# Choice Numbers of Multi-Bridge Graphs 

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#### Abstract

Suppose $\operatorname{ch}(G)$ and $\chi(G)$ denote, respectively, the choice number and the chromatic number of a graph $G=(V, E)$. If $\operatorname{ch}(G)=\chi(G)$, then $G$ is said to be chromatic-choosable. Here, we find the choice numbers of all multi-bridge or $l$-bridge graphs and classify those that are chromatic-choosable for all $l \geq 2$.


Keywords: List coloring, chromatic-choosable, l-bridge

## 1 Preliminaries

In this paper, $G=(V, E)$ denotes a simple connected graph, where $V=V(G)$ and $E=E(G)$ denote, respectively, the set of vertices and the set of edges of $G$. An edge $e \in E$ with endpoints $u, v \in V$ is denoted by $u v$. Also, we denote by $N(u)=N_{G}(u)=\{x \in V \mid u x \in E\}$ the (open) neighbor set in $G$ of $u \in V . \Delta=\Delta(G), K_{n}$ and $C_{n}$ denote, respectively, the maximum degree of $G$, a complete graph and a cycle on $n$ vertices. The join of two graphs $G_{1}$ and $G_{2}$, denoted by $G_{1} \vee G_{2}$, is the graph $G$ whose vertex set is $V(G)=V\left(G_{1}\right) \cup V\left(G_{2}\right)$, a disjoint union, and whose edge set is $E(G)=E\left(G_{1}\right) \cup E\left(G_{2}\right) \cup\left\{u_{1} u_{2} \mid u_{1} \in\right.$ $\left.V\left(G_{1}\right), u_{2} \in V\left(G_{2}\right)\right\}$. For other basic notions of graphs, see [15].

A list assignment to the graph $G=(V, E)$ is a function $L$ which assigns a finite set (list) $L(v)$ to each vertex $v \in V$. A proper $L$-coloring of $G$ is a function $\phi: V \rightarrow \cup_{v \in V} L(v)$ satisfying, for every $u, v \in V$, (i) $\phi(v) \in L(v)$ and (ii) $u v \in E \rightarrow \phi(v) \neq \phi(u)$.

The choice number of $G$, denoted by $\operatorname{ch}(G)$, is the smallest integer $k$ such that there is always a proper $L$-coloring of $G$ if $L$ satisfies $|L(v)| \geq$ $k$ for every $v \in V$. We define $G$ to be $k$-choosable if it admits a proper

[^0]$L$-coloring whenever $|L(v)| \geq k$ for all $v \in V$; so $\operatorname{ch}(G)$ is the smallest integer $k$ such that $G$ is $k$-choosable. The following theorem is useful in the estimation of choice number.

Theorem A. (Erdős, Rubin and Taylor [3) If $G$ is a connected graph that is neither a complete graph nor an odd cycle, then ch $(G) \leq$ $\Delta(G)$.

Corollary A. For any graph $G, \operatorname{ch}(G) \leq \Delta(G)+1$.
The proof of Corollary A follows from a "greedy coloring" argument.
Clearly, $\chi(G) \leq \operatorname{ch}(G)$ since the chromatic number $\chi(G)$ is similarly defined with the restriction that the list assignment is to be constant and there are many graphs whose choice number exceeds (sometimes greatly) their chromatic number. The two planar graphs in Figure 1 are some examples, where it is not too hard to see that, given the list assignment for each graph $G, \operatorname{ch}(G)=3>2=\chi(G)$.


Figure 1: Two graphs with two list assignments.

Any graph $G$ for which the extremal case $\chi(G)=\operatorname{ch}(G)$ holds is said to be chromatic-choosable. Cycles, cliques and trees are some examples of chromatic-choosable graphs.

Historically, the topic of list colorings is believed to be first introduced by Vizing [9] and independently by Erdős, Rubin and Taylor [3]. Ever since, many researchers (see for e.g., [1], [4]-7]) have sought to
classify chromatic-choosable graphs. It is worth noting that the problem of finding chromatic-choosable graphs contains the famous list coloring conjecture [9: the line graph of any graph is chromatic-choosable. In fact, this conjecture has been partially proved by Galvin [4] in

Theorem B.(Galvin (4) The line graph of any bipartite multigraph is chromatic-choosable.

Recently, Reed et al. [6] settled the well-known Ohba's conjecture [7]. We state their result (or Ohba's conjecture) without proof, in the next theorem.

Theorem C.(Noel, Reed and Wu [6]) If $|V(G)| \leq 2 \chi(G)+1$, then $G$ is chromatic-choosable.

Because the proposed bound is obviously weak in characterizing chromatic-choosable graphs with low chromatic numbers, we classify a class of acyclic graphs with low chromatic number $(\chi \leq 3)$ and arbitrarily large $\Delta$.

## 2 Choice number of some $l$-bridge graphs

The length of a path is the number of its edges and two paths are said to be internally disjoint if they have no common internal vertex.

An l-bridge (or multi-bridge) graph $\Theta\left(a_{1}, \ldots, a_{l}\right)$ is the graph obtained by connecting two distinct vertices $u$ and $v$ with $l$ internally disjoint paths $P_{a_{i}}$ of lengths $a_{i} \geq 1$. It is customary to assume $l \geq 3$ since when $l=2, \Theta\left(a_{1}, a_{2}\right)$ is a cycle on $a_{1}+a_{2}$ vertices; the trivial case when $l=1, \Theta\left(a_{1}\right) \cong u v$, an edge. $L$-bridge graphs are planar and when $l=3$, figure $1(\mathrm{~A})$ depicts an example of $\Theta(1,3,3)$. For the rest of this article, it causes no confusion to denote $P_{a_{i}}:=u v$ if some $a_{i}=1$, and $P_{a_{i}}:=u x_{i_{1}} x_{i_{2}} \ldots x_{i_{a_{i}-1}} v$, a sequence of edges for all $a_{i} \geq 2$.

Recall, the core of a connected graph is the graph obtained by deleting all vertices of degree 1 , and then all vertices of degree 1 in what remains, and so forth, until there are no vertices of degree 1 remaining; except that, in case of $K_{2}$, delete only one vertex. Erdős,

Rubin and Taylor have described the structure of 2-choosable graphs (which are necessarily bipartite) in the next theorem.

Theorem D.(Erdős, Rubin and Taylor [3]). A connected graph $G$ is 2-choosable if and only if the core of $G$ is $K_{1}$, an even cycle, or of the form $\Theta(2,2,2 t)$, where $t$ is a positive integer.

Not surprisingly, there has been no characterization of $k$-choosable graphs, $k \geq 3$. Alon and Tarsi [2] showed that every bipartite planar graph is 3 -choosable and there has been several attempts at characterizing triangle free planar graphs in order to strengthen Alon and Tarsi's result. See for instance, [8], [10-[14]. Clearly, since each graph $G$ in Figure 1 is bipartite and planar, it follows from Alon and Tarsi's result that $\operatorname{ch}(G)=3$, given the list assignment. It is important to point out that $l$-bridge graphs are not necessarily bipartite as they may contain odd cycles. Here, we show that they are 3 -choosable and later, we classify them based on their choice number.

Proposition 1. If $G=\Theta\left(a_{1}, \ldots, a_{l}\right)$, then $G$ is 3 -choosable.
Suppose $L$ is a list assignment to $G$ satisfying $|L(w)| \geq 3$ for each $w \in V(G)$. Because every path is 2-choosable, color properly the vertices (including $u, v$ ) of some path $P_{a_{i}}$. Suppose, in coloring $P_{a_{i}}$, $\phi(u)=c_{1}$ and $\phi(v)=c_{2}$, where $c_{1}$ and $c_{2}$ are not necessarily distinct colors. For each vertex $y \in V\left(G \backslash P_{a_{i}}\right)$, define $L^{\prime}(y)=L(y)-\left\{c_{1}, c_{2}\right\}$. If $\left|L^{\prime}(y)\right| \geq 2$ for each $y \in V\left(G \backslash P_{a_{i}}\right)$, color properly the vertices on each independent path $P_{a_{j}}-u v, j \neq i$. Or else, there exists a vertex $z \in V\left(G \backslash P_{a_{i}}\right)$ such that, for some $k \neq i,\left|L^{\prime}(z)\right| \geq 1$. This implies that $N(u)=z=N(v)$, i.e., $P_{a_{k}}:=u z v$. In this case, color $z$ with the color left in its palette, giving a proper $L$-coloring of $G$.

Theorem 1. Suppose $G=\Theta\left(a_{1}, \ldots, a_{l}\right)$ is any l-bridge graph with $l \geq 3$. ch $(G)=3$ if and only if $G$ is not $\Theta(2,2,2 t)$, for all $t \geq 1$.

Proof. Clearly if $G=\Theta(2,2,2 t)$, then it follows from Theorem D that $\operatorname{ch}(G)=2$. Now, if $G$ is an $l$-bridge that contains an odd cycle, then the result follows from Proposition 1 Thus, to complete the proof,

## Choice Numbers of Multi-Bridge Graphs

we can assume that $G$ is neither $\Theta(2,2,2 t)$ nor contains an odd cycle and show that $\operatorname{ch}(G)>2$. In each upcoming claim we present a list assignment which is left up to the reader to verify in order to establish the result.

Claim A. If $G_{1}=\Theta\left(a_{1}, \ldots, a_{l}\right)$ and each $a_{i}$ is odd, then $G_{1}$ is not 2-choosable, for all $l \geq 3$.

Let $H=\Theta\left(a_{1}, a_{2}, a_{3}\right)$ such that each $a_{i}$ is odd, for $i=1,2,3$. Clearly $H$ contains no odd cycle. Define a list assignment $L_{1}$ satisfying, for each $w \in V(H)$
(i) $L_{1}(u)=L_{1}(v)=L_{1}\left(x_{1_{j}}\right)=\{a, b\}$ for $1 \leq j \leq a_{1}-1$
(ii) $L_{1}\left(x_{2_{1}}\right)=\ldots=L_{1}\left(x_{2_{a_{2}-2}}\right)=L_{1}\left(x_{3_{a_{3}-1}}\right)=\{a, c\}$
(iii) $L_{1}\left(x_{3_{1}}\right)=\ldots=L_{1}\left(x_{3_{a_{3}-2}}\right)=L_{1}\left(x_{2_{a_{2}-1}}\right)=\{b, c\}$.

It is easy to see that every proper $L_{1}$-coloring of $P_{a_{1}}$ will require distinct colors $a, b$ for the vertices $u, v$, forcing $L_{1}\left(x_{i_{j}}\right)=\emptyset$, for some $i \neq 1$ and $1 \leq j \leq a_{i}-1$. Hence, $\operatorname{ch}(H)>2$. Because $H \subseteq G_{1}$, for all $l \geq 3, G_{1}$ is not 2-choosable.

Claim B. If $G_{2}=\Theta(2 r, 2 s, 2 t)$, then $G_{2}$ is not 2-choosable for all $r \geq 1$, and $s, t \geq 2$.

Denote $x_{1}, x_{2}$ and $y_{1}, y_{2}$ the vertices on the paths $P_{2 s}$ and $P_{2 t}$, respectively, such that $x_{1}=N(u), x_{2}=N\left(x_{1}\right), y_{1}=N(u)$, and $y_{2}=$ $N\left(y_{1}\right)$. Then for each $w \in V\left(G_{3}\right)$, define the list assigment $L_{2}$ such that
(i) $L_{2}(u)=L_{2}(v)=L_{2}(z)=\{a, b\}$ for $z \notin\left\{x_{1}, x_{2}, y_{1}, y_{2}\right\}$
(ii) $L_{2}\left(x_{1}\right)=L_{2}\left(y_{2}\right)=\{b, c\}, L_{2}\left(y_{1}\right)=L_{2}\left(x_{2}\right)=\{a, c\}$.

It is easy to see that $G_{2}$ does not admit a proper $L_{2}$-coloring.
Observe that the previous claim completely resolves the case of $l$ bridge graphs (with even paths) that are not of the form $\Theta(2,2,2 t)$ for $l=3$.

Claim C. If $G_{3}=\Theta\left(a_{1}, \ldots, a_{l}\right)$ and each $a_{i}$ is even, then $G_{3}$ is not 2 -choosable, for all $l \geq 4$.

For $l \geq 4$, we present a list assignment $L_{3}$ to $G_{3}$ when $a_{i}=a_{j}$, for each $i \neq j$. A similar list assignment can easily be derived when some $a_{i} \neq a_{k}$ by letting $L_{3}\left(x_{k_{a_{k} / 2}}\right)$ be a specific 2 -subset of $\{a, b, c, d\}$.

Let $H=\Theta\left(a_{1}, a_{2}, a_{3}, a_{4}\right)$ such that $a_{1}=a_{2}=a_{3}=a_{4}$. Now define $L_{3}$ to be a list assignment satisfying, for each $w \in V(H)$ :
(i) $L_{3}(u)=\{a, b\}$ and $L_{3}(v)=\{c, d\}$
(ii) $L_{3}\left(x_{1_{a_{1 / 2}}}\right)=\{a, c\}, L_{3}\left(x_{2_{a_{2} / 2}}\right)=\{a, d\}, L_{3}\left(x_{3_{a_{3} / 2}}\right)=\{b, c\}$, $L_{3}\left(x_{4_{a_{4} / 2}}\right)=\{b, d\}$
(iii) $L_{3}\left(x_{i_{j}}\right)=\{a, b\}$ for $1 \leq j<a_{i} / 2$ and $L_{3}\left(x_{i_{j}}\right)=\{c, d\}$
for $a_{i} / 2<j \leq a_{i}-1$
It is easy to verify that $H \subseteq G_{3}$ admits no proper $L_{3}$-coloring.

Thus, if $G$ contains only even cycles and $G$ is not $\Theta(2,2,2 t), G$ must satisfy one of the previous claims. The result follows for all $l$-bridge graphs, with $l \geq 3$.

Corollary 1. Suppose $G=\Theta\left(a_{1}, \ldots, a_{l}\right), l \geq 3 . G$ is chromaticchoosable if and only if $G$ contains an odd cycle or $G$ is of the form $\Theta(2,2,2 t)$, where $t$ is a positive integer.

Proof. Suppose $G=\Theta\left(a_{1}, \ldots, a_{l}\right)$ is chromatic-choosable. It follows from Proposition 1 that, either (i) $\chi(G)=2=\operatorname{ch}(G)$ or $(i i) \chi(G)=$ $3=\operatorname{ch}(G)$. Case $(i)$ follows from Theorem D. In which case $G \cong$ $\Theta(2,2,2 t)$ while in case ( $i i$ ) it is clear that $G$ must contain an odd cycle. Conversely, if $G$ contains an odd cycle, then $\chi(G)=3$. It follows from Theorem that $G$ is chromatic-choosable. Moreover, if $G \cong \Theta(2,2,2 t)$, then $G$ contains no odd cycle and $\chi(G)=2$. It follows from Theorem D that $G$ is chromatic-choosable.

We end this article with the next lemma which gives an estimate on the choice number of any connected graph. A graph $G$ of order greater than $r$ is said to be $r$-connected if $G$ remains connected whenever fewer than any $r$ number of vertices of $G$ are removed.

Lemma 1. Suppose $G$ is an $r$-connected graph with components $G_{1}, \ldots, G_{m}$. If $k=\max _{1 \leq i \leq m}\left\{\operatorname{ch}\left(G_{i}\right)\right\}$, then $G$ is $(k+r)$-choosable for all $k, r \geq 1$ and $m \geq 2$.

Proof. Suppose $L$ is a list assignment to $v \in V(G)$ satisfying $|L(v)| \geq$ $k+r$ with $k, r \geq 1$. Denote $S \subset V(G)$ a set of $r$ vertices whose deletion produces the non-empty components $G_{1}, \ldots, G_{m}, m \geq 2$. Color each element of $S$ using distinct $r$ colors, and remove those colors from the palette of each vertex $u \in V(G) \backslash S$. Let $L^{\prime}$ be the resulting list assigment for each vertex $u$. It follows that $\left|L^{\prime}(u)\right| \geq k$ for each $u \in$ $V\left(G_{i}\right), 1 \leq i \leq m$. By the hypothesis, each $G_{i}$ is $k$-choosable so we color each vertex $u \in V\left(G_{i}\right)$. Because $G$ is $r$-connected, together with the $r$-colorings of $S$, we have a proper $L$-coloring of $G$.

Notice that this bound is sharp for some 1-connected cyclic graphs. See for instance, Figure 1(B). From this proposition follows
Corollary 2. Suppose $S$ is a clique on $r$ vertices and for some graphs $H_{i}, k=\max _{1 \leq i \leq m}\left\{\operatorname{ch}\left(H_{i}\right)\right\}$. If $G=S \vee\left\{H_{i}\right\}_{i=1}^{m}$, then $\operatorname{ch}(G)=k+r$.
Proof. Because every proper coloring of $S \subset G$ uses exactly $r$ colors, the result follows from similar steps as in Lemma 1 .

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# A remark on the weak Turán's Theorem 

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#### Abstract

A subset $S$ of vertices of a graph $G$ is an independent set if no pair of vertices of $S$ are adjacent. The independence number, $\alpha(G)$ of $G$, is the maximum cardinality of an independent set of $G$. In this note, we present an improvement of the weak Turán's theorem.


Keywords: Independent set, Turán's Theorem, Probabilistic methods.

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## 1 Introduction

For graph theory notation and terminology not given here we refer to [4], and for the probabilistic methods notation and terminology we refer to [1], [5]. We consider finite, undirected and simple graphs $G$ with vertex set $V=V(G)$ and edge set $E(G)$. The number of vertices of $G$ is called the order of $G$ and is denoted by $n=n(G)$, and the number of edges of $G$ is called the size of $G$. The open neighborhood of a vertex $v \in V$ is $N(v)=N_{G}(v)=\{u \in V \mid u v \in E\}$ and the closed neighborhood of $v$ is $N[v]=N_{G}[v]=N(v) \cup\{v\}$. The degree of a vertex $v$, denoted by $\operatorname{deg}(v)\left(\operatorname{or~}_{\operatorname{deg}}^{G}(v)\right.$ to refer to $\left.G\right)$, is the cardinality of its open neighborhood. We denote by $\delta(G)$ and $\Delta(G)$, the minimum and maximum degrees among all vertices of $G$, respectively. For a subset $S$ of vertices of $G$, we denote by $G[S]$ the subgraph of $G$ induced by $S$. A subset $S$ of vertices of $G$ is an independent set if $G[S]$ has no edge. The independence number, $\alpha(G)$ of $G$, is the maximum cardinality of an independent set.

[^1]Turán [6] proved his best-known result, namely Turán's Graph Theorem or just Turán's Theorem, by determining those graphs of order $n$, not containing the complete graph $K_{k}$ of order $k$, and extremal with respect to size (that is, with as many edges as possible). Much have been written about Turán's Theorem, see for example [1], [2] and [5]. The Turán's Theorem states that if $G$ is a graph with $n$ vertices such that $G$ is $K_{r+1}$-free, then the number of edges in $G$ is at most $(1-1 / r) \frac{n^{2}}{2}$. There is an equivalent theorem referred as the dual version of the Turán's Theorem, (or sometimes the Turán's Theorem, too) that states that any graph $G$ of order $n$ and size $m$ contains an independent set of size at least $\frac{n}{d+1}$, where $d=\frac{2 m}{n}$ is the average degree of $G$. A weak version of Turán's Theorem has been proved by several authors by probabilistic methods.

Theorem 1 (A weak Turán's theorem, [1], [5]) If G is a graph of order $n$, and size $m$, and $d=\frac{2 m}{n} \geq 1$ is the average degree, then $\alpha(G) \geq \frac{n}{2 d}$.

In this note, we present an improvement of the weak Turán's Theorem by the same probabilistic methods. We use the following.

Theorem 2 (Caro [3] and Wei [7]) For any graph $G$,

$$
\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{1+\operatorname{deg}(v)}
$$

## 2 Main result

Theorem 3 If $G$ is a graph of order $n$, size $m$, maximum degree $\Delta$, minimum degree $\delta$, and $d=\frac{2 m}{n} \geq 1$ is the average degree, then $\alpha(G) \geq$ $\frac{n}{2 d}+\frac{n}{\Delta+1}\left[\frac{1}{d}+\left(1-\frac{2}{d}\right)\left(1-\frac{1}{d}\right)^{\delta}\right]$.

Proof. Select a random subset of vertices $S \subseteq V(G)$ in such a way that we insert every vertex into $S$ independently with probability $p=\frac{1}{d}$. Let $A$ be the set of all non-isolated vertices of $G[S], G^{\prime}=G[A]$,
and $I_{S}$ be a maximum independent set in $G^{\prime}$. Let $X=\{v \in V(G)-S$ : $\left.N_{G}[v] \cap S=\emptyset\right\}$, and $J_{S}$ be a maximum independent set in $G[X]$. From Theorem 2, we find that

$$
\left|I_{S}\right| \geq \frac{|A|}{\Delta\left(G^{\prime}\right)+1} \geq \frac{|A|}{\Delta(G)+1}=\frac{|A|}{\Delta+1}
$$

and

$$
\left|J_{S}\right| \geq \frac{|X|}{\Delta(G[X]))+1} \geq \frac{|X|}{\Delta(G)+1}=\frac{|X|}{\Delta+1} .
$$

Let $X=|S|, Y$ denotes the number of edges of $G\left[S-I_{S}\right]$, and $Z=\left|J_{S}\right|$. We compute the expectation of $X-Y+Z$. Clearly $E(X)=n p$. Any vertex of $I_{S}$ is incident with at least one edge in $G[S]$. Thus, the number of edges of $G\left[S-I_{S}\right]$ is bounded above by the number of edges of $G[S]$ minus the number of vertices of $I_{S}$. Let $Y_{1}$ denotes the number of edges of $G[S]$, and $Y_{2}$ denotes the number of vertices of $\left|I_{S}\right|$. Then $Y \leq Y_{1}-Y_{2}$. Observe that $E\left(Y_{1}\right)=m p^{2}$, and $E\left(Y_{2}\right)=E\left(\left|I_{S}\right|\right) \geq$ $E\left(\frac{|A|}{\Delta+1}\right)=\frac{1}{\Delta+1} E(|A|)$. For a vertex $v, \operatorname{Pr}(v \in A)=p\left(1-(1-p)^{\operatorname{deg}(v)}\right)$, and $\operatorname{Pr}(v \in X)=(1-p)^{1+\operatorname{deg}(v)}$. Thus,

$$
\begin{aligned}
E(Y) & \leq E\left(Y_{1}\right)-E\left(Y_{2}\right) \\
& \leq m p^{2}-\frac{1}{\Delta+1} E(|A|) \\
& \leq m p^{2}-\frac{1}{\Delta+1} n p\left(1-(1-p)^{\delta}\right) .
\end{aligned}
$$

Moreover,

$$
E\left(\left|J_{S}\right|\right) \geq E\left(\frac{|X|}{\Delta+1}\right)=\frac{1}{\Delta+1} E(|X|) \geq \frac{n}{\Delta+1}(1-p)^{1+\delta} .
$$

Now,

$$
\begin{aligned}
E(X-Y+Z) & \geq \\
n p-m p^{2}+\frac{1}{\Delta+1} n p\left(1-(1-p)^{\delta}\right)+\frac{n}{\Delta+1}(1-p)^{1+\delta} & = \\
=\frac{n}{2 d}+\frac{n}{\Delta+1}\left[\frac{1}{d}+\left(1-\frac{2}{d}\right)\left(1-\frac{1}{d}\right)^{\delta}\right] . &
\end{aligned}
$$

Thus there exists a specific set $S$ for which the number of vertices of $S$ minus the number of edges in $G\left[S-I_{S}\right]$ plus the number of vertices of $J_{S}$ is at least $\frac{n}{2 d}+\frac{n}{\Delta+1}\left[\frac{1}{d}+\left(1-\frac{2}{d}\right)\left(1-\frac{1}{d}\right)^{\delta}\right]$. Select one vertex from each edge of $G\left[S-I_{S}\right]$ and delete it. This leaves a set $S^{*}$ with at least $\frac{n}{2 d}+\frac{n}{\Delta+1}\left[\frac{1}{d}+\left(1-\frac{2}{d}\right)\left(1-\frac{1}{d}\right)^{\delta}\right]$ vertices which is an independent set.

Remark: Although our main result is weaker than Theorem 2, it would be interesting for researchers interested to weak Turán's Theorem.

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# On the locating matrix of a graph and its spectral analysis 

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#### Abstract

We introduce a new matrix representation for a graph by defining the locating matrix $\mathbf{L o}(G)$ of $G$. We define the locating eigenvalues, the locating spectrum, and locating energy of the graph and we calculate them for some standard graphs. We also obtain bounds for the locating energy for regular and strongly regular graphs.

Keywords: Locating eigenvalues (of graph), Locating Spectrum (of graph), Locating energy (of graph).


## 1 Introduction

A graph is completely determined by either its adjacencies or its incidences. This information can be conveniently stated in matrix form. It is often possible to make use of these matrices in order to identify certain properties of a graph. The adjacency matrix $A(G)=A=\left[a_{i j}\right]$ of a labeled graph $G$ with $p$ points is the $p \times p$ matrix in which $a_{i j}=1$ if $v_{i}$ is adjacent with $v_{j}$ and $a_{i j}=0$ otherwise. Thus, there is a one-toone correspondence between labeled graphs with $p$ vertices and $p \times p$ symmetric binary matrices with zero diagonal. The eigenvalues are the roots of the characteristic polynomial

$$
\begin{gathered}
\phi(G ; \lambda)=\operatorname{det}(\lambda I-A) \\
=\lambda^{n}+c_{1} \lambda^{n-1}+c_{2} \lambda^{n-2}+\cdots+c_{n-1} \lambda+c_{n}
\end{gathered}
$$

where $I$ is the $n \times n$ identity matrix. The eigenvalues of $A(G)$ are eigenvalues of $G$. Since $A$ is a real symmetric matrix with zero trace,
these eigenvalues are all real with sum equal to zero. The Spectrum of a graph is the list of distinct eigenvalues $\lambda_{1}>\lambda_{2}>\ldots>\lambda_{r}$ of $G$, with multiplicities $m_{1}, m_{2}, \ldots, m_{r}$, represented by

$$
\operatorname{Spec}(\mathrm{G})=\left(\begin{array}{cccc}
\lambda_{1} & \lambda_{2} & \cdots & \lambda_{r} \\
m_{1} & m_{2} & \cdots & m_{r}
\end{array}\right) .
$$

The energy of the graph $G$ is defined in [8] as the sum of the absolute values of its eigenvalues:

$$
E(G)=\sum_{i=1}^{n}\left|\gamma_{i}\right| .
$$

Details on the theory of graph energy can be found in the book [20], whereas details on its chemical applications in the book [14] and in the review [11].

