BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(52), 2006, Pages 3–16
ISSN 1024–7696

Lie algebras of operators and invariant $GL(2,\mathbb{R})$-integrals for Darboux type differential systems

O.V. Diaconescu, M.N. Popa

Abstract. In this article two-dimensional autonomous Darboux type differential systems with nonlinearities of the i^{th} $(i=\overline{2,7})$ degree with respect to the phase variables are considered. For every such system the admitted Lie algebra is constructed. With the aid of these algebras particular invariant $GL(2,\mathbb{R})$-integrals as well as first integrals of considered systems are constructed. These integrals represent the algebraic curves of the $(i-1)^{th}$ $(i=\overline{2,7})$ degree. It is showed that the Darboux type systems with nonlinearities of the 2^{nd}, the 4^{th} and the 6^{th} degree with respect to the phase variables do not have limit cycles.

Mathematics subject classification: 34C05, 34C14.
Keywords and phrases: The Darboux type differential system, comitant, invariant $GL(2,\mathbb{R})$-integrating factor, invariant $GL(2,\mathbb{R})$-integral, limit cycle.

Consider the system of differential equations

$$\frac{dx^j}{dt}=a^j_\alpha x^\alpha + a^j_{\alpha_1\alpha_2\ldots\alpha_m}x^{\alpha_1}x^{\alpha_2}\ldots x^{\alpha_m} \quad (j,\alpha,\alpha_1,\alpha_2,\ldots,\alpha_m=1,2;\ m\geq 2), \tag{1}$$

where coefficient tensor $a^j_{\alpha_1\alpha_2\ldots\alpha_m}$ is symmetrical in lower indices, in which the complete convolution holds. The system (1) will be considered with the action of the group $GL(2,\mathbb{R})$ of center-affine transformations [1].

We shall consider the following center-affine invariants and comitants [1] of the system (1) written in the tensorial form

$$I_1=a^\alpha_\alpha,\quad I_2=a^\alpha_\beta a^\beta_\alpha,\quad K_2=a^\alpha_\beta x^\beta x^\gamma \varepsilon_{\alpha\gamma},\ \widetilde{K}_{m-1}=a^\alpha_{\alpha\alpha_1\alpha_2\ldots\alpha_{m-1}}x^{\alpha_1}x^{\alpha_2}\ldots x^{\alpha_{m-1}},$$

$$\widetilde{K}_{m+1}=a^\alpha_{\alpha_1\alpha_2\ldots\alpha_m}x^{\alpha_1}x^{\alpha_2}\ldots x^{\alpha_m}x^\beta\varepsilon_{\alpha\beta}, \tag{2}$$

where $\varepsilon_{\alpha\beta}$ is the unit bi-vector with coordinates $\varepsilon_{11}=\varepsilon_{22}=0,\ \varepsilon_{12}=-\varepsilon_{21}=1$.

It is easy to see that when the condition $\widetilde{K}_{m+1}\equiv 0$ holds, the system (1) takes the form

$$\frac{dx^j}{dt}=a^j_\alpha x^\alpha + mx^jR(x^1,x^2)\equiv P^j(x^1,x^2) \quad (j,\alpha=1,2), \tag{3}$$

where $R(x^1,x^2)$ is a homogeneous polynomial of the $(m-1)^{th}$ order. As it is well known, the system (3) is called a Darboux type differential system (see, for example, [2,3]).

A series of papers is devoted to the problem of the investigation of systems of the form (3) from different points of view (see, for example, [2–7]).

Note that the family of systems (3) is a subset of the family of systems (1) defined via center-affine invariant conditions. Indeed, one can verify easily that for the system (3) the conditions $\widetilde{K}_{m+1} \equiv 0$, $\widetilde{K}_{m-1} = (m+1)R(x^1, x^2)$ hold. Therefore we have the next

Lemma 1. *A system (1) belongs to a family of the Darboux type differential systems (3) with* $R(x^1, x^2) \not\equiv 0$ *if and only if* $\widetilde{K}_{m+1} \equiv 0$, $\widetilde{K}_{m-1} \not\equiv 0$.

For the system (1) with $\widetilde{K}_{m+1} \equiv 0$ and $m = 2, 3, ..., 7$ or, that is the same, for (3) with $m = 2, 3, ..., 7$, two algebraic curves of the form

$$\sum_{j=0}^{k} A_j (x^1)^{k-j} (x^2)^j = B_k \quad (k = \overline{2, m-1}), \tag{4}$$

where $B_2 = 0$, and $B_{m-1} \neq 0$ and A_j are polynomials in the coefficients of this system, are particular invariant $GL(2, \mathbb{R})$-integrals.

Remark 1. The construction of particular invariant $GL(2, \mathbb{R})$-integrals (4) is remarkable, because as it is shown in [2], the system (3) can have only one limit cycle and if it exists, it represents an algebraic curve of the form (4) with $k = m - 1$ and $B_k \neq 0$, surrounding the origin of coordinates.

Lemma 2. *If the factorization over* $\mathbb{C}[x, y]$ *of the left-hand side of the algebraic curve of the form (4) with* $B_{m-1} \neq 0$ *contains at least one real linear factor, then this algebraic curve cannot be of the ellipsoidal form.*

Proof. Suppose that some algebraic curve of the form (4) with $B_{m-1} \neq 0$ can be written as

$$(Ax^1 + Bx^2) \sum_{j=0}^{m-2} A'_j (x^1)^{m-j-2} (x^2)^j = B_{m-1},$$

where the linear factor $Ax^1 + Bx^2$ is real. Suppose that the last equation has the ellipsoidal form, surrounding the origin of coordinates, it means, that any line, passing through the origin, has to intersect the curve in two points. Particularly, this holds for the line $Ax^1 + Bx^2 = 0$. However, in this case we get the contradiction: at the intersection points we have $0 = B_{m-1}$, i.e. the assertion of Lemma 2 is true.

Theorem 3. *System (1) with* $\widetilde{K}_{m+1} \equiv 0$ *has the particular invariant* $GL(2, \mathbb{R})$*-integral*

$$K_2 = 0,$$

where K_2 *is from (2).*

Proof. According to Lemma 1, the system (1) with $\widetilde{K}_{m+1} \equiv 0$ has the form (3). Denote by Λ the operator

$$P^1(x^1, x^2)\frac{\partial}{\partial x^1} + P^2(x^1, x^2)\frac{\partial}{\partial x^2}, \tag{5}$$

where P^j $(j = 1, 2)$ is from (3). It is easy to see that

$$\Lambda(K_2) = K_2\left(I_1 + \frac{2m}{m+1}\widetilde{K}_{m-1}\right),$$

where I_1, K_2 and $\widetilde{K}_{m-1}$ are from (2). This identity shows that K_2 is a particular integral of the system (3) or, that is the same, of the system (1) with $\widetilde{K}_{m+1} \equiv 0$. Theorem 3 is proved.

Consider the differential operator

$$X = \xi^1\frac{\partial}{\partial x^1} + \xi^2\frac{\partial}{\partial x^2}, \tag{6}$$

where ξ^1 and ξ^2 are polynomials in variables x^1, x^2 and in coefficients of the system (3).

According to [8], we can show that the system (3) admits the operator (6) if and only if its coordinates satisfy the system of constitutive equations

$$\xi^j_{x^\alpha}P^\alpha = \xi^\beta P^j_{x^\beta} \quad (j, \alpha, \beta = 1, 2), \tag{7}$$

where $\xi^j_{x^\alpha} = \dfrac{\partial \xi^j}{\partial x^\alpha}$ and $P^j_{x^\beta} = \dfrac{\partial P^j}{\partial x^\beta}$.

As well, according to [8] we have that if the system (3) admits the operator (6), then we can apply Lie theorem on integrating factor: *The system (3) admits a group with the operator (6) if and only if the function μ of the form*

$$\mu^{-1} = \xi^1 P^2 - \xi^2 P^1 \tag{8}$$

is an integrating factor of the equation

$$P^2 dx^1 - P^1 dx^2 = 0. \tag{9}$$

In what follows we shall say that μ is an integrating factor of the system (3) if it is an integrating factor of the equation (9).

Theorem 4. *The system (1) with $m = 2$ and $\widetilde{K}_3 \equiv 0$ has the invariant $GL(2, \mathbb{R})$-integrating factor μ of the form*

$$\mu^{-1} = K_2\Phi_1,$$

where K_2 *is from (2) and*

$$\Phi_1 \equiv 8I_1\widetilde{K}_1 - 12K_3 + 3(I_1^2 - I_2) = 0 \tag{10}$$

is a particular invariant $GL(2,\mathbb{R})$*-integral of this system.*

In (10) invariants and comitants $I_1,\ K_2,\ \widetilde{K}_1 = a^{\alpha}_{\alpha\beta}x^{\beta}$ *are taken from (2), and*

$$K_3 = a^{\alpha}_{\beta}a^{\beta}_{\alpha\gamma}x^{\gamma}$$

are defined in [1].

Proof. Consider the system (3) with $m = 2$ and $x^1 = x, x^2 = y$, written in the form

$$\begin{aligned} \frac{dx}{dt} &= cx + dy + 2x(gx + hy) \equiv P^1(x,y),\\ \frac{dy}{dt} &= ex + fy + 2y(gx + hy) \equiv P^2(x,y). \end{aligned} \tag{11}$$

where $c, d, e, f, g, h \in \mathbb{R}$.

Considering (7) it is easy to verify that the system (11) admits the two-dimensional commutative Lie algebra of operators of the form

$$\begin{aligned} &Z_1 = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)[2(fg - eh)x + 2(ch - dg)y + cf - de],\\ &Z_2 = \{[h(cf - de) + c(dg - ch)]\,x + d(dg - ch)y + 2\,[g(dg - ch) + h(fg-\\ &-eh)x^2]\}\,\frac{\partial}{\partial x} + \{e(dg - ch)x + d(fg - eh)y + 2\,[g(dg - ch) + h(fg - eh)]\,xy\}\,\frac{\partial}{\partial y}. \end{aligned}$$

Using any one of these operators and (8) we obtain up to a constant factor an integrating factor of the system (11), in the form

$$\mu^{-1} = [2(fg - eh)x + 2(ch - dg)y + cf - de]\,[-ex^2 + (c - f)xy + dy^2]. \tag{12}$$

Remark 2. In what follows we will use invariants I_1 and I_2 and comitant K_2 from (2) for the system (3) with $a^1_1 = c,\ a^1_2 = d,\ a^2_1 = e,\ a^2_2 = f$

$$I_1 = c + f,\quad I_2 = c^2 + 2de + f^2,\quad K_2 = -ex^2 + (c - f)xy + dy^2. \tag{13}$$

Besides I_1, I_2, K_2, calculating for the system (11) the comitants $\widetilde{K}_1 = a^{\alpha}_{\alpha\beta}x^{\beta}$ and K_3 we obtain

$$\widetilde{K}_1 = 3(gx + hy),\quad K_3 = [g(2c + f) + eh]x + [h(c + 2f) + dg]y. \tag{14}$$

We observe that the second factor from (12) exactly coincides with K_2. Moreover, considering (13),(14) we obtain the first factor from (12) in the form (10) up to a constant factor.

Using the operator (5), one can verify that the first factor from (12) as well as the second one (Theorem 3) is a particular integral for the system (11), or, that is the same, for the system (1) with $m = 2$ and $\widetilde{K}_3 \equiv 0$. Theorem is proved.

Theorem 5. *The system (1) with $m = 3$ and $\widetilde{K}_4 \equiv 0$ has an invariant $GL(2, \mathbb{R})$-integrating factor μ of the form*

$$\mu^{-1} = K_2\Phi_2,$$

and

$$\Phi_2 \equiv 3(4I_1Q_2 - 3I_1^2\widetilde{K}_2 + 2J_7K_2) - 4I_1(I_1^2 - I_2) = 0 \tag{15}$$

is a particular invariant $GL(2, \mathbb{R})$-integral of this system.

In the last expression the invariants and comitants $I_1,\ I_2, K_2,\ \widetilde{K}_2 = a^{\alpha}_{\alpha\beta\gamma}x^{\beta}x^{\gamma}$ are taken from (2), and

$$J_7 = a^{\alpha}_{p}a^{\beta}_{\alpha\beta q}\varepsilon^{pq},\ Q_2 = a^{\alpha}_{\beta}a^{\beta}_{\alpha\gamma\delta}x^{\gamma}x^{\delta} \tag{16}$$

are defined in [9] $(\varepsilon^{11} = \varepsilon^{22} = 0,\ \varepsilon^{12} = -\varepsilon^{21} = 1)$.

Proof. Consider the system (3) with $m = 3$ and $x^1 = x, x^2 = y$, written in the form

$$\begin{aligned} \frac{dx}{dt} &= cx + dy + 3x(gx^2 + hxy + iy^2) \equiv P^1(x, y),\\ \frac{dy}{dt} &= ex + fy + 3y(gx^2 + hxy + iy^2) \equiv P^2(x, y), \end{aligned} \tag{17}$$

where $c, d, e, f, g, h, i \in \mathbb{R}$.

Then it is easy to verify with the aid of constitutive equations (7) that this system admits the two-dimensional commutative Lie algebra of operators

$$Z_1 = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\{(c + f)(cf - de) + 3[(cf - de)g + f^2g - efh + e^2i]x^2-$$

$$-6(dfg - cfh + cei)xy + 3[d^2g + c(c + f)i - d(ch + ei)]y^2\},$$

$$Z_2 = \{[defh+c(-2dfg+deh-f^2h)+c^2(fh-2ei)]x+3[dg(-2fg+eh)+cg(fh-2ei)+$$
$$+h(-f^2g+efh-e^2i)]x^3-2d(dfg-cfh+cei)y+3[-d^2gh+ci(-ch+fh-2ei)+d(ch^2-$$
$$-2fgi+ehi)]xy^2\}\frac{\partial}{\partial x}+\{-2e(dfg-cfh+cei)x+(d(-2f^2g+ceh+efh)+cf(-ch+$$
$$+fh-2ei))y+3(dg(-2fg+eh)+cg(fh-2ei)+h(-f^2g+efh-e^2i))x^2y+3(-d^2gh+$$
$$+ci(-ch + fh - 2ei) + d(ch^2 - 2fgi + ehi))y^3\}\frac{\partial}{\partial y}.$$

Using any of these operators and equality (8) we obtain up to a constant an integrating factor of the system (17) in the form

$$\mu^{-1} = \{(c+f)(cf-de)+3[(cf-de)g+f^2g-efh+e^2i]x^2-6(dfg-cfh+cei)xy+ +3[d^2g+c(c+f)i-d(ch+ei)]y^2\}[-ex^2+(c-f)xy+dy^2]. \qquad (18)$$

Calculating J_7, Q_2 and $\widetilde{K}_2 = a^\alpha_{\alpha\beta\gamma}x^\beta x^\gamma$ for the system (17) we obtain

$$\widetilde{K}_2 = 4(gx^2+hxy+iy^2),\ J_7 = -4dg+2ch-2fh+4ei,\ Q_2 = (3cg+fg+eh)x^2+ +2(dg+ch+fh+ei)xy+(dh+ci+3fi)y^2. \qquad (19)$$

We observe that the second factor from (18) exactly coincides with K_2. Moreover, considering (13) and (19) we obtain the first factor from (18) in form (15) up to a constant factor.

Using the operator (5), one can verify that the first factor from (18) as well as the second one (Theorem 3) is a particular integral for the system (17) or, that is the same, for the system (1) with $m = 3$ and $\widetilde{K}_4 \equiv 0$. Theorem is proved.

Remark 3. In [7] it is shown that for the existence of a limit cycle for the system (1) with $m = 3$ and $\widetilde{K}_4 \equiv 0$, surrounding the origin, it is necessary and sufficient that the following conditions hold

$$2I_2 - I_1^2 < 0;\quad I_1^2J_4 + 2J_7^2 > 0;\quad I_1(4I_1Q_2 - 3I_1^2\widetilde{K}_2 + 2J_7K_2)|_{y=0} > 0,$$

where $I_1, I_2, J_7, K_2, \widetilde{K}_2, Q_2$ are from (2) and (16) and $J_4 = a^\alpha_{\alpha pr}a^\beta_{\beta qs}\varepsilon^{pr}\varepsilon^{rs}$. Moreover, the limit cycle is unique and it is stable (unstable) if $I_1 > 0$ ($I_1 < 0$) and has the form (15).

Theorem 6. *Differential system (1) with $m = 4$ and $\widetilde{K}_5 \equiv 0$ has the invariant $GL(2,\mathbb{R})$-integrating factor μ of the form*

$$\mu^{-1} = K_2\Phi_3,$$

and

$$\Phi_3 \equiv 8(5I_1^2-I_2)(4I_1\widetilde{K}_3-5M_1)+96K_2(M_3-2I_1M_2)+15(5I_1^2-I_2)(I_1^2-I_2) = 0 \quad (20)$$

is a particular invariant $GL(2,\mathbb{R})$-integral of this system.

Here invariants and comitants $I_1,\ I_2, K_2,\ \widetilde{K}_3 = a^\alpha_{\alpha\beta\gamma\delta}x^\beta x^\gamma x^\delta$ are from (2), and

$$M_1 = a^\alpha_\beta a^\beta_{\alpha\gamma\delta\mu}x^\gamma x^\delta x^\mu,\ M_2 = a^\alpha_\beta a^\gamma_{\delta\alpha\gamma\mu}x^\mu\varepsilon^{\beta\delta},\quad M_3 = a^\alpha_\beta a^\gamma_\delta a^\mu_{\mu\gamma\nu}x^\delta\varepsilon^{\beta\nu}.$$

Proof. Consider the system (3) with $m = 4$ and $x^1 = x,\ x^2 = y$, written in the form

$$\begin{aligned} \frac{dx}{dt} &= cx + dy + 4x(gx^3 + hx^2y + ixy^2 + jy^3), \\ \frac{dy}{dt} &= ex + fy + 4y(gx^3 + hx^2y + ixy^2 + jy^3), \end{aligned} \tag{21}$$

where $c, d, e, f, g, h, i, j \in \mathbb{R}$.

This system admits a two-dimensional commutative Lie algebra with one of operators in the form

$$Z_1 = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \right) \times \varphi_3(x, y),$$

where

$$\varphi_3(x, y) = \{-(cf - de)[2(c + f)^2 + (cf - de)] - 4[2c^2fg + de(-5fg + eh)-$$
$$-c[2deg+f(-5fg+eh)]+2(f^3g-ef^2h+e^2fi-e^3j)]x^3-12[d^2eg-d(cfg+2f^2g+ceh)+$$
$$+c(cfh+2f^2h-2efi+2e^2j)]x^2y-12[2d^2fg+c(2c+f)(fi-ej)+d[-2cfh+e(-fi+$$
$$+ej)]]xy^2+4[2d^3g-d^2(2ch+ei)-c(2c^2+5cf+2f^2)j+d(2c^2i+cfi+5cej+2efj)]y^3\}.$$

Using this operator and equality (8) we obtain up to a constant an integrating factor of the system (21) in the form

$$\mu^{-1} = \varphi_3(x, y) \times [-ex^2 + (c - f)xy + dy^2]. \tag{22}$$

Calculating $M_1,\ M_2,\ M_3$ and $\widetilde{K}_3$ for the system (21) we obtain

$$\begin{aligned} &\widetilde{K}_3 = 5(gx^3 + hx^2y + ixy^2 + jy^3),\ M_1 = (4cg + fg + eh)x^3 + (3dg + 3ch + 2fh+ \\ &+ 2ei)x^2y + (2dh + 2ci + 3fi + 3ej)xy^2 + (di + cj + 4fj)y^3,\ M_2 = -\frac{5}{3}(3dg- \\ &ch + fh - ei)x - \frac{5}{3}(dh - ci + fi - 3ej)y,\ M_3 = \frac{5}{3}(-3cdg + c^2h - deh - cfh+ \\ &+ 2cei - efi + 3e^2j)x + \frac{5}{3}(-3d^2g + cdh - 2dfh + dei + cfi - f^2i + 3efj)y. \end{aligned} \tag{23}$$

The second factor from (22) exactly coincides with K_2. Moreover considering (13) and (23) we obtain the first factor from (22) in the form (20) up to a constant.

Using the operator (5), one can verify that the first factor from (22) as well as the second one (Theorem 3) is a particular integral for the system (21) or, that is the same, for the system (1) with $m = 4$ and $\widetilde{K}_5 \equiv 0$. The theorem is proved.

Theorem 7. *The differential system (1) with* $m = 5$ *and* $\widetilde{K}_6 \equiv 0$ *has the invariant* $GL(2,\mathbb{R})$*-integrating factor* μ *of the form*

$$\mu^{-1} = K_2\Phi_4,$$

and

$$\Phi_4 \equiv 5(5I_1^2 - 2I_2)(5I_1^2\widetilde{K}_4 - 6I_1N_1) - 60K_2(3I_1^2N_2 - 2I_1N_3 - K_2N_4)+$$
$$+12I_1(I_1^2 - I_2)(5I_1^2 - 2I_2) = 0 \qquad (24)$$

is a particular invariant $GL(2,\mathbb{R})$*-integral of this system.*

Here invariants and comitants $I_1,\ I_2, K_2,\ \widetilde{K}_4 = a^{\alpha}_{\alpha\beta\gamma\delta\mu}x^\beta x^\gamma x^\delta x^\mu$ *are from (2), and*

$$N_1 = a^\alpha_\beta a^\beta_{\alpha\gamma\delta\mu\nu}x^\gamma x^\delta x^\mu x^\nu,\ \ N_2 = a^\alpha_p a^\beta_{q\alpha\beta\gamma\delta}x^\gamma x^\delta \varepsilon^{pq},\quad N_3 = a^\alpha_p a^\beta_\delta a^\gamma_{\alpha\beta\gamma\mu q}x^\delta x^\mu \varepsilon^{pq},$$
$$N_4 = a^\alpha_p a^\beta_r a^\gamma_{\alpha\beta\gamma sq}\varepsilon^{pq}\varepsilon^{rs}.$$

Proof. Consider the system (3) with $m = 5$ and $x^1 = x,\ x^2 = y$ in the form

$$\begin{aligned}\frac{dx}{dt} &= cx + dy + 5x(gx^4 + hx^3y + ix^2y^2 + jxy^3 + ky^4),\\ \frac{dy}{dt} &= ex + fy + 5y(gx^4 + hx^3y + ix^2y^2 + jxy^3 + ky^4),\end{aligned} \qquad (25)$$

where $c, d, e, f, g, h, i, j, k \in \mathbb{R}$. This system admits a two-dimensional commutative Lie algebra with one of operators

$$Z_1 = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \times \varphi_4(x, y),$$

where

$$\varphi_4(x, y) = (c + f)(cf - de)[3(c + f)^2 + 4(cf - de)] + 5[3d^2e^2g + 3c^3fg-$$
$$-c^2[3deg + f(-13fg + eh)] - de(13f^2g - 4efh + e^2i) + c[de(-16fg + eh) + f(13f^2g-$$
$$-4efh+e^2i)]+3(f^4g-ef^3h+e^2f^2i-e^3fj+e^4k)]x^4+20[df(4de-3f^2)g+c^3fh-c^2(dfg+$$
$$+deh-4f^2h+efi)+c(d^2eg-4df^2g-4defh+3f^3h+de^2i-3ef^2i+3e^2fj-3e^3k)]x^3y-$$
$$-10[3d^3eg-d^2[9f^2g+3c(fg+eh)+e^2i]-c(3c+f)(cfi+3f^2i-3efj+3e^2k)+d[3c^2(fh+$$
$$+ei)+cf(9fh+2ei)+3e(f^2i-efj+e^2k)]]x^2y^2-20[3d^3fg-d^2f(3ch+ei)-c(3c^2+4cf+$$
$$+f^2)(fj-ek)+d[3c^2fi+ef(fj-ek)+c(f^2i+4efj-4e^2k)]]xy^3+5[3d^4g-d^3(3ch+$$
$$+ei)+c(3c^3+13c^2f+13cf^2+3f^3)k+d^2(3c^2i+cfi+4cej+efj+3e^2k)-d(c+f)(3c^2j+$$
$$+cfj + 13cek + 3efk)]y^4.$$

Using this operator and equality (8) we obtain up to a constant an integrating factor of the system (25), in the form

$$\mu^{-1} = \varphi_4(x,y) \times [-ex^2 + (c-f)xy + dy^2]. \tag{26}$$

Calculating $I_1,\ I_2,\ K_2,\ \widetilde{K}_4 = a^{\alpha}_{\alpha\beta\gamma}x^{\beta}x^{\gamma}$ and $N_1,\ N_2,\ N_3,\ N_4$ for the system (25) we obtain the expression (26) in the invariant form. Theorem is proved.

In the same way for the system (3) with $m = 6$ and $x^1 = x,\ x^2 = y$ written in the form

$$\begin{aligned} \frac{dx}{dt} &= cx + dy + 6x(gx^5 + hx^4y + ix^3y^2 + jx^2y^3 + kxy^4 + ly^5),\\ \frac{dy}{dt} &= ex + fy + 6y(gx^5 + hx^4y + ix^3y^2 + jx^2y^3 + kxy^4 + ly^5) \end{aligned} \tag{27}$$

a two-dimensional commutative Lie algebra is obtained with one of operators in the form

$$Z_1 = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \times \varphi_5(x,y),$$

where

$$\begin{aligned}
\varphi_5(x,y) &= (cf-de)[4(c+f)^2+9(cf-de)][6(c+f)^2+cf-de]+6\,[24c^4fg + d^2e^2(97fg-\\
&-9eh) - 2c^3[12deg + f(-77fg+3eh)] + c^2[2de(-103fg+3eh) + f(269f^2g - 37efh+\\
&+4e^2i)]+2de(-77f^3g+29ef^2h-11e^2fi+3e^3j)+2c[26d^2e^2g+de(-183f^2g+23efh-\\
&-2e^2i)+f(77f^3g-29ef^2h+11e^2fi-3e^3j)]+24(f^5g-ef^4h+e^2f^3i-e^3f^2j+e^4fk-\\
&-e^5l)\,]\,x^5+30\,[\,d(-9d^2e^2+58def^2-24f^4)g+6c^4fh-c^3[6d(fg+eh)+f(-37fh+4ei)]+\\
&+c^2(6d^2eg-37df^2g-46defh+58f^3h+4de^2i-22ef^2i+6e^2fj)+c[d^2e(46fg+9eh)-\\
&-2d(29f^3g+29ef^2h-11e^2fi+3e^3j)+24(f^4h-ef^3i+e^2f^2j-e^3fk+e^4l)]]x^4y+\\
&+30[8c^4fi-2c^3[4d(fh+ei)+f(-23fi+6ej)]+c^2[8d^2(fg+eh)+59f^3i-51ef^2j-\\
&-4d(11f^2h+12efi-3e^2j)+48e^2fk-48e^3l]+d[-44d^2efg+d(48f^3g+11e^2fi-3e^3j)+\\
&+12e(-f^3i+ef^2j-e^2fk+e^3l)]-2c[4d^3eg-d^2(22f^2g+22efh+e^2i)+df(24f^2h+11efi-\\
&-3e^2j)+6f(-f^3i+ef^2j-e^2fk+e^3l)]]x^3y^2+30[12d^4eg-3d^3[16f^2g+4c(fg+eh)+e^2i]+\\
&+d^2[12c^2(fh+ei)+2ef(6fi+ej)+c(48f^2h+6efi+11e^2j)]+c(12c^2+11cf+2f^2)(cfj+\\
&+4f^2j-4efk+4e^2l)-d[12c^3(fi+ej)+c^2f(51fi+22ej)+8ef(f^2j-efk+e^2l)+4c(3f^3i+\\
&+12ef^2j-11e^2fk+11e^3l)]]x^2y^3+30[24d^4fg-6d^3f(4ch+ei)+c(24c^3+58c^2f+37cf^2+\\
&+6f^3)(fk-el)-2d[12c^3fj+3ef^2(fk-el)+c^2(11f^2j+29efk-29e^2l)+cf(2f^2j+\\
&+23efk-23e^2l)]+d^2[24c^2fi+2cf(3fi+11ej)+e(4f^2j+9efk-9e^2l)]]xy^4-6[24d^5g-
\end{aligned}$$

$-6d^4(4ch+ei)+d^3(24c^2i+6cfi+22cej+4efj+9e^2k)-c(24c^4+154c^3f+269c^2f^2+$
$+154cf^3+24f^4)l+d[24c^4k+24ef^3l+2c^3(29fk+77el)+2cf^2(3fk+103el)+c^2f(37fk+$
$+366el)]-d^2[24c^3j+c^2(22fj+58ek)+2ef(3fk+26el)+c(4f^2j+46efk+97e^2l)]]y^5.$

Using this operator and equality (8) we obtain up to a constant an integrating factor of the system (27), in the form

$$\mu^{-1}=\varphi_5(x,y)\times[-ex^2+(c-f)xy+dy^2].$$

Therefore we have the next

Theorem 8. *Differential system (1) with $m=6$ and $\widetilde{K}_7\equiv 0$ has the invariant $GL(2,\mathbb{R})$-integrating factor μ of the form*

$$\mu^{-1}=K_2\Phi_5,$$

and

$$\Phi_5\equiv 12(17I_1^2-9I_2)(13I_1^2-I_2)(6I_1\widetilde{K}_5-7O_1)-480(13I_1^2-I_2)K_2(4I_1O_2-3O_3)+$$
$$+5760K_2^2(3I_1O_4-O_5)+35(I_1^2-I_2)(17I_1^2-9I_2)(13I_1^2-I_2)=0 \qquad (28)$$

is a particular invariant $GL(2,\mathbb{R})$-integral of this system.

Here invariants and comitants $I_1,\ I_2, K_2,\ \widetilde{K}_5=a^{\alpha}_{\alpha\beta\gamma\delta\mu\nu}x^\beta x^\gamma x^\delta x^\mu x^\nu$ are from (2), and

$$O_1=a^\alpha_\beta a^\beta_{\alpha\gamma\delta\mu\nu\eta}x^\gamma x^\delta x^\mu x^\nu x^\eta,\ O_2=a^\alpha_p a^\beta_{q\alpha\beta\gamma\delta\mu}x^\gamma x^\delta x^\mu\varepsilon^{pq},\ \ O_3=a^\alpha_p a^\beta_\delta a^\gamma_{\alpha\beta\gamma\mu\nu q}x^\delta x^\mu x^\nu\varepsilon^{pq},$$
$$O_4=a^\alpha_p a^\beta_r a^\gamma_{\alpha\beta\gamma\delta qs}x^\delta\varepsilon^{pq}\varepsilon^{rs},\quad O_5=a^\alpha_p a^\beta_\mu a^\gamma_r a^\delta_{\alpha\beta\gamma\delta qs}x^\mu\varepsilon^{pq}\varepsilon^{rs}.$$

For the system (3) with $m=7$ and $x^1=x,\ x^2=y$ written in the form

$$\begin{aligned}\frac{dx}{dt}&=cx+dy+7x(gx^6+hx^5y+ix^4y^2+jx^3y^3+kx^2y^4+lxy^5+ny^6),\\ \frac{dy}{dt}&=ex+fy+7y(gx^6+hx^5y+ix^4y^2+jx^3y^3+kx^2y^4+lxy^5+ny^6)\end{aligned} \qquad (29)$$

a two-dimensional commutative Lie algebra is also found , for which one of operators has the form

$$Z_1=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\times\varphi_6(x,y),$$

where

$\varphi_6(x,y)=(c+f)(cf-de)[2(c+f)^2+cf-de][5(c+f)^2+16(cf-de)]+7[-10d^3e^3g+$
$+10c^5fg+c^4[-10deg+f(87fg-2eh)]+d^2e^2(101f^2g-16efh+2e^2i)+c^3[2de(-61fg+$
$+eh)+f(227f^2g-17efh+e^2i)]+c^2[35d^2e^2g-de(353f^2g-23efh+e^2i)+f(227f^3g-$

$$-42ef^2h+8e^2fi-e^3j)]+de(-87f^4g+37ef^3h-17e^2f^2i+7e^3fj-2e^4k)+c[2d^2e^2(68fg-$$
$$-3eh)+de(-328f^3g+58ef^2h-10e^2fi+e^3j)+f(87f^4g-37ef^3h+17e^2f^2i-7e^3fj+$$
$$+2e^4k)]+10(f^6g-ef^5h+e^2f^4i-e^3f^3j+e^4f^2k-e^5fl+e^6n)]x^6+42[df(-16d^2e^2+$$
$$+37def^2-10f^4)g+2c^5fh-c^4[2d(fg+eh)+f(-17fh+ei)]+c^3[2d^2eg+d(-17f^2g-$$
$$-23efh+e^2i)+f(42f^2h-8efi+e^2j)]+c^2[d^2e(23fg+6eh)-d(42f^3g+58ef^2h-10e^2fi+$$
$$+e^3j)+f(37f^3h-17ef^2i+7e^2fj-2e^3k)]+c[-6d^3e^2g+2d^2e(29f^2g+8efh-e^2i)+$$
$$+d(-37f^4g-37ef^3h+17e^2f^2i-7e^3fj+2e^4k)+10(f^5h-ef^4i+e^2f^3j-e^3f^2k+e^4fl-$$
$$-e^5n)]]x^5y+21[5c^5fi-c^4[5d(fh+ei)+f(-41fi+5ej)]+c^3[5d^2(fg+eh)+d(-40f^2h-$$
$$-52efi+5e^2j)+f(93f^2i-36efj+10e^2k)]+c^2[-5d^3eg+67f^4i+d^2(40f^2g+50efh+$$
$$+11e^2i)-57ef^3j+52e^2f^2k-d(85f^3h+103ef^2i-37e^2fj+10e^3k)-50e^3fl+50e^4n]+$$
$$+d[10d^3e^2g-d^2(85ef^2g+2e^3i)+d(50f^4g+17e^2f^2i-7e^3fj+2e^4k)-10e(f^4i-ef^3j+$$
$$+e^2f^2k-e^3fl+e^4n)]-c[10d^3e(5fg+eh)+d^2(-85f^3g-85ef^2h-12e^2fi+e^3j)+$$
$$+2df(25f^3h+17ef^2i-7e^2fj+2e^3k)-10f(f^4i-ef^3j+e^2f^2k-e^3fl+e^4n)]]x^4y^2+$$
$$+14[10c^5fj-c^4[10d(fi+ej)+f(-77fj+20ek)]+2c^3[5d^2(fh+ei)+75f^3j-57ef^2k-$$
$$-2d(18f^2i+21efj-5e^2k)+50e^2fl-50e^3n]+c^2[-10d^3(fg+eh)+d^2(70f^2h+74efi+$$
$$+7e^2j)-2df(57f^2i+50efj-14e^2k)+f(77f^3j-72ef^2k+70e^2fl-70e^3n)]+df[70d^3eg-$$
$$-2d^2(50f^2g+7e^2i)+de(20f^2i+7efj-2e^2k)+10e(-f^3j+ef^2k-e^2fl+e^3n)]+$$
$$+2c[5d^4eg-d^3(35f^2g+35efh+e^2i)+d^2(50f^3h+14ef^2i+25e^2fj-7e^3k)+5f^2(f^3j-$$
$$-ef^2k+e^2fl-e^3n)+d(-10f^4i-42ef^3j+37e^2f^2k-35e^3fl+35e^4n)]]x^3y^3-21[10d^5eg-$$
$$-2d^4[25f^2g+5c(fg+eh)+e^2i]+d^3[10c^2(fh+ei)+c(50f^2h+4efi+7e^2j)+e(10f^2i+efj+$$
$$+2e^2k)]-c(10c^3+17c^2f+8cf^2+f^3)(cfk+5f^2k-5efl+5e^2n)-d^2[10c^3(fi+ej)+$$
$$+cf(10f^2i+37efj+12e^2k)+c^2(52f^2i+14efj+17e^2k)+e(5f^3j+11ef^2k-10e^2fl+$$
$$+10e^3n)]+d[10c^4(fj+ek)+c^3f(57fj+34ek)+5ef^2(f^2k-efl+e^2n)+cf(5f^3j+$$
$$+52ef^2k-50e^2fl+50e^3n)+c^2(36f^3j+103ef^2k-85e^2fl+85e^3n)]]x^2y^4-42[10d^5fg-$$
$$-2d^4f(5ch+ei)+d^3f(10c^2i+2cfi+7cej+efj+2e^2k)-c(10c^4+37c^3f+42c^2f^2+17cf^3+$$
$$+2f^4)(fl-en)+d(c+f)[10c^3fk+2ef^2(fl-en)+c^2(7f^2k+37efl-37e^2n)+cf(f^2k+$$
$$+21efl-21e^2n)]-d^2[10c^3fj+c^2f(7fj+17ek)+ef(f^2k+6efl-6e^2n)+c(f^3j+$$
$$+10ef^2k+16e^2fl-16e^3n)]]xy^5+7[10d^6g-2d^5(5ch+ei)+d^4(10c^2i+2cfi+7cej+efj+$$
$$+2e^2k)+c(10c^5+87c^4f+227c^3f^2+227c^2f^3+87cf^4+10f^5)n-d(c+f)[10c^4l+10ef^3n+$$
$$+3c^3(9fl+29en)+2cf^2(fl+56en)+c^2f(15fl+241en)]-d^3[10c^3j+c^2(7fj+$$

$$+17ek)+c(f^2j+10efk+16e^2l)+e(f^2k+6efl+10e^2n)]+d^2(c+f)[10c^3k+ \\ +c^2(7fk+37el)+ef(2fl+35en)+c(f^2k+21efl+101e^2n)]]y^6.$$

Using this operator and equality (8) we obtain up to a constant an integrating factor of the system (29), in the form

$$\mu^{-1}=\varphi_6(x,y)\times[-ex^2+(c-f)xy+dy^2]. \tag{30}$$

Analogously to previous cases we have the next

Theorem 9. *Differential system (1) with $m=7$ and $\widetilde{K}_8\equiv 0$ has the invariant $GL(2,\mathbb{R})$-integrating factor μ of the form*

$$\mu^{-1}=K_2\Phi_6,$$

and

$$\Phi_6\equiv 7I_1(13I_1^2-8I_2)(5I_1^2-I_2)(7I_1\widetilde{K}_6-8S_1)-210I_1(5I_1^2-I_2)K_2(5I_1S_2-4S_3)+ \\ +840K_2^2(6I_1^2S_4-3I_1S_5-K_2S_6)+24I_1(I_1^2-I_2)(13I_1^2-8I_2)(5I_1^2-I_2)=0 \tag{31}$$

is a particular invariant $GL(2,\mathbb{R})$-integral of this system.

Here invariants and comitants $I_1,\ I_2,K_2,\ \widetilde{K}_6=a^\alpha_{\alpha\beta\gamma\delta\mu\nu\eta}x^\beta x^\gamma x^\delta x^\mu x^\nu x^\eta$ are from (2), and

$$S_1=a^\alpha_\beta a^\beta_{\alpha\gamma\delta\mu\nu\eta\varphi}x^\gamma x^\delta x^\mu x^\nu x^\eta x^\varphi,\ S_2=a^\alpha_p a^\beta_{q\alpha\beta\gamma\delta\mu\nu}x^\gamma x^\delta x^\mu x^\nu\varepsilon^{pq},$$

$$S_3=a^\alpha_p a^\beta_\delta a^\gamma_{\alpha\beta\gamma\mu\nu\eta q}x^\delta x^\mu x^\nu x^\eta\varepsilon^{pq},\quad S_4=a^\alpha_p a^\beta_r a^\gamma_{\alpha\beta\gamma\delta\mu qs}x^\delta x^\mu\varepsilon^{pq}\varepsilon^{rs},$$

$$S_5=a^\alpha_p a^\beta_\mu a^\gamma_r a^\delta_{\alpha\beta\gamma\delta\nu qs}x^\mu x^\nu\varepsilon^{pq}\varepsilon^{rs},\quad S_6=a^\alpha_p a^\beta_r a^\gamma_k a^\delta_{\alpha\beta\gamma\delta qsl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}.$$

It is shown in [2, 3] that all singular points of Darboux type differential system (3), different from the origin, are located on its integral straight lines, coinciding with integral straight lines of this system for $R_{m-1}\equiv 0$. Therefore the necessary condition for the existence of a limit cycle for the Darboux type differential system (3) is the condition that the eigenvalues of the matrix of linear terms should be imaginary, i.e. the condition [1] $2I_2-I_1^2<0$.

We observe that the expression Φ_{m-1} from (10), (15), (20), (24), (28) and (31) with $m=\overline{2,7}$ are only algebraic integrals of the form (4) for the Darboux type system (3) with $m=\overline{2,7}$. To prove this remark it is sufficient to examine the explicit form of first integrals for the system (3) with $m=\overline{2,7}$.

One can verify easily that holds

Theorem 10. *The Darboux type differential system (3) with $2I_2-I_1^2<0$ has the first real integral in the form*

$$\frac{I_1}{\sqrt{I_1^2-2I_2}}\arctan\frac{2a_1^2x^1+(a_2^2-a_1^1)x^2}{|x^2|\sqrt{I_1^2-2I_2}}+\frac{1}{2}\ln|K_2|-\frac{1}{m-1}\ln|\Phi_{m-1}|=C\ \ (m=\overline{2,7}), \tag{32}$$

where K_2 is from (2), Φ_{m-1} $(m=\overline{2,7})$ are from (10), (15), (20), (24), (28) and (31).

It is clear from (32) that $\Phi_{m-1}\ \ (m = \overline{2,7})$ is the only one algebraic integral of the form (4).

As for differential systems (11), (21) and (27) the corresponding algebraic invariant integrals (10), (20) and (28) have the homogeneities of odd degree with respect to x^1 and x^2, than with the aid of Remark 1 and Lemma 2 we prove

Theorem 11. *The differential system (1) with* $m = 2l$ *and* $\widetilde{K}_{2l+1} \equiv 0,\ \ (l = 1, 2, 3)$ *does not have limit cycles.*

The main idea of this theorem allow us to suppose that systems of the form (1) with $m = 2l$ and $\widetilde{K}_{2l+1} \equiv 0$ where $l \geq 4$ also do not have limit cycles.

It is easy to prove the next

Theorem 12. *For a system (1) with* $K_2 \not\equiv 0$ *and* $\widetilde{K}_{m+1} \equiv 0, (m = \overline{2,7})$ *to have a first invariant* $GL(2,\mathbb{R})$*-integral of the Darboux type [10] in the form*

$$K_2^{1-m}\Phi_{m-1}^2 = C \quad (m = \overline{2,7})$$

it is necessary and sufficient that $I_1 = 0$*, where* $K_2,\ \widetilde{K}_{m+1}, I_1$ *are from (2), and* $\Phi_{m-1}\ \ (m = \overline{2,7})$ *are from (10), (15), (20), (24), (28) and (31).*

The proof of Theorem 12 results from the identity

$$\Lambda(K_2^{1-m}\Phi_{m-1}^2) = (1-m)I_1K_2^{1-m}\Phi_{m-1}^2 \quad (m = \overline{2,7}),$$

where Λ is from (5).

There exists the supposition that Theorem 12 holds for $m \geq 8$.

The following question remains open: Are all first invariant $GL(2,\mathbb{R})$-integrals of the differential system (3) with $(m = \overline{2,7})$ encapsulated by Theorem 12 or not?

The authors tender thanks to Professor N.I.Vulpe for effective discussions of the results of this paper.

References

[1] Sibirsky K.S. *Introduction to the algebraic theory of invariants of differential equations.* Kishinev, Shtiintsa, 1982 (Manchester University Press, Manchester, 1988).

[2] Lucashevich N.A. *The integral curves of Darboux equations.* Differential equations, 1966, **2(5)**, p. 628–633 (in Russian).

[3] Vulpe N.I., Costas S.I. *The center-affine invariant conditions of topological distinctions of the Darboux differential systems with cubic non-linearities.* Preprint, Chisinau, 1989 (in Russian).

[4] Dedok N.N. *On the singular points of differential equation Darboux.* Differential Equations, 1972, **8**, N 10, p. 1880–1881 (in Russian).

[5] Amelkin V.V., Lucashevich N.A., Sadovski A.P. *Nonlinear variation in systems of the second order.* Minsk, 1982 (in Russian).

[6] Gorbuzov V.N., Samodurov A.A. *Darboux equation and its analogues: Optional course manual*. Grodno, 1985 (in Russian).

[7] Vulpe N.I., Costas S.I. *The center-affine invariant conditions for existence of the limit cycle for one Darboux system*. Mat. issled., 1987, vyp. 92, p. 38–42 (in Russian).

[8] Popa M.N. *Algebraic methods for differential systems*. Seria Matematica Aplicata si Industriala, Nr. 15, Flower Power Edit., Pitesti Univers., Romania, 2004 (in Romanian).

[9] Chebanu V.M. *Minimal polynomial basis of comitants of cubic differential systems*. Differential equations, 1985, **21**, N 3, p. 541–543 (in Russian).

[10] Darboux G. *Memoire sur les equations differentielles algebriques du premier order et du premier degree*. Bull. Sciences Math., 1878, Ser.(2), 2(1), p. 60–96, 123–144, 151–200.

Institute of Mathematics and Computer Sciences
Academy of Sciences of Moldova
str. Academiei, 5
MD-2028, Chisinau, Moldova

E-mail: *odiac@math.md, popam@math.md*

Received August 21, 2006

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(52), 2006, Pages 17–26
ISSN 1024–7696

Measure of stability for a finite cooperative game with a generalized concept of equilibrium *

V.A. Emelichev, E.E. Gurevsky, A.A. Platonov

Abstract. We consider a finite cooperative game in the normal form with a parametric principle of optimality (the generalized concept of equilibrium). This principle is defined by the partition of the players into coalitions. In this situation, two extreme cases of this partition correspond to the lexicographically optimal situation and the Nash equilibrium situation, respectively. The analysis of stability for a set of generalized equilibrium situations under the perturbations of the coefficients of the linear payoff functions is performed. Upper and lower bounds of the stability radius in the l_∞-metric are obtained. We show that the lower bound of the stability radius is accessible.

Mathematics subject classification: 91A12, 90C29, 90C31.
Keywords and phrases: Cooperative game, lexicographic optimality, Nash equilibrium, stability radius.

1 Introduction

Let us consider a finite game of several players in the normal form [1, 2], in which each player $i \in N_n = \{1, 2, \dots, n\}$, $n \geq 2$, has a finite number of options for the selection of a strategy $X_i \subset \mathbf{R}$, $2 \leq |X_i| < \infty$. The realization of the game and its result is uniquely determined by the choice of each player. Assume that, on the set of the situations $X = \prod_{i \in N_n} X_i$ of the game, linear payoff functions of the players

$$f_i(x) = C_i x, \quad i \in N_n$$

are defined. Here C_i is the i-th row of the matrix $C = [c_{ij}]_{n \times n} \in \mathbf{R}^{n \times n}$, $x = (x_1, x_2, \dots, x_n)^T$, $x_j \in X_j$, $j \in N_n$. In the course of the game, which is called the game with matrix C, each player i receives the payoff $f_i(x)$, which he or she wants to maximize by using certain relationships of preference. For any game in normal form, the cooperative and noncooperative principles of optimality (equilibrium concepts) are used, which usually leads to different situations (results). In this paper a parametric principle of optimality is considered. Such principle leads to the set of generalized equilibrium situations. The parameter of this principle is the partition of players into coalitions, for which two extreme cases (one coalition of all players

*This work is supported by program of the Ministry of Education "Fundamental and application studies" of the Republic of Belarus (Grant 492/28).

and the set of one-player coalitions) correspond to the lexicographically optimal situation and the Nash equilibrium situation, respectively. The analysis of stability for the set of situations, optimal for a given partition under the perturbations of coefficients of the linear payoff functions is performed. Lower and upper bounds of the stability radius for the problem of finding the set of generalized equilibrium situations are obtained. Note that back in [3–6], formulas of the stability radius of the optimal situation with various generalizations of the concept of equilibrium was obtained.

2 Basic definitions and properties

Now we introduce the binary relation of lexicographic order $\underset{L}{\prec}$ in the space $\mathbf{R}^d$ of any dimension $d \in \mathbf{N}$, assuming that, for any different vectors $y = (y_1, y_2, \ldots, y_d)$ and $y' = (y'_1, y'_2, \ldots, y'_d)$ of the space, the formula

$$y \underset{L}{\prec} y' \Leftrightarrow y_k < y'_k$$

holds, where $k = \min\{i \in N_d : y_i \neq y'_i\}$.

The following property is obvious.

Property 1. Let $y, y' \in \mathbf{R}^d$, $d \in \mathbf{N}$. If $y_1 < y'_1$, then $y \underset{L}{\prec} y'$.

We will call any nonempty subset $J \subseteq N_n$ of players a coalition. Here and below, x_J is the projection of the vector $x \in X$ onto the coordinate axes of the space $\mathbf{R}^n$ with the numbers of coalition J. For any coalition $J \subseteq N_n$ we introduce a binary relation $\Omega(C, J)$ on a set of situations X as follows:

$$x\ \Omega(C, J)\ x' \ \Leftrightarrow \ \begin{cases} C_J x \underset{L}{\prec} C_J x' \ \ \& \ \ x_{N_n \setminus J} = x'_{N_n \setminus J}, & \text{if } J \neq N_n, \\ Cx \underset{L}{\prec} Cx', & \text{if } J = N_n, \end{cases}$$

where C_J is the submatrix of C consisting of the rows with the numbers of the coalition J.

