BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(82), 2016, Pages 3–11
ISSN 1024–7696

On (σ-δ)-rings over Noetherian rings

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Abstract. For a ring R, an endomorphism σ of R and a σ-derivation δ of R, we introduce (σ-δ)-ring and (σ-δ)-rigid ring which are the generalizations of $\sigma(*)$-rings and δ-rings, and investigate their properties. Moreover, we prove that a (σ-δ)-ring is 2-primal and its prime radical is completely semiprime.

Mathematics subject classification: 16N40, 16P40, 16W20.
Keywords and phrases: Endomorphism, derivation, $\sigma(*)$-rings, δ-ring, 2-primal ring, (σ-δ)-ring, (σ-δ)-rigid ring.

1 Introduction and preliminaries

A ring R always means an associative ring with identity $1 \neq 0$, unless otherwise stated. The prime radical and the set of nilpotent elements of R are denoted by $P(R)$ and $N(R)$ respectively. The ring of integers is denoted by $\mathbb{Z}$, the field of rational numbers by $\mathbb{Q}$, the field of real numbers by $\mathbb{R}$, and the field of complex numbers by $\mathbb{C}$, unless otherwise stated.

Let R be a ring. This article concerns endomorphisms and derivations of a ring and we also discuss certain types of rings involving endomorphisms and derivations. We begin with the following:

Definition 1 (see Krempa [10])**.** An endomorphism σ of a ring R is said to be rigid if $a\sigma(a) = 0$ implies that $a = 0$, for all $a \in R$. A ring R is said to be σ-rigid if there exists a rigid endomorphism σ of R.

Example 1. Let $R = \mathbb{C}$ and $\sigma : \mathbb{C} \to \mathbb{C}$ be defined by $\sigma(a + ib) = a - ib$, for all $a, b \in R$. Then σ is a rigid endomorphism of R.

We recall a ring R is σ-rigid if there exists a rigid endomorphism σ of R and σ-rigid rings are reduced rings by Hong et. al. [6]. Properties of σ-rigid rings have been studied in Krempa [10], Hong et al. [6] and Hirano [5].

Definition 2 (see Kwak [12])**.** Let R be a ring and σ an endomorphism of R. Then R is said to be a $\sigma(*)$-ring if $a\sigma(a) \in P(R)$ implies that $a \in P(R)$, for $a \in R$.

The authors would like to express sincere thanks to the referee for valuable suggestions.

Example 2 (see Example 1 of Kwak [12])**.** Let $\mathbb{F}$ be a field, and $R = \begin{pmatrix} \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{F} \end{pmatrix}$. Then $P(R) = \begin{pmatrix} 0 & \mathbb{F} \\ 0 & 0 \end{pmatrix}$. Let $\sigma : R \to R$ be defined by

$$\sigma\Big(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}.$$

Then it can be seen that σ is an endomorphism of R and R is a $\sigma(*)$-ring.

We note that the above ring is not σ-rigid. Let $0 \neq a \in \mathbb{F}$. Then

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \sigma\Big(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\Big) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ but } \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Example 3. Let $\mathbb{F}$ be a field, and $R = \mathbb{F}[x]$. Let $\sigma : R \to R$ be an endomorphism defined by $\sigma(f(x)) = f(0)$. Then R is not a $\sigma(*)$-ring.

Definition 3 (see [13])**.** An ideal I of a ring R is said to be completely semi-prime if $a^2 \in I$ implies that $a \in I$, for $a \in R$.

Definition 4. A ring R is said to be 2-primal if and only if $P(R) = N(R)$.

Example 4 (see Bhat [4])**.**

1. Let $R = \mathbb{F}[x]$ be the polynomial ring over a field $\mathbb{F}$. Then R is 2-primal with $P(R) = \{0\}$.
2. Let $M_2(\mathbb{Q})$ be the set of 2×2 matrices over $\mathbb{Q}$. Then $R[x]$ is a prime ring with non-zero nilpotent elements and so it cannot be 2-primal.

2-primal rings have been studied in recent years and are being treated by authors for different structures. We know that a ring R is 2-primal if the prime radical is completely semi-prime. Note that a reduced ring is 2-primal and a commutative ring is also 2-primal. For further detail on 2-primal rings refer to [2, 3, 7, 8, 9, 13, 15]. Furthermore, the concept of completely semi-prime ideals is also studied in this area. Kwak in [12] establishes a relation between a 2-primal ring and a $\sigma(*)$-ring. It is also known that if R is a Noetherian ring and σ an endomorphism of R, then R a $\sigma(*)$-ring implies that R is 2-primal (Proposition (2.4) of [4]), but the converse need not be true. For example, we have:

Example 2.5 of [4]: Let $R = F[x]$ be the polynomial ring over a field F. Then R is 2-primal with $P(R) = \{0\}$. Let $\sigma : R \to R$ be an endomorphism defined by

$$\sigma(f(x)) = f(0).$$

Then R is not a $\sigma(*)$-ring. For this consider $f(x) = xa, a \neq 0$.

Also if R is a Noetherian ring and σ an endomorphism of R, then R a $\sigma(*)$-ring implies that $P(R)$ is completely semi-prime (Proposition (1) of [11]), but the converse need not be true. For example, we have

Example [12]: Let $\mathbb{F}$ be a field, $R = \mathbb{F} \times \mathbb{F}$. Let $\sigma : R \to R$ be an automorphism defined as

$$\sigma((a, b)) = (b, a), a, b \in \mathbb{F}.$$

Here $P(R) = \{0\}$ is a completely semi-prime ring, as R is a reduced ring. But R is not a $\sigma(*)$-ring. Since $(1, 0)\sigma((1, 0)) = (0, 0)$, but $(1, 0)$ does not belong to $P(R)$.

Definition 5 (see [14])**.** Let R be a ring, σ an endomorphism of R and $\delta : R \to R$ an additive map such that

$$\delta(ab) = \delta(a)\sigma(b) + a\delta(b), \text{ for all } a, b \in R.$$

Then δ is a σ-derivation of R.

Example 5. Let $R = \mathbb{Z}[\sqrt{2}]$. Then $\sigma : R \to R$ defined as

$$\sigma(a + b\sqrt{2}) = (a - b\sqrt{2}), \text{ for } a + b\sqrt{2} \in R.$$

is an endomorphism of R. For any $s \in R$, define $\delta_s : R \to R$ by

$$\delta_s(a + b\sqrt{2}) = (a + b\sqrt{2})s - s\sigma(a + b\sqrt{2}), \text{ for } a + b\sqrt{2} \in R.$$

Then δ_s is a σ-derivation of R.

Definition 6 (see Bhat [1])**.** Let R be a ring. Let σ be an automorphism of R and δ a σ-derivation of R. Then R is a δ-ring if $a\delta(a) \in P(R)$ implies that $a \in P(R)$.

Note that a δ-ring is without identity, as $1\delta(1) = 0$, but $1 \neq 0$.

Example 6. Let S be a ring without identity and $R = S \times S$ with $P(R) = \{0\}$ (for example we take $S = 2\mathbb{Z}$).

Then $\sigma : R \to R$ is an endomorphism defined by

$$\sigma((a, b)) = (b, a).$$

For any $s \in R$, define $\delta_s : R \to R$ by

$$\delta_s(a, b) = (a, b)s - s\sigma(a, b), \text{ for } (a, b) \in R.$$

Let $(a, b)\delta_s(a, b) \in P(R)$, then $(a, b)\{(a, b)s - s\sigma(a, b)\} \in P(R)$ or $(a, b)\{(a, b)s - s(b, a)\} \in P(R)$, i.e. $(a, b)(as - bs, bs - sa) \in P(R)$. Therefore, $(a(as - bs), b(bs - sa)) \in P(R) = \{0\}$ which implies that $a = 0$, $b = 0$, i.e. $(a, b) = (0, 0) \in P(R)$. Thus R is a δ-ring.

It is known that if R is a δ-ring, σ an endomorphism of R, δ a σ-derivation of R such that $\delta(P(R)) \in P(R)$, then R is 2-primal (Theorem 2.2 of [1]).

In this note we generalize the $\sigma(*)$-rings and δ-rings as follows:

Definition 7. Let R be a ring. Let σ be an endomorphism of R and δ a σ-derivation of R. Then R is said to be a (σ-δ)-ring if $a(\sigma(a)+\delta(a)) \in P(R)$ implies that $a \in P(R)$, for $a \in R$.

Example 7. Let $\mathbb{F}$ be a field, and $R = \begin{pmatrix} \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{F} \end{pmatrix}$. Then $P(R) = \begin{pmatrix} 0 & \mathbb{F} \\ 0 & 0 \end{pmatrix}$. Let $\sigma : R \to R$ be defined by

$$\sigma\Big(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}.$$

Then it can be seen that σ is an endomorphism of R. For any $s \in R$, define $\delta_s : R \to R$ by

$$\delta_s(a) = as - s\sigma(a), \text{ for } a \in R.$$

Let $s = \begin{pmatrix} p & q \\ 0 & r \end{pmatrix}, \quad x = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \quad y = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}$.

Now $\delta_s(xy) = (xy)s - s\sigma(xy) = \begin{pmatrix} 0 & aa_1q + ab_1r + bc_1r - cc_1q \\ 0 & 0 \end{pmatrix}$.

Also $\delta_s(x)\sigma(y) + x\delta_s(y) = \begin{pmatrix} 0 & aa_1q + ab_1r + bc_1r - cc_1q \\ 0 & 0 \end{pmatrix}$.

Hence $\delta_s(xy) = \delta_s(x)\sigma(y) + x\delta_s(y)$. Thus δ_s is a σ-derivation on R.

Now let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \quad s = \begin{pmatrix} p & q \\ 0 & r \end{pmatrix}$.

$A[\sigma(A) + \delta(A)] \in P(R)$ which implies that

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big\{\sigma\Big(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big) + As - s\sigma(A)\Big\} \in P(R),$$

i.e. $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big\{\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} + \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} p & q \\ 0 & r \end{pmatrix} - \begin{pmatrix} p & q \\ 0 & r \end{pmatrix}\sigma(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix})\Big\} \in P(R)$

or $\begin{pmatrix} a^2 & a^2q + abr + bc - acq \\ 0 & c^2 \end{pmatrix} \in P(R) = \begin{pmatrix} 0 & \mathbb{F} \\ 0 & 0 \end{pmatrix}$ which implies that

$a^2 = 0, \quad c^2 = 0$, i.e. $a = 0, \quad c = 0$.

Therefore, $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in P(R)$. Hence $P(R)$ is a (σ-δ)-ring.

Remark 1. 1. If $\delta(a) = 0$, then a (σ-δ)-ring is a $\sigma(*)$-ring.

2. If $\sigma(a) = 0$, then a (σ-δ)-ring is a δ-ring.
3. If $\sigma(a) = a$, $\delta(a) = 0$, then a (σ-δ)-ring is completely semi-prime.

Definition 8. Let R be a ring. Let σ be an endomorphism of R and δ a σ-derivation of R. Then R is said to be a (σ-δ)-rigid ring if

$$a(\sigma(a) + \delta(a)) = 0 \text{ implies that } a = 0, \text{ for } a \in R.$$

Example 8. Let $R = \mathbb{C}$ and $\sigma : \mathbb{C} \to \mathbb{C}$ be defined by

$$\sigma(a + ib) = a - ib, \text{ for all } a, b \in R.$$

Then σ is an endomorphism on R.
Define a σ-derivation δ on R as

$$\delta(A) = A - \sigma(A),$$

i.e. $\delta(a + ib) = a + ib - \sigma(a + ib) = a + ib - (a - ib) = 2ib$.
Now $A[\sigma(A) + \delta(A)] = 0$ which implies that $(a + ib)[\sigma(a + ib) + \delta(a + ib)] = 0$, i.e. $(a + ib)[(a - ib) - 2ib] = 0$ or $(a + ib)(a + ib) = 0$ which implies that $a = 0, b = 0$. Therefore, $A = a + ib = 0$. Hence R is a (σ-δ)-rigid ring.

With this we prove the following

Theorem A: Let R be a Noetherian integral domain which is also an algebra over $\mathbb{Q}$. σ an automorphism on R and δ a σ-derivation of R. If R is a (σ-δ)-ring, then R is 2-primal. (This has been proved in Theorem 2.2).

Theorem B: Let R be a Noetherian integral domain which is also an algebra over $\mathbb{Q}$, σ an automorphism on R and δ a σ-derivation of R. If R is a (σ-δ)-ring, then $P(R)$ is completely semi-prime. (This has been proved in Theorem 2.5).

Example of a ring satisfying the hypothesis of Theorem A and Theorem B is $R = \mathbb{Z}$. It is a Noetherian integral domain which is also an algebra over $\mathbb{Q}$. Let $\sigma : R \to R$ be defined by

$$\sigma(a) = 2a.$$

Then it can be seen that σ is an endomorphism of R.
For any $s \in R$, define $\delta_s : R \to R$ by

$$\delta_s(a) = as - s\sigma(a), \text{ for } a \in R.$$

Then δ_s is a σ-derivation on R. Also R is a (σ-δ)-ring.

2 Proof of the main results

For the proof of the main result, we need the following

Proposition 1. *Let R be a ring, σ an automorphism of R and δ a σ-derivation of R. Then for $u \neq 0$, $\sigma(u) + \delta(u) \neq 0$.*

Proof. Let $0 \neq u \in R$, we show that $\sigma(u) + \delta(u) \neq 0$. Let for $0 \neq u$, $\sigma(u) + \delta(u) = 0$ which implies that

$$\delta(u) = -\sigma(u). \tag{1}$$

We know that for $a, b \in R$, $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$. By using (2.1), this implies that $\delta(ab) = -\sigma(a)\sigma(b) + a(-\sigma(b))$ or $-\sigma(ab) = -[a + \sigma(a)]\sigma(b)$. Since σ is an endomorphism of R, this gives $-\sigma(a)\sigma(b) = -[a + \sigma(a)]\sigma(b)$, i.e. $\sigma(a) = a + \sigma(a)$. Therefore, $a = 0$, which is not possible. Hence the result is proved. $\square$

We now state and prove the main results of this paper in the form of the following Theorems:

Theorem 1. *Let R be a Noetherian integral domain which is also an algebra over $\mathbb{Q}$, σ an automorphism of R and δ a σ-derivation of R. If R is a (σ-δ), then R is 2-primal.*

Proof. R is a (σ-δ)-ring. We know that a reduced ring is 2-primal. We use the principle of Mathematical Induction to prove that R is a reduced ring. Let for $x \in R$, $x^n = 0$. We use induction on n and show that $x = 0$. The result is trivially true for $n = 1$, as $x^n = x^1 = a(\sigma(a) + \delta(a)) = 0$. Now Proposition 1, implies that $a = 0$, hence $x = 0$. Therefore, the result is true for $n = 1$. Let us assume that the result is true for $n = k$, i.e. $x^k = 0$ implies that $x = 0$. Let $n = k + 1$. Then $x^{k+1} = 0$ which implies that

$$a^{k+1}(\sigma(a) + \delta(a))^{k+1} = 0.$$

Again by Proposition 1 we get $a = 0$. Hence $x = 0$. Therefore, the result is true for $n = k + 1$ too. Thus the result is true for all n by the principle of Mathematical Induction. Hence the theorem is proved. $\square$

The converse of the above is not true.

Example 9. Let $R = F(x)$ be the field of rational polynomials in one variable x. Then R is 2-primal with $P(R) = \{0\}$.
Let $\sigma : R \to R$ be an endomorphism defined by

$$\sigma(f(x)) = f(0).$$

For $r \in R$, $\delta_r : R \to R$ be a σ-derivation defined as

$$\delta_r(a) = ar - r\sigma(a).$$

Then R is not a $(\sigma - \delta)$-ring.
Take $f(x) = xa + b, r = \frac{-b}{xa}$. Then

$$\begin{aligned} f(x)\Big\{\sigma(f(x)) + \delta_r(f(x))\Big\} &= f(x)\Big\{b + (xa+b)(\frac{-b}{xa}) - (\frac{-b}{xa})\sigma(f(x))\Big\} \\ &= f(x)\Big\{b - b - \frac{b^2}{xa} + \frac{b}{xa}b\Big\} \\ &= f(x)\Big\{b - b - \frac{b^2}{xa} + \frac{b^2}{xa}\Big\} = 0 \in P(R). \end{aligned}$$

But $f(x) \neq 0$. Therefore, $f(x)$ is not an element of $P(R)$. Hence R is not a (σ-δ)-ring.

For the proof of the next theorem, we require the following:

J. Krempa [10] has investigated the relation between minimal prime ideals and completely prime ideals of a ring R. With this he proved the following:

Theorem 2. *For a ring R the following conditions are equivalent:*
(1) R is reduced.
(2) R is semiprime and all minimal prime ideals of R are completely prime.
(3) R is a subdirect product of domains.

Theorem 3. *Let R be a Noetherian integral domain which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R. If R is a (σ-δ)-ring, then $P(R)$ is completely semi-prime.*

Proof. As proved in Theorem 1, R is a reduced ring and by using Theorem 2, the result follows. □

The converse of the above is not true.

Example 10. Let $\mathbb{F}$ be a field, $R = \mathbb{F} \times \mathbb{F}$. Let $\sigma : R \to R$ be an automorphism defined as

$$\sigma((a, b)) = (b, a), a, b \in \mathbb{F}.$$

Here $P(R)$ is a completely semi-prime ring, as R is a reduced ring.
For $r \in F$, define $\delta_r : R \to R$ by

$$\delta_r((a,b)) = (a,b)r - r\sigma((a,b)) \text{ for } a, b \in F.$$

Then δ_r is a σ-derivation on R. Take $A = (1,-1), r = \frac{1}{2}$.

Now $A\Big\{\sigma(A) + \delta_r(A)\Big\} = (1,-1)\Big\{\sigma((1,-1)) + (1,-1)\frac{1}{2} - \frac{1}{2}\sigma((1,-1))\Big\} = (1,-1)\Big\{(-1,1) + (\frac{1}{2}, \frac{-1}{2}) - \frac{1}{2}(-1,1)\Big\} = (0,0) \in P(R) = \{0\}$. But $(1,-1) \neq 0$. Hence it is not a $(\sigma$-$\delta)$-ring.

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Received January 12, 2015

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BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(82), 2016, Pages 12–23
ISSN 1024–7696

On invariants and canonical form of matrices of second order with respect to semiscalar equivalence

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Abstract. We indicate a complete system of invariants and suggest a canonical form for one class of polynomial matrices of second order with respect to semiscalar equivalence.

Mathematics subject classification: 15A21.
Keywords and phrases: semiscalar equivalence, canonical matrix, complete system of invariants, classification .

The notion of semiscalar equivalence of polynomial matrices is introduced and considered first in [1] (see also [2]). Related results are obtained in [3],[4]. These researches take on further development in [5] – [8]. The most important components of the problem of semiscalar equivalence are the search of invariants and the construction of normal forms for matrices with respect to such equivalence. Large difficulties in this problem arise already for matrices of second order. In this paper, some classes of order two polynomial matrices are singled out for which complete system of invariants is obtained and canonical form with respect to semiscalar equivalence is indicated. This form enables one to solve the classification problem for some polynomial matrices up to semiscalar equivalence.

We consider a ring $M(2,\ C[x])$ of order two polynomial matrices over the field of complex numbers C. According to [1] the matrices $A(x),\ B(x) \in M(2,\ C[x])$ are called semiscalarly equivalent if $CA(x)Q(x) = B(x)$ for some invertible matrices $C \in GL(2,\ C)$, $Q(x) \in GL(2,\ C[x])$. The determinant $|A(x)|$ is called the characteristic polynomial of $A(x)$ and its roots are called the characteristic roots of matrices $A(x)$. By Theorem 1 [1] (see also Theorem 1 §1, Section IV [2]) every matrix of full rank is semiscalarly equivalent to lower triangular form with invariant polynomials on the main diagonal. Without loss of generality, we can assume that first invariant polynomial of considered matrix is identity.

In this paper we use the standard notations. In particular, $c^{(t)}(\alpha)$ is the value at $x = \alpha$ of the t-th derivative of the polynomial $c(x)$.

Proposition 1. *Let be given a matrix*

$$A(x) = \begin{Vmatrix} 1 & 0 \\ a(x) & \Delta(x) \end{Vmatrix}, \quad \deg a(x) < \deg \Delta(x), \tag{1}$$

and a partition

$$M = M_1 \cup \ \dots \ \cup M_w, \quad M_u \cap M_v = \emptyset, \quad u \neq v, \tag{2}$$

of the set M of characteristic roots of matrix $A(x)$ into subsets M_u such that $\alpha, \beta \in M_u$ if $a(\alpha) = a(\beta)$. Subsets M_u are uniquely defined by a class of semiscalarly equivalent matrices $\{CA(x)Q(x)\}$.

Proof. Let a matrix $A(x)$ be semiscalarly equivalent to a matrix

$$B(x) = \left\| \begin{matrix} 1 & 0 \\ b(x) & \Delta(x) \end{matrix} \right\|, \quad \deg b(x) < \deg \Delta(x). \tag{3}$$

Then there exists

$$\left\| \begin{matrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{matrix} \right\| \in GL(2,\ C), \left\| \begin{matrix} r_{11}(x) & r_{12}(x) \\ r_{21}(x) & r_{22}(x) \end{matrix} \right\| \in GL(2,\ C[x]),$$

such that

$$\left\| \begin{matrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{matrix} \right\| \left\| \begin{matrix} 1 & 0 \\ a(x) & \Delta(x) \end{matrix} \right\| = \left\| \begin{matrix} 1 & 0 \\ b(x) & \Delta(x) \end{matrix} \right\| \left\| \begin{matrix} r_{11}(x) & r_{12}(x) \\ r_{21}(x) & r_{22}(x) \end{matrix} \right\|. \tag{4}$$

On the basis of (4) we can write the relation

$$s_{21} + s_{22}a(x) = b(x)r_{11}(x) + \Delta(x)r_{21}(x). \tag{5}$$

Setting $x = \alpha$ and $x = \beta$ in (5), we obtain the relations

$$s_{21} + s_{22}a(\alpha) = b(\alpha)r_{11}(\alpha), \tag{6}$$

$$s_{21} + s_{22}a(\beta) = b(\beta)r_{11}(\beta). \tag{7}$$

From (4) it follows that $r_{11}(x) = s_{11} + s_{12}a(x)$. Since $a(\alpha) = a(\beta)$, then $r_{11}(\alpha) = r_{11}(\beta)$ and from (6) and (7) we have $r_{11}(\alpha)(b(\alpha) - b(\beta)) = 0$. Equality (4) implies that $r_{12}(x) = s_{12}\Delta(x)$. Therefore $r_{11}(\alpha) \neq 0$ and $b(\alpha) = b(\beta)$. The notion of semiscalar equivalence is a symmetrical relation. Then from $b(\alpha) = b(\beta)$ a similar argument yields $a(\alpha) = a(\beta)$. This completes the proof. □

Consider now the case in which in (2) $w = 1$, i.e., $a(\alpha) = a(\beta)$ for arbitrary roots $\alpha,\ \beta \in M$. We may assume (without loss of generality) that $a(\alpha) = 0$.

Let $M = \{\alpha_i,\ i = 1,\ \dots,\ p\}$, n_i and m_i be the multiplicities of root α_i in the polynomials $\Delta(x)$ and $a(x)$, respectively. Since $\deg a(x) < \deg \Delta(x) = s$, for some root $\alpha_j \in M$ multiplicities n_j and m_j satisfy the condition $m_j < n_j$. Let it be the roots α_j, $j = 1,\ \dots,\ q$, $1 \leq q \leq p$ and $m_{q+l} \geq n_{q+l}$, $l = 1,\ \dots,\ p - q$ (the case in which $a(x) \equiv 0$ is trivial).

Theorem 1. *Let every characteristic root $\alpha_i \in M$ of matrix $A(x)$ of the form (1) satisfy the condition $a(\alpha_i) = 0$. Let also multiplicities m_j and n_j of root $\alpha_j \in M$ in the polynomials $a(x)$ and $\Delta(x)$, respectively, satisfy the inequality $m_j < n_j$. Then multiplicities m_j are uniquely defined by a class of semiscalarly equivalent matrices $\{CA(x)Q(x)\}$ and rows $\left\| \begin{array}{cccc} a_{j0} & a_{j1} & \dots & a_{j,\, l_j - m_j - 1} \end{array} \right\|$, $l_j = \min(2m_j,\ n_j)$, of coefficients from decompositions*

$$a(x) = \sum_{t=0}^{s-m_j-1} a_{jt}(x-\alpha_j)^{m_j+t} \tag{8}$$

are determined up to constant factor independent of $j = 1,\ \dots,\ q$.

Proof. Let matrices (1) and (3) be semiscalarly equivalent. If $a(\alpha_i) = b(\alpha_i) = 0$ then from relation (5) it follows that $s_{21} = 0$. Then

$$s_{22}a(x) - s_{11}b(x) - s_{12}a(x)b(x) = \Delta(x)r_{21}(x), \tag{9}$$

where $s_{11} \neq 0$, $s_{22} \neq 0$. Let for multiplicities m_j, m'_j, n_j of root $x = \alpha_j$ in the polynomials $a(x)$, $b(x)$, $\Delta(x)$, respectively, inequalities $m'_j < m_j < n_j$ be valid. Differentiating both members of equality (9) m'_j times at $x = \alpha_j$, we obtain $s_{11}b^{(m'_j)}(\alpha_j) = 0$. It is impossible, since $s_{11} \neq 0$ and $b^{(m'_j)}(\alpha_j) \neq 0$. Then $m'_j \geq m_j$. Considering that semiscalar equivalence is a symmetric relation, we have $m'_j \leq m_j$. Therefore $m'_j = m_j$. The first part of the theorem is proved.

By analogy to (8), write decomposition for the entry $b(x)$ of matrix (3):

$$b(x) = \sum_{t=0}^{s-m_j-1} b_{jt}(x-\alpha_j)^{m_j+t}. \tag{10}$$

Comparing the coefficients of equal degrees of binomial $x - \alpha_j$ on both sides of equality (9), we obtain

$$\begin{cases} \ \ s_{22}a_{j0} - s_{11}b_{j0} = 0, \\ \ \ s_{22}a_{j1} - s_{11}b_{j1} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ s_{22}a_{j,\, l_j-m_j-1} - s_{11}b_{j,\, l_j-m_j-1} = 0, \end{cases} \tag{11}$$

where $l_j = \min(2m_j,\ n_j)$, $j = 1,\ \dots,\ q$, $s_{11} \neq 0$, $s_{22} \neq 0$. From equalities (11) it follows that $a_{j0} = kb_{j0}$, $a_{j1} = kb_{j1}$, $\dots$, $a_{j,\, l_j-m_j-1} = kb_{j,\, l_j-m_j-1}$, where $k = s_{11}s_{22}^{-1}$. This completes the proof of the theorem. □

Corollary 1. *Matrix (1) in the class $\{CA(x)Q(x)\}$ of semiscalarly equivalent matrices is determined up to a constant factor if multiplicities n_j and m_j in polynomials $\Delta(x)$ and $a(x)$ of every its characteristic root α_i, $i = 1,\ \dots,\ p$, satisfy the inequality $2m_i \geq n_i$.*

Proof. Let matrices (1) and (3) be semiscalarly equivalent. By Theorem 1 we have $a(\alpha_i) = b(\alpha_i) = 0$, $a^{(s_i)}(\alpha_i) = b^{(s_i)}(\alpha_i) = 0$, $s_i = 1, \ldots, m_i - 1$, $i = 1, \ldots, p$. From theorem we have also $a^{(h_i)}(\alpha_i) = kb^{(h_i)}(\alpha_i)$, $h_i = m_i, \ldots, n_i - 1$. Then the values of polynomial $a(x)$ and values of its derivative at α_i, $i = 1, \ldots, p$, of order $1, \ldots, n_i - 1$ are proportional to corresponding values of polynomial $b(x)$ and to corresponding values of the derivative of this polynomial. Since $\deg a(x)$, $\deg b(x) < \sum n_i = s$, then polynomials $a(x)$ and $b(x)$ differ from each other by a constant factor. Corollary is proved. □

Consider now the case when the conditions of the corollary are not satisfied, i.e., for some root α_i the inequality $2m_i < n_i$ is fulfilled.

Theorem 2. *Let n_j be the multiplicity of the root α_j in the characteristic polynomial $\Delta(x)$, $\deg \Delta(x) = s$, of the matrices (1) and (3). Besides, let $w = 1$ in the partition (2) of set M of theirs characteristic roots and*

$$a(x) = \sum_{t=0}^{s-m_j-1} a_{jt}(x-\alpha_j)^{m_j+t}, \quad b(x) = \sum_{t=0}^{s-m_j-1} b_{jt}(x-\alpha_j)^{m_j+t}$$

be binomial decompositions of the entries $a(x)$, $b(x)$ of these matrices. Matrices (1) and (3) are semiscalarly equivalent if and only if for every characteristic root α_j such that $m_j < n_j$ and for every pair of characteristic roots α_i, α_l such that $2m_i < n_i$, $2m_l < n_l$, there exists the same number $k \neq 0$, the following conditions hold:

1) $\left\| \begin{matrix} a_{j0} & a_{j1} & \ldots & a_{j,\, l_j-m_j-1} \end{matrix} \right\| = k \left\| \begin{matrix} b_{j0} & b_{j1} & \ldots & b_{j,\, l_j-m_j-1} \end{matrix} \right\|$, $l_j = \min(2m_j,\ n_j)$;

2)

$$\begin{vmatrix} a_{j1} & a_{j2} & \ldots & a_{j,\, s_j-1} & a_{js_j} \\ a_{j0} & a_{j1} & \ddots & a_{j,\, s_j-2} & a_{j,\, s_j-1} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & a_{j1} & a_{j2} \\ 0 & & & a_{j0} & a_{j1} \end{vmatrix} = k^{s_j} \begin{vmatrix} b_{j1} & b_{j2} & \ldots & b_{j,\, s_j-1} & b_{js_j} \\ b_{j0} & b_{j1} & \ddots & b_{j,\, s_j-2} & b_{j,\, s_j-1} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & b_{j1} & b_{j2} \\ 0 & & & b_{j0} & b_{j1} \end{vmatrix}, \tag{12}$$

$s_j = 1, \ldots, m_j - 1, m_j + 1, \ldots, n_j - m_j - 1$;

3)

$$a_{im_i} a_{i0}^{-2} - a_{lm_l} a_{l0}^{-2} = k^{-1}(b_{im_i} b_{i0}^{-2} - b_{lm_l} b_{l0}^{-2}). \tag{13}$$

Proof. Necessity. Let matrices (1) and (3) be semiscalarly equivalent. The condition 1) follows from Theorem 1. If for characteristic root α_j such that $m_j < n_j$ satisfies the inequality $2m_j \geq n_j$, then the condition 2) follows from the condition 1). In the opposite case such that $2m_j < n_j$ from the equality (9) we obtain the systems

$$\begin{cases} s_{22}a_{j0} - s_{11}b_{j0} = 0, \\ s_{22}a_{j1} - s_{11}b_{j1} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ s_{22}a_{j,\, m_j-1} - s_{11}b_{j,\, m_j-1} = 0, \end{cases} \tag{14}$$

$$\left\{\begin{array}{l} s_{22}a_{jm_j} - s_{11}b_{jm_j} - s_{12}a_{j0}b_{j0} = 0, \\ s_{22}a_{j,\,m_j+1} - s_{11}b_{j,\,m_j+1} - s_{12}(a_{j0}b_{j1} + a_{j1}b_{j0}) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ s_{22}a_{j,\,n_j-m_j-1} - s_{11}b_{j,\,n_j-m_j-1} - s_{12}\sum_{u=0}^{n_j-2m_j-1} a_{ju}b_{j,\,n_j-2m_j-u-1} = 0. \end{array}\right. \tag{15}$$

Since $s_{11},\ s_{22} \neq 0$, from (14) we can write

$$a_{j0} = kb_{j0}, a_{j1} = kb_{j1}, \dots, a_{j,\,m_j-1} = kb_{j,\,m_j-1}, k = s_{11}s_{22}^{-1}. \tag{16}$$

From this is follows that equality (12) is satisfied for $s_j = 1,\ \dots,\ m_j - 1$. As appears from (15), if $a_{jm_j} = kb_{jm_j}$, that $s_{12} = 0$ and $a_{js_j} = kb_{js_j}$ for $s_i = m_i + 1,\ \dots,\ n_i - m_i - 1$. From this it follows that equality (12) is valid for the same $s_i = m_i + 1,\ \dots,\ n_i - m_i - 1$. For this reason we think in what follows $a_{jm_j} \neq kb_{jm_j}$, $k = s_{11}s_{22}^{-1}$. From the first and second equations (15) by excluding s_{12} we obtain

$$a_{j0}a_{j,\,m_j+1} - a_{jm_j}(kb_{j1} + a_{j1}) = k^2b_{j0}b_{j,\,m_j+1} - kb_{jm_j}(kb_{j1} + a_{j1}). \tag{17}$$

If $m_j = 1$, then $a_{j0}a_{j2} - a_{j1}^2 = k^2(b_{j0}b_{j2} - b_{j1}^2)$. This means that conditions (12) are fulfilled for $s_j = m_j + 1$. If $m_j > 1$, then $a_{j1} = kb_{j1}$ and from (17) by multiplication $a_{j0}^{m_j-1} = k^{m_j-1}b_{j0}^{m_j-1}$ can be obtained

$$a_{j0}^{m_j}a_{j,\,m_j+1} - 2a_{j0}^{m_j-1}a_{j1}a_{jm_j} = k^{m_j+1}(b_{j0}^{m_j}b_{j,\,m_j+1} - 2b_{j0}^{m_j-1}b_{j1}b_{jm_j}). \tag{18}$$

Denote by A_{juv}, B_{juv} submatrices obtained, respectively, from matrices

$$\left\|\begin{array}{ccccc} a_{j1} & a_{j2} & \dots & a_{jm_j} & a_{j,\,m_j+1} \\ a_{j0} & a_{j1} & \ddots & a_{j,\,m_j-1} & a_{jm_j} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & a_{j1} & a_{j2} \\ 0 & & & a_{j0} & a_{j1} \end{array}\right\|, \quad \left\|\begin{array}{ccccc} b_{j1} & b_{j2} & \dots & b_{jm_j} & b_{j,\,m_j+1} \\ b_{j0} & b_{j1} & \ddots & b_{j,\,m_j-1} & b_{jm_j} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & b_{j1} & b_{j2} \\ 0 & & & b_{j0} & b_{j1} \end{array}\right\|, \tag{19}$$

by obliterating of two last columns and u-th and v-th rows. Denote also by $\delta_{j,\,m_j+1}(A)$, $\delta_{j,\,m_j+1}(B)$ the determinants of matrices (19) respectively. Decompose them for minors of order two that are contained in the last two columns. Because $|A_{juv}| = |B_{juv}| = 0$ for $u \neq m_j + 1$, we have

$$\delta_{j,\,m_j+1}(A) = (-1)^{m_j+1}\left(\left|\begin{array}{cc} a_{jm_j} & a_{j,\,m_j+1} \\ a_{j0} & a_{j1} \end{array}\right| \left|A_{j,\,1,\,m_j+1}\right| - \right.$$

$$\left. - \left|\begin{array}{cc} a_{j,\,m_j-1} & a_{jm_j} \\ a_{j0} & a_{j1} \end{array}\right| \left|A_{j,\,2,\,m_j+1}\right| + \ \dots + \left|\begin{array}{cc} a_{j1} & a_{j2} \\ a_{j0} & a_{j1} \end{array}\right| \left|A_{j,\,m_j,\,m_j+1}\right|\right),$$

$$\delta_{j,\, m_j+1}(B) = (-1)^{m_j+1}\left(\begin{vmatrix} b_{jm_j} & b_{j,\, m_j+1} \\ b_{j0} & b_{j1} \end{vmatrix} \left|B_{j,\, 1,\, m_j+1}\right| -\right.$$

$$\left. -\begin{vmatrix} b_{j,\, m_j-1} & b_{jm_j} \\ b_{j0} & b_{j1} \end{vmatrix} \left|B_{j,\, 2,\, m_j+1}\right| + \ \dots + \begin{vmatrix} b_{j1} & b_{j2} \\ b_{j0} & b_{j1} \end{vmatrix} \left|B_{j,\, m_j,\, m_j+1}\right| \right).$$

Since the rows $\| \ a_{j0} \quad a_{j1} \quad \dots \quad a_{j,\, m_j-1} \ \|$, $\| \ b_{j0} \quad b_{j1} \quad \dots \quad b_{j,\, m_j-1} \ \|$ differ by a multiplier k (see (16)), each summand of expression in parenthesis for $\delta_{j,\, m_j+1}(A)$, except first two, differs from the corresponding summand for $\delta_{j,\, m_j+1}(B)$ by a multiplier k^{m_j+1}. From this fact and from the equality (18) follows equality (12) for $s_j = m_j + 1$.

Denote by $\delta_{js_j}(A)$ and $\delta_{js_j}(B)$ the determinants in left and right parts of equality (12), respectively. Suppose by induction $\delta_{jr}(A) = k^r \delta_{jr}(B)$ for all r such that $m_j < r < n_j - m_j - 1$. Accept for the sake of determinacy $r > 2m_j$. In the case where $r \le 2m_j$ the proof radically is not different. From first r-th equality (15) exclude s_{12} and by sufficiently evident transformations we obtain

$$\left\{\begin{array}{c}
(a_{j,\, m_j+1} - (a_{j0}b_{j0})^{-1} a_{jm_j} \sum_{u=0}^{1} a_{ju} b_{j,\, 1-u})(-a_{j0})^{m_j} \delta_{j,\, r-m_j}(A) = \\
= k^{r+1}(b_{j,\, m_j+1} - (a_{j0}b_{j0})^{-1} b_{jm_j} \sum_{u=0}^{1} a_{ju} b_{j,\, 1-u})(-b_{j0})^{m_j} \delta_{j,\, r-m_j}(B), \\
(a_{j,\, m_j+2} - (a_{j0}b_{j0})^{-1} a_{jm_j} \sum_{u=0}^{2} a_{ju} b_{j,\, 2-u})(-a_{j0})^{m_j+1} \delta_{j,\, r-m_j-1}(A) = \\
k^{r+1}(b_{j,\, m_j+2} - (a_{j0}b_{j0})^{-1} b_{jm_j} \sum_{u=0}^{2} a_{ju} b_{j,\, 2-u})(-b_{j0})^{m_j+1} \delta_{j,\, r-m_j-1}(B), \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
(a_{j,\, r-m_j+1} - (a_{j0}b_{j0})^{-1} a_{jm_j} \sum_{u=0}^{r-2m_j+1} a_{ju} b_{j,\, r-2m_j-u+1} - \\
-a_{j,\, r-m_j+1} + (a_{j0}b_{j0})^{-1} a_{jm_j} \sum_{u=0}^{r-2m_j+1} a_{ju} b_{j,\, r-2m_j-u+1})(-a_{j0})^{r-m_j} \delta_{jm_j}(A) = \\
k^{r+1}(b_{j,\, r-m_j+1} - (a_{j0}b_{j0})^{-1} b_{jm_j} \sum_{u=0}^{r-2m_j+1} a_{ju} b_{j,\, r-2m_j-u+1} - \\
-b_{j,\, r-m_j+1} + (a_{j0}b_{j0})^{-1} b_{jm_j} \sum_{u=0}^{r-2m_j+1} a_{ju} b_{j,\, r-2m_j-u+1})(-b_{j0})^{r-m_j} \delta_{jm_j}(B), \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
(a_{jr} - (a_{j0}b_{j0})^{-1} a_{jm_j} \sum_{u=0}^{r-m_j} a_{ju} b_{j,\, r-m_j-u})(-a_{j0})^{r-1} \delta_1(A) = \\
k^{r+1}(b_{jr} - (a_{j0}b_{j0})^{-1} b_{jm_j} \sum_{u=0}^{r-m_j} a_{ju} b_{j,\, r-m_j-u})(-b_{j0})^{r-1} \delta_1(B), \\
(a_{j,\, r+1} - (a_{j0}b_{j0})^{-1} a_{jm_j} \sum_{u=0}^{r-m_j+1} a_{ju} b_{j,\, r-m_j-u+1})(-a_{j0})^{r} = \\
k^{r+1}(b_{j,\, r+1} - (a_{j0}b_{j0})^{-1} b_{jm_j} \sum_{u=0}^{r-m_j+1} a_{ju} b_{j,\, r-m_j-u+1})(-b_{j0})^{r}.
\end{array}\right. \tag{20}$$

If we add left parts of equality (20) and separately right parts we obtain

$$(-a_{j0})^r a_{j,\, r+1} + (-a_{j0})^{r-1} a_{jr} \delta_{j1}(A) + \ \dots \ + (-a_{j0})^{r-m_j} a_{j,\, r-m_j+1} \delta_{jm_j}(A) + \ \dots \ +$$
$$+(-a_{j0})^{m_j} a_{j,\, m_j+1} \delta_{j,\, r-m_j}(A) + (-a_{j0})^{m_j-1} a_{jm_j} \delta_{j,\, r-m_j+1}(A) -$$
$$-(a_{j0}b_{j0})^{-1} a_{jm_j} (b_{j1} \delta_{j,\, r-m_j}(A)(-a_{j0})^{m_j+1} + b_{j2} \delta_{j,\, r-m_j-1}(A)(-a_{j0})^{m_j+2} + \ \dots \ +$$
$$+b_{j,\, r-m_j} \delta_{j1}(A)(-a_{j0})^{r}) + (a_{j0}b_{j0})^{-1} a_{jm_j} (b_{j1} \delta_{j,\, r-m_j}(A)(-a_{j0})^{m_j+1} +$$
$$+b_{j2} \delta_{j,\, r-m_j-1}(A)(-a_{j0})^{m_j+2} + \ \dots \ + b_{j,\, r-m_j} \delta_{j1}(A)(-a_{j0})^{r} + b_{j,\, r-m_j+1}(-a_{j0})^{r+1}) -$$

$$-(a_{j,\,r-m_j+1}-(a_{j0}b_{j0})^{-1}a_{jm_j}\sum_{u=0}^{r-2m_j+1}a_{ju}b_{j,\,r-2m_j-u+1})(-a_{j0})^{r-m_j}\delta_{jm_j}(A)=$$
$$k^{r+1}((-b_{j0})^r b_{j,\,r+1}+(-b_{j0})^{r-1}b_{jr}\delta_{j1}(B)+\ \dots\ +(-b_{j0})^{r-m_j}b_{j,\,r-m_j+1}\delta_{jm_j}(B)+\ \dots\ +$$
$$+(-b_{j0})^{m_j}b_{j,\,m_j+1}\delta_{j,\,r-m_j}(B)+(-b_{j0})^{m_j-1}b_{jm_j}\delta_{j,\,r-m_j+1}(B)-$$
$$-(a_{j0}b_{j0})^{-1}b_{jm_j}(a_{j1}\delta_{j,\,r-m_j}(B)(-b_{j0})^{m_j+1}+a_{j2}\delta_{j,\,r-m_j-1}(B)(-b_{j0})^{m_j+2}+\ \dots\ +$$
$$+a_{j,\,r-m_j}\delta_{j1}(B)(-b_{j0})^r)+(a_{j0}b_{j0})^{-1}b_{jm_j}(a_{j1}\delta_{j,\,r-m_j}(B)(-b_{j0})^{m_j+1}+$$
$$+a_{j2}\delta_{j,\,r-m_j-1}(B)(-b_{j0})^{m_j+2}+\ \dots\ +a_{j,\,r-m_j}\delta_{j1}(B)(-b_{j0})^r+a_{j,\,r-m_j+1}(-b_{j0})^{r+1})-$$
$$-(b_{j,\,r-m_j+1}-(a_{j0}b_{j0})^{-1}b_{jm_j}\sum_{u=0}^{r-2m_j+1}a_{ju}b_{j,\,r-2m_j-u+1})(-b_{j0})^{r-m_j}\delta_{jm_j}(B))\ . \quad (21)$$

Group similar terms in both parts of obtained equality to have

$$(-a_{j0})^r a_{j,\,r+1}+(-a_{j0})^{r-1}a_{jr}\delta_{j1}(A)+\ \dots\ +(-a_{j0})^{r-m_j}a_{j,\,r-m_j+1}\delta_{jm_j}(A)+$$
$$+\ \dots\ +(-a_{j0})^{m_j}a_{j,\,m_j+1}\delta_{j,\,r-m_j}(A)+(-a_{j0})^{m_j-1}a_{j,\,m_j}\delta_{j,\,r-m_j+1}(A)+$$
$$+(a_{j0}b_{j0})^{-1}a_{jm_j}b_{j,\,r-m_j+1}(-a_{j0})^{r+1}-(a_{j,\,r-m_j+1}-$$
$$-(a_{j0}b_{j0})^{-1}a_{jm_j}\sum_{u=1}^{r-2m_j+1}a_{ju}b_{j,\,r-2m_j-u+1})(-a_{j0})^{r-m_j}\delta_{jm_j}(A)=$$
$$k^{r+1}((-b_{j0})^r b_{j,\,r+1}+(-b_{j0})^{r-1}b_{jr}\delta_{j1}(B)+\ \dots\ +(-b_{j0})^{r-m_j}b_{j,\,r-m_j+1}\delta_{jm_j}(B)+$$
$$+\ \dots\ +(-b_{j0})^{m_j}b_{j,\,m_j+1}\delta_{j,\,r-m_j}(B)+(-b_{j0})^{m_j-1}b_{j,\,m_j}\delta_{j,\,r-m_j+1}(B)+$$
$$+(a_{j0}b_{j0})^{-1}b_{jm_j}a_{j,\,r-m_j+1}(-b_{j0})^{r+1}-(b_{j,\,r-m_j+1}-$$
$$-(a_{j0}b_{j0})^{-1}b_{jm_j}\sum_{u=1}^{r-2m_j+1}a_{ju}b_{j,\,r-2m_j-u+1})(-b_{j0})^{r-m_j}\delta_{jm_j}(B). \quad (22)$$

It follows from (15) that

$$a_{j,\,r-m_j+1}+(a_{j0}b_{j0})^{-1}a_{jm_j}\sum_{u=0}^{r-2m_j+1}a_{ju}b_{j,\,r-2m_j-u+1}=$$
$$=k(b_{j,\,r-m_j+1}+(a_{j0}b_{j0})^{-1}b_{jm_j}\sum_{u=0}^{r-2m_j+1}a_{ju}b_{j,\,r-2m_j-u+1}).$$

From this relation it is easy to be sure that the following equality is true

$$(a_{j0}b_{j0})^{-1}a_{jm_j}b_{j,\,r-m_j+1}(-a_{j0})^{r+1}-(a_{j,\,r-m_j+1}-$$
$$-(a_{j0}b_{j0})^{-1}a_{jm_j}\sum_{u=0}^{r-2m_j+1}a_{ju}b_{j,\,r-2m_j-u+1})(-a_{j0})^{r-m_j}\delta_{jm_j}(A)=$$
$$=k^{r+1}((a_{j0}b_{j0})^{-1}b_{jm_j}a_{j,\,r-m_j+1}(-b_{j0})^{r+1}-(b_{j,\,r-m_j+1}-$$

$$-(a_{j0}b_{j0})^{-1}b_{jm_j}\sum_{u=0}^{r-2m_j+1}a_{ju}b_{j,\,r-2m_j-u+1})(-b_{j0})^{r-m_j}\delta_{jm_j}(B). \tag{23}$$

From (16) and induction hypothesis we can write

$$(-a_{j0})^{m_j-2}a_{j,\,m_j-1}\delta_{j,\,r-m_j+2}(A)+\ \dots\ +(-a_{j0})a_{j2}\delta_{j,\,r-1}(A)+a_{j1}\delta_{jr}(A)=$$

$$=k^{r+1}((-b_{j0})^{m_j-2}b_{j,\,m_j-1}\delta_{j,\,r-m_j+2}(B)+\ \dots\ +(-b_{j0})b_{j2}\delta_{j,\,r-1}(B)+b_{j1}\delta_{jr}(B). \tag{24}$$

Comparing (22), (23) and (24), we obtain equality

$$(-a_{j0})^{r}a_{j,\,r+1}+(-a_{j0})^{r-1}a_{jr}\delta_{j1}(A)+\ \dots\ +(-a_{j0})a_{j2}\delta_{j,\,r-1}(A)+a_{j1}\delta_{jr}(A)=$$

$$=k^{r+1}((-b_{j0})^{r}b_{j,\,r+1}+(-b_{j0})^{r-1}b_{jr}\delta_{j1}(B)+\ \dots\ +(-b_{j0})b_{j2}\delta_{j,\,r-1}(B)+b_{j1}\delta_{jr}(B)), \tag{25}$$

i.e., $\delta_{j,\,r+1}(A)=k^{r+1}\delta_{j,\,r+1}(B)$, $k=s_{11}s_{22}^{-1}$. The necessity of conditions 2) of the theorem is proved.