Lemma 1. [6] For the standard graphs $K_{p}, K_{m, n}$ and $C_{n}$, we have

- $\operatorname{Spec}\left(K_{p}\right)=\left(\begin{array}{cc}n-1 & -1 \\ 1 & n-1\end{array}\right)$.
- $\operatorname{Spec}\left(K_{m, n}\right)=\left(\begin{array}{ccc}\sqrt{m n} & -\sqrt{m n} & 0 \\ 1 & 1 & m+n-2\end{array}\right)$.
- $\operatorname{Spec}\left(C_{n}\right)=\left\{\begin{array}{rrrr}\left(\begin{array}{rrrrr}2 & 2 \cos \frac{2 \pi}{n} & \cdots & 2 \cos \frac{2(n-1) \pi}{2^{n}} \\ 1 & 2 & \cdots & , & \text { if } n \text { is odd; } \\ \left(\begin{array}{ccccc}2 & 2 \cos \frac{2 \pi}{n} & \cdots & 2 \cos \frac{2(n-2) \pi}{2^{n}} & -2 \\ 1 & 2 & \cdots & 1\end{array}\right), \text { if } n \text { is even. }\end{array} \text {. } 10 .\right.\end{array}\right.$


## 2 The Locating Spectrum of a Graph

Definition 1. Let $G=(V, E)$ be a connected graph with vertex set $V=\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$. A locating function of $G$ denoted by $\mathbf{L}(G)$ is a function $\mathbf{L}(G): V(G) \rightarrow \mathbb{R}^{n}$ such that $\mathbf{L}\left(v_{i}\right)=\overrightarrow{v_{i}}=$ $\left(d\left(v_{1}, v_{i}\right), d\left(v_{2}, v_{i}\right), \ldots, d\left(v_{n}, v_{i}\right)\right)$, where $d\left(v_{i}, v_{j}\right)$ is the distance between the vertices $v_{i}$ and $v_{j}$ in $G$. The vector $\overrightarrow{v_{i}}$ is called the locating vector
corresponding to the vertex $v_{i}$.
The locating product of two locating vectors $\overrightarrow{v_{i}}$ and $\overrightarrow{v_{j}}$ in a graph $G$ is denoted by $L\left(\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}\right)$ and defined as:

$$
L\left(\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}\right)= \begin{cases}\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}, & \text { if } i \neq j \text { and } v_{i} \text { adjacent to } v_{j} \\ 0 & \text { otherwise }\end{cases}
$$

where $\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}$ is the dot product of the vectors $\overrightarrow{v_{i}}$ and $\overrightarrow{v_{j}}$ in the Euclidean space $\mathbb{R}^{n}$.

The locating matrix of $G$ is then $\mathbf{L o}=\mathbf{L o}(G)=\left[l_{i j}\right]$, where

$$
l_{i j}=L\left(\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}\right)
$$

The characteristic polynomial $\operatorname{det}(\gamma \boldsymbol{I}-\mathbf{L o}(G))$ of $\mathbf{L o}(G)$ is called the Lo-characteristic polynomial of $G$ and is denoted by $P_{L o}(G)=$ $\sum_{i=0}^{n} a_{i} \gamma^{n-i}$. The eigenvalues of the matrix $\mathbf{L o}(G)$, which are the zeros of $|\gamma \boldsymbol{I}-\mathbf{L o}(G)|$ are called the Lo-eigenvalues of $G$ and form its Spectrum denoted by $\operatorname{Spec}_{L o}(G)$. If the distinct Lo-eigenvalues of $G$ are $\gamma_{1}, \gamma_{2} \ldots, \gamma_{m}$ with multiplicities $t_{1}, t_{2}, \ldots, t_{m}$ respectively, then, $\operatorname{Spec}_{L o}(G)$ is written as: $\left(\begin{array}{llll}\gamma_{1} & \gamma_{2} & \ldots & \gamma_{m} \\ t_{1} & t_{2} & \ldots & t_{m}\end{array}\right)$.

By the above definition, the locating matrix is a real symmetric $n \times n$ matrix. Therefore its eigenvalues $\gamma_{1}, \gamma_{2}, \ldots, \gamma_{m}$ are real numbers. Since the trace of $\mathbf{L o}(G)$ is zero, the sum of its eigenvalues is also equal to zero.
In this paper, by graph, we mean a simple, finite, undirected, connected graph and for short by $\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}$, we mean the locating product of the two locating vectors $\overrightarrow{v_{i}}$ and $\overrightarrow{v_{j}}$ in $G$. For graph theoretic terminology we refer to Charatrand and Lesniak [5].

Lemma 2. Let $G$ be a connected graph with $n$ vertices and let $\gamma_{1}, \gamma_{2}, \ldots, \gamma_{n}$ be its Lo-eigenvalues. Then

1. $\sum_{i=1}^{n} \gamma_{i}=0$
2. $\sum_{i=1}^{n} \gamma_{i}^{2}=2 \sum_{1 \leq i<j \leq n}\left(\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}\right)^{2}$

Proof. (1) $\sum_{i=1}^{n} \gamma_{i}=\operatorname{trace}(\operatorname{Lo}(G))=\sum_{i=1}^{n} a_{i i}=0$.
(2) For $i=1,2, \ldots, n$, the $(i, i)$ entry of $(L o(G))^{2}$ is equal to the $\operatorname{trace}(\operatorname{Lo}(G))^{2}$

$$
\begin{aligned}
\operatorname{trace}[\operatorname{Lo}(G)]^{2} & =\sum_{i=1}^{n} \sum_{j=1}^{n}\left(\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}\right)^{2} \\
& =2 \sum_{1 \leq i<j \leq n}\left(\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}\right)^{2} .
\end{aligned}
$$

Definition 2. The locating energy of the graph $G$ is

$$
E_{L o}=E_{L o}(G)=\sum_{i=1}^{n}\left|\gamma_{i}\right|
$$

Theorem 1. For the complete graph $K_{n}$ of order $n \geq 2$,

$$
\operatorname{Spec}_{L o}\left(K_{n}\right)=\left(\begin{array}{cc}
(n-1)(n-2) & -(n-2) \\
1 & n-1
\end{array}\right),
$$

and $E_{L o}\left(K_{n}\right)=2(n-1)(n-2)$.
Proof. Let $G=K_{n}$ with vertices $v_{1}, v_{2}, \ldots, v_{n}$ and let $\overrightarrow{v_{i}}$ be the locating vector corresponding to the vertex $v_{i}$. Then $\overrightarrow{v_{i}}=\left(a_{1}, \ldots, a_{n}\right)$, where $a_{i}=0$ and $a_{j}=1$. Thus for any two vectors $\overrightarrow{v_{i}}, \overrightarrow{v_{j}}$, where $i \neq j$, we have

$$
\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}=n-2 .
$$

Therefore, $\mathbf{L o}\left(K_{n}\right)=(n-2) \mathbf{A}\left(K_{n}\right)$, were $A\left(K_{n}\right)$ is the adjacency matrix of $K_{n}$, and by Lemma 1 it is easy to see that $\operatorname{Spec}_{L o}\left(K_{n}\right)=$ $\left(\begin{array}{cc}(n-1)(n-2) & -(n-2) \\ 1 & n-1\end{array}\right)$. Hence $E_{L o}\left(K_{n}\right)=2(n-1)(n-2)$.

We now determine the Lo-spectrum and Lo-energy of any cycle $C_{n}$.
Theorem 2. Let $n \geq 2$ be an even integer. Then for the cycle $C_{n}$, we have

$$
\operatorname{Spec}_{L o}\left(C_{n}\right)=
$$

$$
=\left(\begin{array}{ccccc}
\frac{n(n-2)^{2}}{6} & \frac{n(n-2)^{2}}{6} \cos \frac{2 \pi}{n} & \cdots & \frac{n(n-2)^{2}}{6} \cos \frac{2(n-2) \pi}{n} & -\frac{n(n-2)^{2}}{6} \\
1 & 2 & \cdots & 2 & 1
\end{array}\right) .
$$

Further $E_{L o}\left(C_{n}\right)=\frac{n(n-2)^{2}}{12} E\left(C_{n}\right)$, where $E\left(C_{n}\right)$ is the energy of $C_{n}$.
Proof. By labelling the vertices of the cycle $C_{n}$ in the anticlockwise direction as $\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$, we observe that,

$$
\begin{aligned}
& \overrightarrow{v_{1}}=\left(0,1,2,3, \ldots, \frac{n}{2}, \frac{n}{2}-1, \frac{n}{2}-2, \ldots, 1\right) \\
& \overrightarrow{v_{2}}=\left(1,0,1,2, \ldots, \frac{n}{2}-1, \frac{n}{2}, \frac{n}{2}-1, \ldots, 2\right) \\
& \overrightarrow{v_{3}}=\left(2,1,0,1, \ldots, \frac{n}{2}-2, \frac{n}{2}-1, \frac{n}{2}, \ldots, 3\right) \\
& \cdot \\
& \overrightarrow{v_{n}}=\left(1,2,3, \ldots, \frac{n}{2}, \frac{n}{2}-1, \frac{n}{2}-2, \frac{n}{2}-3, \ldots, 0\right) .
\end{aligned}
$$

Then, by symmetry,

$$
\begin{aligned}
\overrightarrow{v_{i}} \cdot \overrightarrow{v_{i+1}} & =2\left((2)(1)+(3)(2)+(4)(3)+\ldots+\frac{n}{2}\left(\frac{n}{2}-1\right)\right) \\
& =2 \sum_{i=2}^{\frac{n}{2}} i(i-1) \\
& =2 \sum_{i=2}^{\frac{n}{2}} i^{2}-2 \sum_{i=2}^{\frac{n}{2}} i \\
& =2\left(\frac{\frac{n}{2}\left(\frac{n}{2}+1\right)(n+1)}{6}-1\right)-2\left(\frac{\frac{n}{2}\left(\frac{n}{2}+1\right)}{2}-1\right) \\
& =\frac{n(n-2)^{2}}{12} .
\end{aligned}
$$

Therefore $\mathbf{L o}\left(C_{n}\right)=\frac{n(n-2)^{2}}{12} \mathbf{A}\left(C_{n}\right)$.

Hence by Lemma 1, we get,

$$
\begin{gathered}
\operatorname{Spec}_{L o}\left(C_{n}\right)= \\
=\left(\begin{array}{ccccc}
\frac{n(n-2)^{2}}{6} & \frac{n(n-2)^{2}}{6} \cos \frac{2 \pi}{n} & \ldots & \frac{n(n-2)^{2}}{6} \cos \frac{2(n-2) \pi}{n} & -\frac{n(n-2)^{2}}{6} \\
1 & 2 & \cdots & 2 & 1
\end{array}\right) .
\end{gathered}
$$

Also clearly

$$
E_{L o}\left(C_{n}\right)=\frac{n(n-2)^{2}}{12} E\left(C_{n}\right),
$$

where $E\left(C_{n}\right)$ is the energy of $C_{n}$.
Theorem 3. Let $n \geq 3$ be an odd integer. Then for the cycle $C_{n}$, we have,

$$
\begin{gathered}
\operatorname{Spec}_{L o}\left(C_{n}\right)= \\
=\left(\begin{array}{cccc}
\frac{(n-1)(n-2)(n+3)}{6} & \frac{(n-1)(n-2)(n+3)}{6} \cos \frac{2 \pi}{n} & \cdots & \frac{(n-1)(n-2)(n+3)}{6} \cos \frac{2(n-1) \pi}{n}
\end{array}\right) .
\end{gathered}
$$

Further $E_{L o}\left(C_{n}\right)=\frac{(n-1)(n-2)(n+3)}{6} E\left(C_{n}\right)$, where $E\left(C_{n}\right)$ is the energy of $E\left(C_{n}\right)$.

Proof. Let $G$ be a cycle $C_{n}$ with odd number $n$ of vertices. By labeling the vertices of $G$ with anticlockwise direction as $\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$, we observe that,

$$
\begin{aligned}
& \overrightarrow{v_{1}}=\left(0,1,2,3, \ldots, \frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2}-1, \frac{n-1}{2}-2, \ldots, 1\right) \\
& \overrightarrow{v_{2}}=\left(1,0,1,2, \ldots, \frac{n-1}{2}-1, \frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2}-1, \ldots, 2\right) \\
& \overrightarrow{v_{3}}=\left(2,1,0,1, \ldots, \frac{n-1}{2}-2, \frac{n-1}{2}-1, \frac{n-1}{2}, \frac{n-1}{2}, \ldots, 3\right) \\
& \cdot \\
& \overrightarrow{v_{n}}=\left(1,2,3, \ldots, \frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2}-1, \frac{n-1}{2}-2, \ldots, 0\right) .
\end{aligned}
$$

Then, by symmetry,

$$
\begin{aligned}
\overrightarrow{v_{i}} \cdot \overrightarrow{v_{i+1}}=2[(2)(1)+(3)(2) & \left.+(4)(3)+\ldots+\frac{n-1}{2}\left(\frac{n-1}{2}-1\right)\right]+ \\
& +\frac{n-1}{2}\left(\frac{n-1}{2}\right) \\
& =2 \sum_{i=2}^{\frac{n-1}{2}} i(i-1)+\frac{(n-1)^{2}}{4} \\
& =2 \sum_{i=2}^{\frac{n-1}{2}} i^{2}-2 \sum_{i=2}^{\frac{n-1}{2}} i+\frac{(n-1)^{2}}{4} \\
& =2\left[\frac{\frac{n-1}{2}\left(\frac{n-1}{2}+1\right)\left(2 \frac{n-1}{2}+1\right)}{6}-1\right]- \\
& -2\left[\frac{\frac{n-1}{2}\left(\frac{n-1}{2}+1\right)}{2}-1\right]+\frac{(n-1)^{2}}{4} \\
& =\frac{(n-1)(n-2)(n+3)}{12} .
\end{aligned}
$$

Therefore,

$$
\begin{gathered}
\operatorname{Spec}_{L o}\left(C_{n}\right)= \\
\left(\begin{array}{cccc}
\frac{(n-1)(n-2)(n+3)}{6} & \frac{(n-1)(n-2)(n+3)}{6} \cos \frac{2 \pi}{n} & \ldots & \frac{(n-1)(n-2)(n+3)}{6} \cos \frac{2(n-1) \pi}{n} \\
1 & 2 & \cdots & 2
\end{array}\right) .
\end{gathered}
$$

Further $E_{L o}\left(C_{n}\right)=\frac{(n-1)(n-2)(n+3)}{12} E\left(C_{n}\right)$.
Theorem 4. Let $G$ be a complete bipartite graph $K_{a, b}$, where $1 \leq a \leq b$. Then

$$
\operatorname{Spec}_{L o}\left(K_{a, b}\right)=\left(\begin{array}{ccc}
(2 a+2 b-4) \sqrt{a b} & -(2 a+2 b-4) \sqrt{a b} & 0 \\
1 & 1 & a+b-2
\end{array}\right) .
$$

Further $E_{L o}\left(K_{a, b}\right)=4(a+b-2) \sqrt{a b}$.
Proof. Let the vertices of $K_{a, b}$ be labelled such that $v_{i}$ are adjacent to $v_{a+j}$ for all $1 \leq i \leq a$ and $1 \leq j \leq b$.

Now, it is obvious that the locating vectors $\overrightarrow{v_{i}}$ of $v_{i}$ are given by:

$$
\begin{aligned}
& \vec{v}_{1}=(0, \underbrace{2, \ldots, 2}_{a-1}, \underbrace{1,1, \ldots, 1}_{b}), \overrightarrow{v_{2}}=(2,0, \underbrace{2, \ldots, 2}_{a-2}, \underbrace{1,1, \ldots, 1}_{b}) \\
& \vec{v}_{3}=(2,2,0, \underbrace{2, \ldots, 2}_{a-3}, \underbrace{1,1, \ldots, 1}_{b}), \ldots, \\
& \vec{v}_{a}=(\underbrace{2, \ldots, 2}_{a-1}, 0, \underbrace{1,1, \ldots, 1}_{b}), \vec{v}_{a+1}=(\underbrace{1, \ldots, 1}_{a}, 0, \underbrace{2, \ldots, 2}_{b-1}), \\
& \vec{v}_{a+2}=(\underbrace{1, \ldots, 1}_{a}, 2,0, \underbrace{2, \ldots, 2}_{b-2}), \ldots, \overrightarrow{v_{a+b}}=(\underbrace{1, \ldots, 1}_{a}, \underbrace{2, \ldots, 2}_{b-1}, 0) .
\end{aligned}
$$

Then it is easy to see that for any two locating vertices $v_{i}, v_{j}$ in $K_{a, b}$, $\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}=2(a+b-2)$. Therefore,

$$
\mathbf{L o}\left(K_{a, b}\right)=2(a+b-2) \mathbf{A}\left(K_{a, b}\right)
$$

Also by using Lemma 1 , we get
$\operatorname{Spec}_{L o}\left(K_{a, b}\right)=\left(\begin{array}{ccc}(2 a+2 b-4) \sqrt{a b} & -(2 a+2 b-4) \sqrt{a b} & 0 \\ 1 & 1 & a+b-2\end{array}\right)$, and hence $E_{L o}\left(K_{a, b}\right)=4(a+b-2) \sqrt{a b}$.

The following results are obtained straightforward from Theorem 4.

1. Let $G$ be a complete bipartite graph $K_{n, n}$, where $n \geq 1$. Then $E_{L o}\left(K_{n, n}\right)=8 n(n-1)$.
2. Let $G$ be any star graph $K_{1, n}$. Then

$$
E_{L o}\left(K_{1, n}\right)=4 \sqrt{n}(n-1)
$$

## 3 Locating Spectrum And Energy of Regular and Strongly Regular Graphs

One of the most important family of regular graphs is the strongly regular graphs (abbreviated SRG), which has so many beautiful properties. There are many SRGs arising from combinatorial concepts such as orthogonal arrays, latin squares, conference matrices, designs and geometric graphs.

A strongly regular graph (SRG) with parameters $(n, k, \lambda, \mu)$ is a graph on $n$ vertices which is regular with valency $k$ and has the following properties:

- any two adjacent vertices have exactly $\lambda$ common neighbours;
- any two nonadjacent vertices have exactly $\mu$ common neighbours.

Theorem 5. [7] Let $G$ be a strongly regular graph with parameters $(n, k, \lambda, \mu)$. Then the eigenvalues of $G$ satisfy the following properties:

1. $G$ has exactly three distinct eigenvalues which are $k, \theta$ and $\tau$ where

$$
\theta=\frac{1}{2}\left(\lambda-\mu+\sqrt{(\lambda-\mu)^{2}+4(k-\mu)}\right),
$$

and

$$
\tau=\frac{1}{2}\left(\lambda-\mu-\sqrt{(\lambda-\mu)^{2}+4(k-\mu)}\right) .
$$

2. The multiplicity of the eigenvalue $k$ is 1 and the multiplicities of $\theta$ and $\tau$ are $f$ and $g$ respectively, where

$$
f=n-1+\frac{(n-1)(\mu-\lambda)-2 k}{\sqrt{(\lambda-\mu)^{2}+4(k-\mu)}}
$$

and

$$
g=n-1-\frac{(n-1)(\mu-\lambda)-2 k}{\sqrt{(\lambda-\mu)^{2}+4(k-\mu)}}
$$

3. If $(n-1)(\mu-\lambda)-2 k \neq 0$, then the eigenvalues $\theta$ and $\tau$ are integers. On the other hand if $(n-1)(\mu-\lambda)-2 k=0$, then $f=g$ and $\theta$ and $\tau$ need not be integers. The strongly regular graph is called a conference graph in this case.

Theorem 6. Let $G$ be a strongly regular graph with parameters $(n, k, \lambda, \mu)$ and $\operatorname{Spec}(G)=\left(\begin{array}{ccc}k & \theta & \tau \\ 1 & f & g\end{array}\right)$. Then

$$
\operatorname{Spec}_{L o}(G)=\left(\begin{array}{ccc}
k \delta & \theta \delta & \tau \delta) \\
1 & f & g
\end{array}\right),
$$

further, $E_{L o}(G)=\delta E(G)$, where $\delta=\lambda+4(n-k-1)$ and $E(G)$ is the energy of $G$.

Proof. Let $G$ be a strongly regular graph with the parameters $(n, k, \lambda, \mu)$. Let $u$ and $v$ be any two adjacent vertices and suppose that $P_{11}^{1}(u, v)$ is the number of vertices which are adjacent to both of the vertices $u$ and $v, P_{12}^{1}(u, v)$ is the number of vertices which are adjacent to $u$ but not adjacent to $v, P_{21}^{1}(u, v)$ the number of vertices which are adjacent to $v$ but not adjacent to $u$ and $P_{22}^{1}(u, v)$ is the number of vertices which are not adjacent to both of the vertices $u$ and $v$. As in [4], we have

$$
P_{12}^{1}(u, v)=P_{21}^{1}(u, v)=n_{1}-P_{11}^{1}(u, v)-1
$$

and

$$
P_{22}^{1}(u, v)=n_{2}-n_{1}+P_{11}^{1}(u, v)+1,
$$

where $n_{1}=k$ and $n_{2}=n-k-1$.
Note that the diameter of $G$ is atmost two. Thus for any vertex $v$ in $G$ there are $k$ vertices that have distance one from $v$ and $n-k-1$ vertices that have distance two from $v$. Suppose that $\vec{v}, \vec{u}$ are the locating vectors corresponding to the adjacent vertices $u$ and $v$ respectively. Then

$$
P_{12}^{1}(u, v)=P_{21}^{1}(u, v)=k-\lambda-1
$$

and

$$
P_{22}^{1}(u, v)=n-2 k+\lambda .
$$

Therefore

$$
\vec{v} \cdot \vec{u}=P_{11}^{1}(u, v)+2 P_{12}^{1}(u, v)+2 P_{21}^{1}(u, v)+4 P_{22}^{1}(u, v) .
$$

Hence

$$
\vec{v} \cdot \vec{u}=\lambda+4(n-k-1) .
$$

Then

$$
L o(G)=(\lambda+4(n-k-1)) A(G) .
$$

If we put $\delta=\lambda+4(n-k-1)$, then,

$$
\operatorname{Spec}_{L o}(G)=\left(\begin{array}{ccc}
k \delta & \theta \delta & \tau \delta \\
1 & f & g
\end{array}\right)
$$

and $E_{L o}(G)=\delta E(G)$

## Notation:

If $\left(\begin{array}{cccc}\lambda_{1} & \lambda_{2} & \cdots & \lambda_{r} \\ m_{1} & m_{2} & \cdots & m_{r}\end{array}\right)$ is the spectrum of a graph $G$, then we write

$$
\delta \operatorname{spec}(G)=\delta\left(\begin{array}{llll}
\lambda_{1} & \lambda_{2} & \cdots & \lambda_{r} \\
m_{1} & m_{2} & \cdots & m_{r}
\end{array}\right)=\left(\begin{array}{cccc}
\delta \lambda_{1} & \delta \lambda_{2} & \cdots & \delta \lambda_{r} \\
m_{1} & m_{2} & \cdots & m_{r}
\end{array}\right),
$$

for any real number $\delta$.
We can generalize the Theorem 6 as the following:
Theorem 7. Let $G=(V, E)$ be a $k$-regular graph of diameter two in which any two adjacent vertices have $t$ common neighbours. If $\operatorname{Spec}(G)=\left(\begin{array}{cccc}k & \lambda_{2} & \ldots & \lambda_{m} \\ 1 & t_{2} & \ldots & t_{m}\end{array}\right)$, then

$$
\operatorname{Spec}_{L o}(G)=\delta \operatorname{Spec}(G) .
$$

Proof. Let $u$ and $v$ be any two adjacent vertices in $G$. We can partition the remaining vertices of $G$ into four sets given by:

1. $A=\{w \in V(G) \mid d(u, w)=1, d(v, w)=1\}$.
2. $B=\{w \in V(G) \mid d(u, w)=1, d(v, w)=2\}$.
3. $C=\{w \in V(G) \mid d(u, w)=2, d(v, w)=1\}$.
4. $D=\{w \in V(G) \mid d(u, w)=2, d(v, w)=2\}$.

Since $G$ is k-regular graph with diameter two, we have, $|A|=P_{11}^{1}(u, v)$, $|B|=P_{12}^{1}(u, v),|C|=P_{21}^{1}(u, v)$ and $|D|=P_{22}^{1}(u, v)$. Since any two adjacent vertices $u, v$ have $t$ common neighbours, then $P_{11}^{1}(u, v)=t$, $P_{12}^{1}(u, v)=P_{21}^{1}(u, v)=k-1-t$ and as the sets $A, B, C$ and $D$ partition the set $V(G)-\{u, v\}$, we have $n-2=2(k-t-1)+t+|D|$. Hence $P_{22}^{1}(u, v)=n-2 k+t$.

