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On overnilpotent radicals of topological rings

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Abstract. For every overnilpotent radical defined on the class of all topological rings every σ-bounded locally bounded topological ring is a subring of some radical topological ring.

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The radical theory of topological rings has developed similarly to the radical theory of discrete rings. Though the presence of topology and the claim for some ideals to be closed in it has made the matter specific. The survey [1] contains rather complete data on radicals of topological rings.

It has been found out that all the overnilpotent radicals defined in the class of all topological rings are rather large. So it was proved in [3] that for every minimal overnilpotent radical defined in the class of all topological rings there exists a radical ring with an identity, and it was proved in [4] that every topological ring is radical for every strictly hereditary overnilpotent radical defined in the class of all topological rings.

The last result became a reason to formulate a hypothesis that for every overnilpotent radical defined in the class of all topological rings every topological ring is a subring of a some radical ring (see [1], the problem I.1.31).

The article contains a particular solution of the problem. Its main result is the theorem asserting that for every overnilpotent radical defined in the class of all topological rings every σ-bounded locally bounded topological ring is a subring of a radical topological ring.

Though the above mentioned result is a step to the positive solution of the hypothesis, one gets less certain that it has the positive general solution while constructing the proof.

1 Remark. *Write:*

1.1. $\mathbb{N}$ *for the set of all naturals;*

1.2. nS *for* $\{\sum\limits_{i=1}^{n} a_i \mid a_i \in S\}$ *where* $n \in \mathbb{N}$ *and* $S \subseteq R$ *and* R *is a ring.*

2 Definition. *In the article:*

2.1. Every topological ring is supposed to be associative and its topology is Hasdorff;

2.2. A radical ρ *defined in the class* $\mathcal{K}$ *of topological rings is said to be overnilpotent if every nilpotent ring* $R \in \mathcal{K}$ *is* ρ*-radical;*

2.3. A subset S of a topological ring (R,τ) is said to be bounded if for every neighbourhood V of zero in it there exists such a neighbourhood U of zero that $U\cdot S = \{u\cdot s | u\in U, s\in S\}\subseteq V$ and $S\cdot U = \{s\cdot u | u\in U, s\in S\}\subseteq V$.

2.4. A topological ring (R,τ) is said to be locally bounded if it contains a bounded zero neighbourhood.

2.5. A topological ring (R,τ) is said to be σ-bounded if R is a union of a countable set of its bounded subsets.

3 Proposition. *Let $\{V_i \mid i = 0,1,\ldots\}$ be a sequence of subsets of the ring R such that $0\in V_i$ and $V_i+V_i\subseteq V_{i-1}$ for every $i\in\mathbb{N}$. Hence $V_n+\sum\limits_{i=1}^{n} V_i\subseteq V_0$ and therefore $\sum\limits_{i=1}^{\infty} V_i\subseteq V_0$.*

The proof can be easily done by the induction on n. □

4 Proposition. *Every locally bounded σ-bounded topological ring (R,τ) is a subring of a locally bounded σ-bounded topological ring $(\tilde R,\tilde\tau)$ with the identity.*

Proof. Write $\mathbb{Z}$ for the ring of integers equipped with the discrete topology and $(\tilde R,\tilde\tau)$ for the semidirect product of topological rings $\mathbb{Z}$ and (R,τ) (see the definition 4.4.2 in [2]). Then $\tilde R = \{(r,k) \mid r\in R,\ k\in\mathbb{Z}\}$,

$(r_1,\ k_1)+(r_2,\ k_2) = (r_1+r_2,\ k_1+k_2)$ and

$(r_1,\ k_1)\cdot(r_2,\ k_2) = (r_1\cdot r_2+k_1\cdot r_2+k_2\cdot r_1,\ k_1\cdot k_2)$.

Hence $\tilde R$ is an associative ring with the identity.

Since the ring $\mathbb{Z}$ is discrete then $R' = \{(r,\ 0) \mid r\in R\}$ is an open subring in $(\tilde R,\tilde\tau)$ and $(R',\tilde\tau|_{R'}$ is topologically isomorphic to the topological ring (R,τ).

Hence $(\tilde R,\tilde\tau)$ is a Hausdorff locally bounded σ-bounded topological ring containing (R,τ) as a subring. □

5 Theorem. *Let ρ be an arbitrary overnilpotent radical defined in the class of all topological rings. If a topological ring (R,τ) is a locally bounded and σ-bounded then there exists a ρ-radical topological ring $(\hat R,\hat\tau)$ such that the topological ring (R,τ) is a subring of the topological ring $(\hat R,\hat\tau)$.*

Proof. Taking into account Proposition 4 assume R to be a ring with the identity e.

5.1. By Theorem 1.6.46 in [2] (R,τ) contains such a bounded neighbourhood U_0 of zero and such a basis $\mathcal{B} = \{V_\omega \mid \omega\in\Omega\}$ of symmetrical neighbourhoods of zero that U_0 is a subsemigroup of a multiplicative group of the ring R and every neighbourhood of zero $V_\omega\in\mathcal{B}$ is an ideal of the semigroup U_0.

5.2. Since a union, a sum and a product of a finite set of bounded sets are bounded in a topological ring (see 1.6.19 and 1.6.22 in [2], then there exists a set $\{\Gamma_i \mid i = 0,1,2,\ldots\}$ of bounded subsets in (R,τ) such that the following assertions hold:

5.2.1. $U_0 \subseteq \Gamma_0$;
5.2.2. $e \in \Gamma_0$;
5.2.3. $-\Gamma_k = \Gamma_k$ and $2^k\Gamma_k \subseteq \Gamma_{k+1}$ for every k.
5.2.4. $\Gamma_k \cdot \Gamma_k \subseteq \Gamma_{k+1}$ for every k.

5.3. For every $\omega \in \Omega$ there exists a sequence $\{U_{i,\ \omega} \mid \ i = 1, 2, \ldots\}$ of neighbourhoods of zero from $\mathcal{B}$ such that the following assertions hold:
5.3.1. $U_{1,\ \omega} + U_{1,\ \omega} \subseteq V_\omega$ for every $\omega \in \Omega$;
5.3.2 $\Gamma_{k+1} \cdot U_{k+1,\ \omega} \subseteq U_{k,\ \omega}$ and $U_{k+1,\ \omega} \cdot \Gamma_{k+1} \subseteq U_{k,\ \omega}$ for every $k \in \mathbb{N}$;
5.3.3. $2^k U_{k+1,\ \omega} \subseteq U_{k,\ \omega}$ for every $k \in \mathbb{N}$.

5.4. Let $X = \{x_2, x_3, \ldots\}$ be a set of variables. Consider the ring $R[X]$ of polynomials over the ring R with the set of variables X which commute with elements of R and each other, i.e. $x_i \cdot x_j = x_j \cdot x_i$ and $r \cdot x_i = x_i \cdot r$ for every $x_i,\ x_j \in X$ and $r \in R$.

Consider the ideal I of the ring $R[X]$ generated by the set $\{x_i^i \ \mid i = 2, 3, \ldots\}$. Let $\hat{R} = R[X]/I$ and $\hat{x}_k = x_k + I$. By identifying the element $r \in R$ with the element $r + I \in \hat{R}$ we may assume that R is a subring of the ring $\hat{R}$ and $\hat{R}$ is a ring of polynomials over R of the set $\{\hat{x}_2, \hat{x}_3, \ldots\}$ commuting with the elements of R and each other and $\hat{x}_k^k = 0$ for every $k \geqslant 2$.

5.5. Given $n \in \mathbb{N}$. Write G_n for a subsemigroup of the multiplicative semigroup of the ring $\hat{R}$ generated by the set $\{e, \hat{x}_2, \ldots, \hat{x}_n\}$. Hence $e \in G_n \subseteq G_{n+1}$ for every $n \in \mathbb{N}$ and $G_1 = \{e\}$.

5.6. Given $n \in \mathbb{N}$ and $\omega \in \Omega$. Write $W_{n,\ \omega}$ for

$$\sum_{i=1}^{\infty} 2^i (U_{n\cdot i+n,\ \omega} \cdot G_{n\cdot i}) + \sum_{s=1}^{\infty} \sum_{j=1}^{s} 2^s \big(\Gamma_{n\cdot s-n} \cdot G_{n\cdot s} \cdot (e - \hat{x}_{n\cdot s+1})^j\big).$$

Prove the set $\hat{\mathcal{B}} = \{W_{k,\ \omega} \mid \ \omega \in \Omega,\ k = 1, 2, \ldots\}$ is a basis of neighbourhoods of zero of some Hausdorff ring topology $\hat{\tau}$ on $\hat{R}$ and $\hat{\tau}\mid_R = \tau$ (i.e. the topological ring $(R,\ \tau)$ is a subring of $(\hat{R},\ \hat{\tau})$).

5.6.1. Since $0 \in U_{k,\ \omega}$ and $0 \in \Gamma_k$ for every $k \in \mathbb{N}$ and $\omega \in \Omega$ then $0 \in W_{n,\ \omega}$ for every n and ω, i.e. the assertion BN1 of Theorem 1.2.5 from [2] holds for the set $\{W_{k,\ \omega} \mid \omega \in \Omega,\ k = 1, 2, \ldots\}$.

5.6.2. Since $2^i M \subseteq 2^{k\cdot i} M$ for every $M \subseteq \hat{R}$ such that $0 \in M$ and naturals i and k then

$$W_{k\cdot n,\ \omega} = \sum_{i=1}^{\infty} 2^i (U_{k\cdot n\cdot i+kn,\ \omega} \cdot G_{k\cdot n\cdot i}) + \sum_{s=1}^{\infty} \sum_{j=1}^{s} 2^s \big(\Gamma_{k\cdot n\cdot s-k\cdot n} \cdot G_{k\cdot n\cdot s} \cdot (e - x_{k\cdot n\cdot s+1})^j\big) \subseteq$$

$$\sum_{i=1}^{\infty} 2^{k\cdot i} (U_{n\cdot k\cdot i+n,\ \omega} \cdot G_{n\cdot k\cdot i}) + \sum_{s=n}^{\infty} \sum_{j=1}^{n} 2^s \big(\Gamma_{n\cdot k\cdot s-n} \cdot G_{n\cdot k\cdot s} \cdot (e - x_{n\cdot k\cdot s+1})^j\big) \subseteq$$

$$\sum_{i=1}^{\infty} 2^i (U_{n\cdot i+n,\ \omega} \cdot G_i) + \sum_{s=1}^{\infty} \sum_{j=1}^{s} 2^s \big(\Gamma_{n\cdot s-n} \cdot G_{n\cdot s} \cdot (e - x_{n\cdot s+1})^j\big) = W_{n,\ \omega}$$

for every $n,\ k \in \mathbb{N}$ and $\omega \in \Omega$ and hence $W_{k\cdot n,\ \omega} \subseteq W_{n,\ \omega} \bigcap W_{k,\ \omega}$, i.e. the assertion BN2 of Theorem 1.2.5 from [2] holds for the set $\{W_{k,\ \omega} \mid \omega \in \Omega,\ k = 1, 2, \ldots\}$.

5.6.3. Since $-U_{i,\ \omega} = U_{i,\ \omega}$ and $-\Gamma_i = \Gamma_i$ for every $i \in \mathbb{N}$ and $\omega \in \Omega$ then $-W_{n,\ \omega} = W_{n,\ \omega}$ for every $n \in \mathbb{N}$ and $\omega \in \Omega$, i.e. the assertion BN3 of Theorem 1.2.5 from [2] also holds for the set $\{W_{k,\ \omega} \mid \omega \in \Omega,\ k = 1, 2, \ldots\}$.

5.6.4. Since $2^{2\cdot k} \geqslant 2^{k+1} = 2^k + 2^k$ for every $k \in \mathbb{N}$ then $W_{2\cdot n,\ \omega} + W_{2\cdot n,\ \omega} =$

$$\sum_{i=1}^{\infty} 2^i (U_{2\cdot n\cdot i+2\cdot n,\ \omega} \cdot G_{2\cdot n\cdot i}) + \sum_{s=1}^{\infty} \sum_{j=1}^{s} 2^s \big(\Gamma_{2\cdot n\cdot s-2\cdot n} \cdot G_{2\cdot n\cdot s} \cdot (e - x_{2\cdot n\cdot s+1})^j\big) +$$

$$\sum_{i=1}^{\infty} 2^i (U_{2\cdot n\cdot i+2n,\ \omega} \cdot G_{2\cdot n\cdot i}) + \sum_{s=1}^{\infty} \sum_{j=1}^{s} 2^s \big(\Gamma_{2\cdot n\cdot s-2\cdot n} \cdot G_{2\cdot n\cdot s} \cdot (e - x_{2\cdot n\cdot s+1})^j\big) =$$

$$\sum_{i=1}^{\infty} (2^i + 2^i)(U_{2\cdot n\cdot i+2\cdot n,\ \omega} \cdot G_{2\cdot n\cdot i}) + \sum_{s=1}^{\infty} \sum_{j=1}^{s} (2^s + 2^s)\big(\Gamma_{2\cdot n\cdot s-2n} \cdot G_{2\cdot n\cdot s} \cdot (e - x_{2\cdot n\cdot s+1})^j\big) \subseteq$$

$$\sum_{i=1}^{\infty} 2^{2\cdot i} (U_{2\cdot n\cdot i+n,\ \omega} \cdot G_{2\cdot n\cdot i}) + \sum_{s=1}^{\infty} \sum_{j=1}^{2\cdot s} 2^{2\cdot s} \big(\Gamma_{2\cdot n\cdot s-n} \cdot G_{2\cdot n\cdot s} \cdot (e - x_{2\cdot n\cdot s+1})^j\big) \subseteq$$

$$\sum_{j=1}^{\infty} 2^j (U_{n\cdot j+n,\ \omega} \cdot G_{n\cdot j}) + \sum_{t=1}^{\infty} \sum_{j=1}^{t} 2^t \big(\Gamma_{n\cdot t-n} \cdot G_{n\cdot t} \cdot (e - x_{n\cdot t+1})^j\big) = W_{n,\ \omega}$$

for every $n \in \mathbb{N}$, i.e. the assertion BN4 of Theorem 1.2.5 from [2] holds for the set $\{W_{k,\ \omega} \mid \omega \in \Omega,\ k = 1, 2, \ldots\}$.

5.6.5. Since $r \cdot x_i = x_i \cdot r$ and $x_i \cdot x_j = x_j \cdot x_i$ for every $r \in R$ and $i,\ j \in \mathbb{N}$ then

$$W_{4\cdot n,\ \omega} \cdot W_{4\cdot n,\ \omega} = \Big(\sum_{i=1}^{\infty} 2^i (U_{4\cdot n\cdot i+4\cdot n,\ \omega} \cdot G_{4\cdot n\cdot i}) +$$

$$\sum_{s=1}^{\infty} \sum_{j=1}^{s} 2^s \big(\Gamma_{4\cdot n\cdot s-4\cdot n} \cdot G_{4\cdot n\cdot s} \cdot (e - x_{4\cdot n\cdot s+1})^j\big)\Big) \times$$

$$\Big(\sum_{i=1}^{\infty} 2^i (U_{4\cdot n\cdot i+4\cdot n,\ \omega} \cdot G_{4\cdot n\cdot i}) + \sum_{s=1}^{\infty} \sum_{j=1}^{s} 2^s \big(\Gamma_{4\cdot n\cdot s-4\cdot n} \cdot G_{4\cdot n\cdot s} \cdot (e - x_{4\cdot n\cdot s+1})^j\big)\Big) =$$

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (2^i \cdot 2^j)\big((U_{4\cdot n\cdot i+4\cdot n,\ \omega} \cdot U_{4\cdot n\cdot j+4\cdot n,\ \omega}) \cdot (G_{4\cdot n\cdot i} \cdot G_{4\cdot n\cdot j})\big) +$$

$$\sum_{i=1}^{\infty} \sum_{s=1}^{\infty} \sum_{j=1}^{s} (2^i \cdot 2^s)\big((U_{4\cdot n\cdot i+4\cdot n,\ \omega} \cdot \Gamma_{4\cdot n\cdot s-4\cdot n}) \cdot (G_{4\cdot n\cdot i} \cdot G_{4\cdot n\cdot s} \cdot (e - x_{4\cdot n\cdot s+1})^j)\big) +$$

$$\sum_{i=1}^{\infty}\sum_{s=1}^{\infty}\sum_{j=1}^{s}(2^s\cdot 2^i)\big((\Gamma_{4\cdot n\cdot s-4\cdot n}\cdot U_{4\cdot n\cdot i+4\cdot n,\ \omega})\cdot(G_{4\cdot n\cdot s}\cdot G_{4\cdot n\cdot i})\cdot(e-x_{4\cdot s+1})^j\big)+$$

$$\sum_{s=1}^{\infty}\sum_{j=1}^{s}\sum_{t=1}^{\infty}\sum_{i=1}^{t}(2^s\cdot 2^t)\big((\Gamma_{4\cdot n\cdot s-4\cdot n}\cdot\Gamma_{4\cdot n\cdot t-4\cdot n})\times$$

$$(G_{4\cdot n\cdot s}\cdot G_{4\cdot n\cdot t})\cdot(e-x_{4\cdot n\cdot s+1})^j\cdot(e-x_{4\cdot n\cdot t+1})^i\big)=$$

$$\sum_{i=1}^{\infty}\sum_{j=1}^{i}(2^{i+j})\big((U_{4\cdot n\cdot i+4\cdot n,\ \omega}\cdot U_{4\cdot n\cdot j+4\cdot n,\ \omega})\cdot(G_{4\cdot n\cdot i}\cdot G_{4\cdot n\cdot j})\big)+$$

$$\sum_{j=2}^{\infty}\sum_{i=1}^{j-1}2^{i+j}\big((U_{4\cdot n\cdot i+4\cdot n,\ \omega}\cdot U_{4\cdot n\cdot j+4\cdot n,\ \omega})\cdot(G_{4\cdot n\cdot i}\cdot G_{4\cdot n\cdot j})\big)+$$

$$\sum_{i=1}^{\infty}\sum_{s=1}^{i-1}\sum_{j=1}^{s}2^{i+s}\Big((U_{4\cdot n\cdot i+4\cdot n,\ \omega}\cdot\Gamma_{4\cdot n\cdot s-4\cdot n})\cdot\big(G_{4\cdot n\cdot i}\cdot G_{4\cdot n\cdot s}\big)\cdot(e-x_{4\cdot n\cdot s+1})^j\Big)+$$

$$\sum_{s=1}^{\infty}\sum_{i=1}^{s}\sum_{j=1}^{s}2^{s+i}\big((U_{4\cdot n\cdot i+4\cdot n,\ \omega}\cdot\Gamma_{4\cdot n\cdot s-4\cdot n})\cdot(G_{4\cdot n\cdot i}\cdot G_{4\cdot n\cdot s})\cdot(e-x_{4\cdot n\cdot s+1})^j\big)+$$

$$\sum_{i=1}^{\infty}\sum_{s=1}^{i-1}\sum_{j=1}^{s}2^{s+i}\big((\Gamma_{4\cdot n\cdot s-4\cdot n}\cdot U_{4\cdot n\cdot i+4\cdot n,\ \omega}\cdot(G_{4\cdot n\cdot s}\cdot G_{4\cdot n\cdot i})\cdot(e-x_{4\cdot n\cdot s+1})^j\big)+$$

$$\sum_{s=1}^{\infty}\sum_{i=1}^{s}\sum_{j=1}^{s}2^{s+i}\big((\Gamma_{4\cdot n\cdot s-4\cdot n}\cdot U_{4\cdot n\cdot i+4\cdot n,\ \omega})\cdot(G_{4\cdot n\cdot s}\cdot G_{4\cdot n\cdot i})\cdot(e-x_{4\cdot n\cdot s+1})^j\big)+$$

$$\sum_{s=1}^{\infty}\sum_{j=1}^{s}\sum_{t=1}^{s-1}\sum_{i=1}^{t}2^{s+t}\Big(\Gamma_{4\cdot n\cdot s-4\cdot n}\cdot\Gamma_{4\cdot n\cdot t-4\cdot n}\Big)\times$$

$$\big(G_{4\cdot n\cdot s}\cdot G_{4\cdot n\cdot t}\cdot(e-x_{4\cdot n\cdot t+1})^i\cdot(e-x_{4\cdot n\cdot s+1})^j\big)\Big)+$$

$$\sum_{t=1}^{\infty}\sum_{j=1}^{t}\sum_{i=1}^{t}2^{t+t}\big((\Gamma_{4\cdot n\cdot t-4\cdot n}\cdot\Gamma_{4\cdot n\cdot t-4\cdot n})\cdot((G_{4\cdot n\cdot t}\cdot G_{4\cdot n\cdot t})\cdot(e-x_{4\cdot n\cdot t+1})^{j+i}\big)+$$

$$\sum_{t=1}^{\infty}\sum_{j=1}^{t}\sum_{s=1}^{t-1}\sum_{i=1}^{s}2^{s+t}\Big((\Gamma_{4\cdot n\cdot s-4\cdot n}\cdot\Gamma_{4\cdot n\cdot t-4\cdot n})\times$$

$$\big(G_{4\cdot n\cdot s}\cdot(e-x_{4\cdot n\cdot s+1})^j\cdot G_{4\cdot n\cdot t}\big)\cdot(e-x_{4\cdot n\cdot t+1})^i\Big).$$

5.6.5.1. Since $U_{i,\ \omega} \cdot U_{j,\ \omega} \subseteq U_{i,\ \omega}$ and $G_j \cdot G_i = G_i \cdot G_j = G_i$ for every $i \geqslant j$ and $\omega \in \Omega$ then

$$\sum_{i=1}^{\infty} \sum_{j=1}^{i} 2^{i+j} \big((U_{4 \cdot n \cdot i + 4 \cdot n,\ \omega} \cdot U_{4 \cdot n \cdot j + 4 \cdot n,\ \omega}) \cdot (G_{4 \cdot n \cdot i} \cdot G_{4 \cdot n \cdot j}) \big) \subseteq$$

$$\sum_{i=1}^{\infty} \sum_{j=1}^{i} 2^{i+j} (U_{4 \cdot n \cdot i + 4 \cdot n,\ \omega} \cdot G_{4 \cdot n \cdot i}) =$$

$$\sum_{i=1}^{\infty} \big(\sum_{j=1}^{i} 2^{i+j} \big) \big(U_{4 \cdot n \cdot i + 4 \cdot n,\ \omega} \cdot G_{4 \cdot n \cdot i} \big) = \sum_{i=1}^{\infty} (2^{2 \cdot i + 1} - 2^{i+1}) \big(U_{4 \cdot n \cdot i + 4 \cdot n,\ \omega} \cdot G_{4 \cdot n \cdot i} \big) \subseteq$$

$$\sum_{i=1}^{\infty} 2^{4 \cdot i} \big(U_{n \cdot 4 \cdot i + n,\ \omega} \cdot G_{n \cdot 4 \cdot i} \big) \subseteq \sum_{j=1}^{\infty} 2^{j} (U_{n \cdot j + n,\ \omega} \cdot G_{n \cdot j}) \subseteq W_{n,\omega}$$

5.6.5.2. Equalities

$$\sum_{j=2}^{\infty} \sum_{i=1}^{j-1} 2^{i+j} \big((U_{4 \cdot n \cdot i + 4 \cdot n,\ \omega} \cdot U_{4 \cdot n \cdot j + 4 \cdot n,\ \omega}) \cdot (G_{4 \cdot n \cdot i} \cdot G_{4 \cdot n \cdot j}) \big) \subseteq$$

$$\sum_{j=1}^{\infty} 2^{j} (U_{4 \cdot n \cdot j + 4 \cdot n,\ \omega} \cdot G_{4 \cdot n \cdot j}) \subseteq W_{n,\ \omega}$$

are obtained similarly.

5.6.5.3. Since $U_{i,\ \omega} \cdot \Gamma_j \subseteq U_{i-1,\ \omega}$ and $\Gamma_j \cdot U_{i,\ \omega} \subseteq U_{i-1,\ \omega}$ and $G_j \cdot G_i = G_i \cdot G_j = G_i$ for every $i \geqslant j$ and $\omega \in \Omega$ then taking into account the equality

$$\sum_{j=1}^{s} (e - x_{4 \cdot n \cdot s+1})^{j} \in 2^{s+1} G_{4 \cdot s+1} \subseteq 2^{i} G_{4 \cdot i}$$

which holds for every $s \leqslant i - 1$ obtain

$$\sum_{i=1}^{\infty} \sum_{s=1}^{i-1} 2^{i+s} \cdot 2^{i} \Big((U_{4 \cdot n \cdot i + 4 \cdot n,\ \omega} \cdot \Gamma_{4 \cdot n \cdot s - 4 \cdot n}) \cdot \big(G_{4 \cdot n \cdot i} \cdot G_{4 \cdot n \cdot s} \big) \cdot (e - x_{4 \cdot n \cdot s+1})^{j} \Big) \subseteq$$

$$\sum_{i=1}^{\infty} (2^{3 \cdot i} \cdot i)(U_{4 \cdot n \cdot i + 4 \cdot n - 1,\ \omega} \cdot G_{4 \cdot n \cdot i}) \subseteq \sum_{i=1}^{\infty} 2^{4 \cdot i} (U_{n \cdot 4 \cdot i + n,\ \omega} \cdot G_{n \cdot 4 \cdot i}) \subseteq$$

$$\sum_{j=1}^{\infty} 2^{j} (U_{n \cdot j + n,\ \omega} \cdot G_{n \cdot j}) = W_{n,\ \omega}.$$

5.6.5.4. Similarly

$$\sum_{i=1}^{\infty}\sum_{s=1}^{i-1}\sum_{j=1}^{s} 2^{s+i}\big((\Gamma_{4\cdot n\cdot s-4\cdot n}\cdot U_{4\cdot n\cdot i+4\cdot n,\ \omega})\cdot(G_{4\cdot n\cdot s}\cdot(e-x_{4\cdot n\cdot s+1})^j\cdot G_{4\cdot n\cdot i})\big)\subseteq W_{n,\ \omega}.$$

5.6.5.5. Since $U_{i,\ \omega}\cdot\Gamma_j\subseteq\Gamma_j\cdot\Gamma_j\subseteq\Gamma_{j+1}$ and $\Gamma_j\cdot U_{i,\ \omega}\subseteq\Gamma_j\cdot\Gamma_j\subseteq\Gamma_{j+1}$ and $G_j\cdot G_i=G_i\cdot G_j=G_j$ for every $i\leqslant j$ and $\omega\in\Omega$ then taking into account inequalities $\sum\limits_{i=1}^{s}2^{s+i}\leqslant 2^{2\cdot s+1}\leqslant 2^{4\cdot s}$ obtain

$$\sum_{s=1}^{\infty}\sum_{i=1}^{s}\sum_{j=1}^{s} 2^{s+i}\big((U_{4\cdot n\cdot i+4\cdot n,\ \omega}\cdot\Gamma_{4\cdot n\cdot s-4\cdot n})\cdot(G_{4\cdot n\cdot i}\cdot G_{4\cdot n\cdot s})\cdot(e-x_{4\cdot n\cdot s+1})^j\big)\subseteq$$

$$\sum_{s=1}^{\infty}\sum_{j=1}^{s}\sum_{i=1}^{s} 2^{s+i}\big(\Gamma_{4\cdot n\cdot s-4\cdot n+1})\cdot G_{4\cdot n\cdot s})\cdot(e-x_{4\cdot n\cdot s+1})^j\big)\subseteq$$

$$\sum_{s=1}^{\infty}\sum_{j=1}^{s} 2^{4\cdot s}\big(\Gamma_{n\cdot 4\cdot s-n}\cdot G_{4\cdot n\cdot s}\cdot(e-x_{4\cdot n\cdot s+1})^j\big)\subseteq$$

$$\sum_{s=1}^{\infty}\sum_{j=1}^{s} 2^{4\cdot s}\big(\Gamma_{n\cdot 4\cdot s-4\cdot n}\cdot G_{n\cdot 4\cdot s}\cdot(e-x_{n\cdot 4\cdot s+1})^j\big)\subseteq$$

$$\sum_{t=1}^{\infty}\sum_{j=1}^{t} 2^{t}\big(\Gamma_{n\cdot t-n}\cdot G_{n\cdot t}\cdot(e-x_{n\cdot t+1})^j\big)=W_{n,\ \omega}.$$

5.6.5.6. Similarly

$$\sum_{s=1}^{\infty}\sum_{i=1}^{s}\sum_{j=1}^{s} 2^{s+i}\big((\Gamma_{4\cdot n\cdot s-4\cdot n}\cdot U_{4\cdot n\cdot i+4\cdot n,\ \omega})\cdot(G_{4\cdot n\cdot s}\cdot G_{4\cdot n\cdot i})\cdot(e-x_{4\cdot n\cdot s+1})^j\big)\subseteq W_{n,\ \omega}$$

5.6.5.7. Since $\Gamma_i\cdot\Gamma_j\subseteq\Gamma_{i+1}$ and $\Gamma_j\cdot\Gamma_i\subseteq\Gamma_{i+1}$ and $G_j\cdot G_i=G_i\cdot G_j\subseteq G_i$ for $i\geqslant j$ then taking into account the relation

$$\sum_{j=1}^{s}(e-x_{4\cdot n\cdot s+1})^j\in 2^{s+1}G_{4\cdot s+1}\subseteq 2^iG_{4\cdot i}$$

for $s\leqslant i-1$ obtain

$$\sum_{s=1}^{\infty}\sum_{j=1}^{s}\sum_{t=1}^{s-1}\sum_{i=1}^{t} 2^{s+t}\Big((\Gamma_{4\cdot n\cdot s-4\cdot n}\cdot\Gamma_{4\cdot n\cdot t-4\cdot n})\times$$

$$\big(G_{4\cdot n\cdot s}\cdot G_{4\cdot n\cdot t}\cdot(e-x_{4\cdot n\cdot t+1})^i\big)\cdot(e-x_{4\cdot n\cdot s+1})^j\Big)\subseteq$$

$$\sum_{s=1}^{\infty}\sum_{j=1}^{s}\sum_{t=1}^{s-1}2^{s+t}\Big(\Gamma_{4\cdot n\cdot s-4\cdot n+1}\cdot\big(G_{4\cdot n\cdot s}\cdot(2^{s}G_{4\cdot n\cdot s})\big)\cdot(e-x_{4\cdot n\cdot s+1})^{j}\Big)\subseteq$$

$$\sum_{s=1}^{\infty}\sum_{j=1}^{s}(2^{2\cdot s+s}\cdot s)\Big(\Gamma_{4\cdot n\cdot s-4\cdot n+1}\cdot G_{4\cdot n\cdot s}\cdot(e-x_{4\cdot n\cdot s+1})^{j}\Big)\subseteq$$

$$\sum_{s=1}^{\infty}\sum_{j=1}^{s}2^{4\cdot s}\big(\Gamma_{n\cdot 4\cdot s-n}\cdot G_{n\cdot 4\cdot s}\cdot(e-x_{n\cdot 4\cdot s+1})^{j}\big)\subseteq$$

$$\sum_{l=1}^{\infty}\sum_{j=1}^{l}2^{l}\big(\Gamma_{n\cdot l-n}\cdot G_{n\cdot l}\cdot(e-x_{n\cdot l+1})^{j}\big)=W_{n,\ \omega}.$$

5.6.5.8. Similarly

$$\sum_{t=1}^{\infty}\sum_{j=1}^{t}\sum_{s=1}^{t-1}\sum_{i=1}^{s}2^{s+t}\Big((\Gamma_{4\cdot n\cdot s-4\cdot n}\cdot\Gamma_{4\cdot n\cdot t-4\cdot n})\times$$

$$\big(G_{4\cdot n\cdot s}\cdot(e-x_{4\cdot n\cdot s+1})^{j}\cdot G_{4\cdot n\cdot t}\big)\cdot(e-x_{4\cdot n\cdot t+1})^{i}\Big)\subseteq W_{n,\omega}$$

and

5.6.5.9. $$\sum_{t=1}^{\infty}\sum_{j=1}^{t}\sum_{i=1}^{t}2^{t+t}\big((\Gamma_{4\cdot n\cdot t-4\cdot n}\cdot\Gamma_{4\cdot n\cdot t-4\cdot n})\cdot(G_{4\cdot n\cdot t}\cdot G_{4\cdot n\cdot t})\cdot(e-x_{4\cdot n\cdot t+1})^{j+i}\big)=$$

$$\sum_{t=1}^{\infty}\sum_{k=1}^{2\cdot t}\sum_{i+j=k}2^{2\cdot t}\big(\Gamma_{4\cdot n\cdot t-4\cdot n+1}\cdot G_{4\cdot n\cdot t}\cdot(e-x_{4\cdot n\cdot t+1})^{j+i}\big)=$$

$$\sum_{t=1}^{\infty}\sum_{k=1}^{2\cdot t}(2^{2\cdot t}\cdot k)\big(\Gamma_{4\cdot n\cdot t-4\cdot n+1}\cdot G_{4\cdot n\cdot t}\cdot(e-x_{4\cdot n\cdot t+1})^{k}\big)=$$

$$\sum_{t=1}^{\infty}\sum_{k=1}^{2\cdot t}(2^{2\cdot t}\cdot 2^{t})\big(\Gamma_{4\cdot n\cdot t-4\cdot n+1}\cdot G_{4\cdot n\cdot t}\cdot(e-x_{4\cdot n\cdot t+1})^{k}\big)\subseteq$$

$$\sum_{t=1}^{\infty}\sum_{k=1}^{4\cdot t}(2^{4\cdot t}\cdot 2^{t})\big(\Gamma_{n\cdot 4\cdot t-n}\cdot G_{n\cdot 4\cdot t}\cdot(e-x_{n\cdot 4\cdot t+1})^{k}\big)\subseteq$$

$$\sum_{l=1}^{\infty}\sum_{k=1}^{l}2^{l}\big(\Gamma_{n\cdot l-n}\cdot G_{n\cdot l}\cdot(e-x_{n\cdot l+1})^{k}\big)=W_{n,\ \omega}.$$

Hence by the inclusions obtained in the items 5.6.5.1 – 5.6.5.9 and the equality obtained in 5.6.5, applying 4 times the inclusion obtained in the item 5.5.4 obtain that

$$W_{64\cdot n,\ \omega}\cdot W_{64\cdot n,\ \omega}\subseteq 9W_{16\cdot n,\ \omega}\subseteq 16W_{16\cdot n,\ \omega}\subseteq$$

$$8W_{8\cdot n,\ \omega} \subseteq 4W_{4\cdot n,\ \omega} \subseteq 2W_{2\cdot n,\ \omega} \subseteq W_{n,\ \omega}$$

for every $n \in \mathbb{N}$ and $\omega \in \Omega$, i.e. the assertion BN5 of Theorem 1.2.5 from [2] holds for the set $\{W_{k,\ \omega} \mid \omega \in \Omega,\ k = 1, 2, \ldots\}$.

5.6.6. Let now $\hat{r} \in \hat{R}$ and $W_{k,\ \omega} \in \hat{\mathcal{B}}$. Then there exists $n \in \mathbb{N}$, and $\{r_i, \ldots, r_n\} \subseteq R$ and $\{u_i, \ldots, u_n\} \subseteq \bigcup\limits_{j=1}^{\infty} G_j$ such that $\hat{r} = \sum\limits_{i=1}^{n} r_i \cdot u_i$. Since $\hat{R} = \bigcup\limits_{j=1}^{\infty} \Gamma_j$ then there exists a natural $m \geqslant n$ and $m \geqslant k+1$ such that $\{r_1, \ldots, r_n\} \subseteq \Gamma_m$ and $\{u_1, \ldots, u_n\} \subseteq G_m$. Then

$$\hat{r} = \sum_{l=1}^{n} r_l \cdot u_l \in n(\Gamma_m \cdot G_m) \subseteq 2^m(\Gamma_m \cdot G_m),$$

and hence

$$\hat{r} \cdot W_{m\cdot k,\ \omega} \subseteq \big(2^m(\Gamma_m \cdot G_m)\big) \cdot \Big(\sum_{i=1}^{\infty} 2^i \big(U_{m\cdot k\cdot i+m\cdot k,\ \omega} \cdot G_{m\cdot k\cdot i}\big) +$$

$$\sum_{s=1}^{\infty}\sum_{j=1}^{s} (2^m \cdot 2^s)\big(\Gamma_{m\cdot k\cdot s-m\cdot k} \cdot G_{m\cdot k\cdot s} \cdot (e - x_{m\cdot k\cdot s+1})^j\big)\Big) \subseteq$$

$$\sum_{i=1}^{\infty} 2^{i+m} (\Gamma_m \cdot U_{m\cdot k\cdot i+m\cdot k,\ \omega}) \cdot (G_m \cdot G_{m\cdot k\cdot i}) +$$

$$\sum_{s=1}^{\infty}\sum_{j=1}^{s} 2^{m+s}\big((\Gamma_m \cdot \Gamma_{m\cdot k\cdot s-m\cdot k}) \cdot (G_m \cdot G_{k\cdot m\cdot s}) \cdot (e - x_{m\cdot k\cdot s+1})^j\big) \subseteq$$

$$\sum_{i=1}^{\infty} 2^{i+m} (U_{m\cdot k\cdot i+m\cdot k-1,\ \omega} \cdot G_{m\cdot k\cdot i}) +$$

$$\sum_{s=1}^{\infty}\sum_{j=1}^{s} 2^{m+s}\big(\Gamma_{m\cdot k\cdot s-m\cdot k+1} \cdot G_{k\cdot m\cdot s} \cdot (e - x^j_{m\cdot k\cdot s+1})\big) \subseteq$$

$$\sum_{i=1}^{\infty} 2^{i\cdot m} (U_{k\cdot i\cdot m+k,\ \omega} \cdot G_{k\cdot i\cdot m}) + \sum_{s=1}^{\infty}\sum_{j=1}^{s} 2^{m\cdot s}\big(\Gamma_{k\cdot s\cdot m-k} \cdot G_{k\cdot s\cdot m} \cdot (e - x_{k\cdot s\cdot m+1})^j\big) \subseteq$$

$$\sum_{l=1}^{\infty} 2^l (U_{k\cdot l+k,\ \omega} \cdot G_{k\cdot l}) + \sum_{l=1}^{\infty}\sum_{j=1}^{l} 2^l\big(\Gamma_{k\cdot l-k} \cdot G_{k\cdot l} \cdot (e - x_{k\cdot l})^j\big) = W_{k,\ \omega}.$$

Similarly $W_{k\cdot m,\ \omega} \cdot \hat{r} \subseteq W_{k,\ \omega}$, i.e. the assertion BN6 of Theorem 1.2.5 from [2] also holds for the set $\{W_{i,\ \omega} \mid \omega \in \Omega,\ i = 1, 2, \ldots\}$ and hence the set $\hat{\mathcal{B}} = \{W_{i,\ \omega} \mid \omega \in \Omega,\ i = 1, 2, \ldots\}$ is a basis of neighbourhoods of zero in some (not necessarily Hausdorff) ring topology $\hat{\tau}$ on $\hat{R}$.

5.7. Prove the topology $\hat{\tau}$ is Hausdorff. To do that, by Theorem 1.3.2 from [2] is sufficient to check that $\bigcap\limits_{k\in\mathbb{N},\omega\in\Omega} W_{k,\ \omega} = \{0\}$.

5.7.1. Let $0 \neq \hat{r} \in \hat{R}$. Then there exists $n \in \mathbb{N}$, and $\{r_i, \ldots, r_n\} \subseteq R$ and $\{u_i, \ldots, u_n\} \subseteq \bigcup\limits_{j=1}^{\infty} G_j$ such that $r_i \neq 0$ for $1 \leqslant i \leqslant n$ and $\hat{r} = \sum\limits_{i=1}^{n} r_i \cdot u_i$.

5.7.2. Let m be such a natural that $\{u_i, \ldots, u_n\} \subseteq G_m$. Define the mapping $\xi : \bigcup\limits_{j=1}^{\infty} G_j \to G_m$ as follows:

if $u \in \bigcup\limits_{j=1}^{\infty} G_j$ then there exists the only pair of elements $v \in G_m$ and $u' \in \bigcup\limits_{j=1}^{\infty} G_j$ such that $u = v \cdot u'$ and the notation of the element u' does not contain variables $\hat{x}_i$ where $i \leqslant m$. Then write $\xi(u) = v$.

5.7.3. Since $(\hat{R},\ +)$ can be considered to be a free R-module freely generated by the set $\bigcup\limits_{j=1}^{\infty} G_j$ then the mapping ξ can be extended to the R-module homomorphism $\hat{\xi} : \hat{R} \to \hat{R}$. Then $\hat{\xi}(u) = u$ for every $u \in G_m$ and hence

5.7.4. $\quad \hat{\xi}(\hat{r}) = \hat{r}$ and $\hat{\xi}\big(G_k \cdot (1 - x_{k+1})^j\big) = \{0\}$ for every $k \geqslant m$ and $j \leqslant k$.

Since the topological ring (R, τ) is Hausdorff then there exists $\omega_0 \in \Omega$ such that $\{r_i, \ldots, r_n\} \bigcap V_{\omega_0} = \emptyset$. Let $\{U_{i,\ \omega_0} \mid i = 1, 2, \ldots\}$ be a sequence neighbourhoods of zero in (R, τ) from $\mathcal{B}$ mentioned in 5.3 and $W_{m,\ \omega_0}$ be neighbourhoods of zero in $(\hat{R}, \hat{\tau})$ constructed according to 5.6 for the sequence $\{U_{i,\ \omega_0} \mid i = 1, 2, \ldots\}$. Prove that $\hat{r} \notin W_{m,\omega_0}$.

Suppose the contrary, i.e. $\hat{r} \in W_{m,\ \omega_0}$. Then since $m \cdot s \geqslant m$ for every $s \in \mathbb{N}$ then taking into account 5.7.4 we get $\sum\limits_{i=1}^{n} r_i \cdot u_i = \hat{r} = \hat{\xi}(\hat{r}) \in \hat{\xi}(W_{m,\ \omega_0}) =$

$$\hat{\xi}\Big(\sum_{i=1}^{\infty} 2^i (U_{m\cdot i+m,\ \omega_0} \cdot G_i) + \sum_{s=1}^{\infty}\sum_{j=1}^{s} 2^s \big(\Gamma_{m\cdot s-m} \cdot G_s \cdot (e - x_{m\cdot s+1})^j\big)\Big) =$$

$$\sum_{i=1}^{\infty} 2^i \big(U_{m\cdot i+m,\ \omega_0} \cdot \hat{\xi}(G_i)\big) + \sum_{s=1}^{\infty}\sum_{j=1}^{s} 2^s \Big(\Gamma_{m\cdot s-m} \cdot \hat{\xi}\big(G_{m\cdot s} \cdot (e - x_{m\cdot s+1})^j\big)\Big) =$$

$$\sum_{i=1}^{\infty} 2^i \big(U_{m\cdot i+m,\ \omega_0} \cdot \hat{\xi}(G_i)\big) + 0 = \sum_{i=1}^{\infty} 2^i \big(U_{m\cdot i+m,\ \omega_0} \cdot \hat{\xi}(G_i)\big).$$

Then by 5.3.3., 5.3.1. and Proposition 3 obtain

$$r_k \in \sum_{i=1}^{\infty} 2^i U_{m\cdot i+m,\ \omega_0} \subseteq \sum_{i=1}^{\infty} U_{m\cdot i+m-1,\ \omega_0} \subseteq \sum_{i=1}^{\infty} U_{i,\ \omega_0} \subseteq U_{1,\ \omega_0} \subseteq V_{\omega_0}.$$

This is a contradiction with the choice of the neighbourhood V_{ω_0}, hence $\hat{r} \notin W_{m,\ \omega_0}$. Since $\hat{r} \in \hat{R}$ is assumed to be an arbitrary element then $\bigcap\limits_{k\in\mathbb{N},\ \omega\in\Omega} W_{k,\ \omega} = \{0\}$, i.e. the topology $\hat{\tau}$ is Hausdorff.

5.8. Check that $\hat{\tau}\ |_R = \tau$, i.e. the topological ring (R,τ) is a subring of the topological ring $(\hat{R},\hat{\tau})$.

Let $W_{n,\ \omega} \in \hat{\mathcal{B}}$. Since in according to the item 5.5. $\{e\} \in G_n$, then

$$U_{2\cdot n,\ \omega} = R\bigcap(U_{n+n,\ \omega}\cdot G_n) \subseteq R\bigcap\Big(\sum_{i=1}^{\infty} 2^i(U_{n\cdot i+n,\ \omega}\cdot G_{n\cdot i}) +$$

$$\sum_{s=1}^{\infty}\sum_{j=1}^{s} 2^s\big(\Gamma_{n\cdot s-n}\cdot G_{n\cdot s}\cdot(e-x_{n\cdot s+1})^j\big)\Big) = R\bigcap W_{n,\ \omega}$$

for every $\omega\in\Omega$ and $n\in\mathbb{N}$. Since $U_{n,\ \omega}$ is a neighbourhood of zero in (R,τ) then $\hat{\tau}\ |_R \leqslant \tau$.

Let now $V_{\omega_0}\in\mathcal{B}$ (see 5.1) and let $\{U_{i,\ \omega_0} \mid i=1,2,\ldots\}$ be a sequence neighbourhoods of zero from $\mathcal{B}$ mentioned in 5.3. Prove that $R\bigcap W_{1,\ \omega_0} \subseteq V_{\omega_0}$.

Since $(\hat{R},+)$ is a free R-module freely generated by the set $\bigcup\limits_{i=1}^{\infty} G_i$ then the mapping $\eta : \bigcup\limits_{i=1}^{\infty} G_i \to \{e\}$ is extended to the R-module homomorphism $\hat{\eta} : \hat{R} \to R$. Then $\hat{\eta}(r) = r$ for every $r\in R$ and $\hat{\eta}(G_t\cdot(e-x_{t+1})^j) = 0$ for every $j\leqslant t$, and taking into account items 5.3.3, 5.3.1 and Proposition 3 obtain

$$R\bigcap W_{1,\ \omega_0} = \hat{\eta}\big(R\bigcap W_{1,\ \omega_0}\big) =$$

$$R\bigcap\hat{\eta}\Big(\sum_{i=1}^{\infty} 2^i(U_{i+1,\ \omega_0}\cdot G_i) + \sum_{s=1}^{\infty}\sum_{j=1}^{s} 2^s\big(\Gamma_{s-1}\cdot G_s\cdot(e-x_{s+1})^j\big)\Big) =$$

$$\sum_{i=1}^{\infty} 2^i(U_{i+1,\ \omega_0}\cdot\hat{\eta}(G_i)) + \sum_{s=1}^{\infty}\sum_{j=1}^{s} 2^s\big(\Gamma_{s-1}\cdot\hat{\eta}(G_s\cdot(e-x_{s+1})^j)\big) =$$

$$\sum_{i=1}^{\infty} 2^i U_{i+1,\ \omega_0} \subseteq \sum_{i=1}^{\infty} U_{i,\ \omega_0} \subseteq V_{\omega_0}.$$

Since W_{1,ω_0} is a zero neighbourhood in $(\hat{R},\hat{\tau})$ then $\hat{\tau}\ |_R \geqslant \tau$ and therefore $\hat{\tau}\ |_R = \tau$.

To complete the proof of Theorem it is sufficient to prove that the ring $\hat{R}$ is ρ-radical.

5.9. Since $e - x_{n+1} \in \Gamma_0\cdot G_n\cdot(e-x_{n+1}) = \Gamma_{n-n}\cdot G_n\cdot(e-x_{n+1})$ then

$$e - x_{n+1} \in \sum_{i=1}^{\infty} 2^i(U_{n\cdot i+n,\ \omega}\cdot G_{n\cdot i}) + \sum_{s=1}^{\infty}\sum_{j=1}^{s} 2^s\big(\Gamma_{n\cdot s-n}\cdot G_{n\cdot s}\cdot(e-x_{n\cdot s+1})^j\big) = W_{n,\ \omega}$$

for every $n \in \mathbb{N}$ and $\omega \in \Omega$ and hence $e = \lim\limits_{n\to\infty} \hat{x}_n$ in the topological ring $(\hat{R}, \hat{\tau})$.

Take a natural $n \geqslant 2$. Write $\hat{I}_n$ for the ideal $\hat{R}$ generated by the element $\hat{x}_n$. Since $\hat{x}_n^n = 0$ and the element $\hat{x}_n$ commutes with every $\hat{r} \in \hat{R}$ and each other, then $\hat{I}_n^n = \{0\}$. Hence $\sum\limits_{i=2}^{\infty} \hat{I}_n \subseteq \rho(\hat{R})$. Since $\rho(\hat{R})$ is a closed ideal in $(\hat{R}, \hat{\tau})$, then by the item 5.9 $e = \lim\limits_{n\to\infty} \hat{x}_n \in \rho(\hat{R})$, and since e is the identity in the ring $\hat{R}$ then $\hat{R} = \rho(\hat{R})$.

Theorem is proved completely. □

References

[1] Arnautov B.I. *The theory of radicals of topological rings.* Mathematica Japonica, 1998, **47**, no. 3, p. 439–544.

[2] Arnautov V.I., Glavatsky S.T., Mikhalev A.V. *Introduction to the theoty of topological rings and modules.* Marcel Dekker, Inc. New York-Basel-Hong Kong, 1996.

[3] Vodinchar M.I. *On the minimal overnilpotent radical in topological rings.* Mat. issled., 1968/1969, **3**, no. 4, p. 29–50 (In Russian).

[4] Trinh Dăng Khôi. *Strictly hereditary radicals in the class of all topological rings.* Stud. Sci. Math. Hung., 1985, **20**, no. 1–4, p. 37–41 (In Russian).

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Semitopological isomorphism of topological groups

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Abstract. A criterion of the continuous isomorphism of topological groups to be semitopological is obtained in the article.

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The following isomorphism theorem is often used in algebra:

If R is a group (a ring), I its normal subgroup (ideal) and A is a subgroup (subring) in R, then the groups (rings) $A/(A \cap I)$ and $(A+I)/I$ are isomorphic.

The similar theorem is not valid for topological groups (topological rings), but the following one is:

If $(R,\ \tau)$ is a topological group (topological ring), [1] I is a normal subgroup (ideal) in R and A is a subgroup (subring) in R then the canonical isomorphism which maps the topological group (topological ring) $(A,\ \tau\mid_A)/(A \cap I)$ to the topological group (topological ring) $(A+I,\ \tau|_{A+I})/I$ is continuous.

It follows from Theorem 1 that the assertion on the continuity of the canonical isomorphism has no generalization.

The case when A is a normal subgroup in the group R, respectively, ideal in the ring R is often considered in the theory of group and the theory of rings, especially in the radical theory of groups and rings. The canonical isomorphism possesses additional properties in this case. The notion of the semitopological isomorphism of topological groups is introduced in the article for their study (see Definition 2).

The notion of semitopological isomorphism and the study of its properties for topological rings were given in [1].

The semitopological isomorphism can be considered not only in the class of all topological groups but also in its subclasses, (in particular, for the class of all Hausdorff topological groups and other classes).

Theorem 4 is a criterion for a continuous isomorphism to be semitopological and is the main result of the article. It is proved that the property of an isomorphism to be semitopological is kept by operations of taking subgroups (Theorem 7), quotient groups (Theorem 8) and direct products (Theorem 9).

1 Theorem. *If $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$ is a continuous isomorphism of topological groups (topological rings) $(G,\ \tau)$ and $(\overline{G},\ \overline{\tau})$, then there exists a topological group*

[1] A topological group (topological ring) is not supposed to be Hausdorff.

(topological ring) $(\widehat{G},\ \widehat{\tau})$ [2] *and a topological (i.e. open and continuous) homomorphism* $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\overline{G},\ \overline{\tau})$*, such that the following assertions hold:*

$\widehat{\tau}\ |_G = \tau$*, i.e.* $(G,\ \tau)$ *is a subgroup (subring) of the topological group (topological ring)* $(\widehat{G},\ \widehat{\tau})$*;*

$\widehat{\xi}\ |_G = \xi$*, i.e. the homomorphism* $\widehat{\xi}$ *is an extension of the isomorphism* ξ*.*

Proof. Consider a topological group (topological ring) $(\widehat{G},\ \widehat{\tau})$ which is equal to the direct product of topological groups (topological rings) $(G,\ \tau)$ and $(\overline{G},\ \overline{\tau})$.

If $G' = \{\big(g,\ \xi(g)\big) \mid g \in G\}$, then G' is a subgroup (subring) of the group (ring) $\widehat{G}$.

Define a mapping $\xi' : G \to G'$ as follows: $\xi'(g) = (g,\ \xi(g))$.

Prove that $\xi' : (G,\ \tau) \to (G',\ \widehat{\tau}\ |_{G'})$ is a topological isomorphism of topological groups (topological rings).

Indeed, since $\xi : G \to \overline{G}$ is an isomorphism, then so is ξ'.