Let

$$a(x)=\sum_{t=0}^{s-m_i-1}a_{it}(x-\alpha_i)^{m_i+t},\quad a(x)=\sum_{t=0}^{s-m_l-1}a_{lt}(x-\alpha_l)^{m_l+t},$$

$$b(x)=\sum_{t=0}^{s-m_i-1}b_{it}(x-\alpha_i)^{m_i+t},\quad b(x)=\sum_{t=0}^{s-m_l-1}b_{lt}(x-\alpha_l)^{m_l+t}$$

$$s_{22}a_{im_i}-s_{11}b_{im_i}-s_{12}a_{i0}b_{i0}=0,$$

be decompositions for entries $a(x)$, $b(x)$ of matrices (1), (3) into degrees of binomials $x-\alpha_i$, $x-\alpha_l$. From (9) it may be written

$$s_{22}a_{im_i}-s_{11}b_{im_i}-s_{12}a_{i0}b_{i0}=0,$$

$$s_{22}a_{lm_l}-s_{11}b_{lm_l}-s_{12}a_{l0}b_{l0}=0.$$

From these equalities exclude s_{12}. Considering that $a_{i0}=kb_{i0}$, $a_{l0}=kb_{l0}$, we have (13). The necessity of the conditions 1) – 3) of theorem is proved.

Sufficiency. For each characteristic root $x=\alpha_j$ of matrix (1) such that $m_j<n_j$ and $2m_j\geq n_j$, from condition 1) of theorem it follows that

$$s_{22}a(x)-s_{11}b(x)-s_{12}a(x)b(x)\equiv 0\ (mod\ (x-\alpha_j)^{n_j}), \tag{26}$$

where $s_{22}=1$, $s_{11}=k=a_{j0}b_{j0}^{-1}$, $s_{12}\in C$.

Let now $x=\alpha_j$ be an arbitrary characteristic root of matrices (1), (3) such that $2m_j<n_j$. Consider equalities (14) and (15) as one system of equations with coefficients a_{ju}, b_{ju}, $u=0,\ 1,\ \dots,\ n_j-m_j-1$, $a_{j0}\neq 0$, $b_{j0}\neq 0$, in three unknowns

s_{22}, s_{11}, s_{12}. We shall show that conditions of theorem imply that there is nonzero solution of this system such that $s_{22} = 1$, $s_{11} = k = a_{j0}b_{j0}^{-1}$ the same for every characteristic root α_j of matrices (1), (3) such that $2m_j < n_j$. We shall prove this fact by induction. The condition 1) implies that system (14) has nonzero solution such that it does not dependent on the choice of the characteristic root α_j. After annihilation of equal summands on the both sides of equality (12) for $s_j = m_j + 1$ and after division by $a_{j0}^{m_j} = k^{m_j} b_{j0}^{m_j}$ with the help of simple transformations we can obtain the following relation

$$a_{j,\, m_j+1} - kb_{j,\, m_j+1} - (a_{j0}b_{j0})^{-1}(a_{jm_j} - kb_{jm_j})(a_{j0}b_{j1} + a_{j1}b_{j0}) = 0.$$

This means that

$$s_{22} = 1, \quad s_{11} = k, \quad s_{12} = (a_{j0}b_{j0})^{-1}(a_{jm_j} - kb_{jm_j}). \tag{27}$$

is a common solution of first two equations of system (15). From (13) it follows that $(a_{i0}b_{i0})^{-1}(a_{im_i} - kb_{im_i}) = (a_{l0}b_{l0})^{-1}(a_{lm_l} - kb_{lm_l})$. This result suggests that this solution (27) of first two equations of system (15) does not depend on the choice of the root α_j such that $2m_j < n_j$.

Assume by induction that (27) satisfies first $r - m_j + 1$ equations of system (15), i.e.,

$$\left\{\begin{array}{c} a_{jm_j} - kb_{jm_j} - (a_{j0}b_{j0})^{-1}(a_{jm_j} - kb_{jm_j})a_{j0}b_{j0} = 0, \\ a_{j,\, m_j+1} - kb_{j,\, m_j+1} - (a_{j0}b_{j0})^{-1}(a_{jm_j} - kb_{jm_j})\sum_{u=0}^{1} a_{ju}b_{j,\, 1-u} = 0, \\ \dotfill \\ a_{jr} - kb_{jr} - (a_{j0}b_{j0})^{-1}(a_{jm_j} - kb_{jm_j})\sum_{u=0}^{r-m_j} a_{ju}b_{j,\, r-m_j-u} = 0. \end{array}\right. \tag{28}$$

In so doing, we may think for the sake of determinacy $r > 2m_j$. In opposite case proof is completely analogous. Taking into account the conditions 1), 2) and inductive assumption we can write equalities (23), (24) and (25). From these equalities we obtain equality (22). This relation implies the equality (21). It is evident that from the second and all following equalities of (28) we find that first $r - m_j$ equalities of (20) are valid. The first $r - m_j$ equalities of (20) along with relation (21) yield the last equality of (20). This equality after shortening in $(-a_{j0})^r = k^r(-b_{j0})^r$ and after some simplifications can be written in the form

$$a_{j,\, r+1} - kb_{j,\, r+1} - (a_{j0}b_{j0})^{-1}(a_{jm_j} - kb_{jm_j}) \sum_{u=0}^{r-m_j+1} a_{ju}b_{j,\, r-m_j-u+1} = 0.$$

This means that (27) is the solution of $(r - m_j + 1)$-th equation of system (15). This solution does not dependent on the choice of the root α_j.

Thus, congruence (26) holds true for each characteristic root α_j of matrices (1), (3) and for the same set of numbers (27), where $s_{22} \neq 0$, $s_{11} \neq 0$. It enables us to write the congruence

$$s_{22}a(x) - s_{11}b(x) - s_{12}a(x)b(x) \equiv 0 \ (mod \ \Delta(x)). \tag{29}$$

We introduce the following notation:

$$r_{11}(x) = s_{11} - s_{12}b(x), r_{12}(x) = s_{12}\Delta(x),$$

$$r_{22}(x) = s_{22} - s_{12}b(x), r_{21}(x) = \frac{s_{22}a(x) - s_{11}b(x) - s_{12}a(x)b(x)}{\Delta(x)}.$$

It is clear that $r_{21}(x) \in C$. With this notations check that equality (4) is true. From this it follows that matrices (1) and (3) are semiscalarly equivalent. The theorem is proved. □

Theorem 3. *In the partition (2) for matrix $A(x)$ of the form (1) let us have $w = 1$; n_i and m_i be the multiplicities of some root $\alpha_i \in M$ in the characteristic polynomial $\Delta(x)$ and in polynomial $a(x)$ of matrix, $A(x)$ respectively, moreover $2m_i < n_i$. Then in the class of semiscalarly equivalent matrices $\{CA(x)Q(x)\}$ there exists a matrix $B(x)$ of the form (3), where entry $b(x)$ satisfies the following conditions: $b(\alpha_i) = 0$, $b^{(m_i)}(\alpha_i) = m_i!$, $b^{(2m_i)}(\alpha_i) = 0$. For a fixed root α_i the matrix $B(x)$ is defined uniquely.*

Proof. **Existence.** We may take, that already the entry $a(x)$ of the matrix $A(x)$ satisfies the condition $a^{(m_i)}(\alpha_i) = m_i!$. In the opposite case, for this purpose we divide the first column of matrix $A(x)$ and multiply its first row by $\frac{a^{(m_i)}(\alpha_i)}{m_i!}$. Let α_j denote an arbitrary characteristic root of matrix $A(x)$ of multiplicity n_j such that in the decomposition

$$a(x) = \sum_{t=0}^{s-m_j-1} a_{jt}(x-\alpha_j)^{m_j+t}, \tag{30}$$

where $s = \deg \Delta(x)$, the index m_j is less than n_j. We set

$$\left\| \ b_{j0} \quad b_{j1} \quad \dots \quad b_{j,\, l_j-m_j-1} \ \right\| = \left\| \ a_{j0} \quad a_{j1} \quad \dots \quad a_{j,\, l_j-m_j-1} \ \right\|,$$

where $l_j = \min(2m_j,\ n_j)$. Let $\alpha_l \in M$, $\alpha_l \neq \alpha_i$, be an arbitrary characteristic root such that $2m_l < n_l$. We write the formal equality $b_{im_i}b_{i0}^{-2} - b_{lm_l}b_{l0}^{-2} = a_{im_i}a_{i0}^{-2} - a_{lm_l}a_{l0}^{-2}$, where $a_{l0},\ a_{i0},\ a_{im_i},\ a_{lm_l}$ are coefficients of the decomposition (30) for $j = i$ and $j = l$. Setting $b_{i0} = a_{i0}$, $b_{l0} = a_{l0}$ and $b_{im_i} = 0$ in this relation, we calculate b_{lm_l}. Using this value b_{lm_l} and determined above $b_{l0} = a_{l0}$, $b_{l1} = a_{l1}$, ..., $b_{l,\, m_l-1} = a_{l,\, m_l-1}$, from formal equalities

$$\begin{vmatrix} b_{l1} & b_{l2} & \dots & b_{l,\,s_l-1} & b_{ls_l} \\ b_{l0} & b_{l1} & \ddots & b_{l,\,s_l-2} & b_{l,\,s_l-1} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & b_{l1} & b_{l2} \\ 0 & & & b_{l0} & b_{l1} \end{vmatrix} = \begin{vmatrix} a_{l1} & a_{l2} & \dots & a_{l,\,s_l-1} & a_{ls_l} \\ a_{l0} & a_{l1} & \ddots & a_{l,\,s_l-2} & a_{l,\,s_l-1} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & a_{l1} & a_{l2} \\ 0 & & & a_{l0} & a_{l1} \end{vmatrix}, \qquad (31)$$

$s_l = m_l + 1,\ \dots,\ n_l - m_l - 1$, we find recurrently $b_{l,\,m_l+1},\ \dots,\ b_{l,\,n_l-m_l-1}$. Setting $l = i$, $b_{im_i} = 0$ and using determined above $b_{i0} = a_{i0}$, $b_{i1} = a_{i1}$, $\dots$, $b_{i,\,m_i-1} = a_{i,\,m_i-1}$, similarly from (31) we find recurrently $b_{i,\,m_i+1},\ \dots,\ b_{i,\,n_i-m_i-1}$. Thus, for every root $\alpha_j \in M$ such that in the decomposition (30) $m_j < n_j$, some numbers $b_{i0},\ b_{i1},\ \dots,\ b_{j,\,n_j-m_j-1} \in C$ are defined. We construct the matrix $B(x)$ of the form (3) whose entry $b(x)$, where $\deg b(x) < s$, satisfies such conditions: $b(\alpha_j) = 0$, $b^{(1)}(\alpha_j) = 0,\ \dots,\ b^{(m_j-1)}(\alpha_j) = 0$, $b^{(m_j)}(\alpha_j) = m_j!b_{j0},\ \dots,\ b^{(n_j-1)}(\alpha_j) = (n_j - 1)!b_{j,\,n_j-m_j-1}$, and $b(\alpha) = 0$, $b^{(1)}(\alpha) = 0,\ \dots,\ b^{(n-1)}(\alpha_j) = 0$ for each root $\alpha \in M$ of multiplicity n which is different from α_j. Since matrix (1) and constructed matrix of the form (3) satisfy the conditions of Theorem 2, they are semiscalarly equivalent. The first part of theorem is proved.

The **uniqueness** of the matrix $B(x)$ of the form (3) whose entry $b(x)$ satisfies the conditions described in theorem follows from the uniqueness of construction of the polynomial $b(x)$, $\deg b(x) < s = \deg \Delta(x)$, by known its values and values of its derivatives of respective orders at roots of the polynomial $\Delta(x)$. The theorem is completely proved.

□

Definition 1. The matrix $B(x)$ of the form (3) whose existence and uniqueness in the class $\{CA(x)Q(x)\}$ are established in theorem 3 is called α_i-*canonical*. The matrix $A(x)$ of the form (1) is called also α_i-*canonical* if for each root $\alpha_j \in M$ of multiplicity n_j in the decomposition (30) of its entry $a(x)$ index m_j satisfies the condition $2m_j \geq n_j$ and for some root $\alpha_i \in M$ we have $m_i < n_i$, $a^{(m_i)}(\alpha_i) = (m_i)!$.

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Received June 2, 2015

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(82), 2016, Pages 24–37
ISSN 1024–7696

Resonant Riemann-Liouville Fractional Differential Equations with Periodic Boundary Conditions

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Abstract. In this paper, by using the coincidence degree theory due to J. Mawhin, we consider the solvability of a class of nonlinear fractional two-point boundary value problems at resonance. An example of application illustrates the existence result.

Mathematics subject classification: 34A08, 37C25, 54H25.
Keywords and phrases: Fractional differential equations, resonance, coincidence degree, Riemann-Liouville fractional derivative.

1 Introduction

Fractional differential equations describe many phenomena in various fields of science and engineering such as physics, chemistry, biology, visco-elasticity, electromagnetics, economy, etc. Several methods have been used to deal with the question of solvability of boundary value problems (BVPs for short) for fractional differential equations; we quote the Laplace transform method, iteration methods, the upper and lower solution method, as well as topological methods (fixed point theory and Leray-Schauder degree theory) (see, e. g., [1, 10], and references therein).

In [1] B. Ahmad and J. Nieto studied the following Riemann-Liouville fractional differential equation with fractional boundary conditions:

$$D_{0^+}^{\alpha} u(t) = f(t, u(t)), \quad t \in [0, T], \ 1 < \alpha \leq 2, \tag{1.1}$$

$$D_{0^+}^{\alpha-2} u\left(0^+\right) = b_0 D_{0^+}^{\alpha-2} u\left(T^-\right), \tag{1.2}$$

$$D_{0^+}^{\alpha-1} u\left(0^+\right) = b_1 D_{0^+}^{\alpha-1} u\left(T^-\right), \tag{1.3}$$

where $D_{0^+}^{\alpha}$ denotes the Riemann-Liouville fractional derivative of order α, $b_0 \neq 1$, $b_1 \neq 1$, and the function $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous. Clearly this is a nonresonant problem, i.e. the associated homogeneous problem admits only the following solution:

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2},$$

where the constants c_1, c_2 satisfy

$$\begin{aligned} c_1\Gamma(\alpha).0 + c_2\Gamma(\alpha-1) &= b_0(c_1\Gamma(\alpha).T + c_2\Gamma(\alpha-1)) \\ c_1\Gamma(\alpha) &= b_1 c_1\Gamma(\alpha), \end{aligned}$$

that is $c_1 = c_2 = 0$ for $b_0 \neq 1$ and $b_1 \neq 1$. Then a corresponding Green's function can be computed. A fixed point theorem was used to show that the operator $P : C_{2-\alpha} \longrightarrow C_{2-\alpha}$ defined by

$$\begin{aligned} (Pu)(t) &= \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s))\, ds + \tfrac{b_1 t^{\alpha-1}}{(1-b_1)\Gamma(\alpha)} \int_0^T f(s, u(s))\, ds \\ &\quad + \tfrac{b_0 t^{\alpha-2}}{(1-b_1)(1-b_0)\Gamma(\alpha-1)} \int_0^T (T-(1-b_1)s) f(s, u(s))\, ds \end{aligned}$$

has at least one fixed point.

By a similar method, G. Wang, W. Liu, and C. Ren investigated in [10], the existence and uniqueness of solutions for the fractional boundary-value problem:

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, u(t)), \quad t \in [0, T], \quad 1 < \alpha \leq 2, \\ I_{0^+}^{2-\alpha} u(t)\,|_{t=0} = 0, \quad D_{0^+}^{\alpha-2} u(T) = \sum\limits_{i=1}^{m} a_i I_{0^+}^{\alpha-1}(\xi_i), \end{cases}$$

where $0 < \xi_i < T$, $a_i \in \mathbb{R}$, $m \geq 2$, and $I_{0^+}^{\alpha}$ stands for the Riemann-Liouville fractional integral. Standard fixed point principles have been employed.

In [11], the authors investigated higher-order fractional derivatives, i. e. for $2 < \alpha \leq 3$.

When the nonlinearity of f also depends on the first derivative, Z. Bai [2] discussed the solvability of m-point fractional BVPs at resonance; the coincidence degree theory as developed by Mawhin in [8] was employed. Concerning papers dealing with fractional-order BVPs at resonance, we refer, for example, to [4–6, 11, 12]. See also [9] for a resonant second-order boundary value problem.

In the present work, Mawhin's coincidence degree theory is used to deal with BVP (1.1), (1.2), (1.3) at the resonance case, i. e. for $b_0 = b_1 = 1$. An existence result illustrated by means of two examples of application is provided in Section 2.

We first present some definitions and auxiliary lemmas about fractional calculus theory.

Definition 1 (see [3, 7])**.** The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $h : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds,$$

where $\Gamma(.)$ refers to the function gamma, provided the right side is pointwise defined on $(0, +\infty)$.

Definition 2 (see [7, 11])**.** The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $h : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{h(s)}{(t-s)^{\alpha-n+1}} ds = \frac{d^n}{dt^n} I_{0^+}^{n-\alpha} h(t),$$

where $n = [\alpha] + 1$, provided the right side is pointwise defined on $(0, +\infty)$. Here $[\alpha]$ denotes the integer part of the real number α.

For $\alpha < 0$, we set by convention $D_{0^+}^{\alpha} h(t) = I_{0^+}^{-\alpha} h(t)$, and if $0 \leq \beta \leq \alpha$, we get $D_{0^+}^{\beta} I_{0^+}^{\alpha} h(t) = I_{0^+}^{\alpha-\beta} h(t)$.

Given these definitions, it can be checked that the Riemann-Liouvelle fractional integration and fractional differentiation operators of the power functions t^{λ} yield power functions of the same form. Indeed, for $\lambda > -1$ and $\alpha \geq 0$, we have

$$I_{0^+}^{\alpha} t^{\lambda} = \tfrac{\Gamma(\lambda+1)}{\Gamma(\lambda+\alpha+1)} t^{\lambda+\alpha} \text{ and } D_{0^+}^{\alpha} t^{\lambda} = \tfrac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha}.$$

Also note that $D_{0^+}^{\alpha} t^{\lambda} = 0$, for all $\lambda = \alpha - i$ with $i = 1, 2, 3, \ldots, n$ (n is the smallest integer greater than or equal to α). Also we have

Lemma 1 (see [4])**.** *Suppose that $h \in L^1(0, +\infty)$ and α, β are positive real numbers. Then*

$$I_{0^+}^{\alpha} I_{0^+}^{\beta} h(t) = I_{0^+}^{\alpha+\beta} h(t) \quad \textit{and} \quad D_{0^+}^{\alpha} I_{0^+}^{\alpha} h(t) = h(t).$$

If, in addition $D_{0^+}^{\alpha} h(t) \in L^1(0, +\infty)$, then

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} h(t) = h(t) + \sum_{i=1}^{i=n} c_i t^{\alpha-i},$$

for some constants $c_i \in \mathbb{R}$ $(1 \leq i \leq n)$.

Finally, notice that the boundary value problem

$$\begin{cases} D_{0^+}^{\alpha} u(t) &= f(t, u(t)), \quad t \in [0, T], \ 1 < \alpha \leq 2 \\ D_{0^+}^{\alpha-2} u(0^+) &= D_{0^+}^{\alpha-2} u(T^-), \\ D_{0^+}^{\alpha-1} u(0^+) &= D_{0^+}^{\alpha-1} u(T^-) \end{cases}$$

is at resonance, i. e., the corresponding homogeneous boundary value problem:

$$\begin{cases} D_{0^+}^{\alpha} u(t) &= 0, \quad t \in [0, T], \ 1 < \alpha \leq 2 \\ D_{0^+}^{\alpha-2} u(0^+) &= D_{0^+}^{\alpha-2} u(T^-), \\ D_{0^+}^{\alpha-1} u(0^+) &= D_{0^+}^{\alpha-1} u(T^-) \end{cases}$$

has $u(t) = ct^{\alpha-2}$ as nontrivial solutions $(c \in \mathbb{R})$.

2 Main result

2.1 Functional framework

Since our main existence result is based on Mawhin's coincidence degree, we first recall some basic facts about this theory; more details can be found in [8].

Let X, Y be two real Banach spaces and $L : \mathrm{dom}(L) \subset X \to Y$ a Fredholm operator of index zero. Then there exist two continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Im}\, P = \mathrm{Ker}\, L$, $\mathrm{Ker}\, Q = \mathrm{Im}\, L$, $X = \mathrm{Ker}\, L \oplus \mathrm{Ker}\, P$, and $Y = \mathrm{Im}\, L \oplus \mathrm{Im}\, \mathrm{Q}$. It follows that the operator

$$L_P = L\left|_{\mathrm{dom}(L)\cap \mathrm{Ker}\, P}\right. : \mathrm{dom}(L) \cap \mathrm{Ker}\, P \to \mathrm{Im}\, L$$

is invertible; we denote its inverse by K_P (i.e. $L_P^{-1} = K_P$). Let Ω be an open bounded subset of X such that $\mathrm{dom}\,(L) \cap \overline{\Omega} \neq \emptyset$. The map $N : X \to Y$ is said to be L-compact on $\overline{\Omega}$ if $QN\left(\overline{\Omega}\right)$ is bounded and the operator $K_{P,Q} = K_P\,(I - Q)\,N : \overline{\Omega} \to X$ is compact. Since $\mathrm{Im}\, Q$ and $\mathrm{Ker}\, L$ have the same dimension, then there exists a linear isomorphism $J : \mathrm{Im}\, Q \to \mathrm{Ker}\, L$. Mawhin [8] established the following existence result for the abstract nonlinear equation $Lu = Nu$:

Theorem 1. *Let $L : X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$ be L-compact operator on $\overline{\Omega}$. Then the equation $Lu = Nu$ has at least one solution in $\mathrm{dom}\,(L) \cap \overline{\Omega}$ if the following conditions are satisfied:*

1. *$Lu \neq Nu$ for each $(u, \lambda) \in [(\mathrm{dom}\,(L) \setminus \mathrm{Ker}\, L) \cap \partial\Omega] \times [0, 1]\,;$*
2. *$Nu \notin \mathrm{Im}\, L$, for each $u \in \mathrm{Ker}\, L \cap \partial\Omega$;*
3. *$\deg\left(QN\left|_{\mathrm{Ker}\, L}\right., \mathrm{Ker}\, L \cap \Omega, 0\right) \neq 0.$*

As usual, $C\,[0, T]$ will denote the Banach space of continuous real valued functions defined on $[0, T]$ with the norm $\|u\| = \sup_{t\in[0,T]} |u\,(t)|$. For all $t \in [0, T]$, we define the function u_r by $u_r\,(t) = t^r u\,(t)$, $r \geq 0$. Let $C_r\,[0, T]$ be the space of all functions u such that $u_r \in C\,[0, T]$. Then

Lemma 2. *$C_r\,[0, T]$ endowed with the norm $\|u\|_r = \sup_{t\in[0,T]} t^r\,|u\,(t)|$ is a real Banach space.*

Let $Y = L^1\,[0, T]$ be the Lebesgue space of measurable functions y such that $s \longmapsto |y(s)|$ is Lebesgue integrable equipped with the norm $\|y\|_1 = \int_0^T |y(s)|\, ds$ and $X = C_{2-\alpha}\,[0, T]$ endowed with the norm $\|u\|_{2-\alpha} = \sup_{t\in[0,T]} t^{2-\alpha}\,|u\,(t)|$. Define the linear operator $L : \mathrm{dom}\,(L) \cap X \longrightarrow Y$ by

$$Lu = D_{0^+}^{\alpha} u, \tag{2.1}$$

where

$$\mathrm{dom}(L) = \left\{u \in X : \, D_{0^+}^{\alpha} u \in Y, u \text{ satisfies conditions } (1.2), (1.3) \text{ with } b_0 = b_1 = 1\right\}.$$

Finally, define the Nemytskii operator $N : X \longrightarrow Y$ by

$$(Nu)(t) = f(t, u(t)), \quad t \in [0, T]. \tag{2.2}$$

Thus, BVP (1.1), (1.2), (1.3) with $b_0 = b_1 = 1$ can be written as

$$Lu = Nu, \ u \in \mathrm{dom}(L).$$

In a series of lemmas, we next investigate the properties of operators L and N.

2.2 Auxiliary lemmas

Lemma 3. *Let L be the operator defined by (2.1); then*

$$\mathrm{Ker}\, L = \left\{ct^{\alpha-2} : c \in \mathbb{R}\right\} \quad \textit{and} \ \mathrm{Im}\ L = \left\{y \in L[0,T] : \int_0^T y(s)ds = 0\right\}.$$

Proof. The equation $D^{\alpha}_{0^+}u(t) = 0$ admits $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2}$ as solutions, where c_1, c_2 are arbitrary constants. Then

$$D^{\alpha-2}_{0^+}u(t) = I^{2-\alpha}_{0^+}u(t) = c_1\Gamma(\alpha)\, t + c_2\Gamma(\alpha-1) \ \text{ and } \ D^{\alpha-1}_{0^+}u(t) = c_1\Gamma(\alpha).$$

Combining this with (1.2) and (1.3), we find that

$$c_2\Gamma(\alpha-1) = c_1\Gamma(\alpha)\, T + c_2\Gamma(\alpha-1)$$

and hence $c_1 = 0$ while c_2 is any constant.

If $y \in \mathrm{Im}\ (L)$, then there exists $u \in \mathrm{dom}(L)$ such that $D^{\alpha}_{0^+}(t) = y(t)$. Hence

$$u(t) = I^{\alpha}_{0^+}y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}$$

and

$$\begin{array}{rcl} D^{\alpha-2}_{0^+}u(t) & = & I^{2}_{0^+}y(t) + c_1\Gamma(\alpha)\, t + c_2\Gamma(\alpha-1), \\ D^{\alpha-1}_{0^+}u(t) & = & I^{1}_{0^+}y(t) + c_1\Gamma(\alpha). \end{array}$$

By the boundary conditions (1.2), (1.3), we infer that

$$c_1 = -\frac{1}{\Gamma(\alpha)\, T}\int_0^T (T-s)\, y(s)ds \ \text{ and } \ \int_0^T y(s)ds = 0.$$

Let $y \in Y$ satisfy $\int_0^T y(s)ds = 0$. If $u(t) = I^{\alpha}_{0^+}y(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)T}\int_0^T (T-s)\, y(s)ds$, then $u \in \mathrm{dom}(L)$ and $D^{\alpha}_{0^+}D^{\alpha}_{0^+}u(t) = y(t)$. As a consequence $y \in \mathrm{Im}\ (L)$. □

Lemma 4.

(a) $L : \mathrm{dom}\,(L) \cap X \longrightarrow Y$ is a Fredholm operator of index 0.

(b) The linear continuous projectors $Q : Y \to Y$ and $P : X \to X$ are such that

$$Qy = \frac{1}{T}\int_0^T y(s)ds \ \textit{ and } \ (Pu)(t) = \frac{1}{\Gamma(\alpha-1)} I^{2-\alpha}_{0^+}u(t)\mid_{t=T} t^{\alpha-2}.$$

Proof. It is easy to see that $Q^2y = Qy$ and $P^2u = Pu$, for $y \in Y$, $u \in X$. For all $y \in Y$, $y_1 = y - Qy \in \mathrm{Im}\ (L)$ because $\int_0^T y_1(s)ds = 0$. Hence $Y = \mathrm{Im}\ (L) + \mathrm{Im}\ (Q)$, $(\mathrm{Im}\,(Q) = \mathbb{R})$. For $m \in \mathrm{Im}\ (L) \cap \mathbb{R}$, we have $\int_0^T mds = Tm = 0$; therefore $m = 0$ and $Y = \mathrm{Im}\ (L) \oplus \mathrm{Im}\ (Q)$. Thus $\dim(\mathrm{Ker}\, L) = co \dim\,(\mathrm{Im}\, L) = \dim(\mathrm{Im}\, Q) = \dim(\mathbb{R}) = 1$. So L is a Fredholm operator of index 0. ☐

Lemma 5. *Let* $L_P = L\mid_{\mathrm{dom}(L)\cap \mathrm{Ker}\, P}: \mathrm{dom}(L) \cap \mathrm{Ker}\, P \to \mathrm{Im}\ (L)$. *The inverse* K_P *of* L_P *is given by*

$$(K_P y)(t) = I_{0+}^{\alpha} y(t) - \frac{t^{\alpha-1}}{T\Gamma(\alpha)} I_{0+}^{2} y(T).$$

Moreover

$$\|K_P y\|_{2-\alpha} \le \frac{2T}{\Gamma(\alpha)} \|y\|_1,$$

for all $y \in \mathrm{Im}\ (L)$.

Proof. For all $y \in \mathrm{Im}\ (L)$, we have

$$(LK_P y)(t) = D_{0+}^{\alpha} \left(I_{0+}^{\alpha} y(t) - \frac{t^{\alpha-1}}{T\Gamma(\alpha)} I_{0+}^{2} y(T) \right) = y(t).$$

Recall that

$$\mathrm{Ker}\, P = \left\{ u \in \mathrm{dom}\,(L) : \ I_{0+}^{2-\alpha} u(t) \mid_{t=T} = 0 \right\}.$$

Thus, for $u \in \mathrm{dom}\,(L) \cap \mathrm{Ker}\, P$, we have

$$\begin{aligned} (K_P L)\, u(t) &= I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) - \tfrac{t^{\alpha-1}}{T.\Gamma(\alpha)} I_{0+}^{2} D_{0+}^{\alpha} u(T) \\ &= u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} - \tfrac{I_{0+}^{2-\alpha} u(T)}{T.\Gamma(\alpha)} t^{\alpha-1}. \end{aligned}$$

Since $u \in \mathrm{dom}(L) \cap \mathrm{Ker}\, P$, then

$$(K_P L)\, u \in \mathrm{dom}(L) \cap \mathrm{Ker}\, P$$

and so

$$I_{0+}^{2-\alpha} u(T) = 0 \ \text{ and } \ c_1 t^{\alpha-1} + c_2 t^{\alpha-2} \in \mathrm{dom}(L) \cap \mathrm{Ker}\, P.$$

Moreover

$$I_{0+}^{2-\alpha} \left(c_1 t^{\alpha-1} + c_2 t^{\alpha-2} \right) = c_1 \Gamma(\alpha)\, t + c_2 \Gamma(\alpha - 1);$$

hence

$$c_2 \Gamma(\alpha - 1) = c_1 \Gamma(\alpha)\, T + c_2 \Gamma(\alpha - 1) = 0.$$

Finally $c_2 = c_1 = 0$ and

$$(K_P L)\, u(t) = u(t),$$

which shows that $K_P = (L_P)^{-1}$.

Keeping in mind that

$$t^{2-\alpha} (K_P y)(t) = \frac{t^{2-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)\, y(s) ds,$$

we deduce that

$$t^{2-\alpha}\left|(K_P y)(t)\right| \le \frac{T^{2-\alpha}}{\Gamma(\alpha)}T^{\alpha-1}\int_0^T |y(s)|\,ds + \frac{T}{T\Gamma(\alpha)}T\int_0^T |y(s)|\,ds = \frac{2T}{\Gamma(\alpha)}\|y\|_1 .$$

Finally

$$\|K_P y\|_{2-\alpha} = \sup_{t\in[0,T]} t^{2-\alpha}\left|(K_P y)(t)\right| \le \frac{2T}{\Gamma(\alpha)}\|y\|_1 .$$

□

Lemma 6. *For all $u \in X$, $t \in [0,T]$, we have*

$$t^{2-\alpha}K_P(I-Q)Nu(t) = \frac{1}{\Gamma(\alpha)}\int_0^T G(t,s)\,f(s,u(s))\,ds,$$

where

$$G(t,s) = \begin{cases} t^{2-\alpha}(t-s)^{\alpha-1} + \frac{ts}{T} - \frac{t}{2} - \frac{t^2}{\alpha T}, & 0 \le s < t \le T,\\ \frac{ts}{T} - \frac{t}{2} - \frac{t^2}{\alpha T}, & 0 \le t < s \le T. \end{cases}$$

Lemma 7. *Let $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that Ω is an open bounded subset from X such that $\operatorname{dom}(L)\cap\overline{\Omega} \neq \emptyset$; then N is L-compact on $\overline{\Omega}$.*

Proof. In order to prove that N is L-compact on $\overline{\Omega}$, we only need to show that $QN\left(\overline{\Omega}\right)$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to Y$ is compact.

Since $f : [0,T]\times\mathbb{R}\to\mathbb{R}$ is continuous, $\overline{\Omega}$ is bounded; therefore there exists a constant $M > 0$ such that $|f(t,u(t))| \le M$, $\forall\, u \in \overline{\Omega}$, $\forall\, t \in [0,T]$. Consequently, for all $u \in \overline{\Omega}$, we have

$$\begin{aligned} \|QN(u)\|_1 &= \int_0^T \left[\frac{1}{T}\left|\int_0^T f(s,u(s))\,ds\right|\right]ds = \left|\int_0^T f(s,u(s))\,ds\right| \\ &\le \int_0^T |f(s,u(s))|\,ds \le MT. \end{aligned}$$

Since $(I-Q)$ and K_P are continuous linear operators, then $(I-Q)N(u)$ and $K_P(I-Q)N(u)$ are bounded. Hence

$$\begin{aligned} \|(I-Q)N(u)\|_1 &\le \|N(u)\|_1 + \|QN(u)\|_1 \le 2TM, \\ \|K_P(I-Q)N(u)\|_{2-\alpha} &\le \frac{2T}{\Gamma(\alpha)}\|(I-Q)N(u)\|_1 \le \frac{4T^2M}{\Gamma(\alpha)}. \end{aligned}$$

For all $t_1 \in [0,T]$, $t_2 \in [0,T]$, $(t_1 < t_2)$, and $u \in \overline{\Omega}$, we have

$$\begin{aligned} &\left|t_2^{2-\alpha}K_P(I-Q)Nu(t_2) - t_1^{2-\alpha}K_P(I-Q)Nu(t_1)\right| \\ &= \frac{1}{\Gamma(\alpha)}\left|\int_0^T G(t_2,s)\,f(s,u(s))\,ds - \int_0^T G(t_1,s)\,f(s,u(s))\,ds\right| \\ &= \frac{1}{\Gamma(\alpha)}\left|\int_0^T (G(t_2,s)-G(t_1,s))f(s,u(s))\,ds\right| \\ &\le \frac{M}{\Gamma(\alpha)}\int_0^T |G(t_2,s)-G(t_1,s)|\,ds. \end{aligned}$$

Next, we distinguish between three different cases:

1. Case $t_1 < t_2 < s$. We have
$$\begin{array}{rcl} \left|G(t_2,s)-G(t_1,s)\right| & = & \left|t_2-t_1\right|\left|\frac{s}{T}-\left(\frac{1}{2}+\frac{t_2+t_1}{\alpha T}\right)\right| \\ & \leq & \left|t_2-t_1\right|\left(\frac{s}{T}+\left(\frac{1}{2}+\frac{t_2+t_1}{\alpha T}\right)\right); \end{array}$$
then
$$\begin{array}{rcl} \int_0^T\left|G(t_2,s)-G(t_1,s)\right|ds & \leq & \left|t_2-t_1\right|\int_0^T\left(\frac{s}{T}+\left(\frac{1}{2}+\frac{t_2+t_1}{\alpha T}\right)\right)ds \\ & = & \left(T+\frac{t_2+t_1}{\alpha}\right)\left|t_2-t_1\right|. \end{array}$$

2. Case $s < t_1 < t_2$. We have
$$\begin{array}{rcl} \left|G(t_2,s)-G(t_1,s)\right| & = & \Big|t_2^{2-\alpha}(t_2-s)^{\alpha-1}-t_1^{2-\alpha}(t_1-s)^{\alpha-1} \\ & & +(t_2-t_1)\left(\frac{s}{T}-\left(\frac{1}{2}+\frac{t_2+t_1}{\alpha T}\right)\right)\Big| \\ & \leq & \left|t_2^{2-\alpha}(t_2-s)^{\alpha-1}-t_1^{2-\alpha}(t_1-s)^{\alpha-1}\right| \\ & & +\left|(t_2-t_1)\left(\frac{s}{T}-\left(\frac{1}{2}+\frac{t_2+t_1}{\alpha T}\right)\right)\right|. \end{array}$$
Note that the function Ψ_s defined by
$$\Psi_s(t)=t^{2-\alpha}(t-s)^{\alpha-1},$$
where $t\in[0,T]$ and $0\leq s<t$, is increasing on $[0,T]$ because its derivative
$$\Psi_s'(t)=(2-\alpha)\left(\frac{t-s}{t}\right)^{\alpha-1}+(\alpha-1)\left(\frac{t}{t-s}\right)^{2-\alpha}$$
is positive. Then
$$t_2^{2-\alpha}(t_2-s)^{\alpha-1}-t_1^{2-\alpha}(t_1-s)^{\alpha-1}>0$$
and
$$\begin{array}{rl} & \int_0^T(t_2^{2-\alpha}(t_2-s)^{\alpha-1}-t_1^{2-\alpha}(t_1-s)^{\alpha-1})ds \\ = & t_2^{2-\alpha}\int_0^{t_2}(t_2-s)^{\alpha-1}ds-t_1^{2-\alpha}\int_0^{t_1}(t_1-s)^{\alpha-1}ds \\ = & \frac{t_2-t_1}{\alpha}. \end{array}$$
Finally
$$\begin{array}{rcl} \int_0^T\left|G(t_2,s)-G(t_1,s)\right|ds & \leq & \frac{t_2-t_1}{\alpha}+\left(T+\frac{t_2+t_1}{\alpha}\right)\left|t_2-t_1\right| \\ & = & \left(T+\frac{t_2+t_1+1}{\alpha}\right)\left|t_2-t_1\right|. \end{array}$$

3. Case $t_1 < s < t_2$. We have
$$\begin{array}{rcl} \left|G(t_2,s)-G(t_1,s)\right| & = & \left|t_2^{2-\alpha}(t_2-s)^{\alpha-1}+(t_2-t_1)\left(\frac{s}{T}-\left(\frac{1}{2}+\frac{t_2+t_1}{\alpha T}\right)\right)\right| \\ & \leq & t_2^{2-\alpha}(t_2-s)^{\alpha-1}+\left|(t_2-t_1)\left(\frac{s}{T}-\left(\frac{1}{2}+\frac{t_2+t_1}{\alpha T}\right)\right)\right| \\ & \leq & t_2^{2-\alpha}(t_2-t_1)^{\alpha-1}+\left(T+\frac{t_2+t_1}{\alpha}\right)\left|t_2-t_1\right|. \end{array}$$

This shows that $K_P(I-Q)N$ is equicontinuous, as claimed. $\square$

2.3 Existence theorem

We are now in position to state and prove our main existence result.

Theorem 2. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous. Assume that*

- (H_1) *there exist two functions a, $r \in L^1[0,T]$ such that for all $t \in [0,T]$ and $x \in \mathbb{R}$, we have $|f(t,x)| \le t^{2-\alpha} a(t)|x| + r(t)$,*
- (H_2) *there exists a constant $M > 0$ such that for all $u \in \operatorname{dom}(L)$, if $|u(t)| > M$ for all $t \in [0,T]$, then $\int_0^T f(s,u(s))\,ds \neq 0$,*
- (H_3) *there exists a constant $M^* > 0$ such that for all $c \in \mathbb{R}$, if $|c| > M^*$, then either*
$$c\int_0^T f\left(s, cs^{\alpha-2}\right) ds < 0 \ \text{ or } \ c\int_0^T f\left(s, cs^{\alpha-2}\right) ds > 0.$$

Then the boundary value problem (1.1), (1.2), (1.3) with $b_0 = b_1 = 1$ has at least one solution $u \in C_{2-\alpha}[0,T]$ provided that $\|a\|_1 < \frac{\Gamma(\alpha)}{2T}$.

Proof. Let
$$\Omega_1 = \{u \in \operatorname{dom}(L) \diagdown \operatorname{Ker} L : \ Lu = \lambda Nu, \ \lambda \in (0,1)\}.$$

For $u \in \Omega_1$, we have $u \in \operatorname{dom}(L) \cap \operatorname{Ker} P$ and $Lu = \lambda Nu$ with $\lambda \neq 0$ because $u \notin \operatorname{Ker} L$; then
$$\begin{aligned} \|u\|_{2-\alpha} &= \|K_P Lu\|_{2-\alpha} \\ &\le \tfrac{2T}{\Gamma(\alpha)} \|Lu\|_1 = \tfrac{2T\lambda}{\Gamma(\alpha)} \|Nu\|_1 \\ &\le \tfrac{2T}{\Gamma(\alpha)} \int_0^T |f(s,u(s))|\,ds. \end{aligned}$$

From condition (H_1), we have
$$|f(s,u(s))| \le s^{2-\alpha} a(s)|u(s)| + r(s) \le a(s) \sup_{s\in[0,T]} s^{2-\alpha}|u(s)| + r(s).$$

Hence
$$\int_0^T |f(s,u(s))|\,ds \le \|a\|_1 \|u\|_{2-\alpha} + \|r\|_1.$$

Then
$$\|u\|_{2-\alpha} \le \frac{2T}{\Gamma(\alpha)} \left(\|a\|_1 \|u\|_{2-\alpha} + \|r\|_1\right).$$

Finally
$$\|u\|_{2-\alpha} \le \frac{2T\|r\|_1}{\Gamma(\alpha) - 2T\|a\|_1} = M_1.$$

Consider the set
$$\Omega_2 = \{u \in \operatorname{Ker} L : \ Nu \in \operatorname{Im} L\}.$$

For $u \in \Omega_2$, we have $u(t) = ct^{\alpha-2}$ and $\int_0^T f\left(s, cs^{\alpha-2}\right) ds = 0$. Then, from the condition (H_2), there exists $t_0 \in [0,T]$ such that $\left|ct_0^{\alpha-2}\right| \leq M$, with $t_0 \neq 0$. Therefore

$$\|u\|_{2-\alpha} = \sup_{t\in[0,T]} t^{2-\alpha}\left|ct^{\alpha-2}\right| = |c| \leq Mt_0^{2-\alpha} = M_2.$$

Let

$$\Omega_3 = \left\{u \in \operatorname{Ker} L:\ -\lambda Ju + (1-\lambda)\, QNu = 0,\ \lambda \in [0,1]\right\},$$

where $J : \operatorname{Ker} L \to \operatorname{Im} Q$ is the linear isomorphism defined by $J(u) = c$.