Thus,

$$
\vec{u} \cdot \vec{v}=P_{11}^{1}(u, v)+2 P_{12}^{1}(u, v)+2 P_{21}^{1}(u, v)+4 P_{22}^{1}(u, v)=t+4(n-k-1) .
$$

Putting $t+4(n-k-1)=\delta$, we obtain $\mathbf{L o}(G)=\delta, \mathbf{A}(G)$. Hence

$$
\operatorname{Spec}_{L o}(G)=\left(\begin{array}{cccc}
k \delta & \delta \lambda_{2} & \ldots & \delta \lambda_{m} \\
1 & t_{2} & \ldots & t_{m}
\end{array}\right) .
$$

Theorem 8. Let $G=(V, E)$ be a regular graph of diameter two and without triangles. Then

\[

\]

Proof. When $G$ has no triangles in Theorem 8, that means $t=0$. Hence we obtain the following:

$$
\begin{aligned}
& \text { If } \operatorname{Spec}(G)=\left(\begin{array}{cccc}
k & \lambda_{2} & \ldots & \lambda_{m} \\
1 & t_{2} & \ldots & t_{m}
\end{array}\right) \text {, then } \\
& \operatorname{Spec}_{L o}(G)= \\
& =\left(\begin{array}{cccc}
4 k(n-k-1)) & 4(n-k-1) \lambda_{2} & \ldots & 4(n-k-1) \lambda_{m} \\
1 & t_{2} & \ldots & t_{m}
\end{array}\right) .
\end{aligned}
$$

## 4 Bounds for the Locating Energy $E_{L o}(G)$

Theorem 9. If $G$ is a graph with locating vectors $\overrightarrow{v_{1}}, \overrightarrow{v_{2}}, \ldots, \overrightarrow{v_{n}}$, then

$$
\sqrt{\alpha} \leq E_{L o}(G) \leq \sqrt{n \alpha}
$$

where $\alpha=2 \sum_{1 \leq i<j \leq n}\left(\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}\right)^{2}$.
Proof Let $\gamma_{1}, \gamma_{2}, \ldots, \gamma_{n}$ be the Lo-eigenvalues of $G$. The CauchySchwarz inequality states that if $\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ and $\left(b_{1}, b_{2}, \ldots, b_{n}\right)$ are $n$-vectors, where $a_{1}, a_{2}, \ldots, a_{n}, b_{1}, b_{2}, \ldots, b_{n} \in \mathbb{R}$, then,

$$
\left(\sum_{i=1}^{n} a_{i} b_{i}\right)^{2} \leq\left(\sum_{i=1}^{n} a_{i}^{2}\right)\left(\sum_{i=1}^{n} b_{i}^{2}\right) .
$$

Now, by setting $a_{i}=1$ and $b_{i}=\left|\gamma_{i}\right|, i=1,2, \ldots, n$, in the above inequality, we obtain

$$
\left(\sum_{i=1}^{n}\left|\gamma_{i}\right|\right)^{2} \leq\left(\sum_{i=1}^{n} 1^{2}\right)\left(\sum_{i=1}^{n}\left|\gamma_{i}\right|^{2}\right) .
$$

Hence by using Lemma 2 we get

$$
\sum_{i=1}^{n}\left|\gamma_{i}\right| \leq \sqrt{n \sum_{i=1}^{n}\left|\gamma_{i}\right|^{2}}=\sqrt{2 n \sum_{1 \leq i<j \leq n}\left(\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}\right)^{2}}
$$

and so

$$
E_{L o}(G) \leq \sqrt{n \alpha}
$$

For the lower bound, we have

$$
\left(E_{L o}(G)\right)^{2}=\left(\sum_{i=1}^{n}\left|\gamma_{i}\right|\right)^{2} \geq \sum_{i=1}^{n}\left(\left|\gamma_{i}\right|\right)^{2}=2 \sum_{1 \leq i<j \leq n}\left(\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}\right)^{2} .
$$

Thus

$$
E_{L o}(G) \geq \sqrt{\alpha}
$$

Theorem 10. For any connected graph of diameter $d$ with $n$ vertices, $n \geq 2$,

$$
(n-2) \sqrt{2(n-1)} \leq E_{L o}(G) \leq n(n-2) d^{2} \sqrt{n-1} .
$$

Proof. Let $G$ be a connected graph with vertices $v_{1}, v_{2}, \ldots, v_{n}$, with diameter $d$, then for any locating vectors $\overrightarrow{v_{i}}, \overrightarrow{v_{j}}$ in $G$ the locating product $\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}$ is at least equal to $n-2$, that means,

$$
\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}} \geq n-2
$$

and hence

$$
\left(\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}\right)^{2} \geq(n-2)^{2} .
$$

The number of $\left(\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}\right)^{2}$ for which $1 \leq i<j \leq n$ and the locating product $\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}$ does not equal to zero is at least $n-1$ because $G$ is connected.

Therefore

$$
\sum_{1 \leq i<j \leq n}^{n}\left(\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}\right)^{2} \geq(n-1)(n-2)^{2}
$$

By using the lower bound found in Theorem 9 we have,

$$
E_{L o}(G) \geq \sqrt{2 \sum_{1 \leq i<j \leq n}\left(\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}\right)^{2}} \geq(n-2) \sqrt{2(n-1)} .
$$

Similarly, to obtain the upper bound, we have

$$
\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}} \leq(n-2) d^{2},
$$

and we have,

$$
\left(\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}\right)^{2} \leq(n-2)^{2} d^{4} .
$$

Obviously, for any connected graph $G$ there exist at most $n(n-1) / 2$ of the locating product $\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}$ which are not zero. Therefore

$$
\sum_{1 \leq i<j \leq n}^{n}\left(\overrightarrow{v_{i}} \cdot \vec{v}_{j}\right)^{2} \leq \frac{1}{2}\left(n(n-1)(n-2)^{2} d^{4}\right)
$$

By using the upper bound which is obtained in Theorem 9, we have,

$$
E_{L o}(G) \leq \sqrt{2 n \sum_{1 \leq i<j \leq n}\left(\overrightarrow{v_{i}} \cdot \vec{v}_{j}\right)^{2}} \leq n(n-2) d^{2} \sqrt{n-1}
$$

Theorem 11. Let $G$ be $k$-regular graph with $n$ vertices and diameter two. Then

$$
E_{L o} G \leq n(4 n-3 k-5) \sqrt{n-1} .
$$

Proof. Let $G$ be a connected $k$-regular graph with diameter two and its vertices are $v_{1}, v_{2}, \ldots, v_{n}$. For any two locating vectors $\overrightarrow{v_{i}}$ and $\overrightarrow{v_{j}}$ the locating product $\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}$ in $G$ has maximum value if the vertices which have distance two from $v_{i}$ also have distance two from $v_{j}$. Since $G$ is regular, there are $n-k-1$ vertices that have distance two from $v_{i}$ and $k$ vertices that have distance one. Hence for any adjacent vertices $v_{i}$ and $v_{j}$, we have $\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}} \leq 4 n-3 k-5$, that implies to

$$
\left(\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}\right)^{2} \leq(4 n-3 k-5)^{2} .
$$

Also since $L o(G)$ contains at most $n(n-1)$ terms that are not zero, therefore, as earlier,

$$
\sum_{1 \leq i<j \leq n}\left(\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}\right)^{2} \leq \frac{1}{2}\left(n(n-1)(4 n-3 k-5)^{2}\right) .
$$

Hence by Theorem 9

$$
E_{L o} G \leq \sqrt{n \alpha}=\sqrt{2 n} \sqrt{\sum_{1 \leq i<j \leq n}\left(\overrightarrow{v_{i}} \cdot \overrightarrow{v_{j}}\right)^{2}}
$$

Hence,

$$
E_{L o} G \leq n(4 n-3 k-5) \sqrt{n-1} .
$$

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# Computing Comprehensive Gröbner Systems: A Comparison of Two Methods 

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#### Abstract

In this paper, we consider two main approaches to compute Gröbner bases for parametric polynomial ideals, namely the DisPGB algorithm developed by Montes [18] and the PGBMain proposed by Kapur, Sun and Wang [11]. The former algorithm creates new branches in the space of parameters during the construction of Gröbner basis of a given ideal in the polynomial ring of variables and the latter computes (at each iteration) a Gröbner basis of the ideal in the polynomial ring of the variables and parameters and creates new branches according to leading coefficients in terms of parameters. Therefore, the latter algorithm can benefit from the efficient implementation of Gröbner basis algorithm in each computer algebra system. In order to compare these two algorithms (in the same platform) we use the recent algorithm namely GVW due to Gao et al. [8] to compute Gröbner bases which makes the use of the $\mathrm{F}_{5}$ criteria proposed by Faugère to remove superfluous reductions [6]. We show that there exists a class of examples so that an incremental structure on the DisPGB algorithm by using the GVW algorithm is faster than the PGBMAIN by applying the same algorithm to compute Gröbner bases. The mentioned algorithms have been implemented in Maple and experimented with a number of examples


Keywords: Comprehensive Gröbner systems, DisPGB algorithm, PGBMAIN algorithm, $\mathrm{F}_{5}$ criteria, GVW algorithm.

[^2]
## 1 Introduction

One of the most important tools in computer algebra is Gröbner bases. This concept along with the first algorithm to compute it, were introduced in 1965 by Buchberger in his PhD thesis (see [3]). His two criteria and the implementation methods [4] transformed Gröbner bases to a powerful tool to tackle many important problems in polynomial ideals theory. However, Buchberger's algorithm was not efficient in practice for large polynomial systems. In 1983, Lazard described a new algorithm to compute Gröbner bases, using linear algebra techniques [14]. In 1988, Gebauer and Möller have installed Buchberger's two criteria on Buchberger's algorithm in an efficient manner (see [9]). In 1999, Faugère described his $\mathrm{F}_{4}$ algorithm to compute Gröbner bases (see [5]). This algorithm (which is an efficient algorithm based on [9], [14]) exploits fast linear algebra on sparse matrices, and has been implemented in Maple and Magma. In 2002, Faugère has described the $F_{5}$ algorithm; a new incremental algorithm which makes the use of the $F_{5}$ criteria to compute Gröbner bases [6] (see also [17]). Ars and Hashemi [1] proposed a non-incremental version of this algorithm by defining new orderings on the signatures to make it independent from the order of the input polynomials. Gao et al. [7] presented $\mathrm{G}^{2} \mathrm{~V}$; a variant of the $\mathrm{F}_{5}$ algorithm which is simpler and more efficient than $\mathrm{F}_{5}$. Finally, Gao et al. [8] proposed a new framework more general than the $\mathrm{G}^{2} \mathrm{~V}$ algorithm, namely GVW to compute simultaneously Gröbner bases for an ideal and its syzygy module.

The concept of comprehensive Gröbner bases can be considered as an extension of Gröbner bases of polynomials over fields to polynomials with parametric coefficients. This extension plays an important role in the applications such as constructive algebraic geometry, robotics, electrical network, automatic theorem proving and so on (see e.g. [15], [16], [18], [19]). Comprehensive Gröbner bases and comprehensive Gröbner systems (for simplification, we employ the term CGS to refer to comprehensive Gröbner system) were introduced in 1992 by Weispfenning [24]. He proved that any parametric polynomial ideal has a finite CGS and described an algorithm to compute it. In 2002,

Montes [18] proposed a more efficient algorithm namely DisPGB for computing CGSs. Suzuki and Sato [21] provided an important improvement for computing CGSs using only computations of reduced Gröbner bases in polynomial rings over ground fields (we subsequently refer to this algorithm as the Suzuki-Sato algorithm). In 2010, Kapur et al. [11] by combining Weispfenning's algorithm [24] with the Suzuki-Sato algorithm, gave a new algorithm (that we refer to as PGBMain algorithm) for computing CGSs (see also [12], [13]). Finally, Montes and Wibmer in [20] presented the GröbnerCover algorithm (see [23]) which computes a finite partition of the space of parameters into locally closed subsets together with polynomial data, from which the reduced Gröbner basis for a given parameter point can immediately be determined.

It is worth noting that PGBMain at each iteration computes the Gröbner basis over a polynomial ring in the variables and parameters. Therefore, it makes the use of a Gröbner basis function in each computer algebra system. On the other hand, DisPGB reduces the computation in a polynomial ring of only variables by creating new branches when a new polynomial with an undecidable coefficient is constructed. So a natural question arises: Which of these two algorithms is more efficient in practice? In this paper, we consider this question by proposing an incremental structure on DisPGB by applying GVW equipped with the $\mathrm{F}_{5}$ criteria. We have implemented in Maple this algorithm and also PGBMAIN by using GVW as the engine of Gröbner bases computation. We compare the performance of these algorithms on a number of polynomial ideals by showing that there exists a class of ideals for which our new variant of DisPGB is more efficient than PGBMain. We shall mention that due to the structure of PGBMAIN, its outputs in general have less number of branches than DisPGB.

Now, we give the structure of the paper. Section 2 contains the basic definitions and notations related to CGSs, and a short description of DisPGB. In Section 3, we present briefly GVW. Section 4 is devoted to the description of our new algorithm namely GVWDisPGB for computing CGSs. In Section 5, we show the performance of this algorithm w.r.t. our implementation of PGBMAIN in Maple and the
function cgsdr of Singular via some examples.

## 2 Comprehensive Gröbner systems and DisPGB algorithm

In this section, we recall the basic definitions and notations concerning CGSs, and describe briefly the DisPGB algorithm.
Let $R=K[\mathbf{x}]$ be a polynomial ring, where $\mathbf{x}=x_{1}, \ldots, x_{n}$ is the sequence of variables and $K$ an arbitrary field. Let $I=\left\langle f_{1}, \ldots, f_{k}\right\rangle$ be the ideal of $R$ generated by the polynomials $f_{1}, \ldots, f_{k}$. Also, let $f \in R$ and let $\prec$ be a monomial ordering on $R$. The leading monomial of $f$ is the greatest monomial (w.r.t. $\prec$ ) appearing in $f$, and we denote it by $\operatorname{LM}(f)$. The leading coefficient of $f$, denoted by $\operatorname{LC}(f)$, is the coefficient of $\mathrm{LM}(f)$. The leading term of $f$ is $\mathrm{LT}(f)=\mathrm{LC}(f) \mathrm{LM}(f)$. The leading term ideal of $I$ is defined to be

$$
\operatorname{LT}(I)=\langle\operatorname{LT}(f) \mid f \in I\rangle .
$$

A finite set $G=\left\{g_{1}, \ldots, g_{k}\right\} \subset I$ is called a Gröbner basis of $I$ w.r.t. $\prec$ if $\operatorname{LT}(I)=\left\langle\operatorname{LT}\left(g_{1}\right), \ldots, \operatorname{LT}\left(g_{k}\right)\right\rangle$. For more details, we refer to [2], pages 213-214.

Now consider $F=\left\{f_{1}, \ldots, f_{k}\right\} \subset S=K[\mathbf{a}, \mathbf{x}]$, where $\mathbf{a}=$ $a_{1}, \ldots, a_{m}$ is the sequence of parameters. Let $\prec_{\mathbf{x}}\left(\right.$ resp. $\left.\prec_{\mathbf{a}}\right)$ be a monomial ordering involving the $x_{i}$ 's (resp. $a_{i}$ 's). We also need a compatible elimination product ordering $\prec_{\mathbf{x}, \mathbf{a}}$. It is defined as follows: For all $\alpha, \gamma \in \mathbb{Z}_{\geq 0}^{n}$ and $\beta, \delta \in \mathbb{Z}_{\geq 0}^{m}$

$$
\mathbf{x}^{\gamma} \mathbf{a}^{\delta} \prec_{\mathbf{x}, \mathbf{a}} \mathbf{x}^{\alpha} \mathbf{a}^{\beta} \text { iff }\left\{\begin{array}{l}
\mathbf{x}^{\gamma} \prec_{\mathbf{x}} \mathbf{x}^{\alpha} \quad \text { or } \\
\mathbf{x}^{\gamma}=\mathbf{x}^{\alpha} \quad \text { and } \mathbf{a}^{\delta} \prec_{\mathbf{a}} \mathbf{a}^{\beta} .
\end{array}\right.
$$

Now, we recall the definition of a CGS for a parametric ideal.
Definition 1. Let $G=\left\{\left(G_{i}, N_{i}, W_{i}\right)\right\}_{i=1}^{\ell}$ be a finite set of triples, where $N_{i}, W_{i} \subset K[\mathbf{a}]$ and $G_{i} \subset S$ are finite for $i=1, \ldots, \ell$. The set $G$ is called a $C G S$ for $\langle F\rangle$ w.r.t. $\prec_{\mathbf{x}, \mathbf{a}}$ if for any specialization $\sigma: K[\mathbf{a}] \rightarrow \bar{K}$ with $\bar{K}$ the algebraic closure of $K$ there exists $i$ such that the following conditions hold

- $\sigma\left(G_{i}\right) \subset \bar{K}[\mathbf{x}]$ is a Gröbner basis for $\sigma(\langle F\rangle) \subset \bar{K}[\mathbf{x}]$ w.r.t. $\prec_{\mathbf{x}}$
- $\sigma(p)=0$ for each $p \in N_{i}$ and $\sigma(q) \neq 0$ for each $q \in W_{i}$.

For each $i$, the set $N_{i}$ (resp. $W_{i}$ ) is called a (resp. non-) null conditions set. Each pair $\left(N_{i}, W_{i}\right)$ is called a specification (for a homomorphism $\sigma$ if both the conditions in the above definition are satisfied).

Now, we describe shortly Montes DisPGB algorithm to compute CGSs for parametric ideals (see [15], [18]). The main idea of DisPGB is based on discussing the nullity or not w.r.t. a given specification $(N, W)$ of the leading coefficients of the polynomials appearing at each step (this process is performed by NewCond subalgorithm). Let us consider a set $F \subset S$ of parametric polynomials. For a given polynomial $f \in F$, and a given specification $(N, W)$, NewCond is called. Three cases are possible: If $\mathrm{LC}(f)$ specializes to zero w.r.t. $(N, W)$, we replace $f$ by $f-\mathrm{LT}(f)$, and then start again. If $\mathrm{LC}(f)$ specializes to a non-zero element, we continue with the next polynomial in $F$. Otherwise (if $\mathrm{LC}(f)$ is not decidable), the subalgorithm Branch is called to create two supplementary cases by assuming $\operatorname{LC}(f)=0$ and $\operatorname{LC}(f) \neq 0$. Therefore, two new disjoint branches with the specifications $(N \cup\{\mathrm{LC}(f)\}, W)$ and $(N, W \cup\{\mathrm{LC}(f)\})$ will be made. This procedure will continue until every polynomial in $F$ has a non-null leading coefficient w.r.t. the current specification. Then, we proceed with CondPGB: This algorithm receives as input a set of parametric polynomials and a specification ( $N, W$ ) and using Buchberger's algorithm, it creates new polynomials. When a new polynomial is generated, NewCond verifies whether its leading coefficient gives a new condition or not. If a new condition is found it stops, and Branch is called to make two new disjoint branches. Otherwise, this continues and computes a Gröbner basis for $\langle F\rangle$, according to the current specification. The collection of these bases, gives a CGS for $\langle F\rangle$.

## $3 \quad \mathbf{F}_{5}$ criteria and GVW algorithm

This section aims to present the $\mathrm{F}_{5}$ theory. After recalling some notations and definitions (used also in the next section), we state the main
theorem of [6] which forms the basis of the $\mathrm{F}_{5}$ algorithm. Finally, we present briefly the GVW algorithm following [8].

Let $R=K[\mathbf{x}]$ be a polynomial ring, where $\mathbf{x}=x_{1}, \ldots, x_{n}$ is the sequence of variables, $K$ is an arbitrary field and $I=\left\langle f_{1}, \ldots, f_{k}\right\rangle$ is the ideal of $R$ generated by the polynomials $f_{1}, \ldots, f_{k}$. Let $R^{k}$ be a $k$-dimensional $R$-module and $\mathbf{f}_{1}, \ldots, \mathbf{f}_{\mathrm{k}}$ its canonical basis. A module monomial is an element of $R^{k}$ of the form $m \mathbf{f}_{\mathbf{i}}$, where $m \in R$ is a monomial. Given two module monomials $m \mathbf{f}_{\mathbf{i}}$ and $m^{\prime} \mathbf{f}_{\mathbf{j}}$, one can extend a monomial ordering $\prec$ on $R$ to a module monomial ordering on $R^{k}$ in different ways. In [8] the authors proposed four different module monomial orderings. Below, we recall one of them under which GVW closely corresponds to the $\mathrm{G}^{2} \mathrm{~V}$ algorithm presented in [7].

$$
m \mathbf{f}_{i}<m^{\prime} \mathbf{f}_{j} \text { if }\left\{\begin{array}{l}
j<i \quad \text { or } \\
i=j \text { and } m \prec m^{\prime} .
\end{array}\right.
$$

For an element $\mathbf{g}=\sum_{i=1}^{k} g_{i} \mathbf{f}_{\mathbf{i}} \in R^{k}$, we define the index of $\mathbf{g}$, index $(\mathbf{g})$ to be the lowest integer $i$ such that $g_{i} \neq 0$. Let index $(\mathbf{g})=i_{0}$, then we call $\operatorname{LM}\left(g_{i_{0}}\right) \mathbf{f}_{\mathbf{i}_{0}}$ the module leading monomial of $\mathbf{g}$ and denote it by $\operatorname{MLM}(\mathbf{g})$. Also, we use $\operatorname{LM}(\mathbf{g})$ to denote $\operatorname{LM}\left(\sum_{i=1}^{k} g_{i} f_{i}\right)$.

The elements of the form $r=\left(m \mathbf{f}_{i}, f\right) \in A=R^{k} \times R$, where $m$ is a monomial, $i$ an integer and $f$ a polynomial are called labelled polynomials. $\mathrm{S}(r)=m \mathbf{f}_{i}$ is called the signature part of $r$ and poly $(r)=f$ the polynomial part of $r$. Denote $\psi$ the map $\psi: R^{k} \rightarrow R$ so that $\psi\left(g_{1}, \ldots, g_{k}\right)=g_{1} f_{1}+\cdots+g_{k} f_{k}$. A labelled polynomial $r=(\mathrm{S}(r), \operatorname{poly}(r))$ is called admissible if there exists $g \in R^{k}$ such that $\psi(g)=\operatorname{poly}(r)$ and $\operatorname{MLM}(g)=\mathrm{S}(r)$. We define the following operations on labelled polynomials: Let $r=\left(m \mathbf{f}_{i}, f\right)$ be a labelled polynomial, $u$ a monomial and $c$ a constant. Then we define $u r=\left(u m \mathbf{f}_{i}, u f\right)$ and $c r=\left(m \mathbf{f}_{i}, c f\right)$. These definitions obviously preserve the admissibility. The special reduction of $\mathrm{F}_{5}$ also preserves it and ensures that during a Gröbner basis computation by $\mathrm{F}_{5}$, the labelled polynomials always take the minimal possible signature and remain admissible. We also need the following definitions to state the main theorem.

Definition 2 ( $\mathrm{F}_{5}$ criterion). An admissible labelled polynomial $r=$
$\left(m \mathbf{f}_{\mathbf{i}}, f\right)$ is called normalized if we have $m \notin \mathrm{LT}\left(\left\langle f_{i+1}, \ldots, f_{k}\right\rangle\right)$. A pair $(r, s)$ of admissible labelled polynomials is normalized if ur and vs are normalized, where $r=\left(m \mathbf{f}_{\mathbf{i}}, f\right), s=\left(m^{\prime} \mathbf{f}_{\mathbf{j}}, g\right), u=\frac{\operatorname{lcm}(\operatorname{LM}(f), \operatorname{LM}(g))}{\operatorname{LM}(f)}$ and $v=\frac{\operatorname{lcm}(\operatorname{LM}(f), \mathrm{LM}(g))}{\operatorname{LM}(g)}$.

Faugère has described $\mathrm{F}_{5}$ as an incremental algorithm to use the $\mathrm{F}_{5}$ criterion, i.e. to compute the Gröbner basis of $I$, it computes respectively the Gröbner bases of the ideals

$$
\left\langle f_{k}\right\rangle,\left\langle f_{k-1}, f_{k}\right\rangle, \ldots,\left\langle f_{1}, \ldots, f_{k}\right\rangle
$$

In the following, we define the concept of $t$-representation for labelled polynomials, imposing additional conditions on the signatures (see [2, page 219]).

Definition 3. Let $P \subset A$ be a finite set of labelled polynomials, and $r, t \in A$ two labelled polynomials with $\operatorname{poly}(r)=f$, where $f \neq 0$. We say that $f=\sum_{p_{i} \in P} h_{i} \operatorname{poly}\left(p_{i}\right)$ is a $t$-representation of $r$ w.r.t. $P$ if for all $p_{i} \in P$ with $\operatorname{poly}\left(p_{i}\right) \neq 0$ we have

$$
\operatorname{LM}\left(h_{i}\right) \mathrm{LM}\left(\operatorname{poly}\left(p_{i}\right)\right) \preceq \operatorname{LM}(\operatorname{poly}(t)) \text { and } \operatorname{LM}\left(h_{i}\right) \mathrm{S}\left(p_{i}\right)<\mathrm{S}(r) .
$$

This property is denoted by $r=\mathcal{O}_{P}(t)$. We write $s=o_{P}(t)$ if there exists labelled polynomial $t^{\prime} \in A$ satisfying $\mathrm{S}\left(t^{\prime}\right)<\mathrm{S}(t)$ and $\mathrm{LM}\left(\operatorname{poly}\left(t^{\prime}\right)\right) \prec \mathrm{LM}(\operatorname{poly}(t))$ such that $s=\mathcal{O}_{P}\left(t^{\prime}\right)$.

Let $f, g \in R$ be two polynomials. The $S$-polynomial of $f$ and $g$ is defined as:

$$
\operatorname{Spoly}(f, g)=\frac{\operatorname{lcm}(\operatorname{LM}(f), \operatorname{LM}(g))}{\operatorname{LT}(f)} f-\frac{\operatorname{lcm}(\operatorname{LM}(f), \operatorname{LM}(g))}{\operatorname{LT}(g)} g
$$

Let $r=(\mathrm{S}(r), f)$ and $s=(\mathrm{S}(s), g)$ be two admissible labelled polynomials such that $v \mathrm{~S}(s)<u \mathrm{~S}(r)$ with $u=\frac{\operatorname{lcm}(\operatorname{LM}(f), \mathrm{LM}(g))}{\operatorname{LM}(f)}$ and $v=$ $\frac{\operatorname{lcm}(\operatorname{LM}(f), \operatorname{LM}(g))}{\operatorname{LM}(g)}$. Then, we define $\operatorname{Spoly}(r, s)=(u \mathrm{~S}(r), \operatorname{Spoly}(f, g))$.
Theorem 1. ([6]) Let $I=\left\langle f_{1}, \ldots, f_{k}\right\rangle \subset R$. Let $G \subset A$ be a finite set of admissible labelled polynomials such that

- for every $i$, we have $f_{i}=\operatorname{poly}\left(r_{i}\right)$ for some $r_{i} \in G$,
- for each $\left(r_{i}, r_{j}\right) \in G \times G$ which is normalized, $\operatorname{Spoly}\left(r_{i}, r_{j}\right)$ is either zero or equal to $o_{G}\left(u_{s} r_{s}\right)$, where
$u_{s}=\frac{\operatorname{lcm}\left(\operatorname{LM}\left(\text { poly }\left(r_{i}\right)\right), \mathrm{LM}\left(\text { poly }\left(r_{j}\right)\right)\right)}{\operatorname{LM}\left(\operatorname{poly}\left(r_{s}\right)\right)}$ for $s \in\{i, j\}$.
Then the set $\{\operatorname{poly}(r) \mid r \in G\}$ is a Gröbner basis for $I$.
Faugère in the $\mathrm{F}_{5}$ algorithm has used another criterion, namely IsRewritten criterion, to detect more useless critical pairs, however he has not declared it explicitly in [6]. We recall this criterion and refer to [10] for more details.

Definition 4 (IsRewritten criterion). With the above notations, let $u \in$ $R$ be a monomial and $r=\left(m \mathbf{f}_{\mathbf{i}}, f\right)$ an admissible labelled polynomial. Then, the pair $[u, r]$ is called rewritable if there exists an admissible labelled polynomial $r^{\prime}=\left(m^{\prime} \mathbf{f}_{\mathbf{i}}, f^{\prime}\right)$ computed after $r$, i.e. $\mathrm{S}(r)<\mathrm{S}\left(r^{\prime}\right)$, such that $m^{\prime}$ divides um. A pair $(r, s)$ of admissible labelled polynomials is rewritable if $[u, r]$ or $[v, s]$ is rewritable, where $r=\left(m \mathbf{f}_{\mathbf{i}}, f\right), s=$ $\left(m^{\prime} \mathbf{f}_{\mathbf{j}}, g\right), u=\frac{\operatorname{lcm}(\operatorname{LM}(f), \operatorname{LM}(g))}{\operatorname{LM}(f)}$ and $v=\frac{\operatorname{cm}(\operatorname{LM}(f), \mathrm{LM}(g))}{\operatorname{LM}(g)}$.

