\eta - \omega)^2 - 2\xi + 2B(\omega_\xi)]\omega_{\xi\xi} - 2\eta\omega_{\xi\eta} = 0$$

and the relation

$$2R_\xi = [R^2 + 2B(\omega_\xi)\omega_{\xi\xi}.$$

It can be written in the form

$$2\frac{dR}{d\Omega} = R^2 + 2B(\Omega)$$

using the notation $\omega_\xi = \Omega$.

Proposition 2. *In the case* $h \neq 0$ *and* $g = g(a,c)$ *the system (3) is equivalent to the equation*

$$\Theta_a\Big(\frac{\Theta_a}{\Theta_c}\Big)_{ccc} - \Theta_c\Big(\frac{\Theta_a}{\Theta_c}\Big)_{acc} = 1 \tag{8}$$

where

$$g = -\frac{\Theta_a}{\Theta_c}, \quad h_c = \frac{1}{\Theta_c}.$$

To integrate this equation we use the presentation $c = \Omega(\Theta, a)$. From the relations

$$1 = \Omega_\Theta \Theta_c, \quad 0 = \Omega_\Theta \Theta_a + \Omega_c$$

we get

$$\Theta_c = \frac{1}{\Omega_\Theta}, \quad \Theta_a = -\frac{\Omega_a}{\Omega_\Theta} \quad \text{and} \quad \frac{\Omega_a}{\Omega_\Theta}(\Omega_a)_{ccc} + \frac{1}{\Omega_\Theta}(\Omega_a)_{cca} = 1.$$

Now we get

$$\Omega_{ac} = \frac{\Omega_{a\Theta}}{\Omega_\Theta} = (\ln \Omega_\Theta)_a = K, \quad \Omega_{acc} = \frac{K_\Theta}{\Omega_\Theta},$$

$$\Omega_{accc} = (\frac{K_\Theta}{\Omega_\Theta})_\Theta \frac{1}{\Omega_\Theta}, \quad (\Omega_{acc})_a = (\frac{K_\Theta}{\Omega_\Theta})_a - \frac{\Omega_a}{\Omega_\Theta}(\frac{K_\Theta}{\Omega_\Theta})_\Theta.$$

As a result the equation (8) takes the form

$$\left[\frac{(\ln \Omega_\Theta)_{a\Theta}}{\Omega_\Theta}\right]_a = \Omega_\Theta \tag{9}$$

and can be integrated by the substitution $\Omega(\Theta, a) = \Lambda_a$. So, we get the equation

$$\Lambda_{\Theta\Theta} = \frac{1}{6}\Lambda_\Theta^3 + \alpha(\Theta)\Lambda_\Theta^2 + \beta(\Theta)\Lambda(\Theta) + \gamma(\Theta) \tag{10}$$

with arbitrary coefficients α, β, γ.

Let us consider the following examples.

1. $\alpha = \beta = \gamma = 0$

The solution of equation (10) is

$$\Lambda = A(a) - 6\sqrt{B(a) - \frac{1}{3}\Theta}$$

and we get

$$c = A' - \frac{3B'}{\sqrt{B - \frac{1}{3}\Theta}} \qquad \text{or} \qquad \Theta = 3B - 27\frac{B'^2}{(c - A')^2}.$$

This solution corresponds to the equation

$$b'' = -\frac{\Theta_a}{\Theta_c} = -\frac{1}{18B'}b'^3 + \frac{A'}{6B'}b'^2 + \left(\frac{B''}{B'} - \frac{A'^2}{6B'}\right)b' + A'' + \frac{A'^3}{18B'} - \frac{A'B''}{B'}$$

cubical in the first derivative b' with arbitrary coefficients $A(a), B(a)$. This equation is equivalent to the equation $b'' = 0$ under a point transformation.

In fact, from the formulas

$$L_1 = \frac{\partial}{\partial y}(a_{4y} + 3a_2a_4) - \frac{\partial}{\partial x}(2a_{3y} - a_{2x} + a_1a_4) - 3a_3(2a_{3y} - a_{2x}) - a_4a_{1x},$$

$$L_2 = \frac{\partial}{\partial x}(a_{1x} - 3a_1a_3) + \frac{\partial}{\partial x}(a_{3y} - 2a_{2x} + a_1a_4) - 3a_2(a_{3y} - 2a_{2x}) + a_1a_{4y}$$

which determine the components of projective curvature of the space of linear elements for the equation in the form

$$y'' + a_1(x,y)y'^3 + 3a_2(x,y)y'^2 + 3a_3(x,y)y' + a_4(x,y) = 0$$

we have

$$a_1(x,y) = \frac{1}{18B'}, \quad a_2(x,y) = -\frac{A'}{18B'}, \quad a_3(x,y) = \frac{A'^2}{18B'} - \frac{B''}{3B'},$$

$$a_4(x,y) = \frac{A'B''}{B'} - \frac{A'^3}{18B'} - A''$$

and conditions $L_1 = 0$, $L_2 = 0$ hold.

This means that our equation determines a projective flat structure in the space of elements (x, y, y').

Remark 3. *The conditions* $L_1 = 0$, $L_2 = 0$ *correspond to the solutions of the equation (3) in the form*

$$g(a,b,b') = A(a,b)b'^3 + 3B(a,b)b'^2 + 3C(a,b)b' + D(a,b).$$

In general case the equation (2) with condition (3) determines the 3-dimensional Einstein-Weyl geometry in the space of linear elements (a, b, b').

For more general classes of the form-invariant equations the notion of dual equation is introduced by analogous way.

For example, for the form-invariant equation of the type

$$P_n(b')b'' - P_{n+3}(b') = 0,$$

where $P_n(b')$ are the polynomials of degree n in b' with coefficients depending on the variables a, b, the dual equation $b'' = g(a,b,b')$ has the right-hand side $g(a,b,b')$ in the form [9]

$$\begin{vmatrix} \psi_{n+4} & \psi_{n+3} & \dots & \psi_4 \\ \psi_{n+5} & \psi_{n+4} & \dots & \psi_5 \\ . & . & \dots & . \\ \psi_{2n+4} & \psi_{2n+3} & \dots & \psi_{n+4} \end{vmatrix} = 0,$$

where the functions ψ_i are determined with the help of the relations

$$4!\psi_4 = -\frac{d^2}{da^2}g_{cc} + 4\frac{d}{da}g_{bc} - g_c(4g_{bc} - \frac{d}{da}g_{cc}) + 3g_bg_{cc} - 6g_{bb},$$

$$i\psi_i = \frac{d}{da}\psi_{i-1} - (i-3)g_c\psi_{i-1} + (i-5)g_b\psi_{i-2}, \quad i > 4.$$

For example, for the equation $2yy'' - y'^4 - y'^2 = 0$ with the solution

$$x = a(t + \sin t) + b, \quad y = a(1 - \cos t)$$

we have the dual equation $b'' = -\tan(b'/2)/a$.

According to the above formulas for $n = 1$ we get

$$4!\psi_4 = \frac{3}{2a^3}\tan\frac{c}{2}(1+\tan^2\frac{c}{2})^3, \qquad 5!\psi_5 = -\frac{15}{4a^4}\tan\frac{c}{2}(1+\tan^2\frac{c}{2})^4,$$

$$6!\psi_6 = \frac{90}{8a^5}\tan\frac{c}{2}(1+\tan^2\frac{c}{2})^5,$$

and the relation

$$\psi_5^2 - \psi_4\psi_6 = 0$$

is satisfied.

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About some equations of the third order with six poles

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Abstract. Investigating ordinary differential equations of the third order on the subject of belonging to P-type (solutions of such equations have no movable critical singular points), Chazy has built an equation (Chazy equation) with 32 coefficients. If these coefficients satisfy the special (S)-system, then Chazy equation belongs to P-type. In this paper we find three solution of the (S)-system and build three classes of Chazy equation of the P-type.

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Having researched nonlinear differential equations of the third order on the subject of belonging to P-type (solutions of such equations have no movable critical singular points), Chazy have obtained the equation [1]

$$w''' = \sum_{k=1}^{6} \frac{(w'-a'_k)(w''-a''_k) + A_k(w'-a'_k)^3 + B_k(w'-a'_k)^2 + C_k(w'-a'_k)}{w-a_k} +$$

$$+Dw'' + Ew' + \prod_{i=1}^{6}(w-a_i)\sum_{k=1}^{6}\frac{F_k}{w-a_k}, \tag{1}$$

32 coefficients of equation (1) $A_k, B_k, C_k, F_k, D, E, a_k$ $(k=\overline{1,6})$ are functions of z.

The aim of this paper is building of three classes of equations (1) of P-type.

Equation (1) is connected quite closely with Painleve equations [2]. Investigation of equation (1) is also connected with the theory of isomonodromy deformation of linear systems, the theory of golonomic quantum fields and nonlinear evolution equations. The necessary and sufficient conditions of belonging of equation (1) to P-type are a system (S) [1], which consists of 31 algebraic and differential equations

$$\sum_{k=1}^{6} A_k = 0, \quad \sum_{k=1}^{6} a_k A_k = 0, \quad \sum_{k=1}^{6} a_k^2 A_k = 0, \tag{2}$$

$$2A_k^2 + \sum_j \frac{A_k - A_j}{a_k - a_j} = 0 \quad (k,j=\overline{1,6};\ j\neq k), \tag{3}$$

$$2D+\sum_{k=1}^{6}(B_k-3a_k'A_k)=0,\quad \sum_{k=1}^{6}F_k=\sum_{k=1}^{6}a_kF_k=\sum_{k=1}^{6}a_k^2F_k=0, \tag{4}$$

$$-(\frac{5}{2}A_k+\sum_j\frac{1}{a_k-a_j})B_k+\sum_j(\frac{1}{2}A_k+\frac{1}{a_k-a_j})B_j=-A_k'+$$

$$+A_k\sum_j\frac{a_k'-a_j'}{a_k-a_j}-3\sum_j A_j\frac{a_k'-a_j'}{a_k-a_j}+\frac{3}{2}A_k\sum_{i=1}^{6}a_i'A_i, \tag{5}$$

$$-(2A_k+\sum_j\frac{1}{a_k-a_j})C_k+\sum_j C_j\frac{1}{a_k-a_j}=B_k^2-B_k'-B_k\sum_j\frac{a_k'-a_j'}{a_k-a_j}-$$

$$-\sum_j\frac{3A_j(a_k'-a_j')^2+2B_j(a_k'-a_j')}{a_k-a_j}+B_kD-E-\sum_j\frac{a_k''-a_j''}{a_k-a_j}, \tag{6}$$

$$-a_k'''-B_kC_k+C_k'+\sum_j\frac{(a_k'-a_j')(a_k''-a_j''-C_k)+A_j(a_k'-a_j')^3}{a_k-a_j}+$$

$$+\sum_j\frac{B_j(a_k'-a_j')^2+C_j(a_k'-a_j')}{a_k-a_j}+E\ a_k'+D(a_k''-C_k)+F_k\ \prod_j(a_k-a_j)=0, \tag{7}$$

where $k,j=\overline{1,6};\ j\neq k$.

Chasy did not investigate the (S) system and therefore did not single out explicitly the classes of equations like (1), which are P-type equations. Prof. N.A. Lukashevich continued the investigation of system (S). In [3] he proved that solution of systems (2), (3) is

$$A_k=-1/a_k\quad (k=\overline{1,6}). \tag{8}$$

The search of solutions of systems (4)-(7) is contained in the papers [4, 5]. Here to simplify calculations we consider the case when a_k $(k=\overline{1,6})$ are constants.

Let us consider system (5). From the relation (4_1) (the first relation of the system (4)) we find

$$D=\sum_{i=1}^{6}(-\frac{1}{2}\ B_i+\frac{3}{2}a_i'A_i). \tag{9}$$

Using relation (9) we rewrite system (5) as

$$-3A_kB_k+\sum_j\frac{B_j-B_k}{a_k-a_j}=-A_k'+\sum_j(A_k-3A_j)\frac{a_k'-a_j'}{a_k-a_j}+A_kD\quad (k,j=\overline{1,6};\ j\neq k). \tag{10}$$

We simplify the sum in the right-hand side of (10) (here we use relations (8))

$$\sum_j \left(-\frac{1}{a_k}+\frac{3}{a_j}\right)\ \frac{a_k'-a_j'}{a_k-a_j}=-\frac{1}{a_k}\ \sum_j \frac{a_k'-a_j'}{a_k-a_j}+3\sum_j \frac{1}{a_j}\frac{a_k'-a_j'}{a_k-a_j}=$$
$$=\frac{2}{a_k}\ \sum_j \frac{a_k'-a_j'}{a_k-a_j}+\frac{3}{a_k}\left(a_k'\sum_j \frac{1}{a_j}-\sum_j \frac{a_j'}{a_j}\right). \tag{11}$$

From equalities (8) it follows that

$$\sigma_1=\sigma_5=0, \tag{12}$$

where $\sigma_1,\ \sigma_5$ are the first and the fifth elementary symmetric polynomials composed of the elements a_k $(k=\overline{1,6})$. Using (12) we get $\sum_j \frac{1}{a_j}=\sum_{i=1}^{6}\frac{1}{a_i}-\frac{1}{a_k}=-\frac{1}{a_k}$. Then expression (11) is

$$\frac{2}{a_k}\sum_j \frac{a_k'-a_j'}{a_k-a_j}-3\frac{a_k'}{a_k^2}-\frac{3}{a_k}\sum_j \frac{a_j'}{a_j}. \tag{13}$$

Using (13) we can write system (10) in the form

$$\frac{3}{a_k}B_k+\frac{B_j-B_k}{a_k-a_j}=-\frac{a_k'}{a_k^2}+\frac{2}{a_k}\sum_j \frac{a_k'-a_j'}{a_k-a_j}-3\frac{a_k'}{a_k^2}-\frac{3}{a_k}\frac{\sigma_6'}{\sigma_6}-\frac{D}{a_k}\quad (k,j=\overline{1,6};\ j\neq k), \tag{14}$$

where $\sigma_6=\prod_{i=1}^{6}a_i$. Let us set

$$B_i=\psi_i-\frac{1}{3}D-2\frac{a_i'}{a_i}-\frac{1}{3}\ \frac{\sigma_6'}{\sigma_6}\quad (i=\overline{1,6}). \tag{15}$$

Then system (15) is

$$\left(\frac{3}{a_k}-\sum_j \frac{1}{a_k-a_j}\right)\ \psi_k+\sum_j \frac{\psi_j}{a_k-a_j}=-3\ (a_k^{-1})'\quad (k,j=\overline{1,6};\ j\neq k), \tag{16}$$

where ψ_k $(k=\overline{1,6})$ are unknown values. A simple calculation shows us that the determinant of the system (16) is equal to zero. Hence according to Kroneker-Capelli criterion for the compatibility of system (16) it is necessary and sufficient that the rank of extended matrix be equal to the rank of matrix of system (16). This condition is true if a_k $(k=\overline{1,6})$ are constants. Then the system (16) has the form

$$\left(\frac{3}{a_k}-\sum_j \frac{1}{a_k-a_j}\right)\ \psi_k+\sum_j \frac{\psi_j}{a_k-a_j}=0\quad (k,j=\overline{1,6};\ j\neq k).$$

Solving this system we obtain

$$\psi_1=\frac{a_1}{a_6}\ \frac{a_2a_3+a_2a_4+a_2a_5+a_2a_6+a_3a_4+a_3a_5+a_3a_6+a_4a_5+a_4a_6+a_5a_6}{a_1a_2+a_1a_3+a_1a_4+a_1a_5+a_2a_3+a_2a_4+a_2a_5+a_3a_4+a_3a_5+a_4a_5}\ \psi_6,$$

$$. \quad . \tag{17}$$

$$\psi_5 = \frac{a_5}{a_6} \; \frac{a_1a_2 + a_1a_3 + a_1a_4 + a_1a_6 + a_2a_3 + a_2a_4 + a_2a_6 + a_3a_4 + a_3a_6 + a_4a_6}{a_1a_2 + a_1a_3 + a_1a_4 + a_1a_5 + a_2a_3 + a_2a_4 + a_2a_5 + a_3a_4 + a_3a_5 + a_4a_5} \; \psi_6.$$

Using elementary symmetric polynomials we rewrite the relations (17) as

$$\psi_i = \frac{\delta_i}{\delta_6} \; \psi_6 \quad (i = \overline{1,5}),$$

where

$$\delta_k = a_k \; (\sigma_2 + a_k^2) \quad (k = \overline{1,6}), \tag{18}$$

ψ_6 is an arbitrary analytical function of z.