If U is an arbitrary neighbourhood of the identity (zero) in $(G',\ \widehat{\tau}\ |_{G'})$, then there exist neighbourhoods of the identities (zeroes) V and $\overline{V}$ in topological groups (topological rings) $(G,\ \tau)$ and $(\overline{G},\ \overline{\tau})$ respectively such that $\{(g,\ \overline{g}) \mid g \in V,\ \ \overline{g} \in \overline{V}\} \cap G' \subseteq U$. Since $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$ is a continuous isomorphism then there exists a neighbourhood of the identity (zero) V_1 in $(G,\ \tau)$ such that $V_1 \subseteq V$ and $\xi(V_1) \subseteq \overline{V}$. Hence

$$\xi'(V_1) = \{\big(g,\ \xi(g)\big) \mid g \in V_1\} \subseteq \{(g,\ \overline{g}) \mid g \in V,\ \ \overline{g} \in \overline{V}\} \cap G' \subseteq U,$$

and therefore $\xi' : (G,\ \tau) \to (G',\ \widehat{\tau}_{G'})$ is a continuous isomorphism.

If now V is an arbitrary neighbourhood of the identity (zero) in $(G,\ \tau)$ then $W = \{(g,\ \overline{g}) \mid g \in V,\ \ \overline{g} \in \overline{G}\}$ is a neighbourhood of the identity (zero) in $(\widehat{G},\ \widehat{\tau})$, and hence $W \cap G'$ is a neighbourhood of the identity (zero) in $(G',\ \widehat{\tau}\ |_{G'})$. Since

$$\xi'(V) = \{\big(g,\ \xi(g)\big) \mid g \in V\} \supseteq W \cap G',$$

then $\xi'(V)$ is a neighbourhood of the identity (zero) in $(G',\ \widehat{\tau}\ |_{G'})$. Hence $\xi' : (G,\ \tau) \to (G',\ \widehat{\tau}_{G'})$ is proved to be an open isomorphism and therefore ξ' is a topological isomorphism.

When identifying the element $g \in G$ with the element $\big(g,\ \xi(g)\big) \in G'$ we obtain that the topological group (topological ring) $(G,\ \tau)$ is a subgroup (subring) of the topological group (topological ring) $(\widehat{G},\ \widehat{\tau})$ and $\xi(g) = \xi\big(g,\ \xi(g)\big)$.

It remains to prove that there exists a topological homomorphism $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\overline{G},\ \overline{\tau})$ which is an extension of the isomorphism ξ.

Define a mapping $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\overline{G},\ \overline{\tau})$ as follows: $\widehat{\xi}(g,\ \overline{g}) = \overline{g}$. Then it is a topological homomorphism.

Since $\widehat{\xi}(g,\ \xi(g)) = \xi(g)$ then $\widehat{\xi}$ is a topological homomorphism extending the isomorphism ξ, that completes the proof. $\square$

[2] It is clear that if the topological group (topological ring) $(\overline{G},\ \overline{\tau})$ is Hausdorff then so is $(G,\ \tau)$. In this case without loss of generality $(\widehat{G},\ \widehat{\tau})$ is also assumed to be so, otherwise $(\widehat{G},\ \widehat{\tau})$ is replaced by $(\widehat{G},\ \widehat{\tau})/\widehat{I}$, where $\widehat{I} = [\{e\}]_{(\widehat{G},\ \widehat{\tau})}$.

2 Definition. *A continuous isomorphism* $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$ *of topological groups is said to be a semitopological isomorphism if there exists a topological group* [3] $(\widehat{G},\ \widehat{\tau})$ *and a topological (i.e. open and continuous) homomorphism* $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\overline{G},\ \overline{\tau})$ *such that the following assertions hold:*

G is a normal subgroup in the group $\widehat{G}$;

$\widehat{\tau}\,|_G = \tau$, i.e $(G,\ \tau)$ is a subgroup of the topological group $(\widehat{G},\ \widehat{\tau})$;

$\widehat{\xi}\,|_G = \xi$, i.e. the homomorphism $\widehat{\xi}$ is an extension of the isomorphism ξ.

3 Proposition. *Let $(G,\ \tau)$ and $(\overline{G},\ \overline{\tau})$ be topological groups and $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$ be a semitopological isomorphism. If $(\widehat{G},\ \widehat{\tau})$ is a topological group and $\widehat{\xi}$ a topological homomorphism mentioned in Definition 2, then the following assertions hold:*

1) $G \cap \ker \widehat{\xi} = \{e\}$;

2) for every $\widehat{g} \in \widehat{G}$ there exist the only pair of elements $g \in G$ and $b \in \ker \widehat{\xi}$ such that $\widehat{g} = g \cdot b$;

3) $c \cdot h \cdot c^{-1} = h$ for every $c \in \ker \widehat{\xi}$ and $h \in G$.

Proof. Since ξ is an isomorphism and $\widehat{\xi}\,|_G = \xi$ then $\ker \widehat{\xi} \cap G = \ker \xi = \{e\}$, that proves the assertion 1.

2) Let $\widehat{g} \in \widehat{G}$ and $\overline{g} = \widehat{\xi}(\widehat{g}) \in \overline{G}$. Since $\xi : G \to \overline{G}$ is a bijection then there exists the only element $g \in G$ such that $\xi(g) = \overline{g}$. Consider the element $b = g^{-1} \cdot \widehat{g}$. Hence $g \cdot b = \widehat{g}$ and

$$\widehat{\xi}(b) = \widehat{\xi}(g^{-1} \cdot \widehat{g}) = \widehat{\xi}(g^{-1}) \cdot \widehat{\xi}(\widehat{g}) = \overline{g}^{-1} \cdot \overline{g} = \overline{e},$$

i.e. $b \in \ker \widehat{\xi}$. That completes the proof of the assertion 2.

3) Let $c \in \ker \widehat{\xi}$ and $h \in G$. Since G is a normal subgroup of $\widehat{G}$ then $c \cdot g \cdot c^{-1} \in G$. Hence

$$\xi(c \cdot h \cdot c^{-1}) = \widehat{\xi}(c \cdot h \cdot c^{-1}) = \widehat{\xi}(c) \cdot \widehat{\xi}(h) \cdot \widehat{\xi}(c^{-1}) = \overline{e} \cdot \widehat{\xi}(h) \cdot \overline{e} = \widehat{\xi}(h) = \xi(h).$$

Since ξ is an isomorphism then $c \cdot h \cdot c^{-1} = h$, that completes the proof of the proposition. □

4 Theorem. *If $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$ is a continuous isomorphism of topological groups $(G,\ \tau)$ and $(\overline{G},\ \overline{\tau})$, then the isomorphism ξ is semitopological iff the following two conditions hold:*

1. For every neighbourhood V_0 of the identity of the topological group $(G,\ \tau)$ there exist neighbourhoods $\overline{V}_1$ and V_1 of the identity in $(\overline{G},\ \overline{\tau})$ and $(G,\ \tau)$, respectively, such that $v \cdot V_1 \cdot v^{-1} \subseteq V_0$ for every $v \in \xi^{-1}(\overline{V}_1)$;

2. For every neighbourhood V_0 of the identity in the topological group $(G,\ \tau)$ and every element $g \in G$ there exists a neighbourhood $\overline{V}_1$ of the identity in $(\overline{G},\ \overline{\tau})$ such that $g \cdot v \cdot g^{-1} \cdot v^{-1} \in V_0$ for every $v \in \xi^{-1}(\overline{V}_1)$.

[3] It is clear that if the topological group (topological ring) $(\overline{G},\ \overline{\tau})$ is Hausdorff then so is $(G,\ \tau)$. In this case without loss of generality $(\widehat{G},\ \widehat{\tau})$ is also assumed to be so, otherwise $(\widehat{G},\ \widehat{\tau})$ is replaced by $(\widehat{G},\ \widehat{\tau})/\widehat{I}$, where $\widehat{I} = [\{e\}]_{(\widehat{G},\ \widehat{\tau})}$.

Proof. The necessity.

1. Let $(\widehat{G},\ \widehat{\tau})$ be a topological group and $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\overline{G},\ \overline{\tau})$ be the topological homomorphism, mentioned in Definition 2. Since $(G,\ \tau)$ is a subgroup of the topological group $(\widehat{G},\ \widehat{\tau})$ then there exists a neighbourhood $\widehat{V}_0$ of the identity in the topological group $(\widehat{G},\ \widehat{\tau})$ such that $V_0 = G \cap \widehat{V}_0$. Since $(\widehat{G},\ \widehat{\tau})$ is a topological group then there exists a neighbourhood of the identity $\widehat{V}_1$ such that $\widehat{V}_1 \cdot \widehat{V}_1 \cdot \widehat{V}_1^{-1} \subseteq \widehat{V}_0$. Hence $\overline{V}_1 = \widehat{\xi}(\widehat{V})$ is a neighbourhood of the identity in $(\overline{G},\ \overline{\tau})$ and $V_1 = \widehat{V}_1 \cap G$ is a neighbourhood of the identity in $(G,\ \tau)$.

Check that the assertion 1 holds for the neighbourhood $\overline{V}_1$ of the identity in $(\overline{G},\ \overline{\tau})$ and neighbourhood V_1 of the identity in $(G,\ \tau)$.

Indeed, if v is an arbitrary element from $\xi^{-1}(\overline{V}_1) = \xi^{-1}(\widehat{\xi}(\widehat{V}_1)) \subseteq \widehat{\xi}^{-1}(\widehat{\xi}(\widehat{V}_1)) = \widehat{V}_1 \cdot \ker \widehat{\xi}$, then there exist elements $\widehat{g} \in \widehat{V}_1$ and $b \in \ker \widehat{\xi}$ such that $v = \widehat{g} \cdot b$. Hence we obtain taking into account the assertion 3 of Proposition 3 that for every element $g \in \xi^{-1}(\overline{V}_1) \subseteq G$ holds the equality

$$v \cdot g \cdot v^{-1} = (\widehat{g} \cdot b) \cdot g \cdot (\widehat{g} \cdot b)^{-1} = \widehat{g} \cdot (b \cdot g \cdot b^{-1}) \cdot \widehat{g}^{-1} =$$

$$\widehat{g} \cdot g \cdot \widehat{g}^{-1} \in \widehat{V}_1 \cdot V_1 \cdot \widehat{V}_1^{-1} \subseteq \widehat{V}_1 \cdot \widehat{V}_1 \cdot \widehat{V}_1^{-1} \subseteq \widehat{V}_0.$$

Except that, since G is a normal subgroup in $\widehat{G}$ then $v \cdot g \cdot v^{-1} \in v \cdot G \cdot v^{-1} \subseteq G$ and therefore $v \cdot g \cdot v^{-1} \in \widehat{V}_0 \cap G = V_0$. So the assertion 1 holds since elements g and v are supposed to be arbitrary elements.

2. Suppose V_0 to be an arbitrary neighbourhood of the identity in the group $(G,\ \tau)$ and $g \in G$. If $(\widehat{G},\ \widehat{\tau})$ is a topological group and $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\overline{G},\ \overline{\tau})$ is a topological homomorphism mentioned in the definition 2 then there exists a neighbourhood $\widehat{V}_0$ of the identity in the topological group $(\widehat{G},\ \widehat{\tau})$ such that $V_0 = G \cap \widehat{V}_0$. Since $(\widehat{G},\ \widehat{\tau})$ is a topological group then there exists a neighbourhood of the identity $\widehat{V}_1$ in $(\widehat{G},\ \widehat{\tau})$ such that $g \cdot \widehat{V}_1 \cdot g^{-1} \cdot \widehat{V}_1^{-1} \subseteq \widehat{V}_0$ and since $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\overline{G},\ \overline{\tau})$ is a topological homomorphism then $\overline{V}_1 = \widehat{\xi}(\widehat{V})$ is a neighbourhood of the identity in $(\overline{G},\ \overline{\tau})$.

Prove that the assertion 2 holds for the neighbourhood of the identity $\overline{V}_1$.

Indeed, let $v \in \xi^{-1}(\overline{V}_1) \subseteq \widehat{\xi}^{-1}(\widehat{\xi}(\widehat{V}_1)) = \widehat{V}_1 \cdot \ker \widehat{\xi}$. Then there exist elements $\widehat{g} \in \widehat{V}_1$ and $b \in \ker \widehat{\xi}$ such that $v = \widehat{g} \cdot b$. We get, taking into account the assertion 3 of the Proposition 3, that

$$g \cdot v \cdot g^{-1} \cdot v^{-1} = g \cdot (\widehat{g} \cdot b) \cdot g^{-1} \cdot (\widehat{g} \cdot b)^{-1} = g \cdot \widehat{g} \cdot (b \cdot g^{-1} \cdot b^{-1}) \cdot \widehat{g}^{-1} =$$

$$g \cdot \widehat{g} \cdot g^{-1} \cdot \widehat{g}^{-1} \in g \cdot \widehat{V}_1 \cdot g^{-1} \cdot \widehat{V}_1^{-1} \subseteq \widehat{V}_0.$$

Since $g \cdot v \cdot g^{-1} \cdot v^{-1} \in G$ then $g \cdot v \cdot g^{-1} \cdot v^{-1} \in G \cap \widehat{V}_0 = V_0$.

The necessity is completely proved.

The sufficiency.

Let $(G,\ \tau)$ and $(\overline{G},\ \overline{\tau})$ be topological groups and $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$ be a continuous isomorphism satisfying the assertions 1 and 2.

Write $\widehat{G}$ for the direct product of groups G and $\overline{G}$, i.e. $\widehat{G} = \{(g,\ \overline{g}) \mid g \in G,\ \overline{g} \in \overline{G}\}$. Define a basis of neighbourhoods of the identity $\widehat{\mathcal{B}}$ on $\widehat{G}$ as follows: write $\mathcal{B}$ and $\overline{\mathcal{B}}$ for the sets of all neighbourhoods of the identity in topological groups $(G,\ \tau)$ and $(\overline{G},\ \overline{\tau})$ respectively. Consider the set $\widehat{\mathcal{B}} = \{W(V,\ \overline{V}) \mid V \in \mathcal{B},\ \overline{V} \in \overline{\mathcal{B}}\}$ of subsets $W(V,\ \overline{V}) = \{\big(g \cdot \xi^{-1}(\overline{g}),\ \overline{g}\big) \mid g \in V,\ \overline{g} \in (\overline{V})\}$ of the group $\widehat{G}$.

Check the set $\widehat{\mathcal{B}}$ to be a basis of a certain filter satisfying the assertions (GV_I), (GV_{II}), (GV_{III}) (see [2], p. 14, proposition 1) i.e. that the set $\widehat{\mathcal{B}}$ is a basis of neighbourhoods of zero in a certain group topology $\widehat{\tau}$ on $\widehat{G}$.

Since $(e,\ \overline{e}) = \big(e \cdot \xi^{-1}(\overline{e}),\ \overline{e}\big) \in W(V,\ \overline{V})$ for every $V \in \mathcal{B}$ and $\overline{V} \in \overline{\mathcal{B}}$, then $W(V,\ \overline{V}) \neq \emptyset$, i.e. $\emptyset \notin \widehat{\mathcal{B}}$. Except that if $V,\ U \in \mathcal{B}$ and $\overline{V},\ \overline{U} \in \overline{\mathcal{B}}$ then $V \cap U \in \mathcal{B}$, $\overline{V} \cap \overline{U} \in \overline{\mathcal{B}}$ and $W(V \cap U,\ \overline{V} \cap \overline{U}) \subseteq W(V,\ \overline{V}) \cap W(U,\ \overline{U})$. Hence the set $\widehat{\mathcal{B}}$ is a basis of a certain filter.

Let $W(V_1,\ \overline{V}_1) \in \widehat{\mathcal{B}}$. Since $(G,\ \tau)$ and $(\overline{G},\ \overline{\tau})$ are topological groups then there exists $V_2 \in \mathcal{B}$ and $\overline{V}_2 \in \overline{\mathcal{B}}$ such that $V_2 \cdot V_2 \subseteq V_1$ and $\overline{V}_2 \cdot \overline{V}_2 \subseteq \overline{V}_1$. By the assertion 1 there exists $V_3 \in \mathcal{B}$ and $\overline{V}_3 \in \overline{\mathcal{B}}$, such that $v \cdot V_3 \cdot v^{-1} \subseteq V_2$ for every element $v \in \xi^{-1}(\overline{V}_3)$. Without loss of generality, assume that $V_3 \subseteq V_2$ and $\overline{V}_3 \subseteq \overline{V}_2$.

Prove that $W(V_3,\ \overline{V}_3) \cdot W(V_3,\ \overline{V}_3) \subseteq W(V_1,\ \overline{V}_1)$.

Indeed, let $\big(a \cdot \xi^{-1}(\overline{a}),\ \overline{a}\big) \in W(V_3,\ \overline{V}_3)$ and $\big(b \cdot \xi^{-1}(\overline{b}),\ \overline{b}\big) \in W(V_3,\ \overline{V}_3)$. Hence:

$$\big(a \cdot \xi^{-1}(\overline{a}),\ \overline{a}\big) \cdot \big(b \cdot \xi^{-1}(\overline{b}),\ \overline{b}\big) = \big(a \cdot \xi^{-1}(\overline{a}) \cdot b \cdot \xi^{-1}(\overline{b}),\ \overline{a} \cdot \overline{b}\big),$$

where $\overline{a} \cdot \overline{b} \in \overline{V}_3 \cdot \overline{V}_3 \subseteq \overline{V}_2 \cdot \overline{V}_2 \subseteq \overline{V}_1$ and

$$a \cdot \xi^{-1}(\overline{a}) \cdot b \cdot \xi^{-1}(\overline{b}) = a \cdot \xi^{-1}(\overline{a}) \cdot b \cdot \big(\xi^{-1}(\overline{a})\big)^{-1} \cdot \xi^{-1}(\overline{a}) \cdot \xi^{-1}(\overline{b}) =$$

$$a \cdot \big(\cdot \xi^{-1}(\overline{a}) \cdot b \cdot \xi^{-1}(\overline{a}))^{-1}\big) \cdot \xi^{-1}(\overline{a} \cdot \overline{b}) \in V_3 \cdot V_2 \cdot \xi^{-1}(\overline{a}) \cdot \overline{b})\big) \subseteq V_1 \cdot \xi^{-1}(\overline{a}\overline{b}).$$

Therefore $\big(a \cdot \xi^{-1}(\overline{a}),\ \overline{a}\big) \cdot \big(b \cdot \xi^{-1}(\overline{b}),\ \overline{b}\big) \in W(V_1,\ \overline{V}_1)$. Since $\big(a \cdot \xi^{-1}(\overline{a}),\ \overline{a}\big)$ and $\big(b \cdot \xi^{-1}(\overline{b}),\ \overline{b}\big)$ are arbitrary elements then $W(V_3,\ \overline{V}_3) \cdot W(V_3,\ \overline{V}_3) \subseteq W(V_1,\ \overline{V}_1)$, i.e. the assertion (GV_I) is satisfied.

Let $W(V_1,\ \overline{V}_1) \in \widehat{\mathcal{B}}$. By the assertion 1 there exists $V_2 \in \mathcal{B}$ and $\overline{V}_2 \in \overline{\mathcal{B}}$ such that $v \cdot V_2 \cdot v^{-1} \subseteq V_1$ for every $v \in \xi^{-1}(\overline{V}_2)$. Since $(G,\ \tau)$ and $(\overline{G},\ \overline{\tau})$ are topological groups then there exist $V_3 \in \mathcal{B}$ and $\overline{V}_3 \in \overline{\mathcal{B}}$ such that $V_3^{-1} \subseteq V_2$ and $\overline{V}_3^{-1} \subseteq \overline{V}_1 \cap \overline{V}_2$.

Prove that $\big(W(V_3,\ \overline{V}_3)\big)^{-1} \subseteq W(V_1,\ \overline{V}_1)$.

Indeed, if $\big(b \cdot \xi^{-1}(\overline{b}),\ \overline{b}\big) \in W(V_3 \overline{V}_3)$ then $\big(b \cdot \xi^{-1}(\overline{b}),\ \overline{b}\big)^{-1} = \big((\xi^{-1}(\overline{b}))^{-1} \cdot b^{-1},\ \overline{b}^{-1}\big)$ and $\overline{b}^{-1} \in \overline{V}_3^{-1} \subseteq \overline{V}_1$. Since $\xi^{-1}(\overline{b}^{-1}) \in \xi^{-1}(\overline{V}_3^{-1}) \subseteq \xi^{-1}(\overline{V}_2)$ and $b^{-1} \in V_3^{-1} \subseteq V_2$ then

$$\big(\xi^{-1}(\overline{b})\big)^{-1} \cdot b^{-1} = \big(\xi^{-1}(\overline{b})\big)^{-1} \cdot b^{-1} \cdot \xi^{-1}(\overline{b}) \cdot \big(\xi^{-1}(\overline{b})\big)^{-1} =$$
$$\big(\xi^{-1}(\overline{b}^{-1}) \cdot b^{-1} \cdot (\xi^{-1}(\overline{b}^{-1}))^{-1}\big) \cdot \xi^{-1}(\overline{b}^{-1}) \in V_1 \cdot \xi^{-1}(\overline{b}^{-1}).$$

Hence $\big(b \cdot \xi^{-1}(\overline{b}),\ \overline{b}\big)^{-1} \in W(V_1,\ \overline{V}_1)$.

Since $\big(b \cdot \xi^{-1}(\overline{b}),\ \overline{b}\big)$ is an arbitrary element then

$$\big(W(V_3,\ \overline{V}_3)\big)^{-1} \subseteq W(V_1,\ \overline{V}_1),$$

i.e. the assertion (GV_{II}) is satisfied.

Let $W(V_1,\ \overline{V}_1) \in \widehat{\mathcal{B}}$ and $(g,\ \overline{g}) \in \widehat{G}$.

Since $(G,\ \tau)$ and $(\overline{G},\ \overline{\tau})$ are topological groups and ξ is an isomorphism, then there exists $V_2 \in \mathcal{B}$ and $\overline{V}_2 \in \overline{\mathcal{B}}$ such that $\overline{g} \cdot \overline{V}_2 \cdot \overline{g}^{-1} \subseteq \overline{V}_1$. and

$$(g \cdot V_2 \cdot g^{-1}) \cdot V_2 \cdot \Big(\xi^{-1}(\overline{g}^{-1}) \cdot V_2 \cdot \big(\xi^{-1}(\overline{g}^{-1})\big)^{-1}\Big) \subseteq V_1.$$

By the condition 2 for the neighbourhood $V_2 \in \mathcal{B}$ and elements g and $\xi^{-1}(\overline{g}^{-1}) \in G$ there exists a neighbourhood $\overline{V}_3 \in \overline{\mathcal{B}}$ such that $g \cdot \overline{h} \cdot g^{-1}\overline{h}^{-1} \in V_2$ and

$$\xi^{-1}(\overline{g}^{-1}) \cdot h \cdot \big(\xi^{-1}(\overline{g}^{-1})\big)^{-1} \cdot h^{-1} \in V_2$$

for every element $h \in \xi^{-1}(\overline{V}_3)$. Without loss of generality assume $\overline{V}_3^{-1} = \overline{V}_3 \subseteq \overline{V}_2$.

Prove that

$$(g,\ \overline{g}) \cdot W(V_2,\ \overline{V}_3) \cdot (g,\ \overline{g})^{-1} \subseteq W(V_1,\ \overline{V}_1).$$

Indeed, if $\big(v \cdot \xi^{-1}(\overline{v}),\ \overline{v}\big) \in W(V_2,\ \overline{V}_3)$, then $v \in V_2$ and $\overline{v} \in \overline{V}_3$. Hence

$(g,\ \overline{g}) \cdot \big(v \cdot \xi^{-1}(\overline{v}),\ \overline{v}\big) \cdot (g,\ \overline{g})^{-1} = \big(g \cdot v \cdot \xi^{-1}(\overline{v}) \cdot g^{-1},\ \overline{g} \cdot \overline{v} \cdot \overline{g}^{-1}\big) =$
$\Big(\big(g \cdot v \cdot \xi^{-1}(\overline{v}) \cdot g^{-1}\big) \cdot \big(\xi^{-1}(\overline{g} \cdot \overline{v}^{-1} \cdot \overline{g}^{-1})\big) \cdot \big(\xi^{-1}(\overline{g} \cdot \overline{v}^{-1} \cdot \overline{g}^{-1})\big)^{-1},\ \overline{g} \cdot \overline{v} \cdot \overline{g}^{-1}\Big) =$
$\Big(\big(g \cdot v \cdot \xi^{-1}(\overline{v}) \cdot g^{-1}\big) \cdot \big(\xi^{-1}(\overline{g} \cdot \overline{v}^{-1} \cdot \overline{g}^{-1})\big) \cdot \big(\xi^{-1}(\overline{g} \cdot \overline{v} \cdot \overline{g}^{-1})\big),\ \overline{g} \cdot \overline{v} \cdot \overline{g}^{-1}\Big)$ where
$\overline{g} \cdot \overline{v} \cdot \overline{g}^{-1} \in \overline{g} \cdot \overline{V}_3 \cdot \overline{g}^{-1} \subseteq \overline{g} \cdot \overline{V}_2 \cdot \overline{g}^{-1} \subseteq \overline{V}_1$ and
$\big(g \cdot v \cdot \xi^{-1}(\overline{v}) \cdot g^{-1}\big) \cdot \big(\xi^{-1}(\overline{g} \cdot \overline{v}^{-1} \cdot \overline{g}^{-1})\big) =$
$\big(g \cdot v \cdot g^{-1} \cdot g \cdot \xi^{-1}(\overline{v}) \cdot g^{-1}\big) \cdot \big((\xi^{-1}(\overline{v}))^{-1} \cdot \xi^{-1}(\overline{v}) \cdot \xi^{-1}(\overline{g}) \cdot \xi^{-1}(\overline{v}^{-1}) \cdot \xi^{-1}(\overline{g}^{-1})\big) \in$
$\big(g \cdot V_2 \cdot g^{-1}\big) \cdot \big(g \cdot \xi^{-1}(\overline{v}) \cdot g^{-1} \cdot (\xi^{-1}(\overline{v}))^{-1}\big) \times$
$\big(\xi^{-1}(\overline{v}) \cdot \xi^{-1}(\overline{g}) \cdot \xi^{-1}(\overline{v}^{-1}) \cdot \xi^{-1}(\overline{g}^{-1})\big) \subseteq$
$\big(g \cdot V_2 \cdot g^{-1}\big) \cdot V_2 \cdot \Big(\xi^{-1}(\overline{g}) \cdot \xi^{-1}(\overline{g}^{-1}) \cdot \xi^{-1}(\overline{v}) \cdot \xi^{-1}(\overline{g}) \cdot \xi^{-1}(\overline{v}^{-1}) \cdot \xi^{-1}(\overline{g}^{-1})\Big) =$
$\big(g \cdot V_2 \cdot g^{-1}\big) \cdot V_2 \cdot \Big(\xi^{-1}(\overline{g}) \cdot \big(\xi^{-1}(\overline{g}^{-1}) \cdot \xi^{-1}(\overline{v}) \cdot \xi^{-1}(\overline{g})\big) \cdot \big((\xi^{-1}(\overline{v}))^{-1}\big) \cdot \xi^{-1}(\overline{g}^{-1})\Big) \subseteq$
$\big(g \cdot V_2 \cdot g^{-1}\big) \cdot V_2 \cdot \big(\xi^{-1}(\overline{g}) \cdot V_2 \cdot \xi^{-1}(\overline{g}^{-1})\big) \subseteq V_1,$

since $\xi^{-1}(\overline{v}) \in \xi^{-1}(\overline{V}_3)$ (see the definition of the neighbourhood $\overline{V}_3$). Hence

$$(g,\ \overline{g}) \cdot (v \cdot \xi^{-1}(\overline{v}),\ \overline{v}) \cdot (g,\ \overline{g})^{-1} =$$

$$\Big(\big(g \cdot v \cdot \xi^{-1}(\overline{v}) \cdot g^{-1}\big) \cdot \big(\xi^{-1}(\overline{g} \cdot \overline{v}^{-1} \cdot \overline{g}^{-1})\big) \cdot \big(\xi^{-1}(\overline{g} \cdot \overline{v} \overline{g}^{-1})\big),\ \overline{g} \cdot \overline{v} \cdot \overline{g}^{-1}\Big) \in W(V_1,\ \overline{V}_1).$$

Since $(v \cdot \xi^{-1}(\overline{v}),\ \overline{v})$ is an arbitrary element then

$$(g,\ \overline{g}) \cdot W(V_2,\ \overline{V}_3) \cdot (g,\ \overline{g})^{-1} \subseteq W(V_1,\ \overline{V}_1),$$

i.e. the assertion (GV_{III}) holds.

Therefore the set $\widehat{\mathcal{B}}$ is a basis of neighbourhoods of the identity in a certain group topology $\widehat{\tau}$ on the group $\widehat{G}$. Prove that the topological group $(\widehat{G},\ \widehat{\tau})$ is the desired one.

One can easily see that $G' = \{(g,\ \overline{e}) \mid g \in G\}$ is a normal subgroup in $\widehat{G}$ and, since $W(V,\ \overline{V}) \cap G' = \{(g,\ \overline{e}) \mid g \in V\}$, then the mapping $\xi' : (G,\ \tau) \to (G',\ \widehat{\tau}\ |_{G'})$ which puts in correspondence the element $(g,\ \overline{e}) \in G'$ to the element $g \in G$ is a topological isomorphism.

Identify the topological group $(G,\ \tau)$ with a subgroup $(G',\ \widehat{\tau}|_{G'})$ of a topological group $(\widehat{G},\ \widehat{\tau})$ with respect to the mapping ξ'.

Note that G' is a normal subgroup in $\widehat{G}$. Hence taking into account the identification given above we get that $\xi(g,\ \overline{e}) = \xi(g)$ and hence the homomorphism $\widehat{\xi} : \widehat{G} \to \overline{G}$ putting $\widehat{\xi}(g,\ \overline{g})$ in correspondence to $\xi(g)$ is an extension of the isomorphism ξ. It remains to check only the homomorphism $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\overline{G},\ \overline{\tau})$ to be topological, i.e. to be continuous and open.

Let $\overline{V} \in \overline{\mathcal{B}}$. Since $(\overline{G},\ \overline{\tau})$ is a topological group then there exists a neighbourhood $\overline{V}_1 \in \overline{\mathcal{B}}$ such that $\overline{V}_1 {\cdot} \overline{V}_1 \subseteq \overline{V}$. Since $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$ is a continuous isomorphism then there exists a neighbourhood $V_1 \in \mathcal{B}$, such that $\xi(V_1) \subseteq \overline{V}_1$. Hence

$$\begin{aligned}
&\widehat{\xi}(W(V_1,\ \overline{V}_1)) = \{\widehat{\xi}\big(g \cdot \xi^{-1}(\overline{g}),\ \overline{g}\big) \mid g \in V_1,\ \overline{g} \in \overline{V}_1\} = \\
&\{\widehat{\xi}\big(g \cdot \xi^{-1}(\overline{g})\big) \mid g \in V_1,\ \overline{g} \in \overline{V}_1\} = \\
&\{\xi(g) \cdot \overline{g}) \mid g \in V_1,\ \overline{g} \in \overline{V}_1\} = \xi(V_1) \cdot \overline{V}_1 \subseteq \overline{V}_1 \cdot \overline{V}_1 \subseteq \overline{V},
\end{aligned}$$

and hence $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\overline{G},\ \overline{\tau})$ is a continuous homomorphism.

Since

$$\widehat{\xi}(W(V,\ \overline{V})) = \{\widehat{\xi}\big(g \cdot \xi^{-1}(\overline{g}),\ \overline{g}\big) \mid g \in V,\ \overline{g} \in \overline{V}\} =$$

$$\{\widehat{\xi}\big(g \cdot \xi^{-1}(\overline{g})\big) \mid g \in V,\ \overline{g} \in \overline{V}\} \supseteq \{\xi(e) \cdot \overline{g}) \mid \overline{g} \in \overline{V}\} = \overline{V}$$

for every neighbourhood $\overline{V} \in \overline{\mathcal{B}}$ then $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\overline{G},\ \overline{\tau})$ is an open homomorphism, that completes the proof of Theorem. □

5 Corollary. *Let $(G,\ \tau)$ be a group equipped with the discrete topology, $(\overline{G},\ \overline{\tau})$ be a topological group and $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$ be a continuous isomorphism. The isomorphism ξ is semitopological iff for every element $g \in G$ there exists a neighbourhood $\overline{V}$ of the identity in $(\overline{G},\ \overline{\tau})$ such that $g \cdot \big(\xi^{-1}(\overline{v})\big) = \big(\xi^{-1}(\overline{v})\big) \cdot g$ for every $\overline{v} \in \overline{V}$.*

Proof. Necessity. Indeed, since the topology τ is discrete, then $V_0 = \{e\}$ is a neighbourhood of the identity in $(G,\ \tau)$. If $\overline{V}_1$ is a neighbourhood of the identity in $(\overline{G},\ \overline{\tau})$ such that its elements satisfy the condition 2 of Theorem 4 for the element $g \in G$ and neighbourhood of the identity V_0 in $(G,\ \tau)$, then $g {\cdot} \xi^{-1}(\overline{v}) {\cdot} (\xi^{-1}(\overline{v}))^{-1} {\cdot} g^{-1} = e$ for every element $\overline{v} \in \overline{V}$, which is equivalent to the assertion $g \cdot \xi^{-1}(\overline{v}) = \xi^{-1}(\overline{v}) \cdot g$ for every element $\overline{v} \in \overline{V}$.

Sufficiency. Let V be an arbitrary neighbourhood of the identity in $(G,\ \tau)$. Since $\{e\}$ a neighbourhood of the identity in $(G,\ \tau)$ and $\xi^{-1}(\bar{g}) \cdot e \cdot (\xi^{-1}\bar{g})^{-1} = e \in V$ for every element $\bar{g} \in \bar{G}$ then the condition 1 of Theorem 4 holds.

Except that since for every element $\bar{g} \in \bar{G}$ there exists a neighbourhood $\bar{V}_1$ of the identity in $(\bar{G},\ \bar{\tau})$ such that $g \cdot \xi^{-1}(\bar{v}) = \xi^{-1}(\bar{v}) \cdot g$ for every $\bar{v} \in \bar{V}_1$ then $g \cdot \xi^{-1}(\bar{v}) \cdot g^{-1} \cdot (\xi^{-1}(\bar{v}))^{-1} = e \in V$ for every $\bar{v} \in \bar{V}_1$. Hence the condition 2 Theorem 4 holds. $\square$

6 Corollary. *Let G and $\bar{G}$ be groups and $f : G \to \bar{G}$ be a certain group isomorphism. If $\{\tau_\gamma |\ \gamma \in \Gamma\}$ and $\{\bar{\tau}_\gamma |\ \gamma \in \Gamma\}$ are such families of group topologies on G and $\bar{G}$, respectively, that for every $\gamma \in \Gamma$ the isomorphism $f : (G,\ \tau_\gamma) \to (\bar{G},\ \bar{\tau}_\gamma)$ is semitopological where $\tau = sup\{\tau_\gamma |\ \gamma \in \Gamma\}$ and $\bar{\tau} = sup\{\bar{\tau}_\gamma |\ \gamma \in \Gamma\}$, then so is the isomorphism $f : (G,\ \tau) \to (\bar{G},\ \bar{\tau})$.*

The corollary follows from Theorem 4 and from the outlook of neighbourhoods of the identity in $sup\{\tau_\gamma |\ \gamma \in \Gamma\}$ (see 1.2.22 in [3]).

7 Theorem. *Let $(G,\ \tau)$, $(\overline{G},\ \overline{\tau})$ be topological groups and $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$ be a semitopological isomorphism. If A is a subgroup of the group G and $\overline{A} = \xi(A)$, then $\xi\ |_A : (A,\ \tau\ |_A) \to (\overline{A},\ \overline{\tau}\ |_{\overline{A}})$ is a semitopological isomorphism.*

Proof. If U is a neighbourhood of the identity in $(A,\ \tau\ |_A)$ then $U = V \cap A$ for a certain neighbourhood V of the identity in $(G,\ \tau)$. Since $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$ is a semitopological isomorphism then there exist neighbourhoods $\overline{V}$ and V_1 of the identity in $(\overline{G},\ \overline{\tau})$ and $(G,\ \tau)$ respectively such that $v \cdot V_1 \cdot v^{-1} \subseteq V$ for every element $v \in \xi^{-1}(\overline{V})$. Hence $(V_1 \cap A)$ and $\overline{A} \cap \overline{V}$ are neighbourhoods of identities in $(A,\ \tau\ |_A)$ and $(\overline{A},\ \overline{\tau}\ |_{\overline{A}})$ respectively. Note that since $\xi : G \to \overline{G}$ is an isomorphism, then $\xi^{-1}(\overline{A}) = \xi^{-1}(\xi(A)) = A$ and hence $v \cdot (V_1 \cap A) \cdot v^{-1} \subseteq V \cap A = U$ for every $v \in \xi^{-1}\big(\overline{A} \cap \overline{V}\big)$. It means that the assertion 1 of Theorem 4 holds for the isomorphism $\xi\ |_A : (A,\ \tau\ |_A) \to (\overline{A},\ \overline{\tau}\ |_{\overline{A}})$.

Let $g \in A$ and U be a neighbourhood of the identity in $(A,\ \tau\ |_A)$. Hence $U = V \cap A$ for a certain neighbourhood V of the identity in $(G,\ \tau)$. Since $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$ is a semitopological isomorphism then for a neighbourhood V of the identity in topological group $(G,\ \tau)$ and for the element $g \in A \subseteq G$ there exists a neighbourhood $\overline{V}_1$ of the identity in $(\overline{G},\ \overline{\tau})$ such that $g \cdot v \cdot g^{-1} \cdot v^{-1} \in V$ for every $v \in \xi^{-1}(\overline{V}_1)$. Since $\xi^{-1}(\overline{A}) = \xi^{-1}(\xi(A)) = A$ then $g \cdot v \cdot g^{-1} \cdot v^{-1} \in V \cap A = U$ for every $v \in \xi^{-1}(\overline{V}_1 \cap \overline{A})$, i.e. the assertion 2 of Theorem 4 holds for the isomorphism $\xi\ |_A : (A,\ \tau\ |_A) \to (\overline{A},\ \overline{\tau}\ |_{\overline{A}})$. Hence $\xi\ |_A : (A,\ \tau\ |_A) \to (\overline{A},\ \overline{\tau}\ |_{\overline{A}})$ is a semitopological isomorphism. $\square$

8 Theorem. *Let $(G,\ \tau)$ and $(\overline{G},\ \overline{\tau})$ be topological groups and $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$ be a semitopological isomorphism. If A is a normal subgroup of the group G and*

$$\begin{tikzcd} G \arrow[r, "\xi"] \arrow[d, "\eta"'] & \overline{G} \arrow[d, "\overline{\eta}"] \\ G/A \arrow[r, "\widehat{\xi}"] & \overline{G}/\xi(A) \end{tikzcd}$$

where:

$\eta : G \to G/A$ *is the canonical homomorphism (i.e.* $\eta(g) = g \cdot A$*);*

$\overline{\eta} : \overline{G} \to \overline{G}/(\xi(A))$ *is the canonical homomorphism (i.e.* $\overline{\eta}(\overline{g}) = \overline{g} \cdot \xi(A)$*);*

$\widehat{\xi} : G/A \to \overline{G}/(\xi(A))$ *is the canonical isomorphism (i.e.* $\widehat{\xi}(g \cdot A) = \xi(g) \cdot \xi(A)$*).*

Hence $\widehat{\xi} : (G,\ \tau)/A \to (\overline{G},\ \overline{\tau})/(\xi(A))$ *is a semitopological isomorphism.*

Proof. If $\widehat{V}_0$ is a neighbourhood of the identity in $(G,\ \tau)/A$ then $V_0 = \eta^{-1}(\widehat{V}_0)$ is a neighbourhood of the identity in $(G,\ \tau)$. By the assertion 1 of Theorem 4 there exist neighbourhoods V_1 and $\overline{V}_1$ of the identity in $(G,\ \tau)$ and $(\overline{G},\ \overline{\tau})$ respectively such that $v \cdot V_1 \cdot v^{-1} \subseteq V_0$ for every $v \in \xi^{-1}(\overline{V}_1)$. Hence $\widehat{V}_1 = \eta(V_1)$ and $\widetilde{V_1} = \overline{\eta}(\overline{V}_1)$ are neighbourhoods of the identity in $(G,\ \tau)/A$ and $(\overline{G},\ \overline{\tau})/\xi(A)$, respectively.

Note that $\widehat{\xi}\big(\eta(\xi^{-1}(\overline{V}_1))\big) = \overline{\eta}(\overline{V}_1) = \widetilde{V_1}$. Hence if $\widehat{v} \in \widehat{\xi}^{-1}(\widetilde{V_1})$ then there exists an element $v \in \xi^{-1}(\overline{V}_1)$ such that $\eta(v) = \widehat{v}$. Hence

$$\widehat{v} \cdot \widehat{V}_1 \cdot \widehat{v}^{-1} = \eta(v) \cdot \eta(V_1) \cdot (\eta(v))^{-1} = \eta(v \cdot V_1 \cdot v^{-1}) \subseteq \eta(V_0) = \eta\big(\eta^{-1}(\widehat{V}_0)\big) = \widehat{V}_0.$$

Hence the assertion 1 of Theorem 4 holds for the isomorphism
$\widehat{\xi} : (G,\ \tau)/A \to (\overline{G},\ \overline{\tau})/(\xi(A))$.

Check the assertion 2 of Theorem 4 to hold for the isomorphism
$\widehat{\xi} : (G,\ \tau)/A \to (\overline{G},\ \overline{\tau})/(\xi(A))$.

Let $\widetilde{g} \in (\overline{G},\ \overline{\tau})/(\xi(A))$ and $\widehat{V}$ be a neighbourhood of the identity in $(G,\ \tau)/A$. Hence $V = \eta^{-1}(\widehat{V})$ is a neighbourhood of the identity in $(G,\ \tau)$ and there exists an element $g \in G$ such that $\widehat{g} = \eta(g)$. Since the assertion 2 of Theorem 4 holds for the homomorphism $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$, then there exists such a neighbourhood $\overline{V}_1$ of the identity in $(\overline{G},\ \overline{\tau})$ that $g \cdot v \cdot g^{-1} \cdot v^{-1} \in V$ for every $v \in \xi^{-1}(\overline{V}_1)$. Hence $\widetilde{V_1} = \overline{\eta}(\overline{V}_1)$ is a neighbourhood of the identity in $(\overline{G},\ \overline{\tau})/(\xi(A))$. Note that $\eta\big(\xi^{-1}(\overline{V}_1)\big) = \widehat{\xi}^{-1}(\overline{\eta}(\overline{V}_1)) = \widehat{\xi}^{-1}(\widetilde{V}_1)$.

If $\widehat{v} \in \widehat{\xi}^{-1}(\widetilde{V}_1))$, then there exists an element $v \in \xi^{-1}(\overline{V}_1)$ such that $\eta(v) = \widehat{v}$. Hence

$$\widehat{g} \cdot \widehat{v} \cdot \widehat{g}^{-1} \cdot \widehat{v}^{-1} = \eta(g) \cdot \eta(v) \cdot \big(\eta(g)\big)^{-1} \cdot \big(\eta(v)\big)^{-1} = \eta(g \cdot v \cdot g^{-1} \cdot v^{-1}) \in \eta(V) = \widehat{V},$$

i.e. the assertion 2 of Theorem 4 holds for the isomorphism
$\widehat{\xi} : (G,\ \tau)/A \to (\overline{G},\ \overline{\tau})/(\xi(A))$. The theorem is completely proved. □

9 Theorem. *Let* $\{(G_\gamma,\ \tau_\gamma) \mid \gamma \in \Gamma\}$ *and* $\{(\overline{G}_\gamma,\ \overline{\tau}_\gamma) \mid \gamma \in \Gamma\}$ *be two families of topological groups and for every* $\gamma \in \Gamma$ *there exists a semitopological isomorphism* $\xi_\gamma : (G_\gamma,\ \tau_\gamma) \to (\overline{G}_\gamma,\ \overline{\tau}_\gamma)$*. If* $(\widehat{G},\ \widehat{\tau}) = \prod\limits_{\gamma\in\Gamma}(G_\gamma,\ \tau_\gamma)$ *and* $(\widetilde{G},\ \widetilde{\tau}) = \prod\limits_{\gamma\in\Gamma}(\overline{G}_\gamma,\ \overline{\tau}_\gamma)$ *are direct products of these families equipped with the Tychonoff topology and* $\widehat{\xi} : \widehat{G} \to \widetilde{G}$ *is a canonical isomorphism (i.e.* $\xi_\gamma(\mathrm{pr}_\gamma(\widehat{g})) = \mathrm{pr}_\gamma\widehat{\xi}(\widehat{g})$*) for any* $\gamma \in \Gamma$*, then* $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\widetilde{G},\ \widetilde{\tau})$ *is a semitopological isomorphism.*

Proof. If $\widehat{V}$ is a neighbourhood of the identity in $(\widehat{G},\ \widehat{\tau})$, then there exists a finite subset $S \subseteq \Gamma$ such that for every $\gamma \in S$ there exists a neighbourhood V_γ in $(G_\gamma,\ \tau_\gamma)$

such that $\bigcap\limits_{\gamma\in S} {\mathrm{pr}_\gamma}^{-1}(V_\gamma) \subseteq \widehat{V}$. Since for every $\gamma \in \Gamma$ the mapping $\xi_\gamma : (G_\gamma,\ \tau_\gamma) \to (\overline{G}_\gamma,\ \overline{\tau}_\gamma)$ is a semitopological isomorphism then there exist neighbourhoods $\overline{V}_\gamma$ and U_γ of the identity in $(\overline{G}_\gamma,\ \overline{\tau}_\gamma)$ and $(G_\gamma,\ \tau_\gamma)$, respectively such that $v_\gamma \cdot U_\gamma \cdot v_\gamma^{-1} \subseteq V_\gamma$ for every elements $v_\gamma \in \xi^{-1}(\overline{V}_\gamma)$. Hence $\widetilde{V} = \bigcap\limits_{\gamma\in S} \mathrm{pr}_\gamma^{-1}(V_\gamma)$ and $\widehat{U} = \bigcap\limits_{\gamma\in S} \mathrm{pr}_\gamma^{-1}(U_\gamma)$ are neighbourhoods of the identity in $(\widetilde{G},\ \widetilde{\tau})$ and $(\widehat{G},\ \widehat{\tau})$ respectively.

If $\widehat{v} \in \widehat{\xi}^{-1}(\widetilde{V})$ then $\mathrm{pr}_\gamma(\widehat{v}) \in \xi_\gamma^{-1}(\overline{V}_\gamma)$ and therefore $\mathrm{pr}_\gamma(\widehat{v}) \cdot U_\gamma \cdot (\mathrm{pr}_\gamma(\widehat{v}))^{-1} \subseteq V_\gamma$ for every $\gamma \in S$. Hence $\widehat{v} \cdot \widehat{U} \cdot \widehat{v}^{-1} \subseteq \widehat{V}$, i.e. the condition 1 of Theorem 4 holds for the isomorphism $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\widetilde{G},\ \widetilde{\tau})$.

If $\widehat{g} \in \widehat{G}$ and $\widehat{V}$ is a neighbourhood of the identity in $(\widehat{G},\ \widehat{\tau})$, then there exists a finite set $S \subseteq \Gamma$ and for every $\gamma \in S$ there exists a neighbourhood V_γ of the identity in $(G_\gamma,\ \tau_\gamma)$ such that $\bigcap\limits_{\gamma\in S} \mathrm{pr}_\gamma^{-1}(V_\gamma) \subseteq \widehat{V}$. Since for every $\gamma \in \Gamma$ the mapping $\xi_\gamma : (G_\gamma,\ \tau_\gamma) \to (\overline{G}_\gamma,\ \overline{\tau}_\gamma)$ is a semitopological isomorphism then for the neighbourhood V_γ of the identity in the topological group $(G_\gamma,\ \tau_\gamma)$ and for the element $g_\gamma = \mathrm{pr}_\gamma(\widehat{g})$ there exists a neighbourhood $\overline{V}_\gamma$ of the identity in $(\overline{G}_\gamma,\ \overline{\tau}_\gamma)$ such that $g_\gamma \cdot v_\gamma \cdot g_\gamma^{-1} \cdot v_\gamma^{-1} \in V_\gamma$ for every $v_\gamma \in \xi_\gamma^{-1}(\overline{V}_\gamma)$. Hence $\widetilde{V} = \bigcap\limits_{\gamma\in S} {\mathrm{pr}_\gamma}^{-1}(\overline{V}_\gamma)$ is a neighbourhood of the identity in $(\widetilde{G},\ \widetilde{\tau})$ and if $\widehat{v} \in \widehat{\xi}^{-1}(\widetilde{V})$ then $v_\gamma = \mathrm{pr}_\gamma(\widehat{v}) \in \xi_\gamma^{-1}(\mathrm{pr}_\gamma(\widetilde{V})) = \xi_\gamma^{-1}(\mathrm{pr}_\gamma(\overline{V}_\gamma))$ for any $\gamma \in S$. Hence $\widehat{g} \cdot \widehat{v} \cdot \widehat{g}^{-1} \cdot \widehat{v}^{-1} \in \bigcap\limits_{\gamma\in S} \mathrm{pr}_\gamma^{-1}(V_\gamma) \subseteq \widehat{V}$, i.e. the assertion 2 of Theorem 4 holds for the isomorphism $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\widetilde{G},\ \widetilde{\tau})$.

Hence $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\widetilde{G},\ \widetilde{\tau})$ is a semitopological isomorphism. □

10 Remark. *Theorem 9 remains valid if groups* $\widehat{G} = \prod\limits_{\gamma\in\Gamma} G_\gamma$ *and* $\widetilde{G} = \prod\limits_{\gamma\in\Gamma} \overline{G}_\gamma$ *are equipped not with the Tychonoff topology but with the topology of* $\mathfrak{m}$*-product (see [3], Definition 4.1.3).*

11 Remark. *The following example proves that the superposition of semitopological isomorphisms needs not to be semitopological. The topological groups mentioned in it are not Hausdorff. An example with Hausdorff topological groups can be obtained by an easy modification of the given one.*

12 Example. *Let G be a nilpotent group of index 2 (i.e. G a noncommutative group such that its quotient group G/Z by its center Z is commutative). Consider the following three group topologies on the group G:*

τ_0 *is the discrete topology, i.e. the set* $\{\{e\}\}$ *is a basis of neighbourhoods of the identity in* $(G,\ \tau_0)$*;*

τ_1 *is the topology such that the set* $\{Z\}$ *is a basis of neighbourhoods of the identity in* $(G,\ \tau_1)$*;*

τ_2 *is the antidiscrete topology, i.e. the set* $\{G\}$ *is a basis of neighbourhoods of the identity in* $(G,\ \tau_2)$*;*

Let $\xi : G \to G$ be an identity mapping. One can easily see that assertions 1 and 2 of Theorem 4 hold for the continuous isomorphisms $\xi : (G,\ \tau_0) \to (G,\ \tau_1)$ *and* $\xi : (G,\ \tau_1) \to (G,\ \tau_2)$ *and hence they are semitopological.*

Prove now that the assertion 2 of Theorem 4 does not hold for the isomorphism $\xi : (G,\ \tau_0) \to (G,\ \tau_2)$, *i.e. it is not semitopological.*

Suppose the contrary, i.e. the assertion 2 of Theorem 4 holds for the isomorphism $\xi : (G,\ \tau_0) \to (G,\ \tau_2)$. *Since the group* G *is not a commutative group, then there exist elements* $g,\ v \in G$ *such that* $g \cdot v \neq v \cdot g$, *i.e.* $g \cdot v \cdot g^{-1} \cdot v^{-1} \neq e$. *Then for an element* $g \in G$ *and the neighbourhood* $\{e\}$ *of the identity in* $(G,\ \tau_0)$ *there exists a neighbourhood* V *of the identity in* $(G,\ \tau_2)$ *such that* $g \cdot u \cdot g^{-1} \cdot u^{-1} \in \{e\}$ *for every element* $u \in \xi^{-1}(V)$. *Since* τ_2 *is the antidiscrete topology then* $V = G$ *and hence we may assume the element* u *to be equal to* v. *Hence* $g \cdot v \cdot v^{-1} \cdot g^{-1} \in \{e\}$ *that contradicts to the choice of elements* $g,\ v \in G$.

13 Problem. *Given a class* $\mathfrak{K}$ *of topological groups (rings) and a group (ring)* G. *What is the group (ring) topology* τ *on* G *such that* $(G,\ \tau) \in \mathfrak{K}$ *and every semitopological homomorphism* $(G,\ \tau) \to (H,\ \mu)$ *is topological, where* $(H,\ \mu) \in \mathfrak{K}$ *(so are known to be the topological rings with no generalized zero divisors, see [1], Theorem 2).*

14 Problem. *What is the group (ring)* G *such that for every group (ring) topology* τ *on it every semitopological isomorphism* $(G,\ \tau) \to (H,\ \mu)$ *is topological (so are known to be the rings with an identity).*

15 Problem. *What are the continuous isomorphisms which are superpositions of semitopological (note that they need not to be semitopological, see the example 12).*

Author does not know whether every continuous isomorphism of topological groups is so.