In case (H_3) is satisfied, assume that $c\int_0^T f\left(s, cs^{\alpha-2}\right) ds < 0$ holds. For all $u \in \Omega_3$, we can write $u = ct^{\alpha-2}$ and

$$\lambda c^2 = \frac{(1-\lambda)}{T} c \int_0^T f\left(s, cs^{\alpha-2}\right) ds.$$

If $\lambda = 1$, then $c = 0$. Otherwise, if Hypothesis $|c| > M^*$, then by (H_3), one has

$$\frac{(1-\lambda)}{T} c \int_0^T f\left(s, cs^{\alpha-2}\right) ds < 0,$$

which contradicts $\lambda c^2 \geq 0$. Thus

$$\|u\|_{2-\alpha} = |c| \leq\ M^*.$$

If $c\int_0^T f\left(s, cs^{\alpha-2}\right) ds > 0$ holds, then Ω_3 can be defined as follows:

$$\Omega_3 = \left\{u \in \operatorname{Ker} L:\ \lambda Ju + (1-\lambda)\, QNu = 0,\ \lambda \in [0,1]\right\}.$$

Next, we shall prove that all conditions of Theorem 1 are fulfilled.

Let Ω be bounded open such that $\overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \overline{\Omega}_3 \subset \Omega$. We have already proved that L is a Fredholm operator of index 0 and that N is L-compact on $\overline{\Omega}$. Also, we have

1. $Lu \neq Nu$, for each $(u, \lambda) \in [(\operatorname{dom}(L) \diagdown \operatorname{Ker} L) \cap \partial\Omega] \times [0,1]$ for $\overline{\Omega}_1 \subset \Omega$.
2. $Nu \notin \operatorname{Im} L$ for each $u \in \operatorname{Ker} L \cap \partial\Omega$ for $\overline{\Omega}_2 \subset \Omega$.
3. In order to take into account the subset Ω_3 in the above two cases, we consider the homotopy $H(u, \lambda) = \pm\lambda Ju + (1-\lambda)\, QNu$. Then $H(u, \lambda) \neq 0$, for each $u \in \operatorname{Ker} L \cap \partial\Omega$. As $\overline{\Omega}_3 \subset \Omega$. By the homotopy property of the degree, we finally deduce that

$$\begin{aligned} \deg\left(QN\mid_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega, 0\right) &= \deg\left(H(u,0), \operatorname{Ker} L \cap \Omega, 0\right) \\ &= \deg\left(H(u,1), \operatorname{Ker} L \cap \Omega, 0\right) \\ &= \deg\left(\pm J, \operatorname{Ker} L \cap \Omega, 0\right) \neq 0, \end{aligned}$$

which completes the proof of Theorem 2. $\square$

2.4 Example 1

Consider the boundary value problem:

$$\left\{\begin{array}{rcl} D_{0^+}^{\frac{3}{2}} u(t) & = & \frac{3\sqrt{t}}{5\pi\sqrt{\pi}} u(t)\left(2\sin u(t) - 3\right) + \pi\sqrt{\pi}\cos t, \quad 0 < t < \frac{\pi}{4}, \\ D_{0^+}^{\frac{1}{2}} u(0^+) & = & D_{0^+}^{\frac{1}{2}} u\left(\frac{\pi}{4}^-\right), \\ I_{0^+}^{\frac{1}{2}} u(0^+) & = & I_{0^+}^{\frac{1}{2}} u\left(\frac{\pi}{4}^-\right). \end{array}\right. \tag{1}$$

In this example,

$$\alpha = \frac{3}{2},\ T = \frac{\pi}{4}, \text{ and } f(t,x) = \frac{3\sqrt{t}}{5\pi\sqrt{\pi}} x\left(2\sin x - 3\right) + \pi\sqrt{\pi}\cos t.$$

In addition, we have

1.
$$|f(t,x)| \le \frac{3\sqrt{t}}{5\pi\sqrt{\pi}} |x|\left(2|\sin x| + 3\right) + \pi\sqrt{\pi}\cos t \le \frac{3}{\pi\sqrt{\pi}}\sqrt{t}\,|x| + \pi\sqrt{\pi}\cos t.$$

   Then
$$a(t) = \frac{3}{\pi\sqrt{\pi}},\ \|a\|_1 = \frac{3}{4\sqrt{\pi}} < \frac{\Gamma\left(\frac{3}{2}\right)}{2\frac{\pi}{4}} = \frac{1}{\sqrt{\pi}}, \text{ and } r(t) = \pi\sqrt{\pi}\cos t.$$

2. Let $M = 80$. For each $u \in \operatorname{dom}(L)$, suppose that $|u(t)| > M$, for all $t \in [0, \frac{\pi}{4}]$. If $u(t) > M$, for all $t \in [0, \frac{\pi}{4}]$, then $2\sin u(t) - 3 \le -1$ and thus
$$f(t, u(t)) \le -\frac{3\sqrt{t}}{5\pi\sqrt{\pi}} u(t) + \pi\sqrt{\pi}\cos t \le -\frac{3\sqrt{t}}{5\pi\sqrt{\pi}} M + \pi\sqrt{\pi}\cos t.$$

   Notice that since $-u(t) < -M$, then
$$\int_0^{\frac{\pi}{4}} f(t, u(t))\,dt \le \int_0^{\frac{\pi}{4}} \left(-\frac{3\sqrt{t}}{5\pi\sqrt{\pi}} M + \pi\sqrt{\pi}\cos t\right) dt = -0.06 < 0.$$

   If $u(t) < -M$, for all $t \in [0, \frac{\pi}{4}]$, then $0 < M < -u(t)$ and
$$\frac{3\sqrt{t}}{5\pi\sqrt{\pi}} M < -\frac{3\sqrt{t}}{5\pi\sqrt{\pi}} u(t) \le \frac{3\sqrt{t}}{5\pi\sqrt{\pi}} u(t)\left(2\sin u(t) - 3\right).$$

   Hence $f(t, u(t)) \ge \frac{3\sqrt{t}}{5\pi\sqrt{\pi}} M + \pi\sqrt{\pi}\cos t$, for all $t \in [0, \frac{\pi}{4}]$. Consequently
$$\int_0^{\frac{\pi}{4}} f(t, u(t))\,dt \ge \int_0^{\frac{\pi}{4}} \left(\frac{3\sqrt{t}}{5\pi\sqrt{\pi}} M + \pi\sqrt{\pi}\cos t\right) ds = 7.93 > 0.$$

   Finally $\int_0^{\frac{\pi}{4}} f(t, u(t))\,dt \neq 0$.

3. Let $M^* = 95$. For every $c \in \mathbb{R}$ with $|c| > M^*$, we have $\left(2\sin\frac{c}{\sqrt{t}} - 3\right) \leq -1$. Then

$$\frac{3}{5\pi\sqrt{\pi}}c^2\left(2\sin\frac{c}{\sqrt{t}} - 3\right) + \pi\sqrt{\pi}c\cos t \leq -\frac{3}{5\pi\sqrt{\pi}}c^2 + \pi\sqrt{\pi}c\cos t.$$

Finally

$$\begin{aligned} c\int_0^{\frac{\pi}{4}} f\left(t, \tfrac{c}{\sqrt{t}}\right)dt &\leq \int_0^{\frac{\pi}{4}}\left(-\tfrac{3}{5\pi\sqrt{\pi}}c^2 + \pi\sqrt{\pi}c\cos t\right)dt \\ &= -\tfrac{3}{20\sqrt{\pi}}c^2 + \tfrac{\pi\sqrt{\pi}}{\sqrt{2}}c < 0, \end{aligned}$$

for all $c \notin \left[0, \frac{20\pi^2}{3\sqrt{2}}\right]$. We conclude that all conditions of Theorem 2 hold, proving that problem 1 has at least one solution u in $C_{\frac{1}{2}}[0, \frac{\pi}{4}]$.

2.5 Example 2

Consider the following boundary value problem

$$\begin{cases} D_{0_+}^{\frac{3}{2}}u(t) = f(t, u(t)), \quad 0 < t < 1, \\ D_{0_+}^{\frac{1}{2}}u(0^+) = D_{0_+}^{\frac{1}{2}}u(1^-), \\ I_{0_+}^{\frac{1}{2}}u(0^+) = I_{0_+}^{\frac{1}{2}}u(1^-), \end{cases} \tag{2}$$

where

$$f(t,x) = \begin{cases} -\frac{\sqrt{t}}{10}, & t \in [0,1],\ x \in (-\infty, 0) \\ \frac{\sqrt{t}}{10}\left(x - 1 + \frac{1}{3}\ln\left(|x|\sqrt{t} + 1\right)\right), & t \in [0,1],\ x \in [0, +\infty). \end{cases}$$

Next, we check all of assumptions of Theorem 2:

1. Since for all $s > 0, \ln s \leq s - 1 < s$, then

$$|f(t,x)| \leq \frac{\sqrt{t}}{10}\left(|x| + \frac{1}{3}\left(|x|\sqrt{t} + 1\right)\right) + \frac{\sqrt{t}}{10} = \sqrt{t}\left(\frac{1}{10} + \frac{\sqrt{t}}{30}\right)|x| + 4\frac{\sqrt{t}}{30}.$$

Then we take

$$a(t) = \left(\frac{1}{10} + \frac{\sqrt{t}}{30}\right) \text{ and } r(t) = 4\frac{\sqrt{t}}{30}$$

with $a, r \in L^1[0,1]$ and

$$\|a\|_1 = \int_0^1\left(\frac{1}{10} + \frac{\sqrt{t}}{30}\right)dt = \frac{1}{10} + \frac{2}{90} = \frac{11}{90} < \frac{\Gamma\left(\frac{3}{2}\right)}{2} \simeq 0.443.$$

2. For $M = 91$, assume that $u(t) > M$, for all $t \in [0,1]$. Then

$$f(s, u(s)) \geq \frac{\sqrt{s}}{10}\left(M - 1 + \frac{1}{3}\ln\left(M\sqrt{s} + 1\right)\right).$$

As a consequence, we derive the estimates:

$$\begin{aligned}
\int_0^1 f(s,u(s))ds &\geq (M-1)\int_0^1 \frac{\sqrt{s}}{10}ds + \frac{1}{30}\int_0^1 \sqrt{s}\ln(M\sqrt{s}+1)ds \\
&= \frac{2}{30}(M-1) + \frac{2}{90}\left(\left(1+\frac{1}{M^3}\right)\ln(M+1) - \frac{(M+1)^3}{3M^3}\right. \\
&\quad \left. + \frac{3(M+1)^2}{2M^3} - \frac{3(M+1)}{M^3} + \frac{11}{6M^3}\right) \\
&\geq \frac{2}{30}(M-1) - \frac{2}{90}\frac{(M+1)^3+9(M+1)}{3M^3} \simeq 5.99.
\end{aligned}$$

Now suppose that $u(t) < -M$, for all $t \in [0,1]$. Then

$$\int_0^1 f(s,u(s))ds = \int_0^1 -\frac{\sqrt{s}}{10}ds = -\frac{2}{30} < 0$$

which shows that

$$\int_0^1 f(s,u(s))ds \neq 0,$$

for all $u \in \mathrm{dom}(L)$ satisfying $|u(t)| > M$, for all $t \in [0,1]$.

3. Let $M^* = \frac{2}{3}$. For all $c > M^*$, we have

$$\begin{aligned}
c\int_0^1 f\left(s,\frac{c}{\sqrt{s}}\right)ds &= \int_0^1 c\frac{\sqrt{s}}{10}\left(\frac{c}{\sqrt{s}} - 1 + \frac{1}{3}\ln\left(\frac{|c|}{\sqrt{s}}\sqrt{s}+1\right)\right)ds \\
&= \frac{c^2}{10} - \frac{2}{30}c + \frac{2}{90}c\ln(|c|+1) \\
&= \frac{c}{10}\left(c - \frac{2}{3} + \frac{2}{9}\ln(|c|+1)\right) > 0,
\end{aligned}$$

while for $c < -M^*$, we have

$$c\int_0^1 f\left(s,\frac{c}{\sqrt{s}}\right)ds = c\int_0^1 -\frac{\sqrt{s}}{10}ds = -\frac{2}{30}c > 0.$$

Therefore we have showed that problem 2 has at least one solution u in $C_{\frac{1}{2}}[0,1]$.

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Received August 23, 2015

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BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(82), 2016, Pages 38–56
ISSN 1024–7696

Cubic systems with degenerate infinity and invariant straight lines of total parallel multiplicity five

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Abstract. In this paper cubic systems which have degenerate infinity and invariant straight lines of total multiplicity five are classified. It is proved that, modulo affine transformations and time rescaling, there are 24 classes of such systems. For every class the qualitative investigation was carried out in the Poincaré disc.

Mathematics subject classification: 34C05.
Keywords and phrases: Cubic differential system, invariant straight line, phase portrait.

1 Introduction and statement of main results

We consider the real cubic differential system

$$\frac{dx}{dt} = \sum_{r=0}^{3} P_r\left(x,y\right) \equiv P\left(x,y\right), \ \frac{dy}{dt} = \sum_{r=0}^{3} Q_r\left(x,y\right) \equiv Q\left(x,y\right), \quad \gcd\left(P,Q\right)=1, \quad (1)$$

where P_r, Q_r are homogeneous polynomials of degree r and $|P_3(x,y)| + |Q_3(x,y)| \not\equiv 0$.

A curve $f(x,y) = 0$, $f \in \mathbb{C}[x,y]$, is said to be an *invariant algebraic curve* of (1) if there exists a polynomial $K_f \in \mathbb{C}[x,y]$ such that the identity $\frac{\partial f}{\partial x}P(x,y)+\frac{\partial f}{\partial y}Q(x,y) \equiv \equiv f(x,y)K_f(x,y)$ holds. We say that an invariant algebraic curve $f(x,y) = 0$ has the *parallel multiplicity* equal to m, if m is the greatest positive integer such that f^{m-1} divides K_f.

The system (1) is called *Darboux integrable* if there exists a non-constant function of the form $F = f_1^{\lambda_1} \cdots f_s^{\lambda_s}$, where f_j is an invariant algebraic curve and $\lambda_j \in \mathbb{C}$, $j = \overline{1,s}$, such that either F is a first integral or is an integrating factor for (1). We will be interested in invariant algebraic curves of degree one, that is invariant straight lines $\alpha x + \beta y + \gamma = 0$, $(\alpha, \beta) \neq (0,0)$.

There are a great number of works dedicated to the investigation of polynomial differential systems with invariant straight lines.

The problem of estimating the number of invariant straight lines which a polynomial differential system can have was considered in [1]; the problem of coexistence of invariant straight lines and limit cycles in [4,5]; the problem of coexistence of invariant straight lines and singular points of center type for cubic systems in [3,10]. The

*Supported by FP7-PEOPLE-2012-IRSES-316338 and 15.817.02.03F

classification of all cubic systems with the maximum number of invariant straight lines, taking into account their multiplicities, is given in [6].

In [1] it was proved that the cubic system (1) can have in the finite part of the phase plane at most eight invariant straight lines. Cubic systems with exactly eight invariant straight lines has been studied in [6, 7] and with total parallel multiplicity of invariant straight lines equal to seven in [11, 13]. A qualitative investigation of systems (1) with six real invariant straight lines along two (three) directions is given in [8] ([9]). In [12] we examined some cubic systems with degenerate infinity that have invariant straight lines of total parallel multiplicity five or six, three of which are parallel. In [14] all canonical forms of the cubic systems with degenerate infinity that have invariant straight line of total parallel multiplicity equal to six were obtained.

In this paper we continue the investigation from [8, 9, 12, 14] and give a full qualitative study of cubic systems (1) with degenerated infinity and invariant straight lines of total multiplicity six.

Theorem 1. *Assume that a cubic system with degenerate infinity possesses invariant straight lines of total parallel multiplicty five. Then via an affine transformation and time rescaling this system can be brought to one of the systems 1)–24). Moreover, up to topological equivalence, its phase portrait on the Poincaré disc corresponds to one of the portraits given in Fig. 1 – Fig. 23. In the table below for each of the systems 1) – 24) the first arrow points to the straight lines and the first integral F (or integrating factor μ) that corresponds to the system.*

$$1)\quad \begin{cases} \dot{x} = x\,(x+1)\,(x-a)\,,\ \ a>0,\ \ c\neq 2, \\ \dot{y} = y\left(-a+c\,x-y+x^2\right),\ a+c>1; \\ \text{Configuration } (3r, 1r, 1r) \end{cases} \qquad \rightarrow\ (2) \quad \rightarrow \text{Fig. 1;}$$

$$2)\quad \begin{cases} \dot{x} = x\,(x+1)\,(x-a)\,,\ a>0,\ b>0, \\ \dot{y} = y\left(b+(b-a)\,x-y+x^2\right),\ b-a\neq 0; \\ \text{Configuration } (3r, 1r, 1r) \end{cases} \qquad \rightarrow\ (3) \quad \rightarrow \text{Fig. 2;}$$

$$3)\quad \begin{cases} \dot{x} = x(x+1)(x-a),\ \ a>0, \\ \dot{y} = y(x+1)(x-a)+x^2+y^2; \\ \text{Configuration } (3r, 1c_1, 1c_1) \end{cases} \qquad \rightarrow\ (4) \quad \rightarrow \text{Fig. 3;}$$

$$4)\quad \begin{cases} \dot{x} = x(x+1)(x-a),\ \ a>0, \\ \dot{y} = (x+1)^2+xy(x-a)+by^2,\ \ b>0; \\ \text{Configuration } (3r, 1c_1, 1c_1) \end{cases} \qquad \rightarrow\ (5) \quad \rightarrow \text{Fig. 4;}$$

$$5)\quad \begin{cases} \dot{x} = (x-a)(x^2+1),\ \ a\in\mathbb{R}, \\ \dot{y} = y(1-ac+cx-y+x^2),\ \ c\neq 0; \\ \text{Configuration } (1r+2c_0, 1r, 1r) \end{cases} \qquad \rightarrow\ (6) \quad \rightarrow \text{Fig. 5;}$$

$$6)\quad \begin{cases} \dot{x} = (x-a)(x^2+1),\ \ a\in\mathbb{R}, \\ \dot{y} = (x-a)^2+y+\frac{1}{b}y^2+x^2y,\ \ b>0; \\ \text{Configuration } (1r+2c_0, 1c_1, 1c_1) \end{cases} \qquad \rightarrow\ (7) \quad \rightarrow \text{Fig. 6;}$$

7) $\begin{cases} \dot{x} = x^2(x+1),\ \ a > 0, \\ \dot{y} = y\left((a+1)x - y + x^2\right); \\ \text{Configuration } (3(2)r, 1r, 1r) \end{cases}$ $\rightarrow$ (8) $\rightarrow$ Fig. 7;

8) $\begin{cases} \dot{x} = x^2(x+1), \\ \dot{y} = y(a + ax - y + x^2),\ \ a \neq 0; \\ \text{Configuration } (3(2)r, 1r, 1r) \end{cases}$ $\rightarrow$ (9) $\rightarrow$ Fig. 8;

9) $\begin{cases} \dot{x} = x^2(x+1),\ \ a > 0, \\ \dot{y} = ax^2 + xy + ay^2 + x^2y; \\ \text{Configuration } (3r, 1c_1, 1c_1) \end{cases}$ $\rightarrow$ (10) $\rightarrow$ Fig. 9;

10) $\begin{cases} \dot{x} = x^2(x+1),\ \ a \neq 0, \\ \dot{y} = a(x+1)^2 + ay^2 + x^2y; \\ \text{Configuration } (3(2)r, 1c_1, 1c_1) \end{cases}$ $\rightarrow$ (11) $\rightarrow$ Fig. 10;

11) $\begin{cases} \dot{x} = x^3,\ \ a > 0, \\ \dot{y} = y(ax - y + x^2); \\ \text{Configuration } (3(3)r, 1r, 1r) \end{cases}$ $\rightarrow$ (12) $\rightarrow$ Fig. 11;

12) $\begin{cases} \dot{x} = x^3,\ \ a > 0, \\ \dot{y} = ax^2 + ay^2 + x^2y; \\ \text{Configuration } (3(3)r, 1c_1, 1c_1) \end{cases}$ $\rightarrow$ (13) $\rightarrow$ Fig. 12;

13) $\begin{cases} \dot{x} = x(x-1)(y+a), \\ \dot{y} = y(y-1)(x+a),\ a \notin \{-1; -1/2; 0\}; \\ \text{Configuration } (2r, 2r, 1r) \end{cases}$ $\rightarrow$ (14) $\rightarrow$ Fig. 13;

14) $\begin{cases} \dot{x} = x^2(y+a), a > 0, b > 0, \\ \dot{y} = y^2(x+b),\ ab \neq 0; \\ \text{Configuration } (2(2)r, 2(2)r, 1r) \end{cases}$ $\rightarrow$ (15) $\rightarrow$ Fig. 14;

15) $\begin{cases} \dot{x} = (x^2+1)(y+a), \\ \dot{y} = (y^2+1)(x+a),\ \ a \neq 0; \\ \text{Configuration } (2c_0, 2c_0, 1r) \end{cases}$ $\rightarrow$ (16) $\rightarrow$ Fig. 15;

16) $\begin{cases} \dot{x} = x(a - 2ay + x^2 + y^2),\ a \notin \{0; 1/2; 1\}, \\ \dot{y} = ay + (a-1)x^2 - (a+1)y^2 + x^2y + y^3; \\ \text{Configuration } (2c_1, 2c_1, 1r) \end{cases}$ $\rightarrow$ (17) $\rightarrow$ Fig. 16;

17) $\begin{cases} \dot{x} = 2(\frac{x}{2} + by + bx^2 - xy - by^2 + x^3 + xy^2), \\ \dot{y} = (2y-1)(2bx - y + x^2 + y^2),\ \ b \neq 0; \\ \text{Configuration } (2c_1, 2c_1, 1r) \end{cases}$ $\rightarrow$ (18) $\rightarrow$ Fig. 17;

18) $\begin{cases} \dot{x} = ax^2 + 2bxy - ay^2 + x^3 + xy^2, \\ \dot{y} = -bx^2 + 2axy + by^2 + x^2y + y^3, \\ \quad |a| + |b| \neq 0,\ a \geq 0; \\ \text{Configuration } (2(2)c_1, 2(2)c_1, 1r) \end{cases}$ $\rightarrow$ (19) $\rightarrow$ Fig. 18;

19) $\begin{cases} \dot{x} = x(x-1)(1 + (a-1)x + (b-1)y), \\ \dot{y} = y(-1 + 2x + y + (a-1)x^2 + (b-1)xy), \\ \quad ab(b-1)(b+1)(a-b) \neq 0; \\ \text{Configuration } (2r, 1r, 1r, 1r) \end{cases}$ $\rightarrow$ (20) $\rightarrow$ Fig. 19;

20) $\begin{cases} \dot{x} = (1 + (x-a)^2)(x + by), \; b \neq 0, \\ \dot{y} = (a^2+1)(y - bx) + (ab-1)x^2 - 2axy- \\ \quad -(ab+1)y^2 + x^2y + bxy^2; \\ \text{Configuration } (2c_0, 1r, 1c_1, 1c_1) \end{cases}$ $\to$ (21) $\to$ Fig. 20;

21) $\begin{cases} \dot{x} = x + cy + (2a+c)x^2 + 2(-1+ac)xy - cy^2+ \\ \qquad +(a^2+b^2-b+ac)x^3 + (-2a-c+a^2c+ \\ \qquad +b^2c)x^2y - (b-1+ac)xy^2, \\ \dot{y} = -cx + y + (b-ac)x^2 + 2(a+c)xy + (b-2+ \\ \qquad +ac)y^2 + (a^2+b^2-b+ac)x^2y + (-2a-c+ \\ \qquad +a^2c+b^2c)xy^2 - (b-1+ac)y^3, \\ \quad bc(|a| + |b^2-1|) \neq 0; \\ \text{Configuration } (1r, 1c_1, 1c_1, 1c_1, 1c_1) \end{cases}$ $\to$ (22) $\to$ Fig. 17, Fig. 21;

22) $\begin{cases} \dot{x} = x(1 + 2ax - 2y + (a^2+b^2-c)x^2- \\ \qquad -2axy - (c-1)y^2), \\ \dot{y} = y + cx^2 + 2axy + (c-2)y^2 + (a^2+b^2- \\ \qquad -c)x^2y - 2axy^2 - (c-1)y^3, \\ \quad bc(b^2-c^2)(|a| + |b^2-1|) \neq 0; \\ \text{Configuration } (1r, 1c_1, 1c_1, 1c_1, 1c_1) \end{cases}$ $\to$ (23) $\to$ Fig. 16;

23) $\begin{cases} \dot{x} = x(1 + (a+b)x - 2y + (ab-c)x^2- \\ \qquad -(a+b)xy + (1-c)y^2) \\ \dot{y} = y + cx^2 + (a+b)xy + (c-2)y^2 + (ab- \\ \qquad -c)x^2y - (a+b)xy^2 + (1-c)y^3, \\ \quad c(b-a) \neq 0; \\ \text{Configuration } (1r, 1r, 1r, , 1c_1, 1c_1) \end{cases}$ $\to$ (24) $\to$ Fig. 22;

24) $\begin{cases} \dot{x} = x(1 + (a+b)x - 2y + abx^2 + (1-a- \\ \qquad -b-c)xy + cy^2), \\ \dot{y} = y(1 + \alpha x - (c+1)y + abx^2 - \alpha xy + cy^2), \\ \quad \alpha = a+b+c-1, \\ \quad ab(a-1)(b-1)(c-1) \neq 0, \; a > b; \\ \text{Configuration } (1r, 1r, 1r, 1r, 1r) \end{cases}$ $\to$ (25) $\to$ Fig. 23.

$$\begin{gathered} l_1 = x+1, \; l_2 = x, \; l_3 = x-a, \; l_4 = y, \; l_5 = (a+c-1)x - y; \\ F = (l_1/l_3)^{a+c-1}(l_4/l_5)^{a+1}; \end{gathered} \tag{2}$$

$$\begin{gathered} l_1 = x+1, \; l_2 = x, \; l_3 = x-a, \; l_4 = y, \; l_5 = b(x+1) - y; \\ F = l_2^b l_3^{-b} l_4^a l_5^{-a}; \end{gathered} \tag{3}$$

$$\begin{gathered} l_1 = x+1, \; l_2 = x, \; l_3 = x-a, \; l_{4,5} = y \pm ibx; \\ \mu(x,y) = 1/(l_1 l_3 l_4 l_5); \end{gathered} \tag{4}$$

$$\begin{gathered} l_1 = x+1, \; l_2 = x, \; l_3 = x-a, \; l_{4,5} = y \pm i\sqrt{b}\,(x+1); \\ \mu(x,y) = 1/(l_2 l_3 l_4 l_5); \end{gathered} \tag{5}$$

$$\begin{gathered} l_1 = x-i, \; l_2 = x-a, \; l_3 = x+i, \; l_4 = y, \; l_5 = cx - y - ac; \\ \mu(x,y) = 1/(l_1 l_3 l_4 l_5), \;\; F = y\exp(-c \cdot arctan(x))/l_5; \end{gathered} \tag{6}$$

$$l_1 = x - i,\ l_2 = x - a,\ l_3 = x + i,\ l_{4,5} = y \pm i(x - a);\\ \mu(x, y) = 1/(l_1 l_3 l_4 l_5); \tag{7}$$

$$l_1 = x + 1,\ l_2 \equiv l_3 = x,\ l_4 = y,\ l_5 = ax - y;\\ F = l_l^{1-a} l_2^{a-1} l_4^{-1} l_5; \tag{8}$$

$$l_1 = x + 1,\ l_2 \equiv l_3 = x,\ l_4 = y,\ l_5 = a + ax - y;\\ F = y \exp(a/x)/(a + ax - y). \tag{9}$$

$$l_1 = x + 1,\ l_2 \equiv l_3 = x,\ l_{4,5} = y \pm ix; \quad \mu(x, y) = 1/(l_1 l_2 l_4 l_5); \tag{10}$$

$$l_1 = x + 1,\ l_2 \equiv l_3 = x,\ l_{4,5} = y \pm i(x + 1); \quad \mu(x, y) = 1/(l_2^2 l_4 l_5); \tag{11}$$

$$l_{1,2,3} = x,\ l_4 = y, l_5 = ax - y; \quad F = y \exp(a/x)/(ax - y); \tag{12}$$

$$l_{1,2,3} = x,\ l_{4,5} = y \pm ix; \quad \mu(x, y) = 1/(l_1^2 l_4 l_5). \tag{13}$$

$$l_1 = x,\ \ l_2 = x - 1,\ \ l_3 = y,\ \ l_4 = y - 1,\ \ l_5 = x - y;\\ F = (l_1/l_3)^a (l_4/l_2)^{a+1}; \tag{14}$$

$$l_1 \equiv l_2 = x,\ \ l_3 \equiv l_4 = y,\ \ l_5 = ax - by;\\ F = l_1 l_3^{-1} exp((ax - by)/(xy)); \tag{15}$$

$$l_{1,2} = x \pm i,\ l_{3,4} = y \pm i,\ l_5 = x - y; \quad \mu(x, y) = 1/(l_1 l_2 l_3 l_4); \tag{16}$$

$$l_{1,2} = y \mp ix,\ l_{3,4} = y \mp ix - 1,\ l_5 = x; \quad \mu(x, y) = 1/(l_1 l_2 l_3 l_4); \tag{17}$$

$$l_{1,2} = y \mp ix,\ l_{3,4} = y \mp ix - 1,\ l_5 = 2y - 1; \quad \mu(x, y) = 1/(l_1 l_2 l_3 l_4); \tag{18}$$

$$l_{1,3} = y - ix,\ l_{2,4} = y + ix,\ l_5 = bx - ay; \quad \mu(x, y) = 1/(l_1 l_2)^2. \tag{19}$$

$$l_1 = x,\ l_2 = x - 1,\ l_3 = y,\ l_4 = x + y - 1,\ l_5 = ax + by;\\ F = l_1 l_2^{-b} l_4^b l_5^{-1}; \tag{20}$$

$$l_{1,2} = x - a \pm i,\ l_{3,4} = y \pm ix,\ l_5 = ax + y - a^2 - 1;\\ \mu(x, y) = 1/(l_1 l_2 l_3 l_4. \tag{21}$$

$$l_{1,2} = y \mp xi,\ l_{3,4} = y - (a \pm bi)x - 1,\ l_5 = 1 + ax - y + by;\\ \mu(x, y) = 1/(l_1 l_2 l_3 l_4); \tag{22}$$

$$l_{1,2} = y \mp xi,\ l_{3,4} = y - (a \pm bi)x - 1,\ l_5 = x;\\ \mu(x, y) = 1/(l_1 l_2 l_3 l_4); \tag{23}$$

$$l_{1,2} = y \mp ix,\ l_3 = y - ax - 1,\ l_4 = y - bx - 1,\ l_5 = x;\\ \mu(x, y) = 1/(l_1 l_2 l_3 l_4); \tag{24}$$

$$l_1 = x,\ l_2 = y,\ l_3 = y - x,\ l_4 = y - ax - 1,\ l_5 = y - bx - 1;\\ F = l_2^{(b-a)} l_3^{(a-b)} l_4^{(c-1)} l_5^{(1-c)}. \tag{25}$$

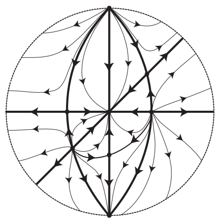


Fig. 1

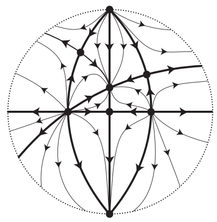


Fig. 2

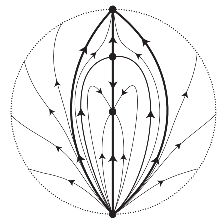


Fig. 3

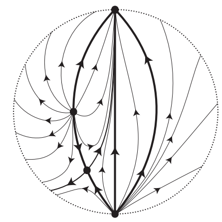


Fig. 4

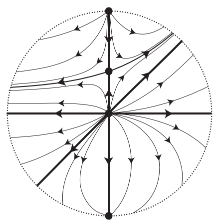


Fig. 5

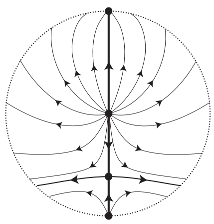


Fig. 6

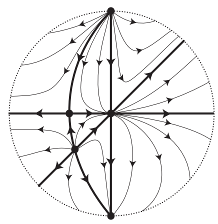


Fig. 7

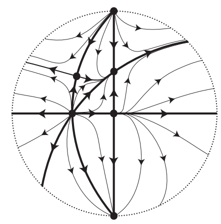


Fig. 8

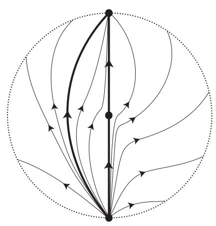


Fig. 9

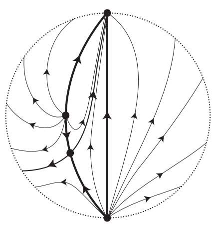


Fig. 10

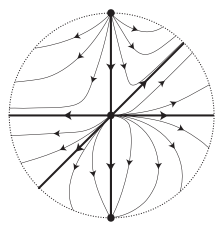


Fig. 11

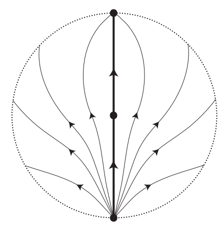
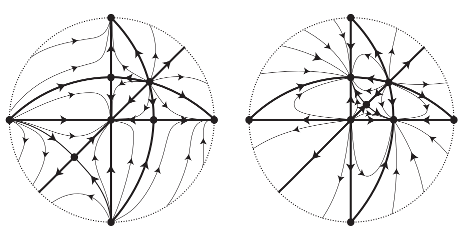


Fig. 12



a) b)

Fig. 13

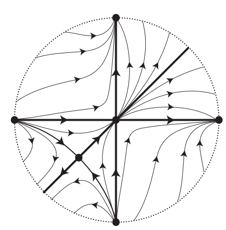


Fig. 14

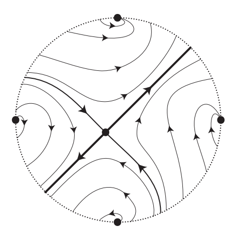
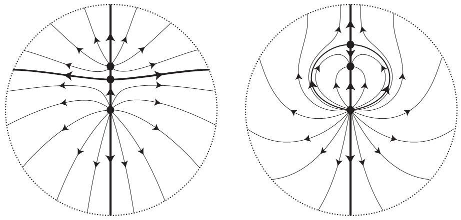


Fig. 15



a) b)

Fig. 16

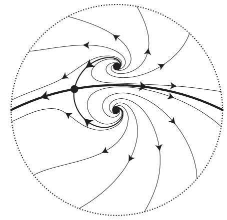


Fig. 17

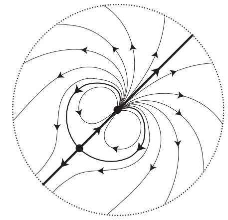
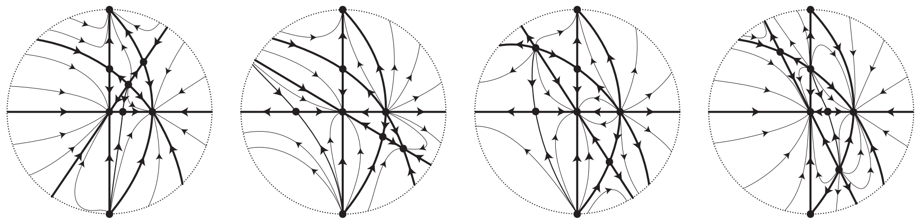
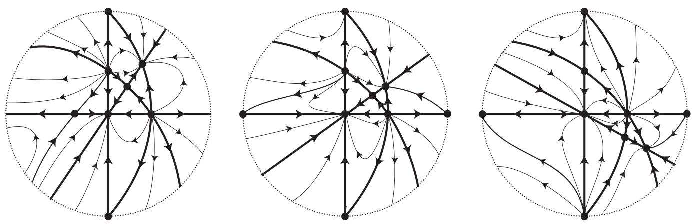


Fig. 18



a) b) c) d)

Fig. 19



e) f) g)

Fig. 19

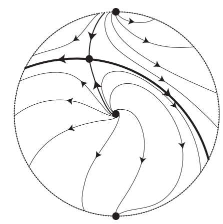


Fig. 20

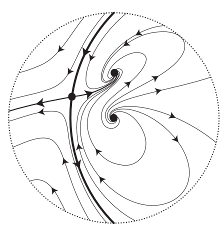
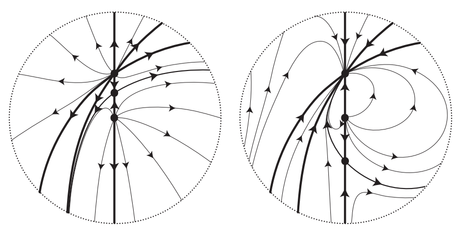
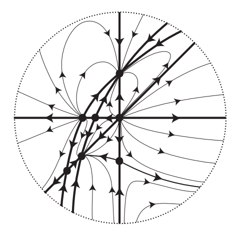


Fig. 21



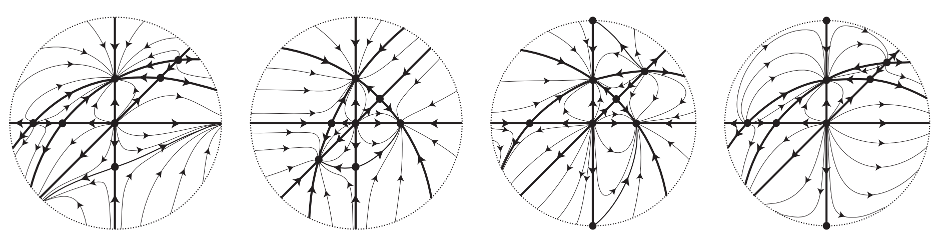
a) b)

Fig. 22



a)

Fig. 23



b) c) d) e)

Fig. 23

2 Some properties of cubic systems with straight lines

By *a configuration of straight lines* we understand the $\mathbb{R}^2$ plane with a certain number of straight lines.

To each two-dimensional differential system (with invariant straight lines) we can associate a configuration consisting of invariant straight lines of this system. It is easy to show that the converse is not always true.

The problem arises to determine for invariant straight lines such properties that allow to construct all realizable configurations of invariant straight lines for (1). Below we shall enumerate these properties. Their proofs are rather easy and we omit them.

Proposition 1. *The system* (1) *has at most nine singular points in the finite part of the phase plane.*

Proposition 2. *There are at most 3 singular points of system* (1) *on any invariant straight line in the finite part of the phase plane.*

A straight line l will be called *complex* if $l \in \mathbb{C}[x,y] \setminus \mathbb{R}[x,y]$.

Proposition 3. *Complex invariant straight lines of system* (1) *occur in complex conjugate pairs (*l *and* $\bar{l}$*).*

Proposition 4. *The intersection point* (x_0, y_0) *of two invariant straight lines* l_1 *and* l_2 *of system* (1) *is a singular point. Moreover, if* $l_1, l_2 \in \mathbb{R}[x,y]$ *or* $l_2 \equiv \bar{l_1}$*, then* $x_0, y_0 \in \mathbb{R}$.

Proposition 5. *A complex straight line* l *can pass through at most one point with real coordinates.*

Proposition 6. *If a straight line passes through two distinct real points or through two complex conjugate points, then this straight line is real.*

A complex straight line passing through a real point will be called *a relative complex straight line* and a complex straight line not passing through any real point – *a purely imaginary straight line.*

Proposition 7. *Through any point of a purely imaginary straight line at most one real straight line can pass.*

Proposition 8. *A complex invariant straight line of system* (1) *is purely imaginary iff this straight line is parallel to its conjugate one (*$l \parallel \bar{l}$*).*

Proposition 9. *Let* l_1 *and* l_2 *be two parallel invariant straight lines of the system* (1)*, then only one of the following properties occurs:*
1. $l_1, l_2 \in \mathbb{R}[x,y]$*;* *2.* l_1 *is real and* l_2 *is purely imaginary;*
3. l_1 *and* l_2 *are purely imaginary;* *4.* l_1 *and* l_2 *are relative complex.*

We say that the cubic system (1) has *degenerate infinity* if the following identity

$$yP_3(x,y) - xQ_3(x,y) \equiv 0 \tag{26}$$

holds. In such a case the infinity consists only of singular points.

Proposition 10. *The identity* (26) *is invariant under any affine transformation of the system* (1).

Proposition 11. *Invariant straight lines of the cubic system* (1) *with degenerate infinity passing through the same point* $M_0\,(x_0, y_0)$*,* $x_0, y_0 \in \mathbb{C}$*, have at most three slopes.*

Proposition 12. *Through any point of a complex invariant straight line of the cubic system with degenerate infinity at most one real straight lines can pass.*

Proposition 13. *A straight line passing through three distinct singular points of system (1) with degenerate infinity is invariant for (1).*

Proposition 14. *The maximum number of invariant straight lines for a differential cubic system with degenerate infinity is equal to six.*

Proposition 15. *Let the cubic system* (1) *have two concurrent invariant straight lines* l_1, l_2. *If* l_1 *has the parallel multiplicity equal to* m, $1 \le m \le 3$, *then this system cannot have more than* $3-m$ *singular points on* $l_2 \setminus l_1$.

We say that three straight lines are in *generic position* if all lines have different slopes and no more than two lines pass through a point.

Proposition 16. *Let the cubic system* (1) *have 3 invariant straight lines in generic position, then their total parallel multiplicity is at most four.*

Proposition 17. *The cubic system* (1) *with degenerate infinity can have at most one triplet of parallel invariant straight lines.*

Proposition 18. *The cubic system* (1) *with degenerate infinity can have at most two pair of parallel invariant straight lines.*

3 The proof of Theorem 1

Using Propositions 17 and 18, the family of cubic systems [(1)][(26)] with six invariant straight lines can be divided in four classes:

A) Systems with a triplet of parallel invariant straight lines;

B) Systems with two pairs of parallel invariant straight lines;

C) Systems with only a pair of parallel invariant straight lines;

D) Systems with invariant straight lines of different slopes.

The class A) was studied in [8, 12] and is characterized by the systems 1)–12) of Theorem 1.

3.1 Class B): two pairs of parallel invariant straight lines

For cubic systems in class B) the following 8 configurations of invariant straight lines are possible:

B1) $(\mathbf{2r}, \mathbf{2r}, \mathbf{1r})$	B2) $(2(2)r, 2r, 1r,)$	**B3)** $(\mathbf{2(2)r}, \mathbf{2(2)r}, \mathbf{1r})$
B4) $(2r, 2c_0, 1r)$	B5) $(2(2)r, 2c_0, 1r)$	**B6)** $(\mathbf{2c_0}, \mathbf{2c_0}, \mathbf{1r})$
B7) $(\mathbf{2c_1}, \mathbf{2c_1}, \mathbf{1r})$	**B8)** $(\mathbf{2(2)c_1}, \mathbf{2(2)c_1}, \mathbf{1r})$	

By $(2r, 2r, 1r)$ we denoted the configuration which consists of five distinct real straight lines $l_1, \dots, l_5 \in \mathbb{R}[x, y]$, of which l_1, l_2 and l_3, l_4 form two pairs of parallel straight lines, i.e. $l_1 \parallel l_2$, $l_3 \parallel l_4$, $l_1 \nparallel l_3$ and $l_j \nparallel l_5$, $j = 1, \dots, 4$. In the case of configuration $(2c_0, 2c_0, 1r)$ we have five straight lines $l_1, \dots, l_5$, where l_1, l_2, l_3 and l_4 are purely imaginary, l_5 is real, l_1, l_2 and l_3, l_4 form two pairs of parallel straight lines. The configuration $(2(2)r, 2r, 1r)$ consists of five real straight lines, where $l_1 \equiv l_2$, $l_3 \parallel l_4$, $l_1 \nparallel l_3$, $l_j \nparallel l_5$, $j = 1, \dots, 4$, and the straight line l_1 (or l_2) has parallel multiplicity equal to two.

Proposition 19. *Cubic systems with degenerate infinity possessing invariant straight lines of the configuration* $(2(2)r, 2r)$ *can not have other invariant straight lines.*

Indeed, a system of this configuration can be brought to the form:

$$\dot{x} = x^2(y + a), \quad \dot{y} = y(y - 1)(x + b).$$

Since this system has only the following singular points: $(0, 0)$, $(0, 1)$, $(-b, -a)$ and $a(a + 1)b \neq 0$, the above proposition follows.

Remark 1. Propositions 2, 7 and 15 (Proposition 19) do not allow the realization of configurations B4) and B5) (configuration B2)) in the class of cubic systems with degenerate infinity.

Configuration B1) $(\mathbf{2r}, \mathbf{2r}, \mathbf{1r})$. Via an affine transformation and time rescaling the system [(1)][(26)] with two pairs of real invariant straight lines can be written in the form:

$$\dot{x} = x(x - 1)(y + a), \quad \dot{y} = y(y - 1)(x + b), \quad a, b \notin \{-1; 0\}. \tag{27}$$

The system (27) has the invariant straight lines $l_1 = x$, $l_2 = x - 1$, $l_3 = y$, $l_4 = y - 1$ and the singular points $(0, 0)$, $(1, 0), (0, 1)$, $(1, 1)$, $(-b, -a)$. Therefore, any other invariant straight line of (27) must pass through the singular points $(0, 0)$ and $(1, 1)$ or through the singular points $(1, 0)$ and $(0, 1)$. When $(0, 0), (1, 1) \in l_5$ and l_5 is invariant for (27), we get $b = a$, i.e. the system 13) of Theorem 1. The case $(1, 0), (0, 1) \in l_5$ provides an affine equivalent system with the system 13).

