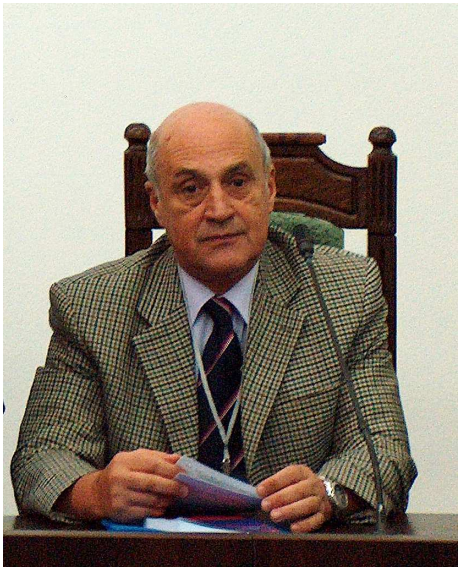
At the 70th anniversary of Academician Florin Gheorghe Filip



With the appearance of the possibilities of establishing scientific relations between the Republic of Moldova and Romania, that is to say, in 1991, together with a group of colleagues from the Institute of Mathematics and Computer Science of the Academy of Sciences of Moldova (IMCS), I was received at the National Institute for Research and Development in Informatics, Bucharest (ICI), by the director of this prestigious institution Dr. Florin Gheorghe Filip.

The research topics of common interest have been already found starting from the first discussions, and a collaborative and collegiate relationship has been established with some joint research projects.

In the course of time, professor and academician F.G. Philip has accepted to be a member of the advisory board of the newly born Computer Science Journal of Moldova (CSJM), and has participated in defence of some habilitation theses of IMCS researchers. Relations

with IMCS have grown to the relations with the Academy of Sciences of Moldova. Now academician Florin Gheorghe Filip is an honorary member of the Academy of Sciences of Moldova.

Researchers, students, master students and doctoral students from the Republic of Moldova study and draw inspiration from the monographs of Professor F.G. Philip, some of which are included in the bibliography recommended in the curricula of disciplines studied by computer science students at several universities.

Academician F.G. Philip has significant results and an appreciated impact in decision support systems, hierarchical large-scale systems of control and optimization, technology management and foresight, and IT application to the cultural domain. The monographs published in prestigious scientific publishing houses, about 300 articles in journals serve as the evidence.

The scientific community appreciates the merits of researcher F.G. Philip, who is being invited as the lecturer at scientific centers in the United Kingdom, Austria, Brazil, France, Sweden, China, Poland, Germany, the Republic of Moldova, Chechia etc. Academician F.G. Filip has the degree honoris causa of Lucian Blaga University of Sibiu, Valahia University, Târgoviste, Ovidius University, Constanta Ecole Centrale de Lille, France, Polytechnic University of Timisoara, Agora University of Oradea, Academy of Economic Studies of Bucharest, Uni-

versity Petrol-Gaze of Ploiesti, University of Pitesti.

Academician F.G.Filip has an intense organizational and managerial activity: for 10 years, from 2000 till 2010, he was vice-president of the Romanian Academy. In 2010, he was elected as the Head of the Section "Information Science and Technology" of the Romanian Academy. Currently he is the director of the Romanian Academy Library. He has an intensive activity on a European and international scale as a member of the program committees of more than 50 scientific conferences in Europe, USA, South America, Asia and Africa.

Acad. Filip is the founder and Editor-in-Chief of the journal "Studies in Informatics and Control" (1991), and co-founder and Editor-in-Chief of "International Journal of Computers Communications & Control" (2006). He is a member of the editorial staff of several scientific journals, such as:

- Systems Analysis, Modeling and Simulation;
- International J. of Critical Infrastructures;
- Computer Science Journal of Moldova (Chisinau, since 1993);
- Information Technologies and Control;
- Romanian Journal of Information Science and Technology;
- Control Engineering and Applied Informatics (SRAIT);
- Romanian Journal of Informatics and Automatics ;
- Romanian Journal of Automatics;
- Technological and Economic Development of Economy
- International Journal of Information Technology & Decision Making (Science Direct);
- Advances in Electrical and Computer Engineering;
- Financial Innovation (Springer, since 2016);

- Journal of System and Management Sciences (Beijing Jiatong Univ., China).

We are glad to have the opportunity to know academician F.G. Philip as a scholar, manager, but also a person with high qualities of character, full of energy, modest, generous man, colleague and friend.

Our heartiest congratulations on his Birthday! We wish him health, happiness and new successes for the prosperity of science!

On behalf of the journal CSJM editorial board and colleagues in IMCS,

Constantin Gaindric

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Collaborative Control Theory and Decision Support Systems

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Abstract

Collaborative Decision Support Systems, CDSS, depend on cost-effective collaboration among the decision participants. Those may include, in addition to human decision makers, non-human entities such as robots, software and hardware agents, sensors, and autonomous instruments. The purpose of this article is to explore the impact that CCT, the Collaborative Control Theory, has on cyber supported augmentation of collaboration in general, and its proven and potential impacts on CDSS in particular. Three recent case studies are discussed. The correlation between CDSS decision process and quality; and the level of CCT-based collaboration augmentation and the resulting level of Collaborative Intelligence, CI, is presented. It is concluded that while there are clear positive impacts of CCT based augmentation and level of CI, they need to be measured and optimized, not maximized. Further research in this area is also described.

Key Words: CCT-based Collaboration Protocols; Co-Insight; Collaboration Augmentation; Collaborative Intelligence; Collaboration Requirements Planning; Error and Conflict Prevention

1 Introduction

The significant research on decision making and taking by Academician Florin G. Filip,e.g., [17,18], and the recent publication of Filip et al. book on CDSS [19], *collaborative decision support systems*, offer an opportunity to analyze the mutual relations between CDSS and

CCT, the *Collaborative Control Theory*. The book discusses in detail the role of collaboration in DSS, *Decision Support Systems*, the various processes and protocols of collaboration among the human decision makers, and the enabling computer, communication, information and cyber technologies that make CDSS increasingly more feasible, and sometimes more effective (e.g., [4], [5], [6], [54], [33], [35], [39], [42], [49], [50], [56]). It also discusses the fact that while the common team of decision makers are human, increasingly the participants are distributed, decentralized, and include software and hardware agents, robots, and machines. In particular, in real-time decision making and control, the heart and brain of smart and autonomous automation, the role of the automated, often autonomous non-human participants carries a larger responsibility.

Several questions arise in this context for control and automation engineers and scientists, and these questions can be presented by two key problems:

- What are the risks and what is the balance of these risks compared with the advantages of CDSS?

- How can such CDSS be designed, operated and maintained to minimize those risks while maximizing the benefits?

These problems are not new, as they have been asked and addressed by researchers and practitioners since computerized DSS first appeared. When additional computational resources at higher levels of cyber sophistication and power are added, these problems become even more acute. With greater advantages in supporting decision processes, come greater risks.

The purpose of this article is to address these problems from the perspective of CCT. The Collaborative Control Theory emerged when it was realized that internetworked, interconnected automation systems become so complex and interdependent that they will collapse unless designed and even optimized for effective and cyber-supported collaboration.

The article includes four sections following this introduction: Risks and advantages of collaboration in CDSS; CCT augmentation to overcome collaboration limits and risks; Collaborative Intelligence (CI) by CCT augmentation of CDSS; conclusions and further research.

2 Risks and Advantages of Collaboration in CDSS

Who collaborates on decision making and decision taking? Why do they need and why would they want to collaborate? Let us consider the functions of collaboration shown in Table 1, and the examples shown in Table 2. They illustrate who collaborates, the purpose of their collaboration, the motivation to reach and implement decisions, and some of the risks involved.

Revolutionizing collaboration by cyber support, including the case of CDSS, carries a large number of advantages (Figure 1). Some of them can be considered *mandatory collaboration requirements*, meaning that without them no good decisions can be analyzed and made. For instance, in a design case, without timely data from customers about the details of their demand, and from suppliers about their capacity availability to deliver, no correct decision can be expected. Errors and conflicts can be expected. Over large supply networks, and with inevitable changes and modification in supplies and demands, these mandatory collaboration requirements scale up and escalate.

Optional collaboration requirements are those that may or may not be beneficial to have, but are not as clearly necessary as the mandatory requirements to collaborate. Typical examples involve the amount of additional information gained by collaboration, but having unclear value to influence a decision, nor its quality. Furthermore, the cost and effort to obtain those additional opinions, or preferences, priorities, etc. may even complicate the decision and damage the entire decision process.

To evaluate the advantages and limits of collaboration for decision support in the context of CCT, five key metrics can be considered:

1. *Decision quality* – quality of the decisions being made now; of future decisions

2. *Information availability* – what information is required and when; what information is not required; what additional information can add benefit to the decision making process and to the decisions' quality.

3. *Timely completion* – Which decisions have to be made, and by when.

4. *Multiple views* – What level of diversity of logic and of motivations are beneficial; negotiated decisions; visibility of the decision process and the decisions made; co-insight, the ability to avail and gain timely collaborative insights of the multiple participants, including overcoming adversarial attitudes.

5. *Multiple engagement* – For a CDSS to be useful and effective, the collaborating participants may or may not need to be engaged during (or during part of) the decision making process, at the decision taking stage, and during (or during part of) the implementation and revisions of the decisions.

These five metrics are interrelated and influence each other. They will be considered in the case studies described later in this article. There are other metrics that can be considered (e.g., see [11], [12], [13], [16], [19]).

3 CCT augmentation to overcome collaboration limits and risks

CCT has been developed, validated and implemented by researchers and engineers worldwide. Its main purpose is to understand, design and optimize collaboration support systems, collaboration protocols, and collaboration algorithms that can augment all aspects of collaboration. Despite the potential risks and failures inherent in complex interactions

Table 1. Characteristics of DSS and CDSS and their role in augmenting collaboration (Source: [19]; adapted from [23])

Type	Functions
Knowledge repository	★ Identify and solve problems
	❖ Facilitate interactions among decision-makers
	➢ Define, document, and regulate the actions of decision-makers
	★ ❖ ➢ Private repositories under the access control of individuals
	★ ❖ ➢ Public repositories; shared access and share knowledge
Requests	❖ Customize requests based on specific needs
	★ Enhance flexibility in timing of requests
Operations	★ Provide knowledge to meet unanticipated demands
	★ Generate knowledge via automated calculation/analysis/reasoning
Presentation	❖ Customized presentation of results based on specific needs
Coordination	❖ Facilitate internal/external communication among decision-makers
	➢ Structure and regulate individual/group decision-making tasks
	➢ Structure and regulate interrelated decisions

❖ Characteristics to support communication among decision-makers
★ Characteristics to support knowledge for decision-making
➢ Characteristics to support decision-making processes

associated with collaboration (Table 2), augmentation by CCT has been developed to overcome them.

As it is shown in Figure 1, cyber support is integrated with common CDSS (Figure 1a), but in addition, cyber support with CCT-augmentation of collaboration processes (Figure 1b) can and is designed to overcome the risks and shortcomings of collaboration processes and systems.

A brief summary of CCT ([35]) is provided in Table 3. CCT comprises seven augmentation principles, listed in the first column. For each of them, its role, collaborative decisions, and examples of collaboration augmentation models, protocols, and algorithms developed to implement it are shown in the last column. One can find details about each of them in the references of Table 3.

The CCT augmentation roles of each principle and its related cyber tools are as follows.

CRP: Collaboration Requirement Planning. It includes advanced

Table 2. Decision making by CDSS based on collaboration among participants

Decision making participants	**Decision examples**	**Risks of Collaborative Decisions**
People	Investments; policies; budgeting; responses; resource allocation; scheduling	- Low or no incentive to collaborate - Potential logic errors - Potential conflicts - Wrong/missing data - Costs of collaboration - Delays - Poor or no compromise - Too late for some or all - Too early for some or all - Other mismatch challenges
People and machines	Activation; recovery; diagnostics	
Software agents	Simulations; calculations; assembly design; service planning	
People-machines-agents	Coordination; priorities; healthcare action alternatives	
Robots-Robots	Navigation; monitoring; co-assembly	
Swarms of robots, drones	Surface treatment; rescue; exploration; security	
Sensors	Health of crops; safety; assessment and prediction of conditions	
Combinations changing over time	Above decision combinations	

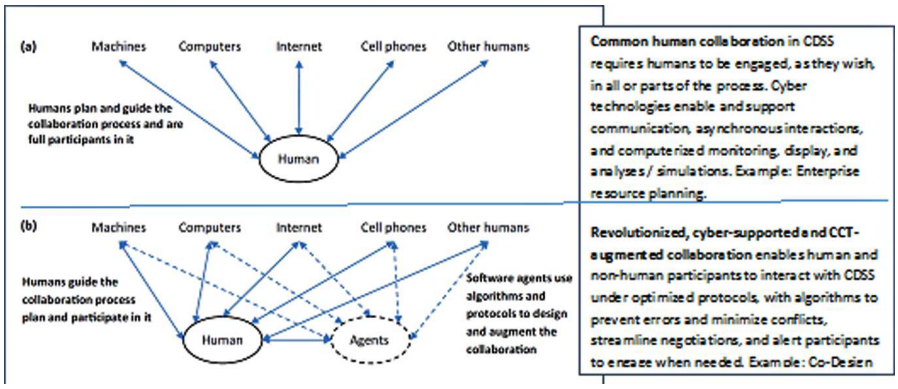

Figure 1. CDSS collaboration with: (a) Cyber support; (b) Cyber support and CCT augmentation (Source: [38])

pre-planning (CRP-I), followed by on-going monitoring and adaptive control/re-planning of collaborating resources (CRP-II). The algorithms, protocols, and multi-agent systems for CRP are designed to create and gain collaborative intelligence (CI) from multiple human and non-human participants for the collaborative decisions. By pre-planning and re-planning the collaboration, there is a greater chance to eliminate gaps and inefficiencies, thus improving the quality of collaboration process and of their outcomes.

EWP: e-Work Parallelism. It implies optimally exploiting the fact that work in cyber workspaces and workflows, and in human workspaces and workflows can and must be allowed to advance in parallel, and should not bottleneck each other. For decision support, it implies that cyber tools, hardware and software agents, can operate at their own speed and in parallel to human decision makers, to prepare, acquire, exchange, analyze, evaluate, an even recommend decisions in support of human decision makers and decisions.

ECR: Errors and Conflicts Resolution. Eliminate or minimize the cost of resolving conflicts among collaborating e-workers and cyber tools by automated, cyber-supported error and conflict detection, prog-

nostics, and prevention systems. Without it, complex, large scale decision systems based only on human-interaction will collapse, as has been proven theoretically and empirically.

CFT: Collaborative Fault Tolerance. Cyber tools, protocols, algorithms, and agent systems are designed to improve the performance results of a team, including team decisions, such that a team of weak collaborators can together reach better results than a single, non-team and even flawless agent.

A-D: Associate-Dissociate (also known as JLR, Join-Leave-Remain). Cyber tools designed under this principle include collaborative control decisions on when, whether, and why to associate, or dissociate from a team, or network, of collaborators, based on on-going cost/benefit evaluations. This evaluations are conducted in parallel to the collaborative network performance. For collaborating decision makers, for instance, it means that some of them (e.g., sensors, knowledge bases, etc.) may or may not need to be engaged in certain portions of a decision process but join later. Or it could mean that for a team of decision makers, they may find out that one or a few of them (e.g., certain robots, or drones, or humans) can be disengaged from the team, at least for a certain period, to eliminate damage in future decisions, or in certain decisions.

ELOCC: Evolutionary Lines of Collaboration and Command. This CCT principle guides the development of evolutionary and machine-learning cyber mechanisms for organizational learning and improvement of both ad-hoc decisions improvisation, on the spot self-reorganization and contact creation, and best matching protocols (BMP), for pairing suppliers (providers) and consumes (clients). For decision support, it implies the same, with emphasis on the evolutionary nature of decisions over time.

BMP: Best Matching Protocols were originally developed as part of ELOCC, and later also as part of all other CCT principles. They are shown in Table 3 under ELOCC and CRP. Their objective is to optimally match sets, either by pairing best analytic tools and agents to given decision requirements, or matching higher dimensional sets of

sensors, robots, instruments, and given planning and control decisions ([31]).

BIC: Bio Inspired Collaboration. These are cyber tools, protocols and algorithms designed to increase the collaborative intelligence (CI), hence the resulting benefits of collaborative decisions and control, by bio inspired and socio inspired collaboration mechanisms observed in nature, e.g., genetic algorithms, ant and other colony protocols and algorithms, and market negotiation games.

CSCW oriented protocols

A major objective in CDSS is to understand and deliver interaction protocols that would structure and improve collaboration processes and the resulting decisions' quality. Protocols of collaboration in CDSS ([19], [41]) include interaction protocols developed extensively by researchers in the area of CSCW, Computer Supported Collaborative Work. Their focus is on social, human factors and psychological aspects of computer supported collaboration. Mostly, they are concerned with the collaboration shown in Figure 1a, and can provide guidance to cyber-supported collaboration shown in Figure 1b.

Steps of collaboration that are addressed by the CSCW protocols are identified as Generate, Reduce, Clarify, Organize, Evaluate, and Build Consensus (e.g., [27]). Typical functions followed by these protocols are:

- Voting methods (e.g., [14], [20])
- Information sharing (e.g., [9])
- Argumentation by groups (e.g., [43])
- Resource sharing and allocation (e.g., [1], [45])
- Mediation and interaction (e.g., [21])
- Crowd sourcing (e.g., [2],[10])

An example of a systematic support system for CDSS to enable collaboration is Thinklet ([3], [17], [22]). Additional details on CSCW oriented protocols can be found in [41]. Several researchers have studied their complexity, e.g., [20, 21].

While the CSCW collaboration and interaction protocols have advanced collaborative decision systems ability by providing protocols for functional collaboration, the CCT augmentation protocols differ in two main aspects:

1. They seek to automate and alleviate known risks and limitations that are typical in computer supported collaborative interactions, as described above and further below.

2. In addition, CCT assumes that any decision system involves, beyond human decision makers and knowledge bases also sensors, robots, and software agents, who may need to make their own decisions autonomously, as well as interact with humans for their decisions

CCT augmentation Protocols

As discussed above and shown in Table 3, the CCT augmentation protocols are designed as cyber-based augmentation of collaborative interactions. They are focused on solving the following typical risks in collaborative interactions:

- Inefficient, ineffective decision processes due to overloaded decision makers and lack of common, workflow based decision process plans. These risks are addressed by the CRP protocols (e.g., [59])

- Unclear assignment of who does what and when in support of the decision interactions. These risks are addressed by the EWP protocols (e.g., [7])

- Errors and conflicts encountered during collaboration, requiring monitoring and detection, and either recovery to overcome them, or better yet, machine learning to prevent and eliminate them. These risks are addressed by the ECR protocols (e.g., [8], [24], [28])

- Errors and conflicts which cannot be resolved, or can only be resolved too late, pose risks to collaboration. Such risks require fault tolerance mechanisms designed in the support systems, and are provided by the CFT protocols (e.g., [25])

- Not all humans, robots, agents, sensors, need to be engaged throughout the entire decision process. On the other hand, they may be needed as active and engaged participants at certain times, weather preferred by them or needed by other participants. For instance, in such case they should be alerted for active participation when needed. Handling this concern is by the AD protocols (e.g., [53])

- Certain participants may or may not be available when they need to or are invited to participate actively. These risks are addressed by the ELOCC protocols ([52], [58])

Examples of the design implementation and applications of the CCT augmentation protocols are given, for example, by [39] in manufacturing and logistics; [36] and [47] in modeling and decision support for sustainability; [26] and [37]in the design of service tasks administration protocols; [46] in complex production facility collaborative management; and [51] in security of supply networks. In many of these research applications, humans are in the loop as collaborating decision makers, including robots, sensors, and software agents. In some of these research applications, only autonomous robots, sensors, and software agents are collaborating to make their own autonomous decisions.

The impact and benefits gained by applying the above CCT augmentation cyber tools are intuitive, as they address directly solutions to critical and common weaknesses of collaboration in CDSS. These impacts and benefits have also been modeled, measured, and validated by researchers, based on the above five metrics and other metrics.

Table 3: CCT principles and PRISM Center discoveries of CCT collaboration augmentation cyber tools developed for them (Source: [30], [61]; Adapted from [35], [38])

Principle* – Rationale	Features	Applied decisions	e-Mfg/e-Service decisión areas	Model/ Algorithm/ Protocol**
CRP-I & CRP-II *"Think before you act"*	Collaboration planning & interaction	Resource planning	Multi-robotic assembly; Multiprocessors	CRP; TAP; BMP
	Multi-agent design	Agent theory	Mfg operations	ABMS
EWP *"Divide and conquer"*	Collaboration protocol design	Telecommunication, adaptive, and exchange protocols	ERP applications; Electronic inspection/testing; Wireless Micro-Electro Mechanical Systems (MEMS); Mfg networks	TIE/P; Test-LAN; TIF; BMP; TAP
	Middleware protocols	Client-server models	Automotive electronics; Flexible assembly	RAP; TOP
	Parallelism	Parallel/grid computing	Global design/mfg; Collaborative decision-making	DPIEM; TAP

Continuation of Table 3

Principle* – Rationale	Features	Applied decisions	e-Mfg/e-Service decisión areas	Model/ Algorithm/ Protocol**
	Resource & task allocation	Local area networks; Internet	Electronic assembly & test; Global mfg networks	TestLAN; MEN; TAP
ECR *"Learn from mistakes"*	Synchronization/ Re-synchronization	Agent theory	Robotic maintenance	ServSim
	Information assurance	Total quality management	Agent-based mfg/service	MERP
	Error detection & prevention	Computer recovery; Multi-agent systems	Robotic assembly; Multi-robot systems	NEFUSER; EDPA; CEDP
CFT *"Team for synergy"*	Fault-tolerant integration	Sensor fusion	Flow MEMS sensors; Wireless MEMS sensors	FTTP; TIE/ MEMS
	Conflict resolution	Telecommunication; Co-assembly	Co-facility design; Multi-robot systems; Assembly/disassembly	FDL; FDL-CR; CRP; BMP

Continuation of Table 3

Principle* – Rationale	Features	Applied decisions	e-Mfg/e-Service decisión areas	Model/ Algorithm/ Protocol**
AD *"Be selective"*	Enterprise integration	Network flow	Distributed & networked mfg/service systems	MEN Opt.; JLR; BMP; CD-CSP; TAP
	Organizational learning	Enterprise computing	Mfg/assembly corp.	CMS
ELOCC *"Trust the backup"*	Workflow integration & harmonization	Data flow; Distributed database; Workflow protocols	Aerospace mfg; CIM	DFI; DAF-Net & AIMIS; BMP; TAP
	Information sharing & collaboration	Virtual environments; Task graphs; Network computing; Internet/Intranet	Mfg cells; Distributed designers; Mfg networks; e-Business/e-Service	FDL; IDM; Co-X Tools; T-C-M; TAP

Continuation of Table 3

Principle* – Rationale	Features	Applied decisions	e-Mfg/e-Service decisión areas	Model/ Algorithm/ Protocol**
	e-Learning/ e-Training	Learning theory; Distributed & collaborative DSS	ERP applications; Emergency response	MERP/C; TSTP
	Viability measures	Virtual mfg	HCI	TIE/A
	e-Work scalability	Distributed computers	Mfg networks	MEN Opt.
BIC *"Follow nature"*	Distributed optimization & control	Agent theory; HMS; Swarm intelligence; Evolutionary algorithms	Mfg process planning & scheduling; Intelligent shop floor control; Collaborative mfg/service processes	GA; AS; NN
	Evolution	Emergent networks; Neural networks; Evolutionary & adaptive behaviors/ patterns in nature	Evolutionary robotics; Mfg networks; Negotiation systems; Self-formation & self-evolution of emergent networks	GA; AS; NN

* CRP: Collaboration Requirement Planning; EWP: e-Work Parallelism; ECR: Error and Conflict Resolution; CFT: Collaborative Fault Tolerance; AD: Association-Dissociation; ELOCC: Emergent Lines of Collaboration and Command; BIC: Bio-Inspired Collaboration; BMP: Best Matching Protocol

** These models, protocols, and algorithms are described in detail in the table references

4 Collaborative Intelligence (CI) by CCT augmentation of CDSS

Two CCT-based developments augmenting collaboration in CDSS are the Co-Insight system, and the Collaborative Intelligence (CI) of participants. Recent research has shown that both help understand better the collaborative decision and control process, and enable reaching better quality decisions (e.g., [15], [60] [61]).

Research on acquiring and accumulating intelligence has been conducted by many researchers (e.g., [32], [40], [44], [55]). See a summary in Table 4.

CCT augmentation of collaborative decisions by the Co-Insight framework is shown in Figure 2. It is designed to enable multiple participants to engage in information and knowledge exchange in a way that incorporates visual analytics through knowledge repositories and exchange protocols. The unique advantage of this framework is that it is built with CCT cyber tools. A Co-Net, a collaborative network of decision participants, enables interactions under best matching protocol of participants, recommending who should be involved at each period of time. The recommendations are generated through a collaborative network optimization protocol (CNO). Another best matching protocol guides the matching of decision analytics tools that are best suitable for each given decision or decision stage.

The Co-Insight framework is developed on a HUB, a powerful computational infrastructure (e.g., Industrial Internet of Things/Internet of Services, or cloud computing) to enable large scale, decentralized

interactions for a small social/group network, or for a wide network of participants.

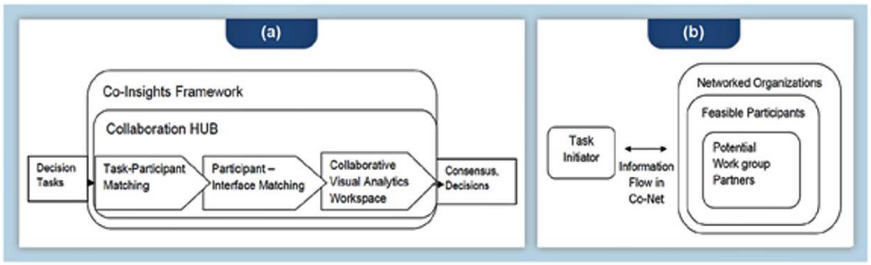

Figure 2. CCT augmentation of collaborative decisions by Co-Insight: (a) The Co-Insight framework with task-participant matching (deciding with whom to collaborate for a specific decision problem); participant-interface matching (deciding which tool to apply for a specific decision analysis); within a collaborative, visual analytics workspace. (b) The role of participating collaborators in a Co-Net for Co-Insight. (Source: [60])

Research has shown that building and augmenting the CI of participants in cyber-physical systems and in CDSS can provide better support for achieving both their individual and their common, organizational objectives. A definition and formal quantitative measure of CI have been developed ([61]). They are based on the definitions of the two key elements, collaboration and intelligence. Three recent case studies of collaborative decision support systems have been analyzed relative to their formal level of CI. The three cases involve limited, though non-finite groups of collaborating human and non-human entities.

Case 1. Collaborative Design

Telerobot-enabled, computer supported collaborative design under CLM, collaborative life-cycle management, was modeled and experimentally studied in a lab. Novice and experienced designers collaborated over a HUB with CAD systems, CAE systems, control software development, and a remote robot to collaboratively design and test

Table 4. Interactions impacting intelligence to improve decision processes and decisions quality features (Source: [61]; adapted from [15])

Type	Definition	Methods	Benefits	Example
Communication intelligence	The ability to communicate between humans, machines, and human - machine systems	Effectively communicate between multiple entities	Increase communication efficiency, and reduce cost and time	Negotiation and making contracts between business partners; process outsourcing
Cumulative intelligence	The ability to accumulate knowledge and form a base for decision-making	Make decisions based on present state and also previous states	Identify positive and negative strategies from historical data	Research activities (standing on the shoulders of giants)
Cooperative intelligence	The ability to deal with cooperation among multiple partners	Prevent conflicts during real time parallel execution	Reduce down time and interruptions due to conflicting actions	Concurrent assembly operations along a conveyor
Collective intelligence	The ability to integrate intelligence from a group/organization and to act, even approximately, as a single, rational agent	Collect and combine knowledge from disparate sources	Increase the probability of better decisions	Trade promotions suggestions for client and customer
CI, Collaborative Intelligence	The ability of an entity or a group to collaborate well with others	Collaborate towards a set of common goals while keeping autonomy	Streamline the workflow of information and operations for overall welfare	Between knowledge intensive clients and e-Service providers; between retailers, manufactures & suppliers (Nestle, Wal-Mart)

an electronic assembly. The decisions they made throughout the experiments with and without CCT augmentation were evaluated. The level of CI in the experiments was measured and evaluated. Metrics of improved collaboration and improved quality of design decisions were analyzed (e.g., [57]).