16 Problem. *Let* G *and* $\overline{G}$ *be groups,* $f : G \to \overline{G}$ *be an isomorphism,* $\{\tau_\gamma \mid \gamma \in \Gamma\}$ *and* $\{\overline{\tau}_\gamma \mid \gamma \in \Gamma\}$ *be families of group topologies on* G *and* $\overline{G}$ *respectively such that* $f : (G, \tau_\gamma) \to (G, \overline{\tau}_\gamma)$ *is a semitopological isomorphism for every* $\gamma \in \Gamma$. *Write* τ *for* $\inf\{\tau_\gamma \mid \gamma \in \Gamma\}$ *and* $\overline{\tau}$ *for* $\inf\{\overline{\tau}_\gamma \mid \gamma \in \Gamma\}$. *Is the isomorphism* $f : (G, \tau) \to (\overline{G}, \overline{\tau})$ *semitopological?*

References

[1] Arnautov V.I. *Semitopological isomorphisms of topological rings.* Mathematical investigations, 1969, **4:2(12)**, p. 3–16 (In Russian).

[2] Bourbaki N. *General topology.* Moscow, Nauka, 1969 (In Russian).

[3] Arnautov V.I., Glavatsky S.T., Mikhalev A.V. *Introduction to the theory of topological rings and modules.* Marcel Dekker, Inc., New York-Basel-Hong Kong, 1996.

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Special radicals of graded rings

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Abstract. We consider the graded radicals of graded rings, and prove that any radical in the category ring graded by a group G can be defined by means of some class of graded modules. We also describe the classes of graded modules for special graded radical.

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1 Introduction

The general theory of radicals of rings and algebras began to develop in papers of A.G. Kurosh and S.A. Amitsur in the 1950s. They observed that the general theory of radicals can be developed in any algebraic systems which the concept of the kernel homomorphism with usual properties takes place in, i.e. in the "good" categories.

For basic notions and terminology on general theory of radicals we refer the reader to the monograph [1].

For given radical $\mathcal{R}$ in the category of associative rings and ring homomorphisms there are different ways of defining a graded version of $\mathcal{R}$:

At first, one may consider a natural definition for graded version of $\mathcal{R}_G$ in category of graded rings and graded-preserving ring homomorphisms.

Secondly, for defining a graded version of $\mathcal{R}$ for a graded ring A one can consider $\mathcal{R}(A)_G$, the largest graded ideal contained in $\mathcal{R}(A)$.

At last, it is possible to consider the largest graded ideal I of A such that $I \bigcap A_e = \mathcal{R}(A_e)$, where A_e is an identity graded component of A.

Besides using the generalized smash product, M. Beattie and P. Stewart [2] introduced a method for defining a so called reflexive radical $\mathcal{R}_{ref}$. They investigated the properties of reflexive radicals and compared them with graded radicals which had been previously studied.

The graded radicals of graded rings have been investigated in papers ([3],[4],[5], [6]).

On the other hand, in 1962 V.A. Andrunakievich and Yu.M. Ryabuhin showed that any special radical of an associative ring can be defined by means of some class of modules.

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The purpose of this paper is to prove analogous results for special radicals of category of the graded rings and to define the classes of graded modules corresponding to classical graded radicals.

2 Preliminaries

Let A be an associative ring (not necessaryly with identity), G a group with identity e.

A ring A is called *G-graded* (or simple *graded*) if there exists a family $\{A_g \mid g \in G\}$ of additive subgroups of A such that $A = \bigoplus_{g\in G} A_g$ and $A_g A_h \subseteq A_{gh}$ for all $g, h \in G$.

If a ring A has an identity 1, then $1 \in A_e$.

The elements of the set $h(A) = \bigcup\limits_{g\in G} A_g$ are called *homogeneous elements* of the ring A. A nonzero element $r_g \in A_g$ is said to be *homogeneous of degree g*.

Any nonzero r has a unique expression as a sum of homogeneous elements, $r = \sum\limits_{g\in S} r_g$, where r_g is nonzero for a finite number of $g \in G$. The nonzero elements r_g in the decomposition of r are called *homogeneous components* of r.

An ideal I is called *graded* (or *homogeneous*) if $I = \bigoplus_{g\in G}(I \cap A_g)$. For any ideal I of A (left, right or two-sided) the largest graded ideal of A contained in I will be denoted by I_G.

Let $A = \bigoplus_{g\in G} A_g$ and $B = \bigoplus_{g\in G} B_g$ be G-graded rings. A ring homomorphism $f : A \longrightarrow B$ is called graded-preserving if $f(A_g) \subseteq B_g$ for all $g \in G$.

The category of G-graded rings $G\mathcal{R}ings$ consists of G-graded rings and graded-preserving homomorphisms.

Let A be a G-graded ring. A right A-module M is called a *G-graded A-module* if there exists a family $\{M_g \mid g \in G\}$ of additive subgroups of M such that $M = \bigoplus_{g\in G} M_g$ and $M_g A_h \subseteq M_{gh}$ for all $g, h \in G$.

Let N and M be G-graded right A-module. A homomorphism $f : N \to M$ is called a *graded morphism of degree h* if $f(N_g) \subseteq M_{hg}$ for any $g \in G$. All graded morphisms of degree h form the additive subgroup $HOM(N_R, M_R)_h$ of the group $Hom(N_R, M_R)$.

A submodule $N \subseteq M$ is called a *graded submodule* if $N = \bigoplus_{g\in G}(N \cap M_g)$. In other words, $N \subseteq M$ is a graded submodule if for any $x \in N$ it follows that N contains all homogeneous components of x.

Let M be a graded module, N its graded submodule. Then M/N may be made into a graded module by putting $(M/N)_g = (M_g + N)/N$ for all $g \in G$. With this definition, the canonical projection $M \to M/N$ is a graded morphism of degree e.

Further details on graded rings and modules may be found in [3]

3 Graded modules and radicals

Let A be a ring, M a right A-module. The set $Ann_A(M) = \{a \in A | Ma = 0\}$ is called an annihilator of a A-module M. Recall that a module M is called faithful if $Ann_A(M) = 0$.

In [8] V.A. Andrunakievich and Yu.M. Ryabuhin defined a general class Σ of modules in the following way. For every associative ring A they denote some class of nontrivial right A-modules (can be empty) by Σ_A . The set

$$Ker\Sigma_A = \bigcap\{Ann_A(M)|M \in \Sigma_A\}$$

is called the kernel of the class Σ_A. If $\Sigma_A = \emptyset$ then let‘s assume $Ker\Sigma_A = A$.

The class Σ of all Σ_A is called a *general class of the modules*, if the following hold:

P1. If $M \in \Sigma_{A/B}$, then $M \in \Sigma_A$.

P2. If $M \in \Sigma_A$ and $B \subseteq Ann_A(M)$, then $M \in \Sigma_{A/B}$.

P3. If $Ker\Sigma_A = 0$, then $\Sigma_B \neq \emptyset$ for any nonzero ideal B of A.

P4. If $\Sigma_B \neq \emptyset$ for any nonzero ideal B of A, then $Ker\Sigma_A = 0$.

Using the general class of modules, they defined Σ-radical

$$\mathcal{R}(\Sigma, A) = Ker\Sigma_A = \bigcap\{Ann_A(M)|M \in \Sigma_A\}$$

for any associative ring A and proved that Σ-radical is a Kurosh-Amitsur radical. Conversely, if $\mathcal{R}$ is a radical in the category of associative rings, then there is a general class of modules Σ such that $\mathcal{R}$ is equal to Σ-radical ([8], Theorem 1).

In this case any Σ-semisimple ring is the subdirect product of a family of rings from $\mathcal{L}(\Sigma)$, where $\mathcal{L}(\Sigma)$ is the class of all rings A that have a faithful A-module from Σ_A ([8], Corollary 4).

Since the category of G-graded rings $G\mathcal{R}ings$ has all necessary properties the general theory of radicals in sense of Kurosh-Amitsur is valid.

It is straightforward to check that if M is a graded A-module, then the annihilator $Ann_A(M)$ is the graded ideal in A. We have the following

Proposition 1. *Let A be a G-graded ring, B a graded ideal of A. If M is a graded A/B-module, then M becomes a graded A-module by setting $xa = x(a + B)$ and $B \subseteq Ann_A(M)$. Conversely, if M is a graded A-module and B is a graded ideal, such that $B \subseteq Ann_A(M)$, then M is a graded A/B-module by setting $x(a+B) = xa$. Any graded submodule of A/B-module M is a graded submodule of A-module M too, and conversely. Moreover, $Ann_{A/B}(M) = Ann_A(M)/B$ (as graded ideals).*

Proof. The proof follows from ([8], Proposition 1) as the given definition is coordinated with grading.

On the other hand, it is possible to define *the general class of G-graded modules* Σ_G as the class all Σ_{GA} satisfying conditions GP1-GP4, which are obtained from the corresponding conditions P1-P4 by replacement of the word "ideal" by the word "graded ideal" (here Σ_{GA} is some class of right graded modules over G-graded ring A.)

Then the *graded Σ_G-radical* of G-graded ring A is defined as

$$\mathcal{R}(\Sigma_G, A) = Ker\Sigma_{GA} = \bigcap\{Ann_A(M)|M \in \Sigma_{GA}\}$$

Theorem 2. *Let Σ_G be a general class of G-graded modules, then Σ_G-radical is a radical in the category of graded rings $G\mathcal{R}ings$. Conversely, if $\mathcal{R}$ is a radical in the category $G\mathcal{R}ings$, then there exists a general class of G-graded modules Σ_G such that $\mathcal{R}$ coincides with the Σ_G-radical.*

Moreover, any Σ_G-semisimple graded ring is the subdirect product of graded rings A which have faithful A-modules from the class Σ_{GA}.

Proof. The proof of this theorem using Proposition 1 and conditions GP1-GP4 is essentially the same as that in the ungraded case (see [8], Theorem 1).

4 Special graded radicals

Recall that a special radical $R_{\mathfrak{M}}(A)$ of an associative ring A is an upper radical, defined by any special class of rings $\mathfrak{M}$, i.e. for any ring A, $R_{\mathfrak{M}}(A)$ is equal to the intersection of all ideals P of A such that $A/P \in \mathfrak{M}$ ([1], chapter 3)

In [7] the general classes of modules, corresponding to special radicals, are called *the special classes of modules* and the special classes of modules for classical special radicals were determined.

Using the concept of a special radical in categories ([1], Chapter 5) we can define a special radical in the category of G-graded rings $G\mathcal{R}ings$.

Let $\mathfrak{M}$ be some class of graded rings, then a graded ideal P of a G-graded ring A is called an $\mathfrak{M}$-ideal if $A/P \in \mathfrak{M}$.

A class $\mathfrak{M}$ of G-graded rings is said to be *a special class* if it satisfies the following conditions:

M1. If B is a graded ideal of a ring A and P is an $\mathfrak{M}$-ideal in A which does not contain of B, then $P\bigcap B$ is a proper $\mathfrak{M}$-ideal in B.

M2. If Q is a proper $\mathfrak{M}$-ideal in B and B is a graded ideal of a ring A, then there exists only one $\mathfrak{M}$-ideal P in A such that $P\bigcap B = Q$.

Proposition 3. *Let $\mathfrak{M}$ be a special class of graded rings, Σ_G a general class of graded modules such that the special radical $R_{\mathfrak{M}}$ is equal to Σ_G-radical. Then a graded ideal P of a ring A is an $\mathfrak{M}$-ideal if and only if $P = Ann_A(M)$ for some graded A-module $M \in \Sigma_{GA}$.*

Proof. Let P be a graded $\mathfrak{M}$-ideal of a ring A. From Theorem 2 we have that there exists a faithful graded A/P-module $M \in \Sigma_{G(A/P)}$. By (GP1), M belongs to Σ_{GA}. Hence $P = Ann_A(M)$ by Proposition 1.

Conversely, if $P = Ann_A(M)$ for some graded A-module $M \in \Sigma_{GA}$, then by Proposition 1 and (GP2) we obtain that M is a faithful graded A/P-module. Therefore $A/P \in \mathfrak{M}$. The proof is complete.

From ([1], Chapter 5) we have that there is the largest special class of graded rings $\mathfrak{P}$.

Recall that a graded ideal P of a graded ring A is said to be *gr-prime* if for any graded ideals I, J such that $IJ \subseteq P$ we have either $I \subseteq P$ or $J \subseteq P$. A graded ring is called *gr-prime* if (0) is the gr-prime ideal.

Proposition 4. *The largest special class $\mathfrak{P}$ of graded rings coincides with the class of all gr-prime rings.*

Proof. Assume that $A \in \mathfrak{P}$ and A is not gr-prime. Then there are nonzero graded ideals I and J such that $IJ = 0$. From ([1], Ch.5, §5, Proposition5) we have that $K = I \bigcap J \neq 0$ is a graded ideal in A, $K^2 \subseteq IJ = 0$ and $K \in \mathfrak{P}$. Now consider $E = K \oplus K$. Then the ideals $P_1 = \{(k, 0)|k \in K\}$, $P_2 = \{(0, k)|k \in K\}$ and $B = \{(k, k)|k \in K\}$ belong to $\mathfrak{P}$, but for $\mathfrak{P}$-ideals P_1 and P_2 we have $P_1 \bigcap B = P_1 \bigcap B = 0$. This contradicts condition M2.

Let‘s show that the class of all gr-prime rings is special.

Let P be a gr-prime ideal of A, B a graded ideal of A, and $IJ \subseteq P \bigcap B$ for some graded ideals I and J in B. Denote by I_A and J_A the ideals in A, generated by I and J respectively. Since $I_A^3 \subseteq I \subseteq I_A$, $J_A^3 \subseteq J \subseteq J_A$, then $I_A^3 J_A^3 \subseteq P$. As an ideal P is gr-prime, then either $I_A^3 \subseteq P$ or $J_A^3 \subseteq P$. Hence either $I \subseteq P \bigcap B$ or $J \subseteq P \bigcap B$, therefore $Q = P \bigcap B$ is gr-prime. Thus condition M1 is carried out.

Let Q be a proper gr-prime ideal of B. Define the set

$$P_0 = \{a \in A|aB \subseteq Q, \quad Ba \subseteq Q\}.$$

As Q is gr-prime it is straightforward to check that P_0 is a gr-prime ideal in A and $P_0 \bigcap B = Q$. Let P be any gr-prime ideal in A such that $P \bigcap B = Q$. Since Q is gr-prime, $P \subseteq P_0$. On the other hand, $BP_0 \subseteq P_0 \bigcap B = Q \subseteq P$. Therefore by primeness of P we have $P_0 \subseteq P$. Thus condition M2 is carried out. The proof is complete.

From Proposition 4 and ([1], Ch. 5, §5, Theorem 4) we shall receive the following description of special classes of graded rings.

A class $\mathfrak{M}$ of graded rings is *special* if and only if the following hold:

GA1. All rings belonging to $\mathfrak{M}$ are gr-prime.

GA2. If $A \in \mathfrak{M}$ and I is a nonzero graded ideal of a ring A, then $I \in \mathfrak{M}$.

GA3. If B is a graded ideal of a gr-prime ring A and $B \in \mathfrak{M}$, then $A \in \mathfrak{M}$.

Consider now special graded radicals.

The graded Jacobson radical. Recall that a graded right A-module M is *gr-irreducible* if $MA = M$ and M does not contain non-trivial graded submodules. A graded ring A is *gr-primitive* (right) if there is a faithful gr-irreducible right A-module.

The graded Jacobson radical of A, $\mathcal{J}_G(A)$, may be defined equivalently :

(1) [3] the intersection of all annihilators of gr-irreducible right A-modules,

(2) [3] $\mathcal{J}_G(A)$ is also the intersection of the maximal graded right ideals of A,

(3) the special radical, defined by the class of all gr-primitive rings.

Thus, if Σ_G is the class of all gr-irreducible right A-modules, then $\mathcal{J}_G(A)$ is a Σ_G-radical.

In [9] G. Abrams and C. Menini defined for semigroup-graded rings A, the graded Jacobson radical $\mathcal{J}_{gr}(A)$ as the intersection of all annihilators of gr-irreducible $*$-graded A-modules. For a semigroup S (possibly with zero ν) a S-graded module M is called $*$-graded if its ν-component M_ν is equal to (0). They provided various conditions on A which imply that $\mathcal{J}_{gr}(A) \subseteq \mathcal{J}(A)$.

Remark 5. Note that definitions (1)–(3) above for graded Jacobson radical are not equivalent if we consider rings graded by arbitrary semigroups, as in this case the annihilator $Ann_A(M)$ of graded A-modules can be an ungraded ideal.

Example 6. Let S be the simplest rectangle band, i.e. a semigroup $S = \{(m,n) \mid m,n = 1,2\}$ with multiplication defined by $(x,y)(z,t) = (x,t)$. Consider a semigroup ring $A = kS$ with coefficients in a field k, S-graded in the usual way.

Let M be a gr-irreducible right A-module, then $M = mA$ for any nonzero homogeneous element m from M. Hence, the element $(1,1) - (2,1)$ belongs to $Ann_A(M)$ for any gr-irreducible A-module M, and thus $(1,1) - (1,2) \in J_{gr}(A)$.

On other hand, $(1,1) \notin Ann_A(P)$ for gr-irreducible A-module $P = (1,1)A$, and consequently $(1,1) \notin J_{gr}(A)$. Thus in this case $\mathcal{J}_{gr}(A)$ is an ungraded ideal of A.

The graded prime radical. The graded prime radical of A $\mathcal{B}_G(A)$ is the intersection of all gr-prime ideals of A ([3],[4]).

In [6] a graded right A-module M is called *gr-prime* if for every nonzero graded submodule N of M and every graded ideal I of A, $NI = 0$ implies $I \subseteq Ann_A(M)$. They defined the graded prime radical $\mathcal{B}_G(A)$ as the intersection of the annihilators of gr-prime modules.

It is straightforward to check that these definitions are equivalent.

Thus $\mathcal{B}_G(A)$ is the smallest graded special radical and it is the Σ_G-radical, generated by the class Σ_G of all gr-prime modules.

The graded Levitzki radical. For a graded ring A the graded Levitzki radical $\mathcal{L}_G(A)$ is the intersection of the gr-prime ideals P of A such that A/P has no nonzero graded locally nilpotent ideal [2].

Since $\mathcal{L}_G(A) = \mathcal{L}(A)_G$ ([2], proposition 3.2), $\mathcal{L}_G(A)$ is the largest locally nilpotent graded ideal.

A gr-prime A-module M is called a *graded Levitzki A-module* if $A/Ann_A(M)$ has no nonzero graded locally nilpotent ideal. Let Σ_G be the class of all graded Levitzki modules. Then the Σ_G-radical $R(\Sigma_G, A)$ coincides with the graded Levitzki radical $\mathcal{L}_G(A)$.

The graded Köthe radical. For a graded ring A the graded Köthe radical $\mathcal{K}_G(A)$ is the largest graded nilideal. It is clear that $\mathcal{K}_G(A) = (\mathcal{K}(A))_G$ and $\mathcal{K}_G(A)$ is the intersection of the gr-prime ideals P of A such that A/P has no nonzero graded nilideal.

A gr-prime A-module M is called a *graded Köthe A-module* if $A/Ann_A(M)$ has no nonzero graded nilideal. Then the graded Köthe radical $\mathcal{K}_G(A)$ is the Σ_G-radical, generated by the class of all graded Köthe modules.

The graded Brown-McCoy radical. For a graded ring A the graded Brown-McCoy radical $\mathcal{U}_G(A)$ is the intersection of the graded ideals P of A such that A/P is a graded simple ring with identity.

For every G-graded ring A, $\mathcal{U}(A)_G \subseteq \mathcal{U}_G(A)$, and this inclusion may be proper. ([2], proposition 3.5).

Like in [7], a graded right A-module M is called *gr-simple* if $MA \neq 0$ and for every graded ideal I of A such that $MI \neq 0$, there exits $b \in I_e$ such that $mb = m$ for all $m \in M$.

It is straightforward to check that the class Σ_G of all gr-simple modules defines the Σ_G-radical that coincides with the graded Brown-McCoy radical $\mathcal{U}_G(A)$.

The graded compressive radical. Recall that $\mathcal{A}(A)$, the compressive radical of a ring A, is the intersection of all ideals I of A such that A/I has no zero divisor.

It is straightforward to check that the class of all graded rings which have no homogeneous zero divisor is special.

For a graded ring A the graded compressive radical $\mathcal{A}_G(A)$ is the intersection of the graded ideals I of A such that A/I has no homogeneous zero divisor.

Recall that a nonzero element $m \in M_A$ is called a zero divisor if there exits $a \in A$, $a \notin Ann_A M$ such that $ma = 0$. The class Σ_G of all graded modules which have no homogeneous zero divisor defines the graded compressive radical $\mathcal{A}_G(A)$.

Remark 7. Note that these results will be true if we consider rings graded by cancellative semigroups. However, for an arbitrary semigroup it does not hold.

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References

[1] Andrunakievich V.A., Ryabuhin Yu.A. *Radicals of algebras and structure theory.* Moskva, Nauka, 1979.

[2] Beattie M.A., Stewart P.N. *Graded radicals of graded rings.* Acta Math. Hung., 1991, **58**, no. 3–4, p. 261–272.

[3] Năstăsescu C., Van Oystaeyen F. *Graded ring theory.* North-Holland, Amsterdam. 1982.

[4] Cohen M., Montgomery S. *Group-graded ring, smash products, and group action.* Trans. Amer. Math. Soc., 1984, **282**, no. 1, p. 237–258.

[5] Beattie M.A., Liu S.-X., Stewart P. *Comparing graded versions of the prime radical.* Canad. Math. Bull., 1991, **34**, no. 2, p. 158–164.

[6] Liu S.-X., Van Oystaeyen F. *Group-graded rings,smash product and additive categories.* Perspectives in ring theory, Kluwer Academ Press, 1988, p. 299–300.

[7] Andrunakievich V.A., Ryabuhin Yu.M. *Special modules and special radicals.* Dokl. AN SSSR, 1962, **147**, no. 6, p. 1274–1277.

[8] Andrunakievich V.A., Ryabuhin Yu.M. *Modules and radicals*. Dokl. AN SSSR, 1964, **156**, no. 5, p. 991–994.

[9] Abrams G., Menini C. *Embedding modules in graded modules over semigroup-graded rings*. Comm. Algebra, 2001, **29**, no. 16, p. 2611–2625.

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Generating properties of biparabolic invertible polynomial maps in three variables

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Abstract. Invertible polynomial map of the standard 1-parabolic form $x_i \to f_i(x_1, \dots, x_{n-1}),\ \ i < n, x_n \to \alpha x_n + h_n(x_1, \dots, x_{n-1})$ is a natural generalization of a triangular map. To generalize the previous results about triangular and bitriangular maps, it is shown that the group of tame polynomial transformations TGA_3 is generated by an affine group AGL_3 and any nonlinear biparabolic map of the form $U_0 \cdot q_1 \cdot U_1 \cdot q_2 \cdot U_2$, where U_i are linear maps and both q_i have the standard 1-parabolic form.

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All invertible polynomial maps of the affine space A_n over a field K form the group GA_n (the affine Cremona group). It represents an important example of so called $Ind-$groups or $\infty-$dimensional algebraic groups (an inductive limit of finite dimensional algebraic varieties, see [1]).The elements of GA_n can be represented as tuples of polynomials

$$g =< f_1(x_1, \dots, x_n), f_2(x_1, \dots, x_n), ..., f_n(x_1, \dots, x_n) >, \tag{1}$$

which action on the volume form $dx_1 \wedge \dots \wedge dx_n$ is a multiplication it by a constant. It leads to the Jacobian condition

$$\det\left(\frac{\partial f_i}{\partial x_j}\right) = const; \tag{2}$$

$const \neq 0$. Remember that $Lie(GA_n) = ga_n$ consists of linear differential operators of the form

$$\sum_{i=1}^{n} a_i(x_1, \dots, x_n)\frac{\partial}{\partial x_i}, \tag{3}$$

where a_i are polynomials under the condition $\sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i} = const \in K$ It is well known (see [2]) that ga_n is a graded irreducible transitive algebra of a polynomial growth: $ga_n = \oplus_{k=-1}^{\infty} ga_n^{(k)}$, where homogeneous components $ga_n^{(k)}$ consist of the operators (3) for which $\deg a_i = k + 1$.

There are important subgroups of GA_n:

(i) the affine group $AGL_n = GL_n \ltimes A_n^+ :\ \ \deg f_i = 1, i = 1, 2, \dots, n;$

(ii) B_n is a subgroup of triangular maps which elements have the form (1), where $f_i = f_i(x_1, \dots, x_i), i = 1, \dots, n$;

(iii) $GA_n^{(0)}$ is a stabilizer of zero and has a chain of normal subgroups $GA_n^{(0)} \rhd GA_n^{(1)} \rhd GA_n^{(2)} \rhd \dots \rhd GA_n^{(k)} \rhd \dots$, which members $GA_n^{(k)}$ consist of the maps (1) of the type $f_i = x_i + \phi_i(x_1, \dots, x_n) + \dots$, where ϕ_i- are homogeneous $k+1-$ forms and $\dots$ means items of higher degrees, by the way, $GA_n^{(0)} = GL_n(K) \ltimes GA_n^{(1)}$;

(iv) the subgroup of tame maps TGA_n which are generated by the elementary transformations: $f_i = x_i,\ \ i \neq j, \quad f_j = x_j + h_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ and AGL_n.

As was shown in [3], $Lie(AGL_n)$ is a maximal subalgebra of the ga_n. The direct application of Shafarevich's theorem (see [1]) about the connection between Lie algebras and correspondent $\infty-$ dimensional algebraic groups leads to the conclusion: AGL_n is a maximal closed subgroup of GA_n. The subgroup B_n (Jonquièar's group) is a maximal solvable subgroup of GA_n and so can be considered as an analog of a Borel's subgroup. Remark that tuples of the form (1), which coordinates are formal power series without constant terms form a group with the composition of tuples as a group operation. It contains GA_n^0 as a subgroup. Moreover, the factors $GA_n^{(0)}/GA_n^{(k)}$ are finite dimensional algebraic groups.

Tame maps give most simple examples of nonlinear invertible polynomial maps. It is easy to see that $TGA_n =< AGL_n, B_n >$. As is well known, GA_2 has the structure of the amalgamated product: $GA_2 = AGL_2 * B_2$ and so $GA_2 = TGA_2$. In the dimension $n = 3$, I. Shestakov and U. Umurbaev in [4] have proved that Nagata's automorphism is wild, so TGA_3 is a proper subgroup of GA_3. Remark that if this automorphism is extended in a natural way to an automorphism of A_n for some $n > 3$ then this extension will be tame. As was mentioned above AGL_n is a maximal closed subgroup of GA_n. On the other hand, as follows from [5], a finite affine group nearly always is a maximal subgroup in the correspondent symmetrical group. So it is natural to investigate intermediate subgroups from the interval $AGL_n < TGA_n$. By using an amalgamated structure of GA_2 it isn't hard to construct such subgroups in the dimension $n = 2$. For example the groups $Q_m =< AGL_2, \sigma^{(m)} >$, where $\sigma^{(m)} =< x_1, x_2 + x_1^{m+1} >\in GA_2^{(m)} \cap B_2$ form an ascending chain $AGL_2 = Q_0 < Q_1 < \dots Q_m < Q_{m+1}, \dots$ and $GA_2 = \cup_m Q_m$. From the uniqueness of element's decomposition in amalgamated products it follows that all maps $\sigma^{(k)}, k > m$ don't belong to Q_m. As is well known, GA_3 has not such structure and to point out an intermediate subgroup isn't a simple task. It is easy to see that TGA_n can be defined also in such a manner $TGA_n =< B_n, AGL_n >$. In fact more strong result holds

Theorem 1. *([6]) Let t be an arbitrary nonlinear triangular map from B_n then*

$$TGA_n =< t, AGL_n > .$$

This theorem may be generalized so called standard 1-parabolic transformations.

Definition 1. *The transformation q of the form (1) is called standard 1-parabolic if there is an affine map A such that*

$$q^A =< f_1(x_1,\dots,x_{n-1}),\dots,f_{n-1}(x_1,\dots,x_{n-1}), x_n + f_n(x_1,\dots,x_{n-1}) > . \quad (4)$$

Theorem 2. *Let q be an arbitrary nonlinear standard 1-parabolic transformation then*

$$TGA_n =< q, AGL_n > .$$

Proof. The result is a direct corollary of Theorem 1. Really, without lost of generality one can suppose that q has the form (4). Let $q^{-1} =< g_1,\dots,g_{n-1}, x_n - h_n(x_1,\dots,x_{n-1}) >$, then

$$g_i(f_1,\dots,f_{n-1}) \equiv x_i, f_n + h_n(f_1,\dots,f_{n-1}) \equiv 0. \quad (5)$$

If all g_i are linear then the map q has the form $U\cdot t, U \in AGL_n, t \in B_n$. Otherwise, for number i such that g_i is nonlinear polynomial let us use the transvection $A_{n,i} = < x_1,\dots,x_{n-1}, x_n+x_i > \in AGL_n$ and get the element $q^{A_{n,i}}\cdot q^{-1} =< x_1,\dots,x_{n-1}, x_n - x_i + g_i >$ which is nonlinear triangular. □

Definition 2. *A map $q \in GA_n$ is called biparabolic if it can be represented as a composition of two standard 1-parabolic maps.*

In particular bitriangular maps, which were defined in [6] as maps of the kind $C_0\cdot t_1\cdot C_1\cdot t_2\cdot C_2,\ \ t_1,t_2 \in B_n, C_k \in AGL_n$, form a subclass of biparabolic ones. Let $G =< AGL_n, q >$, where q is a biparabolic map. Without lost of generality one can suppose that $q \in G$ has the form $q = q_1\cdot q_2^A,\ A \in GL_n$. It is clear that standard 1- parabolic maps are permutable with the translations along the last coordinate $c_n : x_i \to x_i, i < n, x_n \to x_n + 1, 1 \in K$. This fact could be used for proving the same result ($G = TGA_n$) for biparabolic maps q. Really, the map q_2^A is permutable with the translation $c = c_n^A \in A_n^+$, so we can get the standard 1-parabolic map $q^c\cdot q^{-1} = q_1^c q_1^{-1} \in G$. Thus for most biparabolic maps the result can be deduced from Theorem 2. But it may happen that q_3 will be a linear map and the application of this theorem is impossible. In [7] (theorem 3) this situation was considered for bitriangular maps in the dimension $n = 3$. Next theorem is a generalization of that result.

Theorem 3. *Let q be an arbitrary nonlinear biparabolic transformation then*

$$TGA_3 =< q, AGL_3 > .$$

Proof. Let $G =< q, AGL_3 >$ As was mentioned above, we can suppose that $q = p_1\cdot p_2^A$, where $p_1, p_2 \in GA_n^{(1)}$ (without linear parts). If $A = B_1\cdot W\cdot B_2$ is a Brua decomposition, where W is a permutation matrix and B_1, B_2 are lower triangular matrices then we have $B_2 q B_2^{-1} = p_1^{B_2^{-1}}\cdot (p_2)^{B_1\cdot W} \in G$. Since the maps $p_1^{B_2^{-1}}, (p_2)^{B_1}$

are standard 1-parabolic transformations also without linear parts, then without lost of generality one can suppose that $q = p_1 \cdot p_2^W$. Moreover, the maps $p_i^{(1,2)}, \ i = 1, 2,$ are standard 1- parabolic ones and so one can suppose that there is an element $q \in G$ of the form

$$q = p \cdot p_1^{(1,3)}. \tag{6}$$

Let $p =< f_1(x_1, x_2), f_2(x_1, x_2), x_3 + f_3(x_1, x_2) >$ and $p^{-1} =< g_1(x_1, x_2), g_2(x_1, x_2), x_3 + g_3(x_1, x_2) >$ and identities (5) hold. If $\deg_{x_1} f_2 < \deg_{x_1} f_1$ then one can remove the map q by $(1, 2) \cdot q$, where $(1, 2) =< x_2, x_1, x_3 >$ is a transposition. So we can suppose that $\deg_{x_1} f_2 \geq \deg_{x_1} f_1$. On the other hand, if p has a decomposition $p = p'g$, where $g =< x_1 + h(x_2), x_2, x_3 >$, and p' has the form (4) then we can rewrite the map q in such a manner $q = p' \cdot (g^{(1,3)} \cdot p_1)^{(1,3)}$. Since $g^{(1,3)}$ is a triangular map then $g^{(1,3)} \cdot p_1$ is a 1-parabolic map. Hence, we can suppose also that p doesn't admit such decomposition $p = p' \cdot g$, where $h \not\equiv 0$.

Since the second factor of (6) is permutable with translation $c_1 =< x_1 + 1, x_2, x_3 >$, one can get an element $q_3 = q^{c_1} \cdot q^{-1} = p_1^{c_1} \cdot p_1^{-1} \in G$. As was mentioned above, the map q_3 has the form (4) and if it isn't a linear one then the result follows from Theorem 2. Let us investigate the situation when $q_3 = \Lambda \cdot x + z \in AGL_n$, here $\Lambda = (\lambda_{i,j}), i, j = 1, 2, 3, z = (z_1, z_2, z_3)$. The equality $p_1^{c_1} \cdot p_1^{-1} = q_3$ leads to the coordinate equalities

$$f_i(g_1 + 1, g_2) = \lambda_{i1}x_1 + \lambda_{i2}x_2 + \lambda_{i3}x_3 + z_i, i = 1, 2;$$
$$x_3 + g_3(x_1, x_2) + f_3(g_1 + 1, g_2) = \lambda_{31}x_1 + \lambda_{32}x_2 + \lambda_{33}x_3 + z_3.$$

By comparing the coefficients of x_3 we can obtain $\lambda_{i3} = 0, \lambda_{33} = 1$. If we take in account the identities (5) and act on the previous equalities by p we get

$$f_1(x_1 + 1, x_2) = \lambda_{11}f_1 + \lambda_{12}f_2 + z_1; \tag{7}$$
$$f_2(x_1 + 1, x_2) = \lambda_{21}f_1 + \lambda_{22}f_2 + z_2; \tag{8}$$

$$f_3(x_1 + 1, x_2) = \lambda_{31}f_1 + \lambda_{32}f_2 + f_3 + z_3.$$

Let us represent the coordinates of p in the form

$$f_i = \sum_{s=0}^{M_i} \phi_s^i(x_2)x_1^s,$$

$\phi_{M_i}^i \not\equiv 0, \ i = 1, 2, 3$. If $M_1 = M_2 = M$ then $M > 0$ and by comparing the coefficients of x_1^M in (7),(8) one can get

$$\phi_M^i = \lambda_{i1}\phi_M^1 + \lambda_{i2}\phi_M^2, \quad i = 1, 2.$$

If ϕ_M^1, ϕ_M^2 are linear independent polynomials over K then $\lambda_{i,j} = \delta_{i,j}$ (Kroneker's symbol). Comparing of the coefficients of x_1^{M-1} leads to the equality $M\phi_M^i + \phi_{M-1}^i = \phi_{M-1}^i$ which implies the contradiction $\phi_M^i \equiv 0, i = 1, 2$. So $\phi_M^2(x_2) = \mu\phi_M^1(x_2)$ for some $\mu \in K$. Let us use the transvection $U =< x_1 - \mu x_2, x_2, x_3 >$ and replace

$q \to U \cdot q$. In such a manner we get a map of the form (6) with $\phi^1_{M_1} \equiv 0$ in p i.e. for this map $M = M_2 > M_1$. Comparing the coefficients of x_1^M in identities (7),(8) leads to the equalities $0 = \lambda_{12}\phi^2_M, \phi^2_M = \lambda_{22}\phi^2_M$ which imply that $\lambda_{12} = 0, \lambda_{22} = 1$. Let us compare the coefficients of $x_1^{M_1-1}$:

$$\phi^1_{M-1} = \lambda_{11}\phi^1_{M-1}, \quad M\phi^2_M + \phi^2_{M-1} = \lambda_{21}\phi^1_{M-1} + \phi^2_{M-1}.$$

It follows that $\phi^1_{M-1} \not\equiv 0$ $(M_1 = M-1)$ and $\lambda_{11} = 1$. It is clear that the highest degree of x_1 which can be present by jacobian of the pair(f_1, f_2) does not exceed $2M-2$. With regard to the equality $M\phi^2_M = \lambda_{21}\phi^1_{M-1}$, the jacobian condition (2) leads to the identity $\phi^2_M \cdot \frac{d\phi^2_M}{dx_2} = 0$, hence, $\phi^2_M = const$. If $M > 2$ then comparing the coefficients of x_1^{M-2} in (7) leads to the contradiction $(M-1)\phi^1_{M-1} + \phi^1_{M-2} = \phi^1_{M-2}$, i.e.$\phi^1_{M-1} = 0$. Hence, $M = 2$ or $M = 1$. In the first case from (7) we have $\phi^1_1 = z_1$. The equalization of monomials without x_1 in (8) leads to the equality

$$\frac{M(M-1)}{2}\phi^2_2 + (M-1)\phi^2_1 + \phi^2_0 = \lambda_{21}\phi^1_0 + \phi^2_0 + z_2,$$

which under $M = 2$ implies $\phi^2_1 = \mu\phi^1_0 + const, \mu \in K$. After all we obtain that

$$f_1 = z_1x_1 + \phi^1_0(x_2), \quad f_2 = \phi^2_2x_1^2 + \left(\mu\phi^1_0(x_2) + const\right)x_1 + \phi^2_0(x_2).$$

This implies that p can be decomposed in such a manner

$$\begin{aligned} p =< z_1x_1, \phi^2_2(x_1-(z_1)^{-1}\phi^1_0(x_2))^2 + \left(\mu\phi^1_0(x_2) + const\right)(x_1-(z_1)^{-1}\phi^1_0(x_2)) + \phi^2_0(x_2),\\ x_3 + f_3(x_1 - (z_1)^{-1}\phi^1_0(x_2), x_2) > \cdot < x_1 + (z_1)^{-1}\phi^1_0(x_2), x_2, x_3 > . \end{aligned}$$

But, as was mentioned above, the map p doesn't admit such decomposition and so $\phi^1_0(x_2) \equiv 0$. Thus $p =< z_1x_1, \phi^2_2x_1^2 + const x_1 + \phi^2_0(x_2), x_3 + f_3(x_1, x_2) >$ is a triangular map. In the case $M = 1$ it is evident that the map $(1,2) \cdot t$ is a triangular one. On the other hand, we can repeat our reasoning for the map $\hat{q} = q^{-(1,3)} = p_1^{-1} \cdot p^{-(1,3)}$ and conclude that p_1 is also triangular. This means that in fact, the situation when $q_3 = q^{c_1} \cdot q^{-1}, q_4 = \hat{q}^{c_1} \cdot \hat{q}^{-1} \in AGL_n$, can be realized when both elements owe triangular ones i.e. when q is bitriangular. So the result follows from Theorem 3 from [7]. $\square$

References

[1] Shafarevich I. *On some Infinite Dimensional Groups II.* Izv. AN USSR, Ser. math., 1981, **45**, p. 214–226.

[2] Kac V. *Simple irreducible graded Lie algebrais of finite growth.* Izv. AN USSR, Ser. math., 1969, **32**, p. 1923–1967.

[3] Bodnarchuk Yu. *Some extreme properties of the affine group as an automorphism group of the affine space.* Contribution to General Algebra, 2001, **13**, p. 15–29.

[4] Shestakov I., Umirbaev U. *The tame and the wild automorphisms of polynomial rings in three variables*. Preprint São Paulo University, 2002, **1**, p. 1–35.

[5] Mortimer B. *Permutation Groups containing Affine Groups of the same degree*. J. of London Math. Soc., 1977, **15**, p. 445–455.

[6] Bodnarchuk Yu. *Generating properties of triangular and bitriangular birational automorphisms of an affine space*. Dopovidi NAN Ukraine, 2002, **11**, p. 7–22.

[7] Bodnarchuk Yu. *On affine-split tame invertible polynomial maps in three variables.* Buletinul A. Ş. a R. M., Matematika, 2002, **2(39)**, p. 37–43.

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Rings over which some preradicals are torsions

I.D. Bunu

Abstract. Let R be an associative ring with identity and z be a pretorsion such that its filter consists of the essential left ideals of the ring R. In this paper, it is proved that every preradical $r \geq z$ of $R-Mod$ is a torsion if and only if the ring R is a finite direct sum of pseudoinjective simple rings.

Mathematics subject classification: 16S90.
Keywords and phrases: Torsion (pretorsion), essential ideal, strongly prime ring, pseudoinjective module (ring).

Let z be the Goldie pretorsion of $R-Mod$ category of left $R-$ modules over the associative ring R with identity, i.e. its filter consists of essential left ideals of this ring.

In this paper some rings are described, over which all preradicals $r \geq z$ are torsions. It is proved that such rings are exactly those that can be decomposed in a finite direct sum of pseudoinjective simple rings.

First of all we present some preliminary notions and definitions.

01. A *preradical* r of $R-Mod$ is a subfunctor of the identity functor of $R-Mod$ [1, 2].

A preradical r is called

– *radical* if $r\left(M/r\left(M\right)\right)=0$ for any $M \in R-Mod$;

– *pretorsion* if $r\left(N\right) = N \cap r\left(M\right)$ for any submodule N of an arbitrary module M;

– *torsion* if r is a radical and pretorsion.

02. An arbitrary preradical r of category $R-Mod$ defines two classes of modules: $\mathcal{R}\left(r\right) = \{M \in R-Mod \mid r\left(M\right)=M\}$ and $\mathcal{P}\left(r\right) = \{M \in R-Mod \mid r\left(M\right)=0\}$. Modules of the class $\mathcal{R}\left(r\right)$ are called *r-torsion*, and of the class $\mathcal{P}\left(r\right)$ are called r-torsion free.

Preradicals 0 and ε for which $\mathcal{P}\left(0\right)=R-Mod$ and $\mathcal{R}\left(\varepsilon\right)=R-Mod$ are called *nul* and *identity*, respectively.

03. If r_1 and r_2 are two arbitrary preradicals, then $r_1 \leq r_2$ means that $r_1\left(M\right) \subseteq r_2\left(M\right)$ for any $M \in R-Mod$.

The intersection of preradicals r_1 and r_2 is the preradical $r_1 \wedge r_2$ determined by the rule: $\left(r_1 \wedge r_2\right)\left(M\right) = r_1\left(M\right) \cap r_2\left(M\right)$ for any $M \in R-Mod$.

The sum of preradicals r_1 and r_2 is the preradical r_1+r_2 defined by the relation $\left(r_1+r_2\right)\left(M\right) = r_1\left(M\right)+r_2\left(M\right)$ for any $M \in R-Mod$.

04. The least pretorsion containing preradical r is denoted by $h(r)$. It always exists and is determined by the equality $h(r)(M) = M \cap r\left(\hat{M}\right)$, where $\hat{M}$ is the injective envelope of an arbitrary module $M \in R - Mod$ ([1], p.23).

For any pretorsion r the least torsion $\bar{r}$ containing it exists and satisfies the property $r(M) \subseteq' \bar{r}(M)$ for any $M \in R - Mod$, ([1], 1.8 item Cyrillic "b").

05. Every nonzero module M determines the radical r_M such that $r_M(A) = \cap Ker f$ for all homomorphism $f \in Hom_R(A, M)$ for every $A \in R - Mod$. The radical r_M is the greatest among all preradicals r with the property $r(M) = 0$ ([1], p.13). If the module M is injective, then the radical r_M is a torsion ([1], p.32). Moreover $r_M(R) = (0 : M)$.

06. A module M is called *pseudoinjective* if for any monomorphism $i : B \to A$ and every homomorphism $f : B \to M$ there are such homomorphisms $\alpha : M \to M$ and $\bar{f} : A \to M$ that $0 \neq \alpha f = i \bar{f}$.

The following conditions are equivalent:

(1) M is a pseudoinjective module.

(2) $r_M = r_{\hat{M}}$.

(3) The radical r_M is a torsion ([3], p.45).

07. The Goldie pretorsion z is a torsion if and only if $z(R) = 0$ ([2], prop. I.10.2).

08. A ring R is called

– *strongly prime* (SP), if $r(R) = 0$ for any proper pretorsion r of $R - Mod$ category;

– *left strongly semiprime* (SSP),if every essential ideal P is cofaithful, i.e. $(0 : P) = \bigcap\limits_{\alpha=1}^{n} (0 : p_\alpha) = 0$ (essential ideal means a two-sided ideal that is essential as a left ideal).

Some descriptions of SSP-rings are obtained in the papers [4–7]. We present only a part of them.

The following conditions are equivalent:

(1) R is a SSP-ring.

(2) All pretorsions $r \geq z$ are torsions.

(3) Every proper pretorsion generates a proper torsion.

(4) R is a semiprime ring every nonzero ideal P of which possesses the property $(0 : P) = \left(0 : \hat{P}\right) = \bigcap\limits_{\alpha=1}^{n} (0 : p_\alpha)$ for some elements $p_\alpha \in P$.

(5) The ring R is a finite subdirect sum of SP-rings.

09. Let $R = R_1 \oplus R_2 \oplus \ldots \oplus R_n$ be a ring direct sum. Denote by f_i the corresponding projections. There is a one-to-one correspondence between preradicals of $R - Mod$ and ordered n - tuples $(r_1, r_2, \ldots, r_n)$, where r_i is a preradical for $R_i - Mod$, given by $r \to (f_1[r], \ldots, f_n[r])$ and $(r_1, r_2, \ldots, r_n) \to \sum \{r_i\} f_i = \cap [r_i] f_i$. This correspondence preserves the elementary properties, intersections, sums, inclusions in both directions ([2], prop. I.9.1).

Now we begin the investigation of rings over which any preradical $r \geq r_R \quad (r \geq z)$ is a radical (torsion).

Proposition 1. *Every SSP-ring is a finite direct sum of indecomposable SSP-rings.*

Proof. We show that a SSP-ring does not contain any infinite direct sums of two-sided ideals ([4], prop. 6). Indeed, let us consider a direct sum $P = \sum\limits_i \oplus P_i$ of ideals of the ring R. Then since R is semiprime we have $P \oplus (0 : P) \subseteq' R$. By assumption, R is a SSP-ring. Then the ideal $P \oplus (0 : P)$ is cofaithful and therefore $(0 : [P \oplus (0 : P)]) = \bigcap\limits_{i=1}^{n} [0 : (p_i + p_i^*)]$ for some $p_i \in P$ and $p_i^* \in (0 : P)$. We show that in this case $P_\alpha = 0$ for any $\alpha = n+1,\ n+2,\ \ldots$. Indeed, from the equality $P_\alpha \cdot P_i = 0$ for any $\alpha \neq i$ we obtain $P_\alpha \cdot p_i = 0$. Besides that, the inclusion $P_\alpha \subseteq P \Rightarrow (0 : P) \subseteq (0 : P_\alpha) \Rightarrow p_i^* \in (0 : P_\alpha) \Rightarrow p_i^*\ P_\alpha = 0$ holds for any $i = \overline{1,\, n}$. Then $P_\alpha p_i^* \cdot\ P_\alpha p_i^* = 0$, therefore since R is semiprime we have $P_\alpha\ \cdot p_i^* = 0$. But then $P_\alpha (p_i + p_i^*) = 0 \Rightarrow P_\alpha \subseteq (0 : (p_i + p_i^*))$ for any $i = \overline{1,\, n}$. This means that $P_\alpha \subseteq \bigcap\limits_{i=1}^{n} (0 : (p_i + p_i^*)) = 0$ and consequentely the considered direct sum is finite: $P = \sum\limits_{i=1}^{n} \oplus P_i$. From this it follows that R does not contain any infinite sets of central and orthogonal indempotents because otherwise it would contain infinite direct sums of ideals. But the latter is equivalent to the decomposability of the ring R in a finite direct sum of SSP-rings. $\square$

A pretorsion r of the category $R - Mod$ is called *superhereditary* (*stable*) if the class of r − torsion modules is closed with respect to direct products (essential extensions).

Superhereditarity of the pretorsion r is equivalent to the condition that its filter contains the least ideal P. It is denoted by $r_{(P)}$ and it is easy to verify that $r_{(P)}(M) = M \Leftrightarrow PM = 0$.

Lemma 2. *The following conditions are equivalent:*

(1) *All superhereditary pretorsions of $R - Mod$ are stable.*

(2) *All left ideals of the ring R are idempotent.*

(3) $(0 : M) = \left(0 : \hat{M}\right)$ *for any module $M \in R$ - Mod.*

Proof. Equivalence of (1) and (2) is proved in [2].

(1)⇒(3). Let M be an arbitrary module M for which $(0 : M) \neq 0$ because otherwise the equality $(0 : M) = \left(0 : \hat{M}\right)$ is obvions. Then the superhereditary pretorsion $r_{((0:M))}$ is stable. Since $r_{((0:M))}(M) = M$, we obtain $r_{((0:M))}\left(\hat{M}\right) = \hat{M}$, i.e. $(0 : M) \cdot \hat{M} = 0$ therefore $(0 : M) \subseteq \left(0 : \hat{M}\right)$. Now from the inclusion $\left(0 : \hat{M}\right) \subseteq (0 : M)$ we obtain $(0 : M) = \left(0 : \hat{M}\right)$.

(3)⇒(1). Let $r_{(P)}$ be an arbitrary superhereditary pretorsion. If $r_{(P)}(M) = M$

then $PM = 0$, therefore $P \subseteq (0 : M) = \left(0 : \hat{M}\right)$. It means $P\hat{M} = 0$ that is equivalent to the equality $r_{(P)}\left(\hat{M}\right) = \hat{M}$. Therefore $r_{(P)}$ is a stable pretorsion. □

Corollary 3. *Every indecomposable SSP-ring over which all left ideals are idempotent is simple.*

Indeed, let us consider an arbitrary essential ideal P of ring R and the torsion $\tau = r_{\widehat{R/P}}$. Then $\tau(R) = \left(0 : \widehat{R/P}\right) = (0 : R/P) = P$ (Lemma 2). Since $z(R) = 0$ it follows that $z < z \vee \tau$ and therefore $\tau(R) \subseteq (z \vee \tau)(R) \subseteq' R$. From the stability of the torsion $z \vee \tau$ (Statement 8) we obtain $(z \vee \tau)\left(\hat{R}\right) = (z + \tau)\left(\hat{R}\right) = z\left(\hat{R}\right) + \tau\left(\hat{R}\right) = \tau\left(\hat{R}\right) = \hat{R}$, therefore $\tau(R) = R$, i.e. $\tau = \varepsilon$. But then from the relations $\tau(R) = P = R$ it follows that the ring R does not contain any proper essential ideal. Let us now show that R is a simple rings. If K is a nonzero ideal of the ring R, then from its semiprimeness (R is an SSP-ring) it follows that the ideal $K \oplus (0 : K) \subseteq' R$. According to those proved earlier the ring $R = K \oplus (0 : K)$ and its indecomposability implies that $K = R$. In this way R is a simple ring. □

Corollary 4. *Any SSP-ring left ideals of which are indepotent is a finite direct sum of simple rings.*

Indeed, if R is a finite direct sum of rings R_α, then R is a SSP-ring left ideals of which are idempotent if in only if each direct summand R_α satisfies the same property. It remains us to use Proposition 1 and Corollary 3. □

Corollary 5. *If all preradicals of $R - Mod$ category are torsions then the ring R is a finite direct sum of simple rings with the same property.*

It is sufficient to show that the ring R satisfies the conditions of the previous Corollary 4.

Let us remark that R is a SSP-ring (Statement 08). Besides that, from the equality $r_M = r_{\hat{M}}$ it follows that any simple module is injective. Consequently, R is a left V-ring and therefore all its left ideals are idempotent ([2], prop. I.11.7). □

Corollary 6. *(Faith theorem). Any semiprime Goldie left V-ring is simple.*

This result follows directly from Corollary 3.

Corollary 7. *Any Goldie left V-ring is a finite direct sum of simple rings.*

It obviously follows from Corollary 4.