Configuration B3) $(\mathbf{2(2)r}, \mathbf{2(2)r}, \mathbf{1r})$. The cubic system with degenerate infinity possessing real invariant straight lines with the configuration $(2(2)r, 2(2)r)$ can be written as:

$$\dot{x} = x^2(y + a), \quad \dot{y} = y^2(x + b), \tag{28}$$

This system has the invariant straight lines $l_{1,2} = x$, $l_{3,4} = y$, $l_5 = ax - by$, i.e. we obtained the system 14) of Theorem 1.

Configuration B6) $(\mathbf{2c_0}, \mathbf{2c_0}, \mathbf{1r})$ In this case the pairs of parallel invariant straight lines can be brought to the form $l_{1,2} = x \pm i$ and $l_{3,4} = y \pm i$. The system [(1)][(26)] with these invariant straight lines has the form

$$\dot{x} = (x^2 + 1)(y + a), \quad \dot{y} = (y^2 + 1)(x + b), \tag{29}$$

with the following singular points: $(-i, -i)$, $(-i, i)$, (i, i), $(i, -i)$, $(-b, -a)$. Any other invariant straight line of system (29) can pass only through the pairs of reciprocally conjugate singular points $(-i, -i)$, (i, i) or $(-i, i)$, $(i, -i)$, therefore it is described by equation $l_5 = x + y$ or $l_5 = x - y$, respectively. The invariance for (29) is conditioned by $b = a$ or $b = -a$. When $b = a$ we have the system 15) of Theorem 1. The case $b = -a$ is affine equivalent with the system 15).

Configuration B7) $(\mathbf{2c_1}, \mathbf{2c_1}, \mathbf{1r})$ Via an affine change of coordinates, the straight lines $l_1, \ldots, l_4$ can be brought to the form $l_{1,2} = y \pm ix$, $l_{3,4} = y \pm ix - 1$. The cubic system [(1)][(26)] with these invariant straight lines has the form:

$$\begin{cases} \dot{x} = ax + by + bx^2 - 2axy - by^2 + x^3 + xy^2, \\ \dot{y} = -bx + ay + (a-1)x^2 + 2bxy - (a+1)y^2 + x^2y + y^3. \end{cases} \tag{30}$$

The obtained system has the following singular points: $(0,0)$, $(-i/2, 1/2)$, $(0,1)$, $(i/2, 1/2)$, $(-b, a)$. Any other real invariant straight line l_5 can pass only through the singular points $(0,0)$, $(0,1)$ or $(-i/2, 1/2)$, $(i/2, 1/2)$, therefore it is described by equation $l_5 = x$ or $l_5 = 2y - 1$, respectively. This straight line is invariant for system (30) iff $b = 0$ or $a = 1/2$. Thus, was obtained the systems 16) and 17) of Theorem 1.

Configuration B8) $(\mathbf{2(2)c_1}, \mathbf{2(2)c_1}, \mathbf{1r})$ Via an affine transformation and time rescaling, we can bring the pair of conjugate complex invariant straight lines to the form $l_{1,2} = y \pm ix$. The cubic system [(1)][(26)] with these invariant straight lines has the form:

$$\begin{cases} \dot{x} = a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + (a_{20} - b_{11})y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2, \\ \dot{y} = -a_{01}x + a_{10}y + (b_{02} - a_{11})x^2 + b_{11}xy + b_{02}y^2 + a_{30}x^2y + a_{21}xy^2 + a_{12}y^3. \end{cases} \tag{31}$$

Each of straight lines $l_{1,2} = y \pm ix$ has parallel multiplicity equal to two iff $a_{01} = a_{10} = a_{21} = 0$, $a_{11} = 2b_{02}$, $b_{11} = 2a_{20}$, $a_{30} = a_{12}$. Via a time rescaling, we can make $a_{12} = 1$. Denoting by $a_{20} = a$ and $b_{02} = b$, we obtain the system 18) of Theorem 1.

3.2 Class C): one pair of parallel invariant straight lines

For cubic systems in class C) the following 6 configurations of invariant straight lines are possible:

C1) $(\mathbf{2r}, \mathbf{1r}, \mathbf{1r}, \mathbf{1r})$	C2) $(2(2)r, 2r, 1r, 1r)$	C3) $(2r, 1r, 1c_1, 1c_1)$
C4) $(2(2)r, 1r, 1c_1, 1c_1)$	C5) $(2c_0, 1r, 1r, 1r)$	**C6)** $(\mathbf{2c_0}, \mathbf{1r}, \mathbf{1c_1}, \mathbf{1c_1})$

Remark 2. Propositions 2, 7 and 15 do not allow the realization of configurations C2) and C4) in the class of cubic systems with degenerate infinity.

Proposition 20. *The configurations C3) and C5) do not realize in the class of cubic systems with degenerate infinity.*

Proof. Let the cubic system [(1)][(26)] has only two distinct parallel invariant straight lines l_1 and l_2. If these straight lines are real, then [(1)][(26)] can be written in the following form:

$$\begin{cases} \dot{x} = x(x-a)(a_{20} + a_{30}x + a_{21}y), \\ \dot{y} = b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + a_{30}x^2y + a_{21}xy^2, \end{cases} \tag{32}$$

and if these straight lines are complex, then we have the system

$$\begin{cases} \dot{x} = (x^2+1)(a_{20}+a_{30}x+a_{21}y), \\ \dot{y} = b_{00}+b_{10}x+b_{01}y+b_{20}x^2+b_{11}xy+b_{02}y^2+a_{30}x^2y+a_{21}xy^2. \end{cases} \tag{33}$$

The invariant straight lines of the system (32) (respectively, (33)) are $l_1 = x$ and $l_2 = x - a$ (respectively, $l_{1,2} = x \pm i$). Taking into account that the right-hand sides of these systems have no common factors, it is easy to see that, for both systems, each straight line l_1 and l_2 can pass through at most two singular points.

Let the system (32) have another real invariant straight line, then via an affine transformation, this system can be brought to the form:

$$\begin{cases} \dot{x} = x(x-a)(a_{20}+a_{30}x+a_{21}y), \\ \dot{y} = y(b_{01}+b_{11}x+b_{02}y+a_{30}x^2+a_{21}xy). \end{cases} \tag{34}$$

The invariant straight lines of (34) are: $l_1 = x$, $l_2 = x-a$, $l_3 = y$. All singular points have real coordinates, thus, considering Proposition 6, all other invariant straight lines must be real, i.e. the configuration C3) is not possible.

The system (33) has at most four invariant straight lines, because of Proposition 7 and the fact that on each invariant straight line l_1, l_2 only two singular points lie. Therefore, the configuration C5) is not realizable. □

Configuration C1) $(\mathbf{2r}, \mathbf{1r}, \mathbf{1r}, \mathbf{1r})$. Let the straight lines l_1, l_2, l_3, l_4 with configuration $(2r, 1r, 1r)$ be invariant for system [(1)][(26)]. These straight lines can be brought to the form $l_1 = x$, $l_2 = x - 1$, $l_3 = y$ and $l_4 = x + y - 1$. Therefore, the system [(1)][(26)] has the following form:

$$\begin{cases} \dot{x} = x(x-1)(b_{01}+b_{11}+a_{30}x+a_{21}y), \\ \dot{y} = y(b_{01}+b_{11}x-b_{01}y+a_{30}x^2+a_{21}xy). \end{cases} \tag{35}$$

The intersection points of the straight lines of the system (35) are $(0,0)$, $(0,1)$ and $(1,0)$. Through the singular point $(1,0)$ the invariant straight lines l_2, l_3 and l_4 pass. According to Proposition 11 any other real invariant straight line must pass through the point $(0,0)$ or $(0,1)$.

Let l_5 be a real straight line for system (35) passing through the point $(0,0)$, i.e. it is described by equation $y = Ax$. This straight line is invariant for the system (35) iff $b_{11} = -2b_{01}$, $A = (a_{30} - b_{01})/(b_{01} - a_{21})$. Without loss of generality we consider $b_{01} = -1$. Let $a_{30} = a - 1$ and $a_{21} = b - 1$, then we obtain the system 19) of Theorem 1. The conditions $ab(b-1)(b+1)(a-b) \neq 0$ will guarantee that the system 19) is not from another class. Similarly, from the system (35), we can obtain a system possessing five invariant straight lines with $(0,1) \in l_5$, but it will be affine equivalent with system 19).

Configuration C6) $(\mathbf{2c_0}, \mathbf{1r}, \mathbf{1c_1}, \mathbf{1c_1})$. Let the system [(1)][(26)] have four invariant straight lines with configuration $(2c_0, 1c_1, 1c_1)$. The straight lines can be written as $l_{1,2} = x - a \pm i$ and $l_{3,4} = y \pm ix$. The system [(1),(26)] with these invariant straight lines looks as

$$\begin{cases} \dot{x} = ((x-a)^2+1)(a_{30}x + a_{21}y), \\ \dot{y} = (a^2+1)(a_{30}y - a_{21}x) + b_{20}x^2 - 2aa_{30}xy + (b_{20} - 2aa_{21})y^2 + \\ \qquad +a_{30}x^2y + a_{21}xy^2 \end{cases} \tag{36}$$

and has the following singular points: $O_1(a-i, 1+ai)$, $O_2(a+i, 1-ai)$, $O_3(a+i, -1+ai)$, $O_4(a-i, -1-ai)$, $O_5(0,0)$, $O_6(a_{21}(1+a^2)/b_{20}, -a_{30}(1+a^2)/b_{20})$, $O_1 = l_1 \cap l_4$, $O_2 = l_2 \cap l_3$, $O_3 = l_2 \cap l_4$, $O_4 = l_1 \cap l_3$. Any other real invariant straight line of the system (36) must pass through one of two pairs of conjugate complex singular points $\{O_1, O_2\}$ or $\{O_3, O_4\}$, therefore, $l_5 = ax + y - a^2 - 1$ or $l_5 = ax - y - a^2 - 1$, respectively. In the first case, $l_5 = ax + y - a^2 - 1$ is invariant for (36) iff $b_{20} = aa_{21} - a_{30}$. Furthermore, if $a_{30} = 0$, then the system (36) has six invariant straight lines. Let $a_{30} \neq 0$ and denote $a_{21} = b \cdot b_{30}$. After rescaling the time $t = 1/a_{30}\tau$, we get the system 20) of Theorem 1. In the second case, $l_5 = ax - y - a^2 - 1$ is invariant for the system (36) iff $b_{20} = aa_{21} + a_{30}$. Moreover, (36) has exactly five invariant straight lines if $a_{30} \neq 0$. The obtained system is affine equivalent with system 20).

3.3 Class D): invariant straight lines with different slopes

For cubic systems in class B) the following three configurations of invariant straight lines are possible:

D1) $(\mathbf{1r}, \mathbf{1c_1}, \mathbf{1c_1}, \mathbf{1c_1}, \mathbf{1c_1})$ **D2)** $(\mathbf{1r}, \mathbf{1r}, \mathbf{1r}, \mathbf{1c_1}, \mathbf{1c_1})$
D3) $(\mathbf{1r}, \mathbf{1r}, \mathbf{1r}, \mathbf{1r}, \mathbf{1r})$

Configuration D1) $(\mathbf{1r}, \mathbf{1c_1}, \mathbf{1c_1}, \mathbf{1c_1}, \mathbf{1c_1})$**.** Let the system [(1)][(26)] have the invariant straight lines $l_j \in \mathbb{C}[x,y] \setminus \mathbb{R}[x,y]$, $j = \overline{1,4}$, $l_j = \overline{l}_{j+1}$, $j = 1,3$, $l_j \not\parallel l_k$, $j \neq k$. Via an affine transformation and time rescaling we can bring them to the form $l_{1,2} \equiv y \pm ix = 0$, $l_{3,4} = y - (a \pm bi)x - 1 = 0$, $a, b \in \mathbb{R}$, $b(|a| + |b \pm 1|) \neq 0$. There are two affine different systems [(1)][(26)] with these invariant straight lines:

$$\begin{cases} \dot{x} = y + x^2 + 2axy - y^2 + (2a - b_{02})x^3 + (a^2 + b^2 - 1)x^2y - b_{02}xy^2, \\ \dot{y} = -x + (b_{02} - 2a)x^2 + 2xy + b_{02}y^2 + (2a - b_{02})x^2y + \\ \qquad +(a^2 + b^2 - 1)xy^2 - b_{02}y^3; \end{cases} \tag{37}$$

$$\begin{cases} \dot{x} = x + cy + (2a + c)x^2 + 2(-1 + ac)xy - cy^2 + (-2 + a^2 + b^2 - b_{02} + \\ \qquad +2ac)x^3 + (-2a - c + a^2c + b^2c)x^2y - (1 + b_{02})xy^2, \\ \dot{y} = -cx + y + (2 + b_{02} - 2ac)x^2 + 2(a + c)xy + b_{02}y^2 + (-2 + a^2 + \\ \qquad b^2 - b_{02} + 2ac)x^2y + (-2a - c + a^2c + b^2c)xy^2 - (1 + b_{02})y^3. \end{cases} \tag{38}$$

Let $O_{j,k}$ be the intersection point of the invariant straight lines l_j and $l_k, j \neq k$. Then, $O_{1,2} = (0,0)$, $O_{1,3} = (-1/(-i + a + bi), 1/(1 - b + ai))$, $O_{1,4} = (-1/(-i + a - bi), 1/(1 + b + ai))$, $O_{3,4} = (0,1)$, $O_{2,3} \equiv \overline{O_{1,4}}$ and $O_{2,4} \equiv \overline{O_{1,3}}$. The straight line passing through singular points $O_{1,3}$ and $O_{2,4}$ ($O_{1,4}$ and $O_{2,3}$) is described by equation $1 + ax - y + by = 0$ ($1 + ax - y - by = 0$). Using only the information provided by singular points we can state that besides the invariant straight lines

$l_{1,2,3,4}$, the systems (37), (38) can have also the invariant straight lines described by equations $1 + ax - y + by = 0$, $1 + ax - y - by = 0$ and $x = 0$.

The straight line $x = 0$ can't be invariant for (37), because the coefficients of the monomials y, $-y^2$ from right-hand side of first equation from system (37) are constant. The straight lines $l_5 = 1+ax-y+by$ and $l_6 = 1+ax-y-by$ are invariant for (37) only simultaneously, therefore this system can't have exactly five invariant straight lines.

The straight line $l_5 = 1 + ax - y + by$ ($l_5 = 1 + ax - y - by$) is invariant for the system (38) iff $b_{02} = ac - 2 + b$ ($b_{02} = ac - 2 - b$), i.e. we obtained the system 21) of Theorem 1 (a system affine equivalent with 21)).

Also, the straight line $x = 0$ is invariant for the system (38) iff $c = 0$. In (38) we take $c = 0$ and denote $b_{02} = c - 2$, where c is a real parameter. This way we get the system 22) of Theorem 1.

Configuration D2) $(\mathbf{1r}, \mathbf{1r}, \mathbf{1r}, \mathbf{1c_1}, \mathbf{1c_1})$**.** The complex invariant straight lines of the system [(1)][(26)], via an affine transformation, can be brought to the form $l_{1,2} = y \pm ix$. According to Proposition 11, two of real invariant straight lines $l_{3,4,5}$ can't pass through the intersection point $(0, 0)$ of l_1 and l_2. Therefore, via a rotation and a contraction $x \to kx$, $y \to ky$, $k \in \mathbb{R}^*$, we can bring the intersection point of the straight lines l_3 and l_4 in $(0, 1)$, i.e. these straight lines are described by $l_3 = y - ax - 1$ and $l_4 = y - bx - 1$, $a, b \in \mathbb{R}$, $a \neq b$. The fifth invariant straight line must pass through the points $(0, 0)$ and $(0, 1)$, i.e. it is described by $l_5 = x$. Asking that these straight lines to be invariant for the system [(1)][(26)] we get the system 23) of Theorem 1.

Configuration D3) $(\mathbf{1r}, \mathbf{1r}, \mathbf{1r}, \mathbf{1r}, \mathbf{1r})$**.** Let the system [(1)][(26)] have at least five real invariant straight lines with diffefrent slopes l_j, $j = \overline{1, 5}$. Via an affine transformation we can bring these straight lines to be described by equations: $x = 0$, $y = 0$, $y = x$, $y = ax + 1$, $y = bx + 1$, $ab(a - 1)(b - 1) \neq 0$, $a < b$. The cubic system with these invariant straight lines has the form 24) of Theorem 1.

3.4 Qualitative investigation of systems 13)-24)

In this section, the qualitative study of systems $13) - 24)$ of Theorem 1 will be done. For this purpose, in order to determine the topological behavior of trajectories, the singular points will be examined. Using also the information provided by the existence of invariant straight lines, we will construct all phase portraits of systems $3) - 11)$ on Poincaré disk.

We denote by SP singular points; λ_1 and λ_2 the characteristic roots of the SP; TSP – type of SP; S – saddle ($\lambda_1\lambda_2 < 0$); N^s – stable node ($\lambda_1, \lambda_2 < 0$); N^i – instable node ($\lambda_1, \lambda_2 > 0$); $DN^{s(i)}$ – improper stable (instable) node ($\lambda_1 = \lambda_2 \neq 0$); C – centre, $P^{i(s)}$– instable (stable) parabolic sector, $F^{i(s)}$ – instable (stable) focus.

In the next tables, the first column will indicate the singular points of the systems; the second column - the eigenvalues corresponding to these singular points and the third column - the types of the singularities. All these points are simple and together with the invariant straight lines, fully determine the phase portrait for each of the systems 13)–24).

Table 1. Systems 13), 15), 16), 17), 19), 20), 21), 22) and 23)

SP	$\lambda_1;\lambda_2$	TSP		SP	$\lambda_1;\lambda_2$	TSP	
System 13); Fig. 15							
$J=\{a \mid a\in(-1,0)\}$		$a<-1$	J			$a<-1$	J
$O_1(0,0)$	$-a\,;-a$	DN^i	DN^i	$O_2(1,0)$	$a;\,-a-1$	S	N^s
$O_3(0,1)$	$a\,;-a-1$	S	N^s	$O_4(1,1)$	$a+1;\,a+1$	DN^s	DN^i
$O_5(-a,-a)$	$\pm a(a+1)$	S	S	$X_\infty(1,0,0)$	$-a;\,-a-1$	N^i	S
$Y_\infty(0,1,0)$	$-a;\,-a-1$	N^s	S				
System 15); Fig. 15							
$O(-a,-a)$	$\pm(a^2+1)$	S		$X_\infty(1,0,0)$	$-a\pm i$	F^s	
$Y_\infty(0,1,0)$	$-a\pm i$	F^s					
System 16); Fig. 16							
$G=\{a \mid a\in(0,1)\}$		G	$a>1$			G	$a>1$
$O_1(0,0)$	$a;\,a$	DN^i	DN^i	$O_2(0,1)$	$1-a;\,1-a$	DN^i	DN^s
$O_3(0,a)$	$\pm a(a-1)$	S	S				
System 17); Fig. 17)							
$O_1(0,0)$	$1\pm 2bi$	F^i		$O_2(0,1)$	$1\pm 2bi$	F^i	
$O_3(-b,\frac{1}{2})$	$\pm\frac{1}{2}\left(4b^2+1\right)$	S					
System 20); Fig. 20							
$O_1(0,0)$	$(a^2+1)(1\pm ib)$	F^i		$O_2\left(\frac{b(a^2+1)}{ab-1},\frac{a^2+1}{1-ab}\right)$	$\pm\frac{(a^2+1)(b^2+1)}{ab-1}$	S	
$I_\infty(0,1,0)$	$1\pm bi$	F^i					
System 21; Fig. 17.a), Fig. 21							
		$b<0$	$b>0$				
$O_1(0,0)$	$1\pm ci$	F^i		$O_3(\frac{-c}{ac-b+1},\frac{1}{a-b+1})$	$\pm\frac{b(c^2+1)}{ac-b+1}$	S	
$O_2(0,1)$	$b(-1\pm ci)$	F^i	F^s				
System 22); Fig. 16							
		$c<0$	$c>0$				
$O_1(0,0)$	$1;\,1$	N^i		$O_3(0,\frac{1}{1-c})$	$\pm\frac{1}{1-c}$	S	
$O_2(0,1)$	$-c;\,-c$	N^i	N^s				
System 23); Fig. 22							
		$c<0$	$c>0$				
$O_1(0,0)$	$1;\,1$	N^i		$O_3(0,\frac{1}{1-c})$	$\pm\frac{1}{1-c}$	S	
$O_2(0,1)$	$-c;\,-c$	N^i	N^s				

Systems 13), 15)–17), 19)–23). All these systems have hyperbolic singular points in the finite part of the phase plane and at the infinity. These singular points, their type and the phase portraits corresponding to each system are shown in Table 1.

System 14). This system has two singular points in the finite part of the phase plane and other two at the infinity. Their coordinates, their types and the phase portraits corresponding to each system are shown in Table 2.

As we see from Table 2, the origin is a nonhyperbolic singular point. Using polar

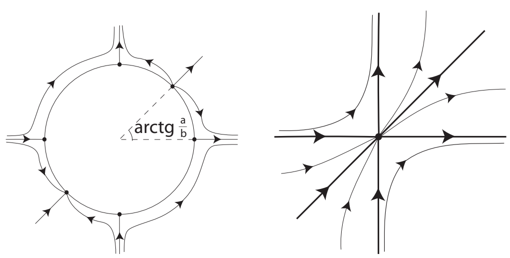
Table 2. System 14); Fig. 14

SP	$\lambda_1;\ \lambda_2$	TSP	SP	$\lambda_1;\ \lambda_2$	TSP
$O_1(0,0)$	0 ;0	HP^sHP^i	$O_2(-b,-a)$	$ab;\ -ab$	S
$X_\infty(1,0,0)$	$-a$;$-a$	DN^s	$Y_\infty(0,1,0)$	$-b;\ -b$	DN^s
Blow-up of the origin $(0,0)$					
$M_1(0,0)$	a ;$-a$	S	$M_2(0,\frac{\pi}{2})$	$b;\ -b$	S
$M_3(0,\pi)$	a ;$-a$	S	$M_4(0,\frac{3\pi}{2})$	$b;\ -b$	S
$M_5(0,\mathrm{arctg}\frac{b}{a})$	$\frac{ab}{\sqrt{a^2+b^2}}$;0	DN^i	$M_6(0,\mathrm{arctg}\frac{b}{a}+\pi)$	$\frac{-ab}{\sqrt{a^2+b^2}};\ \frac{-ab}{\sqrt{a^2+b^2}}$	DN^s

coordinates and after rescaling the time $t=\tau/\rho$, this systems takes the form:

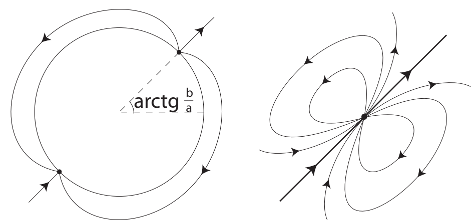
$$\begin{cases} \dot{\rho}=\rho\left(a\cos^3\theta+\rho\cos^3\theta\sin\theta+b\sin^3\theta+\rho\cos\theta\sin^3\theta\right), \\ \dot{\theta}=\cos\theta\sin\theta(a\cos\theta-b\sin\theta). \end{cases}$$

We get six singular points of the form $M_i(0,\theta_i)$, their coordinates and types are given in Table 2. Using this information, we get Fig. 24a) and after "compressing" all these points to the origin we obtain Fig. 24b), i.e. the origin can be described as HP^sHP^i singular point.



a) b)

Fig. 24



a) b)

Fig. 25

System 18). The system has only two singular points in the finite part of the phase plane (Table 3). To study neighborhood of the origin of coordinates we will use the blow-up method. In polar coordinates the system has the form:

$$\begin{cases} \dot{\rho}=\rho\left(a\cos\theta+b\sin\theta+\rho\right) \\ \dot{\theta}=a\sin\theta-b\cos\theta \end{cases}$$

Table 3. System 18); Fig. 18

SP	$\lambda_1;\ \lambda_2$	TSP	SP	$\lambda_1;\ \lambda_2$	TSP
$O_1(0,0)$	0 ;0	EP^iP^iE	$O_2(-a,-b)$	$\pm\left(a^2+b^2\right)$	S

Solving the equation $a\sin\theta-b\cos\theta=0$ gives us the information that $O_1(0,0)$ consists of two hyperbolic singular points: $M_1(0,\mathrm{arctg}\frac{b}{a})$ – instable improper node and $M_2(0,\mathrm{arctg}\frac{b}{a}+\pi)$ – stable improper node.

Compressing these points to the origin of coordinates we get that the neighborgood of the origin consists of two eliptic sectors separated by a separatrix (Fig. 25).

System 19). If $a \neq 1$, then this system has the singular points $O_1(0,0)$, $O_2(0,1)$, $O_3(1,0)$, $O_4(-1/(a-1),\ 0)$, $O_5(1,-a/b)$, $O_6(b/(b-a),-a/(b-a))$.

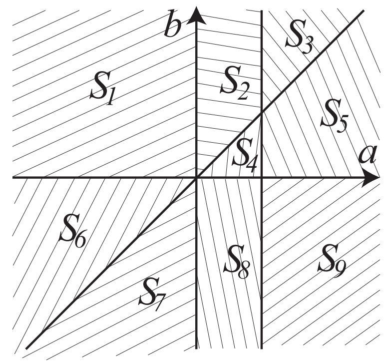


Fig. 26

The straight lines $a=0$, $b=0$, $a=1$ and $a-b=0$ divide the plane of coefficients (a,b) in 9 sectors (Fig. 26). Using relative positions of the singular points and the invariant straight lines, also the qualitative structure of these points, we notice that some systems with coefficients from the different sectors have the same trajectories. In particular, the phase portraits of systems with coefficients from S_6 and S_7 are topologically equivalent, and the phase portraits of systems S_2 (respectively, S_3, S_8) and S_5 (respectively, S_4, S_9) are equivalent. Therefore we obtain Table 4 which contains information about sectors S_1, S_3, S_5, S_6 and S_8.

Table 4. System 19), $a \neq 1$.

<table>
<tr><td>S.P.</td><td>$O_1(0,0)$</td><td>$O_2(0,1)$</td><td>$O_3(1,0)$</td><td>$O_4(-\frac{1}{a-1},0)$</td><td>$O_5(1,-\frac{a}{b})$</td><td>$O_6(\frac{b}{b-a},-\frac{a}{b-a})$</td><td>$I_\infty(0,1,0)$</td><td rowspan="3">Fig.</td></tr>
<tr><td>λ_1</td><td>-1</td><td>1</td><td>a</td><td>$-\frac{a}{a-1}$</td><td>$-a$</td><td>$\frac{a}{a-b}$</td><td>-1</td></tr>
<tr><td>λ_2</td><td>-1</td><td>$-b$</td><td>a</td><td>$\frac{a}{a-1}$</td><td>$\frac{a}{b}$</td><td>$\frac{ab}{a-b}$</td><td>$-b$</td></tr>
<tr><td>S_1</td><td rowspan="5">ND^s</td><td rowspan="2">S</td><td>ND^s</td><td rowspan="5">S</td><td rowspan="2">S</td><td>N^i</td><td rowspan="2">N^s</td><td>19a)</td></tr>
<tr><td>S_3</td><td rowspan="2">ND^i</td><td>N^s</td><td>19b)</td></tr>
<tr><td>S_5</td><td rowspan="3">N^i</td><td>N^s</td><td rowspan="3">S</td><td rowspan="3">S</td><td>19c)</td></tr>
<tr><td>S_6</td><td>ND^s</td><td>N^i</td><td>19d)</td></tr>
<tr><td>S_8</td><td>ND^i</td><td>N^s</td><td>19e)</td></tr>
</table>

If $a=1$, then the singular point $O_4(-1/(a-1),0)$ goes to the infinity. We note that the cases $b \in (0,1)$ and $b \in (1,+\infty)$ are topologically equivalent, therefore we have Table 5.

Table 5. System 19), $a = 1$.

<table>
<tr><td>S.P.</td><td>$O_1(0,0)$</td><td>$O_2(0,1)$</td><td>$O_3(1,0)$</td><td>$O_5(1,-\frac{1}{b})$</td><td>$O_6(\frac{b}{b-1},-\frac{1}{b-1})$</td><td>$I_\infty(1,0,0)$</td><td>$I_\infty(0,1,0)$</td><td rowspan="3">Fig.</td></tr>
<tr><td>λ_1</td><td>-1</td><td>1</td><td>1</td><td>-1</td><td>$\frac{a}{a-b}$</td><td>-1</td><td>-1</td></tr>
<tr><td>λ_2</td><td>-1</td><td>$-b$</td><td>1</td><td>$\frac{1}{b}$</td><td>$\frac{ab}{a-b}$</td><td>1</td><td>$-b$</td></tr>
<tr><td>$b<0$</td><td rowspan="2">ND^s</td><td>N^i</td><td rowspan="2">ND^i</td><td>N^s</td><td>N^i</td><td rowspan="2">S</td><td>S</td><td>19f)</td></tr>
<tr><td>$b>1$</td><td>S</td><td>S</td><td>N^s</td><td>N^s</td><td>19g)</td></tr>
</table>

System 24). This system has three real parameters under conditions $ab(a-1)(b-1)(c-1) \neq 0$, $a > b$, so the space of coefficients must be three-dimensional. We can simplify this by restraining the parameter c and obtaining three simpler cases. If $c \neq 0$ then the system has seven singular point in the finite part of the phase plane and if $c = 0$, then the system has six singular points (see Table 6).

Using the above conditions and the information provided by characteristic roots of singular points, we get six sectors $S_1, \ldots, S_6$ ilustrated in Fig. 27.

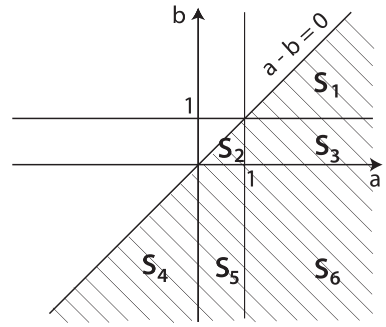


Fig. 27

Table 6. System 24)

<table>
<tr><th colspan="2">SP</th><th>$(0,0)$</th><th>$(0,1)$</th><th>$\left(0,\frac{1}{c}\right)$</th><th>$\left(-\frac{1}{a},0\right)$</th><th>$\left(-\frac{1}{b},0\right)$</th><th>$\left(\frac{1}{1-a},\frac{1}{1-a}\right)$</th><th>$\left(\frac{1}{1-b},\frac{1}{1-b}\right)$</th><th rowspan="2">Fig.</th></tr>
<tr><td colspan="2">λ_1
λ_2</td><td>1
1</td><td>$c-1$
$c-1$</td><td>$\frac{c-1}{c}$
$-\frac{c-1}{c}$</td><td>$\frac{b-a}{a}$
$\frac{1-c}{b}$</td><td>$\frac{a-b}{a}$
$\frac{1-c}{b}$</td><td>$\frac{b-a}{a-1}$
$\frac{c-1}{a-1}$</td><td>$\frac{a-b}{b-1}$
$\frac{c-1}{b-1}$</td></tr>
<tr><td rowspan="6">$c<1$
$c\neq 0$</td><td>S_1</td><td rowspan="18">N^i</td><td rowspan="6">N^s</td><td rowspan="12">S</td><td rowspan="6">S</td><td rowspan="3">N^i</td><td>N^s</td><td rowspan="6">S</td><td>23a)</td></tr>
<tr><td>S_2</td><td>N^i</td><td>23b)</td></tr>
<tr><td>S_3</td><td>N^s</td><td rowspan="3">23a)</td></tr>
<tr><td>S_4</td><td rowspan="3">N^i</td><td rowspan="2">N^i</td></tr>
<tr><td>S_5</td></tr>
<tr><td>S_6</td><td>N^s</td><td rowspan="2">23c)</td></tr>
<tr><td rowspan="6">$c>1$</td><td>S_1</td><td rowspan="6">N^i</td><td rowspan="3">N^s</td><td rowspan="6">S</td><td rowspan="6">S</td><td>N^i</td></tr>
<tr><td>S_2</td><td rowspan="5">N^s</td><td rowspan="2">23a)</td></tr>
<tr><td>S_3</td></tr>
<tr><td>S_4</td><td>N^i</td><td>23c)</td></tr>
<tr><td>S_5</td><td rowspan="2">N^s</td><td rowspan="2">23a)</td></tr>
<tr><td>S_6</td></tr>
<tr><td rowspan="6">$c=0$</td><td>S_1</td><td rowspan="6">N^s</td><td rowspan="6">$I_\infty(0,1,0)$
S</td><td rowspan="6">S</td><td rowspan="3">N^i</td><td>N^s</td><td rowspan="6">S</td><td>23d)</td></tr>
<tr><td>S_2</td><td>N^i</td><td>23e)</td></tr>
<tr><td>S_3</td><td>N^s</td><td rowspan="3">23d)</td></tr>
<tr><td>S_4</td><td rowspan="3">N^i</td><td rowspan="2">N^i</td></tr>
<tr><td>S_5</td></tr>
<tr><td>S_6</td><td>N^s</td><td>23e)</td></tr>
</table>

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Received May 27, 2016

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BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(82), 2016, Pages 57–71
ISSN 1024–7696

Invariant integrability conditions for ternary differential systems with quadratic nonlinearities of the Darboux form

Natalia Neagu

Abstract. The general integral for ternary differential system with quadratic nonlinearities of the Darboux form was constructed by using the Lie theorem on integrating factor. The case is achieved when the comitant of the linear part of differential system, which is a $GL(3,\mathbb{R})$-invariant particular integral, describes an invariant variety.

Mathematics subject classification: 34C20, 34C45.
Keywords and phrases: Center-affine group, comitant, Lie algebra, integrating factor, ternary Darboux differential system, general integral.

1 Preliminaries

Consider the ternary differential system with quadratic nonlinearities

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta \equiv P^j(x) \quad (j, \alpha, \beta = \overline{1,3}), \tag{1}$$

where $a^j_{\alpha\beta}$ is a symmetric tensor in lower indices, in which the complete convolution is done and $x = (x^1, x^2, x^3)$ is the vector of phase variables. The expressions $a^j_\alpha x^\alpha$ represent the linear part of the system (1) and $a^j_{\alpha\beta} x^\alpha x^\beta$ represent the quadratic part of this system. The coefficients and the variables take values from the field of real numbers $\mathbb{R}$. We will use the center-affine group $GL(3,\mathbb{R})$ given by substitutions

$$\bar{x}^j = q^j_\alpha x^\alpha (det(q^j_\alpha) \neq 0) \ \ (j, \alpha = \overline{1,3}).$$

It is well known that $F(x) = C$ is *a first integral* of system (1) if and only if $\Lambda(F) = 0$, where

$$\Lambda = P^j \frac{\partial}{\partial x^j} \quad (j = \overline{1,3}), \tag{2}$$

and in index j the complete convolution is done.

The system (1) has two functional-independent first integrals, which form *the general integral* of this system.

Suppose system (1) admits a two-dimensional commutative Lie algebra of operators [1]

$$X_\alpha = \xi^i_\alpha \frac{\partial}{\partial x^i} \quad (\alpha = 1, 2;\ j = \overline{1,3}), \tag{3}$$

where $\xi^i_\alpha(x)$ $(j = \overline{1,3})$ are polynomials in the coordinates of the vector $x = (x^1, x^2, x^3)$. This means that the coordinates of the operators (3) satisfy the determinant equations

$$\begin{gathered}(\xi^1_\alpha)_{x^1}P^1 + (\xi^1_\alpha)_{x^2}P^2 + (\xi^1_\alpha)_{x^3}P^3 = \xi^1_\alpha P^1_{x^1} + \xi^2_\alpha P^1_{x^2} + \xi^3_\alpha P^1_{x^3},\\ (\xi^2_\alpha)_{x^1}P^1 + (\xi^2_\alpha)_{x^2}P^2 + (\xi^2_\alpha)_{x^3}P^3 = \xi^1_\alpha P^2_{x^1} + \xi^2_\alpha P^2_{x^2} + \xi^3_\alpha P^2_{x^3},\\ (\xi^3_\alpha)_{x^1}P^1 + (\xi^3_\alpha)_{x^2}P^2 + (\xi^3_\alpha)_{x^3}P^3 = \xi^1_\alpha P^3_{x^1} + \xi^2_\alpha P^3_{x^2} + \xi^3_\alpha P^3_{x^3} \ \ (\alpha = 1, 2).\end{gathered} \tag{4}$$

Denote by

$$\Delta = \begin{vmatrix} \xi^1_1 & \xi^2_1 & \xi^3_1 \\ \xi^1_2 & \xi^2_2 & \xi^3_2 \\ P^1 & P^2 & P^3 \end{vmatrix} \tag{5}$$

the determinant of coordinates of the operators (2) and (3). From [1] the following assertion follows for system (1).

Theorem 1. *Suppose the ternary polynomial system* (1) *admits the two-dimensional commutative Lie algebra with operators* (3)*. Then the function* $\mu = \Delta^{-1}$ *is the Lie integrating factor for the Pfaff's equations*

$$(\xi^3_\alpha P^2 - \xi^2_\alpha P^3)dx^1 + (\xi^1_\alpha P^3 - \xi^3_\alpha P^1)dx^2 + (\xi^2_\alpha P^1 - \xi^1_\alpha P^2)dx^3 = 0 \quad (\alpha = 1, 2),$$

which define the general integral of the system (1)*, where* $\Delta \neq 0$ *has the form* (5)*.*

Consider the comitant of system (1) from [2] with respect to the center-affine group. It depends on two cogradient vectors $x = (x^1, x^2, x^3)$ and $y = (y^1, y^2, y^3)$ defined in [3], whose tensorial form is

$$\eta = a^\alpha_{\beta\gamma}x^\beta x^\gamma x^\delta y^\mu \varepsilon_{\alpha\delta\mu},$$

where $\varepsilon_{\alpha\delta\mu}$ is the unit trivector with coordinates $\varepsilon_{123} = -\varepsilon_{132} = \varepsilon_{312} = -\varepsilon_{321} = \varepsilon_{231} = -\varepsilon_{213} = 1$ and $\varepsilon_{\alpha\delta\mu} = 0$ $(\alpha, \delta, \mu = \overline{1,3})$ in the other cases.

In [2] the following assertions were proved:

Theorem 2. *The system* (1) *with* $\eta \equiv 0$ *can be written in the form*

$$\frac{dx^j}{dt} = \alpha^j_\alpha x^\alpha + 2x^j(gx^1 + hx^2 + kx^3) \equiv P^j(x) \ \ (j = \overline{1,3}) \tag{6}$$

and will be called the ternary differential system with quadratic nonlinearities of the Darboux form.

Theorem 3. *The system* (6) *has the* $GL(3,\mathbb{R})$*-invariant particular integral*

$$\sigma_1 = a^\alpha_\mu a^\beta_\delta a^\gamma_\alpha x^\delta x^\mu x^\nu \varepsilon_{\beta\gamma\nu}, \tag{7}$$

where σ_1 *is the comitant of* (1) *with respect to the center-affine group* $GL(3,\mathbb{R})$*.*

Remark 1. Let $\varkappa_2$ be the mixt comitant from [4] of system (6) with respect to the center-affine group

$$\varkappa_2 = a^{\alpha}_{\beta} x^{\beta} u_{\alpha}, \tag{8}$$

which depends on coordinates of the contravariant vector $x = (x^1, x^2, x^3)$ and of the covariant vector $u = (u_1, u_2, u_3)$ defined in [3]. If $\varkappa_2 \neq 0$, then at least one coefficient of the linear part of system (6) is not equal to zero. Otherwise, from $\varkappa_2 \equiv 0$ it follows that $a^j_{\alpha} = 0$ $(j, \alpha = \overline{1,3})$ and the system (6) can be reduced to a trivial homogeneous quadratic system.

Remark 2. Let q_1 be the mixt comitant from [2] of system (1) with respect to the center-affine group

$$q_1 = a^{\alpha}_{\beta\gamma} x^{\beta} x^{\gamma} u_{\alpha}, \tag{9}$$

which depends on coordinates of the contravariant vector $x = (x^1, x^2, x^3)$ and of the covariant vector $u = (u_1, u_2, u_3)$ defined in [3]. If $q_1 \neq 0$, then at least one coefficient of the quadratic part of system (1) and hence of system (6) is not equal to zero. Otherwise, from $q_1 \equiv 0$ it follows that $a^j_{\alpha\beta} = 0$ $(j, \alpha, \beta = \overline{1,3})$ and the system (1) and hence the system (6) can be reduced to a linear system.

As it follows from [2], the following assertions hold

Lemma 1. *Assume in* (7) *that* $\sigma_1 \equiv 0$. *Then under the center-affine transformation*

$$\bar{x}^1 = x^2, \quad \bar{x}^2 = x^1 + \frac{a^3_2}{a^3_1} x^2, \quad \bar{x}^3 = x^3$$

where $a^3_1 \neq 0$, *the quadratic part of system* (6) *preserves the form and the coefficients from the linear part of the system obey one of the following conditions:*

$$a^1_2 = a^1_3 = a^2_1 = a^2_3 = a^3_1 = a^3_2 = 0; \;\; a^3_3 = a^2_2; \tag{10}$$

$$a^1_2 = a^1_3 = a^2_1 = a^2_3 = a^3_1 = a^3_2 = 0; \;\; a^3_3 = a^1_1; \tag{11}$$

$$a^1_2 = a^1_3 = a^2_1 = a^2_3 = a^3_1 = a^3_2 = 0; \;\; a^2_2 = a^1_1; \tag{12}$$

$$a^1_2 = a^1_3 = a^2_1 = a^3_1 = a^3_2 = 0; \;\; a^2_3 \neq 0; \;\; a^3_3 = a^1_1; \tag{13}$$

$$a^1_2 = a^1_3 = a^2_1 = a^3_1 = a^3_2 = 0; \;\; a^2_2 = a^1_1; \;\; a^2_3 \neq 0; \tag{14}$$

$$a^1_2 = a^2_1 = a^2_3 = a^3_1 = a^3_2 = 0; \;\; a^1_3 \neq 0; \;\; a^3_3 = a^2_2; \tag{15}$$

$$a^1_2 = a^2_1 = a^3_1 = a^3_2 = 0; \;\; a^1_3 \neq 0; \;\; a^2_2 = a^1_1; \tag{16}$$

$$a_1^2 = a_3^2 = a_1^3 = a_2^3 = 0; \;\; a_2^1 \neq 0; \;\; a_3^3 = a_2^2; \tag{17}$$

$$a_1^2 = a_1^3 = a_2^3 = 0; \;\; a_2^1 \neq 0; \;\; a_3^2 = \frac{a_3^1(a_2^2 - a_1^1)}{a_2^1}; \;\; a_3^3 = a_1^1; \tag{18}$$

$$a_1^3 = a_2^3 = 0; \;\; a_1^2 \neq 0; \;\; a_2^1 = \frac{(a_1^1 - a_3^3)(a_2^2 - a_3^3)}{a_1^2}; \;\; a_3^1 = \frac{a_3^2(a_1^1 - a_3^3)}{a_1^2}; \tag{19}$$

$$a_1^2 = a_1^3 = 0; \;\; a_2^3 \neq 0; \;\; a_3^1 = \frac{a_2^1(a_3^3 - a_1^1)}{a_2^3}; \;\; a_3^2 = \frac{(a_1^1 - a_2^2)(a_1^1 - a_3^3)}{a_2^3}. \tag{20}$$

Lemma 2. *Suppose for linear part of system* (1) *or* (6) *we have* $\sigma_1 \equiv 0$*, where* σ_1 *is from* (7)*. Then the characteristic equation of these systems has real roots.*

Proof. The characteristic equation of the systems (1) and (6) looks

$$\lambda^3 - n\lambda^2 - m\lambda - l = 0, \tag{21}$$

where l, m and n are the center-affine invariants of these systems

$$l = \frac{1}{6}(\theta_1^3 - 3\theta_1\theta_2 + 2\theta_3), \qquad m = \frac{1}{2}(\theta_2 - \theta_1^2), \qquad n = \theta_1 \tag{22}$$

with

$$\theta_1 = a_\alpha^\alpha, \qquad \theta_2 = a_\beta^\alpha a_\alpha^\beta, \qquad \theta_3 = a_\gamma^\alpha a_\alpha^\beta a_\beta^\gamma. \tag{23}$$

According to [5], the discriminant of the equation (21) can be written

$$D = -27l^2 - 18lmn + 4m^3 - 4ln^3 + m^2n^2 \tag{24}$$

and it is a center-affine invariant of the systems (1) and (6).

