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Hospital-Scale Chest X-Ray Database Visualization Using RAWGraph Technique

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Abstract

For identification and screening of several lung diseases, the chest X-ray is just one of the very most often obtainable radiological tests. Most modern-day hospitals' Photo Archiving and Communication Systems (PACS) collect thousands of X-ray imaging scientific studies, followed by radiological stories that are collected and saved. In this paper, we employ the graph strategy to collect and store the dataset of those X-ray pictures. The RAWGraph can be an open source web tool for its production of inactive information visualizations, which can be changed to become further altered. Initially designed for picture artists to extend a succession of responsibilities, maybe not available in combination with different applications, it has developed into a stage that offers easy tactics to map information measurements on visual factors. That poses a more chart-based way of information visualization; every visual version will be an unaffiliated module displaying distinct visual factors that may be utilized to map information measurements. Thus, end-users may develop complex information visualizations. We assess the correlation and relationship among different aspects of this data set. We now provide a chest X-ray database, particularly "ChestX-ray8", that contains 108,948 frontal perspective X-rays of 32,717 specific and distinct patients, with all containing a written text created from eight disorder image tags (where just about every image could have multi-labels), in the related reports utilizing standard language processing. In this paper, we use diverse methods for visualization, which can be Circle Packing, Bee Swarm Plot, Convex Hull, Boxplot, and Circular Dendrogram. We image the dataset more accurately and examine the terms of these various arrangements of features precisely.

Keywords: data visualization, x-rays, graphs, visualization tools, visual interface

1 Introduction

Using information visualization is standard procedure in most medical areas. Even though most visual variations are understood and have been used from the last several decades, their production remains difficult without human end users involvement. Although the current evolution of visualization libraries has empowered the production of innovative and very personalized options, continual programming comprehension and also a great deal of time are necessary to perfect the equipment [1]. The theory that guides RAW forging would be essential to supply an infrastructure to automatically create graphs where only minimal programming abilities are necessary, and the graphs can be reused together with their data. Beginning with the knowledge from design laboratory, then further researching together through papers, we understood that the production visualization procedure is not linear. Ergo, it cannot be solved utilizing one software [1]. Therefore, when planning visualizations, it is reasonable to proceed from device to software based on the use and process, however, we need to get a specific outcome. So, we then created a stage to accomplish this difficult endeavor with no code for the design of information measurements on rare visual variations. The results from this stage must be understood as easily available and alterable; this also usually means they are specifically built to become additionally altered and enhanced with secondary applications, such as vector images editors, including Adobe Illustrator, Inkscape and vectr.com [2].

Initially imagined as something for both designers and vis geeks, raw data is utilized to make a connection involving commonly available software such as Microsoft Excel, the Apple library Open Refine, along with vector image editors such as Adobe Illustrator and Inkscape. This is predicated around the SVG format [3], where visualizations are readily erased and edited, and then used by vector images software for additional refinements or embedded into website pages. We recognize the importance of dealing with and protecting using sensitive information. The information is routed raw from and is processed solely from the Internet browser without any matter data storage or operation that has been completed. In this case, we will probably amend, copy, or otherwise alter our data. RAW can also be exceptionally customizable and flexible, enabling new, custom-made graphs characterized by end-users. To learn more regarding ways to edit or add graphs, visit the developer manual [4].

2 Literature Review

There have been efforts to generate publicly accessible, post-secondary health image databases using number of individuals who have been tested amount that ranges from a few hundred to two million records. However, no qualitative disorder detection answers have been already reported [5]. Our freshly suggested chest X-ray database, in term of size order, will be a minimally larger than OpenI. To attain the improved clinical significance, we now concentrate on exploiting the organizational operation of weakly-supervised, multi-label picture classification and disorder localization of frequent sinus ailments; into this standard measure the “discovering characteristics” or “visible modeling” are also included. Chest X-rays [6], entire lung tomograms, and surgical findings are associated with 152 patients having an extra thoracic malignancy who experienced 182 thoracotomies for their test of pulmonary nodules. Several pulmonary nodules are revealed through complete lung tomography, including one of 25 patients using a typical chest X-ray. Only two of 25 patients experienced carcinoma together with bilateral nodules, illustrated by tomography. Of all 64 patients having unilateral nodules found by traditional chest X-ray,” 10 of 32 exhibited sarcoma, and two more with melanoma [7]. To make the conclusion in the scope of sinus metastasis in patients, lung tomograms or something similar with tomographic examinations [8] are the most efficient method of observation.

A sizable level screen with moderate amount of resolution with TV-fluoroscopic technique is using a 43 cm $\times$ 43 cm center, which has been utilized to track patients undergoing radiation treatment with mega-voltage remedy beams. The following report reveals some preliminary outcomes made for 60Co and 6 MV x-rays. The brain, the supraclavicular region, both the chest, along with lymph areas, are imaged. The stay video graphics show lung and heart, along with diaphragm movement. Except for the gut, different filing arrangements have been also displayed. Permanent files are manufactured on videotape or video disc drive. Moreover, the corresponding confirmation films have been displayed in every case [9]. Increased picture quality can readily be accessed with hardware. The movie graphics can easily be enriched by exclusive optical circuitry that's available at an affordable price. In this paper, we provide a way to extract bronchus spots from 3D CT pictures of lung shots from a helical CT scanner and then to describe them as 3D-shaded pictures. The extraction treatment comprises a 3D region expanding with all the parameters corrected mechanically and will be achieved immediately using a 3D painting algorithm. The result can be envisioned by PC images workstations, and also, the bronchus is seen out of the interior the same as using the simulated bronchus endoscope, openly and with no pain. We predict that manner of updating "navigation" [10].

From the current analysis, the feasibility of employing high-performance microtomography (Micro CT) for discovery of lung cancer has been researched in mice that were living in a heightened phase of cyst progress. The chest field of anesthetized mice had been reimaged by X-ray Micro CT [11]. In mice having a slight and significant chemical loading, Micro CT was always a quick and noninvasive imaging apparatus for the discovery of lung cysts. After the identification of their CT statistics by histologic sectioning, it had been shown that the majority of microbes could be differentiated from the rebuilt digital bits received by Micro CT. The info from Micro CT was additionally proved when supported by a visible review of their excised lungs post-mortem. Micro CT opens great perspectives for imaging enzyme development, and also its particular development as a noninvasive method [12]. The

Micro CT also enables for a routine test of lung cancer from medication. Many X-ray CT scanners demand only a couple seconds to create one two-dimensional (2D) picture of the cross-section of a human body. The truth of full-scale three-dimensional (3D) graphics of this human body synthesized by an adjacent collection of 2D pictures developed by successive CT scans of adjoining human anatomy pieces are tied to number of functions, such as: *i*) slice-to-slice enrollment (placement of the affected individual); *ii*) slit thickness; and also *iii*) movement, both voluntary and involuntary (which happens through the entire time necessary to scan all of pieces. For that reason, this way is insufficient for legitimate energetic 3D imaging of organs, including the lungs, heart, and flow. For resolving these issues, the Dynamic Spatial Reconstructor (DSR) was created from the Biodynamics Research Unit in the Mayo Clinic to present uninterrupted volumes of imaging. These are stop action (1/100 therefore), high-repetition-rate (up-to 60/s), a simultaneous scan of several concurrent sparse cross segments (up to 240, just about every 0.45-millimeter-thick, thick 0.9 mm aside) crossing the whole anatomic degree of the physiological organ(s) of attention [13]. These capacities are accomplished using multiple X-ray resources and several 2D fluoroscopic video-camera assemblies onto a continually rotating gantry. The desired trade-offs between temporal, spatial, and frequency settlement might be performed by retrospective processing and selection of typical subsets of their overall data listed via an ongoing DSR scanning arrangement.

3 Construction of Hospital-scale Chest X-ray Database

In this paper, we clarify the approach of constructing a hospital-scale chest X-ray image database, particularly “ChestX-ray8”, created in our magician’s PACS technique. To begin with, we shortlisted eight shared nasal pathology keywords, which are many times identified and observed on numerous occasions, i.e., Atelectasis, Cardiomegaly, Effusion, Infiltration, Mass, Nodule, Pneumonia, along with Pneumothorax, dependent on radiologists' suggestions. Considering these eight keywords, we hunt the PACS strategy to extract all of the linked radiological stories (along with graphics) as our database corpus [14]. Several pure Natural Language Processing (NLP) methods have been accommodated for discovering the pathology like stop word removal, frequent word removal, rare word removal [15]. Every single record will probably undoubtedly be linked to a couple of keywords or marked using "Normal" in the desktop classification. As a consequence of that search, the ChestX-ray8 database consists of all 108,948 frontal-view X-ray pictures (resulting from 32,717 patients), and just about every picture is tagged with multiple or one pathology keywords.

3.1 Stop words removal

We have discussed stop words removal earlier in basic feature extraction from database. In basic pre-processing we have followed the same earlier routine. We have used a predefined library and also used a list of stop words.

3.2 Frequent words removal

In the previous step, we have just removed stop words, but in this step, we have removed common words. We have collected the ten most frequently occurring words, then take a call to retain or remove. We have removed those words that are not used in classification of the database.

3.3 Rare words removal

In the prior step we have removed the most common words, and in the next step we have removed the most rare words from the database. Due to the rarity, the association between them and other words is dominated by noise. We can replace the rare words with the general word form to increase the count of the words.

3.4 Labeling Disease Names by Text Mining

In general, our strategy generates tags by employing the accounts from two moves. From the very first iteration, we discover most of the disorder theory from the corpus. The most crucial figure of just about every chest X-ray report is commonly ordered as “Replies”, “Indication”, “Findings”, and “Perception” segments. We give attention to discovering illness theories from the Indication and Perception segments. When an account comprises neither of the two segments, the full report is then going to be viewed. At the second pass, we follow the accounts. As a rule, they should not consist of any disorders (maybe not confined by two adjoining pathologies).

4 RawGraph Visualization Results

In this section, we focuse on different sets of visualization. It presents a chart-based approach to data visualization; each visual model is an independent module exposing different visual variables that can be used to map data dimensions. Consequently, users can create complex data visualizations. We have the lung disease dataset, which has 5607 rows of the dataset. In each row we have any specific data on males and females. In the first set of visualization, we upload the first 500 rows of diseases in RAWGraphs that are as shown in Figure 1.

4.1 RAWGraph Interface

In the interface of the RAWGraph, we can map the dimension of our interest in work more appropriately. After uploading the dataset, we can set our dimensions in different perspectives. We can set the hierarchy where we can drag number strings and dates, strings that we can drag here could use the gender and age of the patient. In section Hierarchy we can set the hierarchy of our data so, that the records to be ordered by patient’s gender or age; in this section we can drag numbers, strings or dates for the data; string data are used for Patient Gender and numbers are used for Patient Age. In section Size we can set the age, where we can drag just the numbers for the age of the patient. In section Color we can set the Patient ID by dragging the

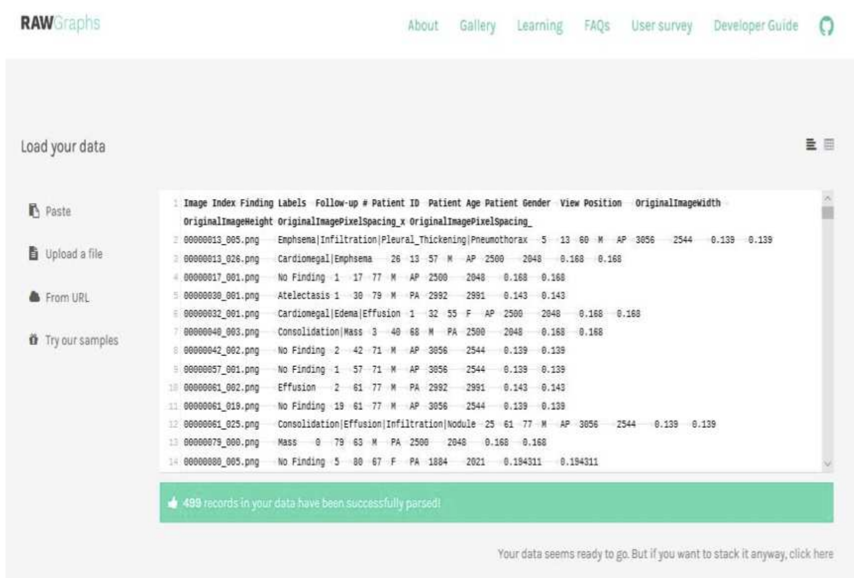

Figure 1. 500 rows of lungs disease data are loading and unstacked

number to separate out some patients from the others. Also, in section Label, we can label the patient's gender and age as it is shown in Figure 2. In the process of mapping our dimensions, we can also map different dimensions that could be a group, X-axis, Y-axis, and radius of the dataset, more precisely.

4.2 Circle packing

At the start of the work, the dataset is going to be unstacked, and the first type of graph will be generating; the name of this type of graph is called Circle Packing. To compare the values and to represent hierarchies, we used nested circles. This circle packing is used to show the proportion of elements through their position and their areas in a hierarchical structure. Two large circles represent genders in the dataset; and size of the smaller circles represents the age. The bigger circles represent those patients whose age is higher than the rest of the patients. The parameters for this result are shown in Figure 3 as

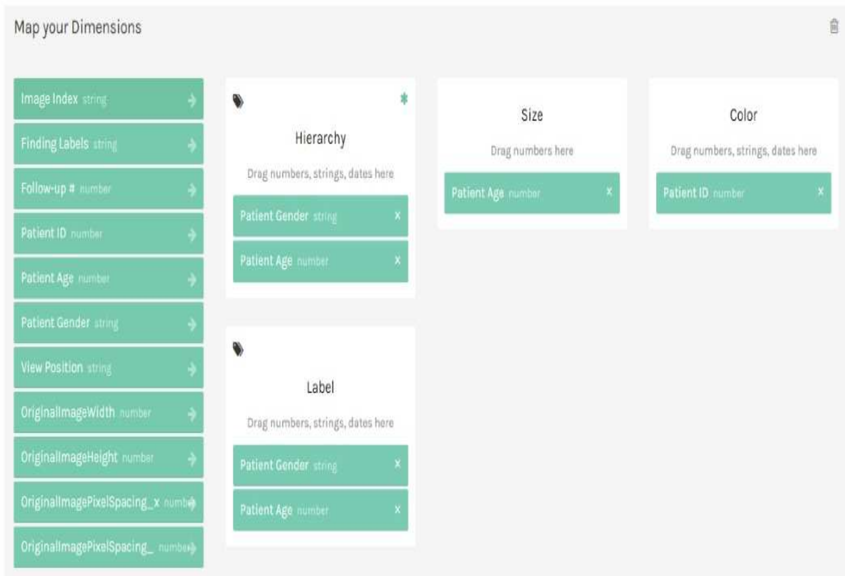

Figure 2. Representation of the dimensions used in RawGraphs

follows.

In this type of graphic representation, the rows that we use for disease from dataset are 3501–5000. To compare the values and to represent hierarchies, we used nested circles to show the proportion of elements through their position and their areas in a hierarchical structure. The graph represents classification of diseases between males and females along with their age groups. As it can be visualized, the disease represented in red circles is most common among males and females, both shown in Figure 4.

4.3 Bee swarm plot

It distributes the elements horizontally, avoiding overlap between them and according to a selected dimension. The x-axis has the age of the patients, and the y-axis has gender. Here, it can be visualized that, in the early ages, the female is supposed to be less likely to be affected by the lung disease as compared to males as shown in Figure 5.

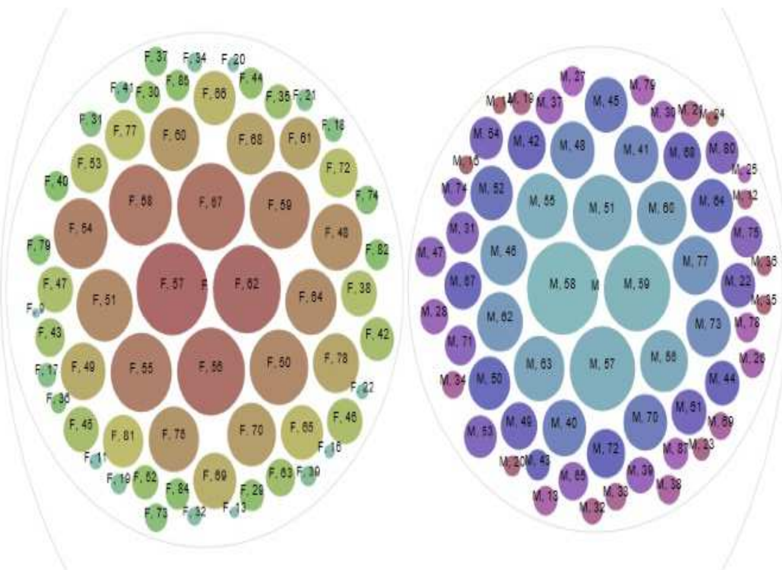

Figure 3. Representation of circle packing Graphs

4.4 Convex hull

In this type of graphic representation, the rows that we use for disease from dataset are 501–1500. In mathematics, the convex hull is the smallest convex shape containing a set of points. It is useful to identify points belonging to the same category when applied to a scatterplot. The shapes represent the same category of disease between females and males of different ages. The x-axis has a patient id, and the y-axis has patient age, as shown in Figure 6. Also, the rows that we use for disease from dataset are 1501–2500; they show the horizontal strokes that represent males and females who have lung disease of the same category, as shown in Figure 7.

4.5 Boxplot

In this type of graphic representation, the rows that we use for disease from dataset are 2501–3000. To summarize a quantitative distribution,

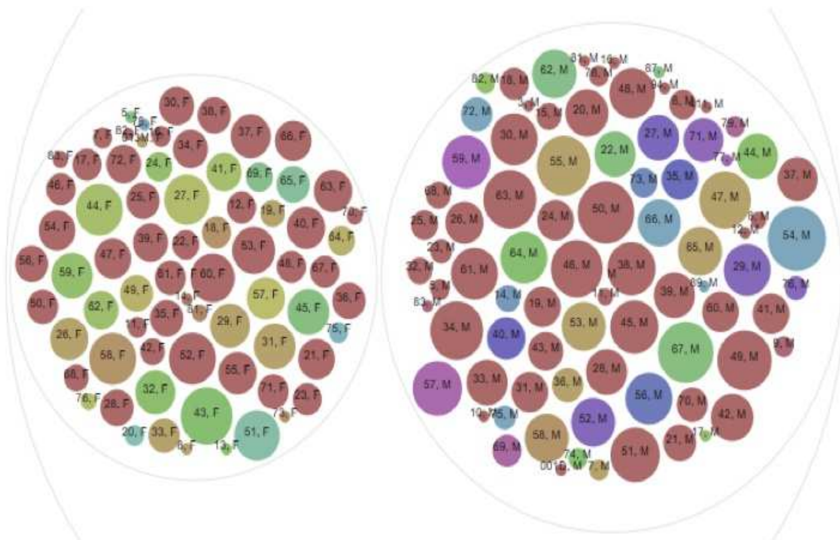

Figure 4. Representation of circle packing Graphs

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we use five official statistics: the most significant value, upper quartile, median, lower quartile, and the smallest value. Different color of bars represents different diseases, and size of the bar represents age of patients; along the y-axis there is the age and along the x-axis there are the findings of diseases as it is shown in Figure 8.

4.6 Circular dendrogram

In this type of graphic representation, the rows that we use for disease from dataset are 5001–5607. We use this to represent the distribution of hierarchical clustering; the tree-like diagrams use Dendrograms. On the x-axis, the different depth levels represented by each node are visualized. The graph represents the classification of diseases between males and females among the selected data set. The nodes contain names of the diseases which are comprised of males and females, respectively, as it is shown in Figure 9.

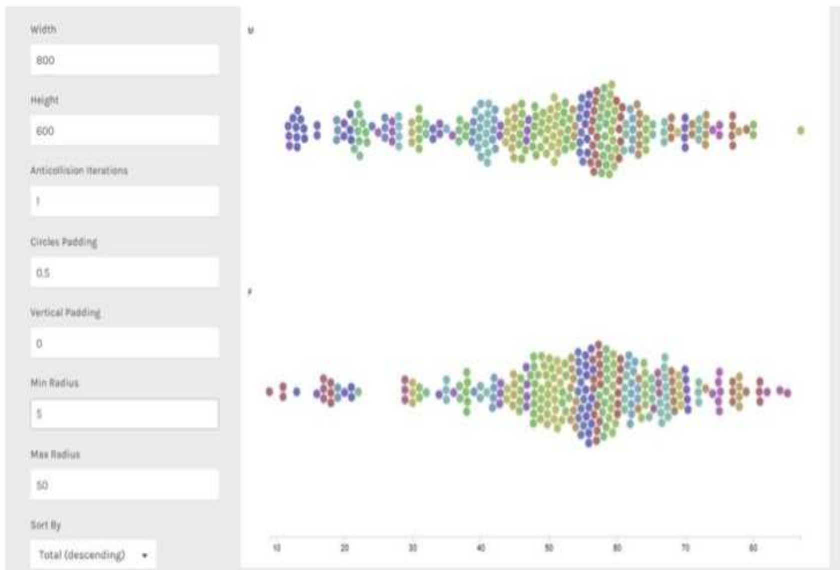

Figure 5. Representation of Bee swarm Plot graphs

5 Conclusion and Future Work

In this paper, we offered RAWGraphs – an Internet program known for its quick production of visualizations out of a data set. The application form is still opensource and can be assumed to become expendable; through, programming skills are also required. We use diverse methods for visualization, which can be circle packing, bee swarm plot, convex hull, boxplot, and circular dendrogram.