Case 2. Supply Network Control

Collaborative product line control in a global supply network of paper and related products was designed based on CCT augmentation of the decision support. Certain decisions were automated for real-time alerts, batch order rescheduling and resequencing, and feedforward process control adjustments. Selectively, some of these control decisions were escalated to human-in-the-loop supervisors. Performance metrics of decision and control processes, of their CI, and decision and control impact on production quality were analyzed and evaluated (e.g., [46]).

Case 3. Demand and Capacity Sharing

Collaborating enterprises can benefit from sharing demands for their products and services, and supply capacities that are available to them. On-going negotiations and interactions about such sharing between those that have, from time to time, excess unused capacity, or temporary decline in demand, can yield significant mutual advantages. For instance, airlines sharing their equipment and passengers dynamically ("code sharing"), and automotive suppliers, one having excess storage capacity, while another is having surplus of unsold vehicles and lacking storage space. Such demand and capacity sharing decisions were designed with CCT augmentation, and the level of CI and corresponding decisions' quality and decision process metrics were evaluated (e.g., [29], [48], [53]).

The five collaboration metrics were assessed in the above three cases, and overall observations are summarized in Table 5. Based on these observations, the benefits based on these metrics have been measured and shown with statistical significant to yield advantages when a higher formal levels of CI are enabled as shown.

Table 5: Measured impacts of the formal level of CI on key CDSS metrics (Source: [38], [61])

Collaboration Metrics	*Case 1. Collaborative Design*	*Case 2. Supply Network Control*	*Case 3. Demand-Capacity Sharing*
Decisions quality, service level, effectiveness, stability	- Improved design quality - Improved design robustness	- Lower throughput variability - Reduced work-in-progress	- Improved stability of interactions - Improved stability of integration decisions
Information availability	Improved through Co-Insight	Improved through Co-Insight	Improved through Co-Insight
Timely completion	Less time to complete design tasks	- Increased throughput - Reduced work-in-progress	- Improved resource utilization - Reduced cost of mismatch
Multiple views	Enabled through Co-Insight	Enabled through Co-Insight	Enabled through Co-Insight

Continuation of Table 5

Collaboration Metrics	*Case 1. Collaborative Design*	*Case 2. Supply Network Control*	*Case 3. Demand-Capacity Sharing*
Multiple engagement	Embedded on HUB-CI with decision support alerts	- Automatic engagement as needed of sensors and knowledge-bases - Alert-based interactions with line supervisors as needed	- Automatic engagement as needed of sensors and knowledge-bases - Alert-based interactions with enterprise agents and supervisors as needed
Cost of Collaboration	Errors and conflicts removed at earlier stages of design	Minimized impact of disruptions	Reduced cost of mismatch

5 Conclusions

For CDSS, Collaborative Decision Support Systems to function effectively and to deliver high quality decisions over time, effective collaboration support is essential ([19]). In this article, the contributing power of CCT, the Collaborative Control Theory and its associated cyber tools to augment collaboration ([38]) by multiple decision participants are explored. Beyond traditional CSCW protocols and methods, that address mostly human decision makers, CCT augmentation of collaboration incorporated multiple human decision makers and multiple software and hardware agents, sensors, robots, and other automated instruments.

The design principles of CCT and their associated collaboration protocols are discussed, with their specific contributions to solve and alleviate risks and weaknesses common in collaboration for CDSS. The CCT-based Co-Insight framework and Collaborative Intelligence (CI) are presented as additional major components that can improve and enable productive and effective CDSS.

Three case studies implementing CTT principles, protocols, and Co-Insight are described based on recent research on the correlation between CI and its impacts on decision process and decision quality. According to these case studies and research results, the correlation is positive, meaning that with greater levels of CI along time, better decision processes and decisions quality can be gained. In addition, this research has provided experimental methods that are available for further research as follows.

While it can be intuitive that higher levels of collaboration and higher levels of CI can lead to better performance based on better decisions, it is still necessary to establish the limits and appropriate levels that are optimal, or best in terms of cost and benefits. Specifically:

1. What are the best ways to create, foster, adaptively adjust, and sustain collaboration processes and level of resulting CI throughout the lifecycle of given decision support systems and the systems those decisions are meant to optimize?

2. It has been proven that optimal performance of the CDSS is typically attained with optimal but selective levels of collaboration and of CI; what are the ways to simplify and optimize, not maximize those levels?

Future research in these directions is anticipated by the CDSS, CSCW, and CCT communities. And already CDSS are implemented and positively influencing large scale, connected enterprises and cyber physical infrastructure and networks.

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The Physical-mathematical Theory of Hyper-random Phenomena

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Abstract

We give the survey of the researches dedicated to the statistical stability phenomenon and the physical-mathematical theory of hyper-random phenomena that takes into account the violation of statistical stability. It is presented the study technique of statistical stability, the results of the theoretical and experimental investigations of statistical stability of various processes, the mathematical apparatus of the theory of hyper-random phenomena, the particularities of mathematical statistics of hyper-random variables (including ones connected with the law of large numbers and the central limit theorem), and the explanation why the accuracy of actual measurements is limited. The description is constructed on the comparison of the theory of hyper-random phenomena with the probability theory.

Keywords: Phenomenon of statistical stability; Probability theory; Theory of hyper-random phenomena; Physical process; Violation of convergence.

1 Introduction

For description of mass physical phenomena in uncertainty conditions different mathematical and physical-mathematical theories are used.

Between these two types' theories there is essential difference: in mathematical theories the physical entity is ignored and in physical-mathematical ones it plays a key role. *Subject matter* of a mathematical theory is *abstract mathematical objects* and *subject matter* of a physical-mathematical theory is *physical phenomena* of the *real physical world*.

Classical probability theory based on Kolmogorov's axioms [1] is a typical example of mathematical theory, the *subject matter* of which is *abstract probability space.* Theories *exploiting uncertainty approach* and *approximate reasoning* (in particular, *imprecise probability theory* [2, 3], *interval analyses* [4], *interval probability theory* [5], *robust Bayesian analysis* [6, 7], *probability box theory* [8], *robust Neyman-Pearson theory* [9], *Huber's robust statistics* [10], etc.) are of *mathematical type* too (Table 1).

Table 1. Theories describing mass physical phenomena in uncertainty conditions

	Mathematical theories	**Physical-mathematical theories**
Probability approach	Probability theory as *mathematical discipline* (A.N. Kolmogorov)	Probability theory as *physical discipline* (D. Hilbert)
Uncertainty approach	*Mathematical theories* based on approximate reasoning	*Physical-mathematical theory of hyper-random phenomena*

Besides mathematical interpretation of the probability theory it is known the alternative one, the follower of which was David Hilbert. He and many other scientists regarded the probability theory as *physical discipline* [11]. Although the physical approach is less popular now among mathematicians but it is very popular among engineers and physicists. The *subject matter* of the physical-mathematical probability theory is *statistical stability of actual mass phenomena.*

The *probability theory* has the centuries-old development history. During this time it has established itself as the most powerful tool solving various statistical tasks. There is even opinion that any statistical problem can be effectively solved within the paradigm of the probability theory. However, as it turned out, it is not so.

Some conclusions of the probability theory do not accord to the experimental data. A typical example concerns the potential accuracy

of measurement. If systematic error absents, according to the probability theory (*Cramer–Rao inequality* [12, 13]), with increasing of the number of measurement results of any physical quantity the error of the averaged estimator follows to zero. But every engineer or physicist knows that the actual measurement accuracy is always limited and to overcome the limit by the statistical averaging of the data is not possible.

Study of the causes of discrepancies between the theory and practice led to the understanding that the problem is related to the *unjustified idealization of the phenomenon of statistical stability*.

The modern probability theory regarded as a *physical-mathematical* one has mathematical and physical components [14]. The mathematical component is based on the A.N. Kolmogorov's classical axioms while the physical component is based on physical hypotheses, in particular the *hypothesis of ideal* (*perfect*) *statistical stability of actual events, variables, processes, and fields* assuming the *convergence of statistics* when the sample size goes to infinity.

The results of numerous experimental studies of various physical quantities and processes over long observation intervals have shown that the hypothesis of perfect statistical stability *is not confirmed experimentally*.

For relatively short temporal, spatial, or spatio-temporal observation intervals, an increase in data volume usually reduces the level of fluctuation in the statistics. However, when the volumes become very large, this tendency is no longer visible, and once a certain level is reached, the fluctuations remain practically unchanged or even grow. This indicates *a lack of convergence for real statistics* (*their inconsistency*).

If the volume of processing data is small, the violation of the convergence practically does not influence on the results, but if this volume is large, the influence is very significant.

The study of violations of statistical stability of physical processes and the development of an effective way for description of the actual world with taking into account such violations has resulted in the construction of the *physical-mathematical theory of hyper-random phenomena* (Table 1).

The *subject matter* of this theory as well as the *physical-mathematical probability theory* is *statistical stability of actual mass phenomena.* The *scope of study* of it is the *violation of statistical* stability among the characteristics and parameters of real physical phenomena.

The theory of hyper-random phenomena consists of physical and mathematical components. Mathematical component is based on the Kolmogorov's axioms and constructed on the scheme of the classical probability theory. However it accumulates knowledge obtained in the framework of a number of adjacent *mathematical theories exploiting approximate reasoning.*

Physical component of the theory is based on the hypotheses that essentially differ from the physical hypotheses of the physical-mathematical probability theory, in particular, on the *hypothesis of limited statistical stability* assuming the *absence of convergence of actual statistics.*

The theory of hyper-random phenomena began to develop at the end of the XX century. Quite a few scientific works concerning this theory are written. The publication list, in particular, includes eight monographs [14–21], two of which [14, 21] are written in English.

The *purpose* of this survey article is to *present main results concerning modern investigation of the phenomenon of statistical stability* and *to compare two approaches for its description* proposed by *probability theory* and *theory of hyper-random phenomena.*

In Sect. 2 we familiarize with the manifestations of the statistical stability phenomenon and two physical hypotheses: perfect and imperfect statistical stability.

Sect. 3 is devoted to description of Hilbert's sixth problem and its solution in the part of statistical stability proposed by the probability theory and the theory of hyper-random phenomena.

Sect. 4 presents the investigation technique of statistical stability on infinite and finite observation intervals as well as the results of theoretical researches of statistical stability of stochastic processes and the experimental investigations of statistical stability of actual processes of various physical nature.

Sect. 5 familiarizes with the *mathematical apparatus* used in the

theory of hyper-random phenomenon for description of real physical events, quantities, processes, and fields in conditions of imperfect statistical stability. The mathematical apparatus is developed for hyper-random events, scalar and vector hyper-random variables, scalar, vector, stationary, and ergodic hyper-random functions, hyper-random differential equations, transformations of hyper-random variables and functions. The special part is devoted to mathematical statistics of hyper-random phenomena. We do not describe all these questions (they are presented in detail, in particular, in the monographs [14, 16, 17]). To obtain general representation about the developed mathematical approaches and main theoretical results we consider briefly only the *description of scalar hyper-random variables*, the particularities of mathematical statistics of hyper-random variables, notions of *generalized limit* and *convergence of sequences in the generalized sense*, *generalized law of large numbers*, and *generalized central limit theorem*.

Sect 6 concerns the *engineering* and *practical questions*. We describe here the *classic determinate–random measurement model* rested upon the probability theory and the *determinate–hyper-random one* based on the theory of hyper-random phenomena, compare these two models, and present estimation results of *potential measurement accuracy of physical quantities* calculated with using these models.

Note, from the issues presented in the article a special place occupies the questions concerning *experimental research of violation of statistical stability of actual processes*. The results of these investigations *give physical grounds* for correct using in practice not only hyper-random mathematical models but *other mathematical models based on approximate reasoning principles*.

2 The Physical Phenomenon of Statistical Stability

2.1 Manifestation of the Phenomenon of Statistical Stability

The statistical stability is manifested in stability of relative frequency of mass events. The first to draw attention to the phenomenon of

statistical stability was the cloth merchant J. Graunt in 1662 [22]. Information about research on statistical stability is fragmentary for the period from the end of the XVII century to the end of the XIX century, e.g., by J. Bernoulli, S.D. Poisson, I.J. Bienayme, A.A. Cournot, L.A.J. Quetelet, J. Venn, etc. [23, 24].

Systematic study of statistical stability began at the end of the XIX century. In 1879, the German statistician W. Lexis made the first attempt to link the concept of statistical stability of the relative frequency with the dispersion [23]. At the turn of the century and in the early XX century, statistical stability was studied by C. Pearson, A.A. Chuprov, L. von Bortkiewicz, A.A. Markov, R.E. von Mises, and others [23, 24].

A lot of well known scientists led experimental investigations of the statistical stability of relative frequency of mass events. It is known, for example, that coin-tossing experiments were studied by P.S. de Laplace, G.L.L. de Buffon, K. Pearson, R.P. Feynman, A. de Morgan, W.S. Jevons, V.I. Romanovskiy, W. Feller, and others. At the first glance, this quite a trivial task per se was not presented for them.

A new stage of experimental research began in the late XX century. The necessity for additional studies is called due to the new applied tasks and the detection of a number of phenomena that can not be satisfactorily explained and described within the framework of the classical probability theory. The new tasks are, in particular, the *ultra-precise measurement* of physical quantities and *ultra-precise forecasting* of developments over large intervals of observation. To the relatively new phenomena can be led, for instance, an *unpredictable* measurement *progressive* (*drift*) *error* [25, 26], as well as a flicker noise [27], which is detected everywhere and *can not be suppressed by averaging the data*.

The phenomenon of statistical stability is manifested also in the stability of the average $y(t)$ of the process $x(t)$ and its sample mean $y_n = \frac{1}{n}\sum_{i=1}^{n} x_i$, where $x_1, \ldots, x_n$ are discrete samples of the process $x(t)$.

Interesting that this phenomenon occurs in case of averaging of the fluctuations that are of different types, in particular, of the *stochastic*, *determinate*, and *actual physical processes*.

Example 1. In Fig. 1**a** and Fig. 1**c** a realization of noise with uniform power spectral density (white noise) and a determinate periodical process are presented. In Fig. 1**b** and Fig. 1**d** the dependencies of the according averages on the averaging interval are shown. As can be seen from Fig. 1**b** and Fig. 1**d**, when the averaging interval increases, fluctuations in the sample mean decrease and the average value gradually stabilizes.

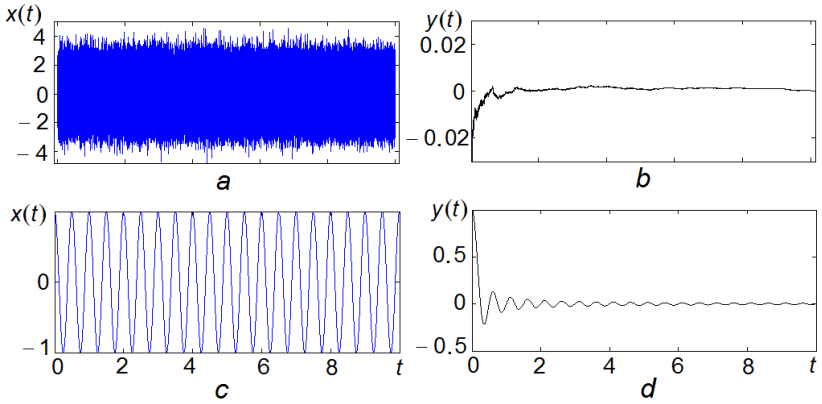


Figure 1. Realization of white Gaussian noise (**a**) and harmonic oscillation (**c**), together with the dependencies of the corresponding sample mean on the average interval (**b**, **d**)

Example 2. Fig. 2**a** and Fig. 2**b** show how the mains voltage in a city fluctuates quickly, while the average changes slowly. As the averaging interval increases from zero to one hour, the average voltage stabilizes (Fig. 2**b**).

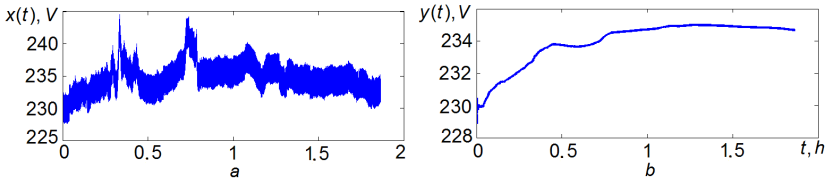


Figure 2. Dependence of the mains voltage (**a**) and the corresponding average (**b**) on time over 1.8 hours

The phenomenon of statistical stability is observed in calculation of other statistics too, in particular, the *sample standard deviation* $z_n = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - y_n)^2} \quad (n = 2, 3, \ldots)$.

2.2 The Hypothesis of Ideal Statistical Stability

Taking into account the statistical stability of the relative frequency of *actual physical events and actual statistics* it seems naturally to assume that if the number of the test n infinitely increases, the fluctuation level of the relative frequency $p_n(A)$ of *any actual event* A tends to zero, and also that in the unlimited increasing of the sample size n (increasing the observation time t) the fluctuation level of the sample mean y_n of any *random or real physical oscillation* $x(t)$ follows to zero too.

In other words, it is *possible to hypothesize* that there is a *convergence* of the sequence of the relative frequencies $p_1(A),\ p_2(A), \ldots$ of *any actual event* A to some *determinate value* $P(A)$ and there is a *convergence* of the sequence of averages $y_1,\ y_2, \ldots$ of *any stochastic or actual process* to *determinate value* m, viz. the limit of the relative frequency $\lim\limits_{n\to\infty} p_n(A) = P(A)$, and the limit of the average $\lim\limits_{n\to\infty} y_n = m$.

The modern probability theory is based on this *hypothesis of ideal* (*perfect*) *statistical stability* or, in other words, on the *assumption of convergence of statistics*.

The value $P(A)$ is interpreted in practice as the *probability* of the event A, and the value m is regarded as the *expectation* of the process $x(t)$.

2.3 The Hypothesis of Imperfect Statistical Stability

For many years it *was believed that the hypothesis of perfect statistical stability adequately reflects the reality.* Although some scholars (even the founder of axiomatic probability theory A.N. Kolmogorov [1, 28, 29] and such famous scientists as A.A. Markov [30], A.V. Skorokhod [31], E. Borel [32], V.N. Tutubalin [33], and others) noticed that, *in the real world, this hypothesis is valid only with certain reservations.*

Pay attention, the convergence of the relative frequency and other statistics is only a hypothesis. It does not follow from any experiments and any logical inferences. Not all processes, even of oscillatory type, have the property of perfect statistical stability.

Experimental studies of various processes of different physical nature over broad observation intervals show that the hypothesis of perfect statistical stability *is not confirmed*. The real world is continuously changing, and changes occur at all levels, including the statistical one. Statistical assessments formed on the basis of relatively small observation intervals are relatively stable. Their stability is manifested through a decrease in the fluctuation of statistical estimators when the volume of statistical data grows. This creates an *illusion* of perfect statistical stability. However, beyond a certain critical volume, the level of fluctuations remains practically unchanged (and sometimes even grows) when the amount of the data is increased. This indicates that the *statistical stability is not perfect*.

Example 3. Non-perfect statistical stability is illustrated in Fig. 3 [14] which presents mains voltage fluctuations over 2.5 days. Note, the fluctuation in Fig. 2**a** shows the beginning part of the fluctuation presented in Fig 3**a**. As can be seen from Fig. 3**b**, the sample average does not stabilize, even for very long averaging intervals.

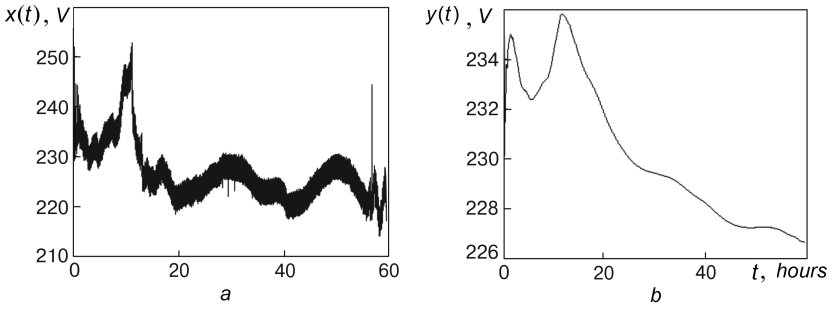


Figure 3. Dependence of the mains voltage (**a**) and the corresponding average (**b**) on time over 60 hours

3 Hilbert's Sixth Problem and Approaches for Its Solving

3.1 Description of the Phenomenon of Statistical Stability in the Framework of Probability Theory

Prior to the early twentieth century, probability theory was regarded as a *physical theory*, which described the phenomenon of statistical stability. Then at the beginning of the last century, the problem of axiomatizing probability theory was raised. In fact, David Hilbert formulated this as part of the problem of axiomatizing physics (the Hilbert's sixth problem) [11].

Many famous scientists tried to solve the problem and various approaches were proposed. Today, the most widely recognized approach is the set-theoretic one [1] developed by A.N. Kolmogorov in 1929 [34]. This approach has even been elevated to the rank of a standard [35].

The basic notion in Kolmogorov's probability theory is the notion of a *random event*. Random events are regarded as mathematical objects, described by means of a *probability space* defined as a triad $(\Omega, \Im, P)$, where Ω is the *space of elementary events* $\omega \in \Omega$, $\Im$ is a *σ-algebra of subsets of events* (*Borel field*), and P is a *probability measure on subsets of events*.

For any random event A the *probability* $P(A)$ is defined by the following three *axioms*:

1) the probability of any event A is a non-negative number, i.e. $P(A) \geq 0$;

2) for pairwise disjoint events A_1, A_2, ... (both finite and *countable*), the probability of their union is the sum of the probabilities of the events, i.e. $P(\bigcup\limits_n A_n) = \sum\limits_n P(A_n)$;

3) the probability of the event Ω is equal unity (i.e. $P(\Omega) = 1$).

A *random variable* X is regarded as a measurable function defined on the space Ω of elementary random events ω, while a *random* (*stochastic*) *function* $X(t)$ is a function of an independent argument t, whose value is a random variable when this argument is fixed.

A *random phenomenon* is understood as a *mathematical object*

(a random event, random variable, or stochastic function), which is exhaustively characterized by some specific probability distribution law. In particular, a random event is described exhaustively by the *probability*, a random variable X – by the *distribution function* $F(x) = P\{X < x\}$, where $P\{X < x\}$ is the probability of the inequality $X < x$, and a scalar random function $X(t)$ – by the *distribution function* $F(\vec{x};\vec{t}) = P\{X(t_1) < x_1, ..., X(t_L) < x_L\}$, where $\vec{x} = (x_1, ..., x_L)$ is the L-dimensional vector of values of the function $X(t)$ at times $t_1, ..., t_L$ represented by the L-dimensional vector $\vec{t} = (t_1, ..., t_L)$.

Note that a *phenomenon* or *mathematical model, not described by specific distribution law is not considered to be random.* This is an extremely important point that must be taken into account.

In probability theory, the probability of an event is a key concept. Note that, in Kolmogorov's definition, it is an *abstract mathematical concept.* Using a more visual statistical definition due to R. von Mises [36], the probability $P(A)$ of a random event A is interpreted as a limit of the relative frequency $p_N(A)$ of the event, when the experiments are carried out under identical statistical conditions and the number N of experiments tends to infinity. When N is small, the relative frequency $p_N(A)$ can fluctuate greatly, but with increasing N, it gradually stabilizes, and as $N \to \infty$, it tends to a definite limit $P(A)$.

All mathematical theories, including the version of probability theory based on Kolmogorov's axioms, are related to *abstract mathematical concepts* which *are not associated with the actual physical world.* In practice, these theories can be successfully applied if we admit certain *physical hypotheses* asserting the adequate description of real world objects by relevant mathematical models. For probability theory, such *physical hypotheses* are as follows [14]:

Hypothesis 1 *For mass phenomena occurring in the real world, the relative frequency of an event has the property of ideal (perfect) statistical stability, i.e., when the sample volume increases, the relative frequency converges to a constant value.*

Hypothesis 2 *Mass phenomena are adequately described by random models which are exhaustively characterized by distribution functions.*

It is often assumed that the hypothesis of perfect statistical stability

is valid for any physical mass phenomena. In other words, a *stochastic concept of world structure is accepted*.

Kolmogorov's axioms with added Hypotheses 1 and 2 solve Hilbert's sixth problem in the part of axiomatizing of the probability theory as physical discipline.

3.2 Description of the Phenomenon of Statistical Stability in the Framework of Theory of Hyper-random Phenomena

In Sect. 2.3, attention was drawn to the fact that the experimental study of real physical phenomena over broad observation intervals does not confirm the hypothesis of perfect statistical stability (Hypothesis 1). For a correct application of the classical probability theory in this case, it is sufficient in principle to replace Hypothesis 1 by the following:

Hypothesis 1' *For real mass phenomena, the relative frequency of an event has the property of limited statistical stability, i.e., when the sample volume increases, the relative frequency does not converge to a constant value.*

The replacement of Hypothesis 1 by Hypothesis 1' leads to considerable mathematical difficulties due to the *violation of convergence*. There are different ways to overcome them. The development of one of these led to the *physical-mathematical theory of hyper-random phenomena* [14].

In classical probability theory, the basic mathematical entities are random events, random variables, and random functions. In the theory of hyper-random phenomena, the analogues of these basic entities are *hyper-random events*, *hyper-random variables*, and *hyper-random functions*, which are *sets* of *non-interconnected* random events, random variables, and stochastic functions, respectively, each regarded as a comprehensive whole.

A *hyper-random event* can be described by a tetrad $(\Omega, \Im, G, P_g)$, where Ω is a space of elementary events $\omega \in \Omega$, $\Im$ is a Borel field, G is a set of conditions $g \in G$, and P_g is a probability measure on subsets of events, depending on the condition g. Thus, the probability measure

is defined for all subsets of events and all possible conditions $g \in G$. Note that the *measure for conditions $g \in G$ is not determined.*

Using a statistical approach, a hyper-random event A can be interpreted as an event whose relative frequency $p_N(A)$ *is not stabilized by growth of the number N, and which has no limit when $N \to \infty$.*

It is essential to understand that the hyper-random events, variables, and functions (*hyper-random phenomena*) are *many-valued objects exhaustively characterized by the sets of non-interconnected probability measures.* Hence,

- a hyper-random event is described exhaustively by the *collection of probabilities*;
- a hyper-random variable $X = \{X_g,\ \ g \in G\}$ is described exhaustively by the *collection of conditional distribution functions* $F(x/g)$ with conditions $g \in G$, forming the *many-valued distribution function* $\tilde{F}(x) = \{F(x/g),\ \ g \in G\}$[1], where $X_g = X/g$ is a random variable subject to the condition g, and the set G can be finite, countably infinite, or uncountable;
- a scalar hyper-random function $X(t) = \{X_g(t),\ \ g \in G\}$ is described exhaustively by the *collection of conditional multidimensional distribution functions* $F(\vec{x};\vec{t}/g)$ with conditions $g \in G$, forming the *many-valued distribution function* $\tilde{F}(\vec{x};\vec{t}) = \left\{F(\vec{x};\vec{t}/g),\ \ g \in G\right\}$, where $X_g(t) = X(t)/g$ is a random function subject to the condition g.

For correct use of the theory of hyper-random phenomena, one must also adopt the following hypothesis, in addition to Hypothesis 1'.

Hypothesis 2' *Mass phenomena are adequately described by hyper-random models which are exhaustively characterized by the sets of distribution functions.*

The assumption that these hypotheses are valid for a wide range of mass phenomena leads to *a world-building concept based on hyper-random principles.*

[1] A tilde under a letter indicates that the object described by the letter is many-valued.

So the *mathematical part of the theory of hyper-random phenomena* is based on the *classical axioms of probability theory*, and the *physical part* – on *Hypotheses 1' and 2'.*

Note, in contrast to the classical Kolmogorov's mathematical probability theory, the theory of hyper-random phenomena is *physical-mathematical* one. Its subject matter is phenomenon of statistical stability and the scope of research is adequate description of it by *hyper-random models* (*hyper-random phenomena*) taking into account the violation of statistical stability.