By Lemma 1, from $\sigma_1 \equiv 0$, without considering the center-affine transformation (1), we have the conditions (10)–(20). Then for each of them, calculating the expressions (22)–(24), we get $D = 0$. □

2 Lie's integrating factor and the general integral of system (6) with $\sigma_1 \equiv 0$ and $\varkappa_2 q_1 \not\equiv 0$

Theorem 4. *Suppose the coefficients of the linear part of system* (6) *satisfy conditions* (10) *with* $\varkappa_2 q_1 \not\equiv 0$ *from* (8)–(9)*. Then the general integral of this system with notations* $x = x^1,\ y = x^2,\ z = x^3$ *consists of the following two first integrals:*

$$F_1 \equiv yz^{-1} = C_1, \;\; F_2 \equiv x^{a_2^2} y^{-a_1^1} \Phi^{a_1^1 - a_2^2} = C_2, \tag{25}$$

where

$$\Phi = a_1^1 a_2^2 + 2[a_2^2 gx + a_1^1(hy + kz)]. \tag{26}$$

Proof. Assume that the coordinates of the operator (3) have the form

$$\xi_\alpha^i = A_{\alpha\beta}^i x^\beta + A_{\alpha\beta\gamma}^i x^\beta x^\gamma \ \ (\alpha \geq 1;\ \beta, \gamma = \overline{1,3}), \tag{27}$$

and satisfy the determinant equations (4). Solving (4) under the conditions $\varkappa_2 q_1 \not\equiv 0$ from (8)–(9) and the expressions (27), we obtain for differential system (6) the following operators ($x = x^1,\ y = x^2,\ z = x^3$):

$$\begin{aligned} Y_1 &= (a_1^1 + 2gx)x\frac{\partial}{\partial x} + 2gxy\frac{\partial}{\partial y} + 2gxz\frac{\partial}{\partial z}, \\ Y_2 &= 2hxy\frac{\partial}{\partial x} + (a_2^2 + 2hy)y\frac{\partial}{\partial y} + 2hyz\frac{\partial}{\partial z}, \\ Y_3 &= 2hxz\frac{\partial}{\partial x} + (a_2^2 + 2hy)z\frac{\partial}{\partial y} + 2hz^2\frac{\partial}{\partial z}, \\ Y_4 &= 2kxy\frac{\partial}{\partial x} + 2ky^2\frac{\partial}{\partial y} + (a_2^2 + 2kz)y\frac{\partial}{\partial z}, \\ Y_5 &= 2kxz\frac{\partial}{\partial x} + 2kyz\frac{\partial}{\partial y} + (a_2^2 + 2kz)z\frac{\partial}{\partial z}. \end{aligned} \tag{28}$$

These operators compose the Lie algebra L_5 with the structure equations

$$\begin{gathered} [Y_1, Y_i] = 0\ (i = \overline{2,5}),\ [Y_2, Y_3] = -a_2^2 Y_3,\ [Y_2, Y_4] = a_2^2 Y_4,\ [Y_2, Y_5] = 0, \\ [Y_3, Y_4] = a_2^2(Y_5 - Y_2),\ [Y_3, Y_5] = -a_2^2 Y_3,\ [Y_4, Y_5] = a_2^2 Y_4. \end{gathered} \tag{29}$$

Using the operators Y_1 and Y_2, which form by (28) and (29) a two-dimensional commutative Lie algebra, we obtain from (5) (making abstraction of a constant) the Lie integrating factor $\mu^{-1} = xyz\Phi$, where Φ is given in (26).

Taking into account this expression and Theorem 1, we obtain the functional-independent integrals (25)–(26) of system (6). The conditions (10) and $\varkappa_2 q_1 \not\equiv 0$ from (8)–(9) imply that not all coefficients in this system are equal to zero. □

Theorem 5. *Assume the coefficients of the linear part of system* (6) *satisfy the conditions* (11) *with* $\varkappa_2 q_1 \not\equiv 0$ *from* (8)–(9). *Then the general integral of this system with notations* $x = x^1,\ y = x^2,\ z = x^3$ *is composed from the following two first integrals:*

$$F_1 \equiv xz^{-1} = C_1;\ \ F_2 \equiv x^{-a_2^2} y^{a_1^1} \Phi^{a_2^2 - a_1^1} = C_2, \tag{30}$$

where

$$\Phi = a_1^1 a_2^2 + 2[a_2^2(gx + kz) + a_1^1 hy]. \tag{31}$$

Proof. We make the substitutions $\bar{x}^1 = x^2,\ \bar{x}^2 = x^1,\ \bar{x}^3 = x^3$ in (6) under the conditions (11). Then we obtain the system (6) with conditions (10) for which the general integral is determined in Theorem 4. Using this result and the above-mentioned notations, we obtain for system (6) the integrals (30)–(31) on the conditions (11). □

Theorem 6. *If the coefficients of the linear part of system* (6) *satisfy the conditions* (12) *with* $\varkappa_2 q_1 \not\equiv 0$ *from* (8)–(9)*, then the general integral of this system with notations* $x = x^1,\ y = x^2,\ z = x^3$ *consists of two first integrals:*

$$F_1 \equiv x^{-1}y = C_1; \quad F_2 \equiv y^{-a^3_3} z^{a^1_1} \Phi^{a^3_3 - a^1_1} = C_2,$$

where

$$\Phi = a^1_1 a^3_3 + 2[a^3_3(gx + hy) + a^1_1 kz].$$

The proof of Theorem 6 is similar to Theorem 5 if we make the substitutions $\bar{x}^1 = x^3,\ \bar{x}^2 = x^2,\ \bar{x}^3 = x^1$ in (6) and take into account the conditions (10).

Theorem 7. *Suppose the coefficients of the linear part of system* (6) *satisfy the conditions* (13) *with* $\varkappa_2 q_1 \not\equiv 0$ *from* (8)–(9)*. Then the general integral of this system with notations* $x = x^1,\ y = x^2,\ z = x^3$ *consists of the following two first integrals:*

$$F_1 \equiv xz^{-1} = C_1; \quad F_2 \equiv x^{-a^2_2}[(a^1_1 - a^2_2)y - a^2_3 z]^{a^1_1} \Phi^{a^2_2 - a^1_1} = C_2, \tag{32}$$

where

$$\Phi = a^1_1 a^2_2 + 2[a^2_2 gx + a^1_1 hy + (a^2_2 k - a^2_3 h)z]. \tag{33}$$

Proof. Assume the coordinates of the operator (3) have the form (27). Solving the system (4) we obtain the following operators ($x = x^1,\ y = x^2,\ z = x^3$) for the differential system (6):

$$\begin{gathered}
Y_1 = (a^1_1 + 2gx)x\frac{\partial}{\partial x} + 2gxy\frac{\partial}{\partial y} + 2gxz\frac{\partial}{\partial z},\\
Y_2 = (a^1_1 + 2gx)z\frac{\partial}{\partial x} + 2gyz\frac{\partial}{\partial y} + 2gz^2\frac{\partial}{\partial z},\\
Y_3 = 2[a^2_3 h + (a^1_1 - a^2_2)k]x^2\frac{\partial}{\partial x} + [a^1_1 a^2_3 + 2(a^2_3 h +\\
+(a^1_1 - a^2_2)k)y]x\frac{\partial}{\partial y} + [(a^1_1 - a^2_2)(a^1_1 + 2kz) + 2a^2_3 hz]x\frac{\partial}{\partial z},\\
Y_4 = 2[a^2_3 h + (a^1_1 - a^2_2)k]xz\frac{\partial}{\partial x} + [a^2_3(a^1_1 + 2hy) +\\
+2(a^1_1 - a^2_2)ky]z\frac{\partial}{\partial y} + [(a^1_1 - a^2_2)(a^1_1 + 2kz) + 2a^2_3 hz]z\frac{\partial}{\partial z},\\
Y_5 = 2[a^1_1 hy - (a^2_3 h - a^2_2 k)z]x\frac{\partial}{\partial x} + \{a^1_1 a^2_2 + 2[a^1_1 hy -\\
-(a^2_3 h - a^2_2 k)z]\}y\frac{\partial}{\partial y} + [a^1_1 a^2_2 + 2(a^1_1 hy - (a^2_3 h - a^2_2 k)z)]z\frac{\partial}{\partial z}.
\end{gathered} \tag{34}$$

These operators form the Lie algebra L_5 with the structure equations

$$\begin{gathered}
[Y_1, Y_2] = -a^1_1 Y_2, \ \ [Y_1, Y_3] = a^1_1 Y_3, \ \ [Y_1, Y_4] = [Y_1, Y_5] = [Y_4, Y_5] = 0,\\
[Y_2, Y_3] = a^1_1[(a^2_2 - a^1_1)Y_1 + Y_4], \ \ [Y_2, Y_4] = a^1_1(a^2_2 - a^1_1)Y_2,\\
[Y_2, Y_5] = -a^1_1 a^2_2 Y_2, \ \ [Y_3, Y_4] = a^1_1(a^1_1 - a^2_2)Y_3, \ \ [Y_3, Y_5] = a^1_1 a^2_2 Y_3.
\end{gathered} \tag{35}$$

Using the operators Y_1 and Y_4, which form by (34) and (35) a two-dimensional commutative Lie algebra, we obtain from (5) the Lie integrating factor of the form

$$\mu^{-1} = xz[(a_1^1 - a_2^2)y - a_3^2 z]\Phi,$$

where Φ is from (33).

Taking into account this expression and Theorem 1, we obtain the functional-independent integrals (32)–(33) of system (6). The conditions (13) and $\varkappa_2 q_1 \not\equiv 0$ from (8)–(9) imply that not all coefficients in this system are equal to zero. $\square$

Theorem 8. *Assume the coefficients of the linear part of system* (6) *satisfy the conditions* (14) *with* $\varkappa_2 q_1 \not\equiv 0$ *from* (8)–(9). *Then the general integral of this system with notations* $x = x^1,\ y = x^2,\ z = x^3$ *consists of two first integrals:*

$$\begin{gathered} F_1 \equiv [(a_1^1 - a_3^3)y + a_3^2 z]x^{-1} = C_1; \\ F_2 \equiv z^{a_1^1}[(a_1^1 - a_3^3)y + a_3^2 z]^{-a_3^3}\Phi^{a_3^3 - a_1^1} = C_2, \end{gathered} \tag{36}$$

where

$$\Phi = a_1^1 a_3^3 + 2[a_3^3(gx + hy) + (a_1^1 k - a_3^2 h)z]. \tag{37}$$

Proof. Let the coordinates of the operator (3) have the form (27). Solving (4) we obtain for differential system (6) the following operators ($x = x^1,\ y = x^2,\ z = x^3$):

$$\begin{gathered} Y_1 = (a_1^1 + 2gx)x\frac{\partial}{\partial x} + 2gxy\frac{\partial}{\partial y} + 2gxz\frac{\partial}{\partial z}, \\ Y_2 = (a_1^1 + 2gx)[(a_1^1 - a_3^3)y + a_3^2 z]\frac{\partial}{\partial x} + 2g[(a_1^1 - a_3^3)y + a_3^2 z]y\frac{\partial}{\partial y} + \\ +2g[(a_1^1 - a_3^3)y + a_3^2 z]z\frac{\partial}{\partial z}, \\ Y_3 = 2hx^2\frac{\partial}{\partial x} + (a_1^1 + 2hy)x\frac{\partial}{\partial y} + 2hxz\frac{\partial}{\partial z}, \\ Y_4 = 2h[(a_1^1 - a_3^3)y + a_3^2 z]x\frac{\partial}{\partial x} + (a_1^1 + 2hy)[(a_1^1 - a_3^3)y + a_3^2 z]\frac{\partial}{\partial y} + \\ +2h[(a_1^1 - a_3^3)y + a_3^2 z]z\frac{\partial}{\partial z}, \\ Y_5 = 2[a_3^3 hy + (a_1^1 k - a_3^2 h)z]x\frac{\partial}{\partial x} + \{a_1^1 a_3^3 + 2[a_3^3 hy + \\ +(a_1^1 k - a_3^2 h)z]\}y\frac{\partial}{\partial y} + [a_1^1 a_3^3 + 2(a_3^3 hy + (a_1^1 k - a_3^2 h)z)]z\frac{\partial}{\partial z}. \end{gathered} \tag{38}$$

These operators form the Lie algebra L_5 with the structure equations

$$\begin{gathered} [Y_1, Y_2] = -a_1^1 Y_2,\ \ [Y_1, Y_3] = a_1^1 Y_3,\ \ [Y_1, Y_4] = [Y_1, Y_5] = [Y_4, Y_5] = 0, \\ [Y_2, Y_3] = a_1^1[(a_3^3 - a_1^1)Y_1 + Y_4],\ \ [Y_2, Y_4] = a_1^1(a_3^3 - a_1^1)Y_2, \\ [Y_2, Y_5] = -a_1^1 a_3^3 Y_2,\ \ [Y_3, Y_4] = a_1^1(a_1^1 - a_3^3)Y_3,\ \ [Y_3, Y_5] = a_1^1 a_3^3 Y_3. \end{gathered} \tag{39}$$

Using the operators Y_1 and Y_5, which form by (38) and (39) a two-dimensional commutative Lie algebra, we obtain from (5) the Lie integrating factor of the form

$$\mu^{-1} = xz[(a_1^1 - a_3^3)y + a_3^2 z]\Phi,$$

where Φ is from (37).

Taking into account this expression and Theorem 1, we obtain the functional-independent integrals (36)–(37) of system (6). The conditions (14) and $\varkappa_2 q_1 \not\equiv 0$ from (8)–(9) imply that not all coefficients in this system are equal to zero. □

Theorem 9. *Suppose the coefficients of the linear part of system* (6) *satisfy the conditions* (15) *with* $\varkappa_2 q_1 \not\equiv 0$ *from* (8)–(9). *Then the general integral of this system with notations* $x = x^1,\ y = x^2,\ z = x^3$ *consists of two first integrals:*

$$\begin{gathered} F_1 \equiv yz^{-1} = C_1; \\ F_2 \equiv y^{-a_1^1}[(a_2^2 - a_1^1)x - a_3^1 z]^{a_2^2}\Phi^{a_1^1 - a_2^2} = C_2, \end{gathered} \tag{40}$$

where

$$\Phi = a_1^1 a_2^2 + 2[a_2^2 g x + a_1^1 h y + (a_1^1 k - a_3^1 g)z]. \tag{41}$$

Proof. Let us make the substitutions $\bar{x}^1 = x^2,\ \bar{x}^2 = x^1,\ \bar{x}^3 = x^3$ in (6) taking into account (15). We obtain the system (6) under the conditions (13) for which the general integral is determined in Theorem 7. Using this result and the above-mentioned notations, we obtain for (6) the integrals (40)–(41) with conditions (15). □

Theorem 10. *Assume the coefficients of the linear part of system* (6) *satisfy the conditions* (16) *with* $\varkappa_2 q_1 \not\equiv 0$ *from* (8)–(9). *Then the general integral of this system with notations* $x = x^1,\ y = x^2,\ z = x^3$ *is composed from the following two first integrals:*

$$\begin{gathered} F_1 \equiv (-a_3^2 x + a_3^1 y)[(a_1^1 - a_3^3)y + a_3^2 z]^{-1} = C_1; \\ F_2 \equiv z^{-a_1^1}[(a_1^1 - a_3^3)y + a_3^2 z]^{a_3^3}\Phi^{a_1^1 - a_3^3} = C_2, \end{gathered} \tag{42}$$

where

$$\Phi = a_1^1 a_3^3 + 2[a_3^3(gx + hy) + (a_1^1 k - a_3^1 g - a_3^2 h)z]. \tag{43}$$

Proof. Let the coordinates of the operator (3) have the form (27). Then solving the system (4) we obtain the operators ($x = x^1,\ y = x^2,\ z = x^3$):

$$Y_1 = (a_1^1 + 2gx)(a_3^2 x - a_3^1 y)\frac{\partial}{\partial x} + 2g(a_3^2 x - a_3^1 y)y\frac{\partial}{\partial y} + 2g(a_3^2 x - a_3^1 y)z\frac{\partial}{\partial z},$$

$$\begin{gathered} Y_2 = (a_1^1 + 2gx)[(a_1^1 - a_3^3)y + a_3^2 z]\frac{\partial}{\partial x} + 2g[(a_1^1 - a_3^3)y + a_3^2 z]y\frac{\partial}{\partial y} + \\ +2g[(a_1^1 - a_3^3)y + a_3^2 z]z\frac{\partial}{\partial z}, \end{gathered}$$

$$\begin{gathered}
Y_3 = \{a_3^3[a_1^1a_3^1 + 2(a_3^1g + a_3^2h)x]y + Wxz\}\frac{\partial}{\partial x} + \\
+\{a_3^3[a_1^1a_3^2 + 2(a_3^1g + a_3^2h)y] + Wz\}y\frac{\partial}{\partial y} + \\
+\{a_3^3[a_1^1a_3^2 + 2(a_3^1g + a_3^2h)y] + Wz\}z\frac{\partial}{\partial z}, \\
Y_4 = \{a_3^3[2(a_3^2)^2hx^2 + (a_3^1)^2(a_1^1 + 2gx)y] + a_3^1Wxz\}\frac{\partial}{\partial x} + \\
+\{a_3^3[(a_3^2)^2(a_1^1 + 2hy)x + 2(a_3^1)^2gy^2] + a_3^1Wyz\}\frac{\partial}{\partial y} + \\
+\{a_3^3[a_3^1(a_1^1a_3^2 + 2a_3^1gy) + 2(a_3^2)^2hx] + a_3^1Wz\}z\frac{\partial}{\partial z}, \\
Y_5 = \{a_3^3[2a_1^1a_3^2kxz - a_3^1[(a_1^1 - a_3^3)(a_1^1 + 2gx)y + 2a_3^2gxz]] - a_1^1Wxz\}\frac{\partial}{\partial x} + \\
+\{a_3^3[a_1^1a_3^2(a_3^2 + 2ky)z - 2a_3^1[(a_1^1 - a_3^3)gy + a_3^2gz]y] - a_1^1Wyz\}\frac{\partial}{\partial y} + \\
+\{a_3^3[a_3^3(a_1^1a_3^2 + 2a_3^1gy) - a_1^1a_3^2(a_1^1 - 2kz) - 2a_3^1g(a_1^1y + a_3^2z)] - a_1^1Wz\}z\frac{\partial}{\partial z},
\end{gathered} \tag{44}$$

where $W = 2a_3^2(a_1^1k - a_3^2h - a_3^1g)$.

These operators form the Lie algebra L_5 with the structure equations

$$\begin{gathered}
[Y_3, Y_5] = a_3^1(a_3^3)^2[Y_1, Y_2] = -a_3^3[Y_1, Y_5] = a_3^1a_3^3[Y_2, Y_3] = -a_1^1a_3^1a_3^2(a_3^3)^2Y_2, \\
[Y_3, Y_4] = -a_3^3[Y_1, Y_4] = -a_1^1a_3^2a_3^3[a_3^1(a_3^3Y_1 - Y_3) + Y_4], \\
[Y_1, Y_3] = [Y_2, Y_5] = 0, \quad [Y_4, Y_5] = a_3^1a_3^3[Y_2, Y_4] = \\
= a_1^1a_3^1a_3^2a_3^3[a_3^3(a_3^3 - a_1^1)Y_1 - a_3^1a_3^3Y_2 + (a_1^1 - a_3^3)Y_3 + Y_5].
\end{gathered} \tag{45}$$

We use the operators Y_1 and Y_3, which form by (44) and (45) a two-dimensional commutative Lie algebra. Then from (5) we obtain the Lie integrating factor of the form

$$\mu^{-1} = (-a_3^2x + a_3^1y)z[(a_1^1 - a_3^3)y + a_3^2z]\Phi,$$

where Φ is from (43).

Taking into account this expression and Theorem 1, we obtain the functional-independent integrals (42)–(43) of system (6). The conditions (16) and $\varkappa_2q_1 \not\equiv 0$ from (8)–(9) imply that not all coefficients in this system are equal to zero. □

Theorem 11. *Let the coefficients of the linear part of system* (6) *satisfy the conditions* (17) *with* $\varkappa_2q_1 \not\equiv 0$ *from* (8)–(9)*. Then the general integral of this system with notations* $x = x^1,\ y = x^2,\ z = x^3$ *consists of two first integrals:*

$$\begin{gathered}
F_1 \equiv yz^{-1} = C_1; \\
F_2 \equiv z^{a_1^1}[(a_1^1 - a_2^2)x + a_2^1y + a_3^1z]^{-a_2^2}\Phi^{a_2^2 - a_1^1} = C_2,
\end{gathered} \tag{46}$$

where

$$\Phi = a_2^2(a_1^1 + 2gx) + 2[(a_1^1h - a_2^1g)y + (a_1^1k - a_3^1g)z]. \tag{47}$$

Proof. Suppose the coordinates of the operator (3) have the form (27). Then from system (4) we obtain the following operators ($x = x^1,\ y = x^2,\ z = x^3$) for differential system (6):

$$\begin{gathered}
Y_1 = \{a_2^1 a_2^2 + 2[a_2^1 g + (a_2^2 - a_1^1)h]x\} y \frac{\partial}{\partial x} + \\
+[2a_2^1 gy + (a_2^2 + 2hy)(a_2^2 - a_1^1)] y \frac{\partial}{\partial y} + 2[a_2^1 g + (a_2^2 - a_1^1)h] yz \frac{\partial}{\partial z}, \\
Y_2 = [a_2^1 a_2^2 + 2(a_2^1 g + (a_2^2 - a_1^1)h)x] z \frac{\partial}{\partial x} + \\
+[2a_2^1 gy + (a_2^2 + 2hy)(a_2^2 - a_1^1)] z \frac{\partial}{\partial y} + 2[a_2^1 g + (a_2^2 - a_1^1)h] z^2 \frac{\partial}{\partial z}, \\
Y_3 = 2(a_2^1 k - a_3^1 h) xy \frac{\partial}{\partial x} + [-a_3^1 a_2^2 + 2(a_2^1 k - a_3^1 h)y] y \frac{\partial}{\partial y} + \\
+[a_2^1 a_2^2 + 2(a_2^1 k - a_3^1 h)z] y \frac{\partial}{\partial z}, \\
Y_4 = 2(a_2^1 k - a_3^1 h) xz \frac{\partial}{\partial x} + [-a_3^1 a_2^2 + 2(a_2^1 k - a_3^1 h)y] z \frac{\partial}{\partial y} + \\
+[a_2^1 a_2^2 + 2(a_2^1 k - a_3^1 h)z] z \frac{\partial}{\partial z}, \\
Y_5 = [a_2^1 a_2^2 (a_1^1 + 2gx) + 2(a_2^1 y + a_3^1 z)(a_1^1 h - a_2^1 g)] x \frac{\partial}{\partial x} + \\
+\{[a_2^1 a_2^2 (a_1^1 + 2gx) + 2(a_2^1 y + a_3^1 z)(a_1^1 h - a_2^1 g)] y + a_1^1 a_3^1 a_2^2 z\} \frac{\partial}{\partial y} + \\
+2[a_2^1 a_2^2 gx + (a_2^1 y + a_3^1 z)(a_1^1 h - a_2^1 g)] z \frac{\partial}{\partial z}.
\end{gathered} \tag{48}$$

These operators form the Lie algebra L_5 with the structure equations

$$\begin{gathered}
[Y_1, Y_2] = a_2^2(a_1^1 - a_2^2) Y_2 \quad [Y_1, Y_3] = a_2^2[a_3^1 Y_1 + (a_2^2 - a_1^1) Y_3], \\
a_1^1 a_2^1 [Y_1, Y_4] = -a_2^1 [Y_1, Y_5] = -a_1^1 a_3^1 [Y_2, Y_4] = a_3^1 [Y_2, Y_5] = a_1^1 a_2^1 a_3^1 a_2^2 Y_2, \\
[Y_2, Y_3] = -a_2^2 [a_2^1 Y_1 + (a_1^1 - a_2^2) Y_4], \\
a_1^1 [Y_3, Y_4] = -[Y_3, Y_5] = a_1^1 a_2^2 (a_2^1 Y_3 + a_3^1 Y_4), \quad [Y_4, Y_5] = 0.
\end{gathered} \tag{49}$$

If we use the operators Y_4 and Y_5, which form by (48) and (49) a two-dimensional commutative Lie algebra, we obtain from (5) the Lie integrating factor of the form

$$\mu^{-1} = z(a_2^1 y + a_3^1 z)[(a_1^1 - a_2^2)x + a_2^1 y + a_3^1 z]\Phi,$$

where Φ is from (47).

Taking into account this expression and Theorem 1, we obtain the functional-independent integrals (46)–(47) of system (6). The conditions (17) and $\varkappa_2 q_1 \not\equiv 0$ from (8)–(9) imply that not all coefficients in this system are equal to zero. □

Theorem 12. *If the coefficients of the linear part of system* (6) *satisfy the conditions* (18) *with* $\varkappa_2 q_1 \not\equiv 0$ *from* (8)–(9)*, then the general integral of this system with notations* $x = x^1,\ y = x^2,\ z = x^3$ *consists of two first integrals:*

$$\begin{aligned} F_1 &\equiv z[(a_1^1 - a_2^2)x + a_2^1 y + a_3^1 z]^{-1} = C_1; \\ F_2 &\equiv (a_2^1 y + a_3^1 z)^{a_1^1}[(a_1^1 - a_2^2)x + a_2^1 y + a_3^1 z]^{-a_2^2}\Phi^{a_2^2 - a_1^1} = C_2, \end{aligned} \tag{50}$$

where

$$\Phi = a_1^1 a_2^1 a_2^2 + 2\{a_2^1[a_2^2 g x + (a_1^1 h - a_2^1 g)y + (a_2^2 k - a_3^1 g)z] + a_3^1(a_1^1 - a_2^2)hz\}. \tag{51}$$

Proof. Assume the coordinates of the operator (3) have the form (27). Then system (4) yields the following operators ($x = x^1,\ y = x^2,\ z = x^3$):

$$\begin{aligned} Y_1 &= (a_1^1 + 2gx)[(a_1^1 - a_2^2)x + a_2^1 y]\frac{\partial}{\partial x} + 2g[(a_1^1 - a_2^2)x + a_2^1 y]y\frac{\partial}{\partial y} + \\ &\quad + 2g[(a_1^1 - a_2^2)x + a_2^1 y]z\frac{\partial}{\partial z}, \\ Y_2 &= (a_1^1 + 2gx)z\frac{\partial}{\partial x} + 2gyz\frac{\partial}{\partial y} + 2gz^2\frac{\partial}{\partial z}, \\ Y_3 &= 2(a_2^1 k - a_3^1 h)xz\frac{\partial}{\partial x} + [-a_1^1 a_3^1 + 2(a_2^1 k - a_3^1 h)y]z\frac{\partial}{\partial y} + \\ &\quad + [a_1^1 a_2^1 + 2(a_2^1 k - a_3^1 h)z]z\frac{\partial}{\partial z}, \\ Y_4 &= \{a_1^1 a_2^1 a_2^2 + 2[a_2^1 a_2^2 g x + (a_1^1 h - a_2^1 g)(a_2^1 y + a_3^1 z)]\}x\frac{\partial}{\partial x} + \\ &\quad + [a_1^1 a_2^2(a_2^1 y + a_3^1 z) + 2[a_2^1 a_2^2 g x + (a_1^1 h - a_2^1 g)(a_2^1 y + a_3^1 z)]y]\frac{\partial}{\partial y} + \\ &\quad + 2[a_2^1 a_2^2 g x + (a_1^1 h - a_2^1 g)(a_2^1 y + a_3^1 z)]z\frac{\partial}{\partial z}, \\ Y_5 &= (a_1^1 a_2^1 a_3^1 a_2^2 + W)x\frac{\partial}{\partial x} + [a_1^1 a_3^1 a_2^2(a_2^2 - a_1^1)x + Wy + a_1^1(a_3^1)^2 a_2^2 z]\frac{\partial}{\partial y} + \\ &\quad + [a_1^1 a_2^1 a_2^2((a_1^1 - a_2^2)x + a_2^1 y) + Wz]\frac{\partial}{\partial z}, \end{aligned} \tag{52}$$

where $W = 2a_2^1 a_3^1 g(a_2^2 x - a_2^1 y - a_3^1 z) - 2a_3^1 h[(a_2^2 x - a_2^1 y)(a_1^1 - a_2^2) - a_1^1 a_3^1 z] - 2a_2^1 a_2^2 k((a_2^2 - a_1^1)x - a_2^1 y)$.

These operators form the Lie algebra L_5 with the structure equations

$$\begin{gathered} [Y_1, Y_2] = a_1^1(a_2^2 - a_1^1)Y_2, \quad [Y_2, Y_5] = a_1^1 a_2^2[-a_2^1 Y_1 + a_2^1 a_3^1 Y_2 + (a_1^1 - a_2^2)Y_3], \\ a_2^2[Y_1, Y_3] = -[Y_1, Y_4] = -a_3^1 a_2^2[Y_2, Y_3] = a_3^1[Y_2, Y_4] = a_1^1 a_2^1 a_3^1 a_2^2 Y_2, \\ [Y_1, Y_5] = a_1^1[a_2^1 a_3^1 a_2^2 Y_1 - a_2^1(a_3^1)^2 a_2^2 Y_2 - a_3^1(a_1^1 - a_2^2)Y_4 + (a_1^1 - a_2^2)Y_5], \\ [Y_3, Y_4] = 0, \quad a_2^2[Y_3, Y_5] = -[Y_4, Y_5] = a_1^1 a_2^1 a_2^2[-a_3^1 a_2^2 Y_3 + a_3^1 Y_4 - Y_5]. \end{gathered} \tag{53}$$

Using the operators Y_3 and Y_4, which form by (52) and (53) a two-dimensional commutative Lie algebra, we obtain from (5) the Lie integrating factor of the form

$$\mu^{-1} = z(a_2^1 y + a_3^1 z)[(a_1^1 - a_2^2)x + a_2^1 y + a_3^1 z]\Phi,$$

where Φ is from (51).

Taking into account this expression and Theorem 1, we obtain the functional-independent integrals (50)–(51) of system (6). The conditions (18) and $\varkappa_2 q_1 \not\equiv 0$ from (8)–(9) ensure that not all coefficients in this system are equal to zero. $\square$

Theorem 13. *Suppose the coefficients of the linear part of system* (6) *satisfy the conditions* (19) *with* $\varkappa_2 q_1 \not\equiv 0$ *from* (8)–(9). *Then the general integral of this system with notations* $x = x^1,\ y = x^2,\ z = x^3$ *consists of two first integrals:*

$$\begin{gathered} F_1 \equiv z[-a_1^2 x + (a_1^1 - a_3^3) y - a_3^2 z]^{-1} = C_1; \\ F_2 \equiv z^{-(a_1^1 + a_2^2 - a_3^3)}[-a_1^2 x - (a_2^2 - a_3^3) y - a_3^2 z]^{a_3^3} \Phi^{a_1^1 + a_2^2 - 2a_3^3} = C_2, \end{gathered} \tag{54}$$

where

$$\begin{gathered} \Phi = a_1^2 a_3^3 (a_1^1 + a_2^2 - a_3^3) + 2\{a_1^2(a_2^2 g - a_1^2 h)x + [a_3^3 g(a_1^1 + a_2^2 - a_3^3) - \\ -a_1^1(a_2^2 g - a_1^2 h)]y + [-a_3^2(g(a_1^1 - a_3^3) + a_1^2 h) + a_1^2 k(a_1^1 + a_2^2 - a_3^3)]z\}. \end{gathered} \tag{55}$$

Proof. Assuming the coordinates of the operator (3) have the form (27), we obtain from (4) the following operators ($x = x^1,\ y = x^2,\ z = x^3$):

$$\begin{gathered} Y_1 = [-2Wx - a_3^3(a_2^2 - a_3^3)]z\frac{\partial}{\partial x} + [a_1^2 a_3^3 - 2Wy]z\frac{\partial}{\partial y} - 2Wz^2\frac{\partial}{\partial z}, \\ Y_2 = (Vx - a_3^2 a_3^3)z\frac{\partial}{\partial x} + Vyz\frac{\partial}{\partial y} + (a_1^2 a_3^3 + Vz)z\frac{\partial}{\partial z}, \\ Y_3 = [a_1^2 a_3^3 Tx + 2a_1^2 Ux^2 + 2(a_3^3 gT - a_1^1 U)xy + a_3^2(a_3^3 T + 2Ux)z]\frac{\partial}{\partial x} + \\ +[a_1^2 a_3^3 T + 2a_1^2 Ux + 2(a_3^3 gT - a_1^1 U)y + 2a_3^2 Uz]y\frac{\partial}{\partial y} + \\ +2[a_1^2 Ux + (a_3^3 gT - a_1^1 U)y + a_3^2 Uz]z\frac{\partial}{\partial z}, \\ Y_4 = \{a_1^2 a_3^3(a_3^3 gT - a_1^1 U)x + a_3^3(-a_1^1 a_2^2 + a_3^3 T)Uy + \\ +2a_3^3[a_3^3 g^2 T - (a_1^1 g + a_1^2 h)U]xy - a_3^2 a_3^3 WTz - 2a_3^2 WUxz\}\frac{\partial}{\partial x} + \\ +[-(a_1^2)^2 a_3^3 Ux + a_1^2 a_3^3(a_3^3 gT - a_2^2 U)y + 2a_3^3[a_3^3 g^2 T - (a_1^1 g + a_1^2 h)U]y^2 - \\ -2a_3^2 UWyz]\frac{\partial}{\partial y} + \{2a_3^3[a_3^3 g^2 T - (a_1^1 g + a_1^2 h)U]yz - 2a_3^2 UWz^2\}\frac{\partial}{\partial z}, \\ Y_5 = \{a_3^3[-a_1^2 a_3^2[(a_1^1 - a_3^3)g + a_1^2 h] + (a_1^2)^2 kT]x - \\ -a_3^2 a_3^3(a_1^1 - a_3^3)Uy + a_3^3[(a_1^1 - a_3^3)g + a_1^2 h]Vxy - a_3^2 a_3^3(a_3^2 g - a_1^2 k)Tz + \\ +a_3^2 UVxz\}\frac{\partial}{\partial x} + \{-a_1^2 a_3^3(a_3^2 g - a_1^2 k)Ty + a_3^3[(a_1^1 - a_3^3)g + \\ +a_1^2 h]Vy^2 + a_3^2 UVyz\}\frac{\partial}{\partial y} + \{-(a_1^2)^2 a_3^3 Ux + a_1^2 a_3^3(a_1^1 - a_3^3)Uy + \\ a_3^3[(a_1^1 - a_3^3)g + a_1^2 h]Vyz + a_3^2 UVz^2\}\frac{\partial}{\partial z}, \end{gathered} \tag{56}$$

where

$$W = (a_2^2 - a_3^3)g - a_1^2 h, V = -2(a_3^2 g - a_1^2 k),$$
$$T = a_1^1 + a_2^2 - a_3^3, U = a_2^2 g - a_1^2 h. \tag{57}$$

These operators form the Lie algebra L_5 with the structure equations

$$\begin{gathered}
[Y_1, Y_2] = -a_1^2 a_3^3 Y_1, \quad [Y_1, Y_3] = a_1^2 a_3^3 T Y_1,\\
[Y_1, Y_4] = a_1^2 (a_3^3)^2 [(a_1^1 - a_3^3)g + a_1^2 h] Y_1,\\
[Y_1, Y_5] = -a_1^2 a_3^3 [-\frac{TV}{2} Y_1 - (T - a_3^3) U Y_2 + W Y_3 + Y_4],\\
[Y_2, Y_3] = 0, \quad [Y_2, Y_4] = a_1^2 a_3^2 a_3^3 U Y_1, \quad [Y_2, Y_5] = a_1^2 a_3^3 (a_3^2 U Y_2 + \frac{V}{2} Y_3 - Y_5),\\
[Y_3, Y_4] = -a_1^2 a_3^2 a_3^3 T U Y_1, \quad [Y_3, Y_5] = a_1^2 a_3^3 (-a_3^2 T U Y_2 - \frac{TV}{2} Y_3 + T Y_5),\\
[Y_4, Y_5] = a_1^2 a_3^3 \{\frac{a_3^2 T U V}{2} Y_1 + a_3^2 T U W Y_2 + [a_3^2 a_3^3 g^2 T - a_3^2 U^2 -\\
-a_1^2 a_3^3 k [(a_1^1 - a_3^3) g + a_1^2 h]] Y_3 - a_3^2 U Y_4 + a_3^3 [(a_1^1 - a_3^3) g + a_1^2 h] Y_5\},
\end{gathered} \tag{58}$$

where T, U and V are from (57).

Using the operators Y_2 and Y_3, which form by (56) and (58) a two-dimensional commutative Lie algebra, we obtain from (5) the Lie integrating factor of the form

$$\mu^{-1} = z(-a_1^2 x + (a_1^1 - a_3^3)y - a_3^2 z)(-a_1^2 x - (a_2^2 - a_3^3)y - a_3^2 z)\Phi,$$

where Φ is from (55).

Taking into account this expression and Theorem 1, we obtain the functional-independent integrals (54)–(55) of system (6). The conditions (19) and $\varkappa_2 q_1 \not\equiv 0$ from (8)–(9) imply that not all coefficients in this system are equal to zero. $\square$

Theorem 14. *Assume the coefficients of the linear part of system* (6) *satisfy the conditions* (20) *with* $\varkappa_2 q_1 \not\equiv 0$ *from* (8)–(9). *Then the general integral of this system with notations* $x = x^1,\ y = x^2,\ z = x^3$ *consists of two first integrals:*

$$F_1 \equiv (a_2^3 x - a_2^1 z)[a_2^3 y + (a_1^1 - a_2^2) z]^{-1} = C_1;$$
$$F_2 \equiv (a_2^3 x - a_2^1 z)^{a_1^1 - a_2^2 - a_3^3} [a_2^3 y - (a_1^1 - a_3^3) z]^{a_1^1} \Phi^{-2a_1^1 + a_2^2 + a_3^3} = C_2, \tag{59}$$

where

$$\Phi = a_2^3 (a_1^1 - a_2^2 - a_3^3)(a_1^1 + 2gx) + 2a_2^3 (a_2^1 g - a_3^3 h + a_2^3 k) y +$$
$$+2[-a_2^1 (a_1^1 - a_3^3) g + a_1^1 (a_1^1 - a_2^2 - a_3^3) h + a_2^2 (a_3^3 h - a_2^3 k)] z. \tag{60}$$

Proof. Let the coordinates of the operator (3) have the form (27). Then solving (4) we obtain for differential system (6) the operators ($x = x^1, y = x^2, z = x^3$):

$$
\begin{gathered}
Y_1 = (a_1^1 + 2gx)(a_2^3 x - a_2^1 z)\frac{\partial}{\partial x} + 2g(a_2^3 x - a_2^1 z)y\frac{\partial}{\partial y} + 2g(a_2^3 x - a_2^1 z)z\frac{\partial}{\partial z},\\
Y_2 = (a_1^1 + 2gx)(a_2^3 y + (a_1^1 - a_2^2)z)\frac{\partial}{\partial x} + 2g(a_2^3 y + (a_1^1 - a_2^2)z)y\frac{\partial}{\partial y} +\\
+2g(a_2^3 y + (a_1^1 - a_2^2)z)z\frac{\partial}{\partial z},\\
Y_3 = (2a_2^3 Wxy + a_1^1 a_2^1 Tz - 2(a_2^2 W - a_1^1 hT)xz)\frac{\partial}{\partial x} +\\
+[a_1^1 a_2^3 T + 2a_2^3 Wy - 2(a_2^2 W - a_1^1 hT)z]y\frac{\partial}{\partial y} + [a_1^1 a_2^3 T +\\
+2a_2^3 Wy - 2(a_2^2 W - a_1^1 hT)z]z\frac{\partial}{\partial z},\\
Y_4 = \{-2a_2^3(a_1^1 hT + a_3^3(a_3^3 h - a_2^3 k) - a_2^1(a_1^1 - a_2^2)g)xy +\\
+a_1^1 a_2^1(a_1^1 - a_2^2)Tz + 2(a_1^1 - a_2^2)[(a_1^1 - a_3^3)(a_3^3 h - a_2^3 k) - a_2^1 a_2^2 g]xz\}\frac{\partial}{\partial x} +\\
+\{a_1^1 a_2^3(a_3^3 - a_2^2)Ty - 2a_2^3[a_1^1 hT + a_3^3(a_3^3 h - a_2^3 k) - a_2^1 g(a_1^1 - a_2^2)]y^2 -\\
-a_1^1(a_1^1 - a_2^2)(a_1^1 - a_3^3)Tz + 2(a_1^1 - a_2^2)[(a_3^3 h - a_2^3 k)(a_1^1 - a_3^3) -\\
-a_2^1 a_2^2 g]yz\}\frac{\partial}{\partial y} + \{-a_1^1(a_2^3)^2 Ty - 2a_2^3[a_1^1 hT + +a_3^3(a_3^3 h - a_2^3 k) -\\
-a_2^1 g(a_1^1 - a_2^2)]yz + 2(a_1^1 - a_2^2)[(a_3^3 h - a_2^3 k)(a_1^1 - a_3^3) - a_2^1 a_2^2 g]z^2\}\frac{\partial}{\partial z},\\
Y_5 = \{-a_1^1 a_2^3[(a_1^1 - a_3^3)h + a_2^3 k]Tx + 2a_2^1 a_2^3 gWxy +\\
+a_1^1 a_2^1[a_2^1 g + (a_1^1 - a_3^3)h + a_2^3 k]Tz - 2a_2^1 g(a_2^2 W - a_1^1 hT)xz\}\frac{\partial}{\partial x} +\\
+[a_1^1 a_2^3(a_1^1 - a_3^3)gTx + a_1^1 a_2^1 a_2^3 gTy + 2a_2^1 a_2^3 gWy^2 -\\
-a_1^1 a_2^1(a_1^1 - a_3^3)gTz - 2a_2^1 g(a_2^2 W - a_1^1 hT)yz]\frac{\partial}{\partial y} +\\
+[a_1^1(a_2^3)^2 gTx + 2a_2^1 a_2^3 gWyz - 2a_2^1 g(a_2^2 W - a_1^1 hT)z^2]\frac{\partial}{\partial z},
\end{gathered}
\tag{61}
$$

where $W = a_2^1 g - a_3^3 h + a_2^3 k$, $T = a_1^1 - a_2^2 - a_3^3$.

These operators form the Lie algebra L_5 with the structure equations

$$
\begin{gathered}
a_2^1 T[Y_1, Y_2] = [Y_1, Y_4] = a_2^1[Y_2, Y_3] = -T[Y_3, Y_4] = -a_1^1 a_2^1 a_2^3 TY_2,\\
[Y_1, Y_3] = 0,\ [Y_1, Y_5] = a_1^1 a_2^3[(a_2^1 g + (a_1^1 - a_3^3)h + a_2^3 k)TY_1 - a_2^1 gY_3 + Y_5,\\
[Y_2, Y_5] = a_1^1 a_2^3[-(2a_1^1 - a_2^2 - a_3^3)gTY_1 - [a_2^1 g + (a_1^1 - a_3^3)h + a_2^3 k]TY_2 +\\
+(a_1^1 - a_2^2)gY_3 - gY_4],\quad [Y_2, Y_4] = a_1^1 a_2^3(a_1^1 - a_3^3)TY_2,\\
[Y_3, Y_5] = a_1^1 a_2^3 T\{-[a_2^1 g + (a_1^1 - a_3^3)h + a_2^3 k]TY_1 + a_2^1 gY_3 - Y_5\},\\
[Y_4, Y_5] = -a_1^1 a_2^3 T\{[a_2^1 g(a_1^1 - a_2^2) - [(a_1^1 - a_3^3)h + a_2^3 k](a_1^1 - a_3^3)]TY_1 +\\
+a_2^1(a_2^1 g + (a_1^1 - a_3^3)h + a_2^3 k)TY_2 + a_2^1(a_2^2 - a_3^3)gY_3 + a_2^1 gY_4 - (a_1^1 - a_3^3)Y_5\}.
\end{gathered}
\tag{62}
$$

If we use the operators Y_1 and Y_3, which form by (61) and (62) a two-dimensional commutative Lie algebra, we obtain from (5) the Lie integrating factor of the form

$$\mu^{-1} = (a_2^3 x - a_2^1 z)[a_2^3 y + (a_1^1 - a_2^2)z][a_2^3 y - (a_1^1 - a_3^3)z]\Phi,$$

where Φ is from (60).

Taking into account this expression and Theorem 1, we obtain the functional-independent integrals (59)–(60) of system (6). The conditions (20) and $\varkappa_2 q_1 \not\equiv 0$ from (8)–(9) imply that not all coefficients in this system are equal to zero. $\square$

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Received June 25, 2016

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(82), 2016, Pages 72–81
ISSN 1024–7696

Nontrivial convex covers of trees

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Abstract. We establish conditions for the existence of nontrivial convex covers and nontrivial convex partitions of trees. We prove that a tree G on $n \geq 4$ vertices has a nontrivial convex p-cover for every p, $2 \leq p \leq \varphi_{cn}^{max}(G)$. Also, we prove that it can be decided in polynomial time whether a tree on $n \geq 6$ vertices has a nontrivial convex p-partition, for a fixed p, $2 \leq p \leq \lfloor \frac{n}{3} \rfloor$.

Mathematics subject classification: 05A18, 05C05, 05C85, 68Q25.
Keywords and phrases: convexity, convex cover, convex partition, tree, graph.

1 Introduction

We denote by G a connected tree with vertex set $X(G)$, $|X(G)| = n$, and edge set $U(G)$, $|U(G)| = m$. We denote by $d(x, y)$ the *distance* between two vertices x and y of G [3]. The *diameter* of G, denoted $diam(G)$, is the length of the shortest path between the most distant vertices of G. The *neighborhood* of a vertex $x \in X$ is the set of all vertices $y \in X$ such that $x \sim y$, and it is denoted by $\Gamma(x)$.

We remind some notions defined in [1, 2]. The *metric segment*, denoted $\langle x, y\rangle$, is the set of all vertices lying on a shortest path between vertices $x, y \in X(G)$. A subset $S \subseteq X(G)$ is called *convex* if $\langle x, y\rangle \subseteq S$, for all $x, y \in S$.

By [6], a family of sets $\mathcal{P}(G)$ is called a *nontrivial convex cover* of a graph G if the following conditions hold:

1) every set of $\mathcal{P}(G)$ is convex in G;
2) every set S of $\mathcal{P}(G)$ satisfies inequalities: $3 \leq |S| \leq |X(G)| - 1$;
3) $X(G) = \bigcup_{Y \in \mathcal{P}(G)} Y$;
4) $Y \not\subseteq \bigcup_{\substack{Z \in \mathcal{P}(G) \\ Z \neq Y}} Z$ for every $Y \in \mathcal{P}(G)$.