Let us set

$$\psi \equiv \psi_k / \delta_k \quad (k = \overline{1,6}). \tag{19}$$

Using the substitution (18), (19), we obtain a solution of the system (5) in the form

$$B_k = a_k(\sigma_2 + a_k^2)\psi - \frac{1}{3} \; D \quad (k = \overline{1,6}), \tag{20}$$

where D is a known function, ψ is an arbitrary analytical function of z. Further we shall use the relations [5]

$$\sum_{k=1}^{6} \frac{a_k^n}{\phi(a_k)} \equiv 0 \quad (n = \overline{0,4}), \quad \sum_{k=1}^{6} \frac{a_k^5}{\phi(a_k)} \equiv 1, \quad \sum_{k=1}^{6} \frac{a_k^6}{\phi(a_k)} \equiv \sigma_1,$$

$$\sum_{k=1}^{6} \frac{a_k^7}{\phi(a_k)} \equiv \sigma_1^2 - \sigma_2, \quad \sum_{k=1}^{6} \frac{a_k^8}{\phi(a_k)} \equiv \sigma_1^3 - 2\sigma_1\sigma_2 + \sigma_3, \tag{21}$$

where $\phi(a_k) \equiv \prod_j \; (a_k - a_j) \; (j \neq k; j,k = \overline{1,6})$, $\sigma_k \quad (k = \overline{1,6})$ is an elementary symmetric polynomial composed of the elements a_k $(k = \overline{1,6})$. For $\sigma_1 = \sigma_5 = 0$ from the identities (21) we have

$$\sum_{k=1}^{6} \frac{a_k^n}{\phi(a_k)} \equiv 0 \quad (n = \overline{0,4}, 6), \quad \sum_{k=1}^{6} \frac{a_k^5}{\phi(a_k)} \equiv 1, \quad \sum_{k=1}^{6} \frac{a_k^7}{\phi(a_k)} \equiv -\sigma_2,$$

$$\sum_{k=1}^{6} \frac{a_k^8}{\phi(a_k)} \equiv \sigma_3, \quad \sum_{k=1}^{6} \frac{a_k^9}{\phi(a_k)} \equiv \sigma_2^2 - \sigma_4, \quad \sum_{k=1}^{6} \frac{a_k^{10}}{\phi(a_k)} \equiv (-2)\sigma_2\sigma_3, \tag{22}$$

$$\sum_{k=1}^{6} \frac{a_k^{11}}{\phi(a_k)} \equiv \sigma_3^2 - \sigma_2^3 + 2\sigma_2\sigma_4 - \sigma_6, \quad \sum_{k=1}^{6} \frac{a_k^{12}}{\phi(a_k)} \equiv 3\sigma_2^2\sigma_3 - 2\sigma_3\sigma_4.$$

Since $a_k' = 0 \quad (k = \overline{1,6})$, the first relation of the system (4) is

$$\sum_{k=1}^{6} B_k = -2D. \tag{23}$$

Hence

$$\sum_{k=1}^{6} B_k = \sigma_2\psi\ \sigma_1 + \psi\sum_{k=1}^{6} a_k^3 - 2D = s_3\psi - 2D. \tag{24}$$

From the equalities (23) and (24) the next theorem follows.

Theorem 1. *System* (4_1)*,* (5) *with respect to constantsa*$_k$ $(k=\overline{1,6})$ *is compatible and it has two solutions:*

1) $B_k = -\frac{1}{3}\ D\ (k=\overline{1,6})$ *(for* $\psi = 0$*),* (25)

or

2) B_k $(k=\overline{1,6})$ *is determined according to (20) (for* $s_3 = 0$*).*

Then we shall find a solution of the systems (6), (7). The system (6) for our case has the form

$$\left(\frac{2}{a_k} - \sum_j \frac{1}{a_k - a_j}\right)\ C_k + \sum_j \frac{C_j}{a_k - a_j} =$$

$$= (\delta_k\psi - \frac{1}{3}\ D)^2 - \delta_k\psi' + \frac{1}{3}D' + \delta_k\psi D - \frac{1}{3}D^2 - E \quad (k,j=\overline{1,6};\ j\neq k). \tag{26}$$

The determinant of system (26) is equal to zero. By setting

$$C_k = \frac{a_k}{3}\ (E - \frac{1}{3}D' + \frac{2}{9}D^2) + \chi_k \quad (k=\overline{1,6}) \tag{27}$$

in the system (26) we get the system

$$\left(\frac{2}{a_k} - \sum_j \frac{1}{a_k - a_j}\right)\ \chi_k + \sum_j \frac{\chi_j}{a_k - a_j} = (\frac{1}{3}D\psi - \psi')\delta_k + \psi^2\delta_k^2 \quad (k,j=\overline{1,6};\ j\neq k). \tag{28}$$

Applying Cramer's rule to the system (28), we find the value of the determinant

$$\begin{vmatrix}
(\frac{1}{3}D\psi-\psi')\delta_1+\psi^2\delta_1^2 & \frac{1}{a_1-a_2} & \frac{1}{a_1-a_3} & \frac{1}{a_1-a_4} & \frac{1}{a_1-a_5} & \frac{1}{a_1-a_6}\\
(\frac{1}{3}D\psi-\psi')\delta_2+\psi^2\delta_2^2 & a_{22} & \frac{1}{a_2-a_3} & \frac{1}{a_2-a_4} & \frac{1}{a_2-a_5} & \frac{1}{a_2-a_6}\\
(\frac{1}{3}D\psi-\psi')\delta_3+\psi^2\delta_3^2 & \frac{1}{a_3-a_2} & a_{33} & \frac{1}{a_3-a_4} & \frac{1}{a_3-a_5} & \frac{1}{a_3-a_6}\\
(\frac{1}{3}D\psi-\psi')\delta_4+\psi^2\delta_4^2 & \frac{1}{a_4-a_2} & \frac{1}{a_4-a_3} & a_{44} & \frac{1}{a_4-a_5} & \frac{1}{a_4-a_6}\\
(\frac{1}{3}D\psi-\psi')\delta_5+\psi^2\delta_5^2 & \frac{1}{a_5-a_2} & \frac{1}{a_5-a_3} & \frac{1}{a_5-a_4} & a_{55} & \frac{1}{a_5-a_6}\\
(\frac{1}{3}D\psi-\psi')\delta_6+\psi^2\delta_6^2 & \frac{1}{a_6-a_2} & \frac{1}{a_6-a_3} & \frac{1}{a_6-a_4} & \frac{1}{a_6-a_5} & a_{66}
\end{vmatrix},$$

where $a_{ii} = \frac{2}{a_i} - \sum_{j\ (j\neq i)} \frac{1}{a_i - a_j}$ $(i=\overline{2,6})$. A simple calculation shows us that this determinant is equal to

$$\frac{12a_1}{\sigma_6}\ (\sigma_3 - a_1\sigma_2 - a_1^3)\ \left(\frac{1}{3}D\psi - \psi' + \sigma_3\psi^2\right).$$

Also this determinant must be equal to zero because the determinant of the system (28) is equal to zero. Hence we obtain the condition for the function ψ

$$\frac{1}{3}D\psi - \psi' + \sigma_3\psi^2 = 0. \tag{29}$$

Now consider the first case when $\psi = 0$. Then we get

$$B_k = -\frac{1}{3}\ D \quad (k = \overline{1,6}). \tag{30}$$

Solving the system

$$\left(\frac{2}{a_k} - \sum_j \frac{1}{a_k - a_j}\right)\ \chi_k + \sum_j \frac{\chi_j}{a_k - a_j} = 0 \quad (k, j = \overline{1,6};\ j \neq k). \tag{31}$$

with unknown functions $\chi_k \ \ (k = \overline{1,6})$, we find

$$\chi_k = \frac{\xi_i}{\xi_6}\ \chi_6 \quad (i = \overline{1,5}), \tag{32}$$

where

$$\xi_1 = a_1(a_2a_3a_4 + a_2a_3a_5 + a_2a_3a_6 + a_2a_4a_5 + a_2a_4a_6 + a_2a_5a_6 + a_3a_4a_5 +$$

$$a_3a_4a_6 + a_3a_5a_6 + a_4a_5a_6) = a_1(\sigma_3 - a_1\sigma_2 - a_1^3),$$

. . . .

$$\xi_6 = a_6(a_1a_2a_3 + a_1a_2a_4 + a_1a_2a_5 + a_1a_3a_4 + a_1a_3a_5 + a_1a_4a_5 + a_2a_3a_4 +$$

$$a_2a_3a_5 + a_2a_4a_5 + a_3a_4a_5) = a_6(\sigma_3 - a_6\sigma_2 - a_6^3),$$

χ_6 is an arbitrary analytical function of z. Let us set

$$\xi_k = a_k(\sigma_3 - a_k\sigma_2 - a_k^3), \quad \chi = \frac{\chi_k}{\xi_k} \quad (k = \overline{1,6}). \tag{33}$$

Using relations (32), (33), for our case we write a solution of the system (6) in the form

$$C_k = \frac{a_k}{3}\ (E - \frac{1}{3}D' + \frac{2}{9}D^2) + \xi_k\ \chi \quad (k = \overline{1,6}), \tag{34}$$

where ξ_k are determined according to formulas (33), χ is any analytical function.

Let us consider the second case, when $s_3 = \sigma_3 = 0$. Taking into account relation (27), we obtain system (26) as

$$\left(\frac{2}{a_k} - \sum_j \frac{1}{a_k - a_j}\right)\ \chi_k + \sum_j \frac{\chi_j}{a_k - a_j} = \psi^2\delta_k^2 \quad (k, j = \overline{1,6};\ j \neq k), \tag{35}$$

where the function ψ, according to (29), has the form

$$\psi = C\ \exp\left(\frac{1}{3}\int D dx\right), \tag{36}$$

(C is an arbitrary constant). We shall seek functions $\chi_k \quad (k=\overline{1,6})$ in the form

$$\chi_k = \psi^2[a_k^3 \left(\frac{1}{3}a_k^4 + \frac{4}{3}\sigma_2 a_k^2 + \sigma_2^2 + \frac{4}{3}\ \sigma_4\right) + \frac{a_k}{3}\ (6\sigma_2\sigma_4 + 2\sigma_6)] + \tilde{\chi_k}\ \ (k=\overline{1,6}). \quad (37)$$

By substituting (37) into the system (35) we obtain the system (31), where functions $\tilde{\chi_k}$ $(k=\overline{1,6})$ are unknown values. Let us consider the solution (32), (33) of the system (31). Because $\sigma_3 = 0$ then solution of the last system is

$$\tilde{\chi_k} = -a_k^2(\sigma_2 + a_k^2)\chi \quad (k=\overline{1,6}),$$

where χ is an arbitrary analytical function. Hence solution of system (35) is

$$\chi_k = \psi^2[a_k^3 \left(\frac{1}{3}a_k^4 + \frac{4}{3}\sigma_2 a_k^2 + \sigma_2^2 + \frac{4}{3}\ \sigma_4\right) + \frac{a_k}{3}\ (6\sigma_2\sigma_4 + 2\sigma_6)] - a_k^2(\sigma_2 + a_k^2)\chi\ \ (k=\overline{1,6}). \quad (38)$$

Taking into account the substitution (38), we obtain

$$C_k = \psi^2[a_k^3 \left(\frac{1}{3}a_k^4 + \frac{4}{3}\sigma_2 a_k^2 + \sigma_2^2 + \frac{4}{3}\ \sigma_4\right) +$$

$$+ \frac{a_k}{3}\ (E - \frac{1}{3}D' + \frac{2}{9}D^2 + 6\sigma_2\sigma_4 + 2\sigma_6)] - a_k^2(\sigma_2 + a_k^2)\chi \quad (k=\overline{1,6}). \quad (39)$$

Thus in the second case we have determined functions C_k $(k=\overline{1,6})$ in the form (39). From Theorem 1 and relations (39) the next theorem follows:

Theorem 2. *System* (6) *with respect to constants* a_k $(k=\overline{1,6})$ *is compatible and it has two solutions:*
1) $C_k = \frac{a_k}{3}(E - \frac{1}{3}D' + \frac{2}{9}D^2) + \xi_k\chi$ *(for* $\psi = 0$*),* (40)
or
2) C_k $(k=\overline{1,6})$ *is determained according to (39) (for* $s_3 = 0$*).*

Now we shall seek a solution of the system (7). By our assumptions this system is

$$-B_kC_k + C_k' - DC_k + F_k \prod_j (a_k - a_j) = 0 \quad (k, j = \overline{1,6};\ j \neq k). \quad (41)$$

Using the notations

$$\phi(a_k) \equiv \prod_j (a_k - a_j) = a_k(6a_k^4 + 4\sigma_2 a_k^2 + 2\sigma_4) \quad (k=\overline{1,6})$$

and (20) in the general case we rewrite system (41) in the form

$$F_k = \frac{(a_k(\sigma_2 + a_k^2)\psi + 2D/3)C_k - C_k'}{\phi(a_k)} \quad (k=\overline{1,6}).$$