As the following proposition yields, if a critical pair is rewritable, its S-polynomial has a standard representation w.r.t. the last computed Gröbner basis, and therefore the $\mathrm{F}_{5}$ algorithm deletes all such pairs.

Proposition 1. ( [10]) Let $I=\left\langle f_{1}, \ldots, f_{k}\right\rangle \subset R$ be an ideal. Let $r_{i}$ and $r_{j}$ be two labelled polynomials treated during an execution of the $F_{5}$ algorithm for computing the Gröbner basis of I. If $\left(r_{i}, r_{j}\right)$ is rewritable, then $\operatorname{Spoly}\left(r_{i}, r_{j}\right)$ is either zero or equal to $o_{G}\left(u_{s} r_{s}\right)$, where $u_{s}=\frac{\operatorname{lcm}\left(\operatorname{LM}\left(\operatorname{poly}\left(r_{i}\right)\right), \mathrm{LM}\left(\operatorname{poly}\left(r_{j}\right)\right)\right)}{\operatorname{LM}\left(\operatorname{poly}\left(r_{s}\right)\right)}$ for $s \in\{i, j\}$ (and so the pair $\left(r_{i}, r_{j}\right)$ can be omitted).

Throughout this paper, by the $F_{5}$ criteria we mean $F_{5}$ criterion and IsRewritten criterion. The main problem with the $\mathrm{F}_{5}$ algorithm is that it is difficult to both understand and implement. Gao et al. in [8] presented the GVW algorithm which seems to be simpler and more efficient than the $\mathrm{F}_{5}$ algorithm. That is why, we use this algorithm to apply the $\mathrm{F}_{5}$ criteria on DisPGB.

To explain more precisely the structure of GVW, let us suppose that we are going to compute a Gröbner basis for the ideal $\left\langle f_{1}, \ldots, f_{k}\right\rangle$ with respect to a monomial ordering $\prec$. The main difference of GVW and $F_{5}$ relies on the non-incremental structure of GVW which makes the use of different kinds of module monomial orderings. Let us consider e.g. module monomial ordering < defined as above (remark that GVW endowed with this ordering is equivalent to $\left.\mathrm{G}^{2} \mathrm{~V}[7]\right)$. Without lose of generality, suppose that for all labelled polynomials $(\mathbf{u}, v), \mathrm{LC}(v)=1$. Given two labelled polynomials $p_{1}=\left(\mathbf{u}_{1}, v_{1}\right)$ and $p_{2}=\left(\mathbf{u}_{2}, v_{2}\right)$, the J-pair of $p_{1}$ and $p_{2}$ is the new pair $t_{i} p_{i}$, where $t_{i}=\frac{\operatorname{lcm}\left(\operatorname{LM}\left(v_{1}\right), \operatorname{LM}\left(v_{2}\right)\right)}{\operatorname{LM}\left(v_{i}\right)}$ and $t_{i} v_{i}=\max _{<}\left\{t_{1} \mathbf{u}_{1}, t_{2} \mathbf{u}_{2}\right\}$, provided that $t_{1} \mathbf{u}_{1} \neq t_{2} \mathbf{u}_{2}$.

At the first step, GVW begins with the initial set of J-pairs $\left\{\left(\mathbf{f}_{1}, f_{1}\right), \ldots,\left(\mathbf{f}_{k}, f_{k}\right)\right\}$. It takes in each step the smallest J-pair (w.r.t. signature) and repeatedly performs only regular top reductions until it is no longer regular top reducible. A labelled polynomial $\left(\mathbf{u}_{1}, v_{1}\right)$ is top reducible by $\left(\mathbf{u}_{2}, v_{2}\right)$ if there exists a monomial $t \in R$ such that $\mathrm{LM}\left(v_{1}\right)=t \mathrm{LM}\left(v_{2}\right)$ and $t \mathbf{u}_{2}<\mathbf{u}_{1}$. The corresponding top reduction is

$$
\left(\mathbf{u}_{1}, v_{1}\right)-t\left(\mathbf{u}_{2}, v_{2}\right)=\left(\mathbf{u}_{1}, v_{1}-t v_{2}\right) .
$$

If $t \mathbf{u}_{2}=\mathbf{u}_{1}$, the top reduction is called super, otherwise it is called regular. Let $(\mathbf{u}, v)$ be the result of the reduction of a labelled polynomial. If $v \neq 0$, we add $(\mathbf{u}, v)$ to the current Gröbner basis, and form the new J-pairs. Otherwise, GVW uses $\mathbf{u}$ to delete useless J-pairs: For any labelled polynomial $\left(\mathbf{u}^{\prime}, v^{\prime}\right)$, if $t \mathbf{u}=\mathbf{u}^{\prime}$ for some monomial $t$, then we can discard ( $\mathbf{u}^{\prime}, v^{\prime}$ ), provided that $t \mathrm{LM}(v) \prec \operatorname{LM}\left(v^{\prime}\right)$. Indeed, this is a special case of super top reduction, where ( $\mathbf{u}^{\prime}, v^{\prime}$ ) is super top reducible by $(\mathbf{u}, 0)$. Furthermore, a J-pair $(\mathbf{u}, v)$ is called covered by $G$ if there is a pair $\left(\mathbf{u}^{\prime}, v^{\prime}\right) \in G$ so that $\mathbf{u}^{\prime}$ divides $\mathbf{u}$ and $t \mathrm{LM}\left(v^{\prime}\right) \prec \mathrm{LM}(v)$ (strictly smaller), where $t=\mathbf{u}^{\prime} / \mathbf{u}$ is a monomial.

Remark 1. The relation of the criteria used in $G V W$ with $F_{5}$ criterion (Theorem 1) and IsRewritten criterion (Proposition 1) is illustrated in [8, Corollaries 2.5 and 2.6], respectively. Furthermore the correctness and termination of $G V W$ are proved in [8, Theorem 3.1].

## 4 Description of the new algorithm

In this section, we show how to combine the GVW algorithm with the DisPGB algorithm to compute CGSs for parametric ideals. For this, we use the improved version of DisPGB described in [15], and an incremental structure on DisPGB to be able to apply the $\mathrm{F}_{5}$ criteria. More precisely, let $I=\left\langle f_{1}, \ldots, f_{k}\right\rangle \subset K[\mathbf{a}, \mathbf{x}]$ be a parametric ideal, where $\mathbf{x}=x_{1}, \ldots, x_{n}$ is the sequence of variables and $\mathbf{a}=a_{1}, \ldots, a_{m}$ is the sequence of parameters. Let $\prec_{\mathrm{x}}\left(\right.$ resp. $\left.\prec_{\mathrm{a}}\right)$ be a monomial ordering involving the $x_{i}$ 's (resp. $a_{i}$ 's). Then, to compute a CGS for $I$, we compute CGSs of the ideals $\left\langle f_{k}\right\rangle,\left\langle f_{k-1}, f_{k}\right\rangle, \ldots,\left\langle f_{1}, \ldots, f_{k}\right\rangle$ respectively and for each $i$, we use the CGS of $\left\langle f_{i+1}, \ldots, f_{k}\right\rangle$ to compute a CGS for $\left\langle f_{i}, \ldots, f_{k}\right\rangle$.

Example 1. In this simple example, we show how an incremental structure may be used to compute a CGS for an ideal. Let $I=\langle a x+1, b y+1\rangle \subset K[\mathbf{a}, \mathbf{x}]$, where $\mathbf{a}=a, b$ and $\mathbf{x}=x, y$. We compute first a CGS for the ideal $\langle b y+1\rangle$ which is equal to $\{(\{1\},\{b\},\{ \}),(\{b y+1\},\{ \},\{b\})\}$. Now, we will discuss the addition of ax +1 to each member of this system according to nullity or not of a. It follows the following CGS for I:
$\{(\{1\},\{b\},\{a\}),(\{1\},\{a, b\},\{ \}),(\{1\},\{a\},\{b\}),(\{b y+1, a x+1\},\{ \},\{b, a\})\}$.
We describe now the main algorithm GVWDisPGB which computes incrementally a CGS for a given ideal.

```
Algorithm 1 GVWDisPGB
Require: \(F\) : finite subset of \(S\)
Ensure: A CGS for \(\langle F\rangle\)
    global: List, Grob, JP
    List:=Null
    \(\left\{f_{1}, \ldots, f_{k}\right\}:=\) InterReduce \(\left(F, \prec_{\mathbf{x}, \mathbf{a}}\right)\)
    \(\operatorname{Branch}\left(\left(1, f_{k}\right),\{ \},\{ \},\{ \},\{ \},\{ \}\right)\)
    for i from \(k-1\) to 1 do
        \(\operatorname{IncDisPGB}\left(f_{i},\{\operatorname{List}\}\right)\)
    end for
    Return (List)
```

Note that the function InterReduce $\left(F, \prec_{\mathbf{x}, \mathbf{a}}\right)$ inter-reduces a list of polynomials $F$ w.r.t. $\prec_{\mathbf{x}, \mathbf{a}}$; i.e. every polynomial in $F$ must be divided by the remaining elements of $F$ such that at the output no monomial of any polynomial of $F$ is divisible by the leading monomials of the other polynomials in $F$. In the above algorithm, List (resp. JP) is a global variable in which (and at each iteration) we save the computed CGS (resp. set of J-pairs). That is why, at the beginning of each iteration we must keep them null (see the next subalgorithm). Indeed, Branch calculates a CGS for $\left\langle f_{k}\right\rangle$ and save it in List. Then, for any $i$ between $k-1$ and 1 , IncDisPGB computes a CGS for the ideal $\left\langle f_{i}, \ldots, f_{k}\right\rangle$ (using the CGS of $\left\langle f_{i+1}, \ldots, f_{k}\right\rangle$ which has already been computed) and saves it in List. Thus, at the end, List is a CGS for the ideal $\left\langle f_{1}, \ldots, f_{k}\right\rangle$. Now, we describe IncDisPGB.

```
Algorithm 2 IncDisPGB
Require: \(\left\{\begin{aligned} f_{i}: & \text { a polynomial with } 1 \leq i \leq k-1 \\ \left\{G_{1}, \ldots, G_{t}\right\}: & \text { a CGS for }\left\langle f_{i+1}, \ldots, f_{k}\right\rangle\end{aligned}\right.\)
Ensure: A CGS for \(\left\langle f_{i}, \ldots, f_{k}\right\rangle\)
    L:=Null
    for \(j\) from 1 to \(t\) do
        JP:=\{ \}, List:=Null
        \((\) Grob, \(N, W):=G_{j}\)
        \(f:=\bar{f}_{i}^{N}\) (the remainder of the division of \(f_{i}\) by \(N\) )
        if \(f=0\) then
            List:=List, \(G_{j}\)
        else
            \(\operatorname{Branch}((1, f), \operatorname{Grob}, N, W,\{ \},\{(0, g) \mid g \in \operatorname{Grob}\})\)
        end if
        \(\mathrm{L}:=\mathrm{L}, \mathrm{List}\)
    end for
    List:=L
```

```
Algorithm 3 BRANCH
Require: \(\left\{\begin{aligned}(u, f): & \text { labelled polynomial } \\ B: & \text { specializing basis } \\ N, W: & \text { where }(N, W) \text { is an specification } \\ J P: & \text { set of J-pairs } \\ R: & \text { set of computed labelled polynomials }(v, g) \text { s.t. } g \in B\end{aligned}\right.\)
Ensure: It stores the refined ( \(B^{\prime}, N^{\prime}, W^{\prime}, J P^{\prime}, R^{\prime}\) ), and either creates two
    new vertices when necessary or marks the vertex as terminal
    \((N, W):=\operatorname{CanSpec}(\mathrm{N}, \mathrm{W})\)
    \((c d, f, N, W):=\operatorname{NewCond}(f, N, W)\)
    if \(c d=\{ \}\) then
        (test, \(\left.\left(u^{\prime}, f^{\prime}\right), B^{\prime}, N^{\prime}, W^{\prime}, J P^{\prime}, R^{\prime}\right):=\operatorname{CondPGB}((u, f), B, N, W, J P, R)\)
        if test then
            if \(J P=\{ \}\) then
                List:=List, \(\left(B^{\prime}, N, W\right)\)
            end if
        else
            \(\operatorname{Branch}\left(\left(u^{\prime}, f^{\prime}\right), B^{\prime}, N^{\prime}, W^{\prime}, J P^{\prime}, R^{\prime}\right)\)
        end if
    else
        \(\operatorname{Branch}((u, f), B, N \cup c d, W, J P, R)\)
        \(\operatorname{Branch}((u, f), B, N, W \cup c d, J P, R)\)
    end if
```

To clarify Branch, suppose that it receives a polynomial $f$ and a specification $(N, W)$. If $\mathrm{LC}(f)$ is not decidable w.r.t. $(N, W)$, this subalgorithm creates two supplementary cases by adding $\mathrm{LC}(f)=0$ and $\mathrm{LC}(f) \neq 0$ to the set of null and non-null conditions set respectively. Therefore, two new disjoint branches with the specifications $(N \cup\{\mathrm{LC}(f)\}, W)$ and $(N, W \cup\{\mathrm{LC}(f)\})$ will be made. We explain here the significance of the variables $B$ and $c d$ in Branch. The variable $B$ contains all polynomials related to the corresponding branch which will be completed, and at the end of the branch it will be the Gröbner basis for the corresponding specification. We explain now the variable $c d$. When we call $\operatorname{NewCond}(f, N, W)$, if $\operatorname{LC}(f)$ is decidable w.r.t the specification $(N, W)$, it returns $(\emptyset, f-\operatorname{LT}(f), N, W)$ in the case that $\mathrm{LC}(f)$ specializes to zero w.r.t. $(N, W)$, and it returns

## A. Hashemi, B. M.-Alizadeh, M. Dehghani Darmian

$(\emptyset, f, N, W)$ in the case that $\mathrm{LC}(f)$ does not specialize to zero w.r.t. $(N, W)$. Otherwise, it returns $\left(c d, f^{\prime}, N, W\right)$, where $c d$ contains one of the non-decidable factors (w.r.t $(N, W)$ ) of $\mathrm{LC}(f)$. We describe below the NewCond subalgorithm in which FacVar factors a polynomial in parameters. Remark that $\operatorname{FACVAR}\left(\operatorname{LC}\left(f^{\prime}\right)\right) \backslash W^{\prime}$ returns only one factor of $\mathrm{LC}\left(f^{\prime}\right)$.

```
Algorithm 4 NewCond
Require: \(\left\{\begin{aligned} f: & \text { a parametric polynomial } \\ N, W: & \text { where }(N, W) \text { is an specification }\end{aligned}\right.\)
Ensure: \(\left\{\begin{aligned} c d: & \text { set of a new condition } \\ f^{\prime}: & \text { a parametric polynomial } \\ N^{\prime}, W^{\prime}: & \text { where }\left(N^{\prime}, W^{\prime}\right) \text { is an specification }\end{aligned}\right.\)
    \(f^{\prime}:=f\)
    test:=true
    \(N^{\prime}:=N\)
    while test do
        if \(\operatorname{LC}\left(f^{\prime}\right) \in \sqrt{\left\langle N^{\prime}\right\rangle}\) then
            \(N^{\prime}:=\) a Gröbner basis for \(\left\langle N^{\prime}, \mathrm{LC}\left(f^{\prime}\right)\right\rangle\) w.r.t. \(\prec_{\mathrm{a}}\)
            \(f^{\prime}:=f^{\prime}-\operatorname{LM}(f)\)
        else
            test:=false
        end if
        \(f^{\prime}:=\overline{f^{\prime}}{ }^{N^{\prime}}\)
        \(W^{\prime}:=\left\{\bar{w}^{N^{\prime}} \mid w \in W\right\}\)
        \(c d:=\operatorname{FacVar}\left(\operatorname{LC}\left(f^{\prime}\right)\right) \backslash W^{\prime}\)
    end while
    Return ( \(c d, f^{\prime}, N^{\prime}, W^{\prime}\) )
```

We shall note that $\sqrt{I}$ denotes the radical of $I$, i.e., the set of all elements for which some positive power lies in $I$. By Hilbert Nullstellensatz, if a polynomial over an algebraically closed field vanishes on the vanishing set of an ideal, then a power of the polynomial belongs to the ideal, see [2, Theorem 7.40]. For a radical membership test (which has been used in the algorithm), we refer to [2, page 268]. The next algorithm is a variant of the GVW algorithm (to apply the $\mathrm{F}_{5}$ criteria) for parametric ideals.

```
Algorithm 5 CondPGB
Require: \(\left\{\begin{aligned}(u, f): & \text { labelled polynomial } \\ B: & \text { specializing basis } \\ N, W: & \text { where }(N, W) \text { is an specification } \\ J P: & \text { set of J-pairs } \\ R: & \text { set of labelled polynomials }\end{aligned}\right.\)
Ensure: (flag, \((v, g), B^{\prime}, N^{\prime}, W^{\prime}, J P^{\prime}, R^{\prime}\) ) with flag=true when \(f\) does not ge-
    nerate no new condition and no new JPair and \(B^{\prime}=B \cup\{f\}\) is a Gröbner ba-
    sis for the corresponding branch; otherwise, flag=false and \(B, N, W, J P, R\)
    are updated
    \(B:=\bar{B}^{N}\); (if a polynomial in \(B\) is reduced to zero, then remove all J-pairs
    in \(J P\) containing this polynomial)
    \(R:=\left\{\left(v, \bar{g}^{N}\right) \mid(v, g) \in R\right\}\)
    \(L M:=\{\operatorname{LM}(g) \mid g \in\) Grob \(\}\)
    \(f:=\bar{f}^{N}\)
    \(f:=\bar{f}^{\text {Grob }}\)
    if \(f=0\) and \(J P=\{ \}\) then
        Return (true, \((0,0), B, N, W,\{ \},\{ \})\)
    end if
    if \(f \neq 0\) then
        \((c d, f, N, W):=\operatorname{NewCond}(f, N, W)\)
        if \(f=0\) then
            Return (true, \((0,0), B, N, W,\{ \},\{ \})\)
        end if
        if \(c d \neq\{ \}\) then
            \(J P:=J P \cup\{\operatorname{JPair}((u, f), r) \mid r \in R\}\)
            add \((u, f)\) into \(R\) and add \(f\) into \(B\)
            sort \(J P\) by increasing signature
        else
            Return (false, \((u, f), B, N, W, J P, R)\)
        end if
    end if
```


## A. Hashemi, B. M.-Alizadeh, M. Dehghani Darmian

```
Algorithm 5 Continuation of CondPGB
    while \(J P \neq\{ \}\) do
        select and remove the first J-pair \(P\) from \(J P\)
        \((v, g):=\) Reduction \((P, R)\) performing only regular top reductions
        if \(g=0\) then
            \(L M:=L M \cup\{v\}\)
            remove any pair in \(J P\) if \(v\) divides its signature
        else
            \(g:=\bar{g}^{N}\)
            \((c d, g, N, W):=\operatorname{NewCond}(g, N, W)\)
            if \(c d=\{ \}\) then
            if \(g \neq 0\) then
                    \(J P:=J P \cup\{\operatorname{JPair}((v, g), r) \mid r \in R\}\)
                    add \((v, g)\) into \(R\) and add \(g\) into \(B\)
                        sort \(J P\) by increasing signature
            end if
            else
            if \(g \neq 0\) then
                    Return (false, \((v, g), B, N, W, J P, R)\)
            end if
            end if
        end if
    end while
    Return (true, \((0,0), B, N, W,\{ \},\{ \})\)
```

Theorem 2. Suppose that we are willing to compute a CGS for the ideal $\langle F\rangle$ with $F=\left\{f_{i+1}, \ldots, f_{k}\right\}$ for some $i$. Let $N$ and $W$ be the null and non-null conditions sets respectively and let $\sigma$ a homomorphism such that $(N, W)$ is its specification. Let also $B$ be the set of polynomials satisfying the following conditions:

- $\sigma(B)$ is a basis of $\langle\sigma(F)\rangle$,
- $\sigma(\mathrm{LC}(g)) \neq 0$ for $g \in B$.

Suppose that the CondPGB algorithm outputs (test, $B^{\prime}, N^{\prime}, W^{\prime}, J P^{\prime}, R^{\prime}$ ). If test $=$ true, then $\sigma\left(B^{\prime}\right)$ is a Gröbner basis of $\langle\sigma(F)\rangle$, where $\left(N^{\prime}, W^{\prime}\right)$ is a specification of $\sigma$. If test $=$ false, then $B^{\prime}$ is an extended set of $B$

Computing Comprehensive Gröbner Systems ...
which contains at least one polynomial for which the actual specification ( $N, W$ ) cannot decide if its leading coefficient specializes to zero or not. The sets $N, W, J P, R$ are also updated to $N^{\prime}, W^{\prime}, J P^{\prime}, R^{\prime}$ respectively.

Proof. The structure of the new CondPGB is similar to that of the old CondPGB in [18]. Therefore, the proof of this theorem may be deduced from [18, Proposition 15] and [8, Theorem 3.1].

Remark 2. We shall note that the finite termination of this algorithm is guaranteed by that of the GVW algorithm [8, Theorem 3.1].

Remark 3. In the above algorithm, if $J P=\emptyset$, it returns the sets $B, N, W$, where $B$ is a Gröbner basis for the ideal $\left\langle f_{i+1}, \ldots, f_{k}\right\rangle$ for some $i$ w.r.t. the specification ( $N, W$ ). To speed-up the computations, we can first minimize $B$; i.e. discard any polynomial in $B$ whose leading monomial (w.r.t $\prec_{\mathrm{x}}$ ) is divisible by the leading monomial of another polynomial in $B$. Then, we may replace $B$ by InterReduce ( $B, \prec_{\mathbf{x}, \mathbf{a}}$ ) to have the reduced Gröbner basis. Indeed, this may reduce the number of J-pairs for the next step.

Below, we describe the CanSpec algorithm from [18] to produce $k$-quasi-canonical representation $\left(N^{\prime}, W^{\prime}\right)$ for a specification $(N, W)$.

Definition 5. A specification $(N, W)$ is called $k$-quasi-canonical if the following conditions hold:

- $N$ is the reduced Gröbner basis of the ideal containing all polynomials that specialize to zero in $K[\mathbf{a}]$, w.r.t. $\prec \mathbf{a}$.
- The polynomials in $W$ specializing to non-zero are reduced modulo $N$ and are irreducible over $K^{\prime}[\mathbf{a}]$, where $K^{\prime}$ is an algebraic extension of $K$.
- $\prod_{q \in W} q \notin \sqrt{\langle N\rangle}$ and the polynomials in $N$ are square-free.
- None of the polynomials in $N$ have an irreducible factor contained in $W$.


## A. Hashemi, B. M.-Alizadeh, M. Dehghani Darmian

For more details we refer to [18].

```
Algorithm 6 CANSPEC
Require: \(\{N, W\) : where \((N, W)\) is an specification
Ensure: \(\left\{\begin{array}{l}\text { test: } \begin{cases}\text { true: } & \text { if } N \text { and } W \text { are compatible } \\ \text { false : otherwise }\end{cases} \\ \left(N^{\prime}, W^{\prime}\right): \text { a k-quasi-canonical representation of }(N, W) .\end{array}\right.\)
    \(W^{\prime}:=\operatorname{FacVAR}\left(\left\{\bar{q}^{N}: q \in W\right\}\right) ;\)
    test:=true
    \(N^{\prime}:=N\) and \(h:=\prod_{q \in W} q\)
    if \(h \in \sqrt{\left\langle N^{\prime}\right\rangle}\) then
        test:=false
        \(N^{\prime}:=\{1\}\)
        Return (test, \(\left(N^{\prime}, W^{\prime}\right)\) )
    end if
    flag:=true
    while flag do
        flag:=false
        \(N^{\prime \prime}:=\) Drop any factor of a polynomial in \(N^{\prime} \cap W^{\prime}\), as well as multiple
        factors
        if \(N^{\prime \prime} \neq N^{\prime}\) then
            flag:=true
            \(N^{\prime}:=\) a Gröbner basis of \(\left\langle N^{\prime \prime}\right\rangle\) w.r.t. \(\prec_{\mathrm{a}}\)
            \(W^{\prime}:=\operatorname{FACVAR}\left(\left\{\bar{q}^{N^{\prime \prime}}: q \in W^{\prime}\right\}\right)\)
        end if
    end while
    Return (test , ( \(\left.N^{\prime}, W^{\prime}\right)\) )
```


## 5 Experiments and results

We have implemented all the algorithms described in this paper in Maple $15^{1}$. In this section, we compare the performance of GVWDisPGB algorithm with DisPGB and PGBMain algorithms. It should be noted that in PGBMain algorithm we use GVW algo-

[^3]rithm for the Gröbner basis computation. We use also the cgsdr function from grobcov.lib library which is a Singular [22] implementation of PGBMAIN. The following parametric ideals in the ring $S=K[a, b, c, d, m, n, r, t][x, y, z, u]$ were chosen, and our aim was to compute a CGS of each ideal w.r.t. the product of the orderings $u \prec_{l e x}$ $z \prec_{\text {lex }} y \prec_{\text {lex }} x$ and $t \prec_{\text {lex }} r \prec_{\text {lex }} n \prec_{\text {lex }} m \prec_{\text {lex }} d \prec_{\text {lex }} c \prec_{\text {lex }} b \prec_{\text {lex }} a$.