Lemma 8. *If all preradicals $r \geq r_R$ over ring R are radicals, then R is left strongly semiprime.*

Proof. We prove that any proper pretorsion r generates a proper torsion $\bar{r}$. Indeed, if $r \neq \varepsilon$ and $\bar{r} = \varepsilon$ then $r(R) \subseteq' R$. Consider the preradical $t = z + r_R + r$. Obviously, $t > r_R$ and $t > z$. By hypothesis, the preradical t is a radical and therefore $t(R/t(R)) = t(R/(z + r)(R)) = 0$. On the other hand, since the preradical $t > z$ and $t(R) = (z + r)(R) \subseteq' R$ we have $t(R/t(R)) = t(R/(z + r)(R)) =$

$R/(z+r)(R)$. For the equality $t(R/t(R)) = R/t(R) = 0$ it follows that $R = t(R) = (z+r)(R) = z(R) + r(R)$. Then $1 = x + y$ where $x \in z(R)$ and $y \in r(R)$. But then $(0:x) \cap (0\ :y) \subseteq (0:(x+y)) = 0$. From this and from $(0:x) \subseteq' R$ we have $(0\ :y) = 0 \in \mathrm{F}\ (r)$ where $\mathrm{F}\ (r)$ is the filter of pretorsion r. Consequently $r = \varepsilon$. The obtained contradiction shows that R is a SSP-ring. □

Lemma 9. *The following rings are simple:*

(1) *Indecomposable ring R over which all preradicals $r \geq r_R$ are radicals.*

(2) *Indecomposable self - injective SSP - rings.*

Prof. (1) Let P be a nonzero ideal of an indecomposable ring R over which all preradicals $r \geq r_R$ are radicals. By Lemma 8, R is a semiprime ring and therefore $P \oplus (0:P) \subseteq' R$. Consider the preradical $\tau = r_{R/P} + r_P$. Then $z = r_{\hat{R}} < r_R \leq r_P \leq \tau$ and $\tau(R) = \left(r_{R/P} + r_P\right)(R) \quad = r_{R/P}(R) + r_P(R) = (0:R/P) + (0:P) = P \oplus (0:P) \subseteq' R$. Since τ is radical $(\tau > r_R)$ we have $\tau(R/\tau(R)) = 0$. On the other hand, the relation $z \leq \tau$ and the inclusion $\tau(R) \subseteq' R$ imply $\tau(R/\tau(R)) = R/\tau(R)$. Then, from the last two equalities $\tau(R/\tau(R)) = 0 = R/\tau(R)$ we obtain that $R = \tau(R) = P \oplus (0:P)$, therefore, $P = R$ (the ring R is indecomposable). Consequently, R is a simple ring.

(2) Repeating proof of item (1) we have $\tau(R) = P \oplus (0:P) \subseteq' R$. According to the construction, we have $\tau \geq z$. Then, from the hypothesis (R is SSP-ring) and Statement 08, it follows that $h(\tau)$ is a stable torsion and therefore $h(\tau)\ (R) = R$. Self-injectivity of the ring R implies $h(\tau)\ (R) = \tau(R) = R = P \oplus (0:P)$, but its indecomposability implies that $P = R$. In this way, R is a simple ring. □

Theorem 10. *For self-injective ringR the following statements are equivalent:*

(1) *All preradicals $r \geq z$ are torsions.*

(2) *All preradicals $r \geq z$ are radicals.*

(3) *All pretorsions $r \geq z$ are torsions.*

(4) *The ring R is a finite direct sum of simple rings.*

(5) *The ring R is a finite direct sum of SP-rings.*

Proof. Implications (1)⇒(2)⇒(3) and (4)⇒(5) are obvious.

(3)⇒(4). According to Statement 08, R is a SSP-ring and, from Proposition 1, $R = \sum\limits_{\alpha=1}^{n} \oplus R_\alpha$, where R_α are indecomposable SSP-rings. Moreover, the rings R_α are self-injective because R itself is self-injective. Then by Lemma 9 item (2) R_α are simple rings, $\alpha = \overline{1,n}$.

(5)⇒(1). By the hypothesis $R = \sum\limits_{\alpha=1}^{n} \oplus R_\alpha$where R_α are self-injective SP - rings. Let K be one of these rings R_α. Consider an arbitrary proper preradical r of the category $R - Mod$ with the property $r \geq z$. Since K is a self-injective SP-ring, we have $r(K) = h(r)(K) = 0$ and, therefore $r \leq h(r) \leq r_K = z \leq r$, i.e. $r = z$ is a torsion. In this way over any direct summand R_α of the ring R every preradical $r \geq z$ is a torsion. Then R itself satisfies this property (Statement 09). □

Theorem 11. *The following conditions are equivalent:*
(1) *All preradicals $r \geq r_R$ of $R - Mod$ are radicals.*
(2) *The ring R is a finite direct sum of simple rings.*

Proof. (1)⇒(2). By Lemma 8, the ring R is a SSP-ring, and in according to Proposition 1 $R = \sum_{\alpha=1}^{n} \oplus R_\alpha$, where R_α are indecomposable SSP-rings for any $\alpha = \overline{1, n}$. Besides that, by hypothesis and Statement 09, over each direct summand R_α all preradicals $r \geq r_{R_\alpha}$ are radicals. Then, according to Lemma 9 item(1), R_α are simple rings.

The implication (2)⇒(1) follows from Statement 09, because over any simple ring R all preradicals $r \geq r_R$ are radicals. □

Theorem 12. *The following conditions are equivalent:*
(1) *All preradicals $r \geq z$ of $R - Mod$ are torsions.*
(2) *The ring R is a finite direct sum of pseudoinjective simple rings.*

Proof. (1)⇒(2). By assumption and by Theorem 11, the ring R is a finite sum of simple rings. Let us show that each direct summand K of the ring R is a pseudoinjective ring i.e. we prove that $r_K = r_{\hat{K}}$. Indeed, since z is a torsions we have $z = r_{\hat{K}} \leq r_K$. From hypothesis, the radical r_K is also a torsion. Then, according to the Statement 06, $r_K = r_{\hat{K}}$ and therefore, K is a pseudoinjective ring.

(2)⇒(1). Let r be an arbitrary preradical of the pseudoinjective simple ring K. Then $r_K = r_{\hat{K}}$ (Statement 06) and every preradical r on the category $K - Mod$ with property $r \geq z$ is a torsion because the equality $r(K) = 0$ implies $r \leq r_K = r_{\hat{K}} = z \leq r$, therefore $r = z$. But then, by Statement 09 overing R, all preradicals $r \geq z$ also are torsions. □

References

[1] Kashu A.I. *Radicals and torsions in modules.* Kishinev, Ştiinţa, 1983 (In Russian).

[2] Bican L., Kepka T., Nemec P. *Rings, modules and preradicals.* Marcel Dekker, 1982.

[3] Kashu A.I. *When the radical associated to a module is a torsion*? Mat. Zametki, 1974, **16**, p. 41–48 (In Russian).

[4] Handelman D.E. *Strongly semiprime rings.* Pacif. J. Math., 1975, **60(1)**, p. 115–122.

[5] Kutami M., Oshiro K. *Strongly semiprime rings and nonsingular quasi-injective modules.* Osaka J. Math., 1980, **17**, p. 41–50.

[6] Bunu I.D. *On the strongly semiprime rings.* Bulet. A.Ş. R.M., Matematica, 1997, no. 1(23), p. 78–83.

[7] Van den Berg J.E. *Primeness Described in the Language of Torsion Preradicals.* Semigroup Forum, 2002, **64**, p. 425–442.

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Transfer properties in radical theory

B.J. Gardner

Abstract. A functor is said to reflect radical classes if under this functor the inverse image of a radical class is always a radical class.Prototypical examples of such functors include polynomial and matrix functors and various forgetful functors.This paper is for the most part a survey of known results concerning radical reflections,but there are a few new results,including a generalization to right alternative rings of a well known result of Andrunakievici on upper radicals of simple associative rings.

Mathematics subject classification: 18E40,16N80,16S90.
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A functor $\phi\colon \mathcal{C} \to \mathcal{D}$ is said to *reflect radical classes* if for every radical class $\mathcal{R}$ in $\mathcal{D}$, the class $\mathcal{R}^* = \phi^{-1}(\mathcal{R}) = \{\mathcal{A}\colon \phi(\mathcal{A}) \in \mathcal{R}\}$ is a radical class in $\mathcal{C}$. This notion was studied systematically in the '70s, but there are many examples in the earlier and later literature, and the concept has been investigated by (in no particular order, and with apologies to those overlooked) Amitsur, Ortiz, Gardner, Stewart, Puczyłowski, Sierpińska, Beattie, Fang, Krempa, Skosyrskii, Widarma, Thedy, McCrimmon, Arnautov, Vodinchar, Slin'ko and Soweiter. (This joke is due to Georges Perec.) From the number of talks at the Chişinău conference which mentioned problems, questions and results which concern examples of radical reflections, it seems that the idea has considerable contemporary relevance for radical theorists.

There are a number of significant ways in which the study of radical reflections (and other methods for transferring radicals from one context to another) can contribute to radical theory.

- As a source of examples.
- By describing interactions between radicals and algebraic constructions (matrix rings, polynomial rings and so on).
- By generalizing particular radicals to new settings (e.g. finding the "correct version" of local nilpotence for varieties of non-associative rings).
- By extending known results concerning radicals in one context to analogous results in another (e.g. existence of hereditary semi-simple classes, lattice properties).
- By transferring a "traditional" radical theory to a non-standard setting, perhaps comparing the transferred theory with some ad hoc version of radical theory set up in the latter.
- By transferring some kind of radical theory to a context where no obvious one exists (as when a category "suitable for radical theory" is equivalent to an "unsuitable" one and an equivalence effects the transfer).

In what follows we shall give examples to illustrate all of these possibilities. While on the whole we are presenting a survey of known results, there are a few novelties. We do not work at a fixed level of generality. For much of the time we work with multioperator groups for the sake of definiteness. Laci Márki suggested that semi-abelian categories in the sense of [1] might provide an appropriate context. Certainly the group-based structures of [2] are sometimes too general (see Example 1.6). On the whole our terminology is consistent with [2] and [3]. In some of the examples, categories are given self-explanatory bold-faced names, but on occasion they are also referred to more informally.

A preliminary version of part of this paper was contained, together with some other topics, in a talk to the Pat Stewart Memorial Session of the 2002 APICS mathematics meeting in Sackville, New Brunswick.

1 Reflected Radicals

Let $\mathcal{C}$ and $\mathcal{D}$ be varieties of multioperator groups. We say that a functor $\phi\colon \mathcal{C} \to \mathcal{D}$ *reflects radical classes* if for every radical class $\mathcal{R}$ in $\mathcal{D}$, the class $\mathcal{R}^* = \phi^{-1}(\mathcal{R})$ is a radical class in $\mathcal{C}$.

Theorem 1.1. *(See [4]). If ϕ is exact and preserves unions of chains of normal subobjects, then ϕ reflects radical classes.*

We list some examples of functors satisfying the conditions of 1.1. In each case the action of the given functor on morphisms is well known.

Example 1.2.

(i)**Rings** $\to$ **Rings**; $A \mapsto A[X]$.

(ii)**Rings** $\to$ **Rings**; $A \mapsto [S]$ (semigroup ring; fixed semigroup S).

(iii)**Rings** $\to$ **Rings**; $A \mapsto M_n(A)$(matrix ring, fixed n).

(iv)**Rings** $\to$ **Jordan Rings**; $(A, +, \cdot) \mapsto (A, +, \odot)$ where $a \odot b = ab + ba$.

(v)**Rings** $\to$ **Lie Rings**;$(A, +, \cdot) \mapsto (A, +, [*, *])$.

(vi)**Rings** $\to$ **Abelian Groups**; $(A, +, \cdot) \mapsto (A, +)$.

(vii)$K-$**Algebras** $\to$ **Rings** (for a commutative ring K with identity); forgetful functor.

(viii)**Rings** $\to$ **Rings**; $A \mapsto A^{op}$ (opposite ring).

(ix)**Differential Rings** $\to$ **Rings**; forgetful functor.

(x)**Rings with Involution** $\to$ **Rings**; forgetful functor.

(xi)**Quasiregular Rings** $\to$ **Groups**; $(A+, \cdot) \mapsto (A, \circ)$.

We shall see later that neither of the conditions of 1.1 is necessary for the reflection of radical classes, but as the following few examples show, neither is sufficient either.

Example 1.3. (See [4].) The functor from **Rings** to **Rings** which associates with each ring A the power series ring $A[[X]]$ is exact but does not reflect radical classes.(For some information on radicals and power series, see [5].)

Example 1.4. The functor ϕ: **Rings** $\to$ **Rings**, where $\phi(A) = A^2$ for each A and ϕ acts on homomorphisms by restriction, is not exact: if

$$0 \to I \to A \to A/I \to 0$$

is exact, then so is

$$0 \to I \cap A^2 \to A^2 \to A^2/I \cap A^2 \cong (A^2 + I)/I = (A/I)^2 \to 0$$

but in general I^2 and $I \cap A^2$ can be quite different. For instance, if I is a ring with $I^2 = 0$ and $A = I * \mathbf{Z}$ is the standard unital extension, then $I^2 = 0$ and $I \cap A^2 = I \cap A = I$. However, as one shows easily, ϕ preserves unions of chains of ideals. Let $\mathcal{R}$ be the (radical)class of boolean rings. If R is a ring with $R^3 = 0 \neq R^2$, then trivially$(R^2)^2 \in \mathcal{R}$ and $(R/R^2)^2 = 0 \in \mathcal{R}$ so R^2 and $R/R^2 \in \mathcal{R}^*$. But $R \notin \mathcal{R}^*$ as $0 \neq R^2 \notin \mathcal{R}$. Thus $\mathcal{R}^*$ is not a radical class. (Note that ϕ preserves quotients.)

Example 1.5. Let ϕ: **Abelian Groups** $\to$ **Abelian Groups** assign the *socle* and act on homomorphisms in the usual way. If

$$0 \to H \to G \to G/H \to 0$$

is exact, then so is

$$0 \to \phi(H) \to \phi(G) \to \phi(G)/\phi(H) \to 0$$

but if, e.g., $H = \mathbf{Z}(p)$ and $G = \mathbf{Z}(p^\infty)$, then $\phi(G)/\phi(H) = 0$, while $\phi(G/H) \cong \phi(\mathbf{Z}(p^\infty) \cong \mathbf{Z}(p)$. On the other hand, if $\{H_\lambda : \lambda \in \Lambda\}$ is a chain of subgroups of G, then

$$\phi(\bigcup_{\lambda\in\Lambda} H_\lambda) = \{x \in \bigcup_{\lambda\in\Lambda} H_\lambda : 0(x) \text{is square-free}\} = \bigcup_{\lambda\in\Lambda} \phi(H_\lambda).$$

Let $\mathcal{T}_p$ be the (radical) class of abelian p-groups, q a prime $\neq p$. Then $\phi(\mathbf{Z}) = 0 \in \mathcal{T}_p$ but $\phi(\mathbf{Z}/q\mathbf{Z}) = \mathbf{Z}/q\mathbf{Z} \notin \mathcal{T}_p$, so $\mathbf{Z} \in \mathcal{T}_p^*$ while $\mathbf{Z}/q\mathbf{Z} \notin \mathcal{T}_p^*$. Hence ϕ does not reflect radical classes. (Note that ϕ takes subgroups to subgroups.)

Though we shall not seriously address the problem of characterizing the functors which reflect radical classes, we note one further pertinent example of one which doesn't. One of our categories is not a variety of multioperators here, but the functor is a forgetful one and provides some contrast with some of our cited examples in 1.2.

Example 1.6. Let ϕ be the forgetful functor from **Hausdorff Topological Groups** to **Abelian Groups** (forget the topology). Let $\mathcal{R}$ be a radical class of abelian groups, $\{A_\lambda : \lambda \in \Lambda\} \subseteq \mathcal{R}$. Give each A_λ the discrete topology and let P denote the cartesian product of the A_λ with the product topology. Then $\bigoplus_{\lambda\in\Lambda} A_\lambda$ with the subspace topology from P is in $\mathcal{R}^*$, so if $\mathcal{R}^*$ is a radical class, $\mathcal{R}^*(P)$ is a closed subgroup containing $\bigoplus_{\lambda\in\Lambda} A_\lambda$. But the latter is dense, so we must have $P \in \mathcal{R}^*$. Hence if $\mathcal{R}^*$ is a radical class, then $\mathcal{R}$ must be closed under direct products.

Closure under direct products is not enough, however. For a prime p, let $\mathbf{Q}(p)$ be the group $\{m/p^n : m, n \in \mathbf{Z}\}$. Let $\mathcal{D}_p$ be the (radical) class of p-divisible groups. Let H be a torsion-free group of rank 2 with $\mathcal{D}_p(H) \cong \mathbf{Q}(p)$ and $H/\mathcal{D}_p(H) \cong \mathbf{Q}(q)$ for a prime $q \neq p$. Then H has no elements of infinite q-height and $\mathcal{D}_p(H)$ is dense in H for the q-adic topology. If $\mathcal{D}_p^*$ were a radical class it would have to contain H. But $H \notin \mathcal{D}_p$.

The analogous question for hausdorff topological *rings* (*algebraic radicals*) has been treated in [6].

2 The Local Effect

Reflection of radical classes as thus far described is a global phenomenon. There is also a local phenomenon, which we can illustrate by first observing that for some ring radical classes $\mathcal{R}$ we have $\mathcal{R}(A[X]) = \mathcal{R}(A)[X]$ for all A, and then asking, if this equation is not universally valid but for *some* ring A we have $\mathcal{R}(A[X]) = I[X]$ for *some* $I \triangleleft A$, what is the nature of I?

We maintain the notation and assumptions of the previous section.

Lemma 2.1. *Let ϕ satisfy the conditions of 1.1. If $\mathcal{R}$ is a radical class in $\mathcal{D}$ then $\phi(\mathcal{R}^*(A)) \subseteq \mathcal{R}(\phi(\mathcal{A}))$ for each $A \in \mathcal{C}$.*

Proof Since $\mathcal{R}^*(A) \in \mathcal{R}^*$ we have $\phi(\mathcal{R}^*(A)) \in \mathcal{R}$. But
$\mathcal{R}^*(A) \triangleleft A$, so $\phi(\mathcal{R}^*(A)) \triangleleft \phi(A)$.

Theorem 2.2. *For $\phi \colon \mathcal{C} \to \mathcal{D}$ as in Theorem 1.1, the following are equivalent for $A \in \mathcal{C}$. (i)$\mathcal{R}(\phi(A)) = \phi(\mathcal{R}(A))$; (ii)$\mathcal{R}(\phi(A)) = \phi(I)$ for some $I \triangleleft A$.*

Proof (ii)⇒(i):If $\mathcal{R}(\phi(A)) = \phi(I)$,where $I \triangleleft A$, then $\phi(I) \in \mathcal{R}$, so $I \in \mathcal{R}^*$ and hence $I \subseteq \mathcal{R}^*(A)$. But then $I \triangleleft \mathcal{R}^*(A)$ so

$$\mathcal{R}(\phi(A)) = \phi(I) \triangleleft \phi(\mathcal{R}^*(A)).$$

The reverse inclusion follows from 2.1.

Corollary 2.3. *For a ring A and a radical class $\mathcal{R}$ of rings, $\mathcal{R}(A[X]) = I[X]$ for some $I \triangleleft A$ if and only if $\mathcal{R}(A[X]) = \mathcal{R}^*(A)[X]$.*

Example 2.4. (See [7].) Let $\mathcal{C} = \mathcal{D} =$the category of rings, $\phi(A) = M_n(A)$ for all A. Then for every A and every $\mathcal{R}$ there is an ideal I of A for which $\mathcal{R}(M_n(A)) = M_n(I)$. Thus

$$\mathcal{R}(M_n(A)) = \mathcal{R}(\phi(A)) = \phi(\mathcal{R}^*(A)) = M_n(\mathcal{R}^*(A)).$$

Example 2.5. (See [8].) Let ϕ be the forgetful functor from rings to abelian groups. If $\mathcal{R}$ is any radical class of abelian groups, then for every G, $\mathcal{R}(G)$ is a fully invariant subgroup of G. Hence for a ring A, $\mathcal{R}((A,+))$ is a fully invariant subgroup of $(A,+)$. Since left and right multiplications are additive endomorphisms, $\mathcal{R}(A,+)$ is an ideal of A, or, more precisely, $\mathcal{R}(A,+) = (I,+)$ for some $I \triangleleft A$. this I must be $\mathcal{R}^*(A)$. Thus $\mathcal{R}(A,+) = (\mathcal{R}^*(A),+)$ for all $\mathcal{R}$, A.

Example 2.6. (See [9].) Let ϕ be the forgetful functor from algebras over a field K to rings. For every radical class $\mathcal{R}$ of rings, $\mathcal{R}(A)$ is an algebra ideal of every K-algebra A. Thus "$\mathcal{R}(A) = \mathcal{R}^*(A)$".

Example 2.7. (See [10].) For the functor ϕ from right alternative algebras over $\mathbf{Q}(2)$ to Jordan algebras over $\mathbf{Q}(2)$ where the multiplication is replaced by $a \odot b = \frac{1}{2}(ab+ba)$, if $\mathcal{R}$ is a *non-degenerate* radical of Jordan algebras (semi-simple algebras have no strong zero- divisors) then the same is true of $\mathcal{R}^*$, and $(\mathcal{R}^*(A), +, \odot) = \mathcal{R}(A, +, \odot)$, since $\mathcal{R}(\phi(A)) = \mathcal{R}(A, +, \odot)$ is $(I, +, \odot)$ for an ideal I of A (for every A). This result is used in [10] to transfer many standard radicals from Jordan to right alternative algebras and show that they retain significant properties. (The same functor can be used to define a substitute for local nilpotence in right alternative algebras [11].) Analogous results for some other types of algebras are given in [12]. On the other hand, the similar notion of reflection of radicals from *Lie* algebras seems not to have attracted much attention.

The conditions of 2.2 are met (for a given A and $\mathcal{R}$) when $\mathcal{R}(A)$ is "highly invariant", maintaining normality when a richer, or at least different structure is imposed (as in the passage from abelian groups to rings or from right alternative to Jordan rings). In the cases of the polynomial and matrix functors, the conditions correspond to "well-behaved ideals"; e.g. ideals of matrix rings which have to be matrix rings over ideals. This piece of unification is perhaps of some independent interest.

3 Properties Preserved by the Lower Radical Construction

We consider a functor ϕ as in Section 1, but with $\mathcal{C} = \mathcal{D}$, and call a class $\mathcal{K} \subseteq \mathcal{C}$ a *ϕ-invariant class* if $\phi(A) \in \mathcal{K}$ for all $A \in \mathcal{K}$. We can then ask whether ϕ-invariance is preserved by the lower radical construction in the sense indicated in the following result.

Proposition 3.1. *Let $\phi\colon \mathcal{C} \to \mathcal{C}$ satisfy the conditions of Section 1. Let $\mathcal{M}$ be a homomorphically closed subclass of $\mathcal{C}$, $L(\mathcal{M})$ its lower radical class. If $\mathcal{M}$ is ϕ-invariant, then $L(\mathcal{M}) \subseteq L(\mathcal{M})^*$ and $L(\mathcal{M})$ is ϕ-invariant.*

Proof If $A \in \mathcal{M}$, then $\phi(A) \in \mathcal{M} \subseteq L(\mathcal{M})$ so $A \in L(\mathcal{M})^*$. Thus $\mathcal{M} \subseteq L(\mathcal{M})^*$ so $L(\mathcal{M}) \subseteq L(\mathcal{M})^*$. Hence for all $B \in L(\mathcal{M})$ we have $B \in L(\mathcal{M})^*$, i.e. $\phi(B) \subseteq L(\mathcal{M})$.

Thus, e.g., if a class of (associative) rings is closed under formation of polynomial rings, then so is its lower radical class ([13]; see [14] for the corresponding result for Jordan rings). Likewise a class closed under formation of $n \times n$ matrix rings forms a lower radical class with the same property.

The following property is also worth looking at.

$$A \in \mathcal{M} \text{ and } \phi(A) \cong \phi(B) \Rightarrow B \in \mathcal{M} - - - - - - - - (\dagger)$$

When is (†) preserved under the lower radical construction? We have just a little information about this.

Proposition 3.2. *If ϕ is a monoid ring functor ($\phi(A) = A[S]$, S fixed) from rings to rings, and if $\mathcal{M}$ is homomorphically closed and ϕ-invariant, then $\mathcal{M}$ satisfies* (†).

Proof If $A \in \mathcal{M}$ and $A[S] \cong B[S]$, then by ϕ-invariance, $A[S] \in \mathcal{M}$, so $B[S] \in \mathcal{M}$ and thus $B \in \mathcal{M}$.

Using 3.1 and 3.2, we get

Corollary 3.3. *If $\mathcal{M}$ is homomorphically closed and ϕ-invariant and satisfies* (†), *then $L(\mathcal{M})$ satisfies* (†).

Example 3.4. When ϕ is the forgetful functor from rings to abelian groups, (†) need not be preserved under the lower radical construction. For instance $\{GF(p), \mathbf{Z}(p)^0\}$ satisfies (†) but its lower radical class excludes $GF(p^2)$, though this field has the same additive group as $GF(p) \oplus \mathbf{Z}(p)^0$.

4 Categorical Equivalence

If $\mathcal{C}$ and $\mathcal{D}$ are varieties of multioperator groups, and $\phi\colon \mathcal{C} \to \mathcal{D}$ is an equivalence, with $\psi\colon \mathcal{D} \to \mathcal{C}$ the complementary equivalence, then for a radical class $\mathcal{R}$ in $\mathcal{D}$ we denote $\phi^{-1}(\mathcal{R})$ by$\mathcal{R}^*$ as before, and for a radical class $\mathcal{U}$ in $\mathcal{C}$ we let $\psi^{-1}(\mathcal{U}) = \mathcal{U}^{\#}$. As ϕ and ψ preserve limits and colimits, $\mathcal{R}^*$ and $\mathcal{U}^{\#}$ are always radical classes. Now

$$\mathcal{R}^{*\#} = \{D \in \mathcal{D} : \psi(D) \in \mathcal{R}^*\} = \{D \in \mathcal{D} : \phi\psi(D) \in \mathcal{R}\},$$

so, since $D \cong \phi\psi(D)$ we have $\mathcal{R}^{*\#} = \mathcal{R}$ for every radical class $\mathcal{R}$ in $\mathcal{D}$, and similarly $\mathcal{U}^{\#*} = \mathcal{U}$ for every radical class $\mathcal{U}$ in $\mathcal{C}$. Thus we have

Proposition 4.1. *A categorical equivalence ϕ between varieties $\mathcal{C}$ and $\mathcal{D}$ of multioperator groups induces a bijection $\mathcal{R} \leftrightarrow \mathcal{R}^*$ between radical classes in $\mathcal{D}$ and $\mathcal{C}$.*

It is easy to see that 4.1 has no converse; if F is a finite field and K an infinite field, then in the categories of $F-$ and $K-$ vector spaces there are only the trivial radical classes, but the categories are not equivalent since all pairs of non-zero K-vector spaces have infinite Hom-sets but this is not so for F-vector spaces.

One feels that equivalent categories (of multioperator groups or not) should be "radically the same". It is possible for a category which supports some kind of radical theory to be equivalent to one which does not (at least not in any obvious sense). In such circumstances it seems reasonable to use the equivalence to *induce* radical notions in the second category. If there is already some kind of radical theory in the second category, a comparison of the two competing versions may prove instructive. For instance radical theory for modules over a ring R can be transferred easily to the category of *affine* R-modules [15], or that of *pointed* R-modules. The categories of affine and pointed modules over certain rings are equivalent to certain categories of quasigroups [16],[17]. We shall consider some of these module-quasigroup connections elsewhere. In the case of *idempotent* quasigroups, there is already a version of radical theory [18], which contributes a further strand to this story. There are also equivalences between categories of MV-algebras and l-rings and abelian l-groups [19],[20],[21]. The l-structures are of course multioperator groups, but MV-algebras are rather different. Would radical theory reflected to MV-algebras by these equivalences produce anything interesting?

5 Transferring Radicals from a Subvariety

A rather different kind of functor from those treated hitherto enables us to reflect radical classes to a variety from a subvariety. If the radical theory of the subvariety is well understood, this technique may provide useful information about radicals in the larger variety. We shall again work with multioperator groups.

For a subvariety $\mathcal{V}$ of a variety $\mathcal{W}$, for each $A \in \mathcal{W}$ we let

$$A(\mathcal{V}) = \bigcap \{I : I \triangleleft A, A/I \in \mathcal{V}\}.$$

If $f\colon A \to B$ is a homomorphism in $\mathcal{W}$ and B/J is in $\mathcal{V}$, then denoting the natural map $B \to B/J$ by p, we have

$$A/Ker(pf) \cong Im(pf) = (Im(f) + J)/J \subseteq B/J \in \mathcal{V},$$

so $A(\mathcal{V}) \subseteq Ker(pf)$ and so $f(A(\mathcal{V})) \subseteq Ker(pf) = J$. This being so for all such J, we have $f(A(\mathcal{V})) \subseteq B(\mathcal{V})$. Thus the correspondence $A \mapsto A(\mathcal{V})$ defines a functor (*subfunctor of the identity*). Now (for f as above) we get a homomorphism $\hat{f}\colon A/A(\mathcal{V}) \to B/B(\mathcal{V})$ by defining $\hat{f}(a + A(\mathcal{V})) = f(a) + B(\mathcal{V})$ for each $a \in A$. This makes a functor of the correspondence $A \mapsto A/A(\mathcal{V})$ (*factor functor of the identity*) and this is the functor we shall use.

We shall denote by $U_{\mathcal{V}}(), U_{\mathcal{W}}()$ the upper radical in $\mathcal{V}, \mathcal{W}$ respectively.

Theorem 5.1. *(See [22].) Let $\mathcal{V}$ be a subvariety of $\mathcal{W}$. For a radical class $\mathcal{R}$ in $\mathcal{V}$ let $\mathcal{R}^* = \{A \in \mathcal{W} : A/A(\mathcal{V}) \in \mathcal{R}\}$. Then if $\mathcal{R}$ has semi-simple class $\mathcal{S}$, we have $\mathcal{R}^* = U_{\mathcal{W}}(\mathcal{S})$. In particular, $\mathcal{R}^*$ is a radical class in $\mathcal{W}$.*

(We note that no matter what $\mathcal{W}$ is, $U_{\mathcal{W}}((\mathcal{S})$ exists, as $\mathcal{S}$ is a regular class in both $\mathcal{V}$ and $\mathcal{W}$.)

The transfer obtained is likely to be useful only if the classes $\mathcal{R}^*$ are not too big. For instance if $\mathcal{W}$ is the variety of alternative rings and $\mathcal{V}$ that of associative rings, there are lots of Cayley-Dickson rings which must belong to every $\mathcal{R}^*$.We impose another condition and get a stronger conclusion than that of 5.1 which is useful.

Theorem 5.2. *(See [22].) Let $\mathcal{V}, \mathcal{W}$ be as in 5.1 and suppose further that $A(\mathcal{V})/A(\mathcal{V})(\mathcal{V}) \in \mathcal{R}$ (i.e. $A(\mathcal{V}) \in \mathcal{R}^*$) for every $A \in \mathcal{W}$. Then*

(i)$\mathcal{R}^*(A)/A(\mathcal{V}) = \mathcal{R}(A/A(\mathcal{V}))$ for all $A \in \mathcal{V}$ and

(ii)$\mathcal{S}$ is the semi-simple class of $\mathcal{R}^*$.

Here we have some possibility of extending a result concerning well-behaved radicals from $\mathcal{V}$ to $\mathcal{W}$, since some semi-simple classes in $\mathcal{V}$ remain semi-simple classes in $\mathcal{W}$, and there are properties of semi-simple classes which make for well-behaved radicals. We give some illustrations of the situation described in 5.2.

Example 5.3. $\mathcal{V}$ is a semi-simple radical class in $\mathcal{W}$ if and only if $A(\mathcal{V}) = A(\mathcal{V})(\mathcal{V})$ for every $A \in \mathcal{W}$ [23], i.e. $A(\mathcal{V}) \in \{0\}^* = U_{\mathcal{W}}(\mathcal{V})$ for every A. If $\mathcal{R}$ is any radical class in $\mathcal{V}$, then each $A(\mathcal{V})$ is in $\mathcal{R}^*$ and the semi-simple class of $\mathcal{R}$ in $\mathcal{V}$ remains a semi-simple class in $\mathcal{W}$.

Example 5.4. If $\mathcal{W}$ is the class of all (not necessarily associative) rings, $\mathcal{V}$ the class of associative rings, then in $\mathcal{W}$ the only hereditary semi-simple classes are those corresponding to A-radicals [24] while all semi-simple classes in $\mathcal{V}$ are hereditary. Hence only A-radicals (in $\mathcal{W}$) satisfy the hypotheses of 5.2.

It is more convenient to have examples of the phenomenon in 5.2 where our starting point is a radical class in $\mathcal{W}$ rather than in $\mathcal{V}$.

Proposition 5.5. ([22])(Notation as in 5.2.) *If $\mathcal{U}$ is a radical class in $\mathcal{W}$ and $A(\mathcal{V}) \in \mathcal{U}$ for all $A \in \mathcal{W}$, then $\mathcal{U} = (\mathcal{U} \cap \mathcal{V})^*$ and $\mathcal{U}$, $\mathcal{U} \cap \mathcal{V}$ are related as $\mathcal{R}^*$ and $\mathcal{R}$ are related in (i),(ii) of 5.2.*

Example 5.6. We illustrate 5.5 by considering the case where $\mathcal{W}$ is the class of *right alternative rings*, $\mathcal{V}$ the class of *alternative rings*. By a result of Skosyrskii [25] $A(\mathcal{V})$, which is called the *alternator* of A, is contained in the McCrimmon radical of A (cf. 2.7). The McCrimmon radical is the upper radical defined by the class of *non-degenerate rings*, i.e. rings with no strong zero-divisors. For this example, "ring" always means "ring in which division by 2 is possible"; in particular, characteristic 2 is avoided. Thus if $\mathcal{U}$ is any non-degenerate radical class in $\mathcal{W}$ (i.e. all $\mathcal{U}$-semi-simple rings are non-degenerate) then $A(\mathcal{V}) \in \mathcal{U}$. Hence, by 5.5, $\mathcal{U}$ (in $\mathcal{W}$) and $\mathcal{U} \cap \mathcal{V}$ (in $\mathcal{V}$) have the same semi-simple class. In particular, non-degenerate radicals of right alternative rings have hereditary semi-simple classes.

This can be improved.

Theorem 5.7. *(See [22].) Let $\mathcal{W}$ be a variety, $\mathcal{V}$ a subvariety, $\mathcal{U}$ a hereditary radical class in $\mathcal{W}$ such that $A(\mathcal{V}) \in \mathcal{U}$ for all $A \in \mathcal{W}$. If every radical class in $\mathcal{V}$ satisfies* ADS *then every radical class $\mathcal{T}$ in $\mathcal{W}$ with $\mathcal{U} \subseteq \mathcal{T}$ also satisfies* ADS*.*

Now all radical classes of alternative rings satisfy ADS so we have

Corollary 5.8. *(See [22].) Every non-degenerate radical class of right alternative rings satisfies* ADS*.*

We conclude with a more detailed result obtained similarly.

Theorem 5.9. *Let $\mathcal{M}$ be a class of simple right alternative rings, $\mathcal{U}$ the upper radical class defined by $\mathcal{M}$ (in the class of right alternative rings). The following conditions are equivalent.*

(i) $\mathcal{U}$ is hereditary and has the intersection property with respect to $\mathcal{M}$.
(ii) All rings in $\mathcal{M}$ are unital.

(This theorem was proved for associative rings by Andrunakievich [26] and can be generalized to alternative rings by means of results of Suliński [27]. More recently, Leavitt [28] has shown that for associative rings (ii) is equivalent to the intersection property alone, and taking account of the fact that non-unital simple alternative rings are associative, one can show straightforwardly that the stronger result is valid in the alternative case too.)

Proof $\neg$ (ii)$\Rightarrow$ $\neg$ (i). If $\mathcal{M}$ contains a non-unital ring S, let S^* be the ring obtained form S by the adjunction of the identity of $\mathbf{Q}$ or $\mathbf{Z}_p$ to match the characteristic of S (so that S^*/S is isomorphic to the appropriate field). The only simple image of S^* is S^*/S. If $S^*/S \notin \mathcal{M}$, then $S^* \in \mathcal{U}$ but $S \notin \mathcal{U}$, so $\mathcal{U}$ is not hereditary. If $S^*/S \in \mathcal{M}$, then S^* is in the semi-simple class of $\mathcal{U}$ but is subdirectly irreducible and non-simple, so that $\mathcal{U}$ does not have the intersection property with respect to $\mathcal{M}$. (This is a familiar argument in the associative case.)

(ii)$\Rightarrow$(i). Suppose all rings in $\mathcal{M}$ are unital. As every right alternative ring has nil alternator, the rings in $\mathcal{M}$ are alternative. Let $\mathcal{S}$ be the class of subdirect products of rings in $\mathcal{M}$. Then $\mathcal{S}$ is a semi-simple class in the universal class of alternative rings and $\mathcal{U}$ is its upper radical class in the universal class of *right* alternative rings. Since the alternator of every ring is in $\mathcal{U}$, 5.5 says that $\mathcal{S}$ is the semi-simple class of $\mathcal{U}$ (in the class of right alternative rings). Hence $\mathcal{U}$ has the intersection property with respect to $\mathcal{M}$.

Now a radical class with ADS is hereditary if and only if its semi-simple class is closed under essential extensions. (This is proved as for the associative case in [29].)Every radical class of alternative rings has ADS, so $\mathcal{S}$ is closed under *alternative* essential extensions. If $A \in \mathcal{S}$, $A \triangleleft^\bullet B$ and B is right alternative, let J be the alternator of B. Then $J \cap A$ is a nil ideal of A and a member of $\mathcal{S}$, so $J \cap A = 0$, whence $J = 0$ and B is alternative. But then B is in $\mathcal{S}$. Thus $\mathcal{S}$ is closed under right alternative essential extensions, so that by 5.8, $\mathcal{U}$ is hereditary.

It's well known that 5.9 is not valid for the class of all (not necessarily associative) rings, and it would be interesting to know how far beyond right alternative rings it extends. It does not extend to power-associative rings. The following example was used by Henriksen [30] for other purposes.

Example 5.10. Let F be a field, and let R be an F-algebra with basis $\{a, b, e\}$, $ab = e = -ba, e^2 = e$ and all other basis products zero. If $\alpha, \beta, \gamma \in F$ then $(\alpha a + \beta b + \gamma e)^2 = \gamma^2 e$, $(\alpha a + \beta b + \gamma e)\gamma^2 e = \gamma^3 e = \gamma^2 e(\alpha a + \beta b + \gamma e)$ and so on, so R is power-associative. If $g \colon R \to F$ is a homomorphism, then $g(a)^2 = 0 = g(b)^2$ so $g(a) = 0 = g(b)$, and then $g(e) = g(ab) = g(a)g(b) = 0$, so $g = 0$. Thus R is in the upper radical class defined by $\{F\}$, but $F \cong Fe \triangleleft R$, so the upper radical class is not hereditary.

References

[1] Janelidze G., Márki L., Tholen W. *Semi-abelian categories.* J. Pure Appl. Algebra, 2002, **168**, p. 367–386.

[2] Gardner B.J. *Radical Theory.* Longman, 1989.

[3] Gardner B.J., Wiegandt R. *Radical Theory of Rings.* Dekker, 2003.

[4] Gardner B.J., Stewart P.N. *Reflected radical classes.* Acta. Math. Acad. Sci. Hungar., 1976, **28**, p. 293–298.

[5] Puczyłowski E.R. *Nil ideals of power series rings.* J. Austral. Math. Soc., 1983, **34**, p. 287–292.

[6] Arnautov V.I. *Algebraic radicals in topological rings.* Mat. issled., 1969, **4**, no. 2, p. 116–122 (In Russian).

[7] Krempa J. *Radicals of semi-group rings.* Fund. Math., 1974, **85**, p. 57–71.

[8] Gardner B.J. *Radicals of abelian groups and associative rings.* Acta Math. Acad. Sci. Hungar., 1973, **24**, p. 259–268.

[9] Divinsky N., Suliński A. *Kurosh radicals of rings with operators.* Canad. J. Math., 1965, **17**, p. 278–280.

[10] Thedy A. *Radicals of right alternative and Jordan rings.* Comm. in Algebra, 1984, **12**, p. 857–887.

[11] Skosyrskii V.G. *Nilpotence in Jordan and right alternative algebras.* Algebra i Logika, 1979, **18**, p. 49–57 (In Russian).

[12] Thedy A. *Jordan radicals of U-algebras.* Comm. in Algebra, 1990, **18**, p. 711–739.

[13] Gardner B.J. *A note on radicals and polynomial rings.* Math. Scand., 1972, **31**, p. 83–88.

[14] McCrimmon K. *Amitsur shrinkage of Jordan radicals.* Comm. in Algebra, 1984, **12**, p. 777–826.

[15] Csákány B. *Varieties of affine modules.* Acta. Sci. Math. (Szeged), 1975, **37**, p. 3–10.

[16] Csákány B., Megyesi L. *Varieties of idempotent medial quasigroups.* Acta. Sci. Math. (Szeged), 1975, **37**, p. 17–23.

[17] Ježek J., Kepka T. *Varieties of abelian quasigroups.* Czechoslovak. Math. J., 1977, **27**, p. 473–503.

[18] Gardner B.J. *Radical decompositions of idempotent algebras.* J. Austral. Math. Soc., Series A, 1984, **36**, p. 213–236.

[19] Belluce L.P., Di Nola A., Georgescu G. *Perfect MV-algebras and l-rings.* J. Applied Non-Classical Logics, 1999, **9**, p. 159–172.

[20] Mundici D. *Interpretation of $AF\,C^*$-algebras in Łukasiewicz sentential calculus.* J. Functional Analysis, 1986, **65**, p. 15–63.

[21] Di Nola A., Lettieri A. *Perfect MV-algebras are categorically equivalent to abelian l-groups.* Studia Logica, 1994, **53**, p. 417–432.

[22] Widarma G. *Nondegenerate radicals of right alternative rings.* Acta Math. Hungar., 2003, **100**, p. 19–30.

[23] Gardner B.J. *Semi-simple radical classes of algebras and attainability of identities.* Pacific J. Math., 1975, **61**, p. 401–416.

[24] Gardner B.J. *Some degeneracy and pathology in non-associative radical theory.* Annales. Univ. Sci. Budapest. Sect. Math., 1979–1980, **22-23**, p. 65–74.

[25] Skosyrskii V.G. *Right alternative algebras*. Algebra i Logika, 1983, **23**, p. 478–489 (In Russian).

[26] Andrunakievich V.A. *Radicals of associative rings II. Examples of special radicals*. Amer. Math. Soc. Translations Series, 1966, **2**, no. 52, p. 129–149.

[27] Suliński A. *The Brown-McCoy radical in categories*. Fund. Math., 1966, **49**, p. 23–41.

[28] Leavitt W.G. *A minimally embeddable ring*. Period. Math. Hungar., 1981, **12**, p. 129–140.

[29] Armendariz E.P. *Closure properties in radical theory*. Pacific J. Math., 1968, **26**, p. 1–7.

[30] Henriksen M. *Conditions that guarantee that all nilpotents commute with every element of an alternative ring*. Algebra Univ., 1977, **7**, p. 119–132.

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Exponent matrices and their quivers

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Abstract. We consider exponent matrices and investigate their connections with tiled orders and quivers, finite partially ordered sets and doubly stochastic matrices.

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1 Introduction

Exponent matrices appeared in the study of tiled orders over discrete valuation rings. Many properties of such orders are formulated using this notion. We think that such matrices are of interest in them own right, in particular, it is convenient to write finite partially ordered sets (posets) and finite metric spaces as special exponent matrices.

Note that when we defined a quiver $Q(\mathcal{E})$ of a reduced exponent matrix $\mathcal{E}$, $\mathcal{E}$ corresponds to a reduced tiled order Λ, a matrix $\mathcal{E}^{(1)}$ corresponds to a Jacobson radical R of Λ, and $\mathcal{E}^{(2)}$ corresponds to R^2. Then the adjacency matrix $[Q] = \mathcal{E}^{(2)} - \mathcal{E}^{(1)}$ defines a structure of the Λ-bimodule $V = R/R^2$.

Note that investigations on tiled orders over discrete valuation rings and finite posets are discussed in [10]. The bibliography about tiled orders see in [2] and [3].

2 Quivers

We recall basic facts about quivers and related topics. Following P. Gabriel a finite directed graph Q is called *a quiver*.

Definition 2.1. *A quiver Q without multiple arrows and multiple loops is called a simply laced quiver.*

Denote by $VQ = \{1, \ldots, s\}$ the set of all vertices of Q and by AQ the set of its all arrows. We shall write $Q = \{AQ, VQ\}$. Denote by $1, \ldots, s$ the vertices of a quiver Q and assume that we have q_{ij} arrows beginning at the vertex i and ending at the vertex j. The matrix

$$[Q] = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1s} \\ q_{21} & q_{22} & \cdots & q_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ q_{s1} & q_{s2} & \cdots & q_{ss} \end{pmatrix}$$

is called *the adjacency matrix* of Q.

Obviously, a quiver Q is simply laced if and only if $[Q]$ is a $(0,1)$-matrix.

Let Q be a quiver. Usually we will denote the vertices of Q by the numbers $1, 2, \dots, s$. If an arrow σ connects a vertex i with a vertex j then i is called its *start vertex* and j its *end vertex*. This will be denoted as $\sigma : i \to j$. A loop at the vertex j is an arrow such that the start vertex j coincides with the end vertex j.

A *path of the quiver* Q from a vertex i to a vertex j is an ordered set of k arrows $\{\sigma_1, \sigma_2, ..., \sigma_k\}$ such that the start vertex of each arrow σ_m coincides with the end vertex of the previous one σ_{m-1} for $1 < m \leq k$, and moreover, the vertex i is the start vertex of σ_1, while the vertex j is the end vertex of σ_k. The number k of these arrows is called the *length of the path*.

The start vertex i of the arrow σ_1 is called the *start of the path* and the end vertex j of the arrow σ_k is called the *end of the path*. We shall say that the path connects the vertex i with the vertex j and it is denoted by $\sigma_1\sigma_2...\sigma_k : i \to j$.

Now we shall give a definition of a diagram $Q(P)$ of a finite poset P.

Definition 2.2. ([1], Ch.1, §3). *By "a covers b" in a poset P, it is meant that $a > x > b$ for no $x \in P$.*

Definition 2.3. ([4], p. 233, see also [6]). *Let $P = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a finite poset with an ordering relation $\leq$. The* diagram of P *is the quiver $Q(P)$ with the set of vertices $VQ(P) = \{1, \dots, n\}$ and the set of arrows $AQ(P)$ such that in $AQ(P)$ there is an arrow $\sigma : i \to j$ if and only if α_j covers α_i.*

Definition 2.4. ([7], §8.4). *A quiver without oriented cycles is called an* acyclic quiver.

Definition 2.5. *An arrow $\sigma : i \to j$ of an acyclic quiver Q is called* extra *if there exists a path from i to j of length greater than 1.*

Theorem 2.6. ([6], [4], §7.7). *Let Q be an acyclic simply laced quiver without extra arrows. Then Q is the diagram of some finite poset P. Conversely, the diagram $Q(P)$ of a finite poset P is an acyclic simply laced quiver without extra arrows.*

3 Exponent matrices

Denote by $M_n(\mathbb{Z})$ the ring of all square $n \times n$-matrices over the ring of integers $\mathbb{Z}$. Let $\mathcal{E} \in M_n(\mathbb{Z})$.

Definition 3.1. *We call a matrix $\mathcal{E} = (\alpha_{ij})$ an* exponent matrix *if $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$ for $i, j, k = 1, \dots, n$ and $\alpha_{ii} = 1, \dots, n$ for $i = 1, \dots, n$. These relations are called* ring inequalities. *An exponent matrix $\mathcal{E}$ is called* reduced *if $\alpha_{ij} + \alpha_{ji} > 0$ for $i, j = 1, \dots, n$.*

Let $\mathcal{E} = (\alpha_{ij})$ be a reduced exponent matrix. Set $\mathcal{E}^{(1)} = (\beta_{ij})$, where $\beta_{ij} = \alpha_{ij}$ for $i \neq j$ and $\beta_{ii} = 1$ for $i = 1, \dots, n$, and $\mathcal{E}^{(2)} = (\gamma_{ij})$, where $\gamma_{ij} = \min\limits_{1 \leq k \leq n} (\beta_{ik} + \beta_{kj})$. Obviously, $[Q] = \mathcal{E}^{(2)} - \mathcal{E}^{(1)}$ is a $(0,1)$-matrix.

Definition 3.2. *The quiver $Q(\mathcal{E})$ shall be called the quiver of the reduced exponent matrix $\mathcal{E}$.*

Definition 3.3. *A strongly connected simply laced quiver shall be called* admissible *if it is a quiver of a reduced exponent matrix.*

Definition 3.4. *A reduced exponent matrix $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbb{Z})$ shall be called* Gorenstein *if there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that $\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)}$ for $i, k = 1, \ldots, n$.*

The permutation σ is denoted by $\sigma(\mathcal{E})$. Notice that $\sigma(\mathcal{E})$ for a reduced Gorenstein exponent matrix $\mathcal{E}$ has no cycles of the length 1.

Definition 3.5. *We shall call two exponent matrices $\mathcal{E} = (\alpha_{ij})$ and $\Theta = (\theta_{ij})$* equivalent *if they can be obtained from each other by transformations of the following two types:*

(1) subtracting an integer from the i-th row with simultaneous adding it to the i-th column;

(2) simultaneous interchanging of two rows and the equally numbered columns.

Proposition 3.6. [3]. *Suppose that $\mathcal{E} = (\alpha_{ij})$ and $\Theta = (\theta_{ij})$ are exponent matrices and Θ is obtained from $\mathcal{E}$ by a transformation of type (1). Then $[Q(\mathcal{E})] = [Q(\Theta)]$. If $\mathcal{E}$ is a reduced Gorenstein exponent matrix with permutation $\sigma(\mathcal{E})$, then Θ is also reduced Gorenstein with $\sigma(\Theta) = \sigma(\mathcal{E})$.*

Proposition 3.7. [3]. *Under transformations of the second type the adjacency matrix $[\tilde{Q}]$ of $Q(\Theta)$ changes according to the formula: $[\tilde{Q}] = P_\tau^T[Q]P_\tau$, where $[Q] = [Q(\mathcal{E})]$. If $\mathcal{E}$ is Gorenstein then Θ is also Gorenstein and for the new permutation π we have: $\pi = \tau^{-1}\sigma\tau$, i.e., $\sigma(\Theta) = \tau^{-1}\sigma(\mathcal{E})\tau$.*

Definition 3.8. *The* index *(in $\mathcal{E}$) of a reduced exponent matrix $\mathcal{E}$ is the maximal real eigenvalue of the adjacency matrix $[Q(\mathcal{E})]$ of $Q(\mathcal{E})$.*

It follows from Proposition 3.6 and Proposition 3.7 that indices of equivalent reduced exponent matrices coincide.

Theorem A. *The matrix $[Q] = \mathcal{E}^{(2)} - \mathcal{E}^{(1)}$ is the adjacency matrix of the strongly connected simply laced quiver $Q = Q(\mathcal{E})$.*

Proof. $[Q]$ is a $(0,1)$-matrix, then it is the adjacency matrix of a simply laced quiver.

We shall show that $[Q]$ is a strongly connected quiver. Suppose the contrary. It means that there is no path from the vertex i to the vertex j in Q. Denote by $VQ(i) = V_1$ the set of all vertices k of Q such that there exists a path beginning at the vertex i and ending at the vertex k. It is obviously that $V_2 = VQ \setminus VQ(i) \neq 0$ $(j \in V(Q) \setminus V(Q)(i))$. Consequently, $VQ = V_1 \cup V_2$ and $V_1 \cap V_2 = 0$. It is clear that there are no arrows from V_1 to V_2. One can assume that $V_1 = \{1, \ldots, m\}$ and $V_2 = \{m+1, \ldots, s\}$. It is obvious, that a simultaneous permutation of rows and columns will take place in the exponent matrix $\mathcal{E}$. Moreover, under transformations of the first type, we can make the elements at the first row of $\mathcal{E}$ equal zero, i.e., $\alpha_{1p} = 0$ for $p = 1, \ldots, s$. So, $\alpha_{pq} \geq 0$ for $p, q = 1, \ldots, s$ and

$$[Q] = \left(\begin{array}{c|c} * & 0 \\ \hline * & * \end{array}\right),$$

$$\mathcal{E} = \left(\begin{array}{c|c} \mathcal{E}_1 & * \\ \hline * & \mathcal{E}_2 \end{array}\right),$$

where $\mathcal{E}_1 \in M_m(\mathbb{Z})$, $\mathcal{E}_2 \in M_{s-m}(\mathbb{Z})$. With the exponent matrix $\mathcal{E}_2$ we connect a poset $P_{\mathcal{E}_2} = \{m+1, \ldots, s\}$ with an ordering relation $i \leq j$ if and only if $\alpha_{ij} = 0$. One can consider that $m+1 \in P_{\mathcal{E}_2}$ is the minimal element. Then $\alpha_{im+1} > 0$ for $i > m+1$. Since, $q_{1m+1} = 0$, then there exists k $(2 \leq k \leq m)$ such that $\alpha_{1m+1} = \alpha_{1k} + \alpha_{km+1}$. Simultaneously interchanging the 2-nd and k-th columns and the 2-nd and k-th rows of $\mathcal{E}$, we obtain that $\alpha_{2m+1} = 0$. Since $q_{2m+1} = 0$, again obtain $\alpha_{2m+1} = 0 = \alpha_{2k} + \alpha_{2m+1}$ for $3 \leq k \leq m$, i.e., one can consider that $\alpha_{23} = 0$ and $\alpha_{3m+1} = 0$. The elements of the matrix $\mathcal{E}^{(1)}$ $\beta_{31} = \alpha_{31}$, $\beta_{32} = \alpha_{32}$, $\beta_{33} = 1$ are nonzero. Again, $q_{3m+1} = 0$ and $\alpha_{3m+1} = 0 = \alpha_{3k} + \alpha_{km+1}$ for $4 \leq k \leq m$. Hence, $\alpha_{4m+1} = 0$. Continuing this process we have that $\alpha_{12} = \alpha_{23} = \ldots = \alpha_{m-1m} = 0$ and $\alpha_{im+1} = 0$ for $i = 1, \ldots, m$, consequently a matrix $\mathcal{E}_1$ is down triangular, and all elements $\beta_{m1}, \ldots, \beta_{mm}$ are natural integers. So, $q_{mm+1} = \min(\beta_{mk} + \beta_{km+1}) - \alpha_{mm+1} = 1 - 0 = 0$. We obtained a contradiction. Theorem is proved. $\square$

4 Gorenstein exponent matrices and entropic quasigroups

In general case a Latin square [5] of order n is a square with rows and columns each of which is a permutation of a set $S = \{s_1, \ldots, s_n\}$. Every Latin square is a Cayley table of a finite quasigroup. In particular, the Cayley table of a finite group is the Latin square. As a set S we will consider $S = \{0, 1, \ldots, n-1\}$.