Let $s \in N_n$, $N_n = \bigcup\limits_{r \in N_s} J_r$ be the partition of the set N_n into s coalitions, i. e. $J_r \neq \emptyset$, $r \in N_s$; $p \neq q \Rightarrow J_p \cap J_q = \emptyset$. Under the game with matrix C we understand the problem $Z^n(C, J_1, J_2, \ldots, J_s)$ of finding the set of generalized equilibrium or, in other words of $(J_1, J_2, \ldots, J_s)$-optimal situations according to the formula

$$Q^n(C, J_1, J_2, \ldots, J_s) = \{x \in X : \forall r \in N_s \, \forall x' \in X \; (x \ \overline{\Omega(C, J_r)} \ x')\},$$

where $\overline{\Omega(C, J_r)}$ denotes the negation of relation $\Omega(C, J_r)$.

Thus, in each coalition the relationships of players are constructed on the basis of the lexicographic principle. Therefore, any N_n-optimal situation $x \in Q^n(C, N_n)$ (all players form one coalition) is lexicographically optimal in the space X of all situations. This means that all players are ordered (enumerated) by importance in such a way that each preceding one is more important than all the next. This situation corresponds to the generic setup of an optimization problem with several criteria (payoffs) applied consecutively [7, 8]. It is easy to see that the set $Q^n(C, N_n)$ of N_n-optimal situations is a lexicographic set

$$L^n(C) = \{x \in X : \quad \forall\, x' \in X \quad (Cx \underset{L}{\overline{\prec}} Cx')\},$$

which is a subset of the Pareto set.

Clearly, in another extreme case, where the game is noncooperative ($s = n$), any individually optimal situation $x \in Q^n(C, \{1\}, \{2\}, \dots, \{n\})$ is the Nash equilibrium situation (or equilibrium) [9] (see also [1, 2]). Indeed, by the definition, situation x is equilibrium if and only if the following formula

$$\nexists\, k \in N_n \;\; \nexists\, x' \in X \;\; (C_k x < C_k x' \;\;\&\;\; x_{N_n \setminus \{k\}} = x'_{N_n \setminus \{k\}})$$

holds. Therefore, the reasonability of equilibrium situation x means that any player does not benefit from a deviation from it (while all others stick to it). We denote by $NE^n(C)$ the set of all Nash equilibrium situations.

In this context, by the parametrization of the principle of optimality we mean introducing a characteristic of binary relation $\Omega(C, J)$ of preference of situations that allows us to relate the classical concepts of lexicographic optimality and Nash equilibrium.

Without loss of generality, below we will assume that the elements of the partition $N_n = \bigcup\limits_{r \in N_s} J_r$ have the form

$$J_r = \{t_{r-1} + 1, t_{r-1} + 2, \dots, t_r\},$$

$$r \in N_s, \;\; t_0 = 0, \;\; t_s = n.$$

By taking into account the separability of the linear payoff functions $C_i x, \; i \in N_n$, we derive the following formula from the definition of the set $(J_1, J_2, \dots, J_s)$-optimal situations

$$Q^n(C, J_1, J_2, \dots, J_s) = \prod_{r=1}^{s} L^{|J_r|}(C^r), \tag{1}$$

where each factor $L^{|J_r|}(C^r)$ is the set of lexicographically optimal solutions of a $|J_r|$-criteria vector problem

$$C^r z \to \operatorname{lex} \max_{z \in X_{J_r}},$$

i. e.

$$L^{|J_r|}(C^r) = \{z \in X_{J_r} \;:\; \forall z' \in X_{J_r} \; (C^r z \underset{L}{\overline{\prec}} C^r z')\}.$$

Here C^r is a square $|J_r| \times |J_r|$ matrix consisting of the entries of matrix C, standing at the intersection of the rows and columns with numbers from J_r; X_{J_r} is the projection of the set X onto J_r, i. e.

$$X_{J_r} = \prod_{j \in J_r} X_j \subset \mathbf{R}^{|J_r|}.$$

It is known [7, 8] that the set $L^{|J_r|}(C^r)$ is the result of solving the sequence of scalar problems

$$L_i^{|J_r|} = \operatorname{Arg}\max\{C_i^r z : \ z \in L_{i-1}^{|J_r|}\}, \ i \in N_{|J_r|}, \tag{2}$$

where $L_0^{|J_r|} = X_{J_r}$; C_i^r is the i-th row of matrix C^r. Thus, $L^{|J_r|}(C^r) = L_{|J_r|}^{|J_r|}$ for each index $r \in N_s$.

Owing to the fact that the set X_{J_r} is finite for any index $r \in N_s$, we conclude that the lexicographic set $L^{|J_r|}(C^r)$ is nonempty for any index $r \in N_s$. Therefore (in view of (1)) the set of $(J_1, J_2, \ldots, J_s)$-optimal situations $Q^n(C, J_1, J_2, \ldots, J_s)$ is nonempty for any matrix $C \in \mathbf{R}^{n\times n}$ and for any partition. In particular, the equilibrium situations exist for any matrix $C \in \mathbf{R}^{n\times n}$ (see Corollary 5).

Under the measure of stability in cooperative game with matrix C we understand the stability radius of the problem $Z^n(C, J_1, J_2, \ldots, J_s)$ of finding the set $Q^n(C, J_1, J_2, \ldots, J_s)$ which analogously to [6, 10, 11] is defined as follows:

$$\rho^n(C, J_1, J_2, \ldots, J_s) = \begin{cases} \sup\ \Phi & \text{if } \Phi \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\Phi = \{\varepsilon > 0 \ : \ \forall\, B \in \Xi(\varepsilon) \quad (Q^n(C+B, J_1, J_2, \ldots, J_s) \subseteq Q^n(C, J_1, J_2, \ldots, J_s))\},$$

$$\Xi(\varepsilon) = \{B \in \mathbf{R}^{n\times n} \ : \ ||B||_\infty < \varepsilon\},$$

$$||B||_\infty = \max\{|b_{ij}| \ : \ (i,j) \in N_n \times N_n\}, B = [b_{ij}]_{n\times n}.$$

In other words, the stability radius determines the limit level of perturbations of the parameters of payoff function in the l_∞-metric, for which new generalized optimal situations do not appear. Obviously, the problem $Z^n(C, J_1, J_2, \ldots, J_s)$ is stable and the stability radius is infinite if the equality $Q^n(C, J_1, J_2, \ldots, J_s) = X$ holds. If the set

$$\overline{Q^n}(C, J_1, J_2, \ldots, J_s) = X \setminus Q^n(C, J_1, J_2, \ldots, J_s)$$

is nonempty, then we say that the problem $Z^n(C, J_1, J_2, \ldots, J_s)$ is non-trivial.

Suppose

$$\overline{L^{|J_r|}}(C^r) = X_{J_r} \setminus L^{|J_r|}(C^r),$$

$$K(C) = \{r \in N_s \ : \ \overline{L^{|J_r|}}(C^r) \neq \emptyset\},$$

$$||a||_1 = \sum_{i=1}^m |a_i|, \ a = (a_1, a_2, \ldots, a_m) \in \mathbf{R}^m.$$

The following properties are obvious.

Property 2. The situation $x^0 \in \overline{Q^n}(C, J_1, J_2, \ldots, J_s)$ if and only if there exists an index $k \in K(C)$ such that $x^0_{J_k} \in \overline{L^{|J_k|}}(C^k)$.

Property 3. The problem $Z^n(C, J_1, J_2, \ldots, J_s)$ is non-trivial if and only if the set $K(C)$ is nonempty.

From formula (1) and property 2 we derive

Property 4. If $r \in K(C)$ and there exists a perturbing matrix $\widehat{B} \in \mathbf{R}^{n\times n}$ such that the following formula holds

$$\forall\ z \in L^{|J_r|}(C^r) \quad \left(z \in \overline{L^{|J_r|}}(C^r + \widehat{B}^r)\right), \tag{3}$$

then we have

$$\forall\ x \in Q^n(C, J_1, J_2, \ldots, J_s) \quad \left(x \in \overline{Q^n}(C + \widehat{B}, J_1, J_2, \ldots, J_s)\right). \tag{4}$$

3 Bounds of the stability radius

Suppose

$$\varphi^n(C, J_1, J_2, \ldots, J_s) = \min_{r\in K(C)} \min_{z\in\overline{L^{|J_r|}}(C^r)} \max_{z'\in L^{|J_r|}(C^r)} \frac{C_1^r(z'-z)}{||z'-z||_1}.$$

Theorem. *The stability radius $\rho^n(C, J_1, J_2, \ldots, J_s)$ of the non-trivial problem $Z^n(C, J_1, J_2, \ldots, J_s)$, $n \geq 2$, $s \geq 1$, has the following bounds*

$$\varphi^n(C, J_1, J_2, \ldots, J_s) \leq \rho^n(C, J_1, J_2, \ldots, J_s) \leq \min\{||C_1^r||_\infty : r \in K(C)\}.$$

Proof. Note that in view of property 3 the non-triviality of the problem $Z^n(C, J_1, J_2, \ldots, J_s)$ implies the non-emptiness of the set $K(C)$.

Let us introduce the notations

$$\varphi := \varphi^n(C, J_1, J_2, \ldots, J_s), \quad \rho := \rho^n(C, J_1, J_2, \ldots, J_s).$$

It is easy to see that $\varphi \geq 0$.

At first we prove the inequality $\rho \geq \varphi$. If $\varphi = 0$, then this inequality is obvious.

Let $\varphi > 0$, $B \in \Xi(\varphi)$, $x^0 \in \overline{Q^n}(C, J_1, J_2, \ldots, J_s)$. Let us show that $x^0 \in \overline{Q^n}(C + B, J_1, J_2, \ldots, J_s)$.

It follows directly from the definition of φ that

$$\forall r \in K(C) \quad \forall z \in \overline{L^{|J_r|}}(C^r) \quad \left(\max_{z'\in L^{|J_r|}(C^r)} \frac{C_1^r(z'-z)}{||z'-z||_1} \geq \varphi\right). \tag{5}$$

According to the property 2 there exists an index $k \in K(C)$ such that $x^0_{J_k} \in \overline{L^{|J_k|}}(C^k)$. Therefore the formula (5) implies the existence of a vector $z' \in L^{|J_k|}(C^k)$, such that the following inequalities

$$\frac{C^k_1(z' - x^0_{J_k})}{||z' - x^0_{J_k}||_1} \geq \varphi > ||B||_\infty \geq ||B^k||_\infty$$

hold. Due to the obvious inequality

$$|uv| \leq ||u||_1 ||v||_\infty,$$

which is valid for all $u = (u_1, u_2, \ldots, u_n) \in \mathbf{R}^n$ and $v = (v_1, v_2, \ldots, v_n)^T \in \mathbf{R}^n$, we obtain

$$(C^k_1 + B^k_1)(z' - x^0_{J_k}) = C^k_1(z' - x^0_{J_k}) + B^k_1(z' - x^0_{J_k}) \geq$$
$$\geq C^k_1(z' - x^0_{J_k}) - ||B^k||_\infty ||z' - x^0_{J_k}||_1 > 0.$$

Thus, according to property 1 we have $(C^k + B^k)x^0_{J_k} \underset{L}{\prec} (C^k + B^k)z'$, i. e. $x^0_{J_k} \in \overline{L^{|J_k|}}(C^k + B^k)$. In view of property 2, we conclude that $x^0 \in \overline{Q^n}(C + B, J_1, J_2, \ldots, J_s)$.

So, the following formula is true

$$\forall B \in \Xi(\varphi) \quad \big(Q^n(C + B, J_1, J_2, \ldots, J_s) \subseteq Q^n(C, J_1, J_2, \ldots, J_s)\big),$$

which means that $\rho \geq \varphi$.

To prove the upper bound we need to show that for any index $r \in K(C)$ the following formula $\rho \leq ||C^r_1||_\infty$ is valid.

Let $r \in K(C)$, $\varepsilon > ||C^r_1||_\infty$, $\psi_i = ||C^r_i||_\infty$, $i \in N_{|J_r|}$. We build a perturbing matrix $\widehat{B} = \left[\widehat{b}_{ij}\right]_{n \times n}$, assuming

$$\widehat{b}_{ij} = \begin{cases} -c_{ij} - \delta c_{i+p-1,j} & \text{if } i = t_{r-1} + 1,\ j \in J_r, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < \delta < \frac{\varepsilon - \psi_1}{\psi_p}$. Here

$$p = \min\{i \in N_{|J_r|} :\ X_{J_r} \neq L^{|J_r|}_i\},$$

and $L^{|J_r|}_i$ is defined by the formula (2). It is easy to see that $\psi_p \neq 0$. After a simple calculation we obtain $||\widehat{B}||_\infty < \varepsilon$, i. e. $\widehat{B} \in \Xi(\varepsilon)$.

Let $z^* \in X \backslash L^{|J_r|}_p$. Then for any vector $z \in L^{|J_r|}(C^r)$ we have

$$C^r_p(z^* - z) < 0.$$

Using this and taking into account the construction of the row $\widehat{B}^r_1$, we derive

$$(C^r_1 + \widehat{B}^r_1)(z - z^*) = \delta C^r_p(z^* - z) < 0.$$

This inequality in view of property 1 is equivalent to the following relation

$$(C^r + \widehat{B}^r)z \underset{L}{\prec} (C^r + \widehat{B}^r)z^*.$$

From this we obtain the formula (3), and therefore by virtue of property 4 we have (4). Hence

$$Q^n(C + \widehat{B}, J_1, J_2, \dots, J_s) \not\subseteq Q^n(C, J_1, J_2, \dots, J_s). \tag{6}$$

Resuming all the said above, we conclude that for any index $r \in K(C)$ and for any number $\varepsilon > ||C_1^r||_\infty$ there exists a matrix $\widehat{B} \in \Xi(\varepsilon)$ such that the formula (6) is true. This means that the stability radius $\rho \le ||C_1^r||_\infty$ for any index $r \in K(C)$. That complete the proof.

4 Some of special cases

The theorem allows us to formulate the following corollaries.

Corollary 1. *If $|X_j| = 2,\ j \in N_n$, then for the stability radius $\rho^n(C, J_1, J_2, \dots, J_s)$ of non-trivial problem $Z^n(C, J_1, J_2, \dots, J_s)$ the formula*

$$\rho^n(C, J_1, J_2, \dots, J_s) = \varphi^n(C, J_1, J_2, \dots, J_s) \tag{7}$$

holds.

Proof. Taking into account the proved inequality $\rho \ge \varphi$ (see theorem) for deriving the formula (7) it remains to show that $\rho \le \varphi$. Let us introduce the notations:

$$X_j = \{x_j^-, x_j^+\}, \quad x_j^-, x_j^+ \in \mathbf{R}, \quad x_j^- < x_j^+,\ j \in N_n.$$

By the definition of number φ, there exist an index $r \in K(C)$ and a vector $z^* \in \overline{L^{|J_r|}}(C^r)$ such that for any vector $z \in L^{|J_r|}(C^r)$ we have

$$C_1^r(z - z^*) \ \le\ \varphi||z - z^*||_1.$$

Then, assuming $\varepsilon > \varphi,\ \widehat{B} = \left[\widehat{b}_{ij}\right]_{n\times n} \in \Xi(\varepsilon)$, where

$$\widehat{b}_{ij} = \begin{cases} -\alpha & \text{if } z^*_{j-t_{r-1}} = x_j^-,\ i = t_{r-1}+1,\ j \in J_r, \\ \alpha & \text{if } z^*_{j-t_{r-1}} = x_j^+,\ i = t_{r-1}+1,\ j \in J_r, \\ 0 & \text{otherwise}, \end{cases}$$

$$\varepsilon > \alpha > \varphi,$$

we derive

$$(C_1^r + \widehat{B}_1^r)(z - z^*) = C_1^r(z - z^*) + \widehat{B}_1^r(z - z^*) =$$

$$= C_1^r(z - z^*) - \alpha||z - z^*||_1 \le \varphi||z - z^*||_1 - \alpha||z - z^*||_1 < 0,$$

i. e. $z \in \overline{L^{|J_r|}}(C^r + \widehat{B}^r)$. Therefore we have $(C^r + \widehat{B}^r)z \underset{L}{\prec} (C^r + \widehat{B}^r)z^*$.

Using this we obtain the formula (3), and therefore by virtue of property 4 we have (4). Hence the formula (6) is true.

Resuming all the information given above, we conclude that for any number $\varepsilon > \varphi$ there exists a matrix $\widehat{B} \in \Xi(\varepsilon)$ such that the formula (6) is true. This means that $\rho \leq \varphi$. That completes the proof of Corollary 1.

Note, that Corollary 1 shows the accessibility of the lower bound of the stability radius ρ.

Corollary 2. *If* $Q^n(C, J_1, J_2, \dots, J_s) = \{x^0\}$, *then*

$$\rho^n(C, J_1, J_2, \dots, J_s) = \min_{r \in N_s} \min_{z \in X_{J_r} \setminus \{x^0_{J_r}\}} \frac{C^r_1(x^0_{J_r} - z)}{||x^0_{J_r} - z||_1}. \tag{8}$$

Proof. Denote the right side of the formula (8) by ζ. It is easy to see that the problem $Z^n(C, J_1, J_2, \dots, J_s)$ is nontrivial and the number ζ is $\varphi = \varphi^n(C, J_1, J_2, \dots, J_s)$. Therefore, in view of inequality $\rho \geq \varphi$ (see the theorem) it remains to show that $\rho \leq \zeta$.

By the definition of number ζ, we have that there exist an index $r \in N_s$ and a vector $z^* \in X_{J_r} \setminus \{x^0_{J_r}\}$ such that

$$C^r_1(x^0_{J_r} - z^*) = \zeta ||x^0_{J_r} - z^*||_1, \quad \{x^0_{J_r}\} = L^{|J_r|}(C^r).$$

Therefore, assuming $\varepsilon > \zeta$ and building a perturbing matrix $\widehat{B} = \left[\widehat{b}_{ij}\right]_{n \times n} \in \Xi(\varepsilon)$ with elements

$$\widehat{b}_{ij} = \begin{cases} -\alpha & \text{if } x^0_j \geq z^*_{j - t_{r-1}},\ i = t_{r-1} + 1,\ j \in J_r, \\ \alpha & \text{if } x^0_j < z^*_{j - t_{r-1}},\ i = t_{r-1} + 1,\ j \in J_r, \\ 0 & \text{otherwise}, \end{cases}$$

where $\varepsilon > \alpha > \zeta$, we obtain

$$(C^r_1 + \widehat{B}^r_1)(x^0_{J_r} - z^*) = C^r_1(x^0_{J_r} - z^*) - \alpha ||x^0_{J_r} - z^*||_1 = \zeta ||x^0_{J_r} - z^*||_1 - \alpha ||x^0_{J_r} - z^*||_1 < 0.$$

Hence, taking into account property 1 we have

$$(C^r + \widehat{B}^r)x^0_{J_r} \underset{L}{\prec} (C^r + \widehat{B}^r)z^*.$$

From here we find the formula (3), and therefore according to the property 4 we have (4), i. e. the formula (6) is true.

Resuming the said above we conclude that for any number $\varepsilon > \varphi$ there exists a perturbing matrix $\widehat{B} \in \Xi(\varepsilon)$ such that the formula (6) is true. This means that $\rho \leq \varphi$. The proof of Corollary 2 is completed.

The theorem implies

Corollary 3. *The stability radius $\rho^n(C, N_n)$ of a non-trivial problem $Z^n(C, N_n)$ of finding the lexicographic set $L^n(C)$ has the following bounds:*

$$\min_{x \in \overline{L^n}(C)} \max_{x' \in L^n(C)} \frac{C_1(x' - x)}{||x' - x||_1} \ \leq \ \rho^n(C, N_n) \ \leq \ ||C_1||_\infty.$$

Here $\overline{L}^n(C) = X \backslash L^n(C) = X \backslash Q^n(C, N_n)$.

Obviously, in case we have a noncooperative game ($s = n$), for any index $r \in N_s$ the inequality $\overline{L^1}(C^r) \neq \emptyset$, where $C^r = c_{rr}$, is equivalent to the inequality $c_{rr} \neq 0$. Therefore, the theorem implies

Corollary 4 [12]. *For the stability radius $\rho^n(C, \{1\}, \{2\}, \ldots, \{n\})$, $n \geq 2$, of the problem $Z^n(C, \{1\}, \{2\}, \ldots, \{n\})$ of finding the set of Nash equilibrium situations $NE^n(C)$ the formula*

$$\rho^n(C, \{1\}, \{2\}, \ldots, \{n\}) = \begin{cases} \min\{|c_{kk}| \ : \ k \in K(C)\} & \text{if } K(C) \neq \emptyset, \\ \infty & \text{if } K(C) = \emptyset \end{cases}$$

holds.

Taking into account the formula (1) we obtain

Corollary 5 [3]. *The situation $x^0 = (x^0_1, x^0_2, \ldots, x^0_n) \in X$ of a noncooperative game with matrix $C \in \mathbf{R}^{n \times n}$ is a Nash equilibrium situation if and only if the strategy of each player $i \in N_n$ has the form*

$$x^0_i = \begin{cases} \max\{\, x_i \ : \ x_i \in X_i\} & \text{if } c_{ii} > 0, \\ \min\{\, x_i \ : \ x_i \in X_i\} & \text{if } c_{ii} < 0, \\ x_i \in X_i & \text{if } c_{ii} = 0. \end{cases}$$

References

[1] Moulen H. *Théorie des jeux pour l'économie et la politique.* Harmann Paris, 1979.

[2] Petrosyan L.A., Zenkevich N.A., Semina E.A. *Games theory.* Moscow, Vysshaya shkola, 1998 (in Russian).

[3] Emelichev V.A., Bukhtoyarov S.E. *Stability of generally efficient situation in finite cooperative games with parametric optimality principle ("from Pareto to Nash").* Computer Science Journal of Moldova, 2003, **11**, N 3, p. 316–323.

[4] Bukhtoyarov S.E., Emelichev V.A., Stepanishina Yu.V. *Stability of discrete vector problems with the parametric principle of optimality.* Cybernetics and Systems Analysis, 2003, **39**, p. 604–614.

[5] Emelichev V.A., Kuzmin K.G. *Finite cooperative games with a parametric concept of equilibrium under uncertainty conditions.* J. of Computer and Systems International, 2006, **45**, N 2, p. 276–281.

[6] Bukhtoyarov S.E., Emelichev V.A. *Measure of stability for a finite cooperative game with a parametric optimality principle (from Pareto to Nash).* Comput. Math. and Mathem. Physics, 2006, **46**, N 7, p. 1193–1199.

[7] Podinovskiy V.V., Gavrilov V.M. *Optimization on sequential applied criteria.* Moskow, Sovetskoe radio, 1975 (in Russian).

[8] Chervak Yu. *Unimprovable choice.* Uzhgorod, Uzhgorod National University, 2002 (in Ukrainian)

[9] Nash J.F. *Non-cooperative games.* Ann. Math., 1951, **54**, N 2, p. 286–295.

[10] Sotskov Yu., Leontev V., Gordeev E. *Some concepts of stability analysis in combinatorial optimization.* Discrete Appl. Math., 1995, **58,** N 2, p. 169–190.

[11] Emelichev V., Girlich E., Nikulin Yu., Podkopaev D. *Stability and regularization of vector problems of integer linear programming.* Optimization, 2002, **51**, N 4, p. 645–676.

[12] Bukhtoyarov S.E., Kuzmin K.G. *On stability in game problems of finding Nash set.* Computer Science Journal of Moldova, 2004, **12**, N 3, p. 381–388.

Belarussian State University
ave. Independence, 4
Minsk, 220050, Belarus

E-mails: *emelichev@bsu.by,* *emelichev@tut.by,*
Eugen_Eugen@tut.by

Received August 25, 2006

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(52), 2006, Pages 27–44
ISSN 1024–7696

Collocation and Quadrature Methods for Solving Singular Integral Equations with Piecewise Continuous Coefficients

Titu Capcelea

Abstract. The computation schemes of collocation and mechanical quadrature methods for approximate solving of the complete singular integral equations with piecewise continuous coefficients and a regular kernel with weak singularity are elaborated. The case when the equations are defined on the unit circumference of the complex plane is examined. The sufficient conditions for the convergence of these methods in the space L_2 are obtained.

Mathematics subject classification: 45E05; 45F15; 45G05; 45L05; 45L10; 65R20.
Keywords and phrases: Cauchy singular integral equations; piecewise continuous coefficients; collocation method; quadrature method.

1 The Problem Formulation

Let Γ_0 be a unit circumference of the complex plane $\mathbb{C}$ with the center at the origin, let D^+ be a domain bounded by Γ_0, $D^- = \mathbb{C} \setminus \{D^+ \cup \Gamma_0\}$, and let $L_2(\Gamma_0)$ be a space of all functions $f : \Gamma_0 \to \mathbb{C}$ that are Lebesgue measurable and square integrable on Γ_0.

We will denote by $PC(\Gamma_0)$ a Banach algebra of all functions $a : \Gamma_0 \to \mathbb{C}$ which are continuous on Γ_0 with exception of a finite number of points in such a way that at each point of discontinuity there exist unilateral finite limits $a(t-0)$, $a(t+0)$ and $a(t-0) = a(t)$.

To each element $a \in PC(\Gamma_0)$ we associate the function $\hat{a} : \Gamma_0 \times [0,1] \to \mathbb{C}$ in the following way $\hat{a}(t,\mu) = \mu a(t+0) + (1-\mu)a(t), t \in \Gamma_0,\ 0 \le \mu \le 1$. The set $\Gamma_{\hat{a}}$ of values of the function $\hat{a}(t,\mu)$ represents a closed curve. This curve is a union of the set of values of the function $a(t)$ and segments $\mu a(t_k+0) + (1-\mu)a(t_k)$ $(0 \le \mu \le 1, k = \overline{1,n})$, where $t_1, ..., t_n$ are all points of discontinuity of the function a. The curve $\Gamma_{\hat{a}}$ can be oriented in a natural way.

We say that the function $a \in PC(\Gamma_0)$ is 2-nonsingular if the curve $\Gamma_{\hat{a}}$ doesn't go through the origin. We denote the number of rotations of the curve $\Gamma_{\hat{a}}$ around the origin by index $ind_2 a$ of the 2-nonsingular function a.

In $L_2(\Gamma_0)$ we consider the following singular integral equation

$$(A\varphi \equiv)\, a_0(t)\varphi(t) + \frac{b_0(t)}{\pi i}\int_{\Gamma_0}\frac{\varphi(\tau)}{\tau - t}d\tau + \frac{1}{2\pi i}\int_{\Gamma_0} h(t,\tau)\varphi(\tau)d\tau = f(t),\ t \in \Gamma_0, \quad (1)$$

where $a_0, b_0, f : \Gamma_0 \to \mathbb{C}$, $h : \Gamma_0 \times \Gamma_0 \to \mathbb{C}$ are known functions, $a_0, b_0 \in PC(\Gamma_0)$, $h(t,\tau) = h_0(t,\tau)|\tau - t|^{-\gamma}\,(0 < \gamma < 1)$, $h_0 \in C(\Gamma_0 \times \Gamma_0), f \in L_2(\Gamma_0)$ and $\varphi : \Gamma_0 \to \mathbb{C}$ is an unknown function.

It is known that operators $K, S : L_2(\Gamma_0) \to L_2(\Gamma_0)$, defined in the following way $(K\varphi)(t) = \dfrac{1}{2\pi i}\int_{\Gamma_0} h(t,\tau)\varphi(\tau)d\tau$, $(S\varphi)(t) = \dfrac{1}{\pi i}\int_{\Gamma_0} \frac{\varphi(\tau)}{\tau - t}d\tau$, are bounded [1, 2]. Taking into account that $\|c\varphi\|_2 \le \|c\|_\infty \|\varphi\|_2$ for all functions $c \in PC(\Gamma_0)$, the operator $A = a_0 I + b_0 S + K$ which describes the left term of equation (1) is bounded in $L_2(\Gamma_0)$.

In [3, 4] the theoretical foundation of the collocation and quadrature methods for equation (1) in the norm of the space $L_2(\Gamma_0)$ was obtained in the case of coefficients that satisfy Holder condition on Γ_0 and in [5] the foundation was obtained in the case of continuous coefficients on Γ_0. In the present paper we will state conditions of convergence of these methods in $L_2(\Gamma_0)$ in the case when coefficients of the equation (1) belong to the space $PC(\Gamma_0)$.

2 The deduction of a computation schemes

We will denote by $\mathcal{P}_n$ the set of all trigonometric polynomials of the form $\sum\limits_{k=-n}^{n} r_k t^k$ $(t \in \Gamma_0)$, where r_k $(k = \overline{-n,n})$ are arbitrary complex numbers. We will consider on Γ_0 the following equidistant points

$$t_j = exp\,(2\pi i j/(2n+1)),\ j = \overline{-n,n}. \tag{2}$$

In the following it is convenient to write equation (1) in the equivalent form

$$(A\varphi \equiv) a(t)(P\varphi)(t) + b(t)(Q\varphi)(t) + (K\varphi)(t) = f(t), t \in \Gamma_0, \tag{3}$$

where $a(t) = a_0(t) + b_0(t), b(t) = a_0(t) - b_0(t), P = (I + S)/2, Q = I - P$, I is the identity operator, and S is a singular operator.

The presence of discontinuity in the kernel of the regular part of equation (1) implies essential difficulties in the practical realization of the calculation scheme of the collocation method applied to it, and the quadrature method cannot be applied.

In order to eliminate this drawback, in an analogous way to [3, 6], we introduce a new equation

$$(A_\rho\varphi \equiv) a(t)(P\varphi)(t) + b(t)(Q\varphi)(t) + (K_\rho\varphi)(t) = f(t), t \in \Gamma_0, \tag{4}$$

in which

$$(K_\rho\varphi)(t) = \frac{1}{2\pi i}\int_{\Gamma_0} h_\rho(t,\tau)\varphi(\tau)d\tau,$$

$$h_\rho(t,\tau) = \begin{cases} h_0(t,\tau)|\tau - t|^{-\gamma}, & for\ \ |\tau - t| \ge \rho \\ h_0(t,\tau)\rho^{-\gamma}, & for\ \ |\tau - t| < \rho \end{cases}, \rho \in (0,1).$$

Equations (3) and (4) have the same characteristic part, and the kernel of the regular part of equation (4) is a continuous function on Γ_0 in both variables.

In the following the collocation and quadrature methods will be applied to equation (4). The obtained approximate solutions will be considered as the approximations of the exact solution of equation (3), and, thus, of equation (1).

According to the collocation method we will seek for an approximate solution of equation (4) in the form of the polynomial

$$\varphi_n(t) = \sum_{k=-n}^{n} \alpha_k^{(n)} t^k \in \mathcal{P}_n, \tag{5}$$

unknown coefficients of which $\alpha_k^{(n)} = \alpha_k \, (k = \overline{-n, n})$ will be determined from the following system of linear algebraic equations (SLAE)

$$a(t_j)\sum_{k=0}^{n} \alpha_k t_j^k + b(t_j)\sum_{k=-n}^{-1} \alpha_k t_j^k + \sum_{k=-n}^{n} \alpha_k \frac{1}{2\pi i} \int\limits_{\Gamma_0} h_\rho(t_j, \tau)\tau^k d\tau = f(t_j), j = \overline{-n, n}. \tag{6}$$

The proposed calculation scheme essentially simplifies the process of its numerical implementation.

If for solving equation (4) the method of quadratures is applied, then we will seek for the approximate solution of this equation in the form (5) and we will determine coefficients $\alpha_k \, (k = \overline{-n, n})$ as solutions of SLAE

$$a(t_j)\sum_{k=0}^{n} \alpha_k t_j^k + b(t_j)\sum_{k=-n}^{-1} \alpha_k t_j^k + \frac{1}{2n+1}\sum_{k=-n}^{n} \alpha_k \sum_{s=-n}^{n} h_\rho(t_j, t_s) t_s^{k+1} = f(t_j), j = \overline{-n, n}. \tag{7}$$

Let a bounded and measurable function $f : \Gamma_0 \to \mathbb{C}$ be given. There exists a unique interpolation polynomial

$$(L_n f)(t) = \sum_{k=-n}^{n} \Lambda_k t^k \in \mathcal{P}_n, \quad \Lambda_k = \frac{1}{2n+1} \sum_{j=-n}^{n} f(t_j) t_j^{-k} \tag{8}$$

such that $(L_n f)(t_j) = f(t_j)$ for each $j = \overline{-n, n}$ [7, p.151]. The operator L_n, for which $L_n^2 = L_n$, is a Lagrange interpolation projector. Besides this nonorthogonal projector, we consider an orthogonal projector $S_n : L_2(\Gamma_0) \to \mathcal{P}_n$, which for each function $\varphi \in L_2(\Gamma_0)$ puts into correspondence a partial sum of order n of the Fourier series after the system of functions $\{t^k\}_{k=-\infty}^{+\infty}$, $(S_n\varphi)(t) = \sum\limits_{k=-n}^{n} \varphi_k t^k$. Taking into account that for functions of the form (5) the following equalities are true $(S_n\varphi_n)(t) = \varphi_n(t)$, we obtain that systems of equations (6), (7) are equivalent to the following operator equations

$$(A_{n,\rho}\varphi_n \equiv) \, L_n(aP + bQ + K_\rho)S_n\varphi_n = L_n f, \tag{9}$$

$$(A'_{n,\rho}\tilde{\varphi}_n \equiv) \, L_n(aP + bQ + \Delta_n)S_n\tilde{\varphi}_n = L_n f, \tag{10}$$

where $(\Delta_n\tilde{\varphi}_n)(t) = \frac{1}{2\pi i}\int_{\Gamma_0} L_n^\tau(h_\rho(t, \tau)\tilde{\varphi}_n(\tau))d\tau$. Notice that here and in what follows L_n^τ denotes the operator L_n, applied with respect to the variable τ. Therefore in the following instead of systems (6) and (7) we will study operator equations (9) and, respectively, (10) which are considered in the subspace $\mathcal{P}_n$, in which the same norm as in $L_2(\Gamma_0)$ is introduced.

In the case of an equation with coefficients from $PC(\Gamma_0)$, in order to apply the methods studied in the paper it is necessary to choose the right term f from a subclass of $L_2(\Gamma_0)$. As such a subclass the set $R(\Gamma_0)$ of all bounded, defined on Γ_0 and integrable by Riemann functions can be chosen. With the norm $\|g\|_\infty = \sup\limits_{t\in\Gamma_0} |g(t)|$ the set $R(\Gamma_0)$ becomes the Banach space.

3 Some preliminary results

In this section we will state some relations between integral operators with kernel $h_0(t,\tau)|\tau - t|^{-\gamma}$ and $h_\rho(t,\tau)$, considered in the space $L_2(\Gamma_0)$. These results, as well as other results from this section, will be used for the theoretical foundation of the elaborated computational schemes.

We will denote by $\chi_\rho(t)$ the function defined on Γ_0 in the following way. If $\varphi(t) \in L_2(\Gamma_0)$, then

$$\chi_\rho(t) = \frac{1}{2\pi i}\int_{\Gamma_0} [h_0(t,\tau)|\tau - t|^{-\gamma} - h_\rho(t,\tau)]\varphi(\tau)d\tau,$$

where $h_0(t,\tau)$ and $h_\rho(t,\tau)$ are the defined above functions.

Lemma 1. *Let $h_0(t,\tau) \in C(\Gamma_0 \times \Gamma_0)$ (in both variables) and $\varphi(t) \in L_2(\Gamma_0)$. Then it is true that*

a) $\|\chi_\rho\|_2 \le d_1\rho^{\frac{1-\gamma}{2}}\|\varphi\|_2$;
b) $(K_\rho\varphi)(t) \in C(\Gamma_0)$;
c) The operator $K_\rho : L_2(\Gamma_0) \to C(\Gamma_0)$ is completely continuous.

Proof. Let $t \in \Gamma_0$ and $\Gamma_\rho := \{\tau \in \Gamma_0 : |\tau - t| < \rho\}$. Then, as $\chi_\rho(t) = 0$ for $|\tau - t| \ge \rho$, we have

$$\|\chi_\rho\|_2^2 = \frac{1}{2\pi}\int_{\Gamma_0} |\chi_\rho|^2|dt| = \frac{1}{2\pi}\int_{\Gamma_0}\left|\frac{1}{2\pi i}\int_{\Gamma_0}[h_0(t,\tau)|\tau - t|^{-\gamma} - h_\rho(t,\tau)]\varphi(\tau)d\tau\right|^2|dt| =$$

$$= \frac{1}{(2\pi)^3}\int_{\Gamma_0}\left|\int_{\Gamma_\rho} h_0(t,\tau)\left[|\tau - t|^{-\gamma} - \rho^{-\gamma}\right]\varphi(\tau)d\tau\right|^2|dt| \le$$

$$\le \frac{1}{(2\pi)^3}\int_{\Gamma_0}\left(\int_{\Gamma_\rho}|h_0(t,\tau)|\left||\tau - t|^{-\gamma} - \rho^{-\gamma}\right||\varphi(\tau)||d\tau|\right)^2|dt|.$$

Since $|\tau - t|^{-\gamma} - \rho^{-\gamma} > 0\,(\tau \in \Gamma_\rho)$, from the last relation we obtain

$$\|\chi_\rho\|_2^2 \le \frac{\|h_0\|_C^2}{(2\pi)^3}\int_{\Gamma_0}\left(\int_{\Gamma_\rho}\frac{|\varphi(\tau)||d\tau|}{|\tau - t|^\gamma}\right)^2|dt|.$$

Estimating the interior integral using the Holder inequality for integrals (see [8, p.496]), we obtain

$$\int_{\Gamma_\rho} \frac{|\varphi(\tau)|}{|\tau-t|^\gamma}|d\tau| = \int_{\Gamma_\rho} \frac{1}{|\tau-t|^{\gamma/2}}\frac{|\varphi(\tau)|}{|\tau-t|^{\gamma/2}}|d\tau| \leq \left(\int_{\Gamma_\rho} \frac{|d\tau|}{|\tau-t|^\gamma}\right)^{\frac{1}{2}} \left(\int_{\Gamma_\rho} \frac{|\varphi(\tau)|^2}{|\tau-t|^\gamma}|d\tau|\right)^{\frac{1}{2}}.$$

Then

$$\|\chi_\rho\|_2^2 \leq \frac{\|h_0\|_C^2}{(2\pi)^3}\int_{\Gamma_0}\left(\int_{\Gamma_\rho}\frac{|d\tau|}{|\tau-t|^\gamma}\right)\left(\int_{\Gamma_\rho}\frac{|\varphi(\tau)|^2}{|\tau-t|^\gamma}|d\tau|\right)|dt|.$$

We estimate integral $\int_{\Gamma_\rho}\frac{|d\tau|}{|\tau-t|^\gamma}$ using the following relation (see [9, p.10])

$$|d\tau| = |ds| \leq \frac{\pi}{2}dr, \tag{11}$$

where ds is a length of the arc of the circumference $\breve{\tau t}$ (the smallest arc from two possible ones), and dr is a length of the chord that subtends the arc $\breve{\tau t}$ ($|\tau - t| = r$). Then when τ passes the arc Γ_ρ, the value r passes the segment $[0;\rho]$. Using relation (11) we obtain

$$\int_{\Gamma_\rho}\frac{|d\tau|}{|\tau-t|^\gamma} \leq \frac{\pi}{2}\int_0^\rho r^{-\gamma}dr = \frac{\pi}{2(1-\gamma)}\rho^{1-\gamma}.$$

Then we have

$$\|\chi_\rho\|_2^2 \leq \frac{\|h_0\|_C^2}{16\pi^2}\frac{1}{(1-\gamma)}\rho^{1-\gamma}\int_{\Gamma_0}\int_{\Gamma_\rho}\frac{|\varphi(\tau)|^2}{|\tau-t|^\gamma}|d\tau||dt| =$$

$$= \frac{\|h_0\|_C^2}{16\pi^2}\frac{1}{1-\gamma}\rho^{1-\gamma}\int_{\Gamma_\rho}|\varphi(\tau)|^2\int_{\Gamma_0}\frac{|dt|}{|\tau-t|^\gamma}|d\tau|.$$

Repeating the above argumentation, we obtain for interior integral the following estimation

$$\int_{\Gamma_0}\frac{|dt|}{|\tau-t|^\gamma} \leq \frac{\pi}{2}\int_0^2 r^{-\gamma}dr = \frac{\pi}{1-\gamma}2^{-\gamma}.$$

Taking this into account, we obtain

$$\|\chi_\rho\|_2^2 \leq \frac{\|h_0\|_C^2}{16\pi}\frac{2^{-\gamma}}{(1-\gamma)^2}\rho^{1-\gamma}\int_{\Gamma_\rho}|\varphi(\tau)|^2|d\tau|,$$

from which results the inequality a), in which $d_1 = \dfrac{2^{(-2-\gamma/2)}}{(1-\gamma)\pi^{1/2}}\|h_0\|_C$.

Now we will show that the function $(K_\rho\varphi)(t) = \dfrac{1}{2\pi i}\int_{\Gamma_0} h_\rho(t,\tau)\varphi(\tau)d\tau$ is continuous on Γ_0. For $\varphi \in L_2(\Gamma_0)$ we have $\|\varphi\|_2 < \alpha$. The function $h_\rho(t,\tau)$, being continuous on the compact $\Gamma_0 \times \Gamma_0$, is uniformly continuous. In such a way for $\varepsilon > 0$ there exists $\delta > 0$ such that the inequalities $|t_2 - t_1| < \delta$, $|\tau_2 - \tau_1| < \delta$ imply the relation $|h_\rho(t_2, \tau_2) - h_\rho(t_1, \tau_1)| < \varepsilon/\alpha$. Taking into account the last inequality and Holder inequalities, we obtain for $|t_2 - t_1| < \delta$

$$|(K_\rho\varphi)(t_2) - (K_\rho\varphi)(t_1)| \le \frac{1}{2\pi}\int_{\Gamma_0} |h_\rho(t_2,\tau) - h_\rho(t_1,\tau)||\varphi(\tau)||d\tau| \le$$

$$\le \frac{1}{2\pi}\left(\int_{\Gamma_0} |h_\rho(t_2,\tau) - h_\rho(t_1,\tau)|^2 |d\tau|\right)^{1/2}\left(\int_{\Gamma_0} |\varphi(\tau)|^2|d\tau|\right)^{1/2} \le \frac{\varepsilon}{\alpha}\|\varphi\|_2 < \varepsilon. \quad (12)$$

In such a way, the function $(K_\rho\varphi)(t)$ is continuous.

The affirmation from point c) is stated using the Arzela-Ascoli theorem. The linearity of the operator K_ρ is evident. Let M be a bounded set in $L_2(\Gamma_0)$. In this way there exists $\alpha > 0$ such that $\|\varphi\|_2 < \alpha$ ($\varphi \in M$). For every $\varphi \in M$, according to inequality (12), we obtain that inequality $|t_2 - t_1| < \delta$ implies $|(K_\rho\varphi)(t_2) - (K_\rho\varphi)(t_1)| < \varepsilon$. This means that the functions of the set $K_\rho(M)$ are equally continuous. Let us show that the set $K_\rho(M)$ is bounded in $C(\Gamma_0)$. Let $\beta = \max\limits_{t,\tau\in\Gamma_0} |h_\rho(t,\tau)|$. We have

$$|(K_\rho\varphi)(t)| \le \frac{1}{2\pi}\int_{\Gamma_0} |h_\rho(t,\tau)||\varphi(\tau)||d\tau| \le$$

$$\le \frac{1}{2\pi}\left(\int_{\Gamma_0} |h_\rho(t,\tau)|^2|d\tau|\right)^{1/2}\left(\int_{\Gamma_0} |\varphi(\tau)|^2|d\tau|\right)^{1/2} \le \beta\|\varphi\|_2.$$

So, for each $K_\rho\varphi \in K_\rho(M)$ we have $\|K_\rho\varphi\|_{C(\Gamma_0)} = \max\limits_{t\in\Gamma_0} |(K_\rho\varphi)(t)| < \alpha\beta$. Therefore, the set $K_\rho(M) \subset C(\Gamma_0)$ is uniformly bounded and functions of this set are equally bounded.

According to the Arzela-Ascoli theorem the set $K_\rho(M)$ is relatively compact in $C(\Gamma_0)$ and in such a way the operator K_ρ is completely continuous. The lemma is proved.

Lemma 2. *Let the operator A, defined by the left term of equation (3), be invertible in the space $L_2(\Gamma_0)$. Then for ρ such that*

$$\varepsilon_\rho := d_1\rho^{(1-\gamma)/2}\left\|A^{-1}\right\|_2 \le q_1 < 1, \quad (13)$$

the operator A_ρ, defined by the left term of equation (4), is invertible in $L_2(\Gamma_0)$ as well and the inequality $\|A_\rho^{-1}\|_2 \le (1-\varepsilon_\rho)^{-1}\|A^{-1}\|_2$ is true. For the solutions $\varphi = A^{-1}f$ and $\varphi_\rho = A_\rho^{-1}f$ of equations (3) and (4), respectively, we have

$$\|\varphi - \varphi_\rho\|_2 \le \varepsilon_\rho(1-\varepsilon_\rho)^{-1}\left\|A^{-1}\right\|_2\|f\|_2.$$

Proof. Using item a) from Lemma 1 we obtain the estimation $\|(A-A_\rho)x\|_2 = \|(K-K_\rho)x\|_2 = \|\chi_\rho\|_2 \le d_1\rho^{(1-\gamma)/2}\|x\|_2$, $\forall x \in L_2(\Gamma_0)$. Then $\|A - A_\rho\|_2 \le d_1\rho^{(1-\gamma)/2}$.

We will show that if inequality (13) holds, then the operator A_ρ is invertible for sufficiently small values of ρ. For this we will use the representation $A_\rho = A-(A-A_\rho) = A(I-A^{-1}(A-A_\rho))$. Since $\|A^{-1}(A-A_\rho\|_2 \le \|A^{-1}\|_2 d_1 \rho^{(1-\gamma)/2} = \varepsilon_\rho \le q_1 < 1$ is true, then according to Banach theorem about small perturbations of an invertible operator, results the existence of the inverse operator $A_\rho^{-1} = (I-A^{-1}(A-A_\rho))^{-1}A^{-1}$ the norm of which satisfies the inequality

$$\left\|A_\rho^{-1}\right\|_2 \le \left\|(I-A^{-1}(A-A_\rho))^{-1}\right\|_2 \left\|A^{-1}\right\|_2 = (1-\varepsilon_\rho)^{-1}\left\|A^{-1}\right\|_2.$$

For solutions φ and φ_ρ of equations (3) and (4), respectively, we have

$$\|\varphi-\varphi_\rho\|_2 \le \left\|A^{-1}-A_\rho^{-1}\right\|_2 \|f\|_2 \le \left\|A^{-1}\right\|_2 \|A_\rho - A\|_2 \left\|A_\rho^{-1}\right\|_2 \|f\|_2 \le$$

$$\le (1-\varepsilon_\rho)^{-1}\left\|A^{-1}\right\|_2^2 d_1\rho^{(1-\gamma)/2}\|f\|_2 = \varepsilon_\rho(1-\varepsilon_\rho)^{-1}\left\|A^{-1}\right\|_2 \|f\|_2.$$

The lemma is proved.

Remark 1. As $\varepsilon_\rho \to 0$ when $\rho \to 0$, it results that $\|\varphi-\varphi_\rho\|_2 \to 0$ when $\rho \to 0$. This fact justifies the made convention with relation to the possibility of approximation of the exact solution of equation (3) with the approximate solution of equation (4), obtained according to the collocation method. In the following we will consider that ρ satisfies condition (13). This is true if ρ is sufficiently small.

It is known from [10, p.5; 11, p.12], that the operator L_n that acts in the space $L_2(\Gamma_0)$ is unbounded, but being looking for as an operator that acts from the space $R(\Gamma_0)$ to $L_2(\Gamma_0)$ it is bounded, and in [11] it is shown that

$$\|L_n f - f\|_2 \to 0,\ \forall f \in R(\Gamma_0). \tag{14}$$

Lemma 3. *Let $\{t_j\}_{j=-n}^n$ be the system of points (2). Then for each integer number m, such that $|m| \le 2n$ the following relation is true:*

$$\frac{1}{2n+1}\sum_{j=-n}^{n} t_j^m = \begin{cases} 1, & if\ \ m=0 \\ 0, & if\ \ m \ne 0 \end{cases}. \tag{15}$$

Proof. For $m=0$ relation (15) is evident. For $m \ne 0,\ |m| \le 2n$, we have $t_m \ne 1$ and $t_m^{2n+1} = 1$. In such a way we obtain $\sum\limits_{j=-n}^{n} t_j^m = \sum\limits_{j=-n}^{n} t_m^j = \dfrac{1-t_m^{2n+1}}{t_m^n(1-t_m)} = 0$. The lemma is proved.

Lemma 4. *For each measurable and bounded function $g : \Gamma_0 \to \mathbb{C}$ and each polynomial $p_n \in \mathcal{P}_n$ the following relation is true*

$$\|L_n g\, p_n\|_2 \le \|g\|_\infty \|p_n\|_2, \tag{16}$$

where $\|g\|_\infty = \sup\limits_{t\in\Gamma_0} |g(t)|$.