- EX. $1=\left[x^{3}+(d-a) x y+m-a,-c b a+a z^{2}+c x-d,(c-a) y^{2}+\right.$ $\left.x n+a, u^{3}+\left(a^{2}-1\right) x+n-m\right]$
- EX. $2=\left[a b u^{4}+b^{2} a^{2}+x y z-1, a y^{2}+n(m t-2) x z+a, b a z^{3}+t(2-\right.$ $\left.\left.b^{7}\right) x y z+x^{2} z-1\right]$
- EX. $3=\left[(c-1) y^{3}+(a c-b) x+d n, r x^{5}+(b a-c) z-n, z^{3}-(c-t) y\right]$
- EX. $4=\left[(a+1) x^{2}+a^{2} b(d-1) y^{2}+a, y^{2}+b x+c\left(c^{2}-4\right),(a-\right.$ d) $\left.z^{3}+a y^{2}+b\left(c^{4}-1\right)-1\right]$
- EX. $5=\left[a x+b y+c z^{6}, a x y+b y z^{6}+c z x, x y z^{6}-1\right]$
- EX. $6=\left[a x^{2}+c u+(a-3) x, b a y x+a y^{3}-c z+1,(t-3) u^{3}+\right.$ $\left.t u, b z^{3}+(1-b) z^{2}+m n x\right]$
- EX. $7=\left[c z^{5}+d a x^{2}+d b y, y z^{5} b+a x y+c z x, d x y z^{5}-b a\right]$
- EX. $8=\left[a m y^{2}+x^{3}-z b-b t-1, a m y^{3}+t x+n-t, z^{3}+t x-a b c d\right]$
- EX. $9=\left[(b a-c) z+r x^{4}-n, y^{2}+(a c-b) x+d, z^{2}-(c b-a) y+t\right]$
- EX. $10=\left[(m n-a) x+y^{2}+a, c x+z^{2}, x^{2}+(a d-c) y x+b a, u^{2}+\right.$ $\left.\left(a^{4}-4\right) x\right]$
A. Hashemi, B. M.-Alizadeh, M. Dehghani Darmian

Table 1. The performance comparison of different algorithms.

| Example | Method | Time (Sec) | Used Memory (GB) | Branch |
| :---: | :---: | :---: | :---: | :---: |
| EX. 1 | GVWDIsPGB | 11.77 | 0.7 | 140 |
|  | DisPGB | 26.59 | 1.65 | 149 |
|  | PGBMAIN | - | - | - |
|  | cgsdr | - | - | - |
|  | FirstGB | - | - | - |
| EX. 2 | GVWDIsPGB | 6.59 | 0.44 | 15 |
|  | DisPGB | 33.45 | 2.21 | 14 |
|  | PGBMAIN | - | - | - |
|  | cgsdr | - | - | - |
|  | FirstGB | - | - | - |
| EX. 3 | GVWDIsPGB | 40.35 | 2.5 | 71 |
|  | DisPGB | 72.81 | 4.85 | 59 |
|  | PGBMAIN | - | - | - |
|  | cgsdr | - | - | - |
|  | FirstGB | 119.15 | 11.38 | - |
| EX. 4 | GVWDIsPGB | 2.19 | 0.11 | 43 |
|  | DisPGB | - | - | - |
|  | PGBMAIN | - | - | - |
|  | cgsdr | - | - | - |
|  | FirstGB | 256.2 | 33.19 | - |
| EX. 5 | GVWDIsPGB | 10.65 | 0.67 | 10 |
|  | DisPGB | - | - | - |
|  | PGBMAIN | - | - | - |
|  | cgsdr | - | - | - |
|  | FirstGB | 3.24 | 0.31 | - |
| EX. 6 | GVWDIsPGB | 4.1 | 0.23 | 51 |
|  | DisPGB | 82.11 | 4.66 | 36 |
|  | PGBMAIN | - | - | - |
|  | cgsdr | 0.5 | - | 24 |
|  | FirstGB | 30.08 | 3.49 | - |
| EX. 7 | GVWDIsPGB | 9.57 | 0.62 | 17 |
|  | DisPGB | - | - | - |
|  | PGBMAIN | - | - | - |
|  | cgsdr | - | - | - |
|  | FirstGB | 3.96 | 0.39 | - |


| Example | Method | Time (Sec) | Used Memory (GB) | Branch |
| :---: | :---: | :---: | :---: | :---: |
| EX. 8 | GVWDIsPGB | 102.3 | 5.85 | 43 |
|  | DisPGB | - | - | - |
|  | PGBMain | - | - | - |
|  | cgsdr | - | - | - |
|  | FirstGB | 5.40 | 0.55 | - |
| EX. 9 | GVWDIsPGB | 10.38 | 6.7 | 119 |
|  | DisPGB | - | - | - |
|  | PGBMain | - | - | - |
|  | cgsdr | - | - | - |
|  | FirstGB | - | - | - |
| EX. 10 | GVWDISPGB | 22.38 | 1.4 | 245 |
|  | DisPGB | 10.11 | 0.67 | 55 |
|  | PGBMain | - | - | - |
|  | cgsdr | - | - | - |
|  | FirstGB | 136.8 | 12.45 | - |

We shall emphasize that from DisPGB we mean the classical DisPGB algorithm due to Montes [18]. The results are shown in the above tables, where the timings were conducted on a personal computer with 7 core, 8 GB RAM and 64 bits under the windows 7 operating system. All the computations are done over $\mathbb{Q}$. In these tables, the third and fourth columns show respectively the CPU time (in second) and the amount of required memory (in gigabytes) of the corresponding method computation. The last column indicates the number of branches of the output CGS. Furthermore, "First GB" method stands for the computation of the reduced Gröbner basis of the corresponding ideal in $K[\mathbf{a}, \mathbf{x}]$ w.r.t. $\prec_{\mathbf{x}, \mathbf{a}}$ using the Maple function Basis. It is worth nothing that, this computation is needed in PGBMain to compute a CGS w.r.t. $\prec_{\mathbf{x}, \mathbf{a}}$. Also, "-" means that the related function can not compute anything and we stopped the computation after 400 seconds. A comparison of the timing columns in the above tables and our test for some other examples show that this first implementation of GVWDISPGB is efficient for many examples.

According to our experiments on more than 50 examples, we may consider two main classes of examples for which the performance of

GVWDisPGB and PGBMAin are quite different. In general, for each ideal $I$ with $I \cap K[\mathbf{a}] \neq\langle 0\rangle$ PGBMAIN has a better performance than GVWDisPGB. For instance, let us consider the ideal $I=\langle a x-a c+$ $\left.a^{2}, a b y^{3}+b y+a b+b^{2} c, a x y-b y+a x-c, a b y^{3}+a x y, b y+b c+c^{2}\right\rangle \subset$ $K[a, b, c][x, y]$. By running both algorithms over $I$ w.r.t. $c \prec_{l e x} b \prec_{\text {lex }} a$ and $y \prec_{l e x} x$, we observe that PGBMAIN takes about 20 seconds of CPU time while GVWDisPGB needs more than 95 seconds. It seems that when PGBMAIN finds a non-empty generating set for $I \cap K[\mathbf{a}]$, then this generating set has a positive impact on the rest of calculation. However, in contrast, if $I \cap K[\mathbf{a}]=\langle 0\rangle$, then GVWDisPGB has a better performance than PGBMain. For instance, one can refer to EX. 1 in Table 5.

The other issue concerning PGBMAIN is that by the structure of this algorithm, one needs to compute successive Gröbner bases by adding, at each step, new polynomials. This may enlarge the size of the computed Gröbner basis at each step. Let us consider the ideal $\left\langle a b y^{3}+b^{2} c+a b+b y, a x y+a x-b y-c, a b y^{3}+a x y\right\rangle$ in $K[a, b, c][x, y]$ with $c \prec_{\text {lex }} b \prec_{\text {lex }} a$ and $y \prec_{\text {lex }} x$. In Figure 5 , the x-axis shows the number of computed branches and the $y$-axis indicates the number of polynomials as the input of Gröbner basis algorithm in each branch.


Figure 1. Comparing the output of GVWDisPGB and PGBMAin
As it is shown, GVWDisPGB computes more branches than PGBMain (17 versus 6) but the maximum number of polynomials as the input of Gröbner basis algorithm in GVWDisPGB is 3 compared with

## 14 in PGBMAIN.

Let us consider the ideal $I=\left\langle f_{1}, f_{2}\right\rangle \subset K[a, b][x, y]$ generated by the polynomials

$$
\begin{aligned}
& f_{1}=\left(a^{12}+a^{5}+6\right) x^{17}+b x^{5}+x^{6}+x^{4}+a^{7} x+a^{5}, \text { and } \\
& f_{2}=\left(a^{13}+a^{6}+3\right) y^{7}+a y^{3}+y+1
\end{aligned}
$$

One can observe that $f_{1}$ and $f_{2}$ have coprime leading monomials w.r.t $y \prec_{\text {lex }} x$ however this does not hold by considering $f_{1}$ and $f_{2}$ in $K[a, b, x, y]$ with the product of $b \prec_{\text {lex }} a$ and $y \prec_{\text {lex }} x$. Therefore, if the leading coefficients of $f_{1}$ and $f_{2}$ in $K[a, b][x, y]$ are non-zero, then $\left\{f_{1}, f_{2}\right\}$ forms a Gröbner basis by Buchberger's first criterion and in turn GVWDIsPGB needs only 0.3 second to compute a CGS for $I$, however just the first Gröbner basis in $K[a, b, x, y]$ (which is the first step in PGBMAIN) takes more than five minutes.

Finally, it is worth noting that PGBMAIn has the advantage of using the Gröbner basis function of a computer algebra system. However, if the number of parameters is high then the first Gröbner basis computation may be hard and GVWDisPGB may have a better performance than PGBMAIn. Furthermore, due to the structure of PGBMAIN, the number of branches in the output CGS by using this algorithm is generally less compared with the one of GVWDisPGB.

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# Stability Analysis of Efficient Portfolios in a Discrete Variant of Multicriteria Investment Problem with Savage's Risk Criteria 

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#### Abstract

We consider a multicriteria discrete variant of investment portfolio optimization problem with Savage's risk criteria. Three combinations of norms in problem parameter spaces are considered. In each combination, one of the three spaces is endowed with Hölder's norm, and the other two spaces are endowed with Chebyshev's norm. The lower and upper attainable bounds on the stability radius of one Pareto optimal portfolio are obtained.


Keywords: Multicriteria problem, Pareto optimal portfolio, Savage's risk criteria, stability radius, Hölder's norms

## 1 Introduction

Modern financial environments require mitigation of the limitations of modern portfolio theory to make portfolio choice easier in the context of long-term and goal-based investing [1]. Investment managing problems are of the type with uncertainty of the initial data (see e.g. [2]). Usually, any separate investment asset has a higher level of risk and less return than the portfolio of those assets, and there is no reason to invest in one particular asset. Creating the portfolio by diversification and mixing a variety of investments an investor reduces the riskiness of the portfolio.

Following classical Markowitz's portfolio theory [3], [4], the investor plots on the graph an efficient frontier depending on various pairs of risk and expected return, and then he chooses portfolio drawing on

[^4]individual risk-return preferences. It gives him a chance to construct a portfolio with the same expected return and less risk.

The model we consider is rather different from the classical models. The risk matrix is constructed for several market states related to each type of the risk. Unlike classic modern portfolio theories, where a portfolio consists of a percentage of each asset, in our model a Boolean decision vector is used to describe feasible portfolios. The problem consists in finding a set of Pareto optimal portfolios with Savage's risk criteria.

The model formulation requires statistical and expert evaluation of risks (e.g. financial or ecological) [5] to be specified as the initial data. To construct an efficient portfolio, the investor must be able to quantify risk and provide the necessary inputs. Usually, the collected data contain computational errors and inaccuracies. It leads to the situation when the initial data representing risk values are inaccurate and uncertain. One of the key points of portfolio choice analysis under uncertainty is estimation of perturbation ranges for initial data the optima. The quantitative measure of the data perturbation level that do not violate optimality is known as the stability radius. The concept is widely presented and analyzed in the recent literature focusing on finding analytical expressions and bounds (see e.g. [6]-[10]). Similar approaches were also developing in parallel in scheduling theory (see [11]-[13]).

Analytic formulas are pairwise comparisons of solutions depending on selected optimality principles. The structure of global perturbation of this problem and the structure of the solution set should be taken into account. The particular definition of the stability radius depends on choosing optimality principles (given the problem was multicriterial), an uncertain data and a type of distance metric used to measure the nearness in problem parameter spaces. Various types of metrics allow considering a specific of problem parameters perturbation. So in the case of the Chebyshev metric $l_{\infty}$ the maximum changes in the initial data is taken into account only. Thus the perturbations are considered to be independent. In the case of the Manhattan metric $l_{1}$ every change of the initial data can be monitored in total. Hölder's metric $l_{p}, 1 \leq p \leq$

Stability Analysis of Efficient Portfolios in a Discrete Variant of. . .
$\infty$, is the metric with the parameter and includes such extreme cases as the Chebyshev metric $l_{\infty}$, Manhattan metric $l_{1}$ and also Euclidean metric $l_{2}$. Thus, $l_{p}$ norm allows controlling the degree and type of admissible perturbations, and therefore gives the decision maker more flexibility. For more details on the issue of using Hölder's metric in portfolio optimization we refer the reader to [14].

Along with quantitative analysis, a qualitative approach is developed in parallel. This approach concentrates on specifying analytical conditions which will guarantee some certain pre-specified behavior of the set of optimal solutions. To highlight the ideas of this approach, it is worth mentioning papers [15], [16], where comparative analysis of five different types of stability is presented for multicriteria integer linear programming problem. Similar results were obtained for multicriteria combinatorial problems with bottleneck criteria [17] as well as with some other nonlinear criteria [18].

In the previous papers (see, e.g. [19]-[24]), some bounds on the stability radii were obtained in the cases where three-dimensional problem parameters space is equipped with different combinations of $l_{1}$ and $l_{\infty}$ norms. In the present paper, we obtain the lower and upper bounds on the stability radius of one Pareto optimal portfolio for the multicriteria investment problem with Savage's risk criteria, where we assume that in one space an arbitrary $l_{p}$ norm is defined with $1 \leq p \leq \infty$. At the same time, we measure distances with $l_{\infty}$ norm in the remaining spaces. This allows the investor to make a more detailed control over changes in the initial data regarding to the three spaces. For example, the Euclidian metric is often used to deal with risks, and $l_{p}$ norm can treat the case once the decision maker needs it.

## 2 Problem formulation and basic definitions

Consider a multicriteria discrete variant of the portfolio optimization problem. We assume the model can be described by the following primitives listed below. Let
$N_{n}=\{1,2, \ldots, n\}$ be a variety of alternatives (investment assets, projects);
$N_{m}$ be a set of possible financial market states (market situations, scenarios);
$N_{s}$ be a set of possible risks;
$r_{i j k}$ be a numerical measure of economic risk of type $k \in N_{s}$ if investor chooses project $j \in N_{n}$ given the market state $i \in N_{m}$;
$R=\left[r_{i j k}\right] \in \mathbf{R}^{m \times n \times s} ;$
$x=\left(x_{1}, x_{2}, \ldots, x_{n}\right)^{T} \in \mathbf{E}^{n}$ be an investment portfolio, where $\mathbf{E}=$ $\{0,1\}$,

$$
x_{j}= \begin{cases}1 & \text { if investor chooses project } j, \\ 0 & \text { otherwise } ;\end{cases}
$$

$X \subset \mathbf{E}^{n}$ be a set of all admissible investment portfolios;
$\mathbf{R}^{m}$ be a financial market state space;
$\mathbf{R}^{n}$ be a portfolio space;
$\mathbf{R}^{s}$ be a risk space.
In our model, we assume that the risk measure is addictive, i.e. the total risk of one portfolio is a sum of risks of the projects included in the portfolio. The risk of each project can be measured, for instance, by means of the associated implementation cost.

The presence of a risk factor is integral feature of financial market functioning. One can find information about risk measurement methods and their classification in [25]. The last trend is to quantify risks using five $R$ : robustness, redundancy, resourcefulness, response and recovery. The natural target of any investor is to minimize different types of risks. It creates a motivation for multicriteria analysis within risk modeling. It leads to the usage of multicriteria decision making tools [26].

Assume that the efficiency of a chosen portfolio (Boolean vector) $x \in X,|X| \geq 2$, is evaluated by a vector objective function

$$
f(x, R)=\left(f_{1}\left(x, R_{1}\right), f_{2}\left(x, R_{2}\right), \ldots, f_{s}\left(x, R_{s}\right)\right),
$$

each partial objective represents minimax Savage's risk criterion [27].

$$
f_{k}\left(x, R_{k}\right)=\max _{i \in N_{m}} R_{i k} x=\max _{i \in N_{m}} \sum_{j \in N_{n}} r_{i j k} x_{j} \rightarrow \min _{x \in X}, \quad k \in N_{s},
$$

where $R_{k} \in \mathbf{R}^{m \times n}-k$-th cut $R=\left[r_{i j k}\right] \in \mathbf{R}^{m \times n \times s}$ with rows $R_{i k}=\left(r_{i 1 k}, r_{i 2 k}, \ldots, r_{i n k}\right) \in \mathbf{R}^{n}, i \in N_{m}$.

If investor chooses Savage's risk criterion [28], then (s)he minimizes the total risk of the selected portfolio in the worst (maximum risk state) case. This approach takes place when the decision maker has the most pessimistic expectations about the market.

The problem of finding Pareto optimal (efficient) portfolios is referred to as the multicriteria investment Boolean problem with Savage's risk criteria and denoted $Z^{s}(R), s \in \mathbf{N}$. The set of Pareto optimal portfolios is defined as follows
$P^{s}(R)=\left\{x \in X: \nexists x^{\prime} \in X \quad\left(g\left(x, x^{\prime}, R\right) \geq 0_{(s)} \& g\left(x, x^{\prime}, R\right) \neq 0_{(s)}\right)\right\}$,
where

$$
\begin{gathered}
g\left(x, x^{\prime}, R\right)=\left(g_{1}\left(x, x^{\prime}, R_{1}\right), g_{2}\left(x, x^{\prime}, R_{2}\right), \ldots, g_{s}\left(x, x^{\prime}, R_{s}\right)\right), \\
\left.g_{k}\left(x, x^{\prime}, R_{k}\right)=f_{k}\left(x, R_{k}\right)-f_{k}\left(x^{\prime}, R_{k}\right)=\min _{i^{\prime} \in N_{m} \in N_{m}} \max _{i k} x-R_{i^{\prime} k} x^{\prime}\right), k \in N_{s}, \\
0_{(s)}=(0,0, \ldots, 0) \in \mathbf{R}^{s} .
\end{gathered}
$$

If $m=1$, then the problem $Z^{s}(R)$ transforms into $s$-criteria linear Boolean programming problem:

$$
\begin{equation*}
Z_{B}^{s}(R): \quad R x \rightarrow \min _{x \in X} \tag{1}
\end{equation*}
$$

where $X \subseteq \mathbf{E}^{n}, R=\left[r_{k j}\right] \in \mathbf{R}^{s \times n}$ is a matrix with rows $R_{k}=$ $\left(r_{k 1}, r_{k 2}, \ldots, r_{k n}\right) \in \mathbf{R}^{n}, k \in N_{s}$. The case $m=1$ can be interpreted as a stable market with one state only.

While solving investment problems, it is necessary to take into account inaccuracy of initial information (statistical and expert risks evaluation errors) which are very common in real life. Under these conditions, it is highly recommended to get numerical bounds about possible changes in initial data preserving efficiency of the original Pareto optimal portfolio for any perturbation. Similarly to [20], [29], the number

$$
\rho=\rho^{s}\left(x^{0}\right)= \begin{cases}\sup \Xi & \text { if } \Xi \neq \emptyset \\ 0 & \text { if } \Xi=\emptyset\end{cases}
$$

is called a stability radius of a Pareto optimal solution $x^{0} \in P^{s}(R)$, where

$$
\begin{gathered}
\Xi=\left\{\varepsilon>0: \forall R^{\prime} \in \Omega(\varepsilon) \quad\left(x^{0} \in P^{s}\left(R+R^{\prime}\right)\right)\right\}, \\
\Omega(\varepsilon)=\left\{R^{\prime} \in \mathbf{R}^{m \times n \times s}:\left\|R^{\prime}\right\|<\varepsilon\right\} .
\end{gathered}
$$

Here $\Omega(\varepsilon)$ is a set of feasible perturbation matrices, $P^{s}\left(R+R^{\prime}\right)$ is a Pareto set of perturbed problem $Z^{s}\left(R+R^{\prime}\right),\left\|R^{\prime}\right\|$ is the norm of the matrix $R^{\prime}=\left[r_{i j k}^{\prime}\right]$. This norm depends on norms specified in portfolio space $\mathbf{R}^{n}$, market state space $\mathbf{R}^{m}$ as well as risk space $\mathbf{R}^{s}$.

Further, we investigate the stability radius in three different cases depending on which of those three spaces $\mathbf{R}^{n}, \mathbf{R}^{m}$ or $\mathbf{R}^{s}$ is equipped with Hölder's $l_{p}$-norm, $1 \leq p \leq \infty$. For any dimension $d$ and $1 \leq p \leq$ $\infty$, the Hölder $l_{p}$ norm of $a=\left(a_{1}, a_{2}, \ldots, a_{d}\right) \in \mathbf{R}^{d}$ in $\mathbf{R}^{d}$ is defined by the following equation

$$
\|a\|_{p}= \begin{cases}\left(\sum_{j \in N_{d}}\left|a_{j}\right|^{p}\right)^{1 / p} & \text { if } 1 \leq p<\infty \\ \max \left\{\left|a_{j}\right|: j \in N_{d}\right\} & \text { if } p=\infty\end{cases}
$$

It is well-known that $l_{p}$ norm, defined in $\mathbf{R}^{d}$, induces conjugated $l_{p^{*}}$ norm in $\left(\mathbf{R}^{d}\right)^{*}$. For $p$ and $p^{*}$, the following relations hold

$$
\frac{1}{p}+\frac{1}{p^{*}}=1, \quad 1<p<\infty
$$

Here as usual, we set $p^{*}=1$ if $p=\infty$, and $p^{*}=\infty$ if $p=1$. Thus, we assume that $p$ and $p^{*}$ vary within the range $[1, \infty]$. We also assume $1 / p=0$ if $p=\infty$.

It is easy to see that

$$
\begin{equation*}
\|z\|_{p}\|z\|_{p^{*}}=\|z\|_{1} \text { for } z \in\{-1,0,1\}^{n}, p \in[1, \infty] . \tag{2}
\end{equation*}
$$

For any $\alpha>0$ and $m \in \mathbf{N}$,

$$
\begin{equation*}
\|(\underbrace{\alpha, \ldots, \alpha}_{m})\|_{p}=m^{1 / p} \alpha . \tag{3}
\end{equation*}
$$

Further, we will use classical Hölder's inequality $a b \leq\|a\|_{p}\|b\|_{p^{*}}$, where $a=\left(a_{1}, a_{2}, \ldots, a_{n}\right) \in \mathbf{R}^{n}, b=\left(b_{1}, b_{2}, \ldots, b_{n}\right)^{T} \in \mathbf{R}^{n}$.

The following lemma can be easily proven by contradiction.
Lemma. Let $x^{0} \in P^{s}(R), \gamma>0$. If for any portfolio $x \in X \backslash\left\{x^{0}\right\}$ and every perturbing matrix $R^{\prime} \in \Omega(\gamma)$ there exists an index $l \in N_{s}$ such that $g_{l}\left(x, x^{0}, R_{l}+R_{l}^{\prime}\right)>0$, then $x^{0}$ is Pareto optimal in any perturbed problem $Z^{s}\left(R+R^{\prime}\right)$, i.e. $x^{0} \in P^{s}\left(R+R^{\prime}\right)$ as $R^{\prime} \in \Omega(\gamma)$.

## 3 Case A: portfolio space $R^{n}$ is endowed with $l_{p}$

We endow portfolio space $\mathbf{R}^{n}$ with an arbitrary Hölder's $l_{p}$ norm, $1 \leq$ $p \leq \infty$, while in market state space $\mathbf{R}^{m}$ and risk space $\mathbf{R}^{s}$ we measure distances by means of $l_{\infty}$. Thus, for any matrix $R=\left[r_{i j k}\right] \in \mathbf{R}^{m \times n \times s}$

$$
\|R\|_{p \infty \infty}=\left\|\left(\left\|R_{1}\right\|_{p \infty},\left\|R_{2}\right\|_{p \infty}, \ldots,\left\|R_{s}\right\|_{p \infty}\right)\right\|_{\infty}=\max _{k \in N_{s}}\left\|R_{k}\right\|_{p \infty},
$$

where $\quad\left\|R_{k}\right\|_{p \infty}=\left\|\left(\left\|R_{1 k}\right\|_{p},\left\|R_{2 k}\right\|_{p}, \ldots,\left\|R_{m k}\right\|_{p}\right)\right\|_{\infty}, \quad k \in N_{s}$. Obviously, $\left\|R_{i k}\right\|_{p} \leq\left\|R_{k}\right\|_{p \infty} \leq\|R\|_{p \infty \infty}, \quad i \in N_{m}, k \in N_{s}$. Additionally, due to Hölder's inequality, for any $x, x^{0} \in X$ we get

$$
\begin{align*}
& R_{i k} x-R_{i^{\prime} k} x^{0} \geq-\left(\left\|R_{i k}\right\|_{p}\|x\|_{p^{*}}+\left\|R_{i^{\prime} k}\right\|_{p}\left\|x^{0}\right\|_{p^{*}}\right) \geq \\
& \geq-\left\|R_{k}\right\|_{p \infty}\left(\|x\|_{p^{*}}+\left\|x^{0}\right\|_{p^{*}}\right), \quad i, i^{\prime} \in N_{m}, \quad k \in N_{s} . \tag{4}
\end{align*}
$$

In this context $\rho_{1}=\rho_{1}^{s}\left(x^{0}, m, p, \infty, \infty\right)$ denotes the stability radius of $x^{0}$. For Pareto optimal portfolio $x^{0}$ in $Z^{s}(R)$, we will use the following notation

$$
\begin{aligned}
& \varphi_{1}=\varphi_{1}^{s}\left(x^{0}, m, p, \infty, \infty\right)=\min _{x \in X \backslash\left\{x^{0}\right\}} \frac{\left\|\left[g\left(x, x^{0}, R\right)\right]^{+}\right\|_{\infty}}{\|x\|_{p^{*}}+\left\|x^{0}\right\|_{p^{*}}}, \\
& \psi_{1}=\psi_{1}^{s}\left(x^{0}, m, p, \infty, \infty\right)=\min _{x \in X \backslash\left\{x^{0}\right\}} \frac{\left\|\left[g\left(x, x^{0}, R\right)\right]^{+}\right\|_{\infty}}{\left\|x-x^{0}\right\|_{p^{*}}} .
\end{aligned}
$$

Obviously, $\psi_{1} \geq \varphi_{1} \geq 0$. Here and henceforth we will use a vector $a=$ $\left(a_{1}, a_{2}, \ldots, a_{s}\right) \in \mathbf{R}^{s}$ projection operator to the nonnegative orthant:

$$
[a]^{+}=\left(a_{1}^{+}, a_{2}^{+}, \ldots, a_{s}^{+}\right),
$$

where sign " + " means the positive projection of the vector, i.e. $a_{k}^{+}=\max \left\{0, a_{k}\right\}, k \in N_{s}$.
Theorem 1. For any $m, s \in \mathbf{N}$ and $p \in[1, \infty]$, the stability radius $\rho_{1}^{s}\left(x^{0}, m, p, \infty, \infty\right)$ of Pareto optimal portfolio $x^{0} \in P^{s}(R)$ in $Z^{s}(R)$ has the following upper and lower bounds

$$
\begin{equation*}
\varphi_{1}^{s}\left(x^{0}, m, p, \infty, \infty\right) \leq \rho_{1}^{s}\left(x^{0}, m, p, \infty, \infty\right) \leq \psi_{1}^{s}\left(x^{0}, m, p, \infty, \infty\right) \tag{5}
\end{equation*}
$$

Proof. Let $x^{0} \in P^{s}(R)$. First we prove $\rho_{1} \geq \varphi_{1}$. The claim is evident if $\varphi_{1}=0$. Assume $\varphi_{1}>0$. According to the definition of $\varphi_{1}$, for any portfolio $x \in X \backslash\left\{x^{0}\right\}$ the inequality

$$
\begin{equation*}
\left\|\left[g\left(x, x^{0}, R\right)\right]^{+}\right\|_{\infty} \geq \varphi_{1}\left(\|x\|_{p^{*}}+\left\|x^{0}\right\|_{p^{*}}\right) \tag{6}
\end{equation*}
$$

holds. Further, we are going to prove by contradiction that

$$
\forall R^{\prime} \in \Omega\left(\varphi_{1}\right) \quad \exists l \in N_{s} \quad\left(g_{l}\left(x, x^{0}, R_{l}+R_{l}^{\prime}\right)>0\right) .
$$

Suppose, there exists a perturbing matrix $R^{0} \in \Omega\left(\varphi_{1}\right)$ with cuts $R_{k}^{0}$, $k \in N_{s}$ such that

$$
g_{k}\left(x, x^{0}, R_{k}+R_{k}^{0}\right) \leq 0, \quad k \in N_{s} .
$$

Then due to (4) for any $k \in N_{s}$, we obtain

$$
\begin{gathered}
0 \geq g_{k}\left(x, x^{0}, R_{k}+R_{k}^{0}\right)=\max _{i \in N_{m}}\left(R_{i k}+R_{i k}^{0}\right) x-\max _{i \in N_{m}}\left(R_{i k}+R_{i k}^{0}\right) x^{0}= \\
=\min _{i^{\prime} \in N_{m}} \max _{i \in N_{m}}\left(R_{i k} x-R_{i^{\prime} k} x^{0}+R_{i k}^{0} x-R_{i^{\prime} k}^{0} x^{0}\right) \geq \\
\geq g_{k}\left(x, x^{0}, R_{k}\right)-\left\|R_{k}^{0}\right\|_{p \infty}\left(\|x\|_{p^{*}}+\left\|x^{0}\right\|_{p^{*}}\right) \geq \\
\geq g_{k}\left(x, x^{0}, R_{k}\right)-\left\|R^{0}\right\|_{p \infty \infty}\left(\|x\|_{p^{*}}+\left\|x^{0}\right\|_{p^{*}}\right)> \\
\quad>g_{k}\left(x, x^{0}, R_{k}\right)-\varphi_{1}\left(\|x\|_{p^{*}}+\left\|x^{0}\right\|_{p^{*}}\right) .
\end{gathered}
$$

From the last, we deduce

$$
\left\|\left[g\left(x, x^{0}, R\right)\right]^{+}\right\|_{\infty}<\varphi_{1}\left(\|x\|_{p^{*}}+\left\|x^{0}\right\|_{p^{*}}\right)
$$

Stability Analysis of Efficient Portfolios in a Discrete Variant of...
and it contradicts to (6). Finally, using Lemma, we get $x^{0} \in P^{s}\left(R+R^{\prime}\right)$ for any $R^{\prime} \in \Omega\left(\varphi_{1}\right)$. Hence, $\rho_{1} \geq \varphi_{1}$.

