# SQS-3-Groupoids with $q(x, x, y)=x$ 

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#### Abstract

A new algebraic structure $(P ; q)$ of a Steiner quadruple systems $\operatorname{SQS}(P ; B)$ called an SQS-3-groupoid with $q(x, x, y)=x$ (briefly: an SQS-3-quasigroup) is defined and some of its properties are described. Sloops are considered as derived algebras of SQS-skeins. Squags and also commutative loops of exponent 3 with $x(x y)^{2}=y^{2}$ given in [7] are derived algebras of SQS-3-groupoids. The role of SQS-3-groupoids in the clarification of the connections between squags and commutative loops of exponent 3 is described.


## 1. Introduction

A Steiner quadruple (triple) system is a pair $(T ; B)$, where $T$ is a finite set and $B$ is a collection of 4 -subsets (3-subsets) called blocks of $T$ such that every 3 -subset (2-subset) of $T$ is contained in exactly one block of $B[4,8]$. Let $\operatorname{SQS}(m)$ denote a Steiner quadruple system (briefly: quadruple system) of cardinality $m$ and $\operatorname{STS}(n)$ denote Steiner triple system (briefly: triple system) of cardinality $n$. It is will known that $\operatorname{SQS}(m)$ exists iff $m \equiv 2$ or $4(\bmod 6)$ and $\operatorname{STS}(n)$ exists iff $m \equiv 1$ or $3(\bmod 6)(c f .[4,8])$.

If we consider $T_{x}=T-\{x\}$ for any point $x \in T$ and delete this point from all blocks which contain it, then the obtaining system $\left(T_{x} ; B(x)\right)$ is a triple system, where $B(x)=\left\{b^{\prime}=b-\{x\}: b \in B\right.$ and $\left.x \in b\right\}$. The system $\left(T_{x} ; B(x)\right)$ is called a derived triple system of $(P ; B)[4,7]$. There are one-to-one correspondences between STSs and each of sloops and squags and also between SQSs and SQS-skeins $[4,11]$.

An SQS-skein is an algebra $\mathbf{T}=\left(T ; q^{\prime}\right)$ with one fundamental ternary operation $q^{\prime}$ satisfying the identities:

[^0]\[

$$
\begin{aligned}
& q^{\prime}(x, y, z)=q^{\prime}(y, x, z)=q^{\prime}(z, x, y) \\
& q^{\prime}(x, x, y)=y \\
& q^{\prime}\left(x, y, q^{\prime}(x, y, z)\right)=z
\end{aligned}
$$
\]

A sloop (or a Steiner loop) is a commutative loop $(T ; \cdot, 1)$ satisfying the Steiner identity $x \cdot(x \cdot y)=y$. A squag (or a Steiner quasigroup) is an idempotent commutative quasigroup $(Q ; \cdot)$ satisfying the Steiner identity. Note that sloops are derived algebras of SQS-skeins, while squags can't be considered as derived algebras of SQS-skeins.

Let $q$ be a ternary operation of a nonempty finite set $T$, then the algebra $(T ; q)$ will be called an SQS-3-groupoid with $q(x, x, y)=x$ (briefly: an SQS3 -groupoid), if the following identities are satisfied:

$$
\begin{aligned}
& q(x, y, z)=q(x, z, y)=q(z, y, x) \\
& q(x, x, y)=x \\
& q(x, y, q(x, y, z))=z \quad \text { if } \quad x \neq y
\end{aligned}
$$

It is clear that $q$ is a totally commutative idempotent ternary operation and satisfies the Steiner equation (the third equation). Moreover, this algebra is a 3 -groupoid but not a 3-quasigroup, because the equation $q(a, a, x)=a$ has no unique solution [10]. Also, the operation $q$ is commutative but not associative [10]. Similarly, it is idempotent, but it doesn't satisfy the generalized idempotent law, i.e. $q(x, x, y) \neq y$ for $y \neq x$.

This algebra does not seem to be a nice algebra because many algebraic constructions can't be made within this class. For example, if $\rho$ is a congruence on an SQS-3-groupoid $(T ; q)$, and if $(x, y) \in \rho$, then $(x, x) \in \rho$ and $(z, z) \in \rho$ for all $z \in T$. Hence $(q(x, x, z), q(x, y, z))=(x, q(x, y, z)) \in \rho$ for all $z \in T$, i.e. $(x, w) \in \rho$ for all $w \in T$. This means that such algebra has no proper congruences.

An SQS is called distributive or medial, if the associated SQS-3-groupoid satisfies the distributive or the medial law for SQSs, respectively (or more precisely, if all derived squags of the associated SQS-3-groupoid are distributive or medial, respectively).

Let $(T ; B)$ be an SQS, we define the ternary operation $q_{B}$ on $T$ putting

$$
q_{B}(x, y, z)=\left\{\begin{array}{lll}
w & \text { if } & \{x, y, z, w\} \in B \\
x & \text { if } & x=y \text { or } x=z \\
y & \text { if } y=z
\end{array}\right.
$$

Obviously such defined $\left(T ; q_{B}\right)$ is an SQS-3-groupoid.

Conversely, let $(T ; q)$ be an SQS-3-groupoid. Consider the set:

$$
B_{q}:=\{\{x, y, z, q(x, y, z)\}: \text { for all }\{x, y, z\} \subseteq T \text { with }|\{x, y, z\}|=3\} .
$$

It is clear that $q(x, y, z) \notin\{x, y, z\}$, otherwise if $q(x, y, z)=x$, then $q(x, y, q(x, y, z))=q(x, y, x)=x$ and from the Steiner equation we have $q(x, y, q(x, y, z))=z$, which is a contradiction. Since $|\{x, y, z\}|=3$, hence $|\{x, y, z, q(x, y, z)\}|=4$. Since $q$ is commutative, then $\left(T ; B_{q}\right)$ is an SQS. Moreover one can deduce that $q_{B q}=q$ and $B_{q_{B}}=B$.

This proves that there is a one-to-one correspondence between SQS's and SQS-3-groupoids.

Carmichael [3] and Lüneburg [9] constructed an $\operatorname{SQS}\left(3^{n}+1\right)$. In section 2, we prove that the associated SQS-3-groupoids with this construction of $\operatorname{SQS}\left(3^{n}+1\right)$ satisfies the medial law for SQSs. Next, in section 3, we prove that a commutative loop of exponent 3 satisfying the identity $x(x y)^{2}=y^{2}$ (i.e. an interior Steiner loop [2]) is also a derived algebra of the SQS-3groupoid.

## 2. Medial SQS-3-groupoids

For any element $a \in T$, we define the derived algebra $\left(T_{a} ; \circ\right)$ of the SQS-3groupoid ( $T ; q$ ) putting: $T_{a}=T-\{a\}$ and $x \circ y=q(a, x, y)$ for all $x, y \in T_{a}$. Since

$$
\begin{aligned}
x \circ x & =x \\
x \circ y & =y \circ x \\
x \circ(x \circ y) & =y,
\end{aligned}
$$

the derived algebra $\left(T_{a} ; \circ\right)$ is the well-known squag.
The class of SQS-3-groupoids is not variety, but the class of all derived algebras forms the well-known variety of squags.

The interesting subclass of squags forms medial squags which are squags satisfying the medial identity:

$$
(x \circ y) \circ(z \circ w)=(x \circ z) \circ(y \circ w) .
$$

The finite medial squags correspond to the class of affine geometries over GF (3) (see Klossek [7] and Guelzow [5]). Medial squags are derived algebras of the subclass of so-called medial SQS-3-groupoids, i.e. SQS-3-groupoids satisfying the following medial law for SQSs:

$$
q(a, q(a, x, y), q(a, z, w))=q(a, q(a, x, z), q(a, y, w)) .
$$

Associated SQSs are called medial.
The smallest nontrivial medial SQS-3-groupoid is the associated SQS3 -groupoid of the quadruple system $\operatorname{SQS}(10)$.

Example 1. Let ( $T(10) ; q_{B}$ ) be the SQS-3-groupoid associated with the quadruple system of order 10 . Any derived squag $\left(T_{a} ; \circ\right)$ of $\left(T(10) ; q_{B}\right)$ is of order 9 and is associated with the triple system $\operatorname{STS}(9)$. The $\operatorname{STS}(9)$ is isomorphic to the affine plane over $\mathrm{GF}(3)$ (cf. [5, 7]) and then the squag $\left(T_{a} ; \circ\right)$ satisfies the medial identity. This implies $\left(T(10) ; q_{B}\right)$ satisfies the medial law for SQSs.

Carmichael [3] and Lüneburg [9] constructed an $\operatorname{SQS}\left(3^{n}+1\right)=\left(K^{*} ; B^{*}\right)$ having a sharp triply transitive automorphism group $\Gamma^{*}$. Namely, $K^{*}$ and $B^{*}$ are defined by:

$$
K^{*}=G F\left(3^{n}\right) \cup\{\infty\}
$$

and

$$
B^{*}=\left\{\Psi(B): B=\{0,1,-1, \infty\} \text { and } \Psi \in \Gamma^{*}\right\},
$$

where

$$
\Gamma^{*}=\left\{\Psi: K^{*} \rightarrow K^{*}: \Psi(x)=(a x+b) /(c x+d), a d-b c \neq 0\right\} .
$$

The following theorem is given in [9], helps us to show that the construction $\left(K^{*} ; B^{*}\right)$ supplies us with an example of a medial SQS-3-groupoid of cardinality $3^{n}+1$ for each positive integer $n$.

Theorem 1. If $(K ; B)$ is a triple system with a sharp doubly transitive automorphism group $\Gamma^{*}$, then $(K ; B)$ is an affine plane over $G F(3)$.

Example 2. For any $p \in K^{*}$, we have the derived STS ( $K_{p}^{*} ; B_{p}^{*}$ ) of ( $K^{*} ; B^{*}$ ) with the automorphism group $\Gamma_{p}^{*}$ defined by:

$$
\Gamma_{P}^{*}=\left\{\Psi: K_{P}^{*} \rightarrow K_{P}^{*}: \Psi \in \Gamma^{*}, \Psi(p)=p\right\},
$$

where $\Gamma_{p}^{*}$ is a sharp doubly transitive automorphism group of ( $K_{p}^{*} ; B_{p}^{*}$ ).
According to the above theorem, the triple system ( $K_{p}^{*} ; B_{p}^{*}$ ) is an affine plane over GF(3). This means that the squag ( $K_{p}^{*} ; \circ$ ) associated with $\left(K_{p}^{*} ; B_{p}^{*}\right)$ satisfies the medial identity. The medial law for SQSs is satisfied, too.

According to the above discussion and the construction $\left(K^{*} ; B^{*}\right)=$ $S Q S\left(3^{n}+1\right)$ given by Carmichael [3] and Lüneburg [9], we can say that any finite medial squag is a derived algebra from a medial SQS-3-groupoid.

In other words, we can say that there are quadruple systems in which the associated squag of any its derived triple systems is medial.

Now, we consider the question about the existence of an quadruple systems in which the squag associated with each derived STS is distributive, i . e. each derived squag of an SQS-3-groupoid satisfies the distributive law:

$$
x \circ(y \circ z)=(x \circ y) \circ(x \circ z)
$$

In other word, is there a non-medial SQS-3-groupoid satisfying the distributive law for SQSs

$$
q(a, x, q(a, y, z))=q(a, q(a, x, y), q(a, x, z)) ?
$$

M. Hall [6] constructed an STS (called now a Hall triple system) in which each three elements generate the affine plane over GF(3). The smallest cardinality for a Hall STS is 81 . The associated squags of Hall triple systems are distributive. A vector-space model of a distributive squag of cardinalty $3^{m}$ is given by Klossek [7].

As a special case, for $m=4$, we obtain the smallest non-medial distributive squag $\left(\mathrm{GF}(3)^{4} ; \circ\right)$, where the binary operation $\circ$ is defined by:

$$
x \circ y=2 x+2 y+\left(0,0,0,\left|\begin{array}{ll}
x_{2} & x_{3} \\
y_{2} & y_{3}
\end{array}\right|\left(x_{1}-y_{1}\right)\right) .
$$

The above question can be formulated in the following way:
Is there SQS-3-groupoid of cardinality 82 all of whose derived squags are distributive but non-medial?
or in the combinatorial language:
Is there $S Q S(82)$ all of whose derived $S T S(81)$ s are Hall STSs but are not isomorphic to the direct power $\operatorname{STS}(3)^{4}$ ?

## 3. SQS-3-Groupoids and commutative loops

A commutative loop $(L ; \cdot, e)$ of exponent 3 is called Moufang, if it satisfies the Moufang identity:

$$
x \cdot(x \cdot(y \cdot z))=(x \cdot y) \cdot(x \cdot z)
$$

Commutative Moufang loops of exponent 3 and distributive squags are polynomially equivalent [7]. There is a one-to-one correspondence between
triple systems and commutative loops of exponent 3 with $x(x y)^{2}=y^{2}$ (cf. [1]). Therefore, as it is proved in [1], commutative loops of exponent 3 with $x(x y)^{2}=y^{2}$ are polynomially equivalent to squags.

Moreover, any commutative loop of exponent 3 with $x(x y)^{2}=y^{2}$ is a derived algebra of the constructed SQS-3-groupoid.

Indeed, let $(T ; q)$ be an SQS-3-groupoid and $a \in T$. Then ( $T-\{a\} ; \circ$ ) is a squag, where $\circ$ is defined by the formula

$$
x \circ y=q(a, x, y) .
$$

Fixing $e \in T-\{a\}$ and putting

$$
x \cdot y=q(a, e, q(a, x, y)),
$$

we can see that $(T-\{a\} ; \cdot, e)$ is a commutative loop of exponent 3 with $x \cdot(x \cdot y)^{2}=y^{2}$.

Moreover, for SQS-3-groupoids, we have

$$
x \cdot y=q(a, e, q(a, x, y)=e \circ(x \circ y)
$$

and

$$
\begin{aligned}
x \circ y & =q(a, x, y)=q\left(a, q\left(a, e, x^{2}\right), q\left(a, e, y^{2}\right)\right) \\
& =q\left(a, e, q\left(a, x^{2}, y^{2}\right)\right)=x^{2} \cdot y^{2}
\end{aligned}
$$

for distributive SQS-3-groupoids.
Thus, using results from [7] and [1], we can see that
(i) if $(T ; q)$ is a distributive SQS-3-groupoid, then for each $a \in T$ and each $e \in T-\{a\}$, the squag $(T-\{a\} ; \circ)$ is distributive and $(T-\{a\} ; \cdot, e)$ is a commutative Moufang loop of exponent 3,
(ii) if $(T ; q)$ is a medial SQS-3-groupoid, then for all $a \in T$ and $e \in T-\{a\}$, the squag $(T-\{a\} ; \circ$ ) is medial and the loop $(T-\{a\} ; \cdot, e)$ is a commutative group of exponent 3 .

This, according to results obtained in [7] and [1], means that the distributive squag $(T-\{a\} ; \circ)$ is polynomially equivalent to the commutative Moufang loop ( $T-\{a\} ; \cdot, e$ ) of exponent 3 with $x \cdot(x \cdot y)^{2}=y^{2}$.

In the next page, we give the diagram presenting some connections between different types of algebras derived from SQS-3-groupoids.

SQS-3-groupoids with $q(x, x, y)=x$


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# Some results on hyper BCK-algebras 

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#### Abstract

In this paper by considering the notion of hyper $B C K$-algebra, we state and prove some theorems which determine the relationship among (weak) hyper BCK-ideals, positive implicative hyper $B C K$-ideals of types $1,3, \ldots, 8$ and hypersubalgebras, under some suitable conditions. Moreover, we define the notions of commutative hyper $B C K$-ideals of types $1,2,3$ and 4 and obtain some results.


## 1. Introduction

The study of $B C K$-algebras was initiated by Y. Imai and K. Iséki [5] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then a great deal of literature has been produced on the theory of $B C K$-algebras. In particular, emphasis seems to have been put on the ideal theory of BCK-algebras. The hyperstructure theory (called also multialgebras) was introduced in 1934 by F. Marty [8] at the 8th congress of Scandinavian Mathematicians. Around 40's, several authors worked on hypergroups, especially in France, United States, Italy, Greece and Iran. Hyperstructures have many applications to several sectors of both pure and applied sciences. In [7], Y.B. Jun, M.M. Zahedi, X. L. Xin and R.A. Borzooei applied the hyperstructures to $B C K$-algebras, and introduced the notion of a hyper $B C K$-algebra which is a generalization of $B C K$-algebra, and investigated some related properties. They also introduced the notions of hyper $B C K$-ideal and weak (strong) hyper $B C K$-ideal, and gave relations among this notions. Now we follow $[3,6,7]$ and obtain some results, as mentioned in the abstract.

2000 Mathematics Subject Classification: 06F35, 03G25
Keywords: hyper $B C K$-algebra, weak hyper $B C K$-ideal, commutative hyper $B C K$ ideal, positive implicative hyper $B C K$-ideal

## 2. Preliminaries

Definition 2.1. By a hyper BCK-algebra we mean a nonempty set $H$ endowed with a hyperoperation $\circ$ and a constant 0 satisfies the following axioms:
$(\mathrm{HK} 1) \quad(x \circ z) \circ(y \circ z) \ll x \circ y$,
(HK2) $(x \circ y) \circ z=(x \circ z) \circ y$,
(HK3) $\quad x \circ H \ll\{x\}$,
(HK4) $\quad x \ll y$ and $y \ll x$ imply $x=y$
for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H, A \ll B$ is defined by $\forall a \in A, \exists b \in B$ such that $a \ll b$. In such case, we call $\ll$ the hyperorder in $H$.

Theorem 2.2 [7]. In any hyper $B C K$-algebra $H$, the following hold:
(i) $0 \circ 0=\{0\}$,
(ii) $0 \ll x$,
(iii) $x \ll x$,
(iv) $A \subseteq B$ implies $A \ll B$,
(v) $0 \circ x=\{0\}$,
(vi) $x \circ y \ll x$,
(vii) $x \circ 0=\{x\}$,
for all $x, y, z \in H$ and for all nonempty subsets $A$ and $B$ of $H$.
Let $I$ be a nonempty subset of a hyper $B C K$-algebra $H$. Then $I$ is said to be a hyper $B C K$-ideal of $H$, if for all $x, y \in H, x \circ y \ll I$ and $y \in I$ imply $x \in I$, weak hyper $B C K$-ideal of $H$, if for all $x, y \in H, x \circ y \subseteq I$ and $y \in I$ imply $x \in I$, strong hyper $B C K$-ideal of $H$, if for all $x, y \in H$, $(x \circ y) \cap I \neq \emptyset$ and $y \in I$ imply $x \circ y \subseteq I$, hyper BCK-subalgebra of $H$, if $I$ is a hyper $B C K$-algebra with respect to the hyperoperation o on $H$.

Clear that, any strong hyper $B C K$-ideal of $H$ is a hyper $B C K$-ideal and any hyper $B C K$-ideal of $H$ is a weak hyper $B C K$-ideal. Moreover, let $I$ be a nonempty subset of a hyper $B C K$-algebra $H$. Then $I$ is a hypersubalgebra of $H$ if and only if $x \circ y \subseteq I$ for all $x, y \in I$.

Definition 2.3. Let $I$ be a nonempty subset of hyper $B C K$ algebra $H$ and $0 \in I$. Then $I$ is said to be a positive implicative hyper BCK-ideal of
(i) type 1 ,
if $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$ imply that $x \circ z \subseteq I$ for all $x, y, z \in H$,
(ii) type 2,
if $(x \circ y) \circ z \ll I$ and $y \circ z \subseteq I$ imply that $x \circ z \subseteq I$ for all $x, y, z \in H$,
(iii) type 3,
if $(x \circ y) \circ z \ll I$ and $y \circ z \ll I$ imply that $x \circ z \subseteq I$ for all $x, y, z \in H$, (iv) type 4,
if $(x \circ y) \circ z \subseteq I$ and $y \circ z \ll I$ imply that $x \circ z \subseteq I$ for all $x, y, z \in H$,
(v) type 5,
if $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$ imply that $x \circ z \ll I$ for all $x, y, z \in H$, (vi) type 6,
if $(x \circ y) \circ z \ll I$ and $y \circ z \ll I$ imply that $x \circ z \ll I$ for all $x, y, z \in H$, (vii) type 7,
if $(x \circ y) \circ z \subseteq I$ and $y \circ z \ll I$ imply that $x \circ z \ll I$ for all $x, y, z \in H$, (viii) type 8 ,
if $(x \circ y) \circ z \ll I$ and $y \circ z \subseteq I$ imply that $x \circ z \ll I$ for all $x, y, z \in H$.
In the following diagram, we can see the relationship among all of types of positive implicative hyper $B C K$-ideals.


Let $H$ be a hyper $B C K$-algebra and for each $a, b \in H,|a \circ b|$ be cardinality of $a \circ b$. An element $a \in H$ is said to be left (resp. right) scalar if $|a \circ x|=1$ (resp. $|x \circ a|=1$ ) for all $x \in H$. If $a \in H$ is both left and right scalar, we say that $a$ is a scalar element.

We say that subset $I$ of $H$ satisfies the closed condition, if $x \ll y$ and $y \in I$ imply $x \in I$, for all $x, y \in H$.

Lemma 2.4. If $I$ is a hyper BCK-ideal and $A$ is a nonempty subset of $H$, then I satisfies the closed condition and if $A \ll I$, then $A \subseteq I$.

Theorem 2.5. Let I be a nonempty subset of $H$ satisfying the closed condition. If $I$ is a positive implicative hyper BCK-ideal of type $i$, then $I$ is a positive implicative hyper BCK-ideal of type $j$, for all $1 \leqslant i, j \leqslant 8$.

Proof. By considering the Lemma 2.4 the proof is straightforward.
Lemma 2.6 [3]. Let $H=\{0,1,2\}$ be a hyper BCK-algebra of order 3. Then the following statements are hold.
(a) If $H$ satisfies the simple condition (that is $1 \nless 2$ and $2 \nless 1$ ), then
(i) $1 \circ 1 \in\{\{0\},\{0,1\}\}$ and $1 \circ 2=\{1\}$,
(ii) $2 \circ 1=\{2\}$ and $2 \circ 2 \in\{\{0\},\{0,2\}\}$.
(b) If $H$ satisfies the normal condition (that is $1 \ll 2$ or $2 \ll 1$ ), then
(iii) $1 \circ 1 \in\{\{0\},\{0,1\}\}$,
(iv) $1 \circ 2 \in\{\{0\},\{0,1\}\}$,
(v) $2 \circ 1 \in\{\{1\},\{2\},\{1,2\}\}$,
(vi) $2 \circ 2 \in\{\{0\},\{0,1\},\{0,2\},\{0,1,2\}\}$.

Theorem 2.7 [3]. Let $H$ be a hyper BCK-algebra of order 3 which satisfies the normal condition. Then $H$ has at most one proper hyper BCK-ideal.

## 3. Positive implicative hyper $B C K$-ideals

In the sequel $H$ denotes a hyper $B C K$-algebra.
Definition 3.1. A nonempty subset $I$ of $H$ is said to be $S$-reflexive if $(x \circ y) \bigcap I \neq \emptyset$ implies that $(x \circ y) \subseteq I$, for all $x, y \in H$.

Theorem 3.2. Let $I$ be a $S$-reflexive nonempty subset of $H$. If $I$ is a positive implicative hyper BCK-ideal of type 1, then I is a strong hyper $B C K$-ideal of $H$ and so is a positive implicative hyper BCK-ideal of type $i$ for all $1 \leqslant i \leqslant 8$.
Proof. Assume that $I$ is a positive implicative hyper $B C K$-ideal of type 1 , $(x \circ y) \bigcap I \neq \emptyset$ and $y \in I$ for $x, y \in H$. Since $I$ is S-reflexive, then $x \circ y \subseteq I$. Hence by Theorem $2.2(v i i),(x \circ y) \circ 0=x \circ y \subseteq I$ and $y \circ 0=\{y\} \subseteq I$. Since $I$ is a positive implicative hyper $B C K$-ideal of type 1 , then $\{x\}=x \circ 0 \subseteq I$ i.e $\quad x \in I$. Thus $I$ is a strong hyper $B C K$-ideal of $H$ and so $I$ is a hyper $B C K$-ideal of $H$. Hence by Lemma 2.4, I satisfy the closed condition and so by Theorem 2.5, $I$ is a positive implicative hyper BCK-ideal of type $i$ for all $1 \leqslant i \leqslant 8$.

Example 3.3. Let $H$ be a hyper $B C K$-algebra which is defined as follows:

| $\circ$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| 1 | $\{1\}$ | $\{0\}$ | $\{0\}$ |
| 2 | $\{2\}$ | $\{1,2\}$ | $\{0,1,2\}$ |

Then $I=\{0,1\}$ is a positive implicative hyper $B C K$-ideal of type $1,3, \ldots, 8$, but it is not a strong hyper $B C K$-ideal and it is not a S-reflexive. Because $2 \circ 1=\{1,2\} \nsubseteq I$, where $(2 \circ 1) \bigcap I \neq \emptyset$. Therefore, the S-reflexive condition is necessary in Theorem 3.2.

Definition 3.4. (i) $H$ is called a positive implicative hyper $B C K$-algebra, if for all $x, y, z \in H,(x \circ y) \circ z=(x \circ z) \circ(y \circ z)$.
(ii) $H$ is called an alternative quasi hyper BCK-algebra, if for all $x, y \in H$, $(x \circ y) \circ y=x \circ(y \circ y)$.

Lemma 3.5. Let $A, B$ and $I$ are nonempty subsets of $H$. If $I$ is a weak hyper $B C K$-ideal of $H, A \circ B \subseteq I$ and $B \subseteq I$, then $A \subseteq I$.

Theorem 3.6. If $H$ is a positive implicative hyper $B C K$-algebra, then any weak hyper BCK-ideal of $H$ is a positive implicative hyper BCK-ideal of types 1 and 5.
Proof. Let $I$ be a weak hyper $B C K$-ideal of $H,(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$, for $x, y, z \in H$. Since $H$ is a positive implicative hyper $B C K$-algebra, then $(x \circ z) \circ(y \circ z)=(x \circ y) \circ z \subseteq I$. Hence by Lemma 3.5, we get that $x \circ z \subseteq I$. Therefore $I$ is a positive implicative hyper $B C K$-ideal of type 1 and so by diagram in section 2, $I$ is a positive implicative hyper $B C K$-ideal of type 5.

Example 3.7. Let $H$ be a hyper $B C K$-algebra which is defined as follows:

| $\circ$ | 0 | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: | :---: |
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| 1 | $\{1\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| 2 | $\{2\}$ | $\{2\}$ | $\{0\}$ | $\{0\}$ |
| 3 | $\{3\}$ | $\{3\}$ | $\{2\}$ | $\{0,2\}$ |

Then $H$ is not a positive implicative hyper $B C K$-algebra. Since ( $3 \circ 2$ ) $\circ 2=$ $0 \neq 2=(3 \circ 2) \circ(2 \circ 2)$. Moreover $I=\{0,1\}$ is a weak hyper $B C K$-ideal of $H$ but it is not a positive implicative hyper $B C K$-ideal of type 5 . Since $(3 \circ 2) \circ 2=\{0\} \subseteq I$ and $2 \circ 2=\{0\} \subseteq I$, but $3 \circ 2=\{2\} \nless I$ and so by diagram in section 2, $I$ is not a positive implicative hyper $B C K$-ideal of type 1. Therefore, positive implicative condition is necessary in Theorem 3.6.

Definition 3.8. A subset $I$ of $H$ is said to be proper if $\{0\} \subset I \subset H$.
Theorem 3.9. Let $H=\{0,1,2\}$ be an alternative quasi hyper BCKalgebra. Then, there is at least one proper weak hyper BCK-ideal of $H$.

Proof. We claim that $I=\{0,1\}$ is a weak hyper $B C K$-ideal of $H$. Let $x \circ y \subseteq I$ and $y \in I$ for $x, y \in H$. We must show that $x \in I$. Let $x \notin I$ (by contrary). Then $x=2$ and so $2 \circ y \subseteq I$. Since $y \in I$ then $y=0$ or 1. If $y=0$ then by Theorem 2.2 (vii), $2 \in\{2\}=2 \circ 0 \subseteq I$, which is a contradiction. Hence $y=1$. By Lemma $2.6,2 \circ 1=\{1\},\{2\}$ or $\{1,2\}$. If $2 \circ 1=\{2\}$ or $\{1,2\}$, then $2 \in 2 \circ 1=x \circ y \subseteq I$, which is impossible. Hence $2 \circ 1=\{1\}$. Moreover, by Lemma $2.6(i i i), 1 \circ 1=\{0\}$ or $\{0,1\}$. If $1 \circ 1=\{0\}$, then by Theorem 2.2 (vii)

$$
(2 \circ 1) \circ 1=1 \circ 1=\{0\} \neq\{2\}=2 \circ 0=2 \circ(1 \circ 1)
$$

which is contradiction by alternative quasi. If $1 \circ 1=\{0,1\}$, then $(2 \circ 1) \circ 1=$ $1 \circ 1=\{0,1\}$. But $2 \in\{2\}=2 \circ 0 \subseteq 2 \circ(1 \circ 1)$ and so $(2 \circ 1) \circ 1 \neq$ $2 \circ(1 \circ 1)$, which is a contradiction by alternative quasi hyper $B C K$-algebra. Therefore, $I=\{0,1\}$ is a weak hyper $B C K$-ideal of $H$.

Theorem 3.10. Let $H=\{0,1,2\}$ be a hyper BCK-algebra of order 3 and $I$ be a proper subset of $H$. Then
(i) I is a positive implicative hyper BCK-ideal of type 3 if and only if $I$ is a hyper BCK-ideal,
(ii) I is a positive implicative hyper BCK-ideal of type 1 if and only if $I$ is a weak hyper $B C K$-ideal of $H$.

Proof. (i) It is easy to check that, any positive implicative hyper $B C K$ ideal of type 3 is a hyper $B C K$-ideal of $H$.

Conversely, let $I$ be a hyper $B C K$-ideal of $H$. We consider two following cases.

Case 1. $H$ satisfies the normal condition. By Theorem 2.7, $H$ has at most one proper hyper $B C K$-ideal which is $I=\{0,1\}$. Now, let $I=\{0,1\}$ be a hyper $B C K$-ideal of $H$. Then $2 \circ 1 \nless I$. Since $1 \in I$, if $2 \circ 1 \ll I$, then $2 \in I$, which is impossible. Hence $2 \in 2 \circ 1$ and so by Lemma $2.6(v)$, $2 \circ 1=\{2\}$ or $\{1,2\}$. Now, let $(x \circ y) \circ z \ll I$ and $y \circ z \ll I$, but $x \circ z \nsubseteq I$. Then $2 \in x \circ z$. By Lemma $2.6(i i i)$ and $(i v), x \neq 1$. Moreover, $x \neq 0$. Since if $x=0$, then $2 \in x \circ z=0 \circ z=\{0\}$, which is impossible. Thus $x=2$. Since $I$ is a hyper $B C K$-ideal of $H$, then

$$
(x \circ y) \circ z \subseteq I \text { and } y \circ z \subseteq I
$$

Now, we considering the following cases:
Case 1.1. If $z=0$, since $\{y\}=y \circ 0=y \circ z \subseteq I$, then $y=0$ or 1. If $y=0$, then $\{2\}=(2 \circ 0) \circ 0=(x \circ y) \circ z \subseteq I$, which is a contradiction. If $y=1$, then $2 \in 2 \circ 1=(2 \circ 1) \circ 0=(x \circ y) \circ z \subseteq I$, which is impossible.

Case 1.2. If $z=1$, then $y \circ 1=y \circ z \subseteq I$. Since $I$ is a hyper $B C K$-ideal of $H$ and $1 \in I$, then $y \in I$ and so $y=0$ or 1 . If $y=0$, then by (HK2)

$$
2 \in 2 \circ 1=(2 \circ 1) \circ 0=(2 \circ 0) \circ 1=(x \circ y) \circ z \subseteq I
$$

which is a contradiction. If $y=1$, then

$$
2 \in 2 \circ 1 \subseteq(2 \circ 1) \circ 1=(x \circ y) \circ z \subseteq I
$$

which is impossible.
Case 1.3. If $z=2$, since $2 \in x \circ z$ and $x=z=2$, then $2 \in 2 \circ 2$. Hence, by Lemma $2.6(v i), 2 \circ 2=\{0,2\}$ or $\{0,1,2\}$. If $y=0$, then

$$
2 \in 2 \circ 2=(2 \circ 0) \circ 2=(x \circ y) \circ z \subseteq I
$$

which is a contradiction. If $y=1$, then by (HK2)

$$
2 \in 2 \circ 1 \subseteq(2 \circ 2) \circ 1=(2 \circ 1) \circ 2=(x \circ y) \circ z \subseteq I
$$

which is impossible. If $y=2$, then

$$
2 \in 2 \circ 2 \subseteq(2 \circ 2) \circ 2=(x \circ y) \circ z \subseteq I
$$

which is impossible. Therefore, $x \circ z \subseteq I$ and so $I$ is a positive implicative hyper $B C K$-ideal of type 3 .

Case 2. $H$ satisfies the simple condition. By Theorem 3.1 [3], there are only three hyper $B C K$-algebras of order 3 which satisfies the simple condition. Now, we can show that the $I_{1}=\{0,1\}$ and $I_{2}=\{0,2\}$ are hyper $B C K$-ideals and positive implicative hyper $B C K$-ideal of type 3 in the this three hyper $B C K$-algebras.
(ii) The proof is similar to the proof of case $(i)$.

Theorem 3.11. Let $H=\{0,1,2\}$ be an alternative quasi hyper $B C K$ algebra. Then there is at least one proper positive implicative hyper BCKideal of type $1,3, \ldots, 8$.

Proof. By the proof of Theorem $3.9, I=\{0,1\}$ is a weak hyper $B C K$ ideal of $H$ and so by Theorem 3.10 (ii), $I$ is a positive implicative hyper $B C K$-ideal of type 1 .

Now, we show that $I$ is a hyper $B C K$-ideal of $H$. Let $x \circ y \ll I$ and $y \in I$ but $x \notin I$ (by contrary). Then $x=2$. Since $y \in I$, then $y=0$ or 1. If $y=0$, then $\{2\}=2 \circ 0 \ll I_{1}$ and so $2 \ll 1$. Hence $0 \in 2 \circ 1$, which
is impossible by Lemma 2.6. If $y=1$, then we consider the following two cases.

Case 1. Let $H$ satisfies the simple condition. Then by Lemma 2.6 (ii), $\{2\}=2 \circ 1 \ll I_{1}=\{0,1\}$ and so $2 \ll 1$, which is a contradiction.

Case 2. Let $H$ satisfies the normal condition. Then by Lemma $2.6(v)$, $2 \circ 1=\{1\},\{2\}$ or $\{1,2\}$. If $2 \circ 1=\{2\}$ or $\{1,2\}$, then $2 \in 2 \circ 1 \ll I_{1}=\{0,1\}$ and so $2 \ll 1$. Hence $0 \in 2 \circ 1$ which is a impossible by Lemma 2.6. If $2 \circ 1=\{1\}$, then $2 \circ 1 \subseteq I$. Since $I$ is a weak hyper $B C K$-ideal of $H$, and $1 \in I$, then $2 \in I=\{0,1\}$ which is impossible. Hence, $I$ is a hyper $B C K$ ideal of $H$. Therefore, by Lemma 2.4 and Theorem 2.5 since $I$ is a positive implicative hyper $B C K$-ideal of type 1 , then $I$ is a positive implicative hyper $B C K$-ideal of type $i$, for all $1 \leqslant i \leqslant 8$.

Theorem 3.12. Let $H$ be a positive implicative and an alternative quasi hyper BCK-algebra. Then every hyper BCK-subalgebra of $H$ is a positive implicative hyper BCK-ideal of type 1.

Proof. Let $I$ be a hyper $B C K$-subalgebra of $H,(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$, for $x, y, z \in H$. Since $H$ is a positive implicative hyper $B C K$-algebra, then $(x \circ z) \circ(y \circ z)=(x \circ y) \circ z \subseteq I$. Then for all $t \in x \circ z$ and $s \in y \circ z$, $t \circ s \subseteq I$. Since by Theorem 2.2 (iii) and (vii), $0 \in s \circ s$ and for all $t \in x \circ z$, $t \in\{t\}=t \circ 0$, hence

$$
t \in t \circ 0 \subseteq t \circ(s \circ s)=(t \circ s) \circ s \subseteq I \circ s \subseteq I,
$$

since $I$ is a hyper $B C K$-subalgebra and $s \in I$. Thus $x \circ z \subseteq I$. Therefore, $I$ is a positive implicative hyper $B C K$-ideal of type 1 .

Example 3.13. Consider the following tables:

| $\circ_{1}$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| 1 | $\{1\}$ | $\{0,1,2\}$ | $\{1\}$ |
| 2 | $\{2\}$ | $\{0,2\}$ | $\{0,2\}$ |


| $\circ_{2}$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| 1 | $\{1\}$ | $\{0,2\}$ | $\{2\}$ |
| 2 | $\{2\}$ | $\{0\}$ | $\{0\}$ |


| $\circ$ | 0 | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: | :---: |
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| 1 | $\{1\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| 2 | $\{2\}$ | $\{2\}$ | $\{0,2\}$ | $\{0\}$ |
| 3 | $\{3\}$ | $\{3\}$ | $\{3\}$ | $\{0,3\}$ |

$\left(H, \circ_{1}\right)$ is a positive implicative and alternative quasi hyper $B C K-$ algebra and $I=\{0,1\}$ is a positive implicative hyper $B C K$-ideal of type 1
but it is not a hyper $B C K$-subalgebra of $H$. Since $1 \in I$, but $1 \circ 1 \nsubseteq I$. Therefore, the converse of Theorem 3.12 is not correct in general.
$\left(H, \mathrm{o}_{2}\right)$ is a hyper $B C K$-algebra but it is not a positive implicative hyper $B C K$-algebra. Since, $(1 \circ 1) \circ 1 \neq(1 \circ 1) \circ(1 \circ 1)$. Moreover, $I=\{0,2\}$ is a hyper $B C K$-subalgebra of $H$, but it is not a positive implicative hyper $B C K$-ideal of type 1 . Since $(1 \circ 2) \circ 0 \subseteq I$ and $2 \circ 0 \subseteq I$ but $1 \circ 0 \nsubseteq I$.
$\left(H, \circ_{3}\right)$ is a hyper $B C K$-algebra but it is not an alternative quasi hyper $B C K$-algebra. Since, $(2 \circ 3) \circ 3 \neq 2 \circ(3 \circ 3)$. Moreover, $I=\{0,1,3\}$ is a hyper $B C K$-subalgebra of $H$, but it is not a positive implicative hyper $B C K$-ideal of type 1 . Since $(2 \circ 3) \circ 0 \subseteq I$ and $3 \circ 0 \subseteq I$ but $2 \circ 0 \nsubseteq I$.

## 4. Commutative hyper $B C K$-ideals

Definition 4.1. Let $I$ be a subset of $H$ such that $0 \in I$. Then $I$ is said to be a commutative hyper BCK-ideal of
(i) type 1, if $(x \circ y) \circ z \subseteq I$ and $z \in I$ imply $x \circ(y \circ(y \circ x)) \subseteq I$,
(ii) type 2, if $(x \circ y) \circ z \subseteq I$ and $z \in I$ imply $x \circ(y \circ(y \circ x)) \ll I$,
(iii) type 3 , if $(x \circ y) \circ z \ll I$ and $z \in I$ imply $x \circ(y \circ(y \circ x)) \subseteq I$,
(iv) type 4 , if $(x \circ y) \circ z \ll I$ and $z \in I$ imply $x \circ(y \circ(y \circ x)) \ll I$, for all $x, y, z \in H$.

Theorem 4.2. Let I be a nonempty subset of $H$. Then the following statements hold:
(i) if $I$ is a commutative hyper $B C K$-ideal of type 3 , then $I$ is a commutative hyper $B C K$-ideal of type 1 and 4,
(ii) if I is a commutative hyper BCK-ideal of type 1 or 4 , then $I$ is a commutative hyper BCK-ideal of type 2 .

Example 4.3. (i) Let $H$ be the hyper $B C K$-algebra which is defined as follows:

| $\circ$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| 1 | $\{1\}$ | $\{0\}$ | $\{1\}$ |
| 2 | $\{2\}$ | $\{0,2\}$ | $\{0,2\}$ |

Thus, $I=\{0,1\}$ is a commutative hyper $B C K$-ideal of type 1,2 and 4 but it is not of type 3 . Because, $(2 \circ 1) \circ 1=\{0,2\} \circ 1=\{0,2\} \ll I$ and $1 \in I$, but $2 \circ(1 \circ(1 \circ 2))=2 \circ(1 \circ 1)=2 \circ 0=\{2\} \nsubseteq I$.
(ii) Let $H=\{0,1,2,3\}$. The following table shows a hyper $B C K$ algebra structure on $H$ :

| $\circ$ | 0 | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: | :---: |
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| 1 | $\{1\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| 2 | $\{2\}$ | $\{1\}$ | $\{0,1\}$ | $\{1\}$ |
| 3 | $\{3\}$ | $\{1\}$ | $\{0\}$ | $\{0,1\}$ |

Then $I=\{0,2\}$ is a commutative hyper $B C K$-ideal of type 2 and 4 , but it is not commutative hyper $B C K$-ideal of type 1 . Since, $(2 \circ 1) \circ 2=1 \circ 2=$ $\{0\} \subseteq I$ and $2 \in I$ but $2 \circ(1 \circ(1 \circ 2))=2 \circ(1 \circ 0)=2 \circ 1=\{1\} \nsubseteq I$.

Moreover, $I=\{0,3\}$ is a commutative hyper $B C K$-ideal of type 2 , but it is not commutative hyper $B C K$-ideal of type 4 . Since, $(2 \circ 0) \circ 3=2 \circ 3=$ $\{1\} \ll I$ and $3 \in I$ but $2 \circ(0 \circ(0 \circ 2))=2 \circ 0=\{2\} \nless K$.

Theorem 4.4. Let I be a nonempty subset of $H$. Then:
(i) if $I$ is a commutative hyper BCK-ideal of type 3, then I is a hyper $B C K$-ideal of $H$,
(ii) if $I$ is a commutative hyper BCK-ideal of type 1 , then $I$ is a weak hyper $B C K$-ideal of $H$.

Proof. (i) Let $I$ be a commutative hyper $B C K$-ideal of type $3, x \circ y \ll I$ and $y \in I$, for $x, y \in H$. Since $(x \circ 0) \circ y=x \circ y \ll I$ and $y \in I$, then by hypothesis we get that $\{x\}=x \circ 0=x \circ(0 \circ(0 \circ x)) \subseteq I$. Therefore, $I$ is a hyper $B C K$-ideal of $H$.
(ii) The proof is similar to the proof $(i)$.