Even though end-users' responses are been mostly favorable, by the comments and opinions. We identify about three enhancements to improve long-term function. To begin with, inside today's edition, the legends (e.g., coloration mapping) are observable just at the modifying surroundings; they are lost when the graph has been already exported. Additionally, tags are obvious things that are difficult to take care of, and also new purposes should be supplied to ease tackling, pruning, and

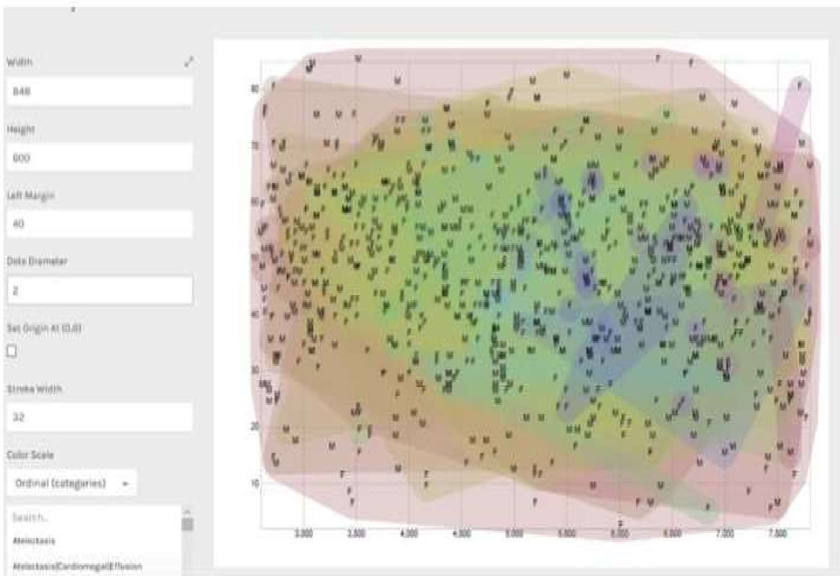

Figure 6. The same category of disease between males and females

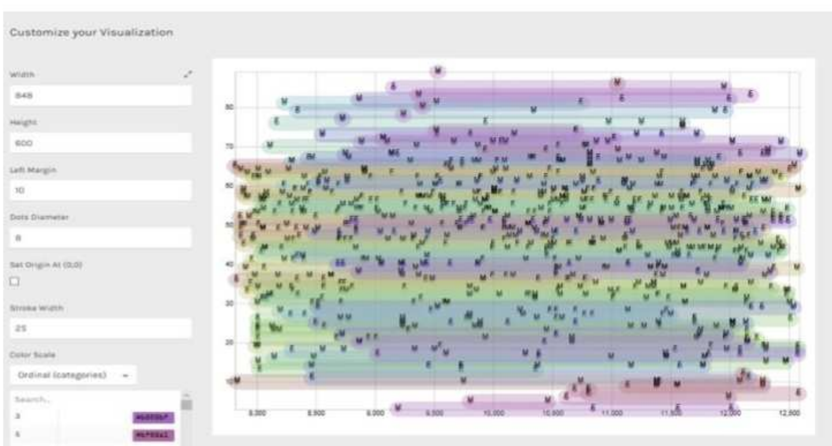

Figure 7. The horizontal strokes represent males and females who have the same category of disease

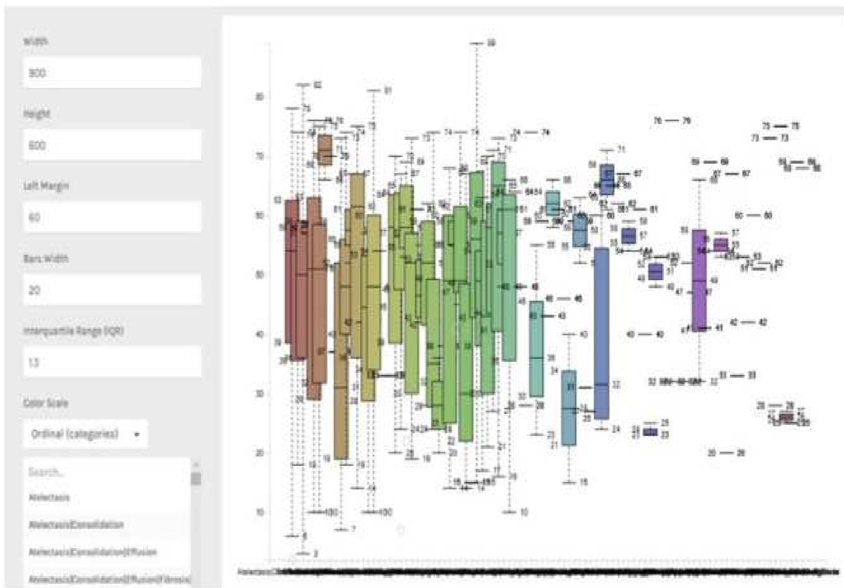

Figure 8. Different color of bars represents different diseases, and size of the bar represents age of patients

filtering. Until now, quite a few subscribers have established their very own customized edition of the application, plus they have added new visual variations; nonetheless, it is still hard for noninvasive end users to authenticate those bits of code. The commencement of advanced visualizations can add a competitive advantage in new graphs. However, it seems to be a beneficial advancement. At length, probably, the engaging obstacle would be to expand the present procedure. Therefore, users may cause interactive visualizations, perhaps not static websites. That necessitates incorporating the capacity to export a package comprising all of the data files (HTML and JavaScript) together with all the mapping. That will allow end-users to manually map measurements, maybe not solely on visible options but in addition to connections (for example, tool-tips or behaviors within an assortment). The comprehension about programming languages is desired; nonetheless,

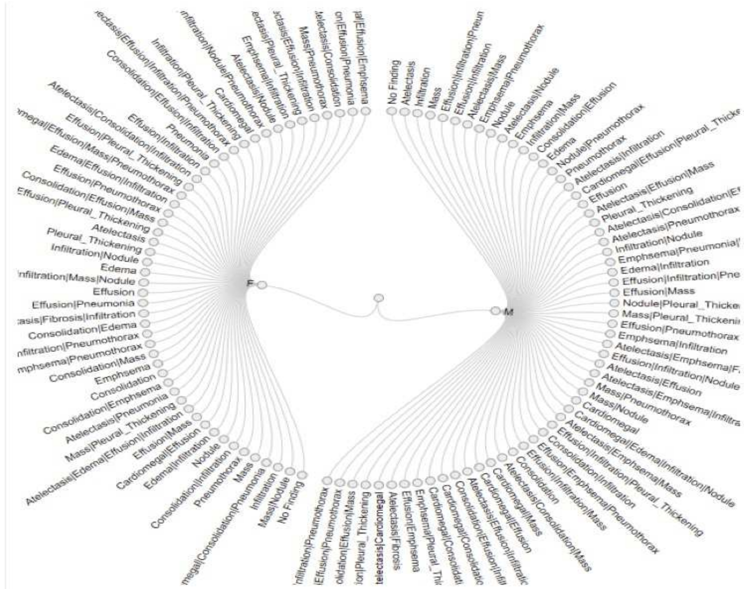

Figure 9. Represents the classification of diseases between males and females

nevertheless, it would be scalable based on the amount of data users wish to alter. The logical approaches adapting by those people already implemented from the present phase of the job could be further researched together to launch new methods for a far more technical direction of developing a design and style, especially inside the discipline of information visualization.

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Solving transportation problems with concave cost functions using genetic algorithms

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Abstract

In this paper we propose a genetic algorithm for solving the non-linear transportation problem on a network with concave cost functions and the restriction that the flow must pass through all arcs of the network. We show that the algorithm can be used in solving large-scale problems. We prove that the complexity of a single iteration of the algorithm is $O(nm)$ and converges to an ϵ-optimum solution. We also present some implementation and testing examples of the algorithm using Wolfram Mathematica.

Keywords: genetic algorithm, population, minimum cost flow, non-linear transport problem, large-scale problem, concave function.

MSC 2010: 05C21, 90C06, 90C26, 90B15, 90B06, 90C59.

1 Introduction

Problems that describe real situations using concave functions are often too complex to be solved by polynomial algorithms; therefore, they can be solved only by checking each possible solution with brute force algorithms. Because such algorithms would be too time consuming, genetic algorithms are an alternative that can find the solution in a reasonable amount of time. These are stochastic and heuristic algorithms, which means that the obtained solutions are not always optimal, but they come close to the optima. The use of these algorithms in favour to other heuristic algorithms is recommended, because they do not need the gradient or Hessian information. They are also resistant to locks in a local minimum and can be used to solve large-scale non-linear optimization problems.

There are several principles [1] that must be followed when designing and implementing a genetic algorithm. We must use several characteristic operations, i.e. selection, crossover and mutation, to improve the final solution. When codifying a solution, we must keep in mind to use as little memory as possible for the chromosomes. The complexity of the evaluation, crossover and mutation has to be of a low order. Each chromosome, mutation or crossover must correspond to an admissible solution, thus we can guarantee the correctness of the algorithm. Although it is possible to decode the chromosomes one-to-one (each chromosome corresponds to a single solution), one-to-n (each chromosome corresponds to n solutions) or n-to-one (n chromosomes correspond to a single solution), the one-to-one decoding is recommended. When searching for a solution to the problem, a balance must be kept between the exploration of as many of the admissible solutions as possible and the exploitation of the solution as close to the optimal solution as possible.

An overview and comparative analysis of the genetic algorithms is given in [3,4]. In [2] an original genetic algorithm with local search is presented.

In this paper we present a modification to the algorithm discussed in [5,6] that can solve the non-linear transport problem with concave cost functions. This new algorithm was tested in Wolfram Mathematica, and some results are presented.

2 Problem formulation. Main results

We consider the transportation problem on a network described by a connected acyclic graph. On the finite set of vertices V the real function of production and consumption $q(v)$ is defined. On the finite set of arcs E the concave non-decreasing piecewise linear functions of cost $\varphi_e(x_e)$ are defined. It is required to solve the non-linear optimization problem that consists in determining a flow x^* that minimizes the function:

$$F(x) = \sum_{e \in E} \varphi_e(x_e)$$

We must solve the non-linear problem:

$$F(x^*) = \min_{x \in X} F(x) \tag{1}$$

$$\sum_{e \in E^+} x_e - \sum_{e \in E^-} x_e = q(v) \tag{2}$$

$$x_e > 0, \forall(e), \tag{3}$$

where X is the set of admissible solutions which satisfies the conditions (2) – (3) of existence of flow in the network. An additional restriction is that there must be a flow passing through every arc of the network. This condition ensures that the algorithm is capable of arriving at the solution, otherwise it would be nearly impossible for it to find a solution with no flow through some arcs. Such a problem is the model of the transportation of a flow through water or electricity networks.

$$q(v) = \begin{cases} p(v) = \sum_{i=1}^{k} p(v_i), & v = v_0, v_i \in V_t, \forall i = 1, ..., k \\ 0, & V/V_t \{v_0\} \\ -p(v_i), & v_i \in V_t, \forall i = 1, ..., k \end{cases} \tag{4}$$

It is preferable to use an elitist model of the genetic algorithm that transfers the best chromosomes to the next population, in order to avoid losing solutions that cannot be restored later.

Definition 1. *The value of the total objective function of the population $F_T(x)$ is the sum of the values of the objective function of the solutions decoded from the chromosomes of the population.*

The value of the total objective function will be smaller (better) after each step, because new chromosomes will have better characteristics.

2.1 Genetic algorithm P2

The first step in the use of a genetic algorithm is the codification of the problem, that is, the description of the chromosomes that each represents the admissible solution of the problem. A population consists of chromosomes which, in fact, represents a set of admissible solutions.

In this algorithm every gene of a chromosome will be a matrix that contains the rate of the flow that passes through outgoing arcs of the vertex i. To save computer memory, the chromosomes will not be described by a weighted matrix of size $n \times n$ which is sparse, but by a table of lists of total size m. Each list i contains the rate of the flow for those arcs coming out of the vertex i.

Description of the genetic algorithm P2:

Step 1.

Initialization. The initial population of $4n$ chromosomes is generated as follows: every chromosome is described by a list of the form $L = \{\{l_{i1}, ..., l_{im_i}\} \,|\forall i = 1, \ldots ,n\}$, m_i – the number of outgoing arcs of the vertex i. Every $l_{ij}, i = 1, \ldots , n, j = 1, \ldots , m_i$ shows the part of the flow that passes through the arc j that comes out of vertex i. For every vertex the condition $\sum_{j=1}^{m_i} l_{ij} = 1, i = 1, \ldots , n$ must be satisfied.

Step 2.

Decoding and Evaluation of the chromosome from the current population. The decoding is the association of each chromosome with an admissible solution based on how the problem is codified. Evaluation is the determination of the value of the objective function for each of the decoded solutions.

Step 3.

Selection of the parent chromosomes that will participate in the creation of the next population. The chromosomes will be sorted in order of increasing value of the objective function. The first $2n$ of them will be transferred to the next population and will participate in crossover and creation of the offsprings.

Step 4.

Crossover of the chromosomes will occur between chromosomes transferred from population $P(i - 1)$ to $P(i)$. The parents will be cut randomly at the same point. Then the chromosomes will be combined in the following way: the left half of the first chromosome with right half of the second chromosome will yield one offspring, then another offspring will be obtained by combining the right half of the first chromosome with the left half of the second one. Thus every pair of parents will have two offsprings, and the total size of the population will remain constant.

Step 5.

Mutation of a chromosome will take place at a rate of $\alpha \in [0.01, 0.1]$ and implies the mutation of a random gene. A new list $\{l_{i1}, ..., l_{im_i}\}$ will be generated for a random vertex i such that the condition $\sum_{j=1}^{m_i} l_{ij} = 1, i = 1, \ldots, n$ is satisfied.

Step 6.

Checking the stopping condition implies stopping the algorithm when the condition $\left|f(x_{P(i)}) - f(x_{P(i-1)})\right| \leq \epsilon$ is satisfied for the solutions associated with the first chromosome of two consecutive populations $P(i-1)$ and $P(i)$ sorted in order of increasing value of the objective function. The solution to the problem will be the solution that corresponds to the first chromosome from the last population. If the stopping condition is not satisfied, the algorithm returns to Step 2.

Observation 1. *One of the following conditions can also be used to stop the algorithm:*

- *Generation of k populations. During our testing there proved to be no advantages to this stopping condition in favor of the condition $\left|f(x_{P(i)}) - f(x_{P(i-1)})\right| \leq \epsilon$, except minor improvements in the final solution or rare major changes. It works better in small tests where the solution set isn't very large and mutation or crossover that leads to a much better solution can easily be found.*

- *Generation of populations until a time limit is especially desirable for large networks. This condition is useful for extremely large networks, where the execution time and the complexity of the problem is too large for the condition $\left|f(x_{P(i)}) - f(x_{P(i-1)})\right| \leq \epsilon$. It is preferable for the generation of k populations, because it is* *not possible to accurately judge the amount of time needed for an iteration.*

Decoding a chromosome of the population to a solution associated with it is done as follows: it is known that a flow $p(v)$ must pass through the transportation network. This flow will be distributed over the arcs of the network using the list L generated in step 1, which contains the part of the total flow available in vertex i that passes through the arc (i,j), $\forall j = 1, \dots, m_i$. The value obtained thus is the flow x_k associated with the arc (i,j). This value will be placed on the position k in the admissible solution of the problem, which has the form $x = (x_1, x_2, ..., x_m)$.

2.2 Theoretical results

The algorithm P2 described above can be applied when the network satisfies the following conditions:

- The graph that describes the network is acyclic;
- Every vertex of the graph, except the destination, has at least one outgoing arc;
- The destination vertex does not have any outgoing arcs.

Theorem 1. *The genetic algorithm P2 requires memory $O(nm)$.*

Proof. The transportation network is described by an adjacency list of size m. Every chromosome consists of a list L of size m. Thus a population of $4n$ chromosomes will be of size $4nm$. This population will be renewed at each iteration of the algorithm and no additional memory will be needed. Therefore, the algorithm requires memory $O(nm)$.$\square$

Theorem 2. *The complexity of a single iteration of the genetic algorithm P2 is $O(nm)$.*

Proof. To fill in the adjacency list that describes the graph, $O(m)$ operations are necessary. The generation of a single chromosome has a *complexity of $O(m)$, thus the generation of a population requires $O(nm)$ operations. The evaluation of a solution associated to a chromosome requires $O(m)$ operations. As there are $4n$ chromosomes in a population, the total complexity is $O(nm)$. The crossover and mutation have a complexity of $O(m)$ for each chromosome and require $O(nm)$ operations in total. Therefore a single iteration of the algorithm has the complexity $O(nm)$.*$\square$

Definition 2. *An ϵ-optimum solution of the non-linear transportation problem on a network is the solution $x_{P(i)}$ generated by a population $P(i)$ that satisfies the condition $\left|f(x_{P(i)}) - f(x_{P(i-1)})\right| \leq \epsilon$.*

Theorem 3. *The genetic algorithm P2 always converges to an ϵ-optimum solution.*

Proof. *The chromosomes whose objective function is the smallest will be transferred from the population $P(i-1)$ to $P(i)$. The value of the total objective function of each new population will be smaller than the previous population. Each new population $P(i)$ will generate a solution whose objective function will be smaller or equal to the solution generated by the population $P(i-1)$. Because the algorithm converges to a local minimum and satisfies the condition $\left|f(x_{P(i)}) - f(x_{P(i-1)})\right| \leq \epsilon$, we say that it converges to an ϵ-optimum solution.*$\square$

Observation 2. *Based on Theorem 2, it can be implied that the execution time of the genetic algorithm P2 is $O(Unm)$, where U is the number of necessary iterations to obtain an ϵ-optimum solution.*

2.3 Practical results

Below we will consider the application of the Genetic Algorithm P2 on a set of test cases.

Example 1. *Let the transportation network be described by a connected acyclic graph.*

The set of the vertices $\{1,2,3,4,5\}$ *is associated with the production and consumption function:* $q(v)=\begin{cases} 15 \text{ if } v=1 \\ 0 \text{ if } v=2,3,4 \\ -15 \text{ if } v=5 \end{cases}.$

Every arc from the set $\{e_1,e_2,e_3,e_4,e_5,e_6,e_7,e_8,e_9\}$ *is associated with a cost function as follows:*

– the cost function $\varphi_1(x)=\begin{cases} x \text{ if } x\le 1 \\ 1 \text{ if } x>1 \end{cases}$ *will be associated with the arcs* $\{e_1,e_3,e_4,e_7\}$*;*

– the cost function $\varphi_2(x)=\begin{cases} 2x \text{ if } x\le 2 \\ 4 \text{ if } x>2 \end{cases}$ *will be associated with the arcs* $\{e_2,e_5,e_6,e_8,e_9\}$*.*

The chromosome mutation will be done with a probability of 0.1.

Step 1. *Initialization of a population of 20 chromosomes, each containing 5 lists. Each list i will have some numbers that describe the part of the flow that passes through each outgoing arc of that vertex.*

The value of the objective function, calculated for the first generated chromosome $\{\{0.39,0.01,0.23,0.37\},\{0.48,0.52\},\{0.18,0.82\},\{1.00\}\}$ *associated with the solution* $x=\{3.4,5.9,5.7,0.11,1.6,1.8,6.1,1.3,12.\}$*, is F(x)=17. This value describes the initial cost of transporting 15 units of the product from vertex 1 to vertex 5.* $F_T(x)=340$ *units.*

Iteration 1 *The chromosomes of the initial population are sorted in increasing order of the value of the objective function for the solution associated with the respective chromosomes. The first half of them is transferred to the next population, and it will be the parents of the second half of that population. The offspring is obtained through crossover and then mutated by choosing a random vertex i and generating a new list* $\{l_{i1},...,l_{im_i}\}$*. The minimum value of the objective function for the generated population is: F(x)=10 and* $F_T(x)=310$ *units.*

Iteration 2 *The chromosomes of the population 1 are sorted in increasing order of the value of the objective function for the solution associated with the respective chromosomes. The first half of them is transferred to the next population, and it will be the parents of the second half of that population. The offspring is obtained through* *crossover and then mutated by choosing a random vertex i and generating a new list* $\{l_{i1},...,l_{im_i}\}$. *The minimum value of the objective function for the generated population is F(x)=10 which is equal to the minimum value of the objective function of the previous population. Because the solutions of two consecutive populations coincide, then* $x = \{3.4, 0.27, 7.7, 3.6, 3.1, 0.37, 3.2, 0.09, 11.\}$ *will be the* ϵ*-optimum solution of the presented problem.* $F_T(x) = 290$ *units. STOP.*

The following examples will be of much larger dimensions (n – vertices, m – arcs), that is why we will present only the value of $F_T(x)$ obtained at each iteration and the value of the objective function for the ϵ-optimum solution *(Table 1.)*.

Table 1. Examples 2-4 of GA

Iteration	**Ex. 2 (u.c.)**	**Ex. 3 (u.c.)**	**Ex. 4 (u.c.)**
	n=30 / m=238	n=100 / m=2392	n=150 / m=5875
1	*74520*	*479731*	*683430*
2	*70750*	*463270*	*658387*
3	*68168*	*461585*	*640905*
4	*65941*	*454880*	*624014*
5	*63955*	*448784*	*608769*
6	*61794*	*442621*	*593624*
7	*59992*	*437814*	*579315*
8	*58840*	*433995*	*564279*
9	*57422*	*430823*	*550445*
10	*56276*	*428015*	*535530*
11	*56276*	-	*525713*
F(x)	430	1000	820

The tests were performed on an Intel i5-2500 machine with 4 Cores and 8GB DDR3 memory in the Wolfram Mathematica 12.

Through practical examples the convergence of the algorithm is evident. By calculating $F_T(x)$ we notice a decrease of this value from one iteration to another.

The algorithm described in Section 2.1 was implemented in Wolfram Language and tested on a set of examples of transportation networks of various dimensions in terms of number of arcs and vertices of the graph that describes the network.

Table 2. Execution time of the GA P2

vertices/ arcs	$t_\epsilon(sec.)$	$t_k(sec.)$	**vertices/ arcs**	$t_\epsilon(sec.)$	$t_k(sec.)$
10 / 33	*0.078*	*0.3125*	**50 / 647**	*9.9687*	*27,7813*
15 / 62	*0.3125*	*0.2300*	**70 / 1274**	*42,9375*	*78,9219*
20 / 105	*0.4843*	*1.7675*	**80 / 1764**	*45,7031*	*130,0320*
25 / 168	*0.9687*	*3.4843*	**90 / 2275**	*216,6410*	*194,5470*
30 / 231	*4.8595*	*5.4218*	**100 / 2572**	*354,4537*	*244,5780*
40 / 402	*7.3593*	*13.4219*	**120 / 4530**	*237,4530*	*430,5310*

The tests (*Table 2*) were performed using two stopping conditions:

1. the algorithm is stopped only when the condition $\left|f(x_{P(i)}) - f(x_{P(i-1)})\right| \leq \epsilon$ is satisfied for solutions corresponding to the value of the minimum objective function from two consecutive populations $P\,(i-1)$ and $P\,(i)$, i.e. an ϵ-optimum solution was found;

2. the algorithm is stopped after $k = 10$ iterations, and the final solution is the solution from the last population with minimum objective function.

3 Conclusions

Genetic algorithms are very good solutions to nonlinear transportation problems with concave cost functions, especially when there are additional restrictions that force the flow to pass through all arcs of the network. From the above we can state that:

- The genetic algorithm P2 is correctly codified because it respects the condition of existence of flow in the network and lets us solve large-scale problems in reasonable time;

- The proposed decoding algorithm always obtains an admissible solution;

- The value of the total objective cost function $F_T(x)$ decreases from one generation to another, and practical tests confirm the convergence to a local solution that is also an ϵ-optimum solution;

- It has been proven practically that the stopping condition $\left|f(x_{P(i)}) - f(x_{P(i-1)})\right| \leq \epsilon$ is correct and lets us obtain a good result much faster than constructing a fixed number of populations.

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Total k-rainbow domination subdivision number in graphs

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Abstract

A total k-rainbow dominating function (TkRDF) of G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1,\dots,k\}$ such that (i) for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u\in N(v)} f(u) = \{1,\dots,k\}$ is fulfilled, where $N(v)$ is the open neighborhood of v, and (ii) the subgraph of G induced by $\{v \in V(G) \mid f(v) \neq \emptyset\}$ has no isolated vertex. The total k-rainbow domination number, $\gamma_{trk}(G)$, is the minimum weight of a TkRDF on G. The total k-rainbow domination subdivision number $\mathrm{sd}_{\gamma_{trk}}(G)$ is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the total k-rainbow domination number. In this paper, we initiate the study of total k-rainbow domination subdivision number in graphs and we present sharp bounds for $\mathrm{sd}_{\gamma_{trk}}(G)$. In addition, we determine the total 2-rainbow domination subdivision number of complete bipartite graphs and show that the total 2-rainbow domination subdivision number can be arbitrary large.

Keywords: total k-rainbow domination, total k-rainbow domination subdivision number, k-rainbow domination.
MSC 2010: 05C69.

1 Introduction

In this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). For every vertex $v \in V$, the *open neighborhood*

$N_G(v) = N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. The *minimum degree* and the *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A *leaf* is a vertex of degree one, a *support vertex* is a vertex adjacent to a leaf and a strong support vertex is a vertex adjacent to at least two leaves. Let L_v denote the set of leaves adjacent to the vertex v.

A subset S of vertices of G is a *dominating (total dominating) set* if $N[S] = V$ $(N(S) = V)$. The *domination (total domination) number* $\gamma(G)$ $(\gamma_t(G))$ is the minimum cardinality of a (total) dominating set of G. A (total) dominating set with cardinality $\gamma(G)$ $(\gamma_t(G))$ is called a $\gamma(G)$-set ($\gamma_t(G)$-set). The domination and its variations have been attracted considerable attention and surveyed in three books [20], [21]. Velammal [26] defined the *domination subdivision number* $sd_\gamma(G)$ to be the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the domination number. The domination subdivision number has been studied by several authors (see for instance [7], [18]). Similar concepts related to connected domination were studied in [17], to total domination in [16], [19], [22], to Roman domination in [5], [6], [8], to rainbow domination in [12], [15], to weakly convex domination in [13] and to convex domination in [14].

Let k be a positive integer, and let $[k] := \{1, 2, \ldots, k\}$. A function $f : V(G) \to 2^{[k]}$ is a *k-rainbow dominating function* (kRDF) of G if for each vertex $v \in V(G)$ with $f(v) = \emptyset$, the condition $\bigcup_{u \in N(v)} f(u) = [k]$ is fulfilled. The *weight* of a kRDF f on G is $\omega(f) = \sum_{v \in V(G)} |f(v)|$. The *$k$-rainbow domination number* of G, $\gamma_{rk}(G)$, is the minimum weight of a kRDF on G. The k-rainbow domination number was introduced by Brešar, Henning and Rall [9] and has been studied by several authors [6], [10], [11], [23]–[25].

A k-rainbow dominating function f on G, is called a *total k-rainbow dominating function* (TkRDF) if the subgraph of G induced by the set $\{v \in V(G) \mid f(v) \neq \emptyset\}$ has no isolated vertex. The *total k-rainbow domination number* of G, $\gamma_{trk}(G)$, is the minimum weight of a $T\text{k}RDF$

of G. A $T\mathrm{k}RDF$ f of G with weight $\gamma_{trk}(G)$ is called a $\gamma_{trk}(G)$-function. Note that $\gamma_{tr1}(G)$ is equal to the classical total domination number, denoted by $\gamma_t(G)$. The total k-rainbow domination has been studied in [1]–[3].

