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# On fixed point subalgebras of some local algebras over a field

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**Abstract.** Fixed point subalgebras of some local algebras obtained as quotients of polynomial algebras over an arbitrary field $F$ with respect to all $F$-algebra automorphisms are described.

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## Introduction

The notion of infinitely near points was initially part of the intuitive foundations of differential calculus. In the simplest terms, two points which lie at an infinitesimal distance apart are considered infinitely near [17]. Charles Ehresmann influenced by language of Taylor polynomials (which precised the infinite nearness in calculus) introduced the concept of $r$-jet in his paper [2] (1951). According to this, jets of smooth mappings are defined as equivalence classes of mappings. Presumably it was Ehresmann's initiative which stimulated the paper of André Weil [16] in which Weil, being experienced from his previous algebraic geometry research in the use of methods of commutative algebra, introduced the concept of infinitely near points on a smooth manifold as algebra homomorphisms from the algebra of smooth real functions on the manifold into a local $\mathbb{R}$-algebra (which is now called the Weil algebra).

The *Weil algebra* is defined as local commutative (and associative) $\mathbb{R}$-algebra $A$ with identity, the nilpotent ideal $\mathfrak{n}_A$ of which has a finite dimension as a vector space and $A/\mathfrak{n}_A = \mathbb{R}$. André Weil in [15] commented on non-semisimple finite dimensional algebras that ". . . *on sait qu'on ne sait rien sur cette sorte d'algèbre*"; and Shafarevich in [13] noted that Weil's observation retains its validity up to this days. We remark also that the ideas about Weil algebras enter into models for synthetic differential geometry. Disentangling structures from geometric phenomena to their categorical formulation was a long process and it is described in [11].

It is well known that the *differential invariant* is defined as a $G^r_n$-equivariant mapping $f\colon Y \to Z$ from a $G^r_n$-manifold $Y$ into a $G^r_n$-manifold $Z$ ( see [4]), where $G^r_n = \operatorname{inv} J^r_0(\mathbb{R}^n, \mathbb{R}^n)_0$ (invertible $r$-th order jets from $\mathbb{R}^n$ into $\mathbb{R}^n$ with source and target in $0 = (0, \dots, 0)$); $G^r_n$ is a Lie group (called usually the *jet group* or the *differential group*), $Y$ and $Z$ are manifolds endowed with the left action of $G^r_n$ and

$f(gy) = gf(y)$. However, $G^r_n$ is (isomorphic to) the group of $\mathbb{R}$-algebra automorphisms of the Weil algebra $\mathbb{D}^r_n = \mathbb{R}[X_1, \dots, X_n]/\mathfrak{m}^{r+1}$, where $\mathfrak{m}$ is the maximal ideal in the algebra of real polynomials in $n$ indeterminates. The group $G^r_n$ can be generalized to $\mathrm{Aut}_{\mathbb{R}} A$ for an arbitrary Weil algebra $A$ and $\mathrm{Aut}_{\mathbb{R}} A$ is, of course, a Lie group, too. The study of differential invariants has many applications: differential invariants completely characterize invariants systems of differential equations as well as invariant variational principles, see the monograph [12] of Peter J. Olver.

The study of the subalgebra $SA = \{a \in A; \phi(a) = a \text{ for all } \phi \in \mathrm{Aut}_{\mathbb{R}} A\}$ of a Weil algebra $A$, is motivated by some classifications problems in differential geometry, in particular, in the classification of all natural operators lifting vector fields from $m$-dimensional manifolds to bundles of Weil contact elements which was solved in [5]. Although in the known geometrically motivated examples is usually $SA = \mathbb{R}$ (such $SA$ is called *trivial*), there are some algebras for which $SA \supsetneqq \mathbb{R}$ and they call attention to the geometry of corresponding bundles. Thus, the fundamental problem is a classification of algebras having $SA$ nontrivial. In this paper, we study only the group of automorphisms of $\mathbb{D}^r_n$; nevertheless we replace $\mathbb{R}$ by an arbitrary field $F$ and obtain new results — we come to a different situation in particular cases: for finite fields the considered algebras are *finite rings* and there is the whole theory about this topic. It is known the *ring automorphism problem* liying in a decision if a finite ring has a non-identical automorphism or not. Results about fixed point subalgebras are also qualitatively totally different from the real case and, for the finite fields, they can have interesting applications in the coding theory and cryptography.

In the first section, we recall the real case and all definitions. The second section is devoted to local algebras of the first order: so called dual numbers and their generalizations plural numbers. Groups in question are general linear groups. The higher order case is studied in the third section. Corresponding groups of automorphisms are called (in the real case) jet groups. Possible applications are mentioned in the last section.

## 1 The real field: Weil algebras and jet groups

We recall that the *Weil algebra* is a local commutative $\mathbb{R}$-algebra $A$ with identity, the nilradical (nilpotent ideal) $\mathfrak{n}_A$ of which has a finite dimension as a vector space and $A/\mathfrak{n}_A = \mathbb{R}$. Then we call the *order* of $A$ the minimum $\mathrm{ord}(A)$ of the integers $r$ satisfying $\mathfrak{n}_A^{r+1} = 0$ and the *width* $\mathrm{w}(A)$ of $A$ the dimension $\dim_{\mathbb{R}}(\mathfrak{n}_A/\mathfrak{n}_A^2)$.

One can assume $A$ is expressed as a finite dimensional quotient of the algebra $\mathbb{R}[X_1, \dots, X_n]$ of real polynomials in several indeterminates. Thus, the main example is

$$\mathbb{D}^r_n = \mathbb{R}[X_1, \dots, X_n]/\mathfrak{m}^{r+1},$$

$\mathfrak{m} = (X_1, \dots, X_n)$ being the maximal ideal of $\mathbb{R}[X_1, \dots, X_n]$ and we observe that $\mathrm{ord}(\mathbb{D}^r_n) = r$ and $\mathrm{w}(\mathbb{D}^r_n) = n$. Every other such algebra $A$ of order $r$ can be expressed in a form

$$A = \mathbb{R}[X_1, \dots, X_n]/\mathfrak{j} = \mathbb{R}[X_1, \dots, X_n]/\mathfrak{i} + \mathfrak{m}^{r+1},$$

where the ideal $\mathfrak{i}$ satisfies $\mathfrak{m}^{r+1} \subsetneqq \mathfrak{i} \subseteq \mathfrak{m}^2$ and is generated by a finite number of polynomials. The fact $\mathfrak{i} \subseteq \mathfrak{m}^2$ implies that the width of $A$ is $n$ as well. Clearly, $A$ can be expressed also as

$$A = \mathbb{D}^r_n/\mathfrak{i},$$

where $\mathfrak{i}$ is an ideal in $\mathbb{D}^r_n$.

As to the group of automorphisms $\operatorname{Aut}_{\mathbb{R}} A$ of the algebra $A$, which is studied in this paper, we recall the well known fact (see [3]) that

$$\operatorname{Aut}_{\mathbb{R}} \mathbb{D}^r_n = G^r_n,$$

the *$n$-dimensional jet (differential) group of the order $r$.*

By a *fixed point* of $A$ we mean every $a \in A$ satisfying $\phi(a) = a$ for all $\phi \in \operatorname{Aut}_{\mathbb{R}} A$. Let

$$SA = \{a \in A; \phi(a) = a \text{ for all } \phi \in \operatorname{Aut}_{\mathbb{R}} A\}$$

be the set of all fixed points of $A$. It is clear, that $SA$ is a subalgebra of $A$ containing constants (of couse, every automorphism sends 1 into 1), i.e. $SA \supseteq \mathbb{R}$. If $SA = \mathbb{R}$, we say that $SA$ is *trivial*. For some classification results, see [8] and [9].

We will use the same terminology below although we will not focus only on the real field [1].

## 2 Dual and plural numbers

### 2.1 Dual numbers

Let $F$ be an arbitrary field and $F[X]$ the ring of polynomials over $F$. Then $F[X]$ is an $F$-algebra thanks to the ring homomorphism mapping elements of $F$ to constant polynomials in $F[X]$. The indeterminate $X$ generates the maximal ideal $(X)$ in $F[X]$. The quotient

$$\mathbb{D}_F = F[X]/(X)^2$$

is also an $F$-algebra and it is usually called the *algebra of dual numbers over $F$*. Then $\mathbb{D}_F$ has the unique maximal ideal generated by $X$ (and so $\mathbb{D}_F$ is local). We can express $\mathbb{D}_F$ by

$$\mathbb{D}_F = \{a_0 + a_1 X;\ a_0, a_1 \in F,\ X^2 = 0\}.$$

We will describe automorphisms of $\mathbb{D}_F$. For every such an automorphism $\phi$

$$\phi(1_F) = 1_F$$

[1] In algebraic literature, there exists also the denotation $A^{\operatorname{Aut}_F A}$ for our $SA$ assuming $A$ is an $F$-algebra over a field $F$. Following [14], we can say that $A$ is a *Galois extension* of $F$ with *Galois group* $\operatorname{Aut}_F A$ if $F = A^{\operatorname{Aut}_F A}$. (This does not quite correspond with the definition in [14] where the Galois group is considered finite.) Then the problem of a triviality of $SA$ identifies with the problem whether $A$ is a Galois extension of $F$ (with Galois group $\operatorname{Aut}_F A$) or not.

is satisfied and thus

$$\phi(a_0) = a_0 \text{ for every } a_0 \in F.$$

Further, in general,

$$\phi(X) = b_0 + b_1 X;\ b_0, b_1 \in F.$$

We compute

$$0_F = \phi(0_F) = \phi(X^2) = \phi(X)\phi(X) = b_0^2 + b_0 b_1 X + b_1 b_0 X + b_1^2 X^2 = b_0(b_0 + 2b_1 X),$$

thus, by a comparing of coefficients standing at 1 at $X$, $b_0 = 0$, then, necessarily, $b_1$ must be invertible and thus non-zero for $\phi$ be a bijection.

**Proposition 1.** *Let $A = \mathbb{D}_F$. Then $SA$ is nontrivial if and only if $F = \mathbb{F}_2$.*

*Proof.* We have derived that every automorphism $\phi$ acts by

$$\phi(a_0 + a_1 X) = a_0 + b_1 a_1 X;\ b_1 \in F - \{0_F\}.$$

Hence elements $a_1 X$ are fixed if and only if $b_1 = 1_F$: so we must have a field with only two elements $0_F$ and $1_F$ for it. $\square$

## 2.2 Plural numbers

It is easy to generalize the concept of dual numbers to the quotient of the polynomial $F$-algebra in $n$ indeterminates. We take the $F$-algebra

$$(\mathbb{D}_F)_n = F[X_1, \ldots, X_n]/(X_1, \ldots, X_n)^2$$

and call this $F$-algebra the *algebra of plural numbers over $F$*.

A general form of endomorphisms of $(\mathbb{D}_F)_n$ is

$$\begin{aligned}
\phi(1) &= 1 \\
\phi(X_1) &= b_{10} + b_{11}X_1 + b_{12}X_2 + \cdots + b_{1n}X_n \\
\phi(X_2) &= b_{20} + b_{21}X_1 + b_{22}X_2 + \cdots + b_{2n}X_n \\
&\cdots \\
\phi(X_n) &= b_{n0} + b_{n1}X_1 + b_{n2}X_2 + \cdots + b_{nn}X_n.
\end{aligned}$$

However, we have

$$\begin{aligned}
0_F = {} & \phi(0_F) = \phi(X_1^2) = b_{10}^2 + b_{11}^2 X_1^2 + \cdots + b_{1n}^2 X_n^2 + 2b_{10}b_{11}X_1 + \cdots + 2b_{10}b_{1n}X_n = \\
& b_{10}(b_{10} + 2b_{11}X_1 + \cdots + 2b_{1n}X_n),
\end{aligned}$$

thus, $b_{10} = 0$, and analogously $b_{20} = \cdots = b_{n0} = 0$. Now, the matrix $\begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ b_{n1} & b_{n2} & \ldots & b_{nn} \end{pmatrix}$ must be invertible for $\phi$ be a bijection. So, automorphisms of $(\mathbb{D}_F)_n$ form exactly the group $\mathrm{GL}(n, F)$.

**Remark 1.** General linear groups are widely studied. Especially, for the finite case, the order of $GL(n, \mathbb{F}_{p^k})$ ($p$ a prime number, $k \in \mathbb{N}$) is

$$\prod_{i=0}^{n-1} \left(p^{nk} - p^{ik}\right).$$

**Proposition 2.** *Let $n \in \mathbb{N}$, $n > 2$, $A = (\mathbb{D}_F)_n$. Then $SA$ is always trivial.*

*Proof.* We show that the element $a = a_1X_1 + \cdots + a_nX_n$ cannot be fixed. Of course, we can assume that one of $a_i$, say $a_1$, is non-zero. Let us consider the (diagonal) automorphism $\phi$

$$\begin{aligned} \phi(1_F) &= 1_F \\ \phi(X_1) &= bX_1 \\ \phi(X_2) &= X_2 \\ &\cdots \\ \phi(X_n) &= X_n, \qquad \text{where } b \neq 0_F. \end{aligned}$$

Let us first suppose that $b \neq 1_F$. Then evidently $\phi(a) \neq a$. However, we have not always a possibility to take $b \neq 1_F$. It occurs in the case $F = \mathbb{F}_2$.

So, in the rest of this proof, let $F = \mathbb{F}_2$. Let $i, j \in \{1, \ldots, n\}$, $i \neq j$. Then $\phi_{(i,j)} \colon (\mathbb{D}_2)_n \to (\mathbb{D}_2)_n$ given by

$$\begin{aligned} \phi_{(i,j)}(1) &= 1 \\ \phi_{(i,j)}(X_i) &= X_i + X_j \\ \phi_{(i,j)}(X_k) &= X_k \text{ for all } k \in \{1, \ldots, n\}, k \neq i \end{aligned}$$

belongs to $\mathrm{Aut}_{\mathbb{F}_2}(\mathbb{D}_{\mathbb{F}_2})_n$ becasue it is clear that $\phi_{(i,j)}$ meets the general form above. First, let us suppose that $a_1 = \cdots = a_n = 1$ and prove that the element

$$X_1 + X_2 + \cdots + X_n$$

is not fixed. For this, it suffices to take some automorphism $\phi_{(i,j)}$, e.g. $\phi_{(1,2)}$ sends $X_1 + X_2 + \cdots + X_n$ onto $X_1 + X_2 + X_2 + X_3 + \cdots + X_n = X_1 + X_3 + \cdots + X_n$. Second, let $\{k_1, \ldots, k_h\}$ be a (non-empty) proper subset of $\{1, \ldots, n\}$, i.e. $h < n$. We prove that the element

$$X_{k_1} + X_{k_2} \cdots + X_{k_h}$$

is not fixed, too. We take $i \in \{k_1, \ldots, k_h\}$ and $j \in \{1, \ldots, n\} - \{k_1, \ldots, k_h\}$ and apply $\phi_{(i,j)}$: it sends $X_{k_1} + X_{k_2} \cdots + X_{k_h}$ onto $X_{k_1} + X_{k_2} \cdots + X_{k_h} + X_j$.

So, $SA = F$ is always trivial. $\square$

# 3 Higher order case

## 3.1 One indeterminate

Of course, the powers of the maximal ideal $(X)$ represent notable class of ideals in $\mathbb{D}_F$. For $r \in \mathbb{N}$, $r > 1$, we will study the algebra

$$(\mathbb{D}_F)^r = F[X]/(X)^{r+1}.$$

Elements of $(\mathbb{D}_F)^r$ have a form

$$a_0 + a_1X + a_2X^2 + \cdots + a_rX^r;\ a_0, a_1, a_2, \ldots, a_r \in F,\ X^{r+1} = 0.$$

We start with the following lemma.

**Lemma 1.** *Automorphisms* $\phi\colon (\mathbb{D}_F)^r \to (\mathbb{D}_F)^r$ *have a form*

$$\begin{aligned} \phi(1) &= 1 \\ \phi(X) &= b_1X + b_2X^2 + \cdots + b_rX^r;\ b_1 \in F - \{0_F\},\ b_2, \ldots, b_r \in F. \end{aligned}$$

*Proof.* It suffices to describe $\phi^{-1}$. We have

$$\begin{aligned} Y &= \phi(X) = b_1X + b_2X^2 + \cdots + b_rX^r \\ Y^2 &= b_1^2X^2 + \text{terms of degree} > 2 \\ &\ldots \\ Y^{r-1} &= b_1^{r-1}X^{r-1} + \text{a term of degree } r \\ Y^r &= b_1^rX^r \end{aligned}$$

The last equation provides $X^r$ as $b_1^{-r}Y^r$, the last but one provides (after the substitution) $X^{r-1}$ and so on.

On the other hand, we cannot allow any more general form of automorphisms: it is evident if we consider an endomorphism

$$\begin{aligned} \phi(1) &= 1 \\ \phi(X) &= b_1X + b_2X^2 + \cdots + b_rX^r \end{aligned}$$

with $b_1 = 0$ that its kernel is nontrivial and hence does not represent an automorphism. □

For an $F$-algebra $A$ in question and its nilradical $\mathfrak{n}_A$, if an element $a \in A$ has the property $au = 0$ for all $u \in \mathfrak{n}_A$, we call $a$ the *socle element* of $A$. It is easy to find that all socle elements constitute an ideal; this ideal is called the *socle* of $A$ and denoted by $\operatorname{soc} A$.

**Lemma 2.** *Let $p$ be a prime number, $k \in \mathbb{N}$, $F = \mathbb{F}_{p^k}$ the finite field, $l \in \mathbb{N}$, $r = l(p^k - 1)$. Then for $A = (\mathbb{D}_F)^r$ all elements in* $\operatorname{soc} A$ *belong to* $SA$.

*Proof.* It is well known that for every $x \in F$, $x \neq 0$ the equality

$$x^{p^k - 1} = 1$$

holds (the generalization of Little's Fermat Theorem for finite fields). As $X^r \in \operatorname{soc} A$ maps onto $b_1^r X^r$, for $r$ which is the $l$-multiple of $p^k - 1$ is $b_1^r = 1$. □

**Example 1.** Let us consider $A = (\mathbb{D}_{\mathbb{F}_2})^3$. Then the element $a = X^2 + X^3$ belongs to $SA$. We compute

$$\phi(X^2 + X^3) = X^2 + b_2 X^3 + b_2 X^3 + X^3 = X^2 + X^3$$

and we see that $a$ is fixed. Hence there exist elements of $SA$ not belonging to $\operatorname{soc} A$, cf. [10], Proposition 2.

**Proposition 3.** *Let $A = (\mathbb{D}_F)^r$. For fields of characteristic 0, $SA$ is trivial. For finite fields, $SA$ is nontrivial and contains* $\operatorname{soc} A$.

*Proof.* The proof follows directly from the previous two lemmas and their proofs. □

### 3.2 More indeterminates

Let us consider the $n$-dimensional $(n > 1)$ case now. Elements of the algebra

$$A = (\mathbb{D}_F)^r_n = F[X_1, \dots, X_n]/(X_1, \dots, X_n)^{r+1}$$

have a form

$$\begin{aligned}
&a_0 + \\
&a_1 X_1 + a_2 X_2 + \cdots + a_n X_n + \\
&a_{11} X_1^2 + a_{12} X_1 X_2 + \cdots + a_{nn} X_n^2 + \\
&\cdots + \\
&a_{\underbrace{1\ldots1}_{r}} X_1^r + a_{\underbrace{1\ldots12}_{r}} X_1^{r-1} X_2 + \ldots a_{\underbrace{n\ldots n}_{r}} X_n^r; \\
&\quad a_0, a_1, \ldots, a_{\underbrace{n\ldots n}_{r}} \in F.
\end{aligned}$$

On basis of previous results we can find out nature of this general case now.

**Proposition 4.** *For $r \in \mathbb{N}$, $n \in \mathbb{N}$, $n > 1$, let $A = (\mathbb{D}_F)^r_n$. Then the subalgebra $SA$ of fixed points of $A$ is always trivial.*

*Proof.* Obviously, elements of $\mathrm{GL}(n, F)$ represent automorphisms also for $(\mathbb{D}_F)^r_n$. Of course, not *all* automorphisms, however, these (*linear*) automorphisms suffice for our following considerations. In the proof, we use formally partial derivations $\frac{\partial}{\partial X_j}$ for an expressing whether elements of $A$ contain $X_j$ in some non-zero power or not.

Let $u \in A$ and let exist $i, j \in \{1, \dots, n\}$ such that $\frac{\partial u}{\partial X_i} \neq 0$ and $\frac{\partial u}{\partial X_j} = 0$. Analogously with the case $r = 1$, $n > 1$, we apply $\phi_{(i,j)}$ for the demonstration that $u$ can not be fixed.

So, let $v \in A$ be not of such a type and let $\sigma$ be a permutation of $n$-tuple $(X_1, \dots, X_n)$ for which $\sigma(v) \neq v$. As permutations of $(X_1, \dots, X_n)$ are also elements of $\mathrm{GL}(n, F)$, we find again that $v$ can not be fixed.

Therefore we take $w \in A$ such that $\frac{\partial w}{\partial X_i} \neq 0$ for all $i \in \{1, \dots, n\}$ and such that does not exist any permutation of $(X_1, \dots, X_n)$ yielding a transformation of $w$. Nevertheless, a "symmetry" of $w$ will be again unbalanced by $\phi_{(i,j)}$, e. g. $\phi_{(1,2)}$. Hence we have an automorphism for which not even $w$ is fixed.

Thus, only zero power elements of $A$ remain fixed with respect to all automorphisms: $SA$ is trivial. $\square$

## 4 Comments to applications

We do not intend go into detail in this section and define at length every mentioned concept; just informative comments are here.

### 4.1 The real case: Weil contact elements

Now, let $M$ be a smooth manifold and let the Weil algebra $A$ have width $\mathrm{w}(A) = k < m = \dim M$ and order $\mathrm{ord}(A) = r$. Every $A$-velocity $V$ (see [3]) determines an underlying $\mathbb{D}^1_k$-velocity $\underline{V}$. We say $V$ is *regular* if $\underline{V}$ is regular, i.e. having maximal rank $k$ (in its local coordinates). Let us denote $\mathrm{reg}\, T^A M$ the open subbbundle of $T^A M$ of regular velocities on $M$. The *contact element of type* $A$ or briefly the *Weil contact element* on $M$ determined by $X \in \mathrm{reg}\, T^A M$ is the equivalence class

$$\mathrm{Aut}_{\mathbb{R}}\, A_M(X) = \{\phi(X); \phi \in \mathrm{Aut}_{\mathbb{R}}\, A\}.$$

We denote by $K^A M$ the set of all contact elements of type $A$ on $M$. Then

$$K^A M = \mathrm{reg}\, T^A M / \mathrm{Aut}_{\mathbb{R}}\, A$$

has a differentiable manifold structure and $\mathrm{reg}\, T^A M \to K^A M$ is a principal fiber bundle with the structure group $\mathrm{Aut}_{\mathbb{R}}\, A$. Moreover, $K^A M$ is a generalization of the bundle of higher order contact elements $K^r_k M = \mathrm{reg}\, T^r_k M / G^r_k$ introduced by Claude Ehresmann. We remark that the local description of regular velocities and contact elements is covered by the paper [6].

We have deduced in [5] and [7] the following results:

*There is a one-to-one correspondence between all natural operators lifting vector fields from $m$-manifolds to the bundle functor $K^A$ of Weil contact elements and the subalgebra of fixed elements $SA$ of $A$.*

*There is a one-to-one correspondence between all natural affinors on $K^A$ and the subalgebra of fixed elements $SA$ of $A$.*

*All natural operators lifting 1-forms from $m$-dimensional manifolds to the bundle functor $K^A$ of Weil contact elements are classified for the case of dwindlable Weil algebras: they represent constant multiples of the vertical lifting.*

### 4.2 The finite case: Cryptography, coding theory

Finite structures are extensively applied in cryptography. The problem of developing new public key cryptosystem had occupied the cryptographic research fields for the last decades. So called multivariate cryptosystems use polynomial automorphisms, in particular, there are known tame transformation methods using for ciphering compositions of affine automorphisms and de Jonquières automorphisms. The security of such systems is based on the difficulties in decomposition of a composed polynomial automorphism.

So, the natural modification of these public key cryptosystems is a use of local (finite) algebras instead polynomial. The role of automorphisms remains unchanged. Surely, it is important to understand the subalgebra of fixed elements (which are not transformed under any automorphism).

**Example 2.** As a toy exercise, we can consider $A = (\mathbb{D}_{\mathbb{F}_4})^2$ and take e.g. polynomials in two indeterminates $Y_1$, $Y_2$ over $A$, i.e. elements of $(\mathbb{D}_{\mathbb{F}_4})^2[Y_1, Y_2]$. In multivariate public key cryptosystems, the cipher procedure is based on composed polynomial automorphisms, which are used as the public key. Let us imagine a simple scheme based on the composition $\pi = \lambda_2 \circ \tau \circ \lambda_1$ of affine ($\lambda_1$ and $\lambda_2$) and de Jonquières ($\tau$) $\mathbb{F}_4$-automorphisms which play a role of a private key. Without a decomposition of $\pi$, it is not easy to find $\pi^{-1}$ which is necessary for decryption. Of course, a descryption of fixed elements is the substantial feature of such a system.

We only remark that local finite algebras are used also in the coding theory, for detail see [1].

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# Stability radius bounds in multicriteria Markowitz portfolio problem with venturesome investor criteria

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**Abstract.** The lower and upper bounds on the stability radius are obtained in multicriteria Boolean Markowitz investment problem with criteria of extreme optimism (MAXMAX) about portfolio return in the case when portfolio and financial market states spaces are endowed with Hölder metric, and criteria space of economical efficiency of investment projects is endowed with Chebyshev metric.

**Mathematics subject classification:** 90C09, 90C29, 90C31, 90C47.
**Keywords and phrases:** Multicriteria Markowitz problem, Boolean investment problem, stability radius bound, efficiency of investment project.

## 1 Introduction

In the papers [1,2], vector investment Boolean problems with Savage and Wald's criteria are formulated based on Markowitz portfolio theory [3]. Stability radius bounds are obtained only for particular cases, when three-dimensional problem parameters space is equipped with different combinations of $l_1$ and $l_\infty$ metric. In the present paper, multicriteria portfolio problem with venturesome investor under the same Markowitz model framework is considered. The investor maximizes various portfolio efficiency types when financial market is in the most favorable state, i.e. with criteria of extreme optimism (MAXMAX). We investigate such a kind of stability of the problem which is a discrete analogue of the property to be semicontinuous from above in Hausdorff's sense of a point-set mapping which transforms any set of parameters of the investment problem into the corresponding Pareto set. As a result of the conducted parametric analysis, power and upper bounds on the stability radius of the problem are obtained in the case when portfolio space and financial market spaces are endowed with Hölder metric $l_p, 1 \leq p \leq \infty$, and criteria space of economical efficiency of investment projects is endowed with Chebyshev metric $l_\infty$.

## 2 Problem statement and definitions

Based on [2,4], consider multicriteria variant of Markowitz investment management problem [3].

Let $m$ be the number of possible financial market states $(A_1, A_2, \dots, A_m)$, $n$ be the number of alternative investment projects $(B_1, B_2, \dots, B_n)$ and $s$ be the number of types (measures) of the project economical efficiency $(C_1, C_2, \dots, C_s)$. Given the expected evaluation of economical efficiency $e_{ijk}$ for an arbitrary investment project $B_j$ of type $C_k$ in the case when market is in the state $A_i$. We denote three-dimensional matrix $[e_{ijk}] \in \mathbf{R}^{m\times n\times s}$ by $E$ and its $k$-th cut by $E_k \in \mathbf{R}^{m\times n}$. Let $x = (x_1, x_2, \dots, x_n)^T \in \mathbf{E}^n$ be an investment portfolio where $\mathbf{E} = \{0, 1\}$, $x_j = 1$ if the investor chooses the project $B_j$ and $x_j = 0$ otherwise; $X \subseteq \mathbf{E}^n$ be the set of all possible investment portfolios, i.e. those realization of which does not exceed investor's initial budget and admissible level of risk.

Note that there are several approaches to evaluate efficiency of investment projects (see e.g. the bibliography in [2]).

In the portfolio space $X$ we introduce vector objective function

$$f(x, E) = (f_1(x, E_1), f_2(x, E_2), \dots, f_s(x, E_s)),$$

components of which are well known in the decision making theory criteria of extreme optimism (MAXMAX)

$$f_k(x, E_k) = \max_{1\le i\le m} e_{ik}x = \max_{1\le i\le m} \sum_{j=1}^{n} e_{ijk}x_j \to \max_{x\in X}, \quad k \in N_s = \{1, 2, \dots, s\},$$

where $e_{ik} = (e_{i1k}, e_{i2k}, \dots, e_{ink})$ is the $i$-th row of the cut $E_k$. Using this criteria venturesome investor optimizes the efficiency $e_{ik}x$ of the portfolio $x$ under the assumption that market is in the most favorable state for him. In other words when a portfolio return is maximal. It is evident that the approach is based on the behavior stereotype of reckless optimism ("make or mar", "who does not risk cannot win" etc.). It is worth to notice that such situations in economics when we have to behave this way are common. Such dealing is inherent in not only optimist but investors with his (her) back to the wall.

Under a multicriteria Boolean investment problem $Z^s(E), s \in \mathbf{N}$ we understand the problem of searching the Pareto set $P^s(E)$, i.e. the set of Pareto optimal investment portfolio

$$P^s(E) = \{x \in X \ : \ X(x, E) = \varnothing\},$$

where

$$X(x, E) = \{x' \in X \ : \ f(x, E) \le f(x', E) \ \& \ f(x, E) \ne f(x', E)\}.$$

It is obvious that $P^s(E) \ne \varnothing$ for any matrix $E \in \mathbf{R}^{m\times n\times s}$.

For any natural number $d$ in the real space $\mathbf{R}^d$ we define Hölder metric $l_p, p \in [1, \infty]$, i.e. under the norm of vector $y = (y_1, y_2, \dots, y_d)^T \in \mathbf{R}^d$ we understand the number

$$\|y\|_p = \begin{cases} (\sum\limits_{i=1}^{d} |y_i|^p)^{1/p} & \text{if } 1 \le p < \infty, \\ \max\limits_{1\le i\le d} |y_i| & \text{if } p = \infty. \end{cases}$$

It is well known that for any vectors $a, b \in \mathbf{R}^n$ the Hölder inequality holds

$$|a^T b| \le \|a\|_p \|b\|_q, \tag{1}$$

and for numbers $p$ and $q$ the following relation is true

$$1/p + 1/q = 1.$$

Here $q = 1$ if $p = \infty$ and $q = \infty$ if $p = 1$. Thereby suppose $1/p = 0$ for $p = \infty$. Further we assume that the domain of variation of $p$ and $q$ is the segment $[1, \infty]$.

In the portfolio space $\mathbf{R}^n$ and financial market states space $\mathbf{R}^m$ define an arbitrary Hölder metric $l_p,\ p \in [1, \infty]$, and in the criteria space of measures of project economical efficiency $\mathbf{R}^s$ define Chebyshev metric $l_\infty$, since under the norm of a matrix $E \in \mathbf{R}^{m\times n\times s}$ we understand the number

$$\|E\|_{pp\infty} = \|(\|E_1\|_{pp}, \|E_2\|_{pp}, \ldots, \|E_s\|_{pp})\|_\infty,$$

where

$$\|E_k\|_{pp} = \|(\|e_{1k}\|_p, \|e_{2k}\|_p, \ldots, \|e_{mk}\|_p)\|_p, \quad k \in N_s.$$

Obviously,

$$\|e_{ik}\|_p \le \|E_k\|_{pp} \le \|E\|_{pp\infty}, \quad i \in N_m,\ k \in N_s. \tag{2}$$

Therefore using Hölder inequality (1), it is easy to see that for any portfolios $x, x' \in X$ and matrix $E \in \mathbf{R}^{m\times n\times s}$ the following inequalities are valid

$$e_{ik}x - e_{i'k}x' \ge -\|E\|_{pp\infty}\|x + x'\|_1^{1/q}, \quad i, i' \in N_m, \quad k \in N_s. \tag{3}$$

Indeed

$$e_{ik}x - e_{i'k}x' \ge -(\|e_{ik}\|_p\|x\|_q + \|e_{i'k}\|_p\|x'\|_q) \ge$$

$$-\|(\|e_{ik}\|_p, \|e_{i'k}\|_p)\|_p\|(\|x\|_q, \|x'\|_q)\|_q \ge -\|E\|_{pp\infty}\|x + x'\|_1^{1/q}.$$

Moreover, it is easy to see that for vector $a = (a_1, a_2, \ldots, a_n)^T \in \mathbf{R}^n$ with conditions $|a_i| = \alpha,\ i \in N_n$, for any number $p \in [1, \infty]$ the following equality holds

$$\|a\|_p = \alpha n^{1/p}. \tag{4}$$

According to $[1, 2, 5]$, with respect to the metrics defined, under the stability radius of the problem $Z^s(E), s \in \mathbf{N}$, we understand the number

$$\rho = \rho(m, n, s, p) = \begin{cases} \sup \Xi_p & \text{if } \Xi_p \ne \varnothing, \\ 0 & \text{if } \Xi_p = \varnothing, \end{cases}$$

where

$$\Xi_p = \{\varepsilon > 0 \ :\ \forall E' \in \Omega_p(\varepsilon) \ \ (P^s(E + E') \subseteq P^s(E))\},$$

$$\Omega_p(\varepsilon) = \{E' \in \mathbf{R}^{m\times n\times s} \ :\ \|E'\|_{pp\infty} < \varepsilon\}$$

be the set of perturbed matrices, $P^s(E+E')$ be the Pareto set of perturbed problem $Z^s(E + E')$. Thus the stability radius of problem $Z^s(E)$ is the limit level of perturbations of matrix $E$ elements in normed vector space $\mathbf{R}^{m\times n\times s}$ which do not lead to appearance of new Pareto optimal portfolios. It is obvious that for $P^s(E) = X$ the stability radius of the problem is supposed to be infinite. The problem for which $P^s(E) \ne X$ is called nontrivial.

## 3 Bounds on the stability radius of the problem

For the nontrivial problem $Z^s(E)$ put

$$\varphi = \varphi(m,n,s,p) = \min_{x \notin P^s(E)} \max_{x' \in P(x,E)} \frac{\gamma(x',x)}{\|x'+x\|_1^{1/q}},$$

$$\psi = \psi(m,n,s) = \min_{x \notin P^s(E)} \max_{x' \in P(x,E)} \frac{\gamma(x',x)}{\|x'-x\|_1},$$

where

$$\gamma(x',x) = \min\{f_k(x',E_k) - f_k(x,E_k) : k \in N_s\},$$
$$P(x,E) = X(x,E) \cap P^s(E).$$

It is easy to see that $\varphi,\ \psi \geq 0$.

**Theorem 1.** *For any $m,n,s \in \mathbf{N}$ and $p \in [1,\infty]$ for the stability radius $\rho(m,n,s,p)$ of the multicriteria nontrivial investment problem $Z^s(E)$ the following bounds are valid*

$$\varphi(m,n,s,p) \leq \rho(m,n,s,p) \leq (mn)^{1/p}\psi(m,n,s). \tag{5}$$

*Proof.* Let us first show that the inequality $\rho \geq \varphi$ is valid. For $\varphi = 0$ it is evident. Let $\varphi > 0$ and the perturbed matrix $E' \in \mathbf{R}^{m\times n\times s}$ with cuts $E'_k$, $k \in N_s$, belongs to the set $\Omega_p(\varphi)$, i.e. $\|E'\|_{pp\infty} < \varphi$. According to the definition of number $\varphi$ for any portfolio $x \notin P^s(E)$ there exists portfolio $x^0 \in P(x,E)$ such that

$$\gamma(x^0,x) \geq \varphi\|x^0+x\|_1^{1/q},$$

i.e. the inequalities

$$f_k(x^0,E_k) - f_k(x,E_k) \geq \varphi\|x^0+x\|_1^{1/q}, \quad k \in N_s$$

hold.

Therefore, taking into account inequality (3), for any index $k \in N_s$ we obtain

$$f_k(x^0,E_k+E'_k) - f_k(x,E_k+E'_k) = \max_{1\leq i\leq m}(e_{ik}+e'_{ik})x^0 - \max_{1\leq i\leq m}(e_{ik}+e'_{ik})x =$$

$$= \min_{1\leq i\leq m}\max_{1\leq i'\leq m}(e_{i'k}x^0 - e_{ik}x + e'_{i'k}x^0 - e'_{ik}x) \geq$$

$$\geq \min_{1\leq i\leq m}\max_{1\leq i'\leq m}(e_{i'k}x^0 - e_{ik}x) - \|E'\|_{pp\infty}\|x^0+x\|_1^{1/q} =$$

$$= f_k(x^0,E_k) - f_k(x,E_k) - \|E'\|_{pp\infty}\|x^0+x\|_1^{1/q} \geq (\varphi - \|E'\|_{pp\infty})\|x^0+x\|_1^{1/q} > 0,$$

where $e'_{ik}$ is the $i$-th row of the cut $E'_k$. Thus, any portfolio $x$, which is not in $P^s(E)$, is not a Pareto optimal portfolio on the perturbed problem $Z^s(E+E')$. Therefore we conclude that for any perturbed matrix $E' \in \Omega_p(\varphi)$ the inclusion $P^s(E+E') \subseteq P^s(E)$ is valid. Hence the inequality $\rho \geq \varphi$ is true.

Further we prove the inequality $\rho \leq (mn)^{1/p}\psi$.

According to the definition of the number $\psi$ there exists portfolio $x^0 \notin P^s(E)$ such that for any portfolio $x \in P(x^0, E)$ there exists $l = l(x) \in N_s$, for which

$$f_l(x, E_l) - f_l(x^0, E_l) \leq \psi \|x - x^0\|_1. \tag{6}$$

Assuming $\varepsilon > (mn)^{1/p}\psi$, we define the $k$-th cut $E_k^0$, $k \in N_s$ elements $e_{ijk}^0$ of the perturbed matrix $E^0$ by the rule

$$e_{ijk}^0 = \begin{cases} \delta & \text{if} \quad i \in N_s,\ x_j^0 = 1 \\ -\delta & \text{otherwise,} \end{cases}$$

where $\varepsilon/(mn)^{1/p} > \delta > \psi$. Then according to (4) we have

$$\|e_{ik}^0\|_p = n^{1/p}\delta, \quad \|E_k^0\|_{pp} = (mn)^{1/p}\delta, \quad i \in N_m, \quad k \in N_s,$$

$$\|E^0\|_{pp\infty} = (mn)^{1/p}\delta.$$

This means that $E^0 \in \Omega_p(\varepsilon)$. Moreover, all rows $e_{ik}^0$, $i \in N_m$, of the cut $E_k^0$, $k \in N_s$, are the same and consist of the components $\delta$ and $-\delta$. Therefore, assuming $A = e_{ik}^0$, $i \in N_m$, $k \in N_s$, we have

$$A(x - x^0) = -\delta \|x - x^0\|_1. \tag{7}$$

Hence, taking into account (6), we conclude that for any portfolio $x \in P(x^0, E)$ there exists $l \in N_s$, satisfying the relations

$$f_l(x, E_l + E_l^0) - f_l(x^0, E_l + E_l^0) = \max_{1 \leq i \leq m} (e_{il} + e_{il}^0)x - \max_{1 \leq i \leq m} (e_{il} + e_{il}^0)x^0 =$$

$$= \min_{1 \leq i \leq m} \max_{1 \leq i' \leq m} (e_{i'l}x - e_{il}x^0 + e_{i'l}^0 x - e_{il}^0 x^0) = f_l(x, E_l) - f_l(x^0, E_l) + A(x - x^0) \leq$$

$$\leq (\psi - \delta)\|x - x^0\|_1 < 0.$$

Thus the formula

$$\forall x \in P(x^0, E) \quad (x \notin X(x^0, E + E^0)) \tag{8}$$

is valid. If $X(x^0, E + E^0) = \emptyset$ then $x^0 \in P^s(E + E^0)$. Recall that $x^0 \notin P^s(E)$.

Now suppose that $X(x^0, E + E^0) \neq \emptyset$.

Then due to the external stability of the set $P^s(E + E^0)$ (see e.g. [6, 7]) there exists portfolio $x^* \in P(x^0, E + E^0)$. We show that $x^* \notin P^s(E)$.

We assume the contrary: $x^* \in P^s(E)$. According to (8) the inclusion

$$x^* \in P^s(E) \setminus P(x^0, E)$$

holds. Therefore only two following cases are possible.

**Case 1.** $f(x^*, E) = f(x^0, E)$. Then for any $k \in N_s$ from equality (7) it follows

$$f_k(x^*, E_k + E_k^0) - f_k(x^0, E_k + E_k^0) = f_k(x^*, E_k) - f_k(x^0, E_k) +$$

$$+A(x^* - x^0) = -\delta\|x^* - x^0\|_1 < 0.$$

**Case 2.** There exists $q \in N_s$ such that $f_q(x^*, E_q) < f_q(x^0, E_q)$. Then again using (7) we obtain

$$f_q(x^*, E_q + E_q^0) - f_q(x^0, E_q + E_q^0) = f_q(x^*, E_q) - f_q(x^0, E_q) + A(x^* - x^0) < 0.$$

Consequently both cases contradict the inclusion $x^* \in P(x^0, E + E^0)$. Therefore it is proved that $x^* \notin P^s(E)$. Recall that $x^* \in P^s(E + E^0)$.

Thus for any number $\varepsilon > (mn)^{1/p}\psi$ it is guaranteed that there exists a perturbing matrix $E^0 \in \Omega_p(\varepsilon)$ such that there exists portfolio ($x^0$ or $x^*$) which is not Pareto optimal portfolio for $Z^s(E)$ but becomes Pareto optimal in the perturbed problem $Z^s(E + E^0)$. Hence the formula

$$\forall \varepsilon > (mn)^{1/p}\psi \quad \exists E^0 \in \Omega_p(\varepsilon) \quad (P^s(E + E^0) \nsubseteq P^s(E))$$

is valid.

Consequently, $\rho \leq (mn)^{1/p}\psi$. □

The well known result follows from Theorem 1.

**Corollary 1 [8].** $\varphi(m, n, s, \infty) \leq \rho(m, n, s, \infty) \leq \psi(m, n, s)$.

The following evident statement confirms attainability on these bounds

**Corollary 2.** *If for any pair $x \notin P^s(E)$ and $x' \in P(x, E)$ the equality*

$$\{j \in N_n \ : \ x_j = x'_j = 1\} = \varnothing$$

*holds then the formula*

$$\rho(m, n, s, \infty) = \varphi(m, n, s, \infty) = \psi(m, n, s)$$

*is valid.*

Attainability of the upper bound in (5) for $m = 1$ and $p = \infty$ follows from the following known theorem.

**Theorem 2 [9].** $\rho(1, n, s, \infty) = \psi(1, n, s), \quad n, s \in \mathbf{N}$.

*Remark* 1. From theorem 1 it follows that the upper bound on the stability radius of the problem $\rho(m, n, s, p)$ decreases $mn$ times with number $p$ increasing from 1 to $\infty$. That is the upper bound decreases from $mn\psi(m, n, s)$ to $\psi(m, n, s)$. At the same time the lower bound also decreases from

$$\varphi(m, n, s, 1) = \min_{x \notin P^s(E)} \max_{x' \in P(x,E)} \gamma(x', x)$$

to

$$\varphi(m, n, s, \infty) = \min_{x \notin P^s(E)} \max_{x' \in P(x,E)} \frac{\gamma(x', x)}{\|x' + x\|_1}.$$

As follows from Corollary 2, when its conditions hold the lower values of the lower and upper bounds on the stability radius are identical:

$$\varphi(m, n, s, \infty) = \psi(m, n, s).$$

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# $l_p(R)$-equivalence of topological spaces and topological modules

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**Abstract.** Let $R$ be a topological ring and $E$ be a unitary topological $R$-module. Denote by $C_p(X, E)$ the class of all continuous mappings of $X$ into $E$ in the topology of pointwise convergence. The spaces $X$ and $Y$ are called $l_p(E)$-equivalent if the topological $R$-modules $C_p(X, E)$ and $C_p(Y, E)$ are topological isomorphisms. Some conditions under which the topological property $\mathcal{P}$ is preserved by the $l_p(E)$-equivalence (Theorems 8 – 11) are given.

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## 1 Preliminaries

Let $G$ be an Abelian group under addition operation and $R$ be a ring. We call $G$ a left $R$-module or, simply, an $R$-module if on it the operation of multiplication between an element of $R$ and an element of $G$ is defined, say $ra \in G$, where $r \in R$ and $a \in G$, with the following properties: $r(a + b) = ra + rb$, $(r + s)a = ra + sa$, $r(sa) = (rs)a$, for any $r, s \in R$ and $a, b \in G$. In other words, an $R$-module is a vector space where the base field is replaced by a base ring $R$. Usually the operation of multiplication $(r, a) \mapsto ra$ is called scalar multiplication. Obviously, any ring $R$ is a module over itself. An $R$-module is unitary [11] if $R$ possesses a multiplicative identity 1 and $1x = x$ for every $x \in G$.

An additive topological group is an additive group $G$ with a topology such that the addition operation $(x, y) \to x + y$ and inverse operation $x \to -x$ are continuous mappings [11]. A topological ring is a ring $R$ with a topology making $R$ into an additive Abelian topological group such that the multiplication is a continuous mapping [11].

Let $R$ be a topological ring. A topological $R$-module is an $R$-module $E$ together with a topology such that $E$ is an additive Abelian topological group and scalar multiplication is a continuous mapping.

Let $E$ and $F$ be $R$-modules. The mapping $\varphi : E \to F$ is a linear mapping if it satisfies the conditions:

(i) $\varphi(x + y) = \varphi(x) + \varphi(y)$, for any $x, y \in E$;

(ii) $\varphi(\alpha x) = \alpha\varphi(x)$, for any $x \in E$ and $\alpha \in R$.

Throughout this paper, by a "space" we will mean a "completely regular space", by a "topological ring" we will mean "topological unitary ring" and by a "topological module" we will mean a "topological unitary module".

Fix a space $X$, a topological ring $R$ and a topological $R$-module $E$. By $C(X,E)$ we will denote the family of all $E$-valued continuous functions with the domain $X$ and by $C_p(X,E)$ we will denote the space $C(X,E)$ endowed with the topology of pointwise convergence. Recall that the family of sets of the form $W(x_1,x_2,...,x_n,U_1,U_2,...,U_n) = \{f : C(X,E) : f(x_i) \in U_i \textit{ for any } \ i \leq n\}$, where $x_1,x_2,...,x_n \in X$, $U_1,U_2,...,U_n$ are open sets of $E$ and $n \in \mathbb{N}$, is an open base of the space $C_p(X,E)$.

Let $E$ be a topological $R$-module. The spaces $X$ and $Y$ are called $l_p(E)$-equivalent if the spaces $C_p(X,E)$ and $C_p(Y,E)$ are linearly homeomorphic and we denote $X \overset{E}{\sim} Y$.

Recall that an embedding of $X$ into $Y$ is a mapping $e : X \to Y$ such that $e$ is a homeomorphism of $X$ onto $e(X) \subseteq Y$.

**Proposition 1.** *Fix a topological $R$-module $E$. Then $C_p(X,E)$ is a topological $R$-module and $E$ is embedded in a natural way in $C_p(X,E)$ as a closed submodule of $C_p(X,E)$.*

*Proof.* $C_p(X,E)$ is a group under operation of pointwise addition and respectively is unitary module over the ring $R$. We put $a_X(x) = a$ for any $a \in E$ and $x \in X$, i.e. $a_X$ is a constant function.

Let $e : E \to C_p(X,E)$, where $e(a) = a_X(x)$ for every $a \in E$. The mapping $e$ is injective, since, if $a,b \in E$, with $a \neq b$, then $a_X(x) = a \neq b = b_X(x)$ for every $x \in X$.

The sets $W(x_1,x_2,...,x_n,U_1,U_2,...,U_n) = \{f \in C_p(X,E) : f(x_i) \in U_i\}$, where $n \in \mathbb{N}$, $x_1,x_2,...,x_n \in X$ and $U_1,U_2,...,U_n$ are open subsets of $E$, form an open base of the space $C_p(X,E)$.

If $x \in X$ and $U$ is open in $E$, then $e(U) = \{a_X : a \in U\} = e(E) \cap W(x_1,x_2,...,x_n,U,U,...,U)$ and $e^{-1}(W(x_1,x_2,...,x_n,U_1,U_2,...,U_n)) = \{a \in E : a_X \in U_1 \cap U_2 \cap ... \cap U_n\} = U_1 \cap U_2 \cap ... \cap U_n$. Hence, $e$ is an embedding of $E$ in $C_p(X,E)$.

Fix $g \in C_p(X,E)$ such that $g \notin e(E)$. There exist two distinct points $x_1,x_2 \in X$ such that $g(x_1) \neq g(x_2)$. Now we fix two open sets $U_1$ and $U_2$ from $R$ such that $g(x_1) \in U_1$, $g(x_2) \in U_2$ and $U_1 \cap U_2 = \emptyset$. Then $g \in W(x_1,x_2,U_1,U_2)$ and $W(x_1,x_2,U_1,U_2) \cap e(R) = \emptyset$. Hence the set $e(E)$ is closed. □

A space $X$ is zero-dimensional if $indX = 0$ (small inductive dimension is zero), i.e. $X$ has a base of clopen (open and closed) subsets.

**Proposition 2.** *If $E$ is a zero-dimensional topological $R$-module, then $C_p(X,E)$ is a zero-dimensional topological $R$-module.*

*Proof.* Since $C_p(X,E)$ is a subspace of $E^X$, the proof is complete. □

## 2 The evaluation mapping

Let $X$ be a space, $R$ be a topological ring and $E$ be a non-trivial topological $R$-module. Fix $x \in X$. Then the mapping $\xi_x : C_p(X,E) \to E$ defined by $\xi_x(f) = f(x)$ is called the evaluation mapping at $x$.

**Proposition 3.** *The evaluation mapping $\xi_x : C_p(X,E) \to E$ is continuous and linear for every point $x \in X$.*

*Proof.* Fix a point $x \in X$. For every open set $U$ of $E$ we have $\xi_x^{-1}(U) = \{f \in C_p(X,E) : f(x) \in U\} = W(x,U)$. But $W(x,U)$ is an element of the subbase of the topology on $C_p(X,E)$, i.e. $\xi_x^{-1}(U)$ is open in $C_p(X,E)$ for every open $U \subseteq E$.

Obviously $\xi_x(f+\alpha g) = (f+\alpha g)(x) = f(x)+\alpha g(x) = \xi_x(f)+\alpha\xi_x(g)$. Hence $\xi_x$ is a linear continuous mapping. □

We now define the canonical evaluation mapping $e_X : X \to C_p(C_p(X,E),E)$, where $e_X(x) = \xi_x$ for any $x \in X$.