Proof. Taking into account the fact that the functions t^n, $t = e^{i\theta}$, $n \in \mathbb{Z}$, form an orthogonal basis in $L_2(\Gamma_0)$ and relations (8) and (15), the norm of the polynomial $L_n f$ (f is measurable and bounded in $L_2(\Gamma_0)$) can be calculated:

$$\|L_n f\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} \left|(L_n f)(e^{i\theta})\right|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{k=-n}^{n} \Lambda_k e^{i\theta k}\right|^2 d\theta = \sum_{k=-n}^{n} |\Lambda_k|^2 =$$

$$= \frac{1}{(2n+1)^2}\sum_{k=-n}^{n}\left(\sum_{j=-n}^{n} f(t_j) t_j^{-k}\right)\left(\sum_{l=-n}^{n}\overline{f(t_l)} t_l^k\right) =$$

$$= \frac{1}{(2n+1)^2}\sum_{j=-n}^{n} f(t_j)\left(\sum_{l=-n}^{n}\overline{f(t_l)}\left(\sum_{k=-n}^{n} t_k^{l-j}\right)\right) = \frac{1}{2n+1}\sum_{j=-n}^{n}|f(t_j)|^2 .$$

In this way we obtain:

$$\|L_n g\, p_n\|_2^2 = \frac{1}{2n+1}\sum_{j=-n}^{n} |g(t_j)|^2\, |p_n(t_j)|^2 \le \|g\|_\infty^2 \|L_n\, p_n\|_2^2 = \|g\|_\infty^2 \|p_n\|_2^2,$$

which implies relation (16). The lemma is proved.

Lemma 5. *Each 2-nonsingular function $a \in PC(\Gamma_0)$ can be represented in the form $a(t) = r_n(t)h(t)$, where r_n is a trigonometric polynomial from $\mathcal{P}_n$ and $h \in PC(\Gamma_0)$ (h is 2-nonsingular and with the same discontinuities as a) such that $\|h - 1\|_\infty = \sup_{t\in\Gamma_0} |h(t) - 1| \le q < 1$.*

Proof. The 2-nonsingular function $a \in PC(\Gamma_0)$ with discontinuity points $t_1, ..., t_n$ can be represented in the form $a(t) = |a(t)| \exp(i\theta(t))$. We set $\rho(t) = |a(t)|$. From the hypothesis it results that $\rho \in PC(\Gamma_0)$ and there exists $\delta > 0$ such that $\rho(t) \ge \delta$ for all $t \in \Gamma_0$. In such a way we can include ρ in the factor h and so we can assume, without loosing generality, that $a(t) = \exp(i\theta(t))$.

We choose an arbitrary point $t_0 \in \Gamma_0$, $t_0 \ne t_j$ ($j = \overline{1,n}$) as an initial point from which the calculation of argument begins. The fact that $\hat{a}(t,\mu) \ne 0$ for all $(t,\mu) \in \Gamma_0 \times [0,1]$ allows us to choose the function θ with real values in such a way that θ is continuous at all points $t \in \Gamma_0$ which are different from t_j ($j = \overline{0,n}$), is left continuous at $t_0, t_1, ..., t_n$ and for $\delta > 0$ the relations $|\theta(t_j) - \theta(t_j + 0)| < \pi - \delta$ ($j = \overline{1,n}$) are true while $a(t_0) - a(t_0 + 0)$ is multiple of 2π. We define the functions $b, c \in PC(\Gamma_0)$ with real values in the following way: $b(t_j) = \theta(t_j)$, $b(t_j + 0) = \theta(t_j + 0)$, $j = \overline{0,n}$, $c(t_0) = \theta(t_0)$, $c(t_0 + 0) = \theta(t_0 + 0)$, $c(t_j) = c(t_j + 0) = \frac{1}{2}(\theta(t_j) + \theta(t_j + 0))$, $j = \overline{1,n}$, and on residual arcs of Γ_0, $b(t)$ and $c(t)$ are defined by linear interpolation. Then the following inequality is true

$$\sup_{t\in\Gamma_0} |b(t) - c(t)| < \frac{1}{2}(\pi - \delta). \tag{17}$$

The mode of choice of functions $b(t)$ and $c(t)$ implies the fact that the functions $\theta(t) - b(t)$ and $\exp(ic(t))$ are continuous on Γ_0. So, the following function

$$f(t) = \exp(i(\theta(t) - b(t) + c(t))) \tag{18}$$

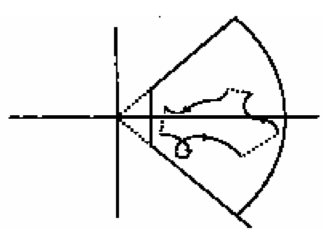
is continuous on Γ_0. It is evident that $|f(t)| = 1$ for all $t \in \Gamma_0$. Therefore, according to the Wierstrass second theorem of approximation, there exists trigonometric polynomial $p_n(t) = \sum_{k=-n}^{n} a_k t^k$ such that $p_n(t) \neq 0$ on Γ_0 and which approximates uniformly the function f, such that f can be represented in the following way $f = p_n(1 - m)$, and for $m \in C(\Gamma_0)$ the following relations are true:

$$\sup_{t \in \Gamma_0} |m(t)| < \frac{1}{2}, \tag{19}$$

$$-\frac{1}{4}\delta < \arg(1 - m(t)) < \frac{\delta}{4}. \tag{20}$$

We mention the fact that relation (20) can be obtained by choosing the polynomial p_n in such a way that for the function m the value $\sup_{t \in \Gamma_0} |m(t)|$ is sufficiently small.

We define the function u in the following way $u(t) = (1 - m(t)) \exp(i(b(t) - c(t)))$, $t \in \Gamma_0$. Then $u \in PC(\Gamma_0)$. As the function f from relation (18) is equal to $p_n(1 - m)$, we conclude that $a(t) = \exp(i\theta(t)) = f(t) \exp(i(b(t) - c(t))) = p_n(t)u(t)$. Since $p_n \in C(\Gamma_0)$, $p_n(t) \neq 0$ on Γ_0, and the function a is 2-nonsingular, from the last relation it results that the function u is 2-nonsingular and it has the same discontinuities as a. From relations (17) and (20) we obtain $|\arg u(t)| < \pi/2 - \delta/4$, and from (19) we obtain $|u(t)| \geq 1/2$. In such a way values of the function u are situated in a semi-plane of the line $Re\, u(t) \geq \delta_0 > 0\, (t \in \Gamma_0)$. More exactly, the values $u(t)$ for all $t \in \Gamma_0$ are situated in the triangular sector as it is indicated on the figure.



Evidently, by the similarity transformation with the coefficient $\gamma\, (> 0)$ this sector can be translated into a sector all points of which are distant from point 1 with the distance which is less than 1. So, a number $\gamma > 0$ can be chosen such that for all $t \in \Gamma_0$ the values $\gamma u(t)$ belong to the unit circle and $\|1 - \gamma u\|_\infty = \sup_{t \in \Gamma_0} |1 - \gamma u| \leq q < 1$. Now we set $r_n(t) = \gamma^{-1} p_n(t)$, $h(t) = \gamma u(t)$. As $a(t) = r_n(t)h(t)$, it results that the lemma is proved.

Corollary 1. *According to Lemma 5, each 2-nonsingular function* $a(t) \in PC(\Gamma_0)$ *can be represented in the form*

$$a(t) = r_n(t)(g(t) + 1), \tag{21}$$

where $r_n \in \mathcal{P}_n$*, and the function* $g \in PC(\Gamma_0)$ *satisfies the condition* $\|g\|_\infty = \sup_{t\in\Gamma_0} |g(t)| \leq q < 1$*. So, we have* $ind_2(g(t) + 1) = 0$*, and, as* $r_n(t)$ *and* $g(t) + 1$ *do not have common discontinuity points, we obtain that* $ind_2 a(t) = ind\, r_n(t)$*.*

4 The formulation and the proof of the convergence theorems

Let equation (3) have a unique solution, i.e. the operator A that describes the left term of the given equation is invertible in $L_2(\Gamma_0)$. We will show that this condition is sufficient for the convergence of the collocation and quadrature methods applied to this equation.

The integral operator K with the weak singularity (see equation (1)) is completely continuous in the space $L_2(\Gamma_0)$ [1].

Let the operator $M = aP + bQ \in L(L_2(\Gamma_0))$ be invertible. Then M is nöetherian and $Ind\, M = 0$, that implies the nöetherian character of the operator $A = M + K$ and the condition $Ind\, A = Ind\, M = 0$ [2, p.145]. Let $\dim\ker A = 0$. Then, as $Ind\, A = \dim\ker A - \dim co\ker A$, we obtain that $\dim co\ker A = 0$, and thus $Im A = L_2(\Gamma_0)$, that implies the invertibility of the operator A in $L_2(\Gamma_0)$.

Taking into account all the mentioned above and the necessary and sufficient conditions of invertibility of the operator M (see [12, 13]), the following results about convergence of the collocation and quadrature methods can be formulated:

Theorem 1. *Let the following conditions be true:*

1) $a_0(t), b_0(t) \in PC(\Gamma_0)$, $f(t) \in R(\Gamma_0)$, $h_0(t, \tau) \in C(\Gamma_0 \times \Gamma_0)$*;*

2) (i) $b(t \pm 0) \neq 0$, $t \in \Gamma_0$*; (ii)* $\hat{c}(t, \mu) \neq 0$, $(t, \mu) \in \Gamma_0 \times [0, 1]$, where $c = ab^{-1}$;

3) *The number* $k := ind_2 c(t) = 0$*;*

4) $\dim\ker A = 0$*;*

5) *Nodes* $t_j\,(j = \overline{-n, n})$ *are calculated according to formula (2).*

Then, for sufficiently small ρ $(\varepsilon_\rho \leq q_1 < 1)$ *and for sufficiently large* n $(n \geq n_0)$*, system (6) has a unique solution* $\alpha_k\,(k = \overline{-n, n})$*. The approximate solutions* $\varphi_n(t)$*, constructed according to formula (5), converge when* $\rho \to 0$ *and* $n \to \infty$ *to exact solution* $\varphi(t)$ *of equation (1) in the norm of the space* $L_2(\Gamma_0)$ $\lim_{\rho\to 0} \lim_{n\to\infty} \|\varphi - \varphi_n\|_2 = 0$.

Theorem 2. *Let all conditions of Theorem 1 be true with the exception of* $h_0(t, \tau) \in H_\alpha(\Gamma_0 \times \Gamma_0)$*, where* H_α *is the Banach space of all functions that satisfy Hölder condition on* Γ_0 *(see, for example [4, 6]). Then the affirmations of Theorem 1 are true with the condition that SLAE (6) is changed with SLAE (7).*

Proof of Theorem 1. According to condition (ii) we have that the function $c \in PC(\Gamma_0)$ is 2-nonsingular. Then according to the above corollary c is represented in the form (21) and $ind_2 c(t) = ind\, r_n(t)$. From condition 3) it results that $ind\, r_n(t) = 0$.

As the trigonometric polynomial $r_n(t) \neq 0$ on Γ_0, it can be represented in the form (see [14, p.30])

$$r_n(t) = \prod_{j=1}^{n+k} \left(1 - t_j^+ t^{-1}\right) t^k \prod_{j=1}^{n-k} \left(t - t_j^-\right), \tag{22}$$

where $k = ind\, r_n(t)$, and t_j^+ $(j = \overline{1, n+k})$ $(t_j^-$ $(j = \overline{1, n-k}))$ are all zeroes (taking into account their multiplicity) of the polynomial $r_n(t)$ which belong to the domain D^+ (the domain D^-). As polynomials

$$r_{n+k}^-(t) = \prod_{j=1}^{n+k} \left(1 - t_j^+ t^{-1}\right), \;\; r_{n-k}^+(t) = \prod_{j=1}^{n-k} \left(t - t_j^-\right) \tag{23}$$

satisfy conditions $r_{n+k}^-(t) \neq 0$, $t \in D^- \cup \Gamma_0$, $r_{n-k}^+(t) \neq 0$, $t \in D^+ \cup \Gamma_0$, and $(r_{n+k}^-)^{\pm 1}$ (respectively $(r_{n-k}^+)^{\pm 1}$) are analytical in D^- (respectively in D^+), and $k = ind\, r_n(t) = 0$, we obtain that equality (22) is a canonic factorization of the polynomial r_n with respect to the closed contour Γ_0

$$r_n(t) = r_n^-(t) r_n^+(t). \tag{24}$$

Taking into account properties of polynomials (23) (for $k = 0$) and the equality $P + Q = I$, we obtain that $P r_n^- Q \varphi_n = Q (r_n^+)^{-1} P \varphi_n = 0$. From this we have

$$P (r_n^+)^{\pm 1} P = (r_n^+)^{\pm 1} P; \;\; P (r_n^-)^{\pm 1} P = P (r_n^-)^{\pm 1}. \tag{25}$$

The condition (i) implies the existence of the inverse of the function $b \in PC(\Gamma_0)$. We have that $b^{-1} \in PC(\Gamma_0)$. Relations (21), (24) imply for the 2-nonsingular function $c = ab^{-1} \in PC(\Gamma_0)$ (a, b are coefficients of equation (3)) the representation

$$c(t) = r_n^-(t) r_n^+(t) h(t), \tag{26}$$

where $h(t) = g(t) + 1$ is a function that possesses properties described above (in Lemma 5).

According to equality (26) equation (4) is equivalent to the equation

$$h(t) r_n^-(t)(P\varphi)(t) + (r_n^+(t))^{-1}(Q\varphi)(t) + (r_n^+(t))^{-1} b^{-1}(t)(K_\rho \varphi)(t) = f_1(t), \tag{27}$$

where $f_1 = (r_n^+)^{-1} b^{-1} f \in R(\Gamma_0)$. Thus, system (6) is equivalent to the system $h(t_j) r_n^-(t_j)(P\varphi_n)(t_j) + (r_n^+(t_j))^{-1}(Q\varphi_n)(t_j) + (r_n^+(t_j))^{-1} b^{-1}(t_j)(K_\rho \varphi_n)(t_j) = f_1(t_j)$, $j = \overline{-n, n}$. As the last system is equivalent to the following operator equation

$$L_n(h r_n^- P + (r_n^+)^{-1} Q + (r_n^+)^{-1} b^{-1} K_\rho) S_n \varphi_n = L_n f_1, \; \varphi_n \in \mathcal{P}_n, \tag{28}$$

equations (9) and (28) are equivalent. Thus, the invertibility of the operator $L_n(aP+bQ+K_\rho)S_n$ implies the invertibility of $L_n(hr_n^-P+(r_n^+)^{-1}Q+(r_n^+)^{-1}b^{-1}K_\rho)S_n$ and vice versa.

Using relations (25), we obtain

$$hr_n^-P+(r_n^+)^{-1}Q = hPr_n^-P+hQr_n^-P+P(r_n^+)^{-1}Q+Q(r_n^+)^{-1}Q =$$

$$= hPr_n^- + Q(r_n^+)^{-1} + hQr_n^-P + P(r_n^+)^{-1}Q,$$

and, as

$$hPr_n^- + Q(r_n^+)^{-1} = (hP+Q)(Pr_n^- + Q(r_n^+)^{-1}) = (I+gP)(Pr_n^- + Q(r_n^+)^{-1})$$

is true, it results that equation (28) has the form

$$L_n\Big((I+gP)(Pr_n^-+Q(r_n^+)^{-1})+hQr_n^-P+P(r_n^+)^{-1}Q+(r_n^+)^{-1}b^{-1}K_\rho\Big)S_n\varphi_n = L_nf_1.$$

Introducing notations $V = (I+gP)(Pr_n^- + Q(r_n^+)^{-1})$, $K_1 = (r_n^+)^{-1}b^{-1}K_\rho$, $K_2 = hQr_n^-P + P(r_n^+)^{-1}Q$, the last equation is written in the following form

$$L_n(V+K_1+K_2)S_n\varphi_n = L_nf_1. \tag{29}$$

We will show that for sufficiently large n, the operator $L_n(V+K_1+K_2)S_n$, defined by the left term of equation (29), is invertible as an operator that acts from $\mathcal{P}_n$ to $\mathcal{P}_n$, and approximate solutions φ_n converge to the solution φ_ρ of equation (4). Toward this end we will show that for sufficiently large values n all conditions of the following known affirmation about the relation between convergence manifolds of operators C and $C+T$, where T is a complete continuous operator (see [4, p.22; 15, p.432]) are true.

Let X, Y be Banach spaces, and $\{P_n\}, \{Q_n\}\,(n=1,2,\dots)$ are two sequences of projectors with domains $D(P_n)\subset X$, $D(Q_n)\subset Y$ and closed images $Im\,P_n\subset X$, $Im\,Q_n\subset Y$. By $L(X,Y)$ we will denote the Banach algebra of all linear and bounded operators that acts from X to Y, and by $\mathcal{K}(X,Y)$ - the ideal of all complete continuous operators that acts from X to Y. By $GL(X,Y)$ we denote the set of all invertible elements of $L(X,Y)$.

Lemma 6. *Let the operator $C\in GL(X,Y)$, for $n\ge n_0$ the relation $C(Im\,P_n)\subset D(Q_n)$ be true and operators $Q_nCP_n\in GL(Im\,P_n, Im\,Q_n)$. Let Z be a Banach space that is continuously embedded in Y, such that $Z\subset\mathcal{L}(C,P_n,Q_n) := \{f\in Y : f\in D(Q_n),\ n\ge n_1(f),\ \|C^{-1}f-(Q_nCP_n)^{-1}Q_nf\|_X\to 0\}$ - the convergence manifold of the operator C after the system of projectors Q_n and P_n. Also, let $T\in\mathcal{K}(X,Z)$ and the following two conditions be true:*

$$1)\dim Ker(C+T)=0; \qquad 2)\ Q_n|_Z\in L(Z,Y).$$

Then the operators $Q_n(C+T)P_n\in GL(Im\,P_n, Im\,Q_n)$ for $n\ge n_2$ and the equality $\mathcal{L}(C,P_n,Q_n)=\mathcal{L}(C+T,P_n,Q_n)$ holds.

We set $X = Y = L_2(\Gamma_0)$, $Q_n = L_n$ $P_n = S_n$ $C = V$ $D(Q_n) = R(\Gamma_0)$, $Z = R(\Gamma_0)$, $T = K_1 + K_2$. Let us show that all conditions of the lemma take place.

Using relations (23) and (25), it can be easily verified that the operator $B = Pr_n^- + Q(r_n^+)^{-1}$ is invertible in $L_2(\Gamma_0)$, with the inverse operator $B^{-1} = P(r_n^-)^{-1} + Qr_n^+$.

As $\|S\|_2 = 1$ and $P = (I+S)/2$ is a projector, it results that $\|P\|_2 = 1$. From here we have $\|gP\|_2 \leq \|g\|_2 \leq \|g\|_\infty$, and, as $\|g\|_\infty < 1$ (see Corollary 1), it results that the operator $D = I + gP$ is invertible in $L_2(\Gamma_0)$. In such a way the invertibility of operators B and D implies the invertibility of the operator V in $L_2(\Gamma_0)$.

Lemma 7. *The inclusion $B\mathcal{P}_n \subseteq \mathcal{P}_n$ takes place.*

Proof. Let $x_n(t) = \sum\limits_{k=-n}^{n} q_k t^k$ be an arbitrary polynomial from $\mathcal{P}_n$. As $r_n^-(t) = \sum\limits_{k=-n}^{0} l_k t^k$ and $\left(r_n^+(t)\right)^{-1} = \sum\limits_{k=0}^{\infty} m_k t^k$. It results that $r_n^-(t)x_n(t) = \sum\limits_{k=-n}^{0} l_k t^k \sum\limits_{j=-n}^{n} q_j t^j = \sum\limits_{k=-2n}^{n} n_k t^k$, and $\left(r_n^+(t)\right)^{-1} x_n(t) = \sum\limits_{k=0}^{\infty} m_k t^k \sum\limits_{j=-n}^{n} q_j t^j = \sum\limits_{k=-n}^{\infty} s_k t^k$. Then we have $P(r_n^- x_n) = \sum\limits_{k=0}^{n} n_k t^k$ and $Q\left((r_n^+)^{-1} x_n\right) = \sum\limits_{k=-n}^{-1} s_k t^k$, from which we obtain $P(r_n^- x_n) + Q\left((r_n^+)^{-1} x_n\right) = \sum\limits_{k=0}^{n} n_k t^k + \sum\limits_{k=-n}^{-1} s_k t^k \in \mathcal{P}_n$. Thus, $B\mathcal{P}_n \subseteq \mathcal{P}_n$ takes place, and the lemma is proved.

On the basis of this result and thanks to the fact that $PC(\Gamma_0)$ is an algebra, we obtain $\forall x_n \in \mathcal{P}_n$, $Vx_n = DBx_n = (I + gP)y_n \in PC(\Gamma_0) \subset R(\Gamma_0) = D(Q_n)$. As the operators $L_n V S_n$ are linear and $dim \mathcal{P}_n < \infty$, it results that they are bounded as operators that act in $\mathcal{P}_n$.

We consider the operator $D_n = L_n(I + gP)S_n \in L(\mathcal{P}_n)$. Using the evident relations $S_n x_n = x_n$, $Px_n \in \mathcal{P}_n$, and $\|Px_n\|_2 \leq \|x_n\|_2$, where $x_n \in \mathcal{P}_n$, as well as relation (16), we obtain $\|L_n(I + gP)S_n x_n\|_2 = \|(S_n + L_n gPS_n)x_n\|_2 = \|x_n + L_n gPx_n\|_2 \geq \|x_n\|_2 - \|L_n gPx_n\|_2 \geq \|x_n\|_2 - \|g\|_\infty \|Px_n\|_2 \geq \|x_n\|_2 - \|g\|_\infty \|x_n\|_2 = (1 - \|g\|_\infty)\|x_n\|_2$, $\forall x_n \in \mathcal{P}_n$. The constant $C = 1 - \|g\|_\infty > 0$, because $\|g\|_\infty \leq q < 1$, therefore the operator D_n is bounded below in $\mathcal{P}_n$. As $Im\, D_n = \mathcal{P}_n$, according to the known criterion of invertibility (see [16, p.209]), the operator D_n is invertible in $\mathcal{P}_n$. At the same time the following inequality is true:

$$\|x_n\|_2 \leq \frac{1}{1 - \|g\|_\infty} \|(S_n + L_n gPS_n)x_n\|_2,\ x_n \in \mathcal{P}_n. \tag{30}$$

The relation $Bx_n \in \mathcal{P}_n\,(x_n \in \mathcal{P}_n)$, implies the representation $L_n V S_n = L_n(I + gP)S_n(Pr_n^- + Q(r_n^+)^{-1}) = D_n B$, and the invertibility of the operators D_n and B in $\mathcal{P}_n$ implies the invertibility of the operator $L_n V S_n$ in $\mathcal{P}_n$.

The Banach space $R(\Gamma_0)$ is included continuously in $L_2(\Gamma_0)$. We will show that $R(\Gamma_0) \subset \mathcal{L}(V, S_n, L_n)$.

As it was shown above, the operators V and $L_n V S_n$ are invertible respectively in $L_2(\Gamma_0)$ and in $\mathcal{P}_n$, with the inverse operators $V^{-1} = (P(r_n^-)^{-1} + Qr_n^+)(I + gP)^{-1}$ and $(L_n V S_n)^{-1} = (P(r_n^-)^{-1} + Qr_n^+)(L_n(I + gP)S_n)^{-1}$. Then for $f_1 \in R(\Gamma_0)$ we have

$$\|V^{-1}f_1 - (L_n V S_n)^{-1} L_n f_1\|_2 \leq$$
$$\leq \|P(r_n^-)^{-1} + Qr_n^+\|_2 \|(I + gP)^{-1} f_1 - (L_n(I + gP)S_n)^{-1} L_n f_1\|_2. \tag{31}$$

Let

$$\psi = (I + gP)^{-1} f_1 \in L_2(\Gamma_0), \tag{32}$$

$$\psi_n = (L_n(I + gP)S_n)^{-1} L_n f_1 \in \mathcal{P}_n. \tag{33}$$

Evidently, the following relation is true:

$$\|\psi_n - \psi\|_2 \leq \|S_n\psi - \psi_n\|_2 + \|S_n\psi - \psi\|_2. \tag{34}$$

As $S_n\psi - \psi_n \in \mathcal{P}_n$, using consecutively inequalities (30) and (33), we obtain for the first term from the right term of inequality (34)

$$\|S_n\psi - \psi_n\|_2 \leq \frac{1}{1 - \|g\|_\infty} \|(S_n + L_n gPS_n)(S_n\psi - \psi_n)\|_2 =$$
$$= \frac{1}{1 - \|g\|_\infty} \|(S_n + L_n gPS_n)\psi - L_n f_1\|_2. \tag{35}$$

From relation (32) we obtain $\psi = f_1 - gP\psi$. Then we have $(S_n + L_n gPS_n)\psi = S_n f_1 - S_n gP\psi + L_n gPS_n\psi$, but in relation (35)

$$\|S_n\psi - \psi_n\|_2 \leq \frac{1}{1 - \|g\|_\infty} \|S_n f_1 - S_n gP\psi + L_n gPS_n\psi - L_n f_1\|_2 \leq \frac{1}{1 - \|g\|_\infty} \times$$
$$\times (\|L_n gPS_n\psi - gP\psi\|_2 + \|S_n gP\psi - gP\psi\|_2 + \|S_n f_1 - f_1\|_2 + \|L_n f_1 - f_1\|_2). \tag{36}$$

Lemma 8. *For every $x \in L_2(\Gamma_0)$*

$$\|L_n gPS_n x - gPx\|_2 \to 0. \tag{37}$$

Proof. For the proof of the lemma we will use the Banach-Steinhaus theorem [16, p.271]. Consecutively using relations (16), $\|Px_n\|_2 \leq \|x_n\|_2$ ($x_n \in \mathcal{P}_n$) and $\|S_n\|_2 = 1$, we obtain for every $x \in L_2(\Gamma_0)$, $\|L_n gPS_n x\|_2 \leq \|g\|_\infty \|PS_n x\|_2 \leq \|g\|_\infty \|S_n x\|_2 \leq \|g\|_\infty \|x\|_2 := c_x < \infty$. Such sequence of operators $L_n gPS_n : L_2(\Gamma_0) \to \mathcal{P}_n$ is simply bounded. As $L_2(\Gamma_0)$ is the Banach space, it results that the sequence $L_n gPS_n$ is uniformly bounded (see [16, p.269, Theorem 1]) $\|L_n gPS_n\|_2 \leq const$, $n = 1, 2, \ldots$.

If $\mathcal{P}_m = \{x_m(t) = \sum_{k=-m}^{m} s_k t^k | s_k \in \mathbb{C}\}$ is the set of trigonometrical polynomials of order m ($m \geq 0$), defined on Γ_0, the set $\bigcup_{m=0}^\infty \mathcal{P}_m$ is dense in $L_2(\Gamma_0)$. If $x \in \bigcup_{k=0}^\infty \mathcal{P}_k$, then there exists m such that $x = x_m \in \mathcal{P}_m$, and it is true that $S_n x_m = x_m$ for

$n \geq m$. The inclusions $g \in PC(\Gamma_0) \subset R(\Gamma_0)$ and $Px_m \in \mathcal{P}_m \subset R(\Gamma_0)$ imply the fact that $gPx_m \in R(\Gamma_0)$. Then according to relation (14) it results:

$$\|L_n gPS_n x_m - gPx_m\|_2 = \|L_n gPx_m - gPx_m\|_2 \to 0, \ \forall x_m \in \bigcup_{k=0}^{\infty} \mathcal{P}_k.$$

On the basis of all the mentioned above, according to the Banach-Steinhaus theorem, we have that

$$\|L_n gPS_n x - gPx\|_2 \to 0, \ \forall x \in L_2(\Gamma_0).$$

The lemma is proved.

As $\psi \in L_2(\Gamma_0)$, relation (37) implies

$$\|L_n gPS_n \psi - gP\psi\|_2 \to 0. \tag{38}$$

Let $L_\infty(\Gamma_0)$ be a Banach algebra of all essentially bounded functions on Γ_0. An alternative characterization for this space is $L_\infty(\Gamma_0) = \{\varphi \in L_2(\Gamma_0) : \varphi f \in L_2(\Gamma_0), \forall f \in L_2(\Gamma_0)\}$ [17, p.39]. Then, as $g \in PC(\Gamma_0) \subset L_\infty(\Gamma_0)$ and $P\psi \in L_2(\Gamma_0)$, we have $gP\psi \in L_2(\Gamma_0)$, and thus (see [11, 16])

$$\|S_n gP\psi - gP\psi\|_2 \to 0. \tag{39}$$

Analogously, as $f_1 \in R(\Gamma_0) \subset L_2(\Gamma_0)$, we have

$$\|S_n f_1 - f_1\|_2 \to 0, \tag{40}$$

and according to relation (14),

$$\|L_n f_1 - f_1\|_2 \to 0. \tag{41}$$

Using relations (38)–(41), we obtain from (36) $\|S_n\psi - \psi_n\|_2 \to 0$. The last relation with $\|S_n\psi - \psi\|_2 \to 0$, implies in (34) $\|\psi_n - \psi\|_2 \to 0$, i.e. $\|(L_n(I + gP)S_n)^{-1}L_n f_1 - (I + gP)^{-1} f_1\|_2 \to 0$. As the operator $P(r_n^-)^{-1} + Qr_n^+$ is bounded in $L_2(\Gamma_0)$, from (31) we obtain:

$$\|V^{-1}f_1 - (L_n V S_n)^{-1} L_n f_1\|_2 \to 0, \ \forall f_1 \in R(\Gamma_0).$$

In such a way the inclusion $R(\Gamma_0) \subset \mathcal{L}(V, S_n, L_n)$ takes place.

It is easy to verify (see [18, p.96]) that for every $x \in L_2(\Gamma_0)$, the functions $Qr_n^- Px$, $P(r_n^+)^{-1}Qx$ are continuous on Γ_0. Taking into account item b) from Lemma 1 we obtain that the bounded operators $K_1 = (r_n^+)^{-1}b^{-1}K_\rho$ and $K_2 = hQr_n^-P + P(r_n^+)^{-1}Q$, where $(r_n^+)^{-1}b^{-1}$, $h \in PC(\Gamma_0)$, act from $L_2(\Gamma_0)$ to $PC(\Gamma_0) \subset R(\Gamma_0)$. As the operator K_ρ is completely continuous (see item c) of Lemma 1) and equalities $Qr_n^-P = \frac{1}{2}(r_n^-S - Sr_n^-)$, $P(r_n^+)^{-1}Q = -\frac{1}{2}((r_n^+)^{-1}S - S(r_n^+)^{-1})$, are true, and r_n^-, $(r_n^+)^{-1}$ are continuous functions on Γ_0, we obtain that the operators K_1, K_2 are completely continuous (see [2, p.33]) $K_1, K_2 \in \mathcal{K}(L_2(\Gamma_0), R(\Gamma_0))$.

The conditions 1)–4) of the convergence theorem assure the invertibility of the operator A of equation (3). Then, according to Lemma 2, if relation (13) is true, the operator A_ρ is invertible as well, which implies the relation $dimKerA_\rho = 0$. As $A_\rho = V + K_1 + K_2$, we obtain that $dimKer(V + K_1 + K_2) = 0$. As it was mentioned above, the operator $L_n \in L(R(\Gamma_0), L_2(\Gamma_0))$.

In such a way all conditions of Lemma 6 about the lineal of convergence of the operator $V + K_1 + K_2$ are verified, and according to it we obtain that the operators $L_n(V + K_1 + K_2)S_n : \mathcal{P}_n \to \mathcal{P}_n$ are invertible for sufficiently large n and $\mathcal{L}(V, S_n, L_n) = \mathcal{L}(V + K_1 + K_2, S_n, L_n)$ is true. Then $R(\Gamma_0) \subset \mathcal{L}(V + K_1 + K_2, S_n, L_n)$ and, as equations (29), (9) and, respectively, (27), (4), are equivalent, we obtain that equation (9) for sufficiently large n has a unique solution, and the approximate solutions φ_n converge to the exact solution φ_ρ of equation (4) $\|\varphi_\rho - \varphi_n\|_2 \to 0$. From here and from Lemma 2, using the relation $\|\varphi - \varphi_n\|_2 \le \|\varphi - \varphi_\rho\|_2 + \|\varphi_\rho - \varphi_n\|_2$, we obtain $\|\varphi - \varphi_n\|_2 \to 0$ when $n \to \infty$ and $\rho \to 0$.

In such a way Theorem 1 is proved.

Proof of Theorem 2. It is easy to verify that equation (10) is equivalent to the following operator equation

$$(A_n + \Gamma_n)\tilde{\varphi}_n = L_n f, \tag{42}$$

where $A_n\tilde{\varphi}_n = L_n(aP + bQ + K_\rho)\tilde{\varphi}_n$, $\Gamma_n\tilde{\varphi}_n = -L_n(K_\rho - \Delta_n)\tilde{\varphi}_n$, and the operators K_ρ and Δ_n were defined above.

Of course, $A_n\tilde{\varphi}_n = L_n f$ is an operator equation of the collocation method studied above. So, equation (42), which describes the quadrature method, can be interpreted as the perturbation of the equation of the collocation method.

In such a way to state the convergence of the quadrature method we will use the following lemma about the stability in the sense of Mikhlin of the approximation method [4, p.31; 15, p.438].

Let X, Y be Banach spaces, and $\{P_n\}, \{Q_n\}$ be sequences of projectors considered in Lemma 6.

Lemma 9. *Let $A \in GL(X, Y)$ and $A_n := Q_nAP_n \in GL(ImP_n, ImQ_n)$ $(n \ge n_0)$, and Z is a Banach space which is continuously embedded in Y in such a way that $ImQ_n \subset Z \subset \mathcal{L}(A, P_n, Q_n), Q_n|_Z \in L(Z, Y)$, and let $y \in Z$.*

Then there exist positive constants p, γ which do not depend on n and y in such a way that for the operator $R_n \in L(ImP_n, ImQ_n)$, which verifies the relation

$$\|R_n\|_{X \to Z} < \gamma, \tag{43}$$

we have

1) The equation

$$(A_n + R_n)\tilde{x}_n = Q_n y \tag{44}$$

has the unique solution $(n \ge n_0)$;

2) For solutions $\tilde{x}_n, x_n \in ImP_n$ of equation (44) and, respectively $A_n x_n = Q_n y$, the estimation

$$\|\tilde{x}_n - x_n\|_X \leq p\|y\|_Z \|R_n\|_{X\to Z} \tag{45}$$

holds.

We set $X = Y = L_2(\Gamma_0)$, $Z = R(\Gamma_0)4$, $Q_n = L_n, P_n = S_n$, $A = A_\rho$, $R_n = \Gamma_n$, $y = f$. The conditions of the last lemma with the exception of relation (43) were already verified in the proof of Theorem 1. Evidently $\Gamma_n \in L(ImS_n, ImL_n)$. We will show that for Γ_n condition (43) holds and even more, $\|\Gamma_n\|_{L_2\to R} \to 0$ when $n \to \infty$. In such conditions, taking into account estimation (45), we have $\|\tilde{x}_n - x_n\|_2 \to 0$ when $n \to \infty$.

Taking into account the identity $\int\limits_{\Gamma_0} L_n^\tau(h_\rho(t,\tau)\varphi_n(\tau))d\tau = \int\limits_{\Gamma_0} \frac{1}{\tau} L_n^\tau[\tau h_\rho(t,\tau)]\varphi_n(\tau)d\tau$, $\forall \varphi_n \in \mathcal{P}_n$ (see [18, p.72]), and the fact that $(\Delta_n\varphi_n)(t) - (K_\rho\varphi_n)(t) \in C(\Gamma_0)$, as well as Hölder inequalities, we obtain $\|(K_\rho\varphi_n)(t) - (\Delta_n\varphi_n)(t)\|_C = (2\pi)^{-1} \max\limits_{t\in\Gamma_0} \left| \int\limits_{\Gamma_0} \tau^{-1}(\tau h_\rho(t,\tau) - L_n^\tau[\tau h_\rho(t,\tau)])\varphi_n(\tau)d\tau \right| \leq (2\pi)^{-1} \max\limits_{t\in\Gamma_0} \int\limits_{\Gamma_0} |\tau h_\rho(t,\tau) - L_n^\tau[\tau h_\rho(t,\tau)]|\,|\varphi_n(\tau)|\,|d\tau| \leq (2\pi)^{-1} \max\limits_{t\in\Gamma_0} \|\tau h_\rho(t,\tau) - L_n^\tau[\tau h_\rho(t,\tau)]\|_2 \|\varphi_n\|_2$. As $t \mapsto \|\tau h_\rho(t,\tau) - L_n^\tau[\tau h_\rho(t,\tau)]\|_2$ is a continuous function on $\Gamma_0, \exists\, t_n \in \Gamma_0$ such that $\max\limits_{t\in\Gamma_0} \|\tau h_\rho(t,\tau) - L_n^\tau[\tau h_\rho(t,\tau)]\|_2 = \|\tau h_\rho(t_n,\tau) - L_n^\tau[\tau h_\rho(t_n,\tau)]\|_2$. As $\tau h_\rho(t,\tau) \in C(\Gamma_0)$ by τ, using the relation $\|g - L_n g\|_2 \leq 2E_n(g), \forall g \in C(\Gamma_0)$ (see [19, p.63]), we obtain $\|\tau h_\rho(t_n,\tau) - L_n^\tau[\tau h_\rho(t_n,\tau)]\|_2 \leq 2E_n^\tau(\tau h_\rho(t_n,\tau))$. According to Jackson theorem in $C(\Gamma_0)$ (see [19, p.43]) we have $E_n^\tau(\tau h_\rho(t_n,\tau)) \leq 12\omega^\tau(\tau h_\rho; \frac{1}{n+1}) \leq 12\omega^\tau(h_\rho; \frac{1}{n+1})$, where $\omega(g;\delta)$ is the modulus of continuity of the function $g(t)$.

In such a way we have $\|(K_\rho\varphi_n)(t) - (\Delta_n\varphi_n)(t)\|_C \leq \frac{12}{\pi}\omega^\tau(h_\rho; \frac{1}{n+1})\|\varphi_n\|_2$. Then taking into account the estimation $\|L_n\|_C \leq d_1 \ln n$ (see [19, p.49]), we have $\|\Gamma_n\varphi_n\|_C \leq \|L_n\|_C \|(K_\rho\varphi_n)(t) - (\Delta_n\varphi_n)(t)\|_C \leq d_2 \ln n\, \omega^\tau(h_\rho; \frac{1}{n+1})\|\varphi_n\|_2$.

Taking into account that $h_\rho(t,\tau) \in H_\delta(\Gamma_0 \times \Gamma_0), \delta = \min(\alpha,\gamma)$ (see [20, p.22; 6, p.10]), we have $\omega^\tau\left(h_\rho; \frac{1}{n+1}\right) = \sup\limits_{|\tau'-\tau''|\leq\frac{1}{n+1}} |h_\rho(t,\tau') - h_\rho(t,\tau'')| \leq \sup\limits_{|\tau'-\tau''|\leq\frac{1}{n+1}} d_3|\tau' - \tau''|^\delta = d_3\frac{1}{(n+1)^\delta}$. Consequently, as $\Gamma_n\varphi_n \in C(\Gamma_0)$, we have $\|\Gamma_n\varphi_n\|_{R(\Gamma_0)} = \|\Gamma_n\varphi_n\|_{C(\Gamma_0)}$ and then $\|\Gamma_n\|_{L_2\to R} \leq d_4 n^{-\delta}\ln n \to 0$ when $n \to \infty$.

In virtue of Theorem 1, the equation $A_n\varphi_n = L_n f$, which describes the collocation method, for all sufficiently large n, has the unique solution φ_n and $\|\varphi_n - \varphi\| \to 0$ when $n \to \infty$ (φ is the solution of equation (1)). Applying Lemma 9 we obtain that equation (42) (equivalent to equation (10)) has the unique solution $\tilde{\varphi}_n$ and $\|\tilde{\varphi}_n - \varphi_n\| \to 0$ ($n \to \infty$).

Then $\|\tilde{\varphi}_n - \varphi\| \leq \|\tilde{\varphi}_n - \varphi_n\| + \|\varphi_n - \varphi\| \to 0$ when $n \to \infty$ and in such a way we convince of the verity of statements of Theorem 2.

References

[1] Mihlin S.G. *Lineinye uravnenia v chiastnyh proizvodnyh.* Moskva, Vysshaia shkola, 1977.

[2] Gohberg I.Ts., Krupnik N.Ya. *Vvedenie v teoriu odnomernyh singuliarnyh integral'nyh operatorov.* Kishinev, Shtiintsa, 1973.

[3] Gabdulhaev B.G. *Priamye metody reshenia nekotoryh operatornyh uravnenii, I.* Izvestia vuzov, Matematika, 1971, N 11(151), p. 33–44.

[4] Prossdorf S., Silbermann B. *Projektionsverfahren und die naherungsweise Losung singularer Gleichungen.* Leipzig, Teubner-Texte, 1977.

[5] Boikov I.V. *O priblijonnom reshenii singuliarnyh integral'nyh uravnenii.* Doklady Akademii Nauk SSSR, 1972, N 3(203).

[6] Zolotarevschi V.A. *O shodimosti kollokatsionnogo metoda i metoda mehanicheskih kvadratur dlia sistem singuliarnyh integral'nyh uravnenii.* Issledovania po functsional'nomu analizu i differentsial'nym uravneniam, Kishinev, Shtiintsa, 1978, p. 3-12.

[7] Ivanov V.V. *Teoria priblijonnyh metodov i ee primenenie k chislennomu resheniu singuliarnyh integral'nyh uravnenii.* Kiev, Naukova Dumka, 1968.

[8] Liusternik L.A., Sobolev V.I. *Elementy funktsional'nogo analiza.* Moskva, Nauka, 1965.

[9] Cibrikova L.I. *Osnovnye granichnye zadachi dlia analiticheskih funktsii.* Kazani, Izd. Kazansk. un-ta, 1977.

[10] Gabdulhaev B.G., Dushkov P.N. *Metod mehanicheskih kvadratur dlia singuliarnyh integral'nyh uravnenii.* Izvestia vuzov, Matematika, 1974, N 12(151), p. 3–14.

[11] Zigmund A. *Trigonometricheskie riady, T. 2.* Moskva, Mir, 1965.

[12] Gohberg I.Ts., Krupnik N.Ya. *O spektre singuliarnyh integral'nyh operatorov v prostranstvah L_p.* Studia Mathematica, 1968, **31**, p. 347–362.

[13] Presdorf Z. *Lineinye integral'nye uravnenia.* Sovremennye problemy matematiki, Fundamentalinie napravlenia, T. 27, Moskva, 1988, p. 5–130.

[14] Gohberg I.Ts., Felidman I.A. *Uravnenia v sviortkah i proektsionnye metody ih reshenia.* Moskva, Nauka, 1971.

[15] Presdorf Z. *Nekotorye klassy singuliarnyh uravnenii.* Moskva, Mir, 1979.

[16] Kantorovici L.V., Akilov G.P. *Funktsional'nyi analiz.* Moskva, Nauka, 1977.

[17] Halmosh P. *Ghil'bertovo prostranstvo v zadaciah.* Moskva, Mir, 1970.

[18] Zolotarevschi V.A. *Konechnomernye metody reshenia singuliarnyh integral'nyh uravnenii na zamknutyh konturah integrirovania.* Kishinev, Shtiintsa, 1991.

[19] Zolotarevschi V.A. *Metode aproximative de rezolvare a ecuatsiilor integrale singulare pe circumferintsa unitate.* Kishinau, CE USM, 1997.

[20] Muskhelishvili N.I. *Singuliarnye integral'nye uravnenia.* Moskva, Nauka, 1968.

State University of Moldova
str. A. Mateevici 60, MD-2009, Chişinău
Moldova
E-mails: ylan12@yahoo.com

Received September 9, 2006

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(52), 2006, Pages 45–50
ISSN 1024–7696

Criterion of parametrical completeness in the 6-element non-chain extension of Intuitionistic logic of A. Heyting

Vadim Cebotari

Abstract. The problem of parametrical completeness in the 6-element non-chain extension of Intuitionistic logic is considered. The conditions permiting to determine the parametrical completeness of an arbitrary system of formulas in mentionted logic are established in terms of 13 parametrical pre-complete classes of formulas.

Mathematics subject classification: 03B45.
Keywords and phrases: Intuitionistic logic, parametrical expressibility, parametrical completeness, pre-complete system.

L.E. Brouwer [1] discarded the Tertiun non datur Law and proclaimed Classical logic doubtful. Gradually it became clear that Intuitionistic logic presents value in diverse aspects, including in the theory of algorithms. A. Heyting (1930) succeeded to represent it by means of well known nowadays Intuitionistic calculus [2].

A.V. Kuznetsov [3] introduced in consideration the notion of *parametrical expressibility* as a generalization of explicit expressibility. He found out the criterion of parametrical completeness in the classical logic, and put the problem to find out conditions for parametrical completeness in the Intuitionistic propositional logic [3, p. 28, problem 16]. In order to approach to the problems for Intuitionistic logic, it is more preferable to solve analogous problems, first, for some more simple logic which approximates it. A. Danil'chenko [4] obtained a criterion of parametrical completeness for the logic of *First Jaskowski's matrix*, generalized later by I. Cucu [5] for the case of the logic of any finite or countable chain.

In the present paper we give the necessary and sufficient conditions for parametrical completeness of any arbitrary system of formulas in the logic of 6-element pseudo-boolean algebra with one atom, and one penultimate element and two incomparable ones. This logic played an essential role in solving the problem of completeness with respect to explicit expressibility in the Intuitionistic logic realized by M. Rata [7] in 1970.

We construct the formulas in usual way [2] with the connectives &, $\vee$, $\supset$, and $\neg$, starting with propositional variables $p, q, r, \ldots$, possibly with indices. The symbols 0, 1, $\perp p$, $(A \sim B)$ and $(A \oplus B)$ denote respective by the formulas

$(p \& \neg p)$, $(p \supset p)$, $(p \vee \neg p)$, $(A \supset B) \& (B \supset A)$ and $((\neg A \& B) \vee (A \& \neg B))$.

The result of substituting formulas $F_1, \ldots, F_n$ in a formula G, respectively, for the propositional variables $\pi_1, \ldots, \pi_n$ is denoted by the symbols $G[\pi_1/F_1, \ldots, \pi_n/F_n]$ or in short, $G[F_1, \ldots, F_n]$.

A formula F is said to be *explicitly expressible in the logic* L *by a system* Σ *of formulas* if F can be obtained from variables and formulas belonging to Σ by means of a finite numbers of *week substitutions* (i.e. transitions from B and C to $B[\pi/C]$, where π is a variable) and *replacements by equivalents in* L (i.e. transitions from B to C such that $(B \sim C) \in L$). If all the transitions consist only in applications of week substitution rule, then they say that F *is directly expressible by* Σ.

A formula F is said to be *parametrically expressible* (in short, p. expressible) in a logic L in terms of a system (of formulas) Σ if there exist numbers l and s, variables $\pi, \pi_1, \ldots, \pi_l$ not occurring in F, pairs a formulas A_i, B_i $(i = 1, \ldots, s)$ that are expressible in L in terms of Σ, and formulas $D_1, \ldots, D_l$ that do not contain the variables $\pi, \pi_1, \ldots, \pi_l$ such that take place the relations

$$L \vdash ((F \sim \pi) \supset (A_1 \sim B_1)\& \ldots \&(A_s \sim B_s)[\pi_1/D_1], \ldots, \pi_l/D_l]),$$

$$L \vdash ((A_1 \sim B_1)\& \ldots \&(A_s \sim B_s) \supset (F \sim \pi)).$$

The relation of *parametrical expressibility is transitive.* But the partial case of this relation when parameters are absent is called *implicit expressibility*, and in general case it is not transitive. A system (of formulas) Σ is said to be *parametrically complete* (in short, p. complete) in a logic L if all formulas of the language of L are p. expressible in L in terms of Σ.

Classical logic, Intuitionistic one, which is intermediate between logics, and also absolute contradictory logic can be united under the general notion of super-Intuitionistic logic. For any of these logic there exist some pseudo-boolean algebra in which the respective logic may be interpreted.

By a *pseudo-boolean algebra* [6] we mean a system $< M; \Omega >$, where $\Omega = \{\&, \vee, \supset, \neg\}$, which is a lattice with respect to & and $\vee$, with relative pseudo-complement $\supset$ and pseudo-complement $\neg$. They say that a formula F is true in a (pseudo-boolean) algebra Λ if F, as on function of Λ, is identically equal to the greatest element 1 of Λ. The set of all formulas true in Λ constitutes a super-intuitionistic logic, called the logic of the algebra Λ and denoted below by the expression $L\Lambda$.

The pseudo-boolean algebra whose diagram is represented in Fig. 1 is denoted by the expression $Z_2 + Z_5$. The logic $L(Z_2 + Z_5)$ played an essential role in solving the problem of completeness relative to explicit expressibility in the Intuitionistic logic and in its super-intuitionistic extensions realized by M.Rata [7].

Let us remark that the chains $\{0, \tau, \omega, 1\}$, $\{0, \rho, \omega, 1\}$ and $\{0, \sigma, \omega, 1\}$ with respect to operations &, $\vee$, $\supset$, $\neg$ constitute isomorphic subalgebras of the algebra $Z_2 + Z_5$, and any of them is the interpretation of one and the same (super-intuitionistic) logic denoted below by the symbol LZ_4.