Since the mathematical part of the theory of hyper-random phenomena uses the system of mathematical axioms of probability theory, from the mathematical standpoint it is a *branch* of classical probability theory. But from the physical point of view, the theory of hyper-random phenomena is a new *physical theory* based on new physical hypotheses.

In general, *the theory of hyper-random phenomena* can be regarded as a *new physical-mathematical theory* constituting a *complete solution of Hilbert's sixth problem in the context of statistical stability.*

4 The Investigation of the Statistical Stability Violation

4.1 Formalization of the Statistical Stability Concept

Curiously enough is that the concept of statistical stability was not formalized until recent time. First of all note, a data statistically stable with respect to some statistics can be unstable with respect to other statistics. This means that the statistical stability is an attribute not only of a data, but also of the statistics. In addition, the level of statistical stability depends on the number of the data and on the sequence of this data.

It was proposed the number of parameters characterizing statistical stability violation. For the *random sequence* $X_1, \ldots, X_N$ most useful are parameters of statistical instability *with respect to the average* γ_N and *respect to the sample standard deviation* Γ_N described by the fol-

lowing expressions: $\gamma_N = \mathrm{E}\left[\bar{D}_{Y_N}\right]/\mathrm{E}\left[\bar{D}_{X_N}\right]$, $\Gamma_N = \mathrm{E}[\bar{D}_{Z_N}]/\mathrm{E}[\bar{D}_{X_N}]$, where $\bar{D}_{Y_N} = \frac{1}{N-1}\sum\limits_{n=1}^{N}(Y_n - \bar{m}_{Y_N})^2$ is the sample variance of the fluctuations in the average $Y_n = \frac{1}{n}\sum\limits_{i=1}^{n} X_i$ $(n = \overline{1,N})$, $\bar{m}_{Y_N} = \frac{1}{N}\sum\limits_{n=1}^{N} Y_n$ is the sample mean of the average fluctuations, $\bar{D}_{Z_N} = \frac{1}{N-2}\sum\limits_{n=2}^{N}(Z_n - \bar{m}_{Z_N})^2$ is the sample variance of the fluctuations in the sample standard deviation $Z_n = \sqrt{\frac{1}{n-1}\sum\limits_{i=1}^{n}(X_i - Y_n)^2}$ $(n = \overline{2,N})$, $\bar{m}_{Z_N} = \frac{1}{N-1}\sum\limits_{n=2}^{N} Z_n$ is the average of the sample standard deviations, $\bar{D}_{X_N} = \frac{1}{N-1}\sum\limits_{n=1}^{N}(X_i - Y_N)^2$ is the sample variance of the initial sequence.

The actual range of the parameters γ_N, Γ_N is $[0,\infty)$. The smaller the values of the parameters γ_N and Γ_N the more stable the sequence with respect to average and standard deviation respectively. Small values for large sample sizes N point to high statistical stability of the sequence, and large values point to statistical instability.

Random samples, for which the *parameters of statistical instability* γ_N and Γ_N do not follow to zero, are considered to be *statistically unstable with respect to the average and standard deviation* respectively.

Any measurement procedure consists in the comparison of the measurement result with some unit. For quantitative characterizing of the degree of instability, the *measurement units* are requested, the comparison with which would allow judging about the degree of instability in respect to the average and standard deviation. As the role of the *measurement unit*, a variable γ_{0N} can play, that is the parameter γ_N calculated for the *standard statistically stable sequence of uncorrelated samples of white Gaussian noise*.

The absolute level of statistical instability with respect to the average and standard deviation in units γ_{0N} characterize the parameters of the statistical instability h_N and H_N described by the following expressions: $h_N = \gamma_N/\gamma_{0N}$, $H_N = \Gamma_N/\gamma_{0N}$. The actual range of theses parameter is $[0,\infty)$. The measurement unit of them is the number

$h_{0N} = 1$ that does not depend on the sample size.

For solving of the practical tasks it is usually not important the behavior of statistics on the infinite observation interval, though it is laid in the basis for a formal definition of statistical stability. More important the behavior of statistics on the *actual observation interval*: the presence or absence of the *trends* indicating a violation of statistical stability. *If on the observation interval these trends are not tracing, the process can be considered as statistically stable, but otherwise as statistically unstable.*

Various statistics and processes, as a rule, have different *statistical stability intervals*. The concepts of the interval of statistical stability with respect to the average τ_{sm} and of the interval of statistical stability with respect to the standard deviation τ_{sd} can be formalized by the *statistical stability borders of the confidence intervals*.

For the parameters of the statistical instability γ_N and Γ_N the statistical stability upper border of the confidence interval is given by $\gamma^+_{0N} = \gamma_{0N} + \varepsilon\sigma_{\gamma^*_{0N}}$, where ε is the confidence parameter that determines the width of the confidence interval and $\sigma_{\gamma^*_{0N}}$ is the standard deviation of the variable $\gamma^*_{0N} = \bar{D}_{Y_N}/\mathrm{E}[\bar{D}_{X_N}]$ calculated for standard statistically stable sequence.

The *criteria of statistical stability violation with respect to the average and with respect to the standard deviation* (that determine the amounts of the *intervals of statistical stability* τ_{sm} and τ_{sd}) can be that the parameters γ_N and Γ_N go beyond the border γ^+_{0N} or the parameters h_N and H_N go beyond the border $h^+_{0N} = \gamma^+_{0N}/\gamma_{0N}$.

In practical work, due to the limited amount of data, instead of the parameters of statistical instability γ_N, h_N and Γ_N, H_N, we have to admit using of the appropriate estimates γ^*_N, h^*_N and Γ^*_N, H^*_N.

4.2 The Statistical Stability of Stochastic Processes

4.2.1 Dependence of the Statistical Stability on the Process's Spectrum

Studies show that the statistical stability of a stochastic sequence (process) with respect to the average and standard deviation is determined

by its *spectrum.*

In particular, for the sequence $X_1, \ldots, X_N$ with zero expectation and power spectral density $S_{x_N}(k)$ the *parameter of statistical instability with respect to the average* γ_N when $N \to \infty$ is described by the following asymptotic formula:

$$\gamma_N = \frac{\sum\limits_{k=2}^{N/2} \frac{1}{(k-1)^2} \left[\frac{\pi^2}{4} + (C + \ln(2\pi(k-1)))^2\right] S_{x_N}(k)}{4\pi^2 \sum\limits_{k=2}^{N/2} S_{x_N}(k)},$$

where k is the spectral sample number ($k = \overline{1, N}$), C is the *Euler–Mascheroni constant* ($C \approx 0.577216$).

4.2.2 Stochastic Processes Whose Spectrum is Described by a Power Function

In many cases, actual noise is well approximated by random processes whose power spectral density is described by a *power function* $1/f^\beta$ for various values of the shape parameter β, where f is frequency. Such noise sometimes is called a *color noise.* One thus speaks of violet, blue (cyan), white, pink, brown (red), and black noise that corresponds to $\beta = -2, -1, 0, 1, 2,$ and > 2.

Flicker noise and *fractal* (*self-similar*) *processes* are other examples of the processes with power spectral density described by power functions.

Taking into account the prevalence of the processes with power spectral density described by a power function the research of their statistical stability was carried out.

Studies show that the *process with power spectral density described by a power function is statistically stable with respect to the average and with respect to standard deviation if* $\beta < 1$ *and statistically unstable if* $\beta \geq 1$.

Since the state of statistical stability of the process changes at the point $\beta = 1$, the process with this particular parameter value can

be regarded as a *limiting unstable process with respect to average and standard deviation* (*limiting unstable in broad sense*).

Investigations show that *if* $\beta \leq 0$, *the process is more stable with respect to the average, than with respect to the standard deviation* ($\gamma_N < \Gamma_N$), and *if* $\beta > 0$, *on the contrary, it is less stable* ($\gamma_N > \Gamma_N$).

Summarizing these results it is possible to mark the following (see Fig. 4):

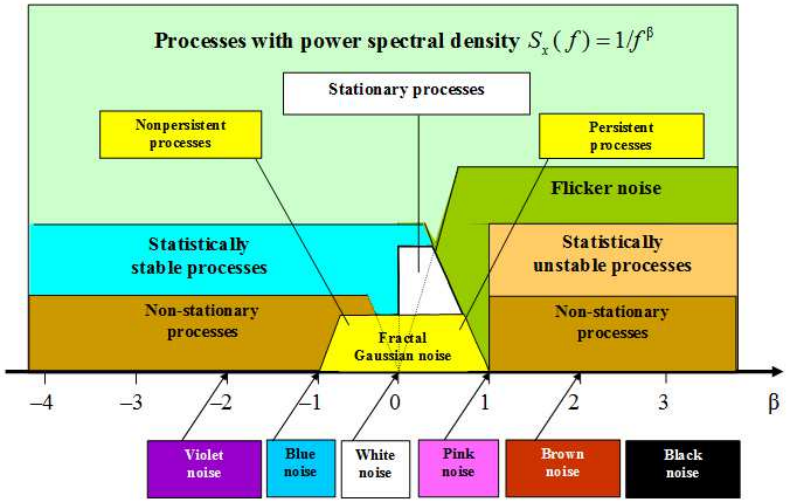


Figure 4. Processes with power spectral density described by a power function

- *statistically stable with respect to the average and standard deviation* (*stable in the broad sense*) are stationary processes, a part of the non-stationary processes, the so called fractal Gaussian noise, a part of the flicker noise, as well as violet, blue, and white noise;

- *statistically unstable with respect to the average and standard deviation* (*unstable in the broad sense*) are a part of the non-stationary processes, a part of the flicker noise, as well as pink, brown, and black noise.

4.2.3 Dependence of Statistical Stability on Other Process's Particularities

Investigation of dependence of statistical stability of the processes on the correlation of the samples shows that a *positive correlation* between the samples leads to a *decrease* in the statistical stability, and a *negative correlation,* to an *increase*.

Studies show [18] that violations of statistical stability occur not only in the case of low-frequency processes, but also for *narrowband stochastic processes* too.

Not only the non-stationary but *stationary in a narrow sense* stochastic processes can be statistically unstable in a broad sense. The statistically unstable processes, for example, are stationary stochastic processes, cut sets of which are described by distributions that do not have any moments or do not have moments higher than of the first order (processes described by Cauchy, Pareto, Fischer–Snedecor, Frechet, and et al. distributions).

Violation of statistical stability can have many causes. These include the *inflow into an open system of matter, energy, and* (*or*) *information* feeding non-equilibrium processes, various *nonlinear transformations*, *low-frequency linear filtering* of special type, etc. It is shown that, as the result of low-frequency filtration, broadband stationary and statistically stable noise can be transformed into a statistically unstable process.

4.3 The Results of Experimental Investigations of the Statistical Stability of Actual Processes of Various Physical Nature

To find out whether the *actual processes* are statistically stable or not, and if they are unstable, on the whole, but at what observation interval they can be considered as stable ones, various actual physical processes were studied over long observation intervals.

For instance, it is investigated the supply-line voltage. The active (effective) voltages were recorded in the computer memory and then analyzed. Recording sessions were conducted over two months, with

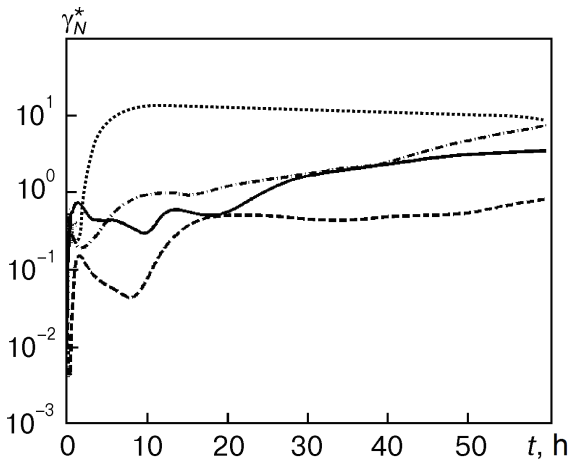


Figure 5. Variations in the estimate γ_N^* of the statistical instability parameter during a 60-hour observation of the mains voltage in four sessions

breaks of a few days. The duration of each session was about 60 hours. One of such records is presented in Fig. 3**a.** The estimates of the statistical instability parameter γ_N^* with respect to the average calculated for four sessions are shown in Fig. 5.

It follows from the figure that, for long observation times, the instability parameter does not show any tendency to fall to zero. Consequently, the *mains voltage is statistically unstable.* The *statistical stability interval with respect to average* τ_{sm} *of the mains voltage is approximately an hour.*

In the same mane it has been investigated statistical stability of a lot of various processes, in particular the Earth's magnetic field, the height and period of waves on the surface of the sea, the temperature and speed of sound in the ocean, the air temperature and atmospheric precipitation in different cities, exchange rates, the X-ray intensity of astrophysical objects, etc. [14, 17]. It has been found that all the processes have limited interval of statistical stability. Table 2 presents the estimation result of these intervals for some real processes.

All these estimates, except for the one in row 9, relate to statistical stability with respect to the average. The estimate in row 9 corresponds to statistical stability with respect to the standard deviation.

Table 2. Estimates of the statistical stability intervals for various real processes

No	Real process	Estimate of the statistical stability interval τ_s
1	Oscillations in the mains voltage	About 1 h
2	Currency rate oscillations	About 1 h
3	Height and period of sea surface waves	About half a day
4	Temperature and sound speed variations in the ocean	Ten hours
5	Radiation oscillations of astrophysical source Cygnus X-1	About a week
6	Variations of air temperature	Several weeks
7	Radiation oscillations of astrophysical source GRS 1915+105	About a month
8	Narrowband fluctuations of water temperature in the ocean with an average period from 2 to 10 hours	Several weeks
9	Radiation oscillations of pulsar PSR J1012+5307	Several months
10	Fluctuations in the wind speed in Chernobyl	Several months
11	Earth's magnetic field variations	Several months
12	Precipitation fluctuations	Many tens of years

It is important to note that *all the processes,* taken intentionally from different fields of knowledge, *are statistically unstable*. This allows us to suggest the following hypothesis: *all real physical phenomena are statistically unstable.* This physical hypothesis becomes the

foundation for constructing of the mathematical part of the theory of hyper-random phenomena.

Note, the violation of statistical stability in the real world means that the *probability concept has no physical interpretation* [14, 17]. *Probability is thus a mathematical abstraction.*

5 The Mathematical Apparatus of the Theory of Hyper-random Phenomena

5.1 Scalar Hyper-random Variables

5.1.1 Conditional Characteristics

To describe the hyper-random variable $X = \{X_g,\ \ g \in G\}$, we use various probabilistic characteristics of the *conditional random variables* X_g ($g \in G$) such as the *conditional distribution functions* (Fig. 6) $F_{x/g}(x)$ and the *conditional probability density functions*[2] $f_{x/g}(x) = \frac{\mathrm{d}F_{x/g}(x)}{\mathrm{d}x}$.

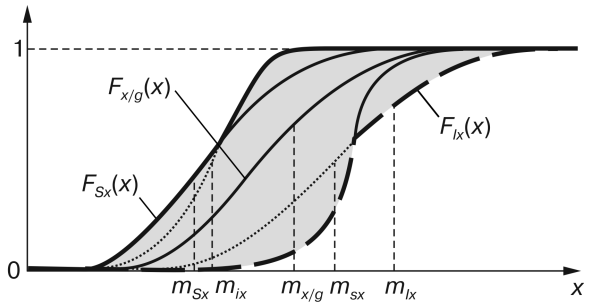


Figure 6. A set of conditional distribution functions $F_{x/g}(x)$ (thin lines) and the bounds of the distribution function $F_{Sx}(x)$, $F_{Ix}(x)$ (bold lines) of the hyper-random variable X

The most complete description of the hyper-random variable X gives its *many-valued distribution function* $\tilde{F}_x(x) = \{F_{x/g}(x),\ g \in G\}$.

[2]It is assumed here and below that all the above distribution functions are continuous or piecewise continuous.

A less complete description supplies the *conditional crude* and *central moments* of the *hyper-random variable* X (*conditional moments*), in particular, the *conditional expectation* $m_{x/g} = \mathrm{E}[X_g] = \int\limits_{-\infty}^{\infty} x\, f_{x/g}(x)\ \mathrm{d}x$, the *conditional variances* $D_{x/g} = \mathrm{E}[(X_g - m_{x/g})^2]$, the *conditional standard deviations* $\sigma_{x/g} = \sqrt{D_{x/g}}$, and others.

In this interpretation, the *expectation*, *variance*, and *standard deviation of the hyper-random variable* X are *many-valued values*, which is analytically described as follows: $\tilde{m}_x = \{m_{x/g}, \quad g \in G\}$, $\tilde{D}_x = \{D_{x/g}, \quad g \in G\}$, $\tilde{\sigma}_x = \{\sigma_{x/g}, \quad g \in G\}$.

The *scalar hyper-random variables* X_1 and X_2 described by the respectively distribution functions $\tilde{F}_{x_1}(x) = \{F_{x_1/g}(x), \quad g \in G\}$ and $\tilde{F}_{x_2}(x) = \{F_{x_2/g}(x), \quad g \in G\}$ are said to be *equal in all conditions*, if under all conditions $g \in G$ for the same g their conditional distribution functions coincide: $F_{x_1/g}(x) = F_{x_2/g}(x)$.

5.1.2 Bounds of the Distribution Function and Their Moments

A general view of the hyper-random variable X is given by the *bounds of the distribution function* $F_{Sx}(x) = \sup\limits_{g \in G} F_{x/g}(x)$, $F_{Ix}(x) = \inf\limits_{g \in G} F_{x/g}(x)$ that are respectively the *upper* and *lower bounds* of probability that $X < x$ (see Fig. 6).

These *bounds can be considered as the distribution functions of some virtual random variables*. Between these bounds there is the *uncertainty area* (shaded area in Fig. 6).

The analogues of the probability density function of the *random variable* are the *probability densities functions of the bounds, viz.* $f_{Sx}(x) = \frac{\mathrm{d}\, F_{Sx}(x)}{\mathrm{d}\, x}$, $f_{Ix}(x) = \frac{\mathrm{d}\, F_{Ix}(x)}{\mathrm{d}\, x}$.

To describe a hyper-random variable, we may use the *moments of the bounds,* in particular, the expectations, variances, and standard deviations of the bounds, and so on.

The *expectations of the bounds* m_{Sx}, m_{Ix} of the hyper-random variable X are described by the formulas $m_{Sx} = \mathrm{E}_S[X] = \int\limits_{-\infty}^{\infty} x f_{Sx}(x)\mathrm{d}x$,

$m_{Ix} = \mathrm{E}_I[X] = \int\limits_{-\infty}^{\infty} x f_{Ix}(x)\mathrm{d}x$ (see Fig. 6).

For a *real* hyper-random variable X the *variances of the bounds* D_{Sx}, D_{Ix} are defined by $D_{Sx} = \mathrm{E}_S\left[(X - m_{Sx})^2\right]$, $D_{Ix} = \mathrm{E}_I\left[(X - m_{Ix})^2\right]$, and the *standard deviations of bounds* – by $\sigma_{Sx} = \sqrt{D_{Sx}}$, $\sigma_{Ix} = \sqrt{D_{Ix}}$.

The *scalar hyper-random variables* X_1 and X_2 described by the distribution functions $\tilde{F}_{x_1}(x)$ and $\tilde{F}_{x_2}(x)$ respectively, are said to be *equal* if their upper and lower bounds of the distribution coincide: $F_{Sx_1}(x) = F_{Sx_2}(x)$, $F_{Ix_1}(x) = F_{Ix_2}(x)$.

5.1.3 Bounds of the Moments

The *bounds of the moments* give a general view of the hyper-random variable X.

The upper and lower bounds of the expectation of the hyper-random variable X are the values $m_{sx} = \mathrm{E}_s[X] = \sup\limits_{g\in G} m_{x/g}$, $m_{ix} = \mathrm{E}_i[X] = \inf\limits_{g\in G} m_{x/g}$ (see Fig. 6).

The upper and lower bounds of the variance of the hyper-random variable X are the values $D_{sx} = \sup\limits_{g\in G} D_{x/g}$, $D_{ix} = \inf\limits_{g\in G} D_{x/g}$. The roots $\sigma_{sx} = \sqrt{D_{sx}}$, $\sigma_{ix} = \sqrt{D_{ix}}$ of these values are the *bounds of the standard deviation.*

In general, the operators $\mathrm{E}_s[\cdot]$, $\mathrm{E}_i[\cdot]$ do not coincide with the operators $\mathrm{E}_S[\cdot]$, $\mathrm{E}_I[\cdot]$, and the bounds of the expectation and variance m_{sx}, m_{ix}, D_{sx}, D_{ix} do not coincide with the expectations and variances of the bounds m_{Sx}, m_{Ix}, D_{Sx}, D_{Ix}.

5.2 Particularities of Statistics of Hyper-random Variables

5.2.1 A Hyper-random Sample

The concepts of mathematical statistics of the theory of hyper-random phenomena are based on the concepts of mathematical statistics of the probability theory.

The *entire assembly* (*general population*) *of the hyper-random variable* $X = \{X_g,\ \ g \in G\}$ is the *infinite set of all its determinate realizations* (*sample elements or components*) observed *under all conditions* $g \in G$. This set can be either countable or uncountable.

It implies from this definition that the general population of the hyper-random variable X is the union of the populations of all its random components $X_g,\ \ g \in G$.

The general population can be described by the *many-valued distribution function* $\tilde{F}_x(x)$ of the hyper-random variable X, the set of conditional distribution functions $F_{x/g}(x)$ $(g \in G)$, the upper and lower bounds of the distribution function $F_{Sx}(x)$, $F_{Ix}(x)$, the moments of the bounds, the bounds of the moments, and other characteristics.

A set of members of the general population

$$\vec{x} = (x_1, ..., x_N) = \{x_{1g}, \ldots, x_{Ng},\ \ g \in G\} = \{\vec{x}_g,\ \ g \in G\}$$

of the hyper-random variable X obtained for a finite number N of experiments in different *fixed* or *non-fixed conditions* $g \in G$ is called the *sample of the population*, and its elements $x_1, ..., x_N$ or $x_{1g}, \ldots, x_{Ng}$ are called the *sampling values* or *realizations*.

Without specifying a condition g each sampling value x_n $(n = \overline{1, N})$ is a *set of determinate values* (*set of numbers*), and with specifying the condition g each sampling value x_{ng} is a *determinate value* (*number*).

Ones believe that the sample $x_1, \ldots, x_N$ belongs to the hyper-random variable $X = \{X_g,\ \ g \in G\}$ described by the conditional distribution functions $F_{x/g}(x)$, $g \in G$ if it is obtained from the general population described under condition g by the distribution function $F_{x/g}(x)$.

Infinite set of the samples $\vec{x} = (x_1, ..., x_N)$ of a volume N taken from a general population without specifying of a condition g forms N-dimensional hyper-random vector

$$\vec{X} = (X_1, ..., X_N) = \{X_{1g}, \ldots, X_{Ng},\ \ g \in G\} = \left\{\vec{X}_g,\ \ g \in G\right\},$$

called *hyper-random sample* and the infinite set of samples $\vec{x}_g = (x_{1g}, \ldots, x_{Ng})$ of the volume N taken from this general population un-

der condition g forms N-dimensional *random vector* (*random sample*) $\vec{X}_g = (X_{1g}, \ldots, X_{Ng})$.

Generally one believes that all elements of hyper-random vector are described by the same many-valued distribution function $\tilde{F}_x(x)$ and each component $X_{ng} \quad (n = \overline{1, N})$ of the random vector $\vec{X}_g$ corresponding to the specific condition g is described by the same single-valued distribution function $F_{x/g}(x)$ (or probability density function $f_{x/g}(x)$).

Ones usually assume that the components X_n of the hyper-random sample $\vec{X}$ are *mutually independent under all conditions*. Then the conditional distribution function $F_{\vec{x}/\vec{g}}(\vec{x})$ of the hyper-random sample $\vec{X}$ under conditions $g \in G$ factorizes: $F_{\vec{x}/g}(\vec{x}) = \prod_{n=1}^{N} F_{x/g}(x_n)$.

In the theory of hyper-random phenomena a *statistics* is *any function* of the *hyper-random sample* $\vec{X}$, *random sample* $\vec{X}_g$ under a fixed condition $g \in G$, *determinate many-valued sample* $\vec{x}$ or *determinate single-valued sample* $\vec{x}_g$ under a fixed condition $g \in G$.

5.2.2 Evaluations of Characteristics and Parameters of a Hyper-random Variable

Using the general population of a hyper-random variable theoretically it is possible to calculate various its *exact determinate characteristics and parameters*, such as the conditional distribution functions $F_{x/g}(x)$, bounds of distribution function $F_{Sx}(x)$, $F_{Ix}(x)$, conditional expectations $m_{x/g}$, expectations of bounds m_{Sx}, m_{Ix}, bounds of expectation m_{sx}, m_{ix}, conditional variances $D_{x/g}$, variances of bound D_{Sx}, D_{Ix}, bounds of variance D_{sx}, D_{ix}, and so on.

Using certain statistics of realizations of the hyper-random variable it is possible to calculate *approximate evaluations* of the same characteristics and parameters, in particular the evaluations of conditional distribution functions $F^*_{x/g}(x)$, bounds of distribution function $F^*_{Sx}(x)$, $F^*_{Ix}(x)$, conditional expectations $m^*_{x/g}$, expectations of bounds m^*_{Sx}, m^*_{Ix}, bounds of expectation m^*_{sx}, m^*_{ix}, conditional variances $D^*_{x/g}$, variances of bound D^*_{Sx}, D^*_{Ix}, bounds of variance D^*_{Sx}, D^*_{Ix}, and so on.

If the sample is *hyper-random*, then the evaluations are the *hyper-*

random estimators, if it is *determinate*, then the evaluations are *determinate estimates*.

The estimates can be made in several steps. First, samples $x_{1g}, \ldots, x_{Ng}$ are formed separately for each condition $g \in G$. Using samples $\vec{x}_g = (x_{1g}, \ldots, x_{Ng})$ for all $g \in G$, one then calculates the conditional characteristic and parameter estimates, in particular, estimates of the conditional distribution functions $F^*_{x/g}(x)$, estimates of the conditional expectations $m^*_{x/g}$, estimates of the conditional variances $D^*_{x/g}$, and others.

From the conditional distribution functions $F^*_{x/g}(x)$ for all $g \in G$, one can calculate estimates of the distribution function bounds: $F^*_{Sx}(x) = \sup\limits_{g \in G} F^*_{x/g}(x)$, $F^*_{Ix}(x) = \inf\limits_{g \in G} F^*_{x/g}(x)$, and estimates of the parameters describing these bounds: estimates m^*_{Sx}, m^*_{Ix} of the expectations of the bounds, estimates D^*_{Sx}, D^*_{Ix} of the variances of the bounds, and so forth.

Using estimates of the conditional variables, one can calculate estimates of the corresponding variable bounds, for example, estimates of the expectation bounds $m^*_{sx} = \sup\limits_{g \in G} m^*_{x/g}$, $m^*_{ix} = \inf\limits_{g \in G} m^*_{x/g}$, estimates of the variance bounds $D^*_{sx} = \sup\limits_{g \in G} D^*_{x/g}$, $\quad D^*_{ix} = \inf\limits_{g \in G} D^*_{x/g}$, etc.

When applying this technique, *certain difficulties can be expected in the first stage*, when the samples $\vec{x}_g$ for all $g \in G$ are formed, because at first glance, it is difficult to control and maintain the conditions g. The situation *is facilitated* by the facts that a lot of actual samples are possessed of *ergodic property* and the calculation of a number of characteristics *do not require information about the specific conditions under which the conditional characteristics have been obtained*.

Most important that, in the sample formation phase, all possible conditions g of the set G are represented, and for every fixed condition g in the sample $\vec{x}_g$, *only the data corresponding to this condition g is used*.

Typically, for actual phenomena occurring in the real world, in the case of a broad observation interval, the latter requirement can be easily provided, because, although the conditions often vary continuously,

they vary sufficiently slowly, and it is possible to evaluate the maximum number of elements N_s for which the conditions can be treated as practically constant.

Therefore one can collect data on a broad observation interval (that is essentially larger than N_s) without taking care about what the statistical conditions are at any given time and in what way they alternate, and then one can separate the resulting data into a number of fragments containing N_s consistent elements. Using these fragments, which represent the variable under different statistical conditions g, one can then calculate the required estimates. The main requirement for this technique is to collect the data for all possible observation conditions in G.