We summarize the Theorems 4.2 and 4.4 in the following diagram:


Lemma 4.5. Let $A, B$ and $I$ are nonempty subsets of a hyper $B C K$-algebra $H$. If $I$ is a hyper $B C K$-ideal of $H$, then $A \circ B \ll I$ and $B \subseteq I$ imply $A \subseteq I$.

Theorem 4.6. Let $H=\{0,1,2\}$ be a hyper $B C K$-algebra of order 3 and $I$ be a nonempty subset of $H$. Then:
(i) I is a commutative hyper BCK-ideal of type 3 if and only if $I$ is a hyper BCK-ideal of $H$,
(ii) $I$ is a commutative hyper $B C K$-ideal of type 1 if and only if $I$ is a weak hyper BCK-ideal of $H$,
(iii) if $I$ is a commutative hyper BCK-ideal of type 1 , then $I$ is a commutative hyper $B C K$-ideal of type 4.

Proof. $(i)(\Longrightarrow)$ The proof follows from Theorem 4.4 (i).
$(\Longleftarrow)$ Let $I=\{0,1\}$ be a hyper $B C K$-ideal of $H,(x \circ y) \circ z \ll I$ and $z \in I$ but $x \circ(y \circ(y \circ x)) \nsubseteq I$, for $x, y, z \in H$. Thus $2 \in x \circ(y \circ(y \circ x))$ and so $x \neq 0$. Because, if $x=0$, then $2 \in 0 \circ(y \circ(y \circ 0))=\{0\}$, which is impossible. Since $z \in I$ and $I$ is a hyper $B C K$-ideal, then by Lemma 4.5 , $x \circ y \subseteq I$. This implies that $2 \notin x \circ y$. If $y \in I$ (i.e. $y=0$ or 1 ), then $x \in I$ and since $x \neq 0$, then $x=1$. Now, if $y=0$, then by hypothesis, $2 \in 1 \circ(0 \circ(0 \circ 1))=1 \circ 0=\{1\}$, which is a contradiction. If $y=1$, since $1 \circ 1=x \circ y \subseteq I$, thus $1 \circ 1=\{0\}$ or $\{0,1\}$ and so $2 \in 1 \circ(1 \circ(1 \circ 1)) \subseteq\{0,1\}$, which is impossible.

Now, let $y=2$. Hence, $x \circ 2=x \circ y \subseteq I$ and $2 \notin x \circ 2$. We consider two cases:

Case 1. Let $H$ satisfies the simple condition. By Lemma 2.6 (a), $x=1$ and $1 \circ 2=\{1\}$ or $x=2$ and $2 \circ 2=\{0\}$. If $x=1$, since by Lemma $2.6(a)$, $2 \circ 1=\{2\}$ and $2 \circ 2=\{0\}$ or $\{0,2\}$, thus

$$
2 \in x \circ(y \circ(y \circ x))=1 \circ(2 \circ(2 \circ 1))=1 \circ(2 \circ 2)=\{1\}
$$

which is a contradiction. If $x=2$, then

$$
2 \in 2 \circ(2 \circ(2 \circ 2))=2 \circ(2 \circ 0)=2 \circ 2=\{0\}
$$

which is impossible.
Case 2. $H$ satisfies the normal condition. If $x=1$, then by Lemma 2.6 (iii) and (iv), for all $t \in H, 2 \notin 1 \circ t$ and so

$$
2 \notin \bigcup_{t \in y \circ(y \circ 1)} 1 \circ t=1 \circ(y \circ(y \circ 1))=x \circ(y \circ(y \circ x)
$$

which contradicts the contrary hypothesis. If $x=2$, since $2 \circ 2=x \circ 2 \subseteq I$, then $2 \circ 2=\{0\}$ or $\{0,1\}$ and so $2 \in 2 \circ(2 \circ(2 \circ 2))=\{0\}$ or $\{0,1\}$, which is a contradiction.

Now, let $I=\{0,2\}$ be a hyper $B C K$-ideal of $H,(x \circ y) \circ z \ll I$ and $z \in I$ but $x \circ(y \circ(y \circ x)) \nsubseteq I$, for $x, y, z \in I$. So, $1 \in x \circ(y \circ(y \circ x))$ and so $x \neq 0$. Since $z \in I$ and $I$ is a hyper $B C K$-ideal of $H$, then by lemma 4.5, $x \circ y \subseteq I$. Thus $1 \notin x \circ y$. If $y \in I$, then $x \in I$ and since $x \neq 0$, thus $x=2$. Now, if $y=0$, then by hypothesis, $1 \in 2 \circ(0 \circ(0 \circ 2))=2 \circ 0=\{2\}$ which is a contradiction. If $y=2$, since $2 \circ 2=x \circ y \subseteq I$, then $2 \circ 2=\{0\}$ or $\{0,2\}$. Hence, $1 \in 2 \circ(2 \circ(2 \circ 2))=\{0\}$ or $\{0,2\}$ which is impossible.

Now let $y=1$. Since $x \circ 1=x \circ y \subseteq I$, then $x \circ 1=\{0\}$ or $\{2\}$ or $\{0,2\}$. We consider the following cases:

Case 1. $H$ satisfies the simple condition. By Lemma 2.6 (a) we have $x=1$ and $1 \circ 1=\{0\}$ or $x=2$ and $2 \circ 1=\{2\}$. If $x=1$, then

$$
1 \in 1 \circ(1 \circ(1 \circ 1))=1 \circ(1 \circ 0)=1 \circ 1=\{0\},
$$

which is a contradiction. If $x=2$, since by Lemma $2.6(a), 1 \circ 1=\{0\}$ or $\{0,1\}$ and $1 \circ 2=\{1\}$, thus

$$
1 \in 2 \circ(1 \circ(1 \circ 2))=2 \circ(1 \circ 1)=\{2\}
$$

which is impossible.
Case 2. H satisfies the normal condition. By Lemma 2.6 (b), we have $x=1$ and $1 \circ 1=\{0\}$ or $x=2$ and $2 \circ 1=\{2\}$. If $x=1$, similar to the preceding case we get a contradiction. If $x=2$, since by Lemma 2.6 (iv), $1 \circ 2=\{0\}$ or $\{0,1\}$, then $1 \in 2 \circ(1 \circ(1 \circ 2))=\{2\}$, which is impossible.
(ii) The proof is similar to the proof $(i)$.
(iii) Let $I$ be a commutative hyper $B C K$-ideal of type $1,(x \circ y) \circ z \ll I$ and $z \in I$ but $x \circ(y \circ(y \circ x)) \nless I$, for $x, y, z \in H$. If $I=\{0,1\}$, thus $2 \in x \circ(y \circ(y \circ x))$ and $2 \nless 1$. Since $(x \circ y) \circ z \ll I$, then $2 \notin(x \circ y) \circ z$ and so $(x \circ y) \circ z=\{0\}$ or $\{1\}$ or $\{0,1\}$. Hence, $(x \circ y) \circ z \subseteq I$. Since $z \in I$ and $I$ is a commutative hyper BCK-ideal of type 1 , then $x \circ(y \circ(y \circ x)) \subseteq I$ and so $x \circ(y \circ(y \circ x)) \ll I$, which is a contradiction.

The proof of the case $I=\{0,2\}$ is similar.
Example 4.7. Let $H=\{0,1,2,3\}$. The following table shows a hyper $B C K$-algebra structure on $H$ :

| $\circ$ | 0 | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: | :---: |
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| 1 | $\{1\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| 2 | $\{2\}$ | $\{2\}$ | $\{0\}$ | $\{0\}$ |
| 3 | $\{3\}$ | $\{3\}$ | $\{3\}$ | $\{0,3\}$ |

Then $I=\{0,1\}$ is a weak hyper $B C K$-ideal and a hyper $B C K$-ideal of $H$, but it is not commutative hyper $B C K$-ideal of type 1 and 3 . Since, $(2 \circ 3) \circ 1=0 \circ 1=\{0\} \subseteq I$ and $1 \in I$ but

$$
2 \circ(3 \circ(3 \circ 2))=2 \circ(3 \circ 3)=2 \circ(3 \circ 3)=2 \circ\{0,3\}=\{0,2\} \nsubseteq I
$$

Hence, $I=\{0,1\}$ is not commutative hyper $B C K$-ideal of type 1 and so is not commutative hyper $B C K$-ideal of type 3 .

Corollary 4.8. Let $H=\{0,1,2\}$ be a hyper BCK-algebra of order 3 and $I$ be a nonempty subset of $H$. Then:
(i) I is a positive implicative hyper BCK-ideal of type 3 if and only if is a commutative hyper BCK-ideal of type 3,
(ii) I is a positive implicative hyper BCK-ideal of type 1 if and only if is a commutative hyper BCK-ideal of type 1 .

Proof. The proof is a consequence of Theorems 3.10 and 4.6.
Theorem 4.9. In any hyper BCK-algebra of order 3, there is at least one commutative hyper BCK-ideal of type 2 and 4 .
Proof. Let $H=\{0,1,2\}$ be hyper $B C K$-algebra of order 3 . We show that $I=\{0,2\}$ is a commutative hyper $B C K$-ideal of type 2 and 4 of $H$. But, by Theorem 4.2 (ii), it is enough to show that $I=\{0,2\}$ is a commutative hyper $B C K$-ideal of type 4 . Let $(x \circ y) \circ z \ll I$ and $z \in I$ but $x \circ(y \circ(y \circ x)) \nless I$, for $x, y, z \in H$. Thus $1 \in x \circ(y \circ(y \circ x))$ and $1 \nless 2$. Moreover, by Theorem $2.2(v), x \neq 0$. Since $z \in I$, thus $z=0$ or $z=2$.

Now we consider two following cases:
Case 1. Let $z=0$. Then $x \circ y=(x \circ y) \circ 0=(x \circ y) \circ z \ll I$. Since $1 \nless 2$, then $1 \notin x \circ y$. Hence $x \circ y=\{0\}$ or $\{2\}$ or $\{0,2\}$.

Case 1.1. Let $x \circ y=\{0\}$. Then by Lemma 2.6, $x=y=1$ or $x=$ $y=2$ or $x=1, y=2$. If $x=y=1$ or $x=y=2$, then by hypothesis $1 \in x \circ(y \circ(y \circ x))=\{0\}$, which is impossible. If $x=1$ and $y=2$, then $1 \circ 2=\{0\}$ and so $1 \ll 2$, which is impossible.

Case 1.2. Let $x \circ y=\{2\}$. Then by Lemma 2.6, $x=2$ and $y=1$. Since $2 \circ 1=\{2\}$, then $2 \nless 1$ and so $H$ satisfies the simple condition. But in this case, $1 \in x \circ(y \circ(y \circ x))=2 \circ(1 \circ(1 \circ 2))=2 \circ(1 \circ 1) \subseteq 2 \circ\{0,1\}=\{2\}$, which is impossible.

Case 1.3. Let $x \circ y=\{0,2\}$. Then by Lemma 2.6, $x=2$ and $y=2$. Hence $2 \circ 2=\{0,2\}$ and so

$$
1 \in x \circ(y \circ(y \circ x))=(2 \circ(2 \circ 2))=2 \circ(2 \circ\{0,2\})=2 \circ\{0,2\}=\{0,2\},
$$

which is impossible.
Case 2. Let $z=2$. Hence $(x \circ y) \circ 2 \ll I$. Since $1 \nless 2$, then by Lemma $2.6,1 \circ 2=\{1\}$ and $1 \notin(x \circ y) \circ 2$.

Case 2.1. Let $y=0$. Then

$$
1 \in x \circ(y \circ(y \circ x))=x \circ(0 \circ(0 \circ x))=x \circ 0=\{x\}
$$

and $1 \notin(x \circ y) \circ 2=(x \circ 0) \circ 2=x \circ 2$. Thus, $x=1$, and so $1 \notin 1 \circ 2=\{1\}$, which is impossible.

Case 2.2. Let $y=1$. Then

$$
1 \in x \circ(y \circ(y \circ x))=x \circ(1 \circ(1 \circ x)) \text { and } 1 \notin(\mathrm{x} \circ \mathrm{y}) \circ 2=(\mathrm{x} \circ 1) \circ 2
$$

If $x=1$, then $1 \in 1 \circ(1 \circ(1 \circ 1))$ and so $1 \circ 1 \neq\{0\}$. Hence, by Theorem $2.6,1 \circ 1=\{0,1\}$. But, in this case, $1 \notin(x \circ 1) \circ 2=\{0,1\} \circ 2=\{0,1\}$, which is impossible.

If $x=2$, then

$$
1 \in 2 \circ(1 \circ(1 \circ 2))=2 \circ(1 \circ 1), \quad 1 \notin(2 \circ 1) \circ 2
$$

By Theorem 2.6, $2 \circ 1=\{1\}$ or $\{2\}$ or $\{1,2\}$. If $2 \circ 1=\{1\}$, then $1 \notin$ $(2 \circ 1) \circ 2=1 \circ 2=\{1\}$, which is impossible. If $2 \circ 1=\{2\}$, then $1 \in$ $2 \circ(1 \circ 1) \subseteq 2 \circ\{0,1\}=\{2\}$, which is impossible. If $2 \circ 1=\{1,2\}$, then $1 \notin(2 \circ 1) \circ 2 \subseteq\{1,2\} \circ 2 \subseteq\{0,1,2\}$, which is impossible.

Case 2.3. Let $y=2$. Then
$1 \in x \circ(y \circ(y \circ x))=x \circ(2 \circ(2 \circ x)), \quad 1 \notin(x \circ y) \circ 2=(x \circ 2) \circ 2$
If $x=1$, then $1 \notin(1 \circ 2) \circ 2=\{1\} \circ 2=\{1\}$, which is impossible. If $x=2$, then $1 \in 2 \circ(2 \circ(2 \circ 2))$ and $1 \notin(2 \circ 2) \circ 2$. If $1 \in 2 \circ 2$, then $\{1\}=1 \circ 2 \subseteq(2 \circ 2) \circ 2$, which is impossible. Since $0 \in 2 \circ 2$, hence $2 \circ 2=\{0\}$ or $\{0,2\}$. If $2 \circ 2=\{0\}$, then $1 \in 2 \circ(2 \circ(2 \circ 2))=\{0\}$, which is impossible. If $2 \circ 2=\{0,2\}$, then $1 \in 2 \circ(2 \circ(2 \circ 2))=\{0,2\}$, which is impossible.

Therefore, $I=\{0,2\}$ is a commutative hyper $B C K$-ideal of type 4 .
Corollary 4.10. Let $H=\{0,1,2\}$ be a hyper BCK-algebra of order 3 and $I$ be a nonempty subset of $H$. Then I is a commutative hyper BCK-ideal of type 2 if and only if I is a commutative hyper BCK-ideal of type 4.
Proof. ( $\Longleftarrow)$ The proof follows by Theorem 4.2 (ii).
$(\Longrightarrow)$ Let $I$ be a commutative hyper $B C K$-ideal of type 2 of $H=$ $\{0,1,2\}$. If $I=\{0,2\}$, then by the proof of Theorem $4.9, I$ is a commutative
hyper $B C K$-ideal of type 4. If $I=\{0,1\}$, then by Theorems 3.1, 3.3, 3.4 and 3.5 of [3], there are only 3 non-isomorphic hyper $B C K$-algebra of order 3 such that $I=\{0,1\}$ is not a hyper $B C K$-ideal of them, which are as follows:

| $\circ_{1}$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| 1 | $\{1\}$ | $\{0\}$ | $\{0\}$ |
| 2 | $\{2\}$ | $\{1\}$ | $\{0\}$ |


| $\circ_{2}$ | 0 | 1 | 2 |
| :--- | :---: | :---: | :---: |
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| 1 | $\{1\}$ | $\{0\}$ | $\{0\}$ |
| 2 | $\{2\}$ | $\{1\}$ | $\{0,1\}$ |


| $\circ_{3}$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| 1 | $\{1\}$ | $\{0,1\}$ | $\{0,1\}$ |
| 2 | $\{2\}$ | $\{1\}$ | $\{0,1\}$ |

Moreover, in the above hyper $B C K$-algebras, $I=\{0,1\}$ is not a commutative hyper $B C K$-ideal of type 2 . Since, in all of them, $(2 \circ 0) \circ 1=2 \circ 1=$ $\{1\} \subseteq\{0,1\}$ and $1 \in\{0,1\}$ but $2 \circ(0 \circ(0 \circ 2))=2 \circ 0=\{2\} \nless\{0,1\}$.

Now, since except of the above 3 hyper $B C K$-algebras, $I=\{0,1\}$ is a hyper $B C K$-ideal of $H$, then by Theorem $4.6(\mathrm{i}), I=\{0,1\}$ is a commutative hyper $B C K$-ideal of type 3 and so by Theorem $4.2(\mathrm{i})$, it is a commutative hyper $B C K$-ideal of type 4 .

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# Quasi p-ideals of quasi BCI-algebras 

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#### Abstract

As a continuation of our previous study of fuzzy subquasigroups and fuzzy ideals of $B C I$-algebras, the notion of a quasi $p$-ideal is introduced. Characterizations of quasi $p$-ideals of the set of all fuzzy points in $B C I$-algebras are obtained. Next, using special chains of reals we determine the number of non-equivalent fuzzy $p$-ideals of some types of $B C I$-algebras (especially $B C I$-algebras which are quasigroups) and give the method of computation of fuzzy $p$-ideals.


## 1. Introduction

The fundamental concept of a fuzzy set, introduced by Zadeh [10] in 1965, provides a natural generalization for treating mathematically the fuzzy phenomena which exist pervasively in our real world and for building new branches of fuzzy mathematics. In the area of fuzzy $B C K / B C I$-algebra, several researches have been carried out since 1991. The connection between some $B C I$-algebras, quasigroups and commutative groups motivated us to study connections between fuzzy ideals of $B C I$-algebras and fuzzy subgroups of the corresponding groups (see for example [2] and [3]).

On the other hand, in [7], Lele et al. used the notion of fuzzy point to study some properties of $B C K$-algebras. Jun and Lele [5] used the notion of fuzzy points for establishing quasi ideal. As a continuation of [5] and our previous study, in this paper, we introduce the notion of quasi $p$-ideal in the set of all fuzzy points of a fixed $B C I$-algebra, and give some characterizations of this ideal.

Next, using special sequences of real numbers, we determine the number of non-equivalent fuzzy $p$-ideals of some types of $B C I$-algebras (especially

2000 Mathematics Subject Classification: 06F35, 03B52
Keywords: quasi $B C I$-algebra, quasi p-ideal
these $B C I$-algebras which are quasigroups) and give the method of computation of such fuzzy $p$-ideals.

## 2. Preliminaries

An algebra $(X, *, 0)$ of type $(2,0)$ is said to be a $B C I$-algebra if for all $x, y, z \in X$ it satisfies:
(1) $((x * y) *(x * z)) *(z * y)=0$,
(2) $(x *(x * y)) * y=0$,
(3) $x * x=0$,
(4) $x * y=0$ and $y * x=0$ imply $x=y$.

A non-empty subset $A$ of a $B C I$-algebra $X$ is called an ideal of $X$ if

- $0 \in A$,
- $\quad x * y \in A$ and $y \in A$ imply $x \in A$.

A non-empty subset $A$ of a $B C I$-algebra $X$ is called a $p$-ideal of $X$ if

- $0 \in A$,
- $\quad(x * z) *(y * z) \in A$ and $y \in A$ imply $x \in A$.

A $p$-ideal is an ideal. The converse is not true [6], but every ideal is a subset of some $p$-ideal (see [11]). In $B C I$-algebras which are quasigroups, i.e. in $B C I$-algebras isotopic to commutative groups (see [1]), these ideals coincide. Such quasigroups are medial and a finite subset of such $B C I$ algebra is an ideal if and only if it is a subgroup of the corresponding group. For infinite ideals it is not true.

A fuzzy set $\mu$ in a $B C I$-algebra $X$ is called a fuzzy ideal of $X$ if for all $x, y \in X$ we have

- $\mu(0) \geqslant \mu(x)$,
- $\mu(x) \geqslant \min \{\mu(x * y), \mu(y)\}$.

A fuzzy set $\mu$ in a $B C I$-algebra $X$ is called a fuzzy $p$-ideal of $X$ if for all $x, y, z \in X$ we have

- $\mu(0) \geqslant \mu(x)$,
- $\mu(x) \geqslant \min \{\mu((x * z) *(y * z)), \mu(y)\}$.

Any fuzzy $p$-ideal is a fuzzy ideal. The converse does not hold in general [6]. But basing on the results obtained in [1] it is not difficult to see that in a $B C I$-quasigroup a fuzzy set $\mu$ is a fuzzy ideal if and only if it is a fuzzy p-ideal.

A fuzzy set $\mu$ in a set $X$ is called a fuzzy point if it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at $x$ is $\alpha \in(0,1]$ we denote this fuzzy point by $x_{\alpha}$, where the point $x$ is called its support.

Let $F P(X)$ denote the set of all fuzzy points in $X$ and define a binary operation $\odot$ on $F P(X)$ by

$$
x_{\alpha} \odot y_{\beta}=(x * y)_{\min \{\alpha, \beta\}}
$$

where $*$ is a binary operation on $X$. If $(X, *)$ is a quasigroup, then $(F P(X), \odot)$ is not a quasigroup in general.

If $(X, *, 0)$ is a $B C I$-algebra, then
$\left(p_{1}\right) \quad\left(\left(x_{\alpha} \odot y_{\beta}\right) \odot\left(x_{\alpha} \odot z_{\gamma}\right)\right) \odot\left(z_{\gamma} \odot y_{\beta}\right)=0_{\min \{\alpha, \beta, \gamma\}}$,
$\left(p_{2}\right) \quad\left(x_{\alpha} \odot\left(x_{\alpha} \odot y_{\beta}\right)\right) \odot y_{\beta}=0_{\min \{\alpha, \beta\}}$,
$\left(p_{3}\right) \quad x_{\alpha} \odot x_{\alpha}=0_{\alpha}$,
for all $x_{\alpha}, y_{\beta}, z_{\gamma} \in F P(X)$. But the following does not hold:
$\left(p_{4}\right) \quad x_{\alpha} \odot y_{\beta}=y_{\beta} \odot x_{\alpha}=0_{\min \{\alpha, \beta\}} \quad$ imply $\quad x_{\alpha}=y_{\beta}$.
Hence we know (see [5]) that $F P(X)$ may not be a $B C I$-algebra, and so we call $F P(X)$ the quasi $B C I$-algebra.

## 3. Quasi p-ideals

For a fuzzy set $\mu$ in a $B C I$-algebra $X$ we define the set $F P(\mu)$ of all fuzzy points in $X$ covered by $\mu$ to be the set

$$
F P(\mu)=\left\{x_{q} \in F P(X) \mid q \leqslant \mu(x), \quad 0<q \leqslant 1\right\} .
$$

Example 3.1. Let $X=\{0, a, b, c, d\}$ be a $B C I$-algebra with the following Cayley table:

| $*$ | 0 | $a$ | $b$ | $c$ | $d$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 0 | 0 | 0 | $d$ | $c$ | $b$ |
| $a$ | $a$ | 0 | $d$ | $c$ | $b$ |
| $b$ | $b$ | $b$ | 0 | $d$ | $c$ |
| $c$ | $c$ | $c$ | $b$ | 0 | $d$ |
| $d$ | $d$ | $d$ | $c$ | $b$ | 0 |

For a fuzzy set $\mu$ in $X$ defined by $\mu(0)=1, \mu(a)=0.6$ and $\mu(b)=$ $\mu(c)=\mu(d)=0.3$, we have

$$
F P(\mu)=\left\{0_{\alpha}, a_{\beta}, b_{\gamma}, c_{\delta}, d_{\sigma} \mid \alpha \in(0,1], \beta \in(0,0.6], \gamma, \delta, \sigma \in(0,0.3]\right\}
$$

Definition 3.2. For a fuzzy set $\mu$ in a $B C I$-algebra $X$, the set $F P(\mu)$ of all fuzzy points in $X$ covered by $\mu$ is called a quasi p-ideal of $F P(X)$ if for all $\delta \in \operatorname{Im}(\mu)$ and $x_{\alpha}, y_{\beta}, z_{\gamma} \in F P(X)$ :
(i) $0_{\delta} \in F P(\mu)$
(ii) $\quad\left(x_{\alpha} \odot z_{\gamma}\right) \odot\left(y_{\beta} \odot z_{\gamma}\right), y_{\beta} \in F P(\mu) \Longrightarrow x_{\min \{\alpha, \beta, \gamma\}} \in F P(\mu)$.

It is not difficult to see that in the above example $F P(\mu)$ is a quasi $p$-ideal of $F P(X)$.

Note that in [5] and [7] Jun and Lele et al. described ideals of $F P(X)$ of the second type which are called quasi ideals.

Definition 3.3. A subset $F P(\mu)$ of $F P(X)$ is called a quasi ideal of $F P(X)$ if $0_{\alpha} \in F P(\mu)$ for all $\alpha \in \operatorname{Im}(\mu)$ and
(iii) $x_{\alpha} \odot y_{\beta}, y_{\beta} \in F P(\mu) \Longrightarrow x_{\min \{\alpha, \beta\}} \in F P(\mu)$
for all $x_{\alpha}, y_{\beta} \in F P(X)$.
Proposition 3.4. Every quasi p-ideal of $F P(X)$ is also a quasi ideal.
Proof. Let $x_{\alpha}, y_{\beta} \in F P(X)$ be such that $x_{\alpha} \odot y_{\beta} \in F P(\mu)$ and $y_{\beta} \in F P(\mu)$. Then $\left(x_{\alpha} \odot y_{\beta}\right) \odot\left(y_{\beta} \odot y_{\beta}\right)=x_{\alpha} \odot y_{\beta} \in F P(\mu)$ and $y_{\beta} \in F P(\mu)$. Since $F P(\mu)$ is a quasi $p$-ideal of $F P(X)$, it follows that $x_{\min \{\alpha, \beta\}} \in F P(\mu)$. Hence $F P(\mu)$ is a quasi ideal of $F P(X)$.

The converse of Proposition 3.4 may not be true as seen in the following example.

Example 3.5. Let $X=\{0, a, b, c, d\}$ be a set with the following Cayley table:

| $*$ | 0 | $a$ | $b$ | $c$ | $d$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | $a$ | 0 | $a$ | 0 | 0 |
| $b$ | $b$ | $b$ | 0 | 0 | 0 |
| $c$ | $c$ | $c$ | $c$ | 0 | 0 |
| $d$ | $d$ | $c$ | $d$ | $a$ | 0 |

Then $(X, *, 0)$ is a BCK-algebra and hence a $B C I$-algebra. Let $\mu$ be a fuzzy set in $X$ defined by

$$
\mu(x)=\left\{\begin{array}{lll}
0.9 & \text { if } & x \in\{0, b\}, \\
0.3 & \text { if } & x \in\{a, c, d\} .
\end{array}\right.
$$

Consider the set

$$
F P(\mu)=\left\{0_{\alpha}, a_{\beta}, b_{\gamma}, c_{\delta}, d_{\sigma} \mid \alpha, \gamma \in(0,0.9], \beta, \delta, \sigma \in(0,0.3]\right\} .
$$

Then $F P(\mu)$ is a quasi ideal of $F P(X)$. Note that
$\left(a_{0.4} \odot c_{0.5}\right) \odot\left(b_{0.7} \odot c_{0.5}\right)=(a * c)_{0.4} \odot(b * c)_{0.5}=0_{0.4} \odot 0_{0.5}=0_{0.4} \in F P(\mu)$
and $b_{0.7} \in F P(\mu)$. But $a_{\min \{0.4,0.5,0.7\}}=a_{0.4} \notin F P(\mu)$. This shows that $F P(\mu)$ is not a quasi $p$-ideal of $F P(X)$.

The converse of Proposition 3.4 is true only in some very limited cases. One of such cases is given in following theorem.

Theorem 3.6. Let $\mu$ be a fuzzy set in a BCI-algebra X. If $F P(\mu)$ is a quasi ideal of $F P(X)$ such that for all $x_{\alpha}, y_{\beta}, z_{\gamma} \in F P(X)$

$$
\left(x_{\alpha} \odot z_{\gamma}\right) \odot\left(y_{\beta} \odot z_{\gamma}\right) \in F P(\mu) \Longrightarrow x_{\alpha} \odot y_{\beta} \in F P(\mu),
$$

then $F P(\mu)$ is a quasi p-ideal of $F P(X)$.
Proof. Let $x_{\alpha}, y_{\beta}, z_{\gamma} \in F P(X)$ be such that $\left(x_{\alpha} \odot z_{\gamma}\right) \odot\left(y_{\beta} \odot z_{\gamma}\right) \in F P(\mu)$ and $y_{\beta} \in F P(\mu)$. Then by hypothesis, we have $x_{\alpha} \odot y_{\beta} \in F P(\mu)$ and $y_{\beta} \in F P(\mu)$, and so $x_{\min \{\alpha, \beta\}} \in F P(\mu)$ since $F P(\mu)$ is a quasi ideal of $F P(X)$. But $\min \{\alpha, \beta, \gamma\} \leqslant \min \{\alpha, \beta\}$ and $x_{\min \{\alpha, \beta\}} \in F P(\mu)$ imply (according to the definition of $F P(\mu)$ ) that $x_{\min \{\alpha, \beta, \gamma\}} \in F P(\mu)$. Hence $F P(\mu)$ is a quasi $p$-ideal of $F P(X)$.

Now we describe the connection between fuzzy $p$-ideals of a $B C I$-algebra $X$ and quasi $p$-ideals of $F P(X)$.

Theorem 3.7. If $\mu$ is a fuzzy p-ideal of a BCI-algebra $X$, then $F P(\mu)$ is a quasi $p$-ideal of $F P(X)$.

Proof. Since $\mu(0) \geqslant \mu(x)$ for all $x \in X$, we have $\mu(0) \geqslant \alpha$ for all $\alpha \in \operatorname{Im}(\mu)$. Hence $0_{\alpha} \in F P(\mu)$.

Let $x_{\alpha}, y_{\beta}, z_{\gamma} \in F P(X)$ be such that $\left(x_{\alpha} \odot z_{\gamma}\right) \odot\left(y_{\beta} \odot z_{\gamma}\right) \in F P(\mu)$ and $y_{\beta} \in F P(\mu)$. Then $\mu((x * z) *(y * z)) \geqslant \min \{\alpha, \beta, \gamma\}$ and $\mu(y) \geqslant \beta$. Since $\mu$ is a fuzzy $p$-ideal of $X$, it follows that

$$
\begin{aligned}
\mu(x) & \geqslant \min \{\mu((x * z) *(y * z)), \mu(y)\} \\
& \geqslant \min \{\min \{\alpha, \beta, \gamma\}, \beta\}=\min \{\alpha, \beta, \gamma\}
\end{aligned}
$$

so that $x_{\min \{\alpha, \beta, \gamma\}} \in F P(\mu)$. This completes the proof.

We now consider the converse of Theorem 3.7.
Theorem 3.8. Let $\mu$ be a fuzzy set in a BCI-algebra $X$ such that $F P(\mu)$ is a quasi p-ideal of $F P(X)$. Then $\mu$ is a fuzzy p-ideal of $X$.

Proof. Obviously $\mu(0) \geqslant \mu(x)$ for all $x \in X$. Let $x, y, z \in X$ be such that $\mu((x * z) *(y * z))=\alpha$ and $\mu(y)=\beta$. Then $y_{\beta} \in F P(\mu)$ and

$$
\left(x_{\alpha} \odot z_{\alpha}\right) \odot\left(y_{\beta} \odot z_{\alpha}\right)=((x * z) *(y * z))_{\min \{\alpha, \beta\}} \in F P(\mu) .
$$

Since $F P(\mu)$ is a quasi $p$-ideal, it follows that $x_{\min \{\alpha, \beta\}} \in F P(\mu)$ so that

$$
\mu(x) \geqslant \min \{\alpha, \beta\}=\min \{\mu((x * z) *(y * z)), \mu(y)\} .
$$

Therefore $\mu$ is a fuzzy $p$-ideal of $X$.
Lemma 3.9. [6] $A$ fuzzy set $\mu$ in a BCI-algebra $X$ is a fuzzy $p$-ideal of $X$ if and only if the level set $L(\mu ; \alpha)=\{x \in X \mid \mu(x) \geqslant \alpha\}$ is a p-ideal of $X$ when it is non-empty.

Combining Lemma 3.9 and Theorems 3.7 and 3.8, we have
Theorem 3.10. Let $\mu$ be a fuzzy set in a BCI-algebra $X$. Then the following statements are equivalent.
(i) $\mu$ is a fuzzy $p$-ideal of $X$,
(ii) $F P(\mu)$ is a quasi p-ideal of $F P(X)$,
(iii) $L(\mu ; \alpha)$ is a $p$-ideal of $X$ for every $\alpha \in \operatorname{Im}(\mu)$.

## 4. Fuzzy $p$-ideals with a finite set of values

Results of this section are motivated by the corresponding results obtained for fuzzy subgroups and different types of fuzzy ideals of algebras connected with logic (cf. for example [2], [4] and [6]).

In the sequel we will consider only fuzzy sets with a finite image, i.e. fuzzy sets for which $2 \leqslant|\operatorname{Im}(\mu)|<\infty$. Similarly as in the group theory, we assume that the empty set $\emptyset$ is a subalgebra (a subgroup, respectively). Moreover, we assume also that every fuzzy set takes value 1 on the empty set. Thus a fuzzy point $x_{\alpha}$ can be defined as a fuzzy set $x_{\alpha}$ on $X$ such that

$$
x_{\alpha}(z)=\left\{\begin{array}{ccc}
1 & \text { for } & z \in \emptyset \\
\alpha & \text { for } & z=x \\
0 & \text { for } & z \neq x
\end{array}\right.
$$

We start with the following.

Proposition 4.1. Let $\left\{X_{\omega}: \omega \in \Omega\right\}$, where $\emptyset \neq \Omega \subseteq[0,1]$, be a collection of p-ideals of a BCI-algebra $X$ such that
(i) $X=\bigcup_{\omega \in \Omega} X_{\omega}$,
(ii) $\alpha>\beta \Longleftrightarrow X_{\alpha} \subset X_{\beta} \quad \forall \alpha, \beta \in \Omega$.

Then a fuzzy set $\mu$ in $X$ defined by

$$
\mu(x)=\sup \left\{\omega \in \Omega: x \in X_{\omega}\right\}
$$

is a fuzzy p-ideal of $X$.
Proof. In view of Lemma 3.9, it is sufficient to show that every nonempty level set $L(\mu ; \alpha)$ is a $p$-ideal of $X$. Assume $L(\mu ; \alpha) \neq \emptyset$ for some $\alpha \in[0,1]$. Then

$$
\alpha=\sup \{\beta \in \Omega: \beta<\alpha\}=\sup \left\{\beta \in \Omega: X_{\alpha} \subset X_{\beta}\right\}
$$

or

$$
\alpha \neq \sup \{\beta \in \Omega: \beta<\alpha\}=\sup \left\{\beta \in \Omega: X_{\alpha} \subset X_{\beta}\right\} .
$$

In the first case we have $L(\mu ; \alpha)=\bigcap_{\beta<\alpha} X_{\beta}$, because

$$
x \in L(\mu ; \alpha) \Longleftrightarrow x \in X_{\beta} \text { for all } \beta<\alpha \Longleftrightarrow x \in \bigcap_{\beta<\alpha} X_{\beta}
$$

In the second case, there exists $\varepsilon>0$ such that $(\alpha-\varepsilon, \alpha) \cap \Omega=\emptyset$. We prove that in this case $L(\mu ; \alpha)=\bigcup_{\beta \geqslant \alpha} X_{\beta}$. Indeed, if $x \in \bigcup_{\beta \geqslant \alpha} X_{\beta}$, then $x \in X_{\beta}$ for some $\beta \geqslant \alpha$, which gives $\mu(x) \geqslant \beta \geqslant \alpha$. Thus $x \in L(\mu ; \alpha)$, i.e. $\bigcup_{\beta \geqslant \alpha} X_{\beta} \subseteq L(\mu ; \alpha)$. Conversely, if $x \notin \bigcup_{\beta \geqslant \alpha} X_{\beta}$, then $x \notin X_{\beta}$ for all $\beta \geqslant \alpha$, which implies that $x \notin X_{\beta}$ for all $\beta>\alpha-\varepsilon$, i.e. if $x \in X_{\beta}$ then $\beta \leqslant \alpha-\varepsilon$. Thus $\mu(x) \leqslant \alpha-\varepsilon$. Therefore $x \notin L(\mu ; \alpha)$. Hence $L(\mu ; \alpha) \subseteq \bigcup_{\beta \geqslant \alpha} X_{\beta}$, and in the consequence $L(\mu ; \alpha)=\bigcup_{\beta \geqslant \alpha} X_{\beta}$. This completes our proof because $\bigcup_{\beta \geqslant \alpha} X_{\beta}$ and $\bigcap_{\beta<\alpha} X_{\beta}$ are $p$-ideals.

Proposition 4.2. Let $\mu$ be a fuzzy set in $X$ and let $\operatorname{Im}(\mu)=\left\{\lambda_{0}, \lambda_{1}, \ldots, \lambda_{n}\right\}$, where $\lambda_{0}>\lambda_{1}>\ldots>\lambda_{n}$. If $X_{0} \subset X_{1} \subset \ldots \subset X_{n}=X$ are $p$-ideals of $X$ such that $\mu\left(X_{k} \backslash X_{k-1}\right)=\lambda_{k}$ for $k=0,1, \ldots, n$, where $X_{-1}=\emptyset$, then $\mu$ is a fuzzy $p$-ideal in $X$.

Proof. Since $X_{0}$ is a $p$-ideal, then $0 \in X_{0}$ and $\mu(0)=\mu\left(X_{0} \backslash X_{-1}\right)=\lambda_{0}$, which gives $\mu(0) \geqslant \mu(x)$ for all $x \in X$.

To prove that $\mu$ satisfies the second condition of the definition of fuzzy $p$-ideals we consider the following four cases:

$$
\begin{array}{lll}
1^{o} & (x * z) *(y * z) \in X_{k} \backslash X_{k-1}, & y \in X_{k} \backslash X_{k-1}, \\
2^{o} & (x * z) *(y * z) \in X_{k} \backslash X_{k-1}, & y \notin X_{k} \backslash X_{k-1}, \\
3^{o} & (x * z) *(y * z) \notin X_{k} \backslash X_{k-1}, & y \in X_{k} \backslash X_{k-1}, \\
4^{o} & (x * z) *(y * z) \notin X_{k} \backslash X_{k-1}, & y \notin X_{k} \backslash X_{k-1} .
\end{array}
$$

In the first case $x \in X_{k}$, because $X_{k}$ is a $p$-ideal. Thus

$$
\mu(x) \geqslant \lambda_{k}=\mu((x * z) *(y * z))=\mu(y)=\min \{\mu((x * z) *(y * z)), \mu(y)\} .
$$

In the second case $y \in X_{k-1} \subset X_{k}$ or $y \in X_{m} \backslash X_{m-1} \subset X_{m} \backslash X_{k}$ for some $m>k$. This together with $(x * z) *(y * z) \in X_{k} \backslash X_{k-1}$ implies $x \in X_{k}$ or $x \in X_{m} \backslash X_{k}$. Thus

$$
\mu(x) \geqslant \lambda_{k}=\mu((x * z) *(y * z))=\min \{\mu((x * z) *(y * z)), \mu(y)\}
$$

for $x \in X_{k}, y \in X_{k-1}$. Similarly

$$
\mu(x) \geqslant \lambda_{m}=\mu(y)=\min \{\mu((x * z) *(y * z)), \mu(y)\}
$$

for $y \in X_{m} \backslash X_{m-1}, x \in X_{m} \backslash X_{k}$.
In the last two cases the process of verification is analogous.
Corollary 4.3. Let $\mu$ be a fuzzy set in $X$ and let $\operatorname{Im}(\mu)=\left\{\lambda_{0}, \lambda_{1}, \ldots, \lambda_{n}\right\}$, where $\lambda_{0}>\lambda_{1}>\ldots>\lambda_{n}$. If $X_{0} \subset X_{1} \subset \ldots \subset X_{n}=X$ are $p$-ideals of $X$ such that $\mu\left(X_{k}\right) \geqslant \lambda_{k}$ for $k=0,1, \ldots, n$, then $\mu$ is a fuzzy $p$-ideal in $X$.
Corollary 4.4. If $\operatorname{Im}(\mu)=\left\{\lambda_{0}, \lambda_{1}, \ldots, \lambda_{n}\right\}$, where $\lambda_{0}>\lambda_{1}>\ldots>\lambda_{n}$, is the the set of values of a fuzzy p-ideal $\mu$ in $X$, then all $L\left(\mu ; \lambda_{k}\right)$ are $p$-ideals of $X$ such that $\mu\left(L\left(\mu ; \lambda_{0}\right)\right)=\lambda_{0}$ and $\mu\left(L\left(\mu ; \lambda_{k}\right) \backslash L\left(\mu ; \lambda_{k-1}\right)\right)=\lambda_{k}$ for $k=1,2, \ldots, n$.
Proposition 4.5. If a fuzzy p-ideal $\mu$ in a BCI-algebra $X$ has the finite set of values, then every descending chain of p-ideals of $X$ terminates at finite step.

Proof. Suppose there exists a strictly descending chain $X_{1} \supset X_{2} \supset X_{3} \supset \ldots$ of $p$-ideals of a $B C I$-algebra $X$ which does not terminate at finite step. We prove that $\mu$ defined by

$$
\mu(x)=\left\{\begin{array}{cll}
\frac{n}{n+1} & \text { for } & x \in X_{n} \backslash X_{n+1}, n=1,2, \ldots \\
1 & \text { for } & x \in \bigcap X_{n}, n=1,2, \ldots
\end{array}\right.
$$

where $X_{1}=X$, is a fuzzy $p$-ideal with an infinite number of values.
Clearly $\mu(0) \geqslant \mu(x)$ for all $x \in X$. Let $x, y, z \in X$. Assume that $(x * z) *(y * z) \in X_{n} \backslash X_{n+1}$ and $y \in X_{k} \backslash X_{k+1}$ for some $k$ and some $n$. (Without loss of generality, we can assume $n \leqslant k$.) Then $y \in X_{n}$, and in the consequence, $x \in X_{n}$ because $X_{n}$ is a $p$-ideal. Hence

$$
\mu(x) \geqslant \frac{n}{n+1}=\min \{\mu((x * z) *(y * z)), \mu(y)\} .
$$

If $(x * z) *(y * z)$ and $y$ are in $\bigcap X_{n}$, then $x \in \bigcap X_{n}$. Thus

$$
\mu(x)=1=\min \{\mu((x * z) *(y * z)), \mu(y)\}
$$

If $(x * z) *(y * z) \notin \bigcap X_{n}$ and $y \in \bigcap X_{n}$, then $(x * z) *(y * z) \in X_{k} \backslash X_{k+1}$ for some $k$. Hence $x \in X_{k}$ and

$$
\mu(x) \geqslant \frac{k}{k+1}=\min \{\mu((x * z) *(y * z)), \mu(y)\} .
$$

If $(x * z) *(y * z) \in \bigcap X_{n}$ and $y \notin \bigcap X_{n}$, then $y \in Y_{t} \backslash X_{t+1}$ for some $t$, which implies $x \in X_{t}$ and

$$
\mu(x) \geqslant \frac{t}{t+1}=\min \{\mu((x * z) *(y * z)), \mu(y)\}
$$

This proves that $\mu$ is a fuzzy $p$-ideal. Obviously $\mu$ has an infinite number of different values. Obtained contradiction completes our proof.