Example 1. The Cayley table of the Klein four-group $(2) \times (2)$ can be written in such form:

$$K = K(4) = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}.$$

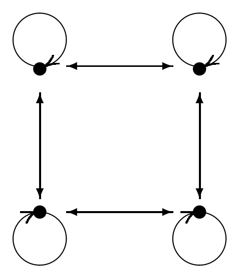
Then $K(4)$ is a reduced Gorenstein exponent matrix with permutation $\sigma = \sigma(K(4)) = (14)(23)$. Obviously,

$$K^{(2)} = \begin{bmatrix} 2 & 2 & 3 & 3 \\ 2 & 2 & 3 & 3 \\ 3 & 3 & 2 & 2 \\ 3 & 3 & 2 & 2 \end{bmatrix}$$

and

$$[Q(K)] = K^{(2)} - K^{(1)} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} = 3 \cdot P_1,$$

where P_1 is a doubly stochastic matrix, and $Q(K)$ is



Obviously, $in\, K = 3$.

Definition 4.1. *A real non-negative $s \times s$-matrix $P = (p_{ij})$ is doubly stochastic if $\sum_{j=1}^{s} p_{ij} = 1$ and $\sum_{i=1}^{s} p_{ij} = 1$ for any $i, j = 1, \ldots, s$.*

Definition 4.2. (see [8], p. 140). *A quasigroup Q which satisfies the identity $(xu)(vy) = (xv)(uy)$ for $x, y, u, v \in Q$ is called entropic.*

Example 2. ([8], p. 141, V. 2.2.1. Example). Let $Q(5) = \{0, 1, 2, 3, 4\}$ be the quasigroup with the following Cayley table

0	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

It is clear, that $Q(5)$ is an entropic quasigroup. The Cayley table

$$\mathcal{E}(5) = \begin{bmatrix} 0 & 4 & 3 & 2 & 1 \\ 1 & 0 & 4 & 3 & 2 \\ 2 & 1 & 0 & 4 & 3 \\ 3 & 2 & 1 & 0 & 4 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

of $Q(5)$ is a reduced Gorenstein exponent matrix with $\sigma(\mathcal{E}(5)) = (12345)$.

Obviously,

$$[Q(\mathcal{E}(5))] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = 2P_2,$$

where P_2 is a doubly stochastic matrix, and $in\,\mathcal{E}(5) = 2$.

Definition 4.3. *A reduced Gorenstein exponent matrix $\mathcal{E}$ is called cyclic if $\sigma(\mathcal{E})$ is a cycle.*

Remark. *Note, that a reduced tiled order Λ is Gorenstein if and only if its reduced exponent matrix $\mathcal{E}(\Lambda)$ is Gorenstein.*

Hence, in view of the Theorem 3.4 [9] we have such theorem.

Theorem B. *Let $\mathcal{E}$ be a cyclic reduced Gorenstein exponent matrix. Then $[Q(\mathcal{E})] = \lambda P$, where λ is a positive integer and P is a doubly stochastic matrix.*

For the Cayley table

$$\mathcal{E}(n) = \begin{bmatrix} 0 & n-1 & n-2 & \dots & 2 & 1 \\ 1 & 0 & n-1 & \dots & 3 & 2 \\ 2 & 1 & 0 & \dots & 4 & 3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ n-2 & n-3 & n-4 & \dots & 0 & n-1 \\ n-1 & n-2 & n-3 & \dots & 1 & 0 \end{bmatrix}$$

of the entropic quasigroup $Q(n)$, we have $[Q(\mathcal{E}(n))] = E_n + J_n^-(0) + e_{1n}$, where $J_n^-(0) = e_{21} + \ldots + e_{nn-1}$ is the lower nilpotent Jordan block.

The next definition is given in ([9], Section IV).

Definition 4.4. *A finite quasigroup Q defined on the set $S = \{0, 1, \ldots, n-1\}$ is called Gorenstein if its Cayley table $C(Q) = (\alpha_{ij})$ has a zero main diagonal and there exists a permutation $\sigma : i \to \sigma(i)$ for $i = 1, \ldots, n$ such that $\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)}$ for $i = 1, \ldots, n$.*

If σ is a cycle then G is a cyclic Gorenstein quasigroup.

Proposition 4.5. *The quasigroup $Q(n)$ is Gorenstein with permutation $\sigma = (12\ldots n)$, i.e. $Q(n)$ is a cyclic Gorenstein quasigroup.*

Proof. Obvious.

Theorem 4.6. *For any permutation $\sigma \in S_n$ without fixed elements there exists a Gorenstein reduced exponent matrix $\mathcal{E}$ with permutation $\sigma(\mathcal{E}) = \sigma$.*

Proof. Suppose that σ has no cycles of length 1 and decomposes into a product of non-intersecting cycles $\sigma = \sigma_1 \cdots \sigma_k$, where σ_i has length m_i. Denote by t the least common multiple of the numbers $m_1 - 1, \ldots, m_k - 1$.

Consider the matrix

$$\mathcal{E}(m_1,\dots,m_s)=\begin{pmatrix} t_1\mathcal{E}(m_1) & tU_{m_1\times m_2} & tU_{m_1\times m_3} & \dots & tU_{m_1\times m_k} \\ 0 & t_2\mathcal{E}(m_2) & tU_{m_2\times m_3} & \dots & tU_{m_2\times m_k} \\ 0 & 0 & t_3\mathcal{E}(m_3) & \dots & tU_{m_3\times m_k} \\ \dots & \dots & \dots & \ddots & \dots \\ 0 & 0 & 0 & \dots & t_k\mathcal{E}(m_k) \end{pmatrix},$$

where $t_j=\frac{t}{m_j-1}$, $U_{m_i\times m_j}$ is an $m_i\times m_j$ - matrix whose entries equal 1; $\mathcal{E}(m)=(\varepsilon_{ij})$, $\varepsilon_{ij}=\begin{cases} i-j, & \text{if } i\geq j; \\ i-j+m, & \text{if } i<j. \end{cases}$

Let us remark that $\varepsilon_{ij}+\varepsilon_{j\sigma(i)}=\varepsilon_{i\sigma(i)}=m-1$ for all i,j.

Evidently, $\mathcal{E}(m_1,\dots,m_s)$ is the reduced Gorenstein exponent matrix with permutation $\pi(A)=(123\dots m_1)(m_1+1\dots m_1+m_2)\cdots(m_1+m_2+\dots+m_{k-1}+1\dots m_1+m_2+\dots+m_{k-1}+m_k)$.

Since the permutations σ and π have the same type, these permutations are conjugate, i.e., there exists a permutation τ such that $\sigma=\tau^{-1}\pi(A)\tau$.

Consequently, by Propositions 3.6 and 3.7, the matrix $P_\tau^T\mathcal{E}(m_1,\dots,m_s)P_\tau$ is the reduced Gorenstein exponent matrix with permutation $\sigma(\mathcal{E})=\sigma$. □

In conclusion of this section we formulate the following question.

Suppose that a Latin square $\mathcal{E}$ [5] defined on $S=\{0,1,\dots,n-1\}$ is an exponent matrix which is doubly symmetric, that is $\mathcal{E}$ is symmetric with respect to the main diagonal and is also symmetric with respect to the secondary diagonal. Suppose also that the first row of $\mathcal{E}$ is $\{0\,1\,2\,\dots\,n-1\}$.

Is it true that $\mathcal{E}$ is necessarily the Cayley table of an elementary abelian 2-group?

5 Reduced exponent $(0,1)$-matrices and finite partially ordered sets

With any finite partially ordered set (poset) P we relate a reduced exponent $(0,1)$-matrix $\mathcal{E}_P=(\lambda_{ij})$ by the following way: $\lambda_{ij}=0\Leftrightarrow i\leq j$, otherwise $\lambda_{ij}=1$.

It is easy to see that $\mathcal{E}_P$ is indeed a reduced exponent matrix.

Conversely, a reduced $(0,1)$-matrix $\mathcal{E}=(\lambda_{ij})$ defines the finite poset $P_\mathcal{E}$ by the rule: $i\leq j$ if and only if $\lambda_{ij}=0$, and $P_{\mathcal{E}_P}=P$.

Denote by P_{max} (resp. P_{min}) the set of the maximal (resp. minimal) elements of P and by $P_{max}\times P_{min}$ their Cartesian product.

From ([2], Theorem 6.12) we have

Theorem C. *The quiver $Q(\mathcal{E}_P)$ can be obtained from the diagram $Q(P)$ by adding the arrows σ_{ij} for all $(p_i,p_j)\in P_{max}\times P_{min}$.*

Definition 5.1. *We shall say that finite posets S and T are Q-equivalent if reduced exponent $(0,1)$-matrices $\mathcal{E}_S$ and $\mathcal{E}_T$ are equivalent.*

Definition 5.2. *An index in P of a finite poset P is the maximal real eigen-value of the adjacency matrix $[Q(\mathcal{E}_P)]$ of $Q(\mathcal{E}_P)$.*

Now we shall give the list of indexes of posets with at most four elements.

I. $(1) = \{\bullet\}$, $in\,(I,1) = 1$.

II. $(1) = \left\{\begin{array}{c}\bullet\\ |\\ \bullet\end{array}\right\}$, $in\,(II,1) = 1$; $(2) = \{\bullet\ \bullet\}$, $in(II,2) = 2$.

III. $(1) = \left\{\begin{array}{c}\bullet\\ |\\ \bullet\\ |\\ \bullet\end{array}\right\}$, $in\,(III,1) = 1$;

$(2) = \left\{\begin{array}{ccccc} & & \bullet & & \\ & / & & \backslash & \\ \bullet & & & & \bullet\end{array}\right\}$, $(3) = \left\{\begin{array}{ccccc}\bullet & & & & \bullet\\ & \backslash & & / & \\ & & \bullet & & \end{array}\right\}$, $in\,(III,2) = in(III,3) = \sqrt{2}$;

$(4) = \left\{\begin{array}{cc} & \bullet\\ & |\\ \bullet & \bullet\end{array}\right\}$, $in\,(III,4) = \frac{1+\sqrt{5}}{2}$; $(5) = \{\bullet\ \bullet\ \bullet\}$, $in(III,5) = 3$.

IV. $(1) = \left\{\begin{array}{c}\bullet\\ |\\ \bullet\\ |\\ \bullet\\ |\\ \bullet\end{array}\right\}$, $in\,(IV,1) = 1$; $(2) = \left\{\begin{array}{ccccc} & & \bullet & & \\ & / & & \backslash & \\ \bullet & & & & \bullet\\ & \backslash & & / & \\ & & \bullet & & \end{array}\right\}$, $in(IV,2) = \sqrt[3]{2}$;

$(3) = \left\{\begin{array}{ccccc}\bullet & & & & \bullet\\ & \backslash & & / & \\ & & \bullet & & \\ & & | & & \\ & & \bullet & & \end{array}\right\}$, $(4) = \left\{\begin{array}{ccccc} & & \bullet & & \\ & & | & & \\ & & \bullet & & \\ & / & & \backslash & \\ \bullet & & & & \bullet\end{array}\right\}$, $in\,(IV,3) = in(IV,4) = \sqrt[3]{2}$;

$(5) = \left\{\begin{array}{ccc} & & \bullet\\ & & |\\ \bullet & & \bullet\\ & \backslash & |\\ & & \bullet\end{array}\right\}$, $(6) = \left\{\begin{array}{ccc} & & \bullet\\ & / & |\\ \bullet & & \bullet\\ & & |\\ & & \bullet\end{array}\right\}$; $\chi_{5,6}(x) = x(x^3 - x - 1)$ and

$1.32 < in\,(IV,5) = in\,(IV,6) < 1.33$;

$(7) = \left\{\begin{array}{cc} \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{array}\right\}$, $in\,(IV,7) = \sqrt{2}$; $(8) = \left\{\begin{array}{cc} & \bullet \\ & | \\ & \bullet \\ & | \\ \bullet & \bullet \end{array}\right\}$, $\chi_8(x) = x(x^3 - x^2 - 1)$

and

$1.46 < in\,(IV,8) < 1.47$;

$(9) = \left\{\begin{array}{ccc} \bullet & & \bullet \\ | & \diagdown & | \\ \bullet & & \bullet \end{array}\right\}$, $in\,(IV,9) = \sqrt{3}$;

$(10) = \left\{\begin{array}{ccccc} & & \bullet & & \\ & \diagup & | & \diagdown & \\ \bullet & & \bullet & & \bullet \end{array}\right\}$, $(11) = \left\{\begin{array}{ccccc} \bullet & & \bullet & & \bullet \\ & \diagdown & | & \diagup & \\ & & \bullet & & \end{array}\right\}$, $in(10) = in(11) = \sqrt{3}$;

$(12) = \left\{\begin{array}{cccccc} & & & \bullet & & \\ & & \diagup & & \diagdown & \\ \bullet & \bullet & & & & \bullet \end{array}\right\}$, $(13) = \left\{\begin{array}{cccccc} \bullet & \bullet & & & & \bullet \\ & & \diagdown & & \diagup & \\ & & & \bullet & & \end{array}\right\}$,

$in\,(IV,12) = in\,(IV,13) = 2$;

$(14) = \left\{\begin{array}{ccc} \bullet & & \bullet \\ | & \times & | \\ \bullet & & \bullet \end{array}\right\}$, $in(IV,14) = 2$;

$(15) = \left\{\begin{array}{ccc} & & \bullet \\ & & | \\ \bullet & \bullet & \bullet \end{array}\right\}$, $\chi_{15}(x) = x^2(x^2 - 2x - 1)$ and $in\,(IV,15) = 1 + \sqrt{2}$;

$(16) = \{\bullet\ \bullet\ \bullet\ \bullet\}$, $in\,(IV,16) = 4$.

Note that posets $(IV,2)$, $(IV,3)$ and $(IV,4)$ are Q-equivalent. For posets $N = (IV,9)$ and $F_4 = (IV,11)$ we have $in\,N = in\,F_4 = \sqrt{3}$, but N and F_4 are non-Q-equivalent.

References

[1] Birkhoff G. *Lattice theory.* Amer. Math. Soc., Providence, 1973.

[2] Chernousova Zh.T., Dokuchaev M.A., Khibina M.A., Kirichenko V.V., Miroshnichenko S.G., Zhuravlev V.N. *Tiled orders over discrete valuation rings, finite Markov chains and partially ordered sets. I.* Algebra and Discrete Math., 2002, **1**, p. 32–63.

[3] Chernousova Zh.T., Dokuchaev M.A., Khibina M.A., Kirichenko V.V., Miroshnichenko S.G., Zhuravlev V.N. *Tiled orders over discrete valuation rings, finite Markov chains and partially ordered sets. II.* Algebra and Discrete Math., 2003, **2**, no. 2, p. 47–86.

[4] Gubareni N.M., Kirichenko V.V. *Rings and Modules.* Czestochowa, 2001.

[5] Keedwell A.D., Denes J. *Latin squares and their applications.* N.Y., Academic Press, 1974.

[6] Kirichenko V.V., Mogileva V.V., Pirus E.M., Khibina M.A. *Semi-perfect rings and picewise domains*. Algebraic Researches, Institute of Mathematics NAS Ukraine, 1995, p. 33–65 (In Russian).

[7] Pierce R.S. *Associative Algebras*. Springer-Verlag, New York, 1982.

[8] Plugfelder H.O. *Quasigroups and loops: Introduction*. Berlin, Heldermann, 1990.

[9] Roggenkamp K.W., Kirichenko V.V., Khibina M.A., Zhuravlev V.N. *Gorenstein tiled orders*. Comm. in Algebra, 2001, **29(9)**, p. 4231–4247.

[10] Simson D. *Linear Representations of Partially Ordered Sets and Vector Categories*. Algebra, Logic Appl. 4, Gordon and Breach, 1992.

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Radicals of rings with involution

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Abstract. The aim of the present paper is to give a survey of the most important features of radicals in associative rings with involution including some new remarks and the most recent results on primitivity.

Mathematics subject classification: 16N80, 16W10.
Keywords and phrases: Associative ring, involution, radical, primitive ring.

1 Basic definitions and important examples

Throughout the paper a *ring with involution* R will be an associative ring endowed with a supplementary operation $* : x \to x^*$ called involution and satisfying the rules

$$\begin{array}{rcl} (x^*)^* & = & x \\ (x+y)^* & = & x^* + y^* \\ (xy)^* & = & y^* x^* \,. \end{array}$$

An element x of R is called *symmetric* if $x^* = x$ and skew if $x^* = -x$; the sets of these elements will be denoted by S resp. K. The element $x + x^*$ is called the *trace* of x and $x - x^*$ the *skew-trace* of x. For every ring R, R^{op} will denote the ring obtained by interchanging the order of the elements in the multiplication.

The most important examples of involutions are:
- the trivial involution $x^* = x$ on commutative rings,
- the conjugate $(x + iy)^* = x - iy$ on the complex numbers,
- the additive inverse $x^* = -x$ on commutative rings,
- the exchange involution $(x, y)^* = (y, x)$ on $R \oplus R^{op}$,
- transposition $A^* = A^T$ on rings M_n of $n \times n$ matrices and
- the symplectic involution $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} D^T & -C^T \\ -B^T & A^T \end{pmatrix}$ on rings M_{2n} of $2n \times 2n$-matrices.

2 Some older results

The first results on rings with involution linked with radicals appear in the sixties and seventies of the 20^{th} century. They go back to J.M. Osborn, C. Lanski and S. Montgomery; the following generalized versions can be found in Herstein's book [6].

Theorem 2.1 ([6], 2.1.8). *A semiprime ring R with involution in which every nonzero trace $x + x^*$ is invertible is of one of the following four types:*

1. R is a commutative ring of characteristic 2 with no nonzero nilpotent elements endowed with the trivial involution;

2. R is a division ring with no restriction on the involution;

3. $R \simeq D \oplus D^{op}$ for some division ring D endowed with the exchange involution;

4. $R \simeq M_2$ over a field with the symplectic involution.

Further results of the same type are:

Theorem 2.2 ([6], 2.1.7). *A semiprime ring R with involution in which every nonzero symmetric element is invertible is of one of the types 2,3 and 4 of the preceding theorem.*

Theorem 2.3 ([6], 2.3.1). *A semiprime noncommutative ring R with involution in which every nonzero skew-trace $x - x^*$ is invertible is of one of the types 2,3 and 4 of theorem 2.1.*

Theorem 2.4 ([6], 2.3.2). *If R is a noncommutative ring with involution which is not nil-semisimple and in which every nonzero skew-trace $x - x^*$ is invertible, then its nil-radical $N(R)$ satisfies*

1. $R/N(R)$ is commutative and

2. $N(R)^2 = 0$.

Theorem 2.5 ([6], 2.3.4). *For a ring R with involution in which every trace $x+x^*$ is nilpotent or invertible (or in which every skew-trace $x-x^*$ is nilpotent or invertible) the factor ring $R/J(R)$ by its Jacobson-radical $J(R)$ is of one of the following four types:*

1. a commutative ring with trivial involution;

2. a division ring with no restriction on the involution;

3. $R/J(R) \simeq D \oplus D^{op}$ with the exchange involution for some division ring D;

4. $R/J(R) \simeq M_2$ over a field endowed with the symplectic involution.

Notice that the fact to have the symplectic involution in the last case (although it is not mentioned in [6]) follows from theorem 1 resp. 3 since in this case $R/J(R)$ is simple and Jacobson-semisimple, hence semiprime.

A careful look to all these results reveals that they are dealing with concepts defined in the variety of associative rings without involution and make use of the supplementary operation of involution just in order to get a more precise description of some objects.

3 Radical theory in the variety of associative rings with involution

The appropriate terms to build up a radical theory in the class of all associative rings with involution are those of homomorphisms compatible as well with the involution and the corresponding ideals.

We call these ring-homomorphisms (fulfilling as well $f(x)^* = f(x^*)$) $*$-*homomorphisms* and the corresponding kernels $*$-*ideals*; the latter are exactly those ideals I of a ring with involution R which satisfy $I^* = I$. R is called $*$-simple if it contains no nontrivial $*$-ideals and $*$-prime resp. $*$-semiprime if $I \cdot J = 0$ for $*$-ideals I and J of R implies $I = 0$ or $J = 0$ (resp.$I^2 = 0$ implies $I = 0$).

The difference to the classical concepts without involution is exhibited by the following

Proposition 3.1 (see for ex. [18]).

1. A ring R with involution is $*$-simple if and only if R is simple or $R \simeq S \oplus S^{op}$ for some simple ring S, the involution being the exchange involution.

2. R is $*$-prime if and only if R contains a prime ideal P satisfying $P \cap P^* = 0$, i.e. if and only if R is prime or a subdirect product of two prime rings.

Remark 3.2. *However, a ring R with involution is $*$-semiprime if and only if it is semiprime.*

Radical theory in the variety of associative rings with involution was introduced in 1977 by Salavova [19]. She pointed out that the general radical theory introduced by Kurosh [7] and Amitsur [2] for Ω-groups resp. by Ryabuhin [17] for certain categories applies to the variety of rings with involution. The obtained radicals in this class will be called $*$-*radicals*. Consequently the following assertions hold:

Theorem 3.3.

1. (see for ex. [21]) *A mapping ρ assigning to every ring R with involution a $*$-ideal ρR is a $*$-radical if and only if the following conditions hold:*

($\rho 1$) $f(\rho R) \subseteq \rho(fR)$ *for every $*$-homomorphisms f defined on R*

($\rho 2$) $\rho(R/\rho R) = 0$

($\rho 3$) *ρ is idempotent:* $\rho(\rho R) = \rho R$

($\rho 4$) *ρ is complete:* $I \lhd^* R, \rho I = I \Rightarrow I \subseteq \rho R$.

2. (see [15]) *A class $\mathbf{R}$ of rings with involution is the radical class of a $*$-radical if and only if the following assertions hold:*

($\mathbf{R}1$) $\mathbf{R}$ *is closed under taking $*$-homomorphic images;*

($\mathbf{R}2$) $\mathbf{R}$ *is closed under taking sums of $*$-ideals within rings with involution;*

($\mathbf{R}3$) $\mathbf{R}$ *is $*$-extension closed, i.e. $I \lhd^* R$ with $I \in \mathbf{R}$ and $R/I \in \mathbf{R}$ implies* $R \in \mathbf{R}$.

3. (see [15]) *A class $\mathbf{S}$ of rings with involution is the semisimple class of a $*$-radical if and only if the following assertions hold:*

($\mathbf{S}1$) $\mathbf{S}$ *is closed under $*$-subdirect products, i.e. $I_\lambda \lhd^* R$ with $\bigcap_{\lambda \in \Lambda} I_\lambda = 0$ and $R/I_\lambda \in \mathbf{S}$ for all $\lambda \in \Lambda$ implies $R \in \mathbf{S}$;*

($\mathbf{S}2$) $\mathbf{S}$ *is $*$-extension closed;*

($\mathbf{S}3$) $\mathbf{S}$ *is $*$-regular, i.e. $0 \neq I \lhd^* R \in \mathbf{S}$ implies that I has a nontrivial $*$-homomorphic image in* $\mathbf{S}$;

($\mathbf{S}4$) $(R\mathbf{S})\mathbf{S} \lhd^* R$, *where $R\mathbf{S}$ denotes the intersection of all $*$-ideals I of R with* $R/I \in \mathbf{S}$.

Remark 3.4. *Notice that by Salavova's example 1.9 we know that the semisimple classes of $*$-radicals are not necessarily $*$-hereditary (i.e. $I \lhd R \in \mathbf{S}$ does not imply $I \in \mathbf{S}$). Hence, condition ($\mathbf{S}3$) cannot be replaced by heredity as in the case of associative rings without involution and ($\mathbf{S}4$) can not be omitted as far as is known.*

By the above remark it is clear that $*$-radicals do not always have the ADS-property, i.e. the $*$-radical of a $*$-ideal I of R is not necessarily an ideal in R (it is obviously closed under the involution). This fact lead to a series of papers by Loi and Wiegandt containing the following main results:

Theorem 3.5 ([13]). *For a $*$-radical ρ on the variety of all algebras with involution over a commutative ring R with identity the following assertions are equivalent:*

(1) ρ has the ADS-property;

(2) if an algebra A with involution belongs to the radical class $\mathbf{R}_\rho$ of ρ and satisfies $A^2 = 0$ then A belongs to $\mathbf{R}_\rho$ when endowed with any other involution;

(3) A with $*$ belongs to $\mathbf{R}_\rho$ if and only if A with involution $x \to -x^*$ belongs to $\mathbf{R}_\rho$ whenever $A^2 = 0$;

(4) A with the trivial involution belongs to $\mathbf{R}_\rho$ if and only if A with the additive inverse involution $x \to -x$ belongs to $\mathbf{R}_\rho$ whenever $A^2 = 0$.

Remark 3.6. *An example constructed in [13] shows that a $*$-radical with a $*$-hereditary semisimple class does not necessarily have the ADS-property.*

Theorem 3.7 ([13]). *All $*$-radicals on the variety of involution algebras over a field K have the ADS-property if and only if char $K = 2$.*

A result of the same kind as above has been proved for algebras with involution over commutative rings R with involution and identity, the difference to the earlier case being the rule

$$(ra)^* = r^*a^* \text{ instead of } (ra)^* = ra^*$$

for all $r \in R$ and $a \in A$.

Theorem 3.8 ([9]). *On the class of all algebras with involution on a commutative ring with involution and identity the assertions (1) and (3) of theorem 3.5 are equivalent.*

Let us recall that a $*$-radical ρ of rings or algebras with involution is called *hypernilpotent* if every nilpotent ring, resp. algebra is in the radical class $\mathbf{R}_\rho$ and *hypoidempotent* if the radical class $\mathbf{R}_\rho$ consists of idempotent rings resp. algebras only.

Loi proved the following

Theorem 3.9 ([10]). *Every $*$-radical of algebras with involution over a field with nontrivial involution is either hypernilpotent or hypoidempotent.*

Using this result, he also obtained

Theorem 3.10 ([10]). *In the variety of all algebras with involution over a field with nontrivial involution every $*$-radical has the ADS-property.*

A description of classes which are both radical and semisimple with respect to suitable radicals can be found in a paper by Loi dating back to 1989:

Theorem 3.11 ([11]). *For radical-semisimple classes of involution algebras over a field K with involution the following assertions hold*

1. *If K is infinite, there are no nontrivial radical-semisimple classes*;
2. *If K is finite, every nontrivial semisimple (and hence every nontrivial radical-semisimple class) consists of all subdirect sums of algebras belonging to some strongly hereditary finite set of simple involution algebras.*

4 The connection between ring radicals and $*$-radicals

The next question arising is whether a radical of associative rings is already a $*$-radical for associative rings with involution. A complete answer was given in 1992 in a paper by Lee and Wiegandt:

Theorem 4.1 ([8]). *For a radical ρ of associative rings the following assertions are equivalent:*

1. *ρ is a $*$-radical, i.e. $\rho R \lhd^* R$ for every ring R with involution*;
2. *$R \in \mathbf{S}_\rho$ implies $R^{op} \in \mathbf{S}_\rho$*;
3. *$R \in \mathbf{R}_\rho$ implies $R^{op} \in \mathbf{R}_\rho$.*

This theorem infers that the ring radicals which are $*$-radicals are exactly the symmetric ones. (Notice that the basic definition needs not to be symmetric as can be seen from the Jacobson radical defined via primitivity). A list of the most important among them is given in

Corollary 4.2 ([8]). *The following ring radicals are $*$-it radicals: the Koethe (nil) radical, the generalized nil radical, the Baer (prime) radical, the Behrens radical, the Brown-McCoy radical, the Jacobson radical, the Levitzki radical, the von Neumann-regular radical, the strongly regular radical, the idempotent radical.*

Remark 4.3. *There are ring radicals which are not $*$-radicals as can be seen from the examples of the right strongly prime resp. the right superprime radical (see [16, 20]).*

In two papers from 1996 resp 1998, Booth and Groenewald looked to the question of constructing $*$-radicals from ring radicals. They introduced a mapping λ assigning to every ring radical ρ a $*$-radical λ_ρ taking for $(\lambda\rho)(R)$ the sum of all $*$-ideals I of R belonging to $\mathbf{R}_\rho$.

Theorem 4.4 ([5]). *The following assertions hold:*

1. *every ring radical ρ induces a $*$-radical λ_ρ*;
2. *λ maps the symmetric ring radicals bijectively onto the invariant $*$-radicals*, i.e. *those $*$-radicals for which $(\lambda\rho)(R)$ is the same for all involutions on R*;
3. *$(\lambda\rho)(R) \subseteq \rho R \cap \rho(R^{op})$ with equality whenever $\mathbf{R}_\rho$ is hereditary.*

Notice that in view of theorem 4.1, λ restricted to the symmetric ring radicals is in fact the identity mapping and yields therefore $*$-radicals with $*$-hereditary semisimple classes.

The dual to the above construction using the semisimple homomorphic images of R instead of the radical ideals has not been considered so far; thus for every ring radical ρ and every ring R with involution let us define

$$(\sigma\rho)(R) = \bigcap(K \lhd^* R | R/K \in \mathbf{S}_\rho).$$

Theorem 4.5. *The following assertions hold:*

1. *every ring radical ρ induces a $*$-radical $\sigma\rho$ with a $*$-hereditary semisimple class;*
2. *σ restricted to the symmetric ring radicals is the identity mapping;*
3. *$\rho R + \rho(R^{op}) \subseteq (\sigma\rho)(R)$ with equality whenever $\mathbf{S}_\rho$ is homomorphically closed.*
4. *$(\lambda\rho)R \subseteq \rho R \subseteq (\sigma\rho)(R)$ with equality if and only if ρ is symmetric.*

Proof.

1. We show that the class $\mathbf{S}_{\sigma\rho}$ is closed under taking subdirect products, extensions and $*$-ideals, the crucial point being the obvious inclusion $\rho R \subseteq (\sigma\rho)(R)$ with equality for $R \in \mathbf{S}_\rho$ since 0 is always a $*$-ideal.

If R is a subdirect product of involution rings $R_i \in \mathbf{S}_{\sigma\rho}(i \in I)$, then R, considered as a ring, is a subdirect product of the rings $R_i \in \mathbf{S}_\rho$ and thus belongs to $\mathbf{S}_\rho$; hence $\mathbf{S}_{\sigma\rho}$ is subdirectly closed.

If R and R/I belong to $\mathbf{S}_{\sigma\rho}$ for some $*$-ideal I of R, then R and R/I are rings belonging to $\mathbf{S}_\rho$; thus R belongs to $\mathbf{S}$ inferring $R \in \mathbf{S}_{\sigma\rho}$.

Heredity of $\mathbf{S}_{\sigma\rho}$ is obtained by a similar argument.

2. This assertion follows from the fact that the symmetric ring radicals are $*$-radicals.

3. $(\sigma\rho)(R)$ being a $*$-ideal, it contains both ρR and $(\rho R)^*$. The isomorphism $(\rho R)^* \simeq (\rho R)^{op} = \rho(R^{op})$ yields $\rho R + \rho(R^{op}) \subseteq (\sigma\rho)(R)$.

If $\mathbf{S}_\rho$ is homomorphically closed, then $R/\rho R + \rho(R^{op})$ belongs to $\mathbf{S}_\rho$, hence to $\mathbf{S}_{\sigma\rho}$ implying $(\sigma\rho)(R) \subseteq \rho R + \rho(R^{op})$.

4. Is a direct consequence of 2 and 3 in theorems 4.3 and 4.5.

Special ring radicals are defined by the upper radicals $\mathcal{U}M = \{R|R$ has no nonzero homomorphic image in $M\}$ of special classes M of rings, where a class M is called *special* if

1. M consists of prime rings,
2. M is hereditary,
3. M is closed under essential extensions, i.e. if an essential ideal I of R belongs to M, then R belongs to M.

Already Salavova introduced the concept of $*$-*special classes* of rings with involution in her paper [19]. Her definition is equivalent to the $*$-analogue of the above one obtained by writing $*$-prime, $*$-hereditary and essential $*$-ideal in 1,2 and 3.

In the definition of $*$-special radicals, we have to use $\mathcal{U}^*M = \{R|R$ has no nonzero $*$-homomorphic image in $M\}$.

In 1996, resp.1998 Booth and Groenewald showed how special classes induce $*$-special classes.

Theorem 4.6 ([4], resp. [5]). *Every special class M of rings induces a $*$-special class M of rings R with involution by*

$$M^* = \{R|\exists P \lhd R \quad \textit{with} \quad P \cap P^* = 0 \quad \textit{and} \quad R/P \in M\}.$$

Moreover, if ρ is a special radical of rings, then

$$\mathcal{U}^*(\mathbf{R}\rho) = \mathbf{R}_{\lambda\rho}$$

where λ is the mapping from theorem 4.4.

5 The role of $*$-biideals

One-sided ideals are the basis for the construction of many radicals of rings. However, in rings with involution $*$, they are never closed under $*$ unless they are two-sided, i.e. $*$-ideals. So, what kind of substructure could replace the one-sided ideals in presence of an involution?

Taking just the $*$-ideals, the situation would become similar to that of commutative rings and we would loose a lot of information. A much better solution is inherited by considering the fact that in the classical case the annihilators of elements of R-modules are one-sided ideals of R. For (one-sided) modules over rings R with involution, it is usual to give the following definition:

$$Ann^*_R m = \{r \in R | rm = 0 = r^*m\} \quad \text{and}$$
$$Ann^*_R M = \{r \in R | rM = 0 = r^*M\}.$$

It is easy to see that for $Ann^*_A M$ instead of an ideal in the classical case we now obtain a $*$-ideal.

$Ann^*_R m$ can be seen either as a $*$-*quasiideal*, i.e. a subgroup Q of $(R,+)$ satisfying $Q^* = Q$ and $QR \cap RQ \subseteq Q$ or as a $*$-*biideal*, i.e. a subgroup B of $(R,+)$ which satisfies $B^* = B$ and $BRB \subseteq B$. Since quasiideals are not necessarily subrings, the suitable structure to replace one-sided ideals in the theory of $*$-radicals seems to be that of $*$-biideals.

The first result underlining this conjecture has been given by Loi in 1990:
Theorem 5.1 ([12]). *A semiprime ring R with involution has d.c.c on principal right ideals (when considered just as a ring) if and only if R has d.c.c. on principal $*$-biideals (i.e. $*$-biideals generated by a single element).*

This theorem was generalised by Aburawash in 1991:

Theorem 5.2 ([1]). *A semiprime ring R with involution has d.c.c. on right ideals (when considered as a ring) if and only if it has d.c.c. on $*$-biideals. Moreover, such a ring always has a.c.c.on $*$-biideals as well.*

In 1993, Beidar and Wiegandt proved

Theorem 5.3 ([3]). *A ring R with involution has d.c.c on $*$-biideals if and only if both R and its Jacobson radical $J(R)$ are right- and left-artinian, i.e. have d.c.c. on right and left ideals.*

Remark 5.4. *In the same paper an example of a ring with involution is given which, considered just as a ring, has both a.c.c. on right and left ideals, but fails to have a.c.c. on $*$-biideals.*

Recently, $*$-primitive rings with involution have been studied. A ring R with involution is called *$*$-primitive* if there is an irreducible R-left-module M satisfying $Ann^*_R M = 0$.

It is well known (see for ex. [18]) that a ring with involution is $*$-primitive if and only if considered without involution it is either a left-primitive ring or the subdirect sum $R/P \oplus_{sub} R/P^*$ of a left and a right primitive ring.

A subdirect sum giving rather poor information, it seemed worth to look for a new description of $*$-primitivity.

Let us recall that a ring R (without involution) is called *primitive* if there is a faithful irreducible R-module, i.e. if R contains a maximal left ideal L such that $Ann_R R/L = 0$. $Ann_R R/L$ being the largest ideal of R contained in L, this means that R contains a maximal left ideal L which does not contain any nonzero ideal of R.

Now, the $*$-analogue is given by:

Theorem 5.5 ([14]). *A ring R with involution is $*$-primitive if and only if it contains a maximal $*$-biideal which does not contain any nonzero $*$-ideal of R.*

Furthermore, in the same paper, the involution analogue of the well known statement that a prime ring with a minimal left ideal is primitive has been proved.

Theorem 5.6 ([14]). *A $*$-prime ring with involution with a minimal $*$-biideal is $*$-primitive.*

Thus, the suitable structure to use in rings with involution instead of one-sided ideals are the $*$-biideals.

References

[1] Aburawash U.A. *Semiprime involution rings and chain conditions.* Proc. Vienna Conf. 1990, Contr. to General Alg. 7, H"older-Pichler-Tempsky, Wien/B.G. Teubner, Stuttgart, 1991, p. 7–11.

[2] Amitsur S.A. *A general theory of radicals II: radicals in rings and bicategories.* Amer. J. Math., 1954, **76**, p. 100–125.

[3] Beidar K.I., Wiegandt R. *Rings with involution and chain conditions.* J. Pure Appl. Alg., 1993, **87**, p. 205–220.

[4] Booth G.L., Groenwald N.J. *Special radicals in rings with involution.* Publ. Math. Debrecen, 1996, **48**, p. 241–251.

[5] BOOTH G.L., GROENEWALD N.J. *Radicals of involution rings.* Algebra Colloquium, 1998, **5**, p. 277–284.

[6] HERSTEIN I.N. *Rings with involution.* The University of Chicago Press, Chicago, 1976.

[7] KUROSH A.G. *Radicals of rings and algebras.* Mat. Sbornik, 1953, **33**. p. 13–26 (In Russian).

[8] LEE P.H., WIEGANDT R. *On radicals and rings with involution.* Publications in mathematics and applications, Ser. A, 1992, **3**, p. 219–224.

[9] LOI N.V. *The ADS-property for radicals of involution K^*-algebras.* Archiv d. Math., 1987, **49**, p. 196–199.

[10] LOI N.V. *A note on the radical theory of involution algebra.* Studia Sci. Math. Hung., 1988, **23**, p. 157–160.

[11] LOI N.V. *Semisimple radical classes of involution algebras.* Proc. Edinburgh Math. Soc., 1989, **32**. p. 1–9.

[12] LOI N.V. *On the structure of semiprime involution rings.* General Algebra, 1988; Proc. Krems Conf., North-Holland Publ. Comp., Amsterdam, 1990, p. 153–161.

[13] LOI N.V., WIEGANDT R. *Involution algebras and the Anderson-Divinsky-Sulinski property.* Acta Sci. Math., 1986, **50**, p. 15–38.

[14] BEIDAR K.I., MÁRKI L., MLITZ R., WIEGANDT R. *On primitivity of involution rings.* Manuscript, submitted.

[15] MLITZ R. *Radicals and semisimple classes of Ω-groups.* Proc. Edinburgh Math. Soc., 1980, **23**, p. 37–41.

[16] PARMENTER M.M., PASSMAN D.S., STEWART P.N. *The strongly prime radical of crossed products.* Comm. Algebra, 1984, **12**, p. 1099–1113.

[17] RJABUHIN YU.M. *Radicals in categories.* Mat. issled., 1967, **2**, vyp. 3(5), p. 107–165 (In Russian).

[18] ROWEN L.H. *Ring Theory I.* Academic Press, New York, 1988.

[19] SALAVOVA K. *Radicals of rings with involution 1,2.* Commentationes. Math. Univ. Carolinae, 1977, **18**, p. 367–381 and p. 455–466 (In Russian).

[20] VELDSMAN S. *The superprime radical.* Proc. Krems Conf. 1985, Contr. to General Alg. 4, H"older-Pichler-Tempsky, Wien B.G. Teubner, Stuttgart, 1987, p. 181–188.

[21] WIEGANDT R. *Radical and semisimple classes of rings.* Queen's papers in pure and applied Math. 37, Queen's University, Kingston, 1974.

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Radicals around Köthe's problem

S. Tumurbat, R. Wiegandt

Abstract. Radicals γ will be studied for which the condition "$A[x] \in \gamma$ for all nil rings A" is equivalent to the positive solution of Köthe's Problem ($A[x]$ is Jacobson radical for all nil rings A, in Krempa's formulation). The closer γ is to the Jacobson radical, the better approximation of the positive solution is obtained. Seeking, however, for a negative solution, possibly large radicals γ are of interest. In this note such large radicals will be studied.

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Keywords and phrases: Primitive and polynomial ring, Köthe's Problem, nil and Jacobson radical.

1 Introduction

We shall work with associative rings (not necessarily with unity element) and Kurosh–Amitsur radicals. For details we refer to [4]. We shall use the following letters for operators acting on classes of rings:

$\mathcal{L}$ lower radical operator;
$\mathcal{U}$ upper radical operator;
h homomorphic closure operator;
$\mathcal{H}$ hereditary closure operator.

Further notations:

$\mathcal{N} = \{\text{all nil rings}\}$, the nil radical class;
$\mathcal{J}$ the Jacobson radical or radical class;
$\mathcal{G}$ the Brown–McCoy radical or radical class;
$\mathcal{B}$ the Behrens radical: the upper radical of rings with nonzero idempotents;
u the upper radical of uniformly strongly prime rings (a ring A is uniformly strongly prime, if there exists a finite subset $F \subseteq A$ such that $xFy \neq 0$ whenever $0 \neq x, y \in A$);
$\mathcal{P} = \{\text{all primitive rings}\}$;
$\mathcal{Q} = \{A[x] \mid A \in \mathcal{N}\}$;
$\ell = \mathcal{L}h\mathcal{Q}$;
$\mathfrak{K} = \mathcal{U}\mathcal{H}(\ell \cap \mathcal{P})$;
$\mathfrak{M} = \mathcal{U}(\ell \cap \mathcal{P})$ may not be a radical class, though homomorphically closed.

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Köthe's Problem (1930) asks as whether the sum of two nil left ideals is always a nil left ideal. Krempa's well-known criterion says that Köthe's Problem is equivalent to the condition: $A[x] \in \mathcal{J}$ for every $A \in \mathcal{N}$, that is, $\mathcal{Q} \subset \mathcal{J}$.

This raises the possibility of approximating Köthe's Problem by radicals. At present, by [1], [2] we know that $A \in \mathcal{N}$ implies $A[x] \in \mathcal{B} \cap u$. Tumurbat [9] gave the exact lower bound ℓ, and a positive solution of Köthe's Problem is equivalent to $\ell(A[x]) \subseteq \mathcal{J}(A[x])$ for all nil rings A.

McConnell and Stokes [6] introduced and investigated a non-hereditary radical $\mathcal{K}$, they proved that $\mathcal{J} \subset \mathcal{K}$ and that Köthe's Problem has a positive solution if and only if $A[x] \in \mathcal{K}$ for every $A \in \mathcal{N}$. Recently Sakhajev [7] announced the negative solution of Köthe's Problem. Thus, solving Köthe's problem in the negative by an explicitly given counterexample, possibly large radicals γ may be of interest for which $\mathcal{J} \subset \gamma$ and $A[x] \in \gamma$ for every $A \in \mathcal{N}$. In this note we shall investigate such large radicals.

2 An interval of radicals

Proposition 2.1. (i) *For a radical γ, $\mathcal{J} \cap \mathcal{Q} = \gamma \cap \mathcal{Q}$ if and only if $\mathcal{J} \cap h\mathcal{Q} = \gamma \cap h\mathcal{Q}$;*

(ii) *$A \in \mathcal{N}$ and $A[x] \in \mathfrak{M}$ implies $A[x] \in \mathcal{J} \cap \ell$;*

(iii) *$\mathfrak{M} \cap \mathcal{Q} = \mathcal{J} \cap \mathcal{Q} = (\mathcal{J} \cap \ell) \cap \mathcal{Q}$.*

Proof. (i) Straightforward.

(ii) If $A \in \mathcal{N}$ and $A[x] \notin \mathcal{J}$, then $A[x]$ has a nonzero homomorphic image in $\ell \cap \mathcal{P}$; so $A[x] \notin \mathfrak{M}$. Hence $A \in \mathcal{N}$ and $A[x] \in \mathfrak{M}$ implies $A[x] \in \mathcal{J}$, whence $A[x] \in \mathcal{J} \cap \ell$.

(iii) Obvious by (ii). $\square$

Proposition 2.2. *Let γ be any radical. Then*

(i) *$\gamma \in [\ell \cap \mathcal{J}, \mathfrak{M}]$ implies $\gamma \cap \mathcal{Q} = \mathcal{J} \cap \mathcal{Q}$;*

(ii) *$\gamma \cap \mathcal{Q} = \mathcal{J} \cap \mathcal{Q}$ implies $\ell \cap \mathcal{J} \subseteq \gamma$;*

(iii) *if γ is hereditary and $\gamma \cap \mathcal{Q} = \mathcal{J} \cap \mathcal{Q}$, then $\gamma \subseteq \mathfrak{M}$.*

Proof. (i) Since $\mathcal{Q} \subset \ell$, the equality $(\ell \cap \mathcal{J}) \cap \mathcal{Q} = \mathcal{J} \cap \mathcal{Q}$ is obvious.

Next, we prove that $\mathfrak{M} \cap \mathcal{Q} = \mathcal{J} \cap \mathcal{Q}$. Clearly $\mathcal{J} \subseteq \mathfrak{M}$, therefore $\mathcal{J} \cap \mathcal{Q} \subseteq \mathfrak{M} \cap \mathcal{Q}$. Assume that there exists a ring $A[x] \in (\mathfrak{M} \cap \mathcal{Q}) \setminus \mathcal{J}$. Then $A[x] \in \ell$ and has a nonzero homomorphic image B in $\mathcal{P}$. Hence $B \in \ell \cap \mathcal{P}$, and so $A[x] \notin \mathfrak{M}$. This contradiction proves that $\mathfrak{M} \cap \mathcal{Q} \subseteq \mathcal{J} \cap \mathcal{Q}$.

Let γ be any radical class in the interval $[\ell \cap \mathcal{J}, \mathfrak{M}]$. Then we have

$$\mathcal{J} \cap \mathcal{Q} = (\ell \cap \mathcal{J}) \cap \mathcal{Q} \subseteq \gamma \cap \mathcal{Q} \subseteq \mathfrak{M} \cap \mathcal{Q} = \mathcal{J} \cap \mathcal{Q}.$$

(ii) Assume that $\ell \cap \mathcal{J} \nsubseteq \gamma$, and $A \in (\ell \cap \mathcal{J}) \setminus \gamma$. Then every nonzero homomorphic image B of A has a nonzero accessible subring C in $h\mathcal{Q} \cap \mathcal{J} = h\mathcal{Q} \cap \gamma \subseteq \gamma$. Hence $A \in \gamma$ follows, contradicting $A \notin \gamma$.

(iii) Suppose that $\gamma \nsubseteq \mathfrak{M}$ and $A \in \gamma \setminus \mathfrak{M}$. Then A has a nonzero homomorphic image $B \in \gamma \cap \ell \cap \mathcal{P}$, and therefore B has a nonzero accessible subring $C \in h\mathcal{Q} \cap \mathcal{P}$.

Since γ is hereditary, also $C \in \gamma \cap h\mathcal{Q} \cap \mathcal{P}$ holds. Hence $\gamma \cap h\mathcal{Q} \nsubseteq \mathcal{J} \cap h\mathcal{Q}$ follows. In view of Proposition 2.1 (i) this is a contradiction. □

Theorem 2.3. *A radical γ is in the interval $[\ell \cap \mathcal{J}, \mathfrak{M}]$ if and only if $\gamma \cap \ell = \mathcal{J} \cap \ell$.*

Proof. Assume that $\gamma \in [\ell \cap \mathcal{J}, \mathfrak{M}]$. Clearly $\ell \cap \mathcal{J} \subseteq \gamma$ and so $\ell \cap \mathcal{J} \subseteq \ell \cap \gamma$. Suppose that $\ell \cap \mathcal{J} \neq \ell \cap \gamma$. Then there exists a ring $A \in (\ell \cap \gamma) \setminus (\ell \cap \mathcal{J})$, and necessarily $A \notin \mathcal{J}$. Hence A has a nonzero homomorphic image $B \in \mathcal{P} \cap \ell$, and so $A \notin \mathfrak{M}$, contradicting $A \in \ell \cap \mathcal{J} \subseteq \gamma \subseteq \mathfrak{M}$.

Conversely, suppose that $\gamma \cap \ell = \mathcal{J} \cap \ell$ for some radical γ. We claim that $\gamma \subseteq \mathfrak{M}$. Assume that this is not true, and $\gamma \nsubseteq \mathfrak{M}$. Then there exists a nonzero homomorphic image of a ring $A \in \gamma$ such that $B \in \gamma \cap (\ell \cap \mathcal{P}) = (\mathcal{J} \cap \ell) \cap \mathcal{P} = \{0\}$, a contradiction. Hence $\gamma \subseteq \mathfrak{M}$. Further, $\mathcal{J} \cap \ell = \gamma \cap \ell \subseteq \gamma$. □

Next, we give conditions equivalent to the positive solution of Köthe's problem.

Theorem 2.4. *The following conditions are equivalent:*

(i) *Köthe's problem has a positive solution;*
(ii) $\ell \subseteq \mathcal{J}$*;*
(iii) $\ell \cap \mathcal{P} = \{0\}$*;*
(iv) $\mathfrak{K} = \{$*all rings*$\} = \mathfrak{M}$*;*
(v) $\mathcal{Q} \subseteq \gamma$ *for any radical* γ *with* $\ell \cap \mathcal{J} \subseteq \gamma$*.*

Proof. The following implications are straightforward:

(i)$\Longrightarrow$(ii)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv),
(ii)$\Longrightarrow$(v)$\Longrightarrow$(i). □

Corollary 2.5. *Let γ be a radical such that $\mathcal{Q} \subset \gamma$.*

Then $\gamma \subseteq \mathfrak{M}$ if and only if Köthe's problem has a positive solution. In particular, γ may be the Behrens, Brown–McCoy, uniformly strongly prime radicals, or the upper radical of von Neumann regular rings.

Proof. $\mathcal{Q} \subset \gamma$ implies $\ell \subseteq \gamma$, and so $\ell \cap \mathcal{J} \subseteq \gamma$. Hence from Theorem 2.3 it follows that $\ell \subseteq \gamma \cap \ell = \mathcal{J} \cap \ell \subseteq \mathcal{J}$. Further, $\mathcal{Q} \subset \mathcal{B} \cap \mathcal{G} \cap u \cap \mathcal{U}\nu$ is well-known (see [1] and [2]). □

A radical γ is said to be *polynomially extensible* if $A \in \gamma$ implies $A[x] \in \gamma$.

Corollary 2.6. *Köthe's problem has a positive solution if and only if the interval $[\ell \cap \mathcal{J}, \mathfrak{M}]$ contains a polynomially extensible radical.*

Proof. If Köthe's problem has a positive solution, then $\mathfrak{M} = \{$all rings$\}$ is polynomially extensible.

Let $\gamma \in [\ell \cap \mathcal{J}, \mathfrak{M}]$ be a polynomially extensible radical. Then $\mathcal{N} \subseteq \ell \cap \mathcal{J} \subseteq \gamma$ implies $\mathcal{Q} \subseteq \gamma$. Hence Theorem 2.4 (v) yields the assertion. □

Remark 2.7. *In Corollaries 2.5 and 2.6 the class $\mathfrak{M}$ can be replaced by the radical $\mathfrak{K}$.*

3 On the radical $\mathfrak{K}$

As we have seen in Theorem 2.4, the radical $\mathfrak{K}$ is not the class of all rings if and only if $\ell \cap \mathcal{P} \neq \{0\}$. This is the case precisely when there exists a polynomial ring $R[x]$ over a nil ring R which has a nonzero primitive homomorphic image. In this section we shall discuss properties of the radical $\mathfrak{K}$ and prove criteria of the positive solution of Köthe's Problem in terms of $\mathfrak{K}$.