**Proposition 4.** *The canonical evaluation mapping $e_X : X \to C_p(C_p(X,E),E)$ is continuous. Moreover, the set $e_X(X)$ is closed in the space $C_p(C_p(X,E),E)$.*

*Proof.* Let $U = W(f_1,f_2,...,f_k,U_1,U_2,...,U_k) \cap e_X(X)$ be a standard open set in $C_p(C_p(X,E),E) \cap e_X(X)$. Without loss of generality, we can assume that $U \subseteq e_X(X)$, i.e. $U = \{\xi_x \in e_X(X) : \xi_x(f_i) \in U_i, x \in X, i = \overline{1,k}\} = \{\xi_x \in e_X(X) : f_i(x) \in U_i, x \in X, i = \overline{1,k}\}$. On the other hand $e_X^{-1}(U) = \{x \in X : f_i(x) \in U_i, i = \overline{1,k}\} = \cap\{f_i^{-1}(U_i) : i = \overline{1,k}\}$. Since for every $i = \overline{1,k}$ and $f_i \in C(X,E)$, the set $e_X^{-1}(U)$ is a finite intersection of open sets, therefore it is open.

Fix $\varphi \in C_p(C_p(X,E),E) \setminus e_X(X)$. There exist $f \in C_p(X,E)$ and $b \in X$ such that $\varphi(f) \neq f(b) = \xi_b(f)$. Fix in $E$ two open sets $V$ and $W$ such that $\varphi(f) \in V$, $f(b) \in W$ and $V \cap W = \emptyset$. The set $U = \{\psi C_p(C_p(X,E),E) : \psi(f) \in V\}$ is open in $C_p(C_p(X,E),E)$ and $U \cap e_X(X) = \emptyset$. The proof is complete. □

Let $X$ and $Y$ be spaces, $\Phi$ a family of functions $f : X \to Y$. We say that $\Phi$ *separates points* of $X$ (or simply is *separating* [1]) if for any $x,y \in X$, $x \neq y$, there exists $f \in \Phi$ such that $f(x) \neq f(y)$. We also say that $\Phi$ *separates points from closed sets* (or is *regular* [1]) if for any closed subset $F$ of $X$ and any point $x \in X \setminus F$ there exists $f \in \Phi$ such that $f(x) \notin cl_Y f(F)$.

**Proposition 5.** *If $C_p(X,E)$ is a separating and regular family, then the canonical evaluation mapping $e_X : X \to C_p(C_p(X,E),E)$ is a homeomorphism from $X$ to the subspace $e_X(X)$ of $C_p(C_p(X,E),E)$.*

*Proof.* Since canonical evaluation mapping is continuous, it is clear that it is surjective and we have only to prove that $e_X$ is injective and the inverse function is continuous.

First, we show that $e_X$ is injective. Let $x,y \in X$, $x \neq y$. By assumption, $C_p(X,E)$ is a separating collection, i.e., we can find a function $f \in C_p(X,E)$ such that $f(x) \neq f(y)$, hence $\xi_x \neq \xi_y$.

Now, we prove that $e_X^{-1}$ is continuous. Let $F$ be a closed subset of $X$. By assumption $C_p(X,E)$ is a regular collection, i.e. for any $x \notin F$ we can find $f \in C_p(X,E)$ such that $f(x) \notin cl_E f(F)$. Therefore $f(x)$ has a neighbourhood $U_{f(x)}$ for which $U_{f(x)} \cap f(F) = \emptyset$. Then $W(f, U_{f(x)})$ is a neighbourhood of $\xi_x$ that is not contained in $e_X(F)$, i.e. $e_X(F)$ is closed. Hence $e_X^{-1}$ is continuous. □

A space $X$ is called:

(i) *$R$-completely regular* if for any closed subset $F$ of $X$ and any point $a \in X \setminus F$ there exists $f \in C(X,R)$ such that $f(a) \notin cl_R f(F)$;

(ii) *$R$-Tychonoff* if for any closed subset $F$ of $X$, any point $a \in X \setminus F$ there exists $g \in C(X,R)$ such that $g(a) = 0$ and $F \subseteq g^{-1}(1)$.

The space $R$ is $R$-completely regular. The Cartesian product of $R$-completely regular spaces is $R$-completely regular and the Cartesian product of $R$-Tychonoff spaces is an $R$-Tychonoff space. A subspace of an $R$-Tychonoff ($R$-completely regular) space is an $R$-Tychonoff ($R$-completely regular) space.

**Remark 1.** Let $X$ be an $R$-Tychonoff space. Then:

(i) $X$ is a Tychonoff space.

(ii) If $E$ is a topological $R$-module, then for each closed set $F$ of $X$, any point $a \in X \setminus F$ and any point $b \in E$, there exists $f \in C(X,E)$ such that $f(a) = 0$ and $f(F) = b$.

(iii) $X$ is $R$-completely regular.

**Remark 2.** Let $E$ be a topological $R$-module and $X$ be an $R$-completely regular space. Then:

(i) $X$ is a Tychonoff space.

(ii) For any closed subset $F$ of $X$ and any point $a \in X \setminus F$ there exists $f \in C(X,E)$ such that $f(a) \notin cl_E f(F)$.

(iii) $C(X,E)$ is a separating and regular family of continuous mappings.

**Proposition 6.** *A space $X$ is $R$-completely regular if and only if the family $C(X,E)$ is separating and regular for any non-trivial topological $R$-module $E$.*

*Proof.* It is obvious. □

**Proposition 7.** *If $indX = 0$, then the space $X$ is $R$-Tychonoff.*

*Proof.* If $C$ is a clopen subset, then $\chi_C$ is continuous, where $\chi_C(C) = 1$ and $\chi_C(X \setminus C) = 0$. Fix a point $x \in X$ and closed subset $F$ of $X$ such that $x \in X \setminus F$. Since $indX = 0$ we can find a clopen subset $C$ such that $C \subseteq X \setminus F$. Then $X \setminus C$ is also clopen and $F \subseteq cl_X F \subseteq X \setminus C$. The characteristic function $g = \chi_{X \setminus C}$ is continuous, $g(x) = 0$ and $F \subseteq g^{-1}(1)$. Hence $X$ is $R$-Tychonoff. □

Let $R$ be a topological ring. A topological $R$-module $E$ is called:

(i) *simple* if it does not contain a non-trivial submodule over $R$;

(ii) *locally simple* if $E$ is not trivial and there exists an open subset $U$ of $E$ such that $0 \in U$ and $U$ does not contain non-trivial $R$-submodules of $E$;

(iii) *R-closed* if there exists a continuous mapping $\varphi_E : E \longrightarrow R$ onto $R$ such that $\varphi_E(x+y) = \varphi_E(x) + \varphi_E(y)$ and $\varphi_E(tx) = t\varphi_E(x)$ for any $t \in R$ and $x, y \in E$.

**Example 1.** Let $\mathbb{R}$ be the field of real numbers and $\mathbb{C}$ be the field of complex numbers. The rings $\mathbb{R}$ and $\mathbb{C}$ are simple rings.

**Example 2.** If $R$ is the field of real numbers or of complex numbers, then any locally convex linear space over $R$ is an $R$-closed module.

**Example 3.** Let $\mathbb{Q}$ be the field of rational numbers. Then $\mathbb{R}$ and $\mathbb{C}$ are locally simple, not simple and not $\mathbb{Q}$-closed $\mathbb{Q}$-modules.

**Example 4.** If $R$ is a locally simple ring, then $R^n$ is a locally simple $R$-closed $R$-module for any natural number $n \geq 1$.

**Example 5.** Let $\mathbb{Z}$ be the ring of integers. Relative to discrete topology $\mathbb{Z}$ is locally simple non-simple ring. If $p \geq 2$ and $\mathbb{Z}_p = p \cdot \mathbb{Z}$, then $\mathbb{Z}_p$ is an ideal and $\{n\mathbb{Z}_p : n \in \mathbb{N}\}$ is a base at 0 of the ring topology. In that topology $\mathbb{Z}$ is not a locally simple ring.

Let $R$ be a ring and $E$ be an $R$-module. For any $a \in E$ we put $E_a = Ra = \{ta : t \in R\}$.

**Lemma 1.** *Let $R$ be a ring and $E$ be an $R$-module. Then $Ra$ is an $R$-module for any $a \in E$.*

*Proof.* Fix $a \in E$, $a \neq 0$. By definition, $Ra = \{xa : x \in R\}$. In the first we will prove that $Ra$ is an Abelian group.

Let $x, y \in Ra$. Then there exist $x_1, y_1 \in R$ such that $x = x_1 a$ and $y = y_1 a$. Then $x + y = x_1 a + y_1 a = (x_1 + y_1)a \in Ra$. Also $0 + x_1 a = x_1 a + 0 = x_1 a$ and $x_1 a + (-x_1)a = (-x_1)a + x_1 a = 0$. Hence $Ra$ is a group under addition operation.

Now we will prove that $Ra$ is an $R$-module. Let $\alpha, \beta \in R$ and $x, y \in Ra$. By definition, $x = x_1 a$ and $y = y_1 a$ for some $x_1, y_1 \in R$. Then $\alpha(x_1 a + y_1 a) = \alpha x_1 a + \alpha y_1 a \in Ra$, $(\alpha + \beta)x_1 a = \alpha x_1 a + \beta x_1 a$ and $\alpha(\beta x) = \alpha(\beta x_1 a) = (\alpha\beta)x_1 a = (\alpha\beta)x$. □

**Remark 3.** It is obvious that, by Lemma 1, for any simple ring $R$ and any $a \in R$, $a \neq 0$, we have $Ra = R$, i. e. $R$ is a field. Moreover, for any simple $R$-module $E$ and any $a \in E$, $a \neq 0$, we have $Ra = E$.

**Proposition 8.** *Let $E$ be a locally simple $R$-closed $R$-module, $a \in E$ and $\varphi_E(a) = 1$. Then $E_a = \{ta : t \in R\}$ is an $R$-submodule of $E$ with the following properties:*

1. *$v_a = \varphi_E | E_a : E_a \longrightarrow R$ is a topological isomorphism of the $R$-module $E_a$ onto $R$-module $R$.*
2. *The mapping $\psi_a : E \longrightarrow E_a$, where $\psi_a(x) = v_a^{-1}(\varphi_E(x))$ for each $x \in E$, is an open continuous homomorphism of the $R$-module $E$ onto the $R$-module $E_a$.*
3. *The space $E$ is homeomorphic to the space $\varphi_E^{-1}(0) \times E_a$.*
4. *The set $E_a$ is closed in $E$.*

*Proof.* The set $\varphi_E^{-1}(1)$ is non-empty. Fix $a \in \varphi_E^{-1}(1)$. If $t \in R$, then $\varphi_E(ta) = t\varphi_E(a) = t \cdot 1 = t$. Thus $\varphi_E(E_a) = R$ and $v_a$ is a one-to-one continuous homomorphism of $E_a$ onto $R$. Since $(t, x) \mapsto tx$ is a continuous mapping of $R \times E$ onto $E$, the mapping $v_a$ is a homeomorphism, i. e. $v_a^{-1} : R \longrightarrow E_a$ is continuous. Assertion 1 is proved. Assertion 2 follows from the Assertion 1.

The mapping $\psi : \varphi_E^{-1}(0) \times E \longrightarrow E$, where $\psi(x, y) = x + y$ for any $x \in \varphi_E^{-1}(0)$ and $y \in E_a$, is a homeomorphism. $\square$

Let $X$ be a space, $R$ be a topological ring and $E$ be a topological $R$-module. We will consider two subsets of $C_p(C_p(X, E), E)$:

(i) $L_p(X, E) = \{\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n : \alpha_i \in R, x_i \in e_X(X), n \in \mathbb{N}\}$.

(ii) $M_p(X, E)$ the subspace of all linear mappings from $C_p(X, E)$ into $E$.

(iii) If $F$ is a topological $R$-module, then $\mathfrak{L}_p(F, E)$ is the space of all linear continuous mappings $\varphi : F \to E$ as a subspace of the space $C_p(F, E)$.

**Proposition 9.** *Let $R$ be a locally simple ring and $X$ be an $R$-Tychonoff space. Then $M_p(X, R) = L_p(X, R)$.*

*Proof.* We will show that every continuous linear mapping $\mu \in M_p(X, R)$ can be written as $\mu = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n$ for some concrete $n \in \mathbb{N}$, $\alpha_i \in R$, $x_i \in e_X(X)$ and $1 \leq i \leq n$.

Fix $\mu \in M_p(X, R)$. Assume that $\mu \neq 0$. Let $f \in C(X, R)$, $f = 0$. Then $\mu(f) = 0$, since $\mu$ is linear. Fix a neighbourhood $U$ of $0 \in R$ which does not contain $R$-submodules. Since $\mu$ is continuous, we can find $n \in \mathbb{N}$, distinct points $x_1, x_2, ..., x_n \in X$, and $V = V_1 = V_2 = ... = V_n \subseteq U$ such that $\mu(W(x_1, x_2, ..., x_n, V_1, V_2, ..., V_n)) \subseteq U$.

Let $g \in C_p(X, R)$ and $g(x_1) = g(x_2) = ... = g(x_n) = 0$. Clearly, $\alpha g \in W(x_1, x_2, ..., x_n, V_1, V_2, ..., V_n)$ for any $\alpha \in R$. Thus $\mu(\alpha g) \in U$ and, since $\mu$ is a linear functional, we have $\alpha\mu(g) \in U$ for every $\alpha \in R$. From Lemma 1 it follows that $R\mu(g)$ is an $R$-submodule of $R$ and $R\mu(g) \subseteq U$, a contradiction. Therefore $R\mu(g) = \{0\}$ and $\mu(g) = 0$.

Fix $g_i \in C(X, R)$ such that $g_i(x_i) = 1$ and $g_i(x_j) = 0$ for $j \neq i$.

We put $\alpha_i = \mu(g_i)$. Consider $\mu' = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n$. Then $\mu'(g) = \alpha_1 g(x_1) + \alpha_2 g(x_2) + ... + \alpha_n g(x_n)$ for any $g \in C(X, R)$.

Let $g \in C_p(X, R)$ and $g' = g - g(x_1)g_1 - g(x_2)g_2 - ... - g(x_n)g_n$. Obviously, $g' \in C_p(X, R)$ and $g'(x_i) = 0$ for each $i \leq n$. Hence $\mu(g') = 0$. Since $\mu$ is a linear mapping, $0 = \mu(g') = \mu(g) - \mu(\Sigma\{g(x_i)g_i : i \leq n\})$ and $\mu(g) = \mu(\Sigma\{g(x_i)g_i : i \leq n\})$ $= \Sigma\{g(x_i)\mu(g_i) : i \leq n\} = \Sigma\{\alpha_i g(x_i) : i \leq n\}$. Hence $\mu = \mu'$ $\square$

**Remark 4.** Let $E$ be a topological $R$-module and $Hom(E)$ be the set of all continuous homomorphisms $\varphi : E \longrightarrow E$. If $\mu \in M_p(X, E)$ and $\varphi \in Hom(E)$, then $\varphi \circ \mu \in M_p(X, E)$. As follows from the next example, there exist a topological ring $R$, a topological $R$-module $E$, a space $X$, $\mu \in L_p(X, E)$ and $\varphi \in Hom(E)$ such that $\varphi \circ \mu \in M_p(X, E) \setminus L_p(X, E)$.

**Example 6.** Let $R$ be a topological ring with the identity 1, $A$ be a non-empty set and $E = R^A$. Then $E$ is an $R$-closed $R$-module and a topological ring. For any subset $B$ of $A$ consider the point $1_B = (t_\alpha(B) : \alpha \in A) \in E$, where $t_\alpha(B) = 1$ for $\alpha \in B$, and $t_\alpha(B) = 0$ for $\alpha \in A \setminus B$. The point $1_B$ generate the homomorphism $\varphi_B \in Hom(E)$, where $\varphi_B(x) = x \cdot 1_B$ for any $x \in E$. If $E_B = \{x \cdot 1_B : x \in E\} = \varphi_B(E)$, then $E_B$ is a a subring of $E$ and $R$-submodule of $E$. The homomorphism $\varphi_B$ is a retraction of $E$ onto $E_B$.

Let $X$ be a non-empty space. We consider two cases.

**Case 1.** $|X| + |R| + \aleph_0 \leq |A|$.

Fix $a \in X$. We put $\psi_B(g) = \varphi_B(g(a))$ for any $g \in C_p(X, E)$. We identify $a = e_X(a)$. Hence $\psi_B = \varphi_B \circ a$. We have $\psi_B(C(X, E)) = E_B$. Therefore $|M_p(X, E)| \geq |\{\psi_B : B \in A\}| = 2^{|A|} > |A|$, where $|Y|$ is the cardinality of the set $Y$. Obviously, $|L_p(X, E)| \leq |X| + |R| + \aleph_0$. Hence, since $|X| + |R| + \aleph_0 \leq |A|$, we have $|L_p(X, E)| < |M_p(X, E)|$ and $M_p(X, E) \setminus L_p(X, E) \neq \emptyset$.

**Case 2.** *$R$ is a simple ring and* $2 \leq |A|$.

In this case $R$ is a field and $E$ is a locally simple $R$-module provided the set $A$ is finite. Fix a non-empty proper subset subset $B$ of $A$ and $\psi \in \mathfrak{L}_p(C_p(X, E), E_B)$, where $\psi \neq 0$. We affirm that $\psi \in M_p(X, E) \setminus L_p(X, E)$. Suppose that we can find $n \in \mathbb{N}$, distinct points $x_1, x_2, ..., x_n \in X$ and $\alpha_1, \alpha_2, .., \alpha_n \in R \setminus \{0\}$ such that $\psi(g) = \alpha_1 g(x_1) + \alpha_2 g(x_2) + ... + \alpha_n g(x_n)$ for any $g \in C(X, R)$. Let $C = A \setminus B$. Fix a function $h \in C(X, E)$ for which $h(x_1) = 1_C$ and $h(x_i) = 0$ for each $i \geq n$. Then $\psi(h) = \alpha_1 h(x_1) \in E_C \setminus E_B$, a contradiction with the condition $\psi(B) \subseteq E_B$. In particular, we have $\varphi_B \in M_p(X, E) \setminus L_p(X, E)$.

**Proposition 10.** *Let $R$ be a ring, $E$ be a topological $R$-module and $X$ be a space. Then for any $g \in C(X, E)$ there exists a unique linear mapping $\overline{g} \in \mathfrak{L}_p((L_p(X, E), E)$ such that $g = \overline{g} \circ e_X$, where $e_X : X \to L_p(X, E)$ is the evaluation mapping.*

*Proof.* Let $E_f = E$ for any $f \in C_p(X, E)$. By definition, $e_X(X) \subseteq L_p(X, E) \subseteq E^{C(X,E)} = \Pi\{E_f : f \in C(X, E)\}$. We consider the projection $\pi_f : E^{C(X,E)} \longrightarrow E_f = E$. Let $\overline{f} = \pi_f | L_p(X, E) : L_p(X, E) \longrightarrow E$. Then $\overline{f}$ and $\pi_f$ are continuous linear mappings. If $x \in X$, then $\overline{f}(e_X(x))) = f(x)$. Hence $f = \overline{f}|_X$ and for any $f \in C(X, E)$ there exists a linear mapping $\overline{f} \in \mathfrak{L}_p(L_p(X, E), E)$ such that $f = \overline{f} \circ e_X$. Since the subspace $e_X(X)$ generates the linear space $L_p(X, E)$, the linear mapping $\overline{f}$ is unique. $\square$

**Theorem 1.** *Let $R$ be a ring, $E$ be a topological $R$-module and $X$ be a space. Consider the space $e_X(X)$, where $e_X : X \to L_p(X, E)$ is the evaluation mapping. Then the linear spaces $C_p(X, E)$, $C_p(e_X(X), E)$ and $\mathfrak{L}_p(L_p(X, E), E)$ are linearly homeomorphic.*

*Proof.* Let $E_f = E$ for any $f \in C_p(X, E)$. By definition, $e_X(X) \subseteq L_p(X, E) \subseteq E^{C_p(X,E)} = \Pi\{E_f : f \in C(X, E)\}$. We consider the projection $\pi_f : E^{C(X,E)} \longrightarrow E_f = E$. Let $\overline{f} = \pi_f | L_p(X, E) : L_p(X, E) \longrightarrow E$. Then $\overline{f}$ and $\pi_f$ are continuous linear mappings.

If $g : e_X(X) \to E$ is a continuous mapping, then $g \circ e_X = f$ for a unique $f \in C(X, E)$. Therefore, $g = \pi_f|e_X(X)$ and the correspondence $f \to \pi_f|e_X(X)$ is a linear homeomorphism of $C_p(X, E)$ onto $C_p(e_X(X), E)$.

Hence, without loss of generality, we can assume that $X = e_X(X) \subseteq L_p(X, E)$.

By virtue of Proposition 10, the correspondence $\psi : C_p(X, E) \to \mathfrak{L}_p(L_p(X, E), E)$, where $\psi(f) = \overline{f}$, is a one-to-one linear mapping of $C(X, E)$ onto $\mathfrak{L}_p(L_p(X, E), E)$.

For each $y \in L_p(X, E)$ there exist the minimal $n = n(y) \in \mathbb{N}$, the unique points $x_1(y), ..., x_n(y) \in X$ and the unique points $\alpha_1(y), ..., \alpha_n(y) \in R$ such that $y = \alpha_1(y)x_1(y) + ... + \alpha_n(y)x_n(y)$. Hence, the correspondence $\psi$ is continuous and linear.

Since $\psi(f)|X = f$, the mapping $\psi^{-1}$ is continuous. □

**Remark 5.** We say that $e_X(X)$ is the $E$-replica of the space $X$. If $X$ is $R$-completely regular, then $X = e_X(X)$.

**Corollary 1.** *Let $X$, $Y$ be spaces and $R$ be a locally simple $R$-module. The spaces $C_p(X, R)$ and $C_p(Y, R)$ are linearly homeomorphic if and only if the spaces $L_p(X, R)$ and $L_p(Y, R)$ are linearly homeomorphic.*

For $n \geq 1$, an $R$-module $E$ and a space $X$ we put $L_{p,n}(X, E) = \{\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n : x_i \in e_X(X), \alpha_i \in R, i \leq n\}$. Obviously, $L_{p,n}(X, E) \subseteq L_{p,n+1}(X, E)$ for each $n$ and $L_p(X, E) = \bigcup\{L_{p,n}(X, E) : n \in \mathbb{N}\}$.

**Proposition 11.** *The mapping $p_n : R^n \times X^n \longrightarrow L_{p,n}(X, E)$, where $p_n(\alpha_1, \alpha_2, ..., \alpha_n, x_1, x_2, ..., x_n) = \alpha_1 e_X(x_1) + \alpha_2 e_X(x_2) + ... + \alpha_n e_X(x_n)$, is a continuous mapping of $R^n \times X^n$ onto $L_{p,n}(X, E)$.*

*Proof.* Follows from Proposition 1. □

Let $E$ be an $R$-module. We say that the pair $(F(X, E), i_X)$ is an $E$-free $R$-module of a space $X$ if it has the following properties:

1. $F(X, E)$ is a submodule of the topological $R$-module $E^\tau$ for some cardinal number $\tau$;

2. $i_X : X \to F(X, E)$ is a continuous mapping and the set $i_X(X)$ algebraically generates the $R$-module $F(X, E)$;

3. For any continuous mapping $f : X \to E$ there exists a continuous homomorphism $\overline{f} : F(X, E) \longrightarrow E$ such that $f = \overline{f} \circ i_X$.

From the property 2 it follows that the homomorphism $\overline{f}$ is unique and is called the homomorphism generated by the mapping $f$.

**Proposition 12.** *For any space $X$ there exists a unique $E$-free $R$-module. The pair $(L_p(X, E), e_X)$ is the $E$-free $R$-module of the space $X$.*

*Proof.* The uniqueness of the $E$-free $R$-module of the space $X$ is well known (see [6, 7]). From the method of construction of the object $(L_p(X, E), e_X)$ and from the definition of the $E$-free $R$-module it follows that $(L_p(X, E), e_X)$ is the $E$-free $R$-module of the space $X$. □

**Proposition 13.** *Let $X$ be an $R$-Tychonoff space and $E$ be an $R$-module. Then $L_{p,n}(X,E)$ is a closed subset of $L_p(X,E)$ for any $n \in \mathbb{N}$.*

*Proof.* We follow very closely the proof of Proposition 0.5.16 in [1].

Since $e_X$ is an embedding of $X$ into $L_p(X,E)$, we can assume that $X = e_X(X) \subseteq L_p(X,E)$. In this case a point $x \in X$ as an element of $E^{C(X,E)}$ has the form $x = (f(x) : f \in C(X,E))$.

Fix $y \in L_p(X,E) \setminus L_{p,n}(X,E)$. Then $y = \alpha_1 y_1 + \alpha_2 y_2 + ... + \alpha_m y_m$, where $m > n$, $y_i \in X$, $\alpha_i \in R$ and $\alpha_i y_i \neq 0$ for any $i \leq m$, $y_i \neq y_j$ provided $i \neq j$. For any $i \leq m$ there exists $h_i \in C(X,E)$ such that $\alpha_i h_i(y_i) \neq 0$. We put $b_i = h_i(y_i)$.

Fix a family of pairwise disjoint open sets $\{V_i : i \leq m\}$ in $X$ and the continuous functions $\{f_i \in C(X,R) i \leq m\}$ such that $y_i \in V_i$, $f_i(y_i) = 1$ and $f_i(X \setminus V_i) = 0$ for any $i \leq m$. Let $g_i(x) = f_i(x)b_i$. By virtue of Proposition 10, each $g_i$ extends to a continuous and linear mapping $\overline{g_i} : L_p(X,E) \longrightarrow E$. The subset $U = \bigcap\{\overline{g_i}^{-1}(E \setminus \{0\})\}$ of $L_p(X,E)$ is open.

We have $\overline{g_i}(\alpha_i y_i) = \alpha_i g(y_i) = \alpha_i b_i \neq 0$. If $j \neq i$, then $\overline{g_i}(\alpha_j y_j) = \alpha_j g(y_j) = \alpha_i 0 = 0$. Hence $\overline{g_i}(y) = \alpha_i b_i \neq 0$ for any $i \leq m$. Therefore $y \in U$.

We will show that $U \cap L_{p,n}(X,E) = \emptyset$. Fix $z \in U$, i.e. for some $k \in \mathbb{N}$, $z_1, z_2, ..., z_k \in X$ and $\beta_1, \beta_2, ..., \beta_k \in R$ we have $z = \beta_1 z_1 + \beta_2 z_2 + ... + \beta_k z_k$, $\beta_i z_i \neq 0$ for any $i \leq k$ and $z_i \neq z_j$ provided $i \neq j$. Then $\overline{g_i}(z) = \beta_1 g_i(z_1) + \beta_2 g_i(xz_2) + ... + \beta_k g_i(xz_k) \neq 0$ for each $i \leq k$. Hence $V_i \cap \{z_1, z_2, ..., z_k\} = V_i' \neq \emptyset$ for each $i \leq k$. As the sets $\{V_i : i \leq m\}$ are pairwise disjoint, it follows that the sets $\{V_i' : i \leq k\}$ are non-empty and pairwise disjoint too. Hence $k \geq m > n$, i.e. $U \cap L_{p,n} = \emptyset$. The proof is complete. □

Let $L^c_{p,n}(X,E) = L_{p,n}(X,E) \setminus L_{p,n-1}(X,E)$ and $H_{p,n}(X,E) = p_n^{-1}(L^c_{p,n}(X,E))$ for any $n \in \mathbb{N}$.

**Proposition 14.** *Let $X$ be an $R$-Tychonoff space, $R$ be a simple ring and $E$ be a topological $R$-module. The following assertions are true:*

*1. The mapping $q_n = p_n|_{H_{p,n}(X,E)} : H_{p,n}(X,E) \longrightarrow L^c_{p,n}(X,E)$ is one-to-one.*

*2. If $R$ is a topological field and the module $E$ is $R$-closed, then the mapping $q_n = p_n|_{H_{p,n}(X,E)} : H_{p,n}(X,E) \longrightarrow L^c_{p,n}(X,E)$ is a homeomorphism.*

*Proof.* By virtue of Propositions 11 and 1, the mapping $q_n$ is continuous. Since $e_X$ is an embedding of $X$ into $L_p(X,E)$, we can assume that $X = e_X(X) \subseteq L_p(X,E)$.

Since $R$ is a simple ring, $R$ is a field and for any $\lambda \in R \setminus \{0\}$ there exists the inverse element $\lambda^{-1}$. The ring $R$ is a topological field provided the mapping ${}^{-1} : R \setminus \{0\} \to R$ is continuous.

We have $\alpha x \neq 0$ for all $\alpha \in R \setminus \{0\}$ and $x \in E \setminus \{0\}$.

**Claim 1.** *Let $n \in \mathbb{N}$, $x_1, x_2, ..., x_n \in X$, $\alpha_1, \alpha_2, ..., \alpha_n \in R$ and $x_i \neq x_j$ provided $i \neq j$. If $\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n = 0$, then $\alpha_i = 0$ for each $i \leq n$.*

Assume that $\alpha_i \neq 0$ for each $i \leq n$. A point $x \in X$ as an element of $E^{C(X,E)}$ has the form $x = (f(x) : f \in C(X,E))$. Hence, for any $i \leq m$ there exists $h_i \in C(X,E)$ such that $\alpha_i h_i(x_i) \neq 0$. We put $b_i = h_i(x_i)$.

Fix a family of pairwise disjoint open sets $\{V_i : i \leq m\}$ in $X$ and the continuous functions $\{f_i \in C(X,R) i \leq m\}$ such that $x_i \in V_i$, $f_i(x_i) = 1$ and $f_i(X \setminus V_i) = 0$ for any $i \leq m$. Let $g_i(x) = f_i(x)b_i$. Then $0 = 0(g_i) = (\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n)(g_i) = \alpha_1 x_1(g_i) + \alpha_2 x_2(g_i) + ... + \alpha_n x_n(g_i) = \alpha_i b_i \neq 0$, a contradiction.

**Claim 2.** *Let $n, m \in \mathbb{N}$, $x_1, x_2, ..., x_n, y_1, y_2, ..., y_m \in X$, $\alpha_1, \alpha_2, ..., \alpha_n, \beta_1, \beta_2, ..., \beta_m \in R$, $x_i \neq x_j$ provided $i \neq j$, and $y_l \neq y_k$ provided $l \neq k$. If $\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n = \beta_1 y_1 + \beta_2 y_2 + ... + \beta_m y_m$, then $m = n$, $\{x_1, x_2, ..., x_n\} = \{y_1, y_2, ..., y_m\}$ and $\alpha_i = \beta_j$ provided $x_i = y_j$.*

Obviously, Claim 2 follows from Claim 1. From Claim 2 it follows that the mapping $q_n$ is one-to-one.

Assume now that $R$ is a topological field and $E$ is an $R$-closed topological $R$-module. Fix the continuous homomorphism $\varphi_E : E \longrightarrow R$ of the topological $R$-module $E$ onto the topological $R$-module $R$. Fix $a \in \varphi_E^{-1}(1)$. If $E_a = Ra$, then the mapping $\varphi_E | E_a$ is a homeomorphism of $E_a$ onto $R$. Hence the mapping $\varphi_E$ is open and continuous as quotient homomorphism of the topological $R$-module $E$ onto the topological $R$-module $R$.

We can fix the non-empty open subsets $V_1, V_2, ..., V_n$ of $R$ and the non-empty open subsets $W_1, W_2, ..., W_n$ of $X$ such that:

– $U = V_1 \times V_2 \times ... \times V_n \times W_1 \times W_2 \times ... \times W_n \subseteq H_{(p,n)}(X, E)$;

– $W_i \cap W_j = \emptyset$ provided $i \neq j$;

– $0 \notin V_i$ for each $i \leq n$.

We affirm that the set $q_n(U)$ is open in $L^c_{p,n}(X, E)$.

Fix a point $y \in q_n(U)$. By definition, $y = \alpha_1 y_1 + \alpha_2 y_2 + ... + \alpha_n y_n$ and $q_n^{-1}(y) y = (\alpha_1, \alpha_2, ..., \alpha_n, y_1, y_2, ..., y_n)$, where $\alpha_i \in V_i \subseteq R \setminus \{0\}$, $y_i \in W_i \subseteq X$ and $y_j \neq y_i$ for each $i \leq n$ and $j \neq i$. Now we fix $f_i \in C(X, E)$, $i \leq n$, such that $f_i(y_i) = \alpha_i^{-1} a$ and $f_i(X \setminus V_i) = 0$. Then $y(f_i) = \alpha_1 y_1(f_i) + \alpha_2 y_2(f_i) + ... + \alpha_n y_n(f_i) = \alpha_1 f_i(y_1) + \alpha_2 f_i(y_2) + ... + \alpha_n f_i(y_n) = \alpha_i f_i(y_i) = a$.

Since $\alpha_i \in V_i$, there exist two open subsets $D(1,i)$ and $D(2,i)$ such that $1 \in D(1,i)$, $\alpha_i^{-1} \in D(2,i)$, $0 \notin D(1,i) \cup D(2,i)$ and $D(1,i) D(2,i)^{-1} \subseteq V_i$. By construction, $p_E(f_i(y_i)) \in D(2,i)$. Hence, we can fix $g_i \in C(X, E)$ for which $g_i(y_i) = a$ and $g_i(X \setminus W_i \cap f_i^{-1}(p_E^{-1}(D(2,i))) = 0$. For each $i \leq n$ there exists the unique continuous linear mappings $\overline{f_i}$, $\overline{g_i} \in \mathfrak{L}_p((L_p(X,E), E)$ such that $f_i = \overline{f_i}|X$ and $g_i = \overline{g_i}|X$.

We put $V = \cap \{\overline{f_i}^{-1}(p_E^{-1}(D(1,i))) \cap \overline{g_i}^{-1}(E \setminus \{0\}) : i \leq n\}$. By construction, $\overline{f_i}(y) = a$ and $\overline{g_i}(y) \neq 0$. Hence, $y \in V$. As in the proof of Proposition 12 we establish that $V \cap L_{p,n-1}(X, E) = \emptyset$. Hence, $V \cap L_{p,n}(X, E) \subseteq L^c_{p,n}(X, E)$.

Fix some $z \in V \cap L_{p,n}(X, E)$. Then $z = \beta_1 z_1 + \beta_2 z_2 + ... + \beta_n z_n$, where $\alpha_i \in R \setminus \{0\}$ and $z_j \neq z_i$ for each $i \leq n$ and $j \neq i$. Since $\overline{f_i}(z) \neq 0$ for all $i \leq n$, we can assume that $\overline{f_i}(z_i) \neq 0$ and $z_i \in V_i$.

By construction, $\overline{g_i}(z) = \beta_i g_i(z_i)$ and $\overline{g_i}(z) \neq 0$. Since $p_E(\overline{f_i}(z_i)) \in D(2,i)$, $p_E(\overline{f_i}(z)) \in D(1,i)$ and $\overline{f}_i(z) = \beta_i f(z_i)$, we have $p_E(f_i(z_i)) \in D(2,i)$ and $\beta_i p_E(f_i(z_i)) \in D(1,i)$. Therefore, $\beta_i p_E(f_i(z_i)) \cdot p_E(f_i(z_i))^{-1} \in D(1,i) \cdot D(2,i)^{-1} \subseteq V_i$ and $\beta_i \in V_i$. Hence, $z \in q_n(U)$ and $V \subseteq q_n(U)$, i.e. $q_n(U)$ is an open subset of $L^c_{p,n}(X, E)$ and the mapping $p_n^{-1}$ is continuous. □

**Example 7.** Let $E = \mathbb{R}$ be the field of reals with the topology generated by the Euclidian distance. Denote by $R$ the topological space $\mathbb{R} \times \mathbb{R}$ with the following operations:

– the additive operation $(\alpha, \beta) + (\delta, \mu) = (\alpha + \delta, \beta + \mu)$;

– the inverse operation $-(\alpha, \beta) = (-\alpha, -\beta)$;

– the multiplicative operation $(\alpha, \beta) \cdot (\delta, \mu) = (\alpha \cdot \delta, \alpha\mu + \beta\delta)$.

Then $R$ is a topological commutative ring with the unity $(1, 0)$. The ring $R$ is a locally simple $R$-module. The multiplicative operation $\cdot : R \times E \longrightarrow E$ is defined as follows $(\alpha, \beta) \cdot t = \alpha t$. In this case $E$ is a simple $\mathbb{R}$-module and a simple $R$-module. Obviously, we have the same subspaces $L_{p,n}(X, E)$ when $E$ is considered an $\mathbb{R}$-module or an $R$-module.

Since $\mathbb{R}$ is a topological field and $E$ is an $\mathbb{R}$-closed topological $\mathbb{R}$-module, the mapping $q_n = p_n|_{H_{p,n}(X,E)} : H_{p,n}(X, E) \longrightarrow L^c_{p,n}(X, E)$ is a homeomorphism if $E$ is considered as an $\mathbb{R}$-module. Hence $indL^c_{p,n}(X, E) = n$ provided $indX = 0$.

Now we consider $E$ as an $R$-module. In this case $indH_{p,n}(X, E) \geq indR^n = 2n$. Hence the mapping $q_n$ is not one-to-one if $E$ is considered as an $R$-module. Moreover, the fibers $q_n^{-1}(y)$ have the dimension equal to $n$ and are homeomorphic to the space $\mathbb{R}^n$. Hence the assumption that $R$ is a simple ring in the conditions of Proposition 14 is essential.

**Remark 6.** For any topological ring ring $R$ and a space $X$ the mapping $q_n = p_n|_{H_{p,n}(X,R)} : H_{p,n}(X, R) \longrightarrow L^c_{p,n}(X, R)$ is one-to-one.

**Lemma 2.** *Let $X$ be an $R$-Tychonoff space, $Z$ be a closed subspace of $X$, $E$ be a topological $R$-module and $g : X \longrightarrow E$ be a continuous mapping. For any finite subset $F$ of $X \setminus Z$ and any function $f : F \longrightarrow E$ there exists a continuous function $\varphi : X \longrightarrow E$ such that $f = \varphi|_F$ and $\varphi|_Z = g|_Z$.*

*Proof.* Fix a family $\{U_x : x \in F\}$ of open subsets of $X$ such that $x \in U_x \subseteq X \setminus Z$ for each $x \in F$ and $U_x \cap U_y = \emptyset$ for each distinct points $x, y \in F$. For each $x \in F$ fix a continuous function $f_x : X \longrightarrow R$ such that $f_x(x) = 1$ and $f_x(X \setminus U_x) = 0$. We put $\varphi_x(y) = f_x(y) \cdot f(x)$ for each $x \in F$ and $y \in X$. Let $\varphi_F(y) = \sum\{\varphi_x(y) : x \in F\}$. By construction, the function $\varphi_F$ is continuous, $\varphi_F|_F = f$ and $\varphi_F(Z) = 0$. Let $g_x(y) = 1 - f_x(y)$ for any $x \in F$ and $y \in X$. We put $g_F(y) = \prod\{g_x(y) : x \in F\}$ for each $y \in X$. The function $g_F$ is continuous, $g_F(F) = 0$ and $g_F(Z) = 1$. Let $\varphi_Z(y) = g_F(y) \cdot g(y)$ for each $y \in Y$. By construction, the function $\varphi_Z$ is continuous, $\varphi_Z(F) = 0$ and $\varphi_Z|_Z = g|_Z$. Obviously, $\varphi = \varphi_F + \varphi_Z$ is the desired function. $\square$

For any subspace $Y$ of a space $X$ we put $C_p(Y|X, E) = \{f|_Y : f \in C(X, E)\}$.

A subspace $Y$ of $X$ is *$E$-full* if $C(Y|X, E) = C(Y, E)$.

A space $X$ is called *compactly $E$-full* if $C(Y|X, E) = C(Y, E)$ for any compact subspace $Y$ of $X$.

**Lemma 3.** *Let $X$ be a zero-dimensional space and $E$ be a metrizable space. Then $X$ is a compactly $E$-full space. Moreover, for any compact subset $Y$ of $X$ and any $f \in C(Y, E)$ there exists $g \in C(X, E)$ such that $g(X) \subseteq f(Y)$ and $f = g|_Y$, i.e. $X$ is compactly $E$-full.*

*Proof.* Fix a metric $d$ on $E$. Let $Y$ be a non-empty compact subspace of the space $X$. Fix $f \in C(Y, E)$. For each point $y \in Y$ and each $n \in \mathbb{N}$ we fix a clopen subset $U_n y$ of $X$ such that $y \in U_n y$ and $d(f(y), f(z)) < 2^{-n-1}$ for each $z \in U_n y \cap Y$.

There exists a sequence $\{Y_n : n \in \mathbb{N}\}$ of finite subsets of $Y$ and a sequence $\{\gamma_n = \{V_n y : y \in Y_n\} : n \in \mathbb{N}\}$ of families of clopen subsets of the space $X$ such that:

– $Y_n \subseteq Y_{n+1}$ for each $n \in \mathbb{N}$;

– $y \in V_{n+1} y \subseteq V_n y \subseteq U_n y$ for any $n \in \mathbb{N}$ and any $y \in Y_n$;

– $Y \subseteq \cup\{V_n y : y \in Y_n\} = \cup \gamma_n$ for each $n \in \mathbb{N}$;

– for each $n \in \mathbb{N}$ and each $y \in Y_{n+1}$ there exists a unique $z(y) \in Y_n$ such that $V_{n+1} y \subseteq V_n z(y)$;

– if $y_1, y_2 \in Y_n$, $y_1 \neq y_2$ and $n \in \mathbb{N}$, then $V_n y_1 \cap V_n y_2 = \emptyset$.

We put $V_n = \cup\{V_n y : y \in Y_n\}$. Fix $a \in f(Y)$. We will construct a sequence $\{g_n : X \longrightarrow E : n \in \mathbb{N}\}$ of continuous mappings with the next properties:

– $d(g_n(y), f(y)) < 2^{-n}$ for each $n \in \mathbb{N}$ and any $y \in Y$;

– $d(g_n(x), g_{n+k}(x)) < 2^{-n}$ for each $n, k \in \mathbb{N}$ and any $x \in X$;

– $g_n(X) \subseteq f(Y)$ for each $n \in \mathbb{N}$.

We put $f_1(x) = a$ for each $x \in X \setminus V_1$ and $f_1(V_1 y) = f(y)$ for each $y \in Y_1$.

Assume that $n \geq 1$ and the function $g_n$ is constructed. We put $g_{n+1}|_{(X \setminus V_{n+1})} = g_n|_{(X \setminus V_{n+1})}$ and $g_{n+1}(V_{n+1} y) = f(y)$ for each $y \in Y_{n+1}$. The sequence $\{g_n : n \in \mathbb{N}\}$ is constructed. Since $\{g_n : n \in \mathbb{N}\}$ is a fundamental sequence and $(f(Y), d)$ is a compact metric space, there exists the continuous limit $g = \lim g_n$. By construction, we have $f = g|_Y$. The proof is complete. □

## 3 About theorem of Nagata

Let $R$ be a simple topological ring. We consider only $R$-Tychonoff spaces. Fix $n \in \mathbb{N}$. A functional $\mu : C(X, R) \longrightarrow R^n$ is called *multiplicative* if it is linear and $\mu(fg) = \mu(f)\mu(g)$ for any $f, g \in C(X, R^n)$.

Denote by $I_{(p,n)}(X, R) = \{\mu \in L_p(X, R, R^n) : \mu \neq 0, \mu \text{ is multiplicative}\}$.

**Theorem 2.** *The spaces $X^n$ and $I_{(p,n)}(X, R)$ are homeomorphic.*

*Proof.* Let $1 = (1, 1, ..., 1)$ be the unity of the ring $R^n$. For each $i \leq n$ we put $R_i = \{(x_1, x_2, ..., x_n) \in R^n : x_j = 0 \text{ for any } j \neq i\}$. Then $R_i$ is a subring of $R^n$ with the unity $1_i = (0, 0, ..., 1, 0, ..., 0) \in R_i$. The ring $R$ and $R_i$ are topologically isomorphic. The mapping $p_i : R^n \longrightarrow R_i$, where $p_i(x_1, x_2, ..., x_n) = 1_i \cdot (x_1, x_2, ..., x_n)$ for each $(x_1, x_2, ..., x_n) \in R^n$ is open, continuous, linear and multiplicative. If $\mu \in I_{(p,n)}(X, R)$, then we put $\pi_i(\mu) = \mu_i \circ p_i$. Then $\pi_i(\mu) \in I_{(p,1)}(X, R)$ and $\mu(f) = (\pi_1(\mu)(f), \pi_2(\mu)(f), ..., \pi_n(\mu)(f))$ for all $f \in C(X, R)$. Hence $I_{(p,n)}(X, R) = I_{(p,1)}(X, R)^n$.

Now is sufficient to prove that $I_p(X, R) = I_{(p,1)}(X, R)$ and $X$ are homeomorphic.

Obviously, $\xi_x \in I_p(X, R)$ for any $x \in X$. Assume that $\mu \in I_p(X, R)$. Then there exists $n \geq 1$, $x_1, x_2, ..., x_n \in X$ and $\alpha_1, \alpha_2, ..., \alpha_n \in R \setminus \{0\}$ such that $\mu = \alpha_1 \xi_{x_1} + \alpha_2 \xi_{x_2} + ... + \alpha_n \xi_{x_n}$. Since $R$ is a simple ring, we have $\alpha_i \beta_i = 1$ for some

$\beta_i \in R$. For each $i \leq n$ fix $f_i \in C(X, R)$ such that $f_i(x_i) = \beta_i$ and $f_i(x_j) = 0$ for each $j \neq i$. By construction, $\mu(f_i) = \alpha_i f(x_i) = \alpha_i \beta_i = 1$ for each $i \leq n$. Assume that $n \geq 2$. Then $f_i \cdot f_2 = 0$ and $0 = \mu(0) = \mu(f_1 \cdot f_2) = \mu(f_1)\mu(f_2) = 1 \cdot 1 = 1$, a contradiction. Hence $n = 1$ and $\mu = \alpha_1 \xi_{x_1}$.

Assume that $\alpha_1 \neq 1$. Then $\beta_1 \neq 1$ and $\beta_1 = 1 \cdot \beta_1 = \alpha_1 \beta_1 \beta_1 = \alpha_1 (f_1 \cdot f_1)(x_1) = \mu(f_1 \cdot f_1) = \mu(f_1) \cdot \mu(f_1) = 1 \cdot 1 = 1$, a contradiction. Hence $\beta_1 = \alpha_1 = 1$ and $\mu = \xi_{x_1}$. The proof is complete. □

**Corollary 2.** *If the rings $C_p(X, R)$ and $C_p(Y, R)$ are topologically isomorphic, then the spaces $X$ and $Y$ are homeomorphic.*

The Corollary 2 for the ring $\mathbb{R}$ of reals was proved by Nagata (see [1], Theorem 0.6.1).

## 4 Algebraical classes of spaces

Fix a topological ring $R$. Assume that $R$ is an $R$-Tychonoff space. A class $\mathcal{P}$ of topological spaces is called *an algebraical $R$-class* of spaces if:

(i) any space $X \in \mathcal{P}$ is $R$-Tychonoff and $Y \in \mathcal{P}$ for any closed subspace $Y$ of $X$;

(ii) if $f : X \longrightarrow Y$ is a continuous mapping of $X$ onto $Y$, $X \in \mathcal{P}$ and $Y$ is an $R$-Tychonoff space, then $Y \in \mathcal{P}$;

(iii) if $\{X_n \in \mathcal{P} : n \in \mathbb{N}\}$ is a sequence of closed subspaces of an $R$-Tychonoff space $X$ and $X = \cup\{X_n : n \in \mathbb{N}\}$, then $X \in \mathcal{P}$;

(iv) if $X, Y \in \mathcal{P}$, then $X \times Y \in \mathcal{P}$;

(v) $R \in \mathcal{P}$.

**Lemma 4.** *Let $\mathcal{P}$ be an algebraical $R$-class of spaces, $\{X_n : n \in \mathbb{N}\}$ be a sequence of subspaces of an $R$-Tychonoff space $X$, $X = \cup\{X_n : n \in \mathbb{N}\}$ and $X_n \in \mathcal{P}$ for any $n \in \mathbb{N}$. Then $X \in \mathcal{P}$.*

*Proof.* Let $Y_n = X_n \times \{n\}$ and $Y$ is the discrete sum of the spaces $\{Y_n : n \in \mathbb{N}\}$. Obviously $Y$ is an $R$-Tychonoff space, $Y_n \in \mathcal{P}$ and $Y_n$ is closed in $Y$ for any $n \in \mathbb{N}$. Hence $Y \in \mathcal{P}$ and $X$ is a continuous image of $Y$. □

**Theorem 3.** *Let $\mathcal{P}$ be an algebraical $R$-class of spaces, $R$ be a topological ring and $E$ be a topological $R$-module. Assume that $E$ is an $R$-Tychonoff space. For an $R$-Tychonoff space $X$ the following assertions are equivalent:*

*(i) $X \in \mathcal{P}$.*

*(ii) $L_p(X, E) \in \mathcal{P}$.*

*Proof.* Assume that $X \in \mathcal{P}$. We consider the mapping $\varphi_n : X^n \times R^n \longrightarrow L_p(X, E)$ where $\varphi_n((x_1, x_2, ..., x_n), (\alpha_1, \alpha_2, ..., \alpha_n)) = \alpha_1 \xi_{x_1} + \alpha_2 \xi_{x_2} + ... + \alpha_n \xi_{x_n}$. The mapping $\varphi_n$ is continuous and we put $L_{p,n}(X, E) = \varphi_n(X^n \times R^n)$.

Since $L_p(X, E) \subseteq E^{C(X,E)}$, the space $L_p(X, E)$ is $R$-Tychonoff. Hence $X^n \times R^n$, $L_{p,n}(X, E) \in \mathcal{P}$ for each $n$. From Proposition 13 we have $L_p(X, E) = \cup\{L_{p,n}(X, E) : n \in \mathbb{N}\}$. Thus $L_p(X, E) \in \mathcal{P}$.

Assume now that $L_p(X,E) \in \mathcal{P}$. Since $X = L_{p,1}(X,E)$, by virtue of Proposition 13, $X$ is a closed subspace of the space $L_p(X,E)$. Then $X \in \mathcal{P}$. $\square$

**Corollary 3.** *Let $\mathcal{P}$ be an algebraical $R$-class of spaces, $R$ be a topological ring and $E$ be a topological $R$-module. Assume that $E$ is an $R$-Tychonoff space. If $C_p(X,E)$ and $C_p(Y,E)$ are topologically homeomorphic and $X \in \mathcal{P}$, then $Y \in \mathcal{P}$.*

**Remark 7.** For the ring $\mathbb{R}$ of reals and $E = \mathbb{R}$ the above assertion is proved in [1], Proposition 0.5.13.

## 5 The support mapping

Fix a topological ring $R$ and a non-trivial locally simple topological $R$-module $E$.

Consider a space $X$ and a functional $\mu \in M_p(X,E)$. We put $\mathcal{S}(\mu) = \{B \subseteq X :$ if $B \subseteq f^{-1}(0)$, then $\mu(f) = 0\})$. Obviously, $X \in \mathcal{S}(\mu)$. Thus the set $\mathcal{S}(\mu)$ is non-empty.

The set $supp_X(\mu)$ is the family of all points $x \in X$ such that for each neighbourhood $U$ of $x$ in $X$ there exists $f \in C_p(X,E)$ such that $f(X \setminus U) = 0$ and $\mu(f) \neq 0$ (see [2,8] for $E = R = \mathbb{R}$, and [3,10] for $R = \mathbb{R}$).

If $f \in C_p(X,E)$ and $U$ is an open neighbourhood of 0 in $E$, then we put $A(f,L,U) = \{g \in C_p(X,E) : f(x) - g(x) \in U$ for any $x \in L\}$. The family $\{A(f,L,U) : f \in C_p(X,E), L$ *is a finite subset of* $X, U$ *is an open neighbourhood of 0 in* $E\}$ is an open base of the space $C_p(X,E)$.