For finite $B C I$-algebras the following proposition is true (cf. [6]).
Proposition 4.6. Let $\mu$ and $\nu$ be a fuzzy p-ideals of a finite BCI-algebra $X$ such that the families of level $p$-ideals of $\mu$ and $\nu$ are identical. Then $\mu=\nu$ if and only if $\operatorname{Im}(\mu)=\operatorname{Im}(\nu)$.

## 5. Equivalences of fuzzy $p$-ideals

Results of this section are motivated by the corresponding results obtained for fuzzy subgroups [9] and by the connection of some BCI-algebras [1] with groups.

In the set $F(X)$ of all fuzzy sets on $X$ we can introduce (sf. [9]) the equivalence relation based on the heuristic principle that the distinction or similarity of fuzzy sets is really based on the relative membership degrees of elements with respect to each other rather than the absolute membership
degree of each element to the fuzzy set under consideration. Thus two fuzzy sets are similar if they maintain the same relative degrees of membership with respect to two elements. This gives the motivation to the following relation [9]:

$$
\mu \sim \nu \Longleftrightarrow\left\{\begin{array}{ccc}
\mu(x)>\mu(y) & \Leftrightarrow \quad \nu(x)>\nu(y) \\
\mu(x)=1 & \Leftrightarrow \quad \nu(x)=1 \\
\mu(x)=0 & \Leftrightarrow \quad \nu(x)=0
\end{array}\right.
$$

for all $x, y \in X$.
It is not difficult to see that this relation is an equivalence relation on $F(X)$ and coincides with the equality of subsets in $2^{X}$.

The condition $\mu(x)=0$ if and only if $\nu(x)=0$ says that the supports of $\mu$ and $\nu$ are equal. This condition cannot be redundant since it is an essential part of the equivalence relation as seen in the example below.

If in the above definition we replace the strict inequality by $\geqslant$ we obtain the new equivalence relation which has the same equivalence classes as the above equivalence.

Example 5.1. Let $K=\{1,-1, i,-i\}$ be a group. Then $(K, \cdot, 1)$ is a $B C I-$ algebra (a $B C I$-quasigroup in fact) with $0=1$. Define two fuzzy sets $\mu$ and $\nu$ putting
$\mu(x)=\left\{\begin{array}{cll}1 & \text { for } & x=1 \\ 0.5 & \text { for } & x=-1 \\ 0.3 & \text { for } & x \in\{i,-i\}\end{array} \quad\right.$ and $\quad \nu(x)=\left\{\begin{array}{cll}1 & \text { for } & x=1 \\ 0.5 & \text { for } & x=-1 \\ 0 & \text { for } & x \in\{i,-i\}\end{array}\right.$
Then these fuzzy sets are fuzzy $p$-ideals satisfying only two first condition of the above definition. Hence $\mu$ and $\nu$ are not equivalent.

Proposition 5.2. If $\mu$ and $\nu$ are equivalent fuzzy p-ideals (fuzzy ideals), then $|\operatorname{Im}(\mu)|=|\operatorname{Im}(\nu)|$.
Proof. The proof is analogous to the proof of Proposition 2.2 in [9].
Note that the converse of Proposition 5.2 is not true.
Example 5.3. Let $X=\{0, a, b, c\}$ be a $B C I$-algebra with the following Cayley table:

| $*$ | 0 | $a$ | $b$ | $c$ |
| :--- | :--- | :--- | :--- | :--- |
| 0 | 0 | 0 | $b$ | $b$ |
| $a$ | $a$ | 0 | $c$ | $b$ |
| $b$ | $b$ | $b$ | 0 | 0 |
| $c$ | $c$ | $b$ | $a$ | 0 |

Define fuzzy sets $\mu$ and $\nu$ in $X$ as follows:

$$
\mu(x)=\left\{\begin{array}{cll}
1 & \text { for } & x=0 \\
0.5 & \text { for } & x=a \\
0.3 & \text { for } & x \in\{b, c\}
\end{array} \quad \nu(x)=\left\{\begin{array}{cll}
1 & \text { for } & x=0 \\
0.5 & \text { for } & x=b \\
0.3 & \text { for } & x \in\{a, c\}
\end{array}\right.\right.
$$

Then these two fuzzy sets are fuzzy ideals with the same supports and the same images. But $\mu$ and $\nu$ are not equivalent because $\mu(a)>\mu(b)$, but $\nu(a) \ngtr \nu(b)$.

Between level $p$-ideals of equivalent fuzzy $p$-ideals there is a one-to-one correspondence. Namely, the following theorem is valid.

Theorem 5.4. Two fuzzy p-ideals $\mu$ and $\nu$ of a BCI-algebra $X$ are equivalent if and only if for each $\alpha>0$ there exists $\beta>0$ such that $L(\mu ; \alpha)=$ $L(\nu ; \beta)$.
Proof. Let $\mu$ and $\nu$ be equivalent. If $\mu(x)=1$ for all $x$, then also $\nu(x)=1$ for all $x$. In this case we put $\beta=\alpha$. Analogously when $\mu(x)=0$ for all $x \in X$. Now, if $|\operatorname{Im}(\mu)| \geqslant 2$, then, according to the Proposition 4.2, fuzzy $p$-ideals $\mu$ and $\nu$ have the same number of values. Thus $\operatorname{Im}(\mu)=\left\{\alpha_{1}, \ldots, \alpha_{n}\right\}$ and $\operatorname{Im}(\nu)=\left\{\beta_{1}, \ldots, \beta_{n}\right\}$ for some $\alpha_{i}<\alpha_{i+1}$ and $\beta_{i}<\beta_{i+1}$. Hence $L\left(\mu ; \alpha_{i}\right) \supsetneq L\left(\mu ; \alpha_{i+1}\right)$ and $L\left(\nu ; \beta_{i}\right) \supsetneq L\left(\nu ; \beta_{i+1}\right)$. This together with the condition $\mu(x)>\mu(y) \Leftrightarrow \nu(x)>\nu(y)$ gives $L\left(\mu ; \alpha_{i}\right)=L\left(\nu ; \beta_{i}\right)$.

Conversely, since by the assumption $|\operatorname{Im}(\mu)| \geqslant 2$, there exists $x \in X$ such that $\mu(x)>0$. Thus $x \in L(\mu ; \alpha)$ for some $\alpha>0$. But by hypothesis there is $\beta>0$ such that $L(\mu ; \alpha)=L(\nu ; \beta)$. Hence $\nu(x) \geqslant \beta>0$. Similarly we can show that $\nu(x)>0$ implies $\mu(x)>0$. Therefore $\mu(x)=0$ if and only if $\nu(x)=0$.

Now let $\alpha=\mu(x)>\mu(y)$ for some $x, y \in X$. In this case, by hypothesis $x \in L(\mu ; \alpha)=L(\nu ; \beta)$. If $\nu(x) \leqslant \nu(y)$, then obviously $\nu(y) \geqslant \beta$ and $y \in L(\nu ; \beta)=L(\mu ; \alpha)$, which is impossible. Thus $\nu(x)>\nu(y)$. Similarly $\nu(x)>\nu(y)$ implies $\mu(x)>\mu(y)$.

If $\mu(x)=1$, then also $\mu(0)=1$, by the definition of fuzzy $p$-ideals, and, in the consequence $0, x \in L(\mu ; 1)=L(\nu ; \beta)$ for some $\beta>0$. Hence $\nu(0)=\nu(x)$ for all $x \in L(\nu ; \beta)=L(\mu ; \alpha)$ because $\nu(0)>\nu(x)$ implies $1=\mu(0)>\mu(x)$. But for $\nu(0)<1=\nu(\emptyset)$ we have also $\mu(0)<\mu(\emptyset)=1$, which is a contradiction. Therefore $\beta=1$. Hence $\mu(x)=1$ if and only if $\nu(x)=1$. This completes the proof.

Now let

$$
\emptyset \subset X_{1} \subset X_{2} \subset \ldots \subset X_{n}=X
$$

be a maximal chain of $p$-ideals of a $B C I$-algebra $X$. Putting $\mu(\emptyset)=1$ and $\mu\left(X_{k} \backslash X_{k-1}\right)=\lambda_{k}$ for all $k=1, \ldots, n$, where

$$
1 \geqslant \lambda_{1} \geqslant \lambda_{2} \geqslant \ldots \geqslant \lambda_{n} \geqslant 0
$$

we can obtain a fuzzy $p$-ideal $\mu$ on $X$. Such fuzzy $p$-ideal can be identified with the sequence

$$
\lambda_{1} \lambda_{2} \ldots \lambda_{n}
$$

It is clear that non-equivalent fuzzy $p$-ideals have distinct sequences.
Example 5.5. Let $(X, *, 0)$ be a $B C I$-algebra induced by $\mathbb{Z}_{5}$, i.e. let $X=$ $\mathbb{Z}_{5}$ and $x * y=(x+4 y)(\bmod 5)$. Then $(X, *, 0)$ is a group-like $B C I$-algebra ( $B C I$-quasigroup) in which all $p$-ideals are subgroups of $\mathbb{Z}_{5}$ (cf. [1]). Thus a maximal chain of $p$-ideals of $X$ has the form $\emptyset \subset X_{1} \subset X_{2}$, where $X_{1}=\{0\}$ and $X_{2}=\mathbb{Z}_{5}$ and corresponds to the sequence $\lambda_{1} \lambda_{2}$.

Using Theorem 5.4 it is not difficult to see that any fuzzy $p$-ideal of $X$ corresponds to a fuzzy $p$-ideal determined by one of the following three sequences: $11,1 \lambda, 10$, where $1>\lambda>0$. The first sequence determines a fuzzy $p$-ideal $\mu_{1}$ such that $\mu_{1}(x)=1$ for all $x \in X$. The second corresponds to $\mu_{2}$ such that $\mu_{2}(0)=1$ and $\mu_{2}(x)=\lambda$ for all $x \neq 0$. The sequence 10 represents $\mu_{3}$ such that $\mu_{3}(0)=1$ and $\mu_{3}(x)=0$ for all $x \neq 0$. (Fuzzy $p$-ideals $\mu_{2}$ and $\mu_{3}$ are non-equivalent because they have different supports.)

Note that the number of fuzzy $p$-ideals of this $B C I$-algebra is $1+2=$ $2^{2}-1$, i.e., one fuzzy $p$-ideal whose support is $X_{1}$ and two whose support is $X_{2}$.

Example 5.6. Now let $(X, *, 0)$ be a $B C I$-algebra induced by $\mathbb{Z}_{4}$. Then $x * y=(x+3 y)(\bmod 4)$ and $\emptyset \subset X_{1} \subset X_{2} \subset X_{3}$, where $X_{1}=\{0\}$, $X_{2}=\{0,2\} \simeq \mathbb{Z}_{2}, X_{3}=\mathbb{Z}_{4}$, is a maximal chain of $p$-ideals of $X$. This chain corresponds to the sequence $\lambda_{1} \lambda_{2} \lambda_{3}$.

Similarly as in the previous case, it is not difficult to see that all nonequivalent fuzzy $p$-ideals of $X$ correspond to one of the following sequences: $111,11 \lambda_{1}, 110,1 \lambda_{1} \lambda_{1}, 1 \lambda_{1} \lambda_{2}, 1 \lambda_{1} 0,100$, where $1>\lambda_{1}>\lambda_{2}>0$.

The sequence $1 \alpha \beta$ represents a fuzzy $p$-ideal

$$
\mu(x)=\left\{\begin{array}{lll}
1 & \text { for } & x \in X_{1} \\
\alpha & \text { for } & x \in X_{2} \backslash X_{1} \\
\beta & \text { for } & x \in X_{3} \backslash X_{2}
\end{array}\right.
$$

In this case the number of fuzzy $p$-ideals is $1+2+2^{2}=2^{3}-1$, i.e. one fuzzy $p$-ideal whose support is $X_{1}, 2$ whose support is $X_{2}$, and $2^{2}$ whose support is $X_{3}$.

Basing on the above two examples we can formulate the following theorem, which can be proved by induction.

Theorem 5.7. A chain $X_{1} \subset X_{2} \subset \ldots \subset X_{n}=X$ of $p$-ideals of a BCIalgebra $X$ induces $\sum_{k=0}^{n-1} 2^{k}=2^{n}-1$ non-equivalent fuzzy $p$-ideals of $X$.

Corollary 5.8. A BCI-algebra $X$ in which all its p-ideals can be ordered in the chain $X_{1} \subset X_{2} \subset \ldots \subset X_{n}=X$ has exactly $2^{n}-1$ non-equivalent fuzzy p-ideals.

## 6. Fuzzy $p$-ideals of group-like $B C I$-algebras

Group-like $B C I$-algebras are described in [1]. Such $B C I$-algebras are quasigroups induced by commutative groups, i.e. for every group-like $B C I$-algebra $(X, *, 0)$ there exists a commutative group $(X,+, 0)$ such that $x * y=$ $x-y$ holds for all $x, y \in X$. The maximal chain of $p$-ideals of BCI-algebra $X$ induced by a cyclic $p$-group $\mathbb{Z}_{p^{n}}$ coincides with the maximal chain of subgroups of $\mathbb{Z}_{p^{n}}$ and has the form $\{0\} \subset X_{1} \subset \ldots \subset X_{n}$, where $X_{k}=\mathbb{Z}_{p^{k}}$. Thus, as a consequence of our Theorem 5.7 or Proposition 3.3 from [9], we obtain

Corollary 6.1. A BCI-algebra induced by a cyclic p-group $\mathbb{Z}_{p^{n}}$ has exactly $2^{n+1}-1$ non-equivalent fuzzy $p$-ideals.

Similarly, as a consequence of our Theorem 5.7 and results obtained in [9] (Theorem 3.4 and Proposition 3.6), we obtain

Corollary 6.2. A BCI-algebra induced by the group $\mathbb{Z}_{p^{n}} \times \mathbb{Z}_{q}$, where $p \neq q$ are primes, has exactly $2^{n+1}(n+2)-1$ non-equivalent fuzzy $p$-ideals.

Corollary 6.3. A BCI-algebra induced by the group $\mathbb{Z}_{q} \times \mathbb{Z}_{q}$, where $q$ is a prime, has exactly $4 q+7$ non-equivalent fuzzy p-ideals.

Thus, for example, $B C I$-algebras induced by $\mathbb{Z}_{p}$, where $p$ is a prime, have only 3 non-equivalent fuzzy $p$-ideals. All these fuzzy $p$-ideals are described in Example 5.5. $B C I$-algebras induced by $\mathbb{Z}_{p^{2}}$ have 7 non-equivalent fuzzy $p$-ideals (see Example 5.6), but BCI-algebras induced by $\mathbb{Z}_{12}, \mathbb{Z}_{18}$ and $\mathbb{Z}_{20}$ have 31 such fuzzy $p$-ideals.

Acknowledgements. The second author was supported by Korea Research Foundation Grant (KRF-2003-005-C00013).

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# On groupoids with identity $x(x y)=y$ 

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#### Abstract

The groupoid identity $x(x y)=y$ appears in defining several classes of groupoids, such as Steiner's loops which are closely related to Steiner's triple systems, the class of cancellative groupoids with property $(2,5)$, Boolean groups, and groupoids which exhibit orthogonality of quasigroups. Its dual identity is one of the defining identities for the variety of quasigroups corresponding to strongly 2-perfect m-cycle systems. In this paper we consider the following varieties of groupoids: $\mathcal{V}=\operatorname{Var}(x(x y)=y), \mathcal{V}_{c}=\operatorname{Var}(x(x y)=$ $y, x y=y x), \mathcal{V}_{u}=\operatorname{Var}(x(x y)=y,(x y) y=x y), \mathcal{V}_{i}=\operatorname{Var}(x(x y)=y,(x y) y=y x)$. Suitable canonical constructions of free objects in each of these varieties are given and several other structural properties are presented. Some problems of enumeration of groupoids are also resolved. It is shown that each $\mathcal{V}_{i}$-groupoid defines a Steiner quintuple system and vice versa, implying existence of Steiner quintuple systems of enough large finite cardinality.


## 1. Preliminaries

A groupoid is a pair $(G, \cdot)$ consisting of a nonempty set $G$ and a binary operation • on G. Some well known classes of groupoids are semigroups Sem i.e. groupoids satisfying the identity $x(y z)=(x y) z$, commutative groupoids Com with the identity $x y=y x$, groupoids with unit $U n$ (satisfying the law $(\exists x)(\forall y) y x=x y=y)$, etc. We note that some of these classes are defined by identities, i.e. they are varieties of groupoids. The class $U n$ is not a variety, but it is functionally equivalent ([10]) to the variety of groupoids determined by the identities $x e=e x=x$, where $e$ is a nullary operation. For that reason we will think of $U n$ as being a variety.

2000 Mathematics Subject Classification: 20N02, 20N05, 08B20, 51E10
Keywords: design, groupoid, identity, free object, quasigroup, Steiner system, variety

In this paper we are mainly interested in varieties of groupoids satisfying the identity $x(x y)=y$ and we consider the following varieties:

$$
\begin{aligned}
& \mathcal{V}=\operatorname{Var}(x(x y)=y), \\
& \mathcal{V}_{e}=\mathcal{V} \cap U n(\text { with extended signature }), \\
& \mathcal{V}_{c}=\mathcal{V} \cap \operatorname{Com}, \\
& \mathcal{V}_{u}=\operatorname{Var}(x(x y)=y,(x y) y=x y), \\
& \mathcal{V}_{i}=\operatorname{Var}(x(x y)=y,(x y) y=y x) .
\end{aligned}
$$

Suitable constructions of free objects in each of these varieties and several other structural properties and properties of freeness are presented in next sections.

The variety

$$
\mathcal{V}_{c s}=\operatorname{Var}(x(x y)=y, x y=y x, x(y z)=(x y) z)
$$

is the variety of Boolean groups (i.e. elementary 2-Abelian groups). Several results on this variety as well as the variety

$$
\mathcal{V}_{\text {sem }}=\operatorname{Var}(x(x y)=y, x(y z)=(x y) z)
$$

are presented in [8].
In the sequel $B \neq \emptyset$ will be an arbitrary set and $T_{B}$ will denote the set of all groupoid terms over $B$ in signature $\cdot T_{B}$ is the absolutely free groupoid with (free) base $B$ where the operation is defined by $(u, v) \mapsto u v$. Length $|u|$ of an element $u \in T_{B}$ is defined inductively by:

$$
u \in B \Longrightarrow|u|=1, \quad u=x y \Longrightarrow|u|=|x|+|y| .
$$

Let $\mathcal{B}\left(T_{B}\right)$ be the boolean of $T_{B}$, i.e. the set of all subsets of $T_{B}$. We define inductively a mapping $P: T_{B} \rightarrow \mathcal{B}\left(T_{B}\right)$ by:

$$
t \in B \Longrightarrow P(t)=\{t\}, \quad t=t_{1} t_{2} \Longrightarrow P(t)=\{t\} \cup P\left(t_{1}\right) \cup P\left(t_{2}\right)
$$

For instance, $P((x y)(x z))=\{x, y, z, x y, x z,(x y)(x z)\}$ for $x, y, z \in B$.
The cardinal number of a base of a free groupoid $F$ is said to be the rank of $F$.

## 2. Variety $\mathcal{V}$

Free objects in $\mathcal{V}$ are defined in [4]. Here we state another description.
Let $F=\left\{t \in T_{B} \mid\left(\forall u, v \in T_{B}\right) u \cdot u v \notin P(t)\right\}$. Then for all $u, v \in F$ we have $\left.u v \notin F \Leftrightarrow\left(\exists w \in T_{B}\right) v=u w\right)$. Define an operation $*$ on $F$ by

$$
u * v= \begin{cases}u v & u v \in F \\ w & v=u w \text { for some } w \in F\end{cases}
$$

for each $u, v \in F$.

The product $u * v$ is well defined since $v=u w_{1}=u w_{2}$ implies $w_{1}=w_{2}$ in the absolutely free groupoid $T_{B}$.

Theorem 1. $(F, *)$ is a free groupoid in the variety $\mathcal{V}$ with free base $B$.
Theorem 2. Every subgroupoid $(G, *)$ of $(F, *)$ is free as well.
Proof. We show that the set $R=(B \cap G) \cup\{u v \in G \mid\{u, v\} \nsubseteq G\}$ is a free base of $(G, *)$.

First, by induction on length of terms we show that $R$ is nonempty and generating for $G$. Let $t \in G$ such that $|t|=\min \{|s| \mid s \in G\}$. If $t \in B$, then $t \in R$. If $t=u v$, then $|u|<|t|,|v|<|t|$, so $\{u, v\} \nsubseteq G$. Hence $t \in R$. Let $u v \in G$. If $\{u, v\} \nsubseteq G$ then $u v \in R$, else $u v=u * v$ and by inductive hypothesis is generated by $R$.

Let $(H, \circ) \in \mathcal{V}$ and let $f: R \longrightarrow H$ be a mapping. Define a mapping $\hat{f}: G \longrightarrow H$ by

$$
\hat{f}(t)= \begin{cases}f(t) & t \in R \\ \hat{f}(u) \circ \hat{f}(v) & t=u v, u, v \in G\end{cases}
$$

Let $u, v \in G$. If $u v \in G$, then $\hat{f}(u * v)=\hat{f}(u v)=\hat{f}(u) \circ \hat{f}(v)$. Otherwise, if $v=u w$, then $\hat{f}(u * v)=\hat{f}(w)=\hat{f}(u) \circ(\hat{f}(u) \circ \hat{f}(w))=\hat{f}(u) \circ \hat{f}(u w)=$ $\hat{f}(u) \circ \hat{f}(v)$.

Hence, the class of free objects in $\mathcal{V}$ is hereditary.
We next give two simple properties concerning the rank of a subgroupoid of a free $\mathcal{V}$-groupoid and the number of all $\mathcal{V}$-groupoids on a finite set.

Proposition 1. Every free $\mathcal{V}$-groupoid $F$ contains a subgroupoid with an infinite rank.

Proof. Let $b$ be an arbitrary element of the free base of $F$. Then the subgroupoid $G$ of $F$ generated by the set $\left\{c_{i} \mid i \in \mathbb{N}\right\}$, where $c_{0}=b b$ and $c_{i+1}=\left(c_{i} b\right) b$ has an infinite rank.

Further on we will use the following lemma.
Lemma 1. The number of permutations whose disjoint cycles representation consists of cycles of length at most 2 on a set with $n$ elements is

$$
\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n!}{2^{k} k!(n-2 k)!}
$$

Proof. Consider a permutation of the wanted type with $f$ fixed elements and $k$ disjoint cycles of length 2 . Then $n=f+2 k$ and $0 \leqslant k \leqslant\left[\frac{n}{2}\right]$. The fixed elements can be chosen on $\binom{n}{n-2 k}$ ways. It can be proved by induction that the number of different disjoint cycles of length 2 that can be made over a set with $2 k$ elements is $(2 k-1)!!$. So, given $k$, there are $\binom{n}{n-2 k}(2 k-1)!!=\frac{n!}{2^{k} k!(n-2 k)!}$ such permutations.

Proposition 2. The number of different $\mathcal{V}$-groupoids on a set with $n$ elements is $\left(\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n!}{2^{k} k!(n-2 k)!}\right)^{n}$.

Proof. Let $G$ be a $\mathcal{V}$-groupoid of cardinality $n$. Note that $x y=z \Longleftrightarrow x z=$ $y$ holds in $G$ and $G$ is left-cancellative, so each row in the multiplication table of $G$ can be considered as a permutation on the set $G$ whose disjoint cycles representation consists of cycles of length at most 2 . The number of such permutations is ordered by Lemma 1 , and there are $n$ rows in the multiplication table of $G$.

For example, there are $64 \mathcal{V}$-groupoids on the set $\{1,2,3\}$, and they can be obtained by suitable arrangements of the strings $123,132,321$ and 213 as rows of their multiplication tables. Here we have that the corresponding permutations are (1)(2)(3), (1)(23), (13)(2) and (12)(3).

## 3. Variety $\mathcal{V}_{e}$

The variety $\mathcal{V}_{e}$ consists of all $\mathcal{V}$-groupoids with unit. Note that each groupoid in this variety is involutory i.e. $x^{2}=e$ is its identity. So, we can use the free object $F$ from $\mathcal{V}$ to obtain a free object in $\mathcal{V}_{e}$. Namely, let $e \notin F$ and let $F_{e}=\left\{t \in F \mid\left(\forall u \in T_{B}\right) u^{2} \notin P(t)\right\} \cup\{e\}$. Define an operation $*$ on $F_{e}$ by

$$
\begin{gathered}
e * u=u * e=u, \quad e * e=e \\
u * v= \begin{cases}u v & u v \in F_{e} \\
e & u=v \\
w & v=u w, w \in F_{e}\end{cases}
\end{gathered}
$$

where $u, v \in F_{e} \backslash\{e\}$.
Theorem 3. $\left(F_{e}, *, e\right)$ is a free groupoid in $\mathcal{V}_{e}$ with a free base $B$.

Proof. One can check that $F_{e} \in \mathcal{V}_{e} . B$ is a generating set of $F_{e}$ since $B$ generates $F$ and $b * b=e$, for each $b \in B \neq \emptyset$. Given $(G, \circ, 1) \in \mathcal{V}_{e}$ and a mapping $f: B \rightarrow G$, in an inductive way we extend it to a homomorphism $\hat{f}: F_{e} \rightarrow G$ as follows: $\hat{f}(e)=1, \hat{f}(b)=f(b)$ for $b \in B$, and $\hat{f}(x y)=$ $\hat{f}(x) \circ \hat{f}(y)$.

Theorem 4. Every subgroupoid of $\left(F_{e}, *, e\right)$ is free as well.
The proof of this theorem is similar to the proof of Theorem 2. Namely, given a subgroupoid $(G, *, e)$ of $\left(F_{e}, *, e\right)$, if $|G|=1$ then $G=\{e\}$ is free with empty base, and if $|G|>1$ then we define the set $R$ as before. $R \neq \emptyset$ since it contains the elements $t$ such that $|t|=\min \{|s| \mid s \in G, s \neq e\}$. Now, the proof follows the same lines as the proof of Theorem 2.

If the rank of $F_{e}$ is 1 , then $F_{e}$ is a two-element groupoid. Therefore, the corresponding property of Theorem 3 for the variety $\mathcal{V}_{e}$ can be stated as follows.

Proposition 3. Every free $\mathcal{V}_{e}$-groupoid $F_{e}$ with a rank greater than one, contains a subgroupoid with an infinite rank.

Proof. Let $B$ be the free base of $F_{e}, a, b \in B, a \neq b$. Then the subgroupoid of $F_{e}$ generated by the set $\left\{c_{i} \mid i \in \mathbb{N}\right\}$, where $c_{0}=a b, c_{i+1}=\left(c_{i} b\right) b$ has an infinite rank.

Proposition 4. The number of different $\mathcal{V}_{e}$-groupoids on a set with $n$ elements, $n>1$, is $\quad n\left(\sum_{k=0}^{\left[\frac{n}{2}\right]-1} \frac{(n-2)!}{2^{k} k!(n-2-2 k)!}\right)^{n-1}$.

Proof. If $G$ is a $\mathcal{V}_{e}$-groupoid with unit $e$, then $x \cdot x=e$ and $x \cdot e=x$, for each $x \in G$. So, in the multiplication table of $G$, the row for the unit $e$ is uniquely defined, and in the row of any other element $x \neq e$ there are two fixed elements, obtained from $x \cdot x=e$ and $x \cdot e=x$. The remaining $n-2$ elements in the row of $x$ correspond to a permutation of order $n-2$ whose disjoint cycles representation consists of cycles of length at most 2 . The total number of such permutations is ordered by Lemma 1 , there are $n-1$ rows that should be suitably fulfilled, and there are $n$ ways a unit to be chosen.

For example, exactly 32 distinct $\mathcal{V}_{e}$-groupoids can be constructed over the set $\{1,2,3,4\}$. Fix a unit, for instance 1. Then, in the multiplication table of the groupoid, the row and the column for 1 are determined, and on the main diagonal it is only 1 . The row for 2 can be completed by
choosing the elements 3 and 4 in two different ways (corresponding to the permutation (3)(4) or the permutation (34)), and so on.

## 4. Variety $\mathcal{V}_{c}$

In this section we focus on the variety $\mathcal{V}_{c}$ containing all $\mathcal{V}$ commutative groupoids.

Proposition 5. Any two of the identities $x \cdot x y=y, y x \cdot x=y, x y=y x$ imply the third one.

Proof. Let $x \cdot x y=y$ and $y x \cdot x=y$ hold. Then

$$
x y=y(y \cdot x y)=y((x \cdot x y) \cdot x y)=y x .
$$

Hence, $\mathcal{V}_{c}$ can be defined by any two of the preceding three identities, and we have that the groupoids in $\mathcal{V}_{c}$ are TS-quasigroups (totally symmetric quasigroups [3]). Further on we describe the free objects in this variety with base $B$.

Let $(G, \cdot)$ be a groupoid. For $x, y, z=x y \in G$, we say that $x$ and $y$ are divisors of $z$. An element is prime if it has no divisors.

Proposition 6. ([2]) A groupoid $(C, \cdot)$ is a free commutative groupoid with free base $B$ if and only if
(i) $(\forall x, y, t, u \in C)(x y=t u \Longrightarrow\{x, y\}=\{t, u\})$;
(ii) $B$ is the set of primes in $(C, \cdot)$ and it generates $(C, \cdot)$.

Let $(C, \cdot)$ denote the free commutative groupoid with base $B$ and $F_{c}=$ $\left\{t \in C \mid(\forall u, v \in C) u(u v) \notin P_{c}(t)\right\}$, where the mapping $P_{c}: C \rightarrow \mathcal{B}(C)$ is defined inductively by: $t \in B \Rightarrow P_{c}(t)=\{t\}, \quad t=u v \Rightarrow P_{c}(t)=$ $\{t\} \cup P_{c}(u) \cup P_{c}(v) . \quad P_{c}$ is well defined by Proposition 6(i). Define an operation $*$ on $F_{c}$ in the following way:

$$
u * v= \begin{cases}u v & u v \in F_{c} \\ w & v=u w \text { or } u=v w \text { in }(C, \cdot)\end{cases}
$$

Theorem 5. $\left(F_{c}, *\right)$ is a free groupoid in the variety $\mathcal{V}_{c}$ with a free base $B$.
Proof. Let $u, v \in F_{c}$ and $u \cdot v \notin F_{c}$. Then $u * v=w$ for some $w \in P_{c}(u) \cup P_{c}(v)$ and since $y \in P_{c}(x) \wedge x \in F_{c} \Longrightarrow y \in F_{c}$, we get $u * v \in F_{c}$. Therefore $\left(F_{c}, *\right)$ is a groupoid and it is commutative by construction. Also, for $u, v \in$
$F_{c}$, if $u * v=u v$ then $u *(u * v)=v$. If $u * v=w, v=u w($ or $u=v w)$ in $(C, \cdot)$ then $u *(u * v)=u * w=u w=v($ or $u *(u * v)=v w *(v w * v)=v w * w=v)$. Hence, $\left(F_{c}, *\right) \in \mathcal{V}_{c}$.

If $(G, \circ)$ is a $\mathcal{V}_{c^{-}}$-groupoid and $f: B \rightarrow G$ a mapping, let $\hat{f}: C \rightarrow G$ be the homomorphism that extends $f$, i.e. $\left.\hat{f}\right|_{B}=f$. Then $\left.\hat{f}\right|_{F_{c}}$ is a homomorphism from $F_{c}$ to $G$ that extends $f$.

By using similar ideas as in the proofs of Theorem 2 and Theorem 4, it can be proved that the property of freeness in $\mathcal{V}_{c}$ is hereditary too:

Theorem 6. Each subgroupoid of a free $\mathcal{V}_{c}$-groupoid is free as well.
Proposition 7. Every free $\mathcal{V}_{c}$-groupoid contains a subgroupoid with infinite rank.

Proof. Define terms $b^{<n>}$ inductively in the following way: $b^{<0>}=b$, $b^{<k+1>}=b^{<k>} \cdot b^{<k>}$. If $b$ is a base element of a free $\mathcal{V}_{c^{\text {-groupoid, }} \text {, then the }}$ subgroupoid generated by the set $\left\{c_{i} \mid i \in \mathbb{N}\right\}$, where $c_{0}=b^{<1>}, c_{i+1}=$ $b^{<i+1>} \cdot b$ has an infinite rank.

Let $G$ be a subgroupoid of a free $\mathcal{V}_{c}$-groupoid and let $t$ be one of its elements with minimal length. Since $\left\{t^{<n>} \mid n \in \mathbb{N}\right\}$ is an infinite set, we conclude that every subgroupoid of a free groupoid in $\mathcal{V}_{c}$ is infinite as well. The same construction can be applied for $\mathcal{V}$ too, i.e. every subgroupoid of a free $\mathcal{V}$-groupoid is not finite.

The problem concerning the enumeration of all TS-quasigroups defined on $n$-element set remains open.

Example 1. Let $(G, \cdot)$ be a commutative group and define an operation * on $G$ by $\quad x * y=c x^{-1} y^{-1}, c \in G$. Then $(G, *) \in \mathcal{V}_{c}$.

Example 2. The following 5-element quasigroup is a TS-quasigroup which can not be obtained by the construction given in Example 1.

|  | 0 | 1 | 2 | 3 | 4 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 0 | 0 | 2 | 1 | 4 | 3 |
| 1 | 2 | 3 | 0 | 1 | 4 |
| 2 | 1 | 0 | 4 | 3 | 2 |
| 3 | 4 | 1 | 3 | 2 | 0 |
| 4 | 3 | 4 | 2 | 0 | 1 |

Note that $\mathcal{V}_{c} \cap U n$ is in fact the variety of Steiner's loops. For constructions of free objects in that variety and some related topics the reader is referred to $[6,7]$.

## 5. Variety $\mathcal{V}_{u}$

We now consider the variety $\mathcal{V}_{u}$ defined by the identities $x(x y)=y,(x y) y=$ $x y$. As it will soon become clear, its groupoids have very simple structure.
Proposition 8. $\mathcal{V}_{u}=\operatorname{Var}\left(x y=y^{2}, x^{2} \cdot x^{2}=x\right)$
Proof. By definition $\mathcal{V}_{u}=\operatorname{Var}(x \cdot x y=y, x y \cdot y=x y)$ so we get first (1) $x y \cdot x y=x y \cdot(x y \cdot y)=y$ and then (2) $y \cdot x y=(x y \cdot x y) \cdot x y=x y \cdot x y=y$. Now (1) and (2) give $x y=(y \cdot x y)(y \cdot x y)=y^{2}$, and (1) gives $x^{2} \cdot x^{2}=x$.

On the other hand, $x y=y^{2}, x^{2} \cdot x^{2}=x$ first imply $x \cdot x y=x \cdot y^{2}=$ $y^{2} \cdot y^{2}=y$ and after that $y x=y x \cdot(y x \cdot y x)=y x \cdot\left(x^{2} \cdot x^{2}\right)=y x \cdot x$.

As a consequence of the previous proposition we get that in the variety $\mathcal{V}_{u}$ despite of $x y=y^{2}$ and $x^{2} \cdot x^{2}=x$, the following identities hold: $x^{2} \cdot y=$ $y^{2}, x \cdot y^{2}=y, x^{2} \cdot y^{2}=y$. (Namely, $x \cdot y^{2}=x \cdot x y=y \Longrightarrow x^{2} \cdot y^{2}=y \Longrightarrow$ $x^{2} \cdot y=x^{2} \cdot\left(x^{2} \cdot y^{2}\right)=y^{2}$.)

Note that $x^{2}=x$ is not an identity, since $(\{0,1\}, *) \in \mathcal{V}_{u}$ where $0 * 0=$ $1 * 0=1,0 * 1=1 * 1=0$.

Let $F_{u}=\left\{b, b^{2} \mid b \in B\right\}$ and define an operation $*$ on $F_{u}$ by $u * b=$ $b^{2}, u * b^{2}=b$ for all $b \in B, u \in F_{u}$. Then we have:
Theorem 7. $\left(F_{u}, *\right)$ is a free groupoid with free base $B$ in $\mathcal{V}_{u}$.
As a result from the last theorem we get that any free groupoid in $\mathcal{V}_{u}$ with finite base of cardinality $n$ is itself finite and of order $2 n$.
Theorem 8. Every subgroupoid of a free groupoid in $\mathcal{V}_{u}$ is free too.
Proof. Let $G$ be a subgroupoid of a free $\mathcal{V}_{u}$-groupoid $F_{u}$ and $B_{1}=B \cap G$, where $B$ is the free base of $F_{u}$. Since $a \in G \subseteq F_{u}$ imply either $a=b$ or $a=b^{2}$ for some $b \in B$, and $y^{2} \cdot y^{2}=y$ is an identity in $\mathcal{V}_{u}$, it follows that $G \backslash B_{1}=\left\{b^{2} \mid b \in B_{1}\right\}$. Hence, $G$ is free in $\mathcal{V}_{u}$ with free base $B_{1}$.

Hence, any subgroupoid of the free groupoid with base $B$ coincides with the free groupoid with some base $B_{1} \subseteq B$ and we get the following corollary.
Corollary 1. Let $|B|=n$. Then the number of all subgroupoids of a free groupoid of $\mathcal{V}_{u}$ with base $B$ is $2^{n}-1$.

Since finite $\mathcal{V}_{u}$-groupoids are exactly those $\mathcal{V}$-groupoids which rows in its multiplication tables are identical and all elements in a row are different $\left(x^{2}=y^{2} \Rightarrow x=y\right.$ in $\left.\mathcal{V}_{u}\right)$, by Lemma 1 we get that the number of different $\mathcal{V}_{u}$-groupoids defined on a set with $n$ elements is $\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n!}{2^{k} k!(n-2 k)!}$.

## 6. Variety $\mathcal{V}_{i}$

The variety $\mathcal{V}_{i}$ is an interesting one, because its finite members are closely connected with the Steiner quintuple systems. Here firstly we give a description of the free objects in $\mathcal{V}_{i}$, and after that we discuss some aspects of the mentioned connection with Steiner quintuple systems.

Proposition 9. Besides the defining identities
(1) $\quad x \cdot x y=y \quad$ and $\quad$ (2) $\quad y x \cdot x=x y$,
the following identities hold in $\mathcal{V}_{i}$ :

| $(3)$ | $x y \cdot x=x \cdot y x$, | $(8)$ |
| :--- | :--- | :--- |
| $(4)$ | $x x=x$, | $(9)$ |
| $(5)$ | $y x \cdot y=x \cdot y x, y x$, |  |
| $(5 y \cdot y x=y$, | $(10)$ | $x y \cdot(x \cdot y x)=x$, |
| $(6)$ | $(x y \cdot x) x=y$, | $(11)$ |
| $(7)$ | $(x y \cdot x) y=x$, | $(12)$ |
| $(x y) \cdot x y=y x$, |  |  |
|  | $(x y \cdot x) \cdot y x=x y$, |  |

as well as the cancellation laws and anticommutativity.
Proof. For any $x, y$ in a $\mathcal{V}_{i}$ - groupoid we have
(3) $x y \cdot x \stackrel{(2)}{=}(y x \cdot x) x \stackrel{(2)}{=} x \cdot y x$;
(4) $x x \stackrel{(2)}{=} x x \cdot x \stackrel{(3)}{=} x \cdot x x \stackrel{(1)}{=} x$;
(5) $x y \cdot y x \stackrel{(2)}{=} x y \cdot(x y \cdot y) \stackrel{(1)}{=} y$;
(6) $(x y \cdot x) x \stackrel{(2)}{=} x \cdot x y \stackrel{(1)}{=} y$;
(7) $(x y \cdot x) y \stackrel{(6)}{=}(x y \cdot x)((x y \cdot x) x) \stackrel{(1)}{=} x$;
(8) $x(y x \cdot y) \stackrel{(1)}{=}(y \cdot y x)(y x \cdot y) \stackrel{(5)}{=} y x$;
(9) $y x \cdot y \stackrel{(1)}{=} x(x(y x \cdot y)) \stackrel{(8)}{=} x \cdot y x$;
(10) $x y \cdot(x \cdot y x) \stackrel{(3)}{=} x y \cdot(x y \cdot x) \stackrel{(1)}{=} x$;
(11) $(x y \cdot x) \cdot x y \stackrel{(3)}{=} x y \cdot(x \cdot x y) \stackrel{(1)}{=} x y \cdot y \stackrel{(2)}{=} y x$;
(12) $(x y \cdot x) \cdot y x \stackrel{(3,9)}{=}(y x \cdot y) \cdot y x \stackrel{(3)}{=} y x \cdot(y \cdot y x) \stackrel{(1)}{=} y x \cdot x \stackrel{(2)}{=} x y$.

Also

$$
\begin{aligned}
& x y=x z \quad \Longrightarrow \quad y=x \cdot x y=x \cdot x z=z \\
& y x=z x \quad \Longrightarrow \quad x y=y x \cdot x=z x \cdot x=x z \\
& x y=y x \quad \Longrightarrow \quad y=x \cdot x y=x \cdot y x \stackrel{(3,9)}{=} y \cdot x y=y \cdot y x=x .
\end{aligned}
$$

From (1), (6) and the cancellation laws we have:
Corollary 2. Any groupoid in $\mathcal{V}_{i}$ is a quasigroup.

Note that in any groupoid of $\mathcal{V}_{i}$ we have $x \cdot y x=x y \cdot x=y x \cdot y=y \cdot x y$ by (3) and (9). Let $\alpha$ be the congruence on $T_{B}$ generated by the preceding equalities. We denote by $u v u$ the class $u(v u) / \alpha$ and use the same operation symbol for $T_{B} / \alpha$ as we did for $T_{B}$. Also, we shall sometimes continue using the notions "term" and "subterm" for the elements of $T_{B} / \alpha$.