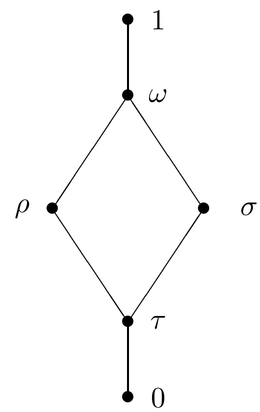


Fig. 1

Analogously, we denote by the symbol LZ_2 the logic of Boolean algebra $Z_2 =< \{0,1\}; \Omega >$.

Following A.V. Kuznetsov [3], we say that a formula $F(p_1, \dots p_n)$ *preserves the predicate* $R(x_1, \dots x_m)$ in the algebra Λ if, for any elements $\alpha_{ij} \in \Lambda$ ($i = 1, \dots, m$; $j = 1, \dots, n$), the truth of propositions

$$R[\alpha_{11}, \alpha_{21}, \dots, \alpha_{m1}], \dots, R[\alpha_{1n}, \alpha_{2n}, \dots, \alpha_{mn}]$$

implies

$$R[F[\alpha_{11}, \dots, \alpha_{1n}], \dots, F[\alpha_{m1}, \dots, \alpha_{mn}]].$$

Proposition 1 [3]. *A system of formulas Σ is p.complete in the classical logic LZ_2 if and only if there are formulas of Σ that do not preserve the predicates*

$$x = 0, x = 1, x = \neg y, x\&y = z, x \vee y = z, ((x \sim y) \sim z) = u \qquad (1)$$

in the algebra Z_2.

Under *the formula centralizer [8] of a function F* we mean the set of formulas permutable with F in a given pseudo-boolean algebra. Let denote it by the symbol $< F >$.

Let us define seven functions $f_1, \dots, f_7$ by means of Tables 1 and 2, and note that these functions cannot be expressed by formulas.

p	0	τ	ω	1
f_1	0	0	-	1
f_2	0	τ	τ	1
f_3	0	1	τ	1

Table 1

p	0	τ	ρ	σ	ω	1
f_4	0	τ	σ	ρ	1	1
f_5	0	τ	τ	τ	1	1
f_6	0	τ	1	1	1	1
f_7	0	1	τ	τ	1	1

Table 2

Theorem 1. *In order that a system Σ of formulas be parametrically complete in the logic $L(Z_2 + Z_5)$ it is necessary and sufficient that Σ be parametrically complete* *in the classical logic LZ_2 and for every $i = 1, \dots, 7$ there exist a formula F_i of Σ which does not belong to the formula centralizer $< f_i >$.*

Let's remind [3] that the formula centralizer $< F >$ coincides with the set of all formulas preserving the *predicate* $f(x_1, \dots, x_n) = y$ in the considered algebra, where the variable y differs from $x_1, \dots, x_n$. Let denote the classes of formulas preserving the predicates of line (1) in Z_2 , respectively, by the symbols $C_0, C_1, \dots, C_5$. Analogously, for any $i = 1, 2, \dots, 7$, we denote the class of formulas preserving the predicate $f_i(x) = y$ by the symbol C_{i+5}

On the base of Proposition 1 Theorem 1 is equivalent with the following

Theorem 2. *In order that a system of formulas Σ be p.complete in the logic $L(Z_2 + Z_5)$ it is necessary and sufficient that Σ be not included in one of the classes $C_0, \dots, C_{12}$.*

The necessity follows from the fact that the classes $C_0, \dots, C_{15}$ are closed with respect to p.expressibility, and they are incomparable two by two relative to the inclusion.

Sufficiency. If the condition holds, then for each $i = 1, 2, \dots, 12$ there exists a formula F_i from system Σ not belonging to the class C_i. Note that the system of six formulas $\{F_0, F_1, \dots, F_5\}$, in accordance with Proposition 1, is p.complete in the classical logic LZ_2.

In following we present twelve lemmas necessary for the proof of Theorem 2. Also we admit, for short, to use the symbol L_6 instead of the expression $L(Z_2 + Z_5)$.

Lemma 1. *The formulas 0 and 1 are explicitly expressible in L_6 by means of F_0, F_1 and F_2.*

Lemma 2. *At least one of three formulas*

$$\neg p, \neg\neg p, or \bot p \tag{2}$$

is explicitly expressible in L_6 by means of the formulas 0, 1 and F_6.

Lemma 3. *The formula $\neg p$ is implicitly expressible in L_6 by means of the formulas 0, 1, F_3, F_4 and F_6.*

Lemma 4. *The formula $\neg\neg(p\&q)$ is explicitly expressible in L_6 by means of the formulas 0, 1, $\neg p$ and F_5.*

Lemma 5. *The formulas $\bot p$ and $\neg p \& \bot q$ are p.expressible in L_6 by means of the formulas 0, 1, $\neg p$, F_9 and F_{11}.*

Lemma 6. *The formulas $\neg p\&q$, $\neg p \vee q$ and $p \oplus q$ are implicitly expressible in L_6 by means of the formulas*

$$\neg p, \neg\neg(p\&q), \neg p \& \bot q. \tag{3}$$

Lemma 7 [3]. *The conjunction $p\&q$ is implicitly expressible in any super-intuitionistic logic by means of the implication $p \supset q$.*

Lemma 8. *At least one of the following three formulas*

$$p\&q, p \sim q, p \supset q \tag{4}$$

is p.expressible in L_6 by means of the formulas of the list

$$0, 1, \neg p, \bot p, \neg p\&q, \neg p \vee q, p \oplus q \tag{5}$$

and the formulas (plus)

$$F_7, F_8, F_9, \ldots, F_{11}. \tag{6}$$

Lemma 9. *The formula $p \supset q$ is p.expressible in L_6 by means of the formulas of list (5) plus the list*

$$p \sim q, F_{12}. \tag{7}$$

Lemma 10. *At least one of three formulas*

$$p \vee q, p \sim q, p \supset q \tag{8}$$

is p.expressible in L_6 by means of the formulas (5) and the formulas

$$p\&q, F_7, F_8, F_9, F_{10}. \tag{9}$$

Lemma 11. *The formula $p \supset q$ is p.expressible in L_6 by means of the formulas of list (5) and the formulas*

$$p\&q, p \vee q, F_7, F_8. \tag{10}$$

Lemma 12. *The formula $p \vee q$ is p.expressible in L_6 by means of the formulas of list (5) and the formulas*

$$p \supset q, F_9. \tag{11}$$

Let us return to the proof of the theorem. We sum up that the formulas of list (5) because of Lemmas 1–6 are p.expressible in L_6 by means of the formulas $F_0, \ldots, F_6, F_9$, and F_{11}.

On the base of Lemma 8 at least one of the formulas of the line (4) is p.expressible in L_6 by means of the formulas of the lists (5) and (6).

In dependence of this fact there are three cases.

CASE 1. Let the formula $p \supset q$ be p.expressible in L_6 by means of the lists (5) and (6). Then in virtue of Lemma 7 the formula $p\&q$ also is p.expressible in L_6 through formulas (5) and (6). It remain to say in analyzed case that third formula $p \sim q$ from line (4) is explicitly expressible in L_6 by means of $p\&q$ and $p \supset q$, because it takes place that $(p \sim q) \sim ((p \supset q)\&(q \supset p))$.

CASE 2. Let the formula $p \sim q$ be p. expressible in L_6 via the formulas of lists (5) and (6). Then on the base of Lemma 9 the formula $p \supset q$ is p. expressible in L_6 by means of formulas from list (7). But in virtue of Lemma 7 the third formula $p\&q$ of list (4) is implicitly expressible in L_6 via the implication $p \supset q$.

CASE 3. Let the formula $p\&q$ be p. expressible in L_6 by means of the formulas of lists (5) and (6). Then on base of Lemma 10 at least one of three formulas of line

(8) is p. expressible in L_6 via formulas of lines (5) and (9). If $p \supset q$ is p.expressible then the subcase falls under the case 1. If $p \sim q$ is p.expressible then it falls under the case 2. Let $p \vee q$ be p. expressible by means of formulas of lists (5) and (9). Then, in accordance with Lemma 11, the formula $p \supset q$ is p.expressible in L_6 via formulas of lines (5) and (10).

So, we can say that all three formulas of list (4) are p. expressible in L_6 by means of the formulas of line (5) and formulas $F_7, \ldots, F_{12}$. On the base of Lemma 12, the formula $p \vee q$ also is p. expressible in L_6 by means of the formulas of lines (5) and (11).

It remained to sum up that any formula of the following system $\{\neg p, p\&q, p \vee q, p \supset q\}$ is p. expressible in L_6 by means of formulas from the hypothesis of theorem, and add that this system is explicitly complete in the logic L_6.

The theorem is proved.

A system (of formulas) Σ is said to be *parametrically pre-complete in a logic* L if Σ is not complete in L, but, for any formula F not belonging to Σ, the system $\Sigma \cup \{F\}$ is p. complete in L.

Theorem 3. *There exist exactly 13 parametrically pre-complete in $L(Z_2 + Z_5)$ classes of formulas.*

Theorem 4. *There exists non-complex algorithm which, for any finite system of formulas, enables to determine whether this system is parametrically complete in the $L(Z_2 + Z_5)$.*

References

[1] Brouwer L.E.J. *De onbetrouwbaarheid der logische principes.* Tijdschrift voor wijsbegeerte, 1908, **2,** p. 152–158.

[2] Kleene S.C.K. *Introduction to metamathematics.* Van Nostrand, Princeton, N.J,. 1952.

[3] Kuznetsov A.V. *On tools for the discovery of nondeducibity or non expressibility.* Logical Deduction. Moscow, Nauka, 1979, p. 5–33 (in Russian).

[4] Danil'chenko A.F. *The criterion for becentral completeness in the 3-valued pseudo-Boolean algebra.* XI All-Union algebraic coll. Short communications and reports, Chişinău, 1971, p. 250–251 (in Russian).

[5] Kuku I.V. *On parametrical completeness for formula systems in the chain logics.* Bull. of the Academy of Sciences of Moldavian S.S.R. The Series of Fizical, Technical and Mathematical Sciences, 1988, N 3, p. 9–14 (in Russian).

[6] Rasiowa H., Sikorski R. *The mathematics of metamathematic.* Monografie Mat., T. 41, PWN, Warsaw, 1963.

[7] Raca M.F. *A criterion for functional completeness in the Intuitionistic Propositional logic.* Soviet Math. Dokl., 1971, **12**, N 6, p. 1732–1737 (in Russian).

[8] Cohn P.M. *Universal algebra.* New York, Evanston, London, 1965.

Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
str. Academiei 5, MD-2028 Chişinău
Moldova
E-mail: *cebotari@math.md*

Received August 25, 2006

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(52), 2006, Pages 51–56
ISSN 1024–7696

Linear stability bounds in a convection problem for variable gravity field

Ioana Dragomirescu, Adelina Georgescu

Abstract. A problem governing the convection-conduction in a horizontal layer bounded by rigid walls of a fluid heated from below for a linearly decreasing across the layer gravity field is reformulated as a variational problem. Stability bounds from the case of classical convection [1] and the case of convection in a linearly decreasing across the layer gravity field are compared. The new criterion, which yields good stability bounds for the stability limit, is shown by the numerical evaluations obtained in [2–4].

Mathematics subject classification: 76E06, 47A55.
Keywords and phrases: Convection, varying gravity field.

1 Problem setting

Variations in the gravity field occur in, on and above the Earth's surface due to the fluid and atmosphere dynamics. In order to study the variable gravity effects on various convection problems and to compare them with the results obtained on a laboratory scale or deduced from the atmospheric models the mathematical model governing the conduction-convection must be investigated. In this paper we analyze the influence of a linearly decreasing gravity field on the stability bounds in a convection problem. The governing mathematical model is that given in [7]. This problem is quite unusual in the linear hydrodynamic stability theory due to the variable coefficients involved in the equations. It is a two-point eigenvalue problem, where the Rayleigh number is the eigenvalue which can be expressed by a functional defined on a Hilbert space of smooth functions satisfying some boundary conditions. The smallest eigenvalue, the only one of interest in applications, corresponds to the neutral stability in the case when the principle of exchange of stability holds. It can be computed as the minimum of that functional in the class H of admissible functions. This variational problem can be solved by means of a Fourier series technique and its solution is the smallest eigenvalue, called the linear stability limit. An alternative approach is to use isoperimetric and algebraic inequalities to provide bounds of this limit. Herein these two types of results are reported.

Consider a horizontal layer of a heat conducting viscous fluid situated between the planes $z = 0$ and $z = h$. For $t > 0$ the conduction and convective motion is governed by the conservation equations of momentum, mass and internal energy [7]

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot \mathrm{grad})\mathbf{v} = -\dfrac{1}{\rho}\mathrm{grad}p + \nu\Delta\mathbf{v} + \mathbf{g}(z)\alpha T, \\ \mathrm{div}\mathbf{v} = 0, \\ \dfrac{\partial T}{\partial t} + (\mathbf{v}\cdot \mathrm{grad})T = k\Delta T, \end{cases} \tag{1}$$

where ν is the coefficient of kinematic viscosity, ρ is the density, α the thermal expansion coefficient, k the thermal diffusivity, p the pressure, T the temperature, $\mathbf{v}$ the velocity and $\mathbf{g}(z) = gH(z)\mathbf{k}$ is the gravity, with g constant and $\mathbf{k}$ the unit vector in the z-direction. The boundary conditions at the rigid boundaries are [7]

$$\begin{cases} \mathbf{v} = 0, \text{ at } z = 0, h, \\ T = T_L, \text{ at } z = 0, \\ T = T_U, \text{ at } z = h, \text{ with } T_L > T_U. \end{cases} \tag{2}$$

The linear stability of the conduction stationary solution of equations (1) characterized by $\mathbf{v} = 0$, $T = -\beta z + T_L$, $\beta = \dfrac{T_L - T_U}{h}$, written in the nondimensional form, against normal mode perturbations is governed by the following two-point problem for the ordinary differential equations [5, 7]

$$\begin{cases} (D^2 - a^2)^2 W = RH(z)a^2\Theta, \\ (D^2 - a^2)\Theta = -RN(z)W, \end{cases} \tag{3}$$

$$W = DW = \Theta = 0 \quad \text{at } z = 0, 1. \tag{4}$$

Here $D = \dfrac{d}{dz}$, R^2 is the Rayleigh number and it represents the eigenvalue of the problem (3)–(4), a is the wavenumber and W and Θ are the amplitudes of the vertical velocity and pressure perturbation. They form the eigenfunction of the eigenvalue problem (3)–(4).

In the sequel we consider that $H(z) = 1 - \varepsilon z$, $N(z) \equiv 1$ and $k = 1$, so $\mathbf{g}(z) = g(1 - \varepsilon z)\mathbf{k}$.

2 Stability criteria

For $\varepsilon \in [0, 1]$, the principle of exchange of stability holds [5] for the eigenvalue problem

$$\begin{cases} (D^2 - a^2)^2 W = R(1 - \varepsilon z)a^2\Theta, \\ (D^2 - a^2)\Theta = -RW, \end{cases} \tag{5}$$

with the boundary conditions (4). Eliminating W between the equations (5) we obtain the following six-order ordinary differential equation

$$(D^2 - a^2)^3\Theta = -R^2 a^2 (1 - \varepsilon z)\Theta,$$

and the boundary conditions, written in Θ only, are

$$\Theta = (D^2 - a^2)\Theta = D(D^2 - a^2)\Theta = 0, \quad \text{at} \quad z = 0, 1.$$

The problem (5)–(4) possesses a non-trivial solution only for particular values of R. So we have an eigenvalue problem for R. For a given a we must determine the lowest value of R. This minimum value with respect to a is the critical Rayleigh number at which the instability sets in. It corresponds to the most unstable mode.

Introduce the new function

$$\Psi = (D^2 - a^2)\Theta. \tag{6}$$

In this way, we have

$$(D^2 - a^2)^2\Psi = -R^2a^2(1 - \varepsilon z)\Theta, \tag{7}$$

with the boundary conditions

$$\Theta = \Psi = D\Psi = 0 \;\; \text{at} \;\; z = 0, 1. \tag{8}$$

Some practical criteria can be derived for the hydrodynamic stability problem using the following three isoperimetric inequalities due to Joseph [6]

$$I_1^2 \geq \lambda_1^2 I_0^2, \quad I_2^2 \geq \lambda_2^2 I_1^2, \quad I_3^2 \geq \lambda_3^2 I_0^2, \tag{9}$$

where $\lambda_1 = \pi$, $\lambda_2 = 2\pi$, $\lambda_3 = (4.73)^2$ and $I_i^2(\Phi) = \displaystyle\int_0^1 (D^i\Phi)^2$. These isoperimetric inequalities are valid in the Hilbert space H_1 of real-valued four times continuously differentiable functions Φ on $[0, 1]$ satisfying the boundary conditions

$$\Phi(0) = \Phi(1) = D\Phi(0) = D\Phi(1) = 0.$$

Here, the functions Ψ and Θ are both indefinitely differentiable functions on the Hilbert space $L^2(0, 1)$. The unknown function Ψ satisfies the necessary boundary conditions so that the isoperimetric inequalities are valid for Ψ. Denote by H_2 the Hilbert subspace of $L^2(0, 1)$ consisting of real-valued four times continuously differentiable functions Φ on $[0, 1]$ satisfying the boundary conditions $\Phi(0) = \Phi(1) = 0$.

Multiplying (7) by Ψ, integrating the result over $[0, 1]$ and taking into account the boundary conditions (8) we obtain

$$I_2^2(\Psi) + 2a^2I_1^2(\Psi) + a^4I_0^4(\Psi) = -R^2a^2\int_0^1 (1 - \varepsilon z)\Theta\Psi. \tag{10}$$

Taking into account the isoperimetric inequalities (9), it is proved that the following stability criterion holds.

Proposition 1 [2]. *For $A \equiv a(a-\varepsilon) > 0$, a stability bound in the two-point problem (5), (4) is $R^2 \geq B(\pi^2 + a^2)^2/(a^2(\pi^2 + A))$, where $B = 4.73^4 + 2a^2\pi^2 + a^4$.*

Proposition 2 [2]. *For $A < 0$ the stability bound in the two-point problem (5), (4) is given by $R^2 \geq B(\pi^2 + a^2)/(a^2(1+\varepsilon))$.*

Let us recall that, for $\varepsilon = 0$, (5) becomes

$$\begin{cases} (D^2 - a^2)^2 W = Ra^2\Theta, \\ (D^2 - a^2)\Theta = -RW, \end{cases}$$

which together with the boundary conditions (4), form the classical Bénard convection problem. Denote by R_c^2 the Rayleigh number for this two-point problem and by R_ε^2 the Rayleigh number for the two-point problem (5), (4) in which the gravity field is linearly decreasing across the layer. Then the following result holds.

Proposition 3. *The domain of stability in the convection problem (5), (4) increases with $\varepsilon > 0$, i.e. $R_c^2 \leq R_\varepsilon^2$.*

Proof. By multiplying (6) by Θ, integrating the obtained result between 0 and 1, (10) is rewritten in the form

$$I_2(\Psi) - 2a^2 I_1^2(\Psi) + a^4 I_0^2(\Psi) = R^2 a^2 \int_0^1 -\Theta\Psi dz + R^2 a^2 \varepsilon \int_0^1 z\Theta\Psi dz. \tag{11}$$

Taking into account (6) projected on Ψ, for $\varepsilon = 0$, from (10) it follows [1] that the lowest characteristic value of the Bénard problem is given as the minimum

$$R_c^2 = \min_{\Psi\in H_1, \Theta\in H_2} \frac{\displaystyle\int_0^1 [(D^2 - a^2)^2 \Psi dz]}{a^2 \displaystyle\int_0^1 [(D\Theta)^2 + a^2\Theta^2] dz}. \tag{12}$$

Further let us come back to the case $\varepsilon \neq 0$. By (6) we have $\Psi = (D^2 - a^2)\Theta$. Then the following equalities

$$\begin{aligned} &\int_0^1 -\Theta\Psi dz = \int_0^1 (D\Theta)^2 + a^2(\Theta)^2 dz = I_1^2(\Theta) + a^2 I_0^2(\Theta) > 0, \\ &\int_0^1 z\Theta\Psi dz = -\int_0^1 z[(D\Theta)^2 + a^2\Theta^2] dz < 0 \end{aligned} \tag{13}$$

hold. Consequently, from (11) it follows that the lowest characteristic value can be obtained by taking the minimum of the functional

$$R_e^2 = \min_{\Psi\in H_1, \Theta\in H_2} \frac{I_2(\Psi) + 2a^2 I_1^2(\Psi) + a^4 I_0^2(\Psi)}{\displaystyle\int_0^1 \Big\{(D\Theta)^2 + a^2\Theta^2 - z[(D\Theta)^2 + a^2\Theta^2]\Big\} dz}. \tag{14}$$

The comparison of (12) and (13) implies immediately that $R_\varepsilon^2 \geq R_c^2$. □

In Fig. 1 we present the neutral curve for the classical case ($\varepsilon = 0$) and some neutral curves for the variable gravity field case. These graphs illustrate the stability criteria too.

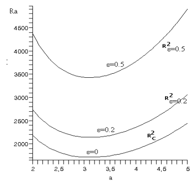


FIG. 1. The function $Ra(a)$

3 Conclusions

In this paper we presented two stability criteria (one from [2] and a new one) for a convection problem with a variable gravity field. They show that when the gravity field is linearly decreasing across the layer (in our case this means that $\varepsilon > 0$), the stability domain enlarges. The numerical results obtained in [2–4] sustain this conclusion.

In [3, 4] we obtained numerical evaluations of the Rayleigh number by using methods based on Fourier series expansions and these results agree very well with the ones obtained by Straughan in [7] using the energy method. In [2], using a variational method (in fact, isoperimetric inqualities), we also obtained numerical evaluations of the stability bounds for this convection problem. Obviously, these bounds are smaller than the limits (even approximate) obtained by methods based on Fourier series expansions. However, the advantage of applying the variational method is its easy use and the quick result obtained.

References

[1] Chandrasekhar S. *Hydrodynamic and hydromagnetic stability*. Clarendon, Oxford, 1961.

[2] Dragomirescu I. *Bounds for the onset of a convection in a variable gravity field by using isoperimetric inequalities*. International Conference "Several Aspects of Biology, Chemistry, Informatics, Mathematics and Physics", Oradea, November 11–13, 2005.

[3] Dragomirescu I. *Linear stability for a convection problem for a variable gravity field*. Proceedings of APLIMAT 2006, Bratislava, p. 231–236.

[4] Dragomirescu I. *Approximate neutral surface of a convection problem for variable gravity field*. Rend. Sem. Mat. Univ. Pol. Torino, 2006, **64**.

[5] Herron I. *On the principle of exchange of stabilities in Rayleigh-Bénard convection.* SIAM J. Appl. Math., 2000, **61**, N 4, p. 1362–1368.

[6] Joseph D.D. *Eigenvalue bounds for the Orr-Sommerfeld equation.* J. Fluid Mech., 1968, **33**, N 3, p. 617–621.

[7] Straughan B. *The energy method, stability, and nonlinear convection*, 2nd ed. Springer, Berlin, 2003.

Ioana Dragomirescu
University "Politechnica" of Timisoara
Romania
E-mail: *ioana_dragomirescu@yahoo.com*

Received September 18, 2006

Adelina Georgescu
University of Pitesti
Romania
E-mail: *adelinageorgescu@yahoo.com*

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(52), 2006, Pages 57–64
ISSN 1024–7696

A new method for computing the number of n-quasigroups

S. Markovski, V. Dimitrova, A. Mileva

Abstract. We use the isotopy classes of quasigroups for computing the numbers of finite n-quasigroups ($n = 1, 2, 3, \dots$). The computation is based on the property that every two isotopic n-quasigroups are substructures of the same number of $n+1$-quasigroups. This is a new method for computing the number of n-quasigroups and in an enough easy way we could compute the numbers of ternary quasigroups of orders up to and including 5 and of quaternary quasigroups of orders up to and including 4.

Mathematics subject classification: 20N05, 20N15, 05B15.
Keywords and phrases: n-quasigroup, isotopism, n-Latin square.

1 Introduction

An n-groupoid ($n \geq 1$) is an algebra (Q, f) on a nonempty set Q as its universe and with one n-ary operation $f : Q^n \to Q$. An n-groupoid (Q, f) is said to be an n-quasigroup if any n of the elements $a_1, a_2, \dots, a_{n+1} \in Q$, satisfying the equality

$$f(a_1, a_2, \dots, a_n) = a_{n+1},$$

uniquely determine the other one [1]. An n-groupoid is said to be a cancellative n-groupoid if it satisfies the cancellation law

$$f(a_1, \dots, a_i, x, a_{i+2}, \dots, a_n) = f(a_1, \dots, a_i, y, a_{i+2}, \dots, a_n) \Rightarrow x = y$$

for each $i = 0, 1, \dots, n-1$ and every $a_j \in Q$. An n-groupoid is said to be a solvable n-groupoid if the equation $f(a_1, \dots, a_i, x, a_{i+2}, \dots, a_n) = a_{n+1}$ has a solution x for each $i = 0, 1, \dots, n-1$ and every $a_j \in Q$.

The definition of an n-quasigroup immediately implies the following.

Lemma 1. *Let (Q, f) be a finite n-quasigroup and let the mapping $\varphi : Q \to Q$ be defined by $\varphi(x) = f(a_1, \dots, a_i, x, a_{i+2}, \dots, a_n)$. Then φ is a permutation on Q.*

Here we consider only finite n-quasigroups (Q, f), i.e., Q is a finite set, and in this case we have the next property.

Proposition 1. *The following statements for a finite n-groupoid (Q, f) are equivalent:*

(*a*) *(Q, f) is an n-quasigroup.*
(*b*) *(Q, f) is a cancellative n-groupoid.*
(*c*) *(Q, f) is a solvable n-groupoid.*

Proof. $(a) \Rightarrow (b)$ follows immediately by the definitions.

$(a) \Rightarrow (c)$ follows by Lemma 1.

Clearly, (b) and (c) imply (a).

$(b) \Rightarrow (c)$: Let (Q, f) be a cancellative n-groupoid. Then

$$\{f(a_1, \dots, a_i, x, a_{i+2}, \dots, a_n)|\ x \in Q\} = Q$$

for any fixed $a_j \in Q$.

$(c) \Rightarrow (b)$: If the groupoid (Q, f) is not cancellative then, for some $a_j \in Q$ and $i \in \{0, \dots, n-1\}$, the equation $f(a_1, \dots, a_i, x, a_{i+2}, \dots, a_n) = a_{n+1}$ has two different solutions $x_1 \neq x_2$. Then there is an element $b \in Q$ such that $b \notin \{f(a_1, \dots, a_i, x, a_{i+2}, \dots, a_n)|\ x \in Q\}$. Hence, the equation $f(a_1, \dots, a_i, x, a_{i+2}, \dots, a_n) = b$ has no solution on x. □

Given n-quasigroups (Q, f) and (Q, h), we say that (Q, f) is isotopic to (Q, h) if there are permutations $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$ on Q such that for every $a_j \in Q$

$$\alpha_{n+1} f(a_1, \dots, a_n) = h(\alpha_1 a_1, \dots, \alpha_n a_n).$$

If (Q, f) is isotopic to (Q, h), then (Q, h) is isotopic to (Q, f) too, since for the permutations $\alpha_1^{-1}, \alpha_2^{-1}, \dots, \alpha_{n+1}^{-1}$ we have $\alpha_{n+1}^{-1} h(a_1, a_2, \dots, a_n) = f(\alpha_1^{-1} a_1, \dots \dots, \alpha_n^{-1} a_n)$. Then we say that the $n+1$-tuple of permutations $(\alpha_1, \dots, \alpha_{n+1})$ is an isotopism between the n-quasigroups (Q, f) and (Q, h). The set of all isotopisms of an n-quasigroup is a group under the operation [3]:

$$(\alpha_1, \alpha_2, \dots, \alpha_{n+1})(\beta_1, \beta_2, \dots, \beta_{n+1}) = (\alpha_1\beta_1, \alpha_2\beta_2, \dots, \alpha_{n+1}\beta_{n+1}).$$

Also, the relation "is isotopic to" is an equivalence relation in the set of all n-quasigroups over a set Q. The equivalence classes are called the classes of isotopism or isotopy classes.

Example 1. A unary quasigroup (Q, f) is in fact a permutation on the set Q. If (Q, f) and (Q, g) are unary quasigroups, then they are isotopic by the isotopism (g^{-1}, f^{-1}). Hence, there is only one isotopy class in the set of unary quasigroups over given universe.

Let $Q = \{1, 2, \dots, r\},\ r > 0$. An $\underbrace{r \times \cdots \times r}_{n}$-matrix $L = [l_{i_1, i_2, \dots, i_n}]$, such that for each $i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_n$ and each j the $(i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_n)$-th row vector $(l_{i_1, \dots, i_{j-1}, 1, i_{j+1}, \dots, i_n}, l_{i_1, \dots, i_{j-1}, 2, i_{j+1}, \dots, i_n}, \dots, l_{i_1, \dots, i_{j-1}, r, i_{j+1}, \dots, i_n})$ of L is a permutation of Q, is said to be an n-Latin square of order r. The main body of the multiplication table of an n-quasigroup (Q, f) is an n-Latin square. Conversely, from an n-Latin square we can obtain an n-quasigroup, by its bordering [2, 5]. (Note that a 1-Latin square is a permutation of Q, and a 2-Latin square is a Latin square [2].)

In this paper we give a new method for computing the number of n-quasigroups, that is based on the main theorem from Section 2. For computing the number of $n+1$-quasigroups one needs the number of elements of each isotopy class of n-quasigroups, and a representative of each isotopy class. We note that the other methods for computing the number of $n+1$-quasigroups of order r use the formula $L_r = r!(r-1)!^n N_r$, where L_r is the number of all $n+1$-quasigroups of order r, and N_r is the number of so called normal $n+1$ quasigroups of order r. Usually, the number N_r is computed by different combinatorial technique, while our approach is algebraically based.

Applications of our method for computing the number of n-quasigroups are given in Section 3. For that aim we introduce a linear ordering of the set of n-quasigroups on the universe set $\{1, 2, \ldots, r\}$. The obtained results are the same as those obtained by other methods.

2 Main theorem

The problem of enumerating the set of quasigroups of given order r is well known. In fact, only the number of binary quasigroups of order $r \leq 11$ is known [4]. Nowadays, one can handle by personal computer only the set of quasigroups of order $r \leq 6$ (or maybe 7), since there are about 8.12×10^8 quasigroups of order 6, 6.14×10^{13} quasigroups of order 7 and 1.08×10^{20} quasigroups of order 8.

The main theorem of this paper allows the numbers of $n+1$-quasigroups (of small orders) to be computed, provided the isotopy classes of n-quasigroups of given order are known.

Given an $n+1$-quasigroup (Q, f) of order $r = |Q|$, $n \geq 1$, we define an (a, i)-projected n-quasigroup $(Q, f_{a,i})$ for each $i = 1, 2, \ldots, n+1$ and each $a \in Q$ by

$$f_{a,i}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) := f(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_{n+1}).$$

We have by Proposition 1 that $f_{a,i}$ is an n-quasigroup operation and that

$$f_{a,i} = f_{b,i} \iff a = b. \tag{1}$$

This implies that the $n+1$-ary operation f is uniquely determined by each of the sets $F_i = \{f_{a,i} | \ a \in Q\}$ of (a, i)-projected n-ary operations ($i = 1, 2, \ldots, n+1$).

Proposition 2. *Let Q be a finite nonempty set and let $\{f_a | \ a \in Q\}$ be a set of n-quasigroup operations on Q such that*

$$a \neq b \Rightarrow f_a(a_1, \ldots, a_n) \neq f_b(a_1, \ldots, a_n) \tag{2}$$

for every $a_1, \ldots, a_n \in Q$. Fix a number $i \in \{1, 2, \ldots, n+1\}$. Then an $n+1$-quasigroup (Q, f) can be defined such that (Q, f_a) are its (a, i)-projected n-quasigroups, i.e. $(Q, f_{a,i}) = (Q, f_a)$ for each $a \in Q$.

Proof. Choose a number $i \in \{1, 2, \ldots, n+1\}$ and define an $n+1$-ary operation f by

$$f(a_1, a_2, \ldots, a_{n+1}) := f_{a_i}(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n+1})$$

for every $a_1, \ldots, a_{n+1} \in Q$. Then (Q, f) is a cancellative $n+1$-groupoid, hence it is an $n+1$-quasigroup by Proposition 1. By the definition of an (a, i)-projected n-quasigroup, we have $f_{a,i} = f_a$. □

Theorem 1. *Let* $Q = \{q_1, q_2, \ldots, q_r\}$, $r \geq 1$, *and let* (Q, g) *and* (Q, h) *be two* n*-quasigroups from the same isotopy class. Fix a number* $i \in \{1, 2, \ldots, n+1\}$. *Then the number of* $n+1$*-quasigroups having* (Q, g) *as its* (q_1, i)*-projected* n*-quasigroup is equal to the number of* $n+1$*-quasigroups having* (Q, h) *as its* (q_1, i)*-projected* n*-quasigroup.*

Proof. Fix a number $i \in \{1, 2, \ldots, n+1\}$. Let $(\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ be an isotopism from (Q, g) to (Q, h), i.e.

$$\alpha_{n+1} g(a_1, \ldots, a_n) = h(\alpha_1 a_1, \ldots, \alpha_n a_n)$$

for each $a_1, \ldots, a_n \in Q$. Let (Q, f) be an $n+1$-quasigroup such that $f_{q_1,i} = h$. Then, for the projected quasigroups, by (1) we have

$$f_{q_s,i} = f_{q_t,i} \Longleftrightarrow s = t. \tag{3}$$

Define a set of n-quasigroups $\{(Q, f'_q) |\ q \in Q\}$ by $f'_{q_1} = g$ and

$$f'_{q_j}(x_1, x_2, \ldots, x_n) := \alpha_{n+1}^{-1} f_{q_j,i}(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n) \tag{4}$$

for $j = 2, 3, \ldots, r$.

The condition (2) of Proposition 2 is satisfied for the set of n-quasigroups $\{(Q, f'_q) |\ q \in Q\}$. Namely, if $f'_{q_s} = f'_{q_t}$, then

$$\alpha_{n+1}^{-1} f_{q_s,i}(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n) = \alpha_{n+1}^{-1} f_{q_t,i}(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n)$$

and that implies

$$f_{q_s,i}(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n) = f_{q_t,i}(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n).$$

Since α_k are permutations, we have $f_{q_s,i} = f_{q_t,i}$, leading to $s = t$ by (3). Now, by Proposition 2, we can define an $n+1$-quasigroup (Q, f') such that $f'_{q_1,i} = g$ and $f'_{q_j,i} = f'_{q_j}$ for $j \geq 2$.

We showed that to any $n+1$-quasigroup (Q, f) satisfying the condition $f_{q_1,i} = h$ we can adjoin an $n+1$-quasigroup (Q, f') satisfying the condition $f'_{q_1,i} = g$. If $(Q, \tilde{f})$ is another $n+1$-quasigroup satisfying the condition $\tilde{f}_{q_1,i} = h$ and if an $n+1$-quasigroup $(Q, \tilde{f}')$ is constructed from $\tilde{f}$ as above, then $\tilde{f}' \neq f'$. Namely, the equality $\tilde{f}' = f'$ implies, by (4),

$$\alpha_{n+1}^{-1} f_{q_j,i}(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n) = \alpha_{n+1}^{-1} \tilde{f}_{q_j,i}(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n),$$

i.e. we have $f_{q_j,i} = \tilde{f}_{q_j,i}$ for each $j = 1, 2, \ldots, r$. Hence, $f = \tilde{f}$.

We proved that the number of $n+1$-quasigroups having (Q, g) as its (q_1, i)-projected n-quasigroup is not smaller than the number of $n+1$-quasigroups having (Q, h) as its (q_1, i)-projected n-quasigroup. Analogously, the number of $n+1$-quasigroups having (Q, h) as its (q_1, i)-projected n-quasigroup is not smaller than the number of $n+1$-quasigroups having (Q, g) as its (q_1, i)-projected n-quasigroup. □

Corollary 1. *Let $Q = \{q_1, q_2, \dots, q_r\}$, $r \geq 1$, and let the isotopy classes of the n-quasigroups on Q be $C_1, C_2, \dots, C_k$. Then the number of $n+1$-quasigroups on Q is equal to*

$$b_1|C_1| + b_2|C_2| + \cdots + b_k|C_k| \tag{5}$$

where b_i denotes the number of $n+1$-quasigroups having as its $(q_1, 1)$-projected n-quasigroup an n-quasigroup from the class C_i.

Example 2. There are 6 unary quasigroups on the set $Q = \{1, 2, 3\}$ and they can be represented as the permutations 123, 132, 213, 231, 312 and 321. They form one class of isotopism C_1 and the unary quasigroup 123 can be (1,1)-projected quasigroup to $b_1 = 2$ binary quasigroups:

$*_1$	1	2	3
1	1	2	3
2	2	3	1
3	3	1	2

$*_2$	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

Consequently, there are $2 \times 6 = 12$ binary quasigroups on the set $\{1, 2, 3\}$.

3 Numerical results

The main theorem of this paper helps us to compute the numbers of n-quasigroups of order r. We could do that only for smaller values of r. For computing purposes we present the set of n-quasigroups of order r linearly and we order them lexicographically as follows. We take that the universe set is $Q = \{1, 2, \dots, r\}$ and that the n-quasigroups are given by their n-Latin squares. The unary quasigroups are linearly presented and lexicographically ordered in a natural way, since its 1-Latin square consist of only one permutation of Q. An $n+1$-quasigroup (Q, f) of order r is uniquely determined by its (q, i)-projected quasigroups $(Q, f_{1,i}), (Q, f_{2,i}), \dots, (Q, f_{r,i})$, for each fixed $i \in \{1, 2, \dots, n+1\}$. We fix $i = 1$ and let $S_1, S_2, \dots, S_r$ be the linear presentations of the quasigroups $(Q, f_{1,1}), (Q, f_{2,1}), \dots, (Q, f_{r,1})$ respectively. Then the linear presentation of the $n+1$-quasigroup (Q, f) is given by

$$S_1 \underbrace{|| \dots |}_{n} S_2 \underbrace{|| \dots |}_{n} \dots \underbrace{|| \dots |}_{n} S_r. \tag{6}$$

Now, the lexicographic ordering of the linear presentations of all n-quasigroups of order r gives the ordering of the quasigroups.

Example 3. On the set $\{1, 2, 3\}$ we have the following linear presentations and lexicographically ordering of the unary, binary and ternary quasigroups.

$123 < 132 < 213 < 231 < 312 < 321,$

$123|231|312 < 123|312|231 < 132|213|321 < 132|321|213 <$

$< 213|132|321 < 213|321|132 < 231|123|312 < 231|312|123 <$
$< 312|123|231 < 312|231|123 < 321|132|213 < 321|213|132,$

$123|231|312||231|312|123||312|123|231 < 123|231|312||312|123|231||231|312|123 <$
$< 123|312|231||231|123|312||312|231|123 < 123|312|231||312|231|123||231|123|312 <$
$< 132|213|321||213|321|132||321|132|213 < 132|213|321||321|132|213||213|321|132 <$
$< 132|321|213||213|132|321||321|213|132 < 132|321|213||321|213|132||213|132|321 <$
$< 213|132|321||132|321|213||321|213|132 < 213|132|321||321|213|132||132|321|213 <$
$< 213|321|132||132|213|321||321|132|213 < 213|321|132||321|132|213||132|213|321 <$
$< 231|123|312||123|312|231||312|231|123 < 231|123|312||312|231|123||123|312|231 <$
$< 231|312|123||123|231|312||312|123|231 < 231|312|123||312|123|231||123|231|312 <$
$< 312|123|231||123|231|312||231|312|123 < 312|123|231||231|312|123||123|231|312 <$
$< 312|231|123||123|312|231||231|123|312 < 312|231|123||231|123|312||123|312|231 <$
$< 321|132|213||132|213|321||213|321|132 < 321|132|213||213|321|132||132|213|321 <$
$< 321|213|132||132|321|213||213|132|321 < 321|213|132||213|132|321||132|321|213.$

Thus, on the set $\{1,2,3\}$, the 5-th binary quasigroup is $213|132|321$ and the 16-th ternary quasigroup is $231|312|123||312|123|231||123|231|312$. One can see that the 14-th ternary quasigroup is built up from the 8-th, 9-th and the first binary quasigroups. □

Trivially, there is only one n-quasigroup of order 1 and $r!$ unary quasigroups of order r. For computing the number of n-quasigroups of order 2 and 3 it is useful to be noted the following. Let in (6) us $S_1 = s_{11}s_{12}\dots s_{1r}|t_{11}t_{12}\dots t_{1r}|\dots$, $S_2 = s_{21}s_{22}\dots s_{2r}|t_{21}t_{22}\dots t_{2r}|\dots$, $S_r = s_{r1}s_{r2}\dots s_{rr}|t_{r1}t_{r2}\dots t_{rr}|\dots$, where $s_{\lambda\mu}, t_{\lambda\mu} \in \{1,2,\dots,r\}$. Then

$$\begin{array}{c} s_{11}s_{21}\dots s_{r1}, \quad s_{12}s_{22}\dots s_{r2}, \dots, \quad s_{1r}s_{2r}\dots s_{rr}, \\ t_{11}t_{21}\dots t_{r1}, \quad t_{12}s_{22}\dots t_{r2}, \dots, \quad t_{1r}t_{2r}\dots t_{rr}, \quad \dots \end{array} \tag{7}$$

are permutations of $\{1,2,\dots,r\}$. Immediately we have:

Proposition 3. *There are only* 2 *n-quasigroups of order* 2.

Proposition 4. *The number of n-quasigroups of order* 3 *is* 3×2^n.

Proof. Let $S_1\underbrace{||\dots|}_{n}S_2\underbrace{||\dots|}_{n}S_3$ be an $n+1$-quasigroup of order 3. S_1 can be any n-quasigroup of order 3. Given S_1, by (7), there are only two choices for S_2; given S_1 and S_2, the quasigroup S_3 is uniquely determined. Since we have 6 1-quasigroups of order 3, the result follows. □

Proposition 5. *The number of n-quasigroups of order* 4 *for* $n=1,\ n=2,\ n=3$ *and* $n=4$ *are* 24, 576, 55 296 *and* 36 972 288 *respectively.*

Isotopy class	Represent of C_i	$\lvert C_i\rvert$	b_i	$b_i\lvert C_i\rvert$
C_1	1234|2143|3412|4321|| 2143|1234|4321|3412|| 3412|4321|1234|2143|| 4321|3412|2143|1234	**864**	2292	1980288
C_2	1234|2143|3421|4312|| 2143|1234|4312|3421|| 3421|4312|2143|1234|| 4312|3421|1234|2143	**2592**	852	2208384
C_3	1234|2143|3412|4321|| 2143|1234|4321|3412|| 3412|4321|2143|1234|| 4321|3412|1234|2143	**2592**	876	2270592
C_4	1234|2143|3412|4321|| 2143|1234|4321|3412|| 3421|4312|1243|2134|| 4312|3421|2134|1243	**2592**	876	2270592
C_5	1234|2143|3412|4321|| 2143|1234|4321|3412|| 3421|4312|2134|1243|| 4312|3421|1243|2134	**2592**	876	2270592
C_6	1432|3241|4123|2314|| 4123|2314|1432|3241|| 3214|4132|2341|1423|| 2341|1423|3214|4132	**2592**	876	2270592
C_7	1432|3241|4123|2314|| 4123|2314|1432|3241|| 3241|1432|2314|4123|| 2314|4123|3241|1432	**2592**	876	2270592
C_8	1432|3241|4123|2314|| 4123|2314|1432|3241|| 3214|1423|2341|4132|| 2341|4132|3214|1423	**2592**	876	2270592
C_9	1234|2341|3412|4123|| 4123|3412|2341|1234|| 3412|1234|4123|2341|| 2341|4123|1234|3412	**5184**	144	746496
C_{10}	1234|2341|3412|4123|| 4321|1432|2143|3214|| 2413|3124|4231|1342|| 3142|4213|1324|2431	**5184**	144	746496
C_{11}	1243|2431|3124|4312|| 3421|4213|1342|2134|| 2314|3142|4231|1423|| 4132|1324|2413|3241	**5184**	144	746496
C_{12}	1234|2143|3412|4321|| 2143|1234|4321|3412|| 3412|4321|1243|2134|| 4321|3412|2134|1243	**20736**	816	16920576

TABLE. ISOTOPY CLASSES OF TERNARY QUASIGROUPS OF ORDER 4

Proof. We use Corollary 1. There is only one isotopy class of unary quasigroups of order 4, and the unary quasigroup 1234 is the first unary quasigroup of 24 binary quasigroups. So, there are $24 \times 24 = 576$ binary quasigroups. There are 2 isotopy classes of binary quasigroups, C_1 with 144 and C_2 with 432 elements. The quasigroup $1234|2143|3412|4321 \in C_1$ is the first quasigroup of $b_1 = 132$ ternary quasigroups, and the quasigroup $1234|2143|3421|4312 \in C_2$ is the first quasigroup of $b_2 = 84$ ternary quasigroups. So, there are $144 \times 132 + 432 \times 84 = 55\,296$ ternary quasigroups of order 4. For the quaternary quasigroups of order 4 the results are presented in Table. □

We remark that the result of Table differs from that given in "On-Line Encyclopedia of Integer Sequences" (see [7]) for the sequence $A099321$ of "Number of isotopy classes of Latin cubes of order n". We note that, by using Table, we correctly computed the number of quaternary quasigroups of order 4 (see also [5, 6]).

Proposition 6. *The numbers of n-quasigroups of order* 5 *for* $n = 1$, $n = 2$ *and* $n = 3$ *are* 120, 161 280 *and* 2 781 803 520 *respectively.*

Proof. We use Corollary 1. There is only one isotopy class of unary quasigroups of order 5, and the unary quasigroup 12345 is the first unary quasigroup of 56 binary quasigroups of the form

$$12345|2a_1b_1c_1d_1|3a_2b_2c_2d_2|4a_3b_3c_3d_3|5a_4b_4c_4d_4.$$

So, 12345 can be the first unary quasigroup of $56 \times 4! = 1\,344$ binary quasigroups. Hence, there are $5! \times 1\,344 = 161\,280$ binary quasigroups of order 5.

There are 2 isotopy classes of binary quasigroups of order 5, C_1 with 17 280 and C_2 with 144 000 elements. The quasigroup $12345|31452|43521\ |54213|25134 \in C_1$ is the first quasigroup of $b_1 = 22\,584$ ternary quasigroups, and the quasigroup $12345|21453|34512|45231|53124 \in C_2$ is the first quasigroup of $b_2 = 16\,608$ ternary quasigroups. So, there are $17\,280 \times 22\,584 + 144\,000 \times 16\,608 = 2\,781\,803\,520$ ternary quasigroups of order 5. □

Acknowledgement The authors are grateful to Prof. Wieslaw Dudek for giving them valuable information, that improved the quality of this paper.

References

[1] Belousov V.D. *n-ary Quasigroups*. Kishinev, Stiinca, 1972 (in Russian).

[2] Dénes J., Keedwell A.D. *Latin Squares and their Applications*. English Univer. Press Ltd., 1974.

[3] Dimitrova V. *Quasigroup Transformations and Their Applications*. MSc thesis, Skopje, 2005.

[4] McKay B.D., Wanless I.M. *On the number of Latin squares*. Ann. Combin., 2005, **9**, p. 335–344.

[5] Mullen G.L., Weber R.E. *Latin cubes of order* $\leq$ 5. Discrete Math., 1980, **32**, N 3, p. 291–297.

[6] Potapov V.N., Krotov D.S. *Asymptotics for the number of n-quasigroups of order* 4. Siberian Math. J., 2006, N 47(4), p. 720—731.

[7] http://www.research.att.com/∼njas/sequences/?q=A099321&sort=0 &fmt=0&language=english&go=Search.

"Ss Cyril and Methodius" University
Faculty of Sciences
Institute of Informatics
Skopje, Rep. of Macedonia, P.O. Box 162
E-mail: *markovski_smile@yahoo.com*
saskak@rgf.ukim.edu.mk

Received September 18, 2006

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(52), 2006, Pages 65–72
ISSN 1024–7696

The $GL(2,\mathbb{R})$-orbits of polynomial differential systems of degree four

Angela Păşcanu

Abstract. In this paper we characterize the $GL(2,\mathbb{R})-$orbits of the differential systems $\dot{x_1} = P(x_1,x_2)$, $\dot{x_2} = Q(x_1,x_2)$, where P, Q are polynomials of degree four, with respects to their dimensions.

Mathematics subject classification: 34C05.
Keywords and phrases: group action, group orbits,dimension of orbits.

1 Center-affine transformations

Consider the system

$$\dot{x_1} = \sum_{k=0}^{4} P_k(x_1,x_2) \equiv P(x_1,x_2), \quad \dot{x_2} = \sum_{k=0}^{4} Q_k(x_1,x_2) \equiv Q(x_1,x_2), \tag{1}$$

where

$$P_k(x_1,x_2) = \sum_{i+j=k} a_{ij}x_1^i x_2^j, \quad Q_k(x_1,x_2) = \sum_{i+j=k} b_{ij}x_1^i x_2^j.$$

Denote by E the space of the coefficients

$$a = (a_{00}, a_{10}, a_{01}, a_{20}, \dots, a_{13}, a_{04}; b_{00}, b_{10}, b_{01}, b_{20}, \dots, b_{13}, b_{04})$$

of system (1) and by $GL(2,\mathbb{R})$ the group of the center-affine transformations of the phase space Ox, $x = (x_1, x_2)$.