Using relations (39) from the last system we find

$$F_k = [3\psi^3 a_k^{10} + 15\sigma_2\psi^3 a_k^8 + \psi(2D\psi - 6\psi' - 9\chi)a_k^7 + \psi^3(9\lambda_1 + 12\sigma_2^2)a_k^6 +$$

$$+2\sigma_2\psi(4D\psi-9\chi-12\psi')a_k^5+(3\psi^3(\lambda_2+3\lambda_1\sigma_2+\gamma)+9\chi'-6D\chi)a_k^4+$$
$$+3\psi(2D\lambda_1\psi-3\sigma_2^2\chi-6\lambda_1\psi')a_k^3+(3\sigma_2\psi^3(\lambda_2+\gamma)+3\sigma_2(3\chi'-2D\chi))a_k^2+ \quad (42)$$
$$+\psi(2D\lambda_2\psi+2D\gamma\psi-3\psi\gamma'-6\lambda_2\psi'-6\gamma\psi')a_k] \ \ (k=\overline{1,6}),$$

where $\gamma \equiv E-\frac{1}{3}D'+\frac{2}{9}D^2$, $\lambda_1 \equiv \sigma_2^2+\frac{4}{3}\sigma_4$, $\lambda_2 \equiv \sigma_2\sigma_4+2\sigma_6$. We find the exact form of the functions F_k $(k=\overline{1,6})$ for each of two cases. Consider the case 1 ($\psi=0$, the functions B_k $(k=\overline{1,6})$ are determined according to relations (30), C_k $(k=\overline{1,6})$ - according to (40)). Taking into account relations (42) we obtain functions F_k $(k=\overline{1,6})$ as

$$F_k=\frac{a_k}{3\phi(a_k)}\ \gamma_1+\frac{\xi_k}{\phi(a_k)}\ (\chi'+\frac{2}{3}D\chi)\ \ (k=\overline{1,6}), \quad (43)$$

where

$$\gamma_1=\frac{1}{3}D''-\frac{2}{3}DD'-E'+\frac{2}{3}DE+\frac{4}{27}D^3. \quad (44)$$

Since equality (5_2) : $\sum_{k=1}^{6}\ F_k=0$ is true, then $\chi'+\frac{2}{3}D\chi=0$ or

$$\chi=C\ \exp(-\frac{2}{3}\int Ddx), \quad (45)$$

where C is an arbitrary constant. From relations (43) and (45) we find

$$F_k=\frac{a_k}{3\phi(a_k)}\ \gamma_1\ \ (k=\overline{1,6}). \quad (46)$$

Thus in the first case equation (1) has the form

$$w'''=\sum_{k=1}^{6}\frac{w'w''-(a_k)^{-1}\ w'^3+B_kw'^2+C_kw'}{w-a_k}+Dw''+Ew'+\prod_{i=1}^{6}(w-a_i)\ \sum_{k=1}^{6}\frac{F_k}{w-a_k}, \quad (47)$$

where B_k $(k=\overline{1,6})$ are determined according to formulas (30), C_k $(k=\overline{1,6})$ - according to formulas (40), F_k $(k=\overline{1,6})$ - according to formulas (46) and χ - according to formula (45).

Consider the second case ($s_3=0$ or, taking into account the relation $s_3=3\sigma_3$, we have $\sigma_3=0$). Functions B_k $(k=\overline{1,6})$ are determined from (20), functions C_k $(k=\overline{1,6})$ are determined from (39). Then relations (22) are

$$\sum_{k=1}^{6}\frac{a_k^n}{\phi(a_k)}\equiv 0\ \ (n=\overline{0,4},6,8,10,12),\ \ \sum_{k=1}^{6}\frac{a_k^5}{\phi(a_k)}\equiv 1,\ \ \sum_{k=1}^{6}\frac{a_k^7}{\phi(a_k)}\equiv -\sigma_2,$$
$$\sum_{k=1}^{6}\frac{a_k^9}{\phi(a_k)}\equiv \sigma_2^2-\sigma_4,\ \ \sum_{k=1}^{6}\frac{a_k^{11}}{\phi(a_k)}\equiv -\sigma_2^3+2\sigma_2\sigma_4-\sigma_6. \quad (48)$$

To find functions $F_k \ \ (k=\overline{1,6})$ we use the identities (48). Substituting values of the functions $F_k \ \ (k=\overline{1,6})$ from (42) in the relation $\sum_{k=1}^{6} F_k = 0$, we obtain

$$3\sigma_2\psi(2D\psi - 6\psi' - 3\chi) = 0. \tag{49}$$

Since $\psi \neq 0$ (otherwise we have the first case), then from (49) it follows:

$$\sigma_2 = 0, \tag{50}$$

or

$$\chi = \frac{2}{3}D\psi - 2\psi'. \tag{51}$$

Substituting the value of function ψ from (36) into (51), we obtain

$$\chi = 0. \tag{52}$$

Substitute (42), (48) and (52) into the relations $\sum_{k=1}^{6} a_k F_k = 0$, $\sum_{k=1}^{6} a_k^2 F_k = 0$. Then the first relation becomes

$$\frac{1}{3}\ \psi^3\ (2D^2 + 3(3E - 3\sigma_2\sigma_4 + 3\sigma_6 - D') = 0, \tag{53}$$

and the second one be the identity. From (53) we find the value of function E

$$E = \frac{1}{9}(3D' - 2D^2) + \sigma_2\sigma_4 - \sigma_6. \tag{54}$$

By substituting (52), (54) into equation (42), we obtain functions $F_k \ \ (k=\overline{1,6})$ as

$$F_k = \frac{a_k^2(\sigma_2 + a_k^2)(a_k^6 + 4\sigma_2 a_k^4 + a_k^2(3\sigma_2^2 + 4\sigma_4) + 7\sigma_2\sigma_4 + \sigma_6)}{3\phi(a_k)}\ \psi^3 \ \ (k=\overline{1,6}). \tag{55}$$

Consider here the subcase $\sigma_2 = 0$. Then from the relation $\sum_{k=1}^{6} a_k F_k = 0$, we find

$$\chi' = \frac{1}{27}(18D\chi + (3D' - 2D^2 - 9E - 9\sigma_6)\psi^3). \tag{56}$$

Substitute (42), (48), (56) and $\sigma_2 = 0$ into the relation $\sum_{k=1}^{6} a_k^2 F_k = 0$, which we can write as

$$9\sigma_4\ \chi\ \psi = 0.$$

From the last equation it follows that $\sigma_4 = 0$ (otherwise we obtain one of two considered above cases: $\psi = 0$ or $\chi = 0$). Then functions $F_k \ \ (k=\overline{1,6})$ have the next form:

$$F_k = \frac{1}{27\phi(a_k)}[a_k\psi(9a_k^9\psi^2 - 27a_k^6\chi + 9a_k^3\sigma_6\psi^2 + \psi(3(D'' - 3E') - 4DD'))]\ (k=\overline{1,6}). \tag{57}$$

Thus, in the second case ($\sigma_3 = 0$) we obtain two equations (47), where $B_k \ \ (k=\overline{1,6})$ are determined by (20), $C_k \ \ (k=\overline{1,6})$ are determined by (39), $F_k \ \ (k=\overline{1,6})$ be (55) or (57). The preceding gives the following two theorems.

Theorem 3. *For constants* a_k $(k = \overline{1,6})$ *the system (7) is compatible and it has three solutions:*
1) F_k $(k = \overline{1,6})$ *are determined by (46) and there are (44), (45) (for* $\psi = 0$*)*
or
2) F_k $(k = \overline{1,6})$ *are determined by (55) and there are (52), (54) (for* $s_3 = 0$*)*
or
3) F_k $(k = \overline{1,6})$ *are determined by (57) and there are (56),* $\sigma_2 = \sigma_4 = 0$ *(for* $s_3 = 0$*).*

Theorem 4. *Differential equations (47), where*
1) B_k $(k = \overline{1,6})$ *are determined by (30),* C_k $(k = \overline{1,6})$ *- by (40),* F_k $(k = \overline{1,6})$ *- by (46),* $\chi = C\ \exp(-\frac{2}{3}\int Ddx)$ *(C is arbitrary constant)*
or
2) B_k $(k = \overline{1,6})$ *are determined by (20),* C_k $(k = \overline{1,6})$ *- by (39),* F_k $(k = \overline{1,6})$ *- by (55) and there are* $\sigma_3 = 0$*, (52), (54)*
or
3) B_k $(k = \overline{1,6})$ *are determined by (20),* C_k $(k = \overline{1,6})$ *- by (39),* F_k $(k = \overline{1,6})$ *- by (57) and there are* $\sigma_2 = \sigma_3 = \sigma_4 = 0$*, (56) belong to P-type.*

By theorem 4 we obtain three classes of differential equations (47) of P-type.

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On rational bases of $GL(2,\mathbb{R})$-comitants of planar polynomial systems of differential equations

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The linear transformations of autonomous planar polynomial systems of differential equations which reduce these systems to the canonical forms with coefficients expressed as rational functions of $GL(2,\mathbb{R})$-comitants and $GL(2,\mathbb{R})$-invariants are established. Such canonical forms for general quadratic and cubic systems are constructed in concrete forms. Using constructed canonical forms for polynomial systems some rational bases of $GL(2,\mathbb{R})$-comitants depending on the coordinates of one vector are obtained.

differential system, comitant, invariant, transformation, canonical form, rational basis.

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Ergodic sets and mixing extensions of topological transformation semigroups

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Abstract. We extend the concept of the ergodic set [1] – [2] from topological transformation groups to topological transformation semigroups. We investigate, in particular, connections between ergodicity, weak ergodicity, topological transitivity and minimality of the Whitney's sum of extensions of topological transformation semigroups.

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1 Basic definitions and notations

In this paper we use terminology and notation generally accepted at present in the theory of topological transformation groups (semigroups). We give only definitions of concepts which are necessary in our opinion; for more detailed discussions the reader is referred to [2] – [5].

A topological transformation semigroup (for short transformation semigroup) is a triple (X, S, π), where X is a nonempty compact Hausdorff topological space with unique uniformity $\mathcal{U}[X]$ (phase space), S is a topological semigroup with the unit element e (phase semigroup) and $\pi : X \times S \to X$ is a continuous mapping satisfying the following conditions:

1) $\forall x \in X,\ \pi(x, e) = x$;

2) $\forall x \in X,\ \forall s, t \in S\ \pi(\pi(x, s), t) = \pi(x, st)$.

We shall refer to (X, S) rather than (X, S, π).

Let (X, S, π) be a transformation semigroup, $s \in S,\ A \subset X$. Usually we shall write π^s for the map $X \to X$ defined by $\pi^s(x) = \pi(x, s)$ $(x \in X)$; $xs = \pi^s(x)$ and $xS = \{\ xs \mid s \in S\ \}$ $(x \in X)$. For $x \in X$ we denote the set $xs^{-1} = \{y \mid y \in X \wedge ys = x\}$ and

$$AS^{-1} = \bigcup_{a \in A, s \in S} as^{-1}.$$

A is called invariant if $AS \subset A$. A is called minimal if $A \neq \emptyset$ and $\overline{xS} = A$ for every $x \in A$. A (X, S) is minimal if the set X is minimal. If for $x \in X$, $\overline{xS}$ is minimal x is called an almost periodic point. We denote by AJ the set of all almost periodic points from A.

An extension (a homomorphism) $\varphi : (X, S, \pi) \to (Y, S, \rho)$ of transformation semigroups is a continuous surjection $\varphi : X \to Y$ such that for $\forall x \in X$ $\forall s \in S$, $\varphi(\pi^s(x)) = \rho^s(\varphi(x))$. A homomorphism $\varphi : (X, S, \pi) \to (Y, S, \rho)$ is called an isomorphism if φ is a homeomorphic map. Let $\varphi : (X, S) \to (Y, S)$, $\psi : (Z, S) \to (Y, S)$ be two extensions. We denote:

$R_{\varphi\psi} = \{(x, z) \mid (x, z) \in X \times Z \wedge \varphi(x) = \psi(z)\}$; $R_\varphi = R_{\varphi\varphi}$; $\Delta(X) = \{(x, x) \mid x \in X\}$;

$$P(R_\varphi) = \bigcap_{\alpha \in \mathcal{U}[X]} \bigcup_{s \in S} \{(x, y) \mid (x, y) \in R_\varphi \wedge (xs, ys) \in \alpha\};$$

$$Q(R_\varphi) = \bigcap_{\alpha \in \mathcal{U}[X]} \overline{\bigcup_{s \in S} \{(x, y) \mid (x, y) \in R_\varphi \wedge (xs, ys) \in \alpha\}}.$$

The set $R_{\varphi\psi}$ is an invariant set in the direct product of (X, S) and (Z, S). Hence are defined the transformation semigroup $(R_{\varphi\psi}, S)$ and the Whitney's sum of the extensions φ and ψ $\eta : (R_{\varphi\psi}, S) \to (Y, S)$, where $\eta(x, y) = \varphi(x) = \psi(y)$ $((x, y) \in R_{\varphi\psi})$.

An extension $\varphi : (X, S) \to (Y, S)$ is called minimal if (X, S) is minimal.

An extension φ is called distal (proximal, regionally distal) if $P(R_\varphi) = \Delta(X)$ ($P(R_\varphi) = R_\varphi$, $Q(R_\varphi) = \Delta(X)$). If Y is a singleton, then the distal (proximal, regionally distal) extension $\varphi : (X, S) \to (Y, S)$ is called transformation semigroup (X, S) distal (proximal, regionally distal).

2 Ergodic transformation semigroups

The transformation semigroup (X, S) is said to be ergodic (weakly ergodic) if $X = \overline{VS^{-1}}$ for any nonempty and open (nonempty, invariant and open) set $V \subset X$. (X, S) is said to be topological transitive if $\overline{xS} = X$ for some $x \in X$.

It is clear that for transformation groups the concepts of the ergodicity and weak ergodicity are the same and every ergodic transformation semigroup is weakly ergodic.

Theorem 1. *If for every nonempty open set $V \subset X$ there exists nonempty, open and invariant set $U \subset VS^{-1}$, then any weakly ergodic transformation semigroup (X, S) is ergodic.*

Proof. Let $V \subset X$ be a nonempty and open set and $U \subset VS^{-1}$ be a nonempty, open and invariant set. Since $X = \overline{US^{-1}} \subset \overline{(VS^{-1})S^{-1}} \subset \overline{VS^{-1}} \subset X$ then $X = \overline{VS^{-1}}$.

Theorem 2. *Let (X, S) be a transformation semigroup. The following assertions are equivalent.*

1) *(X, S) is ergodic.*

2) *X does not contain an invariant closed proper subset with the nonempty interior.*

3) *$X = \overline{US}$ for any nonempty open set $U \subset X$.*

4) *$U \cap Vs \neq \emptyset$ for any nonempty open sets U and V from X and some $s \in S$.*

5) *$Us^{-1} \cap V \neq \emptyset$ for any nonempty open sets U and V from X and some $s \in S$.*

Proof. Suppose 1) holds, $B \subset X$ is a closed and invariant set, $V = int\ B \neq \emptyset$ and $U \subset X$ is nonempty and open. Since $X = \overline{US^{-1}}$ then $V \cap Us^{-1} \neq \emptyset$ for some $s \in S$, hence $U \cap VS \neq \emptyset$. Then $X = \overline{VS}$. $B = X$ because $X = \overline{(int\ B)S} \subset \overline{BS} \subset B \subset X$. We proved 1) $\Longrightarrow$ 2). Suppose 2) holds and $U \subset X$ is nonempty and open. Then $X = \overline{US}$, because X contains the nonempty, closed and invariant subset $\overline{US}$ with the nonempty interior . We proved 2) $\Longrightarrow$ 3). Suppose 3) holds and U and V are nonempty open sets from X. Then $X = \overline{VS}$ and $U \cap VS \neq \emptyset$, hence $U \cap Vs \neq \emptyset$ for some $s \in S$. We proved 3) $\Longrightarrow$ 4). Suppose 4) holds and U and V are nonempty open sets from X. Then $U \cap Vs \neq \emptyset$ for some $s \in S$. Therefore there is an $x \in U$ such that $x \in Vs$. Then $x = ys$ for some $y \in V$, hence $y \in Us^{-1}$ and $Us^{-1} \cap V \neq \emptyset$. We proved 4) $\Longrightarrow$ 5). Suppose 5) holds and $V \subset X$ is a nonempty open set, $x \in X$ and U is an open neighborhood of x. Then $Vs^{-1} \cap U \neq \emptyset$ for some $s \in S$. Therefore $x \in \overline{VS^{-1}}$ and $X = \overline{VS^{-1}}$. We proved 5) $\Longrightarrow$ 1).