Theorem 3.1. *Köthe's Problem has a positive solution if and only if the radical $\mathfrak{K}$ is hereditary.*

Proof. If Köthe's Problem has a positive solution, then $\ell \cap \mathcal{P} = \{0\}$ and $\mathfrak{K}$ is the class of all rings, which is trivially hereditary.

Conversely, suppose that $\mathfrak{K}$ is hereditary. Let us consider an arbitrary nonzero ring A and its Dorroh extension A^1. We are going to prove that $A^1 \in \mathfrak{K}$. Suppose the contrary, that $A^1 \notin \mathfrak{K}$. Then A^1 has a nonzero homomorphic image $B^1 \in \mathcal{H}(\ell \cap \mathcal{P})$. Hence B^1 is an accessible subring of a ring $C \in \ell \cap \mathcal{P}$, and so has a nonzero idempotent, the unity element e of B^1. By a Zorn lemma argument C has a subdirectly irreducible homomorphic image $C/M \in \ell$ possessing a nonzero idempotent $e + M$ in its heart. Thus C/M is in the Behrens semisimple class $\mathcal{SB}$. Taking into account that $C/M \in \ell$, we conclude that there exists a nonzero accessible subring D of C/M which is in $\mathcal{SB} \cap h\mathcal{Q}$. Thus there exists a polynomial ring $E[x]$ over a nil ring E such that $D \cong E[x]/K$. But by Beidar, Fong and Puczyłowski [1], $E \in \mathcal{N}$ implies $E[x] \in \mathcal{B}$ and also $D \in \mathcal{B}$, a contradiction. Hence $A^1 \in \mathfrak{K}$. Thus by $A \triangleleft A^1$ the hereditariness of $\mathfrak{K}$ yields $A \in \mathfrak{K}$, which means that $\mathfrak{K}$ is the class of all rings, and so $\ell \cap \mathcal{P} = \{0\}$ and $\ell \subseteq \mathcal{J}$ follows. Hence $A \in \mathcal{N}$ implies $A[x] \in \mathcal{J}$. $\square$

A ring A is said to be an *s-ring* if every primitive homomorphic image of A is a reduced ring or has a homomorphic image with nonzero idempotent. Recall that the class $\mathcal{L}$ of locally nilpotent rings is the *Levitzki radical class*. In the proof of the next Proposition and in Theorem 4.5 we shall make use of the radicals ϱ and δ which are *the upper radicals of the classes*

$$\{A \in \mathcal{SL} \mid \text{every nil subring of } A \text{ is in } \mathcal{L}\}$$

and

$$\{A \in \mathcal{SN} \mid \text{the nilpotent elements of } A \text{ form a subring}\},$$

respectively.

Proposition 3.2. *All s-rings are in the radical class $\mathfrak{K}$.*

Proof. Suppose that A is an s-ring and $A \notin \mathfrak{K}$. Then A has a nonzero homomorphic image B which is an accessible subring of a ring $C \in \ell \cap \mathcal{P}$. Hence also B is a primitive ring. Suppose that B is a reduced ring. We choose an ideal I of C which is maximal relative to $I \cap B = 0$. Then by $B \cong (B + I)/I$, we may assume that B is an accessible subring in $D = C/I$. By induction we can see that B is an essential accessible subring in D. Since B is primitive, we conclude that also D is primitive.

By an iterated application of the Andrunakievich Lemma we get that a power J of the ideal of D generated by B, is contained in B. Since B is primitive, necessarily $J \neq 0$. Thus D has a nonzero ideal J contained in B. J is a reduced ring as B is so. We show that also D is a reduced ring. Assume that $a^2 = 0$ for a nonzero element $a \in D$. Since D is primitive, necessarily $aJa \neq 0$ and so $0 \neq aja \in J$ with a suitable element $j \in J$. Hence $0 = aja \cdot aja \in J$ follows, a contradiction. Thus D is a reduced ring and $D \in \ell$ by $C \in \ell \cap \mathcal{P}$. As proved in [9, Theorem 2.9], $\ell \subset \varrho \cap \delta$, and so $D \in \ell \subset \varrho$. Hence D has a nonzero locally nilpotent ideal or D has a nil subring which is not locally nilpotent, whence D is not reduced, a contradiction. Thus B is not reduced, but B has a homomorphic image possessing a nonzero idempotent as well as a subdirectly irreducible homomorphic image B/K which has a nonzero idempotent in its heart H/K. Let us consider the ideal $\langle K \rangle_D$ of D generated by K. Now we have $K \subseteq \langle K \rangle_D \cap H$. By the simplicity of H/K either $\langle K \rangle_D \cap H = H$ or $\langle K \rangle_D \cap H = K$. In the first case there exists a natural number $n \geq 3$ such that $H = H^n = \langle K \rangle_D^n \cap H \subseteq \langle K \rangle_D^n \subseteq K$, contradicting $H/K \neq 0$. So $\langle K \rangle_D \cap H = K$. Using the Zorn Lemma there exists an ideal M of D which is maximal relative to $M \cap H = K$. Then the factor ring D/M is subdirectly irreducible with heart $(H+M)/M \cong H/K$. Hence $D/M \in \ell \setminus \mathcal{B}$, contradicting $\ell \subset \mathcal{B}$ (cf. [1]). Thus $A \in \mathfrak{K}$ has been established. $\square$

Applying Proposition 3.2 to some special cases of s-rings, we get

Corollary 3.3. *All rings with unity element, all commutative rings and all rings with d.c.c. on principal left ideals are in* $\mathfrak{K}$.

For a ring A we denote by $[A, A]$ the ideal of A generated by the commutators $[a, b] = ab - ba$ for all $a, b \in A$.

Theorem 3.4. *The following conditions are equivalent:*

(i) *Köthe's Problem has a positive solution,*

(ii) *if a finitely generated Jacobson semisimple ring A is in $\mathfrak{K}$, then also its commutator ideal $[A, A]$ is in $\mathfrak{K}$.*

Proof. (i)$\Longrightarrow$(ii) Trivial by Theorem 2.4 (iv).

(ii)$\Longrightarrow$(i) Let F be a finitely generated free ring.

Clearly, also the unital extension F^1 of F is finitely generated, and both F and F^1 are Jacobson semisimple. Hence by Corolary 3.3 the ring F^1 is in $\mathfrak{K}$, and by (ii) we have that $[F^1, F^1] \in \mathfrak{K}$. Again by Corolary 3.3 the commutative ring $F/[F, F]$ is in $\mathfrak{K}$. But $[F, F] = [F^1, F^1] \in \mathfrak{K}$, so also $F \in \mathfrak{K}$.

Suppose that Köthe's Problem has a negative solution. Then there exists a nil ring A such that $A[x] \notin \mathcal{J}$. Hence there exists a polynomial $f(x) = \sum_{i=0}^{n} a_i x^i \in A[x]$ such that $f(x)$ has no quasi-inverse in $A[x]$. Let B denote the subring of A generated by the elements $a_0, \dots, a_n \in A$. By $B \subseteq A$, also B is a nil ring, further, also the ring C generated by B and x is finitely generated. Since every finitely generated free ring is in $\mathfrak{K}$, we have that $C \in \mathfrak{K}$. Further, from

$$C/B[x] \cong \{x\} \in \mathcal{SJ}$$

it follows that $\mathcal{J}(C) \subseteq \mathcal{J}(B[x])$, and from $B[x] \lhd C$ we conclude that $\mathcal{J}(B[x]) \subseteq \mathcal{J}(C)$. So $D = C/\mathcal{J}(B[x])$ is a finitely generated Jacobson semisimple ring in $\mathfrak{K}$. Applying condition (ii) we get that $[D, D] \in \mathfrak{K}$. Obviously we have

$$[D, D] = [B[x]/\mathcal{J}(B[x]), B[x]/\mathcal{J}(B[x])]\,.$$

Thus, we infer from Corollary 3.3 that

$$\frac{B[x]/\mathcal{J}(B[x])}{[D, D]} \in \mathfrak{K}.$$

Hence $B[x]/\mathcal{J}(B[x]) \in \mathfrak{K}$ and by $\mathcal{J}(B[x]) \in \mathfrak{K}$ also $B[x] \in \mathfrak{K}$ follows. Moreover, using the fact that the Jacobson radical has the Amitsur property, we have

$$\left(\frac{B}{B \cap \mathcal{J}(B[x])}\right)[x] \cong \frac{B[x]}{(B \cap \mathcal{J}(B[x])[x]} = \frac{B[x]}{\mathcal{J}(B[x])} \lhd \frac{C}{\mathcal{J}(B[x])} \in \mathcal{SJ},$$

and so $(B/(B \cap \mathcal{J}(B[x]))[x] \in \mathcal{SJ}$. Thus, taking into account that $B \in \mathcal{N}$, there exists a nonzero homomorphic image E of $(B/B \cap \mathcal{J}(B[x]))[x]$ such that $E \in \mathcal{P} \cap h\mathcal{Q} \subseteq \mathcal{P} \cap \ell$. Hence $E \notin \mathfrak{K}$, contradicting $B[x] \in \mathfrak{K}$. □

A finitely generated nil ring L is said to be *strongly nil*, if

i) L can be embedded into a ring A as a left ideal,

ii) $A = L + K$ where K is a finitely generated nil left ideal of A and $L \cap K = 0$,

iii) A is generated by two nilpotent elements $x \in L$ and $y \in K$.

Theorem 3.5. *Köthe's Problem has a positive solution if and only if $L[x] \in \mathfrak{K}$ for every strongly nil ring L.*

Proof. Suppose that $L[x] \in \mathfrak{K}$ for every strongly nil ring L, but Köthe's problem has a negative solution. Then, as is well-known (cf. Krempa [5] and Sands [8]), there exists a nil ring B such that the 2×2 matrix ring $M_2(B)$ is not nil. Hence there exists an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(B)$ which is not nilpotent. Nevertheless, the elements $x = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$ are nilpotent, as one readily sees. Let A denote the subring of $M_2(B)$ generated by x and y, and L and K the left ideals of A generated by x and y, respectively. Obviously, L as a ring is generated by elements $y^i x^k$ ($i \geq 0$, $k \geq 1$) and K is generated by $x^k y^i$ ($k \geq 0$, $i \geq 1$). Since x and y are nilpotent elements, both L and K are finitely generated. Clearly $A = L + K$ and $L \cap K = 0$. Thus both L and K are strongly nil rings, and so by the assumption $L[x] \in \mathfrak{K}$ and $K[x] \in \mathfrak{K}$. Hence by Proposition 2.1 (ii) we have $L[x] \in \mathcal{J}$, $K[x] \in \mathcal{J}$, and therefore $A[x] = (L + K)[x] = L[x] + K[x] \in \mathcal{J}$. But then A is a nil ring, a contradiction.

The opposite implication is obvious. □

4 Radicals in $[\mathcal{J}, \mathfrak{M}]$

Seeking for a positive solution of Köthe's Problem, it is of interest to find radicals γ for which $A \in \mathcal{N}$ implies $A[x] \in \gamma$ and γ is "close" to $\mathcal{J}$. Sakhajev [7], however, states that Köthe's Problem has a negative solution. Searching for a counterexample, one has to find a nil ring A such that $A[x] \notin \gamma$ where γ is a possibly large radical in the interval $[\mathcal{J}, \mathfrak{M}]$. The main goal of this section is to construct a rather big radical Ξ such that $\mathcal{J} \subset \Xi \subset \mathfrak{M}$.

McConnell and Stokes [6] considered the following generalization of the Jacobson radical class

$$\mathcal{K} = \{A \mid (A, \circ) \text{ is a simple semigroup}\}$$

where $\circ$ denotes the adjoint operation $a \circ b = a + b + ab$ and simplicity means that the semigroup has no proper ideals. In [6] it was proved, among others, that

(1) *$\mathcal{K}$ is a non-hereditary radical;*

(2) *$\mathcal{J} \subset \mathcal{K} \subset \mathcal{G}$ and $\mathcal{J} = \mathcal{K} \cap \mathcal{B}$;*

(3) *Köthe's problem has a positive solution if and only if $A \in \mathcal{N}$ implies $A[x] \in \mathcal{K}$.*

Notice that by (2), (3) requires seemingly less than Krempa's criterion $A \in \mathcal{N} \Rightarrow A[x] \in \mathcal{J}$. Moreover, looking at the original definition of $\mathcal{K}$ in [6], one sees that $\mathcal{K}$ *is a polynomial but not a multiplicative radical in the sense of Drazin and Roberts* [3].

Proposition 4.1. *$\mathcal{K} \in [\ell \cap \mathcal{J}, \mathfrak{K}]$.*

Proof. By (2) the containment $\ell \cap \mathcal{J} \subset \mathcal{K}$ is clear. We show that $\mathcal{J} \cap \mathcal{Q} = \mathcal{K} \cap \mathcal{Q}$. Let $A[x] \in \mathcal{K} \cap \mathcal{Q}$. Then by [1] we have $A[x] \in \mathcal{B}$, and so $A[x] \in \mathcal{K} \cap \mathcal{B} = \mathcal{J}$. Thus $\mathcal{K} \cap \mathcal{Q} \subseteq \mathcal{J} \cap \mathcal{Q}$. The opposite inclusion is trivial. Applying Proposition 2.2 (iii), we get that $\mathcal{K} \subseteq \mathfrak{M}$ and also $\mathcal{K} \subseteq \mathfrak{K}$. □

For a radical γ we consider the classes

$$\mu_\gamma = \{A \in \mathcal{S}\gamma \mid A \text{ is a prime ring with a minimal left ideal}\}$$

and

$$\nu_\gamma = \left\{ A \in \mathcal{S}\gamma \,\middle|\, \begin{array}{l} \text{every nonzero prime homomorphic image of } A \\ \text{which is in } \mathcal{S}\gamma, \text{ has no minimal left ideals} \end{array} \right\}.$$

Proposition 4.2. *If γ is a special radical, then $\gamma = \mathcal{U}(\mu_\gamma \cup \nu_\gamma) = \mathcal{U}\mu_\gamma \cap \mathcal{U}\nu_\gamma$.*

Proof. The inclusion $\gamma \subseteq \mathcal{U}(\mu_\gamma \cup \nu_\gamma)$ is obvious. For proving $\mathcal{U}(\mu_\gamma \cup \nu_\gamma) \subseteq \gamma$, suppose the contrary. Then there exists a ring $A \in \mathcal{U}(\mu_\gamma \cup \nu_\gamma) \setminus \gamma$. Since γ is a special radical, A has a nonzero prime homomorphic image $B \in \mathcal{S}\gamma$. Certainly $B \notin \nu_\gamma$. Hence B has a nonzero prime homomorphic image C in $\mathcal{S}\gamma$ which has a minimal left ideal. Thus $C \in \mu_\gamma$, a contradiction.

The proof of $\mathcal{U}(\mu_\gamma \cup \mathcal{U}\nu_\gamma) = \mathcal{U}\mu_\gamma \cap \mathcal{U}\nu_\gamma$ is straightforward. □

Let m stand for the class of all subdirectly irreducible rings with minimal left ideals. Then in view of Proposition 4.2 the heart $H(A)$ of any $A \in m$ has a minimal left ideal, and so by the Litoff Theorem $H(A)$ is a locally matrix ring. Hence $H(A)$ contains a nonzero idempotent for every $A \in m$. Moreover the Weyl algebra

$W = \mathbb{Q}\langle x, y\rangle$ of rational polynomials with non-commuting indeterminates subject to $xy - yx = 1$ is a simple ring with unity element which does not contain minimal left ideals. From these considerations we conclude that $\mathcal{B} \subset \mathcal{U}m$.

Let ϱ and δ stand for the radicals introduced before Proposition 3.2, and put $\kappa = \mathcal{B} \cap u \cap \varrho \cap \delta$.

Proposition 4.3. ([9, Theorem 2.9]) *If $A \in \mathcal{N}$, then $A[x] \in \kappa$.*

We shall denote by Ξ the upper radical class of the class

$$\pi = \{A \mid A \text{ is an accessible subring of a primitive ring in } \kappa\}.$$

Proposition 4.4. *$\mathcal{K} \subset \Xi \not\subseteq \mathcal{U}\{S\}$ for every simple ring S with unity element.*

Proof. Suppose that $\mathcal{K} \not\subseteq \Xi$ and there exists a ring $A \in \mathcal{K}\backslash\Xi$. Then A has a nonzero primitive homomorphic image B in $\pi \cap \mathcal{K}$. Since $\kappa \subset \mathcal{B}$ and the hereditariness of $\mathcal{B}$ implies $\pi \subset \mathcal{B}$, by $B \in \mathcal{K}$ we get that $B \in \mathcal{K} \cap \mathcal{B} = \mathcal{J}$, contradicting the primitivity of B. Thus $\mathcal{K} \subseteq \Xi$. The left ideal $L = Wy$ of the Weyl algebra W is a simple domain without nonzero idempotents, as it is well known. So $L \in \mathcal{B} \cap \mathcal{S}\mathcal{J}$ and $L \notin u$. Hence $L \notin \mathcal{J} = \mathcal{K} \cap \mathcal{B}$ and $L \in \Xi$ follow, implying $L \in \Xi \setminus \mathcal{K}$ and $\mathcal{K} \subset \Xi$.

Since every simple ring with unity element is in Ξ, we have $\Xi \not\subseteq \mathcal{U}\{S\}$. □

Theorem 4.5. *If A is a nil ring and $A[x] \in \Xi$, then $A[x] \in \mathcal{J}$.*

Proof. Since A is a nil ring, by [1] we have $A[x] \in \mathcal{B} \subset \mathcal{U}m$.

We shall show that $A[x] \in \mathcal{U}\nu_\gamma$. Assume that $A[x] \notin \mathcal{U}\nu_\gamma$. Then $A[x]$ has a nonzero homomorphic image B in $\mathcal{S}\mathcal{J}$, and so B has a nonzero primitive homomorphic image C. By Proposition 4.3 we have $A[x] \in \kappa$ and also $C \in \kappa$. Hence $A[x] \notin \Xi$, a contradiction. Thus

$$A[x] \in \mathcal{U}m \cap \mathcal{U}\nu_{\mathcal{J}} \subseteq \mathcal{U}\mu_{\mathcal{J}} \cap \mathcal{U}\nu_{\mathcal{J}} = \mathcal{J}$$

in view of Proposition 4.2. □

To attempt the finding of an explicit counterexample, the following may be helpful.

Corollary 4.6. *The following assertiong are equivalent:*

i) *Köthe's Problem has a positive solution;*
ii) *$A[x] \in \Xi$ for every nil ring A;*
iii) *$\ell(A[x]) = \Xi(A[x])$ for every nil ring A.*

Proof. i)$\Longleftrightarrow$ii) If Köthe's Problem has a positive solution, then we have $A[x] \in \mathcal{J} \subset \Xi$ for every nil ring A.

Suppose that $A[x] \in \Xi$ for every nil ring. Then by Theorem 4.5 we have $A[x] \in \mathcal{J}$.

ii)$\Longleftrightarrow$iii) This is obvious by Theorems 2.4 and 4.5. □

References

[1] Beidar K.I., Fong Y., Puczyłowski E.R. *A polynomial ring over a nil ring cannot be homomorphically mapped onto a ring with nonzero idempotents.* J. Algebra, 2001, **238**, p. 389–399.

[2] Beidar K.I., Puczyłowski E.R., Wiegandt R. *Radicals and polynomial rings.* J. Austral. Math. Soc., 2002, **71**, p. 1–7.

[3] Drazin M.P., Roberts M.L. *Polynomial, multiplicative and special radicals.* Comm. in Algebra, 2000, **28**, p. 3073–3093.

[4] Gardner B.J., Wiegandt R. *Radical theory of rings.* Marcel Dekker, 2004.

[5] Krempa J. *Logical connections among some open problems in non-commutative rings.* Fund. Math., 1972, **76**, p. 121–130.

[6] McConnell N.R., Stokes T. *Rings having simple adjoint semigroup.* Contemporary Math., 2001, **273**, p. 203–208.

[7] Sakhajev I.I. *Negative solution hypothesis of Koethe and some problems connected with it.* Abstract of Talks. Algebra Conference Venezia, 2002, p. 37–38.

[8] Sands A.D. *Radicals and Morita contexts.* J. Algebra, 1973, **24**, p. 335–345.

[9] Tumurbat S. *The radicalness of polynomial rings over nil rings.* Math. Pannonica, 2002, **13**, p. 191–199.

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Wedderburn decomposition of LCM-rings

M.I. Ursul, I. Fechete

Abstract. In this paper we extend in this paper a result of Zelinsky to the class of linearly compact, monocompact rings of prime characteristic.

Mathematics subject classification: 16W80.
Keywords and phrases: Monocompact ring, linearly compact ring, hereditarily linearly compact ring, topologically locally finite ring, topologically nilpotent ring, Wedderburn decomposition in the category of topological rings.

1 Introduction

A subtle fact of the theory of algebras over a field is the Wedderburn–Mal'cev Theorem (see, e.g., [3, 4]). This Theorem was extended also to classes of topological rings (see, e.g., [1, 10, 13]). The aim of this paper is an extension of the Wedderburn Theorem to the class of bounded, linearly compact, monocompact rings.

2 Notation and conventions

All topological ring are assumed to be Hausdorff and associative (and not necessarily with identity). If R is a topological ring and $S \subseteq R$, then by $\langle S \rangle$ the closed subring of R generated by S is denoted.

A *monocompact* ring [11] is a topological ring R which is the reunion of its compact subrings (equivalently, for each element $x \in R$ the subring $\langle x \rangle$ is compact).

A topological ring R is *called linearly compact* [8] if it has a fundamental system of neighborhoods of zero consisting of left ideals and every filter base consisting of cosets relative to closed left ideals has a non-empty intersection.

A topological ring R is called *hereditarily linearly compact* [1] if every closed subring is a linearly compact ring.

The class of hereditarily linearly compact rings is intermediate between compact totally disconnected rings and linearly compact rings.

A topological ring R is called *topologically locally finite* [11] provided for every finite subset F the subring $\langle F \rangle$ is compact.

Recall that an element of a topological ring is called *topologically nilpotent* provided $x^n \to 0$.

The connected component of zero of a topological Abelian group R is denoted by R_0.

As usual, a *local* ring is a ring with identity having a unique maximal left ideal.

A *semiprimitive* ring is a ring with identity whose Jacobson radical is zero.

If n is a natural number and R is a ring, then $M(n, R)$ denotes the ring of $n \times n$ matrices over R.

The symbol $A \cong_{\text{top}} B$ means that topological rings A and B are isomorphic.

3 Semiprimitive LCM-rings

Definition 3.1. *A* LCM*-ring is a linearly compact, monocompact ring, having a fundamental system of neighborhoods of zero consisting of ideals.*

Example 3.2. *Let $\mathbb{F}$ be any field which is an infinite algebraic extension of a finite field. Then the ring $\begin{bmatrix} \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{F} \end{bmatrix}$ is a discrete* LCM*-ring.*

Lemma 3.3. *For any topologically nilpotent element a of a compact ring R and any integers $c_n \in \mathbb{Z}$, the series $\sum\limits_{n=1}^{\infty} c_n a^n$ converges.*

Proof. We may consider without loss of generality that R is quasi-regular. We may consider that $R = \langle a \rangle$.

The ring R/R_0 has a a fundamental system of neighborhood of zero consisting of ideals. Evidently, R/R_0 is topologically nilpotent. By theorem of Kaplansky [11, Theorem 2.5.7], $R_0 R = RR_0 = 0$, hence R is quasi-regular and so is topologically nilpotent. Since R is complete, by the Cauchy criterion, the series $\sum\limits_{n=1}^{\infty} c_n a^n$ is convergent. $\square$

Recall that a ring R is called SBI-*ring* [6] if for any $a \in J(R)$ there exists an $x \in J(R)$ such that:

(i) $x^2 + x = a$;

(ii) for all $z \in J(R)$, $az = za$ implies $xz = zx$.

Theorem 3.4. *Any topological ring R whose Jacobson radical $J(R)$ is monocompact is a* SBI*-ring.*

Proof. Let $a \in J(R)$. Then $\langle a \rangle \subseteq J(R)$ and $\langle a \rangle$ is compact. Consider the sequence

$$c_1 = 1, \quad c_k = -\sum_{i=1}^{k-1} c_i c_{k-i}, \quad k = 2, 3, ...,$$

of integers. Then, by Lemma 3.3, $x = \sum\limits_{n=1}^{\infty} c_n a^n$ exists. Evidently, $x^2 + x = a$ and for all $z \in J(R)$, $az = za$ implies $xz = zx$. $\square$

Corollary 3.5. *(see [6, p. 125]). Any compact ring is a* SBI*-ring.*

Corollary 3.6. *Any countably compact ring with identity is a* SBI*-ring.*

Lemma 3.7. *If R is a* LCM-*ring, R' is a topological ring having a fundamental system of neighborhoods of zero consisting of left ideals and $f : R \to R'$ is a surjective continuous homomorphism, then R' is a* LCM-*ring, too.*

Proof. Indeed, R' is linearly compact. Since R is the union of its compact subrings, R is the union of its compact subrings, too. Therefore R' is monocompact, hence R' is LCM-ring. □

Corollary 3.8. *If R is a* LCM-*ring and V is an open ideal, then the quotient ring R/V is a discrete* LCM-*ring.*

Lemma 3.9. *Any discrete* LCM-*ring R is locally finite.*

Proof. For every $x \in R$, the subring $\langle x \rangle$ is finite, hence $J(R)$ is a nilring. By [11, Theorem 2.9.30], $J(R)$ is a locally nilpotent ideal and so $J(R)$ is locally finite.

The ring $R/J(R)$ is isomorphic to a finite product $M(n_1, \Delta_1) \times \cdots \times M(n_k, \Delta_k)$, where $\Delta_1, \cdots, \Delta_k$ are algebraic extensions of finite fields. It follows that $R/J(R)$ is locally finite. Since the class of locally finite rings is closed under extensions, R is a locally finite ring. □

Problem 3.10. *Let R be a linearly compact ring and I be a closed left topological nilideal. Is then I locally topologically nilpotent?*

In the case when R has a fundamental system of neighborhoods of zero consisting of ideals, the Problem 3.10 has a positive answer, according to [11, Theorem 2.9.30].

Theorem 3.11. *If R is a* LCM-*ring then R is topologically locally finite.*

Proof. By Corollary 3.8 and Lemma 3.9, for every open ideal V, the quotient ring R/V is a locally finite ring. Then $R \cong_{\text{top}} \varprojlim R/V$ is a topologically finite ring. □

Corollary 3.12. *Any* LCM-*ring is a* SBI-*ring.*

Lemma 3.13. *The Jacobson radical $J(R)$ of a* LCM-*ring R is monocompact.*

Proof. By Leptin's Theorem [7], $J(R)$ is a closed ideal of R. □

Theorem 3.14. *If R is a* LCM-*ring with identity, $R/J(R)$ is topologically isomorphic to $M(n, \Delta)$ where Δ is a division ring, then $R \cong_{\text{top}} M(n, P)$ where P is a* LCM-*ring and $P/J(P)$ is topologically isomorphic to Δ.*

Proof. By Lemma 3.13, $J(R)$ is monocompact and by Theorem 3.4, R is a SBI-ring.

By [5, Theorem 3.8.1], $R \cong M(n, P)$, where P is a ring with identity and $P/J(P) \cong S$. We identify R with $M(n, P)$. By [11, Theorem 2.6.65], there exists a topology $\mathfrak{T}_0$ on P such that the ring $M(n, P)$ is equipped with the canonical topology of a matrix ring. Since $(P, \mathfrak{T}_0)$ is topologically isomorphic to $e\, M(n, P)\, e$ for some idempotent e, $(P, \mathfrak{T}_0)$ is a LCM-ring. □

Lemma 3.15. *If R is a left linearly compact discrete ring, then any family $\{e_\alpha : \alpha \in \Omega\}$ of orthogonal idempotents of R is finite.*

Proof. Indeed, if there exists a sequence $(e_{\alpha_n})_{n\geq 1}$ of pairwise different, non-zero elements of $\{e_\alpha : \alpha \in \Omega\}$, then $\sum\limits_{n=1}^{\infty} Re_{\alpha_n}$ is an infinite direct sum of left ideals, a contradiction. □

Lemma 3.16. *If R is a* LCM*-ring then each family $\{e_\alpha : \alpha \in \Omega\}$ of orthogonal idempotents is summable.*

Proof. Since R has a fundamental system of neighborhoods $\mathcal{B}$ of zero consisting of ideals,

$$R \cong_{\text{top}} \varprojlim \{R/V : V \in \mathcal{B}\} \subseteq \prod_{V\in\mathcal{B}} R/V.$$

For each $W \in \mathcal{B}$ denote by pr_W the canonical projection of $\prod\limits_{V\in\mathcal{B}} R/V$ on R/W. Since, $\{\text{pr}_V(e_\alpha) : \alpha \in \Omega\}$ is a family of orthogonal idempotents, by Lemma 3.15, this family is finite, therefore it is summable. By [2, Proposition 3.5.4], the family $\{e_\alpha : \alpha \in \Omega\}$ is summable in $\prod\limits_{V\in\mathcal{B}} R/V$. Since $\varprojlim \{R/V : V \in \mathcal{B}\}$ is a closed subring in $\prod\limits_{V\in\mathcal{B}} R/V$, this family is summable in $\varprojlim \{R/V : V \in \mathcal{B}\}$,too. □

Theorem 3.17. *An* LCM*-ring R is semiprimitive if and only if*

$$R \cong_{\text{top}} \prod_{\alpha\in\Omega} M(n_\alpha, R_\alpha),$$

where each R_α is an algebraic extension of a finite field.

Proof. Suppose that R is semiprimitive. By Leptin's Theorem [7],

$$R \cong_{\text{top}} \prod_{\alpha\in\Omega} M(n_\alpha, R_\alpha),$$

where each R_α is a division ring. Since each R_α is monocompact, every subring of R_α generated by one element is finite. Therefore, each R_α has a finite characteristic and can be regarded as an algebra over a finite field F_α. Since R_α is monocompact, R_α is an algebraic algebra over F_α and by Theorem of Jacobson [5, Theorem 7.12.2], R_α is commutative.

Conversely, let $R \cong_{\text{top}} \prod\limits_{\alpha\in\Omega} M(n_\alpha, R_\alpha)$ where each R_α is an algebraic extension of a finite field. Let $x = (x_\alpha)_{\alpha\in\Omega} \in \prod\limits_{\alpha\in\Omega} M(n_\alpha, R_\alpha)$. Then for every $\beta \in \Omega$,

$$\text{pr}_\beta \langle x\rangle \subseteq \left\langle \text{pr}_\beta(x)\right\rangle = \langle x_\beta\rangle,$$

hence $\langle x\rangle \subseteq \prod\limits_{\alpha\in\Omega} \langle \text{pr}_\alpha(x)\rangle$. Since every subring $\langle \text{pr}_\alpha(x)\rangle$ is finite, by Theorem of Tihonov, $\prod\limits_{\alpha\in\Omega} \langle \text{pr}_\alpha(x)\rangle$ is compact and so $\langle x\rangle$ is compact. □

4 Wedderburn decomposition of LCM-rings

We say that a topological ring R *admits a Wedderburn decomposition in the category of topological rings* [1] if the Jacobson radical $J(R)$ is closed and there exists a closed subring S such that:

(i) $R = S + J(R)$;

(ii) $S \cap J(R) = 0$;

(iii) the restriction of the canonical homomorphism $\varphi : R \to R/J(R)$ to S is a topological isomorphism.

We will use below the following Theorem of Zelinsky:

Theorem 4.1 (13)**.** . *If R is a compact ring of prime characteristic p, then there exists a compact subring S of R such that $R = S + J(R)$.*

Lemma 4.2. *If R is a local* LCM*-ring, $R/J(R)$ is finite and* $\operatorname{char} R = p$ *is a prime number, then there exists a finite subring F of R which is a field, such that $R = F + J(R)$.*

Proof. The group of units $U(R/J(R))$ of the field $R/J(R)$ is cyclic. Denote by ϕ the canonical homomorphism of R onto $R/J(R)$. Let $\theta \in R$ such that $\phi(\theta)$ is a generator of $U(R/J(R))$. The subring $\langle\theta\rangle$ is compact; evidently, $R = \langle\theta\rangle + J(R)$. By Theorem **??**, there exists a subfield F of $\langle\theta\rangle$ such that $\langle\theta\rangle = F + J(\langle\theta\rangle)$. Since $J(\langle\theta\rangle)$ is topologically nil, $J(\langle\theta\rangle) \subseteq J(R)$, hence $R = F + J(R)$. □

Remark 4.3. *If K is a compact subring of R, then $J(K) = J(R) \cap K$.*

Indeed, since $J(R) \cap K$ is a topologically nil ideal of K, we obtain that $J(R) \cap K \subseteq J(K)$. Conversely, since $K/(J(R) \cap K) \cong_{\text{top}} (K + J(R))/J(R)$ and $(K + J(R))/J(R)$ is a subfield of $R/J(R)$ or 0, we obtain that $J(K) \subseteq J(R) \cap K$.

Lemma 4.4. *Let R be a local* LCM*-ring of prime characteristic p. Then there exists a finite subring F of R which is a field, such that $R = F + J(R)$.*

Proof. Denote by φ the canonical homomorphism of R onto $R/J(R)$.

Consider that $R/J(R) = \bigcup_{i=1}^{\infty} K_i$, where each K_i is a finite subfield of $R/J(R)$ and $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n \subseteq \cdots$.

Let $K_1 = \langle\varphi(x_1)\rangle$ and consider the subring $\langle x_1\rangle$ of R. By Theorem of Zelinsky, there exists a finite subring S_1 of $\langle x_1\rangle$ such that

$$\langle x_1\rangle = S_1 + J(\langle x_1\rangle).$$

Assume that we have constructed for a positive integer n, a set $\{S_1, \cdots, S_n\}$ of subrings of R which are finite fields, such that:

(*i*) $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n$;

(*ii*) $\varphi(S_i) = K_i$, $i = 1, ..., n$.

Since K_{n+1} is a finite field, there exists $x_{n+1} \in R$ such that $K_{n+1} = \langle \varphi(x_{n+1}) \rangle$. Again, by Theorem of Zelinsky there exists a finite subring P of $\langle x_1 \rangle$ which is a field such that

$$\langle x_{n+1} \rangle = P + J(\langle x_{n+1} \rangle).$$

Evidently, $\varphi(P) = \varphi(\langle x_{n+1} \rangle) = K_{n+1}$. Therefore P is isomorphic to K_{n+1}. We note that P contains a subfield Q isomorphic to K_n. Consider the subring $\langle Q, S_n \rangle$ of R. By Theorem 3.11, $\langle Q, S_n \rangle$ is compact. Since $R/J(R)$ contains a unique subfield isomorphic to K_n, we obtain that $\varphi(Q) = \varphi(S_n) = K_n$. It follows that $\varphi(\langle Q, S_n \rangle) \subseteq \langle \varphi(Q), \varphi(S_n) \rangle = K_n$. Since $\varphi(\langle Q, S_n \rangle) \supseteq \varphi(S_n) = K_n$, we obtain that $\varphi(\langle Q, S_n \rangle) = K_n$. Since $\varphi(Q) = \varphi(S_n) = K_n$,

$$\langle Q, S_n \rangle = Q + J(\langle Q, S_n \rangle) = S_n + J(\langle Q, S_n \rangle).$$

By Theorem of Mal'cev (see, for example [11, Theorem 2.10.3]), there exists $a \in J(\langle Q, S_n \rangle) \subseteq J(R)$, such that

$$S_n = (1+a)^{-1} Q\ (1+a).$$

Then

$$(1+a)^{-1} P\ (1+a) \supseteq (1+a)^{-1} Q\ (1+a) = S_n.$$

Put

$$S_{n+1} = (1+a)^{-1} P\ (1+a).$$

Then S_{n+1} is a field, $S_n \subseteq S_{n+1}$ and

$$\varphi(S_{n+1}) = (\varphi(1+a))^{-1}\ \varphi(P)\ \varphi(1+a) = \varphi(P) = K_{n+1}.$$

We constructed a sequence

$$S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq \cdots$$

of subrings of R which are finite fields and $\varphi(S_n) = K_n$, for each $n \in \mathbb{N}$. Then $S = \bigcup_{i=1}^{\infty} S_i$ is a subring of R which is a field and

$$\varphi\left(\bigcup_{i=1}^{\infty} S_i\right) = \bigcup_{i=1}^{\infty} \varphi(S_i) = \bigcup_{i=1}^{\infty} K_i = R/J(R).$$

Therefore $R = S + J(R)$. Since $S \cap J(R) = 0$ and $J(R)$ is open in R, we obtain that R is a topological direct sum of R and $J(R)$. $\square$

Lemma 4.5. *Let R be a* LCM*-ring with identity such that $R/J(R) \cong M(n, \Delta)$ where Δ is a division ring. Then there exists a subring S of R isomorphic to $M(n, \Delta)$ such that $R = S + J(R)$.*

Proof. By Theorem 3.14, there exists a local LCM-ring P such that $R \cong_{\text{top}} M(n,P)$. By Lemma 4.4, there exists a subring S of P such that $P = S + J(P)$ a topological direct sum of S and $J(P)$. We identify R with $M(n,P)$. Then

$$M(n,P) = M(n,S) + M(n,J(P))$$

and

$$M(n,S) \cap M(n,J(P)) = 0.$$

The subring $M(n,S)$ is discrete and $J(M(n,P)) = M(n,J(P))$. $\square$

Theorem 4.6. *Let $f : R \to R'$ be a continuous homomorphism of a* LCM*-ring R with identity e on a* LCM*-ring R' with identity e' and $Ker f \subseteq J(R)$. If $\{e'_\alpha : \alpha \in \Omega\}$ is a family of orthogonal idempotents, $e' = \sum_{\alpha\in\Omega} e'_\alpha$, then there exists a family $\{e_\alpha : \alpha \in \Omega\}$ of orthogonal idempotents such that $e = \sum_{\alpha\in\Omega} e_\alpha$, $f(e_\alpha) = e'_\alpha$, $\alpha \in \Omega$.*

The proof of this Theorem is analogous to the proof of Theorem 2.6.57 from [11].

The following Theorem was proved for compact rings by Z.S. Lipkina [9].

Theorem 4.7. *Let R be an arbitrary* LCM*-ring. Then there exists a closed subring A, topologically isomorphic to a product of primary* LCM*-rings such that $R = A + J(R)$.*

The proof of this Theorem is analogous to the proof of Theorem 2.6.58 from [11].

Theorem 4.8. *Let R be a* LCM*-ring of prime characteristic. Then there exists a closed subring S such that $R = S \oplus J(R)$ (a topological direct group sum).*

Proof. By Theorem 4.7, there exists a closed subring A, such that $A \cong_{\text{top}} \prod_{\alpha\in\Omega} R_\alpha$, where each R_α is a primary ring and $R = A + J(R)$. By Lemma 4.5, for each $\alpha \in \Omega$, there exists a subring S_α, such that $R_\alpha = S_\alpha + J(R_\alpha)$. Since $J\left(\prod_{\alpha\in\Omega} R_\alpha\right) = \prod_{\alpha\in\Omega} J(R_\alpha)$, there exists a subring S of the ring A, topologically isomorphic to $\prod_{\alpha\in\Omega} S_\alpha$, such that $A = S + J(A)$.

We note that $J(A) \subseteq J(R)$. Indeed, since

$$R/J(R) = (A + J(R))/J(R) \cong A/(A \cap J(R)),$$

$A/(A \cap J(R))$ is semiprimitive, hence $J(A) \subseteq J(R)$.

Therefore

$$R = A + J(R) = S + J(A) + J(R) = S + J(R)$$

and, evidently, $S \cap J(R) = 0$.

We affirm that this sum is a topological direct sum. Indeed, let $\varphi : R \to R/J(R)$ the canonical homomorphism. Since $\varphi|_S : S \to R/J(R)$ is a continuous isomorphism of semiprimitive linearly compact rings, $\varphi|_S$ is a topological isomorphism. By [1, Lemma 13], this sum is a topological. $\square$

References

[1] Andrunakievič V.A., Arnautov V.I., Ursul M.I. *Wedderburn decomposition of hereditarily linearly compact rings*. Dokl. AN SSSR, 1973, **211(1)**, p. 15–18.

[2] Bourbaki N. *Éléments de mathématique. Topologie générale*. Herman, Paris, 1969.

[3] Curtis C.W., Reiner I. *Representation theory of finite groups and associative algebras*. Interscience Publishers, New York, London, 1962.

[4] Jacobson N. *The theory of rings*. Amer. Math. Soc. Surveys, Vol. 2, Providence, 1943.

[5] Jacobson N. *Structure of rings*, Amer. Math. Soc. Colloq. Publ., Vol. 37, Providence, 1964 (revised edition).

[6] Kaplansky I. *Fields and Rings*. The University of Chicago Press, 1969.

[7] Leptin H. *Linear kompakte Moduln und Ringe*. Mathematische Zeitschrift, 1955, **62**, p. 241–267.

[8] Leptin H. *Linear kompakte Moduln und Ringe, II*. Mathematische Zeitschrift, 1957, **66**, p. 289–327.

[9] Lipkina Z.S. *Structure of compact rings*. Sib. Mat. Zhurnal, 1973, **17**, p. 1346–1348.

[10] Numakura K. *A note on Wedderburn decompositions of compact rings*. Nihon Gakushiin. Proceedings, 1959, **35**, p. 313–315.

[11] Ursul M.I. *Compact rings satisfying compactness conditions*. Kluwer Academic Publishers, 2002.

[12] Warner S. *Topological rings*, North-Holand Math. Studies 178, 1993.

[13] Zelinsky D. *Raising idempotents*. Duke Math Journal, 1954, **21(2)**, p. 315–322.

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Totally bounded rings and their groups of units

M.I. Ursul, A. Tripe

Abstract. We will present here some recent results concerning totally bounded topological rings. Most results will be presented but not proved.

Mathematics subject classification: 16W80.
Keywords and phrases: Pseudo-compact space, countably compact space, precompact group, the pointwise topology, atomic boolean ring, Bohr compactification of a topological ring.

All topological spaces are assumed to be Tychonoff. Topological groups are assumed to be Hausdorff. Topological rings are assumed to be associative and Hausdorff. The Jacobson radical of a ring R will be denoted $J(R)$. The symbol $R = A \oplus B$ means that the group R is a topological direct sum of its subgroups A and B.

A topological space X is called:
pseudo-compact provided each real-valued function on it is bounded;
countably compact provided each countable open cover has a finite subcover.

The closure of a subset A of a topological space X will be denoted by $\overline{A}$. If R is a ring and A its subset, then $\langle A\rangle$ stands for the subring of R generated by A.

We will examine endomorphisms of linear spaces over finite fields by using of pointwise topology.

Let k be a finite field and V be a linear k-space. Recall that *the pointwise topology* on End$V$ is given by a fundamental system of neighbourhoods of zero consisting of subsets of the form $T(K) = \{\alpha : \alpha \in \mathrm{End}V, \alpha(K) = 0\}$, where K runs all finite subsets of V. We will consider End$V$ as a topological ring with the pointwise topology. Below $GL(V)$ stands for the topological group of all invertible elements of End$V$ with respect to the pointwise topology.

The pointwise topology allows to study some endomorphisms of V.

Definition 1. *An element α of EndV is called: topologically nilpotent provided it is a topologically nilpotent element of EndV; compact provided the subring $\overline{\langle\alpha\rangle}$ is compact; topologically unipotent provided the element $1-\alpha$ is topologically nilpotent; semisimple provided the subring $\overline{\langle\alpha\rangle}$ is a compact semiprimitive ring.*

Recall that an element $\alpha \in \mathrm{End}V$ is called *locally finite* provided V is decomposed in a direct sum of α-invariant finite-dimensional subspaces.

Remark 1. It follows from ([18], Theorem 19.4) that if k is a finite field, V a left vector k-space, then $\alpha \in \mathrm{End}V$ is compact $\Leftrightarrow$ for every $v \in V$, the subset $\langle\alpha\rangle\, v$ is finite.

We note that every locally finite element of End$V$ is compact. The following example shows that the reverse affirmation is not true:

Let k be any finite field and V a linear k-space of countable infinite dimension. Fix a countable base $\{v_i : i \in \omega\}$. Let $\alpha \in \mathrm{End}V, \alpha(v_0) = 0$, and $\alpha(v_i) = v_{i-1}$ for $i > 0$. Each nonzero α-invariant linear subspace of V contains kv_0, hence V cannot be decomposed in a direct sum of finite-dimensional α-invariant subspaces, i.e., α is not locally finite. It follows from Remark 1 that α is compact.

Proposition 1. *Let V be a linear space over a finite field k and $\alpha \in \mathrm{End}V$ a compact element. Then:*

i) there exist unique elements α_s, α_n such that $\alpha = \alpha_s + \alpha_n$, where α_s is semisimple, α_n is topologically nilpotent, and $\alpha_s\alpha_n = \alpha_n\alpha_s$;

ii) if $\alpha \in GL(V)$, then there exist unique elements $\alpha_s, \alpha_u \in GL(V)$ satisfying the conditions: α_s is semisimple, α_u topologically unipotent, $\alpha = \alpha_s\alpha_u$, and $\alpha_s\alpha_u = \alpha_u\alpha_s$.

In analogy with the theory of linear algebraic groups we shall call the decomposition $\alpha = \alpha_s + \alpha_n$ *the additive Jordan decomposition* of α and the decomposition $\alpha = \alpha_s\alpha_u$ for an invertible α *the multiplicative Jordan decomposition* for α.

Theorem 2. *Let $R = S \oplus J(R)$ be Wedderburn-Mal'cev decomposition of a compact ring with identity of prime characteristic. Then $U(R) = U(S) \cdot (1 + J(R))$, $U(S) \cap (1 + J(R)) = 1$, i.e., $U(R)$ is a semidirect topological product of $U(S)$ and $1 + J(R)$.*

Theorem 3. *Let V be a linear space over a finite field k. If H is a closed subgroup of $GL(V), x \in H, x$ is compact, $x = x_sx_u$ its multiplicative Jordan decomposition then $x_s \in H$ and $x_u \in H$.*

Theorem 4 [15]. *Let R be a countably compact ring with identity. The following conditions are equivalent:*

1) $U(R)$ is a torsion group;

2) R has a finite characteristic and there exist two different positive integers n and k such that the ring R satisfies the identity $x^n = x^k$;

3) R is a locally finite ring;

4) for every $x \in R$ the subring $\langle x \rangle$ is finite.

Theorem 5 [15]. *Let R be a compact ring with identity. Then the following conditions are equivalent:*

1) $U(R)$ is a torsion group;

2) $R(+)$ is a torsion group and $R/J(R) \cong_{top} L_1^{m_1} \times \cdots \times L_n^{m_n}$, where $L_1, \cdots, L_n$ are finite simple rings and $m_1, \ldots, m_n$ are arbitrary cardinal numbers.

In 1988–1997 there appeared a number of interesting papers of Jo-Ann Cohen, Kwangil Koh and I. W. Lorimer concerning groups of units of compact rings with identity [1–11].

A topological ring R is a *semidirect product* of a subring S and an ideal I provided R is a topological group sum of S and I.

The following Theorem of A.Tripe generalizes a result obtained by Jo-Ann Cohen and K.Koh [10]:

Theorem 6 [14]. *Let R be a countably compact ring with identity. Then the group $U(R)$ is simple iff R is a boolean ring or it is topologically isomorphic to one of the following rings:*

1) $A_0 \times I$ where A_0 is a finite field of cardinality 3 or 2^m where $2^m - 1$ is a prime number (=a prime number of Mersenne);

2) $A_0 \times I$ where A_0 is the ring of $n \times n$ matrices over $\mathbb{Z}/(2), n \geq 3$;

3) a semidirect product of I and $\mathbb{Z}/(4)$;

4) a semidirect product of I and $\mathbb{Z}/(2)[x]/(x^2)$;

5) a semidirect product of I and $M(2, \mathbb{Z}/(2))$,

where in all cases I is a countably compact boolean ring.

There are examples showing that in 3), 4), 5) the semidirect product cannot be replaced by direct products.

There is a gap between pseudo-compactness and countable compactness:

Theorem 7 [16]. *Let k be a finite field and X a set of cardinality $2^{\omega \cdot}$ Then the ring $k[X]$ of polynomials over X with coefficients from k admits a pseudo-compact ring topology.*

As follows from Chevalley's Theorem ([19], Chapter II, Theorem13) if $(R, \mathcal{T})$ is a commutative compact Noetherian ring and $\mathcal{T}_1$ is a ring topology on R such that $(R, \mathcal{T}_1)$ has a fundamental system of neighbourhoods of zero then $\mathcal{T} \leq \mathcal{T}_1$. We extend this assertion to the noncommutative case:

Theorem 8. *Let $(R, \mathcal{T})$ be a compact left Noetherian ring with identity. If $\mathcal{T}_1$ is a ring topology on R and $(R, \mathcal{T}_1)$ has a fundamental system of neighbourhoods of zero consisting of left ideals then $\mathcal{T} \leq \mathcal{T}_1$.*

Proof. Any left ideal of R is closed in $(R, \mathcal{T})$. Let $\{V_\alpha\}_{\alpha \in \Omega}$ be a fundamental system of neighbourhoods of zero of $(R, \mathcal{T}_1)$ consisting of left ideals. If V is an open ideal of $(R, \mathcal{T})$, then $\{0\} \subseteq \cap_{\alpha \in \Omega} V_\alpha \subseteq V$. By compactness of $(R, \mathcal{T})$ there exist $\alpha_1 \ldots, \alpha_n \in \Omega$ such that $V_{\alpha_1} \cap \cdots \cap V_{\alpha_n} \subseteq V$. Therefore V is an open ideal of $(R, \mathcal{T}_1)$. Since V was arbitrarily, $\mathcal{T} \leq \mathcal{T}_1$. □

Recall that if a, b are two elements of a boolean ring R, then put $a \leq b$ if $ab = a$. An element a of a boolean ring R is called an *atom* provided $a \neq 0$ and for each $x \in R$, $x \leq a$, $x \neq a$, $x = 0$. A boolean ring R is called *atomic* provided for each $x \in R$, $x \neq 0$, there exists at least one atom a such that $a \leq x$.

Theorem 9. *Let R be an atomic boolean ring. Then there exists a totally bounded ring topology $\mathcal{T}_0$ on R such that $\mathcal{T}_0 \leq \mathcal{T}_1$ for each Hausdorff ring topology $\mathcal{T}_1$ on R.*

Proof. Consider the family $\mathfrak{B}$ consisting of ideals of the form $\mathrm{Ann}(a)$, $a \in R$. Each element of B is a maximal ideal of R, hence it is cofinite. We note that B is a filter base. Indeed, let $a_1, a_2, ..., a_n \in R$. Then there exists $a \in R$ such that $Ra_1 + Ra_2 + ... + Ra_n = Ra$. Evidently, $\mathrm{Ann}(a) = \mathrm{Ann}(a_1) \cap \mathrm{Ann}(a_2) \cap ... \cap \mathrm{Ann}(a_n)$. We affirm that $\cap\mathfrak{B} = 0$: Let $0 \neq x \in R$. If a is an atom of R, $a \leq x$, then $xa = a$, hence $x \notin \mathrm{Ann}(a) = R(1-a) \in B$. It follows that $\mathfrak{B}$ gives a totally bounded ring topology $\mathcal{T}_0$ on R. If $\mathcal{T}_1$ is another $\mathcal{T}_1$-ring topology on R, then each $\mathrm{Ann}(a)$ is closed in $(R,\mathcal{T}_1)$ and cofinite, hence $\mathrm{Ann}(a)$ is open in $(R,\mathcal{T}_1)$ and so $\mathcal{T}_0 \leq \mathcal{T}_1$. □

Corollary 10. *The quasicomponent of any atomic topological $\mathcal{T}_1$-ring is equal to zero.*

The notion of the Bohr compactification of a topological ring was introduced by Holm [12, 13].

Definition 2. *Let $(R, \mathcal{T})$ be a topological ring. A pair $((bR, b\mathcal{T}), b_R)$ with the following properties is called a Bohr compactification of $(R, \mathcal{T})$:*

1) $(bR, b\mathcal{T})$ is a compact ring;

2) b_R is a continuous homomorphism from $(R, \mathcal{T})$ onto a dense subring $(bR, b\mathcal{T})$;

3) for every continuous homomorphism α of $(R, \mathcal{T})$ into a compact ring C there exists a continuous homomorphism $\hat{\alpha} : (bR, b\mathcal{T}) \to C$ such that $\hat{\alpha} \circ b_R = \alpha$.

Theorem 11. *Every topological ring $(R, \mathcal{T})$ has a Bohr compactification unique up to an isomorphism.*

It is interesting to calculate the Bohr compactification of concrete topological rings.

Theorem 12 [12]. *The Bohr compactification $b(\mathbb{Z}, \mathcal{T}_d)$ of the ring of integers is isomorphic to $\prod_{p \in P} \mathbb{Z}_p$ as a topological ring.*

Theorem 13 [12]. *Let R be a ring furnished with the discrete topology, bR its Bohr compactification, $P(R)$ the lattice of all precompact ring topologies on R, and $C(bR)$ the lattice of closed ideals of bR. Then there is a lattice antiisomorphism $\Phi : P(R) \to C(bR)$ such that $bR/\Phi(\mathcal{T})$ is isomorphic to the completion of $(R, \mathcal{T})$.*

In [12] was introduced the concept of a van der Waerden ring: A compact ring $(R, \mathcal{T})$ is called a vdW-*ring* provided each ring homomorphism $h : (R, \mathcal{T}) \to (K, \mathcal{U})$ with $(K, \mathcal{U})$ is continuous.