Let $F_{i} \subseteq T_{B} / \alpha$ be the set of all terms that do not contain as a subterm a left-hand side of (i) - (viii):

| (i) $\quad s s=s$, | $(v)$ | $s \cdot s t s=t s$, |
| :--- | :--- | :--- |
| $(i i)$ | $s \cdot s t=t$, | $(v i)$ |
| $s t \cdot s t s=s$, |  |  |
| $(i i i)$ | $s t \cdot t=t s$, | $(v i i)$ |
| sts $\cdot s=t$, |  |  |
| $($ iv $)$ | $s t \cdot t s=t$, | $($ viii $)$ |
| sts $\cdot s t=t s$, |  |  |

where $s, t \in T_{B}$.
The next proposition justifies the definition of the set $F_{i}$ as well as the use of the notions "term" and "subterm".

Proposition 10. If the term $u(v u) \in T_{B}$ for some $u, v \in T_{B}$ does not contain as a subterm a term of the following forms: ss, s•st, st $\cdot t$, st $\cdot t s$, $s \cdot s(t s), s \cdot(s t) s, s \cdot(t s) t, s \cdot t(s t), s t \cdot s(t s), s t \cdot(s t) s, s t \cdot(t s) t, s t \cdot t(s t),(s t) s \cdot s$, $s(t s) \cdot s, t(s t) \cdot s,(t s) t \cdot s,(s t) s \cdot s t, s(t s) \cdot s t, t(s t) \cdot s t,(t s) t \cdot s t$, then the same holds for the terms (uv)u, (vu)v and $v(u v)$.

Proof. By checking all the possibilities it is easy to see that (vu)v does not contain such a subterm. Namely, each assumption that the term has such a subterm, means that the term is of the given form (having in mind that the statement holds for $u, v$ and $v u$ ) which always leads to contradiction for $u, v, v u$ or $u(v u)$. For instance, $(v u) v=(s t) s \cdot s t \Longrightarrow u(v u)=s \cdot(s t) s$. In the same way, it can be shown in all the cases for $u v$ and then finally for $v(u v)$ as well.

Define an operation $*$ on $F_{i}$ in the following way. For $u, v \in F_{i}$, if $u v \in F_{i}$ then $u * v=u v$. Otherwise, if $u v$ has the form of a left-hand side of some of $(i)-(v i i i)$ define $u * v$ to be the corresponding right-hand side of the identity, except in the case of (iii) i.e. when $u=w v$, then we put $u * v=v * w$. It can be shown, by induction on length of terms, that $*$ is well defined. Note that, by the previous proposition if sts $\in F_{i}$ then also $t s \in F_{i}$.

Theorem 9. $\left(F_{i}, *\right)$ is free in $\mathcal{V}_{i}$ with free base $B$.
Proof. First, we show that $\left(F_{i}, *\right)$ satisfies (1). Let $u, v \in F_{i}$. If $u v \in F_{i}$ then $u *(u * v)=u *(u v) \stackrel{(i i)}{=} v$. Otherwise, we consider several cases.
$\left(i^{\prime}\right) \quad u=v: u *(u * v)=u *(u * u) \stackrel{(i)}{=} u * u \stackrel{(i)}{=} u=v ;$
$\left(i i^{\prime}\right) \quad v=u t: u *(u * v)=u *(u * u t) \stackrel{(i i)}{=} u * t=u t=v$;
(iii') $u=t v$ and
0. $v t \in F_{i}: u *(u * v)=t v *(t v * v) \stackrel{(i i i)}{=} t v *(v * t)=t v * v t \stackrel{(i v)}{=} v ;$

1. $v=t$ is impossible case since we would have $u=t v=t t \notin F_{i}$;
2. $t=v p: u *(u * v)=t v *(v * t)=v p v *(v * v p) \stackrel{(i i)}{=} p v p * p \stackrel{(v i i)}{=} v$;
3. $v=p t: u *(u * v)=t v *(v * t)=t p t *(p t * t)=t p t *(t * p)=$ $t p t * t p \stackrel{(v i i i)}{=} p t=v$;
4. $v=q s, t=s q$;
5. $t=v p v$;
6. $v=p q, t=p q p ;$
7. $v=t p t$;
8. $v=p q p, t=p q$.

All the cases 4.-8. are impossible since they lead to $u=s q \cdot q s, u=$ $v p v \cdot v, u=p q p \cdot p q, u=t \cdot t p t, u=p q \cdot p q p$, respectively, contradicting $u \in F_{i}$.
$\left(i v^{\prime}\right) \quad u=t p, v=p t: u *(u * v)=t p *(t p * p t) \stackrel{(i i)}{=} t p * p \stackrel{(i v)}{=} p * t=p t=v ;$
$\left(v^{\prime}\right) \quad v=u t u: u *(u * v)=u *(u * u t u) \stackrel{(v)}{=} u * t u=u t u=v$;
$\left(v i^{\prime}\right) \quad u=t p, v=t p t: u *(u * v)=t p *(t p * t p t) \stackrel{(v i)}{=} t p * t=t p t=v ;$
$\left(v i i^{\prime}\right) \quad u=v t v: u *(u * v)=v t v *(v t v * v) \stackrel{(v i i)}{=} v t v * t=v$;
(viií) $u=t p t, v=t p: u *(u * v)=t p t *(t p t * p t)=t p t * p t=t p=v$.
So we have shown that $\left(F_{i}, *\right)$ satisfies (1) and continue for (2). If $u, v \in F_{i}$ and $u * v=u v \in F_{i}$, then $(u * v) * v=u v * v=v * u$. Otherwise, we have the cases:
$\left(i^{\prime \prime}\right) \quad u=v:(u * v) * v=(u * u) * u=u * u=v * u$;
$\left(i i^{\prime \prime}\right) \quad v=u t$, and in this case $t u t \in F_{i}$ i.e. no other case is possible and we get $(u * v) * v=(u * u t) * u t=t * u t=t u t=u t u=u t * u=v * u$;
$\left(i i i^{\prime \prime}\right) \quad u=t v:(u * v) * v=(t v * v) * v=(v * t) * v$ and there are several possibilities:
0. $v t \in F_{i}\left(v t v \in F_{i}\right):(u * v) * v=v t * v=v t v=v * u$;

1. $v=t$ is impossible case;
2. $t=v p:(u * v) * v=(v * t) * v=(v * v p) * v=p * v=p v$ since $u=t v=v p v \in F_{i}$, so $p v \in F_{i}$, and on the other hand $v * u=v * v p v=p v$;
3. $v=p t:(u * v) * v=(v * t) * v=(p t * t) * p t=(t * p) * p t$ and since $u=t v=t p t \in F_{i}$ also $t p \in F_{i}$ and $(t * p) * p t=t p * p t=p$ and $v * u=p t * t p t=p ;$
4. $v=p q, t=q p$;
5. $\quad t=v p v ;$
6. $v=p q, t=p q p ;$
7. $v=t p t ; 0$
8. $v=p q p, t=p q$.

All the cases 4.-8. are impossible.
$\left(i v^{\prime \prime}\right) \quad u=p t, v=t p:(u * v) * v=(p t * t p) * t p=t * t p=p=t p * p t=v * u$;
$\left(v^{\prime \prime}\right) \quad v=u p u:(u * v) * v=(u * u p u) * u p u=p u * u p u=p=u p u * u=v * u$;
$\left(v i^{\prime \prime}\right) u=p t, v=p t p:(u * v) * v=(p t * p t p) * p t p=p * p t p=t p=p t p * p t=$ $v * u$;
$\left(v i i^{\prime \prime}\right) \quad u=v p v:(u * v) * v=(v p v * v) * v=p * v=p v=v * v p v=v * u ;$ $\left(v i i i^{\prime \prime}\right) u=p t p, v=p t:(u * v) * v=(p t p * p t) * p t=t p * p t=p=p t * p t p=$ $v * u$.

Thus we have shown that $F_{i} \in \mathcal{V}_{i}$.
By induction on length of terms one can show that $B$ is a base for $F_{i}$. Namely, $B \subseteq F_{i}$ and if $u v \in F_{i}$ then $u v=u * v$ is generated by $B$ if $u$ and $v$ are.

Let $(G, \circ) \in \mathcal{V}_{i}$ and $f: B \rightarrow G$. Define a mapping $\hat{f}: F_{i} \rightarrow G$ inductively by $\hat{f}(b)=f(b), b \in B$ and $\hat{f}(u v)=\hat{f}(u) \circ \hat{f}(v)$ for $u v \in F_{i} \backslash B$. We show that $\hat{f}$ is a homomorphism and an extension of $f$. If $u, v \in F_{i}$ and $u v \in F_{i}$ the statement is clear by definition of $\hat{f}$. Otherwise one of the same eight cases might occur. We check here only the third case when $u=t v$, because the others can be checked as earlier. Now, $\hat{f}(u * v)=$ $\hat{f}(t v * v)=\hat{f}(v * t)$ which by induction on length of $u$ equals to $\hat{f}(v) \circ \hat{f}(t)=$ $(\hat{f}(t) \circ \hat{f}(v)) \circ \hat{f}(v)=\hat{f}(t v) \circ \hat{f}(v)=\hat{f}(u) \circ \hat{f}(v)$.

Note that $|B|=1 \Longrightarrow\left|F_{i}\right|=1$ and $|B|=2 \Longrightarrow\left|F_{i}\right|=5$. It is clear that in each $\mathcal{V}_{i}$-groupoid every two distinct elements generate a subgroupoid with five elements. In fact, $\mathcal{V}_{i}$ is the class of cancellative groupoids with property $(2,5)([11])$. (A class K is said to have the property ( $\mathrm{k}, \mathrm{n}$ ) if every algebra in K generated by k distinct elements has exactly n elements.)

Let $(G, \cdot)$ belongs to $\mathcal{V}_{i}$ and define a groupoid by $x * y=y x, x, y \in G$. It is easy to verify that the quasigroup $(G, *)$ is an orthogonal mate of $G$.

Let $F_{i}$ be a free groupoid in $\mathcal{V}_{i}$, such that its free base contains three distinct elements $a, b, c$. Then the subgroupoid of $F_{i}$ generated by the set $\left\{d_{i} \mid i \in \mathbb{N}\right\}$, where $d_{0}=a b$ and $d_{3 i+1}=\left(d_{3 i} \cdot c\right) a, d_{3 i+2}=\left(d_{3 i+1} \cdot b\right) c, d_{3 i+3}=$ $\left(d_{3 i+2} \cdot a\right) b$, for $i \in \mathbb{N}$, has an infinite rank. Hence, we get the following result.

Proposition 11. Every free $\mathcal{V}_{i}$-groupoid with rank greater than two has a subgroupoid with an infinite rank.

So, unlike the free $\mathcal{V}_{i}$-groupoids with rank one or two, the free $\mathcal{V}_{i^{-}}$ groupoids with rank greater than two are infinite. Also, apart from the groupoids of the previous varieties, there is no $\mathcal{V}_{i}$-groupoid with $n \leqslant 20$ elements, $n \neq 1,5$. Here we present the table of a $\mathcal{V}_{i}$-groupoid with 21 elements.

|  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 2 | 1 | 4 | 3 | 6 | 5 | 8 | 7 | 10 | 9 | 12 | 11 | 14 | 13 | 16 | 15 | 18 | 17 | 20 | 19 |
| 1 | 3 | 1 | 4 | 0 | 2 | 9 | 12 | 10 | 11 | 5 | 7 | 8 | 6 | 17 | 20 | 18 | 19 | 13 | 15 | 16 | 14 |
| 2 | 4 | 3 | 2 | 1 | 0 | 11 | 10 | 12 | 9 | 8 | 6 | 5 | 7 | 20 | 17 | 19 | 18 | 14 | 16 | 15 | 13 |
| 3 | 2 | 4 | 0 | 3 | 1 | 12 | 9 | 11 | 10 | 6 | 8 | 7 | 5 | 18 | 19 | 17 | 20 | 15 | 13 | 14 | 16 |
| 4 | 1 | 0 | 3 | 2 | 4 | 10 | 11 | 9 | 12 | 7 | 5 | 6 | 8 | 19 | 18 | 20 | 17 | 16 | 14 | 13 | 15 |
| 5 | 7 | 13 | 15 | 16 | 14 | 5 | 8 | 0 | 6 | 17 | 18 | 19 | 20 | 1 | 4 | 2 | 3 | 9 | 10 | 11 | 12 |
| 6 | 8 | 15 | 13 | 14 | 16 | 7 | 6 | 5 | 0 | 19 | 20 | 17 | 18 | 2 | 3 | 1 | 4 | 11 | 12 | 9 | 10 |
| 7 | 6 | 16 | 14 | 13 | 15 | 8 | 0 | 7 | 5 | 20 | 19 | 18 | 17 | 3 | 2 | 4 | 1 | 12 | 11 | 10 | 9 |
| 8 | 5 | 14 | 16 | 15 | 13 | 0 | 7 | 6 | 8 | 18 | 17 | 20 | 19 | 8 | 1 | 3 | 2 | 10 | 9 | 12 | 11 |
| 9 | 11 | 17 | 18 | 19 | 20 | 13 | 14 | 15 | 16 | 9 | 12 | 0 | 10 | 5 | 6 | 7 | 8 | 1 | 2 | 3 | 4 |
| 10 | 12 | 19 | 20 | 17 | 18 | 14 | 13 | 16 | 15 | 11 | 10 | 9 | 0 | 6 | 5 | 8 | 7 | 3 | 4 | 1 | 2 |
| 11 | 10 | 20 | 19 | 18 | 17 | 15 | 16 | 13 | 14 | 12 | 0 | 11 | 9 | 7 | 8 | 5 | 6 | 4 | 3 | 2 | 1 |
| 12 | 9 | 18 | 17 | 20 | 19 | 16 | 15 | 14 | 13 | 0 | 11 | 10 | 12 | 8 | 7 | 6 | 5 | 2 | 1 | 4 | 3 |
| 13 | 15 | 9 | 10 | 11 | 12 | 17 | 20 | 18 | 19 | 1 | 2 | 3 | 4 | 13 | 16 | 0 | 14 | 5 | 7 | 8 | 6 |
| 14 | 16 | 11 | 12 | 9 | 10 | 18 | 19 | 17 | 20 | 3 | 4 | 1 | 2 | 15 | 14 | 13 | 0 | 7 | 5 | 6 | 8 |
| 15 | 14 | 12 | 11 | 10 | 9 | 19 | 18 | 20 | 17 | 4 | 3 | 2 | 1 | 16 | 0 | 15 | 13 | 8 | 6 | 5 | 7 |
| 16 | 13 | 10 | 9 | 12 | 11 | 20 | 17 | 19 | 18 | 2 | 1 | 4 | 3 | 0 | 15 | 14 | 16 | 6 | 8 | 7 | 5 |
| 17 | 19 | 5 | 7 | 8 | 6 | 1 | 4 | 2 | 3 | 13 | 15 | 16 | 14 | 9 | 12 | 10 | 11 | 17 | 20 | 0 | 18 |
| 18 | 20 | 6 | 8 | 7 | 5 | 4 | 1 | 3 | 2 | 16 | 14 | 13 | 15 | 11 | 10 | 12 | 9 | 19 | 18 | 17 | 0 |
| 19 | 18 | 7 | 5 | 6 | 8 | 2 | 3 | 1 | 4 | 14 | 16 | 15 | 13 | 12 | 9 | 11 | 10 | 20 | 0 | 19 | 17 |
| 20 | 17 | 8 | 6 | 5 | 7 | 3 | 2 | 4 | 1 | 15 | 13 | 14 | 16 | 10 | 11 | 9 | 12 | 0 | 19 | 18 | 20 |

The most interesting characteristic of the $\mathcal{V}_{i}$ variety is due to its $(2,5)$ property and reflects the connection between $\mathcal{V}_{i}$ and the Steiner quintuple systems.

Let $(Q, \cdot)$ be an $n$-element quasigroup in $\mathcal{V}_{i}$. Consider the set $\hat{Q}=$ $\{K \mid(K, \cdot)$ is a 5 -element subquasigroup of $(Q, \cdot)\}$. It follows by the $(2,5)$ property that for any two elements $a, b \in Q$ there exists a unique $K$ in $Q$ such that $a, b \in K$. Hence, $\hat{Q}$ is a $2-(\mathrm{n}, 5,1)$ design, i.e. a Steiner quintuple system.

Example 3. From the preceding 21 -element quasigroup we have the following Steiner quintuple system:

$$
\begin{array}{llll}
\{0,1,2,3,4\}, & & \\
\{0,5,6,7,8\}, & \{0,9,10,11,12\}, & \{0,13,14,15,16\}, & \{0,17,18,19,20\}, \\
\{1,5,9,13,17\}, & \{1,6,12,15,18\}, & \{1,7,10,16,19\}, & \{1,8,11,14,20\}, \\
\{2,5,11,15,19\}, & \{2,6,10,13,20\}, & \{2,7,12,14,17\}, & \{2,8,9,16,18\}, \\
\{3,5,12,16,20\}, & \{3,6,9,14,19\}, & \{3,7,11,13,18\}, & \{3,8,10,15,17\}, \\
\{4,5,10,14,18\}, & \{4,6,11,16,17\}, & \{4,7,9,15,20\}, & \{4,8,12,13,19\} .
\end{array}
$$

On the other hand, let $S=\left\{B_{1}, B_{2}, \ldots, B_{k}\right\}$ be a Steiner quintuple system. Clearly, for each $i \in\{1,2, \ldots, k\}$, a $\mathcal{V}_{i}$-quasigroup ( $B_{i}, *_{i}$ ) can be constructed. Now, put $Q=\bigcup B_{i}$ and $*=\cup *_{i}$. For arbitrary $a, b \in Q$ there
is a unique $i$, such that $a$ and $b$ both belong to $B_{i}$. By the construction of $*, a *(a * b)=a *_{i}\left(a *_{i} b\right)=b$, and similarly the other identity can be checked, so $(Q, *)$ is in $\mathcal{V}_{i}$.

We have shown that every $\mathcal{V}_{i}$-quasigroup induces a Steiner quintuple system and vice versa. Note that the first procedure was deterministic, unlike the second one. Namely, on each 5 -element set six different $\mathcal{V}_{i^{-}}$ quasigroups can be defined, which means that for one Steiner quintuple system $\left\{B_{1}, B_{2}, \ldots, B_{k}\right\}, 6^{k}$ different $\mathcal{V}_{i}$-quasigroups can be constructed, in the way presented above. By the preceding discussion we have proved the following result.
Theorem 10. Each n-element $\mathcal{V}_{i}$-quasigroup give rise to an $n$-element Steiner quintuple system, i.e. $2-(n, 5,1)$ design, and each $n$-element Steiner quintuple system give rise to $6^{n}$ different $n$-element $\mathcal{V}_{i}$-quasigroups.

Let $(Q, \cdot)$ and $\left(Q^{\prime}, *\right)$ be isomorphic $\mathcal{V}_{i}$-quasigroups and $S$ and $S^{\prime}$ be their corresponding Steiner quintuple systems. Let $f: Q \longrightarrow Q^{\prime}$ be an isomorphism. Since $f$ preserves subquasigroups and for any subquasigroup $\left(K^{\prime}, *\right)$ of $\left(Q^{\prime}, *\right)$ there is a unique subquasigroup $(K, \cdot)$ of $(Q, \cdot)$ satisfying $f(K)=K^{\prime}, f$ is an isomorphism from $S$ to $S^{\prime}$.

For the opposite, let $f$ be an isomorphism from a Steiner quintuple system $S=\left\{B_{1}, \ldots, B_{k}\right\}$ to a Steiner quintuple system $S^{\prime}=\left\{B_{1}^{\prime}, \ldots, B_{k}^{\prime}\right\}$ and $Q=\bigcup B_{i}, Q^{\prime}=\bigcup B_{i}^{\prime}$. Let $(Q, \cdot)$ be one of the quasigroups arising from $S$. Define an operation * in $Q^{\prime}$ by

$$
a * b=c \Longleftrightarrow f^{-1}(a) \cdot f^{-1}(b)=f^{-1}(c .)
$$

Then $\left(Q^{\prime}, *\right)$ is a quasigroup arising from $S^{\prime}$ and $f$ is an isomorphism from $(Q, \cdot)$ to $\left(Q^{\prime}, *\right)$.

Denote by $\mathcal{F} \mathcal{V}_{i}$ the class of all finite $\mathcal{V}_{i}$-quasigroups, and by $\mathcal{S}$ the class of all Steiner quintuple systems. An equivalence on $\mathcal{F} \mathcal{V}_{i}$ can be defined by

$$
(Q, \cdot) \alpha\left(Q^{\prime}, *\right) \Longleftrightarrow \hat{Q}=\hat{Q}^{\prime}
$$

where $\hat{Q}$ is defined as before. The reasoning above leads us to the following result.
Theorem 11. There is one to one correspondence between $\mathcal{F} \mathcal{V}_{i} / \alpha$ and $\mathcal{S}$.
Corollary 3. A necessary condition for existence of $n$-element $\mathcal{V}_{i}$-quasigroup is $n=20 k+1$ or $n=20 k+5$ for some nonnegative integer $k$.

Proof. Given an $n$-element $\mathcal{V}_{i}$-quasigroup, we construct an $n$-element Steiner quintuple with $b$ blocks. Since there are $x=n(n-1) / 2$ different pairs of
elements and each block contains $y=5 \cdot 4 / 2=10$ such pairs, we have $b=x / y=n(n-1) / 20$. On the other hand, $n=4 m+1$ where $m$ is the number of occurrences of fixed element in the blocks.

We do not know whether for each $n$ such that $n=20 k+1$ or $n=20 k+5$ there exists an $n$-element $\mathcal{V}_{i}$-quasigroup.

Since a direct product of $\mathcal{V}_{i}$-quasigroups is a $\mathcal{V}_{i}$-quasigroup, we have possibility to construct Steiner quintuple systems of enough large finite cardinality. It follows from the next property:

Corollary 4. The existence of $n$-element and m-element Steiner quintuple systems implies existence of $n m$-element Steiner quintuple system.

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# $\mathbf{m}$-Systems and $\mathbf{n}$-systems in ordered semigroups 

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#### Abstract

The aim of this short note is to introduce the concepts of $m$-systems and $n$-systems in ordered semigroups. These concepts are related to the concepts of weakly prime and weakly semiprime ideals, play an important role in studying the structure of ordered semigroups, so it seems to be interesting to study them.


There were several attempts to define the $m$-systems and $n$-systems in ordered semigroups. These concepts being related to the concepts of weakly prime and weakly semiprime ideals, play an important role in studying the structure of ordered semigroups. The aim of this note is to introduce the concepts of $m$-systems and $n$-systems in ordered semigroups. We begin our consideration by proving the relation between $m$-systems and weakly prime ideals, $n$-systems and weakly semiprime ideals. We prove that if $S$ is an ordered semigroup, $I$ a weakly prime (resp. weakly semiprime) proper ideal of $S$, then the complement $S \backslash I$ of $I$ to $S$ is an $m$-system (resp. an $n$-system) of $S$. "Conversely", if $I$ is an ideal and $S \backslash I$ an $m$-system (resp. an $n$-system) of $S$, then $I$ is weakly prime (resp. weakly semiprime). Thus a proper ideal $I$ of an ordered semigroup $S$ is weakly prime (resp. weakly semiprime) if and only if the complement $S \backslash I$ of $I$ to $S$ is an $m$-system (resp. an $n$-system). Moreover, each $n$-system of an ordered semigroup $S$ containing an element $a$ of $S$, contains an $m$-system that contains the same element $a$.

If $(S, \cdot)$ is a semigroup or an ordered semigroup, a subset $T$ of $S$ is called weakly prime if for all ideals $A, B$ of $S$ such that $A B \subseteq T$, we have $A \subseteq T$ or $B \subseteq T$. The subset $T$ of $S$ is called weakly semiprime if for every ideal $A$ of $S$ such that $A^{2} \subseteq T$, we have $A \subseteq T$ [2]. For $H \subseteq S$, we define

$$
(H]:=\{t \in S \mid t \leqslant h \text { for some } h \in H\} .
$$

[^1]A non-empty subset $A$ of an ordered semigroup $S$ is called an ideal of $S$ if 1) $A S \subseteq A, S A \subseteq A, 2) a \in A, S \ni b \leqslant a$ implies $b \in A[2]$. An ideal $I$ of $S$ is called proper if $I \neq S$. A non-empty subset $B$ of an ordered semigroup $S$ is called a bi-ideal of $S$ if 1) $B S B \subseteq B$ and 2) if $a \in B$ and $S \ni c \leqslant a$, then $c \in B$ [3].

If $(S, \cdot)$ is a semigroup, a non-empty subset $A$ of $S$ is called an $m$-system of $S$ if for each $a, b \in A$ there exists $x \in S$ such that $a x b \in A[4]$.

The set $A$ is called an $n$-system of $S$ if for each $a \in A$ there exists $x \in S$ such that $a x a \in A$.

In ordered semigroups the $m$-system and the $n$-system are defined as follows:

Definition 1. Let $S$ be an ordered semigroup and $\emptyset \neq A \subseteq S$. The set $A$ is called an $m$-system of $S$ if for each $a, b \in A$ there exist $c \in A$ and $x \in S$ such that $c \leqslant a x b$.

Equivalent Definition: For each $a, b \in A$ there exists $c \in A$ such that $c \in(a S b]$.

Definition 2. Let $S$ be an ordered semigroup and $\emptyset \neq A \subseteq S$. The set $A$ is called an $n$-system of $S$ if for each $a \in A$ there exist $c \in A$ and $x \in S$ such that $c \leqslant a x a$.

Equivalent Definition: For each $a \in A$ there exists $c \in A$ such that $c \in(a S a]$.

Remark 1. Each $m$-system is an $n$-system. Each bi-ideal is an $m$-system.
Remark 2. If $(S, \cdot)$ is a semigroup, we endow $S$ with the order relation defined by $\leqslant:=\{(a, b) \mid a=b\}$. Then $(S, \cdot, \leqslant)$ is an ordered semigroup. Moreover, the set $A$ is an $m$-system (resp. an $n$-system) of $(S, \cdot)$ if and only if $A$ is an $m$-system (resp. an $n$-system) of $(S, \cdot, \leqslant)$.

Lemma. (cf. [1]). Let $S$ be an ordered semigroup and $I$ an ideal of $S$. Then $I$ is weakly prime if and only if for each $a, b \in S$ such that $a S b \subseteq I$, we have $a \in I$ or $b \in I$.

We remark that since $I$ is an ideal of $S$, we have

$$
(a S b] \subseteq I \text { if and only if } a S b \subseteq I
$$

Proposition 1. Let $S$ be an ordered semigroup and $I$ an ideal of $S$. Then:

1) if $I$ is weakly prime and $S \backslash I \neq \emptyset$, then $S \backslash I$ is an $m$-system,
2) if $S \backslash I$ is an m-system, then $I$ is weakly prime.

Proof. 1) Clearly, $\emptyset \neq S \backslash I \subseteq S$. Let $a, b \in S \backslash I$. Then, there exists $c \in S \backslash I$ such that $c \in(a S b]$. In fact, let $c \notin(a S b]$ for every $c \in S \backslash I$. We prove that $a S b \subseteq I$. Then, since $I$ is weakly prime, by the Lemma, we have $a \in I$ or $b \in I$, which is impossible.

Let $a S b \nsubseteq I$. Then, there exists $y \in S$ such that $a y b \notin I$, so $a y b \in S \backslash I$. For the element $a y b \in S \backslash I$, we have $a y b \in a S b \subseteq(a S b]$.
2) Let $a, b \in S, a S b \subseteq I$. Then $a \in I$ or $b \in I$. Indeed, let $a, b \in S \backslash I$. Since $S \backslash I$ is an $m$-system, there exist $c \in S \backslash I$ and $x \in S$ such that $c \leqslant$ $a x b \in a S b \subseteq I$. Since $I$ is an ideal of $S$, we have $c \in I$. Impossible.

Corollary 1. An ideal I of an ordered semigroup $S$ is weakly prime if and only if either $S \backslash I=\emptyset$ or the set $S \backslash I$ is an $m$-system. A proper ideal $I$ of an ordered semigroup $S$ is weakly prime if and only $S \backslash I$ is an $m$-system.

In a similar way, we prove the following:
Proposition 2. Let $S$ be an ordered semigroup and $I$ an ideal of $S$. Then:

1) if $I$ is weakly semiprime and $S \backslash I \neq \emptyset$, then $S \backslash I$ is an n-system,
2) if $S \backslash I$ is an n-system, then $I$ is weakly semiprime.

Corollary 2. An ideal I of an ordered semigroup $S$ is weakly semiprime if and only if either $S \backslash I=\emptyset$ or the set $S \backslash I$ is an n-system. A proper ideal $I$ of an ordered semigroup $S$ is weakly semiprime if and only if $S \backslash I$ is an n-system.

According to Remark 2, by Propositions 1 and 2, we get the Corollaries 3 and 4 below which are referred to semigroups without order.
Corollary 3. Let $S$ be a semigroup and $I$ an ideal of $S$. Then:

1) if $I$ is weakly prime and $S \backslash I \neq \emptyset$, then $S \backslash I$ is an $m$-system,
2) if $S \backslash I$ is an $m$-system, then $I$ is weakly prime.

Corollary 4. Let $S$ be a semigroup and $I$ an ideal of $S$. Then:

1) if $I$ is weakly semiprime and $S \backslash I \neq \emptyset$, then $S \backslash I$ is an $n$-system,
2) if $S \backslash I$ is an $n$-system, then $I$ is weakly semiprime.

In the rest of this note we prove that, in ordered semigroups, each $n$ system containing an element $a$, contains an $m$-system which contains the same element. As a consequence, the same result is true for semigroups without order, as well (Corollary 5). This interesting result has been first noticed by R. D. Giri and A. K. Wazalwar in [1] extending the corresponding known result of rings.

Proposition 3. Let $S$ be an ordered semigroup. If $N$ is an $n$-system of $S$ and $a \in N$, then there exists an $m$-system $M$ of $S$ such that $a \in M \subseteq N$.

Proof. Since $N$ is an $n$-system and $a \in N$, there exists $c_{1} \in N$ such that $c_{1} \in(a S a]$, then $(a S a] \cap N \neq \emptyset$. Take $a_{1} \in(a S a] \cap N$. Since $N$ is an $n$-system and $a_{1} \in N$, there exists $c_{2} \in N$ such that $c_{2} \in\left(a_{1} S a_{1}\right]$, then $\left(a_{1} S a_{1}\right] \cap N \neq \emptyset$. Take $a_{2} \in\left(a_{1} S a_{1}\right] \cap N$. We continue this way. Take $a_{i} \in\left(a_{i-1} S a_{i-1}\right] \cap N$. Since $N$ is an $n$-system and $a_{i} \in N$, there exists $c_{i+1} \in N$ such that $c_{i+1} \in\left(a_{i} S a_{i}\right]$, then $\left(a_{i} S a_{i}\right] \cap N \neq \emptyset$.
We put $a_{0}=a, \quad M=\left\{a_{0}, a_{1}, a_{2}, \ldots, i, a_{i+1}, \ldots.\right\}$. We have $a_{0} \in M$ and $a_{n} \in N$ for all $n=0,1,2, \ldots, i, \ldots$, so $a \in M \subseteq N$.
The set $M$ is an $m$-system. Indeed: $\emptyset \neq M \subseteq S(a \in M)$.Let $a_{i}, a_{j} \in M$.
If $i=j$ then, for the element $a_{i+1} \in S$, we have $a_{i+1} \in\left(a_{i} S a_{i}\right]=\left(a_{i} S a_{j}\right]$. If $i<j$ then, for the element $a_{j+1} \in S$, we have

$$
a_{j+1} \in\left(a_{j} S a_{j}\right] \subseteq\left(\left(a_{j-1} S a_{j-1}\right] S a_{j}\right] \subseteq\left(a_{j-1} S a_{j}\right] \subseteq \ldots \subseteq\left(a_{i} S a_{j}\right]
$$

If $j<i$ then, for the element $a_{i+1} \in S$, we have

$$
a_{i+1} \in\left(a_{i} S a_{i}\right] \subseteq\left(a_{i} S\left(a_{i-1} S a_{i-1}\right)\right] \subseteq\left(a_{i} S a_{i-1}\right] \subseteq \ldots \subseteq\left(a_{i} S a_{j}\right]
$$

which completes the proof.
Corollary 5. Let $S$ be a semigroup. If $N$ is an n-system of $S$ and $a \in N$, then there exists an m-system $M$ of $S$ such that $a \in M \subseteq N$.

Acknowledgement. I would like to express my warmest thanks to Professor Wieslaw A. Dudek for his interest in my work and to the referee for his useful comments.

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# Fuzzy congruences on groups 

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#### Abstract

In this paper we define a fuzzy congruence on a group which is a new notion and consider their fundamental properties. We show that there is a lattice isomorphism between the set of fuzzy normal subgroups of a group and the set of fuzzy congruences on this group.


## 1. Introduction

While there are many papers about fuzzy group theory, we can not find papers about fuzzy congruences on groups. In the theory of crisp group theory, there exist close relationships between normal subgroups and congruences. It is a natural question to extend these relationships to the case of fuzzy group theory. In this paper we define fuzzy congruences on groups and fuzzy quotient groups by fuzzy congruences and investigate their properties. In [4], Rosenfeld defined fuzzy subgroupoids and proved that a homomorphic image of a fuzzy subgroupoid with the sup property was a fuzzy subgroupoid, and hence that a homomorphic image of a fuzzy subgroup with the sup property was a fuzzy subgroup. This theorem needs the sup property. But in this paper we can show the theorem without the sup property, that is, a homomorphic image of a fuzzy subgroup is a fuzzy subgroup. And in [3], Mukherjee and Bhattacharya showed that if $\bar{A}$ is a fuzzy subgroup of a finite group $G$ is such that all the level subgroups of $G$ are normal subgroups then $\bar{A}$ is a fuzzy normal subgroup. We can also prove the theorem without finiteness using the transfer principle which is a fundamental tool we have developped here.

In this paper we show that

[^2]1. The lattice $F N S(G)$ of all fuzzy normal subgroups of a group $G$ is isomorphic to the lattice $F C o n(G)$ of all fuzzy congruences on $G$;
2. $F N S_{\alpha}(G)$ forms a modular lattice for every $\alpha \in[0,1]$;
3. Let $G$ and $G^{\prime}$ be groups and $f: G \rightarrow G^{\prime}$ be a homomorphism. If $\bar{A}$ is a fuzzy (normal) subgroup of $G$ then $f[\bar{A}]$ is a fuzzy (normal) subgroup of $G^{\prime}$;
4. Let $G, G^{\prime}$ and $f$ be as above. If $\bar{A}$ is a fuzzy normal subgroup of $G^{\prime}$ then $G / f^{-1}(\bar{A}) \cong f(G) / \bar{A}$.

Of course, these results are not completely new, but these are obtained by the crisp group theory with the so-called transfer principle. This means that the transfer principle is a very important tool to investigate the fuzzy theory.

## 2. Fuzzy groups

Let $G$ be any group. By a fuzzy subgroup $\bar{A}$ of $G$ we mean a function $\bar{A}: G \rightarrow[0,1]$ such that

$$
\bar{A}\left(x y^{-1}\right) \geqslant \bar{A}(x) \wedge \bar{A}(y)=\min \{\bar{A}(x), \bar{A}(y)\}
$$

for all $x, y \in G$. Moreover a fuzzy normal subgroup $\bar{A}$ of $G$ is defined as a fuzzy subgroup satisfying the condition

$$
\bar{A}(x y) \geqslant \bar{A}(y x) .
$$

For the sake of simplicity, we denote by $F S(G)(F N S(G))$ the class of all fuzzy (normal) subgroups of $G$.

For every fuzzy subgroup $\bar{A}$ of $G$ we have the following (see [3, 4]):
Proposition 1. Let $G$ be a group with the unit element e and $\bar{A}$ be a fuzzy subgroup of $G$. For all $x, y \in G$,

1. $\bar{A}(x) \leqslant \bar{A}(e)$,
2. $\bar{A}(x)=\bar{A}\left(x^{-1}\right)$,
3. $\bar{A}\left(x y^{-1}\right)=\bar{A}(e)$ implies $\bar{A}(x)=\bar{A}(y)$.

As to the converse problem whether $\bar{A}(x)=\bar{A}(y)$ implies $\bar{A}\left(x y^{-1}\right)=$ $\bar{A}(e)$, we have a counter-example. Let $G=\{e, a, b, c\}$ be a Klein's group defined by the following table.

|  | $e$ | $a$ | $b$ | $c$ |
| :---: | :---: | :---: | :---: | :---: |
| $e$ | $e$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $e$ | $c$ | $b$ |
| $b$ | $b$ | $c$ | $e$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $e$ |

Let $\bar{A}$ be a map from $G$ to $[0,1]$ as $\bar{A}(e)=1, \bar{A}(a)=0.5, \bar{A}(b)=$ $\bar{A}(c)=0$. We see that $\bar{A}$ is a fuzzy normal subgroup of $G$. Then, while $\bar{A}(b)=\bar{A}(c)=0$, we have $\bar{A}\left(b c^{-1}\right)=\bar{A}(b c)=\bar{A}(a)=0.5 \neq 1=\bar{A}(e)$. Thus the converse problem above does not hold.

## 3. Transfer principle

To express the transfer principle in group theory exactly we shall first of all define several terms in the theory of groups.

Let $G$ be any group, $V$ be a countable set $\{x, y, z, \ldots\}$ of syntactic variables which range over the elements of $G$. A term of $G$ is defined recursively:
(1) The unit element $e \in G$ is a term;
(2) Each variable of $V$ is a term;
(3) If $s$ and $t$ are terms, then st is a term.

Thus, for example, $x, e x$, and $x(e y)$ are terms of $G$. We use the symbols $a, b, c, \ldots$ as elements of $G$ and $x, y, z, \ldots$ as variables of $V$. We have to distinguish the elements of $G$ and the terms of $G$.

Let $A$ be a subset of $G$ which satisfies the following property $\mathcal{P}$ expressed by a first-order formula:

$$
\mathcal{P}: \forall x \ldots \forall y\left(t_{1}(x, \ldots, y) \in A \wedge \ldots \wedge t_{n}(x, \ldots, y) \in A \rightarrow t(x, \ldots, y) \in A\right)
$$

where $t_{1}(x, \ldots, y), \ldots, t_{n}(x, \ldots, y)$, and $t(x, \ldots, y)$ are terms of $G$ constructed by variables $x, \ldots, y$. We say that the subset $A$ satisfies the property $\mathcal{P}$ if, for all elements $a, \ldots, b \in G, t(a, \ldots, b) \in A$ whenever $t_{1}(a, \ldots, b), \ldots, t_{n}(a, \ldots, b) \in A$. For the subset $A$ which satisfies $\mathcal{P}$ we define a fuzzy subset $\bar{A}$ which satisfies the following property $\overline{\mathcal{P}}$ :

$$
\overline{\mathcal{P}}: \forall x \ldots \forall y\left(\bar{A}(t(x, \ldots, y)) \geqslant \bar{A}\left(t_{1}(x, \ldots, y)\right) \wedge \ldots \wedge \bar{A}\left(t_{n}(x, \ldots, y)\right)\right)
$$

For any $\alpha \in[0,1]$, we put

$$
U(\bar{A}: \alpha)=\{x \in G \mid \bar{A}(x) \geqslant \alpha\} .
$$

The closed interval $[0,1]$ has a lattice structure with the usual order, so we have $\alpha \wedge \beta=\min \{\alpha, \beta\}$ for $\alpha, \beta \in[0,1]$.

Now the following is established. We call the theorem the transfer principle.
Theorem 1. (Transfer principle) $A$ fuzzy subset $\bar{A}$ satisfies property $\overline{\mathcal{P}}$ if and only if for any $\alpha \in[0,1]$ if $U(\bar{A}: \alpha) \neq \emptyset$ then the crisp set $U(\bar{A}: \alpha)$ satisfies the property $\mathcal{P}$.

We simply denote this result by

$$
\bar{A}: \overline{\mathcal{P}} \Longleftrightarrow \forall \alpha(U(\bar{A}: \alpha) \neq \emptyset \Longrightarrow U(\bar{A}: \alpha): \mathcal{P})
$$

Proof. $\Rightarrow$ : Suppose that $\bar{A}$ satisfies $\overline{\mathcal{P}}$. If there is $\alpha \in[0,1]$ such that $U(\bar{A}: \alpha) \neq \emptyset$ but $U(\bar{A}: \alpha)$ does not satisfy $\mathcal{P}$, then we have $t_{1}(a, \ldots, b) \in$ $U(\bar{A}: \alpha), \ldots, t_{n}(a, \ldots, b) \in U(\bar{A}: \alpha)$ but $t(a, \ldots, b) \notin U(\bar{A}: \alpha)$ for some $a, \ldots, b \in G$. Since $\bar{A}\left(t_{1}(a, \ldots, b)\right) \geqslant \alpha, \ldots, \bar{A}\left(t_{n}(a, \ldots, b)\right) \geqslant \alpha$ but $\bar{A}(t(a, \ldots, b)) \ngtr \alpha$, it follows that

$$
\bar{A}(t(a, \ldots, b)) \not \equiv \bar{A}\left(t_{1}(a, \ldots, b)\right) \wedge \ldots \wedge \bar{A}\left(t_{n}(a, \ldots, b)\right)
$$

for some $a, \ldots, b \in G$. This means that $\bar{A}$ does not satisfy $\overline{\mathcal{P}}$. This is a contradiction.
$\Leftarrow$ : Suppose that $\bar{A}$ does not satisfy $\overline{\mathcal{P}}$. Since

$$
\bar{A}(t(a, \ldots, b)) \nsupseteq \bar{A}\left(t_{1}(a, \ldots, b)\right) \wedge \ldots \wedge \bar{A}\left(t_{n}(a, \ldots, b)\right)
$$

for some $a, \ldots, b \in G$, if we put $\alpha=\bigwedge_{i} \bar{A}\left(t_{i}(a, \ldots, b)\right)$ then we have $\alpha \in[0,1]$ and $U(\bar{A}: \alpha) \neq \emptyset$ because $t_{i}(a, \ldots, b) \in U(\bar{A}: \alpha)$. On the other hand, in this case we have $t(a, \ldots, b) \notin U(\bar{A}: \alpha)$. This means that $U(\bar{A}: \alpha)$ does not satisfy $\mathcal{P}$.

The above implies that if we define a fuzzy subset $\bar{A}$ to have the property $\overline{\mathcal{P}}$ whenever a crisp subset $A$ satisfies $\mathcal{P}$ then the transfer principle holds generally.