Applying in (1) the transformation $X = qx$, where $X = (X_1, X_2)$, $q \in GL(2,\mathbb{R})$, i.e.

$$q = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}, \alpha_{ij} \in \mathbb{R}, \Delta = det(q) \neq 0, \ q^{-1} = \frac{1}{\Delta}\begin{pmatrix} \alpha_{22} & -\alpha_{12} \\ -\alpha_{21} & \alpha_{11} \end{pmatrix},$$

we obtain the system

$$\dot{X}_1 = \sum_{i+j=0}^{4} a^*_{ij} X_1^i X_2^j, \quad \dot{X}_2 = \sum_{i+j=0}^{4} b^*_{ij} X_1^i X_2^j. \tag{2}$$

The coefficients a^* of (2) are expressed linearly by coefficients of system (1): $a^* = \Lambda_{(q)}(a)$, $det\Lambda_{(q)} \neq 0$. The set $\Lambda = \{\Lambda_{(q)} | q \in GL(2,\mathbb{R})\}$ forms a 4-parameter group with the operation of composition. Λ is called the representation of the $GL(2,\mathbb{R})$ group of the center-affine transformations of the phase space Ox in the space of coefficients E of system (1).

The set $O(a)=\{\Lambda_{(q)}(a)|\, q\in GL(2,\mathbb{R})\}$ is called a $GL(2,\mathbb{R})-$orbit of the point $a\in E$ or of the differential system (1) corresponding to this point.

Let

$$q_1^t=\begin{pmatrix} e^t & 0\\ 0 & 1\end{pmatrix},\; q_2^t=\begin{pmatrix} 1 & t\\ 0 & 1\end{pmatrix},\; q_3^t=\begin{pmatrix} 1 & 0\\ t & 1\end{pmatrix},\; q_4^t=\begin{pmatrix} 1 & 0\\ 0 & e^t\end{pmatrix}$$

and $G_l=\{q_l^t|\, t\in\mathbb{R}\}\subset GL(2,\mathbb{R})$, $l=\overline{1,4}$. Denote $g_l^t=\Lambda_{(q_l^t)}$ and $a^{*l}=g_l^t(a)\in E$. Then $\Lambda_l=\{g_l^t\}$, $l=\overline{1,4}$, are representations in E of the subgroups G_l, respectively. Each of the pairs $(E,\{g_l^t\})$, $l=\overline{1,4}$, is a differential flow. They define in E the following differential system of linear equations

$$\frac{da}{dt}=\left(\frac{dg_l^t(a)}{dt}\right)\Big|_{t=0}=A^{(l)}\cdot a,\quad l=\overline{1,4}. \tag{3}$$

Let

$$v_l=\sum_{i+j=0}^{4}\Big(A_{ij}^{(l)}\frac{\partial}{\partial a_{ij}}+B_{ij}^{(l)}\frac{\partial}{\partial b_{ij}}\Big),\quad l=\overline{1,4},$$

be the vector fields defined in E by systems (3). The coordinates of the vectors v_l, $l=\overline{1,4}$, are given by the formulas

$$\begin{aligned}
&A_{ij}^{(1)}=(1-i)a_{ij}, && B_{ij}^{(1)}=-ib_{ij};\\
&A_{i0}^{(2)}=b_{i0}, && A_{ij}^{(2)}=b_{ij}-(i+1)a_{i+1,j-1};\\
&B_{i0}^{(2)}=0, && B_{ij}^{(2)}=-(i+1)b_{i+1,j-1},\quad j\neq 0;\\
&A_{0j}^{(3)}=0, && A_{ij}^{(3)}=-(j+1)a_{i-1,j+1};\\
&B_{0j}^{(3)}=a_{0j}, && B_{ij}^{(3)}=a_{ij}-(j+1)b_{i-1,j+1},\ i\neq 0;\\
&A_{ij}^{(4)}=-ja_{ij}, && B_{ij}^{(4)}=(1-j)b_{ij}.
\end{aligned}$$

If we denote by L_v the derivative with respect to the vector v and we set $w=[u,v]$, where $L_w=L_uL_v-L_vL_u$, it is easy to determine that the vector fields v_l, $l=\overline{1,4}$, generate a Lie algebra. The dimension of the orbit $O(a)$ is equal to the dimension of this algebra, i.e. with the rank of the matrix of dimension 4×30 [1,2]:

$$M=\begin{pmatrix} A_{00}^{(1)} & A_{10}^{(1)} & A_{01}^{(1)} & A_{20}^{(1)} & \dots & A_{04}^{(1)} & B_{00}^{(1)} & \dots & B_{04}^{(1)}\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ A_{00}^{(4)} & A_{10}^{(4)} & A_{01}^{(4)} & A_{20}^{(4)} & \dots & A_{04}^{(4)} & B_{00}^{(4)} & \dots & B_{04}^{(4)}\end{pmatrix}. \tag{4}$$

The purpose of this paper consists in the classification of systems (1) according to the dimensions of their $GL(2,\mathbb{R})-$orbits.

We notice that such classification was done for some particular cases of system (1) in [2–9].

From [10] follows

Lemma 1. *Let $O(a)$ be a $GL(2,\mathbb{R})-$orbit of the system (1). Then*

1) $dimO(a) = 0$ if and only if (1) has the form

$$\dot{x_1} = bx_1, \quad \dot{x_2} = bx_2, \quad b = const; \tag{5}$$

2) $dimO(a) \neq 1$, $\forall a \in E$.

By Lemma 1, $dimO(a) > 1$, i.e. $dimO(a)$ is equal to one of the numbers 2,3 or 4, if and only if

$$|P(x_1,x_2) - a_{10}x_1| + |Q(x_1,x_2) - a_{10}x_2| \not\equiv 0.$$

Therefore, if the right-hand sides of the system (1) have either at least one constant term a_{00}, b_{00} or one nonlinear term, then the dimension of the $GL(2,\mathbb{R})-$orbit is at least two.

For the linear system

$$\dot{x_1} = a_{10}x_1 + a_{01}x_2, \quad \dot{x_2} = b_{10}x_1 + b_{01}x_2 \tag{6}$$

the matrix (4) has the form:

$$M_1 = \begin{pmatrix} 0 & a_{01} & -b_{10} & 0 \\ b_{10} & b_{01} - a_{10} & 0 & -b_{10} \\ -a_{01} & 0 & a_{10} - b_{01} & a_{01} \\ 0 & -a_{01} & b_{10} & 0 \end{pmatrix}. \tag{7}$$

It is easy to determine that $rank\, M_1 \leq 2$. So, the linear system has the orbit's dimension equal to zero only if it has the form (5) and $dim\, O(a) = 2$ in other cases, i.e. when

$$\dot{x_1} = a_{10}x_1 + a_{01}x_2, \quad \dot{x_2} = b_{10}x_1 + b_{01}x_2, \; |a_{10} - b_{01}| + |a_{01}| + |b_{10}| \neq 0. \tag{8}$$

Applying in (1) the transformation of coordinates

$$x_1 \longrightarrow x_2, \quad x_2 \longrightarrow x_1, \tag{9}$$

we obtain

$$\dot{x_1} = Q(x_2,x_1), \quad \dot{x_2} = P(x_2,x_1). \tag{10}$$

Denote by v_l^*, $l = \overline{1,4}$, the vectorial fields associated to the differential system (10).

Remark 1. The equalities $\alpha v_1 + \beta v_2 + \gamma v_3 + \delta v_4 = 0$ and $\delta v_1^* + \gamma v_2^* + \beta v_3^* + \alpha v_4^* = 0$ $(\alpha, \beta, \gamma, \delta \in \mathbb{R})$ are equivalent.

Talking into consideration Remark 1, in order to determine the orbits of dimension two and three it is enough to examine the following two cases:

$$\alpha v_1 + v_4 = 0, \tag{11}$$

$$\alpha\, v_1 + v_2 + \gamma\, v_3 + \delta\, v_4 = 0. \tag{12}$$

2 The case $\alpha v_1 + v_4 = 0$

The equality (11) written in the coordinates of v_1 and v_4 represents a homogeneous linear algebraic system in coefficients a_{ij} of (1).

If $v_1 = v_4 = 0$, then (1) is of the form $\dot{x_1} = a_{10}x_1$, $\dot{x_2} = b_{01}x_2$ and it is a particular case of (6).

In the case when at least one of the vectors v_1 and v_4 is nonzero, this algebraic system has nontrivial solutions only for the following values of the parameter α :

$$\alpha = \pm 2;\ 3;\ 4;\ \pm 1;\ 0;\ \pm\frac{1}{2};\ \frac{1}{3};\ \frac{1}{4}.$$

According to Remark 1, it is enough to examine only the cases:

$$\alpha = 2;\ 3;\ 4;\ -1;\ -2;\ 0;\ 1.$$

To this values of α the following solutions correspond respectively:

1) $a_{ij} = 0,\ (i,j) \neq (1,0),\ (0,2),\ b_{ij} = 0,\ (i,j) \neq (0,1);$

2) $a_{ij} = 0,\ (i,j) \neq (1,0),\ (0,3),\ b_{ij} = 0,\ (i,j) \neq (0,1);$

3) $a_{ij} = 0,\ (i,j) \neq (1,0),\ (0,4),\ b_{ij} = 0,\ (i,j) \neq (0,1);$

4) $a_{ij} = 0,\ (i,j) \neq (1,0),\ (2,1),\ b_{ij} = 0,\ (i,j) \neq (0,1),(1,2);$

5) $a_{ij} = 0,\ (i,j) \neq (1,0),\ (2,2),\ b_{ij} = 0,\ (i,j) \neq (0,1),(1,3);$

6) $a_{ij} = 0,\ j \neq 0,\ b_{ij} = 0,\ j \neq 1;$

6a) $a_{ij} = b_{ij} = 0,\ i + j \neq 1.$

Notice that in the case $6a$) we obtain the linear system (6).

Denote

$i_1 = |a_{00}| + |b_{01} - a_{10}| + |b_{11} - a_{20}| + |b_{21} - a_{30}| + |a_{40}| + |b_{31}|,$

$i_2 = |b_{01} - a_{10}| + |a_{20}| + |a_{30}| + |a_{40}| + |b_{11}| + |b_{21}| + |b_{31}|,$

$i_3 = |a_{00}| + |a_{20}| + |a_{30}| + |a_{40}| + |b_{11}| + |b_{21}| + |b_{31}|.$

In order to separate the orbits of dimension two from those of dimension three, we will determine the conditions on the coefficients of system (1) such that in each of the cases 1) $-$ 6) all the minors of order three of matrix (4) should be equal to zero. We have respectively:

$1')$ $a_{02}(a_{10} - b_{01}) = 0;$ $2')$ $a_{03}(a_{10} - b_{01}) = 0;$

$3')$ $a_{04}(a_{10} - b_{01}) = 0;$ $4')$ $|a_{21}| + |b_{12}| = 0;$

$5')$ $|a_{22}| + |b_{13}| = 0;$ $6')$ $i_1 \cdot i_2 \cdot i_3 = 0.$

The cases $[1), 1'),\ a_{02} = 0]$; $[2), 2'),\ a_{03} = 0]$; $[3), 3'),\ a_{04} = 0]$; $[4), 4'), a_{21} = b_{12} = 0]$; $[5), 5'), a_{22} = b_{13} = 0]$ and $[6), 6'), i_3 = 0]$ lead us to a system of the form (6). Later on, assuming that $\alpha v_1 + v_4 = 0$, we have the following distribution by dimensions of orbits of the system (1) (the systems (5) and (6) are not included here):

dim O(a)=2

$$\dot{x_1} = a_{10}x_1 + a_{02}x_2^2, \quad \dot{x_2} = a_{10}x_2, \quad a_{02} \neq 0; \tag{13}$$

$$\dot{x_1} = a_{10}x_1 + a_{03}x_2^3, \quad \dot{x_2} = a_{10}x_2, \quad a_{03} \neq 0; \tag{14}$$

$$\dot{x_1} = a_{10}x_1 + a_{04}x_2^4, \quad \dot{x_2} = a_{10}x_2, \quad a_{04} \neq 0; \tag{15}$$

$$\dot{x_1} = x_1 \cdot F,\ \dot{x_2} = x_2 \cdot F,\ F = a_{10} + a_{20}x_1 + a_{30}x_1^2,\ |a_{20}| + |a_{30}| \neq 0; \tag{16}$$

$$\dot{x_1} = a_{00} + a_{10}x_1, \quad \dot{x_2} = a_{10}x_2, \quad a_{00} \neq 0. \tag{17}$$

dim O(a)=3

$$\dot{x_1} = a_{10}x_1 + a_{02}x_2^2, \quad \dot{x_2} = b_{01}x_2, \quad a_{02}(a_{10} - b_{01}) \neq 0; \tag{18}$$

$$\dot{x_1} = a_{10}x_1 + a_{03}x_2^3, \quad \dot{x_2} = b_{01}x_2, \quad a_{03}(a_{10} - b_{01}) \neq 0; \tag{19}$$

$$\dot{x_1} = a_{10}x_1 + a_{04}x_2^4, \quad \dot{x_2} = b_{01}x_2, \quad a_{04}(a_{10} - b_{01}) \neq 0; \tag{20}$$

$$\dot{x_1} = x_1(a_{10} + a_{21}x_1x_2), \quad \dot{x_2} = x_2(b_{01} + b_{12}x_1x_2),\ |a_{21}| + |b_{12}| \neq 0; \tag{21}$$

$$\dot{x_1} = x_1(a_{10} + a_{22}x_1x_2^2), \quad \dot{x_2} = x_2(b_{01} + b_{13}x_1x_2^2), \quad |a_{22}| + |b_{13}| \neq 0; \tag{22}$$

$$\begin{cases} \dot{x_1} = a_{00} + a_{10}x_1 + a_{20}x_1^2 + a_{30}x_1^3 + a_{40}x_1^4, \\ \dot{x_2} = x_2(b_{01} + b_{11}x_1 + b_{21}x_1^2 + b_{31}x_1^3),\ i_1 \cdot i_2 \cdot i_3 \neq 0. \end{cases} \tag{23}$$

3 The case $\alpha\, v_1 + v_2 + \gamma\, v_3 + \delta\, v_4 = 0$.

In this section we will need the following notations:
$\alpha_i = (\delta - i\alpha)/(i+1), \delta_i = (\alpha - i\delta)/(i+1), i = \overline{1,4}, \nu_1 = \delta + 2\alpha, \nu_2 = \alpha + 2\delta;$
$j_1 = |a_{12} - 3a_{03}\alpha| + |\alpha + \delta| + |a_{00}| + |a_{11}| + |a_{02}| + |a_{13}| + |a_{04}|,$
$j_2 = |a_{00}| + |a_{01}| + |a_{02}| + |a_{03}| + |a_{04}|,$
$j_3 = |a_{11} - 2\alpha\, a_{02}| + |\alpha + \delta| + |a_{00}| + |a_{01}| + |a_{12}| + |a_{03}| + |a_{13}| + |a_{04}|,$
$j_4 = |a_{00}| + |a_{11}| + |a_{02}| + |a_{12}| + |a_{03}| + |a_{13}| + |a_{04}|,$
$j_5 = |a_{13} - 4\alpha\, a_{04}| + |\alpha + \delta| + |a_{00}| + |a_{01}| + |a_{11}| + |a_{02}| + |a_{12}| + |a_{03}|,$
$j_6 = |a_{01}| + |a_{03}| + |\gamma + \delta^2|,$
$j_7 = |a_{12} + 3\delta\, a_{03}| + |\gamma + \delta^2| + |a_{01}|,$
$j_8 = |a_{12}| + |a_{03}|,$
$j_9 = |a_{13}| + |a_{04}|,$
$j_{10} = |\alpha + \delta| + |a_{01}| + |a_{04}|,$
$j_{11} = |a_{13} + 4\delta\, a_{04}| + |\alpha + \delta| + |a_{01}|.$

The equality (12) holds if and only if at least one of the following seven series of conditions is realized:

7) $\gamma = \alpha_2 \cdot \delta_2,\ a_{20} = \delta_2^2 a_{02},\ a_{11} = 2\delta_2 a_{02},\ b_{10} = \alpha_2\delta_2 a_{01},\ b_{01} = a_{10} - 2\delta_1 a_{01},$ $b_{20} = -\delta_2^3 a_{02},\ b_{11} = -2\delta_2^2 a_{02},\ b_{02} = -\delta_2 a_{02},\ a_{ij} = b_{ij} = 0,\ i + j = 0, 3, 4;$

8) $\gamma = \alpha_3 \cdot \delta_3$, $a_{30} = a_{03}\delta_3^3$, $a_{21} = 3a_{03}\delta_3^2$, $a_{12} = 3a_{03}\delta_3$, $b_{10} = a_{01}\alpha_3\delta_3$, $b_{01} = a_{10} - 2\delta_1 a_{01}$, $b_{30} = -a_{03}\delta_3^4$, $b_{21} = -3a_{03}\delta_3^3$, $b_{12} = -3a_{03}\delta_3^2$, $b_{03} = -a_{03}\delta_3$, $a_{ij} = b_{ij} = 0$, $i+j = 0,2,4$;

9) $\gamma = \alpha_4 \cdot \delta_4$, $a_{40} = a_{04}\delta_4^4$, $a_{31} = 4a_{04}\delta_4^3$, $a_{22} = 6a_{04}\delta_4^2$, $a_{13} = 4a_{04}\delta_4$, $b_{10} = a_{01}\alpha_4\delta_4$, $b_{01} = a_{10} - 2\delta_1 a_{01}$, $b_{40} = -a_{04}\delta_4^5$, $b_{31} = -4a_{04}\delta_4^4$, $b_{22} = -6a_{04}\delta_4^3$, $b_{13} = -4a_{04}\delta_4^2$, $b_{04} = -a_{04}\delta_4$, $a_{ij} = b_{ij} = 0$, $i+j = 0,2,3$;

10) $\gamma = \alpha \cdot \delta$, $a_{20} = -\delta(a_{11} + \delta a_{02})$, $a_{30} = \delta^2(a_{12} + 2\delta a_{03})$, $a_{21} = -\delta(2a_{12} + 3a_{03}\delta)$, $a_{40} = -\delta^3(a_{13} + 3\delta a_{04})$, $a_{31} = \delta^2(3a_{13} + 8a_{04}\delta)$, $a_{22} = -3\delta(a_{13} + 2\delta a_{04})$, $b_{00} = -\alpha a_{00}$, $b_{10} = a_{01}\alpha\delta$, $b_{01} = a_{10} - 2\delta_1 a_{01}$, $b_{20} = -a_{02}\alpha\delta^2$, $b_{11} = \delta(4\delta_1 a_{02} - a_{11})$, $b_{02} = a_{11} - 3\delta_2 a_{02}$, $b_{30} = a_{03}\alpha\delta^3$, $b_{21} = \delta^2(a_{12} - 6a_{03}\delta_1)$, $b_{12} = \delta(9a_{03}\delta_2 - 2a_{12},)$ $b_{03} = a_{12} - 4\delta_3 a_{03}$, $b_{40} = -\alpha\delta^4 a_{04}$, $b_{31} = \delta^3(8a_{04}\delta_1 - a_{13})$, $b_{22} = 3\delta^2(a_{13} - 6a_{04}\delta_2)$, $b_{13} = \delta(16a_{04}\delta_3 - 3a_{13}), b_{04} = a_{13} - 5\delta_4 a_{04}$;

11) $\alpha = -\delta$, $a_{30} = -\gamma(a_{12} + 2\delta a_{03})$, $a_{21} = -2a_{12}\delta - 4a_{03}\delta^2 - \gamma a_{03}$, $b_{10} = \gamma a_{01}$, $b_{01} = a_{10} + 2\delta a_{01}$, $b_{30} = -\gamma^2 a_{03}$, $b_{21} = -\gamma(a_{12} + 6\delta a_{03})$, $b_{12} = -2\delta a_{12} - 8\delta^2 a_{03} + \gamma a_{03}$, $b_{03} = a_{12} + 4\delta a_{03}$, $a_{ij} = b_{ij} = 0$, $i+j = 0,2,4$;

12) $\gamma = \nu_1 \cdot \nu_2$, $b_{10} = \nu_1\nu_2 a_{01}$ $b_{01} = a_{10} - 2\delta_1 a_{01}$, $a_{40} = \nu_1\nu_2^2(a_{13} + 3\delta a_{04})$, $a_{31} = -\nu_2(3\alpha a_{13} - 8\delta_1^2 a_{04})$, $a_{22} = -3(a_{04}\alpha^2 + a_{13}\delta + 2\alpha\delta a_{04} + 3\delta^2 a_{04})$, $b_{40} = \nu_1^2\nu_2^3 a_{04}$, $b_{31} = \nu_1\nu_2^2(a_{13} - 8\delta_1 a_{04})$, $b_{22} = -3\nu_2(a_{13}\alpha - a_{04}\alpha^2 + 6a_{04}\alpha\delta + a_{04}\delta^2)$, $b_{13} = -3\delta a_{13} + 4a_{04}\delta_1(\alpha + 5\delta)$, $b_{04} = a_{13} - 5a_{04}\delta_4$, $a_{ij} = b_{ij} = 0$, $i+j = 0,2,3$;

13) $a_{10} = \alpha b_{10} b_{01} - \delta b_{10}$, $b_{10} = \gamma a_{01}$, $a_{ij} = b_{ij} = 0$, $i+j = 0,2,3,4$.

Notice that in conditions 13) we have a system of the form (6).

Equating to zero the minors of order three of the matrix (4) in each of the cases 7) − 12), we obtain respectively:

$7')$ $a_{01} \cdot a_{02} = 0$;
$8')$ $a_{01} \cdot a_{03} = 0$;
$9')$ $a_{01} \cdot a_{04} = 0$;
$10')$ $j_1 \cdot j_2 \cdot j_3 \cdot j_4 \cdot j_5 = 0$;
$11')$ $j_6 \cdot j_7 \cdot j_8 = 0$;
$12')$ $j_9 \cdot j_{10} \cdot j_{11} = 0$.

The relations [7), 7′)] − [12), 12′)] lead us to the following distribution of the $GL(2,\mathbb{R})-$orbits of the system (1) (the cases which lead us to the system (6) are not considered here):

dim O(a)=2

$$\dot{x_1} = a_{10}x_1 + F, \quad \dot{x_2} = a_{10}x_2 - \delta_2 \cdot F, \quad F = a_{02}(\delta_2 x_1 + x_2)^2 \not\equiv 0; \tag{24}$$

$$\dot{x_1} = a_{10}x_1 + F, \quad \dot{x_2} = a_{10}x_2 - \delta_3 \cdot F, \quad F = a_{03}(\delta_3 x_1 + x_2)^3 \not\equiv 0; \tag{25}$$

$$\dot{x_1} = a_{10}x_1 + F, \quad \dot{x_2} = a_{10}x_2 - \delta_4 \cdot F, \quad F = a_{04}(\delta_4 x_1 + x_2)^4 \not\equiv 0; \tag{26}$$

$$\begin{cases} \dot{x_1} = x_1 \cdot F, \quad \dot{x_2} = x_2 \cdot F, \\ F = a_{10} - a_{11}(\delta\, x_1 - x_2) + a_{12}(\delta\, x_1 - x_2)^2 - a_{13}(\delta\, x_1 - x_2)^3, \\ |a_{11}| + |a_{12}| + |a_{13}| \neq 0; \end{cases} \tag{27}$$

$$\dot{x_1} = a_{00} + a_{10}x_1, \quad \dot{x_2} = -\alpha\, a_{00} + a_{10}x_2, \quad a_{00} \neq 0. \tag{28}$$

dim O(a)=3

$$\begin{cases} \dot{x_1} = a_{10}x_1 + a_{01}x_2 + F, \\ \dot{x_2} = \alpha_2\delta_2 a_{01}x_1 + (a_{10} - 2\delta_1 a_{01})x_2 - \delta_2 \cdot F, \\ F = a_{02}(\delta_2 x_1 + x_2)^2, \quad a_{01} \cdot a_{02} \neq 0; \end{cases} \tag{29}$$

$$\begin{cases} \dot{x_1} = a_{10}x_1 + a_{01}x_2 + F, \\ \dot{x_2} = \alpha_3\delta_3 a_{01}x_1 + (a_{10} - 2\delta_1 a_{01})x_2 - \delta_3 \cdot F, \\ F = a_{03}(\delta_3 x_1 + x_2)^3, \quad a_{01} \cdot a_{03} \neq 0; \end{cases} \tag{30}$$

$$\begin{cases} \dot{x_1} = a_{10}x_1 + a_{01}x_2 + F, \\ \dot{x_2} = \alpha_4\delta_4 a_{01}x_1 + (a_{10} - 2\delta_1 a_{01})x_2 - \delta_4 \cdot F, \\ F = a_{04}(\delta_4 x_1 + x_2)^4, \quad a_{01} \cdot a_{04} \neq 0; \end{cases} \tag{31}$$

$$\begin{cases} \dot{x_1} = a_{00} + a_{10}x_1 + a_{01}x_2 - ((a_{11} + a_{02}\delta)x_1 + a_{02}x_2) \cdot F + \\ \qquad + ((a_{12} + 2a_{03}\delta)x_1 + a_{03}x_2) \cdot F^2 - \\ \qquad - ((a_{13} + 3a_{04}\delta)x_1 + a_{04}x_2) \cdot F^3, \\ \dot{x_2} = -\alpha a_{00} + \alpha\delta a_{01}x_1 + (a_{10} - 2\delta_1 a_{01})x_2 - \\ \qquad - (\alpha\,\delta a_{02}x_1 + (a_{11} - 3a_{02}\delta_2)x_2) \cdot F + \\ \qquad + (\alpha\,\delta a_{03}x_1 + (a_{12} - 4a_{03}\delta_3)x_2) \cdot F^2 - \\ \qquad - (\alpha\,\delta a_{04}x_1 + (a_{13} - 5a_{04}\delta_4)x_2) \cdot F^3, \\ F = \delta\, x_1 - x_2, \quad j_1 \cdot j_2 \cdot j_3 \cdot j_4 \cdot j_5 \neq 0; \end{cases} \tag{32}$$

$$\begin{cases} \dot{x_1} = a_{10}x_1 + a_{01}x_2 - ((a_{12} + 2a_{03})x_1 + a_{03}x_2) \cdot F, \\ \dot{x_2} = \gamma\, a_{01}x_1 + (a_{10} - 2a_{01}\delta_1)x_2 + (a_{03}\gamma x_1 + (a_{12} + 4a_{03})x_2) \cdot F, \\ F = \gamma\, x_1^2 + 2\delta\, x_1x_2 - x_2^2, \quad j_6 \cdot j_7 \cdot j_8 \neq 0; \end{cases} \tag{33}$$

$$\begin{cases} \dot{x_1} = a_{10}x_1 + a_{01}x_2 + ((a_{13} + 3\delta\, a_{04})x_1 + a_{04}x_2) \cdot F, \\ \dot{x_2} = \nu_1\,\nu_2\, a_{01}x_1 + (a_{10} - 2\delta_1 a_{01})x_2 + \\ \qquad + (\nu_1\,\nu_2\, a_{04}x_1 + (a_{13} - 5\delta_4\, a_{04})x_2) \cdot F, \\ F = (\nu_1\, x_1 + x_2)(\nu_2\, x_1 - x_2)^2, \quad j_9 \cdot j_{10} \cdot j_{11} \neq 0. \end{cases} \tag{34}$$

Remark 2. It is easy to see that the systems (13) − (15), (17) are particular cases of the systems (24) − (26), (28) respectively. The (16) by substitution (9) can be reduced to a system of the form (27).

The results obtained above are gathered in the following theorem:

Theorem. *Up to a transformation (9), the dimension of the $GL(2,\mathbb{R})-$orbit of the system (1) is equal to*

0 *if it has the form (5);*
2 *if it has one of the forms (8), (24) − (28);*
3 *if it has one of the forms (18) − (23), (29) − (34);*
4 *in other cases.*

References

[1] Ovsyanikov L.V. *Group analysis of differential equations.* Moscow, Nauka, 1978 (English transl. by Academic press, 1982.)

[2] Popa M.N. *Applications of algebras to differential systems.* Academy of Sciences of Moldova, Chişinău, 2001 (in Russian).

[3] Braicov A.V., Popa M.N. *The $GL(2,\mathbb{R})-$orbits of differential system with homogeneites second order.* The Internationals Conference "Differential and Integral Equations", Odessa, September 12–14, 2000, p. 31.

[4] Boularas D., Braicov A.V., Popa M.N. *Invariant conditions for dimensions of $GL(2,\mathbb{R})-$orbits for quadratic differential system.* Bul. Acad. Sci. Rep. Moldova, Math., 2000, N 2(33), p. 31–38.

[5] Boularas D., Braicov A.V., Popa M.N. *The $GL(2,\mathbb{R})-$orbits of differential system with cubic homogeneites.* Bul. Acad. Sci. Rep. Moldova, Math., 2001, N 1(35), p. 81–82.

[6] Naidenova E.V., Popa M.N. *On a classification of Orbits for Cubic Differential Systems.* Abstracts of "16th International Symposium on Nonlinear Acoustics", section "Modern group analysis" (MOGRAN-9), August 19–23, 2002, Moscow, p. 274.

[7] Naidenova E.V., Popa M.N. *$GL(2,\mathbb{R})-$orbits for one cubic system.* Abstracts of "11th Conference on Applied and Industrial Mathematics", May 29–31, 2003, Oradea, Romania, p. 57.

[8] Starus E.V. *Invariant conditions for the dimensions of the $GL(2,\mathbb{R})-$orbits for one differential cubic system*, Bul. Acad. Sci. Rep. Moldova, Math., 2003, N 3(43), p. 58–70.

[9] Starus E.V. *The classification of the $GL(2,\mathbb{R})-$orbit's dimensions for the system $s(0,2)$ and a factorsystem $s(0,1,2)/GL(2,\mathbb{R})$.* Bul. Acad. Sci. Rep. Moldova, Math., 2004, N 1(44), p. 120–123.

[10] Păşcanu A., Şubă A. *$GL(2,\mathbb{R})-$orbits of the polynomial systems of differential equation.* Bul. Acad. Sci. Rep. Moldova, Math., 2004, N 3(46), p. 25–40.

Department of Mathematics
State University of Tiraspol
MD-2069, Chişinău
Moldova

E-mail: *pashcanu@mail.ru*

Received October 6, 2006

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(52), 2006, Pages 73–86
ISSN 1024–7696

A Linear Parametrical Programming Approach for Studying and Solving Bilinear Programming Problem *

Dmitrii Lozovanu, Maria Fonoberova

Abstract. An approach for studying and solving a bilinear programming problem, based on linear parametrical programming, is proposed. Using duality principle for the considered problem we show that it can be transformed into a problem of determining the compatibility of a system of linear inequalities with a right-hand member that depends on parameters, admissible values of which are defined by another system of linear inequalities. Some properties of this auxiliary problem are obtained and a conical algorithm for its solving is proposed. We show that this algorithm can be used for finding the exact solution of bilinear programming problem as well as its approximate solution.

Mathematics subject classification: 65K05,68W25.
Keywords and phrases: Bilinear Programming, Linear Parametrical Programming, Duality Principle for Parametrical Systems, Conical Algorithms.

1 Introduction and Problem Formulation

We consider the following bilinear programming problem (BPP) [1,9]:
to minimize the object function

$$z = xCy + c'x + c''y \tag{1}$$

on subject

$$Ax \leq a, \quad x \geq 0; \tag{2}$$

$$By \leq b, \quad y \geq 0, \tag{3}$$

where C, A, B are matrices of size $n \times m_1,\ m_2 \times n,\ k \times m_1$, respectively, and c', $x \in R^n;\ c'', y \in R^{m_1};\ a \in R^{m_2},\ b \in R^k$. In order to simplify the notations we will omit transposition symbol for vectors.

This problem generalizes a large class of practical and theoretical combinatorial optimization problems [6,9]. For example, a linear boolean programming problem, resource allocation problem, and determining Nash equilibria in bimatrix games, can be formulated as BPP (1)–(3).

It is easy to show that all local and global minima of the considered problem belong to basic solutions of systems (2), (3) and can be found in finite time. But it

*Supported by CRDF-MRDA 008 award MOM2-3049-CS-03

is well-known that BPP is NP-hard and therefore the elaboration of efficient polynomial-time algorithms for its solving looks to be unrealizable. Nevertheless in this paper we stress the attention to a general approach for studying and solving BPP, which is based on linear parametrical programming. The proposed approach allows us to elaborate such an exact algorithm that in the case of long time calculation it can be interrupted and an admissible solution, which is appropriate to an optimal one, can be obtained. Some classes of problems, for which the proposed approach can be used, are given.

2 Parametrical programming approach for studying and solving BPP

Let L be the size of BPP (1)-(3) with integer coefficients of the matrices C, A, B and vectors a, b, c', c''. So, $L = L_1 + L_2 + \log H + 2$, where

$$L_1 = \sum_{i=1}^{m_2} \sum_{j=1}^{n} \log(|a_{ij}| + 1) + \sum_{i=1}^{m_2} \log(|a_i| + 1) + \log m_2 n + 1;$$

$$L_2 = \sum_{i=1}^{k} \sum_{j=1}^{m_1} \log(|b_{ij}| + 1) + \sum_{i=1}^{k} \log(|b_i| + 1) + \log k m_1 + 1;$$

$$H = \max\{|c_{ij}|,\ |c'_i|,\ |c''_j|,\ i = \overline{1, n},\ j = \overline{1, m_1}\}.$$

In [7] the following lemma is proved.

Lemma 1. *If A, B, C and a, b, c', c'' are integer, then for nonempty and bounded solution sets of systems (2) and (3) the optimal value of the object function of BPP (1)–(3) is a quantity of the form t/r, where t and r are integer and $|t|, |r| \le 2^L$.*

On the basis of this lemma and results from [4, 5] we may conclude that if BPP (1)–(3) has solution then it can be solved by varying the parameter $h \in [-2^L, 2^L]$ in the problem of determining the compatibility of the system

$$\begin{cases} Ax \le a; \\ xCy + c'x + c''y \le h; \\ By \le b; \\ x \ge 0,\ y \ge 0. \end{cases} \tag{4}$$

If there exists an algorithm T for determining the compatibility of such a system then we can find the optimal value h^* of the object function and the solution of BPP (1)-(3) by using the dichotomy method on the segment $[-2^L, 2^L]$, checking at every step the compatibility of system (4) with $h = h_k$, where h_k is a current value of parameter h at the kth step of the method. On the basis of results from [4, 5, 7] we can conclude that using $3L + 2$ steps we obtain the optimal value h^* with the precision 2^{-2L-2}. As it is shown in [4, 5] if an approximate solution for h^* is known with the precision 2^{-2L-2} then an exact solution can be found in polynomial time by using a special approximate procedure.

In the following we will reduce the problem of the compatibility of system (4) to the problem of finding the compatibility of the system of linear inequalities with a right-hand member depending on parameters. So, we prove the following theorems.

Theorem 2. *Let solution sets X and Y of systems (2) and (3) be nonempty. Then system (4) has no solution if and only if the following system of linear inequalities*

$$\begin{cases} -A^T u \le Cy + c'; \\ au < c''y - h; \\ u \ge 0 \end{cases} \tag{5}$$

is compatible with respect to u for every y satisfying (3).

Proof. $\Rightarrow$ Let us assume that system (4) has no solution. This means that for every $y \in Y$ the following system of linear inequalities

$$\begin{cases} Ax \le a, \\ x(Cy + c') \le h - c''y, \\ x \ge 0 \end{cases} \tag{6}$$

has no solution with respect to x. Then according to theorem 2.14 from [2] the incompatibility of system (6) involves the solvability with respect to u and t of the following system of linear inequalities

$$\begin{cases} A^T u + (Cy + c')t \ge 0; \\ au + (h - c''y)t < 0; \\ u \ge 0, \ t \ge 0, \end{cases} \tag{7}$$

for every $y \in Y$.

Note that for every fixed $y \in Y$ in obtained system (7) for an arbitrary solution (u^*, t^*) the condition $t^* > 0$ holds. Indeed, if $t^* = 0$, then it means that the system

$$\begin{cases} A^T u \ge 0; \\ au < 0, \ u \ge 0, \end{cases}$$

has solution, what, according to theorem 2.14 from [2], involves the incompatibility of initial system (2) that is contrary to the initial assumption. Consequently, $t^* > 0$.

Since $t > 0$ in (7) for every $y \in Y$, then, dividing every of inequalities of this system by t and denoting $z = (1/t)u$, we obtain the following system of linear inequalities

$$\begin{cases} -A^T z \le Cy + c'; \\ az < c''y - h; \\ z \ge 0, \end{cases}$$

which has solution with respect to z for every $y \in Y$.

$\Leftarrow$ Let system (5) have solution with respect to u for every $y \in Y$. Then the following system of linear inequalities

$$\begin{cases} A^T u + (Cy + c')t \geq 0; \\ au + (h - c''y)t < 0; \\ u \geq 0, \ t > 0, \end{cases}$$

is compatible with respect to u and t for every $y \in Y$. However this system is equivalent to system (7), as it was shown that for every solution (u, t) of system (7) the condition $t > 0$ holds. Again using theorem 2.14 from [2], we obtain from the solvability of system (7) with respect to u and t for every $y \in Y$ that system (6) is incompatible with respect to x for every $y \in Y$. This means that system (4) has no solution. □

Theorem 3. *The minimal value of the object function in BPP (1)-(3) is equal to the maximal value h^* of the parameter h in the system*

$$\begin{cases} -A^T u \leq Cy + c'; \\ au \leq c''y - h; \\ u \geq 0 \end{cases} \tag{8}$$

for which it is compatible with respect to u for every $y \in Y$. An arbitrary point $y^* \in Y$, for which system (5) with $h = h^*$ and $y = y^*$ has no solution with respect to u, corresponds to one of optimal points for BPP (1)-(3).

Proof. Let h^* be a maximal value of parameter h, for which system (8) with $h = h^*$ has solution with respect to u for every $y \in Y$. Then system (5) with $h = h^*$ has solution with respect to u not for every $y \in Y$. From this on the basis of the previous theorem it results that system (4) with $h = h^*$ is compatible. Using the previous theorem we can see that if for every fixed $h < h^*$ system (5) has solution with respect to u for every $y \in Y$, then system (4) with $h < h^*$ has no solution. Consequently, the maximal value h^* of parameter h, for which system (8) has solution with respect to u for every $y \in Y$, is equal to the minimum value of the object function of BPP (1)-(3).

Now let us prove the second part of the theorem. Let $y^* \in E^{m_1}$ be an arbitrary point for which system (5) with $h = h^*$ and $y = y^*$ has no solution with respect to u. Then on the basis on the duality principle the following system

$$\begin{cases} Ax \leq a; \\ x(Cy^* + c') \leq h^* - c''y; \\ x \geq 0 \end{cases}$$

has solution with respect to x. Consequently, system (4) with $h = h^*$ is compatible and the point $y^* \in Y$ together with the certain $x^* \in X$ represents its solution, i.e. y^* is one of sought optimal points for BPP (1)-(3). □

So, the problem of determining the compatibility of system (4) is equivalent to the problem of determining the compatibility of system (8) for every y satisfying (3). If an algorithm for solving this problem is elaborated, then we will obtain an algorithm for solving BPP (1)-(3).

3 Main properties of systems of linear inequalities with a right-hand member depending on parameters

The systems of linear inequalities with a right-hand member depending on parameters have been studied in [6–8].

3.1 Duality principle for parametrical systems of linear inequalities

Let the following system of linear inequalities be given

$$\begin{cases} \sum_{j=1}^{n} a_{ij}u_j \leq \sum_{s=1}^{k} c_{is}y_s + c_{i0}, \ i = \overline{1,m}; \\ u_j \geq 0, \ j = \overline{1,p} \ (p \leq n) \end{cases} \tag{9}$$

with the right-hand member depending on parameters $y_1, y_2, \dots, y_k$. We consider the problem of determining the compatibility of system (9) for every $y_1, y_2, \dots, y_k$ satisfying the following system

$$\begin{cases} \sum_{s=1}^{k} b_{is}y_s + b_{i0} \leq 0, \ i = \overline{1,r}; \\ y_s \geq 0, \ s = \overline{1,q} \ (q \leq k). \end{cases} \tag{10}$$

In [6,7] the following theorem has been proved.

Theorem 4. *System (9) is compatible with respect to $u_1, u_2, \dots, u_n$ for every $y_1, y_2, \dots, y_k$ satisfying (10) if and only if the following system*

$$\begin{cases} -\sum_{i=1}^{r} b_{is}v_i \leq \sum_{i=1}^{m} c_{is}z_i, \ s = \overline{0,q}; \\ -\sum_{i=1}^{r} b_{is}v_i = \sum_{i=1}^{m} c_{is}z_i, \ s = \overline{q+1,k}; \\ v_i \geq 0, \ i = \overline{1,r} \end{cases}$$

is compatible with respect to $v_1, v_2, \dots, v_r$ for every $z_1, z_2, \dots, z_m$ satisfying the following system

$$\begin{cases} -\sum_{i=1}^{m} a_{ij}z_i \leq 0, \ j = \overline{1,p}; \\ -\sum_{i=1}^{m} a_{ij}z_i = 0, \ j = \overline{p+1,n}; \\ z_i \geq 0, \ i = \overline{1,m}. \end{cases}$$

3.2 Two special cases of the parametrical problem

Note that if $r = 0$ and $q = k$ in system (10) then we obtain the problem of determining the compatibility of system (9) for every nonnegative values of parameters $y_1, y_2, \ldots, y_k$. It is easy to observe that in this case system (9) is compatible for every nonnegative values of parameters $y_1, y_2, \ldots, y_k$ if and only if each of the following $k+1$ systems ($s = \overline{0,k}$)

$$\begin{cases} \sum_{j=1}^{n} a_{ij} u_j \leq c_{is}, \ i = \overline{1,m}; \\ u_j \geq 0, \ j = \overline{1,p} \end{cases}$$

is compatible.

Another special case of the problem is the one when $n = 0$. This case can be reduced to the previous one using the duality problem for it.

In such a way, our problem can be solved in polynomial time for the mentioned above cases.

3.3 General approach for determining the compatibility property for parametrical systems

Let us assume that the solution sets UY and Y of systems (9) and (10) are bounded. Then it is easy to observe that the compatibility property of system (9) for all admissible values of parameters $y_1, y_2, \ldots, y_k$ satisfying (10) can be verified by checking the compatibility of system (9) for every basic solution of system (10). This fact follows from the geometrical interpretation of the problem. The set $\overline{Y} \subseteq R^k$ of vectors $y = (y_1, y_2, \ldots, y_k)$, for which system (9) is compatible, corresponds to the orthogonal projection on R^k of the set $UY \subseteq R^{n+k}$ of solutions of system (9) with respect to variables $u_1, u_2, \ldots, u_n$, $y_1, y_2, \ldots, y_k$. Therefore $Y \subseteq \overline{Y}$ if and only if system (9) is compatible with respect to $u_1, u_2, \ldots, u_n$ for every basic solution of system (10) (see Fig.1).

Another general approach which can be argued on the basis of the mentioned above geometrical interpretation is the following.

We find the system of linear inequalities

$$\sum_{j=1}^{r} c'_{ij}\ y_j + c'_{i0} \leq 0, \ i = \overline{1,m'}, \tag{11}$$

which determines the orthogonal projection $\overline{Y}$ of the set $UY \subseteq R^{n+k}$ on R^k; then we solve the problem from Section 3.2. System (11) can be found by using method of elimination of variables $u_1, u_2, \ldots, u_n$ from system (9). Such a method of elimination of variables can be found in [2]. Note that in final system (11) the number of inequalities m' can be too big. Therefore such an approach for solving our problem can be used only for a small class of problems.

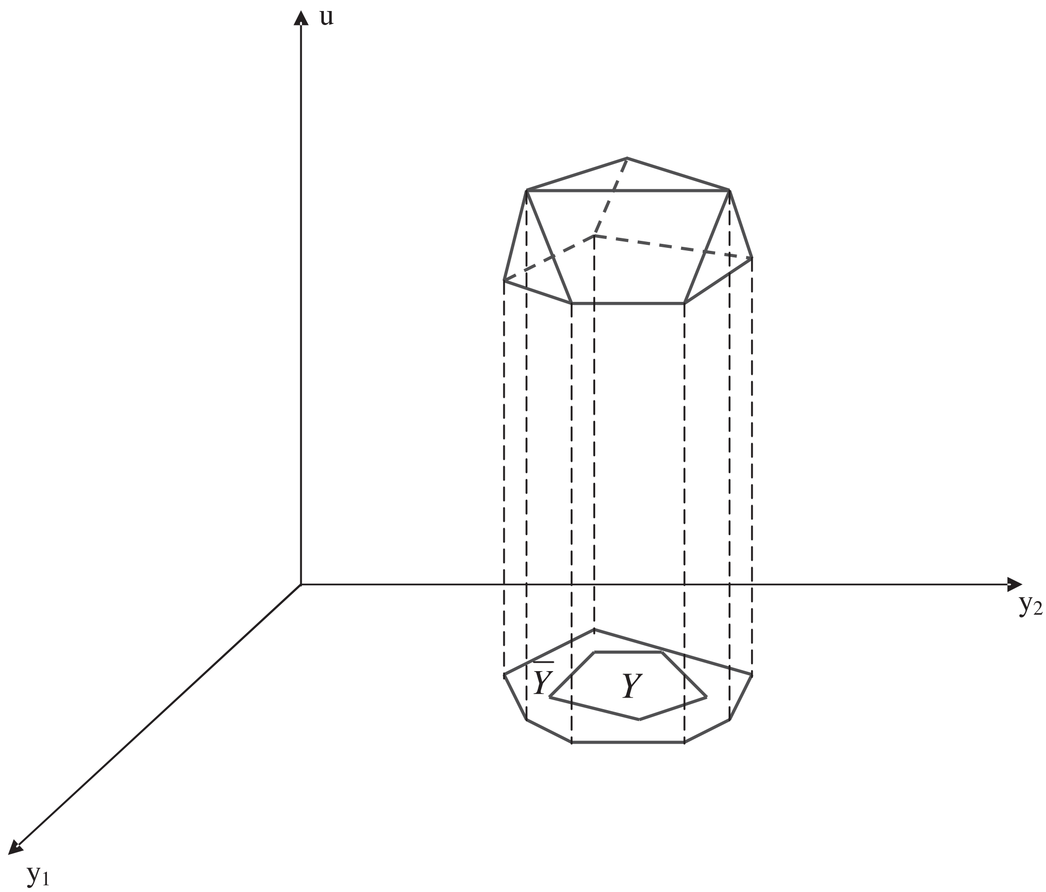


Fig.1

4 An algorithm for determining the compatibility of a parametrical problem

Let us assume that $h = h_k \in [M^1, M^2]$, where $M^1 \leq -2^L$, $M^2 \geq 2^L$. We propose an algorithm for determining the compatibility of system (8) with $h = h_k$ for every y satisfying (3). This algorithm works in the case when the solution sets of the considered systems are bounded. The case of the problem with unbounded solution sets can be easily reduced to the bounded one.

The proposed approach is based on the idea of conical algorithms from [3,9,10].

Algorithm 1.

Step 1. Choose an arbitrary basic solution y^0 of system (3). This solution corresponds to a solution of the system of linear equations

$$\sum_{j=1}^{m_1} b_{i_s j} y_j + b_{i_s 0} = 0, \ s = \overline{1, m_1}. \tag{12}$$

The matrix $\overline{B} = (b_{i_s j})_{m_1 \times m_1}$ of this system represents a submatrix of the matrix

$$B' = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m_1} \\ b_{21} & b_{22} & \dots & b_{2m_1} \\ \dots & \dots & \dots & \dots \\ b_{k1} & b_{k2} & \dots & b_{km} \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

and the vector $(b_{i_1 0}, b_{i_2 0}, \dots, b_{i_{m_1} 0})^T$ is a "subvector" of $(b_1, b_2, \dots, b_k, 0, 0, \dots, 0)^T$.

If for $y = y^0$ system (8) is not compatible with respect to u then we fix $z = y^0$ and STOP.

If for $y = y^0$ system (8) is compatible with respect to u then we find the minimal cone Y^0, originating in y^0, which contains the polyhedral solution set Y of system (3) (see Fig.2).

It is easy to observe that the system of inequalities

$$\sum_{j=1}^{m_1} b_{i_s j} y_j + b_{i_s 0} \leq 0, \;\; s = \overline{1, m_1},$$

which corresponds to system (12), determines in R^{m_1} the cone Y^0 with the following generating rays $y^s = y^0 + \overline{b}^s t, \;\; s = \overline{1, m_1}, \;\; t \geq 0$.

Here $\overline{b}^1, \overline{b}^2, \dots, \overline{b}^{m_1}$ represent directing vectors of respective rays originating in y^0. These directing vectors correspond to columns of the matrix $\overline{B}^{-1}$.