It is clear that the minimal transformation semigroup is ergodic and the topological transitive transformation group is ergodic. The following example shows that for the transformation semigroups the notions of ergodicity and weak ergodicity are not the same and topological transitive transformation semigroups are not obligatory ergodic.

Let $S = \{0, 1, 2, 3\}$, $S(\cdot)$ be a discrete semigroup with respect to modulo 4 multiplication, i.e. $s \cdot t = r$ where r is the remainder by the division of the product of the numbers s and t by 4. If $\pi(s,t) = s \cdot t$ $(s,t \in S)$, then (S, S, π) is a topological transitive but not ergodic and not weakly ergodic transformation semigroup. There is also the following general proposition.

Theorem 3. *If $Ss \subset sS$ for $\forall s \in S$, then every topological transitive transformation semigroup (X,S) is weakly ergodic.*

Proof. Let (X,S) be a topological transitive transformation semigroup and U be a nonempty, invariant and open subset of X. And let $X = \overline{xS}$ for some $x \in X$. Then there is $s \in S$ with $xs \in U$. Let $t \in S$. Then $st = tp$ for some $p \in S$. Since $xtp = xst \in U$ then $xt \in US^{-1}$, hence $\overline{xS} \subset \overline{US^{-1}}$ and $X = \overline{US^{-1}}$.

Theorem 4. *Let (X,S) be an ergodic transformation semigroup, X be a metric space. Then (X,S) is topological transitive. Furthermore, there is $M \subset X$ such that $\overline{M} = X$ and for $\forall x \in M$, $\overline{xS} = X$.*

Proof. Let $B = \{V_i \mid i = 1, 2, \ldots\}$ be a countable base of the topology of X and U be any nonempty open subset of X. By ergodicity of (X,S) for every natural number i we have $X = \overline{V_i S^{-1}}$. Let $M = \cap_{i=1}^{\infty} V_i S^{-1}$. By Baire's theorem the set M is nonempty and $\overline{M} = X$. Let $\forall x \in M$. Then $x \in V_i S^{-1}$ for every natural number i. Since $V_k \subset U$ for some natural number k then $x \in US^{-1}$. Whence it follows that $xS \cap U \neq \emptyset$ and $\overline{xS} = X$.

The next example will demonstrate the existence of a weakly ergodic but not topological transitive transformation semigroup with metric phase space.

Let X be a compact metric space, $f: X \to X$ be a constant mapping, S be a semigroup of nonnegative integer numbers by addition, $\pi: X \times S \to X$ be a map by the definition: $\pi(x, s) = f^s(x)$ where f^s is the constant mapping $X \to X$ if $s = 0$ and $f^s = f$ if $s \neq 0$. Then (X, S, π) is weakly ergodic with metric phase space but not topological transitive.

Theorem 5. *A distal and ergodic transformation semigroup* (X, S) *is minimal.*

Proof. By Theorem 4 and Corollary 3 from [4] the transformation semigroup (X, S) is inclosed into some transformation group (X, T) and $E(X, S) = E(X, T)$ where $E(X, S)$ and $E(X, T)$ are Ellis groups of (X, S) and (X, T) accordingly. By the definition of the inclosure of a transformation semigroup into a transformation group and by Theorem 2 it follows that the transformation group (X, T) is ergodic. In this case by Ellis theorem [2] (X, T) is minimal. Since $E(X, S) = E(X, T)$ then (X, S) is minimal, too.

Theorem 6. *Let* $\varphi : (X, S) \to (Y, S)$ *be an extension. If* (X, S) *is ergodic (weakly ergodic), then* (Y, S) *is ergodic (weakly ergodic), too.*

Proof. Let $U \subset Y$ be nonempty and open (invariant, nonempty and open). Since $A_1 = \varphi^{-1}(U) \subset X$ is nonempty and open (invariant, nonempty and open) then $X = \overline{A_1 S^{-1}}$. Because $Y = \varphi(\overline{A_1 S^{-1}}) \subset \overline{\varphi(A_1 S^{-1})} = \overline{\varphi(\varphi^{-1}(U) S^{-1})} \subset \overline{US^{-1}} \subset Y$ then $Y = \overline{US^{-1}}$ and (Y, S) is ergodic (weakly ergodic).

Theorem 7. *Let* $\varphi : (X, S) \to (Y, S)$ *be a proximal extension. If* (Y, S) *is ergodic and* $\overline{XJ} = X$, *then* (X, S) *is ergodic, too.*

Proof. We suppose that A is an invariant and closed subset of X with $V = int\, A \neq \emptyset$ and will prove that $A = X$. First we will prove that the set $B = X \setminus V$ is invariant. It is sufficient to show that $VS^{-1} = V$. For the latter is sufficient to show that $VS^{-1} \subset A$. Let $y \in VS^{-1}$ and $y \notin A$. At this point $Ut_0 \subset V$ and $U \cap A = \emptyset$ for some point $t_0 \in S$ and some neighborhood U of y. There is an almost periodic point x belonging to U. Then $\overline{xS} = \overline{xt_0 S} \subset A$ and consequently $x \in A$. But this contradicts $U \cap A = \emptyset$. The contradiction proved that $VS^{-1} \subset A$. Thus B is an invariant and closed subset of X and $B \neq X$, too. If $B = \emptyset$, then $A = X$. Suppose $B \neq \emptyset$. It is clear the set $Y \setminus \varphi(B)$ is open and $Y \setminus \varphi(B) \subset \varphi(A)$. Suppose that $Y \setminus \varphi(B) = \emptyset$ then $Y = \varphi(B)$. Let $x \in XJ$, then $\varphi(x) = \varphi(b)$ for some $b \in B$ and $\overline{xS} \cap \overline{bS} \neq \emptyset$ by proximality of φ. From the latter we have $x \in \overline{bS}$ by minimality of the set $\overline{xS}$. Hence $x \in B$, $XJ \subset B$ and $X = \overline{XJ} \subset B$. At this point $X = B$. But this contradicts $B \neq X$. Therefore $Y \setminus \varphi(B)$ is nonempty. Thus Y contains an invariant and closed subset $\varphi(A)$ with the nonempty interior. At this point $Y = \varphi(A)$. By the same argument as in the proof of the equality $X = B$ we have that $A = X$.

Corollary 1. *Every proximal transformation semigroup (X, S) with $\overline{XJ} = X$ is ergodic.*

Theorem 8. *Let X, Y be metric spaces and $\varphi : (X, S) \to (Y, S)$ be a distal extension with $YJ = Y$. If (X, S) is ergodic, then it is minimal.*

Proof. By Teorem 4 (X, S) is topological transitive. Since $YJ = Y$ and φ is distal, then $XJ = X$. At this point (X, S) is minimal.

Theorem 9. *Let $\varphi : (X, S) \to (Y, S)$ be a regionally distal extension with (Y, S) minimal. If (X, S) is ergodic, then it is minimal.*

Proof. Let X' be a minimal subset of X, $x \in X$, $x' \in X'$ and $\varphi(x) = \varphi(x')$. Since (X, S) is ergodic, then by Theorem 2 $x\alpha \cap x'\alpha s_\alpha^{-1} \neq \emptyset$ for any open index $\alpha \in \mathcal{U}[X]$ and some $s_\alpha \in S$. Therefore $x_\alpha s_\alpha \in x'\alpha$ for some $x_\alpha \in x\alpha$. Without loss of generality we may suppose that $\lim_\alpha x_\alpha = x$ and $\lim_\alpha x_\alpha s_\alpha = x'$. Since the restriction of φ to X' is an open map and $\lim_\alpha \varphi(x_\alpha)s_\alpha = \varphi(x')$, then for α there is some point $x'_\alpha \in X'$ with $\varphi(x'_\alpha) = \varphi(x_\alpha)$ and $\lim_\alpha x'_\alpha s_\alpha = x'$. Suppose that $\lim_\alpha x'_\alpha = z \in X'$. Then $(x, z) \in Q(R_\varphi)$ and $x = z$ because φ is regionally distal. Therefore $x \in X'$ and $X' = X$.

3 Mixing extensions

We shall say that the pair (φ, ψ) of the extensions $\varphi : (X, S) \to (Y, S)$ and $\psi : (Z, S) \to (Y, S)$ is disjoint (weakly disjoint, mixing, weakly mixing) if $(R_{\varphi\psi}, S)$ is minimal (topological transitive, ergodic, weakly ergodic). The extension $\varphi : (X, S) \to (Y, S)$ is called mixing (weakly mixing) if the pair (φ, φ) is mixing (weakly mixing). We denote the disjointness (weak disjointness) of pair (φ, ψ) by $\varphi \perp \psi$ ($\varphi \mathrel{\widetilde{\perp}} \psi$).

Theorem 10. *Let X, Y, Z be metric spaces, $\varphi : (X, S) \to (Y, S)$ be a distal extension and $\psi : (Z, S) \to (Y, S)$ be an extension with $ZJ = Z$. If the pair (φ, ψ) is mixing, then $\varphi \perp \psi$.*

Proof. Since the projection map $R_{\varphi\psi} \to Z$ is a distal extension $(R_{\varphi\psi}, S) \to (Z, S)$, then $\varphi \perp \psi$ by Theorem 8.

Corollary 2. *Let X, Y, Z be metric spaces, $\varphi : (X, S) \to (Y, S)$ and $\psi : (Z, S) \to (Y, S)$ be distal extensions with $YJ = Y$. If the pair (φ, ψ) is mixing, then $\varphi \perp \psi$.*

Proof. Since ψ is distal and $YJ = Y$, then $ZJ = Z$ and by Theorem 10 $\varphi \perp \psi$.

Corollary 3. *Let X, Y be metric spaces and $\varphi : (X, S) \to (Y, S)$ be a distal extension with $YJ = Y$. If φ is mixing, then it is minimal and it is an isomorphism.*

Corollary 4. *A distal transformation semigroup with metric phase space is trivial if it is mixing.*

Theorem 11. *Let $\varphi : (X, S) \to (Y, S)$ be regionally distal and $\psi : (Z, S) \to (Y, S)$ be minimal. If the pair (φ, ψ) is mixing, then $\varphi \perp \psi$.*

Proof. Since the projection map $R_{\varphi\psi} \to Z$ is a regionally distal extension $(R_{\varphi\psi}, S) \to (Z, S)$, then $\varphi \perp \psi$ by Theorem 9.

Theorem 12. *Let $\varphi : (X, S) \to (Y, S)$ and $\psi : (Z, S) \to (Y, S)$ be regionally distal extensions with (Y, S) minimal. If the pair (φ, ψ) is mixing, then $\varphi \perp \psi$.*

Proof. Since φ and ψ are regionally distal then the Whitney's sum $(R_{\varphi\psi}, S) \to (Y, S)$ of φ and ψ is regionally distal, then $\varphi \perp \psi$ by Theorem 9.

Corollary 5. *Let $\varphi : (X, S) \to (Y, S)$ be regionally distal and (Y, S) be minimal. If φ is mixing, then it is minimal and it is an isomorphism.*

Corollary 6. *A regionally distal transformation semigroup is trivial if it is mixing.*

Let u be a fixed idempotent from a fixed minimal right ideal I of the Ellis enveloping semigroup of a universal minimal transformation semigrup of the class of all minimal transformation semigroups with the fixed phase semigroup S, $\mathcal{E} = Iu$ [5]. Henceforth it is assumed that $\varphi : (X, S) \to (Y, S)$ and $\psi : (Z, S) \to (Y, S)$ are minimal extensions; $x_0 \in Xu$, $y_0 \in Yu$ and $z_0 \in Zu$ such that $\varphi(x_0) = \psi(z_0) = y_0$; $\mathcal{A} = \{p \mid p \in \mathcal{E} \wedge x_0p = x_0\}$, $\mathcal{B} = \{p \mid p \in \mathcal{E} \wedge z_0p = z_0\}$, $\mathcal{F} = \{p \mid p \in \mathcal{E} \wedge y_0p = y_0\}$ are the Ellis groups of (X, S), (Z, S) and (Y, S), respectively [5].

The regionally distal extension $\psi : (Z, S) \to (Y, S)$ is called an RD-factor of the extension $\varphi : (X, S) \to (Y, S)$ if $\varphi = \eta \circ \psi$ for some extension $\eta : (X, S) \to (Z, S)$. The pair (φ, ψ) is called RD-prime if every common RD-factor η of φ and ψ, $\eta \neq \varphi$ and $\eta \neq \psi$, is an isomorphism. The extension φ is called RD-prime if the pair (φ, φ) is RD-prime.

The pair (φ, ψ) is called B-pair if $R_{\varphi\psi} = \overline{R_{\varphi\psi}J}$. The extension φ is called B-extension if $R_{\varphi} = \overline{R_{\varphi}J}$. The transformation semigroup (X, S) is called B-transformation semigroup if $X \times X = \overline{(X \times X)J}$.

Theorem 13. *Every mixing pair of the extensions is RD-prime.*

Proof. Let η be a maximal RD-factor of the mixing pair (φ, ψ), δ is an extension such that $\varphi = \delta \circ \eta$; $\theta = \delta \times id\, Z$; $q : R_{\eta\psi} \to Z$ is a projection map. Since the set $R_{\varphi\psi}$ is ergodic and $\theta(R_{\varphi\psi}) = R_{\eta\psi}$ then by Theorem 6 $R_{\eta\psi}$ is ergodic. Because η is regionally distal then q is regionally distal, too. At this point by Theorem 9 the set $R_{\eta\psi}$ is minimal, hence η is an isomorphism.

Corollary 7. *Every mixing transformation semigroup is RD-prime.*

From theorems 4.4.5 and 4.1.12 from [5] we obtain the following two theorems.