Now we prove that $\rho_{1} \leq \psi_{1}$. According to definition of $\psi_{1}>0$, there exists a portfolio $x^{*} \in X \backslash\left\{x^{0}\right\}$ such that

$$
\begin{align*}
g_{k}\left(x^{*}, x^{0}, R_{k}\right) & \leq\left[g_{k}\left(x^{*}, x^{0}, R_{k}\right)\right]^{+} \leq \\
\leq\left\|\left[g\left(x^{*}, x^{0}, R\right)\right]^{+}\right\|_{\infty} & =\psi_{1}\left\|x^{*}-x^{0}\right\|_{p^{*}}, \quad k \in N_{s} . \tag{7}
\end{align*}
$$

Assuming $\varepsilon>\psi_{1}$, consider a perturbing matrix $R^{0}=\left[r_{i j k}^{0}\right] \in \mathbf{R}^{m \times n \times s}$ with elements

$$
r_{i j k}^{0}=\delta \frac{x_{j}^{0}-x_{j}^{*}}{\left\|x^{*}-x^{0}\right\|_{p}}, \quad i \in N_{m}, \quad j \in N_{n}, \quad k \in N_{s},
$$

where $\psi_{1}<\delta<\varepsilon$. Since in any cuts $R_{k}^{0} \in \mathbf{R}^{m \times n}, k \in N_{s}$, all the rows $R_{i k}^{0}, i \in N_{m}$, are the same (let $A$ denotes such a row), we have

$$
\begin{equation*}
A=\delta \frac{\left(x^{0}-x^{*}\right)^{T}}{\left\|x^{*}-x^{0}\right\|_{p}} \tag{8}
\end{equation*}
$$

Therefore, $\left\|R^{0}\right\|_{p \infty \infty}=\left\|R_{k}^{0}\right\|_{p \infty}=\left\|R_{i k}^{0}\right\|_{p}=\|A\|_{p}=\delta, i \in N_{m}, k \in$ $N_{s}$, and, hence $R^{0} \in \Omega(\varepsilon)$ for any $\varepsilon>\delta$. Further, due to (2) and (8), for any $p \in[1, \infty]$ the chain of equalities is true

$$
A\left(x^{*}-x^{0}\right)=-\delta \frac{\left\|x^{*}-x^{0}\right\|_{1}}{\left\|x^{*}-x^{0}\right\|_{p}}=-\delta\left\|x^{*}-x^{0}\right\|_{p^{*}}
$$

Finally, using the above equalities along with (7), we conclude that for any $k \in N_{s}$ the following is true

$$
\begin{aligned}
& g_{k}\left(x^{*}, x^{0}, R_{k}+R_{k}^{0}\right)=\max _{i \in N_{m}}\left(R_{i k}+A\right) x^{*}-\max _{i \in N_{m}}\left(R_{i k}+A\right) x^{0}= \\
& =g_{k}\left(x^{*}, x^{0}, R_{k}\right)+A\left(x^{*}-x^{0}\right)=g_{k}\left(x^{*}, x^{0}, R_{k}\right)-\delta\left\|x^{*}-x^{0}\right\|_{p^{*}}< \\
& <g_{k}\left(x^{*}, x^{0}, R_{k}\right)-\psi_{1}\left\|x^{*}-x^{0}\right\|_{p^{*}} \leq 0 .
\end{aligned}
$$

Therefore, $x^{0} \notin P^{s}\left(R+R^{0}\right)$. And hence, $\rho_{1} \leq \psi_{1}$.

Attainability of the upper and lower bounds specified in (5) when $p=\infty$ follows also from the following evident statement, which is a direct consequence of Theorem 1 .
Corollary 1 If for any investment portfolio $x \neq x^{0}$ the set $\left\{j \in N_{n}\right.$ : $\left.x_{j}^{0}=x_{j}=1\right\}$ is empty, then for any number $m \in \mathbf{N}$ the formula

$$
\begin{aligned}
\rho_{1}^{s}\left(x^{0}, m, \infty, \infty, \infty\right) & =\varphi_{1}^{s}\left(x^{0}, m, \infty, \infty, \infty\right)=\psi_{1}^{s}\left(x^{0}, m, \infty, \infty, \infty\right)= \\
& =\min _{x \in X \backslash\left\{x^{0}\right\}} \frac{\left\|\left[g\left(x, x^{0}, R\right)\right]^{+}\right\|_{\infty}}{\left\|x+x^{0}\right\|_{1}}
\end{aligned}
$$

holds.
From Theorem 1 it also follows the corollary below.
Corollary $\mathbf{2}[$ [21] ] For any $m \in \mathbf{N}$, the following bounds take place

$$
\varphi_{1}^{s}\left(x^{0}, m, \infty, \infty, \infty\right) \leq \rho_{1}^{s}\left(x^{0}, m, \infty, \infty, \infty\right) \leq \psi_{1}^{s}\left(x^{0}, m, \infty, \infty, \infty\right)
$$

The following theorem gives evidence about attainability of lower bound specified in Corollary 3, i.e. lower bound (5) while $p=\infty$.

Theorem 2. There exists a class of problems $Z^{s}(R)$, such that for portfolio $x^{0} \in P^{s}(R)$ the following relations are valid

$$
\begin{equation*}
0<\rho_{1}^{s}\left(x^{0}, m, \infty, \infty, \infty\right)=\varphi_{1}^{s}\left(x^{0}, m, \infty, \infty, \infty\right)<\psi_{1}\left(x^{0}, m, \infty, \infty, \infty\right) \tag{9}
\end{equation*}
$$

Proof. Let $\varphi_{1}>0$. To fulfill the inequality $\varphi_{1}<\psi_{1}$ it is sufficient that $\left\|x+x^{0}\right\|_{1}>\left\|x-x^{0}\right\|_{1}$ holds for any $x \in X \backslash\left\{x^{0}\right\}$. To prove $\rho_{1}=\varphi_{1}$, according to Theorem 1 , it is sufficient to specify a class of problems with $\rho_{1} \leq \varphi_{1}$. So, the rest of the proof is about this. From the definition of $\varphi_{1}>0$ there exists $x^{*} \in X \backslash\left\{x^{0}\right\}$ with

$$
\begin{equation*}
\varphi_{1}\left\|x^{*}+x^{0}\right\|_{1} \geq g_{k}\left(x^{*}, x^{0}, R_{k}\right), \quad k \in N_{s} . \tag{10}
\end{equation*}
$$

Further conclusions are true for any $k \in N_{s}$. Denote

$$
i\left(x^{0}\right)=\arg \max \left\{R_{i k} x^{0}: i \in N_{m}\right\},
$$

$$
\begin{aligned}
& i\left(x^{*}\right)=\arg \max \left\{R_{i k} x^{*}: i \in N_{m}\right\}, \\
& \Delta=\left\|x^{*}+x^{0}\right\|_{1}-\left\|x^{*}-x^{0}\right\|_{1}>0 .
\end{aligned}
$$

Further assume

$$
\begin{equation*}
\left(R_{i\left(x^{*}\right) k}-R_{i\left(x^{0}\right) k}\right) x^{*}>\varphi_{1} \Delta, \tag{11}
\end{equation*}
$$

which implies $i\left(x^{0}\right) \neq i\left(x^{*}\right)$ since $\varphi_{1} \Delta>0$. For any $\varepsilon>\varphi_{1}$ the elements of the cut $R_{k}^{0}$ in the perturbing matrix $R^{0}$ we define as below

$$
r_{i j k}^{0}= \begin{cases}\delta & \text { if } i=i\left(x^{0}\right), \quad x_{j}^{0}=1,  \tag{12}\\ -\delta & \text { otherwise }\end{cases}
$$

where

$$
\begin{equation*}
\min \left\{\varepsilon, \frac{1}{\Delta}\left(R_{i\left(x^{*}\right) k}-R_{i\left(x^{0}\right) k}\right) x^{*}\right\}>\delta>\varphi_{1} . \tag{13}
\end{equation*}
$$

Notice also that the last inequalities are valid due to (11). Due to the construction specific of $R_{k}^{0}$ we have

$$
\begin{gather*}
R_{i k}^{0} x^{*}=-\delta\left\|x^{*}\right\|_{1}, \quad i \in N_{m} \backslash\left\{i\left(x^{0}\right)\right\}  \tag{14}\\
R_{i\left(x^{0}\right) k}^{0} x^{0}=\delta\left\|x^{0}\right\|_{1}  \tag{15}\\
\left\|R_{k}^{0}\right\|_{p \infty}=\left\|R^{0}\right\|_{p \infty \infty}=\delta, \quad R^{0} \in \Omega(\varepsilon)
\end{gather*}
$$

Additionally,

$$
\begin{equation*}
R_{i\left(x^{0}\right) k}^{0} x^{*}=\delta\left(\Delta-\left\|x^{*}\right\|_{1}\right) . \tag{16}
\end{equation*}
$$

Indeed, let

$$
\begin{gathered}
Q_{1}=\left\{j \in N_{n}: \quad x_{j}^{*}=x_{j}^{0}=1\right\} \\
Q_{2}=\left\{j \in N_{n}: \quad x_{j}^{*}=1, x_{j}^{0}=0\right\} .
\end{gathered}
$$

Then

$$
\begin{gathered}
\left|Q_{1}\right|=\Delta / 2, \\
\left|Q_{2}\right|=\left\|x^{*}\right\|_{1}-\Delta / 2, \\
R_{i\left(x^{0}\right) k}^{0} x^{*}=\delta\left(\left|Q_{1}\right|-\left|Q_{2}\right|\right),
\end{gathered}
$$

and (16) follows.

Further we prove $g_{k}\left(x^{*}, x^{0}, R_{k}+R_{k}^{0}\right)<0$. According to (15) we have

$$
\begin{equation*}
f_{k}\left(x^{0}, R_{k}+R_{k}^{0}\right)=\max _{i \in N_{m}}\left(R_{i k}+R_{i k}^{0}\right) x^{0}=f_{k}\left(x^{0}, R_{k}\right)+\delta\left\|x^{0}\right\|_{1} . \tag{17}
\end{equation*}
$$

Now we show

$$
\begin{equation*}
f_{k}\left(x^{*}, R_{k}+R_{k}^{0}\right)=f_{k}\left(x^{*}, R_{k}\right)-\delta\left\|x^{*}\right\|_{1} . \tag{18}
\end{equation*}
$$

Using (14), we yield

$$
\begin{gathered}
f_{k}\left(x^{*}, R_{k}+R_{k}^{0}\right)=\max \left\{\left(R_{i\left(x^{*}\right) k}+R_{i\left(x^{*}\right) k}^{0}\right) x^{*}, \max _{i \neq i\left(x^{*}\right)}\left(R_{i k}+R_{i k}^{0}\right) x^{*}\right\}= \\
=\max \left\{\left(f_{k}\left(x^{*}, R_{k}\right)-\delta\left\|x^{*}\right\|_{1}\right), \max _{i \neq i\left(x^{*}\right)}\left(R_{i k}+R_{i k}^{0}\right) x^{*}\right\} .
\end{gathered}
$$

Therefore, taking into account that

$$
f_{k}\left(x^{*}, R_{k}\right)-\delta\left\|x^{*}\right\|_{1} \geq\left(R_{i k}+R_{i k}^{0}\right) x^{*}, \quad i \in N_{m} \backslash\left\{i\left(x^{0}\right), i\left(x^{*}\right)\right\},
$$

in order to prove (18) it is sufficient to check

$$
f_{k}\left(x^{*}, R_{k}\right)-\delta\left\|x^{*}\right\|_{1} \geq\left(R_{i\left(x^{0}\right) k}+R_{i\left(x^{0}\right) k}^{0}\right) x^{*} .
$$

To do this, we use (13) and (16)

$$
\begin{gathered}
f_{k}\left(x^{*}, R_{k}\right)-\delta\left\|x^{*}\right\|_{1}-\left(R_{i\left(x^{0}\right) k}+R_{i\left(x^{0}\right) k}^{0}\right) x^{*}= \\
=\left(R_{i\left(x^{*}\right) k}-R_{i\left(x^{0}\right) k}\right) x^{*}-\delta\left\|x^{*}\right\|_{1}-R_{i\left(x^{0}\right) k}^{0} x^{*}> \\
>\delta\left(\Delta-\left\|x^{*}\right\|_{1}\right)-R_{i\left(x^{0}\right) k}^{0} x^{*}=0 .
\end{gathered}
$$

Finally, sequentially applying (17), (18), (10) and (13), for any index $k \in N_{s}$ we get
$g_{k}\left(x^{*}, x^{0}, R_{k}+R_{k}^{0}\right)=g_{k}\left(x^{*}, x^{0}, R_{k}\right)-\delta\left\|x^{*}+x^{0}\right\|_{1} \leq\left(\varphi_{1}-\delta\right)\left\|x^{*}+x^{0}\right\|_{1}<0$.
And hence, the formula below holds

$$
\forall \varepsilon>\varphi_{1} \quad \exists R^{0} \in \Omega(\varepsilon) \quad\left(x^{0} \notin P^{s}\left(R+R^{0}\right)\right),
$$

which due to $x^{0} \in P^{s}(R)$ produces $\rho_{1} \leq \varphi_{1}$. Summarizing, the correctness of (9) now becomes clear.

Stability Analysis of Efficient Portfolios in a Discrete Variant of...

Now consider a numerical example illustrating Theorem 2.
Example 1. Let $m=2, n=3, s=1, X=\left\{x^{0}, x^{*}\right\}, x^{0}=(1,1,0)^{T}$, $x^{*}=(0,1,1)^{T}$,

$$
R=\left(\begin{array}{ccc}
-5 & 2 & 2 \\
1 & -1 & 0
\end{array}\right)
$$

Then $f\left(x^{0}, R\right)=0, f\left(x^{*}, R\right)=4, x^{0} \in P^{1}(R),\left\|x^{*}+x^{0}\right\|_{1}=4, \| x^{*}-$ $x^{0} \|_{1}=2, i\left(x^{0}\right)=2, i\left(x^{*}\right)=1$. Therefore $\varphi_{1}=1, \psi_{2}=2$, $\left(R_{i\left(x^{*}\right) k}-\right.$ $\left.R_{i\left(x^{0}\right) k}\right) x^{*}=5>2=\varphi_{1}\left(\left\|x^{*}+x^{0}\right\|_{1}-\left\|x^{*}-x^{0}\right\|_{1}\right)$.

If the perturbing matrix $R^{0}$ is defined according to (12)

$$
R^{0}=\left(\begin{array}{ccc}
0 & -\delta & -\delta \\
\delta & \delta & -\delta
\end{array}\right), \quad 1<\delta<2.5
$$

it is easy to see that $\left\|R^{0}\right\|=\delta$ and $f\left(x^{0}, R+R^{0}\right)=2 \delta>4-2 \delta=$ $f\left(x^{*}, R+R^{0}\right)$. from the last and from the relations $\left\|R^{0}\right\|>1, x^{0} \in$ $P^{1}(R)$ it follows that $\rho_{1} \leq 1$. Therefore due to Theorem 1 we conclude $\rho_{1}=\varphi_{1}=1<\psi_{1}=2$.

The following known result gives us the evidence about attainability of the upper bound on stability radius of $x^{0} \in P^{s}(R)$ in $Z^{s}(R)$ for the case $m=1$ (see (1)). In this context $\mathbf{R}^{n}$ is endowed with $l_{p}$, and $\mathbf{R}^{s}$ is endowed with $l_{\infty}$.

Theorem 3 ([29]). For any $p \in[1, \infty]$ and $s \in \mathbf{N}$, the stability radius of $x^{0} \in P^{s}(R)$ in the linear Boolean programming problem $Z_{B}^{s}(R)$, $R \in \mathbf{R}^{s \times n}$ is expressed by the formula

$$
\rho_{1}^{s}\left(x^{0}\right)=\min _{x \in X \backslash\left\{x^{0}\right\}} \frac{\left\|\left[R\left(x-x^{0}\right)\right]^{+}\right\|_{\infty}}{\left\|x-x^{0}\right\|_{p^{*}}}
$$

## 4 Case B: market state space $\mathbf{R}^{m}$ is endowed with $l_{p}$

Now consider the case when portfolio space $\mathbf{R}^{n}$ and risk space $\mathbf{R}^{s}$ are endowed with $l_{\infty}$, whereas market state space $\mathbf{R}^{m}$ is equipped with
arbitrary Hölder's $l_{p}$ norm, $1 \leq p \leq \infty$. Thus, the norm of matrix is defined by

$$
\|R\|_{\infty p \infty}=\left\|\left(\left\|R_{1}\right\|_{\infty p},\left\|R_{2}\right\|_{\infty p}, \ldots,\left\|R_{s}\right\|_{\infty p}\right)\right\|_{\infty}=\max _{k \in N_{s}}\left\|R_{k}\right\|_{\infty p}
$$

where

$$
\left\|R_{k}\right\|_{\infty p}=\left\|\left(\left\|R_{1 k}\right\|_{\infty},\left\|R_{2 k}\right\|_{\infty}, \ldots,\left\|R_{m k}\right\|_{\infty}\right)\right\|_{p}, \quad k \in N_{s} .
$$

Obviously,

$$
\left\|R_{i k}\right\|_{\infty} \leq\left\|R_{k}\right\|_{\infty p} \leq\|R\|_{\infty p \infty}, \quad i \in N_{m}, k \in N_{s} .
$$

Additionally, due to Hölder's inequality, for any $x, x^{0} \in X$ we have

$$
\begin{gather*}
R_{i k} x-R_{i^{\prime} k} x^{0} \geq-\left(\left\|R_{i k}\right\|_{\infty}\|x\|_{1}+\left\|R_{i^{\prime} k}\right\|_{\infty}\left\|x^{0}\right\|_{1}\right) \geq \\
\geq-\left\|R_{k}\right\|_{\infty p}\left\|x+x^{0}\right\|_{1}, \quad i, i^{\prime} \in N_{m}, k \in N_{s} . \tag{19}
\end{gather*}
$$

In this context, $\rho_{2}^{s}\left(x^{0}\right)=\rho_{2}^{s}\left(x^{0}, m, \infty, p, \infty\right)$ is the stability radius of $x^{0}$. For Pareto optimal portfolio $x^{0}$ in $Z^{s}(R)$ we use the following notations

$$
\begin{aligned}
& \varphi_{2}=\varphi_{2}^{s}\left(x^{0}, m\right)=\min _{x \in X \backslash\left\{x^{0}\right\}} \frac{\left\|\left[g\left(x, x^{0}, R\right)\right]^{+}\right\|_{\infty}}{\left\|x+x^{0}\right\|_{1}}, \\
& \psi_{2}=\psi_{2}^{s}\left(x^{0}, m\right)=\min _{x \in X \backslash\left\{x^{0}\right\}} \frac{\left\|\left[g\left(x, x^{0}, R\right)\right]^{+}\right\|_{\infty}}{\left\|x-x^{0}\right\|_{1}} .
\end{aligned}
$$

Evidently, $\psi_{2} \geq \varphi_{2} \geq 0$.
Theorem 4. For any $m, s \in \mathbf{N}$ and $p \in[1, \infty]$ the stability radius $\rho_{2}^{s}\left(x^{0}, m\right)$ of Pareto optimal portfolio $x^{0} \in P^{s}(R)$ in $Z^{s}(R)$ the following bounds are valid

$$
\varphi_{2}^{s}\left(x^{0}, m\right) \leq \rho_{2}^{s}\left(x^{0}\right) \leq m^{1 / p} \psi_{2}^{s}\left(x^{0}, m\right)
$$

Stability Analysis of Efficient Portfolios in a Discrete Variant of...

Proof. Let $x^{0} \in P^{s}(R)$. First we prove $\rho_{2} \geq \varphi_{2}$. It is evident if $\varphi_{2}=0$. Let $\varphi_{2}>0$. According to the definition of $\varphi_{2}$, for any portfolio $x \in X \backslash\left\{x^{0}\right\}$ the inequality

$$
\begin{equation*}
\left\|\left[g\left(x, x^{0}, R\right)\right]^{+}\right\|_{\infty} \geq \varphi_{2}\left\|x+x^{0}\right\|_{1} \tag{20}
\end{equation*}
$$

is true. Further, by contradiction, we show the correctness of the formula given below

$$
\forall R^{\prime} \in \Omega\left(\varphi_{2}\right) \quad \exists l \in N_{s} \quad\left(g_{l}\left(x, x^{0}, R_{l}+R_{l}^{\prime}\right)>0\right) .
$$

From contrary, let it be so that there exists a perturbing matrix $R^{0} \in$ $\Omega\left(\varphi_{2}\right)$ with cuts $R_{k}^{0}, k \in N_{s}$ such that

$$
g_{k}\left(x, x^{0}, R_{k}+R_{k}^{0}\right) \leq 0, \quad k \in N_{s} .
$$

Then according to (19) for any index $k \in N_{s}$ we obtain

$$
\begin{gathered}
0 \geq g_{k}\left(x, x^{0}, R_{k}+R_{k}^{0}\right)=\max _{i \in N_{m}}\left(R_{i k}+R_{i k}^{0}\right) x-\max _{i \in N_{m}}\left(R_{i k}+R_{i k}^{0}\right) x^{0}= \\
=\min _{i^{\prime} \in N_{m}} \max _{i \in N_{m}}\left(R_{i k} x-R_{i^{\prime} k} x^{0}+R_{i k}^{0} x-R_{i^{\prime} k}^{0} x^{0}\right) \geq \\
\geq g_{k}\left(x, x^{0}, R_{k}\right)-\left\|R_{k}^{0}\right\|_{\infty p}\left\|x+x^{0}\right\|_{1} \geq \\
\geq g_{k}\left(x, x^{0}, R_{k}\right)-\left\|R^{0}\right\|_{\infty p \infty}\left\|x+x^{0}\right\|_{1}>g_{k}\left(x, x^{0}, R_{k}\right)-\varphi_{2}\left\|x+x^{0}\right\|_{1} .
\end{gathered}
$$

From the last we deduce inequality

$$
\left\|\left[g\left(x, x^{0}, R\right)\right]^{+}\right\|_{\infty}<\varphi_{2}\left\|x+x^{0}\right\|_{1},
$$

which contradicts to (20). Finally, applying Lemma, we have $x^{0} \in$ $P^{s}\left(R+R^{\prime}\right)$ for every perturbing matrix $R^{\prime} \in \Omega\left(\varphi_{2}\right)$. Hence, $\rho_{2} \geq \varphi_{2}$.