Step 2. For each $s = \overline{1, m_1}$, solve the following problem:

$$\textit{maximize } t$$

on subject

$$\begin{cases} By \leq b; \\ y \geq 0; \\ y = y^0 + \overline{b}^s t, \;\; t \geq 0 \end{cases}$$

and find m_1 points $\overline{y}^1, \overline{y}^2, \dots, \overline{y}^{m_1}$, which correspond to m_1 basic solutions of system (3), i.e. $\overline{y}^1, \overline{y}^2, \dots, \overline{y}^{m_1}$ represent neighboring basic solutions for y^0. If system (8) is compatible with respect to u for each $y = \overline{y}^1, y = \overline{y}^2, \dots, y = \overline{y}^{m_1}$, then go to step 3; otherwise system (8) is not compatible for every y satisfying (3) and STOP.

Step 3. For each $s = \overline{1, m_1}$, solve the following problem:

$$\textit{maximize } t$$

on subject

$$\begin{cases} -A^T u \leq Cy + c'; \\ au \leq c'' y - h; \\ u \geq 0 \\ y = y^0 + \overline{b}^s t, \;\; t \geq 0 \end{cases}$$

and find m_1 solutions $t'_1, t'_2, \ldots, t'_{m_1}$. Then fix m_1 points $\widehat{y}^s = y^0 + \overline{b}^s t'_s,\ s = \overline{1, m_1}$. (On Fig.2 we can see $\widehat{y}^1$ and $\widehat{y}^2$.)

Step 4. Find the hyperplane Γ (see Fig.2), determined by the following equation $\sum_{j=1}^{m_1} a'_j y_j + a'_0 = 0$, which passes through the points $\widehat{y}^1, \widehat{y}^2, \ldots, \widehat{y}^{m_1}$.

Consider that the basic solution $y^0 = (y_1^0, y_2^0, \ldots, y_{m_1}^0)$ satisfies the following condition $\sum_{j=1}^{m_1} a'_j y_j^0 + a'_0 \leq 0$. Then add to system (3) the inequality $-\sum_{j=1}^{m_1} a'_j y_j - a'_0 \leq 0$. If after that the obtained system is not compatible, then conclude that system (8) is compatible for every y satisfying initial system (3) and STOP; otherwise change the initial system with the obtained one and go to step 1.

Note that this algorithm works well when the polyhedral set Y is a small part of the orthogonal projection $\overline{Y}$ (see Fig.3) or when the intersection of Y and $\overline{Y}$ is a small part of $\overline{Y}$ (see Fig.4). In the case when the polyhedral set Y is a big part of the orthogonal projection $\overline{Y}$ the algorithm may work too long (see Fig.5).

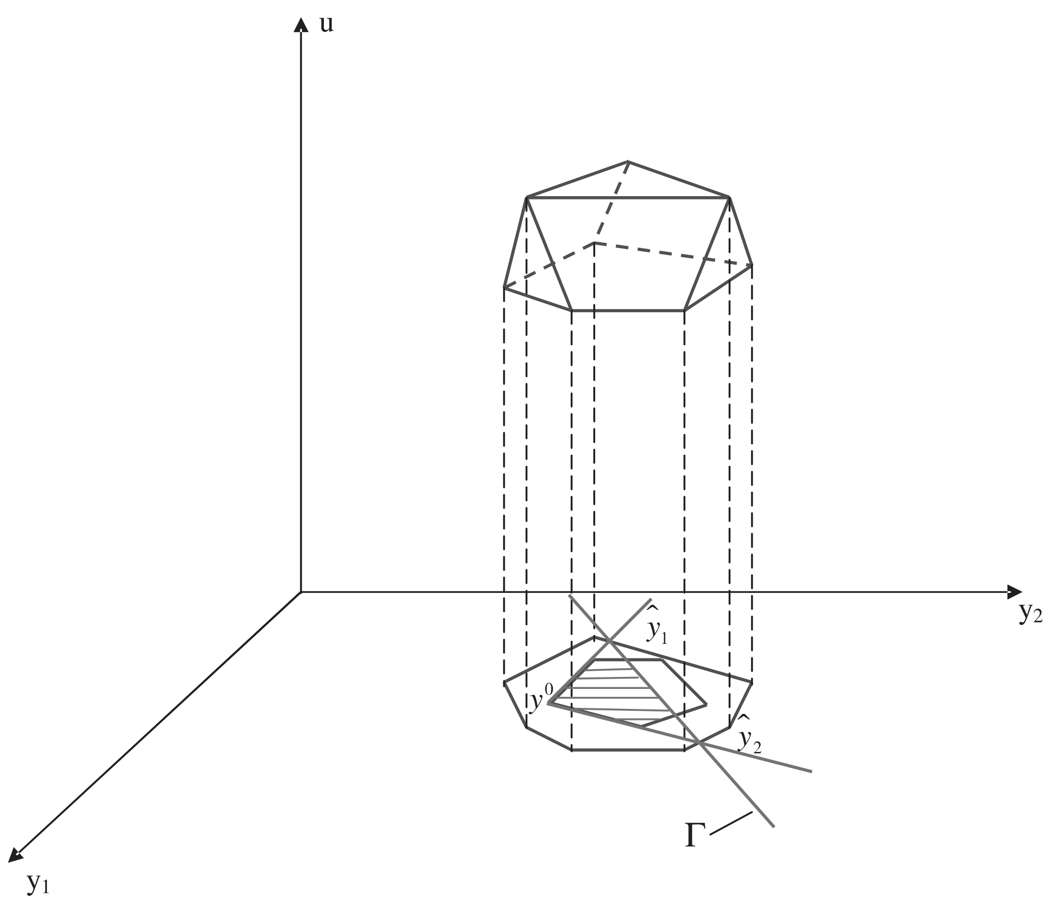


Fig.2

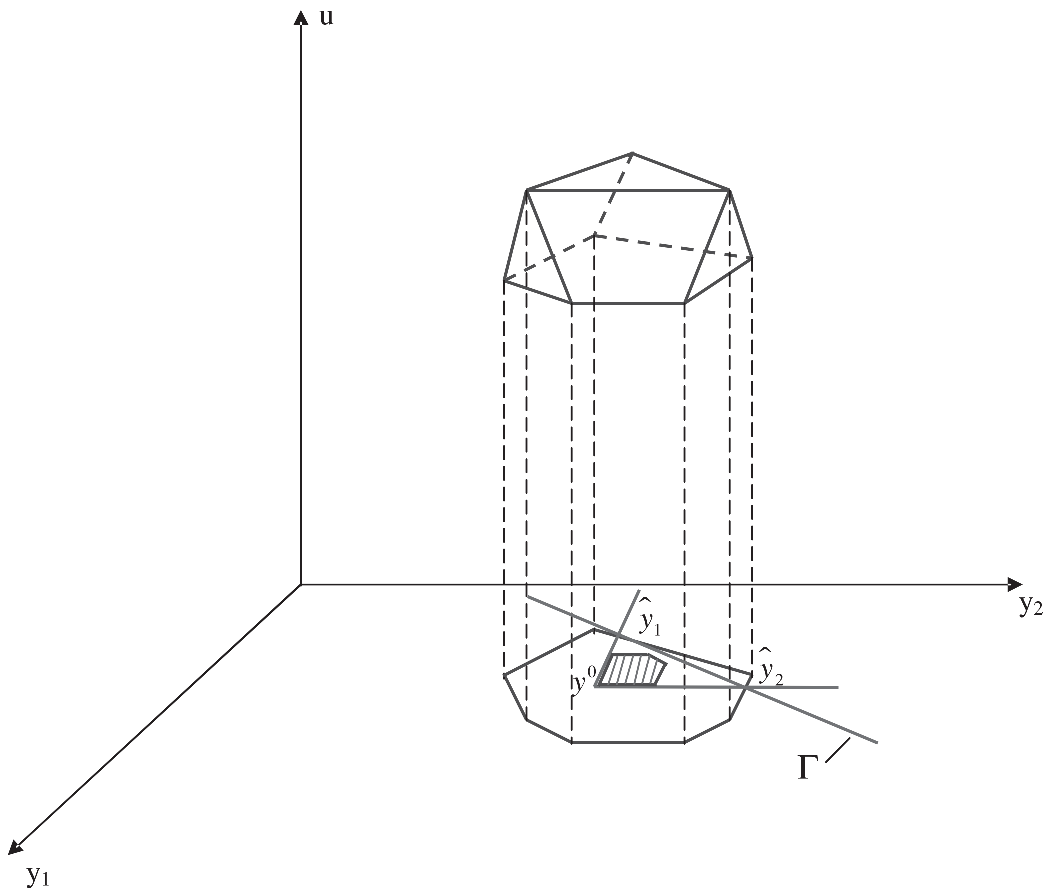


Fig.3

5 An algorithm for solving BPP

Using algorithm 1 for determining the compatibility of system (8) for every y satisfying (3) when $h = h_k$ is fixed, we can now propose an algorithm for solving BPP (1)-(3).

Algorithm 2.

Preliminary step (step 0). Fix an arbitrary basic solution $z = y^0$ of system (3) and put $M^1 = -2^L$, $M^2 = 2^L$, $h_0 = M^1$.

General step (step k). Find $\varepsilon = M^2 - M^1$. If $\varepsilon < \frac{1}{2^{2L+2}}$ then fix $y^k = z$ and STOP, otherwise find $h_k = \frac{M^1+M^2}{2}$. Then apply algorithm 1 with $h = h_k$ and determine if system (8) is compatible with respect to u for every y satisfying (3). If this condition is satisfied then change M^2 by $\frac{M^1+M^2}{2}$ and go to the next step; otherwise fix the basic solution $y^0 = z$ for which system (8) has no solution with respect to u, change M^1 by $\frac{M^1+M^2}{2}$ and go to the next step.

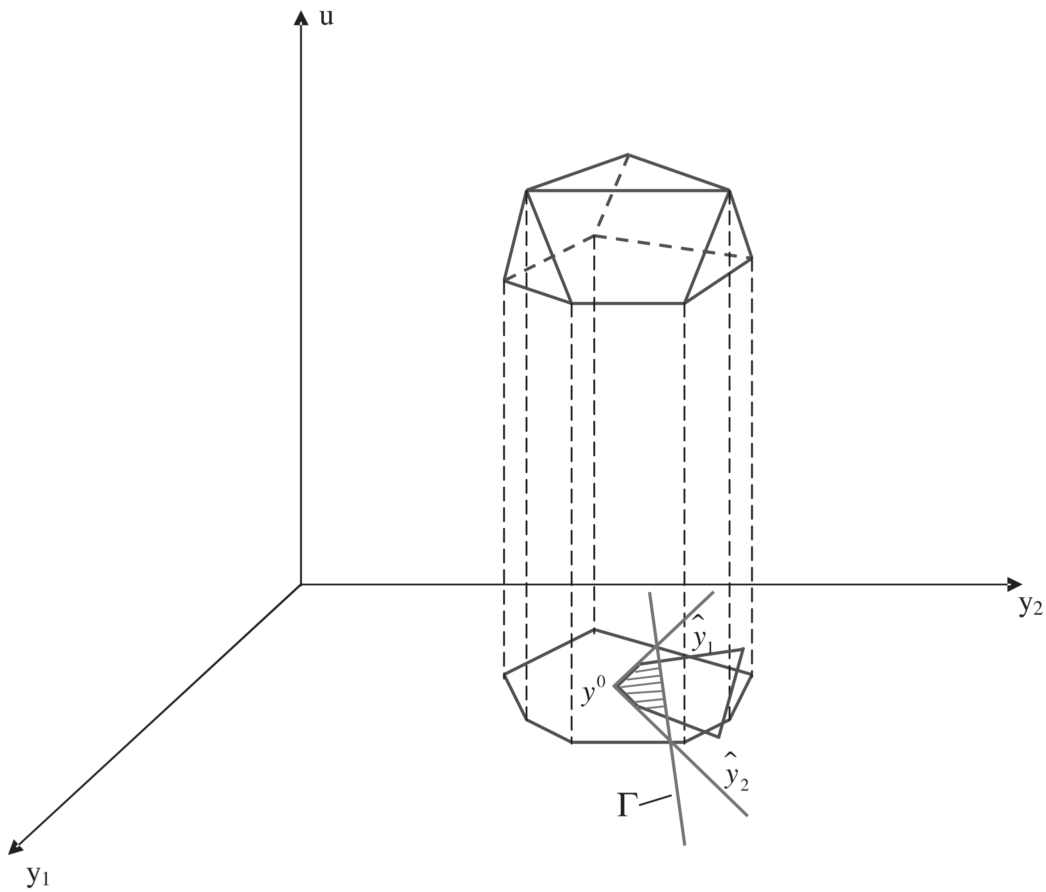


Fig.4

In general, this algorithm finds the exact solution of BPP (1)–(3). But if the process of calculation takes too much time then we can stop it and we obtain an admissible solution of BPP (1)-(3), which is appropriate to an optimal one.

Taking into account the geometrical interpretation of the auxiliary parametrical programming problem we may conclude that the proposed algorithm will work efficiently if BPP (1)-(3) has a global minimum with the corresponding value of the object function, which differs essentially from the values of local minima. Namely in this case for the auxiliary problem the set Y of the parametrical problem is a small part of the orthogonal projection $\overline{Y}$. In the case when BPP has many local minima with not essential deviations of the corresponding values of the object function the algorithm may work too much time.

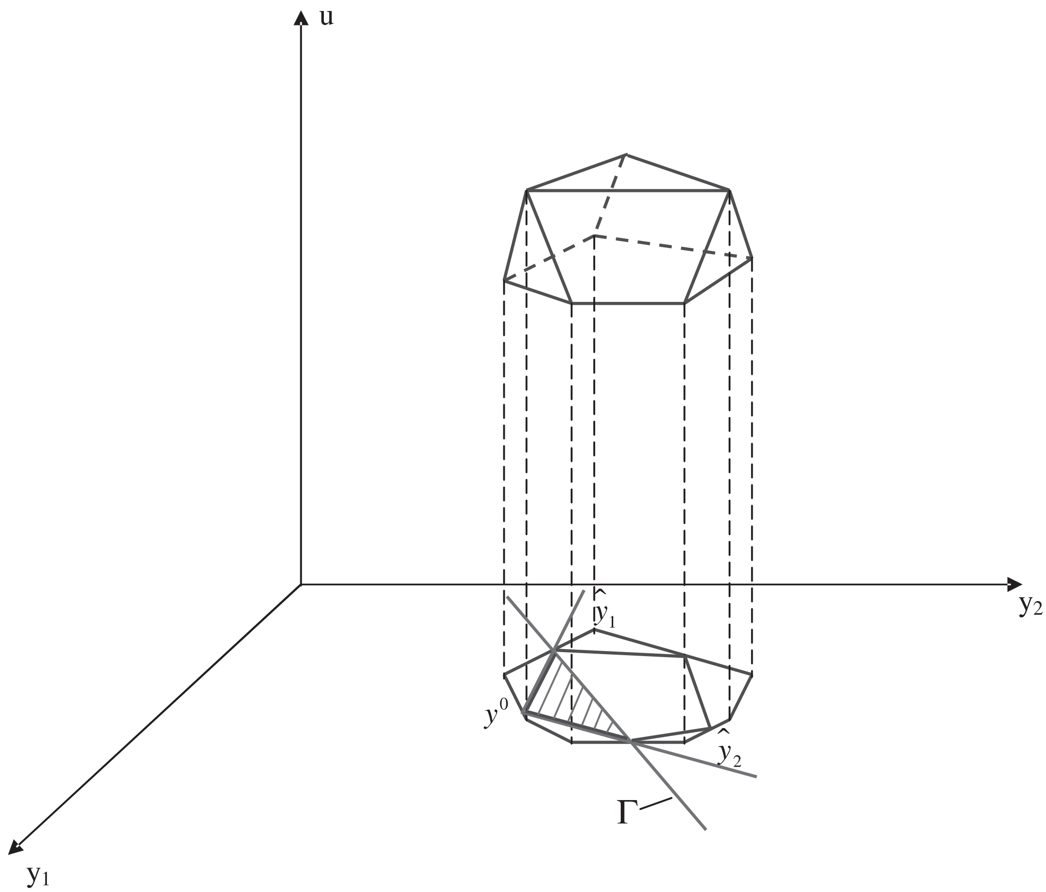


Fig.5

6 Applications

In this section we show that the proposed approach can be used for studying and solving the linear boolean programming problem and the resource allocation problem.

Let us consider the following linear boolean programming problem:
to minimize

$$z = \sum_{j=1}^{n} c_j x_j$$

on subject

$$\begin{cases} \sum_{j=1}^{n} a_{ij} x_j \le a_i, \ i = \overline{1, m_2}; \\ x_j \in \{0, 1\}, \ j = \overline{1, n}. \end{cases}$$

It is easy to observe that if this problem has solution then it is equivalent to the following concave programming problem *to minimize*

$$z = \sum_{j=1}^{n} c_j x_j + M \sum_{j=1}^{n} \min\{x_j, 1 - x_j\}$$

on subject

$$\begin{cases} \sum_{j=1}^{n} a_{ij} x_j \le a_i, \ i = \overline{1, m_2}; \\ 0 \le x_j \le 1, \ j = \overline{1, n}, \end{cases}$$

where $M > \sum_{j=1}^{n} |c_j|$.

In the following we represent this problem as BPP:
to minimize

$$z = \sum_{j=1}^{n} c_j x_j + M \sum_{j=1}^{n} (x_j y_j + (1 - x_j)(1 - y_j))$$

on subject

$$\begin{cases} \sum_{j=1}^{n} a_{ij} x_j \le a_i, \ i = \overline{1, m_2}; \\ 0 \le x_j \le 1, \ j = \overline{1, n}; \\ 0 \le y_j \le 1, \ j = \overline{1, n}. \end{cases}$$

So, we obtain BPP (1)-(3), where

$$B = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

This means that BPP (1)-(3) is NP-hard even in such a case.

In [7] it is shown that the proposed approach can be used for studying and solving the following concave programming problem:
to minimize

$$z = \sum_{i=1}^{q} \min\{c^{il} x + c_0^{il}, \ l = \overline{1, r_i}\} \tag{13}$$

on subject

$$\begin{cases} Ax \le a; \\ x \ge 0, \end{cases} \tag{14}$$

where $x \in R^n$, $c^{il} \in R^n$, $c_0^{il} \in R^1$, A is an $m_2 \times n$-matrix, $a \in R^{m_2}$. This problem arises as an auxiliary one when solving a large class of resource allocation problems [7, 9].

Problem (13)-(14) can be transformed into BPP:
to minimize

$$z = \sum_{i=1}^{q} \sum_{l=1}^{r_i} (c^{il} x + c_0^{il}) y_{il}$$

on subject

$$\begin{cases} Ax \le a, \quad x \ge 0; \\ \sum\limits_{l=1}^{r_i} y_{il} = 1, \ i = \overline{1,q}; \\ y_{il} \ge 0, \ l = \overline{1,r_i}, \ i = \overline{1,q}. \end{cases}$$

In a more detailed form the algorithm for solving this problem is described in [7].

References

[1] Altman M. *Bilinear Programming*. Bul. Acad. Polan. Sci., Ser. Sci. Math. Astron. et Phis, 1968, N 16(9), p. 741–746.

[2] Chernicov S.N. *Linear Inequalities*. Moscow, Nauka, 1968.

[3] Horst R., Tuy H. *Global Optimization. Deterministic Approaches* Springer Verlag, Berlin, 1993.

[4] Khachian L.G. *Polynomial time algorithm in linear programming*. USSR, Computational Mathematics and Mathematical Physics, 1980, **20**, p. 51–58.

[5] Khachian L.G. *On exact solution of the system of linear inequalities and linear programming problem*. USSR, Computational Mathematics and Mathematical Physics, 1982, **22**, p. 999–1002.

[6] Lozovanu D. *Duality Principle for Systems of Linear Inequalities with a Right-Hand Member that Depends on Parameters*. Izv. AN SSSR, Ser. Techn. Cyb., 1987, **6**, p. 3–13 (in Russian).

[7] Lozovanu D. *Extremal-Combinatorial Problems and Algorithms for their Solving*. Chisinau, Stiintsa, 1991 (in Russian).

[8] Lozovanu D. *Parametrical approach for studying and solving bilinear programming problem*. Proceedings of the International Workshop on Global Optimization, San Jose, Almeria, Spain, September 18-22th, 2005, p. 165–170.

[9] Pardalos P., Rosen J. *Methods for Global Concave Minimization. A bibliographical Survey*. SIAM Review, 1986, N 28(3), p. 367–379.

[10] Tuy H. *Normal Conical Algorithm for Concave Minimization over Polytopes*. Math. Progr., 1991, **51**, p. 229–245.

Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
Academiei str. 5, MD-2028 Chişinău
Moldova
E-mail:*lozovanu@math.md, fonoberov@math.md*

Received October 9, 2006

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(52), 2006, Pages 87–100
ISSN 1024–7696

On topological torsion LCA groups with commutative ring of continuous endomorphisms

Valeriu Popa

Abstract. In this paper, we determine for some classes $\mathcal{S}$ of topological torsion LCA (locally compact abelian) groups the structure of those groups in $\mathcal{S}$ which have a commutative ring of continuous endomorphisms.

Mathematics subject classification: Primary: 22B05; Secondary: 16W80.
Keywords and phrases: LCA groups, ring of continuous endomorphisms, commutativity.

1 Introduction

Given an LCA group X, let $E(X)$ denote the ring of all continuous endomorphisms of X. The very pleasant facts that, with respect to the compact-open topology, $E(X)$ is a complete Hausdorff topological ring and the evaluation map $(u, x) \to u(x)$ from $E(X) \times X$ to X is continuous, where $E(X) \times X$ is taken with the product topology, provide a felicitous setting for the study of interconnections between the algebraic-topological properties of X and those of $E(X)$. Similar problems for discrete X constituted the subject of an enormous number of investigations.

The present paper is concerned with the following question:

For which LCA groups X is the ring $E(X)$ commutative?

The prototype of this problem, corresponding to the case when X is discrete, is listed in Fuchs' book [6] as problem 46(a), and was studied for the first time by T. Szele and J. Szendrei. In [14], they have completely solved the case of torsion groups and have obtained some partial results for the case of mixed groups. In the case of torsionfree groups a solution, due to L. C. A. van Leeuwen [9], has been obtained only for very special groups.

This paper is intended to be the first of several investigating the structure of LCA groups X with a commutative ring $E(X)$. We begin our study by examining the case of topological torsion LCA groups, which represent a natural generalization of discrete torsion abelian groups within the class of all LCA groups. In contrast with the case of discrete torsion groups, this new situation is much more complicated and we do not settle it completely. Though we are unable to give a full description of topological torsion LCA groups X having a commutative ring $E(X)$, we give such a description for certain important special cases of this kind of groups.

2 Notation

In the following, $\mathbb{P}$ is the set of prime numbers, $\mathbb{N}$ is the set of natural numbers (including zero), and $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$.

For $p \in \mathbb{P}$, we denote by $\mathbb{Q}_p$ the group of p-adic numbers, by $\mathbb{Z}_p$ the group of p-adic integers (both with their usual topologies), by $\mathbb{Z}(p^\infty)$ the quasi-cyclic group corresponding to p and by $\mathbb{Z}(p^n)$, where $n \in \mathbb{N}$, the cyclic group of order p^n (all with the discrete topology).

We let $\mathcal{L}$ denote the class of locally compact abelian groups, and $\mathcal{L}_p$, where $p \in \mathbb{P}$, the subclass of $\mathcal{L}$ consisting of all topological p-primary groups.

Let X be a group in $\mathcal{L}$. For any closed subgroup C of X, X/C will indicate the quotient group of X by C, equipped with the quotient topology.

We let 1_X, $c(X)$, $d(X)$, $k(X)$, $m(X)$, $t(X)$, and X^* denote, respectively, the identity map on X, the connected component of X, the maximal divisible subgroup of X, the subgroup of compact elements of X, the smallest closed subgroup K of X such that the quotient group X/K is torsionfree, the torsion subgroup of X, and the character group of X.

For $n \in \mathbb{N}$, we let

$$X[n] = \{x \in X \mid n \cdot x = 0\} \qquad \text{and} \qquad n \cdot X = \{n \cdot x \mid x \in X\}.$$

If $p \in \mathbb{P}$, X_p stands the topological p-primary component of X, i. e.

$$X_p = \{x \in X \mid \lim_{n \to \infty} p^n x = 0\}.$$

If X is a topological torsion group, we let

$$S(X) = \{p \in \mathbb{P} \mid X_p \neq 0\}.$$

For $a \in X$ and $S \subset X$, $o(a)$ is the order of a, $\langle a \rangle$ is the subgroup of X generated by a, $\overline{S}$ is the closure of S in X, and

$$A(X^*, S) = \{\gamma \in X^* \mid \gamma(x) = 0 \quad \text{for all} \quad x \in S\}.$$

For $u \in E(X)$, we let u^* be the transpose of u, i.e. the endomorphism $u^* \in E(X^*)$ defined by the rule $u^*(\gamma) = \gamma \circ u$ for all $\gamma \in X^*$.

If Y is another group in $\mathcal{L}$, then $H(X, Y)$ stands for the group of all continuous homomorphisms from X into Y. For $h \in H(X, Y)$, we denote by $\mathrm{im}(h)$ the image of h and by $\ker(h)$ the kernel of h.

Also, we write $X = A \oplus B$ in case X is a topological direct sum of its subgroups A and B.

Let $(X_i)_{i \in I}$ be a collection of topological groups (rings) indexed by a set I. We write $\prod_{i \in I} X_i$ for the direct product of the family $(X_i)_{i \in I}$, taken with the product topology. In case each X_i is a discrete abelian group, $\bigoplus_{i\in I} X_i$ denotes the external direct sum of the family $(X_i)_{i\in I}$, taken with the discrete topology. If each $X_i = X$ for some fixed X, then $\prod_{i\in I} X_i$ is denoted by X^I and $\bigoplus_{i\in I} X_i$ by $X^{(I)}$.

Suppose, in addition, that for each $i \in I$ we are given an open subgroup (subring) U_i of X_i. The local direct product of the family $(X_i)_{i\in I}$ with respect to $(U_i)_{i\in I}$ will be indicated by $\prod_{i\in I}(X_i; U_i)$. Recall that the group (ring) $\prod_{i\in I}(X_i; U_i)$ consists of all $(x_i)_{i\in I} \in \prod_{i\in I} X_i$ such that $x_i \in U_i$ for all but finitely many i and is topologized by declaring all neighborhoods of zero in the topological group (ring) $\prod_{i\in I} U_i$ to be a fundamental system of neighborhoods of zero in $\prod_{i\in I}(X_i; U_i)$.

The symbol $\cong$ denotes topological group (ring) isomorphism.

3 Some technical lemmas

We collect here several facts which will be frequently used in the sequel.

Lemma 3.1. *For any $X \in \mathcal{L}$, the mapping $u \to u^*$ is a topological ring antiisomorphism from $E(X)$ onto $E(X^*)$.*

Lemma 3.2. *Let X be a group in $\mathcal{L}$ such that $X = A \oplus B$ for some subgroups A, B of X. If $\varepsilon_A \in E(X)$ is the canonical projection of X onto A, then $E(A) \cong \varepsilon_A E(X)\varepsilon_A$, where $\varepsilon_A E(X)\varepsilon_A$ carries the induced topology.*

Definition 3.3. *A closed subgroup C of a group $X \in \mathcal{L}$ is said to be topologically fully invariant in X if $u(C) \subset C$ for all $u \in E(X)$.*

Lemma 3.4. *Let $(X_i)_{i\in I}$ be an indexed collection of groups in $\mathcal{L}$, and, for each $i \in I$, let Y_i be a compact open subgroup of X_i. If $X = \prod_{i\in I}(X_i; Y_i)$ and if every subgroup*

$$X'_j = \{(x_i)_{i\in I} \in X \mid x_i = 0 \quad \text{for all} \quad i \neq j\}, j \in I,$$

is topologically fully invariant in X, then $E(X)$ is topologically isomorphic with $\prod_{i\in I}(E(X_i); \Omega_{X_i}(Y_i, Y_i))$.

Proof. See [11, (2.2)] □

The following lemma provides us with a tool of constructing noncommuting continuous endomorphisms.

Lemma 3.5. *Let X be a group in $\mathcal{L}$ admitting a continuous endomorphism w such that $\overline{\mathrm{im}(w)} = A \oplus B$ for some nonzero subgroups A, B of X with $w(A) \subset A$ and $w(B) \subset B$. If there exists $h \in H(A, B)$ satisfying $w(A) \not\subset \ker(h)$, then $E(X)$ fails to be commutative.*

Proof. Let $\pi_A : \overline{\mathrm{im}(w)} \to A$ and $\pi_B : \overline{\mathrm{im}(w)} \to B$ denote the canonical projections corresponding to the above decomposition of $\overline{\mathrm{im}(w)}$. If $\eta_A : A \to X$ and $\eta_B : B \to X$ are the canonical injections, define $u, v \in E(X)$ by setting $u = \eta_B \circ h \circ \pi_A \circ w$ and $v = \eta_A \circ \pi_A \circ w$. We cannot have

$$h \circ \pi_A \circ w \circ \eta_A \circ \pi_A \circ w = 0,$$

since otherwise it would follow that

$$(h \circ \pi_A \circ w \circ \eta_A \circ \pi_A)(\overline{\mathrm{im}(w)}) \subset \overline{(h \circ \pi_A \circ w \circ \eta_A \circ \pi_A \circ w)(X)} = \{0\}$$

[2, Ch. 1, §2, Theorem 1], which would imply

$$h(w(A)) = (h \circ \pi_A \circ w \circ \eta_A \circ \pi_A)(A) \subset (h \circ \pi_A \circ w \circ \eta_A \circ \pi_A)(\overline{\mathrm{im}(w)}) = \{0\},$$

a contradiction. Thus $h \circ \pi_A \circ w \circ \eta_A \circ \pi_A \circ w \neq 0$. It then follows that $uv \neq 0$, and since $vu = 0$, the proof is complete. □

4 Discrete and compact groups

As we have mentioned in Introduction, T. Szele and J. Szendrei characterized in [14] the major classes of discrete abelian groups with commutative endomorphism ring.

For torsion groups, the characterization of [14] may be paraphrased as follows:

Theorem 4.1. [14] *The endomorphism ring $E(X)$ of a discrete torsion group $X \in \mathcal{L}$ is commutative if and only if*

$$X \cong \bigoplus_{p \in S_1} \mathbb{Z}(p^\infty) \times \bigoplus_{p \in S_2} \mathbb{Z}(p^{n_p}),$$

where $S_1 \cup S_2 = S(X)$, $S_1 \cap S_2 = \varnothing$ and $n_p \in \mathbb{N}_0$ for all $p \in S_2$.

As a first application of this result, we obtain the description of compact totally disconnected groups in $\mathcal{L}$ with commutative ring of continuous endomorphisms.

Corollary 4.2. *The endomorphism ring $E(X)$ of a compact totally disconnected group $X \in \mathcal{L}$ is commutative if and only if*

$$X \cong \prod_{p \in S_1} \mathbb{Z}_p \times \prod_{p \in S_2} \mathbb{Z}(p^{n_p}),$$

where $S_1 \cup S_2 = S(X)$, $S_1 \cap S_2 = \varnothing$ and $n_p \in \mathbb{N}_0$ for all $p \in S_2$.

Proof. Since the rings $E(X)$ and $E(X^*)$ are topologically antiisomorphic, and since X is discrete and torsion if and only if X^* is compact and totally disconnected [8, (23.17) and (24.26)], the assertion follows from Theorem 4.1 by taking duals. □

5 Topological torsion groups

Theorem 4.1, due to T. Szele and J. Szendrei, gives a complete description of torsion discrete abelian groups X whose ring $E(X)$ is commutative. In the present section, which contains our main results, we will be concerned with a natural generalization within $\mathcal{L}$ of this class of groups, namely, with the class of topological torsion groups.

Definition 5.1. *A group $X \in \mathcal{L}$ is said to be a topological torsion group in case, for each $x \in X$, $\lim_{n\to\infty}(n!)x = 0$.*

Our first goal will be to describe the p-groups in $\mathcal{L}$ with commutative ring of continuous endomorphisms.

Theorem 5.2. *Let $p \in \mathbb{P}$, and let X be a p-group in $\mathcal{L}$. The ring $E(X)$ is commutative if and only if X is topologically isomorphic with either $\mathbb{Z}(p^\infty)$ or $\mathbb{Z}(p^n)$ for some $n \in \mathbb{N}$.*

Proof. Let $E(X)$ be commutative. We first consider the case when X is nonreduced. As is well known, X contains then a closed subgroup D topologically isomorphic with $\mathbb{Z}(p^\infty)$ [1, Proposition 4.22]. Let us fix an isomorphism $j : \mathbb{Z}(p^\infty) \to D$. Since the discrete divisible groups are splitting in the class of totally disconnected LCA groups [1, Proposition 6.21], we can write $X = D \oplus X_0$ for some closed subgroup X_0 of X. Assume by way of contradiction that $X_0 \neq \{0\}$, and let U be a nonzero compact open subgroup of X_0. By the structure theorem for torsion compact groups [8, (25.9)], there is a topological isomorphism φ from U onto a group of the form $\prod_{i\in I}\mathbb{Z}(p^{m_i})$, where I is a nonempty set and the m_i's are nonzero natural numbers not exceeding a fixed $N \in \mathbb{N}$. Picking any $i_0 \in I$, let π denote the canonical projection of $\prod_{i\in I}\mathbb{Z}(p^{m_i})$ onto $\mathbb{Z}(p^{m_{i_0}})$ and ρ the canonical injection of $\mathbb{Z}(p^{m_{i_0}})$ into $\mathbb{Z}(p^\infty)$. Since D is divisible and U is open, $j \circ \rho \circ \pi \circ \varphi \in H(U, D)$ extends [8, (A.7)] to a nonzero homomorphism $h \in H(X_0, D)$ [3, Ch. III, §2, Proposition 23]. Then applying Lemma 3.5 to $w = 1_X$ and our $h \in H(X_0, D)$ leads to a contradiction. Consequently, we must have $X_0 = \{0\}$, and hence $X \cong \mathbb{Z}(p^\infty)$.

Next we dispose of the case when X is reduced. Our first goal will be to prove that X is of bounded order. Pick an arbitrary compact open subgroup V of X. In view of the earlier mentioned structure theorem for torsion compact groups, we know that V is of bounded order. Therefore, the desired fact that X is of bounded order will follow if we show that X/V is of bounded order.

It is not difficult to see that X/V is reduced. Indeed, since V is open, X/V is a discrete p-group. If X/V were nonreduced, we could write $X/V = D_1 \oplus G$, where $D_1 \cong \mathbb{Z}(p^\infty)$ and G is a subgroup of X/V. Since $A(X^*; V) \cong (X/V)^*$ [8, (23.25)] it would then follow from [1, Corollary 6.10] and [8, (25.2)] that $A(X^*; V) = \Delta \oplus \Gamma$, where $\Delta \cong \mathbb{Z}_p$. Let $\psi \in H(A(X^*; V), \Delta)$ denote the canonical projection with kernel Γ and choose any nonzero $\eta \in H(\Delta, \mathbb{Q}_p)$. Since $A(X^*; V)$ is open in X^* (because V is compact) and $\mathbb{Q}_p$ is divisible, $\eta \circ \psi$ extends to a nonzero $\chi \in H(X^*, \mathbb{Q}_p)$, and so the transpose map χ^* would be a nonzero member of $H(\mathbb{Q}_p, X)$, which would imply that X is nonreduced, a contradiction. Consequently, X/V must be reduced.

Having established this, we are ready to prove that X/V is of bounded order. Let

$$n_V = \min\{n \in \mathbb{N} \mid p^n V = \{0\}\}.$$

It is easily seen that $p^{n_V} V^* = \{0\}$ as well. Since $V^* \cong X^*/A(X^*; V)$ [8, (24.5)], it then follows that

$$p^{n_V} X^* \subset A(X^*; V). \tag{5.1}$$

If X/V were not of bounded order, it would follow that X/V has cyclic direct summands of arbitrarily high orders [7, Chapter V, §27, Exercise 1]. Hence we could write $X/V = A \oplus B \oplus C \oplus F$, where $A \cong \mathbb{Z}(p^{n_A})$, $B \cong \mathbb{Z}(p^{n_B})$, $C \cong \mathbb{Z}(p^{n_C})$ and $n_C \geq n_B \geq n_A \geq 2n_V + 1$. By [1, Corollary 6.10] and [8, (23.25)], we then would obtain $A(X^*; V) = A_1 \oplus B_1 \oplus C_1 \oplus F_1$, where $A_1 \cong A$, $B_1 \cong B$ and $C_1 \cong C$. Letting $\alpha \in A_1$, $\beta \in B_1$ and $\gamma \in C_1$ be generators, define $f \in H(C_1, B_1)$ and $g \in H(B_1, A_1)$ by the rule $f(\gamma) = \beta$ and $g(\beta) = \alpha$. Further, letting $\xi \in H(A(X^*; V), B_1)$ and $\zeta \in H(A(X^*; V), C_1)$ be the canonical projections, $\sigma \in H(B_1, X^*)$ and $\tau \in H(A_1, X^*)$ the canonical injections, and taking account of (5.1), define $u, v \in E(X^*)$ by setting

$$u = \tau \circ g \circ \xi \circ p^{n_V} 1_{X^*} \qquad \text{and} \qquad v = \sigma \circ f \circ \zeta \circ p^{n_V} 1_{X^*}.$$

Then $(u \circ v)(\gamma) = u(p^{n_V} \beta) = p^{2n_V} \alpha \neq 0$ and $(v \circ u)(\gamma) = v(0) = 0$, so that $uv \neq vu$. This is a contradiction because, in view of Lemma 3.1, $E(X^*)$ must be commutative. In summary, X/U is a group of bounded order, and hence so is X.

Finally, since in a group of bounded order every cyclic subgroup generated by an element of maximal order splits topologically [10, (3.8)], we can write $X = L \oplus M$, where $L \cong \mathbb{Z}(p^n)$ for some $n \in \mathbb{N}$ and M is a subgroup of X. Again we must have $M = \{0\}$ since otherwise it would follow that $H(L, M) \neq \{0\}$, contradicting by Lemma 3.5 the commutativity of $E(X)$. Hence $X \cong \mathbb{Z}(p^n)$.

Since the converse is clear, the proof is complete. □

As a direct consequence of Theorem 5.2, we obtain the following result.

Corollary 5.3. *Let X be a topological torsion group in $\mathcal{L}$ with torsion primary components. Then $E(X)$ is commutative if and only if*

$$X \cong \prod_{p \in S_1} (\mathbb{Z}(p^\infty); \mathbb{Z}(p^\infty)[p^{k_p}]) \times \prod_{p \in S_2} (\mathbb{Z}(p^{n_p}); \mathbb{Z}(p^{n_p})[p^{k_p}]),$$

where $S_1 \cup S_2 = S(X)$, $S_1 \cap S_2 = \varnothing$, and $n_p, k_p \in \mathbb{N}$ for all $p \in S(X)$.

Proof. By [4, Ch. III, §1, Th*éorème* 1] we have

$$X \cong \prod_{p \in S(X)} (X_p; U_p),$$

where, for each $p \in S(X)$, U_p is a compact open subgroup of X_p. Since the X_p's are topologically fully invariant in X, it follows from Lemma 3.4 that

$$E(X) \cong \prod_{p \in S(X)} (E(X_p); \Omega(U_p, U_p)).$$

Consequently, the commutativity of $E(X)$ is equivalent to the commutativity of all the $E(X_p)$'s.

Now, since X has torsion topological primary components, Theorem 5.2 shows that this last condition is equivalent to saying that, for each $p \in S(X)$, X_p is topologically isomorphic with either $\mathbb{Z}(p^\infty)$ or $\mathbb{Z}(p^{n_p})$ for some $n_p \in \mathbb{N}$. It remains to put

$$S_1 = \{p \in S(X) \mid X_p \cong \mathbb{Z}(p^\infty)\} \qquad \text{and} \qquad S_2 = S(X) \setminus S_1. \qquad \square$$

To dualize the preceding corollary, a few definitions are in order.

Definition 5.4. *A group $X \in \mathcal{L}$ is said to be compact-by-bounded order in case X admits a compact subgroup K such that X/K is of bounded order.*

Definition 5.5. *Let $X \in \mathcal{L}$. The subgroup $\bigcap_{n\in\mathbb{N}_0}\overline{nX}$ of X is called the subgroup of elements of infinite topological height of X. If $\bigcap_{n\in\mathbb{N}_0}\overline{nX} = \{0\}$, X is said to have no elements of infinite topological height.*

It is easy to see that if $X \in \mathcal{L}_p$ for some prime p, then X is compact-by-bounded order if and only if $\overline{p^m X}$ is compact for some $m \in \mathbb{N}$, and X has no elements of infinite topological height if and only if $\bigcap_{n\in\mathbb{N}} \overline{p^n X} = \{0\}$.

Corollary 5.6. *Let X be a topological torsion group in $\mathcal{L}$ such that its primary components are compact-by-bounded order and have no elements of infinite topological height. Then $E(X)$ is commutative if and only if*

$$X \cong \prod_{p\in S_1} (\mathbb{Z}_p; p^{k_p}\mathbb{Z}_p) \times \prod_{p\in S_2} (\mathbb{Z}(p^{n_p}); p^{k_p}\mathbb{Z}(p^{n_p})),$$

where $S_1 \cup S_2 = S(X)$, $S_1 \cap S_2 = \varnothing$, and $n_p, k_p \in \mathbb{N}_0$ for all $p \in S(X)$.

Proof. Observing that a group $X \in \mathcal{L}$ is compact-by-bounded order and has no elements of infinite topological height if and only if X^* is torsion, the assertion follows from [8, (23.33)], Lemma 3.1 and Corollary 5.3. $\square$

Specializing Theorem 5.2 to torsion groups, we arrive at the following corollary, which sharpens Theorem 4.1.

Corollary 5.7. *The following are equivalent for a group $X \in \mathcal{L}$:*

(i) *X is discrete and torsion, and $E(X)$ is commutative.*

(ii) *X is torsion, and $E(X)$ is commutative.*

(iii) *$X \cong \bigoplus_{p\in S_1} \mathbb{Z}(p^\infty) \times \bigoplus_{p\in S_2} \mathbb{Z}(p^{n_p})$, where $S_1 \cup S_2 = S(X)$, $S_1 \cap S_2 = \varnothing$, and $n_p \in \mathbb{N}_0$ for all $p \in S_2$.*

Proof. Clearly, (i) implies (ii), and (iii) implies (i). Assuming (ii), we deduce from Corollary 5.3 that

$$X \cong \prod_{p\in S_1} (\mathbb{Z}(p^\infty); \mathbb{Z}(p^\infty)[p^{k_p}]) \times \prod_{p\in S_2} (\mathbb{Z}(p^{n_p}); \mathbb{Z}(p^{n_p})[p^{k_p}]),$$

where $S_1 \cup S_2 = S(X)$, $S_1 \cap S_2 = \varnothing$, and $n_p, k_p \in \mathbb{N}$ for all $p \in S(X)$, so that in particular

$$\prod_{p\in S_1} \mathbb{Z}(p^\infty)[p^{k_p}] \times \prod_{p\in S_2} \mathbb{Z}(p^{n_p})[p^{k_p}] \qquad \Big(\cong \prod_{p\in S(X)} \mathbb{Z}(p^{k_p}),$$

since we may assume that $k_p \le n_p$ for all $p \in S_2\Big)$ has to be torsion. It then follows that $\{p \in S(X) \mid k_p \neq 0\}$ is finite, so

$$\prod_{p\in S_1} (\mathbb{Z}(p^\infty); \mathbb{Z}(p^\infty)[p^{k_p}]) \times \prod_{p\in S_2} (\mathbb{Z}(p^{n_p}); \mathbb{Z}(p^{n_p})[p^{k_p}])$$

is discrete by [8, (6.16)(d)], and hence (iii) holds. □

Corollary 5.8. *The following are equivalent for a group $X \in \mathcal{L}$:*

(i) *X is compact and totally disconnected, and $E(X)$ is commutative.*

(ii) *X is a compact-by-bounded order topologically torsion group with no elements of infinite topological height, and $E(X)$ is commutative.*

(iii) *$X \cong \prod_{p\in S_1} \mathbb{Z}_p \times \prod_{p\in S_2} \mathbb{Z}(p^{n_p})$, where $S_1 \cup S_2 = S(X)$, $S_1 \cap S_2 = \varnothing$, and $n_p \in \mathbb{N}_0$ for all $p \in S_2$.*

We next show that Corollary 5.6 can be improved by dropping the assumption that the considered groups do not contain elements of infinite topological height.

Theorem 5.9. *Let X be a topological torsion group in $\mathcal{L}$ with compact-by-bounded order topological primary components. The following are equivalent:*

(i) *$E(X)$ is commutative.*

(ii) *For each $p \in S(X)$, X_p is topologically isomorphic with either $\mathbb{Z}_p$ or $\mathbb{Z}(p^{n_p})$ for some $n_p \in \mathbb{N}_0$.*

Proof. As we already mentioned in the proof of Corollary 5.3, for a topological torsion group $X \in \mathcal{L}$, the commutativity of $E(X)$ is equivalent to the commutativity of all the $E(X_p)$'s.

Pick any $p \in S(X)$, and assume that $E(X_p)$ is commutative. Since X_p is compact-by-bounded order, there is a compact subgroup K of X_p such that X_p/K is of bounded order. Hence $p^{n_0}(X_p/K) = \{0\}$ for some $n_0 \in \mathbb{N}$. It follows that $\overline{p^{n_0}X_p}$ is a closed subgroup of K, so that $\overline{p^{n_0}X_p}$ is compact, and hence $(\overline{p^{n_0}X_p})^*$ is a discrete p-group. But then $(\overline{p^{n_0}X_p})^*$ admits a direct summand isomorphic with either $\mathbb{Z}(p^\infty)$ or $\mathbb{Z}(p^m)$ for some $m \in \mathbb{N}$ [7, Corollary 27.3]. Since every decomposition of $(\overline{p^{n_0}X_p})^*$ as a direct sum produces a decomposition into a topological direct sum of $\overline{p^{n_0}X_p}$ [1, Corollary 6.10], we can write $\overline{p^{n_0}X_p} = A \oplus B$, where A is topologically isomorphic with either $\mathbb{Z}_p$ or $\mathbb{Z}(p^m)$. We must have $H(A, B) = \{0\}$, for otherwise we would obtain a contradiction by applying Lemma 3.5 with $\omega = p^{n_0}1_{X_p}$ and any nonzero $h \in H(A, B)$.

Assume $A \cong \mathbb{Z}_p$. Since for every $x \in X_p$ there exists $f \in H(\mathbb{Z}_p, X_p)$ such that $x \in \mathrm{im}(f)$ [1, Lemma 2.10], the equality $H(A, B) = \{0\}$ can occur only if $B = \{0\}$. It follows that $\overline{p^{n_0}X_p} \cong \mathbb{Z}_p$, so that $\bigcap_{n\in\mathbb{N}} p^n X_p = \{0\}$, and hence $X_p \cong \mathbb{Z}_p$ by Corollary 5.6.

Now assume $A \cong \mathbb{Z}(p^m)$. Since $H(A, B) = \{0\}$, we must clearly have $t(B) = \{0\}$, so that $B \cong \mathbb{Z}_p^\nu$ for some cardinal number ν [8, 25.8]. But in view of Lemma 3.5 $H(B, A) = \{0\}$ too, which can only occur if $\nu = 0$. It follows that $(\overline{p^{n_0}X_p})^* \cong \mathbb{Z}(p^m)$, so X_p is of bounded order, and hence, by Theorem 5.2, $X \cong \mathbb{Z}(p^{n_p})$ for some $n_p \in \mathbb{N}_0$.

Since $E(\mathbb{Z}_p)$ and $E(\mathbb{Z}(p^{n_p}))$ are clearly commutative, the proof is complete. □

To dualize the preceding theorem, we need a new definition.

Definition 5.10. *A group $X \in \mathcal{L}$ is said to be bounded order-by-discrete in case X contains an open subgroup of bounded order.*

The following extends Corollary 5.3.

Corollary 5.11. *Let X be a topological torsion group in $\mathcal{L}$ with bounded order-by-discrete topological primary components. The following are equivalent:*

(i) *$E(X)$ is commutative.*

(ii) *For each $p \in S(X)$, X_p is topologically isomorphic with either $\mathbb{Z}(p^\infty)$ or $\mathbb{Z}(p^{n_p})$ for some $n_p \in \mathbb{N}_0$.*

Next we describe the nonreduced torsionfree topological p-primary groups $X \in \mathcal{L}$ with commutative ring $E(X)$.