Theorem 14. *For the RD-prime B-pair (φ, ψ) of the extensions the following assertions are valid:*

1) *If $Ss \subset sS$ for any $s \in S$ and $\mathcal{A}\mathcal{B}$ is a group, then the pair (φ, ψ) is weakly mixing.*

2) *If X, Y, Z are metric spaces and $\mathcal{A} = \mathcal{B}$ or $\mathcal{A}$ $(\mathcal{B})$ is an invariant subgroup of $\mathcal{F}$, then $\varphi \mathrel{\widetilde{\perp}} \psi$.*

Theorem 15. *For the RD-prime B-extension φ we have the following assertions:*

1) *If $Ss \subset sS$ for any $s \in S$, then φ is weakly mixing.*

2) *If X and Y are metric spaces, then $\varphi \mathrel{\widetilde{\perp}} \varphi$.*

From Theorems 13 – 15 we obtain the following results.

Theorem 16. *Let (φ, ψ) be a B-pair with the conditions: $\mathcal{A}\mathcal{B}$ is a group; for $\forall s \in S$ $Ss \subset sS$ and for any nonempty and open set $V \subset R_{\varphi\psi}$ there exists a nonempty, open and invariant set $U \subset VS^{-1}$ (in partucular S is a group). Then the following statements are equivalent.*

1) *(φ, ψ) is RD-prime.*

2) *(φ, ψ) is weakly mixing.*

3) *(φ, ψ) is mixing.*

Theorem 17. *Let φ be a B-extension such that $Ss \subset sS$ $(s \in S)$ and for any nonempty and open set $V \subset R_\varphi$ there exists a nonempty, open and invariant set $U \subset VS^{-1}$ (in partucular S is a group). Then the following statements are equivalent.*

1) *φ is RD-prime.*

2) *φ is weakly mixing.*

3) *φ is mixing.*

Theorem 18. *Let S be a group, X, Y, Z be metric spaces and (φ, ψ) be a B-pair such that $\mathcal{A}$ or $\mathcal{B}$ is an invariant subgroup of $\mathcal{F}$. Then the following statements are equivalent.*

1) *(φ, ψ) is RD-prime.*

2) *$\varphi \mathrel{\widetilde{\perp}} \psi$.*

3) *(φ, ψ) is mixing.*

Theorem 19. *Let S be a group, X and Y be metric spaces and φ be a B-extension. Then the following statements are equivalent.*

1) φ *is RD-prime.*

2) $\varphi \; \widetilde{\perp} \; \varphi$.

3) φ *is mixing.*

Corollary 8. *Let* (X, S) *be a RD-prime B-transformation semigroup. Then:*

1) *If for* $\forall s \in S$, $Ss \subset sS$, *then* (X, S) *is weakly mixing.*

2) *If* X *is metric, then* $(X \times X, S)$ *is topological transitive.*

Corollary 9. *Let* S *be a group,* X *be a metric space and* (X, S) *be a B-transformation semigroup. Then the following statements are equivalent.*

1) (X, S) *is RD-prime.*

2) (X, S) *is mixing.*

3) $(X \times X, S)$ *is topological transitive.*

Remark 1. 1) (φ, ψ) *is a B-pair, in particular, if* φ *or* ψ *is a RIC-extension* [5].

2) $\mathcal{A}$ *or* $\mathcal{B}$ *is an invariant subgroup of* $\mathcal{F}$, *in particular, if* φ *or* ψ *is regular* (*an extension* φ *is called regular if for* $\forall (x, y) \in R_\varphi J$ *there exists a homomorphism* $\alpha : (X, S) \to (X, S)$ *such that* $\alpha(x) = y$).

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Linear singular perturbations of hyperbolic-parabolic type

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Abstract. We study the behavior of solutions of the problem $\varepsilon u''(t)+u'(t)+Au(t) = f(t), u(0) = u_0$, $u'(0) = u_1$ in the Hilbert space H as $\varepsilon \to 0$, where A is a linear, symmetric, strong positive operator.

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1 Introduction

Let V and H be the real Hilbert spaces endowed with the norm $||\cdot||$ and $|\cdot|$, respectively, such that $V \subset H$, where the embedding is defined densely and continuously. By $(,)$ denote the scalar prodact in H. Let $A : V \to H$ be a linear, closed, symmetric operator and

$$(Au, u) \geq \omega ||u||^2, \quad \forall u \in V, \quad \omega > 0. \tag{1}$$

In this paper we shall study the behavior of the solutions of the problem

$$\begin{cases} \varepsilon u''(t) + u'(t) + Au(t) = f(t), \quad t > 0, \\ u(0) = u_0, \, u'(0) = u_1 \end{cases} \tag{P_ε}$$

as $\varepsilon \to 0$, where ε is a small positive parameter. Our aim is to show that $u \to v$ as $\varepsilon \to 0$, where v is the solution of the problem

$$\begin{cases} v'(t) + Av(t) = f(t), \quad t > 0 \\ v(0) = u_0. \end{cases} \tag{P_0}$$

The main tool of our approach is the relation between the solutions of the problems (P_ε) and (P_0).

For $k \in \mathbb{N}, p \in [1, \infty)$ and $(a, b) \subset (-\infty, +\infty)$ we denote by $W^{k,p}(a, b; H)$ the usual Sobolev spaces of vectorial distributions $W^{k,p}(a, b; H) = \{f \in D'(a, b; H); f^{(l)} \in L^p(a, b; H), l = 0, 1, \ldots, k\}$ with the norm

$$||f||_{W^{k,p}(a,b;H)} = (\sum_{l=0}^{k} ||f^{(l)}||^p_{L^p(a,b;H)})^{1/p}.$$

For each $k \in \mathbb{N}$, $W^{k,\infty}(a,b;H)$ is the Banach space equipped with the norm

$$||f||_{W^{k,\infty}(a,b;H)} = \max_{0 \le l \le k} ||f^{(l)}||_{L^\infty(a,b;H)}$$

For $s \in \mathbb{R}$, $k \in \mathbb{N}$ and $p \in [1,\infty]$ we denote the following Banach space $W_s^{k,p}(a,b;H) = \{f : (a,b) \to H; f^{(l)}(t)e^{-st} \in L^p(a,b;H)\}$ with the norm

$$\|f\|_{W_s^{k,p}(a,b;H)} = \max_{0 \le l \le k} ||f^{(l)}(\cdot)e^{-st}||_{L^p(a,b;H)}.$$

2 A *priori* estimates for solutions of the problem (P_ε)

In this section we shall prove the *a priori* estimates for the solutions of the problem (P_ε) which are uniform relative to the small values of parameter ε. First of all we shall remind the existence theorems for the solutions of the problems (P_ε) and (P_0).

Theorem A. *[1] For any $T > 0$ suppose that $f \in W^{1,1}(0,T;H), u_0, u_1 \in V$ and the operator A satisfies the condition (1). Then there exists a unique function $u \in C(0,T;H) \cap L^\infty(0,T;V)$ satisfying the problem (P_ε) and the conditions: $Au \in L^\infty(0,T;H)$, $u' \in L^\infty(0,T;V)$, $u'' \in L^\infty(0,T;H)$.*

Theorem B. *[1] If $f \in W^{1,1}(0,T;H), u_0 \in V$ and A satisfies the condition (1), then there exists a unique strong solution $v \in W^{1,\infty}(0,T;H)$ of the problem (P_0) and estimates*

$$|v(t)| \le e^{-\omega t}\Big(|u_0| + \int_0^t e^{\omega\tau}|f(\tau)|d\tau\Big),$$

$$|v'(t)| \le e^{-\omega t}\Big(|Au_0 - f(0)| + \int_0^t e^{\omega\tau}|f'(\tau)|d\tau\Big)$$

are true for $0 \le t \le T$.

Before to prove the estimates for solutions of problem (P_ε) we recall the following well-known lemma.

Lemma A. *[2] Let $\psi \in L^1(a,b)(-\infty < a < b < \infty)$ with $\psi \ge 0$ a. e. on (a,b) and let c be a fixed real constant. If $h \in C([a,b])$ verifies*

$$\frac{1}{2}h^2(t) \le \frac{1}{2}c^2 + \int_a^t \psi(s)h(s)ds, \ \forall t \in [a,b],$$

then

$$|h(t)| \le |c| + \int_a^t \psi(s)ds, \ \forall t \in [a,b]$$

also holds.

Denote by

$$E_1(u,t)=\varepsilon|u'(t)|+|u(t)|+\Big(\varepsilon\Big(Au(t),u(t)\Big)\Big)^{1/2}+\Big(\varepsilon\int_0^t|u'(\tau)|^2d\tau\Big)^{1/2}+$$

$$+\Big(\int_0^t\Big(Au(\tau),u(\tau)\Big)d\tau\Big)^{1/2}.$$

Lemma 1. *Suppose that for any $T>0$ $f\in W^{1,1}(0,T;H),u_0,u_1\in V$ and the operator A satisfies the condition (1). Then there exist positive constants γ and C depending on ω such that for the solutions of the problem (P_ε) the following estimates*

$$E_1(u,t)\le C\Big(E_1(u,0)+\int_0^t\Big|f(\tau)\Big|d\tau\Big),\quad 0\le t\le T, \tag{2}$$

$$E_1(u',t)\le C\Big(E_1(u',0)+\int_0^t|f'(\tau)|d\tau\Big),\quad 0\le t\le T \tag{3}$$

are true.

Proof. Denote by

$$E(u,t)=\varepsilon^2|u'(t)|^2+\frac{1}{2}|u(t)|^2+\varepsilon\Big(Au(t),u(t)\Big)+\varepsilon\int_0^t|u'(\tau)|^2d\tau+$$

$$+\varepsilon\Big(u(t),u'(t)\Big)+\int_0^t\Big(Au(\tau),u(\tau)\Big)d\tau.$$

The direct computations show that for every solution of the problem (P_ε) the following equality

$$\frac{d}{dt}E(u,t)=\Big(f(t),u(t)+2\varepsilon u'(t)\Big) \tag{4}$$

is fulfilled. From (4) it follows that

$$\frac{d}{dt}E(u,t)\le|f(t)|\Big(|u(t)|+2\varepsilon|u'(t)|\Big). \tag{5}$$

As $E(u,t)\ge 0$ and $|u(t)|+2\varepsilon|u'(t)|\le C(E(u,t))^{1/2}$, then from (5) we have

$$\frac{d}{dt}\Big(E(u,t)\Big)\le C\Big|f(t)\Big|\Big(E(u,t)\Big)^{1/2}.$$

Integrating the last inequality we obtain

$$\frac{1}{2}E(u,t)\le\frac{1}{2}E(u,0)+C\int\Big(E(u,\tau)\Big)^{1/2}\Big|f(\tau)\Big|d\tau.$$

From the last inequality using Lemma A we get the estimate

$$\Big(E(u,t)\Big)^{1/2}\le C\Big[\Big(E(u,0)\Big)^{1/2}+\int_0^t\Big|f(\tau)\Big|d\tau\Big]. \tag{6}$$

It is easy to see that there exist positive constants C_0, C_1 such that

$$C_0\Big(E(u,t)\Big)^{1/2} \leq E_1(u,t) \leq C_1\Big(E(u,t)\Big)^{1/2}. \tag{7}$$

Using the inequality (7) from (6) we obtain the estimate (2).

To prove the estimate (3) let us denote by

$$E_h(u,t) = \varepsilon^2|u'(t+h)-u'(t)|^2 + \varepsilon\Big(A(u(t+h)-u(t)), u(t+h)-u(t)\Big)+$$

$$+\frac{1}{2}|u(t+h)-u(t)|^2 + \varepsilon\Big(u'(t+h)-u'(t), u(t+h)-u(t)\Big)+$$

$$\varepsilon\int_0^t |u'(\tau+h)-u'(\tau)|^2 d\tau+$$

$$+\int_0^t \Big(A(u(\tau+h)-u(\tau)), u(\tau+h)-u(\tau)\Big)d\tau,\ h>0, t\geq 0.$$

For any solution of the problem (P_ε) we have

$$\frac{d}{dt}E_h(u,t) = \Big(2\varepsilon(u'(t+h)-u'(t)) + u(t+h)-u(t), f(t+h)-f(t)\Big),\ t\geq 0.$$

Dividing the last equality by h^2 and then passing to the limit as $h\to 0$ we get

$$\frac{d}{dt}E(u',t) = \Big(f'(t), 2\varepsilon u''(t) + u'(t)\Big). \tag{8}$$

Since $u'(0) = u_1, \varepsilon u''(0) = f(0) - u_1 - Au_0$, then the estimate (3) follows from (8) in the same way as the estimate (2) follows from (4). Lemma 1 is proved.

3 Relation between the solutions of the problems (P_ε) and (P_0)

In this section we shall give the relation between the solutions of the problems (P_ε) and (P_0). This relation was inspired by the work [3]. At first we shall prove some properties of the kernel $K(t,\tau)$ of transformation which realizes this connection.

For $\varepsilon > 0$ denote

$$K(t,\tau) = \frac{1}{2\varepsilon\sqrt{\pi}}\Big(K_1(t,\tau) + 3K_2(t,\tau) - 2K_3(t,\tau)\Big),$$

where

$$K_1(t,\tau) = \exp\Big\{\frac{3t-2\tau}{4\varepsilon}\Big\}\lambda\Big\{\frac{2t-\tau}{2\sqrt{\varepsilon t}}\Big\}, \tag{9}$$

$$K_2(t,\tau) = \exp\Big\{\frac{3t+6\tau}{4\varepsilon}\Big\}\lambda\Big(\frac{2t+\tau}{2\sqrt{\varepsilon t}}\Big), \tag{10}$$

$$K_3(t,\tau) = \exp\Big\{\frac{\tau}{\varepsilon}\Big\}\lambda\Big(\frac{t+\tau}{2\sqrt{\varepsilon t}}\Big), \tag{11}$$

and $\lambda(s) = \int_s^\infty e^{-\eta^2}d\eta$.