Now we prove $\rho_{2} \leq m^{1 / p} \psi_{2}$. According to the definition $\psi_{2}>0$, there exists a portfolio $x^{*} \in X \backslash\left\{x^{0}\right\}$ such that

$$
\begin{align*}
g_{k}\left(x^{*}, x^{0}, R_{k}\right) & \leq\left[g_{k}\left(x^{*}, x^{0}, R_{k}\right)\right]^{+} \leq \\
\leq\left\|\left[g\left(x^{*}, x^{0}, R\right)\right]^{+}\right\|_{\infty} & =\psi_{2}\left\|x^{*}-x^{0}\right\|_{1}, \quad k \in N_{s} . \tag{21}
\end{align*}
$$

Assuming $\varepsilon>m^{1 / p} \psi_{2}$, consider a perturbing matrix $R^{0}=\left[r_{i j k}^{0}\right] \in$ $\mathbf{R}^{m \times n \times s}$ whose elements are defined as follows

$$
r_{i j k}^{0}=\delta\left(x_{j}^{0}-x_{j}^{*}\right), \quad i \in N_{m}, \quad j \in N_{n}, \quad k \in N_{s},
$$

where $\varepsilon / m^{1 / p}>\delta>\psi_{2}$. Since all the rows $R_{i k}^{0}, i \in N_{m}$ in the cut $R_{k}^{0} \in \mathbf{R}^{m \times n}, k \in N_{s}$ are the same in the matrix $\mathbf{R}^{0}$, then we have (let $A \in \mathbf{R}^{m}$ denote such a row)

$$
\begin{gather*}
A=\delta\left(x^{0}-x^{*}\right)^{T} .  \tag{22}\\
\left\|R_{i k}^{0}\right\|_{\infty}=\|A\|_{\infty}=\delta, \quad i \in N_{m}, k \in N_{s}
\end{gather*}
$$

From the last and (3), we get

$$
\begin{gathered}
\left\|R_{k}^{0}\right\|_{\infty p}=m^{1 / p} \delta, \quad k \in N_{s} \\
\left\|R^{0}\right\|_{\infty p \infty}=m^{1 / p} \delta \geq m^{1 / p} \psi_{2}
\end{gathered}
$$

Thus $R^{0} \in \Omega(\varepsilon)$ for any $\varepsilon>m^{1 / p} \psi_{2}$. Further due to (22), we have

$$
A\left(x^{*}-x^{0}\right)=-\delta\left\|x^{*}-x^{0}\right\|_{1} .
$$

Finally, combining the equality above and (21), we conclude that for any $k \in N_{s}$ the following relations are true

$$
\begin{gathered}
g_{k}\left(x^{*}, x^{0}, R_{k}+R_{k}^{0}\right)=\max _{i \in N_{m}}\left(R_{i k}+A\right) x^{*}-\max _{i \in N_{m}}\left(R_{i k}+A\right) x^{0}= \\
=g_{k}\left(x^{*}, x^{0}, R_{k}\right)+A\left(x^{*}-x^{0}\right)=g_{k}\left(x^{*}, x^{0}, R_{k}\right)-\delta\left\|x^{*}-x^{0}\right\|_{1}< \\
<g_{k}\left(x^{*}, x^{0}, R_{k}\right)-\psi_{2}\left\|x^{*}-x^{0}\right\|_{1} \leq 0 .
\end{gathered}
$$

Thus $x^{0} \notin P^{s}\left(R+R^{0}\right)$. Hence, $\rho_{2} \leq m^{1 / p} \psi_{2}$.
The following known result confirms attainability on the upper bound of the stability radius of $x^{0} \in P^{s}(R)$ in $Z^{s}(R)$ for the case $m=1$ (see (1)). In this context, both $\mathbf{R}^{n}$ and $\mathbf{R}^{s}$ are equipped with $l_{\infty}$.
Theorem 5 ( [30]). For the stability radius of $x^{0} \in P^{s}(R)$ in the Boolean linear programming problem $Z_{B}^{s}(R), R \in \mathbf{R}^{s \times n}$, and $s \in \mathbf{N}$ the following analytical expression holds

$$
\rho_{2}^{s}\left(x^{0}\right)=\min _{x \in X \backslash\left\{x^{0}\right\}} \frac{\left\|\left[R\left(x-x^{0}\right)\right]^{+}\right\|_{\infty}}{\left\|x-x^{0}\right\|_{1}}
$$

## 5 Case C: risk space $\mathbf{R}^{s}$ is endowed with $l_{p}$

Now assume we measure distances by means of $l_{\infty}$ in portfolio space $\mathbf{R}^{n}$ and market state space $\mathbf{R}^{m}$ while in risk space $\mathbf{R}^{s}$ we use $l_{p}, 1 \leq p \leq \infty$. In this case under the norm of the matrix $R$ we understand the number

$$
\|R\|_{\infty \infty p}=\left\|\left(\left\|R_{1}\right\|_{\infty \infty},\left\|R_{2}\right\|_{\infty \infty}, \ldots,\left\|R_{s}\right\|_{\infty \infty}\right)\right\|_{p},
$$

where

$$
\left\|R_{k}\right\|_{\infty \infty}=\left\|\left(\left\|R_{1 k}\right\|_{\infty},\left\|R_{2 k}\right\|_{\infty}, \ldots,\left\|R_{m k}\right\|_{\infty}\right)\right\|_{\infty}, \quad k \in N_{s} .
$$

Obviously,

$$
\left\|R_{i k}\right\|_{\infty} \leq\left\|R_{k}\right\|_{\infty \infty} \leq\|R\|_{\infty \infty p}, \quad i \in N_{m}, k \in N_{s} .
$$

It is easy to check that for any portfolios $x$ and $x^{\prime}$ the inequalities hold

$$
\begin{equation*}
R_{i k} x-R_{i^{\prime} k} x^{\prime} \geq-\left\|R_{k}\right\|_{\infty \infty}\left\|x+x^{\prime}\right\|_{1}, \quad i, \quad i^{\prime} \in N_{m}, \quad k \in N_{s} . \tag{23}
\end{equation*}
$$

In this context, $\rho_{3}=\rho_{3}^{s}\left(x^{0}, m, \infty, \infty, p\right)$ denotes the stability radius of $x^{0}$. For Pareto optimal portfolio $x^{0}$ in $Z^{s}(R)$, we introduce the notation

$$
\begin{aligned}
& \varphi_{3}=\varphi_{3}^{s}\left(x^{0}, m, \infty, \infty, p\right)=\min _{x \in X \backslash\left\{x^{0}\right\}} \frac{\left\|\left[g\left(x, x^{0}, R\right)\right]^{+}\right\|_{p}}{\left\|x+x^{0}\right\|_{1}}, \\
& \psi_{3}=\psi_{3}^{s}\left(x^{0}, m, \infty, \infty, p\right)=\min _{x \in X \backslash\left\{x^{0}\right\}} \frac{\left\|\left[g\left(x, x^{0}, R\right)\right]^{+}\right\|_{p}}{\left\|x-x^{0}\right\|_{1}} .
\end{aligned}
$$

Evidently, $\psi_{3} \geq \varphi_{3} \geq 0$.
Theorem 6. For any $m, s \in \mathbf{N}$ and $p \in[1, \infty]$, the stability radius $\rho_{3}^{s}\left(x^{0}, m, \infty, \infty, p\right)$ of portfolio $x^{0} \in P^{s}(R)$ in $Z^{s}(R)$ has the following lower and upper bounds

$$
\varphi_{3}^{s}\left(x^{0}, m, \infty, \infty, p\right) \leq \rho_{3}^{s}\left(x^{0}, m, \infty, \infty, p\right) \leq \psi_{3}^{s}\left(x^{0}, m, \infty, \infty, p\right) .
$$

Proof. Let $x^{0} \in P^{s}(R)$. First we prove $\rho_{3} \geq \varphi_{3}$. Without loss of generality, assume $\varphi_{3}>0$ (otherwise inequality $\rho_{3} \geq \varphi_{3}$ is obvious). According to the definition of $\varphi_{3}$, for any $x \neq x^{0}$ the following is true

$$
\begin{equation*}
\left\|\left[g\left(x, x^{0}, R\right)\right]^{+}\right\|_{p} \geq \varphi_{3}\left\|x+x^{0}\right\|_{1} . \tag{24}
\end{equation*}
$$

To prove the lower bound, it is necessary to show that the formula below is true

$$
\begin{equation*}
\forall R^{\prime} \in \Omega\left(\varphi_{3}\right) \quad \exists l \in N_{s} \quad\left(g_{l}\left(x, x^{0}, R_{l}+R_{l}^{\prime}\right)>0\right) . \tag{25}
\end{equation*}
$$

From contrary, assume there exists a perturbing matrix $R^{0} \in \Omega\left(\varphi_{3}\right)$ such that

$$
g_{k}\left(x, x^{0}, R_{k}+R_{k}^{0}\right) \leq 0, \quad k \in N_{s}
$$

Then using (23), we easily deduce
$0 \geq g_{k}\left(x, x^{0}, R_{k}+R_{k}^{0}\right)=\min _{i^{\prime} \in N_{m}} \max _{i \in N_{m}}\left(R_{i k} x-R_{i^{\prime} k} x^{0}+R_{i k}^{0} x-R_{i^{\prime} k}^{0} x^{0}\right) \geq$

$$
\geq g_{k}\left(x, x^{0}, R_{k}\right)-\left\|R_{k}^{0}\right\|_{\infty \infty}\left\|x+x^{0}\right\|_{1}
$$

i.e.

$$
\left[g_{k}\left(x, x^{0}, R_{k}\right)\right]^{+} \leq\left\|R_{k}^{0}\right\|_{\infty \infty}\left\|x+x^{0}\right\|_{1}, \quad k \in N_{s}
$$

Thus, due to $R^{0} \in \Omega\left(\varphi_{3}\right)$ while $p \in[1, \infty]$ we have

$$
\left\|\left[g\left(x, x^{0}, R\right)\right]^{+}\right\|_{p} \leq\left\|R^{0}\right\|_{\infty \infty p}\left\|x+x^{0}\right\|_{1}<\varphi_{3}\left\|x+x^{0}\right\|_{1} .
$$

This contradicts to (24), and hence (25) is true. From here, according to the Lemma, $x^{0} \in P^{s}\left(R+R^{\prime}\right)$ for any $R^{\prime} \in \Omega\left(\varphi_{3}\right)$. Hence, $\rho_{3}^{s}\left(x^{0}, m, \infty, \infty, p\right) \geq \varphi_{3}^{s}\left(x^{0}, m, \infty, \infty, p\right)$.

Further, we prove that $\rho_{3} \leq \psi_{3}$ holds for any $p \in[1, \infty]$. Let $\varepsilon>\psi_{3}>0$, and portfolio $x^{*} \neq x^{0}$ is such that

$$
\left\|\left[g\left(x^{*}, x^{0}, R\right)\right]^{+}\right\|_{p}=\psi_{3}\left\|x-x^{0}\right\|_{1} .
$$

Then, taking into account that the norm $l_{p}$ depends on vector continuously, we take $\delta \in \mathbf{R}^{s}$ with positive components such that

$$
\begin{equation*}
\delta_{k}\left\|x^{*}-x^{0}\right\|_{1}>\left[g_{k}\left(x^{*}, x^{0}, R_{k}\right)\right]^{+}, \quad k \in N_{s}, \tag{26}
\end{equation*}
$$

and $\varepsilon>\|\delta\|_{p}>\psi_{3}$. Then we construct a perturbing matrix $R^{0} \in \Omega(\varepsilon)$, where $\varepsilon>\|\delta\|_{p}$, with cuts $R_{k}^{0}, k \in N_{s}$ such that for every $k \in N_{s}$ the inequality

$$
\begin{equation*}
g_{k}\left(x^{*}, x^{0}, R_{k}+R_{k}^{0}\right)<0 \tag{27}
\end{equation*}
$$

holds. Using components of vector $\delta$, we define the elements of any $k$-th cut $R_{k}^{0}=\left[r_{i j k}^{0}\right] \in \mathbf{R}^{m \times n}$ of the perturbing matrix $R^{0}=\left[r_{i j k}^{0}\right] \in \mathbf{R}^{m \times n \times s}$ using the formula

$$
r_{i j k}^{0}=\left\{\begin{array}{lll}
\delta_{k} & \text { if } i \in N_{m}, & x_{j}^{0} \geq x_{j}^{*}, \\
-\delta_{k} & \text { if } i \in N_{m}, & x_{j}^{0}<x_{j}^{*} .
\end{array}\right.
$$

Then, we have

$$
\left\|R_{k}^{0}\right\|_{\infty \infty}=\delta_{k}, \quad k \in N_{s} .
$$

Therefore, it is easy to see that $\left\|R^{0}\right\|_{\infty \infty p}=\|\delta\|_{p}<\varepsilon$. Additionally, all the rows $R_{i k}^{0}\left(i \in N_{m}\right)$ in the cut $R_{k}^{0}, k \in N_{s}$ are the same and contain components $\delta_{k}$ and $-\delta_{k}$ only. Denoting such a row $A_{k}$, we obtain

$$
A_{k}\left(x^{*}-x^{0}\right)=-\delta_{k}\left\|x^{*}-x^{0}\right\|_{1}, \quad k \in N_{s} .
$$

From this for any $k \in N_{s}$ due to (26) we get (27):

$$
\begin{aligned}
& g_{k}\left(x^{*}, x^{0}, R_{k}+R_{k}^{0}\right)=g_{k}\left(x^{*}, x^{0}, R_{k}\right)+A\left(x^{*}-x^{0}\right)= \\
& =g_{k}\left(x^{*}, x^{0}, R_{k}\right)-\delta_{k}\left\|x^{*}-x^{0}\right\|_{1} \leq \\
& \leq\left[g_{k}\left(x^{*}, x^{0}, R_{k}\right)\right]^{+}-\delta_{k}\left\|x^{*}-x^{0}\right\|_{1}<0, \quad k \in N_{s} .
\end{aligned}
$$

Thus, while $\varepsilon>\psi_{3}$ there exists a perturbing matrix $R^{0} \in \Omega(\varepsilon)$ such that $x^{0} \in P^{s}(R)$ is not Pareto optimal in the perturbed problem $Z^{s}(R+$ $\left.R^{0}\right)$. This implies that for any $\varepsilon>\psi_{3}$ we have $\rho_{3}<\varepsilon$. Hence, $\rho_{3} \leq \psi_{3}^{s}$, and then $p \in[1, \infty]$.

The following statement gives the evidence about attainability of the lower and upper bounds specified in Theorem 6.
Corollary 3 If for any $x \neq x^{0}$ the set $\left\{j \in N_{n}: x_{j}^{0}=x_{j}=1\right\}$ is empty, then for any $m \in \mathbf{N}$ any $p \in[1, \infty]$ the following holds

$$
\rho_{3}^{s}\left(x^{0}, m, \infty, \infty, p\right)=\varphi_{3}^{s}\left(x^{0}, m, \infty, \infty, p\right)=
$$

$$
=\psi_{3}^{s}\left(x^{0}, m, \infty, \infty, p\right)=\min _{x \in X \backslash\left\{x^{0}\right\}} \frac{\left\|\left[g\left(x, x^{0}, R\right)\right]^{+}\right\|_{p}}{\left\|x+x^{0}\right\|_{1}}
$$

If $m=1$, as it was pointed out before, $Z^{s}(R)$ transforms into $s$ criteria Boolean linear programming problem $Z_{B}^{s}(R), R \in \mathbf{R}^{s \times n}$ (see (1)). In this context, $\mathbf{R}^{n}$ is equipped with $l_{\infty}$, and $\mathbf{R}^{s}$ is equipped with $l_{p}, 1 \leq p \leq \infty$. The following known result illustrates the fact that the upper bound specified in Theorem 6 is right.

Theorem $\mathbf{7}$ ([29]). For any $p \in[1, \infty]$ and $s \in \mathbf{N}$, the stability radius of $x^{0} \in P^{s}(R)$ in $Z_{B}^{s}(R), R \in \mathbf{R}^{s \times n}$ is expressed by the formula

$$
\rho_{3}^{s}\left(x^{0}\right)=\min _{x \in X \backslash\left\{x^{0}\right\}} \frac{\left\|\left[R\left(x-x^{0}\right)\right]^{+}\right\|_{p}}{\left\|x-x^{0}\right\|_{1}} .
$$

## 6 Conclusion

While composing a portfolio, the investor's intention to minimize different types of risks motivates the use of multicriteria environment within the corresponding mathematical and economical models. This approach allows using a variety of multicriteria decision making methods [26], [31]. In this paper, to maintain the different types of risks, we used the bottleneck partial objectives, forcing the investor to choose a portfolio with the minimal total aggregated risk in the worst case scenario, i.e. in the situation where the values of risks are at their maximum.

Another type of uncertainty is related to inaccuracy of statistical observations and expert evaluations while measuring different risks. In this context there is a necessity to conduct post-optimal analysis in order to quantify the extreme level of initial data changes not violating the portfolio optimality. In this work different cases are analyzed depending on a type of the metric used in problem parameter spaces. In all the cases analyzed, the lower and upper bounds on stability radius of an efficient portfolio are presented.

The results give an investor information on reliability of chosen optimal portfolio and prevent the situation, when his/her portfolio will

Stability Analysis of Efficient Portfolios in a Discrete Variant of...
lose its property with unexpected changes in the initial data. Using Hölder's norm the control of changes in the initial data can vary depending on the space of the problem parameters. The results could be potentially interesting for construction the investment models in which the investor wants to merge different types of risks at the presence of unreliable information and forecasted financial market states.

The straightforward application of the results to practical calculation is limited due to enumerating structure of analytical expressions which may need a number of comparisons growing exponentially with $n$ and $s$. In the case when direct calculation is time consuming (it may happen if $n \geq 40$ and $s \geq 3$ ), getting the values should be calculated heuristically, for example some multicriteria genetic algorithms can be used.

It is also important to note that sometimes the stability radius does not give us complete information about the quality of a given solution in the case when problem data are located outside of the stability region. Some attempts to study a quality of the problem solution in this case are connected with concepts of stability and accuracy functions. These functions were first introduced in [32],[33] for the scalar combinatorial optimization problem. In [34], the results were later extended to the vector linear discrete optimization problem with Pareto and lexicographic optimality principles. Similar results were obtained for Boolean linear programming [35], game theory problem formulations [36] and some scheduling models [37]. Moreover, as it was shown recently (see, e.g. [38], [39]), calculating stability and accuracy functions is closely related to analyzing problem robustness. Robust optimization in that context is understood as a process aiming to produce solutions that optimize an additionally constructed objective. The objective must assure that the optimal solution will remain feasible under worst case realization of uncertain problem input parameters. Robust optimization is also known as worst-case or minmax regret optimization, and optimal solutions of worst case optimization are often referred to as robust solutions (see, e.g. [40]). Conducting similar research for investment models could be an interesting direction for further investigations.

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# A linear model for multidimensional Big Data visualization 

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#### Abstract

The author introduces and analyzes a model that allows organizing visualization of primary linear constructs such as interval, simplex and polygonal lines in multidimensional space.

Keywords: computer science, big data, data visualization, multi-dimensional data, exploratory data research.


## 1 Introduction

There are a number of well-known methods to visualize multidimensional data. There are Andrews plots, Bergeron's or Wong's model, Zinoviev model as well as Klaft, Barrett and Kleiner-Hartigan, and every one of them introduces their own unique mechanism for data visualization [1]. However, every method has its own limitations, narrowing the field of direct applicability. For instance, the Bergeron's model visualizes the wave lines and the time interval for a single frequency [2].

Graphical methods are especially helpful during the Exploratory Data Research (EDR) of the large sets of multi-dimensional data and the clustering problems, enabling the analyst to discover patterns and relationships hidden in the data set. The main advantage of the modeling data as a multi-dimensional set of points or observations is the convenience and effectiveness of analyzing a big volume of data, particularly when applied to a time-series. The problem with such model is its bulkiness and poor suitability for simple tasks of the operational data processing [3].

[^5]
## 2 Model Definition

Theorem 1. As a basis for visualization of the multidimensional data a linear modification of a multidimensional observation $\mathcal{H}$ into twodimensional curved line $\mathfrak{L}_{\mathcal{H}}(t)$ is used, so $\mathcal{H}$ approximates $\mathfrak{L}_{\mathcal{H}}(t): \mathcal{H} \leftrightarrow$ $\mathfrak{L}_{\mathcal{H}}(t)$, with the provable condition that the values of the dimensional attributes of observations $\mathcal{H}$ and $\mathcal{X}$ correspond to graphics $\mathfrak{L}_{\mathcal{H}}(t)$ and $\mathfrak{L}_{\mathcal{X}}(t)$ that visually appear near each other, reflecting the relative closeness of $\mathcal{H}$ and $\mathcal{X}$. Conversely when these values are relatively distant, the graphical lines will appear to be far apart.

Proof Theorem 1. For the analysis of the proposed method we will use the most general system of data presentation. Let's pick a vector $H$ in $P n$ - a space with finite number of dimensions.

$$
\begin{equation*}
H=(h 0, h 1, h 2, h 3, \ldots h n-1) \in P n . \tag{1}
\end{equation*}
$$

To create the visualization of the vector we have to create a basis for transformation as a set of orthogonal functions $\{\varphi(t)\} \rightarrow \infty$. Legendre orthogonal polynomials can be applied on a 0 to 1 interval, set of which can be shown as $\zeta(t) \rightarrow \infty$. In this case the vector H with coordinates ( $h 0, h 1, h 2, h 3, \ldots h n-1$ ) $\in P n$ corresponds to the following function:

$$
\begin{equation*}
E_{H(t)}=\sum_{i=0}^{n-1} H_{i} L_{i}(t) \tag{2}
\end{equation*}
$$

Conversion of the vector $H$ is accomplished by conversion of its multidimensional data. In order to characterize the observable multidimensional object its coordinate values play a significant role. In the extreme cases, each coordinate should have its own measurement defined, and its value should affect the appearance of the $E h(t)$ function. To exclude the influence of the individual measurement types over the $E h(t)$ function, it is necessary to switch to a neutral set of values, by using one of the known methods.

It should be noted that the of inclusion of the dimensional values in vector $H$ also can influence the look of the $E h(t)$ function. To justify
the order of inclusion of these characteristics in certain applications, an expertise determining the "informativeness" (the degree of influence) of each individual one can be performed, accompanied by an analysis of the optimal sequence of inclusion of these characteristics into the vector $H$ [4].

Let's introduce a second vector into the model:

$$
\begin{equation*}
X=(x 0, x 1, x 2, x 3, \ldots x n-1) \in P n \tag{3}
\end{equation*}
$$

and its corresponding function:

$$
\begin{equation*}
E x(t)=\sum_{i=0}^{n-1} H_{i} L_{i}(t) \tag{4}
\end{equation*}
$$

And now we can transform two points $H \& X$ from the Pn space, into the graphical view of their representative functions $E h(t)$ and $E x(t)$ (Fig.1).


Figure 1. Visualization of $H \& X$ from the $P n$ space.
When we consider $H \& X$ to be vectors, with the beginning located at the beginning of the coordinate system selected for the $P n$ space

- then the relative proximity between all points in the Pn space becomes definitively tied to the graphical representations of their corresponding $E h(t)$ and $E x(t)$ functions, with axes values defined as $h 0, h 1, h 2, h 3, \ldots h n-1$. By introducing a variable, we can create an equation:

$$
\begin{array}{r}
C(C)=(1-c) H+c X=((1-c) h 0+c x 0,(1-c) h 1+c x 1, \\
\ldots(1-c) h n 1+c x n 1 . \tag{5}
\end{array}
$$

From which obviously follows $C(0)=H$ and $C(1)=X$, which can be viewed as a definition of a multidimensional "straight" line connecting $H \& X$ in the $P n$ space, and we can use the expression like (5) to represent a multidimensional segment $H X$ :

$$
\begin{equation*}
H X=(1-c) H+c X, \text { where } \in[0,1] . \tag{6}
\end{equation*}
$$

Assuming "c" represents the distance in the $P n$ space, the equation (6) can be shown as the proposed model:

$$
\begin{equation*}
E_{H X}(c)=\sum_{i=0}^{n-1}(1-c) H_{i} L_{i}(t)+c x_{i} L_{i}(t) \tag{7}
\end{equation*}
$$

This function has two arguments $\{c, t\}$, which allows us to get a graphical function $E h x(c)=E h x\{c, t\}$ that visually represents the $H X$ segment, as shown in Figure 2.

When defined over the $[0,1] /[0,1]$ square, it is possible to produce a smooth surface based on (7), that corresponds to an analytical expression (6) that represents a multidimensional segment $H X$ [5] QED.

## 3 Sample Application

To test the model, we will apply it to a sample set of multidimensional objects with the following values: $H 1=\{1,0,0,0\}, H 2=$ $\{0,1,0,0\}, H 3=\{0,0,1,0\}, H 4=\{0,0,0,1\}$, and transform them


Figure 2. Visualization of smooth surface, corresponding to the HX segment from the $P n$ space
using polynomial matrices

$$
\begin{aligned}
& {\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right] \rightarrow 1 * l 0(t)+0 * l 0(t)+0 * l 2(t)+0 * l 3(t)} \\
& {\left[\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right] \rightarrow 1 * l 0(t)+0 * l 0(t)+0 * l 2(t)+0 * l 3(t)} \\
& {\left[\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right] \rightarrow 1 * l 0(t)+0 * l 0(t)+0 * l 2(t)+0 * l 3(t)} \\
& {\left[\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right] \rightarrow 1 * l 0(t)+0 * l 0(t)+0 * l 2(t)+0 * l 3(t)}
\end{aligned}
$$

and get the general formula

$$
E=\left[\begin{array}{ll}
f 0 & f 1 \\
f 2 & f 3
\end{array}\right] \rightarrow f 0 * l 0(t)+f 1 * l 0(t)+f 2 * l 2(t)+f 3 * l 3(t) .
$$

The polynomial argument $\{t\}$ is the characterization of the composite representation and has a value, but no measure. Vector $E$ cannot
be shown in 3D, and therefore it is being substituted with a 2 D line $E 1(t)$ [6].

Let's see what the graphic would look like for the following 10dimensional values.


Figure 3. Visualization of $H: 54,1,18,2.6,6.4,0.2,4.7,8,3.3,2$ in the $P n$ space

When these two graphics are joined, it becomes very clear that they not only look very similar, but are very close to each other in the dimensional points, indicating that the original raw observations are in close proximity in the Pn space as shown in Figure 5.

The more indistinguishable are the graphical representations of the raw observations, the closer to each other are these observations in their original space, as the multidimensional points are bijected into their corresponding graphics.

It is possible to visualize many interesting characteristics, by reproducing the Figure 5 graphics in 3D, by defining a $Z$-order as the distance between the points in the $P n$-space or the time-interval between the observations. The $P n$-space distance or the time interval can be measured in any applicable way and scaled to fit the relative distance into the graphic, which makes it possible not only evaluate


Figure 4. Visualization of $X: 50,1,19,2.4,6,1,5,3.8,8,3,2.5$ in the $P n$ space


Figure 5. Proximity visualization of $H \& X$ from the $P n$ space.
the static characteristics of the observation data, but to view some of the changes dynamically [7].

## 4 Model Optimization

In linear transformation of $H \leftrightarrow F_{H}(t)$, using the segment between the multidimensional observations $H \& X$, we obtain a corresponding surface, that ties the projected observations. Every line representing observations with intermediate values (observations that belong to the [ $H ; X]$ segment in $P n$ space) will appear on that surface [6]. Let's consider Figure 6.


Figure 6. Proximity visualization of $H \& X$ from the $P n$ space.
In order to compare the observations in greater detail, particularly in the case of the heterogeneity of the units of measurement of the various data characteristics, a traditional modification mechanism should be applied:

- Normalization - to be applied to express the results in a coherent system of measurements;
- Standardization - to be applied to enable comparison of the data characteristics with variant attribute distributions and/or different units of measure [8];

Currently there are numerous programmatic and algorithmic visualization tools for multidimensional data structures. However, quite often basic visualization technique cannot be directly applied to a task at hand, since the researchers are usually interested in some very specific properties of the data that cannot be identified using standard approaches. Cases like these, call for development of the specialized types of presentation, focusing on the specific requirements of task at hand.

So, the developed model (6) of multidimensional data visualization demonstrates that the proposed approach holds promise in the area of analysis and representation of raw multidimensional data.

The particulars in the model selection process and the characteristics of its generation depend on the specific expectations of the outcome of the research. To formulate the task, it would be more reasonable to produce a limited and compact model. For example, an exponential function or a spline with two or three junction points. If we consider a forecasting model, there are no severe restrictions on the function's look and feel, with the one and only requirement being the authenticity and immutability of the prediction, when extrapolating the multidimensional data. However, the same principle of searching for the best model by the way of self-organization applies in all cases.