Of course a number of questions arise. What are the conditions under which the hyper-random evaluations converge to the exact characteristics and parameters? What are types of these parameters and characteristics? What are their distribution laws? The *generalized law of large numbers* and the *generalized central limit theorem* help to obtain answer to these and other questions.

To understand this material, one should be familiar with some mathematical concepts, such as the *generalized limit* and the *convergence of sequences in the generalized sense.*

5.3 Generalized Limit and the Convergence of Sequences in the Generalized Sense

5.3.1 Generalized Limit

According to classical concepts, the *numerical sequence* $x_1, x_2, ..., x_n$ is considered as a *convergent sequence* if there is a limit $a = \lim\limits_{n\to\infty} x_n$. If the limit exists, then it is *unique.* The sequence which has not the limit is considered as a *divergent sequence.*

From every infinite sequence one can form the set of *partial sequences* (*subsequences*) derived from the original sequence by discarding part of its members, while *maintaining the order of the remaining members.*

It is proved that when the sequence converges, all its partial sequences converge too. If the sequence diverges, then all its partial sequences

do not necessary diverge. Some of them can converge to certain limits (*limit points*). The set of all limit points a_m, $m = 1, 2, \ldots$ of the sequence $x_1, x_2, ..., x_n$ also called *partial limits*, form the *spectrum of limit points* $\tilde{S}_x$.

The spectrum of limit points $\tilde{S}_x$ is a generalization of the limit concept on any sequence, including divergent. If the sequence converges, the spectrum of the limit points consists of a single element (number), and if it is divergent, it consists of a set of numbers. The spectrum of limit points can be described by the expression $\tilde{S}_x = \underset{n\to\infty}{\mathrm{LIM}}\, x_n$, where, unlike the conventional limit $\lim\limits_{n\to\infty}$ the symbol of the *generalized limit* $\underset{n\to\infty}{\mathrm{LIM}}$ is used.

This expression can be interpreted as the *convergence of the sequence to the spectrum of limit points*. The spectrum may be discrete, continuous, or mixed (discrete-continuous). If the spectrum forms a continuous interval, they say that the *sequence converges to the interval*.

A divergent sequence can be characterized by not only the spectrum of limit points, but also by a *set* (in general) *of the measures* described by the *many-valued* (in general) *distribution function of the limit points* $\tilde{F}_x(x) = \underset{n\to\infty}{\mathrm{LIM}} \frac{m_n(x)}{n}$, where $m_n(x)$ is the number of terms of the sequence $x_1, x_2, ..., x_n$ that are less than x.

If the sequence converges in the usual sense to the number a, the distribution function of limit points is described by the unique distribution function $F_x(x)$ in the form of a unit step function at the point a (Fig. **7a**) (then the measure equals to one at the point a and zero at all other points).

If the sequence diverges (converges to the set of numbers (in the particular case converges to the interval)), the distribution function is either a single-valued non-decreasing function $F_x(x)$ that differs from the unit step function (Fig. **7b**), or a many-valued function $\tilde{F}_x(x)$ (Fig. **7c**). Note that the *special case of hyper-random variable is the interval variable*, the distribution function of which is described by a rectangle of unit height (Fig. **7d**).

Using the terminology of the theory of hyper-random phenomena,

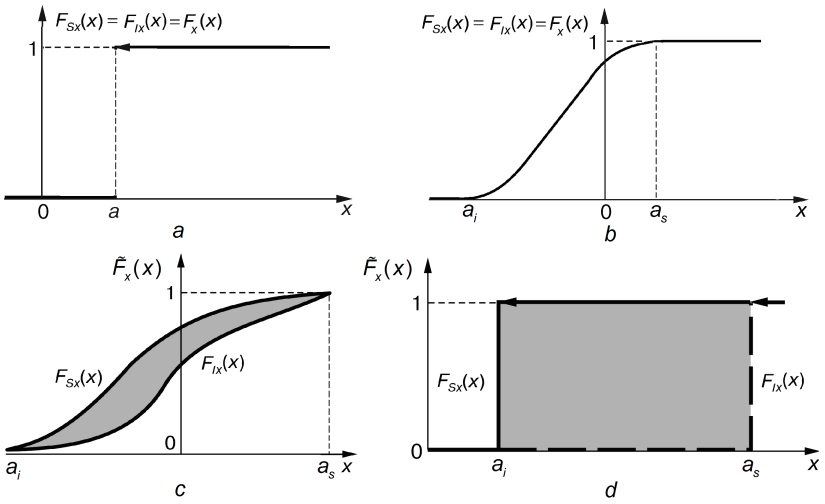


Figure 7. Single-valued $F_x(x)$ (**a**, **b**) and many-valued $\tilde{F}_x(x)$ (**c**, **d**) distribution functions of the limit points and their bounds $F_{Ix}(x)$, $F_{Sx}(x)$ for sequences converge to the number a (**a**) and to the interval $[a_i,\ a_s]$ (**b**–**d**)

we can say that the *spectrum of the limit points of a numerical sequence* can be

- a *number* (interpreted by the set of real numbers with the unit measure at the point $x = a$ and zero measure at all other points) (Fig. 7**a**),
- a *random variable* (Fig. 7**b**),
- a *hyper-random variable* (Fig. 7**c**) (in the degenerated case an *interval variable* (Fig. 7**d**)).

In other words, the numerical sequence may converge to a *number* or to a *set of numbers* (in the particular case to an *interval*). If it converges to a set of numbers, the spectrum of limit points may be either a *random variable* or a *hyper-random variable*.

5.3.2 Convergence of Sequences of Hyper-random Variables in the Generalized Sense

By analogy with the convergence of a *sequence of random variables*, in the theory of hyper-random phenomena the concept of *convergence* (*in generalized sense*) of a *sequence of hyper-random variables* is introduced. There is convergence *in distribution function, in mean-square, almost surely* (*with probability one*), *and in probability* (*in some measure*).

Consider the convergence of the sequence in the generalized sense in probability and in distribution function.

Suppose we have a sequence of hyper-random variables $\mathrm{X} = \{X_1, ..., X_N\}$ and a hyper-random variable X, where $X_n = \{X_{ng},\ \ g \in G\}$ $(n = \overline{1, N})$ and $X = \{X_g,\ \ g \in G\}$. For all $X_1, ..., X_N$ and X, there are distribution functions $\tilde{F}_{x_1}(x) = \{F_{x_1/g}(x),\ \ g \in G\}, \ldots, \tilde{F}_{x_N}(x) = \{F_{x_N/g}(x),\ \ g \in G\}$ and $\tilde{F}_x(x) = \{F_{x/g}(x),\ \ g \in G\}$.

Then the sequence of hyper-random variables X *converges in the generalized sense* to the hyper-random variable X *in probability* $(P(|X_N - X| > \varepsilon) \to 0)$ if for all conditions $g \in G$ and $\varepsilon > 0$, when $N \to \infty$, $P(|X_{Ng} - X_g| > \varepsilon) \to 0$, i.e., for all $g \in G$, the random sequence $X_{1g}, ..., X_{Ng}$ converges in probability to the random variable X_g.

The sequence of hyper-random variables X *converges in the generalized sense* to the hyper-random variable X *in distribution* $(\tilde{F}_{x_N}(x) \to \tilde{F}_x(x))$ if for each point x, where $F_{x/g}(x)$ is continues, for all conditions $g \in G$, when $N \to \infty$, $F_{x_N/g}(x) \to F_{x/g}(x)$.

As in the case of the sequences of random variables, convergence in distribution is *weaker* than convergence in probability, i.e. the sequence of hyper-random variables that converges in probability converges in distribution too. The converse is not always true.

It follows from the definitions that, as well as a numerical sequence, the *hyper-random sequence* can converge to a *number* (determinate variable, the distribution function of which is a unit step function), to a *random variable* or to a *hyper-random variable*. It is obvious that a *random sequence* can also converge to a *number*, to a *random variable*

or to a *hyper-random variable*.

5.4 Generalized Law of Large Numbers

Several variants of the law of large numbers for *random sequences* are known. Let us dwell on one of them formulated and proved by P.L. Chebyshev in 1867.

Chebyshev theorem. Let $X_1,\ \ldots, X_N$ be a sequence of pairwise independent random variables with expectations $m_1, \ldots, m_N$ and bounded variances. Then, when the sample size N goes to infinity, the average $m^*_{xN} = \frac{1}{N}\sum\limits_{n=1}^{N} X_n$ of the sample values $X_1,\ \ldots, X_N$ tends in probability to the average $m_{xN} = \frac{1}{N}\sum\limits_{n=1}^{N} m_{x_n}$ of the expectations $m_1, \ldots, m_N$: $\lim\limits_{N\to\infty} P\left\{|m^*_{xN} - m_{xN}| > \varepsilon\right\} = 0$.

In typical for the probability theory interpretation the law of large numbers consists in that the average m^*_{xN} converges in probability to some number m_x that is a conventional limit of the average m_{xN} of the expectations $m_{x_1}, \ldots, m_{x_N}$.

The analysis of the proof of this assertion (which we will not present herein) shows that in the proof *it is not applied the assumption* that the average m^*_{xN} of the random samples and the average m_{xN} of the expectations have the conventional limits. This means that the sequences $\{m^*_{xN}\} = m^*_{x1}, \ldots, m^*_{xN}$ and $\{m_{xN}\} = m_{x1}, \ldots, m_{xN}$ *may not have limits in the conventional sense*, i.e. the sequences *may be divergent*.

But if they do not converge in the conventional sense, they *can converge in the generalized sense to the many-valued variables*: to random or hyper-random ones.

Hereafter, following the above mentioned agreement concerning designations of single-valued and many-valued variables and functions, the *single-valued* limits of the sequences $\{m^*_{xN}\}$ and $\{m_{xN}\}$ we shall denote by m^*_x and m_x, and a *many-valued* ones by the same manner but with tilde above: $\tilde{m}^*_x$ and $\tilde{m}_x$.

Whether the considered limits are single-valued or many-valued, ac-

cording to the law of large numbers, when the sample size N increases, the *sample mean* m^*_{xN} *gradually approaches the average of the expectations* m_{xN}.

When $N \to \infty$, there are two possibilities:

Case 1 The variable m^*_{xN} converges to the *single-valued* average of the expectations m_x (number).

Case 2 The variable m^*_{xN}, becoming a many-valued variable $\tilde{m}^*_x$ in the limit, converges in the general sense to a *many-valued variable* $\tilde{m}_x$.

Case 1 is the idealized case considered in probability theory. In this case, the limit m_x of the average of the expectations is described by the distribution function $F_{m_x}(x)$, which is a unit step function at the point m_x. The distribution function $F_{m^*_{xN}}(x)$ of the sample mean m^*_{xN} tends to it when $N \to \infty$ (see Fig. 8**a**).

Case 2 is more realistic. Here the limit sample mean $\tilde{m}^*_x$ and the limit average of the expectations $\tilde{m}_x$ are described respectively by the many-valued spectra $\tilde{S}_{m^*_x}$ and $\tilde{S}_{m_x}$. In this case there may be two variants:

Case 2.1 The limit of the sample mean $\tilde{m}^*_x$ and the limit of the average expectations $\tilde{m}_x$ are *variables of random type.* Then the spectra $\tilde{S}_{m^*_x}$ and $\tilde{S}_{m_x}$ are characterized by the *single-valued distribution functions* $F_{m^*_x}(x)$ and $F_{m_x}(x)$ (see Fig. 8 **b**).

Case 2.2 The limit of the sample mean $\tilde{m}^*_x$ and the limit of the average expectations $\tilde{m}_x$ are *variables of hyper-random type.* Then the spectra $\tilde{S}_{m^*_x}$ and $\tilde{S}_{m_x}$ are characterized by the *many-valued distribution functions* $\tilde{F}_{m^*_x}(x)$ and $\tilde{F}_{m_x}(x)$ (see Fig. 8**c**).

Since the *convergence in distribution* of a sequence of *random variables* is weaker than the *convergence in probability*, in Case 2.1, the limit distribution function $F_{m^*_x}(x)$ coincides with the limit distribution function $F_{m_x}(x)$.

For *hyper-random variables*, convergence of the sequence *in distribution* is also weaker than convergence *in probability.* Therefore, in Case 2.2, the limit distribution function $\tilde{F}_{m^*_x}(x)$ coincides with the limit distribution function $\tilde{F}_{m_x}(x)$. In this case, the lower bound $F_{Im^*_x}(x)$ of the limit distribution function $\tilde{F}_{m^*_x}(x)$ coincides with the lower bound $F_{Im_x}(x)$ of the limit distribution function $\tilde{F}_{m_x}(x)$, and the upper bound

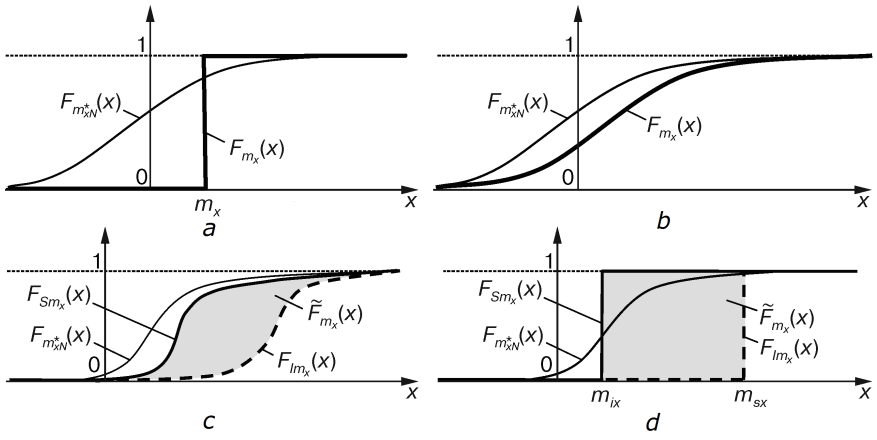


Figure 8. Formation of the limit distribution function $\tilde{F}_{m_x^*}(x)$ of the sample mean in the case of a random sequence: the limit sample mean and the limit average of expectations are a number (**a**), a random variable (**b**), and a hyper-random variable (**c**, **d**) (**c** is the general case and **d** is a special case)

$F_{Sm_x^*}(x)$ of the limit distribution function $\tilde{F}_{m_x^*}(x)$ coincides with the upper bound $F_{Sm_x}(x)$ of the limit distribution function $\tilde{F}_{m_x}(x)$.

The uncertainty area located between the specified bounds is shown in Fig. 8**c** by the shaded area. It is proved that, if the distribution function describing the spectrum of the sequence of averages of determinate values is many-valued, then the corresponding uncertainty area is continuous. So the *uncertainty area of the distribution function* $\tilde{F}_{m_x}(x)$ *is continuous.*

The interval in which the sample mean m_{xN}^* fluctuates when $N \to \infty$ is described by the lower bound m_{ix}^* when the function $F_{Sm_x^*}(x)$ begins to rise from zero and the upper bound m_{sx}^* when the function $F_{Im_x^*}(x)$ reaches unity. Naturally, these bounds coincide with the corresponding bounds m_{ix}, m_{sx} of the functions $F_{Sm_x}(x)$, $F_{Im_x}(x)$: $m_{ix}^* = m_{ix}$, $m_{sx}^* = m_{sx}$. These bounds can be either finite or infinite.

Note that Case 2.2 includes the special case when the limit sample

mean $\tilde{m}_x^*$ and the limit average of the expectations $\tilde{m}_x$ are of *interval type* (Fig. 8**d**).

Systematizing the results of the present section, we may note that the sample mean m_{xN}^* of a random sample can converge to a *number* m_x (finite or infinite) or *fluctuate within a certain interval* $[m_{ix}, m_{sx}]$.

In the latter case, we shall say that there is *convergence of the sample mean to the interval*. Theoretically the limit of the sample mean $\tilde{m}_x^*$ and the limit average of the expectations $\tilde{m}_x$ can be *numbers*, *random variables*, *intervals*, or *hyper-random variables.* The spectra $\tilde{S}_{m_x^*}$ and $\tilde{S}_{m_x}$ can be *numbers* or *intervals*. The limit distribution functions $F_{m_x^*}(x)$ and $F_{m_x}(x)$ can be of *unit step type*, *single-valued functions*, or *many-valued functions with a continuous uncertainty area*.

Convergence of the sample mean to a number is not corroborated by the experiments and convergence to an interval is corroborated by a lot of them. We shall return to the question concerning the type of the limit distribution function after study of the generalized central limit theorem.

5.5 Generalized Central Limit Theorem

In the probability theory it is known the *central limit theorem.* There are many variants of it. One of them can be formulated with some simplification by the following manner.

Lindeberg-Feller theorem. Let $X_1, \ldots, X_N$ be, in general, a non-uniform random sample with mutually independent terms described by distribution functions $F_{x_n}(x)$ with expectations m_{x_n} and variances D_{x_n} ($n = \overline{1, N}$). We assume the so called *Lindeberg condition*. Then the distribution function $F_{m_{xN}^*}(x)$ of the sample mean m_{xN}^* *converges uniformly* to a *Gaussian distribution function* $F(x/m_{xN}, D_{xN}) = \Phi\left((x - m_{xN})/\sqrt{D_{xN}}\right)$ with expectation $m_{xN} = \frac{1}{N}\sum\limits_{n=1}^{N} m_{x_n}$ and variance $D_{xN} = \frac{1}{N^2}\sum\limits_{n=1}^{N} D_{x_n}$, viz.

$$\lim_{N\to\infty} F_{m_{xN}^*}(x) = \lim_{N\to\infty} F(x/m_{xN}, D_{xN}), \tag{1}$$

where $\Phi(x)$ is *Laplace function*.

According to this theorem, with increasing of the sample size the random variable m^*_{xN} becomes a *Gaussian random variable*.

Using the technique devised to obtain (1), a more general statement can be proved: if the conditions specified in Lindeberg-Feller theorem are satisfied, the difference between the distribution function $F_{m^*_{xN}}(x)$ of the sample mean m^*_{xN} and the Gaussian distribution function $F(x/m_{xN}, D_{xN})$ converges uniformly to zero

$$\lim_{N\to\infty}\left[F_{m^*_{xN}}(x) - F(x/m_{xN}, D_{xN})\right] = 0. \tag{2}$$

There is a significant difference between (1) and (2). The expression (1) implies that the sample mean m^*_{xN} has a single-valued limit distribution function $F_{m^*_x}(x)$ to which the distribution function $F_{m^*_{xN}}(x)$ tends when $N \to \infty$, and there is a *single-valued* Gaussian limit distribution function $F_{m_x}(x) = F(x/m_x, D_x)$ to which the distribution function $F(x/m_{xN}, D_{xN})$ tends, where m_x and D_x are the expectation and the variance of the limit distribution function, respectively.

The formula (2), on the other hand, allows the given limit distribution functions to be *many-valued*. The many-valuedness of the limit distribution function to which the function $F(x/m_{xN}, D_{xN})$ tends is stipulated by the many-valuedness of the expectation and (or) variance. Therefore, in the expression $\tilde{F}_{m_x}(x) = \tilde{F}(x/\tilde{m}_x, \tilde{D}_x)$ representing the limit distribution function of the average of the expectations, the many-valued parameters $\tilde{m}_x$ and $\tilde{D}_x$ appear. In general these parameters are hyper-random variables. Therefore the function $\tilde{F}(x/\tilde{m}_x, \tilde{D}_x)$ is a hyper-random function. It can be interpreted as a set of single-valued Gaussian distribution functions. Each of these is described by a single-valued expectation $m_x \in \tilde{m}_x$ and variance $D_x \in \tilde{D}_x$.

The relation $F_{m^*_{xN}}(x) \to \tilde{F}(x/\tilde{m}_x, \tilde{D}_x)$ follows from (2), implying that there is convergence in distribution of the sequence of determinate functions $F_{m^*_{xN}}(x)$ to the hyper-random function $\tilde{F}(x/\tilde{m}_x, \tilde{D}_x)$. In other words, the many-valued limit distribution functions $\tilde{F}_{m^*_x}(x)$, $\tilde{F}(x/\tilde{m}_x, \tilde{D}_x)$ are described by identical sets of single-valued conditional distribution functions.

When $\tilde{m}_x = m_x$ and $\tilde{D}_x = D_x$ (i.e. the both parameters are numbers) and $D_x = 0$, the limit Gaussian distribution function $F_{m_x}(x) = F(x/m_x, D_x)$ is the unit step function shown in Fig. 8**a** by the bold line; and when $\tilde{m}_x = m_x$ and $\tilde{D}_x = D_x$ are numbers but $D_x \neq 0$, this distribution function is described by the single-valued Gaussian curve shown in Fig. 8**b** by the bold line.

When the limit expectation $\tilde{m}_x$, the limit variance $\tilde{D}_x$ or both these parameters are many-valued variables, the limit distribution function $\tilde{F}_{m_x}(x)$ is a many-valued function. In Fig. 8**c**, **d**, it is displayed by the shaded areas.

Note, the analogues results concerning the law of large numbers and the central limit theorem are generalized on hyper-random sequences.

5.6 Experimental Study of the Convergence of the Sample Mean

The theoretical research presented in Sects. 5.4 and 5.5 indicates that with increasing of the sample size the sample means are *not necessarily normalized* (i.e. they do not necessarily take on the Gaussian character) and *tend to a certain fixed value*. This result is quite different from the conclusion of the classical probability theory. It raises a very important question: how do the actual sample means behave?

To answer this question, we return to investigation of the mains voltage oscillations (see Fig. 3a) and present some results of additional experimental studies of the process.

Studies consisted in calculation and analysis of the estimates of the distribution functions of the voltage fluctuations $F_g^*(x)$ on adjacent observation intervals, each lasting about one hour ($g = \overline{1,64}$) (Fig. 9**a**), and the estimate of the distribution function of the sample mean $F^*_{m^*_{xN}}(x)$ (Fig. 9**b**).

The curves of the distribution functions $F_g^*(x)$ corresponding to different values of the parameter g differ essentially from one another (primarily by their location) (see Fig. 9**a**), and this confirms the claimed *nonstationarity* of the oscillations.

The calculation results of the estimate of the sample mean distri-

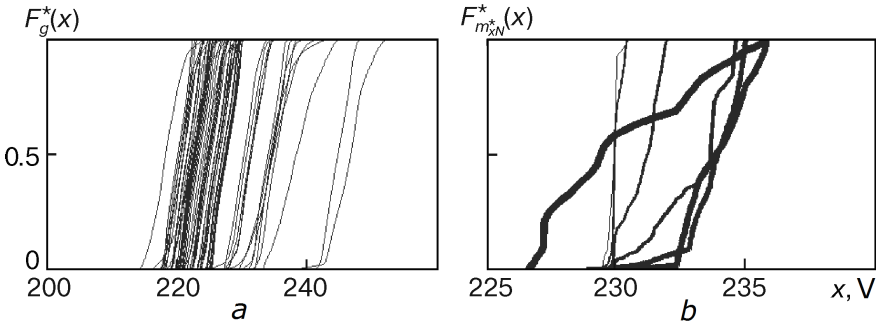


Figure 9. Estimates of the distribution functions of mains voltage oscillations on 64 contiguous observation intervals (**a**) and estimates of the distribution function of the voltage sample mean $F^*_{m^*_{xN}}(x)$ for sample sizes $N = 2^r, \quad r = 8, 10, 12, 14, 16, 18, 20$ (**b**) (the line thickness increases with the value of the parameter r)

bution function $F^*_{m^*_{xN}}(x)$ for exponentially growing sample size (see Fig. 9**b**) show that $F^*_{m^*_{xN}}(x)$ *does not tend to a certain limit distribution function* $F_{m_x}(x)$, and the sample mean m^*_{xN} *does not tend to a certain limit value* m_x.

On the basis of the curves for the estimate of the distribution function of the sample mean $F^*_{m^*_{xN}}(x)$ for small values of the parameter r (8 and 10) (see Fig. 9 **b**), we may with some level of skepticism conclude that it is tending to a Gaussian distribution with decreasing variance, as probability theory would predict. However, for large values of r (starting from 10 to 20), the assumed trend is not confirmed.

When the sample size increases, the variance of the sample mean m^*_{xN} sometimes increases (for values of r from 8 to 14 and from 18 to 20) and sometimes decreases (for r from 14 to 18). In general, as one moves from small to large sample sizes, the variance does not manifest any tendency to go to zero, as would have been predicted by probability theory (see Fig. 8**a**), but in fact increases, even by a significant factor (the range of the sample mean increases approximately from $1V$ to $8V$).

It follows from these results that the *distribution function of the sample mean tends to a many-valued function* $\tilde{F}_{m_x}(x)$ *of general form* (see Fig. 8**c**).

Studies of the distribution functions of the sample means of a lot of processes show that when the data volume is large *there is not the aspiration* of the estimate $F^*_{m^*_{xN}}(x)$ of the distribution function of the sample mean to any specific distribution law, and more so to a Gaussian distribution with variance that tends to zero.

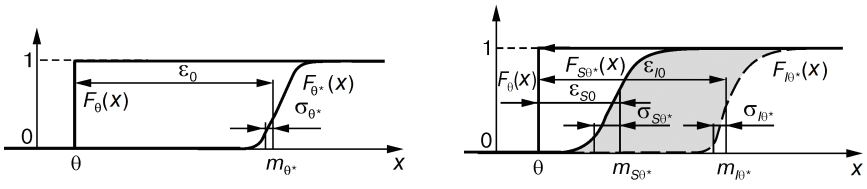
Thus, the experimental studies of the actual physical processes show that in case of a *small data volume* ones observe the *trends of normalization and stabilization of the sample means* and in case of a *large amount of data* such tendencies *are not fixed*.

The changing in the character of the behavior of the sample means can be explained by a violation of statistical stability of the actual processes on large observation intervals. These disorders lead to restriction of the accuracy of measurement and prediction of real physical quantities.

6 Accuracy and Measurement Models

6.1 Measurement Models

Any measurement is based on some models. It is usually suggested that the *measurand* (*measurement quantity*) *has a determinate character, while its estimator is random*. Modern classical measurement theory uses this paradigm.



a b

Figure 10. The classical determinate-random (**a**) and determinate–hyper-random (**b**) measurement models

When measuring a scalar quantity, the measurand θ can be represented by a unit step distribution function $F_\theta(x)$ and the measurement result Θ^* – by a distribution function $F_{\theta^*}(x)$ (Fig. 10**a**). Such a measurement model may naturally be referred to as *determinate–random*.

The foundations of this model were laid out by *Galileo Galilei*, who introduced the *concepts of systematic and random errors.* The systematic error is described by the bias of the estimator $\varepsilon_0 = m_{\theta^*} - \theta$ and the random error is often presented by the standard deviation σ_{θ^*} of the estimator Θ^*.

Modern metrology is based on the following hypotheses: the ideal value of a physical quantity is determinate, single-valued, and is not changed during the measurement time; the measure does not change its characteristics during the measurement; the statistical conditions are constant during the measurement time; and the result of a concrete measurement is unique.

All of these items, to put it mildly, not very reasonable. All actual physical objects and physical quantities describing them are subjected to change over time (except perhaps some universal constants). *Everything is changed: the object of measurement* (*measurand*), *the measure, and the measurement conditions.*

Any measurement is carried out not instantaneously, but over some time interval. Therefore the *measurement result is an average value representing over this interval the various states of the measuring object, the different states of the measure, and different measurement conditions.*

Of course, it is very convenient to represent the measurand by a determinate, unique, and unchanging value, and the measurement result – by a random variable. But this primitive model does not reflect many nuances of the real situation.

The theory of hyper-random phenomenon proposes different hyper-random mathematical models, taking into account some of them. *Determinate–hyper-random model* (Fig. 10**b**) describes, for instance, the measurand by a determinate model and the estimator by a hyper-random variable. In the figure, $F_{S\theta^*}(x)$ and $F_{I\theta^*}(x)$ are the upper and lower bounds of the distribution function of the hyper-random estima-

tor Θ^*; $\varepsilon_{I0} = m_{S\theta^*} - \theta$ and $\varepsilon_{I0} = m_{I\theta^*} - \theta$ are the biases of the upper and lower bounds of the distribution function of the hyper-random estimator with respect to the measurand; $m_{S\theta^*}$, $m_{I\theta^*}$ are the expectations of the upper and lower bounds of the hyper-random estimator; and $\sigma_{S\theta^*}$, $\sigma_{I\theta^*}$ are the standard deviations of the appropriate bounds of the hyper-random estimator. The uncertainty area of the hyper-random estimator is shown by the shaded area.