Lemma 2. *The function $K(t,\tau)$ possesses the following properties:*

(i) $K \in C(\overline{R}_+ \times \overline{R}_+) \cap C^2(R_+ \times R_+)$;

(ii) $K_t(t,\tau) = \varepsilon K_{\tau\tau}(t,\tau) - K_\tau(t,\tau), \quad t > 0, \tau > 0;$

(iii) $\varepsilon K_\tau(t,0) - K(t,0) = 0, \quad t \geq 0$;

(iv) $K(0,\tau) = \dfrac{1}{2\varepsilon} \exp\Big\{ -\dfrac{\tau}{2\varepsilon}\Big\}, \quad \tau \geq 0$;

(v) *For each fixed $t > 0$, there exist constants $C_1(t,\varepsilon) > 0$ and $C_2(t) > 0$ such that*

$$|K(t,\tau)| \leq C_1(t,\varepsilon)\exp\{-C_2(t)\tau/\varepsilon\}, \quad |K_t(t,\tau)| \leq C_1(t,\varepsilon)\exp\{-C_2(t)\tau/\varepsilon\},$$

$$|K_\tau(t,\tau)| \leq C_1(t,\varepsilon)\exp\{-C_2(t)\tau/\varepsilon\}, \quad |K_{\tau\tau}(t,\tau)| \leq C_1(t,\varepsilon)\exp\{-C_2(t)\tau/\varepsilon\}$$

for $\tau > 0$;

(vi) $K(t,\tau) > 0, \quad t \geq 0, \quad \tau \geq 0$;

(vii) *For any $\varphi : [0,\infty) \to H$ continuous on $[0,\infty)$ such that $|\varphi(t)| \leq M\exp\{Ct\}$ for $t \geq 0$, the relation*

$$\lim_{t\to 0} \int_0^\infty K(t,\tau)\varphi(\tau)d\tau = \int_0^\infty e^{-\tau}\varphi(2\varepsilon\tau)d\tau$$

is valid in H for each fixed ε, $0 < \varepsilon \ll 1$;

(viii) $\int_0^\infty K(t,\tau)d\tau = 1, \quad t \geq 0$;

(ix) *Let $\rho : [0,\infty) \to \mathbb{R}, \rho \in C^1[0,\infty), \rho$ and ρ' be increasing functions and $|\rho(t)| \leq Me^{ct}, |\rho'(t)| \leq Me^{ct}$, for $t \in [0,\infty)$. Then there exist positive constants C_1 and C_2 such that*

$$\int_0^\infty K(t,\tau)|\rho(t) - \rho(\tau)|d\tau \leq C_1\sqrt{\varepsilon}e^{C_2 t}, \quad t > 0;$$

(x) *Let $f(t)e^{-Ct}, f'(t)e^{-Ct} \in L^\infty(0,\infty;H)$ with some $C \geq 0$. Then there exist positive constants C_1, C_2 such that*

$$\Big|f(t) - \int_0^\infty K(t,\tau)f(\tau)d\tau\Big|_H \leq C_1\sqrt{\varepsilon}e^{C_2 t}\|f'\|_{L^\infty_C(0,\infty;H)}, \quad t \geq 0, \quad 0 < \varepsilon \ll 1;$$

(xi) *There exists $C > 0$ such that*

$$\int_0^t \int_0^\infty K(\tau,\theta)\exp\Big\{-\frac{\theta}{\varepsilon}\Big\}d\theta d\tau \leq C\varepsilon, \quad t \geq 0, \quad \varepsilon > 0.$$

Proof. The properties **(i)-(iv)** can be verified by direct calculation.

Proof **(v).** From (9), (10) and (11) we have

$$K_t(t,\tau)=\frac{1}{8\pi\varepsilon^2}\Big[3K_1(t,\tau)+9K_2(t,\tau)-6\sqrt{\frac{\varepsilon}{t}}\exp\Big\{-\frac{(t-\tau)^2}{4\varepsilon t}\Big\}\Big], t>0,\tau>0, \quad (12)$$

$$K_\tau(t,\tau)=\frac{1}{4\pi\varepsilon^2}\Big[-K_1(t,\tau)+9K_2(t,\tau)-4K_3(t,\tau)\Big],\quad t>0,\tau>0, \quad (13)$$

$$K_{\tau\tau}(t,\tau)=\frac{1}{8\pi\varepsilon^3}\Big[K_1(t,\tau)+27K_2(t,\tau)-8K_3(t,\tau)-$$
$$-6\sqrt{\frac{\varepsilon}{t}}\exp\Big\{-\frac{(t-\tau)^2}{4\varepsilon t}\Big\}\Big],\quad t>0,\tau>0. \quad (14)$$

As $|\lambda(s)|\le\sqrt{\pi}$ for $s\in\mathbb{R}$ and $|\exp\{s^2\}\lambda(s)|\le C$ for $s\in[0,\infty)$, then

$$\Big|K_1(t,\tau)\Big|\le\exp\Big\{\frac{t-2\tau}{4\varepsilon}\Big\},\quad \tau>0,t>0, \quad (15)$$

$$\Big|K_2(t,\tau)\Big|\le C\exp\Big\{-\frac{(t-\tau)^2}{4\varepsilon t}\Big\}\quad t>0,\tau>0, \quad (16)$$

$$\Big|K_3(t,\tau)\Big|\le C\exp\Big\{-\frac{(t-\tau)^2}{4\varepsilon t}\Big\}\quad t>0,\tau>0. \quad (17)$$

Using (15), (16) and (17) from (12), (13) and (14) we get the estimates from property **(v)**. The property **(v)** is proved.

Proof **(vi).** We shall prove property **(vi)** using the maximum principle for the solutions of equation **(ii)**. It is easy to see that

$$K(t,0)=\frac{1}{\varepsilon\sqrt{\pi}}\Big[2\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big)-\lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big],\quad t\ge 0. \quad (18)$$

We intend to prove that

$$K(t,0)>0,\quad t\ge 0. \quad (19)$$

To this end we consider the function $f(s)=2q(s)-q(s/2)$, where $q(s)=\exp\{s^2\}\lambda(s), s\in[0,\infty)$. Because $K(t,0)=(\sqrt{\varepsilon\pi})^{-1}\exp\{-t/4\varepsilon\}f(\sqrt{t/\varepsilon})$, to prove (19) it is sufficient to show that $f(s)>0$ for $s\in[0,\infty)$. At first we shall prove that $q'(s)<0$ for $s\in[0,\infty)$. Since

$$q'(s)=2sq(s)-1,\ q''(s)=2(2s^2+1)q(s)-2s,\ q'''(s)=(8s^3+12s)q(s)-4(s^2+1)$$

and $\lim_{s\to+\infty}2sq(s)=1$, then $q'(0)=-1$ and $\lim_{s\to+\infty}q'(s)=0$. Suppose that there exists $s_1\in(0,\infty)$ such that $q''(s_1)=0$, i. e. $q(s_1)=s_1(2s_1^2+1)^{-1}$. As $q'''(s_1)=-4(2s_1^2+1)^{-1}$, then s_1 is the point of maximum for $q'(s)$, and $q'(s_1)<0, s_1\in[0,\infty)$ and consequently the function $q(s)$ is decreasing on $(0,\infty)$. Further, we note that

$$f(0)=q(0)=\frac{\sqrt{\pi}}{2},\quad \lim_{s\to+\infty}f(s)=0. \quad (20)$$

Suppose that $s_1 \in (0,\infty)$ is any critical point for function $f(s)$, i. e. $f'(s_1)=0$, then we have: $4s_1q(s_1) - 2^{-1}s_1q(s_1/2) - 3/2 = 0$, from which follows

$$f(s_1) = 2q(s_1) - q\Big(\frac{s_1}{2}\Big) = \frac{3}{s_1} - 6q(s_1). \tag{21}$$

As $q'(s) < 0$ for $s \in (0,\infty)$, then $2s_1q(s_1) < 1$. Hence from (21) it follows that $f(s_1) > 0$. The last condition and conditions (20) permit us to conclude that $f(s) > 0$ for $s \in [0,\infty)$, i. e. $K(t,0) > 0$ for $t \geq 0$. Finally, from **(ii)**, **(iv)**, **(v)** and (18) it follows that the function $V(t,\tau) = \exp\{(t-2\tau)/4\varepsilon\}K(t,\tau)$ is the bounded solution of the problem

$$\begin{cases} V_t(t,\tau) = \varepsilon V_{\tau\tau}(t,\tau), \quad t>0, \tau>0 \\ V(0,\tau) = \dfrac{1}{2\varepsilon}\exp\Big\{-\dfrac{\tau}{\varepsilon}\Big\}, \quad \tau \geq 0, \\ V(t,0) = \dfrac{1}{\varepsilon\sqrt{\pi}} f\Big(\sqrt{\dfrac{t}{\varepsilon}}\Big), \quad t \geq 0, \end{cases} \tag{$P.V$}$$

in $Q_T = \{(t,\tau) : \tau \geq 0, 0 \leq t \leq T\}$, for any $T > 0$. Using the maximum principle for the solutions of problem $(P.V)$ we conclude that $V(t,\tau) > 0$ and consequently $K(t,\tau) > 0$. The property **(vi)** is proved.

Proof **(vii).** For any fixed $C > 0$ and for any fixed $\varepsilon > 0$, we get

$$\int_0^\infty K_2(t,\tau)e^{C\tau}d\tau = \frac{2\varepsilon}{3+2C\varepsilon}\Big[\exp\Big\{C(1+C\varepsilon)t\Big\}\lambda\Big(-\frac{1+2C\varepsilon}{2}\sqrt{\frac{t}{\varepsilon}}\Big) -$$

$$-\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big)\Big] = \frac{2\varepsilon}{3+2C\varepsilon}\Big[\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big)\Big(1-\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\Big) + \int\limits_{-\frac{1+2C\varepsilon}{2}\sqrt{\frac{t}{\varepsilon}}}^{\sqrt{\frac{t}{\varepsilon}}} e^{-\eta^2}d\eta -$$

$$-\Big(1-\exp\Big\{C(1+C\varepsilon)t\Big\}\Big)\lambda\Big(-\frac{1+2C\varepsilon}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big] = O(\sqrt{t}), \quad t \to 0. \tag{22}$$

If $\varphi : [0,\infty) \to H$, and $|\varphi(t)|_H \leq Me^{Ct}, t \geq 0$, then from (22) we have

$$\Big|\int_0^\infty K_2(t,\tau)\varphi(\tau))d\tau\Big|_H \leq M\int_0^\infty K_2(t,\tau)e^{C\tau}d\tau \leq MC(\varepsilon)\sqrt{t}, \quad 0<t\ll 1, \tag{23}$$

for any fixed $\varepsilon > 0$. Similarly as was obtained (22) we get

$$\int_0^\infty K_3(t,\tau)e^{C\tau}d\tau = \frac{\varepsilon}{1+C\varepsilon}\Big[\exp\{C(1+C\varepsilon)t\Big\}\lambda\Big(-\frac{1+2C\varepsilon}{2}\sqrt{\frac{t}{\varepsilon}}\Big) - \lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big] =$$

$$= \frac{\varepsilon}{1+C\varepsilon}\Big[\Big(\exp\Big\{C(1+C\varepsilon)t\Big\}-1\Big)\lambda\Big(-\frac{1+2C\varepsilon}{2}\sqrt{\frac{t}{\varepsilon}}\Big) +$$

$$+\int\limits_{-\frac{1+2C\varepsilon}{2}\sqrt{\frac{t}{\varepsilon}}}^{\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}} e^{-\eta^2}d\eta\Big] = O(\sqrt{t}), \quad t\to 0, \tag{24}$$

for any fixed $\varepsilon > 0$. If $\varphi : [0,\infty) \to H$, and $|\varphi(t)|_H \le M\exp\{Ct\}, t \ge 0$, then from (24) it follows that

$$\Big|\int_0^\infty K_3(t,\tau)\varphi(\tau)d\tau\Big|_H \le M\int_0^\infty K_3(t,\tau)\exp\{C\tau\}d\tau \le C(\varepsilon)M\sqrt{t} \tag{25}$$

for $0 < t \ll 1$. For $\varphi : [0,\infty) \to H$, $\varphi \in C(0,\infty;H)$ and $|\varphi(t)|_H \le M\exp\{Ct\}$, $t \ge 0$, we have

$$\int_0^\infty K_1(t,\tau)\varphi(\tau)d\tau = \exp\Big\{\frac{3t}{4\varepsilon}\Big\}\int_0^\infty \exp\Big\{-\frac{\tau}{2\varepsilon}\Big\}\Big[\lambda\Big(\frac{2t-\tau}{2\sqrt{\varepsilon t}}\Big) - \lambda\Big(-\frac{\tau}{2\sqrt{\varepsilon t}}\Big)\Big]\varphi(\tau)d\tau +$$

$$+\Big(\exp\Big\{\frac{3t}{4\varepsilon}\Big\}-1\Big)\int_0^\infty \exp\Big\{-\frac{\tau}{2\varepsilon}\Big\}\lambda\Big(-\frac{\tau}{2\sqrt{\varepsilon t}}\Big)\varphi(\tau)d\tau +$$

$$+\int_0^\infty \exp\Big\{-\frac{\tau}{2\varepsilon}\Big\}\lambda\Big(-\frac{\tau}{2\sqrt{\varepsilon t}}\Big)\varphi(\tau)d\tau = I_1 + I_2 + I_3. \tag{26}$$

Let us evaluate the integrals I_i, $i = 1,2,3$, from (26). For any fixed $0 < \varepsilon < (2C)^{-1}$ we have

$$\begin{aligned} |I_1|_H &\le M\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\int_0^\infty \exp\Big\{-\frac{\tau}{4\varepsilon}+C\tau\Big\}\int_{-\frac{\tau}{2\sqrt{\varepsilon t}}}^{\frac{2t-\tau}{2\sqrt{\varepsilon t}}} \exp\Big\{-\eta^2\Big\}d\eta d\tau \le \\ &\le \frac{2M}{1-2C\varepsilon}\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\sqrt{\varepsilon t} \le C(\varepsilon)\sqrt{t}, \quad 0<t\ll 1, \end{aligned} \tag{27}$$

and

$$\begin{aligned} |I_2|_H &\le M\Big|\exp\Big\{\frac{3t}{4\varepsilon}\Big\}-1\Big|\sqrt{\pi}\int_0^\infty \exp\Big\{-\frac{\tau}{2\varepsilon}+C\tau\Big\}d\tau \le \\ &\le C(\varepsilon)t, \quad 0<t\ll 1. \end{aligned} \tag{28}$$

At last, let us investigate the behaviour of integral I_3 as $t \to 0$. I_3 can be represented in the form

$$I_3 = \int_0^\infty \exp\Big\{-\frac{\tau}{2\varepsilon}\Big\}\Big[\lambda\Big(-\frac{\tau}{2\sqrt{\varepsilon t}}\Big)-\sqrt{\pi}\Big]\varphi(\tau)d\tau + \sqrt{\pi}\int_0^\infty \exp\Big\{-\frac{\tau}{2\varepsilon}\Big\}\varphi(\tau)d\tau. \tag{29}$$

The first term of the right side of (29) can be evaluated as follows

$$\Big|\int_0^\infty \exp\Big\{-\frac{\tau}{2\varepsilon}\Big\}\Big[\lambda\Big(-\frac{\tau}{2\sqrt{\varepsilon t}}\Big)-\sqrt{\pi}\Big]\varphi(\tau)d\tau\Big|_H \le$$

$$\leq M\int_0^\infty \exp\Big\{-\frac{\tau}{2\varepsilon}+C\tau\Big\}\lambda\Big(\frac{\tau}{2\sqrt{\varepsilon t}}\Big)d\tau =$$

$$=\frac{2M\varepsilon}{1-2C\varepsilon}\Big[\lambda(0)-\exp\Big\{\frac{(1-2C\varepsilon)^2t}{4\varepsilon}\Big\}\lambda\Big(\frac{1-2C\varepsilon}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big] =$$

$$=\frac{2M\varepsilon}{1-2C\varepsilon}\Big[\Big(1-\exp\Big\{\frac{(1-2C\varepsilon)^2t}{4\varepsilon}\Big\}\Big)\lambda(0)+$$

$$+\exp\Big\{\frac{(1-2C\varepsilon)^2t}{4\varepsilon}\Big\}\int\limits_0^{\frac{(1-2C\varepsilon)^2}{2}\sqrt{\frac{t}{\varepsilon}}}\exp\Big\{-\eta^2\Big\}d\eta\Big]\leq C(\varepsilon)\sqrt{t},\quad 0<t\ll 1. \tag{30}$$

From (29) and (30) follows the estimate

$$\Big|I_3-\sqrt{\pi}\int_0^\infty \exp\Big\{-\frac{\tau}{2\epsilon}\Big\}\varphi(\tau)d\tau\Big|_H\leq C(\varepsilon)\sqrt{t},\quad 0<t\ll 1. \tag{31}$$

Hence due to (26), (27), (28) and (31) we have

$$\Big|\int_0^\infty K_1(t,\tau)\varphi(\tau)d\tau-2\varepsilon\sqrt{\pi}\int_0^\infty e^{-\tau}\varphi(2\varepsilon\tau)d\tau\Big|_H\leq C\sqrt{t},\quad 0<t\ll 1, \tag{32}$$

for any fixed ε, $0<\varepsilon\ll 1$. Finally, from (23), (25) and (32) we get the proof of the property **(vii)**.