If $|\mathcal{P}(G)| = p$, then this family is called a *nontrivial convex p-cover* of G. In particular, $\mathcal{P}(G)$ is called a *nontrivial convex partition* of G if it is a nontrivial convex cover of G and any two sets of $\mathcal{P}(G)$ are disjoint [6]. A nontrivial convex p-cover of G is called a *nontrivial convex p-partition* if it is a nontrivial convex partition of G.

Generally, convex p-covers and convex p-partitions of graphs are examined in [4–8]. Particularly, nontrivial convex p-cover and nontrivial convex p-partition are defined in [6], where it is proved that it is NP-complete to decide whether a graph has a nontrivial convex p-partition or a nontrivial convex p-cover for a fixed $p \geq 2$. Also, in [8] it is proved that it is NP-complete to decide whether a graph has any nontrivial convex partition. Further, there is specific interest in studying nontrivial convex p-covers and nontrivial convex p-partitions for different classes of graphs. In this paper we study nontrivial convex cover problem of trees.

The greatest $p \geq 2$ for which a graph G has a nontrivial convex p-cover is said to be the *maximum nontrivial convex cover number* $\varphi_{cn}^{max}(G)$. Similarly, we define the *maximum nontrivial convex partition number* $\theta_{cn}^{max}(G)$. A nontrivial convex cover that corresponds to $\varphi_{cn}^{max}(G)$ is denoted by $\mathcal{P}_{\varphi_{cn}^{max}}(G)$. In the same way we denote by $\mathcal{P}_{\theta_{cn}^{max}}(G)$ a nontrivial convex partition that corresponds to $\theta_{cn}^{max}(G)$.

A vertex $x \in X(G)$ is called *resident* in $\mathcal{P}(G)$ if x belongs to only one set of $\mathcal{P}(G)$. Let $L = [x^1, x^2, \ldots, x^k]$ be a vertex path of a tree G. By $R_L(x)$ we denote the set of vertices $v \in X(G)$ for which there is a path $L' = [x, \ldots, v]$ such that L' has no elements of L except x, where $x \in L$.

2 Existence of nontrivial convex covers

Recall that a *terminal vertex* of a tree G is a vertex of degree 1.

Lemma 1. *A tree G with $diam(G) \geq 3$ has a nontrivial convex cover.*

Proof. We know from [7] that a tree on $n \geq 4$ vertices has a nontrivial convex 2-cover. Since a tree with $diam(G) \geq 3$ has at least $n \geq 4$ vertices, we obtain that G with $diam(G) \geq 3$ has a nontrivial convex cover. □

Theorem 1. *Let G be a tree with $diam(G) \geq 3$. There exists a maximum nontrivial convex cover $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ such that every terminal vertex of G is resident in $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ and any two terminal vertices do not belong to the same set of $\mathcal{P}_{\varphi_{cn}^{max}}(G)$.*

Proof. From Lemma 1 we know that G has a nontrivial convex cover. Let $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ be a maximum nontrivial convex cover of G, where there is at least one terminal vertex x that is not resident in $\mathcal{P}_{\varphi_{cn}^{max}}(G)$. Since x is a terminal vertex of G and $diam(G) \geq 3$, we see that there is a vertex y adjacent to x that is adjacent to the set of nonterminal vertices S and to the set of terminal vertices S' of G such that $S \neq \varnothing$ and $S' \neq \varnothing$.

We consider two cases.

1) Suppose that S contains a vertex z that is not resident in $\mathcal{P}_{\varphi_{cn}^{max}}(G)$. Firstly, we replace vertex x by vertex z in every set of $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ that contains x. Secondly, we add a convex set $\{x, y, z\}$ to $\mathcal{P}_{\varphi_{cn}^{max}}(G)$. Further, we obtain a new nontrivial convex cover $\mathcal{P}(G)$ in which x is resident, where $|\mathcal{P}(G)| > |\mathcal{P}_{\varphi_{cn}^{max}}(G)|$. Hence, we get a contradiction.

2) Now suppose that every vertex of S is resident in $\mathcal{P}_{\varphi_{cn}^{max}}(G)$. Firstly, we choose a vertex z of S and a set Z of $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ that contains z. Secondly, we replace vertex x by vertex z in every set of $\mathcal{P}_{\varphi_{cn}^{max}}(G)\backslash\{Z\}$ which contains x. After, we add x and y to set Z. Finally, we get a new nontrivial convex cover $\mathcal{P}(G)$ in which x is resident, where $|\mathcal{P}(G)| = |\mathcal{P}_{\varphi_{cn}^{max}}(G)|$. On the other hand, if now set S' contains one more vertex that is not resident in $\mathcal{P}(G)$, then taking into account case 1) we obtain a contradiction.

Consequently, there exists a maximum nontrivial convex cover $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)$ such that every terminal vertex of G is resident in $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)$.

Now suppose that there are at least two terminal vertices x and y which belong to the same set S of $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)$.

Let us consider two cases.

1) Assume that $|S| \geq 4$. In this case, we replace set S in $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)$ by two convex sets $S' = S \backslash \{x\}$, $|S'| \geq 3$, and $S'' = S \backslash \{y\}$, $|S''| \geq 3$. Further, we obtain a new nontrivial convex cover $\boldsymbol{\mathcal{P}}(G)$ in which x and y belong to different sets, where $|\boldsymbol{\mathcal{P}}(G)| > |\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)|$. Whence, we have a contradiction.

2) Assume now that $|S| = 3$. In our case $S = \{x, y, z\}$, where $\Gamma(x) = \Gamma(y) = \{z\}$. As above, note that set $\Gamma(z) \backslash \{x, y\}$ contains at least one nonterminal vertex h.

If h is not resident in $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)$, then we replace S by two convex sets $\{x, z, h\}$ and $\{y, z, h\}$. Further, we obtain a new nontrivial convex cover $\boldsymbol{\mathcal{P}}(G)$ in which x and y belong to different sets, where $|\boldsymbol{\mathcal{P}}(G)| > |\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)|$. Whence, we have a contradiction.

If all nonterminal vertices of $\Gamma(z) \backslash \{x, y\}$ are resident in $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)$, then we choose a set H that contains h. Further, we subtract x from S and add it to H. Also, we add h to S and z to H. Consequently, we obtain a new nontrivial convex cover $\boldsymbol{\mathcal{P}}(G)$ in which x and y belong to different sets, where $|\boldsymbol{\mathcal{P}}(G)| = |\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)|$.

It follows that any two terminal vertices do not belong to the same set of $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)$. $\square$

As a consequence of Theorem 1, we obtain 3 corollaries.

Corollary 1. *Let G be a tree with $diam(G) \geq 3$ and p terminal vertices. Then, $\varphi_{cn}^{max}(G) \geq p$.*

Corollary 2. *Let G be a tree with $diam(G) \geq 3$ and p terminal vertices, where every nonterminal vertex of G is adjacent to at least one terminal vertex. Then, $\varphi_{cn}^{max}(G) = p$.*

Corollary 3. *Let G be a tree with $3 \leq diam(G) \leq 5$ and p terminal vertices. Then, $\varphi_{cn}^{max}(G) = p$.*

Theorem 2. *A tree G on $n \geq 4$ vertices has a nontrivial convex p-cover, for every p, $2 \leq p \leq \varphi_{cn}^{max}(G)$.*

Proof. It is know that a tree on $n \geq 4$ vertices has a nontrivial convex cover [7]. Let G be a tree on $n \geq 4$ vertices and let $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)$ be a maximum nontrivial convex cover of G. If $\varphi_{cn}^{max}(G) = 2$, then the theorem is proved. Let us analyze case $\varphi_{cn}^{max}(G) \geq 3$. We use the following procedure. We select two sets X_1 and X_2 of $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)$ such that $x_1 \in X_1$ and $x_2 \in X_2$, where x_1 is adjacent to x_2. Since union of sets X_1 and X_2 is convex in G, excluding from $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)$ sets X_1, X_2 and adding set $X_1 \cup X_2$, we obtain a new family $\boldsymbol{\mathcal{P}}(G)$ that covers G by $p = \varphi_{cn}^{max}(G) - 1$ nontrivial convex sets. If $p = 2$, then the theorem is correct. Conversely, if $p \geq 3$, then repeating $\varphi_{cn}^{max}(G) - 3$ times this procedure for $\boldsymbol{\mathcal{P}}(G)$ we obtain a nontrivial convex 2-cover of G. Consequently, the theorem is proved. $\square$

Next, we analyze nontrivial convex partitions of trees. The following two families of trees $\mathcal{A}$ and $\mathcal{B}$ are needed for the sequel.

$\mathcal{A}$ is a family of trees G which satisfy the following conditions:

1) $X(G) = \{x, y, x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_{k'}\}$, where $k, k' \geq 2$;

2) $U(G) = \{(x, y)\} \cup \bigcup_{i=1}^{k}\{(x, x_i)\} \cup \bigcup_{i=1}^{k'}\{(y, y_i)\}$.

$\mathcal{B}$ is a family of trees G which are constructed as follows:

1) We choose $k \geq 0$, $k' \geq 2$, $k_1 \geq 2$ and for every i, $2 \leq i \leq k'$, we select $k_i \geq 1$;

2) If $k \geq 1$, then we get $X = \{x_0\} \cup \bigcup_{i=1}^{k}\{x_i\}$ and $U = \bigcup_{i=1}^{k}\{(x_0, x_i)\}$, otherwise we get $X = \{x_0\}$ and $U = \varnothing$;

3) We obtain sets $X(G) = X \cup \bigcup_{i=1}^{k'} \bigcup_{j=0}^{k_i}\{x_i^j\}$ and $U(G) = U \cup \bigcup_{i=1}^{k'}\{(x_0, x_i^0)\} \cup \bigcup_{i=1}^{k'} \bigcup_{j=1}^{k_i}\{(x_i^0, x_i^j)\}$.

It can easily be checked that diameter of all trees of $\mathcal{A}$ is 3, and diameter of all trees of $\mathcal{B}$ is 4. Moreover, every tree of $\mathcal{A}$ and every tree of $\mathcal{B}$ has at least 6 vertices.

Algorithm 1. Determines whether a tree belongs to one of families: $\mathcal{A}$, $\mathcal{B}$.

Input: *A tree* G*.*

Output: *YES-*$\mathcal{A}$*:* G *belongs to* $\mathcal{A}$*, or YES-*$\mathcal{B}$*:* G *belongs to* $\mathcal{B}$*, or NO:* G *does not belong to any of the families.*

Step 1) If $|X(G)| \leq 5$*, then return NO.*

Step 2) Compute $diam(G)$*. If* $diam(G) \leq 2$ *or* $diam(G) \geq 5$*, then return NO; otherwise, if* $diam(G) = 4$*, then go to Step 4).*

Step 3) Choose two different vertices $x, y \in X(G)$ *such that* $|\Gamma(x)| \geq 2$ *and* $|\Gamma(y)| \geq 2$*. Next, if* $|\Gamma(x)| \geq 3$ *and* $|\Gamma(y)| \geq 3$*, then return YES–*$\mathcal{A}$*; otherwise return NO.*

Step 4) Check whether there exist two different terminal vertices $x, y \in X(G)$ *such that* $\Gamma(x) \cap \Gamma(y) \neq \varnothing$ *and there is a terminal vertex* $z \in X(G)$*, where* $d(x, z) = diam(G)$*. If there exist such vertices* $x, y \in X(G)$*, then return YES–*$\mathcal{B}$*; otherwise return NO.*

Theorem 3. *Algorithm 1 determines in time* $O(n^3)$ *whether a tree* G *belongs to one of families:* $\mathcal{A}$*,* $\mathcal{B}$*.*

Proof. Correctness of the algorithm results from structure of trees of families $\mathcal{A}$ and $\mathcal{B}$. Step 1) runs in constant time. If we use Floyd–Warshall algorithm for finding the diameter of a graph, then the complexity of step 2) is $O(n^3)$. It is clear that step 3) is executed in $O(n)$ time. Since Floyd–Warshall algorithm is executed in the step 2), we know all pairs of vertices for which distance is equal to $diam(G)$. Further, step 4) runs in $O(n^2)$ time. Based on the mentioned facts, the execution time of the algorithm is $O(n^3)$. ∎

Theorem 4. *A tree G has a nontrivial convex 2-partition if and only if one of the following conditions holds:*

1) $diam(G) \geq 5$;

2) $G \in \mathcal{A}$;

3) $G \in \mathcal{B}$.

Proof. It is clear that if a tree G has a nontrivial convex 2-partition, then inequality $n \geq 6$ holds. Let us analyze nontrivial convex 2-partition of G in dependency on its diameter.

Suppose $diam(G) = 2$. Here G is a star graph. It can simply be verified that a star graph has no nontrivial convex 2-partition.

Suppose $diam(G) = 3$. We choose two vertices $x, x' \in X(G)$ such that there is a path $L = [x, y, z, x']$ and length of L is equal to diameter of G. Evidently, L is a unique path between vertices x and x' and vertices x, x' are terminal, i.e., $\Gamma(x) = y$ and $\Gamma(x') = z$. From relation $n \geq 6$, it follows that G contains at least two vertices different from x, y, z, x'. Assume that $v \in X(G)$ is different from vertices x, y, z, x', and $v \in R_L(y)$ such that $d(y, v) \geq 2$, or $v \in R_L(z)$ and $d(z, v) \geq 2$. Further, we obtain a contradiction, because $d(y, x') = d(z, x) = 2$ and length of paths $L^1 = [x', z, y, \dots, v]$, $L^2 = [x, y, z, \dots, v]$ is greater then or equal to 4. Consequently, all vertices of G different from x, y, z, x' are adjacent only to y or to z. It can easily be checked that if y is adjacent only to x and z, or z is adjacent only to x' and y, then G has no nontrivial convex 2-partition. In the converse case G has a nontrivial convex 2-partition:

$$\mathcal{P}(G) = \{\{x, y\} \cup R_L(y), \{z, x'\} \cup R_L(z)\}.$$

In other words, if $diam(G) = 3$, then G has a nontrivial convex 2-partition if and only if $G \in \mathcal{A}$.

Suppose $diam(G) = 4$. We choose two vertices $x, x' \in X(G)$ such that there is a path $L = [x, y, z, h, x']$. Length of the L is equal to diameter of G and vertices x and x' are terminal. Since $n \geq 6$, tree G contains at least one vertex v different from x, y, z, h, x'. If v is adjacent to y or to h, then G has a nontrivial convex 2-partition:

$$\mathcal{P}(G) = \{\{x, y\} \cup R_L(y), \{z, h, x'\} \cup R_L(z) \cup R_L(h)\} \text{ or}$$

$$\mathcal{P}(G) = \{\{x, y, z\} \cup R_L(y) \cup R_L(z), \{h, x'\} \cup R_L(h)\}, \text{ respectively.}$$

Assume that there are no vertices different from x, y, z, h, x' which are adjacent to y or to h. Then, there exist vertices z' different from y and h which are adjacent to z. If we have $|\Gamma(z')| = 1$ or $|\Gamma(z')| = 2$, for all such z', then it is not hard to check that G has no nontrivial convex 2-partition. Now assume that there are at least two vertices z'' and z''' different from z and adjacent to z', i.e., $|\Gamma(z')| \geq 3$. In this case, we obtain a path $L = [z'', z', z, y, x]$. As mentioned above, it follows that G has a nontrivial convex 2-partition. Equivalently, if $diam(G) = 4$, then G has a nontrivial convex 2-partition if and only if $G \in \boldsymbol{\mathcal{B}}$.

Suppose $diam(G) \geq 5$. There are two vertices x and x' in G such that $d(x, x') = diam(G)$. Let $L = [x, x^1, x^2, \ldots, x^k, x']$, $k \geq 4$, be a path between x and x'. L contains at least 6 vertices. Moreover, L is a unique path between x and x'. Hence, paths $[x, x^1, x^2]$ and $[x^3, \ldots, x^k, x']$ generate a nontrivial convex 2-partition of G:

$$\boldsymbol{\mathcal{P}}(G) = \{\{x\} \cup \bigcup_{i=1}^{2} R_L(x^i), \{x'\} \cup \bigcup_{i=3}^{k} R_L(x^i)\}.$$

The theorem is proved. □

Theorem 5. *If a tree G on $n \geq 6$ vertices has a nontrivial convex partition, then G has a nontrivial convex p-partition, for every p, $2 \leq p \leq \theta_{cn}^{max}(G)$.*

Proof. If a tree G has a nontrivial convex partition, then there is a maximum nontrivial convex partition $\boldsymbol{\mathcal{P}}_{\theta_{cn}^{max}}(G)$. If $\theta_{cn}^{max}(G) = 2$, then the theorem is proved. If $\theta_{cn}^{max}(G) \geq 3$, then repeating $\theta_{cn}^{max}(G) - 2$ times the procedure described in proof of Theorem 2 we obtain a nontrivial convex 2-partition of G. Hence, G has a nontrivial convex p-partition, for every p, $2 \leq p \leq \theta_{cn}^{max}(G)$. □

The following corollaries are true.

Corollary 4. *If a tree G on $n \geq 6$ vertices has a nontrivial convex partition, then G has a nontrivial convex 2-partition.*

Corollary 5. *A tree G has a nontrivial convex p-partition, for every p, $2 \leq p \leq \theta_c^{max}(G)$, if and only if one of the following conditions holds:*

1) $diam(G) \geq 5$;

2) $G \in \boldsymbol{\mathcal{A}}$;

3) $G \in \boldsymbol{\mathcal{B}}$.

3 Determination of nontrivial convex partitions

Let C be the set of all terminal vertices of G. Let x be a vertex of G for which $|\Gamma(x) \cap C| \geq 2$ or there is another vertex $y \in \Gamma(x)$ such that $\Gamma(y) = \{x, z\}$, $z \in C$.

For x that satisfies the announced properties we define the set:

$$S_x = \{x\} \cup \{v \in X(G) : v \in \Gamma(x) \cap C\} \cup \{v_1, v_2 \in X(G) : \Gamma(v_1) = \{x, v_2\}, v_2 \in C\}.$$

The set S_x is called a *nontrivial terminal set* of G. Note that S_x is a nontrivial convex set of G. We say that a terminal vertex z of a tree G corresponds to a nontrivial terminal set S_x of G if S_x contains z.

Let $\boldsymbol{\mathcal{S}}(G)$ be the family of all nontrivial terminal sets of G.

Lemma 2. *All nontrivial terminal sets of* $\boldsymbol{\mathcal{S}}(G)$ *are disjoint.*

Proof. Suppose that there are at least two different nontrivial terminal sets S_x and S_y of $\boldsymbol{\mathcal{S}}(G)$ such that $S_x \cap S_y \neq \varnothing$. By the definition of nontrivial terminal set, we have $x = y$ and consequently $S_x = S_y$. Whence, we obtain a contradiction. □

Lemma 3. $\boldsymbol{\mathcal{S}}(G)$ *is unique for* G.

Proof. Correctness of the lemma results from the definition of nontrivial terminal set and Lemma 2. □

Lemma 4. *Every set of* $\boldsymbol{\mathcal{S}}(G)$ *belongs to exactly one set of* $\boldsymbol{\mathcal{P}}_{\theta_{cn}^{max}}(G)$ *such that any two nontrivial terminal sets of* $\boldsymbol{\mathcal{S}}(G)$ *do no belong to the same set of* $\boldsymbol{\mathcal{P}}_{\theta_{cn}^{max}}(G)$.

Proof. From the definition of nontrivial terminal set and definition of nontrivial convex partition, it follows that every set of $\boldsymbol{\mathcal{S}}(G)$ belongs to exactly one set of $\boldsymbol{\mathcal{P}}_{\theta_{cn}^{max}}(G)$. Suppose that there is a set C of $\boldsymbol{\mathcal{P}}_{\theta_{cn}^{max}}(G)$ that contains at least two different nontrivial terminal sets of G. Let $\boldsymbol{\mathcal{S}}_C$ be the family of all nontrivial terminal sets which are in C and $k = |\boldsymbol{\mathcal{S}}_C| \geq 2$. By Lemmas 2 and 3, we know that $\boldsymbol{\mathcal{S}}(G)$ is unique for G and all nontrivial terminal sets are disjoint. Further, we separate C into disjoint nontrivial convex sets $S_1, S_1, ..., S_k$, where every set contains exactly one nontrivial terminal set of $\boldsymbol{\mathcal{S}}_C$. We select a vertex x from all vertices of C which remain uncovered by new nontrivial convex sets such that x is adjacent to a vertex y, $y \in S$, $S \in \{S_1, S_1, ..., S_k\}$, and further add x to S. If some uncovered vertices remain, then we repeat the above procedure. Since $k \geq 2$, we get a new convex cover $\boldsymbol{\mathcal{P}}(G)$ of G such that $|\boldsymbol{\mathcal{P}}(G)| > |\boldsymbol{\mathcal{P}}_{\theta_{cn}^{max}}(G)|$. Hence, we have a contradiction. □

Lemma 5. *A tree* G *on* $n \geq 3$ *vertices with* $2 \leq diam(G) \leq 4$ *has at least one nontrivial terminal set.*

Proof. From the definition of nontrivial terminal set, we get that every tree G of order $n \geq 3$ with $diam(G) = 2$ contains exactly one nontrivial terminal set $S_x = X(G)$. It can easily be checked that a tree $G \in \boldsymbol{\mathcal{A}}$ has exactly two nontrivial terminal sets, and a tree $G \in \boldsymbol{\mathcal{B}}$ has at least two nontrivial terminal sets. Similarly, if a tree G with $diam(G) = 3$ does not belong to $\boldsymbol{\mathcal{A}}$, or $diam(G) = 4$ and $G \notin \boldsymbol{\mathcal{B}}$, then G has exactly one nontrivial terminal set $S_x = X(G)$. □

Lemma 6. *A tree* G *with* $diam(G) \geq 5$ *has at least two nontrivial terminal sets.*

Proof. Let G be a tree with $diam(G) \geq 5$. Let x and y be two terminal vertices such that $d(x, y) = diam(G)$. Assume that x does not correspond to any nontrivial terminal set. By the definition of nontrivial terminal set, we see that x is adjacent to a vertex z that is adjacent to at least two vertices different from x and all of them are nonterminal. Let $z^1, z^2, \ldots, z^k$, where $k \geq 2$, be vertices different from x and adjacent to z. Path between x and y contains exactly one vertex $z' \in \{z^1, z^2, \ldots, z^k\}$. Since $z^1, z^2, \ldots, z^k$ are nonterminal vertices, to every vertex $z'' \in \{z^1, z^2, \ldots, z^k\} \backslash \{z'\}$ corresponds a vertex z^* different from z such that z^* is adjacent to z''. Since for every two vertices of G there is only one path that connects them, this yields that for every z^* we get $d(z^*, y) > diam(G)$. Consequently, we obtain a contradiction. Similarly, we get a contradiction if assume that y does not correspond to any nontrivial terminal set. Since $diam(G) \geq 5$, vertices x and y correspond to different nontrivial terminal sets. Hence, a connected tree G with $diam(G) \geq 5$ has at least two nontrivial terminal sets. $\square$

Algorithm 2. Determines $\boldsymbol{\mathcal{S}}(G)$ for a tree G.

Input: *A tree G.*

Output: $\boldsymbol{\mathcal{S}}(G)$.

Step 1) Fix set $\boldsymbol{\mathcal{S}}(G) = \varnothing$.

Step 2) Determine all terminal vertices C of G.

Step 3) Go through all vertices $x \in X(G)\backslash C$. If for a vertex x of G we have $|\Gamma(x) \cap C| \geq 2$ or there is another vertex $y \in \Gamma(x)$ such that $\Gamma(y) = \{x, z\}$, where $z \in C$, then we define the set $S_x = \{x\} \cup \{v \in X(G) : v \in \Gamma(x) \cap C\} \cup \{v_1, v_2 \in X(G) : \Gamma(v_1) = \{x, v_2\}, v_2 \in C\}$ and then add it to $\boldsymbol{\mathcal{S}}(G)$.

Step 4) Return $\boldsymbol{\mathcal{S}}(G)$.

Theorem 6. *Algorithm 2 determines family of nontrivial terminal sets $\boldsymbol{\mathcal{S}}(G)$ of a tree G in time $O(n^2)$.*

Proof. Correctness of the algorithm results from Lemmas 2, 3, 5 and 6. Clearly, steps 1) and 4) run in constant time. The step 2) operates in $O(n)$ and the step 3) is executed in $O(n^2)$ time. Further, the execution time of the algorithm is $O(n^2)$. $\square$

Let $\boldsymbol{\mathcal{F}}(G)$ be a family of subtrees that is obtained after elimination of all nontrivial terminal sets of $\boldsymbol{\mathcal{S}}(G)$ from a tree G.

Theorem 7. *The following relation holds:*

$$\theta_{cn}^{max}(G) = \begin{cases} |\boldsymbol{\mathcal{S}}(G)| + \sum_{G' \in \boldsymbol{\mathcal{F}}(G)} \theta_{cn}^{max}(G'), & \text{if } |X(G)| \geq 3; \\ 0, & \text{if } 0 \leq |X(G)| \leq 2. \end{cases}$$

Proof. By Lemma 4, we conclude that through the elimination of all nontrivial terminal sets of $\boldsymbol{\mathcal{S}}(G)$ from G, in fact, we eliminate minimal nontrivial convex sets of G which contain nontrivial terminal sets. Besides, after elimination of all nontrivial terminal sets of $\boldsymbol{\mathcal{S}}(G)$ from G we obtain a family of subtrees $\boldsymbol{\mathcal{F}}(G)$ such that some of them also contain nontrivial terminal sets.

If $0 \leq |X(G)| \leq 2$, then evidently $\theta_{cn}^{max}(G) = 0$. In the contrary case, if $|X(G)| \geq 3$, then taking into account Lemmas 2 – 6, we obtain:

$$\theta_{cn}^{max}(G) = |\boldsymbol{\mathcal{S}}(G)| + \sum_{G' \in \boldsymbol{\mathcal{F}}(G)} \theta_{cn}^{max}(G').$$

The theorem is proved. $\square$

Next, we propose recursive procedure $Max\theta(G)$ that determines the number $\theta_{cn}^{max}(G)$ of a tree G. After, we prove that this procedure executes in polynomial time.

$Max\theta(G)$

Input: *A tree G.*

Output: *$\theta_{cn}^{max}(G)$.*

Step 1) If $0 \leq |X(G)| \leq 2$, then return 0.

Step 2) Apply Algorithm 2, i.e., determine $\boldsymbol{\mathcal{S}}(G)$, remove every nontrivial terminal set of $\boldsymbol{\mathcal{S}}(G)$ from G and obtain $\boldsymbol{\mathcal{F}}(G)$.

Step 3) For every tree G' of $\boldsymbol{\mathcal{F}}(G)$ apply procedure $Max\theta(G')$ and after return the number $\theta_{cn}^{max}(G) = |\boldsymbol{\mathcal{S}}(G)| + \sum_{G' \in \boldsymbol{\mathcal{F}}(G)} Max\theta(G')$.

Theorem 8. *Procedure $Max\theta(G)$ determines the number $\theta_{cn}^{max}(G)$ of a tree G in time $O(n^3)$.*

Proof. From Theorem 7, we know that for a tree G procedure $Max\theta(G)$ returns the number $\theta_{cn}^{max}(G)$. By Theorem 6 we obtain that in general case the processing time of procedure $Max\theta(G)$ is:

$$T(n) = \sum_{i=1}^{k} T(n_i) + O(n^2),$$

where $\sum_{i=1}^{k} n_i \leq n - 6$ and $k \geq 1$.

The worst behavior of procedure $Max\theta(G)$ occurs when in every examined tree there are exactly two nontrivial terminal sets which consist of three elements such that after their elimination a single subtree remains. In this case, processing time of $Max\theta(G)$ is:

$$T(n) = T(n-6) + O(n^2).$$

Using arithmetic progression, we get $T(n) = O(n^3)$. Finally, the procedure $Max\theta(G)$ determines number $\theta_{cn}^{max}(G)$ in time $O(n^3)$. $\square$

Corollary 6. *It can be decided in time $O(n^3)$ whether a tree G on $n \geq 6$ vertices has a nontrivial convex p-partition, for a fixed p, $2 \leq p \leq \lfloor \frac{n}{3} \rfloor$.*

4 Conclusion

In this paper we establish conditions for the existence of nontrivial convex covers and nontrivial convex partitions of trees. We prove that a tree G on $n \geq 4$ vertices has a nontrivial convex p-cover for every p, $2 \leq p \leq \varphi_{cn}^{max}(G)$. In addition, we prove that if a tree G has a nontrivial convex partition, then G has a nontrivial convex p-partition for every p, $2 \leq p \leq \theta_{cn}^{max}(G)$. Also, we propose polynomial algorithm that recognizes whether a tree belongs to one of families $\boldsymbol{\mathcal{A}}$ or $\boldsymbol{\mathcal{B}}$. Finally, we develop polynomial algorithm for determining the number $\theta_{cn}^{max}(G)$ of a tree G. But the general convex cover problem of trees remains the task of further research.

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Received July 27, 2016

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(82), 2016, Pages 82–98
ISSN 1024–7696

An investment problem under multicriteriality, uncertainty and risk

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Abstract. The strong stability radius of the multicriteria investment Boolean problem with the Savage risk criteria is investigated. The problem is to find the set of Pareto optimal portfolios. Upper and lower bounds of such a radius are derived for the case where different Hölder metrics are defined in the three problem parameters spaces.

Mathematics subject classification: 90C09, 90C29, 90C31, 90C47.
Keywords and phrases: Multicriteria optimization, Savage's risk criteria, set of Pareto optimal portfolios, strong stability radius, Hölder metric.

1 Introduction

Most of business and management decisions are made within uncertain and risky environment. Investment managing problems are as a type of problems with uncertainty of the initial data. Any separate investment asset has higher level of risk and less return than the portfolio of those assets and there is no reason to invest in one particular asset. Creating the portfolio by diversification and mixing variety of investments an investor reduces the riskiness of the portfolio. Following Markowitz's portfolio theory the investor plotting on the graph an efficient frontier depending on various pairs risk and expected return chooses portfolio drawing on individual risk-return preferences. It gives ability to construct a portfolio with the same expected return and less risk.

Based on Markowitz's portfolio optimization concept [1, 2] a multicriteria Boolean discrete variant of portfolio optimization holding constant expected return and minimize risk of portfolios consisting of the investment projects is considered. This problem is viewed as a problem of finding the Pareto optimal portfolios set using Savage's risk criteria. It means that a portfolio is a Pareto optimal one, when its total level of risk, i.e. the sum of all risks of the projects included in the portfolio is minimal in the worst market situation for one type of the risk. Unlike classical modern portfolio theory where a portfolio consisting of percentage of each asset there are several investment projects composes the portfolio. This model can be considered as a discrete variant of Markowitz problem with encoding a portfolio selection where for each project the risk matrix is constructed for several market states related to each type of the risk.

The model formulation requires statistical and expert evaluation of risks (e.g. financial or ecological) [3] to be specified as the initial data. The collected data usually contain computational errors and inaccuracies. It leads to the situation when the initial data representing risk values are inaccurate and uncertain. One of the key questions while analyzing an uncertain data is about the limiting level of the initial data changes (perturbations) which do not violate the optima. The quantitative measure of the data perturbation level is known as the stability radius, which concept is widely presented and analyzed in the recent literature focusing on finding analytical expressions and bounds (see e.g. [4–8]). Similar approaches were also developed in parallel in scheduling theory (see [9]). Analytical formulas are pairwise comparisons of solutions that reflect the specific of the selected principle of optimality, the structure of global perturbation of this problem and the structure of the solution set, namely Boolean portfolios. The evaluation of the stability properties is a global property itself. The particular definition of the stability radius concept depends on chosen optimality principles (if the problem is multicriteria), uncertain data and a type of distance metric used to measure the closeness in problem parameter spaces. Various types of metrics allow to consider a specific of problem parameters perturbation. So in the case of Chebyshev metric l_∞ the maximum changes in the initial data take into account only that allow perturbations to be independent. In the case of Manhattan metric l_1 every change of the initial data can be monitored in total. Hölder metric $l_p, 1 \leq p \leq \infty$, is the metric with a parameter and includes such extreme cases as Chebyshev metric l_∞, Manhattan metrics l_1 and also Euclidean metric l_2. Thus, using Hölder metric l_p for obtaining the stability radius depending on the properties of the initial data the control of perturbations can be varied.

Along with a quantitative approach to analyzing admissible level of the initial data perturbations, a qualitative approach is developed in parallel. This approach concentrates on specifying analytical conditions which will guarantee some certain pre-specified behavior of the optimal solutions set. Within this approach authors focus on finding necessary and sufficient conditions of different types of the problem stability (see the monograph [10], the reviews [11, 12], and the articles [13–17]), on revealing relations between different types of stability [18, 19], and also on finding and describing the stability region of an optimal solution [20].

This work continues started in [21–29] researches of different types of stability of vector nonlinear investment problems. Thus the work follows the approach of obtaining qualitative characteristics of stability. One of such characteristics, called commonly a stability radius of a problem, is defined as a limit level of problem parameters perturbations in the metric space such that pre-specified property of the problem solution set is preserved. Perturbing parameters usually are coefficients of the scalar or vector criteria.

Stability of a multicriteria discrete optimization problem of finding the Pareto set is commonly understood (see e.g. [10]) as discrete analog of the Hausdorff upper semicontinuity property of the point-to-set mapping that defines the Pareto choice function. Thus, the stability property means that there exists a neighborhood of the initial problem parameters in which appearance of a new Pareto optimum is impossible. In other words, the Pareto set inside this neighborhood can only narrow in the result of the problem parameters perturbations. Relaxation of this requirements leads to a new stability type. It is understood as existence of such neighborhood of the initial problem parameters in which appearance of new Pareto optimums is possible; but at least one Pareto optimal solution (not necessarily one and the same) preserves its optimality for any perturbation. Following the terminology of [30–33], we call such a stability strong.

Strong stability was first time investigated in [34] for a one-criterion (scalar) linear trajectorial problem. Later in [32, 35, 36] the lower and upper bounds of this type stability radius were derived for the multicriteria linear Boolean integer programming problem. The article [37] is devoted to obtaining similar bounds for the vector investment problem with the Wald criteria. We also point out the work [30] where necessary and sufficient conditions of the strong stability are found for the multicriteria problem of threshold functions minimization. The mentioned results were obtained in the case of the Chebyshev metric l_∞ in the problem parameter spaces.

In this paper the lower and upper bounds of the strong stability radius are found for the multicriteria investment problem with the Savage risk criteria in the case of different Hölder metrics in the three problem parameter spaces. Separately we investigate a particular case of the investment problem with the linear criteria, i.e. the case when the state of the financial market does not doubt the investor.

2 Problem formulation and basic definitions

Consider a mutlicriteria discrete analogue of the Markowitz portfolio management problem [1], which is based on diversification as a tool of risk minimization. Let

$N_n = \{1, 2, \ldots, n\}$ be a variety of alternative investment projects (assets);

N_m be a set of possible financial market states (market situations, scenarios);

N_s be a set of possible risks;

r_{ijk} be a numerical measure of economic risk of type $k \in N_s$, which the investor may face if (s)he chooses project $j \in N_n$ assuming that the market state is $i \in N_m$;

$R = [r_{ijk}] \in \mathbf{R}^{m\times n\times s}$;

$x = (x_1, x_2, \ldots, x_n)^T \in \mathbf{E}^n$ be an investment portfolio, where $\mathbf{E} = \{0, 1\}$,

$$x_j = \begin{cases} 1, & \text{if investor chooses project } j, \\ 0 & \text{otherwise;} \end{cases}$$

$X \subset \mathbf{E}^n$ be a set of all admissible investment portfolios, i.e. those realizations which provide expected total income and do not exceed the budget;

$\mathbf{R}^m$ be a financial market state space;

$\mathbf{R}^n$ be a project space;

$\mathbf{R}^s$ be a risk space.

The presence of a risk factor is integral feature of financial market functioning. One can find information about risk measurement methods and their classification in [38]. The last trend is to quantify risks using five R: robustness, redundancy, resourcefulness, response and recovery. The natural target of any investor is to minimize different types of risks. It creates a motivation for multicriteria analysis within risk modelling. It leads to the usage of multicriteria decision making tools [39, 40].

Assume that the efficiency of a chosen portfolio (Boolean vector) $x = (x_1, x_2, ..., x_n)^T \in X,\ |X| \geq 2$, is evaluated by a vector objective function

$$f(x,R) = (f_1(x,R_1), f_2(x,R_2)\ldots, f_s(x,R_s))^T,$$

each partial objective represents minmax Savage's risk criterion (extreme pessimism) [41]

$$f_k(x,R_k) = \max_{i\in N_m} r_{ik}x = \max_{i\in N_m} \sum_{j\in N_n} r_{ijk}x_j \to \min_{x\in X}, \quad k \in N_s,$$

where $R_k \in \mathbf{R}^{m\times n}$ is the k-th cut $R = [r_{ijk}] \in \mathbf{R}^{m\times n\times s}$ with rows $r_{ik} = (r_{i1k}, r_{i2k}, \ldots, r_{ink}) \in \mathbf{R}^n,\ i \in N_m$.

Thus, if an investor chooses the Savage risk (bottleneck) criterion [42, 43], then (s)he optimizes the total profit of the selected portfolio in the worst (maximum risk) case. This approach takes place when the decision maker has pessimistic expectations and wants to achieve the guaranteed result. In other words, the investor adhere to the wise rule that suggests to expect the worst case.

A problem of finding the Pareto optimal (efficient) portfolios is referred to as a multicriteria investment Boolean problem with the Savage risk criteria and is denoted $Z^s_m(R), s \in \mathbf{N}$. The set of Pareto optimal portfolios is defined as follows

$$P^s(R) = \{x \in X\ :\ X(x,R) = \emptyset\},$$

where

$$X(x,R) = \{x' \in X\ :\ f(x,R) \geq f(x',R)\ \&\ f(x,R) \neq f(x',R).$$

It is evident that $P^s(R) \neq \emptyset$ for any matrix $R \in \mathbf{R}^{m\times n\times s}$. Let us note that the problem $Z^s_m(R)$ can be interpreted as "the worst case optimization".

Let the Hölder metrics (generally speaking different) l_p, l_q, and $l_r,\ p,q,r \in [1,\infty]$, be defined in the spaces $\mathbf{R}^n, \mathbf{R}^m$, and $\mathbf{R}^s$ correspondingly. It means that the norm of the matrix $R \in \mathbf{R}^{m\times n\times s}$ is the number

$$\|R\|_{pqr} = \left\|(\|R_1\|_{pq},\ \|R_2\|_{pq}, \ldots,\ \|R_s\|_{pq})\right\|_r,$$

$$\|R_k\|_{pq} = \left\|(\|r_{1k}\|_p,\ \|r_{2k}\|_p, \ldots,\ \|r_{mk}\|_p)\right\|_q,\ k \in N_s.$$

Recall that the Hölder norm l_p in the space $\mathbf{R}^n$ is defined as follows

$$||a||_p = \begin{cases} \left(\sum_{j\in N_n} |a_j|^p\right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \max\{|a_j|\ :\ j \in N_n\}, & \text{if } p = \infty, \end{cases}$$

where $a = (a_1, a_2, \dots, a_n)^T \in \mathbf{R}^n$.

It is easy to see that for any $p, q, r \in [1, \infty]$, the following inequalities hold

$$||r_{ik}||_p \leq ||R_k||_{pq} \leq ||R||_{pqr}, \quad i \in N_m, \quad k \in N_s. \tag{1}$$

Following [30–37, 44, 45], the strong stability radius (in terminology of [10] – T_1-stability radius) of the problem $Z^s_m(R)$, $s \in \mathbf{N}$, with the Hölder norms l_p, l_q, and l_r in the spaces $\mathbf{R}^m, \mathbf{R}^n$, and $\mathbf{R}^s$ correspondingly is the number

$$\rho = \rho^s_m(p, q, r) = \begin{cases} \sup \Xi_{pqr}, & \text{if } \Xi_{pqr} \neq \emptyset, \\ 0, & \text{if } \Xi_{pqr} = \emptyset, \end{cases}$$

where

$$\Xi_{pqr} = \{\varepsilon > 0 \ : \ \forall R' \in \Omega_{pqr}(\varepsilon) \quad (P^s(R) \cap P^s(R + R') \neq \emptyset)\},$$

$$\Omega_{pqr}(\varepsilon) = \{R' \in \mathbf{R}^{m \times n \times s} \ : \ ||R'||_{pqr} < \varepsilon\}.$$

Here $\Omega_{pqr}(\varepsilon)$ is the set of perturbing matrixes R' with cuts $R'_k \in \mathbf{R}^{m \times n}$, $k \in N_s$; $P^s(R + R')$ is the Pareto set of the perturbed problem $Z^s(R + R')$; $||R'||_{pqr}$ is the norm of the matrix $R' = [r'_{ijk}]$.

Thus, the strong stability radius of the problem $Z^s_m(R)$ is a limit level of the matrix R elements perturbations in the metric space $\mathbf{R}^{m \times n \times s}$ such that for each of those perturbations at least one (not necessary one and the same) optimal portfolio of the problem $Z^s_m(R)$ preserves its optimality in the perturbed problem $Z^s_m(R + R')$.

It is obvious that if $P^s(R) = X$, then the set $P^s(R) \cap P^s(R + R')$ is not empty for any perturbing matrix $R \in \Omega_{pqr}(\varepsilon)$ and any number $\varepsilon > 0$. That is why the strong stability radius of such problem is not upper limited. Hereafter, a problem with $P^s(R) \neq X$ is called non-trivial.

3 Auxiliary statements

Let u be any of the numbers p, q, r introduced earlier. For the number u, define a conjugate number u' by the following relations

$$1/u + 1/u' = 1, \quad 1 < u < \infty.$$

Moreover, let $u' = 1$ when $u = \infty$; and $u' = \infty$ when $u = 1$. Thus, the acceptable range of the numbers u and u' is the interval $[1, \infty]$; and the numbers are tied by the relations above. Also we assume $1/u = 0$ if $u = \infty$.

Further we use the known Hölder inequality

$$|a^T b| \leq ||a||_u ||b||_{u'}, \tag{2}$$

valid for any vectors a and b of the same dimension.

Lemma. *For any portfolios $x, x^0 \in X$, indexes $i, i' \in N_n$, $k \in N_s$, and numbers $p, q \in [1, \infty]$, the following inequality is valid*

$$r_{i'k}x^0 - r_{ik}x \geq -||R_k||_{pq}||(||x^0||_{p'}, ||x||_{p'})||_v,$$

where

$$v = \min\{p', q'\}.$$

Indeed, if $i \neq i'$ then applying the Hölder inequality (2), get

$$r_{i'k}x^0 - r_{ik}x \geq -(||r_{i'k}||_p||x^0||_{p'} + ||r_{ik}||_p||x||_{p'}) \geq$$

$$\geq -||(||r_{i'k}||_p, ||r_{ik}||_p)||_q \ ||(||x^0||_{p'}, ||x||_{p'})||_{q'} \geq$$

$$\geq -||R_k||_{pq} \ ||(||x^0||_{p'}, ||x||_{p'})||_{q'} \geq -||R_k||_{pq} \ ||(||x^0||_{p'}, ||x||_{p'})||_v.$$

If $i = i'$ then we apply (1), the Hölder inequality (2) and derive

$$r_{i'k}x^0 - r_{ik}x \geq -||r_{ik}||_p \ ||x^0 - x||_{p'} \geq -||R_k||_{pq} \ ||x^0 - x||_{p'} \geq$$

$$\geq -||R_k||_{pq} \ ||(||x^0||_{p'}, ||x||_{p'})||_{q'} \geq -||R_k||_{pq} \ ||(||x^0||_{p'}, ||x||_{p'})||_v.$$

Moreover, for a vector $a = (a_1, a_2, \dots, a_n)^T \in \mathbf{R}^n$ with $|a_j| = \alpha$, $j \in N_n$, and any number $p \in [1, \infty]$, easily obtain

$$||a||_p = \alpha n^{1/p}. \tag{3}$$

4 The strong stability radius bounds

For a non-trivial problem $Z_m^s(R)$ we denote

$$\varphi = \varphi^s(p, q) = \min_{x \notin P^s(R)} \ \max_{x' \in P(x,R)} \ \min_{k \in N_s} \ \frac{g_k(x.x', R_k)}{||(||x||_{p'}, ||x'||_{p'})||_v},$$

$$\psi = \psi^s(p, q, r) = \max_{x' \in P^s(R)} \ \min_{x \notin P^s(R)} \ \frac{||[g(x, x', R)]^+||_r}{||(||x||_{p'}, ||x'||_{p'})||_v},$$

$$\chi = \chi^s(p, q, r) = n^{1/p}m^{1/q}s^{1/r} \min_{x \notin P^s(R)} \ \max_{x' \in P^s(R)} \ \max_{k \in N_s} \ \frac{g_k(x, x', R_k)}{||x - x'||_1}.$$

Here

$$P(x, R) = X(x, R) \cap P^s(R),$$

$$g(x, x', R) = (g_1(x, x', R_1), g_2(x, x', R_2), \dots, g_s(x, x', R_s))^T,$$

$$g_k(x, x', R_k) = f_k(x, R_k) - f_k(x', R_k), \quad k \in N_s,$$

$$v = \min\{p', q'\},$$

$$[y]^+ = (y_1^+, y_2^+, \dots, y_s^+)^T$$

is a positive cutoff of a vector $y = (y_1, y_2, \dots, y_s)^T \in \mathbf{R}^s$, i.e. $y_k^+ = \max\{0, y_k\}$, $k \in N_s$.