The *total k-rainbow domination subdivision number* $\mathrm{sd}_{\gamma_{trk}}(G)$ of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the total k-rainbow domination number of G. (An edge $uv \in E(G)$ is subdivided if the edge uv is deleted, but a new vertex x is added, along with two new edges ux and vx. The vertex x is called a subdivision vertex). Observation 1 below shows that the total k-rainbow domination number of graphs cannot decrease when an edge of graph is subdivided.

The purpose of this paper is to initiate the study of the total k-rainbow domination subdivision number in graphs. We first present some sharp bounds on $\mathrm{sd}_{\gamma_{trk}}(G)$, and then determine the total 2-rainbow domination subdivision number of complete bipartite graphs. In addition, we show that the total 2-rainbow domination subdivision number can be arbitrary large. Although it may not be immediately obvious that the total k-rainbow domination subdivision number is defined for all graphs without isolated vertices, we will show this shortly (see Corollary 4).

We make use of the following results in this paper.

Proposition A. *[2] For any graph G of order n without isolated vertices*

$$min\{n, k, \gamma_{rk}(G), \gamma_t(G)\} \leq \gamma_{trk}(G) \leq k\gamma_t(G).$$

Proposition B. *[2] Let $k \geq 2$ be an integer, and let G be a graph of order $n \geq k$. Then $\gamma_{trk}(G) = k$ if and only if $n = k$ and there exists a set $A = \{v_1, v_2, \ldots, v_t\} \subseteq V(G)$ with $2 \leq t \leq k$ such that the induced subgraph $G[A]$ has no isolated vertex and $V(G) - A \subseteq N(v_i)$ for each $1 \leq i \leq t$.*

Proposition C. *[2]* For $n \geq 3$, $\gamma_{tr2}(C_n) = \lceil \frac{2n}{3} \rceil$.

Corollary 1. For $n \geq 3$,

$$\mathrm{sd}_{\gamma_{tr2}}(C_n) = \begin{cases} 1 & \text{if} \quad n \equiv 0, 1 \pmod 3 \\ 2 & \text{if} \quad n \equiv 2 \pmod 3. \end{cases}$$

Proposition D. *[2]* For $n \geq 2$, $\gamma_{tr2}(P_n) = \lceil \frac{2n+2}{3} \rceil$.

Corollary 2. For $n \geq 2$,

$$\mathrm{sd}_{\gamma_{tr2}}(P_n) = \begin{cases} 1 & \text{if} \quad n \equiv 0, 2 \pmod 3 \\ 2 & \text{if} \quad n \equiv 1 \pmod 3. \end{cases}$$

Proposition E. *[2]* If $k \geq 3$ and $n \geq 3$, then $\gamma_{trk}(P_n) = \gamma_{trk}(C_n) = n$.

Corollary 3. *If $k \geq 3$ and $n \geq 3$, then* $\mathrm{sd}_{\gamma_{trk}}(P_n) = \mathrm{sd}_{\gamma_{trk}}(C_n) = 1$.

Observation 1. Let G be a graph and $u \in V(G)$ be a support vertex with a leaf neighbor v. If f is a $\gamma_{trk}(G)$-function, then $|f(u)| + |f(v)| \geq 2$.

2 Bounds and exact values

In this section we present basic results on the total k-rainbow domination subdivision number in graphs. Our first result shows that the total k-rainbow domination number of a graph can not be decreased by subdividing an edge.

Proposition 1. Let G be a simple connected graph of order $n \geq 3$ and $e = uv \in E(G)$. If G' is obtained from G by subdividing the edge $e = uv$ with vertex x, then $\gamma_{trk}(G') \geq \gamma_{trk}(G)$.

Proof. Let f be a $\gamma_{trk}(G')$-function. If $f(x) = \emptyset$, then $f|_{V(G)}$ is a TkRDF of G. Let $f(x) \neq \emptyset$. Since f is a TkRDF of G', we have $\max\{|f(u)|, |f(v)|\} \geq 1$. Suppose without loss of generality that $|f(v)| \geq 1$. Define $g : V(G) \to 2^{[k]}$ by $g(u) = f(u) \cup f(x)$, and $g(x) = f(x)$ otherwise. Obviously g is a TkRDF of G with weight $\gamma_{trk}(G')$ and so $\gamma_{trk}(G') \geq \gamma_{trk}(G)$. $\square$

Theorem 1. *Let G be a graph and $u \in V(G)$ be a vertex with degree at least two. Then* $\mathrm{sd}_{\gamma_{trk}}(G) \le \deg(u)$.

Proof. Let $N(u) = \{u_1, u_2, \ldots, u_r\}$ and let G' be the graph obtained from G by subdividing the edges $uu_1, \ldots, uu_r$ with subdivision vertices $x_1, \ldots, x_r$, respectively. Suppose f is a $\gamma_{trk}(G)$-function. If $f(u) = \{1, 2, \ldots, k\}$, then in order that u to be totally rainbow dominated, we may assume that $|f(x_1)| \ge 1$, and the function $g : V(G) \to 2^{[k]}$ defined by $g(u) = \{1\}$, $g(u_i) = f(u_i) \cup f(x_i)$ for $1 \le i \le r$, and $g(x) = f(x)$ otherwise, is a TkRDF of G of weight less than $\gamma_{trk}(G')$. If $1 \le |f(u)| \le k-1$, then in order that u to be totally rainbow dominated, we can assume that $|f(x_1)| \ge 1$, and the function $g : V(G) \to 2^{[k]}$ defined by $g(u) = f(x_1)$, $g(u_i) = f(u_i) \cup f(x_i)$ for $2 \le i \le r$, and $g(x) = f(x)$ otherwise, is a TkRDF of G of weight less than $\gamma_{trk}(G')$. Henceforth we assume that $f(u) = \emptyset$. In order that u to be rainbow dominated, we must have $\sum_{i=1}^{r} |f(x_i)| \ge k$, and in order that x_i to be totally rainbow dominated, we must have $|f(u_i)| \ge 1$ for each i. Then the function $g : V(G) \to 2^{[k]}$ defined by $g(u) = \{1\}$, and $g(x) = f(x)$ otherwise, is a TkRDF of G of weight less than $\gamma_{trk}(G')$, and this implies that $\mathrm{sd}_{\gamma_{trk}}(G) \le \deg(u)$. □

The following results are immediate consequences of Theorem 1.

Corollary 4. If $k \ge 2$ is an integer and G is a connected graph of order $n \ge 2$, then
$$\mathrm{sd}_{\gamma_{trk}}(G) \le \Delta(G).$$
Furthermore, this bound is sharp for C_n when $n \equiv 2 \pmod 3$ and $k = 2$.

Corollary 4 shows that the total k-rainbow domination subdivision number is well-defined for all non-trivial graphs when $k \ge 2$.

Corollary 5. If $k \ge 2$ is an integer and G is a connected graph with $\delta(G) \ge 2$, then
$$\mathrm{sd}_{\gamma_{trk}}(G) \le \delta(G).$$
This bound is sharp for C_n when $n \equiv 2 \pmod 3$ and $k = 2$.

Corollary 6. If G is a graph and u, v are two adjacent vertices each of degree at least two, then $\mathrm{sd}_{\gamma_{trk}}(G) \le \deg(u)+\deg(v)-|N(u)\cap N(v)|-1$.

Corollary 7. If G is a connected graph of $n \ge 3$ and $v \in V(G)$ is a support vertex, then $\mathrm{sd}_{\gamma_{trk}}(G) \le \deg(v)$.

Ahangar et al. [2] proved that for any connected graph G of order $n \ge 3$, $\gamma_{trk}(G) \le n - \delta(G) + 2$. Using this bound and Theorem 1 we obtain the next result.

Corollary 8. For any connected graph G with $\delta(G) \ge 2$,

$$\mathrm{sd}_{\gamma_{trk}}(G) \le n - \gamma_{trk}(G) + 2.$$

Moreover, Ahangar et al. [2] showed that for any connected graph G of order $n \ge 3$, $\gamma_{trk}(G) \ge \lceil \frac{kn}{\Delta(G)+k-1} \rceil$. Applying this lower bound and Corollary 8, the next result follows.

Corollary 9. If G is a connected graph with $n \ge 3$, then $\mathrm{sd}_{\gamma_{trk}}(G) \le n - \left\lceil \frac{kn}{\Delta(G)+k-1} \right\rceil + 2$.

Now we provide some sufficient conditions to have small total k-rainbow domination number.

Proposition 2. If $k \ge 2$ is an integer and G contains a strong support vertex, then

$$\mathrm{sd}_{\gamma_{trk}}(G) = 1.$$

Proof. Let v be a strong support vertex of G and let $v_1, v_2 \in L_v$. Assume that G' is the graph obtained from G by subdividing the edge vv_1 with vertex x. Suppose f is a $\gamma_{trk}(G')$-function. By Observation 1, we have $|f(x_1)| + |f(v_1)| \ge 2$, $|f(v)| + |f(v_2)| \ge 2$ and $|f(v)| \ge 1$. Define $g : V(G) \to 2^{[k]}$ by $g(v_1) = \{1\}$, and $g(x) = f(x)$ otherwise. Clearly, g is a TkRDF of G with weight smaller than $w(f)$, implying that $\mathrm{sd}_{\gamma_{trk}}(G) = 1$. □

Proposition 3. Let $n > k \ge 2$ be integers and G a simple graph with $\gamma_{trk}(G) = k$. Then $\mathrm{sd}_{\gamma_{trk}}(G) = 1$.

Proof. Suppose $e = uv$ is an edge of G and G' is the graph obtained from G by subdividing the edge uv with subdivision vertex x. We show that $\gamma_{trk}(G') > \gamma_{trk}(G)$. Suppose, to the contrary, that $\gamma_{trk}(G') = \gamma_{trk}(G) = k$. By Theorem B, there exists a set $A = \{v_1, v_2, \ldots, v_t\} \subseteq V(G')$ with $2 \leq t \leq k$ such that the induced subgraph $G'[A]$ has no isolated vertex and $V(G') - A \subseteq N(v_i)$ for each $1 \leq i \leq t$. It follows that $x \in V(G) \setminus A$ and $A = \{u, v\}$. Since $G[A]$ has no isolated vertex, there must exist another edge $e' = uv$ in G which leads to a contradiction because G is simple. Thus $\gamma_{trk}(G') > \gamma_{trk}(G)$ and so $\mathrm{sd}_{\gamma_{trk}}(G) = 1$. □

Proposition 4. *Let $k \geq 2$ be an integer and G be a connected graph of order $n \geq k + 2$ with $\gamma_{trk}(G) = k + 1$. Then* $\mathrm{sd}_{\gamma_{trk}}(G) \leq 2$.

Proof. The result is immediate for $n = 4$. Assume that $n \geq 5$. If G is a star, then by Proposition 2 we have $\mathrm{sd}_{\gamma_{trk}}(G) = 1$. Assume that G is not a star, and let $M = \{u_1v_1, u_2v_2\}$ be a matching in G. Let G' be the graph obtained from G by subdividing the edges u_1v_1, u_2v_2 with vertices x, y, and let f be a $\gamma_{trk}(G')$-function. We show that $\gamma_{trk}(G') \geq k+2$. If $f(x) = f(y) = \emptyset$, then we must have $f(u_i) \cup f(v_i) = \{1, 2, \ldots, k\}$ for $i = 1, 2$, and this implies that $\gamma_{trk}(G') \geq 2k \geq k + 2$ as desired. Suppose without loss of generality that $|f(x)| \geq 1$. Then, in order that x to be totally dominated, we may assume that $|f(u_1)| \geq 1$. Now if $f(y) = \emptyset$, then we have $f(u_2) \cup f(v_2) = \{1, 2, \ldots, k\}$, implying that $\gamma_{trk}(G') \geq k + 2$. Suppose $|f(y)| \geq 1$. If $f(z) \neq \emptyset$ for each $z \in V(G)\setminus\{u_1, u_2, v_1, v_2\}$, then, clearly, $\gamma_{trk}(G') \geq k+2$; and if $f(z) = \emptyset$ for some $z \in V(G)\setminus\{u_1, u_2, v_1, v_2\}$, then we have $\cup_{u \in N(z)} f(u) = \{1, \ldots, k\}$, and this implies that $\gamma_{trk}(G') \geq k + 2$ because $x, y \notin N(z)$. Thus, $\mathrm{sd}_{\gamma_{trk}}(G) \leq 2$, and the proof is complete. □

Lemma 1. *Let $k \geq 2$ be an integer and G be a connected graph containing a triangle uvw. If G' is obtained from G by subdividing the edges uv, vw, wu with vertices x_1, x_2, x_3, respectively, then for any $\gamma_{trk}(G')$-function, $|f(u)|+|f(v)|+|f(w)|+\sum_{i=1}^{3} |f(x_i)| \geq \min\{5, k+2\}$.*

Proof. Let f be a $\gamma_{trk}(G')$-function such that $f(u)$ is as large as possible. By the choice of f we may assume that $f(u) \neq \emptyset$. If $f(u) = \{1, 2, \ldots, k\}$, then in order that x_2 to be totally rainbow dominated, we must have $|f(w)|+|f(x_2)|+|f(v)| \geq 2$, and this leads to the result. Suppose $|f(u)| < k$. If $f(x_1), f(x_3) \neq \emptyset$, then in order that x_2 to be totally rainbow dominated, we must have $|f(w)| + |f(x_2)| + |f(v)| \geq 2$, implying that $|f(u)| + |f(v)| + |f(w)| + \sum_{i=1}^{3} |f(x_i)| \geq 5$ as desired. Without loss of generality, assume that $f(x_1) = \emptyset$. Then $|f(u)| + |f(v)| \geq k$. If $f(x_2), f(x_3) \neq \emptyset$, then obviously $|f(u)|+|f(v)|+|f(w)|+\sum_{i=1}^{3} |f(x_i)| \geq k+2$. Assume that $f(x_3) = \emptyset$ (the case $f(x_2) = \emptyset$ is similar). In order that x_3 to be rainbow dominated, we must have $|f(u)| + |f(w)| \geq k$. Note that $|f(u) \cap f(w)| \geq 1$. If $f(x_2) \neq \emptyset$, then, clearly, $|f(u)|+|f(v)|+|f(w)| + \sum_{i=1}^{3} |f(x_i)| \geq k+2$, and if $f(x_2) = \emptyset$, then in order that x_2 to be rainbow dominated, we have $f(v) \cup f(w) = \{1, 2, \ldots, k\}$, and so $|f(u)| + |f(v)| + |f(w)| + \sum_{i=1}^{3} |f(x_i)| \geq k+2$.

□

Proposition 5. *Let G be a simple connected graph of order at least three. If G has a vertex $v \in V(G)$ which is contained in a triangle vuw such that $N(u) \cup N(w) \subseteq N[v]$, then $\mathrm{sd}_{\gamma_{trk}}(G) \leq 3$.*

Proof. Let $N(v) = \{v_1 = u, v_2 = w, v_3, \ldots, v_{\deg(v)}\}$ and G' be obtained from G by subdividing the edges vu, vw, uw with vertices x_1, x_2, x_3, respectively. By Lemma 1, the inequality $|f(u)| + |f(v)| + |f(w)| + \sum_{i=1}^{3} |f(x_i)| \geq \min\{5, k+2\}$ holds; and the function $g : V \to \mathcal{P}(\{1, 2, \ldots, k\})$ defined by $g(v) = \{1, 2, \ldots, k\}$, $g(u) = \{1\}$, $g(w) = \emptyset$, and $g(x) = f(x)$ otherwise, is a TkRDF of G of weight less than $\gamma_{trk}(G')$, implying that $\mathrm{sd}_{\gamma_{trk}}(G) \leq 3$. □

3 The special case $k = 2$

In this section we focus on the case $k = 2$.

3.1 An upper bound

Here we present an upper bound on $\mathrm{sd}_{\gamma_{tr2}}(G)$.

Theorem 2. For any connected graph G of order $n \geq 3$ with $\delta(G) = 1$,

$$\mathrm{sd}_{\gamma_{tr2}}(G) \leq \min\{\gamma_{tr2}(G) - 1, \alpha'(G) + 1\}$$

where $\alpha'(G)$ is the matching number of G. This bound is sharp for complete graphs.

Proof. The result is immediate for $\gamma_{tr2}(G) = 2$ or 3 by Propositions 3 and 4. Assume that $\gamma_{tr2}(G) \geq 4$. Let $u \in V$ be vertex of degree one, $uv \in E(G)$ and $N(v) = \{u = v_1, v_2, \ldots, v_k\}$. By the proof of Theorem 1, subdividing all edges adjacent to u increases the total 2-rainbow domination number. First, we prove that $\mathrm{sd}_{\gamma_{tr2}}(G) \leq \gamma_{tr2}(G) - 1$. Now let S be a largest subset of $N(v)$ containing u, such that subdividing the edges uv_i for $v_i \in S$ does not increase the total 2-rainbow domination number. If $|S| \leq 2$, then $\mathrm{sd}_{\gamma_{tr2}}(G) \leq 3 \leq \gamma_{tr2}(G) - 1$. Let $|S| \geq 3$ and assume without loss of generality that $S = \{v_1, \ldots, v_r\}$. Let G' be the graph obtained from G by subdividing the edges $vv_1, \ldots, vv_r$ with vertices $x_1, \ldots, x_r$. By the choice of S, we have $\gamma_{tr2}(G) = \gamma_{tr2}(G')$. Let f be a $\gamma_{tr2}(G')$-function. Clearly, $|f(v_1)| + |f(x_1)| \geq 2$. If $|f(v)| \geq 1$, then the function $g : V(G) \to \mathcal{P}(\{1, 2\})$ defined by $g(v_1) = \{1\}$, $g(v_i) = f(v_i) \cup f(x_i)$ for $1 \leq i \leq r$, and $g(x) = f(x)$ otherwise, is a T2RDF of G of weight less than $\gamma_{tr2}(G)$, which is a contradiction. Hence we assume that $f(v) = \emptyset$. In order that x_i to be totally rainbow dominated, we must have $|f(x_i)| + |f(v_i)| \geq 2$ for each $1 \leq i \leq r$. Then we have $\gamma_{tr2}(G) = \gamma_{tr2}(G') \geq 2s > s + 1 \geq \mathrm{sd}_{\gamma_{tr2}}(G)$. Thus $\mathrm{sd}_{\gamma_{tr2}}(G) \leq \gamma_{tr2}(G) - 1$.

Next we show that $\mathrm{sd}_{\gamma_{tr2}}(G) \leq \alpha'(G)+1$. If $\mathrm{sd}_{\gamma_{tr2}}(G) \leq 2$, then the result is immediate. Suppose $\mathrm{sd}_{\gamma_{tr2}}(G) \geq 3$. By Corollary 7, we may have $\deg(v) \geq \alpha' + 1$. Let S be a smallest subset of $N(v)$ containing u, such that subdividing the edges uv_i for $v_i \in S$ increases the total 2-rainbow domination number. We may assume without loss of generality that $S = \{vv_1, \ldots, vv_r\}$. By assumption we have $r \geq 3$. Let G' be the graph obtained from G by subdividing the edges $vv_1, vv_2, \ldots, vv_{r-1}$ with vertices $x_1, x_2, \ldots, x_{r-1}$, respectively. Then $\gamma_{tr2}(G) = \gamma_{tr2}(G')$. Let f be $\gamma_{tr2}(G)$-function. By Observation 1, we have $|f(v_1)|+|f(x_1)| \geq 2$. As above we may assume that $f(v) = \emptyset$. If $|f(v_1)| + |f(x_1)| \geq 3$, then the function $g : V(G) \to \mathcal{P}(\{1, 2\})$ defined by $g(v) = g(v_1) = \{1\}$, $g(v_i) = f(v_i) \cup f(x_i)$ for $2 \leq i \leq r$, and $g(x) = f(x)$ otherwise, is a T2RDF of G of weight less than $\gamma_{tr2}(G')$ which leads to a contradiction.

Assume that $|f(v_1)|+|f(x_1)|=2$ and so $|f(v_1)|=|f(x_1)|=1$. Suppose without loss of generality that $f(v_1)=f(x_1)=\{1\}$. If $|f(x_i)|\geq 1$ for some $2\leq i\leq r-1$, say $i=2$, then the function $g:V(G)\rightarrow \mathcal{P}(\{1,2\})$ defined by $g(v_1)=\{1\}$, $g(v)=f(x_2)$, $g(v_i)=f(v_i)\cup f(x_i)$ for $3\leq i\leq r$, and $g(x)=f(x)$ otherwise, is a T2RDF of G of weight less than $\gamma_{tr2}(G')$ which leads to a contradiction. Suppose that $f(x_i)=\emptyset$ for each $i\in\{2,\ldots,r\}$. Then in order that x_i to be totally 2-rainbow dominated, we must have $f(v_i)=\{1,2\}$ for each $i\in\{2,\ldots,r-1\}$. If there is some v_i $(2\leq i\leq r-1)$, say $i=2$, such that $f(w)\neq\emptyset$ or $\cup_{x\in N(w)\setminus\{v_i\}}f(x)=\{1,2\}$ for each $w\in N_G(v_i)\setminus\{v\}$, then function $g:V(G)\rightarrow\mathcal{P}(\{1,2\})$ defined by $g(v)=\{1\}$, $g(v_2)=\{1\}$, and $g(x)=f(x)$ otherwise, is a T2RDF of G of weight smaller than $\gamma_{tr2}(G')$, a contradiction. Thus for each $2\leq i\leq r-1$, v_i has a private neighbor w_i with respect to $\{v_2,\ldots,v_{r-1}\}$. Clearly, the set $\{vv_1,v_2w_2,\ldots,v_{r-1}w_{r-1}\}$ is a matching of G and this implies that $\mathrm{sd}_{\gamma_{tr2}}(G)\leq r+1\leq\alpha'(G)+1$ as desired. This completes the proof. □

3.2 A family of graphs with large total 2-rainbow domination subdivision number

In the section we will show that the total 2-rainbow domination subdivision number can be arbitrary large. Haynes et al. in [19] introduced the following graph to prove a similar result on $\mathrm{sd}_{\gamma_t}(G)$.

Let $X=\{1,2,\ldots,3(k-1)\}$, and let $\mathcal{Y}$ be the set that consists of all k-subsets of X. Clearly, $|\mathcal{Y}|=\binom{3(k-1)}{k}$. Let $\mathcal{G}$ be the graph with vertex set $X\cup\mathcal{Y}$ and with edge set constructed as follows: add an edge joining every two distinct vertices of X and for each $x\in X$ and $Y\in\mathcal{Y}$, add an edge joining x and Y if and only if $x\in Y$. Then, $\mathcal{G}$ is a connected graph of order $n=\binom{3(k-1)}{k}+3(k-1)$. We observe that the set X induces a clique in $\mathcal{G}$, the set $\mathcal{Y}$ is an independent set and each vertex of $\mathcal{Y}$ has degree k in $\mathcal{G}$. It is proved in [19] that $\gamma_t(\mathcal{G})=2k-2$ and $\mathrm{sd}_{\gamma_t}(\mathcal{G})=k$.

Lemma 2. *For any integer $k\geq 3$, $\gamma_{tr2}(\mathcal{G})=4(k-1)$.*

Proof. By Proposition A and the fact $\gamma_t(\mathcal{G})=2k-2$ we have $\gamma_{tr2}(\mathcal{G})\leq 4(k-1)$. To prove the inverse inequality, let f be a $\gamma_{tr2}(\mathcal{G})$-function such that $|Z = \{v \in V : |f(v)| = 1\}|$ is as small as possible. We proceed with two claims.

Claim 1. For each $Y \in \mathcal{Y}$, $|f(Y)| \leq 1$.

Suppose, to the contrary, that $f(Y) = \{1, 2\}$ for some $Y \in \mathcal{Y}$. Since f is a T2RDF of $\mathcal{G}$, Y has a neighbor $x \in X$ with $|f(x)| \geq 1$. We assume without loss of generality that $1 \in f(x)$. Let $z \in Y - \{x\}$ and define the function $g : V(\mathcal{G}) \to \mathcal{P}(\{1, 2\})$ by $g(Y) = \emptyset$, $g(z) = \{2\} \cup f(z)$ and $g(x) = f(x)$ for all $x \in V(\mathcal{G}) - \{Y, z\}$. Since $\mathcal{Y}$ is independent and $\mathcal{G}[X]$ is a clique, g is a T2RDF of $\mathcal{G}$ with smaller weight than $\gamma_{tr2}(\mathcal{G})$ which is a contradiction.