**Theorem 4.** *Let $X$ be an $R$-Tychonoff space, $E$ be a non-trivial locally simple topological $E$-module, $\mu \in M_p(X,E)$ and $\mu \neq 0$. Then:*

1. *There exists a finite set $K \in \mathcal{S}(\mu)$ such that $supp_X(\mu) \subseteq K$.*

2. *$supp_X(\mu) \in \mathcal{S}(\mu)$ and $supp_X(\mu)$ is a finite subset of $X$.*

*Proof.* Fix an open subset $V_0$ of $E$ such that $0 \in V_0$ and $V_0$ does not contain non-trivial $R$-submodules of $E$.

There exists a finite subset $K$ of $X$ such that $\mu(f) \in V_0$ for each $f \in A(0,K,V_0)$. Let $f \in C_p(X,E)$ and $f(K) = 0$. Then $\alpha f \in A(0,K,V_0)$ for each $\alpha \in R$. Hence $\mu(\alpha f) \in W_0$ for each $\alpha \in R$. Thus $E \cdot \mu(f) \subseteq V_0$ and $E \cdot \mu(f)$ is the trivial $R$-submodule. Thus $\mu(f) = 0$ and $K \in \mathcal{S}(\mu)$. In this case $supp_X(\mu) \subseteq K$. Hence $supp_X(\mu)$ is a finite set and $K$ is a finite set from $\mathcal{S}(\mu)$.

Let $L \in \mathcal{S}(\mu)$ be a finite set and $x_0 \in L \setminus supp(\mu)$. Then $L_1 = L \setminus \{x_0\} \in \mathcal{S}(\mu)$. Really, since $x_0 \notin supp_X(\mu)$, there exists an open subset $H$ of $X$ such that $x_0 \in H$ and $\mu(f) = 0$ provided $f(X \setminus H) = 0$. Fix $h \in C(X,R)$ such that $h(x_0) = 1$ and $h(X \setminus H) = 0$. Let $f \in C_p(X,E)$ and $f(L_1) = 0$. We put $f_1(x) = h(x)f(x)$ for any $x \in X$ and $f_2 = f - f_1$. Since $f_1(X \setminus H) = 0$, we have $\mu(f_1) = 0$. By construction, $f_2(L) = 0$ and $\mu(f_2) = 0$. Hence $f = f_1 + f_2$ and $\mu(f) = \mu(f_1 + f_2) = \mu(f_1) + \mu(f_2) = 0$. Hence $L_1 \in \mathcal{S}(\mu)$. Since $K \setminus supp_X(\mu)$ is a finite set, we have $supp_X(\mu) \in \mathcal{S}(\mu)$. $\square$

**Proposition 15.** *If $x \in X$ and $\xi_x(f) = f(x)$ for each $f \in C_p(X,E)$, then $\xi_x \in L_p(X,E)$.*

*Proof.* Obviously, $\xi_x = \alpha x(f)$, where $\alpha \in R$ and $\alpha = 1$. Thus $\xi_x \in L_p(X,E)$. □

**Remark 8.** Let $R$ be a locally simple ring, $X$ be an $R$-Tychonoff space, $\mu \in M_p(X,R)$ and $supp(\mu) = \{x_1, x_2, ..., x_n\}$. Then, by virtue of Proposition 9, there exists $\alpha_1, \alpha_2, ..., \alpha_n \in R \setminus \{0\}$ such that $\mu = \Sigma\{\alpha_1 \xi_{x_i} : i \leq n\}$.

As was mentioned in Remark 4 and Example 6, as a rule $M_p(X,E) \neq L_p(X,E)$. The following result specifies the form of linear functionals for locally simple $R$-modules.

**Theorem 5.** *Let $X$ be an $R$-Tychonoff space, $E$ be a non-trivial topological $E$-module, $\mu \in M_p(X,E)$, $\mu \neq 0$, $supp_X(\mu) \in \mathcal{S}(\mu)$ and $supp_X(\mu)$ is a finite subset of $X$. Then $\mu = \varphi \circ \eta$ for some $\varphi \in Hom(E)$ and $\eta \in L_p(X,E)$.*

*Proof.* Assume that $n \geq 1$ and $supp_X(\mu) = \{b_1, b_2, ..., b_n\}$, where $b_i \neq b_j$ for $i \neq j$. We put $\eta_i = b_i \in L_p(X,E)$ and $\eta = \eta_1 + \eta_2 + ... + \eta_n$. Obviously, $\eta \in L_p(X,E)$ and $supp_X(\eta) = \{b_1, b_2, ..., b_n\} = supp_X(\mu)$.

We can fix the non-empty open subsets $V_1, V_2, ..., V_n$ of $X$ and the functions $f_1, f_2, ..., f_n \in C(X,R)$ such that:

– $V_i \cap V_j = \emptyset$ provided $i \neq j$;

– $b_i \in V_i$, $f_i(b_i) = 1$ and $f_i(X \setminus V_i) = 0$ for each $i \leq n$.

Fix $i \leq n$. For each $y \in E$ we put $\varphi_i(y) = \mu(f_i \cdot y)$, where $(f_i \cdot y)(x) = f_i(x)y$ for each $x \in X$. If $g \in C(X,E)$, then $\mu_i(g) = \mu(f_i \cdot g)$.

**Claim 1**. *$\varphi_i \in Hom(E)$ for each $i \leq n$.*

The mapping $\psi_i : E \longrightarrow C_p(X,E)$, where $\psi_i(y) = f_i \cdot y$ for each $y \in E$, is continuous and linear. The equality $\varphi_i = \mu \circ \psi_i$ completes the proof of the Claim 1.

**Claim 2**. *$\mu_i \in M_p(X,E)$ and $supp_X(\mu_i) = \{b_i\}$ for each $i \leq n$.*

It is clear that $\mu_i \in M_p(X,E)$. If $g \in C(X,E)$, then $g(b_i) = (f_i \cdot g)(b_i)$ and $(f_i \cdot g)(b_j) = 0$ provided $i \neq j$. Hence $\mu_i(g) = \mu_i(f_i \cdot g) = \mu(f_i \cdot g)$. Claim 2 is proved.

**Claim 3**. *$\mu = \mu_1 + \mu_2 + ... + \mu_n$.*

Let $g \in C(X,E)$ and $h = f_1 g + f_2 g + ... + f_n g$. Then $g|supp_X(\mu) = h|supp_X(\mu)$. Hence $\mu(g) = \mu(h) = \mu(f_1 g + f_2 g + ... + f_n g) = \mu(f_1 g) + \mu(f_2 g) + ... + \mu(f_n g) = \mu_1(g) + \mu_2(g) + ... + \mu_n(g)$. Claim 3 is proved.

**Claim 4**. *$\mu = \varphi \circ \eta$.*

By construction, $supp_X(\mu) = supp_X(\varphi \circ \eta)$. If $g \in C(X,E)$ and $h = f_1 g + f_2 g + ... + f_n g$, then $(\varphi \circ \eta)(g) = (\varphi \circ \eta)(h)$. Since $(\varphi \circ \eta)(f_i g) = ((\varphi_1 + \varphi_2 + ... + \varphi_n) \circ (\eta_1 + \eta_2 + ... + \eta_n))(f_i g)) = \Sigma\{(\varphi_i \circ \eta_j)(f_i g) : i, j \leq n\} = (\varphi_i \circ \eta_i)(f_i g) = \varphi_i(\eta_j(f_i g))$ $= \varphi_i(g(b_i)) = \mu(f_i g)$, we have $(\varphi \circ \eta)(g) = (\varphi \circ \eta)(h) = (\varphi \circ \eta)(f_1 g + f_2 g + ... + f_n g)$ $= \Sigma\{(\varphi \circ \eta)(f_i g) : i \leq n\} = \Sigma\{\mu(f_i g) : i \leq n\} = (\mu(f_1 g + f_2 g + ... + f_n g) = \mu(g)$. Claim is proved. The proof is complete. □

## 6 Topological properties of the mapping $supp_X$

Fix a topological ring $R$, a non-trivial locally simple $R$-module $E$ and an $R$-Tychonoff space $X$.

Recall that a set-valued mapping $f : X \to 2^Y$ is lower semicontinuous (l. s. c) if for every open subset $U$ of $Y$ the inverse image of $U$, $f^{-1}(U) = \{x \in X : f(x) \cap U \neq \emptyset\}$ is open in $X$.

**Proposition 16.** *The set-valued mapping $supp_X : M_p(X, E) \to X$ is l. s. c.*

*Proof.* We follow very closely the proof of [3], Property 4.2, and [8], Lemma 6.8.2(4).

Let $U$ be an open subset of $X$, and put $V = supp_X^{-1}(U)$, i.e., $V = \{\mu \in M_p(X, E, F) : supp_X(\mu) \cap U \neq \emptyset\}$. Let $\mu \in V$, and take $x \in supp_X(\mu) \cap U$. Fix an open subset $W$ of $X$ such that $x \in W \subseteq cl_X W \subseteq U$. The there exists $f \in C(X, E)$ such that $f(X \setminus W) = \{0\}$ and $\mu(f) \neq 0$.

Let $H = \{\eta \in M_p(X, E, F) : \eta(f) \neq 0\}$. Since the set $\{0\}$ is closed in $E$, $H$ is the prebasic open set $W(f, E \setminus \{0\}) = \{\eta \in M_p(X, E) : \eta(f) \in E \setminus \{0\}\}$ and $\mu \in W(f, E \setminus \{0\})$.

We claim that $H \subseteq V$. By contradiction, suppose that $\eta \in H \setminus V$, i. e. $\eta(f) \neq 0$ and $supp_X(\eta) \cap U = \emptyset$. Then $X \setminus cl_X W$ is an open neighbourhood of $supp_X(\eta)$ and since $f(X \setminus cl_X W) = \{0\}$, applying Theorem 4, we get that $\eta(f) = 0$. A contradiction, hence $V$ is open in $M_p(X, E)$. $\square$

A subset $L$ of a space $X$ is *bounded* if any continuous real-valued function $f : X \longrightarrow \mathbb{R}$ is bounded on $L$.

A subset $L$ of a topological $R$-module $E$ is called:

(i) *precompact* or *totally a-bounded* if for any neighbourhood $U$ of 0 in $E$ there exists a finite subset $A$ of $E$ such that $L \subseteq A + U = U + A$;

(ii) *a-bounded* if for any neighbourhood $U$ of the 0 in $E$ there exists $n \in \mathbb{N}$ such that $L \subseteq nU$.

Any bounded set is precompact. In a topological vector space over field of reals any precompact set is $a$-bounded.

A topological $R$-module $E$ is called *locally bounded* if there exists an $a$-bounded neighbourhood $U$ of 0 in $E$ such that $E = \cup\{nU : n \in \mathbb{N}\}$ and for any $a \in E$, $a \neq 0$, and any $n \in \mathbb{N}$ there exists $t \in R$ such that $ta \notin nU$. In this case the set $U$ does not contain $R$-submodules of $E$ and $E$ is a locally simple $R$-module.

**Example 8.** Let $E$ be a normed vector space over reals $\mathbb{R}$. Then $E$ is a locally bounded $\mathbb{R}$-module.

**Example 9.** Let $E$ be a topological vector space over reals $\mathbb{R}$ and there exists a number $q > 0$ and a functional $||\cdot|| : E \longrightarrow \mathbb{R}$ such that:

1. $0 < q \leq 1$.
2. $||x|| \geq 0$ for any $x \in E$.
3. If $||x|| = 0$, then $x = 0$.
4. $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in E$.

5. $||\lambda x|| \leq |\lambda|^q ||x||$ for all $x \in E$ and $\lambda \in \mathbb{R}$.

6. If $x \neq 0$ then $lim_{\lambda \to +\infty} ||\lambda x|| = +\infty$.

The functional $||\cdot||$ is called a *q-norm* if the family $\{V(0,r) = \{x : ||x|| < r\} : r > 0\}$ is a base of $E$ at 0. Any $q$-normed space is locally bounded.

**Theorem 6.** *Let $E$ be a non-trivial locally bounded topological $R$-module, $X$ be an $R$-Tychonoff space and for any non-bounded subset $L$ of $X$ there exists $f \in C(X,E)$ such that the set $f(L)$ is not a-bounded. Then:*

*(i) The set $supp_X(H)$ is bounded in $X$ for any a-bounded subset $H$ of $M_p(X,E)$.*

*(ii) The set $supp_X(H)$ is bounded in $X$ for any totally a-bounded subset $H$ of $M_p(X,E)$.*

*(iii) The set $supp_X(H)$ is bounded in $X$ for any bounded subset $H$ of $M_p(X,E)$.*

*Proof.* Fix an $a$-bounded open neighbourhood $W_1$ of 0 in $E$ such that $E = \bigcup\{nW_1 : n \in \mathbb{N}\}$ and for any $a \in E$, $a \neq 0$ and any $n \in \mathbb{N}$ there exists $t \in R$ such that $ta \notin nW_1$.

Now fix two open neighbourhoods $W$ and $W_0$ of 0 in $E$ such that $W_0 + W_0 + W_0 \subseteq W = -W \subseteq W_1$ and $W_0 = -W_0$.

By construction, $W_1 \subseteq kW_0$ for some $k \in \mathbb{N}$.

Hence the sets $W$ and $W_0$ have the following properties:

– $W$ and $W_0$ are $a$-bounded subsets of $E$;

– $E = \bigcup\{nW : n \in \mathbb{N}\} = \bigcup\{nW_0 : n \in \mathbb{N}\}$;

– if $L$ is a bounded or a precompact subset of $E$, then $L \subseteq nW_0$ for some $n \in \mathbb{N}$;

– if $a \in E$, $a \neq 0$, then for any $n \in \mathbb{N}$ there exists $t \in R$ such that $ta \notin nW$.

Suppose that the set $H$ is $a$-bounded or precompact in $L_p(X,E)$ and the set $supp_X(H)$ is not bounded in $X$. Fix $f \in C(X,E)$ such that the set $f(supp_X(H))$ is not $a$-bounded in $E$.

By induction, we shall construct a sequence $\{\mu_n : n \in \mathbb{N}\} \subseteq H$, a sequence $\{U_k : k \in \mathbb{N}\}$ of open subsets of $X$, a sequence $\{x_n \in supp_X(\mu_n) : n \in \mathbb{N}\}$ and a sequence $\{h_k \in C(X,E) : n \in \mathbb{N}\}$ with properties:

1. $x_i \in U_i$, $h_i(X \setminus U_i) = 0$ for any $i \in \mathbb{N}$;
2. $\{U_n : n \in \mathbb{N}\}$ is a discrete family of subsets of $X$;
3. $\mu_n(h_n) \notin nW$;
4. $supp_X\{\mu_1, \mu_2, ..., \mu_n\} \cap cl_X U_{n+1} = \emptyset$;
5. $f(U_n) \subseteq f(x_n) + W_0$ and $f(x_{n+1}) \notin \bigcup\{f(x_i) + W : i \leq n\}$ for each $n \in \mathbb{N}$.

Fix $\mu_1 \in H$ and $x_1 \in supp_X(\mu_1)$. There exists an open subset $U_1$ of $X$ and $g_1 \in C(X,E)$ such that $f(U_1) \subseteq W_0 + f(x_1)$, $g_1(X \setminus U_1) = 0$ and $\mu_1(g_1) \neq 0$. There exists $\alpha_1 \in R$ such that $\alpha_1 \mu_1(g) \notin W$. We put $h_1 = \alpha_1 g_1$.

Assume that $n \geq 1$ and the objects $\{h_i, x_i, U_i, \mu_i : i \leq n\}$ are constructed. We put $M_n = \bigcup\{supp_X(\mu_i) : i \leq n\}$. The set $M_n$ is finite. Hence the set $f(supp_X(H)) \setminus f(M_n)$ is not $a$-bounded in $E$. For some $m_n \in \mathbb{N}$ we have $f(M_n) \subseteq m_n W_0$.

Fix $\mu_{n+1} \in H$ and $x_{n+1} \in supp_X(H)$ such that $f(x_{n+1}) \in E \setminus m_n W$. There exists an open subset $U_{n+1}$ of $X$ and $g_{n+1} \in C(X,E)$ such that $x_{n+1} \in U_{n+1}$, $f(U_{n+1}) \subseteq f(x_{n+1}) + W_0$, $g_{n+1}(X \setminus U_{n+1}) = 0$, $cl_X U_{n+1} \cap M_n = \emptyset$ and $M_{n+1}(g_{n+1}) \neq 0$. There exists $\alpha_{n+1} \in R$ such that $\alpha_{n+1}\mu_{n+1}(g_{n+1}) \notin (n+1)W$. We put $h_{n+1} =$

$\alpha_{n+1}g_{n+1}$. That completes the inductive construction. The objects $\{x_m, \mu_n, h_n, U_n\}$ are constructed for all $n \in \mathbb{N}$. Let $h = \Sigma\{h_n : n \in \mathbb{N}$. Since $\{U_n : n \in \mathbb{N}\}$ is a discrete family and $h_n(X \setminus U_n) = 0$ for any $n \in \mathbb{N}$, we have $h \in C(X, E)$. By construction $\mu_n(h) = \mu_n(h_n) \notin nW_0$ for any $n$. Then $\{\mu_n(h) : n \in \mathbb{N}\}$ is not $a$-bounded subset of $E$. Since the set $H$ is $a$-bounded, the set $\mu(h) : \mu \in H\}$ is $a$-bounded too, a contradiction. The proof is complete. □

**Remark 9.** If $R$ is the fields of real or complex numbers and $E$ is a locally bounded $R$-module, then:

– $E$ is a metrizable linear space;

– $E$ is a locally simple $R$-module;

– any precompact set is $a$-bounded in $E$.

**Remark 10.** Any normed space is a locally bounded $\mathbb{R}$-module. If $E$ is a non-trivial normed space, then for any non-bounded subset $L$ of the space $X$ there exists $f \in C(X, E)$ such that the set $f(L)$ is not bounded in $E$. For a normed space $E$ Theorem 6 was proved by V. Valov in [10].

A space $X$ is *$\mu$-complete* if any closed bounded subset of $X$ is compact.

A space $X$ is *Dieudonné complete* if the maximal uniformity on $X$ is complete. Any Dieudonné complete space is $\mu$-complete.

Denote by $PX$ the space $X$ with the $G_\delta$-topology generated by the $G_\delta$-subsets of $X$. The set $\delta\text{-}cl_X H = cl_{PX} H$ is called the $G_\delta$-closure of the set $H$ in $X$. If $\delta\text{-}cl_X H$ $= H$, then we say the set $H$ is $G_\delta$-closed.

If the space $X$ is $\mu$-complete, then any $G_\delta$-closed subspace of $X$ is $\mu$-complete.

A tightness of a space $X$ is the minimal cardinal number $\tau$ for which for any subset $L \subseteq X$ and any point $x \in cl_X L$ there exists a subset $L_1 \subseteq L$ such that $|L_1| \leq \tau$ and $x \in cl_X L_1$.

We denote by $t(X)$ and $l(X)$ the tightness and the Lindelöf numbers respectively of a space $X$.

The following assertion for $E = \mathbb{R}$ was proved by A. V. Arhangel'skii and E. G. Pytkeev (see [1], Theorem II.1.1).

**Proposition 17.** *Assume that $E$ is a metrizable space and $l(X^n) \leq \tau$ for any $n \in \mathbb{N}$. Then $t(C_p(X, E)) \leq \tau$.*

*Proof.* The proof is as in [1]. We show only the scheme of the proof.

Fix a metric $d$ on $E$. Let $A \subseteq C_p(X, E)$ and $f \in clA$. Let $\varepsilon_n = 2^{-n}$. For any $x = (x_1, x_2, ..., x_n) \in X^n$ there exists $g_x \in A$ such that $d(g_x(x_i), f(x_i)) < \varepsilon_n$, $i \leq n$. Since $g_x$ and $f$ are continuous, there exists $O_x = \Pi\{Ox_i : i \leq n\}$ such that $d(g_x(y), f(y)) < \varepsilon_n$ for all $y \in O_x$. The $\{O_x : x \in X^n\}$ is a cover of $X^n$. Fix $B_n \subseteq X^n$, $|B_n| \leq \tau$ and $\bigcup\{O_x : x \in B_n\} = X^n$. Let $A_n = \{f_x : x \in B_n\} \subseteq A$. Then $f \in cl(\bigcup\{A_n : n \in \mathbb{N}\})$. □

**Proposition 18.** *Let $X$ and $E$ be spaces and $t(X) \leq \aleph_0$. Then $C_p(X, E)$ is a $G_\delta$-closed subspace of the space $E^X$. Moreover, if $E$ is $\mu$-complete, then the space $C_p(X, E)$ is $\mu$-complete too.*

*Proof.* Since the product of $\mu$-complete spaces is $\mu$-complete, the space $E^X$ is $\mu$-complete provided the space $E$ is $\mu$-complete.

Assume that $g \in E^X \setminus C(X,E)$. Then there exists a point $x_0 \in X$ and an open subset $U$ of $E$ such that $g(x_0) \in U$ and $x_0 \in cl_X\{x \in X : g(x) \notin U\}$. Since $t(X) \leq \aleph_0$ there exists a countable subset $L \subseteq \{x \in X : g(x) \notin U\}$ such that $x_0 \in cl_X L$. Fix an open subset $V$ of $E$ such that $g(x_0) \in V \subseteq cl_E V \subseteq U$.

For each $y \in L$ we put $H_y = \{f \in E^X : f(x_0) \in V, f(y) = E \setminus cl_E V\}$. The set $H_y$ is open in $E^X$ and $g \in H_y$. Let $H = \cap\{H_y : y \in L\}$. Then $H$ is a $G_\delta$-subset of $E^X$.

Assume that $f \in C(X,E)$. If $f(x_0) \notin V$, then $f \notin H_y$ for each $y \in L$. Suppose that $f(x_0) \in V$. There exists an open subset $W$ of $X$ and a point $y \in L$ such that $x_0 \in W$, $y \in L \cap W$ and $f(W) \subseteq V_0$. Then $f \notin H_y$. Hence $H \cap C(X,E) = \emptyset$ and $g \notin \delta\text{-}cl_{E^X} C(X,E)$. Therefore $C(X,E)$ is a $G_\delta$-closed subset of $E^X$. Any $G_\delta$-closed subset of a $\mu$-complete space is $\mu$-complete. □

**Proposition 19.** *Let $F$ and $E$ be topological $R$-modules and $\mathfrak{L}_p(F,E)$ be the space of all linear continuous mappings of $F$ into $E$. Then $\mathfrak{L}_p(F,E)$ is a closed subspace of the space $C_p(F,E)$.*

*Proof.* Fix $g \in C_p(F,E) \setminus \mathfrak{L}_p(F,E)$. Then we have one of the following two cases.

**Case 1.** *There exist $a, b \in F$ such that $g(a+b) \neq g(a) + g(b)$.*

In this case there exist four open subsets $V_1$, $V_2$, $V$ and $W$ of $E$ such that $g(a) \in V_1$, $g(b) \in V_2$, $g(a+b) \in W$, $V_1 + V_2 \subseteq V$ and $V \cap W = \emptyset$. The set $H = \{f \in C_p(F,E) : f(a+b) \in W, f(a) \in V_1, f(b) \in V_2\}$ is open in $C_p(F,E)$ and $H \cap \mathfrak{L}_p(F,E) = \emptyset$.

**Case 2.** *There exist $a \in F$ and $\lambda \in R$ such that $g(\alpha a) \neq \alpha g(a)$.*

In this case there exist three open subsets $V_1$, $V$ and $W$ of $E$ such that $g(a) \in V_1$, $g(\alpha a) \in W$, $\alpha V_1 \subseteq V$ and $V \cap W = \emptyset$. The set $H = \{f \in C_p(F,E) : f(\alpha a) \in W, f(a) \in V_1\}$ is open in $C_p(F,E)$ and $H \cap \mathfrak{L}_p(F,E) = \emptyset$. The proof is complete. □

**Corollary 4.** *Let $E$ and $F$ be topological $R$-modules and $t(F) \leq \aleph_0$. Then $\mathfrak{L}_p(F,E)$ is a $G_\delta$-closed subset of $E^F$. In particular, if $E$ is $\mu$-complete, then the space $\mathfrak{L}_p(F,E)$ is $\mu$-complete too.*

For any subspace $Y$ of a space $X$ we view $C_p(Y|X,E) = \{f|_Y : f \in C(X,E)\}$ as a subspace of the space $C_p(Y,E)$.

**Proposition 20** ([1], Proposition 0.4.1 for $E = \mathbb{R}$). *Let $Y$ be a subspace of the space $X$, $E$ be a non-trivial topological $R$-module, $X$ be an $R$-Tychonoff space and $p_Y(f) = f|_Y$ for each $f \in C_p(X,E)$. Then the mapping $p_Y : C_p(X,E) \longrightarrow C_p(Y|X,E)$ has the following properties:*

*(i) $p_Y$ is a continuous mapping.*

*(ii) If the set $Y$ is closed in $X$, then the mapping $p_Y$ is open.*

*(iii) If $Y$ is dense in $X$, then $p_Y$ is a one-to-one correspondence.*

*(iv) The subspace $C_p(Y|X,E)$ is dense in $C_p(Y,E)$.*

*Proof.* Let $x_1, x_2, ..., x_n$ be a finite subset of $X$. The we can assume that there exists $k \leq n$ such that $x_1, x_2, ..., x_{k-1} \in Y$ and $x_k, ..., x_n \in X \setminus Y$. Let $f \in C(X, E)$ and $U_1, U_2, ..., U_n$ be open subsets of $E$ such that $f(x_i) \in U_i$ for each $i \leq n$. We put $W(f, x_1, x_2, ..., x_n, U_1, ..., U_n) = \{g \in C(X, E) : g(x_i) \in U_i \text{ for each } i \leq n\}$ and $W_Y(f, x_1, ..., x_{k-1}, U_1, ..., U_{k-1}) = \{g|_Y : g \in W(f, x_1, ..., x_{k-1}, U_1, ..., U_{k-1})\}$. We have $p_Y(W(f, x_1, x_2, ..., x_n, U_1, ..., U_n)) \subseteq W_Y(f, x_1, ..., x_{k-1}, U_1, ..., U_{k-1})$. Thus the mapping $p_Y$ is continuous. If $Y$ is closed in $X$, then from Lemma 2 it follows that $p_Y(W(f, x_1, x_2, ..., x_n, U_1, ..., U_n)) = W_Y(f, x_1, ..., x_{k-1}, U_1, ..., U_{k-1})$. Hence the mapping $p_Y$ is open.

Assertion (iii) is obvious. Assertion (iv) follows from Lemma 2. The proof is complete. □

**Theorem 7.** *Let $E$ be a locally bounded metrizable $R$-module, $X$ be an $R$-Tychonoff compactly $E$-full space and for any non-bounded subset $L$ of $X$ there exists $f \in C(X, E)$ such that the set $f(L)$ is not a-bounded in $E$. Then the space $X$ is $\mu$-complete if and only if the space $M_p(X, E)$ is $\mu$-complete.*

*Proof.* By virtue of Proposition 5, we can assume that $X = e_X(X)$ is a subspace of the space $M_p(X, E)$. From Proposition 4 it follows that the subspace $X$ is closed in $M_p(X, E)$.

Let $M_p(X, E)$ be a $\mu$-complete space. Since $X$ is a closed subspace of $M_p(X, E)$, the space $X$ is $\mu$-complete too.

Assume that $X$ is a $\mu$-complete space. Let $\Phi$ be a closed bounded subset of $M_p(X, E)$. Then the closure $Y$ of the set $\cup\{supp_X(\mu) : \mu \in \Phi\}$ is a compact subset of $X$.

The restriction mapping $p_Y : C_p(X, E) \longrightarrow C_p(Y, E)$ is an open continuous linear mapping of the $R$-module $C_p(X, E)$ onto the $R$-module $C_p(Y, E)$.

**Claim 1.** *The dual mapping $\varphi : E^{C(Y,E)} \longrightarrow E^{C_p(X,E)}$ is a linear embedding and the set $\varphi(E^{C(Y,E)})$ is closed in $E^{C(X,E)}$.*

The proof of this fact is similar with the prof of Proposition 0.4.6 from [1].

By construction, we have $\Phi \subseteq \varphi(M_p(Y, E)) \subseteq M_p(X, E)$.

**Claim 2.** *$\varphi(M_p(Y, E))$ is a closed subset of the subspaces $M_p(X, E)$ and $C_p(C_p(X, E), E)$ of the space $E^{C(X,E)}$.*

Follows from Claim 1 and Proposition 19.

**Claim 3.** *$\varphi(C_p(C_p(Y, E), E)) \subseteq C_p(C_p(X, E), E)$.*

Follows from the continuity of the mapping $p_Y$.

**Claim 4.** *The sets $\varphi(M_p(X, E))$ and $\varphi(C_p(C_p(Y, E), E))$ are $G_\delta$-closed in $E^{C(X,E)}$.*

Since $Y$ is compact, from Proposition 17 it follows that $t(C_p(Y, E)) = \aleph_0$. Then, from Proposition 18 it follows that $C_p(C_p(Y, E), E)$ is a $G_\delta$-closed subset of the space $E^{C(Y,E)}$. From Claim 1 it follows that $\varphi(C_p(C_p(Y, E), E))$ is $G_\delta$-closed in $E^{C(X,E)}$. Corollary 4 completes the proof of the claim.

Let $G$ be the $G_\delta$-closure of the set $C_p(C_p(X,E),E))$ in $E^{C(X,E)}$. We have $M_p(X,E) \subseteq G$. Hence $\Phi$ is a bounded subset of the space $G$.

**Claim 5.** *The sets $\varphi(M_p(X,E))$ and $\varphi(C_p(C_p(Y,E),E))$ are closed in $G$.*

Follows from Claim 4.

Since $E$ is a metrizable space, $E$ is a $\mu$-complete space. Thus $\Phi$ is a closed bounded subset of the $\mu$-complete space $G$. Therefore the set $\Phi$ is compact. The proof is complete. □

## 7 Relations between spaces generated by a linear homeomorphism

Let $R$ be a topological ring and $E$ be a non-trivial locally bounded topological $R$-module. Then the $R$-module $E$ is locally simple.

Fix two non-empty $R$-Tychonoff spaces $X$ and $Y$ with the properties:

– for any non-bounded subset $L$ of $X$ there exists $f \in C(X,E)$ such that the set $f(L)$ is not $a$-bounded in $E$;

– for any non-bounded subset $L$ of $Y$ there exists $f \in C(Y,E)$ such that the set $f(L)$ is not $a$-bounded in $E$.

Fix now a continuous linear homeomorphism $u : C_p(X,E) \longrightarrow C_p(Y,E)$. Then the dual mapping $v : M_p(Y,E) \longrightarrow M_p(X,E)$, where $v(\eta) = \eta \circ u$ for each $\eta \in M_p(Y,E)$, is a linear homeomorphism. For each $x \in X$ we put $\varphi(x) = supp_Y(v^{-1}(\xi_x))$ and for any $y \in Y$ put $\psi(y) = supp_X(v(\xi_y))$.

**Property 1.** *$\varphi : X \to Y$ and $\psi : Y \to X$ are l.s.c. set-valued mappings and $\varphi(x)$, $\psi(y)$ are finite sets for all points $x \in X$, $y \in Y$.*

*Proof.* Follows from Proposition 16 and Theorem 4. □

**Property 2.** *Let $y_0 \in Y$, $f \in C(X,E)$ and $f(\psi(y_0)) = 0$. Then $u(f)(y_0) = 0$.*

*Proof.* For any $\eta \in M_p(Y,E)$ and $g \in C(X,E)$ we have $v(\eta)(g) = \eta(u(g))$. Since $f(supp_X(v(\xi_{y_0}))) = f(\psi(y_0)) = 0$, we have $u(f)(y_0) = \xi_{y_0}(u(f)) = v(\xi_{y_0})(f) = f(supp_X(v(\xi_{y_0}))) = 0$. The proof is complete. □

**Corollary 5.** *If $f, g \in C(X,E)$ and $f|_{\phi(y)} = g|_{\phi(y)}$, then $u(f)(y) = u(g)(y)$.*

**Property 3.** *$x \in cl_X\psi(\varphi(x))$ for every point $x \in X$ and $y \in cl_Y\varphi(\psi(y))$ for every point $y \in Y$.*

*Proof.* Assume that $x_0 \in X$ and $x_0 \notin cl_X\psi(\varphi(x_0)) = F$. Fix $f \in C(X,E)$ such that $f(x_0) = b \neq 0$ and $f(F) = f(\psi(\varphi(x_0))) = 0$. Since $\psi(y) \subseteq F$ and $f(F) = 0$ for any $y \in \varphi(x_0)$ by virtue of Property 2, we have $u(f)(y) = 0$ for each $y \in \varphi(x_0)$. Since $u(f)(y) = 0$ for each $y \in \varphi(x_0)$, by virtue of Property 2, we have $f(x_0) = u^{-1}(u(f))(x_0) = 0$. By construction, we have $f(x_0) \neq 0$, a contradiction. □

**Property 4.** *$x \in \psi(\varphi(x))$ for every point $x \in X$.*

*Proof.* For every $x \in X$, $\varphi(x)$ is finite set and $\psi(\varphi(x))$ is compact. Property 3 completes the proof. $\square$

**Property 5.** *If $H$ is a dense subset of $Y$, then $\psi(H)$ is a dense subset of $X$.*

*Proof.* Assume that $x_0 \notin cl_X\psi(H)$. Then there exists $f \in C(X,E)$ such that $f(x_0) \neq 0$ and $f(\psi(H)) = 0$. Since $f(\psi(H)) = 0$ for any $y \in Y$, by virtue of Property 2, we have $u(f)(y) = 0$ for any $y \in Y$. Thus $u(f) = 0$. Hence $f = 0$, a contradiction. $\square$

**Corollary 6.** *The space $X$ is separable if and only if the space $Y$ is separable. In general, $d(X) = d(Y)$.*

**Property 6.** *$\varphi(F)$ is a bounded set of $Y$ for each bounded set $F$ of $X$.*

*Proof.* Let $F$ be a bounded subset of $X$. Then $F$ is a bounded subset of $M_p(X,E)$ and respectively $v^{-1}(F)$ is a bounded subset of $M_p(Y,E)$. By Theorem 6 the set $supp_Y(v^{-1}(F))$ is a bounded subset of $Y$. The proof is complete. $\square$

**Property 7.** *Let $E$ be a metrizable space, $X$ and $Y$ be compactly $E$-full spaces. Then the space $X$ is $\mu$-complete if and only if the space $Y$ is $\mu$-complete.*

*Proof.* Let $X$ be a $\mu$-complete space. Then $M_p(X,E)$ and $M_p(Y,E)$, by virtue of Theorem 7, are $\mu$-complete spaces. By Theorem 7 the space $Y$ is $\mu$-complete too. The proof is complete. $\square$

As in [3, 4] we say that the pair of set-valued mappings $\theta : X \longrightarrow Y$ and $\pi : Y \longrightarrow X$ is called lower-reflective if it satisfies the following conditions:

$1l$. $\theta$ and $\pi$ are l.s.c.

$2l$. $\theta(x)$ and $\pi(x)$ are finite sets for all points $x \in X$ and $y \in Y$.

$3l$. $x \in \pi(\theta(x))$ and $y \in \theta(\pi(y))$ for all points $x \in X$ and $y \in Y$.

Also, as in [3, 4] we say that the pair of set-valued mappings $\theta : X \longrightarrow Y$ and $\pi : Y \longrightarrow X$ is called upper-reflective if it satisfies the following conditions:

$1u$. $\theta(F)$ is a bounded subset of $Y$ for each bounded subset $F$ of $X$.

$2u$. $\pi(\Phi)$ is a bounded subset of $X$ for each bounded subset $\Phi$ of $Y$.

$3u$. $x \in cl_X\pi(\theta(x))$ and $y \in cl_Y\theta(\pi(y))$ for all points $x \in X$ and $y \in Y$.

From the above properties follows

**Corollary 7.** *The space $X$ is separable if and only if the space $Y$ is separable. In general, $d(X) = d(Y)$.*

**General conclusion:** The set-valued mappings $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow X$ form an equivalence of $X$ and $Y$ in sense of articles [3,4]. Thus the general theorems from [3] can be extended for the mappings in topological $R$-modules. In the following sections we formulate the general theorems for that case.

## 8 Application to perfect properties

We say that the property $\mathcal{P}$ is a *perfect property* if for any continuous perfect mapping $f : X \longrightarrow Y$ of $X$ onto $Y$ we have $X \in \mathcal{P}$ if and only if $Y \in \mathcal{P}$. We say that the property $\mathcal{P}$ is a *strongly perfect property* if it is perfect and any space with property $\mathcal{P}$ is $\mu$-complete.

**Example 10.** By virtue of Example 6.2 from [3] (see also [4]), the following properties are perfect:

1. To be a compact space.
2. To be a paracompact $p$-space.
3. To be a paracompact space.
4. To be a metacompact space.
5. To be a $k$-scattered space.
6. To be a monotonically $p$-space.
7. To be a monotonically Čech complete space.
8. To be a Čech complete space.
9. To be a Lindelöf space.
10. To be a Lindelöf $\Sigma$-space.
11. To be a subparacompact space.
12. To be a locally compact space.

**Example 11.** The following properties are strongly perfect:

1. To be a compact space.
2. To be a paracompact $p$-space.
3. To be a paracompact space.
4. To be a $\mu$-complete metacompact space.
5. To be a $k$-scattered $\mu$-complete space.
6. To be a $\mu$-complete monotonically $p$-space.
7. To be a $\mu$-complete monotonically Čech complete space.
8. To be a $\mu$-complete Čech complete space.

9. To be a Lindelöf space.

10. To be a Lindelöf $\Sigma$-space.

11. To be a $\mu$-complete subparacompact space.

12. To be a $\mu$-complete locally compact space.

A space $X$ is called a *wq-space* if for any point $x \in X$ there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets of $X$ such that $x \in \cap\{U_n : n \in \mathbb{N}\}$ and each set $\{x_n \in U_n : n \in \mathbb{N}\}$ is bounded in $X$.

A space $X$ is *pseudocompact* if the set $X$ is bounded in the space $X$. A pseudocompact space is a $\mu$-complete space if and only if it is compact. Any pseudocompact space is a $wq$-space.

**Theorem 8.** *Let $R$ be a topological ring and $E$ be a non-trivial locally bounded topological $R$-module. Fix two non-empty $R$-Tychonoff spaces $X$ and $Y$ with the properties:*

*– for any non-bounded subset $L$ of $X$ there exists $f \in C(X, E)$ such that the set $f(L)$ is not a-bounded in $E$;*

*– for any non-bounded subset $L$ of $Y$ there exists $f \in C(Y, E)$ such that the set $f(L)$ is not a-bounded in $E$.*

*Assume that $u : C_p(X, E) \longrightarrow C_p(Y, E)$ is a linear homeomorphism. Then:*

*1. $X$ is a pseudocompact space if and only if $Y$ is a pseudocompact space.*

*2. If $\mathcal{P}$ is a perfect property and $X$, $Y$ are $\mu$-complete wq-spaces, then $X \in \mathcal{P}$ if and only if $Y \in \mathcal{P}$.*

*Proof.* Consider the set-valued mappings $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow X$ constructed in Section 7.

Let $X$ be a pseudocompact space. Then $X$ is a bounded subset of the space $X$. Hence $Y = \varphi(X)$ is a bounded subset of $Y$ and $Y$ is a pseudocompact space. Assertion 1 is proved.

Assume that $\mathcal{P}$ is a perfect property and $X, Y$ are $\mu$-complete $wq$-spaces. Suppose that $X \in \mathcal{P}$. By virtue of Theorem 2.5 from [3], there exist a space $Z$ and two perfect single-valued mappings $f : Z \longrightarrow X$ and $g : Z \longrightarrow Y$ onto $X$ and $Y$, respectively. Hence, $Y, Z \in \mathcal{P}$. Assertion 2 is proved. The proof is complete. $\square$

**Theorem 9.** *Let $R$ be a topological ring and $E$ be a non-trivial metrizable locally bounded topological $R$-module. Fix two non-empty $R$-Tychonoff compactly $E$-full spaces $X$ and $Y$ with the properties:*

*– for any non-bounded subset $L$ of $X$ there exists $f \in C(X, E)$ such that the set $f(L)$ is not a-bounded in $E$;*

*– for any non-bounded subset $L$ of $Y$ there exists $f \in C(Y, E)$ such that the set $f(L)$ is not a-bounded in $E$.*

*Assume that $u : C_p(X, E) \longrightarrow C_p(Y, E)$ is a linear homeomorphism. Then:*

*1. The space $X$ is $\mu$-complete if and only if the space $Y$ is $\mu$-complete.*

*2. $X$ is a compact space if and only if $Y$ is a compact space.*

*3. If $\mathcal{P}$ is a strongly perfect property and $X$, $Y$ are $wq$-spaces, then $X \in \mathcal{P}$ if and only if $Y \in \mathcal{P}$.*

*Proof.* Consider the set-valued mappings $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow X$ constructed in Section 7. Assertion 1 follows from Property 7.

Assume that $\mathcal{P}$ is a strongly perfect property and $X$, $Y$ are $wq$-spaces. Suppose that $X \in \mathcal{P}$. By definition of a strongly perfect property, $X$ is a $\mu$-complete space. From Assertion 1 it follows that $Y$ is a $\mu$-complete space too. By virtue of Theorem 2.5 from [3], there exist a space $Z$ and two perfect single-valued mappings $f : Z \longrightarrow X$ and $g : Z \longrightarrow Y$ onto $X$ and $Y$, respectively. Hence, we have $Y, Z \in \mathcal{P}$. Assertion 3 is proved.

Let $X$ be a compact space. By virtue of Theorem 8, $Y$ is a pseudocompact space. Hence $X$ and $Y$ are $wq$-spaces. Assertion 3 completes proof of Assertion 2. The proof is complete. □

## 9 Application to open properties

We say that the property $\mathcal{P}$ is an *$of$-property* (*open-finite property*) if for any continuous open finite-to-one mapping $f : X \longrightarrow Y$ and any subspace $Z$ of $X$ we have $Z \in \mathcal{P}$ if and only if $f(Z) \in \mathcal{P}$.

**Example 12.** From the results from [3],[4] and [5] the following properties are $of$-properties:

1. To be hereditarily Lindelöf.
2. To be $\sigma$-space.
3. To be hereditarily separable.
4. To be $\sigma$-metrizable.
5. To be $\sigma$-scattered.
6. To be $\sigma$-discrete space.

**Example 13.** Let $\tau$ be an infinite cardinal. Consider the properties:

1. $X \in e(\tau)$ if and only if $e(X) \leq \tau$;
2. $X \in d(\tau)$ if and only if $d(X) \leq \tau$;
3. $X \in hd(\tau)$ if and only if $hd(X) \leq \tau$;
4. $X \in hl(\tau)$ if and only if $hl(X) \leq \tau$.

Then $e(\tau), d(\tau), hd(\tau), hl(\tau)$ are $of$-properties.

**Theorem 10.** *Let $R$ be a topological ring and $E$ be a non-trivial locally bounded topological $R$-module. Fix two non-empty $R$-Tychonoff spaces $X$ and $Y$ with the properties:*

*– for any non-bounded subset $L$ of $X$ there exists $f \in C(X, E)$ such that the set $f(L)$ is not a-bounded in $E$;*

*– for any non-bounded subset $L$ of $Y$ there exists $f \in C(Y, E)$ such that the set $f(L)$ is not a-bounded in $E$.*

*Assume that $u : C_p(X, E) \longrightarrow C_p(Y, E)$ is a linear homeomorphism. If $\mathcal{P}$ is an $of$-property, then $X \in \mathcal{P}$ if and only if $Y \in \mathcal{P}$.*

*Proof.* Consider the set-valued mappings $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow X$ constructed in the Section 7. As in [3] (see Theorem 2.1 from [3]) we put $Z = \cup\{\{x\} \times \varphi(x) : x \in X\}$ and $S = \cup\{\psi(y) \times \{y\} : y \in Y\}$ as subspaces of the spaces $X \times Y$, $f(x, y) = x$ and $g(x, y) = y$ for any point $(x, y) \in X \times Y$. Then $f : Z \longrightarrow X$ and $g : S \longrightarrow Y$ are continuous open finite-to-one mappings. If $D = Z \cap S$, then from Property 4 it follows that $f(D) = X$ and $g(D) = Y$. Hence $X \in \mathcal{P}$ if and only if $Y \in \mathcal{P}$. The proof is complete. □

## 10 $l_p(E)$-equivalence and metrizability

**Theorem 11.** *Let $R$ be a topological ring and $E$ be a non-trivial metrizable locally bounded topological $R$-module. Fix two non-empty $R$-Tychonoff compactly $E$-full spaces $X$ and $Y$ with the properties:*

*– for any non-bounded subset $L$ of $X$ there exists $f \in C(X, E)$ such that the set $f(L)$ is not a-bounded in $E$;*

*– for any non-bounded subset $L$ of $Y$ there exists $f \in C(Y, E)$ such that the set $f(L)$ is not a-bounded in $E$.*

*Let $X$ and $Y$ be $l_p(E)$-equivalent spaces. Then:*

*1. $X$ is a compact metrizable space if and only if $Y$ is a compact metrizable space.*

*2. If $X$ is a metrizable space, then the space $Y$ is metrizable if and only if $Y$ is a $wq$-space.*

*Proof.* Any metrizable space is a $wq$-space.

Let $X$ be a metrizable space and $Y$ be a $wq$-space. Since $X$ is metrizable, by virtue of Theorem 8, $Y$ is a paracompact $p$-space. From Theorem 10 it follows that $Y$ is a $\sigma$-space. If a paracompact space $Y$ is a $\sigma$-space and a $p$-space, then $Y$ is metrizable [9]. Assertion 2 is proved.

Assertion 1 follows from the Assertion 2 and Theorem 8. The proof is complete. □

## 11 Final remarks and examples

The requirements on spaces $R$, $E$ and $X$ in the conditions of Theorems 8, 9 and 10 are essential.

First, the space $X$ must have a sufficient number of continuous mappings into $R$ and $E$. Moreover, these mappings must determine the topology of the space $X$ and should feel certain properties of subsets relative to their position in the space $X$. These are explained the requirements:

– the space $X$ is a non-empty $R$-Tychonoff space;

– for any non-bounded subset $L$ of $X$ there exists $f \in C(X, E)$ such that the set $f(L)$ is not $a$-bounded in $E$.

Obviously, the requirement "$X$ is a non-empty $R$-Tychonoff space" may be changed by the requirement "$X$ is a non-empty $E$-Tychonoff space". A space $X$ is an $E$-Tychonoff space if for each closed set $F$ of $X$, any point $a \in X \setminus F$ and any point $b \in E$, there exists $f \in C(X, E)$ such that $f(a) = 0$ and $f(F) = b$.

Second, the space $E$ should have unbounded sets and some neighborhoods of zero should be able to distinguish boundedness and unboundedness of sets from $E$.

**Example 14.** Let $\mathbb{Z}$ be the discrete ring of integers, $E = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be the unit circle with the binary operation $(x, y) + (u, v) = (xu - yv, xv + yu)$. Then any zero-dimensional space is $\mathbb{Z}$-Tychonoff and $E$ is a locally simple compact $\mathbb{Z}$-module. Any Tychonoff space is an $E$-Tychonoff space. For any space $X$ the $\mathbb{Z}$-modules $C_p(X, E)$ and $M_p(X, E)$ are $a$-bounded. In this case the assertions (i) and (ii) of Theorem 6 are not true.

**Example 15.** Let $R$ be a finite ring, $0 \neq 1$, and $E = R^A$, where $A$ is an infinite discrete space. Then the ring $\mathbb{R}$ is locally simple and compact. The $R$-module $E$ is compact and not locally simple. Any zero-dimensional space is $R$-Tychonoff. The spaces $C_p(X, E)$, $C_p(X \times A, E)$, $C_p(X, E)^A$, $C_p(X, R)^A$, $C_p(X, E)^A$, and $C_p(X \times A, R)$ are linearly homeomorphic. The space $X$ may be compact and the space $X \times A$ is not pseudocompact. If the space $A$ is countable, the module $E$ is metrizable.

**Example 16.** Let $E = \mathbb{R}^A$, where $A$ is an infinite discrete space. The ring of reals $\mathbb{R}$ is a locally simple and locally compact topological field. The $\mathbb{R}$-module $E$ is not locally simple and not locally $a$-bounded. Any Tychonoff space is an $\mathbb{R}$-Tychonoff space. The spaces $C_p(X, E)$, $C_p(X, \mathbb{R})$, $C_p(X, E)^A$, $C_p(X, \mathbb{R})^A$, $C_p(X, E)^A$, and $C_p(X \times A, \mathbb{R})$ are linearly homeomorphic. The space $X$ may be compact and the space $X \times A$ is not pseudocompact. If the space $A$ is countable, the module $E$ is metrizable.

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# One subfamily of cubic systems with invariant lines of total multiplicity eight and with two distinct real infinite singularities

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**Abstract.** In this article we classify a subfamily of differential real cubic systems possessing eight invariant straight lines, including the line at infinity and including their multiplicities. This subfamily of systems is characterized by the existence of two distinct infinite singularities, defined by the linear factors of the polynomial $C_3(x, y) = yp_3(x, y) - xq_3(x, y)$, where $p_3$ and $q_3$ are the cubic homogeneities of these systems. Moreover we impose additional conditions related with the existence of triplets and/or couples of parallel invariant lines. This classification, which is taken modulo the action of the group of real affine transformations and time rescaling, is given in terms of affine invariant polynomials. The invariant polynomials allow one to verify for any given real cubic system whether or not it has invariant straight lines of total multiplicity eight, and to specify its configuration of straight lines endowed with their corresponding real singularities of this system. The calculations can be implemented on computer and the results can therefore be applied for any family of cubic systems in this class, given in any normal form.

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## 1 Introduction and the statement of the Main Theorem

We consider here real polynomial differential systems

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y), \tag{1}$$

where $P,\ Q$ are polynomials in $x,\ y$ with real coefficients, i. e. $P, Q \in \mathbb{R}[x, y]$. We say that systems (1) are *cubic* if $\max(\deg(P), \deg(Q)) = 3$.

Let

$$\mathbf{X} = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}$$

be the polynomial vector field associated to systems (1).

A straight line $f(x, y) = ux + vy + w = 0$, $(u, v) \neq (0, 0)$ satisfies

$$\mathbf{X}(f) = uP(x, y) + vQ(x, y) = (ux + vy + w)R(x, y)$$

for some polynomial $R(x, y)$ if and only if it is *invariant* under the flow of the systems. If some of the coefficients $u$, $v$, $w$ of an invariant straight line belong to $\mathbb{C}\setminus\mathbb{R}$, then we say that *the straight line is complex*; otherwise *the straight line is real.* Note that, since systems (1) are real, if a system has a complex invariant straight line $ux + vy + w = 0$, then it also has its conjugate complex invariant straight line $\bar{u}x + \bar{v}y + \bar{w} = 0$.