Theorem 1. *For any $s, m \in \mathbf{N}$ and $p, q, r \in [1, \infty]$, for the strong stability radius $\rho^s_m(p, q, r)$ of the non-trivial problem $Z^s_m(R)$ the following bounds are valid*

$$0 < \max\{\varphi^s(p, q),\ \psi^s(p, q, r)\} \leq \rho^s_m(p, q, r) \leq \min\{\chi^s(p, q, r),\ ||R||_{pqr}\}.$$

Proof. From the evident formula

$$\forall x' \in P^s(R) \quad \forall x \notin P^s(R) \quad \exists k \in N_s \quad (f_k(x, R_k) > f_k(x', R_k)),$$

we easily get the inequality

$$\psi = \psi^s(p, q, r) > 0,$$

which shows that lower bound of the radius $\rho^s_m(p, q, r)$ and the radius itself are positive numbers.

Now let us show validity of the lower bound

$$\rho = \rho^s_m(p, q, r) \geq \varphi^s(p, q) = \varphi. \tag{4}$$

Suppose that $\varphi > 0$ (otherwise the inequality is evident).

Let $R' = [r'_{ijk}] \in \mathbf{R}^{m\times n\times s}$ be a perturbing matrix with cuts R'_k, $k \in N_s$, from the set $\Omega_{pqr}(\varphi)$. By the definition of φ and inequality (1), we get the formula

$$\forall x \notin P^s(R) \quad \exists x^0 \in P(x, R) \quad \forall k \in N_s$$

$$\left(\frac{f_k(x, R_k) - f_k(x^0, R_k)}{||(||x||_{p'}, ||x^0||_{p'})||_v} \geq \varphi > ||R'||_{pqr} \geq ||R'_k||_{pq}\right).$$

Using the lemma, for any $k \in N_s$ derive

$$f_k(x, R_k + R'_k) - f_k(x^0, R_k + R'_k) = \max_{i\in N_m}(r_{ik} + r'_{ik})x - \max_{i\in N_m}(r_{ik} + r'_{ik})x^0 =$$

$$= \min_{i\in N_m} \max_{i'\in N_m} (r_{i'k}x + r'_{i'k}x - r_{ik}x^0 - r'_{ik}x^0) =$$

$$= f_k(x, R_k) - f_k(x^0, R_k) - ||R'_k||_{pq}\ ||(||x||_{p'}, ||x^0||_{p'})||_v > 0,$$

where r'_{ik} is the i-th row of the k-th cut R'_k of R'. This means that $x \notin P^s(R + R')$. Resuming, we conclude that any non-efficient portfolio of the problem $Z^s_m(R)$ preserves optimality in the perturbed problem $Z^s_m(R + R')$. Therefore, the following relations are valid

$$\emptyset \neq P^s(R + R') \subseteq P^s(R).$$

Hence, $P^s(R) \cap P^s(R+R') \neq \emptyset$ for any perturbing matrix $R' \in \Omega_{pqr}(\varphi)$, i.e. inequality (4) is true.

Now we pass to the proof of the lower bound

$$\rho = \rho^s_m(p, q, r) \geq \psi^s(p, q, r) = \psi.$$

As in the previous case, let $R' = [r'_{ijk}] \in \mathbf{R}^{m\times n\times s}$ be a perturbing matrix from the set $\Omega_{pqr}(\psi)$. As it was established earlier, ψ is a positive number. To prove inequality $\rho > \psi$ it is sufficient to show that there exists portfolio x^* that belongs to the set $P^s(R) \cap P^s(R+R')$.

By the definition of ψ, there exists a portfolio $x^0 \in P^s(R)$ such that for any portfolio $x \notin P^s(R)$ the following inequalities hold

$$0 < \psi ||(||x||_{p'}, ||x^0||_{p'})||_v \leq ||[g(x, x^0, R)]^+||_r. \tag{5}$$

Let us now prove the formula

$$\forall x \notin P^s(R) \quad \forall R' \in \Omega_{pqr}(\psi) \quad \left(x \notin X(x^0, R+R')\right). \tag{6}$$

We prove it by contradiction. Supposing to the contrary, obtain the formula

$$\exists \tilde{x} \notin P^s(R) \quad \exists \tilde{R} \in \Omega_{pqr}(\psi) \quad \Big(\tilde{x} \in X(x^0, R+\tilde{R})\Big).$$

It implies that for any index $k \in N_s$ we get the inequality

$$g_k(\tilde{x}, x^0, R_k + \tilde{R}_k) \leq 0,$$

where $\tilde{R}_k$ is the k-th cut of the matrix $\tilde{R} = [\tilde{r}_{ijk}]$. Hence, taking into account the lemma and inequality (1), we get relations

$$\begin{gathered} 0 \geq g_k(\tilde{x}, x^0, R_k + \tilde{R}_k) = f_k(\tilde{x}, R_k + \tilde{R}_k) - f_k(x^0, R_k + \tilde{R}_k) = \\ = \max_{i \in N_m}(r_{ik} + \tilde{r}_{ik})\tilde{x} - \max_{i \in N_m}(r_{ik} + \tilde{r}_{ik})x^0 = \\ = \min_{i \in N_m} \max_{i' \in N_m} \left(r_{ik}\tilde{x} - r_{i'k}x^0 + \tilde{r}_{ik}\tilde{x} - \tilde{r}_{i'k}x^0\right) \geq \\ \geq g_k(\tilde{x}, x^0, R_k) - ||\tilde{R}_k||_{pq} \; ||(||\tilde{x}||_{p'}, ||x^0||_{p'})||_v. \end{gathered}$$

Having them, we derive

$$g_k(\tilde{x}, x^0, R_k) \leq ||\tilde{R}_k||_{pq} \; ||(||\tilde{x}||_{p'}, ||x^0||_{p'})||_v$$

and then conclude that

$$[g_k(\tilde{x}, x^0, R_k)]^+ \leq ||\tilde{R}_k||_{pq} \; ||(||\tilde{x}||_{p'}, ||x^0||_{p'})||_v.$$

As a result, we get the following contradiction with inequality (5)

$$||[g_k(\tilde{x}, x^0, R_k)]^+||_r \leq ||\tilde{R}||_{rpq} \; ||(||\tilde{x}||_{p'}, ||x^0||_{p'})||_v < \psi \; ||(||\tilde{x}||_{p'}, ||x^0||_{p'})||_v.$$

Hence, formula (6) is proved.

Now we show the way of choosing the required portfolio

$$x^* \in P^s(R) \cap P^s(R+R'),$$

where $R' \in \Omega_{pqr}(\psi)$. If $x^0 \in P^s(R+R')$ then $x^* = x^0$. Suppose $x^0 \notin P^s(R+R')$. Due to the external stability property of the Pareto set $P^s(R+R')$ (see e.g. [46], p. 39) we can choose a portfolio $x^* \in P^s(R+R')$ such that $x^* \in X(x^0, R+R')$. Using the proved formula (6), we easily find out that $x^* \in P^s(R)$. Thus, the inequality $\rho \geq \psi$ is proved.

Further we show correctness of the upper bound

$$\rho_m^s(p,q,r) \leq \chi^s(p,q,r) = \chi. \tag{7}$$

By definition of χ, there exists a portfolio $x^0 \notin P^s(R)$ such that for any efficient portfolio $x \in P^s(R)$ and any index $k \in N_s$ the following inequality holds

$$\chi||x^0 - x||_1 \geq n^{1/p} m^{1/q} s^{1/r} g_k(x^0, x, R_k). \tag{8}$$

Let $\varepsilon > \chi$. We set the elements of the perturbing matrix $R^0 = [r^0_{ijk}] \in \mathbf{R}^{m\times n\times s}$ with cuts $R^0_k,\ k \in N_s$, by the rule

$$r_{ijk} = \begin{cases} -\delta, & \text{if} \quad i \in N_m, \quad x^0_j = 1, \quad k \in N_s, \\ \delta, & \text{if} \quad i \in N_m, \quad x^0_j = 0, \quad k \in N_s. \end{cases}$$

Here the number δ is chosen to satisfy the inequality

$$\chi < \delta n^{1/p} m^{1/q} s^{1/r} < \varepsilon. \tag{9}$$

Therefore, with proved (3) we derive

$$||r^0_{ik}||_p = \delta n^{1/p}, \quad i \in N_m, \quad k \in N_s,$$

$$||R^0_k||_{pq} = \delta n^{1/p} m^{1/q}, \quad k \in N_s,$$

$$||R^0||_{pqr} = \delta n^{1/p} m^{1/q} s^{1/r}.$$

This means that $R^0 \in \Omega_{pqr}(\varepsilon)$. Moreover, all the rows $r^0_{ik},\ i \in N_m$, of any k-th cut $R^0_k,\ k \in N_s$, are equal and consist of the components δ and $-\delta$. So, denoting $c = r^0_{ik},\ i \in N_m,\ k \in N_s$, we obtain the relations

$$c(x^0 - x) = -\delta||x^0 - x||_1 < 0$$

valid for any portfolio $x \neq x^0$. Therefore, taking into account (8) and (9), for any portfolio $x \in P^s(R)$ and any index $k \in N_s$, we derive

$$g_k(x^0, x, R_k + R^0_k) = \min_{i\in N_m}(r_{ik} + c)x^0 - \min_{i\in N_m}(r_{ik} + c)x =$$

$$= \min_{i\in N_m} r_{ik}x^0 - \min_{i\in N_m} r_{ik}x + c(x^0 - x) = g_k(x^0, x, R_k) + c(x^0 - x) \leq$$

$$\leq (\chi(n^{1/p} m^{1/q} s^{1/r})^{-1} - \delta)||x^0 - x||_1 < 0.$$

Thus, any portfolio $x \in P^s(R)$ of the problem $Z^s_m(R)$ does not belong to the Pareto set of the pertubed problem $Z^s_m(R + R^0)$. In other words, for any number $\varepsilon > \chi$, there exists a matrix $R^0 \in \Omega_{pqr}(\varepsilon)$ such that $P^s(R+R^0) \cap P^s(R) = \emptyset$, i.e. $\rho < \varepsilon$ for any $\varepsilon > \chi$. Inequality (7) is proved.

Now we must only verify the inequality $\rho \leq ||R||_{pqr}$. Suppose $x^0 \notin P^s(R)$ and $\varepsilon > ||R||_{pqr}$. Choose a number δ such that

$$0 < \delta n^{1/p} m^{1/q} < \varepsilon - ||R||_{pqr}. \tag{10}$$

We build an auxiliary matrix $V = [v_{ij}] \in \mathbf{R}^{m \times n}$ with components

$$v_{ij} = \begin{cases} -\delta, & \text{if} \quad i \in N_m, \quad x_j^0 = 1, \\ \delta, & \text{if} \quad i \in N_m, \quad x_j^0 = 0. \end{cases}$$

Using (3), calculate

$$||V||_{pq} = \delta n^{1/p} m^{1/q}. \tag{11}$$

It is evident that all the rows $v_i,\ i \in N_m$, of the matrix V are the same and consist of the components δ and $-\delta$. Denoting $d = v_i,\ i \in N_m$, we get the relation

$$d(x^0 - x) = -\delta ||x^0 - x||_1 < 0 \tag{12}$$

valid for any portfolio $x \neq x^0$ and, in particular, for the efficient portfolio $x \in P_m^s(R)$.

Let $R^0 \in \mathbf{R}^{m \times n \times s}$ be a perturbing matrix with cuts $R_k^0,\ k \in N_s$, set by the rule

$$R_k^0 = \begin{cases} V - R_1, & \text{if} \quad k = 1, \\ -R_k, & \text{if} \quad k \neq 1. \end{cases}$$

Applying (10) and (11), get

$$||R^0||_{pqr} \leq ||V||_{pq} + ||R||_{pqr} = \delta n^{1/p} m^{1/q} + ||R||_{pqr} < \varepsilon.$$

Furthermore, taking into account the structure of the matrix V we derive

$$f_1(x^0, V) - f_1(x, V) = d(x^0 - x),$$

what with (12) gives

$$g_1(x^0, x, R_1 + R_1^0) = f_1(x^0, R_1 + R_1^0) - f_1(x, R_1 + R_1^0) =$$
$$= f_1(x^0, V) - f_1(x, V) = d(x^0 - x) = -\delta ||x^0 - x||_1 < 0.$$

Additionally, it is evident that

$$g_k(x^0, R_k + R_k^0) = 0, \quad k \in N_s \backslash \{1\}$$

Finally, we conclude that

$$x^0 \in X(x, R + R^0).$$

Hence, $x \notin P^s(R+R^0)$ if $x \in P^s(R)$. That is the set $P^s(R) \cap P^s(R+R^0)$ is empty. Resuming, we have $\rho_m^s(p,q,r) < \varepsilon$ for any number $\varepsilon > ||R||_{pqr}$. Consequently, $\rho_m^s(p,q,r) \leq ||R||_{pqr}$. $\square$

From Theorem 1 the known result follows.

Corollary 1 [37]. *If $p = q = r = \infty$ then, for any $s, m \in \mathbf{N}$, the following bounds of the strong stability radius of the problem $Z^s_m(R)$ hold*

$$0 < \max_{x' \in P^s(R)} \min_{x \notin P^s(R)} \max_{k \in N_s} \frac{g_k(x, x', R_k)}{||x + x'||_1} \leq$$

$$\leq \rho^s_m(\infty, \infty, \infty) \leq \min_{x \notin P^s(R)} \max_{x' \in P^s(R)} \max_{k \in N_s} \frac{g_k(x, x', R_k)}{||x - x'||_1}.$$

5 Case of linear criteria ($m - 1$)

When $m = 1$ our investment problem becomes a vector (s-criteria) linear Boolean programming problem. We rewrite the problem in more convenient form

$$Z^s_1(R): \quad r_k x \to \max_{x \in X}, \quad k \in N_s,$$

where $x = (x_1, x_2, ..., x_n)^T \in X \subset \mathbf{R}^n$; $r_k \in \mathbf{R}^n$ is the k-th row of the matrix $R = [r_{kj}] \in \mathbf{R}^{s \times n}$. Such a case can be interpreted as a situation when the financial market state does not doubt the investor. As previously, we assume that the Hölder norms l_p and l_r, $p, r \in [1, \infty]$, are defined correspondingly in the project space $\mathbf{R}^n$ and in the criterial risk space $\mathbf{R}^s$. For the problem $Z^s_1(R)$ we will use the previous notations $P^s(R), P(x, R)$ etc.

In this linear case the lower bound of the problem $Z^s_1(R)$ strong stability radius $\rho^s_1(p, r)$ can be improved.

Theorem 2. *For any $p, r \in [1, \infty]$ and $s \in \mathbf{N}$, for the strong stability radius $\rho^s_1(p, r)$ of the non-trivial problem $Z^s_1(R)$ the following bounds are valid*

$$0 < \max\{\varphi^*, \psi^*\} \leq \rho^s_1(p, r) \leq \min\{\chi^*, ||R||_{pr}\},$$

where

$$\varphi^* = \varphi^*(p) = \min_{x \notin P^s(R)} \max_{x' \in P^s(x, R)} \min_{k \in N_s} \frac{r_k(x - x')}{||x - x'||_{p'}},$$

$$\psi^* = \psi^*(p, r) = \max_{x' \in P^s(R)} \min_{x \notin P^s(R)} \frac{||[R(x - x')]^+||_r}{||x - x'||_{p'}},$$

$$\chi^* = \chi^*(p, r) = n^{1/p} s^{1/r} \min_{x \notin P^s(R)} \max_{x' \in P^s(R)} \max_{k \in N_s} \frac{r_k(x - x')}{||x - x'||_1},$$

$$||R||_{pr} = ||(||r_1||_p, ||r_2||_p, ..., ||r_s||_p)||_r.$$

Proof. The upper bounds follow directly from Theorem 1.

From the evident formula

$$\forall x' \in P^s(R) \quad \forall x \notin P^s(R) \quad \exists k \in N_s \quad (r_k(x - x') > 0),$$

we conclude that

$$\psi^* = \psi^*(p, r) > 0.$$

Thus, the lower bound of the strong stability radius and the radius itself are positive numbers.

Now let us show that $\rho_1^s(p, r) \geq \varphi^*$. Suppose $\varphi^* > 0$ (otherwise the inequality is evident).

Let $R' \in \mathbf{R}^{s \times n}$ be a perturbing matrix with rows $r'_k \in \mathbf{R}^n, \ k \in N_s$ and the norm

$$||R'||_{pr} = ||(||r'_1||_p, ||r'_2||_p, ..., ||r'_s||_p)||_r < \varphi^*,$$

i.e. $R' \in \Omega_{pr}(\varphi^*)$. By the definition of φ^*, for any portfolio $x \notin P^s(R)$ there exists a portfolio $x^0 \in P(x, R)$ such that

$$\frac{r_k(x - x^0)}{||x - x^0||_{p'}} \geq \varphi^* > ||R'||_{pr} \geq ||r'_k||_p, \quad k \in N_s.$$

Having these inequalities and Hölder's inequality (2), derive

$$(r_k + r'_k)(x - x^0) \geq r_k(x - x^0) - ||r'_k||_p ||x - x^0||_{p'} > 0, \quad k \in N_s,$$

and, as a result, deduce

$$x \notin P(x, R + R').$$

Therefore, any non-efficient portfolio of the problem $Z_1^s(R)$ retains this non-efficiency in any perturbed problem $Z_1^s(R + R')$ with $R' \in \Omega_{pq}(\varphi^*)$ or, strictly, $\emptyset \neq P^s(R + R') \subseteq P^s(R)$. Thus, $P^s(R) \cap P^s(R + R') \neq \emptyset$ for any perturbing matrix $R' \in \Omega_{pr}(\varphi)$, i.e. $\rho_1^s(p, r) \geq \varphi^*$.

Further, remembering that $\psi^* > 0$, we show the inequality $\rho_1^s(p, r) \geq \psi^*$.

As earlier, let $R' \in \mathbf{R}^{s \times n}$ be a perturbing matrix with rows $r'_k \in \mathbf{R}^n, \ k \in N_s$ and the norm $||R'||_{pr} < \psi^*$, i.e $R' \in \Omega_{pq}(\psi^*)$.

By the definition of ψ^*, there exists a portfolio $x^0 \in P^s(R)$ such that for any portfolio $x \notin P^s(R)$

$$0 < \psi^* ||x - x^0||_{p'} \leq ||[R(x - x^0)]^+||_r. \tag{13}$$

First, let us show that

$$\forall x \notin P^s(R) \quad \forall R' \in \Omega_{pr}(\psi^*) \quad (x \notin X(x^0, R + R')). \tag{14}$$

Suppose that there exist a portfolio $\widetilde{x} \notin P^s(R)$ and a perturbing matrix $\widetilde{R} \in \Omega_{pr}(\psi^*)$ with rows $\widetilde{r_k}, \ k \in N_s$, such that $\widetilde{x} \in X(x^0, R + \widetilde{R})$. Then for any $k \in N_s$ we have

$$(r_k + \widetilde{r_k})\widetilde{x} \leq (r_k + \widetilde{r_k})x^0,$$

and, consequently,

$$r_k(\widetilde{x} - x^0) \leq \widetilde{r_k}(x^0 - \widetilde{x}).$$

Having this, easily get the inequality

$$[r_k(\widetilde{x} - x^0)]^+ \le |\widetilde{r_k}(x^0 - \widetilde{x})|,$$

that with Hólder's inequality (2) gives us

$$[r_k(\widetilde{x} - x^0)]^+ \le ||\widetilde{r_k}||_p ||\widetilde{x} - x^0||_{p'}.$$

This means that

$$||[R(\widetilde{x} - x^0)]^+||_r \le ||\widetilde{R}||_{pr} ||\widetilde{x} - x^0||_{p'} < \psi^* ||\widetilde{x} - x^0||_{p'}.$$

This derived contradiction to (13) proves (14).

Next, we show that there exists a portfolio $x^* \in P^s(R) \cap P^s(R+R')$ in the case where $R' \in \Omega_{pr}(\psi^*)$.

If the portfolio $x^0 \in P^s(R)$ from (13) is in the Pareto set $P^s(R+R')$ then $x^* = x^0$. If $x^0 \notin P^s(R+R')$ then due to the external stability property of the Pareto set $P^s(R+R')$ (see, e.g., [46], p. 39) we can choose a portfolio $x^* \in P^s(R+R')$ such that $x^* \in X(x^0, R+R')$. Using the proved formula (14), we easily find out that $x^* \in P^s(R)$. Therefore, the inequality $\rho_1^s(p,r) \ge \psi^*$ is proved. □

From Theorem 2 the two known results follow.

Corollary 2 **[36]** *(see also* **[10]***). If $p = r = \infty$ then for any $s \in \mathbf{N}$ the following bounds of the strong stability radius of the linear non-trivial problem $Z_1^s(R)$ hold*

$$\psi^*(\infty,\infty) = \max_{x' \in P^s(R)} \min_{x \notin P^s(R)} \max_{k \in N_s} \frac{r_k(x - x')}{||x - x'||_1} \le$$

$$\le \rho_1^s(\infty,\infty) \le \chi^*(\infty,\infty) = \min_{x \notin P^s(R)} \max_{x' \in P^s(R)} \max_{k \in N_s} \frac{r_k(x - x')}{||x - x'||_1}.$$

Corollary 3 **[34].** *If $p = \infty$ then for any $r \in [1,\infty]$ the following bounds of the strong stability radius of the linear scalar (single criterion) non-trivial problem $Z_1^1(R)$, $R \in \mathbf{R}^{1\times n}$, hold*

$$\rho_1^1(\infty,r) = \varphi^*(\infty) = \chi^*(\infty,r) = \min_{x \notin P^1(R)} \max_{x' \in P^1(R)} \frac{R(x - x')}{||x - x'||_1}.$$

In another particular case the lower bound takes the following form.

Corollary 4. *If $p = 1$, $r \in [1,\infty]$, and $s \in \mathbf{N}$ then*

$$\rho_1^s(1,r) \ge \max\{\varphi^*(1), \quad \psi^*(1,r),$$

where

$$\varphi^*(1) = \min_{x \notin P^s(R)} \max_{x' \in P(x,R)} \min_{k \in N_s} r_k(x - x'),$$

$$\psi^*(1,r) = \max_{x' \in P^s(R)} \min_{x \notin P^s(R)} ||[R(x-x')]^+||_r.$$

Here is one more case where a formula is valid for the strong stability radius.

Consider a linear problem $Z_1^s(R),\ s \in \mathbf{N}$, with the Hölder norms l_p and l_r in the spaces $\mathbf{R}^n$ and $\mathbf{R}^s$. A stability radius of an efficient portfolio $x^0 \in P^s(R)$ of the problem $Z_1^s(R)$ is the number

$$\rho_1^s(x^0,p,r) = \begin{cases} \sup \Theta_{pr}, & \text{if} \quad \Theta_{pr} \neq \emptyset, \\ 0, & \text{if} \quad \Theta_{pr} = \emptyset. \end{cases}$$

where

$$\Theta_{pr} = \{\varepsilon > 0 :\ \forall R' \in \Omega_{pr}(\varepsilon)\ (x^0 \in P^s(R+R'))\}.$$

For the case $P^s(R) = \{x^0\}$, it is easy to see that

$$\rho_1^s(p,r) = \rho_1^s(x^0,p,r).$$

Therefore, using the known formula (see [47, 48]) for the stability radius of an efficient solution of the linear boolean programming problem with the Hölder norms, we state the following

Corollary 5. *If $P^s(R) = \{x^0\}$ then for any $p,\ r \in [1,\infty]$ and $s \in \boldsymbol{N}$ the strong stability radius of the problem $Z_1^s(R)$ is calculated by the formula*

$$\rho_1^s(p,r) = \min_{x \in X \setminus \{x^0\}} \frac{||[R(x-x^0)]^+||_r}{||(x-x^0)||_{p'}}.$$

The results presented in the work were partially reported at the 28th European Conference on Operational Research (EURO-2016) [49].

In conclusion we remark that in [8] similar bounds of the stability radius are found for the multicriteria linear Boolean problem $Z_1^s(R)$ with the Hölder metrics in the parameter spaces.

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Received January 10, 2016

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(82), 2016, Pages 99–120
ISSN 1024–7696

Uniqueness of certain power of a meromorphic function sharing a set with its differential monomial

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Abstract. In this paper we are mainly devoted to find out the specific form of a meromorphic function when it shares a set of small functions with its differential monomial counterpart. Our results will improve and extend some of the recent results due to Zhang-Yang [J. L. Zhang and L. Z. Yang, A power of a meromorphic function sharing a small function with its derivative, Ann. Acad. Sci. Fenn. Math. 34(2009), 249–260] and Xu-Yi-Yang [H. Y. Xu, C. F. Yi and H. Wang, On a conjecture of R. Bruck concerning meromorphic function sharing small functions, Revista de Matematica Teoria y Aplicaciones, 23(1)(2016), 291-308]. We provide some examples to show that certain conditions used in the paper can not be removed.

Mathematics subject classification: 30D35.
Keywords and phrases: Meromorphic function, uniqueness, differential polynomial, shared set.

1 Introduction, Definitions and Results

Let f be a non-constant meromorphic function in the whole complex plane $\mathbb{C}$. We shall use the following standard notations of the value distribution theory:

$$T(r,f), m(r,f), N(r,f), \overline{N}(r,f), \ldots$$

([11, 19, 23]). We denote by $S(r,f)$ any quantity satisfying

$$S(r,f) = o(T(r,f)),$$

as $r \to +\infty$, possibly outside of a set of finite measure. A meromorphic function $a \equiv a(z)$ is called a small function with respect to f if $T(r,a) = S(r,f)$. Let $\mathcal{S}(f)$ be the set of meromorphic functions in the complex plane $\mathbb{C}$ which are small functions with respect to f.

Let f be a non-constant meromorphic function and $a \in \mathcal{S}(f) \cup \{\infty\}$ and $\mathcal{S} \subset \mathcal{S}(f) \cup \{\infty\}$. Define

$$E(\mathcal{S},f) = \bigcup_{a\in\mathcal{S}} \{z : f(z) - a = 0, \text{Counting Multiplicity}\},$$

$$\overline{E}(\mathcal{S},f) = \bigcup_{a\in\mathcal{S}} \{z : f(z) - a = 0, \text{Ignoring Multiplicity}\},$$

If $E(\mathcal{S}, f) = E(\mathcal{S}, g)$, we say that f and g share the set $\mathcal{S}$ CM; if $\overline{E}(\mathcal{S}, f) = \overline{E}(\mathcal{S}, g)$, we say that f and g share the set $\mathcal{S}$ IM. Especially, when $\mathcal{S} = \{a\}$, we say that f and g share the value a CM if $E(\mathcal{S}, f) = E(\mathcal{S}, g)$; and we say that f and g share the value a IM if $\overline{E}(\mathcal{S}, f) = \overline{E}(\mathcal{S}, g)$ [11].

Nowadays the problems relative to a meromorphic function f and its derivative $f^{(k)}$ sharing some value or small functions have been studied rigorously by many researchers. Readers are requested to make a glance at [9, 15, 24, 27].

In 1996, *Brück* [7] proposed the following famous conjecture.

Conjecture 1.1. *Let f be a non-constant entire function. Suppose that $\rho_1(f)$ is not a positive integer or infinite. If f and f' share one finite value a CM, then*

$$\frac{f' - a}{f - a} = c,$$

for some non-zero constant c, where $\rho_1(f)$ is the first iterated order of f which is defined by

$$\rho_1(f) = \limsup_{r\to\infty} \frac{\log\log T(r, f)}{\log r}.$$

In 1996, *Brück* [7] proved that the conjecture is true when $a = 0$ or $N(r, 1/f') = S(r, f)$ and later many researchers like *Gundersen* and *Yang* [10] proved that the conjecture is true when f is of finite order [10]. A few years later, *Chen* and *Shon* [8] proved that the conjecture is true for entire function of first order $\rho_1(f) < \frac{1}{2}$. However, the conjecture fails in general for meromorphic functions, shown by *Gundersen* and *Yang* [10], while it remains true in the case that $N(r, 1/f') = S(r, f)$, shown by *Al-Kahaladi* [1].

In 2008, *Yang* and *Zhang* [20] obtained the following results.

Theorem 1.1 (see [20])**.** *Let f be a non-constant entire function, $n \geq 7$ be an integer. Denote $\mathcal{F} = f^n$. If $\mathcal{F}$ and $\mathcal{F}'$ share 1 CM, then $\mathcal{F} \equiv \mathcal{F}'$, and f assumes the form*

$$f(z) = ce^{\frac{z}{n}},$$

where c is a non-zero constant.

Theorem 1.2 (see [20])**.** *Let f be a non-constant meromorphic function and $n \geq 12$ be an integer. Denote $\mathcal{F} = f^n$. If $\mathcal{F}$ and $\mathcal{F}'$ share 1 CM, then $\mathcal{F} \equiv \mathcal{F}'$, and f assumes the form*

$$f(z) = ce^{\frac{z}{n}},$$

where c is a non-zero constant.

In 2009, *Zhang* and *Yang* [25] improved Theorem 1.1 and Theorem 1.2 to a large extent and obtained the following results.

Theorem 1.3 (see [25]). *Let f be a non-constant entire function, n, k be positive integers and $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 CM and $n \geq k+2$, then $f^n \equiv (f^n)^{(k)}$ and f assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and $\lambda^k = 1$.

Theorem 1.4 (see [25]). *Let f be a non-constant meromorphic function, n, k be positive integers and $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 CM and $n > k + 1 + \sqrt{k+1}$, then $f^n \equiv (f^n)^{(k)}$ and f assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and $\lambda^k = 1$.

Theorem 1.5 (see [25]). *Let f be a non-constant entire function, n, k be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 IM and $n > 2k+3$, then $f^n \equiv (f^n)^{(k)}$ and f assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant with $\lambda^k = 1$.

Theorem 1.6 (see [25]). *Let f be a non-constant meromorphic function, n, k be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 IM and*

$$n > 2k + 3 + \sqrt{(k+3)(2k+3)},$$

then $f^n \equiv (f^n)^{(k)}$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant with $\lambda^k = 1$.

Though the standard definitions and notations of the value distribution theory are available in [3, 22], we explain the following definitions and notations which are used in the paper.

Definition 1.1 (see [3, 22]). *When f and g share 1 IM, we denote by $\overline{N}_L(r, 1; f)$ the counting function of the 1-points of g. Similarly, we have $\overline{N}_L(r, 1; g)$. Let z_0 be a zero of $f - 1$ of multiplicity p and a zero of $g - 1$ of multiplicity q, we also denote by $N_{11}(r, 1; f)$ the counting function of those 1-points of f where $p = q = 1$; $\overline{N}^{(2}_E(r, 1; f)$ denotes the counting function of those $1-$points of f where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way, one can define $N_{11}(r, 1; g)$, $\overline{N}^{(2}_E(r, 1; g)$.*

Definition 1.2 (see [5]). *For $a \in \mathbb{C} \cup \{\infty\}$ and p a positive integer, let f be a non-constant meromorphic function, we denote by $N(r,a;f \mid= 1)$ the counting function of simple a-points of f, denote by $N(r,a;f \mid\leq p)$ $(N(r,a;f \mid\geq p))$ the counting functions of those a-points of f whose multiplicities are not greater (less) than p where each a-point is counted according to its multiplicities. $\overline{N}(r,a;f \mid\leq p)$ $(\overline{N}(r,a;f \mid\geq p))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.*

Definition 1.3 (see [5]). *For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by*

$$N_p(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geq 2) + \ldots + \overline{N}(r,a;f \mid\geq p).$$

Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.

Next we recall the following definition of weighted sharing of values which generally measures how closed a shared value is to being sharing IM or CM, as follows.

Definition 1.4 (see [13, 14]). *Let p be a non-negative integer or infinity. For $c \in \mathbb{C} \cup \{\infty\}$, we denote by $E_f(a,p)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq p$ and $p+1$ times if $m > p$. If $E_f(a,p) = E_g(a,p)$, we say that f, g share the value a with weight p.*

We write f, g share (a,p) to mean that f, g share the value a with weight p. Clearly if f, g share (a,p), then f, g share (a,q) for all integer q $(0 \leq q < p)$. Also, we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ and (a,∞) respectively.

Let $\mathcal{S}$ be a subset of $\mathcal{S}(f) \cup \{\infty\}$, we can get the definition of $E_f(\mathcal{S},p)$ as

$$E_f(\mathcal{S},p) = \bigcup_{a \in \mathcal{S}} E_f(a,p).$$

Very recently in [18], for further investigations, *Xu*, *Yi* asked the following questions:

Question 1.1 (see [18]). *Can the nature of sharing 1 or $a(z)$ CM be further relaxed in Theorem 1.1 and Theorem 1.3?*

Question 1.2 (see [18]). *What will happen when 1 or $a(z)$ are replaced by the set*

$$\mathcal{S}_m = \left\{ a(z), a(z)\zeta, a(z)\zeta^2, ..., a(z)\zeta^{m-1} \right\}$$

of small functions in Theorems 1.1 – 1.4, where $\zeta = \cos\dfrac{2\pi}{m} + i\sin\dfrac{2\pi}{m}$ and m is a positive integer?

To answer their question *Xu*, *Yi* and *Wang* [18] obtained the following two results which in turn improve Theorem 1.3 and Theorem 1.4.

Theorem 1.7 (see [18]). *Let f be a non-constant entire function, n, k, p, m be positive integers and $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $E_{f^n}(\mathcal{S}_m, p) = E_{(f^n)^{(k)}}(\mathcal{S}_m, p)$ and*

$$n > \max\left\{k+1, k+\frac{\eta}{pm}\right\},$$

where $\eta = k+p+2$, then $f^n \equiv t(f^n)^{(k)}$ with $t^m = 1$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and $\lambda^{km} = 1$.

Theorem 1.8 (see [18]). *Let f be a non-constant meromorphic function, n, k, p, m be positive integers and $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $E_{f^n}(\mathcal{S}_m, p) = E_{(f^n)^{(k)}}(\mathcal{S}_m, p)$ and*

$$n > \max\left\{k+1, \frac{p(m+1)k+2\eta}{2pm} + \frac{\sqrt{4\eta(\eta+pk)+(m-1)^2p^2k^2}}{2pm}\right\},$$

where $\eta = k+p+2$, then $f^n \equiv t(f^n)^{(k)}$ with $t^m = 1$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and $\lambda^{km} = 1$.

We observe from the above discussions that the research have gradually been shifted towards finding the relation between the power of a meromorphic function and its certain derivative. Since derivative's natural extension is a differential monomial it will be quite natural to expect the extension and improvement of Theorems 1.1 – 1.8 up to a relation between a power of a meromorphic function and a general differential monomial sharing set of small functions.

Next we present the following well known definition.

Definition 1.5 (see [5]). *Let $n_{0j}, n_{1j}, \ldots, n_{kj}$ be nonnegative integers and $g = f^n$.*

The expression $M_j[g] = (g)^{n_{0j}}(g^{(1)})^{n_{1j}} \ldots (g^{(k)})^{n_{kj}}$ is called a differential monomial generated by g of degree $d_{M_j} = d(M_j) = \sum_{i=0}^{k} n_{ij}$ and weight $\Gamma_{M_j} = \sum_{i=0}^{k}(i+1)n_{ij}$.

The sum $\mathcal{P}[g] = \sum_{j=1}^{t} b_j M_j[g]$ is called a differential polynomial generated by g of degree $\overline{d}(\mathcal{P}) = max\{d(M_j) : 1 \le j \le t\}$ and weight $\Gamma_{\mathcal{P}} = max\{\Gamma_{M_j} : 1 \le j \le t\}$, where $T(r, b_j) = S(r, g)$ for $j = 1, 2, \ldots, t$.

The numbers $\underline{d}(\mathcal{P}) = min\{d(M_j) : 1 \le j \le t\}$ and k (the highest order of the derivative of g in $\mathcal{P}[g]$) are called respectively the lower degree and order of $\mathcal{P}[g]$.

$\mathcal{P}[g]$ is said to be homogeneous if $\overline{d}(\mathcal{P})=\underline{d}(\mathcal{P})$. $\mathcal{P}[g]$ is called a linear differential polynomial generated by g if $\overline{d}(\mathcal{P}) = 1$. Otherwise $\mathcal{P}[g]$ is called a non-linear differential polynomial.

We denote by $Q = max\ \{\Gamma_{M_j} - d(M_j) : 1 \leq j \leq t\} = max\ \{n_{1j} + 2n_{2j} + \ldots + kn_{kj} : 1 \leq j \leq t\}$.

Also for the sake of convenience for a differential monomial $M[g]$ we denote by $d_{_M} = d(M)$ and $Q_{_M} = \Gamma_{_M} - d_{_M}$.

Next we pose the following questions which have great significance towards the further extension and improvement of all the above mentioned theorems.

Question 1.3. *Is it possible to extend $(f^n)^{(k)}$ to a differential monomial $M[f^n]$ to get the same conclusion as in Theorem 1.7 and Theorem 1.8?*

Question 1.4. *Like Theorem 1.7 and Theorem 1.8, is it possible to find out the specific form of the function f?*

Question 1.5. *Can the lower bound of n be further reduced in Theorem 1.7 and Theorem 1.8?*

Our main intention of writing this paper is to find out the possible affirmative answer of all the above questions such that Theorems 1.1 – 1.8 can be accommodated under a single theorem which extends and improves all of them. Henceforth we need the following notations throughout the paper for the sake of convenience.

Let

$$\alpha = 2Q_{_M} + 3, \quad \beta = mQ_{_M} + (k+1)d_{_M} + 2 \quad \text{and} \quad \gamma_m^p = mQ_{_M} + 1 + \frac{1}{p},$$

where p, m and k are three positive integers.

The following two theorems are the main results of this paper answering all the above mentioned questions affirmatively.

Theorem 1.9. *Let f be a non-constant meromorphic function, n, k, p, m be positive integers and $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $E_{f^{nd_M}}(\mathcal{S}_m, p) = E_{M[f^n]}(\mathcal{S}_m, p)$ and if*

1. *$p \geq 2$ and $n > \dfrac{\gamma_m^p + \gamma_1^p + \sqrt{(\gamma_m^p - \gamma_1^p)^2 + 4C}}{2md_{_M}}$, or if*
2. *$p = 0$ and $n > \dfrac{\alpha + \beta + \sqrt{(\alpha - \beta)^2 + 4D}}{2md_{_M}}$,*

where $C = \dfrac{(p+1)(p(k+1)d_{_M} + 1)}{p^2}$ and $D = (Q_{_M} + 3)(2(k+1)d_{_M} + 1)$,

then $f^{nd_M} \equiv tM[f^n]$ with $t^m = 1$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant with $\lambda^{mQ_M} = 1$.

Theorem 1.10. *Let f be a non-constant entire function, n, k, p, m be positive integers and $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $E_{f^{nd_M}}(\mathcal{S}_m, p) = E_{M[f^n]}(\mathcal{S}_m, p)$ and if*

1. *$p \geq 2$ and $n > \dfrac{pmQ_M + p + 1}{pmd_M}$, or if*
2. *$p = 0$ and $n > \dfrac{mQ_M + (k+1)d_M + 2}{md_M}$,*

then $f^{nd_M} \equiv tM[f^n]$ with $t^m = 1$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant with $\lambda^{mQ_M} = 1$.

2 Some Corollaries

In Theorem 1.9 and Theorem 1.10, if we take $M[f^n] = (f^n)^{(k)}$, where $n > k$, then it is clear that $d_M = 1$, $Q_M = k$. The following are some corollaries of the main results of this paper. What worth noticing here is that the lower bound of n is reduced as compare to Theorem 1.7 and Theorem 1.8.

Corollary 1. *Let f be a non-constant meromorphic function and n, m, p, k be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z) \neq 0, \infty$. If $E_{f^n}(\mathcal{S}_m, p) = E_{(f^n)^{(k)}}(\mathcal{S}_m, p)$ and if*

1. *$p \geq 2$ and $n > \dfrac{2p + p(m+1)k + 2}{2pm} + \dfrac{\sqrt{4(p+1)(pk+p+1) + (m-1)^2p^2k^2}}{2pm}$, or if*
2. *$p = 0$ and $n > \dfrac{(m+3)k + 6}{2m} + \dfrac{\sqrt{4(k+3)(2k+3) + (m-1)^2k^2}}{2m}$,*

then $f^n \equiv t(f^n)^{(k)}$, where $t^m = 1$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and $\lambda^{mk} = 1$.

Corollary 2. *Let f be a non-constant entire function and n, m, p, k be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z) \neq 0, \infty$. If $E_{f^n}(\mathcal{S}_m, p) = E_{(f^n)^{(k)}}(\mathcal{S}_m, p)$ and if*

1. *$p \geq 2$ and $n > k + \dfrac{p+1}{pm}$, or if*
2. *$p = 0$ and $n > k + \dfrac{k+3}{m}$,*

then $f^n \equiv t(f^n)^{(k)}$, *where* $t^m = 1$ *and* f *assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c *is a non-zero constant and* $\lambda^{mk} = 1$.

Corollary 3. *Let* f *be a non-constant entire function and* n, p, k *be positive integers and* $a \equiv a(z)$ *is a small meromorphic function of* f *and* $E_{f^n}(\mathcal{S}_1, p) = E_{(f^n)^{(k)}}(\mathcal{S}_1, p)$, *then if*

1. $p \geq 2$ *and* $n > k + \dfrac{p+1}{p}$, *or if*
2. $p = 0$ *and* $n > 2k + 3$,

then $f^n \equiv (f^n)^{(k)}$ *and* f *assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c *is a non-zero constant and* $\lambda^k = 1$.

3 Examples

The following examples show that conditions 1. and 2. in Corollary 1 and Corollary 2 are essential in order to get the conclusions.

Example 3.1. *For* $n \geq 2$, *let the principal branch of* f *be given by* $f(z) = \left(e^{\theta z} + 2a\right)^{\frac{1}{n}}$, *where* $a \neq 0$ *is a constant and* θ *is a root of the equation* $z^n + 1 = 0$. *Let* $\mathcal{S}_m = \{a\}$ *and* $M[f^n] = (f^n)^{(n)}$. *Clearly* $f^n = e^{\theta z} + 2a$ *and* $M[f^n] = -e^{\theta z}$ *and* $d_{_M} = 1$. *Therefore we see that* $E_{f^{nd_M}}(\mathcal{S}_m, \infty) = E_{M[f^n]}(\mathcal{S}_m, \infty)$ *and*

$$n \leq \min\left\{k + \frac{p+1}{pm}, k + \frac{k+3}{m}\right\} = \max\left\{n+1, 2n+3\right\} = n+1.$$

Here it is clear that

$$f^n \not\equiv tM[f^n]$$

with $t^m = 1$. *Also we see that* f *does not assume the form*

$$f(z) = ce^{\frac{\lambda}{n}z}$$

with $\lambda^{mQ_M} = 1$.

The following example shows that the conditions 1. and 2. used in Corollary 1 and Corollary 2 are not necessary but sufficient.

Example 3.2. *Let* $\mathcal{S}_m = \{-1, 1, -i, i\}$ *and* f *be given by* $f(z) = e^{\frac{\lambda}{3}z}$, *where* λ *is a root of the equation* $z^3 + 1 = 0$. *Let* $M[f^3] = (f^3)^{(3)}$. *It is clear that* $f^3(z) = e^{\lambda z}$ *and* $M[f^3] = -e^{\lambda z}$. *Also* $E_{f^{nd_M}}(\mathcal{S}_m, \infty) = E_{M[f^n]}(\mathcal{S}_m, \infty)$ *and*

$$n \leq \min\left\{k + \frac{p+1}{pm},\ k + \frac{k+3}{m}\right\} = \min\left\{\frac{13}{4},\ \frac{9}{2}\right\} = \frac{13}{4}.$$

But we see that $f^3 \equiv tM[f^3]$ *with* $t^m = (-1)^4 = 1$. *Also here* f *assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where $c = 1$ *and* $\lambda^{mQ_M} = \lambda^{12} = 1$.

The following examples show that the set $\mathcal{S}_m$ in Theorems *1.9 – 1.10* can not be replaced by an arbitrary set.

Example 3.3. *Let* $\mathcal{S}_m = \left\{\frac{a\omega}{2}, \frac{a\omega}{3}, \frac{2a\omega}{3}\right\}$, *where* a *is an arbitrary non-zero complex number. Let* $f^n = \mathcal{B}e^{\theta z} + a\omega$, *where* $n \leq 16$ *is a positive integer and* θ *and* ω *are roots of the equations* $z^{n-5} + 1 = 0$ *and* $z^3 - 1 = 0$ *respectively and* $\mathcal{B} \in \mathbb{C} - \{0\}$. *Let* $M[f^n] = (f^n)^{(n-5)}$, *then we see that* $M[f^n] = -\mathcal{B}e^{\theta z}$. *It is clear that* $E_{f^{nd_M}}(\mathcal{S}_m, \infty) = E_{M[f^n]}(\mathcal{S}_m, \infty)$ *and*

$$n > \max\left\{k + \frac{p+1}{pm},\ k + \frac{k+3}{m}\right\}.$$

But we see that $f^n \not\equiv tM[f^n]$ *with* $t^m = 1$*and hence* f *does not assume the form*

$$f(z) = ce^{\frac{\lambda}{n}z}$$

with $\lambda^{mQ_M} = 1$.