Claim 2. $|Z \cap \mathcal{Y}| = 0$.

Suppose, to the contrary, that $|Z \cap \mathcal{Y}| \geq 1$. Let $Y_1 \in \mathcal{Y}$ such that $|f(Y_1)| = 1$. Since f is a T2RDF of $\mathcal{G}$, Y_1 must have a neighbor $x_1 \in X$, with $|f(x_1)| \geq 1$. Assume that Y_2 is a k-subset of X not containing x_1. In order that Y_2 to be totally rainbow dominated, it has a neighbor $x_2 \in X$ with $|f(x_2)| \geq 1$. Now the function g defined by $g(x_1) = \{1, 2\}$, $g(Y_1) = \emptyset$, and $g(x) = f(x)$ otherwise, is a $\gamma_{tr2}(\mathcal{G})$-function which contradicts the choice of f, and the claim follows.

Let X_i (i=1,2) be the set of vertices of X such that $f(x) = \{i\}$ and let X_3 be the set of vertices of X assigned $\emptyset$ by f. If $|X_1|+|X_3| \geq k$, then no k-subset of $X_1 \cup X_3$ is rainbow dominated under f, a contradiction. Hence, $|X_1|+|X_3| \leq k-1$. Likewise, we have $|X_2|+|X_3| \leq k-1$. Note that the other vertices of X are assigned $\{1, 2\}$ under f. Let $X_{1,2} = X - \{X_1, X_2, X_3\}$. Clearly, the following integer linear programming

$$\begin{array}{ll} Min & |X_1| + |X_2| + 2|X_{1,2}| \\ s.t. & |X_1| + |X_3| \leq k - 1 \\ & |X_2| + |X_3| \leq k - 1 \\ & |X_1| + |X_2| + |X_{1,2}| + |X_3| = 3k - 3 \\ & |X_i| \in \mathbb{Z} \geq 0 \end{array}$$

has the optimal value $4(k - 1)$, and this completes the proof. □

Theorem 3. *For any integer $k \geq 4$, $\mathrm{sd}_{\gamma_{tr2}}(\mathcal{G}) = k$.*

Proof. Let $F = \{e_1, e_2, \ldots, e_{k-1}\}$ be an arbitrary subset of $k-1$ edges of $\mathcal{G}$. Assume H is obtained from $\mathcal{G}$ by subdividing each edge in F.

We show that $\gamma_{tr2}(H) = \gamma_{tr2}(\mathcal{G})$. Since $\mathrm{sd}_{\gamma_t}(\mathcal{G}) = k$, we have $\gamma_t(\mathcal{G}) = \gamma_t(H) = 2(k-1)$, and we deduce from Proposition A and Lemma 2 that $\gamma_{tr2}(H) = \gamma_{tr2}(\mathcal{G}) = 4(k-1)$. Hence, $\mathrm{sd}_{\gamma_{tr2}}(\mathcal{G}) \geq k$. Now the result follows by Theorem 1. $\square$

3.3 Complete bipartite graphs $K_{m,n}$

In this subsection we determine the total 2-rainbow domination subdivision number of complete bipartite graphs.

Proposition F. *If $G = K_{n,m}$ is the complete bipartite graph with $m \geq n \geq 1$, then*

$$\mathrm{sd}_{\gamma_{tr2}}(G) = \begin{cases} 1 & \text{if} \quad n = 1, 2 \\ 2 & \text{if} \quad n \geq 3. \end{cases}$$

Proof. Let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ be the bipartite sets of $K_{m,n}$. The result is trivial for $m = n = 1$. If $n = 1$ and $m \geq 2$, then, clearly, $\gamma_{tr2}(K_{1,m}) = 3$, and it follows from Proposition 2 that $\mathrm{sd}_{\gamma_{tr2}}(K_{1,m}) = 1$.

Assume next that $n = 2$. If $m = 2$, then the result follows from Corollary 1. Suppose $m \geq 3$. Then we have $\gamma_{tr2}(K_{2,m}) = 3$. Let G' be the graph obtained from $G = K_{2,m}$ by subdividing the edge x_1y_1 with vertex x and let f be a $\gamma_{tr2}(G')$-function. In order that x to be totally rainbow dominated, we must have $|f(x_1)| + |f(x)| + |f(y_1)| \geq 2$.
First, let $f(x) \neq \emptyset$. Then in order that x to be totally rainbow dominated, we may assume that $|f(x_1)| \geq 1$. If $|f(y_1)| \geq 1$, then the function f restricted to $G = K_{2,m}$ is a T2RDF of $K_{2,m}$ of weight less than $\omega(f)$, and so $\mathrm{sd}_{\gamma_{tr2}}(K_{2,m}) = 1$. Assume that $f(y_1) = \emptyset$. Then in order that y_1 to be rainbow dominated, we must have $|f(x)| + |f(x_2)| \geq 2$. If $f(x_2) = \emptyset$, then in order that x_2 to be rainbow dominated, we have $\sum_{i=2}^{m} |f(y_i)| \geq 2$, implying that $\gamma_{tr2}(G') \geq 4$, and if $|f(x_2)| \geq 1$, then in order that x_2 to be totally dominated, we have $\sum_{i=2}^{m} |f(y_i)| \geq 1$, yielding $\gamma_{tr2}(G') \geq 4$ again.
Now let $f(x) = \emptyset$. If $f(x_1) \neq \emptyset$ and $f(y_1) \neq \emptyset$, then in order that x_1, y_1 to be totally dominated, we must have $|f(x_2)| \geq 1$ and

$\sum_{i=2}^{m}|f(y_i)| \geq 1$, and so $\gamma_{tr2}(G') \geq 4$. Assume without loss of generality that $f(y_1) = \emptyset$. Now in order that x, y_1 to be rainbow dominated, we must have $f(x_1) = \{1,2\}$ and $f(x_2) = \{1,2\}$, yielding $\gamma_{tr2}(G') \geq 4$. This implies that $\mathrm{sd}_{\gamma_{tr2}}(K_{2,m}) = 1$.

Finally, let $n \geq 3$. Clearly, $\gamma_{tr2}(K_{n,m}) = 4$ in this case. First, we show that $\mathrm{sd}_{\gamma_{tr2}}(K_{n,m}) \geq 2$. Let $e = u_iv_j$ be an arbitrary edge of $K_{n,m}$. We may assume without loss of generality that $i = j = 1$. Assume G' is obtained from $K_{n,m}$ by subdividing the edge e. Then the function $g : V(G') \to \mathcal{P}(\{1,2\})$ defined by $g(x_1) = g(y_2) = \{1\}$, $g(y_1) = g(x_2) = \{2\}$, and $g(x) = 0$ otherwise, is a total 2-rainbow dominating function of G' of weight 4, and so $\mathrm{sd}_{\gamma_{tr2}}(K_{n,m}) \geq 2$.

Next we show that $\mathrm{sd}_{\gamma_{tr2}}(K_{n,m}) \leq 2$. Assume that G' is the graph obtained from $K_{n,m}$ by subdividing the edges x_1y_1 and x_1y_2 with vertices z_1 and z_2, respectively, and let f be $\gamma_{tr2}(G')$-function. If $|f(x_1)| + \sum_{i=1}^{2}|f(z_i)| \geq 3$ and $|f(x_i)| \geq 1$ for each $2 \leq i \leq n$, then we have $\gamma_{tr2}(G') = \omega(f) \geq 5$, and if $|f(x_1)| + \sum_{i=1}^{2}|f(z_i)| \geq 3$ and $f(x_i) = \emptyset$ for some $2 \leq i \leq m$, then in order that x_i to be rainbow dominated, we must have $\sum_{j=1}^{n}|f(y_j)| \geq 2$, yielding $\gamma_{tr2}(G') = \omega(f) \geq 5$. Henceforth, we assume that $|f(x_1)| + \sum_{i=1}^{2}|f(z_i)| \leq 2$. We consider the following cases.

Case 1. $f(x_1) = \{1,2\}$.
Then $f(z_1) = f(z_2) = \emptyset$. In order that x_1 to be totally dominated, we must have $|f(y_j)| \geq 1$ for some $j \geq 3$, say $j = 3$. If $f(y_j) = \emptyset$ for some $j \in \{1,2\}$, then in order that y_j to be rainbow dominated, we must have $\sum_{i=2}^{m}|f(x_i)| \geq 2$, implying that $\gamma_{tr2}(G') = \omega(f) \geq 5$. Otherwise we have $|f(y_1)| \geq 1$ and $|f(y_2)| \geq 1$ and again $\gamma_{tr2}(G') = \omega(f) \geq 5$.

Case 2. $f(x_1) = \emptyset$.
Then in order that z_1, z_2 to be totally rainbow dominated, we must have $|f(z_1)|+|f(y_1)| \geq 2$ and $|f(z_2)|+|f(y_2)| \geq 2$. If $|f(y_3)| \geq 1$, then, clearly, $\gamma_{tr2}(G') = \omega(f) \geq 5$. Assume that $f(y_3) = \emptyset$. Then in order that y_3 to be rainbow dominated, we must have $\sum_{i=2}^{m}|f(x_i)| \geq 2$, and this implies that $\gamma_{tr2}(G') = \omega(f) \geq 6$.

Case 3. $|f(x_1)| = 1$.
Suppose without loss of generality that $f(x_1) = \{1\}$. Then in order

that z_i to be rainbow dominated, we must have $|f(z_i)| + |f(y_i)| \geq 1$ for each $i \in \{1, 2\}$. If $|f(z_1)| + |f(y_1)| + |f(z_2)| + |f(y_2)| \geq 3$ and $|f(y_i)| \geq 1$ for some $i \geq 3$, then we have $\gamma_{tr2}(G') = \omega(f) \geq 5$, and if $|f(z_1)|+|f(y_1)|+|f(z_2)|+|f(y_2)| \geq 3$ and $f(y_i) = \emptyset$ for some i, then in order that y_i to be rainbow dominated, we must have $\sum_{i=2}^{m} |f(x_i)| \geq 1$, implying that $\gamma_{tr2}(G') = \omega(f) \geq 5$. Hence, we assume that $|f(z_1)| + |f(y_1)| + |f(z_2)| + |f(y_2)| = 2$. We distinguish the following situations.

- $f(z_1) = f(z_2) = \emptyset$.
  Considering our assumption, in order that z_1, z_2 to be rainbow dominated, we have $f(y_1) = f(y_2) = \{2\}$. In order that y_1 to be totally dominated, we may assume without loss of generality that $|f(x_2)| \geq 1$. If $|f(x_i)| \geq 1$ for each $i \geq 3$, then, clearly, $\gamma_{tr2}(G') = \omega(f) \geq 5$. Assume that $f(x_i) = \emptyset$ for some $i \geq 3$, say $i = 3$. Then in order that x_3 to be rainbow dominated, we must have $1 \in f(y_j)$ for some $j \geq 3$, and so $\gamma_{tr2}(G') = \omega(f) \geq 5$.

- $f(z_1) = \emptyset$ and $|f(z_2)| = 1$.
  By assumption, we have $f(y_1) = \{2\}$ and $f(y_2) = \emptyset$. As above, we may assume that $|f(x_2)| \geq 1$. If $|f(x_i)| \geq 1$ for some $i \geq 3$, then, clearly, $\gamma_{tr2}(G') = \omega(f) \geq 5$. Otherwise, in order that x_3 to be rainbow dominated, we must have $\sum_{i=3}^{m} |f(y_i)| \geq 1$, yielding $\gamma_{tr2}(G') = \omega(f) \geq 5$ again.

- $|f(z_1)| = |f(z_2)| = 1$.
  Then $f(y_1) = f(y_2) = \emptyset$. In order that y_1, y_2 to be rainbow dominated, we may assume that $|f(x_2)| \geq 1$. Now in order that x_2 to be totally dominated, we must have $\sum_{i=3}^{m} |f(y_i)| \geq 1$, yielding $\gamma_{tr2}(G') = \omega(f) \geq 5$ again.

Thus $\mathrm{sd}_{\gamma_{tr2}}(K_{n,m}) = 2$ when $n \geq 3$. $\square$

4 Conclusion

In this paper, we initiated the study of the total k-rainbow domination subdivision number in graphs and presented some sharp bounds on the total k-rainbow domination subdivision number in terms of the order, maximum degree and total k-rainbow domination number. In the special case of $k = 2$, we proved that the total 2-rainbow domination subdivision number can be arbitrary large. For further study we pose the following open problems.

Problem 1. *Is it true that for any integer $k \geq 2$ and a connected graph G with $\delta(G) \geq 2$, $\mathrm{sd}_{\gamma_{tr2}}(G) \leq \alpha'(G) + 1$?*

Problem 2. *Is it true that for any integer $k \geq 2$ and a connected graph G with $\delta(G) \geq 2$, $\mathrm{sd}_{\gamma_{tr2}}(G) \leq \gamma_{tr2}(G) - 1$?*

By Theorem 1 and Proposition 2 we have that for any tree T of order $n \geq 3$, $\mathrm{sd}_{\gamma_{tr2}}(T) \leq 2$.

Problem 3. *Characterize all tree T with $\mathrm{sd}_{\gamma_{tr2}}(T) = 2$*

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Chromatic Spectrum of K_s-WORM Colorings of K_n*

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Abstract

An *H-WORM* coloring of a simple graph G is the coloring of the vertices of G such that no copy of $H \subseteq G$ is monochrome or rainbow. In a recently published article by one of the authors [3], it was claimed that the number of r-partitions in a K_s-WORM coloring of K_n is $\zeta_r = \left\{ {n \atop r} \right\}$, where $\left\{ {n \atop r} \right\}$ denotes the Stirling number of the second kind, for all $3 \le r \le s < n$. We found that $\zeta_r = \left\{ {n \atop r} \right\}$ if and only if $\lceil \frac{n+3}{2} \rceil < s \le n$ with $r < s$. Further investigations into ζ_2, given any K_3-WORM coloring of K_n, show its relation with the number of spanning trees of cacti and the Catalan numbers. Moreover, we extend the notion of H-WORM colorings to $(H_1; H_2)$-*mixed* colorings, where H_1 and H_2 are distinct subgraphs of G; these coloring constraints are closely related to those of mixed hypergraph colorings.

Keywords: Catalan numbers, Chromatic spectrum, Mixed hypergraph coloring, Stirling numbers.
MSC 2020: 05C15, 05C30, 05C35, 05C65.

1 Preliminaries

A *partition* $\mathscr{P}$ of a set A is a set of nonempty subsets of A such that each element of A is in exactly one subset of A. The elements of $\mathscr{P}$ are often called *blocks* and an *r-partition* is a partition with r number of *blocks*.

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A *coloring* of a set S is a mapping $c : S \rightarrow [r]$, where $[r] = \{1, 2, \ldots, r\}$ and an r-coloring of S is a coloring of the elements of S using exactly r colors. As such, a coloring $c(S)$ is a partition of the set S since all of the elements of S are assigned a color; elements that share the same color (monochrome subset) belong to the same block and elements with different colors (rainbow subset) belong to distinct blocks. A set $A \subseteq S$ is said to be *monochrome* if all of its elements share the same color and A is *rainbow* if all of its elements have different colors.

Let $G = (V, E)$ denote a simple graph and let H be a subgraph of G; we write $H \subseteq G$. An *H-WORM* (vertex) coloring of G is the coloring of the vertices of G such that no copy of $H \subseteq G$ is monochrome or rainbow. This coloring constraint was first introduced by W. Goddard, K. Wash and H. Xu [9], [10], and independently studied by Cs. Bujtás and Zs. Tuza [4], [6], [9], [10]. In [2], it was extended to the notion of $\mathscr{F}$-WORM colorings, where $\mathscr{F}$ represents a collection of distinct subgraphs of G instead of a single subgraph of G. Given any H-WORM coloring of G, the sequence $(\zeta_\alpha, \ldots, \zeta_\beta)$ is called a *chromatic spectrum*, where α and β are known as *lower* and *upper chromatic numbers*, respectively. Each chromatic spectral value, ζ_r with $\alpha \leq r \leq \beta$, counts the number of *proper* r-partitions which are the r distinct partitions that satisfy the coloring constraint. We note that the term *partition vector* was used in [2] to describe chromatic spectrum. The integer set $F = \{r : \zeta_r > 0\}$ with $\alpha \leq r \leq \beta$ is called a *feasible set* and it has been the subject of numerous research publications (see e.g., [5], [7], [8], [16], [22], [23]). In general, (see [6], for instance) it is NP-hard to determine α and it is NP-complete to decide whether or not a graph G admits a K_3-WORM coloring. Moreover, it is a far more difficult problem to find the chromatic spectral values ζ_r, since one must first determine the feasible set. Recently, the chromatic spectra of some 2-trees given any K_3-WORM coloring have been found [2], [3]. Specifically, in [2], the chromatic spectral values in any K_s-WORM coloring of K_n have been determined. Unfortunately, there was an oversight in the estimates of some of these spectral values. In this article, we provide a result (Theorem 2) that resolves the issue by classifying these spectral values in relation to the well-know Stirling number of the second kind. Further, given any K_3-

WORM coloring of K_n, we found that the lower spectral values, ζ_2, are closely related to the well-known Catalan numbers. Other relations between the lower spectral value and the number of spanning trees of some cacti are also shown. In the last section, we extend the notion of H-WORM colorings to $(H_1; H_2)$-mixed colorings. In particular, some related results between $(K_r; K_s)$-mixed colorings of K_n and complete (r, s)-uniform mixed hypergraph colorings are established.

2 K_s-WORM Colorings

The *Stirling number of the second kind* (see for e.g., [17]) which we denote by $\left\{ {n \atop r} \right\}$, counts the r-partitions of a set of order n. Clearly $\left\{ {n \atop 1} \right\} = 1 = \left\{ {n \atop n} \right\}$ for $n \geq 1$. It is often computed with the identity $\left\{ {n \atop r} \right\} = \dfrac{1}{r!} \displaystyle\sum_{j=0}^{r} (-1)^n \binom{r}{j} (r-j)^n$. There are several other well-known combinatorial identities and generating functions on the Stirling numbers of the second kind that can be found in [11]. Recently ([3]), the Stirling numbers of the second kind appeared in the chromatic spectrum of some K_s-WORM colorings of K_n. However, many of the proposed spectral values turned out to be significantly less than the Stirling numbers of the second kind. The main result of this section outlines the spectral values that are equal to the Stirling numbers of the second kind and those that are not.

We begin with a restatement of the original work in [3] concerning the spectral values.

Theorem 1. *[[3],* ***Theorem 2.1****] The partition vector in a K_s-WORM coloring of K_n is $(\zeta_{\lfloor \frac{n}{s-1} \rfloor}, \ldots, \zeta_{s-1})$, where $\zeta_r = \left\{ {n \atop r} \right\}$ for all $3 \leq s < n$.*

Here is one simple counterexample to Theorem 1.

Let $n = 4$ and $s = 3$. Clearly, $3 \leq s < n$. Suppose there is a K_3-WORM coloring of K_4 and let $V = \{a, b, c, d\}$, where $V = V(K_4)$. The total number of 2-partitions (with no constraint) of V is $\left\{ {4 \atop 2} \right\} = 7$. Now, consider ζ_2, the number of 2-partitions.

Claim. $\zeta_2 < \left\{ {4 \atop 2} \right\}$ (when $r = 2$).

Proof. It suffices to show that there is a 2-partition that fails to satisfy the K_3-WORM coloring condition of K_4: no monochrome K_3 and no rainbow K_3. Suppose $V = \{a, b, c, d\}$. Clearly, $\{\{a, b, c\}, \{d\}\}$ is a 2-partition of V that contains a subset of size 3, giving a monochrome K_3. Thus, it is not a proper 2-partition of V. So, the number of all proper 2-partitions of V must be smaller than $\left\{ {4 \atop 2} \right\}$, the number of all 2-partitions of V. □

Later, in Corollary 6, we show that $\zeta_2 = \frac{4(4-1)}{4} = 3$, in which case the 2-partitions of V that satisfy the K_3-WORM coloring condition are $\{\{a, b\}, \{c, d\}\}$, $\{\{a, c\}, \{b, d\}\}$ and $\{\{a, d\}, \{b, c\}\}$. The four remaining 2-partitions that fail to satisfy the K_3-WORM coloring condition are: $\{\{a\}, \{b, c, d\}\}$, $\{\{b\}, \{a, c, d\}\}$, $\{\{c\}, \{a, b, d\}\}$, $\{\{d\}, \{a, b, c\}\}$.

Lemma 1. *There is a K_s-WORM vertex coloring of K_n if and only if $n \leq (s-1)^2$ for all $3 \leq s \leq n$.*

Proof. Suppose there is a K_s-WORM coloring of K_n. There is an r-partition of $[n]$ such that no block contains s or more elements from $[n]$. It follows that $n \leq r(s-1)$. Moreover, $r \leq s-1$, or else some subgraph $K_s \subseteq K_n$ is rainbow. Conversely, suppose that there is a K_s-WORM vertex coloring of K_n and $n \geq (s-1)^2+1$. Let $A = [n]$. Partition the elements of A into $(s-1)$-blocks, each containing $(s-1)$ elements. Any remaining element, since there is at least one, say $x \in A$, must be added to the elements of one of blocks, giving a monochrome s-block, or else, at least one extra block is needed for x, giving a rainbow s-set. Hence, $n \leq (s-1)^2$.

□

Remark 1.

A result similar to Lemma 1 had been first proved by Goddard, Wash, and Xu in [9]. Also, from the previous two results, it is clear that not every K_n admits a K_s-WORM coloring for some $s < n$. For example, there is no K_2-WORM coloring of K_5 even though there is a

2-partition of K_5. Moreover, it is easy to see that an $(s-1)$-partition of $[n]$ is a K_s-WORM coloring of K_n if every block is of size $s-1$ or less. Thus, it is clear that an r-partition of $[n]$ is a K_s-WORM coloring of K_n if and only if $r \leq s-1$ and every block is of size $s-1$ or less. Such partition will be said to be *representative* of a K_s-WORM coloring of K_n.

The next result is simply a restatement of Lemma 1.

Corollary 1. *Suppose $K_s \subseteq K_n$. There is a K_s-WORM vertex coloring of K_n if and only if $\lceil \sqrt{n} \rceil < s \leq n$ for all $n \geq 3$.*

Corollary 2. *The feasible set in a K_s-WORM vertex coloring of K_n is $F = \{\lceil \frac{n}{s-1} \rceil, \ldots, s-1\}$, for all $s > \lceil \sqrt{n} \rceil$.*

Proof. Suppose there is a K_s-WORM vertex coloring of K_n, in which case $\lceil \sqrt{n} \rceil < s \leq n$. It follows that each block is of size at most $s-1$, giving at least $\lceil \frac{n}{s-1} \rceil$ distinct blocks for all $n \geq 3$. $\square$

Here, we correct Theorem 1 with Theorem 2 which identifies the spectral values that are actually equal to the Stirling numbers of the second kind, given any K_s-WORM coloring of K_n, $n \geq 3$.

Theorem 2. *Suppose ζ_i denotes a spectral value in a K_s-WORM coloring of K_n. Then $\zeta_i \leq \left\{ {n \atop i} \right\}$ with equality if and only if $n \leq s+i-2$, for $2 \leq i < s \leq n$.*

Proof. Suppose there is a K_s-WORM coloring of K_n. It is clear that $\zeta_r \leq \left\{ {n \atop r} \right\}$, $r < s \leq n$ since every K_s-WORM coloring of K_n is a partition of $[n]$. Consider all r-partitions of $[n]$, for some $r \leq s-1$. If all partitions have only blocks of size less than s, in which case they are representatives of a K_s-WORM coloring of K_n, then there are exactly $\left\{ {n \atop r} \right\}$ such partitions.

Now, consider any r-partition of $[n]$, with $r \leq s-1$. If $r < s-1$, we simultaneously remove elements from some existing blocks to form new blocks of the partition until we attain the maximum number of allowable blocks, when $r = s-1$; this is always possible since $r < s \leq n$. Let $\mathscr{P} = \dot{\bigcup}_{j=1}^{r} A_j$ denote the newly formed r-partition of $[n]$, with $r = s-1$, such that $|A_i| \leq |A_k|$, for $1 \leq i < k \leq r$. $\mathscr{P}$ is representative of a K_s-WORM coloring of K_n and without loss of generality, we can assume that there is a block $A_r \in \mathscr{P}$ such that $|A_r| = s-1$; otherwise, the result follows easily from the next argument.