To a line $f(x, y) = ux + vy + w = 0$, $(u, v) \neq (0, 0)$ we associate its projective completion $F(X, Y, Z) = uX + vY + wZ = 0$ under the embedding $\mathbb{C}^2 \hookrightarrow \mathbf{P}_2(\mathbb{C})$, $(x, y) \mapsto [x : y : 1]$. The line $Z = 0$ in $\mathbf{P}_2(\mathbb{C})$ is called *the line at infinity* of the affine plane $\mathbb{C}^2$. It follows from the work of Darboux (see, for instance [10]) that each system of differential equations of the form (1) over $\mathbb{C}$ yields a differential equation on the complex projective plane $\mathbf{P}_2(\mathbb{C})$ which is the compactification of the differential equation $Qdx - Pdy = 0$ in $\mathbb{C}^2$. The line $Z = 0$ is an invariant manifold of this complex differential equation.

**Definition 1** (see [27])**.** We say that an invariant affine straight line $f(x, y) = ux + vy + w = 0$ (respectively the line at infinity $Z = 0$) for a cubic vector field $\mathbf{X}$ has multiplicity $m$ if there exists a sequence of real cubic vector fields $X_k$ converging to $\mathbf{X}$, such that each $\mathbf{X}_k$ has $m$ (respectively $m-1$) distinct invariant affine straight lines $f_k^j = u_k^j x + v_k^j y + w_k^j = 0$, $(u_k^j, v_k^j) \neq (0, 0)$, $(u_k^j, v_i^k, w_k^j) \in \mathbb{C}^3$ $(j \in \{1, \dots m\})$, converging to $f = 0$ as $k \to \infty$ (with the topology of their coefficients), and this does not occur for $m + 1$ (respectively $m$).

We mention here some references on polynomial differential systems possessing invariant straight lines. For quadratic systems see [11, 24, 25, 27–30] and [31]; for cubic systems see [15–18, 26, 34] and [35]; for quartic systems see [33] and [36]; for some more general systems see [13, 21, 22] and [23].

According to [2] the maximum number of invariant straight lines taking into account their multiplicities for a polynomial differential system of degree $m$ is $3m$ when we also consider the infinite straight line. This bound is always reached if we consider the real and the complex invariant straight lines, see [9].

So the maximum number of the invariant straight lines (including the line at infinity $Z = 0$) for cubic systems is 9. A classification of all cubic systems possessing the maximum number of invariant straight lines taking into account their multiplicities have been made in [16]. We also remark that a subclass of the family of cubic systems with eight invariant lines was discussed in [34] and [35].

It is well known that for a cubic system (1) with finite number of infinite singularities there exist at most 4 different slopes for invariant affine straight lines, for more information about the slopes of invariant straight lines for polynomial vector fields, see [1].

**Definition 2** (see [31])**.** Consider a planar cubic system (1). We call *configuration of invariant straight lines* of this system, the set of (complex) invariant straight lines (which may have real coefficients) of the system, each endowed with its own multiplicity and together with all the real singular points of this system located on these invariant straight lines, each one endowed with its own multiplicity.

*Remark* 1. In order to describe the various kinds of multiplicity for infinite singular points we use the concepts and notations introduced in [24]. Thus we denote by '$(a, b)$' the maximum number $a$ (respectively $b$) of infinite (respectively finite) singularities which can be obtained by perturbation of the multiple point.

Suppose that a cubic system (1) possesses 8 distinct invariant straight lines (including the line at infinity). We say that these lines form a *configuration of type* $(3, 3, 1)$ if there exist two triplets of parallel lines and one additional line, every set with different slopes. And we say that these lines form a *configuration of type* $(3, 2, 1, 1)$ if there exist one triplet and one couple of parallel lines and two additional lines, every set with different slopes. Similarly *configurations of types* $(3, 2, 2)$ and $(2, 2, 2, 1)$ are defined and these four types of the configurations exhaust all possible configurations formed by 8 invariant lines for a cubic system.

Note that in all configurations the invariant straight line which is omitted is the infinite one.

Suppose a cubic system (1) possesses 8 invariant straight lines, including the infinite one, and taking into account their multiplicities. We say that these lines form a *potential configuration of type* $(3, 3, 1)$ (respectively, $(3, 2, 2)$; $(3, 2, 1, 1)$; $(2, 2, 2, 1)$) if there exists a sequence of vector fields $\mathbf{X}_k$ as in Definition 1 having 8 distinct lines of type $(3, 3, 1)$ (respectively, $(3, 2, 2)$; $(3, 2, 1, 1)$; $(2, 2, 2, 1)$).

It is well known that the infinite singularities (real or complex) of cubic systems are determined by the linear factors of the polynomial $C_3(x, y) = yp_3(x, y) - xq_3(x, y)$ where $p_3$ and $q_3$ are the cubic homogeneities of these systems.

In this paper we consider the family of cubic systems possessing two distinct infinite singularities defined by one triple and one simple factors of the invariant polynomial $C_3(x, y)$. This family univocally is determined by affine invariant criteria (see Lemma 7). Moreover we impose some additional conditions related with the existence of triplets and/or couples of parallel invariant lines of these systems (see Theorem 1 and Main Theorem). As a result we investigate the obtained subfamily of cubic systems and determine necessary and sufficient affine invariant conditions for the existence of eight invariant straight lines, including the line at infinity and taking into account their multiplicities.

Our results are stated in the following theorem.

**Main Theorem.** *We consider here the family of cubic systems for which the conditions* $\mathcal{D}_1 = \mathcal{D}_3 = \mathcal{D}_4 = 0$, $\mathcal{D}_2 \neq 0$ *hold, i.e. the infinite singularities of these systems are determined by one triple and one simple factors of the invariant polynomial* $C_3(x, y)$. *Moreover we assume in addition that for this family the condition* $\mathcal{V}_1 = \mathcal{V}_3 = 0$ *is satisfied. Then:*

***(A)*** *This family of cubic systems could be brought via an affine transformation and time rescaling to the systems*

$$\dot{x} = a + cx + dy + 2hxy + ky^2 + x^3, \quad \dot{y} = b + ex + fy + lx^2 + 2mxy + ny^2, \qquad (2)$$

*which could possess one of the 16 possible configurations Config. 8.23 – Config. 8.38 of invariant lines given in Figure 1.*

***(B)*** *The condition $\mathcal{K}_5 = N_1 = 0$ is necessary for a system* (2) *to have invariant lines of total multiplicity 8, including the line at infinity. Assuming this condition to be satisfied, a system* (2) *possesses the specific configuration Config. 8.j* ($j \in \{23, 24, \dots, 38\}$) *if and only if the corresponding additional conditions included below are fulfilled. Moreover this system can be brought via an affine transformation and time rescaling to the canonical form, written below next to the configuration:*

- *Config.8.23* $\Leftrightarrow N_2N_3 \neq 0,\ N_4 = N_5 = N_6 = N_7 = 0$ : $\begin{cases} \dot{x} = (x-1)x(1+x), \\ \dot{y} = x - y + x^2 + 3xy; \end{cases}$

- *Config. 8.24 - 8.27* $\Leftrightarrow$ $N_2 \neq 0, N_3 = 0,\ N_4 = N_6 = N_8 = 0,\ N_9 \neq 0$:
$$\begin{cases} \dot{x} = x(r + 2x + x^2), \\ \dot{y} = (r + 2x)y,\ r(9r - 8) \neq 0; \end{cases} \quad \begin{cases} \textit{Config.8.24} \ \Leftrightarrow \ N_{11} < 0 \ (r < 0); \\ \textit{Config.8.25} \ \Leftrightarrow \ N_{10} > 0, N_{11} > 0 \ (0 < r < 1); \\ \textit{Config.8.26} \ \Leftrightarrow \ N_{10} = 0 \ (r = 1); \\ \textit{Config.8.27} \ \Leftrightarrow \ N_{10} < 0 \ (r > 1); \end{cases}$$

- *Config. 8.28 - 8.30* $\Leftrightarrow$ $N_2 \neq 0, N_3 = 0,\ N_5 = N_8 = N_{12} = 0,\ N_{13} \neq 0$:
$$\begin{cases} \dot{x} = x(r - 2x + x^2), (9r - 8) \neq 0 \\ \dot{y} = 2y(x - r), r(r - 1) \neq 0; \end{cases} \quad \begin{cases} \textit{Config.8.28} \ \Leftrightarrow \ N_{15} < 0 \ (r < 0); \\ \textit{Config.8.29} \Leftrightarrow N_{14} < 0, N_{15} > 0 \ (0 < r < 1); \\ \textit{Config.8.30} \ \Leftrightarrow \ N_{14} > 0 \ (r > 1); \end{cases}$$

- *Config. 8.31, 8.32* $\Leftrightarrow$ $N_2 = N_3 = 0,\ N_{17} = N_{18} = 0,\ N_{10}N_{16} \neq 0$:
$$\begin{cases} \dot{x} = x(r + x^2), \\ \dot{y} = x - 2ry,\ r \in \{-1, 1\}; \end{cases} \quad \begin{cases} \textit{Config.8.31} \ \Leftrightarrow \ N_{10} < 0 \ (r = -1); \\ \textit{Config.8.33} \Leftrightarrow \ N_{10} > 0,\ (r = 1); \end{cases}$$

- *Config. 8.33* $\Leftrightarrow$ $N_2 = N_3 = 0,\ N_{10} = N_{17} = N_{18} = 0, N_{16} \neq 0$: $\begin{cases} \dot{x} = x^3, \\ \dot{y} = 1 + x; \end{cases}$

- *Config.8.34 - 8.38* $\Leftrightarrow$ $N_2 = N_3 = 0,\ N_{16} = N_{19} = 0,\ N_{18} \neq 0$:
$$\begin{cases} \dot{x} = x(r + x + x^2), \\ \dot{y} = 1 + ry, (9r - 2) \neq 0; \end{cases} \quad \begin{cases} \textit{Config. 8.34} \ \Leftrightarrow \ N_{21} < 0 \ (r < 0); \\ \textit{Config. 8.35} \ \Leftrightarrow \ N_{20} > 0, N_{21} > 0 \ (0 < r < 1/4); \\ \textit{Config. 8.36} \ \Leftrightarrow \ N_{20} = 0 \ (r = 1/4); \\ \textit{Config. 8.37} \ \Leftrightarrow \ N_{20} < 0 \ (r > 1/4); \\ \textit{Config. 8.38} \ \Leftrightarrow \ N_{21} = 0 \ (r = 0). \end{cases}$$

*Remark* 2. If in a configuration an invariant straight line has multiplicity $k > 1$, then the number $k$ appears near the corresponding straight line and this line is in bold face. Real invariant straight lines are represented by continuous lines, whereas complex invariant straight lines are represented by dashed lines. We indicate next to the real singular points of the system, located on the invariant straight lines, their corresponding multiplicities.

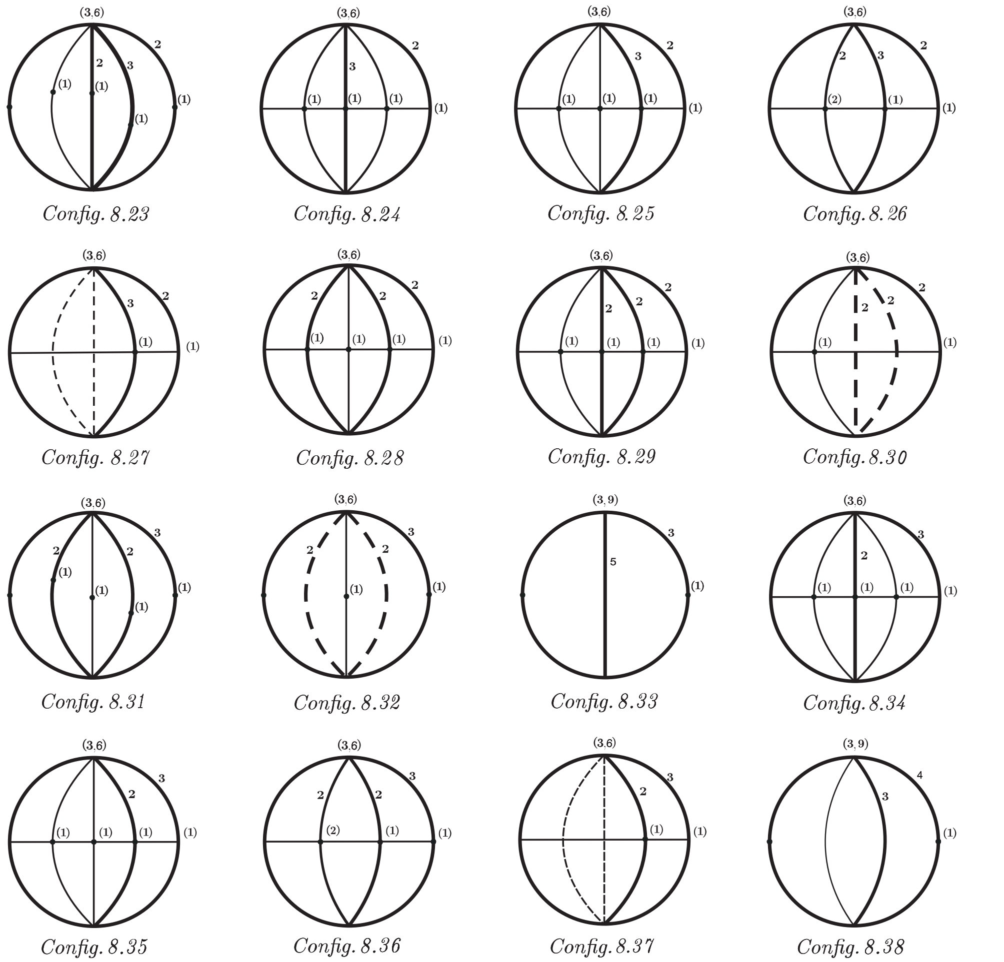



Figure 1. **The configurations of invariant straight lines of cubic systems** (2)

## 2 Preliminaries

Consider real cubic systems, i.e. systems of the form:

$$\begin{aligned} \dot{x} &= p_0 + p_1(x,y) + p_2(x,y) + p_3(x,y) \equiv p(x,y), \\ \dot{y} &= q_0 + q_1(x,y) + q_2(x,y) + q_3(x,y) \equiv q(x,y) \end{aligned} \tag{3}$$

with real coefficients and variables $x$ and $y$. The polynomials $p_i$ and $q_i$ $(i = 0,1,2,3)$ are homogeneous polynomials of degree $i$ in $x$ and $y$:

$$\begin{aligned} &p_0 = a_{00}, \quad p_3(x,y) = a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3, \\ &p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ &q_0 = b_{00}, \quad q_3(x,y) = b_{30}x^3 + 3b_{21}x^2y + 3b_{12}xy^2 + b_{03}y^3, \\ &q_1(x,y) = b_{10}x + b_{01}y, \quad q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{aligned}$$

Let $a = (a_{00}, a_{10}, a_{01}, \dots, a_{03}, b_{00}, b_{10}, b_{01}, \dots, b_{03})$ be the 20-tuple of the coefficients of systems (3) and denote $\mathbb{R}[a, x, y] = \mathbb{R}[a_{00}, a_{10}, a_{01}, \dots, a_{03}, b_{00}, b_{10}, b_{01}, \dots, b_{03}, x, y]$.

## 2.1 The main invariant polynomials associated to configurations of invariant lines

It is known that on the set **CS** of all cubic differential systems (3) the group $Aff(2, \mathbb{R})$ of affine transformations acts on the plane [27]. For every subgroup $G \subseteq Aff(2, \mathbb{R})$ we have an induced action of $G$ on **CS**. We can identify the set **CS** of systems (3) with a subset of $\mathbb{R}^{20}$ via the map $\mathbf{CS} \longrightarrow \mathbb{R}^{20}$ which associates to each system (3) the 20-tuple $a = (a_{00}, a_{10}, a_{01}, \dots, a_{03},\ b_{00}, b_{10}, b_{01}, \dots, b_{03})$ of its coefficients.

For the definitions of an affine or $GL$-comitant or invariant as well as for the definition of a $T$-comitant and $CT$-comitant we refer the reader to [27]. Here we shall only construct the necessary $T-$ and $CT-$comitants associated to configurations of invariant lines for the family of cubic systems mentioned in the statement of Main Theorem.

Let us consider the polynomials

$$\begin{aligned} C_i(a, x, y) &= y p_i(a, x, y) - x q_i(a, x, y) \in \mathbb{R}[a, x, y],\ i = 0, 1, 2, 3, \\ D_i(a, x, y) &= \frac{\partial}{\partial x} p_i(a, x, y) + \frac{\partial}{\partial y} q_i(a, x, y) \in \mathbb{R}[a, x, y],\ i = 1, 2, 3, \end{aligned}$$

which in fact are $GL$-comitants, see [32]. Let $f,\ g \in \mathbb{R}[a, x, y]$ and

$$(f, g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h}\ \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

$(f, g)^{(k)} \in \mathbb{R}[a, x, y]$ is called *the transvectant* of index $k$ of $(f, g)$ (cf. [12],[19])

We apply a translation $x = x' + x_0,\ y = y' + y_0$ to the polynomials $p(a, x, y)$ and $q(a, x, y)$ and we obtain $\tilde{p}(\tilde{a}(a, x_0, y_0), x', y') = p(a, x' + x_0, y' + y_0)$, $\tilde{q}(\tilde{a}(a, x_0, y_0), x', y') = q(a, x' + x_0, y' + y_0)$. Let us construct the following polynomials:

$$\begin{aligned} &\Omega_i(a, x_0, y_0) \equiv \operatorname{Res}_{x'} \Big( C_i\big(\tilde{a}(a, x_0, y_0), x', y'\big), C_0\big(\tilde{a}(a, x_0, y_0), x', y'\big) \Big) / (y')^{i+1}, \\ &\tilde{\mathcal{G}}_i(a, x, y) = \Omega_i(a, x_0, y_0)|_{\{x_0 = x,\ y_0 = y\}} \in \mathbb{R}[a, x, y] \quad (i = 1, 2, 3). \end{aligned}$$

*Remark* 3. We note that the constructed polynomials $\tilde{\mathcal{G}}_1(a, x, y)$, $\tilde{\mathcal{G}}_2(a, x, y)$ and $\tilde{\mathcal{G}}_3(a, x, y)$ are affine comitants of systems (3) and are homogeneous polynomials in the coefficients $a_{00}, \dots, b_{02}$ and non-homogeneous in $x, y$ and

$$\begin{array}{lll} \deg_a \mathcal{G}_1 = 3, & \deg_a \mathcal{G}_2 = 4, & \deg_a \mathcal{G}_3 = 5, \\ \deg_{(x,y)} \mathcal{G}_1 = 8, & \deg_{(x,y)} \mathcal{G}_2 = 10, & \deg_{(x,y)} \mathcal{G}_3 = 12. \end{array}$$

*Notation* 1. *Let* $\mathcal{G}_i(a,X,Y,Z)$ $(i=1,2,3)$ *be the homogenization of* $\tilde{\mathcal{G}}_i(a,x,y)$*, i.e.*

$$\mathcal{G}_1(a,X,Y,Z)=Z^8\tilde{\mathcal{G}}_1(a,X/Z,Y/Z),\quad \mathcal{G}_2(a,X,Y,Z)=Z^{10}\tilde{\mathcal{G}}_2(a,X/Z,Y/Z),$$
$$\mathcal{G}_3(a,X,Y,Z)=Z^{12}\tilde{\mathcal{G}}_3(a,X/Z,Y/Z),$$

*and* $\mathcal{H}(a,X,Y,Z) = \gcd\Big(\mathcal{G}_1(a,X,Y,Z),\ \mathcal{G}_2(a,X,Y,Z),\ \mathcal{G}_3(a,X,Y,Z)\Big)$ *in* $\mathbb{R}[a,X,Y,Z]$.

The geometrical meaning of the above defined affine comitants is given by the two following lemmas (see [16]):

**Lemma 1.** *The straight line* $f(x,y)\equiv ux+vy+w=0$, $u,v,w\in\mathbb{C}$, $(u,v)\neq(0,0)$ *is an invariant line for a cubic system* (3) *if and only if the polynomial* $f(x,y)$ *is a common factor of the polynomials* $\tilde{\mathcal{G}}_1(a,x,y)$, $\tilde{\mathcal{G}}_2(a,x,y)$ *and* $\tilde{\mathcal{G}}_3(a,x,y)$ *over* $\mathbb{C}$*, i.e.* $\tilde{\mathcal{G}}_i(a,x,y)=(ux+vy+w)\widetilde{W}_i(x,y)$ $(i=1,2,3)$*, where* $\widetilde{W}_i(x,y)\in\mathbb{C}[x,y]$.

**Lemma 2.** *Consider a cubic system* (3) *and let* $a\in\mathbb{R}^{20}$ *be its 20-tuple of coefficients.*

*1) If* $f(x,y)\equiv ux+vy+w=0$, $u,v,w\in\mathbb{C}$, $(u,v)\neq(0,0)$ *is an invariant straight line of multiplicity* $k$ *for this system then* $[f(x,y)]^k \mid \gcd(\tilde{\mathcal{G}}_1,\tilde{\mathcal{G}}_2,\tilde{\mathcal{G}}_3)$ *in* $\mathbb{C}[x,y]$*, i.e. there exist* $W_i(a,x,y)\in\mathbb{C}[x,y]$ $(i=1,2,3)$ *such that*

$$\tilde{\mathcal{G}}_i(a,x,y)=(ux+vy+w)^kW_i(a,x,y),\quad i=1,2,3. \tag{4}$$

*2) If the line* $l_\infty: Z=0$ *is of multiplicity* $k>1$ *then* $Z^{k-1}\mid\gcd(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3)$*, i.e. we have* $Z^{k-1}\mid H(a,X,Y,Z)$.

Consider the differential operator $\mathcal{L}=x\cdot\mathbf{L}_2-y\cdot\mathbf{L}_1$ constructed in [4] and acting on $\mathbb{R}[a,x,y]$, where

$$\begin{aligned}\mathbf{L}_1 =&3a_{00}\frac{\partial}{\partial a_{10}}+2a_{10}\frac{\partial}{\partial a_{20}}+a_{01}\frac{\partial}{\partial a_{11}}+\frac{1}{3}a_{02}\frac{\partial}{\partial a_{12}}+\frac{2}{3}a_{11}\frac{\partial}{\partial a_{21}}+a_{20}\frac{\partial}{\partial a_{30}}+\\ &3b_{00}\frac{\partial}{\partial b_{10}}+2b_{10}\frac{\partial}{\partial b_{20}}+b_{01}\frac{\partial}{\partial b_{11}}+\frac{1}{3}b_{02}\frac{\partial}{\partial b_{12}}+\frac{2}{3}b_{11}\frac{\partial}{\partial b_{21}}+b_{20}\frac{\partial}{\partial b_{30}},\\ \mathbf{L}_2 =&3a_{00}\frac{\partial}{\partial a_{01}}+2a_{01}\frac{\partial}{\partial a_{02}}+a_{10}\frac{\partial}{\partial a_{11}}+\frac{1}{3}a_{20}\frac{\partial}{\partial a_{21}}+\frac{2}{3}a_{11}\frac{\partial}{\partial a_{12}}+a_{02}\frac{\partial}{\partial a_{03}}+\\ &3b_{00}\frac{\partial}{\partial b_{01}}+2b_{01}\frac{\partial}{\partial b_{02}}+b_{10}\frac{\partial}{\partial b_{11}}+\frac{1}{3}b_{20}\frac{\partial}{\partial b_{21}}+\frac{2}{3}b_{11}\frac{\partial}{\partial b_{12}}+b_{02}\frac{\partial}{\partial b_{03}}.\end{aligned}$$

Using this operator and the affine invariant $\mu_0=\text{Resultant}_x\big(p_3(a,x,y),q_3(a,x,y)\big)/y^9$ we construct the following polynomials

$$\mu_i(a,x,y)=\frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\ \ i=1,..,9,$$

where $\mathcal{L}^{(i)}(\mu_0)=\mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$ and $\mathcal{L}^{(0)}(\mu_0)=\mu_0$.

These polynomials are in fact comitants of systems (3) with respect to the group $GL(2,\mathbb{R})$ (see [4]). The polynomial $\mu_i(a,x,y)$, $i\in\{0,1,\dots,9\}$ is homogeneous of degree 6 in the coefficients of systems (3) and homogeneous of degree $i$ in the variables $x$ and $y$. The geometrical meaning of these polynomial is revealed in the next lemma.

**Lemma 3** (see [3, 4]). *Assume that a cubic system $(S)$ with coefficients $\tilde{a}$ belongs to the family* (3). *Then:*

*(i) The total multiplicity of all finite singularities of this system equals $9-k$ if and only if for every $i \in \{0, 1, \dots, k-1\}$ we have $\mu_i(\tilde{a}, x, y) = 0$ in the ring $\mathbb{R}[x, y]$ and $\mu_k(\tilde{a}, x, y) \neq 0$. In this case the factorization $\mu_k(\tilde{a}, x, y) = \prod_{i=1}^{k}(u_i x - v_i y) \neq 0$ over $\mathbb{C}$ indicates the coordinates $[v_i : u_i : 0]$ of those finite singularities of the system $(S)$ which "have gone" to infinity. Moreover the number of distinct factors in this factorization is less than or equal to four (the maximum number of infinite singularities of a cubic system) and the multiplicity of each one of the factors $u_i x - v_i y$ gives us the number of the finite singularities of the system $(S)$ which have collapsed with the infinite singular point $[v_i : u_i : 0]$.*

*(ii) The system $(S)$ is degenerate (i.e. $\gcd(P, Q) \neq \mathrm{const}$) if and only if $\mu_i(\tilde{a}, x, y) = 0$ in $\mathbb{R}[x, y]$ for every $i = 0, 1, \dots, 9$.*

In order to define the needed invariant polynomials we first construct the following comitants of second degree with respect to the coefficients of the initial system:

$$
\begin{array}{lll}
S_1 = (C_0, C_1)^{(1)}, & S_{10} = (C_1, C_3)^{(1)}, & S_{19} = (C_2, D_3)^{(1)}, \\
S_2 = (C_0, C_2)^{(1)}, & S_{11} = (C_1, C_3)^{(2)}, & S_{20} = (C_2, D_3)^{(2)}, \\
S_3 = (C_0, D_2)^{(1)}, & S_{12} = (C_1, D_3)^{(1)}, & S_{21} = (D_2, C_3)^{(1)}, \\
S_4 = (C_0, C_3)^{(1)}, & S_{13} = (C_1, D_3)^{(2)}, & S_{22} = (D_2, D_3)^{(1)}, \\
S_5 = (C_0, D_3)^{(1)}, & S_{14} = (C_2, C_2)^{(2)}, & S_{23} = (C_3, C_3)^{(2)}, \\
S_6 = (C_1, C_1)^{(2)}, & S_{15} = (C_2, D_2)^{(1)}, & S_{24} = (C_3, C_3)^{(4)}, \\
S_7 = (C_1, C_2)^{(1)}, & S_{16} = (C_2, C_3)^{(1)}, & S_{25} = (C_3, D_3)^{(1)}, \\
S_8 = (C_1, C_2)^{(2)}, & S_{17} = (C_2, C_3)^{(2)}, & S_{26} = (C_3, D_3)^{(2)}, \\
S_9 = (C_1, D_2)^{(1)}, & S_{18} = (C_2, C_3)^{(3)}, & S_{27} = (D_3, D_3)^{(2)}.
\end{array}
$$

We shall use here the following invariant polynomials constructed in [16] to characterize the family of cubic systems possessing the maximal number of invariant straight lines:

$$
\begin{aligned}
\mathcal{D}_1(a) &= 6S_{24}^3 - \left[(C_3, S_{23})^{(4)}\right]^2, \ \mathcal{D}_2(a, x, y) = -S_{23}, \\
\mathcal{D}_3(a, x, y) &= (S_{23},\, S_{23})^{(2)} - 6C_3(C_3,\, S_{23})^{(4)}, \ \mathcal{D}_4(a, x, y) = (C_3, \mathcal{D}_2)^{(4)}, \\
\mathcal{V}_1(a, x, y) &= S_{23} + 2D_3^2, \ \mathcal{V}_2(a, x, y) = S_{26}, \ \mathcal{V}_3(a, x, y) = 6S_{25} - 3S_{23} - 2D_3^2, \\
\mathcal{V}_4(a, x, y) &= C_3\left[(C_3, S_{23})^{(4)} + 36\,(D_3, S_{26})^{(2)}\right], \\
\mathcal{V}_5(a, x, y) &= 6T_1(9A_5 - 7A_6) + 2T_2(4T_{16} - T_{17}) - 3T_3(3A_1 + 5A_2) + 3A_2T_4 + \\
&\quad + 36T_5^2 - 3T_{44}, \\
\mathcal{L}_1(a, x, y) &= 9C_2\,(S_{24} + 24S_{27}) - 12D_3\,(S_{20} + 8S_{22}) - 12\,(S_{16}, D_3)^{(2)} \\
&\quad - 3\,(S_{23}, C_2)^{(2)} - 16\,(S_{19}, C_3)^{(2)} + 12\,(5S_{20} + 24S_{22}, C_3)^{(1)},
\end{aligned}
$$

$$\begin{aligned}
\mathcal{L}_2(a,x,y) =\ & 32\,(13S_{19}+33S_{21},D_2)^{(1)}+84\,(9S_{11}-2S_{14},D_3)^{(1)}+\\
&+8D_2\,(12S_{22}+35S_{18}-73S_{20})-448\,(S_{18},C_2)^{(1)}-\\
&-56\,(S_{17},C_2)^{(2)}-63\,(S_{23},C_1)^{(2)}+756D_3S_{13}-1944D_1S_{26}+\\
&+112\,(S_{17},D_2)^{(1)}-378\,(S_{26},C_1)^{(1)}+9C_1\,(48S_{27}-35S_{24})\,,\\
\mathcal{U}_1(a) =\ & T_{31}-4T_{37},\\
\mathcal{U}_2(a,x,y) =\ & 6\,(T_{30}-3T_{32},T_{36})^{(1)}-3T_{30}\,(T_{32}+8T_{37})-\\
&-24T_{36}^2+2C_3\,(C_3,T_{30})^{(4)}+24D_3\,(D_3,T_{36})^{(1)}+24D_3^2T_{37}.\\
\mathcal{K}_2(a,x,y) =\ & T_{74},\quad \mathcal{K}_4(a,x,y)=T_{13}-2T_{11},\\
\mathcal{K}_1(a,x,y) =\ & \big(3223T_2^2T_{140}+2718T_4T_{140}-829T_2^2T_{141},T_{133}\big)^{(10)}/2,\\
\mathcal{K}_5(a,x,y) =\ & 45T_{42}-T_2T_{14}+2T_2T_{15}+12T_{36}+45T_{37}-45T_{38}+30T_{39},\\
\mathcal{K}_6(a,x,y) =\ & 4T_1T_8(2663T_{14}-8161T_{15})+6T_8(178T_{23}+70T_{24}+555T_{26})+\\
&+18T_9(30T_2T_8-488T_1T_{11}-119T_{21})+5T_2(25T_{136}+16T_{137})-\\
&-15T_1(25T_{140}-11T_{141})-165T_{142},\\
\mathcal{K}_8(a,x,y) =\ & 10A_4T_1-3T_2T_{15}+4T_{36}-8T_{37}.
\end{aligned}$$

However these invariant polynomials are not sufficient to characterize the cubic systems with invariant lines of the total multiplicity 8. So we construct here the following new invariant polynomials:

$$\begin{aligned}
N_1(a,x,y) =\ & S_{13},\quad N_2(a,x,y)=C_2D_3+3S_{16},\quad N_3(a,x,y)=T_9,\\
N_4(a,x,y) =\ & -S_{14}^2-2D_2^2(3S_{14}-8S_{15})-12D_3(S_{14},C_1)^{(1)}+\\
&+D_2(-48D_3S_9+16(S_{17},C_1)^{(1)}),\\
N_5(a,x,y) =\ & 36D_2D_3(S_8-S_9)+D_1(108D_2^2D_3-54D_3(S_{14}-8S_{15}))+\\
&+2S_{14}(S_{14}-22S_{15})-8D_2^2(3S_{14}+S_{15})-9D_3(S_{14},C_1)^{(1)}-16D_2^4,\\
N_6(a,x,y) =\ & 40D_3^2(15S_6-4S_3)-480D_2D_3S_9-20D_1D_3(S_{14}-4S_{15})+\\
&+160D_2^2S_{15}-35D_3(S_{14},C_1)^{(1)}+8\big((S_{23},C_2)^{(1)},C_0\big)^{(1)},\\
N_7(a,x,y) =\ & 18C_2D_2(9D_1D_3-S_{14})-2C_1D_3(8D_2^2-3S_{14}-74S_{15})-\\
&-432C_0D_3S_{21}+48S_7(8D_2D_3+S_{17})-51S_{10}S_{14}+\\
&+6S_{10}(12D_2^2+151S_{15})-162D_1D_2S_{16}+864D_3(S_{16},C_0)^{(1)},\\
N_8(a,x,y) =\ & -32D_3^2S_2-108D_1D_3S_{10}+108C_3D_1S_{11}-18C_1D_3S_{11}-\\
&-27S_{10}S_{11}+4C_0D_3(9D_2D_3+4S_{17})+108S_4S_{21},\\
N_9(a,x,y) =\ & 11S_{14}^2-2592D_1^2S_{25}+88D_2(S_{14},C_2)^{(1)}-\\
&-16D_1D_3(16D_2^2+19S_{14}-152S_{15})-8D_2^2(7S_{14}+32S_{15}),\\
N_{10}(a,x,y) =\ & -24D_1D_3+4D_2^2+S_{14}-8S_{15},\\
N_{11}(a,x,y) =\ & S_{14}^2+8D_1D_3[2D_2^2-(S_{14}-8S_{15})]-2D_2^2(5S_{14}-8S_{15})+\\
&+8D_2(S_{14},C_2)^{(1)},
\end{aligned}$$

$$
\begin{aligned}
N_{12}(a,x,y) =\ & 135D_1D_3[8D_2^2-(S_{14}-20S_{15})]-5D_2^2(39S_{14}-32S_{15})+\\
&+5S_{14}^2-160D_2^4-1620D_3^2S_3+85D_2(S_{14},C_2)^{(1)}+\\
&+81\big((S_{23},C_2)^{(1)},C_0\big)^{(1)},\\
N_{13}(a,x,y) =\ & 2(136D_3^2S_2-126D_2D_3S_4+60D_2D_3S_7+63S_{10}S_{11})-\\
&-18C_3D_1(S_{14}-28S_{15})-12C_1D_3(7S_{11}-20S_{15})+\\
&+4C_0D_3(21D_2D_3+17S_{17})+3C_2(S_{14},C_2)^{(1)}-192C_2D_2S_{15},\\
N_{14}(a,x,y) =\ & -6D_1D_3-15S_{12}+2S_{14}+4S_{15},\\
N_{15}(a,x,y) =\ & 216D_1D_3(63S_{11}-104D_2^2-136S_{15})+4536D_3^2S_6+\\
&+4096D_2^4+120S_{14}^2+992D_2(S_{14},C_2)^{(1)}+\\
&+135D_3\big[28(S_{17},C_0)^{(1)}+5(S_{14},C_1)^{(1)}\big],\\
N_{16}(a,x,y) =\ & 2C_1D_3+3S_{10},\ \ N_{17}(a,x,y)=\ 6D_1D_3-2D_2^2-(C_3,C_1)^{(2)},\\
N_{18}(a,x,y) =\ & 2D_2^3-6D_1D_2D_3-12D_3S_5+3D_3S_8,\\
N_{19}(a,x,y) =\ & C_1D_3(18D_1^2-S_6)-3C_0D_3(4D_1D_2+6S_5-3S_8)+\\
&+6C_2D_1S_82D_2(9D_3S_1-4D_2S_2)+2D_1(12D_3S_2-9C_3S_6)+\\
&+4C_0D_2^3-18D_3(S_4,C_0)^{(1)},\\
N_{20}(a,x,y) =\ & 3D_2^4-8D_1D_2^2D_3-8D_3^2S_6-16D_1D_3S_{11}+16D_2D_3S_9,\\
N_{21}(a,x,y) =\ & 2D_1D_2^2D_3-4D_3^2S_6+D_2D_3S_8+D_1(S_{23},C_1)^{(1)}
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 =\ & S_{24}/288,\quad A_2=S_{27}/72,\quad A_3=(72D_1A_2+(S_{22},D_2)^{(1)}/24,\\
A_4 =\ & [9D_1(S_{24}-288A_2)+4(9S_{11}-2S_{14},D3)^{(2)}+8(3S_{18}-S_{20}-4S_{22},D_2)^{(1)}]/2^73^3
\end{aligned}
$$

are affine invariants, whereas the polynomials

$$
\begin{aligned}
T_1 =\ & C_3,\quad T_2=\ D_3,\quad T_3=\ S_{23}/18,\quad T_4=\ S_{25}/6,\quad T_5=\ S_{26}/72,\\
T_6 =\ & \big[2C_3(2D_2^2-S_{14}+8S_{15})-3C_1M_1-2C_2M_2\big]/2^4/3^2,\\
T_8 =\ & \big[5D_2(D_3^2+27T_3-18T_4)+20D_3S_{19}+12\big(S_{16},D_3\big)^{(1)}-8D_3S_{17}\big]/5/2^5/3^3,\\
T_9 =\ & \big[9D_1M_1+2D_2(D_2D_3-3S_{17}-S_{19}-9S_{21})+18\big(S_{15},C_3\big)^{(1)}-\\
&-6C_2(2S_{20}-3S_{22})+18C_1S_{26}+2D_3S_{14}\big]/2^4/3^3,\\
T_{11} =\ & \big[6\big(M_1,D_2\big)^{(1)}-\big(M_1,C_2\big)^{(2)}-12\big(S_{26},C_2\big)^{(1)}+12D_2S_{26}+\\
&+432(A_1-5A_2)C_2\big]/2^7/3^4,\\
T_{13} =\ & \big[27(T_3,C_2)^{(2)}-18(T_4,C_2)^{(2)}+48D_3S_{22}-216(T_4,D_2)^{(1)}+36D_2S_{26}-\\
&-1296C_2A_1-7344C_2A_2+(D_3^2,C_2)^{(2)}\big]/2^7/3^4,\\
T_{14} =\ & \big[\big(8S_{19}+9S_{21},D_2\big)^{(1)}-D_2(8S_{20}+3S_{22})+18D_1S_{26}+1296C_1A_2\big]/2^4/3^3,
\end{aligned}
$$

$$
\begin{aligned}
T_{15} = {} & 8\big(9S_{19}+2S_{21}, D_2\big)^{(1)}+3\big(M_1, C_1\big)^{(2)}-4\big(S_{17}, C_2\big)^{(2)}+\\
& +4\big(S_{14}-17S_{15}, D_3\big)^{(1)}-8\big(S_{14}+S_{15}, C_3\big)^{(2)}+432C_1(5A_1+11A_2)+\\
& +36D_1S_{26}-4D_2(S_{18}+4S_{22})\big]/2^6/3^3,\\
T_{21} = {} & \big(T_8, C_3\big)^{(1)}, \quad T_{23} = \big(T_6, C_3\big)^{(2)}/6, \quad T_{24} = \big(T_6, D_3\big)^{(1)}/6,\\
T_{26} = {} & \big(T_9, C_3\big)^{(1)}/4, \quad T_{30} = \big(T_{11}, C_3\big)^{(1)}, \quad T_{31} = \big(T_8, C_3\big)^{(2)}/24,\\
T_{32} = {} & \big(T_8, D_3\big)^{(1)}/6, \quad T_{36} = \big(T_6, D_3\big)^{(2)}/12, \quad T_{37} = \big(T_9, C_3\big)^{(2)}/12,\\
T_{38} = {} & \big(T_9, D_3\big)^{(1)}/12, \quad T_{39} = \big(T_6, C_3\big)^{(3)}/2^4/3^2, \quad T_{42} = \big(T_{14}, C_3\big)^{(1)}/2,\\
T_{44} = {} & \big((S_{23}, C_3)^{(1)}, D_3\big)^{(2)}/5/2^6/3^3,\\
T_{74} = {} & \big[27C_0M_1^2 - C_1\big(2^83^5T_{11}C_3 + 3M_1M_2\big) + 2^83^4T_{11}C_2^2+\\
& + C_2M_1(8D_2^2 + 54D_1D_3 - 27S_{11} + 27S_{12} - 4S_{14} + 32S_{15})-\\
& - 54D_1M_1S_{16} - 54C_3M_1(2D_1D_2 - S_8 + 2S_9) - 2^63^2T_6M_2\big]/2^8/3^4,\\
T_{133} = {} & (T_{74}, C_3)^{(1)}, \quad T_{136} = \big(T_{74}, C_3\big)^{(2)}/24, \quad T_{137} = \big(T_{74}, D_3\big)^{(1)}/6,\\
T_{140} = {} & \big(T_{74}, D_3\big)^{(2)}/12, \; T_{141} = \big(T_{74}, C_3\big)^{(3)}/36, \; T_{142} = \big((T_{74}, C_3)^{(2)}, C_3\big)^{(1)}/72
\end{aligned}
$$

where $M_1 = 9T_3-18T_4-D_3^2,\ M_2 = 2D_2D_3-S_{17}+2S_{19}-6S_{21}$ and $T_i,\ i = 1, ..., 142$ are $T$-comitants of cubic systems (3) (see for details [27]). We note that these invariant polynomials are the elements of the polynomial basis of $T$-comitants up to degree six constructed by Iu. Calin [8].

## 2.2 Preliminary results

In order to determine the degree of the common factor of the polynomials $\tilde{\mathcal{G}}_i(a, x, y)$ for $i = 1, 2, 3$, we shall use the notion of the $k^{th}$ *subresultant* of two polynomials with respect to a given indeterminate (see for instance [14, 19]).

Following [16] we consider two polynomials $f(z) = a_0z^n+a_1z^{n-1}+\cdots+a_n,\ g(z) = b_0z^m + b_1z^{m-1} + \cdots + b_m$, in the variable $z$ of degree $n$ and $m$, respectively. Thus the $k$–th *subresultant* with respect to variable $z$ of the two polynomials $f(z)$ and $g(z)$ will be denoted by $R_z^{(k)}(f, g)$.

We say that the $k$–th *subresultant* with respect to variable $z$ of the two polynomials $f(z)$ and $g(z)$ is the $(m + n - 2k) \times (m + n - 2k)$ determinant

$$
R_z^{(k)}(f, g) = \begin{array}{l} \left| \begin{array}{llll} a_0a_1 & a_2\dots & \dots a_{m+n-2k-1} \\ \;0a_0 & a_1\dots & \dots a_{m+n-2k-2} \\ \;00 & a_0\dots & \dots a_{m+n-2k-3} \\ \dots\dots & \dots\dots & \dots\dots\ \dots\ \dots \\ \;00 & b_0\dots & \dots b_{m+n-2k-3} \\ \;0b_0 & b_1\dots & \dots b_{m+n-2k-2} \\ b_0b_1 & b_2\dots & \dots b_{m+n-2k-1} \end{array} \right| \begin{array}{l} \left.\vphantom{\begin{array}{l}a\\a\\a\end{array}}\right\}(m-k)-times \\ \\ \left.\vphantom{\begin{array}{l}a\\a\\a\end{array}}\right\}(n-k)-times \end{array} \end{array} \qquad (5)
$$

in which there are $m-k$ rows of $a$'s and $n-k$ rows of $b$'s, and $a_i = 0$ for $i > n$, and $b_j = 0$ for $j > m$.

For $k = 0$ we obtain the standard resultant of two polynomials. In other words we can say that the $k$–th subresultant with respect to the variable $z$ of the two polynomials $f(z)$ and $g(z)$ can be obtained by deleting the first and the last $k$ rows and the first and the last $k$ columns from its resultant written in the form (5) when $k = 0$.

The geometrical meaning of the subresultant is based on the following lemma.

**Lemma 4** (see [14, 19])**.** *Polynomials $f(z)$ and $g(z)$ have precisely $k$ roots in common (considering their multiplicities) if and only if the following conditions hold:*

$$R_z^{(0)}(f,g) = R_z^{(1)}(f,g) = R_z^{(2)}(f,g) = \cdots = R_z^{(k-1)}(f,g) = 0 \neq R_z^{(k)}(f,g).$$

For the polynomials in more than one variables it is easy to deduce from Lemma 4 the following result.

**Lemma 5.** *Two polynomials $\tilde{f}(x_1, x_2, ..., x_n)$ and $\tilde{g}(x_1, x_2, ..., x_n)$ have a common factor of degree $k$ with respect to the variable $x_j$ if and only if the following conditions are satisfied:*

$$R_{x_j}^{(0)}(\tilde{f},\tilde{g}) = R_{x_j}^{(1)}(\tilde{f},\tilde{g}) = R_{x_j}^{(2)}(\tilde{f},\tilde{g}) = \cdots = R_{x_j}^{(k-1)}(\tilde{f},\tilde{g}) = 0 \neq R_{x_j}^{(k)}(\tilde{f},\tilde{g}),$$

*where $R_{x_j}^{(i)}(\tilde{f},\tilde{g}) = 0$ in $\mathbb{R}[x_1, \ldots x_{j-1}, x_{j+1}, \ldots, x_n]$.*

In paper [16] 23 configurations of invariant lines (one more configuration is constructed in [5]) are determined in the case, when the total multiplicity of these lines (including the line at infinity) equals nine. For this purpose in [16] the authors proved some lemmas concerning the number of triplets and/or couples of parallel invariant straight lines which could have a cubic system. In [6] these results have been completed.

**Theorem 1** (see [6])**.** *If a cubic system* (3) *possesses a given number of triplets or/and couples of invariant parallel lines real or/and complex, then the following conditions are satisfied, respectively:*

| | | |
|---|---|---|
| *(i)* | *2 triplets* | $\Rightarrow \ \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$; |
| *(ii)* | *1 triplet and 2 couples* | $\Rightarrow \ \mathcal{V}_3 = \mathcal{V}_4 = \mathcal{U}_2 = 0$; |
| *(iii)* | *1 triplet and 1 couple* | $\Rightarrow \ \mathcal{V}_4 = \mathcal{V}_5 = \mathcal{U}_2 = 0$; |
| *(iv)* | *one triplet* | $\Rightarrow \ \mathcal{V}_4 = \mathcal{U}_2 = 0$; |
| *(v)* | *3 couples* | $\Rightarrow \ \mathcal{V}_3 = 0$; |
| *(vi)* | *2 couples* | $\Rightarrow \ \mathcal{V}_5 = 0$. |

In papers [6] and [7] all the possible configurations of invariant straight lines of total multiplicity 8, including the line at infinity with its own multiplicity are determined for cubic systems with at least three distinct infinite singularities. In particular the next result is obtained.

**Lemma 6** (see [6]). *A cubic system with four distinct infinite singularities could not possess configuration of invariant lines of type* $(3,2,2)$. *And it possesses a configuration or potential configuration of a given type if and only if the following conditions are satisfied, respectively*

$$\begin{array}{ccl}(3,3,1) & \Leftrightarrow & \mathcal{V}_1=\mathcal{V}_2=\mathcal{L}_1=\mathcal{L}_2=\mathcal{K}_1=0,\ \mathcal{K}_2\neq 0;\\ (3,2,1,1) & \Leftrightarrow & \mathcal{V}_5=\mathcal{U}_2=\mathcal{K}_4=\mathcal{K}_5=\mathcal{K}_6=0,\ \ \mathcal{D}_4\neq 0;\\ (2,2,2,1) & \Leftrightarrow & \mathcal{V}_3=\mathcal{K}_4=\mathcal{K}_2=\mathcal{K}_8=0,\ \ \mathcal{D}_4\neq 0.\end{array}$$

Let $L(x,y)=Ux+Vy+W=0$ be an invariant straight line of the family of cubic systems (3). Then, we have

$$UP(x,y)+VQ(x,y)=(Ux+Vy+W)(Ax^2+2Bxy+Cy^2+Dx+Ey+F),$$

and this identity provides the following 10 relations:

$$\begin{aligned}Eq_1=&(a_{30}-A)U+b_{30}V=0,\ Eq_2=(3a_{21}-2B)U+(3b_{21}-A)V=0,\\ Eq_3=&(3a_{12}-C)U+(3b_{12}-2B)V=0,\ Eq_4=(a_{03}-C)U+b_{03}V=0,\\ Eq_5=&(a_{20}-D)U+b_{20}V-AW=0,\\ Eq_6=&(2a_{11}-E)U+(2b_{11}-D)V-2BW=0,\\ Eq_7=&a_{22}U+(b_{22}-E)V-CW=0,\ Eq_8=(a_{10}-F)U+b_{10}V-DW=0,\\ Eq_9=&a_{01}U+(b_{01}-F)V-EW=0,\ Eq_{10}=a_{00}U+b_{00}V-FW=0.\end{aligned}\tag{6}$$

As it was mentioned earlier, the infinite singularities (real or complex) of systems (3) are determined by the linear factors of the polynomial $C_3$. So in the case of two distinct infinite singularities they are determined either by one triple and one simple real or two double real (or complex) factors of the polynomial $C_3(x,y)$. We consider here the first case.

**Lemma 7** (see [20]). *A cubic system* (3) *possesses the infinite singularities determined by one triple and one simple factors of the invariant polynomial* $C_3(x,y)$ *if and only if the conditions* $\mathcal{D}_1=\mathcal{D}_3=\mathcal{D}_4=0,\ \mathcal{D}_2\neq 0$ *hold. Moreover the cubic homogeneities of this system could be brought via a linear transformation to the canonical form*

$$\begin{aligned}x'=&(u+1)x^3+vx^2y+rxy^2,\\ y'=&ux^2y+vxy^2+ry^3,\quad \textit{with}\quad C_3=x^3y.\end{aligned}\tag{7}$$

## 3 The proof of the Main Theorem

Assume that a cubic system possesses two distinct infinite singularities which are determined by one simple and one triple real factors of the polynomial $C_3$. Then considering Lemma 7 we obtain that systems (3) via a linear transformation become:

$$\begin{aligned}x'&=p_0+p_1(x,y)+p_2(x,y)+(u+1)x^3+vx^2y+rxy^2,\\ y'&=q_0+q_1(x,y)+q_2(x,y)+ux^2y+vxy^2+ry^3\end{aligned}\tag{8}$$

with $C_3 = x^3y$. Hence, the infinite singular points are located at the "ends" of the following straight lines: $x = 0$ and $y = 0$.

The proof of the Main Theorem proceeds in 4 steps.

First we construct the cubic homogeneous parts $(\tilde{P}_3, \tilde{Q}_3)$ of systems for which the corresponding necessary conditions provided by Theorem 1 in order to have the given number of triplets or/and couples of invariant parallel lines in the respective directions are satisfied.

Secondly, taking cubic systems $\dot{x} = \tilde{P}_3,\ \dot{y} = \tilde{Q}_3$ we add all quadratic, linear and constant terms and using the equations (6) we determine these terms in order to get the needed number of invariant lines in the needed configuration. Thus the second step ends with the construction of the canonical systems possessing the needed configuration.

The third step consists in the determination of the affine invariant conditions necessary and sufficient for a cubic system to belong to the family of systems (constructed at the second step) which possess the corresponding configuration of invariant lines.

And finally, in the case of the existence of multiply invariant lines in a potential configuration we construct the corresponding perturbed systems possessing 8 distinct invariant lines (including the line at infinity).

## 3.1 Construction of the corresponding cubic homogeneities

In what follows we construct the cubic homogeneous parts of systems (8) for each one of the possible configurations mentioned in Lemma 6.