Conversely we can show that we have to define a fuzzy subset $\bar{A}$ as in the form $\overline{\mathcal{P}}$ if the transfer principle is too hold.
Theorem 2. If the transfer principle holds for a subset $A$ with the property $\mathcal{P}$, then the fuzzy subset $\bar{A}$ has the property $\overline{\mathcal{P}}$.
Proof. Suppose that the transfer principle holds for $A$ but the fuzzy subset $\bar{A}$ does not have the property $\overline{\mathcal{P}}$. Thus there exist $a, \ldots, b \in G$ such that

$$
\bar{A}(t(a, \ldots, b)) \nsupseteq \bar{A}\left(t_{1}(a, \ldots, b)\right) \wedge \ldots \wedge \bar{A}\left(t_{n}(a, \ldots, b)\right) .
$$

We take $\alpha=\bigwedge_{i} \bar{A}\left(t_{i}(a, \ldots, b)\right)$. It is clear that $\alpha \in[0,1]$ and $U(\bar{A}: \alpha) \neq \emptyset$ because of

$$
\bar{A}\left(t_{i}(a, \ldots, b)\right) \geqslant \alpha
$$

Since $\bar{A}(t(a, \ldots, b)) \not \ni \alpha$, we have $t(a, \ldots, b) \notin U(\bar{A}: \alpha)$ but $t_{i}(a, \ldots, b) \in$ $U(\bar{A}: \alpha)$. This means that $U(\bar{A}: \alpha)$ does not satisfy the property $\mathcal{P}$. This is a contradiction.

Since any concept defined for groups so far has the forms of $\mathcal{P}$ and the corresponding fuzzy subsets are defined by the form of $\overline{\mathcal{P}}$, it is easy to get the relation between a crisp set $A$ with $\mathcal{P}$ and its fuzzy set $\bar{A}$ with $\overline{\mathcal{P}}$. For example, a non-empty subset $A$ of $G$ is called a subgroup if $x \in A$ and $y \in A$ then $x y^{-1} \in A$. So if we define a fuzzy subgroup $\bar{A}$ as $\bar{A}\left(x y^{-1}\right) \geqslant$ $\bar{A}(x) \wedge \bar{A}(y)$ then the transfer principle holds generally. Thus from the above we can verify the next theorem immediately. The theorem is proved already Theorem 3.9 in [3] under the restriction of finiteness of the group $G$. We note that this restriction is not needed to get the theorem.
Theorem 3. $\bar{A}$ is a fuzzy (normal) subgroup of $G$ if and only if for all $\alpha$ $U(\bar{A}: \alpha) \neq \emptyset \rightarrow U(\bar{A}: \alpha)$ is a (normal) subgroup of $G$.

Let $G, G^{\prime}$ be groups and $f: G \rightarrow G^{\prime}$ be a homomorphism, that is, $f(x y)=f(x) f(y)$. For any fuzzy subgroup $\bar{A}$ of $G^{\prime}$, we define a map $f^{-1}(\bar{A})$ from $G$ to $[0,1]$ by

$$
f^{-1}(\bar{A})(x)=\bar{A}(f(x))
$$

for all $x \in G$. We call $f^{-1}(\bar{A})$ a preimage of fuzzy subgroup $\bar{A}$ under $f$. For any fuzzy subgroup $\bar{A}$ of $G$ we define an image $f[\bar{A}]$ of $\bar{A}$ under $f$ by

$$
f[\bar{A}](y)=\bigvee_{u \in f^{-1}(y)} \bar{A}(u)
$$

for all $y \in G^{\prime}$. It follows from definition that
Proposition 2. If $f: G \rightarrow G^{\prime}$ is a homomorphism from $G$ to $G^{\prime}$ and $\bar{A}$ is a fuzzy (normal) subgroup of $G^{\prime}$, then the preimage $f^{-1}(\bar{A})$ is a fuzzy (normal) subgroup of $G$.

Considering the image $f[\bar{A}]$ of the fuzzy (normal) subgroup $\bar{A}$ of $G$ under $f$, we have the following theorem by use of the transfer principle. The following result is proved in Proposition 4.2 [4] under the restriction of the sup property:

A homomorphic image of a fuzzy subgroupoid which has the sup property is a fuzzy subgroupoid.

A fuzzy set $\bar{A}$ of $G$ has the sup property if for any subset $S \subseteq G$ there exists $s_{0} \in S$ such that

$$
\bar{A}\left(s_{0}\right)=\bigvee_{s \in S} \bar{A}(s)
$$

We note here that the restriction is not essential, that is, we can prove the theorem without such a restriction. To prove the theorem we need a lemma. Let $G, G^{\prime}$ be groups and $f: G \rightarrow G^{\prime}$ be a homomorphism.

Lemma 1. For all $\alpha \in[0,1], U(f[\bar{A}]: \alpha)=\bigcap_{\epsilon>0} f(U(\bar{A}: \alpha-\epsilon))$.
Proof.

$$
\begin{aligned}
y \in U(f[\bar{A}]: \alpha)) & \Longleftrightarrow f[\bar{A}](y) \geqslant \alpha \\
& \Longleftrightarrow \bigvee_{u \in f^{-1}(y)} \bar{A}(u) \geqslant \alpha \\
& \Longleftrightarrow \forall \epsilon>0 \exists u \in f^{-1}(y) \text { s.t. } \bar{A}(u) \geqslant \alpha-\epsilon \\
& \Longleftrightarrow \forall \epsilon>0 \exists u \in f^{-1}(y) \text { s.t. } u \in U(\bar{A}: \alpha-\epsilon) \\
& \Longleftrightarrow \forall \epsilon>0 y=f(u) \in f(U(\bar{A}: \alpha-\epsilon)) \\
& \Longleftrightarrow y \in \bigcap_{\epsilon>0} f(U(\bar{A}: \alpha-\epsilon))
\end{aligned}
$$

Using this lemma we can prove the next theorem.
Theorem 4. Let $f: G \rightarrow G^{\prime}$ be a surjective homomorphism and $\bar{A}$ be a fuzzy (normal) subgroup of $G$, then the image $f[\bar{A}]$ is a (normal) subgroup of $G^{\prime}$.

Proof. It follows from the transfer principle that
$f[\bar{A}]$ is a fuzzy (normal) subgroup of $G^{\prime}$ if and only if $\forall \alpha$
$U(f[\bar{A}]: \alpha) \neq \emptyset \rightarrow U(f[\bar{A}]: \alpha)$ is a (normal) subgroup of $G^{\prime}$.
It is sufficient to show that $U(f[\bar{A}]: \alpha)$ is a (normal) subgroup of $G^{\prime}$ if $U(f[\bar{A}]: \alpha) \neq \emptyset$. Since $\bar{A}$ is a fuzzy (normal) subgroup, it follows from the transfer principle that $U(\bar{A}: \alpha-\epsilon)$ is also a (normal) subgroup of $G$ for all $\epsilon>0$ if $U(\bar{A}: \alpha-\epsilon) \neq \emptyset$. From $f$ being surjective, we have $f(U(\bar{A}: \alpha-\epsilon))$ is a (normal) subgroup of $G^{\prime}$ for all $\epsilon>0$ by well-known results about crisp group theory. Hence $\bigcap_{\epsilon>0} f(U(\bar{A}: \alpha-\epsilon))=U(f[\bar{A}]: \alpha)$ is the (normal) subgroup of $G^{\prime}$.

It is well known that the class of all normal subgroups of $G$ forms a modular lattice. Similarly we may expect that the class $F N S(G)$ of all fuzzy normal subgroups of $G$ is a modular lattice. In the following, we shall show that $F N S(G)_{\alpha}$ forms a modular lattice for every $\alpha \in[0,1]$. Before doing, we define an order $\leqslant$ on $\operatorname{FNS}(G)$ by

$$
\bar{A} \leqslant \bar{B} \Longleftrightarrow \bar{A}(x) \leqslant \bar{B}(x)
$$

for all $x \in G$.
A fuzzy subset $\bar{A} \wedge \bar{B}$ of $G$ is defined by

$$
(\bar{A} \wedge \bar{B})(x)=\bar{A}(x) \wedge \bar{B}(x)
$$

for $x \in G$. It is obvious that $\bar{A} \wedge \bar{B} \in F N S(G)$ and

$$
\bar{A} \wedge \bar{B}=\inf \{\bar{A}, \bar{B}\}
$$

with respect to the order.
As for $\sup \{\bar{A}, \bar{B}\}$, we consider a fuzzy subset $\bar{A} \bar{B}$ defined by

$$
(\bar{A} \bar{B})(x)=\bigvee_{a b=x}(\bar{A}(a) \wedge \bar{B}(b)) .
$$

For this fuzzy subset $\bar{A} \bar{B}$, it follows from definition that

$$
\begin{equation*}
\bar{A} \bar{B}(x)=\bigvee_{a} \bar{A}(a) \wedge \bar{B}\left(a^{-1} x\right)=\bigvee_{b} \bar{A}\left(x b^{-1}\right) \wedge \bar{B}(b) \tag{*}
\end{equation*}
$$

First of all we show that $\bar{A} \bar{B} \in F N S(G)$ for $\bar{A}, \bar{B} \in F N S(G)$ by using the transfer principle.

Lemma 2. For $\bar{A}, \bar{B} \in F N S(G)$, we have $\bar{A} \bar{B} \in F N S(G)$
Proof. By transfer principle we have that $\bar{A} \bar{B}$ is a fuzzy normal subgroup if and only if for all $\alpha U(\bar{A} \bar{B}: \alpha) \neq \emptyset \rightarrow U(\bar{A} \bar{B}: \alpha)$ is a normal subgroup.

It is sufficient to show that $U(\bar{A} \bar{B}: \alpha)$ is a normal subgroup of $G$ provided that $U(\bar{A} \bar{B}: \alpha) \neq \emptyset$. Let $\alpha$ be such that $U(\bar{A} \bar{B}: \alpha) \neq \emptyset$ and $x \in U(\bar{A} \bar{B}: \alpha)$. For every $y \in G$ we have to show that $y x y^{-1} \in U(\bar{A} \bar{B}: \alpha)$.

Since

$$
\begin{aligned}
\bar{A} \bar{B}\left(y x y^{-1}\right) & =\bigvee_{u} \bar{A}(u) \wedge \bar{B}\left(u^{-1} y x y^{-1}\right) \\
& =\bigvee_{v} \bar{A}\left(v y^{-1}\right) \wedge \bar{B}\left(y v^{-1} y x y^{-1}\right) \quad(\text { by }(*)) \\
& =\bigvee_{v} \bar{A}\left(y^{-1} v\right) \wedge \bar{B}\left(v^{-1} y x y^{-1} y\right) \\
& =\bigvee_{v} \bar{A}\left(y^{-1} v\right) \wedge \bar{B}\left(v^{-1} y x\right) \\
& =\bigvee_{v} \bar{A}\left(y^{-1} v\right) \wedge \bar{B}\left(\left(y^{-1} v\right)^{-1} x\right) \\
& =\bar{A} \bar{B}(x) \geqslant \alpha
\end{aligned}
$$

we have $y x y^{-1} \in U(\bar{A} \bar{B}: \alpha)$, that is, $U(\bar{A} \bar{B}: \alpha)$ is the normal subgroup of $G$ if $U(\bar{A} \bar{B}: \alpha) \neq \emptyset$. So we can conclude that $\bar{A} \bar{B} \in F N S(G)$.

As to $\bar{A} \bar{B}$ we also have
Lemma 3. $\bar{A} \bar{B}=\sup \{\bar{A}, \bar{B}\}$ for $\bar{A}, \bar{B} \in F N S(G)$ such that $\bar{A}(e)=\bar{B}(e)$.
Proof. It is easy to show that $\bar{A}, \bar{B} \leqslant \bar{A} \bar{B}$. Let $\bar{C}$ be any fuzzy normal subgroup such that $\bar{A}, \bar{B} \leqslant \bar{C}$. For every $x \in G$, since

$$
\begin{aligned}
\bar{A} \bar{B}(x) & =\bigvee_{u} \bar{A}(u) \wedge \bar{B}\left(u^{-1} x\right) \\
& \leqslant \bigvee_{u} \bar{C}(u) \wedge \bar{C}\left(u^{-1} x\right) \\
& \leqslant \bigvee_{u} \bar{C}\left(u u^{-1} x\right)=\bar{C}(x)
\end{aligned}
$$

it follows that $\bar{A} \bar{B}=\sup \{\bar{A}, \bar{B}\}$ when $\bar{A}(e)=\bar{B}(e)$.
By the above we denote $\sup \{\bar{A}, \bar{B}\}$ by $\bar{A} \vee \bar{B}$.
Let $\alpha$ be an element of $[0,1]$ and $F N S_{\alpha}(G)$ be the class of all fuzzy normal subgroups of $G$ satisfying $\bar{A}(e)=\alpha$ for all $\bar{A} \in F N S(G)$, that is, $F N S_{\alpha}(G)=\{\bar{A} \in F N S(G) \mid \bar{A}(e)=\alpha\}$. We can prove that

Theorem 5. $F N S_{\alpha}(G)$ forms a modular lattice for any $\alpha \in[0,1]$.
Proof. Let $\bar{A}, \bar{B}, \bar{C} \in F N S_{\alpha}(G)$ and $\bar{A} \leqslant \bar{C}$. It is sufficient to prove that
$(\bar{A} \vee \bar{B}) \wedge \bar{C} \leqslant \bar{A} \vee(\bar{B} \wedge \bar{C})$. For all $x \in G$, since

$$
\begin{aligned}
((\bar{A} \vee \bar{B}) \wedge \bar{C})(x) & =\left(\bigvee_{u}\left(\bar{A}(u) \wedge \bar{B}\left(u^{-1} x\right)\right)\right) \wedge \bar{C}(x) \\
& =\bigvee_{u}\left(\bar{A}(u) \wedge \bar{B}\left(u^{-1} x\right) \wedge \bar{C}(x)\right) \\
& =\bigvee_{u}\left(\bar{A}(u) \wedge \bar{C}(u) \wedge \bar{B}\left(u^{-1} x\right) \wedge \bar{C}(x)\right) \\
& =\bigvee_{u}\left(\bar{A}(u) \wedge \bar{B}\left(u^{-1} x\right) \wedge \bar{C}(u) \wedge \bar{C}(x)\right) \\
& \leqslant \bigvee_{u}\left(\bar{A}(u) \wedge \bar{B}\left(u^{-1} x\right) \wedge \bar{C}\left(u^{-1} x\right)\right) \\
& =(\bar{A} \vee(\bar{B} \wedge \bar{C}))(x),
\end{aligned}
$$

it follows that if $\bar{A} \leqslant \bar{C}$ then $(\bar{A} \vee \bar{B}) \wedge \bar{C} \leqslant \bar{A} \vee(\bar{B} \wedge \bar{C})$. This means that $F N S_{\alpha}(G)$ is a modular lattice.

## 4. Fuzzy congruences

We define a fuzzy congruence on an arbitrary group $G$. A binary function $\theta$ from $G \times G$ to $[0,1]$ is called a fuzzy congruence on $G$ if for all elements $x, y, z, u \in G$ it satisfies the conditions:

1. $\bar{\theta}(e, e)=\bar{\theta}(x, x)$,
2. $\bar{\theta}(x, y)=\bar{\theta}(y, x)$,
3. $\bar{\theta}(x, z) \geqslant \bar{\theta}(x, y) \wedge \bar{\theta}(y, z)$,
4. $\bar{\theta}(x u, y u), \bar{\theta}(u x, u y) \geqslant \bar{\theta}(x, y)$.

Lemma 4. If $\bar{\theta}$ satisfies the conditions (2), (3) and (4) above, then (1) is equivalent to (1)' $\bar{\theta}(e, e) \geqslant \bar{\theta}(x, y)$ for all $x, y \in G$.
Proof. Suppose that $\bar{\theta}(e, e)=\bar{\theta}(x, x)$. Since $\bar{\theta}$ satisfies the conditions (2) and (3), we have $\bar{\theta}(e, e)=\bar{\theta}(x, x) \geqslant \bar{\theta}(x, y) \wedge \bar{\theta}(y, x)=\bar{\theta}(x, y)$.

Conversely, from (4) we have $\bar{\theta}(e, e) \leqslant \bar{\theta}(x e, x e)=\bar{\theta}(x, x)$.
Proposition 3. If $\bar{\theta}$ is a fuzzy congruence on $G$, then $\bar{\theta}(x, y)=\bar{\theta}\left(x y^{-1}, e\right)$ for all $x, y \in G$.
Proof. $\bar{\theta}(x, y) \leqslant \bar{\theta}\left(x y^{-1}, y y^{-1}\right)=\bar{\theta}\left(x y^{-1}, e\right) \leqslant \bar{\theta}\left(x y^{-1} y, e y\right)=\bar{\theta}(x, y)$. Hence $\bar{\theta}(x, y)=\bar{\theta}\left(x y^{-1}, e\right)$

For every element $a \in G$, we define a subset

$$
a / \bar{\theta}=\{b \in G \mid \bar{\theta}(a, b)=\bar{\theta}(e, e)\}
$$

of $G$ and $G / \bar{\theta}=\{a / \bar{\theta} \mid a \in G\}$. We also define an operator "." on the set $\{a / \bar{\theta} \mid a \in G\}$ by

$$
a / \bar{\theta} \cdot b / \bar{\theta}=(a b) / \bar{\theta} .
$$

This operator is well-defined. Because, if $a / \bar{\theta}=a^{\prime} / \bar{\theta}$ and $b / \bar{\theta}=b^{\prime} / \bar{\theta}$, then we have $\bar{\theta}\left(a, a^{\prime}\right)=\bar{\theta}\left(b, b^{\prime}\right)=\bar{\theta}(e, e)$. Since $\bar{\theta}(e, e)=\bar{\theta}\left(a, a^{\prime}\right) \leqslant \bar{\theta}\left(a b, a^{\prime} b\right)$ and $\bar{\theta}(e, e)=\bar{\theta}\left(b, b^{\prime}\right) \leqslant \bar{\theta}\left(a^{\prime} b, a^{\prime} b^{\prime}\right)$, we have $\bar{\theta}(e, e) \leqslant \bar{\theta}\left(a b, a^{\prime} b\right) \wedge \bar{\theta}\left(a^{\prime} b, a^{\prime} b^{\prime}\right) \leqslant$ $\bar{\theta}\left(a b, a^{\prime} b^{\prime}\right) \leqslant \bar{\theta}(e, e)$. This means that $\bar{\theta}\left(a b, a^{\prime} b^{\prime}\right)=\bar{\theta}(e, e)$ and $a b / \bar{\theta}=a^{\prime} b^{\prime} / \bar{\theta}$ hence that the operator is well-defined. It is easy to show that $G / \bar{\theta}$ forms a group with respect to this operator. So we omit its proof.

Proposition 4. If $\bar{\theta}$ is a fuzzy congruence on $G$, then $G / \bar{\theta}$ is a group with the unit element $e / \bar{\theta}$.

Proposition 5. If $\bar{A}$ is a fuzzy normal subgroup of $G$, then the fuzzy relation $\theta_{\bar{A}}(x, y)$ defined by $\theta_{\bar{A}}(x, y)=\bar{A}\left(x y^{-1}\right)$ is a fuzzy congruence.

Proof. We only show that $\theta_{\bar{A}}$ satisfies the conditions (3) and (4). For the case of (3), we have

$$
\begin{aligned}
\theta_{\bar{A}}(x, z)=\bar{A}\left(x z^{-1}\right) & =\bar{A}\left(x y^{-1} y z^{-1}\right) \\
& \geqslant \bar{A}\left(x y^{-1}\right) \wedge \bar{A}\left(y z^{-1}\right)=\theta_{\bar{A}}(x, y) \wedge \theta_{\bar{A}}(y, z)
\end{aligned}
$$

For the case of (4), it follows that

$$
\begin{aligned}
\bar{\theta}(x u, y u) & =\bar{A}\left((x u)(y u)^{-1}\right)=\bar{A}\left((x u)\left(u^{-1} y^{-1}\right)\right) \\
& =\bar{A}\left(x y^{-1}\right)=\theta_{\bar{A}}(x, y)
\end{aligned}
$$

The case of $\theta_{\bar{A}}(u x, u y) \geqslant \theta_{\bar{A}}(x, y)$ is similar.
Conversely,
Proposition 6. If $\bar{\theta}$ is a fuzzy congruence, then the function $A_{\bar{\theta}}$ from $G$ to $[0,1]$ defined by $A_{\bar{\theta}}(x)=\bar{\theta}(x, e)$ is a fuzzy normal subgroup of $G$.

Proof. We have $A_{\bar{\theta}}(x) \wedge A_{\bar{\theta}}(y)=\bar{\theta}(x, e) \wedge \bar{\theta}(y, e)=\bar{\theta}(x, e) \wedge \bar{\theta}(e, y) \leqslant$ $\bar{\theta}(x, y)=\bar{\theta}\left(x y^{-1}, e\right)=A_{\bar{\theta}}\left(x y^{-1}\right)$. Thus $A_{\bar{\theta}}$ is a fuzzy subgroup of $G$.

Moreover,

$$
\begin{aligned}
A_{\bar{\theta}}(x y)=\bar{\theta}(x y, e) & \leqslant \bar{\theta}\left(x y y^{-1}, e y^{-1}\right)=\bar{\theta}\left(x, y^{-1}\right) \\
& \leqslant \bar{\theta}\left(x^{-1} x, x^{-1} y\right)=\bar{\theta}\left(e, x^{-1} y^{-1}\right) \\
& \leqslant \bar{\theta}\left(e y, x^{-1} y^{-1} y\right)=\bar{\theta}\left(y, x^{-1}\right) \\
& \leqslant \bar{\theta}\left(y x, x^{-1} x\right)=\bar{\theta}(y x, e)=A_{\bar{\theta}}(y x)
\end{aligned}
$$

Hence $A_{\bar{\theta}}$ is a fuzzy normal subgroup of $G$.
It is natural to ask whether there is a one-to-one correspondence between the class $F N S(G)$ of all fuzzy normal subgroups of $G$ and the class $F C o n(G)$ of all fuzzy congruences on $G$. We can answer the question "Yes". We see that both sets are (complete) lattices with respect to the order of set inclusion.

Theorem 6. $F N S(G) \cong \operatorname{FCon}(G)$ as lattices. In particular, $\bar{A}=A_{\theta_{\bar{A}}}$ and $\bar{\theta}=\theta_{A_{\bar{\theta}}}$.

Proof. It is easy to see that the map $\xi: F N S(G) \rightarrow F \operatorname{Con}(G)$ defined by $\xi(\bar{A})=\theta_{\bar{A}}$ is an isomorphism.

## 5. Homomorphism theorem

Since $\theta_{\bar{A}}$ is a fuzzy congruence when $\bar{A}$ is a fuzzy normal subgroup of $G$, we conclude that $G / \theta_{\bar{A}}$ is a group. We denote the group simply by $G / \bar{A}$ and call it a fuzzy quotient group induced by a fuzzy normal subgroup $\bar{A}$.

Let $G, G^{\prime}$ be groups and $f$ be a homomorphism from $G$ to $G^{\prime}$. If $\bar{A}$ is a fuzzy normal subgroup of $G^{\prime}$, then the map $f^{-1}(\bar{A})$ defined by

$$
f^{-1}(\bar{A})(x)=\bar{A}(f(x))
$$

for all $x \in G$ is a fuzzy normal subgroup of $G$ as proved above. Thus $G / f^{-1}(\bar{A})$ and $f(G) / \bar{A}$ are groups. In this case we can show the following result which is an extension of homomorphism theorem.
Theorem 7. Let $G, G^{\prime}$ be groups, $f$ a homomorphism, and $\bar{A}$ a fuzzy normal subgroup of $G^{\prime}$. Then there is an isomorphism from $G / f^{-1}(\bar{A})$ onto $f(G) / \bar{A}$, that is,

$$
G / f^{-1}(\bar{A}) \cong f(G) / \bar{A} .
$$

Proof. We define a map $h$ from $G / f^{-1}(\bar{A})$ to $f(G) / \bar{A}$ by

$$
h\left(x / f^{-1}(\bar{A})\right)=f(x) / \bar{A}
$$

for all $x \in G$. The map $h$ is well-defined. Because, if $x / f^{-1}(\bar{A})=$ $y / f^{-1}(\bar{A})$, since $f^{-1}(\bar{A})\left(x y^{-1}\right)=f^{-1}(\bar{A})(e)$ by definition, then we have $\bar{A}\left(f(x)(f(y))^{-1}\right)=\bar{A}(f(e))=\bar{A}\left(e^{\prime}\right)$, where $e^{\prime}$ is the unit element of $G^{\prime}$, and thus $f(x) / \bar{A}=f(y) / \bar{A}$. This implies that $h$ is well-defined.

For injectiveness of $h$, we suppose that $h\left(x / f^{-1}(\bar{A})\right)=h\left(y / f^{-1}(\bar{A})\right)$, that is, $f(x) / \bar{A}=f(y) / \bar{A}$. It follows from definition that $\bar{A}\left(f(x)(f(y))^{-1}\right)=$ $\bar{A}\left(e^{\prime}\right)$ and hence $f^{-1}(\bar{A})\left(x y^{-1}\right)=f^{-1}(\bar{A})(e)$. This means that $x / f^{-1}(\bar{A})=$ $y / f^{-1}(\bar{A})$. Therefore $h$ is injective.

It is easy to show that $h$ is a surjective homomorphism.
Thus we can conclude that $G / f^{-1}(\bar{A}) \cong f(G) / \bar{A}$.

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# Quotient groups induced by fuzzy subgroups 

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#### Abstract

We construct a quotient group induced by a fuzzy normal subgroup and prove the corresponding isomorphism theorems. Obtained results are used to the characterization of selected classes of quotient groups.


## 1. Introduction

In [16] L. A. Zadeh introduced the concept of fuzzy sets and fuzzy set operations. A. Rosenfeld [14] applied this concept to the theory of groupoids and groups. The various constructions of fuzzy quotient groups and fuzzy subgroup isomorphisms have been investigated by several researchers (see e.g. $[1,3,6,9,11,13])$. In this paper we give a new method of construction of quotient groups by fuzzy normal subgroups and apply this construction to the characterization of selected classes of quotient groups.

## 2. Preliminaries

A fuzzy subset of a group $G$, i.e. a function $\mu$ from $G$ into $[0,1]$, is called a fuzzy subgroup of $G$ if
$\left(\mathrm{F}_{1}\right) \quad \mu(x y) \geqslant \min \{\mu(x), \mu(y)\}$ for all $x, y \in G$, and
( $\left.\mathrm{F}_{2}\right) \quad \mu\left(x^{-1}\right) \geqslant \mu(x)$ for all $x \in G$,
or, equivalently, if $\mu\left(x y^{-1}\right) \geqslant \min \{\mu(x), \mu(y)\}$ for all $x, y \in G$.
A fuzzy subgroup $\mu$ of a group $G$ is called normal if for all $x, y \in G$ it satisfies one of the following equivalent conditions (cf. [15]):

2000 Mathematics Subject Classification: 20E10, 94D05
Keywords: Fuzzy (normal) subgroup, fuzzy isomorphism theorem
$\left(\mathrm{F}_{3}\right) \quad \mu\left(x y x^{-1}\right) \geqslant \mu(y)$,
$\left(\mathrm{F}_{4}\right) \mu\left(x y x^{-1}\right)=\mu(y)$,
$\left(\mathrm{F}_{5}\right) \quad \mu(x y)=\mu(y x)$.
It is not difficult to see that for all fuzzy subgroups $\mu$ of a group $G$ and all $x, y \in G$
(i) $\mu(e) \geqslant \mu(x)$,
(ii) $\mu\left(x^{-1}\right)=\mu(x)$,
(iii) $\mu\left(x y^{-1}\right)=\mu(e)$ implies $\mu(x)=\mu(y)$.

Fuzzy subgroups of $G$ can be characterized by the collection of levels, i.e. sets of the form $\mu_{t}=\{g \in G \mid \mu(g) \geqslant t\}$, where $t \in[0,1]$. Namely, as it is proved in [15], a fuzzy subset $\mu$ of a group $G$ is a fuzzy (normal) subgroup of $G$ if and only if for all $t \in[0,1], \mu_{t}$ is either empty or a (normal) subgroup of $G$.

The image $f(\eta)$ of a fuzzy subset $\eta$ of $G$ and preimage $f^{-1}(\mu)$ of a fuzzy subset $\mu$ of $G^{\prime}$ and a map $f: G \rightarrow G^{\prime}$ are defined as

$$
f(\eta)(y)= \begin{cases}\sup _{x \in f^{-1}(y)} \eta(x) & \text { if } f^{-1}(y) \neq \emptyset \\ 0 & \text { otherwise }\end{cases}
$$

and

$$
f^{-1}(\mu)(x)=\mu(f(x)), \quad x \in G .
$$

It is not difficult to see that $f(\eta)$ and $f^{-1}(\mu)$ are fuzzy subsets.

## 3. Quotient groups induced by fuzzy subgroups

Let $\mu$ be a fuzzy normal subgroup of a group $G$. For any $x, y \in G$, define a binary relation $\sim$ on $G$ by

$$
x \sim y \Longleftrightarrow \mu\left(x y^{-1}\right)=\mu(e)
$$

where $e$ is the unit of $G$.
Lemma 1. $\sim$ is a congruence of $G$.
Proof. The reflexivity and symmetry are obvious. To prove the transitivity let $x \sim y$ and $y \sim z$. Then $\mu\left(x y^{-1}\right)=\mu\left(y z^{-1}\right)=\mu(e)$ and $\mu\left(x z^{-1}\right)=$ $\mu\left(x y^{-1} y z^{-1}\right) \geqslant \min \left\{\mu\left(x y^{-1}\right), \mu\left(y z^{-1}\right)\right\}=\mu(e)$. Hence $\mu\left(x z^{-1}\right)=\mu(e)$, which proves that $\sim$ is an equivalence relation.

Now, if $x \sim y$, then $\mu\left(x y^{-1}\right)=\mu(e)$. Thus for all $z \in G$ we have $\mu\left((x z)(y z)^{-1}\right)=\mu\left(x z z^{-1} y^{-1}\right)=\mu\left(x y^{-1}\right)=\mu(e)$. Hence $x z \sim y z$. Since $\mu$ is a fuzzy normal subgroup, we have $\mu\left((z x)(z y)^{-1}\right)=\mu\left(z x y^{-1} z^{-1}\right)=$ $\mu\left(z^{-1} z x y^{-1}\right)=\mu\left(x y^{-1}\right)=\mu(e)$. This gives $z x \sim z y$.

Using these facts it is not difficult to see that $\sim$ is a congruence.
The equivalence class containing $x$ is denoted by $\mu_{x}$. $G / \mu$ denotes the corresponding quotient set.
Proposition 1. If $\mu$ is a fuzzy normal subgroup of a group $G$, then $G / \mu$ is a group with the operation $\mu_{x} \mu_{y}=\mu_{x y}$.
Example. Let $G$ be the additive group of all integers and let $\mu(x)=t_{1}$ if $2 \mid x$, and $\mu(x)=t_{0}$ if $2 \nmid x$, where $0 \leqslant t_{0}<t_{1} \leqslant 1$. Then $\mu$ is a fuzzy normal subgroup of $G$ and $G / \mu=\left\{\mu_{0}, \mu_{1}\right\}$ is a quotient group induced by $\mu$.
Lemma 2. [13] If $f: G \rightarrow G^{\prime}$ is an epimorphism of groups and $\mu$ a fuzzy normal subgroup of $G$, then $f(\mu)$ is a fuzzy normal subgroup of $G^{\prime}$.

Basing on this Lemma and Proposition 4.2 in [7] we can proved
Lemma 3. Let $f: G \rightarrow G^{\prime}$ be a homomorphism of groups, $\mu$ a fuzzy subgroup of $G$ and $\nu$ a fuzzy subgroup of $G^{\prime}$.
(i) If $f$ is an epimorphism, then $f\left(f^{-1}(\nu)\right)=\nu$.
(ii) If $\mu$ is a constant on kerf, then $f^{-1}(f(\mu))=\mu$.

Let $G_{\mu}=\mu_{\mu(0)}=\{x \in G \mid \mu(x)=\mu(0)\}$. It is obvious that if $\mu$ is a fuzzy (normal) subgroup of $G$, then $G_{\mu}$ is a (normal) subgroup of $G$.
Theorem 1. Let $f: G \rightarrow G^{\prime}$ be an epimorphism of groups and $\mu$ a fuzzy normal subgroup of $G$ with $\operatorname{ker} f \subseteq G_{\mu}$. Then $G / \mu \cong G^{\prime} / f(\mu)$.
Proof. By Proposition 1 and Lemma 2, $G / \mu$ and $G^{\prime} / f(\mu)$ are groups.
Let $\eta: G / \mu \rightarrow G^{\prime} / f(\mu)$, where $\eta\left(\mu_{x}\right)=(f(\mu))_{f(x)}$. If $\mu_{x}=\mu_{y}$, then $\mu\left(x y^{-1}\right)=\mu(e)$. Since $\operatorname{ker} f \subseteq G_{\mu}$, then $\mu$ is a constant on kerf, and by Lemma 3 (ii) we have $f^{-1}(f(\mu))=\mu$. Thus $\left(f^{-1}(f(\mu))\right)\left(x y^{-1}\right)=$ $\left(f^{-1}(f(\mu))\right)(e)$, i.e. $f(\mu)\left(f\left(x y^{-1}\right)\right)=f(\mu)(f(e))$, then $f(\mu)\left(f(x)(f(y))^{-1}\right)$ $=f(\mu)\left(e^{\prime}\right)$, and so $(f(\mu))_{f(x)}=(f(\mu))_{f(y)}$. Hence $\eta$ is well-defined.

It is also a homomorphism because $\eta\left(\mu_{x} \mu_{y}\right)=\eta\left(\mu_{x y}\right)=(f(\mu))_{f(x y)}=$ $(f(\mu))_{f(x) f(y)}=(f(\mu))_{f(x)}(f(\mu))_{f(y)}=\eta\left(\mu_{x}\right) \eta\left(\mu_{y}\right)$. Since $f$ is an epimorphism, for any $(f(\mu))_{y} \in G^{\prime} / f(\mu)$, there exists $x \in G$ such that $f(x)=y$. So $\eta\left(\mu_{x}\right)=(f(\mu))_{f(x)}=(f(\mu))_{y}$, which means that $\eta$ is an epimorphism.

Moreover, $(f(\mu))_{f(x)}=(f(\mu))_{f(y)} \Rightarrow f(\mu)\left(f(x)(f(y))^{-1}\right)=f(\mu)\left(e^{\prime}\right) \Rightarrow$ $f(\mu)\left(f\left(x y^{-1}\right)\right)=f(\mu)(f(e)) \Rightarrow\left(f^{-1}(f(\mu))\right)\left(x y^{-1}\right)=\left(f^{-1}(f(\mu))\right)(e) \Rightarrow$ $\mu\left(x y^{-1}\right)=\mu(e) \Rightarrow \mu_{x}=\mu_{y}$, which proves that $\eta$ is an isomorphism.

Hence $G / \mu \cong G^{\prime} / f(\mu)$.
Corollary 1. Let $f: G \rightarrow G^{\prime}$ be an epimorphism of groups and $\nu$ a fuzzy normal subgroup of $G^{\prime}$. Then $G / f^{-1}(\nu) \cong G^{\prime} / \nu$.

Proof. Since $f^{-1}(\nu)$ is a fuzzy normal subgroup (cf. [12]), $G / f^{-1}(\nu)$ and $G^{\prime} / \nu$ are groups. Moreover, by Lemma 3, we have $\nu=f\left(f^{-1}(\nu)\right)$.

If $x \in \operatorname{kerf}$, then $f(x)=e^{\prime}=f(e)$, and so $\nu(f(x))=\nu(f(e))$, i.e. $f^{-1}(\nu)(x)=f^{-1}(\nu)(e)$. Hence $x \in G_{f^{-1}(\nu)}$, i.e. $\operatorname{ker} f \subseteq G_{f^{-1}(\nu)}$.

Theorem 1 completes the proof.
Proposition 2. Let $\chi_{S}$ be a characteristic function of a subset $S$ of a group $G$. Then $\chi_{S}$ is a fuzzy normal subgroup of $G$ if and only if $S$ is a normal subgroup of $G$.
Proof. If $x, y \in S$, where $S$ is a normal subgroup of $G$, then $\chi_{S}\left(x y^{-1}\right)=$ $\chi_{S}(x)=\chi_{S}(y)=1$. Hence $\chi_{S}\left(x y^{-1}\right)=\min \left\{\chi_{S}(x), \chi_{S}(y)\right\}$. If at least one of $x$ and $y$ is not in $S$, then at least one of $\chi_{S}(x)$ and $\chi_{S}(y)$ is 0 . Therefore $\chi_{S}\left(x y^{-1}\right) \geqslant \min \left\{\chi_{S}(x), \chi_{S}(y)\right\}$. Hence $\chi_{S}$ is a fuzzy subgroup of $G$. Moreover, for any $x, y \in G$, if $y \in S$, then $x y x^{-1} \in S$ and $\chi_{S}\left(x y x^{-1}\right)=$ $1=\chi_{S}(y)$. If $y \notin S$, then $\chi_{S}(y)=0$, so $\chi_{S}\left(x y x^{-1}\right) \geqslant \chi_{S}(y)$. Hence $\chi_{S}$ is a fuzzy normal subgroup of $G$.

Conversely, if $\chi_{S}$ be a fuzzy normal subgroup of $G$, then for any $x, y \in S$, we have $\chi_{S}\left(x y^{-1}\right) \geqslant \min \left\{\chi_{S}(x), \chi_{S}(y)\right\}=1$. Thus $\chi_{S}\left(x y^{-1}\right)=1$ and $x y^{-1} \in S$. Similarly for any $y \in S, x \in G$ we have $\chi_{S}\left(x y x^{-1}\right) \geqslant \chi_{S}(y)=1$. Hence $\chi_{S}\left(x y x^{-1}\right)=1$ and $x y x^{-1} \in S$. This proves that $S$ is a normal subgroup of $G$.
Corollary 2. $G / \chi_{\text {kerf }} \cong G^{\prime}$ for any epimorphism $f: G \rightarrow G^{\prime}$ of groups.
Proof. It follows from the fact that $\chi_{\{e\}} f=\chi_{\text {kerf }}$ and $G^{\prime} / \chi_{\{e\}} \cong G^{\prime}$.
Let $N$ be a normal subgroup of a group $G$. Recall that a quotient group $G / N$ induced by a normal subgroup $N$ is determined by an equivalent relation $\sim$, where $x \sim y$ is defined by $x y^{-1} \in N$. For no confusion, we write $x \sim y(N)$ if $x$ is equivalent to $y$ with respect to $N$, and $x \sim y\left(\chi_{N}\right)$ if $x$ is equivalent to $y$ with respect to the fuzzy normal subgroup $\chi_{N}$.

Lemma 4. If $N$ is a normal subgroup of a group $G$, then $x \sim y(N)$ if and only if $x \sim y\left(\chi_{N}\right)$.

Corollary 3. Let $f: G \rightarrow G^{\prime}$ be an epimorphism of groups and $N$ be a normal subgroup of $G$ such that $\operatorname{ker} f \subseteq N$. Then $G / \chi_{N} \cong G^{\prime} / \chi_{f(N)}$.
Proof. By Proposition 2, $\chi_{N}$ and $\chi_{f(N)}$ are fuzzy normal subgroups of $G$ and $G^{\prime}$, respectively. Putting $\mu=\chi_{N}$ in Theorem 1, we obtain $G_{\mu}=$
$G_{\chi_{N}}=N \supseteq \operatorname{kerf}$. Since $f$ is an epimorphism, for any $x^{\prime} \in G^{\prime}$, there exists $x \in G$ such that $x^{\prime}=f(x)$. If $x^{\prime} \in f(N)$, then $x \in N$, which by Lemma 3 (ii) gives $f(\mu)\left(x^{\prime}\right)=f\left(\chi_{N}\right)\left(x^{\prime}\right)=f\left(\chi_{N}\right)(f(x))=\chi_{N}(x)=1=\chi_{f(N)}\left(x^{\prime}\right)$. If $x^{\prime} \notin f(N)$, then $x \notin N$ and $f(\mu)\left(x^{\prime}\right)=f\left(\chi_{N}\right)\left(x^{\prime}\right)=\chi_{N}(x)=0=\chi_{f(N)}\left(x^{\prime}\right)$. Hence $G / \chi_{N} \cong G^{\prime} / \chi_{f(N)}$.

Observe that by Lemma 4, we obtain $G / \chi_{N} \cong G / N$ and $G^{\prime} / \chi_{f(N)} \cong$ $G^{\prime} / f(N)$. This together with Corollary 3 implies the First Isomorphism Theorem for groups.

Moreover, if $f: G \rightarrow G^{\prime}$ is an epimorphism of groups and $K$ is a normal subgroup of $G^{\prime}$, then, by Proposition 2, we see that $\chi_{f^{-1}(K)}$ and $\chi_{K}$ are fuzzy normal subgroups of $G$ and $G^{\prime}$, respectively.

Putting $\nu=\chi_{K}$, we have $f^{-1}(\nu)=f^{-1}\left(\chi_{K}\right)=\chi_{f^{-1}(K)}$. Indeed, if $x \in f^{-1}(K)$, then $f(x) \in K, f^{-1}\left(\chi_{K}\right)(x)=\chi_{K} f(x)=1=\chi_{f^{-1}(K)}(x)$. If $x \notin f^{-1}(K)$, then $f(x) \notin K, f^{-1}\left(\chi_{K}\right)(x)=\chi_{K} f(x)=0=\chi_{f^{-1}(K)}(x)$.

Thus for $\nu=\chi_{K}$, as a consequence of Corollary 1, we obtain
Corollary 4. If $f: G \rightarrow G^{\prime}$ is an epimorphism of groups and $K$ is a normal subgroup of $G^{\prime}$, then $G / \chi_{f^{-1}(K)} \cong G^{\prime} / \chi_{K}$.
Lemma 5. If $N$ is a normal subgroup and $\mu$ is a fuzzy normal subgroup of a group $G$, then $\mu$ restricted to $N$ is a fuzzy normal subgroup of $N$ and $N / \mu$ is a normal subgroup of $G / \mu$.
Proof. Indeed, if $\mu_{a}, \mu_{b} \in N / \mu$, where $a, b \in N$, then $\mu_{a}\left(\mu_{b}\right)^{-1}=\mu_{a} \mu_{b^{-1}}=$ $\mu_{a b^{-1}} \in N / \mu$. If $\mu_{a} \in N / \mu, \mu_{x} \in G / \mu$, where $a \in N$ and $x \in G$, then $x a x^{-1} \in N$ and $\mu_{x} \mu_{a}\left(\mu_{x}\right)^{-1}=\mu_{x} \mu_{a} \mu_{x^{-1}}=\mu_{x a x^{-1}} \in N / \mu$. Thus $N / \mu$ is a normal subgroup of $G / \mu$.
Theorem 2. If $\mu$ and $\nu$ are two fuzzy normal subgroups of a group $G$ such that $\mu(e)=\nu(e)$, then $G_{\mu} G_{\nu} / \nu \cong G_{\mu} /(\mu \cap \nu)$.
Proof. By Lemma $5, \nu$ is a fuzzy normal subgroup of $G_{\mu} G_{\nu}$. By [11] $\mu \cap \nu$ is a fuzzy normal subgroup of $G_{\mu}$. Thus $G_{\mu} G_{\nu} / \nu$ and $G_{\mu} /(\mu \cap \nu)$ are groups.