Suppose that $|A_t| \geq 2$, for some t, with $1 \leq t \leq r-1$. Let $x_t \in A_t$ and define $A'_t = A_t - \{x_t\}$, and $A'_r = A_r \cup \{x_t\}$. Let $\mathscr{P}' = \dot{\bigcup}_{j=1}^{r-2} A_j \cup A'_t \cup A'_r$ is an r-partition. This implies $|A'_r| = s$ in which case $\mathscr{P}'$ is not representative of a K_s-WORM coloring of K_n. Thus, given some $|A_j| = s-1$, every r partitions of $[n]$ are representives of K_s-WORM colorings of K_n if and only if $|A_1| = \ldots = |A_{r-1}| = 1$, for each $A_j \in \mathscr{P}$, $1 \leq j \leq r-1$. Hence, it must be that $n \leq s+r-2$ whenever the number of r-partitions in a K_s-WORM coloring of K_n is $\left\{ n \atop r \right\}$, giving the result for all $2 \leq r < s \leq n$. □

The next result is a special case of Theorem 2 when $n = s$.

Corollary 3. *Given a K_n-WORM coloring of K_n, the chromatic spectrum is $(\zeta_2, \ldots, \zeta_{n-1})$, where $\zeta_i = \left\{ n \atop i \right\}$ with $2 \leq i \leq n-1$.*

Proof. By definition, every r-partition of $[n]$ represents a coloring of K_n. Further, with $2 \leq r \leq n-1$, no r-partition of $[n]$ contains a block of size n, by the pigeonhole principle. Clearly, if either $r = 1$ or $r = n$, then $[n]$ becomes monochrome or rainbow, respectively. □

Corollary 4. *Given any K_s-WORM coloring of K_n, if $s \geq \lceil \frac{n+3}{2} \rceil$, then the chromatic spectrum is $(\zeta_2, \ldots, \zeta_{s-1})$, where $\zeta_i = \left\{ n \atop i \right\}$ with $2 \leq i \leq s-1$.*

Proof. The result follows from Theorem 2 when $n \leq s+r-2$ and the fact that $r \leq s-1$. □

The next result follows from Corollary 2 and Corollary 4, for all $n \geq 4$.

Corollary 5. *Given any K_s-WORM coloring of K_n, if $\lceil \sqrt{n} \rceil < s < \lceil \frac{n+3}{2} \rceil$, then the chromatic spectrum is $(\zeta_{\lceil \frac{n}{s-1} \rceil}, \ldots, \zeta_{s-1})$, where $\zeta_i < \left\{ {n \atop i} \right\}$ for all $\lceil \frac{n}{s-1} \rceil \leq i \leq s-1$.*

In the case when $n = 3$ it is clear that we obtain $\zeta_2 = \left\{ {3 \atop 2} \right\}$, showing that the bounds on s in the previous two results are tight.

In Figure 1, we summarize the results on the feasible sets, the chromatic spectral values given any K_s-WORM coloring of K_n, $n \geq 3$.

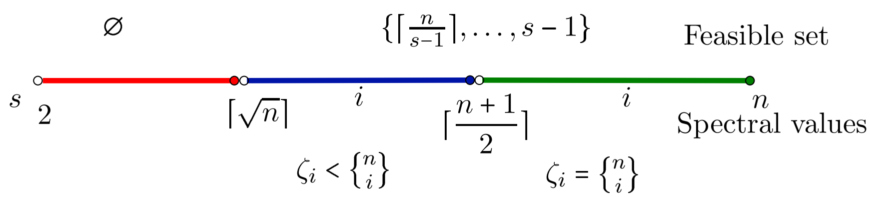

Figure 1. K_s-WORM coloring of K_n, for $3 \leq s \leq n$

In the next result we give the lower spectral values, ζ_2, when $\zeta_2 < \left\{ {n \atop 2} \right\}$; these values are related to the Catalan numbers as shown in a remark after the result. We note here that, for $r \geq 3$, the exact values of ζ_r when $s < \lceil \frac{n+3}{2} \rceil$ remain to be found.

Corollary 6. *Suppose there is a K_s-WORM coloring of K_n. If $s = \lceil \frac{n}{2} \rceil + 1$, then the chromatic spectrum is $(\zeta_2, \ldots, \zeta_{s-1})$, where*

$$\zeta_2 = \begin{cases} \binom{n}{\frac{n}{2}}/2, & \text{if } n \text{ is even} \\ \binom{n}{\lceil \frac{n}{2} \rceil}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Suppose there is a K_s-WORM coloring of K_n. Because $\lceil \frac{n}{2} \rceil + 1 < \lceil \frac{n+3}{2} \rceil$ for all $n \geq 4$, it is clear from Corollary 5 that $0 < \zeta_2 < \left\{ {n \atop 2} \right\}$. Now, consider a 2-partition of $[n]$ that is representative of a K_s-WORM coloring. Since $s = \lceil \frac{n}{2} \rceil + 1$, the blocks of the partition must be of sizes $\lceil \frac{n}{2} \rceil$ and $\lfloor \frac{n}{2} \rfloor$. When n is even, the result follows by symmetry, when considering all $\frac{n}{2}$-subsets of $[n]$ to form one block of the partition, leaving the remaining (half) elements of $[n]$ for the other block. On the other hand, when n is odd, each $\lceil \frac{n}{2} \rceil$ subset of $[n]$ paired with the remaining $\lfloor \frac{n}{2} \rfloor$ elements of $[n]$ form distinct 2-partitions, giving the result.

$\square$

Remark 2.

The k^{th} *Catalan number* is known to be given by $C_k = \frac{1}{k+1}\binom{2k}{k}$; besides having numerous combinatorial meanings (see for e.g., [19]), C_k also counts the number of triangulations of a convex $(k+2)$-gon. Further, from Corollary 6 we establish the following relation between 2-partitions in a K_{s+1}-WORM coloring of K_{2s}, and the k^{th} Catalan number. Namely,

$$\zeta_2 = \frac{(s+1)}{2} C_s.$$

Thus, the number of 2-partitions of a K_{s+1}-WORM coloring of K_{2s} is $\dfrac{(2s)!}{2(s!)^2}$, $s \geq 2$. In the case when $s = 2$ we obtain $\zeta_2 = 3$, a value that is supported by both Corollary 6 and the remark following the counterexample given at the beginning of this section.

Suppose $\tau(G)$ denotes the number of spanning trees of a graph G. It is clear that $\tau(G) = 1$ for any acyclic graph, and when $G = C^m$, which we denote a cycle on m vertices, $\tau(G) = m$. Further, it is well-known (see for e.g., [1]) that if $G = K_{m,n}$, a complete bipartite graph, then $\tau(G) = m^{n-1}n^{m-1}$. Here, we show a close relation between K_3-WORM colorings and spanning trees. We recall that the *girth* of a graph is the length of its smallest cycle and a *cactus* is a simple connected graph in which every pair of cycles share at most one vertex. Figure 2 shows some cacti with girth 3.

Proposition 1. *Suppose G is a cactus with k cycles of length $m_1, m_2, \dots, m_k$. Then the number of its spanning trees is $\tau(G) = \prod_{i=1}^{k} m_i$, for each $m_i \geq 3$.*

Proof. If $G = C^{m_1}$, then $\tau(G) = m_1$, each spanning tree is obtained by deleting exactly one edge from C^{m_1}. Because every two cycles share exactly one vertex, the argument follows by induction on $k \geq 1$. $\square$

Suppose G denotes a simple graph and $\mathscr{F} = \{F_1, \dots, F_k\}$ is a collection of distinct subgraphs $F_i \subseteq G$, $1 \leq i \leq k$. An *$\mathscr{F}$-WORM* coloring of G is the coloring of the vertices of G such that no copy of $F_i \subseteq G$ is monochrome or rainbow. This notion was first introduced in [2] as a generalization of F-WORM colorings. For the next two results, given a cactus G, we denote by $\mathscr{C} = \{C^1, C^2, \dots, C^k\}$ the collection of all distinct cycles $C^i \subseteq G$.

Proposition 2. *Let G denote a cactus of order n. If there is a $\mathscr{C}$-WORM coloring of G, then the feasible set is $F = \{2, \dots, n-k\}$, with $1 \leq i \leq k$.*

Proof. Suppose there is a $\mathscr{C}$-WORM coloring of G. It follows that no C^i is rainbow or monochrome for each $1 \leq i \leq k$. We give the colorings or partitions that produce the infimum and the supremum of F. Color the vertices of G in such a way that, for each of the k cycles, exactly two vertices are monochrome while all other vertices of G are kept rainbow. This gives the supremum. On the other hand, select a pair of vertices from each of the k cycles. Color all such pairs with a single color, and any other (remaining) vertices of G with another color. This gives the infimum of G. In both cases it is easy to verify that each coloring is representative of a proper $\mathscr{C}$-WORM coloring. $\square$

Proposition 3. *Suppose G denotes a cactus with k cycles, each of length $m_1, m_2, \dots, m_k$, $m_i \geq 3$. The number of 2-partitions in a $\mathscr{C}$-WORM coloring of G is at least $\prod_{i=1}^{k} m_i$, the number of its spanning trees.*

Proof. In every spanning tree of a graph G, exactly one edge $e_i \in E(C^{m_i})$, for each $1 \le i \le k$, is removed to create an acyclic connected graph. Let A_1 be the block whose elements are the endpoints of e_i, $1 \le i \le k$, and let A_2 be the block that contains any remaining vertices of G. Clearly no $C^{m_j} \subseteq G$ is monochrome or rainbow, and the collection $\{A_1, A_2\}$ is a 2-partition of G. Hence the number of 2-partitions $\zeta_2(G) \ge \tau(G)$, and the result follows from Proposition 1.

□

Proposition 4. *Suppose G is a cactus with k cycles each of length 3. If ζ_2 counts the 2-partitions of G in a K_3-WORM coloring, then $\zeta_2 \ge 3^k$ with equality if and only if G is bridgeless.*

Proof. Suppose G is a bridgeless cactus with k cycles each of length 3 and consider a K_3-WORM coloring. It follows from Proposition 3 that $\zeta_2 \ge 3^k$. Further, a 2-coloring of each $K_3 \subseteq G$ is a 2-coloring of G since no other vertex of G lies outside some K_3. This implies that $\zeta_2 \le 3^k$. Hence the equality.

□

Remark 3.

Proposition 4 points out that a K_3-WORM coloring of G is equivalent to a graph operation that yields spanning trees provided G is bridgeless (see Figure $2(c)$)–In every 2-coloring of $K_3 \subseteq G$, exactly one pair of vertices share the same color, in which case the edge incident to those vertices can be considered "deleted"–Because no edge is a bridge, the resulting graph G' remains connected. Also since G is a cactus, no edge is shared by two or more cycles, and the removal of an edge does not induce any (larger) cycle as a subgraph. Therefore the resulting graph G' contains no cycle and yet, it includes all vertices of G, giving a spanning tree.

Figure 2 helps illustrate this previous remark. Both Figures $2(a)$ and $2(b)$ show some K_3-WORM 2-colorings that do not represent spanning trees; in both cases $\zeta_2 \ge 3^3$. Further, in Figure $2(c)$, G is bridgeless and $\zeta_2 = 3^4$.

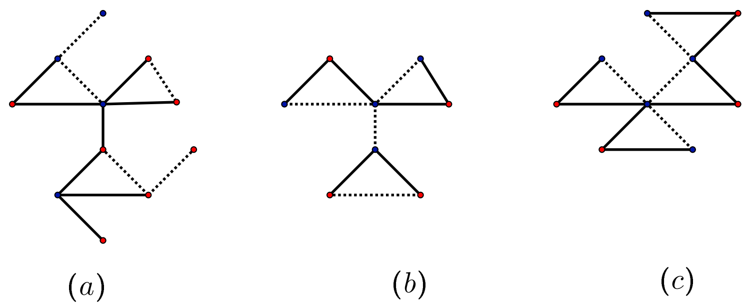

Figure 2. Spanning trees and 2-colorings of some cacti

Remark 4.

It is shown in [2] that when $G = \theta(1, 2, \ldots, 2)$, an $(n-1)$-bridge, then $\zeta_2(G) = 2^{n-2} + 1$. Further when $G = F_n$, a Fan on $n \geq 3$ vertices, $\zeta_2 = \frac{1}{\sqrt{5}}\Big[\frac{\beta - 2}{\beta}\alpha^n - \frac{\alpha - 2}{\alpha}\beta^n\Big]$, a shifted Fibonacci number, with $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Although, given any K_3-WORM coloring of G, $\tau(G) = \zeta_2(G)$ when G is a bridgeless cactus of girth 3, it is also true that, for any acyclic graph G, $1 = \tau(G) < \zeta_2(G)$. With this observation, it is tempting to claim that, for any graph G with girth 3, the number of its spanning trees $\tau(G) \leq \zeta_2(G)$, the number of its 2-partitions in an K_3-WORM coloring. However, the next proposition shows that it is not always the case. In particular, $\zeta_2(G) < \tau(G)$ when $G = \theta(1, 2, \ldots, 2)$.

Proposition 5. *If* $G = \theta(1, 2, \ldots, 2)$*, an* $(n-1)$*-bridge on* $n \geq 3$ *vertices, then* $\tau(G) = n2^{n-3}$.

Proof. Consider the path $\{u, v\} \subset E(G)$ where every other vertices of G are adjacent to both u and v. In every spanning tree of G, either (i) the edge uv is deleted or (ii) uv is kept, in which case, for each K_3 which necessarily includes uv, exactly one of the edges incident to either u or v is deleted. From case (i), it follows that the resulting graph is $K_{n-2,2}$

where u and v are the vertices of part size 2 and $\tau(K_{n-2,2})$ counts the number of its spanning trees. In case (ii), this is equivalent to a 2-partition of G, given a K_3-WORM coloring. However, $\zeta_2(G)$ includes the case when $c(u) = c(v)$, and there is exactly one such partition which we remove since it falls under case (i). Together, we have

$$\begin{aligned} \tau(G) &= \tau(K_{n-2,2}) + \zeta_2(G) - 1 \qquad (1) \\ &= (n-2)2^{n-3} + 2^{n-2} \\ &= n2^{n-3}, \end{aligned}$$

giving the result for all $n \geq 3$. □

It is clear from equation 1 that when $n = 3$, the equality $\zeta_2(G) = 3 = \tau(G)$ holds since $\tau(K_{1,2}) = 1$ and for all $n \geq 4$, the inequality $\zeta_2(G) < \tau(G)$ holds.

Example 1.

Given Proposition 5, the case when $n = 4$, i.e., when $G = \theta(2,1,1)$, is illustrated by Figure 3. We show all spanning trees of G (after the arrows), and $\tau(G) = 8$. The spanning trees in case (ii), when $c(u) = c(v)$, are not representative of K_3-WORM colorings; each pair of deleted edges would yield a corresponding monochrome K_3. Also, $\zeta_2(G) = 5$ which are obtained from case (i), when $c(u) \neq c(v)$, and one additional graph from case (ii) which has exactly one edge deleted; this produces a C_4, which is trivially counted as a 2-partition in a K_3-WORM coloring of G.

3 $(K_r; K_s)$-mixed coloring

A *hypergraph* $\mathscr{H}$ is an ordered pair $(\mathscr{V}, \mathscr{E})$, where $\mathscr{V}$ is a finite set of vertices with *order* $|\mathscr{V}| = n$ and $\mathscr{E}$ is a collection of nonempty subsets of $\mathscr{V}$, called *hyperedges*. When $\mathscr{E}$ is a collection of all nonempty s-subsets of $\mathscr{V}$, then $\mathscr{H}$ is called a *complete s-uniform* hypergraph.

Hypergraphs are extensively used in machine learning techniques, data mining and information retrieval tasks such as clustering and classification. See for instance [12]–[14].

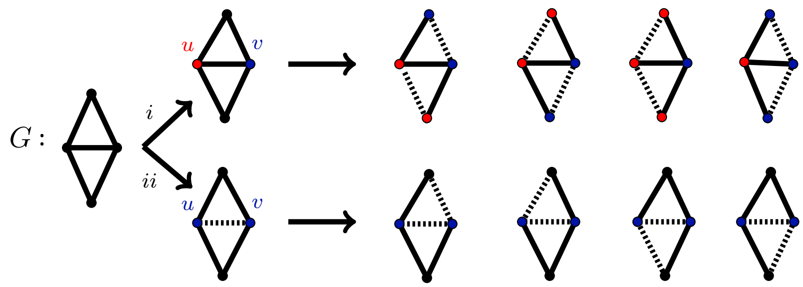

Figure 3. K_3-WORM 2-colorings and the spanning trees of $G = \theta(2,1,1)$

Given any vertex coloring of $\mathscr{H}$, if no hyperedge $e \in \mathscr{E}$ is monochrome, $\mathscr{H} = (\mathscr{V}, \mathscr{D})$ is called a *$\mathscr{D}$-hypergraph* which is the classic hypergraph vertex coloring. When no hyperedge $e \in \mathscr{E}$ is rainbow, $\mathscr{H} = (\mathscr{V}, \mathscr{C})$ is called a *cohypergraph*. In the event no hyperedge $e \in \mathscr{E}$ is monochrome or rainbow, $\mathscr{H} = (\mathscr{V}, \mathscr{B})$ is called a *bihypergraph*, where $\mathscr{B}$ is a nonempty intersection of $\mathscr{C}$ and $\mathscr{D}$ or when $\mathscr{C} = \mathscr{D}$. As a generalization of these different coloring constraints on the vertices of $\mathscr{V}$, it is customary to define a *mixed hypergraph* $\mathscr{H} = (\mathscr{V}, \mathscr{C}, \mathscr{D})$ as a triple such that $\mathscr{C}$ and $\mathscr{D}$ are (not necessarily distinct) subsets of $\mathscr{E}$. Mixed hypergraph colorings were first introduced by Voloshin (see for e.g., [8], [20]–[23]). They are often used to encode partitioning constraints and also to construct cyber security models [15], [18].

A mixed hypergraph $\mathscr{H}$ is said to be *uncolorable* if its feasible set $F = \varnothing$, in which case $\mathscr{H}$ admits no proper coloring. It is obvious that when $|e| \leq 2$ for some $e \in \mathscr{B}$, $\mathscr{H}$ is uncolorable, so we assume $|e| \geq 3$ for every $e \in \mathscr{B}$. Various classes of uncolorable mixed hypergraphs have been extensively discussed, including *complete (r,s)-uniform mixed hypergraphs* [7], [20], [23] which are complete uniform mixed hypergraphs such that, for every hyperedges $d \in \mathscr{D}$ and $c \in \mathscr{C}$, $|d| = r$ and $|c| = s$.

Remark 5.

With the concept of mixed hypergraph colorings, it is natural to define an $(H_1; H_2)$-*mixed (vertex) coloring* of a graph G as the coloring of the vertices of G such that no $H_1 \subseteq G$ is rainbow and no $H_2 \subseteq G$ is monochrome. In particular when $H_1 = H_2 = H$, an $(H; H)$-mixed coloring of G is an H-*WORM* coloring. Thus, by definition, a $(K_r; K_s)$-mixed coloring of K_n is equivalent to a proper coloring of a complete (r, s)-uniform mixed hypergraphs, and when $r = s$, a K_s-WORM coloring is a complete s-uniform bihypergraph. Within these contexts, we later state equivalent results without offering any additional proof. Further, it is clear that, if $\mathscr{P} = \dot{\bigcup}_{j=1}^{k} A_j$ is a k-partition of $[n]$, then $\mathscr{P}$ is *representative* of a $(K_r; K_s)$-mixed coloring of K_n if and only if $k < r$ and $|A_j| < s$.

Proposition 6. *There exists a set of k positive integers $t_1, \ldots, t_k$ such that $\sum_{\substack{i=1 \\ t_i < s}}^{r-1} t_i = n$ if and only if $n \leq (s-1)(r-1)$, $2 \leq r \leq s \leq n$.*

Proof. Take $r - 1$ integers, say t_i's, such that each $t_i \leq s - 1$. They add up to at most $(s-1)(r-1)$, giving the result. □

Lemma 2. *There is a (K_r, K_s)-mixed coloring of K_n if and only if $n \leq (s-1)(r-1)$, $2 \leq r \leq s \leq n$.*

Proof. Let $\mathscr{P} = \dot{\bigcup}_{j=1}^{k} A_j$ denote an k-partition of $[n]$ such that $|A_i| < s$. For all $k < r$, let $|A_i| = t_i$ with $3 \leq t_i < s$ and the result follows from Proposition 6. □

Lemma 3. *A complete (r, s)-uniform mixed hypergraph of order n is colorable if and only if $n \leq (s-1)(r-1)$, $2 \leq r \leq s \leq n$.*

We note here that Zs. Tuza and V. Voloshin ([20], Theorem 8) were first to prove the negation of the statement in Lemma 3, and Lemma 1 is a special case when $r = s$. We state the equivalent statement of this special case in the next Lemma.

Lemma 4. *A complete s-uniform bihypergraph of order $n \geq 3$ is colorable if and only if $n \leq (s-1)^2$, for $3 \leq s \leq n$.*

Corollary 7. *Suppose C_s denotes the s^{th} Catalan number for all $s \geq 2$. The number of 2-colorings of a complete $(s+1)$-uniform bihypergraph of order $2s$, from a list of $r \geq 2$ colors is $\binom{r}{2}(s+1)C_s$.*

Proof. Following Remark 2, and the fact that there are exactly $r(r-1)$ ways of colorings the elements of each 2-partition, we have a total of $\frac{r(r-1)(s+1)}{2}C_s$ such colorings. □

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On the Computational Complexity of Optimization Convex Covering Problems of Graphs

Radu Buzatu

Abstract

In this paper we present further studies of convex covers and convex partitions of graphs. Let G be a finite simple graph. A set of vertices S of G is convex if all vertices lying on a shortest path between any pair of vertices of S are in S. If $3 \leq |S| \leq |X| - 1$, then S is a nontrivial set. We prove that determining the minimum number of convex sets and the minimum number of nontrivial convex sets, which cover or partition a graph, is in general NP-hard. We also prove that it is NP-hard to determine the maximum number of nontrivial convex sets, which cover or partition a graph.

Keywords: NP-hardness, convex cover, convex partition, graph.

MSC 2010: 05A18, 05C35; 68Q25; 68R10

1 Introduction

We denote by $G = (X; U)$ a simple undirected graph with vertex set X, $X(G)$, and edge set U, $U(G)$. The *neighborhood* of a vertex $x \in X$ is the set of all vertices $y \in X$ such that $y \sim x$ (i.e., adjacent to x), and it is denoted by $\Gamma(x)$. Let S be a subset of X. If every two vertices of S are adjacent in G, then it is called a *clique*. If every vertex of $X \backslash S$ is adjacent to at least one vertex of S, then S is a dominating set of G. If $3 \leq |S| \leq |X| - 1$, then S is a *nontrivial* set. The distance $d(x, y)$ between two vertices $x, y \in X$ is the length of the shortest path between x and y. The diameter of G, denoted by $diam(G)$, is the distance between two farthest vertices of G. We denote by $G[S]$ the subgraph of G induced by S.

We remind some notions defined in [1]. A set $S \subseteq X$ is called *convex* if the inclusion $\{z \in X : d(x, z) + d(z, y) = d(x, y)\} \subseteq S$ holds for any two vertices $x, y \in S$. The *convex hull* of $S \subseteq X$, denoted by $d - conv(S)$, is the smallest convex set containing S.

By [4], the family of sets $\mathcal{P}(G)$ is called the *convex cover* of a graph $G = (X;\, U)$ if the following statements hold:

1) each set of $\mathcal{P}(G)$ is convex in G;

2) $X = \bigcup_{S \in \mathcal{P}(G)} S$;

3) $S \not\subseteq \bigcup_{C \in \mathcal{P}(G), C \neq S} C$ for each $S \in \mathcal{P}(G)$.

If $|\mathcal{P}(G)| = p$, then we say that $\mathcal{P}(G)$ is a *convex p-cover* of G. If any two sets of $\mathcal{P}(G)$ are disjoint, then this family is called a *convex partition* of G. The family $\mathcal{P}(G)$ is said to be the *nontrivial convex cover* of G if each set of $\mathcal{P}(G)$ is nontrivial and convex. A vertex $x \in X$ is called *resident* in $\mathcal{P}(G)$ if x belongs to only one set of $\mathcal{P}(G)$.

We know from [4], [5] and [9] that it is NP-complete to decide whether a graph has a convex p-cover or a convex p-partition for a fixed $p \geq 2$. If the nontrivial sets are considered as elements of convex p-covers or convex p-partitions of a graph, the problems also remain NP-complete for a fixed $p \geq 2$.