Theorem 5.12. *Let $p \in \mathbb{P}$, and let X be a nonreduced torsionfree group in $\mathcal{L}_p$. The ring $E(X)$ is commutative if and only if $X \cong \mathbb{Q}_p$.*

Proof. Assume $E(X)$ is commutative. Since X is a nonreduced torsionfree topological p-primary group, it follows from [1, Theorem 4.23] that X contains a closed subgroup D topologically isomorphic with $\mathbb{Q}_p$. We shall show that X must coincide with D. Suppose this is not the case. Then, taking into account that $\mathbb{Q}_p$ is splitting in the class of torsionfree LCA groups [1, Proposition 6.23], we can write $X = D \oplus Y$ for some nonzero subgroup Y of X. Let U be an arbitrary compact open subgroup of Y. Since Y is torsionfree, it follows that $U \cong \mathbb{Z}_p^\nu$ for some cardinal number $\nu \geq 1$ [4, Ch. III, §1, Proposition 3]. Combining an arbitrary topological isomorphism from U onto $\mathbb{Z}_p^\nu$ with a projection of $\mathbb{Z}_p^\nu$ onto $\mathbb{Z}_p$ and with an arbitrary continuous monomorphism from $\mathbb{Z}_p$ into D, we obtain a nonzero $h \in H(U, D)$. Since D is divisible and U is open in Y, h extends to a nonzero homomorphism $h_0 \in H(Y, D)$, contradicting by Lemma 3.5 our assumption that $E(X)$ is commutative. Therefore we must have $X \cong \mathbb{Q}_p$.

The converse is clear. □

As a consequence we have the following two corollaries.

Corollary 5.13. *Let X be a torsionfree topological torsion group in $\mathcal{L}$ such that, for each $p \in S(X)$, X_p is nonreduced. The ring $E(X)$ is commutative if and only if $X \cong \prod_{p\in S(X)}(\mathbb{Q}_p; \mathbb{Z}_p)$.*

Corollary 5.14. *Let X be a topological torsion densely divisible group in $\mathcal{L}$ such that, for each $p \in S(X)$, $m(X_p) \neq X_p$. The ring $E(X)$ is commutative if and only if $X \cong \prod_{p\in S(X)}(\mathbb{Q}_p; \mathbb{Z}_p)$.*

We now turn our attention to the case of topological torsion groups in $\mathcal{L}$ with mixed topological primary components. As we saw, the key argument used in proving Theorem 5.12 was the fact, due to L. C. Robertson [12], that, for each $p \in \mathbb{P}$, $\mathbb{Q}_p$ is splitting in the class of torsionfree LCA groups. In order to do with mixed nonreduced topological p-primary groups, we first extend Robertson's result to more general groups.

Lemma 5.15. *Let X be a group in $\mathcal{L}$ satisfying $c(X) \subset m(X) \neq X$, and let D be a closed subgroup of X such that $D \cong \mathbb{Q}_p$ for some $p \in \mathbb{P}$ and $D \cap m(X) = \{0\}$. Then D splits topologically from X.*

Proof. It is clear from the very definition of $m(X)$ that $m(X) \subset k(X)$. Since by hypothesis $c(X) \subset m(X)$, it follows that $c(X)$ is compact [5, Proposition 3.3.6], so that $m(X)/c(X)$ is closed in $X/c(X)$ [8, (5.18)]. Taking into account that $X/c(X)$ is totally disconnected [8, (7.3)], and

$$X/m(X) \cong (X/c(X))/(m(X)/c(X)) \quad [8, (5.35)],$$

we then deduce from [8, (7.11)] that $X/m(X)$ is totally disconnected as well. Let π denote the canonical projection of X onto $X/m(X)$. Fixing an arbitrary topological isomorphism f from $\mathbb{Q}_p$ onto D, set $h = \pi \circ f$. Since $D \cap m(X) = \{0\}$, it follows that π acts injectively on D, so that h is injective too. Remembering that $X/m(X)$ is totally disconnected, we conclude from [1, Proposition 4.21] that $h(\mathbb{Q}_p)$ is a closed subgroup of $X/m(X)$ and that h establishes a topological isomorphism from $\mathbb{Q}_p$ onto $h(\mathbb{Q}_p)$. Therefore, taking account of the above mentioned fact that $\mathbb{Q}_p$ is splitting in the class of torsionfree LCA groups, we can write

$$X/m(X) = h(\mathbb{Q}_p) \oplus \Gamma$$

for some closed subgroup Γ of $X/m(X)$. Let $G = \pi^{-1}(\Gamma)$. We shall show that $X = D \oplus G$. Clearly, G is a closed subgroup of X containing $m(X)$, $\pi(G) = \Gamma$ and $\pi(D) = h(\mathbb{Q}_p)$. If there existed a nonzero $a \in D \cap G$, it would follow that

$$\pi(a) \in \pi(D) \cap \pi(G) = h(\mathbb{Q}_p) \cap \Gamma = \{0\}.$$

This would imply that $a \in m(X)$, contradicting our assumption that $D \cap m(X) = \{0\}$. Thus we must have $D \cap G = \{0\}$. To see that also $X = D + G$, pick an arbitrary $x \in X$. Since

$$X/m(X) = h(\mathbb{Q}_p) \oplus \Gamma = \pi(D) \oplus \pi(G),$$

there exist $y \in D$ and $z \in G$ such that $\pi(x) = \pi(y) + \pi(z)$, so that $x = y + z + t$ for some $t \in m(X)$. But $z + t \in G$ because $m(X) \subset G$, and since $x \in X$ was chosen arbitrarily, this shows that $X = D + G$. Consequently, X decomposes as an algebraic direct sum of D and G. To conclude that the obtained decomposition is in fact topological, it remains to observe [1, Proposition 6.5] that D, being topologically isomorphic to $\mathbb{Q}_p$, is σ-compact. □

Corollary 5.16. *Let X be a totally disconnected group in $\mathcal{L}$ having closed torsion subgroup. If X contains a closed subgroup D topologically isomorphic with $\mathbb{Q}_p$ for some $p \in \mathbb{P}$, then D splits topologically from X.*

Proof. Since $c(X) = \{0\}$, $m(X) = t(X)$ and $\mathbb{Q}_p$ is torsionfree, the assertion follows from Lemma 5.15. □

We approach the description of mixed nonreduced topological p-primary groups $X \in \mathcal{L}$ with commutative ring $E(X)$ through two lemmas.

Lemma 5.17. *Let $p \in \mathbb{P}$, and let X be a mixed group in $\mathcal{L}_p$. If $E(X)$ is commutative, then $t(X)$ is reduced.*

Proof. If $t(X)$ were nonreduced, X would contain a copy D of $\mathbb{Z}(p^\infty)$. Since $\mathbb{Z}(p^\infty)$ is splitting in the class of totally disconnected LCA groups, it would then follow that $X = D \oplus T$ for some nonzero (because $X \neq t(X)$) closed subgroup T of X. Letting U be an arbitrary nonzero compact open subgroup of T, then U^* would be a nonzero discrete p-group, and so U^* would admit by [7, Corollary 27.3] a direct summand isomorphic with either $\mathbb{Z}(p^\infty)$ or $\mathbb{Z}(p^n)$ for some $n \in \mathbb{N}$. We would then conclude from [8, (23.18)] that U has a topological direct summand topologically isomorphic with either $\mathbb{Z}_p$ or $\mathbb{Z}(p^n)$, which would imply that $H(U, D) \neq \{0\}$. Extending the elements of $H(U, D)$, we would obtain that $H(T, D) \neq \{0\}$, so that by Lemma 3.5 $E(X)$ could not be commutative. □

Lemma 5.18. *Let $p \in \mathbb{P}$, and let X be a group in $\mathcal{L}_p$ such that $d(X) \not\subset m(X)$. If $E(X)$ is commutative, then $d(X) \cong \mathbb{Q}_p$ and $X = d(X) \oplus m(X)$.*

Proof. Fix any $a \in d(X) \setminus m(X)$, and let D denote the minimal divisible subgroup of X containing a. It is clear from the definition of $m(X)$ that $t(X) \subset m(X)$, so $a \notin t(X)$, and hence D is algebraically isomorphic to $\mathbb{Q}$. It follows that $\overline{D}$ is divisible, because every group in $\mathcal{L}$ containing a dense divisible subgroup of finite rank is itself divisible [1, (5.39)(e)]. We assert that $\overline{D} \cong \mathbb{Q}_p$. Indeed, since $X \in \mathcal{L}_p$, it follows from [1, Lemma 2.11] that $\overline{\langle a \rangle} \cong \mathbb{Z}_p$. Pick a topological isomorphism φ from $\mathbb{Z}_p$ onto $\overline{\langle a \rangle}$. Since $\overline{D}$ is divisible, $\eta \circ \varphi$ extends to homomorphism $f \in H(\mathbb{Q}_p, \overline{D})$, where η is the canonical injection of $\overline{\langle a \rangle}$ into $\overline{D}$. To show that f is injective, pick any $x \in \ker(f)$. Since $\mathbb{Q}_p$ is the minimal divisible extension of $\mathbb{Z}_p$, we can find an $l \in \mathbb{N}$ such that $p^l x \in \mathbb{Z}_p$. It follows that $p^l x \in \ker(\eta \circ \varphi)$, so $p^l x = 0$, whence $x = 0$ because $\mathbb{Q}_p$ is torsionfree. Thus our claim is established. As every group in $\mathcal{L}_p$ is totally disconnected, it then follows from [1, Proposition 4.21] that $f(\mathbb{Q}_p)$ is closed in $\overline{D}$ and f is a topological isomorphism from $\mathbb{Q}_p$ onto $f(\mathbb{Q}_p)$. But $\overline{\langle a \rangle} \subset f(\mathbb{Q}_p)$ and

$f(\mathbb{Q}_p)$ is divisible, so $D \subset f(\mathbb{Q}_p)$, and hence $f(\mathbb{Q}_p) = \overline{D}$, proving that $\overline{D} \cong \mathbb{Q}_p$. Next we show that $\overline{D} \cap m(X) = \{0\}$. Assume the contrary, and let $U = \overline{D} \cap m(X)$. Then U is open in $\overline{D}$. Since $\lim_{n\to\infty} p^n a = 0$, there exists $k \in \mathbb{N}$ such that $p^k a \in U$, and so

$$a + m(X) \in t(X/m(X)).$$

As $X/m(X)$ is torsionfree, it follows that $a \in m(X)$, contradicting the choice of a. This proves that $\overline{D} \cap m(X) = \{0\}$.

Now, according to Lemma 5.15, we can write $X = D \oplus G$ for some closed subgroup G of X. Since, in view of Lemma 3.5, $H(D, G)$ and $H(G, D)$ cannot be nonzero groups, we must have $d(G) = \{0\}$ and $m(G) = G$, so that $D = d(X)$ and $G = m(X)$. □

Theorem 5.19. *Let $p \in \mathbb{P}$, and let $X \in \mathcal{L}_p$ be a nonreduced mixed group having closed torsion subgroup. The ring $E(X)$ is commutative if and only if X is topologically isomorphic to $\mathbb{Q}_p \times \mathbb{Z}(p^{n_p})$ for some $n_p \in \mathbb{N}_0$.*

Proof. Assume $E(X)$ is commutative. Since $t(X)$ is closed in X, we clearly have $m(X) = t(X)$. But $t(X)$ is reduced by Lemma 5.17, so that $d(X) \not\subset m(X)$. It then follows from Lemma 5.18 that $X = D \oplus t(X)$, where $D \cong \mathbb{Q}_p$. As, by Lemma 3.2, $E(t(X))$ is also commutative, we deduce from Theorem 5.2 that $t(X) \cong \mathbb{Z}(p^{n_p})$ for some $n \in \mathbb{N}_0$.

Assume the converse. We have $X = d(X) \oplus t(X)$, where $d(X) \cong \mathbb{Q}_p$ and $t(X) \cong \mathbb{Z}(p^{n_p})$. Since $d(X)$ is torsionfree and $t(X)$ is reduced, it follows that $d(X)$ and $t(X)$ are topologically fully invariant subgroups of X, so that $E(X) \cong E(\mathbb{Q}_p) \times E(\mathbb{Z}(p^{n_p}))$. □

We prove now

Theorem 5.20. *Let X be a topological torsion group in $\mathcal{L}$ such that its topological primary components have closed torsion subgroup and compact-by-bounded order quotient modulo the subgroup of elements of infinite topological height. The following are equivalent:*

(i) *$E(X)$ is commutative.*

(ii) *For each $p \in S(X)$, X_p is topologically isomorphic with one of the groups $\mathbb{Z}(p^\infty)$, $\mathbb{Z}(p^{n_p})$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ or $\mathbb{Q}_p \times \mathbb{Z}(p^{n_p})$, where $n_p \in \mathbb{N}$.*

Proof. As we know, $E(X)$ is commutative if and only if all the $E(X_p)$'s have this property.

Pick any $p \in S(X)$, and assume that $E(X_p)$ is commutative. If $X_p = t(X_p)$, we deduce from Theorem 5.2 that either $X \cong \mathbb{Z}(p^\infty)$ or $X \cong \mathbb{Z}(p^{n_p})$ for some $n_p \in \mathbb{N}$. Let us suppose further that $X_p \neq t(X_p)$. Since

$$\bigcap_{n\in\mathbb{N}} \overline{p^n X_p^*} = A(X_p^*, t(X_p)) \quad [8, (24.24)]$$

and $t(X_p)$ is closed in X_p, it then follows from [8, (23.24)(b)] that $\bigcap_{n\in\mathbb{N}} \overline{p^n X_p^*} \neq \{0\}$. But

$$\big(X_p/t(X_p)\big)^* \cong \bigcap_{n\in\mathbb{N}} \overline{p^n X_p^*} \quad [8, (23.25)]$$

and since $X_p/t(X_p)$ is torsionfree, we conclude by a theorem of Robertson [13, Theorem 5.2] that $\bigcap_{n\in\mathbb{N}} \overline{p^n X_p^*}$ is densely divisible, so that X_p^* is nonreduced. Now, if $X_p^* = t(X_p^*)$, we use Theorem 5.2 again to deduce that $X_p^* \cong \mathbb{Z}(p^\infty)$, whence $X_p \cong \mathbb{Z}_p$. Further, if $t(X_p^*) = \{0\}$, we conclude from Theorem 5.12 that $X_p^* \cong \mathbb{Q}_p$, so $X_p \cong \mathbb{Q}_p$ because $\mathbb{Q}_p$ is self-dual. Thus, it only remains to consider the case when X_p^* is mixed. As $X_p/\bigcap_{n\in\mathbb{N}} \overline{p^n X_p}$ is compact-by-bounded order, it is then easily seen that $t(X_p^*)$ is closed in X_p^*, so that by Theorem 5.19 $X_p^* \cong \mathbb{Q}_p \times \mathbb{Z}(p^{m_p})$ for some $m_p \in \mathbb{N}$, whence $X_p \cong \mathbb{Q}_p \times \mathbb{Z}(p^{m_p})$.

On the other hand, it is clear that the groups $\mathbb{Z}(p^\infty)$, $\mathbb{Z}(p^{n_p})$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{Q}_p \times \mathbb{Z}(p^{m_p})$ have commutative ring of continuous endomorphisms. □

Remark. Observe that by dualizing Theorem 5.20 we would obtain nothing new because the class $\mathcal{S}$ of topological torsion groups in $\mathcal{L}$ whose topological primary components have closed torsion subgroup and compact-by-bounded order quotient by the subgroup of elements of infinite topological height is self-dual, i.e. if $X \in \mathcal{S}$, then $X^* \in \mathcal{S}$ too.

References

[1] Armacost D.L. *The structure of locally compact abelian groups.* Pure and Applied Mathematics Series, Vol. 68. New York, Marcel Dekker, 1981.

[2] Bourbaki N. *Topologie generale, Chapter 1-2, Éléments de mathematique.* Moscow, Nauka, 1968.

[3] Bourbaki N. *Topologie generale, Chapter 3-8, Éléments de mathematique.* Moscow, Nauka 1969.

[4] Braconnier J. *Sur les groupes topologiques localement compact.* J. Math. Pures Apl., 1948, **27**, N 9, p. 1–85.

[5] Dikranjan D., Prodanov I., Stoyanov L. *Topological groups.* Pure and Applied Mathematics Series, Vol. 130. New York and Basel, Marcel Dekker, 1990.

[6] Fuchs L. *Abelian groups.* Publishing House of the Hungarian Academy of Science, Budapest, 1958.

[7] Fuchs L. *Infinite abelian groups, Vol. 1.* New York and London, Academic Press, 1970.

[8] Hewitt E., Ross K. *Abstract Harmonic Analysis, Vol. 1.* Moscow, Nauka, 1975.

[9] Van Leeuwen L.C.A. *Remarks on endomorphism rings of torsion-free abelian groups.* Acta Scient. Math., 1971, **32**, N 3-4, p. 309–324.

[10] Popa V. *Units, idempotents and nilpotents of an endomorphism ring, II.* Bul. Acad. Şti. R. Moldova, Matematica, 1997, N 1(23), p. 93–105.

[11] Popa V. *On LCA groups with compact rings of continuous endomorphisms.* Bul. Acad. Ştiinţe R. Moldova, Matematica, 2000, N 1(32), p. 17–32.

[12] Robertson L.C. *Transfinite torsion, p-constituents, and splitting in locally compact abelian groups* (1968, unpublished).

[13] Robertson L.C. *Connectivity, divisivility, and torsion.* Trans. Amer. Math. Sos., 1967, **128**, p. 482–505.

[14] Szele T., Szendrei J. *On abelian groups with commutative endomorphism ring.* Acta Math. Acad. Sci. Hung., 1951, **2**, p. 309–324.

Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
Academiei str. 5, MD-2028 Chişinău
Moldova

E-mail: *vpopa@math.md*

Received October 11, 2006

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(52), 2006, Pages 101–108
ISSN 1024–7696

Natural classes and torsion free classes in categories of modules

A.I. Kashu

Abstract. The relation between natural classes and torsion free classes of modules is studied. The mapping ϕ: R-nat $\to \mathcal{P}$ between corresponding lattices is defined and some properties of ϕ are shown, in particular, the compatibility of ϕ with operations of unions in lattices.

Mathematics subject classification: 16S90, 16D90, 16D80.
Keywords and phrases: Torsion, torsion free class, natural class, lattice of torsions, frame.

1 Preliminaries

The abstract class of modules $\mathcal{K} \subseteq R$-Mod, (i.e. the class closed under isomorphisms) is called *natural* (or *saturated*) if it is closed with respect to submodules, direct sums and essential extensions (or injective envelopes). This type of classes of modules was studied from diverse points of view in a series of works, for example in [1–4]. The purpose of this note is to elucidate the relation between the natural classes and torsions ($\equiv$ hereditary radicals) of R-Mod, in special, torsion free classes of R-modules. It is well known that every torsion r of R-Mod determines two classes of modules:

$$\mathcal{T}_r = \{{}_RM \mid r(M) = M\}, \quad \mathcal{F}_r = \{{}_RM \mid r(M) = 0\}.$$

The class of the form $\mathcal{T}_r$, where r is a torsion, is called *torsion class* and is characterized as a class closed under submodules, direct sums, homomorphic images and extensions. Dually, the class of the form $\mathcal{F}_r$, where r is a torsion, is called *torsion free class* and can be described as a class closed under submodules, direct products and essential extensions (or injective envelopes). We note that every torsion free class is closed also under extensions. These and other facts on torsions can be found in the books [5–8].

In such a way all results on torsions (and on radicals) can be expounded by classes of modules, using the classes of the form $\mathcal{T}_r$ and $\mathcal{F}_r$. The relation between these two types of classes can be expressed by the following *operators of Hom-orthogonality*:

$$\mathcal{K} \subseteq R\text{-Mod}, \qquad \mathcal{K}^{\uparrow} = \{{}_RX \mid Hom_R(X,Y) = 0 \quad \forall\, Y \in \mathcal{K}\},$$
$$\mathcal{K}^{\downarrow} = \{{}_RY \mid Hom_R(X,Y) = 0 \quad \forall\, X \in \mathcal{K}\}.$$

For every torsion r of R-Mod the following relations are true:

$$\mathcal{T}_r = \mathcal{F}_r^{\uparrow}, \quad \mathcal{F}_r = \mathcal{T}_r^{\downarrow}.$$

In the following statement we give an account of elementary properties of the operators of Hom-orthogonality [6, 8].

Lemma 1.1. (1) *The operators* $(\uparrow)$ *and* $(\downarrow)$ *are anti-monotone, i.e. they converte the inclusions of classes: if* $\mathcal{K}_1 \subseteq \mathcal{K}_2$*, then*

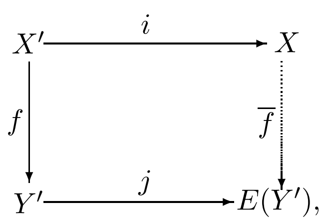
$$\mathcal{K}_1^{\uparrow} \supseteq \mathcal{K}_2^{\uparrow}, \quad \mathcal{K}_1^{\downarrow} \supseteq \mathcal{K}_2^{\downarrow}.$$

(2) *For every* $\mathcal{K} \subseteq R$*-Mod the class* $\mathcal{K}^{\uparrow}$ *is a* **radical class***, i.e. it is closed under homomorphic images, direct sums and extensions.*
(3) *For every* $\mathcal{K} \subseteq R$*-Mod the class* $\mathcal{K}^{\downarrow}$ *is a* **semisimple class***, i.e. it is closed under submodules, direct products and extensions.*
(4) *For every* $\mathcal{K} \subseteq R$*-Mod the class* $\mathcal{K}^{\downarrow\uparrow}$ *is the smallest radical class containing* $\mathcal{K}$*.*
(5) *For every* $\mathcal{K} \subseteq R$*-Mod the class* $\mathcal{K}^{\uparrow\downarrow}$ *is the smallest semisimple class containing* $\mathcal{K}$*.*

The abstract class $\mathcal{K} \subseteq R$-Mod is called *hereditary class* if it is closed under submodules, and $\mathcal{K}$ is called *stable class* if it is closed under essential extensions (if $\mathcal{K}$ is hereditary, then the last condition is equivalent to the closeness under injective envelopes). It is known that if $(\mathcal{T}, \mathcal{F})$ is a torsion theory in the sense of S.E. Dickson (i.e. $\mathcal{T} = \mathcal{F}^{\uparrow}$ and $\mathcal{F} = \mathcal{T}^{\downarrow}$), then the class $\mathcal{T}$ is hereditary if and only if $\mathcal{F}$ is stable. This statement is a corollary of the following facts [6, 8].

Lemma 1.2. (1) *If* $\mathcal{K}$ *is a hereditary and stable class, then* $\mathcal{K}^{\uparrow}$ *is hereditary.*
(2) *If* $\mathcal{K}$ *is a hereditary class, then* $\mathcal{K}^{\downarrow}$ *is a stable class.*

Proof. 1). Let $X \in \mathcal{K}^{\uparrow}$ and $X' \subseteq X$. If $X' \notin \mathcal{K}^{\uparrow}$, then there exists $0 \neq f : X' \to Y$, $Y \in \mathcal{K}$ and denoting $Y' = Im\, f \neq 0$, we have $Y' \in \mathcal{K}$. Now we consider the diagram:



where $E(Y')$ is the injective envelope of $Y' \in \mathcal{K}$ and i, j are inclusions. Then $E(Y') \in \mathcal{K}$ and there exists $\overline{f} : X \to E(Y')$ which extends f. From $f \neq 0$ it follows $\overline{f} \neq 0$, a contradiction with $X \in \mathcal{K}^{\uparrow}$.

2) Let $Y \in \mathcal{K}^{\downarrow}$ and $Y \subseteq^* Z$ (where $\subseteq^*$ is the essential inclusion). If $Hom_R(X, Z) \neq 0$ for some $X \in \mathcal{K}$, then there exists $0 \neq f : X \to Z$ with $0 \neq Im\, f \subseteq Z$. From $Y \subseteq^* Z$ it follows $Y \cap Im\, f \neq 0$. Denoting $X' = f^{-1}(Y \cap Im\, f)$, we have $X' \in \mathcal{K}$ and the restriction of f to X' is a non-zero homomorphism

$0 \neq f' : X' \to Y \cap Im\, f = Y'$, where $Y' \in \mathcal{K}^{\downarrow}$ (since $\mathcal{K}^{\downarrow}$ is hereditary), so $Hom_R(X', Y') \neq 0$, a contradiction. Therefore $Hom_R(X, Z) = 0$ for every $X \in \mathcal{K}$, i.e. $Z \in \mathcal{K}^{\downarrow}$. □

Corollary 1.3. (1) *If* $\mathcal{K}$ *is a natural class, then* $\mathcal{K}^{\uparrow}$ *is a radical and hereditary class, i.e. a torsion class.*
(2) *If* $\mathcal{K}$ *is a natural class, then the class* $\mathcal{K}^{\uparrow\downarrow}$ *is semisimple and stable, i.e. a torsion free class.*

Further we will use the following notations:
R-tors – the set (lattice) of all torsions of R-Mod;
R-nat – the set (lattice) of all natural clases of R-Mod;
$\boldsymbol{\mathcal{R}}$ – the set of all torsion classes of R-Mod;
$\boldsymbol{\mathcal{P}}$ – the set of all torsion free classes of R-Mod.

It is known that R-nat can be transformed in a lattice and this lattice is boolean [1, 3, etc]. Similarly, the sets $\boldsymbol{\mathcal{R}}$ and $\boldsymbol{\mathcal{P}}$ are transformed in a natural way in lattices, where the order relation is the inclusion and the lattice operations "$\wedge$" and "$\vee$" are defined as follows:

$$\text{in } \boldsymbol{\mathcal{R}}: \quad \bigwedge_{\alpha \in \mathfrak{A}} \mathcal{R}_\alpha = \bigcap_{\alpha \in \mathfrak{A}} \mathcal{R}_\alpha, \quad \bigvee_{\alpha \in \mathfrak{A}} \mathcal{R}_\alpha = \cap \{\mathcal{R} \in \boldsymbol{\mathcal{R}} \,|\, \mathcal{R} \supseteq \mathcal{R}_\alpha \;\; \forall \alpha \in \mathfrak{A}\};$$

$$\text{in } \boldsymbol{\mathcal{P}}: \quad \bigwedge_{\alpha \in \mathfrak{A}} \mathcal{P}_\alpha = \bigcap_{\alpha \in \mathfrak{A}} \mathcal{P}_\alpha, \quad \bigvee_{\alpha \in \mathfrak{A}} \mathcal{P}_\alpha = \cap \{\mathcal{P} \in \boldsymbol{\mathcal{P}} \,|\, \mathcal{P} \supseteq \mathcal{P}_\alpha \;\; \forall \alpha \in \mathfrak{A}\}.$$

Since there exists a monotone bijection between torsions and torsion free classes, we have a lattice isomorphism $\boldsymbol{\mathcal{R}} \cong R$-tors. The anti-monotone bijection between $\boldsymbol{\mathcal{R}}$ and $\boldsymbol{\mathcal{P}}$ is established by the operators of Hom-orthogonality ($\uparrow$) and ($\downarrow$), and these operators are compatible with lattice operations in the following sense.

Proposition 1.4. *For every sets* $\{\mathcal{R}_\alpha \,|\, \alpha \in \mathfrak{A}\} \subseteq \boldsymbol{\mathcal{R}}$ *and* $\{\mathcal{P}_\alpha \,|\, \alpha \in \mathfrak{A}\} \subseteq \boldsymbol{\mathcal{P}}$ *the following relations are true:*

$$\text{a)} \quad \Big(\bigwedge_{\alpha \in \mathfrak{A}} \mathcal{P}_\alpha\Big)^{\uparrow} = \bigvee_{\alpha \in \mathfrak{A}} (\mathcal{P}_\alpha^{\uparrow}); \qquad \text{b)} \quad \Big(\bigwedge_{\alpha \in \mathfrak{A}} \mathcal{R}_\alpha\Big)^{\downarrow} = \bigvee_{\alpha \in \mathfrak{A}} (\mathcal{R}_\alpha^{\downarrow});$$

$$\text{c)} \quad \Big(\bigvee_{\alpha \in \mathfrak{A}} \mathcal{P}_\alpha\Big)^{\uparrow} = \bigwedge_{\alpha \in \mathfrak{A}} (\mathcal{P}_\alpha^{\uparrow}); \qquad \text{d)} \quad \Big(\bigvee_{\alpha \in \mathfrak{A}} \mathcal{R}_\alpha\Big)^{\downarrow} = \bigwedge_{\alpha \in \mathfrak{A}} (\mathcal{R}_\alpha^{\downarrow}).$$

Therefore, the lattice $\boldsymbol{\mathcal{R}}$ is anti-isomorphic to the lattice $\boldsymbol{\mathcal{P}}$.

Remark. Some results on the lattice of natural classes are contained in [9] and [10]. In particular, the lattice R-nat is described as a skeleton (boolean part) of the frame of closed classes of R-Mod.

2 Natural classes and torsion free classes

From the definitions of §1 it is clear that every torsion free class (i.e. of the form $\mathcal{F}_r$, where r is a torsion) is natural, so we have the inclusion $i : \mathcal{P} \to R$-nat. Now we define an inverse mapping $\phi : R\text{-nat} \to \mathcal{P}$, considering that for every $\mathcal{K} \in R$-nat the class $\phi(\mathcal{K})$ is the smallest torsion free class containing $\mathcal{K}$ (i.e. the intersection of all torsion free classes of R-Mod which contain $\mathcal{K}$).

From the Lemma 1.1 and Corollary 1.3 it follows

Proposition 2.1. *For every natural class $\mathcal{K}$ the following relation is true:*

$$\phi(\mathcal{K}) = \mathcal{K}^{\uparrow\downarrow}.$$

The next specification of this relation follows from the fact that every class $\mathcal{K} \in R$-nat is hereditary (to compare with Prop. 2.5, chapter VI of [5]).

Proposition 2.2. *For every natural class $\mathcal{K}$ of R-Mod we have*:

$$\phi(\mathcal{K}) = \{{}_RY \mid \forall\ 0 \neq Y' \subseteq Y,\ \exists \text{epi}\ 0 \neq f : Y' \to Y'',\ Y'' \in \mathcal{K}\},$$

i.e. $\phi(\mathcal{K})$ consists of all modules Y such that for every non-zero submodule $Y' \subseteq Y$ there exists a non-zero epimorphism $f : Y' \to Y''$ with $Y'' \in \mathcal{K}$.

Proof. Denote by $\overline{\mathcal{K}}$ the class of right part of this relation.
$\phi(\mathcal{K}) \subseteq \overline{\mathcal{K}}$: From definitions we have:

$$\begin{aligned} \mathcal{K}^{\uparrow\downarrow} &= \{{}_RY \mid Hom_R(X, Y) = 0\ \forall\ X \in \mathcal{K}^{\uparrow}\} = \\ &= \{{}_RY \mid Hom_R(X, Z) = 0\ \forall\ Z \in \mathcal{K} \Rightarrow Hom_R(X, Y) = 0\} = \\ &= \{{}_RY \mid Hom_R(X, Y) \neq 0 \Rightarrow \exists\, Z \in \mathcal{K},\ Hom_R(X, Z) \neq 0\}. \end{aligned}$$

Let $Y \in \phi(\mathcal{K}) = \mathcal{K}^{\uparrow\downarrow}$ and $0 \neq Y' \subseteq Y$. Then $Hom_R(Y', Y) \neq 0$, therefore there exists $Z \in \mathcal{K}$ such that $Hom_R(Y', Z) \neq 0$. For $0 \neq f : Y' \to Z$ and $Y'' = Im\, f$, we obtain a non-zero epimorphism $\overline{f} : Y' \to Y'' \subseteq Z$, where $Y'' \in \mathcal{K}$ (since $\mathcal{K}$ is hereditary), therefore $Y \in \overline{\mathcal{K}}$.
$\phi(\mathcal{K}) \supseteq \overline{\mathcal{K}}$: Let $Y \in \overline{\mathcal{K}}$ and we will prove that $Hom_R(X, Y) = 0$ for every $X \in \mathcal{K}^{\uparrow}$. Suppose the contrary: there exists an $X \in \mathcal{K}^{\uparrow}$ such that $Hom_R(X, Y) \neq 0$. Then we have $0 \neq f : X \to Y$ and denote $0 \neq Y' = Im\, f \subseteq Y$. Since $Y \in \overline{\mathcal{K}}$, there exists a non-zero epimorphism $0 \neq g : Y' \to Y''$, $Y'' \in \mathcal{K}$. Therefore we have a non-zero epimorphism $0 \neq gf : X \to Y' \to Y''$, $Y'' \in \mathcal{K}$, in contradiction with $X \in \mathcal{K}^{\uparrow}$. □

Now we will show another description of the class $\phi(\mathcal{K})$ for $\mathcal{K} \in R$-nat, using the closeness properties. Comparing the respective definitions, it is clear that for the natural class $\mathcal{K}$ to be torsion free class it is necessary in addition to be closed under direct products. In continuation we will prove that to obtain the class $\phi(\mathcal{K})$ for $\mathcal{K} \in R$-nat it is sufficient to close the class $\mathcal{K}$ with respect to submodules and direct products. For that we consider the class of all modules of R-Mod *cogenerated* by the natural class $\mathcal{K}$:

$$Cog\,(\mathcal{K}) = \{{}_R M \mid \exists \text{ mono } 0 \to M \to \prod_{\alpha \in \mathfrak{A}} M_\alpha,\ M_\alpha \in \mathcal{K}\},$$

i.e. $Cog\,(\mathcal{K})$ is the smallest class of R-Mod, which contains $\mathcal{K}$ and is closed under submodules and direct products.

Proposition 2.3. *For every class $\mathcal{K} \in R$-nat the following relation is true*:

$$\phi(\mathcal{K}) = Cog\,(\mathcal{K}).$$

Proof. Firstly we verify that the class $Cog\,(\mathcal{K})$ is torsion free. From definition it follows that the class $Cog\,(\mathcal{K})$ is closed under submodules and direct products, so it remains to prove that $Cog\,(\mathcal{K})$ is a stable class.

Let $M \in Cog\,(\mathcal{K})$ and $E(M)$ be the injective envelope of M. Then there exists a monomorphism $0 \to M \xrightarrow{\phi} \prod_{\alpha \in \mathfrak{A}} M_\alpha,\ M_\alpha \in \mathcal{K}$. Since $\prod_{\alpha \in \mathfrak{A}} E(M_\alpha)$ is an injective module, the inclusion $\prod_{\alpha \in \mathfrak{A}} M_\alpha \subseteq \prod_{\alpha \in \mathfrak{A}} E(M_\alpha)$ can be extended to a monomorphism $\psi : E\Big(\prod_{\alpha \in \mathfrak{A}} M_\alpha\Big) \to \prod_{\alpha \in \mathfrak{A}} E(M_\alpha)$. Now we consider the diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & M & \xrightarrow{\ \ \varphi\ \ } & \prod_{\alpha \in \mathfrak{A}} M_\alpha & & \\
& \Big\downarrow j & & \Big\downarrow i & & \\
& E(M) & \xrightarrow{\ \ \overline{\varphi}\ \ } & E(\prod_{\alpha \in \mathfrak{A}} M_\alpha) & \xrightarrow{\ \ \psi\ \ } & \prod_{\alpha \in \mathfrak{A}} E(M_\alpha),
\end{array}$$

where i, j are inclusions. By injectivity of $E\Big(\prod_{\alpha \in \mathfrak{A}} M_\alpha\Big)$ the monomorphism $i\varphi$ can be extended to a $\overline{\varphi} : E(M) \to E\Big(\prod_{\alpha \in \mathfrak{A}} M_\alpha\Big)$ and, since $M \subseteq^* E(M)$, $\overline{\varphi}$ is a monomorphism. So we obtain a monomorphism $\psi\overline{\varphi} : E(M) \to \prod_{\alpha \in \mathfrak{A}} E(M_\alpha)$, where $E(M_\alpha) \in \mathcal{K}$ for every $\alpha \in \mathfrak{A}$, since $\mathcal{K}$ is a stable class. Therefore $E(M) \in Cog\,(\mathcal{K})$ and so the class $Cog\,(\mathcal{K})$ is stable.

Taking into account that $\mathcal{K} \subseteq Cog\,(\mathcal{K})$, from the preceding result it follows the inclusion $\mathcal{K}^{\uparrow\downarrow} \subseteq Cog\,(\mathcal{K})$, since $\mathcal{K}^{\uparrow\downarrow}$ is the smallest torsion free class containing $\mathcal{K}$ (Corollary 1.3).

It remains to prove that $Cog\,(\mathcal{K}) \subseteq \mathcal{K}^{\uparrow\downarrow}$. Let $M \in Cog\,(\mathcal{K})$, i.e. we have a monomorphism $0 \to M \xrightarrow{\varphi} \prod_{\alpha \in \mathfrak{A}} M_\alpha$, $M_\alpha \in \mathcal{K}$ for every $\alpha \in \mathfrak{A}$. We will verify that $Hom_R(X, M) = 0$ for every $X \in \mathcal{K}^{\uparrow}$.

Suppose the contrary: there exists $X \in \mathcal{K}^{\uparrow}$ such that $Hom_R(X, M) \neq 0$. Then we have $0 \neq f : X \to M$ and since φ is mono, there exists $\beta \in \mathfrak{A}$ such that $p_\beta \varphi f \neq 0$:

$$X \xrightarrow{f} M \xrightarrow{\varphi} \prod_{\alpha\in\mathfrak{A}} M_\alpha \xrightarrow{p_\beta} M_\beta,$$

where p_β is the canonical projection. Therefore, $Hom_R(X, M_\beta) \neq 0$, where $M_\beta \in \mathcal{K}$ and $X \in \mathcal{K}^{\uparrow}$, a contradiction. $\square$

The studied mapping $\phi : R\text{-nat} \to \boldsymbol{\mathcal{P}}$ can be extended to a mapping $\psi : R\text{-nat} \to \boldsymbol{\mathcal{R}}$, where $\boldsymbol{\mathcal{R}}$ is the lattice of torsion classes of R-Mod, taking by definition:

$$\psi(\mathcal{K}) = [\phi(\mathcal{K})]^{\uparrow}$$

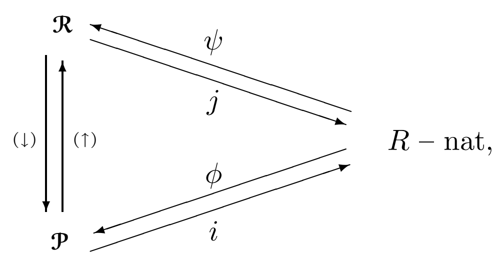
(since $\phi(\mathcal{K})$ is stable, $[\phi(\mathcal{K})]^{\uparrow}$ is a torsion class by Corollary 1.3). By Prop. 2.1 $\phi(\mathcal{K}) = \mathcal{K}^{\uparrow\downarrow}$, so we have:

$$\psi(\mathcal{K}) = (\mathcal{K}^{\uparrow\downarrow})^{\uparrow} = \mathcal{K}^{\uparrow}.$$

Moreover, we can define the mapping $j : \boldsymbol{\mathcal{R}} \to R$-nat by the rule:

$$j\,(\mathcal{R}) = \mathcal{R}^{\downarrow},\ \mathcal{R} \in \boldsymbol{\mathcal{R}}.$$

So we obtain the diagram:



where i is the inclusion. By definitions it is clear that ϕ is a monotone mapping, while ψ and j are anti-monotone. The following relations (commutativity of the diagram) are obvious:

$$j \cdot \psi = i \cdot \phi, \quad i = j \cdot (\uparrow), \quad \psi \cdot i = (\uparrow), \quad \psi \cdot j = (\downarrow).$$

As we have seen above, the operators $(\uparrow)$ and $(\downarrow)$ are compatible with lattice operations of $\boldsymbol{\mathcal{R}}$ and $\boldsymbol{\mathcal{P}}$ (Prop. 1.4), i.e. these mappings convert the lattice operations. Now we will study the similar question for the mappings ϕ and ψ. We begin with the following remark.

Lemma 2.4. *The mapping ϕ preserves the lattice operations if and only if the mapping ψ converts these operations.*

Proof. Let, for example, ϕ preserves the unions:

$$\phi\Big(\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha\Big) = \bigvee_{\alpha\in\mathfrak{A}} \phi(\mathcal{K}_\alpha).$$

Then applying Prop. 1.4 we obtain:

$$\Big(\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha\Big)^{\uparrow\downarrow} = \bigvee_{\alpha\in\mathfrak{A}} (\mathcal{K}_\alpha^{\uparrow\downarrow}) \Leftrightarrow \Big(\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha\Big)^{\uparrow} = \Big(\big(\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha\big)^{\uparrow\downarrow}\Big)^{\uparrow} =$$

$$= \Big(\bigvee_{\alpha\in\mathfrak{A}} (\mathcal{K}_\alpha^{\uparrow\downarrow})\Big)^{\uparrow} = \bigwedge_{\alpha\in\mathfrak{A}} (\mathcal{K}_\alpha^{\uparrow\downarrow\uparrow}) = \bigwedge_{\alpha\in\mathfrak{A}} (\mathcal{K}_\alpha^{\uparrow}),$$

i.e. ϕ preserves the unions if and only if ψ transforms the unions of classes in intersections. □

From this statement it follows that it is sufficient to prove the respective relations only for one of the mappings ϕ or ψ. Now we will show that the mapping ψ converts the unions of the lattice R-nat in the intersections of the lattice $\mathfrak{R}$. For that we would remind that the unions of classes of R-nat can be characterized as follows:

$$\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha = \{{}_RM \mid \exists \bigoplus_{\alpha\in\mathfrak{A}} M_\alpha \subseteq^* M, \quad M_\alpha \in \mathcal{K}_\alpha\} \text{ (see [1, Theor. 2.15])},$$

where $\mathcal{K}_\alpha \in R$-nat for every $\alpha \in \mathfrak{A}$ and $\subseteq^*$ is the essential inclusion.

Theorem 2.5. *For every set of natural classes* $\{\mathcal{K}_\alpha \mid \alpha \in \mathfrak{A}\}$ *the following relation is true*:

$$\Big(\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha\Big)^{\uparrow} = \bigwedge_{\alpha\in\mathfrak{A}} (\mathcal{K}_\alpha^{\uparrow}),$$

i.e. ψ *converts the unions of* R*-nat in the intersections of* $\mathfrak{R}$.

Proof. ($\subseteq$). From $\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha \supseteq \mathcal{K}_\alpha$ for every $\alpha \in \mathfrak{A}$ it follows $\Big(\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha\Big)^{\uparrow} \subseteq \mathcal{K}_\alpha^{\uparrow}$ for every $\alpha \in \mathfrak{A}$, therefore $\Big(\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha\Big)^{\uparrow} \subseteq \bigwedge_{\alpha\in\mathfrak{A}} (\mathcal{K}_\alpha^{\uparrow})$.

($\supseteq$). Let $X \in \bigwedge_{\alpha\in\mathfrak{A}} (\mathcal{K}_\alpha^{\uparrow})$. We must prove that $Hom_R(X, M) = 0$ for every $M \in \bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha$. Suppose the contrary: there exists $M \in \bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha$ such that $Hom_R(X, M) \neq 0$. From the description of the class $\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha$ indicated above, it follows that there exists a direct sum $\bigoplus_{\alpha\in\mathfrak{a}} M_\alpha \subseteq^* M$ with $M_\alpha \in \mathcal{K}_\alpha$ for every $\alpha \in \mathfrak{A}$. Then we have a non-zero homomorphism $0 \neq f : X \to M$, $0 \neq Im\, f \subseteq M$ and the essential inclusion $\bigoplus_{\alpha\in\mathfrak{A}} M_\alpha \subseteq^* M$ implies $\big(\bigoplus_{\alpha\in\mathfrak{A}} M_\alpha\big) \cap Im\, f \neq 0$. Therefore, there exists an element $0 \neq m_{\alpha_1} + \cdots + m_{\alpha_k} \in Im\, f$ with $m_{\alpha_i} \in M_{\alpha_i}$. Then $0 \neq Rm_{\alpha_1} + \cdots + Rm_{\alpha_k} \subseteq Im\, f$ and it is obvious that there exists $\alpha_i \in \mathfrak{A}$ such that $0 \neq Rm_{\alpha_i} \subseteq Im\, f$. So we obtain a non-zero homomorphism from $f^{-1}(Rm_{\alpha_i})$ in M_{α_i}:

$$X \supseteq f^{-1}(Rm_{\alpha_i}) \xrightarrow{f} \bigoplus_{\alpha \in \mathfrak{A}} M_\alpha \xrightarrow{p_{\alpha_i}} M_{\alpha_i},\ M_{\alpha_i} \in \mathcal{K}_{\alpha_i}.$$

On the other hand, since $X \in \bigwedge_{\alpha \in \mathfrak{A}} (\mathcal{K}_\alpha^{\uparrow})$, we obtain $X \in \mathcal{K}_{\alpha_i}^{\uparrow}$, where $\mathcal{K}_\alpha^{\uparrow}$ is a hereditary class, so $f^{-1}(Rm_{\alpha_i}) \in \mathcal{K}_{\alpha_i}^{\uparrow}$. This means that $f^{-1}(Rm_{\alpha_i})$ has no non-zero homomorphism in the modules of $\mathcal{K}_{\alpha_i}$, a contradiction. □

From Theorem 2.5 and Lemma 2.4 immediatly follows

Corollary 2.6. *The mapping ϕ preserves the unions, i.e.*

$$\Big(\bigvee_{\alpha \in \mathfrak{K}} \mathcal{K}_\alpha \Big)^{\uparrow\downarrow} = \bigvee_{\alpha \in \mathfrak{K}} (\mathcal{K}_\alpha^{\uparrow\downarrow})$$

for every set $\{\mathcal{K}_\alpha \,|\, \alpha \in \mathfrak{A}\}$ of natural classes.

References

[1] Dauns J. *Module types.* Rocky Mountain J. of Math., 1997, **27**, N 2, p. 503–557.

[2] Dauns J. *Lattices of classes of modules.* Commun. in Algebra, 1999, **27**, N 9, p. 4363–4387.

[3] Zhou Y. *The lattice of natural classes of modules.* Commun. in Algebra, 1996, **24**, N 5, p. 1637–1648.

[4] Garcia A.A., Rincon H., Montes J.R. *On the lattices of natural and conatural classes in R-Mod.* Commun. in Algebra, 2001, **29**, N 2, p. 541–556.

[5] Stenström B. *Rings of quotients.* Springer Verlag, Berlin, 1975.

[6] Bican L., Kepka P., Nemec P. *Rings, modules and preradicals.* Marcell Dekker, New York, 1982.

[7] Golan J.S. *Torsion theories.* Longman Sci. Techn., New York, 1986.

[8] Kashu A.I. *Radicals and torsions in modules.* Kishinev, Ştiinţa, 1983 (in Russian).

[9] Kashu A.I. *On natural classes of modules.* Bul. A.Ş.R.M., Matematica, 2004, N 2(45), p. 95–101.

[10] Kashu A.I. *On the lattice of closed classes of modules.* Bul. A.Ş.R.M., Matematica, 2005, N (48), p. 43–50.

Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
Academiei str. 5, MD-2028, Chişinău
Moldova
E-mail: *kashuai@math.md*

Received October 16, 2006

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(52), 2006, Pages 109–116
ISSN 1024–7696

Finite difference schemes for problems of mixture of two component elastic materials

Ghenadie Bulgac

Abstract. In this paper we consider the numerical approximation of the solution of the 2D unsteady equations of mixture on a rectangular domain using the operator-splitting schemes for solving unsteady elasticity problems. Its major peculiarity is that transition to the next time level is performed by solving separate elliptic problems for each component of the displacement vector. The previous results make it possible to design efficient numerical algorithms for two component mixture elasticity equations.

Mathematics subject classification: 74H15, 93A30.
Keywords and phrases: Two component mixture model, finite difference method, unsteady elasticity equations.

1 Introduction

The continuum theory of mixtures has been a subject of study in recent years. The linearized theory of elasticity for the indicated medium was given by T.R. Steel. [1] The two-dimensional problems for the isotropic mixture are considered by T.R. Steel [2] and M.O. Basheleishvili [3]. Some three-dimensional basic problems for indicated medium are considered by D.G. Natroshvili, A.J. Jagmaidze and M.J. Svanadze [4].

In this work, we develop our study using the finite difference methodology for spaces discretization. For dynamic problems of continuum mechanics the unsteady system of elastic mixture equations is used. These equations constitute a hyperbolic system of equations of second order. Stability analysis of the proposed schemes is made in framework of the general theory of stability for operator-difference schemes [5]. Discretization in space is performed in such a way that all basic properties of the differential operator are preserved in the corresponding grid Hilbert spaces. Finally, an additive scheme (of predictor-corrector type) is constructed using a triangular splitting for the discrete matrix operator.

2 Differential problem

For simplicity let us treat the transient problem of elasticity of mixture where there is no dependence on the longitudinal coordinate. Let us then consider the stressed state of an elastic isotropic body of mixture with rectangular section Ω. In the two-dimensional case the basic equations of the theory of the elastic mixture have the form [6–8]:

$$\begin{aligned}
&\rho_{11}\frac{\partial^2 u}{\partial t^2}-\rho_{12}\frac{\partial^2 v}{\partial t^2}+\alpha\left(\frac{\partial u}{\partial t}-\frac{\partial v}{\partial t}\right)-\\
&-\left(a_1\,\Delta u+b_1\,grad\,div\,u+c\,\Delta v+d\,grad\,div\,v\right)=f_1\left(x,t\right),\\
&\rho_{22}\frac{\partial^2 v}{\partial t^2}-\rho_{12}\frac{\partial^2 u}{\partial t^2}-\alpha\left(\frac{\partial u}{\partial t}-\frac{\partial v}{\partial t}\right)-\\
&-\left(c\,\Delta u+d\,grad\,div\,u+a_2\,\Delta v+b_2\,grad\,div\,v\right)=f_2\left(x,t\right),
\end{aligned}\tag{1}$$

where $u=(u_1,u_2)$, $v=(v_1,v_2)$ are partial displacements, a_1, b_1, c, d, a_2, b_2 are the known constants characterizing the physical properties of the mixture, Δ is the two-dimensional Laplacian, f is the vector of volumetric forces, *grad* and *div* are the operators on the field theory, ρ_1 and ρ_2 are the partial densities (positive constants), $\alpha\geq 0$,

$$a_j=\mu_j-\lambda_5,\quad b_j=\mu_j+\lambda_j+\lambda_5+\frac{(-1)^j\,\rho_{3-j}\alpha_2}{\rho_1+\rho_2},$$

$$\rho_{jj}=\rho_j+\rho_{12},\quad j=1,2,\quad c=\mu_3+\lambda_5,$$

$$d=\mu_3+\lambda_3+\lambda_5-\frac{\rho_1\alpha_2}{\rho_1+\rho_2}=\mu_3+\lambda_4-\lambda_5+\frac{\rho_2\alpha_2}{\rho_1+\rho_2},\quad \alpha_2=\lambda_3-\lambda_4,$$

μ_1, μ_2, μ_3, λ_1, λ_2, ..., λ_5 are elastic constants of the mixture [1, 6, 10].