Proof **(viii).** Integrating by parts we have

$$\int_0^\infty K_1(t,\tau)d\tau = 2\varepsilon\Big[\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big)+\lambda\Big(-\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big],$$

$$\int_0^\infty K_2(t,\tau)d\tau = \frac{2\varepsilon}{3}\Big[\lambda\Big(-\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)-\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big)\Big],$$

$$\int_0^\infty K_3(t,\tau)d\tau = \varepsilon\Big[\lambda\Big(-\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)-\lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big],$$

from which follows the proof of the property **(viii)**.

Proof **(ix).** As ρ is increasing and $|\rho(t)|\leq M\exp(Ct)$, then integrating by parts and using the property **(v)** we get

$$\int_0^\infty K_1(t,\tau)|\rho(t)-\rho(\tau)|d\tau = \exp\Big\{\frac{3t}{4\varepsilon}\Big\}\Big[\int_0^t \exp\Big\{-\frac{\tau}{2\varepsilon}\Big\}\lambda\Big(\frac{2t-\tau}{2\sqrt{\varepsilon t}}\Big)\Big(\rho(t)-\rho(\tau)\Big)d\tau+$$

$$+\int_t^{\infty}\exp\Big\{-\frac{\tau}{2\varepsilon}\Big\}\lambda\Big(\frac{2t-\tau}{2\sqrt{\varepsilon t}}\Big)\Big(\rho(\tau)-\rho(t)\Big)d\tau\Big]=2\varepsilon\Big(\rho(t)-\rho(0)\Big)\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big)+$$

$$+\sqrt{\frac{\varepsilon}{t}}\int_0^{\infty}\exp\Big\{-\frac{(t-\tau)^2}{4\varepsilon t}\Big\}\Big|\rho(t)-\rho(\tau)\Big|d\tau-2\varepsilon\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\times$$

$$\times\int_0^{\infty}\exp\Big\{-\frac{\tau}{2\varepsilon}\Big\}\rho'(\tau)\lambda\Big(\frac{2t-\tau}{2\sqrt{\varepsilon t}}\Big)\,\mathrm{sign}(t-\tau)d\tau. \qquad (33)$$

Similarly can be obtained the equalities

$$\int_0^{\infty}K_2(t,\tau)|\rho(t)-\rho(\tau)|d\tau=-\frac{2\varepsilon}{3}\Big(\rho(t)-\rho(0)\Big)\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big)+$$

$$+\frac{1}{3}\sqrt{\frac{\varepsilon}{t}}\int_0^{\infty}\exp\Big\{-\frac{(t-\tau)^2}{4\varepsilon t}\Big\}\Big|\rho(t)-\rho(\tau)\Big|d\tau+$$

$$+\frac{2\varepsilon}{3}\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\int_0^{\infty}\exp\Big\{\frac{3\tau}{2\varepsilon}\Big\}\rho'(\tau)\lambda\Big(\frac{2t+\tau}{2\sqrt{\varepsilon t}}\Big)\,\mathrm{sign}(t-\tau)d\tau, \qquad (34)$$

and

$$\int_0^{\infty}K_3(t,\tau)|\rho(t)-\rho(\tau)|d\tau=-\varepsilon\Big(\rho(t)-\rho(0)\Big)\lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)+$$

$$+\frac{1}{2}\sqrt{\frac{\varepsilon}{t}}\int_0^{\infty}\exp\Big\{-\frac{(t-\tau)^2}{4\varepsilon t}\Big\}\Big|\rho(t)-\rho(\tau)\Big|d\tau+$$

$$+\varepsilon\int_0^{\infty}\exp\Big\{\frac{\tau}{\varepsilon}\Big\}\rho'(\tau)\lambda\Big(\frac{t+\tau}{2\sqrt{\varepsilon t}}\Big)\,\mathrm{sign}(t-\tau)d\tau, \qquad (35)$$

As a consequence from (33), (34) and (35) we get

$$\int_0^{\infty}K(t,\tau)|\rho(t)-\rho(\tau)|d\tau=\frac{1}{\sqrt{\pi}}\Big[\lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big(\rho(t)-\rho(0)\Big)+$$

$$+\frac{1}{2\sqrt{\varepsilon t}}\int_0^{\infty}\exp\Big\{-\frac{(t-\tau)^2}{4\varepsilon t}\Big\}\Big|\rho(t)-\rho(\tau)\Big|d\tau+$$

$$+\int_0^{\infty}\rho'(\tau)\Big[\exp\Big\{\frac{3t+6\tau}{4\varepsilon}\Big\}\lambda\Big(\frac{2t+\tau}{2\sqrt{\varepsilon t}}\Big)-\exp\Big\{\frac{3t-2\tau}{4\varepsilon}\Big\}\lambda\Big(\frac{2t-\tau}{2\sqrt{\varepsilon t}}\Big)-$$

$$-\exp\Big\{\frac{\tau}{\varepsilon}\Big\}\lambda\Big(\frac{t+\tau}{2\sqrt{\varepsilon t}}\Big)\Big]\,\mathrm{sign}(t-\tau)d\tau\Big], \qquad (36)$$

Since $\rho'(t)$ is increasing and $|\rho'(t)| \leq M\exp(Ct)$, then it follows that

$$\begin{aligned} &\lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big(\rho(t)-\rho(0)\Big) \leq \lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)Mt\exp\{Ct\} \leq \\ &\leq C_1 t\exp\Big\{-\frac{t}{4\varepsilon}+Ct\Big\} \leq C_1\varepsilon\exp\{C_2 t\}, \quad t\geq 0, \quad \varepsilon\leq\frac{1}{8C}. \end{aligned} \tag{37}$$

Further we have

$$\int_0^\infty \exp\Big\{-\frac{(t-\tau)^2}{4\varepsilon t}\Big\}|\rho(t)-\rho(\tau)|d\tau \leq$$

$$\leq M\int_0^\infty \exp\Big\{-\frac{(t-\tau)^2}{4\varepsilon t}+C\max\{t,\tau\}\Big\}|t-\tau|d\tau =$$

$$= 4M\varepsilon t\int\limits_{-\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}}^{\infty}|\eta|\exp\Big\{-\eta^2+C\max\{t,t+2\eta\sqrt{\varepsilon t}\}\Big\}d\eta =$$

$$= 4M\varepsilon t\exp\{Ct\}\Big(\int\limits_{-\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}}^{0}|\eta|\exp\{-\eta^2\}d\eta+\int_0^\infty \eta\exp\Big\{-\eta^2+2C\sqrt{\varepsilon t}\eta\Big\}d\eta\Big) \leq$$

$$\leq C_1\varepsilon t\exp\{C_2 t\}, \quad t\geq 0. \tag{38}$$

As $|\lambda(s)\exp\{s^2\}| \leq C$, for $s\geq 0$, then we have

$$\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\int_0^\infty |\rho'(\tau)|\exp\Big\{\frac{3\tau}{2\varepsilon}\Big\}\lambda\Big(\frac{2t+\tau}{2\sqrt{\varepsilon t}}\Big)d\tau \leq$$

$$\leq M\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\int_0^\infty \exp\Big\{C\tau+\frac{3\tau}{2\varepsilon}\Big\}\lambda\Big(\frac{2t+\tau}{2\sqrt{\varepsilon t}}\Big)d\tau \leq C_1\int_0^\infty \exp\Big\{C\tau-\frac{(t-\tau)^2}{4\varepsilon\tau}\Big\} =$$

$$= C_1\sqrt{\varepsilon t}\exp\{Ct\}\int\limits_{-\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}}^{\infty}\exp\Big\{2C\sqrt{\varepsilon t}\eta-\eta^2\Big\}d\eta \leq C_1\sqrt{\varepsilon t}\exp\Big\{C_2 t\Big\}, \quad t\geq 0. \tag{39}$$

Similarly we get the estimates

$$\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\int_0^\infty \exp\Big\{-\frac{\tau}{2\varepsilon}\Big\}\lambda\Big(\frac{2t-\tau}{2\sqrt{\varepsilon t}}\Big)|\rho'(\tau)|d\tau \leq C_1\sqrt{\varepsilon t}\exp\Big\{C_2 t\Big\}, \quad t\geq 0, \tag{40}$$

and

$$\int_0^\infty \exp\Big\{\frac{\tau}{\varepsilon}\Big\}\lambda\Big(\frac{\tau+t}{2\sqrt{\varepsilon t}}\Big)|\rho'(\tau)|d\tau \leq C_1\sqrt{\varepsilon t}\exp\Big\{C_2 t\Big\}, \quad t\geq 0. \tag{41}$$

Finally from (36) and the estimates (37)-(41) follows the estimate from property (**ix**).

Proof **(x).** From the properties (**viii**) and (**ix**) it follows that

$$\Big|f(t)-\int_0^\infty K(t,\tau)f(\tau)d\tau\Big|_H \leq \int_0^\infty K(t,\tau)|f(t)-f(\tau)|_H\, d\tau \leq$$
$$\leq \int_0^\infty K(t,\tau)\Big|\int_\tau^t |f'(\theta)|_H\, d\theta\Big| \leq M\int_0^\infty K(t,\tau)\,|e^{C\tau}-e^{Ct}|d\tau \leq$$
$$\leq C_1\sqrt{\varepsilon}\, e^{C_2 t}\|f'\|_{L^\infty_C(0,\infty;H)},$$

for $t\geq 0, 0\leq\varepsilon\ll 1$. Property (**x**) is proved.

Proof **(xi).** Denote by $\mathcal{K}(t,\tau)=K(t,\tau)|_{\varepsilon=1}, \mathcal{K}_i(t,\tau)=K_i(t,\tau)|_{\varepsilon=1}, i=1,2,3.$ Then

$$I=\int_0^t\int_0^\infty K(\tau,\theta)\exp\Big\{-\frac{\theta}{\varepsilon}\Big\}d\theta d\tau=\varepsilon\int_0^{\frac{t}{\varepsilon}}\int_0^\infty \mathcal{K}(\tau,\theta)\exp\{-\theta\}d\theta d\tau=$$
$$=\frac{\varepsilon}{2\sqrt{\pi}}\Big(I_1+3I_2-2I_3\Big). \tag{42}$$

As $0<\mathcal{K}_i(\tau,\theta)\leq C\exp\Big\{-\frac{(\tau-\theta)^2}{4\tau}\Big\}, i=2,3,$ then

$$I_i\leq\int_0^{\frac{t}{\varepsilon}}\int_0^\infty\exp\Big\{\frac{(\tau+\theta)^2}{4\tau}\Big\}d\theta d\tau\leq C,\quad t\geq 0, i=2,3. \tag{43}$$

For I_1 we have the estimate

$$I_1=\int_0^{\frac{t}{\varepsilon}}\int_0^\infty\mathcal{K}_1(\tau,\theta)e^{-\theta}d\theta d\tau=\int_0^{\frac{t}{\varepsilon}}\exp\Big\{-\frac{9\tau}{4}\Big\}\int_{-\infty}^{\sqrt{\tau}}\exp\Big\{3\eta\sqrt{\tau}\Big\}\lambda(\eta)d\eta d\tau=$$
$$=\frac{1}{3}\int_0^{\frac{t}{\varepsilon}}\tau^{-1/2}\exp\Big\{\frac{3\tau}{4}\Big\}\lambda(\sqrt{\tau})d\tau-\frac{1}{3}\int_0^{\frac{t}{\varepsilon}}\tau^{-1/2}\lambda(\frac{\sqrt{\tau}}{2})d\tau\leq C,\quad t\geq 0. \tag{44}$$

From (42), (43) and (44) follows the property (**xi**). Lemma 2 is proved.

Now we are ready to establish the relation between the solutions of the problem (P_ε) and the corresponding solutions of the problem (P_0).

Theorem 1. *Let $A: D(A)\subset H\to H$ be a linear and closed operator, $f\in W^{1,\infty}_C(0,\infty;H)$ for some $C\geq 0$. If u is a solution of the problem (P_ε) such that $u\in W^{2,\infty}_C(0,\infty;H)$ with some $C\geq 0$, then the function v_0 which is defined by*

$$v_0(t)=\int_0^\infty K(t,\tau)u(\tau)d\tau$$

satisfies the following conditions:

$$\begin{cases} v_0'(t)+Av_0(t)=F_0(t,\varepsilon), & t>0,\\ v_0(0)=\varphi_\varepsilon, \end{cases} \tag{$P.v_0$}$$

where

$$F_0(t,\varepsilon) = \frac{1}{\sqrt{\pi}}\Big[2\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big) - \lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big]u_1 + \int_0^\infty K(t,\tau)f(\tau)d\tau,$$

$$\varphi_\varepsilon = \int_0^\infty e^{-\tau}u(2\varepsilon\tau)d\tau.$$

Proof. Integrating by parts and using the properties (**i**) – (**iii**) and (**v**) of Lemma 2 we get

$$v_0'(t) = \int_0^\infty K_t(t,\tau)u(\tau)d\tau = \int_0^\infty \Big(\varepsilon K_{\tau\tau}(t,\tau) - K_\tau(t,\tau)\Big)u(\tau)d\tau =$$

$$= \int_0^\infty K(t,\tau)\Big(\varepsilon u''(\tau) + u'(\tau)\Big)d\tau + \varepsilon K(t,0)u_1 - Av_0(t) + \int_0^\infty K(t,\tau)f(\tau)d\tau.$$

Thus $v_0(t)$ satisfies the equation from $(P.v_0)$. From property (**viii**) of Lemma 2 follows the validity of the initial condition of $(P.v_0)$. Theorem 1 is proved.

4 The limit of the solutions of the problem (P_ε) as $\varepsilon \to 0$

In this section we shall study the behavior of the solutions of the problem (P_ε) as $\varepsilon \to 0$.

Theorem 2. *Suppose $f \in W_C^{1,\infty}(0,\infty;H)$, with some $C \geq 0$, $u_0, u_1 \in H, Au_0, Au_1 \in H$ and the operator A satisfies the condition (1). Then*

$$|u(t) - v(t)| \leq C_1 M e^{C_2 t}\sqrt{\varepsilon}, \quad t \geq 0, \quad 0 \leq \varepsilon \ll 1, \tag{45}$$

where u and v are the solutions of the problems (P_ε) and $(P.v)$, respectively,

$$M = |f(0)| + |u_0| + |Au_0| + |u_1| + ||f'||_{L_C^\infty(0,\infty;H)},$$

and C_1 and C_2 are independent of M and ε.