For any $x \in G_{\mu} G_{\nu}, x=a b$, where $a \in G_{\mu}$ and $b \in G_{\nu}$, we define $g: G_{\mu} G_{\nu} / \nu \rightarrow G_{\mu} /(\mu \cap \nu)$ putting $g\left(\nu_{x}\right)=(\mu \cap \nu)_{a}$.

If $\nu_{x}=\nu_{y}$, where $y=a_{1} b_{1}, a_{1} \in G_{\mu}$ and $b_{1} \in G_{\nu}$, then

$$
\nu\left(a b\left(a_{1} b_{1}\right)^{-1}\right)=\nu\left(a b b_{1}^{-1} a_{1}^{-1}\right)=\nu\left(a_{1}^{-1} a b b_{1}^{-1}\right)=\nu\left(a_{1}^{-1} a\left(b_{1} b^{-1}\right)^{-1}\right)=\nu(e) .
$$

Hence $\nu\left(a_{1}^{-1} a\right)=\nu\left(b_{1} b^{-1}\right)=\nu(e)$. Thus

$$
\begin{aligned}
(\mu \cap \nu)\left(a a_{1}^{-1}\right) & =\min \left\{\mu\left(a a_{1}^{-1}\right), \nu\left(a a_{1}^{-1}\right)\right\}=\min \left\{\mu(e), \nu\left(\left(a_{1}^{-1} a\right)^{-1}\right)\right\} \\
& =\min \{\mu(e), \nu(e)\}=(\mu \cap \nu)(e),
\end{aligned}
$$

i.e. $(\mu \cap \nu)_{a}=(\mu \cap \nu)_{a_{1}}$. Hence $g$ is well-defined.

If $\nu_{x}, \nu_{y} \in G_{\mu} G_{\nu} / \nu$, where $x=a b, y=a_{1} b_{1}, a, a_{1} \in G_{\mu}$ and $b, b_{1} \in G_{\nu}$, then $x y=a b a_{1} b_{1}$. Since $G_{\mu}$ is normal, $b a_{1} b_{1} \in G_{\mu}$. Hence $g\left(\nu_{x} \nu_{y}\right)=$ $g\left(\nu_{x y}\right)=(\mu \cap \nu)_{a\left(b a_{1} b_{1}\right)}=(\mu \cap \nu)_{a}(\mu \cap \nu)_{b a_{1} b_{1}}$ and $(\mu \cap \nu)\left(\left(b a_{1} b_{1}\right) a_{1}^{-1}\right)=$ $\min \left\{\mu\left(b a_{1} b_{1} a_{1}^{-1}\right), \nu\left(b a_{1} b_{1} a_{1}^{-1}\right)\right\}=\min \left\{\mu\left(\left(b a_{1} b_{1}\right) a_{1}^{-1}\right), \nu\left(b\left(a_{1} b_{1} a_{1}^{-1}\right)\right)\right\}=$ $\min \{\mu(e), \nu(e)\}=(\mu \cap \nu)(e)$. Hence $(\mu \cap \nu)_{b a_{1} b_{1}}=(\mu \cap \nu)_{a_{1}}$, i.e. $g\left(\nu_{x} \nu_{y}\right)=$ $(\mu \cap \nu)_{a}(\mu \cap \nu)_{a_{1}}=g\left(\nu_{x}\right) g\left(\nu_{y}\right)$, which shows that $g$ is a homomorphism.

It is also endomorphism since for $(\mu \cap \nu)_{a} \in G_{\mu} /(\mu \cap \nu)$ and $b \in G_{\nu}$, we have $x=a b \in G_{\mu} G_{\nu}$ and $g\left(\nu_{x}\right)=(\mu \cap \nu)_{a}$.

Moreover, if $x, y \in G_{\mu} G_{\nu}$, where $x=a b, y=a_{1} b_{1}, a, a_{1} \in G_{\mu}$, $b, b_{1} \in G_{\nu}$, and $(\mu \cap \nu)_{a}=(\mu \cap \nu)_{a_{1}}$, then $(\mu \cap \nu)\left(a a_{1}^{-1}\right)=(\mu \cap$ $\nu)(e)$, i.e, $\min \left\{\mu\left(a a_{1}^{-1}\right), \nu\left(a a_{1}^{-1}\right)\right\}=\min \{\mu(e), \nu(e)\}$. But $\mu(e)=\nu(e)$ and $\mu\left(a a_{1}^{-1}\right)=\mu(e)$ imply $\nu\left(a a_{1}^{-1}\right)=\nu(e)$. Therefore $\nu\left(x y^{-1}\right)=$ $\nu\left(a b\left(a_{1} b_{1}\right)^{-1}\right)=\nu\left(a b b_{1}^{-1} a_{1}^{-1}\right)=\nu\left(a_{1}^{-1} a b b_{1}^{-1}\right) \geqslant \min \left\{\nu\left(a_{1}^{-1} a\right), \nu\left(b b_{1}^{-1}\right)\right\}=$ $\min \left\{\nu\left(\left(a a_{1}^{-1}\right)^{-1}\right), \nu\left(b b_{1}^{-1}\right)\right\}=\min \{\nu(e), \nu(e)\}=\nu(e)$. Thus $\nu_{x}=\nu_{y}$.

Hence $G_{\mu} G_{\nu} / \nu \cong G_{\mu} /(\mu \cap \nu)$.
Corollary 5. Let $N, K$ be two normal subgroups of a group $G$. Then $N K / \chi_{K} \cong N / \chi_{N \cap K}$.

Proof. By Proposition 2, $\chi_{N}$ and $\chi_{K}$ are fuzzy normal subgroups of $G$. Putting $\mu=\chi_{N}$ and $\nu=\chi_{K}$ in Theorem 2, we obtain $G_{\mu}=N, G_{\nu}=$ $K, \mu \cap \nu=\chi_{N} \cap \chi_{K}=\chi_{N \cap K}$ and $\mu(e)=1=\nu(e)$. Hence $N K / \chi_{K} \cong$ $N / \chi_{N \cap K}$.

Since $N K / \chi_{K} \cong N K / K$ and $N / \chi_{N \cap K} \cong N / N \cap K$, as a consequence of the above two lemmas we obtain the Second Isomorphism Theorem of groups. The Third Isomorphism Theorem is a consequence of the following
Theorem 3. Let $\mu$ and $\nu$ be two fuzzy normal subgroups of a group $G$ with $\nu \leqslant \mu$ and $\nu(e)=\mu(e)$. Then $(G / \nu) /\left(G_{\mu} / \nu\right) \cong G / \mu$.
Proof. By Lemma 5, $G_{\mu} / \nu$ is a normal subgroup of $G / \nu$.
Putting $f\left(\nu_{x}\right)=\mu_{x}$ for all $x \in G$, we define $f: G / \nu \rightarrow G / \mu$ such that $\nu\left(x y^{-1}\right)=\nu(e)=\mu(e)$ for all $\nu_{x}=\nu_{y}$. Because $\nu \leqslant \mu$, we have $\mu\left(x y^{-1}\right) \geqslant \nu\left(x y^{-1}\right)=\mu(e)$, and so $\mu\left(x y^{-1}\right)=\mu(e)$, i.e. $\mu_{x}=\mu_{y}$, which means that $f$ is well-defined. Since $f\left(\nu_{x} \nu_{y}\right)=f\left(\nu_{x y}\right)=\mu_{x y}=\mu_{x} \mu_{y}=$ $f\left(\nu_{x}\right) f\left(\nu_{y}\right), f$ is a homomorphism. By the definition, it is an epimorphism, too. But kerf $=\left\{\nu_{x} \in G / \nu \mid f\left(\nu_{x}\right)=\mu_{e}\right\}=\left\{\nu_{x} \in G / \nu \mid \mu_{x}=\mu_{e}\right\}=\left\{\nu_{x} \in\right.$ $G / \nu \mid \mu(x)=\mu(e)\}=\left\{\nu_{x} \in G / \nu \mid x \in G_{\mu}\right\}=G_{\mu} / \nu$. Thus $\operatorname{ker} f=G_{\mu} / \nu$ and $(G / \nu) /\left(G_{\mu} / \nu\right) \cong G / \mu$.
Corollary 6. $\left(G / \chi_{K}\right) /\left(N / \chi_{K}\right) \cong G / \chi_{N}$ for any normal subgroups $N \subseteq K$ of a group $G$.

Finally we consider fuzzy abelian subgroups, i.e. fuzzy subgroups $\mu$ of a group $G$ satisfying the identity $\mu\left(x y x^{-1} y^{-1}\right)=\mu(e)$.

Proposition 3. A fuzzy subgroup $\mu$ of a group $G$ is abelian if and only if $G / \mu$ is abelian.
Proof. If $\mu$ is a fuzzy abelian subgroup, then $\mu\left(x y x^{-1} y^{-1}\right)=\mu(e)$, and hence $\mu(x y)=\mu(y x)$. Thus $\mu$ is fuzzy normal. Since $\mu\left(x y(y x)^{-1}\right)=$ $\mu\left(x y x^{-1} y^{-1}\right)=\mu(e)$, we have $\mu_{x y}=\mu_{y x}$, i.e. $\mu_{x} \mu_{y}=\mu_{y} \mu_{x}$. Hence $G / \mu$ is an abelian group.

Conversely, if $G / \mu$ is abelian, then $\mu_{x y}=\mu_{y x}$ and $\mu\left(x y(y x)^{-1}\right)=\mu(e)$. So $\mu\left(x y x^{-1} y^{-1}\right)=\mu(e)$.

Let $\mu$ be a fuzzy subgroup of a group $G$. The smallest positive integer $n$ (if it exists) such that $\mu\left(x^{n}\right)=\mu(e)$ is called the fuzzy order of $x$ with respect to $\mu$ and is denoted by $F O_{\mu}(x)$ (cf. [4]). If $F O_{\mu}(x)$ is finite for every $x \in G$, then $\mu$ is called fuzzy torsion. In the case when for all $x \in G$ $F O_{\mu}(x)$ is a power of a prime number $p$, we say that $\mu$ is a fuzzy $p$-subgroup of $G$.

Proposition 4. A fuzzy normal subgroup $\mu$ of a group $G$ is a fuzzy psubgroup if and only if $G / \mu$ is a p-group.
Proof. If $\mu$ is a fuzzy $p$-subgroup of $G$, then for any $\mu_{x} \in G / \mu$ there is a nonnegative integer $s$ such that $\mu\left(x^{p^{s}}\right)=\mu(e)$, i.e. $\mu_{x^{p^{s}}}=\mu_{e}$. Hence $\left(\mu_{x}\right)^{p^{s}}=\mu_{e}$. Conversely, if $G / \mu$ is a $p$-group of $G$, then for any $x \in G$ and some nonnegative integer $t$ we have $\left(\mu_{x}\right)^{p^{t}}=\mu_{e}$, i.e. $\mu_{x^{p^{t}}}=\mu_{e}$. Thus $\mu\left(x^{p^{t}}\right)=\mu(e)$, which completers the proof.

Proposition 5. A fuzzy subgroup $\mu$ of an abelian group $G$ is fuzzy torsion if and only if $G / \mu$ is torsion.

Proof. Because $G$ is an abelian group, $\mu$ is normal. Let $G / \mu$ be torsion. For any $x \in G$, there is a positive integer $n$ such that $\left(\mu_{x}\right)^{n}=\mu_{e}$, i.e. $\mu_{x^{n}}=\mu_{e}$, and so $\mu\left(x^{n}\right)=\mu(e)$. Hence $F O_{\mu}(x)$ is finite and $\mu$ is fuzzy torsion.

The converse is obvious.

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# Zeroids and idempoids in AG-groupoids 

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#### Abstract

Clifford and Miller (Amer. J. Math. 70, 1948) and Dawson (Acta Sci. Math. 27, 1966) have studied semigroups having left or right zeroids in a semigroup. In this paper, we have investigated AG-groupoids, and AG-groupoids with weak associative law, having zeroids or idempoids. Some interesting characteristics of these structures have been explored.


An Abel-Grassman's groupoid [8], abbreviated as $A G$-groupoid, is a groupoid $G$ whose elements satisfy the left invertive law: $(a b) c=(c b) a$. It is also called a left almost semigroup $[4,5,6,7]$. In [3], the same structure is called a left invertive groupoid. In this note we call it an AG-groupoid. It is a useful non-associative algebraic structure, midway between a groupoid and a commutative semigroup, with wide applications in the theory of flocks.

AG-groupoid is medial [5], that is, $(a b)(c d)=(a c)(b d)$ for all $a, b, c, d$ in $G$. It has been shown in [5] that if an AG-groupoid contains a left identity then it is unique. It has been proved also that an AG-groupoid with right identity is a commutative monoid, that is, a semigroup with identity element. An element $a_{\circ}$ of an AG-groupoid $G$ is called a left zero if $a_{\circ} a=a_{\circ}$ for all $a \in G$.

It has been shown in [5] that if $a b=c d$ then $b a=d c$ for all $a, b, c, d$ in an AG-groupoid with left identity. If for all $a, b, c$ in an AG-groupoid $G$, $a b=a c$ implies that $b=c$, then $G$ is called left cancellative. Similarly, if $b a=c a$ implies that $b=c$, then $G$ is called right cancellative. It is known [5] that every left cancellative AG-groupoid is right cancellative but the converse is not true. However, every right cancellative AG-groupoid with left identity is left cancellative.

Clifford and Miller [1] have defined an element $z_{l}$ as a left zeroid in a semigroup $G$ if for each element $x$ in $G$, there exists $a$ in $G$ such that $a x=z_{l}$.

A right zeroid is similarly defined. An element is a zeroid in $G$ if it is both left and right zeroid.

Dawson [2] has studied semigroups having left or right zeroid elements and investigated some of their properties. In this paper we introduce the concept of left idempoids in AG-groupoid and investigate some of their properties.

Next we prove the following result.
Theorem 1. An AG-groupoid $G$ is a semigroup if and only if $a(b c)=(c b) a$ for all $a, b, c \in G$.

Proof. Let $a(b c)=(c b) a$. Since $G$ is an AG-groupoid, $(a b) c=(c b) a$. As the right hand sides of the two equations are equal, we conclude that $(a b) c=$ $a(b c)$. Thus $G$ is a semigroup.

Conversely, suppose that an AG-groupoid $G$ is a semigroup. This means that $(a b) c=(c b) a$ and $(a b) c=a(b c)$. Since the left hand sides of these equations are equal, we get $a(b c)=(c b) a$ for all $a, b, c \in G$.

An element $z_{r}$ of an AG-groupoid $G$ is called a right idempoid if, for each $x \in G$, there exists $a \in G$ such that $(x a) a=z_{r}$.

Note that $G$ contains a right idempoid because for any $x, y \in G$ there exists $a \in G$ such that $a x, a y \in G$. So $(a x)(a y)=(a a)(x y)=(a a) z=$ $(z a) a$, where $z=x y$ is an arbitrary element in $G$, implies that $G$ contains a right idempoid.

Proposition 1. An AG-groupoid $G$ is a semigroup if and only if $z_{r}=a(a x)$ is a right idempoid for some fixed $a$ and any $x \in G$.

Proof. The proof follows directly from Theorem 1.
Theorem 2. An $A G$-groupoid $G$ with $G^{2}=G$ is a commutative semigroup if and only if $(a b) c=a(c b)$ for all $a, b, c \in G$.

Proof. Suppose $(a b) c=a(c b)$. Since $G$ is an AG-groupoid, $(c b) a=(a b) c$. Combining the two equations we obtain $(c b) a=a(c b)$ implying that $G$ is commutative. Thus $(a b) c=(c b) a=a(c b)=a(b c)$ shows that $G$ is a commutative semigroup.

The converse follows immediately.
Corollary 1. An $A G$-groupoid is a commutative semigroup if and only if $z_{r}=x a^{2}$ is a right idempoid for fixed $a \in G$ and any $z \in G$.

Proof. The proof follows immediately from Theorem 2.

Proposition 2. The square of every left zeroid in an $A G$-groupoid $G$ with an idempotent is a right idempoid.

Proof. Let $x$ be an idempotent and $z_{l}$ a left zeroid in $G$. Since $z_{l}$ is a left zeroid, there exists $a$ in $G$ such that $a x=z_{l}$. Therefore

$$
z_{l} z_{l}=(a x)(a x)=(a a)(x x)=(a a) x=(x a) a=z_{r},
$$

which completes the proof.
Corollary 2. In an $A G$-groupoid $G$ there exists a left zeroid element.
Proof. If we define a mapping $l_{a}: G \rightarrow G$ by $(x) l_{a}=a x$ by for all $x$ in $G$, then obviously these mappings are related to left zeroids in a natural way.

In the following we shall examine the necessary and sufficient conditions for $l_{a}$ to be an epimorphism, endomorphism, automorphism, monomorphism and anti-homomorphism.
Theorem 3. If in a left cancellative $A G$-groupoid $G$ we define for a fixed $a$ and some $x$, a mapping $l_{a}: x \mapsto a x$, from $G$ onto $G$, then the following statements are equivalent:
(i) $l_{a}$ is an epimorphism,
(ii) $a$ is an idempotent in $G$,
(iii) $l_{a}$ is an automorphism.

Proof. Suppose ( $i$ ) holds. Then there exists $x$ in $G$ such that for some fixed $a, a x=y$, in $G$. This implies that for some $x$ in $G$ and a fixed $a$ in $G$, there exists an element $y$ in $G$ such that $y=(x) l_{a}$. Now $(a) l_{a} y=(a) l_{a}(x) l_{a}=$ $(a a)(a x)$ and $(a) l_{a}(x) l_{a}=(a x) l_{a}=a(a x)=a y$ imply that $(a) l_{a}=a$, that is, $a$ is an idempotent in $G$. Hence ( $i$ ) implies ( $i i$ ).

Also $(x) l_{a}(y) l_{a}=(a x)(a y)=(a a)(x y)=a(x y)$ because $a$ is idempotent. This implies that $(x) l_{a}(y) l_{a}=(x y) l_{a}$, which further implies that $l_{a}$ is an endomorphism. In order to show that $l_{a}$ in an automorphism it is sufficient to show that $l_{a}$ is one-to-one. But this is obvious since $(x) l_{a}=(y) l_{a}$ and $a x=a y$ implies that $x=y$ by virtue of left cancellation. Thus (ii) implies (iii).

Since $l_{a}$ is an automorphism, (iii) implies (i).
Theorem 4. In an $A G$-groupoid $G$ the following statements are equivalent:
(i) G has a right zero,
(ii) $l_{a}: x \longmapsto a x$ an automorphism and $G$ has an idempotent element,
(iii) $G$ has a zero.

Proof. If $x$ is a right zero of $G$, then $a x=x$ for some $a \in G$. But $x=a x=$ $(x) l_{a}$ for every $x$ in $G$. This implies that $l_{a}$ is the identity mapping, which is an automorphism and, in particular, $a=(a) l_{a}$. It follows that $a=a a$, that is, $a$ is an idempotent. Thus ( $i$ ) implies ( $i i$ ).

Further, for any $x$ and some $a$ in $G$, we have $a(x x)=(x x) l_{a}=x x$ and $(x x) a=(a x) x=(x) l_{a} x=x x$. This implies that $a(x x)=(x x) a=x x$, showing that $x x$ is a zero in $G$. Hence (ii) implies (iii).
(iii) obviously implies (i).

Theorem 5. If $(G) l_{a}=\left\{(x) l_{a}: x \in G\right\}$, where $a$ is a fixed idempotent of an $A G$-groupoid $G$, then $(G) l_{a}$ is an $A G$-groupoid with an idempotent $a$.

Proof. Let $(x) l_{a},(y) l_{a}$ belong to $(G) l_{a}$. Then

$$
(x) l_{a}(y) l_{a}=(a x)(a y)=(a a)(x y)=a(x y)=(x y) l_{a} .
$$

This implies that $(x) l_{a}(y) l_{a} \in(G) l_{a}$. Now

$$
(x) l_{a}(y) l_{a}(z) l_{a}=((a x)(a y))(a z)=((a z)(a y))(a x)=\left((z) l_{a}(y)(x) l_{a} .\right.
$$

Hence $(G) l_{a}$ is an AG-groupoid.
Theorem 6. If $(G) l_{a}=\left\{(x) l_{a}: x \in G\right\}$, where $a$ is a fixed element of $a$ right cancellative $A G$-groupoid $G$, then $l_{a}$ is an endomorphism if and only if $a$ is an idempotent of $G$.

Proof. Let $l_{a}$ be an endomorphism. Then $\left(x x^{\prime}\right)=(x) l_{a}\left(x^{\prime}\right) l_{a}$. Hence

$$
a\left(x x^{\prime}\right)=(a x)\left(a x^{\prime}\right)=(a a)\left(x x^{\prime}\right)
$$

imply that $a=a a$.
Conversely, if $a=a a$ then

$$
(x) l_{a}\left(x^{\prime}\right) l_{a}=(a x)\left(a x^{\prime}\right)=(a a)\left(x x^{\prime}\right)=a\left(x x^{\prime}\right)=\left(x x^{\prime}\right) l_{a},
$$

which completes our proof.
Theorem 7. If $G$ is an $A G$-groupoid with an idempotent $a$ and $l_{a}$ is an anti-homomorphism, then a commutes with every element of $G$.

Proof. Let $x$ be an arbitrary element of $G$. Then there exists $x^{\prime} \in G$ such that $\left(x^{\prime}\right) l_{a}=x$. Consider $x a$ for any $x$ and some idempotent $a$ in $G$. Then

$$
x a=x(a a)=x(a) l_{a}=\left(x^{\prime}\right) l_{a}(a) l_{a}=\left(a x^{\prime}\right) l_{a}=a\left(a x^{\prime}\right)=a\left(x^{\prime}\right) l_{a}=a x .
$$

This implies that $a$ commutes with every $x$ in $G$.

Theorem 8. In a right cancellative $A G$-groupoid $G$ with an idempotent $a$, if $l_{a}: x \mapsto a x$ is an anti-homomorphism, then the following statements are equivalent:
(i) $l_{a}$ is an anti-epimorphism,
(ii) $G$ is a commutative monoid,
(iii) $l_{a}$ is an anti-automorphism.

Proof. Suppose ( $i$ ) holds. Then for a fixed $a \in G$, there exist $x$ and $y$ in $G$ such that, $y=a x=(x) l_{a}$. Now

$$
y a=y(a a)=(x) l_{a}(a) l_{a}=(a x) l_{a}=a(a x)=a(x) l_{a}=a y
$$

because $l_{a}$ is an anti-epimorphism.
Further $a y=(a a) y=(y a) a$, which implies that $y a=(y a) a$. So $y=$ $y a=a y$. Hence $a$ is the identity of $G$. But an AG-groupoid with right identity is a commutative monoid by a result in [5]. Hence ( $i$ ) implies(ii).

Now, since $a$ is the identity in $G$, then for any $x$ in $G$, we have $a x=x$ implying that $(x) l_{a}=x$ and so $l_{a}$ is the identity mapping. This implies that $l_{a}$ is an anti-automorphism. It follows that (ii) implies (iii).

Also, (iii) implies (i), follows immediately since an anti-automorphism must necessarily be an anti-epimorphism.

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# Characterization of division $\mu$-LA-semigroups 

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#### Abstract

Let $G$ be a left almost semigroup (LA-semigroup), also known as Abel-Grassman's groupoid and a left invertive groupoid. In this paper we have shown that $G$ is a division $\mu$-LA-semigroup if and only if it has a linear form. Characterization of division $\mu$-LA-semigroups is also done by using permutations.


## 1. Introduction

A left almost semigroup [2], abbreviated as LA-semigroup, is an algebraic structure midway between a groupoid and a commutative semigroup. Although the structure is non-associative and non-commutative, nevertheless, it possesses many interesting properties which we usually find in associative and commutative algebraic structures.

Kazim and Naseerudin have introduced the concept of an LA-semigroup and have investigated some basic but important characteristics of this structure in [2]. They have generalized some useful results of semigroup theory. Relationships between LA-semigroups and quasigroups, semigroups, loops, monoids, and groups have been established.

Later, Mushtaq and others in [1], [5], [6], [7], [8], and [10] have studied the structure further and added many results to the theory of LAsemigroups. Holgate [1], has called the same structure as left invertive groupoid. It is also known as Abel-Grassman's groupoid or AG-groupoid. In this paper we shall call it LA-semigroup.

Kepka [4] has done extensive study of quasigroups satisfying some weak forms of the medial law. In this paper we have extended some of his results to LA-semigroups.

2000 Mathematics Subject Classification: 20M10, 20N99
Keywords: LA-semigroup, invertive law, medial law, division $\mu$-LA-semigroup

A groupoid $\mathcal{G}=(G, \cdot)$ is called a left almost semigroup, abbreviated as $L A$-semigroup, if its elements satisfy the left invertive law: $(a b) c=(c b) a$. Examples of LA-semigroups can be found in [5] and [6].

An element $e \in G$ is called a left identity if $e a=a$ for all $a \in G$. An element $a^{\prime} \in G$ is called a left inverse of $a$ if $\mathcal{G}$ contains left identity $e$ and $a^{\prime} a=e$. As in the case of semigroups, both $e$ and $a^{\prime}$ are unique [5]. In [5] it is proved also that if $\mathcal{G}$ contains a left identity then $a b=c d$ implies $b a=d c$ for all $a, b, c, d \in G$. As in the case of semigroups, an element $a$ of an LA-semigroup $\mathcal{G}$ is called left cancellative if $a b=a c$ implies $b=c$. Similarly, it is right cancellative if $b a=c a$ implies $b=c$. If it is both left and right, it is called cancellative.

It is also known [2] that in an LA-semigroup $\mathcal{G}$, the medial law: $(a b)(c d)$ $=(a c)(b d)$ holds for every $a, b, c, d \in G$. An LA-semigroup with a left identity is called an LA-monoid. In [9] an LA-monoid with a left inverse is called an $L A$-group. Because in an LA-group every left inverse is a right inverse, therefore, we can re-define an LA-group as follows: An LA-monoid $G$ is called a left almost group, abbreviated as LA-group, if it contains inverses.

Suppose that $(G, \cdot)$ is a commutative group. Then it is easy to see that $(G, *)$, where $a * b=b a^{-1}$, is an example of an LA-group.

Let $\mathcal{G}$ be an LA-semigroup and $a \in G$. A mapping $L_{a}: G \rightarrow G$, defined by $L_{a}(x)=a x$, is called the left translation by $a$. Similarly a mapping $R_{a}: G \rightarrow G$, defined by $R_{a}(x)=x a$ is called the right translation by $a$. An LA-semigroup $\mathcal{G}$ is called a division LA-semigroup if the mappings $L_{a}$ and $R_{a}$ are onto for all $a \in G$.

An LA-semigroup $\mathcal{G}$ is called a $\mu$-LA-semigroup if there are two mappings $\alpha, \beta$ of the set $G$ onto $G$ and an LA-monoid ( $G, \circ$ ) such that $a b=$ $\alpha(a) \circ \beta(b)$ for all $a, b \in G$. Note that if we take $\alpha, \beta$ to be identity maps and $(G, \circ)=(G, \cdot)$, then an LA-monoid $(G, \cdot)$ is trivially a $\mu$-LA-monoid.

Let $\mathcal{G}$ be a division $\mu$-LA-semigroup. Then $((G, \circ), \alpha, \psi, g)$ is said to be a right linear form of $\mathcal{G}$ if $(G, \circ)$ is an LA-group, $\alpha$ a mapping of $G$ onto $G, \psi$ an endomorphism of $(G, \circ), g \in G$ and $a b=\alpha(a) \circ(g \circ \psi(b))$ for all $a, b \in G$. Similarly $((G, \circ), \psi, \alpha, g)$ is said to be a left linear form of $\mathcal{G}$ if $a b=\psi(a) \circ(g \circ \alpha(b))$ for all $a, b \in G$. If $\varphi=\alpha$ is an endomorphism of $\mathcal{G}$, then $((G, \circ), \varphi, \psi, g)$ is a called a linear form of $\mathcal{G}$.

## 2. Division LA-semigroups

Having set the terminology and given the basic definitions we are now in a position to prove the following results.
Proposition 2.1. Every LA-group is a division $\mu$-LA-group.
Proof. Let $\mathcal{G}$ be an LA-group and $L_{a}$ its left translation. Then

$$
a b=(e a) b=(b a) e
$$

yields

$$
\begin{aligned}
L_{a}\left((x e) a^{-1}\right) & =a\left((x e) a^{-1}\right)=\left(\left((x e) a^{-1}\right) a\right) e=\left(\left(a a^{-1}\right)(x e)\right) e \\
& =(e(x e)) e=(x e) e=(e e) x=e x=x
\end{aligned}
$$

Thus for every $x \in G$ there exists $(x e) a^{-1} \in G$ such that $L_{x}\left((x e) a^{-1}\right)=$ $x$. Hence $L_{a}$ is onto. Also $R_{a}$ is onto because $R_{a}\left(x a^{-1}\right)=\left(x a^{-1}\right) a=$ $\left(a a^{-1}\right) x=e x=x$ for every $x \in G$. Hence $\mathcal{G}$ is a division LA-group. Thus, the observation that every LA-monoid is trivially a $\mu$-LA-monoid, and Theorem 9 in [3], imply that $\mathcal{G}$ is in fact a division $\mu$-LA-group.

Let $C(G, \circ)$ denote the centre of LA-semigroup ( $G, \circ$ ).
Theorem 2.2. If $\mathcal{G}$ is an $L A$-semigroup, then the following statements are equivalent:
(i) $\mathcal{G}$ is a division $\mu$-LA-semigroup,
(ii) $\mathcal{G}$ has a linear form $((G, \circ), \varphi, \psi, g)$ such that $\varphi \psi(a) \circ g=g \circ \psi \varphi(a)$ for every $a \in G$. In this case $C(G, \circ)=G$.

Proof. $\quad(i) \Rightarrow(i i)$. Since $\mathcal{G}$ is a division $\mu$-LA-semigroup satisfying the medial law, by Theorem 15 in [3], $((G, \circ), \varphi, \psi, g)$ is the linear form of $\mathcal{G}$ such that $\varphi \psi(a) \circ h=h \circ \psi \varphi(a)$ for all $a \in G$, where $h=\psi \varphi(x) \circ g$ for some $x \in G$. But by Theorem 15 in [3], we can assume that $x$ is the left identity of $(G, \circ)$. Thus $h=x \circ g=g$.
$(i i) \Rightarrow(i)$. Since $\mathcal{G}$ has a linear form $((G, \circ), \varphi, \psi, g)$, therefore by the definition, $\mathcal{G}$ is a division $\mu$-LA-semigroup and so $a b=\varphi(a) \circ(g \circ \psi(b))$ for all $a, b \in G$, where $(G, \circ)$ is an LA-group. If $e$ is the left identity in $(G, \circ)$, then this last equation can be written as $\varphi(a) \circ(e \circ \psi(b)=\varphi(a) \circ \psi(b)$, which implies that $\mathcal{G}$ is a division $\mu$-LA-semigroup.

Let $x \in C(G, \circ)$. We wish to show that $x \in G$. Let $a, b, c \in G$, then

$$
\begin{aligned}
(a x)(b c) & =(\varphi(a) \circ(g \circ \psi(x))(\varphi(b) \circ(g \circ \psi(c)) \\
& =\varphi(\varphi(a) \circ(g \circ \psi(x)) \circ(g \circ \psi(\varphi(b) \circ(g \circ \psi(c)))) \\
& =\varphi^{2}(a) \circ(\varphi(g) \circ \varphi \psi(x)) \circ\left(g \circ\left(\psi \varphi(b) \circ\left(\psi(g) \circ \psi^{2}(c)\right)\right) .\right.
\end{aligned}
$$

Since $(G, \circ)$ is an LA-group, we can apply the medial and the left invertive laws (which hold in $(G, \circ)$ ) to the above identity. Hence

$$
(a x)(b c)=\left(\varphi^{2}(a) \circ g\right) \circ\left(\left(\psi \varphi(x) \circ\left(\psi(g) \circ \psi^{2}(c)\right)\right) \circ(\varphi(g) \circ \varphi \psi(b))\right) .
$$

Since $(\varphi \psi(a) \circ g) \circ(g \circ \psi \varphi(b))=(\psi \varphi(b) \circ g) \circ(g \circ \psi \varphi(a))$, therefore

$$
\begin{aligned}
(a x)(b c) & =\left(\varphi^{2}(a) \circ g\right) \circ\left((\psi \varphi(b) \circ \psi(g)) \circ\left(\left(\psi(g) \circ \psi^{2}(c)\right) \circ \varphi \psi(x)\right)\right) \\
& =\left(\varphi^{2}(a) \circ g\right) \circ\left(( \varphi ( g ) \circ \varphi \psi ( b ) ) \circ \left(\left(\psi \varphi(x) \circ\left(\psi(g) \circ \psi^{2}(c)\right)\right)\right.\right.
\end{aligned}
$$

Applying the medial law again, we get

$$
\begin{aligned}
(a x)(b c) & =\left(\varphi^{2}(a) \circ(\varphi(g) \circ \varphi \psi(b))\right) \circ\left(g \circ\left(\psi \varphi(x) \circ\left(\psi(g) \circ \psi^{2}(c)\right)\right)\right) \\
& =\varphi(\varphi(a) \circ(g \circ \psi(b))) \circ(g \circ \psi(\varphi(x) \circ(g \circ \psi(c)))) \\
& =(\varphi(a) \circ(g \circ \psi(b)))(\varphi(x) \circ(g \circ \psi(c)))=(a b)(x c) .
\end{aligned}
$$

Thus $x \in G$, and so $C(G, \circ) \subseteq G$.
Conversely, let $y \in G$. Then

$$
(\varphi \psi(a) \circ g) \circ \psi \varphi(y)=(\psi \varphi(y) \circ g) \circ \varphi \psi(a)
$$

Since $\varphi \psi(a) \circ g=g \circ \psi \varphi(a)$, therefore the above identity gives

$$
(g \circ \psi \varphi(a)) \circ \psi \varphi(y)=(g \circ \varphi \psi(y)) \circ \varphi \psi(a),
$$

i.e.

$$
(\psi \varphi(y) \circ \psi \varphi(a)) \circ g=(\varphi \psi(a) \circ \varphi \psi(y)) \circ g
$$

Since $(G, \circ)$ is cancellative, $\psi \varphi(y) \circ \psi \varphi(a)=\varphi \psi(a) \circ \varphi \psi(y)$. But $\psi \varphi=\varphi \psi$, by Theorem 16 in [3]. So $\psi \varphi(y) \circ \psi \varphi(a)=\psi \varphi(a) \circ \psi \varphi(y)$. Thus $\psi \varphi(y) \in$ $C(G, \circ)$. This together with the fact that $\psi \varphi: G \rightarrow G$ is a homomorphism, imply $y \in G$. Hence $G \subseteq C(G, \circ)$, and in consequence $G=C(G, \circ)$.

Corollary 2.3. A division $\mu$-LA-semigroup $\mathcal{G}$ is commutative if it has a linear form $((G,+), \varphi, \psi, g)$ such that $(G,+)$ is a commutative group and $\psi \varphi=\varphi \psi$.

Proof. If a division $\mu$-LA-semigroup $\mathcal{G}$ has a linear form as above, then $\varphi \psi(a)+g=\psi \varphi(a)+g=g+\psi \varphi(a)$. Therefore $G=C(G, \circ)$ by Theorem 2.2.

Theorem 2.4. For any division $\mu$-LA-semigroup $\mathcal{G}$ there are mappings $\alpha, \beta$ of $G$ onto $G$ such that $\alpha(a) \beta(b)=\alpha(b) \beta(a)$ for every $a, b \in G$.

Proof. Since $\mathcal{G}$ is a division $\mu$-LA-semigroup, therefore $\alpha=L_{c}$ and $\beta=R_{c}$ are onto mappings (for all $c \in G$ ), and $\alpha(a) \beta(b)=L_{c}(a) R_{c}(b)=(c a)(b c)=$ $(b c)(c a)=(c b)(a c)=L_{c}(b) R_{c}(a)=\alpha(b) \beta(a)$.

Theorem 2.5. A division $\mu$-LA-semigroup $\mathcal{G}$ is commutative if and only if the mapping $a \mapsto a a$ is an endomorphism of $\mathcal{G}$.

Proof. If $a \mapsto a a$ is an endomorphism of $\mathcal{G}$. Then $(a b)(a b)=(a a)(b b)$ for every $a, b \in G$, because $\mathcal{G}$ is medial, and so $G=C(G, \circ)$ by Theorem 2.2.

Conversely, if $\mathcal{G}$ is commutative, then $(a b)(a b)=(a a)(b b)$ implies that the mapping $a \mapsto a a$ is an endomorphism of $\mathcal{G}$.

Proposition 2.6. The mapping $a \mapsto a a$ is an endomorphism of $\mathcal{G}$ if $\mathcal{G}$ is an $L A$-semigroup.
Proof. The proof is a trivial consequence of the medial law.
Note here that the converse is not true because there are medial groupoids, which are not LA-semigroups.

An LA-semigroup $\mathcal{G}$ is called idempotent if $a a=a$ for all $a \in G$. An LA-semigroup $\mathcal{G}$ in which $a a=b b$ for all $a, b \in G$ is called unipotent.
Proposition 2.7. Let $\mathcal{G}$ be a left cancellative $L A$-semigroup. Then:
(i) $\alpha$ and $\psi$ are permutations of $G$, if $((G, \circ), \alpha, \psi, g)$ is a right linear form of $\mathcal{G}$,
(ii) $\varphi$ and $\beta$ are permutations of $G$, if $((G, \circ), \varphi, \beta, g)$ is a left linear form of $\mathcal{G}$.
Proof. (i) Since $((G, \circ), \alpha, \psi, g)$ is a right linear form of a left cancellative LA-semigroup $\mathcal{G}$, therefore $\alpha$ is a mapping from $G$ onto $G$ and $\psi$ is an endomorphism of $\mathcal{G}$. We prove that $\alpha$ and $\psi$ are one-to-one.

Let $\alpha(a)=(a j) \circ g^{-1}=R_{j}(a) \circ g^{-1}$. If $\alpha(a)=\alpha(b)$, then $R_{j}(a) \circ g^{-1}=$ $R_{j}(b) \circ g^{-1}$. Since ( $G, \circ$ ) is cancellative, therefore $R_{j}(a)=R_{j}(b)$, which by Theorem 2.6 from [5], implies $a=b$. Hence $\alpha$ is one-to-one.

Let $\psi(a)=L_{y}(a)$, where $y=\alpha^{-1}\left(g^{-1}\right)$. Since $\alpha(a)=R_{j}(a) \circ g^{-1}$, therefore $\alpha(y)=R_{j}(y) \circ g^{-1}$. But $\alpha(y)=g^{-1}$ implies $g^{-1}=R_{j}(y) \circ g^{-1}$, i.e. $y=\alpha^{-1}\left(R_{j}(y) \circ g^{-1}\right)=\alpha^{-1}\left(g^{-1}\right)$. Now $\psi(a)=L_{y}(a)=\alpha^{-1}\left(R_{j}(y) \circ g^{-1}\right) a$. If $\psi(a)=\psi(b)$, then $\alpha^{-1}\left(y j \circ g^{-1}\right) a=\alpha^{-1}\left(y j \circ g^{-1}\right) b$. Since $\alpha$ is one-toone, therefore $\left(y j \circ g^{-1}\right) a=\left(y j \circ g^{-1}\right) b$, which by Theorem 2.6 from [5] implies $a=b$. Thus $\psi$ is one-to-one.
(ii) Analogously as (i).

Theorem 2.8. Let $\mathcal{G}$ be an LA-semigroup. Then the following conditions are equivalent:
(i) $\mathcal{G}$ is a division $\mu$-LA-semigroup,
(ii) $\mathcal{G}$ has a linear form $((G,+), \sigma, \psi, g)$ such that $(G,+)$ is a commutative group and $\sigma(\psi(a)+g)=\sigma(g)+\psi \sigma(a)$.

Proof. Since a division LA-semigroup $\mathcal{G}$ is medial, by Theorem 16 in [3], $\mathcal{G}$ has a linear form $((G,+), \sigma, \psi, g)$ such that $(G,+)$ is a commutative group and $\sigma \psi=\psi \sigma$. Thus $\sigma(\psi(a)+g)=\sigma(g)+\sigma \psi(a)=\sigma(g)+\psi \sigma(a)$ because $\sigma$ is an endomorphism.

Conversely, if an LA-semigroup $\mathcal{G}$ has a linear form as in (ii), then $a b=\sigma(a)+g+\psi(b)$, which for $g=0$ shows that $\mathcal{G}$ is a division $\mu$-LAsemigroup.

Theorem 2.9. Let an LA-semigroup $\mathcal{G}$ has a linear form $((G, \circ), \varphi, \psi, g)$. Then $\mathcal{G}$ is a commutative group, if $\varphi, \psi$ are central automorphism of ( $G, \circ$ ) and $\varphi \psi=\psi \varphi$.
Proof. If $\varphi, \psi$ are central automorphisms of $(G, \circ)$ such $\varphi \psi=\psi \varphi$, then $\varphi(a), \psi(a) \in C(G, \circ)$ for every $a \in G$. Thus $\varphi \psi(a) \in C(G, \circ)$ and $\varphi \psi(a) \circ$ $g=g \circ \psi \varphi(a)$ for every $g \in G$. Theorem 2.2 completes the proof.

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# A note on Salem numbers and Golden mean 

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#### Abstract

It is known that every Pisot number is a limit of Salem numbers. At present there are 47 known Salem numbers less than 1.3 and the list is known to be complete through degree 40. There is a well known relationship between Coxeter systems, Salem numbers, and Golden mean. In this short note, we have discovered the existence of Golden mean in the action of $P S L_{2}(Z)$ on $Q(\sqrt{5} \cup\{\infty\}$ and investigated some interesting properties of these.


## 1. Introduction

An algebraic integer $\lambda>1$ is a Pisot number if its conjugates (other than $\lambda$ itself) satisfy $\left|\lambda^{\prime}\right|<1$. Similarly, an algebraic integer $\lambda>1$ is a Salem number if its conjugates (other than $\lambda$ itself) satisfy $\left|\lambda^{\prime}\right| \leqslant 1$ and include $\frac{1}{\lambda}$.