*a) The case of the configuration* $(3, 3, 1)$. In this case we have two triplets of parallel invariant straight lines and according to Theorem 1 the condition $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$ is necessary for systems (8). A straightforward computation of the value of $\mathcal{V}_1$ provides $\mathcal{V}_1 = 16\sum_{j=0}^{4}\mathcal{V}_{1j}x^{4-j}y^j$, where

$$\begin{aligned} &\mathcal{V}_{10} = u(2u+3), \quad \mathcal{V}_{12} = 4ru + 3r + 2v^2, \\ &\mathcal{V}_{11} = v(4u+3), \quad \mathcal{V}_{13} = 4vr, \quad \mathcal{V}_{14} = 2r^2. \end{aligned}$$

Therefore from $\mathcal{V}_1 = 0$ it results $v = r = 0$ and $u(2u + 3) = 0$, and we consider two subcases: $u = 0$ and $u = -3/2$. For $u = 0$ we get the cubic homogeneous system:

$$\dot{x} = x^3, \quad \dot{y} = 0 \tag{9}$$

whereas for $u = -3/2$, after the time rescaling $t \to -2t$, we have

$$\dot{x} = x^3, \quad \dot{y} = 3x^2y. \tag{10}$$

It has to be underlined that for systems (9) and (10) the relation $\mathcal{V}_2 = \mathcal{U}_1 = 0$ holds.

*b) The case of the configuration* $(3,2,1,1)$. According to Theorem 1, if a cubic system possesses 7 invariant straight lines in the configuration $(3,2,1,1)$, then necessarily the conditions $\mathcal{V}_4=\mathcal{V}_5=\mathcal{U}_2=0$ hold.

We consider again systems (8). A straightforward computation of the value of $\mathcal{V}_5$ yields: $\mathcal{V}_5=\dfrac{9}{32}\sum\limits_{j=0}^{4}\mathcal{V}_{5j}x^{4-j}y^j$, where

$$\begin{aligned}&\mathcal{V}_{50}=-u(3r+ru-v^2),\quad \mathcal{V}_{52}=6r^2u,\\&\mathcal{V}_{51}=4ruv,\quad \mathcal{V}_{53}=0,\quad \mathcal{V}_{54}=-r^3.\end{aligned}$$

Hence $r=0$ which gives $\mathcal{V}_4=0$ and $\mathcal{U}_2=-12288v^2x^2(ux+vy)^2$. So the condition $\mathcal{U}_2=0$ is equivalent to $v=0$ and in this case we have $\mathcal{V}_5=\mathcal{V}_4=\mathcal{U}_2=0$. As a result we get the family of systems

$$\dot{x}=(u+1)x^3,\quad \dot{y}=ux^2y \tag{11}$$

if $u\neq 0$, whereas if $u=0$ we arrive at system (9).

*c) The case of the configuration* $(2,2,2,1)$. According to Theorem 1 if a cubic system possesses 7 invariant straight lines in the configuration (2, 2, 2, 1), then necessarily the condition $\mathcal{V}_3=0$ holds.

So we shall consider the family of systems (8) and we force the condition $\mathcal{V}_3=0$ to be satisfied. We have:

$$\mathcal{V}_{30}=u(3+u),\quad \mathcal{V}_{31}=2uv,\quad \mathcal{V}_{32}=-3r+2ru+v^2,\quad \mathcal{V}_{33}=2rv,\quad \mathcal{V}_{34}=r^2$$

where $\mathcal{V}_{3j}$ are the elements of $\mathcal{V}_3=-32\sum\limits_{j=0}^{4}\mathcal{V}_{3j}x^{4-j}y^j$. So the condition $\mathcal{V}_{34}=0$ is equivalent to $r=0$ and, in consequence, $\mathcal{V}_{33}=\mathcal{V}_{34}=0$ and $\mathcal{V}_{32}=v^2$. So $v=0$ and the condition $\mathcal{V}_{30}=0$ gives $u(u+3)=0$. Therefore if $u=-3$ due to the time rescaling $t\to -t$ we arrive at the cubic homogeneities

$$\dot{x}=2x^3,\quad \dot{y}=3x^2y \tag{12}$$

whereas in the case $u=0$ we get system (9).

So we get three specific systems (9), (10) and (12) and one-parameter family of systems (11). As it can be observed, the first three systems belong to this family for some values of the parameter $u$: system (9) for $u=0$, system (10) for $u=-3/2$ (after the time rescaling $t\to -2t$) and system (12) for $u=-3$ (after the time rescaling $t\to -t$).

On the other hand for systems (11) we have $\mathcal{V}_1=16u(3+2u)x^4$, $\mathcal{V}_3=-32u(3+u)x^4$ and therefore we arrive at the next proposition.

**Proposition 1.** *Assume that for a cubic homogeneous system the conditions $\mathcal{D}_1=\mathcal{D}_3=\mathcal{D}_4=0$ and $\mathcal{D}_2\neq 0$ hold. Then this system can be brought to one of the canonical systems indicated below if and only if the corresponding conditions are satisfied, respectively:*

*(i)* $\mathcal{V}_1 = \mathcal{V}_3 = 0, \Rightarrow$ *system* (9), *(ii)* $\mathcal{V}_1 = 0,\ \mathcal{V}_3 \neq 0 \Rightarrow$ *system* (10),

*(iii)* $\mathcal{V}_1 \neq 0,\ \mathcal{V}_3 = 0 \Rightarrow$ *system* (12), *(iv)*$\mathcal{V}_1\mathcal{V}_3 \neq 0,\ \mathcal{V}_5 = \mathcal{U}_2 = 0 \Rightarrow$ *system* (11).

Thus for the further investigation four different homogeneous systems remain: (9), (10), (11) and (12). However in this article, we will consider only the cubic systems with cubic homogeneities of the form (9), as in the statement of the Main Theorem we assume the additional condition $\mathcal{V}_1 = \mathcal{V}_3 = 0$.

We observe that if for perturbed systems some condition $K(x, y) = 0$ holds, where $K(x, y)$ is an invariant polynomial, then this condition must hold also for the initial (unperturbed) systems. So considering Lemma 6 we arrive at the next remark.

*Remark* 4. Assume that a cubic system with two distinct infinite singularities possesses a potential configuration of a given type. Then for this system the following conditions must be satisfied, respectively:

$$\begin{array}{lll}
(a_1) & (\,3,\,3,\,1\,) & \Rightarrow\ \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{K}_1 = 0;\\
(a_2) & (\,3,\,2,\,1,\,1\,) & \Rightarrow\ \mathcal{V}_5 = \mathcal{U}_2 = \mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = 0;\\
(a_3) & (\,2,\,2,\,2,\,1\,) & \Rightarrow\ \mathcal{V}_3 = \mathcal{K}_4 = \mathcal{K}_2 = \mathcal{K}_8 = 0.
\end{array}$$

## 3.2 Construction of the configurations and of the corresponding normal forms

In this case, considering (8) and (9) via a translation of the origin of coordinates we can consider $g = 0$ and hence we get the cubic systems

$$\dot{x} = a + cx + x^3 + dy + 2hxy + ky^2, \quad \dot{y} = b + ex + lx^2 + fy + 2mxy + ny^2 \qquad (13)$$

for which we have $H(X, Y, Z) = Z$ (see Notation 1).

Now we force the necessary conditions given in Remark 4 which correspond to each type of configuration. We claim that if any of the conditions $(a_1)$, $(a_2)$ or $(a_3)$ are satisfied for a system (13) then $k = h = n = 0$ and this condition is equivalent to $\mathcal{K}_5 = 0$. We divide the proof of this claim in three subcases defined by $(a_1)$–$(a_3)$.

$(a_1)$. For systems (13) we calculate: $\mathcal{L}_1 = 0$ and

$$\begin{aligned}
&\text{Coefficient}[\mathcal{L}_2, xy] = -20736(12h^2 + 7km - 6hn + 3n^2) = 0,\\
&\text{Coefficient}[\mathcal{K}_1, y^2] = 3967 \cdot 2^{18}3^9 5^4 7^3 19 k^6 = 0.
\end{aligned}$$

Therefore we get $k = 0$ and as the discriminant of the binary form $4h^2 - 2hn + n^2$ is negative we obtain $h = n = 0$ (and this implies $\mathcal{L}_2 = \mathcal{K}_1 = 0$).

$(a_2)$. In the same manner in the case of the configuration $(3, 2, 1, 1)$ we determine $\mathcal{K}_4 = \mathcal{K}_6 = 0$ and

$$\mathcal{K}_5 = -180m(h-n)x^4 + 60(4h^2 - 3km - 2hn + n^2)x^3y - 240k^2xy^3.$$

From $\mathcal{K}_5 = 0$ it results $k = 0$ and we get the same binary form $4h^2 - 2hn + n^2$ which leads to $h = n = 0$. Consequently $\mathcal{K}_5 = 0$ if and only if $k = h = n = 0$.

$(a_3)$. We calculate $\mathcal{K}_4 = 0$ and Coefficient$[\mathcal{L}_2, x^2y^7] = 2k^3 = 0$, i.e. $k = 0$. Then calculations yield

$$\text{Coefficient}[\mathcal{K}_2, x^5y^4] = -2n(h-n)^2 = 0, \ \text{Coefficient}[\mathcal{K}_8, x^3y] = 2(4h^2 + 14hn + n^2) = 0$$

and evidently we obtain $h = n = 0$ (then $\mathcal{K}_2 = \mathcal{K}_8 = 0$) and this completes the proof of the claim.

*Remark* 5. Since infinite singularities of systems (8) are located on the "ends" of the axis $x = 0$ and $y = 0$, the invariant affine lines must be either of the form $Ux + W = 0$ or $Vy + W = 0$. Therefore we can assume $U = 1$ and $V = 0$ (for the direction $x = 0$) and $U = 0$ and $V = 1$ (for the direction $y = 0$). In this case, considering $W$ as a parameter, six equations among (6) become linear with respect to the parameters $\{A, B, C, D, E, F\}$ (with the corresponding non–zero determinant) and we can determine their values, which annulate some of the equations (6). So in what follows we will examine only the non-zero equations containing the last parameter $W$.

Since for systems (8) the condition $k = h = n = 0$ is equivalent to $\mathcal{K}_5 = 0$ we assume this condition to be fulfilled.

We begin with the examination of the direction $x = 0$ $(U = 1,\ V = 0)$. So, considering (6) and Remark 5 for systems (13) we have: $Eq_9 = d,\ Eq_{10} = a - cW - W^3$. So in the direction $x = 0$ we could have three invariant lines (which could coincide) and this occurs if and only if $d = 0$. Thus we arrive at the family of systems

$$\dot{x} = a + cx + x^3, \quad \dot{y} = b + ex + lx^2 + fy + 2mxy \tag{14}$$

for which we calculate

$$H(X, Z) = Z(X^3 + cXZ^2 + aZ^3). \tag{15}$$

*Remark* 6. Any invariant line of the form $x + \alpha = 0$ (i.e. in the direction $x = 0$) of cubic systems (3) must be a factor of the polynomials $P(x, y)$, i.e. $(x + \alpha) \mid P(x, y)$.

Indeed, according to the definition, for an invariant line $ux + vy + w = 0$ we have $uP + vQ = (ux + vy + w)R(x, y)$, where the cofactor $R(x, y)$ generically is a polynomial of degree two. In our particular case (i.e. $u = 1, v = 0, w = \alpha$) we obtain $P(x) = (x + \alpha)R(x)$, which means that $(x + \alpha)$ divides $P(x)$.

This remark could be applied for any cubic systems when we examine the direction $x = 0$. Similarly, for an invariant line $y + \beta = 0$ in the the direction $y = 0$ it is necessary $(y + \beta = 0) \mid Q(x, y)$.

Considering systems (14) we calculate

$$
\begin{aligned}
\mathcal{G}_1/H =& lX^4 + X^3\big[4mY + 2(e-lm)Z)\big] + X^2\big[(3f-4m^2)YZ + (3b-cl-lf-\\
&+-2em)Z^2\big] + X\big[-4fmYZ^2 + (-2al-ef-2bm)Z^3\big] + (cf-f^2-\\
&-2am)YZ^3 + (bc-ae-bf)Z^4 \equiv F_1(X,Y,Z),\\
\mathcal{G}_2/H =& (X^3+cXZ^2+aZ^3)\Big\{2lX^3 + \big[X^2(6mY+(3e-2lm)Z\big] + X\big[(3f-\\
&-4m^2)YZ + (3b-cl-2em)Z^2\big] - 2fmYZ^2 + (-al-2bm)Z^3\Big] \equiv\\
&\equiv P^*(X,Z)F_2(X,Y,Z),\\
\mathcal{G}_3/H =& 24(lX^2+2mXY+eXZ+fYZ+bZ^2)(X^3+cXZ^2+aZ^3)^2 \equiv\\
&\equiv 24Q^*(X,Y,Z)\,[P^*(X,Z)]^2,
\end{aligned}
\tag{16}
$$

where $P^*(X,Z)$ and $Q^*(X,Y,Z)$ are the homogenization of the polynomials $P(x)$ and $Q(x,y)$ of systems (14). It is clear that these systems are degenerate if and only if the polynomials $P(x)$ and $Q(x,y)$ have a nonconstant common factor (depending on $x$) and this implies the existence of such a common factor (depending on $X$ and $Z$) of the polynomials $P^*(X,Z)$ and $Q^*(X,Y,Z)$. So for non-degenerate systems the condition

$$R_X^{(0)}\left(P^*(X,Z), Q^*(X,Y,Z)\right) \neq 0 \tag{17}$$

must hold. We have the next lemma.

**Lemma 8.** *For a non-degenerate system* (14) *the polynomial $P^*(X,Z)$ could not be a factor of $\mathcal{G}_1/H$, i.e. $P^*(X,Z)$ does not divide $F_1(X,Y,Z)$.*

*Proof.* Suppose the contrary that $P^*(X,Z)$ divides $F_1(X,Y,Z)$. Then considering the form of the polynomial $P^*(X,Z)$ (which contains the term $X^3$) by Lemma 5 the following conditions are necessary and sufficient: $R_X^{(0)}(F_1,P^*) = R_X^{(1)}(F_1,P^*) = R_X^{(2)}(F_1,P^*) = 0$. We calculate $R_X^{(2)}(F_1,P^*) = \big[(3f-4m^2)Y + (3b-2cl-lf-2em)\big]Z = 0$ and this implies $f = 4m^2/3$ and $b = 2(3cl+3em+2lm^2)/9$. Then we obtain

$$R_X^{(1)}(F_1,P^*) = \frac{Z^4}{81}\big[12m(3c+4m^2)Y + (27al+18ce-6clm+24em^2+8lm^3)Z\big]^2 = 0$$

and we consider two cases: $m \neq 0$ and $m = 0$.

**1)** If $m \neq 0$ then we may assume $m = 1$ and $e = 0$ due to the change $(x,y,t) \to (mx, y-e/2m, t/m^2)$ and in this case the above condition gives us $c = -4/3$ and $a = -16/27$. However in this case we have $R_X^{(0)}(P^*,Q^*) = 0$, i.e. we get a contradiction with the condition (17).

**2)** Assume now $m = 0$. In this case we obtain

$$
\begin{aligned}
&R_X^{(1)}(F_1,P^*) = (3al+2ce)^2Z^6 = 0,\\
&R_X^{(0)}(F_1,P^*) = (27a^2+4c^3)\big[27a^2l^3 + 27ae(cl^2-e^2) + 2c^2l(cl^2+9e^2)\big]Z^{12}/27 = 0,\\
&R_X^{(0)}(P^*,Q^*) = \big[27a^2l^3 + 27ae(cl^2-e^2) + 2c^2l(cl^2+9e^2)\big]/27 \neq 0
\end{aligned}
$$

and this implies $c \neq 0$, otherwise the second equality yields $a = 0$ and then $R_X^{(0)}(P^*, Q^*) = 0$. So $c \neq 0$ and the first equation gives $e = -3al/(2c)$ and then we arrive at the contradiction:

$$R_X^{(0)}(F_1, P^*) = \frac{l^3 Z^{12}}{216c^3}(27a^2 + 4c^3)^3 = 0, \quad R_X^{(0)}(P^*, Q^*) = \frac{l^3 Z^6}{216c^3}(27a^2 + 4c^3)^2 \neq 0.$$

This completes the proof of the lemma. □

Now we examine the direction $y = 0$. The following proposition holds.

**Proposition 2.** *For the existence of an invariant line of systems* (14) *in the direction $y = 0$ it is necessary and sufficient*

$$l = 0, \quad ef - 2bm = 0, \quad f^2 + m^2 \neq 0. \tag{18}$$

*Proof.* Indeed, considering the equations (6) for a system (14) we obtain

$$Eq_5 = l, \ Eq_8 = e - 2mW, \ Eq_{10} = b - fW.$$

Clearly, $Eq_5 = 0$ is equivalent to $l = 0$. On the other hand in order to have a line in the direction $y = 0$ the condition $f^2 + m^2 \neq 0$ is necessary. Therefore the condition $\operatorname{Res}_W(Eq_8, Eq_{10}) = ef - 2bm = 0$ is necessary and sufficient for the existence of a common solution $W = W_0$ of the equations $Eq_8 = 0$ and $Eq_{10} = 0$. This completes the proof of the proposition. □

### 3.2.1 The case $m \neq 0, \ l \neq 0$

By Proposition 2 we could not have invariant line in the direction $y = 0$. So after the transformation $(x, y, t) \to (mx, -e/2m + ly, t/m^2)$ we can consider $l = m = 1$ and $e = 0$. As a result we arrive at the family of systems

$$\dot{x} = a + cx + x^3 \equiv P(x), \quad \dot{y} = \quad b + x^2 + fy + 2xy \equiv Q(x, y). \tag{19}$$

**Proposition 3.** *Systems* (19) *possess invariant lines of total multiplicity 8 if and only if*

$$a = 0, \quad f = c = -\frac{4}{9}, \ b = \frac{4}{27}. \tag{20}$$

*Proof. Sufficiency.* Assume that (20) are satisfied. Then for the system (19) we calculate $H(X, Y, Z) = -3^{-8} X^2 (3X - 2Z)^3 Z (3X + 2Z)$ and hence, we have 8 invariant straight lines (including the line at infinity).

*Necessity.* Consider systems (19) for which the polynomial $H$ has the form (15). The degree of this polynomial equals four, but should be seven . Therefore we have to find out the conditions to increase the degree of the polynomial $H$ up to seven, namely we have to find out additionally a common factor of degree three of the polynomials $\mathcal{G}_i, \ i = 1, 2, 3$ (see Lemma 2 and Notation 1).

Considering (16) for systems (19) we obtain $\mathcal{G}_1/H|_{Z=0} = X^3(X + 4Y)$. Therefore we conclude that all three polynomials could only have common factors of the form

$X+\alpha=0$, which by Remark 6 must be factors of the polynomial $P^*(X,Z)$. We observe that $P^*(X,Z)$ is a common factor of the polynomials $\mathcal{G}_2/H$ and $\mathcal{G}_3/H$ and, moreover, in the last one this factor is of the second degree.

According to Lemma 8 the polynomial $P^*(X,Z)$ could not be a factor of $\mathcal{G}_1/H$, i.e. of the polynomial $F_1(X,Y,Z)$. Thus not all the factors of the polynomial $P^*(X,Z)$ are also the factors in $F_1(X,Y,Z)$. This leads us to the conclusion that the polynomial $F_2(X,Y,Z)$ must have a common factor with $P^*(X,Z)$, i.e. the condition

$$R_X^{(0)}(F_2,P^*)=(8+27a+18c)Z^3R_X^{(0)}(P^*,Q^*)=0$$

has to be fulfilled. Due to (17) this gives $c=-(8+27a)/18$ and we obtain that the polynomial $\psi=(3X-2Z)$ is a common factor of the polynomials $F_2(X,Y,Z)$ and $P^*(X,Z)$. On the other hand it must be a factor in $F_1(X,Y,Z)$. We calculate

$$\begin{aligned} R_X^{(0)}(F_1,\psi)=&-(8+27a+18f)Z^3(12Y+9fY+4Z+9bZ)/2=0,\\ R_X^{(0)}(P^*,Q^*)=&(12Y+9fY+4Z+9bZ)\Psi(Y,Z)\neq 0,\end{aligned}$$

where $\Psi(Y,Z)$ is a polynomial. So the above conditions give us the equality $a=-2(4+9f)/27$ and then we obtain $f=c$. In this case calculations yield

$$\begin{aligned}\mathcal{G}_1/H=&\frac{1}{27}(3X-2Z)\big[9X^3+12X^2(3Y-Z)+3(9c-4)XYZ+\\ &+(27b-18c-8)XZ^2-2(4+9c)YZ^2\big]\equiv\frac{1}{27}(3X-2Z)F_1'(X,Y,Z),\\ \mathcal{G}_2/H=&\frac{1}{729}(3X-2Z)^2\big[18X^2+54XY-6XZ+27cYZ+(27b-9c-4)Z^2\big]\times\\ &\times(9X^2+6XZ+4Z^2+9cZ^2)\equiv\frac{1}{729}(3X-2Z)^2F_2'(X,Y,Z)\tilde{P}(X,Y,Z)\end{aligned}$$

and we obtain

$$\begin{aligned} R_X^{(0)}\left(F_1',\tilde{F}_2'\right)=&-729Z^2\big[36Y^2-3(4+9c)YZ+(4-27b+9c)Z^2\big]\Gamma(Y,Z),\\ R_X^{(0)}\left(F_1',\tilde{P}\right)=&729(4+9c)Z^4\Gamma(Y,Z),\\ R_X^{(0)}(P^*,Q^*)=&\frac{1}{729}Z^3(12Y+9cY+4Z+9bZ)\Gamma(Y,Z),\end{aligned}$$

where $\Gamma(Y,Z)$ is a polynomial. Since $R_X^{(0)}\left(F_1',\tilde{F}_2'\right)\neq 0$ due to $R_X^{(0)}(P^*,Q^*)\neq 0$, we deduce that for the existence of a common factor of degree 3 of the polynomials $\mathcal{G}_1/H$ and $\mathcal{G}_2/H$ the condition $R_X^{(0)}\left(F_1',\tilde{P}\right)=0$ is necessary, i.e. $c=-4/9$ and we get $c=f=-4/9$ and $a=0$. In this case we obtain

$$\begin{aligned}&\mathcal{G}_1/H=\frac{1}{9}X(3X-2Z)(3X^2+12XY-4XZ-8YZ+9bZ^2)\equiv\frac{1}{9}X(3X-2Z)F_1'',\\ &P^*(X,Z)=X(3X-2Z)(3X+2Z)/9\end{aligned}$$

and since $X$ could not be a factor of $F_1''(X,Y,Z)$ and, moreover, as it was proved earlier the polynomial $P^*(X,Z)$ could not divide $\mathcal{G}_1/H$, we deduce that the factor of $F_1''(X,Y,Z)$ must be $3X-2Z$. So the condition

$$R_X^{(0)}\left(F_1'', 3X-2Z\right) = 3(27b-4)Z^2 = 0$$

is necessary and this implies $b = 4/27$, i.e. we arrive at the conditions (20) and this completes the proof of Proposition 3. ☐

Considering the conditions (20) we obtain the family of systems which after the suitable transformation $(x,y,t) \to (2x/3, y+1/3, 9t/4)$ becomes

$$\dot{x} = (x-1)x(1+x), \quad \dot{y} = x - y + x^2 + 3xy \tag{21}$$

with $H(X,Y,Z) = -X^2(X-Z)^3Z(X+Z)$. We observe that these systems possess 3 finite singularities: $(0,0)$, $(1,-1)$ and $(-1,0)$. On the other hand considering Lemma 3 for systems (21) we calculate:

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0, \quad \mu_6 = 8x^6 \neq 0.$$

So by Lemma 3 all other 6 finite singular points have gone to infinity and collapsed with the singular point $[0,1,0]$ located on the "end" of the invariant line $x=0$.

Thus this system possesses 3 real distinct invariant affine lines (besides the double infinite line) and namely: one triple, one double and one simple, all real and distinct. Therefore we obtain the configuration *Config. 8.23*.

### 3.2.2 The case $m \neq 0$, $l = 0$

As it was mentioned earlier we may assume $m=1$ and $e=0$ due to the change $(x,y,t) \to (mx, y-e/2m, t/m^2)$. So we get the family of systems

$$\dot{x} = a + cx + x^3, \quad \dot{y} = b + fy + 2xy \tag{22}$$

which by Proposition 2 possess invariant line in the direction $y=0$ if and only if $b=0$.

**1) The subcase $b \neq 0$.** We claim that in this case the above systems could not have invariant lines of total multiplicity 8. Indeed, due to the rescaling $y \to by$ we can consider $b=1$ and we obtain that for systems (22) the polynomial $H$ of the form (15) has the degree 4, but should be 7. Moreover we have $\mathcal{G}_1/H|_{Z=0} = 4X^3Y$ and hence the polynomials $\mathcal{G}_k/H$, $k=1,2,3$ (see their values (16) for $m=b=1$ and $l=e=0$) could have only the common factors of the form $X+\alpha Z$.

Considering Remark 6 and Lemma 8 we arrive again at the conclusion that the polynomial $F_2(X,Y,Z)$ must have a common factor with $P^*(X,Z)$. We determine that for systems (22) $F_2(X,Y,Z) = (3X-2Z)P^*(X,Z)Q^*(X,Y,Z)$ and hence due to the condition (17) and according to Lemma 8 (which says that $P^*(X,Z)$ could not divide $\mathcal{G}_1/H$) we conclude that $3X-2Z$ must be a double factor in $\mathcal{G}_1/H$. However we obtain

$$R_X^{(1)}\left((3X-2Z)^2, \mathcal{G}_1/H\right) = 162Z^3 \neq 0,$$

i. e. for systems (22) we could not increase the degree of $H(X,Y,Z)$ up to 7 and this completes the proof of our claim.

**2) The subcase $b=0$.** We obtain the family of systems

$$\dot{x} = a + cx + x^3 \equiv P(x), \quad \dot{y} = y(f+2x) \equiv y\tilde{Q}(x). \tag{23}$$

**Proposition 4.** *Systems* (23) *possess invariant lines of total multiplicity 8 if and only if one of the following sets of conditions holds:*

$$f = c,\ a = -\frac{2(4+9c)}{27},\ (4+3c)(4+9c) \neq 0; \tag{24}$$

$$f = \frac{-2(3c+2)}{3},\ a = \frac{2(4+9c)}{27},\ \ (4+3c)(4+9c) \neq 0. \tag{25}$$

*Proof. Sufficiency.* Assume that (24) $\big($respectively (25)$\big)$ are satisfied. Then considering systems (23) we calculate $H(X,Y,Z) = 3^{-8}Y(3X-2Z)^3Z(9X^2+6XZ+4Z^2+9cZ^2)$ ( respectively $H(X,Y,Z) = 3^{-9}2YZ(3X+2Z)(9X^2-6XZ+4Z^2+9cZ^2)^2$) and hence, we have 8 invariant straight lines, including the line at infinity. Moreover for the corresponding systems we calculate $R_X^{(0)}(\mathcal{G}_2/H, \mathcal{G}_1/H) = 3^{11}2(4+3c)^2(4+9c)Z^3$ (respectively $R_X^{(0)}(\mathcal{G}_2/H, \mathcal{G}_1/H) = -3^{15}(4+3c)^2(4+9c)Z^3$) and this leads to the condition $(4+3c)(4+9c)\neq 0$ which does not allow us to have 9 invariant lines.

*Necessity.* For systems (23) we have $H(X,Y,Z) = YZ(X^3+cXZ^2+aZ^3)$. Thus according to Lemma 2 we conclude that we need additionally a non-constant factor of the second degree of $H$. For systems (23) we calculate (see Notation 1)

$$\begin{aligned}
\mathcal{G}_1/H =& 4X^3 - (4-3f)X^2Z - 4fXZ^2 - (2a - cf + f^2)Z^3,\\
\mathcal{G}_2/H =& (3X-2Z)(2X+fZ)(X^3+cXZ^2+aZ^3) \equiv (3X-2Z)\tilde{Q}^*(X,Z)P^*(X,Z),\\
\mathcal{G}_3/H =& 24(2X+fZ)(X^3+cXZ^2+aZ^3)^2 \equiv \tilde{Q}^*(X,Z)\left[P^*(X,Z)\right]^2,
\end{aligned}$$

where $P^*(X,Z)$ and $\tilde{Q}^*(X,Z)$ are the homogenization of the polynomial $P(x)$ and $\tilde{Q}(x)$ from (23).

We observe that $\mathcal{G}_1/H|_{Z=0} = 4X^3$ and we conclude that all three polynomials could not have as a common factor $Z$. On the other hand these polynomials do not depend on $Y$. So common factors of the above polynomials could be only factors of the form $X+\alpha Z$, which by Remark 6 must be also factors in $P^*(X,Z)$. So considering this remark and Lemma 8 we arrive at the two possibilities: the linear form $3X-2Z$ either is a common factor of the polynomials $\mathcal{G}_1/H$ and $P^*(X,Z)$ or it is not.

***a)*** Assume first that $3X-2Z$ is a common factor of $\mathcal{G}_1/H$ and $P^*(X,Z)$. Then the following condition must be satisfied:

$$R_X^{(0)}(3X-2Z, P^*) = (8+27a+18c)Z^3 = 0$$

and this implies $a = -2(4+9c)/27$. Herein we have

$$\begin{aligned} R_X^{(0)}\,(3X-2Z, \mathcal{G}_1/H,) =& 9(c-f)(4+3f)Z^3 = 0,\\ R_X^{(0)}\,(P^*(X,Z), Q^*(X,Z)) =& (4+3f)(16+36c-12f+9f^2)Z^3/27 \neq 0 \end{aligned}$$

and hence the condition $f = c$ must hold, which leads to the first two conditions (24).

***b)*** Suppose now that $3X-2Z$ is not a common factor of $\mathcal{G}_1/H$ and $P^*(X,Z)$. Then clearly these polynomials must have a common factor of the second degree. So the conditions

$$\begin{aligned} &R_X^{(0)}\,(P^*, \mathcal{G}_1/H) = (8a-4cf-f^3)\Phi_1(a,c,f)Z^9 = 0, \quad R_X^{(1)}\,(P^*, \mathcal{G}_1/H) = \Phi_2(a,c,f)^4 = 0,\\ &R_X^{(0)}\,(P^*, Q^*) = (4cf+f^3-8a)Z^3 \neq 0 \end{aligned}$$

must hold, where $\Phi_1 = 8a+27a^2+4c^2+4c^3+18af-f^3-cf(4+3f)$, $\Phi_2 = 16c^2+2c(8+6f+3f^2)+3(6af-8a+4f^2+f^3)$. Due to $R_X^{(0)}\,(P^*,Q^*) \neq 0$ we must have $\Phi_1 = \Phi_2 = 0$ and we calculate

$$R_a^{(0)}\,(\Phi_1, \Phi_1) = 3(4+6c+3f)^2(4c+3f^2)(16+16c+3f^2) = 0.$$

We claim that the condition $4+6c+3f = 0$ has to be satisfied for non-degenerate systems (23). Indeed assuming $c = -3f^2/4$ (respectively $c = -(16+3f^2)/16)$) we get that $4a+f^3$ (respectively $32a+16f-f^3$) is a common factor of $\Phi_1$ and $\Phi_2$, however in this case the polynomial $R_X^{(0)}\,(P^*,Q^*)$ gives the value $-2(4a+f^3)Z^3 \neq 0$ (respectively $-(32a+16f-f^3)Z^3/4 \neq 0$).

So $4+6c+3f = 0$, i.e $f = -2(2+3c)/3$ and in this case the common factor of $\Phi_1$ and $\Phi_2$ is $(8-27a+18c)$. Hence the condition $\Phi_1 = \Phi_2 = 0$ implies $a = 2(4+9c)/27$ and this leads to the conditions (25). □

Next we construct the respective canonical forms of systems (23) when either the conditions (24) or (25) of Proposition 4 are satisfied.

*(i) Conditions* (24). We observe that in this case due to a translation and an additional notation, namely $r = (4+3c)/3$, we arrive at the family of systems

$$\dot{x} = x(r+2x+x^2), \quad \dot{y} = (r+2x)y \tag{26}$$

for which we have $H(X,Y,Z) = X^3YZ(X^2+2XZ+rZ^2)$. So the polynomial $H(X,Y,Z)$ has the degree 7 and by Lemma 2 the above systems possess invariant lines of total multiplicity 8 (including the line at infinity, which is double). Now we need an additional condition under the parameter $r$ which conserves the degree of the polynomial $H(X,Y,Z)$. For systems (26) we calculate $R_X^{(0)}\,(\mathcal{G}_3/H, \mathcal{G}_1/H) = 48r^3(8-9r)^2Z^5 \neq 0$. Consequently we get the condition $r(8-9r) \neq 0$ which for systems (23) is equivalent to $(4+3c)(4+9c) \neq 0$ (see the last condition from (24)).

Besides the infinite line $Z = 0$ (which is double) systems (26) possess six affine invariant lines, namely:

$$L_{1,2,3} = x,\ L_4 = y,\ L_{5,6} = r + 2x + x^2.$$

We detect that the lines $L_{5,6} = 0$ are either complex or real distinct or real coinciding, depending on the sign of the discriminant of the polynomial $x^2+2x+r$, which equals $\Delta = 4(1-r)$. We also observe that systems (26) possess 3 finite singularities: $(0,0)$ and $(-1 \pm \sqrt{1-r}, 0)$ which are located on the invariant line $y = 0$. On the other hand considering Lemma 3 for systems (26) we calculate:

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0, \quad \mu_6 = r^3x^6 \neq 0.$$

So by Lemma 3 all other 6 finite singular points have gone to infinity and collapsed with the singular point $[0,1,0]$ located on the "end" of the invariant line $x = 0$. Moreover by this lemma systems (26) became degenerate only if $r = 0$, and we observe that in this case the system indeed is degenerate.

We consider the three possibilities given by the value of the discriminant $\Delta$.

***a***) *The possibility* $\Delta > 0$. Then $1 - r > 0$, i. e. $r < 1$. We set the notation $1 - r = u^2$ (i. e. $r = 1 - u^2$) which leads to the systems

$$\dot{x} = (1 - u + x)x(1 + u + x), \quad \dot{y} = (1 - u^2 + 2x)y$$

possessing one triple and three simple distinct real invariant lines. Comparing the line $x = \mp u - 1$ with $x = 0$ we conclude that if $|u| > 1$ (i. e. $r < 0$) then in the direction $x = 0$ the triple invariant line is situated in the domain between two simple ones, whereas in the case $|u| < 1$ (i. e. $0 < r < 1$) the triple line is located outside this domain. As a result we get *Config. 8.24* in the case of $r < 0$ and *Config. 8.25* in the case of $0 < r < 1$.

***b***) *The possibility* $\Delta = 0$. Then $r = 1$ and we obtain the configuration *Config. 8.26.*

***c***) *The possibility* $\Delta < 0$. In this case $r > 1$ and we get systems possessing two complex, one simple and one triple real all distinct invariant lines and this leads to the configuration *Config. 8.27.*

*(ii) Conditions* (25). In this case after the translation of the origin of coordinates to the singular point $(-2/3, -e/2)$ and setting a new parameter $r = (4 + 3c)/3$ we obtain the systems

$$\dot{x} = (r - 2x + x^2)x, \quad \dot{y} = 2(x - r)y. \tag{27}$$

For these systems we have $H(X,Y,Z) = 2XYZ(X^2 - 2XZ + rZ^2)^2$. Besides the double infinite line systems (27) possess 4 affine invariant lines:

$$L_1 = x,\ L_2 = y,\ L_{3,4} = x^2 - 2x + r,$$

where the lines $L_{3,4}=0$ are double ones. We denote by $\Delta = 4(1-r)$ the discriminant of the polynomial $x^2-2x+r$ and we observe that for $\Delta=0$ (i.e. $r=1$) the systems become degenerate.

We also observe that systems (27) possess 3 finite singularities: $(0,0)$ and $(1\pm\sqrt{1-r},0)$ which are located on the invariant line $y=0$. On the other hand considering Lemma 3 for systems (26) we calculate:

$$\mu_0=\mu_1=\mu_2=\mu_3=\mu_4=\mu_5=0, \quad \mu_6=8(1-r)r^2x^6.$$

If $r(r-1)\neq 0$ by Lemma 3 all other 6 finite singular points have gone to infinity and collapsed with the singular point $[0,1,0]$ located on the "end" of the invariant line $x=0$. Moreover by this lemma systems (27) became degenerate only if either $r=0$ or $r=1$ and in both cases we get degenerate systems.

Thus we have the following two possibilities:

***a)*** *The possibility* $\Delta>0$. Then $r<1$ and denoting $r=1-v^2$ we obtain the systems

$$\dot{x}=(1+v-x)x(1-v-x), \quad \dot{y}=2(v^2-1+x)y \tag{28}$$

with $H(X,Y,Z)=2XYZ(X-Z-vZ)^2(X-Z+vZ)^2$. Examining the lines $x=1\pm v$ and $x=0$ we conclude that if $|v|>1$ then we get a simple invariant line between two double real lines in the directions $x=0$ and consequently we arrive at *Config. 8.28*. In the case of $|v|<1$ these two double real lines are located on the right–hand side of the simple invariant line. So we get *Config. 8.29*.

***b)*** *The possibility* $\Delta>0$. In this case $r>1$ and systems (27) possess 2 real simple, 2 complex double invariant lines, all distinct $\Rightarrow$ *Config. 8.30*.

### 3.2.3 The case $m=0,\ l\neq 0$

We claim that in this case systems (14) could not possess invariant lines of total multiplicity 8.

Indeed, since $l\neq 0$ by Proposition 2 we could not have a line in the direction $y=0$. Via the rescaling $(x\to x, y\to ly, t\to t)$ we can consider $l=1$ and therefore we arrive at the systems

$$\dot{x}=a+cx+x^3, \quad \dot{y}=\ b+ex+x^2+fy \tag{29}$$

for which the polynomial $H$ has the form (15) and $\mathcal{G}_i/H$ $(i=1,2,3)$ are the polynomials (16) for the particular case $m=0$ and $l=1$. We observe that $\mathcal{G}_1/H|_{Z=0}=X^4$ and hence $Z$ could not be a common factor of these polynomials. Since we have no invariant lines in the direction $y=0$, in what follows we shall examine only the conditions given by resultants with respect to $X$. According to (16) and condition (17) the polynomial $F_1(X,Y,Z)$ must have a common factor of degree 3 with $\left[P^*(X,Y)\right]^2$. For systems (29) we calculate Coefficient$[R_X^{(2)}\left(F_1,\left[P^*\right]^2\right), Y^4Z^4]=81f^4$. Clearly the condition $f=0$ is necessarily to get a common factor of the degree 3. Then we have

$$R_X^{(0)}\left(F_1,\left[P^*\right]^2\right)=(27a^2+4c^3)^2[\Phi(a,b,c,e)]^2Z^{24}=0,\ R_X^{(0)}(Q^*,P^*)=\Phi(a,b,c,e)Z^6\neq 0$$

where $\Phi(a,b,c,e)$ is a polynomial. So the above conditions imply $27a^2+4c^3=0$. First we examine the possibility $a=0$ and we get $c=0$. Then we calculate

$$R_X^{(0)}(Q^*,P^*)=b^3Z^6\neq 0,\ R_X^{(2)}\left(F_1,[P^*]^2\right)=81b^4Z^8=0$$

and we arrive at the contradictory condition $(0\neq b=0)$. So it remains to examine the case when $a\neq 0$. Since in this case $c\neq 0$ we denote $a=2a_1c$ which implies $c=-27a_1^2$. We calculate

$$\begin{aligned}R_X^{(0)}(Q^*,P^*)=&(9a_1^2+b-3a_1e)^2(36a_1^2+b+6a_1e)Z^6\neq 0,\\ R_X^{(1)}\left(F_1,[P^*]^2\right)=&2^33^{10}a_1^5(9a_1^2+b-3a_1e)^3(36a_1^2+b+6a_1e)^2Z^{15}=0\end{aligned}$$

and we also get a contradiction which completes the proof of our claim.

### 3.2.4 The case $m=0,\ l=0$

We divide our examination in two subcases: $e\neq 0$ and $e=0$.

**1) The subcase $e\neq 0$.** Then due to the rescaling $(x,y,t)\to(ex,y,t/e^2)$ we can consider $e=1$ and therefore we arrive at the systems

$$\dot{x}=a+cx+x^3,\quad \dot{y}=b+x+fy. \tag{30}$$

**Proposition 5.** *Systems* (30) *possess invariant lines of total multiplicity 8 if and only if the following conditions hold:*

$$f=-2c,\quad a=0. \tag{31}$$

*Proof.* *Sufficiency.* Assume that (31) is satisfied. Then considering systems (30) we calculate $H(X,Y,Z)=XZ^2(X^2+cZ^2)^2$ and hence, we have invariant straight lines of total multiplicity 8 (including the line at infinity). On the other hand we could not have 9 lines, because $R_X^{(0)}(G_2/H,G_1/H)=-27(2cY-bZ)^3=0$ if and only if $b=c=0$. However in this case we get a degenerate system.

*Necessity.* For systems (30) we have $H(X,Y,Z)=Z^2(X^3+cXZ^2+aZ^3)$ and we observe that the degree of the polynomial $H$ is 5. So we have to increase the degree of $H$ up to 7. In other words we have to determine the conditions under which the three polynomials $G_1/H$, $G_2/H$ and $G_3/H$ have a common factor of degree 2. For these systems we calculate

$$\begin{aligned}\mathcal{G}_1/H=&2X^3+3fX^2Y+3bX^2Z-fXZ^2+f(c-f)YZ^2+(bc-a-bf)Z^3,\\ \mathcal{G}_2/H=&3X(X+fY+bZ)(X^3+cXZ^2+aZ^3)\equiv 3XQ^*P^*,\\ \mathcal{G}_3/H=&24(X+fY+bZ)(X^3+cXZ^2+aZ^3)^2\equiv 24Q^*[P^*]^2.\end{aligned}$$

We observe that $G_1/H|_{Z=0}=2X^3+3fX^2Y$ and hence $Z$ could not be a common factor of these polynomials. For systems (30) we get $R_Y^{(0)}(G_3/H,G_1/H)=-24f(X^3+cXZ^2+aZ^3)^3$ which vanishes if and only if $f=0$ and since $m=0$, considering Proposition 2, we conclude that in this case we could not have a line in the direction $y = 0$. Thus all three mentioned polynomials could only have common factors of the form $X + \alpha = 0$, which by Remark 6 must be factors of the polynomial $P^*(X, Z)$. So considering this remark and Lemma 8 we arrive at the two possibilities: the linear form $X$ either is not a common factor of the polynomials $\mathcal{G}_1/H = F_1(X, Y, Z)$ and $P^*(X, Z)$ (i.e. $a \neq 0$) or it is (i.e. $a = 0$).

***a)*** Assume first that $X$ is not a factor of $P^*(X, Z)$, i.e. we have to consider $a \neq 0$. According to (16) and condition (17) the polynomial $F_1(X, Y, Z)$ must have a common factor of degree 2 with $P^*(X, Y)$. Then considering systems (30) the following conditions must be satisfied:

$$R_X^{(0)}(F_1, P^*) = [27a^2 + (c - f)(2c + f)^2]Z^6\Psi(Y, Z) = 0,\ R_X^{(0)}(Q^*, P^*) = \Psi(Y, Z) \neq 0$$

where $\Psi(Y, Z)$ is a polynomial. So the condition $27a^2 + (c - f)(2c + f)^2 = 0$ is necessary for the existence of a common factor of the polynomials $F_1$ and $P^*$. Then $(c - f)(2c + f) \neq 0$ (due to $a \neq 0$) and denoting $u = 2c + f \neq 0$ (i.e. $f = u - 2c$) we obtain $c = u/3 - 9a^2/u^2$ and $f = u - 2c = (54a^2 + u^3)/(3u^2)$. In this case we obtain

$$F_1 = (uX + 3aZ)F_1^*(X, Y, Z)/(3u^4),\ P^* = (uX + 3aZ)(3uX^2 - 9aXZ + u^2Z^2)/(3u^2)$$

where $F_1^*(X, Y, Z)$ is a polynomial of the second degree. Assume first that $uX + 3aZ$ is a factor in $F_1^*$. In this case it must be a factor in $3uX^2 - 9aXZ + u^2Z^2$ and therefore the following condition must hold:

$$R_X^{(0)}(uX + 3aZ, 3uX^2 - 9aXZ + u^2Z^2) = u(54a^2 + u^3)Z^2 = 0.$$

Since $u \neq 0$ we can set $a = a_1u$ and thus, we get $u = -54a_1^2$. Then

$$R_X^{(0)}(F_1^*, uX + 3aZ) = 18a_1(3a_1 - b)Z^2 = 0,\ R_X^{(0)}(P^*, Q^*) = (b - 3a_1)^2(6a_1 + b)Z^3 \neq 0$$

and we arrive at the contradiction.

Now we consider that $uX + 3aZ$ is not a factor in $F_1^*$. Then the polynomials $F_1^*$ and $3uX^2 - 9aXZ + u^2Z^2$ must have a common factor, i.e. the following conditions hold:

$$R_X^{(0)}(F_1^*, 3uX^2 - 9aXZ + u^2Z^2) = 27u^5Z^2F_1^{**}(Y, Z) = 0,$$

$$R_X^{(0)}(P^*, Q^*) = [(3a - bu)3uZ - (54a^2 + u^3)Y]F_1^{**}(Y, Z)/(27u^6) \neq 0$$

where $F_1^{**}(Y, Z)$ is a polynomial of the second degree. Since $c \neq 0$ in this case we also arrive at the contradictory condition.

***b)*** Assume now that $X$ is a common factor of $P^*(X, Z)$, i.e. we have the condition $a = 0$ which implies $\mathcal{G}_2/H = 3X^2(X^2 + cZ^2)Q^*$. Therefore either $X^2$ or $X^2 + cZ^2$ must be a factor of $F_1$. In order to have $X^2$ as a common factor of the mentioned polynomial the condition $R_X^{(0)}(X^2, F_1) = R_X^{(1)}(X^2, F_1) = 0$ must be satisfied. We calculate

$$R_X^{(1)}(X^2, F_1) = -fZ^2 = 0,\ R_X^{(0)}(X^2, F_1) = (c - f)^2Z^4(fY + bZ)^2 = 0$$

and $R_X^{(0)}(P^*,Q^*)\mid_{\{c=f=0\}}= -b(b^2+c)Z^3$. It is evident that in order to have $X^2$ as a factor of the polynomial $F_1$ it is necessary the conditions $f=c=0$ and $b\neq 0$ to be satisfied, i.e. we get a particular case of the conditions (31). Since $b\neq 0$, due to the rescaling $\{x\to bx, y\to y/b, t\to t/b^2\}$ we can consider $b=1$. So we arrive at the system

$$\dot{x}=x^3,\quad \dot{y}=\quad 1+x \tag{32}$$

for which $H(X,Z)=X^5Z^2$. This system possesses the affine invariant line of the multiplicity 5 in the direction $x=0$ and the infinite invariant line is of the multiplicity 3. Considering Lemma 3 for these systems we get

$$\mu_0=\mu_1=\mu_2=\mu_3=\mu_4=\mu_5=\mu_6=\mu_7=\mu_8=0,\quad \mu_9=9x^9\neq 0.$$

Therefore by Lemma 3 all 9 finite singular points have gone to infinity and collapsed with the singular point $[0,1,0]$ located on the "end" of the invariant line $x=0$. Consequently we get the configuration *Config. 8.33.*

Now we assume that $X^2+cZ^2$ is a factor of the polynomial $F_1$, i.e. the condition $R_X^{(0)}\big(X^2+cZ^2, F_1\mid_{\{a=0\}}\big)=R_X^{(1)}\big(X^2+cZ^2, F_1\mid_{\{a=0\}}\big)=0$ must hold. We calculate

$$R_X^{(1)}\big(X^2+cZ^2, F_1\mid_{\{a=0\}}\big)=-(2c+f)Z^2=0$$

from which it results $f=-2c\neq 0$ and we obtain the conditions (31). Since $c\neq 0$ we may assume $b=0$ (applying the translation of the origin of coordinates at the point $x_0=0,\ y_0=b/2c$). Therefore we arrive at non-degenerate systems depending on the parameter $c=\{-1,1\}$ (applying a rescaling)

$$\dot{x}=x(c+x^2),\quad \dot{y}=\quad x-2cy. \tag{33}$$

For the above systems we have $H(X,Z)=XZ^2(X^2+cZ^2)^2$. Thus beside the triple infinite invariant line systems (37) possess 5 invariant affine lines. More precisely, we have one simple and two double, all real and distinct if $c=-1$ and one simple real and two double complex if $c=1$.

On the other hand we observe that systems (33) possess 3 finite singularities: $(0,0)$ and $\big(\pm\sqrt{-c},\mp 1/(2\sqrt{-c})\big)$. Considering Lemma 3 for these systems we calculate:

$$\mu_0=\mu_1=\mu_2=\mu_3=\mu_4=\mu_5=0,\quad \mu_6=-8c^3x^6\neq 0.$$

Therefore by Lemma 3 all other 6 finite singular points have gone to infinity and collapsed with the singular point $[0,1,0]$ located on the "end" of the invariant line $x=0$. Thus we get *Config. 8.31* if $c=-1$ and *Config. 8.32* if $c=1$. □

**2) The subcase $e=0$.** Then we get the family of systems

$$\dot{x}=a+cx+x^3,\quad \dot{y}=\quad b+fy \tag{34}$$

for which $H(a,X,YZ) = Z^2(fY+bZ)(X^3+cXZ^2+aZ^3)$. So the degree of $H$ is six but should be seven. Therefore we need an additional common factor of $\mathcal{G}_i,\ i=1,2,3$. We calculate

$$\begin{aligned}\mathcal{G}_1/H =&3X^2+cZ^2-fZ^2, \quad \mathcal{G}_2/H = 3X(X^3+cXZ^2+aZ^3),\\ \mathcal{G}_3/H =&24(X^3+cXZ^2+aZ^3)^2\end{aligned}$$

and we observe that these polynomials could not have as a common factor neither Z nor Y. So we examine their resultants with respect to X. We calculate

$$\begin{aligned}R_X^{(0)}(\mathcal{G}_1/H,P^*) =&[27a^2+(c-f)(2c+f)^2]Z^6=0,\\ R_X^{(0)}(P^*,Q^*) =&(fY+bZ)^3 \neq 0,\end{aligned}$$

which implies $27a^2+(c-f)(2c+f)^2=0$. We observe that $(c-f)(2c+f)\neq 0$, otherwise we get $a=0$ and this leads to systems with invariant lines of total multiplicity 9.

Denoting $u=2c+f\neq 0$ (i.e. $f=u-2c$) we obtain $c=u/3-9a^2/u^2$ and $f=u-2c=(54a^2+u^3)/(3u^2)$. So we get the family of systems

$$\dot{x} =\frac{1}{3u^2}(3a+ux)(u^2-9ax+3ux^2), \quad \dot{y} = \quad b+\frac{54a^2+u^3}{3u^2}y. \tag{35}$$

Without loss of generality we may assume $b\neq 0$, because in the case $b=0$ we must have $54a^2+u^3\neq 0$ (otherwise we get degenerate systems) and then via a translation $y\to y+y_0$ (with $y_0\neq 0$) we obtain $b\neq 0$. So applying the translation of the origin of coordinates at the point $(-3a/u,0)$, after the suitable rescaling $\{x\to -(9ax)/u, y\to bu^2y/(81a^2), t\to tu^2/81a^2\}$ systems (35) become

$$\dot{x} =rx+x^2+x^3, \quad \dot{y} = \quad 1+ry, \tag{36}$$

where $r=(54a^2+u^3)/(243a^2)$. For these systems we calculate $H=X^2(rY+Z)Z^2(X^2+XZ+rZ^2)$ and $R_X^{(0)}(\mathcal{G}_2/H,\mathcal{G}_1/H)=3(9r-2)Z^3\neq 0$ and this leads to the condition $9r-2\neq 0$ which guarantee the non-existence of nine invariant lines. We observe that the infinite invariant line Z=0 is triple if $r\neq 0$ and it has multiplicity four in the case $r=0$.