If

$$u_0, Au_0, u_1, f(0) \in V, f \in W_C^{2,\infty}(0,\infty;H), \quad \textit{with some} \quad C \geq 0, \tag{46}$$

then

$$\Big|u'(t) - v'(t) + h\exp\Big\{-\frac{t}{\varepsilon}\Big\}\Big| \leq C_1 M_1 e^{C_2 t}\sqrt{\varepsilon}, \quad t \geq 0, \quad 0 \leq \varepsilon \ll 1, \tag{47}$$

where $h = f(0) - u_1 - Au_0$, $M_1 = |f'(0)| + |Ah| + ||f''||_{L_C^\infty(0,\infty;H)}$, and C_1 and C_2 are independent of M_1 and ε.

If

$$u_0, Au_0, Au_1 \in V, Af \in W_C^{1,\infty}(0,\infty;H), \quad \textit{with some} \quad C \geq 0, \tag{48}$$

then

$$||u(t) - v(t)|| \leq C_1 M_2 e^{C_2 t}\sqrt{\varepsilon}, \quad t \geq 0, \quad 0 \leq \varepsilon \ll 1, \tag{49}$$

where $M_2 = |Af(0)| + |Au_0| + |Au_1| + |A^2u_0| + ||Af'||_{L_C^\infty(0,\infty;H)}$, and C_1 and C_2 are independent of M_2 and ε.

Proof. Under the conditions of the theorem from (3) follows the estimate

$$|u'(t)| \leq CM, \quad t \geq 0. \tag{50}$$

According to Theorem 1 the function w which is defined by

$$w(t) = \int_0^\infty K(t,\tau)u(\tau)d\tau$$

is a solution of the problem

$$\begin{cases} w'(t) + Aw(t) = F(t,\varepsilon), \\ w(0) = w_0, \end{cases} \tag{P.w}$$

where

$$F(t,\varepsilon) = F_0(t,\varepsilon) + \int_0^\infty K(t,\tau)f(\tau)d\tau,$$

$$F_0(t,\varepsilon) = \frac{1}{\sqrt{\pi}}\Big[2\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big) - \lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big]u_1, \quad w_0 = \int_0^\infty e^{-\tau}u(2\varepsilon\tau)d\tau.$$

Using the property (**x**) of Lemma 2 and the estimate (50) we get

$$|u(t) - w(t)| \leq C_1 M e^{c_2 t}\sqrt{\varepsilon}, \quad t \geq 0. \tag{51}$$

Let us denote $R(t) = v(t) - w(t)$, where v is the solution of the problem $(P.v)$ and w is the solution of the problem $(P.w)$. Then $R(t)$ is the solution of the problem

$$\begin{cases} R'(t) + AR(t) = \mathcal{F}(t,\varepsilon), \quad t \geq 0, \\ R(0) = R_0, \end{cases}$$

where $R_0 = u_0 - w_0$ and

$$\mathcal{F}(t,\varepsilon) = f(t) - \int_0^\infty K(t,\tau)f(\tau)d\tau - F_0(t,\varepsilon).$$

As

$$\frac{d}{dt}|R(t)|^2 = -2\Big(AR(t),R(t)\Big) + 2\Big(\mathcal{F}(t,\varepsilon),R(t)\Big) \leq$$
$$\leq -2\omega|R(t)|^2 + 2|\mathcal{F}(t,\varepsilon)||R(t)|, \quad t \geq 0,$$

and hence

$$\frac{1}{2}|R(t)|^2 e^{2\omega t} \leq \frac{1}{2}|R_0|^2 + \int_0^t |\mathcal{F}(\tau,\varepsilon)||R(\tau)|e^{2\omega\tau}d\tau, \quad t \geq 0,$$

then using Lemma A we obtain the estimate

$$|R(t)| \leq e^{-\omega t}\Big(|R_0| + \int_0^t |\mathcal{F}(\tau,\varepsilon)|e^{\omega\tau}d\tau\Big), \quad t \geq 0. \tag{52}$$

From (50) follows the estimate

$$|R_0| \leq \int_0^\infty e^{-\tau}|u(2\varepsilon\tau) - u_0|d\tau \leq \int_0^\infty e^{-\tau}\int_0^{2\varepsilon\tau}|u'(s)|dsd\tau \leq CM\varepsilon \tag{53}$$

for $0 < \varepsilon \ll 1$. Now let us estimate $|\mathcal{F}(t,\varepsilon)|$. Using the property (**x**) of Lemma 2 we have

$$\Big|f(t) - \int_0^\infty K(t,\tau)f(\tau)d\tau\Big| \leq C_1 M\sqrt{\varepsilon}e^{C_2 t}, \quad t \geq 0. \tag{54}$$

As

$$\int_0^t \exp\Big\{\frac{3\tau}{4\varepsilon} + \omega\tau\Big\}\lambda\Big(\sqrt{\frac{\tau}{\varepsilon}}\Big)d\tau = \varepsilon\int_0^{\frac{t}{\varepsilon}} \exp\Big\{\frac{3\tau}{4} + \omega\tau\Big\}\lambda\Big(\sqrt{\tau}\Big)d\tau$$
$$\leq C\int_0^\infty e^{\tau}\lambda(\sqrt{\tau}) \leq C\varepsilon, \quad t \geq 0, \quad 0 < \varepsilon \ll 1,$$

and

$$\int_0^t e^{\omega\tau}\lambda\Big(\frac{1}{2}\sqrt{\frac{\tau}{\varepsilon}}\Big)d\tau \leq C\varepsilon, \quad t \geq 0, \quad 0 < \varepsilon \ll 1,$$

then

$$\int_0^t e^{\omega\tau}|F_0(\tau,\varepsilon)|d\tau \ \leq C\varepsilon|u_1| \leq C\varepsilon M, \quad t \geq 0, \quad 0 < \varepsilon \ll 1. \tag{55}$$

From (54) and (55) follows the estimate

$$\int_0^t e^{\omega\tau}|\mathcal{F}(\tau,\varepsilon)|d\tau \leq C_1 M e^{\omega t}\sqrt{\varepsilon}, \quad t \geq 0, \quad 0 < \varepsilon \ll 1. \tag{56}$$

From (52), using the estimates (53) and (56) we get

$$|R(t)| \leq C_1 M e^{C_2 t}\sqrt{\varepsilon}, \quad t \geq 0, \quad 0 < \varepsilon \ll 1. \tag{57}$$

Finally from estimates (51) and (57) we have

$$|u(t) - v(t)| \leq |u(t) - w(t)| + |R(t)| \leq C_1 M e^{C_2 t}\sqrt{\varepsilon}, \quad t \geq 0, \quad 0 < \varepsilon \ll 1.$$

The estimate (45) is proved.

Let us prove the estimate (47). Denote by $z(t) = u'(t) + h\exp\Big\{-\dfrac{t}{\varepsilon}\Big\}$. If u_0, u_1 and f satisfy the conditions (46) and A satisfies the condition (1), then $z(t)$ is a solution of the problem

$$\begin{cases} \varepsilon z''(t) + z'(t) + Az(t) = f'(t) + \exp\Big\{-\dfrac{t}{\varepsilon}\Big\}h, \quad t \geq 0, \\ z(0) = f(0) - Au_0, \quad z'(0) = 0. \end{cases}$$

According to Theorem 1 the function $w_1(t)$ which is defined by

$$w_1(t) = \int_0^\infty K(t,\tau)z(\tau)d\tau$$

is a solution of the problem

$$\begin{cases} w_1'(t) + Aw_1(t) = \mathcal{F}_1(t,\varepsilon), \quad t \geq 0, \\ w_1(0) = \displaystyle\int_0^\infty \exp\Big\{-\tau\Big\} z(2\varepsilon\tau)d\tau, \end{cases}$$

where

$$\mathcal{F}_1(t,\varepsilon) = \int_0^\infty K(t,\tau)\Big[f'(\tau) - \exp\Big\{-\frac{t}{\varepsilon}\Big\}Ah\Big]d\tau.$$

Further denote by $v_1(t) = v'(t)$, where $v(t)$ is the solution of the problem $(P.v)$. Then $v_1(t)$ is the solution of the problem

$$\begin{cases} v_1'(t) + Av_1(t) = f'(t), \quad t \geq 0, \\ v_1(0) = f(0) - Au_0. \end{cases}$$

Let $R_1(t) = w_1(t) - v_1(t)$. Then $R_1(t)$ is the solution of the problem

$$\begin{cases} R_1'(t) + AR(t) = \mathcal{F}_1(t,\varepsilon) - f'(t), \quad t \geq 0, \\ R_1(0) = \displaystyle\int_0^\infty \exp\Big\{-\tau\Big\}\int_0^{2\varepsilon\tau} z'(\theta)d\theta d\tau. \end{cases}$$

Using Theorem B we obtain the estimate

$$|R_1(t)| \leq e^{-\omega t}\Big(|R_1(0)| + \int_0^t e^{\omega t}|\mathcal{F}_1(\tau,\varepsilon) - f'(\tau)|d\tau\Big), \quad t \geq 0. \tag{58}$$

Using the estimate (3) we get

$$|z'(t)| \leq C_1\Big(|f'(0) + Ah| + \int_0^t \Big|f''(t) - \frac{1}{\varepsilon}\exp\Big\{-\frac{t}{\varepsilon}\Big\}Ah\Big|d\tau\Big) \leq C_1 e^{C_2 t} M_1 \tag{59}$$

for $t \geq 0$. Then from (59) follows the estimate

$$|R(0)| \leq C_1\varepsilon, \quad 0 < \varepsilon \ll 1. \tag{60}$$

Due to the property (**x**) of Lemma 2 we get the estimate

$$|f'(t) - \int_0^\infty K(t,\tau)d\tau| \leq C_1 e^{C_2 t}\sqrt{\varepsilon}||f''||_{L^\infty_C(0,\infty;H)}, \quad t \geq 0, \quad 0 < \varepsilon \ll 1. \tag{61}$$

Further using the property (**xi**) of Lemma 2 we have

$$\Big|\int_0^t\int_0^\infty K(\tau,\theta)\exp\Big\{-\frac{\theta}{\varepsilon}\Big\}Ahd\theta d\tau\Big| \leq C\varepsilon M_1, \quad t \geq 0. \tag{62}$$

Using the estimates (60), (61) and (62) from (58) follows the estimate

$$|R_1(t)| \leq C_1 e^{C_2 t}\sqrt{\varepsilon}M_1, \quad t \geq 0, 0 < \varepsilon \ll 1. \tag{63}$$

From the property (**xi**) of Lemma 2 and the estimates (59) we get

$$|w_1(t) - z(t)| \le \int_0^\infty K(t,\tau)\Big|\int_\tau^t z'(\theta)d\theta\Big|d\tau \le$$

$$\le C_1 e^{C_2 t}\sqrt{\varepsilon}M_1, \quad t \ge 0, 0 < \varepsilon \ll 1. \tag{64}$$

Finally, from the estimates (63) and (64) we obtain

$$|z(t) - v_1(t)| \le |z(t) - w_1(t)| + |R_1(t)| \le C_1 e^{C_2 t}\sqrt{\varepsilon}M_1, \quad t \ge 0, 0 < \varepsilon \ll 1,$$

i. e. the estimate (47).

Let us prove the estimate (49). Denote by $y(t) = Au(t)$, $y_1(t) = Av(t)$. Then under conditions (48) $y(t)$ is the solution of the problem

$$\begin{cases} \varepsilon y''(t) + y'(t) + Ay(t) = Af(t), \quad t \ge 0, \\ y(0) = Au_0, \quad y'(0) = Au_1, \end{cases}$$

and $y_1(t)$ is the solution of the problem

$$\begin{cases} y_1'(t) + Ay_1(t) = Af(t), \\ y_1(0) = Au_0. \end{cases}$$

From (45) follows the estimate

$$|Au(t) - Av(t)| \le C_1 e^{C_2 t}\sqrt{\varepsilon}M_2, \quad t \ge 0, 0 < \varepsilon \ll 1. \tag{65}$$

As from (1) it follows that

$$|Au(t) - Av(t)| \ge \omega||u(t) - v(t)||,$$

then using (65) we obtain the estimate (48). Theorem 2 is proved.

Remark 1. *The relation (47) shows that the function $u'(t)$ possesses the boundary function in the neighborhood of the line $t = 0$. But, if $h = 0$, then the function $u'(t)$ like $u(t)$ does not have a boundary function.*

Finally let us give one simple example. Consider the following initial boundary problems

$$\begin{cases} \varepsilon u_{tt}(x,t) + u_t(x,t) + L(x,\partial_x)u(x,t) = f(x,t), \quad x \in \Omega, t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \overline{\Omega}, \\ u(x,t) = 0, \quad (x,t) \quad \text{on} \quad \partial\Omega \times [0,\infty), \end{cases} \tag{66}$$

$$\begin{cases} v_t(x,t) + L(x,\partial_x)v(x,t) = f(x,t), \quad x \in \Omega, t > 0, \\ v(x,0) = u_0(x), \quad x \in \overline{\Omega}, \\ u(x,t) = 0, \quad (x,t) \quad \text{on} \quad \partial\Omega \times [0,\infty), \end{cases} \tag{67}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega$. The operator

$$L(x, \partial_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\cdot\Big) + a(x)\cdot$$

is uniformly elliptic in $\overline{\Omega}$, i.e. $a, a_{ij} : \overline{\Omega} \to \mathbb{R}$, $a, a_{ij} \in C(\overline{\Omega})$, $a_{ij}(x) = a_{ji}(x)$, and

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \omega|\xi|^2, \quad \xi \in \mathbb{R}^n, x \in \overline{\Omega},$$

where $\omega > 0, a(x) \geq 0$ for $x \in \overline{\Omega}$. Let us put $H = L^2(\Omega), V = H_0^1(\Omega)$. In this conditions the problems (P_ε) and $(P.v)$ represent the functional analytical statement of the problems (66) and (67) respectively, where A is the closure of the operator L in $L^2(\Omega)$. Under suitable conditions on the functions u_0, u_1 and f which follow from conditions (46) and (48) from Theorem 2 for the variational solutions of the problems (66), (67) we get

$$u = v + O(\sqrt{\varepsilon}) \quad \text{in} \quad C(0,T;L^2(\Omega)), \quad \varepsilon \to 0,$$

$$u_t = v_t + h\exp\Big\{-\frac{t}{\varepsilon}\Big\} + O(\sqrt{\varepsilon}) \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)), \quad \varepsilon \to 0,$$

$$u = v + O(\sqrt{\varepsilon}) \quad \text{in} \quad L^\infty(0,T;H_0^1(\Omega)), \quad \varepsilon \to 0,$$

where $h(x) = u_1(x) + L(x,\partial_x)u_0(x) - f(x,0)$.

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