It is known that the Pisot numbers form a closed subset $P \subset R$, where $R$ is a field of real numbers, and that every Pisot number is a limit of Salem numbers [4]. The smallest Pisot number $\lambda_{P}$, equivalent to 1.324717 , is a root of $x^{3}-x-1=0$, while the smallest accumulation point in $P$ is the Golden mean, $\lambda_{G}=\frac{1+\sqrt{5}}{2}$ equivalent to 1.61803. Note that $\lambda_{G}^{2}=\frac{3+\sqrt{5}}{2}$ is equivalent to 2.61803...

## 2. Golden mean

Theorem. In an action of the modular group on $Q\left(\sqrt{5} \cup\{\infty\}, \lambda_{G}\right.$ is the fixed point of the commutator of the modular group.

2000 Mathematics Subject Classification: 11E04, 20G15
Keywords: Salem number, Pisot number, Golden mean, modular group, commutator

Proof. It is well known that the modular group $P S L_{2}(Z)$ is generated by the linear fractional transformations $x: z \mapsto \frac{-1}{z}$ and $y: z \mapsto \frac{z-1}{z}$ which obviously satisfy the relations $x^{2}=y^{3}=1$.

Then $\quad \lambda_{G} x=\frac{1-\sqrt{5}}{2}, \quad \lambda_{G} x y=\frac{3+\sqrt{5}}{2}, \quad \lambda_{G} x y^{2}=\frac{-1+\sqrt{5}}{2}$, $\lambda_{G} x y^{2} x=\frac{-1-\sqrt{5}}{2}, \quad \lambda_{G} x y^{2} x y^{2}=\frac{3-\sqrt{5}}{2} \quad$ and $\quad \lambda_{G} x y^{2} x y=\frac{1+\sqrt{5}}{2}=$ $\lambda_{G}$.

Corollary 1. $\lambda_{G}^{2}-\lambda_{G}-1=0$.
Proof. $\quad \lambda_{G} x y^{2} x y=\left(\lambda_{G}+1\right) y x y=\left(\frac{\lambda_{G}+1-1}{\lambda_{G}+1}\right) x y=\frac{\lambda_{G}+1-1}{\lambda_{G}+1}+1$.
Therefore $\quad \lambda_{G} x y^{2} x y=\lambda_{G}, \quad$ and so $\quad \frac{\lambda_{G}+1-1}{\lambda_{G}+1}+1=\lambda_{G} \quad$ yields $\lambda_{G}^{2}-\lambda_{G}-1=0$.

Corollary 2. Let $\bar{\lambda}_{G}$ denote the algebraic conjugate of $\lambda_{G}$. Then:
(i) $\quad \lambda_{G} x=\bar{\lambda}_{G}, \quad \lambda_{G} x y=\lambda_{G}^{2}, \quad \lambda_{G} x y^{2}=-\bar{\lambda}_{G}$,
(ii) $\quad\left(\lambda_{G} x y^{2}\right) x=-\lambda_{G}, \quad\left(\lambda_{G} x y^{2}\right) x y=\lambda_{G}, \quad\left(\lambda_{G} x y^{2}\right) x y^{2}=\left(\bar{\lambda}_{G}\right)^{2}$.

Proof. The proof follows directly from Corollary 1.
All Pisot numbers $\lambda, \lambda_{G}+\epsilon$ are known [1]. The Salem numbers are less well understood. The catalog of 39 Salem numbers given in [1] includes all Salem numbers $\lambda<1.3$ of degree less than or equal to 20 over the field of rationals. At present there are 47 known Salem numbers $\lambda<1.3$, and the list of such is known to be complete through degree 40 [2] and [3].

Next we give approximation of the golden mean. The Golden mean $\lambda_{G}=$ $\frac{1+\sqrt{5}}{2}$ is the quadratic irrationality, which is hardest to approximate by rational numbers, that is, $\lambda_{G}-\frac{p}{q} \neq 0$, where $p$ and $q$ are co-prime integers. We make $\left|\lambda_{G}-\frac{p}{q}\right|$ as small as possible for a fixed $q$, i.e., $\left|\lambda_{G}-\frac{p}{q}\right|<\varepsilon_{q}\left(\lambda_{G}\right)$, when $\varepsilon_{q}\left(\lambda_{G}\right)$ tends to zero as $q$ tends to infinity. Trivially, $\varepsilon_{q}\left(\lambda_{G}\right)<\frac{1}{2 q}$. We can, in fact, for any irrational $\alpha$, choose a sequence $q_{1}, q_{2}, \ldots, q_{n}, \ldots$ tending to infinity such that $\varepsilon_{q_{i}}(\alpha)<\frac{1}{q_{i}^{2}}$. For the number $\lambda_{G}=\frac{1+\sqrt{5}}{2}$,
we cannot do better than this. If $\beta=\frac{a \alpha+b}{c \alpha+d}$, where $a d-b c= \pm 1$ and $a, b, c, d$ are integers then by Liouvelli's Theorem approximation by rational integers is roughly the same for $\alpha$ as for $\beta$. In other words, if $\alpha$ is nearly $\frac{p}{q}$ then $\frac{a \frac{p}{q}+b}{c^{p}+d}$ is a good approximation to $\beta$.

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[^3]
# Abel-Grassmann's bands 

Petar V. Protić and Nebojša Stevanović


#### Abstract

Abel-Grassmann's groupoids or shortly $A G$-groupoids have been considered in a number of papers, although under the different names. In some papers they are named $L A$-semigroups [3] in others left invertive groupoids [2]. In this paper we deal with $A G$-bands, i.e., $A G$-groupoids whose all elements are idempotents. We introduce a few congruence relations on $A G$-band and consider decompositions of Abel-Grassmann's bands induced by these congruences. We also give the natural partial order on AbelGrassmann's band.


## 1. Introduction

A groupoid $S$ in which the following

$$
\begin{equation*}
(\forall a, b, c \in S) \quad a b \cdot c=c b \cdot a, \tag{1}
\end{equation*}
$$

is true is called an Abel-Grassmann's groupoid, [5]. It is easy to verify that in every $A G$-groupoid the medial law $a b \cdot c d=a c \cdot b d$ holds.

Abell-Grassmann's groupoids are not associative in general, however they have a close relation with semigroups and with commutative structures. Introducing a new operation on $A G$-groupoid makes it a commutative semigroup. On the other hand introducing a new operation on a commutative inverse semigroup turns it into an $A G$-groupoid.

Abel-Grassmann's groupoid satisfying $(\forall a, b, c \in S) a b \cdot c=b \cdot c a$ (called weak associative law in [4]) is an $A G^{*}$-groupoid. It is easy to prove that any $A G^{*}$-groupoid satisfies the permutation identity of a next type

$$
a_{1} a_{2} \cdot a_{3} a_{4}=a_{\pi(1)} a_{\pi(2)} \cdot a_{\pi(3)} a_{\pi(4)}
$$

2000 Mathematics Subject Classification: 20N02
Keywords: AG-groupoid, antirectangular $A G$-band, AG-band decompositions
Supported by Grant 1379 of Ministry of Science through Math. Inst.SANU
where $\pi$ is any permutation on a set $\{1,2,3,4\}$, [5].
Let ( $S, \cdot$ ) be $A G$-groupoid, $a \in S$ be a fixed element. We can define the "sandwich" operation on $S$ as follows:

$$
x \circ y=x a \cdot y, \quad x, y \in S
$$

It is easy to verify that $x \circ y=y \circ x$ for any $x, y \in S$, in other words ( $S, \circ$ ) is a commutative groupoid. If $S$ is $A G^{*}$-groupoid and $x, y, z \in S$ are arbitrary elements, then

$$
(x \circ y) \circ z=((x a \cdot y) a) z=z a \cdot(x a \cdot y)
$$

and

$$
x \circ(y \circ z)=x a \cdot(y \circ z)=x a \cdot(y a \cdot z)=z a \cdot(y a \cdot x)=z a \cdot(x a \cdot y),
$$

whence $(x \circ y) \circ z=x \circ(y \circ z)$. Consequently $(S, \circ)$ is a commutative semigroup.

Let $S$ be the commutative inverse semigroup. We define a new operation on $S$ as follows:

$$
a \bullet b=b a^{-1}, \quad a, b \in S .
$$

It has been shown in [3] that $(S, \bullet)$ is Abel-Grassmann's groupoid. Connections mentioned above makes $A G$-groupoid to be among the most interesting nonassociative structures. Same as in Semigroup Theory bands and band decompositions appears as the most fruitful methods for research on $A G$-groupoids.

If in $A G$-groupoid $S$ every element is an idempotent, then $S$ is an $A G$ band.

An $A G$-groupoid $S$ is an $A G$-band $Y$ of $A G$-groupoids $S_{\alpha}$ if

$$
S=\bigcup_{\alpha \in Y} S_{\alpha},
$$

$Y$ is an $A G$-band, $S_{\alpha} \cap S_{\beta}=\emptyset$ for $\alpha, \beta \in Y, \alpha \neq \beta$ and $S_{\alpha} S_{\beta} \subseteq S_{\alpha \beta}$.
A congruence $\rho$ on $S$ is called band congruence if $S / \rho$ is a band.

## 2. Some decompositions of $A G$-bands

Let $S$ be a semigroup and for each $a \in S, a^{2}=a$. That is, let $S$ be an associative band. If for all $a, b \in S, a b=b a$, then $S$ is a semilattice. If for all $a, b \in S, a=a b a$, then $S$ is the rectangular band. It is a well known
result in Semigroup Theory that the associative band $S$ is a semilattice of rectangular bands. It is not hard to prove that a commutative $A G$-band is a semilattice.

Let us now introduce the following notion.
Definition 2.1. Let $S$ be an $A G$-band, we say that $S$ is an antirectangular $A G$-band if for every $a, b \in S, a=b a \cdot b$.

Let us remark that in that case it holds

$$
\begin{equation*}
a=b a \cdot b=b a \cdot b b=b b \cdot a b=b \cdot a b . \tag{2}
\end{equation*}
$$

From above it follows that each antirectangular $A G$-band is a quasigroup.
Example 2.1. Let a groupoid $S$ be a given by the following table.

| $\cdot$ | 1 | 2 | 3 | 4 |
| :--- | :--- | :--- | :--- | :--- |
| 1 | 1 | 4 | 2 | 3 |
| 2 | 3 | 2 | 4 | 1 |
| 3 | 4 | 1 | 3 | 2 |
| 4 | 2 | 3 | 1 | 4 |

Then $S$ is an antirectangular $A G$-band and a quasigroup. Let us remark that $S$ is the unique $A G$-band of order 4 and we shall see below that it appears frequently in band decompositions both as an $A G$-band into which other bands can be decomposed and like a component. For this reasons from now on we shall call this band Traka 4 or simply T4. We also remark that nonassociative $A G$-bands of order $\leqslant 3$ do not exist.

An $A G$-band is anticommutative if for all $a, b \in S, a b=b a$ impies that $a=b$.

Lemma 2.1. Every antirectangular $A G$-band is anticommutative.
Proof. Let $S$ be an antirectangular band, $a, b \in S$ and $a b=b a$. Then

$$
a=b a \cdot b=a b \cdot b=b b \cdot a=b a=a b=a a \cdot b=b a \cdot a=a b \cdot a=b \text {. }
$$

Theorem 2.1. If $S$ is an $A G$-band, then $S$ is an $A G$-band $Y$ of, in general case nontrivial, antirectangular $A G$-bands $S_{\alpha}, \alpha \in Y$.

Proof. Let $S$ be an $A G$-band. Then we define the relation $\rho$ on $S$ as

$$
\begin{equation*}
a \rho b \Longleftrightarrow a=b a \cdot b, b=a b \cdot a . \tag{3}
\end{equation*}
$$

Clearly, the relation $\rho$ is reflexive and symmetric. If $a \rho b, b \rho c$, then by (2) and (3) we have

$$
\begin{aligned}
a c \cdot a & =a c \cdot(b a \cdot b)=((b a \cdot b) c) a=(c b \cdot b a) a \\
& =(a \cdot b a) \cdot c b=b \cdot c b=c .
\end{aligned}
$$

Similarly, $a=c a \cdot c$ thus the relation $\rho$ is transitive. Hence, $\rho$ is an equivalence relation.

Let $a \rho b$ and $c \in S$. Then by (1) and the medial law we have

$$
\begin{aligned}
a c & =(b a \cdot b) c=c b \cdot b a=(c c \cdot b) \cdot b a=(b c \cdot c) \cdot b a \\
& =(b a \cdot c) \cdot b c=(b a \cdot c c) \cdot b c=(b c \cdot a c) \cdot b c .
\end{aligned}
$$

Dually, $b c=(a c \cdot b c) \cdot a c$ and so $a c \rho b c$. Also,

$$
\begin{aligned}
c a & =c c \cdot a=a c \cdot c=((b a \cdot b) c) c=(c b \cdot b a) c=(c \cdot b a) \cdot c b \\
& =(c c \cdot b a) \cdot c b=(c b \cdot c a) \cdot c b .
\end{aligned}
$$

Dually, $c b=(c a \cdot c b) \cdot c a$ and so $c a \rho c b$. Hence, $\rho$ is a congruence on $S$.
Since $S$ is a band we have that $\rho$ is a band congruence on $S$. From $a \rho b$ we have $a=a^{2} \rho a b$, whence it follows that $\rho$-classes are closed under the operation. By the definition of $\rho$ it follows that $\rho$-classes are antirectangular $A G$-bands. By Lemma 2.1, $\rho$ classes are anticommutative $A G$-bands.

In Example 2.1. we have $\rho=S \times S$.
Example 2.2. Let $A G$-band $S$ be given by the following table.

| $\cdot$ | 1 | 2 | 3 | 4 | 5 | 6 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 1 | 2 | 2 | 5 | 6 | 4 |
| 2 | 2 | 2 | 2 | 5 | 6 | 4 |
| 3 | 2 | 2 | 3 | 5 | 6 | 4 |
| 4 | 6 | 6 | 6 | 4 | 2 | 5 |
| 5 | 4 | 4 | 4 | 6 | 5 | 2 |
| 6 | 5 | 5 | 5 | 2 | 4 | 6 |

Now, $S=S_{\alpha} \cup S_{\beta} \cup S_{\gamma}$ where $S_{\alpha}=\{1\}, S_{\beta}=\{3\}, S_{\gamma}=\{2,4,5,6\}$ are equivalence classes $\bmod \rho$ and $Y=\{\alpha, \beta, \gamma\}$ is a semilattice. Obviously, $S_{\alpha}, S_{\beta}$ are trivial $A G$-bands and $S_{\gamma}$ is anti-isomorphic with $A G$-band $T 4$ (as is Example 2.1.).
Lemma 2.2. Let $S$ be an $A G$-band and $e, a, b \in S$. Then $e a=e b$ implies that $a e=b e$ and conversely.

Proof. Suppose that $e a=e b$, then

$$
\begin{aligned}
a e & =a a \cdot e=e a \cdot a=e b \cdot a=e b \cdot a a=e a \cdot b a=e b \cdot b a \\
& =(e e \cdot b) \cdot b a=(b e \cdot e) \cdot b a=(b a \cdot e) \cdot b e=(e a \cdot b) \cdot b e \\
& =(e b \cdot b) \cdot b e=(b b \cdot e) \cdot b e=b e \cdot b e=b e
\end{aligned}
$$

Conversely, suppose that $a e=b e$, then

$$
e a=e e \cdot a=a e \cdot e=b e \cdot e=e e \cdot b=e b .
$$

Remark 2.1. As a consequence of Lemma 2.2, $e=e f$ and so $e=f e$ and conversely.

Theorem 2.2. Let $S$ be an $A G$-band. Then the relation $\nu$ defined on $S$ by

$$
a \nu b \Longleftrightarrow(\exists e \in S) e a=e b
$$

is a band congruence relation on $S$.
Proof. Reflexivity and symmetry is obvious. Suppose that $a \nu b$ and $b \nu c$ for some $a, b, c \in S$. Then there exist elements $e, f \in S$ such that $e a=e b$ and $f b=f c$. According to the Lemma 2.2 we also have $a e=b e, b f=c f$. Now

$$
\begin{aligned}
f e \cdot a & =a e \cdot f=b e \cdot f=b e \cdot f f=b f \cdot e f=c f \cdot e f \\
& =c e \cdot f f=c e \cdot f=f e \cdot c,
\end{aligned}
$$

implies that $\nu$ is transitive.
It remains to prove compatibility. Suppose $a \nu b$ and let $c \in S$ be an arbitrary element. Then there exists $e \in S$ such that $e a=e b$. We have, now

$$
c \cdot e a=c \cdot e b \Longrightarrow c c \cdot e a=c c \cdot e b \Longrightarrow c e \cdot c a=c e \cdot c b,
$$

so $a \nu c b$. Similarly

$$
e a \cdot c=e b \cdot c \Longrightarrow e a \cdot c c=e b \cdot c c \Longrightarrow e c \cdot a c=e c \cdot b c,
$$

so acıbc.
In Example 2.1 we have $\nu \equiv \triangle$, since $S$ is a quasigroup. In Example 2.2, $S=S_{\alpha} \cup S_{\beta} \cup S_{\gamma} \cup S_{\delta}$, where $S_{\alpha}=\{1,2,3\}, S_{\beta}=\{4\}, \quad S_{\gamma}=\{5\}$, $S_{\delta}=\{6\}$ are the equivalence classes $\bmod \nu$. Let us remark that $A G$-band $Y=\{\alpha, \beta, \gamma, \delta\}$ is anti-isomorphic with $T 4$.

Lemma 2.3. Let $S$ be an $A G$-groupoid. Then the relation $\sigma$ on $S$ defined by the formula

$$
a \sigma b \Longleftrightarrow a b=b a
$$

is reflexive, symmetric and compatible.

Proof. Clearly $\sigma$ is reflexive and symmetric. If $a \sigma b$ and $c \in S$, then by medial law we have

$$
\begin{aligned}
a c \cdot b c & =a b \cdot c c=b a \cdot c c=b c \cdot a c, \\
c a \cdot c b & =c c \cdot a b=c c \cdot b a=c b \cdot c a .
\end{aligned}
$$

Hence $a c \sigma b c, c a \sigma c b$, and so $\sigma$ is left and right compatible. This means that $\sigma$ is compatible.

Definition 2.2. Let $S$ be an $A G$-band. Then $S$ is transitively commutative if for every $a, b, c \in S$ from $a b=b a$ and $b c=c b$ it follows that $a c=c a$.

It is easy to verify that $A G$-bands in examples 2.1 and 2.2 are transitively commutative.

Theorem 2.3. Let $S$ be a transitively commutative $A G$-band. Then $S$ is an $A G$-band $Y$ of, in general case nontrivial, semilattices $S_{\alpha}, \alpha \in Y$.

Proof. In this way the relation $\sigma$ defined by (3) is transitive. Now, by Lemma 2.3 we have that relation $\sigma$ is a band congruence on $S$. Clearly, $\sigma$-classes are commutative $A G$-bands, i.e., semilattices.

In Example 2.2 we have that $S=S_{\alpha} \cup S_{\beta} \cup S_{\gamma} \cup S_{\delta}, A G$-band $Y=$ $\{\alpha, \beta, \gamma, \delta\}$ is anti-isomorphic with $A G$-band $T 4, S_{\alpha}=\{1,2,3\}$ is nontrivial semilattice and $S_{\beta}=\{4\}, S_{\gamma}=\{5\}, S_{\delta}=\{6\}$ are trivial semilattices.

Now, let $S$ be a transitively commutative $A G$-band, and let $a \sigma b \Longleftrightarrow$ $a b=b a$. Then from

$$
\begin{aligned}
a b \cdot a & =b a \cdot a=a a \cdot b=a a \cdot b b=a b \cdot a b, \\
a b \cdot b & =b b \cdot a=b b \cdot a a=b a \cdot b a=a b \cdot a b
\end{aligned}
$$

it follows that $a b \cdot a=a b \cdot b$, and so $a \nu b$. Hence, if $S$ is an transitively commutative $A G$-band, then $\sigma \subseteq \nu$.

## 3. The natural partial order of AG-band

Theorem 3.1. If $S$ is $A G$-band, then the relation $\leqslant$ defined on $E(S)$

$$
e \leqslant f \Longleftrightarrow e=e f
$$

is a (natural) partial order relation and $\leqslant$ is compatible with the right and with the left with multiplication.

Proof. Clearly, $e \leqslant e$ and relation $\leqslant$ is reflexive. Let $e \leqslant f, f \leqslant e$, then $e=e f, f=f e$ and by the Remark 2.1 we have $e=f$ so relation $\leqslant$ is antisymmetric. If $e \leqslant f, f \leqslant g$ then $e=e f, f=f g$ also by the Remark 2.1 it holds that $f=g f$. Now by (1) it follows that

$$
e g=e f \cdot g=g f \cdot e=f e=e .
$$

Hence, $e \leqslant g$ and relation $\leqslant$ is transitive thus $\leqslant$ is a partial order relation. Now, $e \leqslant f \Longleftrightarrow e=e f$ and $g \in S$ yields

$$
\begin{aligned}
& e g=e f \cdot g=e f \cdot g g=e g \cdot f g, \\
& g e=g \cdot e f=g g \cdot e f=g e \cdot g f
\end{aligned}
$$

so $e g \leqslant f g, g e \leqslant g f$. Hence, the relation $\leqslant$ is left and right compatible with multiplication.

In Example 2.1, $\leqslant \equiv \triangle$. In Example 2.2 we have $2<1,2<3$ while other elements are uncomparable.

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# Hyper I-algebras and polygroups 

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#### Abstract

In this note first we give the notion of hyper $I$-algebra, which is a generalization of $B C I$-algebra and also it is a generalization of hyper $K$-algebra. Then we obtain some fundamental results about this notion. Finally we give some relationships between the notion of hyper $I$-algebra and the notions of hypergroup and polygroup. In particular we study these connections categorically. In other words by considering the categories of hyper $I$-algebrs, hypergroups and commutative polygroups, we give some full and faithful functors.


## 1. Introduction

The hyperalgebraic structure theory was introduced by F.Marty [8] in 1934. Imai and Iseki [7] in 1966 introduced the notion of a $B C K$-algebra. Recently [2], [9] Borzooei, Jun and Zahedi et.al. applied the hypersrtucture to $B C K$-algebras and introduced the concepts of hyper $K$-algebra which is a generalization of $B C K$-algebra. In [5] 1988 Dudek obtained some connections between $B C I$-algebras and (quasi)groups. Bonansinga and Corsini [1] in 1982 introduced the notion of quasi-canonical hypergroup, called polygroup by Comer [3]. Now in this note we consider all of the above referred papers and introduce the notion of hyper $I$-algebra and then we obtain some results as mentioned in the abstract.

## 2. Preliminaries

By a hyperstructure ( $H, \circ$ ) we mean a nonempty set $H$ with a hyperoperation ○, i.e. a function $\circ$ from $H \times H$ to $\mathcal{P}(H) \backslash\{\emptyset\}$.

Definition 2.1. A hyperstructure ( $H, \circ$ ) is called hypergroup if:
(i) $(x \circ y) \circ z=x \circ(y \circ z)$ for all $x, y, z \in H$,
(ii) $a \circ H=H \circ a=H \quad$ for all $a \in H$,
(i.e. for all $a, b \in H$ there exist $c, d \in H$ such that $b \in c \circ a$ and $b \in a \circ d$ ).

Definition 2.2. A hyperstructure ( $H, \circ$ ) is called quasi-canonical hypergroup or polygroup if it satisfies the following conditions:
(i) $(x \circ y) \circ z=x \circ(y \circ z)$ for all $x, y, z \in H$ (associative law),
(ii) there exists $e \in H$ such that $e \circ x=\{x\}=x \circ e$ for all $x \in H$ (identity element),
(iii) for all $x \in H$ there exists a unique element $x^{\prime} \in H$ such that $e \in\left(x \circ x^{\prime}\right) \bigcap\left(x^{\prime} \circ x\right)$, we denote $x^{\prime}$ by $x^{-1}$ (inverse element),
(iv) for all $x, y, z \in H$ we have: $z \in x \circ y \Longrightarrow x \in z \circ y^{-1} \Longrightarrow y \in x^{-1} \circ z$ (reversibility property).
If $(H, \circ)$ is a polygroup and $x \circ y=y \circ x$ holds for all $x, y \in H$, then we say that $H$ is a commutative polygroup.

If $A \subseteq H$, then by $A^{-1}$ we mean the set $\left\{a^{-1}: a \in A\right\}$.
Lemma 2.3. Let $(H, \circ)$ be a polygroup. Then for all $x, y \in H$, we have:
(i) $\left(x^{-1}\right)^{-1}=x$,
(ii) $e=e^{-1}$,
(iii) $e$ is unique,
(iv) $(x \circ y)^{-1}=y^{-1} \circ x^{-1}$.

Proof. See [4].
Lemma 2.4. Let $(H, \circ)$ be a polygroup. Then $(A \circ B) \circ C=A \circ(B \circ C)$ for all nonempty subsets $A, B$ and $C$ of $H$.

## 3. Hyper $I$-algebra

Definition 3.1. A hyperstructure ( $H, \circ$ ) is called a hyper I-algebra if it contains a constant 0 and satisfies the following axioms:
(HK1) $(x \circ z) \circ(y \circ z)<x \circ y$,
(HK2) $(x \circ y) \circ z=(x \circ z) \circ y$,
(HK3) $x<x$,
(HK4) $x<y, y<x \Longrightarrow x=y$,
(HI5) $x<0 \Longrightarrow x=0$,
for all $x, y, z \in H$, where $x<y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H, A<B$ is defined by $\exists a \in A, \exists b \in B$ such that $a<b$.

A simple example of a hyper $I$-algebra is a $B C I$-algebra $(H, *, 0)$ with the hyperopration $\circ$ defined by $x \circ y=\{x * y\}$. Also it is not difficult to see that a hyper $I$-algebra is a generalization of hyper $K$-algebras considered in [2] and [9]. The following example shows that there are hyper $I$-algebras which are not a hyper $K$-algebras.
Example 3.2. Let $H=\{0,1,2\}$. Then the following tables show the hyper $I$-algebra structures on $H$.

| $\circ$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | $\{0\}$ | $\{0\}$ | $\{2\}$ |
| 1 | $\{1\}$ | $\{0\}$ | $\{2\}$ |
| 2 | $\{2\}$ | $\{2\}$ | $\{0,2\}$ |


| $\circ$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | $\{0\}$ | $\{0,1\}$ | $\{2\}$ |
| 1 | $\{1\}$ | $\{0\}$ | $\{2\}$ |
| 2 | $\{2\}$ | $\{0\}$ | $\{0,1,2\}$ |

Note that none of the above hyper $I$-algebras is not a hyper $K$-algebra, because $0 \nless 2$.

Theorem 3.3. Let $(H, \circ, 0)$ be a hyper I-algebra. Then for all $x, y, z \in H$ and for all non-empty subsets $A, B$ and $C$ of $H$ the following hold:
(i) $x \circ y<z \Longleftrightarrow x \circ z<y$, (vi) $A<A$,
(ii) $(x \circ z) \circ(x \circ y)<y \circ z, \quad($ vii $) \quad(A \circ C) \circ(A \circ B)<B \circ C$,
(iii) $x \circ(x \circ y)<y, \quad(v i i i) \quad(A \circ C) \circ(B \circ C)<A \circ B$,
(iv) $(A \circ B) \circ C=(A \circ C) \circ B, \quad(i x) \quad A \circ B<C \Leftrightarrow A \circ C<B$.
(v) $A \subseteq B \Longrightarrow A<B$,

Proof. The proof is similar to the proof of Proposition 2.5 of [2].
Example 3.4. Let $H=\{0,1,2\}$. Then the following table shows a hyper $I$-algebra structure on $H$ such that $x \circ y \nless x$, because $1 \circ 2=2 \nless 1$.

| $\circ$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | $\{0\}$ | $\{0\}$ | $\{2\}$ |
| 1 | $\{1\}$ | $\{0,1\}$ | $\{2\}$ |
| 2 | $\{2\}$ | $\{2\}$ | $\{0\}$ |

Lemma 3.5. Let $H$ be a hyper I-algebra. Then for all $x$ in $H$ we have:
(i) $x \circ 0<x$,
(ii) $x \in x \circ 0$.

Proof. (i) We have $0 \in 0 \circ 0 \subseteq(x \circ x) \circ 0=(x \circ 0) \circ x$. So there exists $t \in x \circ 0$ such that $0 \in t \circ x$. Thus $t<x$, and hence $x \circ 0<x$.
(ii) By (i) $x \circ 0<x$. So there exists $t \in x \circ 0$ such that $t<x$. Since $t \in x \circ 0$, then $x \circ 0<t$ and hence $x \circ t<0$, by Theorem 3.3(i). Thus there exists $h \in x \circ t$ such that $h<0$, so by (HI5) we have $h=0$. Therefore $0 \in x \circ t$ and hence $x<t$. Since $t<x$, then by (HK4) we get that $t=x$. Therefore $x \in x \circ 0$.

Definition 3.6. Let $(H, \circ, 0)$ be a hyper $I$-algebra. We define

$$
H^{+}=\{x \in H \mid 0 \in 0 \circ x\} .
$$

Note that $H^{+} \neq \emptyset$ because $0 \in 0 \circ 0$.
Proposition 3.7. Let $(H, \circ, 0)$ be a hyper I-algebra. Then $\left(H^{+}, \circ, 0\right)$ is a hyper $K$-algebra if and only if $x \circ y \subseteq H^{+}$, for all $x, y$ in $H^{+}$.

Proof. Straightforward.
Example 3.8. (i) Let $H=\{0,1,2\}$. Then the following tables show two different hyper $I$-algebra structures on $H$ :

| $\circ$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | $\{0\}$ | $\{0\}$ | $\{2\}$ |
| 1 | $\{1\}$ | $\{0,1\}$ | $\{2\}$ |
| 2 | $\{2\}$ | $\{2\}$ | $\{0,1\}$ |


| $\circ$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | $\{0\}$ | $\{0\}$ | $\{2\}$ |
| 1 | $\{1\}$ | $\{0,1\}$ | $\{0,2\}$ |
| 2 | $\{2\}$ | $\{2\}$ | $\{0,1,2\}$ |

We can seen that $H^{+}=\{0,1\}$ and it is a hyper $K$-algebra.
(ii) The following table shows a hyper $I$-algebra structure on $H=\{0,1,2\}$, where $H^{+}=\{0,1\}$ and it is not a hyper $K$-algebra, since $1 \in H^{+}$but $1 \circ 1 \nsubseteq H^{+}$.

| $\circ$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | $\{0\}$ | $\{0\}$ | $\{2\}$ |
| 1 | $\{1\}$ | $\{0,2\}$ | $\{0,2\}$ |
| 2 | $\{2\}$ | $\{2\}$ | $\{0,2\}$ |

Theorem 3.9. Let $(H, \circ, e)$ be a commutative polygroup. Then $(H, \diamond, e)$ is a hyper I-algebra, where the hyperopration $\diamond$ is defined by $x \diamond y=x \circ y^{-1}$. Furthermore we have:
(i) $H^{+}=\{e\}$,
(ii) $e \diamond(e \diamond x)=x$ for all $x$ in $H$.

Proof. (HK1) Let $A=(x \diamond y) \diamond(z \diamond y)$. Then by considering Lemma 2.3 we have $A=(x \diamond y) \diamond(z \diamond y)=\bigcup_{\substack{a \in x \diamond y \\ b \in z \diamond y}} a \diamond b=\bigcup_{\substack{a x \circ y-1 \\ b \in z o y^{-1}}} a \circ b^{-1}=\bigcup_{\substack{a \in x, y-1 \\ b^{-1} \in y \circ z^{-1}}} a \circ b^{-1}$.
Thus, by Lemma 2.4, we get that

$$
A=\left(x \circ y^{-1}\right) \circ\left(y \circ z^{-1}\right)=x \circ\left(y^{-1} \circ\left(y \circ z^{-1}\right)\right)=x \circ\left(\left(y^{-1} \circ y\right) \circ z^{-1}\right) .
$$

By Lemma 2.3 we have

$$
A \diamond(x \diamond z)=\bigcup_{\substack{a \in A \\ b \in x \diamond z}} a \diamond b=\bigcup_{\substack{a \in A \\ b \in x \circ z^{-1}}} a \circ b^{-1}=A \circ\left(z \circ x^{-1}\right) .
$$

Since $e \in y^{-1} \circ y$, hence $e \circ z^{-1} \subseteq\left(y^{-1} \circ y\right) \circ z^{-1}$, so

$$
x \circ\left(e \circ z^{-1}\right) \subseteq x \circ\left(\left(y^{-1} \circ y\right) \circ z^{-1}\right)=A
$$

Thus we get that
$\left(x \circ z^{-1}\right) \circ\left(z \circ x^{-1}\right)=\left(x \circ\left(e \circ z^{-1}\right)\right) \circ\left(z \circ x^{-1}\right) \subseteq A \circ\left(z \circ x^{-1}\right)=A \diamond(x \diamond z)$.
Now, by Definition 2.2 and Lemma 2.4 we have

$$
\begin{aligned}
& x \circ\left(\left(z^{-1} \circ z\right) \circ x^{-1}\right) \\
&=x \circ\left(z^{-1} \circ\left(z \circ x^{-1}\right)\right)=\left(x \circ z^{-1}\right) \circ\left(z \circ x^{-1}\right) \subseteq A \diamond(x \diamond z) .
\end{aligned}
$$

Since $e \in z^{-1} \circ z$ and $e \in x \circ x^{-1}$, then we have $e \in A \diamond(x \diamond z)$, so $A<x \diamond z$. Therefore $(x \diamond y) \diamond(z \diamond y)<x \diamond z$.
(HK2) By Definition 2.2 and hypothesis we get that $(x \diamond y) \diamond z=\left(x \circ y^{-1}\right) \diamond z=$ $\left(x \circ y^{-1}\right) \circ z^{-1}=x \circ\left(y^{-1} \circ z^{-1}\right)=x \circ\left(z^{-1} \circ y^{-1}\right)=\left(x \circ z^{-1}\right) \circ y^{-1}=(x \diamond z) \diamond y$. Therefore (HK2) holds.
(HK3) Since $e \in x \circ x^{-1}=x \diamond x$ we conclude that $x<x$ and hence (HK3) holds.
(HK4) To show that (HK4) holds, we prove that $x<y$ implies that $x=y$. Let $x<y$. Then $e \in x \diamond y=x \circ y^{-1}$. By Definition 2.2 (vi) we have $y \in e^{-1} \circ x=e \circ x=\{x\}$, thus $y=x$.
(HI5) Let $x<e$. Then by the proof of (HK4) we get that $e=x$, and hence (HI5) holds.

Therefore ( $H, \diamond, e$ ) is a hyper $I$-algebra.
The proofs of the statements (i) and (ii) are routine.

## Category of commutative polygroups: $\mathcal{C P G}$

Consider the class of all polygroups. For any two polygroups ( $H_{1}, \circ_{1}, e_{1}$ ) and ( $H_{2}, \mathrm{o}_{2}, e_{2}$ ) we define a morphism $f: H_{1} \longrightarrow H_{2}$ as a strong homomorphism between $H_{1}$ and $H_{2}$ (i.e. $\left.f\left(x \circ_{1} y\right)=f(x) \circ_{2} f(y) \forall x, y \in H\right)$, which satisfies $f\left(e_{1}\right)=e_{2}$. Then it can easily checked that the class of all polygroups and the above morphisms construct a category which is denoted by $\mathcal{C P G}$.

Remark 3.10. It is well known that if $f \in \mathcal{C P} \mathcal{G}\left(H_{1}, H_{2}\right)$, then $f\left(x^{-1}\right)=$ $(f(x))^{-1}$ for all $x \in H_{1}$.

## Category of hyper I-algebras: $\mathcal{I} \mathcal{A} \mathcal{L} G$

Consider the class of all hyper $I$-algebras. For any two $I$-algebras ( $H_{1}, \circ_{1}, 0_{1}$ ) and ( $H_{2}, \mathrm{o}_{2}, 0_{2}$ ) we define a morphism $f: H_{1} \longrightarrow H_{2}$ as a strong homomorphism between $H_{1}$ and $H_{2}$, which satisfies the condition $f\left(0_{1}\right)=0_{2}$. Then it can easily checked that the class of all hyper $I$-algebras and the above morphisms construct a category which is denoted by $\mathcal{I A} \mathcal{L G}$.

Theorem 3.11. $F: \mathcal{C P G} \longrightarrow \mathcal{I A L G}$ is a faithful functor, where $F(H, \circ, e)=$ $(H, \diamond, e)$ and $F(f)=f$ for all $H \in \mathcal{C P G}$ and $f \in \mathcal{C P G}\left(H_{1}, H_{2}\right)$.
Proof. Let $(H, \circ, e)$ be a polygroup. Then by Theorem $3.9(H, \diamond, e)$ is a hyper $I$-algebra, hence $F(H)$ is an object in $\mathcal{I} \mathcal{A L G}$. Now let $f \in \mathcal{C P G}\left(H_{1}, H_{2}\right)$ we prove that $F f \in \mathcal{I} \mathcal{A} \mathcal{L}\left(F\left(H_{1}\right), F\left(H_{2}\right)\right)$. By Theorem 3.9 we have

$$
\begin{aligned}
F f\left(x \diamond_{1} y\right) & =f\left(x \diamond_{1} y\right)=f\left(x \circ_{1} y^{-1}\right)=f(x) \circ_{2} f\left(y^{-1}\right) \\
& =f(x) \circ_{2}(f(y))^{-1}=f(x) \diamond_{2} f(y)=(F f)(x) \diamond_{2}(F f)(y) .
\end{aligned}
$$

Now it is easy to see that $F$ satisfies to the other conditions of a functor. Since $F$ maps $\mathcal{C P} \mathcal{G}\left(H_{1}, H_{2}\right)$ injectively to $\mathcal{I} \mathcal{A} \mathcal{L}\left(F H_{1}, F H_{2}\right)$, hence $F$ is faithful.

Problem: Is the functor $F$ (defined in Theorem 3.11) full embedding ?
Definition 3.12. A hyperstructure ( $H, \circ$ ) is called a semipolygroup if it satisfies the following axioms:
(i) $(x \circ y) \circ z=x \circ(y \circ z)$ for all $x, y, z \in H$,
(ii) there exists $e \in H$ such that $e \circ x=\{x\}=x \circ e$ for all $x \in H$,
(iii) for all $x \in H$ there exists a unique element $x^{\prime} \in H$ such that $e \in\left(x \circ x^{\prime}\right) \bigcap\left(x^{\prime} \circ x\right)$, we denote $x^{\prime}$ by $x^{-1}$.

Example 3.13. Let $H=\{0,1,2\}$ and the hyperopration $\circ$ on $H$ is given by the following table:

| $\circ$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | $\{0\}$ | $\{1\}$ | $\{2\}$ |
| 1 | $\{1\}$ | $\{2\}$ | $\{0,1\}$ |
| 2 | $\{2\}$ | $\{0,1\}$ | $\{1,2\}$ |

Then $H$ is a semipolygroup, but it is not a polygroup because the reversibility does not hold. Indeed, $1 \in 1 \circ 2=\{0,1\}$ but $1 \notin 1 \circ 2^{-1}=1 \circ 1=\{2\}$.

Lemma 3.14. Any group can be cosidered as a semipolygroup.
Lemma 3.15. Let $(H, \circ, 0)$ be a hyper I-algebra. If $H^{+} \neq\{0\}$, then $0 \circ(0 \circ x) \neq x$ for all nonzero elements $x \in H^{+}$.

Proof. Let $x \neq 0$ be in $H^{+}$. Then $0 \in(0 \circ x)$. Thus $0 \in(0 \circ 0) \subseteq 0 \circ(0 \circ x)$, hence $0 \in 0 \circ(0 \circ x)$. Since $x \neq 0$, so $0 \circ(0 \circ x) \neq x$.

Note that the following example shows that if $H^{+}=\{0\}$, then it may be that the equality $0 \circ(0 \circ x)=x$ holds or does not hold.

Example 3.16. (i) Let $H=\{0,1,2\}$. Then the following table shows a hyper $I$-algebra structure on $H$ such that $H^{+}=\{0\}$, while $0 \circ(0 \circ 2)=$ $0 \circ 1=1 \neq 2$.

| $\circ$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | $\{0\}$ | $\{1\}$ | $\{1\}$ |
| 1 | $\{1\}$ | $\{0,1\}$ | $\{0,1\}$ |
| 2 | $\{2\}$ | $\{1\}$ | $\{0,1,2\}$ |

(ii) The following table shows a hyper $I$-algebra structure on $H=\{0,1,2\}$. Then $H^{+}=\{0\}$ and $0 \circ(0 \circ x)=x$ for all $x \in H$.

| $\circ$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | $\{0\}$ | $\{2\}$ | $\{1\}$ |
| 1 | $\{1\}$ | $\{0,1\}$ | $\{2\}$ |
| 2 | $\{2\}$ | $\{1,2\}$ | $\{0,1\}$ |

Theorem 3.17. Let $(H, \circ, 0)$ be a hyper $I$-algebra. If $H^{+}=\{0\}$ and $0 \circ(0 \circ x)=x$ for all $x \in H$, then $(H, \odot, 0)$ is a commutative semipolygroup, where the hyperopration $\odot$ is defined by $x \odot y=x \circ(0 \circ y)$.
Proof. By Theorem 3.3(iv) we get that $x \odot y=x \circ(0 \circ y)=(0 \circ(0 \circ x)) \circ(0 \circ y)=$ $(0 \circ(0 \circ y)) \circ(0 \circ x)=y \circ(0 \circ x)=y \odot x$, namely $(H, \odot)$ is commutative.

Now we show that $(H, \odot)$ is associative. We have

$$
\begin{aligned}
(x \odot y) \odot z & =(x \circ(0 \circ y)) \circ(0 \circ z) & & \\
& =(x \circ(0 \circ z)) \circ(0 \circ y) & & \text { by Theorem } 3.3 \text { (iv) } \\
& =((0 \circ(0 \circ x)) \circ(0 \circ z)) \circ(0 \circ y) & & \text { by hypothesis } \\
& =((0 \circ(0 \circ z)) \circ(0 \circ x)) \circ(0 \circ y) & & \text { by Theorem } 3.3 \text { (iv) } \\
& =(z \circ(0 \circ y)) \circ(0 \circ x) & & \text { by Theorem } 3.3 \text { (iv) } \\
& =(z \odot y) \odot x & & \\
& =x \odot(z \odot y) & & \text { by commutativity } \\
& =x \odot(y \odot z) & & \text { by commutativity }
\end{aligned}
$$

Thus $(H, \odot)$ is associative.
Now, we prove that $0 \circ x$ has only one element for all $x \in H$. On the contrary, let $x_{1}, x_{2} \in 0 \circ x$ and $x_{1} \neq x_{2}$. Then by hypothesis we have $0 \circ x_{1} \subseteq 0 \circ(0 \circ x)=x$, hence $0 \circ x_{1}=x$ and similarly $0 \circ x_{2}=x$. Thus $0 \circ\left(0 \circ x_{1}\right)=x_{1}$ and $0 \circ x_{1}=x$ imply that $0 \circ x=x_{1}$. Since $x_{2} \in 0 \circ x$, hence $x_{1}=x_{2}$ which is a contradiction.