Example 3.4. *Let* $\mathcal{S}_m = \left\{\frac{1}{r}\mathcal{A}, \frac{r-1}{r}\mathcal{A} : 2 \leq r \leq \frac{m+3}{2}\right\}$, *where* $\mathcal{A}$ *is an arbitrary non-zero complex number and* $m, r \in \mathbb{N}$ *where* m *is odd and* $m > n+2$. *Let* $f^n = \mathcal{A}e^{\theta z} + \mathcal{A}$, *where* $n \geq 2$ *is a positive integer and* θ *is a root of the equation* $z^{n-1} + 1 = 0$ *and* $\mathcal{A} \in \mathbb{C} - \{0\}$. *Let* $M[f^n] = (f^n)^{(n-1)}$, *then we see that* $M[f^n] = -\mathcal{A}e^{\theta z}$. *It is clear that* $E_{f^{nd_M}}(\mathcal{S}_m, \infty) = E_{M[f^n]}(\mathcal{S}_m, \infty)$ *and*

$$n > \max\left\{k + \frac{p+1}{pm}, k + \frac{k+3}{m}\right\}.$$

But we see that $f^n \not\equiv tM[f^n]$ *with* $t^m = 1$. *Also we see that* f *does not assume the form*

$$f(z) = ce^{\frac{\lambda}{n}z}$$

with $\lambda^{mQ_M} = 1$.

The following example shows that if the conditions of Theorem 1.9 and Theorem 1.10 are satisfied, then the conclusions hold.

Example 3.5. *Let $\mathcal{S}_m = \{-1, 1, -i, i\}$ and f be given by $f(z) = e^{\frac{\lambda}{5}z}$, where λ is a root of the equation $z^3 + 1 = 0$. Let $M[f^n] = (f^n)^{(k)}$. It is clear that $f^n(z) = e^{\lambda z}$ and $M[f^n] = -e^{\lambda z}$ with $n = 5$, $k = 3$, $m = 4$ and $d_M = 1$. Also we see that $E_{f^{nd_M}}(\mathcal{S}_m, \infty) = E_{M[f^n]}(\mathcal{S}_m, \infty)$ and*

$$n > \max\left\{k + \frac{p+1}{pm}, k + \frac{k+3}{m}\right\} = \max\left\{\frac{13}{4}, \frac{9}{2}\right\} = \frac{9}{2}.$$

Here we see that $f^{nd_M} \equiv tM[f^n]$ with $t^m = (-1)^4 = 1$. Also here f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where $c = 1$ and $\lambda^{mQ_M} = \lambda^{12} = 1$.

Example 3.6. *For a non-zero complex number a, let $S = \left\{a, a\zeta, a\zeta^2, a\zeta^3, a\zeta^4\right\}$, where ζ is the non-real 5^{th} root of unity and f is given by $f(z) = e^{\zeta^{\frac{1}{nk}}z}$. It is clear that $f^n(z) = e^{\zeta^{\frac{1}{k}}z}$ and $M[f^n] = \zeta\, e^{\zeta^{\frac{1}{k}}z}$, where $M[f^n] = (f^n)^{(k)}$ with $n = 10$, $k = 7$, $m = 5$ and $d_M = 1$. Also we see that $E_{f^{nd_M}}(\mathcal{S}_m, \infty) = E_{M[f^n]}(\mathcal{S}_m, \infty)$ and*

$$n > \max\left\{k + \frac{p+1}{pm}, k + \frac{k+3}{m}\right\} = \max\left\{\frac{36}{5}, 9\right\} = 9.$$

Here we see that $f^{nd_M} \equiv tM[f^n]$ with $t^m = (\frac{1}{\zeta})^5 = 1$. Also here f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where $c = 1$ and $\lambda^{mQ_M} = \lambda^{12} = 1$.

4 Lemmas

In this section we present some Lemmas which will be needed in the sequel. Let $\mathcal{F}$, $\mathcal{G}$ be two non-constant meromorphic functions. Henceforth we shall denote by $\mathcal{H}$ the following function

$$\mathcal{H} = \left(\frac{\mathcal{F}''}{\mathcal{F}'} - \frac{2\mathcal{F}'}{\mathcal{F} - 1}\right) - \left(\frac{\mathcal{G}''}{\mathcal{G}'} - \frac{2\mathcal{G}'}{\mathcal{G} - 1}\right). \tag{4.1}$$

$$\mathcal{V} = \left(\frac{\mathcal{F}'}{\mathcal{F} - 1} - \frac{\mathcal{F}'}{\mathcal{F}}\right) - \left(\frac{\mathcal{G}'}{\mathcal{G} - 1} - \frac{\mathcal{G}'}{\mathcal{G}}\right) \tag{4.2}$$

and

$$\mathcal{U} = \frac{\mathcal{F}'}{\mathcal{F} - 1} - \frac{\mathcal{G}'}{\mathcal{G} - 1}. \tag{4.3}$$

Lemma 1 (see [18])**.** *Let f be a non-constant meromorphic function and k, p are positive integers. Then*

$$N_p(r,0;f^{(k)}) \leq T(r,f^{(k)}) - T(r,f) + N_{p+k}(r,0;f) + S(r,f).$$

$$N_p(r,0;f^{(k)}) \leq k\overline{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f).$$

Lemma 2. *Let f be a non-constant meromorphic function and $M[f^n]$ be a differential monomial of degree d_M and weight Γ_M. Then*

$$N(r,0;M[f^n]) \leq T(r,M) - nd_M T(r,f) + nd_M N(r,0;f) + S(r,f).$$

Proof. This can be proved in the line of the proof of ([6, Lemma 2.3]). □

Lemma 3. *Let f be a non-constant meromorphic function and $M[f^n]$ be a differential monomial of degree d_M and weight Γ_M. Then*

$$N(r,0;M[f^n]) \leq nd_M N(r,0;f) + Q_M \overline{N}(r,\infty;f) + S(r,f).$$

Proof. This can be proved in the line of the proof of ([6, Lemma 2.4]). □

Lemma 4. *For the differential monomial $M[f^n]$,*

$$N_p(r,0;M[f^n]) \leq d_M N_{p+k}(r,0;f^n) + Q_M \overline{N}(r,\infty;f) + S(r,f).$$

Proof. This can be proved in the line of the proof of ([6, Lemma 2.9]). □

Lemma 5 (see [21])**.** *Let f be a non-constant meromorphic function and $P(f) = a_n f^n + a_{n-1}f^{n-1} + ... + a_0$, where $a_0, a_1, ..., a_n$ are constants with $a_n \neq 0$. Then*

$$T(r,P(f)) = nT(r,f) + S(r,f).$$

Lemma 6 (see [21])**.** *Let $\mathcal{H}$ be given by (4.1), $\mathcal{F}$ and $\mathcal{G}$ be two non-constant meromorphic functions. If $\mathcal{H} \not\equiv 0$, then*

$$N_{11}(r,1;\mathcal{F}) \leq N(r,\mathcal{H}) + S(r,\mathcal{F}) + S(r,\mathcal{G}).$$

Lemma 7. *Let f be a non-constant meromorphic function and $a \equiv a(z)$ be a small meromorphic functions of f such that $a(z) \not\equiv 0, \infty$ and let $\mathcal{F}_1 = \dfrac{f^{nd_M}}{a}$ and $\mathcal{G}_1 = \dfrac{M[f^n]}{a}$. Let $\mathcal{V}$ be given by (4.2) and $\mathcal{F} = \mathcal{F}_1^m$ and $\mathcal{G} = \mathcal{G}_1^m$. If n, m and k are positive integers such that $n > k$ and $\mathcal{V} \equiv 0$, then $f^{nd_M} \equiv tM[f^n]$, where $t^m = 1$ and f assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and $\lambda^{mQ_M} = 1$.

Proof. Let $\mathcal{V} \equiv 0$. Then we get

$$1 - \frac{1}{\mathcal{F}_1^m} \equiv \mathcal{A} - \frac{\mathcal{A}}{\mathcal{G}_1^m}, \tag{4.4}$$

where $\mathcal{A}$ is a non-zero constant. We now consider the following cases.
Case 1. Let $N(r,\infty;f) = S(r,f)$. If $\mathcal{A} \neq 1$, then from (4.4) we have

$$\overline{N}\left(r, \frac{1}{1-\mathcal{A}}; \mathcal{F}_1^m\right) = \overline{N}(r,\infty;\mathcal{G}_1^m) = S(r,f).$$

By the Second Fundamental Theorem and definitions of $\mathcal{F}_1, \mathcal{G}_1$, we have

$$\begin{aligned} & T(r,\mathcal{F}_1^m) \\ \leq\ & \overline{N}(r,\infty;\mathcal{F}_1^m) + \overline{N}(r,0;\mathcal{F}_1^m) + \overline{N}\left(r, \frac{1}{1-\mathcal{A}}; \mathcal{F}_1^m\right) + S(r,f). \end{aligned}$$

i.e.,

$$mnd_{_M} T(r,f) \leq \overline{N}(r,0;f) + S(r,f),$$

which is not possible.
Case 2. Let $N(r,\infty;f) \neq S(r,f)$. Then there exists a z_0 which is not a zero or pole of $a(z)$ such that $\dfrac{1}{f(z_0)} = 0$, so $\dfrac{1}{\mathcal{F}_1(z_0)} = \dfrac{1}{\mathcal{G}_1(z_0)} = 0$. Therefore, from (4.4) we get $\mathcal{A} = 1$.

Thus, by (4.4) and $\mathcal{A} = 1$, then $\mathcal{F}_1^m = \mathcal{G}_1^m$, i.e.,

$$f^{nd_M} \equiv tM[f^n], \tag{4.5}$$

where $t^m = 1$. Now if z_0 be a zero of f with multiplicity q, then z_0 is a zero of f^{nd_M} with multiplicity $nqd_{_M}$ and a zero of $M[f^n]$ with multiplicity $nqd_{_M} - Q_{_M}$. Therefore,

$$nqd_{_M} = nqd_{_M} - Q_{_M},$$

which is not possible. Thus it is obvious that 0 is a Picard exceptional value of f. Similarly we can get that ∞ is also a Picard exceptional value of f. Then from (4.5) we have

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and $\lambda^{nQ_M} = 1$. $\square$

Lemma 8. *Let $\mathcal{V}$ be given by (4.2) and $\mathcal{F}, \mathcal{G}, \mathcal{F}_1$ and $\mathcal{G}_1$ be given by Lemma 7 and n, m be positive integers. If $\mathcal{V} \not\equiv 0$, then*

$$(mnd_{_M} - 1)\overline{N}(r,\infty;f) \leq N(r,\infty;\mathcal{V}) + S(r,f).$$

Proof. From (4.2) and the definitions of $\mathcal{F}, \mathcal{G}$, we see that if z_0 is a pole of f with the multiplicity q such that $a(z_0) \neq 0$ and $a(z_0) \neq \infty$, then z_0 is a zero of $\frac{\mathcal{F}'}{\mathcal{F}-1} - \frac{\mathcal{F}'}{\mathcal{F}}$ with the multiplicity $mnqd_{_M} - 1$ and a zero of $\frac{\mathcal{G}'}{\mathcal{G}-1} - \frac{\mathcal{G}'}{\mathcal{G}}$ with the multiplicity $m(nqd_{_M} + Q_{_M}) - 1$. Therefore z_0 is zero of $\mathcal{V}$ with multiplicity

$$p \geq \min\left\{mnd_{_M} - 1, m(nd_{_M} + Q_{_M}) - 1\right\} = mnd_{_M} - 1.$$

Also note that $m(r, \mathcal{V}) = S(r, f)$. Therefore

$$\begin{aligned} & (mnd_{_M} - 1)\overline{N}(r, \infty; f) \\ \leq\ & N(r, 0; \mathcal{V}) + S(r, f) \\ \leq\ & T(r, \mathcal{V}) + S(r, f) \\ \leq\ & N(r, \infty; \mathcal{V}) + S(r, f). \end{aligned}$$

□

Lemma 9. *Let $\mathcal{U}$ be given by (4.3) and $\mathcal{F}, \mathcal{G}, \mathcal{F}_1$ and $\mathcal{G}_1$ be given by Lemma 7. If n, m are psotive integers such that $n > k$ and $\mathcal{U} \equiv 0$, then*

$$f^{nd_M} \equiv tM[f^n],$$

where $t^m = 1$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}}z,$$

where c is a non-zero constant and $\lambda^{mQ_M} = 1$.

Proof. Since $\mathcal{U} = 0$, we get

$$\mathcal{F} \equiv \mathcal{B}\mathcal{G} + 1 - \mathcal{B}, \tag{4.6}$$

where $\mathcal{B}$ is a non-zero constant. By the definitions of $\mathcal{F}, \mathcal{G}, \mathcal{F}_1$ and $\mathcal{G}_1$, we get $N(r, \infty; f) = S(r, f)$. We discuss the following cases.
Case 1. Let $\mathcal{B} = 1$. Then we see that $\mathcal{F} \equiv \mathcal{G}$, i.e., $\mathcal{F}_1^m \equiv \mathcal{G}_1^m$. Then we have

$$f^{nd_M} \equiv tM[f^n],$$

where $t^m = 1$. Then f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant with $\lambda^{mQ_M} = 1$.
Case 2. Let $\mathcal{B} \neq 1$. If $N(r, 0; f) \neq S(r, f)$, then there exists a point z_0 for which $f(z_0) = 0$ but $a(z_0) \neq 0$. Since $n > k$, then it is clear that $F(z_0) = 0 = G(z_0)$. Now from (4.6), we get $B = 1$, which is clearly absurd.

Again if $N(r,0;f)=S(r,f)$, then from (4.6) and using Lemma 3, we get

$$\begin{aligned}&\overline{N}(r,1-\mathcal{B};\mathcal{F})=\overline{N}(r,0;\mathcal{G})\\ \leq\;& nd_{_M}N(r,0;f)+Q_{_M}\overline{N}(r,\infty;f)\\ \leq\;& S(r,f).\end{aligned}$$

Now using Second Fundamental Theorem and $N(r,0;f)=N(r,\infty;f)=S(r,f)$, we have

$$\begin{aligned}&mnd_{_M}T(r,f)\leq T(r,\mathcal{F})+S(r,f)\\ \leq\;& \overline{N}(r,\infty;\mathcal{F})+\overline{N}(r,0;\mathcal{F})+\overline{N}(r,1-\mathcal{B};\mathcal{F})+S(r,f)\\ \leq\;& \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;\mathcal{G})+S(r,f)\\ \leq\;& S(r,f),\end{aligned}$$

which is not possible. □

Lemma 10. *Let $\mathcal{U}$ be given by* (4.3) *and $\mathcal{F},\mathcal{G},\mathcal{F}_1$ and $\mathcal{G}_1$ be given by Lemma 7. If n,m and k are positive integers such that $n>k$ and $\mathcal{U}\not\equiv 0$, then*

$$[(nd_{_M}-Q_{_M})m-1]\overline{N}(r,0;f)\leq N(r,\infty;\mathcal{U})+S(r,f).$$

Proof. Let z_0 is a zero of f with multiplicity $q(\geq 1)$ such that $a(z_0)\neq 0,\infty$. Then z_0 is a zero of $\dfrac{\mathcal{F}'}{\mathcal{F}-1}$ with the multiplicity $nmqd_{_M}-1$ and z_0 is also a zero of $\dfrac{\mathcal{G}'}{\mathcal{G}-1}$ of multiplicity $(nqd_{_M}-Q_{_M})m-1$. Therefore z_0 is a zero of $\mathcal{U}$ of multiplicity at least $(nqd_{_M}-Q_{_M})m-1$. Since $m(r,\mathcal{U})=S(r,f)$, we have

$$\begin{aligned}&[(nd_{_M}-Q_{_M})m-1]\overline{N}(r,0;f)\leq N(r,0;\mathcal{U})+S(r,f)\\ \leq\;& T(r,\mathcal{U})+S(r,f)\\ \leq\;& N(r,\infty;\mathcal{U})+S(r,f).\end{aligned}$$

□

Lemma 11. *Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{F}_1$, $\mathcal{G}_1$ be as in Lemma 7 and $\mathcal{V}$ as in (4.2). Now if $n>k$ and $E_p(1,\mathcal{F})=E_p(1;\mathcal{G})$ and $\mathcal{V}\not\equiv 0$, then the following hold:*

1. *When $p\geq 2$, then*

$$\left\{mnd_{_M}-1-Q_{_M}-\frac{1}{p}\right\}\overline{N}(r,\infty;f)\leq\left\{(k+1)d_{_M}+\frac{1}{p}\right\}\overline{N}(r,0;f)+S(r,f). \tag{4.7}$$

2. *When $p=0$, then*

$$\left\{mnd_{_M}-1-2(Q_{_M}+1)\right\}\overline{N}(r,\infty;f)\leq\left\{2(k+1)d_{_M}+1\right\}\overline{N}(r,0;f)+S(r,f). \tag{4.8}$$

Proof. Let $p \geq 2$ and $\mathcal{V} = \dfrac{\mathcal{F}'}{\mathcal{F}(\mathcal{F}-1)} - \dfrac{\mathcal{G}'}{\mathcal{G}(\mathcal{G}-1)}$. Now since $E_p(1;\mathcal{F}) = E_p(1;\mathcal{G})$, so we have

$$N(r,\infty;\mathcal{V}) \leq \overline{N}(r,0;\mathcal{G}) + \overline{N}_{(p+1}(r,1;\mathcal{F}) + S(r,f),$$

where

$$\begin{aligned} &\overline{N}_{(p+1}(r,1;\mathcal{F}) \leq \frac{1}{p} N\left(r,\frac{\mathcal{F}}{\mathcal{F}'}\right) \\ \leq\ & \frac{1}{p} N\left(r,\frac{\mathcal{F}'}{\mathcal{F}}\right) + S(r,f) \\ \leq\ & \frac{1}{p}\overline{N}(r,\infty;\mathcal{F}) + \frac{1}{p}\overline{N}(r,0;\mathcal{F}) + S(r,f) \\ \leq\ & \frac{1}{p}\overline{N}(r,\infty;f) + \frac{1}{p}\overline{N}(r,0;f) + S(r,f). \end{aligned}$$

Now by applying Lemma 8 and Lemma 4 we get

$$\begin{aligned} &(mnd_{_M}-1)\overline{N}(r,\infty;f) \leq \frac{1}{p}\overline{N}(r,0;f) + \frac{1}{p}\overline{N}(r,\infty;f) + \overline{N}(r,0;\mathcal{G}) + S(r,f) \\ \leq\ & \frac{1}{p}\overline{N}(r,0;f) + \frac{1}{p}\overline{N}(r,\infty;f) + d_{_M}N_{k+1}N(r,0;f^n) + Q_{_M}\overline{N}(r,\infty;f) + S(r,f), \end{aligned}$$

i.e.,

$$\left\{mnd_{_M} - 1 - Q_{_M} - \frac{1}{p}\right\}\overline{N}(r,\infty;f) \leq \left\{(k+1)d_{_M} + \frac{1}{p}\right\}\overline{N}(r,0;f) + S(r,f).$$

Let $p = 0$, then

$$N(r,\infty;\mathcal{V}) \leq \overline{N}(r,0;\mathcal{G}) + \overline{N}_L(r,1;\mathcal{F}) + \overline{N}_L(r,1;\mathcal{G}) + S(r,f),$$

where

$$\begin{aligned} &\overline{N}_L(r,1;\mathcal{F}) \leq N\left(r,\frac{\mathcal{F}}{\mathcal{F}'}\right) \leq N\left(r,\frac{\mathcal{F}'}{\mathcal{F}}\right) + S(r,f) \\ \leq\ & \overline{N}(r,\infty;\mathcal{F}) + \overline{N}(r,0;\mathcal{F}) + S(r,f) \\ \leq\ & \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + S(r,f). \end{aligned}$$

Similarly, applying Lemma 4 and proceeding as above, we get

$$\begin{aligned} &\overline{N}_L(r,1;\mathcal{G}) \leq \overline{N}(r,\infty;\mathcal{G}) + \overline{N}(r,0;\mathcal{G}) + S(r,f) \\ \leq\ & (Q_{_M}+1)\overline{N}(r,\infty;f) + (k+1)d_{_M}\overline{N}(r,0;f) + S(r,f). \end{aligned}$$

Now by Lemma 8 and Lemma 4, we get

$$(mnd_{_M}-1)\overline{N}(r,\infty;f) \leq \{2(k+1)d_{_M}+1\}\overline{N}(r,0;f) + 2(Q_{_M}+1)\overline{N}(r,\infty;f) + S(r,f).$$

i.e.,

$$\left\{mnd_{M}-1-2(Q_{M}+1)\right\}\overline{N}(r,\infty;f)\leq\left\{2(k+1)d_{M}+1\right\}\overline{N}(r,0;f)+S(r,f).$$

□

Lemma 12. *Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{F}_1$, $\mathcal{G}_1$ be as in Lemma 7 and $\mathcal{U}$ as in (4.3). Now if $n>k$ and $E_p(1,\mathcal{F})=E_p(1;\mathcal{G})$ and $\mathcal{U}\not\equiv 0$, then the following holds:*

1. *When $p\geq 2$, then*
$$\left\{(nd_{M}-Q_{M})m-1-\frac{1}{p}\right\}\overline{N}(r,0;f)\leq\left\{1+\frac{1}{p}\right\}\overline{N}(r,\infty;f)+S(r,f).\quad(4.9)$$
2. *When $p=0$, then*
$$\left\{(nd_{M}-Q_{M})m-(k+1)d_{M}-2\right\}\overline{N}(r,0;f)\leq\left\{Q_{M}+3\right\}\overline{N}(r,\infty;f)$$
$$+S(r,f).\quad(4.10)$$

Proof. Let $p\geq 2$, then we have

$$\begin{aligned} & N(r,\infty;\mathcal{U})\leq\overline{N}(r,\infty;\mathcal{F})+\overline{N}_{(p+1}(r,1;\mathcal{F})+S(r,f)\\ \leq\; & \overline{N}(r,\infty;f)+\left\{\frac{1}{p}\overline{N}(r,0;f)+\frac{1}{p}\overline{N}(r,\infty;f)\right\}+S(r,f)\\ \leq\; & \frac{1}{p}\overline{N}(r,0;f)+\left(1+\frac{1}{p}\right)\overline{N}(r,\infty;f)+S(r,f).\end{aligned}$$

Now by applying Lemma 10 we get

$$\left\{(nd_{M}-Q_{M})m-1\right\}\overline{N}(r,0;f)\leq\frac{1}{p}\overline{N}(r,0;f)+\left(1+\frac{1}{p}\right)\overline{N}(r,\infty;f)+S(r,f),$$

i.e.,

$$\left\{(nd_{M}-Q_{M})m-1-\frac{1}{p}\right\}\overline{N}(r,0;f)\leq\left(1+\frac{1}{p}\right)\overline{N}(r,\infty;f)+S(r,f).$$

Let $p=0$, by applying Lemma 10 and Lemma 4 and proceeding in the same way as done in the proof of Lemma 11, we get

$$\begin{aligned} & N(r,\infty;\mathcal{U})\leq\overline{N}(r,\infty;\mathcal{F})+\overline{N}_L(r,1;\mathcal{F})+\overline{N}_L(r,1;\mathcal{G})+S(r,f)\\ \leq\; & \overline{N}(r,\infty;f)+\left\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\right\}+\big\{(Q_{M}+1)\overline{N}(r,\infty;f)\\ & +(k+1)d_{M}\overline{N}(r,0;f)\big\}+S(r,f),\end{aligned}$$

i.e.,

$$\left\{(nd_{M}-Q_{M})m-(k+1)d_{M}-2\right\}\overline{N}(r,0;f)\leq\left\{Q_{M}+3\right\}\overline{N}(r,\infty;f)+S(r,f).$$

□

Lemma 13. *Let $\mathcal{F}$ and $\mathcal{G}$ be two non-constant meromorphic functions such that $E_p(1,\mathcal{F}) = E_p(1,\mathcal{G})$ and $\mathcal{H} \not\equiv 0$ and $p = 0$, then*

$$\begin{aligned} & T(r,\mathcal{F}) + T(r,\mathcal{G}) \\ \leq\ & 2N_2(r,0;\mathcal{F}) + 2N_2(r,0;\mathcal{G}) + 6\overline{N}(r,\infty;\mathcal{F}) + 3\overline{N}_L(r,1;\mathcal{F}) + 3\overline{N}_L(r,1;\mathcal{G}) + S(r,\mathcal{F}). \end{aligned}$$

Proof. Noting that $S(r,\mathcal{F}) = S(r,\mathcal{G})$, the lemma can be proved by using Lemma 2.1, Lemma 2.2 and Lemma 2.3 of [4]. □

Lemma 14. *Let $\mathcal{F}$ and $\mathcal{G}$ be two non-constant meromorphic functions such that $E_p(1,\mathcal{F}) = E_p(1,\mathcal{G})$ and $\mathcal{H} \not\equiv 0$ and $p \geq 2$, then*

$$T(r,\mathcal{F}) + T(r,\mathcal{G}) \leq 2N_2(r,0;\mathcal{F}) + 2N_2(r,0;\mathcal{G}) + 6\overline{N}(r,\infty;\mathcal{F}) + S(r,\mathcal{F}).$$

Proof. Since $\mathcal{F}$ and $\mathcal{G}$ share $(1,p)$ where $p \geq 2$, so it is clear that $\mathcal{F}$ and $\mathcal{G}$ share $(1,2)$. Then the lemma can be obtained from Lemma 13 of [2]. □

Lemma 15. *Let $\mathcal{H}$ be given by (4.1) and $\mathcal{F},\mathcal{G},\mathcal{F}_1$ and $\mathcal{G}_1$ be given by Lemma 7. If n,m and k are positive integers such that $n > k$ and*

$$\overline{N}(r,\infty;f) = N(r,0;f) = S(r,f)$$

and $\mathcal{H} \equiv 0$, then

$$f^{nd_M} \equiv tM[f^n],$$

where $t^m = 1$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and $\lambda^{nd_M} = 1$.

Proof. Since $\mathcal{H} \equiv 0$, by integration we obtain

$$\frac{1}{\mathcal{F}-1} \equiv \frac{\mathcal{C}}{\mathcal{G}-1} + \mathcal{D}, \tag{4.11}$$

where $\mathcal{C}(\neq 0)$ and $\mathcal{D}$ are constants. Now from (4.11) we have

$$\mathcal{G} \equiv \frac{(\mathcal{D}-\mathcal{C})\mathcal{F} + (\mathcal{C}-\mathcal{D}-1)}{\mathcal{D}\mathcal{F} - (\mathcal{D}+1)},$$

i.e.,

$$\mathcal{G}_1^m \equiv \frac{(\mathcal{D}-\mathcal{C})\mathcal{F}_1^m + (\mathcal{C}-\mathcal{D}-1)}{\mathcal{D}\mathcal{F}_1^m - (\mathcal{D}+1)}. \tag{4.12}$$

Now we discuss the following cases.

Case 1. Let $\mathcal{D} \neq 0,-1$. Therefore from (4.12) we have

$$\overline{N}\left(r,\frac{\mathcal{D}+1}{\mathcal{D}};\mathcal{F}_1^m\right) = \overline{N}(r,\infty;\mathcal{G}_1^m).$$

By applying the Second Fundamental Theorem with $S(r,\mathcal{F}) = S(r,f)$, we get

$$\begin{aligned}
& mnd_{_M}T(r,f) = T(r,\mathcal{F}_1^m) + S(r,f) \\
\leq\ & \overline{N}(r,\infty;\mathcal{F}_1^m) + \overline{N}(r,0;\mathcal{F}_1^m) + \overline{N}\left(r,\frac{\mathcal{D}+1}{\mathcal{D}};\mathcal{F}_1^m\right) + S(r,f) \\
\leq\ & \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \overline{N}(r,\infty;\mathcal{G}_1^m) + S(r,f) \\
\leq\ & S(r,f),
\end{aligned}$$

which is not possible.
Case 2. Suppose $\mathcal{D} = 0$. Then from (4.12), we have

$$\overline{N}\left(r,\frac{\mathcal{C}-1}{\mathcal{C}};\mathcal{F}_1^m\right) = \overline{N}(r,0;\mathcal{G}_1^m).$$

Subcase 2.1. Let $\mathcal{C} \neq 1$. Now by the Second Fundamental Theorem and using Lemma 3, we get

$$\begin{aligned}
& mnd_{_M}T(r,f) = T(r,\mathcal{F}_1^m) + S(r,f) \\
\leq\ & \overline{N}(r,\infty;\mathcal{F}_1^m) + \overline{N}(r,0;\mathcal{F}_1^m) + \overline{N}\left(r,\frac{\mathcal{C}-1}{\mathcal{C}};\mathcal{F}_1^m\right) + S(r,f) \\
\leq\ & \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \overline{N}(r,0;\mathcal{G}_1^m) + S(r,f) \\
\leq\ & (nd_{_M}+1)N(r,0;f) + (Q_{_M}+1)\overline{N}(r,\infty;f) + S(r,f) \\
\leq\ & S(r,f),
\end{aligned}$$

which is not possible.
Subcase 2.2. Let $\mathcal{C} = 1$. Then we have $\mathcal{F}_1^m \equiv \mathcal{G}_1^m$, i.e.,

$$f^{nd_M} \equiv tM[f^n].$$

Then f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and λ^{mQ_M}.
Case 3. Let $\mathcal{D} = -1$, then from (4.12), we get

$$\mathcal{G}_1^m \equiv \frac{(\mathcal{C}+1)\mathcal{F}_1^m - \mathcal{C}}{\mathcal{F}_1^m}.$$

Now proceeding exactly the same way as in Case 2, we get $\mathcal{F}_1^m\mathcal{G}_1^m \equiv 1$, i.e., $f^{nd_M}M[f^n] \equiv ta^2$, where $t^m = 1$. Again since $\overline{N}(r,\infty;f) = S(r,f) = N(r,0;f)$, so

$$\begin{aligned}
& 2T\left(r,\frac{f^{nd_M}}{a}\right) = T\left(r,\frac{ta^2}{f^{2nd_M}}\right) + O(1) \\
\leq\ & T\left(r,\frac{M[f^n]}{f^{nd_M}}\right) + O(1)
\end{aligned}$$

$$\begin{aligned}
&\leq m\left(r, \frac{M[f^n]}{f^{nd_M}}\right) + N\left(r, \frac{M[f^n]}{f^{nd_M}}\right) + O(1) \\
&\leq N\left(r, \infty; M[f^n]\right) + N\left(r, 0; f^{nd_M}\right) + O(1) \\
&\leq (nd_M + Q_M)\overline{N}(r, \infty; f) + nd_M N(r, 0; f) + O(1) \\
&\leq S(r, f),
\end{aligned}$$

which is not possible. □

5 Proofs of the Theorems

Proof of Theorem 1.9. Let $\mathcal{F}_1 = \dfrac{f^{nd_M}}{a}$ and $\mathcal{G}_1 = \dfrac{M[f^n]}{a}$ and $\mathcal{F} = \mathcal{F}_1^m$, $\mathcal{G} = \mathcal{G}_1^m$, where f is a non-constant meromorphic function. Now we discuss the following cases.

Case 1. If $\mathcal{UV} \equiv 0$, then by using Lemma 7 and Lemma 9, we get the conclusions of the Theorem 1.9.

Case 2. Let $\mathcal{UV} \not\equiv 0$, then from the assumption of Theorem 1.9, we see that $E_p(1, \mathcal{F}) = E_p(1, \mathcal{G})$.

Subcase 2.1. When $p \geq 2$, then by using Lemma 11 and Lemma 12, we get

$$\begin{aligned}
&\left\{mnd_M - 1 - Q_M - \frac{1}{p}\right\}\left\{(nd_M - Q_M)m - 1 - \frac{1}{p}\right\}\overline{N}(r, \infty; f) \qquad (5.1) \\
\leq\ &\left\{(k+1)d_M + \frac{1}{p}\right\}\left\{1 + \frac{1}{p}\right\}\overline{N}(r, \infty; f) + S(r, f)
\end{aligned}$$

and

$$\begin{aligned}
&\left\{mnd_M - 1 - Q_M - \frac{1}{p}\right\}\left\{(nd_M - Q_M)m - 1 - \frac{1}{p}\right\}\overline{N}(r, 0; f) \qquad (5.2) \\
\leq\ &\left\{(k+1)d_M + \frac{1}{p}\right\}\left\{1 + \frac{1}{p}\right\}\overline{N}(r, 0; f) + S(r, f).
\end{aligned}$$

Now from the equations (5.1) and (5.2), we get

$$\left\{(mnd_M - \gamma_1^p)(mnd_M - \gamma_m^p) - C\right\}\overline{N}(r, \infty; f) \leq S(r, f) \qquad (5.3)$$

and

$$\left\{(mnd_M - \gamma_1^p)(mnd_M - \gamma_m^p) - C\right\}\overline{N}(r, 0; f) \leq S(r, f), \qquad (5.4)$$

where $\gamma_m^p = mQ_M + 1 + \dfrac{1}{p}$ and $C = \left\{(k+1)d_M + \dfrac{1}{p}\right\}\left\{1 + \dfrac{1}{p}\right\}$.

Since

$$\left\{mnd_M - \gamma_1^p\right\}\left\{mnd_M - \gamma_m^p\right\} - C$$

$$
\begin{aligned}
&= m^2d_{_M}^2n^2 - md_{_M}\Big\{\gamma_1^p + \gamma_m^p\Big\}n + \Big\{\gamma_1^p\gamma_m^p - C\Big\}\\
&= m^2d_{_M}^2\Big\{n - \frac{\gamma_m^p + \gamma_1^p + \sqrt{(\gamma_m^p - \gamma_1^p)^2 + 4C}}{2md_{_M}}\Big\}\Big\{n - \frac{\gamma_m^p + \gamma_1^p - \sqrt{(\gamma_m^p - \gamma_1^p)^2 + 4C}}{2md_{_M}}\Big\},
\end{aligned}
$$

in view of the assumptions of Theorem 1.9, we get a contradiction from (5.3) and (5.4).

Thus we obtained from above

$$\overline{N}(r,0;f) = S(r,f) = \overline{N}(r,\infty;f). \tag{5.5}$$

We now consider the following two cases:

Case 2.1.1. Let $\mathcal{H} \not\equiv 0$. Using Lemma 13 and Lemma 14 and (5.5), we get $T(r,f) = S(r,f)$, which is a contradiction.

Case 2.1.2. Let $\mathcal{H} \equiv 0$. Then from Lemma 15, we get the conclusion of Theorem 1.9. **Subcase 2.2.** When $p = 0$, using Lemma 11 and Lemma 12, we get

$$
\begin{aligned}
&\Big\{mnd_{_M} - 1 - 2(Q_{_M} + 1)\Big\}\Big\{(nd_{_M} - Q_{_M})m - (k+1)d_{_M} - 2\Big\}\overline{N}(r,\infty;f) \qquad (5.6)\\
\leq\ &\Big\{2(k+1)d_{_M} + 1\Big\}\Big\{Q_{_M} + 3\Big\}\overline{N}(r,\infty;f) + S(r,f)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big\{mnd_{_M} - 1 - 2(Q_{_M} + 1)\Big\}\Big\{(nd_{_M} - Q_{_M})m - (k+1)d_{_M} - 2\Big\}\overline{N}(r,0;f) \qquad (5.7)\\
\leq\ &\Big\{2(k+1)d_{_M} + 1\Big\}\Big\{Q_{_M} + 3\Big\}\overline{N}(r,0;f) + S(r,f).
\end{aligned}
$$

Now using equations (5.6) and (5.7) and proceeding the same way as done in Subcase 2.1, the rest of the proof can be carried out. So we omit the detail. □

Proof of Theorem 1.10. Since f is an entire function, we have $N(r,\infty;f) = S(r,f)$. Now if $\mathcal{U} \equiv 0$, then using Lemma 9, we get the conclusion of Theorem 1.10.

If $\mathcal{U} \not\equiv 0$, then using Lemma 10 for $p \geq 2$ we get from (5.2) that

$$(mnd_{_M} - \gamma_1^p)(mnd_{_M} - \gamma_m^p)\overline{N}(r,0;f) \leq S(r,f).$$

Since $n > \dfrac{pmQ_{_M} + p + 1}{pmd_{_M}}$, we get a contradiction.

Again when $p = 0$, using Lemma 10 we get from (5.7)

$$\Big\{mnd_{_M} - [2Q_{_M} + 3]\Big\}\Big\{mnd_{_M} - [mQ_{_M} + (k+1)d_{_M} + 2]\Big\}\overline{N}(r,0;f) \leq S(r,f),$$

which is a contradiction since $n > \dfrac{mQ_{_M} + (k+1)d_{_M} + 2}{md_{_M}}$.

Therefore $\overline{N}(r,0;f) = S(r,f)$. Now the rest of the proof follows Case 1 and Case 2 of the proof of Theorem 1.9. □

6 Some Open Questions

Question 6.1. *Can we replace f^n by a general linear expression $P(f)$ in anyway in Theorem 1.9 and Theorem 1.10 to get the same specific form the function?*

Question 6.2. *Can we replace the differential monomial $M[f^n]$ by a differential polynomial $\mathcal{P}[f^n]$ in anyway in Theorem 1.9 and Theorem 1.10 to get the same specific form the function?*

Question 6.3. *Can the lower bound of n be further reduced in Theorem 1.9 and Theorem 1.10 to get the same conclusions?*

Acknowledgement. This research work is supported by the Council of Scientific and Industrial Research, Extramural Research Division, CSIR Complex, Pusa, New Delhi-110012, India, under the sanction project no. 25(0229)/14/EMR-II.

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Received October 15, 2016

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BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(82), 2016, Pages 121–124
ISSN 1024–7696

On a solution to equation with discrete multiplicative -additive derivative

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Abstract. As it is well known, a discrete differential equation (basically, with additive derivative) is called a difference equation [1–3]. The Cauchy problem for such kind of equations is considered in [4]. Several initial and boundary value problems for additive derivatives are also considered in [5].

Mathematics subject classification: 39A12, 34N05.
Keywords and phrases: Discrete additive analysis, multiplicative discrete analysis, additive-multiplicative and multiplicative-additive derivatives.

The subject of this paper is to study the solution to non-linear differential problems. The domain of solution determination is divided into a grid with step h for discretization of the problem for ordinary differential equations. Here, we accept $h = 1$. Therefore, we do not need the result as a continuous process.

As well as the equation, additional conditions can also be non-linear. It is based on discrete additive derivative, discrete multiplicative derivative and discrete integrals.

Let consider the equation with differentiation as follows:

$$y_{i+3} = y_{i+2} + \frac{f_i \cdot (y_{i+2} - y_{i+1})^2}{y_{i+1} - y_i}, \quad i \geq 0. \tag{1}$$

In order to solve this equation, we firstly find the equality describing how this sequence is obtained by giving values to i. If $i = 0$, then we obtain from (1)

$$y_3 = y_2 + f_0 \cdot \frac{(y_2 - y_1)^2}{y_1 - y_0}, \tag{2}$$

if $i = 1$, then

$$y_4 = y_3 + f_1 \cdot \frac{(y_3 - y_2)^2}{y_2 - y_1} = y_2 + f_0 \frac{(y_2 - y_1)^2}{y_1 - y_2} + f_1 \frac{(y_3 - y_2)^2}{y_2 - y_1} =$$

$$= y_2 + f_0 \frac{(y_2 - y_1)^2}{y_1 - y_0} + f_1 f_0^2 \frac{(y_2 - y_1)^2}{y_1 - y_0}, \tag{3}$$

if $i = 2$, then

$$y_5 = y_4 + f_2\frac{(y_4-y_3)^2}{y_3-y_2} = y_2 + f_0\frac{(y_2-y_1)^2}{y_2-y_0} + f_1 f_0^2\frac{(y_2-y_1)^2}{(y_1-y_0)^2} + f_2\frac{f_1^2(y_3-y_2)^3}{(y_2-y_1)^2} =$$

$$= y_2 + f_0\frac{(y_2-y_1)^2}{y_1-y_0} + f_1 f_0^2\frac{(y_2-y_1)^3}{(y_1-y_0)^2} + f_2 f_1^2 f_0^3\frac{(y_2-y_1)^4}{(y_1-y_0)^3} =$$

$$= y_2 + \sum_{k=0}^{2}\left(\prod_{p=1}^{k+1} f_{k+1-p}^p\right)\frac{(y_2-y_1)^{k+2}}{(y_2-y_0)^{k+1}}. \tag{4}$$

So we obtain

$$y_{i+3} = y_2 + \sum_{k=0}^{i}\frac{(y_2-y_1)^{k+2}}{(y_1-y_0)^{k+1}}\cdot\prod_{p=1}^{k+1} f_{k+1-p}^p, \quad i \geq 0. \tag{5}$$

Now let prove the relation obtained in (5) by mathematical induction.

We just proved that the statement (5) holds for $i=2$. Let show that if (5) holds for $i \leq q-1$, then also it holds for $i=q$:

$$y_{q+3} = y_{q+2} + \frac{(y_{q+2}-y_{q+1})^2}{y_{q+1}-y_q}\cdot f_q = y_2 + \sum_{k=0}^{q-1}\frac{(y_2-y_1)^{k+2}}{(y_1-y_0)^{k+1}}\cdot\prod_{p=1}^{k+1} f_{k+1-p}^p +$$

$$+ f_q\frac{\left[y_2 + \sum\limits_{k=0}^{q-1}\frac{(y_2-y_1)^{k+2}}{(y_1-y_0)^{k+1}}\cdot\prod\limits_{p=1}^{k+1} f_{k+1-p}^p - y_2 - \sum\limits_{k=0}^{q-2}\frac{(y_2-y_1)^{k+2}}{(y_2-y_0)^{k+1}}\cdot\prod\limits_{p=1}^{k+1} f_{k+1-p}^p\right]^2}{y_2 + \sum\limits_{k=0}^{q-2}\frac{(y_2-y_1)^{k+2}}{(y_1-y_0)^{k+1}}\cdot\prod\limits_{p=1}^{k+1} f_{k+1-p}^p - y_2 - \sum\limits_{k=0}^{q-3}\frac{(y_2-y_1)^{k+2}}{(y_2-y_0)^{k+1}}\cdot\prod\limits_{p=1}^{k+1} f_{k+1-p}^p} =$$

$$= y_2 + \sum_{k=0}^{q-1}\frac{(y_2-y_1)^{k+2}}{(y_1-y_0)^{k+1}}\cdot\prod_{p=1}^{k+1} f_{k+1-p}^p + f_q\frac{\left[\frac{(y_2-y_1)^{q+1}}{(y_1-y_0)^q}\prod\limits_{p=1}^{q} f_{q-p}^p\right]^2}{\frac{(y_2-y_1)^q}{(y_2-y_0)^{q-1}}\prod\limits_{p=1}^{q-1} f_{q-1-p}^p} =$$

$$= y_2 + \sum_{k=0}^{q-1}\frac{(y_2-y_1)^{k+2}}{(y_1-y_0)^{k+1}}\cdot\prod_{p=1}^{k+1} f_{k+1-p}^p + f_q\frac{(y_2-y_1)^{q+2}}{(y_1-y_0)^{q+1}}\cdot\frac{f_{q-1}^2 f_{q-2}^1 \dots f_0^2}{f_{q-2} f_{q-3}^2 \dots f_0^{q-1}} =$$

$$= y_2 + \sum_{k=0}^{q-1}\frac{(y_2-y_1)^{k+2}}{(y_1-y_0)^{k+1}}\cdot\prod_{p=1}^{k+1} f_{k+1-p}^p + \frac{(y_2-y_1)^{q+2}}{(y_1-y_0)^{q+1}}\cdot f_q\cdot f_{q-1}^2\cdot f_{i-2}^3\dots\times$$

$$\times f_1^q \dots f_0^{q+1} = y_2 + \sum_{k=0}^{q}\frac{(y_2-y_1)^{k+2}}{(y_1-y_0)^{k+1}}\cdot\prod_{p=1}^{k+1} f_{k+1-p}^p \tag{6}$$

thereby showing that indeed (5) holds for $i = q$. Since both the basis and the inductive step have been performed, by mathematical induction, the statement (5) holds for all natural numbers i. So we proved the following theorem.

Theorem 1. *If there is an order-bounded sequence with true values f_i for the given non-linear third order difference equation (1), then the general solution to this equation has the form (5), where y_0, y_1 and y_2 are arbitrary constants.*

Cauchy problem. If the Cauchy problem is considered for the equation (1), then the following initial conditions shall be provided

$$y_k = \alpha_k, \quad \alpha = 0, 1, 2. \tag{7}$$

Given these conditions, the solution to the problem (1), (7) takes the form

$$y_{i+3} = \alpha_2 + \sum_{k=0}^{i} \frac{(\alpha_2 - \alpha_1)^{k+2}}{(\alpha_1 - \alpha_0)^{k+1}} \cdot \prod_{p=1}^{k+1} f_{k+1-p}^{p}, \quad i = \overline{0, N-3}.$$

Boundary value problem. If we consider the boundary value problem for equation (1) with boundary conditions

$$y_2 - y_1 = \alpha_1, \quad y_1 - y_0 = \alpha_0, \quad y_N = \alpha_N, \tag{8}$$

then in accordance with (5) we obtain the following solution

$$y_{i+3} = y_2 + \sum_{k=0}^{i} \frac{\alpha_1^{k+2}}{\alpha_0^{k+1}} \cdot \prod_{p=1}^{k+1} f_{k+1-p}^{p}, \quad i = \overline{0, N-3},$$

$$\alpha_N = y_2 + \sum_{k=0}^{N-3} \frac{\alpha_1^{k+2}}{\alpha_0^{k+1}} \cdot \prod_{p=1}^{k+1} f_{k+1-p}^{p},$$

Substituting the last expression into the equation, one can get the following

$$y_2 = \alpha_N - \sum_{k=0}^{N-3} \frac{\alpha_1^{k+2}}{\alpha_0^{k+1}} \cdot \prod_{p=1}^{k+1} f_{k+1-p}^{p} \quad i = \overline{0, N-3}.$$

Thus, a single-valued solution could be obtained.

Conclusion. We have studied the Cauchy problem and the boundary value problem for the third order difference equation with discrete derivative and the analytical expressions for their solutions were obtained. Once the form of the general solution to the equation was determined, it was proved by means of mathematical induction. Finally, the constants included in the general solution were studied and defined.

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Received December 10, 2016