***a)*** *The possibility* $r\neq 0$. In this case the geometry of the configuration depends on the sign of the discriminant $\Delta$ of the polynomial $x^2+x+r$, i.e. $\Delta=1-4r$. Accordingly we conclude that besides the double infinite invariant line the systems (35) possess 5 affine lines which are as follows:

| | | | |
|---|---|---|---|
| $\Delta>0$ | (i.e. $0\neq r<1/4$) | $\Rightarrow$ | 3 simple, 1 double, all real and distinct, |
| $\Delta=0$ | (i.e. $r=1/4$) | $\Rightarrow$ | 1 simple, 2 double, all real and distinct, |
| $\Delta<0$ | (i.e. $r>1/4$) | $\Rightarrow$ | 2 real simple, 1 complex double. |

On the other hand considering Lemma 3 we calculate:

$$\begin{gathered}\mu_0=\mu_1=\mu_2=\mu_3=\mu_4=\mu_5=0,\ \mu_6=r^3x^6,\ \mu_7=r^2x^6(3x-ry),\\ \mu_8=rx^6(3x^2-2rxy+r^3y^2),\ \mu_9=9x^7(x^2-rxy+r^3y^2).\end{gathered}$$

Since $r \neq 0$ by Lemma 3 only 6 finite singular points have gone to infinity and collapsed with the singular point $[0,1,0]$ located on the "end" of the invariant line $x=0$. Other three finite points are $(0,-1/r)$ and $\big((-1\pm\sqrt{1-4r})/2,-1/r\big)$ (located on the invariant line $ry+1=0$).

Moreover, in the case of $\Delta>0$, denoting $1-4r=v^2$ (i.e. $r=(1-v^2)/4$ ) we obtain the systems

$$\dot{x} =(1-v+2x)x(1+v+2x)/4, \quad \dot{y} = \ 1+(1-v^2)y/4.$$

We compare the lines $x=(-1\pm v)/2$ with $x=0$ and conclude that if $|v|>1$ , i.e. $r<0$ (respectively $0<|v|<1/4$, i.e. $0<r<1/4$) then the double real invariant line is located (respectively is not located) between two simple ones and we arrive at the configuration *Config. 8.34* (respectively *Config. 8.35.*).

Additionally, we have the configuration *Config. 8.36* in the case of $\Delta=0$ ( i.e. $r=1/4$) and *Config. 8.37* in the case of $\Delta<0$ ( i.e. $r>1/4$).

***b)*** *The possibility* $r=0$. In this case we get the system

$$\dot{x}=x^2(x+1), \quad \dot{y}=1 \tag{37}$$

with $H(X,Z)=X^3Z^3(X+Z)$. Therefore besides the infinite line of the multiplicity four this system possesses 2 distinct affine invariant lines (one of the multiplicity 3 and one simple), and namely: $L_{1,2,3}=x, \quad L_4=x+1$.

Since in this case we obtain $\mu_i=0$ $(i=0,1,\dots,8)$ and $\mu_9=9x^9\neq 0$, by Lemma 3 all 9 finite singular points have gone to infinity and collapsed with the the same singular point $[0,1,0]$. As a result we get the configuration *Config. 8.38* Thus considering the above results we arrive at the following proposition.

**Proposition 6.** *The systems* (34) *possess invariant lines of total multiplicity eight if and only if*

$$27a^2+(c-f)(2c+f)^2=0, \quad a\neq 0. \tag{38}$$

## 3.3 Invariant conditions for the configurations *Config. 8.23*–*Config. 8.38*

By Lemma 7 the conditions $\mathcal{D}_1=\mathcal{D}_3=\mathcal{D}_4=0,\ \mathcal{D}_2\neq 0$ are necessary and sufficient for a cubic system to have two real distinct infinite singularities, and namely they are determined by one triple and one simple factors of $C_3(x,y)$. After a linear transformation a cubic system could be brought to the form (8). According to Proposition 1, the condition $\mathcal{V}_1=\mathcal{V}_3=0$ gives systems (13) (via a linear transformation and time rescaling). At the beginning of Subsection 3.2 it was proved that for these systems the condition $\mathcal{K}_5=0$ is equivalent to $k=h=n=0$. Moreover, for the existence of invariant lines in the direction $x=0$ the additional condition $d=0$ has to be satisfied. So considering the condition $\mathcal{K}_5=0$ (i.e. $k=h=n=0$)

for systems (13) we calculate $N_1 = 12d$ and evidently $N_1 = 0$ is equivalent to $d = 0$ and we arrive at systems (14). For these systems we calculate

$$N_2 = -m^2x^4, \ N_3 = -12lx^5.$$

We remark that in the previous subsections the examination of systems (14) was divided in the cases determined by the parameters $m$ and $l$. In addition it was proved (see Subsection 3.2.3) that in the case $m = 0$ and $l \neq 0$ (i.e. $N_2 = 0$ and $N_3 \neq 0$) systems (14) could not have invariant lines of total multiplicity 8. So in what follows we split our examination here in three cases, defined by the invariant polynomials $N_2$ and $N_3$:

$$(i) \ N_2N_3 \neq 0; \quad (ii) \ N_2 \neq 0, N_3 = 0; \quad (iii) \ N_2 = N_3 = 0.$$

### 3.3.1 The case $N_2N_3 \neq 0$

Then $l \cdot m \neq 0$ and as it was shown earlier systems (14) could be brought via an affine transformation to systems (19). According to Proposition 3 the last systems possess invariant lines of total multiplicity 8 if and only if the conditions (3) are satisfied. We prove that these conditions are equivalent to $N_4 = N_5 = N_6 = N_7 = 0$, i.e.

$$a = 0, \ f = c = -\frac{4}{9}, \ b = \frac{4}{27} \quad \Leftrightarrow \quad N_4 = N_5 = N_6 = N_7 = 0.$$

Indeed, for systems (19) we calculate

$$N_4 = 5184(c - f)x^4 \quad \text{and} \quad N_5 = 2592(4 + 6c + 3f)x^4$$

and clearly the condition $N_4 = N_5 = 0$ is equivalent to $f = c = -4/9$. Then considering the last conditions we calculate $N_6 = 8640ax^4$ and hence $N_6 = 0$ gives $a = 0$. It remains to determine the invariant condition which governs the parameter $b$. Considering the obtained conditions for systems (19) we calculate $N_7 = 288(27b - 4)x^6 = 0$ which is equivalent to $b = \frac{4}{27}$. So if for systems (14) the conditions $N_2N_3 \neq 0, \ N_4 = N_5 = N_6 = N_7 = 0$ are satisfied then we arrive at the system (21) possessing the configuration *Config. 8.23*.

### 3.3.2 The case $N_2 \neq 0, \ N_3 = 0$

These conditions imply $m \neq 0$ and $l = 0$, and as it was proved in Subsection 3.2.2 the condition $ef - 2bm = 0$ is necessary to be fulfilled for systems (17) in order to have invariant lines of total multiplicity 8. On the other hand for these systems we calculate $N_8 = 1296(ef - 2bm)x^6$ and the last condition is equivalent to $N_8 = 0$. Due to a rescaling we may assume $m = 1$ and then we get $b = ef/2$ and this leads to systems (23). By Proposition 4 these systems possess invariant lines of total multiplicity 8 if and only if either the conditions (24) or (25) are satisfied.

In what follows we consider each one of these sets of conditions and construct the corresponding equivalent invariant conditions as well as the additional invariant conditions for the realization of the respective configurations.

*(a) Conditions* (24). We clam that for a system (23) the following conditions are equivalent:

$$f = c,\ a = -\frac{2(4+9c)}{27},\ (4+3c)(4+9c) \neq 0 \quad \Leftrightarrow \quad N_4 = N_6 = 0,\ N_9 \neq 0.$$

Indeed, for systems (23) we calculate $N_4 = 5184(c-f)x^4$ and therefore $N_4 = 0$ gives $f = c$. Then we have $N_6 = 320(27a + 18c + 8)x^4$ and $N_9 = 2304(4 + 3c)(4 + 9c)x^4$ which imply the condition $a = -\dfrac{2(4+9c)}{27}$ if $N_6 = 0$ and $(4 + 3c)(4 + 9c) \neq 0$ if $N_9 \neq 0$.

Thus if the conditions $N_4 = N_6 = 0$ are satisfied then systems (23) via a translation and a suitable notation can be brought to systems (26), for which the condition $N_9 = 6912r(9r - 8)x^4 \neq 0$ holds. Now for these systems we need to determine the invariant polynomials which govern the conditions under parameter $r$ in order to get different configurations of invariant straight lines.

We calculate $N_{10} = 144(1 - r)x^2$ and $N_{11} = 3456rx^4$. Therefore, considering the obtained earlier for systems (26) configurations (see page 70) we conclude that if for a system (14) the conditions $N_3 = N_4 = N_6 = N_8 = 0,\ N_2N_9 \neq 0$ are satisfied then we get the configuration *Config. 8.24* if $N_{11} < 0$; *Config. 8.25* if $N_{10} > 0$ and $N_{11} > 0$; *Config. 8.26* if $N_{10} = 0$ and *Config. 8.27* in the case $N_{10} < 0$.

*(b) Conditions* (25). We clam that for a system (23) the next conditions are equivalent:

$$f = -\frac{2(2+3c)}{3},\ a = \frac{2(4+9c)}{27},\ (4+3c)(4+9c) \neq 0 \quad \Leftrightarrow \quad N_5 = N_{12} = 0,\ N_{13} \neq 0.$$

Indeed, for (23) we calculate $N_5 = 2592(4+6c+3f)x^4$ and hence $N_5 = 0$ implies $f = -\dfrac{2(2+3c)}{3}$. Then we have $N_{12} = 3240(27a - 18c - 8)x^4$ and, clearly, $N_{12} = 0$ gives $a = \dfrac{2(4+9c)}{27}$. For $N_5 = N_{12} = 0$ we calculate $N_{13} = 1008(4 + 3c)(4 + 9c)x^5y$ and therefore $N_{13} \neq 0 \Leftrightarrow (4 + 3c)(4 + 9c) \neq 0$.

So, considering the above relations among the parameters $a, c$ and $f$ of systems (23) it was shown earlier that these systems can be brought (via a translation and additional notation) to systems (27).

It remains to determine the invariant polynomial which gives the expression of the discriminant $\Delta = 4(1 - r)$ . For these systems we calculate $N_{14} = 288(r - 1)x^2$ and $N_{15} = 2^9 3^7 rx^4$.

Therefore if for a system (14) the conditions $N_3 = N_5 = N_8 = N_{12} = 0,\ N_2N_{13} \neq 0$ are satisfied then we get the configuration *Config. 8.28* if $N_{15} < 0$; the configuration *Config. 8.29* if $N_{14} < 0,\ \ N_{15} > 0$ and the configuration *Config. 8.30* if $N_{14} > 0$.

### 3.3.3 The case $N_2 = N_3 = 0$

Then $l = m = 0$ and we get systems for which we calculate $N_{16} = -12ex^4$. In what follows we split our examination here in two subcases, defined by the polynomial $N_{16}$.

**1)** *The subcase* $N_{16} \neq 0$. Then $e \neq 0$ and systems (14) could be brought via a rescaling (i. e. assuming $e = 1$) to systems (30). According to Proposition 5 the last systems possess invariant lines of total multiplicity 8 if and only if the conditions (31) are satisfied. We prove that these conditions are equivalent to $N_{17} = N_{18} = 0$, i. e.

$$f = -2c, \quad a = 0 \ \Leftrightarrow \ N_{17} = N_{18} = 0.$$

Indeed, for the corresponding systems we calculate $N_{17} = 12(2c + f)x^2 = 0,\ N_{18} = 216ax^3 = 0$ and evidently, the above equalities are equivalent to $f = -2c,\ a = 0$.

It remains to determine the invariant condition which governs the value of $c$. For the last systems we determine $N_{10} = 72cx^2$. Next we split our examinations according to the parameter $c$.

***a)*** *The possibility* $N_{10} \neq 0$. Then $c \neq 0$ and assuming $b = 0$ after a translation we arrive at the system (33). So, if for systems (14) the conditions $N_2 = N_3 = N_{17} = N_{18} = 0,\ N_{10}N_{16} \neq 0$ are satisfied then we get the configuration *Config. 8.31* if $N_{10} < 0$ and *Config. 8.32* if $N_{10} > 0$.

***b)*** *The possibility* $N_{10} = 0$. Then $f = c = 0$ and after a rescaling we assume $b = 1$ and we get the systems (32). So, if for systems (14) the conditions $N_2 = N_3 = N_{10} = N_{17} = N_{18} = 0,\ N_{16} \neq 0$ are satisfied then we get the configuration *Config. 8.33*.

**2)** *The subcase* $N_{16} = 0$. Then $e = 0$ and systems (14) became of the form (34).

According to Proposition 6 the last systems possess invariant lines of total multiplicity 8 if and only if the conditions (38) hold. We prove that these conditions are equivalent to $N_{19} = 0,\ N_{18} \neq 0$, i.e.

$$27a^2 + (c - f)(2c + f)^2 = 0,\ a \neq 0\ \Leftrightarrow N_{19} = 0,\ N_{18} \neq 0.$$

Indeed, for systems (34) we have $N_{19} = 24[27a^2 + (c - f)(2c + f)^2]x^3y$ and, evidently, $N_{19} = 0$ implies $27a^2 + (c - f)(2c + f)^2 = 0$. On the other hand we have $N_{18} = 216ax^3$ and thus, the condition $N_{18} \neq 0$ is equivalent to $a \neq 0$. Therefore if the conditions $N_{19} = 0,\ N_{18} \neq 0$ are satisfied then systems (34) via a transformation and a suitable notation (see page 76) can be brought to systems (36). For these systems we calculate $N_{20} = 48(1 - 4r)x^4, \quad N_{21} = 48rx^4$.
Therefore if for a system (14) the conditions $N_2 = N_3 = N_{16} = N_{19} = 0$ and $N_{18} \neq 0$ hold then we obtain the configuration *Config. 8.34* if $N_{21} < 0$; *Config. 8.35* if $N_{20} > 0, N_{21} > 0$; *Config. 8.36* if $N_{20} = 0$ and *Config. 8.37* in the case $N_{20} < 0$. Moreover if $N_{21} = 0$, i.e. $r = 0$ we obtain *Config. 8.38*.

### 3.3.4 Perturbations of normal forms

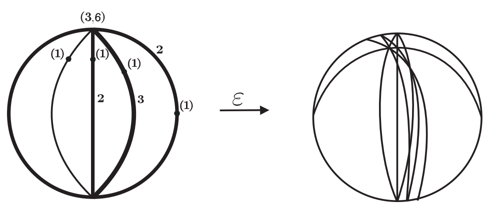
To finish the proof of the Main Theorem it remains to construct for the normal forms given in this theorem the corresponding perturbations, which prove that the respective invariant straight lines have the indicated multiplicities. In this section we construct such perturbations and for each configuration *Configs. 8.j*, $j = 23, 24, \ldots, 38$ we give: $(i)$ the corresponding normal form and its invariant straight lines; $(ii)$ the respective perturbed normal form with its invariant straight lines and $(iii)$ the configuration *Configs. 8.j*$_\varepsilon$, $j = 23, 24, \ldots, 38$ corresponding to the perturbed system.

$$\textit{Config. 8.23} \begin{cases} \dot{x} = (x-1)x(1+x), \\ \dot{y} = x - y + x^2 + 3xy; \end{cases}$$

*Invariant lines:* $L_{1,2} = x,\ L_{3,4,5} = x - 1,\ L_6 = x + 1,\ L_7:\ Z = 0;$

$$\textit{Config. 8.23}_\varepsilon\text{:} \begin{cases} \dot{x} = x(1+x)(x + 3x\varepsilon - 1), \\ \dot{y} = (1 + 3\varepsilon y)(x + x^2 - y + 3xy - 3\varepsilon y + 3\varepsilon xy - 6\varepsilon y^2 - 9\varepsilon^2 y^2); \end{cases}$$

$$\textit{Invariant lines:} \begin{cases} L_1 = x, \quad L_2 = x - 3\varepsilon y, \quad L_3 = x + 3\varepsilon x - 1,\ L_4 = x - 3\varepsilon y - 1, \\ L_5 = x - 3\varepsilon - 6\varepsilon y - 9\varepsilon^2 y - 1,\ L_6 = 1 + x,\ L_7 = 1 + 3\varepsilon y. \end{cases}$$



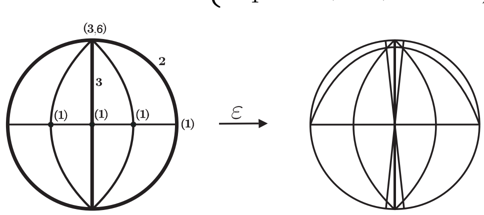

*Config. 8.23* *Config. 8.23*$_\varepsilon$

$$\textit{Config. 8.24-8.26} \begin{cases} \dot{x} = x(1-u+x)(1+u+x), \\ \dot{y} = (1 - u^2 + 2x)y, \quad |u| \neq 1, \end{cases} \begin{cases} |u| > 1 & \Rightarrow \textit{Config. 8.24}; \\ |u| < 1 & \Rightarrow \textit{Config. 8.25}; \\ u = 0 & \Rightarrow \textit{Config. 8.26}; \end{cases}$$

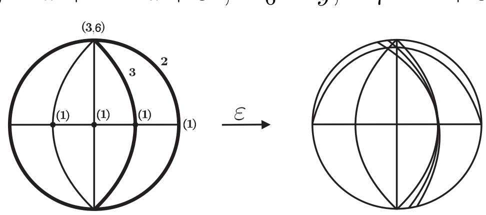
*Invariant lines:* $L_{1,2,3} = x,\ L_4 = x + 1 + u,\ L_5 = x + 1 - u,\ L_6 = y,\ L_7:\ Z = 0;$

$$\textit{Config. 8.24}_\varepsilon\textit{-8.26}_\varepsilon\text{:} \begin{cases} \dot{x} = x(1 - u + \varepsilon^2 + x)(1 + u - \varepsilon^2 + x), \\ \dot{y} = y(1 + \varepsilon y)\big[1 - (u - \varepsilon^2)^2 + 2x + \big(\varepsilon^2(u - \varepsilon^2)^2 - \varepsilon^2\big)y\big]; \end{cases}$$

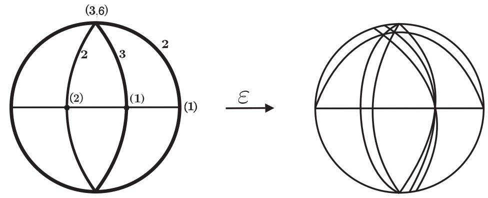
$$\textit{Invariant lines:} \begin{cases} L_1 = x, \quad L_2 = x - \varepsilon(u+1)y, \quad L_3 = x - \varepsilon(u-1)y - y\varepsilon^3, \\ L_4 = x + 1 + u - \varepsilon^2,\ L_5 = x + 1 - u + \varepsilon^2,\ L_6 = y,\ L_7 = 1 + \varepsilon y. \end{cases}$$




*Config. 8.24* *Config. 8.24*$_\varepsilon$




*Config. 8.25* *Config. 8.25*$_\varepsilon$



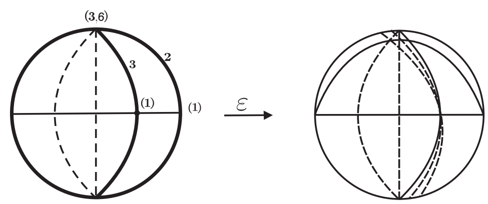

*Config. 8.26* *Config. 8.26*$_\varepsilon$

*Config. 8.27*: $\begin{cases} \dot{x} = x\big[(x+1)^2 + u^2\big], \\ \dot{y} = (1 + u^2 + 2x)y, \quad u \neq 0; \end{cases}$

*Invariant lines:* $L_{1,2,3} = x,\ L_4 = x+1+iu,\ L_5 = x+1-iu,\ L_6 = y,\ L_7:\ Z = 0;$

*Config. 8.27*$_\varepsilon$: $\begin{cases} \dot{x} = x\big[(x+1)^2 + u^2\big], \\ \dot{y} = y(1-y\varepsilon)(1 + u^2 + 2x + y\varepsilon + u^2 y\varepsilon); \end{cases}$

*Invariant lines:* $L_1 = x,\ L_{2,3} = x+\varepsilon y \pm iu\varepsilon y,\ L_{4,5} = x+1 \pm iu,\ L_6 = y,\ L_7 = -1+y\varepsilon.$



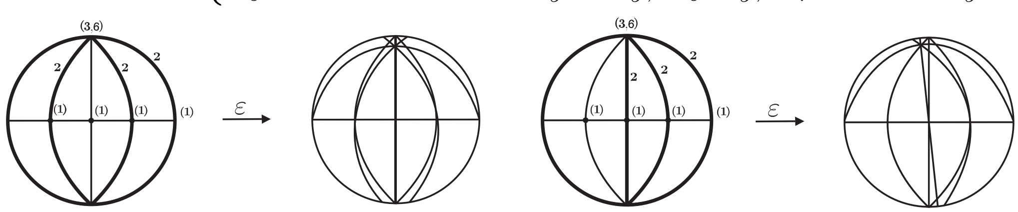

*Config. 8.27* *Config. 8.27*$_\varepsilon$

*Config. 8.28, 8.29* $\begin{cases} \dot{x} = (1-x+u)x(1-x-u), \\ \dot{y} = 2(u^2 + x - 1)y,\ |u| \neq 1, \end{cases}$ $\begin{cases} |u| > 1 \quad \Rightarrow \quad \textit{Config. } 8.28; \\ |u| < 1 \quad \Rightarrow \quad \textit{Config. } 8.29; \end{cases}$

*Invariant lines:* $L_1 = x,\ L_{2,3} = 1-x+u,\ L_{4,5} = 1-x-u,\ L_6 = y,\ L_7:\ Z = 0;$

*Config. 8.28*$_\varepsilon$, *8.29*$_\varepsilon$: $\begin{cases} \dot{x} = (1-x+u)x(1-x-u), \\ \dot{y} = y(1+u-\varepsilon y)(2u^2 + 2x + \varepsilon y - u\varepsilon y - 2)/(1+u); \end{cases}$

*Invariant lines:* $\begin{cases} L_1 = x, \quad L_2 = 1-x+u, \quad L_3 = 1-x+u-\varepsilon y,\ L_4 = 1-x-u, \\ L_5 = x - 1 + u^2 + ux + \varepsilon y - u\varepsilon y,\ L_6 = y,\ L_7 = 1+u-\varepsilon y. \end{cases}$



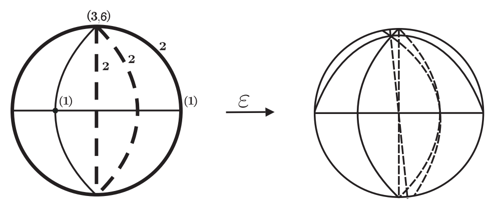

*Config. 8.28* *Config. 8.28*$_\varepsilon$ *Config. 8.29* *Config. 8.29*$_\varepsilon$

*Config. 8.30*: $\begin{cases} \dot{x} = x(1 + u^2 - 2x + x^2), \\ \dot{y} = 2y(x - 1 - u^2), \quad u \neq 0; \end{cases}$

*Invariant lines:* $L_1 = x,\ L_{2,3} = x-1-iu,\ L_{4,5} = x-1+iu,\ L_6 = y,\ L_7:\ Z = 0;$

*Config. 8.30*$_\varepsilon$: $\begin{cases}\dot{x} = x(1+u^2-2x+x^2),\\ \dot{y} = y(1-\varepsilon y)(2x-2-2u^2+\varepsilon y+u^2\varepsilon y);\end{cases}$

*Invariant lines:* $\begin{cases}L_1 = x,\ L_2 = x-1-iu,\ L_3 = x-1-iu+y\varepsilon+iu\varepsilon y,\\ L_4 = x-1+iu,\ L_5 = x-1+iu+\varepsilon y-iu\varepsilon y,\ L_6 = y,\ L_7 = \varepsilon y-1.\end{cases}$

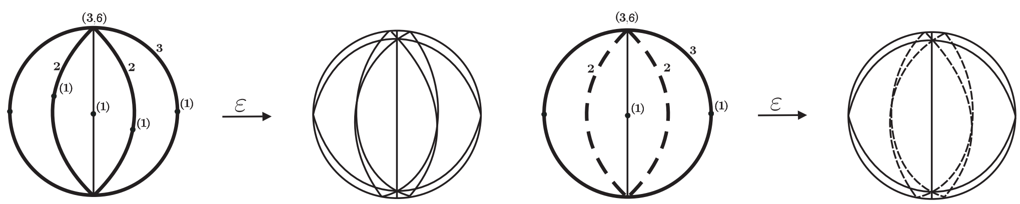



*Config. 8.30* *Config. 8.30*$_\varepsilon$

*Config. 8.31, 8.32* $\begin{cases}\dot{x} = x(x^2+r),\\ \dot{y} = x-2ry,\end{cases}$ $\begin{cases}r=-1 \ \Rightarrow\ \textit{Config. 8.31};\\ r=1 \ \Rightarrow\ \textit{Config. 8.32};\end{cases}$

*Invariant lines:* $L_1 = x,\ L_{2,3} = x-\sqrt{-r},\ L_{4,5} = x+\sqrt{-r},\ L_6 = y,\ L_{6,7}: Z=0;$

*Config. 8.31*$_\varepsilon$*, 8.32*$_\varepsilon$: $\begin{cases}\dot{x} = (2r-\varepsilon^4+\varepsilon^6)(4r+4x^2-4r\varepsilon^2-3\varepsilon^4+6\varepsilon^6-3\varepsilon^8)\times\\ (x-x\varepsilon+6ry\varepsilon+2ry\varepsilon^2-3y\varepsilon^5-y\varepsilon^6+3y\varepsilon^7+y\varepsilon^8)/(8r),\\ \dot{y} = (x-2ry+\varepsilon^4y-y\varepsilon^6)(4r-4r\varepsilon^2+16r^2\varepsilon^2y^2-3\varepsilon^4+\\ +6\varepsilon^6-16r\varepsilon^6y^2-3\varepsilon^8+16r\varepsilon^8y^2+4\varepsilon^{10}y^2-8\varepsilon^{12}y^2+4\varepsilon^{14}y^2)/(4r);\end{cases}$



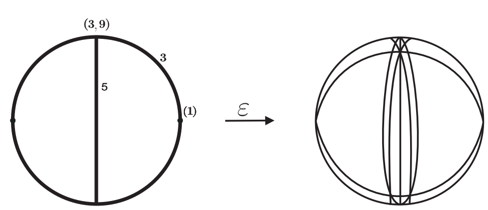

*Config. 8.31* *Config. 8.31*$_\varepsilon$ *Config. 8.32* *Config. 8.32*$_\varepsilon$

*Config. 8.33*: $\{\dot{x} = x^3, \quad \dot{y} = 1+x;$

*Invariant lines:* $L_{1,2,3,4,5} = x,\ L_{6,7}:\ Z=0;$

*Config. 8.33*$_\varepsilon$: $\begin{cases}\dot{x} = x(9x-6\varepsilon+4\varepsilon^2)(9x+6\varepsilon-10\varepsilon^2+4\varepsilon^3)/81,\\ \dot{y} = (3-2\varepsilon+y\varepsilon^2)(3-2\varepsilon-y\varepsilon^2)(9+9x-15\varepsilon+6\varepsilon^2-\varepsilon^2y+\varepsilon^3y)/81;\end{cases}$

*Invariant lines:* $\begin{cases}L_1 = x,\ L_2 = x-6\varepsilon+4\varepsilon^2,\ L_3 = x+6\varepsilon-10\varepsilon^2+4\varepsilon^3,\\ L_4 = x-3\varepsilon+2\varepsilon^2+\varepsilon^3y,\ L_5 = x+3\varepsilon-5\varepsilon^2+2\varepsilon^3-\varepsilon^3y+\varepsilon^4y,\\ L_6 = 3-2\varepsilon+\varepsilon^2y,\ L_7 = -3+2\varepsilon+\varepsilon^2y.\end{cases}$



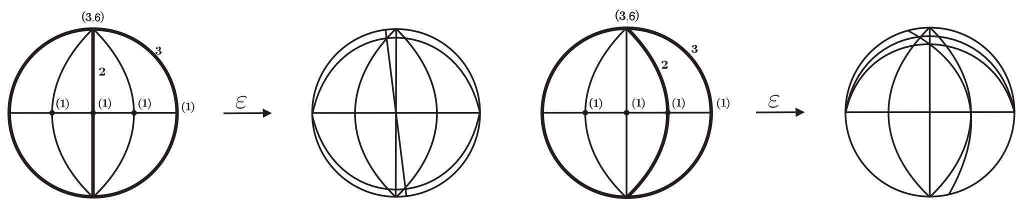

*Config. 8.33* *Config. 8.33*$_\varepsilon$

*Config. 8.34-8.37* $\begin{cases}\dot{x}=x(r+x+x^2),\\ \dot{y}=ry, \quad r\neq 0,\end{cases}$ $\begin{cases} r<0 & \Rightarrow \textit{Config. } 8.34;\\ 0<r<1/4 & \Rightarrow \textit{Config. } 8.35;\\ r=1/4 & \Rightarrow \textit{Config. } 8.36;\\ r>1/4 & \Rightarrow \textit{Config. } 8.37;\end{cases}$

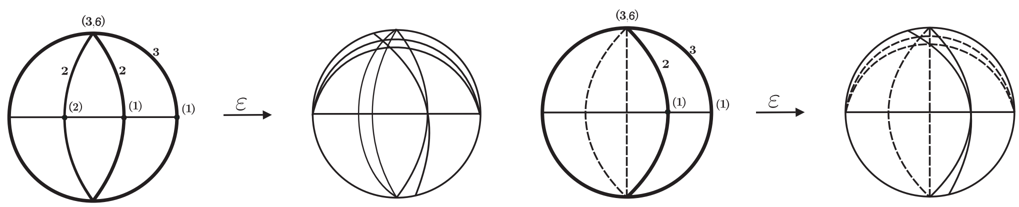
*Invariant lines:* $L_{1,2}=x,\ L_{3,4}=r+x+x^2,\ L_5=y,\ L_{6,7}:\ Z=0;$

*Config. 8.34$_\varepsilon$-8.37$_\varepsilon$:* $\begin{cases}\dot{x}=x(r-\varepsilon^2+x+x^2),\\ \dot{y}=y(r-\varepsilon^2-\varepsilon y+\varepsilon^2y^2);\end{cases}$

*Invariant lines:* $\begin{cases}L_1=x,\quad L_2=x-\varepsilon y,\quad L_{3,4}=r+x+x^2-\varepsilon^2,\\ L_{5,6}=r-\varepsilon y-\varepsilon^2+\varepsilon^2y^2,\ L_7=y.\end{cases}$




*Config. 8.34* *Config. 8.34$_\varepsilon$* *Config. 8.35* *Config. 8.35$_\varepsilon$*



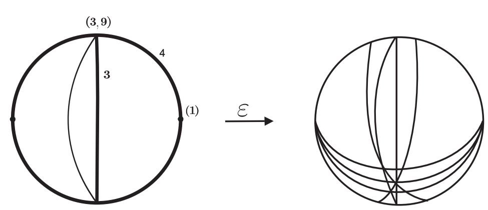

*Config. 8.36* *Config. 8.36$_\varepsilon$* *Config. 8.37* *Config. 8.37$_\varepsilon$*

*Config. 8.38*: $\{\,\dot{x}=x^2(x+1),\quad \dot{y}=1;$

*Invariant lines:* $L_{1,2,3}=x,\ L_4=x+1,\ L_{5,6,7}:\ Z=0;$

*Config. 8.38$_\varepsilon$:* $\begin{cases}\dot{x}=x(x-\varepsilon)(1+x+\varepsilon-2\varepsilon y),\\ \dot{y}=(\varepsilon y-1)(2\varepsilon y-1)(1-2\varepsilon y+2\varepsilon^2 y);\end{cases}$

*Invariant lines:* $\begin{cases}L_1=x,\ L_2=x-\varepsilon,\ L_3=x+\varepsilon-2y\varepsilon^2, L_4=1+x-\varepsilon-2y\varepsilon+2y\varepsilon^2,\\ L_5=y\varepsilon-1,\ L_6=2y\varepsilon-1,\ L_7=1-2y\varepsilon+2y\varepsilon^2.\end{cases}$




*Config. 8.38* *Config. 8.38$_\varepsilon$*

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# Primary decomposition of general graded structures

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**Abstract.** In this paper we discuss the primary decomposition in the case of general graded modules – moduloids, a generalization of already done work for general graded rings – anneids. These structures, introduced by Marc Krasner are more general than graded structures of Bourbaki since they do not require the associativity nor the commutativity nor the unitarity in the set of grades. After proving the existence and uniqueness of primary decomposition of moduloids, we breafly turn our attention to Krull's Theorem and to the existence of the primary decomposition of Krasner–Vuković paragraded rings.

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## 1 Introduction

The graded primary decomposition of graded modules graded by torsion free Abelian groups can be found in Bourbaki [1]. The more general case of graduations by finitely generated Abelian groups is covered in papers of M. Perling, S. D. Kummar [16], and S. D. Kummar, S. Behara [15]. However, M. Krasner introduced the theory of general graded structures (groups, rings, modules) where nothing is assumed for the set of grades except its nonemptiness, since, in his definitions, additive graduation and structures of rings and modules will imply operations (generally partial) in the set of grades. It all started when M. Krasner defined the notion of a corpoid while he was studying valued division rings [4]. Corpoid is actually the homogeneous part of a division ring, graded by an arbitrary set, with induced operations among which the induced addition is, naturally, a partial operation, since the sum of two homogeneous elements does not have to be homogeneous. As a generalization of a corpoid, we have the notion of an anneid – the homogeneous part of a graded ring, and finally, the notion of a moduloid over an anneid. The general graded theory continued to develop [2, 3, 5–9, 14]. Particularly, M. Chadeyras considered the existence and uniqueness of the primary decomposition of commutative anneids in [2]. Unlike the abstract case, an irreducible ideal of a Noetherian anneid is not in general a primary ideal, but it is under certain assumptions. In this paper we will extend these results to the case of moduloids. Since proofs of propositions in the case of moduloids are similar to ones in the case of anneids discussed in [2], which were inspired by those given by O. Zariski, P. Samuel in [19], we will give proofs only of some results. Krull's Theorem for moduloids is followed by the brief observation of the existence of primary decomposition of Noetherian paragraded rings, actually of their homogeneous parts. Paragraduations were introduced by M. Krasner and M. Vuković [11–14] and these structures are closed with respect to the direct sum (direct product) in the sense that the homogeneous part of obtained structure is the direct sum (direct product) of homogeneous parts of components (see [14, 18]), which is not necessarily true for their graded counterparts.

## 2 Preliminaries

According to the definition of Bourbaki–Krasner [10, 14], a graded group $G$, with the neutral element $e$, is a group with *graduation* $\gamma : \Delta \to \mathrm{Sg}(G)$, $\gamma(\delta) = G_\delta$ ($\delta \in \Delta$), such that $G = \bigoplus_{\delta\in\Delta} G_\delta$, where $\Delta$ is a nonempty set, called the *set of grades*, and $\mathrm{Sg}(G)$ the set of subgroups of $G$. We also say that $G$ has graduation $(\Delta, \gamma)$. The set $H = \bigcup_{\delta\in\Delta} G_\delta$ is called the *homogeneous part* of $G$ and elements from $H$ are called *homogeneous*. The grade $\xi \in \Delta$ for which $e \neq x \in G_\xi$ is called the *grade* of a homogeneous element $x$ and is denoted by $\delta(x)$. The neutral element $e$ does not have a grade but we may associate a grade to it from $\Delta \setminus \Delta^*$, where $\Delta^* = \{\delta \in \Delta \mid G_\delta \neq \{e\}\}$, and call it the *zero grade* and write $\delta(e) = 0$. If $\Delta = \Delta^* \cup \{0\}$, a graded group $G = \bigoplus_{\delta\in\Delta} G_\delta$ is called *proper*. We will assume graduations throughout the paper to be proper. Multiplicative operation on $G$ induces the partial operation on $H$. If $x, y \in H$, then $xy$ is defined in $H$ if and only if $xy \in G$ is the element from $H$, and in that case the result is the same and we write it in the same way. If this situation occurs, we say that the elements $x, y$ are *composable* or *multipliable* (*addible* in the case of an additive operation) and we write $x\#y$ [10, 14]. Elements $x, y$ are composable if and only if they come from the same subgroup $G(a) = \{x \in H \mid ax \in H\}$, $a \in H^* = H \setminus \{e\}$. In [10, 14] it is proved that when $H$ with the operation induced from $G$ is given, then we may reconstruct $G$ up to $H$-isomorphism, namely $G = \bigoplus_{G(a)\in D^*} G(a)$, where $D^* = \{G(a) \mid a \in H^*\}$. Also, from [10, 14] we know that $H$ is the homogeneous part of some graded group $G$, with operation induced from that group, if and only if:

*i*) $(\exists e \in H)(\forall x \in H)\ x\#e \wedge xe = x$;

*ii*) $(\forall x \in H)\ x\#x$;

*iii*) $(\forall x, y, z \in H)\ x\#y \wedge y\#z \wedge y \neq e \Rightarrow x\#z$;

*iv*) for all $a \in H^*$, $H(a) = \{x \in H \mid a\#x\}$ is a group.

The structure that satisfies formentioned axioms is called a *homogroupoid* [10].

If $R$ is a ring whose additive group has a proper graduation $(\Delta, \gamma)$, it is called a *graded ring* if for all $\xi, \eta \in \Delta$ there exists $\zeta \in \Delta$ such that $R_\xi R_\eta \subseteq R_\zeta$ [10]. Clearly, this $\zeta$ is unique if $R_\xi R_\eta \neq \{0\}$. According to Krasner's Lemma [3], if $(a, a')$ and $(b, b')$ are two pairs of elements of the same grade and if $ab \neq 0$, $a'b' \neq 0$, then $ab$ and $a'b'$ are of the same grade. Hence, we may define a multiplicative operation on

$\Delta$ in the following manner [10, 14]: if $\xi, \eta \in \Delta$ we put

$$\xi\eta = \begin{cases} 0, & R_\xi R_\eta = \{0\}; \\ \zeta, & \{0\} \neq R_\xi R_\eta \subseteq R_\zeta. \end{cases}$$

This multiplication does not have to be associative. However, if $R_\xi R_\eta R_\zeta \neq \{0\}$, then $(\xi\eta)\zeta = \xi(\eta\zeta)$. Thus, this definition represents a generalization of the definition of a graded ring given by Bourbaki [1]. If $A = \bigcup_{\delta\in\Delta} R_\delta$ – the homogeneous part of $R$, then $A$, with respect to operations induced by $R$, satisfies the following axioms [10, 14]:

*i)* $A$ is a multiplicative semigroup with a biabsorbing element 0, i.e. with an element 0 for which $a0 = 0a = 0$, for all $a \in A$;

*ii)* $(\forall a, b, c \in A)\ a\#b \wedge b\#c \wedge b \neq 0 \Rightarrow a\#c$;

*iii)* $(\forall a \in A)\ a\#0$;

*iv)* $(\forall a, b \in A)\ a\#b \Rightarrow b\#a$;

*v)* $(\forall a \in A)\ a\#a$;

*vi)* $(\forall a \in A \setminus \{0\})$ the set $A(a) = \{x \in A \mid x\#a\}$ is a group with respect to addition;

*vii)* distributivity of multiplication with respect to addibility and additivity holds.

An *anneid* [10] is a nonempty set $A$ which satisfies the axioms $i) - vii)$. If $A$ is an anneid, then we put $\Delta = \Delta^* \cup \{0\}$, where $\Delta^* = \{A(a) \mid a \in A \setminus \{0\}\}$. Then $\overline{A} = \bigoplus_{A(a)\in\Delta^*} A(a)$ is a graded ring and is called an *associated graded ring* to an anneid $A$ or a *linearization* of $A$ [2, 3, 10].

Let $R$ be a graded ring with graduation $(\Delta, \gamma)$, $(M, +)$ a commutative graded group with graduation $(D, \Gamma)$, and let $M$ be an $R$-module with external multiplication $(a, x) \to ax$ $(a \in R,\ x \in M)$. $M$ is a *graded $R$-module* if $(\forall \xi \in \Delta)(\forall s \in D)(\exists t \in D)\ R_\xi M_s \subseteq M_t$ [10].

Let $M$ be a commutative additive homogroupoid, $A$ an anneid, and let the external multiplication $(a, x) \to ax$ $(a \in A, x \in M, ax \in M)$ have the following properties: $i)$ $a\#b \Rightarrow ax\#bx$ and $(a + b)x = ax + bx$; $ii)$ $x\#y \Rightarrow ax\#ay$ and $a(x + y) = ax + ay$; $iii)$ $a(bx) = (ab)x$. Then $M$ is called an *$A$-moduloid* [10]. If $A$ is unitary, then $M$ is unitary if $1x = x$, for all $x \in M$. We will assume all moduloids unitary. A graded $\overline{A}$-module $\overline{M} = \bigoplus M(x)$, where $M(x)$ runs through the set of addibility groups of a homogroupoid $M$, is called an *associated graded module* to a moduloid $M$. A nonempty subset $N$ of an $A$-moduloid $M$ is called a *submoduloid* if $x - y \in N$ for all addible $x, y \in N$, and $ax \in N$ for all $a \in A$ and $x \in N$.

## 3 Primary decomposition of submoduloids

Like in the abstract case, we begin with the definition of a primary submoduloid of a moduloid. The notion of a primary decomposition of a submoduloid is clear enough. Also, the notion of a Noetherian moduloid is analogous to the ungraded case.

**Definition 1.** A submoduloid $N$ of a moduloid $M$ over an anneid $A$ is called *primary* if $N \neq M$ and whenever $a \in A$, $x \in M$ and $ax \in N$ implies $x \in N$ or $a^n M \subseteq N$, for some $n \in \mathbb{N}$.

The following result is straightforward.

**Lemma 1.** *If $N$ is a primary submoduloid of an $A$-moduloid $M$, then $\sqrt{N : M} = \{a \in A \mid (\exists n \in \mathbb{N})\ a^n M \subseteq N\}$ is a prime ideal of $A$. If $P = \sqrt{N : M}$, then we say that $N$ is $P$-primary.*

Before we proceed, we need to make few observations which are analogous to what Krasner [10] and Chadeyras [2] did in the case of anneids. First, let $X$ be a subset of a moduloid $M$ over an anneid $A$. Denote by $X_+$ an additive homogroupoid generated by $X$ and if $X$ and $Y$ are subsets of $M$, let $X + Y$ be the set $\{x + y \mid x \in X \wedge y \in Y \wedge x \# y\}$. If $AX \subseteq X$ and $AY \subseteq Y$, then it is easy to prove that $A(X + Y) \subseteq X + Y$. Also, if $X$ and $Y$ are additive subhomogroupoids of $M$, then $X + Y$ is also an additive subhomogroupoid and so, if $X$ and $Y$ are submoduloids of $M$, $X + Y$ is also a submoduloid of a moduloid $M$. If $AX \subseteq X$, then $AX_+ \subseteq X_+$ and $X_+$ is a submoduloid of a moduloid $M$. We are particularly interested in the case $X = \{m\}$. We denote $(m) = (Am)_+$. Let $M$ and $M'$ be two $A$-moduloids, where $A$ is an anneid. The mapping $f : M \to M'$ is called a *quasihomomorphism* if $x \# y \Rightarrow f(x) \# f(y)$ and in that case $f(x+y) = f(x) + f(y)$, and also $f(ax) = af(x)$, where $x, y \in M$ and $a \in A$ (see [2, 10, 14] for more details). In [2] M. Chadeyras observed *agglutinations* $M^{(f)} = \bar{f}^{-1}(M')$, where $\bar{f} : \overline{M} \to \overline{M'}$, $f = \bar{f}|_M$, is a *quasihomogeneous homomorphism* of graded modules, that is, it is a homomorphism of modules and $f(M) \subseteq M'$ [2]. In particular, for $a \in A$, the mapping $f_a : M \to M$, $f(x) = ax$ $(x \in M)$ is a quasihomomorphism and let $M^{(a)} = \overline{f_a}^{-1}(M)$. M. Krasner in [10] proved that if $M$ is Noetherian with every element from $\Delta$ being semiregular, that is, if $M$ is *strong Noetherian*, then the chain $M^{(a)} \subseteq M^{(a^2)} \subseteq \dots$ is stationary, i. e. there exists $n$ such that $M^{(a^n)} = M^{(a^{n+1})}$; the smallest such $n$ is called an *exponent of semiregularity*. An element $\delta \in \Delta$ is called *semiregular* if the sequence $(\epsilon^{(a^n)})$ is finite, where $\epsilon^{(a)}$ is an equivalence of grades defined by $d_1 \sim d_2 \Leftrightarrow \delta(a) d_1 = \delta(a) d_2$, $d_1, d_2 \in D$, where $D$ is the set of grades of $\overline{M}$.

**Lemma 2.** *Each submoduloid of a Noetherian moduloid is the intersection of finitely many irreducible submoduloids.*

*Proof.* This follows from Zorn's Lemma. □

The following lemma is crucial in our discussion, since an irreducible submoduloid is not in general a primary submoduloid. The assumption of strongness imposed on a Noetherian moduloid removes this issue.

**Lemma 3.** *Let $N$ be a submoduloid of a strong Noetherian $A$-moduloid $M$, where $A$ is an anneid. If $N$ is irreducible, then it is primary.*

*Proof.* Suppose $N$ is not primary. Then there exist $m \in M \setminus N$ and $a \in A$ such that $am \in N$ and $a^n M \nsubseteq N$, for every $n \in \mathbb{N}$. Then we have an ascending chain of submoduloids $(N : \{a^n\}) = \{x \in M \mid a^n x \in N\}$. Since $M$ is Noetherian, there exists $s \in \mathbb{N}$ such that $(N : \{a^s\}) = (N : \{a^{s+1}\})$. Also, since $M$ is strong Noetherian, by M. Krasner [10] there exists $r \in \mathbb{N}$ such that $M^{(a^r)} = M^{(a^{r+1})}$. Let $n = r + s$. Submoduloids $N_1 = N + (m)$ and $N_2 = N + (a^n)$ strictly contain $N$. Let $0 \neq x \in N_1 \cap N_2$. Then there exist $\alpha, \beta \in N$, $\xi \in M^{(a^n)}$ and $\eta \in A$ such that $x = \alpha + \xi a^n = \beta + \eta m$. All elements are mutually addible. Now, $ax = a\beta + \eta am$ and since $am \in N$, we have $ax \in N$ and so $\xi a^{n+1} \in N$. Let $\zeta = \xi a^r$. Then $\zeta \in M$ and $\zeta a^{s+1} = \xi a^{n+1} \in N$ and from $(N : \{a^s\}) = (N : \{a^{s+1}\})$ we have that $\xi a^n = \zeta a^s \in N$. Hence, $x \in N$. ☐

The notion of a *reduced primary decomposition* is defined as in the case of abstract modules [19].

**Corollary 1.** *Each submoduloid of a strong Noetherian moduloid has a reduced primary decomposition.*

**Corollary 2.** *A Noetherian module has a reduced primary decomposition.*

*Proof.* Let $M$ be a Noetherian $R$-module. Then $R$ may be viewed as a graded ring via trivial graduation, and as its homogeneous part coincides with $R$, it may be regarded as an anneid. Analogously, $M$ is a moduloid over an anneid $R$, if observed as a graded module with trivial graduation. Hence, $M$ is a strong Noetherian $R$-moduloid with the exponent of semiregularity equal to 1 and it admits a primary decomposition. ☐

**Definition 2.** Let $M$ and $M'$ be two unitary moduloids over an anneid $A$, $f : M \to M'$ a quasihomomorphism, and $N, N'$ submoduloids of $M, M'$, respectively. Then $(N')^c := f^{-1}(N')$ is a submoduloid of $M$ called *a contraction* of $N'$ and $N^e := \langle f(N) \rangle$ is a submoduloid of $M'$ called *an extension* of $N$.

It is easy to prove the following

**Lemma 4.** *Let $M$ and $M'$ be two unitary moduloids over an anneid $A$ and $f : M \to M'$ a quasihomomorphism. If $N'$ is a $P'$-primary submoduloid of $M'$, then $(P')^c$ is a prime ideal and $(N')^c$ is $(P')^c$-primary.*

**Corollary 3.** *Let $M$ be an $A$-moduloid. If $\overline{M}$ is Noetherian, then $M$ admits a reduced primary decomposition.*

*Proof.* Let $N$ be a submoduloid of $M$ and let $\overline{N}$ be its linearization, that is a homogeneous submodule of a graded module $\overline{M}$. Then, since $\overline{M}$ is Noetherian, $\overline{N}$ has a primary decomposition. The assertion now follows from the previous lemma and the fact that $N = \overline{N} \cap M$. $\square$

*Remark* 1. It should be noted that if a moduloid admits a primary decomposition, then this does not imply that its linearization has the same property.

Let us now give the uniqueness theorems.

**Theorem 1.** *Let $M$ be an $A$-moduloid and $N$ a submoduloid with a reduced primary decomposition $N = \bigcap_i N_i$ and let $P_i = \sqrt{(N_i : M)}$. Then $P_i$'s are prime ideals $P$ in $A$ for which there exists $x \in M$, $x \notin N$, such that $(N : \{x\})$ is a $P$-primary ideal.*

*Proof.* The proof is similar to the classical case [19]. Namely, if $x \in \bigcap_{j \neq i} N_j$, $x \in N_i$, then $(N : \{x\})$ contains the annihilator of $M/N_i$, and hence it can be proved that $(N : \{x\})$ is $P_i$-primary. The converse is easy as well. $\square$

**Theorem 2.** *If $N$ is a submoduloid of an $A$-moduloid $M$ which has a reduced primary decomposition $N = \bigcap_i N_i$, $N_i$ a $P_i$-primary, then the minimal elements of the family of all prime ideals $P_i$ are also the minimal elements of the family of all prime ideals $P$ which contain the annihilator of $M/N$.*

*Proof.* Let $Q_i = \operatorname{ann}(M/N_i)$. Then $Q_i$ is a $P_i$-primary ideal and $\operatorname{ann}(M/N) = \bigcap_i Q_i$. Since $\bigcap_i Q_i$ represents the primary decomposition of $\operatorname{ann}(M/N)$, the assertion follows from the known result for anneids [2] which claims that a prime ideal of an anneid $A$ contains an ideal $I$, which has a reduced primary decomposition $\bigcap_i Q_i$, $Q_i$ a $P_i$-primary, if and only if it contains one of the $P_i$'s. $\square$

**Theorem 3.** *Let $N$ be a submoduloid of an $A$-moduloid $M$ which has a reduced primary decomposition $N = \bigcap_i N_i$, $N_i$ a $P_i$-primary. The set $N_i'$ of all elements $x \in M$ for which there exists $a \notin P_i$ such that $ax \in N$ is a submoduloid of $M$ which is contained in $N_i$, and, if $P_i$ is a minimal element of the family $\{P_i\}$, then $N_i' = N_i$.*

*Proof.* Let $x_1, x_2 \in N_i'$ and $x_1 \# x_2$. Then there exist $a_1, a_2 \notin P_i$ such that $a_1x_1, a_2x_2 \in N$, and hence, $a_1a_2(x_1 - x_2) \in N$ while $a_1a_2 \notin P_i$, which proves that $N$ is a submoduloid of $M$. The inclusion $N_i' \subseteq N_i$ is clear enough. If $P_i$ is a minimal element of the family $\{P_i\}$, then for all $j \neq i$, we have $P_j \nsubseteq P_i$. Let $a_j \in P_j \setminus P_i$, $n(j) \in \mathbb{N}$ be such that $a_j^{n(j)}M \subseteq N_j$ and $a = \prod_{j \neq i} a_j^{n(j)}$. Then $a \notin P_i$ and, if $x \in N_i$, then $ax \in N$, which means that $x \in N_i'$. $\square$

## 4 Krull's Theorem

The proof of the following result runs exactly as in the abstract case [19].