Since $0 \circ x$ has only one element for all $x \in H$, hence $0 \in 0 \circ 0$, thus we conclude that $0 \circ 0=0$. By Theorem 3.3 (iv) and hypothesis we get that $x \circ 0=(0 \circ(0 \circ x)) \circ 0=(0 \circ 0) \circ(0 \circ x)=0 \circ(0 \circ x)=x$. Hence $x \circ 0=x$. Therefore $0 \odot x=x \odot 0=x \circ(0 \circ 0)=x \circ 0=x$. So $(H, \odot)$ satisfies in condition (ii) of Definition 3.12.

Since $H^{+}=\{0\}$ hence $0 \notin 0 \circ x$ for all $x \neq 0$. Therefore for all $0 \neq x \in H$ there exists $0 \neq x^{\prime} \in H$ such that $0 \circ x=x^{\prime}$. By Theorem 3.3 (vi) we have $0 \in(0 \circ x) \circ(0 \circ x)=x^{\prime} \circ(0 \circ x)=x^{\prime} \odot x=x \odot x^{\prime}$. Thus the condition (iii) of Definition 3.12 holds. Therefore $(H, \odot)$ is a commutative semipolygroup.

Theorem 3.18. Let $(H, \circ, 0)$ be a hyper I-algebra such that $H^{+}=\{0\}$. If $0 \circ(0 \circ x)=x$ and $x \circ x=0$ hold for all $x \in H$, then $(H, \odot, 0)$ is an abelian group.
Proof. By considering Theorem 3.17 it is sufficient to show that $x \circ y$ has only one element for all $x, y \in H$. On the contrary let $x_{1} \neq x_{2}$ and $x_{1}, x_{2} \in x \circ y$. Then by the proof of Theorem 3.17 we conclude that there are $x^{\prime}, y^{\prime} \in H$ such that $0 \circ x=x^{\prime}, 0 \circ y=y^{\prime}, 0 \circ x^{\prime}=x$ and $0 \circ y^{\prime}=y$. By (HK2) and $x \circ x=0$ we get that $y^{\prime}=0 \circ y=(x \circ x) \circ y=(x \circ y) \circ x$. Since $x_{1}, x_{2} \in x \circ y$, hence $x_{1} \circ x=y^{\prime}$ and $x_{2} \circ x=y^{\prime}$. Thus $y^{\prime} \circ x_{1}=$ $\left(x_{1} \circ x\right) \circ x_{1}=\left(x_{1} \circ x_{1}\right) \circ x=0 \circ x=x^{\prime}$ and also $y^{\prime} \circ x_{2}=x^{\prime}$. By (HK2) and hypothesis we get that $\left(y^{\prime} \circ x^{\prime}\right) \circ x_{1}=\left(y^{\prime} \circ x_{1}\right) \circ x^{\prime}=x^{\prime} \circ x^{\prime}=\{0\}$,
similarly $\left(y^{\prime} \circ x^{\prime}\right) \circ x_{2}=\{0\}$. Since $0 \in\left(y^{\prime} \circ x^{\prime}\right) \circ x_{1}$ so there exists $t \in y^{\prime} \circ x^{\prime}$ such that $0 \in t \circ x_{1}$. By (HK2) we have $\left(t \circ x_{1}\right) \circ t=(t \circ t) \circ x_{1}=0 \circ x_{1}$. Since $0 \in t \circ x_{1}$ hence $0 \circ t \subseteq 0 \circ x_{1}$. By the proof of Theorem $3.170 \circ x_{1}$ has only one element so we get that $0 \circ t=0 \circ x_{1}$. By hypothesis we have $t=0 \circ(0 \circ t)=0 \circ\left(0 \circ x_{1}\right)=x_{1}$. Therefore $x_{1} \in y^{\prime} \circ x^{\prime}$. Since $\left(y^{\prime} \circ x^{\prime}\right) \circ x_{2}=0$, then $x_{1} \circ x_{2}=0$ and similarly $x_{2} \circ x_{1}=0$. Thus (HK4) implies that $x_{1}=x_{2}$, which is a contradiction. So $x \circ y$ has only one element. Therefore Theorem 3.17 implies that $(H, \odot, 0)$ is an abelian group.

Since every group is a polygroup hence $(H, \odot)$ in Theorem 3.18 is a commutative polygroup. The following example shows that in Theorem 3.18 the condition $x \circ x=0$ for all $x \in H$ is necessary.

Example 3.19. Let $H=\{0,1,2\}$ be a hyper $I$-algebra, in which the hyperopration $\circ$ is given by the following table:

| $\circ$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: | :---: |
| 0 | $\{0\}$ | $\{2\}$ | $\{1\}$ |
| 1 | $\{1\}$ | $\{0,1\}$ | $\{2\}$ |
| 2 | $\{2\}$ | $\{1,2\}$ | $\{0,1\}$ |

Then $H^{+}=\{0\}, 0 \circ(0 \circ x)=x$ for all $x \in H$ and $1 \circ 1 \neq 0$. But $(H, \odot, 0)$ is not a group since $1 \odot 2=\{0,1\}$.

Note that the above example also shows that if we omit the condition $x \circ x=0$, in Theorem 3.18, then $(H, \odot)$ is not necessary to be a polygroup. Because the reversibility property does not hold. Indeed, in this example we have $1 \in 1 \odot 2=1 \circ(0 \circ 2)=1 \circ 1=\{0,1\}$, but $1 \notin 1 \odot 2^{-1}=$ $1 \circ\left(0 \circ 2^{-1}\right)=1 \circ(0 \circ 1)=1 \circ 2=2$.

Theorem 3.20. Let $(H, \circ, 0)$ be a hyper I-algebra. If $H^{+}=\{0\}$ and $0 \circ(0 \circ x)=x$ for all $x \in H$, then $(H, \odot, 0)$ is a commutative hypergroup.
Proof. The proof of Theorem 3.17 shows that $(H, \odot, 0)$ is commutative and associative. Let $a, b \in H$ be arbitrary. By the proof of Theorem 3.17 there exists $a^{\prime} \in H$ such that $0 \in a^{\prime} \odot a$ and $b \odot 0=b$. Thus $b \in b \odot 0 \subseteq b \odot\left(a^{\prime} \odot a\right)=\left(b \odot a^{\prime}\right) \odot a$. So there exists $t \in b \odot a^{\prime}$ such that $b \in t \odot a=a \odot t$, namely $a \odot H=H \odot a=H$.

Hence $(H, \odot, 0)$ is a commutative hypergroup.
Notation: Let $\mathcal{I}^{+} \mathcal{A L G}$ be a subcategory of $\mathcal{I} \mathcal{A} \mathcal{L G}$ in which for every object $H$ we have $H^{+}=\{0\}$ and $0 \circ(0 \circ x)=0$ for all $x \in H$. Similarly, let $\mathcal{C H G}$ be the category of commutative hypergroups with strong morphisms.

Theorem 3.21. $G: \mathcal{I}^{+} \mathcal{A L G} \longrightarrow \mathcal{C H G}$ is a faithful functor, where $G(H, \circ, 0)$ $=(H, \odot, 0)$ for $H \in \mathcal{I}^{+} \mathcal{A} \mathcal{L G}$ and $G(f)=f$ for $f \in \mathcal{I}^{+} \mathcal{A} \mathcal{L} \mathcal{G}\left(H_{1}, H_{2}\right)$.
Proof. Let $(H, \circ, 0)$ be an object in $\mathcal{I}^{+} \mathcal{A L G}$. Then by Theorem 3.20 we have $G(H)=(H, \odot, 0)$ is an object in $\mathcal{C H} \mathcal{G}$.

Let $f \in \mathcal{I}^{+} \mathcal{A} \mathcal{L} \mathcal{G}\left(H_{1}, H_{2}\right)$. We prove that $G f=f \in \mathcal{C H} \mathcal{G}\left(G\left(H_{1}\right), G\left(H_{2}\right)\right)$. By Theorem 3.20 we have

$$
\begin{aligned}
G f\left(x \odot_{1} y\right) & =f\left(x \odot_{1} y\right)=f\left(x \circ_{1}\left(0_{1} \circ_{1} y\right)\right)=f(x) \circ_{2}\left(f\left(0_{1}\right) \circ_{2} f(y)\right) \\
& =f(x) \circ_{2}\left(0_{2} \circ_{2} f(y)\right)=f(x) \odot_{2} f(y)=(G f)(x) \odot_{2}(G f)(y)
\end{aligned}
$$

So it is easy to see that $G$ satisfies to the other condition of a functor. Since $G$ maps $\mathcal{I}^{+} \mathcal{A} \mathcal{L} \mathcal{G}\left(H_{1}, H_{2}\right)$ injectively to $\mathcal{C H} \mathcal{H}\left(G H_{1}, G H_{2}\right)$, hence $G$ is faithful.

Remark 3.22. Let $F: \mathcal{C P G} \longrightarrow \mathcal{I} \mathcal{A L G}$ and $G: \mathcal{I}^{+} \mathcal{A} \mathcal{L} \mathcal{G} \longrightarrow \mathcal{C H}$ Ge the functors which are defined in Theorem 3.11 and 3.21 respectively. By Theorem 3.9, we have $H^{+}=\{0\}$ and $0 \diamond(0 \diamond x)=x$ for all $H \in F(\mathcal{C P G})$ and $x \in H$. Hence $F(\mathcal{C P G}) \subseteq \mathcal{I}^{+} \mathcal{A L G}$. Since $x \odot y=x \diamond(0 \diamond y)=x \diamond\left(0 \circ y^{-1}\right)=$ $x \diamond\left(y^{-1}\right)=x \circ\left(y^{-1}\right)^{-1}=x \circ y$. We get that $G F(H)=G(F H)=G(H)=$ $H$ for all $H \in \mathcal{C P G}$ and $(G F)(f)=G(F f)=G(f)=f$ for all $f \in \mathcal{C P} \mathcal{G}\left(H_{1}, H_{2}\right)$. Therefore $G F=I$.

Let $\mathcal{C S P G}$ be the category of commutative semipolygroups. Then $f \in \mathcal{C S P G}\left(\left(H_{1}, \circ_{1}, 0_{1}\right),\left(H_{2}, \circ_{2}, 0_{2}\right)\right)$ if and only if $f\left(x \circ_{1} y\right)=f(x) \circ_{2} f(y)$ and $f\left(e_{1}\right)=e_{2}$.

Proposition 3.23. $K: \mathcal{I}^{+} \mathcal{A} \mathcal{L G} \longrightarrow \mathcal{C S P G}$ is a full embedding functor, where $K(H, \circ, 0)=(H, \odot, 0)$ for all $H \in \mathcal{I}^{+} \mathcal{A} \mathcal{L G}$ and $K(f)=f$ for all $f \in \mathcal{I}^{+} \mathcal{A} \mathcal{L} \mathcal{G}\left(H_{1}, H_{2}\right)$.
Proof. The proof of Theorem 3.21 shows that $K$ is a faithful functor. Now we show that it is full, i.e. $K\left(\mathcal{I}^{+} \mathcal{A} \mathcal{L G}\left(H_{1}, H_{2}\right)\right)=\mathcal{C S P G}\left(K H_{1}, K H_{2}\right)$. By the proof of Theorem 3.17, for all $y \in H$ there exists a unique $y^{\prime}=y^{-1} \in H$ such that $0_{1} \circ_{1} y=y^{-1}$ and $0_{1} \circ_{1} y^{-1}=y$. Hence for all $f \in \mathcal{C S P G}\left(H_{1}, H_{2}\right)$ we get that

$$
f\left(x \circ_{1} y\right)=f\left(x \circ_{1}\left(0_{1} \circ_{1} y^{-1}\right)\right)=f\left(x \odot_{1} y^{-1}\right)=f(x) \odot_{2} f\left(y^{-1}\right)
$$

Since $0_{2} \in f\left(0_{1}\right) \subseteq f\left(y \odot_{1} y^{-1}\right)=f(y) \odot f\left(y^{-1}\right)$, hence by Definition 3.12 (iii) we get that $f\left(y^{-1}\right)=(f(y))^{-1}$. Thus we have

$$
f\left(x \circ_{1} y\right)=f(x) \odot_{2}(f(y))^{-1}=f(x) \circ_{2}\left(0_{2} \circ_{2}(f(y))^{-1}\right)=f(x) \circ_{2} f(y)
$$

Hence $K$ is full functor. Since $K$ maps $\mathcal{I}^{+} \mathcal{A} \mathcal{L} \mathcal{G}\left(H_{1}, H_{2}\right)$ injectively to $\mathcal{C S P G}\left(K H_{1}, K H_{2}\right)$, then $K$ is faithful. Since $K$ is full and faithful and one-to-one on objects so is full embedding. Thus $K\left(\mathcal{I}^{+} \mathcal{A L G}\right)$ is a full subcategory of $\mathcal{C S P G}$.

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# Extensions of Latin subsquares and local embeddability of groups and group algebras 

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#### Abstract

We will show that any "self-adjoint" Latin subsquare with constant diagonal can be extended to a Latin square with the same property. As a consequence, every loop with inverses satisfying the identity $(x y)^{-1}=y^{-1} x^{-1}$ (an IAA loop for short) is locally embeddable into finite IAA loops, and its loop algebra is locally embeddable into loop algebras of finite IAA loops. The IAA property enables to extend this result to loop algebras with the natural involution arising from the inverse map on the loop. In particular, this is true for groups and their group algebras.


## 1. Introduction

This paper arises from the study of groups locally embeddable into finite groups (LEF groups) and algebras locally embeddable into finite dimensional algebras (LEF algebras). Both notions were introduced and investigated by Gordon and Vershik in [8]. A relation between local embeddability of a group and its group algebra was established by the present author in [12], solving a problem formulated in [8].

A more general notion of approximability of topological groups by finite ones was introduced by E. Gordon in connection with his study of approximation of operators in spaces of functions on topological groups (cf. $[6,7]$ ). However, not all topological groups are approximable by finite ones, in particular, by far not all (discrete) groups are LEF. This raises the issue of approximation of groups by some finite grupoids, retaining as much of the group structure as possible. L. Glebsky and E. Gordon, in [5], proved that the approximability of locally compact groups by finite semigroups is equivalent to their approximability by finite groups. This indicates that in order

[^4]to extend the class of LEF groups one has to sacrifice the associativity of the binary operation. In the mentioned paper the study of approximability of groups by finite quasigroups was commenced.

We will show that every group is even locally embeddable into finite loops with inverses satisfying the identity $(x y)^{-1}=y^{-1} x^{-1}$, which we call IAA loops for short. The last property enables to extend the above mentioned result from [12] to group algebras with involution. In fact, we will be working within a slightly more general scope. Given an IAA loop $L$ and a field $K$ with an involutive automorphism, we will prove that $L$ is locally embeddable into finite IAA loops, and its loop algebra $K L$, with the natural involution arising from the inverse map on $L$, is locally embeddable into loop algebras of finite loops with natural involution.

The proof utilizes the well known relation between quasigroups and Latin squares. Its key ingredient is a kind of embedding theorem for Latin subsquares (Theorem 2.4). It gives some sufficient conditions guaranteeing the extendability of a Latin subsquare, symmetric with respect to some involutive permutation of the set of its elements and with constant diagonal, to a Latin square with the same property.

## 2. $\alpha$-symmetric Latin squares

A $p \times q$ matrix $R=\left(r_{i j}\right)$ with elements from a set $A$ is called a Latin rectangle of size $p \times q$ over $A$ if every element of $A$ occurs at most once in each row as well as in each column. If $p=q$ then the Latin rectangle is called a Latin subsquare of order $p$. If $p=q$ equals the number $n$ of elements of the finite set $A$ then the Latin rectangle is called a Latin square of order $n$ over $A$.

Definition 2.1. Let $\alpha: A \rightarrow A$ be an involutive permutation of the set $A$, i.e., $\alpha^{2}=i d$. A Latin (sub)square $R=\left(r_{i j}\right)$ over $A$ is called $\alpha$-symmetric if $\alpha\left(r_{i j}\right)=r_{j i}$ for all $i, j$.

Obviously, if $\left(r_{i j}\right)$ is an $\alpha$-symmetric Latin (sub)square then $\alpha\left(r_{i i}\right)=r_{i i}$, in other words, all the diagonal elements are fixed by $\alpha$.

We will make use of the following results. The number of occurrences of an element $a \in A$ in a Latin rectangle $R$ will be denoted by $N_{R}(a)$.

Lemma 2.2. [11, Ch. 6, Theorem 2.2] A Latin rectangle $R$ of size $p \times q$ over an $n$ element set $A$ can be extended to a Latin square of over $A$ if and only if $N_{R}(a) \geq p+q-n$ for all $a \in A$.

Lemma 2.3. [4, Corollary II.10.9] Let $m \leq n, \mathcal{U}=\left\{U_{1}, U_{2}, \ldots, U_{n}\right\}$ and $\mathcal{V}=\left\{V_{1}, V_{2}, \ldots, V_{m}\right\}$ be two collections having systems of distinct representatives $(S D R)$. Then some $S D R s \hat{U}$ of $\mathcal{U}$ and $\hat{V}$ of $\mathcal{V}$, satisfying $\hat{V} \subseteq \hat{U}$, exist if and only if

$$
\left|\mathcal{U}^{\prime}\right|+\left|\mathcal{V}^{\prime}\right| \leq m+\left|\bigcup \mathcal{U}^{\prime} \cap \bigcup \mathcal{V}^{\prime}\right|
$$

for all $\mathcal{U}^{\prime} \subseteq \mathcal{U}$ and $\mathcal{V}^{\prime} \subseteq \mathcal{V}$.

The next theorem is a partial case, for $\alpha=i d$, of Cruse theorem on extensions of commutative Latin squares - cf. [3, Theorem 1] or [9, Theorem 4.1]. The other way round, it can be regarded as a generalization of the special case $\left(r_{i i}=1\right)$ of the quoted result from commutative Latin squares to the $\alpha$-symmetric ones.

Theorem 2.4. Let $n$ be even, $\alpha$ be an involutive permutation of the set $A=\{1, \ldots, n\}$ with $\alpha(1)=1$, and $R=\left(r_{i j}\right)$ be an $\alpha$-symmetric Latin subsquare over $A$ of order $m<n$ such that $r_{i i}=1$ for all $i \leq m$. Then $R$ can be extended to an $\alpha$-symmetric Latin square $S=\left(s_{i j}\right)$ over A satisfying $s_{i i}=1$ for all $i \leq n$ if and only if $N_{R}(k) \geq 2 m-n$ for all $k \in A$.

Proof. Obviously, the inequality is necessary by Lemma 2.2. In the reversed direction we will proceed by induction, showing that the Latin subsquare $R$ satisfying the assumptions can be extended to an $\alpha$-symmetric Latin subsquare $\tilde{R}=\left(r_{i j}\right)$ of order $m+1$ over $A$ such that $N_{\tilde{R}}(k) \geq 2(m+1)-n$ for all $k \in A$ and $r_{i i}=1$ for all $i \leq m+1$ (the elements of the extension $\tilde{R}$ will be still denoted by $r_{i j}$ ). This way $R$ can be extended to an $\alpha$-symmetric Latin square $S$ of order $n$ with the desired property, in $n-m$ steps.

The case $m=n-1$ is trivial. So we can assume $m<n-1$.
Let $U_{i}(i=1,2, \ldots m)$ be the set of elements of $A$, not occuring in the $i$ th row of $R$, and $\mathcal{U}=\left\{U_{1}, U_{2}, \ldots, U_{m}\right\}$. Set

$$
\begin{aligned}
\mathcal{V}_{0} & =\left\{\{k\} ; N_{R}(k)=2 m-n\right\} \\
\mathcal{V}_{1} & =\left\{\{k, \alpha(k)\} ; N_{R}(k)=2 m-n+1\right\} \\
\mathcal{V} & =\mathcal{V}_{0} \cup \mathcal{V}_{1}
\end{aligned}
$$

Now it suffices to show that there exist $\operatorname{SDRs} \hat{U}=\left\{u_{1}, u_{2}, \ldots, u_{n}\right\}$ of $\mathcal{U}$ and $\hat{V}$ of $\mathcal{V}$ such that $\hat{V} \subseteq \hat{U}$. Indeed, adding $\left(u_{1}, u_{2}, \ldots, u_{m}\right)^{T}$ to $R$ as a new last column and $\left(\alpha\left(u_{1}\right), \alpha\left(u_{2}\right), \ldots, \alpha\left(u_{m}\right), 1\right)$ as a new last row, we get the
following matrix of order of order $m+1$ :

$$
\tilde{R}=\left(\begin{array}{ccccc}
r_{11} & r_{12} & \cdots & r_{1 m} & u_{1} \\
r_{21} & r_{22} & \cdots & r_{2 m} & u_{2} \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
r_{m 1} & r_{m 2} & \cdots & r_{m m} & u_{m} \\
\alpha\left(u_{1}\right) & \alpha\left(u_{2}\right) & \cdots & \alpha\left(u_{m}\right) & 1
\end{array}\right)
$$

As $u_{i}$ is a representative of the set $U_{i}$, it does not occur in the $i$ th row of $R$, hence by the $\alpha$-symmetry $\alpha\left(u_{i}\right)$ does not occur in its $i$ th column. Thus $\tilde{R}$ is a Latin subsquare over $A$. The $\alpha$-symmetry and $r_{i i}=1$ for all $i \leq m+1$ are clear from the construction.

The inequality $N_{\tilde{R}}(k) \geq 2(m+1)-n$ is automatically satisfied for the elements of $A \backslash \bigcup \mathcal{V}$. The same will be verified for the elements of $\bigcup \mathcal{V}_{0}$ and $\bigcup \mathcal{V}_{1}$ separately.

The $\alpha$-symmetry of $R$ implies $N_{R}(k)=N_{R}(\alpha(k))$ for all $k \in A$. Then

$$
\alpha(k) \in \mathcal{V}_{0} \Leftrightarrow k \in \mathcal{V}_{0}
$$

As $\bigcup \mathcal{V}_{0} \subseteq \hat{V} \subseteq \hat{U}$, we have $\{k, \alpha(k)\} \subseteq \hat{U} \cap \alpha(\hat{U})$, consequently $N_{\tilde{R}}(k)=$ $N_{R}(k)+2=2 m-n+2$ for all $k \in \bigcup \mathcal{V}_{0}$.

If $\{k, \alpha(k)\} \in \mathcal{V}_{1}$ then eighter $k \in \hat{V}$ or $\alpha(k) \in \hat{V}$. In any case $\{k, \alpha(k)\} \subseteq \hat{U} \cup \alpha(\hat{U})$, hence $N_{\tilde{R}}(k) \geq N_{R}(k)+1 \geq 2 m-n+2$ for all $k \in \bigcup \mathcal{V}_{1}$.

Finally, it remains to prove the existence of suitable SDRs $\hat{U}$ and $\hat{V}$. To this end we use Lemma 2.3, thus we have to verify its assumptions.

We show $|\mathcal{V}| \leq|\mathcal{U}|=m$ first. Assume that $|\mathcal{V}|=m+x$, where $x \geq 1$ is an integer.

If $\alpha(k)=k$ and $k \neq 1$ then, by $\alpha$-symmetry, $N_{R}(k)$ is even. As $n$ is even, too, $N(k) \neq 2 m-n+1$. The last inequality is true for $k=1$, as well, because $N_{R}(1)=m \neq 2 m-n+1$. (Recall that $m<n-1$.) Hence $|V|=2$ for all $V \in \mathcal{V}_{1}$. Then the number of fields of the Latin subsquare $R$ which can be filled by elements of $A$ is at most

$$
M=\left|\mathcal{V}_{0}\right|(2 m-n)+2\left|\mathcal{V}_{1}\right|(2 m-n+1)+\left(n-2\left|\mathcal{V}_{1}\right|-\left|\mathcal{V}_{0}\right|\right) m
$$

Then $\left|\mathcal{V}_{1}\right|=m+x-y$, where $y=\left|\mathcal{V}_{0}\right|$. Thus

$$
\begin{aligned}
M & =M(x, y) \\
& =y(2 m-n)+2(m+x-y)(2 m-n+1)+(n-2(m+x-y)-y) m \\
& =(n-m-2) y-2(n-m-1) x+m(2 m-n+2)
\end{aligned}
$$

can be regarded as a function of the arguments $x$ and $y$, decreasing in $x$ and nondecreasing in $y$. As $y$ takes the values from the set $\{0,1, \ldots, m+x\}$, only,

$$
M(x, m+x)=(m-n) x+m^{2}
$$

is the maximal value of $M(x, y)$ for a fixed $x$. This is still a decreasing function of $x$, hence its maximum is

$$
M(1, m+1)=m-n+m^{2}<m^{2}
$$

In other words, not all fields of $R$ can be filled by elements of $A$. Thus the assumption $|\mathcal{V}|>|\mathcal{U}|$ leads to a contradiction, and we have $|\mathcal{V}| \leq|\mathcal{U}|$.

As $V_{1} \cap V_{2}=\emptyset$ for any distinct $V_{1}, V_{2} \in \mathcal{V}$, the collection $\mathcal{V}$ has some SDR. The existence of an SDR for $\mathcal{U}$ follows from Lemma 2.2.

It remains to show the inequality

$$
\begin{equation*}
\left|\mathcal{U}^{\prime}\right|+\left|\mathcal{V}^{\prime}\right| \leq m+\left|\bigcup \mathcal{U}^{\prime} \cap \bigcup \mathcal{V}^{\prime}\right| \tag{1}
\end{equation*}
$$

for all $\mathcal{U}^{\prime} \subseteq \mathcal{U}$ and $\mathcal{V}^{\prime} \subseteq \mathcal{V}$. Take some fixed $\mathcal{U}^{\prime}, \mathcal{V}^{\prime}$, and consider the bipartite graph $\Gamma=\left(\bigcup \mathcal{V}^{\prime}, \mathcal{U}^{\prime}, E\right)$ with the edge set $E=\left\{(v, U) \in \bigcup \mathcal{V}^{\prime} \times \mathcal{U} ; v \in U\right\}$. One can readily see that the degrees of its vertices satisfy the following conditions:

$$
\begin{aligned}
\operatorname{deg}(v) & =n-m, & & v \in \bigcup \mathcal{V}_{0}^{\prime} ; \\
\operatorname{deg}(v) & =n-m-1, & & v \in \bigcup \mathcal{V}_{1}^{\prime} ; \\
\operatorname{deg}(U) & \leq n-m, & & U \in \mathcal{U},
\end{aligned}
$$

where $\mathcal{V}_{i}^{\prime}=\mathcal{V} \cap \mathcal{V}_{i}$ for $i=0,1$. Denoting

$$
p=\left|\bigcup \mathcal{V}_{0}^{\prime} \cap \bigcup \mathcal{U}^{\prime}\right|, \quad q=\left|\bigcup \mathcal{V}_{1}^{\prime} \cap \bigcup \mathcal{U}^{\prime}\right|
$$

we have $\left|\bigcup \mathcal{U}^{\prime} \cap \bigcup \mathcal{V}^{\prime}\right|=p+q$. Now, one can give an upper bound for the number of edges ending in $\mathcal{U} \backslash \mathcal{U}^{\prime}$ :

$$
\left(\left|\bigcup \mathcal{V}_{0}^{\prime}\right|-p\right)(n-m)+\left(\left|\bigcup \mathcal{V}_{1}^{\prime}\right|-q\right)(n-m-1) \leq\left(m-\left|\mathcal{U}^{\prime}\right|\right)(n-m)
$$

Realizing $\left|\bigcup \mathcal{V}_{0}^{\prime}\right|=\left|\mathcal{V}_{0}^{\prime}\right|$ and $\left|\bigcup \mathcal{V}_{1}^{\prime}\right|=2\left|\mathcal{V}_{1}^{\prime}\right|$, the last inequality can be written in the following form

$$
\left(\left|\mathcal{V}_{0}^{\prime}\right|-p\right)(n-m)+\left(2\left|\mathcal{V}_{1}^{\prime}\right|-q\right)(n-m-1) \leq\left(m-\left|\mathcal{U}^{\prime}\right|\right)(n-m)
$$

An elementary computation shows that this one is equivalent to

$$
\left|\mathcal{V}_{0}^{\prime}\right|+\left|\mathcal{V}_{1}^{\prime}\right|+|\mathcal{U}| \leq m+p+\frac{n-m-1}{n-m} q-\frac{n-m-2}{n-m}\left|\mathcal{V}_{1}^{\prime}\right| .
$$

As $\left|\mathcal{V}_{0}^{\prime}\right|+\left|\mathcal{V}_{1}^{\prime}\right|=\left|\mathcal{V}^{\prime}\right|, \frac{n-m-1}{n-m}<1$ and $\frac{n-m-2}{n-m}\left|\mathcal{V}_{1}^{\prime}\right| \geq 0$, the last inequality implies (1).

Hence, by Lemma 2.3, there exist SDRs $\hat{U}$ of $\mathcal{U}$ and $\hat{V}$ of $\mathcal{V}$ such that $\hat{V} \subseteq \hat{U}$.

The idea of the presented proof of Theorem 2.4, based on Lemma 2.3 and the proof of the above mentioned Cruse Theorem [3], was suggested by the referee. The core of author's original, and considerably longer, proof consisted of an algorithm written in a computer-like language. Its entry was an arbitrary extension of the original Latin subsquare $R$ to a Latin square $R^{\prime}$ over $A$, existing by the virtue of Lemma 2.2. The algorithm transformed the $(m+1) \times(m+1)$ upper left corner of $R^{\prime}$ into a Latin subsquare $\tilde{R}$ extending $R$, still satisfying the assumptions of the theorem. Having checked the extendability of $\tilde{R}$, the desired Latin square $S$ could have been obtained by repeating the algorithm $n-m$ times, again.

## 3. IAA loops and groups

A quasigroup is a grupoid $Q$ satisfying both the left and the right cancellation law, i.e.,

$$
\left(x y_{1}=x y_{2} \vee y_{1} x=y_{2} x\right) \Rightarrow y_{1}=y_{2}
$$

for all $x, y_{1}, y_{2} \in Q$. A quasigroup with a unit 1 (which is necessarily unique) is called a loop. If a loop $L$ possess two-sided inverses then, due to the cancellation, they are uniquely determined, so that the notation $x^{-1}$ is unambiguous.

Definition 3.1. A loop $L$ with (two-sided) inverses has the inverse antiautomorphism property if the mapping $x \mapsto x^{-1}$ is an antiautomorphism of $(L, \cdot)$, i.e.,

$$
\begin{equation*}
(x y)^{-1}=y^{-1} x^{-1} \tag{2}
\end{equation*}
$$

for every $x, y \in L$.
A loop with the inverse antiautomorphism property is briefly called an IAA loop. Obviously, every group is an IAA loop. On the other hand, an IAA loop does not necessarily satisfy the conditions $x^{-1}(x y)=y$ and $(x y) y^{-1}=x$.

The following definition goes back to Mal'tsev [10], where it can be found in a more general universal-algebraic setting.

Definition 3.2. Let $Q$ be a grupoid and $\boldsymbol{F}$ be some class of grupoids. Then $Q$ is said to be locally embeddable into the class $\boldsymbol{F}$ if for any finite set $M \subseteq Q$ there is an $F \in \boldsymbol{F}$ and an injective $\operatorname{map} \varphi:\left(M \cup M^{2}\right) \rightarrow F$ such that $\varphi(x y)=\varphi(x) \varphi(y)$ for every $x, y \in M$.

In this section we will prove that every IAA loop, in particular every group, is locally embeddable into the class of finite IAA loops. To this end we will exploit the representation of quasigroups by Latin squares: Enumerating the elements of a finite quasigroup $Q$, its multiplication table can readily be turned into a Latin square over $Q$. Fixing an element 1 of a quasigroup $Q$ and changing the order of some rows and columns, if necessary, we can transform its Latin square into the multiplication table of some loop with the unit 1. Expressed in the quasigroups terminology: Every quasigroup is isotopic to a loop (cf. [1]).

For technical convenience we will formulate the results on embeddability of IAA loops, announced in the introduction within a more general framework of "partial IAA loops with a root".

Definition 3.3. A structure $(L, \sqrt{L}, \cdot)$, where $\cdot$ is a partial binary operation on $L$ and $\sqrt{L} \subseteq L$, is said to be a partial IAA loop with the root $\sqrt{L}$ if
(a) The operation $\cdot$ satisfies the cancellation law, whenever defined.
(b) There exists an element $1 \in \sqrt{L}$ such that $x \cdot 1=1 \cdot x=x$ for all $x \in L$.
(c) The product $x y$ is defined for all $x, y \in \sqrt{L}$ and $L=(\sqrt{L})^{2}$, i.e., each $z \in L$ has the form $z=x y$ for some $x, y \in \sqrt{L}$.
(d) For every $x \in L$ there exists an $x^{-1} \in L$ such that $x x^{-1}=x^{-1} x=1$.
(e) If $x y$ is defined then so is $y^{-1} x^{-1}$ and $(x y)^{-1}=y^{-1} x^{-1}$.

Theorem 3.4. Let $(L, \sqrt{L}, \cdot)$ be a partial IAA loop with a finite root $\sqrt{L}$. Then there exists a finite IAA loop $F$ and an injective map $\varphi: L \rightarrow F$ such that $\varphi(1)=1$ and $\varphi(x y)=\varphi(x) \varphi(y)$ for all $x, y \in \sqrt{L}$.

In other words, every finite root of a partial IAA loop can be extended to a finite IAA loop. It can be easily seen that such a partial embedding $\varphi$ satisfies the condition $\varphi\left(x^{-1}\right)=\varphi(x)^{-1}$, as well.

Proof. Denote the elements of $L$ by $1,2, \ldots, m^{\prime}$ (with 1 denoting the unit). We can assume $\sqrt{L}=\{1,2, \ldots m\}$ for some $m \leq m^{\prime}$. For $i, j=1,2, \ldots, m$
we put $r_{i j}=k$ if and only if $i \cdot j=k$. Choose an even $n \geq \max \left(2 m, m^{\prime}\right)$ and define a permutation $\alpha$ of $A=\{1,2, \ldots, n\}$ as follows:

$$
\alpha(k)= \begin{cases}k^{-1}, & \text { if } k \leq m^{\prime} \\ k, & \text { if } k>m^{\prime}\end{cases}
$$

Without loss of generality we can assume that $R=\left(r_{i j}\right)$ is an $\alpha$ symmetric Latin subsquare over $A$ of order $m$ satisfying the assumptions of Theorem 2.4 (if not, we can always achieve this by changing the order of some rows in $R$ ).

Hence there is an $\alpha$-symmetric Latin square $S=\left(s_{i j}\right)$ over $A$ of order $n$, extending $R$ such that $s_{i i}=1$ for all $i \leq n$.

Define a binary operation $\cdot$ on the set $F=A$ by putting $p \cdot q=k$ if and only if $p=s_{i 1}, q=s_{1 j}, k=s_{i j}$ for some (uniquely determined) $i, j \leq n$. This definition is independent of the order of rows and columns in $S$. The fact that $(F, \cdot)$ is a loop with unit 1 could be visualized by interchanging the order of some rows an columns in $S$ yielding the multiplication table of $F$. Moreover we have $k^{-1}=\alpha(k)$ for each $k \in F$. So it suffices to verify the IAA property, i.e.,

$$
\alpha(p q)=\alpha(q) \alpha(p)
$$

for all $p, q \in F$.
Let $p=s_{i 1}$ and $q=s_{1 j}$. Then $p q=s_{i j}$. By the $\alpha$-symmetry of $S$ we have $\alpha(p q)=s_{j i}, \alpha(q)=s_{1 i}$ and $\alpha(p)=s_{j 1}$. Hence $\alpha(q) \alpha(p)=s_{j i}=\alpha(p q)$.

Now it is enough to take for $\varphi: L \rightarrow F$ the identity map.
Corollary 3.5. Every IAA loop, in particular, every group, is locally embeddable into the class of finite IAA loops.

Proof. Given an IAA loop $L$ and a finite $M \subseteq L$, put $\bar{M}=M \cup M^{-1} \cup\{1\}$. Then $\left(\bar{M} \cup \bar{M}^{2}, \bar{M}, \cdot\right)$ is the partial IAA loop with the root $\bar{M}$.

Applying a standard model-theoretic compactness argument to the last corollary we get (see, e.g., [2])

Corollary 3.6. Every IAA loop, in particular, every group, can be embedded into an ultraproduct of a system of finite IAA loops.

## 4. Quasialgebras and loop algebras

A linear space $A$ over a field $K$ with a bilinear (not necessarily associative) binary operation • will be called a quasialgebra over $K$. We avoid the wide spread term non-associative algebra, as the operation • may (but need not) be associative. A quasialgebra with a unit element 1 is called unitary.

The definition of a quasigroup algebra $K Q$ of a quasigroup $Q$ over $K$ is analogous to that of a group algebra: It is the linear space over $K$ formed by formal linear combinations $\sum_{x \in Q} a_{x} x$ of elements of $Q$ with just finitely many nonzero coefficients $a_{x} \in K$. Their product is defined by the usual convolution formula

$$
\left(\sum_{x \in Q} a_{x} x\right) \cdot\left(\sum_{y \in Q} b_{y} y\right)=\sum_{x, y \in Q}\left(a_{x} b_{y}\right) x y=\sum_{z \in Q} \sum_{x y=z} a_{x} b_{y} z .
$$

A quasigroup algebra of a loop $L$ will be called a loop algebra; it is obviously unitary, with the unit $1 \in L$.

Given an involutive automorphism $a \mapsto \bar{a}$ of the field $K$, a unary operation * on a quasialgebra $A$ is called an involution if for all $u, v \in A$ and $a, b \in K$ we have
(a) $(a u+b v)^{*}=\bar{a} u^{*}+\bar{b} v^{*}$;
(b) $\left(u^{*}\right)^{*}=u$;
(c) $(u v)^{*}=v^{*} u^{*}$.

In what follows $K$ will be some field with an involutive automorphism $a \mapsto \bar{a}$, and we will be dealing just with quasialgebras over $K$.

The following observation can be verified by some straightforward computations.

Proposition 4.1. Let $L$ be an IAA loop. Then
(i) the operation $\left(\sum_{x \in L} a_{x} x\right)^{*}=\sum_{x \in L} \bar{a}_{x} x^{-1}$ is an involution on $K L$;
(ii) $x x^{*}=x^{*} x=1 \quad$ for all $x \in L$.

The above defined operation $u \mapsto u^{*}$ will be referred to as the natural involution of the loop algebra $K L$.

The notion of local embeddability from Definition 3.2 can be modified to quasialgebras (with involution) as follows:

Definition 4.2. Let $A$ be a quasialgebra with involution and $\boldsymbol{H}$ be some class of quasialgebras with involution. Then $A$ is said to be locally embeddable into the class $\boldsymbol{H}$ if for any finite set $M \subseteq A$ there is an $H \in \boldsymbol{H}$ and an injective linear map $\psi: \operatorname{span}\left(M \cup M^{*} \cup M^{2}\right) \rightarrow H$ such that for every $u, v \in M$ we have
(a) $\psi(u v)=\psi(u) \psi(v)$;
(b) $\psi\left(u^{*}\right)=\psi(u)^{*}$.

Theorem 4.3. Let $A=K L$ be the loop algebra of an IAA loop L, endowed with the natural involution. Then $K L$ is locally embeddable into the class of loop algebras of finite IAA loops, with natural involution.

Proof. Let $M \subseteq A$ be finite. Then there is a finite set $M_{0} \subseteq L$ such that $M \subseteq \operatorname{span}\left(M_{0}\right)$ and $M_{0}=M_{0}{ }^{-1}$. By Corollary 3.5, there is an injective map $\varphi: M_{0} \cup M_{0}^{2} \rightarrow F$ into some finite IAA loop $F$.

Let $H=K F$ be the loop algebra of $F$. As $M_{0} \cup M_{0}^{2} \subseteq L$, it is linearly independent in $H$. Hence the map $\varphi$ can be extended to an injective linear map $\lambda: \operatorname{span}\left(M_{0} \cup M_{0}^{2}\right) \rightarrow H$. Now it suffices to take the restriction $\psi$ of $\lambda$ to $\operatorname{span}\left(M \cup M^{*} \cup M^{2}\right) \subseteq \operatorname{span}\left(M_{0} \cup M_{0}^{2}\right)$. Then one can readily see that $\psi: \operatorname{span}\left(M \cup M^{*} \cup M^{2}\right) \rightarrow H$ is an injective linear map, satisfying the conditions (a) and (b) of Definition 4.2.

Corollary 4.4. Let $A=K G$ be the group algebra of a group $G$, endowed with the natural involution. Then $K G$ is locally embeddable into the class of loop algebras with natural involution of finite IAA loops.

Similarly as in Corollary 3.6 one can obtain from Theorem 4.3
Corollary 4.5. Every loop algebra of an IAA loop, in particular, every group algebra, can be embedded into an ultraproduct of a system of loop algebras of finite IAA loops with natural involution.

The question whether Theorem 4.3 can be extended beyond the class of loop algebras of IAA loops remains open. Let us close with the following conjecture.

Conjecture. Let $A$ be a unitary quasialgebra with involution which is spanned by a set $U(A)=\left\{u \in A ; u^{*} u=u u^{*}=1\right\}$. Then $A$ is locally embeddable into the class of finite dimensional quasialgebras with involution.

Obviously, if the set $U(A) \subseteq A$ is closed under multiplication then it forms an IAA loop with the inverse $x^{-1}=x^{*}$. It is not clear if the above
conjecture is true under this additional assumption. If $A$ is an algebra (i.e., it is associative) then $U(A)$ is a group. Even this special case of our conjecture remains open.

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[^0]:    2000 Mathematics Subject Classification: 05B30, 08A99, 05B07, 20N05
    Keywords: Steiner quadruple system, distributive and medial quadruple system,
    SQS-3-groupoid, distributive and medial SQS-3-groupoid

[^1]:    2000 Mathematics Subject Classification: 06F05
    Keywords: $m$-system, $n$-system in ordered semigroups, ideal, weakly prime ideal, weakly semiprime ideal in ordered semigroups

[^2]:    2000 Mathematics Subject Classification: 20N25, 06B10
    Keywords: fuzzy congruence, fuzzy (normal) subgroup, homomorphism theorem, transfer principle

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[^4]:    2000 Mathematics Subject Classification: 20N05, 20E25, 16S34, 05B15
    Keywords: group, loop, Latin square, group algebra, loop algebra, local embeddability
    The paper was supported by VEGA - the grant agency of Slovak Republic