Fix a faithful indexing $\{R_n : n \in \omega\}$ of all matrix rings over finite fields. By Theorem of Kaplansky a semiprimitive ring R is of the form $R = \prod_{n \in I} R_n^{\alpha_n}$ for suitable $I = I(R) \subseteq \omega$ and cardinal numbers $\alpha_n = \alpha_n(R_n)$.

Theorem 14 [12]. *Let R be a compact semiprimitive ring with Kaplansky representation $R = \prod_{n \in I} R_n^{\alpha_n}$. In order that R be a vdW-ring it is necessary and sufficient that each α_n be finite.*

A compact ring with identity is a vdW-ring iff every cofinite ideal is open.

Theorem 15 [12, 17]. *A compact semisimple ring admits a unique pseudo-compact topology iff it is metrizable.*

References

[1] Cohen J., Koh K. *The group of units in a compact ring.* J. Pure Appl. Algebra, 1988, **54**, p. 167–179.

[2] Cohen J., Koh K. *On the group of units in a compact ring.* J. Pure Appl. Algebra, 1988, **51**, p. 231–239.

[3] Cohen J., Koh K. *Half-transitive group actions in a compact ring.* J. Pure Appl. Algebra, 1989, **60**, p. 139–153.

[4] Cohen J., Koh K. *The subgroup generated by the involutions in a compact ring.* J. Pure Appl. Algebra, 1991, **19**, p. 2923–2954.

[5] Cohen J., Koh K. *Involutions in a compact ring.* J. Pure Appl. Algebra, 1989, **9**, p. 151–168.

[6] Cohen J., Koh K. *The structure of compact rings.* J. Pure Appl. Algebra, 1992, **77**, p. 117–129.

[7] Cohen J., Koh K. *A characterization of the p-adic integers.* Comm. in Algebra, 1989 **17**(3), p. 631–636.

[8] Cohen J., Koh K. *The subring generated by the units in a compact ring.* Comm. in Algebra, 1990, **18**(5), p. 1617–1620.

[9] Cohen J., Koh K. *Topological and algebraic properties of the subgroup generated by involutions in a compact ring.* Papers on General Top. and Appl., Eight Summer Conference at Queens College, 1994, p. 310–317.

[10] Cohen J., Koh K. *Compact rings having a finite simple group of units.* J. Pure Appl. Algebra, 1997, **119**, p. 13–26.

[11] Cohen J., Koh K., Lorimer J.W. *On the primitive representations of the group of units of a ring to the symmetric group of nonunits.* J. Pure Appl. Algebra, 1994, **94**, p. 285–306.

[12] Comfort W.W., Remus D., Szambien H. *Extending ring topologies.* J.Algebra, 2000, **232**, p. 21–47.

[13] Holm P. *On the Bohr compactification.* Math. Ann., 1964, **156**, p. 34–46.

[14] Kothe G. *Uber maximale nilpotente Unterringe und Nilringe.* Math.Ann., 1931, **105**, p. 15–39.

[15] Tripe A. *Compact Rings with Simple Groups of Units.* Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, 2002, **2(39)**, p. 93–99.

[16] Tripe A., Ursul M. *Countably compact rings with periodic groups of units.* Acta Math.Hungar., 2001, **90**(1-2), p. 65–73.

[17] Tripe A., Ursul M. *Units in pseudo-compact rings.* Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, 2002, **2(39)**, p. 44–50.

[18] Ursul M.I. *Short communication 512.56.* Proceedings of the 5th Tiraspolian Symposium on General Topology and Its Applications, Kishinev, 1985, p. 231–232.

[19] Ursul M.I. *Topological rings satisfying compactness conditions.* Kluwer, 2002.

[20] Zariski O., Samuel P. *Commutative algebra, Vol. II.* D. Van Nostrand Company, INC, 1960.

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The radical theory of convolution rings

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Abstract. Convolution rings have been defined as a unifying approach to a number of ring constructions, e.g. polynomials, matrices, necklace rings and incidence algebras. Here the radical theory of convolution rings will be investigated.

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Convolution rings were defined and studied in [22] as a unifying procedure to describe a wide variety of ring constructions. Every convolution ring is determined by a convolution type. This type is then imposed on a ring A to give the corresponding convolution ring $C(A)$. For example, a polynomial convolution type is defined which leads to the polynomial rings $A[x]$. Other examples include the direct product of a ring with itself, matrices (finite, infinite or structural), incidence algebras, necklace rings, quaternion rings, etc.

Here we will study the radical theory of convolution rings. The language of convolution rings will enable us to formulate that which is common to all the ring constructions under consideration. But it will also enable us to isolate those properties of convolution types which will enforce certain properties on the radicals of the convolution rings.

1 Introduction

Convolution types have been defined for classes of R-algebras (R any ring), but here we restrict ourselves to the class of all rings ($\mathbb{Z}$-algebras). We recall from [22]:

Definition 1. *A convolution type $\mathcal{T}$ is a quadruple $\mathcal{T} = (X, \mathcal{S}, \sigma, \tau)$ where X is a non-empty set, $\mathcal{S}$ is a non-empty set of subsets of X with $\mathcal{S} \neq \{X\}$, for every $x \in X$, $\sigma(x)$ is a non-empty subset of $X \times X$ and τ is a function $\tau : X \times X \to \mathbb{Z}$ subject to:*

(C1) $Y_1, Y_2 \in \mathcal{S}$ implies there exist a $Y \in \mathcal{S}$ with $Y \subseteq Y_1 \cap Y_2$.

(C2) $Y_1, Y_2 \in \mathcal{S}$ implies there exist a $Y \in \mathcal{S}$ such that for all $(s,t) \in \sigma(y)$, $y \in Y$, either $s \in Y_1$ or $t \in Y_2$.

(C3) For all $Y_1, Y_2 \in \mathcal{S}$, $x \in X$, the set $\{(s,t) \in \sigma(x) \mid s \in X \setminus Y_1$ and $t \in X \setminus Y_2\}$ is finite.

(A1) For all $(s,t) \in \sigma(x), (p,q) \in \sigma(s)$ there exists a unique $v \in X$ with $(p,v) \in \sigma(x), (q,t) \in \sigma(v)$ and such that $\tau(s,t)\tau(p,q) = \tau(p,v)\tau(q,t)$.

(A2) For all $(s,t) \in \sigma(x), (p,q) \in \sigma(t)$ there exists a unique $u \in X$ with $(u,q) \in \sigma(x), (s,p) \in \sigma(u)$ and such that $\tau(s,t)\tau(p,q) = \tau(u,q)\tau(s,p)$.

Let $\mathcal{T}$ be a convolution type and let A be a ring. Let $C(A,\mathcal{T}) = \{f : X \to A \mid$there exists a $Y \in \mathcal{S}$, in general depending on f, such that $f(y) = 0$ for all $y \in Y\}$. This set Y associated with $f \in C(A,\mathcal{T})$ is called a *zero-set* for f (it need not be unique) and when necessary denoted by Y_f. On the set $C(A,\mathcal{T})$ we define the following operations: For $f, g \in C(A,\mathcal{T})$ and $x \in X$,

componentwise addition $(f + g)(x) = f(x) + g(x)$ and

convolution product $(fg)(x) = \sum_{(s,t)\in\sigma(x)} \tau(s,t)f(s)g(t)$.

Then $C(A,\mathcal{T})$ is a ring with respect to these operations. Usually we will write $C(A)$ for $C(A,\mathcal{T})$. Many examples were given in [22], as well as some first results on ideals and homomorphisms of convolution rings. We recall one which will often be used. Let I be an ideal of a ring A and let $\theta : A \to A/I$ be the corresponding surjective homomorphism. Then $C(\theta) : C(A) \to C(A/I)$, defined by $(C(\theta))(f) := \theta \circ f$ for all $f \in C(A)$, is a surjective homomorphism with $\ker C(\theta) = C(I)$, i.e. $C(A/I) \cong C(A)/C(I)$.

For a given convolution type and a radical, the single most important problem is to determine the radical of the convolution ring $C(A)$. Preferably one would like to express it in terms of the radical of the underlying ring A. For this to be possible, some connection between A and $C(A)$ will be required. To ensure this, we will impose further conditions on the convolution type. These conditions (except Example 3.1 for infinite sets X) will be in force for the remainder of this paper and all the examples discussed below will satisfy these conditions.

Let $T = \{t \in X \mid (t,t) \in \sigma(t)$ and $\tau(t,x) = 1 = \tau(x,t)$ for all $x \in X\}$. This set could be empty, but the first of the next three conditions, which we require the convolution type to satisfy, will ensure that $T \neq \varnothing$.

(T1) For every $x \in X$, there exists unique $l_x \in T$ and $r_x \in T$ such that $(l_x, x) \in \sigma(x)$ and $(x, r_x) \in \sigma(x)$.

(T2) If $(p,q) \in \sigma(x)$ and $p \in T$ (respt. $q \in T$), then $q = x$ (respt. $p = x$).

(T3) There exists $Y_T \in \mathcal{S}$ such that $T \subseteq X \setminus Y_T$.

It then follows from [22] that the mapping $\iota : A \to C(A)$ defined by

$$\iota(a) = \iota_a : X \to A \text{ with}$$
$$\iota_a(x) = \begin{cases} a \text{ if } x \in T \\ 0 \text{ if } x \notin T, \end{cases}$$

is a well-defined injective ring homomorphism. If the ring A has an identity 1_A, then $C(A)$ has an identity $e := \iota_{1_A}$ and the ideal in $C(A)$ generated by A coincides with $C(A)$ since $e \in A$.

We should point out that the embedding of A in $C(A)$ need not be unique. Suppose $T \neq \varnothing$ and choose $t_0 \in T$ fixed. The mapping $\varsigma : A \to C(A)$ defined by

$$\varsigma(a) = \varsigma_a : X \to A \text{ with}$$
$$\varsigma_a(x) = \begin{cases} a \text{ if } x = t_0 \\ 0 \text{ if } x \neq t_0 \end{cases}$$

is also an embedding of A into $C(A)$. In this case, however, an identity in A need not ensure that $C(A)$ has an identity. When $|T| = 1$, this distinction falls away, since then $\iota = \varsigma$. Without further notice we will regard ι, as defined above, as our canonical embedding of A into $C(A)$.

2 Radical theory

Throughout this section, $\mathcal{T} = (X, \mathcal{S}, \sigma, \tau)$ is a convolution type. Unless mentioned explicitly otherwise, all radicals will be in the sense of Kurosh-Amitsur and when we say that α is a radical, α will denote both the class of radical rings as well as the radical map which assigns to a ring A its radical $\alpha(A)$. For any class of rings $\mathcal{A}$, $\mathcal{SA}$ will denote the class $\mathcal{SA} = \{A \mid 0 \neq I \lhd A \Rightarrow I \notin \mathcal{A}\}$. In particular, if α is a radical class, $\mathcal{S}\alpha$ is the *semisimple class* of α.

For a given convolution type, the best possible scenario is $\alpha(C(A)) = C(\alpha(A))$ for all rings A and all radicals α. This can be realized, but only in a few very special cases. For example, for any non-empty set X, let $C(A) = \oplus_{x \in X} A$, the discrete direct sum of $|X|$-copies of A (see Example 3.1 below). However, for most convolution types one could have some radical α for which $\alpha(C(A)) = C(\alpha(A))$ holds for all rings A, but for some other radicals these two subsets of $C(A)$ need not even be comparable. For a given radical α, it could also happen that $\alpha(C(A)) = C(\alpha(A))$ for a certain convolution type, but for another convolution type, this equality need no longer be true.

A radical α is said to be $\mathcal{T}$*-invariant* if $\alpha(C(A)) = C(\alpha(A))$ for all rings A. There are two (trivial) $\mathcal{T}$-invariant radicals, namely $\alpha = \{0\}$ and α the class of all rings. In general, invariance will depend on the convolution type as well as the properties of the radical.

Recall, an ideal K of a convolution ring $C(A)$ is called $\mathcal{T}-$*homogeneous* if there is an ideal I of the ring A such that $K = C(I)$. This is equivalent to requiring the equality $C(K \cap A) = K$. Although homogeneity provides a useful link between the ideals of $C(A)$ and those of A, its real value only comes to the fore if an explicit description of the ideal $K \cap A$ of A is known. We also have a need for the following: The ideal K of $C(A)$ is called $\mathcal{T}-$ *weakly homogeneous* if $C(K \cap A) \subseteq K$. The motivation for these notions comes from the work of Amitsur [1] and subsequently Krempa [5] on the radicals of polynomial rings. For polynomial rings, the homogeneity of $\alpha(A[x])$, i.e. $\alpha(A[x]) = (\alpha(A[x]) \cap a)[x]$, is sometimes referred to as the *Amitsur Condition*. We will say the radical α is $\mathcal{T}-$*homogeneous* if $\alpha(C(A)) = C(\alpha(C(A)) \cap A)$ for all rings A and $\mathcal{T}-$ *weakly homogeneous* if $C(\alpha(C(A)) \cap A) \subseteq \alpha(C(A))$ for all rings A. Usually we will drop the reference to the convolution type.

Let P be a function which assigns to each ring A and $f \in C(A)$ a subset $P(f, A)$ of A subject to $P(0, A) = \{0\}$. In most cases we write $P(f)$ for $P(f, A)$. The most frequent definition of P is: Let $\emptyset \neq W \subseteq X$ and let $P(f) = f(W)$, but other choices will also be of some significance. When $P(f) = f(W)$ for all f, we sometimes write P as P_W. For $I \lhd A$, let $(I : P)_{C(A)} = \{f \in C(A) \mid P(f) \subseteq I\}$. When $P = P_W$ for some W, we use the usual notation $(I : W)_{C(A)} = \{f \in C(A) \mid f(W) \subseteq I\}$ in stead of $(I : P_W)_{C(A)}$. To ensure that $(I : P)_{C(A)}$ is an ideal of $C(A)$, it is sufficient to require:

(i) For all $f, g \in (I : P)_{C(A)}$, $P(f - g) \subseteq \{a - b \mid a \in P(f), b \in P(g)\}$.

(ii) For all $f \in (I : P)_{C(A)}$ and $h \in C(A), P(fh) \subseteq P(f)P(h)$ and $P(hf) \subseteq P(h)P(f)$.

In particular, $(I : W)_{C(A)}$ will be an ideal of $C(A)$ provided $\sigma(w) \subseteq W \times W$ for all $w \in W$. If $W = X$, then this condition is trivially fulfilled and $(I : X)_{C(A)} = C(I)$ is an ideal of $C(A)$ as we already know.

The radical α will be called $\mathcal{T}-$*accessible* if for all rings A there is an ideal I of A and a function P such that $\alpha(C(A)) = (I : P)_{C(A)}$ for all rings A. Note that if α is accessible, $P = P_W$ for all f and $W \cap T \neq \emptyset$, then $I = \alpha(C(A)) \cap A$. Indeed, from $\alpha(C(A)) = (I : W)_{C(A)}$ it follows that $\alpha(C(A)) \cap A = (I : W)_{C(A)} \cap A = I$. When $I = \alpha(A)$ for all A, we say α is *directly* $\mathcal{T}-$*accessible*, i.e. $\alpha(C(A)) = (\alpha(A) : P)_{C(A)}$ for all A. Any invariant radical α is directly accessible with $\alpha(C(A)) = (\alpha(A) : X)_{C(A)}$. For a homogeneous radical α, we know that $\alpha(C(A)) = (\alpha(C(A)) \cap A : X)_{C(A)}$, but this does not necessarily mean that α is directly accessible.

We recall from [22]: Let $D = \{x \in X \mid \sigma(x) = \{(x, x)\}$. If $D \neq \emptyset$, then there is a surjective homomorphism $\theta : C(A) \to (A/\alpha(A))^D$ with $\ker\theta = (\alpha(A) : D)_{C(A)}$. Since $(A/\alpha(A))^D \in \mathcal{S}\alpha$, we have $\alpha(C(A)) \subseteq (\alpha(A) : D)_{C(A)}$. Moreover, for a fixed $d_0 \in D$, there is a surjective homomorphism $\gamma : C(A) \to A$ defined by $\gamma(f) = f(d_0)$. Thus we have:

Proposition 1. *Let $\mathcal{T}$ be a convolution type with $D \neq \emptyset$. For any radical α and ring A,*

(1) $\alpha(C(A)) \subseteq (\alpha(A) : D)_{C(A)}$ and

(2) $C(A) \in \alpha \Rightarrow A \in \alpha$.

Sometimes it is possible to embed a ring A as an ideal in $C(A)$. More specifically

Proposition 2. *Let $\mathcal{T}$ be a convolution type which satisfies the condition:*

(T4) If $t_0 \in T$ such that $(t_0, x) \in \sigma(x)$ or $(x, t_0) \in \sigma(x)$, then $x = t_0$.

Then A can be embedded as an ideal in $C(A)$.

Proof. Define $\eta : A \to C(A)$ by $\eta(a) = \eta_a : X \to A$

with $\eta_a(x) = \begin{cases} a \text{ if } x = t_0 \\ 0 \text{ if } x \neq t_0 \end{cases}$.

Then η is an injective homomorphism. We show $\eta(A)$ is an ideal in $C(A)$. Let $a \in A$ and $f \in C(A)$. Then

$(\eta_a f)(x) = \sum_{(p,q)\in\sigma(x)} \tau(p, q)\eta_a(p)f(q)$. Let $b := f(t_0)$. Now $\eta_a(p) = 0$ for all p unless $p = t_0$. But $(t_0, q) \in \sigma(x)$ implies $q = x$ by condition (T2) and then by (T4) we have $x = t_0$. Thus

$$\begin{aligned}(\eta_a f)(x) &= \begin{cases} \tau(t_0, t_0)\eta_a(t_0)f(t_0) \text{ if } x = t_0 \\ \quad 0 \text{ otherwise} \end{cases} \\ &= \begin{cases} ab \text{ if } x = t_0 \\ 0 \text{ otherwise} \end{cases} \\ &= \eta_{ab}(x).\end{aligned}$$

Hence $\eta_a f = \eta_{ab} \in \eta(A)$. Likewise $f\eta_a \in \eta(A)$. □

Since semisimple classes are hereditary, we have

Proposition 3. *Let $\mathcal{T}$ be a convolution type which satisfies condition (T4). Then* $C(A) \in \mathcal{S}\alpha \Rightarrow A \in \mathcal{S}\alpha$.

Proposition 4. *Let $\mathcal{T}$ be a convolution type and let α be a radical.*

Then:

(1) $\alpha(C(A)) \subseteq C(\alpha(A))$ *for all* A
$\Leftrightarrow (A \in \mathcal{S}\alpha \Rightarrow C(A) \in \mathcal{S}\alpha)$
$\Rightarrow (C(A) \in \alpha \Rightarrow A \in \alpha)$

and if α is homogeneous, then $(A \in \mathcal{S}\alpha \Rightarrow C(A) \in \mathcal{S}\alpha) \Leftrightarrow (C(A) \in \alpha \Rightarrow A \in \alpha)$.

(2) $C(\alpha(A)) \subseteq \alpha(C(A))$ *for all* A
$\Leftrightarrow (A \in \alpha \Rightarrow C(A) \in \alpha)$
$\Rightarrow (C(A) \in \mathcal{S}\alpha \Rightarrow A \in \mathcal{S}\alpha)$

and if α is homogeneous, then $(A \in \alpha \Rightarrow C(A) \in \alpha) \Leftrightarrow (C(A) \in \mathcal{S}\alpha \Rightarrow A \in \mathcal{S}\alpha)$.

Proof. (1) The equivalence is clear. Suppose $A \in \mathcal{S}\alpha \Rightarrow C(A) \in \mathcal{S}\alpha$. Let $C(A) \in \alpha$. Then $C(A)/C(\alpha(A)) \in \alpha$. But $A/\alpha(A) \in \mathcal{S}\alpha$ implies $C(A)/C(\alpha(A)) \cong C(A/\alpha(A)) \in \mathcal{S}\alpha$ which gives $A = \alpha(A) \in \alpha$. Suppose α is homogeneous and $C(A) \in \alpha \Rightarrow A \in \alpha$. Let $A \in \mathcal{S}\alpha$. Then $C(\alpha(C(A)) \cap A) = \alpha(C(A)) \in \alpha$ implies $\alpha(C(A)) \cap A \in \alpha$ by the assumption. Thus $\alpha(C(A)) \cap A \subseteq \alpha(A) = 0$. This means $\alpha(C(A)) = C(\alpha(C(A)) \cap A) = 0$, i.e. $C(A) \in \alpha$.

(2) Both the equivalence and implication are clear. We only show the converse of the last implication under the assumption of homogeneity. Suppose α is homogeneous and $C(A) \in \mathcal{S}\alpha \Rightarrow A \in \mathcal{S}\alpha$. Let $A \in \alpha$. Then $\alpha(C(A)) = C(\alpha(C(A)) \cap A)$ and $C(A/\alpha(C(A)) \cap A) \cong C(A)/C(\alpha(C(A)) \cap A) = C(A)/\alpha(C(A)) \in \mathcal{S}\alpha$. By our assumption $A/\alpha(C(A)) \cap A \in \mathcal{S}\alpha$ and thus $A = \alpha(A) \subseteq \alpha(C(A)) \cap A$, i.e. $A \subseteq \alpha(C(A))$. Thus $\alpha(C(A)) = C(\alpha(C(A)) \cap A) = C(A)$ and so $C(A) \in \alpha$. □

Proposition 5. *Let $\mathcal{T}$ be a convolution type and let α be a radical. The following five conditions are equivalent:*

(1) *α is invariant (i.e. $\alpha(C(A)) = C(\alpha(A))$ for all A)*

(2) *(a) $\alpha(C(A)) \subseteq C(\alpha(A))$ for all A and*
(b) $C(\alpha(A)) \subseteq \alpha(C(A))$ for all A.

(3) *(a) $A \in \mathcal{S}\alpha \Rightarrow C(A) \in \mathcal{S}\alpha$ and*
(b) $A \in \alpha \Rightarrow C(A) \in \alpha$.

(4) *(a) α is homogeneous and*
(b) $A \in \alpha \Leftrightarrow C(A) \in \alpha$.

(5) *(a) α is homogeneous and*
(b) $A \in \mathcal{S}\alpha \Leftrightarrow C(A) \in \mathcal{S}\alpha$.

We next investigate the homogeneity condition. Krempa [4] has shown that for polynomial rings $A[x]$ this is equivalent to the condition $\alpha(A[x]) \cap A = 0 \Rightarrow \alpha(A[x]) = 0$. This equivalence does not extend to convolution rings in general, which necessitates more terminology: A radical α is said to satisfy the *Krempa Condition* with respect to the convolution type $\mathcal{T}$ if $\alpha(C(A)) \cap A = 0 \Rightarrow \alpha(C(A)) = 0$.

Proposition 6. *Let $\mathcal{T}$ be a convolution type and let α be a radical. The following three conditions are equivalent:*

(1) *α is homogeneous*

(2) *(a) α is weakly homogeneous and*
(b) α satisfies the Krempa Condition

(3) *(a) α is weakly homogeneous and*
(b) $C(A) \in \mathcal{S}\alpha$ for all rings A which has no non-zero ideals I . with $C(I) \in \alpha$

Proof. Suppose (1) holds. We show the validity of (3). The first part is obvious, so we only verify (b). Let A be a ring which has no non-zero ideals I with $C(I) \in \alpha$. Then $C(\alpha(C(A)) \cap A) = \alpha(C(A) \in \alpha$ implies $\alpha(C(A)) \cap A = 0$. Thus $\alpha(C(A) = 0$, i.e. $C(A) \in \mathcal{S}\alpha$. Next we show (3) $\Rightarrow$ (2). Let A be a ring with $\alpha(C(A)) \cap A = 0$. If I is an ideal of A with $C(I) \in \alpha$, then $C(I) \subseteq \alpha(C(A))$ and thus $I \subseteq C(I) \cap A \subseteq \alpha(C(A)) \cap A = 0$. From (3)$(b)$ we get $\alpha(C(A) = 0$.

(2) $\Rightarrow$ (1). Let A be a ring and let $B := \alpha(C(A)) \cap A$. Then $C(B) \subseteq \alpha(C(A))$. Let $\overline{A} = \dfrac{A}{B}$. Then $\overline{A} \hookrightarrow C(\overline{A}) \cong \dfrac{C(A)}{C(B)}$ and under this isomorphism, $\overline{A} = \dfrac{A}{B} \cong \dfrac{A + C(B)}{C(B)} \hookrightarrow \dfrac{C(A)}{C(B)}$. Since $C(B) \subseteq \alpha(C(A))$, $\alpha(C(\overline{A})) = \dfrac{\alpha(C(A))}{C(B)}$. Thus

$$\alpha(C(\overline{A})) \cap \overline{A} = \frac{\alpha(C(A))}{C(B)} \cap \frac{A + C(B)}{C(B)} =$$

$$= \frac{(\alpha(C(A)) \cap A) + C(B)}{C(B)} = \frac{B + C(B)}{C(B)} = 0.$$

From (2)(a) we get $\alpha(C(\overline{A})) = 0$ which gives $\alpha(C(A)) \subseteq C(\alpha(C(A)) \cap A)$. The converse inclusion is given by (2)(b). $\square$

Below we shall see that weakly homogeneity is often a consequence of the properties of the convolution type. In such cases, homogeneity is equivalent to the Krempa Condition which in turn is equivalent to condition (3)(b). This latter condition has been considered by Tumurbat and Wiegandt [16] for polynomial rings.

Proposition 7. *Let $\mathcal{T}$ be a convolution type such that for every ring R with identity, all ideals of $C(R)$ are homogeneous. Then every radical α is $\mathcal{T}$-homogeneous.*

Proof. Let $D(A)$ be the Dorroh extension of the ring A (i.e. the canonical unital extension of A). By the ADS-property, $\alpha(C(A))$ is an ideal of $C(D(A))$. The assumption implies $\alpha(C(A)) = C(I)$ for some ideal I of $D(A)$. But $C(I) = \alpha(C(A)) \subseteq C(A)$ implies $I \subseteq A$. Thus $\alpha(C(A))$ is a homogeneous ideal of $C(A)$. $\square$

The ideal $\alpha(C(A)) \cap A$ plays an important role in the homogeneous requirement, and we next explore this and related properties. Here the work of Amitsur for polynomial rings [1], Krempa for semi-group rings [5] as well as the generalization considered by Ortiz [6] serves as motivation for our considerations.

For the radical α, we define two classes of rings α^c and $\overline{\alpha}$ by

$\alpha^c := \{R \mid C(R) \in \alpha\}$ and
$\overline{\alpha} := \{R \mid R \subseteq \alpha(C(R))\}$ and three ideals of a ring R by
$\alpha^c(R) := \sum(I \triangleleft R \mid I \in \alpha^c) = \sum(I \triangleleft R \mid C(I) \in \alpha)$,
$\overline{\alpha}(R) := \sum(I \triangleleft R \mid I \in \overline{\alpha}) = \sum(I \triangleleft R \mid I \subseteq \alpha(C(I)))$ and
$\alpha^*(R) := \alpha(C(R)) \cap R$.

All these depend, of course, on the convolution type $\mathcal{T}$, so when necessary, it will be emphasized by adding a subscript $\mathcal{T}$, as for example in $\alpha^c_{\mathcal{T}}$.

It can be verified that $\alpha^c \subseteq \overline{\alpha}$ and for any ring R, $\alpha^c(R) \subseteq \overline{\alpha}(R) \subseteq \alpha^*(R)$. If A is a ring such that the ideal generated by A coincides with $C(A)$, then $A \in \alpha^c \Leftrightarrow A \in \overline{\alpha}$. In particular, as we know from [22], if A is a ring with identity, then $A \in \alpha^c \Leftrightarrow A \in \overline{\alpha}$. If $\alpha(C(A))$ is weakly homogeneous for all rings A, then $\alpha^c = \overline{\alpha}$. Indeed, let $A \in \overline{\alpha}$. Then $A = \alpha(C(A)) \cap A$ and so $C(A) = C(\alpha(C(A)) \cap A) \subseteq \alpha(C(A))$. Thus $C(A) \in \alpha$, i.e. $A \in \alpha^c$. Also note that $C(\alpha^*(A)) \in \alpha$ for all A implies that α is weakly homogeneous and the converse holds if α is hereditary. It is clear that $\alpha^c \subseteq \alpha \Leftrightarrow (A \in \alpha \Rightarrow C(A) \in \alpha)$, $\alpha \subseteq \alpha^c \Leftrightarrow (C(A) \in \alpha \Rightarrow A \in \alpha)$ and thus $\alpha = \alpha^c \Leftrightarrow (C(A) \in \alpha \Leftrightarrow A \in \alpha)$. If $D = \{x \in X \mid \sigma(x) = \{(x,x)\}\} \neq \emptyset$, then $\overline{\alpha} \subseteq \alpha$. Indeed, as mentioned earlier, if $D \neq \emptyset$, then there is a surjective homomorphism $\theta : C(A) \to \dfrac{A}{\alpha(A)}$ with $\dfrac{C(A)}{\ker\theta} \cong \dfrac{A}{\alpha(A)} \in \mathcal{S}\alpha$, $\ker\theta \cap A = \alpha(A)$ and $C(\alpha(A)) \subseteq \ker\theta$. So, if $A \in \overline{\alpha}$, then $A \subseteq C(\alpha(A)) \cap A \subseteq \ker\theta \cap A = \alpha(A)$ which gives $A \in \alpha$.

A last remark on the coincidence of the classes under discussion here, is the following. If $D \neq \emptyset$ and X is U-bounded (cf. [22]) for some finite set U with $\emptyset \neq U \subseteq X$ and $\sigma(u) \subseteq U \times U$ for all $u \in U$, then $\alpha = \alpha^c = \overline{\alpha}$ for any hypernilpotent radical α (i.e. all nilpotent rings are radical). Indeed, $(0 : U)_{C(A)}$ is a nilpotent ideal of $C(A)$ with $\dfrac{C(A)}{(0 : U)_{C(A)}} \cong A^U$ (cf. Proposition 7 in [22]). Since α is hypernilpotent, $(0 : U)_{C(A)} \in \alpha$. Thus, if $A \in \alpha$, we have $A^U \in \alpha$ (since U is finite) and hence $C(A) \in \alpha$, i.e. $A \in \alpha^c$. Since $\alpha^c \subseteq \overline{\alpha} \subseteq \alpha$, we can conclude that $\alpha = \alpha^c = \overline{\alpha}$.

An ideal K of $C(A)$ has the *Summation Property* [22] if it satisfies: Whenever I_p is an ideal of A with $C(I_p) \subseteq K$ for all $p \in \Lambda$, Λ is some index set, then $C(\sum_{p\in\Lambda} I_p) \subseteq K$. Any weakly homogeneous ideal K has the summation property. The status of the converse is not clear. What is known is that an ideal which satisfies the summation property need not be homogeneous. Indeed, if $A = 2\mathbb{Z}$, the ring of even integers, let $C(A) = M_2(A)$, the ring of 2×2 matrices over A. Then $K = \begin{bmatrix} 4\mathbb{Z} & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{bmatrix}$ is weakly homogeneous (and thus satisfies the summation property), but it is not homogeneous.

Proposition 8. *If α is a radical class such that $\alpha(C(A))$ has the summation property for all rings A, then α^c is a radical class. Conversely, if α^c is a radical class and α is hereditary, then $\alpha(C(A))$ has the summation property for all rings A.*

Proof. For a surjective homomorphism $\theta : A \to B$, we have a surjective homomorphism $C(\theta) : C(A) \to C(B)$ defined by $C(\theta)(f) = \theta \circ f$ for all $f \in C(A)$. From this the homomorphic closure of α^c follows.

Suppose every non-zero homomorphic image of the ring A has a non-zero ideal which is in α^c. We show $A \in \alpha^c$. Let $J := \sum(I_p \lhd A \mid C(I_p) \subseteq \alpha(C(A)))$. Then $C(J) \subseteq \alpha(C(A))$ by the summation property. If $J \neq A$, then there is a non-zero ideal $\dfrac{I}{J}$ of $\dfrac{A}{J}$ with $\dfrac{I}{J} \in \alpha^c$. Now $C(I) \nsubseteq \alpha(C(A))$, for if $C(I) \subseteq \alpha(C(A))$, then $I = J$, a contradiction. Hence $0 \neq \dfrac{C(I)}{C(I) \cap \alpha(C(A))} \cong \dfrac{C(I) + \alpha(C(A))}{\alpha(C(A))} \lhd \dfrac{C(A)}{\alpha(C(A))} \in \mathcal{S}\alpha$. But $C(J) \subseteq C(I) \cap \alpha(C(A))$ and $\dfrac{C(I) \cap \alpha(C(A))}{C(J)} \lhd \dfrac{C(I)}{C(J)} \cong C\left(\dfrac{I}{J}\right) \in \alpha$ which means $\dfrac{C(I)}{C(I) \cap \alpha(C(A))} \in \alpha \cap \mathcal{S}\alpha = 0$, a contradiction. Thus $J = A$ and so $C(A) = C(J) \subseteq \alpha(C(A))$, i.e. $C(A) \in \alpha^c$.

Conversely, suppose α^c is a radical and α is hereditary. Let $I_p \lhd A$ with $C(I_p) \subseteq \alpha(C(A))$ for all $p \in \Lambda$, Λ some index set. Since α is hereditary, we get $I_p \in \alpha^c$ for all p. Thus $\sum_{p\in\Lambda} I_p \subseteq \alpha^c(A)$ and so $C(\sum_{p\in\Lambda} I_p) \subseteq C(\alpha^c(A)) \in \alpha$ since $\alpha^c(A) \in \alpha^c$. From the hereditariness of α we get $C(\sum_{p\in\Lambda} I_p) \subseteq \alpha(C(A))$ which shows that the summation property holds. □

Next we investigate when $\overline{\alpha}$ will be a radical class. From Ortiz [6] we know that if for any $I \lhd A$, $C(I) \subseteq IC(D(A))$ where $D(A)$ denotes the Dorroh extension of A, then $\overline{\alpha}$ is a radical class. We will weaken this requirement. Since $\alpha\left(\dfrac{C(A)}{C(\alpha^*(A))}\right) \lhd \dfrac{C(A)}{C(\alpha^*(A))}$, we have $\alpha\left(\dfrac{C(A)}{C(\alpha^*(A))}\right) = \dfrac{B}{C(\alpha^*(A))}$ for some $B = B_A \lhd A$. Then $\alpha^*(A) \subseteq C(\alpha^*(A)) \subseteq B$ and so $\alpha^*(A) \subseteq B \cap A$. We say that α has the *Intersection Property* if $\alpha^*(A) = B_A \cap A$ for all rings A, i.e. $\alpha(C(A)) \cap A = B_A \cap A$ for all rings A. Note that

(1) $B_A = \alpha(C(A)) \Leftrightarrow \alpha(C(A))$ is weakly homogeneous. Indeed, if $B_A = \alpha(C(A))$, then $C(\alpha(C(A)) \cap A) = C(\alpha^*(A)) \subseteq B_A = \alpha(C(A))$. Conversely, if $\alpha(C(A))$ is weakly homogeneous, then $C(\alpha^*(A)) \subseteq \alpha(C(A))$. Then $\dfrac{B_A}{C(\alpha^*(A))} = \alpha\left(\dfrac{C(A)}{C(\alpha^*(A))}\right) = \dfrac{\alpha(C(A))}{C(\alpha^*(A))}$ which gives $B_A = \alpha(C(A))$.

(2) If α satisfies the Krempa Condition and the Intersection Property, then $\alpha(C(A)) \subseteq C(\alpha^*(A))$ for all A. This follows from $\alpha\left(C\left(\dfrac{A}{\alpha^*(A)}\right)\right) \cong \alpha\left(\dfrac{C(A)}{C(\alpha^*(A))}\right)$, $\alpha\left(C\left(\dfrac{A}{\alpha^*(A)}\right)\right) \cap \dfrac{A}{\alpha^*(A)} = \dfrac{(B_A \cap A) + C(\alpha^*(A))}{C(\alpha^*(A))} = 0$ (by the Intersection Property) and the Krempa Condition.

Proposition 9. *If the radical α satisfies the Intersection Property, then $\overline{\alpha}$ is a radical class.*

Proof. Let $\theta : A \to B$ be a surjective homomorphism with $A \in \overline{\alpha}$. Then $A \subseteq \alpha(C(A))$ and since $C(\theta) : C(A) \to C(B)$ is a surjective homomorphism, $B = \theta(A) = (C(\theta))(A) \subseteq C(\theta)(\alpha(C(A))) \subseteq \alpha(C(B))$. Thus $B \in \overline{\alpha}$ which shows that $\overline{\alpha}$ is homomorphically closed.

Note that since $\dfrac{A+C(\alpha^*(A))}{C(\alpha^*(A))}$ is the isomorphic image of $\dfrac{A}{\alpha^*(A)}$ under the isomorphism $C\left(\dfrac{A}{\alpha^*(A)}\right) \cong \dfrac{C(A)}{C(\alpha^*(A))}$, we get

$$\alpha^*\left(\frac{A}{\alpha^*(A)}\right) = \alpha\left(C\left(\frac{A}{\alpha^*(A)}\right)\right) \cap \frac{A}{\alpha^*(A)} \cong \alpha\left(\frac{C(A)}{C(\alpha^*(A))}\right) \cap \frac{A}{\alpha^*(A)} =$$

$$= \frac{B_A}{C(\alpha^*(A))} \cap \frac{A+C(\alpha^*(A))}{C(\alpha^*(A))} = \frac{(B_A \cap A)+C(\alpha^*(A))}{C(\alpha^*(A))} = \frac{\alpha^*(A)+C(\alpha^*(A))}{C(\alpha^*(A))} = 0.$$

Suppose now A is a ring such that every non-zero homomorphic image of A has a non-zero ideal which is in $\overline{\alpha}$. We show $A \in \overline{\alpha}$. Suppose to the contrary that $A \notin \overline{\alpha}$. Then $\alpha^*(A) \subsetneqq A$. By assumption, there is an ideal $0 \neq \dfrac{I}{\alpha^*(A)} \lhd \dfrac{A}{\alpha^*(A)}$ with $\dfrac{I}{\alpha^*(A)} \in \overline{\alpha}$. Then $\dfrac{I}{\alpha^*(A)} \subseteq \alpha\left(C\left(\dfrac{I}{\alpha^*(A)}\right)\right) \subseteq \alpha\left(C\left(\dfrac{A}{\alpha^*(A)}\right)\right)$. Thus $\dfrac{I}{\alpha^*(A)} \subseteq \alpha\left(C\left(\dfrac{A}{\alpha^*(A)}\right)\right) \cap \dfrac{A}{\alpha^*(A)} = \alpha^*\left(\dfrac{A}{\alpha^*(A)}\right) = 0$, a contradiction. Hence $A \in \overline{\alpha}$. $\square$

Next we investigate the properties of the ideal-mapping $\alpha^*(A) = \alpha(C(A)) \cap A$.

Proposition 10. *For any radical α, α^* is a complete pre-radical. It is a Hoehnke radical if and only if α satisfies the Intersection Property and it is idempotent if and only if $\alpha^*(A) \in \overline{\alpha}$. Thus α^* is a Kurosh-Amitsur radical map if and only if $\alpha^*(A) \in \overline{\alpha}$ for all rings A and α satisfies the Intersection Property. In this case, $\alpha^*(A) = \overline{\alpha}(A)$ for all rings A.*

Proof. α^* is a pre-radical: Let $\theta : A \to B$ be a surjective homomorphism. Then $\theta(\alpha^*(A)) = \theta(\alpha(C(A)) \cap A) \subseteq \alpha(C(\theta)(C(A)) \cap B = \alpha(C(B)) \cap B = \alpha^*(B) = \alpha^*(\theta(A))$.

α^* is complete: Let $\alpha^*(I) = I \lhd A$. Then $I = \alpha(C(I)) \cap I \subseteq \alpha(C(A)) \cap A = \alpha^*(A)$.

α^* is idempotent $\Leftrightarrow \alpha^*(\alpha^*(A)) = \alpha^*(A) \Leftrightarrow \alpha(C(\alpha^*(A))) \cap \alpha^*(A) = \alpha^*(A) \Leftrightarrow \alpha^*(A) \subseteq \alpha(C(\alpha^*(A))) \Leftrightarrow \alpha^*(A) \in \overline{\alpha}$.

Next we show that $\alpha^*\left(\dfrac{A}{\alpha^*(A)}\right) = 0$ if and only if α satisfies the Intersection Property:

$$\alpha^*\left(\frac{A}{\alpha^*(A)}\right) = \alpha\left(C\left(\frac{A}{\alpha^*(A)}\right)\right) \cap \frac{A}{\alpha^*(A)} \cong \alpha\left(\frac{C(A)}{C(\alpha^*(A))}\right) \cap \frac{A}{\alpha^*(A)} =$$

$$= \frac{B_A}{C(\alpha^*(A))} \cap \frac{A+C(\alpha^*(A))}{C(\alpha^*(A))} = \frac{(B_A \cap A)+C(\alpha^*(A))}{C(\alpha^*(A))} = 0 \Leftrightarrow B_A \cap A \subseteq C(\alpha^*(A)).$$

This inclusion holds if and only if $B_A \cap A = \alpha^*(A)$ (i.e. the Intersection Property is satisfied). Indeed, suppose $B_A \cap A \subseteq C(\alpha^*(A))$. Then $B_A \cap A = (B_A \cap A) \cap A \subseteq C(\alpha^*(A)) \cap A = \alpha^*(A)$ and $C(\alpha^*(A)) \subseteq B_A$ implies $\alpha^*(A) \subseteq B_A \cap A$. Thus $B_A \cap A = \alpha^*(A)$. The converse is clear since $\alpha^*(A) \subseteq C(\alpha^*(A))$.

Lastly we show that if α^* is a Kurosh-Amitsur radical, then $\alpha^*(A) = \overline{\alpha}(A)$. Suppose thus that α^* is a Kurosh-Amitsur radical. From the above, we know that the Intersection Property is satisfied and so $\overline{\alpha}$ is a radical class (Proposition 9). Since α^* is idempotent, $\alpha^*(A) \in \overline{\alpha}$. Thus $\alpha^*(A) \subseteq \overline{\alpha}(A)$. Since $\overline{\alpha}(A) \subseteq \alpha^*(A)$ always hold, we get $\overline{\alpha}(A) = \alpha^*(A)$. □

As mentioned earlier, weakly homogeneity of α often comes for free as a consequence of properties of the convolution type. We now investigate this and related concepts. We recall from [22]:

A convolution type $\mathcal{T}$ is said to satisfy the *Ortiz Condition* if $C(N) \subseteq NC(D(A))$ for every ring A and subring N of A (remember $D(A)$ denotes the Dorroh extension of A). The origins of the Ortiz condition is to be found in [6], playing a key role in the generalization of certain radicals classes determined by the radicals of polynomial rings. $\mathcal{T}$ is said to satisfy the *Finite Complement Property* if $X \setminus Y$ is finite for all $Y \in \mathcal{S}$. It was shown in [22] that the Finite Complement Property implies the validity of the Ortiz Condition which in turn implies that every radical is weakly homogeneous.

A case that often occurs in the examples is the following: α is a radical which is weakly homogeneous and which satisfies $A \in \alpha \Leftrightarrow C(A) \in \alpha$. In such a case, α is invariant if and only if α is homogeneous if and only if α satisfies the Krempa Condition.

3 Examples

In the examples below, we will not recall or summarize all that is known about the radical theory of the particular convolution type. We will only recall or proof results which will bring certain aspects of the radical theory of convolution rings to the fore.

3.1. Discrete direct sums. Let X be any non-empty set, $\mathcal{S} = \{Y \subseteq X \mid X \setminus Y$ is finite$\}$, $\sigma(x) = \{(x,x)\}$ for all $x \in X$ and $\tau(s,t) = 1$ for all $s,t \in X$. Then $T = X = D$. The corresponding convolution ring $C(A) = \bigoplus_{x\in X} A$, the discrete direct sum of $|X|$-copies of A. For any radical α and ring A, $\alpha(C(A)) = \alpha\left(\bigoplus_{x\in X} A\right) = \bigoplus_{x\in X} \alpha(A) = C(\alpha(A))$; the best possible scenario and there is nothing further to report.

Note that for infinite sets X, conditions (T1) and (T2) are satisfid but not (T3).

3.2. Direct products. Let X be any infinite set, $\mathcal{S} = \{\varnothing\}$, $\sigma(x) = \{(x,x)\}$ for all $x \in X$ and $\tau(s,t) = 1$ for all $s,t \in X$. Then $T = X = D$ and the convolution ring $C(A)$ coincides with the direct product A^X of $|X|$-copies of the ring A.

We know that A can be embedded as an ideal in A^X and that A is a homomorphic image of A^X. Since radical classes are homomorphically closed and semisimple classes are hereditary and closed under subdirect products (and thus also direct products), we have for any radical class $\alpha : A \in \mathcal{S}\alpha \Leftrightarrow A^X \in \mathcal{S}\alpha$ and $A^X \in \alpha \Rightarrow A \in \alpha$. This means that the salient properties of the radical of a direct product depend only on the validity of the converse of the above implication. In fact, we have: $\alpha(A^X) = (\alpha(A))^X \Leftrightarrow \alpha$ is homogenous $\Leftrightarrow (A \in \alpha \Rightarrow A^X \in \alpha) \Leftrightarrow (\alpha(A))^X \in \alpha$ for all rings A. Furthermore, $\alpha^c \subseteq \overline{\alpha} \subseteq \alpha$ and for every ring A, we have $\alpha^c(A) \subseteq \overline{\alpha}(A) \subseteq \alpha^*(A) \subseteq \alpha(A)$ with equality if and only if $\alpha^c = \alpha$.

In general, A^X need not be radical even though A is radical. Also, neither α^c nor $\overline{\alpha}$ need to be Kurosh-Amitsur radicals and α^* need not even be a Hoehnke radical. The example which follows will show all these (negative) properties. In addition, it also shows that $A \in \mathcal{S}\alpha \Leftrightarrow A^X \in \mathcal{S}\alpha$ need not be equivalent to $A^X \in \alpha \Leftrightarrow A \in \alpha$. Let α be the nil radical and let R be the Zassenhaus algebra (see for example Divinsky [3], Chapter 2, Example 3). This ring R is constructed as follows. Let F be any field. The elements of R are the formal (finite) sums $\sum_t a_t x_t$ where $a_t \in F$ and $x_t \in (0,1)$. Multiplication is done according to the rule $x_t x_s = \begin{cases} x_{t+s} \text{ if } t+s<1 \\ 0 \text{ if } t+s \geq 1 \end{cases}$. As is well known, R is a nil ring. Let $X = \mathbb{N}$ be the set of positive integers. Then R^X is not radical, for the element $x = (x_{\frac{1}{2}}, x_{\frac{1}{4}}, x_{\frac{1}{8}}, ..., x_{\frac{1}{2^n}}, ...)$ of R^X is not nilpotent. Next we show that α does not satisfy the Summation Property (and thus $\overline{\alpha}$ is not Kurosh-Amitsur radical). For every $t \in (0,1)$, let I_t be the ideal in R generated by x_t. Then I_t is nilpotent with $I_t^k = 0$ for any $k \in \mathbb{N}$ with $k > \frac{1}{t}$. Thus $C(I_t) = (I_t)^X$ is nilpotent and so $C(I_t) \subseteq \alpha(C(R))$ for all $t \in (0,1)$. But $C(\sum_t I_t) = C(R) \nsubseteq \alpha(C(R))$. Also, α^c is not a Kurosh-Amitsur radical, for if it were, then $R = \sum_t I_t \supseteq \sum(I \lhd R \mid C(I) \in \alpha) = \alpha^c(R)$. This means $R \in \alpha^c$, i.e. $R^X = C(R) \in \alpha$; a contradiction. In addition, it also shows that α does not satisfy the Intersection Property. From Proposition 10 it follows that α^* is a complete pre-radical, but not a Hoehnke radical.

Examples of radicals which do satisfy the condition $A^X \in \alpha \Leftrightarrow A \in \alpha$ can be obtained from the following. Let $k, n \in \mathbb{N}$ be fixed with $1 \leq n < k$. Let $\phi(x_1, x_2, ..., x_k)$ be a fixed element from $\mathbb{Z}\{x_1, x_2, ..., x_k\}$, the ring of polynomials in k non-commuting indeterminates over the integers $\mathbb{Z}$. For a ring A, let $\phi_A : A^k \to A$ be the corresponding evaluation map. The ring A is called an $\phi - ring$ if for all $a_1, a_2, ..., a_n \in A$ there exists $a_{n+1}, a_{n+2}, ..., a_k \in A$ such that $\phi_A(a_1, a_2, ..., a_n, a_{n+1}, a_{n+2}, ..., a_k) = 0$. Let Φ be the class of all ϕ-rings and let $\pi_x : A^X \to A$ be the $x-th$ projection. We suppose that $\pi_x(\phi_{A^X}(a_1, a_2, ..., a_k)) = \phi_A(\pi_x(a_1), \pi_x(a_2), ..., \pi_x(a_k))$ for all $a_1, a_2, ..., a_k \in A^X$ and $x \in X$. Under this assumption, we get $A \in \Phi \Leftrightarrow A^x \in \Phi$. Indeed, let $A \in \Phi$. Let $a_1, a_2, ..., a_n \in A^X$. For any $x \in X$, $\pi_x(a_1), \pi_x(a_2), ..., \pi_x(a_k) \in A$ and by assumption there exist $a'_{n+1}, a'_{n+2}, ..., a'_k \in A$ such that

$\phi_A(\pi_x(a_1), \pi_x(a_2), ..., \pi_x(a_n), a'_{n+1}, a'_{n+2}, ..., a'_k) = 0$.

Each of these a'_{n+j} 's depends on x, so when we want to emphasize this, we write $a'_{n+j} = a'_{n+j}(x)$. For each $j = n+1, n+2, ..., k$, define $a_j : X \to A$ by $a_j(x) = a'_j(x)$ for all $x \in X$. Then

$\pi_x(\phi_{A^X}(a_1, a_2, ..., a_k))$
$= \phi_A(\pi_x(a_1), \pi_x(a_2), ..., \pi_x(a_k))$
$= \phi_A(\pi_x(a_1), \pi_x(a_2), ..., \pi_x(a_n), a'_{n+1}, a'_{n+2}, ..., a'_k)$

$= 0$ for all $x \in X$. Thus $\phi_{A^X}(a_1, a_2, ..., a_k) = 0$ and $A^X \in \Phi$. Conversely, suppose $A^X \in \Phi$ and let $a'_1, a'_2, ..., a'_n \in A$. Choose $x_0 \in X$ fixed. For each $i = 1, 2, ..., n$, define $a_i : X \to A$ by $a_i(x) = \begin{cases} a'_i \text{ if } x = x_0 \\ 0 \text{ otherwise} \end{cases}$. Since $A^X \in \Phi$, there are $a_{n+1}, a_{n+2}, ..., a_k \in A^X$ such that $\phi_{A^X}(a_1, a_2, ..., a_k) = 0$. Thus

$\phi_A(a'_1, a'_2, ..., a'_n, \pi_{x_0}(a_{n+1}), \pi_{x_0}(a_{n+2}), ..., \pi_{x_0}(a_k))$
$= \phi_A(\pi_{x_0}(a_1), \pi_{x_0}(a_2), ..., \pi_{x_0}(a_n), \pi_{x_0}(a_{n+1}), \pi_{x_0}(a_{n+2}), ..., \pi_{x_0}(a_k))$
$= \pi_{x_0}(\phi_{A^X}(a_1, a_2, ..., a_k))$
$= 0$. Thus $A \in \Phi$.

When $\alpha := \Phi$ is a radical class, we will have $\alpha(A^X) = (\alpha(A))^X$ for all A. As examples we may mention for $n = 1$ and $k = 2$, the polynomials $\phi(x, y) = x + y - xy$ and $\phi(x, y) = x - xyx$ which give the Jacobson radical class and the von Neumann regular radical class respectively.

3.3. Polynomials. Let $X = \mathbb{N}_0 := \{0, 1, 2, 3, ...\}$, $\mathcal{S} = \{Y_k \mid k \in \mathbb{N}_0\}$ where $Y_k = \{k+1, k+2, k+3, ...\}$, $\sigma(n) = \{(i, j) \mid i, j \in \mathbb{N}_0, i+j = n\}$ and $\tau(n, m) = 1$ for all $n, m \in \mathbb{N}_0$. Here $T = D = \{0\}$ and the convolution ring $C(A)$ is the polynomial ring $A[x]$ in one indeterminate. The radical theory of this convolution type is one of the classical cases (the other being matrices which will be discussed below). The polynomial convolution type satisfies the Finite Complement Property which means that any radical α is weakly homogeneous, i.e. $(\alpha(A[x]) \cap A)[x] \subseteq \alpha(A[x])$ for any ring A. We also have $A[x] \in \alpha \Rightarrow A \in \alpha$, $\alpha(A[x]) \subseteq \{f \in A[x] \mid f(0) \in \alpha(A)\}$ and $\alpha^c = \overline{\alpha} \subseteq \alpha$. Furthermore, the radical α will be homogeneous (i.e. satisfy the Amitsur condition) if and only if it satisfies the Krempa Condition.