The essence of the development process of the model of optimal complexity by means of model's self-organizing is contained in its gradual organizational identification, i.e. setting of the model's optimal structure and isochronous analysis of its characteristics. In the cases like that, the specific sets of models of varied complexity are generated, and the best of them are identified based on a given rational indicator of regularization.

Figure 7 shows the selection of the criteria for the optimization of multidimensional data [9].

An important question in the research of multidimensional data visualization is the analysis of the relationships between data point's

```
For identification of subsections of multidimensional data in the polynomials and cubic
splines apply the Bergeron's model using the medium-risk criterion;
```

For the multidimensional data sets built on Wong's method of "inclusions with exclusions" model using individual subsets of fragment-candidates;

For the generic data set approximations use Ivakhnenko's Group Method of Data Handling (GMDH) with a wide set of criteria; [10]

Figure 7. Selection of models for development of multidimensional data set optimizations for application in visualization techniques
individual characteristics and its effect on the overall informational content of the data set.

When considering the Bergeron and Wong models (Figure 7) it is important to note that when applied they produce the same effect on a given data-set. Because of it the following statement can be referred to as the "Bergeron-Wong" theorem.

Theorem 2. Full linear big data model, used for the synthesis of the selected multidimensional predictive collections is dependent on the processes characterized by the multicollinearity, neural networks and robust statistical methods applied against large volumes of data for targeted selection of the most informationally significant data attributes.

Proof Theorem 2. The specifics of the Bergeron Model lie in its applicability to the large volumes of data calculation methods with multicollinearity. The model looks as follows:

$$
\begin{gather*}
\Delta v=\frac{a \cdot * v m}{g} \\
a \cdot=a b+D h \\
\Delta v=\frac{a b \cdot * v m}{g} \\
b \cdot=-(a b+D h) \tag{8}
\end{gather*}
$$

This model demonstrates that the visualization problems of information spaces may be resolved by application of the self-organizing modeling described in Figure 7: through various transformations with targeted selection of the informational value of the attributes, breeding records' observed indicators, neural networks and robust statistical methods applied against large volumes of data for targeted selection of the most informationally significant data attributes [2].

Wong's model corresponds to the same representation however it works from the large data sets to the minimalistic ones [11]. The Wong's model can be represented as follows:

$$
\begin{equation*}
\Delta=-g m * \Delta t=\frac{-g t-\Delta r+g t}{2} \Delta t . \tag{9}
\end{equation*}
$$

The formulae for Bergeron's and Wong's models shown above represent a complete linear spatial model for the large sets of data, used for the synthesis of the selected multidimensional predictive collections. Thus the self-organizing approach based on the above statements allows constructing methods and models developed for a specific set of large multidimensional observations. Also it is representable as a collection of the high-level polynomials, through the application of the model connection technique [2].

$$
\begin{gather*}
\Delta=a_{0}+\sum_{i=1}^{m} a_{i} g_{i}+ \\
\sum_{i=1}^{m} \sum_{j=1}^{m} a_{i} a_{j} g_{i} g_{j}+\sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} a_{i} a_{j} a_{k} g_{i} g_{j} g_{k}+m \ldots \tag{10}
\end{gather*}
$$

The key issue in such complex structures based on the very large data sets is to cull the (9) by removing the low-information data attributes, that prove largely irrelevant and to leave a necessary and sufficient number of most meaningful attributes.

The complexity of the model being analyzed is considered optimal when the model remains adequate for the stated purpose with the fewest number of attributes used to comprise the transformed model [7]. Let's illustrate this statement with the Figure 8


Figure 8. Graphical representation of the combined model

## 5 Conclusion

In conclusion, it is important to point out that the data visualization methods are necessary to answer the data interpretation questions, however to determine all the characteristics of the data, manage its size and complexity and eventually reduce the time it takes to develop the answers, these visualizations methods must be preceded and often augmented by the non-visual means of analysis and interpretation of a given set of data.

The more general outcome of this analysis is a methodology for accounting of the group's characteristics, that can be viewed as the linkage between a variety of methodological concepts and can be depicted through modern methods of artificial intelligence [12]. In continuation of this research an original program module will be developed. That model shall implement a generalized multi-faceted algorithm for visualizations of big data with its attributes described and optimized through a set of linear functions, implementing the reflecting "distances" between the attributes at every step of the subset selection.

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# Non-commutative finite associative algebras of 2-dimensional vectors 

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#### Abstract

In this paper properties of the non-commutative finite associative algebra of two-dimensional vectors are presented. Interesting features of algebra are mutual associativity of all modifications of the defined parameterized multiplication operation and existing of a large set of single-side unit elements. In the ordinary case one unique two-side unit element is connected with each element of the algebra, except the elements that are square roots from zero element. There are also presented four different variants of defining commutative associative algebras of 2-dimension vectors. For the case of commutativity the algebra has common unit element for all its elements.


Keywords: finite algebra; ring; Galois field; vector; associative multiplication; parameterized multiplication; cryptoscheme

AMS: 16U60, 11G20, 11T71

## 1 Introduction

Finite non-commutative associative algebras (FNAA) are interesting for applications in the desin of the public-key cryptoschemes characterized in using the hidden conjugacy search problem (called also discrete logarithm problem in hidden commutative subgroup) [1]-[3]. In the literature there are considered different FNAA defined over the finite vector spaces with dimensions $m=4,6$, and 8 . The main attention was paid to the case $m=4$ that provides lower computational difficulty of the multiplication operation in the FNAA, while defining the vector spaces over the same finite field $G F(p)$.

[^6]In the present paper it is shown that the FNAA can be defined over the vector spaces of the dimensions less than 4 . There are introduced two possible variants of defining the FNAA of two-dimensional vectors and investigated some properties of such FNAA. There are also described in brief four possible variants of defining the commutative associative algebras of two-dimensional vectors.

Suppose e and i be some formal basis vectors and $a, b \in G F(p)$, where prime $p \geq 3$, be coordinates. The two-dimensional vectors are denoted as $a \mathbf{e}+b \mathbf{i}$ or as $(a, b)$. The terms $\tau \mathbf{v}$, where $\tau \in G F(p)$ and $\mathbf{v} \in\{\mathbf{e}, \mathbf{i}\}$ are called components of the vector.

The addition of two vectors $(a, b)$ and $(x, y)$ is defined as addition of the corresponding coordinates, i.e. by the following formula $(a, b)+$ $(x, y)=(a+x, b+y)$.

The multiplication of two vectors $a \mathbf{e}+b \mathbf{i}$ and $x \mathbf{e}+y \mathbf{i}$ is defined by the following formula

$$
(a \mathbf{e}+b \mathbf{i}) \circ(x \mathbf{e}+y \mathbf{i})=a x \mathbf{e} \circ \mathbf{e}+b x \mathbf{i} \circ \mathbf{e}+a y \mathbf{e} \circ \mathbf{i}+b y \mathbf{i} \mathbf{i},
$$

where $\circ$ denotes the vector multiplication operation and each product of two basis vectors is to be replaced by some basis vector or by a one-component vector in accordance with the so called basis-vector multiplication table (BVMT) which defines associative (commutative and non-commutative) multiplication of the two-dimensional vectors. In the paper there are considered two variants of the BVMT presented in Table 1 (Section 2) and Table 2 (Section 3) for defining FNAA and four variants of the BVMT presented in Tables 3, 4, 5, and 6 (Section 4) for defining commutative finite algebras.

## 2 Algebra with unique local right-side unit elements

The multiplication of two-dimensional vectors defined by Table 1, where $\mu \neq 0$ and $\tau \neq 0$, is a parameterized operation, different modifications of which correspond to different pairs of values of the so called structural coefficients $\mu$ and $\tau$. As compared with the case of

Table 1. The basis-vector multiplication table for the case $m=2$

| $\circ$ | $\vec{e}$ | $\vec{\imath}$ |
| :---: | :---: | :---: |
| $\vec{e}$ | $\mu \mathbf{e}$ | $\mu \mathbf{i}$ |
| $\vec{\imath}$ | $\tau \mathbf{e}$ | $\tau \mathbf{i}$ |

the commutative finite algebra of the 2-dimensional vectors [4], the defined non-commutative multiplication operation is characterized in the mutual associativity of all its modifications.

Statement 1. Suppose $\circ$ and $\star$ are two arbitrary modifications of the vector multiplication operation, which correspond to different pairs of structural coefficients $\left(\mu_{1}, \tau_{1}\right)$ and $\left(\mu_{2}, \tau_{2}\right) \neq\left(\mu_{1}, \tau_{1}\right)$. Then for arbitrary three vectors $A, B$, and $C$ the following formula $(A \circ B) \star C=$ $A \circ(B \star C)$ holds.

Proof of this statement consists of straightforward calculations using the definition of the multiplication operation and Table 1.

To find the right unit element of the considered FNAA, one can solve the following vector equation

$$
\begin{equation*}
(a \mathbf{e}+b \mathbf{i} \circ(x \mathbf{e}+y \mathbf{i})=(a \mathbf{e}+b \mathbf{i}), \tag{1}
\end{equation*}
$$

where $V=(a \mathbf{e}+b \mathbf{i})$ is an arbitrary vector and $X=(x \mathbf{e}+y \mathbf{i})$ is the unknown one.

Equation (1) can be reduced to solving the following system of two linear equations in $G F(p)$ :

$$
\left\{\begin{array}{l}
(a \mu+b \tau) x=a  \tag{2}\\
(a \mu+b \tau) y=b .
\end{array}\right.
$$

In the case $a \mu+b \tau \neq 0$ this system has a unique solution

$$
\left\{\begin{array}{l}
x=\frac{a}{a \mu+b \tau}  \tag{3}\\
y=\frac{b}{a \mu+b \tau}
\end{array}\right.
$$

All vectors $(a, b)$ such that $a \mu+b \tau \neq 0$ have only one right unit element. In the general case the right unit elements corresponding to different vectors are different, therefore these unit elements can be called local, since they act only in frame of some sufficiently restricted subset of the two-dimensional vectors. There does not exist global right unit element, i.e. right unit acting over the whole two-dimensional vector space. The following is evident:

Statement 2. Suppose $V=(a, b)$ be a vector such that $a \mu+b \tau \neq$ 0 . Then the vector

$$
E_{r}=\left(\begin{array}{ll}
\frac{a}{a \mu+b \tau}, & \frac{b}{a \mu+b \tau} \tag{4}
\end{array}\right)
$$

acts as local right unit in the following subset of two-dimensional vectors $V, V^{2}, \ldots, V^{i}, \ldots$, where $i$ is an arbitrary integer.

Let us consider the sequence $V, V^{2}, \ldots, V^{i}$ (for $i=1,2,3, \ldots$ ). If the vector $V$ is not a zero-divisor relatively some its power (zero-divisors are considered below and it is shown that vectors satisfying condition $a \mu+b \tau \neq 0$ are not zero-divisors), then for some two integers $h$ and $k>h$ we have $V^{k}=V^{h}$ and $V^{k}=V^{k-h} \circ V^{h}=V^{h} \circ V^{k-h}$, i.e. the mentioned sequence is periodic and for some integer $\omega$ (that can be called order of the vector $V$ ) it holds that $V^{\omega}=E^{\prime}$, where $E^{\prime}$ is bi-side local unit such that $V^{i} \circ E^{\prime}=E^{\prime} \circ V^{i}=V^{i}$ holds for all integers $i$. Thus, taking into account that the local right unit element corresponding to the vector $V$ is unique one can conclude the following:

Statement 3. Suppose $V=(a, b)$ be a vector such that $a \mu+b \tau \neq$ 0 . Then the vector $E_{r}$ described by formula (4) acts as a unique bi-side local unit element $E^{\prime}$ in the subset $\left\{V, V^{2}, \ldots, V^{i}, \ldots,\right\}$ and the value $E^{\prime}$ can be computed as some power of $V$.

The following computational example illustrates this fact: for $p=$ 16832914260232697023 and $\mu=276474637$; $\tau=948576254546$ we have

$$
\begin{align*}
& N=(a, b)= \\
& (17235252752952,29124252511124) . \tag{5}
\end{align*}
$$

Computation of the value $E^{\prime}$ as $E^{\prime}=N^{p-1}$ and by using formula (3)
$\underline{\text { Alexander Moldovyan and Nicolay Moldovyan and Victor Shcherbacov }}$
gives the same result

$$
\begin{align*}
& E^{\prime}= \\
& (12597150130467515608,9876457378547066970) . \tag{6}
\end{align*}
$$

To find the left unit elements of the considered FNAA one can solve the following vector equation:

$$
\begin{equation*}
(x \mathbf{e}+y \mathbf{i}) \circ(a \mathbf{e}+b \mathbf{i})=(a \mathbf{e}+b \mathbf{i}) . \tag{7}
\end{equation*}
$$

Equation (7) can be reduced to solving the following system of two linear equations in $G F(p)$ :

$$
\left\{\begin{array}{l}
a \mu x+a \tau y=a  \tag{8}\\
b \mu x+b \tau y=b
\end{array}\right.
$$

The last system defines the following set of the left unit elements:

$$
\begin{equation*}
E_{l}=(x, y)=\left(x, \tau^{-1}(1-x)\right), \tag{9}
\end{equation*}
$$

where $x$ takes on all possible values in $G F(p)$. Each element of the last set acts on all elements of the considered FNAA as the left unit, i.e. elements of set (9) are global left unit elements. Substituting the value $x=a(a \mu+b \tau)^{-1}$ in (9) one can show that all local right units are contained in the set of the (global) left unit elements. This is in compliance with Statement 3.

Let us consider the question of existence of the right and left zerodivisors. The first case is connected with solving the vector equation

$$
\begin{equation*}
(a \mathbf{e}+b \mathbf{i}) \circ(x \mathbf{e}+y \mathbf{i})=(0,0), \tag{10}
\end{equation*}
$$

where $V=(a \mathbf{e}+b \mathbf{i})$ is an arbitrary vector different from $(0,0)$ and $X=(x \mathbf{e}+y \mathbf{i})$ is the unknown one.

Equation (10) can be reduced to solving the following system of two linear equations in $G F(p)$ :

$$
\left\{\begin{array}{l}
(a \mu+b \tau) x=0  \tag{11}\\
(a \mu+b \tau) y=0 .
\end{array}\right.
$$

In the case of the vectors $V$, the coordinates of which satisfy condition $a \mu+b \tau \neq 0$, this system has a unique solution $(x, y)=(0,0)$ that represents zero of the considered FNAA. Each two-dimensional vector acts on the vectors $V$ such that $a \mu+b \tau=0$ as the right zero-divisor.

Consideration of the case of the left zero-divisors is connected with solving the vector equation

$$
\begin{equation*}
(x \mathbf{e}+y \mathbf{i}) \circ(a \mathbf{e}+b \mathbf{i})=(0,0), \tag{12}
\end{equation*}
$$

that can be reduced to the following system of two linear equations in $G F(p)$ :

$$
\left\{\begin{array}{l}
a \mu x+a \tau y=0  \tag{13}\\
b \mu x+b \tau y=0
\end{array}\right.
$$

One can see that each of the vectors

$$
D_{l}=\left(x,-\tau^{-1} \mu x\right),
$$

where $x$ takes on all values in $\operatorname{GF}(p)$, acts on each element of the considered FNAA as the left zero-divisor.

Some zero-divisor $D$ satisfying equation

$$
D^{2}=D \circ D=(0,0)
$$

can be called square root from zero of the FNAA. Finding such elements is connected with solving the vector equation

$$
(x \mathbf{e}+y \mathbf{i}) \circ(x \mathbf{e}+y \mathbf{i})=(0,0),
$$

connected with the following system of two linear equations in $G F(p)$

$$
\left\{\begin{array}{l}
\mu x^{2}+\tau x y=0  \tag{14}\\
\mu x y+\tau y^{2}=0 .
\end{array}\right.
$$

For the last system we have the following solutions that define the set of the square roots from zero element $(0,0)$ :

$$
\begin{equation*}
D=(x, y)=\left(x,-\mu \tau^{-1} x\right), \tag{15}
\end{equation*}
$$

where $x=0,1, \ldots, p-1$. Taking into account the condition of Statement 2 one can conclude that elements, to which no right unit element corresponds, are square roots from the zero vector $(0,0)$.

## 3 Algebra with unique local left-side unit elements

The FNAA of two-dimensional vectors with the multiplication operation defined by Table 2 , where $\mu \neq 0$ and $\tau \neq 0$, has properties analogous to the properties of the FNAA described in Subsection 2.1, for example Statement 1 is valid.

Table 2. Alternative BVMT for the case $m=2$

| $\circ$ | $\vec{e}$ | $\vec{\imath}$ |
| :---: | :---: | :---: |
| $\vec{e}$ | $\mu \mathbf{e}$ | $\tau \mathbf{e}$ |
| $\vec{\imath}$ | $\mu \mathbf{i}$ | $\tau \mathbf{i}$ |

Consideration of the vector equations defining the right and left unit elements, the right and left zero divisors, and square roots from zero $(0,0)$ have given the following statements.

Statement 4. Each two-dimensional vector from the set

$$
\begin{equation*}
E_{r}=(x, y)=\left(x, \tau^{-1}(1-x)\right), \tag{16}
\end{equation*}
$$

where $x$ takes on all possible values in $G F(p)$, represents a global rightside unit element.

Statement 5. Suppose $V=(a, b)$ be a vector such that $a \mu+b \tau \neq$ 0 . Then the vector

$$
E_{l}=\left(\begin{array}{ll}
\frac{a}{a \mu+b \tau}, & b  \tag{17}\\
a \mu+b \tau
\end{array}\right)
$$

is a unique local left-side unit for all vectors from the following set $\left\{V, V^{2}, \ldots, V^{i}, \ldots\right\}$, where $i$ is an arbitrary integer.

Statement 6. A unique local bi-side unit element $E^{\prime}=E_{l}$ acts in the set $\left\{V, V^{2}, \ldots, V^{i}, \ldots\right\}$, where $i$ is an arbitrary integer and vector $V=(a, b)$ is such that $a \mu+b \tau \neq 0$. The value $E^{\prime}$ can be computed as $E^{\prime}=V^{\omega}$ for some integer $\omega$.

Statement 7. Each two-dimensional vector acts on the vectors $V=(a, b)$ such that $a \mu+b \tau=0$ as the left zero-divisor.

Statement 8. Each of the vectors

$$
D_{r}=\left(x,-\tau^{-1} \mu x\right),
$$

where $x$ takes on all values in $G F(p)$, acts on each element of the considered FNAA as the right-side zero-divisor.

## 4 Commutative finite algebras of two-dimensional vectors

Finite commutative associative algebras (FCAA) of two-dimensional vectors can be defined using the following BVMT presented in Tables 3, 4,5 , and 6 , where $\mu \neq 0$ and $\tau \neq 0$.

Table 3. The BVMT defining FCAA with the unit element $(1,0)$

| $\circ$ | $\vec{e}$ | $\vec{\imath}$ |
| :---: | :---: | :---: |
| $\vec{e}$ | $\mathbf{e}$ | $\mathbf{i}$ |
| $\vec{\imath}$ | $\mathbf{i}$ | $\tau \mathbf{e}$ |

The case relating to Table 3 was described in [4], where it has been shown that the algebra represents a finite ring with the unit element $(1,0)$, if the structural coefficient $\tau$ is a quadratic residue modulo $p$, or finite field $G F\left(p^{2}\right)$, if $\tau$ is a quadratic non-residue.

Table 4. The BVMT defining FCAA with the unit element $\left(0, \tau^{-1}\right)$

| $\circ$ | $\vec{e}$ | $\vec{\imath}$ |
| :---: | :---: | :---: |
| $\vec{e}$ | $\mu \mathbf{i}$ | $\tau \mathbf{e}$ |
| $\vec{\imath}$ | $\tau \mathbf{e}$ | $\tau \mathbf{i}$ |

Table 5. Unbalanced BVMT defining FCAA with the unit element $\left(0, \tau^{-1}\right)$

| $\circ$ | $\vec{e}$ | $\vec{\imath}$ |
| :---: | :---: | :---: |
| $\vec{e}$ | $\mu \mathbf{e}$ | $\tau \mathbf{e}$ |
| $\vec{\imath}$ | $\tau \mathbf{e}$ | $\tau \mathbf{i}$ |

Table 6. Unbalanced BVMT defining commutative algebra with the unit element $\left(0, \mu^{-1}\right)$

| $\circ$ | $\vec{e}$ | $\vec{\imath}$ |
| :---: | :---: | :---: |
| $\vec{e}$ | $\mu \mathbf{e}$ | $\mu \mathbf{i}$ |
| $\vec{\imath}$ | $\mu \mathbf{i}$ | $\tau \mathbf{i}$ |

Let us consider the case relating to Table 4. The vector equation for finding the unit element of the considered FCAA is as follows

$$
\begin{equation*}
(a \mathbf{e}+b \mathbf{i}) \circ(x \mathbf{e}+y \mathbf{i})=(a \mathbf{e}+b \mathbf{i}), \tag{18}
\end{equation*}
$$

where $X=(x \mathbf{e}+y \mathbf{i})$ is unknown.
Using Table 4, equation (18) can be reduced to solving the following system of two linear equations in $G F(p)$ :

$$
\left\{\begin{array}{l}
\tau b x+\tau a y=a  \tag{19}\\
\mu a x+\tau b y=b .
\end{array}\right.
$$

In the case $\Delta=\tau^{2} b^{2}-\tau \mu a^{2} \neq 0$ the system has a unique solution $(x, y)=\left(0, \tau^{-1}\right)$. The indicated inequality takes place for all elements $(a, b) \neq(0,0)$ in the following two cases
i) $\tau$ is a quadratic non-residue and $\mu$ is a quadratic residue; ii) $\tau$ is a quadratic residue and $\mu$ is a quadratic non-residue.

In the last two cases all two-dimensional vectors $V=(a, b) \neq(0,0)$ are invertible and the considered FCAA represents the finite field $G F\left(p^{2}\right)$.

If conditions i) and ii) do not take place, for some vectors $V=(a, b)$ we have $\Delta=\tau^{2} b^{2}-\tau \mu a^{2}=0$. Such vectors are not invertible and the FCAA represents a finite ring.

For the non-invertible vector $(a, b) \neq(0,0)$ we have the following set of local unit elements: $E_{x}=\left(x, \tau^{-1} a^{-1}(a-\tau b x)\right)$, where $x=0,1,2, \ldots, p-1$. Except one non-invertible, all other local unit elements are invertible, and $E_{0}=E=\left(0, \tau^{-1}\right)$ represents the global unit element of the FCAA, i.e. the vector acting as a unit element for all elements of the FCAA. The non-invertible local unit element is defined by the following formula:

$$
\begin{equation*}
E_{n}=\left(\frac{a}{\tau b+a \sqrt{\mu \tau}}, \frac{1}{\tau}-\frac{b}{\tau b+a \sqrt{\mu \tau}}\right) . \tag{20}
\end{equation*}
$$

Statement 9. The local unit element $E_{n}$ acts in the set $\left\{V, V^{2}, \ldots, V^{i}, \ldots\right\}$, where $i$ is an arbitrary integer and vector $V=(a, b)$ is such that $\tau^{2} b^{2}-\mu \tau a^{2}=0$, and the value $E_{n}$ can be computed as $E_{n}=V^{\omega}$ for some integer $\omega$.

Each non-invertible vector $(a, b) \neq(0,0)$ devides zero element $(0,0)$, i.e. for some element $D_{x} \neq(0,0)$ we have $(a, b) \circ D_{x}=(0,0)$. Zero divisors $D_{x}=(x, y)$ connected with the non-invertible vector $(a, b) \neq$ $(0,0)$ can be computed from the following system of equations

$$
\left\{\begin{array}{l}
\tau b x+\tau a y=0  \tag{21}\\
\mu a x+\tau b y=0 .
\end{array}\right.
$$

The set of the zero divisors $D_{x}$ is described as follows

$$
D_{x}=\left(x,-b a^{-1} x\right) .
$$

All values $D_{x}$ are non-invertible elements of the FCAA.
In the case of defining FCAA by Table 5 we have the folowing system of equations for computing the unit elements:

$$
\left\{\begin{array}{l}
(\mu a+\tau b) x+\tau a y=a  \tag{22}\\
\tau b y=b .
\end{array}\right.
$$

For all vectors $(a, b) \neq(0,0)$, such that $\mu a+\tau b \neq 0$, system (22) has the same solution $(x, y)=\left(0, \tau^{-1}\right)=E$. The element $E$ is the global unit element of the FCAA defined by Table 5. This FCAA represents a ring with $p$ non-invertible elements $N$ that can be described with the following formular

$$
N=\left(x,-\frac{\mu}{\tau} x\right)
$$

where $x=0,1,2, \ldots p-1$.
Each element from the set

$$
E_{x}=\left(x, \tau^{-1}\right)
$$

acts as local unit element on all non-invertible elements of the FCAA defined by Table 5.

In the case of defining FCAA by Table 6 we have the following system of equations for computing the unit elements:

$$
\left\{\begin{array}{l}
\mu a x=a  \tag{23}\\
\mu b x+(\mu a+\tau b) y=b .
\end{array}\right.
$$

For all vectors $(a, b) \neq(0,0)$, such that $\mu a+\tau b \neq 0$, system (23) has the same solution $(x, y)=\left(\mu^{-1}, 0\right)=E$. The element $E$ is the global unit element of the FCAA defined by Table 6 . This FCAA represents a ring with $p$ non-invertible elements $N$ that can be described with the following formula

$$
N=\left(x,-\frac{\mu}{\tau} x\right),
$$

where $x=0,1,2, \ldots p-1$.
Each element from the set

$$
E_{x}=\left(\mu^{-1}, y\right)
$$

acts as local unit element on all non-invertible elements of the FCAA defined by Table 6 .

## 5 Conclusion

It has been introduced the associative FNAA of the two-dimensional vectors defined over the field $G F(p)$. One of the interesting properties of the investigated FNAA is mutual associativity of all modifications of the parameterized non-commutative multiplication operation. The known in the literature parameterized commutative multiplication operation for the case $m=2$ [4] do not possess such property.

There are also considered FCAAs defined by four different BVMT, three of them being considered for the first time. The considered six BVMT (two for the non-commutativity case and four for the commutativity case) cover possible variants of defining finite associative algebras of the dimension 2. Other variants of BVMT define non-associative algebras, except some modification of Table 3 in which an additional structural coefficient can be inserted.

Future research in frame of the concerned topic is connected with investigation properties of the associative FNAAs of $m$-dimensional vectors for cases $m=3$ and $m=5$.

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Alexander Moldovyan and Nicolay Moldovyan and Victor Shcherbacov
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[^3]:    ${ }^{1}$ The Maple codes of the algorithms are available at http://amirhashemi.iut.ac.ir/softwares

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