6.2 Comparison of the measurement models

In the *determinate–random measurement model*, the error has a *random nature.* It is described by *systematic and random components*, and characterized by two parameters: the *bias* ε_0 and the *standard deviation of the estimator* σ_{θ^*} (Fig. 10**a**). In the *determinate–hyper-random measurement model*, the error has a *hyper-random nature.* It has an *uncertainty area* and is described by four parameters $\varepsilon_{S0}, \varepsilon_{I0}, \sigma_{S\theta^*}, \sigma_{I\theta^*}$ defining the location and size of the uncertainty area on the error axis (Fig. 10**b**).

Techniques of statistical measurement according to the comparing models are well known. They are described for instance in [14, 18]. Here we do not describe them and present only the calculation results for the parameters characterizing the mains voltage (Fig. 3**a**) at the end of 100-second and 60-hour observation intervals (Fig. 11).

The left side of the figure is obtained with using the classic determinate–random measurement model based on probability theory and the right side presents the parameters obtained with using the determinate–hyper-random measurement model based on the theory of hyper-random phenomena (except the parameter marked with a thin arrow).

For the 60-hour observation interval, the sample range and the range of the sample mean are obtained from the data of Figs. 3**a** and 9**b**. The confidence interval (THRP) marked with a bold arrow and the estimate (THRP) are calculated using the technique of the theory of hyper-random phenomena. The confidence interval (PT) marked by a thin arrow is calculated using the classic technique of the probability theory.

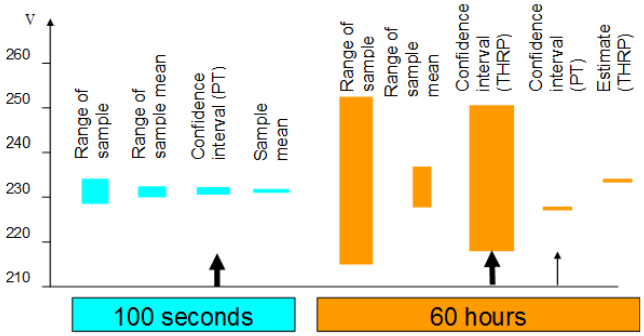


Figure 11. Calculation results for the parameters characterizing the city mains voltage over 100-second and 60-hour observation intervals, using calculation techniques based on probability theory (PT) and the theory of hyper-random phenomena (THRP)

The results shown in the figure for the 100-second and 60-hour observation intervals differ considerably. The parameters on the left side of the figure reflect the state of the electrical supply network under the *specific statistical conditions* that occurred during the relevant 100-second observation interval. The parameters on the right side (except for the one marked by a thin arrow) represent the state of the network for the *varying set of statistical conditions* that succeeded one another *unpredictably* during the relevant 60-hour observation period. The parameter marked by a thin arrow characterizes the state of the network for the *set of different but very specific statistical conditions* that succeeded each other over the same 60-hour period of observation.

For the *100-second observation interval, the most informative parameter is the confidence interval calculated using the classic technique of probability theory,* and for the *60-hour interval, it is the confidence interval calculated using the technique based on the theory of hyper-random phenomena* (in Fig. 11 these parameters are marked by two bold arrows).

For the 60-hour observation interval, the confidence interval, with width 50 mV and average value 229.4 V (calculated in accordance with

probability theory and indicated in the figure by a thin arrow) is *not informative*, because it takes into consideration the *concrete sequence* of changes in the conditions which, in the next 60-hour observation interval, is likely to be something quite different. The confidence interval, with width 33 V and average 233.5 V (calculated using the theory of hyper-random phenomena and marked by a thick arrow) contains useful practical information about the average dynamics of the voltage changes in the power supply.

The loss of useful information in the first case and the fact that it is kept in the second arise because, when there are violations of statistical stability, the classic determinate–random measurement model reflects the real situation with considerable distortion, while the determinate–hyper-random measurement model is able to present it adequately. It follows from the above example that, ignoring the violation of statistical stability can lead to absurd results, and in particular, to an unjustifiable overstatement of measurement accuracy estimators by factors of hundreds or more.

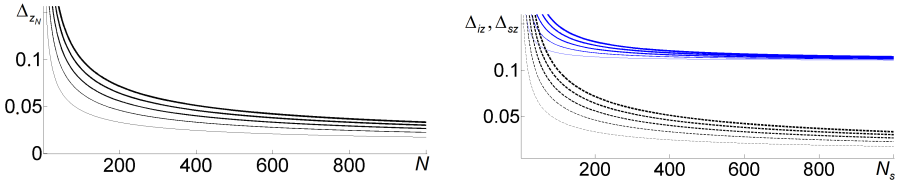
The conclusion is obvious: *when statistical stability is violated, the determinate–random measurement model and the measuring techniques based on it cannot be used.* In this case, *other models and measurement techniques must be used,* and in particular the *determinate–hyper-random measurement model and techniques based on it, which take into consideration the violations of statistical stability.*

6.3 Potential Measurement Accuracy

In case of determinate–random measurement model the error $\Delta_{z_N} = \sqrt{\varepsilon_0^2 + \sigma_{\theta_N^*}^2}$ is determined by the bias ε_0 and the standard deviation of the estimator $\sigma_{\theta_N^*}$. With increasing sample size N, theoretically this magnitude tends to the square of the bias ε_0^2. Let the estimator Θ_N^* be the average of the sample $(X_1^;..., X_N)$ and the sample elements are independent and have identical variance D_x. Then the variance of the estimator $\sigma_{\theta_N^*}^2 = D_x/N$ and the error is described by the expression $\Delta_{z_N}^{=} \sqrt{\varepsilon_0^2 + D_x/N}$. The dependence of the magnitude Δ_{z_N} on the defining parameters is shown in Fig. 12**a**.

It is clear from the figure that, when $N \to \infty$, the error tends to the bias ε_0 (the systematic error). If the bias ε_0 is *negligible*, the magnitude Δ_{z_N} is in inverse proportion to the root of the sample size N. It follows from this that, theoretically, by increasing N, the accuracy of the measurement can grow without limit, and as $N \to \infty$, it should become *infinitely large*.

Probability theory does not give a satisfactory explanation as to why, at low bias, an ultra-high measurement accuracy cannot be achieved by statistical processing of a large number of real data. The explanation of this effect gives the theory of hyper-random phenomena.



a b

Figure 12. Dependence of the error Δ_{z_N} (**a**) and the error bounds Δ_{iz}, Δ_{sz} (**b**) on the sample size N and the variance D_x accordingly for the determinate–random and the determinate–hyper-random measurement models. In case **a** $\varepsilon_0 = 0.01$ and in case **b** $\varepsilon_0 = \varepsilon_{S0} = 0.01$, $\Delta\varepsilon_0 = 0.1$. Thicker lines correspond to large values of the variance $D_x = 0.2;\ 0.4;\ 0.6;\ 0.8;\ 1$

Suppose the measurand θ is determinate and the estimator Θ^* is a hyper-random variable. The elements of the hyper-random sample $(X_1, \ldots, X_N)$ are independent. The statistical conditions change slowly and this allows us to divide the observation interval into G fragments of identical length corresponding to nearly constant statistical conditions. The elements of the sample are taken with uniform step. In any fragment, the number of samples is N_s.

The distribution law of the random elements $X_{1g}, \ldots, X_{N_s g}$ under the fixed condition g is fixed. Under different conditions g, the distribution laws of the elements are different, however all of them have the same variance D_x and differ from each other only in the expectation

value. Then the error bounds can be written as $\Delta_{iz} = \sqrt{\varepsilon_i^2 + D_x/N_s}$, $\Delta_{sz} = \sqrt{\varepsilon_s^2 + D_x/N_s}$, where $\varepsilon_i^2 = \inf\limits_{g\in G}[\varepsilon_{0/g}^2]$ and $\varepsilon_s^2 = \sup\limits_{g\in G}[\varepsilon_{0/g}^2]$ are the lower and upper bounds of the square of the bias.

The dependence of the error bounds Δ_{iz}, Δ_{sz} on the defining parameters is shown in Fig. 12**b**. The dotted lines represent the lower error bounds and the solid ones the upper error bounds.

It is clear from the figure that, with increasing sample size N_s, the upper bound of the error Δ_{sz} tends to $\varepsilon_s = \varepsilon_0 + \Delta\varepsilon_0$ (ε_0 is the systematic error and $\Delta\varepsilon_0$ is the length of the uncertainty area).

Therefore, even if we make the unlikely assumption that the value N_s tends to infinity, the determining upper bound of the error Δ_{sz} will never be less than the value $\Delta\varepsilon_0 \neq 0$. When the bias ε_0 is negligible, the magnitude $\Delta_{sz} \to \Delta\varepsilon_0 \neq 0$.

So with the determinate–hyper-random measurement model, we can explain the *inability in practice to achieve infinitely high accuracy*, even with an unlimited amount of data.

7 Conclusions

Summing up the consideration of the issues it is drawn attention to the following key points.

1. *Statistical stability is a physical phenomenon* manifested in stability of *relative frequency of the actual mass events, sample means* and *other statistics*.

2. There are two theories describing statistical stability phenomenon: the *probability theory* and the *theory of hyper-random phenomena*. The *probability theory* is based on the assumption that the *phenomenon of statistical stability is perfect* (*statistics are converged* and *estimators are consistent*). The *theory of hyper-random phenomena* is based on the assumption that the *phenomenon of statistical stability is not perfect* (*statistics are not converged* and *estimators are not consistent*).

3. Numerous experimental studies of real phenomena of different physical nature indicate that *statistics obtained from actual samples do not demonstrate the tendency to convergence*. The trend towards convergence is observed only when the sample volume is small. In case of large sample volume such trend is not registered.

4. The violation of convergence of the relative frequency of actual events implies that the *probability, the basic concept of the probability theory is an abstract mathematical concept* that *does not have a physical interpretation*.

5. It is formulated and proved for *divergent sequences* the *generalized law of large numbers* and the *generalized central limit theorem*.

6. The results of the experimental studies conform the opinions of some scholars (including A.N. Kolmogorov, A.A. Markov, A.V. Skorokhod, E. Borel, and others) that the hypothesis of perfect statistical stability is valid in the actual world only in certain reservations. Apparently, the actual world really is obeyed to three *types of laws*: *determinate*, *statistically predicted* (*random, stochastic or otherwise probabilistic*), and *statistically unpredictable*.

7. For the small sample size the influence of statically unpredictable laws does not reflect essentially on the results of the measurement of physical quantities. This gives possibility to use the classical models and statistical methods of probability theory in a lot of important cases. *For the large sample size when the violation of statistical stability manifests itself clearly, the using of classical stochastic models leads to unacceptably large measurement errors*. Then the *hyper-random models have obvious advantages over the stochastic models*.

8. The hyper-random models, unlike the random ones theoretically can be used both in case of large and small observation intervals as in large and small samples. However, the hyper-random

models are more complicated. Therefore *for not very large sample sizes the stochastic models are preferred*. The using of the hyper-random models is justified when the stochastic models do not provide an adequate description of the reality.

9. The *limited accuracy of any statistical measurement of actual physical quantities* and the *limited accuracy of the temporal progress forecasting of actual events* can be explained by the presence of a statistically unpredictable laws.

10. The limited nature of statistical stability suggests that it may be necessary to review the postulates of a number of physical disciplines, in which the probability concept and convergence play a key role, in particular, *statistical mechanics, statistical physics, and quantum mechanics*. Taking into account statistical stability violations may lead to new scientific results that will be interesting for both theory and practice.

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Nominative data with ordered set of names

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Abstract

In the paper we analyze the set of nominative sets, which can be considered as some mathematical model for data used in computing systems, under assumption that the set of names is linearly ordered. We design algorithms, implemented for execution of basic set-theoretic operations on this set of nominative sets under assumption that nominative sets are presented by doubly linked lists with the order of names in increasing strength. The worst-case time complexity under logarithmic weight for the designed algorithms is investigated in detail. Applications of presented results for table algebras, which are mathematical models, intended for developing and theoretic analysis of relational databases, as well as of associated query languages, are proposed. The obtained results can be used in formal software development.

Keywords: nominative set, nominative data, linear ordering, set-theoretic operations, time complexity of algorithms.

1 Introduction

It's well known that research of deep internal links "between the theory of programming and the products of software engineering practice" [1] is one of the main challenges of the 21^{st} century. The significance and complexity of this problem is caused by variety of theoretical and application-oriented approaches, that are hardly comparable with each other [2–11].

We deal with composition-nominative approach to program formalization [12–14]. Informally speaking, any software is regarded as some data processor. Mathematical model intended to present different data structures, used in computing systems (arrays, lists, tables, trees, etc.),

in a unified form is *a nominative set*. This model is based on the notion of *a name-value relation*. The cost of this universality is high complexity in elaboration of formal theory for nominative sets, intended to automate software design, starting with creation of formal specifications, and finishing with resolving problems of verification and testing. The main reason of this complexity is caused by the factor that it is necessary to deal with algebraic structures, which significantly differ from the classic ones.

This situation has been illustrated accurately in [15], where the most general case, arising under the assumption that the sets of names and data are abstract ones, has been investigated. Indeed, it has been established that for algebraic system in which the carrier is any fixed set of all nominative sets, and the set of operations is the set of set-theoretic operations over these nominative sets, there exist the following different types of subalgebras: commutative and non-commutative semigroups, non-commutative non-associative magma, and semi-rings. Also it has been established that any fixed partially ordered set of all nominative sets is the union of the set of overlapping isomorphic maximal closed intervals. The mappings that define isomorphism between two intervals differ significantly from each other, and the family of these mappings has sufficiently complicated structure.

The present paper is further development of investigations, that has been started in [15]. We study any fixed set of all nominative sets under assumption that the set of names is linearly ordered. The paper is organized as follows: in Section 2 we recall necessary notions and definitions; in Section 3 we investigate time complexity of set-theoretic operations on any fixed set of all nominative sets; in Section 4 we illustrate how results, established in Section 3 can be applied in table algebras; in Section 5 we give conclusions.

2 Basic notions

Let V ($|V| \geq 2$) and A ($|A| \geq 2$) be finite sets of names and data, respectively. The set $\mathfrak{F}_{V,A}$ of all (possibly, partial) mappings from V to A is called the set of nominative sets. Basic set-theoretic operations

on the set $\mathfrak{F}_{V,A}$ are defined as follows:

$$f_1 \cap f_2 = f \Leftrightarrow graph(f_1) \cap graph(f_2) = graph(f), \quad (1)$$

$$f_1 \setminus f_2 = f \Leftrightarrow graph(f_1) \setminus graph(f_2) = graph(f), \quad (2)$$

$$f_1 \cup f_2 = f \Leftrightarrow graph(f_1) \cup graph(f_2) = graph(f), \quad (3)$$

$$f_1 \oplus f_2 = f \Leftrightarrow graph(f_1) \oplus graph(f_2) = graph(f), \quad (4)$$

$$f_1 \triangleright f_2 = f \Leftrightarrow graph(f_1) \cup graph(f_2|_{\mathrm{Dom} f_2 \setminus \mathrm{Dom} f_1}) = graph(f), \quad (5)$$

$$f_1 \boxplus f_2 = f \Leftrightarrow$$

$$\Leftrightarrow graph(f_1|_{\mathrm{Dom} f_1 \setminus \mathrm{Dom} f_2}) \cup graph(f_2|_{\mathrm{Dom} f_2 \setminus \mathrm{Dom} f_1}) = graph(f). \quad (6)$$

We recall that $\cup$ and $\oplus$ are partial operations on the set $\mathfrak{F}_{V,A}$.

It is supposed that some linearly ordering relation $<_V$ is fixed on the set of the names V (this is true, for example, when the set V reflects different types of memory addressing, such as real, physical, flat, or absolute addressing). Thus, we get the possibility to present any nominative set $f = \{(v_i, a_i) | i = 1, \ldots, k\} \in \mathfrak{F}_{V,A}$ $(k = 0, 1, \ldots, |V|)$ by such doubly linked list L_f:

$$\begin{array}{c} \\ i_1 \\ i_2 \\ \ldots \\ i_{k-1} \\ i_k \end{array} \begin{array}{c} \begin{array}{cccc} \text{previous} & \text{name} & \text{data} & \text{next} \end{array} \\ \left(\begin{array}{cccc} * & v_{r_1} & a_{r_1} & i_2 \\ i_1 & v_{r_2} & a_{r_2} & i_3 \\ \ldots & \ldots & \ldots & \ldots \\ i_{k-2} & v_{r_{k-1}} & a_{r_{k-1}} & i_k \\ i_{k-1} & v_{r_k} & a_{r_k} & * \end{array} \right), \end{array}$$

that the inequalities $v_{r_1} <_V v_{r_2} <_V \ldots <_V v_{r_{k-1}} <_V v_{r_k}$ hold.

In the next section we will show that for basic set-theoretic operations on the set $\mathfrak{F}_{V,A}$ the order of names in increasing strength also remains in the resulting doubly linked list.

For doubly linked list L_f we set:

$$\mathsf{L}_f(i_j) = \begin{cases} (*, v_{r_1}, a_{r_1}, i_2), & if \ \ j = 1 \\ (i_{j-1}, v_{r_j}, a_{r_j}, i_{j+1}), & if \ \ j = 2, \ldots, k-1 \\ (i_{k-1}, v_{r_k}, a_{r_k}, *), & if \ \ j = k \end{cases}$$

and $length(\mathsf{L}_f) = k$. The last equality implies that

$$length(\mathsf{L}_f) \leq |V| \tag{7}$$

for any nominative set $f \in \mathfrak{F}_{V,A}$.

3 Time complexity of set-theoretic operations on the set $\mathfrak{F}_{V,A}$

We will investigate time complexity of set-theoretic operations (1)-(6) on the set $\mathfrak{F}_{V,A}$ under the following assumptions:

1) initial nominative sets $f_1, f_2 \in \mathfrak{F}_{V,A}$ are presented by the doubly linked lists, respectively L_{f_1} and L_{f_2};

2) the result of operation $f_1 \diamond f_2$ ($\diamond \in \{\cap, \setminus, \cup, \oplus, \triangleright, \boxplus\}$) is the doubly linked list $\mathsf{L}_{f_1 \diamond f_2}$;

3) the address parameter is denoted:

– by $i_j^{(1)}$ ($j = 1, \ldots, length(\mathsf{L}_{f_1})$), for doubly linked list L_{f_1};

– by $i_j^{(2)}$ ($j = 1, \ldots, length(\mathsf{L}_{f_2})$), for doubly linked list L_{f_2};

– by $i_j^{(3)}$ ($j = 1, \ldots, length(\mathsf{L}_{f_1 \diamond f_2})$), for doubly linked list $\mathsf{L}_{f_1 \diamond f_2}$.

Analyzing time complexity of set-theoretic operations on the set $\mathfrak{F}_{V,A}$, we deal with asymptotic worst-case time complexity of algorithms under logarithmic weight [16]. The last factor implies that:

1) time $T = O(\log |V|)$ ($|V| \to \infty$) is needed to check for any names $v_1, v_2 \in V$, what of formulae, either $v_1 = v_2$, $v_1 \neq v_2$, or $v_1 <_V v_2$, holds;

2) time $T = O(\log |A|)$ ($|A| \to \infty$) is needed to check for any data $a_1, a_2 \in A$, what of formulae, either $a_1 = a_2$, or $a_1 \neq a_2$, holds;

3) time $T = O(\log length(\mathsf{L}_f))$ ($length(\mathsf{L}_f) \to \infty$) is needed for transition (if it is possible) from any current element of any doubly linked list L_f, either to its previous element, or to its next element.

If initial nominative sets $f_1, f_2 \in \mathfrak{F}_{V,A}$ are presented by doubly linked lists, then the idea of how to compute the result of operation $f_1 \diamond f_2$ ($\diamond \in \{\cap, \setminus, \cup, \oplus, \triangleright, \boxplus\}$), is rather simple: sequentially moving through the doubly linked lists L_{f_1} and L_{f_2}, from their beginnings to their ends, we create the doubly linked list $\mathsf{L}_{f_1 \diamond f_2}$. However, despite

transparency of this idea, there are many subtle aspects in case of its implementation. For this reason, we will design appropriate algorithms in an explicit form.

The operation $\cap$ on the set $\mathfrak{F}_{V,A}$ can be implemented as follows.

ALGORITHM 1.
Input: doubly linked lists L_{f_1} and L_{f_2}.
Output: doubly linked list $\mathsf{L}_{f_1 \cap f_2}$.
Step 1. $\mathsf{L}_{f_1 \cap f_2} := \emptyset$, $j_3 := 0$.
Step 2. If $length(\mathsf{L}_{f_1}) = 0$ or $length(\mathsf{L}_{f_2}) = 0$,
then HALT, else $j_1 := 1$, $j_2 := 1$.
Step 3. $\mathbf{u}_1 := \mathsf{L}_{f_1}(i^{(1)}_{j_1})$, $\mathbf{u}_2 := \mathsf{L}_{f_2}(i^{(2)}_{j_2})$.
Step 4. If $\mathrm{pr}_2\mathbf{u}_1 = \mathrm{pr}_2\mathbf{u}_2$, then go to step 7.
Step 5. If $\mathrm{pr}_2\mathbf{u}_1 <_V \mathrm{pr}_2\mathbf{u}_2$, then $r := 1$, else $r := 2$.
Step 6. Call *Procedure1.1*(r).
Step 7. If $\mathrm{pr}_3\mathbf{u}_1 = \mathrm{pr}_3\mathbf{u}_2$, then Call *Procedure1.2*
Step 8. If $\mathrm{pr}_4\mathbf{u}_1 = *$ or $\mathrm{pr}_4\mathbf{u}_2 = *$,
then HALT, else $j_1 := j_1 + 1$, $j_2 := j_2 + 1$,
and go to step 3.

Procedure1.1(r)
begin;
If $\mathrm{pr}_4\mathbf{u}_r = *$, then HALT,
else $j_r := j_r + 1$, $\mathbf{u}_r := \mathsf{L}_{f_r}(i^{(r)}_{j_r})$, and go to step 4;
end;

Procedure1.2
begin;
If $j_3 \neq 0$, then go to M1;
$j_3 := j_3 + 1$, $\mathsf{L}_{f_1 \cap f_2}(i^{(3)}_{j_3}) := (*, \mathrm{pr}_2\mathbf{u}_1, \mathrm{pr}_3\mathbf{u}_1, *)$,
go to step 8;
M1: $\mathrm{pr}_4\mathsf{L}_{f_1 \cap f_2}(i^{(3)}_{j_3}) := j_3 + 1$,
$\mathsf{L}_{f_1 \cap f_2}(i^{(3)}_{j_3+1}) := (j_3, \mathrm{pr}_2\mathbf{u}_1, \mathrm{pr}_3\mathbf{u}_1, *)$, $j_3 := j_3 + 1$,
go to step 8;
end;

Correctness of algorithm 1 is justified by the following two factors. Firstly, algorithm 1 always halts. Secondly, the list $\mathsf{L}_{f_1 \cap f_2}$ consists of those and only those pairs $(v, a) \in V \times A$, which are elements of the set $graph(f_1 \cap f_2)$.

Theorem 1. *For any nominative sets $f_1, f_2 \in \mathfrak{F}_{V,A}$ time complexity of algorithm 1 is*

$$T = O((length(\mathsf{L}_{f_1}) + length(\mathsf{L}_{f_2}))(\log |V| + \log |A|)) \ (|V| \to \infty). \quad (8)$$

Proof. Let us estimate time complexity of steps of algorithm 1 under the assumption that we are shifting through the doubly linked lists L_{f_1} and L_{f_2}, from their beginnings to their ends, no more, than on one line item.

Time complexity for each of steps 1, and 2 is

$$T = O(1) \ (|V| \to \infty). \quad (9)$$

Time complexity for each of steps 3, 6, and 7 is

$$T = O(\log |V| + \log |A|) \ (|V| \to \infty). \quad (10)$$

Time complexity for each of steps 4, 5, and 8 is

$$T = O(\log |V|) \ (|V| \to \infty). \quad (11)$$

The number of cycles via operation of algorithm 1 doesn't exceed the value

$$length(\mathsf{L}_{f_1}) + length(\mathsf{L}_{f_2}) \quad (12)$$

Formulae (9)-(12) imply that formula (8) holds. □

Corollary 1.1. *For any nominative sets $f_1, f_2 \in \mathfrak{F}_{V,A}$ the following estimation for time complexity of algorithm 1 is true*

$$T = O(|V|(\log |V| + \log |A|)) \ (|V| \to \infty). \quad (13)$$

Proof. Substituting (7) in (8), we get that formula (13) holds. □

Corollary 1.2. *Let $|V| \to \infty$. If either $|A| = o(|V|)$, or $|A| = O(|V|)$, then for any nominative sets $f_1, f_2 \in \mathfrak{F}_{V,A}$ the following estimation of time complexity for algorithm 1 is true*

$$T = O(|V| \log |V|) \ (|V| \to \infty). \tag{14}$$

Proof. Suppose, that $|V| \to \infty$. Substituting either $|A| = o(|V|)$, or $|A| = O(|V|)$ in (13), we get that formula (14) holds. □

Corollary 1.3. *Let $|V| \to \infty$. If either $|V| = o(|A|)$, or $|V| = O(|A|)$, then for any nominative sets $f_1, f_2 \in \mathfrak{F}_{V,A}$ the following estimation for time complexity of algorithm 1 is true*

$$T = O(|A| \log |V|) \ (|V| \to \infty). \tag{15}$$

Proof. Suppose, that $|V| \to \infty$. Substituting either $|V| = o(|A|)$, or $|V| = O(|A|)$ in (14), we get that formula (15) holds. □

The operation $\setminus$ on the set $\mathfrak{F}_{V,A}$ can be implemented as follows.

Algorithm 2.
Input: doubly linked lists L_{f_1} and L_{f_2}.
Output: doubly linked list $\mathsf{L}_{f_1 \setminus f_2}$.
Step 1. $\mathsf{L}_{f_1 \setminus f_2} := \emptyset$, $j_3 := 0$.
Step 2. If $length(\mathsf{L}_{f_1}) = 0$, then HALT.
Step 3. If $length(\mathsf{L}_{f_2}) = 0$, then $\mathsf{L}_{f_1 \setminus f_2} := \mathsf{L}_{f_1}$, and HALT.
Step 4. $j_1 := 1$, $j_2 := 1$, $\mathbf{u}_1 := \mathsf{L}_{f_1}(i^{(1)}_{j_1})$, $\mathbf{u}_2 := \mathsf{L}_{f_2}(i^{(2)}_{j_2})$.
Step 5. If $\mathrm{pr}_2\mathbf{u}_1 = \mathrm{pr}_2\mathbf{u}_2$, then go to step 9.
Step 6. If $\mathrm{pr}_2\mathbf{u}_1 <_V \mathrm{pr}_2\mathbf{u}_2$,
then $r := 0$, and Call *Procedure2.1*(r).
Step 7. If $\mathrm{pr}_4\mathbf{u}_2 = *$, then $r := 1$, and Call *Procedure2.1*(r).
Step 8. $j_2 := j_2 + 1$, $\mathbf{u}_2 := \mathsf{L}_{f_2}(i^{(2)}_{j_2})$, and go to step 5.
Step 9. If $\mathrm{pr}_3\mathbf{u}_1 \neq \mathrm{pr}_3\mathbf{u}_2$,
then $r := 0$, and Call *Procedure2.1*(r).
Step 10. If $\mathrm{pr}_4\mathbf{u}_1 = *$, then HALT,
else $j_1 := j_1 + 1$, $\mathbf{u}_1 := \mathsf{L}_{f_1}(i^{(1)}_{j_1})$,
and go to step 7.

Procedure2.1(r)
begin;
M3: If $j_3 = 0$,
then $j_3 := j_3 + 1$, $\mathsf{L}_{f_1 \setminus f_2}(i^{(3)}_{j_3}) := (*, \mathrm{pr}_2\mathbf{u}_1, \mathrm{pr}_3\mathbf{u}_1, *)$,
and go to M1;
M2: $\mathrm{pr}_4\mathsf{L}_{f_1 \setminus f_2}(i^{(3)}_{j_3}) := j_3 + 1$,
$\mathsf{L}_{f_1 \setminus f_2}(i^{(3)}_{j_3+1}) := (j_3, \mathrm{pr}_2\mathbf{u}_1, \mathrm{pr}_3\mathbf{u}_1, *)$, $j_3 := j_3 + 1$;
M1: If $\mathrm{pr}_4\mathbf{u}_1 = *$, then HALT,
else $j_1 := j_1 + 1$, $\mathbf{u}_1 := \mathsf{L}_{f_1}(i^{(1)}_{j_1})$;
If $r = 0$, then go to step 7, else go to M3;
end;

Correctness of algorithm 2 is justified by the following two factors. Firstly, algorithm 2 always halts. Secondly, the list $\mathsf{L}_{f_1 \setminus f_2}$ consists of those and only those pairs $(v, a) \in V \times A$, which are elements of the set $graph(f_1 \setminus f_2)$.