Some of the well-known radicals are invariant, for example the Baer (= prime) radical as well as the Levitzky (= local nilpotent) radical. Several others are homogeneous, for example the Jacobson radical, nil radical, Brown-McCoy radical, uniformly strongly prime radical and any strongly hereditary radical (i.e. a radical such that any subring of a radical ring is radical). For these homogeneous radicals, $\alpha^*(A) \in \alpha^c = \overline{\alpha}$; hence α^* is a Kurosh-Amitsur radical with $\alpha^*(A) = \alpha^c(A) = \overline{\alpha}(A) \subseteq \alpha(A)$ for all rings A and the inclusion is in general strict. By the Krempa Condition, these radicals satisfy $A \in \mathcal{S}\alpha \Rightarrow A[x] \in \mathcal{S}\alpha$. Smoktunowicz [14] has given an example of a nil ring A for which $A[x]$ is not nil. Thus for α the nilradical, which is homogeneous, we have $A[x]$ nil implies A nil, but the converse implication is not true in general. This situation can also be realized for subidempotent radicals (hereditary and all nilpotent rings are semisimple): Let ν be the von Neumann regular radical. For any ring A, $\upsilon(A[x]) = 0$ (cf. [16]) which means υ is homogeneous, $\nu^c = \overline{\upsilon} = \{0\}, \upsilon^*(A) = 0$ for all A and $A \in \upsilon$ does not necessarily imply $A[x] \in \upsilon$.

The major outstanding problem regarding the radicals of polynomial rings is to characterize the ideal $\alpha^*(A) = \alpha(A[x]) \cap A$ of A in terms of properties of the ring A without reference to $\alpha(A[x])$.

In striking contrast to most of the other convolution types, the Jacobson radical $\mathcal{J}(A[x])$ of $A[x]$ is in general not directly accessible. It is known that $\mathcal{J}(A[x]) = N[x]$

where $N := \mathcal{J}(A[x]) \cap A = \mathcal{J}^*(A)$ is a nil ideal of A. We will now describe the elements of N and start with:

3.3.1. *Let $a \in A$. If ax^k is right quasi-regular in $A[x]$ for some $k \geq 1$, then a is nilpotent. Conversely, if $a \in A$ is nilpotent, then ax^k is right quasi-regular in $A[x]$ for all $k \geq 0$.*

Proof. Let $q(x) = q_0 + q_1x + q_2x^2 + ... + q_nx^n \in A[x], q_n \neq 0$, be such that $ax^k + q(x) - ax^kq(x) = 0$, i.e. $ax^k + (q_0 + q_1x + q_2x^2 + ... + q_nx^n) - ax^k(q_0 + q_1x + q_2x^2 + ... + q_nx^n) = 0$. Comparing constant terms, we get $q_0 = 0$. If $k > n$, then the coefficient of x^k on the left hand side is a which gives $a = 0$ and we are done. Suppose thus $k \leq n$. Comparing coefficients gives $q_i = 0$ for $i = 1, 2, 3, ..., k-1$, $a + q_k = 0$, $q_{k+i} - aq_i = 0$ for $i = 1, 2, 3, ..., n-k$ and $aq_{n-i} = 0$ for $i = 0, 1, 2, ..., k-1$. Since $k \leq n$, we have $n = mk + i$ for some $m \geq 1$ and $0 \leq i < k$. Now $q_{mk} = -a^m$ and so $0 = aq_{n-i} = aq_{mk} = -a^{m+1}$. Thus a is nilpotent.

Conversely, suppose a is nilpotent, say $a^{p+1} = 0$. If $k = 0$, let $q(x) := -a^p$. If $k \neq 0$, let $q(x) = q_kx^k + q_{2k}x^{2k} + ... + q_{pk}x^{pk}$ where $q_{ik} = -a^i$ for $i = 1, 2, 3, ..., p$. Then $q(x) \in A[x]$ and $ax^k + q(x) - ax^kq(x) = 0$, i.e. ax^k is right quasi-regular in $A[x]$. □

For a ring A and elements $c_1, c_2, ..., c_k \in A, k \geq 1$, define a sequence $h = (h_1, h_2, h_3, ...)$ by $h_1 := -c_1$ and if h_{i-1} has been defined, let $h_i := \sum_{j=1}^{i-1} c_jh_{i-j} - c_i$ for $i = 2, 3, 4, ...$ where we take $c_{k+1} = c_{k+2} = c_{k+3} = ... = 0$. Since this sequence depends on the c_i's, if necessary we will denote it by $h = h[c_1, c_2, ..., c_k] = (h_1, h_2, h_3, ...)$.

An element $a \in A$ is called *rqr-nilpotent* if for any $k \geq 1$ and $b_1, b_2, ..., b_k \in A$, the sequence $h = h[ab_1, ab_2, ..., ab_k]$ is ultimately 0, i.e. there exists an $n \geq 1$ such that $h_{n+1} = h_{n+2} = h_{n+3} = ... = 0$.

It can be verified that if a is rqr-nilpotent, then a is nilpotent and so is ab (and hence also ba) for any $b \in A$. If A is commutative, then $a \in A$ is rqr-nilpotent if and only if a is nilpotent.

3.3.2. *An element $a \in A$ is rqr-nilpotent in A if and only if $a(b_1x + b_2x^2 + ... + b_kx^k)$ is right quasi-regular in $A[x]$ for all $b_1, b_2, ..., b_k \in A, k \geq 1$.*

Proof. Suppose a is rqr-nilpotent and let $b_1, b_2, ..., b_k \in A, k \geq 1$. By definition the sequence $h = h[ab_1, ab_2, ..., ab_k] = (h_1, h_2, h_3, ...)$ is ultimately 0, say $h_{n+1} = h_{n+2} = ... = 0$ for some $n \geq 1$. Then $h(x) := h_1x + h_2x^2 + ... + h_nx^n \in A[x]$ and $a(b_1x + b_2x^2 + ... + b_kx^k) + h(x) - a(b_1x + b_2x^2 + ... + b_kx^k)h(x) = 0$. Thus $a(b_1x + b_2x^2 + ... + b_kx^k)$ is right quasi-regular in $A[x]$.

Conversely, suppose $a(b_1x+b_2x^2+...+b_kx^k)$ is right quasi-regular in $A[x]$ for any $b_1, b_2, ..., b_k \in A, k \geq 1$. Choose $b_1, b_2, ..., b_k \in A$ and let $b(x) := b_1x+b_2x^2+...+b_kx^k$. Consider the sequence $h = h[ab_1, ab_2, ..., ab_k] = (h_1, h_2, h_3, ...)$. By assumption there is a $f(x) = f_0+f_1x+f_2x^2+...+f_nx^n \in A[x]$ such that $ab(x)+f(x)-ab(x)f(x) = 0$. Comparing coefficients will then give $f_i = h_i$ for all $i = 1, 2, 3, ..., n$ and $h_{n+i} = 0$ for all $i = 1, 2, 3, ...$ .Thus a is rqr-nilpotent. □

3.3.3. *Let $\mathcal{J}$ denote the Jacobson radical. For any ring A,*
$\mathcal{J}(A[x]) \cap A = \{a \in A \mid a$ is rqr-nilpotent$\}$.

Proof. Let $a \in J(A[x]) \cap A$. Then $ab(x)$ is right quasi-regular in $A[x]$ for any $b(x) \in A[x]$. In particular, this is true for any $b(x)$ of the form $b(x) = b_1x + b_2x^2 + ... + b_kx^k$. From 3.3.2 above we then know that a is rqr-nilpotent. Conversely, suppose a is rqr-nilpotent. Then ac is nilpotent for any $c \in A$ and thus acx^n is right quasi-regular in $A[x]$ for all $n \geq 0$. To get a in $\mathcal{J}(A[x])$, we need to show that $ab(x)$ is right quasi-regular for any $b(x) = b_0 + b_1x + b_2x^2 + ... + b_kx^k \in A[x]$. For such a $b(x) \in A[x]$ we know ab_0 is nilpotent and hence right quasi-regular, say $ab_0 + u - ab_0u = 0$ for some $u \in A$. Let $b'(x) := b_1x + b_2x^2 + ... + b_kx^k$. Since a is rqr-nilpotent, we know that $a(b'(x) - ub'(x))$ is right quasi-regular in $A[x]$, say $a(b'(x) - ub'(x)) + q(x) - a(b'(x) - ub'(x))q(x) = 0$ for some $q(x) \in A[x]$. Let $f(x) := u + q(x) - uq(x)$. Then $f(x) \in A[x]$ and $ab(x) + f(x) - ab(x)f(x) = ab_0 + ab'(x) + u + q(x) - uq(x) - (ab_0 + ab'(x))(u + q(x) - uq(x)) = 0$ which shows that $a \in \mathcal{J}(A[x])$. $\square$

If a ring is called *qr-nil* if all its elements are qr-nilpotent, then the radical class $\mathcal{J}^* = \{A \mid A \text{ is qr-nil}\}$. The question whether all the elements of a nil ring are rqr-nilpotent is equivalent to the Köthe Conjecture.

The radical theory of related convolution rings like the ring of polynomials in n commuting indeterminates $C(A) = A[x_1, x_2, ..., x_n]$, the polynomial ring in n non-commuting indeterminates $C(A) = A\{x_1, x_2, ..., x_n\}$, the ring of formal power series $C(A) = A[[x]]$, the ring of Laurent series $C(A) = A\langle x\rangle$, etc., is not as well-developed as for the polynomial rings $A[x]$. This is except for $C(A) = A[x_1, x_2, ..., x_n]$ where the results of the one indeterminate case carries over mutatis mutandis. For results on the radicals of these convolution rings, one could consult Amitsur [1] and [2], Sierpińska [13] and Puczyłowski [9] and [10].

3.4. Necklace rings. Let $X = \mathbb{N}$, $\mathcal{S} = \{\varnothing\}$, $\sigma(n) = \{(i,j) \mid i,j \in \mathbb{N}, lcm(i,j) = n\}$ and $\tau(n,m) = \gcd(n,m)$ where lcm and gcd denote the least common multiple and greatest common divisor respectively. Then $T = D = \{1\}$ and $C(A)$ is just the necklace ring $N(A)$ over A, see for example [7]. Necklace rings can also be defined over finite subsets $X = \{1, 2, 3, ..., k\}$ of $\mathbb{N}$ with a similar convolution type as above, cf [20]. In this case the convolution ring will be denoted by $N_k(A)$. All results on the radical theory of necklace rings can be found in [20].

We will consider the radical theory of this latter case first. In this case, since X is finite the convolution type has the Finite Complement Property which means that any radical α is weakly homogeneous. We also know that $\alpha(N_k(A)) \subseteq (\alpha(A) : P)_{N_k(A)}$ where $P(f) = \{nf(n) \mid n = 1, 2, 3, ..., k\}$ for $f \in N_k(A)$. Thus $N_k(A) \in \alpha \Rightarrow A \in \alpha$, $\alpha^c = \overline{\alpha}$ is a Kurosh-Amitsur radical, and $\alpha^c(A) = \overline{\alpha}(A) \subseteq \alpha^*(A) = \alpha(N_k(A)) \cap A \subseteq (\alpha(A) : P)_{N_k(A)} \cap A \subseteq \alpha(A)$. If α is supernilpotent (i.e. α is hereditary and contains all the nilpotent rings), then $\alpha(N_k(A)) = (\alpha(A) : P)_{N_k(A)} \supseteq N_k(\alpha(A))$ which gives $N_k(A) \in \alpha \Leftrightarrow A \in \alpha$. So, for these radicals we get $\alpha = \alpha^c$ and $\alpha^c(A) = \overline{\alpha}(A) = \alpha^*(A) = \alpha(A)$ for all rings A. Furthermore, α will be homogeneous if and only if α is invariant. Indeed, if α is homogeneous, then $\alpha(N_k(A)) = N_k(\alpha(N_k(A)) \cap A) \subseteq N_k((\alpha(A) : P)_{N_k(A)} \cap A) = N_k(\alpha(A))$ and so $\alpha(N_k(A)) = N_k(\alpha(A))$. In general, a radical α need not be homogeneous: Let α

be the nilradical and let $R = \mathbb{Z}_8$, the ring of integers $mod 8$. Let $f(1) = 2, f(2) = 5, f(3) = f(5) = 0$ and $f(4) = 3$. Then $f \in (\alpha(R) : P)_{N_5(R)} = \alpha(N_5(R))$ where $\alpha(R) = \{0, 2, 4, 6\}$, but $f \notin N_5(I)$ for any proper ideal I of R. To summarize, here we have an example of a (supernilpotent) radical α which is weakly homogeneous, $C(A) \in \alpha \Leftrightarrow A \in \alpha$, $\alpha^c(A) = \overline{\alpha}(A) = \alpha^*(A) = \alpha(A)$ for all rings A (so $C(\alpha^*(A)) \in \alpha$), the radical is directly accessible, but the Krempa Condition is not satisfied (and hence the radical is not homogeneous).

For the general necklace ring (i.e. $X = \mathbb{N}$), we have the following results: For any radical α, $\alpha(N(A)) \subseteq (\alpha(A) : P)_{N(A)}$ where P is as above. If $\alpha = \mathcal{J}$ is the Jacobson radical, the results are much stronger, for $\mathcal{J}(N(A)) = (\mathcal{J}(A) : P)_{N(A)} \supseteq N(\mathcal{J}(A))$ and thus

$$\begin{aligned} N(\mathcal{J}(N(A)) \cap A) &= N((\mathcal{J}(A) : P)_{N(A)} \cap A) \\ &= N(\mathcal{J}(A)) \subseteq \mathcal{J}(N(A)) \end{aligned}$$

which shows that $\mathcal{J}$ is weakly homogeneous. But from [20] we know that $\mathcal{J}$ is not homogeneous. Furthermore, $A \in \mathcal{J} \Leftrightarrow N(A) \in \mathcal{J}$ and hence $\mathcal{J}^c(A) = \overline{\mathcal{J}}(A) = \mathcal{J}^*(A) = \mathcal{J}(A)$ for all rings A. We thus see that the Jacobson radical enjoys the same properties for the general necklace ring as any supernilpotent radical does for the finite necklace ring, except in the general case the convolution type does not satisfy the Finite Complement Property.

3.5. Matrices. Let $n \geq 1$ be fixed and let $X = \{(i, j) \mid i, j = 1, 2, 3, ..., n\}, \mathcal{S} = \{\varnothing\}$, $\sigma(i, j) = \{(i, t), (t, j) \mid t = 1, 2, 3, ..., n\}$ and $\tau((i, j), (s, t)) = 1$ for all $(i, j), (s, t) \in X$. Then $T = \{(i, i) \mid i = 1, 2, 3, ..., n\}$ and $D = \emptyset$. In this case, the convolution ring $C(A)$ is isomorphic to $M_n(A)$, the complete $n \times n$ matrix ring over A. As is well-known, if R is a ring with an identity, then every ideal of $M_n(R)$ is homogeneous. From Proposition 7 we then know that any radical α is homogeneous. The only outstanding issues regarding the radical theory of finite matrix rings is thus the validity of the following two implications:

(i) $A \in \alpha \Rightarrow M_n(A) \in \alpha$ or, equivalently, $M_n(A) \in \mathcal{S}\alpha \Rightarrow A \in \mathcal{S}\alpha$ and

(ii) $M_n(A) \in \alpha \Rightarrow A \in \alpha$ or, equivalently, $A \in \mathcal{S}\alpha \Rightarrow M_n(A) \in \mathcal{S}\alpha$.

The invariance of a radical α is thus equivalent to $(A \in \alpha \Leftrightarrow M_n(A) \in \alpha)$. In case a radical does satisfy this property, it is said to be *matrix extensible*. It is known that most of the well-known radicals are invariant. There is, however, one notable exception: For α the nilradical, it is known that $M_n(A) \in \alpha \Rightarrow A \in \alpha$, but the validity of the converse is equivalent to the well-known Köthe Conjecture (which is still open).

For infinite matrix rings, there are only a few limited results for which Patterson [8] and Sands [11] can be consulted.

3.6. Structural matrix rings. Let J be a non-empty set and let ρ be a non-empty reflexive and transitive relation on J such that the set $\{z \in J \mid (x, z) \in \rho$ and $(z, y) \in \rho\}$ is finite. Put $X = J \times J, \mathcal{S} = \{X \setminus \rho\}, \sigma(i, j) = \{((i, t), (t, j)) \mid t \in J\}$ and $\tau((i, j), (s, t)) = 1$ for all $(i, j), (s, t) \in X$. It can be shown that $T = \{(a, a) \mid a \in J\}$ and $D = \emptyset$. The convolution ring for this convolution type gives the structural matrix ring $M_J(A, \rho)$ over the ring A. We restrict our attention here to the finite case, i.e. we take $J = \{1, 2, 3, ..., n\}$ and denote the corresponding structural matrix ring by $M_n(A, \rho)$. For the radical theory of structural matrix rings, van Wyk [17], Sands [12] or Veldsman [18] can be consulted. Since X is finite, every radical α is weakly homogeneous. The radicals of these types of rings have been determined successfully, albeit in most cases only for radicals α which are invariant with respect to the finite matrix convolution type (i.e. radicals which are matrix extensible).

For a hypernilpotent radical α (i.e. all nilpotent rings are radical) which is matrix extensible, $\alpha(M_n(A, \rho)) = M_n(\alpha(A), \rho_s) + M_n(A, \rho_a)$ where ρ_s is the symmetric part and ρ_a the anti-symmetric part of ρ (cf. [12]). Thus $\alpha^*(A) = (M_n(\alpha(A), \rho_s) + M_n(A, \rho_a)) \cap A = \alpha(A)$ (remember, our canonical embedding of A into $M_n(A, \rho)$ is the mapping which assigns to every $a \in A$, the structural matrix which has a in every position $(i, i), i \in J$, and 0 elsewhere). Hence $M_n(\alpha(M_n(A, \rho)) \cap A, \rho) = M_n(\alpha(A), \rho)$ which means α is invariant if and only if α is homogeneous (this is still for α hypernilpotent with the matrix extension property). And this will be the case if and only if $\rho_a = \emptyset$. Indeed, if $\rho_a = \emptyset$, then $M_n(A, \rho) = M_n(A, \rho_s)$ is a finite direct sum of complete matrix rings $\oplus_{n_t} M_{n_t}(A)$. Then $\alpha(M_n(A, \rho)) = \alpha(\oplus_{n_t} M_{n_t}(A)) = \oplus_{n_t} \alpha(M_{n_t}(A)) = \oplus_{n_t} M_{n_t}(\alpha(A)) = M_n(\alpha(A), \rho)$. Conversely, suppose

$$M_n(\alpha(A), \rho) = \alpha(M_n(A, \rho)) = M_n(\alpha(A), \rho_s) + M_n(A, \rho_a).$$

If $(i, j) \in \rho_a$, then $M_n(\alpha(A), \rho)$ has an element from $\alpha(A)$ in position (i, j), while the right hand side can have any element from A in position (i, j). Choosing $0 \neq A \in \mathcal{S}\alpha$ then leads to a contradiction which means $\rho_a = \emptyset$.

Next we let α be a subidempotent radical (i.e. α is hereditary and all nilpotent rings are semisimple) which is matrix extensible. From [18] we know that $\alpha(M_n(A, \rho)) = M_n(\alpha(A), (\rho^*)_s)$ where $(\rho^*)_s = \{(i, j) \in \rho \mid \text{for } t \in \{i, j\}$ and $k \in \{1, 2, 3, ..., n\}, (t, k) \in \rho \Leftrightarrow (k, t) \in \rho\}$. If $(\rho^*)_s = \emptyset$, then $\alpha(M_n(A, \rho)) = 0$ for all rings A and α is homogeneous for such a convolution type. Suppose thus $(\rho^*)_s \neq \emptyset$. Then α is homogeneous if and only if $\rho = (\rho^*)_s$. Indeed, if $\rho = (\rho^*)_s$ then the homogeneity follows as in the above case. Conversely, suppose $(i, j) \in \rho \setminus (\rho^*)_s$. By the assumption we have $M_n(\alpha(A), (\rho^*)_s) = \alpha(M_n(A, \rho)) = M_n(\alpha(M_n(A, \rho)) \cap A, \rho)$. The $(i, j)-th$ entry on the left hand side is 0, while on the right hand side it is from $\alpha(M_n(A, \rho)) \cap A$. Thus $M_n(\alpha(A), (\rho^*)_s) = \alpha(M_n(A, \rho)) \cap A = 0$. Since $(\rho^*)_s \neq \emptyset$, this means $\alpha(A) = 0$ which is not necessarily true for all rings A. Thus $\rho = (\rho^*)_s$.

3.7. Incidence algebras. Let $(J, \leq)$ be a locally finite partially ordered set (i.e., each interval $[x, y] = \{z \in J \mid x \leq z \leq y\}$ is finite). Let $X = \{(x, y) \mid x, y \in J, x \leq y\}$, $\mathcal{S} = \{\varnothing\}$, $\sigma(x, y) = \{((x, z), (z, y)) \mid x, y, z \in J \text{ with } x \leq z \leq y\}$ and $\tau((x, y), (s, t)) = 1$ for all $(x, y), (s, t) \in X$. Here $T = \{(x, x) \mid x \in J\}, D = \emptyset$ and $C(A) = I_J(A)$, the incidence algebra over A. For more information on incidence algebras, see [15] and for their radicals [19].

For any radical α, $\alpha(I_J(A)) \subseteq (\alpha(A) : P)_{I_J(A)}$ where $P(f) := \{f(x, x) \mid x \in J\}$. From this it follows that $I_J(A) \in \alpha \Rightarrow A \in \alpha$ and thus $\alpha^*(A) \subseteq \alpha(A)$ for all rings A. The strongest results are for $\alpha = \mathcal{J}$, the Jacobson radical. For any ring A, $\mathcal{J}$ is directly accessible since $\mathcal{J}(I_J(A)) = (\mathcal{J}(A) : P)_{I_J(A)}$ and thus $A \in \mathcal{J} \Leftrightarrow I_J(A) \in \mathcal{J}$, $\mathcal{J}^c(A) = \overline{\mathcal{J}}(A) = \mathcal{J}^*(A) = \mathcal{J}(A)$ for all rings A. Moreover, $\mathcal{J}$ is weakly homogeneous. Since incidence algebras can be regarded as infinite structural matrix rings, the next statement does not come as a surprise, namely, $\mathcal{J}$ is homogeneous if and only if $x = y$ for all $(x, y) \in X$ if and only if $I_J(A) \cong A^J$. Indeed, if $\mathcal{J}$ is homogeneous, then $(\mathcal{J}(A) : P)_{I_J(A)} = \mathcal{J}(I_J(A)) = I_J(\mathcal{J}(I_J(A)) \cap A) = I_J(\mathcal{J}(A))$ for all rings A. Choose $(x_0, y_0) \in X$ with $x_0 \neq y_0$. For any $a \in A$, define $f : X \to A$ by $f(x,y) = \begin{cases} a \text{ if } (x,y) = (x_0, y_0) \\ 0 \text{ otherwise} \end{cases}$. Then $f \in (\mathcal{J}(A) : P)_{I_J(A)} = I_J(\mathcal{J}(A))$ which means $a \in \mathcal{J}(A)$. But $A = \mathcal{J}(A)$ does not hold for all rings A; hence $x = y$ for all $x, y \in J$. The other implications are straightforward (just remember, the Jacobson radical is invariant with respect to arbitrary direct products).

3.8. Splitting extensions. Let $(G, \cdot)$ be the cyclic group with four elements $\{e, a, a^2, a^3\}$. Let $d \in \{1, -1\}$ be fixed, $X = \{e, a\}, \mathcal{S} = \{\varnothing\}$, $\sigma(x) = \{(s,t) \mid s, t \in X, st = x \text{ or } st = a^2 x\}$ and $\tau(x,y) = \begin{cases} 1 \text{ if } xy \in X \\ d \text{ if } xy \notin X \end{cases}$ for all $x, y \in X$. Then $T = \{e\}$. For a ring A, this convolution type gives a splitting extension of A. Such rings and their radicals have been considered in [21]. We identify an element $f \in C(A)$ with the ordered pair $f = (f_1, f_2) = (f(e), f(a))$. This means the product of two elements f, g of $C(A)$ is given by $fg = (f_1, f_2)(g_1, g_2) = (f_1 g_1 + d f_2 g_2, f_1 g_2 + f_2 g_1)$.

Let $P(f) := \{f_1 + df_2, f_1 - df_2\} = \{f_1 + f_2, f_1 - f_2\}$. For a hypernilpotent radical α, $\alpha(C(A)) = (\alpha(A) : P)_{C(A)}$ for all rings A if and only if α satisfies:

(i) $R \in \alpha \Rightarrow C(R) \in \alpha$

(ii) $\alpha(C(R)) \cap R \in \alpha$ for all rings R.

The Jacobson radical $\mathcal{J}$ satisfies these two conditions, so we have $\mathcal{J}(C(A)) = (\mathcal{J}(A) : P)_{C(A)}, A \in \mathcal{J} \Leftrightarrow C(A) \in \mathcal{J}$, $\mathcal{J}$ is homogeneous and $\mathcal{J}^c(A) = \overline{\mathcal{J}}(A) = \mathcal{J}^*(A) = \mathcal{J}(A)$ for all rings A. But $\mathcal{J}$ need not be invariant: Let $A = \mathbb{Z}_4$, the ring of integers $mod 4$. If $\mathcal{J}(C(A)) = C(\mathcal{J}(A))$, then we have $(3,1) \in (\mathcal{J}(A) : P)_{C(A)} = \mathcal{J}(C(A)) = C(\mathcal{J}(A))$. But this is not possible since $3 \notin \{0, 2\} = \mathcal{J}(A)$.

For $d = 1$, $\alpha(C(A)) \subseteq (\alpha(A) : P)_{C(A)}$ holds for all rings A and all radicals α. If α is supernilpotent, then we have equality. For hypoidempotent radicals α (i.e. all nilpotent rings are semisimple), $\alpha(C(A)) \subseteq C(\alpha(A))$ for all rings A with equality if and only if $R \in \alpha \Rightarrow C(R) \in \alpha$. If α is subidempotent, then $C(R) \in \alpha \Leftrightarrow (R \in \alpha$ and $(0 : P)_{CR)} = 0)$. This means, for subidempotent radicals α, $\alpha(C(A)) = C(\alpha(A)) \Leftrightarrow \alpha(A) \cap (0 : P)_{C(R)} = 0$.

References

[1] Amitsur S.A. *Radicals of polynomial rings.* Canad. J. Math., 1956, **8**, p. 335–361.

[2] Amitsur S.A. *Nil radicals; Historical notes and some new results.* Colloq. Math. Soc. Janos Bolyai 6: Rings, Modules and Radicals, Keszthely (Hungary), 1971, p. 46–65.

[3] Divinsky N.J. *Rings and Radicals.* George Allen and Unwin Ltd., London, 1965.

[4] Krempa J. *On radical properties of polynomial rings.* Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys., 1972, **20**, p. 545–548.

[5] Krempa J. *Radicals of semi-group rings.* Fund. Math., 1974, **85**, p. 57–71.

[6] Ortiz A. *A construction in general radical theory.* Canad. J. Math., 1970, **22**, p. 1097–1100.

[7] Parmenter M.M., Spiegel E. *Algebraic properties of necklace rings.* Comm. Algebra., 1998, **26**, p. 1625–1632.

[8] Patterson E.M. *On the radicals of rings of row-finite matrices.* Proc. Royal Soc. Edinburgh., 1962, **66**, p. 42–46.

[9] Puczyłowski E.R. *Radicals of polynomial rings power series rings and tensor products.* Comm. Algebra., 1980, **8**, p. 1699–1709.

[10] Puczyłowski E.R. *Radicals of polynomial rings in non-commuting indeterminates.* Math. Z., 1983, **182**, p. 63–67.

[11] Sands A.D. *On radicals of infinite matrix rings.* Proc. Edinburgh Math. Soc., 1969, **16**, p. 195–203.

[12] Sands A.D. *Radicals of structural matrix rings.* Quaest. Math., 1990, **13**, p. 77–81.

[13] Sierpińska A. *Radicals of rings of polynomials in non-commutative indeterminates.* Bull. Acad. Pol. Sci., 1973, **21**, p. 805–808.

[14] Smoktunowich A. *Polynomial rings over nil rings need not be nil.* J. Algebra., 2000, **233**, p. 427–436.

[15] Spiegel E., O'Donell C.J. *Incidence Algebras.* Marcel Dekker Inc., New York, 1997.

[16] Tumurbat S., Wiegandt R. *Radicals of polynomial rings.* Manuscript.

[17] Van Wyk L. *Special radicals in structural matrix rings.* Comm. Algebra., 1988, **16**, p. 421–435.

[18] Veldsman S. *On the radicals of structural matrix rings.* Monat. Math., 1996, **122**, p. 227–238.

[19] Veldsman S. *The general radical theory of incidence algebras.* Comm. Algebra., 1999, **27**, p. 3659–3673.

[20] Veldsman S. *Necklace rings and their radicals.* Math. Pannonica., 2001, **12**, p. 269–289.

[21] Veldsman S. *Radicals and splitting extensions of rings.* Quaest. Math., 2000, **23**, p. 21–36.

[22] Veldsman S. *Convolution Rings.* Submitted.

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Artinal special Lie superalgebras

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Abstract. The artinian special Lie superalgebras are studied in the paper. It is proved, that the gr-prime radical of a artinian special Lie superalgebra is solvable.

Mathematics subject classification: 17B05.
Keywords and phrases: Lie superalgebra, (gr)-prime radical, artinian algebra.

All algebras are supposed to be algebras over a field F.

In 1963 V. Latyshev defined special Lie algebra [1].

We say that an algebra L is called a special Lie algebra or SPI-algebra, if there exists an associative PI-algebra A such that L is included in $A^{(-)}$ as a Lie algebra, where $A^{(-)}$ is a Lie algebra with respect to the operation of commutation $[x, y] = xy - yx$.

The structural theory of special Lie algebras was studied in [2-8] and others.

Let L be a Lie algebra, $a \in L$. By ad_a we shall denote the linear transformation $\mathrm{ad}_a : L \longrightarrow L$, defined by the formula $(x)\mathrm{ad}_a = [x, a]$. We shall denote by $\mathrm{Ad}(L)$ the associative algebra generated in $\mathrm{End}(L)$ by the set $\{\mathrm{ad}_a \mid a \in L\}$.

We say that the algebra is prime if the following assertion holds for every its ideals U, V: if $UV = 0$ then either $U = 0$ or $V = 0$. The definition is given similarly for associative and Lie algebras.

We say that the ideal P of an algebra L is prime if the factor-algebra L/P is prime.

We define the prime radical $P(L)$ as the intersection of all prime ideals of a Lie algebra L.

It was proved in [6] that the prime radical of a special Lie algebra is locally soluble.

As it is impossible to construct a good structural theory for all special Lie algebras, it is necessary to investigate classes of special Lie algebras, for which such a theory exists. For associative algebras there is a good theory for artinian algebras.

By analogy to associative algebras we say that a Lie algebra is artinian if every non-empty descending chain of its ideals is stabilized.

We remark that unlike associative algebras, for which right or left ideals are considered, for Lie algebras there is no necessity to speak about right artinian or left artinian algebras.

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The following theorem was obtained in [9].

Theorem 1. *Let L be an artinian special Lie algebra and $P(L)$ be its prime radical. Then the ideal $P(L)$ is soluble.*

This theorem is true also for Lie superalgebras.

Lie superalgebra L over a field F is a $\mathbb{Z}_2$-graded vector space $L = L_0 \oplus L_1$ over the field F on which the bilinear operation $[x, y]$ is defined and for homogeneous components the following identities are valid

$$\alpha([x, y]) = \alpha(x) + \alpha(y),$$
$$[x, y] = (-1)^{\alpha(x)\alpha(y)+1}[y, x],$$
$$[x, [y, z]](-1)^{\alpha(x)\alpha(z)} + [z, [x, y]](-1)^{\alpha(-z)\alpha(y)} + [y, [z, x]](-1)^{\alpha(y)\alpha(x)} = 0,$$

where $\alpha(x)$ is the number of the homogeneous component [10].

We call a $\mathbb{Z}_2$-graded associative algebra an associative superalgebra.

The ideal I of an associative superalgebra or Lie superalgebra is called graded if $I = I_0 \oplus I_1$, where $I_i = I \cap A_i, i = 0, 1$. The factor-algebra by a graded two-sided ideal is an associative superalgebra or Lie superalgebra respectively.

We shall use also the concept of graded modulus over associative superalgebra or Lie superalgebra.

We say that an associative superalgebra A is a PI-superalgebra if it satisfies to polynomial identity as an algebra without graduation. It is known that a sufficient condition of the fulfilment of identity in the graded by finite group associative algebra is the fulfilment of identity in a unit component of algebra [11]. In [12] the estimate of the degree of such identity was given.

We call a Lie superalgebra L over a field F special if there exists an associative PI-superslgebra A such that $L \subseteq [A]$, where $[A]$ is the algebra A in respect to operation of the commutation, defined on homogeneous components by the formula

$$[x, y] = xy - (-1)^{\alpha(x)\alpha(y)}yx,$$

where $\alpha(x)$ is the number of the homogeneous component. The associative superalgebra A with respect to the relation to this operation of commutation a is Lie superalgebra $[A]$. The concept of special color Lie superalgebras was studied in [13].

Let L be a Lie superalgebra, $a \in L$. By ad_a we shall denote the linear transformation $\mathrm{ad}_a : L \longrightarrow L$, defined by the formula $x\mathrm{ad}_a = [x, a]$. We shall denote by $\mathrm{Ad}(L)$ the associative algebra generated in $\mathrm{End}(L)$ by set $\{\mathrm{ad}_a \mid a \in L\}$.

The algebra $\mathrm{Ad}(L)$ is an associative superalgebra.

The definition of the solubility is given for Lie superalgebras in the same way as for Lie algebras.

We shall define for Lie superalgebras the concept of a prime algebra and a prime graded ideal in the same way as for associative algebras.

We shall call a gr-prime radical of special Lie superalgebra the intersection of all it gr-prime graded ideals.

For special Lie superalgebras the following result is proved.

Theorem 2. *Let L be an artinian special Lie superalgebra and $P_{gr}(L)$ be its its gr-prime radical. Then the ideal $P_{gr}(L)$ is soluble.*

Proof. We shall consider the sequence of commutators

$$P = P_{gr}(L),\ P^{(1)} = P,\ P^{(2)} = [P^{(1)},\ P^{(1)}], \dots, P^{(k+1)} = [P^{(k)},\ P^{(k)}] \dots$$

Then the inclusions take place $P^{(1)} \supseteq P^{(2)} \supseteq, \dots, \supseteq P^{(k)} \supseteq, \dots$ It is well known [14], that all commutators $P^{(k)}$ are quite characteristic subalgebras and, hence, are ideals of superalgebra L.

As the Lie superalgebra L is artinian then $R^{(m)} = R^{(m+1)}$ for some natural m. We want to prove, that $R^{(k)} = 0$ for some k.

Let d be the degree of polynomial identity in algebra $\mathrm{Ad}(L)$, which exists according to [15].

Let $n = \max(m, [d/2]^2)$.

We shall denote by W the centrelizer of $P^{(n)}$, i.e.

$$W = \{x \mid x \in L, [x, P^{(n)}] = 0\}.$$

In the book [16] it was proved that the centre of an associative superalgebra A graded by an abelian group is graded. It is possible to prove the same is true for the center and the centrelizer of a graded ideal of a Lie superalgebra.

The set W is the graded ideal.

If $W \supseteq P^{(n)}$ then $P^{(n+1)} = 0$. The theorem is proved.

Let's assume that W does not contain $P^{(n)}$. Then we will obtain the contradiction.

We shall consider the factor-superalgebra $\bar{L} = L/W$. By the natural homomorphism $L \to \bar{L}$ the ideal $P^{(n)}$ is mapped to $\bar{P}^{(n)}$. From the assumption it follows that $\bar{P}^{(n)} \neq 0$. We shall obtain from the artinian property of of superalgebra $\bar{L}$ that $\bar{P}^{(n)}$ contains the minimal ideal $\bar{\rho}$.

Then either $[\bar{\rho}, \bar{P}] = 0$, or the ideal $\bar{\rho}$ is irreducible as a module. In the latter case the algebra $\bar{L}$ generates the primitive graded associative superalgebra B in a ring $End(\bar{\rho})$, which is a homomorphic image of superalgebra $\mathrm{Ad}(L)$. According to the graded analogue of the theorem of Kaplansky [17] the algebra B is central simple, finite dimensional over the graded center Z, which dimension is not higher than $[d/2]^2$. Hence, the dimension of algebra $\bar{P}$ over Z is not higher than n. An ideal $\bar{P}$ is locally soluble. Then the image of an ideal $\bar{P}^{(n)}$ in the ring of endomorphisms $End(\bar{\rho})$ is equal to zero. Hence, $[\bar{\rho}, \bar{P}^{(n)}] = 0$.

Passing to inverse images in algebra L we shall obtain $[\rho, P^{(n)}] \subseteq W$. Hence, $[[\rho, P^{(n)}], P^{(n)}] = 0$. Then $[\rho, P^{(n+1)}] = 0$. From the equality $P^{(n)} = P^{(n+1)}$ the equality $[\rho, P^{(n)}] = 0$ follows. Hence $\rho \subseteq W$, that contradicts the definition of ρ. The obtained contradiction proves the theorem.

References

[1] Latyshev V. *On Lie algebras with a polynomial identical.* Siberian Math. J., 1963, **4**. no. 4, p. 821–829.

[2] Razmyslov Yu. *On Jacobson radical in PI-algebras.* Algebra i logica, 1974, **13**. no. 3, p. 337–360.

[3] Bahturin Yu. *On Lie subalgebras of associiative PI-algebras.* J. Algebra, 1980, **67**, no. 2, p. 257–271.

[4] Bahturin Yu. *On artinian special Lie algebras.* Algebra. Moscow, MSU, 1982, p. 24–26.

[5] Pikhtilkov S. *On special Lie algebras.* Uspekhi math. nauk, 1981, **36**, no. 6, p. 225–226.

[6] Beidar K., Pikhtilkov S. *On a prime radical of special Lie algebras.* Uspekhi math. nauk, 1994, no. 1, p. 233.

[7] Beidar K., Pikhtilkov S. *A ptime radical of special Lie algebras.* Fundamental and Applied Mathematics, 2000, **6**, no. 3, p. 643–648.

[8] Pikhtilkov S.A. *Locally Nilpotent Ideals of Special Lie Algebras.* Comm. in Algebra, 2001, **29**, no. 10, p. 3781–3786.

[9] Pikhtilkov S. *Artinian special Lie algebras.* Algorithmic problems of the theory of groups and Semigroups. Tula, TSPU, 2001, p. 189–194.

[10] Berezin F. *Introducing in algebra and analysis with anticommutative variables.* Moskov, MSU, 1983.

[11] Bergen J., Cohen M. *Action of Commutative Hopf Algebras.* Bull. of Lond. Math. Soc., 1986, **18**, p. 159–164.

[12] Bahturin Y., Giambruno A. Riley D. *Group-graded algebras with polynomial identity.* Isr. J. Math., 1998, **104**, p. 145–155.

[13] Bahturin Y., Montgomery S. *PI-envelopes of Lie syperalgebras.* Proc. of the American Math. Soc., 1999, **127**, no. 10, p. 2829–2939.

[14] Burbaki N. *Lie groups and algebras (chapter I-III).* Moskov, Mir, 1976.

[15] Balaba I., Pikhtilkov S. *A prime radical of special Lie superalgebras.* Fundamental and applied Mathematics, 2003, **9**, no. 1, p. 51–60.

[16] Nastasescu C., Van Oystayen F. *Graded ring theory.* Amsterdam, North-Holland, 1982.

[17] Zhu. *Bin Graded primitive rings and Kaplansky's theorem.* Beijing Shifan Daxue Xuebao, 1988, **34**, no. 1, p. 1–5.

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The classification of $GL(2,R)$-orbits' dimensions for system $s(0,2)$ and the factorsystem $s(0,1,2)/GL(2,R)$ *

E.V. Starus

Abstract. Two-dimensional systems of two autonomous polynomial differential equations with homogeneities of the zero, first and second orders are considered with respect to the group of center-affine transformations $GL(2,R)$. The problem of the classification of $GL(2,R)$-orbits' dimensions is solved completely for system $s(0,2)$ with the help of Lie algebra of operators corresponding to $GL(2,R)$ group, and algebras of invariants and comitants. A factorsystem $s(0,1,2)/GL(2,R)$ for system $s(0,1,2)$ is built and with its help two invariant $GL(2,R)$-integrals are obtained for the system $s(1,2)$ in some necessary conditions for the existence of singular point of the type "center".

Mathematics subject classification: 34C14, 34C05.
Keywords and phrases: Differential system, $GL(2,R)$-orbit, factorsystem, invariant integral.

Consider the real system of differential equations

$$\frac{dx^j}{dt} = a^j + a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta, \quad (j,\alpha,\beta = 1,2), \tag{1}$$

which will be denoted by $s(0,1,2)$, where the coefficient tensor $a^j_{\alpha\beta}$ is symmetrical in lower indexes, in which the complete convolution takes place, and the group of center-affine transformations $GL(2,R)$, given by the equalities $\bar{x}^r = q^r_j x^j$, $\Delta_q = = det(q^r_j) \neq 0, (r,j = 1,2)$.

Consider the invariants and comitants of the system (1) with respect to the group $GL(2,R)$, found in [1], which will be used further:

$$K_1 = a^\alpha_{\alpha\beta} x^\beta,\ K_2 = a^p_\alpha x^\alpha x^q \varepsilon_{pq},\ K_5 = a^p_{\alpha\beta} x^\alpha x^\beta x^q \varepsilon_{pq},\ K_6 = a^\alpha_{\alpha\beta} a^\beta_{\gamma\delta} x^\gamma x^\delta,$$

$$K_7 = a^\alpha_{\beta\gamma} a^\beta_{\alpha\delta} x^\gamma x^\delta,\ K_9 = a^\alpha_{p\alpha} a^\beta_{q\gamma} a^\gamma_{\beta\delta} x^\delta \varepsilon^{pq},\ K_{21} = a^p x^q \varepsilon_{pq},\ K_{23} = a^p a^q_{\alpha\beta} x^\alpha x^\beta \varepsilon_{pq},$$

$$K_{25} = a^\alpha a^\beta a^p_{\alpha\beta} x^q \varepsilon_{pq},\ I_1 = a^\alpha_\alpha,\ I_2 = a^\alpha_\beta a^\beta_\alpha,\ I_4 = a^\alpha_p a^\beta_{\beta q} a^\gamma_{\alpha\gamma} \varepsilon^{pq},\ I_5 = a^\alpha_p a^\beta_{\gamma q} a^\gamma_{\alpha\beta} \varepsilon^{pq},$$

$$I_6 = a^\alpha_p a^\beta_\gamma a^\gamma_{\alpha q} a^\delta_{\beta\delta} \varepsilon^{pq},\ I_7 = a^\alpha_{pr} a^\beta_{q\alpha} a^\gamma_{s\beta} a^\delta_{\gamma\delta} \varepsilon^{pq} \varepsilon^{rs},\ I_8 = a^\alpha_{pr} a^\beta_{q\alpha} a^\gamma_{s\delta} a^\delta_{\beta\gamma} \varepsilon^{pq} \varepsilon^{rs},$$

$$I_9 = a^\alpha_{pr} a^\beta_{q\beta} a^\gamma_{s\gamma} a^\delta_{\alpha\delta} \varepsilon^{pq} \varepsilon^{rs},\ I_{13} = a^\alpha_p a^\beta_{qr} a^\gamma_{\gamma s} a^\delta_{\alpha\beta} a^\mu_{\delta\mu} \varepsilon^{pq} \varepsilon^{rs},$$

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$$I_{15} = a^{\alpha}_{pr} a^{\beta}_{qk} a^{\gamma}_{\alpha s} a^{\delta}_{\delta l} a^{\mu}_{\beta\gamma} a^{\nu}_{\mu\nu} \varepsilon^{pq} \varepsilon^{rs} \varepsilon^{kl},\ I_{17} = a^{\alpha} a^{\beta}_{\alpha\beta},\ I_{25} = a^{\alpha} a^{\beta}_{\beta p} a^{\gamma}_{\delta q} a^{\delta}_{\alpha\gamma} \varepsilon^{pq}. \quad (2)$$

where ε^{pq} and ε_{pq} are unit bivectors ($\varepsilon^{11} = \varepsilon^{22} = 0,\ \varepsilon^{12} = -\varepsilon^{21} = 1,\ \varepsilon_{11} = \varepsilon_{22} = 0,$ $\varepsilon_{12} = -\varepsilon_{21} = 1$).

Remark 1. For $I_1 = 0, K_2 \equiv 0$ the system (1) takes the form (it will be denoted by s(0,2) further)

$$\frac{dx^j}{dt} = a^j + a^j_{\alpha\beta} x^\alpha x^\beta, \quad (j,\alpha,\beta = 1,2). \quad (3)$$

I. The proof of the next theorem is based on the classification of $GL(2,R)$-orbits' dimensions for system $s(2)$ from [2]:

Theorem 1. *If $I_1 = 0, K_2 \equiv 0$, the $GL(2,R)$-orbit of the system (3) has the dimension*

4 for $K_1K_5 \not\equiv 0,\ F_1 + K_9 + \beta \not\equiv 0$, *or*
$K_5 \not\equiv 0,\ K_1 \equiv 0,\ F_2 + K_9 + \beta \not\equiv 0$;
3 for $K_1K_5 \not\equiv 0,\ F_1 + K_9 + \beta \equiv 0$, *or*
$K_5 \not\equiv 0,\ K_1 \equiv 0,\ F_2 + K_9 + \beta \equiv 0,\ K_7 + K_{21} \not\equiv 0$, *or*
$K_5 \equiv 0,\ K_1K_{21} \not\equiv 0$;
2 for $K_{21} \equiv 0, K_1 + K_5 \not\equiv 0, K_5(K_1 + K_7) \equiv 0$, *or*
$K_5 \equiv 0, K_1K_{21} \equiv 0, K_1^2 + K_{21}^2 \not\equiv 0$;
0 for $K_1 \equiv K_5 \equiv K_{21} \equiv 0$,
where $\beta = 27I_8 - I_9 - 18I_7$, $F_1 = K_5[-2I_{17}K_5 + K_1(2K_1K_{21} - 3K_{23})]$,
$F_2 = K_{21}^2(3K_1^2 - 2K_6 - 3K_7) + 2K_5K_{25}$, *and* K_1, K_5, K_6, K_7, K_9, K_{21}, K_{23}, I_7, I_8, I_9, I_{17} *are taken from (2).*

For the system $s(0,3)$ the similar problem was considered in [3]. Remark that in (51) only the sets M_1, $M_4 - M_6$, $M_8 - M_{13}$ should be considered as $GL(2,R)$-invariant nonintersecting sets.

II. According to [4] the classification of $GL(2,R)$-orbits' dimensions could be considered as a division of the set $E^{14}(x,a)$ of the coefficients and variables of the system (1) into invariant manifolds, and the maximal dimension orbit is a nonsingular invariant manifold of the $GL(2,R)$ group.

Remark 2. The condition $K_1K_5K_9 \not\equiv 0$ follows from the condition $I_9(I_9 - I_7) \neq 0$, both of them define nonsingular invariant manifolds (see definition in [4]).

The proof is based on the facts that $Rez(K_1, K_5) = I_9$ and $Rez(K_1, K_9) = I_9 - I_7$.

Theorem 2. *On the nonsingular invariant manifold $I_9(I_9 - I_7) \neq 0$ the system (1) has the following factorsystem (see [4]) $s(0,1,2)/GL(2,R)$*

$$\dot{\bar{x}} = I_{17} + \left[\frac{1}{2}I_1 + \frac{-I_1I_7 - 2I_{13}}{2I_9} - \frac{I_4I_{15}}{I_9(I_9 - I_7)}\right]\bar{x} - \frac{I_4}{|I_9 - I_7|^{1/2}}\bar{y} +$$

$$+\left[\frac{I_7+I_9}{2I_9}+\frac{I_{15}^2}{I_9(I_9-I_7)^2}\right]\bar{x}^2+2\frac{I_{15}}{|I_9-I_7|^{3/2}}\bar{x}\bar{y}+\frac{I_9}{(I_9-I_7)}\bar{y}^2,$$

$$\dot{\bar{y}}=\frac{I_{25}}{|I_9-I_7|^{1/2}}+\frac{1}{|I_9-I_7|^{1/2}}\left[\frac{I_4I_{15}^2}{I_9^2|I_9-I_7|^2}-\frac{I_4(I_7^2+I_9^2)}{2I_9^2}+I_5\right]\bar{x}+$$

$$+\left[\frac{1}{2}I_1+\frac{I_1I_7+2I_{13}}{2I_9}+\frac{I_4I_{15}}{I_9(I_9-I_7)}\right]\bar{y}-\frac{I_{15}(I_7+I_9)}{2I_9^2|I_9-I_7|^{1/2}}\bar{x}^2-$$

$$-\frac{I_{15}^3}{I_9^2|I_9-I_7|^{3/2}}\bar{x}^2+2\left[\frac{I_9-I_7}{2I_9}-\frac{I_{15}^2}{I_9(I_9-I_7)^2}\right]\bar{x}\bar{y}-\frac{I_{15}}{|I_9-I_7|^{3/2}}\bar{y}^2, \tag{4}$$

for which $K_1=\bar{x}, K_9=\bar{y}$, and K_1, K_9, I_1, I_4, I_5, I_7, I_9, I_{13}, I_{15}, I_{17}, I_{25} are taken from (2).

III. Consider the center conditions from [5] for the system (1) with $a^j=0$ $(j=1,2)$:

$$I_2<0,\ I_1=I_6=I_{13}=0,\ I_4\neq 0. \tag{5}$$

Taking into account the last four conditions from (5) and $I_{17}=I_{25}=0$, and the syzygies from [6], we conclude that the factorsystem (4) will take the form

$$\dot{\bar{x}}=-\frac{I_4}{|I_9-I_7|^{1/2}}\bar{y}+\frac{I_7+I_9}{2I_9}\bar{x}^2+\frac{I_9}{I_9-I_7}\bar{y}^2,$$

$$\dot{\bar{y}}=\frac{1}{|I_9-I_7|^{1/2}}\left[I_9-\frac{I_4(I_7^2+I_9^2)}{2I_9^2}\right]\bar{x}+\frac{I_9-I_7}{I_9}\bar{x}\bar{y}, \tag{6}$$

for which $I_9(I_9-I_7)\neq 0$. We obtain with the help of (6)

Proposition 1. *The system (1) has the following two invariant $GL(2,R)$-integrals on the nonsingular invariant $GL(2,R)$-manifold $I_9(I_9-I_7)\neq 0$ for $I_{17}=I_{25}=0$ and for necessary center conditions $I_1=I_6=I_{13}=0,\ I_4\neq 0$*

$$\mathcal{F}_1\equiv 2I_5I_9^2-I_4(I_7^2+I_9^2)+2I_9(I_9-I_7)K_9=0,$$

$$\mathcal{F}_2\equiv I_7(I_9+I_7)[(I_9-I_7)^2(I_9-3I_7)K_1^2-2I_9^2K_9^2]+[I_5I_9^2+I_4I_7(-2I_9+I_7)]\cdot$$

$$\cdot[-2I_5I_9^2+I_4(I_7^2+I_9^2)-2I_9(I_9+I_7)K_9]=0.$$

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References

[1] Boularas D., Calin Iu.F., Timochouk L.A., Vulpe N.I. *T-comitants of quadratic systems: a study via the translation invariants.* Report 96-90 Delft, University of Technology, The Netherlands, 1996.

[2] Popa M.N. *Applications of algebras to differential systems.* Academy of Sciences of Moldova, Chisinau, 2001 (In Russian).

[3] Starus E.V. *Invariant conditions for the dimensions of the $GL(2, R)$-orbits for one differential cubic system.* Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, 2003, No. 3(43), p. 58–70.

[4] Ovsyannikov L.V. *Group analysis of the differential equations.* Moscow, Nauka, 1978 (In Russian, published in English in 1982).

[5] Sibirsky K.S., Lunkevichi V.A. *Integrals of common quadratic differential system in the centers' cases.* Differential Equations, 1982, **18**, no. 5, p. 786–792 (In Russian).

[6] Sibirsky K.S. *Introduction to the Algebraic Theory of Invariants of Differential Equations.* Kishinev, Shtiintsa, 1982 (In Russian, published in English in 1988).

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