Since the general convex p-cover problem is NP-complete, several classes of graphs for which there exist polynomial algorithms for deciding whether a graph can be covered or partitioned by a fixed number $p \geq 2$ of convex sets were identified [4], [5], [8], [11].

Note that there exist graphs for which there are no nontrivial convex covers or nontrivial convex partitions or both. For example, a convex simple graph (a graph that does not contain any nontrivial convex sets [3]) can not be covered by nontrivial convex sets. The problem of determining whether a graph G can be partitioned into an arbitrary number of nontrivial convex sets is NP-complete, but it can be established in polynomial time whether G can be covered by an arbitrary number of nontrivial convex sets [12].

In our previous works, we have studied six different invariants that consistently help to determine the existence of convex covers and partitions of graphs. The least $p \geq 2$ for which a graph G has a convex p-cover is said to be the *minimum convex cover number* $\varphi_c^{min}(G)$. Similarly, the least $p \geq 2$ for which G has a convex p-partition is said to be the *minimum convex partition number* $\theta_c^{min}(G)$. In the same way, *minimum nontrivial convex cover number* $\varphi_{cn}^{min}(G)$, *minimum nontrivial convex partition number* $\theta_{cn}^{min}(G)$, *maximum nontrivial convex cover number* $\varphi_{cn}^{max}(G)$ and *maximum nontrivial convex partition number* $\theta_{cn}^{max}(G)$ are defined in the case when the nontrivial convex sets are considered. For supplementary information about estimation of these invariants the papers [9], [10], [11] and [12] can be consulted.

It is obvious that for any graph G we have $\varphi_c^{min}(G) \leq \theta_c^{min}(G)$. As before, if G can be partitioned into nontrivial convex set, then $\theta_c^{min}(G) \leq \theta_{cn}^{min}(G)$ and:

$$\varphi_{cn}^{min}(G) \leq \theta_{cn}^{min}(G) \leq \theta_{cn}^{max}(G) \leq \varphi_{cn}^{max}(G).$$

Anyway, if graph G can be covered by nontrivial convex sets, then $\varphi_c^{min}(G) \leq \varphi_{cn}^{min}(G)$.

2 NP-hardness

In this section, we show that it is NP-hard to determine the values of the invariants $\varphi_c^{min}(G)$, $\theta_c^{min}(G)$, $\varphi_{cn}^{min}(G)$, $\theta_{cn}^{min}(G)$, $\varphi_{cn}^{max}(G)$ and $\theta_{cn}^{max}(G)$ for a graph G. For each problem, firstly, we formulate the corresponding decision problem.

Determination of minimum convex cover number $\varphi_c^{min}(G)$ has the following decision problem:

PROBLEM: Minimum convex cover (MinCC).
INSTANCE: Graph $G = (X;\, U)$, integer p, $2 \leq p \leq |X|$.
QUESTION: Is there a convex q-cover of G such that $2 \leq q \leq p$?

As for invariants $\theta_c^{min}(G)$, $\varphi_{cn}^{min}(G)$, $\theta_{cn}^{min}(G)$, their decision problems are defined in the same manner and only the appropriate specification of the type of convex cover (convex q-partition, nontrivial convex q-cover, nontrivial convex q-partition) in the questions is required.

The clique partitioning problem is defined as follows:

PROBLEM: Clique partition (CP).
INSTANCE: Graph $G = (X; U)$, integer p, $3 \le p \le |X|$.
QUESTION: Is there a partition of X into p disjoint sets $X_1, \dots, X_p$, such that the subgraph induced by X_i is a complete graph for each i, $1 \le i \le p$?

In the sequel, we show that MinCC, minimum convex partition (MinCP), minimum nontrivial convex cover (MinNCC) and minimum nontrivial convex partition (MinNCP) problems are NP-complete. In order to achieve this goal, we reduce the CP that is a well-known NP-complete problem [2] to the problems of interest.

Theorem 1. *The MinCC problem is NP-complete.*

Proof. Verifying whether a set of vertices is convex can be done in polynomial time [6]. Hence, MinCC problem is in NP.

Let $G = (X; U)$ be a generic graph of CP problem and p be an integer, $3 \le p \le |X|$. Without loss of generality, it can be assumed that X is not a clique. We obtain a particular graph $G' = (X'; U')$ of MinCC problem from G by adding auxiliary sets $Y = \{y_1, y_2, \dots, y_p\}$ and $Z = \{z_1, z_2, \dots, z_p\}$ to X such that $X' = X \cup Y \cup Z$, where $\Gamma(y_i) = X \cup \{z_i\}$ and $\Gamma(z_i) = X \cup \{y_i\}$ for each i, $1 \le i \le p$. Obviously, this construction of G' can be done in polynomial time.

The MinCC instance is defined by the graph G' and the number p. In Figure 1 it is shown how a particular graph G' of MinCC problem is obtained from a graph G of CP problem.

It can easily be checked that for every two nonadjacent vertices $a, b \in X'$ we get $X' \subseteq d - conv(\{a, b\})$. In consequence, a set $S \subset X'$ is convex in G' if and only if S is a clique, and further G' cannot be covered by k, $k < p$, convex sets.

If G can be partitioned into p disjoint cliques $X_1, X_2, \dots, X_p$, then we obtain a convex p-cover $\boldsymbol{\mathcal{P}}(G') = \{X'_1, X'_2, \dots, X'_p\}$, where $X'_i = X_i \cup \{y_i, z_i\}$ for each i, $1 \le i \le p$.

Let $\boldsymbol{\mathcal{P}}(G')$ be a convex cover of G' such that $|\boldsymbol{\mathcal{P}}(G')| \le p$. For the above reason, $|\boldsymbol{\mathcal{P}}(G')| = p$. We define a family of convex sets $\boldsymbol{\mathcal{P}} = \varnothing$.

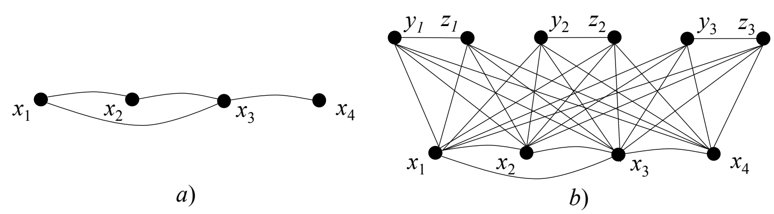

Figure 1. a) Graph G of CP problem with $p = 3$ b) Particular graph G' of MinCC problem obtained from G.

For each set $S \in \mathcal{P}(G')$, if $S \backslash (Y \cup Z) \neq \varnothing$, we add set $S \backslash (Y \cup Z)$ to $\mathcal{P}$. Then, by removing from $\mathcal{P}$ all sets contained in the union of other sets of the family $\mathcal{P}$ we obtain a convex k-cover $\mathcal{P}'(G)$ of G, $2 \leq k \leq |\mathcal{P}| \leq |\mathcal{P}(G')|$. Note that if any graph H can be covered by k cliques and there exists a set S of this cover such that $|S| \geq 2$, then by removing every element of S from other cliques and by splitting S into two cliques, we obtain a cover of H by $k + 1$ cliques. Thus, G can be covered by p cliques. It stands to reason that G can be partitioned into p cliques.

So, G can be partitioned into p disjoint cliques if and only if there exists a cover of G' by at most p convex sets. Thus, it is proved that the MinCC problem is NP-complete. □

In view of demonstration of the Theorem 1, we obtain the correctness of Corollaries 1, 2 and 3.

Corollary 1. *The MinCP problem is NP-complete.*

Corollary 2. *The MinNCC problem is NP-complete.*

Corollary 3. *The MinNCP problem is NP-complete.*

Determination of maximum nontrivial convex partition number $\theta_{cn}^{max}(G)$ has the following decision problem:

PROBLEM: Maximum nontrivial convex partition (MaxNCP)
INSTANCE: Graph $G = (X; U)$, integer p, $2 \leq p \leq |X|$.

QUESTION: Is there a nontrivial convex q-partition of G such that $q \geq p$?

The partition into triangles problem is defined as follows:

PROBLEM: Partition into triangles (PIT).
INSTANCE: A graph $G = (X;\, U)$ with $|X| = 3k$, where $k \in N$.
QUESTION: Is there a partition of X into k disjoint subsets $X_1, X_2, \ldots, X_k$ of three vertices each such that the three possible edges between vertices of every X_i, $1 \leq i \leq k$, are in U?

We reduce PIT, the well known NP-complete problem [2], to MaxNCP problem. Therefore, we prove that MaxNCP problem is NP-complete too.

Theorem 2. *The MaxNCP problem is NP-complete.*

Proof. Notice that MaxNCP problem is in NP because verifying whether a set of vertices is convex can be done in polynomial time [6].

Let $G = (X;\, U)$ be an instance of PIT problem, $|X| = 3k$, $k \in N$. Firstly, we determine the structure of a particular graph $G' = (X';\, U')$ of MaxNCP problem that corresponds to G. We know that the PIT problem remains NP-complete even if the input graph G is tripartite [7]. Note also that every tripartite graph has no cliques with $r \geq 4$ vertices. Here and in the sequel we consider that G has no cliques with $r \geq 4$ vertices.

We construct the graph $G' = (X';\, U')$ as follows:

1) $X' = X \cup \{a, b, c, d, e, f\}$;

2) $U' = U \cup \{\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, e\}, \{d, f\}\}$
$\cup \{\{a, x\}, \{b, x\} : x \in X\}$.

The MaxNCP instance is defined by the graph G' and a number $p = k + 2$. It is easy to see that G' can be constructed in polynomial time. We exhibit in Figure 2 how a particular graph G' of MaxNCP problem is obtained from a graph G of PIT problem.

We have to show that there exists a partition of X into triangles if and only if there exists a nontrivial convex partition of G' of size at least $k + 2$.

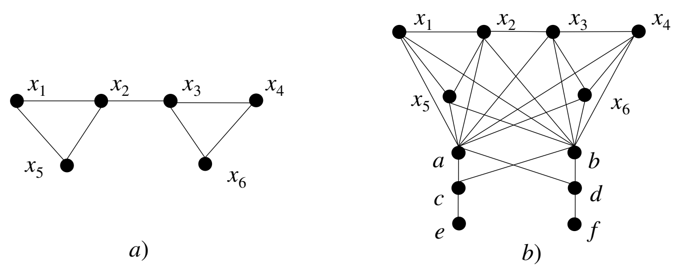

Figure 2. a) Graph G of PIT problem b) Particular graph G' of MaxNCP problem obtained from G.

Let $\mathcal{P}(G) = \{X_1, X_2, \ldots, X_k\}$ be a family of triangles that partitions G. Since every triangle is a clique in G, it follows that each X_i, $1 \le i \le k$, is nontrivial and convex in G' and the set $\{a, b, c, d, e, f\}$ remains uncovered in G'. Observe that $d - conv(\{a, c, e\}) = \{a, c, e\}$ and $d - conv(\{b, d, f\}) = \{b, d, f\}$. For this reason, the family of sets $\mathcal{P}(G) \cup \{\{a, c, e\}, \{b, d, f\}\}$ generates a partition of G' into $k + 2$ nontrivial convex sets.

Let $\mathcal{P}(G')$ be a partition of G' into nontrivial convex sets and let S be a set of $\mathcal{P}(G')$. We distinguish some properties of S:

1) $\{a, b\} \not\subset S$. Assuming the contrary, namely that $\{a, b\} \subset S$, we see that $d - conv(\{a, b\}) = X \cup \{a, b, c, d\}$ and further we obtain $X' \backslash d - conv(\{a, b\}) = \{e, f\}$. Note that the set $\{e, f\}$ is not nontrivial and convex. Hence, $\mathcal{P}(G')$ cannot partition G' into nontrivial convex sets. We get the required contradiction.

2) $\{c, d\} \not\subset S$. Assuming the converse, $\{a, b\} \subset d - conv(\{c, d\})$. Therefore, the property 1) is not satisfied and we obtain a contradiction.

3) $\{e, f\} \not\subset S$. Conversely, we have $\{a, b, c, d\} \subset d - conv(\{e, f\})$ and consequently the properties 1) and 2) are not satisfied. This implies a contradiction.

4) $\{x, y\} \not\subset S$ for every vertex $x \in X$ and $y \in \{c, d\}$. Assuming the converse, there exist $x \in X$ and $y \in \{c, d\}$ such that $\{x, y\} \subset S$. Considering that vertices a and b belong to $d - conv(\{x, y\})$, we obtain

a contradiction.

5) $\{x, y\} \not\subset S$ for every two nonadjacent vertices $x, y \in X$. In the converse case, there are two nonadjacent vertices x and y of X for which $\{x, y\} \subset S$. And it follows that $\{a, b\} \subset d - conv(\{x, y\})$. Have a contradiction.

Let $S_1 = \{a, c, e\}$, $S_2 = \{b, d, f\}$, $S_3 = \{b, c, e\}$ and $S_4 = \{a, d, f\}$. Taking into account the properties 1) – 5) and the fact that each vertex of X' belongs exactly to one set of $\boldsymbol{\mathcal{P}}(G')$, it is seen that $\boldsymbol{\mathcal{P}}(G')$ contains strictly a pair of sets of the following two: S_1, S_2 or S_3, S_4. Each pair of sets covers vertices a, b, c, d, e and f. Hence, vertices of $X' \backslash \{a, b, c, d, e, f\}$ remain to be partitioned into nontrivial convex sets. By property 5), all of these sets are cliques. As mentioned above, G has no cliques with $r \geq 4$ vertices. Further, all of these sets are triangles and by elimination of a pair of sets S_1, S_2 or S_3, S_4 from $\boldsymbol{\mathcal{P}}(G')$ we obtain a family of triangles $\boldsymbol{\mathcal{P}}(G)$ that partitions G. $\boldsymbol{\mathcal{P}}(G')$ contains exactly $k+2$ sets and thus if G has a nontrivial convex partition $\boldsymbol{\mathcal{P}}(G')$ of size at leas $k+2$, then there exists a partition of X into triangles. $\square$

The decision problem for maximum nontrivial convex cover number $\varphi_{cn}^{max}(G)$ is formulated similarly to MaxNCP, but with different question: Is there a nontrivial convex q-cover of G such that $q \geq p$?

The 3-Satisfiability problem is defined as follows:

PROBLEM: 3-Satisfiability (3SAT).

INSTANCE: Given a boolean expression E in conjunctive normal form that is the conjunction of clauses, each of which is the disjunction of three distinct literals.

QUESTION: Is there a satisfying truth assignment for E?

Now we prove that MaxNCC problem is NP-complete. For this purpose, we reduce the 3SAT problem that is NP-complete [2] to MaxNCC problem.

Theorem 3. *The MaxNCC problem is NP-complete.*

Proof. The MaxNCC problem is in NP because verifying whether a set of vertices is convex can be done in polynomial time [6].

The following reduction from 3SAT to MaxNCC will establish that MaxNCC problem is NP-complete. Let E be an instance of the 3SAT problem with n variables $V_1, V_2, \dots, V_n$ and m clauses $K_1, K_2, \dots, K_m$. Given this instance, we construct a graph $G = (X; U)$ with $3n+m+16$ vertices. Vertices v_i, $\overline{v_i}$, y_i correspond to variable V_i, $1 \le i \le n$. One vertex k_j corresponds to clause K_j, $1 \le j \le m$. There are supplimentary vertices grouped in the four sets: $A = \{a, a_1, a_2, a_3\}$, $B = \{b, b_1, b_2, b_3\}$, $C = \{c, c_1, c_2, c_3\}$ and $\{d, e, f, h\}$. Denote by V, $\overline{V}$ and Y sets of all vertices v_i, $\overline{v_i}$ and respectively y_i, $1 \le i \le n$. By K we denote the set of all vertices k_j, $1 \le j \le m$.

The graph G has $12n + 6m + 27$ edges. The three vertices corresponding to each variable are connected by an edge (i.e., v_i, $\overline{v_i}$, y_i form a triangle). Each clause vertex is connected to its component terms (that is, the clause vertex k_l corresponding to the clause $\overline{V_i} \vee V_j \vee V_q$ is connected by edges to vertices $\overline{v_i}, v_j, v_q$). The three additional vertices a, b, c are connected to v_i, $\overline{v_i}$, y_i for all i, $1 \le i \le n$, and to k_j for all j, $1 \le j \le m$. The vertex d is connected by an edge to each vertex $r \in (A \cup B \cup \{h\}) \backslash \{a_1, b_1\}$, the vertex e is connected by an edge to each vertex $r \in (B \cup C \cup \{h\}) \backslash \{b_1, c_1\}$ and f is connected to each vertex $r \in (A \cup C \cup \{h\}) \backslash \{a_1, c_1\}$. Finally, a_1 is connected to a_2 and a_3, b_1 is connected to b_2 and b_3, c_1 is connected to c_2 and c_3.

The MaxNCC instance is defined by the graph G and a number $p = 2n + m + 3$. It should be clear that this construction of G can be done in polynomial time.

For example, consider the 3SAT instance:

$$E = (V_1 \vee V_2 \vee \overline{V_3}) \wedge (V_2 \vee V_3 \vee \overline{V_4}) \wedge (V_1 \vee \overline{V_2} \vee V_4).$$

Then the graph G corresponding to E is presented in Figure 3.

Without loss of generality, we consider that E has no clauses which contain a variable and its negation.

Let us distinguish some properties of $G = (X; U)$:

Property 1: For any two nonadjacent vertices $x, y \in V \cup \overline{V} \cup Y \cup K \cup \{a, b, c\}$, we have $d - conv(\{x, y\}) = X$.

Property 2: For any two vertices $x \in V \cup \overline{V} \cup Y \cup K$, $y \in (A \cup B \cup C \cup \{d, e, f, h\}) \backslash \{a, b, c\}$, we have $d - conv(\{x, y\}) = X$.

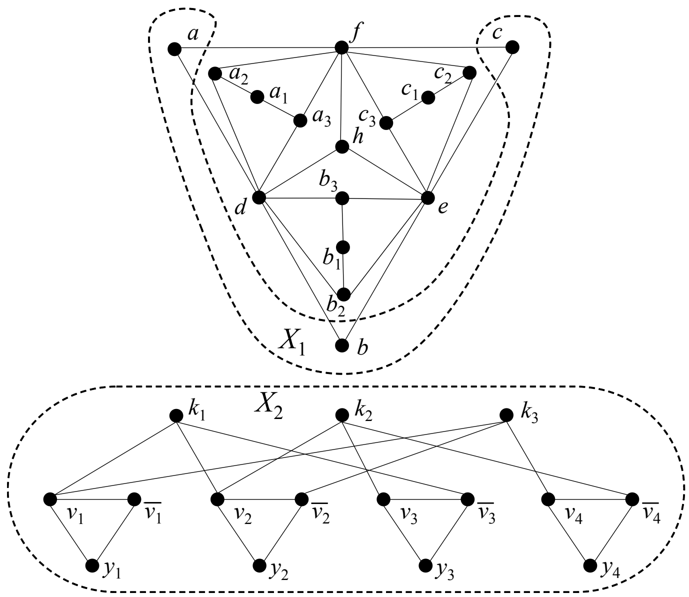

Figure 3. Graph G corresponds to the instance $E = (V_1 \vee V_2 \vee \overline{V_3}) \wedge (V_2 \vee V_3 \vee \overline{V_4}) \wedge (V_1 \vee \overline{V_2} \vee V_4)$. Every vertex of set X_1 is adjacent to every vertex of set X_2. Dashed line marks areas of sets X_1 and X_2.

Property 3: Let $\mathcal{P}(G)$ be a nontrivial convex cover or G. Then, $A^* = A \cup \{d, f, h\}$, $B^* = B \cup \{d, e, h\}$ and $C^* = C \cup \{e, f, h\}$ are in $\mathcal{P}(G)$.

Correctness of the first two properties can be easily verified. We will show that the Property 3 is correct too. Let $S \subset X$ be a nontrivial convex set that contains the vertex a_1. Since $|S| \geq 3$, the structure of G yields that S also contains vertices a_2 and a_3. Hence, we get the equality $d\!-\!conv(\{a_2, a_3\}) = A^*$. Note that for each $x \in X \backslash A^*$ we have $d - conv(A^* \cup \{x\}) = X$. Further, A^* is a unique nontrivial convex set, different from X, that contains a_1. For the same reason, B^* and C^* are unique nontrivial convex sets, different from X, which contain vertices b_1 and c_1, respectively.

We have to show that E is satisfiable if and only if there exists a nontrivial convex cover of G of size at least $2n+m+3$.

Assume E is satisfiable. Then there exists a truth assignment of variables such that all clauses evaluate to true. We will form a nontrivial convex cover of $\mathcal{P}(G)$ as follows. By the Property 3, $\mathcal{P}(G)$ includes the sets A^*, B^* and C^*. We denote by M the set of all vertices v_i for which V_i are true in the assignment, and all $\overline{v_i}$ for which V_i are false. $\mathcal{P}(G)$ includes sets $\{x, y, a\}$ for all $x \in M$, where $y \in Y$ and $y \sim x$. Also, $\mathcal{P}(G)$ includes sets $\{v_i, \overline{v_i}, a\}$ for each i, $1 \leq i \leq n$. Moreover, $\mathcal{P}(G)$ includes one set $\{x, k_j, a\}$ for each k_j, where $x \in M$, $x \sim k_j$, and the existence of such a vertex x yields from the fact that E is satisfiable. The obtained nontrivial convex cover $\mathcal{P}(G)$ contains exactly $2n+m+3$ sets.

Assume there exists a nontrivial convex cover $\mathcal{P}(G)$ of size at least $2n+m+3$. Taking into account the Property 3, $\mathcal{P}(G)$ already includes the sets A^*, B^* and C^*. In view of the Property 2, it remains to analyze a nontrivial convex cover $\mathcal{P}(G')$ resulted from $\mathcal{P}(G)$ after elimination of sets A^*, B^* and C^*, where $G' = G[V \cup \overline{V} \cup Y \cup K \cup \{a, b, c\}]$. Clearly, $|\mathcal{P}(G)| = |\mathcal{P}(G')| + 3$. It follows from the Property 1 and the structure of G that every set $S \in \mathcal{P}(G')$ is a clique, $3 \leq |S| \leq 4$, and it can be classified into one of three types:

(i) $S = \{x, y, z\}$, where $x, y \in V \cup \overline{V} \cup Y \cup K$, $x \sim y$, and $z \in \{a, b, c\}$;
(ii) $S = \{v_i, \overline{v_i}, y_i\}$ for any i, $1 \leq i \leq n$;
(iii) $S = \{v_i, \overline{v_i}, y_i, x\}$ for any i, $1 \leq i \leq n$, where $x \in \{a, b, c\}$.

Now we define a family of convex sets $\mathcal{P}(G'') = \varnothing$, obtained from $\mathcal{P}(G')$, that will cover $G'' = G'[V \cup \overline{V} \cup Y \cup K]$.

We examine each set $S \in \mathcal{P}(G')$ and consider two cases.

1) If S is of the first type (i), then we add set $S \backslash \{a, b, c\}$ to $\mathcal{P}(G'')$.

2) If S is of the second (ii) or the third (iii) type, then taking into account the fact that vertices a, b and c are already covered by sets A^*, B^* and C^*, we analyze two options. Suppose S contains only one resident vertex r in $\mathcal{P}(G')$. In this case, we add set $\{r, x\}$ to $\mathcal{P}(G'')$, where $x \in S \backslash \{a, b, c\}$, $x \neq r$. Suppose S contains at least two resident vertices r_1, r_2 in $\mathcal{P}(G')$. Then, we add sets $\{r_1, x\}$ and $\{r_2, x\}$ to $\mathcal{P}(G'')$,

where $x \in S \backslash \{a,b,c\}$, $x \neq r_1$, $x \neq r_2$.

Let us remark that for a set $S \in \mathcal{P}(G')$ of the type (ii) or (iii), any set $S' \in \mathcal{P}(G')$, $S' \neq S$, $S' \cap S \backslash \{a,b,c\} \neq \varnothing$, is of the first type (i), and thus there are no uncovered vertices in G'', i.e. $\mathcal{P}(G'')$ is a convex cover of G''. It is obvious that $|\mathcal{P}(G'')| \geq |\mathcal{P}(G')|$ and furthermore $|\mathcal{P}(G'')| \geq |\mathcal{P}(G)| - 3 = 2n + m$. We need to take a closer look at the family $\mathcal{P}(G'')$. Every set of $\mathcal{P}(G'')$ has exactly two adjacent vertices. We choose one resident vertex of each set $S \in \mathcal{P}(G'')$ and form the set W as a union of these vertices. It is clear that $D = X(G'') \backslash W$ is a dominating set of G'' such that $|\mathcal{P}(G'')| + |D| = |V \cup \overline{V} \cup Y \cup K| = 3n + m$. If we combine this with the previous inequality, we get $|D| \leq n$.