Theorem 2. *For any nominative sets $f_1, f_2 \in \mathfrak{F}_{V,A}$ time complexity of algorithm 2 is*

$$T = O((length(\mathsf{L}_{f_1}) + length(\mathsf{L}_{f_2}))(\log|V| + \log|A|)) \ (|V| \to \infty). \quad (16)$$

Proof is similar to proof of theorem 1.

Corollary 2.1. *For any nominative sets $f_1, f_2 \in \mathfrak{F}_{V,A}$ the following estimation for time complexity of algorithm 2 is true*

$$T = O(|V|(\log|V| + \log|A|)) \ (|V| \to \infty). \quad (17)$$

Proof is similar to proof of corollary 1.1.

Corollary 2.2. *Let $|V| \to \infty$. If either $|A| = o(|V|)$, or $|A| = O(|V|)$, then for any nominative sets $f_1, f_2 \in \mathfrak{F}_{V,A}$ the following estimation of time complexity for algorithm 2 is true*

$$T = O(|V| \log|V|) \ (|V| \to \infty). \quad (18)$$

Proof is similar to proof of corollary 1.2.

Corollary 2.3. *Let $|V| \to \infty$. If either $|V| = o(|A|)$, or $|V| = O(|A|)$, then for any nominative sets $f_1, f_2 \in \mathfrak{F}_{V,A}$ the following estimation for time complexity of algorithm 2 is true*

$$T = O(|A| \log |V|) \ (|V| \to \infty). \tag{19}$$

Proof is similar to proof of corollary 1.3.

The operation $\cup$ on the set $\mathfrak{F}_{V,A}$ can be implemented as follows.

Algorithm 3.

Input: doubly linked lists L_{f_1} and L_{f_2}.

Output: $\alpha \in \{0, 1\}$, where $\alpha = 0$, if operation $\cup$ is not defined for $f_1, f_2 \in \mathfrak{F}_{V,A}$, and $\alpha = 1$, if doubly linked list $\mathsf{L}_{f_1 \cup f_2}$ is computed.

Step 1. $\mathsf{L}_{f_1 \cup f_2} := \emptyset$, $\alpha := 0$, $j_3 := 0$.

Step 2. If $length(\mathsf{L}_{f_1}) = 0$,
then $\mathsf{L}_{f_1 \cup f_2} := \mathsf{L}_{f_2}$, $\alpha := 1$, and HALT.

Step 3. If $length(\mathsf{L}_{f_2}) = 0$,
then $\mathsf{L}_{f_1 \cup f_2} := \mathsf{L}_{f_1}$, $\alpha := 1$, and HALT.

Step 4. $j_1 := 1$, $j_2 := 1$, $\mathbf{u}_1 := \mathsf{L}_{f_1}(i^{(1)}_{j_1})$, $\mathbf{u}_2 := \mathsf{L}_{f_2}(i^{(2)}_{j_2})$.

Step 5. If $\mathrm{pr}_2\mathbf{u}_1 <_V \mathrm{pr}_2\mathbf{u}_2$,
then $r := 1$, and Call *Procedure3.1*(r).

Step 6. If $\mathrm{pr}_2\mathbf{u}_2 <_V \mathrm{pr}_2\mathbf{u}_1$,
then $r := 2$, and Call *Procedure3.1*(r).

Step 7. If $\mathrm{pr}_3\mathbf{u}_1 \neq \mathrm{pr}_3\mathbf{u}_2$, then $\alpha := 0$, and HALT.

Step 8. If $j_3 = 0$, then $j_3 := j_3 + 1$,
$\mathsf{L}_{f_1 \cup f_2}(i^{(3)}_{j_3}) := (*, \mathrm{pr}_2\mathbf{u}_1, \mathrm{pr}_3\mathbf{u}_1, *)$,
and go to step 10.

Step 9. $\mathrm{pr}_4\mathsf{L}_{f_1 \cup f_2}(i^{(3)}_{j_3}) := j_3 + 1$,
$\mathsf{L}_{f_1 \cup f_2}(i^{(3)}_{j_3+1}) := (j_3, \mathrm{pr}_2\mathbf{u}_1, \mathrm{pr}_3\mathbf{u}_1, *)$, $j_3 := j_3 + 1$.

Step 10. If $\mathrm{pr}_4\mathbf{u}_1 = *$ and $\mathrm{pr}_4\mathbf{u}_2 = *$, then $\alpha := 1$, and HALT.

Step 11. If $\mathrm{pr}_4\mathbf{u}_1 = *$ and $\mathrm{pr}_4\mathbf{u}_2 \neq *$, then go to step 14.

Step 12. If $\mathrm{pr}_4\mathbf{u}_1 \neq *$ and $\mathrm{pr}_4\mathbf{u}_2 = *$, then go to step 16.

Step 13. $j_1 := j_1 + 1$, $j_2 := j_2 + 1$,
$\mathbf{u}_1 := \mathsf{L}_{f_1}(i^{(1)}_{j_1})$, $\mathbf{u}_2 := \mathsf{L}_{f_2}(i^{(2)}_{j_2})$, and go to step 5.

Step 14. $j_2 := j_2 + 1$, $\mathbf{u}_2 := \mathsf{L}_{f_2}(i^{(2)}_{j_2})$, $\mathrm{pr}_4\mathsf{L}_{f_1 \cup f_2}(i^{(3)}_{j_3}) := j_3 + 1$,

$\mathsf{L}_{f_1\cup f_2}(i^{(3)}_{j_3+1}) := (j_3, \mathrm{pr}_2\mathbf{u}_2, \mathrm{pr}_3\mathbf{u}_2, *)$, $j_3 := j_3 + 1$.

Step 15. If $\mathrm{pr}_4\mathbf{u}_2 = *$, then $\alpha := 1$, and HALT, else go to step 14.

Step 16. $j_1 := j_1 + 1$, $\mathbf{u}_1 := \mathsf{L}_{f_1}(i^{(1)}_{j_1})$, $\mathrm{pr}_4\mathsf{L}_{f_1\cup f_2}(i^{(3)}_{j_3}) := j_3 + 1$,

$\mathsf{L}_{f_1\cup f_2}(i^{(3)}_{j_3+1}) := (j_3, \mathrm{pr}_2\mathbf{u}_1, \mathrm{pr}_3\mathbf{u}_1, *)$, $j_3 := j_3 + 1$.

Step 17. If $\mathrm{pr}_4\mathbf{u}_1 = *$, then $\alpha := 1$, and HALT, else go to step 16.

Procedure3.1(r)

begin;

If $j_3 = 0$,

then $j_3 := j_3 + 1$, $\mathsf{L}_{f_1\cup f_2}(i^{(3)}_{j_3}) := (*, \mathrm{pr}_2\mathbf{u}_r, \mathrm{pr}_3\mathbf{u}_r, *)$,

and go to M1;

$\mathrm{pr}_4\mathsf{L}_{f_1\cup f_2}(i^{(3)}_{j_3}) := j_3 + 1$,

$\mathsf{L}_{f_1\cup f_2}(i^{(3)}_{j_3+1}) := (j_3, \mathrm{pr}_2\mathbf{u}_r, \mathrm{pr}_3\mathbf{u}_r, *)$, $j_3 := j_3 + 1$;

M1: If $\mathrm{pr}_4\mathbf{u}_r \neq *$,

then $j_r := j_r + 1$, $\mathbf{u}_r := \mathsf{L}_{f_r}(i^{(r)}_{j_r})$, and go to step 5.

$\mathsf{L}_{f_1\cup f_2}(i^{(3)}_{j_3+1}) := (j_3, \mathrm{pr}_2\mathbf{u}_{3-r}, \mathrm{pr}_3\mathbf{u}_{3-r}, *)$, $j_3 := j_3 + 1$;

M2: If $\mathrm{pr}_4\mathbf{u}_{3-r} = *$, then $\alpha := 1$, and HALT;

$j_{3-r} := j_{3-r}+1$, $\mathbf{u}_{3-r} := \mathsf{L}_{f_{3-r}}(i^{(3-r)}_{j_{3-r}})$, $\mathrm{pr}_4\mathsf{L}_{f_1\cup f_2}(i^{(3)}_{j_3}) := j_3+1$,

$\mathsf{L}_{f_1\cup f_2}(i^{(3)}_{j_3+1}) := (j_3, \mathrm{pr}_2\mathbf{u}_{3-r}, \mathrm{pr}_3\mathbf{u}_{3-r}, *)$, $j_3 := j_3+1$, go to M2;

end;

Correctness of algorithm 3 is justified by the following two factors. Firstly, algorithm 3 always halts. Secondly, $\alpha = 1$ if and only if operation $\cup$ is defined for $f_1, f_2 \in \mathfrak{F}_{V,A}$, and doubly linked list $\mathsf{L}_{f_1\cup f_2}$ is computed.

Theorem 3. *For any nominative sets $f_1, f_2 \in \mathfrak{F}_{V,A}$ time complexity of algorithm 3 is*

$$T = O((length(\mathsf{L}_{f_1}) + length(\mathsf{L}_{f_2}))(\log|V| + \log|A|)) \ (|V| \to \infty). \quad (20)$$

Proof is similar to proof of theorem 1.

Corollary 3.1. *For any nominative sets $f_1, f_2 \in \mathfrak{F}_{V,A}$ the following estimation for time complexity of algorithm 3 is true*

$$T = O(|V|(\log|V| + \log|A|)) \ (|V| \to \infty). \quad (21)$$

Proof is similar to proof of corollary 1.1.

Corollary 3.2. *Let $|V| \to \infty$. If either $|A| = o(|V|)$, or $|A| = O(|V|)$, then for any nominative sets $f_1, f_2 \in \mathfrak{F}_{V,A}$ the following estimation of time complexity for algorithm 3 is true*

$$T = O(|V| \log |V|) \ (|V| \to \infty). \tag{22}$$

Proof is similar to proof of corollary 1.2.

Corollary 3.3. *Let $|V| \to \infty$. If either $|V| = o(|A|)$, or $|V| = O(|A|)$, then for any nominative sets $f_1, f_2 \in \mathfrak{F}_{V,A}$ the following estimation for time complexity of algorithm 3 is true*

$$T = O(|A| \log |V|) \ (|V| \to \infty). \tag{23}$$

Proof is similar to proof of corollary 1.3.

The operation $\oplus$ on the set $\mathfrak{F}_{V,A}$ can be implemented as follows.

Algorithm 4.

Input: doubly linked lists L_{f_1} and L_{f_2}.

Output: $\alpha \in \{0, 1\}$, where $\alpha = 0$, if operation $\oplus$ is not defined for $f_1, f_2 \in \mathfrak{F}_{V,A}$, and $\alpha = 1$, if doubly linked list $\mathsf{L}_{f_1 \oplus f_2}$ is computed.

Step 1. $\mathsf{L}_{f_1 \oplus f_2} := \emptyset$, $\alpha := 0$, $j_3 := 0$.

Step 2. If $length(\mathsf{L}_{f_1}) = 0$, then $\mathsf{L}_{f_1 \oplus f_2} := \mathsf{L}_{f_2}$, $\alpha := 1$, and HALT.

Step 3. If $length(\mathsf{L}_{f_2}) = 0$, then $\mathsf{L}_{f_1 \oplus f_2} := \mathsf{L}_{f_1}$, $\alpha := 1$, and HALT.

Step 4. $j_1 := 1$, $j_2 := 1$, $\mathbf{u}_1 := \mathsf{L}_{f_1}(i^{(1)}_{j_1})$, $\mathbf{u}_2 := \mathsf{L}_{f_2}(i^{(2)}_{j_2})$.

Step 5. If $\mathrm{pr}_2\mathbf{u}_1 <_V \mathrm{pr}_2\mathbf{u}_2$,
then $r := 1$, and Call *Procedure4.1*(r).

Step 6. If $\mathrm{pr}_2\mathbf{u}_2 <_V \mathrm{pr}_2\mathbf{u}_1$,
then $r := 2$, and Call *Procedure4.1*(r).

Step 7. If $\mathrm{pr}_3\mathbf{u}_1 \neq \mathrm{pr}_3\mathbf{u}_2$, then $\alpha := 0$, and HALT.

Step 8. If $\mathrm{pr}_4\mathbf{u}_1 = *$ and $\mathrm{pr}_4\mathbf{u}_2 = *$, then $\alpha := 1$, and HALT.

Step 9. If $\mathrm{pr}_4\mathbf{u}_1 = *$ and $\mathrm{pr}_4\mathbf{u}_2 \neq *$,
then $r := 2$ and Call *Procedure4.3*(r).

Step 10. If $\mathrm{pr}_4\mathbf{u}_1 \neq *$ and $\mathrm{pr}_4\mathbf{u}_2 = *$,
then $r := 1$ and Call *Procedure4.3*(r).

Step 11. $j_1 := j_1 + 1$, $j_2 := j_2 + 1$,
$\mathbf{u}_1 := \mathsf{L}_{f_1}(i^{(1)}_{j_1})$, $\mathbf{u}_2 := \mathsf{L}_{f_2}(i^{(2)}_{j_2})$, and go to step 5.

Procedure4.1(r)
begin;
If $j_3 = 0$, then $j_3 := j_3 + 1$,
$\mathsf{L}_{f_1 \oplus f_2}(i^{(3)}_{j_3}) := (*, \mathrm{pr}_2\mathbf{u}_r, \mathrm{pr}_3\mathbf{u}_r, *)$,
and go to M1;
$\mathrm{pr}_4\mathsf{L}_{f_1 \oplus f_2}(i^{(3)}_{j_3}) := j_3 + 1$,
$\mathsf{L}_{f_1 \oplus f_2}(i^{(3)}_{j_3+1}) := (j_3, \mathrm{pr}_2\mathbf{u}_r, \mathrm{pr}_3\mathbf{u}_r, *)$, $j_3 := j_3 + 1$;
M1: If $\mathrm{pr}_4\mathbf{u}_r \neq *$,
then $j_1 := j_1 + 1$, $\mathbf{u}_r := \mathsf{L}_{f_r}(i^{(r)}_{j_r})$, and go to step 5;
$\mathsf{L}_{f_1 \oplus f_2}(i^{(3)}_{j_3+1}) := (j_3, \mathrm{pr}_2\mathbf{u}_{3-r}, \mathrm{pr}_3\mathbf{u}_{3-r}, *)$, $j_3 := j_3 + 1$;
Call *Procedure4.2*$(3 - r)$;
end;

Procedure4.2(r)
begin;
M1: If $\mathrm{pr}_4\mathbf{u}_r = *$, then $\alpha := 1$, and HALT;
$j_r := j_r + 1$, $\mathbf{u}_r := \mathsf{L}_{f_r}(i^{(r)}_{j_r})$;
If $j_3 = 0$,
then $j_3 := j_3 + 1$, $\mathsf{L}_{f_1 \oplus f_2}(i^{(3)}_{j_3}) := (*, \mathrm{pr}_2\mathbf{u}_r, \mathrm{pr}_3\mathbf{u}_r, *)$,
and go to M1;
$\mathrm{pr}_4\mathsf{L}_{f_1 \oplus f_2}(i^{(3)}_{j_3}) := j_3 + 1$, $\mathsf{L}_{f_1 \oplus f_2}(i^{(3)}_{j_3+1}) := (j_3, \mathrm{pr}_2\mathbf{u}_r, \mathrm{pr}_3\mathbf{u}_r, *)$,
$j_3 := j_3 + 1$, and go to M1;
end;

Procedure4.3(r)
begin;
M1: $j_r := j_r + 1$, $\mathbf{u}_r := \mathsf{L}_{f_r}(i^{(r)}_{j_r})$;
If $j_3 = 0$,
then $j_3 := j_3 + 1$, $\mathsf{L}_{f_1 \oplus f_2}(i^{(3)}_{j_3}) := (*, \mathrm{pr}_2\mathbf{u}_r, \mathrm{pr}_3\mathbf{u}_r, *)$,
and go to M2;
$\mathrm{pr}_4\mathsf{L}_{f_1 \oplus f_2}(i^{(3)}_{j_3}) := j_3 + 1$,
$\mathsf{L}_{f_1 \oplus f_2}(i^{(3)}_{j_3+1}) := (j_3, \mathrm{pr}_2\mathbf{u}_r, \mathrm{pr}_3\mathbf{u}_r, *)$, $j_3 := j_3 + 1$;

M2: If $\mathrm{pr}_4\mathbf{u}_2 = *$, then $\alpha := 1$, and HALT, else go to M1;
end;

Correctness of algorithm 4 is justified by the following two factors. Firstly, algorithm 4 always halts. Secondly, $\alpha = 1$ if and only if operation $\oplus$ is defined for $f_1, f_2 \in \mathfrak{F}_{V,A}$, and doubly linked list $\mathsf{L}_{f_1 \oplus f_2}$ is computed.

Theorem 4. *For any nominative sets $f_1, f_2 \in \mathfrak{F}_{V,A}$ time complexity of algorithm 4 is*

$$T = O((length(\mathsf{L}_{f_1}) + length(\mathsf{L}_{f_2}))(\log|V| + \log|A|)) \ (|V| \to \infty). \quad (24)$$

Proof is similar to proof of theorem 1.

Corollary 4.1. *For any nominative sets $f_1, f_2 \in \mathfrak{F}_{V,A}$ the following estimation for time complexity of algorithm 4 is true*

$$T = O(|V|(\log|V| + \log|A|)) \ (|V| \to \infty). \quad (25)$$

Proof is similar to proof of corollary 1.1.

Corollary 4.2. *Let $|V| \to \infty$. If either $|A| = o(|V|)$, or $|A| = O(|V|)$, then for any nominative sets $f_1, f_2 \in \mathfrak{F}_{V,A}$ the following estimation of time complexity for algorithm 4 is true*

$$T = O(|V| \log|V|) \ (|V| \to \infty). \quad (26)$$

Proof is similar to proof of corollary 1.2.

Corollary 4.3. *Let $|V| \to \infty$. If either $|V| = o(|A|)$, or $|V| = O(|A|)$, then for any nominative sets $f_1, f_2 \in \mathfrak{F}_{V,A}$ the following estimation for time complexity of algorithm 4 is true*

$$T = O(|A| \log|V|) \ (|V| \to \infty). \quad (27)$$

Proof is similar to proof of corollary 1.3.

It is evident that to compute the result of each of operations, $\triangleright$ and $\boxplus$, defined by formulae (5) and (6), it is needed some algorithm, which for any given nominative sets $f, g \in \mathfrak{F}_{V,A}$, presented by doubly linked

lists L_f and L_g, computes the doubly linked list $\mathsf{L}_{f|_{Dom\ f \setminus Dom\ g}}$. Using this algorithm, we can apply the algorithm 3 to compute the result of each of operations $\triangleright$ and $\boxplus$.

An algorithm, which for any given nominative sets $f, g \in \mathfrak{F}_{V,A}$, presented by doubly linked lists L_f and L_g, computes the doubly linked list $\mathsf{L}_{f|_{Dom\ f \setminus Dom\ g}}$, can be designed as follows.

ALGORITHM 5.
Input: doubly linked lists L_f and L_g.
Output: doubly linked list $\mathsf{L}_{f|_{Dom\ f \setminus Dom\ g}}$.
Step 1. $\mathsf{L}_{f|_{Dom\ f \setminus Dom\ g}} := \emptyset$, $j_3 := 0$.
Step 2. If $length(\mathsf{L}_f) = 0$, then HALT.
Step 3. If $length(\mathsf{L}_g) = 0$, then $\mathsf{L}_{f|_{Dom\ f \setminus Dom\ g}} := \mathsf{L}_f$, and HALT.
Step 4. $j_1 := 1$, $j_2 := 1$, $\mathbf{u}_1 := \mathsf{L}_f(i^{(1)}_{j_1})$, $\mathbf{u}_2 := \mathsf{L}_g(i^{(2)}_{j_2})$.
Step 5. If $\mathrm{pr}_2\mathbf{u}_1 <_V \mathrm{pr}_2\mathbf{u}_2$, then Call *Procedure5.1.*
Step 6. If $\mathrm{pr}_2\mathbf{u}_2 <_V \mathrm{pr}_2\mathbf{u}_1$, then Call *Procedure5.2.*
Step 7. If $\mathrm{pr}_4\mathbf{u}_1 = *$,
then HALT,
else $j_1 := j_1 + 1$, $\mathbf{u}_1 := \mathsf{L}_f(i^{(1)}_{j_1})$, and go to step 5.

Procedure5.1.
begin;
If $j_3 = 0$ then $j_3 := j_3 + 1$,
$\mathsf{L}_{f|_{Dom\ f \setminus Dom\ g}}(i^{(3)}_{j_3}) := (*, \mathrm{pr}_2\mathbf{u}_1, \mathrm{pr}_3\mathbf{u}_1, *)$,
and go to M1;
$\mathrm{pr}_4\mathsf{L}_{f|_{Dom\ f \setminus Dom\ g}}(i^{(3)}_{j_3}) := j_3 + 1$,
$\mathsf{L}_{f|_{Dom\ f \setminus Dom\ g}}(i^{(3)}_{j_3+1}) := (j_3, \mathrm{pr}_2\mathbf{u}_1, \mathrm{pr}_3\mathbf{u}_1, *)$, $j_3 := j_3 + 1$;
M1: If $\mathrm{pr}_4\mathbf{u}_1 \neq *$,
then $j_1 := j_1 + 1$, $\mathbf{u}_1 := \mathsf{L}_f(i^{(1)}_{j_1})$, and go to step 5,
else HALT;
end;

Procedure5.2.
begin;
If $\mathrm{pr}_4\mathbf{u}_2 \neq *$,

then $j_2 := j_2 + 1$, $\mathbf{u}_2 := \mathsf{L}_g(i^{(2)}_{j_2})$, and go to step 5;
M1: If $j_3 = 0$ then $j_3 := j_3 + 1$,
$\mathsf{L}_{f|_{Dom\, f \setminus Dom\, g}}(i^{(3)}_{j_3}) := (*, \mathrm{pr}_2\mathbf{u}_1, \mathrm{pr}_3\mathbf{u}_1, *)$,
and go to M2;
$\mathrm{pr}_4\mathsf{L}_{f|_{Dom\, f \setminus Dom\, g}}(i^{(3)}_{j_3}) := j_3 + 1$,
$\mathsf{L}_{f|_{Dom\, f \setminus Dom\, g}}(i^{(3)}_{j_3+1}) := (j_3, \mathrm{pr}_2\mathbf{u}_1, \mathrm{pr}_3\mathbf{u}_1, *)$, $j_3 := j_3 + 1$;
M2: If $\mathrm{pr}_4\mathbf{u}_1 \neq *$,
then $j_1 := j_1 + 1$, $\mathbf{u}_1 := \mathsf{L}_f(i^{(1)}_{j_1})$, and go to M1,
else HALT;
end;

Correctness of algorithm 5 is justified by the following two factors. Firstly, algorithm 5 always halts. Secondly, the list $\mathsf{L}_{f|_{Dom\, f \setminus Dom\, g}}$ consists of those and only those pairs $(v, a) \in V \times A$, which are elements of the set $graph(f|_{Dom\, f \setminus Dom\, g})$.

Theorem 5. *For any nominative sets $f, g \in \mathfrak{F}_{V,A}$ time complexity of algorithm 5 is*

$$T = O((length(\mathsf{L}_f) + length(\mathsf{L}_g))(\log|V| + \log|A|)) \ (|V| \to \infty). \quad (28)$$

Proof is similar to proof of theorem 1.

Corollary 5.1. *For any nominative sets $f, g \in \mathfrak{F}_{V,A}$ the following estimation for time complexity of algorithm 5 is true*

$$T = O(|V|(\log|V| + \log|A|)) \ (|V| \to \infty). \quad (29)$$

Proof is similar to proof of corollary 1.1.

Corollary 5.2. *Let $|V| \to \infty$. If either $|A| = o(|V|)$, or $|A| = O(|V|)$, then for any nominative sets $f, g \in \mathfrak{F}_{V,A}$ the following estimation of time complexity for algorithm 5 is true*

$$T = O(|V|\log|V|) \ (|V| \to \infty). \quad (30)$$

Proof is similar to proof of corollary 1.2.

Corollary 5.3. *Let $|V| \to \infty$. If either $|V| = o(|A|)$, or $|V| = O(|A|)$, then for any nominative sets $f, g \in \mathfrak{F}_{V,A}$ the following estimation for time complexity of algorithm 5 is true*

$$T = O(|A| \log |V|) \ (|V| \to \infty). \tag{31}$$

Proof is similar to proof of corollary 1.3.

The operations $\triangleright$ and $\boxplus$ on the set $\mathfrak{F}_{V,A}$ can be implemented as follows.

ALGORITHM 6.
Input: doubly linked lists L_{f_1} and L_{f_2}.
Output: doubly linked list $\mathsf{L}_{f_1 \triangleright f_2}$.
Step 1. Applying algorithm 5 to doubly linked lists L_{f_2} and L_{f_1}, we design doubly linked list $\mathsf{L}_{f_2|_{Dom\ f_2 \setminus Dom\ f_1}}$.
Step 2. Applying algorithm 3 to doubly linked lists L_{f_1} and $\mathsf{L}_{f_2|_{Dom\ f_2 \setminus Dom\ f_1}}$, we design doubly linked list $\mathsf{L}_{f_1 \triangleright f_2}$.

ALGORITHM 7.
Input: doubly linked lists L_{f_1} and L_{f_2}.
Output: doubly linked list $\mathsf{L}_{f_1 \boxplus f_2}$.
Step 1. Applying algorithm 5 to doubly linked lists L_{f_1} and L_{f_2}, we design doubly linked list $\mathsf{L}_{f_1|_{Dom\ f_1 \setminus Dom\ f_2}}$.
Step 2. Applying algorithm 5 to doubly linked lists L_{f_2} and L_{f_1}, we design doubly linked list $\mathsf{L}_{f_2|_{Dom\ f_2 \setminus Dom\ f_1}}$.
Step 3. Applying algorithm 3 to doubly linked lists $\mathsf{L}_{f_1|_{Dom\ f_1 \setminus Dom\ f_2}}$ and $\mathsf{L}_{f_2|_{Dom\ f_2 \setminus Dom\ f_1}}$, we design doubly linked list $\mathsf{L}_{f_1 \boxplus f_2}$.

Estimations for time complexity of algorithms 3 and 5 (i.e. theorems 3 and 5, and corresponding corollaries) imply that the following estimations for time complexity of algorithms 6 and 7 hold.

Theorem 6. *For any nominative sets $f_1, f_2 \in \mathfrak{F}_{V,A}$ time complexity for each of algorithms 6 and 7 is*

$$T = O((length(\mathsf{L}_{f_1}) + length(\mathsf{L}_{f_2}))(\log |V| + \log |A|)) \ (|V| \to \infty). \tag{32}$$

Corollary 6.1. *For any nominative sets $f_1, f_2 \in \mathfrak{F}_{V,A}$ the following estimation of time complexity for each of algorithms 6 and 7 is true*

$$T = O(|V|(\log|V| + \log|A|)) \ (|V| \to \infty). \tag{33}$$

Corollary 6.2. *Let $|V| \to \infty$. If either $|A| = o(|V|)$, or $|A| = O(|V|)$, then for any nominative sets $f_1, f_2 \in \mathfrak{F}_{V,A}$ the following estimation of time complexity for each of algorithms 6 and 7 is true*

$$T = O(|V| \log|V|) \ (|V| \to \infty). \tag{34}$$

Corollary 6.3. *Let $|V| \to \infty$. If either $|V| = o(|A|)$, or $|V| = O(|A|)$, then for any nominative sets $f_1, f_2 \in \mathfrak{F}_{V,A}$ the following estimation of time complexity for each of algorithms 6 and 7 is true*

$$T = O(|A| \log|V|) \ (|V| \to \infty). \tag{35}$$

The obtained results justify the factor, that in the case of linear ordering on the set of names, all basic set-theoretic operations over nominative sets, presented by doubly linked lists with ordering of names in increasing strength, can be implemented by fast algorithms. The same is also true for operations of inserting elements in any nominative set, and of deleting elements from any nominative set.