In the sequel it will be assumed that the following conditions are fulfilled [1, 6, 10]:

$$\begin{aligned}
&\mu_1>0,\quad \mu_1\mu_2>\mu_3^2,\quad \lambda_1-\frac{\rho_2\alpha_2}{\rho_1+\rho_2}+\frac{2}{3}\mu_1>0,\\
&\lambda_5\leq 0,\quad \rho_{11}>0,\quad \rho_{11}\rho_{22}>\rho_{12}^2,\\
&\left(\lambda_1-\frac{\rho_2\alpha_2}{\rho_1+\rho_2}+\frac{2}{3}\mu_1\right)\left(\lambda_2+\frac{\rho_1\alpha_2}{\rho_1+\rho_2}+\frac{2}{3}\mu_2\right)>\left(\lambda_3-\frac{\rho_1\alpha_2}{\rho_1+\rho_2}+\frac{2}{3}\mu_3\right)^2.
\end{aligned}\tag{2}$$

The system of equations (1) is supplemented with the corresponding boundary and initial conditions. Namely, assume that the boundary $\partial\Omega$ is fixed, i.e. there is no displacement

$$u\left(x,t\right)=0,\quad v(x,t)=0,\quad x\in\partial\Omega.\tag{3}$$

The initial state is specified by

$$u\left(x,t\right)=u^0\left(x\right),\quad v\left(x,t\right)=v^0\left(x\right),\quad x\in\Omega,\tag{4}$$

$$\frac{\partial u}{\partial t}\left(x,0\right)=u^1(x),\quad \frac{\partial v}{\partial t}\left(x,0\right)=v^1(x),\quad x\in\Omega.\tag{5}$$

To formulate the operator for (1)-(5), we first introduce appropriate functional spaces and operators. Let us consider the standard Hilbert space $L_2(\Omega)$ the set of square-integrable scalar valued functions defined on Ω, with the scalar product and the corresponding norm

$$(u,v) = \int_\Omega u(x)\, v(x)\, dx, \quad \|u\| = (u,u)^{\frac{1}{2}},$$

and the Hilbert space $\mathrm{H} = (L_2(\Omega))^4$ with the inner product for 4D vector valued functions u and v, given by

$$(u,v) = \sum_{i=1}^{4} (u_i, v_i)$$

$\overset{0}{W}{}_2^1(\Omega)$ denotes the usual Sobolev space of functions vanishing at the boundary $\partial\Omega$, with the inner product and norm defined by

$$(u,v)_{\overset{0}{W}{}_2^1(\Omega)} = \sum_{\alpha=1}^{2} \int_\Omega \frac{\partial u}{\partial x_\alpha}\frac{\partial v}{\partial x_\alpha} dx, \quad \|u\|_{\overset{0}{W}{}_2^1(\Omega)} = (u,u)^{\frac{1}{2}}_{\overset{0}{W}{}_2^1(\Omega)},$$

and let $V = \left(\overset{0}{W}{}_2^1(\Omega)\right)^4$.

On the H we consider the unbounded operator written in operator matrix form as

$$A\,v = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} v. \tag{6}$$

Where (see (1))

$$A_{11} = -(a_1+b_1)\frac{\partial^2}{\partial x_1^2} - b_1\frac{\partial^2}{\partial x_2^2}, \quad A_{12} = A_{21} = -b_1\frac{\partial^2}{\partial x_1 \partial x_2},$$

$$A_{22} = -b_1\frac{\partial^2}{\partial x_1^2} - (a_1+b_1)\frac{\partial^2}{\partial x_2^2}, \quad A_{13} = A_{31} = -(c+d)\frac{\partial^2}{\partial x_1^2} - d\frac{\partial^2}{\partial x_2^2},$$

$$A_{14} = A_{23} = A_{32} = A_{4'1} = -d\frac{\partial^2}{\partial x_1 \partial x_2}, \quad A_{24} = A_{42} = -d\frac{\partial^2}{\partial x_1^2} - (c+d)\frac{\partial^2}{\partial x_2^2},$$

$$A_{33} = -(a_2+b_2)\frac{\partial^2}{\partial x_1^2} - b_2\frac{\partial^2}{\partial x_2^2}, \quad A_{34} = A_{43} = -b_2\frac{\partial^2}{\partial x_1 \partial x_2},$$

$$A_{44} = -b_2\frac{\partial^2}{\partial x_1^2} - (a_2+b_2)\frac{\partial^2}{\partial x_2^2}$$

The operator A has the domain $D(A) = \{v \in V \,|Av \in H\}$ dense in H.

We have $(Av, v) \geq 0$. In this situation we will write $A \geq 0$ in H.

Besides, it is known that A is maximal monotone, and

$$(Av, u) = (v, Au)\,,$$

i.e., A is selfadjoint in H.

Finally, the following energetic equivalence holds

$$-b\left(\widetilde{\Delta}\, v, v\right) \leq (Av, v) \leq -\,(a+b)\left(\widetilde{\Delta}\, v, v\right), \tag{7}$$

where

$$\widetilde{\Delta} = \begin{pmatrix} \Delta & 0 & 0 & 0 \\ 0 & \Delta & 0 & 0 \\ 0 & 0 & \Delta & 0 \\ 0 & 0 & 0 & \Delta \end{pmatrix}$$

and $b = \min\{b_1, d, b_2\}$, $(a+b) = \max\{(a_1+b_1), (c+d), (a_2+b_2)\}$.

Problem (1)-(5) can be written in differential operator form as the abstract initial value problem

$$\rho\frac{d^2v}{dt^2} + \alpha\frac{dv}{dt} + Av = f, \tag{8}$$

$$v(0) = v^0, \quad \frac{dv}{dt}(0) = v^1, \tag{9}$$

with the unique solution if $v^0 \in D(A)$ and $v^1 \in H$.

The operator A is selfadjoint and positive on space H and, moreover, is energetically equivalent to the analog for Laplace operator. The construction of discrete analogs for A will be oriented to the fulfillment of the same important properties.

3 Space discretization

In considering difference schemes for the solution of problem (1)-(5), we begin with making space approximation. We consider the problem on the rectangle

$$\Omega = \{x \mid x = (x_1, x_2)\,, \quad 0 < x_\alpha < l_a,\ \alpha = 1, 2\}$$

discretized by a uniform rectangular grid mesh steps h_α, $\alpha = 1, 2$. Let ω be the set of internal nodes of the grid

$$\omega = \{x \mid x = (x_1, x_2)\,,\, x_\alpha = i_\alpha h_\alpha,\, i_\alpha = 1,\, 2...,\, N_\alpha - 1,\, N_\alpha h_\alpha = l_\alpha,\, \alpha = 1, 2\}\,,$$

and the $\partial\omega$ the set of boundary nodes. The finite difference solution of problem (1)-(4) will be denoted by $v_h(x,t)$, $x \in \omega \cup \partial\omega$, $0 < t \leq T$. Using the standard index-free notation of the theory of difference schemes [8], for the right and left difference derivatives we write

$$w_x = \frac{w(x+h) - w(x)}{h}, \quad w_{\overline{x}} = \frac{w(x) - w(x-h)}{h},$$

and the second difference derivative is given by the expression

$$w_{\overline{x}x} = \frac{1}{h}\left(w_x - w_{\overline{x}}\right) = \frac{w\left(x+h\right) - 2w\left(x\right) + w\left(x-h\right)}{h^2}.$$

For grid functions equal to zero on $\partial\omega$ we define the Hillbert space $L_2\left(\omega\right)$ where the inner product and norm are as follows

$$\left(y, w\right) = \sum_{x\in\omega} y\left(x\right) w\left(x\right) h_1 h_2, \quad \|y\| = \left(y, y\right)^{\frac{1}{2}}.$$

For the vector grid functions $u\left(x\right)$, $v\left(x\right)$ equal to zero on $\partial\omega$ we introduce $\tilde{H} = \left(L_2\left(\omega\right)\right)^4$ with the inner product and norm given by

$$\left(u, v\right) = \left(u_1, v_1\right) + \left(u_2, v_2\right), \quad \|u\| = \left(u, u\right)^{\frac{1}{2}}.$$

Also, given a self-adjoint and positive definite operator C, $\tilde{H}_C$ denotes the space $\tilde{H}$ provided by the scalar product $\left(u, v\right)_C = \left(Cu, v\right)$ and norm $\|u\|_C = \left(Cu, u\right)^{\frac{1}{2}}$.

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} & \tilde{A}_{14} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} \\ \tilde{A}_{31} & \tilde{A}_{32} & \tilde{A}_{33} & \tilde{A}_{34} \\ \tilde{A}_{41} & \tilde{A}_{42} & \tilde{A}_{43} & \tilde{A}_{44} \end{pmatrix}, \tag{10}$$

$$\tilde{A}_{11} y = -a_1 y_{\overline{x_1}x_1} - b_1 \Delta_h y, \quad \tilde{A}_{12} y = \tilde{A}_{21} y = -\frac{b_1}{2}\left(y_{\overline{x_1}x_2} + y_{x_1\overline{x_2}}\right),$$

$$\tilde{A}_{22} y = -b_1 \Delta_h y - a_1 y_{\overline{x_2}x_2},$$

$$\tilde{A}_{13} y = \tilde{A}_{31} y = -c y_{\overline{x_1}x_1} - d\Delta_h y,$$

$$\tilde{A}_{14} y = \tilde{A}_{23} y = \tilde{A}_{32} y = \tilde{A}_{41} y = -\frac{d}{2}\left(y_{\overline{x_1}x_2} + y_{x_1\overline{x_2}}\right),$$

$$\tilde{A}_{24} y = \tilde{A}_{42} y = -d\Delta_h y - c y_{\overline{x_2}x_2},$$

$$\tilde{A}_{33} y = -a_2 y_{\overline{x_1}x_1} - b_2 \Delta_h y, \quad \tilde{A}_{34} y = \tilde{A}_{43} = -\frac{b_1}{2}\left(y_{\overline{x_1}x_2} + y_{x_1\overline{x_2}}\right),$$

$$\tilde{A}_{44} y = -b_1 \Delta_h y - a_1 y_{\overline{x_2}x_2}.$$

Here, we use the standard 5-point approximation of the Laplace operator

$$\Delta_h y = y_{\overline{x_1}x_1} + y_{x_2\overline{x_2}}.$$

For the grid functions $u(x)$ and $v(x)$ from $\tilde{H}$ we have

$$\left(\tilde{A}v, u\right) = \left(v, \tilde{A}u\right),$$

i.e., the operator $\tilde{A}$ is selfadjoint.

Besides, we have

$$-b\left(\tilde{\Delta}_h v, v\right) \leq \left(\tilde{A}v, v\right) \leq -\left(a+b\right)\left(\tilde{\Delta}_h v, v\right), \tag{11}$$

where

$$\tilde{\Delta}_h = \left(\begin{array}{cccc} \Delta_h & 0 & 0 & 0 \\ 0 & \Delta_h & 0 & 0 \\ 0 & 0 & \Delta_h & 0 \\ 0 & 0 & 0 & \Delta_h \end{array}\right).$$

The relation (11) is a discrete analog of (7) given for the differential operator $\tilde{A}$. We approximate the differential operator A by the difference operator $\tilde{A}$, a self-adjoint and positive definite operator.

After approximation in space and denoting by $u\left(x,t\right)$, $x \in \omega \cup \partial\omega$, $0 < t \leq T$, the semi-discrete solution at time t, we have the initial value problem

$$\rho\frac{d^2u}{dt^2} + \alpha\frac{du}{dt} + \tilde{A}u = f\left(x,t\right), \quad x \in \omega, \quad 0 < t \leq T, \tag{12}$$

$$u\left(0\right) = v_o\left(x\right), \quad \frac{du}{dt}\left(x,0\right) = v_1(x), \quad x \in \omega. \tag{13}$$

4 Approximation in time

For simplicity, we consider a uniform grid in $[0,T]$, with step $\tau > 0$. Let $u_n\left(x\right) = u\left(x, t_n\right)$, $t_n = n\tau$, $n = 0, 1, \ldots, N$, $N\tau = T$. The simplest second-order scheme for problem (12),(13) is

$$\rho\frac{u_{n+1} - 2u_n + u_{n-1}}{\tau^2} + \alpha\frac{u_{n+1} - u_{n-1}}{2\tau} + \tilde{A}u_n = f_n, \quad n = 1, 2, \ldots, N, \tag{14}$$

with prescribed u_0, u_1.

Let us highlight the class of additive schemes called alternating triangular methods. The schemes of this type for evolutionary equations of the first order have been proposed and investigated by A.A. Samarskii in [11]. Here we consider the possibilities of using this approach to construct additive schemes for system of second-order equations.

The alternating triangular method is constructed on the basis of the operator splitting:

$$\tilde{A} = \tilde{A}^{(1)} + \tilde{A}^{(2)}, \quad \left(\tilde{A}^{(1)}\right)^* = \tilde{A}^{(2)}, \tag{15}$$

where, taking into account (10), we define

$$\tilde{A}^{(1)} = \begin{pmatrix} \frac{1}{2}\tilde{A}_{11} & 0 & 0 & 0 \\ \tilde{A}_{21} & \frac{1}{2}\tilde{A}_{22} & 0 & 0 \\ \tilde{A}_{31} & \tilde{A}_{32} & \frac{1}{2}\tilde{A}_{33} & 0 \\ \tilde{A}_{41} & \tilde{A}_{42} & \tilde{A}_{43} & \frac{1}{2}\tilde{A}_{44} \end{pmatrix},$$

$$\tilde{A}^{(1)} = \begin{pmatrix} \frac{1}{2}\tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} & \tilde{A}_{14} \\ 0 & \frac{1}{2}\tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} \\ 0 & 0 & \frac{1}{2}\tilde{A}_{33} & \tilde{A}_{34} \\ 0 & 0 & 0 & \frac{1}{2}\tilde{A}_{44} \end{pmatrix}. \quad (16)$$

Let us consider a simple predictor-corrector scheme for the numerical solution of problem (12), (13). At the predictor stage we calculate $\tilde{u}_{n+1}$ from

$$\rho\frac{u_{n+1} - 2u_n + u_{n-1}}{\tau^2} + \alpha\frac{u_{n+1} - u_{n-1}}{2\tau} + \tilde{A}^{(1)}\frac{\tilde{u}_{n+1} - u_{n-1}}{2} + \tilde{A}^{(2)}u_n = f_n. \quad (17)$$

After that, at the corrector stage, we improve the solution for the next time level:

$$\rho\frac{u_{n+1} - 2u_n + u_{n-1}}{\tau^2} + \alpha\frac{u_{n+1} - u_{n-1}}{2\tau} + $$
$$+\tilde{A}^{(1)}\frac{\tilde{u}_{n+1} - u_{n-1}}{2} + \tilde{A}^{(2)}\frac{u_{n+1} - u_{n-1}}{2} = f_n. \quad (18)$$

Schemes (17), (18) can be written as follows

$$\left(\rho E + \frac{\tau^2}{2}\tilde{A}^{(1)}\right)\frac{1}{\rho}\left(\rho E + \frac{\tau^2}{2}\tilde{A}^{(2)}\right)\frac{u_{n+1} - 2u_n + u_{n-1}}{\tau^2} +$$
$$+\alpha\frac{u_{n+1} - u_{n-1}}{2\tau} + \tilde{A}u_n = f_n, \quad n = 1, 2, \ldots, N,$$

where E denotes the single operator.

The generalization of this scheme is the factorized scheme

$$D\frac{u_{n+1} - 2u_n + u_{n-1}}{\tau^2} + \alpha\frac{u_{n+1} - u_{n-1}}{2\tau} + \tilde{A}u_n = f_n, \quad n = 1, 2, \ldots, N, \quad (19)$$

$$D = \left(\rho E + \sigma\tau^2\tilde{A}^{(1)}\right)\frac{1}{\rho}\left(\rho E + \sigma\tau^2\tilde{A}^{(2)}\right). \quad (20)$$

This schemes is second order in time, since $D = \rho E + \mathrm{O}\left(\tau^2\right)$. To advance to a next time-level, its implementation requires to solve four grid elliptic problems, one for each component of the solution.

References

[1] Steel T.R. *Applications of a theory of interacting continua.* Quart. J. Mech. and Appl. Math., 1967, **20**, N 1, p. 57–72.

[2] Steel T.R. *Linearized theory of plane strain of mixture of two solids.* Int. J. Eng. Scienc., 1967, **5**, N 10, p. 775–790.

[3] Basheleishvili M.O. *Two-Dimensional Boundary Value Problems of Statics of the Elastic Mixtures.* Memories on Diff. Eq. and Math. Physics, Tbilisi, 1995, **6**, p. 59–105.

[4] Natroshvili D.G., Jagmaidze A.J., Svanadze M.Zh. *Some problems of the theory of elastic mixtures.* Tbilisi Univ. Press, Tbilisi, 1986 (in Russian).

[5] Samarskii A.A., Vabishchevich P.N. *Additive Schemes for Problems of Mathematical Physics.* Moskva, Nauka, 1999 (in Russian).

[6] Rushchitskii Ya.Ya. *Elements of mixture theory.* Kiev, Naukova Dumka, 1991 (in Russian).

[7] Lempriere B. *On practice ability of analyzing waves in composites by the theory of mixtures.* Lockheed Palo Alto Research Laboratory. Report No.LMSC-6-78-69-21(1969), p. 76–90.

[8] McNiven H.D., Mengi Y.A. *A mathematical model for the linear dynamic behavior of two-phase periodic materials.* Int. J. Solids and Struct., 1979, **15**, N 4, p. 571–580.

[9] Khoroshunand L.P., Soltanov N.S. *Thermoelasticity of two-component mixtures.* Kiev, Naukova Dumka, 1984 (in Russian).

[10] Filippov I.G. *Dynamical theory of a relative flow of multicomponent media.* Prikl. Mekhanika, 1971, **7**, N 10, p. 92–99 (in Russian).

[11] Samarskii A.A. *An economical algorithm for the numerical solution of system of differential and algebraic equations.* U.S.S.R. Comput. Math., Math. Phys., 1964, **4**, N 3, p. 263–271.

Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
Academiei str. 5, MD-2028 Kishinev
Moldova

E-mail: *bulgac@gmail.com*

Received November 10, 2006

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(52), 2006, Pages 117–120
ISSN 1024–7696

Characteristic functions of Markovian random evolutions in $\mathbb{R}^m$

Alexander D. Kolesnik

Abstract. The recurrent and integral relations for characteristic functions of Markovian random evolution in $\mathbb{R}^m$, $m \geq 2$, are presented.

Mathematics subject classification: Primary 60K99. Secondary 62G30, 60K35, 60J60, 60H30.
Keywords and phrases: Random evolution, characteristic functions, convolutions, Volterra integral equation.

The particular models of random evolutions in various Euclidean spaces of lower dimensions were studied in [1-5]. In this note we announce the recent results on the characteristic functions for the most general m-dimensional random evolution.

The subject of our interests is the following stochastic motion. A particle starts its motion from the origin $x_1 = \cdots = x_m = 0$ of the space $\mathbb{R}^m$, $m \geq 2$ at time $t = 0$. The particle is endowed with constant, finite speed c. The initial direction is a random m-dimensional vector with uniform distribution on the unit m-sphere

$$S_1^m = \left\{ \mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m : \ x_1^2 + \cdots + x_m^2 = 1 \right\}.$$

The particle changes direction at random instants which form a homogeneous Poisson process of rate $\lambda > 0$. At these moments it instantaneously takes on the new direction with uniform distribution on S_1^m, independently of its previous motion.

Let $\mathbf{X}(t) = (X_1(t), \dots, X_m(t))$ be the position of the particle at an arbitrary time $t > 0$. At first, we concentrate our attention on the conditional distributions

$$Pr\{\mathbf{X}(t) \in d\mathbf{x} \mid N(t) = n\} =$$

$$= Pr\{X_1(t) \in dx_1, \dots, X_m(t) \in dx_m \mid N(t) = n\}, \quad n \geq 1$$

where $N(t)$ is the number of Poisson events that have occurred in the interval $(0, t)$ and $d\mathbf{x} = dx_1 \dots dx_m$ is the infinitesimal volume in the space $\mathbb{R}^m$.

Consider the conditional characteristic functions:

$$H_n(t) = E\left\{ e^{i(\boldsymbol{\alpha}, \mathbf{X}(t))} \mid N(t) = n \right\}, \qquad n \geq 1, \tag{1}$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ is the real m-dimensional vector of inversion parameters and $(\boldsymbol{\alpha}, \mathbf{X}(t))$ denotes the scalar (inner) product of the vectors $\boldsymbol{\alpha}$ and $\mathbf{X}(t)$.

Computing the expectation in (1) we obtain

$$H_n(t) = \frac{n!}{t^n} \int_0^t d\tau_1 \int_{\tau_1}^t d\tau_2 \dots \int_{\tau_{n-1}}^t d\tau_n \times$$
$$\times \left\{ \prod_{j=1}^{n+1} \left[2^{(m-2)/2} \Gamma\left(\frac{m}{2}\right) \frac{J_{(m-2)/2}(c(\tau_j - \tau_{j-1})\|\boldsymbol{\alpha}\|)}{(c(\tau_j - \tau_{j-1})\|\boldsymbol{\alpha}\|)^{(m-2)/2}} \right] \right\}. \qquad (2)$$

For the particular cases $m = 2$ (planar motion) and $m = 4$ (four-dimensional motion) the conditional characteristic functions (2) were explicitly computed in [2] (see formula (18) therein), and in [1] (see formula (15) therein), respectively.

We introduce the function

$$\varphi(t) = 2^{(m-2)/2}\, \Gamma\left(\frac{m}{2}\right) \frac{J_{(m-2)/2}(ct\|\boldsymbol{\alpha}\|)}{(ct\|\boldsymbol{\alpha}\|)^{(m-2)/2}}, \qquad m \geq 2. \qquad (3)$$

Then (2) can rewritten in the following form

$$H_n(t) = \frac{n!}{t^n} \int_0^t d\tau_1 \int_{\tau_1}^t d\tau_2 \dots \int_{\tau_{n-1}}^t d\tau_n \left\{ \prod_{j=1}^{n+1} \varphi(\tau_j - \tau_{j-1}) \right\}, \qquad n \geq 1. \qquad (4)$$

Denote the integral factor in (4) as follows

$$\mathcal{I}_n(t) = \int_0^t d\tau_1 \int_{\tau_1}^t d\tau_2 \dots \int_{\tau_{n-1}}^t d\tau_n \left\{ \prod_{j=1}^{n+1} \varphi(\tau_j - \tau_{j-1}) \right\}, \qquad n \geq 1. \qquad (5)$$

The following theorem states that, for different $n \geq 1$, the functions (5) are connected with each other by a convolution-type recurrent relation.

Theorem 1. *For any $n \geq 1$ the following recurrent relation holds*

$$\mathcal{I}_n(t) = \int_0^t \varphi(t-\tau)\, \mathcal{I}_{n-1}(\tau)\, d\tau = \int_0^t \varphi(\tau)\, \mathcal{I}_{n-1}(t-\tau)\, d\tau, \qquad n \geq 1, \qquad (6)$$

where, by definition, $\mathcal{I}_0(x) = \varphi(x)$.

Note that formula (6) can be rewritten in the following convolution form

$$\mathcal{I}_n(t) = \varphi(t) * \mathcal{I}_{n-1}(t), \qquad n \geq 1. \qquad (7)$$

Corollary 1.1. *For any $n \geq 1$ the following relation holds*

$$\mathcal{I}_n(t) = [\varphi(t)]^{*(n+1)}, \qquad n \geq 1, \qquad (8)$$

where the symbol $*(n+1)$ means the $(n+1)$-multiple convolution.

Application of the Laplace transform

$$\mathcal{L}\,[f(t)]\,(s) = \int_0^\infty e^{-st} f(t)\, dt, \qquad \text{Re}\, s > 0,$$

to the equality (8) leads to the following important result.

Corollary 1.2. *For any $n \geq 1$ the Laplace transform of functions (5) has the form*

$$\mathcal{L}\left[\mathcal{I}_n(t)\right](s) = \left(\mathcal{L}\left[\varphi(t)\right](s)\right)^{n+1}, \qquad n \geq 1. \tag{9}$$

These results show that the function $\varphi(t)$ given by (3) plays a key role in our analysis. The reason is that $\varphi(t)$ is exactly the characteristic function (Fourier transform) of the uniform distribution on the surface of the m-sphere S^m_{ct} of the radius ct.

From both the Theorem 1 and its corollaries we see that the conditional characteristic functions $H_n(t)$ and their Laplace transforms, in fact, are expressed in terms of function $\varphi(t)$. Formula (9) shows that the possibility of obtaining the explicit form of the conditional characteristic functions (4) entirely depends on whether the exact Laplace transform of the function $\varphi(t)$ and its inverse Laplace transform can be explicitly computed.

Our next result presents a general formula for the conditional characteristic functions $H_n(t)$ in terms of inverse Laplace transform.

Theorem 2. *For any $n \geq 1$ and any $t > 0$ the conditional characteristic functions (4) are given by*

$$H_n(t) = \frac{n!}{t^n}\,\mathcal{L}^{-1}\left[\left(\frac{1}{\sqrt{s^2+(c\|\boldsymbol{\alpha}\|)^2}}\,F\left(\frac{1}{2},\frac{m-2}{2};\frac{m}{2};\frac{(c\|\boldsymbol{\alpha}\|)^2}{s^2+(c\|\boldsymbol{\alpha}\|)^2}\right)\right)^{n+1}\right](t), \tag{10}$$

where $\mathcal{L}^{-1}$ means the inverse Laplace transform and

$$F(\xi,\eta;\zeta;z) = {}_2F_1(\xi,\eta;\zeta;z) = \sum_{k=0}^{\infty}\frac{(\xi)_k(\eta)_k}{(\zeta)_k}\frac{z^k}{k!}$$

is the standard hypergeometric function.

In view of (4), the characteristic function of $\mathbf{X}(t),\ t \geq 0$, is given by the uniformly converging series

$$H(t) = e^{-\lambda t}\sum_{n=0}^{\infty}\lambda^n\,\mathcal{I}_n(t). \tag{11}$$

The following theorem presents the integral equation for the function $H(t)$.

Theorem 3. *The characteristic function $H(t),\ t \geq 0$, satisfies the following convolution-type Volterra integral equation of second kind with the kernel $e^{-\lambda t}\varphi(t)$:*

$$H(t) = e^{-\lambda t}\varphi(t) + \lambda\int_0^t e^{-\lambda(t-\tau)}\varphi(t-\tau)H(\tau)\,d\tau, \qquad t \geq 0. \tag{12}$$

The integral equation (12) can be rewritten in the following convolution form

$$H(t) = e^{-\lambda t}\varphi(t) + \lambda\left[\left(e^{-\lambda t}\varphi(t)\right) * H(t)\right], \qquad t \geq 0. \tag{13}$$

From this we immediately obtain the general formula for the Laplace transform of the characteristic function $H(t)$:

$$\mathcal{L}\left[H(t)\right](s) = \frac{\mathcal{L}\left[\varphi(t)\right](s+\lambda)}{1-\lambda\,\mathcal{L}\left[\varphi(t)\right](s+\lambda)}\,, \qquad \operatorname{Re} s > 0. \tag{14}$$

The explicit form of (14) is

$$\mathcal{L}\left[H(t)\right](s) = \frac{F\left(\frac{1}{2}, \frac{m-2}{2}; \frac{m}{2}; \frac{(c\|\boldsymbol{\alpha}\|)^2}{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2}\right)}{\sqrt{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2} - \lambda\, F\left(\frac{1}{2}, \frac{m-2}{2}; \frac{m}{2}; \frac{(c\|\boldsymbol{\alpha}\|)^2}{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2}\right)}. \tag{15}$$

From (11) and (8) it follows that the solution of equation (13) has the form

$$H(t) = e^{-\lambda t}\sum_{n=0}^{\infty}\lambda^n \; [\varphi(t)]^{*(n+1)} \; . \tag{16}$$

One should emphasize that, although formula (16) gives a general form of the characteristic function $H(t)$, the multiple convolutions of the function $\varphi(t)$ with itself can scarcely be explicitly evaluated for arbitrary dimension.

From (12) we can see that

$$H(t)\big|_{t=0} = 1, \qquad \left.\frac{\partial H(t)}{\partial t}\right|_{t=0} = 0,$$

and, therefore, the transition density $f(\mathbf{x},t), \; \mathbf{x}\in\mathbb{R}^m, \; t\ge 0$, of the process $\mathbf{X}(t)$ satisfies the following initial conditions

$$f(\mathbf{x},t)\big|_{t=0} = \delta(\mathbf{x}), \qquad \left.\frac{\partial f(\mathbf{x},t)}{\partial t}\right|_{t=0} = 0,$$

where $\delta(\mathbf{x})$ is the m-dimensional Dirac delta-function.

References

[1] Kolesnik A.D. *A four-dimensional random motion at finite speed.* J. Appl. Prob., 2006, **43**.

[2] Kolesnik A.D., Orsingher E. *A planar random motion with an infinite number of directions controlled by the damped wave equation.* J. Appl. Prob., 2005, **42**, p. 1168–1182.

[3] Masoliver J., Porrá J.M., Weiss G.H. *Some two and three-dimensional persistent random walks.* Physica A, 1993, **193**, p. 469–482.

[4] Stadje W. *The exact probability distribution of a two-dimensional random walk.* J. Stat. Phys., 1987, **46**, p. 207–216.

[5] Stadje W. *Exact probability distributions for non-correlated random walk models.* J. Stat. Phys., 1989, **56**, p. 415–435.

Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
Academiei str. 5, MD-2028 Kishinev
Moldova

E-mail: *kolesnik@math.md*

Received November 3, 2006

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(52), 2006, Pages 121–123
ISSN 1024–7696

Nearly simple elementary divisor domains

B.V. Zabavsky, T.N. Kysil'

Abstract. It is proved that a nearly simple Bezout domain is an elementary divisor ring if and only if it is 2-simple.

Mathematics subject classification: 16S50, 15A21.
Keywords and phrases: Bezout domain, elementary divisor ring, 2-simple domain.

1 Introduction

According to Kaplansky's definition [1], a ring R is an elementary divisor ring if every matrix over R is equivalent to a diagonal matrix with condition of complete divisibility of the diagonal elements. In [2] Zabavsky proved that a simple Bezout domain is an elementary divisor ring if and only if it is 2-simple. Nearly simple domains were constructed in [3–6]. We prove that a nearly simple Bezout domain is an elementary divisor ring if and only if it is 2-simple.

2 Definitions

Throughout R will always denote a ring (associative, but not necessarily commutative) with $1 \neq 0$. We shall write R_n for the ring of $n \times n$ matrices with elements in R. By a unit of ring we mean an element with two-sided inverse. We'll say that matrix is unimodular if it is the unit of R_n. We denote by $GL_n(R)$ the group of units of R_n. The Jacobson radical of a ring R is denoted by $J(R)$.

An n by m matrix $A = (a_{ij})$ is said to be diagonal if $a_{ij} = 0$ for all $i \neq j$. We say that a matrix A admits a diagonal reduction if there exist unimodular matrices $P \in GL_n(R)$, $Q \in GL_m(R)$ such that PAQ is a diagonal matrix. We shall call two matrices A and B over a ring R equivalent (and write $A \sim B$) if there exist unimodular matrices P, Q such that $B = PAQ$. If every matrix over R is equivalent to a diagonal matrix (d_{ij}) with the property that every d_{ii} is a total divisor of $d_{i+1,i+1}$ ($Rd_{i+1,i+1}R \subseteq d_{ii}R \cap Rd_{ii}$), then R is an elementary divisor ring. We recall that a ring R is said to be right (left) Hermite if every 1 by 2 (2 by 1) matrix admits a diagonal reduction, and if both, R is an Hermite ring. By a right (left) Bezout ring we mean a ring in which all finitely generated right (left) ideals are principal, and by a Bezout ring a ring which is both right and left Bezout [1].

In any simple ring the property $RaR = R$ holds for every element $a \in R \setminus \{0\}$ and some depends on a. As R is a ring with identity then there exist elements

$u_1, \dots u_k, v_1, \dots, v_k$ such that $u_1 a v_1 + \dots + u_k a v_k = 1$. If the same integer n can be chosen for all nonzero elements a with $u_1 a v_1 + \dots + u_n a v_n = 1$ we say that a ring R is n-simple [3]. For example the full n by n matrix ring over a field K (even a skew field) is n-simple. A nearly simple ring is a ring in which case R, $J(R)$ and (0) are its only ideals.

3 Main result

Main result is the next theorem.

Theorem. *Let R be a nearly simple Bezout domain. Then R is an elementary divisor domain if and only if R is 2-simple domain.*

Proof. If $J(R) = (0)$ then R is a simple domain and the result follows by [2].

If $J(R) \neq (0)$ and R is an elementary divisor domain then it is enough to consider the matrix A of the form

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},$$

where $a \in J(R) \setminus \{0\}$. Since R is an elementary divisor domain there exist matrices $P = (p_{ij}) \in GL_2(R)$ and $Q = (q_{ij}) \in GL_2(R)$ such that

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} P = Q \begin{pmatrix} z & 0 \\ 0 & b \end{pmatrix}, \tag{1}$$

where $RbR \subseteq zR \cap Rz$ for some $z, b \in R$.

Let's consider the ideal RbR. Since R is a nearly simple domain, we obtain three chances:

1) $RbR = \{0\}$;
2) $RbR = R$;
3) $RbR = J(R)$.

1) Let $RbR = \{0\}$ then $b = 0$. From (1) we have

$$ap_{12} = q_{12}b, \quad ap_{22} = q_{22}b. \tag{2}$$

Since $b = 0$ and (2),

$$ap_{12} = 0, \quad ap_{22} = 0. \tag{3}$$

As $a \neq 0$ and R is a domain then $p_{12} = p_{22} = 0$, this case is impossible.

2) Let $RbR = R$. Since $RbR \subseteq zR \cap Rz$, z is a unit of the domain R. Then from (1) we obtain that $z \in RaR$. Since $a \in J(R)$, $z \in J(R)$. And this case is impossible too.

3) Let $RbR = J(R)$. Since $RbR \subseteq zR \cap Rz$, $a \in J(R)$ and (1), $z \in J(R)$. Then $J(R) = zR = Rz$. Also $z^2R = Rz^2$ takes place. Then $z^2R = Rz^2 = J(R) = zR = Rz$, that is impossible as R is a domain and $z \in J(R)$.

The proof is completed.

References

[1] KAPLANSKY I. *Elementary divisors and modules.* Trans. Amer. Math. Soc., 1949, **66**, p. 464–491.

[2] ZABAVSKY B.V. *Simple elementary diviser rings.* Matematicni studii, 2004, **22**, N 2, p. 129–133 (in Russian).

[3] BRUNGS H.H., SCHÖDER M. *Prime segments of skew fields.* Canad. J. Math., 1995, **47**, p. 1148–1167.

[4] BRUNGS H.H., TÖRNER G. *Extensions of chain rings.* Math. Zeit., 1984, **185**, p. 93–104.

[5] MATHIAK K. *Valuations of skew fields and projective Hjelmslev spaces.* Lectures Notes in Math., 1986, Springer–Verlag, 1175.

[6] DUBROVIN N.I. *Chain domains.* Moscow Univ. Math. Bull., 1980, Ser.1, **37**, p. 51–54.

[7] BRUNGS H.H., DUBROVIN N.I. *A classification and examples of rank one chain domains.* Trans. Amer. Math. Soc., 2003, **355**, N 7, p. 2733–2753.

[8] COHN P.M. *On n-simple rings.* Alg. univers., 2005, **53**, p. 301–305.

Ukraine
Lviv National University
E-mail: *b_zabava@franko.lviv.ua*

Received November 3, 2006

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(52), 2006, Pages 124–127
ISSN 1024–7696

Factorization theorems for some spaces of analytic functions

R.F. Shamoyan

Abstract. We provide several factorization theorems for different subspaces of the space of all analytic functions in the unit disk, in particular we prove a strong factorization theorem for Classical Hardy classes with Muckenhoupt weights. Proofs are based on a new weighted version of Coifman–Meyer–Stein theorem on factorization of tent spaces and on properties of an extremal outher function,which was constructed by E. Dynkin.

Mathematics subject classification: 46B20, 46E40, 47B35.
Keywords and phrases: Weighted Tent spaces, strong factorization theorems, Muckenhoupt weights, Hardy Spaces.

1 Introduction

The aim of this note is to provide several factorization theorems for different subspaces of the $H(\mathbb{D})$ space, where $\mathbb{D}$ is the unit disk on the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ is the space of all holomorphic in the unit disk functions. Let us mention several vital known results in that direction.

In [1] such theorem was proved by Gorowitz for B^p_α-Bergman spaces. Much later similar result was proven in [2] by W. Cohn and in much more general form by W. Cohn and I. Verbitsky in [3]. Such theorems are playing very important role in different questions in the theory of analytic functions.

2 Definition and main results

In order to formulate the main results of the paper we will give several definitions. Let Z, X and Y be subspaces of $H(\mathbb{D})$. We will say that Z admits strong factorization g from Z can be represented as a product $g = f_1 f_2$, where $f_1 \in X$, $f_2 \in Y$, and the reverse is also true: for any $f_1 \in X$ and $f_2 \in Y$ we have $f_1 f_2 \in Z$, so $Z = XY$.

Let $T = \{z : |z| = 1\}$ be the boundary of $\mathbb{D}$,

$$\Gamma_\alpha(\xi) = \{z : |1 - \bar{\xi} z| \le \alpha(1 - |z|)\}, \quad \alpha > 1,$$

$dm(\xi)$ and $dm_2(z)$ are normalized Lebesgue measures on the boundary T and in the unit disk $\mathbb{D}$,

$$H^p(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \sup_{r \in (0,1)} M_p(f,r) < \infty, p \in (0,\infty] \right\}.$$

Hardy spaces for $p \in (0,\infty]$, where

$$M_p^p(f,r) = \int_T |\ f(r\xi)\ |^p\ dm(\xi), \quad r \in [0,1),$$

$$\text{Let} : \square I = \big\{z = r\xi : \xi \in I, 1 - |I| \le r < 1\big\},$$

where I is an arc on T, $|I|$ is a length of the arc. Let further $T(E) = \big\{(x,t) \in \in \mathbb{R}_+^{n+1}, B(x,t) \subset E\big\}$ be a tent on $E, E \subset \mathbb{R}^n$ (see [3]), for example E can be a ball $E = B(x,r)$ in $\mathbb{R}^n$ with center at x and radius r. Denote as usual by $A^p(\mathbb{R}^n)$, $p \in [1,\infty)$ all measurable functions in $\mathbb{R}^n w(x)$ such that w is belonging to Muckenhoupt class (see [5]). $A^p(T)$ is a Muckenhoupt class on T. Further let

$$(A_q f)(\xi) = \bigg(\int_{\Gamma_\alpha(\xi)} \frac{|f(z)|^q}{(1-|z|)^2} dm_2(z)\bigg)^{\frac{1}{q}},$$

$$\big(\tilde{T}_q^\infty(w)\big) = \bigg\{f \text{ measurable in } \mathbb{D} : \sup_{I \subset T} \bigg(\int_I w^{\frac{q}{q-p}} dy\bigg)\bigg(\int_{T(I)} \frac{|f(z)|^q}{1-|z|} dm_2(z)\bigg) < \infty\bigg\},$$

$$\big(C_q^q f\big)(\xi) = \sup_{\xi \in I} \bigg(\frac{1}{|I|}\bigg)\bigg(\int_{\square I} \frac{|f(z)|^q}{(1-|z|)} dm_2(z)\bigg),$$

$$\big(A_\infty f\big)(\xi) = \sup_{\Gamma_\alpha(\xi)} \big\{|f(z)| : z \in \Gamma_\alpha(\xi)\big\},\ I \subset T, I \text{ is an arc.}$$

Theorem 1. *Let* $p < q$, $s > 0$, $w_1 = w^{\frac{q}{q-p}}$, $w_1 \in L^1_{loc}$ *and* $w_1 \in A^1(T)$. *Then* $(HT^p_{s,q})(w) = (H^p(w_1))(HT^\infty_{s,q}(w))$ *where*

$$\big(HT^p_{s,q}\big)(w) = \big\{f \in H(\mathbb{D}) : \|A_q\big(f(z)(1-|z|)^s\big)\|_{L^p(w)} < \infty\big\},$$

$$\big(HT^\infty_{s,q}\big)(w) = \big\{f \in H(\mathbb{D}) : f(z)(1-|z|)^s \in \tilde{T}_q^\infty(w)\big\},$$

$$\big(H^p\big)(w_1) = \bigg\{f \in H(\mathbb{D}) : \int_T |\big(A_\infty f\big)(\xi)|^p w_1(\xi) d\xi < \infty, \quad 0 < p < \infty\bigg\},$$

and moreover if $F = F_1 F_2$, *then* $\|F_1\|_{H^p(w_1)} \le c\|F\|_{HT^p_{s,q}}$ *and* $\|F_2\|_{HT^\infty_{s,q}} \le 1$.

Remark 1. The pair $\big(\omega^{-\frac{p}{q-p}}, \omega^{\frac{q}{q-p}}\big)$ can be changed in Theorem 1 to $\big(\omega^{\tau_1}, \omega^{\tau_2}\big)$, $\tau_1 + \tau_2 = 1$.

The proof of Theorem 1 relies on the following extension of Coifman–Meyer–Stein theotem on factorization of tent spaces and some ideas from the article of W. Cohn and I. Verbitsky.

Let $\Gamma(\xi)$ be Luzin cone in $\mathbb{R}^n$ [3].

Theorem 2. *Let* $0 < p < q, \ \ \omega_1 = \omega^{\frac{q}{q-p}}, \ \ \omega_1 \in L^1_{loc}, \ \ \omega_1 \in A^1(\mathbb{R}^n)$. *Then the following equality holds*

$$\tilde{T}^p_q(\omega) = \tilde{T}^p_\infty\big(\omega^{\frac{q}{q-p}}\big)\big(\tilde{T}^\infty_q(\omega)\big),$$

where

$$\tilde{T}^\infty_q(\omega) = \Bigg\{f \ \ \text{is measurable in} \ \ \mathbb{R}^n : $$
$$\sup_B \bigg(\int_B \omega^{\frac{q}{q-p}} dy\bigg)^{-1} \int_{T(B)} \frac{|f(x,t)|^q}{t} dx \ dt < \infty\Bigg\},$$

$$\tilde{T}^p_\infty(\phi) = \Bigg\{f \ \ \text{is measurable in} \ \ \mathbb{R}^n : \big\|\big(A_\infty f\big)(x)\big(\phi(x)\big)^{\frac{1}{p}}\big\|_{L^p(\mathbb{R}^n)} < \infty\Bigg\},$$

$$\tilde{T}^p_q(\omega) = \Bigg\{f \ \ \text{is measurable in} \ \ \mathbb{R}^n : $$
$$\int_{\mathbb{R}^n} \bigg(\int_{\Gamma(\xi)} \frac{|f(y,t)|^q}{t^{n+1}} dy \ dt\bigg)^{\frac{p}{q}} \omega(\xi) d\xi < \infty, \quad 0 < p, q < \infty\Bigg\},$$

where ω *is a locally integrable function,* $\omega \in L^1_{loc}(\mathbb{R}^n)$.

Remark 2. Theorem 0.2 for $\omega = const$ is known and was proved in [3] and [4].

Remark 3. Note that many known spaces of holomorphic functions can be represented by $T^p_q, \ \ 0 < p, q < \infty,$ spaces. So such factorization are very useful in different problems, connected with the theory of spaces of analytic functions [4, 5].

We are going to formulate two theorems in similar direction. The proof relies on the existence of extremal outer function, that was constructed by Dynkin in [6].

Let

$$\big(F^{\infty,q}_{s,p,k}\big) = \big\{f \in H(\mathbb{D}) : |\tilde{\mathbb{D}}^k f(z)|^q \big(1-|z|\big)^{(k-s)q-1} - p \ \ \text{is Carleson measure} \ \ \big\},$$

where $k \in \mathbb{R}, \ \ k > s, \ \ s \in \mathbb{R}, \ \ q \in (0, \infty)$,

$$\big(\tilde{\mathbb{D}}^\alpha f\big)(z) = \sum_{k \geq 0} (k+1)^\alpha a_k z^k, \quad \alpha \in \mathbb{R}, \qquad f(z) = \sum_{k \geq 0} a_k z^k$$

is a fractional derivate and a positive Borel measure μ in $\mathbb{D}$ is a p-Carleson measure if

$$\bigg\|\sup_{\xi \in I} \frac{1}{|I|^p} \int_{\square I} d\mu(z)\bigg\|_{L^\infty(T)} = \big\|\phi(\xi)\big\|^{\infty(T)} < \infty, \quad 0 < p \leq 1.$$

Below $L^{p,q}(T)$ are Lorentz spaces on T. In order to formulate our next theorem we need the following notation. We will write $\|f\|_X \subseteq Y \cdot Z, \ \ X, Y, Z$ are subspaces of $H(\mathbb{D})$, if any functions $f, \ \ \|f\|_X < \infty$ can be writen in the following form $f = (f_1)(f_2), \ \ f_1 \in Y, \ f_2 \in Z$.

Let

$$\Lambda^s \overset{def}{=} \left\{ f \in H(\mathbb{D}) : \sup_{|z|<1} |f'(z)| \big(1-|z|\big)^{1-s} < \infty \right\}, \quad s \in (0,1) \text{ be the Goelder space.}$$

Theorem 3. *Let Y, Z be subspaces of $H(\mathbb{D})$.*

$$\textit{Let} \ \ (Y)\Big(F^{\infty,q}_{-\frac{s}{q},1-q,k}\Big) \subset Z, \quad q \in (0,1), \ \ s \in (0,1),$$

then $Y \subset (\Lambda^s)(Z)$.

Theorem 4.

(*i*) *Let* $s>0$, $q>1$, $f \in H(\mathbb{D})$, $\frac{1}{q}+\frac{1}{q'}=1$. *Then*

$$\left\| \left(\int_{\Gamma_\alpha(\xi)} \Big|f(z)\Big|^{2q'} \big(1-|z|\big)^{s-2)} dm_2(z) \right)^{\frac{1}{2}} \right\|_{L^{q,1}} \subseteq (\Lambda^s)\Big(\tilde{S}^{2(q')^{-1},2(q')^{-1}}_{\frac{sq'}{2}}\Big), \quad s<1.$$

(*ii*) *Let* $v>0$, $q>1$, $f \in H(\mathbb{D})$, $t \geq 0$ *and* $v-t=s \in (0,1)$. *Then*

$$\left\| \sup_{z \in \Gamma_\alpha(\xi)} |f(z)|^{q'} \big(1-|z|\big)^t \right\|_{L^{q,1}} \subseteq (\Lambda^s)\Big(\tilde{S}^{(q')^{-1},(q')^{-1}}_{vq'}\Big),$$

where

$$\tilde{S}^{p,q}_s = \left\{ f \in H(\mathbb{D}) : \int_0^1 \big(M_p(f,|z|)\big)^q \big(1-|z|\big)^{sq-1} d|z| < \infty \right\}, \quad p,q,s, \in (0,\infty).$$

The proofs of these theorems will be presented elsewhere. Here we indicate that some ideas from [3] are being used.

References

[1] Gorowitz Ch. *Factorization theorems for functions in the Bergman space.* Duke Math. Journal, 1977, **44**, p. 201–213.

[2] Cohn W. *A factorization theorem for the derivative of a function on Hp.* Proc. AMS, 1999, **127**, N 2, p. 509–517.

[3] Cohn W., Verbitsky I. *Factorization of Tent spaces and Hankel operators.* Journal of Functional Analysis, 2000, **175**, p. 308–329.

[4] Coifman R.., Meyer Y., Stein E. *Some new functional spaces and their application to harmonic analysis.* Journal of Functional Analysis, 1985, p. 304–335.

[5] Grafakos L. *Classical and Modern Fourier Analysis.* Prentice Hall, 2003.

[6] Dynkin E. *Sets of Free interpolation for Hoelder classes.* Mat. Sbornik, 1979, **109**, N 1, p. 107–128.

Bryansk State University
241050 Bryansk, Russia
E-mail: *rsham@mail.ru*

Received June 19, 2006