**Lemma 5.** *If $A$ is a strong Noetherian anneid and $M$ a Noetherian $A$-moduloid, then if $Q$ is an ideal of $A$ and $N$ a submoduloid of $M$, then there exist an integer $s$ and a submoduloid $N'$ of $M$ such that $QN = N \cap N'$ and $N' \supseteq Q^sM$.*

**Lemma 6.** *Let $Q$ be an ideal of a unitary anneid $A$ and let $N$ be a submoduloid of $M$. If $N = QN$ and if $N$ is finitely generated, then for all $0 \neq x \in N$, $x \in Qx$.*

*Proof.* Let $\{x_1 = x, x_2, \dots, x_n\}$ be the generators of $N$. From $N = QN$ we have that each $x_i$ can be written as a linear combination of $x_1, \dots, x_n$ over $Q$. Thus, we have $n$ equations

$$-\mu_1^i x_1 - \cdots - (\mu_i^i - 1)x_i - \cdots - \mu_j^i x_j - \cdots = 0 \qquad i = 1, \dots, n,$$

where $\mu_j^i \in \overline{Q}$. If we do the calculations in $\overline{M}$, we will obtain a determinant of coefficients equal to $1 - \mu$, where $\mu \in \overline{Q}$, such that $(1-\mu)x_1 = (1-\mu)x = 0$. By regrouping the addible elements of $M$ in the development of $(1-\mu)x$, we get $(1-\alpha)x + \beta_1 x + \cdots + \beta_s x = 0$, $\alpha, \beta_1, \dots, \beta_s \in \overline{Q}$. Since $\overline{M}$ is the direct sum of groups of addibility, we have that $(1-\alpha)x = 0$ which implies that $\alpha = \sum_k a_k \in \overline{Q}$, where $x$ and $a_k x$ are mutually addible. □

Now we may formulate and prove Krull's Theorem for moduloids.

**Theorem 4** (Krull's Theorem)**.** *Let $A$ be a Noetherian strong anneid with unity, and $Q$ an ideal of $A$. If $M$ is a Noetherian $A$-moduloid, then $\bigcap_{n=1}^{\infty} Q^n M = \{0\}$ if and only if $x \notin Qx$, for all $0 \neq x \in M$.*

*Proof.* Let $x \neq 0$ and $x \in Qx$. Then there exists $\alpha = \sum_k a_k$ $(a_k \in Q)$ such that $a_k$ are mutually addible as well as $a_i x \# a_j x$ $(i \neq j)$ and $x = \alpha x = \sum_k a_k x$. Hence,

$$x = \sum_k a_k (\sum_{k'} a_{k'} x) = \sum_k a_k (\sum_{k'} a_{k'} (\sum_{k''} a_{k''} x)) = \dots$$

and so $x \in Q^n M$ for all $n \in \mathbb{N}$. Thus, $\bigcap_{n=1}^{\infty} Q^n M \neq \{0\}$. Conversely, let $\bigcap_{n=1}^{\infty} Q^n M = N \neq \{0\}$. Then there exist an integer $s$ and a submoduloid $N'$ such that $QN = N \cap N'$ with $N' \supseteq Q^s M$. So, $QN \supseteq N \cap Q^s M = N$, which means that $N = QN$, and so for all $0 \neq x \in N \subseteq QM$, $x \in Qx$. □

## 5 Primary decomposition of quasianneids

We start by giving less known notions introduced in [11–14]. The mapping $\pi : \Delta \to \mathrm{Sg}(G)$, $\pi(\delta) = G_\delta$ $(\delta \in \Delta)$, of partially ordered set $(\Delta, <)$, which is from below a complete semi-lattice and from beyond inductively ordered, to the set $\mathrm{Sg}(G)$ of subgroups of the group $G$, is called a *paragraduation* if it satisfies the following six-axiom system:

*i)* $\pi(0) = G_0 = \{e\}$, where $0 = \inf \Delta$; $\delta < \delta' \Rightarrow G_\delta \subseteq G_{\delta'}$;

*Remark* 2. $H = \bigcup_{\delta \in \Delta} G_\delta$ is called *the homogeneous part* of $G$ with respect to $\pi$.

*Remark* 3. If $x \in H$, we say that $\delta(x) = \inf\{\delta \in \Delta \mid x \in G_\delta\}$ is the grade of $x$. We have $\delta(x) = 0$ if and only if $x = e$. Elements $\delta(x)$, $x \in H$, are called *principal grades* and they form a set which we will denote by $\Delta_p$.

*ii*) $\theta \subseteq \Delta \Rightarrow \bigcap_{\delta\in\theta} G_\delta = G_{\inf\theta}$;

*iii*) If $x, y \in H$ and $yx = zxy$, then $z \in H$ and $\delta(z) \leq \inf\{\delta(x), \delta(y)\}$;

*iv*) Homogeneous part $H$ is a generating set of $G$;

*v*) Let $A \subseteq H$ be a subset such that for all $x, y \in A$ there exists an upper bound for $\delta(x), \delta(y)$. Then there exists an upper bound for all $\delta(x)$, $x \in A$;

*vi*) $G$ is generated by $H$ with the set of $H$-inner and left commutation relations:

1. $xy = z$ ($H$-inner relations);
2. $yx = z(x, y)xy$ (left commutation relations).

A group with paragraduation is called a *paragraded group.* A ring $R$ is called *paragraded* if its additive group is paragraded and if for all $\xi, \eta \in \Delta$ there exists $\zeta \in \Delta$ such that $R_\xi R_\eta \subseteq R_\zeta$. If $R$ is a paragraded ring with homogeneous part $H$, then we may observe restrictions of operations on $R$ to $H$. Induced addition is partial and we write $x\#y$ if and only if $x+y \in H$. The obtained structure is called a *paraanneid* [14]. If $x \in H$, let $H(x) = \{y \in H \mid x\#y\}$. Paraanneid certainly satisfies the following axioms:

*i*) There exists an element $0 \in H$ such that $H = H(0)$ and such that for all $x \in H$ we have $0 + x = x$;

*ii*) If $a \in H$, $x + y$ is always defined on $g(a) = \{x \in H \mid H(x) \supseteq H(a)\}$ and $(g(a), +)$ is an Abelian group;

*iii*) If $A \subseteq H$ is such that for all $x, y \in A$ we have $x\#y$, then there exists $g \subseteq H$ such that $x + y \in g$ for all $x, y \in g$, $x \in g$ implies $g(x) \subseteq g$ and $A \subseteq g$;

*iv*) $H^2 \subseteq H$;

*v*) $x\#x'$ and $y\#y'$ imply $xy\#x'y'$.

Structure $(H, +, \cdot)$ which satisfies axioms $i) - v)$ is called a *quasianneid* [14]. A quasianneid however does not have to be a paraanneid; it is under few more assumptions [14]. Let us notice that *iv*) and *v*) imply

*vi*) If $x\#y$ then $z(x + y) = zx + zy$ and $(x + y)z = xz + yz$.

Let us now suppose that a paraanneid $H$ is commutative, and let us consider the mapping $\varphi_a : x \to ax$ $(x \in H)$, where $a \in H$. It is a quasiendomorphism [14] of $H$ (definition is analogous to the notion of a quasiendomorphism for anneids) and let $H^{[a]} = \varphi_a^{-1}(\hat{H})$, where $\hat{H} = \langle\varphi_a(H)\rangle$. The mapping $\varphi_a$ defines the equivalence $\epsilon_a$ on the set of grades $\Delta$ of $H$ in the following manner: $\delta_1 \sim \delta_2 \Leftrightarrow \delta(a)\delta_1 = \delta(a)\delta_2$. Obviously, $(H^{[a]}, +) = (H, +)$ implies discrete equivalence $\theta[x \sim y \Rightarrow x = y]$. Since $\varphi_{ab} = \varphi_a\varphi_b$, we have $H^{[ab]} \supseteq H^{[b]}$, and we write $H^{[ab]} \geq H^{[b]}$. Also, the equivalence $\epsilon_b$ is finer than $\epsilon_{ab}$ and we write $\epsilon_{ab} \geq \epsilon_b$. Hence $H \leq H^{[a]} \leq H^{[a^2]} \leq \ldots$ and

$\theta \le \epsilon_a \le \epsilon_{a^2} \le \dots$. If the sequence $(H^{[a^n]})$ resp. $(\epsilon_{a^n})$ is stationary, then we say that $a$ is a semiregular element resp. semiregular grade [14]. A paraanneid $H$ is called *strong* [14] if every $a \in H$ is semiregular. The notion of a Noetherian paraanneid is clear. A nonempty subset $Q$ of a paraanneid $H$ is called an *ideal* if $x - y \in Q$ for all addible $x, y \in Q$, and if $ax \in Q$ for all $a \in H$ and $x \in Q$. The notion of a *primary ideal* is analogous to the abstract case. Now, as in the case of anneids and moduloids, one can prove the following

**Theorem 5.** *Each ideal of a Noetherian paraanneid is the intersection of finitely many irreducible ideals. If $Q$ be an irreducible ideal of a strong Noetherian paraanneid $H$, $Q$ is primary. A strong Noetherian paraanneid has a primary decomposition.*

If a ring $R$ is *extragraded* [13, 14], that is, if $(R, +)$ is an extragraded group, i.e. if instead of the axiom $vi)$ we have the following axiom:

$vi')$ If $\delta_1, \dots, \delta_s \in \Delta_p$ are incomparable in pairs and if $x_i, x_i' \in H$ $(i = 1, \dots, s)$ are such that $x_1 + \dots + x_s = x_1' + \dots + x_s'$ and $x_i, x_i' \in R_{\delta_i}$ for all $i = 1, \dots, s$, then $\delta(-x_i + x_i') < \delta_i$,

then the ascending chain condition on $R$ implies the existence of a primary decomposition in the corresponding homogeneous part, which we call an *extraanneid*.

**Theorem 6.** *If $R$ is an extragraded Noetherian ring, then its extraanneid has a primary decomposition.*

*Proof.* Let $H$ be a homogeneous part of $R$. Since $R$ is Noetherian, every ideal, and particularly, every homogeneous ideal $Q$, has a primary decomposition. Since $R$ is extragraded, $Q \cap H$ is an ideal in an extraanneid $H$ and $Q$ is generated exactly by $Q \cap H$ [14]. Also, if $Q$ is a primary ideal in $R$, then $Q \cap H$ is a primary ideal in $H$ and the claim follows. □

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# On the distinction between one-dimensional Euclidean and hyperbolic spaces

Alexandru Popa

**Abstract.** The difference between Euclidean and hyperbolic spaces is clear starting with dimension two. However, the difference between elliptic space and both Euclidean and hyperbolic ones can be described also for dimension one. Does it mean that there is no difference between one-dimensional Euclidean and hyperbolic lines, or it is necessary to better define the difference between them? This paper proposes one possible way to draw clear distinction between one-dimensional Euclidean and hyperbolic lines.

**Mathematics subject classification:** 51N25, 51N15.
**Keywords and phrases:** Points connectability, angle measurability, strong and weak separability.

## 1 Introduction

The difference between hyperbolic (named also Lobachevsky in Russian literature) and Euclidean spaces is space curvature — zero in Euclidean case and negative in hyperbolic case. The space curvature is intrinsic space property starting with dimension two [1], it cannot be used to distinguish one-dimensional Euclidean and hyperbolic spaces.

Another approach is to distinguish geometries. Hyperbolic geometry differs from Euclidean one by Parallel axiom [2]:
**Euclidean Parallel axiom:** On the plane with given line $l$, through a point $P \notin l$ *exactly one* line $a$ goes so that $a \cap l = \varnothing$.
**Hyperbolic Parallel axiom:** On the plane with given line $l$, through a point $P \notin l$ *at least two* lines $a, b$ go so that $a \cap l = \varnothing$, $b \cap l = \varnothing$.

These axioms, as well as all their equivalents, assume the existence of two parallel lines, triangles or other figures, that are essentially two-dimensional.

On the other hand, there exists a clear distinction between one-dimensional elliptic and both Euclidean and hyperbolic spaces. Define points of some line *separable* if among any three different points $A, B, C$ one (let it be $B$) divides the line into two half–lines, and remaining two points $A, C$ lie on different half–lines. In this case we can speak that $B$ lies *between* $A$ and $C$. Otherwise, we call points *non-separable.*

The elliptic points are non-separable, because no point devides elliptic line into two half–lines and among any three points no one lies between two others [3]. Euclidean and hyperbolic points are separable. In order to make the difference between them, we refine the point separability property.

## 2 Uniform model of elliptic, Euclidean and hyperbolic lines

Before we can speak about the tuning of points separability property, we need one universal model for all three one-dimensional spaces constructed in spirit of [7].

**Definition 1.** Define a *characteristic* to be a number $k \in \{1, 0, -1\}$. The characteristic is *elliptic* if $k = 1$, *linear* or *parabolic* if $k = 0$ and *hyperbolic* if $k = -1$.

**Definition 2.** For $x, y \in \mathbb{R}^2$ define $x \odot y = x_0 y_0 + k x_1 y_1$. Define the *metaplane* $\mathbb{M}^2 = \{\mathbb{R}^2, \odot\}$.

**Definition 3.** Define the *line* $\mathbb{B}^1 = \{x \in \mathbb{M}^2 \mid x \odot x = 1,\ -x \equiv x\}$ (Figure 1).

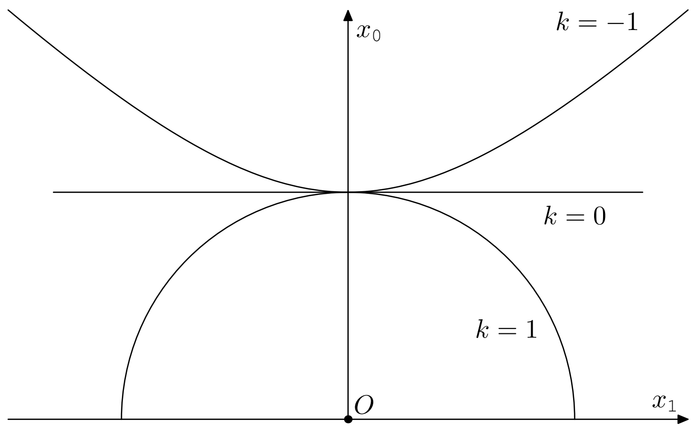


Figure 1. One-dimensional models of elliptic, Euclidean and hyperbolic spaces.

**Definition 4.** Define generalized by $k$ *cosine*, *sine* and *tangent* functions as:

$$C(t) = \sum_{n=0}^{\infty} (-k)^n \frac{t^{2n}}{(2n)!} = \begin{cases} \cos t, & k = 1, \\ 1, & k = 0, \\ \cosh t, & k = -1; \end{cases}$$

$$S(t) = \sum_{n=0}^{\infty} (-k)^n \frac{t^{2n+1}}{(2n+1)!} = \begin{cases} \sin t, & k = 1, \\ t, & k = 0, \\ \sinh t, & k = -1; \end{cases}$$

$$T(t) = \frac{S(t)}{C(t)} = \begin{cases} \tan t, & k = 1, \\ t, & k = 0, \\ \tanh t, & k = -1. \end{cases}$$

**Definition 5.** Define the *translation by* $\varphi$ in $\mathbb{B}^1$ to be the transformation with the matrix

$$\mathfrak{T}(\varphi) = \begin{pmatrix} C(\varphi) & -kS(\varphi) \\ S(\varphi) & C(\varphi) \end{pmatrix}.$$

In these definitions we obtain circle model of one-dimensional elliptic space when $k = 1$. When $k = 0$, the model is identical to one-dimensional Euclidean space identified by the equation $x_0 = 1$ in the metaspace $\mathbb{M}^2$. When $k = -1$ we have hyperbola model of hyperbolic one-dimensional space. This model is equivalent to Beltrami–Klein model of hyperbolic space if instead of coordinates $x_0, x_1$ use one:

$$x' = \frac{x_1}{x_0},$$

and is equivalent to Poincaré model in a disk if:

$$x' = \frac{x_1}{1 + x_0}.$$

It is important to mention that whatever model or coordonate system is used for one-dimensional space it is always possible to reconstruct its metaplane $\mathbb{M}^2$ by fixing some point $O$ as origin with homogeneous coordonates $(1 : 0)$ and for some line point $X$ coordonates will be $(C(x) : S(x))$, where $x$ is the signed distance $|OX|$.

## 3 Point unconnectability and angle unmeasurability notions

Because a metaspace $\mathbb{M}^2$ is not Euclidean unless $k = 1$, we need several more important notions. These notions belong to geometry, not to space model constructions. In order to see it, we obtain them from axioms of two-dimensional elliptic, Euclidean and hyperbolic geometries using duality operation. We can generalize Parallel axiom in the following way (Figure 2):

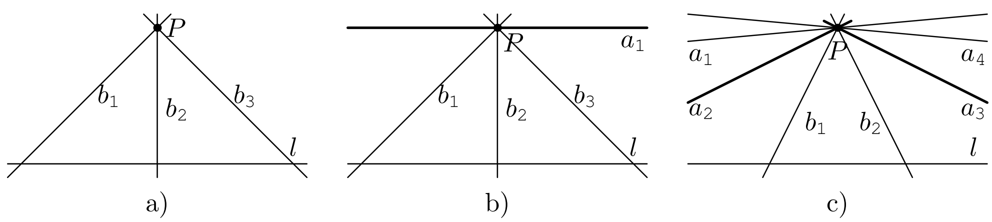


Figure 2. Parallel axiom: a) elliptic, b) Euclidean and c) hyperbolic.

**Generalized Parallel axiom:** On the plane with given line $l$, through a point $P \notin l$ $0^k$ lines $\{a_i\}$ go so that $a_i \cap l = \varnothing$.

*Remark.* The symbol $0^k$ is not used in calculus. Its value is:

$$0^k = \begin{cases} 0, & k = 1, \\ 1, & k = 0, \\ \infty, & k = -1. \end{cases}$$

Duality operation on a plane means exchanging the following relations:

$$\begin{aligned}
\text{Point } P &\longleftrightarrow \text{Line } p,\\
P \in l &\longleftrightarrow p \ni L,\\
P \notin l &\longleftrightarrow p \not\ni L,\\
l = AB &\longleftrightarrow L = a \cap b,\\
|AB| = \varphi &\longleftrightarrow \measuredangle ab = \varphi,\\
a \parallel b &\longleftrightarrow A, B \text{ have no common line.}
\end{aligned}$$

The relation "$A, B$ have no common line" is dual to line parallelism. Such geometries were proposed in [4, 5]. Several of them are described in [6–9].

**Definition 6.** Two points $A, B$ are *unconnectable* if they have no common line.

In order to see different types of points unconnectability, we need new axiom. Let formulate Connectability axiom, dual to Parallel axiom (Figure 3):
**Connectability axiom:** On the plane with given point $L$, in the line $p \not\ni L$ $0^k$ points $\{A_i\}$ lie so that $A_i, L$ are unconnectable.

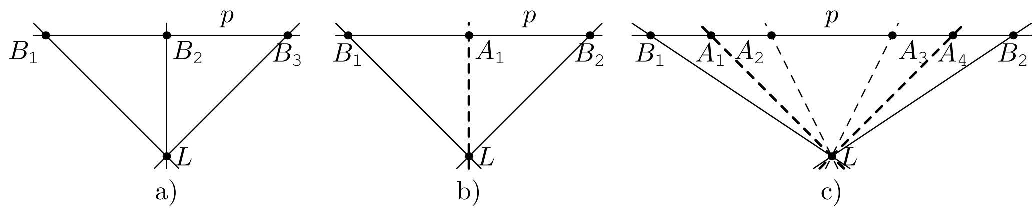


Figure 3. Connectability axiom: a) elliptic, b) parabolic and c) hyperbolic.

*Remark.* As in the case of hyperbolic Parallel axiom (Figure 2, c), the limit case between non-intersected and intresected lines is two parallel lines (bold ones), for hyperbolic Connectability axiom (Figure 3, c), the limit case between connectable and unconnectable points is two unconnectable points (also marked with bold).

*Remark.* Elliptic variant of Connectability axiom is equivalent to the following statement: "Any two different points can be connected by a line", that holds for elliptic, Euclidean and hyperbolic geometries.

**Definition 7.** Define some angle to be *measurable* if any point from its interior (including the rays) is either connectable or unconnectable with the vertex. Define an angle to be *unmeasurable* if its interior (including the rays) contains both connectable and unconnectable points with the vertex.

## 4 Points separability in a line

In order to draw the difference between Euclidean and hyperbolic cases of separable points, give more precise definition [10]. This definition is based only on points connectability notion. Although the Connectability axiom also assumes at least two-dimensional plane, this plane is nothing more than extended space of one-dimensional line — its metaplane. No geometric objects are involved other than objects of an one-dimensional line with its structure.

**Definition 8.** We call points on a line *non-separable* if all points on this line are connectable with any point on the metaplane. We call points on a line *separable* if for any three points $A, B, C$ on this line and some point $D$ on the metaplane, connectable with $A, C$ and unconnectable with $B$, the angle $\angle ADC$ is unmeasurable (Figure 4).

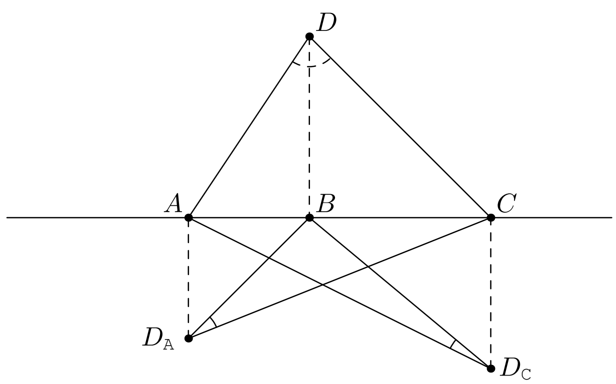


Figure 4. Points separability on a line.

*Remark.* For separable points $A, B, C$ only a single point ($B$) has the described property. For other points ($A, C$) and some unconnectable with them points $D_A, D_C$, the angles $\angle BD_AC$ and $\angle AD_CB$ are measurable.

If points of some line are separable, then the point B devides the line into two half–lines and points $A$ and $C$ lie on different half–lines defined by $B$. When points of some line are non-separable, then no point devides the line into half–lines.

**Definition 9.** In the case of separable points we say that the point $B$ *lies between* points $A$ and $C$.

*Remark.* In the case of non-separable points on a line, among any three points no one divides the line into half–lines, and it is impossible to talk about the position of some point between other two.

**Definition 10.** We call points on some line *weak separable* (Figure 5, left) if any point $D$ of the line metaplane, being unconnectable with point $B$ (that lies between $A$ and $C$) and connectable with both $A, C$, is also connectable with all points from some neighborhood of $B$. We call points on some line *strong separable* (Figure 5, right) if in the same conditions any point $D$ is unconnectable not only with $B$, but also with all points from some its neighborhood.

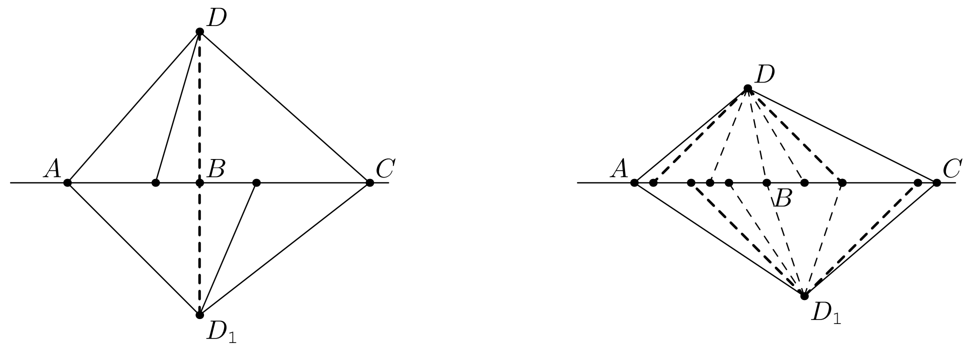


Figure 5. Points separability on a plane: weak (left) and strong (right).

In this definitions, points of elliptic line are non-separable, points of Euclidean line are weak separable, and points of hyperbolic line are strong separable.

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# On the number of ring topologies on countable rings

V. I. Arnautov, G. N. Ermakova

**Abstract.** For any countable ring $R$ and any non-discrete metrizable ring topology $\tau_0$, the lattice of all ring topologies admits:
– Continuum of non-discrete metrizable ring topologies stronger than the given topology $\tau_0$ and such that $\sup\{\tau_1, \tau_2\}$ is the discrete topology for any different topologies;
– Continuum of non-discrete metrizable ring topologies stronger than $\tau_0$ and such that any two of these topologies are comparable;
– Two to the power of continuum of ring topologies stronger than $\tau_0$, each of them being a coatom in the lattice of all ring topologies.

**Mathematics subject classification:** 22A05.
**Keywords and phrases:** Countable ring, ring topology, Hausdorff topology, basis of the filter of neighborhoods, number of ring topologies, lattice of ring topologies, Stone-Čech compacification.

## 1 Introduction

The study of possibility to set a non-discrete Hausdorff topology on infinite algebraic systems in which existing operations are continuous was begun in [1]. In this article for any countable group a method of constructing such group topologies was given.

For countable rings the problem of the possibility to set non-discrete Hausdorff ring topologies was studied in [2, 3]. In these articles for any countable ring a method of obtaining any ring metrizable topology was given and it was proved that any countable ring admits such topology.

The present article is a continuation of research in this direction. The main result of this paper is Theorem 3.1, in which for any countable ring $R$ and any non-discrete metrizable ring topology $\tau_0$, the number of topologies which have some properties in the lattice of all ring topologies is specified.

For countable groups similar result was obtained in [4, 5].

## 2 Notations and preliminaries

To present the main results we remind the following well-known result (see, for example, [2], Proposition 1.2.2 and Theorem 1.2.5).

**Theorem 2.1.** *A set $\Omega$ of subsets of a ring $R$ is a basis of filter of neighborhoods of zero for some Hausdorff ring topology on the ring $R$ if and only if the following conditions are satisfied:*

**1)** $\bigcap\limits_{V\in\Omega} V = \{0\}$;

**2)** *For any subsets* $V_1$ *and* $V_2 \in \Omega$ *there exists a subset* $V_3 \in \Omega$ *such that* $V_3 \subseteq V_1 \cap V_2$;

**3)** *For any subset* $V_1 \in \Omega$ *there exists a subset* $V_2 \in \Omega$ *such that* $V_2 + V_2 \subseteq V_1$;

**4)** *For any subset* $V_1 \in \Omega$ *there exists a subset* $V_2 \in \Omega$ *such that* $-V_2 \subseteq V_1$;

**5)** *For any subset* $V_1 \in \Omega$ *and any element* $r \in R$ *there exists a subset* $V_2 \in \Omega$ *such that* $R \cdot V_2 \subseteq V_1$ *and* $V_2 \cdot r \subseteq V_1$;

**6)** *For any subset* $V_1 \in \Omega$ *there exists a subset* $V_2 \in \Omega$ *such that* $V_2 \cdot V_2 \subseteq V_1$.

**Definition 2.2.** A subset $V$ of the ring $R$ is called *symmetric* if $-V = V$.

**Notation 2.3.** If $V_1, V_2, \dots$ and $S_1, S_2, \dots$ are non-empty symmetric subsets of a ring $R$, then for any natural number $k$ we define by induction the subset $F_k\big(S_1, \dots, S_k; V_1, \dots, V_k\big)$ of the ring $R$:

We take $F_1(S_1; V_1) = V_1 + V_1 + V_1 \cdot V_1 + S_1 \cdot V_1 + V_1 \cdot S_1$ and

$$F_{k+1}(S_1, S_2, \dots, S_{k+1}; V_1, V_2, \dots, V_k) = F_1(S_1; V_1 \cup F_k\big(S_2, \dots, S_{k+1}; V_2, \dots, V_{k+1}\big)).$$

**Proposition 2.4.** *If* $V_1, V_2, \dots$ *and* $S_1, S_2, \dots$ *are some sequences of non-empty finite symmetric subsets of a ring* $R$ *and* $0 \in V_i$ *for any natural numbers* $i$*, then for any natural number* $k$ *the following statements are true:*

**2.4.1.** $V_1 + V_1 \subseteq F_1(S_1; V_1)$, $V_1 \cdot V_1 \subseteq F_1(S_1; V_1)$, $S_1 \cdot V_1 \subseteq F_1(S_1; V_1)$ *and* $V_1 \cdot S_1 \subseteq F_1(S_1; V_1)$;

**2.4.2.** *For any natural number* $k$ *the set* $F_k(S_1, \dots, S_k; V_1, \dots, V_k)$ *is a finite symmetric set;*

**2.4.3.** $F_k(S_1, \dots, S_k; \{0\}, \dots, \{0\}) = \{0\}$ *for any natural number* $k$*;*

**2.4.4.** *If* $k$ *is a natural number and* $U_i \subseteq V_i \subseteq R$ *and* $T_i \subseteq S_i \subseteq R$ *for any natural number* $1 \le i \le k$*, then*

$$F_k(T_1, \dots, T_k; U_1, \dots, U_k) \subseteq F_k(S_1, \dots, S_k; V_1, \dots, V_k);$$

**2.4.5.** *If* $k$ *and* $p$ *are natural numbers and* $V_{k+j} = \{0\}$ *for any natural number* $1 \le j \le p$*, then*

$$F_k(S_1, \dots, S_k; V_1, \dots, V_k) = F_{k+p}(S_1, \dots, S_{k+p}; V_1, \dots, V_{k+p});$$

**2.4.6.** *For any natural number* $k \ge 2$ *the following equality is true*

$$F_k(S_1, \dots, S_k; V_1, \dots, V_k) =$$

$$F_k(S_1, \dots, S_k; V_1 \cup F_{k-1}\ (S_2, \dots, S_k; V_2, \dots, V_k), \dots, V_{k-1} \cup F_1(S_k; V_k), V_k);$$

**2.4.7.** $V_t \subseteq F_k(S_1, \dots, S_k; V_1, \dots, V_k)$ *for any natural numbers* $1 \le t \le k$;

**2.4.8.** $F_{k+1}(S_s, \dots, S_{k+s}; V_s, \dots, V_{k+s}) \subseteq F_{k+s-t+1}(S_t, \dots, S_{k+s}; V_t, \dots, V_{k+s})$ *for any natural numbers* $k, s, t$ *and* $t \le s$.

*Proof.* For proof of Statements 2.4.1 – 2.4.7 see in [2] the proof of Proposition 5.3.2.

We prove Statement 2.4.8 by induction on the number $s-t$.

If $s-t=0$, then $t=s$, and then

$$F_{k+1}(S_s,\ldots,S_{k+s};V_s,\ldots,V_{k+s}) = F_{k+(s-t)+1}\big(S_t,\ldots,S_{k+s};V_t,\ldots,V_{k+s}\big).$$

Assume that the required inclusion is proved for the number $s-t=n$ and any natural numbers $k,s$ and let $s-t=n+1$. Then, from the induction assumption and Statement 4.7 it follows

$$F_{k+1}(S_s,\ldots,S_{k+s};V_s,\ldots,V_{k+s}) \subseteq V_s \cup F_{k+n+1}(S_s,\ldots,S_{n+s};V_2,\ldots,V_{k+n+s}) \subseteq$$

$$F_1(S_s;V_s \cup F_{k+n}(S_{s+1},\ldots,S_{k+n+s};V_{s+1},\ldots,V_{k+n+s})) =$$

$$F_{k+n+1}\big(S_s,\ldots,S_{k+n+s};V_s,\ldots,V_{k+n+s}\big) = F_{k+s-t+1}\big(S_s,\ldots,S_{k+s-t};V_s,\ldots,V_{k+s-t}\big)$$

for any natural numbers $s,k$.

Thus Statement 2.4.8 is proved, and hence, Proposition 2.4 is proved. $\square$

**Definition 2.5.** If $R$ is a ring and $x$ is some variable, then we denote by $R[x]$ the free ring generated by the set $R\bigcup\{x\}$.

We call elements of the ring $R[x]$ the generalized polynomials over the ring $R$ in variable $x$.

**Definition 2.6.** For any element $a\in R$, we consider:

– The mapping $\varphi_a : R\bigcup\{x\}\to R$ such that $\varphi_a(x)=a$ and $\varphi_a(b)=b$ for any $b\in R$;

– The ring homomorphism $\tilde{\varphi}_a : R[x]\to R$, which is an extension of the mapping $\varphi_a : R\bigcup\{x\}\to R$;

– If $f(x)\in R[x]$, then we denote by $f(a)$ the element $\tilde{\varphi}_a(f(x))$ of the ring $R$;

– We call the element $f(0)$ of the ring $R$ the *free term of generalized polynomial* $f(x)\in R[x]$;

– An element $b$ of the ring $R$ is called *a root of a generalized polynomial* $f(x)\in R[x]$ if $f(b)=0$.

**Notation 2.7.** If $R=\{0,\pm r_1,\pm r_2,\ldots\}$ is a countable ring, then for any natural number $k$ let $S_k=\{\pm r_1,\pm r_2,\ldots,\pm r_k\}$.

**Theorem 2.8.** *If $\tau$ is a non-discrete Hausdorff ring topology on a countable ring $R=\{0,\pm r_{1,},\pm r_2,\ldots\}$ and $f(x)$ is a generalized polynomial over the ring $R$ with nonzero free term, then there exists a neighborhood $W$ of zero such that each element $r\in W$ is not a root of this polynomial.*

*Proof.* Since $(R,\tau_0)$ is a Hausdorff space, then there exists a countable basis $\{V_1,V_2,\ldots\}$ of the filter neighborhoods of zero such that $-V_k=V_k$ and $V_k\cap S_k=\emptyset$ (definition of the set $S_k$ see above) and

$$F_1(S_{k+1};V_{k+1}) = V_{k+1}+V_{k+1}+V_{k+1}\cdot V_{k+1}+S_{k+1}\cdot V_{k+1}+V_{k+1}\cdot S_{k+1}\subseteq V_k$$

for any natural number $k$.

Using induction on $k$ it is easy to prove that $F_k(S_{i+1},\dots,S_{i+k};V_{i+1},\dots,V_{i+k}) \subseteq V_i$ for any natural numbers $i,k$.

Since $f(0)\neq 0$, then $f(0)\notin V_{t_0}$ for some natural number $t_0$.

If $S$ is the set of all nonzero elements of the ring $R$ which are included in the expression of the polynomial $f(x)-f(0)$, then the finiteness of the set $S$ implies that $S\subseteq S_{i_0}$ for some natural number $i_0$, and hence, $S\subseteq S_i$ for all natural numbers $i\geq i_0$. Besides, if $n$ is a natural number such that the ring operations include not more than $n$ times in the expression of the polynomial $f(x)-f(0)$ and $n\geq i_0$, then from the definition of sets $F_k(S_{i+1},\dots,S_{i+k};V_{i+1},\dots,V_{i+k})$ it follows that

$$f(x)-f(0)\in F_n\big(S_{t_0+1},\dots,S_{t_0+n};\{0\},\dots,\{0\},\{x,0,-x\}\big).$$

Then

$$f(r)-f(0)\in F_n\big(S_{t_0+1},\dots,S_{t_0+n};\{0\},\dots,\{0\},\{r,0,-r\}\big)\subseteq$$
$$F_n\big(S_{t_0+1},\dots,S_{t_0+n};V_{t_{t_0}+1},\dots,V_{t_0+n}\big)\subseteq V_{t_0}$$

for any $r\in V_{t_0+n}$. And since $F(0)\notin V_{t_0}$, then $f(r)\neq 0$ for any $r\in V_{t_0+n}$.

The theorem is proved. □

Since the intersection of a finite number of neighborhoods of zero is a neighborhood of zero, then from Theorem 8 follows

**Corollary 2.9**. *If $\tau$ is a non-discrete Hausdorff ring topology of a countable ring $R$, then for any finite set of generalized polynomials over the ring $R$ with nonzero free terms there exists a neighborhood $W$ of zero such that each element $r\in W$ is not a root of each of these polynomials.*

**Proposition 2.10.** *The following statements are true:*

**2.10.1.** *There exists a set $\tilde{\mathbb{N}}$ of subsets of the set $\mathbb{N}$ of natural numbers such that the cardinality of the set $\tilde{\mathbb{N}}$ is continuum and $A\cap B$ is a finite set for any different sets $A$ and $B$* (see. [6], the proof of example 3.6.18);

**2.10.2.** *If $(\beta\mathbb{N},\tau)$ is the Stone–Čech compactification of the set $\mathbb{N}$ of natural numbers with the discrete topology, then $\mathbb{N}$ is a dense subset of the Hausdorff space $(\beta\mathbb{N},\tau)$ and the cardinality of the set $\beta\mathbb{N}$ is two to the power of continuum* (see [6], Corollary 3.6.12).

## 3 Basic results

**Theorem 3.1.** *If $R=\{0,\pm r_{1,},\pm r_2,\dots\}$ is a countable ring and $\tau_0$ is a non-discrete Hausdorff ring topology such that the topological ring $(R,\tau_0)$ has a countable basis of the filter of neighborhoods of zero, then the following statements are true:*

**3.1.1.** *For any infinite set $A$ of natural numbers there is a metrizable ring topology $\tau(A)$ such that $\tau_0\leq\tau(A)$;*

**3.1.2.** $\sup\{\tau(A),\tau(B)\}$ *is the discrete topology for any infinite sets $A$ and $B$ of natural numbers such that $A\cap B$ is a finite set;*

**3.1.3.** *There are continuum of ring topologies stronger than $\tau_0$ and such that any two of them are comparable to each other;*

**3.1.4.** *There exist two to the power of continuum of ring topologies such that* $\sup\{\tau_1, \tau_2\}$ *is the discrete topology for any two different topologies $\tau_1$ and $\tau_2$;*

**3.1.5.** *There exist two to the power of continuum of coatoms in the lattice of ring topologies of the ring $R$.*

*Proof.* Since $(R, \tau_0)$ is a Hausdorff space, then there exists a countable basis $\{V_1, V_2, \ldots\}$ of the filter of neighborhoods of zero such that $-V_k = V_k$ and $V_k \cap S_k = \emptyset$ and

$$F_1(S_{k+1}; V_{k+1}) = V_{k+1} + V_{k+1} + V_{k+1} \cdot V_{k+1} + S_{k+1} \cdot V_{k+1} + V_{k+1} \cdot S_{k+1} \subseteq V_k$$

for any natural number $k$.

Then by induction on $n$ it is easy to prove that

$$F_n(S_{i+1}, \ldots, S_{i+n}; V_{i+1} \ldots, V_{i+n}) \subseteq V_i$$

for any natural numbers $i$ and $n$.

Further the proof of Statement 3.1.1 will be realized in several steps.

**Step I.** By induction we construct a sequence $k_1, k_2, \ldots$ of natural numbers such that $k_i \geq i$, for any positive integer number $i$ and a sequence $h_1, h_2, \ldots$ of nonzero elements of the ring $R$ such that $\{-h_i, h_i\} \subseteq V_{k_i}$ and

$$F_n(S_1, \ldots, S_k; U_{A,1}, \ldots, U_{A,n}) \bigcap F_n(S_1, \ldots, S_k; U_{B,1}, \ldots, U_{B,n}) = \{0\}$$

for all subsets $A$ and $B$ of the set of all natural numbers such that $A \cap B = \emptyset$, where $U_{C,i} = \{h_i, 0, -h_i\}$ if $i \in C$ and $U_{C,i} = \{0\}$ if $i \notin C$, for any set $C$ of natural numbers.

We take $k_1 = 2$, and as $h_1$ we take an arbitrary element of the set $V_2 \backslash \{0\}$.

If $A$ and $B$ are some sets of natural numbers such that $A \bigcap B = \emptyset$, then $k_1 \notin A$ or $k_1 \notin B$, and hence, $U_{A,1} = \{0\}$ or $U_{B,1} = \{0\}$. Then $F_1(S_1; U_{A,1}) \cap F_1(S_1; U_{B,1}) = \{0\}$ for any sets $A$ and $B$ of natural number such that $A \cap B = \emptyset$.

Suppose that we defined natural numbers $k_1 < k_2 < \ldots < k_n$ such that $k_i \geq i$ and nonzero elements $h_1, h_2, \ldots, h_n$ of the ring $R$ such that $\{h_i, -h_i\} \subseteq V_{k_i}$ and

$$F_n(S_1, \ldots, S_k; U_{A,1}, \ldots, U_{A,n}) \cap F_n(S_1, \ldots, S_k; U_{B,1}, \ldots, U_{B,n}) = \{0\}$$

for any sets $A$ and $B$ of natural numbers such that $A \cap B = \emptyset$.

For any subsets $A' \subseteq \{1, \ldots, n\}$ and $B' \subseteq \{1, \ldots, n\}$ we consider a finite set

$$\Omega_{(A',B')} = F_{n+1}(S_1, \ldots, S_{n+1}; U_{A',1} \ldots, U_{A',n}, \{x, 0, -x\}) -$$

$$\big(F_n(S_1, \ldots, S_{n+1}; U_{B',1} \ldots, U_{B',n}) \backslash \{0\}\big)$$

of generalized polynomials over the ring $R$ in variable $x$.

Since, according to Statement 4.5,

$$F_{n+1}(S_1,\dots,S_{n+1};U_{A',1}\dots,U_{A',n},\{0\})=F_n(S_1,\dots,S_{n+1};U_{A',1},\dots,U_{A',n})$$

and, according to inductive assumption,

$$F_n(S_1,\dots,S_n;U_{A',1},\dots,U_{A',n})\cap F_n(S_1,\dots,S_n;U_{B',1},\dots,U_{B',n})=\{0\}$$

then the free term of generalized polynomial from $\Omega_{(A',B')}$ is nonzero.

Since the set $\{1,\dots,n\}$ has a finite number of subsets, then the set $\Phi_n = \bigcup\limits_{A',B'\subseteq\{1,\dots,n\},A'\bigcap B'=\emptyset}\Omega_{(A',B')}$ is a finite set of generalized polynomials with nonzero free term.

Then, by Corollary 2.9, there exists a neighborhood $W$ of zero in the topological ring $(R,\tau_0)$ such that any element $r\in W$ is not a root of any polynomial of the set $\Phi_{n+1}(x)$.

Then there exists a natural number $k_{n+1}$ such that $k_{n+1}>n+1$ and $V_{k_{n+1}}\subseteq W$. We take $h_{n+1}$, any element of the set $V_{k_{n+1}}\backslash\{0\}$.

We prove that

$$F_{n+1}(S_1,\dots,S_{n+1};U_{A,1},\dots,U_{A,n+1})\cap F_{n+1}(S_1,\dots,S_{n+1};U_{B,1},\dots,U_{B,n+1})=\{0\}$$

for all sets $A$ and $B$ of natural numbers such that $A\cap B=\emptyset$.

Assume the contrary, and let

$$0\neq r\in F_{n+1}(S_1,\dots,S_{n+1};U_{A,1},\dots,U_{A,n+1})\cap F_{n+1}(S_1,\dots,S_{n+1};U_{B,1},\dots,U_{B,n+1}).$$

Since $A\cap B=\emptyset$ then from inductive assumption it follows that

$$F_{n+1}(S_1,\dots,S_{n+1};U_{A,1},\dots,U_{A,n},\{0\})\cap F_{n+1}(S_1,\dots,S_{n+1};U_{B,1},\dots,U_{B,n},\{0\})=$$

$$F_n(S_1,\dots,S_{n+1};U_{A,1},\dots,U_{A,n})\cap F_{n+1}(S_1,\dots,S_{n+1};U_{B,1},\dots,U_{B,n})=\{0\},$$

and hence $U_{A,n+1}=\{h_{n+1},0,-h_{n+1}\}$ or $U_{B,n+1}=\{h_{n+1},0,-h_{n+1}\}$ and from the definition of sets $U_{C,i}$ it follows that $U_{A,n+1}=\{0\}$ or $U_{B,n+1}=\{0\}$.

Assume, for definiteness, that $U_{A,n+1}=\{0\}$ and $U_{B,n+1}=\{h_{n+1},0,-h_{n+1}\}$.

Then

$$0\neq r\in F_n(S_1,\dots,S_n;U_{A,1},\dots,U_{A,n})\bigcap F_{n+1}(S_1,\dots,S_{n+1};U_{B,1},\dots,U_{B,n+1}),$$

and hence, $r=f(h_{n+1})$ for some generalized polynomial

$$f(x)\in F_{n+1}(S_1,\dots,S_{n+1};U_{B,1},\dots,U_{B,n},\{x,0,-x\}).$$

Since $r\notin F_n(S_1,\dots,S_n;U_{B,1}\dots,U_{B,n})$, the free term of the generalized polynomial $f(x)-r$ is nonzero, and the element $h_{n+1}$ is a root of the generalized polynomial $f(x)-r$, and

$$f(x)-r\in F_{n+1}(S_1,\dots,S_{n+1};U_{A,1},\dots,U_{A,n},\{x,0,-x\})-$$

$$(F_n(S_1,\dots,S_{n+1};U_{B,1}\dots,U_{B,n})\backslash\{0\}).$$

As $U_{C,i} = U_{C\bigcap\{1,\dots,n\},i}$ for any natural number $1 \le i \le n$ and any set $C$ of natural numbers, then $f(h) - r \in F_{n+1}(S_1,\dots,S_{n+1};U_{A',1},\dots,U_{A',n},\{h,0,-h\})-(F_n(S_1,\dots,S_{n+1};U_{B',1}\dots,U_{B',n})\backslash\{0\})$ for some subsets $A', B' \subseteq \{1,\dots,n\}$.

We have contradiction with the definition of the element $h_{n+1}$. Therefore

$$F_{n+1}(S_1,\dots,S_{n+1};U_{A,1}\dots,U_{A,n+1}) \cap F_{n+1}(S_1,\dots,S_{n+1};U_{B,1}\dots,U_{B,n+1}) = \{0\}.$$

So, we defined the sequence $k_1, k_2,\dots$ of natural numbers such that $k_i \ge i$ for any number $i$ and the sequence $h_1, h_2,\dots$ of nonzero elements of the ring $R$ such that $\{-h_i, h_i\} \subseteq V_{k_i}$ for any natural number $i$ and

$$F_n(S_1,\dots,S_k;U_{A,1},\dots,U_{A,n}) \cap F_n\ (S_1,\dots,S_k;U_{B,1},\dots,U_{B,n}) = \{0\}$$

for all sets $A$ and $B$ of natural numbers such that $A \cap B = \emptyset$ and any natural number $n$.

**Step II.** For any pair $(i, j)$ of natural numbers we consider the set

$$U_{(i,j),A} = F_j(U_{i+1,A},\dots,U_{i+j,A};S_{i+1},\dots,S_{i+j}),$$

where $U_{i,A} = \{0\}$ if $i \notin A$ and $U_{i,A} = \{0, h_i, -h_i\}$ if $i \in A$.

We show that for the sets $U_{(i,j),A}$ the following inclusions are true:

**1**. From Statement 2.4.3 it follows that $0 \in U_{(i,j),A}$ for any natural numbers $i, j$ and

$$U_{(i,n),A} = F_n(S_{i+1},\dots,S_{i+n};U_{i+1,A},\dots,U_{i+n,A}) \subseteq$$
$$F_n(S_{i+1},\dots,S_{i+n};V_{i+1},\dots,V_{i+n}) \subseteq V_i$$

for any natural numbers $i$, $n$ and any set $A$ of natural numbers.

**2.** From Statements 2.4.4 and 2.4.5 it follows that $U_{(k,j),A} \subseteq U_{(k,n),A}$ for any natural numbers $j \le n$.

**3.** From Statement 2.4.8 it follows that $U_{(i,j),A} \subseteq U_{(k,j),A}$ for any natural numbers $k \le i$ and $j$.

**4.** From Statement 2.4.2 it follows that $U_{(i,j),A}$ is a symmetric set, i.e. $-U_{(I,j),A} = U_{(i,j),A}$ for any natural numbers $i, j$.

**5.** $U_{(i+1,j),A} \cdot U_{(i+1,j),A} \subseteq U_{(i,j),A}$ and $U_{(i+1,j),A} + U_{(i+1,j),A} \subseteq U_{(i,j),A}$ for any natural numbers $i$ and $j > 1$.

**6.** $r_n \cdot U_{(i+n,j),A} \subseteq U_{(i,j),A}$ and $U_{(i+n,j),A} \cdot r_n \subseteq U_{(i,j),A}$ for any natural numbers $i, j, n$ and any set $A$ of natural numbers.

We prove the inclusion **5** by induction on the number $j$.

In fact, if $j = 2$, then, from the definition of sets $U_{(i,j),A}$, Statements 2.4.1 and 2.4.4, it follows:

$$U_{(i+1,2),A} \cdot U_{(i+1,2),A} = F_1(S_{i+2};U_{i+2,A}) \cdot F_1(S_{i+2};U_{i+2,A}) \subseteq$$

$$F_1\big(S_{i+1};F_1(S_{i+2};U_{i+2,A})\big) \subseteq F_1(S_{i+1};U_{i+1,A} \cup F_1(S_{i+2};U_{i+2,A})) =$$

$$F_2(S_{i+1}, S_{i+2}; U_{i+1,A}, U_{i+2,A}) = U_{(i,2),A}$$

and

$$U_{(i+1,2),A} + U_{(i+1,2),A} = F_1(S_{i+2}; U_{i+2,A}) + F_1(S_{i+2}; U_{i+2,A}) \subseteq$$
$$F_1\big(S_{i+1}; U_{i+1,A} \cup F_1(S_{i+2}; U_{i+2,A})\big) = F_1\big(S_{i+1}; F_1(S_{i+2}; U_{i+2,A})\big) \subseteq$$
$$F_2(S_{i+1}, S_{i+2}; U_{i+1,A}, U_{i+2,A}) = U_{(i,2),A}$$

for any natural number $i$ and any set $A$ of natural numbers.