Consider the graph G''. For any i, $1 \leq i \leq n$, y_i is either in D or adjacent to a vertex of D, and y_i is connected by edges only to v_i and $\overline{v_i}$. It follows that for every i, $1 \leq i \leq n$, either v_i, $\overline{v_i}$, or y_i is in D. This already specifies n vertices, so exactly one vertex for each variable is included in D. We create a truth assignment as follows. V_i will be assigned to true if v_i is in D. Otherwise V_i will be assigned to false. Consider clause K_j. The vertex k_j is not in the dominating set. So k_j is adjacent to some v_i or $\overline{v_l}$ in the dominating set. If k_j is adjacent to v_i in D, then since V_i is set to true, it follows that k_j will be true. If k_j is adjacent to $\overline{v_l}$ in D, then v_l is not in D and V_l will be false, so K_j will be true. It follows that this assignment is a solution for E and E is satisfiable. Thus, if G has a nontrivial convex cover $\mathcal{P}(G)$ of size at least $2n + m + 3$, then E is satisfiable.

So, this completes the proof of the correctness of the reduction and we conclude that MaxNCC is NP-complete. $\square$

3 Conclusion

We have proved that MinCC, MinCP, MinNCC, MinNCP, MaxNCC and MaxNCP problems are NP-complete. This yields that the problems of determining the values of the invariants $\varphi_c^{min}(G)$, $\theta_c^{min}(G)$, $\varphi_{cn}^{min}(G)$, $\theta_{cn}^{min}(G)$, $\varphi_{cn}^{max}(G)$ and $\theta_{cn}^{max}(G)$ for a general graph G are NP-hard.

Of course, it is of interest to develop approximate algorithms, heuristics, and establish other classes of graphs for which the above mentioned invariants can be determined in polynomial time. All these are issues for further research.

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On α-spectral theory of a directed k-uniform hypergraph

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Abstract

In this paper, we study a k-uniform directed hypergraph in general form and introduce its adjacency tensor, Laplacian tensor and signless Laplacian tensor. For the k-uniform directed hypergraph $\mathcal{H}$ and $0 \leq \alpha < 1$ the convex linear combination of $\mathcal{D}$ and $\mathcal{A}$ has been defined as $\mathcal{A}_\alpha = \alpha\mathcal{D} + (1-\alpha)\mathcal{A}$, where $\mathcal{D}$ and $\mathcal{A}$ are the degree tensor and the adjacency tensor of $\mathcal{H}$, respectively. We propose some spectral properties of $\mathcal{A}_\alpha$. We also introduce power directed hypergraph and cored directed hypergraph and investigate their α-spectral properties.

Keywords: Directed Hypergraph, Adjacency tensor, Laplacian tensor, Signless Laplacian tensor, Eigenvalue, α-spectral theory, Odd-bipartite Hypergraph.

MSC 2010: 05C65, 15A18.

1 Introduction

Directed hypergraphs are deeply used as a successful data structure in modeling the problems arising in computer science [3] and operations research, and in recent years have found applications in data mining, clustering, association rules [13], image processing [1] and optical network communications [8]. On the other hand, spectral theory of hypergraphs gives useful and important information about them. In 2005 eigenvalues and eigenvectors of real tensor are defined [9], [14]. Qi [14] introduced the spectral theory of supersymmetric real tensor. In [15] the spectral theory of undirected hypergraphs was presented via eigenvalues and eigenvectors of the adjacency tensor, Laplacian tensor and signless Laplacian tensor. Recently a number of papers appeared in different aspects of spectral theory of hypergraphs.

On the other hand, Nikiforov in [11] proposed the spectral theory of the convex combination of the adjacency matrix and the degree matrix of a graph (see also [4],[12]) and then Lin et.al. [10] expanded it for the hypergraph. Let $\mathcal{H}$ be a hypergraph, $\mathcal{A}(\mathcal{H})$ and $\mathcal{D}(\mathcal{H})$ be the adjacency tensor and the degree tensor of $\mathcal{H}$, respectively. For $0 \leq \alpha < 1$, the convex linear combination, $\mathcal{A}_\alpha$, of $\mathcal{D}$ and $\mathcal{A}$ is defined by

$$\mathcal{A}_\alpha(\mathcal{H}) = \alpha\mathcal{D}(\mathcal{H}) + (1-\alpha)\mathcal{A}(\mathcal{H}) \qquad \forall \; 0 \leq \alpha < 1.$$

The spectral radius of $\mathcal{A}_\alpha(\mathcal{H})$ is called α-spectral radius of $\mathcal{H}$. Lin et.al. [10] gave an upper bound for the α-spectral radius in an n-vertex connected irregular k-uniform hypergraph $\mathcal{H}$ using number of vertices, maximum degree and diameter. Then Gue et.al. [5] studied α-spectral radius of uniform hypergraphs and also proposed some transformations that increase α-spectral radius and determine the unique hypergraphs with maximum α-spectral radius in some classes of uniform hypergraphs.

In spite of a lot of researches in spectral theory and α-spectral theory of undirected hypergraphs, there is almost a blank for (α-)spectral directed hypergraph theory. A special case of the k-uniform directed hypergraph, with one tail node and $k-1$ head nodes, and some its spectral properties were studied in [17]. In this paper, we present the α-spectral properties of the generalized directed hypergraphs and extend some classical results of undirected hypergraphs. We also introduce power directed hypergraphs and cored directed hypergraphs and propose some their α-spectral properties.

In Section 2, we discuss the needed fundamental results of tensors and introduce k-uniform directed hypergraphs in general form with their adjacency tensors, Laplacian tensors and signless Laplacian tensors. In Section 3, the H-eigenvalues of $\mathcal{A}_\alpha$ of k-uniform directed hypergraph are studied. We also introduce power directed hypergraphs and cored directed hypergraphs in Section 4. Finally, we conclude in Section 5.

2 Preliminaries

We first present some basic definitions of tensors. Then we introduce the general k-uniform directed hypergraph with its adjacency tensor, Laplacian tensor and signless Laplacian tensor.

2.1 Tensors and some related subjects

A real tensor $\mathcal{T} = (t_{i_1 \cdots i_k})$ of order k and dimension n, for integers k≥3 and n≥2, is a multi-dimensional array with entries $t_{i_1 \cdots i_k} \in \mathbb{R}$, for $i_j \in [n] := \{1, 2, \cdots, n\}$ and $j \in [k]$ (see [14]).

Definition 1. *[16]: Let $\mathcal{T}$ be a k order n dimension tensor and P and Q be $n \times n$ matrices. The tensor $\mathcal{S} = P\mathcal{T}Q^{k-1}$ is a k order n dimension tensor with the entries*

$$s_{i_1 \cdots i_k} = \sum_{j_1, \cdots, j_k = 1}^{n} t_{j_1 \cdots j_k} p_{i_1 j_1} q_{j_2 i_2} \cdots q_{j_k i_k}.$$

Let $x = (x_1, \cdots, x_n)^T \in \mathbb{C}^n$, we write x^k as a k order n dimension tensor with $(i_1, \cdots, i_k)$-th entry $x_{i_1} x_{i_2} \cdots x_{i_k}$. Then $\mathcal{T}x^{k-1}$ is an n dimensional vector whose i-th component is

$$(\mathcal{T}x^{k-1})_i = \sum_{i_2, \cdots, i_k = 1}^{n} t_{i i_2 \cdots i_k} x_{i_2} \cdots x_{i_k}.$$

The identity tensor of order k and dimension n, $\mathcal{I} = (i_{i_1 \cdots i_k})$, is defined as $i_{i_1 \cdots i_k} = 1$ iff $i_1 = \cdots = i_k \in [n]$, and zero otherwise.

Definition 2. *[14]: Let $\mathcal{T}$ be a nonzero k order n dimension tensor. Then $\lambda \in \mathbb{C}$ is called an eigenvalue of $\mathcal{T}$ if the polynomial system $(\lambda\mathcal{I} - \mathcal{T})x^{[k-1]} = 0$ has a nonzero solution $x \in \mathbb{C}^n$, where $x^{[k-1]} = (x_1^{k-1}, \cdots, x_n^{k-1})^T$. In this case x is called an eigenvector of $\mathcal{T}$ corresponding to λ and (λ, x) is called an eigenpair of $\mathcal{T}$.*

If $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n/\{0\}$, then λ is called an H-eigenvalue and x is called an H-eigenvector of $\mathcal{T}$ [14].

The set of all eigenvalues of $\mathcal{T}$, denoted by $Spec(\mathcal{T})$, is called the spectrum of $\mathcal{T}$. The H-spectrum of $\mathcal{T}$, denoted by $Hspec(\mathcal{T})$, is defined as follows:

$$Hspec(\mathcal{T}) = \{\lambda \in \mathbb{R} | \lambda \text{ is an H-eigenvalue of } \mathcal{T}\}.$$

2.2 K-uniform directed hypergraph

In this subsection we present some needed concepts and definitions of directed hypergraphs and then we introduce adjacency tensor of a k-uniform directed hypergraph in general form. The following definition of the k-uniform directed hypergraph was presented in [1].

Definition 3. *A k-uniform directed hypergraph $\mathcal{H}$ is a pair $= (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = [n]$ is a set of elements called vertices and $\mathcal{E} = \{\vec{e}_1, \cdots, \vec{e}_m\}$ is the set of arcs. Each $\vec{e}_i$, $(i = 1, \cdots, m)$, is considered as an ordered pair (e_i^+, e_i^-), where e_i^+, e_i^- are two nonempty subsets of $\mathcal{V}$ such that $e_i^+ \cap e_i^- = \phi$, $|e_i^+ \cup e_i^-| = k$. e_i^+ is called the tail of $\vec{e}_i$ and e_i^- is its head.*

Note that we assume that in the k-uniform directed hypergraph for any k vertices there exists at most one arc containing them.
The out-degree of a vertex $j \in \mathcal{V}$ is defined as $d_j^+ = |E_j^+|$, where $E_j^+ = \{\vec{e} \in \mathcal{E} | j \in e^+\}$ and the in-degree of a vertex $j \in \mathcal{V}$ is defined as $d_j^- = |E_j^-|$, where $E_j^- = \{\vec{e} \in \mathcal{E} | j \in e^-\}$. The degree of j is defined as $d_j = d_j^+ + d_j^-$. The hypergraph $\mathcal{H}$ is r-out-regular (or r-in-regular or r-regular, respectively) if for each $j \in \mathcal{V}$, $d_j^+ = r$ (or $d_j^- = r$ or $d_j = r$, respectively).

Let $i, j \in \mathcal{V}$ and $i \neq j$. Two vertices i and j are called weak-connected, if there is a sequence of arcs $\vec{e}_1, \cdots, \vec{e}_l$ such that $i \in e_1^+ \cup e_1^-$, $j \in e_l^+ \cup e_l^-$ and $(e_s^+ \cup e_s^-) \cap (e_{s+1}^+ \cup e_{s+1}^-) \neq \phi$ for all $s \in [l-1]$. Two vertices i and j are called strong-connected, denoted by $i \to j$, if there is a sequence of arcs $\vec{e}_1, \cdots, \vec{e}_l$ such that $i \in e_1^+$, $j \in e_l^-$ and $e_s^- \cap e_{s+1}^+ \neq \phi$ for all $s \in [l-1]$. A directed hypergraph $\mathcal{H}$ is called weak-connected, if every pair of different vertices of $\mathcal{H}$ is weakly-connected and $\mathcal{H}$ is called strong-connected, if $i \to j$ and $j \to i$ for all $i, j \in \mathcal{V}$ and $i \neq j$. A directed hypergraph is complete if $\mathcal{E}$ contains all possible arcs with different number of vertices in their tails.

Now we introduce adjacency tensor of a k-uniform directed hypergraph. In [17], authors discussed the case that each arc has only one tail and introduce the adjacency tensor, Laplacian tensor and signless Laplacian tensor. In this paper we consider general form of a k-uniform directed hypergraph and present the following definition of its adjacency tensor:

Definition 4. *The adjacency tensor of a k-uniform directed hypergraph $\mathcal{H}$ is the k order n dimension tensor $\mathcal{A} = (a_{i_1 \cdots i_k})$ whose entries are as follows:*

$$a_{i_1,\dots,i_k} = \begin{cases} \frac{1}{(l_{\vec{e}}-1)!(k-l_{\vec{e}})!} & \textit{if} \,\exists\, \vec{e} = (e^+, e^-) \in \mathcal{E} \;\; s.t \;\; e^+ = \{i_1, \cdots, i_{l_{\vec{e}}}\}, \\ & \qquad\qquad\qquad\qquad\qquad e^- = \{i_{l_{\vec{e}}+1}, \cdots, i_k\} \\ 0 & \textit{otherwise.} \end{cases}$$

Similar to [17] the degree tensor $\mathcal{D}$ defined as the k order n dimension diagonal tensor whose diagonal element $d_{i \cdots i}$ is d_i^+, the out-degree of vertex i, for all $i \in [n]$. Also the Laplacian tensor of $\mathcal{H}$ is $\mathcal{L} = \mathcal{D} - \mathcal{A}$ and $\mathcal{Q} = \mathcal{D} + \mathcal{A}$ is the signless Laplacian tensor of $\mathcal{H}$.

As it has been said before, Lin in [3] defined the convex linear combinations of $\mathcal{A}_\alpha$ of $\mathcal{D}$ and $\mathcal{A}$ as $\mathcal{A}_\alpha = \alpha\mathcal{D} + (1-\alpha)\mathcal{A}$, where $0 \leq \alpha < 1$.

Now the following definition of an odd bipartite directed hypergraph is presented (just as in undirected hypergraph [6]).

Definition 5. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a k-uniform directed hypergraph. $\mathcal{H}$ is called an odd bipartite if k is even and there exists a partition of $\mathcal{V}$ so that $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, $\mathcal{V}_1 \neq \phi$ and*

$$\forall\, \vec{e} = (e^+, e^-) \in \mathcal{E} \qquad |(e^+ \cup e^-) \cap \mathcal{V}_1| \quad \textit{is an odd number.}$$

3 H-Eigenvalues of $\mathcal{A}_\alpha$

Throughout this article, let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph with n vertices, tensors $\mathcal{A}$,$\mathcal{D}$, $\mathcal{L} = \mathcal{D} - \mathcal{A}$ and $\mathcal{Q} = \mathcal{D} + \mathcal{A}$ are the adjacency tensor, the degree tensor, Laplacian tensor and signless Laplacian tensor of $\mathcal{H}$, respectively. For $0 \leq \alpha \leq 1$, let $\mathcal{A}_\alpha$ be defined as $\mathcal{A}_\alpha = \alpha\mathcal{D} + (1-\alpha)\mathcal{A}$.

The notation of weakly irreducible nonnegative tensors was introduced in [2].

Definition 6. *Let $\mathcal{T} = (t_{i_1 \cdots i_k})$ be a k order n dimension nonnegative tensor and $G(\mathcal{T}) = (V, E(\mathcal{T}))$ be a directed graph, where $V = [n]$ and a directed edge $(i, j) \in E(\mathcal{T})$ if there exists $\{i_2, \cdots, i_k\} \in [n]$ such that $j \in \{i_2, \cdots, i_k\}$ and $t_{ii_2 \dots i_k} > 0$. Now $\mathcal{T}$ is called weakly irreducible if $G(\mathcal{T})$ is strongly connected.*

Let $\mathcal{H}$ be a k-uniform undirected hypergraph, then the adjacency of $\mathcal{H}$, $\mathcal{A}$ is weakly irreducible iff $\mathcal{H}$ is connected [2]. For k-uniform directed hypergraph $\mathcal{H}$ if each arc has only one tail, then $\mathcal{A}_\alpha$ is weakly irreducible iff $\mathcal{H}$ is strongly connected, i.e. the strongly connectivity of $\mathcal{H}$ is equivalent to strongly connectivity of $G(\mathcal{A}_\alpha)$. But we have just the sufficient condition in general:

Lemma 1. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a k-uniform directed hypergraph with adjacency tensor $\mathcal{A}$ and degree tensor $\mathcal{D}$. Then, $\mathcal{A}_\alpha = \alpha\mathcal{D} + (1 - \alpha)\mathcal{A}$ is weakly irreducible if $\mathcal{H}$ is strongly connected.*

Proof. Suppose that $\mathcal{H}$ is strongly connected. By Definition 6, we should show that $G(\mathcal{A}_\alpha)$ is strongly connected. Let $i, j \in V$ and $i \neq j$. Since $\mathcal{H}$ is strongly connected, there exists a sequence of vertices and arcs in $\mathcal{H}$ such that:

$$i = j_1 \quad \vec{e}_1 \quad j_2 \quad \vec{e}_2 \quad j_3 \quad \cdots \quad \vec{e}_{q-1} \quad j_q \quad \vec{e}_q \quad j_{q+1} = j,$$

where $j_2, \cdots, j_q \in \mathcal{V}$, $\vec{e}_1, \cdots, \vec{e}_q \in \mathcal{E}$ and $j_t \in e_t^+$, $j_{t+1} \in e_t^-$ for all $t = 1, \cdots, q$. On the other hand, $a_{e_t^+ e_t^-} > 0$ for $t = 1, \cdots, q$, since $\vec{e}_t = (e_t^+, e_t^-) \in \mathcal{E}$, then $a^{(\alpha)}_{e_t^+ e_t^-} > 0$. Hence $e_t = (j_t, j_{t+1})$ is a directed edge in $G(\mathcal{A}_\alpha)$, for all $t = 1, \cdots, q$. Therefore there exists a sequence of vertices and directed edges in $G(\mathcal{A}_\alpha)$:

$$i = j_1 \quad e_1 \quad j_2 \quad e_2 \quad j_3 \quad \cdots \quad e_{q-1} \quad j_q \quad e_q \quad j_{q+1} = j,$$

i.e. $i \to j$ in $G(\mathcal{A}_\alpha)$. Similarly it can be proved that $j \to i$ in $G(\mathcal{A}_\alpha)$. Thus $G(\mathcal{A}_\alpha)$ is strongly connected and then $\mathcal{A}_\alpha$ is weakly irreducible. $\square$

Now we study the H-eigenvalues of $\mathcal{A}_\alpha$. We have the following lemma:

Lemma 2. *Let $\mathcal{H}$ be a k-uniform directed hypergraph. Suppose that $X \in \mathbb{R}^n$, then we have:*

$$(\mathcal{A}_\alpha \mathbf{x}^{[k-1]})_i = \alpha d_i^+ + (1-\alpha) \sum_{\vec{e} \in E_i^+} \prod_{s \in (e^+ \cup e^-) \setminus \{i\}} x_s.$$

Proof.

$$\begin{aligned}(\mathcal{A}_\alpha \mathbf{x}^{[k-1]})_i = & \sum_{i_2, \cdots, i_k = 1}^{n} a^{(\alpha)}_{i i_2 \cdots i_k} x_{i_2} \cdots x_{i_k} \\ & = \alpha \sum_{i_2, \cdots, i_k = 1}^{n} d_{i i_2 \cdots i_k} x_{i_2} \cdots x_{i_k} + (1-\alpha) \sum_{i_2, \cdots, i_k = 1}^{n} a_{i i_2 \cdots i_k} x_{i_2} \cdots x_{i_k} \\ & = \alpha d_i^+ x_i^{k-1} + (1-\alpha) \sum_{\substack{\vec{e} = (e^+, e^-) \in \mathcal{E} \\ i \in e^+,\ |e^+| = l_{\vec{e}}}} \frac{(l_{\vec{e}} - 1)!(k - l_{\vec{e}})!}{(l_{\vec{e}} - 1)!(k - l_{\vec{e}})!} \prod_{s \in (e^+ \cup e^-) \setminus \{i\}} x_s \\ & = \alpha d_i^+ x_i^{k-1} + (1-\alpha) \sum_{\vec{e} \in E_i^+} \prod_{s \in (e^+ \cup e^-) \setminus \{i\}} x_s.\end{aligned}$$

□

Now we have the following theorems.

Theorem 1. *Let $\mathcal{H}$ be a k-uniform directed hypergraph with n vertices. Then each $(\alpha d_j^+, \mathbf{1}_j)$ is an H-eigenpair of $\mathcal{A}_\alpha$ for $j = 1, \cdots, n$.*

Proof. Let $i \in [n]$. By Lemma 2, if $i = j$, then we have:

$$(\mathcal{A}_\alpha \mathbf{1}_j^{[k-1]})_i = \alpha d_j^+ 1 + (1-\alpha) \sum_{\vec{e} \in E_i^+} 0 = \alpha d_j^+$$

and for $i \neq j$, we have:

$$(\mathcal{A}_\alpha \mathbf{1}_j^{[k-1]})_i = \alpha d_j^+ 0 + (1-\alpha) \sum_{\vec{e} \in E_i^+} 0 = 0.$$

By Definition 2, the result follows. □

Theorem 2. *Let $\mathcal{H}$ be a k-uniform directed hypergraph and $\mathcal{A}_\alpha = \alpha\mathcal{D} + (1-\alpha)\mathcal{A}$, where $\frac{1}{2} \leq \alpha < 1$. If λ is an H-eigenvalue of $\mathcal{A}_\alpha$, then we have:*

$$(2\alpha - 1)\delta^+ \leq \lambda \leq \Delta^+,$$

where δ^+ and Δ^+ are the minimum and maximum out-degree in $\mathcal{H}$, respectively.

Proof. Suppose that $\mathbf{x}$ is an H-eigenvector of $\mathcal{A}_\alpha$ associated with λ and $|x_i| = \max\{|x_1|, \cdots, |x_n|\}$. By Lemma 2, we have:

$$\begin{aligned}
&\lambda x_i^{k-1} = \alpha d_i^+ x_i^{k-1} + (1-\alpha) \sum_{\vec{e}\in E_i^+} \prod_{s\in(e^+\cup e^-)\setminus\{i\}} x_s \\
\Longrightarrow\ & (\lambda - \alpha d_i^+) x_i^{k-1} = (1-\alpha) \sum_{\vec{e}\in E_i^+} \prod_{s\in(e^+\cup e^-)\setminus\{i\}} x_s \\
\Longrightarrow\ & |\lambda - \alpha d_i^+||x_i|^{k-1} = (1-\alpha) \sum_{\vec{e}\in E_i^+} \prod_{s\in(e^+\cup e^-)\setminus\{i\}} |x_s| \leq (1-\alpha)\sum_{\vec{e}\in E_i^+} |x_i| \\
\Longrightarrow\ & |\lambda - \alpha d_i^+| \leq (1-\alpha) d_i^+ \\
\Longrightarrow\ & (2\alpha - 1) d_i^+ \leq \lambda \leq d_i^+ \\
\Longrightarrow\ & (2\alpha - 1)\delta^+ \leq \lambda \leq \Delta^+.
\end{aligned}$$

□

Lemma 3. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be an n-vertex k-uniform complete directed hypergraph and $i \in \mathcal{V}$ be an arbitrary vertex. Then $d_i = \binom{n-1}{k-1}$.*

Proof. Since $\mathcal{H}$ is complete, then $\mathcal{E}$ contains all possible arcs. Therefore vertex i has common arcs with any $k-1$ vertices that is $\binom{n-1}{k-1}$.Then $d_i = d_i^+ + d_i^- = \binom{n-1}{k-1}$. □

Theorem 3. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be an n-vertex k-uniform complete directed hypergraph. If $d_i^+ = \binom{n-1}{k-1}$ for each $i \in \mathcal{V}$, then the largest H-eigenvalue of tensor $\mathcal{A}_\alpha$, $\lambda(\mathcal{A}_\alpha)$, is $\binom{n-1}{k-1}$.*

Proof. We show that $\lambda = \binom{n-1}{k-1}$ with $\mathbf{x} = \mathbf{1}$ is an H-eigenpair of $\mathcal{A}_\alpha$. By Lemma 2, we have:

$$\begin{aligned}(\mathcal{A}_\alpha \mathbf{x}^{[k-1]})_i &= \alpha d_i^+ x_i^{k-1} + (1-\alpha) \sum_{\vec{e}\in E_i^+} \prod_{s\in(e^+\cup e^-)\setminus\{i\}} x_s \\ &= \alpha \binom{n-1}{k-1} + (1-\alpha) \sum_{\vec{e}\in E_i^+} 1 = \alpha \binom{n-1}{k-1} + (1-\alpha)\binom{n-1}{k-1} \\ &= \binom{n-1}{k-1} = \lambda x_i^{k-1}.\end{aligned}$$

On the other hand, by Lemma 3, $\Delta^+ = \binom{n-1}{k-1}$, then the result follows from Theorem 2. $\square$

The next theorem characterizes the extreme weak-connected directed hypergraphs with respect to the upper bound of the largest $\mathcal{A}_\alpha$ H-eigenvalue.