Above, it has been investigated the case, when data are elements of an abstract set. Obviously that these results can be easily elaborated in detail for any case, when this or that structure is defined on the set of data. One of such examples will be considered in the next section.

4 Applications to table algebra

Relational databases are widely used in modern software systems. It is well known, that any relational database deals, in essence, with finite relations, defined on some Cartesian products [17].

Mathematical model, intended for developing and theoretic analysis of relational databases, as well as of associated query languages, is some table algebra. It can be characterized as follows.

For any relation its scheme is defined, which is a set of attributes. Any line of a relation is defined as a set of ordered pairs (attribute, value of attribute), where attribute transverses all values according to the scheme of the relation. A relation itself is defined as a set of lines. Proceeding from this set-theoretic representation, formal theory of relations can be implemented easily into table algebra. Thus, analysis in detail of set-theoretic operations over tables is essential for any table algebra.

Unfortunately, in investigation of table algebras all efforts are bent on development of descriptive theory, while there are practically no researches devoted to algorithms elaboration and analysis of their complexity.

Due to this factor, it is worth to point the paper [18], where there have been investigated worst-case and average-case time complexity of algorithms implemented for execution of three main set-theoretic operations over tables, namely: intersection, union and difference.

It is worth to note that in [18] time complexity is considered as the number of elementary steps. As the result, all estimations, established in [18], are typical estimations, that can be established for set-theoretic operations on abstract sets.

Let us consider, how the results, established in Section 3, can be applied effectively to the analysis of complexity of set-theoretic operations in table algebras.

Let $\mathcal{A} = \{A_i | i = 1, \ldots, k\}$ be the set of all attributes that are used in the given table algebra. The active domain of an attribute A_i $(i = 1, \ldots, k)$ is denoted by $Dom\, A_i$. It is evident that any set $Dom\, A_i$ $(i = 1, \ldots, k)$ can be linearly ordered. Due to this factor, we fix some linear ordering $<_{A_i}$ on each set $Dom\, A_i$ $(i = 1, \ldots, k)$. Thus, each Cartesian product $\mathcal{D}_{i_1,\ldots,i_l} = Dom\, A_{i_1} \times \cdots \times Dom\, A_{i_l}$, where $i_j \in \{1, \ldots, k\}$ for all $j \in \{1, \ldots, l\}$, is linearly ordered by lexicographic order $\prec_{i_1,\ldots,i_l}$.

The following two approaches, based on the theory of nominative sets with linearly ordered set of names, can be applied in table algebras for implementing set-theoretic operations on the set of relations of the form $\rho \subseteq \mathcal{D}_{i_1,\ldots,i_l}$, where $i_1, \ldots, i_l \in \{1, \ldots, k\}$ are fixed integers.

Approach I. It is supposed that the following two assumptions hold:

1) the set of all nominative sets is $\mathfrak{F}_{\mathcal{D}_{i_1,\ldots,i_l},\{1\}}$;

2) each nominative set $f_\rho \in \mathfrak{F}_{\mathcal{D}_{i_1,\ldots,i_l},\{1\}}$ is presented by doubly linked list L_{f_ρ} with ordering of names in the increasing strength.

Under these assumptions all algorithms, designed in Chapter 3, can be applied directly for implementation in table algebras operations $\cap$, $\setminus$, $\cup$, $\oplus$, $\triangleright$, and $\boxplus$, defined by formulae (1)-(6). Moreover, the following theorem holds

Theorem 7. *For any nominative sets $f_{\rho_1}, f_{\rho_2} \in \mathfrak{F}_{\mathcal{D}_{i_1,\ldots,i_l},\{1\}}$, presented by doubly linked lists with ordering of names in the increasing strength, for each of algorithms 1-7, implemented for execution of operations $\cap$, $\setminus$, $\cup$, $\oplus$, $\triangleright$, and $\boxplus$, defined by formulae (1)-(6), the worst-case time complexity under logarithmic weight is*

$$T = O((|\rho_1| + |\rho_2|) \log \prod_{j=1}^{l} |Dom\, A_{i_j}|) \ \ (|V| \to \infty). \tag{36}$$

It is worth to point out that in the considered case operations $\cup$ and $\oplus$ are usual total set-theoretic operations on the set $\mathfrak{F}_{\mathcal{D}_{i_1,\ldots,i_l},\{1\}}$.

Approach II. It is supposed that the following three assumptions hold:

1) the set of all nominative sets is $\mathfrak{F}_{V,\mathcal{D}_{i_1,\ldots,i_l}}$, where V is some set of names, ordered by linear ordering relation $<_V$;

2) each relation $\rho \subseteq \mathcal{D}_{i_1,\ldots,i_l}$ is presented by the nominative set $f_\rho \in \mathfrak{F}_{V,\mathcal{D}_{i_1,\ldots,i_l}}$, such that the following condition holds:

$$(v_1, (a_{i_1}^{(1)}, \ldots, a_{i_l}^{(1)})), (v_2, (a_{i_1}^{(2)}, \ldots, a_{i_l}^{(2)})) \in f_\rho \,\&\, v_1 <_V v_2 \Rightarrow$$
$$\Rightarrow (a_{i_1}^{(1)}, \ldots, a_{i_l}^{(1)}) \preceq_{i_1,\ldots,i_l} (a_{i_1}^{(2)}, \ldots, a_{i_l}^{(2)}),$$

i.e. each nominative set $f_\rho \in \mathfrak{F}_{V,\mathcal{D}_{i_1,\ldots,i_l}}$ is a nondecreasing (possibly partial) mapping from V to $\mathcal{D}_{i_1,\ldots,i_l}$;

3) each nominative set $f_\rho \in \mathfrak{F}_{V,\mathcal{D}_{i_1,\ldots,i_l}}$ is presented by doubly linked list L_{f_ρ} with ordering of names in the increasing strength.

Under these assumptions all algorithms, designed in Chapter 3, can be applied directly for implementation in table algebras operations $\cap$,

$\setminus$, $\cup$, $\oplus$, $\triangleright$, and $\boxplus$, defined by formulae (1)-(6). Moreover, the following theorem holds

Theorem 8. *For any nominative sets $f_{\rho_1}, f_{\rho_2} \in \mathfrak{F}_{V,\mathcal{D}_{i_1,\ldots,i_l}}$, presented by doubly linked lists with ordering of names in the increasing strength, for each of algorithms 1-7, implemented for execution of operations $\cap$, $\setminus$, $\cup$, $\oplus$, $\triangleright$, and $\boxplus$, defined by formulae (1)-(6), the worst-case time complexity under logarithmic weight is*

$$T = O((|\rho_1| + |\rho_2|)(\log |V| + \log \prod_{j=1}^{l} |Dom\, A_{i_j}|)) \ (|V| \to \infty). \quad (37)$$

It is worth to note that in the considered case operations $\cup$ and $\oplus$ are partial operations on the set $\mathfrak{F}_{V,\mathcal{D}_{i_1,\ldots,i_l}}$. For table algebras this factor means that some of these or the others additional conditions are associated with the set V of names.

Besides, if we take into account the factor, that each nominative set $f_\rho \in \mathfrak{F}_{V,\mathcal{D}_{i_1,\ldots,i_l}}$ is a nondecreasing (possibly partial) mapping from V to $\mathcal{D}_{i_1,\ldots,i_l}$, then we can speed-up execution of algorithms 1-7.

5 Conclusions

In the given paper it has been formed a strong base, sufficient for effective implementation of application-oriented algorithms theory in theory of nominative sets. Developing these results for hierarchical types of data, multidimensional arrays, lists, trees, algebraic data types, etc. forms some trend for future research.

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On Digitization of Romanian Cyrillic Printings of the 17th–18th Centuries

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Abstract

The paper describes in details recognition of Romanian texts of the 17th–18th centuries printed in the Cyrillic script, and their conversion to the modern Latin script. The challenges are discussed, and solutions of problems are proposed.

The elaborated technology and a tool pack include historical alphabets, sets of recognition patterns, and spelling dictionaries in the corresponding orthographies for ABBYY Finereader. In addition, virtual keyboards, fonts, a transliteration utility, and the user manual were developed.

This permits successful recognition of old Romanian texts in the Cyrillic script. Transliteration to the Latin script grants no-barrier access to historical documents.

1 Introduction

OCR of old books is a sophisticated task. The problems arose from peculiarities of historical typography, non-standardized spelling, and physical degradation of the documents due to their aging and usage.

This paper describes digitization of the Romanian texts of the 17th–18th centuries that were printed in the old Romanian Cyrillic script (RC). We need to OCR them, to present the recognized text in the fonts of the corresponding period, to transliterate then to the modern Romanian Latin script (MRL). Sometimes the reverse transliteration is also useful. We use the existing programs and develop our own pack of additional programs and data.

In the literature, we met the opinion that commercial software like ABBYY Finereader (AFR) is not fully suitable for OCR of old printings. It is so due to the big variability of old fonts, and because such software is internally trained with modern fonts and can't be fully retrained by users. A good introduction into the problem and further references can be found in [1, p. 2−4]. Authors of [1] prefer free "fully trainable" OCR tools like Tesseract or Ocropus.

Old Cyrillic fonts, especially of the selected epoch, are much less variable than Latin ones. The usage of the Cyrillic script is connected with the Slavonic liturgical language of the Orthodox church. For example, Orthodox Serbians use the Cyrillic alphabet while Catholic Croats speaking the same language use the Latin one. Therefore, the Cyrillic alphabet is used not so widely. Additionally, the Latin alphabet exists in blackletter and Antiqua typefaces that adds variability.

Our approach is based on the use of AFR. We produced sets of annexes to AFR containing templates for recognition collected after OCR training, alphabets, and spelling dictionaries (word lists).

We found that successful OCR supposes usage of many such sets corresponding not only to the epochs of Romanian Cyrillic printing but even to the specific typographies. It means that the said variability shows itself up but doesn't prevent obtaining good results with AFR.

It is possible that AFR work for old Cyrillic fonts better than for Latin ones due to some subtle similarity of old and modern Cyrillic fonts.

2 Digitization and its Problems

Romanian printing of the 16th, 17th and a big part of the 18th centuries used the 47-letter Romanian Cyrillic script and thoroughly imitated look and feel of the manuscripts.

We take as our study case the *New Testament of Belgrad* printed in 1648. Belgrad was a typography site; now this Romanian city is named Alba-Iulia.

This book of 682 pages is the very first edition of the full text of New Testament in the Romanian language. Till now we processed the

Gospel according to Matthew (78 pages) and continue the work.

One of the most common peculiarities of these printings is the positioning of some letters above the precedent letter (Fig. 1; a: scan; b: text in RC; c: the same in MRL; d: currently used Romanian variant; e: English, King James version). In manuscripts, this was made regularly to save place. Almost each consonant could be placed overline. At word end, the following **ь** was omitted; sometimes, the following vowel was omitted. This manner of writing was adopted from the Church Slavonic manuscripts [2].

a 

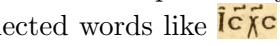
b **... Ирѡд, че пре алтѫ кале ѣѫ ѣѫ дтоаркѫ ла цара лѡрь.**

c ... Irod, ce pre altă cale să să întoarcă la țara lor.

d ... Irod, pe altă cale s-au dus în țara lor.

e ... Herod, they departed into their own country another way.

Figure 1. Overline letters in RC in 1648 (part of Matthew 2:12)

Another peculiarity is the usage of standard abbreviations for selected words like  for **Іисѹсъ Христосъ** (**Jesus Christ**). These abbreviations are composed from several letters and overline signs.

The third peculiarity is denoting numbers by letters with overline signs, like **ѕ** for 5.

The fourth is the stress signs. They were different for vowels at the word end as in **а̂флà** (**afla**, Eng. **learn**) and in another position as in **тóатѣ** (**toate**, **all**). A special sign was set over each vowel at the word beginning as in the same **а̂флà**; this sign replaced the stress if it was the case as in **а̂лтѣ** (**alte**, **other**).

To OCR these peculiarities, we need to train these combinations as ligatures, a feature available with AFR. Fig. 2 shows several trained ligatures in the form of AFR recognition templates (patterns). See [3] for further examples of recognition patterns.

There are more peculiarities, for example, writing several words

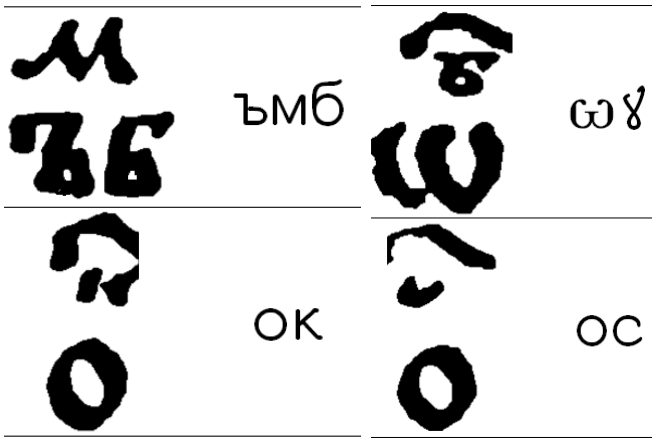


Figure 2. Examples of AFR patterns for ligatures of the 17th century

in one (mostly prepositions and articles with nouns, adjectives, and numerals), writing in lowercase only, etc.

A big problem is the absence of the spelling dictionary for the old Romanian language. It is different from the modern one, both due to the language development, and due to absence of fixed spelling rules; some words may be spelled differently in different epochs and in different places.

The importance of the proper dictionary can be illustrated by the following experiment. We took one page from a book of the late 18th century. Recognition with training but without dictionary resulted in 13% of erroneous words. Then we created a list of words from the page and repeated OCR with only 5% of erroneous words. More pages from the same book but with the dictionary restricted by this one page showed the rate of erroneous words of 8.5%.

We used a spelling dictionary created manually after OCR and partial correction of the result, of approx. 1.600 words for the 17th century at the moment.

To work with old Romanian texts more comfortably, a virtual keyboard was developed (Fig. 3).

Figure 3. Virtual keyboard for RC

We implemented a tool pack collecting the tools necessary to digitize the Romanian Cyrillic printings [4]. It includes: AFR add-ons stored from AFR during training with alphabets, OCR patterns, and spelling dictionaries; the AAConv transliteration utility; virtual keyboards; a shell for selection and uploading the proper add-on into AFR; font covering rare Cyrillic glyphs; user manual.

Developing this approach, we found that the OCR works with better accuracy when we train and use separate templates not only for different epochs of Romanian historical typography but for each typography or a group of typographies that had used, presumably, fonts from the same source.

3 Transliteration to the Latin Script

Once the scanned image was processed and the editable and intelligible Cyrillic text was obtained, the transliteration process takes place.

Here is an example of recognized Romanian text (17th century):

> картѣ деꙟтъи алѹи Самонл ,17,стих 35. нече нѹман съле стръжѹꙗскѫ ши съле пъꙁѣскѫ ꙁѹа ши ноаптѣ,кѹмь пъꙁїѧ Ѩковь патрїꙗрхѹлъ ѡиле лѹи Лаван. бытїе, 31 стих 40 нече съле цїе ꙟкисе ꙟстаѹль

As we got the editable text, we can apply conversion rules to get the text in MRL. The conversion rules can be "one-to-one" like а→**a**

or **ҭ→t**, and "one-to-many". Rules may be context dependent. Examples: **ѧ→îm** (**ѧпѫрѧҭ→împărat**); **ѧ→în** (**ѧдоѧлѫ→îndoială**); **ѧ→î** (**ѧнєѫ→însă**). As we see, there are three different cases of conversion for letter **ѧ**, and just a single one for **ѧ** and **ҭ**.

See Tab. 1 for examples of context dependent rules.

Table 1. Examples of RC→MRL rules

Cyrillic	Latin
ѧ	**îm** before **б**, **п**
ѧ	**î** before **м**, **и**
ѧ	**în** all other cases
ѧ	**a** at the beginning of word; after **ї**, **ц**
ѧ	**e** after **ч**
ѧ	**ea** after another consonant; at the end of word
ѣ	**e** after **ч**; exception **чѣ→cea**
ѣ	**ea** all other cases

All these and many other rules were inplemented in the AAConv utility. Once opened or inserted in AAConv, the text is automatically converted to the modern Romanian Latin script. Below we present the previous Romanian Cyrillic text transliterated to MRL:

> **cartea deîntăi alui Samoil ,17,stih 35. nece numai săle strâjuiască şi săle păzească zua şi noaptea,cum păzea Iacov patriarhul oile lui Lavan. bîtie, 31 stih 40 nece săle ţie închise**

The transliteration utility has many settings and features accessible by the user. One of the functionalities we developed gives the possibility to convert a Cyrillic text in two different modes:

Transliteration with actualization. In addition to the conversion to the Latin script, some words and archaic letters will be changed to the modern ones to allow the text to be more understandable. At the same time it takes away some specifics of the period. For

example, the old word **nece** will be replaced by its modern version **nici**.

Transliteration without actualization. The text will be converted to the MRL preserving archaic words and syntactic structures, as we saw above.

4 Reverse transliteration

AAConv gives also the possibility of backward transliteration, namely, MRL→RC. The conversion rules for the backward transliteration are "one-to-one", "many-to-one", and "one-to-many", and can be context dependent.

This feature was added because there are several old books that were manually transliterated in MRL. Applying the reverse transliteration, we can use them to replenish the dictionaries (word lists) for AFR, to evaluate OCR result, and to present the text in its original form permitting to fix text authenticity.

One-to-one rules are obvious, being mostly the reversed one-to-one rules of the direct transliteration. Examples: **u**→ѹ (**lucru**→лѹкрѹ); **o**→**о** (**proporție**→пропорцїе), etc.

Some one-to-one rules are context dependent. There are also rules of the kind "many-to-one", and (rarely) "one-to-many". Examples of such rules are presented in Tab. 2.

We present below an example of MRL→RC transliteration. We do not use overline signs in the transliteration.

> **Bine au înțeles acestea cei Crai sfinți de demult, că nui numai aceasta deregătoria lor, să poarte grije de oamenii ce sînt sub biruința lor numai trupește, ce mai vărtos să aibă şi săsă vestească că cuvăntul lui Dumnezău întru Ei, din casele să știe voia lui Dumnezeu şi să înțeleagă lucrul spăseniei lor.**

> Бине ѧѹ дцелес ачестѣ чеи Краи сфинци де демѹлт, кѫ нѹи нѹман ачѣста дерегѫторїа лор, гѫ полрте гриже де оаменїн че

Table 2. Examples of MRL→RC context dependent rules

Latin	Cyrillic
z	**ꙁ** if preceded by **ă**
z	**ꙁ̆** if surrounded by **u** from both sides
z	**ѕ** if preceded by e and succeeded by **i** or **ă**
z	**з** in all other cases
i	**иѧ** if preceded by **r** and succeded by **a**
i	**ѧ** if preceded by **a** is at the end of the word
i	**ѩ** if preceded by **a** and succeeded by **r** or **c**
i	**ї** if succeeded by one of the vowels **o**, **e**, **ă**
i	**и** in all other cases

сѫнт сѵпт бирѵинца лор нѵман трѵпѣце, че ман вѫртос сѫ линвѫ ши сѫсѫ вестѣкж кѫ кѵвѫнтѵлѫ лѵи Дѵмнезѫѵ лтрѵ Ен, дин кастеле сѫ цїе воїл лѵи Дѵмнезѣѵ ши сѫ лцелѣгѫ лѵкрѵл спѫсенїен лор.

5 Conclusion

We developed a tool pack containing, in particular, OCR templates and other additions to AFR to recognize Romanian Cyrillic printings of the 17th–18th centuries. We found that better results can be achieved when we use separate OCR templates for each typography. For further processing, we developed the transliteration utilities that convert the recognized text to the Latin script and vice versa. The more ambitious task for the future is the exact electronic reproduction of look and feel of the original text with all overline letters and signs.

With our Romanian colleagues, we use the recognized historical texts and the collected dictionaries in the development of several projects: a diachronic corpus; lexicon for a POS-tagger; PROIEL (Pragmatic Resources in Old Indo-European Languages), etc.

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Comparison of the maximal inaccuracies for two experiments

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Abstract

In this paper we refine and generalize some previous our results on the inaccuracy (error) theory. We define conditions, which characterize different types of functions. Via these functions an indirectly measurable variable Y can be analytically represented. We also present criteria for comparison of the maximal absolute and relative inaccuracies of the indirectly measurable variable Y in the first and in the second order for two experiments. We correct some of our previous conclusions regarding the application of the dimensionless scale for evaluation of the quality of an experiment. Furthermore we give two numerical contra examples.

Keywords: indirectly measurable variable; maximal absolute inaccuracy; maximal relative inaccuracy; dimensionless scale.

1 Introduction

Let an indirectly measurable variable Y be represented as a function of a finite number of directly measurable variables $X_1, X_2, ..., X_n$, i.e. $Y = f(X_1, X_2, ..., X_n)$ and let f be a differentiable function of each of its real variables. If in an experiment we have k number of observations $x_{i1}, x_{i2}, ..., x_{ik}$ of the directly measurable varialbe $X_i \quad (i = 1, 2, ..., n)$, then it is assumed that the arithmetic mean $\bar{x}_i = \frac{1}{k} \sum_{m=1}^{k} x_{im}$ is the most probable (the most reliable) value of X_i. We denote $|\Delta x_{im}| = |x_{im} - \bar{x}_i|$, $i = 1, 2, ..., n$, $m = 1, 2, ..., k$.

The value of the maximal absolute inaccuracy $\Delta^1 Y$ of an indirectly measurable variable Y according to the classical method is

$$\Delta^1 Y = \frac{1}{k} \sum_{m=1}^{k} \sum_{i=1}^{n} \left| \frac{\partial f}{\partial X_i} (x_{1m}, ..., x_{nm}) \right| |\Delta x_{im}|, \tag{1}$$

and the value of the maximal relative inaccuracy of Y is $\frac{\Delta^1 Y}{Y}$, where $\Delta^1 Y$ is defined by (1) and

$$Y = \frac{1}{k} \sum_{m=1}^{k} |f (x_{1m}, ..., x_{nm})|, \tag{2}$$

[6, 7].

The value of the maximal absolute inaccuracy $\Delta^1 Y$ according to our method [1] is

$$\Delta^1 Y = \sum_{i=1}^{n} A_i |\Delta X_i|, \tag{3}$$

where

$$A_i = \frac{1}{k} \sum_{m=1}^{k} \left| \frac{\partial f}{\partial X_i} (x_{1m}, ..., x_{nm}) \right|, \ i = 1, ..., n \tag{4}$$

and

$$|\Delta X_i| = \frac{1}{k} \sum_{j=1}^{k} |\Delta x_{ij}| \ , \ i = 1, 2, ..., n. \tag{5}$$

The value of the maximal relative inaccuracy $\frac{\Delta^1 Y}{Y}$ according to our method [2, 3] is

$$\frac{\Delta^1 Y}{Y} = \sum_{i=1}^{n} B_i \left| \frac{\Delta X_i}{X_i} \right|, \tag{6}$$

where

$$B_i = \frac{1}{k} \sum_{m=1}^{k} \left| \frac{x_{im}}{f (x_{1m}, ..., x_{nm})} \frac{\partial f}{\partial X_i} (x_{1m}, ..., x_{nm}) \right|, \ i = 1, ..., n \tag{7}$$

and

$$\left|\frac{\Delta X_i}{X_i}\right| = \frac{1}{k}\sum_{j=1}^{k}\left|\frac{\Delta x_{ij}}{x_{ij}}\right|\ ,\ i = 1, 2, ..., n. \tag{8}$$

We note, that in (4) and in (7)

$$\frac{\partial f}{\partial X_i}(x_{1m}, ..., x_{nm}) \text{ and } \frac{x_{im}}{f(x_{1m}, ..., x_{nm})}\frac{\partial f}{\partial X_i}(x_{1m}, ..., x_{nm})$$

are respectively the values of $\frac{\partial f}{\partial X_i}$ and $\frac{X_i}{f}\frac{\partial f}{\partial X_i}$, calculated on the m-th observation. A_i and B_i are the arithmetic means of these values for $m = 1, 2, ..., k$.

In [4, 5] we denote the values of the maximal absolute inaccuracy $\Delta^2 Y$ and of the maximal relative inaccuracy $\frac{\Delta^2 Y}{Y}$ of second order of $Y = f(X_1, X_2, ..., X_n)$ respectively by

$$\Delta^2 Y = \sum_{i,j=1}^{n} A_{ij}\left|\Delta X_i\right|\left|\Delta X_j\right| \text{ and } \frac{\Delta^2 Y}{Y} = \sum_{i,j=1}^{n} B_{ij}\left|\frac{\Delta X_i}{X_i}\right|\left|\frac{\Delta X_j}{X_j}\right| \tag{9}$$

where A_{ij} and B_{ij} for $\Delta^2 Y$ and $\frac{\Delta^2 Y}{Y}$ are defined as follows:

$$A_{ij} = \frac{1}{k}\sum_{m=1}^{k}\left|\frac{\partial^2 f}{\partial X_i \partial X_j}(x_{1m}, ..., x_{nm})\right|,\ \ i, j = 1, 2, ..., n \tag{10}$$

and

$$\begin{aligned} B_{ij} \ &= \ \frac{1}{k}\sum_{m=1}^{k}\left|\frac{x_{im}x_{jm}}{f(x_{1m}, ..., x_{nm})}\frac{\partial^2 f}{\partial X_i \partial X_j}(x_{1m}, ..., x_{nm})\right|, \\ &\quad i, j = 1, 2, ..., n. \end{aligned} \tag{11}$$

We note, that in (10) and in (11)

$$\frac{\partial^2 f}{\partial X_i \partial X_j}(x_{1m}, ..., x_{nm}) \text{ and } \frac{x_{im}x_{jm}}{f(x_{1m}, ..., x_{nm})}\frac{\partial^2 f}{\partial X_i \partial X_j}(x_{1m}, ..., x_{nm})$$

are respectively the values of $\frac{\partial^2 f}{\partial X_i \partial X_j}$ and $\frac{X_i X_j}{f}\frac{\partial^2 f}{\partial X_i \partial X_j}$, calculated at the m-th observation. A_{ij} and B_{ij} are the arithmetic means of these values for $m = 1, 2, ..., k$.

The maximum absolute inaccuracy ΔY of an indirectly measurable variable Y in the second degree of approximation, according to [4, 5], is

$$\Delta Y = \Delta^1 Y + \frac{1}{2}\Delta^2 Y, \tag{12}$$

and the maximum relative inaccuracy $\frac{\Delta Y}{|Y|}$ of Y in the second degree of approximation is

$$\frac{\Delta Y}{|Y|} = \frac{\Delta^1 Y}{|Y|} + \frac{1}{2}\frac{\Delta^2 Y}{|Y|}. \tag{13}$$

In this paper we give some conditions that characterize some type of functions. An indirectly measurable variable can be analytically represented via these functions. Thus we obtain some necessary and sufficient conditions for comparison of the values of the maximal inaccuracies for two experiments. We correct some of our previous conclusions regarding the dimensionless scale application for evaluation of the quality of an experiment. We show two numerical counterexamples.

2 Conditions that characterize different types of functions by which an indirectly measurable variable can be represented analytically

Theorem 1. If $f(x_1, ..., x_n)$ is a function with domain R^n and there exist the first partial derivatives of f in respect to all its variables, then the following holds:

$$\frac{\partial f}{\partial x_i} = a_i,\ a_i \in \mathrm{R},\ i = 1, ..., n \tag{14}$$

if and only if

$$f(x_1, ..., x_n) = a_1x_1 + ... + a_nx_n + c, \quad c, a_i \in \mathrm{R}, \ i = 1, ..., n. \tag{15}$$

Proof. If (15) is true, then obviously (14) holds true.

Contrariwise, let (14) is true. Then from $\frac{\partial f}{\partial x_i} = a_i$ it follows $\partial f = a_i \partial x_i$, $a_i \in \mathrm{R}$. Therefore

$$f = a_i x_i + c_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n), \tag{16}$$

where $c_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$ is a real function of $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$.

We will prove that

$$f(x_1, ..., x_n) = a_1x_1 + ... + a_ix_i + c_i(x_{i+1}, ..., x_n), \tag{17}$$

by induction on i, $1 \leq i \leq n$.