Assume that the required inclusion is proved for natural number $j = n \geq 2$ and any natural number $i$. Then

$$U_{(i+1,i+n+1),A} \cdot U_{(i+1,i+n+1),A} = F_n(S_{i+2}, \dots, S_{i+n+1};$$
$$U_{i+2,A}, \dots, U_{i+n+1,A}) \cdot F_n(S_{i+2}, \dots, S_{i+n+1}; U_{i+2,A}, \dots, U_{i+n+1,A}) \subseteq$$
$$F_1(S_{i+1}; U_{i+1} \cup F_n(S_{i+2}, \dots, S_{i+n+1}; U_{i+2,A}, \dots, U_{i+n+1,A})) \subseteq$$
$$F_1\big(S_{i+1}; U_{i+1,A} \cup F_n(S_{i+2} \dots, S_{i+n+1}; U_{i+2,A}, \dots, U_{i+n+1,A})\big) =$$
$$F_{n+1}(S_{i+1} \dots, S_{i+n+1}; U_{i+1,A}, \dots, U_{i+n+1,A}) = U_{(i,n+1),A}$$

and

$$U_{(i+1,i+n+1),A} + U_{(i+1,i+n+1),A} =$$
$$F_n(S_{i+2}, \dots, S_{i+n+1}; U_{i+2,A}, \dots, U_{i+n+1,A}) +$$
$$F_n(S_{i+2}, \dots, S_{i+n+1}; U_{i+2,A}, \dots, U_{i+n+1,A}) \subseteq$$
$$F_1(S_{i+1}; U_{i+1} \cup F_n(S_{i+2}, \dots, S_{i+n+1}; U_{i+2,A}, \dots, U_{i+n+1,A})) \subseteq$$
$$F_1(S_{i+1}; U_{i+1,A} \cup F_n(S_{i+2}, \dots, S_{i+n+1}; U_{i+2,A}, \dots, U_{i+n+1,A})) =$$
$$F_{n+1}(S_{i+1} \dots, S_{i+n+1}; U_{i+1,A}, \dots, U_{i+n+1,A}) = U_{(i,n+1),A}.$$

Proof of inclusion **6**. In fact,

$$r_n \cdot U_{(i+n,j),A} \subseteq S_{i+n} \cdot F_{n+i+j}(S_{n+i+1}, \dots, S_{n+i+j}; U_{n+i+1,A}, \dots, U_{n+i+j,A}) \subseteq$$
$$F_1(S_{n+i}; U_{n+i,A} \cup F_{n+i+j}(S_{n+i+1}, \dots, S_{n+i+j}; U_{n+i+1,A}, \dots, U_{n+i+j,A})) =$$

$U_{(i+n-1,j),A} \subseteq U_{(i,j),A}$ and

$$U_{(i+n,j),A} \cdot r_n \subseteq F_{n+i+j}(S_{n+i+1}, \dots, S_{n+i+j}; U_{n+i+1,A}, \dots, U_{n+i+j,A}) \cdot S_{i+n} \subseteq$$
$$F_1(S_{n+i}; U_{n+i,A} \cup F_{n+i+j}(S_{n+i+1}, \dots, S_{n+i+j}; U_{n+i+1,A}, \dots, U_{n+i+j,A})) =$$

$U_{(i+n-1,j),A} \subseteq U_{(i,j),A}$ for any natural numbers $i, j, n$, and any set $A$ of natural numbers.

**Step III.** For every infinite set $A$ of natural numbers and any natural number $i$ we take $\hat{U}_i(A) = \bigcup\limits_{j=1}^{\infty} U_{(ij),A}$ and show that the set $\{\hat{U}_i(A) | i \in \mathbb{N}\}$ satisfies the conditions of Theorem 2.1, and hence, this set is a basis of the filter of neighborhoods of zero for a ring topology $\tau(A)$ on the ring $R$.

In fact, since

$$U_{(i,n+1),A} = F_{n+1}(S_{i+1}, \dots, S_{i+n+1}; U_{i+1,A}, \dots, U_{i+n+1,A}) \subseteq$$

$$F_{n+1}(S_{i+1} \dots, S_{i+n+1}; V_{i+1}, \dots, V_{i+n+1}) \subseteq V_i$$

for any natural numbers $i$ and $n$, then $\hat{U}_i(A) = \bigcup\limits_{j=1}^{\infty} U_{(ij),A} \subseteq V_i$. Then $\{0\} \subseteq \bigcap\limits_{i=1}^{\infty} \hat{U}_i(A) \subseteq \bigcap\limits_{i=1}^{\infty} V_i = \{0\}$, and hence, the condition 1 of Theorem 2.1 is satisfied.

From inclusions 2 and 3 (see Step II), it follows

$$\hat{U}_i(A) \bigcap \hat{U}_k(A) = (\bigcup_{j=1}^{\infty} (U_{(i,j),A}) \bigcap (\bigcup_{l=1}^{\infty} U_{(k,l),A}) =$$

$$\bigcup_{j=1}^{\infty} \bigcup_{l=1}^{\infty} (U_{(i,j),A} \bigcap U_{(k,l),A}) = \bigcup_{j=1}^{\infty} U_{(t,j),A} = \hat{U}_t(A),$$

where $t = \max\{i, k\}$, and hence, the condition 2 of Theorem 2.1 is satisfied.

From inclusions 2 and 5 (see Step II) it follows

$$\hat{U}_i(A) + \hat{U}_k(A) = (\bigcup_{j=1}^{\infty} U_{(i,j),A}) + (\bigcup_{l=1}^{\infty} U_{(i,l),A}) =$$

$$\bigcup_{j=1} \bigcup_{j=1} (U_{(i,j),A} + U_{(i,l),A}) = \bigcup_{t=1}^{\infty} U_{(i-1,t),A} = \hat{U}_{i-1}(A)$$

and

$$\hat{U}_i(A) \cdot \hat{U}_k(A) = (\bigcup_{j=1}^{\infty} U_{(i,j),A}) \cdot (\bigcup_{l=1}^{\infty} U_{(i,l),A}) =$$

$$\bigcup_{j=1}^{\infty} \bigcup_{j=1}^{\infty} (U_{(i,j),A} \cdot U_{(i,l),A}) = \bigcup_{t=1}^{\infty} U_{(i-1,t),A} = \hat{U}_{i-1}(A)$$

for any natural number $i > 1$, and hence, the conditions 3 and 6 of Theorem 2.1 are satisfied.

From inclusion 3 (see Step II) it follows

$$-\hat{U}_i(A) = -(\bigcup_{j=1}^{\infty} U_{(i,j),A}) = \bigcup_{j=1}^{\infty} \bigcup_{l=1}^{\infty} (-U_{(i,j),A}) = \bigcup_{j=1}^{\infty} U_{j,A} = \hat{U}_i(A)$$

for any natural number $i$, and hence, the condition 4 of Theorem 2.1 is satisfied.

Now, let $r \in R$.

If $r = 0$, then $r \cdot \hat{U}_i(A) = \{0\} \subseteq \hat{U}_i(A)$ and $\hat{U}_i(A) \cdot r = \{0\} \subseteq \hat{U}_i(A)$ for any natural number $i$ and any set $A$ of natural numbers.

If $r \neq 0$, then $r = r_n$ or $r = -r_n$ for some natural number $n$. Then, from the inclusion of 6, it follows $r_n \cdot \hat{U}_{i+n}(A) \subseteq \hat{U}_i(A)$ and $\hat{U}_{i+n}(A) \cdot r_n \subseteq \hat{U}_i(A)$ for any natural number $i$, and hence, the condition 5 of Theorem 2.1 is satisfied.

Thus, we have shown that the set $\{\hat{U}_i(A) | i \in \mathbb{N}\}$ satisfies conditions 1 – 6 of Theorem 2.1, and hence, this set is a basis of the filter neighborhoods of zero for a ring topology $\tau(A)$ on the ring $R$.

Since $\hat{U}_i(A) = \bigcup\limits_{j=1}^{\infty} U_{(i,j),A} \subseteq V_i$ for any natural number $i$, then $\tau_0 \leq \tau(A)$.

Thus Statement 3.1.1 is proved.

**Proof of Statement 3.1.2.**

For any subset $A \in \tilde{\mathbb{N}}$ (definition of the set $\tilde{\mathbb{N}}$ see in Statement 2.10.1) we consider the ring topology $\tau(A)$, constructed in the proof of Step III of Statement 3.1.1.

Since the set $\tilde{\mathbb{N}}$ has cardinality of continuum, then to finish the proof of this Statement, it remains to verify that $\sup\{\tau(A), \tau(B)\}$ is the discrete topology for different sets $A, B \in \tilde{\mathbb{N}}$.

Let $A, B \in \mathbb{N}$. Then there exists a natural number $n$ such that

$$(A \backslash \{1, \ldots, n\}) \bigcap (B \backslash \{1, \ldots, n\}) = \emptyset.$$

If $A' = A \backslash \{1, \ldots, n\}$ and $B' = B \backslash \{1, \ldots, n\}$, then (see proof of Step I of Statement 3.1.1)

$$F_k(S_1, \ldots, S_k; U_{A',1}, \ldots, U_{A',k}) \bigcap F_k(S_1, \ldots, S_k; U_{B',1}, \ldots, U_{B',k}) = \{0\}$$

for any natural number $k$.

Since $U_{i,A} = U_{i,A'}$ and $U_{i,B} = U_{i,B'}$ for any natural number $i > n$, then $U_{(i,j),A} = U_{(i,j),A'}$ and $U_{(i,j),B} = U_{(i,j),B'}$ for any natural numbers $i, j$ such that $i > n$, and hence, $U_{(i,j),A} \bigcap U_{(i,j),B} = U_{(i,j),A'} \bigcap U_{(i,j),B'} = \{0\}$ for any natural numbers $i, j$ such that $i > n$. Then, from the inclusion of 2 (see step II), it follows that

$$\hat{U}_{n+1}(A) \bigcap \hat{U}_{n+1}(B) = \bigcup_{j=1}^{\infty} \big(U_{(n+1,j),A} \bigcap U_{(n+1,j),B}\big) = \{0\}.$$

Since $\hat{U}_{n+1}(A)$ and $\hat{U}_{n+1}(B)$ are neighborhoods of zero in topological rings $(R, \tau(A))$ and $(R, \tau(B))$, respectively, then $\{0\}$ is a neighborhood of zero in the topological ring $(R, \sup\{\tau(A), \tau(B)\})$, and hence, $\sup\{\tau(A), \tau(B)\}$ is the discrete topology.

Statement 3.1.2 is proved.

**Proof of Statement 3.1.3.** If $\mathbb{Q}$ is the set of all rational numbers and $\mathbb{N}$ is the set of all natural numbers, then there is a bijection $\xi : \mathbb{Q} \to \mathbb{N}$.

For each real number $r$ we consider the infinite set $A_r = \{\xi(q)|q \in \mathbb{Q}, r \leq q\}$ of natural numbers and let $\tau(A_r)$ be the ring topology on the ring $R$, constructed in the proof of step III of Statement 3.1.1.

We show that the set $\{\tau(A_r)|r$ is a real number$\}$ of ring topologies is as required.

Since the set $\{\hat{U}_i(A_r)|i \in \mathbb{N}\}$ is a basis of the filter of neighborhoods of zero for the ring topology $\tau(A_r)$, then the topological ring $(R, \tau(A_r))$ has a countable basis of filter of neighborhoods of zero.

We show that for any two distinct real numbers $r, r'$ topologies $\tau(A_r)$ and $\tau(A_{r'})$ are different and comparable.

In fact, if $r < r'$, then $A_r \backslash A_{r'}$ is an infinite set, and then for any natural number $n$ there exists a natural number $k \in A_r \backslash A_{r'}$ such that $k > n$. Then $h_k \in U_{(k,1),A_r} \subseteq U_{(n,1),A_r}$ and $h_k \notin U_{(1,s),A_{r'}}$ for any natural number $s$, and hence, $h_k \in \hat{U}_n(A_r)$ and $h_k \notin \hat{U}_1(A_{r'})$. The arbitrariness of the number $n$ implies that $\tau(A_r) \neq \tau(A_{r'})$, and hence, the set $\{\tau(A_r)|r \in R\}$ has the cardinality of continuum.

In addition, since $A_{r'} = \xi(\{q|q \in \mathbb{Q}, r' \leq q\} \subseteq \xi(\{q|q \in \mathbb{Q}, r \leq q\} = A_r$, then (see the definition of the sets $U_{(i,j),A}$ in the proof of Step II of Statement 3.1.1) $U_{(i,j),A_{r'}} \subseteq U_{(i,j),A_r}$ for any natural numbers $i, j$. Then $\hat{U}_{n,A_{r'}} \subseteq \hat{U}_{n,A_r}$ for any natural number $n$, and since the sets of $\{U_{n,A_{r'}}|n \in \mathbb{N}\}$ and $\{U_{n,A_r}|n \in \mathbb{N}\}$ are basis of the filters of neighborhoods of zero in topological rings $(R, \tau(A_{r'}))$ and $(R, \tau(A_r))$, respectively, then $\tau(A_r) \leq \tau(A_{r'})$.

Statement 3.1.3 is proved.

**Proof of Statement 3.1.4.** If (see Statement 2.10.2) $\hat{a} \in \beta\mathbb{N} \backslash \mathbb{N}$, then $\hat{U} \bigcap N$ is an infinite set of natural numbers for any neighborhood $\hat{U}$ of the element $\hat{a}$ in the topological space $(\beta\mathbb{N}, \tau)$.

Let $\tau(\hat{U} \bigcap \mathbb{N})$ be the ring topology, defined according to Statement 3.1.1, and let $\hat{\tau}_{\hat{a}} = \sup\Big\{\tau(\hat{U} \bigcap N)|\hat{U}$ is a neighborhood of element $\hat{a}$ in the topological space $(\beta\mathbb{N}, \tau)\Big\}$.

Since the cardinality of the set $\beta\mathbb{N} \backslash \mathbb{N}$ is equal to two to the power of the continuum, then it suffices to prove that $\sup\{\hat{\tau}_{\hat{a}}, \hat{\tau}_{\hat{b}}\}$ is a discrete topology for any deferent elements $\hat{a}, \hat{b} \in \beta\mathbb{N} \backslash \mathbb{N}$.

So, let $\hat{a}, \hat{b} \in \beta\mathbb{N} \backslash \mathbb{N}$ and $\hat{a} \neq \hat{b}$. Since the space $(\beta\mathbb{N}, \tau)$ is a Hausdorff space, then there exist neighborhoods $\hat{U}$ and $\hat{V}$ of elements $\hat{a}$ and $\hat{b}$ in the topological space $(\beta\mathbb{N}, \tau)$, respectively, such that $\hat{U} \bigcap \hat{V} = \emptyset$. Then, according to Statement 11.2, $\sup\{\tau(\mathbb{N} \cap \hat{U}), \tau(\mathbb{N} \cap \hat{V})\}$ is the discrete topology, and hence, $\sup\{\hat{\tau}_{\hat{a}}, \hat{\tau}_{\hat{b}}\}$ is the discrete topology.

Statement 3.1.4 is proved.

**Proof of Statement 3.1.5.** If T is the set of all non-discrete, ring topologies on the ring $R$, then by the theorem of Kuratowski-Zorn, for any non-discrete ring topology $\hat{\tau}_{\hat{a}}$, constructed in Statement 3.1.4, there is a maximal element $\tau^*_{\hat{a}}$ such that $\tau^*_{\hat{a}} \geq \hat{\tau}_{\hat{a}}$. Then the set $\{\tau^*_{\hat{a}} \; |\hat{a} \in \beta\mathbb{N} \backslash N\}$ is as required.

The theorem is proved. $\square$

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# Determining the Optimal Evolution Time for Markov Processes with Final Sequence of States

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**Abstract.** This paper describes a class of dynamical stochastic systems that represents an extension of classical Markov decision processes. The Markov stochastic systems with given final sequence of states and unitary transition time, over a finite or infinite state space, are studied. Such dynamical system stops its evolution as soon as given sequence of states in given order is reached. The evolution time of the stochastic system with fixed final sequence of states depends on initial distribution of the states and probability transition matrix. The considered class of processes represents a generalization of zero-order Markov processes, studied in [3]. We are seeking for the optimal initial distribution and optimal probability transition matrix that provide the minimal evolution time for the dynamical system. We show that this problem can be solved using the signomial and geometric programming approaches.

**Mathematics subject classification:** 65C40, 60J22, 90C39, 90C40.
**Keywords and phrases:** Markov Process, Final Sequence of States, Evolution Time, Geometric Programming, Signomial Programming, Posynomial Function.

## 1 Introduction and Problem Formulation

Let $L$ be a stochastic discrete system with finite set of states $V$, $|V| = \omega$. At every discrete moment of time $t \in \mathbb{N}$ the state of the system is $v(t) \in V$. The system $L$ starts its evolution from the state $v$ with the probability $p^*(v)$, for all $v \in V$, where $\sum\limits_{v \in V} p^*(v) = 1$. Also, the transition from one state $u$ to another state $v$ is performed according to given probability $p(u,v)$ for every $u \in V$ and $v \in V$, where $\sum\limits_{v \in V} p(u,v) = 1$, $\forall u \in V$ and $p(u,v) \geq 0$, $\forall u, v \in V$. Additionally we assume that a sequence of states $x_1, x_2, \ldots, x_m \in V$ is given and the stochastic system stops transitions as soon as the sequence of states $x_1, x_2, \ldots, x_m$ is reached in given order. The time $T$ when the system stops is called evolution time of the stochastic system $L$ with given final sequence of states $x_1, x_2, \ldots, x_m$.

Various classes of such systems have been studied in [1] and [5], where polynomial algorithms for determining the main probabilistic characteristics (expectation, variance, mean square deviation, $n$-order moments) of evolution time of the given stochastic systems were proposed. Another interpretations of these Markov processes were analyzed in 1981 by Leo J. Guibas and Andrew M. Odlyzko in [9] and by G. Zbaganu in 1992 in [8]. First article considers the evolution of these stochastic systems as a string, composed from the states of the systems, and studies the periods in this string. In the second paper the author considers that the evolution of Markov process is similar with a poem written by an ape. The evolution time of the system is associated with the time that needs for the ape to write that poem (the final sequence of states of the system).

Next, we consider that the distributions $p$ and $p^*$ are not fixed. So, we have the Markov process $L(p^*, p)$ with final sequence of states $X$, distribution of the states $p^*$ and transition matrix $p$, for every parameters $p$ and $p^*$. The problem is to determine the optimal distribution $p^* = \overline{p}^*$ and optimal transition matrix $p = \overline{p}$ that minimize the expectation of the evolution time $T(p^*, p)$ of the stochastic system $L(p^*, p)$.

Based on the results mentioned above, efficient methods for minimizing the expectation of the evolution time of zero-order Markov processes with final sequence of states and unitary transition time were obtained in [3]. The main idea was that the expectation of the evolution time can be written as a posynomial minus one unit. The geometric programming approach was applied and the problem was reduced to the case of convex optimization and solved using the interior-point methods.

In this paper we consider a generalization of this problem where the evolution time is minimized for Markov processes of order 1.

## 2 Preliminary Results

In order to determine the minimal evolution time for Markov processes with final sequence of states we will use the geometric and signomial programming approaches [6].

### 2.1 Geometric Programming

The geometric programming was introduced in 1967 by Duffin, Peterson, and Zener. Wilde and Beightler in 1967 and Zener in 1971 contributed with several results referred to many extensions and sensitivity analysis. A geometric program represents a type of optimization problem, described by objective and constraint functions that have a special form. A good tutorial on geometric programming was presented in [6].

First numerical methods, based on solving a sequence of linear programs, were elaborated by Avriel et al., Duffin, Rajpogal and Bricker. Nesterov and Nemirovsky in 1994 described the first interior-point method for geometric programs and proved the polynomial time complexity. Recent numerical approaches were presented by Andersen and Ye, Boyd and Vandenberghe, Kortanek.

In the context of geometric programming, a monomial represents a function $f : \mathbb{R}^s \to \mathbb{R}$ of the form $f(x_1, x_2, \ldots, x_s) = c x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_s^{\alpha_s}$, where $c > 0$ and $\alpha_i \in \mathbb{R}$, $i = \overline{1, s}$. An arbitrary sum of monomials, $f(x_1, x_2, \ldots, x_s) = \sum\limits_{k=1}^{K} c_k x_1^{\alpha_{1k}} x_2^{\alpha_{2k}} \ldots x_s^{\alpha_{sk}}$, where $c_k > 0$, $k = \overline{1, K}$ and $\alpha_{ik} \in \mathbb{R}$, $i = \overline{1, s}$, $k = \overline{1, K}$, represents a posynomial. Posynomials are closed under addition, multiplication, and nonnegative scaling. A geometric program is an optimization problem of the form

$$f_0(x_1, x_2, \ldots, x_s) \to \min,$$

$$\begin{cases} f_i(x_1, x_2, \ldots, x_s) \leq 1, & i = \overline{1, r} \\ g_j(x_1, x_2, \ldots, x_s) = 1, & j = \overline{1, m} \\ x_l > 0, & l = \overline{1, s} \end{cases},$$

where $f_i(x_1, x_2, \ldots, x_s)$, $i = \overline{0, r}$, are posynomials and $g_j(x_1, x_2, \ldots, x_s)$, $j = \overline{1, m}$, are monomials.

In order to efficiently solve a geometric program we need to convert it to a convex optimization problem. The conversion is based on a logarithmic change of variables $y_l = \ln x_l$, $l = \overline{1, s}$ and a logarithmic transformation of the objective and constraint functions. The obtained convex optimization problem has the form

$$\ln f_0(e^{y_1}, e^{y_2}, \ldots, e^{y_s}) \to \min,$$

$$\begin{cases} \ln f_i(e^{y_1}, e^{y_2}, \ldots, e^{y_s}) \leq 0, & i = \overline{1, r} \\ \ln g_j(e^{y_1}, e^{y_2}, \ldots, e^{y_s}) = 0, & j = \overline{1, m} \end{cases}$$

and can be efficiently solved using standard interior-point methods (see [6] and [7]).

## 2.2 Signomial Programming

In the context of signomial programming, a signomial monomial represents a function $f : \mathbb{R}^s \to \mathbb{R}$ of the form $f(x_1, x_2, \ldots, x_s) = c x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_s^{\alpha_s}$, where $c \in \mathbb{R}$ and $\alpha_i \in \mathbb{R}$, $i = \overline{1, s}$. An arbitrary sum of signomial monomials of the form $f(x_1, x_2, \ldots, x_s) = \sum\limits_{k=1}^{K} c_k x_1^{\alpha_{1k}} x_2^{\alpha_{2k}} \ldots x_s^{\alpha_{sk}}$, where $c_k \in \mathbb{R}$, $k = \overline{1, K}$ and $\alpha_{ik} \in \mathbb{R}$, $i = \overline{1, s}$, $k = \overline{1, K}$, represents a signomial. Signomials are closed under addition, substraction, multiplication, and scaling. A signomial program is an optimization problem of the form:

$$f_0(x_1, x_2, \ldots, x_s) \to \min,$$

$$\begin{cases} f_i(x_1, x_2, \ldots, x_s) \leq 1, & i = \overline{1, r} \\ g_j(x_1, x_2, \ldots, x_s) = 1, & j = \overline{1, m} \\ x_l > 0, & l = \overline{1, s} \end{cases},$$

where $f_i(x_1, x_2, \ldots, x_s)$, $i = \overline{0, r}$ and $g_j(x_1, x_2, \ldots, x_s)$, $j = \overline{1, m}$, are signomials.

So, a signomial has the same form as a posynomial, but the coefficients are allowed to be also negative. There is a huge difference between a geometric program and a signomial program. The global optimal solution of a geometric program can always be determined, but only a local solution of a signomial program can be calculated efficiently.

## 2.3 Geometric Programs with Posynomial Equality Constraints

In several particular cases the signomial programs can be handled as geometric programs. In [6] it was shown that the geometric programs with posynomial equality constraints represent such particular case, i.e. can be solved using geometric programming method. A geometric program with posynomial equality constraints is a signomial program of the form:

$$f_0(x_1, x_2, \dots, x_s) \to \min,$$

$$\begin{cases} f_i(x_1, x_2, \dots, x_s) \le 1, & i = \overline{1, r} \\ g_j(x_1, x_2, \dots, x_s) = 1, & j = \overline{1, m} \\ h_k(x_1, x_2, \dots, x_s) = 1, & k = \overline{1, n} \\ x_l > 0, & l = \overline{1, s} \end{cases},$$

where $f_i(x_1, x_2, \dots, x_s)$, $i = \overline{0, r}$ and $h_k(x_1, x_2, \dots, x_s)$, $k = \overline{1, n}$, are posynomials and $g_j(x_1, x_2, \dots, x_s)$, $j = \overline{1, m}$, are monomials.

Suppose that for each posynomial equality constraint $h_k(x_1, x_2, \dots, x_s)$, $k = \overline{1, n}$, we can find a different variable $x_{l(k)}$ with the following properties:

- The variable $x_{l(k)}$ does not appear in any of the monomial equality constraint functions;
- The posynomial $h_k(x_1, x_2, \dots, x_s)$ is monotone strictly
  - increasing in $x_{l(k)}$, case in which we denote $\lambda(x_{l(k)}) = -1$ or
  - decreasing in $x_{l(k)}$, case in which we denote $\lambda(x_{l(k)}) = 1$;
- The functions $f_i(x_1, x_2, \dots, x_s)$, $i = \overline{0, r}$, are all
  - monotone decreasing in $x_{l(k)}$ if $\lambda(x_{l(k)}) = -1$;
  - monotone increasing in $x_{l(k)}$ if $\lambda(x_{l(k)}) = 1$.

We first form the geometric program relaxation:

$$f_0(x_1, x_2, \dots, x_s) \to \min,$$

$$\begin{cases} f_i(x_1, x_2, \dots, x_s) \le 1, & i = \overline{1, r} \\ g_j(x_1, x_2, \dots, x_s) = 1, & j = \overline{1, m} \\ h_k(x_1, x_2, \dots, x_s) \le 1, & k = \overline{1, n} \\ x_l > 0, & l = \overline{1, s} \end{cases}.$$

If $f^*$ is the optimal value of the relaxed problem, then any optimal solution of the auxiliary problem

$$\prod_{k=1}^{n} (x_{l(k)})^{\lambda(x_{l(k)})} \to \min,$$

$$\begin{cases} f_i(x_1, x_2, \ldots, x_s) \leq 1, & i = \overline{1, r} \\ g_j(x_1, x_2, \ldots, x_s) = 1, & j = \overline{1, m} \\ h_k(x_1, x_2, \ldots, x_s) \leq 1, & k = \overline{1, n} \\ f_0(x_1, x_2, \ldots, x_s) \leq f^* & \\ x_l > 0, & l = \overline{1, s} \end{cases},$$

is an optimal solution of the original problem.

## 3 The Main Results

### 3.1 Stochastic Systems with Final Sequence of States Independent States

In this subsection we briefly describe the main results referred to the problem of optimization of the evolution time of stochastic systems with final sequence of states and independent states. These systems are also called zero order Markov processes with final sequence of states or strong memoryless stochastic systems with final sequence of states and are analyzed and studied in [2] and [3]. This problem was reduced to a geometric program using the main properties of homogeneous recurrent linear sequences and generating function, presented in [3–5] and [1].

The zero order Markov processes with final sequence of states represent a particular case of stochastic systems with final sequence of states studied in this paper. In this case the states of the system are independent, so, the rows of the transition matrix $p$ are equal to initial distribution $p^*$. The expectation of the evolution time can be determined using the following theorem.

**Theorem 1.** *The expectation of the evolution time $T(p^*)$ of zero-order Markov process $L(p^*)$ is $\mathbb{E}(T(p^*)) = -1 + (m + w_m^{-1}) + \dfrac{1}{w_m} \sum\limits_{k=0}^{m-1} (k+1) z_{mk}$, where $m$ is the length of final sequence of states $X = (x_1, x_2, \ldots, x_m)$, $\pi_s = p^*(x_s)$, $w_s = \prod\limits_{j=1}^{s} \pi_j$, $t(s) = \min(\{t \in \{2, 3, \ldots, s+1\} \mid x_{t-1+j} = x_j,\ j = \overline{1, s+1-t}\})$, $s = \overline{1, m}$ and for each $s = \overline{1, m}$ and $k = \overline{0, s-1}$ the following relation holds:*

$$z_{sk} = \begin{cases} 0 & if\ 0 \leq k \leq t(s) - 3 \\ -w_{t(s)-1} & if\ k = t(s) - 2 \\ w_{t(s)-1}(1 - \pi_1) & if\ t(s - t(s) + 1) = 2 \text{ and } k = t(s) - 1 \\ w_{t(s)-1} & if\ t(s - t(s) + 1) \geq 3 \text{ and } k = t(s) - 1 \\ w_{t(s)-1} z_{s-t(s)+1,\ k-t(s)+1} & if\ t(s) \leq k \leq s - 1 \end{cases}.$$

The following theorem shows how the problem of optimization of the evolution time can be reduced to the geometric program

$$\mathbb{E}(T(p^*)) + 1 \to \min,$$

$$\begin{cases} \sum\limits_{x \in Y} p^*(x) \leq 1 \\ p^*(x) > 0,\ \forall x \in Y \end{cases},$$

where $Y = \{x_1, x_2, \ldots, x_m\}$. If $\pi^* = (\pi^*(x))_{x\in Y}$ represents the optimal solution of this geometric program, then $\overline{p}^* = (\overline{p}^*(x))_{x\in V}$ represents the optimal solution of the initial problem, where

$$\begin{cases} \overline{p}^*(x) = \pi^*(x), & x \in Y \\ \overline{p}^*(x) = 0, & x \in V\backslash Y \end{cases} .$$

**Theorem 2.** *The expression* $\mathbb{E}(T(p^*)) + 1$ *represents a posynomial in the variables* $\pi_1,\ \pi_2,\ \ldots,\ \pi_m$.

Also, several particular cases were analyzed and presented in [3] and the explicit optimal solutions were obtained.

**Theorem 3.** *If* $t(m) = 2$, *then the optimal solution is* $\overline{p}^* = (\overline{p}^*(x))_{x\in V}$, *where* $\overline{p}^*(x_1) = 1$ *and* $\overline{p}^*(y) = 0$, *for all* $y \in V\backslash\{x_1\}$, *and the minimal value of the expectation of evolution time is* $\mathbb{E}(T(\overline{p}^*)) = m - 1$.

**Theorem 4.** *If* $t(m) = m + 1$, *then the components* $\overline{p}^*(y)$, $y \in V$, *of the optimal solution* $\overline{p}^*$ *are direct by proportional to the multiplicities* $m(y)$, $y \in V$, *of the respective states in final sequence of states* $X$ *and the minimal value of the expectation of evolution time is* $\mathbb{E}(T(\overline{p}^*)) = -1 + \prod\limits_{y\in Y} \left(\dfrac{m}{m(y)}\right)^{m(y)}$.

## 3.2 Stochastic Systems with Final Sequence of States and Interdependent States

In this subsection we study the problem of optimization of the evolution time of stochastic systems with final sequence of states and interdependent states. The optimal initial distribution and optimal transition matrix are obtained, using signomial and geometric programming approaches.

Theorem 5 offers us the way for determining the optimal initial distribution of the system.

**Theorem 5.** *The optimal initial distribution of the states is* $\overline{p}^*$, *where* $\overline{p}^*(x_1) = 1$ *and* $\overline{p}^*(x) = 0$, $\forall x \in V\backslash\{x_1\}$.

*Proof.* For finishing the evolution of the system it is necessary to pass consecutively through the final states $x_1,\ x_2,\ \ldots,\ x_m$. So, the evolution time will be minimal when the state $x_1$ will be reached as soon as possible. For this reason, it is optimal to start the evolution of the system from the state $x_1$, i.e. $\overline{p}^*(x_1) = 1$. Since $\sum\limits_{x\in V} \overline{p}^*(x) = 1$, we have $\overline{p}^*(x) = 0$, $\forall x \in V\backslash\{x_1\}$. □

Theorem 6 describes several important properties of the optimal transit matrix.

**Theorem 6.** *We consider the set of active final states* $\overline{X} = \{x_1, x_2, \ldots, x_{m-1}\}$, *the set of final transitions* $\overline{Y} = \{(x_1, x_2), (x_2, x_3), \ldots, (x_{m-1}, x_m)\}$ *and the set of branch states* $\overline{Z} = \{y \in \overline{X}\backslash\{x_1\} \mid \exists x \in \overline{X}, \exists z \in \overline{X} \cup \{x_m\}, z \neq y :\ (x, y) \in \overline{Y}, (x, z) \in \overline{Y}\}$. *The optimal transition matrix* $\overline{p}$ *has the following properties:*

1. $\overline{p}(x, x_1) = 1$ *if* $(x, x_1) \in \overline{Y}$ *and* $(x, z) \notin \overline{Y}$, $\forall z \neq x_1$;

2. $\overline{p}(x, x_1) = 1$, $\forall x \notin \overline{X}$;

3. $\overline{p}(x, x_1) > 0$, $\forall x \in \overline{Z}$ *and* $\overline{p}(x, x_1) = 0$ *if* $(x, x_1) \notin \overline{Y}$, $x \in \overline{X} \backslash \overline{Z}$;

4. $\overline{p}(x, y) = 0$ *if* $(x, y) \notin \overline{Y}$ *and* $y \neq x_1$;

5. $\overline{p}(x, y) > 0$, $\forall (x, y) \in \overline{Y}$;

6. $\sum\limits_{(x,y) \in \overline{Y} \cup \{(x,x_1)\}} \overline{p}(x, y) = 1, \ \forall x \in \overline{X}$.

*Proof.* Let $\overline{X} = \{x_1, x_2, \ldots, x_{m-1}\}$ be the set of the states from which it is possible to perform an optimal transition, $\overline{Y} = \{(x_1, x_2), (x_2, x_3), \ldots, (x_{m-1}, x_m)\}$ – the set of the optimal transitions (that follow optimal realization of the final sequence of states), $\overline{Z} = \{y \in \overline{X} \backslash \{x_1\} \mid \exists x \in \overline{X}, \exists z \in \overline{X} \cup \{x_m\}, z \neq y : \ (x, y) \in \overline{Y}, (x, z) \in \overline{Y}\}$ – the set of branch states, in which the stochastic system, having as goal the realization of the final sequence of states, can make a mistake and need to have a chance to return in the state $x_1$.

1. If $(x, x_1) \in \overline{Y}$ and $(x, z) \notin \overline{Y}$, $\forall z \neq x_1$, then $x \in \overline{X}$ and the transition $(x, x_1)$ is the unique possible transition from the state $x$ that belongs to the set $\overline{Y}$. For ensuring the realization of this transition when the system is in the state $x \in \overline{X}$, it is necessary to have $\overline{p}(x, x_1) = 1$.

2. For finishing the evolution of the system it is necessary to pass consecutively through the final states $x_1$, $x_2$, $\ldots$, $x_m$. So, for minimizing the evolution time of the system it is necessary that the state $x_1$ to be reached as soon as possible. So, if the system is in the state $x \notin \overline{X}$, we need to have $\overline{p}(x, x_1) = 1$.

3. Since $\overline{Z}$ represents the set of branch states, in which the stochastic system, having as goal the realization of the final sequence of states, can make a mistake, we need to give as soon as possible a chance to return in the state $x_1$ for retrying from the beginning the realization of the final sequence of states. So, we can assume that $\overline{p}(x, x_1) > 0$, $\forall x \in \overline{Z}$ and $\overline{p}(x, x_1) = 0$ if $(x, x_1) \notin \overline{Y}$ and $x \in \overline{X} \backslash \overline{Z}$;

4. If the state $x \in \overline{X}$, then $\exists y_1, y_2, \ldots, y_k \in \overline{X} \cup \{x_m\}$ such that $(x, y_j) \in \overline{Y}$, $j = \overline{1, k}$, where $k \geq 1$. For ensuring the realization of one of these transitions when the system is in the state $x \in \overline{X}$ or return to the initial state $x_1$ when it is necessary, we need the nonexistence of another transition $(x, y) \notin \overline{Y}$ with $y \neq x_1$, i.e. need to have $\overline{p}(x, y) = 0$. If $x \notin \overline{X}$, from Property 2 of this Theorem, since $\sum\limits_{y \in V} \overline{p}(x, y) = 1$ and $\overline{p}(x, y) \geq 0$, $\forall x, y \in V$, we have $\overline{p}(x, y) = 0$, $\forall y \neq x_1$.

5. We have $\overline{p}(x,y) > 0$, $\forall (x,y) \in \overline{Y}$, because, otherwise we have $\overline{p}(x,z) = 0$ for at least one transition $(x,z) \in \overline{Y}$, i.e. this transition is not realizable, which implies that the evolution time is infinite (non optimal), contradiction with our minimization goal.

6. The relation $\sum\limits_{(x,y)\in\overline{Y}\cup\{(x,x_1)\}} \overline{p}(x,y) = 1,\ \forall x \in \overline{X}$ is obtained from the formula $\sum\limits_{y\in V} \overline{p}(x,y) = 1$, $\forall x \in \overline{X}$ and the Property 4 from this Theorem.

Such we proved these six properties of the optimal transition matrix $\overline{p}$. □

Theorem 7 offers us the way for determining the optimal transition matrix of the system.

**Theorem 7.** *If $\delta_{i,j}(p) \not\equiv 0$, $i,j = \overline{1,2}$, then the optimal transition matrix can be determined by solving the following geometric programs with posynomial equality constraints:*

$$\mathbb{E}(T(p)) = d_1 d_2^{-1} \to \min, \tag{1}$$

$$\begin{cases} (2a): & \sum\limits_{(x,y)\in\overline{Y}\cup\{(x,x_1)\}} p(x,y) = 1,\ \forall x \in \overline{X} \\ (2b): & d_{1,1}^{-1} d_1 + d_{1,1}^{-1} d_{1,2} = 1 \\ (2c): & d_{2,1}^{-1} d_2 + d_{2,1}^{-1} d_{2,2} = 1 \\ (2d): & d_{1,1}^{-1} \delta_{1,1}(p) = 1 \\ (2e): & d_{1,2}^{-1} \delta_{1,2}(p) = 1 \\ (2f): & d_{2,1}^{-1} \delta_{2,1}(p) = 1 \\ (2g): & d_{2,2}^{-1} \delta_{2,2}(p) = 1 \\ (2h): & d_i > 0,\ i = \overline{1,2} \\ (2i): & d_{i,j} > 0,\ i,j = \overline{1,2} \\ (2j): & p(x,y) > 0,\ \forall (x,y) \in \overline{Y} \\ (2k): & \overline{p}(x,x_1) > 0,\ \forall x \in \overline{Z} \end{cases} \tag{2}$$

*and (1) subject to*

$$\begin{cases} (3a): & \sum\limits_{(x,y)\in\overline{Y}\cup\{(x,x_1)\}} p(x,y) = 1,\ \forall x \in \overline{X} \\ (3b): & d_{1,1}^{-1} d_1 + d_{1,1}^{-1} d_{1,2} = 1 \\ (3c): & d_{2,1}^{-1} d_2 + d_{2,1}^{-1} d_{2,2} = 1 \\ (3d): & d_{1,1}^{-1} \delta_{1,2}(p) = 1 \\ (3e): & d_{1,2}^{-1} \delta_{1,1}(p) = 1 \\ (3f): & d_{2,1}^{-1} \delta_{2,2}(p) = 1 \\ (3g): & d_{2,2}^{-1} \delta_{2,1}(p) = 1 \\ (3h): & d_i > 0,\ i = \overline{1,2} \\ (3i): & d_{i,j} > 0,\ i,j = \overline{1,2} \\ (3j): & p(x,y) > 0,\ \forall (x,y) \in \overline{Y} \\ (3k): & \overline{p}(x,x_1) > 0,\ \forall x \in \overline{Z} \end{cases} \tag{3}$$

*according to the properties described by Theorems 5 and 6, where* $\delta_{i,j}(p)$, $i,j=\overline{1,2}$, *are the posynomials from the decomposition*

$$\mathbb{E}(T(p)) = (\delta_{1,1}(p) - \delta_{1,2}(p))(\delta_{2,1}(p) - \delta_{2,2}(p))^{-1} \tag{4}$$

*which follows from the algorithm developed in [1]. The signomial programs* $(1)-(2)$ *and* $(1)-(3)$ *can be handled as geometric programs using the way followed in [6] and described in Section 2.3. If* $\overline{p}^1$ *is the optimal solution of the problem* $(1)-(2)$ *and* $\overline{p}^2$ *is the optimal solution of the problem* $(1)-(3)$, *then the optimal transition matrix is* $\overline{p} \in \{\overline{p}^1, \overline{p}^2\}$ *for which* $\mathbb{E}(T(\overline{p}))$ *is minimal. If there exists at least one posynomial* $\delta_{i^*,j^*}(p) \equiv 0$, *then in* $(2)$ *and* $(3)$ *the corresponding posynomial equality constraints just disappear and the corresponding substitution* $d_{i^*,j^*}=0$ *is performed in* $(2)$ *and substitution* $d_{i^*,3-j^*}=0$ *is performed in* $(3)$.

*Proof.* From Theorem 5 and theoretical argumentation of the algorithm developed and presented in [1], which determines the generating vector of the distribution of the evolution time, we can observe that the components of generating vector $q(p)$ of the distribution $a = rep(T(p))$ of the evolution time $T(p)$ represent signomials in the variables $p(x,y)$, $x,y \in V$. Since $\mathbb{E}(T(p)) = G^{[a]\prime}(1)$, we obtain that $\mathbb{E}(T(p))$ represents a fraction with signomial numerator and denominator. Because every signomial can be written as a difference between two posynomials, we have the relation (4), where $\delta_{i,j}(p)$, $i,j=\overline{1,2}$, are posynomials.

If we denote $d_{i,1} = \max\{\delta_{i,1}(p), \delta_{i,2}(p)\}$, $d_{i,2} = \min\{\delta_{i,1}(p), \delta_{i,2}(p)\}$, $i=\overline{1,2}$ and $d_i = d_{i,1} - d_{i,2} > 0$, $i=\overline{1,2}$, we obtain $0 < \mathbb{E}(T(p)) = d_1 d_2^{-1}$, $d_{i,1}^{-1} d_i + d_{i,1}^{-1} d_{i,2} = 1$, $i=\overline{1,2}$ and $d_{i,j}^{-1}\delta_{i,j}(p) = 1$ or $d_{i,j}^{-1}\delta_{i,3-j}(p) = 1$, $i,j=\overline{1,2}$. The relations $\sum\limits_{(x,y)\in\overline{Y}\cup\{(x,x_1)\}} p(x,y) = 1$, $\forall x \in \overline{X}$ and $p(x,y) > 0$, $\forall (x,y) \in \overline{Y}$, follow from Theorem 6.

Also, applying the same Theorem 6, we can eliminate the variables $p(x,y)$ for which $p(x,y)=0$ or $p(x,y)=1$ performing the corresponding substitutions. In such way, we obtain the signomial programs $(1)-(2)$ and $(1)-(3)$. Because all constraints $(2a)-(2g)$ and $(3a)-(3g)$ are posynomial equality constraints, these signomial programs are geometric programs with posynomial equality constraints.

Next we will illustrate how these geometric programs with posynomial equality constraints can be handled as geometric programs using the way described in Section 2.3. We will consider only the problem $(1)-(2)$, the argumentation for the problem $(1)-(3)$ can be performed in similar way.

So, if the posynomials $\delta_{i,j}(p)$, $i,j=\overline{1,2}$, are not monomials, we can fix the variable $d_{1,1}$ for constraint $(2b)$, $d_2$ for $(2c)$, $d_{1,2}$ for $(2e)$, $d_{2,1}$ for $(2f)$, $d_{2,2}$ for $(2g)$, an arbitrary variable $p(x^*,y^*)$ that appears in the posynomial $\delta_{1,1}(p)$ for constraint $(2d)$ and an arbitrary variable $p(x,y^*(x)) \neq p(x^*,y^*)$ that appears in the posynomial from $(2a)$ for every $x \in \overline{X}$ for the constraints $(2a)$. These selected variables verify the properties described in Section 2.3, i.e. the problem $(1)-(2)$ can be handled as geometric programs.

If the posynomial $\delta_{1,2}(p)$, $\delta_{2,1}(p)$ or $\delta_{2,2}(p)$ is a monomial, then the respective constraint $(2e)$, $(2f)$ or $(2g)$ just disappears and the respective substitution $d_{i^*,j^*} = \delta_{i^*,j^*}(p)$ is performed in the signomial program $(1) - (2)$. The selected variables for the rest of constraints are not changed. So, the problem $(1) - (2)$ can be handled as geometric programs.

If the posynomial $\delta_{1,1}(p)$ is a non-constant monomial, then the corresponding constraint $(2d)$ just disappears and the corresponding substitution $d_{1,1} = \delta_{1,1}(p)$ is performed in the signomial program $(1) - (2)$. The selected variables for the constraints $(2a)$, $(2c)$, $(2e)$, $(2f)$ and $(2g)$ are not changed. Additionally, the variable $p(x^*, y^*)$ that appears in the posynomial $\delta_{1,1}(p)$ is selected for constraint $(2b)$. These selected variables verify the properties described in Section 2.3, i.e. the problem $(1) - (2)$ can be handled as geometric programs.

If the posynomials $\delta_{1,1}(p)$ and $\delta_{1,2}(p)$ are two constants, then also $d_1$ is a constant. In this case the constraints $(2b)$, $(2d)$ and $(2e)$ just are eliminated. The selected variables for the rest of constraints are not changed. So, in this way, the problem $(1) - (2)$ can be handled as geometric programs.

If the posynomial $\delta_{1,1}(p)$ is a constant and $\delta_{1,2}(p)$ is not a constant, then the constraint $(2d)$ is eliminated and substitution $d_{1,1} = \delta_{1,1}(p)$ is performed in the signomial program $(1) - (2)$. We can fix the variable $d_{1,2}$ for constraint $(2b)$, an arbitrary variable $p(x^{**}, y^{**})$ that appears in the posynomial $\delta_{1,2}(p)$ for constraint $(2e)$ and an arbitrary variable $p(x, y^{**}(x)) \neq p(x^{**}, y^{**})$ that appears in the posynomial from $(2a)$ for every $x \in \overline{X}$ for the constraints $(2a)$. The selected variables for the rest of constraints are not changed. These selected variables verify the properties described in Section 2.3, i.e. the problem $(1) - (2)$ can be handled as geometric programs.

In this way, we analyzed all the possible cases. So, the problems $(1) - (2)$ and $(1) - (3)$ can be handled as geometric programs. $\square$

## 4 Particular cases and generalizations

In the previous section a method for determining the optimal evolution time of stochastic systems with final sequence of states, based on geometric and signomial programming approaches, was theoretically grounded. Theorems 5 and 6 present the main properties of the optimal distribution and optimal transition matrix. From these theorems we can easy remark several particular cases and generalizations.

We consider the particular case $x_1 = x_2 = \ldots = x_m$. From Theorem 6 the optimal transition matrix is obtained. The following formula holds:

$$\overline{p}(x, y) = \begin{cases} 1 & if \quad y = x_1 \\ 0 & if \quad y \neq x_1 \end{cases}, \ \forall x, y \in V.$$

So, the expectation of the evolution time is minimal (equal to $m - 1$) when the stochastic system starts the evolution from the state $x_1$ and remains with probability 1 at every moment of time in this state.

Also, in the case $x_i \neq x_j$, $\forall i, j,\ 1 \leq i < j \leq m$, we have

$$\overline{p}(x,y) = \begin{cases} 1 & if \quad x \notin \{x_1, x_2, \ldots, x_{m-1}\} \ and \ y = x_1 \\ 1 & if \quad \exists\, i,\ 1 \leq i < m,\ such \ that \ x = x_i \ and \ y = x_{i+1} \\ 0 & otherwise \end{cases}, \ \forall x, y \in V.$$

The expectation of the evolution time is minimal (equal to $m-1$) when the stochastic system starts the evolution from the state $x_1$, passes with probability 1 in the state $x_2$, next, in similar way, passes in the state $x_3$, ..., until it reaches the state $x_m$.

Another particular case is when $\forall i, j,\ 1 \leq i < j \leq m$, if $x_i = x_j$ then $j = m$ or $x_{i+1} = x_{j+1}$. This case is an extension of the previous particular case. We consider the minimal values $i^*$ and $j^*$, $i^* < j^*$, for which $x_{i^*} = x_{j^*}$. We have $x_{i^*+k} = x_{j^*+k}$, $k = \overline{0, m - j^*}$. So, $\{x_1, x_2, \ldots, x_m\} = \{x_1, x_2, \ldots, x_{j^*-1}\}$, which implies

$$\overline{p}(x,y) = \begin{cases} 1 & if \quad x \notin \{x_1, x_2, \ldots, x_{j^*-1}\} \ and \ y = x_1 \\ 1 & if \quad \exists\, i,\ 1 \leq i < j^* - 1,\ such \ that \ x = x_i \ and \ y = x_{i+1} \\ 1 & if \quad x = x_{j^*-1} \ and \ y = x_{i^*} \\ 0 & otherwise \end{cases}, \ \forall x, y \in V.$$

The expectation of the evolution time is minimal (equal to $m-1$) when the stochastic system starts the evolution from the state $x_1$, passes with probability 1 in the state $x_2$, next, in similar way, passes in the state $x_3$, ..., until it reaches the state $x_m \in \{x_1, x_2, \ldots, x_{j^*-1}\}$.

Next we present a generalization of the problem studied in this paper for the case in which the number of the states of the system is not finite, i.e. we have $\omega = |V| = \infty$. This case cannot be handled in the same way as finite case, because it is not known any formula and any algorithm for determining the expectation of the evolution time of stochastic system with final sequence of states and interdependent states when the number of the states is not finite and the transition matrix and initial distribution are fixed and given. Nevertheless, the optimal distribution and optimal transition matrix can be determined using the result obtained above.

Indeed, we observed above that the given stochastic system, with finite or infinite number of states, can be reduced to a new stochastic system with maximum $m$ states, $x_1, x_2, \ldots, x_m$, preserving the optimal solution. This reduction is possible thanks to Theorems 5 and 6, from which, in optimal case, the excluded states cannot be reached by system at any moment of time. So, if $\overline{p}$ is the optimal transition matrix for the stochastic system with infinite number of states and $\overline{p_r}$ is the optimal transition matrix for the reduced stochastic system, then

$$\overline{p}(x,y) = \begin{cases} \overline{p_r}(x,y) & if \quad x, y \in \{x_1, x_2, \ldots, x_m\} \\ 1 & if \quad x \notin \{x_1, x_2, \ldots, x_m\} \ and \ y = x_1 \\ 0 & otherwise \end{cases}, \ \forall x, y \in V.$$

## 5 Conclusions

In this paper the following results related to stochastic systems with final sequence of states and unitary transition time were established:

- The given stochastic system, with finite or infinite number of states, can be reduced to a new stochastic system with maximum $m$ states, $x_1, x_2, \ldots, x_m$, preserving the optimal solution;
- The evolution time of the stochastic system with fixed final sequence of states depends on initial distribution of the states and probability transition matrix;
- In the case when the states of the system are independent, the expectation of the evolution time represents a posynomial minus one unit, that offers the possibility to minimize it using geometric programming approach;
- In the case when the states of the system are interdependent, the expectation of the evolution time can be minimized by solving two geometric programs with posynomial equality constraints, that represents signomial programs which can be handled as geometric programs using the models developed in this paper;
- In several particular cases, which were described in Section 4, the optimal initial distribution and optimal probability transition matrix are trivial.

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