Theorem 4. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a weak-connected k-uniform directed hypergraph. Then $\lambda(\mathcal{A}_\alpha) = \Delta^+$ if and only if $\mathcal{H}$ is out-regular.*

Proof. Suppose that $\mathcal{H}$ is out-regular. It is easy to see that $\lambda = \Delta^+$ with the H-eigenvector $\mathbf{x} = \mathbf{1}$ is an H-eigenvalue of $\mathcal{A}_\alpha$, then $\lambda(\mathcal{A}_\alpha) = \Delta^+$. On the other hand, assume that $\lambda(\mathcal{A}_\alpha) = \Delta^+$ and $x \in \mathbb{R}^n$ is its corresponding H-eigenvector. Let $|x_i| = max\ \{|x_j| \big| j \in [n]\}$. By Definition 2, we have:

$$\begin{aligned}&\Delta^+ x_i^{k-1} = \alpha d_i^+ x_i^{k-1} + (1-\alpha) \sum_{\vec{e}\in E_i^+} \prod_{s\in(e^+\cup e^-)\setminus\{i\}} x_s \\ &\Rightarrow \Delta^+ |x_i^{k-1}| \le \alpha d_i^+ |x_i^{k-1}| + (1-\alpha) \sum_{\vec{e}\in E_i^+} \prod_{s\in(e^+\cup e^-)\setminus\{i\}} |x_s| \\ &\Rightarrow \Delta^+ = \alpha d_i^+ + (1-\alpha) \sum_{\vec{e}\in E_i^+} \prod_{s\in(e^+\cup e^-)\setminus\{i\}} \frac{|x_s|}{|x_i|} \le \alpha d_i^+ + (1-\alpha) \sum_{\vec{e}\in E_i^+} 1 = d_i^+ \\ &\Rightarrow \Delta^+ = d_i^+.\end{aligned}$$

and we must have $|x_i| = |x_j|$ for all $j \in e^+ \cup e^-$, where $\vec{e} = (e^+, e^-) \in E_i^+$. Applying the same augment for all such j, we have that $\Delta^+ = d_j^+$ and $|x_i| = |x_j| = |x_l|$ for all $l \in e^+ \cup e^-$, where $\vec{e} = (e^+, e^-) \in E_j^+$. Since $\mathcal{H}$ is weak-connected, we see that $d_j^+ = \Delta^+$ for all $j \in \mathcal{V}$, then $\mathcal{H}$ is out-regular. □

Suppose that x is an H-eigenvector of the $\mathcal{A}_\alpha$ of a k-uniform directed hypergraph corresponding to H-eigenvalue λ. The following theorem gives a sufficient condition for equality of some components of x.

Theorem 5. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a k-uniform directed hypergraph and $i, j \in \mathcal{V}$ such that $E_i^+ = E_j^+$. Then $d_i^+ = d_j^+ = d$. Now suppose that (λ, x) is an H-eigenpair of $\mathcal{A}_\alpha$, such that $\lambda \neq \alpha d$. Then $|x_i| = |x_j|$ and if k is odd, then $x_i = x_j$.*

Proof. Clearly, $d_i^+ = d_j^+ = d$ by the definition of E_i^+. Now suppose that (λ, x) is an H-eigenpair of $\mathcal{A}_\alpha$, such that $\lambda \neq \alpha d$. By Definition 2, we have:

$$\lambda x_i^{k-1} = \alpha d x_i^{k-1} + (1-\alpha) x_j \sum_{\vec{e} \in E_i^+} \prod_{s \in (e^+ \cup e^-) \setminus \{i,j\}} x_s$$

and

$$\lambda x_j^{k-1} = \alpha d x_j^{k-1} + (1-\alpha) x_i \sum_{\vec{e} \in E_j^+} \prod_{s \in (e^+ \cup e^-) \setminus \{i,j\}} x_s.$$

Hence,

$$(\lambda - \alpha d) x_i^k = (\lambda - \alpha d) x_j^k \quad \overset{\lambda \neq \alpha d}{\Longrightarrow} \quad x_i^k = x_j^k.$$

The conclusions follow from the last equality. □

4 Cored directed hypergraphs and Power directed hypergraphs

In this section we introduce two classes of k-uniform directed hypergraphs: 1. Cored directed hypergraphs and 2. Power directed hypergraphs. Hu, Qi and Shao in [7] introduced these two classes in undirected hypergraphs and investigated their spectral properties. We extend their definitions and analyze the α-spectral properties of power directed hypergraphs and cored directed hypergraphs.

4.1 Cored directed hypergraphs

We begin with the definition of Cored directed hypergraphs.

Definition 7. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a directed hypergraph. $\mathcal{H}$ is a cored directed hypergraph if there exists in each arc $\vec{e} = (e^+, e^-)$ a vertex $i \in e^+$ such that $d_i^+ = 1$ and $d_i^- = 0$. Such vertex is called core vertex and a vertex with out-degree greater than one is called intersection vertex.*

By Theorem 5, we have the following lemma:

Lemma 4. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a cored k-uniform directed hypergraph and $(\lambda, \mathbf{x})$ be an H-eigenpair of $\mathcal{A}_\alpha$ and $\lambda \neq \alpha$. If i and j are two core vertices in arc $\vec{e}$, then $x_i = x_j$ when k is odd and $|x_i| = |x_j|$ when k is even.*

In the following we study a special cored directed hypergraph.

Definition 8. *Let $\mathcal{S} = (\mathcal{V}, \mathcal{E})$ be a cored k-uniform directed hypergraph. We call it a directed squid if $\mathcal{V} = \{1, 1_1, 2_1, \cdots, k_1, \cdots, 1_{(k-1)}, 2_{(k-1)}, \cdots, k_{(k-1)}\}$ and the arc set $\mathcal{E} = \{\vec{e}_i \mid i = 0, \cdots, k-1\}$ in which*

$$\begin{aligned} \vec{e}_0 &= (\{1\}, \{1_1, 1_2 \cdots, 1_{(k-1)}\}), \\ \vec{e}_i &= (\{1_i\}, \{2_i, 3_i, \cdots, k_i\}), \qquad\qquad i = 1, \cdots, k-1. \end{aligned}$$

By the Definition 8, it's straightforward that $d_1^+ = d_{1_1}^+ = d_{1_2}^+ = \cdots, d_{1_{(k-1)}}^+ = 1$, and $d_i^+ = 0$ otherwise.

The following theorem determines $Hspec(\mathcal{A}_\alpha)$ of the directed squid $\mathcal{S}$.

Theorem 6. *Let $\mathcal{S} = (\mathcal{V}, \mathcal{E})$ be a k-uniform directed squid, then $Hspec(\mathcal{A}_\alpha) = \{0, \alpha\}$.*

Proof. It is easy to see that $(0, \mathbf{x})$ is an H-eigenpair of $\mathcal{A}_\alpha$, where

$$x_t = \begin{cases} 1 & t = 1, \\ 0 & t = 1_i \quad (\text{for } i = 1, \cdots, k-1), \\ 1 & t = 2_i \quad (\text{for } i = 1, \cdots, k-1), \\ 0 & t = j_i \quad (\text{for } i = 1, \cdots, k-1\ ,\ j = 3, \cdots, k). \end{cases}$$

Now let $\mathbf{x}$ be an H-eigenvector of $\mathcal{A}_\alpha$ corresponding to H-eigenvalue $\lambda \neq 0$. By Lemma 2, we have:

$$(\lambda - \alpha) x_1^{k-1} = (1 - \alpha) \prod_{i=1}^{k-1} x_{1_i}, \tag{1}$$

$$(\lambda - \alpha) x_{1_i}^{k-1} = (1 - \alpha) \prod_{j=2}^{k} x_{j_i}, \qquad i = 1, 2, \cdots, k-1, \tag{2}$$

$$\lambda x_{j_i}^{k-1} = 0, \qquad i = 1, 2, \cdots, k-1, j = 2, \cdots, k. \tag{3}$$

By (3), $x_{j_i} = 0$ for all i, j. By taking it in (2), we have $(\lambda - \alpha) x_{1_i}^{k-1} = 0$. Now three cases are considered:
(**i**) : $x_{1_i} \neq 0$ for $i = 1, 2, \cdots, k-1$, then $\lambda = \alpha$ and by (1), $\prod_{i=1}^{k-1} x_{1_i} = 0$ that is a contradiction.
(**ii**) : $x_{1_i} = 0$ for $i = 1, 2, \cdots, k-1$, then by 1, $(\lambda - \alpha) x_1^{k-1} = 0$. Thus $\lambda = \alpha$ and $x_1 \neq 0$.
(**iii**) : $x_{1_i} = 0$ and $x_{1_j} \neq 0$ for some $i, j = 1, 2, \cdots, k-1$. Then $\lambda = \alpha$ and $x_1 \in \mathbb{R}$.
Therefore, $\lambda = \alpha$ is the only nonzero H-eigenvalue of $\mathcal{A}_\alpha$. □

4.2 Power directed hypergraphs

Definition 9. *Let $G = (V, E)$ be a directed graph and $k \geq 3$. The k-th power of G, $\mathcal{G}^k = (\mathcal{V}, \mathcal{E})$ is defined as the k-uniform directed hypergraph with the set of arcs*

$$\mathcal{E} = \{\vec{e} = (e^+, e^-) \mid e \in E\},$$

where if $e = (i_1^e, i_2^e) \in E$, then $e^+ = \{i_1^e, i_{e,1}, i_{e,2}, \cdots, i_{e,k-2}\}$ and $e^- = \{i_2^e\}$, and the set of vertices $\mathcal{V} = \left(\bigcup_{e \in E} \{i_{e,1}, i_{e,2}, \cdots, i_{e,k-2}\}\right) \cup V$.

It is easy to see that each power directed hypergraph is a cored directed hypergraph, but on the contrary, it is not generally correct, for example, the directed squid which was studied in the previous subsection.

The next theorem gives some basic results about an ordinary arc in a power directed hypergraph.

Theorem 7. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a power k-uniform directed hypergraph and $\mathbf{x}$ be an H-eigenvector of $\mathcal{A}_\alpha$, corresponding to $\lambda \neq \alpha$. If $\vec{e} = (e^+, e^-) \in \mathcal{E}$ is an arbitrary arc with $e^+ = \{i_1^e, i_{e,1}, i_{e,2}, \cdots, i_{e,k-2}\}$ and $e^- = \{i_2^e\}$, then we have:*

(1) If $d^+_{i_1^e} > 1$, $d^+_{i_2^e} \geq 1$ and $x_{i_{e,1}} = \beta \neq 0$, then $x_{i_1^e} x_{i_2^e} = \frac{(\lambda-\alpha)\beta^2}{(1-\alpha)}$ when k is odd and $x_{i_1^e} x_{i_2^e} = \frac{(\lambda-\alpha)\beta^2}{(1-\alpha)}$ or $-\frac{(\lambda-\alpha)\beta^2}{(1-\alpha)}$ when k is even.

(2) If $d^+_{i_1^e} = 1$, $d^+_{i_2^e} \geq 1$ and $x_{i_{e,1}} = \beta \neq 0$, then $x_{i_2^e} = \frac{(\lambda-\alpha)\beta}{(1-\alpha)}$ when k is odd and $x_{i_2^e} = \frac{(\lambda-\alpha)\beta}{(1-\alpha)}$ or $-\frac{(\lambda-\alpha)\beta}{(1-\alpha)}$ when k is even.

(3) If $d^+_{i_2^e} = 0$, then $x_j = 0$ for $j \in \{i_{e,1}, i_{e,2}, \cdots, i_{e,k-2}, i_2^e\}$.

Proof. By Lemma 4, $x_{i_{e,j}} = \beta$ for $j = 2, \cdots, k-2$ when k is odd and $|x_{i_{e,j}}| = \beta$ for $j = 2, \cdots, k-2$ when k is even.
For (1), by Definition 2, we have:

$$(1-\alpha)\beta^{k-3} x_{i_1^e} x_{i_2^e} = (\lambda - \alpha)\beta^{k-1} \qquad \textit{if } k \textit{ is odd,}$$

$$\left\{ \begin{array}{rl} & (1-\alpha)\beta^{k-3} x_{i_1^e} x_{i_2^e} = (\lambda - \alpha)\beta^{k-1} \\ & \textit{or} \\ - & (1-\alpha)\beta^{k-3} x_{i_1^e} x_{i_2^e} = (\lambda - \alpha)\beta^{k-1} \end{array} \right. \qquad \textit{if } k \textit{ is even.}$$

The result follows from $\beta \neq 0$.

For (2), by Lemma 4, $x_{i_1^e} = \beta$ or $x_{i_1^e} = -\beta$. By Definition 2, we have:

$$(1-\alpha)\beta^{k-2}x_{i_2^e} = (\lambda-\alpha)\beta^{k-1} \qquad \textit{if } k \textit{ is odd,}$$

$$\begin{cases} \quad (1-\alpha)\beta^{k-2}x_{i_2^e} = (\lambda-\alpha)\beta^{k-1} \\ \quad \textit{or} \\ - \quad (1-\alpha)\beta^{k-2}x_{i_2^e} = (\lambda-\alpha)\beta^{k-1} \end{cases} \qquad \textit{if } k \textit{ is even.}$$

The result follows from $\beta \neq 0$.
For (3), since $d^+_{i_2^e} = 0$, then $x_{i_2^e} = 0$. Thus by Definition 2 and Lemma 4, $x_j = 0$ for $j \in \{i_{e,1}, i_{e,2}, \cdots, i_{e,k-2}\}$. $\square$

In the following we study a special power directed hypergraph which is a called directed hyperwheel.

Definition 10. *Let $\mathcal{W}_d = (\mathcal{V}, \mathcal{E})$ be a power k-uniform directed hypergraph. We call it a directed hyperwheel if $\mathcal{V} = V_0 \cup V_1 \cup \cdots V_d \cup \bar{V}_1 \cup \bar{V}_2 \cup \cdots \cup \bar{V}_d$ is a disjoint partition of $\mathcal{V}$ in which $V_0 = \{1\}$, $V_i = \{1_i, 2_i, \cdots, (k-1)_i\}$ and $\bar{V}_i = \{1^i, 2^i, \cdots, (k-2)^i\}$ for $i = 1, 2, \cdots, d$ and the arc set $\mathcal{E} = \{\vec{e}_i,\ \vec{a}_i |\ i = 1, \cdots, d\}$ in which*

$$\vec{e}_i = (\{1, 1_i, \cdots, (k-2)_i\}, \{(k-1)_i\}), \qquad i = 1, \cdots, d,$$
$$\vec{a}_i = (\{(k-1)_i, 1^i, \cdots, (k-2)^i\}, \{(k-1)_{i+1}\}), \qquad i = 1, \cdots, d-1,$$
$$\vec{a}_d = (\{(k-1)_d, 1^d, \cdots, (k-2)^d\}, \{(k-1)_1\}).$$

By Definition 10, it can be shown easily.

Lemma 5. *Let $\mathcal{W}_d = (\mathcal{V}, \mathcal{E})$ be a directed k-uniform hyperwheel, then $d^+_1 = d$, $d^+_j = 1$ for $j \neq 1$ and $d^-_{(k-1)_i} = 2$ for $i = 1, \cdots, d$, $d^-_j = 0$ for $i = 1, \cdots, d$ and $j \neq (k-1)_i$.*

In the following theorems the H-spectrum of $\mathcal{A}_\alpha$ of $\mathcal{W}_d$ are determined.

Theorem 8. *Let $\mathcal{W}_d = (\mathcal{V}, \mathcal{E})$ be a directed k-uniform hyperwheel. Then $Hspec(\mathcal{A}_\alpha) = \{\alpha, \alpha d, 1\}$ when d and k are odd, and $Hspec(\mathcal{L}) = \{\alpha, \alpha d, 1, 2\alpha - 1\}$ otherwise.*

Proof. By Theorem 1 and Lemma 5, $\alpha, \alpha d \in Hspec(\mathcal{A}_\alpha)$. Now suppose that $\mathbf{x}$ is an H-eigenvector of $\mathcal{A}_\alpha$ corresponding to H-eigenvalue $\lambda \neq \alpha, \alpha d$. The proof is divided into two cases, which contain several subcases respectively:

1: k is odd.

By Lemma 4, we have:

$$\begin{aligned} &x_{1_i} = x_{2_i} = \cdots = x_{(k-2)_i} = \alpha_i, && i = 1, \cdots, d,\\ &x_{1^i} = x_{2^i} = \cdots = x_{(k-2)^i} = x_{(k-1)_i} = \beta_i, && i = 1, \cdots, d. \end{aligned}$$

Now by Definition 2, we have:

$$(\lambda - \alpha d)x_1^{k-1} = (1-\alpha)\sum_{i=1}^{d} \beta_i \alpha_i^{k-2}, \tag{4}$$

$$(\lambda - \alpha)\alpha_i^{k-1} = (1-\alpha)x_1\alpha_i^{k-3}\beta_i, \qquad i = 1, \cdots, d, \tag{5}$$

$$(\lambda - \alpha)\beta_i^{k-1} = (1-\alpha)\beta_i^{k-2}\beta_{i+1}, \qquad i = 1, \cdots, d-1, \tag{6}$$

$$(\lambda - \alpha)\beta_d^{k-1} = (1-\alpha)\beta_d^{k-2}\beta_1. \tag{7}$$

By (6) and (7), if $\beta_i = 0$ for some $i = 1, \cdots, d$, then all $\beta_i = 0$ and thus by (5) and (4), $\mathbf{x} = 0$ that is a contradiction. Therefore, $\beta_i \neq 0$ for $i = 1, \cdots, d$. Then by (6) and (7), $\frac{(\lambda-\alpha)}{(1-\alpha)} = \frac{\beta_{i+1}}{\beta_i} = \frac{\beta_1}{\beta_d}$ for $i = 1, \cdots, d-1$, then we have:

$$\begin{aligned} \beta_1 = \frac{(\lambda-\alpha)^d}{(1-\alpha)^d}\beta_1 \quad &\Longrightarrow \quad (\lambda-\alpha)^d = (1-\alpha)^d \\ &\Longrightarrow \quad \begin{cases} \lambda = 1, 2\alpha - 1 & \textit{if } d \textit{ is even} \\ \lambda = 1 & \textit{if } d \textit{ is odd} \end{cases} \end{aligned}$$

2: k is even.

By Lemma 4, we have:

$$\begin{aligned} &|x_{1_i}| = |x_{2_i}| = \cdots = |x_{(k-2)_i}|, && i = 1, \cdots, d,\\ &|x_{1^i}| = |x_{2^i}| = \cdots = |x_{(k-2)^i}| = |x_{(k-1)_i}|, && i = 1, \cdots, d. \end{aligned}$$

Now let $x_{1_i} = \alpha_i$ and $x_{(k-1)_i} = \beta_i$ for $i = 1, \cdots, d$. With a little modification in (4), (5), (6) and (7) and by similar argument in the

previous case, $\beta_i \neq 0$ for $i = 1, \cdots, d$. Now we consider two subcases:
(i) d is even. There are two cases:

- $\beta_d = \frac{(\lambda-\alpha)^{d-1}}{(1-\alpha)^{d-1}}\beta_1$, then we have:

$$\begin{aligned} &\textit{if } \lambda < \alpha \;\Rightarrow\; \beta_1 \textit{ and } \beta_d \textit{ have different signs} \\ &\qquad \Rightarrow\; \beta_1 = \frac{(\lambda-\alpha)}{(1-\alpha)}\beta_d \;\Rightarrow\; \frac{(\lambda-\alpha)^d}{(1-\alpha)^d} = 1 \;\Rightarrow\; \lambda = 2\alpha - 1; \\ &\textit{if } \lambda > \alpha \;\Rightarrow\; \beta_1 \textit{ and } \beta_d \textit{ have the same sign} \\ &\qquad \Rightarrow\; \beta_1 = \frac{(\lambda-\alpha)}{(1-\alpha)}\beta_d \;\Rightarrow\; \frac{(\lambda-\alpha)^d}{(1-\alpha)^d} = 1 \;\Rightarrow\; \lambda = 1. \end{aligned}$$

- $\beta_d = -\frac{(\lambda-\alpha)^{d-1}}{(1-\alpha)^{d-1}}\beta_1$, then we have:

$$\begin{aligned} &\textit{if } \lambda > \alpha \;\Rightarrow\; \beta_1 \textit{ and } \beta_d \textit{ have different signs} \\ &\qquad \Rightarrow\; \beta_1 = -\frac{(\lambda-\alpha)}{(1-\alpha)}\beta_d \;\Rightarrow\; \frac{(\lambda-\alpha)^d}{(1-\alpha)^d} = 1 \;\Rightarrow\; \lambda = 1; \\ &\textit{if } \lambda < \alpha \;\Rightarrow\; \beta_1 \textit{ and } \beta_d \textit{ have the same sign} \\ &\qquad \Rightarrow\; \beta_1 = -\frac{(\lambda-\alpha)}{(1-\alpha)}\beta_d \;\Rightarrow\; \frac{(\lambda-\alpha)^d}{(1-\alpha)^d} = 1 \;\Rightarrow\; \lambda = 2\alpha - 1. \end{aligned}$$

(ii) d is odd. There are two cases:

- $\beta_d = \frac{(\lambda-\alpha)^{d-1}}{(1-\alpha)^{d-1}}\beta_1$, then β_1 and β_d have the same sign and we have:

$$\begin{aligned} &\textit{if } \lambda < \alpha \;\Rightarrow\; \beta_1 = -\frac{(\lambda-\alpha)}{(1-\alpha)}\beta_d \;\Rightarrow\; \frac{(\lambda-\alpha)^d}{(1-\alpha)^d} = -1 \Rightarrow \lambda = 2\alpha - 1; \\ &\textit{if } \lambda > \alpha \;\Rightarrow\; \beta_1 = \frac{(\lambda-\alpha)}{(1-\alpha)}\beta_d \;\Rightarrow\; \frac{(\lambda-\alpha)^d}{(1-\alpha)^d} = 1 \;\Rightarrow\; \lambda = 1. \end{aligned}$$

- $\beta_d = -\frac{(\lambda-\alpha)^{d-1}}{(1-\alpha)^{d-1}}\beta_1$, then β_1 and β_d have different signs and we have:

$$\textit{if } \lambda < \alpha \;\Rightarrow\; \beta_1 = \frac{(\lambda-\alpha)}{(1-\alpha)}\beta_d \;\Rightarrow\; \frac{(\lambda-\alpha)^d}{(1-\alpha)^d} = -1 \;\Rightarrow\; \lambda = 2\alpha - 1;$$

$$\textit{if } \lambda > \alpha \;\Rightarrow\; \beta_1 = -\frac{(\lambda-\alpha)}{(1-\alpha)}\beta_d \;\Rightarrow\; \frac{(\lambda-\alpha)^d}{(1-\alpha)^d} = 1 \;\Rightarrow\; \lambda = 1.$$

□

5 Conclusion

In this paper we consider a k-uniform directed hypergraph in general form and introduce its adjacency tensor, Laplacian tensor and signless Laplacian tensor. Then we propose theorems in spectral theory of the convex linear combination of $\mathcal{D}$ and $\mathcal{A}$ that has been defined as $\mathcal{A}_\alpha = \alpha\mathcal{D} + (1-\alpha)\mathcal{A}$, where $\mathcal{D}$ and $\mathcal{A}$ are the degree tensor and the adjacency tensor of $\mathcal{H}$, respectively. Cored directed hypergraphs and power directed hypergraphs are introduced, and some their α-spectral properties are presented.

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C. Gaindric – National Prize Laureate of Moldova

Constantin Gaindric, corresponding member of the Academy of Sciences of RM, is a recognized personality in the field of informatics. He carried out research focused on the mathematical modeling of some economic processes evolving towards the elaboration of decision support systems in diagnosis. He was the first in Moldova who initiated research dedicated to edification of the Information Society, proposing the system of indicators that help to assess grounding of integration in the information society and to monitor digital inequality overcoming.

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For all his prodigious activity, for remarkable results in the field of mathematics and informatics, Professor C. Gaindric is awarded the National Prize in the field of informatics.

Congratulations and wishes of success in his future scientific work!

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