Indeed, for $i = 1$ the equality (17) follows from (16). Assume the equality (17) is true for $i - 1 \geq 1$, i.e.

$$f(x_1, ..., x_n) = a_1x_1 + ... + a_{i-1}x_{i-1} + c_{i-1}(x_i, ..., x_n). \tag{18}$$

Since from (14) and (18) it follows $a_i = \frac{\partial f}{\partial x_i} = \frac{\partial c_{i-1}}{\partial x_i}$, then $\partial c_{i-1} = a_i \partial x_i$. Therefore

$$c_{i-1}(x_i, ..., x_n) = a_i x_i + c_i(x_{i+1}, ..., x_n). \tag{19}$$

As we substitute $c_{i-1}(x_i, ..., x_n)$ from (19) in (18), then we obtain the equality (17). Therefore formula (17) is proved by induction on i, $1 \leq i \leq n$.

Let $i = n$. Then from (17) we have

$$f = a_1x_1 + ... + a_nx_n + c,$$

where $c = c_n \in \mathrm{R}$.

The theorem is prooved. □

Theorem 2. If $f(x_1, ..., x_n)$ is a function with domain R^n and there exist the first partial derivatives of f in respect to all its variables, then the following holds

$$\frac{x_i}{f}\frac{\partial f}{\partial x_i} = k_i,\ k_i \in \mathrm{R},\ i = 1, ..., n \tag{20}$$

if and only if

$$f = cx_1^{k_1}...x_n^{k_n},\quad c, k_i \in \mathrm{R}^+,\ i = 1, ..., n. \tag{21}$$

Proof. Let (21) holds true. Then

$$\frac{x_i}{f}\frac{\partial f}{\partial x_i} = \frac{x_i\, k_i c x_1^{k_1}...x_{i-1}^{k_{i-1}} x_i^{k_i-1} x_{i+1}^{k_{i+1}}...x_n^{k_n}}{cx_1^{k_1}...x_n^{k_n}} = k_i,$$

i.e. (20) holds true.

Contrariwise, let (20) holds true. Let us denote $y = f(x_1, ..., x_n)$. Then from (20) it follows that $\frac{dy}{y} = \frac{k_i}{x_i}\partial x_i$. We obtain $\ln|y| = k_i \ln|x_i| + \ln|c_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)|$, where $c_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$. Then

$$y = \pm x_i^{k_i} c_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n). \tag{22}$$

We will prove by induction for i, $1 \le i \le n$, that

$$y = \pm x_1^{k_1}...x_i^{k_i} c_i(x_{i+1}, ..., x_n). \tag{23}$$

Indeed for $i = 1$ the equality (23) is the proved formula (22). Let us assume, that (23) holds true for $i - 1 \ge 1$, i.e.

$$y = \pm x_1^{k_1}...x_{i-1}^{k_{i-1}} c_{i-1}(x_i, ..., x_n). \tag{24}$$

From formulas (20) and (24) we have

$$k_i = \frac{x_i}{y}\frac{\partial y}{\partial x_i} = \frac{x_i\, x_1^{k_1}...x_{i-1}^{k_{i-1}} \frac{\partial c_{i-1}}{\partial x_i}}{x_1^{k_1}...x_{i-1}^{k_{i-1}} c_{i-1}(x_i, ..., x_n)}.$$

Therefore we obtain the following formulas

$$\frac{\partial c_{i-1}}{c_{i-1}} = \frac{k_i}{x_i}\partial x_i,\ \ln|c_{i-1}| = k_i \ln|x_i| + \ln|c_i(x_{i+1},...,x_n)|,$$

$$c_{i-1} = \pm x_i^{k_i} c_i(x_{i+1},...,x_n).$$

We substitute the last formula in (23) and we get

$$y = \pm x_1^{k_1}...x_{i-1}^{k_i} c_i(x_{i+1},...,x_n).$$

Thus formula (23) is proved by induction on i, $1 \le i \le n$.

Let $i = n$. From (23) we have

$$y = \pm x_1^{k_1}...x_n^{k_n} c,$$

where $c = c_n \in \mathrm{R}$.

The theorem is prooved. □

Theorem 3. If $f = f(x_1,...,x_n)$ is a second degree polynomial with unknown quantities $x_1,...,x_n$, represented in the form

$$\begin{aligned} f(x_1,...,x_n) &= \sum_{i,j=1}^{n} a_{ij}x_ix_j + \sum_{i=1}^{n} a_ix_i + a, a_{ji} = a_{ij}, \\ &\quad a_j, a_i, a \in \mathrm{R}, \end{aligned} \tag{25}$$

then for each $i, j = 1,...,n$ the equality $\frac{\partial^2 f}{\partial x_i \partial x_j} = 2a_{ij}$ holds.

Proof. Let us denote f in the form

$$f(x_1,...,x_n) = \sum_{i=1}^{n} a_{ii}^2 x_i^2 + 2\sum_{i=1}^{n-1}\sum_{j=2}^{n} a_{ij}x_ix_j + \sum_{i=1}^{n} a_ix_i + a.$$

Then

$$\frac{\partial f}{\partial x_i} = 2a_{ii}x_i + 2\sum_{j>i} a_{ij}x_j + a_i.$$

For $j \ne i$ we have $\frac{\partial^2 f}{\partial x_i \partial x_j} = 2a_{ij}$, and for $j = i$ it follows $\frac{\partial^2 f}{\partial x_i \partial x_i} = 2a_{ii}$.

The theorem is prooved. □

Theorem 4. If the function $f = f(x_1, ..., x_n)$ has the form

$$f = cx_1^{k_1}...x_i^{k_i}...x_j^{k_j}...x_n^{k_n}, \quad c, k_i \in \mathrm{R}, \tag{26}$$

then for each i, j the following holds true:

$$\frac{x_i x_j}{f}\frac{\partial^2 f}{\partial x_i \partial x_j} = \begin{cases} k_i k_j, \text{ if } (i)\, j \neq i, \\ k_i(k_i - 1), \text{ if } (ii)\ j = i \text{ and } k_i \neq 1, \\ 0, \text{ if } j = i \text{ and } k_i = 1. \end{cases} . \tag{27}$$

Proof. If $j \neq i$, then the following equalities are true

$$\frac{x_i x_j}{f}\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{x_i x_j}{cx_1^{k_1}...x_n^{k_n}} ck_i k_j x_1^{k_1}...x_i^{k_i-1}...x_j^{k_j-1}...x_n^{k_n} = k_i k_j.$$

If $j = i$ and $k_i \neq 1$, then

$$\frac{x_i^2}{f}\frac{\partial^2 f}{\partial x_i \partial x_i} = \frac{x_i^2}{cx_1^{k_1}...x_n^{k_n}} ck_i(k_i - 1)x_1^{k_1}...x_i^{k_i-2}...x_n^{k_n} = k_i(k_i - 1).$$

If $j = i$ and $k_i = 1$, then obviously the third part of (27) holds true. □

3 Some necessary and sufficient conditions for comparison of the values of the maximal inaccuracies for two experiments

1) Let $\Delta^1 Y$ and $\Delta^1 \tilde{Y}$ be the maximal absolute inaccuracies of the first order for two experiments, i.e.

$$\Delta^1 Y = \sum_{i=1}^{n} A_i |\Delta X_i|, \ \ \Delta^1 \tilde{Y} = \sum_{i=1}^{n} \tilde{A}_i \left|\Delta \tilde{X}_i\right|, \tag{28}$$

where $|\Delta X_i|$ and $\left|\Delta \tilde{X}_i\right|$ are defined from formula (5).

1.1) If in (28) $A_i = \frac{\partial f}{\partial x_i}$ are constant values, $i = 1, ..., n$, then according to formula (3)

$$\Delta^1 Y = \sum_{i=1}^{n} A_i \left|\Delta X_i\right|, \;\; \Delta^1 \tilde{Y} = \sum_{i=1}^{n} A_i \left|\Delta \tilde{X}_i\right|. \tag{29}$$

Thus obviously the following statement is true.

Criterion 1. If $A_i = \frac{\partial f}{\partial x_i} = const$, $i = 1, 2, ..., n$, then the first experiment of the maximal absolute inaccuracy of Y is more accurate than the second one if and only if

$$\sum_{i=1}^{n} A_i \left(\left|\Delta \tilde{X}_i\right| - \left|\Delta X_i\right|\right) \geq 0. \tag{30}$$

Both experiments have equal accuracy if and only if

$$\sum_{i=1}^{n} A_i \left(\left|\Delta \tilde{X}_i\right| - \left|\Delta X_i\right|\right) = 0.$$

In this case for the inaccuracy of the experiments, calculated by the classical way from (1) and (28) we have

$$\Delta^1 Y = \frac{1}{k} \sum_{i=1}^{n} \sum_{m=1}^{k} A_i \left|\Delta x_{im}\right| = \sum_{i=1}^{n} A_i \left|\Delta \bar{X}_i\right| = \sum_{i=1}^{n} A_i \left|\Delta X_i\right|.$$

Therefore this result match with our result from (29).

In particular, by $n = 1$ the first experiment is more accurate than the second one if and only if $\left|\Delta X_1\right| \leq \left|\Delta \tilde{X}_1\right|$.

Both experiments have equal accuracy if and only if $\left|\Delta X_1\right| = \left|\Delta \tilde{X}_1\right|$.

As an example for this case we can consider the function from Theorem 1

$$f\left(x_1, ..., x_n\right) = \sum_{i=1}^{n} a_i X_i + c\,, \quad a_i, c \in \mathrm{R}.$$

1.2) Let in (28) $\Delta X_1, ..., \Delta X_n$ are constant values.

Criterion 2. If $\Delta X_i = const \quad (i = 1, 2, ..., n)$, then the first experiment of the maximal absolute inaccuracy of Y is more accurate than the second one if and only if

$$\sum_{i=1}^{n} \left(\tilde{A}_i - A_i \right) |\Delta X_i| \geq 0. \tag{31}$$

Both experiments have equal accuracy if and only if

$$\sum_{i=1}^{n} \left(\tilde{A}_i - A_i \right) |\Delta X_i| = 0.$$

2) Let $\frac{\Delta^1 Y}{Y}$ and $\frac{\Delta^1 \tilde{Y}}{\tilde{Y}}$ be the maximal relative inaccuracies of the first order for two experiments, i.e.

If $\frac{X_i}{f} \frac{\partial f}{\partial X_i} = B_i \quad (i = 1, 2, ..., n)$ are constant values, then according to formula (6)

$$\frac{\Delta^1 Y}{Y} = \sum_{i=1}^{n} |B_i| \left| \frac{\Delta X_i}{X_i} \right|, \quad \frac{\Delta^1 \tilde{Y}}{\tilde{Y}} = \sum_{i=1}^{n} \left| \tilde{B}_i \right| \left| \frac{\Delta \tilde{X}_i}{\tilde{X}_i} \right|, \tag{32}$$

where $\left| \frac{\Delta X_i}{X_i} \right|$ and $\left| \frac{\Delta \tilde{X}_i}{\tilde{X}_i} \right|$ are defined from (8).

Criterion 3. If $\frac{x_i}{f} \frac{\partial f}{\partial x_i} = const \quad (i = 1, 2, ..., n)$, then the first experiment of the maximal relative inaccuracy of Y is more accurate than the second one if and only if

$$\sum_{i=1}^{n} |B_i| \left(\left| \frac{\Delta \tilde{X}_i}{\tilde{X}_i} \right| - \left| \frac{\Delta X_i}{X_i} \right| \right) \geq 0. \tag{33}$$

Both experiments have equal accuracy if and only if

$$\sum_{i=1}^{n} |B_i| \left(\left| \frac{\Delta \tilde{X}_i}{\tilde{X}_i} \right| - \left| \frac{\Delta X_i}{X_i} \right| \right) = 0.$$

In particular, for $n = 1$ the first experiment is more accurate than the second one if and only if $\left| \frac{\Delta X_1}{X_1} \right| \leq \left| \frac{\Delta \tilde{X}_1}{\tilde{X}_1} \right|$. Both experiments have equal accuracy if and only if $\left| \frac{\Delta X_1}{X_1} \right| = \left| \frac{\Delta \tilde{X}_1}{\tilde{X}_1} \right|$.

As an example for this case we can consider the function from Theorem 2

$$f\left(X_1,...,X_n\right)=cX_1^{k_1}...X_n^{k_n}\,,\quad c,k_i\in \mathrm{R}^+\,,\ i=1,...,n,\quad k_1\neq 0.$$

3) Let $\Delta^2 Y$ and $\Delta^2\tilde{Y}$ are the maximal absolute inaccuracies of the second order of two experiments.

3.1) If $\frac{\partial^2 f}{\partial X_i \partial X_j}=A_{ij}$ are constants, then according to formula (9)

$$\Delta^2 Y=\sum_{i,j=1}^{n}A_{ij}\left|\Delta X_i\right|\left|\Delta X_j\right|,\ \ \Delta^2\tilde{Y}=\sum_{i,j=1}^{n}A_{i,j}\left|\Delta\tilde{X}_i\right|\left|\Delta\tilde{X}_j\right|. \quad (34)$$

Criterion 4. If $\frac{\partial^2 f}{\partial x_i \partial x_j}=A_{ij}=const\quad (i,j=1,2,...,n)$, then the first experiment of the maximal absolute inaccuracy of the second order of Y is more accurate than the second one if and only if

$$\sum_{i,j=1}^{n}A_{ij}\left(\left|\Delta\tilde{X}_i\right|\left|\Delta\tilde{X}_j\right|-\left|\Delta X_i\right|\left|\Delta X_j\right|\right)\geq 0. \quad (35)$$

Both experiments have equal accuracy if and only if

$$\sum_{i,j=1}^{n}A_{ij}\left(\left|\Delta\tilde{X}_i\right|\left|\Delta\tilde{X}_j\right|-\left|\Delta X_i\right|\left|\Delta X_j\right|\right)=0.$$

As an example for this case we can consider the function from Theorem 3

$$f\left(X_1,...,X_n\right)=\sum_{i,j=1}^{n}a_{ij}X_iX_j+\sum_{i=1}^{n}a_iX_i+a\,,\quad a_{ij},a_i,a\in \mathrm{R}.$$

3.2) Let $\Delta X_1,...,\Delta X_n$ are constant values and

$$\Delta^2 Y=\sum_{i,j=1}^{n}A_{ij}\left|X_i\right|\left|X_j\right|,\ \ \Delta^2\tilde{Y}=\sum_{i,j=1}^{n}\tilde{A}_{ij}\left|X_i\right|\left|X_j\right|.$$

Criterion 5. If $\Delta X_i = const$, $(i = 1, 2, ..., n)$, then the first experiment of the maximal absolute inaccuracy of the second order of Y is more accurate than the second one if and only if

$$\sum_{i,j=1}^{n} \left(\tilde{A}_{ij} - A_{ij} \right) |\Delta X_i| \, |\Delta X_j| \geq 0. \tag{36}$$

Both experiments have equal accuracy if and only if

$$\sum_{i,j=1}^{n} \left(\tilde{A}_{ij} - A_{ij} \right) |\Delta X_i| \, |\Delta X_j| = 0.$$

4) Let $\frac{\Delta^2 Y}{Y}$ and $\frac{\Delta^2 \tilde{Y}}{\tilde{Y}}$ are the maximal relative inaccuracies of the second order of two experiments. If $\frac{X_i X_j}{f} \frac{\partial^2 f}{\partial X_i \partial X_j} = B_{ij}$ are constant values, then from (9) we have

$$\frac{\Delta^2 Y}{Y} = \sum_{i,j=1}^{n} B_{ij} \left| \frac{\Delta X_i}{X_i} \right| \left| \frac{\Delta X_j}{X_j} \right|, \quad \frac{\Delta^2 \tilde{Y}}{\tilde{Y}} = \sum_{i,j=1}^{n} B_{ij} \left| \frac{\Delta \tilde{X}_i}{\tilde{X}_i} \right| \left| \frac{\Delta \tilde{X}_j}{\tilde{X}_j} \right|. \tag{37}$$

Criterion 6. If $\frac{x_i x_j}{f} \frac{\partial^2 f}{\partial x_i \partial x_j} = const \quad (i = 1, 2, ..., n)$, then the first experiment of the maximal relative inaccuracy of the second order of Y is more accurate than the second one if and only if

$$\sum_{i,j=1}^{n} B_{ij} \left(\left| \frac{\Delta \tilde{X}_i}{\tilde{X}_i} \right| \left| \frac{\Delta \tilde{X}_j}{\tilde{X}_j} \right| - \left| \frac{\Delta X_i}{X_i} \right| \left| \frac{\Delta X_j}{X_j} \right| \right) \geq 0. \tag{38}$$

Both experiments have equal accuracy if and only if

$$\sum_{i,j=1}^{n} B_{ij} \left(\left| \frac{\Delta \tilde{X}_i}{\tilde{X}_i} \right| \left| \frac{\Delta \tilde{X}_j}{\tilde{X}_j} \right| - \left| \frac{\Delta X_i}{X_i} \right| \left| \frac{\Delta X_j}{X_j} \right| \right) = 0.$$

4 Counterexamples to the dimensionless scale and improvement of its application

In [1] we considered $\Delta X_1, \Delta X_2, ..., \Delta X_n, \pm Y$ as a system of generalized orthogonal coordinates. Then for $n \geq 2$ we get an $(n+1)$-dimensional Euclidean space, where (3) is an equation of a plane that passes through the origin of the coordinate system.

Thus we take ε for *sample plane in the space of the absolute inaccuracy* which represents an imaginary ideal perfectly accurate experiment.

If $\alpha : \Delta Y = A_1 \Delta X_1 + A_2 \Delta X_2 + ... + A_n \Delta X_n$, then ε is determined by $A_1 = A_2 = ... = A_n = 0$, i.e.

$$\varepsilon : \Delta Y = 0.$$

In [2, 3] we considered the angle between the normal vectors $\overrightarrow{n_\alpha}(A_1, A_2, ..., A_n, -1)$ of the plane α of the real experiment and $\overrightarrow{n_\varepsilon}(0, 0, ..., 0, -1)$ of the palne ε. Then the value of the cosine

$$k_\alpha = \cos \angle (\overrightarrow{n_\alpha}, \overrightarrow{n_\varepsilon}) = \frac{1}{\sqrt{A_1^2 + A_2^2 + ... + A_n^2 + 1}} \tag{39}$$

of this angle can be chosen for a coefficient of accuracy in a dimensionless scale, i.e. for a numerical characteristic of the quality of the experiment.

Since $k_\alpha = \cos \angle (\overrightarrow{n_\alpha}, \overrightarrow{n_\varepsilon})$, then the scale for evaluating the quality of the experiment is the interval $[0, 1]$. The value $k_\alpha = 1$ represents the ideal perfectly accurate experiment and the value $k_\alpha = 0$ represents the ideal absolutely inaccurate experiment. The conclusions we have made in [1, 2, 3] regarding the application of the scale are not absolutely correct. We will prove this with the following numerical examples, applying the criteria from section 3.

Example 1) Let $S = f(t) = gt$ be the distance that the uniformly moving object passes with constant velocity v during time t. Thus $f(t)$ has the form from Theorem 1.

For the first experiment we choose $t_{11} = 4$, $t_{12} = 2$. Then $\bar{t}_1 = 3$, $|\Delta t_{11}| = 1$, $|\Delta t_{12}| = 1$, $|\Delta t_1| = 1$. Since $\frac{df}{dt} = v$, then according to

formula (4)

$$A_1 = \frac{1}{2}\sum_{m=1}^{2}\left|\frac{df}{dt}(t_{1m})\right| = \frac{1}{2}\sum_{m=1}^{2}|v| = v.$$

From (3) we find the value of the maximal absolute inaccuracy for the first experiment

$$\Delta^1 Y = \Delta^1 f = A_1\left|\Delta t_1\right| = v.1 = v.$$

For the second experiment we choose $\tilde{t}_{11} = 3,6$, $\tilde{t}_{12} = 2,2$. Then $\bar{t}_1 = 2,9$, $\left|\Delta\tilde{t}_{11}\right| = 0,7$, $\left|\Delta\tilde{t}_{12}\right| = 0,7$, $\left|\Delta\tilde{t}_1\right| = 0,7$. From (4), since $\frac{df}{dt} = v$, we calculate

$$A_2 = \frac{1}{2}\sum_{m=1}^{2}\left|\frac{df}{dt}(t_{1m})\right| = \frac{1}{2}\sum_{m=1}^{2}|v| = v.$$

From (3) we find the value of the maximal absolute inaccuracy for the second experiment

$$\Delta^1\tilde{Y} = \Delta^1\tilde{f} = A_2\left|\Delta\tilde{t}_1\right| = 0,7v.$$

Since $A_1 = A_2$, then from formula (38) we have the following relationship between the coefficients of accuracy:

$$k_1 = \frac{1}{\sqrt{A_1^2+1}} = \frac{1}{\sqrt{A_2^2+1}} = k_2,$$

i.e. regarding [1, 2, 3] we can conclude that both experiments have the same accuracy. But

$$\Delta^1 Y = \Delta^1 f = g > 0,7g = \Delta^1\tilde{Y} = \Delta^1\tilde{f}.$$

Therefore the second experiment is more accurate than the first one. This counterexample contradicts the conclusions in [1, 2, 3] for the dimensionless scale.

From the necessary and sufficient conditions we have presented in section 4, for $A_1 = A_2$, according to Criterion 1, it follows that the

second experiment is more accurate than the first one, because $\left|\Delta\tilde{t}_1\right| < |\Delta t_1|$. Therefore Criterion 1 gives us more precise conclusion.

Example 2) Let $S = f(t) = \frac{gt^2}{2}$ be the distance that free falling object passes during time t (in vacuum) and $g = 9,8\ m/s^2$ is the earth gravitational acceleration. Thus $f(t)$ has the form from Theorem 2.

For the first experiment we choose $t_{11} = 2$, $t_{12} = 1,6$. Then $\bar{t}_1 = 1,8$, $|\Delta t_{11}| = 0,2$, $|\Delta t_{12}| = 0,2$ and $|\Delta t_1| = 0,2$. Since $\frac{df}{dt} = gt$, then from formula (4) we find $A_1 = \frac{1}{2}\sum\limits_{m=1}^{2}\left|\frac{df}{dt}(t_{1m})\right| = \frac{1}{2}\sum\limits_{m=1}^{2}|gt| = \frac{1}{2}g\sum\limits_{m=1}^{2}|t| = \frac{1}{2}g(2+1,6) = 1,8g$. From formula (3) we calculate the value of the maximal absolute inaccuracy for the first experiment

$$\Delta^1 Y = \Delta^1 f = A_1 |\Delta t_1| = 1,8g \times 0,2 = 0,36g.$$

For the second experiment we choose $\tilde{t}_{11} = 1,8$, $\tilde{t}_{12} = 1,9$. Then $\bar{\tilde{t}}_1 = 1,85$, $\left|\Delta\tilde{t}_{11}\right| = 0,05$, $\left|\Delta\tilde{t}_{12}\right| = 0,05$, $\left|\Delta\tilde{t}_1\right| = 0,05$. From formula (4) we find

$$A_2 = \frac{1}{2}\sum_{m=1}^{2}\left|\frac{df}{dt}(t_{1m})\right| = \frac{1}{2}\sum_{m=1}^{2}|gt| = \frac{1}{2}g\sum_{m=1}^{2}|t| =$$

$$= \frac{1}{2}g(1,8+1,9) = 1,85g.$$

From formula (3) we find the value of the maximal absolute inaccuracy for the second experiment

$$\Delta^1\tilde{Y} = \Delta^1\tilde{f} = A_2\left|\Delta\tilde{t}_1\right| = 1,85g \times 0,05 = 0,0925g.$$

Since $A_1 < A_2$, then from formula (39) we have the following relationship between the coefficients of accuracy:

$$k_1 = \frac{1}{\sqrt{A_1^2+1}} > \frac{1}{\sqrt{A_2^2+1}} = k_2,$$

i.e. according to [2, 3] the value of the maximal absolute inaccuracy $\Delta^1 Y$ for the first experiment is more accurate than the value $\Delta^1\tilde{Y}$ of

the second one. But

$$\Delta^1\tilde{Y} = \Delta^1\tilde{f} = 0,0925g < 0,36g = \Delta^1 Y = \Delta^1 f.$$

Therefore we can conclude that the second experiment is more accurate than the first one. This counterexample contradicts the conclusions in [1, 2, 3] for the dimensionless scale.

Both examples show that the conclusions we have made in [1, 2, 3] regarding the dimensionless scale and the sample plane in the spaces of the absolute and relative inaccuracies, have to be improved.

For correct application of the dimensionless scale in [1, 3], we give the following supplements.

Definition 5. We will say that the vector $A = (A_1, A_2, ..., A_n)$ is less than or equal to the vector $B = (B_1, B_2, ..., B_n)$ (coordinate by coordinate) and we will denote with $\bar{A} \leq \bar{B}$, if $A_i \leq B_i$ for each $i = 1, 2, ..., n$.

Let for fixed values of $\Delta X_1, \Delta X_2, ..., \Delta X_n$ for an experiment we have two different forms for representation of the maximal absolute inaccuracy ΔY, i.e.:

$\Delta^1 Y = A_1\Delta X_1 + A_2\Delta X_2 + ... + A_n\Delta X_n$ and $\Delta^1\tilde{Y} = B_1\Delta X_1 + B_2\Delta X_2 + ... + B_n\Delta X_n$.

Then obviously the following conclusion is true:

Theorem 6. For fixed values of $\Delta X_1, \Delta X_2, ..., \Delta X_n$ between two experiments with planes $\alpha:\ \Delta^1 Y = A_1\Delta X_1 + A_2\Delta X_2 + ... + A_n\Delta X_n$ and $\beta:\ \Delta^1\tilde{Y} = B_1\Delta X_1 + B_2\Delta X_2 + ... + B_n\Delta X_n$ the more accurate is that one, the normal vector of which is less than or equal to the other.

If there are two vectors $A = (A_1, A_2, ..., A_n)$, $B = (B_1, B_2, ..., B_n)$ and $\bar{A} \leq \bar{B}$, then $k_\alpha \geq k_\beta$ and for the fixed $\Delta X_1, \Delta X_2, ..., \Delta X_n$ it follows that $\Delta^1 Y \leq \Delta^1\tilde{Y}$. However it is not true the statement that we formulated in [1, 3], that from $k_\alpha \geq k_\beta$ it follows $\Delta^1 Y \leq \Delta^1\tilde{Y}$.

Let us consider that the maximal absolute inaccuracy ΔY has the same representation $\Delta Y = A_1\Delta X_1 + A_2\Delta X_2 + ... + A_n\Delta X_n$ for two provided experiments, i.e. the values of the coefficients $A_1, A_2, ..., A_n$ are fixed. Then obviously for different experiments with measured values $x_{11}, x_{12}, ..., x_{1n}$ and $x_{21}, x_{22}, ..., x_{2n}$ of $\Delta X_1, \Delta X_2, ..., \Delta X_n$ the following conclusion is true:

Theorem 7. For the fixed values $A_1, A_2, ..., A_n$ between experiments with measured values $x_{11}, x_{12}, ..., x_{1n}$ and $x_{21}, x_{22}, ..., x_{2n}$ of $\Delta X_1, \Delta X_2, ..., \Delta X_n$, the more accurate experiment is that one, the vector of which is less than or equal (coordinate by coordinate) to the other.

Thus, if $x_1 = (x_{11}, x_{12}, ..., x_{1n})$, $x_2 = (x_{21}, x_{22}, ..., x_{2n})$ and $\bar{x}_1 \leq \bar{x}_2$, then $\Delta^1 Y \leq \Delta^1 \tilde{Y}$. In this case the reverse statement is not true.

The most accurate experiment will be that one, where the values of the variables and the normal vector (coordinate by coordinate) are the least possible.

Analogical conclusions as Theorem 6 and Theorem 7 can be formulated also for the maximal relative inaccuracy $\frac{\Delta Y}{Y}$ of an indirectly measurable variable Y.

5 Discussion

The suggested by us method for determining the numerical values of the maximal and relative inaccuracy of an indirectly measurable variable is of great importance for every experimental science, in which the studied processes can be modelled via functions. The values of the maximal inaccuracies can be compared very easily when we have two experiments.

6 Conclusion

In this paper we give necessary and sufficient conditions for comparison of the values of the maximal inaccuracies for two experiments. We consider some of the most common in the practice classes of functions. We give numerical counterexamples regarding the introduced by us dimensionless scale in [1, 2, 3] for evaluation of two experiments. We also give some conditions for the correct application of the scale. Thus we improve the conclusions we have made in [1, 2, 3].

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